Bounds for the state-modulated resolvent of a linear Boltzmann generator

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Abstract
We study a generalized resolvent for the generator of a Markovian semigroup. The Markovian generator appears in a linear Boltzmann equation modeling a one-dimensional test particle in a periodic potential and colliding elastically with particles from an ideal background gas. We obtain bounds for the state-modulated resolvent that are relevant in the regime where the mass ratio between the test particle and a particle from the gas is large. These bounds relate to the typical amount of time that the particle spends in different regions of phase space before arriving at a region around the origin.

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1. Introduction

1.1. Model and result

Denote \( \Sigma = \mathbb{T} \times \mathbb{R} \), where \( \mathbb{T} \) is the one-dimensional torus identified with the unit interval \([0, 1)\). Let \( B(\Sigma) \) be the Banach space of all bounded measurable functions on \( \Sigma \) with the supremum norm \( \| \cdot \|_\infty \). Let \( \mathcal{L}_\lambda \) be the backward Kolmogorov generator which acts on a dense domain \( D \subset B(\Sigma) \) such that for \( \Psi \in D \),

\[
(\mathcal{L}_\lambda \Psi)(x, p) = p \frac{\partial}{\partial x} \Psi(x, p) - \frac{dV}{dx}(x) \frac{\partial}{\partial p} \Psi(x, p) + \int_{\mathbb{R}} dp' J_\lambda(p, p') (\Psi(x, p') - \Psi(x, p)),
\]

(1.1)

where the function \( V : \mathbb{T} \to \mathbb{R}^+ \) is continuously differentiable, and the kernel \( J_\lambda(p, p') \) takes the form

\[
J_\lambda(p, p') = (1 + \lambda)|p - p'| e^{-\frac{1}{2}(\frac{1}{\lambda} p^2 - \frac{1}{\lambda} p'^2)}.
\]

(1.2)

The operator \( \mathcal{L}_\lambda \) generates a transition semigroup \( \Phi_{t, \lambda} : B(\Sigma) \to B(\Sigma) \). The Markovian dynamics associated with the semigroup \( \Phi_{t, \lambda} \) models a test particle of mass \( \lambda^{-1} \) in dimension 1 that feels an external, spatially periodic force \( \frac{dV}{dx}(x) \) and receives elastic collisions from a gas
reservoir of particles having mass 1. The spatial degree of freedom has been dilated by a factor \( \lambda^{-1} \), and consequently the kinetic term in (1.1) has the form \( p \frac{\partial}{\partial p} \Psi(x, p) = \frac{\partial}{\partial p} \left( \frac{1}{2} p^2 \right) \Psi(x, p) \) rather than \( \lambda p \frac{\partial}{\partial p} \Psi(x, p) = \frac{\partial}{\partial p} \left( \frac{1}{2} \lambda^2 p^2 \right) \Psi(x, p) \). Also, since our focus is on the dynamical behavior for the momentum variable, the spatial degree of freedom for the test particle has been contracted to the torus. Kernel (1.2) matches equation (8.118) from [11] when the test particle has the mass \( \lambda^{-1} \), a single particle from the gas has mass 1, the temperature of the gas is 1 and the spatial density for the gas is \( 2(2\pi)^{3/2} \).

Consider the generalized resolvent \( U_h^{(k)} \) that operates on elements in \( B(\Sigma) \) and is given formally by

\[
U_h^{(k)} = (M_h - \mathcal{L}_h)^{-1},
\]

where \( M_h : B(\Sigma) \to B(\Sigma) \) acts as multiplication by a bounded measurable function \( h : \Sigma \to \mathbb{R}^+ \). When \( h \) is a constant function, \( U_h^{(k)} \) is a standard resolvent. We will refer to \( U_h^{(k)} \) as the \( h \)-modulated resolvent or, as in [8], the state-modulated resolvent for non-specific \( h \). The operators \( U_h^{(k)} \) were introduced in [9, section 7]. For \( s \in \Sigma \) and \( f \in B(\Sigma) \), we will use the kernel notation \( U_h^{(k)}(s, f) \) to denote the value \( (U_h^{(k)} f)(s) \). We take the following expression for our definition of \( U_h^{(k)} \):

\[
U_h^{(k)}(s, f) = \mathbb{E}^{(k)}_s \left[ e^{-\int_0^t h \, ds(t)} f(S_t) \right], \quad s \in \Sigma,
\]

where \( S_t \in \Sigma \) is the Markov process associated with the backward generator \( \mathcal{L}_h \). The operator \( U_h^{(k)} \) satisfies (1.3) on an appropriate class of functions \( f \in B(\Sigma) \). For a measurable set \( A \subset \Sigma \), the value \( U_h^{(k)}(s, 1_A) \) corresponds to the expected amount of time, when starting from \( s \), that the test particle will spend in \( A \subset \Sigma \) before the expiration of an exponential random time whose rate depends on the trajectory of the particle through the function \( h \). This interpretation becomes clearer by seeing other representations of \( U_h^{(k)} \). The following theorem is used in [2] and is the main result of this paper.

**Theorem 1.1.** Let \( h : \Sigma \to \mathbb{R}^+ \) be a bounded measurable function with \( h \neq 0 \). There is a \( c > 0 \) such that for all bounded measurable functions \( f : \Sigma \to \mathbb{R}^+ \), \( \lambda < 1 \) and \( s \in \Sigma \),

\[
U_h^{(k)}(s, f) \leq c \left( \sup_{s' \in \Sigma} A^{(k)}(s, s') f(s') + \int_{\Sigma} ds' B^{(k)}(s, s') f(s') \right),
\]

where \( A^{(k)}(s, s') \) and \( B^{(k)}(s, s') \) are defined as

\[
A^{(k)}(x, p, x', p') = 1 + \min(|p|, \lambda^{-1}) \chi(|p'| \geq \lambda^{-1}),
\]

\[
B^{(k)}(x, p, x', p') = (1 + \min(|p|, |p'|)) \chi(|p| \leq \lambda^{-1}).
\]

**1.2. Discussion**

Periodic potentials acting on atoms can be generated in experimental settings using counter-propagating laser beams [1]. The interaction between the light and the atom is based on the dipole force, and the resulting periodic potentials can be designed as one, two or three dimensional. The jump rates (1.2) modeling the collisions from the gas are somewhat artificial, since the one-dimensional linear Boltzmann equation does not arise in a low-density limit from a microscopic Hamiltonian model for a test particle interacting with an ideal gas. The techniques in this paper extend to higher dimensional models with one-dimensional periodic potentials, although the notation and some arguments would become messier. If the periodic potential is also of dimension greater than 1, then a different approach is required.

In the limiting regime that we consider, the periodic force plays a smaller role in driving the momentum of the test particle in comparison to the kicks arising from collisions with
the background gas. Without the potential, the test particle’s momentum in the large mass limit behaves roughly as a random walk with a momentum-dependent bias toward the origin. The motivation here is to gain some control on the perturbative contribution to the test particle’s transport induced by the periodic potential. At the typical speeds for the test particle associated with being at equilibrium with the gas reservoir, the kinetic energy outweighs the potential energy, and the test particle passes quickly through the period cells between collisions. The influence from the potential forcing is consequently diminished due to self-cancellation. However, the fluctuations in momentum due to the potential forcing can accumulate over time periods when the test particle travels with lower speeds such that the kinetic and potential components of the energy have roughly the same scale. The contribution of the periodic potential is thus related to the amount of time that the random walk in momentum spends in the region around the origin. The generalized resolvent \( U(\lambda) \) for \( f : \Sigma \to \mathbb{R}^+ \) is part of the technical apparatus for gaining control (e.g. through moment bounds) of integrals of the process \( f(S_t) \) over excursion time periods away from the low-energy region of phase space. In [2, lemma 4.10], the results for the state-modulated resolvent in theorem 1.1 are applied to bounding the predictable quadratic variations for martingales approximating the cumulative drift in momentum generated by the potential forcing: \( \int_0^t dr \frac{dV}{dt}(X_r) \). Thus, treating the generalized resolvent helps in bounding the typical momentum fluctuations resulting from the periodic force.

The operator \( U_h^{(\lambda)} \) appears in the literature on Harris recurrence for Markov processes [9, 8, 5], on limit theorems for null-recurrent Markov processes [12, 4, 7], and on Nummelin splitting for Markov processes [6]. We discuss some alternative representations for \( U_h^{(\lambda)} f \) in section 2.

The dynamics described by (1.1) includes a deterministic part driven by the Hamiltonian \( H = \frac{1}{2}p^2 + V(x) \) and a noisy part determined by the jump kernel \( J_\lambda \). In the statement of theorem 1.1, the potential \( V(x) \) does not have a role in determining the kernels \( A^{(\lambda)} \) and \( B^{(\lambda)} \). This is because the inequality in theorem 1.1 is mainly concerned with bounding \( U_h^{(\lambda)}(x, p, f) \) when \( |p| \gg 1 \), and the influence of the force \( \frac{dV}{dx}(X_t) \) is averaged out when \( |P_t| \gg 1 + \sup_{x} V(x) \) as the particle revolves with high frequency around the torus. In whichever direction the test particle travels, collisions with the gas will, in an average sense, diminish the particle’s movement in that direction. For \( \lambda \ll 1 \), this frictional effect takes on different characteristics depending on the scale of the momentum. The following list characterizes the influence of collisions at different momentum scales relative to \( \lambda^{-1} \).

**List 1.2.**

1. **Contractive regime.** When \( |p| \gg \lambda^{-1} \), the momentum undergoes a super-exponential contraction in which the collisions occur with exponential rate \( \approx \lambda |p| \), and the result of a collision contracts a momentum \( p \) to a value in the vicinity of \( \frac{1}{1+\lambda} p \).

2. **Drift regime.** When \( |p| \) is of the order of \( \lambda^{-1} \), then the collisions occur of the order of 1 per unit time and the resulting momentum due to a collision has a bias in the direction of momentum 0. The bias has the same order as the standard deviation of the momentum jump.

3. **Random walk regime.** When \( |p| \ll \lambda^{-1} \), the collisions generate a nearly unbiased random walk in momentum with Lévy density

\[
    j(q) = |q| e^{-q^2 / 2}.
\]

Over time periods of length \( \gg 1 \), the drift toward momentum 0 is visible.
Besides the multi-scale behavior for the jump rates (1.2) emerging for \( \lambda \ll 1 \), the main source of technical difficulty for proving theorem 1.1 is the perturbation of the dynamics due to the presence of the potential \( V(x) \). Since the potential is bounded, the particle revolves quickly around the torus at high energies. This gives rise to an effective homogenized behavior at high energy which is quasi one dimensional. The homogenized dynamics can be formulated as a Markov process having states that can be identified with connected components of the level curves for the Hamiltonian \( H = \frac{1}{2}p^2 + V(x) \). We refer to the homogenized process as the Freidlin–Wentzell process, since it is similar to processes that arise in Freidlin–Wentzell limits [3], although it is not of a diffusive form. The Freidlin–Wentzell process is more tractable, since there is no drift between collisions and the torus degree of freedom is replaced by a finite labeling. Our strategy for handling the potential is to prove an analog of theorem 1.1 for the corresponding Freidlin–Wentzell process (see lemma 3.5), and then to show in lemma 4.1 that the state-modulated resolvent for the original process satisfies the same integral equation as the state-modulated resolvent for the Freidlin–Wentzell process except for an error that can be controlled. Although the behavior of the original dynamics and the homogenized dynamics diverges at low energy, the cumulative effect of the divergence for the state-modulated resolvent still conforms to the bounds that we consider in theorem 1.1.

The bounds in theorem 1.1 are not optimal. Our analysis does not take advantage of the central drift described in (2) of list 1.2, which should allow for tighter bounds. There are also smaller kernels than \( A^{(\lambda)} \) and \( B^{(\lambda)} \) possible over the domain of \((x, p, x', p')\) in which \( p \) and \( p' \) have opposite signs\(^1\), although we are not interested in this here. Finally, by a slightly different analysis for the contractive regime (3) of list 1.2, the kernel \( A^{(\lambda)} \) can be replaced by the kernel \( A^{(\lambda),'} \) defined as

\[
A^{(\lambda),'}(x, p, x', p') = \left( 1 + \min(|p|, \lambda^{-1} \log(1 + \lambda |p|)) \chi(|p'| \geq \lambda^{-1}) \right) \frac{1}{1 + \lambda |p'|},
\]

This alternative is not strictly stronger than the choice of \( A^{(\lambda)} \).

The remainder of the paper is arranged as follows: in the next section, we give a few examples of inequalities for the state-modulated resolvent of simpler processes. Section 2 contains some general remarks on state-modulated resolvents and also contains some technical preliminaries specific to our dynamics. Section 3 discusses the Freidlin–Wentzell process, and section 4 bridges the analysis of the Freidlin–Wentzell process with the original process. The proof of theorem 1.1 is placed in section 5.

1.3. Examples of inequalities for state-modulated resolvents

The bound for \( U^{(\lambda)}_h \) given in theorem 1.1 is especially complicated due to the different scales described in list 1.2. In particular, the inequality in theorem 1.1 involves two kernels \( A^{(\lambda)}(s, s') \) and \( B^{(\lambda)}(s, s') \) that are used in supremum and integral norms, respectively. The first two examples below only involve random walk behavior (i.e. (3) of list 1.2), and the kernel \( B^{(\lambda)}(s, s') \) is sufficient. Examples 1.4 and 1.5 follow by a simpler analysis than contained in section 3.

**Example 1.3.** Let \( B \) be a one-dimensional standard Brownian motion and \( f : \mathbb{R} \to \mathbb{R}^+ \) be integrable. Define the kernel \( U_h \) such that for \( p \in \mathbb{R} \),

\[
U_h(p, f) = E_p \left[ \int_0^\infty dt f(B_t) e^{-bt} \right],
\]

\(^1\) This can be understood from example 1.3.
where $\eta > 0$ and $t_0$ is the local time at zero of $B$. In other words, $U_\eta$ is the $h$-modulated resolvent of $f$, where $h$ is the $\delta$-function $h(p) = \eta \delta_0(p)$. The function $U_\eta f$ satisfies the differential equation

$$ f(p) = \eta \delta_0(p)U_\eta(p, f) - \frac{1}{2} \Delta U_\eta(p, f). $$

A closed form of the solution is given by

$$ U_\eta(p, f) = \frac{1}{\eta} \int_R f(p) dp + 2 \left\{ \int_0^\infty dq \min(q, p)f(q), \quad p \geq 0, \right\} \left\{ \int_0^\infty dq \min(q, -p)f(-q), \quad p < 0. \right\} $$

Trivially, there exists a $c > 0$ such that for $B(p, p') := 1 + \min(|p'|, |p|)$ and all $p \in \mathbb{R}$ and integrable $f : \mathbb{R} \to \mathbb{R}^+$,

$$ U_\eta(p, f) \leq c \int_{-\infty}^{\infty} dq B(p, p')f(p'). $$

**Example 1.4.** Let the backward Kolmogorov generator $\mathcal{L}$ be defined such that for $\Psi \in B(\mathbb{R})$,

$$ (\mathcal{L}\Psi)(p) = \int_{\mathbb{R}} dp' j(p' - p)(\Psi(p') - \Psi(p)), $$

where $j : \mathbb{R} \to \mathbb{R}^+$ is integrable, continuous and its first two moments satisfy $\int_{\mathbb{R}} dp p j(p) = 0$ and $\int_{\mathbb{R}} dp p^2 j(p) < \infty$. For measurable $h : \mathbb{R} \to \mathbb{R}^+$ with $h \neq 0$, there is a $c > 0$ such that for $B(p, p')$ defined as in example (1.3) and all $p \in \mathbb{R}$ and integrable $f : \mathbb{R} \to \mathbb{R}^+$,

$$ U_\eta(p, f) \leq c \int_{\mathbb{R}} dp' B(p', p)f(p'). $$

**Example 1.5.** Consider the backward Markov generator $\mathcal{L}_\lambda$ that acts on a dense domain of $B(\mathbb{R})$ as

$$ (\mathcal{L}_\lambda\Psi)(p) = \int_{\mathbb{R}} dp' J_\lambda(p, p')(\Psi(p') - \Psi(p)), $$

where $J_\lambda$ is defined as in (1.2). Let $h : \mathbb{R} \to \mathbb{R}^+$ be measurable and $h \neq 0$. There is a $c > 0$ such that for all $f \in B(\mathbb{R})$ with $f \geq 0$, $\lambda < 1$ and $p \in \mathbb{R},$

$$ U^{(\lambda)}_\eta(p, f) \leq c \left( \sup_{p' \in \Sigma} A^{(\lambda)}(p, p')f(p') + \int_\Sigma dp' B^{(\lambda)}(p, p')f(p') \right), $$

where $A^{(\lambda)}(p, p')$ and $B^{(\lambda)}(p, p')$ are defined as

$$ A^{(\lambda)}(p, p') = 1 + \min(|p|, \lambda^{-1})\chi(|p'| \geq \lambda^{-1}), $$

$$ B^{(\lambda)}(p, p') = (1 + \min(|p|, |p'|))\chi(|p'| \leq \lambda^{-1}). $$

## 2. Some basic facts for the state-modulated resolvent

Proposition 2.1 gives alternative representations for $U^{(\lambda)}_\eta(s, f)$ where (2) is from [9, section 7], (1) is from [8, section 2] and (3) is from the proof of [4, proposition 3.4] (for $h = 1$).

**Proposition 2.1.** Let $f, h \in B(\Sigma)$, where $h$ is non-negative and $h \neq 0$. Pick $h \geq \sup_{s \in \Sigma} h(s)$. 
(1) Let $R$ be the stopping time with infinitesimal exponential rate at a time $t < R$ given by $h(S_t)$, i.e. for all $t \in \mathbb{R}^+$ and $\delta \ll 1$,
\[ \mathbb{P}_\delta^{(s)}[R \in [t, t + \delta)] | R \geq t, S_t \text{ for } r \in [0, t]] = h(S_t)\delta + o(\delta). \]

The function $U_h^{(s)} f$ can be written as
\[ U_h^{(s)}(s, f) = \mathbb{E}_s^{(s)} \left[ \int_0^R \mathrm{d}r f(S_t) \right]. \]

(2) Let $M_h : B(\Sigma) \to B(\Sigma)$ be multiplication by $h'(s) = h(s) - h(S_t)$ and $U_h^{(s)}$ be the standard resolvent evaluated at $h \in \mathbb{R}^+$. The operator $U_h^{(s)} f$ can be written as
\[ U_h^{(s)} f = \sum_{n=0}^{\infty} U_h^{(s)}(M_h U_h^{(s)})^n f. \]

(3) Let $e_n$ be independent, mean-$\mathbf{h}^{-1}$ exponentials which are independent of $S_t$. For $\tau_n = \sum_{n=1}^\infty e_m$, the function $U_h^{(s)} f$ can be written as
\[ U_h^{(s)}(s, f) = \mathbb{E}_s^{(s)} \left[ \sum_{n=1}^{\infty} \left( 1 - \frac{h(S_{\tau_n})}{h} \right) \cdots \left( 1 - \frac{h(S_{\tau_{n-1}})}{h} \right) f(S_{\tau_n}) \right]. \]

(4) If coins with head weight $\frac{h(S_{\tau_n})}{h}$ are flipped at every time $\tau_n$ and $\bar{n} \in \mathbb{N}$ is the count of the first head, then $U_h^{(s)} f$ can be written as
\[ U_h^{(s)}(s, f) = \mathbb{E}_s^{(s)} \left[ \sum_{n=1}^{\bar{n}} f(S_{\tau_n}) \right]. \]

The following lemma is specific to our dynamics, and part (3) implies that it is sufficient for us to prove theorem 1.1 for a function $h : \Sigma \to \mathbb{R}^+$ of our choice as long as it has compact support. In later sections of this paper, we will always take
\[ h(s) = \chi(H(s) \leq l), \tag{2.1} \]
where $l = 1 + 2 \sup V(x)$. We pick $l$ primarily to ensure that the particle is not trapped by the potential when $H(S_t) > l$ and revolves around the torus with speed $> 1$ over the time period up to the next collision.

For the proof of (1) from lemma 2.2 below, we use that our dynamics is exponentially ergodic, which was proven in [2, theorem A.1]. Let $T_{\lambda, \mathbf{k}} : B(\Sigma) \to B(\Sigma)$ be the operator
\[ T_{\lambda, \mathbf{k}} f := \mathbf{k} \int_0^\infty \mathrm{d}r e^{-(\lambda + \mathbf{k})r} f(U_h^{(s)}). \]

The kernel $T_{\lambda, \mathbf{k}}(s, ds')$ is the transition kernel for the resolvent chain $S_n$ from (3) of proposition 2.1. To prove (2) and (3) of lemma 2.2, we use that there is a $c_{L,\mathbf{k}} > 0$ such that the forward transition operator $T_{\lambda, \mathbf{k}}$ satisfies
\[ T_{\lambda, \mathbf{k}}(s, ds') \geq c_{L, \mathbf{k}} ds' \tag{2.2} \]
for all $s, s' \in \Sigma$ with $H(s), H(s') \leq L$ and all $\lambda < 1$. This was shown in the proof of [2, proposition 4.3].

In the proof of [2, theorem A.1], $\mathbf{k} = 1$ and $L = 1 + 2 \sup V(x)$, although this does not matter for the argument.
Lemma 2.2. Let \( h, h', f \in B(\Sigma) \) be non-negative and \( h, h' \neq 0 \).

1. The kernel \( U_h^{(1)} \) defines a bounded map on \( B(\Sigma) \) (i.e., with respect to the supremum norm).
2. Let \( h \) be as in (2.1) and pick \( L > 0 \). There is a \( c_L > 0 \) such that for all \( f \) and \( \lambda < 1 \),
\[
\left| \int_{H(\lambda) \leq L} ds u_h^{(1)}(s, f) \right| \leq c_L \int_\Sigma ds f(s) e^{-\lambda H(s)} \leq 2c_L \sup_{H(\lambda) > \frac{1}{\lambda} - 2} f(s) + c_L \int_{H(\lambda) \leq \frac{1}{\lambda} - 2} ds f(s).
\]

3. Suppose that \( h' \) has compact support. There are \( c, L > 0 \) such that for all \( f \) and \( \lambda < 1 \),
\[
U_h^{(1)} f \leq c \sup_{H(\lambda) \leq L} U_h^{(1)}(s, f) + U_h^{(1)} f.
\]

Proof.

Part (1). Recall that \( T_{\lambda, \frac{1}{h}} = hU_h^{(1)} \). By part (2) of proposition 2.1, we can write
\[
U_h^{(1)} = \frac{1}{h} \sum_{n=0}^{\infty} T_{\lambda, \frac{1}{h}} (M_{\lambda} \otimes T_{\lambda, \frac{1}{h}})^n.
\]

Let \( \Phi_{\lambda, t} : B(\Sigma) \rightarrow B(\Sigma) \) be the transition semigroup associated with the backward Kolmogorov equation (1.1). By [2, theorem A.1], \( \Phi_{\lambda, t} \) converges exponentially in the operator norm to the equilibrium projection \( P_\lambda = 1_\Sigma \otimes \Psi_{\infty, \lambda} \) as \( t \rightarrow \infty \). It follows that the operators \( T_{\lambda, \frac{1}{h}} ^n \) also converge exponentially to \( P_\lambda \) for large \( n \). In other terms, there are \( C, \alpha > 0 \) such that for all \( f \in B(\Sigma) \),
\[
\| T_{\lambda, \frac{1}{h}} ^n f - P_\lambda f \|_\infty = \| T_{\lambda, \frac{1}{h}} ^n f - (\Psi_{\infty, \lambda}(f)) 1_\Sigma \|_\infty \leq C e^{-\alpha n} \| f \|_\infty.
\]

Pick an \( N > 0 \) such that \( C e^{-\alpha N} \leq \frac{1}{2} \Psi_{\infty, \lambda}(h) \). Using the form (2.3) and that \( M_{\lambda} \) is a positive multiplication operator with norm \( \leq 1 \), we obtain the first inequality below:
\[
\| U_h^{(1)} f \|_\infty \leq \frac{N}{h} \| f \|_\infty \sum_{n=0}^{\infty} \left( T_{\lambda, \frac{1}{h}} ^n (M_{\lambda} \otimes T_{\lambda, \frac{1}{h}}) \right)^n \leq \frac{N}{h} \| f \|_\infty \sum_{n=0}^{\infty} \left| \frac{T_{\lambda, \frac{1}{h}} ^n h - h}{h} \right| ^n \leq \frac{2N \| f \|_\infty}{\Psi_{\infty, \lambda}(h)} \tag{2.4}
\]

where \( \| \cdot \| \) denotes the operator norm. The third inequality above uses that
\[
0 \leq \left| \frac{T_{\lambda, \frac{1}{h}} ^n h - h}{h} \right| _\infty \leq 1 - \frac{1}{h} \| P_\lambda h \|_\infty + \frac{1}{2h} \Psi_{\infty, \lambda}(h) = 1 - \frac{1}{2h} \Psi_{\infty, \lambda}(h).
\]

Thus, \( U_h^{(1)} = \frac{1}{h} \sum_{n=0}^{\infty} (T_{\lambda, \frac{1}{h}} ^n M_{\lambda} \otimes T_{\lambda, \frac{1}{h}}) \) is a bounded operator.

Part (2). Let \( \nu(ds) = \frac{dr}{r} \delta_1 \) be the normalization of the Lebesgue measure over the set \( A = \{ s \in \Sigma \mid H(s) \leq L \} \) and \( h' : \Sigma \rightarrow \mathbb{R}^+ \) be the function \( h' = (f_A ds)c_{L,2} 1_A \). By (2.2), the transition kernel for \( T_{\lambda, \frac{1}{h}} \) satisfies
\[
T_{\lambda, \frac{1}{h}} (s, ds') \geq h'(s') \nu(ds') \tag{2.5}
\]
for all \( s, s' \in \Sigma \) with \( H(s), H(s') \leq L \). By (2.5),
\[
M_{\lambda, \frac{1}{h}} = M_{1, \frac{1}{h}} M_{\sqrt{1-\frac{1}{\lambda}}, \frac{1}{h}} \leq M_{1, \sqrt{1-\frac{1}{\lambda}}} T_{\lambda, \frac{1}{h}} \leq M_{1, \sqrt{1-\frac{1}{\lambda}} - \frac{1}{10} h'} \otimes \nu.
\]
since we have \((1 - \frac{1 - \frac{h}{2}}{2})^2 \leq \frac{1}{10}\) when \(h \leq 1\). Using (2.3) and the fact that \(1 - \frac{h}{2} \geq \frac{1}{2}\),

\[ U_h^{(k)} \leq 2M \sqrt{1 - \frac{h}{2}} U_h^{(k)} M \sqrt{1 - \frac{h}{2}} = 2 \sum_{n=0}^{\infty} \left( M \sqrt{1 - \frac{h}{2}} T_{\lambda, \frac{h}{2}} M \sqrt{1 - \frac{h}{2}} \right)^2 \leq 2 \sum_{n=0}^{\infty} \left( T_{\lambda, \frac{h}{2}} - \frac{1}{10} h' \otimes v \right)^n. \]

Thus, the inequality below holds:

\[ \nu U_h^{(k)} f \leq 2v \sum_{n=0}^{\infty} \left( T_{\lambda, \frac{h}{2}} - \frac{1}{10} h' \otimes v \right)^n f \]

\[ = 20 \int_\Sigma \frac{d\nu}{\Sigma} \psi_{\infty, \lambda}(s) f(s) \]

\[ = 20 \int_\Sigma \frac{d\nu}{\Sigma} \psi_{\infty, \lambda}(s) h(s) = 20 \int_\Sigma \frac{d\nu}{\Sigma} \psi_{\infty, \lambda}(s) e^{-J(s)} f(s). \quad (2.6) \]

The first equality is an identity from [10, theorem 3]. However,

\[ \nu U_h^{(k)} f \geq \frac{1}{2} \nu T_{\lambda, \frac{h}{2}} U_h^{(k)} f \geq \frac{cL2}{2} \int_{H \leq L} ds U_h^{(k)} (s, f), \]

where the second inequality is by (2.2). The first inequality in (2.7) follows by the relations

\[ U_h^{(k)} = T_{\lambda, \frac{h}{2}} + T_{\lambda, \frac{h}{2}} M_{1-\frac{h}{2}} U_h^{(k)} \geq \frac{1}{2} T_{\lambda, \frac{h}{2}} U_h^{(k)} \]

where the equality is equivalent to part (2) of proposition 2.1 and the inequality is from \(1 - \frac{h}{2} \geq \frac{1}{2}\). Combining inequalities (2.6) and (2.7) gives that \(U_h^{(k)} f\) is bounded by a multiple of \(\int_\Sigma ds \psi_{\infty, \lambda}(s) e^{-J(s)} f(s)\).

Finally,

\[ \int_\Sigma ds \psi_{\infty, \lambda}(s) f(s) \leq \left( \sup_{H > \frac{1}{2} \lambda^{-2}} f(s) \right) \int_{H > \frac{1}{2} \lambda^{-2}} ds e^{-J(s)} + \int_{H \leq \frac{1}{2} \lambda^{-2}} ds f(s), \]

\[ \leq 2 \sup_{H > \frac{1}{2} \lambda^{-2}} f(s) + \int_{H \leq \frac{1}{2} \lambda^{-2}} ds f(s), \]

where we have split the integration into the domains \(H' > \frac{1}{2} \lambda^{-2}\) and \(H' \leq \frac{1}{2} \lambda^{-2}\), and the second inequality is for \(\lambda\) small enough.

**Part (3).** Since \(U_h^{(k)} f \leq U_h^{(k)} f\) when \(h_1 \leq h_2\), we can assume without loss of generality that \(h\) also has compact support. For the same reason, we can take \(h'\) to be of the form \(h' = h1_A\) for some \(h > 0\) and where \(A = \{s \mid H(s) \leq L\}\) for some \(L > 0\).

Define the kernels \(\partial_s, \partial_{sl} : \Sigma \to \Sigma\) such that

\[ \partial_{s \lambda} = \sum_{n=0}^{\infty} (T_{\lambda, \frac{h}{2}} M_{1-\frac{h}{2}})^n T_{\lambda, \frac{h}{2}} M_{1-\frac{h}{2}}, \]

\[ \partial_{s l} = \sum_{n=1}^{\infty} \partial_{s l} \left( M_{1-\frac{h}{2}} \partial_{s l} \right)^n. \]

For each \(s \in \Sigma\), \(\partial_{s l}(s, ds')\) is a probability measure supported on \(A \subset \Sigma\), and \(\partial_{s l}(s, ds')\) is a measure with total weight bounded by

\[ \sup_{s \in \Sigma} \partial_{s l}(s, \Sigma) \leq \sum_{n=1}^{\infty} \sup_{s \in \Sigma} \partial_{s l}(s, \Sigma) \leq 2 + \sum_{n=1}^{\infty} \left( \sup_{s \in A} \partial_{s l} M_{1-\frac{h}{2}}(s, \Sigma) \right)^n \]

\[ \leq 2 + \sum_{n=1}^{\infty} \left( 1 - cLh \int_A ds \frac{h(s')}{h} \right)^n \leq 2 + \frac{h}{cLh} \int_A ds' h(s') := C. \quad (2.8) \]

where the third inequality uses remark (2.2). The second inequality above follows since \(\partial_{s l}(s, ds')\) is supported in \(A\).
The resolvent $U_h^{(k)}$ can be written in terms of $U_h^{(1)}$ and $q_h$ as
\[
U_h^{(k)}(s, f) = \int_{\Sigma} q_h(s, ds') U_h^{(k)}(s', f) + U_h^{(1)}(s, f) \\
\leq \left( \sup_{r \in \Sigma} \int_{\Sigma} q_h(s', ds') \right) \sup_{s \in \Lambda} U_h^{(1)}(s', f) + U_h^{(1)}(s, f) \\
\leq C \sup_{r \in \Lambda} U_h^{(1)}(s', f) + U_h^{(1)}(s, f).
\]

The second inequality is by (2.8).

3. The Freidlin–Wentzell dynamics

We will now define a homogenized dynamics for which there is no deterministic evolution between jumps in phase space due to collisions. The homogenized dynamics behaves similarly to the original dynamics, except that its state-modulated resolvent is more analytically tractable.

For the original dynamics, between collisions, the particle follows an orbit over a connected segment of a level curve of the Hamiltonian $H = \frac{1}{2}p^2 + V(x)$. If the particle starts at $(x, p) \in \Sigma$ with $|p| \gg 1$, then the particle will likely pass through over that curve of the order of $C(1 + \lambda |p|)$ times before the next collision, since the escape rates satisfy $E_\nu(p) \leq C(1 + \lambda |p|)$ for some $C > 0$ and all $\lambda < 1$ and $p \in \mathbb{R}$. This is suggestive of a Freidlin–Wentzell limit [3] in which a Markovian process emerges on the set of connected level curves of a Hamiltonian for a dynamics driven by a Hamiltonian evolution perturbed by a comparatively slow-acting noise (which the authors take to be a white noise). Since $V(x)$ is bounded, the level curves of $H(x, p)$ are almost flat when $|p| \gg 1$ (and thus essentially like those of $H(x, p) = \frac{1}{2}p^2$ for large energies). We do not discuss Freidlin–Wentzell limits further, and we proceed with defining the formalism relevant for us.

First, we define a state space $\Gamma_V$ identified with the set of connected components of level curves of $H(x, p)$ determined by the potential $V : T \to \mathbb{R}^+$. We define $\Gamma_V$ as the image of a map $\Sigma \to \mathbb{R}^+ \times \mathbb{Z}$ given by $G_V(s) = (2^s H^2(s), n(s))$ in which the component $n(s) \in \mathbb{Z}$ is a labeling of the connected components of the level curves corresponding to the energy $H(s)$. When the particle has the energy $H(s) > \sup V(x)$, the Hamiltonian evolution drives the particle to revolve around the torus in one direction or another. For $s \in \Sigma$ with $H(s) > \sup V(x)$, we make the convention that the associated level curves are labeled with $\pm 1$ depending on the sign of $p$, and the remainder of the labeling at lower energies is arbitrary.

**Definition 3.1.**

1. We place a measure on $\Gamma_V \subset \mathbb{R}^+ \times \mathbb{Z}$ through the Lebesgue measure on the preimage in $\Sigma$ of the map $s \to G_V(s) = (2^s H^2(s), n(s)) \in \Gamma_V$. We refer to this measure by $d\gamma$, where the dummy variable $\gamma$ is identified as an element in $\Gamma_V$.

2. For $\gamma \in \Gamma_V$, we define the probability measure $\eta_\gamma$ on $\Sigma$ as the normalization of Lebesgue measure over the preimage of $G_V^{-1}(\gamma)$. Also, we define the probability measure $\kappa_\gamma$ as
\[
\kappa_\gamma(ds) = \frac{\eta_\gamma(ds)}{\int_{\Sigma} \eta_\gamma(ds') \left( |p'|^2 + \left| \frac{dV}{dt}(x') \right|^2 \right)^{-\frac{1}{2}}},
\]
where $s = (x, p)$ and $s' = (x', p')$.

3. Define the map $B(\Sigma) \to B(\Gamma_V)$ sending $f : \Sigma \to \mathbb{R}^+$ to $\hat{f} : \Gamma_V \to \mathbb{R}^+$ given by
\[
\hat{f}(\gamma) = \int_{\Sigma} \eta_\gamma(ds) f(s).
\]
(4) We define the jump kernel \( \tilde{\mathcal{J}}_h : \Gamma_V \to \mathbb{R}^+ \) as
\[
\tilde{\mathcal{J}}_h(y, y') = \int_\Sigma \kappa_y(dx \, dp) \eta_y(dx' \, dp') \delta_0(x - x') \mathcal{J}_h(p, p').
\] (3.1)

We will sometimes use \( \hat{f} \) to denote an arbitrary element of \( B(\Gamma_V) \) without reference to a specific preimage \( f \). The kernel \( \tilde{\mathcal{J}}_h(y, y') \) defines a Markov process \( \Gamma_t \in \Gamma_V \) that has the same essential features as described in list 1.2. For this comparison, the value \( q(y) = \epsilon \rho_l \rho_\epsilon \rho \sqrt{\gamma} \) can be identified as the momentum of the element \( y = (\rho, \epsilon) \in \Gamma_V \). For \( \tilde{h}, \tilde{f} \in B(\Gamma_V) \) with \( \tilde{h} \) non-negative and \( \tilde{h} \neq 0 \), we define the kernel \( \tilde{U}^{(h)}_{\tilde{h}}(y, \tilde{f}) \) as
\[
\tilde{U}^{(h)}_{\tilde{h}}(y, \tilde{f}) = E^{(h)}_y \left[ \int_0^\infty \gamma dt \gamma \mathcal{G}_t(\tilde{f}) e^{-\int_0^t \gamma dt \gamma} \right].
\]
The analogous statements of section 2 all hold for \( \tilde{U}^{(h)}_h \). With \( \gamma : \Sigma \to \mathbb{R}^+ \) defined as in (2.1), we define \( \gamma(h, \epsilon) = \chi(\rho \leq \sqrt{2l}) \). In future, we will drop the subscript from \( \tilde{U}^{(h)}_h \) and take the form of \( \gamma \) as above.

**Remark 3.2.**

1. We can recover the Lebesgue measure from the \( \eta_y \)s as the integral
\[
ds = \int_{\Gamma_V} d\gamma \eta_y(ds).
\]
2. The kernel \( \tilde{\mathcal{J}}_h(y, y') \) can be written as
\[
\tilde{\mathcal{J}}_h(y, y') = \int_\Sigma \kappa_y(dx \, dp) \times \sum_{p' \epsilon \sqrt{|p'|^2 - 2V(x)}} \chi((x, p') \in G^{-1}_V(y')) \mathcal{J}_h(p, p') \left( \frac{|p'|^2 + |\gamma(x)|^2}{|p'|^2} \right)^{\frac{1}{2}}.
\]
3. Let \( I_y \subseteq T \) be the range of the torus component of the set \( G^{-1}_V(y) \). For \( x' \in I_y \),
\[
\int_\Sigma \kappa_y(dx \, dp) \delta_0(x - x') = \frac{\rho^2 - 2V(x')}{(\rho^2 - 2V(x'))^2}.
\]

When \( \rho > 2 \sup_x V(x) \), \( I_y = T \).

For facility, we list some notations below.

- \( y = (\rho, \epsilon) \) state in \( \Gamma_V \subset \mathbb{R}^+ \times \mathbb{Z} \)
- \( G_V(x) = y(s) \) state in \( \Gamma_V \) associated with the element \( s \in \Sigma \)
- \( q(\gamma) \) the quasi-momentum: \( q(\rho, \epsilon) = \rho \in 1_{\rho \leq \sqrt{2l}} \)
- \( g_\gamma = (r_\rho, e_\rho) \) skeleton chain for the Freidlin–Wentzell process
- \( \tilde{\mathcal{J}}_h(y, y') \) jump kernel for the Freidlin–Wentzell process
- \( \tilde{E}_h(y) \) escape rates for the Freidlin–Wentzell process
- \( \tilde{\mathcal{T}}_h(y, y') \) transition kernel for the skeleton chain
- \( U^{(h)} f \) the \( h \)-modulated resolvent of \( f \in B(\Sigma) \) for the original process
- \( \tilde{U}^{(h)} \tilde{f} \) the \( \tilde{h} \)-modulated resolvent of \( \tilde{f} \in B(\Gamma_V) \) for the Freidlin–Wentzell process.

The skeleton chain for the Freidlin–Wentzell process is the sequence of states at collision times and has the transition kernel \( \tilde{\mathcal{T}}_h(y, y') = \frac{\tilde{\mathcal{J}}_h(y, y')}{\tilde{E}_h(y')} \).
Proposition 3.3 lists some characteristics of the jump rates for the original process and Freidlin–Wentzell process that we will use. The proof relies on elementary techniques, and we do not include it here. Parts (1) and (2) of proposition 3.3 give bounds for the rate of collisions, and parts (3) and (4) are consequences of the contractive nature of the jump rates at high momentum.

**Proposition 3.3.** There are \( c, C > 0 \) such that the following hold:

1. For all \( p \in \mathbb{R} \),
   \[
   c \max(1, \lambda |p|) \leq \mathcal{E}_r(p) \leq C(1 + \lambda |p|).
   \]
2. For all \( (\rho, \epsilon) \in \Gamma_V \),
   \[
   c \max(1, \lambda \rho) \leq \widehat{\mathcal{E}}_r(\rho, \epsilon) \leq C(1 + \lambda \rho).
   \]
3. Let \( W : \Sigma \to [0, 1] \) be defined as
   \[
   W(s) = \frac{\tilde{W}(s)}{1 + H^2(s)}. \]
   For all \( \lambda < 1 \) and \( s = (x, p) \) with \( |p| > \lambda^{-1} \),
   \[
   \int_{\Sigma} \text{d}' f_\lambda(s, s')(W(s) - W(s')) \geq c \lambda.
   \]
4. Let \( W_r : \Gamma_V \to \mathbb{R}^+ \) be defined as \( W_r(\rho, \epsilon) = \lambda^{-1} \log(1 + \lambda \rho) \). For all \( \lambda < 1 \) and \( \gamma = (\rho, \epsilon) \) with \( \rho > \lambda^{-1} \),
   \[
   \int_{\Gamma_V} \text{d}y' \tilde{T}_\gamma(\gamma', \gamma')(W_r(\gamma) - W_r(\gamma')) \geq c.
   \]

In the lemma below, we set \( l = 1 + 2 \sup_x V(x) \) as in the definition of \( h(2.1) \).

**Lemma 3.4.** Let \( g_n = (r_n, e_n) \in \Gamma_V \) be the skeleton chain for the Freidlin–Wentzell process starting from \( \gamma = (\rho, \epsilon) \). Also, let \( \tilde{N} \) be the hitting time that \( r_n \) jumps below \( \rho - 1 \).

1. There is a \( C > 0 \) such that for all \( \gamma \) with \( \rho > \sqrt{2}l \) and all non-negative \( \tilde{f} \in B(\Gamma_V) \),
   \[
   \mathbb{E}^{(\lambda)}_\gamma \left[ \sum_{n=1}^{\tilde{N}-1} \tilde{f}(g_n) \right] \leq C \sup_{\rho' > \lambda^{-1}} \tilde{f}(\gamma') + \int_{\rho' < \rho' < \lambda^{-1}} \text{d}y' \tilde{f}(\gamma').
   \]
2. Define the density \( F^{(\lambda)}_\gamma(\gamma') := \mathbb{E}^{(\lambda)}_\gamma[\delta(g_{\tilde{N}} - \gamma')] \). There is a \( C > 0 \) such that for all \( \lambda < 1 \) and \( \gamma \) with \( \sqrt{2}l \leq \rho \leq \lambda^{-1} \),
   \[
   F^{(\lambda)}_\gamma(\gamma') \leq C e^{-\frac{2}{\lambda^2}} \chi(\rho' \leq \rho - 1),
   \]
   where \( \gamma' = (\rho', \epsilon) \).
3. For \( \gamma \) with \( \rho > \lambda^{-1} \), let \( N \) be the hitting time that \( r_n \) jumps below \( \lambda^{-1} \). There is a \( C > 0 \) such that for all \( \gamma, \lambda < 1 \) and non-negative \( \tilde{f} \in B(\Gamma_V) \),
   \[
   \mathbb{E}^{(\lambda)}_\gamma \left[ \sum_{n=0}^{N-1} \frac{\tilde{f}(g_n)}{\mathcal{E}_r(g_n)} \right] \leq C \lambda^{-1} \log(1 + \lambda \rho) \sup_{\rho > \lambda^{-1}} \frac{\tilde{f}(\gamma)}{\mathcal{E}_r(\gamma)}.
   \]
4. Pick \( L > 0 \): there is a \( C_L > 0 \) such that for all \( \lambda < 1 \) and \( \tilde{f} \in B(\Gamma_V) \),
   \[
   \int_{\rho' \leq \sqrt{2}l} \text{d}y' U^{(\lambda)}_\gamma(\gamma', \tilde{f}) \leq C_L \int_{\Gamma_V} \text{d}y' e^{-\frac{2}{\lambda^2}(\rho')^2} \tilde{f}(\gamma').
   \]
Proof.

Part (1). Define the measure $\mu^{(t)}_y$ on $\Gamma_V$ such that for $\hat{f} \in B(\Gamma_V)$,

$$
\mathbb{E}^{(t)}_y \left[ \sum_{n=1}^{N-1} \hat{f}(g_n) \right] = \sum_{n=1}^{\infty} \int_{\rho - 1 \leq \rho_n} \mathcal{D}y_1 \cdots \mathcal{D}y_n \hat{T}_n(g, y_1) \prod_{m=1}^{n-1} \hat{T}_m(g_m, y_{m+1}) \hat{f}(y_n)
$$

$$
\hat{f}(y') = \int_{\Gamma_V} \mathcal{D}y_1 \cdots \mathcal{D}y_n \hat{T}_n(g, y_1) \prod_{m=1}^{n-1} \hat{T}_m(g_m, y_{m+1}) \hat{f}(y_n),
$$

(3.2)

where $y_m = (\rho_m, \epsilon_m)$. The measure $\mu^{(t)}_y$ has its support on the set of $(\rho', \epsilon') \in \Gamma_V$ with $\rho' \geq \rho - 1$:

$$
\int_{\Gamma_V} \mathcal{D}y_1 \cdots \mathcal{D}y_n \hat{T}_n(g, y_1) \prod_{m=1}^{n-1} \hat{T}_m(g_m, y_{m+1}) \hat{f}(y_n)
$$

$$
\leq \int_{\Gamma_V} \mathcal{D}y_1 \cdots \mathcal{D}y_n \hat{T}_n(g, y_1) \prod_{m=1}^{n-1} \hat{T}_m(g_m, y_{m+1}) \hat{f}(y_n)
$$

$$
\leq \left( \sup_{\rho' > \lambda^{-1}} \hat{f}(y') \right) \int_{\rho' > \lambda^{-1}} \mathcal{D}y_1 \cdots \mathcal{D}y_n \hat{T}_n(g, y_1) \prod_{m=1}^{n-1} \hat{T}_m(g_m, y_{m+1}) \hat{f}(y_n)
$$

$$
+ e^\frac{1}{2} \int_{\rho' \leq \lambda^{-1}} \mathcal{D}y_1 \cdots \mathcal{D}y_n \hat{T}_n(g, y_1) \prod_{m=1}^{n-1} \hat{T}_m(g_m, y_{m+1}) \hat{f}(y_n).
$$

(3.3)

The first inequality in (3.3) is from the detailed balance-type inequality

$$
\hat{T}_n(g, y) \leq e^{\frac{1}{2} - (\rho')^2 + \rho^2} \hat{T}_n(g, y'),
$$

(3.4)

which we apply for each instance of $\hat{T}_n$. Equation (3.4) follows by the formula defining the jump rates $\hat{J}_n(g, y')$ and the following three facts: $\hat{T}_n(g, y') = \frac{\hat{J}_n(g, y')}{\hat{E}_n(y')}$ by definition of $\hat{T}_n$, $\hat{E}_n(y') \geq (1 + \lambda)\hat{E}_n(y) = (1 + \lambda)\hat{E}_n(y)$, and

$$
\hat{J}_n(g, y') = (1 + \lambda) \left| p - p' \right| e^{\frac{1}{2} - (\rho')^2 + \rho^2} e^{-\frac{1}{2} (p + p')^2 - \frac{1}{2} (p - p')^2}
$$

$$
\leq (1 + \lambda) e^{\frac{1}{2} - (\rho')^2 + \rho^2} \hat{J}_n(g, y').
$$

The second inequality in (3.3) is Holder’s for the domain $\rho' > \lambda^{-1}$, and for the domain $\rho' \leq \lambda^{-1}$, we use that

$$
e^{\frac{1}{2} - (\rho')^2 + \rho^2} \leq e^\frac{1}{2},
$$

since $\mu^{(t)}_y(y')$ has support over $y'$ with $\rho' \geq \rho - 1$ and $\rho \leq \lambda^{-1}$.

However, we claim that there is a $c > 0$ such that for all $y$ with $\rho > \sqrt{M}$,

$$
\mu^{(t)}_y(dy') \leq c \chi (\rho' > \rho - 1) dy'.
$$

(3.5)

Let us assume this now and return to it at the end of the proof. Plugging (3.5) into (3.3) gives the first inequality below,

$$
\mathbb{E}^{(t)}_y \left[ \sum_{n=1}^{N-1} \hat{f}(g_n) \right] \leq c \left( \sup_{\rho' > \lambda^{-1}} \hat{f}(y') \right) \int_{\rho' > \lambda^{-1}} dy' e^{\frac{1}{2} - (\rho')^2 + \rho^2} + c e^\frac{1}{2} \int_{\rho' \leq \rho' \leq \lambda^{-1}} dy' \hat{f}(y')
$$

$$
\leq 8c \sup_{\rho' > \lambda^{-1}} \hat{f}(y') + c e^\frac{1}{2} \int_{\rho' \leq \rho' \leq \lambda^{-1}} dy' \hat{f}(y').
$$

(3.6)

The second inequality is from

$$
\int_{\rho' > \lambda^{-1}} dy' e^{\frac{1}{2} - (\rho')^2 + \rho^2} \leq 4 e^\frac{1}{2} \int_{\rho' > \lambda^{-1}} d\rho' e^{\frac{1}{2} - (\rho')^2} \leq 4 e^\frac{1}{2} \lambda \int_{\rho' > \lambda^{-1}} d\rho' \rho' e^{-\frac{1}{2} (\rho')^2} \leq 8.
$$
In the first inequality, we have bounded \( d\nu' \leq 4d\rho' \), since the measure \( d\nu' \) is close to \( d\rho' \) for regions with \( \rho' \gg 1 \). The factor of 4 is used because there are two branches corresponding to the positive and negative momentum, and we have multiplied by an extra factor of 2 to cover the error with the dominant term.

Now we show (3.5). The measure \( \mu_\nu^{(0)} \) can be written as

\[
\mu_\nu^{(0)}(d\nu') = \chi(\rho' \geq \rho - 1)\hat{T}_0(\nu', \nu') + \int_{\rho' \geq \rho - 1} d\nu'' \hat{T}_0(\nu', \nu'') E^{(\lambda)}_{\gamma}(\sum_{n=1}^{N-1} \delta(g_n - \nu')).
\] (3.7)

Define the density \( w_\gamma(\nu') \) on \( \Gamma_\nu \) as

\[
w_\gamma(\nu') = \int_{\rho' \leq \rho - 1} d\nu'' \hat{T}_0(\nu'', \nu') \chi(\rho' \geq \rho - 1).
\]

The flat measure \( d\nu' \) is invariant with respect to the transition rates \( \hat{T}_0(\nu', \nu') \), and thus for \( \gamma' = (\rho', \epsilon') \in \Gamma_\nu \),

\[
\chi(\rho' \geq \rho - 1) = w_\nu(\nu') + \chi(\rho' \geq \rho - 1) \int_{\rho' \geq \rho - 1} d\nu'' w_\nu(\nu'') \hat{T}_0(\nu'', \nu')
\]

\[
+ \int_{\rho' \geq \rho - 1} d\nu'' w_\nu(\nu'') \int_{\rho' \geq \rho - 1} d\nu''' \hat{T}_0(\nu'', \nu''') E^{(\lambda)}_{\gamma} \left( \sum_{n=1}^{N-1} \delta(g_n - \nu') \right).
\] (3.8)

This formula treats the influx of mass jumping from the set \( \{\nu'| \rho' \leq \rho - 1\} \) as a source, and sums the expected occupation density before the mass leaves the set \( \{\nu'| \rho' > \rho - 1\} \). However, we can find a \( c \) such that

\[
\hat{T}_0(\nu', \nu') \leq c \int_{\rho' \geq \rho - 1} d\nu'' w_\nu(\nu'') \hat{T}_0(\nu'', \nu')
\] (3.9)

for all \( \nu \) with \( \rho > \sqrt{L} \) and all \( \nu' \) with \( \rho' > \rho - 1 \). If (3.9) holds, then plugging into (3.8) and throwing away the first term on the right-hand side gives

\[
\chi(\rho' \geq \rho - 1) \geq \frac{1}{c} \chi(\rho' \geq \rho - 1) \hat{T}_0(\nu', \nu')
\]

\[
+ \frac{1}{c} \int_{\rho' \geq \rho - 1} d\nu'' \hat{T}_0(\nu'', \nu'') E^{(\lambda)}_{\gamma} \left( \sum_{n=1}^{N-1} \delta(g_n - \nu') \right).
\]

We can employ this inequality in (3.7) to reach (3.5).

To see (3.9), first observe that the transition kernel \( \hat{T}_0(\nu', \nu') \) has the simpler form

\[
\hat{T}_0(\nu', \nu') = \frac{1}{8} \int_{\Sigma^2} \kappa_\nu(dx \ d\rho) \eta_{\nu'}(dx' \ d\rho') \delta_0(x - x') \mathcal{J}_0(p, p')
\]

\[
= \frac{1}{8} \int_{\mathbb{R}^2} \frac{\chi(\rho'^2 - 2V(x) + \|x\|^2)^{\frac{1}{2}}}{\rho'^2 - 2V(x)} \mathcal{J}_0(\epsilon(\rho^2 - 2V(x))^{\frac{1}{2}}, \epsilon'(|\rho'|^2 - 2V(x))^{\frac{1}{2}})
\]

\[
= \frac{1}{8} \int_{\mathbb{R}^2} \frac{\chi(\rho'^2 - 2V(x) + \|x\|^2)^{\frac{1}{2}}}{\rho'^2 - 2V(x)} \mathcal{J}_0(\epsilon(\rho^2 - 2V(x))^{\frac{1}{2}}, \epsilon'(|\rho'|^2 - 2V(x))^{\frac{1}{2}})
\]

\[
\leq \frac{1}{8} \int_{\mathbb{R}^2} \frac{\chi(\rho'^2 - 2V(x) + \|x\|^2)^{\frac{1}{2}}}{\rho'^2 - 2V(x)} \mathcal{J}_0(\epsilon(\rho^2 - 2V(x))^{\frac{1}{2}}, \epsilon'(|\rho'|^2 - 2V(x))^{\frac{1}{2}})
\]

since the escape rates \( \mathcal{E}_0(\nu) = 8 \) are constant. The second equality only holds when \( \rho' \geq \text{sup} \nu_0 \nu \), and otherwise there are two terms. For \( \gamma = (\rho, \epsilon) \) with \( \rho \geq \sqrt{L} \), the label component is \( \epsilon = \pm 1 \), and we can identify \( \nu \) with the quasi-momentum value \( q(\nu) = \epsilon \rho \).

The rates describe what is nearly an unbiased random walk for the quasi-momentum. The function \( w_\nu(\nu'') \) is uniformly bounded away from zero over any finite region of \( \nu'' \) with \( \rho - 1 \leq \nu'' \leq \rho + L \) for \( L > 0 \). It is sufficient to take, say, \( L = 1 \). For large enough
\[ c' > 0, \] we thus have the first inequality below:
\[
\int_{\rho \geq \rho - 1} dy'' w_\gamma (y'') \tilde{T}_0(y'', y') \geq \frac{1}{c'} \int_{\rho - 1 \leq \rho' \leq \rho + 1} dy'' \tilde{T}_0(y'', y') \\
\geq \frac{1}{c'} \tilde{T}_0(y, y').
\]

Finally, we can choose \( c > 0 \) large enough to make the second inequality hold for all \( \gamma, \gamma' \) with \( \sqrt{2T} < \rho \) and \( \rho' \leq \rho - 1 \).

**Part (2).** We have the closed formula

\[
\mathbb{E}_\gamma^{(\lambda)} [\delta(g_{N} - \gamma')] = \mathbb{E}_\gamma^{(\lambda)} \left[ \sum_{n=0}^{N-1} \tilde{T}_\lambda (g_n, \gamma') \right].
\]  

(3.10)

This follows formally by the optional stopping theorem with stopping time \( \tilde{N} \) and ‘martingale’

\[
\sum_{n=1}^{m} \delta(g_n - \gamma') - \tilde{T}_\lambda (g_{n-1}, \gamma').
\]

The values \( g_n \in \Gamma_\gamma \) are bounded away from \( \gamma' \) for \( n < \tilde{N} \), and thus \( \sum_{n=1}^{\tilde{N}-1} \delta(g_n - \gamma') = 0 \). To be more rigorous, we should replace \( \delta(-\gamma') \) by a family of functions approximating it.

With (3.10), we can apply part (1) with \( f'(\gamma) = \tilde{T}_\lambda (\gamma, \gamma') \) to obtain the inequality below for some \( C' > 0 \):

\[
\mathbb{E}_\gamma^{(\lambda)} \left[ \sum_{n=0}^{\tilde{N}-1} \tilde{T}_\lambda (g_n, \gamma') \right] = \tilde{T}_\lambda (\gamma, \gamma') + \mathbb{E}_\gamma^{(\lambda)} \left[ \sum_{n=1}^{\tilde{N}-1} \tilde{T}_\lambda (g_n, \gamma') \right] \\
\leq \tilde{T}_\lambda (\gamma, \gamma') + C \left( \sup_{\rho' > \lambda^{-1}} \tilde{T}_\lambda (\gamma'', \gamma') + \int_{\rho \leq \rho' \leq \lambda^{-1}} dy'' \tilde{T}_\lambda (y'', \gamma') \right).
\]  

(3.11)

However, there is a \( c > 0 \) such that for all \( \lambda < 1 \) and \( \gamma, \gamma' \) with \( \rho, \rho' \leq \lambda^{-1} \),

\[
\tilde{T}_\lambda (\gamma, \gamma') \leq c e^{-\frac{1}{\lambda} |\rho - \rho'|} \quad \text{and} \quad \sup_{\rho' > \lambda^{-1}} \tilde{T}_\lambda (\gamma, \gamma') \leq c e^{-\frac{1}{\lambda} |\rho - \lambda^{-1}|}.
\]

Plugging these into (3.11) gives the uniform bound.

**Part (3).** We begin with the inequality

\[
\mathbb{E}_\gamma^{(\lambda)} \left[ \sum_{n=1}^{N-1} \tilde{T}_\lambda (g_n, \gamma') \right] \leq \left( \sup_{\rho' > \lambda^{-1}} \tilde{T}_\lambda (\gamma') \right) \mathbb{E}_\gamma^{(\lambda)} [N].
\]

Let \( W_n : \Gamma_\gamma \to \mathbb{R}^+ \) be as in (4) of proposition 3.3. It follows by (4) of proposition 3.3 that \( cn + W_n (g_n) \) is a supermartingale over the time interval \( n \in [0, N] \). We have the inequalities

\[
\mathbb{E}_\gamma^{(\lambda)} [N] \leq \frac{1}{c} \mathbb{E}_\gamma^{(\lambda)} [W_n (\gamma) - W_n (g_n)] \leq \frac{1}{c} W_n (\gamma) = \frac{1}{c\lambda} \log(1 + \lambda \rho),
\]

where the first inequality is by the optional stopping theorem, and the second inequality is since \( W_n \geq 0 \).

**Part (4).** This follows analogously to part (2) of lemma 2.2.

The inequality in part (2) of the lemma below is analogous to theorem 1.1.

**Lemma 3.5.** Let \( \hat{U}^{(\lambda)} \) be the state-modulated resolvent of the function \( \hat{h} \).
(1) \( \tilde{U}^{(\lambda)} \) satisfies the integral equation
\[
\hat{f}(\gamma) = \tilde{h}(\gamma)(\tilde{U}^{(\lambda)}(\gamma, \tilde{f})) + \int_{\Gamma_{\gamma}} d\gamma' \tilde{h}_{\lambda}(\gamma, \gamma') (\tilde{U}^{(\lambda)}(\gamma', \tilde{f}) - \tilde{U}^{(\lambda)}(\gamma', \tilde{f})).
\]

(2) There is a \( c > 0 \) such that for all measurable \( \tilde{f} : \Gamma_{\gamma} \to \mathbb{R}^+, \lambda < 1 \) and \( \gamma \in \Gamma_{\gamma} \),
\[
\tilde{U}^{(\lambda)}(\gamma, \tilde{f}) \leq c \left( \sup_{\gamma' \in \Gamma_{\gamma}} A^{(\lambda)}(\gamma, \gamma') \hat{f}(\gamma') + \int_{\Gamma_{\gamma}} d\gamma' B^{(\lambda)}(\gamma, \gamma') \tilde{f}(\gamma') \right),
\]
where \( A^{(\lambda)}(\gamma, \gamma') \) and \( B^{(\lambda)}(\gamma, \gamma') \) are defined as
\[
A^{(\lambda)}(\rho, \epsilon, \rho', \epsilon') = (1 + \min(\rho, \lambda^{-1}) \log(1 + \lambda \rho)) \chi(\rho' \geq \lambda^{-1}) \frac{1}{\tilde{E}_{\lambda}(\rho', \epsilon')},
\]
\[
B^{(\lambda)}(\rho, \epsilon, \rho', \epsilon') = (1 + \min(\rho, \rho')) \chi(\rho \leq \lambda^{-1}) \frac{1}{\tilde{E}_{\lambda}(\rho', \epsilon')},
\]

**Proof.** Part (1) follows easily from the definition of \( \tilde{U}^{(\lambda)} \), so we focus on part (2). By rearranging the integral equation from part (1), we have the equation
\[
\tilde{U}^{(\lambda)}(\gamma, \tilde{f}) = \frac{\hat{f}(\gamma)}{\tilde{h}(\gamma) + \tilde{E}_{\lambda}(\gamma)} + \int_{\Gamma_{\gamma}} d\gamma' \frac{\tilde{h}_{\lambda}(\gamma, \gamma')}{\tilde{h}(\gamma) + \tilde{E}_{\lambda}(\gamma)} \tilde{U}^{(\lambda)}(\gamma', \tilde{f})
\]
\[
= \tilde{C}_{\lambda}(\gamma) \frac{\hat{f}(\gamma)}{\tilde{E}_{\lambda}(\gamma)} + \tilde{C}_{\lambda}(\gamma) \int_{\Gamma_{\gamma}} d\gamma' \tilde{C}_{\lambda}(\gamma', \gamma') \tilde{U}^{(\lambda)}(\gamma', \tilde{f}).
\]
(3.12)

where \( \tilde{C}_{\lambda} \) and \( \tilde{C}_{\lambda} \) are defined as
\[
\tilde{C}_{\lambda}(\gamma, \gamma') = \frac{\tilde{h}_{\lambda}(\gamma, \gamma')}{\tilde{E}_{\lambda}(\gamma)} \quad \text{and} \quad \tilde{C}_{\lambda}(\gamma) = \frac{\tilde{E}_{\lambda}(\gamma)}{\tilde{h}(\gamma) + \tilde{E}_{\lambda}(\gamma)}.
\]
Consider the chain \( g_n = (r_n, e_n) \in \Gamma_{\gamma} \) starting at \( \gamma \) and making jumps with the transition kernel \( \tilde{T}_{\lambda} \). The kernel for \( \tilde{U}^{(\lambda)} \) can be written as
\[
\tilde{U}^{(\lambda)}(\gamma, \tilde{f}) = \sum_{n=0}^{\infty} \mathbb{E}^{(\lambda)} \left[ \left( \prod_{r=0}^{n} \tilde{C}_{\lambda}(g_{r}) \right) \frac{\tilde{f}(g_{n})}{\tilde{E}_{\lambda}(g_{n})} \right] \quad \text{by (3.13)}
\]

First, we will show that the bound for \( \tilde{U}^{(\lambda)}((\rho, \epsilon), \tilde{f}) \) when \( \rho > \lambda^{-1} \) follows from the bound for \( \tilde{U}^{(\lambda)}((\rho, \epsilon), \tilde{f}) \) when \( \rho \leq \lambda^{-1} \). For \( \gamma = (\rho, \epsilon) \) with \( \rho > \lambda^{-1} \), let \( N_{\gamma} \in \mathbb{N} \) be the hitting time that \( r_n \) jumps below \( \lambda^{-1} \). The form (3.13) allows us to write
\[
\tilde{U}^{(\lambda)}(\gamma, \tilde{f}) = \mathbb{E}^{(\lambda)} \left[ \sum_{n=0}^{N_{\gamma}-1} \left( \prod_{r=0}^{n} \tilde{C}_{\lambda}(g_{r}) \right) \frac{\tilde{f}(g_{n})}{\tilde{E}_{\lambda}(g_{n})} \right] + \mathbb{E}^{(\lambda)} \left[ \left( \prod_{r=0}^{N_{\gamma}} \tilde{C}_{\lambda}(g_{r}) \right) \tilde{U}^{(\lambda)}(g_{N_{\gamma}}, \tilde{f}) \right]
\]
\[
\leq C_\lambda^{-1} \log(1 + \lambda \rho) \left( \sup_{\rho' > \lambda^{-1}} \tilde{f}(\gamma') \right) + \sup_{\rho' \leq \lambda^{-1}} \tilde{U}^{(\lambda)}(\gamma', \tilde{f}).
\]
(3.14)

The first inequality uses that \( \tilde{C}_{\lambda}(\gamma) \leq 1 \), and the second inequality uses part (3) of lemma 3.4 for the first term and the definition of the hitting time \( N_{\gamma} \) for the second. Thus, it is sufficient for us to prove the statement of this lemma for the domain of \( \gamma = (\rho, \epsilon) \) with \( \rho \leq \lambda^{-1} \).

Next, we focus on the domain \( \sqrt{M} < \rho \leq \lambda^{-1} \). For \( (r_0, e_0) = (\rho, \epsilon) \) with \( \rho > \sqrt{M} \), let \( N_0 \) be the sequence of hitting times such that \( N_0 = 0 \) and
\[
N_n = \inf \{ m > N_{n-1} \ | \ r_m \leq r_{N_{n-1}} - 1 \}, \quad n \geq 1.
\]
In the above, we can take the infimum of the empty set to be \( \infty \), and clearly there can be at most \( [\rho] \) of the hitting times \( \hat{N}_n \) that are not infinite. Also let \( T \in \mathbb{N} \) be the first time \( \hat{N}_n \) such that \( r_n \leq \sqrt{T} \), and \( I \) be the number of \( \hat{N}_n \) in \([1, T]\). Analogously to (3.14), we have the inequality

\[
\bar{U}^{(i)}(\gamma', \hat{f}) \leq \frac{\hat{f}(\gamma')}{E_\lambda(\gamma')} + E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] + E_\gamma^{(i)} \left[ \bar{U}^{(i)}(g, \hat{f}) \right]. \tag{3.15}
\]

By breaking the time step interval \([1, T]\) into subintervals \([N_{m-1} + 1, N_m]\) for \( m \in [1, I] \) and using nested conditional expectations along with the strong Markov property,

\[
E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] = E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] - \hat{f}(\gamma') \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)}
\]

\[
= E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] - \hat{f}(\gamma') \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)}.
\tag{3.16}
\]

To bound \( \bar{U}^{(i)}(\gamma', \hat{f}) \), we must bound the terms

(i) \( E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] \),
(ii) \( E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] \),
(iii) \( E_\gamma^{(i)} [\bar{U}^{(i)}(g, \hat{f})] \).

By part (1) of lemma 3.4, there is a \( C > 0 \) such that (i) is smaller than

\[
E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] \leq C E_\gamma^{(i)} [\bar{U}^{(i)}(g, \hat{f})] \sup_{\rho > \gamma} \frac{\hat{f}(\gamma')}{E_\lambda(\gamma')} + C E_\gamma^{(i)} \left[ \int_{\rho_{N_{m-1}}}^{\rho^*} \hat{f}(\gamma') \frac{E_\lambda(\gamma')}{\hat{E}_\lambda(\gamma')} \right]
\]

\[
\leq C \rho \sup_{\rho > \gamma} \frac{\hat{f}(\gamma')}{E_\lambda(\gamma')} + C \int_{\rho_{N_{m-1}}}^{\rho^*} \hat{f}(\gamma')(1 + \min(\rho', \rho)) \frac{E_\lambda(\gamma')}{\hat{E}_\lambda(\gamma')}.
\tag{3.17}
\]

For both terms in the second inequality, we have used that the sequence \( r_{N_k} \) decreases by increments \( \geq 1 \) for \( m = 1, \ldots, I \). Thus, \( I \leq \rho \), and the number of \( m \) such that \( r_{N_m} - 1 \) is smaller than some value \( \rho' \leq \lambda^{-1} \) is less than \( 1 + \min(\rho, \rho') \).

For (ii), we can write

\[
E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \hat{f}(g_n) \frac{\hat{E}_\lambda(g_n)}{E_\lambda(g_n)} \right] = \int_{\gamma'} \nu_\gamma^{(i)}(d\gamma') \frac{\hat{f}(\gamma')}{E_\lambda(\gamma')},
\tag{3.18}
\]

where \( \nu_\gamma^{(i)}(d\gamma') = E_\gamma^{(i)} \left[ \sum_{m=1}^{T-1} \delta(\gamma_{N_{m+1}} - \gamma') \right] \). By nested conditional expectations and the strong Markov property, we have the equalities below,

\[
\nu_\gamma^{(i)} = E_\gamma^{(i)} \left[ \sum_{m=0}^{T-2} \hat{E}_\lambda(g_{n+m+1} - \gamma') \right] = E_\gamma^{(i)} \left[ \sum_{m=0}^{T-2} \delta(\gamma_{N_{m+1}} - \gamma') \right]
\]

\[
\leq C \rho \sum_{m=0}^{T-2} \sum_{n=0}^{T_m-1} e^{-\pi^2 \chi_n} \left( \rho' \leq r_{N_m} - 1 \right)
\]

\[
< C \rho \leq \rho - 1 \sum_{n=0}^{\infty} e^{-\pi n} < 16C \rho \leq \lambda^{-1},
\tag{3.19}
\]

\[
\leq C \rho \sum_{m=0}^{T-2} \sum_{n=0}^{T_m-1} e^{-\pi^2 \chi_n} \left( \rho' \leq r_{N_m} - 1 \right)
\]

\[
= C \rho \sum_{m=0}^{T-2} \sum_{n=0}^{T_m-1} e^{-\pi^2 \chi_n} \left( \rho' \leq r_{N_m} - 1 \right)
\]

\[
< C \rho \leq \rho - 1 \sum_{n=0}^{\infty} e^{-\pi n} < 16C \rho \leq \lambda^{-1},
\tag{3.19}
\]

\[
\leq C \rho \sum_{m=0}^{T-2} \sum_{n=0}^{T_m-1} e^{-\pi^2 \chi_n} \left( \rho' \leq r_{N_m} - 1 \right)
\]

\[
= C \rho \sum_{m=0}^{T-2} \sum_{n=0}^{T_m-1} e^{-\pi^2 \chi_n} \left( \rho' \leq r_{N_m} - 1 \right)
\]

\[
< C \rho \leq \rho - 1 \sum_{n=0}^{\infty} e^{-\pi n} < 16C \rho \leq \lambda^{-1},
\tag{3.19}
\]

\[
\leq C \rho \sum_{m=0}^{T-2} \sum_{n=0}^{T_m-1} e^{-\pi^2 \chi_n} \left( \rho' \leq r_{N_m} - 1 \right)
\]

\[
= C \rho \sum_{m=0}^{T-2} \sum_{n=0}^{T_m-1} e^{-\pi^2 \chi_n} \left( \rho' \leq r_{N_m} - 1 \right)
\]

\[
< C \rho \leq \rho - 1 \sum_{n=0}^{\infty} e^{-\pi n} < 16C \rho \leq \lambda^{-1},
\tag{3.19}
\]
where $F^{(k)}_{\gamma}(\gamma') = \mathbb{E}_{\gamma}^{(k)}[\delta(\mathbf{g}_{\gamma} - \gamma')]$ is defined as in part (2) of proposition 3.4, and the first inequality is for some $C' > 0$ by part (2) of proposition 3.4. The second inequality uses that $\mathbf{r}_{\gamma}$ decreases by at least 1 at each time step. With (3.18) and (3.19),

$$\mathbb{E}_{\gamma}^{(k)} \left[ \sum_{m=1}^{\gamma - 1} \hat{f}(\mathbf{g}_{\mathbf{r}_{\gamma}}) \right] \leq C \int_{\rho < \lambda - 1} d\gamma' \frac{\hat{f}(\gamma')}{\hat{E}_x(\gamma')}$$

for $C = 16C'$.

For (iii), we have the following relations:

$$\mathbb{E}_{\gamma}^{(k)}[\mathbf{U}^{(k)}(\mathbf{g}_{\mathbf{T}}, \hat{f})] = \int_{\gamma} d\gamma' \mathbf{U}^{(k)}(\gamma', \hat{f}) \mathbb{E}_{\gamma}^{(k)}[\delta(\mathbf{g}_{\mathbf{T}} - \gamma')] \leq C' \int_{\rho' \leq \sqrt{\gamma}} d\gamma' \mathbf{U}^{(k)}(\gamma', \hat{f})$$

$$\leq C' \int_{\Gamma_{\gamma}} d\gamma' e^{-\frac{1}{2}(\gamma')^2} \hat{f}(\gamma') \leq C' \int_{\Gamma_{\gamma}} d\gamma' e^{-\frac{1}{2}(\gamma')^2} \frac{\hat{f}(\gamma')}{\hat{E}_x(\gamma')}$$

$$\leq C' \int_{\rho' \leq \lambda - 1} \hat{f}(\gamma') + C' \sup_{\rho' > \lambda - 1} \hat{f}(\gamma')$$

(3.20)

For the first inequality, the density $d\gamma' = \mathbb{E}_{\gamma}^{(k)}[\delta(\mathbf{g}_{\mathbf{T}} - \gamma')]$ is smaller than some $c' > 0$ by part (2) of proposition 3.4. The second equality is part (4) of proposition 3.4 and the third by the bounds for $\hat{E}_x$ from part (2) of proposition 3.3. For the last inequality, we have split the integration into the domains of $\gamma = (\rho, \epsilon)$ with $\rho \leq \lambda - 1$ and $\rho > \lambda - 1$ similarly to the proof of part (2) of lemma 2.2.

With (i)–(iii), we have shown that there is a $C > 0$ such that for all $\lambda < 1$ and all $\gamma = (\rho, \epsilon)$ with $\rho > \sqrt{2}$,

$$\mathbf{U}^{(k)}(\gamma, \hat{f}) \leq \left\| \frac{\hat{f}}{\hat{E}_x} \right\|_{\infty} + C \rho \sup_{\rho' > \lambda - 1} \frac{\hat{f}(\gamma')}{\hat{E}_x(\gamma')} + C \int_{\rho' \leq \lambda - 1} d\gamma' (1 + \min(\rho, \rho')) \frac{\hat{f}(\gamma')}{\hat{E}_x(\gamma')}$$

(3.21)

We can use (3.21) to extend our bound to the domain of $\gamma = (\rho, \epsilon)$ with $\rho \leq \sqrt{2}$. Starting with the integral equation (3.12),

$$\mathbf{U}^{(k)}(\gamma, \hat{f}) = \hat{C}_x(\gamma) \frac{\hat{f}(\gamma)}{\hat{E}_x(\gamma)} + \hat{C}_x(\gamma) \int_{\Gamma_{\gamma}} d\gamma' \hat{T}_x(\gamma, \gamma') \mathbf{U}^{(k)}(\gamma', \hat{f})$$

$$\leq \left\| \frac{\hat{f}}{\hat{E}_x} \right\|_{\infty} + \left( \sup_{\lambda < 1} \hat{T}_x(\gamma, \gamma') \right) \int_{\rho' \leq \sqrt{2}} d\gamma' \mathbf{U}^{(k)}(\gamma', \hat{f})$$

$$+ \int_{\rho' > \sqrt{2}} d\gamma' \left( \sup_{\lambda < 1} \hat{T}_x(\gamma, \gamma') \right) \mathbf{U}^{(k)}(\gamma', \hat{f})$$

$$\leq \left\| \frac{\hat{f}}{\hat{E}_x} \right\|_{\infty} + C \left( \sup_{\rho' > \lambda - 1} \frac{\hat{f}(\gamma')}{\hat{E}_x(\gamma')} + \int_{\rho' \leq \lambda - 1} d\gamma' \frac{\hat{f}(\gamma')}{\hat{E}_x(\gamma')} \right)$$

(3.22)

for large enough constant $C' > 0$. For the second term in the second inequality, the supremum of $\hat{T}_x(\gamma, \gamma')$ over $\lambda < 1$ and $\gamma, \gamma' \in \Gamma_{\gamma}$ is bounded, and we bound the integral $\int_{\rho' \leq \sqrt{2}} d\gamma' \mathbf{U}^{(k)}(\gamma', \hat{f})$ by the argument in (3.20). For the third term in the second inequality of (3.22), we have bounded $\mathbf{U}^{(k)}(\gamma', \hat{f})$ with inequality (3.21) and used that $\sup_{\rho' > \lambda - 1} \hat{T}_x(\gamma, \gamma')$ has a Gaussian tail in $\rho'$ for $\gamma' = (\rho', \epsilon')$. □
4. Linking the original and the Freidlin–Wentzell dynamics

Define the linear map \( \hat{U}^{(i)} : B(\Sigma) \to B(\Gamma_V) \) to act as \( \hat{U}^{(i)} f = \overline{U^{(i)} f} \), where the map \( \overline{B(\Sigma) \to B(\Gamma_V)} \) is from part (3) of definition 3.1. The lemma below states that \( \hat{U}^{(i)} f \) satisfies the same integral equation as \( \overline{U^{(i)} f} \) in part (1) of lemma 3.5 with an error term for which part (3) gives a bound.

**Lemma 4.1.** Let \( f \in B(\Sigma) \) be non-negative.

(1) \( \hat{U}^{(i)} f : \Gamma_V \to \mathbb{R}^+ \) satisfies the equation

\[
\hat{f}(\gamma) = \hat{h}(\gamma)\hat{U}^{(i)}(\gamma, f) + \int_{\Gamma_V} d\gamma' \hat{J}_u(\gamma, \gamma')(\hat{U}^{(i)}(\gamma', f) - \hat{U}^{(i)}(\gamma, f)) + E_\lambda(\gamma),
\]

where \( E_\lambda(\gamma) \) has the form

\[
E_\lambda(\gamma) = \int_{\Gamma_V} d\gamma' \hat{J}_u(\gamma, \gamma')(\hat{U}^{(i)}(\gamma', f) - (\hat{J}_u U^{(i)} f)(\gamma)),
\]

and the operator \( J_u \) acts on \( B(\Sigma) \) with the kernel density \( J_u(p, p') \). The above is equivalent to the statement

\[
\hat{U}^{(i)} f = \overline{U^{(i)} f} - E_\lambda.(\hat{f} - E_\lambda). (4.2)
\]

(2) There is a \( C > 0 \) such that for all \( \lambda < 1 \), \( f \in B(\Sigma) \) and \( H(x, p) > 1 \),

\[
|U^{(i)}(x, p, f) - U^{(i)}(\gamma(x, p), f)| \leq C\max\left(\frac{1}{1 + |p|}, \lambda\right)\tilde{U}^{(i)}(\gamma(x, p), f).
\]

(3) For \( \gamma = (\rho, \epsilon) \in \Gamma_V \), let \( A_\gamma \subset \Gamma_V \) be the set

\[
A_\gamma = \left\{ \{(\rho', \epsilon')|\frac{1}{2} \rho \leq \rho' \leq 2 \rho \text{ and } \epsilon' = \epsilon\} \right\} \rho > \sqrt{2l}, \rho \leq \sqrt{2l},
\]

and define the function \( M_\lambda : \Gamma_V^2 \to \mathbb{R}^+ \),

\[
M_\lambda(\gamma, \gamma') = \max\left(\frac{1}{1 + \rho'}, \lambda\right)\left(\frac{1}{\rho^2}(1 + (\rho' - \rho)^2 + \lambda^2 \rho^2)\chi(\gamma' \in A_\gamma) + \chi(\gamma' \notin A_\gamma)\right).
\]

The error \( E_\lambda \) from part (1) is a sum of parts \( E_\rho' \) and \( E_\lambda - E_\rho' \) for which there is a \( C > 0 \) such that for all \( \lambda < 1 \), \( f \in B(\Sigma) \), \( \gamma = (\rho, \epsilon) \in \Gamma_V \),

\[
|E_\lambda(\gamma) - E_\rho'(\gamma)| \leq C e^{-\rho} \left( \sup_{H > \frac{1}{4}\lambda^{-2}} f(s') + \int_{H < \frac{1}{4}\lambda^{-2}} d\gamma f(s') \right),
\]

\[
|E_\rho'(\gamma)| \leq C \int_{\Gamma_V} d\gamma' \hat{J}_u(\gamma, \gamma')M_\lambda(\gamma, \gamma')\tilde{U}^{(i)}(\gamma', f).
\]

**Proof.**

**Part (1).** Consider the altered escape and jump rates given by the following:

\[
\mathcal{E}_\lambda(x, p) = \max_{x \in \mathbb{T}} E_\lambda\left(\sqrt{2H(x, p) - 2V(x)}\right),
\]

\[
\mathcal{J}_\lambda^{(x, p', p')} = J_u(p, p') + \delta_0 (p' - p) (E_\lambda(x, p) - E_\lambda(\sqrt{2H(x, p) - 2V(x)})).
\]

With the above constructions, \( \mathcal{J}_\lambda^{(x, p', p')} = E_\lambda(x, p) \) for each \( x \in \mathbb{T} \). Replacing the jump rates \( J_u \) by \( \mathcal{J}_\lambda^{(x, p')} \) makes no difference for the underlying process, since it merely adds
a spatially dependent rate of vacuous jumps \( p \to p \) so that the escape rate is invariant of the Hamiltonian evolution. 

Let \( \tau \) be a mean-\( l \) exponential time and \( t_1 \) be the first ‘collision’ time according to our new jump rates. By considering the stopping time \( \min(\tau, t_1) \), we are led to the integral equation 

\[
U^{(i)}(x, p, f) = \frac{f(x, p)}{1 + E\hat{\gamma}(x, p)} + (1 - h(x, p)) \int_0^\infty \mathrm{d}t \ e^{-t - E\hat{\gamma}(x, p)} U^{(i)}((x, p), f) \\
+ \int_0^\infty \mathrm{d}t \ e^{-t - E\hat{\gamma}(x, p)} \int_{\mathbb{R}} \mathrm{d}p' J^{(i)}(x, p') U^{(i)}((x, p'), f),
\]

where \((x, p)\) is the phase space point at time \( t \) when evolving according to the Hamiltonian \( H \) starting from the point \((x, p)\). The above equation can be reshuffled to give 

\[
f(x, p) = h(x, p)(1 + E\hat{\gamma}(x, p)) \int_0^\infty \mathrm{d}t \ e^{-t - E\hat{\gamma}(x, p)} U^{(i)}((x, p), f) \\
+ \left( U^{(i)}((x, p), f) - (1 + E\hat{\gamma}(x, p)) \int_0^\infty \mathrm{d}t \ e^{-t - E\hat{\gamma}(x, p)} U^{(i)}((x, p), f) \right) \\
+ (1 + E\hat{\gamma}(x, p)) \int_0^\infty \mathrm{d}t \ e^{-t - E\hat{\gamma}(x, p)} \int_{\mathbb{R}} \mathrm{d}p' J^{(i)}(x, p') \\
\times ((U^{(i)}((x, p), f) - (U^{(i)} f))(x, p')).
\]

The jump rates \( J^{(i)}(x, p) \) in the last term can be replaced by the original rates \( J\), since the difference is merely the vacuous jumps. Moreover, by integrating both sides over \((x, p) \in \Sigma \) against \( \delta(2^{\frac{1}{2}} H\hat{\gamma}(x, p) - \rho)\chi(n(x, p) = \epsilon) \), we obtain 

\[
\hat{f}(\gamma) = \hat{h}(\gamma) \hat{U}^{(i)}(\gamma, f) + \int_{\Gamma_v} \mathrm{d}y' \frac{\delta}{\hat{\gamma}}(\gamma, y')((\hat{U}^{(i)}(\gamma, f)) - (\hat{U}^{(i)}(\gamma', f)) \\
+ \left( \int_{\Gamma_v} \mathrm{d}y' \frac{\delta}{\hat{\gamma}}(\gamma, y')((\hat{U}^{(i)}(\gamma', f)) - (\hat{U}^{(i)}(\gamma', f))) \right).
\]

for \( \gamma = (\rho, \epsilon) \). We will illustrate the computation for the first term on the right-hand side of (4.3):

\[
\int_{\Sigma} \mathrm{d}x \delta(2^{\frac{1}{2}} H\hat{\gamma}(x, p) - \rho)\chi(n(x, p) = \epsilon) \int_0^\infty \mathrm{d}t e^{-t - E\hat{\gamma}(x, p)} U^{(i)}((x, p), f) \\
= \int_0^\infty \mathrm{d}t \int_{\Sigma} \mathrm{d}x \delta(2^{\frac{1}{2}} H\hat{\gamma}(x, p) - \rho)\chi(n(x, p) = \epsilon) \\
\times (1 + E\hat{\gamma}(x, p)) e^{-t - E\hat{\gamma}(x, p)} U^{(i)}((x, p), f) \\
= \int_{\Sigma} \mathrm{d}x \delta(2^{\frac{1}{2}} H\hat{\gamma}(x, p) - \rho)\chi(n(x, p) = \epsilon) U^{(i)}((x, p), f) = \hat{U}^{(i)}((\rho, \epsilon), f).
\]

The first equality uses Fubini’s theorem to pull out the integral \( \int_0^\infty \mathrm{d}t \), and then employs a change of variables over \( \Sigma \) with the dynamical transformation map \((x, p) \to (x', p', \cdots) \) (i.e. backward time evolution according to the Hamiltonian \( H \) for a time interval of length \( t \)). Thus, \((x, p)\) maps to \((x, p)\), and other expressions do not change since \( H\hat{\gamma}, \rho, E\hat{\gamma} \) are functions of the energy. The second equality uses Fubini to compute the integral \( \int_0^\infty \mathrm{d}t e^{-t - E\hat{\gamma}(x, p)} = (1 + E\hat{\gamma}(x, p))^{-1} \).

The equality \( \hat{U}^{(i)} f = \hat{U}^{(i)}(\hat{f} - E\hat{\gamma}) \) follows from part (1) of lemma 3.5.

**Part (2).** As in part (1), let \( t_1 \) be the first ‘collision’ time with the vacuous jumps included. If the particle begins at \((x, p)\) with \( H(x, p) > l \), then the final time \( R \) from part (1) of
prop. 2.1 cannot occur over the interval \([0, t_1]\), since the modulating function \(h\) has support on the set \(H(x, p) \leq I\). The value \(U^{(k)}(x, p, f)\) can be written as

\[ U^{(k)}(x, p, f) = \int_{\Sigma} \kappa^{(k)}_{(x, p)}(dx') \int_{\mathbb{R}} dp' \frac{\mathcal{J}^{(k)}(p', p')}{E_{x'}^{(k)}(x', p')} U^{(k)}((x', p''), f), \]

where the measure \(\kappa^{(k)}_{(x, p)}\) is supported on the set of \((x', p')\) with \(y(x', p') = y(x, p) \in \Gamma_y\) and is defined by

\[ \kappa^{(k)}_{(x, p)}(dx' dp') = \mathbb{E}_{(x, p)}^{(k)}[\delta_{y(x', p')}(X_1, P_{\gamma})]. \]

On the other hand,

\[ \tilde{U}^{(k)}(y(x, p), f) = \int_{\Sigma} \kappa_{y(x, p)}(dx' dp') \int_{\mathbb{R}} dp'' \frac{\mathcal{J}^{(k)}(p', p')}{E_{x'}^{(k)}(x', p')} U^{(k)}((x', p''), f), \] \hspace{1cm} (4.4)

where \(\kappa_y\) is the normalized measure from def. 3.1. By the bounds on the escape rates in part (1) of prop. 3.3, the random variable \(t_1\) is exponential with mean \(\geq c \min(1, (\lambda |p|)^{-1})\) for some \(c > 0\) and all \(\lambda < 1 + p\). Since the particle is traveling with velocity \(p\), it will typically revolve around the level curve of the order of \(\min(|p|, \lambda^{-1})\) times before \(t_1\) occurs. The Radon–Nikodym derivative \(\frac{d\kappa^{(k)}_{y(x, p)}}{d\kappa_{y(x, p)}}\) satisfies

\[ \sup_{s \in \mathbb{R}} \left| \frac{d\kappa^{(k)}_{y(x, p)}}{d\kappa_{y(x, p)}}(s') - 1 \right| \leq c' \max(|p|^{-1}, \lambda) \] \hspace{1cm} (4.5)

for some \(c'\). Thus,

\[ \left| U^{(k)}(x, p, f) - \tilde{U}^{(k)}(y(x, p), f) \right| \leq \tilde{U}^{(k)}(y(x, p), f) - U^{(k)}(x, p, f), \]

where we have applied inequality (4.5) and used formula (4.4) for \(\tilde{U}^{(k)}(y(x, p), f)\).

Part (3). The expression \(E_y(y)\) can be written as

\[ E_y(y) = \int_{\Sigma} \kappa_y(dx dp) \int_{\mathbb{R}} dp' \mathcal{J}_y(p, p')(U^{(k)}((x, p'), f) - \tilde{U}^{(k)}(y(x, p'), f)) \]

\[ = \int_{\Gamma_y} dy' \mathcal{J}_y(y, y') \int_{\Sigma} \eta_{y'}(dx dp) \int_{\Sigma} \kappa_y(dx dp) \delta_0(x - x') \]

\[ \times \frac{\mathcal{J}_y(p, p')}{\mathcal{J}_y(y, y')} (U^{(k)}((x, p'), f) - \tilde{U}^{(k)}(y(x, p'), f)). \]

The second equality is by commuting integrals and using (2) of rem. 3.2:

\[ \int_{\mathbb{R}} dp' = \int_{\Sigma} dx' \delta_0(x - x') = \int_{\Gamma_y} dy' \int_{\Sigma} \eta_{y'}(dx dp) \delta_0(x - x'). \]

We define \(E'_y(y')\) to be the analogous expression with the integration \(\int_{\Gamma_y} dy'\) replaced by the restricted integration \(\int_{\rho > 1} dy'\). The value \(E'_y(y')\) is bounded by

\[ |E'_y(y')| \leq \int_{\rho > 1} dy' \mathcal{J}_y(y, y') \sup_{x, x' \in E_y(y')} \left| U^{(k)}(s, f) - U^{(k)}(x', f) \right| \]

\[ \times \int_{\Sigma} \eta_{y'}(dx dp) \int_{\Sigma} \kappa_y(dx dp) \delta_0(x - x') \frac{\mathcal{J}_y(p, p')}{\mathcal{J}_y(y, y')} - 1 \] \hspace{1cm} (4.6)
For $E_s(\gamma) - E_s^c(\gamma)$, note that
\[
|E_s(\gamma) - E_s^c(\gamma)| \leq \int \kappa_p(d\gamma)p' \int_{\gamma(x, p) \leq l} \mathcal{J}_s(p, p')(U^{(k)}(x, p'), f) \\
+ \sup_{(x, p) \in G_{\gamma}(\gamma')} \mathcal{J}_s(p, p') \int_\gamma dx \int_{\gamma(x, p) \leq l} d\gamma
\]
\[
\leq I^2 \left( \sup_{(x, p) \in G_{\gamma}(\gamma')} \mathcal{J}_s(p, p') \right) \int_\gamma dx \int_{\gamma(x, p) \leq l} d\gamma
\]
\[
\times (U^{(k)}(x, p'), f) + \sup_{(x, p) \in G_{\gamma}(\gamma')} \mathcal{J}_s(p, p') \int_\gamma dx \int_{\gamma(x, p) \leq l} d\gamma
\]
\[
\leq C e^{-\rho} \int_{H \leq l} \mathcal{J}_s(p, p') + \int_{H \leq l} d\gamma f(s') \right),
\]
where $H = H(s)$ and $H' = H(s')$. The second inequality uses (3) of remark 3.2 and that for $\rho^2 > l = 1 + 2 \sup_x V(x)$,
\[
\frac{\rho^2 - 2V(x')}{2} \leq \sup_x \left( \rho^2 - 2V(x) \right) \frac{1}{\max_x \left( \rho^2 - 2V(x) \right)} \leq I^2.
\]
The supremum over the values of $\mathcal{J}_s(p, p')$ in the second line of (4.7) decays super-exponentially for large $\rho$, and we bounded it by a multiple of $e^{\rho}$. For the third inequality in (4.7), we have also used that $\int_\gamma dx \int_{\gamma(x, p) \leq l} d\gamma = \int_{H \leq l} d\gamma$, and the fourth inequality is by part (2) of lemma 2.2.

The following two statements hold, where (I) is by part (2), and we prove (II) below.

(I) There is a $C > 0$ such that for all $\lambda < 1$, $\gamma' = (\rho', \epsilon') \in \Gamma_\rho$ with $\rho' > \sqrt{2l}$ and non-negative $f \in \mathcal{B}(\Gamma_\rho)$,
\[
\sup_{s, s' \in G_{\gamma}(\gamma')} |U^{(k)}(s, f) - U^{(k)}(s', f)| \leq C \max \left( \frac{1}{1 + \rho'}, \lambda \right) \mathcal{J}_s^{(k)}(\gamma', f).
\]

(II) There is a $C > 0$ such that for all $\lambda \leq 1$ and $\gamma, \gamma' \in \Gamma_\rho$,
\[
\int_{\gamma} \eta_{\gamma'}(d\gamma') \left\| \int_{\gamma} \kappa_{\gamma'}(d\gamma)p \delta_0(x - x') \mathcal{J}_s(p, p') - 1 \right\| \leq \chi(\gamma' \notin A_\gamma) + \frac{C}{\rho^2} (1 + |\rho' - \rho|^2 + \lambda^2 \rho^2) \chi(\gamma' \in A_\gamma).
\]

For the right-hand side of (I), we have used that $\rho'(x, p) = (2H(x, p))^{\frac{1}{2}}$ is close to $|p|$ for $|p| \geq 1$.

Given (I) and (II), by (4.6)
\[
|E_s^c(\gamma)| \leq 2C \int_{\gamma} d\gamma' \mathcal{J}_s(\gamma, \gamma') \mathcal{J}_s^{(k)}(\gamma', f)
\]
\[
\times \max \left( \frac{1}{1 + \rho'}, \lambda \right) \left( \chi(\gamma' \notin A_\gamma) + \frac{C}{\rho^2} (1 + |\rho' - \rho|^2 + \lambda^2 \rho^2) \chi(\gamma' \in A_\gamma) \right).
\]

This would complete the proof.

Now we will prove (II). Note that the expression in (II) is $\leq 2$ for all $\gamma, \gamma'$, since by the definition of $\mathcal{J}_s$,
\[
\int_{\gamma} \eta_{\gamma'}(d\gamma') \int_{\gamma} \kappa_{\gamma'}(d\gamma)p \delta_0(x - x') \mathcal{J}_s(p, p') = 1.
\]

For $\gamma = (\rho, \epsilon)$ with $\rho \geq 1$, the following bounds hold.
(i) There is a $c > 0$ such that for all $\gamma$ with $\rho > \sqrt{2l}$,

$$\sup_{x', x'' \in \Sigma} \left| \int_{x} k_{\gamma}(dx \, dp) (\delta_{0}(x - x') - \delta_{0}(x - x'')) \right| \leq \frac{c}{(1 + \rho)^{2}}.$$ 

(ii) There is a $c > 0$ such that for all $\gamma_{1} \in \Gamma_{V}$ and $\gamma_{2} \in A_{\gamma_{1}}$

$$\sup_{\gamma_{1} = \gamma(x, p_{1}) = \gamma(x', p'_{1}) \atop \gamma_{2} = \gamma(x, p_{2}) = \gamma(x', p'_{2})} |\mathcal{J}_{\rho}(p_{1}, p_{2}) - \mathcal{J}_{\rho}(p'_{1}, p'_{2})| \leq \frac{c}{\rho_{1}} \sup_{\gamma_{1} = \gamma(x, p_{1})} |\rho_{2} - \rho_{1}| \left(1 + |\rho_{2} - \rho_{1}|^{2} + \lambda^{2} \rho_{1}^{2}\right) \inf_{\gamma_{2} = \gamma(x, p_{2})} \mathcal{J}_{\rho}(p_{1}, p_{2}).$$

Statement (i) concerns only the level curves of the Hamiltonian. The value $\int_{\Sigma} \eta_{\gamma}(dx \, dp) \delta_{0}(x - x')$ is the density (normalized to 1) for the amount of time that the particle will spend at the point $x'$ when revolving once around the level curve $\gamma$. By (3) of remark 3.2, we have the following closed formula for $\rho > \sqrt{2l}$:

$$\int_{\Sigma} k_{\gamma}(dx \, dp) \delta_{0}(x - x') = \left(\frac{\rho^{2} - 2V(x')}{\rho^{2} - 2V(x)}\right)^{-\frac{1}{2}} \int_{\Sigma} dx (\rho^{2} - 2V(x))^{-\frac{1}{2}}.$$ 

Thus, (i) is smaller than

$$\sup_{x', x'' \in \Sigma} \left| \int_{x} k_{\gamma}(dx \, dp) (\delta_{0}(x - x') - \delta_{0}(x - x'')) \right| \leq \frac{2 \sup_{\gamma} V(x)}{\rho^{2}} - 2 \sup_{\gamma} V(x) \leq \frac{4 \sup_{\gamma} V(x)}{\rho^{2}}.$$

where the inequalities have used the restriction $\rho^{2} > 2l > 4 \sup_{\gamma} V(x)$.

For statement (ii), let $\gamma_{1} \in \Gamma_{V}$ and $\gamma_{2} \in A_{\gamma_{1}}$. For $0 \leq V \leq 2 \sup_{\gamma} V(x)$,

$$\left| \frac{d}{dV} \mathcal{J}_{\rho}(\sqrt{\rho_{1}^{2} - V}, \sqrt{\rho_{2}^{2} - V}) \right| \leq \left| \frac{1}{\rho_{1}^{2} - V} \right| - \left| \frac{1}{\rho_{2}^{2} - V} \right| \left| e^{-\frac{i}{\rho_{1}^{2}} \sqrt{\rho_{2}} \sqrt{\rho_{1}^{2} - V} - \frac{i}{\rho_{2}^{2}} \sqrt{\rho_{1}} \sqrt{\rho_{2}^{2} - V}} \right|^2 + |\rho_{2} - \rho_{1}| \left| \frac{1}{\sqrt{\rho_{2}^{2} - V}} - 1 \right| \frac{1}{\rho_{1}^{2} - V} \left| 1 - \frac{1}{\rho_{2}^{2} - V} \right| \left| 1 - \frac{1}{\rho_{1}^{2} - V} \right| \left| 1 - \frac{1}{\rho_{2}^{2} - V} \right|$$

$$\leq \frac{c}{\rho_{1}} \left(1 + |\rho_{2} - \rho_{1}|^{2} + \lambda^{2} \rho_{1}^{2}\right) \mathcal{J}_{\rho}(\sqrt{\rho_{1}^{2} - V}, \sqrt{\rho_{2}^{2} - V}).$$

(4.8)

With the bound of the derivative relation (4.8), it follows that for any $0 \leq V, V' \leq 2 \sup_{\gamma} V(x)$,

$$|\mathcal{J}_{\rho}(\sqrt{\rho_{1}^{2} - V'}, \sqrt{\rho_{2}^{2} - V'}) - \mathcal{J}_{\rho}(\sqrt{\rho_{1}^{2} - V}, \sqrt{\rho_{2}^{2} - V})| \leq e^{c'} \frac{1}{\rho_{1}} \left(1 + |\rho_{2} - \rho_{1}|^{2} + \lambda^{2} \rho_{1}^{2}\right) \mathcal{J}_{\rho}(\sqrt{\rho_{1}^{2} - V}, \sqrt{\rho_{2}^{2} - V}).$$

where we have used that $\frac{1}{\rho_{1}} \left(1 + |\rho_{2} - \rho_{1}|^{2} + \lambda^{2} \rho_{1}^{2}\right) \leq 2$ by our constraints on $\rho_{1}, \rho_{2}$. This proves (ii) with $c = e^{c'}$. 22
By the triangle inequality and supremizing over everything,
\[
\sup_{y' = y(x', p')} \left| \int \kappa_y'(d\gamma d\delta_0(x - x')) J_s(p, p') J_s(y, y') \left( \sup_{y = y(x, p_1)} J_s(p_1, p_2) \right) \right| \leq \sup_{x', x'' \in \mathcal{T}} \left| \int \kappa_y'(d\gamma d\delta_0(x - x')) \left( \sup_{y = y(x, p_1), y' = y(x, p_2)} J_s(p_1, p_2) \right) \right|
\]

\[
+ \left( \sup_{y = y(x, p_1), y' = y(x, p_2)} J_s(p_1, p_2) - J_s(p'_1, p'_2) \right) \left( \sup_{x' \in \mathcal{T}} \kappa_y'(d\gamma d\delta_0(x - x')) \right)
\]

\[
\leq \frac{2c}{\rho^2} + \frac{2c}{\rho^2} (1 + |\rho' - \rho|^2 + \lambda^2 \rho^2) \leq \frac{4c}{\rho^2} (1 + |\rho' - \rho|^2 + \lambda^2 \rho^2).
\]

\[\square\]

**Lemma 4.2.** There is a \( C > 0 \) such that for all non-negative \( f \in B(\Sigma) \) and \( \lambda < 1 \),
\[
\sup_{H \geq \frac{1}{4} \lambda^{-2}} U^{(k)}(s, f) \leq C \lambda^{-1} \sup_{H \geq \frac{1}{4} \lambda^{-2}} f(s) + \sup_{H \leq \frac{1}{4} \lambda^{-2}} U^{(k)}(s, f), \text{ or}
\]
\[
\sup_{H > \frac{1}{4} \lambda^{-2}} U^{(k)}(s, f) \leq C \lambda^{-1} \sup_{H \leq \frac{1}{4} \lambda^{-2}} f(s) + \sup_{H \leq \frac{1}{4} \lambda^{-2}} U^{(1)}(s, f) + C e^{-\lambda^{-1}} \|f\|_{\infty},
\]
where \( H = H(s) \).

**Proof.** For \( s \in \Sigma \) with \( H(s) \geq \frac{1}{2} \lambda^{-2} \), we can write \( U^{(k)}(s, f) \) as
\[
U^{(k)}(s, f) = E_s^{(k)} \left[ \int_0^\infty e^{-\int_0^t \delta(S) f(S_t) dt} \right]
\]
\[
\leq \|f\|_{\infty} E_s^{(k)}[\omega] + E_s^{(k)}[U^{(k)}(S_\omega, f)] \leq C \lambda^{-1} \|f\|_{\infty} \left( \sup_{H \leq \frac{1}{4} \lambda^{-2}} U^{(k)}(s', f) + E_s^{(k)}[U^{(k)}(S_\omega, f) \chi(H(S_\omega) \leq 1)] \right)
\]

(4.9)

where \( \omega \) is the hitting time that \( H(S_t) \) jumps below \( \frac{1}{2} \lambda^{-2} \). For the second inequality, we have used that \( E_s^{(k)}[\omega] \) is bounded by a constant multiple of \( \lambda^{-1} \) for all \( \lambda < 1 \) and \( s \) with \( H(s) \geq \frac{1}{4} \lambda^{-2} \), which we show below.

Let the function \( W : \Sigma \to [0, 1] \) be defined as in part (3) of proposition 3.3. By part (3) of proposition 3.3, there is a \( c > 0 \) such that the process \( W(S_t) + c \lambda t \) is a supermartingale over the time interval \([0, \omega]\). Thus, the optional stopping theorem gives the first inequality below:
\[
E_s^{(k)}[\omega] \leq \frac{1}{c \lambda} E_s^{(k)}[W(s) - W(S_\omega)] \leq \frac{1}{c \lambda}.
\]

For the second inequality, we have used that \( 0 \leq W \leq 1 \). Thus, \( E_s^{(k)}[\omega] \) is bounded by \( C \frac{1}{\lambda} \) for \( C = \frac{1}{\lambda} \), and plugging this into (4.9) gives the first inequality in the statement of the lemma.

Now we will show that the last term on the right-hand side of (4.10) is \( \|f\|_{\infty} O(e^{-\lambda^{-1}}) \).

Let \( \omega' \) be the collision time that precedes \( \omega \), and \( \beta^{(k)}_{\omega'}(s') \) be the conditional probability density for \( s' = S_{\omega'} \) when given the value \( s = S_\omega \). By the strong Markov property,
\[
E_s^{(k)}[U^{(k)}(S_\omega, f) \chi(H(S_\omega) \leq 1)] = E_s^{(k)} \left[ \int_{H \leq \omega} d\gamma \beta^{(k)}_{S_\omega, \omega'}(s') U^{(k)}(s', f) \right]
\]

(4.10)
\[ \left( \sup_{H > \frac{1}{\lambda^2}, \ H' \leq \lambda} \beta^{(\lambda)}_s(s') \right) \int_{H' \leq \lambda} ds' U^{(\lambda)}(s', f) \]

\[ \leq c \lambda^{-2} \| f \|_\infty \sup_{H > \frac{1}{\lambda^2}, \ H' \leq \lambda} \beta^{(\lambda)}_s(s'), \]  

where \( H = H(s), \ H' = H(s') \). The second inequality is by part (2) of lemma 2.2 and that \( \int_{\Sigma_e} ds \ e^{-\lambda H(s)} \propto \lambda^{-\frac{1}{2}} \) for \( \lambda \ll 1 \). To make use of (4.11), we must bound the values of \( \beta^{(\lambda)}_s(s') \).

Let \((x(s), p(s))\) be the phase space point at time \( t > 0 \) when evolving according to the Hamiltonian evolution from the point \( s \in \Sigma \). The density \( \beta^{(\lambda)}_s(s') \) can be written as

\[ \beta^{(\lambda)}_s(s') = \kappa^{(\lambda)}_s(s') \int_{\Sigma} R_s(p', p) \chi(H(s') \leq \frac{1}{2} \lambda^{-2}) \]

where \( \kappa^{(\lambda)}_s(s') \) is the probability measure on \( T \) given by

\[ \kappa^{(\lambda)}_s(s') = \int_0^\infty \! dt \! e^{-\int_0^t \! \delta (p(s')) \int_{\Sigma} \! \delta (s', s) \delta (s(s) - s') \int_0^\infty \! \delta (p(s), p') \frac{\partial}{\partial p} \mathcal{J}_s(p, p') \! \frac{\partial}{\partial p} \mathcal{J}_s(p, p') \! \frac{\partial}{\partial p} \mathcal{J}_s(p, p') \}

However, there is a \( c > 0 \) such that for all \( \lambda < 1 \),

(i) \[ \sup_{H > \frac{1}{\lambda^2}, \ H' \leq \lambda} \kappa^{(\lambda)}_s(s') < c, \] (ii) \[ \sup_{H > \frac{1}{\lambda^2}, \ H' \leq \lambda} \frac{\mathcal{J}_s(p, p')}{\mathcal{J}_s(p, p')} \leq c e^{-\frac{1}{2} \lambda^{-2}}. \]

Plugging (4.12) into (4.11) and using (i) and (ii) gives the result. Statement (ii) follows from the Gaussian decay in the rates \( \mathcal{J}_s(p, p') \) and that \( V(x) \) is bounded. For statement (i), we actually have that \( \sup_{x} | \kappa^{(\lambda)}_s(s') - 1 | = O(\lambda) \) for \( H(s) > \frac{1}{2} \lambda^{-2} \) and \( \lambda \ll 1 \). We will sketch why. Note that by the conservation of energy, \( H(s) = H(x(s), p(s)) \). For \( |p| > \lambda^{-1} > \sqrt{2 \sup_{x} V(x)} \), \( p(s) \) and \( p \) have the same sign and

\[ |p(s) - p| = |\sqrt{p^2 + 2V(x)} - 2V(x(s)) - |p| | \leq 2 \sup_{x} V(x) \frac{|p|}{|p|}. \]

This implies that there is very little deviation of the values \( p(s) \) from \( p \) when \( |p| > \lambda^{-1} \). Consequently, \( x(s) \) revolves around the torus with nearly uniform speed \( |p| \), and the terms \( \mathcal{E}_s(p(s)) \) and \( \mathcal{J}_s(p(s), p) \) in expression (4.13) are approximately equal to \( \mathcal{E}_s(p) \) and \( \mathcal{J}_s(p, p') \), respectively. Moreover, the starting location \( x \) on the torus only makes a difference in \( \kappa^{(\lambda)}_s(x') \) on the order of \( O(\lambda) \), since the decay factor in (4.13) satisfies \( e^{-\int_0^\infty \delta (p(s)), p) \frac{\partial}{\partial p} \mathcal{J}_s(p, p') \frac{\partial}{\partial p} \mathcal{J}_s(p, p') \}

Putting parts (1) and (3) of lemma 4.1 together gives an inequality including the values of the function \( \hat{U}^{(\lambda)} f \) and weighted integrals of those values. This suggests using a Gronwall-type recursive scheme to obtain bounds for \( \hat{U}^{(\lambda)} f \). However, it is useful to bring the results of lemma 4.1 into a more tailored form that is amenable to recursion, and this is the purpose of the following lemma.

**Lemma 4.3.** There exist \( C, C', \ L > 0 \) such that for all \( \lambda < 1 \), \( \gamma = (p, \epsilon) \) with \( \rho \ll \lambda^{-1} \) and non-negative \( f \in B(\Sigma) \),

\[ \hat{U}^{(\lambda)}(\gamma, f) \leq C \| f \|_\infty + C \rho \sup_{\rho' > \lambda^{-1}} f(\gamma') + C \int_{\rho' < \lambda^{-1}} \! \frac{\partial}{\partial \rho'} \! (1 + \min(\rho, \rho')) \frac{\partial}{\partial \rho'} \! \hat{U}^{(\lambda)}(\gamma', f) \]

\[ + C \int_{\rho' \leq \rho'' \leq \lambda^{-1}} \! \frac{\partial}{\partial \rho'} \! (1 + \min(\rho, \rho')) (1 + \rho')^3 \hat{U}^{(\lambda)}(\gamma', f), \]

(4.14)
and \( C \int_{\sqrt{U} \leq \rho' \leq L^k} dy' \frac{1}{(1+\rho')^2} \leq \frac{1}{2}. \)

**Proof.** By part (1) of lemma 4.1, we have the equality \( \tilde{U}^{(k)} f = U^{(k)} (\hat{f} - E_k). \) Applying part (2) of lemma 3.5 gives the inequality

\[
\tilde{U}^{(k)}(y, f) = U^{(k)}(y, \hat{f} - E_k) \leq U^{(k)}(y, \hat{f}) + U^{(k)}(y, |E_k|) + U^{(k)}(y, |E_k - E_k'|)
\]

\[
\leq C \left( \left\| \frac{\hat{f}}{\xi \tilde{E}_k} \right\|_{\infty} + \left\| \frac{|E_k|}{\xi \tilde{E}_k} \right\|_{\infty} + \left\| \frac{|E_k - E_k'|}{\xi \tilde{E}_k} \right\|_{\infty} \right)
\]

\[
+ c \rho \sup_{\rho' > \frac{1}{2} L^k} \left( \frac{\hat{f}(y')}{\xi \tilde{E}_k(y')} + \frac{|E_k(y')|}{\xi \tilde{E}_k(y')} + \frac{|E_k(y') - E_k'(y')|}{\xi \tilde{E}_k(y')} \right)
\]

\[
+ c \int_{\rho' \leq L^k} dy' (1 + \min(\rho, \rho')) \frac{\hat{f}(y')}{(1 + \rho')^3}
\]

for some \( c > 0, \) where \( E_k' \) is defined as in the proof of part (3) of proposition 4.1. We will show that there is \( C' > 0 \) such that

\[
\tilde{U}^{(k)}(y, f) \leq C' \| f \|_\infty + C' \rho \sup_{H > \frac{1}{2} L^k} f(s') + C' \lambda \sup_{\rho' \leq \frac{1}{2} L^k} \tilde{U}^{(k)}(y', f)
\]

\[
+ C' \int_{\rho' \leq \frac{1}{2} L^k} dy' (1 + \min(\rho, \rho')) \hat{f}(y')
\]

\[
+ C' \int_{\rho' \leq \frac{1}{2} L^k} dy' (1 + \min(\rho, \rho')) \frac{\hat{f}(y')}{(1 + \rho')^3} \tilde{U}^{(k)}(y', f).
\]

Given (4.16), we can split the integral \( \int_{\rho' \leq \frac{1}{2} L^k} \) of the last term into two parts \( \int_{\sqrt{U} \leq \rho' \leq \frac{1}{2} L^k} \) and \( \int_{\rho' \leq \sqrt{U}} \) with \( L > 0 \) large enough so that

\[
C' \int_{\sqrt{U} \leq \rho' \leq \frac{1}{2} L^k} dy' \frac{1}{(1 + \rho')^2} \leq C' \frac{C}{\sqrt{2} L} \leq \frac{1}{2}.
\]

We can bound the remainder \( \int_{\rho' \leq \sqrt{U}} \) through the inequalities

\[
\int_{\rho' \leq \sqrt{U}} dy' (1 + \min(\rho', \rho)) \frac{\hat{f}(y', f)}{(1 + \rho')^3} \tilde{U}^{(k)}(y', f) \leq \int_{\rho' \leq \sqrt{U}} dy' \tilde{U}^{(k)}(y', f) = \int_{H \leq L} ds \tilde{U}^{(k)}(s, f)
\]

\[
\leq a \| f \|_\infty + a \int_{H \leq \frac{1}{2} L^k} ds f(s)
\]

\[
= a \| f \|_\infty + a \int_{H \leq \frac{1}{2} L^k} dy' \hat{f}(y'),
\]

where \( H = H(s). \) The second inequality above holds for some \( a > 0 \) by part (2) of lemma 2.2. Thus, with (4.16) and (4.17), there are \( C', C'' L > 0 \) with \( \frac{C}{\sqrt{2} L} \leq \frac{1}{2} \) such that

\[
\tilde{U}^{(k)}(y, f) \leq C' \| f \|_\infty + C' \rho \sup_{H > \frac{1}{2} L^k} f(s') + C'' \lambda \sup_{\rho' \leq \frac{1}{2} L^k} \tilde{U}^{(k)}(y', f)
\]

\[
+ C' \int_{\rho' \leq \frac{1}{2} L^k} dy' (1 + \min(\rho, \rho')) \hat{f}(y')
\]

\[
+ C' \int_{\rho' \leq \sqrt{U}} dy' (1 + \min(\rho, \rho')) \frac{\hat{f}(y')}{(1 + \rho')^3} \tilde{U}^{(k)}(y', f).
\]
By supremizing both sides of (4.18) over \( \gamma = (\rho, \epsilon) \) with \( \rho \leq \lambda^{-1} \), we obtain
\[
\sup_{\rho' \leq \lambda^{-1}} \tilde{U}^{(1)}(\gamma', f) \leq \frac{C'}{\lambda - \lambda} \left( \|f\|_{\infty} + \lambda^{-1} \sup_{\rho' \geq \lambda^{-1}} f(\gamma') + \int_{\rho' \leq \lambda^{-1}} dy' (1 + \min(\rho', \lambda^{-1})) f(\gamma') \right).
\]
Plugging this bound back into (4.16) gives inequality (4.14) for small enough \( \lambda \) (\( \lambda \) bounded away from zero does not pose a problem).

Now, we work to prove (4.16) starting from (4.15). Since \( \tilde{E}_{\lambda} \) is bounded away from zero by part (2) of proposition 3.3, the expressions on the right-hand side of (4.15) with \( \tilde{f} \) and \( E_{\lambda} - E'_{\lambda} \) are not problematic, since in particular, we can bound \( E_{\lambda} - E'_{\lambda} \) with part (3) of lemma 4.1. For the term \( E'_{\lambda} \), we have the following expressions to bound:
\[
(i) \|E'_{\lambda}\|_{\infty}, \quad (ii) \rho \sup_{\rho' \geq \lambda^{-1}} \|E'_{\lambda}(\gamma')\|_{\infty}, \quad (iii) \int_{\rho' \leq \lambda^{-1}} dy' (1 + \min(\rho, \rho')) \|E'_{\lambda}(\gamma')\|_{\infty}.
\]
We will discuss (ii) and (iii), since (i) is handled similarly. In each case, we seek a bound using a linear combination of the terms on the right-hand side of (4.16).

By part (3) of lemma 4.1 and \( \rho \leq \lambda^{-1} \),
\[
\rho \sup_{\rho' \geq \lambda^{-1}} \|E'_{\lambda}(\gamma')\|_{\infty} \leq C \lambda^{-1} \sup_{\rho' \geq \lambda^{-1}} \int_{\Gamma'} dy' \tilde{T}_3(\gamma', \gamma') M_3(\gamma, \gamma') \tilde{U}^{(1)}(\gamma', f)
\leq C \lambda^{-1} \left( \sup_{\rho' \geq \lambda^{-1}, \gamma' \in \Gamma} \tilde{T}_3(\gamma', \gamma') M_3(\gamma, \gamma') \right) \left( \sup_{\gamma'} \tilde{U}^{(1)}(\gamma', f) \right)
\leq C \lambda^2 \sup_{\gamma' \in \Gamma} \tilde{U}^{(1)}(\gamma', f)
\leq C' \lambda^2 \|f\|_{\infty} + C' \lambda \sup_{H' > \lambda^{-2}} f(s') + C' \lambda^2 \sup_{\rho' \leq \lambda^{-1}} \tilde{U}^{(1)}(\gamma', f). \tag{4.19}
\]

The fourth inequality in (4.19) follows since
\[
\sup_{\rho' \geq \lambda^{-1}} \tilde{U}^{(1)}(\gamma', f) \leq \sup_{H' > \lambda^{-2}} U^{(1)}(s', f)
\leq c e^{-\lambda^{-1}} \|f\|_{\infty} + \lambda^{-1} \sup_{H' > \lambda^{-2}} f(s') + \sup_{l < H(s)} U^{(1)}(s', f)
\leq c \|f\|_{\infty} + \lambda^{-1} \sup_{H' > \lambda^{-2}} f(s') + c' \sup_{\rho' \leq \lambda^{-1}} \tilde{U}^{(1)}(\gamma', f), \tag{4.20}
\]
where \( H' = H(s') \) and \( \gamma' = (\rho', \epsilon') \). For the first inequality above, the values \( \tilde{U}^{(1)} \) are averages of the values for \( U^{(1)} \). The second inequality is for some \( c > 0 \) by lemma 4.2, and the third uses that \( U^{(1)}(s, f) \) is bounded by a multiple \( c' \) of \( \tilde{U}^{(1)}(\gamma(s), f) \) for \( s \in \Sigma \) with \( l < H(s) \) by part (2) of lemma 4.1.

Next, we bound term (iii). By part (3) of lemma 4.1,
\[
\int_{\rho' \leq \lambda^{-1}} dy' (1 + \min(\rho, \rho')) |E'_1(\gamma')|.
\]
\[
\leq C \int_{\rho' \leq \lambda^{-1}} dy' (1 + \min(\rho, \rho')) \int_{\Gamma'} dy'' \tilde{T}_3(\gamma', \gamma'') M_3(\gamma', \gamma'') \tilde{U}^{(1)}(\gamma'', f)
\leq C \left( \sup_{\rho' \geq \lambda^{-1}} \tilde{U}^{(1)}(\gamma'', f) \right) \int_{\rho' \leq \lambda^{-1}} dy' (1 + \min(\rho, \rho'))
\]
\[ \times \int_{\rho' \geq \lambda^{-1}} dy'' \hat{f}_k(y', y'') M_k(y', y'') + C \int_{\rho' \leq \lambda^{-1}} dy'' \hat{U}^{(k)}(y', y') f(y') \int_{\rho' \geq \lambda^{-1}} dy' (1 + \min(\rho, \rho')) \hat{f}_k(y', y'') M_k(y', y'') \leq C\lambda \sup_{\rho > \lambda^{-1}} f(y) + C\lambda^2 \sup_{\rho \leq \lambda^{-1}} \hat{U}^{(k)}(y, f) + C' \int_{\rho' \leq \lambda^{-1}} dy'' \frac{(1 + \min(\rho'', \rho'))}{(1 + \rho'')^3} \hat{U}^{(k)}(y'', f). \] (4.21)

The last inequality in (4.21) follows from (I)–(III) below, where (I) and (II) are for the first term and (III) is for the second term.

(I) There is \( c > 0 \) such that for all \( \lambda < 1 \),

\[ \sup_{\rho \geq \lambda^{-1}} \hat{U}^{(k)}(y, f) \leq c\lambda^{-1} \sup_{\rho > \lambda^{-1}} f(y) + c \sup_{\rho \leq \lambda^{-1}} \hat{U}^{(k)}(y, f) + c\|f\|_{\infty}. \]

(II)

\[ \sup_{\lambda < 1} \sup_{\rho, \rho' \leq \lambda^{-1}} \lambda^{-2} \int_{\rho' \leq \lambda^{-1}} dy' (1 + \min(\rho, \rho')) \int_{\rho'' \geq \lambda^{-1}} dy'' \hat{f}_k(y', y'') M_k(y', y'') < \infty. \]

(III)

\[ \sup_{\lambda < 1} \sup_{\rho, \rho' \leq \lambda^{-1}} (1 + \rho'')^3 \int_{\rho' \leq \lambda^{-1}} dy' \frac{(1 + \min(\rho, \rho'))}{(1 + \min(\rho, \rho''))} \hat{f}_k(y', y'') M_k(y', y'') < \infty. \]

Statement (I) is from (4.20). Statements (II) and (III) use the decay from \( M_k(y', y'') \) and that the transition kernels \( \hat{T}_k(y', y') \) have uniformly bounded Gaussian tails in the quasi-momentum \( |q(y) - q(y')| \) for \( y = (\rho, \epsilon) \) with \( \rho \leq \lambda^{-1} \). We do not go through the details of these inequalities.

5. Proof of theorem 1.1

For \( s \in \Sigma \) with \( H(s) > \frac{1}{4} \lambda^{-2} \), we have the inequality

\[ U^{(k)}(s, f) \leq c\lambda^{-1} \sup_{H > \frac{1}{4} \lambda^{-2}} f(s') + \sup_{H \leq \frac{1}{4} \lambda^{-2}} U^{(k)}(s', f) \] (5.1)

for some \( c > 0 \) by lemma 4.2. Thus, it is sufficient to prove the statement of the theorem for the domain \( H(s) \leq \frac{1}{4} \lambda^{-2} \).

For \( s \in \Sigma \) with \( l < H(s) \leq \frac{1}{4} \lambda^{-2} \), there is a \( C > 0 \) such that for all \( s \) and \( \lambda < 1 \),

\[ U^{(k)}(s, f) \leq C\hat{U}^{(k)}(y(s), f) \]

by part (2) of lemma 4.1. Hence, for the domain \( l < H(s) \leq \frac{1}{4} \lambda^{-2} \), it is sufficient to bound the values of \( \hat{U}^{(k)}(y, f) \) for \( y = (\rho, \epsilon) \) with \( \sqrt{2}l < \rho \leq \lambda^{-1} \). By lemma 4.3, there are \( c, c', L > 0 \) such that for all \( \lambda < 1 \) and \( y = (\rho, \epsilon) \) with \( \rho \leq \lambda^{-1} \),

\[ \hat{U}^{(k)}(y, f) \leq \|f\|_{\infty} + c\rho \sup_{H > \frac{1}{4} \lambda^{-2}} f(s') + c \int_{\rho' \leq \lambda^{-1}} dy' (1 + \min(\rho', \rho')) \hat{f}(y') + c' \int_{\sqrt{2}l \leq \rho' \leq \lambda^{-1}} dy' \hat{U}^{(k)}(y', f) \frac{1 + \min(\rho', \rho')}{(1 + \rho')^3}. \] (5.2)
where

\[
c' \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' \frac{1}{(1 + \rho')^2} \leq \frac{1}{2}.
\]  

(5.3)

It immediately follows that

\[
\sup_{\rho \leq \lambda^{-1}} \tilde{U}^{(k)}(y, f) \leq 2c \|f\|_{L_\infty} + 2c \lambda^{-1} \sup_{H > \frac{1}{\lambda^2}} f(s') + 2c \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' (1 + \rho') f'(y').
\]  

(5.4)

However, we need a more refined upper bound than (5.4) given by (5.5) below. By recursively applying inequality (5.2) as in the proof of Gronwall’s inequality, we obtain a series bound

\[
\tilde{U}^{(k)}(y, f) \leq c \|f\|_{L_\infty} + c \rho \sup_{H > \frac{1}{\lambda^2}} f(s') + c \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' (1 + \min(\rho, \rho')) \tilde{f}(y')
\]

\[+ c \sum_{n=1}^{\infty} (c')^n \int \sqrt{\mathcal{L}} \leq \rho_n \leq \lambda^{-1} \ 1 \leq m \leq n \ dy_1 \cdots dy_n \prod_{m=0}^{n-1} \frac{1 + \min(\rho_{m+1}, \rho_m)}{(1 + \rho_{m+1})^3} \]

\[\times (\|f\|_{L_\infty} + \rho_n \sup_{H > \frac{1}{\lambda^2}} f(s')) \]

\[\quad + c \sum_{n=1}^{\infty} (c')^n \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' \int \sqrt{\mathcal{L}} \leq \rho_n \leq \lambda^{-1} \ 1 \leq m \leq n \ dy_1 \cdots dy_n \tilde{f}(y')(1 + \min(\rho', \rho_n)) \]

\[\times \prod_{m=0}^{n-1} \frac{1 + \min(\rho_{m+1}, \rho_m)}{(1 + \rho_{m+1})^3} \]

\[\leq 2c \|f\|_{L_\infty} + 2c \rho \sup_{H > \frac{1}{\lambda^2}} f(s') + 2c \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' (1 + \min(\rho', \rho)) \tilde{f}(y'),
\]  

(5.5)

where we have denoted \(\rho_0 := \rho\) in the argument of the products. The second inequality uses that

\[
c' \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' (1 + \min(\rho, \rho'))(1 + \min(\rho'', \rho')) \leq \frac{1}{2}(1 + \min(\rho', \rho)),
\]  

(5.6)

which follows by (5.3).

The series bound in (5.5) holds, since the error after the \(n\)th iteration of inequality (5.2) is

\[
(c')^{n-1} \int \sqrt{\mathcal{L}} \leq \rho_n \leq \lambda^{-1} \ dy_1 \cdots dy_n \tilde{U}^{(k)}(y_n, f) \frac{1 + \min(\rho_1, \rho) \prod_{m=0}^{n-1} \frac{1 + \min(\rho_{m+1}, \rho_m)}{(1 + \rho_{m+1})^3}}{(1 + \rho')^3} \]

\[\leq \frac{1}{2n-1} \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' \tilde{U}^{(k)}(y', f) \frac{1 + \min(\rho', \rho)}{(1 + \rho')^3} \]

\[\leq \frac{1}{2n-1} \sup_{\rho \leq \lambda^{-1}} \tilde{U}^{(k)}(y, f) \]

\[\leq \frac{c}{2n-2} \left( \|f\|_{L_\infty} + \lambda^{-1} \sup_{H > \frac{1}{\lambda^2}} f(s') + \int \sqrt{\mathcal{L}} \leq \rho' \leq \lambda^{-1} \ dy' (1 + \rho') \tilde{f}(y') \right),
\]

which goes to zero for large \(n\). The first inequality is from (5.6), and the third is from (5.4). Inequality (5.5) implies that there is a \(c'' > 0\) such that for all \(H(x, p) \geq 1\),

\[
\tilde{U}^{(k)}(y(x, p), f) \leq c'' \|f\|_{L_\infty} + c'' |p| \sup_{H > \frac{1}{\lambda^2}} f(s')
\]

\[+ c'' \int_{H \leq \frac{1}{\lambda^2}} dp' \hat{d}x' (1 + \min(|p'|, |p|)) f(x', p'),
\]  

(5.7)
where we have used that \(|p - q| \leq 2 \hat{H}(s)\) is bounded since \(2 \hat{H}(s) = \rho(s)\). This proves our bound for the domain \(l < H(s) \leq \frac{1}{2} \lambda^{-2}\).

Next, we bound \(U^{(k)}(s', \lambda)\) in the domain \(H(s) \leq 1\). Let \(T_{n} = U^{(k)}_{h'}\) for \(h' = 1\Sigma\). The operator \(T_{n} : B(\Sigma) \rightarrow B(\Sigma)\) is the transition kernel for a Markov chain \(\sigma_{n}\) (i.e. the resolvent chain). We have the following identity, which is closely related to part (3) of proposition 2.1:

\[
U^{(k)}(s, \lambda) = \sum_{n=1}^{\infty} \int_{H(s_{n}) \leq l} \int_{H(s_{n-1}) \leq l} T_{n}(s, ds_{1}) \cdots T_{s}(s_{n-1}, ds_{n})
\]

\[
\times \left( \sum_{n=1}^{\infty} (1 - h(s_{1})) \cdots (1 - h(s_{n-1})) f(s_{n}) + (1 - h(s_{1})) \cdots (1 - h(s_{n-1})) U^{(k)}(s_{n}, f) \right).
\]

The integration variable \(s_{n}\) corresponds to the first time that the chain \(\sigma_{n}\) jumps out of the set \(H(s) \leq 1\). The above gives the inequalities

\[
U^{(k)}(s, \lambda) < \sum_{n=1}^{\infty} \int_{H(s_{n}) \leq l} \int_{H(s_{n-1}) \leq l} T_{n}(s, ds_{1}) \cdots T_{s}(s_{n-1}, ds_{n}) (n\|f\|_{\infty} + U^{(k)}(s_{n}, f))
\]

\[
\leq \sup_{\lambda < 1} \sup_{\|f\|_{\infty} \leq 1} \int_{H(s') \leq l} T_{s}(s, ds') (1 - \sup_{\lambda < 1} \sup_{\|f\|_{\infty} \leq 1} \int_{H(s') \leq l} T_{s}(s, ds'))^{2}
\]

\[
\times \left(\|f\|_{\infty} + \sup_{\lambda < 1} \sup_{\|f\|_{\infty} \leq 1} \int_{H(s') \leq l} T_{s}(s, ds') U^{(k)}(s', f)\right).
\]

The second inequality uses Holder’s inequality, that \(U^{(k)}(s_{n}, f) \leq nU^{(k)}(s_{n}, f)\) for \(n \geq 1\), and a geometric sum formula. The probability \(\int_{H(s') \leq l} T_{s}(s, ds')\) can be easily shown to be bounded away from zero for all \(\lambda < 1\) and \(s \in \Sigma\) by considering the event that a single collision occurs over the time interval \([0, l]\) and the particle jumps to an energy \(> l\). Moreover, the jump measures \(T_{s}(s, ds')\) have uniformly bounded exponential tails for \(H(s) \leq l\) and \(\lambda < 1\), since the collision rates \(J_{n}(p, p')\) have Gaussian tails. Thus, we can apply our bound for the values of \(U^{(k)}(s, f)\) over the domain \(H(s') > l\) to obtain our required bound.

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