A Unified View of RS Braneworlds

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There are various different descriptions of Randall-Sundrum (RS) braneworlds. Here we present a unified view of the braneworld based on the gradient expansion approach. In the case of the single-brane model, we reveal the relation between the geometrical and the AdS/CFT approach. It turns out that the high energy and the Weyl term corrections found in the geometrical approach merge into the CFT matter correction found in the AdS/CFT approach. We also clarify the role of the radion in the two-brane system. It is shown that the radion transforms the Einstein theory with Weyl correction into the conformally coupled scalar-tensor theory where the radion plays the role of the scalar field.

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I. INTRODUCTION

Nowadays, the most promising and seemingly a unique candidate for quantum theory of gravity is the string theory. Remarkably, it can be consistently formulated only in 10 dimensions [1]. This fact requires a mechanism to fill the gap between our real world and the higher dimensions. Conventionally, the extra dimensions are considered to be compactified to a small compact space of the Planck scale. However, recent developments of superstring theory invented a new idea, the so-called braneworld. The brane world scenario has been the subject of intensive investigation for the past few years. Although there are many braneworld models, there are similarities in those models. Hence, we study a simple toy model constructed by Randall and Sundrum as a representative [2, 3].

There are various views of RS braneworlds depending on the approach one uses. The purpose of this paper is to unify the various views using the low energy gradient expansion method [4, 5] and give insights into the physics of the braneworld.

The organization of this paper is as follows: In Sec.II, we summarize various views of the braneworlds obtained by different methods. In Sec.III, key questions are presented. In Sec.IV, we explain the gradient expansion method. In Sec.V and VI, the single-brane model and the two-brane model are analyzed separately. The final section is devoted to the answers to the key questions.

II. VARIOUS VIEWS OF BRANEWORLDS

Randall and Sundrum proposed a simple model where the four-dimensional brane with tension $\sigma$ is embedded in the five-dimensional asymptotically anti-de Sitter (AdS) bulk with a curvature scale $\ell$. This single-brane model is described by the action [3]

$$ S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( R + \frac{12}{\ell^2} \right) - \sigma \int d^4x \sqrt{-h} + \int d^4x \sqrt{-h} L_{\text{matter}}, $$

where $R$ and $\kappa^2$ are the scalar curvature and gravitational constant in five-dimensions, respectively. We impose $Z_2$ symmetry on this spacetime, with the brane at the fixed point. The matter $L_{\text{matter}}$ is confined to the brane. Remarkably, the internal dimension is non-compact in this model. Hence, we do not have to care about the stability problem.

Originally, they proposed the two-brane model as a possible solution of the hierarchy problem [2]. The action reads

$$ S = \int d^5x \sqrt{-g} \left( R + \frac{12}{\ell^2} \right) - \sum_{i=\oplus, \ominus} \sigma_i \int d^4x \sqrt{-h_i} + \sum_{i=\oplus, \ominus} \int d^4x \sqrt{-h_i} L_{\text{matter}}, $$

where $\oplus$ and $\ominus$ represent the positive and the negative tension branes, respectively.

Both models are prototypes of other models. Here, we list up the various approaches to understand these prototypes.

A. Cosmological Approach

The homogeneous cosmology of the single-brane model is the simplest case to be studied. It is easy to deduce the effective Friedmann equation as

$$ H^2 = \frac{\kappa^2}{\ell} \rho + \kappa^4 \rho^2 + \frac{C}{a_0^4}, $$

where $H$, $a_0$ and $\rho$ are, respectively, the Hubble parameter, the scale factor and the total energy density.
of each brane. The Newton’s constant can be identified as $8\pi G_N = \kappa^2 / \ell$. Here, $C$ is a constant of integration associated with the mass of a black hole in the bulk. This constant $C$ is referred to as the dark radiation in which the effect of the bulk is encoded.

There are two kinds of corrections, the high energy correction $\rho^2$ and the bulk correction $C$ which exists even in the low energy regime. Thus, the deviation from the conventional Einstein theory is expected even in the low energy regime.

As to the two-brane model, the same effective Friedmann equation can be expected because this equation can be deduced without referring to the bulk equations of motion. In the two-brane case, however, the meaning of $C$ is obscure.

### B. Linear Perturbation Approach

The other useful method to investigate the braneworld is the linearized analysis.

In the case of the single-brane model, it was shown that the gravity is localized on the brane in spite of the noncompact extra dimension. Consequently, it turned out that the conventional linearized Einstein equation approximately holds at scales large compared with the curvature scale $\ell$. It should be stressed that this result can be attained by imposing the outgoing boundary conditions.

In the case of the two-brane model, Garriga and Tanaka analyzed linearized gravity and have shown that the gravity on the brane behaves as the Brans-Dicke theory at low energy [6]. Thus, the conventional linearized Einstein equations do not hold even on scales large compared with the curvature scale $\ell$ in the bulk. Charmousis et al. have clearly identified the Brans-Dicke field as the $\text{AdS}/\text{CFT}$ correspondence reads

$$S_{\text{ct}} = S_{\text{brane}} - S_{4d} - \left[ R^2 \text{terms} \right] ,$$

where $S_{\text{brane}}$ and $S_{4d}$ are the brane action and the 4-dimensional Einstein-Hilbert action, respectively. Here, the higher curvature terms $[R^2\text{terms}]$ should be understood symbolically.

In the case of the braneworld, the brane acts as the cutoff. Therefore, there is no divergences in the above expressions. Hence, we can freely rearrange the terms as follows

$$S_{5d} + S_{\text{brane}} = S_{4d} + S_{\text{CFT}} + \left[ R^2 \text{terms} \right]$$

This tells us that the brane models can be described as the conventional Einstein theory with the cutoff CFT and higher order curvature terms. In terms of the equations of motion, the $\text{AdS}/\text{CFT}$ correspondence reads

$$G_{\mu\nu} = \frac{\kappa^2}{\ell} \left( T_{\mu\nu} + T_{\text{CFT}}^\mu \right) + \left[ R^2 \text{terms} \right] ,$$

where the $R^2$ terms represent the higher order curvature terms and $T_{\text{CFT}}^\mu$ denotes the energy-momentum tensor of the cutoff version of conformal field theory.

### D. Geometrical Approach

Here, let us review the geometrical approach [3]. In the Gaussian normal coordinate system:

$$ds^2 = dy^2 + g_{\mu\nu}(y, x^\alpha) dx^\mu dx^\nu ,$$

we can write the 5-dimensional Einstein tensor $G_{\mu\nu}^{(5)}$ in terms of the 4-dimensional Einstein tensor $G_{\mu\nu}^{(4)}$ and the extrinsic curvature as

$$G_{\mu\nu} = G_{\mu\nu}^{(4)} + K_{\mu\nu,y} - g_{\mu\nu} K_{yy} - K K_{\mu\nu} + 2 K_{\mu\lambda} K_{\nu}^{\lambda} \nu
+ \frac{1}{2} g_{\mu\nu} \left( K^2 + K^{\alpha \beta} K_{\alpha \beta} \right)$$

$$= \frac{6}{\ell^2} g_{\mu\nu} ,$$

where we have introduced the extrinsic curvature

$$K_{\mu\nu} = -\frac{1}{2} g_{\mu\nu,y} ,$$

and the last equality comes from the 5-dimensional Einstein equations. Taking into account the $\mathbb{Z}_2$ symmetry, we also obtain the junction condition

$$[K^{\mu \nu} - \delta^{\mu \nu} K] \bigg|_{y=0} = \frac{\kappa^2}{2} (-\sigma \delta^{\mu \nu} + T^{\mu \nu}) .$$

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$$[K^{\mu \nu} - \delta^{\mu \nu} K] \bigg|_{y=0} = \frac{\kappa^2}{2} (-\sigma \delta^{\mu \nu} + T^{\mu \nu}) .$$
Here, $T_{\mu\nu}$ represents the energy-momentum tensor of the matter. Evaluating Eq. (10) at the brane and substituting the junction condition into it, we have the “effective” equations of motion

$$G_{\mu\nu} = \kappa^2 \frac{\ell}{T_{\mu\nu}} + \kappa^4 \pi_{\mu\nu} - E_{\mu\nu}$$  \hspace{1cm} (13)$$

where

$$\pi_{\mu\nu} = -\frac{1}{4} T^{\lambda}_{\mu} T_{\lambda\nu} + \frac{1}{12} TT_{\mu\nu} + \frac{1}{8} g_{\mu\nu} \left( T^{\alpha\beta} T_{\alpha\beta} - \frac{1}{3} T^2 \right)$$

$$E_{\mu\nu} = C_{\mu\nu\rho|y=0} ,$$

Here $C_{\mu\nu\rho}$ is the Weyl tensor. We assumed the relation

$$\kappa^2 \sigma = 6 \frac{\ell}{\ell}$$  \hspace{1cm} (14)$$

so that the effective cosmological constant vanishes.

The geometrical approach is useful to classify possible corrections to the conventional Einstein equations. One defect of this approach is the fact that the projected Weyl tensor can not be determined without solving the equations in the bulk.

### III. KEY QUESTIONS

As a landmark, we set a sequence of questions. We consider the single-brane model and two-brane model, separately.

#### A. Single-brane model

**Is the Einstein theory recovered even in the non-linear regime?**

In the case of the linear theory, it is known that the conventional Einstein theory is recovered at low energy.

On the other hand, the cosmological consideration suggests the deviation from the conventional Friedmann equation even in the low energy regime. This is due to the dark radiation term.

Therefore, we need to clarify this discrepancy.

**How does the AdS/CFT come into the braneworld?**

It was argued that the cutoff CFT comes into the braneworld. However, no one knows what is the cutoff CFT. It is a vague concept at least from the point of view of the classical gravity. Moreover, it should be noted that the AdS/CFT correspondence is a specific conjecture. Indeed, originally, Maldacena conjectured that the supergravity on $AdS_5 \times S^5$ is dual to the four-dimensional $N = 4$ super Yang-Mills theory \[9\]. Nevertheless, the AdS/CFT correspondence seems to be related to the brane world model as has been demonstrated by several people \[10\].

Hence, it is important to reveal the role of the AdS/CFT correspondence starting from the 5-dimensional general relativity.

**How are the AdS/CFT and geometrical approach related?**

The geometrical approach gives

$$G_{\mu\nu} = \frac{\kappa^2}{\ell} T_{\mu\nu} + \kappa^4 \pi_{\mu\nu} - E_{\mu\nu} .$$

On the other hand, the AdS/CFT correspondence yields

$$G_{\mu\nu} = \frac{\kappa^2}{\ell} (T_{\mu\nu} + T_{\mu\nu}^{CFT}) + [ R^2 \text{terms} ] .$$

An apparent difference is remarkable.

It is an interesting issue to clarify how these two descriptions are related. Shiromizu and Ida tried to understand the AdS/CFT correspondence from the geometrical point of view \[11\]. They argued that $\pi_{\mu\nu}$ corresponds to the trace anomaly of the cutoff CFT on the brane. However, this result is rather paradoxical because there exists no trace anomaly in an odd dimensional brane although $\pi_{\mu\nu}$ exists even in that case. Thus, the more precise relation between the geometrical and the AdS/CFT approaches remains to be understood.

Since both the geometrical and AdS/CFT approaches seem to have their own merit, it would be beneficial to understand the mutual relationship.

#### B. Two-brane model

**How is the geometrical approach consistent with the Brans-Dicke picture?**

Irrespective of the existence of other branes, the geometric approach gives the effective equations \[13\]. The effect of the bulk geometry comes into the brane world only through $E_{\mu\nu}$. In this picture, the two-brane system can be regarded as the Einstein theory with some corrections due to the Weyl tensor in the bulk.

On the other hand, the linearized gravity on the brane behaves as the Brans-Dicke theory on scales large compared with the curvature scale $\ell$ in the bulk \[6\]. Therefore, the conventional linearized Einstein equations do not hold at low energy.

In the geometrical approach, no radion appears. While, from the linear analysis, it turns out that the system can be described by the Brans-Dicke theory where the extra scalar field is nothing but the radion. How can we reconcile these seemingly incompatible pictures?
What replaces the AdS/CFT correspondence in the two-brane model?

In the single-brane model, there are continuum Kaluza-Klein (KK)-spectrum around the zero mode. They induce the CFT matter in the 4-dimensional effective action.

In the two-brane system, the spectrum become discrete. Hence, we can not expect CFT matter on the brane. Nevertheless, KK-modes exist and affect the physics on the brane.

So, it is still interesting to know what kind of 4-dimensional theory mimics the effect of the KK-modes.

IV. GRADIENT EXPANSION METHOD

Our claim in this paper is that the gradient expansion method gives the answers to all of the questions presented in the previous section. Here, we give the formalism developed by us [4].

We use the Gaussian normal coordinate system (9) to describe the geometry of the brane world. Note that the metric varies with the coordinate system.

Decomposing the extrinsic curvature into the traceless part and the trace part

\[ K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{1}{4} h_{\mu\nu} K, \quad K = -\frac{\partial}{\partial y} \log \sqrt{-g}, \]  

we obtain the basic equations which hold in the bulk;

\[ \Sigma^\mu_{\nu,y} - K \Sigma^\mu_\nu = - \left[ \frac{(4)}{R^\mu_\nu} - \frac{1}{4} \delta^\mu_\nu (4) \right], \]  

\[ \frac{3}{4} K^2 - \Sigma^\alpha_\beta \Sigma^\beta_\alpha = \left[ \frac{(4)}{R} \right] + \frac{12}{\ell^2}, \]  

\[ K_{,y} - \frac{1}{4} K^2 - \Sigma^\alpha_\beta \Sigma_{\alpha\beta} = -\frac{4}{\ell^2}, \]  

\[ \nabla_\lambda \Sigma^\lambda_\mu - \frac{3}{4} \nabla_\mu K = 0, \]  

where \( R^\mu_\nu \) is the curvature on the brane and \( \nabla_\mu \) denotes the covariant derivative with respect to the metric \( g_{\mu\nu} \). One also have the junction condition

\[ [K^\mu_\nu - \delta^\mu_\nu K]_{y=0} = \frac{\kappa^2}{2} (-\sigma \delta^\mu_\nu + T^\mu_\nu) . \]  

Along the normal coordinate \( y \), the metric varies with a characteristic length scale \( \ell \); \( g_{\mu \nu, y} \sim g_{\mu \nu} / \ell \). Denote the characteristic length scale of the curvature fluctuation on the brane as \( L \); then we have \( R \sim g_{\mu \nu} / L^2 \). For the reader’s reference, let us take \( \ell = 1 \) mm, for example. Then, the relation (14) give the scale, \( \kappa^2 = (10^8 \text{ GeV})^{-3} \) and \( \sigma = 1 \text{ TeV}^4 \). In this paper, we will consider the low energy regime in the sense that the energy density of matter, \( \rho \), on the brane is smaller than the brane tension, i.e., \( \rho / \sigma < 1 \). In this regime, a simple dimensional analysis implies that the curvature on the brane can be neglected compared with the extrinsic curvature at low energy. Here, we have used the relation (14) and Einstein’s equation on the brane, \( R \sim g_{\mu \nu} / L^2 \sim \kappa^2 \rho / \ell \). Thus, the anti-Newtonian or gradient expansion method used in the cosmological context is applicable to our problem [2].

At zeroth order, we can neglect the curvature term. Then we have

\[ \Sigma^{(0)}_{\mu \nu} - K^{(0)} \Sigma^{(0)}_\mu = 0, \]  

\[ \frac{3}{4} K^{(0)}^2 - \Sigma^{(0)}_\alpha \Sigma^{(0)}_\beta \Sigma^{(0)}_{\alpha\beta} = \frac{12}{\ell^2}, \]  

\[ K_{,y}^{(0)} - \frac{1}{4} K^{(0)}_\mu - \Sigma^{(0)}_\alpha \Sigma^{(0)}_{\alpha\beta} = -\frac{4}{\ell^2}, \]  

\[ \nabla_\lambda \Sigma^{(0)}_\lambda_\mu - \frac{3}{4} \nabla_\mu K^{(0)} = 0. \]  

Equation (22) can be readily integrated into

\[ \Sigma^{(0)}_{\mu \nu} = \frac{C^{(0)}_{\mu \nu}}{\sqrt{-g}}, \quad C^{(0)}_{\mu \nu} = 0, \]  

where \( C^{(0)}_{\mu \nu} \) is the constant of integration. Equation (22) also requires \( C^{(0)}_{\nu \mid \mu \mid} = 0 \). If it could exist, it would represent a radiation like fluid on the brane and hence a strongly anisotropic universe. In fact, as we see soon, this term must vanish in order to satisfy the junction condition. Therefore, we simply put \( C^{(0)}_{\mu \nu} = 0 \), hereafter. Now, it is easy to solve the remaining equations. The result is

\[ K^{(0)} = \frac{4}{\ell}. \]  

Using the definition of the extrinsic curvature

\[ K^{(0)}_{\mu \nu} = -\frac{1}{2} \frac{\partial}{\partial y} g^{(0)}_{\mu \nu}, \]  

we get the zeroth order metric as

\[ ds^2 = dy^2 + a^2(y) h_{\mu \nu} (x^\mu) dx^\mu dx^\nu, \quad a(y) = e^{-2y}, \]  

where the tensor \( h_{\mu \nu} \) is the induced metric on the brane.
From the zeroth order solution, we obtain
\[ \left[ K^{\mu\nu} - \delta^{\mu\nu} K \right]_{y=0} = \frac{3}{\ell} \delta^{\mu\nu} = -\frac{\kappa^2}{2} \sigma^{\mu\nu}. \] (30)

Then we get the well known relation \( \kappa^2 \sigma = 6/\ell \). Here, we will assume that this relation holds exactly. It is apparent that \( C^{\mu\nu} \) is not allowed to exist.

The iteration scheme consists in writing the metric \( g_{\mu\nu} \) as a sum of local tensors built out of the induced metric on the brane, the number of gradients increasing with the order. Hence, we will seek the metric as a perturbative series
\[ g_{\mu\nu}(y, x^\mu) = a^2(y) \left[ h_{\mu\nu}(x^\mu) + (1) g_{\mu\nu}(y, x^\mu) + (2) g_{\mu\nu}(y, x^\mu) + \cdots \right], \] (31)
where \( a^2(y) \) is extracted and we put the Dirichlet boundary condition
\[ g_{\mu\nu}(y = 0, x^\mu) = 0, \] (32)
so that \( g_{\mu\nu}(y = 0, x) = h_{\mu\nu}(x) \) holds at the brane. Other quantities can be also expanded as
\[ K^{\mu\nu} = \frac{1}{\ell} \delta^{\mu\nu} + (1) K^{\mu\nu} + (2) K^{\mu\nu} + \cdots \]
\[ \Sigma^{\mu\nu} = + (1) \Sigma^{\mu\nu} + (2) \Sigma^{\mu\nu} + \cdots . \] (33)

In our scheme, in contrast to the AdS/CFT correspondence where the Dirichlet boundary condition is imposed at infinity, we impose it at the finite point \( y = 0 \), the location of the brane. Furthermore, we carefully consider the constants of integration, i.e., homogeneous solutions. These homogeneous solutions are ignored in the calculation of AdS/CFT correspondence. However, they play the important role in the braneworld.

V. SINGLE BRANE MODEL (RS2)

A. Einstein Gravity at Lowest Order

The next order solutions are obtained by taking into account the terms neglected at zeroth order. At first order, Eqs. (10) - (13) become
\[ \begin{align*}
(1) \Sigma^{\mu\nu}_{,y} - \frac{4}{\ell} \Sigma^{\mu\nu} &= -\left[ (4) R^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} R \right], \] (34)
\[ \begin{align*}
\frac{6}{\ell} (1) K &= \left[ \begin{array}{c}
(4) R \\
1
\end{array} \right], \quad (35)
\begin{align*}
(1) K^{,y} - \frac{2}{\ell} (1) K &= 0, \quad (36)
(1) \Sigma^\lambda_{\mu|\lambda} - \frac{3}{4} (3) K_{|\mu} &= 0 . \quad (37)
\end{align*}
\end{align*}
\]
where the superscript (1) represents the order of the derivative expansion and \( | \) denotes the covariant derivative with respect to the metric \( h_{\mu\nu} \). Here, \( (4) R^{\mu\nu} \) means that the curvature is approximated by taking the Ricci tensor of \( a^2 h_{\mu\nu} \) in place of \( R^{\mu\nu} \). It is also convenient to write it in terms of the Ricci tensor of \( h_{\mu\nu} \), denoted by \( R^{\mu\nu} \).

Substituting the zeroth order metric into \( (4) \), we obtain
\[ \frac{1}{\ell} K = \frac{\ell}{6a^2} R(h). \] (38)

Hereafter, we omit the argument of the curvature for simplicity. Simple integration of Eq. (38) also gives the traceless part of the extrinsic curvature as
\[ (1) \Sigma^{\mu\nu} = \frac{\ell}{2a^2} (R^{\mu\nu} - \frac{1}{6} \delta^{\mu\nu} R) + \frac{\kappa^2}{a^2} \chi^{\mu\nu}(x), \] (39)
where the homogeneous solution satisfies the constraints
\[ \chi^{\mu\nu} = 0 \quad \text{and} \quad \chi^{\mu\nu|\mu} = 0 . \] (40)
As we see later, this term corresponds to dark radiation at this order. The metric can be obtained as
\[ g_{\mu\nu}(y = 0, x^\mu) = 0. \]
Let us focus on the role of \( \chi^{\mu\nu} \) in this part. At this order, the junction condition can be written as
\[ \left. \left[ (1) K^{\mu\nu} - \delta^{\mu\nu} (1) K \right] \right|_{y=0} = \frac{\ell}{2} \left( R^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} R \right) + \chi^{\mu\nu} = \frac{\kappa^2}{2} T^{\mu\nu} . \] (42)
Using the solutions Eqs. (38), (39) and the formula
\[ E^{\mu\nu} = K^{\mu\nu} - \delta^{\mu\nu} K - K^{\mu\lambda} K^{\lambda\nu} + \delta^{\mu\nu} K^{\alpha\beta} K^{\alpha\beta} - \frac{3}{\ell^2} \delta^{\mu\nu} , \] (43)
we calculate the projective Weyl tensor as
\[ (1) E^{\mu\nu} = -\frac{2}{\ell} \chi^{\mu\nu} . \] (44)
Then we obtain the effective Einstein equation
\[ R^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} R = \kappa^2 T^{\mu\nu} - (1) E^{\mu\nu} . \] (45)
At this order, we do not have the conventional Einstein equations. Recall that the dark radiation exists even in
the low energy regime. Indeed, the low energy effective Friedmann equation becomes

\[ H^2 = \frac{\kappa^2}{3\ell} a^3 + \frac{C}{a_0(t)^4} . \] (46)

This equation can be obtained from Eq. (1) by imposing the maximal symmetry on the spatial part of the brane world and the conditions [10]. Hence, we observe that \( \chi^\mu\nu \) is the generalization of the dark radiation found in the cosmological context.

The nonlocal tensor \( \chi^\mu\nu \) must be determined by the boundary conditions in the bulk. The natural choice is asymptotically AdS boundary condition. For this boundary condition, \( \chi^\mu\nu = 0 \). It is this boundary condition that leads to the conventional Einstein theory in linearized gravity. Assuming this, we have

\[ R^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} R = \frac{\kappa^2}{\ell} T^{\mu\nu} . \] (47)

Thus, Einstein theory is recovered at the leading order!

**B. AdS/CFT Emerges**

In this subsection, we do not include the \( \chi \) field because we have adopted the AdS boundary condition. Of course, we have calculated the second order solutions with the contribution of the \( \chi \) field. It merely adds extra terms such as \( \chi^\mu\nu \chi^{\alpha\beta} \), etc.

At second order, the basic equations can be easily deduced. Substituting the solution up to first order into the Ricci tensor and picking up the second order quantities, we obtain the solutions at second order. The trace part is deduced algebraically as

\[ (2) K = \frac{\ell^3}{8a^2} \left( R^{\alpha\beta} R^{\alpha\beta} - \frac{2}{9} R^2 \right) - \frac{\ell^3}{12a^2} \left( R^{\alpha\beta} R^{\alpha\beta} - \frac{1}{6} R^2 \right) . \] (48)

By integrating the equation for the traceless part, we have

\[ (2) \Sigma^{\mu\nu} = -\frac{\ell^2}{2} \left( \frac{y}{a^4} + \frac{\ell}{2a^2} \right) S^{\mu\nu} - \frac{\ell^3}{24a^2} \left( R R^{\mu\nu} - \frac{1}{4} \delta^{\mu\nu} R^2 \right) + \frac{\ell^3}{a^4} \delta^{\mu\nu} , \] (49)

where \( S^{\mu\nu} \) is defined by

\[ \delta \int d^4x \sqrt{-h} \left[ R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 \right] = \int d^4x \sqrt{-h} S_{\mu\nu} \delta g^{\mu\nu} . \] (50)

The tensor \( S^{\mu\nu} \) is transverse and traceless,

\[ S^{\mu\nu}_{\mid\mu} = 0 , \quad S^{\nu\mu} = 0 . \] (51)

The homogeneous solution \( t^{\mu\nu} \) must be traceless. Moreover, it must satisfy the momentum constraint. To be more precise, we must solve the constraint equation

\[ t^{\mu\nu}_{\mid\mu} - \frac{1}{16} R^{\alpha\beta} R_{\alpha\beta}^{\mu\nu} + \frac{1}{48} R R_{\mu\nu} - \frac{1}{24} R_{\mid\lambda} R^{\lambda}_{\mu\nu} = 0 . \] (52)

As one can see immediately, there are ambiguities in integrating this equation. Indeed, there are two types of covariant local tensor whose divergences vanish:

\[ \delta \int d^4x \sqrt{-h} \left[ h R^{\mu\nu} - h K^{\mu\nu} / 3 \right] = \delta \int d^4x \sqrt{-h} K^{\mu\nu} \delta g^{\mu\nu} . \] (54)

Notice that \( S^{\mu\nu} = H^{\mu\nu} - K_{\mu\nu} / 3 \). Hence, only \( S^{\mu\nu} \) and \( K^{\mu\nu} \) are independent. Thanks to the Gauss-Bonnet topological invariant, we do not need to consider the Riemann squared term. In addition to these local tensors, we have to take into account the nonlocal tensor \( \tau^{\mu\nu} \) with the property \( \tau^{\mu\nu}_{\mid\mu} = 0 \). Thus, we get

\[ t^{\mu\nu} = \frac{1}{32} \delta^{\mu\nu} \left( R^{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} R^2 \right) + \frac{1}{24} \left( R R^{\mu\nu} - \frac{1}{4} \delta^{\mu\nu} R^2 \right) + \tau^{\mu\nu} + \left( \frac{\alpha + 1}{4} \right) S^{\mu\nu} + \frac{\beta}{3} K^{\mu\nu} , \] (55)

where the constants \( \alpha \) and \( \beta \) represents the freedom of the gravitational wave in the bulk. The condition \( t^{\mu\nu} = 0 \) leads to

\[ \tau^{\mu\nu} = -\frac{1}{8} \left( R^{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} R^2 \right) - \beta \Box R . \] (56)

This expression is the reminiscent of the trace anomaly of the CFT. It is possible to use the result of CFT at this point. For example, we can choose the \( N = 4 \) super Yang-Mills theory as the conformal matter. In that case, we simply put \( \beta = 0 \). This is the way the AdS/CFT correspondence comes into the brane world scenario.

Up to the second order, the junction condition gives

\[ R^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} R + 2\ell^2 \left[ \tau^{\mu\nu} + \alpha S^{\mu\nu} + \frac{\beta}{3} K^{\mu\nu} \right] = \frac{\kappa^2}{\ell} T^{\mu\nu} . \] (57)

If we define

\[ T_{\mu\nu}^{CFT} = -2\frac{\ell^3}{\kappa^2} T_{\mu\nu} , \] (58)

we can write

\[ G_{\mu\nu} = \frac{\kappa^2}{\ell} \left( T_{\mu\nu} + T_{\mu\nu}^{CFT} \right) - 2\ell^2 \alpha S_{\mu\nu} - \frac{2\ell^2}{3} \beta K_{\mu\nu} . \] (59)

Let us try to arrange the terms so as to reveal the geometrical meaning of the above equation. We can calculate the projective Weyl tensor as

\[ E^{\mu\nu} = \ell^2 \left[ P^{\mu\nu} + 2\tau^{\mu\nu} + 2\alpha S^{\mu\nu} + \frac{\beta}{3} K^{\mu\nu} \right] , \] (60)
where
\[ P = -\frac{1}{4} R \lambda R \alpha + \frac{1}{6} R R \nu \]
\[ + \frac{1}{2 \delta^\mu R^\alpha R^\beta - \frac{1}{16} \delta^\mu R^2. \] (61)

Substituting this expression into Eq. (62) yields our main result
\[ (4) G_{\mu \nu} = \frac{\kappa^2}{\ell} T_{\mu \nu} + \ell^2 P_{\mu \nu} - \frac{\ell}{2} E_{\mu \nu}. \] (62)

Notice that \( E_{\mu \nu} \) contains the nonlocal part and the free parameters \( \alpha \) and \( \beta \). On the other hand, \( P_{\mu \nu} \) is determined locally. One can see the relationship in a more transparent way. Within the accuracy we are considering, we can get \( P_{\mu \nu} = \pi_{\mu \nu} \) using the lowest order equation \( R_{\mu \nu} = \kappa^2 / \ell(T_{\mu \nu} - 1/2 \delta^\nu T) \). Hence, we can rewrite Eq. (62) as
\[ (4) G_{\mu \nu} = \frac{\kappa^2}{\ell} T_{\mu \nu} + \kappa^4 \pi_{\mu \nu} - \ell E_{\mu \nu}. \] (63)

Now, the similarity between Eqs. (13) and (63) is apparent. Thus we get an explicit relation between the geometrical approach and the AdS/CFT approach. However, we note that our Eq. (63) is a closed system of equations provided that the specific conformal field theory is chosen.

Now we can read off the effective action as
\[ S_{\text{eff}} = \frac{\ell}{2 \kappa^2} \int d^4 x \sqrt{-h} R + S_{\text{matter}} + S_{\text{CFT}} \]
\[ + \frac{\alpha \ell^2}{\kappa^2} \int d^4 x \sqrt{-h} \left[ R^\mu R_{\mu \nu} - \frac{1}{3} R^2 \right] \]
\[ + \frac{\beta \ell^2}{6 \kappa^2} \int d^4 x \sqrt{-h} R^2, \] (64)

where we have used the relations Eqs. (60), (63) and (64) and we denoted the nonlocal effective action constructed from \( \tau_{\mu \nu} \) as \( S_{\text{CFT}} \).

VI. TWO-BRANE MODEL (RS1)

A. Scalar-Tensor Theory Emerges

We consider the two-brane system in this section. Without matter on the branes, we have the relation \( g_{\text{brane}} = e^{-2d/\ell} g_{\text{brane}} = \Omega^2 g_{\text{brane}} \) where \( d \) is the distance between the two branes. Although \( \Omega \) is constant for vacuum branes, it becomes the function of the 4-dimensional coordinates if we put the matter on the brane.

Adding the energy momentum tensor to each of the two branes, and allowing deviations from the pure AdS\(_5\) bulk, the effective (non-local) Einstein equations on the branes at low energies take the form\([18, 19]\),
\[ G_{\mu \nu}(h) = \frac{\kappa^2}{\ell} T_{\mu \nu} - \frac{2}{\ell} \chi_{\mu \nu}, \] (65)
\[ G_{\mu \nu}(f) = -\frac{\kappa^2}{\ell} T_{\mu \nu} - \frac{2}{\ell} \chi_{\mu \nu}. \] (66)

where \( h_{\mu \nu} = g_{\text{brane}}, f_{\mu \nu} = g_{\text{brane}} = \Omega^2 h_{\mu \nu} \) and the terms proportional to \( \chi_{\mu \nu} \) are 5-dimensional Weyl tensor contributions which describe the non-local 5-dimensional effect. Although Eqs. (65) and (66) are non-local individually, with undetermined \( \chi_{\mu \nu} \), one can combine both equations to reduce them to local equations for each brane. Since \( \chi_{\mu \nu} \) appears only algebraically, one can easily eliminate \( \chi_{\mu \nu} \) from Eqs. (65) and (66). Defining a new field \( \Psi = 1 - \Omega^2 \), we find
\[ G_{\mu \nu}(h) = \frac{\kappa^2}{\ell \Psi} T_{\mu \nu} + \frac{\kappa^2}{\ell \Psi} T_{\mu \nu} \]
\[ + \frac{1}{\Psi} \left( \Psi_{|\mu \nu} - \delta_{\mu \nu} \Psi_{|\alpha \beta} \right) \]
\[ + \frac{3}{2(1 - \Psi)} \left( \Psi_{|\mu \nu} - \frac{1}{2} \delta_{\mu \nu} \Psi_{|\alpha \beta} \right) \] (67)
\[ \Box \Psi = \frac{\kappa^2}{3\ell(1 - \Psi)} \left( \frac{\partial}{\partial \tau} - \frac{1}{2} \frac{\Psi}{1 - \Psi} \right) \]
\[ - \frac{1}{2(1 - \Psi)} \Psi_{|\mu \nu} \] (68)

where \( | \) denotes the covariant derivative with respect to the metric \( h_{\mu \nu} \). Since \( \Omega \) (or equivalently \( \Psi \)) contains the information of the distance between the two branes, we call \( \Omega \) (or \( \Psi \)) the radion.

We can also determine \( \chi_{\mu \nu} \) by eliminating \( G_{\mu \nu} \) from Eqs. (65) and (66). Then,
\[ \chi_{\mu \nu} = \frac{\kappa^2(1 - \Psi)}{2 \Psi} \left( \frac{\partial}{\partial \tau} T_{\mu \nu} + (1 - \frac{\partial}{\partial \tau} T_{\mu \nu} \right) \]
\[ - \frac{\ell}{2 \Psi} \left[ \left( \Psi_{|\mu \nu} - \delta_{\mu \nu} \Psi_{|\alpha \beta} \right) \right] \]
\[ + \frac{3}{2(1 - \Psi)} \left( \Psi_{|\mu \nu} - \frac{1}{2} \delta_{\mu \nu} \Psi_{|\alpha \beta} \right) \] (69)

Note that the index of \( \frac{\partial}{\partial \tau} \) is to be raised or lowered by the induced metric on the \( \ominus \)-brane, \( f_{\mu \nu} \).

The effective action for the \( \ominus \)-brane which gives Eqs. (67) and (68) is
\[ S_{\ominus} = \frac{\ell}{2 \kappa^2} \int d^4 x \sqrt{-h} \left[ \Psi R - \frac{3}{2(1 - \Psi)} \Psi_{|\alpha \beta} \right] + \int d^4 x \sqrt{-h} \left[ L^\ominus + \int d^4 x \sqrt{-h} (1 - \Psi)^2 L^\ominus \right. \] (70)

It should be stressed that the radion has the conformal coupling.

B. AdS/CFT in two-brane system?

In the two-brane case, it is difficult to proceed to the next order calculations. Hence, we need to invent a new method\([18]\). For this purpose, we shall start with the effective Einstein equation obtained by Shromov, Maeda, and Sasaki
\[ G_{\mu \nu} = T_{\mu \nu} + \pi_{\mu \nu} - E_{\mu \nu} \] (71)
where \( \pi_{\mu\nu} \) is the quadratic of energy momentum tensor \( T_{\mu\nu} \) and \( E_{\mu\nu} \) represents the effect of the bulk geometry. Here we have set \( 8\pi G = 1 \). This geometrical projection approach can not give a concrete prediction, because we do not know \( E_{\mu\nu} \) without solving the equations of motion in the bulk. Fortunately, in the case of the homogeneous cosmology, the property \( E^\mu_{\mu} = 0 \) determines the dynamics as

\[
H^2 = \frac{1}{3}(\rho + \rho^2 + \frac{C}{a_0^2}) .
\] (72)

This reflects the interplay between the bulk and the brane dynamics on the brane.

What we want to seek is an effective theory which contains the information of the bulk as finite number of constant parameters like \( C \) in the homogeneous universe. When we succeed to obtain it, the cosmological perturbation theory can be constructed in a usual way. Although the concrete prediction can not be made, qualitative understanding of the evolution of the cosmological fluctuations can be obtained. This must be useful to make observational predictions.

In the two-brane system, the mass spectrum is known from the linear analysis \([3]\). At low energy, the propagator for the KK mode with the mass \( m \) can be expanded as

\[
\frac{-1}{\Box - m^2} = \frac{1}{m^2} \left[ 1 + \frac{\Box}{m^2} + \frac{\Box^2}{m^4} + \cdots \right].
\] (73)

However, massless modes can not be expanded in this way, hence we must take into account all of the massless modes to construct braneworld effective action. It seems legitimate to assume this consideration is valid even in the non-linear regime. Thus, at low energy, the action can be expanded by the local terms with increasing orders of derivatives of the metric \( g_{\mu\nu} \) and the radion \( \Psi \) \( [4] \).

Let us illustrate our method using the following action truncated at the second order derivatives:

\[
S_{\text{eff}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \Psi R - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right] ,
\] (74)

which is nothing but the scalar-tensor theory with coupling function \( \omega(\Psi) \) and the potential function \( \Lambda(\Psi) \). Note that this is the most general local action which contains up to the second order derivatives and has the general coordinate invariance. It should be stressed that the scalar-tensor theory is, in general, not related to the braneworld. However, we know a special type of scalar-tensor theory corresponds to the low energy braneworld \([4, 3, 14]\). Here, we will present a simple derivation of this known fact.

For the vacuum brane, we can put \( T_{\mu\nu} + \pi_{\mu\nu} = -\lambda g_{\mu\nu} \). Hence, the geometrical effective equation reduces to

\[
G_{\mu\nu} = -E_{\mu\nu} - \lambda g_{\mu\nu} .
\] (75)

First, we must find \( E_{\mu\nu} \). The above action \( [74] \) gives the equations of motion for the metric as

\[
G_{\mu\nu} = -\frac{\Lambda}{\Psi} g_{\mu\nu} + \frac{1}{\Psi} \left( \nabla_\mu \nabla_\nu \Psi - g_{\mu\nu} \Box \Psi \right) + \frac{\omega}{\Psi^2} \left( \nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \Psi \nabla_\alpha \Psi \right) .
\] (76)

The right hand side of this Eq. \( [76] \) should be identified with \(-E_{\mu\nu} - \lambda g_{\mu\nu} \). Hence, the condition \( E^\mu_{\mu} = 0 \) becomes

\[
\Box \Psi = -\frac{\omega}{3\Psi} \nabla^\mu \Psi \nabla_\mu \Psi - \frac{4}{3} (\Lambda - \lambda \Psi) .
\] (77)

This is the equation for the radion \( \Psi \). However, we also have the equation for \( \Psi \) from the action \( [74] \) as

\[
\Box \Psi = \left( \frac{1 - \omega'}{2\omega} \right) \nabla^\alpha \Psi \nabla_\alpha \Psi - \Psi \frac{\omega}{\omega' R} + \frac{\Psi}{\omega} \Lambda' ,
\] (78)

where the prime denotes the derivative with respect to \( \Psi \). In order for these two Eqs. \( [77] \) and \( [78] \) to be compatible, \( \Lambda \) and \( \omega \) must satisfy

\[
\frac{\omega}{3\Psi} = \frac{1 - \omega'}{2\omega} = \frac{4}{3} (\Lambda - \lambda \Psi) ,
\] (79)

where we used \( R = 4\Lambda \) which comes from the trace part of Eq. \( [75] \). Eqs. \( [76] \) and \( [78] \) can be integrated as

\[
\Lambda(\Psi) = \lambda + \lambda \gamma (1 - \Psi)^2 , \quad \omega(\Psi) = \frac{3}{2} \frac{\Psi}{1 - \Psi} ,
\] (81)

where the constant of integration \( \gamma \) represents the ratio of the cosmological constant on the negative tension brane to that on the positive tension brane. Here, one of constants of integration is absorbed by rescaling of \( \Psi \). In doing so, we have assumed the constant of integration is positive. We can also describe the negative tension brane if we take the negative signature.

Thus, we get the effective action

\[
S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Psi R - \frac{3}{4(1 - \Psi)} \nabla^\mu \Psi \nabla_\mu \Psi - \lambda - \lambda \gamma (1 - \Psi)^2 \right] .
\] (82)

Surprisingly, this completely agrees with the previous result \( [70] \). Our simple symmetry principle \( E^\mu_{\mu} = 0 \) has determined the action completely.

As we have shown in \( [12] \), if \( \gamma < -1 \) there exists a static deSitter two-brane solution which turns out to be unstable. In particular, two inflating branes can collide at \( \Psi = 0 \). This process is completely smooth for the observer on the brane. This fact led us to the born-again scenario. The similar process occurs also in the ekpyrotic (cyclic) model \( [16] \) where the moduli approximation is used. It can be shown that the moduli approximation is nothing but the lowest order truncation of the low energy
gradient expansion method developed by us [4]. Hence, it is of great interest to see the leading order corrections due to KK modes to this process.

Let us apply the procedure explained above to the higher order case:

\[ S_{\text{eff}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \Psi R - 2\Lambda(\Psi) - \frac{\omega(\Psi)}{\Psi} \nabla^\mu \Psi \nabla_\mu \Psi \right] \\
+ \int d^4x \sqrt{-g} \left[ A(\Psi) (\nabla^\mu \Psi \nabla_\mu \Psi)^2 + B(\Psi) (\Box \Psi)^2 \\
+ C(\Psi) \nabla^\mu \Psi \nabla_\mu \Psi \Box \Psi + D(\Psi) R \Box \Psi \\
+ E(\Psi) R \nabla^\mu \Psi \nabla_\mu \Psi + F(\Psi) R_{\mu
u} \nabla_\mu \Psi \nabla_\nu \Psi \\
+ G(\Psi) R^2 + H(\Psi) R_{\mu
u} R_{\mu\nu} \\
+ I(\Psi) R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} + \cdots \right] . \] 

(83)

Now we impose the conformal symmetry on the fourth order derivative terms in the action (83) as we did in the previous example. Starting from the action (83), one can read off the equation for the metric from which \( E_{\mu\nu} \) can be identified. The compatibility between the equations of motion for \( \Psi \) and the equation \( \mathcal{E}_{\mu\nu} = 0 \) determines the coefficient functionals in the action (83).

Thus, we find the 4-dimensional effective action with KK corrections as

\[ S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Psi R - \frac{3}{4(1-\Psi)^2} \nabla^\mu \Psi \nabla_\mu \Psi \\
- \lambda - \lambda(1-\Psi^2) \\
+ \ell^2 \int d^4x \sqrt{-g} \left[ \frac{1}{4(1-\Psi)^2} (\nabla^\mu \Psi \nabla_\mu \Psi)^2 \\
+ \frac{1}{(1-\Psi)^2} (\Box \Psi)^2 + \frac{1}{(1-\Psi)^3} \nabla^\mu \Psi \nabla_\mu \Psi \Box \Psi \\
+ \frac{2}{3(1-\Psi)} R \Box \Psi + \frac{1}{3(1-\Psi)^2} R \nabla^\mu \Psi \nabla_\mu \Psi \\
+ \frac{1}{2} R^2 + k R_{\mu\nu} R_{\mu\nu} \right] , \] 

(84)

where constants \( j \) and \( k \) can be interpreted as the variety of the effects of the bulk gravitational waves. It should be noted that this action becomes non-local after integrating out the radion field. This fits the fact that KK effects are non-local usually. In principle, we can continue this calculation to any order of derivatives.

VII. CONCLUSION

We have developed the low energy gradient expansion scheme to give insights into the physics of the braneworld such as the black hole physics and the cosmology. In particular, we have concentrated on the specific questions such as the black hole physics and the cosmology. In other words, we can continue this calculation to any order of derivatives.

A. Single-brane model

Is the Einstein theory recovered even in the non-linear regime?

We have obtained the effective theory at the lowest order as

\[ G^{\mu\nu}_{(4)} = \frac{k^2}{\ell} T^{\mu\nu}_{(4)} - \frac{2}{\ell} \chi^{\mu\nu} . \] 

(85)

Here we have the correction \( \chi^{\mu\nu} \) which can be interpreted as the dark radiation in the cosmological situation.

On the other hand, in the linearized gravity, the conventional Einstein theory is recovered at low energy. This is because the out-going boundary condition is imposed. In other words, the asymptotic AdS boundary condition is imposed. In the nonlinear case, this corresponds to the requirement that the dark radiation term \( \chi^{\mu\nu} \) must be zero. For this condition, the conventional Einstein theory is recovered. Hence, the standard Friedmann equation holds.

In this sense, the answer is yes.

How does the AdS/CFT come into the braneworld?

The CFT emerges as the constant of integration which satisfies the trace anomaly relation

\[ \tau^{\mu\nu} = -\frac{1}{8} \left( R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 \right) - \beta \Box R . \] 

(86)

This constant cannot be determined a priori. Here, the AdS/CFT correspondence could come into the braneworld. Namely, if we identify some CFT with \( \tau^{\mu\nu} \), then we can determine the boundary condition.

How are the AdS/CFT and geometrical approach related?

The key quantity in the geometric approach is obtained as

\[ E^{(2)}_{\mu\nu} = \ell^2 \left[ P_{(2),\mu\nu} + 2 \tau^{\mu\nu} + 2 \alpha \mathcal{D}^{\mu\nu} + \frac{2}{3} \beta K^{\mu\nu} \right] . \] 

(87)

The above expression contains \( \tau^{\mu\nu} \) which can be interpreted as the CFT matter. Hence, once we know \( E_{\mu\nu} \), no enigma remains. In particular, \( P_{(2),\mu\nu} \approx \pi_{\mu\nu} \) is independent of the \( \tau_{\mu\nu} \). In odd dimensions, there exists no trace anomaly, but \( P_{(2),\mu\nu} \) exists. In 4-dimensions, \( \pi_{\mu\nu} \) accidentally coincides with the trace anomaly in CFT.

It is interesting to note that the high energy and the Weyl term corrections found in the geometrical approach merge into the CFT matter correction found in the AdS/CFT approach.
B. Two-brane model

How is the geometrical approach consistent with the Brans-Dicke picture?

In the geometrical approach, no radion seems to appear. On the other hand, the linear theory predicts the dark radiation consists of the radion and the matter. Hence, we can not expect CFT matter on the brane. Instead, the radion controls the bulk/brane correspondence in two-brane model. In fact, the higher derivative terms of the radion mimics the effect of the bulk geometry (KK-effect) as we have shown explicitly.

Hence, the AdS/CFT correspondence does not exist. Instead, the AdS/radion correspondence exists.

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What replaces the AdS/CFT correspondence in the two-brane model?

In the case of the single-brane model, the out-going boundary condition at the Cauchy horizon is assumed. This conforms to AdS/CFT correspondence. Indeed, the continuum KK-spectrum are projected on the brane as CFT matter.

On the other hand, the boundary condition in the two-brane system allows only the discrete KK-spectrum. Hence, we can not expect CFT matter on the brane. Instead, the radion controls the bulk/brane correspondence in two-brane model. In fact, the higher derivative terms of the radion mimics the effect of the bulk geometry (KK-effect) as we have shown explicitly.

Hence, the AdS/CFT correspondence does not exist. Instead, the AdS/radion correspondence exists.

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