On Consistent Hypothesis Testing

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We explore conditions of existence of consistent, uniformly consistent and discernible (strong consistent) tests. We establish that the existence of discernible tests follows from the existence of pointwise consistent tests. We show that, if there are consistent tests, then the set of alternatives can be represented as countable union of nested subsets such that there are uniformly consistent tests for these subsets of alternatives. Implementing these results we explore both sufficient conditions and necessary conditions for existence of consistent, uniformly consistent and discernible tests for hypothesis testing on a probability measure of independent sample, on a mean measure of Poisson process, on a solution of linear ill-posed problems in Gaussian noise, on a solution of deconvolution problem and for the problem of signal detection in Gaussian white noise. In the last three cases the necessary conditions and sufficient conditions coincide.

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1. Introduction

The problems of consistent estimation and consistent classification were rather well studied. The universal consistency was established for the basic statistical procedures. The necessary and sufficient conditions for uniform consistency were obtained. In hypothesis testing the situation is more complicated. Results have a disordered character. The goal of the paper is to represent a systematic viewpoint to this problem.

In the study of consistency of tests different approaches are implemented. The consistency of tests is established as usual consistency or uniform consistency or discernibility (strong consistency). A sequence of tests is called discernible (Dembo and Peres [11], Devroye and Lugosi [13]) or strong consistent (van der Vaart [36]) if the tests make almost surely only finitely number of errors.

It is clear that the uniform consistency or discernibility implies the consistency. It turns out that in some sense the inverse statements hold.

The existence of consistent tests implies the existence of discernible tests. There is a consistent sequence of tests iff the set of alternatives can be represented as countable
unions of nested subsets such that, for these nested subsets of alternatives, there are uniformly consistent sequences of tests.

Thus, for each consistent tests, it seems natural to search for nested subsets of alternatives such that the tests are uniformly consistent on these subsets. For these subsets of alternatives one can say about "distinguishability" of hypotheses and alternatives for finite sample sizes.

Therefore the problems of description of sets of hypotheses admitting the consistent or discernible tests are reduced to the similar problem for the uniformly consistent tests.

From this viewpoint we study both sufficient conditions and necessary conditions for existence of consistent, uniformly consistent and discernible sequences of tests in the problems of hypotheses testing

- on a probability measure of i.i.d.r.v.'s,
- on a mean measure of Poisson process,
- on a solution of linear ill-posed problem in Hilbert space if the noise is Gaussian,
- on a solution of deconvolution problem
  - and in the problem of signal detection in Gaussian white noise.

The sufficient conditions are rather obvious. The goal of the paper is to try to understand to what extent such sufficient conditions are necessary.

For the problems of signal detection in Gaussian white noise, hypothesis testing on a solution of linear ill-posed problem in Gaussian noise and on a solution of deconvolution problem we find necessary and sufficient conditions for existence of consistent and uniformly consistent tests.

In the problems of hypothesis testing on probability measure of i.i.d.r.v.'s the necessary and sufficient conditions of existence of uniformly consistent tests (distinguishability of hypotheses) have been established by Le Cam and Schwartz [31] if the set of densities of probability measures is uniformly integrable for some probability measure. However it was not clear how to check Le Cam and Schwartz conditions and these conditions did not mentioned in the subsequent works. A convenient form of necessary and sufficient conditions for existence of discernible tests has been proposed Dembo and Peres [11]. In the paper we develop a simple approach allowing to study the distinguishability conditions and show that these conditions can be provided in the same convenient form as in Dembo and Peres [11].

We explore also the hypotheses testing on a value of functional of probability measure of i.i.d.r.v.'s. The necessary and sufficient conditions for existence of consistent, uniformly consistent and discernible sequences of tests are provided if the functional satisfies some weak differentiability assumptions. For this setup the problem of discernibility of a sample mean has been studied only (Dembo and Peres [11]).

We show that the necessary conditions of distinguishability obtained for hypothesis testing on a probability measure of i.i.d.r.v.'s are extended to similar problem for mean measure of Poisson process.

All necessary conditions are provided in terms of convergence in weak topologies. Such an approach to the study of existence of uniformly consistent and discernible tests has been developed Le Cam and Schwarts [31], Dembo and Peres [12] and Kulkarni and Zeitouni [29].
The other approaches based on the distance method and using estimators as test statistics are widely implemented in hypothesis testing. In subsection 4.3 we discuss the form of sets of hypotheses and alternatives admitting consistent, uniformly consistent and discernible hypothesis testing for these approaches.

The paper is organized as follows. The general setup and the definitions of consistency, pointwise consistency, distinguishability and discernibility are provided in section 2. Section 2 contains also the review of previous works and the basic technique implemented to the proof of results. The links between the conditions of distinguishability, discernibility and existence of consistent tests are studied in section 3. The necessary conditions and sufficient conditions for existence of consistent, uniformly consistent and discernible tests for hypothesis testing on a probability measure of i.i.d.r.v.'s are explored in section 4. The results for other problems of hypotheses testing are provided in section 5. If the proof of Theorem or Lemma does not provided in section, this proof one can be found to the Appendix.

We shall denote by letters $c$ and $C$ generic constants. Denote $[a]$ the integer part of $a \in \mathbb{R}$. For any measures $P_1, P_2$ denote $P_1 \otimes P_2$ the product of measures $P_1, P_2$.

2. Preliminaries

2.1. General setup. Definitions of consistency, point-wise consistency, discernibility, uniform consistency

Let $\mathfrak{E}_n = (\Omega_n, \mathfrak{B}_n, \mathfrak{P}_n)$ be a sequence of statistical experiments where $(\Omega_n, \mathfrak{B}_n)$ are sample spaces with $\sigma$-fields of Borel sets $\mathfrak{B}_n$ and $\mathfrak{P}_n = \{P_{\theta,n}, \theta \in \Theta\}$ are families of probability measures. We wish to test a hypothesis $H_0 : \theta \in \Theta_0 \subset \Theta$ versus alternative $H_1 : \theta \in \Theta_1 \subset \Theta$.

For any test $K_n$ denote $\alpha_{\theta}(K_n), \theta \in \Theta_0$, and $\beta_{\theta}(K_n), \theta \in \Theta_1$, its type I and type II error probabilities respectively.

Denote

$$\alpha(K_n) = \sup_{\theta \in \Theta_0} \alpha_{\theta}(K_n) \quad \text{and} \quad \beta(K_n) = \sup_{\theta \in \Theta_1} \beta_{\theta}(K_n).$$

A sequence of tests $K_n$ is pointwise consistent ( Lehmann and Romano [32]) if

$$\limsup_{n \to \infty} \alpha_{\theta_0}(K_n) = 0 \quad \text{and} \quad \limsup_{n \to \infty} \beta_{\theta_1}(K_n) = 0$$

for all $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$.

A sequence of tests $K_n$ is consistent ( van der Vaart [36]) if

$$\lim_{n \to \infty} \alpha(K_n) = 0 \quad \text{and} \quad \limsup_{n \to \infty} \beta_{\theta_1}(K_n) = 0$$

for all $\theta_1 \in \Theta_1$. 


A sequence of tests $K_n, \alpha(K_n)$ is uniformly consistent (Hoefding and Wolfowitz [21]) if
\[
\lim_{n \to \infty} \alpha(K_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \beta(K_n) = 0.
\]
Hypothesis $H_0$ and alternative $H_1$ are called distinguishable (Hoefding and Wolfowitz [21]) if there is uniformly consistent sequence of tests.

Hypotheses $H_0$ and alternative $H_1$ are called indistinguishable (Hoefding and Wolfowitz [21]) if for each $n$ and for each test $K_n$, the inequality
\[
\alpha(K_n) + \beta(K_n) \geq 1
\]
holds.

In subsections 5.1 and 5.2 we consider families of statistical experiments $\mathcal{E}_\epsilon = (\Omega_\epsilon, \mathcal{B}_\epsilon, \mathbb{P}_\epsilon)$ depending on continuous parameter $\epsilon > 0$. For this setups the modification of all definitions mentioned above is traditional and is omitted.

A sequence of tests $K_n$ is called discernible (Devroye, Lugosi [13] and Dembo, Peres [11]) or strong consistent (van der Vaart [36]) if
\[
P(K_n = 1 \text{ for only finitely many } n) = 1 \text{ for all } P \in \Theta_0
\]
and
\[
P(K_n = 0 \text{ for only finitely many } n) = 1 \text{ for all } P \in \Theta_1.
\]

2.2. Previous works

The most part of results has been established for hypotheses testing on a probability measure of i.i.d.r.v.’s.

For this setup the distinguishability of hypotheses has been studied Kraft [28], Berger [4], Hoefding and Wolfowitz [21] and Le Cam and Schwartz [31]. Kraft [28] has proved very natural lower bound for the sum of type I and type II error probabilities (see Theorem 4.6). On the base of this lower bound Kraft showed that two sets of probability measures in $\mathbb{R}^\infty$ are distinguishable if their $n$-dimensional projections on the convex hulls of sets of probability measures of hypotheses and alternatives are in some sense asymptotically orthogonal. Berger [4] gave necessary and sufficient conditions of distinguishability of hypotheses. The Berger [4] distinguishability conditions are akin to the conditions of Proposition 2.1 given below but they are more cumbersome.

Hoefding and Wolfowitz [21] showed that two sets of hypotheses are distinguishable if the Kolmogorov-Smirnov distance between their distribution functions is positive. The necessary distinguishability conditions in [21] were provided in terms of variational metric on the set of probability measures with some assumptions of finiteness of number of intervals of monotonicity of differences of distribution functions of hypotheses and alternatives.

Le Cam and Schwartz [31] established necessary and sufficient conditions for the existence of uniformly consistent estimators. The conditions are provided in terms of all
n-fold products of probability measures and difficult to verify. The problem of hypothesis testing can considered as a particular case of this setup. In the paper we discuss one dimensional version of these conditions.

The problem of discernibility of hypotheses have been comprehensively studied Fisher and van Ness [18], Cover [10], Dembo and Peres [11], Kulkarni and Zeitouni [29], Devroye and Lugosi [13] and Nobel [33]. Sufficient conditions and necessary conditions of discernibility have been found. The discernible tests have been proposed for many interesting problems of nonparametric hypothesis testing.

Let us review in more details the work of Dembo and Peres [11] that is closely related to this paper. Dembo and Peres established convenient necessary conditions and sufficient conditions for discernibility of hypotheses on probability measure of i.i.d.r.v.’s. These conditions were provided in terms of convergence in weak topology. The necessary conditions were obtained under the assumption that the density of each probability measure is in $L^p$ for some $p > 1$. The necessary and sufficient conditions for discernibility of hypotheses on a sample mean have been also established.

We almost do not touch the problems related to semiparametric hypothesis testing although the works with such interesting and unexpected results deserve to be mentioned (see Bahadur and Savage [2], Donoho [14] and Devroye and Lugosi [13]).

The study of distinguishability of approaching sets of hypotheses is now very popular problem (Ingster and Suslina [25], Comminges and Dalalyan [9], Baraud, Huet and Laurent [5], Butucea, Matias, and Pouet [8] and numerous other works). For approaching sets of hypotheses the distinguishability conditions should satisfy the conditions of distinguishability for fixed sets of hypotheses and alternatives.

For hypothesis testing in functional spaces the first result on indistinguishability has been obtained Le Cam [30]. For hypothesis testing on a density Le Cam [30] has proved that the center of the ball and the interior of the ball in $L_1$ are indistinguishable. The further results were mostly related to the problem of signal detection in Gaussian white noise. Ibragimov and Hasminski [22] have shown that the center of the ball and the interior of a ball in $L_2$ are indistinguishable. Burnashev [7] has proved the indistinguishability of the interior of a ball in $L^p$-spaces, $p > 0$. For the problem of testing of simple hypothesis Janssen [27] has showed that any test can achieve high asymptotic power only on at most finite dimensional space of alternatives. Ermakov [17] found necessary and sufficient conditions of distinguishability in $L_2$ for arbitrary bounded composite sets of hypotheses and alternatives. For other setups we mention Ingster [23] and Ingster and Kutoyants [24] works. Ingster [23] and Ingster and Kutoyants [24] showed that we could not distinguish the center of the ball and the interior of ball in $L_2$ in the problems of hypothesis testing on a density and on an intensity function of Poisson process respectively.

### 2.3. Basic technique

The proof of distinguishability is based on the following reasoning (similar ideas one can find in Le Cam [30], proof of Lemma 4).
Let one need to test the hypotheses on a distribution of random variable $X$ defined on probability space $(\Omega_n, \mathcal{B}_n, P)$. For the test $K(X) \equiv \alpha, 0 < \alpha < 1$, we get

$$\alpha P(K) + \beta Q(K) = 1$$

for all $P \in \Theta_0$ and $Q \in \Theta_1$.

Thus, it is of interest, to search for the test $K$ such that

$$\alpha P(K) + \beta Q(K) < 1 - \delta, \quad \delta > 0$$

for all $P \in \Theta_0$ and $Q \in \Theta_1$ or, other words,

$$\int KdP + \int (1 - K) dQ < 1 - \delta. \quad (2.3)$$

In this case we say that the hypotheses and alternatives are weakly distinguishable.

For any $\epsilon > 0$ we can approximate the function $K$ by simple function

$$K_0(x) = \sum_{i=1}^{k} c_i 1_{A_i}(x), \quad x \in \Omega$$

such that

$$|K(x) - K_0(x)| < \epsilon, \quad x \in \Omega. \quad (2.4)$$

Here $\{A_1, \ldots, A_k\}$ is a partition of $\Omega_n$.

Substituting $K_0$ in (2.3) and using (2.4), we get

$$\sum_{i=1}^{k} c_i (Q(A_i) - P(A_i)) \geq \delta - 2\epsilon. \quad (2.5)$$

Hence we get the following Proposition.

**Proposition 2.1.** The hypothesis $H_0$ and alternative $H_1$ are weakly distinguishable iff there is a partition $A_1, \ldots, A_k$ of $\Omega_n$ such that the sets

$$V_0 = \{v = (v_1, \ldots, v_k) : v_1 = P(A_1), \ldots, v_k = P(A_k), P \in \Theta_0\} \subset R^k$$

and

$$V_1 = \{v = (v_1, \ldots, v_k) : v_1 = Q(A_1), \ldots, v_k = Q(A_k), Q \in \Theta_1\} \subset R^k$$

have disjoint closures.

Suppose we wish to test a hypothesis on probability measure of i.i.d.r.v.’s. By Proposition 2.1, if the hypothesis and alternative are weakly distinguishable, the problem can be reduced to hypothesis testing for the multinomial distribution. For this problem we immediately get the distinguishability. Thus the weak distinguishability implies the distinguishability (Hoefding and Wolfowitz [21]).
For the problem of hypothesis testing on multinomial distribution the likelihood ratio
tests have exponential decay of type I and type II error probabilities. Hence the distin-
guishability implies also (Le Cam [30] and Schwartz [34]) the existence of sequence of
tests $K_n$ and constant $n_0$ such that

$$\alpha(K_n) \leq \exp\{-cn\} \quad \text{and} \quad \beta(K_n) \leq \exp\{-cn\}$$  \hspace{1cm} (2.6)

for all $n > n_0$.

The exponential decay of type I and type II error probabilities was studied in a large
number of papers (see Hoeffding and Wolfowitz [21], Dembo, Zeitouni [12], Barron [3],
Ermakov [16] and references therein).

### 3. Link of consistency, uniform consistency and
discernibility

The results and the proofs are provided for the problem of hypothesis testing on a proba-
bility measure of i.i.d.r.v.’s. For other setups the results are obtained by easy modification
of the reasoning.

Let $X_1, \ldots, X_n$ be i.i.d.r.v.’s on a probability space $(\Omega, \mathcal{B}, P)$ where $\mathcal{B}$ is $\sigma$-field of
Borel sets on topological space $\Omega$. Denote $\Lambda$ the set of all probability measures on $(\Omega, \mathcal{B})$.

**Theorem 3.1.** There is consistent sequence of tests iff there are nested subsets $\Theta_{1i} \subseteq \Theta_{1,i+1}, 1 \leq i \leq \infty$ such that

$$\Theta_1 = \bigcup_{i=1}^\infty \Theta_{1i},$$

for each $i$ the hypothesis $H_0 : P \in \Theta_0$ and the alternative $H_{1i} : P \in \Theta_{1i}$ are distin-
guishable.

**Theorem 3.2.** There is pointwise consistent sequence of tests iff there are nested sub-
sets $\Theta_{0i} \subseteq \Theta_{0,i+1}$ and $\Theta_{1i} \subseteq \Theta_{1,i+1}, 1 \leq i \leq \infty$ such that

$$\Theta_0 = \bigcup_{i=1}^\infty \Theta_{0i} \quad \text{and} \quad \Theta_1 = \bigcup_{i=1}^\infty \Theta_{1i},$$

for each $i$ the hypothesis $H_{0i} : P \in \Theta_{0i}$ and the alternative $H_{1i} : P \in \Theta_{1i}$ are distin-
guishable.

**Proof of Theorem 3.1.** Let $K_i$ be consistent sequence ofttests. Let $0 < \alpha, \beta < 1$ be
such that $\alpha + \beta < 1$. For each $i$ define the subsets $\Theta_{1i} = \{ P : \beta(K_i, P) \leq \beta, \alpha(K_i) < \alpha, P \in \Theta_1 \}$. The sets $\Theta_0$ and $\Theta_{1i}$ are weakly distinguishable and therefore they are distinguishable. Hence the hypothesis $H_0$ and the alternative $H_{1i} : P \in \Theta_{1i} = \bigcup_{j=1}^i \Theta_{1j}$
are distinguishable by Lemma 3.1 given below.

**Lemma 3.1.** Let hypothesis $H_0 : P \in \Theta_0$ be distinguishable for alternatives $H_{11} : P \in \Theta_{11}$ and $H_{12} : P \in \Theta_{12}$. Then $H_0 : P \in \Theta_0$ is distinguishable for the alternative $H_1 : P \in \Theta_{11} \cup \Theta_{12}$.
This completes the proof of Theorem 3.1. The proof of Theorem 3.2 is similar.

Theorem 3.3. There is a discernible sequence of tests iff there is a pointwise consistent sequence of tests.

The exponential decay of type I and type II error probabilities (see (2.6)) allows to prove uniform discernibility of distinguishable sets of hypotheses and alternatives.

We say that a sequence of tests $K_n$ is uniformly discernible if

$$
\lim_{k \to \infty} \sup_{P \in \Theta_0} P(K_n = 1 \text{ for all } n > k) = 0 \tag{3.1}
$$

and

$$
\lim_{k \to \infty} \sup_{P \in \Theta_1} P(K_n = 0 \text{ for all } n > k) = 0.
$$

If there is uniformly discernible sequence of tests, we say that the sets of hypotheses and alternatives are uniformly discernible.

Theorem 3.4. If hypothesis $H_0$ and alternative $H_1$ are distinguishable, then they are uniformly discernible. For a sequence of tests $K_n$ satisfying (2.6), there are positive constants $c$ and $C$ such that

$$
\sup_{P \in \Theta_0} P(K_n = 1 \text{ for all } n > k) \leq C \exp\{-ck\} \tag{3.2}
$$

and

$$
\sup_{P \in \Theta_1} P(K_n = 0 \text{ for all } n > k) \leq C \exp\{-ck\}. \tag{3.3}
$$

Corollary 3.1. The hypothesis $H_0 : P \in \Theta_0$ and alternative $H_1 : P \in \Theta_1$ are discernible iff the sets $\Theta_0$ of hypotheses and $\Theta_1$ of alternatives can be represented as countable unions of nested subsets $\Theta_{0i} \subseteq \Theta_{0i+1}, \Theta_0 = \bigcup_{i=1}^{\infty} \Theta_{0i}$ and $\Theta_{1i} \subseteq \Theta_{1i+1}, \Theta_1 = \bigcup_{i=1}^{\infty} \Theta_{1i}$ such that the sets $\Theta_{0i}$ of hypotheses and $\Theta_{1i}$ of alternatives are uniformly discernible for each $i$.

Theorem 3.5. If there is a consistent sequence of tests, then there is a discernible sequence of tests satisfying (3.1).

4. Hypothesis testing on a probability measure of independent sample.

Different approaches were implemented to the study of existence of consistent, uniformly consistent and discernible tests, among them implementation of estimators as tests statistics,
convergence of probability measures in weak topologies generated continuous or measurable functions,
distance approach.
The implementation of estimators as test statistics we discuss in the framework of distance approach.

4.1. Weak topologies
Let $\Psi$ be a set of measurable functions $f : \Omega \rightarrow \mathbb{R}$. The coarsest topology in $\Lambda$ providing the continuous mapping

$$P \rightarrow \int f \, dP, \quad P \in \Lambda$$

for all $f \in \Psi$ is called the $\tau_\Psi$-topology.

If $\Psi$ is the set of all bounded continuous functions, the $\tau_\Psi$-topology is the weak topology.

If $\Psi$ is the set of all indicator functions of measurable Borel sets, the $\tau_\Psi$-topology is called the $\tau$-topology (see Groeneboom, Oosterhoff, Ruymgaart [20] and Dembo, Zeitouni [12]) or the topology of setwise convergence on all Borel sets (see Ganssler [19] and Bogachev [6]). In the papers the $\tau$-topology is often replaced with the $\tau_\Phi$-topology with the set $\Phi$ of bounded measurable functions. All results given below coincide for these topologies.

For any set $A \subset \Lambda$ denote $\text{cl}_{\tau_\Psi}(A)$, $\text{cl}_{\tau}(A)$ and $\text{cl}_{\tau_\Phi}(A)$ the closures of $A$ in weak, $\tau$ and $\tau_\Phi$-topologies respectively. We shall write $\tau_\Psi$ if the topology may be chosen arbitrary: weak, $\tau$ or $\tau_\Phi$.

**Theorem 4.1.** i. Let the set $\Theta_0$ be relatively compact in the $\tau_\Psi$-topology. Then the hypothesis $H_0$ and the alternative $H_1$ are distinguishable if $\text{cl}_{\tau_\Psi}(\Theta_0) \cap \text{cl}_{\tau_\Psi}(\Theta_1) = \emptyset$.

ii. If $\Theta_0$ and $\Theta_1$ are relatively compact in the $\tau$-topology, then the condition $\text{cl}_{\tau}(\Theta_0) \cap \text{cl}_{\tau}(\Theta_1) = \emptyset$ is necessary.

**Example 2.1.** Let $\nu$ is Lebesgue measure in $(0, 1)$ and let we wish to test a hypothesis on a density $f$ of probability measure $P$. Let $H_0 : f(x) = 1$, $x \in (0, 1)$ and $H_1 : f \in \Theta_1 = \{f_1, f_2, \ldots\}$ with $f_i(x) = 1 + \sin(2\pi ix)$, $x \in (0, 1), 1 \leq i < \infty$.

For any measurable set $B \in \mathcal{B}$ we have

$$\lim_{i \to \infty} \int_B f_i(x) \, dx = \int_B dx.$$ 

Therefore the hypothesis $H_0$ and the alternative $H_1$ are indistinguishable.

**Proof of ii. Theorem 4.1.** The main problem in the proof of distinguishability is that the partition $\{A_1, \ldots, A_k\}$ satisfying Proposition 2.1 may exist only for the set $\Omega^d$ with $d > 1$. If $\Theta$ is relatively compact, the map $\Theta \rightarrow \Theta \otimes \Theta$ is continuous in $\tau_\Phi$ topology (Le Cam and Schwartz [31] Proposition 1). Then the existence of the partition for $\Omega^d$ implies the existence of such a partition for $\Omega$. 
The proof of i. Theorem 4.1 is akin to the proof of i. Theorem 2 in Dembo and Peres [11] and is omitted.

Le Cam and Schwartz [31] p.141 provided the following necessary and sufficient conditions of distinguishability.

**Theorem 4.2.** Let sets $A \subset \Lambda$ and $B \subset \Lambda$ be relatively compact in the $\tau_{\Phi}$-topology. Then the set $A$ of hypotheses and the set $B$ of alternatives are distinguishable iff "there is a finite family $\{f_j, j = 1, \ldots, m\}$ of measurable bounded functions on $\Omega$ such that

$$\sup \left| \int f_j dP - \int f_j dQ \right| < 1$$

(4.1)

implies that either both $P$ and $Q$ are elements of $A$ or both are elements of $B"."

If we define functions $f_j, 1 \leq j \leq m$ as the indicator functions of the sets of the partition $\{A_1, \ldots, A_k\}$ multiplied on some constants, we get that (4.1) holds if $\text{cl}_\tau(A) \cap \text{cl}_\tau(B) = \emptyset$.

If $\Omega$ is a metric space, then the weak topology and the $\tau$ - topology coincide on compacts in the $\tau$-topology ( Ganssler [19] Lemma 2.3).

Set $\Psi$ is called $F_\sigma$-set if $\Psi$ is countable union of closed sets. Set $\Psi$ is called $\sigma$-compact set if $\Psi$ is countable union of compacts.

**Example.** Let probability measures of set $\Psi \subset \Lambda$ be absolutely continuous for probability measure $\mu$ and let, for each measure $P \in \Psi$, there is some $p > 1$ such that $dP/d\mu \in L_p(d\mu)$. Then $\Psi$ is $\sigma$-compact set in the $\tau$-topology.

The other examples of $\sigma$-compact sets can be provided using Orlich spaces.

Implementing Theorems 3.1 - 3.3, we can easily derive from Theorem 4.1 sufficient conditions and necessary conditions for existence of point-wise consistent, consistent and discernible tests.

**Theorem 4.3.** i. Let $\Theta_0$ be relatively compact in the $\tau_{\Psi}$-topology. Then there is a consistent sequence of tests if $\Theta_0$ and $\Theta_1$ are contained respectively in disjoint closed set and $F_\sigma$-set in the $\tau_{\Psi}$-topology.

ii. For the $\tau$-topology the converse holds if we suppose additionally that $\Theta_1$ is contained in some $\sigma$-compact set.

**Theorem 4.4.** i. There are pointwise consistent sequence of tests if $\Theta_0$ and $\Theta_1$ are contained respectively in disjoint $\sigma$-compact set and $F_\sigma$- set in the $\tau_{\Psi}$-topology.

ii. For the $\tau$-topology the converse holds if we suppose additionally that $\Theta_1$ is contained in some $\sigma$-compact set.

A sequence of probability measures $P_m$ is relatively compact in the $\tau$-topology ( Ganssler (1971)). Thus the condition of indistinguishability can be given in the following form.

For any set $\Psi \in \Theta$ denote $\text{cl}_{\tau}(\Psi)$ the sequential closure of $\Psi$ in the $\tau$-topology.
Theorem 4.5. For any sets $\Theta_0, \Theta_1 \subset \Theta$,

$$\text{cl}_\tau(\Theta_0) \cap \text{cl}_\tau(\Theta_1) \neq \emptyset$$

implies indistinguishability of $H_0$ and $H_1$.

Remark. We could not prove the indistinguishability of any sets $\Theta_0, \Theta_1 \subset \Theta$ such $\text{cl}_\tau(\Theta_0) \cap \text{cl}_\tau(\Theta_1) \neq \emptyset$. The map $P \rightarrow P \times P$ of $\Theta \rightarrow \Theta \times \Theta$ is not continuous in the $\tau$-topology (see 8.10.116 in Bogachev (2000)).

4.2. Necessary conditions of distinguishability and distance on variation

The distance on variation is the standard tool for the study of distinguishability of hypotheses (Hoefding and Wolfowitz [21], Le Cam [30], Lehmann and Romano [32], van der Vaart [36], Devroye and Lugosi [13], Ingster and Suslina [25] and other numerous papers).

For any probability measures $P \ll \nu$ and $Q \ll \nu$ define the variational distance

$$\text{var}(P, Q) = \frac{1}{2} \int_\Omega |dP/d\nu - dQ/d\nu| d\nu$$

For any sets of probability measures $A$ and $B$ denote

$$\text{var}(A, B) = \inf \{\text{var}(P, Q) : P \in A, Q \in B\}.$$

Denote $[A]$ the convex hull of set $A \subset \Theta$.

The proof of distinguishability is based usually on the following Theorem (Kraft [28]).

Theorem 4.6. Let the probability measures in $\Theta_0 \cup \Theta_1$ be absolutely continuous with respect to measure $\nu$. Then, for any test $K$, there holds

$$\alpha(K, \Theta_0) + \beta(K, \Theta_1) \geq 1 - \text{var}([\Theta_0], [\Theta_1]).$$

(4.2)

A sequence of densities $f_k$ converges to density $f_0$ in $L_1(\nu)$ (see Dunford and Schwarz [15], Th.12 sec.8 Ch IV, or Iosida, [26] Th.5 sec.1 Ch. V), if for any measurable set $B \in \mathfrak{B}$, there hold

$$\lim_{k \to \infty} \int_B f_k d\nu = \int_B f_0 d\nu$$

(4.3)

and $f_k$ converges to $f_0$ in measure.

If (4.3) holds and sequence $f_k$ does not converges to $f_0$ in measure we could not distinguish the set of hypotheses $\Theta_0 = \{f_0\}$ and the set of alternatives $\{f_1, f_2, \ldots\}$. By Mazur Theorem (see Iosida [26], Th.2, sec.1, Ch.5), the weak convergence $f_k$ to $f_0$ implies the convergence of convex combinations of $f_k$ to $f_0$ in $L_1(\nu)$. Therefore the right-hand side of (4.2) equals one.
4.3. Distance approach

In distance approach the test statistics are defined the distances between the empirical probability measures $\hat{P}_n$ and the sets of hypotheses and alternatives. Hoeffding and Wolfowitz [21] proposed the classification of distances on consistent and uniformly consistent.

Let $\rho$ be a distance on the set $\Lambda$ of all probability measures. The distance $\rho$ is consistent in $\Theta$, if, for each $\epsilon > 0$ and each $P \in \Theta$, there holds

$$\lim_{n \to \infty} P(\rho(\hat{P}_n, P) > \epsilon) = 0.$$  \hfill (4.4)

The distance $\rho$ is uniformly consistent in $\Theta$ if the convergence in (4.4) is uniform for $P$ with $P \in \Theta$.

Hoeffding and Wolfowitz proved the following Theorem (see Th 3.1 [21]).

**Theorem 4.7.** Let $\rho$ be uniformly consistent in $\Theta$. Then, for sets $\Theta_0 \subset \Theta$ and $\Theta_1 \subset \Theta$ satisfying

$$\rho(\Theta_0, \Theta_1) = \inf\{\rho(P, Q) : P \in \Theta_0, Q \in \Theta_1\} > 0,$$  \hfill (4.5)

the hypothesis and the alternative are distinguishable.

As wellknown the Kolmogorov-Smirnov distance and the distance corresponding omega-squared tests are uniformly consistent.

The conditions of existence of pointwise consistent and consistent sequences of tests can be also provided.

**Theorem 4.8.** Let the distance $\rho$ be consistent in $\Theta \subset \Lambda$. Then

i. for any two subsets $\Theta_0$ and $\Theta_1$ of $\Theta$ containing in disjoint open sets there is pointwise consistent sequence of tests.

ii. Let $\rho$ be uniformly consistent in closed set $\Psi \subset \Theta$ additionally. Then there is consistent sequence of tests if $\Theta_0 \subseteq \Psi$ and $\Theta_1$ is contained in open subset $\Omega_1$ of $\Theta$ such that $\Psi \cap \Omega_1 = \emptyset$.

**Remark.** Theorems 4.7 and 4.8 hold in the case of semimetric $\rho$ as well. Let $X_1, \ldots, X_n$ be i.i.d.r.v.’s in $R^d$. Let $\rho(P, Q) = |E_P[X] - E_Q[Y]|$ where random variables $X$ and $Y$ have the probability measures $P$ and $Q$ respectively. Let $\Theta = \{P : E_P[|X|] < \infty, P \in \Lambda\}$. Then i. in Theorem 4.8 is necessary and sufficient conditions of discernibility in the problem of hypothesis testing on a sample mean (Theorem 1 in Dembo and Peres [11]).

Let $\Theta$ be the set of all probability measures having the densities for probability measure $\nu$. Denote $p = \frac{dP}{d\nu}, P \in \Theta$.

In Theorem 4.9 given below we consider the problem of hypothesis testing on a density: $H_0 : p = dP/d\nu \in \Psi_0$ versus $H_1 : p = dP/d\nu \in \Psi_1$. This theorem 4.9 follows from Theorem 4.8.

We say that an estimator $\hat{p}_n$ of density $p$ is universally consistent (Stone [35]) if, for any $\epsilon > 0$, for all $P \in L_1(\nu)$, we have

$$\lim_{n \to \infty} P\left(\int_{\Omega} |\hat{p}_n - p| d\nu > \epsilon\right) = 0.$$
and \( \hat{p}_n \) is uniformly consistent in \( \Psi_0 \) if, for any \( \epsilon > 0 \), we have

\[
\lim_{n \to \infty} \sup_{p \in \Psi_0} P \left( \int_\Omega |\hat{p}_n - p| \, d\nu > \epsilon \right) = 0.
\]

**Theorem 4.9.**  

i. Let there be universally consistent estimator. Then there is pointwise consistent sequence of tests if \( \Psi_0 \) and \( \Psi_1 \) lie in disjoint open subsets of \( L_1(d\nu) \).

ii. If \( \hat{p}_n \) is uniformly consistent in \( \Psi_0 \) additionally and \( \Psi_1 \) is contained in open set \( \Phi \subset L_1(d\nu) \) such that \( \text{cl}(\Psi_0) \cap \Phi = \emptyset \), then there is consistent sequence of tests.

Here \( \text{cl}(\Theta_0) \) denotes the closure of \( \Theta_0 \) in \( L_1(d\nu) \).

The discernibility of nonparametric family of univariate densities from its complements comprehensively has been studied Devroye and Lugosi [13]. The proofs were based on the convergence of minimax risks on the sets of nonparametric hypotheses and on the universal consistency of estimators. They proved also that the proposed tests satisfy (3.1) (Remark 2 [13]). Theorem 4.9 i. shows that the discernibility can be studied for other wide nonparametric sets of hypotheses and alternatives. Theorems 4.9 ii. and 3.5 show that the convergence of minimax risks can be replaced the weaker assumption of uniform consistency of distances.

### 4.4. Hypothesis testing on a value of functional

Let \( \Psi \) be a metric space with metric \( \rho_1 \). Let map \( T : \Lambda \to \Psi \) be uniformly continuous for uniformly consistent distance \( \rho \) on \( \Lambda \).

We wish to test the hypothesis \( T(P) \in \Theta_0 \subset \Psi, P \in \Lambda \) versus alternative \( H_1 : T(P) \in \Theta_1 \subset \Psi, P \in \Lambda \).

We say that the set \( V \subset \Psi \) is saturated, if for any sequence \( \eta_n \in V, \eta_n \to \eta \in \Psi \) as \( n \to \infty \), there is a sequence probability measures \( P_n \in \Lambda \) and \( P \in \Lambda \) such that \( T(P_n) = \eta_n, T(P) = \eta \) and \( P_n \to P \) in the \( \tau \)-topology.

If the sets \( \Theta_0 \) and \( \Theta_1 \) are saturated, we can implement the distinguishability criteria of Theorem 4.5.

For any set \( A \subset \Psi \) denote \( \text{cl}(A) \) the closure of \( A \) in metric \( \rho_1 \).

**Theorem 4.10.** i. Let \( \Theta_0 \) be relatively compact. Then the hypothesis \( H_0 \) and the alternative \( H_1 \) are distinguishable if \( \text{cl}(\Theta_0) \cap \text{cl}(\Theta_1) = \emptyset \).

ii. The converse holds if the sets \( \Theta_0 \) and \( \Theta_1 \) are saturated additionally.

The functional \( T : \Lambda \to \mathbb{R}^d \) is saturated if the following sufficient condition fulfilled.

For each \( P \in \Theta \) there are non-singular matrix \( D \), signed measures \( G_1, \ldots, G_d \) and \( \delta > 0 \) such that \( P + \sum_{i=1}^d u_i G_i \in \Lambda \) for all \( \vec{u} = (u_1, \ldots, u_d) : |\vec{u}| < \delta, \delta > 0 \) and

\[
|T \left( P + \sum_{i=1}^d u_i G_i \right) - T(P) - D\vec{u}| = o(|\vec{u}|) \quad (4.6)
\]
as $|\vec{u}| \to 0$.

If (4.6) holds, then $T(P)$ is inner point of $V_\delta = \{T(P + \sum_{i=1}^d u_i G_i) : \vec{u} \in \mathbb{R}^d, |u| < \delta\}, \delta > 0$ and $P + \sum_{i=1}^d u_i G_i \to P$ as $|\vec{u}| \to 0$ in the $\tau$-topology.

Example. Kolmogorov tests. Let $\Omega = (0, 1)$ and let $T(P) = \max_{x \in (0, 1)} |F(x) - x|$ where $F(x)$ is distribution function of probability measure $P \in \Lambda$. Define the probability measures $P_u = P_0 + uG, 0 < u < 1$ where $P_0$ is Lebesgue measure and the signed measure $G$ has the density $dG/dP_0(x) = -1$ if $x \in (0, 1/2)$ and $dG/dP_0(x) = -1$ if $x \in (1/2, 1)$.

Then, for any $u, 0 \leq u < 1/2$ and $u_n \to u$ as $n \to \infty$ one can put $P = P_u$ and $P_n = P_{u_n}$ in the definition of saturated set.

Theorem 4.11. 

i. Let $\Theta_0$ be relatively compact. Then there is a consistent sequence of tests if the sets $\Theta_0$ and $\Theta_1$ are contained in disjoint closed set and $F_\sigma$- set $\Phi_1 = \bigcup_{i=1}^\infty \Phi_{1i}$ with closed sets $\Phi_{1i}$, respectively.

ii. The converse holds if the sets $\Theta_0$ and $\Phi_{1i}, 1 \leq i < \infty$ are saturated additionally.

Theorem 4.12. 

i. There is a pointwise consistent sequence of tests if the sets $\Theta_0$ and $\Theta_1$ are contained respectively in disjoint $\sigma$-compact set $\Phi_0 = \bigcup_{i=1}^\infty \Phi_{0i}$ and $F_\sigma$- set $\Phi_1 = \bigcup_{i=1}^\infty \Phi_{1i}$ with compacts $\Phi_{0i}$ and closed sets $\Phi_{1i}, 1 \leq i < \infty$.

ii. Suppose additionally that the sets $\Phi_{0i}$ and $\Phi_{1i}, 1 \leq i < \infty$ are saturated. Then the converse holds.

Theorems 4.11 and 4.12 follow from Theorem 4.10 and Theorems 3.1 and 3.2.

5. Signal detection. Hypothesis testing on a solution of ill-posed problem, on a solution of deconvolution problem and on a mean measure of Poisson random process

5.1. Hypothesis testing on a mean measure of Poisson random process

Let we be given $n$ independent realizations $\kappa_1, \ldots, \kappa_n$ of Poisson random process with mean measure $P$ defined on Borel sets $\mathcal{B}$ of Hausdorff space $\Omega$. The problem is to test a hypothesis $H_0 : P \in \Theta_0 \subset \Theta$ versus alternative $H_1 : P \in \Theta_1 \subset \Theta$ where $\Theta$ is the set of all measures $P, P(\Omega) < \infty$.

Denote $N_n$ the number of atoms of Poisson random process $\kappa_1 + \ldots + \kappa_n$.

We have ((6.2) in Arcones [1])

\begin{align*}
P(|N_n - n\lambda| > x) &\leq \exp\{-n(\lambda + x) \log(1 + x/\lambda) + nx\} \\
&+ \exp\{-n(\lambda - x) \log(1 - x/\lambda) - nx\} \\
&\quad (5.1)
\end{align*}
with $\lambda = P(\Omega)$.

The conditional distribution of $P(\kappa_1 + \ldots + \kappa_n | N_n = k)$ coincide with the distribution of empirical probability measure $\hat{Q}_k$ of i.i.d.r.v.'s $X_1, \ldots, X_k$ having the probability measure $Q(A) = P(A)/P(\Omega), A \in \mathcal{B}$.

This statement and inequality (5.1) allows to extend the results of section 4 on the problem of hypothesis testing on mean measure of Poisson process. Below a version of Theorem 4.1 only will be provided.

**Theorem 5.1.**

i. Let $\Theta_0$ be relatively compact in the $\tau_\Psi$- topology. Then the hypothesis $H_0$ and alternative $H_1$ are distinguishable if $\text{cl}_{\tau_\Psi}(\Theta_0) \cap \text{cl}_{\tau_\Psi}(\Theta_1) = \emptyset$.

ii. If $\Theta_0$ and $\Theta_1$ are relatively compact in the $\tau$- topology, then the condition $\text{cl}_{\tau}(\Theta_0) \cap \text{cl}_{\tau}(\Theta_1) = \emptyset$ is necessary.

**Proof ii.** Theorem 5.1. By Theorem 2.6 in Ganssler [19], the sets $\Theta_0$ and $\Theta_1$ are relatively sequentially compact.

Suppose that the sets $\Theta_0$ and $\Theta_1$ have common limit point $P$ and are not indistinguishable. Then there exist sequences $P_k \in \Theta_0$ and $Q_k \in \Theta_1$ converging to $P \in \Theta$.

For any test $K_n$, for any $l$, we have

$$
\lim_{k \to \infty} E_{P_k}(K_n | N_n = l) = E_P(K_n | N_n = l),
$$

$$
\lim_{k \to \infty} E_{Q_k}(1 - K_n | N_n = l) = E_P(1 - K_n | N_n = l)
$$

and

$$
\lim_{k \to \infty} P_k(N_n = l) = \lim_{k \to \infty} Q_k(N_n = l) = P(N_n = l).
$$

Hence the sets $\Theta_0$ and $\Theta_1$ are indistinguishable and we have a contradiction.

The proof of $i.$ is akin to the proof of $i.$ in Theorem 4.1 and is omitted.

### 5.2. Signal detection in $L_2$

Suppose we observe a realization of stochastic process $Y_\epsilon(t), t \in (0, 1)$, defined by the stochastic differential equation

$$
dY_\epsilon(t) = S(t)dt + \epsilon dw(t), \quad \epsilon > 0
$$

where $S \in L_2(0, 1)$ is unknown signal and $dw(t)$ is Gaussian white noise.

We wish to test a hypothesis $H_0 : S \in \Theta_0 \subset L_2(0, 1)$ versus alternative $H_1 : S \in \Theta_1 \subset L_2(0, 1)$.

The results are provided in terms of the weak topology in $L_2(0, 1)$.

**Theorem 5.2.**

i. Let $\Theta_0$ and $\Theta_1$ be bounded sets in $L_2$. Then $H_0$ and $H_1$ are distinguishable iff the closures of $\Theta_0$ and $\Theta_1$ are disjoint.

ii. Let $\Theta_0$ be bounded set in $L_2$. Then there are consistent tests iff $\Theta_0$ and $\Theta_1$ are contained in disjoint closed set and $F_\tau$- set respectively.
iii. There are point-wise consistent tests iff the sets $\Theta_0$ and $\Theta_1$ are contained in disjoint $F_\sigma$-sets.

Theorem 5.2 i. follows from Theorem 5.3 given below.

For any subspace $\Gamma \subset L_2(\nu)$ denote $\Pi_\Gamma$ the projection operator on the subspace $\Gamma$. For any set $\Psi \subset L_2(\nu)$ denote $\overline{\Psi}$ the closure of $\Psi$ in $L_2(\nu)$.

**Theorem 5.3.** Suppose the sets $\Theta_0$ and $\Theta_1$ are bounded in $L_2$. Then hypothesis $H_0$ and alternative $H_1$ are distinguishable iff there exists a finite dimensional subspace $\Gamma$ such that $\Pi_\Gamma \Theta_0 \cap \Pi_\Gamma \Theta_1 = \emptyset$.

Theorem 5.3 was proved Ermakov [17]. In Appendix we give a sketch of this proof.

**Proof of Theorem 5.2.** ii. and iii. Theorem 5.3 implies that bounded sets $\Theta_0$ and $\Theta_1$ are distinguishable only if there are uniformly consistent tests depending on a finite number of linear statistics

$$\int S_1(t)dY_1(t), \ldots, \int S_k(t)dY_k(t)$$

with $S_1, \ldots, S_k \in L_2(0,1)$.

This allows to prove versions of Theorems 3.1 and 3.2 for discrete values of parameter $\epsilon = \epsilon_n = Cn^{-1/2}$ and to obtain ii. and iii. in Theorem 5.2 for such values of parameter $\epsilon = \epsilon_n$. This proves necessary conditions. It remains only to note that test statistics constructed for the parameters $\epsilon_n$ work for arbitrary $\epsilon > 0$. This implies the sufficiency in ii. and iii. Theorem 5.2.

### 5.3. Hypothesis testing on a solution of ill-posed problem

In Hilbert space $H$ we wish to test a hypothesis on a vector $\theta \in \Theta \subset H$ from the observed Gaussian random vector

$$Y = A\theta + \epsilon \xi.$$  

Hereafter $A : H \to H$ is known operator and $\xi$ is Gaussian random vector having known covariance operator $R : H \to H$ and $E\xi = 0$.

For any operator $U : H \to H$ denote $\mathcal{R}(U)$ the rangespace of $U$.

Suppose that the nullspaces of $A$ and $R$ equal zero and $\mathcal{R}(A) \subset \mathcal{R}(R)$.

**Theorem 5.4.** Let the operator $R^{-1/2}A$ be bounded. Then the statements i.-iii. of Theorems 5.2 hold for the weak topology in $H$.

Proof of Theorem 5.4. The statements i.-iii. of Theorems 5.3 are valid for the sets of hypotheses $R^{-1/2}A\Theta_0$ and alternatives $R^{-1/2}A\Theta_1$. Thus it suffices to implement the inverse map $A^{-1}R^{1/2}$ to obtain Theorem 5.4.

The problem of signal detection in the heteroscedastic Gaussian white noise can be considered as a particular case of Theorem 5.4.
Let we observe a random process $Y(t), t \in (0, 1)$ defined the stochastic differential equation

$$dY(t) = S(t)dt + ch(t)dw(t)$$

where $S \in L_2(0, 1)$ is unknown signal, $h(t)$ is a weight function and $dw(t)$ is Gaussian white noise.

One needs to test a hypothesis on a signal $S$.

**Theorem 5.5.** Let $0 < c < h(t) < C < \infty$ for all $t \in (0, 1)$. Then i.-iii. of Theorems 5.2 hold for the weak topology in $L_2(0, 1)$.

### 5.4. Hypothesis testing on a solution of deconvolution problem

Let we observe i.i.d.r.v.’s $Z_1, \ldots, Z_n$ having density $h(z), z \in R^1$ with respect to Lebesgue measure. It is known that $Z_i = X_i + Y_i, 1 \leq i \leq n$ where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are i.i.d.r.v.’s with densities $f(x), x \in R^1$ and $g(y), y \in R^1$ respectively. The density $g$ is known.

Let $P$ be the probability measure of $f$. We wish to test the hypothesis $H_0 : P \in \Theta_0$ versus the alternative $H_1 : P \in \Theta_1$ where $\Theta_0, \Theta_1 \subset \Lambda$.

Suppose $g \in L_2(R^1)$.

Denote

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} \exp\{i\omega x\}g(y)dy, \quad \omega \in R^1.$$ 

Define the sets $\Psi_i = \{f : f = dP/dx, P \in \Theta_i\}$ with $i = 0, 1$.

**Theorem 5.6.** Suppose the sets $\Theta_0$ and $\Theta_1$ are tight and the sets $\Psi_0$ and $\Psi_1$ are bounded in $L_2(R^1)$. Let

$$\text{essinf}_{\omega \in (-a,a)}|\hat{g}(\omega)| \neq 0$$

for all $a > 0$. Then i. Theorem 5.2 holds for the weak topology in $L_2(R^1)$.

**Proof of Theorem 5.6.** Note that the sets of probability measures having the densities from $L_2(R^1)$ are equicontinuous. Therefore we can implement the same reasoning as in the proof of Theorem 5.4.

### 6. Appendix

**Proof of Lemma 3.1.** By Proposition 2.1, there are $m_i, i = 1, 2$, and partitions $A_{i1}, \ldots, A_{ik_i}$ of $\Omega^{m_i}$ such that the sets

$$V_{0i} = \{v = (v_1, \ldots, v_k) : v_1 = P^{m_i}(A_{i1}), \ldots, v_k = P^{m_i}(A_{ik_i}), P \in \Theta_0\} \subset R^{m_i}$$

and

$$V_{1i} = \{v = (v_1, \ldots, v_k) : v_1 = P^{m_i}(A_{i1}), \ldots, v_k = P^{m_i}(A_{ik_i}), P \in \Theta_1\} \subset R^{m_i}$$
have disjoint closures for each $i = 1, 2$. Hereafter $P^{m_i}$ denotes $m_i$-fold product of probability measure $P$.

Since there is uniformly consistent estimator of $P^{m_1}(A_1), \ldots, P^{m_1}(A_{k_1}), P^{m_2}(A_{21}), \ldots, P^{m_2}(A_{2k_2})$, then there is uniformly consistent sequence of tests for testing the hypothesis $H_0 : P \in \Theta_0$ versus the alternative $H_1 : P \in \Theta_{11} \cup \Theta_{12}$.

**Proof of Theorem 3.3.** Let $\Theta_{0i}$ and $\Theta_{1i}, i = 1, 2, \ldots$ be the subsets of $\Theta_0$ and $\Theta_1$ such that there are uniformly consistent sequences of tests for these subsets of hypotheses. By (2.6), for each $i$ there are tests $K_{ni}$ such that for $n > n_0$,

$$
\alpha(K_{ni}) \leq \exp\{-c_i n\} \quad \text{and} \quad \beta(K_{ni}) \leq \exp\{-c_i n\}.
$$

(6.1)

By Borel-Cantelli Lemma, for the proof of Theorem 3.3, it suffices to define tests $K_n$ such that

$$
\sum_{n=1}^{\infty} \alpha(K_n, P) < \infty \quad \text{for each} \quad P \in \Theta_0
$$

(6.2)

and

$$
\sum_{n=1}^{\infty} \beta(K_n, Q) < \infty \quad \text{for each} \quad Q \in \Theta_1.
$$

(6.3)

Define the numbers $l_i$ such that $l_1 = 1$ and

$$
\exp\{-c_i l_i\}(1 - \exp\{-c_i\})^{-1} \leq i^{-2}
$$

and choose an increasing sequence $l_{i_1} < l_{i_2} < l_{i_3} < \ldots$ with $i_1 = 1$.

We put

$$
K_1 = K_{11}, \ldots, K_{l_{i_1}} = K_{l_{i_1}},
$$

$$
K_{l_{i_1}+1} = K_{l_{i_1}+1}, \ldots, K_{l_{i_2}} = K_{l_{i_2}}, K_{l_{i_2}+1} = K_{l_{i_2}+1}, \ldots
$$

Let $P \in \Theta_{0i_1}$. Then

$$
\sum_{n=l_{i_t}}^{\infty} \alpha(K_n, P) \leq \sum_{s=t}^{l_{i_{t+1}}} \sum_{n=l_{i_s}+1}^{l_{i_{s+1}}} \alpha(K_{i_s n}) \leq \sum_{s=t}^{\infty} l_{i_s}^{-2} < \infty.
$$

In the case of the alternative the reasoning is the same.

The tests $K_n$ satisfies (6.2) and (6.3).

**Proof of Theorem 3.4.** By (2.6), we get

$$
\sup_{P \in \Theta_0} P(K_n = 1 \text{ for all } n > k) \leq \sum_{n=k}^{\infty} \sup_{P \in \Theta_0} P(K_n = 1)
$$

$$
\leq \sum_{n=k}^{\infty} \exp\{-cn\} \leq C \exp\{-ck\}.
$$

(6.4)

The proof of (3.3) is similar.
Proof of Theorem 3.5. It suffices to show that there are tests $K_n$ such that
\[ \sum_{n=1}^{\infty} \alpha(K_n) < \infty. \] (6.5)
The tests $K_n$ defined in the proof of Theorem 3.3 with $\Theta_{0i} = \Theta_0$ satisfy (6.5).

Proof of ii. Theorem 4.8. The reasoning is akin to the proof of Theorem 3.1 in Hoefding and Wolfowitz [21].

For each $\epsilon > 0$ define the sets $\Theta_{1\epsilon} = \{P : \rho(P, \Psi) > \epsilon, P \in \Theta_1\}$. It is clear that $\Theta_1 = \bigcup_{\epsilon > 0} \Theta_{1\epsilon}$.

There exists $\epsilon_n \to 0$ as $n \to \infty$ such that
\[ \lim_{n \to \infty} \sup_{P \in \Theta_0} P(\rho(\hat{P}_n, P) \geq \epsilon_n / 3) = 0. \] (6.6)
Define the tests $K_n = 1$ or $0$ according as $\rho(\hat{P}_n, \Theta_0) - \rho(\hat{P}_n, \Theta_{1\epsilon_n}) > \epsilon_n / 3$ or $\leq \epsilon_n / 3$.

Let $P \in \Theta_0$. We have
\[ \rho(\hat{P}_n, \Theta_0) - \rho(\hat{P}_n, \Theta_{1\epsilon_n}) \leq -\rho(\Theta_0, \Theta_{1\epsilon_n}) + 2\rho(\hat{P}_n, P) = -\epsilon_n + 2\rho(\hat{P}_n, P). \] (6.7)
Let $P \in \Theta_1$. Then $P \in \Theta_{1\epsilon_n}$ for some $\epsilon_n > 0$. We have
\[ \rho(\hat{P}_n, \Theta_0) - \rho(\hat{P}_n, \Theta_{1\epsilon_n}) \geq \rho(P, \Theta_0) - 2\rho(\hat{P}_n, P). \] (6.8)
By (6.6) - (6.8), we get ii.

For i. the reasoning is similar and is omitted.

Proof of Theorem 4.10. Suppose $d(\Theta_0) \cap d(\Theta_1) = \emptyset$. Then the distinguishability will follow from Theorem 4.7 if we show that
\[ \inf \{\rho(P, Q) : T(P) \in \Theta_0, T(Q) \in \Theta_1\} > 0. \]
Suppose the contrary. Then there are sequence $P_k \in \Lambda, T(P_k) \in \Theta_0$ and $Q_k \in \Lambda, T(Q_k) \in \Theta_1$ such that $\rho(P_k, Q_k) \to 0$ as $k \to \infty$. Since $T$ is uniformly continuous, this implies that $\rho(T(P_k), T(Q_k)) \to 0$ as $k \to \infty$. Since $\Theta_0$ is relatively compact there are $\eta \in \Psi$ and subsequence $T(P_{k_l})$ such that $T(P_{k_l}) \to \eta$ and $T(Q_{k_l}) \to \eta$ as $l \to \infty$. We have a contradiction.

Proof of Theorem 5.3. For any $S_1, S_2 \in L_2(0, 1)$ denote
\[ (S_1, S_2) = \int_0^1 S_1(t)S_2(t)dt. \]

Lemma 6.1. Let $\Theta_0 = \{S_0\}, S_0 \in L_2(0, 1)$ and let $\Theta_1 \subset L_2(0, 1)$ be bounded. Assume the hypothesis and set of alternatives are distinguishable. Then there exists a finite dimensional subspace $\Gamma \subset \Theta$ such that $\Pi_{\Gamma} S_0 \notin \Pi_{\Gamma} \Theta_1$. 

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Lemma 6.1 was proved Ermakov [17], Lemma 2.1. This lemma follows also straightforwardly from Theorem 2.1, Janssen [27].

Suppose the contrary. Then we show that there are sequences of vectors \( \tau_k \in \Theta_0 \) and \( \eta_k \in \Theta_1 \) such that

\[
\tau_k = z_k + a_{1k} \phi_k + b_{1k} \xi_{1k} \quad \text{and} \quad \eta_k = z_k + a_{2k} \phi_k + b_{2k} \xi_{1k} + c_{2k} \xi_{2k} \tag{6.9}
\]

with

\[
z_k = \sum_{s=0}^{k-1} a_s \phi_s
\]

where \( \phi_i, \xi_i, \xi_2, 1 \leq i < \infty \) are orthonormal functions and \( a_s, a_{1k}, a_{2k}, b_{1k}, b_{2k}, c_{2k} \) are constants.

We define Bayes a priori measures \( \mu_{0m} \) and \( \mu_{1m} \) such that \( \mu_{0m}(\pi_i) = m^{-1} \) and \( \mu_{1m}(\eta_i) = m^{-1} \) with \( 1 \leq i \leq m \) and denote \( \pi_{tm}, t = 0, 1 \) a posteriori likelihood ratios with respect to the measure of the Gaussian white noise.

We have

\[
\pi_{0m} - \pi_{1m} = \frac{1}{m} \sum_{i=1}^{m} J_i \tag{6.10}
\]

where

\[
J_k = \exp \left\{ -\epsilon^2 \sum_{i=1}^{k-1} \left( a_i \psi_i - \frac{1}{2} a_i^2 \right) \right\} \left\{ \exp \left\{ -\epsilon^2 \left( a_{1k} \psi_k + b_{1k} \xi_{1k} - \frac{1}{2} (a_{1k}^2 + b_{1k}^2) \right) \right\} - \exp \left\{ -\epsilon^2 \left( a_{2k} \psi_k + b_{2k} \xi_{1k} + c_{2k} \xi_{2k} - \frac{1}{2} (a_{2k}^2 + b_{2k}^2 + c_{2k}^2) \right) \right\} \right\}.
\]

Here \( \psi_k, \xi_{1k}, \xi_{2k} \) are independent random variables, \( E[\psi_k] = E[\xi_{1k}] = E[\xi_{2k}] = 0, E[\psi_k^2] = E[\xi_{1k}^2] = E[\xi_{2k}^2] = 1 \).

For \( k_1 \neq k_2 \) we get

\[
E[J_{k_1} J_{k_2}] = e^{-2(\tau_{k_1}, \eta_{k_2})} - e^{-2(\tau_{k_1}, \tau_{k_2})} - e^{-2(\eta_{k_1}, \tau_{k_2})} - e^{-2(\eta_{k_1}, \eta_{k_2})} + e^{-2(\psi_{k_1}, \psi_{k_2})} + e^{-2(\psi_{k_1}, \xi_{1k})} + e^{-2(\psi_{k_1}, a_{1k})} - e^{-2(a_{1k}, a_{1k})} - e^{-2(\psi_{k_1}, a_{2k})} + e^{-2(\psi_{k_1}, a_{2k})} = 0.
\tag{6.11}
\]

We also have

\[
E J_k^2 \leq 2 e^{-2\sum_{s=1}^{k} a_s^2} \{ e^{-2(a_{1k}^2 + b_{1k}^2)} + e^{-2(a_{2k}^2 + b_{2k}^2 + c_{2k}^2)} \}. \tag{6.12}
\]

By (6.11) and (6.12), applying the Cauchy inequality, we get

\[
\lim_{m \to \infty} (E|\pi_{1m} - \pi_{2m}|)^2 \leq \lim_{m \to \infty} E(\pi_{1m} - \pi_{2m})^2 = 0.
\]
Since \( m \) does not depend on \( \epsilon \) we come to contradiction.

The sequences \( \tau_k \) and \( \eta_k \) are defined on the base of the following reasoning.

For any \( \tau \in \Theta_0 \) and \( \eta \in \Theta_1 \) denote \( \Gamma\tau \) and \( \Gamma\eta \) finite dimensional subspaces such that \( \Pi\Gamma\tau \notin \Pi\Gamma\Theta_1 \) and \( \Pi\Gamma\eta \notin \Pi\Gamma\Theta_0 \). Denote \( \Gamma\tau\eta = \Gamma\tau \oplus \Gamma\eta \) where \( \oplus \) means direct sum. Denote \( \Upsilon\tau \) and \( \Upsilon\eta \) the linear spaces generated the vectors \( \tau \) and \( \eta \) respectively.

For any \( \tau \in \Theta_0 \) and \( \eta \in \Theta_1 \) there exists \( \gamma \in \Pi\Gamma\tau\Theta_0 \cap \Pi\Gamma\eta\Theta_1 \). Let us show that there exist sequences of points \( \tau_i \in \Theta_0, \eta_i \in \Theta_1 \) and finite dimensional subspaces \( \Gamma_i \) such that

1. \( \Gamma_i = \Gamma_{i-1} \oplus \Gamma\tau_i \eta_i \oplus \Upsilon\tau_{i-1} \oplus \Upsilon\eta_{i-1} \).
2. there exists a sequence of points \( \gamma_{ij} = \Pi\Gamma\tau_i \gamma_{i+1} = \Pi\Gamma\eta_{j+1} \).
3. for each \( i \) there exists \( z_i \in \Gamma_i \) such that \( \gamma_{ij} = \Pi\Gamma\gamma_{jj} \rightarrow z_i \) as \( j \rightarrow \infty \).

Here \( \Gamma_0 = \Upsilon_\tau = \Upsilon_\eta = \emptyset \).

If \( i.-ii \)-iii hold, then nonorthogonality of \( \tau_i - \gamma_{ii}, \eta_i - \gamma_{ii} \) and \( \tau_j - \gamma_{jj}, \eta_j - \gamma_{jj}, 1 \leq i < j \leq k \) are negligible in further estimates. This allows to prove (6.9).

Denote \( \Gamma_{ci} = \Gamma_i \oplus \Gamma\eta_i \oplus \Upsilon\tau_{i-1} \oplus \Upsilon\eta_{i-1} \).

We can define sequences \( \tau_i \in \Theta_0 \) and \( \eta_i \in \Theta_1 \) satisfying i.-iii. by induction.

Let \( \tau_1 \in \Theta_0 \) and \( \eta_1 \in \Theta_1 \). Denote \( \Gamma_1 = \Gamma\tau_1 \eta_1 \). Let \( \gamma_{11} = \Pi\Gamma\tau_1 \Theta_0 \cap \Pi\Gamma\eta_1 \Theta_1 \). Define \( \tau_2 \in \Theta_0 \) and \( \eta_2 \in \Theta_1 \) as arbitrary points such that \( \gamma_{11} = \Pi\Gamma\tau_2 = \Pi\Gamma\eta_2 \) and so on.

Using these sequences \( \tau_i, \eta_i \) satisfying i.-ii. we find a subsequence satisfying i.-ii. on the base of the following procedure. We choose a subsequence \( \gamma_{i_{11}i_{11}} \) such that \( \gamma_{i_{11}i_{11}} \) converges to some point \( z_1 \in \Gamma_1 \). After that we choose from these subsequence a subsequence \( \gamma_{i_{22}i_{22}} \) such that \( \gamma_{i_{22}i_{22}} \) converges to some point \( z_2 \in \Gamma_{2_1} \) and so on. The sequences of points \( \tau_{i_{11}}, \eta_{i_{11}}, \gamma_{i_{11},i_{11}}, \) and subspaces \( \Gamma_{i_{11}i_{11}} \) satisfy i.-iii.

By an appropriate choice of subsequence \( i_k \) we can make the differences \( \gamma_{ij} - z_{ii} \) negligible for further estimates. Thus we shall assume that \( \gamma_{ij} = z_i, 1 \leq i < \infty, i \leq j < \infty \) in the further reasoning. This allows us to choose a system of coordinates such that (6.9) holds.

This completes the proof of Theorem 5.3.

References

[1] Arcones, M. (2004). The large deviation principle for stochastic processes. II. *Theory.Probab.Appl.*, 48 19-44.
[2] Bahadur, R.R. and Savage R.G. (1956). The nonexistence of certain statistical procedures in nonparametric problems. *Ann. Math. Statist.* 27 1115-1122.
[3] Barron, A.R. (1989). Uniformly powerful goodness of fit tests. *Ann.Statist.* 17 107-124.
[4] Berger, A. (1951). On uniformly consistent tests. *Ann.Math.Statist.*, 18 289-293.
[5] Baraud, Y., Huet, S., and Laurent, B. (2005). Testing convex hypothesis on the mean of a Gaussian vector. Application to testing qualitative hypotheses on a regression function. *Ann.Statist.* 33 214-257.
[6] Bogachev, V.I. (2000). *Measure Theory*. Springer, NY.
[7] Burnashev, M.V. (1979). On the minimax solution of inaccurately known signal in a white Gaussian noise. Background. Theory Probab. Appl. 24 107-119.
[8] Butucea, C., Matias, C. and Pouet, C. (2009). Adaptive goodness-of-fit testing from indirect observations. Ann. Inst. Henri Poincare. Probab. Stat. 45 352-372.
[9] Comminges, L. and Dalalyan A.S. (2013). Minimax testing of a composite null hypothesis defined via a quadratic functional in the model of regression. Electronic Journal of Statistics 7 146-190.
[10] Cover, T.M. (1973). On determining irrationality of the mean of a random variable. Ann. Statist. 1 862-871.
[11] Dembo, A. and Peres, Y. (1994) A topological criterion for hypothesis testing. Ann. Statist. 22 106-117.
[12] Dembo, A. and Zeitouni, O. (1993). Large Deviations Techniques and Applications. Jones and Bartlett, Boston.
[13] Devroye, L. and Lugosi, G. (2002). Almost sure classification of densities. J. Nonpar. Statist. 14 675-698.
[14] Donoho, D.L. (1988). One-sided inference about functionals of a density. Annals of Statistics. 16 1390-1420.
[15] Dunford, N. and Schwartz, J.T.(1958). Linear Operators, part I, NY Interscience Publishers, New York.
[16] Ermakov, M.S.(1993). Large deviations of empirical measures and statistical tests. Zapiski Nauchn. Semin. POMI RAN, 207 37-59 (English translation: Journal of Mathematical Sciences, (1996), 81:2379-2390).
[17] Ermakov, M.S. (2000). On distinguishability of two nonparametric sets of hypothesis. Statist. Probab. Letters. 48 275-282.
[18] Fisher, L. and Van Ness, J.W. (1969). Distinguishability of probability measures. Ann. Math. Statist. 40 381-399.
[19] Ganssler, P.(1971). Compactness and sequential compactness on the space of measures. Z.Wahrsch. Verw. Gebiete 17 124-146.
[20] Groeneboom, P., Oosterhoff, J. and Ruymgaart, F.H. (1979). Large deviation theorems for empirical probability measures. Ann. Probab. 7 553-586.
[21] Hoeffding, W and Wolfowitz, J.(1958). Distinguishability of sets of distributions. Ann. Math. Statist. 29 700-718.
[22] Ibragimov, I.A. and Khasminskii, R.Z. (1977). On the estimation of infinitely dimensional parameter in Gaussian white noise. Dokl. AN USSR 236 1053-1055.
[23] Ingster, Yu. I. (1993). Asymptotically minimax hypothesis testing for nonparametric alternatives. I, II, III. Mathematical Methods of Statistics, 2 85–114, 171–189, 249–268.
[24] Ingster, Yu. I. and Kutoyants, Yu.A.(2007). Nonparametric hypothesis testing for intensity of the Poisson process. Mathematical Methods of Statistics, 16 218–246.
[25] Ingster Yu.I., Suslina I.A. (2002). Nonparametric Goodness-of-fit Testing under Gaussian Models. Lecture Notes in Statistics 169 Springer N.Y.
[26] Iosida K. (1965). Functional Analysis Springer Verlag, New York.
[27] Janssen A. (2000). Global power function of goodness of fit tests. Annals of Statistics, 28 239-253.
[28] Kraft, C. (1955). Some conditions for consistency and uniform consistency of statistical procedures. *Iniv. Californ. Publ. Stat.* 2, 125-142.

[29] Kulkarni, S.R. and Zeitouni, O. (1995). A general classification rule for probability measures. *Ann. Statist.* 23, 1393-1407.

[30] Le Cam, L. (1973). Convergence of estimates under dimensionality restrictions. *Ann. Statist.* 1, 38-53.

[31] Le Cam, L. and Schwartz, L. (1960). A necessary and sufficient conditions for the existence of consistent estimates. *Ann. Math. Statist.* 31, 140-150.

[32] Lehmann, E.L. and Romano, J.P. (2005). *Testing Statistical Hypothesis.* Springer Verlag, NY.

[33] Nobel, A.B. (2006). Hypothesis testing for families of ergodic processes. *Bernoulli.* 12, 251-269.

[34] Schwartz, L. (1965). On Bayes procedures. *Z. Wahrsch. Verw. Gebiete* 4, 10-26.

[35] Stone, C. (1977). Consistent nonparametric regression. *Ann. Stat.* 5, 595-645.

[36] van der Vaart, A.W. (1998). *Asymptotic Statistics.* Cambridge University Press, Cambridge.

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