Optimality Conditions for Higher Order Polyhedral Discrete and Differential Inclusions

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Abstract. The problems considered in this paper are described in polyhedral multi-valued mappings for higher order(s-th) discrete (PDSIs) and differential inclusions (PDFIs). The present paper focuses on the necessary and sufficient conditions of optimality for optimization of these problems. By converting the PDSIs problem into a geometric constraint problem, we formulate the necessary and sufficient conditions of optimality for a convex minimization problem with linear inequality constraints. Then, in terms of the Euler-Lagrange type PDSIs and the specially formulated transversality conditions, we are able to obtain conditions of optimality for the PDSIs. In order to obtain the necessary and sufficient conditions of optimality for the discrete-approximation problem PDFIs, we reduce this problem to the form of a problem with higher order discrete inclusions. Finally, by formally passing to the limit, we establish the sufficient conditions of optimality for the problem with higher order PDFIs. Numerical approach is developed to solve a polyhedral problem with second order polyhedral discrete inclusions.

1. Introduction

This paper is generally concerned with discrete and differential optimization of polyhedral inclusions. In our first approach, we study the following problem labeled (PD) for higher order(s-th) polyhedral discrete inclusions (PDSIs)

\begin{equation}
\begin{aligned}
\text{minimize} & \quad \sum_{t=0}^{T-s} f(x_t, t), \\
\text{subject to} & \quad x_{t+s} \in F(x_t, x_{t+1}, \ldots, x_{t+s-1}) \quad t = 0, 1, \ldots, T-s, \\
& \quad x_k = \tilde{\theta}_k, \quad k = 0, 1, \ldots, s-1, \\
& \quad F(x, v_1, v_2, \ldots, v_{s-1}) = \{v_s : P_0 x + \sum_{r=1}^{s-1} P_r v_r - Q v_s \leq d\}.
\end{aligned}
\end{equation}

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where \( F : (\mathbb{R}^n)^s \to P(\mathbb{R}^n) \) is a polyhedral multi-valued mapping, \( P(\mathbb{R}^n) \) is a set of all non-empty subsets of \( \mathbb{R}^n \). In addition, \( P_0, P_1, \ldots, P_{s-1} \) are \( m \times n \) dimensional matrices with rows \( P_i^0, P_i^1, \ldots, P_i^{s-1} \) \((i = 1, 2, \ldots, m)\) and \( Q \) is \( m \times n \) dimensional matrix with rows \( Q_i \) \((i = 1, 2, \ldots, m)\). Let \( d \) be a \( m \)-dimensional column vector with \( d_i, i = 1, 2, \ldots, m \) and \( \theta_k, k = 0, 1, \ldots, s - 1 \) are fixed vectors. Moreover \( f(\cdot, t) : \mathbb{R}^n \to \mathbb{R} \) is a polyhedral function, i.e., \( \text{epi} f(\cdot, t) \subset \mathbb{R}^{n+1} \) is a polyhedral set.

The problem is to find a sequence of vectors \( \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_T \equiv \{\tilde{x}_t\}_{t=0}^T \) of problem (1) - (3) that minimizes
\[
\sum_{t=0}^{T-1} f(x_t, t). \quad \text{To this end, we will deduce necessary and sufficient conditions of optimality for the solution } \{\tilde{x}_t\}_{t=0}^T \text{ to the problem (PD).}
\]

In the second part of the paper the problem (PC) given by s-th order polyhedral differential inclusions (PDFIs) is studied:

\[
\begin{align*}
\text{minimize} & \quad f(x) = \int_0^1 f(x(t), t)dt + q_0(x(1), x'(1), \ldots, x^{(s-1)}(1)) \\
\text{(PC) subject to} & \quad x^{(i)}(t) \in F(x(t), t', \ldots, x^{(s-1)}(t)) \quad \text{a.e. } t \in [0, 1], \\
& \quad x^{(i)}(0) = \theta_k, \quad k = 0, 1, \ldots, s - 1.
\end{align*}
\]

Here \( F \) is a multi-valued mapping, \( f(\cdot, t) \) and \( q_0 : \mathbb{R}^m \to \mathbb{R} \) are polyhedral functions.

Convex optimization has a wide range of applications in many areas, such as combinatorial optimization and global optimization, where it is used to find bounds on optimal value as well as approximate solutions. However, it is commonly used in the fields of economy and engineering, electronic process automation, automatic control systems and optimum design problems in electrical, chemical, mechanical and aerospace engineering [1], [7], [8], [10], [12], [30]-[35].

In the study of the so-called Von Neumann economic dynamics model, the graph of which is a polyhedral cone, is the main application of mathematical methods to economic problems [14]. The emphasis of studies related to the Von Neumann model has recently been noted that the stochastic version of the Von Neumann system can be applied successfully to the study of fundamental problems in mathematical finance. In addition, the problems (PD) and (PC) can be extended to the linear discrete or linear differential optimal control problem where the control domain is a polyhedral set.

Optimal control theory is considered to be one of the key areas for the application of differential inclusions [2], [4], [6], [9], [13], [28], [29]. Essentially, the optimal control problems with ordinary and partial differential inclusions consist of intensive areas of development in the applied mathematical theory of analysis. Note that the reader can consult on the various problems described in the multi-valued mappings [15]-[27], [29], [37].

The problem with PDFIs and PDFIs considered in this paper is more complicated due to the higher order discrete approximation problem associated with s-th order difference operators expressed by binomial coefficients. As a result, it is very difficult to establish an adjoint PDFIs and PDFIs. The discretization method is important in what follows to avoid this difficulty in polyhedral optimization problems with higher order
derivatives. Optimization of higher order discrete and differential inclusions were first developed by Mahmudov [20], [26] and [24]. To the best of our knowledge, there is no paper that considers the conditions of optimality for these problems but only the qualitative problems of second order differential inclusions. Most of them have been the subject of different mathematical competitions, for example many of the second order papers PDFIs concern the existence or viability of the results over the last few years (see [3], [5], [11], [38] and references therein).

Therefore, this paper discusses a specific type of optimization problem in which the constraints are defined by the PDSIs and PDFIs. Conditionally, the paper can be divided into three parts; in the first part, the optimization of the s-th order PDSIs is investigated; in the second part, the optimization of the s-th order PDFIs is studied. The third part of the paper deals with the s-th order discrete approximation problem, which allows us to bridge the gap between PDSIs and PDFIs problems.

The paper is organized as follows:

In Section 2, by reducing s-th order discrete polyhedral optimization problem (PD) into a problem with geometric constraints and by applying Farkas Theorem 1.13 [15, p.22], we formulate the necessary and sufficient conditions of optimality for a convex minimization problem with linear inequality constraints. In addition, in terms of the Euler-Lagrange polyhedral discrete inclusions (ELPDSIs) and the derived transversality conditions, we conclude conditions of optimality for the s-th order PDSIs.

In Section 3 at first using the discretization method, i.e., s-th order difference operators expressed by binomial coefficients, we define s-th order discrete-approximation problem (PDA) associated with s-th order polyhedral optimization problem (PC). Then by applying the Theorem 2.2 to the (PDA) problem and converting this problem to the form of (PD), we derive the necessary and sufficient conditions of optimality for the s-th order discrete-approximation problem (PDA). Note that the special proven equivalence Theorem 3.1 for subdifferential inclusions, which plays an important role in constructing conditions of optimality for the (PDA) problem, is necessary for the transition to the (PDA) problem.

In Section 4, by passing the limit procedure as a discrete step tends to be zero, we establish sufficient conditions of optimality for the PDFIs.

Some interesting application of Theorem 2.2 is described in Section 5. Namely, the necessary and sufficient conditions of optimality for second order polyhedral discrete inclusions are derived. In particular, it has been shown that this method can also play an important role in numerical procedures for computing the numerical solution.

2. Convex Mathematical Programming and The Problem With PDSIs

In this section, based on convex mathematical programming, we study the optimization of the s-th order PDSIs problem. In what follows, the following lemma plays a key role in the optimization of the s-th order problem with PDSIs.

Lemma 2.1. Let \( M_t \) be a polyhedral set that is defined as

\[
M_t = \left\{ w = (x_0, \ldots, x_T) : \sum_{k=0}^{s-1} P_k x_{t+k} - Q x_{t+s} \leq d \right\}, \quad t = 0, \ldots, T - s.
\]

Then

\[
K^*_M(\partial) = \left\{ w^*(t) : x^*_{t+k}(t) = -P_k^* \lambda_k, \quad k = 0, 1, \ldots, s - 1, \quad \lambda_i \geq 0, \quad \lambda_t \in \mathbb{R}^m, \quad x^*_{t+s}(t) = Q^* \lambda_t \right\}
\]
where \( \hat{w} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_T) \) and \( K^*_M(\hat{w}) \) is the dual cone of tangent directions.

Proof. By the definition of the tangent directions, we obtain

\[
K_M(\hat{w}) = \left\{ \hat{w} : \sum_{k=0}^{s-1} P_k(\hat{x}_{t+k} + \mu \hat{x}_{t+k}) - Q(\hat{x}_{t+s} + \mu \hat{x}_{t+s}) \leq d, \quad \text{for a small } \mu > 0 \right\},
\]

t = 0, 1, \ldots, T-s, \quad \text{where } \hat{x}_{t+s} \in F(\hat{x}_0, \hat{x}_{t+1}, \ldots, \hat{x}_{t+s-1}) \text{ is satisfied and follows from this formula that the following inequalities hold}

\[
\sum_{k=0}^{s-1} P^*_k(\hat{x}_{t+k} + \mu \hat{x}_{t+k}) - Q(\hat{x}_{t+s} + \mu \hat{x}_{t+s}) \leq d, \quad t = 0, \ldots, T-s
\]

as

\[
\sum_{k=0}^{s-1} P^*_k \hat{x}_{t+k} - Q \hat{x}_{t+s} \leq 0, \quad i \in I(\hat{w}) = \left\{ i : \sum_{k=0}^{s-1} P^*_k \hat{x}_{t+k} - Q \hat{x}_{t+s} = d, \quad i = 1, \ldots, m \right\}. \tag{7}
\]

It is easy to see that the inequalities before (7) hold strongly for small \( \mu \), regardless of choosing \((\hat{x}_t, \hat{x}_{t+1}, \ldots, \hat{x}_{t+s})\) if \( i \) is not active indices, i.e., \( i \notin I(\hat{w}) \). Then, because of the arbitrary nature of \( \hat{x}_t \), \( l \neq t, t+1, t+2, \ldots, t+s \), applying the Farkas Theorem 1.13 (see, for example, [15, p.22]) it follows from the inequalities (7) that \( \hat{w}(t) = (x^*_t(t), x_{k}^{T}(t), \ldots, x^*_l(t)) \in K^*_M(\hat{w}) \) if and only if

\[
x^*_t(t) = - \sum_{i \notin I(\hat{w})} P^*_k \lambda^*_i, \quad k = 0, 1, \ldots, s-1, \quad x^*_{t+s}(t) = \sum_{i \notin I(\hat{w})} Q^*_i \lambda^*_i, \quad \lambda^*_i \geq 0, \tag{8}
\]

where \( P^*_k, Q^*_i \) are transposed vectors of \( P_k, Q_i, k = 0, 1, \ldots, s-1 \) respectively. Finally, taking \( \lambda^*_i = 0 \) for \( i \notin I(\hat{w}) \) and denoting \( \lambda_i \) for a vector with \( \lambda^*_i \) components, we have the desired result. Only that can be taken into account here

\[
\left\{ \sum_{k=0}^{s-1} P_k(\hat{x}_{t+k} - \hat{x}_{t+s} - d, \quad \lambda_i) = 0, \quad t = 0, \ldots, T-s. \right\}
\]

Now let’s convert the (PD) problem to a convex mathematical problem. Suppose \( A \) is partitioned into submatrices \( P_0, P_1, \ldots, P_{s-1}, -Q \) and \( m \times n \) zero matrices \( 0 \) and \( D \) is \( m(T-s+1) \times n(T+1) \) dimensional column vector. Obviously \( A \) is a matrix with a size of \( m(T-s+1) \times n(T+1) \), i.e.,

\[
A = \begin{pmatrix}
\begin{array}{ccccccc}
P_0 & P_1 & \cdots & P_{s-1} & -Q & 0 & 0 & \cdots & 0 \\
0 & P_0 & \cdots & P_{s-1} & -Q & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & P_0 & P_1 & \cdots & P_{s-1} -Q
\end{array}
\end{pmatrix}, \quad D = \begin{pmatrix}
d \\
\vdots \\
d
\end{pmatrix}
\]

In addition \( M \) and \( N_k \) are defined as follows: \( M = \{w = (x_0, \ldots, x_T) : Aw \leq D\} \) and \( N_k = \{w = (x_0, \ldots, x_T) : x_k = \hat{\theta}_k, k = 0, 1, \ldots, s-1\} \).

Then it’s not hard to see that the problem posed by (PD) can be transformed into a convex mathematical programming problem:

\[
\text{minimize } \quad g(w) = \sum_{t=0}^{T-1} f(x_t, t), \quad w = (x_0, x_1, \ldots, x_T) \in \mathbb{R}^{n(T+1)} \tag{9}
\]

subject to \( w \in M \cap N_k, k = 0, 1, \ldots, s-1, \quad M = \bigcap_{t=0}^{T-s} M_t. \)
This transformation allows us to rigorously prove that if \( \{\tilde{x}_t\}_{t=0}^1 \) is the optimal solution to the problem (1)-(3), then \( \tilde{w} \) is the solution to the problem (9).

Let \( K_M^*(\tilde{w}) = \{w^* : \langle \tilde{w}, w^* \rangle \geq 0, \forall \tilde{w} \in K_M(\tilde{w}) \} \) is the dual cone to the cone of tangent directions \( K_M(\tilde{w}) = \{\tilde{w} = (\tilde{x}_0, \ldots, \tilde{x}_T) : \tilde{w} + \mu \tilde{w} \in M, \mu \geq 0 \} \). Cone of tangent directions \( K_M(\tilde{w}), t = 0, \ldots, T-s, \) are polyhedral cones and so by Lemma 1.22 [15, p.23], we have \( K_M^*(\tilde{w}) = \sum_{t=0}^{T-s} K_M^*(\tilde{w}) \). By Theorem 3.4 [15, p.99] there exist vectors \( w^*_t \in \partial_\omega g(\tilde{w}), w^*_m \in K_M^*(\tilde{w}) \) and \( w^*_k \in K^s_\mu(\tilde{w}), k = 0, 1, \ldots, s-1 \) such that

\[
\begin{align*}
   w^*_t &= \sum_{k=0}^{s-1} w^*_{k} + \sum_{t=0}^{T-s} w^*_{m} \in K^*_M(\tilde{w}).
\end{align*}
\]  

(10)

This means that the \( \tilde{w} \) solution of the problem (9) has a representation (10) and vice versa. Clearly \( w^*_t = (x^*_{0t}, x^*_{1t}, \ldots, x^*_{tt}) \in \partial_\omega g(\tilde{w}) \) implies that \( x^*_{0t} \in \partial f(\tilde{x}, t), t = s, \ldots, T \).

Besides the definition of the tangent direction cone \( K^s_\mu(\tilde{w}) = \{\tilde{w} = (\tilde{x}_0, \ldots, \tilde{x}_T) : \tilde{x}_k = 0 \} \) and we have

\[
K^s_\mu(\tilde{w}) = \left\{ w^* = (x^*_{0t}, \ldots, x^*_{tt}) : x^*_{t} = 0, t \neq k \right\}, k = 0, 1, \ldots, s-1.
\]  

(11)

The main effort in this section is to formulate the conditions of optimality for the (PD) problem. We give the following valuable theorem to achieve this goal.

**Theorem 2.2.** For optimality of the trajectory \( \{\tilde{x}(t)\}_{t=0}^1 \) in the problem (PD) with PDSIs, it is necessary and sufficient that there are vectors \( x^*_{0t}, \ t = 0, \ldots, T-1 \) not all equal to zero satisfying the ELPDSIs

\[
\begin{align*}
   x^*_{0t} &= \sum_{k=0}^{s-1} P^*_r \lambda_{r-k} + u^*_t, \quad u^*_t \in \partial f(\tilde{x}, t), \ t = 0, \ldots, T-s, \\
   \partial f(\tilde{x}, k) &= \{0\}, k = 0, 1, \ldots, s-1, \\
   x^*_{r+s} &= Q^* \lambda_{r}, \lambda_{r} \geq 0, \lambda_{r-s} \equiv 0, \ r = 1, 2, \ldots, s-1, \\
   \left\{ \sum_{k=0}^{s-1} P^*_k \tilde{x}_{r+k} - Q \tilde{x}_{r+s} - d, \lambda_r \right\} &= 0,
\end{align*}
\]

and transversality conditions

\[
\begin{align*}
   x^*_{T-s+r} - \sum_{r=0}^{s-1} P^*_r \lambda_{T-s-r} \in \partial f(\tilde{x}_{T+s}^*), \ T-s+r, \ r = 1, 2, \ldots, s-1, \\
   x^*_{T-s} &= 0.
\end{align*}
\]

**Proof.** According to the formula (11) and Proposition 2.1, we can write

\[
\begin{align*}
   w_0^* &= (x^*_{0t}, 0, \ldots, 0), \ w_1^* = (0, x^*_{1t}, 0, \ldots, 0), \ldots, \ w_{s-1}^* = (0, 0, \ldots, x^*_{(s-1)(t-1)}, 0, \ldots, 0), \\
   w^*(t) &= (0, \ldots, 0, x^*_t(t), x^*_{t+1}(t), \ldots, x^*_{t+s}(t), 0, \ldots, 0), \ t = 0, \ldots, T-s.
\end{align*}
\]

Now, using the latter relationship from (10) we have

\[
\begin{align*}
   x^*_{00} + x^*_0(0) &= 0, \\
   x^*_{11} + x^*_1(0) + x^*_1(1) &= 0, \\
   \ldots \ldots \\
   x^*_{(s-1)(t-1)} + x^*_{s-1}(0) + x^*_{s-1}(1) + \cdots + x^*_{s-1}(s-1) &= 0,
\end{align*}
\]  

(12)
\[ x_{gt}^r = x_t^r(t-s) + x_t^r(t-(s-1)) + x_t^r(t-(s-2)) + \cdots + x_t^r(t), \quad t = s, \ldots, T-s, \quad (13) \]

\[ \sum_{j=0}^{s-r} x_{T-s+j}^r(T-s-j) = x_{gt(T-s+r)}^r, \quad r = 1, 2, \ldots, s-1, \]

\[ x_T^r(T-s) = 0. \quad (14) \]

For convenience, by denoting \( x_{t+s}^r(t) \equiv x_{t+s}^r, \ t = 1, \ldots, T-s, \ x_{gt}^r \equiv u_t^r \) and taking into account the formula (13) and the Proposition 2.1, we deduce the ELPDSIs of theorem

\[ x_t^r = \sum_{k=0}^{s-1} P_k^r \lambda_{t-k} + u_t^r, \quad u_t^r \in \partial f(\tilde{x}_t, t), \ x_{t+s}^r = Q^r \lambda_t, \quad \lambda_t \geq 0, \quad t = 0, \ldots, T-s, \]

under the condition

\[ \left\{ \sum_{k=0}^{s-1} P_k \tilde{x}_{t+k} - Q \tilde{x}_{t+s} - d, \ \lambda_t \right\} = 0. \]

In addition, by virtue of (12) and setting \( f(\tilde{x}_t, k) \equiv 0, \ x_{t+k}^r \equiv x_t^r, \ k = 0, 1, \ldots, s-1, \ \lambda_{-r} \equiv 0, \ r = 1, 2, \ldots, s-1, \) the formula (13) remains true for \( t = 0, 1, \ldots, s-1. \) On the other hand, using the first relationship of (14) we have

\[ x_{t-s+r}^r - \sum_{j=0}^{s-1-r} P_r^{sh} \lambda_{T-s-j} \in \partial f(\tilde{x}_{T-s+r}, T-s+r), \ r = 1, 2, \ldots, s-1. \]

Since \( \partial f(\tilde{x}_t, T) \equiv \{0\} \) indicates that \( x_t^r = 0. \)

3. Necessary and Sufficient Conditions of Optimality For Higher Order Polyhedral Discrete-Approximation Problem

Let us introduce, first of all, the following s-th order difference operators

\[ \Delta^s x(t) = \frac{1}{\delta^s} \left( \sum_{j=0}^{s} (-1)^j \binom{s}{j} x(t+(s-j)\delta) \right), \quad t = 0, \delta, \ldots, 1 - \delta, \]

where \( \delta \) is a step on the \( t \)-axis, \( x(t) \) is a grid function on \([0, 1]\) and \( \binom{s}{j} = \frac{s!}{j!(s-j)!} \) is a binomial coefficient.

Let us explain the main method that we use to obtain the sufficient conditions of optimality for the \((PC)\) problem. This is a direct method based on discrete approximations. Therefore, the basic idea is to substitute the continuous problem \((PC)\) with a discrete-approximation problem that can be effectively tested. Then, by formally passing the limit on the discrete-approximation problem, we formulate sufficient conditions of optimality for the original problem with s-th order derivatives. As a result, according to the problem \((PC)\), we associate the following s-th order discrete approximation problem \((PDA)\):

\[
\text{minimize} \quad \sum_{i=0}^{s-1} \delta f(x(t), t) + \varphi_0 \left( x(1-(s-1)\delta), \Delta x(1-(s-1)\delta), \ldots, \Delta^{s-1} x(1-(s-1)\delta) \right),
\]

subject to

\[ (PDA) \quad \sum_{k=0}^{s-1} P_k \Delta^s x(t) - Q \Delta^s x(t) \leq d, \quad t = 0, \delta, \ldots, 1 - s\delta, \]

\[ \Delta^s x(0) = \theta_k, \quad k = 0, 1, \ldots, s-1. \quad (15) \]
The method used in this paper requires some special equivalence theorem, which allows us to bridge the gap between (PD) and (PC).

**Theorem 3.1.** Suppose \( \phi : (\mathbb{R}^n)^s \to \mathbb{R} \) is a function defined by the relationship \( \phi(x, v_1, \ldots, v_{s-1}) \equiv \varphi_0(x, \eta_1, \ldots, \eta_{s-1}) \) where \( \eta_1 = \frac{1}{\delta_1} \sum_{j=0}^{r_1-1} (-1)^j v_{j+1} + (-1)^0 x \), \( r_1 = 1, 2, \ldots, s - 1 \), \( (x^0, v^0_1, \ldots, v^0_{s-1}) \in \text{dom} \phi \), \( (x^0, \eta^0_1, \ldots, \eta^0_{s-1}) \in \text{dom} \varphi_0 \). The following subdifferential inclusions are equivalent:

(i) \( \langle \vec{x}^*, \vec{v}_1, \ldots, \vec{v}_{s-1} \rangle \in \partial \varphi_0(x^0, v^0_1, \ldots, v^0_{s-1}) \)

(ii) \( \langle \vec{x}^* + \sum_{j=1}^{s-1} \vec{v}_j^* + v^*_1, v^*_2, \ldots, v^*_s \rangle \in \partial \varphi_0(x^0, \eta^0_1, \ldots, \eta^0_{s-1}) \)

where \( v^*_r = \delta^r \left[ \sum_{j=0}^{s-r-1} (-1)^j \vec{v}_j^* \right] \), \( r = 1, 2, \ldots, s - 1 \).

**Proof.** Clearly \( \partial \varphi_0(z_0) \) is a convex closed set and for \( z_0 = (x^0, v^0_1, \ldots, v^0_{s-1}) \in \text{ri(dom} \phi) \) is bounded. Let’s denote \( y_0 = (x^0, \eta^0_1, \eta^0_2, \ldots, \eta^0_{s-1}) \), by the classical definition of subdifferential sets we get

\[
\partial \varphi_0(y_0) = \left\{ (x^*, v^*_1, \ldots, v^*_s) : \varphi_0(y) - \varphi_0(y_0) \geq \langle x^*, x - x^0 \rangle + \sum_{j=1}^{s-1} \left( \frac{1}{\delta^j} \sum_{j=0}^{s-1} v_{j+1} - v^0_{j+1} \right) \right\}, \quad \forall y \in \mathbb{R}^m.
\]

The last relation for \( \partial \varphi_0(y_0) \) implies,

\[
\partial \varphi_0(y_0) = \left\{ (x^*, v^*_1, \ldots, v^*_s) : \varphi_0(y) - \varphi_0(y_0) \geq \langle x^* + \sum_{j=1}^{s-1} (-1)^j v^*_j, x - x^0 \rangle \right\}
+ \sum_{j=1}^{s-2} \left( \frac{1}{\delta^j} \sum_{j=0}^{s-2} (-1)^j v^*_j v^0_{j+1} + v^*_1 v^0_1 \right) + \sum_{j=0}^{s-2} \left( \frac{1}{\delta^{j+2}} \sum_{j=0}^{s-2} (-1)^j v^*_j v^0_{j+2} + v^*_2 v^0_2 \right)
+ \sum_{j=0}^{s-2} \left( \frac{1}{\delta^{j+2}} \sum_{j=0}^{s-2} (-1)^j v^*_j v^0_{j+2} v^0_{s-2} + v^*_2 v^0_{s-1} \right), \quad \forall y \in \mathbb{R}^m.
\]

Rewriting this inequality, we have

\[
\partial \varphi_0(y_0) = \left\{ (x^*, v^*_1, \ldots, v^*_s) : \varphi_0(y) - \varphi_0(y_0) \geq \langle x^* + \sum_{j=1}^{s-1} (-1)^j v^*_j, x - x^0 \rangle \right\}
+ \sum_{j=1}^{s-1} \left( \frac{1}{\delta^j} \sum_{j=0}^{s-1} (-1)^j v^*_j v^0_{j+1} \right), \quad \forall y \in \mathbb{R}^m.
\]
On the basis of (16) and (17), it can be claimed that
\[
\bar{x}^r = x^r + \sum_{j=1}^{s-1} \frac{(-1)^j}{\delta^j} v_{j,r}, \quad \bar{v}_r^* = \sum_{j=0}^{s-(s+1)} \frac{(-1)^j}{\delta^{j+r}} (j + r) \bar{v}_{j+r}, \quad r = 1, 2, \ldots, s - 1.
\]
\[
(18)
\]
Now, starting with the last equation, by sequentially substituting in (18), we derive
\[
x^r = \bar{x}^r + \sum_{j=1}^{s-1} \bar{v}_{j,r}, \quad v_r^* = \bar{v}_r^* = \delta^r \sum_{j=0}^{s-(s+1)} (j + r) \bar{v}_{j+r}, \quad r = 1, 2, \ldots, s - 1. \quad \Box
\]

**Theorem 3.2.** For optimality of the trajectory \(\{\bar{x}(t)\}_{t=0}^1\) in the problem (PDA), it is necessary and sufficient that there is an adjoint trajectory of vectors \(\{x^*(t)\}_{t=0}^1\) simultaneously not all equal to zero satisfying the approximate ELPDSIs

1. \((-1)^k \Delta^k x^*(t) \in \sum_{k=0}^{s-1} (-1)^k P_k \Delta^k \lambda(t - k \delta) + \partial f(\bar{x}(t), t),\)
2. \(\left( \sum_{k=0}^{s} P_k \Delta^k \bar{x}(t) - Q \Delta^k \bar{x}(t) - d, \lambda(t) \right) = 0, \quad \lambda(t) \geq 0, \quad t = s \delta, \ldots, 1 - s \delta,
\]

and transversality condition

3. \(\left( \xi_1, \xi_2, \ldots, \xi_s \right) \in \partial \varphi_0 \left( \bar{x}(1 - (s - 1) \delta), \Delta \bar{x}(1 - (s - 1) \delta), \ldots, \Delta^{s-1} \bar{x}(1 - (s - 1) \delta) \right),\)
\[
\xi_k = (-1)^k \Delta^{s-k} x^*(1 - (s - k) \delta) + \sum_{j=1}^{s-k} (-1)^j P_j \Delta^{j-1} \lambda(1 - (j + s - k) \delta), \quad k = 1, 2, \ldots, s.
\]

**Proof.** We use the result of Theorem 2.2 to formulate conditions of optimality for problem (15), that’s why we transform this problem into a (PD) form problem:

\[
\text{minimize } \sum_{t=0}^{1-s \delta} \delta f(x(t), t) + \varphi_0 \left( \bar{x}(1 - (s - 1) \delta), \Delta \bar{x}(1 - (s - 1) \delta), \ldots, \Delta^{s-1} \bar{x}(1 - (s - 1) \delta) \right).
\]

subject to
\[
\left[ \sum_{k=0}^{s-1} (-1)^k \delta^{s-k} P_k - (-1)^s Q \right] x(t) + \left[ \sum_{k=1}^{s-1} (-1)^{k+1} \left( \frac{k}{1} \right) \delta^{s-k} P_k - (-1)^s \left( \frac{s}{1} \right) Q \right] \lambda(t + \delta) + \sum_{k=2}^{s-1} (-1)^k \left( \frac{k}{2} \right) \delta^{s-k} P_k - (-1)^s \left( \frac{s}{2} \right) Q \Delta x(t) \leq \delta^s d, \quad t = 0, \delta, \ldots, 1 - s \delta,
\]
\[
\lambda(t) \geq 0, \quad t = s \delta, \ldots, 1 - s \delta,
\]
\[
x(0) = \sum_{j=0}^{k} \left( \frac{k}{j} \right) \delta^j \theta_j, \quad k = 0, 1, \ldots, s - 1.
\]
\[
(19)
\]
Let $\{\bar{x}(t)\}$, $t = 0, \delta, \ldots, 1$ be the optimal solution to the problem (19). It is not hard to see that the adjoint discrete inclusions for $s$-th order polyhedral problems have the forms

$$x'(t) \in \left[ \sum_{k=0}^{s-1} (-1)^{k} \delta^{-k} P_{k} - (-1)^{s} Q \right] \lambda(t) + \left[ \sum_{k=1}^{s-2} (-1)^{k+1} \left( \frac{k}{1} \right) \delta^{-k} P_{k} + (-1)^{s} \left( \frac{s}{1} \right) Q \right] \lambda(t - \delta)$$

$$+ \left[ \sum_{k=2}^{s-1} (-1)^{k} \delta^{-k} P_{k} - (-1)^{s} Q \right] \lambda(t - 2\delta) + \cdots + \left[ \delta P_{s-1} + sQ \right] \lambda(t - (s-1)\delta) + \delta \partial f(\bar{x}(t), t),$$

$t = 0, \delta, \ldots, 1 - \delta$, $x'(t + s\delta) = Q' \lambda(t)$, $\lambda(t) \geq 0$, (20)

$$-Q\bar{x}(t + s\delta) - \delta d, \lambda(t) = 0.$$ (21)

Rewriting the inclusion (20), we have

$$\left( x'(t) + (-1)^{s} Q' \lambda(t) - (-1)^{s} \left( \frac{s}{1} \right) Q' \lambda(t - \delta) + (-1)^{s} \left( \frac{s}{2} \right) Q' \lambda(t - 2\delta) + \cdots + (-s) Q' \lambda(t - (s-1)\delta) \right)$$

$$\in \left[ \sum_{k=0}^{s-1} (-1)^{k} \delta^{-k} P_{k} \right] \lambda(t) + \left[ \sum_{k=1}^{s-2} (-1)^{k+1} \left( \frac{k}{1} \right) \delta^{-k} P_{k} \right] \lambda(t - \delta) + \cdots + \delta P_{s-1} \lambda(t - (s-1)\delta) + \delta \partial f(\bar{x}(t), t).$$

Recall that $x'(t + s\delta) = Q' \lambda(t)$, then we deduce from the last inclusion that

$$\left( x'(t) + (-1)^{s} x'(t + s\delta) - (-1)^{s} \left( \frac{s}{1} \right) x'(t + (s-1)\delta) + (-1)^{s} \left( \frac{s}{2} \right) x'(t + \delta(s-2)) + \cdots + (-s) x'(t + \delta) \right)$$

$$\in \delta P_{s} \lambda(t) + \delta^{-1} \left[ - \lambda(t) + \lambda(t - \delta) \right] P_{1} + \delta^{-2} \left[ \lambda(t) - 2\lambda(t - \delta) + \lambda(t - 2\delta) \right] P_{2}$$

$$+ \cdots + \delta \left[ (-1)^{s-1} \lambda(t) + (-1)^{s-1} \left( \frac{s-1}{1} \right) \lambda(t - \delta) - (-1)^{s-1} \left( \frac{s-1}{2} \right) \lambda(t - 2\delta) \right] P_{s-1} + \delta \partial f(\bar{x}(t), t).$$

Dividing the left hand side and the right hand side of this inclusion by $\delta$ (here $\delta^{-1} x'(t)$, $\delta^{-1} \lambda(t)$ are again denoted by $x'(t)$ and $\lambda(t)$ respectively), we obtain

$$(-1)^{s} \Delta x'(t) \in \sum_{k=0}^{s-1} (-1)^{k} \delta^{-k} \lambda(t - k\delta) + \partial f(\bar{x}(t), t).$$
Similarly, the equation (21) implies
\[
\left\langle \sum_{k=0}^{s} P_{1}A^{k}\xi(t) - QA\xi(t) - d, \lambda(t) \right\rangle = 0, \lambda(t) \geq 0, \ t = s\delta, \ldots, 1 - s\delta.
\]

Then, by virtue of Theorem 2.2, the transversality conditions of problem (19) are derived:
\[
x'(1 - (s + 1)\delta) - \sum_{j=0}^{s-1} \left[ \sum_{k=1+j}^{s-1} (-1)^{k+j+1} \left( \frac{k}{1+j} \right) \delta^{s-j-k}P_{k} + (-1)^{s-j} \left( \frac{s}{1+j} \right) Q \right] \lambda(1 - (s + j)\delta) \\
\in \ \delta P_{s-1} + Q'(1 - (s + 1)\delta), 1 - (s + 1)\delta,
\]
\[
x'(1 - (s + 2)\delta) - \sum_{j=0}^{s-2} \left[ \sum_{k=2+j}^{s-1} (-1)^{k+j+1} \left( \frac{k}{2+j} \right) \delta^{s-j-k}P_{k} + (-1)^{s-j} \left( \frac{s}{2+j} \right) Q \right] \lambda(1 - (s + j)\delta) \\
\in \ \delta P_{s-1} + Q'(1 - (s + 2)\delta), 1 - (s + 2)\delta,
\]
\[
.........
\]
\[
x'(1 - s\delta) - \sum_{j=0}^{s-1} \left[ \sum_{k=s+j}^{s-1} (-1)^{k+j+1} \left( \frac{k}{s+j} \right) \delta^{s-j-k}P_{k} + (-1)^{s-j} \left( \frac{s}{s+j} \right) Q \right] \lambda(1 - (s + j)\delta) \\
\in \ \delta P_{s-1} + sQ'(1 - (s - 1)\delta), 1 - (s - 1)\delta,
\]
\[
x'(1) \in \ \delta P_{s-1} + Q'(1 - (s + 1)\delta), 1 - (s + 1)\delta.
\]

Now, remembering that under the conditions (22) \( x'(t + s\delta) = Q' \lambda(t) \), we deduce
\[
x'(1 - (s + 1)\delta) - \sum_{j=0}^{s-1} \left[ \sum_{k=1+j}^{s-1} (-1)^{k+j+1} \left( \frac{k}{1+j} \right) \delta^{s-j-k}P_{k} \right] \lambda(1 - (s + j)\delta) \\
\in \ \delta P_{s-1} + Q'(1 - (s + 1)\delta), 1 - (s + 1)\delta,
\]
\[
x'(1 - (s + 2)\delta) + \sum_{j=0}^{s-2} \left[ \sum_{k=2+j}^{s-1} (-1)^{k+j+1} \left( \frac{k}{2+j} \right) \delta^{s-j-k}P_{k} \right] \lambda(1 - (s + j)\delta) \\
\in \ \delta P_{s-1} + Q'(1 - (s + 2)\delta), 1 - (s + 2)\delta,
\]
\[
.........
\]
\[
x'(1 - s\delta) - \sum_{j=0}^{s-1} \left[ \sum_{k=s+j}^{s-1} (-1)^{k+j+1} \left( \frac{k}{s+j} \right) \delta^{s-j-k}P_{k} \right] \lambda(1 - (s + j)\delta) \\
\in \ \delta P_{s-1} \lambda(1 - s\delta) - sx'(1) \in \ \delta P_{s-1} \lambda(1 - s\delta) - sx'(1), 1 - (s + 1)\delta,
\]
\[
x'(1) \in \ \delta P_{s-1} + Q'(1 - (s + 1)\delta), 1 - (s + 1)\delta.
\]
Note that the function \( \varphi_0 \) in the problem (19) should be defined as follows

\[
\varphi_0(\tilde{x}(1 - (s - 1)\delta), \Delta \tilde{x}(1 - (s - 1)\delta), \ldots, \Delta^{s-1} \tilde{x}(1 - (s - 1)\delta))
\]

\[
\equiv \phi(\tilde{x}(1 - (s - 1)\delta), \tilde{x}(1 - (s - 2)\delta), \ldots, \tilde{x}(1))
\]

\[
= \delta(\tilde{f}(\tilde{x}(1 - (s - 1)\delta), 1 - (s - 1)\delta) + f(\tilde{x}(1 - (s - 2)\delta), 1 - (s - 2)\delta) + \cdots + f(\tilde{x}(1), 1))
\]

and then the transversality conditions (23) have the following forms

\[
\left(x'(1 - (s - 1)\delta) - \sum_{j=0}^{s-2} (-1)^{s-j}\left(\frac{s}{1 + j}\right)x'(1 - j\delta) - \sum_{j=0}^{s-2} \sum_{k=1+j}^{s-1} (-1)^{k+j+1} \left(\frac{k}{1 + j}\right) \delta^{s-k} P_k \right) \lambda(1 - (s + j)\delta),
\]

\[
x'(1 - (s - 2)\delta) + \sum_{j=0}^{s-2} (-1)^{s-j}\left(\frac{s}{2 + j}\right)x'(1 - j\delta) - \sum_{j=0}^{s-2} \sum_{k=2+j}^{s-1} (-1)^{k+j} \left(\frac{k}{2 + j}\right) \delta^{s-k} P_k \right) \lambda(1 - (s + j)\delta)
\]

\[
\cdots, x'(1 - \delta) - \delta P_{s-1}^* \lambda(1 - s\delta) - s x'(1), x'(1)
\]

\[
\in \partial \varphi(\tilde{x}(1 - (s - 1)\delta), \tilde{x}(1 - (s - 2)\delta), \ldots, \tilde{x}(1)).
\]

Dividing both sides of this inclusion again by \( \delta^{s-1} \), we have

\[
\left(\frac{1}{\delta^{s-1}} \sum_{j=0}^{s-2} (-1)^{s-j}\left(\frac{s}{1 + j}\right)x'(1 - j\delta) - \sum_{j=0}^{s-2} \sum_{k=1+j}^{s-1} (-1)^{k+j+1} \left(\frac{k}{1 + j}\right) \delta^{s-k} P_k \right) \lambda(1 - (s + j)\delta),
\]

\[
\frac{1}{\delta^{s-1}} \sum_{j=0}^{s-2} (-1)^{s-j}\left(\frac{s}{2 + j}\right)x'(1 - j\delta) - \sum_{j=0}^{s-2} \sum_{k=2+j}^{s-1} (-1)^{k+j} \left(\frac{k}{2 + j}\right) \delta^{s-k} P_k \right) \lambda(1 - (s + j)\delta),
\]

\[
\cdots, \frac{x'(1 - \delta) - s x'(1)}{\delta^{s-1}} - \frac{P_{s-1}^* \lambda(1 - s\delta)}{\delta^{s-2}} - \frac{x'(1)}{\delta^{s-1}}
\]

By applying the Theorem 3.1 and using the combinatorial identity, we can express the last inclusion in the subdifferential term \( \varphi_0 \):

\[
(-1)^{j+s}\Delta^{s-2} x'(1 - (s - 1)\delta) + \sum_{j=1}^{s-1} (-1)^j P_j^* \Delta^{j-1} \lambda(1 - (j + s - 1)\delta),
\]

\[
(-1)^j \Delta^{s-2} x'(1 - (s - 2)\delta) + \sum_{j=1}^{s-2} (-1)^j P_j^* \Delta^{j-1} \lambda(1 - (j + s - 2)\delta),
\]

\[
\cdots, -\Delta x'(1 - \delta) - P_2^* \lambda(1 - 3\delta), x'(1)
\]

\[
\in \partial \varphi(\tilde{x}(1 - (s - 1)\delta), \Delta \tilde{x}(1 - (s - 1)\delta), \ldots, \Delta^{s-1} \tilde{x}(1 - (s - 1)\delta)).
\]

The transversality condition of the problem (19) can be rewritten as follows

\[
\left(\xi_1, \xi_2, \ldots, \xi_s \right) \in \partial \varphi(\tilde{x}(1 - (s - 1)\delta), \Delta \tilde{x}(1 - (s - 1)\delta), \ldots, \Delta^{s-1} \tilde{x}(1 - (s - 1)\delta)),
\]
Obviously, for a feasible solution \( x \) and \( \lambda \geq 0 \), for \( t \in [0, 1] \) it can be written
\[
\langle \sum_{k=0}^{s} P_k \frac{d^k x(t)}{dt^k}, \lambda \rangle \leq \langle Q \frac{d^s x(t)}{dt^s} + d, \lambda \rangle.
\]
By using the theorem’s second condition (b), we have

\[
\left\langle \sum_{k=0}^{s-1} P_k \frac{d^k \bar{x}(t)}{dt^k}, \lambda(t) \right\rangle = \left\langle Q \frac{d^s \bar{x}(t)}{dt^s} + d, \lambda(t) \right\rangle.
\]

Then taking into account that \( x'(t) = Q' \lambda(t) \) from the last two relations, we derive that

\[
\left\langle \sum_{k=0}^{s-1} P_k \frac{d^k (x(t) - \bar{x}(t))}{dt^k}, \lambda(t) \right\rangle \leq \left\langle \frac{d^s (x(t) - \bar{x}(t))}{dt^s}, x'(t) \right\rangle.
\]

Then, the relations (26) and (27) imply that

\[
\left\langle (1)^{s-k} d^s x'(t) - \sum_{k=0}^{s-1} (1)^k P_k \frac{d^k \lambda(t)}{dt^k}, x(t) - \bar{x}(t) \right\rangle \geq \left\langle (1)^{s-k} \frac{d^s x'(t)}{dt^s}, x(t) - \bar{x}(t) \right\rangle
\]

\[
- \left\langle \frac{d^s (x(t) - \bar{x}(t))}{dt^s}, x'(t) \right\rangle + \sum_{i=1}^{s-1} \left\langle (1)^{s-i+1} P_k \frac{d^k \lambda(t)}{dt^k}, x(t) - \bar{x}(t) \right\rangle + \sum_{i=1}^{s-1} \left\langle P_k \lambda(t), \frac{d^i (x(t) - \bar{x}(t))}{dt^i} \right\rangle.
\]

Denoting \( \Omega_i(t) = \sum_{k=1}^{i-1} (1)^{s-i+1} P_k \frac{d^k \lambda(t)}{dt^k}, i = 1, 2, \ldots, s - 1 \), in view of the following equation

\[
\sum_{k=1}^{s-1} \left\langle (1)^{s-k+1} P_k \frac{d^k \lambda(t)}{dt^k}, x(t) - \bar{x}(t) \right\rangle + \sum_{k=1}^{s-1} \left\langle P_k \lambda(t), \frac{d^k (x(t) - \bar{x}(t))}{dt^k} \right\rangle
\]

the relation (28) can be rewritten as follows

\[
\left\langle (1)^{s-k} d^s x'(t) - \sum_{k=0}^{s-1} (1)^k P_k \frac{d^k \lambda(t)}{dt^k}, x(t) - \bar{x}(t) \right\rangle \geq \left\langle (1)^{s-k} \frac{d^s x'(t)}{dt^s}, x(t) - \bar{x}(t) \right\rangle
\]

\[
- \left\langle \frac{d^s (x(t) - \bar{x}(t))}{dt^s}, x'(t) \right\rangle + \sum_{i=1}^{s-1} \frac{d}{dt} \left[ \left\langle \Omega_i(t), \frac{d^{i-1} (x(t) - \bar{x}(t))}{dt^{i-1}} \right\rangle \right].
\]

Finally, from the inequalities (24) and (29), we conclude that

\[
f(x(t), t) - f(\bar{x}(t), t) \geq \left\langle (1)^{s-k} d^s x'(t), x(t) - \bar{x}(t) \right\rangle - \left\langle \frac{d^s (x(t) - \bar{x}(t))}{dt^s}, x'(t) \right\rangle
\]

\[
+ \sum_{i=1}^{s-1} \frac{d}{dt} \left[ \left\langle \Omega_i(t), \frac{d^{i-1} (x(t) - \bar{x}(t))}{dt^{i-1}} \right\rangle \right].
\]

Recall that \( x(\cdot), \bar{x}(\cdot) \) are feasible solutions \( \bar{x}^{(k)}(0) = \bar{x}^{(k)}(0) = \theta_k, k = 0, 1, \ldots, s - 1 \), then integrating the inequality (30), we have

\[
\int_0^1 \left( f(x(t), t) - f(\bar{x}(t), t) \right) dt \geq \int_0^1 \left[ \left\langle (1)^{s-k} d^s x'(t), x(t) - \bar{x}(t) \right\rangle - \left\langle \frac{d^s (x(t) - \bar{x}(t))}{dt^s}, x'(t) \right\rangle \right] dt
\]
\[
\sum_{i=1}^{s-1} \left( \Omega_i(1), \frac{d^{s-1}(x(t) - \bar{x}(1))}{dt^{s-1}} \right).
\]  
(31)

Let us denote the expression in the square brackets on the right hand side of (31) by \( \Psi \)

\[
\Psi = \left\langle (-1)^{y} \frac{d^{x}}{dt^x}, x(t) - \bar{x}(t) \right\rangle - \left\langle \frac{d^{x}(x(t) - \bar{x}(1))}{dt}, x(t) \right\rangle
\]

and transform it. The first term of \( \Psi \) can be converted as follows

\[
\left\langle (-1)^{y} \frac{d^{x}}{dt^x}, x(t) - \bar{x}(t) \right\rangle = \frac{d}{dt} \left\langle \left( (-1)^{y} \frac{d^{x-1}}{dt^{x-1}}, x(t) - \bar{x}(t) \right) \right\rangle + \left\langle (-1)^{y+1} \frac{d^{x-1}}{dt^{x-1}}, x(t) - \bar{x}(t) \right\rangle
\]

and the second term of \( \Psi \) can be rewritten in the following form

\[
\frac{d}{dt} \left\langle \left( \frac{d^{x-1}}{dt^{x-1}}, x(t) - \bar{x}(t) \right), x(t) \right\rangle = \frac{d}{dt} \left\langle \left( \frac{d^{x-2}}{dt^{x-2}}, \frac{d^{x}}{dt} \right), x(t) \right\rangle
\]

\[
+ \frac{d}{dt} \left\langle \left( \frac{d^{x-3}}{dt^{x-3}}, \frac{d^{2}x(t)}{dt^2} \right), x(t) \right\rangle + \cdots + (-1)^{y+1} \frac{d}{dt} \left\langle \left( \frac{d}{dt}, \frac{d^{x-2}}{dt^{x-2}} \right), x(t) \right\rangle
\]

Thus by subtracting (32) and (33) we derive

\[
\Psi = \frac{d}{dt} \left\langle \left( (-1)^{y} \frac{d^{x-1}}{dt^{x-1}}, x(t) - \bar{x}(t) \right), x(t) \right\rangle + \frac{d}{dt} \left\langle \left( \frac{d^{x-2}}{dt^{x-2}}, \frac{d^{x}}{dt} \right), x(t) \right\rangle
\]

\[
- \frac{d}{dt} \left\langle \left( \frac{d^{x-3}}{dt^{x-3}}, \frac{d^{2}x(t)}{dt^2} \right), x(t) \right\rangle + \cdots + (-1)^{y+1} \frac{d}{dt} \left\langle \left( \frac{d}{dt}, \frac{d^{x-2}}{dt^{x-2}} \right), x(t) \right\rangle.
\]

Then we need to calculate the following integral,

\[
\int_{0}^{1} \Psi dt = (-1)^{y} \left\langle \frac{d^{x-1}x(t)}{dt^{x-1}}, x(t) - \bar{x}(t) \right\rangle - \left\langle \frac{d^{x-1}x(1)}{dt^{x-1}}, x(1) \right\rangle + \left\langle \frac{d^{x-2}x(t)}{dt^{x-2}}, x(t) \right\rangle
\]

\[
- \left\langle \frac{d^{x-2}x(1)}{dt^{x-2}}, x(1) \right\rangle.
\]

By using (34) from the inequality (31) we deduce

\[
\int_{0}^{1} \left( f(x(t), t) - f(\bar{x}(t), t) \right) dt \geq (-1)^{y} \left\langle \frac{d^{x-1}x(t)}{dt^{x-1}}, x(t) - \bar{x}(t) \right\rangle - \left\langle \frac{d^{x-1}x(1)}{dt^{x-1}}, x(1) \right\rangle
\]

\[
+ \left\langle \frac{d^{x-2}x(t)}{dt^{x-2}}, x(t) \right\rangle + \cdots + (-1)^{y+1} \left\langle \frac{d^{x-2}x(t)}{dt^{x-2}}, x(t) \right\rangle + \left\langle \Omega_{1}(1), x(t) - \bar{x}(t) \right\rangle
\]

\[
+ \left\langle \Omega_{2}(1), x(1) \right\rangle + \left\langle \Omega_{3}(1), \frac{d^{x}}{dt} \right\rangle + \cdots + \left\langle \Omega_{s-1}(1), \frac{d^{x-2}}{dt^{x-2}} \right\rangle.
\]
As a result of rewriting this inequality that we have
\[
\int_0^1 \left( f(x(t), t) - f(\bar{x}(t), t) dt + \left( (-1)^{s-1} \frac{d^{s-1} x'(1)}{dt^{s-1}} + \Omega_1(1), \ x(1) - \bar{x}(1) \right) \right.
\]
\[+ \left( (-1)^{s+1} \frac{d^{s-2} x'(1)}{dt^{s-2}} + \Omega_2(1), \ x'(1) - \bar{x}'(1) \right) + \left( (-1)^{s+2} \frac{d^{s-3} x'(1)}{dt^{s-3}} + \Omega_3(1), \ x''(1) - \bar{x}''(1) \right) \]
\[+ \cdots + \left( \frac{dx(1)}{dt} + \Omega_{s-1}(1), \ x^{(s-2)}(1) - \bar{x}^{(s-2)}(1) \right) - \left( x'(1), \ x^{(s-1)}(1) - \bar{x}^{(s-1)}(1) \right). \]

Therefore, taking into account the expression \( \Omega_i(1), i = 1, 2, \ldots, s-1 \) we can write
\[
\int_0^1 \left( f(x(t), t) - f(\bar{x}(t), t) dt \geq \left( (-1)^{s-1} \frac{d^{s-1} x'(1)}{dt^{s-1}} + \sum_{k=1}^{s-1} (-1)^{k+1} p_k \frac{d^{k-1} \lambda(1)}{dt^{k-1}}, \ x(1) - \bar{x}(1) \right) \right.
\]
\[+ \left( (-1)^{s+1} \frac{d^{s-2} x'(1)}{dt^{s-2}} + \sum_{k=2}^{s-1} (-1)^{k+2} p_k \frac{d^{k-2} \lambda(1)}{dt^{k-2}}, \ x'(1) - \bar{x}'(1) \right) \]
\[+ \left( (-1)^{s+2} \frac{d^{s-3} x'(1)}{dt^{s-3}} + \sum_{k=3}^{s-1} (-1)^{k+3} p_k \frac{d^{k-3} \lambda(1)}{dt^{k-3}}, \ x''(1) - \bar{x}''(1) \right) \]
\[+ \cdots + \left( \frac{dx(1)}{dt} + p_{s-1} \lambda(1), \ x^{(s-2)}(1) - \bar{x}^{(s-2)}(1) \right) - \left( x'(1), \ x^{(s-1)}(1) - \bar{x}^{(s-1)}(1) \right). \]

Finally by summing up the inequalities (25) and (35), we derive that
\[
\int_0^1 f(x(t), t) dt + \varphi_0(x(1), x'(1), \ldots, x^{(s-1)}(1)) \geq \int_0^1 f(\bar{x}(t), t) dt + \varphi_0(\bar{x}(1), \bar{x}'(1), \ldots, \bar{x}^{(s-1)}(1)).
\]

As a result of all feasible solutions \( x(t), J(x(\cdot)) - J(\bar{x}(\cdot)) \geq 0 \) and therefore \( \bar{x}(t) \) is the optimal trajectory. \( \square \)

**Corollary 4.2.** Consider the Bolza problem with cost functional (4) and differential inclusion (5) with the following boundary value conditions
\[
x^{(k)}(0) \in G_k, \quad x^{(k)}(1) \in H_k, \quad k = 0, 1, \ldots, s-1,
\]
where \( G_k \) and \( H_k \) are polyhedral sets. Then for the optimality of the trajectory \( \bar{x}(t) \) in the Bolza problem the transversality conditions at points 0 and 1 should be as follows:

\[
\text{(d)} \quad (-1)^{s-j} \frac{d^{j-s} x(0)}{dt^{j-s}} + \sum_{i=j}^{s-1} (-1)^{i-j-1} p_i \frac{d^{j-i} \lambda(0)}{dt^{j-i}} \in K_{G_{s-i}}(\bar{x}^{(j-i)}(0)), \quad j = 1, 2, \ldots, s-1,
\]
\[
x'(0) \in K_{G_{s-1}}(\bar{x}^{(s-1)}(0)),
\]
\[
\text{(e)} \quad \sum_{i=j}^{s-1} (-1)^{i-j} p_i \frac{d^{j-i} \lambda(1)}{dt^{j-i}} \in K_{H_{s-i}}(\bar{x}^{(j-i)}(1)), \quad j = 1, 2, \ldots, s-1,
\]
\[
-x'(1) \in K_{H_{s-1}}(\bar{x}^{(s-1)}(1)),
\]
\[
\left( -1 \right)^{s-1} \frac{d^{s-1} x(1)}{dt^{s-1}}, \ (-1)^{s-2} \frac{d^{s-2} x'(1)}{dt^{s-2}}, \ (-1)^{s-3} \frac{d^{s-3} x''(1)}{dt^{s-3}}, \ldots, \ -\frac{dx'(1)}{dt}, \ x'(1) \right) \in \partial \varphi_0(\bar{x}(1), \bar{x}'(1), \ldots, \bar{x}^{(s-1)}(1)).
\]
Proof. Integrating inequality (30) we have

\[
\int_0^1 \left( f(x(t), t) - f(\bar{x}(t), t) \right) dt \geq \int_0^1 \left[ \left\langle (\Omega(0), \frac{d^{-1}x(0) - \bar{x}(0)}{dt^{-1}}) \right\rangle + \sum_{i=1}^{s-1} \left\langle \Omega(i), \frac{d^{-1}(x(i) - \bar{x}(i))}{dt^{-1}} \right\rangle + \sum_{i=1}^{s-1} \left\langle \Omega(i), \frac{d^{-1}(x(i) - \bar{x}(i))}{dt^{-1}} \right\rangle + \sum_{i=1}^{s-1} \left\langle \Omega(i), \frac{d^{-1}(x(i) - \bar{x}(i))}{dt^{-1}} \right\rangle \right] dt
\]

Similarly, by denoting the expression in square brackets on the right hand side of (36) by \( \Psi \), we calculate the following integral

\[
\int_0^1 \Psi dt = (-1)^s \left\langle \frac{d^{-1}x'(1)}{dt^{-1}}, x(1) - \bar{x}(1) \right\rangle + \left\langle \frac{d^{-1}(x(1) - \bar{x}(1))}{dt^{-1}}, x'(1) \right\rangle + \left\langle \frac{d^{-2}(x(1) - \bar{x}(1))}{dt^{-1}}, x'(1) \right\rangle
\]

By using the formula (37) and by rewriting the inequality (36), we deduce

\[
\int_0^1 \left( f(x(t), t) - f(\bar{x}(t), t) \right) dt \geq (-1)^s \left\langle \frac{d^{-1}x'(1)}{dt^{-1}}, x(1) - \bar{x}(1) \right\rangle + \left\langle \frac{d^{-1}(x(1) - \bar{x}(1))}{dt^{-1}}, x'(1) \right\rangle
\]

and by writing the definition of \( \Omega(t) \), \( i = 1, 2, \ldots, s - 1 \), in this inequality, we have

\[
\int_0^1 \left( f(x(t), t) - f(\bar{x}(t), t) \right) dt \geq \left( (-1)^s \frac{d^{-1}x'(1)}{dt^{-1}} + \sum_{k=1}^{s-1} (-1)^{k+1} p_k \frac{d^{k-1}x(1) - \bar{x}(1)}{dt^{-1}} \right) + \left\langle \Omega_{s-1}(1), \frac{d^{-2}(x(1) - \bar{x}(1))}{dt^{-2}} \right\rangle
\]

where

\[
\int_0^1 \left( f(x(t), t) - f(\bar{x}(t), t) \right) dt \geq \left( (-1)^s \frac{d^{-1}x'(1)}{dt^{-1}} + \sum_{k=1}^{s-1} (-1)^{k+1} p_k \frac{d^{k-1}x(1) - \bar{x}(1)}{dt^{-1}} \right) + \left\langle \Omega_{s-1}(1), \frac{d^{-2}(x(1) - \bar{x}(1))}{dt^{-2}} \right\rangle
\]
Taking into account the other transversality conditions we have

\[
\begin{aligned}
&\ldots + \left(\frac{dx'}{dt} + P_{s-1}^s(1), x^{(e-2)}(1) - \overline{x}^{(e-2)}(1)\right) - \left(x'(1), x^{(e-1)}(1) - \overline{x}^{(e-1)}(1)\right) \\
&- \left(1\right)^{s-1} \frac{d^{e-1}x'(0)}{dt^{e-1}} + \sum_{k=1}^{s-1} \left(-1\right)^{k+1} P_k^s \frac{d^{k-1}\lambda(0)}{dt^{k-1}}, x(0) - \overline{\lambda}(0) \\
&- \left(1\right)^{s+1} \frac{d^{e-2}x'(0)}{dt^{e-2}} + \sum_{k=2}^{s-1} \left(-1\right)^{k+2} P_k^s \frac{d^{k-2}\lambda(0)}{dt^{k-2}}, x'(0) - \overline{\lambda}'(0) \\
&- \ldots - \left(\frac{dx'(0)}{dt} + P_{s-1}\lambda(0), x^{(e-2)}(0) - \overline{x}^{(e-2)}(0)\right) + \left(x'(0), x^{(e-1)}(0) - \overline{x}^{(e-1)}(0)\right).
\end{aligned}
\]

For all feasible solutions, the transversality condition (e) can be written in the form

\[
\varphi_0\left(x(1), x'(1), \ldots, x^{(e-1)}(1)\right) - \varphi_0\left(\overline{x}(1), \overline{x}'(1), \ldots, \overline{x}^{(e-1)}(1)\right) \\
\geq \left(1\right)^{s-1} \frac{d^{e-1}x'(1)}{dt^{e-1}}, x(1) - \overline{x}(1) + \left(1\right)^{s} \frac{d^{e-2}x'(1)}{dt^{e-2}}, x'(1) - \overline{x}'(1) \\
+ \ldots + \left(-\frac{dx'(1)}{dt}, x^{(e-2)}(1) - \overline{x}^{(e-2)}(1)\right) + \left(x'(1), x^{(e-1)}(1) - \overline{x}^{(e-1)}(1)\right).
\]

Then, by adding the inequalities (38) and (39) we derive

\[
\int_0^1 \left(f(x(t), t) - f(\overline{x}(t), t)\right)dt + \varphi_0\left(x(1), x'(1), \ldots, x^{(e-1)}(1)\right) - \varphi_0\left(\overline{x}(1), \overline{x}'(1), \ldots, \overline{x}^{(e-1)}(1)\right) \\
\geq \left(\sum_{k=1}^{s-1} \left(-1\right)^{k+1} P_k^s \frac{d^{k-1}\lambda(1)}{dt^{k-1}}, x(1) - \overline{x}(1)\right) + \left(\sum_{k=2}^{s-1} \left(-1\right)^{k+2} P_k^s \frac{d^{k-2}\lambda(1)}{dt^{k-2}}, x'(1) - \overline{x}'(1)\right) \\
+ \ldots + \left(P_{s-1}\lambda(1), x^{(e-2)}(1) - \overline{x}^{(e-2)}(1)\right) - \left(1\right)^{s} \frac{d^{e-2}x'(0)}{dt^{e-2}} + \sum_{k=1}^{s-1} \left(-1\right)^{k+1} P_k^s \frac{d^{k-1}\lambda(0)}{dt^{k-1}}, x(0) - \overline{x}(0) \\
- \left(1\right)^{s+1} \frac{d^{e-2}\lambda(0)}{dt^{e-2}} + \sum_{k=2}^{s-1} \left(-1\right)^{k+1} P_k^s \frac{d^{k-2}\lambda(0)}{dt^{k-2}}, x'(0) - \overline{x}'(0) \\
- \ldots - \left(\frac{dx'(0)}{dt} + P_{s-1}\lambda(0), x^{(e-2)}(0) - \overline{x}^{(e-2)}(0)\right) + \left(x'(0), x^{(e-1)}(0) - \overline{x}^{(e-1)}(0)\right).
\]

Taking into account the other transversality conditions we have

\[
\int_0^1 f(x(t), t)dt + \varphi_0(x(1), x'(1), \ldots, x^{(e-1)}(1)) \geq \int_0^1 f(\overline{x}(t), t)dt + \varphi_0(\overline{x}(1), \overline{x}'(1), \ldots, \overline{x}^{(e-1)}(1)),
\]

this means that for all feasible solutions \(x(t), f(x(\cdot)) - f(\overline{x}(\cdot)) \geq 0\) and therefore \(\overline{x}(t)\) is the optimal trajectory. \(\square\)

By using the results obtained in this section, we are able to formulate sufficient conditions of optimality for the continuous problem (4)-(6) given by second order \((s = 2)\) polyhedral differential inclusions.

\[S. \text{ Demir Sağlam, E. Mahmudov} / \text{Filomat} 34:13 (2020), 4533–4553\]
Corollary 4.3. [22] For trajectory $\mathbf{x}(t)$, $t \in [0, 1]$, lying interior to $\text{dom} \mathbf{F}$ to be optimal in Bolza problem with second order ($s = 2$) polyhedral differential inclusion (PC), it is sufficient that there is an absolutely continuous function $x^*(t)$ satisfying the following Euler-Lagrange differential inclusion almost everywhere

\[ \begin{align*}
(i) \quad & \frac{d^2 x(t)}{dt^2} \in P_0^* \lambda(t) - P_1^* \frac{d \lambda(t)}{dt} + \partial f(\mathbf{x}(t), t), \quad \text{a.e. } t \in [0, 1], \\
& x^*(t) = Q^* \lambda(t), \quad \lambda(t) \geq 0, \\
(ii) \quad & \left\{ P_0^* \mathbf{x}(t) + P_1^* \frac{d \mathbf{x}(t)}{dt} - Q^* \frac{d^2 \mathbf{x}(t)}{dt^2} - d, \lambda(t) \right\} = 0, \quad \text{a.e. } t \in [0, 1], \\
\text{and transversality condition} \quad & (iii) \left( - \frac{dx^*(1)}{dt} - P_1^* \lambda(1), x^*(1) \right) \in \partial \mathbf{p}_0(\mathbf{x}(1), \mathbf{x}'(1)).
\end{align*} \]

5. Numerical Application

This section describes some of the interesting applications of the Theorem 2.2. It should be noted that according to Theorem 2.2, the numerical solution of the second order polyhedral discrete problem can be calculated. Let $T = 15$, $f(x_T, T) = x_T$, $f(x, t) = 0$, $t = 2, 3, \ldots, 14$ be given for this purpose. Let’s look at the following example:

Example 5.1.

\[ \begin{align*}
\text{minimize} & \quad \sum_{t=2}^{15} f(x_t, t) \\
x_{t+2} & \in \mathbf{F}(x_t, x_{t+1}), \quad t = 0, 1, \ldots, 13, \\
x_0 & = 0, x_1 = 1,
\end{align*} \]

(41)

where $\mathbf{F}(x, v_1) = \left\{ v_2 : \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} x + \begin{pmatrix} -0.1 \\ -0.2 \\ -0.3 \end{pmatrix} v_1 - \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_2 \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ is a polyhedral multi-valued mapping. In fact, the graph of $\mathbf{F}$ is a cone.

We assume that the objective function has a form $\sum_{t=2}^{15} f(x_T, T)$ in the problem (41) with second-order polyhedral discrete inclusions where $f(x_T, T)$ is not identically zero, i.e., $f(x_T, T) \neq 0$. This means that $x_T^* \neq 0$ and then the transversality condition for $T$ consists of the following inclusion $x_T^* \in \partial f(x_T, T)$.

For optimality of the trajectory $\{\mathbf{x}_i\}_{i=0}^{15}$ of polyhedral discrete inclusions in the second order discrete polyhedral optimization problem (41), it is necessary and sufficient that there are vectors $x_t^*$, $t = 0, \ldots, 15$ not all equal to zero satisfying the Euler-Lagrange discrete inclusion

\[ \begin{align*}
x_t^* & = P_0^* \lambda_t + P_1^* \lambda_{t-1} + u_t^*, \quad u_t^* \in \partial f(\mathbf{x}_i, t), \\
x_{t+2}^* & = Q^* \lambda_t, \quad \lambda_t \geq 0, \quad t = 0, \ldots, 13, \partial f(\mathbf{x}_0, 0) = \partial f(\mathbf{x}_1, 1) = \{0\}, \lambda_{-1} \equiv 0, \\
\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \mathbf{x}_t + \begin{pmatrix} -0.1 \\ -0.2 \\ -0.3 \end{pmatrix} \mathbf{x}_{t+1} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathbf{x}_{t+2}, \lambda_t \right\} = 0,
\end{align*} \]

(42)

and transversality conditions

\[ \begin{align*}
x_{14}^* - P_1^* \lambda_{13} & \in \partial f(\mathbf{x}_{14}, 14), \\
x_{15}^* & \in \partial f(\mathbf{x}_{15}, 15).
\end{align*} \]
Here $P_0^* = (1 - 2 1)$, $P_1^* = (-0.1 - 0.2 - 0.3)$ and $Q^* = (1 - 1 1)$ are transposed matrices. Since $f$ is continuously differentiable function, we have

$$x_t^* = P_0^* \lambda_t + P_1^* \lambda_{t-1}, \quad t = 0, 1, \ldots, 13,$$

and transversality condition

$$x_{14}^* - P_1^* \lambda_{13} = 0, \quad x_{15}^* = 1.$$

By sequentially resolving these equations, it is not difficult to calculate $\lambda_t$ ($t = 0, \ldots, 13$) and $x_t^*$ ($t = 0, \ldots, 15$).

We have

$$\lambda_{13} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_{12} = \begin{pmatrix} 0 \\ 0.1 \\ 0 \end{pmatrix}, \quad \lambda_{11} = \begin{pmatrix} 0.98 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix} 0 \\ 0.298 \\ 0 \end{pmatrix}, \quad \lambda_9 = \begin{pmatrix} 0 \\ 0.9204 \\ 0 \end{pmatrix}.$$

$$\lambda_8 = \begin{pmatrix} 0 \\ 0.688 \\ 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0.7828 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 1.4543 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0.4919 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 \\ 0 \\ 2.9579 \end{pmatrix}.$$

$$\lambda_3 = \begin{pmatrix} 0 \\ 0.0996 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 5.9357 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 1.3864 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_0 = \begin{pmatrix} 0 \\ 12.1488 \end{pmatrix}.$$

The values of the adjoint variables $\{x_t^*\}_{t=0}^{15}$ which are calculated and the optimal trajectory $\{\tilde{x}_t\}_{t=0}^{15}$ obtained by using these values are given in Table 1. In addition, the graphical representation of the optimal trajectory $\{\tilde{x}_t\}_{t=0}^{15}$ is given in detail in Figure 1.

| $t$ | $x_t^*$ |
|-----|---------|
| 15  | 1       |
| 14  | -0.1    |
| 13  | 0.98    |
| 12  | -0.298  |
| 11  | 0.9204  |
| 10  | -0.6884 |
| 9   | 0.78279 |
| 8   | -1.45435|
| 7   | 0.49192 |
| 6   | -2.85791|
| 5   | -0.09966|
| 4   | -5.93575|
| 3   | -1.38647|
| 2   | -12.1488|
| 1   | -5.2027 |
| 0   | -24.2976|

| $t$ | $\tilde{x}_t$ |
|-----|---------------|
| 0   | 0             |
| 1   | 1             |
| 2   | 0.2           |
| 3   | 2.09          |
| 4   | 0.808        |
| 5   | 4.2416       |
| 6   | 2.46432      |
| 7   | 3.99516      |
| 8   | 5.72767      |
| 9   | 3.4224       |
| 10  | 12.14982     |
| 11  | 2.20841      |
| 12  | 24.72133     |
| 13  | -0.26717     |
| 14  | 49.3899      |
| 15  | -5.2027     |

Table 1: The values of adjoint variables $\{x_t^*\}_{t=0}^{15}$ and the values of optimal trajectory $\{\tilde{x}_t\}_{t=0}^{15}$
6. Conclusions

This paper presents a new method of discretization to solve the optimization problem described by multi-valued polyhedral mappings for discrete and differential inclusions of higher order that are often used to describe different processes in science and engineering. The problem of higher-order discreteapproximation inclusions is investigated according to the proposed discretization approach. This approach plays a much larger role in the derivation of discrete and differential inclusions of higher-order adjoints. Equivalence theorems for subdifferential inclusions are basic tools for the analysis of conditions of optimality for discrete and discrete-approximation problems. Therefore, necessary and sufficient conditions of optimality are deduced for such problems. Finally, the numerical approach is presented with a second order polyhedral discrete inclusion to solve the optimal control problem. Besides, it is clear that the investigation of the conditions of optimality for problems with polyhedral discrete and differential inclusions will make a significant contribution to modern development of the optimal control theory with polyhedral differential inclusions. Thus, we can conclude that the proposed method is effective in solving various optimization problems with higher order discrete and differential inclusions.

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