Physical intuition and time-dependent generalizations of the Coulomb and Biot-Savart laws

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February 9, 2022

Abstract

An analysis is performed of the role played by physical intuition in expressing the electromagnetic field in terms of its sources in the time-dependent case. The conclusion is that it is dangerous to dissociate physical intuition from the mathematical description of the phenomena.
Intuition is an element of paramount importance for the construction of physical theories and the solution of physical problems. In the case of electrodynamics, the finiteness of the speed of propagation of electromagnetic influences plays a crucial role in our understanding of electromagnetic phenomena.

The standard argument for the retarded potentials runs as follows. Consider the contribution to the potential at point \( r \) from the charges and currents in the volume element \( d\tau' \) about point \( r' \). Since electromagnetic influences travel at speed \( c \), the potentials at \( r \) at time \( t \) must have originated in the charges and currents present in \( d\tau' \) at an instant \( t_r \) previous to \( t \), so providing electromagnetic influences with the time interval \( t - t_r \) to propagate from \( r' \) to \( r \). The distance between \( r \) and \( r' \) is the magnitude of the vector \( \mathbf{R} = r - r' \), that is, \( R = |r - r'| \). Thus, the retarded time \( t_r \) is determined by \( c(t - t_r) = R \), or

\[
t_r = t - \frac{R}{c} = t - \frac{|r - r'|}{c} \quad \text{(1)}
\]

Physical intuition suggests that the potentials be given in terms of their sources by the same expressions valid in electrostatics and magnetostatics except for the replacement of the charge and current densities by their values at the retarded time. One is led, therefore, to introduce the retarded potentials

\[
V(\mathbf{r}, t) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{R} \, d\tau' \quad , \quad A(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t_r)}{R} \, d\tau' \quad \text{(2)}
\]

Straightforward computations [1] show that the retarded potentials satisfy their corresponding inhomogeneous wave equations and meet the Lorentz condition. The fields are found from the retarded potentials by means of

\[
\mathbf{E} = -\nabla V - \frac{\partial A}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A} \quad \text{(3)}
\]

and a direct calculation [1] yields

\[
\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi \epsilon_0} \int \left[ \frac{\rho(\mathbf{r}', t_r)}{R^2} \mathbf{R} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{cR} \mathbf{R} - \frac{\mathbf{j}(\mathbf{r}', t_r)}{c^2 R} \right] d\tau' \quad , \quad \text{(4)}
\]

and

\[
\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[ \frac{\mathbf{j}(\mathbf{r}', t_r)}{R^2} + \frac{\dot{\mathbf{j}}(\mathbf{r}', t_r)}{cR} \right] \times \mathbf{R} \, d\tau' \quad . \quad \text{(5)}
\]

These equations, in which \( \mathbf{R} = \mathbf{R}/R \) and the dot means partial derivative with respect to time, are time-dependent generalizations of the Coulomb and Biot-Savart laws and appear to have been first published by Jefimenko in 1966, in the first edition of his textbook [3]. These equations have been shown [4] to be equivalent to other seemingly different equations derived by Panofsky and Phillips [5].

It is often emphasized [1, 2] that the same “logic” that worked for the potentials leads to wrong answers for the fields. Indeed, as Griffiths [1] remarks, “to get the retarded potentials, all you have to do is replace \( t \) by \( t_r \) in the electrostatic and magnetostatic formulas, but in the case of the fields not only is time replaced by retarded time, but completely new terms (involving derivatives of \( \rho \) and \( \mathbf{j} \)) appear.” This state of affairs has been called a conundrum by McDonald [4]. Saying, as he does, that the conundrum is resolved by radiation is hardly a satisfying explanation of why intuition seems to have betrayed us in the case of the fields.
Let us take a closer look at the origin of our intuition about the potentials. In the Lorentz gauge the scalar and vector potentials obey the inhomogeneous wave equation

\[ \nabla^2 \phi(r, t) - \frac{1}{c^2} \frac{\partial^2 \phi(r, t)}{\partial t^2} = -f(r, t) . \]  

(6)

Outside the sources, that is, wherever \( f(r, t) = 0 \), the potential \( \phi \) obeys the homogeneous wave equation, and \( \phi \) travels at speed \( c \). But the potential that propagates in vacuum emanates from the sources, which leads us to believe that the propagation of the influence from the cause (source) to produce the effect (potential) takes place at speed \( c \). Thus, the state of the potential at the present time must depend on the state of the sources at the past instant when electromagnetic “information” left them. This expectation derives from the fact that, outside the sources, the potential satisfies the homogeneous wave equation, whose solutions are known to travel at speed \( c \). Therefore, coherence demands that in order to apply the same physical intuition to the fields one must search for equations of the form (6) for \( E \) and \( B \).

From Maxwell’s equations, with the help of the identity \( \nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A \), one easily gets

\[ \nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{\nabla \rho}{\epsilon_0} + \mu_0 \mathbf{J} \]  

(7)

and

\[ \nabla^2 B - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = -\mu_0 \nabla \times \mathbf{J} \]  

(8)

Now the same heuristic argument invoked to justify the retarded potentials suggests that

\[ E(r, t) = -\frac{1}{4\pi \epsilon_0} \int \frac{(\nabla \rho)(r', t_r)}{R} d\tau' - \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(r', t_r)}{R} d\tau' , \]  

(9)

and

\[ B(r, t) = \frac{\mu_0}{4\pi} \int \frac{(\nabla \times \mathbf{J})(r', t_r)}{R} d\tau' . \]  

(10)

One must be careful with the notation: \( (\nabla \rho)(r', t_r) \) denotes the gradient of \( \rho(r, t) \), calculated keeping \( t \) fixed, evaluated at \( r = r' \) and \( t = t_r \); the same goes for \( (\nabla \times \mathbf{J})(r', t_r) \).

The above expressions for \( E \) and \( B \) are not new. It is not obvious that these fields satisfy all of Maxwell’s equations, since they are solutions to second order equations, whereas Maxwell’s equations are of the first order. In Lorrain and Corson the proof that these fields coincide with those obtained from the retarded potentials is left to the reader, who is asked to neglect retardation. Here we show directly that equations (9) and (10) are completely equivalent to Jefimenko’s equations (4) and (5).

Consider the gradient of \( \rho(r', t_r) \) with respect to \( r' \) but now taking into account both the explicit and the implicit dependences:

\[ \nabla' \rho(r', t_r) = (\nabla \rho)(r', t_r) + \frac{\partial \rho(r', t_r)}{\partial t_r} \nabla' t_r = (\nabla \rho)(r', t_r) + \frac{\dot{r}(r', t_r)}{c} \mathbf{R} , \]  

(11)

where we used \( \nabla' R = -\mathbf{R} \) and \( \partial t_r / \partial t = 1 \). Making use of (11) and recalling that \( \mu_0 \epsilon_0 = 1/c^2 \), equation (9) can be recast as
\[ E(r, t) = \frac{1}{4\pi\varepsilon_0} \int \left[ -\nabla'\rho(r', t_r) + \rho(r', t_r)\dot{R} \right. \left. + \frac{\dot{J}(r', t_r)}{cR} \right] d\tau', \quad (12) \]

But

\[
\int \frac{\nabla'\rho(r', t_r)}{R} d\tau' = \int \left[ \nabla'\frac{\rho(r', t_r)}{R} - \rho(r', t_r)\nabla'\left(\frac{1}{R}\right) \right] d\tau'
\]

\[
= \int_{S_\infty} \rho(r', t_r) d\mathbf{a}' - \int \rho(r', t_r) \frac{\dot{R}}{R^2} d\tau' = -\int \rho(r', t_r) \frac{\dot{R}}{R^2} d\tau'
\]

(13)

for localized sources (we have taken advantage of the integral theorem \( \int_V \nabla T d\tau = \oint_S T d\mathbf{a} \) and have denoted by \( S_\infty \) the surface of a sphere at infinity). With the above result, equation (12) becomes identical to Jefimenko’s equation (11).

Similarly,

\[
\nabla' \times \mathbf{J}(r', t_r) = (\nabla \times \mathbf{J})(r', t_r) + \nabla' t_r \times \frac{\partial \mathbf{J}(r', t_r)}{\partial t_r} = (\nabla \times \mathbf{J})(r', t_r) - \frac{1}{c} \mathbf{J}(r', t_r) \times \dot{\mathbf{R}},
\]

(14)

and (10) takes the form

\[
\mathbf{B}(r, t) = \frac{\mu_0}{4\pi} \int \left[ \frac{\nabla' \times \mathbf{J}(r', t_r)}{R} + \frac{\dot{\mathbf{J}}(r', t_r) \times \dot{\mathbf{R}}}{cR} \right] d\tau'.
\]

(15)

Making an integration by parts with the help of

\[
\nabla' \times \left( \frac{\mathbf{J}(r', t_r)}{R} \right) = \nabla' \times \frac{\mathbf{J}(r', t_r)}{R} - \mathbf{J}(r', t_r) \times \nabla'\left(\frac{1}{R}\right) = \nabla' \times \frac{\mathbf{J}(r', t_r)}{R} - \mathbf{J}(r', t_r) \times \frac{\dot{\mathbf{R}}}{R^2}
\]

(16)

and dropping the surface integral that arises from the use of the integral theorem \( \int_V \nabla \times \mathbf{A} d\tau = -\oint_S \mathbf{A} \times d\mathbf{a} \), one finds that equation (15) reduces to Jefimenko’s equation (11).

Physical intuition has not led us astray, after all. In spite of its fallibility, physical intuition is invaluable in the investigation of physical phenomena. The situation here discussed reveals, however, that the appeal to intuitive arguments requires caution. In particular, it is not possible or, at least, it is dangerous to dissociate the physical intuition from the mathematical description of the phenomena.

ACKNOWLEDGMENT

The author is thankful to Jorge Simões de Sá Martins for a critical reading of the manuscript.
References

[1] D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, NJ, 1999), 3rd ed., Sec. 10.2.

[2] D. J. Griffiths and M. A. Heald, “Time-dependent generalizations of the Biot-Savart and Coulomb laws,” Am. J. Phys. 59, 111 (1991).

[3] O. D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966), Sec. 15-7; same section in 2nd ed. (Electret Scientific, Star City, WV, 1989).

[4] K. T. McDonald, “The relation between expressions for time-dependent electromagnetic fields given by Jefimenko and by Panofsky and Phillips,” Am. J. Phys. 65, 1074 (1997).

[5] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1962), 2nd ed., Sec. 14-3.

[6] P. Lorrain and D. Corson, *Electromagnetic Fields and Waves* (Freeman, San Francisco, 1970), 2nd ed., Sec. 10.10.