Numerical analysis of the generalized Starobinsky inflationary model

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In this work we study numerically one kind of generalization of the Starobinsky inflationary model (power-law type), which is characterized by the parameter $p$. In order to find the parameter $p$ that fixes with observations, we compute the cosmological parameters $A_S$, $n_S$, and $r$ for several values of $p \simeq 1$. We have found that the value of $p = 1.0004$ reproduces the value of $A_S$, $n_S$, and $r$ in agreement with current observational data.

Keywords: Cosmological Perturbations; Starobinsky inflationary model.

1. Introduction

Inflation is a stage in the evolution of the Universe where the expansion was accelerated with the original motivation of solving the problems of the big-bang theory, like the flatness problem, the horizon problem, and the problem of magnetic monopoles. Soon was discovered that inflation has another important property, this theory predicts curvature perturbations with an almost scale invariant power spectrum, which causes the Cosmic Microwave Background (CMB) anisotropies and the large scale structure of the universe. Also, it predicts primordial gravitational waves.

A lot of models of inflation have been proposed which are possible to classify by the number of free parameters. Each inflationary model is defined by a potential

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$V(\phi)$ that depends on a scalar field $\phi$. Those inflationary models have been tested with the observational data, and most of them have been ruled out by current observations.\cite{foot0}

The Starobinsky model\cite{staro} is the inflationary model that is currently supported by observations.\cite{foot0} This model is a specific case of $n = 2$ for inflation in $f(R)$ theories.\cite{foot0, foot1}

Recently, there has been interest in a particular class of power-law models that generalized the Starobinsky model\cite{foot2, foot3, foot4, foot5, foot6} we call this the generalized Starobinsky model. This model depends on a free parameter $p$, a real number close to unity, which can be tuned. For recovering the original form of the Starobinsky potential, we need to use $p = 1$. Motohashi\cite{foot2} has found that the parameter $p$ is constrained to be $1.92 \lesssim 2p \leq 2$. Renzi et al.\cite{foot3} have constrained the cosmological parameters using the Planck data and have found that the parameter $p$ must be in the range $0.962 \leq p \leq 1.016$. In these references the analysis has been done using the slow-roll approximation.

The most used technique in cosmological inflation is the slow-roll approximation, which consists that the potential $V(\phi)$ term will dominate over the kinetic term. In this work, we use the slow-roll equations to set up the initial and final value of the scalar field, considering a number of e-folding of $N = 60$. After that, we proceed to calculate numerically the exact expression of the scalar field $\phi$ and the scale factor $a$. Once obtained both expressions, they are used as initial conditions at $t = 0$ in the numerical integration of the perturbations equations. Values of $p$ were taken between $0.995 \leq p \leq 1.005$, and the cosmological parameters $A_S$, $n_S$ and $r$ were computed in all cases in order to compare with the current observational data.

The article is structured as follows. In Section 2, we present the basic equations used in our calculations. In Section 3, we show the generalized Starobinsky model. In Section 4, we describe the method used to calculate the observables. Section 5, shows the results that we have obtained. Finally, in Section 6 we present the conclusions.

2. Basic equations

The equation of motion for an Universe dominated by a scalar field $\phi$ are given by:

\begin{align}
H^2 &= \frac{1}{3} \left[ V(\phi) + \frac{1}{2} \dot{\phi}^2 \right], \\
\ddot{\phi} + 3H \dot{\phi} &= -V_{,\phi},
\end{align}

where dots means derivative respect to the physical time $t$ and $V_{,\phi}$ derivative of the potential respect to the scalar field $\phi$.

Into the slow-roll approximation, Eqs. (1) and (2) reduce to
\[ H^2 \simeq \frac{1}{3} V(\phi), \quad (3) \]
\[ 3H \dot{\phi} \simeq -V_{,\phi}. \quad (4) \]

The slow-roll parameter \( \epsilon \) is given by
\[ \epsilon = \frac{1}{2} \left( \frac{V_{,\phi}}{V} \right)^2, \quad (5) \]
and the number of e-foldings between the horizon crossing of modes of interest and the end of inflation is given by
\[ N \simeq \int_{\phi_i}^{\phi} \frac{V}{V_{,\phi}} d\phi. \quad (6) \]

The scale factor \( a \) and the scalar field \( \phi \) exhibit a simpler form in the physical time \( t \) than in the conformal time \( \eta \), then we write the equations for the scalar and tensor perturbations in variable \( t \). The relation between \( t \) and \( \eta \) is given via the equation \( dt = a d\eta \). Using this change of variable, equations for scalar and tensor perturbations can be written as
\[ \ddot{u}_k + \frac{\dot{a}}{a} \dot{u}_k + \frac{1}{a^2} \left[ k^2 - \frac{\dot{a} z_S + a \ddot{z}_S}{z_S} a \right] u_k = 0, \quad (7) \]
\[ \ddot{v}_k + \frac{\dot{a}}{a} \dot{v}_k + \frac{1}{a^2} \left[ k^2 - (a^2 + a\ddot{a}) \right] v_k = 0, \quad (8) \]
where \( z_S = a\dot{\phi}/H \). Considering the limits \( k^2 \gg |z''_S/z_S| \) (subhorizon scales) and \( k^2 \ll |z''_S/z_S| \) (superhorizon scales), we have the solutions to Eq. (7) exhibit the following asymptotic behavior:
\[ u_k \rightarrow \frac{e^{-ik\eta}}{\sqrt{2k}} \left( k^2 \gg |z''_S/z_S|, -k\eta \rightarrow \infty \right), \quad (9) \]
\[ u_k \rightarrow A_k z \quad \left( k^2 \ll |z''_S/z_S|, -k\eta \rightarrow 0 \right). \quad (10) \]
Equation (10) is used as the initial condition for the perturbations. The same asymptotic conditions hold for tensor perturbations.

Once the solutions for \( u_k \) and \( v_k \) are known, the power spectra for scalar and tensor perturbations are given by the expressions
\[ P_S(k) = \lim_{k t \rightarrow \infty} \frac{k^3}{2\pi^2} \left| \frac{u_k(t)}{z_S(t)} \right|^2, \quad (11) \]
\[ P_T(k) = \lim_{k t \rightarrow \infty} \frac{k^3}{2\pi^2} \left| \frac{v_k(t)}{a(t)} \right|^2, \quad (12) \]
these expressions are calculated at the superhorizon $-k\eta \to 0$.

The spectral index for scalar perturbations is defined by:

$$n_S(k) = 1 + \frac{d \ln P_S(k)}{d \ln k}.$$  \hspace{1cm} (13)

In addition, the tensor-to-scalar ratio $r$ is defined as:

$$r = 8 \frac{P_T(k)}{P_S(k)}.$$  \hspace{1cm} (14)

3. The model

The generalized Starobinsky potential is given by:

$$V(\phi) = V_0 e^{-2\sqrt{2} \sqrt{\phi} \left(e^{-\sqrt{2} \sqrt{\phi}} - 1\right)^{2p-1}},$$  \hspace{1cm} (15)

where $p$ is a real number close to unity, and

$$V_0 = 6 \left(\frac{2p-1}{4p}\right) M^2 \left(\frac{1}{2p}\right)^\frac{1}{2p-1}.$$  \hspace{1cm} (16)

At $p = 1$, equation (15) reduces to the Starobinsky potential. The value of $M$ is fixed in $M = 1.30 \times 10^{-5}$ in order to obtain the parametrization of the amplitude for the scalar power spectrum at the pivot scale $k = 0.05 \text{ Mpc}^{-1}$ in the Starobinsky inflationary model ($p = 1$).

In Fig. 1 we show the form of the generalized Starobinsky potential for three different values of $p$.

4. Methods

We want to obtain the solution of Eqs. (1) and (2) to find the scalar field $\phi$ and the scale factor $a$. As initial conditions, we use the solution to the slow-roll equations (3) and (4).

From Eq. (3) we obtain the scale factor into the slow-roll approximation $a_{sr}$

$$\frac{da}{a} = \frac{1}{\sqrt{3}} \sqrt{V(t)} \, dt \to a_{sr} = e^{h(t)},$$  \hspace{1cm} (17)

where

$$h(t) = \frac{1}{\sqrt{3}} \int_0^t \sqrt{V(t)} \, dt.$$  \hspace{1cm} (18)

From Eq. (4) we obtain the scalar field into the slow-roll approximation $\phi_{sr}$,
Fig. 1. Generalized Starobinsky potential for $p = 0.995$ (dotted-line), $p = 1$ (Starobinsky potential, solid line), and $p = 1.005$ (dashed line)

$$\frac{dt}{\sqrt{\frac{V(\phi)}{V,\phi}}} \rightarrow t = g(\phi) \rightarrow g(\phi) - t = 0,$$

where

$$g(\phi) = -\sqrt{\frac{3}{V,\phi}} \int_{\phi_i}^{\phi} \frac{V(\phi)}{V,\phi} d\phi$$

Doing the integration of Eq. (20) we obtain

$$g(\phi) = -\frac{3}{M} \frac{(-1 + 2p)^{3/2}}{(-2 + 3p)} \frac{\sqrt{3}}{2 + \frac{3}{1 + 2p}} 2^{\frac{3-3p}{1 + 2p}}$$

$$\times \left\{ \left( -1 + e^{\sqrt{\frac{3p}{3}} \phi} \right)^{\frac{2-3p}{1 + 2p}} \right\}$$

$$\times 2F_1 \left[ \frac{1}{1 - 2p}, \frac{1}{1 - 2p}; \frac{1}{1 + 2p} \left( -1 + e^{\sqrt{\frac{3p}{3}} \phi} \right) \right]$$

$$- \left( -1 + e^{\sqrt{\frac{3p}{3}} \phi_i} \right)^{\frac{2-3p}{1 + 2p}}$$

$$\times 2F_1 \left[ \frac{1}{1 - 2p}, \frac{1}{1 - 2p}; \frac{1}{1 + 2p} \left( -1 + e^{\sqrt{\frac{3p}{3}} \phi_i} \right) \right] \right\}.$$

Using Mathematica® we solve Eq. (19) and obtain $\phi_{sr}$ numerically, we call this
\[ \phi_{sr}(p, \phi_i, t). \]

From Eq. (5) we calculate the slow-roll parameter:

\[
\epsilon = \frac{4}{3} \left[ \frac{1 - 2p + (p - 1)e^\sqrt{2/3} \phi}{(1 - 2p)^2 \left( e^\sqrt{2/3} \phi - 1 \right)} \right], \tag{22}
\]

when \( \epsilon = 1 \) we obtain the value of the scalar field at the end of inflation, \( \phi_f \). Note that we recover the Starobinsky model when \( p = 1 \).

From Eq. (6), we obtain the number of e-folding \( N \):

\[
N \simeq \frac{\sqrt{6}}{4} (\phi - \phi_i) - \frac{3}{4} \frac{p}{(p - 1)} \ln \left[ \frac{1 - 2p + (p - 1)e^\sqrt{2/3} \phi_i}{1 - 2p + (p - 1)e^\sqrt{2/3} \phi_i} \right]. \tag{23}
\]

In the limit when \( p = 1 \), Eq. (23) reduces to

\[
N \simeq \frac{\sqrt{6}}{4} (\phi - \phi_i) - \frac{3}{4} \left( e^\sqrt{2/3} \phi - e^\sqrt{2/3} \phi_i \right). \tag{24}
\]

From Eq. (23) and (24) we can obtain numerically the dependence of \( \phi \) with \( N \), \( \phi_{sr}(N) \). Fixing \( N = 60 \), we obtain the initial value of the scalar field \( \phi_i \) for each value of \( \phi_f \). The values obtained are shown in Table 1.

The values of \( \phi_{sr}(p, \phi_i, 0) \) and \( \dot{\phi}_{sr}(p, \phi_i, 0) \) are used in the numerical code as initial conditions to solve the system of equations (1) and (2) in order to find the exact function of the scale factor \( a_{ex}(t) \) and the exact function of the scalar field \( \phi_{ex}(t) \) in function of the cosmic time \( t \). Figs. 2 and 3 show the scale factor and the scalar field in function \( t \). When the scalar field begins to oscillate the inflation ends.

Note that in this model the expansion is quasi exponential, unlike the de Sitter Universe.

The functions \( a_{ex}(t) \) and \( \phi_{ex}(t) \) are used to do the numerical integration of equations (7) and (8), and then using Eqs. (11) and (12) we obtain \( P_S(k) \) and \( P_T(k) \). By using the value of the scalar power spectrum, we calculate from Eq. (13) the scalar spectral index at the pivot scale of \( k = 0.05 \) Mpc\(^{-1} \). With the help of the tensor power spectrum using Eq. (14), we calculate the scalar-to-tensor ratio at the pivot scale of \( k = 0.002 \) Mpc\(^{-1} \). In the following we call \( P_S \) as \( A_S \).
Results and discussion

We do slight variations of the parameter $p$ around $p = 1$, from $p = 0.995$ to $p = 1.005$. In Table 2 we show the results for $p = 1$, note that the values of the cosmological parameters are in good agreement with those reported in the literature when $p = 1^{14}$

We have found that the value of $p$ that reproduces the value supported by the current observational data $^{13}$ is $p = 1.0004$. The results are shown in Table 3 note that the values of $A_S$, $n_S$ and $r$ are in agreement with the current observational

| $p$  | $\phi_i$ | $\phi_f$ |
|------|----------|----------|
| 0.995 | 5.7246   | 0.9473   |
| 0.996 | 5.6684   | 0.9459   |
| 0.997 | 5.6132   | 0.9444   |
| 0.998 | 5.5589   | 0.9430   |
| 0.999 | 5.5056   | 0.9416   |
| 1.000 | 5.4531   | 0.9402   |
| 1.001 | 5.4016   | 0.9388   |
| 1.002 | 5.3510   | 0.9374   |
| 1.003 | 5.3013   | 0.9360   |
| 1.004 | 5.2526   | 0.9346   |
| 1.005 | 5.2046   | 0.9332   |

Fig. 2. Scalar field $\phi$ in function of the cosmic time $t$ for $p = 0.995$ (dotted-line), $p = 1$ (solid line), and $p = 1.005$ (dashed line)
Fig. 3. Scale factor $a$ in function of the cosmic time $t$ for $p = 0.995$ (dotted-line), $p = 1$ (solid line), and $p = 1.005$ (dashed line).

| Parameter | Value           |
|-----------|-----------------|
| $A_S$     | $2.1442 \times 10^{-9}$ |
| $\ln (10^{10} A_S)$ | 3.065          |
| $n_S$     | 0.9638          |
| $r_{0.002}$ | 0.00345       |

In Fig. 4 we can observe the dependence of $\ln (10^{10} A_S)$ for $k = 0.05$ Mpc$^{-1}$ with the parameter $p$, which increases as $p$ increases. Fig. 5 shows the behavior of $n_S$ for $k = 0.05$ Mpc$^{-1}$ in function of the parameter $p$, and we can observe that $n_S$ decreases as $p$ increases. In Fig. 6 we present the scalar-tensor ratio $r$ for $k = 0.002$ Mpc$^{-1}$ in function of the parameter $p$. This value decreases when $p$ increases.

As an additional analysis, we adjusted the energy scale $M$ of the inflationary model for each value of $p$ in order to satisfy the observational constraint on $A_S$. In
Fig. 4. $\ln(10^{10} A_S)$ for $k = 0.05$ Mpc$^{-1}$ in function of the parameter $p$.

Fig. 5. Scalar spectral index $n_S$ for $k = 0.05$ Mpc$^{-1}$ in function of the parameter $p$.

Fig. 7 we can observe that for small values of $p$ it is necessary to increase the value of $M$ for obtaining an accuracy with the observational data. For large values of $p$, it is needed to decrease the value of $M$ for matching with reported data.
6. Conclusions

In this work, we have studied a particular class of power-law models that generalize the Starobinsky model through a parameter $p$. We made a sweep over parameter $p$ between $0.999 < p < 1.0005$ and found that the value of $p = 1.0004$ reproduces the value of the cosmological parameters according to observations. On the other hand,
when we fit the energy scaling $M$ for each value of $p$ we obtain that the value of $M$ decreases as $p$ increases.

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