A SURVEY ON RANK AND INERTIA OPTIMIZATION PROBLEMS OF THE MATRIX-VALUED FUNCTION $A + BXB^*$

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ABSTRACT. This paper is concerned with some rank and inertia optimization problems of the Hermitian matrix-valued functions $A + BXB^*$ subject to restrictions. We first establish several groups of explicit formula for calculating the maximum and minimum ranks and inertias of matrix sum $A + X$ subject to a Hermitian matrix $X$ that satisfies a fixed-rank and semi-definiteness restrictions by using some discrete and matrix decomposition methods. We then derive formulas for calculating the maximum and minimum ranks and inertias of the matrix-valued function $A + BXB^*$ subject to a Hermitian matrix $X$ that satisfies a fixed-rank and semi-definiteness restrictions, and show various properties $A + BXB^*$ from these ranks and inertias formulas. In particular, we give necessary and sufficient conditions for the equality $A + BXB^* = 0$ and the inequality $A + BXB^* \succ 0$ ($\succeq 0, \prec 0, \preceq 0$) to hold respectively for these specified Hermitian matrices $X$.

1. Introduction. Throughout this paper,

$\mathbb{C}^{m \times n}$ and $\mathbb{C}^{m}_H$ stand for the sets of all $m \times n$ complex matrices and $m \times m$ complex Hermitian matrices, respectively;

$A^*$, $A^T$, $r(A)$, and $\mathfrak{R}(A)$ stand for the conjugate transpose, transpose, rank, and range (column space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively;

$I_m$ denotes the identity matrix of order $m$;

$[A, B]$ denotes a row block matrix consisting of $A$ and $B$;

the Moore–Penrose generalized inverse of $A \in \mathbb{C}^{m \times n}$, denoted by $A^\dagger$, is defined to be the unique solution $X$ satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$;

the symbols $E_A$ and $F_A$ stand for $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$, their ranks are $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$;

$i_+(A)$ and $i_-(A)$, called the positive and negative index of inertia of $A \in \mathbb{C}_H$, are defined to be the numbers of the positive and negative eigenvalues of $A$ counted with multiplicities, respectively;

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A > 0, A ≳ 0, A ≲ 0, and A ≦ 0 mean that A is Hermitian positive definite, positive semi-definite, negative definite, and negative semi-definite, respectively;
A, B ∈ ℂ^n_H are said to satisfy the inequalities A > B, A ≳ B, A < B, and A ≲ B in the Löwner partial ordering if A - B is positive definite, positive semi-definite, negative definite, and negative semi-definite, respectively.

Let
\[ \phi(X) = A + BXB^* \]
be a linear matrix-valued function (LMF) over the field of complex numbers, where A ∈ ℂ^n_H and B ∈ ℂ^{m×n} are given, and X ∈ ℂ^n_H is a variable matrix. When X runs over ℂ^n_H, φ(X) varies with respect to the choice of X. Correspondingly, the rank and inertia of φ(X) may vary with respect to X ∈ ℂ^n_H as well. The LMF in (1.1) is a representative of all LMFs with single variable matrix, and is the starting point in dealing with various complicated matrix-valued functions with symmetric patterns. In an earlier book [16], the LMF in (1.1) and its applications in system and control theory were studied by making use of SVD and generalized inverses of matrices. As usual, variable matrices in matrix-valued functions, however, do not necessarily vary over the whole matrix spaces, but are often assumed to satisfy some restrictions on their ranks, ranges, norms, semi-definiteness, orthogonality, etc. If the rank of BXB* in (1.1) is quite small in comparison with the size of A, this BXB* is called a low-rank approximation (perturbation) of the given matrix A; and was hot object of study in the past two decades; see, e.g., [1, 4, 12, 13, 28].

It is well known that the rank and inertia of (real symmetric or complex Hermitian) matrix are basic concepts associated with the matrix, while the characterization and formulation of ranks/inertias of matrices are one of the oldest questions in the field of linear algebra. Recall from the definitions of rank/inertia of matrix that they can only take finite nonnegative integers, and are the functions of the entries in the matrix. Hence, the ranks/inertias of matrix-valued functions can also take finite nonnegative integers no matter what the variable matrices are chosen in the matrix-valued functions. In particular, the ranks/inertias of BXB* and A + BXB* in (1.1) may vary with respect to the choice of the variable matrix X, and thus it is desirable to characterize relations between the rank/inertia of φ(X) and the variable matrix X in it. This paper aims at solving the following two groups of optimization problem on the rank/inertia of φ(X) when X runs the set of all Hermitian matrices and the closed convex cone of positive semi-definite matrices in ℂ^n_H, respectively.

**Problem 1.1.** For the given A + BXB* in (1.1), solve the following constrained rank/inertia optimization problems

- maximize \( r(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H \) and \( r(X) = q \) \hspace{1cm} (1.2)
- minimize \( r(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H \) and \( r(X) = q \) \hspace{1cm} (1.3)
- maximize \( i_+(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H \) and \( r(X) = q \) \hspace{1cm} (1.4)
- minimize \( i_+(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H \) and \( r(X) = q \) \hspace{1cm} (1.5)
- maximize \( r(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H, \pm X \succ 0 \) and \( r(X) = q \) \hspace{1cm} (1.6)
- minimize \( r(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H, \pm X \succ 0 \) and \( r(X) = q \) \hspace{1cm} (1.7)
- maximize \( i_+(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H, \pm X \succ 0 \) and \( r(X) = q \) \hspace{1cm} (1.8)
- minimize \( i_+(A + BXB^*) \) \hspace{1cm} s.t. \( X \in ℂ^n_H, \pm X \succ 0 \) and \( r(X) = q \) \hspace{1cm} (1.9)
Problem 1.2. For the given $A + BXB^*$ in (1.1), solve the following constrained rank/inertia optimization problems

maximize $r(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$ and $p \leq r(X) \leq q$, \hspace{1cm} (1.10)
minimize $r(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$ and $p \leq r(X) \leq q$, \hspace{1cm} (1.11)
maximize $i_{\pm}(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$ and $p \leq r(X) \leq q$, \hspace{1cm} (1.12)
minimize $i_{\pm}(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$ and $p \leq r(X) \leq q$, \hspace{1cm} (1.13)
maximize $r(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$, $\pm X \succ 0$ and $p \leq r(X) \leq q$, \hspace{1cm} (1.14)
minimize $r(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$, $\pm X \succ 0$ and $p \leq r(X) \leq q$, \hspace{1cm} (1.15)
maximize $i_{\pm}(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$, $\pm X \succ 0$ and $p \leq r(X) \leq q$, \hspace{1cm} (1.16)
minimize $i_{\pm}(A + BXB^*)$ s.t. $X \in \mathbb{C}_H^n$, $\pm X \succ 0$ and $p \leq r(X) \leq q$. \hspace{1cm} (1.17)

The LMF $\phi(X)$ in (1.1) and its variations were extensively studied in the literature from theoretical and applied points of view due to its simple and symmetric pattern, and many results on behaviors of $\phi(X)$ were obtained. Some recent work done by the present author and his collaborators on $\phi(X)$ is summarized below:

(i) expansion formulas for calculating the rank/inertia of $\phi(X)$, as well as algebraic methods for finding maximum and minimum rank/inertias of $\phi(X)$ when $X$ running over $\mathbb{C}_H^n$ [8, 18, 27];

(ii) characterizations of algebraic properties of $\phi(X)$, such as, the nonsingularity, positive definiteness, rank invariance, inertia invariance, range invariance, etc., of $\phi(X)$ [18, 27];

(iii) canonical forms of $\phi(X)$ under generalized singular value decompositions and their algebraic properties [8];

(iv) solutions and least-squares solutions of the matrix equation $\phi(X) = 0$ and their algebraic properties [7, 8, 11, 21, 23, 25];

(v) solutions of the matrix inequalities $\phi(X) \succ 0$ $(\succ 0, \prec 0, \preceq 0)$ and their properties [18, 22];

(vi) formulas for calculating the maximum and minimum ranks/inertias of $\phi(X)$ under $X \in \mathbb{C}_H^n$, $r(X) \leq q$ and/or $\pm X \succ 0$ [18, 27];

(vii) formulas for calculating the maximum and minimum ranks/inertias of $\phi(X)$ subject to Hermitian solutions of one or two consistent matrix equations [9, 26];

(viii) formulas for calculating the maximum and minimum ranks/inertias of the Schur complement $A + BC^{-}B^*$, where $C^{-}$ is a Hermitian generalized inverse of a Hermitian matrix $C$, [9, 24].

The maximization and minimization of ranks/inertias of LMFs could be regarded as a special kind of continuous-integer optimization problems. However, this kind of optimization problems cannot be handled by various known methods for solving continuous or discrete optimization problems because matrix multiplications occurred in matrix-valued functions are not necessarily commutative. In fact, there does not exist a rigorous mathematical theory for solving optimization problems on ranks/inertias of LMFs, but we are really able to solve some special cases, as mentioned above, by using matrix decompositions, generalized inverses of matrices, and some tricky matrix operations.

Note that for any two integers $p$ and $q$ satisfying $0 \leq p \leq q \leq m$, the following decompositions of the generalized Stiefel manifolds

$$\{X \in \mathbb{C}_H^n \mid p \leq r(X) \leq q\} = \{X \in \mathbb{C}_H^n \mid r(X) = p\} \cup \{X \in \mathbb{C}_H^n \mid r(X) = p + 1\}$$
and
\[
\cup \cdots \cup \{ X \in \mathbb{C}_n^m \mid r(X) = q \},
\] 
(1.18)
and
\[
\{ 0 \preceq X \in \mathbb{C}_n^m \mid p \leq r(X) \leq q \} = \{ 0 \preceq X \in \mathbb{C}_n^m \mid r(X) = p \} \cup \{ 0 \preceq X \in \mathbb{C}_n^m \mid r(X) = p+1 \}
\cup \cdots \cup \{ 0 \preceq X \in \mathbb{C}_n^m \mid r(X) = q \}
\] 
(1.19)
hold. Once Problem 1.1 is solved, solutions of Problem 1.2 can be obtained consequently from the above matrix set decompositions. The constrained optimization problems formulated in (1.2)–(1.17) consist of determining the global maximum and minimum numbers of ranks/inertias of \(\phi(X)\), and finding the constrained variable matrix \(X\) such that the corresponding \(\phi(X)\) attains the maximum and minimum numbers, respectively. The Hermitian matrix \(X\) with \(r(X) = q\) is not unique, and can be characterized by the following canonical decomposition under the *-congruence transformation:
\[
X = U \text{diag}(I_s, -I_{q-s}, 0) U^*,
\] 
(1.20)
where \(U\) is any nonsingular matrix of order \(n\), and \(0 \leq s \leq q\) with \(r(X) = q\). This canonical decomposition shows that a Hermitian matrix \(X\) with rank \(q\) is characterized by both a variable integer \(s\) with \(0 \leq s \leq q\) and an arbitrary nonsingular matrix \(U\). So that both \(BXB^*\) and \(A + BXB^*\) depend on the choices of both \(s\) and \(U\). In this setting, (1.2)–(1.17) can be classified as some special types of constrained integer optimization problem, and thus we can only derive the maximum and minimum ranks/inertias of \(\phi(X)\) via pure algebraic operations of matrices. The tasks described in (1.2)–(1.17) are challenging, but fortunately, we now are able to derive closed-form solutions by using many known and new formulas on ranks/inertias of matrices and some tricky matrix operations.

2. Preliminary results. The following are some known assertions on ranks/inertias of matrices, which will be used in the latter part of this paper for solving the previous problems.

\textbf{Lemma 2.1} ([18]). Let \(\mathcal{S}\) and \(\mathcal{H}\) be two matrix sets over \(\mathbb{C}^{m \times n}\) and \(\mathbb{C}_n^m\), respectively. Then, the following assertions hold.

\begin{itemize}
  \item[(a)] Under \(m = n\), \(\mathcal{S}\) has a nonsingular matrix if and only if \(\max_{X \in \mathcal{S}} r(X) = m\).
  \item[(b)] Under \(m = n\), all \(X \in \mathcal{S}\) are nonsingular if and only if \(\min_{X \in \mathcal{S}} r(X) = m\).
  \item[(c)] \(0 \in \mathcal{S}\) if and only if \(\min_{X \in \mathcal{S}} r(X) = 0\).
  \item[(d)] \(\mathcal{S} = \{0\}\) if and only if \(\max_{X \in \mathcal{S}} r(X) = 0\).
  \item[(e)] \(\mathcal{H}\) has a matrix \(X \succ 0\) \((X \prec 0\) if and only if \(\max_{X \in \mathcal{H}} i_+(X) = m\)

\[
(\max_{X \in \mathcal{H}} i_-(X) = m).
\]
  \item[(f)] All \(X \in \mathcal{H}\) satisfy \(X \succ 0\) \((X \prec 0\) if and only if \(\min_{X \in \mathcal{H}} i_+(X) = m\)

\[
(\min_{X \in \mathcal{H}} i_-(X) = m).
\]
  \item[(g)] \(\mathcal{H}\) has a matrix \(X \succeq 0\) \((X \preceq 0\) if and only if \(\min_{X \in \mathcal{H}} i_+(X) = 0\)

\[
(\min_{X \in \mathcal{H}} i_+(X) = 0).
\]
  \item[(h)] All \(X \in \mathcal{H}\) satisfies \(X \succeq 0\) \((X \preceq 0\) if and only if \(\max_{X \in \mathcal{H}} i_-(X) = 0\)

\[
(\max_{X \in \mathcal{H}} i_-(X) = 0).
\]
\end{itemize}

This lemma indicates that rank/inertia of matrix can be used as quantitative tools to characterize some fundamental algebraic properties of a given matrix set.

In particular, whether a given matrix-valued function is semi-definite everywhere is ubiquitous in matrix theory and applications. \textbf{Lemma 2.1} (e)–(h) assert that if
Lemma 2.2. Let $A, B \in \mathbb{C}_n^m$. Then,

$$r(A + B) \leq r(A) + r(B),$$  \hspace{1cm} (2.1)

$$i_\pm(A + B) \leq i_\pm(A) + i_\pm(B),$$  \hspace{1cm} (2.2)

$$r(A + B) \geq r(A) - r(B),$$  \hspace{1cm} (2.3)

$$i_\pm(A + B) \geq i_\pm(A) - i_\mp(B).$$  \hspace{1cm} (2.4)

If $B \succ 0$, then

$$r(A + B) \geq i_+(A + B) \geq i_+(A),$$  \hspace{1cm} (2.5)

$$r(A - B) \geq i_-(A - B) \geq i_-(A).$$  \hspace{1cm} (2.6)

Proof. Eq. (2.1) is a well-known rank inequality in elementary linear algebra. Eq. (2.2) was given in [3, 15]. Applying (2.1) and (2.2) to $A = (A + B) + (-B)$ yields (2.3) and (2.4), respectively. Eqs. (2.5) and (2.6) follow from (2.4), respectively. \hfill \Box

Lemma 2.3 ([18]). Let $A \in \mathbb{C}_n^m$, $B \in \mathbb{C}_n^m$, and $Q \in \mathbb{C}^{m \times n}$, and assume that $P \in \mathbb{C}^{m \times m}$ is nonsingular. Then,

$$i_\pm(PAP^*) = i_\pm(A) \quad \text{(Sylvester’s law of inertia)},$$  \hspace{1cm} (2.7)

$$i_\pm(A^\dagger) = i_\pm(A),$$  \hspace{1cm} (2.8)

$$i_\pm(\lambda A) = \begin{cases} i_\pm(A) & \text{if } \lambda > 0 \\ i_\mp(A) & \text{if } \lambda < 0 \end{cases},$$  \hspace{1cm} (2.9)

$$i_\pm \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_\pm(A) + i_\pm(B),$$  \hspace{1cm} (2.10)

$$i_\pm \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = i_- \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q).$$  \hspace{1cm} (2.11)

Lemma 2.4 ([14]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, and $C \in \mathbb{C}^{l \times n}$. Then, the following rank expansion formulas hold

$$r \begin{bmatrix} A & B \end{bmatrix} = r(A) + r(EB) = r(B) + r(EA),$$  \hspace{1cm} (2.12)

$$r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF) = r(C) + r(AF).$$  \hspace{1cm} (2.13)

Hence,

$$r \begin{bmatrix} A & B \end{bmatrix} = r(A) \Leftrightarrow EAB = 0 \Leftrightarrow \mathcal{R}(A) \supseteq \mathcal{R}(B),$$  \hspace{1cm} (2.14)

$$r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow CF = 0 \Leftrightarrow \mathcal{R}(A^*) \supseteq \mathcal{R}(C^*).$$  \hspace{1cm} (2.15)

Lemma 2.5 ([18]). Let $A \in \mathbb{C}_n^m$, $B \in \mathbb{C}^{m \times n}$, and $D \in \mathbb{C}_n^m$, and let

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$
Then, the following inertia/rank expansion formulas hold
\[ i_{\pm}(M_1) = r(B) + i_{\pm}(E_BAE_B), \]
\[ r(M_1) = 2r(B) + r(E_BAE_B), \] (2.16)
\[ i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}\left[ \begin{array}{cc} 0 & E_AB \\ B^*E_A & D - B^*A^\dagger B \end{array} \right], \] (2.17)
\[ r(M_2) = r(A) + r\left[ \begin{array}{cc} 0 & E_AB \\ B^*E_A & D - B^*A^\dagger B \end{array} \right]. \] (2.18)

In particular, the following results hold.
(a) If \( A \succeq 0 \), then
\[ i_{\pm}(M_1) = r[A, B], \quad i_{-}(M_1) = r(B), \quad r(M_1) = r[A, B] + r(B). \] (2.20)
(b) If \( A \preceq 0 \), then
\[ i_{+}(M_1) = r(B), \quad i_{-}(M_1) = r[A, B], \quad r(M_1) = r[A, B] + r(B). \] (2.21)
(c) If \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \), then
\[ i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}(D - B^*A^\dagger B), \quad r(M_2) = r(A) + r(D - B^*A^\dagger B). \] (2.22)

The following three lemmas will play essential roles in solving Problems 1.1 and 1.2.

**Lemma 2.6.** Let \( X \in \mathbb{C}^{m \times n} \) and \( Y \in \mathbb{C}^n \) be two variable matrices, and let
\[ \phi(X, Y) = \begin{bmatrix} 0 & X \\ X^\dagger & Y \end{bmatrix}. \] (2.23)

Then, the following results hold.
(a) The rank/inertia of \( \phi(X, Y) \) satisfy
\[ \max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^n} r[\phi(X, Y)] = \min\{m + n, 2n\}, \] (2.24)
\[ \min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^n} r[\phi(X, Y)] = 0, \] (2.25)
\[ \max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^n} i_{\pm}[\phi(X, Y)] = n, \] (2.26)
\[ \min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^n} i_{\pm}[\phi(X, Y)] = 0. \] (2.27)
(b) For any integer \( q \) with \( 0 \leq q \leq \min\{ m + n, \ 2n \} \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \in \mathbb{C}^{n} \) such that

\[
r[\phi(X, Y)] = q. \tag{2.28}
\]

(c) For any integer \( q \) with \( 0 \leq q \leq n \), there exist \( X_1, X_2 \in \mathbb{C}^{m \times n} \) and \( Y_1, Y_2 \in \mathbb{C}^{n} \) such that

\[
i_+[\phi(X_1, Y_1)] = q, \quad i_-[\phi(X_2, Y_2)] = q \tag{2.29}
\]

hold, respectively.

(d) The rank/inertia of \( \phi(X, Y) \) satisfy

\[
\begin{align*}
\max_{X \in \mathbb{C}^{m \times n}, Y > 0} r[\phi(X, Y)] &= \min\{ m + n, \ 2n \}, \tag{2.30} \\
\min_{X \in \mathbb{C}^{m \times n}, Y > 0} r[\phi(X, Y)] &= 0, \tag{2.31} \\
\max_{X \in \mathbb{C}^{m \times n}, Y > 0} i_+[\phi(X, Y)] &= n, \tag{2.32} \\
\min_{X \in \mathbb{C}^{m \times n}, Y > 0} i_+[\phi(X, Y)] &= 0, \tag{2.33} \\
\max_{X \in \mathbb{C}^{m \times n}, Y > 0} i_-[\phi(X, Y)] &= \min\{ m, \ n \}, \tag{2.34} \\
\min_{X \in \mathbb{C}^{m \times n}, Y > 0} i_-[\phi(X, Y)] &= 0. \tag{2.35}
\end{align*}
\]

(e) For any integer \( q \) with \( 0 \leq q \leq \min\{ m + n, \ 2n \} \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( 0 \not\asymp Y \in \mathbb{C}^{n} \) such that

\[
r[\phi(X, Y)] = q. \tag{2.36}
\]

(f) For any integer \( q \) with \( 0 \leq q \leq n \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( 0 \not\asymp Y \in \mathbb{C}^{n} \) such that

\[
i_+[\phi(X, Y)] = q. \tag{2.37}
\]

(g) For any integer \( q \) with \( 0 \leq q \leq \min\{ m, \ n \} \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( 0 \not\asymp Y \in \mathbb{C}^{n} \) such that

\[
i_-[\phi(X, Y)] = q. \tag{2.38}
\]

Proof. Let \( r(X) = \min\{ m, \ n \} \) and \( Y = I_n \) in (2.23). Then, we find from (2.20) that

\[
r[\phi(X, Y)] = r[X, Y] + r(X) = n + r(X) = \min\{ m + n, \ 2n \},
\]

establishing (2.24). Setting \( X = 0 \) and \( Y = 0 \) in (2.23) leads to (2.25). Setting \( Y = I_n \) in (2.23), we find from (2.20) that

\[
i_+[\phi(X, Y)] = r[X, Y] = n;
\]

setting \( Y = -I_n \) in (2.23), we find from (2.20) that

\[
i_-[\phi(X, Y)] = r[X, Y] = n,
\]

establishing (2.26). Set \( X = 0 \) and \( Y = 0 \) in (2.23) leads to

\[
r[\phi(X, Y)] = i_+[\phi(X, Y)] = i_-[\phi(X, Y)] = 0,
\]

establishing (2.27).

For any integer \( 0 \leq q \leq n \), setting \( X = 0 \) and \( r(Y) = q \) in (2.23) leads to

\[
r[\phi(X, Y)] = r(Y) = q;
\]
Lemma 2.7. Let $A \in \mathbb{H}^m$ be given, $X \in \mathbb{C}^{m \times n}$ and $Y \in \mathbb{C}^n_H$ be two variable matrices, and let
\[
\phi(X, Y) = \begin{bmatrix} A & X \\ X^* & Y \end{bmatrix}.
\] (2.39)

Then, the following results hold.

(a) The rank/inertia of $\phi(X, Y)$ satisfy
\[
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^n_H} r[\phi(X, Y)] = \min\{m + n, \ r(A) + 2n\},
\] (2.40)
\[
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}^n_H} r[\phi(X, Y)] = r(A),
\] (2.41)
For any integer $d$, $i_\pm d = i_\pm (A) + n$.

\begin{align}
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_\pm [\phi(X, Y)] &= i_\pm (A) + n, \\
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_\pm [\phi(X, Y)] &= i_\pm (A).
\end{align}

(b) For any integer $q$ with $r(A) \leq q \leq \min\{m + n, r(A) + 2n\}$, there exist $X \in \mathbb{C}^{m \times n}$ and $Y \in \mathbb{C}_H^n$ such that $r[\phi(X, Y)] = q$.

c) For any integer $q$ with $i_+(A) \leq q \leq i_+(A) + n$, there exist $X \in \mathbb{C}^{m \times n}$ and $Y \in \mathbb{C}_H^n$ such that $i_+[\phi(X, Y)] = q$.

d) For any integer $q$ with $i_-(A) \leq q \leq i_-(A) + n$, there exist $X \in \mathbb{C}^{m \times n}$ and $Y \in \mathbb{C}_H^n$ such that $i_-[\phi(X, Y)] = q$.

e) The rank/inertia of $\phi(X, Y)$ satisfy
\begin{align}
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} r[\phi(X, Y)] &= \min\{m + n, r(A) + 2n\}, \\
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} r[\phi(X, Y)] &= r(A), \\
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_+[\phi(X, Y)] &= i_+(A) + n, \\
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_+[\phi(X, Y)] &= i_+(A), \\
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_-[\phi(X, Y)] &= \min\{m, i_-(A) + n\}, \\
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_-[\phi(X, Y)] &= i_-(A).
\end{align}

(f) For any integer $q$ with $r(A) \leq q \leq \min\{m + n, r(A) + 2n\}$, there exist $X \in \mathbb{C}^{m \times n}$ and $0 \neq Y \in \mathbb{C}_H^n$ such that $r[\phi(X, Y)] = q$.

g) For any integer $q$ with $i_+(A) \leq q \leq i_+(A) + n$, there exist $X \in \mathbb{C}^{m \times n}$ and $0 \neq Y \in \mathbb{C}_H^n$ such that $i_+[\phi(X, Y)] = q$.

(h) For any integer $q$ with $i_-(A) \leq q \leq \min\{m, i_-(A) + n\}$ there exist $X \in \mathbb{C}^{m \times n}$ and $0 \neq Y \in \mathbb{C}_H^n$ such that $i_-[\phi(X, Y)] = q$.

(i) Under $A \succ 0$, the following equalities hold
\begin{align}
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} r[\phi(X, Y)] &= \min\{m + n, r(A) + 2n\}, \\
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} r[\phi(X, Y)] &= r(A), \\
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_+[\phi(X, Y)] &= r(A) + n, \\
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_+[\phi(X, Y)] &= r(A), \\
\max_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_-[\phi(X, Y)] &= n, \\
\min_{X \in \mathbb{C}^{m \times n}, Y \in \mathbb{C}_H^n} i_-[\phi(X, Y)] &= 0,
\end{align}
Applying (2.18) and (2.19) to (2.65) leads to the following expansion formulas

\[ \min_{X \in \mathbb{C}^{m \times n}, Y \geq 0} i_- [\phi(X, Y)] = 0. \]  

(j) Under \( A \geq 0 \), for any integer \( q \) with \( r(A) \leq q \leq \min\{m + n, r(A) + n\} \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \in \mathbb{C}^n_+ \) such that \( r[\phi(X, Y)] = q \).

(k) Under \( A \geq 0 \), for any integer \( q \) with \( r(A) \leq q \leq r(A) + n \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \in \mathbb{C}^n_+ \) such that \( i_+ [\phi(X, Y)] = q \).

(l) Under \( A \geq 0 \), for any integer \( q \) with \( 0 \leq q \leq n \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \in \mathbb{C}^n_+ \) such that \( i_- [\phi(X, Y)] = q \).

(m) Under \( A \geq 0 \), for any integer \( q \) with \( r(A) \leq q \leq \min\{m + n, r(A) + n\} \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \geq 0 \) such that \( r[\phi(X, Y)] = q \).

(n) Under \( A \geq 0 \), for any integer \( q \) with \( r(A) \leq q \leq r(A) + n \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \geq 0 \) such that \( i_+ [\phi(X, Y)] = q \).

(o) Under \( A \geq 0 \), for any integer \( q \) with \( 0 \leq q \leq \min\{m, n\} \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \geq 0 \) such that \( i_- [\phi(X, Y)] = q \).

(p) Under \( A \geq 0 \), the following equalities hold

\[
\max_{\phi(X, Y) \geq 0} \rho[\phi(X, Y)] = r(A) + n, \quad (2.62)
\]

\[
\min_{\phi(X, Y) \geq 0} \rho[\phi(X, Y)] = r(A). \quad (2.63)
\]

(q) Under \( A \geq 0 \), for any integer \( q \) with \( r(A) \leq q \leq r(A) + n \), there exist \( X \in \mathbb{C}^{m \times n} \) and \( Y \geq 0 \) such that \( \phi(X, Y) \geq 0 \) and \( r[\phi(X, Y)] = q \).

Proof. Without lost generality, we assume that \( A \) is given by

\[ A = \text{diag} (I_s, -I_t, 0). \] (2.64)

Correspondingly, \( \phi(X, Y) \) in (2.39) can be written as

\[
\phi(X, Y) = \begin{bmatrix} I_s & 0 & 0 & X_1 \\ 0 & -I_t & 0 & X_2 \\ 0 & 0 & 0 & X_3 \\ X_1^* & X_2^* & X_3^* & Y \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ Y \end{bmatrix}. \] (2.65)

Applying (2.18) and (2.19) to (2.65) leads to the following expansion formulas

\[
r[\phi(X, Y)] = r \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \\ 0 & 0 & 0 & X_3 \\ 0 & 0 & X_3^* & Y - X_1^* X_1 + X_2^* X_2 \end{bmatrix} = s + t + r \begin{bmatrix} 0 \\ X_3^* \\ Y - X_1^* X_1 + X_2^* X_2 \end{bmatrix}, \] (2.66)

\[
i_+ [\phi(X, Y)] = \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \\ 0 & 0 & 0 & X_3 \\ 0 & 0 & X_3^* & Y - X_1^* X_1 + X_2^* X_2 \end{bmatrix} = s + i_+ \begin{bmatrix} X_3 \\ X_3^* \\ Y - X_1^* X_1 + X_2^* X_2 \end{bmatrix}, \] (2.67)

\[
i_- [\phi(X, Y)] = \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \\ 0 & 0 & 0 & X_3 \\ 0 & 0 & X_3^* & Y - X_1^* X_1 + X_2^* X_2 \end{bmatrix}.
\]
Applying Lemma 2.6(a) to (2.66)–(2.68) leads to

\[ t + i = \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix}. \]  (2.68)

Substituting (2.73)–(2.77) into (2.66)–(2.68) yields (2.44)–(2.49), establishing (2.62) and (2.63). Result (q) follows from (2.79) and (2.80).

Applying Lemma 2.6(d) to (2.66)–(2.68) leads to

\[
\begin{align*}
\max_{x \in C^{m \times n}, y \in R^n} r & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = \min \{ m - r(A) + n, \ 2n \}, \quad (2.69) \\
\min_{x \in C^{m \times n}, y \in R^n} r & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = 0, \quad (2.70) \\
\max_{x \in C^{m \times n}, y \in R^n} i & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = n, \quad (2.71) \\
\min_{x \in C^{m \times n}, y \in R^n} i & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = 0. \quad (2.72)
\end{align*}
\]

Substituting (2.69)–(2.72) into (2.66)–(2.68) yields (2.40)–(2.43).

Applying Lemma 2.6(b) and (c) to (2.40)–(2.43) leads to (b)–(d).

Applying Lemma 2.6(d) to (2.66)–(2.68) leads to

\[
\begin{align*}
\max_{x \in C^{m \times n}, y \succ 0} r & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = \max r \begin{bmatrix} 0 & X_3 \\ X_3^* & I_n \end{bmatrix} = \min \{ m - r(A) + n, \ 2n \}, \quad (2.73) \\
\min_{x \in C^{m \times n}, y \succ 0} r & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = 0, \quad (2.74) \\
\max_{x \in C^{m \times n}, y \succ 0} i & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = n, \quad (2.75) \\
\max_{x \in C^{m \times n}, y \succ 0} i & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = \max r \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = \min \{ m - t, \ n \}, \quad (2.76) \\
\min_{x \in C^{m \times n}, y \succ 0} i & \begin{bmatrix} 0 & X_3 \\ X_3^* & Y - X_1^*X_1 + X_2^*X_2 \end{bmatrix} = 0. \quad (2.77)
\end{align*}
\]

Substituting (2.73)–(2.77) into (2.66)–(2.68) yields (2.44)–(2.49).

Applying Lemma 2.6(e)–(g) to (2.44)–(2.49) leads to (f)–(h).

Setting \( A \succ 0 \) in (2.40)–(2.49) leads to (2.50)–(2.61).

Results (i)–(o) follow from (2.50)–(2.61).

It is easy to verify that under \( A \succ 0 \) with \( r(A) = s \),

\[
\phi(X, Y) = \begin{bmatrix} I_s & 0 & X_1 \\ 0 & 0 & X_3 \\ X_1^* & X_3^* & Y \end{bmatrix} \succ 0 \iff X_3 = 0 \text{ and } Y \succ X_1^*X_1. \quad (2.78)
\]

So that

\[
\begin{align*}
\max_{\phi(X, Y) \succ 0} r[\phi(X, Y)] & = \max_{\phi(X, Y) \succ 0} r \begin{bmatrix} I_s & X_1 \\ X_1^* & Y \end{bmatrix} = s + \max_{Y \succ X_1^*X_1} r(Y - X_1^*X_1) \\
& = r(A) + n, \quad (2.79) \\
\min_{\phi(X, Y) \succ 0} r[\phi(X, Y)] & = \min_{\phi(X, Y) \succ 0} r \begin{bmatrix} I_s & X_1 \\ X_1^* & Y \end{bmatrix} = s + \min_{Y \succ X_1^*X_1} r(Y - X_1^*X_1) \\
& = r(A), \quad (2.80)
\end{align*}
\]

establishing (2.62) and (2.63). Result (q) follows from (2.79) and (2.80). \( \Box \)
Lemma 2.8 ([10]). Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times n}$ be given. Then, there exist a unitary matrix $U \in \mathbb{C}^{n \times n}$ and a nonsingular matrix $P \in \mathbb{C}^{m \times m}$ such that

$$A = P^* D_A P \quad \text{and} \quad B = P^* D_B U,$$

where

$$D_A = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & f \\ I & 0 & 0 & 0 & 0 & 0 & f \\ 0 & 0 & \Delta_1 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & \Delta_2 & 0 & 0 & l-t \\ 0 & 0 & 0 & 0 & 0 & 0 & m-(2f+l+k) \\ f & f & t & k & l-t & m-(2f+l+k) & \end{bmatrix},$$

$$D_B = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & 0 & 0 & f \\ 0 & \Sigma & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & k & l-t \\ 0 & 0 & I & 0 & 0 & 0 & m-(2f+l+k) \\ f & t & l-t & n-(f+l) & \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} I_{s_1} & 0 \\ 0 & -I_{t-s_1} \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} I_{s_2} & 0 \\ 0 & -I_{k-s_2} \end{bmatrix},$$

$$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_{s_1}, \lambda_{s_1+1}, \ldots, \lambda_t), \quad \sigma_1 \geq \ldots \geq \sigma_{s_1} > 0, \quad \lambda_{s_1+1} \geq \ldots \geq \lambda_t > 0,$$

and

$$f = r(B) + r[A, B] - r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},$$

$$k = r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B),$$

$$i = r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r[A, B],$$

$$t = r(A) + r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r[A, B],$$

$$f + s_1 + s_2 = i_(A),$$

$$f + t + k - s_1 - s_2 = i_-(A),$$

$$f + l + s_2 = i_+ \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},$$

$$f + l + k - s_2 = i_- \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}.$$

The structures of the blocks in (2.81)–(2.85) and the scalars in (2.86)–(2.93) are formulated more explicitly, which improve the results in [29] and [6]. In fact, the blocks in (2.81)–(2.85) can be derived from those of Lemma 2.1 in [6] through a series of elementary transformations, and (2.90)–(2.93) follow from applying Lemma 2.3 to (2.82)–(2.85).
3. Formulas for the rank/inertia of $A + X$ subject to rank and semi-definite restrictions. One of the special cases in (1.1) is the ordinary sum $A + X$. Many results on equalities and inequalities of rank/inertia of sum of two Hermitian matrices were established in the literature; see, e.g., [2, 5, 18]. Note that the rank/inertia of $A + X$ may vary with respect to the choice of the variable Hermitian matrix $X$. In this section, we derive explicit formulas for calculating the extremum ranks/inertias $A + X$ subject to $X \in \mathbb{C}_H^m$ and $X \preceq 0$, respectively. The formulas obtained will be used in Sections 4 and 5 for deriving general rank/inertia formulas of $A + BXB^*$.

**Theorem 3.1.** Let $A \in \mathbb{C}_H^m$ be given, $X \in \mathbb{C}_H^m$ be a variable matrix, and assume that $p$ and $q$ are two integers satisfying

$$0 \leq p \leq q \leq m.$$ (3.1)

Then, The following results hold.

(a) The rank/inertia of $A + X$ satisfy

$$\max_{X \in \mathbb{C}_H^m, r(X) = q} r(A + X) = \min\{m, r(A) + q\},$$ (3.2)

$$\min_{X \in \mathbb{C}_H^m, r(X) = q} r(A + X) = |r(A) - q|,$$ (3.3)

$$\max_{X \in \mathbb{C}_H^m, r(X) = q} i_+(A + X) = \min\{m, i_+(A) + q\},$$ (3.4)

$$\min_{X \in \mathbb{C}_H^m, r(X) = q} i_+(A + X) = \max\{0, i_+(A) - q\},$$ (3.5)

$$\max_{X \in \mathbb{C}_H^m, r(X) = q} i_-(A + X) = \min\{m, i_-(A) + q\},$$ (3.6)

$$\min_{X \in \mathbb{C}_H^m, r(X) = q} i_-(A + X) = \max\{0, i_-(A) - q\}.$$ (3.7)

(b) The following statements hold.

(i) For any integer $t_1$ between the two quantities on the right-hand sides of (3.2) and (3.3), there exists an $X \in \mathbb{C}_H^{m \times m}$ with $r(X) = q$ such that $r(A + X) = t_1$.

(ii) For any integer $t_2$ between the two quantities on the right-hand sides of (3.4) and (3.5), there exists an $X \in \mathbb{C}_H^{m \times m}$ with $r(X) = q$ such that $i_+(A + X) = t_2$.

(iii) For any integer $t_3$ between the two quantities on the right-hand sides of (3.6) and (3.7), there exists an $X \in \mathbb{C}_H^{m \times m}$ with $r(X) = q$ such that $i_-(A + X) = t_3$.

(iv) There exists an $X \in \mathbb{C}_H^m$ with $r(X) = q$ such that $A + X$ is nonsingular if and only if $r(A) \geq m - q$.

(v) There exists an $X \in \mathbb{C}_H^m$ with $r(X) = q$ such that $A + X = 0$ if and only if $r(A) = q$.

(vi) There exists an $X \in \mathbb{C}_H^m$ with $r(X) = q$ such that $A + X \succ 0$ if and only if $i_+(A) \geq m - q$.

(vii) There exists an $X \in \mathbb{C}_H^m$ with $r(X) = q$ such that $A + X \succ 0$ if and only if $i_+(A) \leq q$.

(viii) There exists an $X \in \mathbb{C}_H^m$ with $r(X) = q$ such that $A + X \prec 0$ if and only if $i_+(A) \geq m - q$.

(ix) There exists an $X \in \mathbb{C}_H^m$ with $r(X) = q$ such that $A + X \preceq 0$ if and only if $i_+(A) \leq q$. 


(c) The rank/inertia of $A + X$ satisfy

\[
\begin{align*}
\max_{X \in \mathbb{C}_H^m, p \leq r(X) \leq q} r(A + X) &= \min\{m, \ r(A) + q\}, \\
\min_{X \in \mathbb{C}_H^m, p \leq r(X) \leq q} r(A + X) &= \max\{0, \ p - r(A), \ r(A) - q\}, \\
\max_{X \in \mathbb{C}_H^m, p \leq r(X) \leq q} i_+(A + X) &= \min\{m, \ i_+(A) + q\}, \\
\min_{X \in \mathbb{C}_H^m, p \leq r(X) \leq q} i_+(A + X) &= \max\{0, \ i_+(A) - q\}, \\
\max_{X \in \mathbb{C}_H^m, p \leq r(X) \leq q} i_-(A + X) &= \min\{m, \ i_-(A) + q\}, \\
\min_{X \in \mathbb{C}_H^m, p \leq r(X) \leq q} i_-(A + X) &= \max\{0, \ i_-(A) - q\}.
\end{align*}
\]

(d) The following statements hold.

(i) For any integer $t_1$ between the two quantities on the right-hand sides of (3.8) and (3.9), there exists an $X \in \mathbb{C}_H^{m \times m}$ with $p \leq r(X) \leq q$ such that $r(A + X) = t_1$.

(ii) For any integer $t_2$ between the two quantities on the right-hand sides of (3.10) and (3.11), there exists an $X \in \mathbb{C}_H^{m \times m}$ with $p \leq r(X) \leq q$ such that $i_+(A + X) = t_2$.

(iii) For any integer $t_3$ between the two quantities on the right-hand sides of (3.12) and (3.13), there exists an $X \in \mathbb{C}_H^{m \times m}$ with $p \leq r(X) \leq q$ such that $i_-(A + X) = t_3$.

(iv) There exists an $X \in \mathbb{C}_H^m$ with $p \leq r(X) \leq q$ such that $A + X$ is nonsingular if and only if $r(A) \geq m - q$.

(v) There exists an $X \in \mathbb{C}_H^m$ with $p \leq r(X) \leq q$ such that $A + X = 0$ if and only if $p \leq r(A) \leq q$.

(vi) There exists an $X \in \mathbb{C}_H^m$ with $p \leq r(X) \leq q$ such that $A + X \geq 0$ if and only if $i_+(A) \geq m - q$.

(vii) There exists an $X \in \mathbb{C}_H^m$ with $p \leq r(X) \leq q$ such that $A + X \geq 0$ if and only if $i_-(A) \leq q$.

(viii) There exists an $X \in \mathbb{C}_H^m$ with $p \leq r(X) \leq q$ such that $A + X < 0$ if and only if $i_-(A) \geq m - q$.

(ix) There exists an $X \in \mathbb{C}_H^m$ with $p \leq r(X) \leq q$ such that $A + X \leq 0$ if and only if $i_+(A) \leq q$.

(e) The rank/inertia of $A + X$ satisfy

\[
\begin{align*}
\max_{X \in \mathbb{C}_H^m, 0 \leq r(X) \leq q} r(A + X) &= \min\{m, \ r(A) + q\}, \\
\min_{X \in \mathbb{C}_H^m, 0 \leq r(X) \leq q} r(A + X) &= \max\{0, \ r(A) - q\}, \\
\max_{X \in \mathbb{C}_H^m, 0 \leq r(X) \leq q} i_+(A + X) &= \min\{m, \ i_+(A) + q\}, \\
\min_{X \in \mathbb{C}_H^m, 0 \leq r(X) \leq q} i_+(A + X) &= \max\{0, \ i_+(A) - q\}, \\
\max_{X \in \mathbb{C}_H^m, 0 \leq r(X) \leq q} i_-(A + X) &= \min\{m, \ i_-(A) + q\}, \\
\min_{X \in \mathbb{C}_H^m, 0 \leq r(X) \leq q} i_-(A + X) &= \max\{0, \ i_-(A) - q\}.
\end{align*}
\]
(f) The rank/inertia of $A + X$ satisfy
\[ \max_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq m} r(A + X) = m, \]
\[ \min_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq m} r(A + X) = \max\{0, p - r(A)\}, \]
\[ \max_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq m} i_+(A + X) = m, \]
\[ \min_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq m} i_+(A + X) = 0, \]
\[ \max_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq m} i_-(A + X) = m, \]
\[ \min_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq m} i_-(A + X) = 0. \]

The Hermitian matrices $X$s satisfying these equalities can be formulated from the canonical form of $A$ under the Hermitian congruence.

**Proof.** It is easy to see from (2.1)–(2.4) that the right-hand sides of (3.2), (3.4), and (3.6) are upper bounds, while the right-hand sides of (3.3), (3.5), and (3.7) are lower bounds. Without loss of generality, we assume that $A$ is of the form
\[ A_1 = \text{diag}(I_t, -I_t, 0) \quad \text{or} \quad A_2 = \text{diag}(-I_t, I_t, 0). \]

The establishments of (3.2)–(3.7) are based on the following assertions:

(i) Let $X = \text{diag}(0_{m-q}, 2I_q)$. If $m \leq r(A) + q$, then $r(A_1 + X) = m - r(A)$; if $r(A) > q$ and $s > q$, then setting $X = \text{diag}(I_s, 0_{m-q})$ gives $r(A_1 + X) = r(A) - q$, so that (3.3) holds; if $r(A) > q \geq s$, then setting $X = \text{diag}(I_s, 0_{m-q})$ gives $r(A_1 + X) = r(A) - q$, so that (3.3) holds.

(ii) If $r(A) \leq q$, then setting $X = \text{diag}(-I_s, I_s, -r(A), 0_{m-q})$ gives $r(A_1 + X) = q - r(A)$; if $r(A) > q$ and $s > q$, then setting $X = \text{diag}(-I_q, 0_{m-q})$ gives $r(A_1 + X) = r(A) - q$, so that (3.3) holds.

(iii) If $m \leq i_+(A) + q$, then setting $X = \text{diag}(0_{m-q}, 2I_q)$ gives $i_+(A_1 + X) = m$; if $m > i_+(A) + q$, then setting $X = \text{diag}(0_{m-q}, 2I_q)$ gives $i_+(A_1 + X) = i_+(A) + r(X) = i_+(A) + q$, so that (3.4) holds.

(iv) If $i_+(A) \leq q$, then setting $X = \text{diag}(-2I_q, 0_{m-q})$ gives $i_-(A_1 + X) = 0$; if $i_+(A) > q$, then setting $X = \text{diag}(I_q, 0_{m-q})$ gives $i_+(A_1 + X) = i_+(A) + q$, so that (3.5) holds.

(v) If $m \leq i_-(A) + q$, then setting $X = \text{diag}(-2I_s, 0_{m-q}, -2I_{q-s})$ gives $i_-(A_1 + X) = m$; if $m > i_-(A) + q$, then setting $X = \text{diag}(-2I_{s-k}, 0_{m-q}, -2I_{q-s+k})$ gives $i_-(A_1 + X) = i_-(A) + r(X) = i_-(A) + q$, so that (3.6) holds.

(vi) If $i_-(A) \leq q$, then setting $X = \text{diag}(0_s, I_q, 0_{m-s-q})$ gives $i_-(A_1 + X) = 0$; if $i_-(A) > q$, then setting $X = \text{diag}(0_s, I_q, 0_{m-s-q})$ gives $i_-(A_1 + X) = i_-(A) + q$, so that (3.7) holds.

It can be seen from (1.18) that
\[
\max_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq q} r(A + X) = \max \left\{ \max_{X \in \mathbb{C}^{m \times m}, r(X) = p} r(A + X), \right. \\
\left. \max_{X \in \mathbb{C}^{m \times m}, r(X) = p+1} r(A + X), \ldots, \max_{X \in \mathbb{C}^{m \times m}, r(X) = q} r(A + X) \right\},
\]
\[
\min_{X \in \mathbb{C}^{m \times m}, \rho \leq r(X) \leq q} r(A + X) = \min \left\{ \min_{X \in \mathbb{C}^{m \times m}, r(X) = p} r(A + X), \right. \\
\left. \min_{X \in \mathbb{C}^{m \times m}, r(X) = p+1} r(A + X), \ldots, \min_{X \in \mathbb{C}^{m \times m}, r(X) = q} r(A + X) \right\}.
\]

(3.27)
Substituting (3.2)–(3.7) for \( r(X) = p, p + 1, \ldots, q \) into (3.27)–(3.32) and making the max-min comparison, we obtain

\[
\min_{X \in C_p^m, r(X) = p} r(A + X) = \min \{ m, r(A) + p \}, \min \{ m, r(A) + p + 1 \}, \ldots, \min \{ m, r(A) + q \}
\]

\[
\max_{X \in C_p^m, p \leq r(X) \leq q} i_+(A + X) = \max \{ m, i_+(A) + p \}, \min \{ m, i_+(A) + p + 1 \}, \ldots, \min \{ m, i_+(A) + q \}
\]

(3.33)
as required for (3.8)–(3.13). Results (e) and (f) follow directly from (c).

When the variable matrix \( X \) in \( A \pm X \) runs over the cone of positive semi-definite matrices, we have the following results.

**Theorem 3.2.** Let \( A \in \mathbb{C}_H^m \) be given, \( X \in \mathbb{C}_H^m \) be a variable matrix, and assume that \( p \) and \( q \) are two integers satisfying \( 0 \leq p \leq q \leq m \). Then, the following results hold.

(a) The rank/inertia of \( A \pm X \) satisfy

\[
\begin{align*}
\max_{0 \leq X, r(X) = q} r(A + X) &= \min\{m, \ r(A) + q\}, \quad (3.39) \\
\min_{0 \leq X, r(X) = q} r(A + X) &= \max\{i_+(A), \ q - i_-(A)\}, \quad r(A) - q\}, \quad (3.40) \\
\max_{0 \leq X, r(X) = q} i_+(A + X) &= \min\{m, \ i_+(A) + q\}, \quad (3.41) \\
\min_{0 \leq X, r(X) = q} i_+(A + X) &= \max\{i_+(A), \ q - i_-(A)\}; \quad (3.42) \\
\max_{0 \leq X, r(X) = q} i_-(A + X) &= i_-(A), \quad (3.43) \\
\min_{0 \leq X, r(X) = q} i_-(A + X) &= \max\{0, \ i_-(A) - q\}, \quad (3.44) \\
\max_{0 \leq X, r(X) = q} r(A - X) &= \min\{m, \ r(A) + q\}, \quad (3.45) \\
\min_{0 \leq X, r(X) = q} r(A - X) &= \max\{i_-(A), \ q - i_+(A)\}, \quad r(A) - q\}, \quad (3.46) \\
\max_{0 \leq X, r(X) = q} i_+(A - X) &= i_+(A), \quad (3.47) \\
\min_{0 \leq X, r(X) = q} i_+(A - X) &= \max\{0, \ i_+(A) - q\}, \quad (3.48) \\
\max_{0 \leq X, r(X) = q} i_-(A - X) &= \min\{m, \ i_-(A) + q\}, \quad (3.49) \\
\min_{0 \leq X, r(X) = q} i_-(A - X) &= \max\{i_-(A), \ q - i_+(A)\}. \quad (3.50)
\end{align*}
\]

(b) The following statements hold.

(i) For any integer \( t_1 \) between the two quantities on the right-hand sides of (3.39) and (3.40), there exists a \( 0 \leq X \in \mathbb{C}_H^{m \times m} \) with \( r(X) = q \) such that \( r(A + X) = t_1 \).

(ii) For any integer \( t_2 \) between the two quantities on the right-hand sides of (3.41) and (3.42), there exists a \( 0 \leq X \in \mathbb{C}_H^{m \times m} \) with \( r(X) = q \) such that \( i_+(A + X) = t_2 \).

(iii) For any integer \( t_3 \) between the two quantities on the right-hand sides of (3.43) and (3.44), there exists a \( 0 \leq X \in \mathbb{C}_H^{m \times m} \) with \( r(X) = q \) such that \( i_-(A + X) = t_3 \).
(iv) For any integer $t_4$ between the two quantities on the right-hand sides of (3.45) and (3.46), there exists a $0 \prec X \in C_H^m$ with $r(X) = q$ such that $r(A - X) = t_4$.

(v) For any integer $t_5$ between the two quantities on the right-hand sides of (3.47) and (3.48), there exists a $0 \prec X \in C_H^m$ with $r(X) = q$ such that $i_+(A - X) = t_5$.

(vi) For any integer $t_6$ between the two quantities on the right-hand sides of (3.49) and (3.50), there exists a $0 \prec X \in C_H^m$ with $r(X) = q$ such that $i_-(A - X) = t_6$.

(vii) There exists a $0 \preceq X \in C_H^m$ with $r(X) = q$ such that $A + X$ is nonsingular if and only if $r(A) \geq m - q$.

(viii) $A + X$ is nonsingular for all $0 \preceq X \in C_H^m$ with $r(X) = q$ if and only if $A \succ 0$, or $A \succ 0$ and $q = m$, or $r(A) = m$ and $q = 0$.

(ix) There exists a $0 \preceq X \in C_H^m$ with $r(X) = q$ such that $A + X = 0$ if and only if $A \preceq 0$ and $r(A) = q$.

(x) There exists a $0 \preceq X \in C_H^m$ with $r(X) = q$ such that $A + X \succ 0$ if and only if $i_-(A) \geq m - q$.

(xi) $A + X \succ 0$ holds for all $0 \preceq X \in C_H^m$ with $r(X) = q$ if and only if $A \succ 0$, or $A \preceq 0$ and $q = m$.

(xii) There exists a $0 \preceq X \in C_H^m$ with $r(X) = q$ such that $A + X \preceq 0$ if and only if $i_+(A) \leq q$.

(xiii) $A + X \preceq 0$ holds for all $0 \preceq X \in C_H^m$ with $r(X) = q$ if and only if $A \not\succ 0$.

(xiv) There exists a $0 \preceq X \in C_H^m$ with $r(X) = q$ such that $A + X \prec 0$ if and only if $A \prec 0$.

(xv) $A + X \prec 0$ holds for all $0 \preceq X \in C_H^m$ with $r(X) = q$ if and only if $A \prec 0$ and $q = 0$.

(xvi) There exists a $0 \preceq X \in C_H^m$ with $r(X) = q$ such that $A + X \approx 0$ if and only if $A \approx 0$ and $r(A) \geq q$.

(xvii) $A + X \approx 0$ holds for all $0 \preceq X \in C_H^m$ with $r(X) = q$ if and only if $A \approx 0$ and $q = 0$.

(xviii) There exists a $0 \approx X \in C_H^m$ with $r(X) = q$ such that $A - X$ is nonsingular if and only if $r(A) \geq m - q$.

(xix) $A - X$ is nonsingular for all $0 \approx X \in C_H^m$ with $r(X) = q$ if and only if $A \approx 0$, or $A \approx 0$ and $q = m$, or $r(A) = m$ and $q = 0$.

(xx) There exists a $0 \approx X \in C_H^m$ with $r(X) = q$ such that $A - X = 0$ if and only if $A \not\approx 0$ and $r(A) = q$.

(xi) There exists a $0 \approx X \in C_H^m$ with $r(X) = q$ such that $A - X \approx 0$ if and only if $A \approx 0$.

(xi) There exists a $0 \approx X \in C_H^m$ with $r(X) = q$ if and only if $A \approx 0$ and $q = 0$.

(xi) There exists a $0 \approx X \in C_H^m$ with $r(X) = q$ if and only if $A \approx 0$ and $r(A) \geq q$.

(xix) $A - X \approx 0$ holds for all $0 \approx X \in C_H^m$ with $r(X) = q$ if and only if $A \approx 0$ and $q = 0$.

(xx) There exists a $0 \approx X \in C_H^m$ with $r(X) = q$ such that $A - X < 0$ if and only if $A < 0$, or $A \approx 0$ and $q = m$.

(xx) There exists a $0 \approx X \in C_H^m$ with $r(X) = q$ if and only if $A \approx 0$ and $q = m$.
There exists a $0 \preceq X \in \mathbb{C}_n^m$ with $r(X) = q$ such that $A - X \preceq 0$ if and only if $i_+(A) \leq q$.

The rank/inertia of $A \pm X$ satisfy

\[
\begin{align*}
\max_{0 \leq X, p \leq r(X) \leq q} r(A + X) &= \min\{m, r(A) + q\}, \quad (3.51) \\
\min_{0 \leq X, p \leq r(X) \leq q} r(A + X) &= \max\{i_+(A), p - i_-(A), r(A) - q\}, \quad (3.52) \\
\max_{0 \leq X, p \leq r(X) \leq q} i_+(A + X) &= \min\{m, i_+(A) + q\}, \quad (3.53) \\
\min_{0 \leq X, p \leq r(X) \leq q} i_+(A + X) &= \max\{i_+(A), p - i_-(A)\}, \quad (3.54) \\
\max_{0 \leq X, p \leq r(X) \leq q} i_-(A + X) &= i_-(A), \quad (3.55) \\
\min_{0 \leq X, p \leq r(X) \leq q} i_-(A + X) &= \max\{0, i_-(A) - q\}, \quad (3.56) \\
\max_{0 \leq X, p \leq r(X) \leq q} r(A - X) &= \min\{m, r(A) + q\}, \quad (3.57) \\
\min_{0 \leq X, p \leq r(X) \leq q} r(A - X) &= \max\{i_-(A), p - i_+(A), r(A) - q\}, \quad (3.58) \\
\max_{0 \leq X, p \leq r(X) \leq q} i_+(A - X) &= i_+(A), \quad (3.59) \\
\min_{0 \leq X, p \leq r(X) \leq q} i_+(A - X) &= \max\{0, i_+(A) - q\}, \quad (3.60) \\
\max_{0 \leq X, p \leq r(X) \leq q} i_-(A - X) &= \min\{m, i_-(A) + q\}, \quad (3.61) \\
\min_{0 \leq X, p \leq r(X) \leq q} i_-(A - X) &= \max\{i_-(A), p - i_+(A)\}. \quad (3.62)
\end{align*}
\]

In consequence, the following results hold.

(i) There exists a $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ such that $A + X$ is nonsingular if and only if $r(A) \geq m - q$.

(ii) $A + X$ is nonsingular for all $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ if and only if $A \succ 0$, or $A \succ 0$ and $p = q = m$, or $r(A) = m$ and $p = q = 0$.

(iii) There exists a $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ such that $A + X = 0$ if and only if $A \preceq 0$ and $p \leq r(A) \leq q$.

(iv) There exists a $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ such that $A + X \succ 0$ if and only if $i_+(A) \geq m - q$.

(v) $A + X \succ 0$ holds for all $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ if and only if $A \succ 0$, or $A \succ 0$ and $p = q = m$.

(vi) There exists a $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ such that $A + X \succ 0$ if and only if $i_-(A) \leq q$.

(vii) $A + X \succ 0$ holds for all $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ if and only if $A \succ 0$.

(viii) There exists a $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ such that $A + X \prec 0$ if and only if $A \prec 0$.

(ix) $A + X \prec 0$ holds for all $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ if and only if $A \prec 0$ and $p = q = 0$.

(x) There exists a $0 \preceq X \in \mathbb{C}_n^m$ with $p \leq r(X) \leq q$ such that $A + X \preceq 0$ if and only if $A \preceq 0$ and $r(A) \geq p$. 


(xi) $A + X \preceq 0$ holds for all $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ if and only if $A \preceq 0$ and $p = q = 0$.

(xii) There exists a $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ such that $A - X$ is nonsingular if and only if $r(A) \geq m - q$.

(xiii) $A - X$ is nonsingular for all $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ if and only if $A \prec 0$, or $A \preceq 0$ and $p = q = m$, or $r(A) = m$ and $p = q = 0$.

(xiv) There exists a $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ such that $A - X = 0$ if and only if $A \succ 0$ and $p \leq r(A) \leq q$.

(xv) There exists a $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ such that $A - X \succ 0$ if and only if $A \succ 0$.

(xvi) $A - X \succ 0$ holds for all $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ if and only if $A \succ 0$ and $p = q = 0$.

(xvii) There exists a $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ such that $A - X \succ 0$ if and only if $A \succ 0$ and $r(A) \geq p$.

(xviii) $A - X \succ 0$ holds for all $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ if and only if $A \succ 0$ and $p = q = 0$.

(xix) There exists a $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ such that $A - X \prec 0$ if and only if $i_-(A) \geq m - q$.

(xx) $A - X \prec 0$ holds for all $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ if and only if $A \prec 0$, or $A \preceq 0$ and $p = q = m$.

(xxi) There exists a $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ such that $A - X \preceq 0$ if and only if $i_+(A) \leq q$.

(xxii) $A - X \preceq 0$ holds for all $0 \preceq X \in \mathbb{C}_H^n$ with $p \leq r(X) \leq q$ if and only if $A \preceq 0$.

(d) The rank/inertia of $A + X$ satisfy

\[ \max_{0 \preceq X, 0 \leq r(X) \leq q} r(A + X) = \min \{m, \ r(A) + q\}, \] (3.63)

\[ \min_{0 \preceq X, 0 \leq r(X) \leq q} r(A + X) = \max \{i_+(A), \ r(A) - q\}, \] (3.64)

\[ \max_{0 \preceq X, 0 \leq r(X) \leq q} i_+(A + X) = \min \{m, \ i_+(A) + q\}, \] (3.65)

\[ \min_{0 \preceq X, 0 \leq r(X) \leq q} i_+(A + X) = i_+(A), \] (3.66)

\[ \max_{0 \preceq X, 0 \leq r(X) \leq q} i_-(A + X) = i_-(A), \] (3.67)

\[ \min_{0 \preceq X, 0 \leq r(X) \leq q} i_-(A + X) = \max \{0, \ i_-(A) - q\}, \] (3.68)

\[ \max_{0 \preceq X, 0 \leq r(X) \leq q} r(A - X) = \min \{m, \ r(A) + q\}, \] (3.69)

\[ \min_{0 \preceq X, 0 \leq r(X) \leq q} r(A - X) = \max \{i_-(A), \ r(A) - q\}, \] (3.70)

\[ \max_{0 \preceq X, 0 \leq r(X) \leq q} i_-(A - X) = i_-(A), \] (3.71)

\[ \min_{0 \preceq X, 0 \leq r(X) \leq q} i_+(A - X) = \max \{0, \ i_+(A) - q\}, \] (3.72)

\[ \max_{0 \preceq X, 0 \leq r(X) \leq q} i_-(A - X) = \min \{m, \ i_-(A) + q\}, \] (3.73)

\[ \min_{0 \preceq X, 0 \leq r(X) \leq q} i_-(A - X) = i_-(A), \] (3.74)
Proof. Without loss of generality, we assume that \( X \) is the variable matrix. The establishments of (3.39)–(3.44) are based on the following choices of bounds. The right-hand sides of (3.40), (3.42), and (3.44) are lower bounds, while the right-hand sides of (3.39), (3.41), and (3.43) are upper bounds. The establishments of (3.39)–(3.44) are based on the following choices of the variable matrix \( X \):

(i) Set \( X = \text{diag}(0_{m-q}, 2I_q) \). If \( m \leq r(A) + q \), then \( r(A + X) = m \); if \( m > r(A) + q \), then \( r(A + X) = r(A) + r(X) = r(A) + q \), so that (3.39) holds.

(ii) Set \( X = \text{diag}(I_q, 0_{m-q}) \). If \( q \leq i_+(A) \), then \( r(A_2 + X) = t - q = r(A) - q \); if \( i_-(A) \leq q \leq r(A) \), then \( r(A_2 + X) = s = i_+(A) \); if \( q \geq r(A) \), then \( r(A_2 + X) = q - t = q - i_-(A) \), establishing (3.40).

(iii) Set \( X = \text{diag}(0_{m-q}, 2I_q) \). If \( m \leq i_+(A) + q \), then \( i_+(A_1 + X) = m \); if \( m > i_+(A) + q \), then \( i_+(A_1 + X) = s + q = i_+(A) + q \), establishing (3.41).

(iv) Set \( X = \text{diag}(I_q, 0_{m-q}) \). If \( q \leq r(A) \), then \( i_+(A_1 + X) = s = i_+(A) \); if \( q \geq r(A) \), then \( i_+(A_1 + X) = q - t = q - i_-(A) \), establishing (3.42).

(v) Set \( X = \text{diag}(I_q/2, 0_{m-q}) \). Then \( i_-(A_1 + X) = t = i_-(A) \), establishing (3.43).

(vi) Set \( X = \text{diag}(I_q, 0_{m-q}) \). If \( q \leq i_-(A) \), then \( i_-(A_2 + X) = t - q = i_-(A) - q \); if \( q \geq i_-(A) \), then \( i_-(A_2 + X) = 0 \), establishing (3.44).

Eqs. (3.45)–(3.50) can be shown similarly.

It following from (1.19) that

\[
\max_{0 \leq X, p \leq r(X) \leq q} r(A + X) = \max \left\{ \max_{0 \leq X, r(X) = p} r(A + X), \ldots, \max_{0 \leq X, r(X) = q} r(A + X) \right\}.
\] (3.87)
\[ \min_{0 \leq X, p \leq r(X) \leq q} r(A + X) = \min_{0 \leq X, r(X) = p} \begin{cases} \min_{0 \leq X, r(X) = p} r(A + X), \\ \min_{0 \leq X, r(X) = p+1} r(A + X), \ldots, \\ \min_{0 \leq X, r(X) = q} r(A + X) \end{cases}, \quad (3.88) \]

\[ \max_{0 \leq X, p \leq r(X) \leq q} i_+(A + X) = \max_{0 \leq X, r(X) = p} \begin{cases} \max_{0 \leq X, r(X) = p} i_+(A + X), \\ \max_{0 \leq X, r(X) = p+1} i_+(A + X), \ldots, \\ \max_{0 \leq X, r(X) = q} i_+(A + X) \end{cases}, \quad (3.89) \]

\[ \min_{0 \leq X, p \leq r(X) \leq q} i_+(A + X) = \min_{0 \leq X, r(X) = p} \begin{cases} \min_{0 \leq X, r(X) = p} i_+(A + X), \\ \min_{0 \leq X, r(X) = p+1} i_+(A + X), \ldots, \\ \min_{0 \leq X, r(X) = q} i_+(A + X) \end{cases}, \quad (3.90) \]

\[ \max_{0 \leq X, p \leq r(X) \leq q} i_-(A + X) = \max_{0 \leq X, r(X) = p} \begin{cases} \max_{0 \leq X, r(X) = p} i_-(A + X), \\ \max_{0 \leq X, r(X) = p+1} i_-(A + X), \ldots, \\ \max_{0 \leq X, r(X) = q} i_-(A + X) \end{cases}, \quad (3.91) \]

\[ \min_{0 \leq X, p \leq r(X) \leq q} i_-(A + X) = \min_{0 \leq X, r(X) = p} \begin{cases} \min_{0 \leq X, r(X) = p} i_-(A + X), \\ \min_{0 \leq X, r(X) = p+1} i_-(A + X), \ldots, \\ \min_{0 \leq X, r(X) = q} i_-(A + X) \end{cases}. \quad (3.92) \]

Substituting (3.39)–(3.44) for \( r(X) = p, p + 1, \ldots, q \) into (3.87)–(3.92) and making the max-min comparison, we obtain (3.51)–(3.56). Eqs. (3.57)–(3.62) hold by symmetry. Results (d) and (e) follow directly from (c).

4. Rank and inertia formulas of \( A + BXB^* \) when \( X \) is Hermitian with a fixed rank. Based on the results in the previous two sections, we are able to derive explicit solutions to Problems 1.1 and 1.2. To do so, we need the following known result on the canonical form of \( A + BXB^* \); see [10].

**Lemma 4.1.** Let \( A \in \mathbb{C}^n_{\mathbb{H}} \) and \( B \in \mathbb{C}^{m \times n} \) be given, and \( X \in \mathbb{C}^n_{\mathbb{H}} \) be a variable matrix. Then,

(i) \( A + BXB^* \) can be decomposed as

\[ A + BXB^* = P^*D_A P + P^*D_B U X U^* D_B^T P = P^*(D_A + D_B Y D_B^T)P, \quad (4.1) \]

where \( P, D_A, \) and \( D_B \) are as given in (2.81), and \( Y = U X^* \) satisfies

\[ X \in \mathbb{C}^n_{\mathbb{H}} \iff Y \in \mathbb{C}^n_{\mathbb{H}}. \quad (4.2) \]

(ii) The inertia/rank of \( A + BXB^* \) satisfy

\[ r(A + BXB^*) = r(D_A + D_B Y D_B^T), \quad (4.3) \]

\[ i_\pm(A + BXB^*) = i_\pm(D_A + D_B Y D_B^T) \quad (4.4) \]

for \( Y = U X U^* \) and \( X \in \mathbb{C}^n_{\mathbb{H}}. \)
(iii) Partition the Hermitian matrix $Y$ as

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12}^* & Y_{22} & Y_{23} & Y_{24} \\ Y_{13}^* & Y_{23}^* & Y_{33} & Y_{34} \\ Y_{14}^* & Y_{24}^* & Y_{34}^* & Y_{44} \end{bmatrix}, \quad f$$

and the following expansion formulas hold

$$D_A + D_B Y D_B^T = \begin{bmatrix} Y_{11} & I & Y_{12} \Sigma & 0 \\ I & 0 & 0 & 0 \\ \Sigma Y_{12}^* & 0 & \Delta_1 + \Sigma Y_{22} \Sigma & 0 \\ 0 & 0 & 0 & \Delta_2 \\ Y_{13}^* & 0 & Y_{23}^* \Sigma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (4.5)$$

where $Y_{ii} = Y_{ii}^*$, $i = 1, 2, 3, 4$. Then,

$$r(A + BXB^*) = 2f + k + r \begin{bmatrix} Y_{22} + \Sigma^{-1} \Delta_1 \Sigma^{-1} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix}$$

$$= 2f + k + r(\hat{A} + \hat{Y}), \quad (4.7)$$

$$i_+(A + BXB^*) = f + s_2 + i_+ \begin{bmatrix} Y_{22} + \Sigma^{-1} \Delta_1 \Sigma^{-1} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix}$$

$$= f + s_2 + i_+(\hat{A} + \hat{Y}), \quad (4.8)$$

$$i_-(A + BXB^*) = f + k - s_2 + i_- \begin{bmatrix} Y_{22} + \Sigma^{-1} \Delta_1 \Sigma^{-1} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix}$$

$$= f + k - s_2 + i_-(\hat{A} + \hat{Y}), \quad (4.9)$$

where

$$\hat{A} = \begin{bmatrix} \Sigma^{-1} \Delta_1 \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} Y_{22} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix}. \quad (4.10)$$

It can be seen from (4.7)–(4.9) that

$$\max_{x \in C_R^H} r(A + BXB^*) = 2f + k + \max_{\hat{Y} \in C_H^L} r(\hat{A} + \hat{Y}), \quad (4.11)$$

$$\min_{x \in C_R^H} r(A + BXB^*) = 2f + k + \min_{\hat{Y} \in C_H^L} r(\hat{A} + \hat{Y}), \quad (4.12)$$

$$\max_{x \in C_R^H} i_+(A + BXB^*) = f + s_2 + \max_{\hat{Y} \in C_H^L} i_+(\hat{A} + \hat{Y}), \quad (4.13)$$

$$\min_{x \in C_R^H} i_+(A + BXB^*) = f + s_2 + \min_{\hat{Y} \in C_H^L} i_+(\hat{A} + \hat{Y}), \quad (4.14)$$

$$\max_{x \in C_R^H} i_-(A + BXB^*) = f + k - s_2 + \max_{\hat{Y} \in C_H^L} i_-(\hat{A} + \hat{Y}), \quad (4.15)$$

$$\min_{x \in C_R^H} i_-(A + BXB^*) = f + k - s_2 + \min_{\hat{Y} \in C_H^L} i_-(\hat{A} + \hat{Y}). \quad (4.16)$$

So that the following result is obvious.
Lemma 4.2 ([18]). Let $A \in \mathbb{C}_H^n$ and $B \in \mathbb{C}^{m \times n}$ be given, and $X \in \mathbb{C}_H^n$ be a variable matrix. Also let

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}.$$ (4.17)

Then, the extremum ranks/inertias of $A + BXB^*$ are given by

$$\max_{X \in \mathbb{C}_H^n, r(X) = q} r(A + BXB^*) = \min\{r[A, B], r(A) + q\},$$ (4.18)

$$\min_{X \in \mathbb{C}_H^n, r(X) = q} r(A + BXB^*) = \max\{2r[A, B] - r(M), 2r[A, B] - r(A) + q - 2n, r(A) - q\},$$ (4.19)

$$\max_{X \in \mathbb{C}_H^n, r(X) = q} i_+(A + BXB^*) = \min\{i_+(M), i_+(A) + q\},$$ (4.20)

$$\min_{X \in \mathbb{C}_H^n, r(X) = q} i_+(A + BXB^*) = \max\{r[A, B] - i_+(M), i_+(A) - q\},$$ (4.21)

$$\max_{X \in \mathbb{C}_H^n, r(X) = q} i_-(A + BXB^*) = \min\{i_-(M), i_-(A) + q\},$$ (4.22)

$$\min_{X \in \mathbb{C}_H^n, r(X) = q} i_-(A + BXB^*) = \max\{r[A, B] - i_-(M), i_-(A) - q\}.$$ (4.23)

In what follows, we derive from (4.7)–(4.9) the extremum rank/inertia of $A + BXB^*$ with rank restrictions on $X$.

Theorem 4.3. Let $A \in \mathbb{C}_H^n$ and $B \in \mathbb{C}^{m \times n}$ be given, $X \in \mathbb{C}_H^n$ be a variable matrix, $M$ be the matrix in (4.17), and assume that $q$ is an integer satisfying $0 \leq q \leq n$. Then, the following equalities hold

$$\max_{X \in \mathbb{C}_H^n, r(X) = n} r(A + BXB^*) = r[A, B],$$ (4.24)

$$\min_{X \in \mathbb{C}_H^n, r(X) = n} r(A + BXB^*) = \max\{2r[A, B] - r(M), 2r[A, B] - r(A) - n\},$$ (4.25)

$$\max_{X \in \mathbb{C}_H^n, r(X) = n} i_+(A + BXB^*) = i_+(M),$$ (4.26)

$$\min_{X \in \mathbb{C}_H^n, r(X) = n} i_+(A + BXB^*) = r[A, B] - i_+(M),$$ (4.27)

$$\max_{X \in \mathbb{C}_H^n, r(X) = n} i_-(A + BXB^*) = \min\{i_-(M), i_-(A) + q\},$$ (4.28)

$$\min_{X \in \mathbb{C}_H^n, r(X) = n} i_-(A + BXB^*) = \max\{r[A, B] - i_-(M), i_-(A) - q\}.$$ (4.29)

In particular,

$$\max_{X \in \mathbb{C}_H^n, r(X) = n} i_+(A + BXB^*) = i_+(M),$$ (4.30)

$$\min_{X \in \mathbb{C}_H^n, r(X) = n} i_+(A + BXB^*) = r[A, B] - i_+(M),$$ (4.31)

$$\max_{X \in \mathbb{C}_H^n, r(X) = n} i_-(A + BXB^*) = \min\{i_-(M), i_-(A) + q\},$$ (4.32)

$$\min_{X \in \mathbb{C}_H^n, r(X) = n} i_-(A + BXB^*) = \max\{r[A, B] - i_-(M), i_-(A) - q\}.$$ (4.33)

In consequence, the following results hold.

(i) There exists an $X \in \mathbb{C}_H^n$ with $r(X) = q$ such that $A + BXB^*$ is nonsingular if and only if $r[A, B] = m$ and $r(A) \geq m - q$. 
There exists an $X \in \mathbb{C}_H^n$ with $r(X) = q$ such that \( A + BXB^* = 0 \) if and only if \( r(A) = m \) and \( B^* A^{-1} B = 0 \), or \( 2r[A, B] - r(A) + q - 2n = m \), or \( r(A) = m \) and \( q = 0 \).

There exists an $X \in \mathbb{C}_H^n$ with $r(X) = q$ such that \( A + BXB^* = 0 \) if and only if \( A \subseteq B \) and \( r(A) \leq q \leq r(A) - 2r(B) + 2n \).

There exists an $X \in \mathbb{C}_H^n$ with $r(X) = q$ such that \( A + BXB^* > 0 \) if and only if $i_+ (M) = m$ and $i_+ (A) \geq m - q$, or $i_+ (M) \geq m$ and $i_+ (A) = m - q$.

$A + BXB^* > 0$ for all $X \in \mathbb{C}_H^n$ with $r(X) = q$ if and only if $A \succ 0$ and $B = 0$, or $A \succ 0$ and $q = 0$.

There exists an $X \in \mathbb{C}_H^n$ with $r(X) = q$ such that \( A + BXB^* \geq 0 \) if and only if $r(A) = r[A, B]$ and $i_-(A) \leq q$.

$A + BXB^* \succeq 0$ for all $X \in \mathbb{C}_H^n$ with $r(X) = q$ if and only if $A \succeq 0$ and $B = 0$, or $A \succeq 0$ and $q = 0$.

There exists an $X \in \mathbb{C}_H^n$ with $r(X) = q$ such that \( A + BXB^* < 0 \) if and only if $i_-(M) = m$ and $i_-(A) \geq m - q$, or $i_-(M) \geq m$ and $i_-(A) = m - q$.

$A + BXB^* \preceq 0$ for all $X \in \mathbb{C}_H^n$ with $r(X) = q$ if and only if $A \preceq 0$ and $B = 0$, or $A \preceq 0$ and $q = 0$.

$A + BXB^* \preceq 0$ for all $X \in \mathbb{C}_H^n$ with $r(X) = q$ if and only if $A \preceq 0$ and $B = 0$, or $A \preceq 0$ and $q = 0$.

\begin{equation}
X \in \mathbb{C}_H^n \quad \text{and} \quad r(X) = q \iff Y \in \mathbb{C}_H^n \quad \text{and} \quad r(Y) = q.
\end{equation}

In this setting, we derive from (4.7)–(4.9) that

\begin{align}
\max_{X \in \mathbb{C}_H^n, r(X) = q} r(A + BXB^*) &= 2f + k + \max_{Y \in \mathbb{C}_H^n, r(Y) = q} r(\hat{A} + \hat{Y}), \quad (4.35) \\
\min_{X \in \mathbb{C}_H^n, r(X) = q} r(A + BXB^*) &= 2f + k + \min_{Y \in \mathbb{C}_H^n, r(Y) = q} r(\hat{A} + \hat{Y}), \quad (4.36) \\
\max_{X \in \mathbb{C}_H^n, r(X) = q} i_+(A + BXB^*) &= f + s_2 + \max_{Y \in \mathbb{C}_H^n, r(Y) = q} i_+(\hat{A} + \hat{Y}), \quad (4.37) \\
\min_{X \in \mathbb{C}_H^n, r(X) = q} i_-(A + BXB^*) &= f + k - s_2 + \min_{Y \in \mathbb{C}_H^n, r(Y) = q} i_-(\hat{A} + \hat{Y}), \quad (4.38) \\
\max_{X \in \mathbb{C}_H^n, r(X) = q} i_-(A + BXB^*) &= f + k - s_2 + \max_{Y \in \mathbb{C}_H^n, r(Y) = q} i_-(\hat{A} + \hat{Y}), \quad (4.39) \\
\min_{X \in \mathbb{C}_H^n, r(X) = q} i_+(A + BXB^*) &= f + k - s_2 + \min_{Y \in \mathbb{C}_H^n, r(Y) = q} i_+(\hat{A} + \hat{Y}). \quad (4.40)
\end{align}

Also note from Lemma 2.7(b) and (4.5) that

\begin{equation}
Y \in \mathbb{C}_H^n \quad \text{and} \quad r(Y) = q \\
\iff Y_{11} = Y_{11}^*, \ Y_{44} = Y_{44}^*, \ \hat{Y} \in \mathbb{C}_H^n, \quad \text{and} \quad q - 2(n - l) \leq r(\hat{Y}) \leq q. \quad (4.41)
\end{equation}

Under $0 \leq q - 2(n - l) \leq q \leq l$, applying Lemma 3.1(c) to $\hat{A} + \hat{Y}$ in (4.35)–(4.40) and simplifying by (2.82) and (2.84), we obtain

\begin{equation}
\max_{Y \in \mathbb{C}_H^n, r(Y) = q} r(\hat{A} + \hat{Y}) = \max_{q - 2(n - l) \leq r(\hat{Y}) \leq q} r(\hat{A} + \hat{Y})
\end{equation}

\begin{equation}
= \min \{ l, r(\hat{A}) + q \} = \min \{ l, t + q \}, \quad (4.42)
\end{equation}
simplifying by (2.82) and (2.84), we obtain (4.22)–(4.27).

$$\min_{Y \in C_{H}, r(Y) = q} i_{+}(\hat{A} + \hat{Y}) = \min_{q - 2(n - l) \leq r(\hat{Y}) \leq q} i_{+}(\hat{A} + \hat{Y}) = \max \left\{ 0, \, q - 2(n - l) - r(\hat{A}), \, r(\hat{A}) - q \right\} = \max \left\{ 0, \, 2l - t + q - 2n, \, t - q \right\}, \quad (4.43)$$

$$\max_{Y \in C_{H}, r(Y) = q} i_{-}(\hat{A} + \hat{Y}) = \max_{q - 2(n - l) \leq r(\hat{Y}) \leq q} i_{-}(\hat{A} + \hat{Y}) = \min \left\{ l, \, i_{-}(\hat{A}) + q \right\} = \min \left\{ l, \, s_1 + q \right\}, \quad (4.44)$$

Substituting (4.42)–(4.47) into (4.35)–(4.40) and simplifying by (2.86)–(2.93), we obtain (4.22)–(4.27).

Under $q - 2(n - l) \leq 0$, applying Theorem 3.1(e) to $\hat{A} + \hat{Y}$ in (4.35)–(4.40) and simplifying by (2.82) and (2.84), we obtain

$$\max_{Y \in C_{H}, r(Y) = q} r(\hat{A} + \hat{Y}) = \max_{0 \leq r(\hat{Y}) \leq q} r(\hat{A} + \hat{Y}) = \min \left\{ l, \, r(\hat{A}) + q \right\} = \min \left\{ l, \, t + q \right\}, \quad (4.48)$$

$$\min_{Y \in C_{H}, r(Y) = q} r(\hat{A} + \hat{Y}) = \min_{0 \leq r(\hat{Y}) \leq q} r(\hat{A} + \hat{Y}) = \max \left\{ 0, \, r(\hat{A}) - q \right\} = \max \left\{ 0, \, t - q \right\}, \quad (4.49)$$

$$\max_{Y \in C_{H}, r(Y) = q} i_{+}(\hat{A} + \hat{Y}) = \max_{0 \leq r(\hat{Y}) \leq q} i_{+}(\hat{A} + \hat{Y}) = \min \left\{ l, \, i_{+}(\hat{A}) + q \right\} = \min \left\{ l, \, s_1 + q \right\}, \quad (4.50)$$

$$\min_{Y \in C_{H}, r(Y) = q} i_{+}(\hat{A} + \hat{Y}) = \min_{0 \leq r(\hat{Y}) \leq q} i_{+}(\hat{A} + \hat{Y}) = \max \left\{ 0, \, i_{+}(\hat{A}) - q \right\} = \max \left\{ 0, \, s_1 - q \right\}, \quad (4.51)$$

$$\max_{Y \in C_{H}, r(Y) = q} i_{-}(\hat{A} + \hat{Y}) = \max_{0 \leq r(\hat{Y}) \leq q} i_{-}(\hat{A} + \hat{Y}) = \min \left\{ l, \, i_{-}(\hat{A}) + q \right\} = \min \left\{ l, \, t - s_1 + q \right\}, \quad (4.52)$$

$$\min_{Y \in C_{H}, r(Y) = q} i_{-}(\hat{A} + \hat{Y}) = \min_{0 \leq r(\hat{Y}) \leq q} i_{-}(\hat{A} + \hat{Y}) = \max \left\{ 0, \, i_{-}(\hat{A}) - q \right\} = \max \left\{ 0, \, t - s_1 - q \right\}. \quad (4.53)$$
Under \(l \leq q\), applying Theorem 3.1(f) to \(\hat{A} + \hat{Y}\) in (4.35)–(4.40) and simplifying by (2.82), we obtain

\[
\begin{align*}
\max_{Y \in C^n_H, r(Y) = q} r(\hat{A} + \hat{Y}) &= \max_{q - 2(n - l) \leq r(\hat{Y}) \leq l} r(\hat{A} + \hat{Y}) = l, \quad (4.54) \\
\min_{Y \in C^n_H, r(Y) = q} r(\hat{A} + \hat{Y}) &= \min_{q - 2(n - l) \leq r(\hat{Y}) \leq l} r(\hat{A} + \hat{Y}) \\
&= \max \left\{ 0, q - 2(n - l) - r(\hat{A}) \right\} \\
&= \max \{0, 2l - t + q - 2n\}, \quad (4.55)
\end{align*}
\]

\[
\begin{align*}
\max_{Y \in C^n_H, r(Y) = q} i_+ (\hat{A} + \hat{Y}) &= \max_{q - 2(n - l) \leq r(\hat{Y}) \leq l} i_+ (\hat{A} + \hat{Y}) = l, \quad (4.56) \\
\min_{Y \in C^n_H, r(Y) = q} i_+ (\hat{A} + \hat{Y}) &= \min_{q - 2(n - l) \leq r(\hat{Y}) \leq l} i_+ (\hat{A} + \hat{Y}) = 0, \quad (4.57) \\
\max_{Y \in C^n_H, r(Y) = q} i_- (\hat{A} + \hat{Y}) &= \max_{q - 2(n - l) \leq r(\hat{Y}) \leq l} i_- (\hat{A} + \hat{Y}) = l, \quad (4.58) \\
\min_{Y \in C^n_H, r(Y) = q} i_- (\hat{A} + \hat{Y}) &= \min_{q - 2(n - l) \leq r(\hat{Y}) \leq l} i_- (\hat{A} + \hat{Y}) = 0. \quad (4.59)
\end{align*}
\]

These two groups of formulas are special cases (4.42)–(4.47), so that (4.22)–(4.27) hold for any \(q\) satisfying \(0 \leq q \leq n\).

\[\square\]

**Corollary 4.4.** Let \(\phi(X)\) and \(M\) be as given in (1.1) and (4.17), and assume that \(p\) and \(q\) are two integer satisfying \(0 \leq p \leq q \leq n\). Then, the following equalities hold

\[
\begin{align*}
\max_{X \in C^n_H, p \leq r(X) \leq q} r(A + BXB^*) &= \min \{ r(A, B), r(A) + q \}, \quad (4.60) \\
\min_{X \in C^n_H, p \leq r(X) \leq q} r(A + BXB^*) &= \min \{ u_p, u_{p+1}, \ldots, u_q \}, \quad (4.61) \\
\max_{X \in C^n_H, p \leq r(X) \leq q} i_+ (A + BXB^*) &= \min \{ i_+ (M), i_+ (A) + q \}, \quad (4.62) \\
\min_{X \in C^n_H, p \leq r(X) \leq q} i_+ (A + BXB^*) &= \max \{ r(A, B) - i_- (M), i_+ (A) - q \}, \quad (4.63) \\
\max_{X \in C^n_H, p \leq r(X) \leq q} i_- (A + BXB^*) &= \min \{ i_- (M), i_- (A) + q \}, \quad (4.64) \\
\min_{X \in C^n_H, p \leq r(X) \leq q} i_- (A + BXB^*) &= \max \{ r(A, B) - i_- (M), i_- (A) - q \}. \quad (4.65)
\end{align*}
\]

where

\[
\begin{align*}
u_p &= \max \{ 2r[A, B] - r(M), 2r[A, B] - r(A) + p - 2n, r(A) - p \}, \\
u_{p+1} &= \max \{ 2r[A, B] - r(M), 2r[A, B] - r(A) + p + 1 - 2n, r(A) - p - 1 \}, \\
\vdots & \quad \vdots \\
u_q &= \max \{ 2r[A, B] - r(M), 2r[A, B] - r(A) + q - 2n, r(A) - q \}.
\end{align*}
\]

In consequence, the following results hold.

(i) There exists an \(X \in C^n_H\) with \(p \leq r(X) \leq q\) such that \(A + BXB^*\) is nonsingular if and only if \(r(A, B) = m\) and \(r(A) \geq m - q\).

(ii) \(A + BXB^*\) is nonsingular for all \(X \in C^n_H\) with \(p \leq r(X) \leq q\) if and only if \(u_p = u_{p+1} = \ldots = u_q = m\).

(iii) There exists an \(X \in C^n_H\) with \(p \leq r(X) \leq q\) such that \(A + BXB^* = 0\) if and only if \(\mathcal{X}(A) \subseteq \mathcal{R}(B)\) and \(r(A) \leq p \leq q \leq r(A) - 2r(B) + 2n\).
(iv) There exists an \( X \in C^n_H \) with \( p \leq r(X) \leq q \) such that \( A + BXB^* \succ 0 \) if and only if \( i_+(M) = m \) and \( i_+(A) \geq m - q \), or \( i_+(M) \geq m \) and \( i_+(A) = m - q \).

(v) \( A + BXB^* \succ 0 \) for all \( X \in C^n_H \) with \( p \leq r(X) \leq q \) if and only if \( A \succ 0 \) and \( p = q = 0 \).

(vi) There exists an \( X \in C^n_H \) with \( p \leq r(X) \leq q \) such that \( A + BXB^* \succeq 0 \) if and only if \( i_+(M) = r[ A, B ] \) and \( i_-(A) \leq q \).

(vii) \( A + BXB^* \succeq 0 \) for all \( X \in C^n_H \) with \( p \leq r(X) \leq q \) if and only if \( A \succeq 0 \) and \( B = 0 \), or \( A \succeq 0 \) and \( p = q = 0 \).

(viii) There exists an \( X \in C^n_H \) with \( p \leq r(X) \leq q \) such that \( A + BXB^* \prec 0 \) if and only if \( i_+(M) = m \) and \( i_-(A) \geq m - q \), or \( i_+(M) \geq m \) and \( i_-(A) = m - q \).

(ix) \( A + BXB^* \prec 0 \) for all \( X \in C^n_H \) with \( p \leq r(X) \leq q \) if and only if \( A \prec 0 \) and \( p = q = 0 \).

(x) There exists an \( X \in C^n_H \) with \( p \leq r(X) \leq q \) such that \( A + BXB^* \preceq 0 \) if and only if \( i_+(M) = r[ A, B ] \) and \( i_+(A) \leq q \).

(xi) \( A + BXB^* \preceq 0 \) for all \( X \in C^n_H \) with \( p \leq r(X) \leq q \) if and only if \( A \preceq 0 \) and \( B = 0 \), or \( A \preceq 0 \) and \( p = q = 0 \).

**Proof.** Note from (1.18) that

\[
\max_{X \in C^n_H, p \leq r(X) \leq q} r(A + BXB^*) = \max \left\{ \max_{X \in C^n_H, r(X) = p} r(A + BXB^*), \right. \\
\left. \max_{X \in C^n_H, r(X) = p+1} r(A + BXB^*), \ldots, \max_{X \in C^n_H, r(X) = q} r(A + BXB^*) \right\}, \quad (4.66)
\]

and

\[
\min_{X \in C^n_H, p \leq r(X) \leq q} r(A + BXB^*) = \min \left\{ \min_{X \in C^n_H, r(X) = p} r(A + BXB^*), \right. \\
\left. \min_{X \in C^n_H, r(X) = p+1} r(A + BXB^*), \ldots, \min_{X \in C^n_H, r(X) = q} r(A + BXB^*) \right\}, \quad (4.67)
\]

Furthermore,

\[
\max_{X \in C^n_H, p \leq r(X) \leq q} i_+(A + BXB^*) = \max \left\{ \max_{X \in C^n_H, r(X) = p} i_+(A + BXB^*), \right. \\
\left. \max_{X \in C^n_H, r(X) = p+1} i_+(A + BXB^*), \ldots, \max_{X \in C^n_H, r(X) = q} i_+(A + BXB^*) \right\}, \quad (4.68)
\]

and

\[
\min_{X \in C^n_H, p \leq r(X) \leq q} i_+(A + BXB^*) = \min \left\{ \min_{X \in C^n_H, r(X) = p} i_+(A + BXB^*), \right. \\
\left. \min_{X \in C^n_H, r(X) = p+1} i_+(A + BXB^*), \ldots, \min_{X \in C^n_H, r(X) = q} i_+(A + BXB^*) \right\}, \quad (4.69)
\]

Substituting (4.22)–(4.27) for \( r(X) = p, p+1, \ldots, q \) into (4.66)–(4.71) and making max-min comparisons, we obtain (4.60)–(4.65).
Corollary 4.5. Let \( \phi(X) \) and \( M \) be as given in (1.1) and (4.17), and assume that \( p \) is an integer satisfying \( 0 \leq p \leq n \). Then, the following equalities hold
\[
\begin{align*}
\max_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq n} r(A + BXB^*) &= r[A, B], \\
\min_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq n} r(A + BXB^*) &= \min\{u_p, u_{p+1}, \ldots, u_n\}, \\
\max_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq n} i_+(A + BXB^*) &= i_+(M), \\
\min_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq n} i_+(A + BXB^*) &= r[A, B] - i_-(M), \\
\max_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq n} i_-(A + BXB^*) &= i_-(M), \\
\min_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq n} i_-(A + BXB^*) &= r[A, B] - i_+(M),
\end{align*}
\]

where
\[
\begin{align*}
u_p &= \max\{2r[A, B] - r(M), 2r[A, B] - r(A) + p - 2n, r(A) - p\}, \\
u_{p+1} &= \max\{2r[A, B] - r(M), 2r[A, B] - r(A) + p + 1 - 2n, r(A) - p - 1\}, \\
&\vdots \\
u_n &= \max\{2r[A, B] - r(M), 2r[A, B] - r(A) - n\}.
\end{align*}
\]

Corollary 4.6. Let \( \phi(X) \) and \( M \) be as given in (1.1) and (4.17). Then,
\[
\begin{align*}
\max_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq q} r(A + BXB^*) &= \min\{r[A, B], r(A) + q\}, \\
\min_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq q} r(A + BXB^*) &= \min\{u_0, u_1, \ldots, u_q\}, \\
\max_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq q} i_+(A + BXB^*) &= \min\{i_+(M), i_+(A) + q\}, \\
\min_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq q} i_+(A + BXB^*) &= \max\{r[A, B] - i_-(M), i_+(A) - q\}, \\
\max_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq q} i_-(A + BXB^*) &= \min\{i_-(M), i_-(A) + q\}, \\
\min_{X \in \mathbb{C}^{n \times n}, \ 0 \leq r(X) \leq q} i_-(A + BXB^*) &= \max\{r[A, B] - i_+(M), i_-(A) - q\},
\end{align*}
\]

where
\[
\begin{align*}
u_0 &= \max\{2r[A, B] - r(M), 2r[A, B] - r(A) - 2n, r(A)\}, \\
u_1 &= \max\{2r[A, B] - r(M), 2r[A, B] - r(A) + 1 - 2n, r(A) - 1\}, \\
&\vdots \\
u_q &= \max\{2r[A, B] - r(M), 2r[A, B] - r(A) + q - 2n, r(A) - q\}.
\end{align*}
\]

5. Rank/inertia formulas of \( A \pm BXB^* \) when \( X \) is positive semi-definite matrix.

Theorem 5.1. Let \( A \in \mathbb{C}^{m \times n}_H \), \( B \in \mathbb{C}^{m \times n} \) be given, \( M \) be the matrix in (4.17), \( X \in \mathbb{C}^{n \times n}_H \) be a variable matrix, and assume that \( q \) is an integer satisfying \( 0 \leq q \leq n \). Then, the following results hold.

(a) The rank/inertia of \( A + BXB^* \) satisfy
\[
\max_{0 \leq X, r(X) = q} r(A + BXB^*) = \min\{r[A, B], r(A) + q\},
\]
\[ \begin{align*}
&\min_{0 \leq X, r(X) = q} \quad r(A + BXB^*) = \max\{i_+(A) + r[A, B] - i_+(M), \\
&\quad 2r(A, B) - i_-(A) - i_+(M) - n + q, \ r(A) - q\}, \\
&\max_{0 \leq X, r(X) = q} \quad i_+(A + BXB^*) = \min\{i_+(M), \ i_+(A) + q\}, \\
&\min_{0 \leq X, r(X) = q} \quad i_+(A + BXB^*) = \max\{i_+(A), \ r(A, B) - i_-(A) - n + q\}, \\
&\max_{0 \leq X, r(X) = q} \quad i_-(A + BXB^*) = i_-(A), \\
&\min_{0 \leq X, r(X) = q} \quad i_-(A + BXB^*) = \max\{r[A, B] - i_+(M), \ i_-(A) - q\}.
\end{align*} \] 

In particular,

\[ \begin{align*}
&\max_{X > 0} r(A + BXB^*) = r[A, B], \\
&\min_{X > 0} r(A + BXB^*) = 2r[A, B] - i_-(A) - i_+(M), \\
&\max_{X > 0} i_+(A + BXB^*) = i_+(M), \\
&\min_{X > 0} i_+(A + BXB^*) = r[A, B] - i_-(A), \\
&\max_{X > 0} i_-(A + BXB^*) = i_-(A), \\
&\min_{X > 0} i_-(A + BXB^*) = r[A, B] - i_+(M).
\end{align*} \]

In consequence, the following results hold.

(i) There exists a \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A + BXB^* \) is nonsingular if and only if \( r[A, B] = m \) and \( r(A) \geq m - q \).

(ii) \( A + BXB^* \) is nonsingular for all \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( i_+(M) = i_+(A) + r[A, B] - m \), or \( 2r[A, B] = i_-(A) + i_+(M) + m + n - q \), or \( r(A) = m \) and \( q = 0 \).

(iii) There exists a \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A + BXB^* \neq 0 \) if and only if \( A \ll 0 \), \( \mathscr{A}(A) \subseteq \mathscr{R}(B) \) and \( n - r(B) \geq q \), \( r(A) \geq 0 \).

(iv) There exists a \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A + BXB^* \neq 0 \) if and only if \( i_+(M) = m \) and \( i_+(A) \geq m - q \), or \( i_+(M) \geq m \) and \( i_+(A) = m - q \).

(v) \( A + BXB^* \neq 0 \) holds for all \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A > 0 \) or \( r[A, B] = i_-(A) + m + n - q \).

(vi) There exists a \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A + BXB^* \neq 0 \) if and only if \( i_+(M) = r[A, B] \) and \( i_-(A) \leq q \).

(vii) \( A + BXB^* \neq 0 \) holds for all \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A > 0 \).

(viii) There exists a \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A + BXB^* \neq 0 \) if and only if \( A \ll 0 \).

(ix) \( A + BXB^* \neq 0 \) holds for all \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A \ll 0 \) and \( B = 0 \), or \( A \ll 0 \) and \( q = 0 \).

(x) There exists a \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A + BXB^* \leq 0 \) if and only if \( A \ll 0 \) and \( r[A, B] \leq r(A) + n - q \).

(xi) \( A + BXB^* \leq 0 \) holds for all \( 0 \ll X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A \ll 0 \) and \( B = 0 \), or \( A \ll 0 \) and \( q = 0 \).

(b) The rank/inertia of \( A - BXB^* \) satisfy

\[ \max_{0 \leq X, r(X) = q} r(A - BXB^*) = \min\{r[A, B], \ r(A) + q\}. \]
\[ \min_{0 \leq X, r(X) = q} r(A - BXB^*) = \max\{i_-(A) + r(A, B) - i_-(M), 2r[A, B] - i_+(A) - i_-(M) - n + q, r(A) - q\}, \]  
(5.14)
\[ \max_{0 \leq X, r(X) = q} i_+(A - BXB^*) = i_+(A), \]  
(5.15)
\[ \min_{0 \leq X, r(X) = q} i_+(A - BXB^*) = \max\{r[A, B] - i_-(M), i_+(A) - q\}, \]  
(5.16)
\[ \max_{0 \leq X, r(X) = q} i_-(A - BXB^*) = \min\{i_-(M), i_-(A) + q\}, \]  
(5.17)
\[ \min_{0 \leq X, r(X) = q} i_-(A - BXB^*) = \max\{i_-(A), r[A, B] - i_+(A) - n + q\}. \]  
(5.18)

In particular,
\[ \max_{X > 0} r(A - BXB^*) = r[A, B], \]  
(5.19)
\[ \min_{X > 0} r(A - BXB^*) = 2r[A, B] - i_+(A) - i_-(M), \]  
(5.20)
\[ \max_{X > 0} i_+(A - BXB^*) = i_+(A), \]  
(5.21)
\[ \min_{X > 0} i_+(A - BXB^*) = r[A, B] - i_-(M), \]  
(5.22)
\[ \max_{X > 0} i_-(A - BXB^*) = i_-(M), \]  
(5.23)
\[ \min_{X > 0} i_-(A - BXB^*) = r[A, B] - i_+(A). \]  
(5.24)

In consequence, the following results hold.
(i) There exists a \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A - BXB^* \) is nonsingular if and only if \( r[A, B] = m \) and \( r(A) \geq m - q \).
(ii) \( A - BXB^* \) is nonsingular for all \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( i_-(M) = i_-(A) + r[A, B] - m \), or \( 2r[A, B] = i_+(A) + i_-(M) + m + n - q \), or \( r(A) = m \) and \( q = 0 \).
(iii) There exists a \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A - BXB^* \neq 0 \) if and only if \( A \gg 0 \), \( \mathcal{R}(A) \subseteq \mathcal{R}(B) \) and \( n - r(B) \geq q - r(A) \).
(iv) There exists a \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A - BXB^* \succ 0 \) if and only if \( A > 0 \).
(v) \( A - BXB^* \succ 0 \) holds for all \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A > 0 \) and \( B = 0 \), or \( A \succ 0 \) and \( q = 0 \).
(vi) There exists a \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A - BXB^* \gg 0 \) if and only if \( A \gg 0 \) and \( r[A, B] \leq r(A) + n - q \).
(vii) \( A - BXB^* \succeq 0 \) holds for all \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A > 0 \) and \( B = 0 \), or \( A \succ 0 \) and \( q = 0 \).
(viii) There exists a \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A - BXB^* \prec 0 \) if and only if \( i_-(M) = m \) and \( i_-(A) \geq m - q \), or \( i_-(M) \geq m \) and \( i_-(A) = m - q \).
(ix) \( A - BXB^* \prec 0 \) holds for all \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A \prec 0 \) or \( r[A, B] = i_+(A) + m + n - q \).
(x) There exists a \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) such that \( A - BXB^* \preceq 0 \) if and only if \( r[A, B] = i_-(M) \) and \( i_+(A) \leq q \).
(xi) \( A - BXB^* \preceq 0 \) holds for all \( 0 \leq X \in \mathbb{C}^n_H \) with \( r(X) = q \) if and only if \( A \preceq 0 \).

Proof. It is obvious from (4.1) and (4.2) that
\[ 0 \leq X \quad \text{and} \quad r(X) = q \Leftrightarrow 0 \leq Y \quad \text{and} \quad r(Y) = q. \]  
(5.25)
In this case, we derive from (4.7)–(4.9) that
\[
\max_{0 \leq X, r(X)=q} r(A + XB^*) = 2f + k + \max_{0 \leq Y, r(Y)=q} r(\hat{A} + \hat{Y}), \tag{5.26}
\]
\[
\min_{0 \leq X, r(X)=q} r(A + XB^*) = 2f + k + \min_{0 \leq Y, r(Y)=q} r(\hat{A} + \hat{Y}), \tag{5.27}
\]
\[
\max_{0 \leq X, r(X)=q} i_+(A + XB^*) = f + s_2 + \max_{0 \leq Y, r(Y)=q} i_+(\hat{A} + \hat{Y}), \tag{5.28}
\]
\[
\min_{0 \leq X, r(X)=q} i_+(A + XB^*) = f + s_2 + \min_{0 \leq Y, r(Y)=q} i_+(\hat{A} + \hat{Y}), \tag{5.29}
\]
\[
\max_{0 \leq X, r(X)=q} i_-(A + XB^*) = f + k - s_2 + \max_{0 \leq Y, r(Y)=q} i_-(\hat{A} + \hat{Y}), \tag{5.30}
\]
\[
\min_{0 \leq X, r(X)=q} i_-(A + XB^*) = f + k - s_2 + \min_{0 \leq Y, r(Y)=q} i_-(\hat{A} + \hat{Y}). \tag{5.31}
\]
Also note from Lemma 2.7(q) and (4.5) that
\[
0 \leq Y \text{ and } r(Y) = q \iff 0 \leq \hat{Y}, \; q - (n - l) \leq r(\hat{Y}) \leq q,
\]
\[
\mathcal{B}\left[\begin{array}{cc}
Y_{12} & Y_{24} \\
Y_{13} & Y_{34}
\end{array}\right] \subseteq \mathcal{B}(\hat{Y}), \text{ and } \left[\begin{array}{cccc}
Y_{11} & Y_{14} \\
Y_{14} & Y_{44}
\end{array}\right] - \left[\begin{array}{cccc}
Y_{12} & Y_{13} \\
Y_{24} & Y_{34}
\end{array}\right] \hat{Y}^1 \left[\begin{array}{cccc}
Y_{12} & Y_{24} \\
Y_{13} & Y_{34}
\end{array}\right] = 0.
\tag{5.32}
\]
Applying (5.32) and Theorem 3.2(c) to \(\hat{A} + \hat{Y}\) in (5.26)–(5.31) and simplifying by
(2.84), we obtain
\[
\max_{0 \leq Y, r(Y)=q} r(\hat{A} + \hat{Y}) = \max_{0 \leq \hat{Y}, q - (n - l) \leq r(\hat{Y}) \leq q} r(\hat{A} + \hat{Y})
= \min\{t, \; r(\hat{A}) + q\} = \min\{l, \; t + q\}, \tag{5.33}
\]
\[
\min_{0 \leq Y, r(Y)=q} r(\hat{A} + \hat{Y}) = \min_{0 \leq \hat{Y}, q - (n - l) \leq r(\hat{Y}) \leq q} r(\hat{A} + \hat{Y})
= \max\{i_+(\hat{A}), \; q - (n - l) - i_-(\hat{A}), \; r(\hat{A}) - q\}
= \max\{s_1, \; q + l - n - t + s_1, \; t - q\}, \tag{5.34}
\]
\[
\max_{0 \leq Y, r(Y)=q} i_+(\hat{A} + \hat{Y}) = \max_{0 \leq \hat{Y}, q - (n - l) \leq r(\hat{Y}) \leq q} i_+(\hat{A} + \hat{Y})
= \min\{t, \; i_+(\hat{A}) + q\} = \min\{l, \; s_1 + q\}, \tag{5.35}
\]
\[
\min_{0 \leq Y, r(Y)=q} i_+(\hat{A} + \hat{Y}) = \min_{0 \leq \hat{Y}, q - (n - l) \leq r(\hat{Y}) \leq q} i_+(\hat{A} + \hat{Y})
= \max\{i_+(\hat{A}), \; q - (n - l) - i_-(\hat{A})\}
= \max\{s_1, \; q + l - n - t + s_1\}, \tag{5.36}
\]
\[
\max_{0 \leq Y, r(Y)=q} i_-(\hat{A} + \hat{Y}) = \max_{0 \leq \hat{Y}, q - (n - l) \leq r(\hat{Y}) \leq q} i_-(\hat{A} + \hat{Y})
= i_-(\hat{A}) = t - s_1, \tag{5.37}
\]
\[
\min_{0 \leq Y, r(Y)=q} i_-(\hat{A} + \hat{Y}) = \min_{0 \leq \hat{Y}, q - (n - l) \leq r(\hat{Y}) \leq q} i_-(\hat{A} + \hat{Y})
= \max\{0, \; i_-(\hat{A}) - q\} = \max\{0, \; t - s_1 - q\}. \tag{5.38}
\]
Substituting (5.33)–(5.38) into (5.26)–(5.31) and simplifying (2.86)–(2.93), we obtain (5.1)–(5.6). Note that

\[ r(A - BXB^*) = r(-A + BXB^*), \quad i_+(A - BXB^*) = i_+(-A + BXB^*). \]

Thus, applying (5.1)–(5.6) to the right-hand sides of the three equalities gives (5.13)–(5.15).

Applying Lemma 2.1 to (5.1)–(5.6) leads to the results in (i)–(xi) of (a); and to (5.13)–(5.18) leads to the results in (i)–(xi) of (b).

**Corollary 5.2.** Let \( A \in \mathbb{C}_n^m \) and \( B \in \mathbb{C}_{m \times n}^n \) be given, \( M \) be the matrix in (4.17), \( X \in \mathbb{C}_n^m \) a variable matrix, and assume that \( p \) and \( q \) are two integers satisfying \( 0 \leq p \leq q \leq n \). Then, the following results hold.

(a) The rank/inertia of \( A + BXB^* \) satisfy

\[
\begin{align*}
\max_{0 \leq X, p \leq r(X) \leq q} r(A + BXB^*) & = \min \{ r[A, B], \ r(A) + q \}, \\
\min_{0 \leq X, p \leq r(X) \leq q} r(A + BXB^*) & = \min \{ u_p, u_{p+1}, \ldots, u_q \}, \\
\max_{0 \leq X, p \leq r(X) \leq q} i_+(A + BXB^*) & = \min \{ i_+(M), \ i_+(A) + q \}, \\
\min_{0 \leq X, p \leq r(X) \leq q} i_+(A + BXB^*) & = \max \{ i_+(A), \ r[A, B] - i_-(A) - n + p \}, \\
\max_{0 \leq X, p \leq r(X) \leq q} i_-(A + BXB^*) & = i_-(A), \\
\min_{0 \leq X, p \leq r(X) \leq q} i_-(A + BXB^*) & = \max \{ r[A, B] - i_+(M), \ i_-(A) - q \},
\end{align*}
\]

where

\[
\begin{align*}
u_p & = \max \{ i_+(A) + r[A, B] - i_+(M), \ 2r[A, B] - i_-(A) - i_+(M) - n + p, \\
r(A) - p \}, \\
u_{p+1} & = \max \{ i_+(A) + r[A, B] - i_+(M), \ 2r[A, B] - i_-(A) - i_+(M) - n + p + 1, \\
r(A) - p - 1 \}, \\
& \vdots \\
u_q & = \max \{ i_+(A) + r[A, B] - i_+(M), \ 2r[A, B] - i_-(A) - i_+(M) - n + q, \\
r(A) - q \}.
\end{align*}
\]

(b) The rank/inertia of \( A - BXB^* \) satisfy

\[
\begin{align*}
\max_{0 \leq X, p \leq r(X) \leq q} r(A - BXB^*) & = \min \{ r[A, B], \ r(A) + q \}, \\
\min_{0 \leq X, p \leq r(X) \leq q} r(A - BXB^*) & = \min \{ v_p, v_{p+1}, \ldots, v_q \}, \\
\max_{0 \leq X, p \leq r(X) \leq q} i_+(A - BXB^*) & = i_+(A), \\
\min_{0 \leq X, p \leq r(X) \leq q} i_+(A - BXB^*) & = \max \{ r[A, B] - i_-(M), \ i_+(A) - q \}, \\
\max_{0 \leq X, p \leq r(X) \leq q} i_-(A - BXB^*) & = \min \{ i_-(M), \ i_-(A) + q \}, \\
\min_{0 \leq X, p \leq r(X) \leq q} i_-(A - BXB^*) & = \max \{ i_-(A), \ r[A, B] - i_+(A) - n + p \},
\end{align*}
\]

where

\[
\begin{align*}
v_p & = \max \{ i_-(A) + r[A, B] - i_-(M), \ 2r[A, B] - i_+(A) - i_-(M) - n + p, \\
r(A) - q \}.
\end{align*}
\]
\begin{align*}
  \text{rank}(A) - p, \\
v_{p+1} = \max\{i_-(A) + r[A, B] - i_-(M), \ 2r[A, B] - i_+(A) - i_-(M) - n + p + 1, \\
  \text{rank}(A) - p - 1\}, \\
\vdots \\
v_q = \max\{i_-(A) + r[A, B] - i_-(M), \ 2r[A, B] - i_+(A) - i_-(M) - n + q, \\
  \text{rank}(A) - q\}.
\end{align*}

(c) The rank/inertia of $A + BXB^*$ satisfy

\begin{align*}
  \max_{0 \leq X, 0 \leq r(X) \leq q} r(A + BXB^*) &= \min\{r[A, B], \ \text{rank}(A) + q\}, \\
  \min_{0 \leq X, 0 \leq r(X) \leq q} r(A + BXB^*) &= \min\{u_0, u_1, \ldots, u_q\}, \\
  \max_{0 \leq X, 0 \leq r(X) \leq q} i_+(A + BXB^*) &= \min\{i_+(M), i_+(A) + q\}, \\
  \min_{0 \leq X, 0 \leq r(X) \leq q} i_+(A + BXB^*) &= i_+(A), \\
  \max_{0 \leq X, 0 \leq r(X) \leq q} i_-(A + BXB^*) &= i_-(A), \\
  \min_{0 \leq X, 0 \leq r(X) \leq q} i_-(A + BXB^*) &= \max\{r[A, B] - i_+(M), i_-(A) - q\},
\end{align*}

where

\begin{align*}
  u_0 &= \max\{i_+(A) + r[A, B] - i_+(M), \ 2r[A, B] - i_-(A) - i_+(M) - n, \ \text{rank}(A)\}, \\
  u_1 &= \max\{i_+(A) + r[A, B] - i_+(M), \ 2r[A, B] - i_-(A) - i_+(M) - n + 1, \\
  \text{rank}(A) - 1\}, \\
\vdots \\
u_q &= \max\{i_+(A) + r[A, B] - i_+(M), \ 2r[A, B] - i_-(A) - i_+(M) - n + q, \\
  \text{rank}(A) - q\}.
\end{align*}

(d) The rank/inertia of $A - BXB^*$ satisfy

\begin{align*}
  \max_{0 \leq X, 0 \leq r(X) \leq q} r(A - BXB^*) &= \min\{r[A, B], \ \text{rank}(A) + q\}, \\
  \min_{0 \leq X, 0 \leq r(X) \leq q} r(A - BXB^*) &= \min\{v_0, v_1, \ldots, v_q\}, \\
  \max_{0 \leq X, 0 \leq r(X) \leq q} i_+(A - BXB^*) &= i_+(A), \\
  \min_{0 \leq X, 0 \leq r(X) \leq q} i_+(A - BXB^*) &= \max\{r[A, B] - i_-(M), i_+(A) - q\}, \\
  \max_{0 \leq X, 0 \leq r(X) \leq q} i_-(A - BXB^*) &= \min\{i_-(M), i_-(A) + q\}, \\
  \min_{0 \leq X, 0 \leq r(X) \leq q} i_-(A - BXB^*) &= i_-(A),
\end{align*}

where

\begin{align*}
  v_0 &= \max\{i_-(A) + r[A, B] - i_-(M), \ 2r[A, B] - i_+(A) - i_-(M) - n, \ \text{rank}(A)\}, \\
  v_1 &= \max\{i_-(A) + r[A, B] - i_-(M), \ 2r[A, B] - i_+(A) - i_-(M) - n + 1, \\
  \text{rank}(A) - 1\}, \\
\vdots \\
v_q &= \max\{i_-(A) + r[A, B] - i_-(M), \ 2r[A, B] - i_+(A) - i_-(M) - n + q, \\
  \text{rank}(A) - q\}.
\end{align*}
The ranks/inertias of 
\[ A + B X B^* \] satisfy

\begin{align*}
\max_{0 \leq X, \ p \leq r(X) \leq n} r(A + B X B^*) &= r[A, B], \\
\min_{0 \leq X, \ p \leq r(X) \leq n} r(A + B X B^*) &= \min\{u_p, u_{p+1}, \ldots, u_n\}, \\
\max_{0 \leq X, \ p \leq r(X) \leq n} i_+(A + B X B^*) &= i_+(M), \\
\min_{0 \leq X, \ p \leq r(X) \leq n} i_+(A + B X B^*) &= \max\{i_+(A), r[A, B] - i_-(A) - n + p\}, \\
\max_{0 \leq X, \ p \leq r(X) \leq n} i_-(A + B X B^*) &= i_-(A), \\
\min_{0 \leq X, \ p \leq r(X) \leq n} i_-(A + B X B^*) &= r[A, B] - i_+(M),
\end{align*}

where

\begin{align*}
\{u_p &= \max\{i_+(A) + r[A, B] - i_+(M), 2r[A, B] - i_-(A) - i_+(M) - n + p, \\
r(A) - p\}, \\
u_{p+1} &= \max\{i_+(A) + r[A, B] - i_+(M), 2r[A, B] - i_-(A) - i_+(M) - n + p + 1, \\
r(A) - p - 1\}, \\
\vdots
\end{align*}

\begin{align*}
\{u_n &= \max\{i_+(A) + r[A, B] - i_+(M), 2r[A, B] - i_-(A) - i_+(M), r(A) - n\}. \\
\end{align*}

The ranks/inertias of 
\[ A - B X B^* \] satisfy

\begin{align*}
\max_{0 \leq X, \ p \leq r(X) \leq n} r(A - B X B^*) &= r[A, B], \\
\min_{0 \leq X, \ p \leq r(X) \leq n} r(A - B X B^*) &= \min\{v_p, v_{p+1}, \ldots, v_n\}, \\
\max_{0 \leq X, \ p \leq r(X) \leq n} i_+(A - B X B^*) &= i_+(A), \\
\min_{0 \leq X, \ p \leq r(X) \leq n} i_+(A - B X B^*) &= r[A, B] - i_-(M), \\
\max_{0 \leq X, \ p \leq r(X) \leq n} i_-(A - B X B^*) &= i_-(M), \\
\min_{0 \leq X, \ p \leq r(X) \leq n} i_-(A - B X B^*) &= \max\{i_-(A), r[A, B] - i_+(A) - n + p\},
\end{align*}

where

\begin{align*}
\{v_p &= \max\{i_-(A) + r[A, B] - i_-(M), 2r[A, B] - i_+(A) - i_-(M) - n + p, \\
r(A) - p\}, \\
v_{p+1} &= \max\{i_-(A) + r[A, B] - i_-(M), 2r[A, B] - i_+(A) - i_-(M) - n + p + 1, \\
r(A) - p - 1\}, \\
\vdots
\end{align*}

\begin{align*}
\{v_n &= \max\{i_-(A) + r[A, B] - i_-(M), 2r[A, B] - i_+(A) - i_-(M), r(A) - n\}'. \\
\end{align*}

The ranks/inertias of 
\[ A \pm B X X^* B^* \] satisfy

\begin{align*}
\max_{X \in \mathbb{C}^{n \times n}} r(A + B X X^* B^*) &= r[A, B], \\
\min_{X \in \mathbb{C}^{n \times n}} r(A + B X X^* B^*) &= i_+(A) + r[A, B] - i_+(M),
\end{align*}
\[
\max_{X \in \mathbb{C}^{n \times n}} (A + BXX^*B^*) = i_+(M), \quad (5.77)
\]
\[
\min_{X \in \mathbb{C}^{n \times n}} (A + BXX^*B^*) = i_+(A), \quad (5.78)
\]
\[
\max_{X \in \mathbb{C}^{n \times n}} i_-(A + BXX^*B^*) = i_-(A), \quad (5.79)
\]
\[
\min_{X \in \mathbb{C}^{n \times n}} i_-(A + BXX^*B^*) = r[A, B] - i_+(M), \quad (5.80)
\]
\[
\max_{X \in \mathbb{C}^{n \times n}} r(A - BXX^*B^*) = r[A, B], \quad (5.81)
\]
\[
\min_{X \in \mathbb{C}^{n \times n}} r(A - BXX^*B^*) = i_-(A) + r[A, B] - i_-(M), \quad (5.82)
\]
\[
\max_{X \in \mathbb{C}^{n \times n}} i_+(A - BXX^*B^*) = i_+(A), \quad (5.83)
\]
\[
\min_{X \in \mathbb{C}^{n \times n}} i_+(A - BXX^*B^*) = r[A, B] - i_-(M), \quad (5.84)
\]
\[
\max_{X \in \mathbb{C}^{n \times n}} i_-(A - BXX^*B^*) = i_-(M), \quad (5.85)
\]
\[
\min_{X \in \mathbb{C}^{n \times n}} i_-(A - BXX^*B^*) = i_-(A). \quad (5.86)
\]

**Proof.** Note from (1.19) that

\[
\max_{0 \leq X, p \leq r(X) \leq q} r(A + BXB^*) = \max \left\{ \max_{0 \leq X, r(X) = p} r(A + BXB^*), \max_{0 \leq X, r(X) = p+1, \ldots, q} r(A + BXB^*) \right\}, \quad (5.87)
\]
\[
\min_{0 \leq X, p \leq r(X) \leq q} r(A + BXB^*) = \min \left\{ \min_{0 \leq X, r(X) = p} r(A + BXB^*), \min_{0 \leq X, r(X) = p+1, \ldots, q} r(A + BXB^*) \right\}, \quad (5.88)
\]
\[
\max_{0 \leq X, p \leq r(X) \leq q} i_+(A + BXB^*) = \max \left\{ \max_{0 \leq X, r(X) = p} i_+(A + BXB^*), \max_{0 \leq X, r(X) = p+1, \ldots, q} i_+(A + BXB^*) \right\}, \quad (5.89)
\]
\[
\min_{0 \leq X, p \leq r(X) \leq q} i_+(A + BXB^*) = \min \left\{ \min_{0 \leq X, r(X) = p} i_+(A + BXB^*), \min_{0 \leq X, r(X) = p+1, \ldots, q} i_+(A + BXB^*) \right\}, \quad (5.90)
\]
\[
\max_{0 \leq X, p \leq r(X) \leq q} i_-(A + BXB^*) = \max \left\{ \max_{0 \leq X, r(X) = p} i_-(A + BXB^*), \max_{0 \leq X, r(X) = p+1, \ldots, q} i_-(A + BXB^*) \right\}, \quad (5.91)
\]
\[
\min_{0 \leq X, p \leq r(X) \leq q} i_-(A + BXB^*) = \min \left\{ \min_{0 \leq X, r(X) = p} i_-(A + BXB^*), \min_{0 \leq X, r(X) = p+1, \ldots, q} i_-(A + BXB^*) \right\}. \quad (5.92)
\]

Applying (5.1)–(5.6) to (5.87)–(5.92) and simplifying yields (5.39)–(5.50). Results (c)–(g) are direct consequences of (5.39)–(5.50). \qed
In the past two decades, many numerical methods for solving rank minimization problems were developed. It seems, however, that there is no solid matrix theory to support the conclusions obtained by various approximation methods on the rank minimization problems. Because all formulas in the previous sections are given in analytical forms, people can utilize the results in this paper as test examples to check the validity of various numerical methods for solving rank minimization problems.

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