COVARIANT GAUGES AT FINITE TEMPERATURE

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Abstract

A prescription is presented for real-time finite-temperature perturbation theory in covariant gauges, in which only the two physical degrees of freedom of the gauge-field propagator acquire thermal parts. The propagators for the unphysical degrees of freedom of the gauge field, and for the Faddeev-Popov ghost field, are independent of temperature. This prescription is applied to the calculation of the one-loop gluon self-energy and the two-loop interaction pressure, and is found to be simpler to use than the conventional one.

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1. INTRODUCTION

In finite-temperature field theory, one calculates a grand partition function

\[ Z = \sum_i \langle i | e^{-\beta H} | i \rangle \]  

and thermal averages

\[ \langle Q \rangle = Z^{-1} \sum_i \langle i | e^{-\beta H} Q | i \rangle. \]

In both cases, the sum is over a complete orthonormal set of physical states. In scalar field theory, these states span the whole Hilbert space, and so one may replace the sums with traces of the operators. But in the case of a gauge theory the Hilbert space contains also unphysical states and so things are more complicated.

In this paper, we reconsider what to do about this for the case of covariant gauges. The traditional solution \[1\] is, in effect, to sum over all states of the gauge particles, and then cancel the unphysical contributions with ghosts. We shall argue that, for calculational purposes, it may be simpler to include in the sums (1.1) only the physical states from the start, and not to introduce unphysical terms that have to be cancelled.

The finite-temperature propagator consists of a vacuum piece, plus a “thermal” part. In real-time perturbation theory, which we consider initially, it is a \(2 \times 2\) matrix\(^2\). Its vacuum piece is

\[
iD^{\mu\nu}(x) = \begin{pmatrix} \langle 0|TA^\mu(x)A^\nu(0)|0 \rangle & \langle 0|A^{\mu(-i\sigma,0)}A^\nu(x)|0 \rangle \\ \langle 0|A^\mu(x^0-i\sigma,0)A^\nu(0)|0 \rangle & \langle 0| \bar{T}A^\mu(x)A^\nu(0)|0 \rangle \end{pmatrix}
\]

with \(\bar{T}\) the anti-time-ordering operator, and \(0 \leq \sigma \leq \beta\). There is a similar matrix ghost-field propagator. The elements of the thermal part of the matrix take account of all the possible states of the heat bath and involve the Bose distribution. In the traditional approach to covariant gauges, all the components of the gauge-field propagator have a thermal part, as also does the ghost propagator. We shall argue that it is simpler to give only the propagators for the physical degrees of freedom a thermal part, so that the unphysical components of the gauge field, and also the ghost field, have only vacuum propagators. In the next section, we give a derivation of this from the Gupta-Bleuler method of quantising the gauge field, though in a sense the result is obvious and can be written down without any long derivation.

This is because the thermal part of a propagator contains a \(\delta\)-function that puts the corresponding particle on shell. The thermal part represents the absorption of a particle from the heat bath or the emission of one into it. That is, the thermal part is directly associated with the particles in the physical states \(|i\rangle\) summed in (1.1). Consequently only the physical degrees of freedom acquire thermal parts for their propagation.

The thermal part of the propagator for the gauge field therefore has the tensor structure

\[
\sum_{\lambda=1}^{2} e_\lambda^\mu(k)e_\lambda^\nu(k)
\]

where \(e_\lambda(k)\) are polarisation vectors and \(k^2 = 0\). There is some freedom in the choice of these, but since the heat bath breaks the Lorentz symmetry already, the obvious choice is the one that causes no additional Lorentz-symmetry breaking. So we choose polarisation vectors orthogonal to the 4-velocity \(u\) of the heat bath, and then (1.3a) is

\[
T^{\mu\nu}(k,u) = -g^{\mu\nu} + \frac{k^\mu u^\nu + u^\mu k^\nu}{u.k} - u^2 \frac{k^\mu k^\nu}{(u.k)^2}.
\]

In the frame where the heat bath is at rest this vanishes if either \(\mu\) or \(\nu\) is zero, and its other elements are

\[
T^{ij}(k) = \delta^{ij} - \frac{k^i k^j}{k^2}.
\]
To summarise what we have said, part of the finite-temperature propagator of the gauge field is the vacuum piece, which in Feynman gauge reads

$$
-g^{\mu\nu} \left( \frac{1}{k^2 + i\epsilon} \left[ 1 + \frac{2\pi i \delta^{(-)}(k^2)}{(k^2 - i\epsilon)(k^2)} \right] + \frac{2\pi i \delta^{(+)}(k^2)}{(k^2 - i\epsilon)(k^2)} \right). 
$$ (1.4a)

We have not written explicitly the colour factor $\delta^{ab}$. We have made the particular choice $\sigma = 0$ in (1.2) so that the vacuum part of the propagator is temperature-independent; for other choices of $\sigma$ it depends on $\beta$, but in a simple way. To (1.4a) must be added the thermal piece of the propagator:

$$
-i T^{\mu\nu}(k) 2\pi \delta(k^2) n(|k_0|) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right), 
$$ (1.4b)

where $n(x) = 1/(e^{\beta x} - 1)$.

There is also the ghost propagator; this has only the vacuum part, which is just equal to the matrix that appears within (1.4a). We derive the propagator (1.4) in section 2 and generalise it to other gauges, both covariant and noncovariant.

In order to show that this prescription can be simpler than the conventional one, we use it in section 3 to calculate the gluon self-energy and the plasma pressure to lowest non-trivial order in the coupling. Section 4 is a discussion of our results. In an Appendix we introduce some alternative versions of the formalism in which the $2 \times 2$ propagator matrix is diagonal and temperature-independent, and the dependence on the temperature is instead transferred to the interaction vertices.

### 2. GUPTA-BLEULER THEORY AT FINITE TEMPERATURE

To evaluate the grand partition function (1.1a) or a thermal average (1.1b), we need to express the Hamiltonian in operator form and to identify the physical states. For this, we use standard Gupta-Bleuler theory. We go through the details only for the case of the Feynman gauge.

Start with the standard QCD Lagrangian $-\frac{1}{4}F^2$, with the addition of a gauge-fixing term but initially no ghost part:

$$
\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial.A)^2 
$$ (2.1)

The canonical momenta conjugate to the space components $A^i$ of the gauge field are

$$
\pi^i = F^{0i}, 
$$ (2.2a)

while that conjugate to $A^0$ is

$$
\pi^0 = -\partial.A. 
$$ (2.2b)

The Hamiltonian density is

$$
H = \pi^0 \dot{A}^0 + \pi^i \dot{A}^i - \mathcal{L} = H_0 + H^{\text{INT}}. 
$$ (2.3)

Here, $H_0$ is the part of $H$, expressed as a function of the fields and the canonical momenta with no explicit time derivatives of the fields, that survives if the coupling $g$ is set equal to zero, and

$$
H^{\text{INT}}(\pi, A) = -g \pi^i.A^0 \wedge A^i + \frac{i}{4}(F^{ij}F^{ij})^{\text{INT}}. 
$$ (2.4)

To derive perturbation theory, one introduces interaction-picture operators $Q_I(t) = \Lambda(t)Q(t)\Lambda^{-1}(t)$, where $\Lambda(t) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$. These operators evolve with time as free fields: they obey Hamilton’s equations of motion with Hamiltonian $H_{0I}$ and the relation between the interaction-picture canonical momenta and the fields is as in (2.2) but with the coupling $g$ set equal to 0:

$$
\pi^i_I = \dot{A}^i_I - \partial^i A^0_I, \quad \pi^0_I = -\partial.A_I. 
$$ (2.2c)
The interaction-picture field equations are just the zero-mass Klein-Gordon equations. Their solutions are

\[ A_I^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|k|} e^{-i k \cdot x} a^\mu(k) + \hbar c \quad (2.5) \]

where \( k^0 = |k| \). The canonical equal-time commutation relations \([A_I^\mu(t,x), \pi_I^\nu(t,y)] = i\delta^{\mu\nu} \delta^{(3)}(x - y)\)

etc give

\[ [a^\mu(k), a^{\nu\dagger}(k')] = -g^{\mu\nu} 2|k|(2\pi)^3 \delta^{(3)}(k - k'), \]

\[ [a^\mu(k), a^{\nu}(k')] = [a^{\mu\dagger}(k), a^{\nu\dagger}(k')] = 0. \quad (2.6) \]

We now have to identify the physical states. For finite values of \( t_0 \), the time at which the interaction picture coincides with the Heisenberg picture, the constraint that they satisfy is non-trivial. However, to derive zero-temperature perturbation theory, or finite-temperature field theory in the real-time formalism, one may switch off the gauge coupling adiabatically and work with the limit \( t_0 \to -\infty \). Then the interaction-picture states are just the \( in\)-states and the condition that picks out the physical states is the same as in abelian gauge theory. Its most general form is that matrix elements of \( \partial A_I \) vanish between physical states, but we may impose the stronger condition

\[ a^0(k) \mid \text{phys, } in \rangle = 0 = k.a(k) \mid \text{phys, } in \rangle. \quad (2.7) \]

In particular, the vacuum is required to satisfy these conditions. Together, (2.5), (2.6) and (2.7) make the vacuum expectation values of \( T \)-products of the fields take the usual Feynman-gauge-propagator forms.

The time-development operator for the interaction-picture states away from \( t = t_0 \) is \( \Lambda(t) \). By converting the differential equation it satisfies into an integral equation and iterating, one finds as usual that the \( S \)-matrix is a time-ordered exponential of an integral of \( H^{\text{INT}}(A_I, \pi_I) \). \( H^{\text{INT}} \) is given in (2.4). At this stage we may replace the \( \pi_I \) with their expressions (2.2c) in terms of \( A_I \) and \( \dot{A}_I \). This gives

\[ H^{\text{INT}} = -L^{\text{INT}}(A_I, \dot{A}_I) + \frac{1}{2} g^2 (A_I^0 \wedge A_I^i)^2. \quad (2.8) \]

The last term here compensates for the presence of \( g\dot{A}_I^i A_I^J \wedge A_I^i \) in the interaction Lagrangian. The Hamiltonian formalism of perturbation theory that we have outlined requires that, when the \( \dot{A}_I \) field propagates between two such neighbouring vertices one should use the propagator \( \langle 0|T\dot{A}_I^j(x_1)\dot{A}_I^k(x_2)|0 \rangle \). However, the usual Feynman rules use instead the double time derivative of the \( A_I \)-field propagator, where the two time differentiations are applied also to the \( \theta \)-functions that appear in the definition of the \( T \)-product. The last term in (2.8) is then simply omitted.

From the differential equation satisfied by \( \Lambda(t) \) it is straightforward to show that the \( S \)-matrix is unitary. However, this does not imply that probability is conserved, because the completeness relation includes the unphysical states. In order to achieve probability conservation one must add to the Lagrangian (2.1) a ghost part. This then guarantees that physical initial states do not scatter into unphysical final states. One may show this directly in the operator formalism, though we shall be content here to accept the usual Faddeev-Popov path integral arguments as justification.

We now extend the perturbation theory to nonzero temperature. The same operator \( \Lambda \) enters again, now with complex argument. Simple algebra gives

\[ Z = \sum_i \langle i | e^{-\beta H_{0I}} \Lambda(t_0 - i\beta)\Lambda^{-1}(t_0) | i \rangle \]

\[ \langle Q(t) \rangle = Z^{-1} \sum_i \langle i | e^{-\beta H_{0I}} \Lambda(t_0 - i\beta)\Lambda^{-1}(t)Q_I(t)\Lambda(t)\Lambda^{-1}(t_0) | i \rangle \quad (2.9) \]

where \( t_0 \) must be the time at which the interaction picture coincides with the Heisenberg picture and the sum is still over all physical states, which we now choose to be interaction-picture \( in \)-states.
In order now to derive the Feynman rules, it is necessary first to check that Wick's theorem may be used as usually in thermal field theory. For this, note that the physical-gluon operators commute with the unphysical gluon and ghost fields. Also, $H_{0I}$ is a sum of commuting parts containing respectively only physical and unphysical operators, $H_{0I} = H_{0I}^{\text{PHYS}} + H_{0I}^{\text{UNPHYS}}$, and $H_{0I}^{\text{UNPHYS}}$ has zero eigenvalue in the physical states. Hence any unphysical operators in the product of operators in (2.9) may be factorised out: only the vacuum expectation value of their product appears, and the normal zero-temperature Wick's theorem applies to this. For the remaining factor, we may replace $e^{-\beta H_{0I}}$ by $e^{-\beta H_{0I}^{\text{PHYS}}}$; the normal finite-temperature Wick's theorem applies to this factor.

So the real-time finite-temperature perturbation theory is much as usual\cite{2}, except that for the unphysical fields the propagators are just vacuum expectation values (1.2), both for the unphysical components of $A$ and for the ghost.

The gauge-field propagator in Feynman gauge therefore reads

$$D_{\mu\nu} = -g_{\mu\nu} \left( e^{-\sigma k_0 \theta(k_0)} \left( \frac{\Delta_F}{\Delta_F - \Delta_F^*} \right) e^{\sigma k_0 \theta(-k_0)} \frac{\left( \Delta_F - \Delta_F^* \right)}{-\Delta_F^*} \right) + T_{\mu\nu} \left( \frac{1}{e^{\beta|k_0|} - 1} \left( \frac{1}{e^{-\sigma k_0}} e^{\sigma k_0} \right) \right)$$

(2.10)

with $0 \leq \sigma \leq \beta$, and $\Delta_F = 1/(k^2 + i\epsilon)$. The choice $\sigma = \beta/2$ is usually made on grounds that this makes the real-time propagator a symmetric $2 \times 2$ matrix. Here this is the case only for the transverse projection $T^{\mu\nu}D_{\mu\nu}T^{\sigma\nu}$. A more natural choice is $\sigma = 0$, for it renders the vacuum piece independent of $\beta$. In the Appendix, we discuss various options for the diagonalisation of (2.10).

Although the derivation was done in Feynman gauge, it seems obvious how the resulting propagator will read in other gauges. In a general linear gauge with quadratic gauge breaking term $\frac{1}{2}(A^\mu f_\mu f_\nu A^\nu)$, where $f_\mu$ is a 4-vector which is either constant or constructed from derivatives $\partial/\partial x$, the vacuum piece generalises by

$$g_{\mu\nu} \rightarrow G_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu f_\nu + f_\mu k_\nu}{f.k} + (f^2 - \xi k^2) \frac{k_\mu k_\nu}{(f.k)^2}.$$  

(2.11)

Here $f = f(ik)$, if the gauge breaking term contains derivatives. With $f_\mu = ik_\mu$, this reproduces the propagator in covariant gauges, while $f_\mu = (0, ik)$ corresponds to the Coulomb gauge. The ghost propagator is determined by $\Delta^{-1}_{\text{ghost}} = f.k$; its $2 \times 2$ matrix structure is analogous to the vacuum part of the gauge propagator. In most gauges, other than Feynman or Coulomb, an infinitesimal nonhermitian piece has to be included in $G_{\mu\nu}$ in order to give meaning to the denominators in (2.11). In this case, the vacuum part of $D_{\mu\nu}$ becomes

$$- \left( e^{-\sigma k_0 \theta(k_0)} \left( \frac{\Delta_F G_{\mu\nu}}{\Delta_F G_{\mu\nu} - \Delta_F^* G_{\mu\nu}^*} \right) e^{\sigma k_0 \theta(-k_0)} \frac{\left( \Delta_F G_{\mu\nu} - \Delta_F^* G_{\mu\nu}^* \right)}{-\Delta_F^* G_{\mu\nu}^*} \right).$$  

(2.12)

Although, strictly speaking, we have given a derivation only for the Feynman gauge, the results we have quoted agree with those derived by James for the $A_0 = 0$ gauge\cite{7}. For general covariant gauges, they agree with those in the literature\cite{8} if one sets the temperature to zero in the non-$T^{\mu\nu}$ part of the propagator.

3. CALCULATIONS

3.1. One-loop gluon self-energy

In a high-temperature expansion, the leading-temperature contributions arise from one-loop diagrams, and they determine the physics at the scale $gT$, where $g$ is the coupling constant. In particular, the one-loop self-energy determines the spectrum of quasiparticles in the plasma, and contains the information on screening and Landau damping. It has been shown by explicit calculations\cite{9} and also by an analysis of the relevant Ward identities\cite{10} that these “hard thermal loops”, as they have...
been called, are independent of the gauge conditions needed to define the gluon propagator. With the usual calculational procedure, this gauge independence arises in a rather nontrivial manner, and in particular in covariant gauges the contributions from the thermalised Faddeev-Popov ghosts are decisive. With our prescription, the latter are absent and it turns out to be much simpler to establish the gauge independence.

With the usual prescription for covariant gauges, the Feynman diagrams containing the hard-thermal-loop contributions to the gluon self-energy are those given in figure 1, where a slashed line denotes the thermal part of a propagator, and a bare line the vacuum part. With both vertices of type 1, loop contributions to the gluon self-energy are those given in figure 1, where a slashed line denotes the Feynman denominator, only the terms with highest degree in \( k \) may further simplify.

On the other hand, with the Feynman-gauge propagator (2.10) one obtains

\[
\Pi_{\mu\nu}^{(a)\text{CG}}(q) = g^2 N\int\frac{d^Dk}{(2\pi)^{(D-1)}}\delta(k^2)n(|k_0|) \left[ -(D-2)\frac{4k_\mu k_\nu - 2k_\mu q_\nu - 2q_\mu k_\nu}{(k-q)^2} - \frac{k_0^2}{(k-q)^2}T_{\mu\nu}(k) \right],
\]

where the principal value of the pole of the integrand is understood. Apart from the piece containing the Feynman denominator, only the terms with highest degree in \( k \) contribute. For this reason, one may further simplify \( k_0^2/(k-q)^2 \to 1 \) in the last term of (3.1).

On the other hand, with the Feynman-gauge propagator (2.10) one obtains

\[
\Pi_{\mu\nu}^{(a)}(q) = g^2 N\int\frac{d^Dk}{(2\pi)^{(D-1)}}\delta(k^2)n(|k_0|) \left[ -(D-2)\frac{4k_\mu k_\nu - 2k_\mu q_\nu - 2q_\mu k_\nu}{(k-q)^2} + \frac{(k+q)^2}{(k-q)^2}T_{\mu\nu}(k) \right].
\]

Because of the presence of the delta-function \( \delta(k^2) \), the numerator in the last term may be rewritten as \( (k+q)^2 \to -(k-q)^2 + 2q^2 \), and the \( q^2 \) dropped because it does not contribute to the hard thermal part. Thus (3.2) gives the same leading-temperature terms as (3.1) does.

Adding the contribution from diagram 1b gives the well-known result \[9\]

\[
\Pi_{\mu\nu}(q) = g^2 N(D-2)\int\frac{d^Dk}{(2\pi)^{(D-1)}}\delta(k^2)n(|k_0|) \left[ g_{\mu\nu} - \frac{4k_\mu k_\nu - 2k_\mu q_\nu - 2q_\mu k_\nu}{(k-q)^2} \right].
\]

In gauges other than the Feynman gauge, the vacuum part of the gluon propagator has to be changed according to (2.11). Diagram 1b is not affected by this, but diagram 1a receives an additional contribution

\[
g^2 N\int\frac{d^Dk}{(2\pi)^{(D-1)}}\delta(k^2)n(|k_0|)T^{\rho\sigma}(k) \left[ \frac{2f_\beta}{f.(k-q)} + \frac{(\xi(k-q)^2 - f^2)(k-q)\delta}{[f.(k-q)]^2} \right] \times \frac{(k-q)^\rho V_{\alpha\rho\sigma}(k-q,q,-k)V_{\rho\nu\beta}(k,-q,q-k)}{f_\mu},
\]

where in general \( f_\mu = f_\mu(k-q) \). \( V \) is the 3-gluon vertex (without the SU(N) colour factor), which obeys

\[
(k-q)^\rho V_{\alpha\rho\sigma}(k-q,q,-k) = g_{\sigma\mu}(k^2 - q^2) - k_\sigma k_\mu + q_\sigma q_\mu.
\]

When this is inserted into (3.4), only the two terms quadratic in the loop momentum \( k \) contribute to the leading-temperature part, but both get cancelled, the \( k^2 \) by the \( \delta \)-function, and the \( k_\sigma k_\mu \) by the transverse projector \( T_{\rho\sigma}(k) \). Therefore the result (3.3) is completely independent of the gauge fixing.

For completeness, we also give the result, in our formalism, for the one-loop contribution to the damping of long-wavelength transverse gluons. In the conventional formalism, the imaginary part
of the gluon self-energy at one-loop order is given by the diagrams of figure 2 and is strongly gauge dependent. In covariant gauges, the result comes with an unphysical sign, which was even taken to be suggestive of a fundamental instability of the perturbative thermal ground state (see reference 11 for a review). In our formalism, only diagram 2a contributes, and the result is obviously gauge independent; it reads

\[ \text{Im}(T_{\mu\nu} \Pi_{11}^{\mu\nu})(k_0, 0) = -\frac{1}{6\pi} N g^2 T^2. \]  

(3.6)

By virtue of \( \text{Im}(T_{\mu\nu} \Pi_{11}^{\mu\nu}) = \tanh(k_0 \beta/2) \text{Im}(T_{\mu\nu} \Pi_{11}^{\mu\nu}) \) this gives a contribution to the damping rate of long-wavelength gluons

\[ \gamma_{\text{one-loop}} = g^2 T N \frac{1}{24\pi}. \]  

(3.7)

This result coincides with the Coulomb-gauge one of the conventional formalism[11], where also only transverse gluons contribute to the imaginary part. It is well-known by now[12] that one-loop results for the gluonic damping rate are incomplete, and the gauge independence of our result (3.7) should not lead one astray. Still, we find it gratifying that in our formalism the unphysical modes are not able to contribute to (3.7), whereas normally in the bare one-loop calculation they even give rise to a negative sign.

3.2. Two-loop gluon-interaction pressure

The thermodynamic pressure of the gluon plasma, \( P = 1/(\beta V) \ln Z \), is a physical quantity and therefore gauge independent. In the following we shall calculate it up to two-loop order in order to perform a further check on our Feynman rules. The diagrammatic rules for calculating the partition function in the real-time formalism have been described in reference 13, to which we refer for more details.

At one-loop order, our formalism leads to exactly the same expression for \( P \) as the conventional one does in Coulomb gauge. Differences occur starting at two-loop order. The diagrams to be considered are given in figure 3, where thermal contributions have to be inserted in all possible ways, keeping at least one of the vertices appearing therein of type 1\[13\].

Diagram 3b potentially gives rise to integrals involving three powers of the Bose distribution function, but these we find to vanish after performing the momentum algebra because of the three delta functions associated with the distribution functions, exactly as in the conventional formalism.

Next consider the diagrams where two lines are thermal. Diagram 3a then identically reproduces the result of conventional Coulomb gauge,

\[ P_2^{(a)} = -\frac{1}{8} N (N^2 - 1) g^2 \int \frac{d^4q d^4k}{(2\pi)^8} n(|q_0|) \delta(q^2) n(|k_0|) \delta(k^2) 2(3 - z^2) \]

\[ = -\frac{1}{216} N (N^2 - 1) g^2 T^4, \]  

(3.8)

where we introduced \( z \equiv k.q/|k||q| \). Diagram 3b with two thermal and one vacuum gluon propagator is different, though. The vacuum piece of the gluon propagator contributes potential gauge dependences. However, because of (2.11), all gauge-dependent terms have one 4-momentum contracted with a 3-gluon vertex, and give rise to integrals of the form

\[ \int d^4q d^4k n(|q_0|) \delta(q^2) n(|k_0|) \delta(k^2) T_{\mu\nu}(k) T_{\sigma\rho}(q)(k - q) I_{\mu\alpha\sigma \sigma\beta\rho}(k - q, q). \]  

(3.9)

\[ \text{Because of} \]

\[ (k - q)^{\alpha} V_{\mu\alpha\sigma} (-k, k - q, q) = g_{\sigma\mu} (q^2 - k^2) - q_\alpha q_\mu + k_\alpha k_\mu \]

\[ \text{it is easy to see that all the integrals of the form (3.9) vanish} \]
Thus the contributions from figure 3b which involve two powers of distribution functions are completely gauge independent. There are actually two such diagrams. The one with both 3-vertices of type 1 gives

\[
P_2^{(b)(11)} = -\frac{1}{4} N(N^2 - 1)g^2 \int \frac{d^4q d^4k}{(2\pi)^8} n(|q_0|)\delta(q^2)n(|k_0|)\delta(k^2) \times -\frac{(1 + z^2)(k + q)^2 + 8(1 - z^2)(k^2 + q^2)}{(k - q)^2},
\]

where the first term can be simplified according to \((k + q)^2/(k - q)^2 \rightarrow -1\) because of the delta functions, and the last term vanishes after performing the integrals, making this contribution coincide with the one of conventional Coulomb gauge. This yields

\[
P_2^{(b)(11)} = -\frac{1}{432} N(N^2 - 1)g^2 T^4.
\]

Because the vacuum part of the gluon propagator (2.10) does have a (12)-component, there is a second diagram of the form 3b, where one vertex is of type 1, and the other of type 2. We find after carrying out the momentum algebra that the latter vanishes because of the presence of three \(\delta\)-functions, \(\delta(q^2)\delta(k^2)\delta((k - q)^2)\).

Finally, there are the diagrams with only one gluon propagator thermal. Those diagrams, however, contain a zero-temperature one-loop subgraph with its external lines put on-shell by the thermal propagator, and so are removed by renormalisation. The additional diagrams containing both a type-1 and a type-2 vertex do not complicate this picture, for they turn out to vanish identically.

Discarding the purely zero-temperature contributions, we have therefore that the gluon-interaction pressure at two-loop order is given by the sum of (3.8) and (3.11),

\[
P_2 = -\frac{1}{144} N(N^2 - 1)g^2 T^4,
\]

reproducing the standard result\cite{13-15}. Notice that the ghost diagram 3c did not contribute to (3.12), although we worked in a general linear gauge.

4. CONCLUSION

We have demonstrated that in the real-time version of finite-temperature perturbation theory it is quite natural to include a thermal part in the gauge propagator only for the two physical degrees of freedom of the gluons, leaving the unphysical gauge-field components and the Faddeev-Popov ghosts unheated. This allows us to combine the simplicity of covariant gauges for the zero-temperature part with the advantages of the noncovariant ghost-free gauges for the thermal contributions. For the thermal contributions we single out the rest frame of the heat bath and do not cause additional violation of Lorentz symmetry. Our derivation has been carried through for the Feynman gauge. The generalization to arbitrary gauges is immediate, though there may be complications from the prescriptions necessary to define the poles at \(f.k = 0\) in (2.11).

We have successfully tested these new Feynman rules in one- and two-loop calculations, and in particular have been able to demonstrate gauge independence of the high-temperature part of the gluon self-energy and for the 2-loop gluon-interaction pressure in a remarkably simple (albeit nontrivial) and general manner. In the conventional formalism, gauge independence of the hard-thermal gluon self-energy has been explicitly checked only in certain classes of gauges\cite{9}, whereas the 2-loop pressure has been calculated so far only in Feynman\cite{13,14} and axial\cite{15,16} gauges.

We have worked throughout in the real-time formalism. This enables us to develop the perturbation theory in terms of \(in\) fields, with the great advantage that not only do the fields obey free-field equations, but also we may ignore the interaction in the condition that picks out the physical states.
We have not been able to translate our prescriptions satisfactorily into an imaginary-time formalism. In order to set up such a formalism, one would need to choose an interaction picture that coincides with the Heisenberg picture at finite time $t_0$. The condition that picks out the physical interaction-picture states is then much more complicated\[^4\]. Even if we were able to implement it, the fact that the unphysical-field propagators are temperature-independent and so not periodic in imaginary time would lead to extra complications\[^6\] and make the formalism difficult to use.

An interesting question, to which we hope to return in future work, is how the present approach could be extended to accommodate a resummation of finite-temperature perturbation theory, as is mandatory for exploring physics at the scale $g^2T$ in high-temperature quantum chromodynamics\[^12\] (and, incidentally, for handling pinch singularities\[^17\]). An important difference between the bare and the resummed theory is that, because of the appearance of an additional collective mode at the scale $gT$, (1.3) no longer covers all of the physical modes in the plasma. Including the latter in a formalism analogous to the one presented here might prove to be of conceptual interest for the resummation program\[^12\], and might perhaps allow an easier verification of its gauge independence.

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### APPENDIX: Diagonalisation of the matrix propagator

The real-time version of finite-temperature field theory can be reformulated by diagonalisation of the $2\times2$ matrix propagators, which allows one to associate all thermal contributions with the vertices.

A well-known possibility\[^2, 13\] is to diagonalise to a matrix constructed from the ordinary Feynman propagator $\Delta_F$,

$$\tilde D = \begin{pmatrix} \Delta_F & 0 \\ 0 & -\Delta_F \end{pmatrix}.$$  \hspace{1cm} (A.1)

In our formalism, the Feynman-gauge propagator may be written as

$$D^{\mu\nu} = T^{\mu\nu} M_T \tilde D M_T - (g^{\mu\nu} + T^{\mu\nu}) M_0 \tilde D M_0$$  \hspace{1cm} (A.2a)

with (when $\sigma = 0$)

$$M_T = \sqrt{\pi(k_0)} \begin{pmatrix} e^{\beta|k_0|/2} & e^{-\beta k_0/2} \\ e^{\beta k_0/2} & e^{\beta|k_0|/2} \end{pmatrix},$$  \hspace{1cm} (A.2b)

and $M_0$ the zero-temperature limit of $M_T$:

$$M_0 = \begin{pmatrix} 1 & \theta(-k_0) \\ \theta(k_0) & 1 \end{pmatrix},$$  \hspace{1cm} (A.2c)

The ghost propagator is $M_0 \tilde D M_0$.

Recently, a different scheme has been proposed by Aurenche and Becherrawy\[^18\], which diagonalises to a matrix composed of retarded and advanced propagators,

$$\tilde D = \begin{pmatrix} \Delta_R & 0 \\ 0 & \Delta_A \end{pmatrix}.$$  \hspace{1cm} (A.3)

instead of Feynman propagators. For this, one has to introduce different matrices to be associated with incoming and outgoing lines\[^18\]. In our formalism for the Feynman gauge

$$D^{\mu\nu} = T^{\mu\nu} U_T \tilde D V_T - (g^{\mu\nu} + T^{\mu\nu}) U_0 \tilde D V_0$$  \hspace{1cm} (A.4a)
with

\[ U_T = \begin{pmatrix} 1 & -n(k_0) \\ 1 & -(1 + n(k_0)) \end{pmatrix}, \quad U_0 = \begin{pmatrix} 1 & \theta(-k_0) \\ 1 & -\theta(k_0) \end{pmatrix}, \]

\[ V_T = \begin{pmatrix} 1 + n(k_0) & n(k_0) \\ 1 & 1 \end{pmatrix}, \quad V_0 = \begin{pmatrix} \theta(k_0) & -\theta(-k_0) \\ 1 & 1 \end{pmatrix}. \]

(A.4b)

The ghost propagator is \( U_0 \bar{D} V_0 \). The matrices \( U \) and \( V \) are removed from the propagators and instead associated with the in- and outgoing lines of the vertices. (Because of

\[ U(k) \tau_1 = V^T(-k), \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

more symmetric Feynman rules result if a factor \( \tau_1 \) is combined with \( U \) and \( \bar{D} \), turning the latter off-diagonal[19].) With (A.4), the analysis of reference 18 carries over to a large extent. However, in addition to retarded and advanced lines one has to distinguish between transverse and non-transverse ones, and in particular there is no simple causality principle when transverse and non-transverse lines come together at a vertex. For example, a 3-vertex connecting solely retarded or solely advanced lines does not vanish, unless the lines are either all transverse or all non-transverse.

Such a reformulation in terms of retarded and advanced Green functions leads to well-defined causal properties, which are not manifest in the usual real-time formalism[20]. Causal Green functions, which are relevant for example in linear response theory, are usually obtained in a direct manner in the imaginary-time formalism. In our prescription, however, this is not straightforward to implement, and so this diagonalization scheme appears to be particularly useful for our approach.
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Figure captions

1 Diagrams which contain the hard-thermal-loop contributions to the gluon self energy. Solid lines are gluons and broken lines Faddeev-Popov ghosts. The slashes denote thermal parts of the propagators; unslashed lines are vacuum propagators. In the formalism described in this paper, diagrams (b) and (c) do not occur.
2 The imaginary part of the gluon self-energy, with the same notation as figure 1. In our formalism, (b) does not occur.
3 Diagrams for the 2-loop gluon interaction pressure. The lines represent complete propagators (vacuum plus thermal parts).