Triangulated category of effective Witt-motives

$DW M^\text{eff}_e(k)$ *

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Abstract

The category of effective Witt-motives $DW M^\text{eff}(k)$ for perfect field $k$, $\text{char}k \neq 2$, with functor $WM : Sm_k \to DW M^\text{eff}(k)$ defining motives of smooth affine varieties is constructed. In the construction Voevodsky-Suslin method is applied to a category of Witt-correspondence between affine smooth varieties $WCor_k$ that morphisms are defined by class in Witt-group of quadratic space $(P, q_P)$ with $P$ being $k[X \times Y]$-module finitely generated projective over $k[X]$ and $q_P : P \to Hom_{k[X]}(P, k[X])$ being $k[X \times Y]$-liner isomorphism. And the natural isomorphism

$$\text{Hom}_{DW M^\text{eff}_e(k)}(WM(X), \mathcal{F}[i]) \simeq H^i_{Nis}(X, \mathcal{F})$$

for any smooth affine $X$ and homotopy invariant Nisnevich sheave $\mathcal{F}$ with Witt-transfers (that is presheave on the category $WCor_k$ such that its restriction on the category $Sm_k$ is a sheave) is proved.

1 Introduction.

This work is devoted to the problem of the construction of the triangulated category of Witt-motives $DW M^\text{eff}(k)$ by the Voevodsky-Suslin method that was originally used for the construction of the category of motives $DM^\text{eff}(k)$.

Let’s explain what is meant under Voevodsky-Suslin method. In [1] V. Voevodsky constructed the triangulated category of motives $DM^\text{eff}(k)$ for perfect field $k$. The construction starts with the category of correspondence between smooth affine varieties $Cor_k$, and proceeds by proving some properties of the presheaves of abelian groups on this category. These properties express certain compatibility between the structure of the category $Cor_k$, the topological structure on the category $Sm_k$ and the structure of the category with an interval specified by the affine line $\mathbb{A}^1_k$. Due to this compatibility the construction results in obtaining the category of motives in sufficiently explicit form as stabilization in the $G^\mathbb{A}^1$ direction from a full subcategory of a derived category.

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So we wish to construct the category of Witt-motives starting from some category of correspondences between smooth affine varieties. It was suggested by I.A. Panin to use so called category of Witt-correspondence $WCor_k$. It is additive category that objects are smooth affine varieties and $Hom$-groups are the Witt-groups of some exact categories. Namely, for two smooth affine varieties $X$ and $Y$, the quadratic spaces $(P, q_P)$ defined by the $k[X \times Y]$-module $P$ finitely generated projective over $k[X]$ and $k[X \times Y]$-linear symmetric isomorphism $q_P: P \simeq Hom_{k[X]}(P, k[X])$ are regarded.

In current text following described above approach we define the category of effective Witt-motives $DW\text{M}^-_{eff}(k)$ as full subcategory of the derived category $D^-(\text{ShNisWtr}_k)$ of the category of Nisnevich sheaves with Witt-transfers, consisting of the motivic complexes, i.e. complexes $A^\bullet$ whose Nisnevich sheaves of cohomologies $h^i_{\text{Nis}}(A)$ are homotopy invariant and define the functor $WM: Sm_k \to DW\text{M}^-_{eff}(k)$ of Witt-motives of smooth affine varieties $WM(X) = \{U \mapsto (\cdots \to WCor_{Nis}(U \times \Delta^i, X) \cdots \to WCor_{Nis}(U \times \Delta^1, X) \to WCor_{Nis}(U, X))\}$ where $WCor_{Nis}(-, Y)$ denotes Nisnevich sheafication of the presheave $WCor(-, Y)$.

We prove

Theorem A. The category $DW\text{M}^-_{eff}(k)$ is equivalent to the localization of the derived category $D^-(\text{ShNisWtr}_k)$ by the morphisms corresponding to the projections $X \times \mathbb{A}^1 \to X$ and

Theorem B. There is natural isomorphism

$$\text{Hom}_{DW\text{M}^-_{eff}(k)}(WM(X), \mathcal{F}[i]) \simeq H^i_{\text{Nis}}(X, \mathcal{F})$$

for any smooth affine varieties $X$ and homotopy invariant sheaves with Witt-transfers $\mathcal{F}$.

For this purpose we strengthen the result of [5] about preservation of homotopy invariance under Nisnevich sheafication for presheaves with Witt-transfers and prove

Theorem C. Nisnevich sheafication $\mathcal{F}_{\text{Nis}}$ of homotopy invariant presheave with Witt-transfers $\mathcal{F}$ is strictly homotopy invariant

and prove

Theorem D. For any presheave with Witt-transfers $\mathcal{F}$ the Nisnevich sheafication $\mathcal{F}_{\text{Nis}}$ and presheaves of cohomologies $H^i_{\text{Nis}}(\mathcal{F}_{\text{Nis}})$ are equipped with Witt-transfers in canonical way that implies that the category $\text{ShNisWtr}$ is abelian and allows to consider the category $D^-(\text{ShNisWtr})$.

Let’s note that to give the definition of the category $DW\text{M}^-_{eff}(k)$ and functor $WM$ it is enough to prove the results on the preservation of Witt-transfers and homotopy invariance only for Nisnevich sheafication of the presheave. But to prove described properties of this objects it requires strictly version of this results, i.e. to prove similar properties of presheaves of Nisnevich cohomologies. (In fact to prove the isomorphism (1) we use more stronger statement (theorem[7] then theorem D.)

Now we describe the contents of the text. In sections 3 and 4 two excision isomorphisms are proved. Namely Zariski excision isomorphism on the affine line over local base

$$\frac{\mathcal{F}(\mathbb{A}^1_U - 0_U)}{\mathcal{F}(\mathbb{A}^1_U)} \simeq \frac{\mathcal{F}(V - 0_U)}{\mathcal{F}(V)}$$

in the section 3 and etale excision isomorphism

$$\frac{\mathcal{F}(X' - Z)}{\mathcal{F}(X')} \simeq \frac{\mathcal{F}(X - Z)}{\mathcal{F}(X)}$$

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in the section 4.

In the section 5 this excision isomorphisms are applied to prove that homotopy invariant Nisnevich sheaf with Witt-transfers is strictly homotopy invariant. In combination with the result proved in [3] this implies that Nisnevich sheafication of homotopy invariant presheave with Witt-transfers is strictly homotopy invariant.

The section 6 is devoted to analyse of the behaviour of Witt-transfer in relation to the Nisnevich topology. In particular we prove theorem D.

In the section 7 the construction of the category of effective Witt-motives with the functor defines Witt-motives of smooth affine varieties is given. And the basic property [1] of Witt-motives for smooth affine schemes is proved.

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2 The Witt-correspondence category

In this section the definition and basic properties of category Witt-correspondence introduced in [5] are given.

At first let’s take next technically useful definition.

**Definition 1** (Proj(p)) For any morphism of affine schemes \( p: S \to U \) we define exact category with duality Proj(p) as follows. The category Proj(p) is full subcategory of the category \( k[S]-\text{mod} \) consisting of modules that are finitely generated projective over \( k[U] \). To define the duality we consider the functor

\[
D_p: k[S]-\text{mod} \to k[S]-\text{mod}
\]

\[
M \mapsto D_p(M) = \text{Hom}_{k[U]}(M, k[U])
\]

(where the structure of \( k[S] \)-module on \( D_p(M) \) is defined by the rule: \( (f \cdot \rho)(m) = \rho(f \cdot m) \) for \( \rho \in D_p(M) \) and \( f \in k[S] \)). Since the functor \( \text{Hom}_{k[U]}(\cdot, k[U]) \) is the duality on the exact category Proj(\( k[U] \)) of finitely generated projective \( k[U] \)-modules and since the forgetful functor \( \text{Proj}(p) \to \text{Proj}(k[U]) \) is exact, \( D_p \) induce on Proj(p) the structure of exact category with duality.

Also we will denote by \( \text{Proj}(X, Y) \) the category \( \text{Proj}(pr_X) \) where \( pr_X: X \times Y \to X \) is canonical projection.

**Remark 1** The scalar restriction along a morphism \( S' \xrightarrow{f} S \) induce a functor \( f_*: \text{Proj}(p \circ \circ f) \to \text{Proj}(p) \) respecting duality for any morphism \( S \xrightarrow{p} U \).

The base change (or reverse image functor) along the morphism \( u: U' \to U \) induce the functor of categories with duality \( u^*: \text{Proj}(p) \to \text{Proj}(p') \) for any morphism \( p: S \to U \) and \( p' = p \times_U U': S \times_U U' \to U \).

Using this definition the additive category WCor_k can be defined as follows.

**Definition 2** (WCor_k)

\( \diamond \) \( \text{Ob} WCor_k = \text{Ob} Sm_k \);
Definition 3 (Presheaves and sheaves with Witt-transfers) Abelian groups presheaf with Witt-transfers is a functor \( F : \text{WCor}_k \to \text{Ab} \) satisfying additivity condition on disjoint unions: \( \mathcal{F}(X_1 \coprod X_2) = \mathcal{F}(X_1) \oplus \mathcal{F}(X_2) \) for any \( X_1 \) and \( X_2 \). The Sheaf with Witt-transfers is a presheaf with Witt-transfers that becomes a sheaf after restriction to \( \text{Sm}_k \).

Remark 2 There is a functor \( i : \text{Sm}_k \to \text{WCor}_k \) that maps the regular map \( f : X \to Y \) to the morphism defined by the module \( k[Y]k[X]_{k[X]} \) and canonical isomorphism \( k[X] \simeq \text{Hom}_{k[X]}(k[X], k[X]) \). Moreover the composition of a morphism with the image of a regular map at the right or left is equal to direct or reverse image of corresponding quadratic space along the \( f \):

\[
 f \circ w = \text{red}_{f \times W}(w), w \in \text{WCor}_k(W, X),
\]

\[
 w \circ f = \text{ind}_f(w), w \in \text{WCor}_k(Y, Z).
\]

Remark 3 For any smooth variety \( X \)

\[
 \text{WCor}(X, \text{pt}) = W(X)
\]

and there is an diagonal embedding

\[
 W(X) \to \text{WCor}(X, X).
\]

So the functor of Witt-groups is a presheave with Witt-transfers and any presheave with Witt-transfers is a presheaf of modules over the Witt ring.

Also let’s note that for any invertible function \( \lambda \in k[X]^* \) we will denote by \( (\lambda)_X \in Q\text{Space}(\text{Proj}(X, X)) \) quadratic spaces of rank one on a diagonal with quadratic form defined by \( \lambda \) i.e. \( (k[X]k[X]_{k[X]}, \lambda) \). And for by regular map \( f : X \to Y \) let’s denote by \( (\lambda)_f \) or \( (\lambda)_{\Gamma} \), quadratic space in \( \text{Proj}(X, Y) \) of rank one that support is graphic \( \Gamma_f \) of map \( f \) and that quadratic form defined by \( \lambda \) i.e. \( (k[Y]k[\Gamma_f]_{k[X]}, \lambda) \).

Remark 4 (Essential smooth schemes) Sometimes we will suppose presheaves with Witt-transfers and bi-functor \( \text{WCor}_k(-, -) \) to be well defined on essential smooth schemes over \( k \).
The category $\text{WCor}_k$ can be defined on essential smooth schemes, i.e. it is possible to define Witt-correspondence between such schemes agreed with the definition of Witt-correspondence between smooth affine schemes, by considering of essential smooth schemes as pro-objects. However, in fact we will use only germs of presheaves on essential smooth schemes and Witt-correspondence between smooth schemes over fields of functions $k(X)$ of smooth varieties and local rings $O_{X,x}$ at smooth point $x$. So we will describe in details only this two constructions.

1) The germ of the presheave $\mathcal{F}$ on any essential smooth variety $U$ is

$$\mathcal{F}(\lim_{i \to \infty} V_i) = \lim_{i \to \infty} \mathcal{F}(V_i).$$

And any morphism $g: U \to X$, $X \in \text{Sm}_k$ induces homomorphism $\mathcal{F}(g): \mathcal{F}(X) \to \mathcal{F}(U)$ agreed with composition with any morphism $f: X \to X'$ $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

2) For any smooth affine variety $S$ and a point $s \in S$ there are functors

$$\text{PreWtr}_k \to \text{PreWtr}_k(S)$$

$$\mathcal{F} \mapsto \mathcal{F}_k(S)$$

$$\text{PreWtr}_k \to \text{PreWtr}_{O_{S,s}}$$

$$\mathcal{F} \mapsto \mathcal{F}_{k[S,s]}$$

such that

$$\mathcal{F}_k(S)(X) \simeq \mathcal{F}(X)$$

($\mathcal{F}(X)$ at the right side is the germ of $\mathcal{F}$ on $X$ considered as essential smooth scheme over $k$), and such that for any morphism $f: X \to Y$ of smooth schemes over open subscheme $S' \subset S$ section group homomorphism

$$\mathcal{F}_k(S)(f_k(S)): \mathcal{F}_k(S)(X) \to \mathcal{F}_k(S)(Y)$$

coincides with germ group homomorphism induced by $f$, and similar for $\mathcal{F}_{k[S,s]}$.

To define these functors it is enough to construct for any Witt-correspondence $\Phi \in \text{WCor}_k(X,Y)$ ($\Phi \in \text{WCor}_k(S,X,Y)$) between smooth affine $k$-schemes ($k[S]$ schemes), some Witt-correspondence $\Phi' \in \text{WCor}_k(X',Y')$ between smooth affine $S'$-schemes, for some open subscheme $S' \subset S$, that goes to $\Phi$ under base chance along the morphism $k[S'] \to k(S)$ ($k[S'] \to k[S]$). To construct such $\Phi'$ we can choose algebraic parametrisation of the data defining $X$, $Y$ and $\Phi$, i.e. parametrisate it by finite set of algebraic parameters from $k(S)$ ($k[S]$), satisfying some algebraic equations. And then set $S' \subset S$ to be open subscheme on that all parameters are well defined.

It is useful for proofs of excision isomorphisms to extend the category $\text{WCor}_k$ to pairs of smooth scheme and its open subscheme.

**Definition 4 (Category $\text{WCor}_k^{\subset \to}$)** The objects of the category $\text{WCor}_k^{\subset \to}$ are the pairs $(X_1, X_2)$ of smooth affine scheme $X_1$ and its open subscheme $X_2$. Any morphism $\Phi \in \text{WCor}_k^{\subset \to}((X_1, X_2) \to (Y_1, Y_2))$ is a pair of morphisms $\Phi_i \in \text{WCor}(X_i, Y_i)$, $i = 1, 2$ such that $\Phi_1 \circ i_X = i_Y \circ \Phi_2$.

It is equivalent to say that $\text{WCor}_k^{\subset \to}$ is full subcategory of the category of arrows of the category $\text{WCor}_k$ consists of the open embeddings.
Definition 5 (Category $\text{WCor}_{k}^{\text{pair}}$) Additive category $\text{WCor}_{k}^{\text{pair}}$ is a factor-category of the additive category $\text{WCor}_{k}$ by the ideal generated by identity morphisms of the objects $(X, X)$ for all varieties $X$.

Remark 5 The last definition is equivalent to say that $\text{WCor}_{k}^{\text{pair}}$ is full subcategory of homotopy category $\mathcal{K}(\text{WCor}_{k})$ of additive category $\text{WCor}_{k}$ consists of the complexes concentrated in 2 adjoin degrees with differential homomorphisms being open embedding $X_2 \to X_1$.

More precisely it means that Hom-groups are defined as follows

$$\text{WCor}_{k}((X_1, X_2), (Y_1, Y_2)) \overset{\text{ref}}{=} H(\text{WCor}_{k}(X_1, Y_2) \overset{i_Y \circ i_X - o_X}{\longrightarrow} \text{WCor}_{k}(X_1, Y_1) \oplus \text{WCor}_{k}(X_2, Y_2) \overset{\circ Y \circ i_Y}{\longrightarrow} \text{WCor}_{k}(Y_2, Y_1)),$$

where $H$ denotes homology group in middle term of the complex of the length 3.

So morphism $\Phi: (X_1, X_2) \to (Y_1, Y_2)$ is defined by pair of morphisms $\Phi_i \in \text{WCor}(X_i, Y_i)$, $i = 1, 2$ such that left diagram is commutative and the pair $(\Phi_1, \Phi_2)$ defines zero morphism if and only if there exists $\Xi \in \text{WCor}_{k}(X_1, Y_2)$ such that right diagram is commutative.

In terms of quadratic spaces to construct the morphism $\Phi: (X_1, X_2) \to (Y_1, Y_2)$ in $\text{WCor}_{k}^{\text{pair}}$ it’s enough to find the quadratic space

$$(P, q_P) \in \text{Proj}(pr_{Y_1 \times X_1 \to X_1}) : \ k[Y_2] \otimes_{k[Y_1]} P \otimes_{k[X_1]} k[X_2] \simeq P \otimes_{k[X_1]} k[X_2]. \quad (2)$$

Because such $(P, q_P)$ defines the morphism in $\text{WCor}_{k}(X_1, Y_1)$. And due to isomorphism $\text{[2]}$ the module $P \otimes_{k[X_1]} k[X_2]$ has the canonical structure of the module over $k[Y_2] \otimes k[Y_1]$. Hence the quadratic space $i_X^*(P, q_P)$ in fact lies in $\text{Proj}(pr_{Y_2 \times X_2 \to X_2})$ and defines the morphism in $\text{WCor}_{k}(X_2, Y_2)$.

It is noteworthy that not every morphism has such representation but all used in the constructions morphisms have.

Remark 6 For any presheaf with Witt-transfers $\mathcal{F}$ one can define the presheaf $\mathcal{F}^{\text{pair}}$ on the category of pairs such that

$$\mathcal{F}^{\text{pair}}(X_1, X_2) = \frac{\mathcal{F}(X_2)}{\mathcal{F}(X_1)}.$$ 

The rest part of the section is devoted to homotopy invariant presheaves with Witt-transfers.

Let’s define the category $\overline{\text{WCor}}_{k}$.

Definition 6 (Category $\overline{\text{WCor}}_{k}$) $\overline{\text{WCor}}_{k}$ is a factor-category of additive category $\text{WCor}_{k}$ such that

$$\overline{\text{WCor}}_{k}(X, Y) = \text{coker}(\text{WCor}_{k}(\mathbb{A}^1 \times X, Y) \overset{(-o_0)(-o_1)}{\longrightarrow} \text{WCor}_{k}(X, Y)),$$

where $i_0, i_1: X \hookrightarrow \mathbb{A}^1 \times X$ are zero and unit sections.

And similar we define factor-category $\overline{\text{WCor}}_{k}^{\text{pair}}$ of the category $\text{WCor}_{k}^{\text{pair}}$. 

Remark 7 (Homotopy invariance)

1) By definition homotopy invariant presheaf with Witt-transfers is a presheaf with Witt-transfers that is homotopy invariant. So such presheaves are exactly the presheaves on the category $\text{WCor}_k$. And this equivalence is agreed with continuation of presheaves to essential smooth schemes in sense of the remark 4.

2) Let’s take used later descriptions of the pairs of homotopy morphisms.

Two morphisms $\Phi_1, \Phi_2: X \to Y$ in $\text{WCor}_k$ represented by the spaces $(P_1, q_1)$ and $(P_2, q_1)$ becomes equal in $\text{WCor}_k$ if and only if exists the quadratic space $(H, q)$, such that the equalities

\[ [j_0^*(H, q_H)] = [(P_0, q_0)], \quad [j_1^*(H, q_H)] = [(P_1, q_1)] \]

holds in $\text{Proj}(pr_{X \times Y} \to Y)$.

For coincidence of two morphisms of pairs in $\text{WCor}_k$ represented by spaces in sense of the remark 5 it is enough the existence of the quadratic spaces $(H, q) \in \text{Proj}(pr_{Y, X_1} \times X)$ and $(G_0, q_0'), (G_1, q_1') \in \text{Proj}(pr_{Y_1, X_1})$, such that

\[ k[Y_2] \otimes_{k[Y_1]} H \otimes_{k[X_1]} k[X_2] = H \otimes_{k[X_1]} k[X_2] \]

\[ [j^*_i(H, q)] = [(P_i, q_i) \oplus (G_i, q'_i)], \quad k[Y_2] \otimes_{k[Y_1]} G_i = G_i, \quad i = 0, 1. \]

3 Excision on $\mathbb{A}^1_U$.

This section is devoted to some particular case of excision isomorphism in Zariski topology on the relative affine line $\mathbb{A}^1_U$ over the local essential-smooth scheme $U$. Namely it’s the case of excision outside the zero section of $\mathbb{A}^1_U$.

Theorem 1 Let $\mathcal{F}$ be homotopy invariant sheave with Witt-transfers and $U = \text{Spec} O_{X,x}$ be spectre of the local ring of a smooth variety $X$ at any point $x$. Then for any Zariski open subvariety $V \subset \mathbb{A}^1_U$ containing zero section $0_U \subset \mathbb{A}^1_U$ restriction homomorphism

\[ i^*: \frac{\mathcal{F}(\mathbb{A}^1_U - 0_U)}{\mathcal{F}(\mathbb{A}^1_U)} \to \frac{\mathcal{F}(V - 0_U)}{\mathcal{F}(V)} \]

is an isomorphism ($i$ denotes embedding of $V$ into $U$).

Remark 8

1) In fact In terms of the remark 6 theorem statement means that $i^*: \mathcal{F}^{\text{pair}}(\mathbb{A}^1_U - 0_U, \mathbb{A}^1_U) \to \mathcal{F}^{\text{pair}}(V - 0_U, V)$ is isomorphism and follows from the following lemma.

2) Obviously theorem is equivalent to the statement that restriction homomorphism

\[ i^*: \frac{\mathcal{F}(V - 0_U)}{\mathcal{F}(V)} \to \frac{\mathcal{F}(V' - 0_U)}{\mathcal{F}(V')} \]

is an isomorphism for any two embedded open subvarieties containing zero section:

\[ 0_U \subset V' \subset V \subset \mathbb{A}^1_U. \]
Lemma 3.1  Let \( i \) denotes embedding of \( V \) into \( \mathbb{A}_1^U \). Then its class \( [i] \in \overline{WCor}_k((V, V - 0_U), (\mathbb{A}_1^U, \mathbb{A}_1^U - 0_U)) \) is isomorphism (as a morphism in \( \overline{WCor}_k \)).

Remark 9

a) The right inverse morphism to \([i]\) in \( \overline{WCor}_K \) is a morphism

\[ \Phi \in WCor_K((\mathbb{A}_1^U, \mathbb{A}_1^U - 0_U), (V, V - 0_U)) : \ [i \circ \Phi] = [id] \in \overline{WCor}_K((\mathbb{A}_1^U - 0_U), (\mathbb{A}_1^U - 0_U)) \]

And it is equivalent to existence of

\[ \Theta \in WCor_K((\mathbb{A}_1^U \times \mathbb{A}_1^1, (\mathbb{A}_1^U - 0_U) \times \mathbb{A}_1^1), (\mathbb{A}_1^U, \mathbb{A}_1^U - 0_U)) : \ [\Theta \circ j_0 = i \circ \Phi, \Theta \circ j_1 = id, \ (3)] \]

(where \( j_0, j_1 : (\mathbb{A}_1^U, \mathbb{A}_1^U - 0_U) \rightarrow (\mathbb{A}_1^U, \mathbb{A}_1^U - 0_U) \times \mathbb{A}_1^1 \) denotes embeddings of zero and unit sections respectively).

b) The left inverse to \([i]\) in \( \overline{WCor}_K \) is an morphism

\[ \Psi \in WCor_K((\mathbb{A}_1^U, \mathbb{A}_1^U - U_0), (V, V - U_0)) : \ [\Psi \circ i] = [id] \in \overline{WCor}_K((V, V - 0_U), (V, V - 0_U)). \]

It means the existence of

\[ \Xi \in WCor_K((V \times \mathbb{A}_1^1, (V - 0_U) \times \mathbb{A}_1^1), (V, V - 0_U)) : \ [\Xi \circ j_0 = \Psi \circ i, \Xi \circ j_1 = id \ (4)] \]

\((j_0, j_1 : (V, V - 0_U) \rightarrow (V, V - 0_U) \times \mathbb{A}_1^1 \) denotes zero and unit sections respectively).

Proof of the lemma [3, 7]

We will construct required morphisms in the category \( WCor_U \) using it’s embedding into \( WCor_k \) extended in sense of remark [4].

a)

(Description of quadratic spaces)

To construct right inverse to \( i \in \overline{WCor}((V, V - 0_U), (\mathbb{A}_1^U, \mathbb{A}_1^U - 0_U)) \) means to find quadratic spaces \((P, q_P)\) and \((H, q_H)\) corresponding to the morphisms \( \Phi \) and \( \Theta \) from remark [9 a). In the terms of the quadratic spaces the properties of \( \Phi \) and \( \Theta \) means following:

1) \( P \in k[V \times U \mathbb{A}_1^1] - mod, \) \( P \) is finitely generated projective over \( k[\mathbb{A}_1^1] \) and \( q_P : P \cong \sim Hom(P, k[\mathbb{A}_1^1]), \) is \( k[V \times U \mathbb{A}_1^1] \)-linear symmetric isomorphism;

2) \( H \in k[\mathbb{A}_1^1, U_1 \mathbb{A}_1^1 \times \mathbb{A}_1^1] - mod, \) \( H \) is finitely generated over \( k[\mathbb{A}_1^1 \times \mathbb{A}_1^1] \) and \( q_H : H \cong \sim Hom(H, k[\mathbb{A}_1^1 \times \mathbb{A}_1^1]), \) is \( k[\mathbb{A}_1^1 \times U \mathbb{A}_1^1 \times \mathbb{A}_1^1] \)-linear symmetric isomorphism;

3) canonical homomorphisms

\[
P \otimes_{k[\mathbb{A}_1^1]} k[\mathbb{A}_1^U - 0_U] \rightarrow k[V - 0_U] \otimes_{k[V]} P \otimes_{k[\mathbb{A}_1^1]} k[\mathbb{A}_1^U - 0_U],
\]

\[
H \otimes_{k[\mathbb{A}_1^1]} k[\mathbb{A}_1^U - 0_U] \rightarrow k[\mathbb{A}_1^U - 0_U] \otimes_{k[\mathbb{A}_1^1]} H \otimes_{k[\mathbb{A}_1^1]} k[\mathbb{A}_1^U - 0_U]
\]

(one of structure of \( k[\mathbb{A}_1^1] \)-module is regarded here as left and another as right) are an isomorphisms (it means that spaces \((P, q_P)\) and \((H, q_H)\) defines the morphisms of pairs);

4) in the Witt group of the category \( Proj(\mathbb{A}_2^U \rightarrow \mathbb{A}_1^U) \) following equalities holds:

\[
[(H, q_H) \otimes_{k[\mathbb{A}_1^1 \times \mathbb{A}_1^1]} k[\mathbb{A}_1^U \times 0)] = [k[\mathbb{A}_1^1]k[V] \otimes_{k[V]} (P, q_P)],
\]

\[
[(H, q_H) \otimes_{k[\mathbb{A}_1^1 \times \mathbb{A}_1^1]} k[\mathbb{A}_1^U \times 1)] = [E_{k[\mathbb{A}_1^1]}],
\]

(it equivalent to the equalities [3]).
(Definition of modules $P$ and $H$)

Let’s now give one lemma (sublemma 3.1.1 from [7]) used in the proof of excision isomorphisms.

**Lemma 3.2** Let $X$ be projective scheme over netherian ring, $Z$ be closed subscheme, $\mathcal{F}$ be a coherent sheave and $\mathcal{L}$ be very ample bundle on $X$. Then for all $n$ larger some $k$ the restriction $\Gamma(\mathcal{F} \otimes \mathcal{L}^\otimes n) \to \Gamma((\mathcal{F} \otimes \mathcal{L}^\otimes n)|_Z)$ is surjective.

And let’s involve one used definition-denotation.

**Definition 7 (sheaf of iseals $I(s)$ and subscheme $Z(I)$)** Any non-zero global section $s \in \Gamma(\mathcal{L})$ of some invertible sheaf $\mathcal{L}$ on any irreducible scheme $X$ defines a sheaf of ideals in $\mathcal{O}(X)$ isomorphic to the sheaf $\mathcal{L}^{-1}$ (The section $s$ defines homomorphism $\mathcal{O}(X) \to \mathcal{L}$). And multiplying it on the sheaf $\mathcal{L}^{-1}$ we get homomorphism $\mathcal{L}^{-1} \to \mathcal{O}(X)$. Or equivalently it is the sheaf of functions $\{f \in \mathcal{O}(X), \text{div}_0 f - \text{div}_0 s < 0\}$. We will denote this sheaf of ideals by $I(s)$. Also for any sheaf of ideals $\mathcal{I} \subset \mathcal{O}(X)$ we will denote by $Z(\mathcal{I})$ corresponding closed subscheme of $X$. And finally we denote the closed subscheme $Z(I(s))$ by $Z(s)$ for any section of line bundle $s$.

Let’s identify $\mathbb{A}^1_U \times_U \mathbb{A}^1_U$ and $V \times_U \mathbb{A}^1_U$ with subsets of $\mathbb{P}^1_{\mathbb{A}^1_U}$

$$\mathbb{A}^1_U \times_U \mathbb{A}^1_U = \mathbb{A}^1_{\mathbb{A}^1_U} \subset \mathbb{P}^1_{\mathbb{A}^1_U}, \quad V \times_U \mathbb{A}^1_U = \mathbb{A}^1_U \subset \mathbb{P}^1_{\mathbb{A}^1_U},$$

Let $T$ and $D$ be its complement and let $\Delta \subset \mathbb{P}^1_{\mathbb{A}^1_U}$ be the graph of embedding $\mathbb{A}^1_U \hookrightarrow \mathbb{P}^1_U$.

$$T = \mathbb{P}^1_{\mathbb{A}^1_U} \setminus \mathbb{A}^1_{\mathbb{A}^1_U}, \quad D = \mathbb{P}^1_{\mathbb{A}^1_U} \setminus \mathbb{A}^1_U \Delta \subset \mathbb{P}^1_{\mathbb{A}^1_U} \Delta \subset \mathbb{P}^1_{\mathbb{A}^1_U} = \Gamma(\mathbb{A}^1_U \hookrightarrow \mathbb{P}^1_U).$$

Then let’s choose a sections

$$\nu, \delta \in \Gamma(\mathbb{P}^1_{\mathbb{A}^1_U}, \mathcal{L}(T)) : \text{div}_0 \nu = 0_{\mathbb{A}^1_U}, \quad \text{div}_0 \delta = \Delta.$$

We can assume in addition that

$$\nu|_T = \delta|_T.$$

Indeed since $T$ is the infinity section of $\mathbb{P}^1_{\mathbb{A}^1_U}$ the fraction $u = \frac{\nu}{\delta}$ can regarded as the function on $\mathbb{A}^1 \times_U$ and since intersections of zero divisors of $\nu$ or $\delta$ with $T$ are empty $u$ is invertible function. So if we multiply $\delta$ by the inverse image of $u$ along the projection

$$\mathbb{P}^1_{\mathbb{A}^1_U} \to \mathbb{A}^1 \times U,$$

we don’t change zero divisor of $\delta$ and make the required equality holds.

Subscheme $V \subset \mathbb{P}^1_{\mathbb{A}^1_U} \times \mathbb{A}^1_U$ contains zero section $0_{\mathbb{A}^1 \times_U}$. Hence the intersection of $D$ with $0_{\mathbb{A}^1 \times_U}$ is empty and by the sublemma 3.2 for sufficiently large $n$ there is a section

$$s_0 \in \Gamma(\mathbb{P}^1_{\mathbb{A}^1_U}, \mathcal{L}(nT)) : s_0|_D = \nu^n, \quad s_0|_{0_{\mathbb{A}^1_U}} = \delta^n.$$

Let $s = s_0 \cdot (1 - t) + \delta^n \cdot \in \Gamma(\mathbb{P}^1_{\mathbb{A}^1_U} \times \mathbb{A}^1_U, \mathcal{L}(nT))$. Then

$$s|_{0_{\mathbb{A}^1 \times \mathbb{A}^1}} = \delta^n, \quad s|_{T_{\mathbb{A}^1 \times \mathbb{A}^1}} = \nu^n (= \delta^n).$$
Since $s$ is invertible on $T_{\mathbb{A}^1}$ and $s_0$ is invertible on $D$, they defines closed subschemes

$$S_0 = Z(s_0)/ \subset V_{\mathbb{A}^1_U}, \quad S = Z(s)/ \subset \mathbb{A}^1_{\mathbb{A}^1_U} \times \mathbb{A}^1,$$

(see definition for $Z(s)$.) Thus we can put

$$P = k[S_0]k[V \times U_{\mathbb{A}^1_U}], \quad H = k[S]k[\mathbb{A}^1_U \times U \times \mathbb{A}^1].$$

(Checking of point 3, i.e. equalities (5))

The condition (5) holds because $s_0|_{0 \times \mathbb{A}^1 \times U} = \delta^n$ and $s|_{0 \times \mathbb{A}^1 \times U \times \mathbb{A}^1} = \delta^n$ and hence

$$S_0 \cap 0 \times \mathbb{A}^1 \times U = 0 \times 0 \times U$$

and

$$S \cap (0 \times \mathbb{A}^1 \times U \times \mathbb{A}^1) = 0 \times 0 \times U \times \mathbb{A}^1.$$

(Definition of quadratic forms $q_P$ and $Q_H$)

To prove that $P$ and $H$ are finitely generated projective over $k[\mathbb{A}^1_U \times \mathbb{A}^1]$ and $k[V \times \mathbb{A}^1_U]$ respectively and to construct quadratic forms $q_P$ and $q_H$ it is useful to consider the morphism of projective smooth schemes over $\mathbb{A}^1_U \times \mathbb{A}^1$

$$\overline{F} = (\langle s : \mu^n \rangle : \mathbb{P}^1_{\mathbb{A}^1_U \times \mathbb{A}^1} \to \mathbb{P}^1_{\mathbb{A}^1_U \times \mathbb{A}^1}).$$

The morphism $\overline{F}$ is finite because it is projective and quasi-finite and it is flat because it is finite morphism of essential smooth schemes of the same dimension.

Next we consider base change of $\overline{F}$ along the embedding $\mathbb{A}^1_{\mathbb{A}^1_U \times \mathbb{A}^1} \hookrightarrow \mathbb{P}^1_{\mathbb{A}^1_U \times \mathbb{A}^1}$. Since $\text{div} \mu = T$ we get the morphism of affine schemes

$$F = \frac{s}{\mu^n} : \mathbb{A}^1_{\mathbb{A}^1_U \times \mathbb{A}^1} \to \mathbb{A}^1_{\mathbb{A}^1_U \times \mathbb{A}^1}$$

that is also finite and flat.

Let’s denote by $B_A$ the algebra corresponding to the morphism $F$, it means that both of $B$ and $A$ as $k$-algebras are isomorphic to the function algebra of relative affine line $k[\mathbb{A}^1_U \times U \times \mathbb{A}^1]$ and homomorphism $A \to B$ is induced by the function $\frac{s}{\mu^n}$. Let’s fix the trivialisation of the canonical class of $\mathbb{A}^1_U \times \mathbb{A}^1 \times U \times \mathbb{A}^1$. Then using the proposition 2.1. from [2] we get $k[B]$-linear isomorphism

$$q_B : k[B] \simeq \text{Hom}_A(k[B], k[A]).$$

Now let’s consider commutative diagram with Cartesian squares
Sublemma 3.2.1 (sublemma 3.3.1 from [6])

Let $B$ are sublagrangian subspaces for $q$ following both of the forms $(k,q)$ of quadratic spaces $P$ affine lines at left and right sides respectively

Then let’s apply to quadratic form $q_B$ base changes along $z$ and $j_0$ and restrictions of scalars along $e$ and $e_0$ and define quadratic forms

$$q_S = z^*(q_B): K[S] \simeq Hom(K[S], K[\mathbb{A}^1_U \times \mathbb{A}^1])$$

$$q_{S_0} = j_0^*(q_S): k[S_0] \simeq Hom_{K[\mathbb{A}^1_U]}(K[S_0], K[\mathbb{A}^1_U])$$

$$q_P = e_0^*(q_{S_0})$$

$$q_H = e_*(q_S).$$

(Checking of point 4, i.e. equalities (6))

The first equality of (6) holds because functoriality of restriction of scalars in respect to base changes

$$i_*(q_P) = (i \circ e_0)_*(q_{S_0}) = (i \circ e_0)_*(j_0^*(q_S)) = j_0^*(e_0)((q_S)) = j_0^*(q_H).$$

The second equality doesn’t necessary true yet. But it becomes true after multiplication booth of the forms $q_H$ and $q_P$ by some invertible function $\lambda \in k[\mathbb{A}^1 \times U]$.

In fact let’s denote $(H, q_H) \otimes_k k[\mathbb{A}^1 \times \mathbb{A}^1 \times U \times 1]$ by $(H_1, q_1)$. Then by definition of $H = k[Z(s)]$ so $H_1 = k[Z(s_{[\mathbb{P}^1 \times \mathbb{A}^1 \times U \times 1]})$ and thus

$$H_1 = H \otimes k[\mathbb{A}^1 \times \mathbb{A}^1 \times U \times 1] \simeq k[\mathbb{A}^1 \times \mathbb{A}^1 \times U]/(\delta^n)$$

($\delta$ is regarded here as function on $\mathbb{A}^1 \times \mathbb{A}^1 \times U$ by trivialising of $\mathcal{L}(T)$ on this subscheme of $\mathbb{P}^1 \times \mathbb{A}^1 \times U$). Then since $q_H$ is $k[\mathbb{A}^1 \times \mathbb{A}^1 \times U \times \mathbb{A}^1]$-linear symmetric isomorphism and $q_{H_1}$ is $k[\mathbb{A}^1 \times \mathbb{A}^1 \times U]$-linear symmetric isomorphism, By the sublemma 3.2.1 for any $k[\mathbb{A}^1 \times \mathbb{A}^1 \times U]$-linear quadratic form $q$ on $k[\mathbb{A}^1 \times \mathbb{A}^1 \times U]/(\delta^n)$ ideals $(\delta^n) \subset k[\mathbb{A}^1 \times \mathbb{A}^1 \times U]$ are sublagrangian subspaces for $i \leq n/2$.

Sublemma 3.2.1 (sublemma 3.3.1 from [6]) Let $B$ be $A$-algebra, and $q: B \simeq Hom_A(B, A)$ be $B$-linear nondegenerate quadratic form on $B$ over $A$. Then for any ideal $I \subset B$ its orthogonal $I^\perp \subset B$ in respect to quadratic form $q$ coincides with annihilator $Ann(I) \subset B$.

So since $n$ is odd by sublagrangian reduction the quadratic space $(k[\mathbb{A}^1 \times \mathbb{A}^1 \times U]/(\delta^n), q_I)$ is equal in Witt-group to some one-ranged quadratic space $(k[\mathbb{A}^1 \times \mathbb{A}^1 \times U]/(\delta^n), \lambda)$ where $\lambda$ is invertible function of diagonal, i.e. $l \in k[\mathbb{A}^1_U]^*.$

Then if we multiply the forms $q_P$ and $q_H$ by $l^{-1}$, we will not violate first equality of (6) and equalities (5) and make the class of $(H_1, q_{H_1})$ in $WCor(\mathbb{A}^1_U, \mathbb{A}^1_U)$ equal to the class of $(1)_{k[\Delta]}$ that is identity morphism of $\mathbb{A}^1_U$.

(Point b))

b)

(Description of quadratic spaces)

To construct the left inverse to $i \in WCor((V, V - 0_U), (\mathbb{A}^1_U, \mathbb{A}^1_U - 0_U))$ means to find quadratic spaces $P$ and $H$ corresponding to $\Psi$ and $\Xi$ from remark 9.b). I.e. to to find following
1) \( P \in k[V \times_U \mathbb{A}_U^1] - \text{mod} \) finitely generated projective over \( k[\mathbb{A}_U^1] \) and \( k[V \times_U \mathbb{A}_U^1] \)-linear symmetric isomorphism \( q_P : P \simeq \text{Hom}(P, k[\mathbb{A}_U^1]) \),

2) \( H \in k[V \times_U \mathbb{A}_U^1] - \text{mod} \) finitely generated over \( k[\mathbb{A}_U^1] \) and \( k[V \times_U \mathbb{A}_U^1] \)-linear symmetric isomorphism \( q_H : H \simeq \text{Hom}(H, k[\mathbb{A}_U^1]) \),

3) canonical homomorphisms

\[
P \otimes_{k[\mathbb{A}_U^1]} k[\mathbb{A}_U^1 - 0_U] \to k[V - 0_U] \otimes_{k[V]} P \otimes_{k[\mathbb{A}_U^1]} k[\mathbb{A}_U^1 - 0_U],
\]

\[
H \otimes_{k[V]} k[V - 0_U] \to k[V - 0_U] \otimes_{k[V]} H \otimes_{k[V]} k[V - 0_U]
\]

(one of structure of \( k[V] \)-module is regarded as left and another as right) are an isomorphism. (It means that spaces \( (P, q_P) \) and \( (H, q_H) \) defines the morphisms of pairs.)

4) in the Witt group of the category \( \text{Proj}(V \times_U V \to V) \) holds the equalities

\[
[(H, q_H) \otimes_{k[V \times_A 1]} k[V^2 \times 0]] = [(P, q_P) \otimes_{k[\mathbb{A}_U^1]} k[V]],
\]

\[
[(H, q_H) \otimes_{k[V \times_A 1]} k[V^2 \times 1]] = [(1) V],
\]

(This equalities are equivalent to the equalities \([4] \).

(Definition of modules \( P \) and \( H \))

Since intersection of \( \Delta \) with \( D_V \) into \( \mathbb{P}_V^1 \) is empty, \( \delta \) is invertible on \( D_V \). Let’s denote by \( \delta^{-1} \in \Gamma(\mathcal{L}(-T)_{D_V}) \) it’s inverse. Next by sublemma \([3, 2] \) for sufficiently large \( n \) there are exist the sections

\[
s' \in \Gamma(\mathcal{L}(n \cdot T), \mathbb{P}_U^1)_{|_{D_V}} = \nu^n, \quad \text{pris}_{|_{0_U}} = \mu^n \cdot \delta
\]

\[
g \in \Gamma(\mathcal{L}((n - 1) \cdot T), \mathbb{P}_U^1 \times V)_{|_{D_V}} = \nu^n \cdot \delta^{-1}, \quad g_{|_{\Delta}} = \mu^{n-1} \cdot g_{|_{0_U \times V}} = \mu^{n-1}
\]

because intersection of \( D_{\mathbb{A}_U^1} \) with \( 0_{\mathbb{A}_U^1} \) into \( \mathbb{P}_U^1 \) is empty and intersection of \( D_V \) with \( \Delta \) into \( \mathbb{P}_V^1 \) is empty too.

Now we can define the sections

\[
s_0 \in \Gamma(\mathcal{L}(n \cdot T), \mathbb{P}_U^1 \times V); \quad s_0 = \mathbb{P}_V \hookrightarrow \mathbb{A}_U^1 \cdot (s')
\]

\[
s_1 \in \Gamma(\mathcal{L}(n \cdot T), \mathbb{P}_U^1); \quad s_1 = g \cdot \delta \in \Gamma(\mathcal{L}(n \cdot T), \mathbb{P}_V)
\]

\[
s \in \Gamma(\mathcal{L}(n \cdot T), \mathbb{P}_U^1 \times \mathbb{A}_U^1); \quad s = s_0 \cdot (1 - t) + s_1 \cdot t.
\]

(\( \text{where in firs line} \mathbb{P}_V \hookrightarrow \mathbb{A}_U^1 \) denotes the embedding \( \mathbb{P}_V \hookrightarrow \mathbb{A}_U^1 \) ) Then by definition of \( s' \), \( g \), \( s_0 \), \( s_1 \) and \( s \)

\[
s_0|_{D_V} = s_1|_{D_V} = s|_{D_V \times \mathbb{A}_U^1} = \nu^n, \quad s_0|_{0_V} = s_1|_{0_V} = s|_{0_V \times \mathbb{A}_U^1} = \mu^{n-1} \cdot \delta, \quad s_1|_{\Delta} = 0, \quad \text{div } s_1 = \text{div } g \bigcup \Delta
\]

Then since \( s \) is invertible on \( D_V \times \mathbb{A}_U^1 \) and \( s_0 \) is invertible on \( D_{\mathbb{A}_U^1} \),

\[
S_0 = Z(s_0) \subset V \times_U \mathbb{A}_U^1, \quad S = Z(s) \subset V \times_U V \times \mathbb{A}^1
\]
So we can put
\[ P = k[S_0]_k[V \times_U \mathbb{A}_{U}^1], \quad H = k[S]_k[V \times_U V \times \mathbb{A}^1]. \]

(Checking of the point 3, i.e. equalities (7))

We should also to define quadratic forms but we can check the equalities (7) just now because it deals only with modules structures of \( P \) and \( H \). To do it first of all note that

\[ P \otimes_{k[\mathbb{A}_{U}^1]} k[\mathbb{A}_{U}^1 - 0_U] \simeq k[S_0 \times_{\mathbb{A}_{U}^1} (\mathbb{A}_{U}^1 - 0_U)], \]
\[ k[V - 0_U] \otimes_{k[V]} P \otimes_{k[\mathbb{A}_{U}^1]} k[\mathbb{A}_{U}^1 - 0_U] \simeq k[(V - 0_U) \times_V S_0 \times_{\mathbb{A}_{U}^1} (\mathbb{A}_{U}^1 - 0_U)]. \]

So first equality of (7) is equivalent to the equality

\[ S_0 \times_{\mathbb{A}_{U}^1} (\mathbb{A}_{U}^1 - 0_U) \simeq (V - 0_U) \times_V S_0 \times_{\mathbb{A}_{U}^1} (\mathbb{A}_{U}^1 - 0_U) \]

that means that
\[ 0_U \times_V S_0 \times_{\mathbb{A}_{U}^1} (\mathbb{A}_{U}^1 - 0_U) = \emptyset. \]

But
\[ s_0 \big|_{0 \times \mathbb{A}_{U}^1} = \delta \]

so
\[ (0 \times \mathbb{A}_{U}^1) \cap S_0 = 0 \times 0 \times U. \]

Similarly second equality is equivalent to

\[ 0_U \times_V S \times_V (V - 0_U) = \emptyset, \]

but
\[ S \cap (0 \times \mathbb{A}_{U}^1 \times \mathbb{A}^1) = 0 \times 0 \times U \times \mathbb{A}^1 \]

because \( s \big|_{0 \times \mathbb{A}_{U}^1 \times \mathbb{A}^1} = \delta. \)

(Definition of quadratic spaces)

Let’s now check that \( P \) and \( H \) are finitely generated projective over \( k[\mathbb{A}_{U}^1] \) and \( k[V \times_U \mathbb{A}^1] \) respectively and define the quadratic forms \( q_P : P \simeq \text{Hom}(P, k[\mathbb{A}_{U}^1]) \) and \( q_H : H \simeq \text{Hom}(H, k[V \times \mathbb{A}^1]) \). It will be done by using the maps

\[ \overline{F'} = [s' : \nu^n] : \mathbb{P}^1_{\mathbb{A}_{U}^1} \to \mathbb{P}^1_{\mathbb{A}_{U}^1}, \quad \overline{F_0} = [s_0 : \nu^n] : \mathbb{P}^1_V \to \mathbb{P}^1_V, \quad \overline{F} = [s : \nu^n] : \mathbb{P}^1_V \times \mathbb{A}^1 \to \mathbb{P}^1_{V \times \mathbb{A}^1}. \]

This maps are finite surjective flat morphisms of essential smooth schemes and are agreed by base changes, i.e \( \overline{F_0} \) coincides with base changes of \( \overline{F'} \) and \( \overline{F} \) along the embeddings \( i : V \hookrightarrow \mathbb{A}_{U}^1 \) and \( j_0 : V^{id_{V} \times 0} V \times \mathbb{A}^1 \) respectively.

Next to construct quadratic forms \( q_P \) and \( q_H \) agreed in sense of (8) we apply to the maps \( \overline{F'}, \overline{F_0} \) and \( \overline{F} \) the same construction as in the point a) simultaneously.

It means following. We consider regular maps \( F', F_0 \) and \( F \) that are base changes of \( \overline{F'}, \overline{F_0} \) and \( \overline{F} \) along the embeddings of affine lines into projective lines

\[ F' = \frac{s'}{\nu^n} : \mathbb{A}_{U}^1 \to \mathbb{A}_{U}^1, \quad F_0 = \frac{s_0}{\nu^n} : \mathbb{A}_V^1 \to \mathbb{A}_V^1, \quad F = \frac{s}{\nu^n} : \mathbb{A}_{V \times \mathbb{A}^1} \to \mathbb{A}_{V \times \mathbb{A}^1}. \]
Thus we get commutative diagram with Cartesian squares

\[
\begin{array}{ccc}
\mathbb{A}^1_{V \times \mathbb{A}^1} & \xrightarrow{F} & \mathbb{A}^1_{V} \\
\downarrow id_V \times i & & \downarrow id_V \\
\mathbb{A}^1_{V \times \mathbb{A}^1} & & \mathbb{A}^1_{V}
\end{array}
\]

where

\[z = 0_{V \times \mathbb{A}^1} : V \times \mathbb{A}^1 \hookrightarrow \mathbb{A}^1_{V \times \mathbb{A}^1}, \quad z_0 = 0_V : V \hookrightarrow \mathbb{A}^1_{V}, \quad z' = 0_{\mathbb{A}^1_U} : \mathbb{A}^1_U \hookrightarrow \mathbb{A}^1_{\mathbb{A}^1_U}.
\]

are embeddings by zero section. Let’s also denote embeddings of subschemes $S$, $S_0$ and $S'$ by

\[e : S \hookrightarrow \mathbb{A}^1_{V \times \mathbb{A}^1}, \quad e_0 : S_0 \hookrightarrow \mathbb{A}^1_V, \quad e' : S' \hookrightarrow \mathbb{A}^1_U.\]

Then we construct symmetric isomorphisms

\[q' : B' \simeq \text{Hom}_{A'}(B', A'), \quad q_0 : B_0 \simeq \text{Hom}_{A_0}(B_0, A_0), \quad q : B \simeq \text{Hom}_A(B, A) : \]

\[j_0^*(q') = q_0 = i^*(q),\]

where $A'$ denotes $k[\mathbb{A}^1_{\mathbb{A}^1_U}]$, $B'_{A'}$ denotes the algebra correspondent to $F'$ and similarly for $A_0$, $B_0_{A_0}$, $A$ and $B_A$ using isomorphism of line bundle of relative canonical class and dual module to function ring for finite flat morphisms and agreed trivialisation of canonical classes.

More detailed, by the proposition 2.1 of [2] there are agreed isomorphisms

\[d' : \omega(F') \simeq \text{Hom}(k[\mathbb{A}^1_{\mathbb{A}^1_U}], k[\mathbb{A}^1_{\mathbb{A}^1_U}]), \quad d_0 : \omega(F_0) \simeq \text{Hom}(k[\mathbb{A}^1_V], k[\mathbb{A}^1_V]), \quad d : \omega(F) \simeq \text{Hom}(k[\mathbb{A}^1_{V \times \mathbb{A}^1}], k[\mathbb{A}^1_{V \times \mathbb{A}^1}]).\]

Thus to find $q$, $q_0$ and $q'$ from [3] it is enough to define agreed trivialisations of $\omega(F')$, $\omega(F_0)$ and $\omega(F)$.

Let’s note that for any smooth schemes $X$, $Y$ and for any relative morphism $f : Y_T \to X_T$ over smooth scheme $T$

\[\omega(f) \simeq \omega(Y_T) \cdot f^*(\omega(X_T))^{-1} \simeq c^*(\omega(Y)) \cdot f^*(c^*(\omega(X)))^{-1}.
\]

where $c$ denoted the projections $X_T \to X$ and $Y_T \to Y$ along $T$. So trivialisation of canonical classes of $Y$ and $X$ defines canonical trivialisation of $\omega(f)$ for all $T$ and $f : Y_T \to X_T$. In our case $X = \mathbb{A}^1$, $Y = \mathbb{A}^1$, $T_1 = \mathbb{A}^1_U$, $T_2 = \mathbb{A}^1_V$ and $T_3 = V \times \mathbb{A}^1$ and fixing trivialisation of canonical class of $\mathbb{A}^1$ we get agreed trivialisations of $\omega(F')$, $\omega(F_0)$ and $\omega(F)$.

get $q_P$ and $q_H$ we apply to $q'$ and $q$ base changes along $z'$, and $z$ and restriction of scalars along $e'$, and $e$:

\[q_P = e'_*(z'^*(q')), \quad q_H = e_*(z^*(q)).\]
(Checking of point 4, i.e. equalities (8))

The first equality of (8) holds due to the existence of \( q_0 \) because

\[
i^*(q_P) = e'_*(z'^*(q')) = e_0^*(z_0^*(q_0))e_*(z^*(q)) = j_0^*(q_H).
\]

To complete the construction it is enough to make the second equality of (8) true (Because it isn’t necessary true yet). Since \( S = Z(s) \) is support of quadratic space \( (H,q_H) \) and \( s|_{\mathbb{P}^1 \times V \times \mathbb{A}^1} = s_1, S_1 = Z(s_1) \) is support of it’s base change \( (H_1,q_{H_1}) \) over unit section \( V \times 1 \subset V \times \mathbb{A}^1 \). Moreover \( H_1 = k[S_1]_{k[\mathbb{A}^1 \times V]} \) (because \( H = k[S]_{k[\mathbb{A}^1 \times V \times \mathbb{A}^1]} \)). Next since \( s_1 = \delta \cdot g \) and \( g|_\Delta \) is invertible,

\[
S_1 = \Delta \coprod Z(g)
\]

So quadratic space \( (H,q_H) \) splits over the unit section \( V \times 1 \subset V \times \mathbb{A}^1 \) into sum

\[
(H_1,q_{H_1}) = q_H \otimes_{k[V \times \mathbb{A}^1]} k[V \times 1] = (E,q_E) \oplus (G,q_G), \ E \simeq k[\Delta \times V \times \mathbb{A}^1], \ G \simeq k[Z(g)].
\]

Since the first summand \( E \) is free module of rank 1 over \( k[U] \), the quadratic form \( q_E \) is defined by some invertible function \( \lambda \) on \( V \). Then let’s multiply \( q_P \) and \( q_H \) by the inverse function \( \lambda^{-1} \) (using the inverse images along the projections of \( V \times_U \mathbb{A}^1_U \) and \( V \times_U V \times \mathbb{A}^1 \) on the first multiplicator), Or equivalently, let’s compose corresponding morphisms in \( WC\) or at the left side with endomorphism of \( V \) defined by \( \lambda^{-1} \):

\[
\Phi \sim (\lambda^{-1})_\Delta \circ \Phi, \ \Theta \sim (\lambda^{-1})_\Delta \circ \Theta.
\]

Then the quadratic form \( q_E \) becomes unit and it doesn’t violate other conditions (i.e. equalities (7) and first equality from (8)).

4 Etale excision

In this section etale excision in an arbitrary dimension \( n \) is proved. It’s generalisation of etale excision on curves proved in [5].

Theorem 2 Let \( \mathcal{F} \) be homotopy invariant presheave with Witt-transfers and \( \pi: X' \to X \) be etale morphism of smooth varieties over the field \( K \) that is field of fractions of some variety over the base field \( k \). Let \( Z \subset X \) be closed subscheme of codimension 1, such that \( \pi \) induces isomorphism between \( Z \) and its preimage \( Z' = \pi^{-1}(Z) \) and let \( z \) be a closed point of \( Z \) and \( z' \) be it’s preimage.

Then \( \pi \) induces the isomorphism

\[
\pi^*: \frac{\mathcal{F}(U - Z)}{\mathcal{F}(U)} \sim \frac{\mathcal{F}(U' - Z')}{\mathcal{F}(U')},
\]

where \( U = \text{Spec}(\mathcal{O}_{X,z}), \ U' = \text{Spec}(\mathcal{O}_{X',z'}) \).

Remark 10 In terms of the remark 6 theorem 2 means that

\[
i^*: \mathcal{F}^{\text{pair}}(U,U - Z) \to \mathcal{F}^{\text{pair}}(U',U' - Z')
\]

is an isomorphism. So it follows from following lemma.
Lemma 4.1 Let $\pi : X \to X'$ be etale morphism of smooth varieties with trivial canonical class, $z$ and $z'$ be closed points of $X$ and $X'$ such that $\pi(z') = z$ and residue fields $k(z)$ and $k(z')$ are isomorphic. Let $Z$ and $Z'$ be closed subschemes of $X$ and $X'$ containing $z$ and $z'$ such that $Z' = \pi^{-1}(Z)$. And let $U = \varprojlim_{x \in V \subset X} V, U' = \varprojlim_{x' \in V' \subset X'} V'$. Then

a) there exists a morphism
$$\Phi \in WCor_K((U, U - Z), (X', X' - Z')) : [\pi \circ \Phi] = [i] \in WCor_K((U, U - Z), (X, X - Z))$$

b) there exists a morphism
$$\Psi \in WCor_K((U, U - Z), (X', X' - Z')) : [\Psi \circ \pi] = [i'] \in WCor_K((U', U' - Z'), (X', X' - Z'))$$

Remark 11 In terms of category $WCor_K$ statement of previous lemma means that there exists

$$\Phi \in WCor((U, U - Z), (X', X' - Z')), \quad \Psi \in WCor((U, U - Z), (X', X' - Z'))$$
$$\Omega \in WCor_K(U, X - Z), \quad \Omega' \in WCor_K(U', X' - Z')$$

such that in the group $WCor_K((U, U - Z), (X, X - Z'))$ following equalities holds:

$$[\pi \circ \Phi] = [i] + [\Omega], \quad [\Psi \circ \pi] = [i'] + [\Omega'].$$

And in terms of the category $WCor_K$ it means the existence of following commutative diagrams

\[ \begin{array}{ccc}
(U - Z) \times \mathbb{A}^1 & \xrightarrow{j_0} & U \times \mathbb{A}^1 \\
\downarrow \pi \circ \Phi & & \downarrow j_1 \\
(U - Z) & \xrightarrow{\pi} & U \\
\downarrow i + \Omega & & \downarrow \Theta \\
X - Z & \xrightarrow{i + \Omega} & X
\end{array} \]

\[ \begin{array}{ccc}
(U' - Z') \times \mathbb{A}^1 & \xrightarrow{j'_0} & U \times \mathbb{A}^1 \\
\downarrow \Psi \circ \pi & & \downarrow j'_1 \\
(U' - Z') & \xrightarrow{\Psi \circ \pi} & U \\
\downarrow i' + \Omega' & & \downarrow \Xi \\
X' - Z & \xrightarrow{i' + \Omega'} & X'
\end{array} \]

such that $j_0, j'_0, j_1$ and $j'_1$ are zero and unit sections respectively.
Proof of the lemma 4.1.

Any morphism from \( U \) to \( X \) (or from \( U' \) to \( X \)) is defined by quadratic space in \( \text{Proj}(X, U) \) (or \( \text{Proj}(X, U') \)) that are categories of projectives modules along the morphisms \( X \times U \to U \) (or \( X \times U' \to U' \)) that has the same relative dimension as \( X \), i.e. \( d \). In the proof of etale excision for curves like as in case of Zariski excision on relative affine line we deals with morphisms of relative dimension 1. To make current situation similar and to reduce it to the case of morphisms of relative dimension 1 we use Quillen’s trick.

(*Construction of relative curves*)

I.e. following Quillen’s trick let’s fix some finite surjective morphism \( p: X \to \mathbb{A}^d_K \) \((d = \text{dim } X)\) and then construct commutative diagram with Cartesian parallelograms

by choosing some linear projection \( pr_{\mathbb{A}} \). (The property that all parallelograms are Cartesian means that \( X' \) is fibred product of \( pr \circ p \) and \( pr \circ p \circ i \), and \( X'' \) is fibred product of \( pr \circ p \circ \pi \) and \( pr \circ p \circ \pi \).) In addition we may choose \( pr_{\mathbb{A}} \) in such way that the projection of closed subscheme \( Z \subset X \to \mathbb{A}^{d-1} \) is finite and unramified at \( z \). (These assumptions isn’t necessary for following construction but allows to simplify it.)

(*Description of quadratic spaces*)

Then since \( X' \) and \( X'' \) have agreed structures of varieties over \( X \times U \) and \( X' \times U \) to construct required morphisms \( \Phi, \Theta \) and \( \Omega \) from the remark 11 it’s enough to find following quadratic spaces:

1) Quadratic space \( P \) in \( \text{Proj}(pr_U) \), where \( pr_U: X' \to U \) is canonical projection. I.e. \( P \in K[X'] - \text{mod} \) finitely generated projective over \( K[U] \) and \( K[X'] \)-linear isomorphism \( q_P: P \cong \text{Hom}_{K[U]}(P, K[U]) \)

2) Quadratic space \( H \) in \( \text{Proj}(pr_{X \times U}) \), where \( pr_{X \times U}: X \times \mathbb{A} \to U \times \mathbb{A} \) is canonical projection. I.e. \( H \in K[X \times \mathbb{A} \times U] - \text{mod} \) finitely generated projective over \( K[U \times \mathbb{A}] \) and \( K[X] \)-linear isomorphism \( q_H: H \cong \text{Hom}_{K[U \times \mathbb{A}]}(H, K[U \times \mathbb{A}]) \),

3) canonical homomorphisms:

\[
P \otimes_{K[U]} K[U - Z] \to K[X' - Z'] \otimes_{K[X']} P \otimes_{K[U]} K[U - Z],
H \otimes_{K[U]} K[U - Z] \to K[X - Z] \otimes_{K[X]} H \otimes_{K[U]} K[U - Z]
\]  

are isomorphisms.

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4) Following isomorphisms of quadratic spaces holds:

\[ j_0^*(H, q_H) \simeq \varpi_*(P, q_P), \]
\[ j_1^*(H, q_H) \simeq (K[\Delta], 1) \oplus (G, q_G), \]  

(11)

where \((K[\Delta], 1)\) denotes space with unit form on \(K[\Delta]\) i.e. the form gotten from unit by isomorphism \(K[\Delta] \simeq K[U]\), and \(G\) is \(K[X]-\text{module}\) such that \(G \simeq K[X - Z] \otimes_{K[X]} G\).

Like as in the proof of etale excision for curves see theorem 1 from [7] and Zariski excision isomorphism on relative affine line made in the previous section the construction of quadratic spaces uses some global sections of line bundles (13) on relative projective curves that are some compactifications of \(X\) and \(X'\).

(Compactification)

Now we construct these projective curves and involve some additional used later definitions and denotations.

Firstly let’s fix an embedding \(e_{A^1} : A^1 \hookrightarrow \mathbb{P}^1\), and consider normalisations of compositions of \(p_U\) and \(\varphi\) with \(e_{A^1}\) that is base change of \(e_h\).

I.e. let \(\overline{p_U} : \overline{X} \to \mathbb{P}^1_U\) be normalisation of the composition \(e_{A^1} \circ p_U : X \to \mathbb{P}^1_U\) and \(\varpi : \overline{X'} \to \overline{X}\) is normalisation of composition of \(\varpi\) with embedding of \(X\) into \(X'\).

\[ \overline{X'} \xrightarrow{\varpi} \overline{X} \xrightarrow{p_U} \mathbb{P}^1_U \quad . \]  

(12)

Let’s note that schemes \(\overline{X}\) are \(\overline{X'}\) contain smooth open subschemes \(X\) and \(X'\) but are not necessarily smooth themselves. Since \(p_U\) and \(\varpi\) are finite, complements to these open subschemes are

\[ D = \overline{X} \setminus \overline{X} = \overline{p_U}^{-1}(\infty_U) \]

(where \(\overline{p_U} = \overline{p_U} \circ \varpi\) is normalisation of \(p'_U \circ \varpi\) and \(\infty_U = \mathbb{P}^1_U \setminus A^1_U\)).

Let’s define relative analogues of closed subsets \(Z\) and \(Z'\) and points \(z\) and \(z' = \pi^{-1}(z)\).

\[
\begin{align*}
\Delta &= \Gamma_{\widetilde{A}^{d-1}}(U \hookrightarrow X) \subset \mathcal{X} \\
\Delta_Z &= \Gamma_{\widetilde{A}^{d-1}}(Z \hookrightarrow X) \subset \mathcal{X} \\
\Delta_{Z'} &= \Gamma_{\widetilde{A}^{d-1}}(Z' \hookrightarrow X') \subset \mathcal{X}' \\
\Delta_z &= \Gamma_{\widetilde{A}^{d-1}}(z \hookrightarrow X) \subset \mathcal{X}, \quad \Delta'_{Z'} &= \Gamma_{\widetilde{A}^{d-1}}(Z' \hookrightarrow X') \subset \mathcal{X}' \\
\Delta_z &= \Gamma_{\widetilde{A}^{d-1}}(z \hookrightarrow X) \subset \mathcal{X}, \quad \Delta'_{Z'} &= \Gamma_{\widetilde{A}^{d-1}}(z' \hookrightarrow X') \subset \mathcal{X}' 
\end{align*}
\]

By definition these are closed subschemes in \(\mathcal{X}\) and \(\mathcal{X}'\) but in fact they are closed subschemes in projective relative curves \(\overline{X}\) and \(\overline{X'}\) because they are finite schemes over \(U\). (Here for schemes \(Z\) and \(Z'\) we use that \(Z\) is finite over \(\widetilde{A}^{d-1}\).)

(Description of sections of line bundles)
In the construction of required quadratic spaces \((P, q_P)\) and \((H, q_H)\) following section of line bundles on \(\mathcal{X}\) and \(\mathcal{X}'\) and \(\mathcal{X} \times \mathbb{A}^1\) are used.

\[
\begin{align*}
s' & \in \mathcal{L}(\mathcal{X}', nD'') : \quad \text{div } s'.Z' = \Delta'_Z, \quad \text{div } s'.D'' = 0, \\
s & \in \mathcal{L}(\mathcal{X} \times \mathbb{A}^1, lD \times \mathbb{A}^1) : \quad \text{div } s.Z \times \mathbb{A}^1 = \Delta_Z \times \mathbb{A}^1, \quad \text{div } s.D \times \mathbb{A}^1 = 0, \\
s_0, s_1 & \in \mathcal{L}(\mathcal{X}, lD) : \quad s|_{\mathcal{X} \times 0} = s_0, \quad s|_{\mathcal{X} \times 1} = s_1, \\
\varpi : Z(s') & \simeq Z(s_0) \simeq \text{div } s_0, \quad s_1|_{\Delta} = 0.
\end{align*}
\tag{13}
\]

Now we assume the existence of such section and construct the required spaces \((P, q_P)\) and \((H, q_H)\). The construction of \(s, s_0\) and \(s_1\) will be given later.

**Construction of quadratic spaces**

Let

\[
\begin{align*}
S' &= Z(s'), \quad S_0 = Z(s_0), \quad S = Z(s), \\
P &= K[S'], \quad H = K[S].
\end{align*}
\]

The quadratic forms \(q_H\) and \(q_P\) are defined by trivialisation of canonical class of \(\mathcal{X} \times \mathbb{A}^1\) and the function

\[
F = (\frac{s}{d^n}, pr_{U \times \mathbb{A}^1}) : \mathcal{X}' \to \mathbb{A}^1 \times U \times \mathbb{A}^1
\]

in following sense.

Since \(\text{div } d = D = \mathcal{X} \setminus \mathcal{X}\), morphism \(F\) is finite. By the same reason as in sublemma 3.1.3 from [7] since \(F\) is finite morphism of varieties of the same dimension, it is surjective. And since \(F\) is finite morphism of smooth schemes of the same dimension it is flat (see for example corollary V.3.9. and theorem II.4.7 in [4]). Then by the proposition 2.1 of [2] there is an isomorphism:

\[
q_B : B \simeq \text{Hom}_A(B, A),
\]

where \(B_A\) denotes algebra corresponding to the morphism \(F\) (i.e. \(B \simeq K[\mathcal{X} \times \mathbb{A}^1], A \simeq K[\mathbb{A}^1 \times U \times \mathbb{A}^1]\) and homomorphism \(A \to B\) is defined by \(F\)). Then let’s put

\[
q_P = e_0* (j_0* (zf^* (q_B))), \quad q_H = e_* (zf^* (q_B)).
\]

**Checking of condition of point 3**, i.e. equation (10)

The condition (10) holds since \(\text{div } s' \cap Z' = \Delta'_Z \subset \mathcal{X}'_Z\) and \(\text{div } s \cap Z \times \mathbb{A}^1 = \Delta_Z \times \mathbb{A}^1 \subset \mathcal{X}_Z\).

**Checking of conditions of point 4**, i.e. equations (11)
This quadratic forms are agreed in the sense of first equality of (11) due to that fact that both of them are gotten by base changes and scalar restrictions from the same isomorphism \( q_B \) and functionality of scalar restriction in respect to base chances.

\[
\text{j}_0^*(q_H) = e_0^* (\text{j}_0^*(q_B)) = \varpi_* (\text{e}'_*(\text{j}_0^*(q_B))) = \varpi_*(q_P).
\]

where \( e_1 : S_1 \hookrightarrow \overline{X} \) denote embedding of closed subscheme \( S_1 = Z(s_1) \).

It can happens that the second equality from (11) doesn’t holds. Nevertheless the restriction of the space \((H, q_H)\) on the unit section \( j_1 : \mathcal{X} \times 1 \hookrightarrow \mathcal{X} \times \mathbb{A}^1 \) splits into direct sum

\[
(H_1, q_{H_1}) = j_1^*(H, q_H) = (E, q_E) \oplus (G, q_G),
\]

\[
G \simeq K[X - Z] \otimes_{K[X]} G,
\]

since it’s support \( S_1 \) splits into disjoint union

\[
S_1 = Z(s_1) = \Delta \coprod R
\]

where \( R \) id closed subscheme of \( X - Z \).

Let’s give formal proof of the last statement. Since \( s_1|_\Delta = 0 \), \( \text{div} \ s_1 > \Delta \). Let \( \text{div} \ s_1 = \Delta + R \) for some effective divisor \( R \) on \( \overline{X} \), and let \( s_1 = \delta \cdot r \) where \( r \in L(\overline{X}, R) \). Then since \( \text{div} \ s_1 \cdot Z = \Delta Z = \Delta, Z, R.Z = 0 \) and \( r \) is invertible on the subscheme \( Z \). Since \( \Delta \) is unique closed point of \( \Delta \) and \( \Delta \subseteq Z, r \) isn’t equal to zero at \( \Delta \) and is invertible on \( \Delta \). Therefore \( r \) is invertible on the subscheme \( \Delta \). Hence on the open subscheme \( \mathcal{X} - Z \) containing \( \Delta \) the sheaf of ideals \( \mathcal{I}(s_1) \) is equal to \( \mathcal{I}(\delta) \), and so \( \Delta \) is connected component of \( S_1 \). The rest part \( R \) of \( S_1 \) is \( Z(r) \) and since \( r \) is invertible on \( Z, R \) is a closed subscheme of \( \mathcal{X} - Z \).

Thus there is isomorphism of algebras \( K[S_1] = K[\Delta] \times K[R] \) and it induce decomposition of quadratic space \( j_1^*(q_B) \) that leads to decomposition of \((H_1, q_{H_1})\) into sum of spaces \((E, q_E)\) and \((G, q_G)\) with the supports \( \Delta \) and \( R \), and modules \( E \) and \( G \) are isomorphic to \( K[\Delta] \) and \( K[R] \) respectively. Then since \( E \) is free module of rank 1 over \( k[U] \), quadratic form \( q_E \) is defined by some invertible function \( \lambda \in K[U]^* \). Let’s multiply quadratic forms \( q_P \) and \( q_H \) on the inverse function \( \lambda^{-1} \). Or equivalently let’s compose Witt-correspondence defined by \((Pmq_P)\) and \((H, q_H)\) with endomorphism of \( U \) in \( WC\text{Cor} \) defined by function \( \lambda^{-1} \) at the right side. Then equation (10) and the first equation of (11) remain true and quadratic form on \( E \) becomes unit. So we get the required equality

\[
(H_1, q_{H_1}) \simeq (1) \Delta \oplus (G, q_G).
\]

(Construction of sections of line bundles from (??))

Now we proceed to construct required sections \( s', s_0, s_1 \) and \( s \) from (??). Like as in proofs of etale excision isomorphism for curves and excision isomorphism in previous section in this construction lemma 3.3.2 direct images of divisors and affine linear homotopy. let’s note that since \( \overline{\mathcal{P}U} \) and \( \varpi \) are finite, and since the sheave \( \mathcal{L}(\infty_U) \) is very ample, sheaves \( \mathcal{L}(D) = \overline{\mathcal{P}}^*(\mathcal{L}(\infty_U)) \mathcal{L}(D'') = \varpi^*(\mathcal{L}(D)) \) are ample too.

We start with construction of some section \( s' \)

\[
s' \in \mathcal{L}(\overline{\mathcal{X}}, nD''): \text{div} \ s' . \mathcal{L}' = \Delta'_Z, \text{div} \ s'.D'' = 0, \varpi : Z(s') \simeq \varpi(Z(s'))
\]

for sufficiently large \( n \). And firstly using sublemma 4.1.1 we construct it on a closed fibre of \( \overline{\mathcal{X}} \), i.e. on closed subscheme \( \overline{\mathcal{X}}_z = \overline{\mathcal{X}} \times_U z \subseteq \overline{\mathcal{X}} \). Namely by sublemma 4.1.1 applying
to morphism \( \varpi_z : \overline{\mathcal{X}}_z \to \mathcal{X}_z \) (that is closed fibre of \( \varpi \)) the sheaf \( \mathcal{L}(D''_z) \) (where \( D''_z = D'' \times_U z \)) closed point \( \Delta_z' \in \overline{\mathcal{X}}_z \) and closed subscheme \( (\mathcal{Z}'_z \cup D'_z) - \Delta'_z \) (it is exactly the set of closed points of \( \mathcal{Z}' \) and \( D' \) distinct to \( \Delta'_z \)) for all \( n \) larger some \( k \) there is a section

\[
\overline{s}' \in \mathcal{L}(\overline{\mathcal{X}}_z, nD''_z): \quad \text{div} \overline{s}' \cdot \mathcal{Z}_z = \Delta'_z, \quad \text{div} \overline{s}' \cdot D''_z = 0, \quad \varpi_z : \text{div} \overline{s}' \simeq \varpi_z^* (\text{div} \overline{s'})
\]

(or equivalently \( \overline{s}'|_{\Delta'_z} = 0, \overline{s}'|_{\xi \neq 0} \forall \xi \in (\mathcal{Z}'_z \cup D'_z) - \Delta'_z ) \).

**Sublemma 4.1.1 (Sublemma 3.1.2 from [7])** Let \( \pi : X' \to X \) be finite morphisms of projective curves over infinite field, \( z \) be a closed point of \( X' \), \( Y \not\cong z \) and \( \mathcal{L} \) ample invertible sheaf on \( X' \). Then for all \( n \) larger some \( k \) there exists global section \( s \) of \( \mathcal{L}^n \) on \( X' \), such that \( s \) is equal to zero at \( z \), \( s \) isn't equal to zero at any point of \( Y \) and such that restriction of \( \pi \) onto \( Z(s) \) is closed embedding.

Then to lift the section \( \overline{s}' \) to global section on \( \overline{\mathcal{X}} \) let’s note that schemes \( \mathcal{Z}' \) and \( D'' \) are finite over local scheme \( U \) and hence they are semi-local. So since any line bundle on semi-local scheme is trivial (and any divisor is prime) there is a section

\[
\delta_{\mathcal{Z}' \cup D''} \in \mathcal{L}(\mathcal{Z}' \cup D''): \quad \text{div} \delta_{\mathcal{Z}' \cup D''} = \Delta'_Z.
\]

Moreover we may assume that

\[
\delta_{\mathcal{Z}' \cup D''}|_{\mathcal{Z}'_z \cup D'_z} = \overline{s}'|_{\mathcal{Z}'_z \cup D'_z},
\]

because \( \Delta'_Z \cap \overline{\mathcal{X}}'_z = \Delta'_Z \mathcal{Z}' \neq 0 \) at another points of \( \mathcal{Z}'_z \cup D'_z \) (Here we use that projection \( Z \to \mathcal{X}_K \) is unramified at \( z \) because we doesn’t state in sublemma 3.1.1 that degree of zero of section \( \overline{s}' \) at \( \Delta'_z \) is one ). And now by lemma 3.2 applying to the sheave \( \mathcal{L}(D'') \) and closed subscheme \( \overline{\mathcal{X}}_z \cup \mathcal{Z}' \cup D'' \) for all \( n \) larger some \( k \) there is a section

\[
s' : s'|_{\overline{\mathcal{X}}'_z} = \overline{s}'|_{\mathcal{Z}' \cup D'}, \quad \delta_{\mathcal{Z}' \cup D'}.
\]

Let’s check the properties (**??**). \( \text{div} s' \cdot \mathcal{Z}' = \Delta'_Z \) by definition of \( \delta_{\mathcal{Z}' \cup D'} \). And \( \text{div} s'.D' = 0 \) because \( \Delta'_Z \) doesn’t intersect with \( D' \) and so \( \delta_{\mathcal{Z}' \cup D'} \) is invertible on \( D \).

The last property states that the restriction of \( \varpi \) onto \( Z(s') \) is closed embedding of schemes. And since it is true at the closed fibre (by definition of \( s' \)), it is true over local base \( U \). In fact this property is equivalent to that morphism of coherent sheaves \( \epsilon_{\varphi} : \mathcal{O}(\overline{\mathcal{X}}) \to \varpi_* (\mathcal{I}(s')) \) induced by \( \varpi \) is surjective. But support of its cokernel \( \text{Supp coker}(\epsilon_{\varphi}) \) is closed subscheme in relative projective scheme over local scheme. So if it isn’t empty then its closed fibre isn’t empty too.

Next since \( \varpi(nD'') = lnD \) (where \( l = deg \text{ovvarpi} \)) there is a section

\[
s_0 \in L(\mathcal{X}, lnD): \quad \text{div} s_0 = \varpi^* (\text{div} s').
\]

Then

\[
\text{div} s_0.\mathcal{Z} = \varpi^* (\text{div} s'.\mathcal{Z}') = \Delta_Z, \quad \text{div} s_0.D = \varpi^* (\text{div} s'.D'') = 0.
\]

Next since \( \Delta \cap \mathcal{Z} = \Delta_\mathcal{Z} \) and \( \Delta \cap D = \emptyset \), by lemma 3.2 applied to sheave \( \mathcal{L}(lnD) \) on \( \overline{\mathcal{X}} \) and closed subscheme \( \Delta \cup \mathcal{Z} \cup D \) for all \( n \) larger some \( k_1 \) there is a section

\[
s_1 \in L(\mathcal{X}, lnD): \quad \text{div} s_1.\mathcal{Z} = \text{div} s_1.\mathcal{Z}, \quad \text{div} s_1.D = \text{div} s_1.D, \quad s_1|_\Delta = 0.
\]
Thus for all $n > \max(k, k_1)$ there are sections $s', s_0$ and $s_1$ described above and finally let’s put
\[ s = s_0 \cdot (1 - t) + s_1 \cdot t \in L(\mathcal{X}, \ln D \times \mathbb{A}^1). \]

**Point b**

Now to construct morphisms between $U'$ and $X'$ we consider fibred products
\[ X'' = X' \times_U U', \overline{X''} = \overline{X'} \times_U U' \]
and their closed subschemes
\[ Z'' = p_U' \circ (Z'), \Delta' = \Gamma_{\mathcal{A}^n}^{-1}(U' \hookrightarrow X'), \Delta_Z'' = \Gamma_{\mathcal{A}^n}^{-1}(Z' \hookrightarrow X') = \partial^{-1}(\Delta_Z), \]
\[ \Delta''_z = \Gamma_{\mathcal{A}^n}^{-1}(z' \hookrightarrow X') = \partial^{-1}(\Delta'_z) \]

**Description of quadratic spaces**

To find required morphisms $\Psi$, $\Omega'$, $\Xi$ from remark 11 it is enough to construct following data

1) The quadratic space $(P, q_P)$ in the category $\text{Proj}(pr_U)$. I.e. $P \in K[\mathcal{X}'] - \text{mod}$ that is finitely generated over $K[U]$ and $K[\mathcal{X}']$-linear isomorphism $P \simeq \text{Hom}_{K[U]}(P, K[U])$,

2) The quadratic space $(H, q_H)$ in the category $\text{Proj}(pr_{U' \times \mathbb{A}^1})$. I.e. $H \in K[\mathcal{X}''] \times \mathbb{A}^1]$ - $\text{mod}$ that is finitely generated over $K[U' \times \mathbb{A}^1]$ and $K[\mathcal{X}'] \times \mathbb{A}^1]$-linear isomorphism $H \simeq \text{Hom}_{K[U' \times \mathbb{A}^1]}(H, K[U' \times \mathbb{A}^1])$,

such that

3) canonical homomorphisms
\[ P \otimes_{K[U]} K[U - Z] \rightarrow K[\mathcal{X}' - Z'] \otimes_{K[\mathcal{X}']} K[\mathcal{X}'], \quad H \otimes_{K[U']} K[U' - Z'] \rightarrow K[\mathcal{X}' - Z'] \otimes_{K[\mathcal{X}']} K[\mathcal{X}'], \quad \] (15)

are isomorphisms,

4) There exists isomorphisms of quadratic spaces
\[ j_0^*(q_H) \simeq (id \times \pi)^*(q_P), \]
\[ j_1^*(q_H) \simeq (1)_{\Delta'} \oplus q_G. \]

(16)
Here \((1)_{\Delta'}\) denotes unit quadratic form on \(K[\Delta']\) (i.e., the form that is gotten from unit by isomorphism \(K[\Delta'] \simeq K[U']\)), and \(G\) is \(K[\mathcal{X}']\)-module equipped with quadratic form \(q_G\) such that
\[
G \simeq K[X' - Z'] \otimes_{K[X']} G.
\]

(Construction of sections \(s', s_0, s_1, s\))

Then since \(\mathcal{L}(\infty_U)\) is ample and \(\mathcal{L}(\infty_U)\) is finite, \(\mathcal{L}(D') = \mathcal{L}(\infty_U)\) is ample. Similar the sheaf \(\mathcal{L}(D'')\) is ample because \(\vartheta\) is finite and \(D'' = \vartheta^{-1}(D')\).

As usual the construction is based on lemma 3.2. We will consequently find the restrictions of section on some closed subschemes and continue them to whole schemes by using of lemma 3.2. Thus to execute the algorithm we should previously choose sufficiently large \(n\) such that all restriction homomorphisms of groups of section of sheaves were surjective.

Let’s give first short description of this process as follows
\[
\Gamma(\mathcal{X}'', \mathcal{L}(nD')) \ni s' \mapsto s'_{|\Delta' \sqcup D'} = \delta_{\Delta' \sqcup D'},
\]
\[
\Gamma(\mathcal{X}'', \mathcal{L}(nD'')) \ni s_0 = \pi X'^*(s')
\]
\[
\Gamma(\mathcal{X}'', \mathcal{L}(nD'')) \ni s_1 \mapsto s_1_{|\Delta'' \sqcup D''} = s_0_{|\Delta'' \sqcup D''},
\]
\[
\Gamma(\mathcal{X}' \times \mathbb{A}^1, \mathcal{L}(nD'' \times \mathbb{A}^1)) \ni s = (1 - t) \cdot s_0 + t \cdot s_1.
\]

Now let’s explain it in detail. In first row we apply lemma 3.2 to line bundle \(\mathcal{L}(D')\) and closed subscheme \(\Delta' \sqcup D' \subset \mathcal{X}'.\) Scheme \(\Delta' \sqcup D'\) is quasi-finite projective and hence projective over local scheme \(U\). Hence it is semi-local. Let’s consider \(\Delta'\) as divisor in \(\mathcal{X}'\) (and in \(\Delta' \sqcup D'\)). Since any divisor on semi-local scheme is prime (and any line bundle is trivial) there is some section
\[
\delta_{\Delta' \sqcup D'} \in \mathcal{L}(nD')_{\mathcal{X}' \sqcup D'}: \quad \text{div} \delta_{\Delta' \sqcup D'} = \Delta'
\]
and we set \(s'\) to be equal to this section on \(\mathcal{X}' \sqcup D'.\) (In particular it implies that \(s'\) is invertible on \(D'\).

Next we define section \(s_0\) explicitly as inverse image of \(s'\). Further we apply lemma 3.2 to the same bundle and to closed subscheme \(\Delta' \sqcup D'\). To check that conditions are compatible it is enough to note that \(\Delta' \cap \Delta' = \Delta''_{\Delta'\Delta'}\) and that \(\Delta''_{\Delta'\Delta'} = \vartheta^{-1}(\Delta'_{\Delta'\Delta'})\) (and hence \(s_0_{|\Delta'_{\Delta'}} = 0\)). Finally we explicitly define \(s\) as homotopy of \(s_0\) and \(s_1\). Then it is immediate follows from definition that
\[
s' \in \Gamma(\mathcal{X}', \mathcal{L}(nD')): \quad \text{div} s'.\Delta' = \Delta'_{\Delta'}, \quad \text{div} s'.D' = 0,
\]
\[
s \in \Gamma(\mathcal{X}' \times \mathbb{A}^1, \mathcal{L}(nD'')): \quad \text{div} s.(\Delta'' \times \mathbb{A}^1) = \Delta''_{\Delta'_{\Delta'}} \times \mathbb{A}^1, \quad \text{div} s.(D'' \times \mathbb{A}^1) = 0,
\]
\[
s_0 \in \Gamma(\mathcal{X}'', \mathcal{L}(nD'')): \quad s_0 = \pi X'^*(s'),
\]
\[
s_1 \in \Gamma(\mathcal{X}'', \mathcal{L}(nD'')): \quad s_1|_{\Delta''_{\Delta'}} = 0.
\]

(Definition of quadratic spaces)

The construction spaces \((P, q_P)\) and \((H, q_H)\) is similar to that one in previous section. In short, divisors of sections \(s'\) and \(s\) will be supports of required modules and to define
quadratic forms, we variate zero divisors of \( s' \) and \( s \) to a families of schemes that constitutes together smooth schemes with trivialized canonical class.

In detail we start with three following morphisms of relative projective curves corresponding to the sections \( s' \), \( s_0 \) and \( s \).

\[
egin{align*}
\mathcal{F}'' &= ([s' : d''], pr_U) : \mathcal{X}'' \to \mathbb{P}^1 \times U, \\
\mathcal{F}_0 &= ([s_0 : d''], pr_U) : \mathcal{X}'' \to \mathbb{P}^1 \times U', \\
\mathcal{F} &= ([s : d''], pr_{U' \times \mathbb{A}^1}) : \mathcal{X}'' \times \mathbb{A}^1 \to \mathbb{P}^1 \times U' \times \mathbb{A}^1
\end{align*}
\]

The fibres of these morphisms over zero section of affine line (at the left side) are \( S \), \( S_0 \), and \( S' \). Then to get a morphisms of smooth varieties we consider neighbourhoods of these zero fibres of \( F \), \( F_0 \) and \( F'' \). I.e. we consider closed subscheme of \( \mathbb{P}^1 \times U \) that is image of \( D' \) along \( \mathcal{F}'' \) and its preimages in \( \mathbb{P}^1 \times U' \) and \( \mathbb{P}^1 \times U' \times \mathbb{A}^1 \)

\[
C' = F''(D'), \quad C_0 = F_0(D'') = (id_{\mathbb{P}^1} \times \pi)^{-1}(C'), \quad C = F(D'' \times \mathbb{A}^1) = (pr_h \times id_{U'})^{-1}(C_0).
\]

And then we consider base changes of \( \mathcal{F}, \mathcal{F}_0 \) and \( \mathcal{F}'' \) over open subschemes that are complements to this closed subschemes

\[

\begin{align*}
F' &= \frac{s'}{d''} : \mathcal{Y} \to \mathbb{P}^1_U - C', \quad \mathcal{Y} = \mathcal{X} \times \mathbb{A}^1 - \mathcal{F}^{-1}(C'), \\
F_0 &= \frac{s_0}{d''} : \mathcal{Y}_0 \to \mathbb{P}^1_U - C_0, \quad \mathcal{Y}_0 = \mathcal{X}'' - \mathcal{F}_0^{-1}(C_0) = \mathcal{Y} \times_U U', \\
F &= \frac{s}{d''} : \mathcal{Y} \to \mathbb{P}^1_{U' \times \mathbb{A}^1} - C, \quad \mathcal{Y} = \mathcal{X} - \mathcal{F}^{-1}(C) = \mathcal{Y}_0 \times \mathbb{A}^1.
\end{align*}
\]

Since \( s'|_{D'}, s_0|_{D''}, \) and \( s|_{D'' \times \mathbb{A}^1} \) are invertible, \( \mathcal{Y}' \subset \mathcal{X}' \), \( \mathcal{Y}_0 \subset \mathcal{X}'' \), \( \mathcal{Y} \subset \mathcal{X}'' \times \mathbb{A}^1 \) and hence schemes \( \mathcal{Y}', \mathcal{Y}_0, \mathcal{Y} \) are smooth. And by the same reason zero sections of relative projective lines doesn’t intersect with \( C', C_0 \) and \( C \).

So we get following commutative diagram with Cartesian squares

where \( zf, zf_0 \) and \( z'f \) denotes zero sections of the affine line at the left side and \( cs, cs_0 \) and \( c' \) denotes closed embeddings of subschemes.

Morphisms \( F, F_0, F'' \) are finite as quasi-finite projective morphisms. Hence \( F, F_0, F' \) are finite too. And since \( \mathcal{X}'' \), \( \mathbb{A}^1 \times U' \), \( \mathcal{X}' \) and \( \mathbb{A} \times U \) are essential smooth schemes of the same dimension \( d + 1 \), morphisms \( F, F_0, F' \) are flat.
Then if we denote by $B_A$, $B_{0A_0}$ and $B'_{A'}$, the algebras corresponding to the finite flat morphisms of essential smooth schemes $F$, $F_0$, and $F'$, then proposition 2.1 from [2], provides isomorphisms

$$q_B: B \simeq \text{Hom}_A(B, A), q_{B_0}: B_0 \simeq \text{Hom}(B_0, A_0), q_{B'}: B' \simeq \text{Hom}(B', A'): \id_{A \times U} \times 0^*(q_B) = q_{B_0}, \id_{A}^1 \times \pi_{X'^*}(q_{B'}) = q_{B_0}.$$ 

Now applying base changes along zero sections $z_f, z_{f_0}$ and $z'_f$ and restrictions of scalars along closed embeddings $c_s, c_{s_0}$ and $c'_s$ we define quadratic forms

$$(K[S], q_s) = zf^*(B, q_B), \quad (K[S_0], q_{s_0}) = zf_0^*(B_0, q_{B_0}), \quad (K[S'], q_{s'}) = z'_f^*(B', q_{B'})$$

$$(H, q_H) = c_{s*}(K[S], q_s), \quad (H_0, q_{H_0}) = c_{s_0*}(K[S_0], q_{s_0}), \quad (P, q_P) = c'_{s*}(K[S'], q_{s'}).$$

In particular it means that

$$H = K[S]_{K[X'' \times A^1]}, \quad P = K[S']_{K[X']}$$

and simultaneously with defining of quadratic forms we have prove that this modules are projective finitely generated over $K[U' \times A^1]$ and $K[U]$.

*(Checking of point 3, i.e. equalities (15))*

Condition (15) holds, because

$$\text{div } s \cap (Z'' \times A^1) = \Delta''_z \times A^1 \subset X''_Z \times A^1, \quad \text{div } s' \cap Z' = \Delta'_z \subset X'_Z$$

where $X'_Z \to Z$ denoted base change of $X' \to U$ along the embedding $Z \hookrightarrow U$ and $X''_Z \to Z'$ denoted base change of $X'' \to U'$ along the embedding $Z' \hookrightarrow U'$.

*(Checking of point 4, i.e. equalities (16))*

The first equality of (16) holds because of existence of form $q_{H_0}$ and functoriality of scalar restrictions in respect to base changes

$$U' \times 0^*(q_H) = q_0 = \pi_{X'^*}(q_P).$$

The second equality from (16) doesn’t necessary holds in full power for defined above spaces. But due to properties of $s_1 = s_{|X'' \times 1}$ it is true that $(H, q_H)$ splits after the restriction on $X'' \times 1$ into direct sum of some spaces

$$(H_1, q_{H_1}) = j_1^*(H, q_H) \simeq (\Lambda, q_\Lambda) \oplus (G, q_G): \Lambda \simeq K[\Delta'], \lambda \in K[\Delta'^*] = K[U'^*], \quad K[\Lambda \otimes_k G] \simeq G. \quad (17)$$

Because

$$H_1 = K[S_1]_{K[X'']}, \quad S_1 = Z(s_1)$$

and $q_{H_1}$ can be considered $K[S_1]$-linear isomorphism. But the equalities

$$s_1|_{\Delta'} = 0, \quad \text{div } s_1, Z'' = \Delta_Z'$$

implies that

$$S_1 = Z(s_1) = \Delta \bigcup \mathbb{R}$$

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where \( R \) is closed subscheme of \( X'' - Z'' \)

The formal proof of last fact can be done for example like as in similar place in point a).

To satisfy the second condition of (\[16\]) we modify constructed quadratic forms in such way to make morphism of pair \((U', Z') \rightarrow (X', Z')\) defined in \( WCor \) by the first summand \((\Lambda, \lambda)\) from (\[17\]) to be equal to the morphism defined by space \((\Lambda, 1)\). Namely we choose a function

\[
\lambda' \in K[U]^*: \lambda'(z) = \lambda(z')
\]

and multiply quadratic forms \(q_H, q_P\) and other forms in the construction on its inverse \(\lambda'^{-1}\) using that fact that all bases of considered spaces (i.e. schemes \(U' \times \mathbb{A}^1, U'\) and \(pbase\)) are schemes over \(U\). After such multiplication the function \(\lambda\) defines quadratic form on the first summand of \((H_1, q_1)\) becomes equal to 1 at the closed point of \(U'\). Then because of proved in point a) existence of right inverse like as in the proof of surjectivity of etale excision on curves (see sublemma 3.1.4 from [7]) morphisms of pairs \((U', Z') \rightarrow (X', Z')\) in \(WCor\) defined by quadratic spaces \((\Lambda, \lambda)\) and \((\Lambda, 1)\) are equal.

**Lemma is proved**

## 5 Strictly homotopy invariance

It was proved in [5] that Nisnevich sheafication of homotopy invariant presheave with Witt-transfer is homotopy invariant.

At this section Zariski excision and etale excision isomorphism proved in previous sections are applied to the prove of strictly homotopy invariance of homotopy invariant sheaves with Witt-transfers.

**Theorem 3** Nisnevich sheafication of the homotopy invariant presheave with Witt-transfers is strictly homotopy invariant. i.e. the presheaves of Nisnevich cohomologies \(H^i_{Nis}(F_{Nis}), i \geq 0\) of the sheafication of homotopy invariant presheave with Witt-transfers \(\mathcal{F}\) are homotopy invariant.

We start with particular case of the theorem for spectres of fields that states that cohomology of affine line over any point in \(Sm_k\) are isomorphic to the cohomology of this point.

**Theorem 4** Let \(\mathcal{F}\) be homotopy invariant presheave with Witt-transfers and \(K = k(X)\) be a field of functions of some smooth variety \(X\) over the base field \(k\). Then

\[
\mathcal{F}_{Nis}(\mathbb{A}_K^1) \cong \mathcal{F}(\mathbb{A}_K^1),
\]

\[
H^i_{Nis}(\mathbb{A}_K^1, \mathcal{F}_{Nis}) \cong 0, \quad i > 0.
\]

**Proof of theorem.** The first equality for global section \(\mathcal{F}(\mathbb{A}_K^1)\) is a particular case of homotopy invariance of \(\mathcal{F}_{Nis}\) proved in [5] (theorem 4).

To compute higher Nisnevich cohomologies let’s consider following flat resolvent of the restriction of \(\mathcal{F}_{Nis}\) on small etale site over \(\mathbb{A}_K^1\)

\[
\mathcal{F}_{Nis}|_{\mathbb{A}_K^1} \rightarrow \eta_*(\mathcal{F}(\eta)) \xrightarrow{d} \sum_{z \in \text{MaxSp}(U)} z_*(\text{coker}(\mathcal{F}(U^h_z) \rightarrow \mathcal{F}(U^h_z - z))), \quad (18)
\]
where $\eta$ is generic point of $\mathbb{A}^1_K$, $\eta*$ denotes direct image along the embedding $\eta \to \mathbb{A}^1_K$, $\text{MaxSp}(\mathcal{U})$ denotes the set of closed points $z$ in any element $\mathcal{U}$ of small etale site, $z*$ denotes direct image along the embedding $z \to \mathcal{U}$ and $U^h_z$ denotes corresponding Hensel local scheme (that is spectrum of Henselisation of local ring at $z$). The first arrow is injective by the injectivity theorem for presheaves with Witt-transfers proved by K.Chepurkin in his diploma work. The second arrow is surjective because $\text{dim}\mathcal{U} = 1$ for all $\mathcal{U}$ in small etale site of $\mathbb{A}^1_K$ and so $U^h_z \text{ is generic point of } U^h_z$. And exactness in middle term follows from definition of Nisnevich sheafication.

Then since this is a flat resolvent of length 2, $H^0_{\text{Nis}}(\mathcal{U}) = \ker(d(\mathcal{U}))$, $H^1_{\text{Nis}}(\mathcal{U}) = \coker(d(\mathcal{U}))$, and higher cohomologies are zero.

To show that $H^1_{\text{Nis}}(\mathcal{U})$ are zero it is enough to check exactness in last term of the sequence combined by global sections of $\mathcal{F}$ and global sections of the resolvent $\{18\}$

$$\mathcal{F}(U) \xrightarrow{i} \mathcal{F}(\eta) \xrightarrow{d^1} \sum_{z \in \text{MaxSp}(\mathcal{U})} \coker(\mathcal{F}(U^h_z) \to \mathcal{F}(U^h_z - z)).$$

But the second arrow is surjective due to surjectivity of the excision homomorphism proved in theorem 2 of $[7]$. Theorem is proved.

Proof of the theorem $[3]$ The proof is based on the main theorems form $[5]$ and theorems $\{1\}$ and $\{2\}$ and it is similar to the proof of the same result for sheaves with Cor-transfers form $[1]$.

We will prove it by induction on $i$ for all homotopy invariant sheaves with Witt-transfers simultaneously. The base of induction is $i = 0$ and it obviously holds. Suppose that the statement of theorem holds for $i - 1$.

Let $p: X \times \mathbb{A}^1 \to X$ denotes projection and $i: X \to X \times \mathbb{A}^1$ denotes embedding of zero section. The composition $i^* \circ p^*: H^i_{\text{Nis}}(X) \to H^i_{\text{Nis}}(X \times \mathbb{A}^1)$ is isomorphism equivalent to prove that kernel of $i^*$ is zero.

Let $a \in H^i_{\text{Nis}}(X \times \mathbb{A}^1, \mathcal{F})$, $i^*(a) = 0$.

Since $H^i_{\text{Nis}}(\eta \times \mathbb{A}^1) = 0$ (where $\eta$ is generic point of $X$), $a|_{\eta \times \mathbb{A}^1} = 0$ and hence $a|_{U \times \mathbb{A}^1} = 0$ for some open affine subscheme $U \subset X$. Let $Z_1 = \text{sign } Z$ is the subset of singular points of $Z$. Let $U_1 = X \setminus Z_1$. By the following lemma applied to $U_1$ and $U$ $a|_{U_1 \times \mathbb{A}^1} = 0$. And by induction we can prove that

$a|_{U_i \times \mathbb{A}^1} = 0$, where $U_i = X \setminus Z_i$, $Z_i = \text{sign } Z_i - 1$.

Since $k$ is perfect

$$\text{dim } Z > \text{dim } Z_1 > \cdots > \text{dim } Z_i > \cdots$$

Hence for some finite $i Z_i = \emptyset$ and $U_i = X$. Thus we get that $a|_{X \times \mathbb{A}^1} = 0$ for any $a \in \text{ker } i^*$.

Lemma 5.1 Let $U$ be open subscheme of smooth affine scheme $X$ and $Z = X \setminus U$ is smooth. Let $a \in H^i_{\text{Nis}}(X \times \mathbb{A}^1, \mathcal{F})$, $i^*(a) = 0$, $a|_{U \times \mathbb{A}^1} = 0$

(where $i$ denotes zero section $X \times Z \times \mathbb{A}^1$). Then $a = 0$. 

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**Proof of lemma.** Let’s consider short exact sequence of presheaves

$$\mathcal{F} \xrightarrow{\epsilon} j_*(j^*(\mathcal{F})) \rightarrow \text{coker}\epsilon$$ (20)

where $j$ denotes embedding of $U \times \mathbb{A}^1$ into $X \times \mathbb{A}^1$. Using theorems 2 and 1 we get

$$\text{coker}\epsilon(V) \overset{\text{def}}{=} \frac{\mathcal{F}(V \setminus Z)}{\mathcal{F}(V)} \cong \frac{\mathcal{F}(\mathbb{G}_m \times (Z \cap U))}{\mathcal{F}(\mathbb{A}^1 \times (Z \cap U))} \overset{\text{def}}{=} \mathcal{F}[1](Z \cap U).$$

Thus there is isomorphism of presheaves $\text{coker}\epsilon \cong \langle i_*(\mathcal{F}[1]) \rangle$ where $i$ denotes the embedding of $Z \times \mathbb{A}^1$ into $X \times \mathbb{A}^1$ and $\mathcal{F}[1]$ is regarded as presheave on $Z \times \mathbb{A}^1$. The sheave $\text{Hom}(\mathbb{G}_m, \mathcal{F})$ splits into direct sum $\mathcal{F} \oplus \mathcal{F}[1]$ because of existence of the projection $\mathbb{G}_m \rightarrow pt$ and unit section $pt \rightarrow \mathbb{G}_m$. Hence $\mathcal{F}[1]$ is the Nisnevich sheave and (20) is the exact sequence of Nisnevich cohomology

$$\ldots H^{i-1}_{\text{Nis}}(X \times \mathbb{A}^1, i_*(\mathcal{F}[1])) \xrightarrow{\delta} H^i_{\text{Nis}}(X \times \mathbb{A}^1, \mathcal{F}) \rightarrow H^i_{\text{Nis}}(X \times \mathbb{A}^1, j_*(j^*(\mathcal{F}))) \rightarrow \text{dots.}$$

Note that $H^{i-1}_{\text{Nis}}(X \times \mathbb{A}^1, i_*(\mathcal{F}[1])) \cong H^{i-1}(Z \times \mathbb{A}^1, \mathcal{F}[1])$ and $H^i_{\text{Nis}}(X \times \mathbb{A}^1, j_*(j^*(\mathcal{F}))) \cong H^i_{\text{Nis}}(U \times \mathbb{A}^1, \mathcal{F})$.

We get following commutative diagram

$$
\begin{array}{cccc}
H^{i-1}(Z \times \mathbb{A}^1, \mathcal{F}[1]) & \xrightarrow{p^*} & H^i_{\text{Nis}}(V \times \mathbb{A}^1, \mathcal{F}) & \xrightarrow{i^*} & H^i_{\text{Nis}}(U \times \mathbb{A}^1, \mathcal{F}) \\
& \uparrow{p_*} & & \uparrow{i_*} & \\
H^{i-1}(Z, \mathcal{F}[1]) & \xrightarrow{p^*} & H^i_{\text{Nis}}(V, \mathcal{F}) & \xrightarrow{i_*} & H^i_{\text{Nis}}(U, \mathcal{F})
\end{array}
$$

Since $a|_{U \times \mathbb{A}^1} = 0$, then

$$a|_{V \times \mathbb{A}^1} = \delta(b), \quad b \in H^{i-1}(V \times \mathbb{A}^1, \mathcal{F}[1]).$$

Since the sheave $\mathcal{F}[1]$ is direct summand of $\text{Hom}(\mathbb{G}_m, \mathcal{F})$, it is homotopy invariant sheave with Witt-transfers, and by induction assumption

$$H^{i-1}(V \times \mathbb{A}^1, [1]) \cong H^{i-1}(X, \mathcal{F}[1]).$$

Hence $b = p^*(i^*(b))$ and since $\delta$ commutes with $p^*$ and $i^*$,

$$a|_{V \times \mathbb{A}^1} = p^*(i^*(a|_{V \times \mathbb{A}^1})) = 0.$$

**Lemma is proved.**

**Theorem is proved.**

### 6 Preserving of Witt-transfers

In current section we prove that Nisnevich sheafification and cohomology of presheave with Witt-transfers has Witt-transfers.

The key fact that provides such behaviour is that any Witt-correspondence form locale Henzel scheme to any other scheme is equal to the sum of Witt-correspondence between locale Henzel schemes.
Theorem 5 There is an unique structure of presheave with Witt-transfers on the Nisnevich sheafication $\mathcal{F}_{Nis}$ such that $\varepsilon: \mathcal{F} \to \mathcal{F}_{Nis}$ is homomorphism of presheaves with Witt-transfers.

Moreover it is natural i.e. there is a functor $PreWtr \to ShNisWtr$ that sends any presheave to it’s Nisnevich sheafication. And this functor is left adjoin to the embedding $ShNisWtr \to PreWtr$.

Before start the proof we give discussion of properties of $Witt$-correspondence in respect to Nisnevich coverings and give some useful definitions.

Definition 8 Let $(P, q_P)$ be quadratic space in the category $Proj(X, Y)$. We call by support of $(P, q_P)$ a closed subscheme of $X \times Y$ corresponding to the $\text{Ann}_{k[X \times Y]} P \subset k[X \times Y]$, where $\text{Ann}_{k[X \times Y]} P \subset k[X \times Y]$ denotes annihilator ideal of $P$ as $k[X \times Y]$-module, i.e.

$$\text{Supp} (P, q_P) = \text{Speck}[X \times Y]/\text{Ann}_{k[X \times Y]} P.$$  

Definition 9 Let $X, Y$ be smooth affine schemes and $u: U \to X$, $v: V \to Y$ be Nisnevich coverings. Let $\Phi$ be quadratic space 'between' $X$, and $Y$ and $\Psi$ its lift along $v$ and $u$, i.e.

$$\Phi \in \text{QuadSpace}(Proj(X, Y)), \Psi \in \text{QuadSpace}(Proj(U, V)) : \quad u^*(\Phi) = v_*(\Psi) \in \text{QuadSpace}(Proj(U, V)).$$

Then $\Psi$ is called a 'good' lift if the morphism $v \times \text{id}_U: V \times U \to Y \times U$ induces isomorphism

$$\text{Supp} \Psi \simeq \text{Supp} u^*(\Phi).$$

Any quadratic space is well defined over the function ring of its support. Now let’s involve denotation for such induced quadratic space on closed subscheme containing support.

Definition 10 (quadratic space $\Phi^Z$) Let $p: S \to X$ be regular map of affine varieties, $\Phi = (P, q_P)$ be quadratic space in $Proj(p)$ and $i: Z \subset S$ be closed embedding and $\text{Supp}\Phi \subset Z$. Then module $P$ and isomorphism $q_P$ are well defined over $k[Z]$. So they define a quadratic space $\Phi^Z$ in the category $Proj(p|_Z)$ corresponding to restriction of $p$ onto $Z p|_Z: S' \to X$, such that $\Phi = i_*(\Phi^Z)$.

Remark 12 'Good' lifts are such lifts that are defined by lifting of the support in following sense.

Let $u: U \to X$ and $v: V \to Y$ be Nisnevich coverings of affine varieties, $\Phi \in \text{QuadSpace}(Proj(X, Y))$, $Z = \text{Supp} u^*(\Phi)$ and $\Phi' = u^*(\Phi)|^Z$ (in sense of definition [17]). Then $\Psi \in \text{QuadSpace}(Proj(U, V))$ is 'good' lift of $\Phi$ if and only if there is a lift $l: Z \to V$ of the morphism of projection $Z \to Y$ such that $\Psi = g_*(\Phi')$ where $g = l \times \text{id}_U: Z \hookrightarrow U \times V$ (that is closed embedding because it is lift of closed embedding of $Z$ into $U \times Y$).

In fact $l: Z \to V$ can be defined as compositions of canonical projections $Z \to V$ with inverse to isomorphisms $Z\Psi \simeq Z$ from definition of 'good' lift. And conversely if $l$ is such
Lemma 6.1 For any affine $X, Y$ and any quadratic space $\Phi$ in $\text{Proj}(X, Y)$ $\text{Supp} \Phi$ is finite over $X$.

Proof of the lemma. Let $\Phi = (P, q_P)$. By definition

$$\text{Supp} \Phi = \text{Spec} k[X \times Y]/\text{Ann} P.$$ 

Let $m_1, \ldots, m_n$ be finite set of generators of $P$ over $k[X]$ then

$$\text{Ann} P = \bigcap_{i=1}^{n} \text{Ann} m_i$$

and the homomorphism of $k[X]$-modules

$$(1, \ldots, 1): R/\text{Ann} P \to \bigoplus_{i=1}^{n} R/\text{Ann} m_i$$

is embedding. The composition of this homomorphism with embeddings $R/\text{Ann} m_i \hookrightarrow P$ gives us the embedding of $R/\text{Ann} P$ into $\bigoplus_{i=1}^{n} P$. So $R/\text{Ann} P$ is isomorphic to the submodule of the finitely generated $k[X]$-module and since $k[X]$ is Noetherian $R/\text{Ann} P$ is finitely generated over $k[X]$. Lemma is proved.

Corollary 1 For any local Hensel $U^h$, smooth affine $Y$ and any $\Phi \in \text{QuadSpace}(\text{Proj}(U^h, Y))$

$$\Phi = \sum_{i=1}^{n} \Phi_i$$

for some $\Phi \in \text{QuadSpace}(\text{Proj}(U^h, Y))$ such that each $\Phi_i$ is represented by quadratic space $(P_i, q_{P_i})$ such that its support is local Hensel scheme.

Proof of the lemma. This immediately follows from previous because finite subscheme over local Hensel scheme splits into disjoint union of local Hensel subschemes. Lemma is proved.

Lemma 6.2 For any smooth affine $X, Y$, Nisnevich covering $v: V \to Y$ and any $\Phi \in \text{QuadSpace}(\text{Proj}(X, Y))$ there are some Nisnevich covering $u: U \to X$ and $\Psi \in \text{QuadSpace}(\text{Proj}(U, V))$ that is a 'good' lift of $\Phi$. 

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Proof of the lemma. We will construct covering \( U \) of \( X \) as disjoint union of Nisnevich neighbourhood for all points \( x \in X \). I.e. \( U \simeq \coprod_{x \in X} U_x \) and for all \( x \in X \) there is a lift 
\( l_x: x \to U_x \) such that composition \( u \circ l_x: x \to X \) is equal to the embedding of \( x \) into \( X \).

So let \( x \in X \) be any point. Let \( u_x^h: U_x^h \to X \) be Hensel neighbourhood of \( X \) at \( x \), i.e. \( k[U_x] \) is henselisation \( O^h_{X,x} \) of local ring \( O_{X,x} \) at \( x \).

By lemma 1 support of \( \Phi \) over \( U_x^h \) splits into finite disjoint union of Hensel local schemes 
\[
Z_x^h = \text{Supp} \left( (u_x^h)^*(\Phi) \right) = \coprod_{y_i} Z_{x,y_i}^h,
\]
where points \( y_i \in Y \) are the image along the projection on \( Y \) of the closed points in the fibre \( Z_x^h \) over \( x \).

Since all \( Z_{x,y_i}^h \) are Hensel local and \( V \to Y \) is Nisnevich covering there are lifts \( l_{x,y_i}: Z_{x,y_i} \to V \) of projections \( Z_{x,y_i} \to Y \). And this defines lifts 
\[
g_{x,y_i}: Z_{x,y_i} \hookrightarrow U_x^h \times V
\]
of closed embeddings \( Z_{x,y_i} \hookrightarrow U_x^h \times Y \).

Quadratic space \( \Phi_x^h = (u_x^h)^*(\Phi) \) splits into direct sum \( \Phi_x^h = \sum_i \Phi_{x,y_i}^h \) of quadratic spaces with supports \( Z_{x,y_i} \). By remark 12 lifts \( l_{x,y_i} \) define a ’good’ lift of quadratic space \( \Phi_x^h \)
\[
\Psi_x^h = \sum_i g_{x,y_i}(\Phi_{x,y_i}^h) \in \text{QuadSpace}(\text{Proj}(U_x^h, V)),
\]
i.e. 
\[
(u_x^h)^*(\Phi) = v_*(\Psi_x^h), \quad \text{Supp} \Psi_x^h \simeq Z_x^h. \tag{21}
\]

Any quadratic space in \( \text{Proj}(U_x^h, V) \) is germ of some quadratic space well defined over some affine scheme \( U_x \). So there are affine schemes \( U_x \) and quadratic spaces \( \Psi_x \), such that \( x \) is closed point of \( U_x \), \( e_{U}^x: U_x^h \to U_x \) is Nisnevich neighbourhood of \( U_x \) at \( x \), and 
\[
\Psi_x \in \text{QuadSpace}(\text{Proj}(U_x, V)): \Psi_x^h = e_{U}^x(\Psi_x).
\]
To make the equalities 
\[
v_*(\Psi_x) \simeq u_x^*(\Phi), \quad \text{Supp} \Psi_x \simeq Z_x
\]
holds, it is enough to change \( U_x \) to some its open subscheme, because over Nisnevich neighbourhood \( U_x^h \) equalities (21) holds.

Lemma is proved.

Lemma 6.3 ’Good’ lifts are closed under base changes and shredding. I.e. if \( X_1, X_2, Y \) are any varieties \( f: X_1 \to X_2 \) is regular map and \( v: V \to Y \), \( u: U \to X_2 \) and \( u': U' \to U \) are Nisnevich coverings then for any \( \Phi \in \text{QuadSpace}(\text{Proj}(X_2, Y)) \) and its ’good’ lift \( \Psi \in \text{QuadSpace}(\text{Proj}(U, V)) \), \( u'^*\Psi \) is ’good’ lift of \( \Phi \) and \( f_{U}^*\Psi \in \text{QuadSpace}(\text{Proj}(U \times_{X_2}}
Remark 13 Statement about pomposition with Nisnevich covering \( \mathcal{U} : U' \to U \) and morphisms \( f : X_2 \to X_1 \) follows immediately from definition and compatibility with composition can be proved similar to lemma 1 by lifting of the diagram of composition firstly at local Nisnevich neigbourhoshos and then continuation of it to some covering. Let’s note in addition that to get a ‘good’ lift over local Hensel scheme we use fixed choice of lifts of images points along covering of \( Y \), i.e. points \( y_i \) in proof of lemma 1 for all ‘preimage’ points in \( X_1 \) and \( X_2 \).

\[
\begin{array}{ccc}
U & \xrightarrow{u} & U' \\
\downarrow{\Phi} & & \downarrow{u'} \\
X_2 & \xrightarrow{\Phi_2} & \Phi_2(X_1, X_2) \\
\end{array}
\]

\[
\begin{array}{ccc}
V & \xrightarrow{\Psi} & Y \\
\downarrow{\Phi} & & \downarrow{f} \\
U & \xrightarrow{u} & X_2 \\
\end{array}
\]

Instead of the proof let’s give following remark

**Lemma 6.4** For any \( \Phi_1, \Phi_2 \in \text{QuadSpace}(\text{Proj}(X, Y)) \) there are Nisnevich coverings \( u : U \to X, v : V \to Y \) and ‘good’ lifts \( \Psi_1, \Psi_2 \in \text{QuadSpace}(\text{Proj}(U, V)) \) of \( \Phi_1, \Phi_2 \) such that \( \Psi_1 \oplus \Psi_2 \) is ‘good’ lift of \( \Phi_1 \oplus \Phi_2 \).

**Proof of the lemma.** By lemma 6.2 for some coverings \( u : U \to X, v : V \to Y \) there is a ‘good’ lift \( \Psi_+ \in \text{QuadS}(\text{Proj}(U, V)) \) of \( \Phi_+ = \Phi_1 \oplus \Phi_2 \) and by terms of the remark 12 it corresponds to some lift \( \Phi_+ \cup \): \( \text{Supp} \, u^*(\Phi_+) \to U \times V \). Since \( \text{Supp} \, u^*(\Phi_1) \subset \text{Supp} \, u^*(\Phi_+) \) quadratic spaces \( u^*(\Phi_1), u^*(\Phi_2) \) are well defined over \( k[\text{Supp} \, u^*(\Phi_+)] \), i.e. they defines some \( \Phi'_1, \Phi'_2 \in \text{QuadS}(\text{Proj}(s_+)) \) where \( s_+ \) is projection of \( \text{Supp} \, u^*(\Phi_+) \) to \( U \). Then \( l^+_* (\Phi'_1) \) and \( l^+_* (\Phi'_2) \) are the required ‘good’ lifts. **Lemma is proved.**

**Lemma 6.5** Let \( \Psi \in \text{QuadSpace}(\text{Proj}(U, V)) \) be a ‘good’ lift of \( \Phi \in \text{QuadSpace}(\text{Proj}(X, Y)) \) along Nisnevich coverings \( u : U \to X, v : V \to Y \). Then if \( \Phi \) is metabolic then \( \Psi \) is metabolic too.

**Proof of the lemma.** Since \( \Phi \) is metabolic \( u^*(\Phi) \) is metabolic. Then \( \Phi' = u^*(\Phi)^2 \) is metabolic (where \( Z = \text{Supp} \, u^*(\Phi) \)). And since in terms of the remark 12 \( \Psi = g^*(\Phi') \), \( \Psi \) is metabolic. **Lemma is proved.**
Definition 11 Let’s define a product for ‘good’ lifts of quadratic spaces, i.e. an operation that for two ‘good’ lifts

\[ \Psi_1, \Psi_2 \in \text{Quad}_S(\text{Proj}(U, V)) \]

of quadratic space \( \Phi \in \text{Quad}(\text{Proj}(X, Y)) \) along Nisnevich coverings \( u: U \to X \) and \( v: V \to Y \), gives a ‘good’ lift

\[ \Psi_1 \times_\Phi \Psi_2 \in \text{Quad}_S(\text{Proj}(U, V \times_Y V)) \]

along coverings \( u \) and \( v \times_Y v \) such that

\[ \text{pr}_i \circ \Psi_1 \times_\Phi \Psi_2 = \Psi_i, \ i = 1, 2 \]

\[ \times_\Phi: (\Psi_1, \Psi_2) \mapsto \Psi_1 \times_\Phi \Psi_2 \]

\[ V \times_Y V \xrightarrow{pr_2} V \xrightarrow{v} Y \]

\[ U \xrightarrow{\Psi_1 \times_\Phi \Psi_2} Z \rightarrow \text{Supp} \Phi \]

To construct it let’s use descriptions of ‘good’ lifts given in remark 12. Let \( Z = \text{Supp} (\Phi \circ u) \subset U \times Y \), \( l_1: Z \to V \) be its lifts corresponding to \( \Psi_1 \) and \( \Psi_2 \). Then the product \( \Psi_1 \times_\Phi \Psi_2 \) of lifts of quadratic spaces is defined by the product of lift of their supports \( (l_1, l_2): Z \to V \times_Y V \). I.e. \( \Psi_1 \times_\Phi \Psi_2 \) is direct image of \( \Phi' \) along \( (l_1, l_2): Z \to V \times_Y V \). where \( \Phi' = \Phi \circ u \in \text{Quad}_S(\text{Proj}(U, Y)) \).

Proof of the theorem.

Let \( a \in \mathcal{F}_{\text{Nis}}(Y) \). Let \( a \) be represented by \( \tilde{a} \in \mathcal{F}(V) \), i.e. \( \epsilon(\tilde{a}) = v^*(a) \) for some Nisnevich covering \( v: V \to Y \). Let \( \Phi \) be some quadratic space in \( \text{Proj}(X, Y) \). By lemma 6.2 for some \( u: U \to X \) exists a ‘good’ lift \( \Psi \in \text{Quad}_S(\text{Proj}(U, V)) \). If sought-for structure of presheave with Witt-transfers on \( \mathcal{F}_{\text{Nis}} \) exists then \( u^*(\Phi^*(a)) \) should be equal to \( \epsilon(\Psi^*(\tilde{a})) \). So we want to put \( \Phi^*(a) \) to be the element \( \mathcal{F}_{\text{Nis}}(X) \) represented by the element \( \Psi^*(\tilde{a}) \) in the group of sections of presheave \( \mathcal{F}(U) \).

To do it first of all we should check that \( \Psi^*(\tilde{a}) \) defines the section of the sheaf, i.e. that

\[ \text{pr}_{u, i}^*(\Psi^*(\tilde{a})) = \text{pr}_{u, 2}^*(\Psi^*(\tilde{a})), \]

where \( \text{pr}_{u, i}: U \times X U \to U \) are canonical projections. By the remark 6.3 the compositions

\[ \text{pr}_{u, i}^*(\Psi) \in \text{Quad}_S(\text{Proj}(U \times_X U, V)), \ i = 1, 2 \]
are two 'good' lifts of $\Phi$ along Nisnevich coverings $v: V \to Y$ and $u^2: U \times_X U \to X$. So due to construction from definition 11 there is

\[
\Psi_3 = pr_{u,1}^*(\Psi) \times_{\Phi} pr_{u,2}^*(\Psi) \in \text{QuadS}(\text{Proj}(U \times_X U, V \times_Y V)): \quad pr_{v,i}^*\Psi_3 = pr_{u,1}^*(\Psi).
\]

Then

\[
pr_{u,1}^*(\Psi^*(\tilde{a})) = \Psi_3^*(pr_{v,1}^*(\tilde{a})) = \Psi_3^*(pr_{v,2}^*(\tilde{a})) = pr_{u,2}^*(\Psi^*(\tilde{a}))
\]

Thus for any $\Phi \in \text{QuadS}(\text{Proj}(X, Y))$ we start with a section $a \in \mathcal{F}_{\text{Nis}}(Y)$ and construct a section $a' \in \mathcal{F}_{\text{Nis}}(X)$ of the sheaf $\mathcal{F}_{\text{Nis}}$ on $X$.

Let's summarize additional data used in this construction. For any set $D = (v: V \to Y, \tilde{a} \in \mathcal{F}(V), u: U \to X, \Psi \in \text{QuadSpace}(\text{Proj}(U, V))):$

- $v^*(a) = \epsilon(\tilde{a}), \quad u^*(\Phi) = v^*(\Psi), \quad \Psi$ is 'good' lift of $\Phi$

we construct

\[
\Phi_D^*(a) \in \mathcal{F}_{\text{Nis}}(X): \quad u^*(\Phi_D^*(a)) = \epsilon(\Psi(v^*(a))) \quad (\Phi_D^*(a) = a')
\]

So to get well defined map $\Phi^*: \mathcal{F}(Y) \to \mathcal{F}(X)$ we should check that this construction doesn't depend on additional data $D$.

Firstly we check independence on the choice of the lift $\Psi$ for fixed $u, v$ and $\tilde{a}$. Let $\Psi_1, \Psi_2 \in \text{QuadS}(\text{Proj}(U, V))$ are two 'good' lifts of $\Phi$. The definition 11 provides a quadratic space

\[
\Psi_1 \times_{\Phi} \Psi_2 \in \text{QuadS}(\text{Proj}(U, V \times_Y V)):
\]

\[
pr_{i*}((\Psi_1 \times_{\Phi} \Psi_2)) = \Psi_i, \quad i = 1, 2
\]

where $pr_i: V \times_Y V \to V$ are canonical projections.

Then

\[
\Psi_1^*(\tilde{a}) = (\Psi_1 \times_{\Phi} \Psi_2)^*(pr_1^*(\tilde{a})) = (\Psi_1 \times_{\Phi} \Psi_2)^*(pr_2^*(\tilde{a})) = \Psi_2^*(\tilde{a})
\]

and

\[
u^*\Phi_{\Psi_1}^*(a) = \epsilon(\Psi_1^*(\tilde{a})) = \epsilon(\Psi_2^*(\tilde{a})) = u^*\Phi_{\Psi_2}^*(a).
\]

So

\[
\Phi_{\Psi_1}^*(a) = \Phi_{\Psi_2}^*(a).
\]
Next we check **independence on the covering** \( u \). Let \( u_1: U_1 \to X, \ u_2: U_2 \to X \) are two Nisnevich covering and \( \Psi_i \in QuadS(Proj(U_i, Y)) \), \( i = 1, 2 \) are 'good' lifts of \( \Phi \).

The compositions

\[
pr_1^*(\Psi_1), \ pr_2^*(\Psi_2) \in QuadS(Proj(U_1 \times_X U_2, V))
\]

are two 'good' lifts of \( \Phi \) along covering \( U_1 \times_X U_2 \to X \) and \( \Phi_{U_i \times U_2, pr_i^*(\Psi_i)}^*(a) = \Phi_{U_i, \Psi_i}^*(a) \) because \( (pr_i^*(\Psi_i))^*(\tilde{a}) = pr_i^*(\Psi_i^*(\tilde{a})) \). Thus by independence on choice of lift for fixed covering

\[
\Phi_{U_1}^*(a) = \Phi_{U_1 \times U_2, pr_1^*(\Psi_1)}^*(a) \Phi_{U_2}^*(a).
\]

The **independence on the choice of the covering** \( V \) and representation \( \tilde{a} \) we check in two steps. First we note that for any Nisnevich covering \( v': V' \to V \)

\[
\Phi_{V', v'^*(\tilde{a})}^*(a) = \Phi_{V, \tilde{a}}^*(a).
\]

Because for some Nisnevich covering \( u': U' \to U \) exists a 'good' lift \( \Psi' \) of \( \Psi \) along \( v' \). And by commutativity of the diagram

\[
(u \circ u')^*(\Phi_{V', \tilde{a}}^*(a)) = \epsilon(u'^*(\Psi^*(\tilde{a}))) = \epsilon(\Psi'^*(v'^*(\tilde{a}))) = (u \circ u')^*(\Phi_{V', v'^*(\tilde{a})}^*(a)).
\]

Secondly for any two representations of section \( a \) along two Nisnevich coverings \( v_i: V_i \to Y, i = 1, 2 \)

\[
\tilde{a}_1 \in \mathcal{F}(V_1), \ \tilde{a}_2 \in \mathcal{F}(V_2): \ v_1^*(a) = \epsilon(\tilde{a}_1), \ v_2^*(a) = \epsilon(\tilde{a}_2)
\]

there is common shredding

\[
v_1'^*(\tilde{a}_1) = \tilde{a} = v_2'^*(\tilde{a}_2).
\]

So by discussion in previous paragraph

\[
\Phi_{V_1, \tilde{a}_1}^*(a) = \Phi_{V, \tilde{a}}^*(a) = \Phi_{V_2, \tilde{a}_2}^*(a).
\]

Thus for any smooth affine \( X \) and \( Y \) and \( \Phi \in QuadS(Proj(X, Y)) \) we get a well defined map \( \Phi^*: \mathcal{F}_{Nis}(Y) \to \mathcal{F}_{Nis}(X) \). To finish the proof we should check following.
1) This maps defines additive homomorphisms $WCor(X, Y) \to \text{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$.

2) This homomorphisms combines the functor $\mathcal{F}_{Nis}: WCor \to Ab$ such that its composition with canonical functor $Sm \to WCor$ is naturally equal to $\mathcal{F}_{Nis}: Sm \to Ab$.

3) This construction is natural on the presheave $\mathcal{F}$. Let for a morphism of presheaves $s: \mathcal{F}_1 \to \mathcal{F}_2$ the morphism $s_{Nis}: \mathcal{F}_{1Nis} \to \mathcal{F}_{2Nis}$ becomes a natural homomorphism of functors from $WCor$ to $Ab$.

Or more precisely:

1.1) $\Phi^*$ is additive for any $\Phi \in QuadS(Proj(X, Y))$,

1.2) $(\Phi_1 \oplus \Phi_2)^* = \Phi_1^* + \Phi_2^*$ for any $\Phi_1, \Phi_2 \in QuadS(Proj(X, Y))$,

1.3) $\Phi^* = 0$ for any metabolic quadratic space $\Phi$.

2.1) $\Phi_1^* \circ \Phi_2^* = (\Phi_2 \circ \Phi_2)^*$ for $\Phi_1 \in WCor(X, Y)$ and $\Phi_2 \in WCor(Y, Z),$

2.2) for Witt-correspondence $\Phi \in WCor(X, Y)$ defined by regular map $f: X \to Y$. $\Phi^* = f^*$.

3) The diagram

$$
\begin{array}{ccc}
\mathcal{F}_2(X) & \xrightarrow{\Phi^*} & \mathcal{F}_2(Y) \\
\downarrow{s_{Nis}} & & \uparrow{s_{Nis}} \\
\mathcal{F}_1(X) & \xrightarrow{\Phi^*} & \mathcal{F}_1(Y)
\end{array}
$$

for any $\Phi \in WCor(X, Y)$ and morphism of presheaves $s: \mathcal{F}_1 \to \mathcal{F}_2$.

We don’t not give detailed checking for all points but write down it for point 1) and explain general scheme of such proof.

Let $a_1, a_2 \in \mathcal{F}_{Nis}(Y)$. Let’s choose a covering $v: V \to Y$ such that $v^*(a_i) = \epsilon(\tilde{a}_i)$, $i = 1, 2$. It can be done by choosing of the coverings $v_i: V_i \to Y$ such that $v_i^*(a_i) = \epsilon(\tilde{a}_i)$, $i = 1, 2$ and putting $V$ to be a product of $V_i$. Then by lemma 6.2 for some Nisnevich cover $u: U \to X$ there is a ‘good’ lift $\Psi \in WCor(U, V)$ of $\Phi$ and we can use it to define $\Phi^*(a_1), \Phi^*(a_2)$ and $\Phi^*(a_1 + a_2)$ to be a sections of $\mathcal{F}_{Nis}$ over $X$ that are represented by sections $\Psi^*(\tilde{a}_1), \Psi^*(\tilde{a}_2)$ and $\Psi^*(\tilde{a}_1 + \tilde{a}_2)$ of $\mathcal{F}$ over $U$. Thus

$$
u^*(\Phi^*(a_1) + \Phi^*(a_2)) = \epsilon(\Psi(\tilde{a}_1 + \tilde{a}_2)) = \epsilon(\Psi(\tilde{a}_1 + \tilde{a}_2)) = u^*(\Phi^*(a_1 + a_2)).$$

Similary for any property of reverse images for presheaves $\Phi^*: \mathcal{F}(X) \to \mathcal{F}(Y)$ we can transfer it to the property of reverse images $\Phi^*: \mathcal{F}_{Nis}(X) \to \mathcal{F}_{Nis}(Y)$ using given upper construction of $\Phi^*$ and proved independence on choice of additional data by the following scheme.

First we lift the diagram of corresponding property for presheaves by consequently choosing compatible Nisnevich coverings for all schemes and lifts for all objects in the formulation of the property like as morphisms and sections. The compatibility means that the property holds for the listed objects. Then we use that fact that lifted objects uniquely determine corresponding objects for the sheaves.
Also let’s note that point 1.2) uses lemma 6.4 and 1.3) uses lemma 6.5. The theorem 5 is proved.

Now we go down to consideration of Nisnevich cohomology groups of the sheave with transfers. One of our aims is to prove ‘strictly’ variant of theorem 5, i.e.

**Theorem 6** For any Nisnevich sheave $\mathcal{F}$ presheaves of cohomologyes $H^i_{\text{Nis}}(\mathcal{F})$ are equipped with canonical structure of preshake with Witt-transfers.

In fact we get it as immediate corollary of following theorem.

**Theorem 7** There is natural isomorphism

$$\text{Ext}^i_{\text{ShNisWtr}}(\mathbb{Z}_{\text{Wtr}}(X), \mathcal{F}) \simeq H^i_{\text{Nis}}(X, \mathcal{F})$$

for all smooth affine $X$ and sheaves with Witt-transfers $\mathcal{F}$.

Note that formulation of the last theorem use that category $\text{ShNisWtr}$ is abelian that follows from the theorem 5.

The last theorem in turn is particular case of adjacency isomorphism of derived functors of adjoin functors

$$\text{ShNis} \rightleftarrows \text{ShNisWtr}$$

that equips the sheaves with the structure of sheave with Witt-transfers and forgetfull functor, and the proof of theorem 7 is given in the end of the section.

Let’s involve following notations.

**Definition 12** Let’s denote by

$$\text{Wtr}_{\text{Pre}}: \text{Pre} \to \text{PreWtr}, \quad \text{Wtr}_{\text{ShN}}: \text{ShNis} \to \text{ShNisWtr}$$

the left Kan extension functor along the functor from additivisation of category $\text{Sm}_k$ into category $\text{WCor}_k$, i.e. the functor that equips any presheave with Witt-transfers by universal way, and the the functor that is composition of the functor of the embedding $\text{ShNis} \to \text{Pre}$, the functor $\text{Wtr}_{\text{Pre}}$ and Nisnevich sheafication functor.

Then let’s denote by

$$L(\text{Wtr}_{\text{Pre}}): D^-(\text{Pre}) \to D^-(\text{PreWtr}), \quad L(\text{Wtr}_{\text{ShN}}): D^-(\text{NisSh}) \to D^-(\text{ShNisWtr})$$

the left derived functors of $\text{Wtr}_{\text{Pre}}$ and $\text{Wtr}_{\text{ShN}}$.

**Remark 14** The functors $\text{Wtr}_{\text{Pre}}$ and $\text{Wtr}_{\text{ShN}}$ are left adjoin to the forgetful functors $F_{\text{Wtr}}: \text{PreWtr} \to \text{Pre}$ and $F_{\text{Wtr}}: \text{ShNisWtr} \to \text{ShNis}$.

**Remark 15** Forgetful functor $F_{\text{Wtr}}: \text{ShNisWtr}: \text{ShNis}$ is exact, hence it induce a functor between derived categories that is booth left and right derived functor for $F_{\text{Wtr}}$. Let’s denote it by the same symbol. Then since left and right derived functors to the left and right adjoin functors are adjoin again, there is an adjacency

$$LWtr: D^-(\text{NisSh}) \rightleftarrows D^-(\text{ShNisWtr}): F_{\text{Wtr}}$$

$$L_{\text{Wtr}} \dashv F_{\text{Wtr}}$$

(22)
Now to apply the adjucency \(^{(22)}\) we want partly calculate the functor \(L_{Wtr}: D^-(ShNisWtr) \rightarrow D^-(ShNis)\). In fact it happens that \(L(Wtr_{ShN})\) coincides with \(L(Wtr_{Pre})\), because derived functor \(L(Wtr_{Pre})\) is exact in respect to exact sequences that goes from Nisnevich topology structure.

More precisely it means the following. Let’s identify by standard equivalence the categories \(D^-(ShNis)\) and \(D^-(ShNisWtr)\) with localisations of categories \(D^-(Pre)\) and \(D^-(PreWtr)\) by Nisnevich sheaf quasi-isomorphisms that are the following.

**Definition 13** The morphism \(q: A^* \rightarrow B^*\) in category \(D^-(Pre)\) or \(D^-(PreWtr)\) is called sheaf quasi-isomorphism if homomorphism \(h^i(q): h^i(A^*) \rightarrow h^i(B^*)\) of Nisnevich sheaf cohomology, i.e. Nisnevich shafication of cohomology presheaf are isomorphisms.

The complex \(A^*\) in category \(D^-(Pre)\) or \(D^-(PreWtr)\) is called sheaf acyclic if all Nisnevich sheaves \(h^i(A^*)\) are zero.

Notice that this identification uses theorem \(^{[5]}\) because by the definition category \(ShNisWtr\) is a full subcategory of \(PreWtr\) but due to theorem \(^{[5]}\) it is equal to localisation of \(PreWtr\) at (Nisnevich-)local isomorphisms and there are commutative diagrams

\[
\begin{align*}
ShNis & \xrightarrow{\text{F}_{Nis}} ShNisWtr & \xrightarrow{\text{F}_{Wtr}} & PreWtr & \xrightarrow{\text{F}_{Wtr}} & D^-\text{(PreWtr)} \\
\text{Wtr}_{Nis} & \xrightarrow{\text{F}_{Wtr}} & \text{Wtr}_{Pre} & \xrightarrow{\text{F}_{Wtr}} \text{Pre} & \underset{\text{L}_{Nis}(Wtr)}{\xrightarrow{\text{F}_{Wtr}}} & D^-(\text{Pre}) \\
\text{ShNis} & \xrightarrow{\text{F}_{Nis}} & \text{Pre} & \underset{\text{L}_{Nis}(Wtr)}{\xrightarrow{\text{F}_{Wtr}}} & D^-(\text{Pre})
\end{align*}
\]

**Theorem 8** The functor \(L(Wtr_{Pre}): D^-(Pre) \rightarrow D^-(PreWtr)\) preserves Nisnevich sheaf quasi-isomorphisms.

**Proof of the theorem.**

As mentioned above this result follows from that fact that any Witt-correspondence from local Hensel scheme splits into the sum of Witt-correspondence between local Hensel schemes. To prove this theorem we use following modification of it.

**Lemma 6.6** For any local Hensel scheme \(U^h\) and affine \(Y\) there is a canonical decomposition

\[
WCor(U^h, Y) = \bigoplus_z WCor_z(U^h, Y)
\]

where \(z\) runs throw all closed points of \(Y\), \(v\) denotes closed point of \(U^h\). And component \(WCor_z(U^h, Y)\) depends only on etale neighbourhood of closed point \(y \in Y\) that is projection on \(Y\) of the point \(z\). I.e. for any etale neighbourhood \(v: (V, y) \rightarrow (Y, y)\), morphism \(v\) induce isomorphism

\[
WCor_z(U^h, V) \simeq WCor_z(U^h, Y).
\]

**Proof of the lemma.**

In fact in proof of the lemma \(^{[6.1]}\) and lemma \(^{[1]}\) it was shown that support of any object in \(Proj(U^h, Y)\) as closed subscheme in \(U^h \times Y\) splits into disjoint union of local
Hensel schemes. Any local Hensel subscheme in $U^h \times Y$ has an unique closed point and all
this points lies in $x \times Y$. Since local Hensel subschemes with different closed points don’t
intersect, $\text{Hom}$-groups between the objects with such supports are zero.

Hence the category $\text{Proj}(U^h, Y)$ splits into direct sum of subcategories $\text{Proj}_z(U^h, Y)$
consisting of the modules that support are local Hensel scheme with closed point $z \in x \times Y$.
This induce required splitting of Witt groups of this categories $W\text{Cor}(U^h, Y) = \bigoplus_z W\text{Cor}_z(U^h, Y)$.

The isomorphism $W\text{Cor}_z(U^h, V) \simeq W\text{Cor}_z(U^h, Y)$ along etale neighbourhoods $(V, y): (Y, y)$
holds on the level of subcategories too and follows from the existence of the lifts of morphisms
from local Hensel schemas along etale neighbourhoods.

In fact for any local Hensel subscheme $Z \subset U^h \times Y$ finite over $U^h$ and such that $z \in Z$
there is a lift $t: Z \rightarrow U^h \times V$ and any local Hensel subscheme $Z \subset U^h \times V$ finite over $U^h$ and
such that $z \in Z$ is isomorphic to the image $v(Z)$. This induce equivalences of subcategories

$$\text{Proj}_Z(U^h, Y) \simeq \text{Proj}_i(Z)(U^h, V), \ Z \subset U^h \times Y$$

$$\text{Proj}_{v(Z)}(U^h, Y) \simeq \text{Proj}_Z(U^h, V), \ Z \subset U^h \times V$$

consists of modules that supports containing in specified closed subchemes. And taking
the limit along closed local Hensel subschemes $Z$ finite over $U^h$ and containing $y$ we get the
equivalences

$$\text{Proj}_z(U^h, Y) \simeq \text{Proj}_z(U^h, V).$$

Lemma is proved.

**Definition 14** Since by previous lemma the group $W\text{Cor}_z(U^h, Y)$ doesn’t change after
changing of $Y$ by its etale neighbourhood at the point $y$ that is projection of the point $z,
we can define the group $W\text{Cor}_z(U^h, V^h)$ where $(V^h, y)$ is Henselisation of $Y$ at $y$. And let’s
define $W\text{Cor}(U^h, V^h) = \bigoplus_{z \times x \times y} W\text{Cor}_z(U^h, Y^h)$.

Let’s note that it is possible to give strong definition of the groups $W\text{Cor}(U^h, V^h)$ as
$\text{Hom}$-groups between the local Hensel schemes (as $\text{Pro}$-objects of the category $W\text{Cor}$) but
here $W\text{Cor}_z(U^h, V^h)$ and $W\text{Cor}(U^h, V^h)$ are only notations.

Preservation of sheaf quasi-isomorphisms is equivalent to the preservation of sheaf acyclicity,
 i.e. that $L(W\text{tr}_{\text{Pre}})$ sends Nisnevich acyclic complexes in the category $D^-(\text{Pre})$ to a
Nisnevich acyclic complexes in $D^-(\text{PreWtr})$. So we show that $L(W\text{tr}_{\text{Pre}})$ sends full triangulated
subcategory of Nisnevich sheaf acyclic complexes in $D^-(\text{Pre})$ into full triangulated
subcategory of Nisnevich sheaf acyclic complexes in $D^-(\text{PreWtr})$.

Now let’s note that left derived functors on the category $D^-(\text{PreWtr})$ can be computed
by using of resolvents consisting of direct sums of representable presheaves in $\mathbb{Z}(\text{Sm})$, i.e.
presheave of free abelian groups corresponding to representable presheave of sets in $\text{Sm}$.

In fact any presheave is direct limit of representable ones, so for any complex in $D^-(\text{PreWtr})$
 there is such resolvent. And since representable presheaves are projective objects in category
$\text{Pre}$, for any complex $\mathcal{F} \in D^-(\text{PreWtr})$ consisting of sums of representable presheaves
$L(W\text{tr})(\mathcal{F}) = W\text{tr}_{\text{Pre}}(\mathcal{F})$, i.e. to compute $L(W\text{tr})$ we can apply $W\text{tr}_{\text{Pre}}$ to each member.

Thus since subcategory of Nisnevich acyclic complexes is closed under quasi-isomorphisms
to prove the theorem it is enough to prove that for any complex of representable presheaves
$\mathcal{F}$ if $\mathcal{F}$ is Nisnevich acyclic then $W\text{tr}_{\text{Pre}}(\mathcal{F})$ is Nisnevich acyclic too.
To finish the proof let’s note that Nisnevich acyclicity is equivalent to acyclicity of the complexes of germs on Hensel local schemes \( Wtr_{Pre}(F^\bullet)(U^h) \). and that lemma 6.6 allows to express \( Wtr_{Pre}(F^\bullet)(U^h) \) in terms of \( F(U^h) \) for any presheave \( F \) and local Hensel \( U^h \). In fact by the formula for left Kan extensions

\[
Wtr_{Pre}(F^\bullet)(U) \simeq \lim_{f: Y_1 \to Y_2} F(Y_2) \otimes WCor(U, Y_1).
\]

By lemma 6.6 for any local Hensel \( U^h \) and any affine \( Y \)

\[
WCor(U^h, Y) = \sum_y WCor_y(U^h, Y).
\]

Therefore by including of direct sums into direct limit and partly computing of direct limit along the morphisms \((V, y) \to (Y, y)\) that are etale neighbourhoods of the closed point \( y \)

\[
\lim_{(V, y)} F(V) \simeq F(Y^h_y)
\]

we get that

\[
Wtr_{Pre}(F^\bullet)(U^h) \simeq \lim_{f: Y_1 \to Y_2} F(Y_2) \otimes WCor(U^h, Y_1) \simeq \lim_{f: Y_1 \to Y_2, z \in x \times Y_1} F(Y_2) \otimes WCor_z(U^h, Y_1) \simeq \lim_{f: V_1^h \to V_2^h, z \in x \times y} F(V_2^h) \otimes WCor(U^h, V_1^h) \simeq \lim_{f: V_1^h \to V_2^h} F(V_2^h) \otimes WCor(U^h, V_1^h).
\]

where \( V_1^h \) and \( V_2^h \) denoted local Hensel schemes, \( x \) denotes closed point of \( U^h \) and \( y \) denotes the projection of \( z \) to \( Y_2 \) or the closed point of \( V^h \). Since presheaves \( F_i \) are direct sums of representable ones that are projective objects of category \( D^-(Pre) \), and since \( F^\bullet(U^h) \) is acyclic complex, complex \( F^\bullet(U^h) \) is contractable. Then since tensor product and direct limit preserves direct sums by equality (17) \( Wtr_{Pre}(F^\bullet)(U^h) \) is acyclic.

**Theorem 8** is proved.

**Remark 16** In the last prove to reduce to the case of the complexes of direct sums of representable presheaves we use that we work with the category \( D^- \) and the existence of resolutions. But in fact it is not essential because even in \( D^b(PreWtr) \) the subcategory of Nisnevich acyclic complexes is generated by the complexes

\[
0 \to \mathbb{Z}(\tilde{U}) \to \mathbb{Z}(U) \oplus \mathbb{Z}(\tilde{X}) \to \mathbb{Z}(X) \to 0
\]

corresponding to the elementary Nisnevich squares

\[
\begin{array}{ccc}
\tilde{U} & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
U & \longrightarrow & X
\end{array}
\]
Proof of the theorem \[7\]
Let’s now substitute in the isomorphism of adjacency \[22\] between \(L(Wtr_{ShN})\) and \(F_{Wtr}\) the Nisnevich sheaf associated to the presheave \(\mathbb{Z}(X)\) for any smooth affine \(X\), that is regarded as complex concentrated in degree zero, and sheaf with Witt-transfers \(\mathcal{F}\) as a complex concentrated in degree \(-i\).

We get natural isomorphism

\[
\text{Hom}_{D^- (ShN)}(\mathbb{Z}(X), \mathcal{F}[i]) \simeq \text{Hom}_{D^- (\text{ShN}isWtr)}(L(Wtr_{ShN})(\mathbb{Z}(X)), \mathcal{F}[i]).
\]

Since \(\mathbb{Z}(X)\) is representable, then \(L(Wtr_{pre})(\mathbb{Z}(X)) \simeq Z_{Wtr}(X)\), and by theorem \[8\]

\[
L(Wtr_{Sh})(\mathbb{Z}(X)) \simeq L(Wtr_{pre})(\mathbb{Z}(X)).
\]

Thus the adjacency \[22\] gives us that

\[
\text{Hom}_{D^- (ShN)}(\mathbb{Z}(X), \mathcal{F}[i]) \simeq \text{Hom}_{D^- (\text{ShN}isWtr)}(\mathbb{Z}(X), \mathcal{F}[i])
\]

and in combination with isomorphisms of \(\text{Hom}\)-groups in derived categories, \(\text{Ext}\)-groups sheaf cohomology groups

\[
H^k_{\text{Nis}}(X, \mathcal{F}) \simeq Ext^i_{\text{ShN}is}(\mathbb{Z}(X), \mathcal{F}) \simeq \text{Hom}_{D^- (ShN)}(\mathbb{Z}(X), \mathcal{F}[i]),
\]

this gives us the required isomorphism from theorem \[7\]

Theorem \[7\] is proved.

7 The construction of category of effective Witt-motives \(DW M^-_{eff}(k)\)

The construction of the category \(DW M^-_{eff}(k)\) is based on the theorems \[3\] and \[4\] and is similar to the construction of \(DM^-_{eff}\).

The category \(DW M^-_{eff}(k)\) can be defined in two ways, in a few informal words as localisation \(DW M^-_{eff,l}(k)\) of the category \(D^- (\text{ShN}isWtr)\) by \(\mathbb{A}^1\)-homotopy equivalences and as full subcategory \(DShNWtr\) of \(D^- (\text{ShNisWtr})\) consists of homotopy invariant objects. And for the first one there is a functor \(p: D^- (\text{ShN}isWtr) \to DW M^-_{eff,l}(k)\) and for the second one there is a functor \(i: DW M^-_{eff,r}(k) \to DShNWtr\).

One of the basic supposed properties of the category of motives is that the functor from the category of varieties to the category of motives is an universal cohomology theory in that sense that any cohomology theory on category of varieties can be passed throw it. So more natural definition (in respect to this universal property) is the first one \((DW M^-_{eff,l}(k))\). However, to compute represented in this category cohomology theories in terms of \(\text{Hom}\)-groups of \(D^- (\text{ShN}isWtr)\), means to define right adjoin to the functor \(p\). And the second definition \((DW M^-_{eff,r}(k))\) just provides computation of this functor as the functor of full embedding \(i\).

The equivalence of this two definition can be proved by showing that structure of category with interval on the category of varieties (where interval is affine line) induce semi-orthogonal decomposition of the category \(D^- (\text{ShN}isWtr)\). That is decomposition to \(\mathbb{A}^1\)-contractable
and $\mathbb{A}^1$-homotopy invariant parts. And both definitions of $DWM_{eff}^{-}(k)$ gives exactly homotopy invariant part of $D^{-}(ShNisWtr)$.

As was mentioned in the introduction the category of motives combines in some sense the structure of category with interval and topological structure on $Sm_k$ (in this case Nisnevich topology). Ability of a good combining of this structures is provided by their coher-ence proved in theorem 3. Therefore firstly we prove that affine line as interval in $Sm$ induces semi-orthogonal decomposition, of derived category of presheaves with Witt-transfers $D^{-}(PreWtr)$, that doesn’t deals with topology, and then we push down this decomposition to the derived category of sheaves $D^{-}(ShNisWtr)$.

As a result we get following commutative diagram

\[
\begin{array}{ccc}
Sm_k & \longrightarrow & Witt_k \\
\downarrow & \searrow & \downarrow \text{com} \\
D^{-}(PreWtr_k) & \longrightarrow & D^{-}(ShNisWtr_k)
\end{array}
\]

where the category $D^{-}(PreWtr_k)$ is homotopy invariant part of the category $D^{-}(PreWtr)$, and functors $l_A$ and $l_{Nis}$ can be regarded as localisation functors by the morphisms corresponding to projections $X \times \mathbb{A}^1 \to X$ for all smooth affine $X$ and by the morphisms $\tilde{X} \to \text{Cone}(U \coprod \tilde{X} \to X)$ where $X$, $\tilde{X}$, $U$, $\tilde{U}$ are vertices of elementary Nisnevich square respectively. The second arrow, i.e. com is the functor that sends any variety $X$ to a complex concentrated in degree zero and defined by the presheave $Z_{wtr}(X)$.

Now we proceed to prove of mentioned above semi-orthogonal decompositions of categories $D^{-}(PreWtr)$ and $D^{-}(ShNisWtr)$.

Let’s give following definition of semi-orthogonal decompositions in triangulated categories.

**Definition 15** A semi-orthogonal decomposition $<A, B>$ of some triangulated category $\mathcal{C}$ is the pair of two full triangulated subcategories $A$ and $B$ that are semi-orthogonal and generates the category $\mathcal{C}$, i.e. such that

$$\text{Hom}_{\mathcal{C}}(B^\bullet, A^\bullet) = 0$$

for all $B^\bullet \in B$ and $A^\bullet \in A$, and for any $C^\bullet \in \mathcal{C}$ there is a the distinguished triangle

$$A^\bullet[-1] \to B^\bullet \to C^\bullet \to A^\bullet$$

with $B^\bullet \in B$ and $A^\bullet \in A$.

**Remark 17** For any semi-orthogonal decomposition $\mathcal{C} = <A, B>$ two pairs of adjoin functors

$$\begin{array}{c}
B \overset{i_B}{\leftarrow} \mathcal{C} \overset{l_B}{\to} A \\
\downarrow \text{i}_A \downarrow \text{l}_A \\
i_B \downarrow i_A \downarrow l_B
\end{array}$$

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where \( i_B \) and \( i_A \) are embedding functors of subcategories \( A \) and \( B \) and \( l_A \) and \( l_B \) are equivalent to localisation functors at subcategories \( A \) and \( B \). And the compositions \( pr_A = i_A \circ l_A \) and \( pr_B = i_B \circ l_B \) are endofunctors-projectors on \( C \), i.e. there are natural transformations \( \epsilon_A : id_C \to pr_A \xi_B : pr_B \to id_C \) such that \( \epsilon(A^\bullet) \) is isomorphism for any \( A^\bullet \in A \) and \( \xi(B^\bullet) \) is isomorphism for any \( B^\bullet \in B \).

Also we will use following standard fact

**Lemma 7.1** Let \( A \) be abelian category with (infinite) direct sums and \( B \) be its full abelian subcategory closed under (infinite) direct sums.

Then thick subcategory of \( D^-(A) \) generated by \( B \) (i.e. by complexes concentrated in degree zero whose zero term is an object of \( B \)) equal to ker\( (D^-(A) \to D^-(A/B)) \) and equal to subcategory of complexes that cohomology lies in \( B \).

And all complexes consisting of objects lying in \( B \) (i.e. lying in \( D^-(B) \)) lies in this category.

Now let’s give also one useful definition

**Definition 16** Let \( C \) be additive category and \( S: C \to C \) be endo-functor with natural transformation \( p: Id_C \to S \) and \( s_0, s_1: S \to Id_C \) such that \( s_1 \circ p = id_S = s_1 \circ p \). The object \( C \in bC \) is called \( S \)-invariant if \( p(C) \) is isomorphism, and it is called \( S \)-contractable if there is a morphism \( h: C \to S(C) \) such that \( s_0(C) \circ h = id_C \) and \( s_0(C) \circ h = 0 \).

The semi-orthogonal decompositions on categories \( D^-(PreWtr) \) and \( D^-(ShNisWtr) \) are induced in some sense by the ‘action’ of affine line on the category of varieties and categories \( WCor, PreWtr, D^-(PreWtr) \) and \( D^-(ShNisWtr) \).

**Definition 17** Let \( Sm \to C \) be any additive category under the category of smooth affine varieties with internal \( Hom \)-functor. Then affine line \( \mathbb{A}^1 \), projection homomorphism \( p: \mathbb{A}^1 \to pt \), and morphism of zero and unit sections \( s_0, s_1: pt \to \mathbb{A}^1 \) considered as objects and morphisms in the category \( D^-(PreWtr) \) induce by applying of internal \( Hom \)-functor an following endo-functor and natural transformations

\[
S = Hom_C(\mathbb{A}^1, -): D^-(PreWtr),
\]

\[
p = Hom_C(p, -): Id_{D^-(PreWtr)} \to S, \quad s_0, s_1 = Hom_{D^-(PreWtr)}(p, -): S \to Id_{D^-(PreWtr)}.
\]

And definition [10] provides a notions of \( \mathbb{A}^1 \)-homotopy invariant and \( \mathbb{A}^1 \)-contractable objects of \( C \).

Finally let’s give following standard definition.

**Definition 18** Let denote by \( \Delta \) the simplicical scheme with

\[
\Delta^n = Speck[x_0, x_1, \ldots, x_n]/(x_0 + x_1 + \ldots x_n - 1)
\]

\[
e_{n,i}: (x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_i, 0, x_{i+1}, \ldots, x_n)
\]

\[
d_{n,i}: (x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_i + x_{i+1}, \ldots, x_n).
\]
**Theorem 9** There is semi-orthogonal decomposition of the category $D^-(PreWtr)$

$$D^-(PreWtr) = \langle D^-_{\bar{\mathbb{A}}-\text{contr}}(PreWtr), D^-_{\bar{\mathbb{A}}-\text{inv}}(PreWtr) \rangle$$

such that $D^-_{\bar{\mathbb{A}}-\text{contr}}(PreWtr)$ as full subcategory of $D^-(PreWtr)$ consists of complexes quasi-isomorphic to complexes of $\mathbb{A}^1$-contractable presheaves and $D^-_{\bar{\mathbb{A}}-\text{inv}}(PreWtr)$ consists of complexes with homotopy invariant cohomologies.

**Proof of the theorem.** Let's consider two full triangulated subcategories of $D^-(PreWtr)$: subcategory $\mathcal{A}$ consisting of homotopy invariant objects of $D^-(PreWtr)$ in sense of definition [17] and subcategory $\mathcal{B}$ that is thick subcategory generated by $\mathbb{A}^1$-contractable objects. And let's show that this categories provides semi-orthogonal decomposition of $D^-(PreWtr)$.

First let's show that $\text{Hom}_C(B^*, A^*) = 0$ for all $B^* \in \mathcal{B}$ and $A^* \in \mathcal{A}$. In fact,

$$s_0(A^*) = s_1(A^*): S(A^*) \to A^*,$$

since $A^*$ is invariant object and $p(A^*)$ induce isomorphism of $A^*$ and $S(A^*)$. And since $B^*$ is $\mathbb{A}^1$-contractable object, there is an $\mathbb{A}^1$-homotopy

$$h \in \text{Hom}_{D^-(PreWtr)}(B^*, \text{Hom}(\mathbb{A}^1, B^*)): s_0(B^*) \circ h = id, s_1(B^*) \circ h = 0.$$

So for any $f \in \text{Hom}(B^*, A^*)$

$$f = f \circ s_1(B^*) \circ h = s_0(A^*) \circ S(f) \circ h = s_1(A^*) \circ S(f) \circ h = f \circ s_0(B^*) \circ h = 0.$$

Next let's consider the functor

$$C^* = \text{Hom}_{D^-(PreWtr)}(\Delta^*, -): D^-(PreWtr) \to D^-(PreWtr).$$

The canonical morphism of simplicial objects $\Delta^* \to pt^* \to pt$ where $pt^*$ denotes constant simplicital object and $pt$ simplicial object concentrated at the degree zero induces natural transformations

$$C^* \xrightarrow{\epsilon} \text{Hom}_{D^-(PreWtr)}(pt^*, -) \to \text{Hom}_{D^-(PreWtr)}(pt^*, -) \simeq \text{Id}_{D^-(PreWtr)}.$$  

Let $\epsilon': C^* \to \text{Id}_{D^-(PreWtr)}$ denotes the composition. Then for any complex $C^*$ there is a distinguished triangle

$$\text{Cone}(\epsilon)[1] \to C \xrightarrow{\epsilon'} C^*(C) \to \text{Cone}(\epsilon).$$

Standard simplicial partition of the cylinders $\Delta^i \times \mathbb{A}^1$ defines the $\mathbb{A}^1$-homotopy between zero and unit section

$$s_0, s_1: C^*(C) \to \text{Hom}(\mathbb{A}^1, C^*(C))$$

that shows that $C^*(C)$ is homotopy invariant. On other side

$$\text{Cone}(\epsilon) =$$

$$\vdots \to \text{Hom}_{PreWtr}(Z_{Wtr}(\Delta^i)/Z_{Wtr}(pt), C) \to \cdots \to \text{Hom}_{PreWtr}(Z_{Wtr}(\Delta^1)/Z_{Wtr}(pt), C) \to 0.$$
And its terms are contractable presheaves, because linear homotopy of affine simplexes

\[ \mathbb{A}^1 \times \Delta^n \to \Delta^n \]
\[ (\lambda, x_0, x_1, \ldots, x_n) \mapsto (x_0 + \lambda \sum_i x_i, (1 - \lambda)x_1, \ldots, (1 - \lambda)x_n) \]

induce a homotopy of \( \text{Hom}_{\text{PreWtr}}(\mathbb{Z}_{\text{Wtr}}(\Delta^n)/\mathbb{Z}_{\text{Wtr}}(pt), C) \). Hence \( \text{Cone}(e) \in D_{\text{K-contr}}^-(\text{PreWtr}) \).

Any contractable presheave defines the contractable object of \( D^-(\text{PreWtr}) \) so any complex consists of contractable presheaves lies in \( D_{\text{K-contr}}^-(\text{PreWtr}) \). So \( D_{\text{K-contr}}^-(\text{PreWtr}) \subset \mathcal{B} \).

Thus \( \text{Cone}(e) \in \mathcal{B} \).

Thus we get semi-orthogonal decomposition

\[ \mathcal{C} = \langle \mathcal{B}, \mathcal{A} \rangle. \]

In addition by the remark \[ \mathcal{B}, \mathcal{A} > \] that fact that the third term in the triangle is the result of the applying functor \( C^* \), implies that \( C^* \) defines the functor form \( D^-(\text{PreWtr}) \) to \( D_{\text{K-inv}}^-(\text{PreWtr}) \) that is left adjoint to the embedding of \( D_{\text{K-inv}}^-(\text{PreWtr}) \) into \( D^-(\text{PreWtr}) \).

To finish the proof it is enough to show that

\[ D_{\text{K-inv}}^-(\text{PreWtr}) = \mathcal{A}, D_{\text{K-contr}}^-(\text{PreWtr}) = \mathcal{B}. \]

To show the first it is enough to note that the functor \( \text{Hom}_{\text{PreWtr}}(\mathbb{A}^1, -) \) is exact and

\[ h^i(b\text{Hom}_{D^-(\text{PreWtr})}(\mathbb{A}^1, C^*)) \simeq \text{Hom}_{\text{PreWtr}}(\mathbb{A}^1, h^i(C^*)). \]

So the equalities

\[ h^i(\text{Hom}_{D^-(\text{PreWtr})}(\mathbb{A}^1, C^*)) \simeq h^i(C^*) \]

are equivalent to the equalities

\[ \text{Hom}_{\text{PreWtr}}(\mathbb{A}^1, h^i(C^*)) \simeq h^i(C^*). \]

Since it is just proved that \( \langle \mathcal{B}, \mathcal{A} > \) is semi-orthogonal decomposition of \( D^-(\text{PreWtr}) \) by remark \[ \mathcal{B}, \mathcal{A} > \] for any \( B^* \in \mathcal{B} \) the morphism \( \text{Cone}(e(B^*))[-1] \to B^* \) is quasi-isomorphism. But as was mentioned above \( \text{Cone}(e(B^*))[-1] \in D_{\text{K-contr}}^-(\text{PreWtr}) \).

Theorem is proved.

**Theorem 10** The category \( D_{\text{K-inv}}^-(\text{PreWtr}) \) is generated as triangulated subcategory by homotopy invariant presheaves (considered as complexes concentrated in degree zero).

The category \( D_{\text{K-contr}}^-(\text{PreWtr}) \) is generated as triangulated subcategory by presheaves \( \mathbb{Z}(\mathbb{A}^1 \times X)/\mathbb{Z}(X) \) (considered as complexes concentrated in degree zero) for all smooth affine \( X \).

**Proof of the theorem.** The first statement is a particular case of the statement from lemma \[ \mathcal{B}, \mathcal{A} > \]

To prove the second it is enough to find a natural resolvent in \( D^-(\text{PreWtr}) \) consisting of infinite direct sums of presheaves \( \mathbb{Z}(\mathbb{A}^1 \times X)/\mathbb{Z}(X) \), because if such resolvent exists then any complex \( B^* \) consisting of contractable presheaves is quasi-isomorphic to totalisation of bi-complex constituted by resolvents of \( B^3 \), and this totalisation is a complex consisting of
direct sums of terms of presheaves \( \mathbb{Z}(\mathbb{A}^1 \times X)/\mathbb{Z}(X) \). For any presheave \( \mathcal{F} \) there is a natural (in \( \mathcal{F} \)) sequence

\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{P}_i & \longrightarrow & \mathcal{P}_{i-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & \longrightarrow & 0 \\
\downarrow & & \varepsilon_i & & \downarrow & & \varepsilon_{i-1} & & \downarrow & & \varepsilon_i & & \downarrow & & \varepsilon \\
\cdots & \longrightarrow & \mathcal{F}_i & \longrightarrow & \mathcal{F}_{i-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & .
\end{array}
\]

\( \mathcal{P}_i = \sum_{U,s \in \mathcal{F}_i(\mathbb{A}^1 \times U)} \mathbb{Z}(\mathbb{A}^1 \times U)/\mathbb{Z}(U) \),

\( j_0^X : 0 \times X \hookrightarrow \mathbb{A}^1 \times X \), \( j_1^X : 1 \times X \hookrightarrow \mathbb{A}^1 \times X \),

\( \varepsilon_{X,s} : \mathbb{Z}(\mathbb{A}^1 \times X)/\mathbb{Z}(X) \simeq \ker(j_0) \xrightarrow{s} \mathcal{F}_i \),

\( \mathcal{F}_i = \ker(\varepsilon_i) \), \( \mathcal{F}_0 = \mathcal{F} \).

(This sequence in fact defines adjoin functor to the embedding functor of subcategory in \( D^{-}(\text{PreWtr}) \) generated by presheaves \( \mathbb{Z}(\mathbb{A}^1 \times X)/\mathbb{Z}(X) \). If \( \mathcal{F} \in D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}) \) then \( C^*(\mathcal{F}) \) is acyclic and in particular \( h^0(C^*(\mathcal{F})) = 0 \). But \( h^0(C^*(\mathcal{F})) = \ker(\varepsilon) \), hence \( \varepsilon \) is surjective. Next since \( \mathbb{Z}(\mathbb{A}^1 \times X)/\mathbb{Z}(X) \in D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}) \) and \( \varepsilon \) is surjective, \( \mathcal{F}_1 = \text{Cone}(\varepsilon)[1] \in D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}) \). Then by induction we get that all \( \varepsilon_i \) are surjective (and all \( \mathcal{F}_i \) lies in \( D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}) \)). Thus sequence (23) gives resolvent of \( \mathcal{F} \). Theorem is proved.

Let’s proceed to the considering of the category \( D^{-}(\text{ShNisWtr}) \) and how semi-orthogonal decomposition consistent with Nisnevich topology.

**Theorem 11** There is a semi-orthogonal decomposition

\[
D^{-}(\text{ShNisWtr}) = < D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}), D^{-}_{\mathbb{A}^{-}\text{inv}}(\text{PreWtr}) >
\]

such that the category \( D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}) \) is thick subcategory generated by Nisnevich sheaves \( Z_{\text{Wtr,Nis}}(\mathbb{A}^1 \times X)/Z_{\text{Wtr,Nis}}(X) \) and the category \( D^{-}_{\mathbb{A}^{-}\text{inv}}(\text{PreWtr}) \) as full subcategory of \( D^{-}(\text{ShNisWtr}) \) consists of the complexes with homotopy invariant cohomologies.

**Proof of the theorem.** We can regard category \( D^{-}(\text{ShNisWtr}) \) as the localisation of \( D^{-}(\text{PreWtr}) \) at Nisnevich sheaf quasi-equivalences. Let’s denote this functor by

\[ l_{\text{Nis}} : D^{-}(\text{PreWtr}) \rightarrow D^{-}(\text{ShNisWtr}) \].

Let’s \( \mathcal{A} \) and \( \mathcal{B} \) be images of subcategories \( D^{-}_{\mathbb{A}^{-}\text{inv}}(\text{PreWtr}) \) and \( D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}) \)

\[ \mathcal{A}, \mathcal{B} \subset D^{-}(\text{ShNisWtr}) : \mathcal{A} = l_{\text{Nis}}(D^{-}_{\mathbb{A}^{-}\text{inv}}(\text{PreWtr})), \mathcal{B} = D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}), \]

i.e. full subcategories that consists of the complexes that are sheaf quasi-isomorphic to the complexes that lies in \( D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}) \) and \( D^{-}_{\mathbb{A}^{-}\text{inv}}(\text{PreWtr}) \) respectively.

Let’s show that

\[ \mathcal{A} = D^{-}_{\mathbb{A}^{-}\text{inv}}(\text{PreWtr}), \mathcal{B} = D^{-}_{\mathbb{A}^{-}\text{contr}}(\text{PreWtr}). \quad (23) \]

The functor \( l_{\text{Nis}} \) sends any presheave as a complex concentrated in degree zero to its sheafification. Then since localisation functor sends generators of thick subcategory to generators
of its image, theorem \[10\] implies that subcategories \( \mathcal{A} \) and \( \mathcal{B} \) are thick subcategories generated by sheafifications of homotopy invariant presheaves and by sheaves \( \mathbb{Z}_{Wtr,Nis}(\mathbb{A}^1 \times X)/\mathbb{Z}_{Wtr,Nis}(X) \) (that are Nisnevich sheafifications of presheaves \( \mathbb{Z}_{Wtr}(\mathbb{A}^1 \times X)/\mathbb{Z}_{Wtr}(X) \) by definition). So we get the equality for \( \mathcal{B} \). To prove the equality for \( \mathcal{A} \) let’s note that by lemma \[3,1\] \( D^\leftarrow_{\text{inv}}(\text{PreWtr}) \) is thick subcategory generated by homotopy invariant sheaves. By theorem 4 form \[5\] sheafification of homotopy invariant presheaves with Witt-transfers is homotopy invariant. Conversely any homotopy invariant sheaf is homotopy invariant presheave. So the set of Nisnevich sheafifications of homotopy invariant presheaves with Witt-transfers is exactly the set of homotopy invariant sheaves with Witt-transfers. Thus we get the equalities \( [23] \).

Then let’s show that theorem \[3\] and theorem \[7\] (about strictly homotopy invariance of Nisnevich sheafication of homotopy invariant presheave with Witt-transfers and about isomorphism of \( \text{Ext} \)-groups in \( ShNisWtr \) and Nisnevich cohomology groups of sheave with Witt-transfers) implies that the categories \( \mathcal{A} \) and \( \mathcal{B} \) are semi-orthogonal. Really to prove that

\[
\text{Hom}_{D^-(ShNisWtr)}(B^\bullet, A^\bullet) = 0: \quad A^\bullet \in \mathcal{A}, B^\bullet \in \mathcal{B}
\]

it is enough to check it on generators of this subcategories, i.e. for

\[
A^\bullet = \mathcal{F}[i], \quad B^\bullet = \mathbb{Z}_{Wtr,Nis}(X \times \mathbb{A}^1)/\mathbb{Z}_{Wtr,Nis}(X)
\]

for any homotopy invariant presheave \( \mathcal{F} \) and smooth affine \( X \). But

\[
\text{Hom}_{D^-(ShNisWtr)}(\mathbb{Z}_{Wtr,Nis}(X \times \mathbb{A}^1)/\mathbb{Z}_{Wtr,Nis}(X), \mathcal{F}[i]) \simeq \text{Ext}^i_{\text{ShNisWtr}}(\mathbb{Z}_{Wtr,Nis}(X \times \mathbb{A}^1)/\mathbb{Z}_{Wtr,Nis}(X), \mathcal{F}_{\text{Nis}}) = 0
\]

and the last group is zero because by the remark to the theorems \[3\] and \[7\]

\[
\text{Ext}^i_{\text{ShNisWtr}}(\mathbb{Z}_{Wtr,Nis}(X \times \mathbb{A}^1), \mathcal{F}_{\text{Nis}}) \simeq H^i_{\text{Nis}}(X \times \mathbb{A}^1, \mathcal{F}_{\text{Nis}}) \simeq H^i_{\text{Nis}}(X, \mathcal{F}_{\text{Nis}}) \simeq \text{Ext}^i_{\text{ShNisWtr}}(\mathbb{Z}_{Wtr,Nis}(X), \mathcal{F}_{\text{Nis}}).
\]

Now to prove that the pair \( < \mathcal{B}, \mathcal{B}, \mathcal{A} > \) provides semi-orthogonal decomposition of \( D^-(ShNisWtr) \) it is enough to show that for any object \( C^\bullet \in D^-(ShNisWtr) \) there is distinguished triangle

\[
A^\bullet[-1] \to B^\bullet \to C^\bullet \to A^\bullet: \quad B^\bullet \in \mathcal{B}, A^\bullet \in \mathcal{A}.
\]

But since \( < D^-_{\text{constr}}(\text{PreWtr}), D^-_{\text{inv}}(\text{PreWtr}) > \) is semi-orthogonal decomposition of \( D^-(\text{PreWtr}) \) for any complex \( C \) there is distinguished triangle in \( D^-(\text{PreWtr}) \) \( A^\bullet[-1] \to B^\bullet \to C^\bullet \to A^\bullet \) with \( B^\bullet \in D^-_{\text{constr}}(\text{PreWtr}) \) and \( A^\bullet \in D^-_{\text{inv}}(\text{PreWtr}) \) and this triangle remains to be distinguished after localisation \( l_{\text{Nis}} \).

\textit{Theorem is proved.}

\textbf{Definition 19} The category of effective motives \( DW_{\text{eff}}(k) \) is homotopy invariant part of \( D^-(ShNisWtr) \), i.e. the category \( D^-_{\text{inv}}(\text{PreWtr}) \).

\textbf{Remark 18} Due to semi-orthogonal decomposition proved in theorem \[11\] there is adjacency

\[
C^\bullet: D^-(ShNisWtr) \rightleftharpoons DW_{\text{eff}}(k): i_{\text{eff}}
\]

\[
C^\bullet \dashv i_{\text{eff}}
\]

of the projection functor defined by \( C^\bullet \) and embedding.
Remark 19 Since the projection functor $D^{-}(Sh_{NisWtr}) \to D^{-}_{inv}(PreWtr)$ is equivalent to the localization functor at the subcategory $D^{-}_{\text{contr}}(PreWtr)$, and since $D^{-}_{\text{contr}}(PreWtr)$ by the theorem [11] is generated by cones of projection $X \times \mathbb{A}^1 \to X$ for all $X$, then the category $DWM_{\text{eff}}(k)$ is equivalent to the localisation of $D^{-}(Sh_{NisWtr})$ at $X \times \mathbb{A}^1 \to X$.

Let’s define Witt-motives of smooth affine varieties.

Definition 20 The functor

$$WM : Sm_k \to DWM_{\text{eff}}(k)$$

defining Witt-motives of smooth affine varieties is composition

$$DWM \overset{\text{def}}{=} l_{\mathbb{A}} \circ -_{\text{Nis}} \circ Wtr \circ \mathbb{Z}(-).$$

Now we get the required property by composing of the adjacency (24) with adjacency (22) from previous paragraph.

Theorem 12 There is natural isomorphism

$$\text{Hom}_{DWM_{\text{eff}}(k)}(WM(X), \mathcal{F}[i]) \simeq H^i_{\text{Nis}}(X, \mathcal{F})$$

for all smooth affine $X$ and homotopy invariant sheaf with Witt-transfers $\mathcal{F}$.

Proof of the theorem. By definition of the functor $WM$

$$WM(X) = C^*(\text{Witt}_{\text{Nis}}(X)).$$

Due to adjacency of $C^* \dashv i_{\mathbb{A}}$ from the remark [18]

$$\text{Hom}_{DWM_{\text{eff}}(k)}(C^*(\mathbb{Z}_{Wtr,Nis}(X)), \mathcal{F}[i]) \simeq \text{Hom}_{D^{-}(Sh_{NisWtr})}(\mathbb{Z}_{Wtr,Nis}(X), \mathcal{F}[i])$$

Then by isomorphism from theorem [7]

$$\text{Hom}_{D^{-}(Sh_{NisWtr})}(\mathbb{Z}_{Wtr,Nis}(X), \mathcal{F}[i]) \simeq \text{Ext}_{\text{NisSh}}(\mathbb{Z}_{Wtr,Nis}(X), \mathcal{F}[i]),$$

And finally

$$\text{Ext}_{\text{NisSh}}(\mathbb{Z}(X), \mathcal{F}[i]) \simeq H^i_{\text{Nis}}(X, \mathcal{F}).$$

Theorem is proved.

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