S.1. PROOFS OF THE RESULTS IN SECTION 2

S.1.1. Preliminary Calculations

Notice that for any integer $1 \leq i \leq n$ and conditional on sample size $N$, such that $1 \leq N \leq n$, we obtain

$$E[R_i] = \frac{N}{n}, \quad \text{var}(R_i) = \frac{N}{n} \left(1 - \frac{N}{n}\right).$$

Also, for any integers $1 \leq j < k \leq n$,

$$\text{var} \left( \sum_{i=1}^{n} R_i \right) = n \text{var}(R_j) + n(n - 1) \text{cov}(R_j, R_k) = 0.$$

This implies

$$\text{cov}(R_j, R_k) = - \frac{\text{var}(R_j)}{n - 1} = - \frac{N}{n(n - 1)} \left(1 - \frac{N}{n}\right).$$

In turn, this implies

$$E[R_i R_j] = \frac{N(N - 1)}{n(n - 1)}.$$

Let

$$\tilde{Y} = \frac{1}{N} \sum_{i=1}^{n} R_i Y_i \quad \text{and} \quad \mu = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$
Now,

\[ E[\bar{Y}] = \frac{1}{N} \sum_{i=1}^{n} E[R_i] Y_i = \mu. \]

Let

\[ S_Y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \mu)^2. \]

Notice that

\[ \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 = \frac{1}{n} \sum_{i=1}^{n} Y_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right)^2 \]

\[ = \frac{n-1}{n^2} \sum_{i=1}^{n} Y_i^2 - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} Y_i Y_j. \]

This implies

\[ nS_Y^2 = \sum_{i=1}^{n} Y_i^2 - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} Y_i Y_j. \]

Therefore,

\[ \text{var}(\bar{Y}) = \frac{1}{N^2} \sum_{i=1}^{n} \text{var}(R_i) Y_i^2 + \frac{2}{N^2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \text{cov}(R_i, R_j) Y_i Y_j \]

\[ = \frac{1}{N^2} \text{var}(R_1) \left( \sum_{i=1}^{n} Y_i^2 - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} Y_i Y_j \right) \]

\[ = \frac{n}{N^2} \text{var}(R_1) S_Y^2 \]

\[ = \frac{S_Y^2}{N} \left( 1 - \frac{N}{n} \right). \]

Let

\[ \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{n} R_i Y_i^2 - \left( \frac{1}{N} \sum_{i=1}^{n} R_i Y_i \right)^2. \]

Then

\[ \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{n} R_i Y_i^2 - \frac{1}{N^2} \sum_{i=1}^{n} R_i Y_i^2 - \frac{2}{N^2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_i R_j Y_i Y_j. \]

Therefore,

\[ E[\hat{\sigma}^2] = \frac{1}{n} \frac{N-1}{N} \sum_{i=1}^{n} Y_i^2 - \frac{2}{n(n-1)} \frac{N-1}{N} \sum_{i=1}^{n} \sum_{j=i+1}^{n} Y_i Y_j = \frac{N-1}{N} S_Y^2. \]
S.1.2. Causal versus Descriptive Estimands

Let

$$\hat{\theta} = \frac{1}{N_1} \sum_{i=1}^{n} R_i X_i Y_i - \frac{1}{N_0} \sum_{i=1}^{n} R_i (1 - X_i) Y_i.$$ 

We will do all the analysis conditional on $N_1, N_0$, for $N_1 > 0, N_0 > 0, n_1 > 0$, and $n_0 > 0$. To economize notation, we will leave this conditioning implicit. Notice that

$$E[\hat{\theta}|X] = \theta^{\text{desc}},$$

where

$$\theta^{\text{desc}} = \frac{1}{n_1} \sum_{i=1}^{n} X_i Y_i - \frac{1}{n_0} \sum_{i=1}^{n} (1 - X_i) Y_i$$

$$= \frac{1}{n_1} \sum_{i=1}^{n} X_i Y^*_i (1) - \frac{1}{n_0} \sum_{i=1}^{n} (1 - X_i) Y^*_i (0).$$

By the law of total variance,

$$\text{var}(\hat{\theta}) = E[\text{var}(\hat{\theta}|X)] + \text{var}(\theta^{\text{desc}}).$$

The expectation of $\theta^{\text{desc}}$ over the randomization distribution is

$$E[\theta^{\text{desc}}] = \frac{1}{n_1} \sum_{i=1}^{n} E[X_i] Y_i - \frac{1}{n_0} \sum_{i=1}^{n} (1 - E[X_i]) Y_i$$

$$= \frac{1}{n_1} \sum_{i=1}^{n} (n_1/n) Y^*_i (1) - \frac{1}{n_0} \sum_{i=1}^{n} (n_0/n) Y^*_i (0)$$

$$= \theta^{\text{causal}}.$$

For the variance of $\theta^{\text{desc}}$, we have to compute $n$ square terms and $n(n-1)$ cross-product terms. Each square term is equal to

$$\frac{\text{var}(X_i)}{n_1^2} Y^*_i (1)^2 + \frac{\text{var}(X_i)}{n_0^2} Y^*_i (0)^2 + 2 \frac{\text{var}(X_i)}{n_1 n_0} Y^*_i (1) Y^*_i (0)$$

$$= \text{var}(X_i) \left( \frac{Y^*_i (1)^2}{n_1^2} + \frac{Y^*_i (0)^2}{n_0^2} + 2 \frac{Y^*_i (1) Y^*_i (0)}{n_1 n_0} \right).$$

Recall from previous calculations that

$$\text{cov}(X_i, X_j) = - \frac{\text{var}(X_i)}{n-1}.$$

Therefore, each of the cross-product terms is equal to

$$- \frac{\text{var}(X_i)}{n-1} \left( \frac{Y^*_i (1) Y^*_j (1)}{n_1^2} + \frac{Y^*_i (1) Y^*_j (0)}{n_1 n_0} + \frac{Y^*_i (0) Y^*_j (1)}{n_1 n_0} + \frac{Y^*_i (0) Y^*_j (0)}{n_0^2} \right).$$
Let $\theta_i = Y_i^*(1) - Y_i^*(0)$; then
\[
nS^2_\theta = \sum_{i=1}^{n} (Y_i^*(1) - Y_i^*(0))^2 - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (Y_i^*(1) - Y_i^*(0))(Y_j^*(1) - Y_j^*(0))
\]
\[
= nS^2_1 + nS^2_0 - 2 \left( \sum_{i=1}^{n} Y_i^*(1)Y_i^*(0) - \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (Y_i^*(1)Y_j^*(0) + Y_i^*(0)Y_j^*(1)) \right).
\]

As a result, we obtain
\[
\text{var}(\theta^{\text{desc}}) = \text{var}(\theta) = \left( \frac{nS^2_1}{n_1^2} + \frac{nS^2_0}{n_0^2} + \frac{nS^2_i}{n_1n_0} + \frac{nS^2_\theta}{n_1n_0} - \frac{nS^2_0}{n_0} \right)
\]
\[
= \frac{n_1n_0}{n^2} \left( \frac{nS^2_1}{n_1^2} + \frac{nS^2_0}{n_0^2} + \frac{nS^2_i}{n_1n_0} + \frac{nS^2_\theta}{n_1n_0} - \frac{nS^2_0}{n_0} \right)
\]
\[
= \frac{n_1n_0}{n^2} \left( \frac{n^2}{n_1n_0}S^2_1 + \frac{n^2}{n_1n_0}S^2_0 - \frac{nS^2_\theta}{n_1n_0} \right)
\]
\[
= \frac{S^2_1}{n_1} + \frac{S^2_0}{n_0} - \frac{S^2_\theta}{n}.
\]

Notice now that (because we condition on $N_1$ and $N_0$)
\[
\text{var}(\theta|X) = \text{var} \left( \sum_{i=1}^{n} R_i X_i \frac{Y_i^*(1)}{N_1} - \sum_{i=1}^{n} R_i (1 - X_i) \frac{Y_i^*(0)}{N_0} \right| X)
\]
\[
= \text{var} \left( \sum_{i=1}^{n} R_i X_i \frac{Y_i^*(1)}{N_1} \right| X) + \text{var} \left( \sum_{i=1}^{n} R_i (1 - X_i) \frac{Y_i^*(0)}{N_0} \right| X).
\]

Let us calculate the first term on the right-hand side of the last equation (the second term will be analogous):
\[
\text{var} \left( \sum_{i=1}^{n} R_i X_i \frac{Y_i^*(1)}{N_1} \right| X)
\]
\[
= \text{var}(R_i|X) = 1 \left[ \sum_{i=1}^{n} X_i \frac{Y_i^*(1)^2}{N_1^2} - \frac{2}{n_1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_i X_j \frac{Y_i^*(1)Y_j^*(1)}{N_1^2} \right].
\]

Taking expectations, the right-hand side becomes
\[
\text{var}(R_i|X) = 1 \left[ \sum_{i=1}^{n} \frac{n_1}{n} \frac{Y_i^*(1)^2}{N_1^2} - \frac{2}{n_1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{n_1(n_1 - 1)}{n(n-1)} \frac{Y_i^*(1)Y_j^*(1)}{N_1^2} \right]
\]
\[
= \frac{1}{nN_1} \left( \frac{n_1 - N_1}{n_1} \right) \left[ \sum_{i=1}^{n} Y_i^*(1)^2 - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} Y_i^*(1)Y_j^*(1) \right] = \frac{n_1 - N_1}{n_1N_1} S^2_1.
\]
This implies
\[ \text{var}(\hat{\theta}) = \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} - \frac{S_\theta^2}{n}. \]

Now, notice that
\[ E[\hat{\theta}|R] = \theta^{\text{causal, sample}}, \]
where
\[ \theta^{\text{causal, sample}} = \frac{1}{N} \sum_{i=1}^{n} R_i(Y_i^*(1) - Y_i^*(0)). \]

Therefore, by the law of total variance,
\[ \text{var}(\hat{\theta}) = E[\text{var}(\hat{\theta}|R)] + \text{var}(\theta^{\text{causal, sample}}). \]

The variance of \( \theta^{\text{causal, sample}} \) is
\[ \frac{\text{var}(R_i)}{N^2} \left[ \sum_{i=1}^{N} (Y_i^*(1) - Y_i^*(0)) - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (Y_i^*(1) - Y_i^*(0))(Y_j^*(1) - Y_j^*(0)) \right] = \frac{1}{N} \left( 1 - \frac{N}{n} \right) S_\theta^2. \]

As a result,
\[ E[\text{var}(\hat{\theta}|R)] = \text{var}(\hat{\theta}) - \text{var}(\theta^{\text{causal, sample}}) \]
\[ = \frac{S_1^2}{N_1} + \frac{S_0^2}{N_0} - \frac{S_\theta^2}{N}. \]

S.1.3. EHW Variance

The EHW variance estimator for \( \hat{\theta} \) is
\[ \hat{\var}(\hat{\theta}) = \frac{N_1 - 1}{N_1^2} \hat{S}_1^2 + \frac{N_0 - 1}{N_0^2} \hat{S}_0^2, \]
where
\[ \hat{S}_1^2 = \frac{1}{N_1 - 1} \sum_{i=1}^{n} R_i X_i \left( Y_i - \frac{1}{N_1} \sum_{i=1}^{n} R_i X_i Y_i \right)^2, \]
and \( \hat{S}_0^2 \) is defined analogously. Using previous results, we obtain
\[ \hat{S}_1^2 = \frac{1}{N_1} \sum_{i=1}^{n} R_i X_i Y_i^2 - \frac{2}{N_1 (N_1 - 1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_i R_j X_i X_j Y_i Y_j. \]
Therefore,
\[
E[\hat{S}^2_1 | X] = \frac{1}{n_1} \sum_{i=1}^{n} X_i Y_i^2 - \frac{2}{n_1(n_1 - 1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_i X_j Y_i Y_j
\]
and
\[
E[\hat{S}^2_0] = \frac{1}{n} \sum_{i=1}^{n} Y_i^*(1)^2 - \frac{2}{n(n - 1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} Y_i^*(1)Y_j^*(1)
= S^2_1.
\]

Let
\[
\hat{V}_{ehw} = \frac{\hat{S}^2_1}{N_1} + \frac{\hat{S}^2_0}{N_0}.
\]

Then
\[
E[\hat{V}_{ehw}] = \frac{S^2_1}{N_1} + \frac{S^2_0}{N_0} \geq \text{var}(\hat{\theta}).
\]

S.1.4. Bootstrap Variance

Consider the bootstrap variance estimator that draws \( N_1 \) treated and \( N_0 \) untreated observations separately,
\[
\hat{V}_B^{\text{boot}} = \frac{1}{B - 1} \sum_{b=1}^{B} (\hat{\theta}^{(b)} - \bar{\theta}_B)^2,
\]
where
\[
\hat{\theta}^{(b)} = \frac{1}{N_1} \sum_{i=1}^{n} K_{1i}^{(b)} R_i X_i Y_i - \frac{1}{N_0} \sum_{i=1}^{n} K_{0i}^{(b)} R_i (1 - X_i) Y_i
\]
and
\[
\bar{\theta}_B = \frac{1}{B} \sum_{b=1}^{B} \hat{\theta}^{(b)}.
\]

Conditional on \( R \) and \( X \), \( K_{1i}^{(b)} \) has a multinomial distribution with parameter \( \{N_1, 1/N_1, \ldots , 1/N_1\} \) for units with \( R_i X_i = 1 \) and \( K_{0i}^{(b)} \) has a multinomial distribution with parameters \( \{N_0, 1/N_0, \ldots , 1/N_0\} \) for units with \( R_i (1 - X_i) = 1 \). The variables \( K_{1i}^{(b)} \) and \( K_{0i}^{(b)} \) are independent of each other and independent across \( b = 1, \ldots , B \). As a result, for \( R_i X_i = R_j X_j = 1 \) with \( i \neq j \), we obtain
\[
E[K_{1i}^{(b)} | R, X] = 1,
\]
\[
E[(K_{1i}^{(b)})^2 | R, X] = (2N_1 - 1)/N_1,
\]
and
\[
E[K_{1i}^{(b)} K_{1j}^{(b)} | R, X] = (N_1 - 1)/N_1.
\]
Conditional on \( Y(1), Y(0), R, \) and \( X, \) the mean of the bootstrap variance is

\[
E[\hat{V}_{\text{boot}}^B|R, X] = \left( \frac{B}{B-1} \right) E[(\hat{\theta}^{(b)} - \tilde{\theta}_B)^2|R, X].
\]

Because of independence of the bootstrap weights between treatment samples, we obtain

\[
E[\hat{V}_{\text{boot}}^B|R, X] = \left( \frac{B}{B-1} \right) E[(\hat{\theta}^{(b)} - \tilde{\theta}_B)^2|R, X] + \left( \frac{B}{B-1} \right) E[(\hat{\theta}^{(b)} - \tilde{\theta}_0B)^2|R, X],
\]

where

\[
\hat{\theta}_1^{(b)} = \frac{1}{N_1} \sum_{i=1}^n K_{1i}^{(b)} R_i X_i Y_i
\]

and

\[
\tilde{\theta}_1 = \frac{1}{B} \sum_{b=1}^B \hat{\theta}_1^{(b)},
\]

with analogous expressions for \( \hat{\theta}_0^{(b)} \) and \( \tilde{\theta}_0. \) Now, let

\[
K_1 = \begin{pmatrix}
K_{11}^{(1)} & K_{11}^{(2)} & \cdots & K_{11}^{(B)} \\
K_{12}^{(1)} & K_{12}^{(2)} & \cdots & K_{12}^{(B)} \\
\vdots & \vdots & \ddots & \vdots \\
K_{1n}^{(1)} & K_{1n}^{(2)} & \cdots & K_{1n}^{(B)}
\end{pmatrix}
\]

and let \( K_{1\pi} \) be a random permutation of the columns of \( K_1. \) Notice that \( \tilde{\theta}_1 \) is fixed conditional on \( R, X, \) and \( K_{1\pi}, \) but \( \hat{\theta}_1^{(b)} \) is not. In addition,

\[
E[\hat{\theta}_1^{(b)}|R, X, K_{1\pi}] = \tilde{\theta}_1.
\]

Therefore,

\[
E[(\hat{\theta}_1^{(b)} - \tilde{\theta}_1)^2|R, X, K_{1\pi}] = E[(\hat{\theta}_1^{(b)})^2|R, X, K_{1\pi}] - \tilde{\theta}_1^2.
\]

Then

\[
E[(\hat{\theta}_1^{(b)} - \tilde{\theta}_1)^2|R, X] = E[(\hat{\theta}_1^{(b)})^2|R, X] - E[\tilde{\theta}_1^2|R, X].
\]

Let

\[
\hat{\theta}_1 = \frac{1}{N_1} \sum_{i=1}^N R_i X_i Y_i.
\]

Notice that for any \( b \) and \( c, \) such that \( 1 \leq b < c \leq B, \) we have

\[
E[\hat{\theta}_1^2|R, X] = \frac{1}{B} E[(\hat{\theta}_1^{(b)})^2|R, X] + \frac{B-1}{B} E[\hat{\theta}_1^{(b)} \hat{\theta}_1^{(c)}|R, X]
\]

\[
= \frac{1}{B} E[(\hat{\theta}_1^{(b)})^2|R, X] + \frac{B-1}{B} \hat{\theta}_1^2.
\]
Therefore,

\[
E[(\hat{\theta}_1^{(b)} - \bar{\theta}_1B)^2 | \mathbf{R}, \mathbf{X}] = \left( \frac{B - 1}{B} \right) \left( E[(\hat{\theta}_1^{(b)})^2 | \mathbf{R}, \mathbf{X}] - \hat{\theta}_1^2 \right).
\]

In addition,

\[
E[(\hat{\theta}_1^{(b)})^2 | \mathbf{R}, \mathbf{X}] = \frac{1}{N_1^2} \left( \sum_{i=1}^{n} \frac{2N_1 - 1}{N_1} R_i X_i Y_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{N_1 - 1}{N_1} R_i R_j X_i X_j Y_i Y_j \right).
\]

Therefore,

\[
E[(\hat{\theta}_1^{(b)})^2 | \mathbf{R}, \mathbf{X}] - \hat{\theta}_1^2 = \frac{1}{N_1^2} \left( \sum_{i=1}^{n} \frac{N_1 - 1}{N_1} R_i X_i Y_i^2 - 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{N_1} R_i R_j X_i X_j Y_i Y_j \right)
\]

\[
= \frac{N_1 - 1}{N_1^2} \left( \frac{1}{N_1} \sum_{i=1}^{n} R_i X_i Y_i^2 - \frac{2}{N_1(N_1 - 1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_i R_j X_i X_j Y_i Y_j \right)
\]

\[
= \frac{N_1 - 1}{N_1^2} \hat{S}_1
\]

and

\[
E[(\hat{\theta}_1^{(b)} - \bar{\theta}_1B)^2 | \mathbf{R}, \mathbf{X}] = \left( \frac{B - 1}{B} \right) \frac{N_1 - 1}{N_1^2} \hat{S}_1^2,
\]

with the analogous result holding for \( E[(\hat{\theta}_0^{(b)} - \bar{\theta}_0B)^2 | \mathbf{R}, \mathbf{X}] \). It follows that

\[
E[\hat{V}_{B}^{\text{boot}} | \mathbf{R}, \mathbf{X}] = \hat{V}_{\text{ehw}}.
\]

**Proof of Lemma 1:** Let

\[
W_{n,i} = \begin{pmatrix} Y_{n,i} \\ X_{n,i} \\ Z_{n,i} \end{pmatrix},
\]

and let \( W_{n,i}^{(k,l)} \) be the \((k, l)\) element of \( W_{n,i} \). Similarly, let \( \tilde{W}_{n,i}^{(k,l)} \), \( W_n^{(k,l)} \), \( \tilde{W}_n^{(k,l)} \), \( \tilde{\Omega}_n^{(k,l)} \), \( \Omega_n^{(k,l)} \) be the \((k, l)\) elements of \( R_{n,i}, W_{n,i}, W_n, \tilde{W}_n, \tilde{\Omega}_n, \) and \( \Omega_n \), respectively. In order to have \( \tilde{W}_n^{(k,l)} \) and \( \tilde{\Omega}_n^{(k,l)} \) well-defined, let \( \tilde{W}_n^{(k,l)} = \Omega_n^{(k,l)} = 0 \) when \( N = 0 \) (this is without loss of generality). Notice that, because \( np_n \to \infty \), for any fixed \( 0 < \epsilon < 1 \), there is \( n_\epsilon \) such that for \( n > n_\epsilon \), we have \( np_n > -\log(\epsilon) \). Therefore, for \( n > n_\epsilon \), we obtain

\[
\Pr(N = 0) = \left( 1 - \frac{np_n}{n} \right)^n < \left( 1 + \frac{\log(\epsilon)}{n} \right)^n < e^{\log(\epsilon)} = \epsilon.
\]

As a result, \( \Pr(N = 0) \to 0 \) and

\[
E[(\tilde{W}_n^{(k,l)} - \Omega_n^{(k,l)})^2 | N = 0] \Pr(N = 0) = (\Omega_n^{(k,l)})^2 \Pr(N = 0) \to 0
\]
by Assumption 5 and Holder’s inequality. Notice that for any integer, $m$, such that $1 \leq m \leq n$, we have
\[
E\left[\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E[W_{n,i}^{(k,l)}]\right|N = m] = 0
\]
and
\[
E\left[(\tilde{W}_{n}^{(k,l)} - \Omega_{n}^{(k,l)})^2|N = m\right] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E[W_{n,i}^{(k,l)}]\right)^2|N = m\right]
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n} E\left[\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)} - E[W_{n,i}^{(k,l)}]\right)^2|N = m\right]
\]
\[
\leq \frac{1}{n^2} \sum_{i=1}^{n} E\left[\left(\frac{n}{N}\right)R_{n,i}W_{n,i}^{(k,l)}\right)^2|N = m\right]
\]
\[
= \frac{1}{m}\left(\frac{1}{n} \sum_{i=1}^{n} E[(W_{n,i}^{(k,l)})^2]\right)
\]
\[
\leq C/m,
\]
for some positive constant, $C$, by Assumption 5. Let
\[
\tilde{\xi}_{m}^{(k,l)} = \begin{cases} 
(\tilde{W}_{n}^{(k,l)} - \Omega_{n}^{(k,l)})^2 & \text{if } m > 0, \\
0 & \text{if } m = 0,
\end{cases} \quad \text{and} \quad \xi_{m} = \begin{cases} 
C/m & \text{if } m > 0, \\
0 & \text{if } m = 0.
\end{cases}
\]
Now,
\[
E[(\tilde{W}_{n}^{(k,l)} - \Omega_{n}^{(k,l)})^2|N > 0] \Pr(N > 0) = E[\tilde{\xi}_{N}^{(k,l)}] \leq E[\xi_{N}].
\]
Applying Chernoff’s bounds, for any $\varepsilon > 0$,
\[
\Pr(\xi_{N} \geq \varepsilon) \leq \Pr(0 < N < C/\varepsilon)
\]
\[
< \Pr(N < C/\varepsilon)
\]
\[
= \Pr\left(N < n\rho_n\left(1 - \frac{n\rho_n - C/\varepsilon}{n\rho_n}\right)\right)
\]
\[
\leq e^{-\left(\frac{n\rho_n - C/\varepsilon}{2n\rho_n}\right)^2} \to 0,
\]
which implies that $\xi_{N}$ converges in probability to zero. Because $\xi_{N}$ is bounded, by the portmanteau lemma we obtain $E[\xi_{N}] \to 0$. As a result,
\[
E\left[(\tilde{W}_{n}^{(k,l)} - \Omega_{n}^{(k,l)})^2\right] \to 0.
\]
For the second result, notice that

\[
E\left[ (\tilde{\Omega}_n^{(k,l)} - \Omega_n^{(k,l)})^2 \mid N = m \right] = E\left[ \left( \sum_{i=1}^{n} \left( \frac{R_{n,i}}{m} - \frac{1}{n} \right) E[W_{n,i}^{(k,l)}] \right)^2 \mid N = m \right]
\]

\[
= \sum_{i=1}^{n} \frac{1}{m^2} E\left[ \left( R_{n,i} - \frac{m}{n} \right)^2 \mid N = m \right] (E[W_{n,i}^{(k,l)}])^2
\]

\[
= \sum_{i=1}^{n} \frac{1}{mn} \left( 1 - \frac{m}{n} \right) (E[W_{n,i}^{(k,l)}])^2
\]

\[
\leq \frac{1}{m} \left( \frac{1}{n} \sum_{i=1}^{n} (E[W_{n,i}^{(k,l)}])^2 \right).
\]

Now, using the same argument as above, we obtain

\[
E\left[ (\tilde{\Omega}_n^{(k,l)} - \Omega_n^{(k,l)})^2 \right] \rightarrow 0.
\]

The proof of the third result is analogous. \textit{Q.E.D.}

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