Study of the Change from Walking to Non-Walking Behavior
in a Vectorial Gauge Theory as a Function of $N_f$

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Based on recent works [1,2], we present the results of calculations for several physical quantities (meson masses, the $S$ parameter, etc.) in a vectorial gauge theory, as a function of the number of fermions, $N_f$. Solutions of the Schwinger-Dyson and the Bethe-Salpeter equations with the improved ladder approximation are used for the calculations. We focus on how the values of physical quantities change as one moves from the QCD-like (non-walking) to walking regimes.

1. Introduction

We consider a (3 + 1)-dimensional vectorial gauge theory (at zero temperature and chemical potential) with the gauge group SU($N_c$) and $N_f$ massless fermions transforming according to the fundamental representation of this group. For $N_c = 3$, if one took $N_f = 2$, this would be an approximation to actual QCD. We restrict here to the range $N_f < (11/2)N_c$ for which the theory is asymptotically free. An analysis using the two-loop beta function and Schwinger-Dyson equation leads to the inference that for $N_f$ in this range, the theory includes two phases: (i) for $0 \leq N_f \leq N_{f,cr}$ a phase with confinement and spontaneous chiral symmetry breaking ($S\chi$SB); and (ii) for $N_{f,cr} \leq N_f \leq (11/2)N_c$ a phase with no spontaneous chiral symmetry breaking (plausibly a non-Abelian Coulomb phase). We shall refer to $N_{f,cr}$, the critical value of $N_f$, as the boundary between these two phases.

For $N_f$ slightly less than $N_{f,cr}$, the theory exhibits an approximate infrared (IR) fixed point with resultant walking behavior. That is, as the energy scale $\mu$ decreases from large values, $\alpha = g^2/(4\pi)$ ($g$ being the SU($N_c$) gauge coupling) grows to be $O(1)$ at a scale $\Lambda$, but increases only rather
slowly as \( \mu \) decreases below this scale, so that there is an extended interval in energy below \( \Lambda \) where \( \alpha \) is large, but slowly varying. Associated with this slowly running behavior, the resultant dynamically generated fermion mass, \( \Sigma \), is much smaller than \( \Lambda \). In addition to its intrinsic field-theoretic interest, this walking behavior has played an important role in theories of dynamical electroweak symmetry breaking [3]-[9]. As \( N_f \) approaches \( N_{f,cr} \) from below, quantities with dimensions of mass vanish continuously; i.e., the chiral phase transition separating phases (i) and (ii) is continuous. Recently, meson masses and other quantities such as the generalized pseudoscalar decay constant \( f_P \) and the \( S \) parameter [10] were calculated in the walking limit of an SU(\( N_c \)) gauge theory [11,12].

It is of interest to investigate how meson masses and other quantities change as one decreases \( N_f \) below \( N_{f,cr} \), moving away from the boundary, as a function of \( N_f \), between phases (i) and (ii), deeper into the confined phase. For this purpose, in Ref. [1], as in Refs. [11,12], we use the Schwinger-Dyson (SD) equation to compute the dynamical fermion mass \( \Sigma \) (generalized constituent quark mass) and then insert this into the Bethe-Salpeter (BS) equation to obtain the masses of the low-lying mesons and other quantities. We restrict to an interval of \( N_f \) values for which the theory has an infrared fixed point. For definiteness, we take \( N_c = 3 \); however, \( N_c \) enters only indirectly, via the dependence of the value of the infrared fixed point \( \alpha_* \) on \( N_c \). Hence, our findings may also be applied in a straightforward way, with appropriate changes in the value of \( \alpha_* \), to an SU(\( N_c \)) gauge theory with a different value of \( N_c \).

In order to study meson masses and other quantities as one moves away from the boundary between phases (i) and (ii), it is first necessary to know as accurately as possible where this boundary lies, as a function of \( N_f \), i.e., to know the value of \( N_{f,cr} \). For sufficiently large \( N_f \), the beta function (calculated to the maximal scheme-independent order, namely two loops) has an IR fixed point at

\[
\alpha_* = \frac{-4\pi(11N_c - 2N_f)}{34N_c^2 - 13N_cN_f + 3N_c^{-1}N_f}.
\]  

(1)

Requiring that \( \alpha_* \) be sufficiently large as to yield spontaneous symmetry breaking in the context of an approximate solution to the SD equation for a fermion yields the condition that \( N_f < N_{f,cr} \), where [9]

\[
N_{f,cr} = \frac{2N_c(50N_c^2 - 33)}{5(5N_c^2 - 3)}.
\]  

(2)

For \( N_c = 3 \) this gives \( N_{f,cr} \approx 11.9 \). These estimates are only rough, in view
of the strongly coupled nature of the physics. Effects of higher-order gluon exchanges and instantons have been studied in Refs. [13].

In our analysis, what we actually vary is the value of the approximate IR fixed point $\alpha_*$, which depends parametrically on $N_f$. Thus, although our SD and BS equations are semi-perturbative, the analysis is self-consistent in the sense that our $\alpha_{cr}$ really is the value at which, in our approximation, one passes from the confinement phase with $S\chi$SB to the chirally symmetric phase, and our values of $\alpha_*$ do span the interval over which there is a crossover from walking to QCD-like (i.e., non-walking) behavior.

2. Schwinger-Dyson Equation

We first use the Schwinger-Dyson equation for the fermion propagator to calculate the dynamically generated mass $\Sigma$ of this fermion. In Fig. 1 (left panel) we show the solution for the dynamical fermion mass $\Sigma$ as a function of $\alpha_*$. A fit to the numerical solution in the walking region $0.89 \leq \alpha_* \leq 1.0$ [11] found agreement with the functional form

$$\Sigma = c\Lambda \exp \left[ -\pi \left( \frac{\alpha_*}{\alpha_{cr}} - 1 \right)^{-1/2} \right],$$

(3)

with $c = 4.0$ (see also Refs. [6,9]). Our calculations for larger $\alpha_*$ show the expected shift away from walking behavior. This shift is evident in Fig. 1 for $\alpha_*$ larger than about 1.2. Our calculation of $\Sigma$, shown in Fig. 1, shows that $\Sigma/\Lambda$ increases substantially, by about a factor of 30, from a value of about 0.01 at $\alpha_* = 1.0$ to 0.32 at $\alpha_* = 2.5$, much closer to the value of $O(1)$ for this ratio in QCD.

Another quantity of interest is the pseudoscalar decay constant $f_P$, the $N_f$-flavor generalization of the pion decay constant, $f_\pi$. In Fig. 1 (right
panel) we show our results for $f_P$ calculated by substituting our solution for $\Sigma(k^2)$ into the Pagels-Stokar formula. In the walking limit, $f_P$ has been shown to satisfy a relation similar to eq. (3), i.e., it is exponentially smaller than the scale $\Lambda$. We display, as the dotted curve, the fit from Ref. [11] for the walking interval $0.89 \leq \alpha_\ast \leq 1.0$, given by eq. (3) with $c = 1.5$. Our results show the change from this walking type of behavior as $\alpha_\ast$ increases above this range; specifically, as $\alpha_\ast$ increases from 1.0 to 2.5, $f_P/\Lambda$ increases substantially, from about $3 \times 10^{-3}$ to about 0.08. This is similar to the factor by which we found that $\Sigma/\Lambda$ increased as $\alpha_\ast$ increased through this interval.

3. Calculation of Meson Masses

We next present the results of the numerical calculations for meson masses, obtained by solving the homogeneous BS equation [1]. (As in Ref. [11], we have checked and confirmed that the flavor-adjoint pseudoscalar meson mass is zero to within the numerical accuracy of our calculation.) In Fig. 2, we show the values of meson masses divided by $\Lambda$ calculated from the SD and BS equations. In Fig. 3 we plot the values of $M_{A,V,S}/f_P$ (left panel) and $M_{A,S}/M_V$ (right panel). Here, the subscripts $V, A$ and $S$ represent vector, axial-vector and scalar, respectively. Our calculations yield a number of interesting results. We summarize these for the changes in these meson masses as $\alpha_\ast$ increases from 0.9 to 2.5. The ratios of the meson masses divided by $\Lambda$ increase dramatically, by factors of order $10^2$, approaching values of order unity at $\alpha_\ast = 2.5$. This amounts to the removal of the exponential suppression of these masses.
which had described the walking limit near $N_{f,cr}$, as one moves away from this limit into the interior of the confined phase. For example, $M_S/f_P$ increases monotonically from about 4 to 7, thereby approaching to within about 35 % of the value 10.7 in QCD for $M_{a_0}/f_\pi$, while $M_V/f_P$ decreases from about 11 to 9, rather close to the value 8.5 for $M_\rho/f_\pi$ and $M_\omega/f_\pi$ in QCD. The ratios $M_A/M_V$ and $M_S/M_V$, which were found in Ref. [11] to have values close to 1.0 and 0.36, respectively, in the walking limit, both increase in the interval of $\alpha_*$ that we study, reaching about 1.2 and 0.74, respectively, at $\alpha_*=2.5$. For comparison, these ratios are approximately 1.6 and 1.3 in QCD.

4. Calculation of the $S$ parameter

In this section we present the results of our calculations of $\hat{S}$ [2] in the crossover region of the theory between the walking limit at $\alpha_* \downarrow \alpha_{cr}$ and larger values of $\alpha_*$ that move toward the QCD-like regime. Here, $\hat{S}$ represents the contribution to the $S$ parameter from one fermion isodoublet. (Studies of $S$ in the walking limit include [12,14].) We calculate $\hat{S}$ via the relation $\hat{S} = 4\pi(\Pi_{VV}(0) - \Pi_{AA}(0))$, where $\Pi_{VV}(q^2)$ and $\Pi_{AA}(q^2)$ are the vector and the axial-vector current-current correlation functions. These correlators are computed by solving the SD equation and the inhomogeneous BS equation [2].

In Fig. 4, as a function of $\alpha_*$, we plot the value of $\hat{S}_n$, the value of $\hat{S}$ normalized by its value at $\alpha_*=1.8$, namely, 0.47. This figure shows that $\hat{S}_n$, and hence also $\hat{S}$, decreases by about 40 % as $\alpha_*$ is reduced from 1.8.
Fig. 4. Plot of $\hat{S}_n$ for several values of $\alpha_*$ in the range of $0.9 \leq \alpha_* \leq 1.8$. As indicated by the subscript $n$, the values are normalized by the value of $\hat{S}$ at $\alpha_* = 1.8$, i.e., 0.47.

to 0.9, or equivalently as $N_f$ is increased from 10.3 to 11.6. Reinserting the factor of the number of fermion isodoublets, $N_D = N_f/2$, to get $S$ itself, we obtain a decrease by about 30% in $S$, since $N_D$ only increases by about 10% over this range. Thus, our calculation shows that for this range of values, $S$ decreases significantly as one moves from the QCD-like to the walking regimes. We recall that the (improved) ladder approximation to the BS equation can overestimate $S$ in QCD by as much as 30% [15]. Hence, in addition to the demonstrated decreasing trend of $\hat{S}$ and $S$ as $\alpha_*$ decreases from 1.8 to 0.9, one may, separately, comment that the absolute magnitude of these quantities could be about 30% smaller than the values yielded by our ladder approximation. Our results thus strengthen the evidence for the reduction of $\hat{S}$ in a walking, as opposed to QCD-like, gauge theory, and are relevant to assessing the impact of the $S$-parameter constraint on technicolor theories.

5. Summary

In summary, using numerical solutions of the Schwinger-Dyson and Bethe-Salpeter equations, we have calculated several physical quantities, including $f_P$, meson masses, and the $S$ parameter, as a function of the approximate infrared fixed point, $\alpha_*$, or equivalently, the number of massless fermions, $N_f$, in a vectorial, confining SU($N$) gauge theory. Our results show the crossover between walking and non-walking behavior in a gauge theory, and demonstrate that $\hat{S}$ and also $S$ decrease significantly as $\alpha_*$ decreases in this range.
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