THE EXISTENCE AND UNIQUENESS OF A POWER PRICE EQUILIBRIUM

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Abstract. We propose a term structure power price model that, in contrast to widely accepted no-arbitrage based approaches, accounts for the non-storable nature of power. It belongs to a class of equilibrium game theoretic models with players divided into producers and consumers. The consumers’ goal is to maximize a mean-variance utility function subject to satisfying an inelastic demand of their own clients (e.g. households, businesses etc.) to whom they sell the power. The producers, who own a portfolio of power plants each defined by a running fuel (e.g. gas, coal, oil...) and physical characteristics (e.g. efficiency, capacity, ramp up/down times...), similarly, seek to maximize a mean-variance utility function consisting of power, fuel, and emission prices subject to production constraints. Our goal is to determine the term structure of the power price at which production matches consumption. In this paper we show that in such a setting the equilibrium price exists and also discuss the conditions for its uniqueness.

Key words. term structure, electricity, game theory, mean-variance, optimization, KKT conditions.

1. Introduction. The electricity price has some unique features which are very distinct from features of stock prices or other commodities. The literature aiming at modeling the electricity price can be broadly divided into non-structural, structural, and game theoretic approaches.

Non-structural approaches model electricity price directly, without considering the underlying reasons that cause the price to change over time. A good overview of regular patterns and statistical properties of electricity prices, together with one and multi-factor Ornstein-Uhlenbeck process models of the spot price, is given in [20]. Since Gaussian distributions are not suitable for modeling spikes, a combination of Ornstein-Uhlenbeck and pure jump-processes have been proposed in [17]. Further developments include using also Levy processes (see [22] for example). Even though such models are good for modeling the spot price, their application for pricing derivatives is sometimes rather questionable. As pointed out in [3], a classical buy-and-hold strategy is not applicable to non-storable commodities. All non-structural models define the price of a forward contract as the expected price of the security at the delivery under the risk-neutral measure, conditioned on some filtration, which contains all available market information. It is often assumed that all available market information is included in the spot price (i.e. the spot price is a martingale). This is a good approximation for stock prices, but definitely false for non-storable commodities. Information about a huge electricity demand increase in a week time, for example, does not have any impact on today’s spot price. The correct filtration must hence include information on weather forecasts, planned power plant outages etc. It can be argued that all the information is contained in prices of forward contracts and other derivatives. A one-factor model in [14] and multi-factor term-structure model in [13] are the first that produce prices that are consistent with observable forward prices.

Another interesting aspect of electricity is its relation to other fuels. Noting that coal, gas, oil, etc. can be used to produce electricity, electricity itself can be treated as a derivative with various fuel and emission prices as the underlying. [2] produced seminal work that used a supply and demand stack to model electricity prices. The supply stack was further extended by [18] and by [9] to include various fuels. Using an exponential bid stack, they calculated spot electricity price as a function of demand and spot fuel prices. These models belong to a class of structural approaches since they try to capture some of the physical properties of the electricity markets.

The third, game theoretic approach, models the physical properties and decisions of market participants more closely. As pointed out in [24], models that include ramp up/down constraints of power plants and study the impact of long term contracts on the spot electricity prices, are needed in order to prevent and explain disastrous events an example of which happened in California in 2001 and cost the state as much as $45 billion. The seminal work of game theoretic models in electricity markets was produced by [5]: a unique relation between a forward and a spot price is given in a two-stage market.
with one producer and one consumer, who each want to maximize their mean-variance objective function. This work has been used to study benefits of derivatives in the electricity market, see [11]. It was extended to a multi-stage setting with a dynamic equilibrium by [8], and [7]. [15] have extended the two-stage mean-variance model to any convex risk measure, while also taking into account liquidity constraints.

The game theoretic approach attempts to model decisions of producers and consumers that participate in the electricity market explicitly. Since a widely used strategy for risk management is delta hedging, it is important to know that the delta hedging strategy can be implemented as a minimum variance strategy in the mean-variance portfolio framework (see [1] for details).

Our work belongs to a class of game theoretic approaches. It extends the model of [7] to more than one producer and consumer, who maximize their mean-variance utility functions. In contrast to other game theoretic models, we include capacity and ramp up/down constraints of power plants. Following ideas from structural approaches, the profit of power plants is modeled as a difference between electricity price and fuel costs (including emissions). Since we model each power plant directly, we do not have to make any assumptions on the bid stack, which is in our case determined by the physical properties of power plants. As in [13], our model is consistent with observable fuel and emission prices. We do not focus on specific emission market schemes, but rather use a simplified version of it. For a detailed treatment of emission market in the game theoretic setting see [10] and [21].

The literature distinguishes between dynamic (i.e. stochastic) and static models. In dynamic models, players adapt to changing environment (i.e. fuel prices, demand etc.) by adapting their decisions. In each stage of their decision making they determine the optimal decisions they have to take now and also the optimal decisions they will take in the future under all possible changes in the environment. Since future decisions affect present decisions this is computationally very demanding. In static models on the other hand, players assume that they know the future state of the environment exactly and can thus stick to an initial plan about future decisions, regardless of the changes in the environment. Such approaches are computationally much more tractable, but they do not reflect the reality very well. The model we propose here can be seen as hybrid of both approaches. The initial optimization problem is static, where players determine all their optimal decisions and assume they will not alter them in the future. However, as the environment changes, players may take recursive actions by calculating new optimal decisions while taking into account the new state of the environment, as well as their decisions taken under the previous state of the environment. Since in such setting the recursive actions are not very dependent, we will mainly focus on each of them separately.

This paper is organized as follows: In Section 2 we give a detailed mathematical description of the model, and in Section 3 we proof that the solution to our model exists and develop the conditions under which it is also unique. We conclude the paper in Section 4.

2. Problem description. The model we propose can be used to determine the term structure of electricity prices. The electricity market is defined by a set of producers $P$ of cardinality $0 < |P| < \infty$, a set of consumers $C$ of cardinality $0 < |C| < \infty$, and a hypothetical market agent. Each of the producers and consumers participates in the electricity market in order to maximize their profit subject to a risk budget under a mean-variance optimization framework. Producers own a number of power plants, which can have different physical characteristics and run on different fuels. The set of all fuels is denoted by $L$. Sets $R^{p,l}$ represent all power plants owned by producer $p \in P$ that run on fuel $l \in L$.

We are interested in delivery times $T_j$, $j \in J = \{1, ..., T\}$, where power for each delivery time $T_j$ can be traded through numerous forward contracts at times $t_i$, $i \in I_j$. The electricity price at time $t_i$ for delivery at time $T_j$ is denoted by $\Pi(t_i, T_j)$. Since contracts with trading time later than delivery time do not exist, we require $t_{\text{max}}(T_j) = T_j$ for all $j \in J$. The number of all forward contracts, i.e. $\sum_{j \in J} |I_j|$, is denoted by $N$. Uncertainty is modeled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t, t \in I\}, \mathbb{F})$, where $I = \cup_{j \in J} I_j$. The $\sigma$-algebra $\mathcal{F}_t$ represents information available at time $t$.

The exogenous variables that appear in our model are (a) aggregate power demand $D(T_j)$ for each delivery period $j \in J$, (b) prices of fuel forward contracts $G_l(t_i, T_j)$ for each fuel $l \in L$, delivery period $j \in J$, and trading period $i \in I_j$, and (c) prices of emissions forward contracts $G_{em}(t_i, T_j)$, $j \in J$, $i \in I_j$. Electricity prices and all exogenous variables are assumed to be adapted to the filtration $\{\mathcal{F}_t\}_{t \in I}$ and have finite second moments.

Let $v_k \in \mathbb{R}^{n_k}$, $n_k \in \mathbb{N}$, $k \in K$, and $K = \{1, ..., |K|\}$ be given vectors. For convenience, we define a
vector concatenation operator as
\[ \| k \in K \| \mathbf{v}_k = [v_1^T, ..., v_{|K|}^T]^T. \]

2.1. Producer. Each producer \( p \in P \) has to decide on the number \( V_p(t_i, T_j) \) of electricity forward contracts and the number \( F_{p,l}(t_i, T_j) \), \( l \in L \) of fuel forward contracts to buy, at trading time \( t_i \), \( i \in I_j \) for delivery at time \( T_j \), \( j \in J \). By \( W_{p,l,r}(T_j) \) we denote the actual production from fuel \( l \in L \) at time \( T_j \) from power plant \( r \in R^{p,l} \), and by \( O_p(t_i, T_j) \) we denote the number of emission forward contracts purchased at time \( t_i \) for delivery at time \( T_j \).

Notation is greatly simplified if decision variables are concatenated into
- electricity trading vectors \( V_p(T_j) = ||_{i \in I_j} V_p(t_i, T_j) \) and \( V_p = ||_{j \in J} V_p(T_j) \),
- fuel trading vectors \( F_p(t_i, T_j) = ||_{l \in L} F_{p,l}(t_i, T_j), \quad F_p(T_j) = ||_{i \in I_j} F_p(t_i, T_j) \), and \( F_p = ||_{j \in J} F_p(T_j) \),
- emission trading vectors \( O_p(T_j) = ||_{i \in I_j} O_p(t_i, T_j) \) and \( O_p = ||_{j \in J} O_p(T_j) \),
- electricity production vectors \( W_{p,l}(T_j) = ||_{r \in R^{p,l}} W_{p,l,r}(T_j), \quad W_p(T_j) = ||_{l \in L} W_{p,l}(T_j) \), and \( W_p = ||_{j \in J} W_p(T_j) \),

and finally \( v_p = [V_p^T, F_p^T, O_p^T, W_p^T]^T \).

Similarly, we define
- electricity price vectors \( \Pi(T_j) = ||_{i \in I_j} \Pi(t_i, T_j) \), and \( \Pi = ||_{j \in J} e^{-rt_j} \Pi(T_j) \), where \( r \in \mathbb{R} \) is a constant interest rate,
- fuel price vectors \( G(t_i, T_j) = ||_{l \in L} G_{l}(t_i, T_j), \quad G(T_j) = ||_{i \in I_j} G(t_i, T_j) \), and \( G = ||_{j \in J} e^{-rt_j} G(T_j) \),
- emission price vector \( G_{em}(T_j) = ||_{i \in I_j} G_{em}(t_i, T_j) \), and \( G_{em} = ||_{j \in J} e^{-rt_j} G_{em}(T_j) \),

and finally \( \pi_p = \left[ \Pi^T, G^T, G_{em}^T, 0, ..., 0 \right]_{dim(W_p)}^T \), where number of zeros matches the dimension of vector \( W_p \).

The profit \( P_p(v_p, \pi_p) \) of producer \( p \in P \) can be calculated as
\[
(2.1) \quad P_p(v_p, \pi_p) = \sum_{j \in J} e^{-rt_j} \left( \sum_{i \in I_j} P_{p,i}^{i,T_j}(v_p, \pi_p) \right)
\]
where the profit \( P_{p,i}^{i,T_j}(v_p, \pi_p) \) for each \( i \in I_j \) and \( j \in J \) can be calculated as
\[
P_{p,i}^{i,T_j}(v_p, \pi_p) = -\Pi(t_i, T_j) V_p(t_i, T_j) - O_p(t_i, T_j) G_{em}(t_i, T_j) - \sum_{l \in L} G_l(t_i, T_j) F_{p,l}(t_i, T_j).
\]

Ramping up and down constraints for each \( j \in \{1, ..., T'-1\} \), where \( T' \) denotes the last delivery period, \( l \in L \) and \( r \in R^{p,l} \) can be expressed as
\[
(2.2) \quad \Delta \mathbf{W}_{p,l,r}^{min} \leq W_{p,l,r}(T_{j+1}) - W_{p,l,r}(T_j) \leq \Delta \mathbf{W}_{p,l,r}^{max}.
\]
For each power plant \( r \in R^{p,l} \), \( \Delta \mathbf{W}_{p,l,r}^{min} \) and \( \Delta \mathbf{W}_{p,l,r}^{max} \) represent maximum rates for ramping up and down, respectively. Similarly, \( \mathbf{W}_{max} \) denotes the maximum production and thus we can write the capacity constraints for each power plant \( r \in R^{p,l} \) as
\[
(2.3) \quad 0 \leq W_{p,l,r}(T_j) \leq \mathbf{W}_{max}.
\]
Additionally, for each \( j \in J \) the electricity sold in the forward market must equal the actually produced electricity, i.e.
\[
(2.4) \quad - \sum_{i \in I_j} V_p(t_i, T_j) = \sum_{l \in L} \sum_{r \in R^{p,l}} W_{p,l,r}(T_j).
\]
and sufficient amount of fuel $l \in L$ must have been bought for each $j \in J$, i.e.
\begin{equation}
(2.5) \quad \sum_{r \in R_p} W_{p,l,r} (T_j) e^{p,l,r} = \sum_{i \in I_j} F_{p,l} (t_i, T_j)
\end{equation}
where $e^{p,l,r} > 0$ is the efficiency of power plant $r$.

The carbon emission obligation constraint can be written as
\begin{equation}
(2.6) \quad \sum_{j \in J} \sum_{i \in I_j} O (t_i, T_j) = \sum_{j \in J} \sum_{i \in L} \sum_{r \in R_p} W_{p,l,r} (T_j) g^i,
\end{equation}
where $g^i > 0$ denotes the carbon emission intensity factor for fuel $l \in L$. This constraint ensures that enough emission certificates have been bought to cover the electricity production over the whole planning horizon.

We bound the number of fuel forward contracts traded for each $i \in I_j$, $j \in J$, and $l \in L$ as
\begin{equation}
(2.7) \quad -F_{\text{trade}} \leq F_{p,l} (t_i, T_j) \leq F_{\text{trade}}
\end{equation}
for some large $F_{\text{trade}} > 0$ and the number of emission forward contracts traded as
\begin{equation}
(2.8) \quad -F_{\text{trade}} \leq O (t_i, T_j) \leq F_{\text{trade}}.
\end{equation}
Additionally, we also bound the number of electricity forward contracts traded for each $i \in I_j$, and $j \in J$ as
\begin{equation}
(2.9) \quad -V_{\text{trade}} \leq V_p (t_i, T_j) \leq V_{\text{trade}}
\end{equation}
for some large $V_{\text{trade}} > 0$. Boundedness of the number of the electricity forward contracts traded enables us to use some of game theoretic theory that only applies to compact sets. If $V_{\text{trade}}$ is set correctly (see Corollary 3.14), then the optimal $V_p (t_i, T_j)$ can never lie on the boundary and hence the constraint does not have any impact on the optimal solution.

Producers would like to maximize their profit subject to a risk budget. Under a mean-variance optimization framework they are interested in the mean-variance utility
\begin{equation}
(2.10) \quad \Phi_p = \max_{v_p} \Psi_p (v_p)
\end{equation}
subject to (2.2), (2.3), (2.4), (2.5), (2.7), (2.8), and (2.9).

\textbf{2.2. Consumer.} We make the assumption that demand is completely inelastic and that each consumer $c \in C$ is responsible for satisfying a proportion $p_c \in [0, 1]$ of the total demand $D (T_j)$ at time $T_j$, $j \in J$. Since $p_c$ is a proportion clearly $\sum_{c \in C} p_c = 1$.

For further argumentation let us define electricity trading vectors $V_c (T_j) = \|_{i \in I_j} V_c (t_i, T_j)$ and $V_c = \|_{j \in J} V_c (T_j)$.

Consumer’s profit can be calculated as
\begin{equation}
(2.11) \quad P_c (V_c, \Pi) = \sum_{j \in J} e^{-\hat{r} T_j} \left( \sum_{i \in I_j} -\Pi (t_i, T_j) V_c (t_i, T_j) + s_c p_c D (T_j) \right),
\end{equation}
where \( \hat{r} \in \mathbb{R} \) denotes a constant interest rate and \( s_c \in \mathbb{R} \) denotes a contractually fixed price that consumer \( c \in C \) receives for selling the electricity further to households or businesses. Demand is expected to be satisfied for each \( T_j \), i.e.

\[
\sum_{i \in I_j} V_c(t_i, T_j) = p_c D(T_j).
\]

At the time of calculating the optimal decisions, consumers assume that they know the future realization of demand \( D(T_j) \) precisely. If the knowledge about the future realization of the demand changes, then players can take recursive actions by recalculating their optimal decisions with the updated demand forecast.

Note further that the contractually fixed price \( s_c \) only affects the optimal objective value of consumer \( c \in C \), but not also his optimal solution. Since we are primarily interested in optimal solutions, we simplify the notation and set \( s_c = 0 \). The correct optimal value can always be calculated via post-processing when an optimal solution is already known. This is sometimes needed for risk management purposes.

We bound the number of the electricity forward contracts for each \( i \in I_j \) and \( j \in J \) as

\[
-V_{\text{trade}} \leq V_c(t_i, T_j) \leq V_{\text{trade}}
\]

for some large \( V_{\text{trade}} > 0 \) for similar reasons as for producers. We will determine the appropriate \( V_{\text{trade}} \) in Corollary 3.14.

Consumers would like to maximize their profit subject to a risk budget. Under a mean-variance optimization framework they are interested in the mean-variance utility

\[
\Psi_c(V_c) = \mathbb{E}^P[P_c(V_c, \Pi)] - \frac{\lambda_c}{2} \text{Var}^P[P_c(V_c, \Pi)]
\]

where \( \lambda_c > 0 \) is their risk preference and \( Q_c := \mathbb{E}^P \left[ (\Pi - \mathbb{E}^P[\Pi]) (\Pi - \mathbb{E}^P[\Pi])^\top \right] \) a covariance matrix. Their objective is to solve the following optimization problem

\[
\Phi_c = \max_{V_c} \Psi_c(V_c)
\]

subject to (2.12) and (2.13).

2.3. The hypothetical market agent. Given the electricity \( \Pi \), fuel \( G \), and emission \( G_{em} \) price vectors, each producer \( p \in P \) and each consumer \( c \in C \) can calculate their optimal electricity trading vectors \( V_p \) and \( V_c \) by solving (2.10) and (2.14), respectively. However, the players are not necessary able to execute their calculated optimal trading strategies because they may not find the counterparty to trade with. In reality each contract consists of a buyer and a seller, which imposes an additional constraint (also called the market clearing constraint) that matches the number of short and long electricity forward contracts for each \( i \in I_j \) and \( j \in J \) as

\[
\sum_{c \in C} V_c(t_i, T_j) + \sum_{p \in P} V_p(t_i, T_j) = 0.
\]

The electricity market is responsible for satisfying this constraint by matching buyers with sellers. The matching is done through sharing of the price and order book information among all market participants. If at the current price there are more long contract than short contracts, it means that the current price is too low and asks will start to be submitted at higher prices. The converse occurs, if there are more short contracts than long contracts. Eventually, the electricity price at which the number of long and short contracts matches is found. At such a price, (2.15) is satisfied “naturally” without explicitly requiring the players to satisfy it. They do so because it is in their best interest, i.e. it maximizes their mean-variance objective functions.
The question is how to formulate such a “natural” constraint in an optimization framework. A
naive approach of writing the market clearing constraint as an ordinary constraint forces the players to
satisfy it regardless of the price. We need a mechanism that models the matching of buyers and sellers
as it is performed by the electricity market. For this purpose, we introduce a hypothetical market
agent that is allowed to slowly change electricity prices to ensure that (2.15) is satisfied.

Assume first that the feasible set of the hypothetical agent must satisfy

\[(2.16) \quad -\Pi_{max} \leq E^p [\Pi (t_i, T_j)] \leq \Pi_{max}\]

for all \(i \in I_j, j \in J, \) and for some large \( \Pi_{max} > 0 \). We write (2.16) in matrix notation as

\[(2.17) \quad B_M E^p [\Pi] \leq b_M\]

for some \( B_M \in \mathbb{R}^{2N \times N} \) and \( b_M \in \mathbb{R}^{2N} \). Boundedness of prices is a reasonable assumption, because
unbounded prices lead to infinite cash flows, which must result in the bankruptcy of one of the counter-
parties involved. Besides limiting the possibility of the bankruptcy of the market players, the
boundedness of prices also allows us to use game theoretic results that only apply to compact sets. If
\( \Pi_{max} \) is set large enough (see Lemma 3.12), then the optimal \( E^p [\Pi] \) can never be on the boundary
of the feasible region. Hence the constraint does not have any practical impact on the equilibrium
electricity price, but, as we will see later, it significantly simplifies the theoretical analysis.

Let the hypothetical market agent have the following profit function

\[(2.18) \quad \Phi_M (\Pi, V) = \sum_{j \in J} e^{-F_j} \left[ \sum_{i \in I_j} \Pi (t_i, T_j) \left( \sum_{c \in C} V_c (t_i, T_j) + \sum_{p \in P} V_p (t_i, T_j) \right) \right]\]

and the expected profit

\[(2.19) \quad \Psi_M (E^p [\Pi], V) = E^p [P_M (V, \Pi)],\]

where \( V = [V_P^T, V_C^T]^T, \) \( V_P = \|_{p \in P} V_p, \) and \( V_C = \|_{c \in C} V_c \) and let the hypothetical market agent attempt to solve

\[(2.20) \quad \Phi_M (V) = \max_{E^p [\Pi]} \Psi_M (E^p [\Pi], V)\]

subject to (2.17). The KKT conditions for (2.20) in the matrix notation read

\[(2.21) \quad \sum_{c \in C} V_c + \sum_{p \in P} V_p - B_M^T \eta_M = 0\]

\[\begin{align*}
(B_M E^p [\Pi] - b_M)^T \eta_M &= 0 \\
B_M E^p [\Pi] - b_M &\leq 0 \\
\eta_M &\geq 0,
\end{align*}\]

where \( \eta_M \) denotes the dual variables of (2.17). Since the optimal \( E^p [\Pi] \) can never be on the boundary
of the feasible region (see Lemma 3.12), we can conclude that \( \eta_M = 0 \). Optimality conditions (2.21) are
then simplified to (2.15). Therefore, we can conclude that (2.15) and (2.20) are equivalent, if \( \eta_M = 0 \)
truly holds. The relationship between (2.15) and (2.20) will be investigated rigorously in Proposition
3.13.

Note, that the equivalence of (2.15) and (2.20) is a theoretical result that has to be applied with
cautions in an algorithmic framework. Formulation (2.20) is clearly unstable since only a small mismatch
in the market clearing constraint sends the prices to \( \pm \Pi_{max} \). A description of an algorithm for the
computation of the equilibrium electricity price exceeds the scope of this paper and will be examined
separately.

For the further argumentation we define \( v_P = \|_{p \in P} v_p \) and \( v = [v_P^T, V_C^T]^T \).
2.4. Construction of the equilibrium electricity price process. In this subsection we take a closer look at the electricity price process $\Pi (t_i, T_j)_{i \in I_j}$, where the expectation of the process is defined internally as a decision of the hypothetical market agent in order to match supply and demand.

Since electricity prices $\Pi (t_i, T_j)_{i \in I_j}$ are adapted for any fixed $j \in J$ and $\mathbb{E}^\mathbb{P} [\Pi] < \infty$ (see Constraint (2.16)), they can be uniquely decomposed, using the Doob Decomposition Theorem, into a sum of a martingale process $M (t_i, T_j)_{i \in I_j}$ and an integrable predictable process $A (t_i, T_j)_{i \in I_j}$, $A (0, T_j) = 0$, such that

$$\Pi (t_i, T_j) = A (t_i, T_j) + M (t_i, T_j)$$

for every $i \in I_j$. Define

$$M (t_i, T_j) := \Pi (t_0, T_j) + \sum_{k=1}^{i} (\Pi (t_k, T_j) - \mathbb{E}^\mathbb{P} [\Pi (t_k, T_j) | \mathcal{F}_{k-1}])$$

and

$$A (t_i, T_j) := \sum_{k=1}^{i} (\mathbb{E}^\mathbb{P} [\Pi (t_k, T_j) | \mathcal{F}_{k-1}] - \Pi (t_{k-1}, T_j)),$$

where $\mathcal{F}_k := \mathcal{F}_{t_k}$. It is easy to see that $M (t_i, T_j)$ is a martingale since

$$\mathbb{E}^\mathbb{P} [M (t_i, T_j) - M (t_{i-1}, T_j) | \mathcal{F}_{i-1}] = 0 \text{ a.s.}$$

Moreover, $A (t_i, T_j)$ is predictable, that is, $\mathcal{F}_{i-1}$-measurable.

Note that

$$\mathbb{E}^\mathbb{P} [\Pi (t_i, T_j) | \mathcal{F}_{i-1}] = A (t_i, T_j) + M (t_{i-1}, T_j),$$

and

$$\mathbb{E}^\mathbb{P} [\Pi (t_i, T_j) | \mathcal{F}_0] = \mathbb{E}^\mathbb{P} [A (t_i, T_j) | \mathcal{F}_0] + M (t_0, T_j).$$

Let us now allow the hypothetical market agent to choose

$$\hat{A}_i (T_j) := \left[\hat{A} (t_{i+1}, T_j), \mathbb{E}^\mathbb{P} [\hat{A} (t_{i+2}, T_j) | \mathcal{F}_i], ..., \mathbb{E}^\mathbb{P} [\hat{A} (t_{\max (I_j)}, T_j) | \mathcal{F}_i]\right]^\top$$

at time $t_i, i \in I_j$ for all $j \in J$. We then model $\Pi (t_i, T_j)$ with a new probability measure $\hat{\mathbb{P}}$ where $A (t_i, T_j)$ is defined internally and not by Doob decomposition of $\Pi (t_i, T_j)$ as before. More formally, we define a new probability measure $\hat{\mathbb{P}} : \mathcal{F} \times \mathbb{R}^N \rightarrow [0, 1]$ such that for any fixed $j \in J$, $i \in I_j$, and for all $D \in \mathcal{F}_{i-1}$

$$\hat{\mathbb{P}} \left(\Pi (t_i, T_j) \in D; \hat{A}_{i-1} (T_j)\right) = \mathbb{P} \left(M (t_i, T_j) + \hat{A} (t_i, T_j) \in D\right)$$

$$= \mathbb{P} \left(M (t_i, T_j) + A (t_i, T_j) + \hat{A} (t_i, T_j) - A (t_i, T_j) \in D\right)$$

$$= \mathbb{P} \left(\Pi (t_i, T_j) + \hat{A} (t_i, T_j) - A (t_i, T_j) \in D\right)$$

$$= \mathbb{P} \left(\Pi (t_i, T_j) \in \varphi (D, \hat{A} (t_i, T_j) - A (t_i, T_j))\right)$$

where $\varphi : \mathcal{F} \times \mathbb{R} \rightarrow \mathcal{F}$ denotes a translation of a set, i.e. $\varphi (D, \Delta d) := \{d : d + \Delta d \in D\}$. Since

$$\mathbb{E}^\mathbb{P} [\Pi (t_k, T_j) | \mathcal{F}_i; \hat{A}_i (T_j)] = \mathbb{E}^\mathbb{P} [\hat{A} (t_k, T_j) | \mathcal{F}_i] + M (t_k, T_j)$$

where $k \in \{i+1, ..., \max \{I_j\}\}$, this selection can be interpreted as determining the term structure of the expected power price relative to the current value of $M (t_i, T_j)$. 

7
At time $t_0$, when players calculate their optimal decisions for the first time, they assume that they will execute their strategies without any future alterations. However, we allow recourse at a later step but this is not taken into account at time $t_0$. Therefore,

$$
(2.24) \quad \bar{A}(t_i, T_j) = \mathbb{E}^\bar{P} \left[ \bar{A}(t_i, T_j) | \mathcal{F}_k \right]
$$

for all $k \in \{0, ..., i - 1\}$, $j \in J$, and $i \in I_j$. In such a setting, it is enough to determine $\bar{A}_0(T_j)$, since all other expected prices can be derived from these using (2.23) and (2.24).

The variance under the new probability measure $\bar{P}$ can be calculated as

$$
(2.25) \quad \text{Var}^\bar{P} \left( \Pi(t_i, T_j) | \mathcal{F}_0 ; \bar{A}_i(T_j) \right) = \text{Var}^\bar{P} \left( \bar{A}(t_i, T_j) + M(t_i, T_j) | \mathcal{F}_0 \right)
$$

We can see that the variance depends only on the process $M(t_i, T_j)$ and can not be influenced by the hypothetical market agent. Using a similar reasoning as in (2.25), we can also conclude

$$
(2.26) \quad \mathbb{E}^\bar{P} \left[ (\pi_p - \mathbb{E}^\bar{P} \left[ \pi_p; \bar{A}_0 \right]) (\pi_p - \mathbb{E}^\bar{P} \left[ \pi_p; \bar{A}_0 \right])^\top ; \bar{A}_0 \right] = Q_p
$$

and

$$
(2.27) \quad \mathbb{E}^\bar{P} \left[ \left( \Pi - \mathbb{E}^\bar{P} \left[ \Pi; \bar{A}_0 \right] \right) \left( \Pi - \mathbb{E}^\bar{P} \left[ \Pi; \bar{A}_0 \right] \right)^\top ; \bar{A}_0 \right] = Q_c,
$$

where $\bar{A}_0 = |i|_{j \in J} \bar{A}_0(T_j)$.

Without loss of generality, we may set $M(t_0, T_j) = 0$. Then from (2.23) and (2.24),

$$
(2.28) \quad \mathbb{E}^\bar{P} \left[ \Pi(t_i, T_j) | \mathcal{F}_0 ; \bar{A}_i(T_j) \right] = \mathbb{E}^\bar{P} \left[ \bar{A}(t_i, T_j) | \mathcal{F}_0 \right] = \bar{A}(t_i, T_j)
$$

for all $i \in I_j$ and $j \in J$. Allowing the hypothetical market agent to choose $\bar{A}(t_i, T_j)$ is thus the same as allowing it to choose $\mathbb{E}^\bar{P} \left[ \Pi(t_i, T_j) | \mathcal{F}_0 ; \bar{A}(t_i, T_j) \right]$. In the rest of the paper, we simplify the notation by writing $\mathbb{E}^\bar{P} \left[ \Pi(t_i, T_j) \right]$ when we actually mean $\mathbb{E}^\bar{P} \left[ \Pi(t_i, T_j) | \mathcal{F}_0 ; \bar{A}(t_i, T_j) \right]$.

The measure $\bar{P}$ corresponds to a physical measure and $\mathbb{E}^\bar{P} \left[ \cdot \right]$ corresponds to a real world expectation. The hypothetical market agent is allowed to choose $\bar{A}(t_i, T_j)$ and consequently the physical measure $\bar{P}$. The aim of the hypothetical market agent could be interpreted as finding the physical measure $\bar{P}$ that is consistent with the facts (e.g. fuel forward prices, emission prices, demand forecasts etc.) observable in the real world.

2.5. Matrix notation. The analysis of the problem is greatly simplified if a more compact notation is introduced. In this subsection we will also rewrite equations using the new probability measure $\bar{P}$ instead of $P$ where applicable.

The profit of producer $p \in P$ can be written as

$$
P_p(v_p, \pi_p) = -\pi_p^\top v_p.
$$

The equality constraints can be expressed as

$$
A_p v_p = 0
$$

and the inequality constraints as

$$
B_p v_p \leq b_p
$$
for some $A_p \in \mathbb{R}^{[J][|L|+1]+1 \times \dim v_p}$, $B_p \in \mathbb{R}^{n_p \times \dim v_p}$ and $b_p \in \mathbb{R}^{n_p}$, where $n_p$ denotes the number of all inequality constraints of producer $p \in P$. Define a feasible set

$$S_p := \{v_p : A_p v_p = a_p \text{ and } B_p v_p \leq b_p\}.$$ 

It is useful to investigate the inner structure of the matrices. By considering equality constraints (2.4), (2.5), and (2.6) we can see that

$$\hat{A}_1 = \begin{bmatrix} \hat{A}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1_{T^p} \end{bmatrix},$$

where $1_{j}, j \in J$ is a row vector of ones of length $|I_j|$. Similarly,

$$\hat{A}_2 = \begin{bmatrix} \hat{A}_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \hat{A}_1 & 0 \\ 0 & \cdots & 0 & 1_{|N|} \end{bmatrix},$$

where the number of rows in the block notation above is $|L| + 1$. The first $|L|$ rows correspond to (2.5) and the last row correspond to (2.6).

In a compact notation, the mean-variance utility of producer $p \in P$ can be calculated as

$$\Psi_p \left( v_p, \mathbb{E}^p [\Pi] \right) = \mathbb{E}^p \left[ -\pi_p^\top v_p - \frac{1}{2} \lambda_p v_p^\top \left( \pi_p - \mathbb{E}^p [\pi_p] \right) \left( \pi_p - \mathbb{E}^p [\pi_p] \right)^\top v_p \right]$$

$$= -\mathbb{E}^p [\pi_p]^\top v_p - \frac{1}{2} \lambda_p v_p^\top Q_p v_p,$$

where (2.6) was used. The inner structure of matrix $Q_p$ is the following

$$Q_p = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 & 0 \\ \hat{Q}_2 & \hat{Q}_3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\hat{Q}_1, \hat{Q}_2$, and $\hat{Q}_3$ do not depend on producer $p \in P$. The size of the larger matrix $Q_p$ depends on producer $p \in P$, because different producers have different number of power plants.

Producer $p \in P$ attempts to solve the following optimization problem

$$\Phi_p \left( \mathbb{E}^p [\Pi] \right) = \max_{v_p \in S_p} -\mathbb{E}^p [\pi_p]^\top v_p - \frac{1}{2} \lambda_p v_p^\top Q_p v_p.$$ 

The profit of consumer $c \in C$ can be written as

$$P_c (V_c, \Pi) = -\Pi^\top V_c.$$ 

Note here that we set $s_c = 0$, w.l.o.g. The equality constraints can be expressed as

$$A_c V_c = a_c.$$
and the inequality constraints as

\[ B_c V_c \leq b_c \]

where \( A_c = A \), \( B_c \in \mathbb{R}^{2N \times N} \), \( a_c \in \mathbb{R}^{|J|} \) and \( b_c \in \mathbb{R}^N \). Define a feasible set

\[ S_c := \{ V_c \in \mathbb{R}^N : A_c V_c = a_c \text{ and } B_c V_c \leq b_c \} \]

In a compact notation, the mean-variance utility of a consumer \( c \in C \) can be calculated as

\[ \Psi_c \left( V_c, \mathbb{E}^\hat{P} [\Pi] \right) = \mathbb{E}^\hat{P} \left[ -\Pi^T V_c - \frac{1}{2} \lambda_c V_c^T (\Pi - \mathbb{E}^\hat{P} [\Pi]) (\Pi - \mathbb{E}^\hat{P} [\Pi])^T V_c \right] = -\mathbb{E}^\hat{P} [\Pi]^T V_c - \frac{1}{2} \lambda_c V_c^T Q_c V_c, \]

where (2.27) was used. Moreover, note that \( Q_c = \hat{Q}_1 \) for all \( c \in C \).

Consumer \( c \in C \) attempts to solve the following optimization problem

\[ \Phi_c \left( \mathbb{E}^\hat{P} [\Pi] \right) = \max_{V_c \in S_c} \mathbb{E}^\hat{P} \left[ -\Pi^T V_c - \frac{1}{2} \lambda_c V_c^T Q_c V_c \right]. \]

The profit function of the hypothetical market agent can be written in a compact notation as

\[ (2.33) \quad P_M (\Pi, V) = \Pi^T \left( \sum_{c \in C} V_c + \sum_{p \in P} V_p \right) \]

and the expected utility as

\[ (2.34) \quad \Psi_M \left( \mathbb{E}^\hat{P} [\Pi], V \right) = \mathbb{E}^\hat{P} \left[ P_M (\Pi, V) \right] = \mathbb{E}^\hat{P} [\Pi]^T \left( \sum_{c \in C} V_c + \sum_{p \in P} V_p \right). \]

Consequently, the hypothetical market agent’s objective is to solve

\[ (2.35) \quad \Phi_M (V) = \max_{\mathbb{E}^\hat{P} [\Pi] \in S_M} \mathbb{E}^\hat{P} [\Pi]^T \left( \sum_{c \in C} V_c + \sum_{p \in P} V_p \right), \]

where the feasible set of the hypothetical market agent is defined as

\[ S_M := \left\{ \mathbb{E}^\hat{P} [\Pi] \in \mathbb{R}^N : -\Pi_{max} \leq \mathbb{E}^\hat{P} [\Pi(t_i, T_j)] \leq \Pi_{max} \text{ for all } j \in J, i \in I_j \right\}. \]

3. Analysis of the model. In this section we analyze the existence and uniqueness of solutions of the model defined in the previous section.

**Definition 3.1. Competitive Equilibrium (CE)**

Decisions \( v_p^* \) and \( \mathbb{E}^\hat{P} [\Pi]^* \) constitute a competitive equilibrium if

1. For every producer \( p \in P \), \( v_p^* \) is a strategy such that

\[ (3.1) \quad \Psi_p \left( v_p, \mathbb{E}^\hat{P} [\Pi]^* \right) \leq \Psi_p \left( v_p^*, \mathbb{E}^\hat{P} [\Pi]^* \right) \]

for all \( v_p \in S_p \);

2. For every consumer \( c \in C \), \( V_c^* \) is a strategy such that

\[ (3.2) \quad \Psi_c \left( V_c, \mathbb{E}^\hat{P} [\Pi]^* \right) \leq \Psi_c \left( V_c^*, \mathbb{E}^\hat{P} [\Pi]^* \right) \]

for all \( V_c \in S_c \);
3. For each \( i \in I_j \) and \( j \in J \)

\[
0 = \sum_{c \in C} V_c(t_i, T_j) + \sum_{p \in \tilde{P}} V_p(t_i, T_j)
\]

must hold.

**Definition 3.2. Nash Equilibrium (NE)**

Decisions \( v^* \) and \( \mathbb{E}^P[\Pi]^* \) constitute a Nash equilibrium if

1. For every producer \( p \in P \), \( v^*_p \) is a strategy such that

\[
\Psi_p \left( v_p, \mathbb{E}^P[\Pi]^* \right) \leq \Psi_p \left( v^*_p, \mathbb{E}^P[\Pi]^* \right)
\]

for all \( v_p \in S_p \);

2. For every consumer \( c \in C \), \( V^*_c \) is a strategy such that

\[
\Psi_c \left( V_c, \mathbb{E}^P[\Pi]^* \right) \leq \Psi_c \left( V^*_c, \mathbb{E}^P[\Pi]^* \right)
\]

for all \( V_c \in S_c \);

3. Price vector \( \mathbb{E}^P[\Pi]^* \) maximizes the objective function of the hypothetical market agent, i.e.

\[
\Psi_M \left( \mathbb{E}^P[\Pi], v^* \right) \leq \Psi_M \left( \mathbb{E}^P[\Pi]^*, v^* \right)
\]

for all \( \mathbb{E}^P[\Pi] \in S_M \).

**Assumption 3.3.** For all \( p \in P \), the exists vector \( v_p \) such that \( A_p v_p = a_p \, a.s. \) and \( B_p v_p < b_p \, a.s. \), for all \( c \in C \), there exists vector \( V_c \) such that \( A_c V_c = a_c \, a.s. \) and \( B_c V_c < b_c \, a.s. \), and the vectors \( V_p \) and \( V_c \) can be chosen so that (3.3) is satisfied.

**3.1. Existence of Solution.** We start the analysis of the existence of solution with stating the following well known theorem.

**Theorem 3.4.** (Debreu, Glickberg, Fan)

Consider a strategic form game \((\mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{\Psi_i\}_{i \in \mathcal{N}})\), where \( \mathcal{N} \) denotes a set of players. Let \( s_{-i} \in S_{-i} = \prod_{i \neq j} S_j \) denote a vector of actions for all players except player \( i \in \mathcal{N} \), and let \( s_i \in S_i \) denote an action for player \( i \in \mathcal{N} \). If for each \( i \in \mathcal{N} \)

1. feasible set \( S_i \) is non-empty, compact and convex;
2. \( \Psi_i(s_i, s_{-i}) \) is continuous in \( s_{-i} \);
3. \( \Psi_i(s_i, s_{-i}) \) is continuous and concave in \( s_i \),

then a pure Nash equilibrium exists.

The proof for the above theorem can be found in [16]. It is straightforward to apply the theorem and show that our problem from Definition 3.2 has a solution.

**Corollary 3.5.** Let Assumption 3.3 hold. Then there exists a pure NE for Problem 3.2.

**Proof.** A set of players \( \mathcal{N} \) is composed of all producers \( p \in P \), consumers \( c \in C \), and the hypothetical market agent. Due to Assumption 3.3 it is clear that the feasible region of each player is non-empty. Since all constraints are affine functions with non-strict inequalities, it is clearly also convex and closed. Boundedness is guaranteed by (2.2), (2.3), (2.7), (2.8), and (2.9) for producers, by (2.13) for consumers, and by (2.16) for the hypothetical market agent. The covariance matrix \( Q_p \) in \( \Psi_p \left( v_p, \mathbb{E}^P[\Pi] \right), v_p \in S_p, \mathbb{E}^P[\Pi] \in S_M \) is by definition positive semidefinite for all \( p \in P \) and thus \( \Psi_p \left( v_p, \mathbb{E}^P[\Pi] \right) \) is concave in \( v_p \). Similarly, we can also see that the covariance matrix \( Q_c \) is positive semidefinite and thus \( \Psi_c \left( V_c, \mathbb{E}^P[\Pi] \right), V_c \in S_c, \mathbb{E}^P[\Pi] \in S_M \) for all \( c \in C \) is concave in \( V_c \). The function \( \Psi_M \left( \mathbb{E}^P[\Pi], v \right) \) that corresponds to the expected utility of the hypothetical market agent is linear in \( \mathbb{E}^P[\Pi] \) and thus concave. Utility functions of producers, consumers, and the hypothetical market agent are all quadratic and therefore continuous. Thus, a pure NE exists. \( \square \)
3.2. Uniqueness of Solution. \textbf{Definition 3.6.} A power plant \( r \in \mathbb{R}^{p,l}, \) \( p \in P, l \in L \) is at the upper bound at delivery time \( T_j, \) \( j \in J, \) if for every \( \epsilon > 0, \) \( W_{p,l,r} (T_j) + \epsilon \) is infeasible. Similarly, a power plant \( r \in \mathbb{R}^{p,l}, \) \( p \in P, l \in L, r \in \mathbb{R}^{p,l} \) is at the lower bound at delivery time \( T_j, \) \( j \in J, \) if for every \( \epsilon > 0, \) \( W_{p,l,r} (T_j) - \epsilon \) is infeasible. Bounds on \( W_{p,l,r} (T_j) \) are defined by constraints (2.2) and (2.3).

Let us start the discussion about the uniqueness of the Nash equilibrium with a direct consequence of Assumption 3.3. 

\textbf{Lemma 3.7.} \textit{If there exists} \( j \in J \) \textit{such that all power plants are at the upper bound simultaneously, then}

\[ 0 > \sum_{c \in C} \sum_{i \in I_j} V_c (t_i, T_j) + \sum_{p \in P} \sum_{i \in I_j} V_p (t_i, T_j). \]

\textit{Similarly, if there exists} \( j \in J \) \textit{such that all power plants are at the lower bound simultaneously, then}

\[ 0 < \sum_{c \in C} \sum_{i \in I_j} V_c (t_i, T_j) + \sum_{p \in P} \sum_{i \in I_j} V_p (t_i, T_j). \]

\textbf{Proof.} 

Using (2.4) and summing over all producers \( p \in P \) we get

\[ \sum_{p \in P} \sum_{i \in I_j} V_p (t_i, T_j) = - \sum_{p \in P} \sum_{l \in L} \sum_{r \in \mathbb{R}^{p,l}} W_{p,l,r} (T_j). \]

Similarly, using (2.12) and the fact that \( \sum_{c \in C} p_c = 1, \) we get

\[ \sum_{c \in C} \sum_{i \in I_j} V_c (t_i, T_j) = \sum_{c \in C} p_c D (T_j) \]

\[ = D (T_j). \]

Assume now that all power plants are at the upper bound simultaneously for some \( j' \in J. \) Then by Assumption 3.3,

\[ D (T_{j'}) < \sum_{p \in P} \sum_{l \in L} \sum_{r \in \mathbb{R}^{p,l}} W_{p,l,r} (T_{j'}) \]

and therefore by (3.7) and (3.8),

\[ 0 > \sum_{c \in C} \sum_{i \in I_j} V_c (t_i, T_j) + \sum_{p \in P} \sum_{i \in I_j} V_p (t_i, T_j). \]

A similar argument also holds for the lower bound \( \blacksquare \)

\textbf{Assumption 3.8.} \textit{None of random variables} \( \Pi, G, \text{and} \ G_{cm} \) \textit{can be written as a linear combination of the others a.s.}

In the rest of this paper we assume that Assumption 3.3 and Assumption 3.8 always hold.

\textbf{Lemma 3.9.} \textit{The mean-variance objective functions of producers} \( p \in P \) \textit{and consumers} \( c \in C \) \textit{are strictly concave in} \( V_p, F_p, \text{and} \ O_p \) \textit{for all} \( p \in P \) \textit{and in} \( V_c \) \textit{for all} \( c \in C. \)

\textbf{Proof.} The objective function of each producer \( p \in P \) can be written as

\[ \Psi_p \left( v_p, E_{\hat{\theta}} [\Pi] \right) = -E_{\hat{\theta}} [\pi_p]^{\top} v_p - \frac{1}{2} \lambda_p v_p^{\top} Q_p v_p, \]

where

\[ Q_p = \begin{bmatrix} \hat{Q} & 0 \\ 0 & 0 \end{bmatrix}, \]

\( \hat{Q} := \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ \hat{Q}_2 & \hat{Q}_3 \end{bmatrix}, \) \textit{and} \( \pi' := [\Pi^{\top}, G^{\top}, G_{cm}^{\top}]^{\top}. \) Under Assumption 3.8, we conclude that \( \hat{Q} > 0. \)

Define \( v'_p := \begin{bmatrix} V_p^{\top}, F_p^{\top}, O_p^{\top} \end{bmatrix}^{\top} \) \textit{and} \( v''_p := W_p. \) Then

\[ \mathcal{D}_{v'_p, \Psi_p} \left( v_p, E_{\hat{\theta}} [\Pi] \right) = -E_{\hat{\theta}} [\pi']^{\top} - \lambda_p \hat{Q} v'_p \]

Under Assumption 3.8, we conclude that \( \hat{Q} > 0. \)
and

\[ D_{v_p} \Psi_p \left( v_p, \mathbb{E}[\Pi] \right) = 0. \]

The second derivative \( D^2_{v_p} \Psi_p \left( v_p, \mathbb{E}[\Pi] \right) = -\lambda_p \hat{Q} < 0 \) since \( 0 < \lambda_p < \infty \). Thus, \( \Psi_p \left( v_p, \mathbb{E}[\Pi] \right) \), \( p \in P \) are strictly concave in \( V_p, F_p, \) and \( O_p \). The proof for consumers \( c \in C \) is similar. \( \square \)

**Lemma 3.10.** Optimal fuel trading strategies \( F_p \) and emission trading strategies \( O_p \) are bounded for all producers \( p \in P \).

**Proof.** Since \( \Psi_p \left( v_p, \mathbb{E}[\Pi] \right) \) is strictly concave and quadratic in \( F_p \) and \( O_p \) for all \( p \in P \), and fuel and emission prices have finite expectation and variance, it is clear that \( \Psi_p \left( v_p, \mathbb{E}[\Pi] \right) \to -\infty \) as \( \|F_p\| \to \infty \) or \( \|O_p\| \to \infty \). This is not optimal and thus we conclude that optimal \( F_p \) and \( O_p \) must be bounded. \( \square \)

**Lemma 3.11.** Consider optimization problem \((2.10)\) without constraints \((2.9)\) for producers \( p \in P \), and optimization problem \((2.14)\) without constraints \((2.13)\) for consumers \( c \in C \). Denote by \( V_k \in \mathbb{R}^N \) a vector of optimal values of any player \( k \in P \cup C \) for a given vector of expected prices \( \mathbb{E}[\hat{F} \Pi] \in \mathbb{R}^N \). Then,

1. if \( \left\| \mathbb{E}^{\hat{F}} [\Pi] \right\| \to \infty \), then \( \left[ \sum_{k \in P \cup C} V_k \right]^T \mathbb{E}^{\hat{F}} [\Pi] \to -\infty \), and
2. if for each delivery period \( j \), \( j \in J \) there exists at least one power plant that is not at the upper or the lower boundary, then \( \|V_k\| \to \infty \) for all \( k \in P \cup C \) simultaneously if and only if \( \left\| \mathbb{E}^{\hat{F}} [\Pi] \right\| \to \infty \).

**Proof.** The necessary and sufficient conditions for \( v_k \) to be a global maximizer of \( \Psi_k \left( v_k, \mathbb{E}[\hat{F} \Pi] \right) \) are, due to Assumption 3.3 that implies the Slater condition, the following

\[
-\mathbb{E}^{\hat{F}} [\pi_k] - \lambda_k Q k v_k - B_k^T \eta_k - A_k^T \mu_k = 0 \\
(B_k v_k - b_k)^T \eta_k = 0 \\
B_k v_k - b_k \leq 0 \\
A_k v_k - a_k = 0 \\
\eta_k \geq 0.
\]

(3.10)

We are only interested in decision variables \( V_k \). By Lemma 3.10, we are allowed to remove constraints \((2.7)\) and \((2.8)\) without affecting the optimal solution. Then, after removing the inequality constraints \((2.9)\) for producers \( p \in P \), and inequality constraints \((2.13)\) for consumers \( c \in C \), there is no inequality constraints that involve variable \( V_k \). Moreover, there is only one equality constraint, i.e. \((2.4)\) for producers and \((2.12)\) for consumers, for each delivery period \( j \) that involve variable \( V_k \). Considering the first equation of \((3.10)\) for each delivery period \( j \) separately, and neglecting all bounded terms, we obtain the following equivalence

\[
\mathbb{E}^{\hat{F}} [\Pi (T_j)] + \lambda_k \hat{Q}_1^j V_k + \mu_{k,j} 1 \sim 0
\]

(3.11)

were \( \mu_{k,j} \in \mathbb{R} \) is the dual variable of the equality constraint \((2.4)\) for producers and \((2.12)\) for consumers, \( 1 \in \mathbb{R}^{|J|} \) is a vector of ones, and \( \hat{Q}_1^j \in \mathbb{R}^{|J| \times N} \) contains only those rows of \( \hat{Q}_1 \) that correspond to \( \mathbb{E}^{\hat{F}} [\Pi (T_j)] \) in \((3.10)\). In the calculation of \((3.11)\) we took into account the construction of the equilibrium price process in Section 2.4, and use the finding that all variances and correlations are finite and can not be affected by the hypothetical market agent. Therefore, \( \|Q_k\| < \infty \). Moreover, \( \left\| \left[ F_p^T, O_p^T \right]^T \right\| < \infty \) due to Lemma 3.10.

Assume that \( \left\| \mathbb{E}^{\hat{F}} [\Pi (T_j)] \right\| \to \infty \). Then by \((3.11)\), \( \|\mu_{k,j}\| \to \infty \) or \( \|V_k\| \to \infty \).

---

1Hence in there exists some \( M_1 \in \mathbb{R} \) such that \( \left\| \left[ F_p^T, O_p^T \right]^T \right\| \leq M_1 < \infty \) for all \( p \in P \). We set \( M_1 \leq F_{trade} \).
1. Assume first that \(\|V_k\|_k < \infty\) for all \(k \in P \cup C\). Let \(E^\#[\Pi(t_i, T_j)] \to \infty\) for some \(i \in I_j\) and \(j \in J\). Then \(\mu_{k,j} \to -\infty\). Since \(\|V_k(T_j)\| < \infty\), it follows from (3.11) that \(E^\#[\Pi(T_j)]\) \(\to \infty\) componentwise and all components of \(E^\#[\Pi(T_j)]\) must be equal up to a constant. From (2.4) and the general interpretation of the dual variables, we can see that as \(\mu_{p,j} \to -\infty, p \in P\), a small increase in \(\sum_{l \in L} \sum_{r \in R_p} W_{p,l,r}(T_j)\) would infinitely improve the objective function \(\Psi_p(v_p, E^\#[\Pi])\). As governed by (2.5) and (2.6), an increase in \(\sum_{l \in L} \sum_{r \in R_p} W_{p,l,r}(T_j)\) would also require that more fuel and emission certificates are bought. Since the fuel and emission prices have a finite expectation and variance, a decrease of the objective function due to the change in fuel and emission buying strategy as \(\mu_{p,j} \to -\infty\) remains bounded. Thus, if \(\mu_{p,j} \to -\infty\), the sum of production of all power plants \(\sum_{l \in I} \sum_{r \in R_p} W_{p,l,r}(T_j)\) of producer \(p \in P\) increases as much as allowed by constraints (2.2) and (2.3). Since all producers share the same electricity, fuel, and emission prices, this holds for all producers \(p \in P\) simultaneously. Then by Lemma 3.7, \(0 > \sum_{c \in C} \sum_{i \in I_j} V_c(t_i, T_j) + \sum_{p \in P} \sum_{i \in I_j} V_p(t_i, T_j)\) and thus \(\left[\sum_{k \in P \cup C} V_k(T_j)\right]^T E^\#[\Pi(T_j)] \to -\infty\). On the other hand, if \(\|E^\#[\Pi(T_j)]\| < \infty\) then also \(\left[\sum_{k \in P \cup C} V_k(T_j)\right]^T E^\#[\Pi(T_j)] < \infty\). Therefore, we can conclude that \(\sum_{k \in P \cup C} V_k(T_j)\) \(\to -\infty\).

2. Assume now that \(\|V_k\|_k \to \infty\) for at least one \(k \in P \cup C\). Then, \(V_k^T \hat{Q} V_k \to \infty\) since \(\hat{Q} > 0\). Assume now that \(E^\#[\Pi] V_k > -\infty\). Then \(\Psi_k(v_k, E^\#[\Pi]) \to -\infty\), which is clearly not optimal for player \(k\). Thus, \(E^\#[\Pi] V_k \to -\infty\), which concludes the proof of point 1.

We continue with the proof of point 2. Assume that for fixed \(j \in J\) not all power plants are at the upper or lower bound simultaneously. We have already seen above that if \(|\mu_{k,j}| \to \infty\) for any \(k \in P \cup C\) and \(j \in J\) then, at delivery time \(T_j\), all power plants are at the upper or lower bound simultaneously. Thus, \(|\mu_{k,j}| < \infty\) for all \(k \in P \cup C\) and \(j \in J\). Rewriting (3.11) without focusing on one delivery period only and taking any norm, we get

\[
\|E^\#[\Pi]\| \sim \|\lambda_k \hat{Q} V_k\|.
\]

Since \(|\lambda_k| < \infty\) and \(\|\hat{Q}\|_\infty < \infty\), it immediately follows from (3.12) that if \(\|E^\#[\Pi(T_j)]\| \to \infty\) then \(\|V_k\| \to \infty\). Because all producers and consumers share the same price this must hold for all of them simultaneously. Since \(\hat{Q}\) is invertible, we can write (3.11) for all delivery periods together as

\[
\|V_k\| \sim \|\lambda_k^{-1} \hat{Q} E^\#[\Pi]\|.
\]

Because \(|\lambda_k^{-1}| < \infty\) and \(\|\hat{Q}^{-1}\|_\infty < \infty\), it follows from (3.13) that if \(\|V_k\| \to \infty\) then \(\|E^\#[\Pi]\| \to \infty\).\]

**Lemma 3.12.** Expected prices \(E^\#[\Pi]\) in a NE are bounded.\(^2\)

**Proof.** Assume that \(\|E^\#[\Pi]\| \to \infty\). Then by Lemma 3.11, \(\left[\sum_{k \in P \cup C} V_k\right]^T E^\#[\Pi] \to -\infty\). Inserting that to (2.34), we get \(\Psi_M(E^\#[\Pi], V) = -\infty\), which is clearly not a NE. Thus, \(\|E^\#[\Pi]\| < \infty\).\]

**Proposition 3.13.** A NE, if it exists, is a CE and vice versa.

**Proof.** The equilibrium conditions for producers and consumers are the same for both the NE and the CE. The remaining part is to show that Point 3 from the CE implies Point 3 from the NE and conversely.

1. Assume the CE holds. Then (3.3) can be inserted into (2.18 - 2.20). Then \(\Psi_M(E^\#[\Pi], V) = 0\) for all \(E^\#[\Pi]\) and thus (3.6) is satisfied.

2. Assume the NE holds. Assume further that (3.3) does not hold for some \(t_i, T_j, i \in I_j, j \in J\). Then according to (2.20) the optimal \(\|E^\#[\Pi(t_i, T_j)]\| \to \infty\). This contradicts Lemma 3.12 and hence such a solution can not be a NE. Thus (3.3) holds for all \(t_i, T_j, i \in I_j, j \in J\).

\(^2\)Hence there exists some \(M_2 \in \mathbb{R}\) such that \(\|E^\#[\Pi(t_i, T_j)]\| \leq M_2 < \infty\) for all \(j \in J\) and \(i \in I_j\). We set \(M_2 < \Pi_{max}\).
Corollary 3.14. The number of forward contracts \( V_k \) in a NE is bounded for all \( k \in P \cup C \).

Proof. Assume that there exists a player \( k \in P \cup C \) such that \( \| V_k \| \to \infty \). By Proposition 3.13 every NE is also a CE and thus (3.3) holds. Thus, we can use Lemma 3.11 and conclude that \( \| V^\emptyset \| \to \infty \). This contradicts Lemma 3.12. Thus, \( \| V_k \| < \infty \) for all \( k \in P \cup C \). \( \square \)

Denote by \( S_M \) the set of prices for which power plants are either all at the upper or all at the lower bound simultaneously for at least one delivery period \( T_j, j \in J \). By Lemma 3.7, we know that the optimal price \( E^\emptyset \) \( \not\in S_M \).

Lemma 3.15. Given an expected price vector \( E^\emptyset \) \( \in R^N \setminus S_M \), the decision vectors \( V_p \) for all \( p \in P \) and the decision vectors \( V_c \) for all \( c \in C \), are unique.

Proof. The objective function of each producer \( p \in P \) can be written as

\[
\Psi_p \left( v_p, E^\emptyset \right) = -E^\emptyset \left[ \pi_p \right]^T v_p - \frac{1}{2} \lambda_p v_p^T Q_p v_p.
\]

Define \( v'_p := V_p \) and \( v''_p := [F_p^T, O_p^T, W_p^T]^T \). Then

\[
D_{v_p} \Psi_p \left( v_p, E^\emptyset \right) = \left[ D_{v_p'} \Psi_p \left( v_p, E^\emptyset \right), D_{v_p''} \Psi_p \left( v_p, E^\emptyset \right) \right]^T.
\]

Due to the strict concavity of the expected utility functions in \( v'_p \), for any \( \hat{v}_p := \left[ \hat{v}'_p^T, \hat{v}''_p^T \right]^T \) and \( \tilde{v}_p := \left[ \tilde{v}'_p^T, \tilde{v}''_p^T \right]^T \) with \( \hat{v}_p \neq \tilde{v}_p \) the following strict inequality holds

\[
(\hat{v}_p - \tilde{v}_p)^T D_{v_p} \Psi_p \left( \hat{v}_p, E^\emptyset \right) + (\hat{v}_p - \tilde{v}_p)^T D_{v_p} \Psi_p \left( \tilde{v}_p, E^\emptyset \right) > 0.
\]

We will continue with a proof by contradiction. Assume that there exist \( \hat{v}_p \neq \tilde{v}_p \) that are both optimal solutions for player \( p \in P \) given the electricity price \( E^\emptyset \). Then both must satisfy the KKT conditions, i.e.

\[
D_{\hat{v}_p} \Psi_p \left( \hat{v}_p, E^\emptyset \right) - B_p^T \hat{\eta}_p - A_p^T \hat{\mu}_p = 0
\]

\[
\hat{\eta}_p^T (B_p \hat{v}_p - b_p) = 0
\]

\[
B_p \hat{v}_p - b_p \leq 0
\]

\[
A_p \hat{v}_p - a_p = 0
\]

\[
\hat{\eta}_p \geq 0
\]

and

\[
D_{\tilde{v}_p} \Psi_p \left( \tilde{v}_p, E^\emptyset \right) - B_p^T \tilde{\eta}_p - A_p^T \tilde{\mu}_p = 0
\]

\[
\tilde{\eta}_p^T (B_p \tilde{v}_p - b_p) = 0
\]

\[
B_p \tilde{v}_p - b_p \leq 0
\]

\[
A_p \tilde{v}_p - a_p = 0
\]

\[
\tilde{\eta}_p \geq 0.
\]

\[\text{3}\text{Hence there exists some } M_3 \in \mathbb{R} \text{ such that } \| V_k (t, T_j) \| \leq M_3 < \infty \text{ for } k \in P \cup C \text{ and for all } j \in J \text{ and } i \in I_j. \text{ We set } M_3 < V_{\text{trade}}.\]
Multiplying the first equation of (3.15) and (3.16) by \((\dot{v}_p - \ddot{v}_p)^\top\) and \((\ddot{v}_p - \dddot{v}_p)^\top\), respectively and summing them up, the following strict inequality

\[
0 = (\dot{v}_p - \ddot{v}_p)^\top D_{\dot{v}_p} \Psi_p \left(\dddot{v}_p, \dddot{v}_p^2 [\Pi]\right) + (\ddot{v}_p - \dddot{v}_p)^\top D_{\ddot{v}_p} \Psi_p \left(\dddot{v}_p, \dddot{v}_p^2 [\Pi]\right)
- (\ddot{v}_p - \dddot{v}_p)^\top B_{\ddot{v}_p} \eta_p - (\dddot{v}_p - \dddot{v}_p)^\top B_{\dddot{v}_p} \eta_p
\]

(3.17)

\[
> - (\ddot{v}_p - \dddot{v}_p)^\top A_{\ddot{v}_p} \mu_p - (\dddot{v}_p - \dddot{v}_p)^\top A_{\dddot{v}_p} \mu_p
\]

is obtained by (3.14).

Rewriting

\[
(\ddot{v}_p - \dddot{v}_p)^\top B_{\ddot{v}_p} \eta_p = (B_{\ddot{v}_p} \dddot{v}_p - B_{\ddot{v}_p} \dddot{v}_p + b_p - b_p)^\top \eta_p
= (B_{\ddot{v}_p} \dddot{v}_p - b_p)^\top \eta_p - (B_{\ddot{v}_p} \dddot{v}_p - b_p)^\top \eta_p
= (B_{\ddot{v}_p} \dddot{v}_p - b_p)^\top \eta_p
\]

and noting \(\eta_p \geq 0\) and \(B_{\ddot{v}_p} \dddot{v}_p - b_p \leq 0\), we can conclude that

\[
(\ddot{v}_p - \dddot{v}_p)^\top B_{\ddot{v}_p} \eta_p \leq 0.
\]

Due to the symmetry also

\[
(\ddot{v}_p - \dddot{v}_p)^\top B_{\ddot{v}_p} \eta_p \leq 0.
\]

Rewriting

\[
(\ddot{v}_p - \dddot{v}_p)^\top A_{\ddot{v}_p} \mu_p = (A_{\ddot{v}_p} \dddot{v}_p - A_{\ddot{v}_p} \dddot{v}_p + a_p - a_p)^\top \mu_p
= (A_{\ddot{v}_p} \dddot{v}_p - a_p)^\top \mu_p - (A_{\ddot{v}_p} \dddot{v}_p - a_p)^\top \mu_p
= 0
\]

(3.20)

and due to the symmetry also

\[
(\ddot{v}_p - \dddot{v}_p)^\top A_{\ddot{v}_p} \mu_p = 0.
\]

(3.21)

Inserting (3.18), (3.19), (3.20), and (3.21) back to (3.17) gives a contradiction. Hence \(\dddot{v}_p = \dddot{v}_p^c\). The proof for consumers \(c \in C\) is similar.

Before we continue with a further analysis of our problem, let us introduce some definitions that are useful for an analysis of piecewise differential functions.

**Definition 3.16.** ([23]) A continuous function \(\hat{Z} : S_1 \to \mathbb{R}^N\) defined on an open set \(S_1 \subseteq \mathbb{R}^N\) is \(P^C^r\) if for every \(x \in S_1\) there exists a finite family of \(C^r\)-functions \(\hat{Z}^i : S_2 \to \mathbb{R}^N\), where \(S_2 \subseteq S_1\) is an open neighborhood of \(x\) and \(i \in I\), such that \(\hat{Z}(z) \in \left\{\hat{Z}^i(z) : i \in I\right\}\) for every \(z \in S_2\). The \(C^r\)-functions \(\hat{Z}^i, i \in I\), are called selection functions of \(\hat{Z}\) at \(x\).

A selection function \(\hat{Z}^i\) of a \(P^C^r\)-function \(\hat{Z}\) at \(x\) is essentially active if

\[
x \in \text{cl int}\left\{z \in S_2 : \hat{Z}^i(z) = \hat{Z}(z)\right\}.
\]

If \(\left\{\hat{Z}^i : i \in I\right\}\) is a family of selection functions for \(\hat{Z}\) at \(x\), then the set of indices \(i\) of essentially active selection functions \(\hat{Z}^i\) at \(x\) is denoted \(I^c(x)\).

**Definition 3.17.** Let \(\hat{Z} : S_1 \to \mathbb{R}^N\) be a continuous \(P^C^1\) function defined on an open set \(S_1 \subseteq \mathbb{R}^N\). Clarke’s generalized Jacobian ([12]) of \(\hat{Z}\) at \(x\) is defined as

\[
\mathcal{D}^C\hat{Z}(x) = \text{conv}\left\{D\hat{Z}^i(x) : i \in I^c(x)\right\}
\]

(3.23)
where \( \text{conv} \) denotes a convex hull.

**Proposition 3.18.** \( \mathcal{D} \hat{Z}^i (x) < 0 \) for all \( i \in I^c (x) \) and all \( x \in S_1 \) if and only if \( \mathcal{J} (x) < 0 \) for all \( \mathcal{J} (x) \in \mathcal{D} \hat{Z} (x) \).

**Proof.** This follows directly from (3.23). For any \( \mathcal{J} (x) \in \mathcal{D} \hat{Z} (x) \) and \( x \in S_1 \), \( \mathcal{J} (x) \) can be written as

\[
\mathcal{J} (x) = \sum_{i \in I^c (x)} \alpha_i \mathcal{D} \hat{Z}^i (x)
\]

for some \( \alpha_i \geq 0 \) such that \( \sum_{i \in I^c (x)} \alpha_i = 1 \). The result then follows trivially. \( \square \)

**Definition 3.19.** A function \( \hat{Z} : S_1 \to \mathbb{R}^N \) defined on an open set \( S_1 \subseteq \mathbb{R}^N \) is strictly decreasing on \( S_1 \) if

\[
(\hat{Z} (x) - \hat{Z} (y))^\top (x - y) < 0
\]

for all \( x, y \in S_1 \) such that \( x \neq y \).

**Proposition 3.20.** Let \( \hat{Z} : S_1 \to \mathbb{R}^N \) be a \( \mathbb{PC}^1 \) function defined on an open set \( S_1 \subseteq \mathbb{R}^N \). If \( \mathcal{D} \hat{Z}^i (x) < 0 \) for all \( i \in I^c (x) \) and \( x \in S_1 \), then \( \hat{Z} \) is strictly decreasing on \( S_1 \).

**Proof.** Assume that \( \mathcal{D} \hat{Z}^i (x) < 0 \) for all \( i \in I^c (x) \). Then by Proposition 3.18 also \( \mathcal{J} (x) < 0 \) for all \( \mathcal{J} (x) \in \mathcal{D} \hat{Z} (x) \). By the extended mean-value theorem (see [19])

\[
\hat{Z} (x) - \hat{Z} (y) = \mathcal{J} (y + \delta (x - y)) (x - y)
\]

for some \( \mathcal{J} (y + \delta (x - y)) \in \mathcal{D} \hat{Z} (y + \delta (x - y)) \) and \( \delta \in (0, 1) \). By multiplying (3.26) from the left by \( (x - y)^\top \) and using the assumption of the negative definiteness of \( \mathcal{J} (x) \), we conclude

\[
(\hat{Z} (x) - \hat{Z} (y))^\top (x - y) < 0.
\]

\( \square \)

**Lemma 3.21.** Let \( k \in P \cup C \). There exists a \( \mathbb{PC}^\infty \) mapping \( \hat{Z}_k : \mathbb{R}^N \setminus \bar{S}_M \to \mathbb{R}^N \) that maps the electricity price vector \( \hat{E}^p [\Pi] \in \mathbb{R}^N \setminus \bar{S}_M \) to a volume vector \( V_k \in \mathbb{R}^N \).

**Proof.** Since the proof for producers and consumers is almost the same we will explicitly write it only for producers. Set \( p = k \) for some \( p \in P \). The optimization problem for a producer \( p \) can be written as

\[
\max_{v_p \in S_p} \Psi_p \left( v_p, \hat{E}^p [\Pi] \right).
\]

For convenience we define \( \hat{\Psi}_p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) as \( \hat{\Psi}_p \left( V_p, \hat{E}^p [\Pi] \right) := \max_{v_p, \hat{E}^p, w_p} \Psi_p \left( v_p, \hat{E}^p [\Pi] \right) \) subject to (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), and (2.8). Using Lemma 3.15 we know that volumes \( V_p \) are unique for a given price \( \hat{E}^p [\Pi] \in \mathbb{R}^N \setminus \bar{S}_M \). Thus there exists a mapping \( \hat{Z}_p : \mathbb{R}^N \setminus \bar{S}_M \to \mathbb{R}^N \) such that \( V_p = \hat{Z}_p \left( \hat{E}^p [\Pi] \right) \). Results from the parametric quadratic programming show that \( \hat{Z}_p \left( \hat{E}^p [\Pi] \right) \) is a continuous and piecewise affine function (and thus \( \mathbb{PC}^\infty \)), see [6] for a strictly convex quadratic objective function and [4] for a convex quadratic objective function. \( \square \)

**Theorem 3.22.** Define a \( \mathbb{PC}^\infty \) mapping \( \hat{Z} \left( \hat{E}^p [\Pi] \right) := \sum_{p \in P} \hat{Z}_p \left( \hat{E}^p [\Pi] \right) + \sum_{c \in C} \hat{E}_c \left( \hat{E}^p [\Pi] \right) \).

If \( \mathcal{D} \hat{Z}^i (x) < 0 \) for all \( i \in I^c (x) \) and \( x \in \mathbb{R}^N \setminus \bar{S}_M \), then electricity prices \( \hat{E}^p [\Pi] \) in the NE are unique.

**Proof.** The market condition (3.3) requires that

\[
0 = \sum_{c \in C} V_c + \sum_{p \in P} V_p.
\]

Using Lemma 3.21 this can be written as

\[
0 = \sum_{p \in P} V_p + \sum_{c \in C} V_c
\]

(3.28)

\[
= \sum_{p \in P} \hat{Z}_p \left( \hat{E}^p [\Pi] \right) + \sum_{c \in C} \hat{E}_c \left( \hat{E}^p [\Pi] \right)
\]

\[
= \hat{Z} \left( \hat{E}^p [\Pi] \right).
\]
If $D \hat{Z}_i(x) < 0$ for all $i \in I^c(x)$ and $x \in \mathbb{R}^N \setminus S_M$, then by Proposition 3.20 $\hat{Z}$ is strictly decreasing on $\mathbb{R}^N \setminus S_M$. Thus, the mapping $\hat{Z}$ must be bijective. By Corollary 3.5, we know that the solution to the problem $\hat{Z} (E^\delta [I]) = 0$ exists and from the bijectivity of $\hat{Z}$, we conclude that it must be unique. \(\square\)

The remaining problem of this section is to show that $D \hat{Z}_i (E^\delta [I]) < 0$ for all $i \in I^c (E^\delta [I])$ and $E^\delta [I] \in \mathbb{R}^N \setminus S_M$ indeed holds. From the proof of Lemma 3.21, we know that $\hat{Z} (E^\delta [I])$, $i \in I$ are all affine functions and thus $D \hat{Z}_i (E^\delta [I])$ are all constants. Therefore, it is enough to show that $D \hat{Z}_i (E^\delta [I]) < 0$ for all $i \in I^c (E^\delta [I])$ and $E^\delta [I] \in \{x : x \in \mathbb{R}^N \setminus S_M \land |I^c(x)| = 1\}$. For such prices $E^\delta [I]$, $D \hat{Z}_k (E^\delta [I])$ exists for all players $k \in P \cup C$. In the rest of this section we simplify the notation and omit writing the dependence on $E^\delta [I] \in \{x : x \in \mathbb{R}^N \setminus S_M \land |I^c(x)| = 1\}$ explicitly. All the statements hold for any such $E^\delta [I]$.

For the further argumentation, we introduce $R(\cdot)$ and $N(\cdot)$ that denote a range and a null space, respectively.

**Lemma 3.23.** Let $x \in \mathbb{R}^N \setminus \{0\}$ and $c \in C$ be any of the consumers. Then $x^T D \hat{Z}_c x \leq 0$. Moreover, $x^T D \hat{Z}_c x = 0$ if and only if $x \in R(\hat{A}_1^\top)$.

**Proof.** Since the optimal power trading strategy $V_c$ can not be at the boundary of the feasible region $S_c$ due to Lemma 3.14, the necessary and due to convexity and the Slater condition in Assumption 3.3 also sufficient KKT conditions for $V_c$ to be a global minimizer given the forward electricity price reads

$$\begin{align*}
-E^\delta [II] - \lambda_c \hat{Q}_1 V_c - \hat{A}_1^\top \mu_c &= 0 \\
\hat{A}_1 V_c &= \alpha_c.
\end{align*}$$

(3.29)

One can then solve for $V_c$ from the first equation of (3.29) by multiplying both sides from the left by $(\lambda_c \hat{Q}_1)^{-1}$ and obtain

$$V_c = -\frac{1}{\lambda_c} \hat{Q}_1^{-1} E^\delta [II] - \frac{1}{\lambda_c} \hat{Q}_1^{-1} \hat{A}_1^\top \mu_c.
\quad \text{(3.30)}$$

By further multiplying both sides of (3.30) from the left by $\hat{A}_1$, we get

$$\hat{A}_1 V_c = -\frac{1}{\lambda_c} \hat{A}_1 \hat{Q}_1^{-1} E^\delta [II] - \frac{1}{\lambda_c} \hat{A}_1 \hat{Q}_1^{-1} \hat{A}_1^\top \mu_c.
\quad \text{(3.31)}$$

Since $\hat{A}_1$ has full row rank $(\hat{A}_1 \hat{Q}_1^{-1} \hat{A}_1^\top)^{-1}$ exists. By using the second equation of (3.29), we get

$$\mu_c = -\left(\hat{A}_1 \hat{Q}_1^{-1} \hat{A}_1^\top\right)^{-1} \left(\lambda_c \alpha_c + \hat{A}_1 \hat{Q}_1^{-1} E^\delta [II]\right).
\quad \text{(3.32)}$$

Inserting that back into (3.30), we can calculate

$$\frac{\partial V_c}{\partial E^\delta [II]} = -\frac{1}{\lambda_c} \hat{Q}_1^{-1} + \frac{1}{\lambda_c} \hat{Q}_1^{-1} \hat{A}_1^\top \left(\hat{A}_1 \hat{Q}_1^{-1} \hat{A}_1^\top\right)^{-1} \hat{A}_1 \hat{Q}_1^{-1}.
\quad \text{(3.33)}$$

Since $\hat{Q}_1$ (and consequently also $\hat{Q}_1^{-1}$) is positive definite and symmetric, we can use the Cholesky decomposition to define a matrix $\hat{Q}_1 \in \mathbb{R}^{N \times N}$ such that $\hat{Q}_1^{-1} = \hat{Q}_1^{-1} \hat{Q}_1^{-\top}$. Since $\hat{Q}_1$ is invertible, we can rewrite (3.33) as

$$\hat{P} := -\lambda_c \hat{Q}_1 \frac{\partial V_c}{\partial E^\delta [II]} \hat{Q}_1^{-\top} = I - \hat{Q}_1^{-\top} \hat{A}_1^\top \left(\hat{A}_1 \hat{Q}_1^{-1} \hat{A}_1^\top\right)^{-1} \hat{A}_1 \hat{Q}_1^{-1}.
\quad \text{(3.34)}$$

It is trivial to check that $\hat{P} = P^2$ (i.e. $\hat{P}$ is idempotent) and symmetric. Thus, $\hat{P}$ must be a projection matrix. It is known that every projection matrix is positive semidefinite and thus $\frac{\partial V_c}{\partial E^\delta [II]} \preceq 0$. Moreover,
for any $x \in \mathbb{R}^N \setminus \{0\}$, $x^T \hat{P} x = 0$ if and only if $x \in R \left( \hat{Q}_1^{-T} \hat{A}_1^T \right)$. Therefore, it follows from (3.34) that $x^T \frac{\partial \tilde{f}}{\partial x} x = 0$ if and only if $x \in R \left( \hat{A}_1^T \right)$.

**Lemma 3.24.** Assume that $x^T \hat{D} \tilde{x}^T \hat{D} x \leq 0$ for all $p \in P$ and all $x \in \mathbb{R}^N \setminus \{0\}$. Then $\hat{D} \tilde{x} \prec 0$ if and only if $\sum_{p \in P} \hat{A}_1 \hat{D} \tilde{x} \hat{A}_1^T$ has a full rank.

**Proof.** We can write

$$ x^T \hat{D} \tilde{x} = \sum_{k \in P \cup C} x^T \hat{D} \tilde{x}_k. $$

By Lemma 3.23 and the assumption that $x^T \hat{D} \tilde{x}_k \leq 0$ for all $p \in P$ and all $x \in \mathbb{R}^N \setminus \{0\}$, we can see that $x^T \hat{D} \tilde{x}_k x \leq 0$ for all $k \in P \cup C$. Thus, $x^T \hat{D} \tilde{x} x < 0$ if and only if for each $x \in \mathbb{R}^N \setminus \{0\}$ there exists at least one player $k' \in P \cup C$ such that $x^T \hat{D} \tilde{x}_k x < 0$. If $x \notin R \left( \hat{A}_1^T \right)$ then $x^T \hat{D} \tilde{x}_k x = 0$ for all $k \in C$. On the other hand, if $x \in R \left( \hat{A}_1^T \right)$, $\hat{D} \tilde{x} \prec 0$ if and only if $\sum_{p \in P} \hat{A}_1 \hat{D} \tilde{x}_k \hat{A}_1^T$ has a full rank.

**Lemma 3.25.** Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times m}$ be real matrices such that $A^{-1}$ and $(C - BC^{-1}B^T)$ exist. Then

$$ \begin{bmatrix} A & B^T \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} $$

and consequently

$$ \begin{bmatrix} A & B^T \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} \begin{bmatrix} A \\ B \end{bmatrix} = A. $$

**Proof.** Since $A^{-1}$ and $(C - BC^{-1}B^T)$ exist, we can use the block matrix inverse formula to calculate

$$ \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B \left( C - B^T A^{-1} B \right)^{-1} B^T A^{-1} & - A^{-1} B \left( C - B^T A^{-1} B \right)^{-1} \\ - \left( C - B^T A^{-1} B \right)^{-1} B^T A^{-1} & \left( C - B^T A^{-1} B \right)^{-1} \end{bmatrix}. $$

The result then follows trivially.

**Theorem 3.26.** For all $x \in \mathbb{R}^N$ and all $p \in P$, $x^T \hat{D} \tilde{x} x \leq 0$ holds. Moreover, if for each delivery period $j \in J$ there exist at least one power plant that has a strictly feasible optimal production, then $\sum_{p \in P} \hat{A}_1 \hat{D} \tilde{x} \hat{A}_1^T$ has a full rank.

**Proof.** We can assume w.l.o.g. that there exists a producer $p \in P$ who owns a set of power plants among which at least one has a strictly feasible optimal production in each delivery periods $j \in J$. In this case, showing that $\sum_{p \in P} \hat{A}_1 \hat{D} \tilde{x} \hat{A}_1^T$ has a full rank simplifies to showing that $\hat{A}_1 \hat{D} \tilde{x} \hat{A}_1^T$ has a full rank.

Assume that the set of active inequality constraints $B_p$ is known. Constraints (2.7) and (2.8) are never active due to Lemma 3.10. Similarly, Constraints (2.9) are never active due to Lemma 3.14. The KKT conditions can thus be written as

$$ -E^{\tilde{P}} \left[ \pi \right] - \lambda_p \hat{Q}_p v_p' - \hat{A}_{12} \mu_p' = 0 $$

$$ - \hat{A}_p \mu_p' - B_p v_p'' = 0 $$

$$ \hat{A}_{12} v_p' + \hat{A}_p v_p'' = 0 $$

$$ B_p v_p'' = b $$

where $v_p' := [V_p^T, F_p^T, O_p^T]^T$ and $v_p'' := W_p$. The internal structure of the matrices is the following

$$ \hat{Q} := \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ \hat{Q}_2 & \hat{Q}_3 \end{bmatrix}, \quad \hat{A}_{12} := \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{bmatrix}, \quad \hat{A}_p := \begin{bmatrix} \hat{A}_{3,p} \\ \hat{A}_{4,p} \end{bmatrix}. $$
Since $\hat{Q} > 0$, we can express $v_p'$ from the first equation of (3.35) as

$$v_p' = -\frac{1}{\lambda_p} \hat{Q}^{-1} \hat{E}^p [\pi] - \frac{1}{\lambda_p} \hat{Q}^{-1} \hat{A}_{12}^{\top} \mu_p'$$

and further $\hat{A}_{12} v_p'$ as

$$\hat{A}_{12} v_p' = -\frac{1}{\lambda_p} \hat{A}_{12} \hat{Q}^{-1} \hat{E}^p [\pi] - \frac{1}{\lambda_p} \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \mu_p'.$$

Using the third equation of (3.35) this reads

$$\hat{A} p v_p'' = \frac{1}{\lambda_p} \hat{A}_{12} \hat{Q}^{-1} \hat{E}^p [\pi] + \frac{1}{\lambda_p} \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \mu_p'.$$

Since $\hat{A}_1$ and $\hat{A}_2$ both have a full rank, also $\hat{A}_{12}$ has a full rank and thus $\left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1}$ exists. Then

$$\mu_p = \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \left( \lambda_p \hat{A}_p v_p'' - \hat{A}_{12} \hat{Q}^{-1} \hat{E}^p [\pi] \right).$$

Multiplying both sides with $\hat{A} p v_p''$ from the left and using the second equation of (3.35), we get

$$B_p v_p'' = \hat{A}_p \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \left( -\lambda_p \hat{A}_p v_p'' + \hat{A}_{12} \hat{Q}^{-1} \hat{E}^p [\pi] \right).$$

It is not possible to express $v_p''$ from (3.41) because rank $\left( \hat{A}_p \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \hat{A}_p \right) < |J| (|L| + 1) + \dim W_p$ and thus $\hat{A}_p \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \hat{A}_p$ is not invertible. We can write (3.41) and the last equation of (3.35) as the following system of linear equations

$$\begin{bmatrix} \lambda_p \hat{A}_p \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \hat{A}_p & B_p \\ 0 & b_p \end{bmatrix} \begin{bmatrix} v_p'' \\ \eta_p \end{bmatrix} = \begin{bmatrix} \hat{A}_p \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \hat{A}_{12} \hat{Q}^{-1} \hat{E}^p [\pi] \end{bmatrix}.$$

Before we attempt to solve (3.42), let us try to evaluate $\frac{\partial v_p''}{\partial \hat{E}^p [\pi]}$ with the currently known facts. We get

$$\frac{\partial v_p''}{\partial \hat{E}^p [\pi]} = -\frac{1}{\lambda_p} \hat{Q}^{-1} - \frac{1}{\lambda_p} \hat{Q}^{-1} \hat{A}_{12}^{\top} \frac{\partial \mu_p'}{\partial \hat{E}^p [\pi]}$$

$$= -\frac{1}{\lambda_p} \hat{Q}^{-1} + \frac{1}{\lambda_p} \hat{Q}^{-1} \hat{A}_{12}^{\top} \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \hat{A}_{12} \hat{Q}^{-1} - \lambda_p \hat{A}_p \frac{\partial \mu_p'}{\partial \hat{E}^p [\pi]},$$

where (3.37) and (3.40) were used to obtain the first and the second equality, respectively. We are interested in the rank of $\hat{A}_1 \hat{D} \hat{Z}_p \hat{A}_1^{\top}$. Define $\hat{A}_1 = \left[ \hat{A}_1 \ 0 \right]$, where $0$ is a matrix of zeros of dimension $|J| \times (N \mid L \mid + 1)$. Then,

$$\hat{A}_1 \hat{D} \hat{Z}_p \hat{A}_1^{\top} = \hat{A}_1 \frac{\partial v_p''}{\partial \hat{E}^p [\pi]} \hat{A}_1^{\top}$$

$$\begin{aligned} & = -\frac{1}{\lambda_p} \hat{A}_1 \hat{Q}^{-1} \hat{A}_1^{\top} + \frac{1}{\lambda_p} \hat{A}_1 \hat{Q}^{-1} \hat{A}_{12}^{\top} \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \hat{A}_{12} \hat{Q}^{-1} \hat{A}_1^{\top} \\ & \quad - \hat{A}_1 \hat{Q}^{-1} \hat{A}_{12}^{\top} \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} \right)^{-1} \hat{A}_p \frac{\partial \mu_p'}{\partial \hat{E}^p [\pi]} \hat{A}_1^{\top}. \end{aligned}$$

Using the internal structure of $\hat{Q}$ and $\hat{A}_{12}$ as defined in (3.36), we evaluate

$$\hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^{\top} = \begin{bmatrix} \hat{A}_1 \hat{Q}_1 \hat{A}_1^{\top} & \hat{A}_1 \hat{Q}_2 \hat{A}_1^{\top} \\ \hat{A}_2 \hat{Q}_1 \hat{A}_2^{\top} & \hat{A}_2 \hat{Q}_2 \hat{A}_2^{\top} \end{bmatrix}$$
and similarly
\begin{equation}
\hat{A}_1 \hat{Q}^{-1} \hat{A}_{12} = [ \hat{A}_1 \hat{Q}_1 \hat{A}_1^\top \quad \hat{A}_1 \hat{Q}_2 \hat{A}_2^\top ].
\end{equation}

Using Lemma 3.25, (3.44), and (3.45), we can rewrite (3.43) as
\begin{equation}
\hat{A}_1 \hat{D} \hat{Z}_p \hat{A}_1^\top = -\hat{A}_1 \hat{Q}^{-1} \hat{A}_{12} \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^\top \right)^{-1} \hat{A}_p \frac{\partial v''}{\partial \hat{E}_p^p [\pi]} \hat{A}_1^\top.
\end{equation}

In order to evaluate (3.46), we return back to the system of linear equations (3.42). Using the generalized Bott-Duffin constrained inverse (see [25]), we can express \( \frac{\partial v''}{\partial \hat{E}_p^p [\pi]} \) from (3.42) as
\begin{equation}
\frac{\partial v''}{\partial \hat{E}_p^p [\pi]} = (\hat{A}_p)^{(+)\top} \hat{A}_p^\top \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^\top \right)^{-1} \hat{A}_{12} \hat{Q}^{-1} + P_{N(\hat{A}_p) \cap S} \frac{\partial z_p}{\partial \hat{E}_p^p [\pi]},
\end{equation}

where \( (\hat{A}_p)^{(+)\top} = P_S (\hat{A}_p P_S + P_{S\perp})^{(+)\top} \), \( \hat{A}_p := \lambda_p \hat{A}_p^\top \left( \hat{A}_{12} \hat{Q}_P^{-1} \hat{A}_{12}^\top \right)^{-1} \hat{A}_p \), \( S \) is a null space of \( B_p \), i.e. \( S = N(B_p) \), \( P_S \) a projection on \( S \), and \( z_p \in \mathbb{R}^{\dim W_p} \) an arbitrary vector. The Moore–Penrose pseudoinverse is denoted by \((+)\). For every \( x \in N(\hat{A}_p) \),
\begin{align}
0 &= \hat{A}_p x \\
&= \hat{A}_p^\top \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^\top \right)^{-1} \hat{A}_p x \\
&= x^\top \hat{A}_p^\top \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^\top \right)^{-1} \hat{A}_p x \\
&= \hat{Q} \hat{A}_p x \\
&= \hat{A}_p x,
\end{align}

where \( \hat{Q} \in \mathbb{R}^{(\dim W_p) \times (\dim W_p)} \) is defined by the Cholesky decomposition such that
\begin{equation}
\hat{Q} \hat{Q}^\top = \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^\top \right)^{-1}.
\end{equation}

Since this holds for every \( x \in N(\hat{A}_p) \), it must hold also for every \( x \in N(\hat{A}_p) \cap S \) and therefore
\begin{equation}
-\hat{A}_1 \hat{Q}^{-1} \hat{A}_{12} \left( \hat{A}_{12} \hat{Q}^{-1} \hat{A}_{12}^\top \right)^{-1} \hat{A}_p P_{N(\hat{A}_p) \cap S} \frac{\partial z_p}{\partial \hat{E}_p^p [\pi]} = 0.
\end{equation}

Properties about the Bott-Duffin inverse (see [25]) show that if \( \hat{A}_p \) is symmetric, then also \( (\hat{A}_p)^{(+)\top} \) is symmetric. Moreover, if \( (v''^\top) \hat{A}_p v'' \geq 0 \) for all \( v'' \) such that \( B_p v'' = b_p \), then also \( (v''^\top) (\hat{A}_p)^{(+)\top} v'' \geq 0 \) for all \( v'' \). Thus, we can see from (3.46), (3.47), and (3.50) that \( x^\top \hat{D} \hat{Z}_p x \leq 0, p \in P \) for all \( x \in \mathbb{R}^N \), which proves the first part of this theorem.

Using (3.50), (3.44), and (3.45), equation (3.46) reads
\begin{equation}
\hat{A}_1 \hat{D} \hat{Z}_p \hat{A}_1^\top = \hat{A}_{3,p} (\hat{A}_p)^{(+)\top} \hat{A}_{3,p}^\top.
\end{equation}

From [25] we know that \( R \left( (\hat{A}_p)^{(+)\top} \right) = R (P_S \hat{A}_p) \). Since \( (\hat{A}_p)^{(+)\top} \) is symmetric and positive semidefinite there exists \( U \in \mathbb{R}^{\dim W_p \times \dim W_p} \) such that \( (\hat{A}_p)^{(+)\top} = U U^\top \). Then,
\begin{align}
R (P_S \hat{A}_p) &= R \left( (\hat{A}_p)^{(+)\top} \right) \\
&= R (U U^\top) \\
&= R(U)
\end{align}
and
\[
\text{rank}\left(\hat{A}_{3,p} \left(\hat{A}_p\right)^{+} \hat{A}^\top_{3,p}\right) = \text{rank}\left(\hat{A}_{3,p}U\right)
\]
(3.53)

\[
= \dim R(U) - \dim N \left(\hat{A}_{3,p}\right) \cap R(U).
\]

The next step is to show that \( R (P_S \hat{A}_p) = R \left( P_S \hat{A}^\top_p \right) \). We know that for any matrices \( \hat{A} \) and \( \hat{B} \) of the same size \( R \left( \hat{A} \right) = R \left( \hat{B} \right) \) if and only if \( N \left( \hat{A}^\top \right) = N \left( \hat{B}^\top \right) \). Since \( \hat{A}_{12}Q^{-1}\hat{A}^\top_{12} \) is positive definite and symmetric, there exists an invertible matrix \( Q \) as defined in (3.49). Then for every \( x \in N \left( (P_S \hat{A}_p)^\top \right) \)
\[
0 = \hat{A}^\top_pQQ^\top \hat{A}_pP_Sx
\]
(3.54)
\[
= x^\top P_S \hat{A}_p^\top QQ^\top \hat{A}_pP_Sx
\]
\[
= \tilde{Q}^\top \hat{A}_pP_Sx
\]
\[
= \hat{A}_pP_Sx,
\]
and thus \( R (P_S \hat{A}_p) = R \left( P_S \hat{A}^\top_p \right) \). It follows from (3.52), (3.53) and (3.54) that
\[
\text{rank}\left(\hat{A}_{3,p} \left(\hat{A}_p\right)^{+} \hat{A}^\top_{3,p}\right) = \text{rank}\left(\hat{A}_{3,p}P_S \left[ \hat{A}^\top_{3,p} \left( \hat{A}^\top_p \right) \right]\right)
\]
(3.55)
\[
= \text{rank}\left(\hat{A}_{3,p}P_S\right),
\]
where the last equality holds since \( P_S \) is symmetric.

A projection matrix \( P_S \) on the subspace \( S = N (B_p) \), can be written as
\[
P_S = I - B_p^\top \left( B_pB_p^\top \right)^{-1} B_p.
\]
(3.56)
By taking into account the structure of \( B_p \) and \( \hat{A}_{3,p} \) is easy to see that \( \hat{A}_{3,p}P_S \) has full rank if and only if for each delivery period \( j \in J \) there exists at least one power plant that has a strictly feasible optimal production (i.e. one of the power plants does not appear in the set of active constraints \( B_p \)).

\[ \square \]

It is possible to establish the following unique closed form relation among the prices \( E^s_i \left[ \Pi(t_i, T_j) \right] \), \( i \in \{1, 2\} \) for any fixed \( j \in J \). Since the claim holds for any fixed \( j \in J \) we simplify the notation and avoid writing \( T_j \).

**Proposition 3.27.** If \( \Pi(t_1) \) and \( E^s_i \left[ \Pi(t_2) \right] \) denote the two-stage equilibrium prices at \( t_1 \), then the following equality
\[
\Pi(t_1) = E^s_i \left[ \Pi(t_2) \right] - \lambda^{-1} \text{Cov}_{t_1} \left[ \Pi(t_2) \cdot \sum_{p \in P} C_p \left[ k_p, F_p(t), G(t) \right] - \left( \sum_{c \in C} s_c p_c \right) D \right]
\]
(3.57)
where \( \lambda^{-1} = \sum_{p=1}^P \frac{1}{\lambda_p} + \sum_{c=1}^C \frac{1}{\lambda_c} \) and \( k_p = V_p(t_1) + V_p(t_2) \), must be satisfied. \( C_p (V, F, G) \) denotes the costs of producing \( V \) units of electricity with the fuel trading strategy \( F \) at fuel prices \( G \).

**Proof.** Profit of a producer \( p \in P \) can in this setting be written as
\[
P_p \left( V_p(t_1), F_p(t), \Pi(t) \right) = -\Pi(t_1) V_p(t_1) - \Pi(t_2) V_p(t_2)
\]
(3.58)
\[
- C_p \left[ V_p(t_1) + V_p(t_2), F_p(t), G(t) \right]
\]
and profit of a consumer \( c \in C \) as
\[
P_c \left( V_c(t), \Pi(t) \right) = s_c p_c D
\]
\[
- \Pi(t_1) V_c(t_1) - \Pi(t_2) V_c(t_2)
\]
Using Lemma 3.14, we can see that for any producer \( p \in P \) there is no constraints on the small changes of \( V_p(t_1) \) and \( V_p(t_2) \), but there is a capacity constraint on changes in the sum \( V_p(t_1) + V_p(t_2) \). Thus by setting \( V_p(t_1) + V_p(t_2) = k_p \) for some fixed \( k_p \in \mathbb{R} \) for all \( p \in P \), constraints can not be violated by small changes in volumes \( V_p(t_1) \) or \( V_p(t_2) \). By inserting (3.58) into a mean-variance objective function and taking a derivative, we obtain the following expression

\[
\frac{\partial \Psi_p(P_p(V_p(t_1), F_p(t_1), I(t)))}{\partial V_p(t_2)} = -\Pi(t_1) + \mathbb{E}_{t_1} \cdot \Pi(t_2) + \lambda_p \left[ \text{Var}_{t_1}(V(t_2)) + \text{Cov}_{t_1}(\Pi(t_2), C_p[k_p, F_p(t), G(t)]) \right].
\]

By setting \( \frac{\partial \Psi_p(P_p(V_p(t_1), B_p(t), I(t)))}{\partial V_p(t_2)} = 0 \) and expressing \( V_p(t_2) \), we obtain

\[
V_p(t_2) = \frac{\Pi(t_1) - \mathbb{E}_{t_1} \cdot \Pi(t_2)}{\lambda_p \text{Var}_{t_1} \Pi(t_2)} + \frac{\text{Cov}_{t_1}(\Pi(t_2), C_p[k_p, F_p(t), G(t)])}{\text{Var}_{t_1} \Pi(t_2)}.
\]

Similarly, for each consumer \( c \in C \)

\[
V_c(t_2) = \frac{\Pi(t_1) - \mathbb{E}_{t_1} \Pi(t_2)}{\lambda_c \text{Var}_{t_1} \Pi(t_2)} + \frac{\text{Cov}_{t_1}(D, \Pi(t_2))}{\text{Var}_{t_1} \Pi(t_2)}.
\]

The market condition requires

\[
0 = \sum_{c \in C} V_c(t_2) + \sum_{p \in P} V_p(t_2).
\]

Inserting expressions (3.59) and (3.60) into (3.61), we obtain the result

\[
\Pi(t_1) = \mathbb{E}_{t_1} \cdot \Pi(t_2) - \lambda^{-1} \text{Cov}_{t_1} \left[ \Pi(t_2), \sum_{p \in P} C_p[k_p, F_p(t), G(t)] - \left( \sum_{c \in C} s_c p_c \right) D \right]
\]

where \( \lambda^{-1} = \sum_{p=1}^{P} \frac{1}{\lambda_p} + \sum_{c=1}^{C} \frac{1}{\lambda_c} \).

The analytical expression obtained in Proposition 3.27 matches expression in [5], [8], and [7], but extends results to more than one producer and consumer. In contrast to the previous work, it also allows capacity constraints.

### 3.3. Mean Maximization

In this subsection we would like to examine a mean maximization problem instead of the mean-variance maximization, in the context of assumptions and theorems presented in the previous subsection.

A mean maximization optimization problem for each producer \( p \in P \) is defined as

\[
\max_{v_p} -\mathbb{E}^\hat{p} \left[ \pi_p \right]^\top v_p
\]

s.t. \( A_p v_p = a_p \)

\( B_p v_p \leq b_p \)

and a mean maximization optimization problem for each consumer \( c \in C \) as

\[
\max_{v_c} -\mathbb{E}^\hat{p} \left[ \Pi \right]^\top V_c
\]

s.t. \( A_c V_c = a_c \)

\( B_c V_c \leq b_c \).

Under Assumption 3.3, we use Corollary 3.5 to conclude that there exists a solution to the mean maximization problem.
However, we are not able to claim the uniqueness of solution. Since $\mathcal{D}_{v_p} \Psi_p \left( v_p, \mathbb{E}^\theta [\Pi] \right) = 0$ for all $p \in P$ and $\mathcal{D}_c \Psi_c \left( V_c, \mathbb{E}^\theta [\Pi]^\top \right) = 0$ for all $c \in C$, expected utility functions of producers and consumers do not satisfy strict concavity. To get some insight into this problem let us examine the most simple case where $|P| = 1$, $|C| = 1$, $T = 1$, and $I_1 = \{1\}$. We simplify the notation by setting $V := V_{p_1}$. The sensitivity analysis of the linear programming shows that the optimal value $V^* \left( \mathbb{E}^\theta [\Pi] \right)$ for a given price $\mathbb{E}^\theta [\Pi]$, is an increasing piecewise constant function. Since we are looking for a price $\mathbb{E}^\theta [\Pi]^*$ such that $V^* \left( \mathbb{E}^\theta [\Pi]^* \right) = D (T_1)$, one can distinguish two cases (see Figure 3.1). In the first case (graph on the left) the price equilibrium is not unique, but the volume for a given price is. In the second case (graph on the right) the price equilibrium is unique, but the volume for a given price is not. In order to ensure uniqueness of both, Lemma 3.15 requires uniqueness of volumes for a given price (i.e. forbids that the graph is vertical), and Theorem 3.22 (by setting $D \hat{Z} \neq 0$) requires uniqueness of prices for a given volume (i.e. forbids that the graph is horizontal). As we have seen in the previous subsection, the first condition depends only on the expected utility function, while the second condition depends on both, the expected utility function and constraints.

![Figure 3.1](image.png)

**Equilibrium types for the mean maximization problems.**

Even though the uniqueness of solution can not be claimed in this setting, it is still possible to establish the following unique relation among the prices $\Pi (t_i, T_j)$, $i \in I_j$ for any fixed $j \in J$.

**Proposition 3.28.** Electricity prices $\mathbb{E}^\theta [\Pi (t_i, T_j)]$ that constitute a CE are equal for all $i \in I_j$ and any fixed $j \in J$.

**Proof.** Let there exist $j \in J$, $i' \in I_j$, $i'' \in I_j$, $i' \neq i''$ such that $\mathbb{E}^\theta [\Pi (t_{i'}, T_j)] < \mathbb{E}^\theta [\Pi (t_{i''}, T_j)]$. Then for each producer $p \in P$, $V_p (t_{i'}, T_j) \geq V_p (t_{i''}, T_j)$, because otherwise they could improve their objective functions by exchanging $V_p (t_{i'}, T_j)$ and $V_p (t_{i''}, T_j)$. The same also holds for all consumers $c \in C$. Note that (3.3) implies

$$0 = \sum_{p \in P} V_p (t_{i'}, T_j) + \sum_{c \in C} V_c (t_{i'}, T_j)$$

and

$$0 = \sum_{p \in P} V_p (t_{i''}, T_j) + \sum_{c \in C} V_c (t_{i''}, T_j).$$

Therefore, $V_p (t_{i'}, T_j) = V_p (t_{i''}, T_j)$ for all $p \in P$ and $V_c (t_{i'}, T_j) = V_c (t_{i''}, T_j)$ for all $c \in C$.

If $|V_p (t_{i'}, T_j)| < V_{\text{trade}}$ then this can not be a CE, because producer $p$ could improve its objective function by decreasing $V_p (t_{i'}, T_j)$ for some $\epsilon > 0$ and increasing $V_p (t_{i''}, T_j)$ for the same $\epsilon > 0$ without changing any other decision variables or violating any constants. The same reasoning also holds for each consumer $c \in C$. Thus, $|V_p (t_{i'}, T_j)| = V_{\text{trade}}$ for all $p \in P$ and $|V_c (t_{i'}, T_j)| = V_{\text{trade}}$ for all $c \in C$. Such solution clearly does not satisfy (2.12). Thus, $\mathbb{E}^\theta [\Pi (t_{i'}, T_j)] \geq \mathbb{E}^\theta [\Pi (t_{i''}, T_j)]$. One can than apply a similar reasoning to $\mathbb{E}^\theta [\Pi (t_{i'}, T_j)] > \mathbb{E}^\theta [\Pi (t_{i''}, T_j)]$ and conclude $\mathbb{E}^\theta [\Pi (t_{i'}, T_j)] = \mathbb{E}^\theta [\Pi (t_{i''}, T_j)].$

**4. Conclusions.** In this paper we propose a new model for modeling the electricity price and its relation to other fuels and emission certificates. The model belongs to a class of game theoretic equilibrium models. In this paper we rigorously show that there exists a solution to the proposed
model and develop the conditions under which the solution is also unique. In the last part of the paper we show why the uniqueness of solution can not be claimed for a mean maximization problem.

REFERENCES

[1] Carol Alexander and Leonardo M. Nogueira, *Hedging options with scale-invariant models*, ICMA Centre Discussion Papers in Finance icma-dp2006-03, Henley Business School, Reading University, June 2006.

[2] M. T. Barlow, *A diffusion model for electricity prices*, Mathematical Finance, 12 (2002), pp. 287–298.

[3] Fred Espen Benth and Thilo Meyer-Brandis, *The information premium for non-storable commodities*, Journal of Energy Markets, (2009).

[4] A.B. Berkelaar, B. Jansen, K. Roos, and T. Terlaky, *Sensitivity analysis in (degenerate) quadratic programming*, tech. report, Jan. 1996.

[5] Hendrik Bessembinder and Michael L. Lemmon, *Equilibrium pricing and optimal hedging in electricity forward markets*, Journal of Finance, 57 (2002), pp. 1347–1382.

[6] J. C. G. Boot, *On sensitivity analysis in convex quadratic programming problems*, Operations Research, 11 (1963), pp. 771–786.

[7] Wolfgang Bühler, *Risk premia of electricity futures: A dynamic equilibrium model*, in Risk Management in Commodity Markets, John Wiley & Sons, Ltd., 2009, pp. 61–80.

[8] Wolfgang Bühler and Jens Möller-Merbach, *Valuation of electricity futures: Reduced-form vs. dynamic equilibrium models*, Mannheim Finance Working Paper No. 2007-07, (2009).

[9] René Carmona, Michael Coulon, and Daniel Schwarz, *Electricity price modeling and asset valuation: a multi-fuel structural approach*, Mathematics and Financial Economics, 7 (2013), pp. 167–202.

[10] R. Carmona, M. Fehr, J. Hinz, and A. Porchet, *Market design for emission trading schemes*, SIAM Review, 52 (2010), pp. 403–452.

[11] Laura Cavallio and Valeria Termini, *Electricity derivatives and the spot market in Italy*, (2005).

[12] Frank H. Clarke, *Optimization and Nonsmooth Analysis*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, 1990.

[13] Les Clewlow and Chris Strickland, *A multi-factor model for energy derivatives*, Research Paper Series 28, Quantitative Finance Research Centre, University of Technology, Sydney, Dec. 1999.

[14] ———, *Valuing energy options in a one factor model fitted to forward prices*, Research Paper Series 10, Quantitative Finance Research Centre, University of Technology, Sydney, Apr. 1999.

[15] Gauthier De Maere d’Aertrycke and Yves Smeers, *Liquidity Risks on Power Exchanges: a Generalized Nash Equilibrium model*, 2012.

[16] Gerard Debreu, *A social equilibrium existence theorem*, Proceedings of the National Academy of Sciences of the United States of America, 38 (1952), p. 886–893.

[17] Ben Hambly, Sam Howison, and Tino Kluge, *Modelling spikes and pricing swing options in electricity markets*, Quantitative Finance, 9 (2009), p. 937–949.

[18] Sam Howison and Michael C. Coulon, *Stochastic behaviour of the electricity bid stack: From fundamental drivers to power prices*, The Journal of Energy Markets, 2 (2009).

[19] V. Jeyakumar and D. T. Luc, *Approximate jacobian matrices for nonsmooth continuous maps and c1-optimization*, SIAM J. Control Optim., 36 (1998), p. 1815–1832.

[20] Julio J. Lucia and Eduardo Schwartz, *Electricity prices and power derivatives: Evidence from the nordic power exchange*, (2000).

[21] M. Ludkovski, *Stochastic switching games and duopolistic competition in emissions markets*, SIAM Journal on Financial Mathematics, 2 (2011), pp. 488–511.

[22] Thilo Meyer-Brandis and Peter Tankov, *Multi-factor jump-diffusion models of electricity prices*, International Journal of Theoretical and Applied Finance (IJTAF), 11 (2008), pp. 503–528.

[23] Daniel Ralph and Stefan Scholtes, *Sensitivity analysis of composite piecewise smooth equations*, Math. Program., 76 (1997), p. 593–612.

[24] Sara Robinson, *Math model explains high prices in electricity markets*, SIAM News, (2005).

[25] Chen Yonglin, *The generalized bott-duffin inverse and its applications*, Linear Algebra and its Applications, 134 (1990), pp. 71 – 91.