Chapter 1

TRANSPORT IN SINGLE CHANNEL QUANTUM WIRES

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Abstract
This tutorial article gives an introduction to the methods needed to treat interacting electrons in a quantum wire with a single occupied band. Since one–dimensional Fermions cannot be described in terms of noninteracting quasiparticles, the Tomonaga–Luttinger model is presented in some detail with an emphasis on transport properties. To achieve a self–contained presentation, the Bosonization technique for one–dimensional Fermions is developed, accentuating features relevant for nonequilibrium systems. The screening of an impurity in the wire is discussed, and the insight gained on the electrostatics of a quantum wire is used to describe the coupling to Fermi–liquid reservoirs. These parts of the article should be readily accessible to students with a background in quantum mechanics including second quantization. To illustrate the usefulness of the methods presented, the current–voltage relation is determined exactly for a spin–polarized quantum wire with a particular value of the interaction parameter. This part requires familiarity with path integral techniques and connects with the current literature.

Keywords: Tomonaga–Luttinger liquid, Bosonization, Electronic transport properties in one dimension, impurity scattering, current–voltage relation.

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1. INTRODUCTION

Within the last decade there has been increased interest in the behavior of quasi one-dimensional Fermionic systems, due to significant advances in the fabrication of single channel quantum wires [1, 2, 3] based on semiconductor heterostructures and the observation of non-Fermi liquid behavior in carbon nanotubes [4, 5, 6]. While the unusual equilibrium properties of Fermions in one dimension have been studied since many decades and are well documented in review articles [7, 8], nonequilibrium quantum wires are an area of active research with many important question remaining to be answered.

In this article, we give a rather elementary introduction to the theoretical framework underlying much of the present studies on transport properties of one-dimensional Fermions. While we do not review extensively features of the Tomonaga–Luttinger model [9, 10] upon which these studies are based, we give an elementary introduction to the Bosonization technique which is an essential ingredient of current theoretical methods. We do not review the rather long history of Bosonization starting with the work by Schotte and Schotte [11] in 1969. Some important articles are contained in a book of reprints collected by Stone [12]. Our approach is based on Haldane’s algebraic Bosonization [13], which can be understood with the usual graduate level background in physics. For a more in-depth discussion of the method, we refer to a recent review by von Delft and Schoeller [14]. The field theoretical approach to Bosonization, which is probably harder to learn but easier to apply, has lately been expounded by Gogolin, Nersesyan and Tsvelik [15].

We employ the Bosonization technique to describe a quantum wire coupled to Fermi–liquid reservoirs. In this connection the electrostatic properties of the wire play an important role. Landauer’s approach [16] to transport in mesoscopic systems, which is based on Fermi liquid theory, is generalized to take the electronic correlations in a single channel quantum wire into account. In the Bosonized version of the model the coupling to reservoirs is shown to be described in terms of radiative boundary conditions [17]. This allows us to use the powerful Bosonization method also for nonequilibrium wires. About a decade ago, Kane and Fisher [18] have noted that transport properties of one-dimensional Fermions are strongly affected by impurities. Even weak impurities have a dramatic effect at sufficiently low temperatures leading to a zero bias anomaly of the conductance. It is the aim of the article to present the theoretical background necessary to study the recent literature on this subject. Again, we do not provide a review of transport properties of
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the Tomonaga–Luttinger model. Rather, the methods developed are illustrated by treating a particular case.

1.1. NONINTERACTING ELECTRONS IN ONE DIMENSION

Let us start by considering first noninteracting electrons of mass \( m \) moving along a one-dimensional wire with a scatterer at \( x = 0 \). For simplicity, the scattering potential is taken as a \( \delta \)-potential

\[
V_{sc}(x) = \frac{\hbar^2}{m} \Lambda \delta(x)
\]

where \( \Lambda \) characterizes the strength. The Schrödinger equation

\[
-\frac{\hbar^2}{2m} \psi''(x) + \frac{\hbar^2}{m} \Lambda \delta(x) \psi(x) = \varepsilon \psi(x)
\]

has for all positive energies

\[
\varepsilon_k = \frac{\hbar^2 k^2}{2m}
\]

a solution \( (k > 0) \)

\[
\psi_k(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} 
\frac{e^{ikx} + r_k e^{-ikx}}{t_k} & , x < 0 \\
\frac{t_k e^{ikx}}{} & , x > 0
\end{cases}
\]

describing a wave incident from the left that is partially transmitted and partially reflected. The transmission amplitude

\[
t_k = \frac{1}{1 + i\Lambda/k} = \sqrt{T_k} e^{i\eta_k}
\]

determines the transmission coefficient \( T_k \) and the phase shift \( \eta_k \). Likewise, there is a solution \( (k > 0) \)

\[
\psi_{-k}(x) = \psi_k(-x)
\]

describing a wave incident from the right.

When the ends of the wire are connected to electrodes with electrical potentials \( \mu_L \) and \( \mu_R \), a voltage

\[
U = (\mu_L - \mu_R)/e
\]

is applied to the wire, where \( e \) is the electron charge, and an electrical current

\[
I = e \int_{-\infty}^{+\infty} dk f_k j_k
\]
flows. Here

\[ j_k(x) = \text{Im} \psi_k^*(x) \frac{\hbar}{m} \frac{\partial}{\partial x} \psi_k(x) = \frac{\hbar k}{2\pi m} T_k \]  

(9)

is the particle current in state \( \psi_k \) which is independent of \( x \), and the

\[ f_k = \begin{cases} f(\epsilon_k - \mu_L), & k > 0 \\ f(\epsilon_k - \mu_R), & k < 0 \end{cases} \]  

(10)

with \( f(\epsilon) = 1/(e^{\beta \epsilon} + 1) \) are state occupation probabilities determined by the Fermi function of the electrode from which the particles come. Both electrodes are assumed to be at the same inverse temperature \( \beta \).

Putting \( \mu_{L,R} = \epsilon_F \pm \frac{1}{2}eU \), we readily find for small voltages \( U \)

\[ I = GU \]  

(11)

with the conductance

\[ G = \frac{e^2}{h} \int_0^\infty dk \frac{\hbar k}{2\pi m} T_k \frac{\partial}{\partial \epsilon} f(\epsilon_k - \epsilon_F) \]  

(12)

\[ = \frac{e^2}{h} \int_0^\infty d\epsilon T(\epsilon) \left[ -\frac{\partial}{\partial \epsilon} f(\epsilon - \epsilon_F) \right]. \]

Provided \( \beta \epsilon_F \) is large, this yields the Landauer formula for a single transport channel

\[ G = \frac{e^2}{h} T_F, \]  

(13)

where \( T_F \) is the transmission coefficient at the Fermi energy \( \epsilon_F \). If we take into account the spin degeneracy of real electrons, the conductance becomes multiplied by 2.

1.2. FANO–ANDERSON MODEL

Of course, to describe electrons in one dimension, we may also start from a tight binding model with localized electronic states at positions \( x_j = a j \) where \( a \) is the lattice constant. The Hamiltonian in the presence of an impurity at \( x = 0 \) then takes the form of the Fano-Anderson model [19, 20]

\[ H = \epsilon_0 a_0^\dagger a_0 - t \sum_{j=-\infty}^{+\infty} (a_{j+1}^\dagger a_j + \text{h.c.}), \]  

(14)

where \( t \) is the hopping matrix element and \( \epsilon_0 > 0 \) is the extra energy needed to occupy the perturbed site at \( x = 0 \). The \( a_j \) are Fermi operators obeying the usual anti-commutation relations. This model is also
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exactly solvable [21] with eigenstates in the energy band
\[ \varepsilon_k = -2t \cos(ka), \ |k| \leq \frac{\pi}{a}, \] (15)
and the transmission amplitude takes the form
\[ t_k = \frac{1}{1 + i/\lambda_k}, \] (16)
where
\[ \lambda_k = \frac{2t}{\varepsilon_0} \sin(ka) = \frac{\hbar v_k}{\varepsilon_0 a}. \] (17)
The last relation follows by virtue of \( v_k = (1/\hbar) \partial \varepsilon_k / \partial k \). It is now easily checked that this model leads to the same result for the linear conductance than the free electron model examined previously, provided we match parameters of the unperturbed models such that the Fermi velocities \( v_F \) at the two Fermi points coincide, and we adjust the strength of the \( \delta \)-function in Eq. (1) such that
\[ \hbar^2 m \Lambda = \varepsilon_0 a. \] (18)
Then, we end up with the same transition coefficient \( T_F \) at the Fermi energy.

1.3. NONINTERACTING TOMONAGA–LUTTINGER MODEL

Apparently, for low temperatures and small applied voltages, the conductance only depends on properties of states in the vicinity of the Fermi energy \( \varepsilon_F \). We can take advantage of this fact by introducing still another model, the noninteracting Tomonaga–Luttinger (TL) model, which has the same properties near the two Fermi points \( \pm k_F \) but is more convenient once we introduce electron interactions. Let us formally decompose the true energy dispersion curve \( \varepsilon_k \) into two branches + and − of right–moving and left–moving electrons, respectively, where these branches comprise states in the energy interval \( [\varepsilon_F - \Delta, \varepsilon_F + \Delta] \). (cf. Fig. 1.1). If \( \Delta \) is chosen large enough, these branches should suffice to describe the low energy physics of the true physical model, since for low temperatures and small applied voltages states with energy below \( \varepsilon_F - \Delta \) are always occupied while states above \( \varepsilon_F + \Delta \) are empty. The Hamiltonian of the noninteracting TL model reads
\[ H = \sum_p \sum_k \varepsilon_{p,k} \left[ c_{p,k}^\dagger c_{p,k} - \langle c_{p,k}^\dagger c_{p,k} \rangle_0 \right]. \] (19)
Figure 1.1. Physical energy dispersion curve $\epsilon_k$ (thin line) and the two branches $+,-$ (thick lines) of the TL model

Here $p = \pm$ labels the two branches. We have introduced a quantization length $L$ such that wave vectors are discrete

$$ k = \frac{2\pi}{L} n_k, \ n_k \text{ integer.} \quad (20) $$

Further, the $\epsilon_{\pm}(k)$ are single particle energies measured relative to the Fermi energy $\epsilon_F$, i.e., $\epsilon_+(k_F) = \epsilon_-(k_F) = 0$. The operators $c_{p,k}^\dagger$ and $c_{p,k}$ are Fermi creation and annihilation operators obeying anti-commutation relations, in particular

$$ \left[ c_{p,k} , c_{p',k'}^\dagger \right]_+ = \delta_{p,p'} \delta_{k,k'} \quad (21) $$

The sum over $k$ states in Eq. (19) is restricted to $k$ values near $\pm k_F$ such that $\epsilon_{\pm}(k) \in [-\Delta, \Delta]$. Finally, the ground state energy is subtracted in Eq. (19) where the ground state $|0,0\rangle_0$ is defined by

$$ c_{+,k} |0,0\rangle_0 = c_{-,k} |0,0\rangle_0 = 0 \quad \text{for } k > k_F, \quad (22) $$

$$ c_{+,k}^\dagger |0,0\rangle_0 = c_{-,k}^\dagger |0,0\rangle_0 = 0 \quad \text{for } k \leq k_F. $$

Frequently, the spectra $\epsilon_{\pm,k}$ are linearized about the Fermi points, i.e.,

$$ \epsilon_{\pm,k} = \pm \hbar v_F (k \mp k_F) \quad (23) $$

\footnote{We use periodic boundary conditions here and consider the limit of large $L$ in the sequel. The same techniques can also be used for other boundary conditions [14, 22].}
and then the cutoff energy $\Delta$ is used as a large energy scale regularizing divergent expressions. Since for low energy systems only inert empty or occupied states are added, an increase of $\Delta$ is admissible. However, the linearization of the spectra is only realistic in the close vicinity of the Fermi points. In fact, some phenomena not discussed here, e.g. the thermopower \cite{23}, depend on band curvature. We shall come back to the limitations of the linearization \eqref{23} below.

Before we discuss transport properties of the TL model, we first introduce methods, the advantage of which becomes apparent only when we pass on to the case of interacting electrons. These methods are independent of the precise dispersion law as long as the $\varepsilon_{p,k}$ are monotous functions of $k$.

2. BOSONIZATION

The noninteracting TL model allows for a formulation in terms of Bose operators. We discuss the Bosonization technique here for spinless Fermions first, and then extend it to the spinful case.

2.1. DENSITY OPERATORS AND THEIR ALGEBRA

While the physical problem of one–dimensional Fermions is described by a single energy dispersion curve $\varepsilon_k$ with empty states at both ends of the range of $k$ values, the TL model introduces two branches with empty states at one end but occupied states at the other end of the $k$ range. This leads to unusual algebraic properties we will discuss now. Since the range of allowed $k$ values plays an important role in this discussion, we keep track of it in detail by introducing

$$W_{p,k} = \begin{cases} 1 & \text{for } \varepsilon_{p,k} \in [-\Delta, \Delta] \\ 0 & \text{else} \end{cases}.$$ \hspace{1cm} (24)

Let us define Fourier components of the densities of $p$–movers ($p = \pm$) by

$$\tilde{\rho}_{p,q} = \sum_k W_{p,k} W_{p,k+q} c_{p,k}^\dagger c_{p,k+q}. $$ \hspace{1cm} (25)

Since Fermi operators on different branches anti–commute, we have

$$[\tilde{\rho}_{p,q}, \tilde{\rho}_{p',q'}]_\pm = 0 \text{ for } p \neq p'. $$ \hspace{1cm} (26)

On the other hand, for the commutator on the same, say the + branch, we find

$$[\tilde{\rho}_{+,q}, \tilde{\rho}_{+,q'}]_- = \sum_{kk'} W_{+,k} W_{+,k+q} W_{+,k'} W_{+,k'+q'} $$ \hspace{1cm} (27)
We now use anti–commutation relations for the second and third operator in each of the two products of four Fermi operators, e.g., $c_{+,k+q}c_{+,k'}c_{+,k'+q'} = \delta_{k+q,k'} - c_{+,k'}c_{+,k+q}$. Then, the remaining products of four Fermi operators are easily seen to cancel by virtue of the anti–commutation relations, and the two terms with two Fermi operators can be written as

$$\left[\hat{\rho}_{+,q}, \hat{\rho}_{+,q'}\right]_\approx \sum_k W_{+,k}W_{+,k+q+q'} \left[W_{+,k+q+q'}^2 - W_{+,k+q}^2\right] c_{+,k}c_{+,k+q+q'}.$$  

The terms of the sum are finite only for $W_{+,k} = W_{+,k+q+q'} = 1$. But then $W_{+,k+q}$ and $W_{+,k+q'}$ are also equal to 1 if $q$ and $q'$ have the same sign. In this case all terms of the sum (28) vanish. Hence, a nontrivial commutator can only arise if $q$ and $q'$ have different signs, say $q > 0$ and $q' < 0$. Then non–vanishing terms of the sum (28) may occur near the upper cutoff where $\varepsilon_{+,k} \approx \Delta$, in particular, when

$$W_{+,k} = W_{+,k+q+q'} = W_{+,k+q} = 1, \text{ but } W_{+,k+q} = 0,$$

as illustrated in Fig. 1.2. However, in this case the operator $c_{+,k}c_{+,k+q+q'}$ tries to annihilate a particle in the state $k + q + q'$ with an energy near $\Delta$. In the low energy sector of the model these states are always empty and $c_{+,k+q+q'}$ can be replaced by zero. Hence, for a low energy system we get no contribution to the commutator from states near the upper cutoff energy.

A contribution near the lower cutoff energy may arise from terms with $W_{+,k} = W_{+,k+q+q'} = W_{+,k+q} = 1$ but $W_{+,k+q'} = 0$ since $\varepsilon_{+,k+q'} < -\Delta$. In this case we use the anti–commutation relations to write Eq. (28) as

$$\left[\hat{\rho}_{+,q}, \hat{\rho}_{+,q'}\right]_\approx \sum_k W_{+,k}W_{+,k+q+q'} \left[W_{+,k+q+q'}^2 - W_{+,k+q}^2\right] \times \left[\delta_{q,q'} - c_{+,k+q+q'}c_{+,k}^\dagger\right].$$

With a similar argument as above, the operator $c_{+,k}^\dagger$ may now be replaced by zero since near the lower cutoff all states are occupied. We thus obtain

$$\left[\hat{\rho}_{+,q}, \hat{\rho}_{+,q'}\right]_\approx = \delta_{q,-q} \sum_{k < k_F} W_{+,k}^2 \left[W_{+,k+q}^2 - W_{+,k-q}^2\right],$$

where the condition $k < k_F$ ensures that we do not get contributions from the region near the upper cutoff where the Fermi operators in
Eq. (29) cannot be dropped. Finally, since

$$\sum_{k<k_F} W^2_{+,k} W^2_{+,k+q} - \sum_{k<k_F} W^2_{+,k-q} W^2_{+,k} = n_q,$$

(31)

where \( q = \frac{2\pi}{L} n_q \) determines the number \( n_q \) of additional non-vanishing terms in the first sum, and with a similar reasoning for the \(-\) branch and other signs of \( q \), we get

$$[\tilde{\rho}_{p,q}, \tilde{\rho}_{p',q'}]_\pm = p \delta_{p,p'} \delta_{q,-q'} n_q.$$

(32)

As we have seen this nontrivial commutator only arises since we have introduced two branches with empty states at one end and occupied states at the other end. The result (32) can also be derived in the same way when the sharp cutoff functions \( W_{\pm,k} \) defined in Eq. (24) are replaced by a smooth cutoff. Furthermore, the relation (32) is not an exact operator relation but holds only in the low energy sector of the Fock space. However, for technical convenience, we may use the linearized spectrum (23) and send the cutoff \( \Delta \) to infinity. We then obtain a model where the relations (32) hold as formally exact commutation relations but have to remember that only the low energy properties of this model are related to the physical problem of one-dimensional electrons.
2.2. BOSE OPERATORS AND BASIS VECTORS

With the result (32) it is now straightforward to introduce Bose annihilation and creation operators

\[ b_q = \frac{-i}{\sqrt{|n_q|}} \sum_p \theta(pq) \tilde{\rho}_{p,q}, \]

\[ b_q^\dagger = \frac{i}{\sqrt{|n_q|}} \sum_p \theta(pq) \tilde{\rho}_{p,-q}, \]

(33)

where \( q \neq 0 \). Inserting (25), we see that for \( q > 0 \) the operator \( b_q \) lowers the wave vector \( k \) of right–movers by \( q \), provided the state \( k - q \) is empty, cf. Fig. 1.3, while \( b_{-q} \) acts accordingly on left–movers. By virtue of Eq. (32), the \( b_q \) and \( b_q^\dagger \) satisfy Bose commutation relations, in particular

\[ [b_q, b_{q'}^\dagger]_- = \delta_{qq'}, \]

(34)

When acting on a \( N \)–particle state, the operators \( b_q^\dagger \) and \( b_q \) create and annihilate electron–hole pairs, respectively, but they conserve of course the particle numbers \( N_p \) of each branch. Consider now \( N_+ \)–particle ground states of right–movers where \( |N_+ = 0 \rangle_0 \) is the many–body state with all single particle states \( k \leq k_F \) occupied and all states \( k > k_F \) empty, while in the state \( |N_+ = 1 \rangle_0 \) also the level \( k = k_F + \frac{2\pi}{L} \) is

\(^2\)As usual, \( \theta(x) \) denotes the step function: \( \theta(x) = 1 \) for \( x > 0 \), \( \theta(x) = 0 \) for \( x < 0 \).
occupied, and so on, as illustrated in Fig. 1.4. One can then show that the states
\[
|N_+, \{m_q\}_{q>0} \rangle = \prod_{q>0} \left( b_q^+ \right)^{m_q} \sqrt{m_q!} |N_+ \rangle_0 ,
\]
with \( N_+ = 0, \pm 1, \pm 2, \ldots \) and \( m_q = 0, 1, 2, \ldots \) form a complete basis in the Fock space of right–movers. It is obvious that the states (35) are all within the Fock space spanned by the standard basis vectors \(|\{N_{+,k}\}\rangle\), where the \( N_{+,k} = 0, 1 \) are the usual Fermi occupation numbers, that is eigenvalues of \( c_{+,k}^+ c_{+,k} \). Haldane [13] has demonstrated completeness of the basis (35) with the help of the model Hamiltonian
\[
h_+ = \sum_k \left( n_k - \frac{1}{2} \right) c_{+,k}^+ c_{+,k} ,
\]
where
\[
: c_{+,k}^+ c_{+,k} : = \begin{cases} 
  c_{+,k}^+ c_{+,k} & \text{for } k > 0 \\
  c_{+,k} c_{+,k}^+ & \text{for } k \leq 0
\end{cases}.
\]
This Hamiltonian assigns positive energies to all states and allows for an explicit evaluation of the partition function \( Z = \sum_\alpha \exp(-\beta E_\alpha) \) for the two sets \( \{\alpha\} \) of basis vectors. Since \( Z \) is a sum of positive terms, and the sum over the states (35) gives the same result as the sum over the states \(|\{N_{+,k}\}\rangle\), the basis \(|N_+, \{m_q\}_{q>0} \rangle\) spans the entire Fock space of right–movers.
Likewise we may introduce a basis

$$|N_-, \{m_q\}_{q<0}\rangle = \prod_{q<0} \left( \frac{(b_q^\dagger)^{m_q}}{\sqrt{m_q!}} \right) |N_\rangle_0$$

(38)

in the Fock space of left–movers and combine both to the basis

$$|N_+, N_-, \{m_q\}\rangle = \prod_{q\neq 0} \left( \frac{(b_q^\dagger)^{m_q}}{\sqrt{m_q!}} \right) |N_+, N_\rangle_0$$

(39)

in the Fock space of one–dimensional particles.

### 2.3. LADDER AND PARTICLE NUMBER OPERATORS

It is clear that the Bose operators $b_q, b_q^\dagger$ cannot generate the whole algebra of operators in the Fock space since they preserve the particle numbers $N_p$. Hence, the Bose operators need to be supplemented by ladder operators removing or adding a particle of branch $p$. The lowering operators $U_p$ are defined by

$$U_p |N_p, N_{-p}\rangle_0 = p^{N_p}|N_p - 1, N_{-p}\rangle_0 ,$$

(40)

where the sign factor $p^{N_p}$ is one possible choice assuring anti–commutation relations for Fermi operators on different branches, and

$$[U_p, b_q]_- = [U_p, b_q^\dagger]_- = 0 .$$

(41)

The adjoint raising operator obeys $U_p^\dagger = U_p^{-1}$. The action of the ladder operator $U_+$ on a state in the Fock space of right–movers is illustrated in Fig. 1.5. Since the ladder operators commute with the Bose operators $b_q, b_q^\dagger$, they preserve the electron–hole pair excitations present in a state.

Now, the particle number $N_p$ is an eigenvalue of the particle operator

$$\hat{N}_p = \sum_k \left( c_{p,k}^\dagger c_{p,k} - \langle c_{p,k}^\dagger c_{p,k}\rangle_0 \right) ,$$

(42)

where the average $\langle \rangle_0$ is over the $|N_p = 0\rangle_0$ ground state. The ladder operators and the particle numbers satisfy the commutation relations

$$\left[ \hat{N}_p, U_{p'} \right]_- = -\delta_{p,p'} U_p ,$$

$$\left[ \hat{N}_p, U_{p'}^\dagger \right]_- = \delta_{p,p'} U_p^\dagger .$$

(43)
Figure 1.5. Illustration of the action of $U_+$ on a many-body state $|\psi\rangle$ of right-movers.

that are easily shown by applying them to an arbitrary vector from the basis set (39). Furthermore, the ladder operators obey

$$\begin{align*} 
\left[ U_+, U_- \right]_+ &= \left[ U_+^\dagger, U_-^\dagger \right]_+ = 0, \\
\left[ U_p, U_p^\dagger \right]_+ &= 2 \delta_{p,p'}, \quad (44) 
\end{align*}$$

which may be demonstrated in the same way taking the sign factor in Eq. (40) into account.

2.4. BOSONIC PHASE FIELDS

With the Bose and ladder operators we should be able to represent all operators in the Fock space of the TL model. To demonstrate this one usually constructs explicitly the Fermi annihilation operators

$$\Psi_p(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} c_{p,k}. \quad (45)$$

It is convenient to first introduce Bosonic phase fields

$$\varphi_p(x) = \frac{1}{2\pi} \sum_{q \neq 0} \theta(pq) \frac{1}{\sqrt{|n_q|}} e^{iqx} b_q, \quad (46)$$

and associated Hermitian phase fields

$$\Phi_p(x) = \varphi_p(x) + \varphi_p^\dagger(x). \quad (47)$$
These are related to the densities

$$\rho_p(x) = \Psi_p^{\dagger}(x) \Psi_p(x) - \langle \Psi_p^{\dagger}(x) \Psi_p(x) \rangle_0 = \frac{1}{L} \left[ \sum_{q \neq 0} e^{iqx} \tilde{\rho}_{p,q} + \tilde{N}_p \right], \quad (48)$$

where the \( \tilde{\rho}_{p,q} = \sum_k c_{p,k}^\dagger c_{p,k+q} \) are the Fourier components studied above.

Using Eq. (33), the phase field (46) can also be written as

$$\varphi_p(x) = -\frac{i}{2\pi} \sum_{q \neq 0} \theta(pq) \frac{1}{|n_q|} e^{iqx} \tilde{\rho}_{p,q}. \quad (49)$$

Because of the factor \( \theta(pq) \), we may replace \( |n_q| \) by \( pn_q = \frac{L}{2\pi} pq \). Then, adding the Hermitian conjugate field, we readily find for the gradient

$$\frac{\partial}{\partial x} \Phi_p(x) = \frac{p}{L} \sum_{q \neq 0} e^{iqx} \tilde{\rho}_{p,q} = p \left( \rho_p(x) - \frac{1}{L} \tilde{N}_p \right). \quad (50)$$

The Bosonic phase fields \( \varphi(x), \varphi^{\dagger}(x) \) obey the commutation relations

$$[\varphi_p(x), \varphi_{p'}^{\dagger}(x')]_- = [\varphi_p^{\dagger}(x), \varphi_{p'}^{\dagger}(x')]_- = 0 \quad (51)$$

while by virtue of Eqs. (34) and (46) the commutator

$$[\varphi_p(x), \varphi_{p'}^{\dagger}(x')]_- = \frac{\delta_{p,p'}}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{ip\frac{2\pi}{L}(x-x')n}. \quad (52)$$

Since

$$\sum_{n=1}^{\infty} \frac{y^n}{n} = -\ln(1 - y), \quad (53)$$

the right hand side of Eq. (52) is logarithmically divergent for \( x = x' \). However, we have to remember that the sum over \( n \) originates from a sum over wave vectors \( q = (2\pi/L)n \) which due to the energy cutoff are restricted to small \( q \). The effect of a cutoff is seen when we multiply the terms of the sum (52) by an exponential cutoff function \( \exp(-aq) = \exp(-\frac{2\pi}{L} an) \) which limits the sum to wave vectors \( q \lesssim 1/a \), where \( a \) is a cutoff length related to a cutoff energy \( \Delta \approx \hbar v_F/a \). We then obtain

$$[\varphi_p(x), \varphi_{p'}^{\dagger}(x')]_- = -\frac{\delta_{p,p'}}{4\pi^2} \ln \left( 1 - \exp \left\{ -\frac{2\pi}{L} \left[ a - ip(x-x') \right] \right\} \right)$$

$$= -\frac{\delta_{p,p'}}{4\pi^2} \ln \left( \frac{2\pi}{L} \left[ a - ip(x-x') \right] \right) \quad (54)$$
where the second equality holds for \( L \gg a, |x-x'| \). For later convenience we note that in particular the commutator for vanishing distance
\[
\left[ \varphi_p(x), \varphi_{p'}(x) \right]_\pm = -\frac{\delta_{pp'}}{4\pi^2} \ln \frac{2\pi a}{L} \tag{55}
\]
is cutoff and size dependent.

Simpler algebraic properties are found for the Hermitian phase field \((47)\). From Eqs. \((51)\) and \((54)\) we readily find
\[
\left[ \Phi_p(x), \Phi_{p'}(x') \right]_\pm = -\frac{\delta_{pp'}}{4\pi^2} \ln \frac{a - ip(x-x')}{a + ip(x-x')} \tag{56}
\]
where \( L \) has dropped out, and we have for \(|x-x'| \gg a\)
\[
\left[ \Phi_p(x), \Phi_{p'}(x') \right]_\pm = \frac{ip\delta_{pp'}}{4\pi} \text{sign}(x-x') \tag{57}
\]
Here \text{sign}(x-x') is the sign of \( x-x' \) which produces a step at \( x=x' \) that in the presence of a cutoff is smeared over a length of order \( a \).

Instead of the fields \( \Phi_{\pm}(x) \) we shall mainly use the linear combinations
\[
\phi(x) = \sqrt{\pi} \left[ \Phi_+(x) + \Phi_-(x) \right] \\
\vartheta(x) = \sqrt{\pi} \left[ \Phi_+(x) - \Phi_-(x) \right] \tag{58}
\]
that are readily seen to obey the commutation relations
\[
\left[ \phi(x), \phi(x') \right]_\pm = \left[ \vartheta(x), \vartheta(x') \right]_\pm = 0 \tag{59}
\]
and
\[
\left[ \phi(x), \vartheta(x') \right]_\pm = \frac{i}{2} \text{sign}(x-x') \tag{60}
\]
which shows that
\[
\Pi_\phi(x) = -\hbar \frac{\partial}{\partial x} \phi(x) \tag{61}
\]
is the field conjugate to \( \vartheta(x) \) with the canonical commutator
\[
\left[ \vartheta(x), \Pi_\phi(x') \right] = i\hbar \delta(x-x') \tag{62}
\]
Likewise \( \Pi_\vartheta(x) = -\hbar \frac{\partial}{\partial x} \vartheta(x) \) is the field canonically conjugate to \( \phi(x) \). Finally, we note that by virtue of Eq. \((50)\)
\[
\frac{1}{\sqrt{\pi}} \frac{\partial}{\partial x} \vartheta(x) = \rho_+(x) + \rho_-(x) - \frac{\hat{N}_+ - \hat{N}_-}{L} \tag{63}
\]
describes the fluctuations of the total density of right– and left–movers.
2.5. **BOSE REPRESENTATION OF FERMI OPERATORS**

To construct the Fermi operators $\Psi_p(x)$ in terms of the Bosonic fields, we start by evaluating the commutator

$$[b_q, \Psi_p(x)]_\pm = \alpha_{p,q}(x) \Psi_p(x), \quad (64)$$

where

$$\alpha_{p,q}(x) = \frac{i}{\sqrt{|n_q|}} \theta(pq) e^{-i qx}. \quad (65)$$

This follows readily by inserting Eqs. (33) and (45) into (64) with the help of the Fermi commutator $[\tilde{\rho}_{p',q}, c_{p,k}]_\pm = -\delta_{p,p'} c_{p,k+q}$. In the same way we find

$$[b_q^\dagger, \Psi_p(x)]_\pm = \alpha^*_{p,q}(x) \Psi_p(x). \quad (66)$$

Operating with $b_q \Psi_p(x)$ onto a $N$–particle ground state $|N_+, N_-\rangle_0$ we have

$$b_q \Psi_p(x) |N_+, N_-\rangle_0 = [b_q, \Psi_p(x)]_\pm |N_+, N_-\rangle_0$$

$$= \alpha_{p,q}(x) \Psi_p(x) |N_+, N_-\rangle_0,$$

where the first relation holds because $b_q |N_+, N_-\rangle_0 = 0$. Now, Eq. (67) shows that $\Psi_p(x) |N_+, N_-\rangle_0$ is an eigenstate of $b_q$ with eigenvalue $\alpha_{p,q}(x)$. Eigenstates of Bose annihilation operators are known as coherent states, and from their familiar properties it follows that $\Psi_p(x) |N_+, N_-\rangle_0$ is of the form

$$\Psi_p(x) |N_+, N_-\rangle = \lambda_p(x) \exp \left( \sum_{q \neq 0} \alpha_{p,q}(x) b_q^\dagger \right) |N_p - 1, N_{-p}\rangle_0, \quad (68)$$

where $\lambda_p(x)$ is an as yet undetermined normalization factor, and we have taken into account that $\Psi_p(x)$ reduces the number $N_p$ of $p$–movers by 1.

With the help of the Bose commutator (34), which implies

$$\left[ b_q, \exp \left( \sum_{q' \neq 0} \alpha_{p,q'}(x) b_{q'}^\dagger \right) \right]_\pm = \alpha_{p,q}(x) \exp \left( \sum_{q' \neq 0} \alpha_{p,q'}(x) b_{q'}^\dagger \right), \quad (69)$$

it is readily seen that the ansatz (68) indeed satisfies Eq. (67). From the definition (46) of the Bosonic phase field $\varphi_p(x)$ and Eq. (65) we find

$$\sum_{q \neq 0} \alpha_{p,q}(x) b_q^\dagger = 2\pi i \varphi_p^\dagger(x), \quad (70)$$
and hence from Eq. (68)

\[ 0\langle N_p - 1, N_- | \Psi_p(x) | N_p, N_- \rangle_0 = \lambda_p(x), \] (71)

where we have made use of

\[ 0\langle N_p - 1, N_- | \exp \left[ 2\pi i \varphi_p^0(x) \right] | N_p - 1, N_- \rangle_0 = 1, \] (72)

which is easily seen by expanding the exponential. We now insert the representation (45) of \( \Psi_p(x) \) into Eq. (71) and use

\[ 0\langle N_p - 1, N_- | c_{p,k} | N_p, N_- \rangle_0 = p^{N_+} \delta_{k,k_p}, \] (73)

where

\[ k_p = p \left( k_F + \frac{2\pi}{L} N_p \right). \] (74)

The sign factor \( p^{N_+} \) arises from the anti–commutation relations of Fermi operators on different branches in accordance with Eq. (40). This yields

\[ \lambda_p(x) = \frac{1}{\sqrt{L}} p^{N_+} e^{ip(k_F + \frac{2\pi}{L} N_p)x}, \] (75)

which can now be combined with the definition (40) of the lowering operator to obtain from Eq. (68)

\[ \Psi_p(x) | N_+, N_- \rangle_0 = \frac{1}{\sqrt{L}} e^{ip(k_F + \frac{2\pi}{L} N_p)x + 2\pi i \varphi_p^0(x)} U_p | N_+, N_- \rangle_0. \] (76)

This represents the action of a Fermi annihilation operator on a \( N \)–particle ground state in terms of Bose and ladder operators. Since particle–hole excitations are created by Bose operators, it is straightforward to generalize Eq. (76) for arbitrary states in the Fock space. For a basis vector (39) we have as a consequence of the commutator (66)

\[ \Psi_p(x) | N_+, N_- , \{ m_q \} \rangle = \sum_{q \neq 0} \frac{[b_q^\dagger - \alpha_{p,q}^*(x)]^{m_q}}{\sqrt{m_q!}} | N_+, N_- \rangle_0. \] (77)

Using Eq. (76), we find

\[ \Psi_p(x) | N_+, N_- , \{ m_q \} \rangle = \frac{1}{\sqrt{L}} U_p e^{ip(k_F + \frac{2\pi}{L} N_p)x + 2\pi i \varphi_p^0(x)} \sum_{q \neq 0} \frac{[b_q^\dagger - \alpha_{p,q}^*(x)]^{m_q}}{\sqrt{m_q!}} | N_+, N_- \rangle_0, \] (78)
since the Bose creation operators commute with $\varphi^\dagger_p(x)$ and $U_p$. In view of
\[ \sum_{q \neq 0} \alpha^*_p q(x) b_q = -2\pi i \varphi_p(x) \] (79)
and the familiar Bose relation $e^{ab} b^\dagger e^{-ab} = b^\dagger + \alpha$ we have
\[ e^{2\pi i \varphi_p(x)} b^\dagger_q e^{-2\pi i \varphi_p(x)} = b^\dagger_q - \alpha^*_p q(x), \] (80)
and Eq. (78) may be written as
\[ \Psi_p(x)|N_+, N_-, \{m_q\}\rangle = \frac{1}{\sqrt{2}} \left( U_p e^{ip(k_F + \frac{2\pi}{L} \hat{N}_p)x} e^{2\pi i \varphi_p^\dagger(x)} e^{2\pi i \varphi_p(x)} \right) |N_+, N_-, \{m_q\}\rangle. \] (81)
Here we have made use of $\varphi_p(x)|N_+, N_-\rangle_0 = 0$, and have replaced in the exponent $N_p$ by the operator $\hat{N}_p$, which is appropriate since $\hat{N}_p$ commutes with the Bose operators and thus acts on an eigenstate with eigenvalue $N_p$. Now, the operator acting on the basis vector on the right hand side of Eq. (81) has the same form for any vector, and we thus obtain the operator identity
\[ \Psi_p(x) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi a}} U_p e^{ip(k_F + \frac{2\pi}{L} \hat{N}_p)x} e^{2\pi i \varphi_p^\dagger(x)} e^{2\pi i \varphi_p(x)}. \] (82)
This is the desired representation of the Fermi operators $\Psi_p(x)$ in terms of Bose and ladder operators. The result (82) is in normal ordered form with all Bose creation operators to the left of the annihilation operators. It should be noted that this remarkable relation is independent of the form of the Hamiltonian, it just relates the form (45) of the Fermi operator with an obvious meaning in the standard occupation number basis $|\{N_+, k\}, \{N_- k\}\rangle$ of the Fock space to a representation with an obvious interpretation in the basis $|N_+, N_-, \{m_q\}\rangle$.

For much of the following discussions another form of $\Psi_p(x)$ is often more convenient. Using the commutator (55) and the operator relation
\[ e^A e^B = e^{A + B} e^{\frac{1}{2}[A, B]}, \] (83)
which holds if $[A, B]$ commutes with $A$ and $B$, we may transform the relation (82) to read
\[ \Psi_p(x) = \frac{1}{\sqrt{2\pi a}} U_p e^{ip[k_F x + \frac{2\pi}{L} \hat{N}_p + \sqrt{\pi} \varphi_p(x)] + i\sqrt{\pi} \varphi_p(x)} \] (84)
where $\Phi_p(x)$ was introduced in Eq. (47). Finally, in terms of the phase fields $\phi(x)$ and $\vartheta(x)$ defined in Eq. (58), we have
\[ \Psi_p(x) = \frac{1}{\sqrt{2\pi a}} U_p e^{ip[k_F x + \frac{2\pi}{L} \hat{N}_p + \sqrt{\pi} \varphi_p(x)] + i\sqrt{\pi} \phi(x)} \] (85)
2.6. BOSE REPRESENTATION OF THE HAMILTONIAN

In the following, we shall make use explicitly of the Hamiltonian (19) with the linearized energy dispersion curves (23). From Eq. (33) we see that the Bose creation operators may be written as

$$b_q^\dagger = \frac{i}{\sqrt{|n_q|}} \sum_{pk} \theta(pq) c_{p,k+q}^\dagger c_{p,k}. \tag{86}$$

With the help of the Fermi anti-commutation relations we then find

$$[H_0, b_q^\dagger] = \hbar v_F |q| b_q^\dagger, \tag{87}$$

where we have also made use of \(\varepsilon_{p,k+q} - \varepsilon_{p,k} = p\hbar v_F q\) which holds for the linear spectrum (23). Now, a \(N\)–particle ground state \(|N_+, N_-\rangle_0\) is an eigenstate of \(H_0\) with the energy

$$E_0(N_+, N_-) = \hbar v_F \frac{\pi}{L} \sum_p N_p(N_p + 1). \tag{88}$$

Since \(|0, 0\rangle_0\) has zero energy by definition and the single particle energies (23) are counted relative to the Fermi energy, the result (88) is obtained readily by adding for \(N_p > 0\) the single particle energies of the states occupied additionally in the \(N\)–particle ground state \(|N_+, N_-\rangle_0\), while for \(N_p < 0\) we have to subtract the (negative) energies of the particles removed. Using the commutator (87) we obtain for an arbitrary basis vector (39)

$$H_0 \prod_{q \neq 0} \left( b_q^\dagger \right)^{m_q} |N_+, N_-\rangle_0$$

$$= \prod_{q \neq 0} \left( b_q^\dagger \right)^{m_q} \left( H_0 + \sum_{q \neq 0} \hbar v_F |q| m_q \right) |N_+, N_-\rangle_0 , \tag{89}$$

which shows that \(|N_+, N_-, \{m_q\}\rangle\) is an eigenstate of \(H_0\) with eigenvalue

$$E_0(N_+, N_-, \{m_q\}) = E_0(N_+, N_-) + \hbar v_F \sum_{q \neq 0} |q|m_q. \tag{90}$$

---

3Some authors use non–standard definitions of the Fermi annihilation operator fields (45). To compare with our notation, one has to make proper replacements, e.g., replace \(x\) by \(-x\). In addition, the fields \(\vartheta(x)\) and \(\phi(x)\) are sometimes defined the other way round.
Since the basis (39) is complete and the basis vectors are clearly eigenstates of
\[ H_0 = \hbar v_F \left[ \sum_{q \neq 0} |q| b_q^\dagger b_q + \frac{\pi}{L} \sum_p \hat{N}_p (\hat{N}_p + 1) \right] \] (91)
with the proper eigenvalues (90), we see that Eq. (91) gives indeed a Bose representation of the Hamiltonian (19).

Combining Eqs. (33) and (50), we may write the gradient of the Bosonic phase field \( \Phi_p(x) \) as
\[ \frac{\partial}{\partial x} \Phi_p(x) = \frac{p}{\sqrt{2\pi L}} \sum_{q \neq 0} \theta(pq) \sqrt{|q|} \left[ i e^{iqx} b_q + \text{h.c.} \right], \] (92)
which gives
\[ \sum_p \int_{-L/2}^{L/2} dx \left( \frac{\partial \Phi_p(x)}{\partial x} \right)^2 = \frac{1}{2\pi} \sum_{q \neq 0} |q| \left( b_q b_q^\dagger + b_q^\dagger b_q \right). \] (93)

The Hamiltonian (91) may thus be written as
\[ H_0 = \pi \hbar v_F \int_{-L/2}^{L/2} dx \sum_p \left[ : \left( \frac{\partial \Phi_p(x)}{\partial x} \right)^2 : + \frac{1}{L^2} \hat{N}_p (\hat{N}_p + 1) \right], \] (94)
where \( : : \) puts the Bose operators in normal order. Further, in terms of the fields (58) this reads
\[ H_0 = \frac{\hbar v_F}{2} \int_{-L/2}^{L/2} dx \left[ : \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \vartheta}{\partial x} \right)^2 : + \frac{2\pi}{L^2} \sum_p \hat{N}_p (\hat{N}_p + 1) \right]. \] (95)

Since \( \Pi_\varphi = -\hbar \partial \varphi / \partial x \) is the conjugate density to the phase field \( \varphi \) with the canonical commutator (62), we finally obtain the \( \varphi \)–representation of the Hamiltonian
\[ H_0 = \frac{\hbar v_F}{2} \int_{-L/2}^{L/2} dx \left[ : \frac{1}{\hbar^2} \Pi_\varphi^2 : + \left( \frac{\partial \varphi}{\partial x} \right)^2 : + \frac{2\pi}{L^2} \sum_p \hat{N}_p (\hat{N}_p + 1) \right]. \] (96)
Likewise, using \( \Pi_\vartheta = -\hbar \partial \vartheta / \partial x \), we can readily write down a \( \vartheta \)–representation of \( H_0 \).
2.7. ACTION FUNCTIONAL

In the usual way, we may introduce a Lagrangian

\[ L_0 = \int_{-L/2}^{L/2} dx \Pi \partial \vartheta \partial t - H_0, \]  

(97)

where the time rate of change of the \( \vartheta \)-field reads

\[ \frac{\partial}{\partial t} \vartheta = \frac{i}{\hbar} [H_0, \vartheta] = -v_F \frac{\partial}{\partial x} \vartheta = \frac{v_F}{\hbar} \Pi \vartheta, \]  

(98)

which follows from the commutation relations (60). In the limit \( L \to \infty \) this gives

\[ L_0 = \hbar \int dx \left[ \frac{1}{2v_F} \left( \frac{\partial \vartheta}{\partial t} \right)^2 - \frac{v_F}{2} \left( \frac{\partial \vartheta}{\partial x} \right)^2 - \pi v_F \left( \overline{\rho}_p^2 + \overline{\rho}_p^2 \right) \right], \]  

(99)

where

\[ \overline{\rho}_p = \frac{N_p}{L} \]  

(100)

is the average density of \( p \)-movers. Note that by definition the densities \( \overline{\rho}_p \) vanish in the ground state \( \vert 0, 0 \rangle_0 \), where the Fermi levels are at \( \pm k_F \). The \( \overline{\rho}_p \) determine a shift of the Fermi points, while the density fluctuations \( \rho_p(x) - \overline{\rho}_p \) arising from electron–hole pair excitations are described by the phase field \( \vartheta \). However, we may define a shifted phase field

\[ \vartheta' = \vartheta + \sqrt{\pi} \left( \overline{\rho}_p^+ + \overline{\rho}_p^- \right) x - \sqrt{\pi} \left( \overline{\rho}_p^+ - \overline{\rho}_p^- \right) v_F t \]  

(101)

with the properties

\[ \frac{1}{\sqrt{\pi}} \frac{\partial \vartheta'}{\partial x} = \rho_+ + \rho_- \]  

(102)

\[ \frac{1}{\sqrt{\pi}} \frac{\partial \vartheta'}{\partial t} = -v_F (\rho_+ - \rho_-). \]

Now, the gradient determines the total density of right– and left–movers including the ground state density \( \overline{\rho}_p^+ + \overline{\rho}_p^- \), while the time rate of change is proportional to the particle current. The Lagrangian then takes the simple form

\[ L_0 = \hbar \int dx \left[ \frac{1}{v_F} \left( \frac{\partial \vartheta}{\partial t} \right)^2 - v_F \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right], \]  

(103)
where we have omitted the prime. The noninteracting TL model with dispersionless spectrum (23) can thus be characterized by the classical action functional

$$S_0 = \frac{\hbar}{2} \int dt \int dx \left[ \frac{1}{v_F} \left( \frac{\partial \vartheta}{\partial t} \right)^2 - v_F \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right], \quad (104)$$

which is the action of a harmonic string with wave velocity $v_F$ and a dimensionless displacement field $\vartheta(x, t)$ measured in units of $(\hbar/v_F \mu)^{1/2}$, where $\mu$ is the mass density in the string. From this mechanical analogue it is obvious that Bosonization provides an alternative description of one-dimensional Fermions in terms of charge density oscillations rather than electron–hole pair excitations.

We remark that in the general case of an arbitrary dispersion law $\varepsilon_{p,k}$ the Bosonization identities, in particular the representation (85) of the Fermi operators remain valid, however, the Hamiltonian $H_0$ is no longer quadratic in the Bose operators $b_q, b_q^\dagger$. As a consequence, the mechanical analogue will be an anharmonic string which may, of course, be treated in the harmonic approximation when we restrict ourselves to low energy excitations. In combination with the Feynman path integral representation the classical action functional (104) can be a convenient starting point for quantum mechanical calculations.

### 2.8. ELECTRON DENSITY OPERATOR

When we use the TL model to make predictions for one-dimensional fermions, we have to remember that the model provides a local approximation to the physical model in the vicinity of the Fermi points, while there is only a single energy dispersion curve for real electrons, cf. Fig. 1.1. This has consequences for observables like the density operator

$$\rho(x) = \frac{1}{L} \sum_{k,k'} e^{-i(k-k')x} \left( a_k^\dagger a_{k'} - \langle a_k^\dagger a_{k'} \rangle_0 \right), \quad (105)$$

where the $a_k, a_k^\dagger$ are Fermi operators for the underlying physical model with a single branch, and we have subtracted the constant density of the ground state with the Fermi points at $\pm k_F$. When we restrict ourselves to low energy states, the operators $a_k^\dagger a_{k'}$ can give a nonvanishing contribution only if both wave vectors $k$ and $k'$ are in the vicinity of the two Fermi points $\pm k_F$. However, for $k \approx pk_F, k' \approx p'k_F, (p, p' = \pm)$, the operators $a_k^\dagger, a_{k'}^\dagger$ may be replaced by the Fermi operators $c_{p,k}^\dagger, c_{p',k'}^\dagger$ of the TL model, and Eq. (105) splits into four terms

$$\rho(x) = \sum_{pp'} \frac{1}{L} \sum_{kk'} e^{-i(k-k')x} \left( c_{p,k}^\dagger c_{p',k'}^\dagger - \langle c_{p,k}^\dagger c_{p',k'}^\dagger \rangle_0 \right)$$
\[
\sum_{pp'} \left( \Psi_p^\dagger(x) \Psi_{p'}(x) - \langle \Psi_p^\dagger(x) \Psi_{p'}(x) \rangle_0 \right),
\] (106)

where we have used Eq. (45) to obtain the second line. Now, the diagonal terms \((p = p')\) just give the densities (48) of \(p\)–movers, while for the non–diagonal terms we employ the representation (85) to find

\[
\Psi_p^\dagger(x) \Psi_{-p}(x) = \frac{1}{2\pi a} e^{-2ipk_F x} e^{-2ip\sqrt{\pi} \vartheta(x)} e^{-ip\frac{2x}{\pi}} (\hat{N}_p + \hat{N}_{-p} + 1) U_p^\dagger U_{-p},
\] (107)

where we have used the commutators (44) and the fact that \(\vartheta(x)\) and \(\phi(x)\) commute at the same position \(x\), which is seen from Eq. (60). Note that in the presence of a cutoff the sign function in Eq. (60) takes the form of the right hand side of Eq. (56) and therefore vanishes for vanishing argument. The density (106) may thus be written as

\[
\rho(x) = \rho_+(x) + \rho_-(x) + \rho_{2k_F}(x),
\] (108)

where

\[
\rho_{2k_F}(x) = \frac{1}{2\pi a} \left( e^{-2i[k_F x + \sqrt{\pi} \vartheta(x)] + \frac{2}{\pi} (N_p + N_{-p} + 1)} \right) U_p^\dagger U_{-p} + \text{h.c.}.
\] (109)

Hence, the density operator for real electrons is not just the sum of the densities in the two branches of the TL model, but there is an additional \(2k_F\)-component \(\rho_{2k_F}(x)\) which comes from the fact that right– and left–movers are propagating in the same channel and interfere.

In the limit \(L \to \infty\) with constant average densities (100), we may introduce the shifted phase field (101) with \(t = 0\) since the expression (108) gives the operator in the Schrödinger picture. We then have in view of Eq. (102)

\[
\rho(x) = \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial x} \vartheta(x) + \rho_{2k_F}(x),
\] (110)

where we have again suppressed the prime on \(\vartheta\) which now contains the terms in the exponent of Eq. (109) that depend on the particle numbers. The operators \(U_p^\dagger U_{-p}\) in Eq. (109) associated with the scattering of an electron from the \((-p)\)– into the \(p\)–branch can often be suppressed since changes of the particle numbers \(N_p\) by 1 can be neglected for \(L \to \infty\). Then \(\rho_{2k_F}(x)\) takes the simple form

\[
\rho_{2k_F}(x) = \frac{k_F}{\pi} \cos \left[ 2k_F x + 2\sqrt{\pi} \vartheta(x) \right],
\] (111)

where we have chosen a cutoff length \(a = k_F^{-1}\) of order a typical microscopic length.
2.9. FERMIONS WITH SPIN

We now briefly summarize the modifications necessary to include the electron spin. Then, the electron spectrum has two branches \( s = \uparrow, \downarrow \) for spin up and spin down particles. For each species \( s \), we may proceed exactly as for spinless fermions and define Bosonic phase fields \( \vartheta_s(x) \) and \( \phi_s(x) \). There are now four branches of the TL model and correspondingly four particle number operators \( \hat{N}_{p,s} \). When defining the ladder operators, we have to include appropriate sign factors in the generalization of Eq. (40) to assure anti-commutation relations

\[
[U_{p,s}, U_{p',s'}]_+ = 2 \delta_{p,p'} \delta_{s,s'} (U_{p,s})^2
\]

\[
[U_{p,s}, U_{p',s'}^+]_+ = 2 \delta_{p,p'} \delta_{s,s'} (U_{p,s}^+)^2
\]

\[
[U_{p,s}, U_{p',s'}^+]_+ = 2 \delta_{p,p'} \delta_{s,s'}
\]

that extend the relations (44) to the case of four branches. The Bose representation (85) of the Fermi operators takes again the same form for each species \( s \), i.e.,

\[
\Psi_{p,s}(x) = \frac{1}{\sqrt{2\pi a}} U_{p,s} e^{i[p_i x + \frac{2\pi}{L} \hat{N}_{p,s} x + \sqrt{\pi \vartheta_s(x)}] + i\sqrt{\pi \phi_s(x)}}.
\]

(113)

It is often convenient to transform to the phase fields

\[
\vartheta_{\rho}(x) = \frac{1}{\sqrt{2}} [\vartheta_{\uparrow}(x) + \vartheta_{\downarrow}(x)]
\]

(114)

\[
\phi_{\rho}(x) = \frac{1}{\sqrt{2}} [\phi_{\uparrow}(x) + \phi_{\downarrow}(x)]
\]

and

\[
\vartheta_{\sigma}(x) = \frac{1}{\sqrt{2}} [\vartheta_{\uparrow}(x) - \vartheta_{\downarrow}(x)]
\]

(115)

\[
\phi_{\sigma}(x) = \frac{1}{\sqrt{2}} [\phi_{\uparrow}(x) - \phi_{\downarrow}(x)]
\]

that satisfy the commutation relations

\[
[\vartheta_{\rho}(x), \vartheta_{\rho}(x')]_+ = [\vartheta_{\sigma}(x), \vartheta_{\sigma}(x')]_+ = 0
\]
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\[ \left[ \phi_\rho(x), \phi_\rho(x') \right]_\text{-} = \left[ \phi_\sigma(x), \phi_\sigma(x') \right]_\text{-} = 0 \]

\[ \left[ \partial_\rho(x), \partial_\sigma(x') \right]_\text{-} = \left[ \phi_\rho(x), \phi_\sigma(x') \right]_\text{-} = 0 \]

\[ \left[ \partial_\rho(x), \phi_\sigma(x') \right]_\text{-} = 0 \]

and

\[ \left[ \partial_\rho(x), \phi_\rho(x') \right] = \left[ \partial_\sigma(x), \phi_\sigma(x') \right] = \frac{i}{2} \text{sign}(x - x'). \]

These fields describe charge and spin density excitations. In particular, instead of Eq. (63) we now have

\[ \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial x} \theta_\rho(x) = \rho_{+,\uparrow}(x) + \rho_{+,\downarrow}(x) + \rho_{-,\uparrow}(x) + \rho_{-,\downarrow}(x) \]

\[ - \frac{\hat{N}_{+,\uparrow} + \hat{N}_{+,\downarrow} + \hat{N}_{-,\uparrow} + \hat{N}_{-,\downarrow}}{L}, \]

which gives the fluctuations of the total particle density of right- and left-movers, while

\[ \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial x} \theta_\sigma(x) = \rho_{+,\uparrow}(x) + \rho_{-,\uparrow}(x) - \rho_{+,\downarrow}(x) - \rho_{-,\downarrow}(x) \]

\[ - \frac{\hat{N}_{+,\uparrow} + \hat{N}_{-,\downarrow} - \hat{N}_{+,\downarrow} - \hat{N}_{-,\uparrow}}{L} \]

determines fluctuations of the spin density. In the limit \( L \to \infty \) with given average densities

\[ \bar{\rho}_{p,s} = \frac{N_{p,s}}{L} \]

we may again introduce shifted phase fields \( \theta'_s \) for each species \( s \) according to Eq. (101) and then obtain the classical action functional of the noninteracting model with dispersionless energy spectrum as

\[ S_0 = \frac{\hbar}{2} \sum_{\alpha} \int dt \int dx \left[ \frac{1}{v_F} \left( \frac{\partial \theta_\alpha}{\partial t} \right)^2 - v_F \left( \frac{\partial \theta_\alpha}{\partial x} \right)^2 \right], \]

where the sum is over the two spin directions \( \alpha = \uparrow, \downarrow \). Then, \( S_0 \) is the sum of the actions (104) for each species \( s \). The expression (121) remains, however, also valid if the sum is over \( \alpha = \rho, \sigma \) with the phase fields introduced in Eqs. (114) and (115). Hence, the action functional for spinful electrons can be split in a charge and spin contribution

\[ S_0 = S_{0,\rho} + S_{0,\sigma}, \]
where each term has the form (104) of the action of a harmonic string.

The representation of the true electron density operator (105) in terms of the Bosonic phase fields has also a straightforward extension to the spinful case. Here we give explicitly only the generalization of the expression (110) valid in the limit \( L \to \infty \). One now obtains

\[
\rho(x) = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial x} \vartheta \rho(x) + \rho_{2k_F}(x),
\]

where

\[
\rho_{2k_F}(x) = \frac{2k_F}{\pi} \cos \left[ 2k_F x + \sqrt{2\pi} \vartheta \rho(x) \right] \cos \left[ \sqrt{2\pi} \vartheta \sigma(x) \right]
\]

is the \( 2k_F \)-contribution from the interference between right- and left-movers. Again we have suppressed the ladder operators that need to be taken into account in general.

3. INTERACTION, VOLTAGE BIAS, AND IMPURITIES IN THE TOMONAGA–LUTTINGER MODEL

In the previous section we have investigated noninteracting Fermions in a single channel quantum wire and seen that the low energy physics of the Fermi gas can be described either in terms of occupation numbers of single electron states or as excitations of Bosonic density waves. In higher dimensions noninteracting quasiparticles, supplemented by an electroneutrality constraint, give a rather accurate description also of the system in presence of Coulomb interaction, provided some parameters are replaced by effective parameters [24]. It has been known since quite some time that in one dimension the Fermi liquid description breaks down. Remarkably, the interaction leads to rather moderate modifications of the action in the Bose representation. The real advantages of Bosonization will thus only emerge in this section where we first present the real, interacting TL model and then extend it to describe a quantum wire with a scatterer in presence of an applied voltage.

3.1. ELECTRON-ELECTRON INTERACTION AND THE TOMONAGA–LUTTINGER MODEL

We now take the interaction into account but restrict ourselves to spinless Fermions first. The electronic charge density in a quantum wire is largely compensated by a homogeneous positive background charge density. Therefore, we now fix the Fermi wave number \( k_F \) so that the
state $|N_+ = 0, N_- = 0\rangle_0$, where all single particle states of $p$-movers with $pk \leq k_F$ are occupied, is electrically neutral. The density $\rho(x)$ introduced in Eq. (105) multiplied by the electron change $e$ is then the excess charge density in the wire, and the interaction may be written

$$H_{\text{int}} = \frac{1}{2} \int dx \int dy \rho(x) U(x - y) \rho(y).$$

(125)

The electron–electron interaction potential

$$U(x) = \frac{1}{L} \sum_q U_q e^{iqx}$$

(126)

has real Fourier components $U_q$ since $U(-x) = U(x)$. Usually, the potential $U(x)$ deviates from a simple Coulomb potential both at small and large distances. At small distances of order the lateral dimensions of the quantum wire, the wave function for the transversal motion of the strictly speaking three–dimensional particles becomes relevant. When these transversal components are integrated out, the resulting effective potential $U(x)$ in the one–dimensional model remains finite for $x \to 0$. On the other hand, at large distances one has to take into account the effect of gate electrodes and other nearby conductors that screen the long–range part of the Coulomb interaction. The effective potential $U(x)$ then has a finite range $R$, which implies that the logarithmic increase for small $q$ of the Fourier transformed Coulomb potential is cut off at $q$–values of order $R^{-1}$. In particular, $U_{q=0}$ then remains finite, with the precise value depending on the geometry of the problem. We assume that this externally screened potential is still sufficiently long ranged so that $U_0 \gg U_{2k_F}$. Then the $2k_F$–component of the electronic density (110) will give a negligible contribution when inserted in the interaction (125), and we find for the Bosonized interaction energy

$$H_{\text{int}} = \frac{1}{2\pi} \int dx \int dy \frac{\partial \vartheta(x)}{\partial x} U(x - y) \frac{\partial \vartheta(y)}{\partial y}.\text{ (127)}$$

When we restrict ourselves to low energy excitations with wavelengths large compared to the range $R$ of the interaction potential, we may replace $U(x)$ by a local interaction $U_0 \delta(x)$, and then obtain the interaction term of the TL model

$$H_{\text{int}} = \frac{U_0}{2\pi} \int dx \left( \frac{\partial \vartheta}{\partial x} \right)^2.\text{ (128)}$$

\footnote{In the limit $L \to \infty$ the sum $\frac{1}{L} \sum_q$ is replaced by $\frac{1}{2\pi} \int dq$.}
We remark that a microscopic local interaction $\sim \delta(x)$ would of course have no effect on spinless Fermions as a consequence of the anti-commutation rules. Essentially, in Eq. (128) one neglects the wave number dependence of $U_q$ for small $q$, but still $U_{2k_F} \ll U_0$. The interaction term (128) can readily be put in the action functional (104), which for the interacting model takes the form

$$S_\rho = \frac{\hbar}{2} \int dt \int dx \left[ \frac{1}{v_F} \left( \frac{\partial \vartheta}{\partial t} \right)^2 - v_F \left( 1 + \frac{U_0}{\pi \hbar v_F} \right) \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right].$$

(129)

and thus remains of the form of the action of a harmonic string. This clearly shows the great advantage of the Bose representation: We still have a model of free Bosonic charge density excitations, only the wave velocity is altered by the interaction, while in the Fermi representation the interaction leads to quartic terms in the Fermi operators.

It is customary to introduce the coupling constant

$$g = \left( 1 + \frac{U_0}{\pi \hbar v_F} \right)^{-1/2}$$

(130)

and the charge density wave velocity

$$v = \frac{v_F}{g}.$$  

(131)

In terms of these quantities the action functional of the TL model reads

$$S_\rho = \frac{\hbar}{2g} \int dt \int dx \left[ \frac{1}{v} \left( \frac{\partial \vartheta}{\partial t} \right)^2 - v \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right].$$

(132)

It would go beyond the scope of this article to demonstrate that the action (132) indeed describes the low energy properties of spinless one-dimensional Fermions correctly. Here we refer to the literature [8, 15]. We would, however, like to point out that the matter is in fact more complex than what our plausible “derivation” of Eq. (132) might suggest. The Coulomb interaction is strong and affects all states not only those near the two Fermi points. Hence, it needs to be introduced in the underlying physical model with a single energy dispersion curve $\varepsilon_k$. Afterwards, one may integrate out states far from the Fermi points until one reaches energy scales sufficiently close to the Fermi energy to allow for a linearization of the spectrum. At this point the Hamiltonian may be re-written in terms of the right- and left-movers of the TL model, but the parameters of the model are then already renormalized by the aforementioned elimination of high energy excitations and additional interaction vertices are generated. One can, however, conclude from a
renormalization group study [7] that the action (132) is indeed a low energy fixpoint of a (spin–polarized) quantum wire. From these remarks it is clear that the parameter $U_0$ in Eq. (128) does not necessarily coincide with the Fourier coefficient at $q = 0$ of the interaction potential. This latter quantity can be considered an estimate of $U_0$ which becomes more accurate for large electron densities at which the effect of the Coulomb interaction is weaker. In the sequel we use $g$ and $v$ as fundamental parameters of the model. For repulsive interaction $U_0 > 0$ and thus $g < 1$.

3.2. SCREENING OF EXTERNAL CHARGES

Let us assume that we perturb the quantum wire by an external charge density

$$\rho_{\text{ext}}(x, t) = eQ(x, t),$$

which interacts with the electronic charge density $e\rho(x, t)$ via the same effective potential $U(x)$ introduced in the previous section. We are interested in the long wavelength response of the quantum wire and may thus disregard the $2k_F$-component of the electronic charge density. With the local approximation $U_0 \delta(x)$, the action (132) is then modified to read

$$S = S_\rho - \frac{U_0}{\sqrt{\pi}} \int dt \int dx Q(x, t) \frac{\partial}{\partial x} \vartheta(x, t).$$

Since the action has only terms linear and quadratic in $\vartheta$, the average electronic density $\langle \rho(x, t) \rangle$ caused by $Q(x, t)$ can be determined from the phase field $\vartheta(x, t)$ minimizing the action (134). The equation of motion

$$\left( \frac{\partial^2}{\partial x^2} - v^2 \frac{\partial^2}{\partial \omega^2} \right) \vartheta(x, t) = \frac{g v U_0}{\sqrt{\pi \hbar}} \frac{\partial}{\partial x} Q(x, t)$$

obeyed by the minimal action field is readily solved in terms of the Fourier representation

$$\vartheta(x, t) = \frac{1}{(2\pi)^2} \int dq \int d\omega \tilde{\vartheta}(q, \omega) e^{iqx-i\omega t}.$$

We find

$$\tilde{\vartheta}(q, \omega) = -\frac{gU_0}{\sqrt{\pi \hbar}} \frac{ivq \tilde{Q}(q, \omega)}{\omega^2 - v^2 q^2},$$

which yields for the electronic density $\langle \rho(x, t) \rangle = \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial x} \vartheta(x, t)$ in Fourier space

$$\langle \tilde{\rho}(q, \omega) \rangle = (1 - g^2) \frac{v^2 q^2 \tilde{Q}(q, \omega)}{\omega^2 - v^2 q^2},$$
where we have expressed $U_0$ in terms of $g$ and $v$ by means of Eqs. (130) and (131).

Now, the relation between the external charge density $eQ(x,t)$ and the resulting screening charge density $e\langle \rho(x,t) \rangle$ is governed by the dielectric function

$$
\varepsilon(q,\omega) = \frac{Q(q,\omega)}{Q(q,\omega) + \langle \tilde{\rho}(q,\omega) \rangle}.
$$

(139)

Combining this with Eq. (138) we find

$$
\varepsilon(q,\omega) = \frac{\omega^2 - v^2q^2}{\omega^2 - g^2v^2q^2}.
$$

(140)

In particular, in the static case $\omega = 0$ we have

$$
\varepsilon = \frac{1}{g^2} = 1 + \frac{U_0}{\pi \hbar v_F},
$$

(141)

which shows that the interaction parameter $g$ is directly related to the dielectric constant of the quantum wire. In a metallic system the dielectric function has a pole for $\omega \to 0$, $q \to 0$ associated with the perfect screening of static charges leading to electroneutrality. However, in the TL model the long range part of the Coulomb interaction is assumed to be screened by other conductors as explained above. Then $U_0$ is finite and there is a finite dielectric constant in the zero frequency and long wavelength limits.

This is in accordance with the fact that the total screening charge in units of $e$

$$
Q_s = \int dx \langle \rho(x) \rangle = \langle \tilde{\rho}(q = 0) \rangle
$$

(142)

accumulated near a static impurity charge $Q$ follows from Eq. (138) as [25]

$$
Q_s = -(1 - g^2)Q.
$$

(143)

Hence a fraction $g^2Q$ of the external charge remains unscreened, and the quantum wire is in general not electroneutral. As we will discuss in greater detail in the next section, the charge $g^2Q$ is screened by the electrode responsible for the finite range of the interaction. Formally, the limit of long range Coulomb interaction corresponds to $g \to 0$ which implies electroneutrality of the wire.

We mention that apart from the long wavelength response of the quantum wire to an impurity charge there is also a $2k_F$–response leading to Friedel oscillations of the charge density. We will not discuss this here but refer to the recent literature [26, 27].
3.3. ELECTROSTATICS OF A QUANTUM WIRE

As we have seen in the preceding section, the electrode screening the long range part of the Coulomb interaction plays an important role in the electrostatic response of a quantum wire to external charges. We can visualize the TL model as a one-dimensional quantum wire screened by a gate coupled to the wire by a distributed capacitance as depicted in Fig. 1.6. The interaction energy (128) can then be interpreted as the charging energy of the wire–gate capacitance

$$H_{\text{int}} = \frac{U_0}{2} \int dx \rho(x)^2 = \int dx \frac{e^2 \rho(x)^2}{2c_0},$$

(144)

where the capacitance per unit length $c_0$ is determined by

$$\frac{e^2}{c_0} = U_0 = \pi \hbar v_F \left( \frac{1}{g^2} - 1 \right).$$

(145)

An electronic charge density $e\rho(x)$ in the wire polarizes the capacitance, and the resulting electric potential

$$\varphi(x) = \frac{e\rho(x)}{c_0}$$

(146)

shifts the band bottom of the quantum wire. This is directly related to the underscreening of external charges discussed previously as can be seen from the following consideration. In a noninteracting system an increase of the Fermi energy by $\Delta E$ shifts the wave number of the Fermi points from $\pm k_F$ to $\pm (k_F + \Delta k)$ where $\Delta k = \Delta E/\hbar v_F$. There are $(L/2\pi)\Delta k$ single particle states in the wave number interval $\Delta k$ and
thus the density of $p$–movers increases by

$$\Delta \rho_p = \frac{\Delta k}{2\pi} = \frac{\Delta E}{2\pi \hbar v_F}.$$  \hfill (147)

Here the factor $1/2\pi\hbar v_F$ gives the density of states of $p$–movers at the Fermi energy. Accordingly, the electronic density increases by

$$\Delta \rho = \Delta \rho_+ + \Delta \rho_- = \frac{\Delta E}{\pi \hbar v_F}.$$  \hfill (148)

On the other hand, in an interacting, electroneutral system the shift of the Fermi energy by $\Delta E$ is accompanied by a shift of the band bottom by the same amount and the electronic density remains unchanged. In the TL model the situation is in between these two extremal cases. The band bottom is shifted by $e\varphi$ where $\varphi$ is the electric potential difference between the wire and the gate electrode. Then, the change of the electronic density is determined by

$$\Delta \rho = \frac{\Delta E - e\varphi}{\pi \hbar v_F}.$$  \hfill (149)

In view of the relations (145) and (146) we have

$$e\varphi = \frac{e^2}{c_0} \Delta \rho = \pi \hbar v_F \left( \frac{1}{g^2} - 1 \right) \Delta \rho,$$  \hfill (150)

which gives

$$\Delta \rho = \frac{g^2 \Delta E}{\pi \hbar v_F}.$$  \hfill (151)

Hence, we see again that a fraction $g^2$ of the “bare” charge density (148) caused by the shift of the Fermi level persists as true electronic charge density in the wire.

### 3.4. VOLTAGE BIAS AND BOUNDARY CONDITIONS

It is not difficult to generalize the preceding considerations to a nonequilibrium quantum wire in presence of an applied voltage. Let us consider a quantum wire which is attached at the ends to two– or three–dimensional Fermi liquid reservoirs. We assume that the contacts between the wire and the reservoirs are adiabatic, which means that at the ends the quantum wire widens sufficiently slowly to avoid any backscattering of outgoing particles into the wire. This is the usual assumption underlying Landauer’s approach [16] to the conductance of
mesoscopic wires. If the contacts are not adiabatic, there will be an additional resistance depending on the precise realization of the contacts and not only on intrinsic properties of the quantum wire.

In equilibrium there is an equal amount of right– and left–movers in the wire and the electrochemical potential is constant. When we attach the wire to reservoirs, the influx of right–movers at the left end of the wire will depend on the electrochemical potential of the left electrode which we assume to be $eU_L$ above the Fermi energy of the equilibrium quantum wire. $U_L$ is then the voltage between the left reservoir and the gate electrode screening the wire. In the absence of interactions, the shift of the Fermi energy by $eU_L$ would increase the density of right–movers near the left end of the wire by

$$
\rho_+^{\text{bare}} = \frac{eU_L}{2\pi\hbar v_F}
$$

as depicted schematically in Fig. (1.7)a. Note that below we will consider quantum wires with impurities. Then, the reservoir determines the density of incoming particles only in the clean section of the wire near the end, where the incoming particles have not yet interacted with the impurities. The density of outflowing particles, on the other hand, will be affected by impurities.

In the presence of Coulomb interaction the excess charge density caused by the reservoirs will charge the distributed wire–gate capacitance leading to a shift of the band bottom by

$$
e\varphi = \frac{e^2}{4\pi}\rho = \frac{\pi\hbar v_F}{g^2} \left( \frac{1}{g^2} - 1 \right) \rho ,
$$

where $e\rho$ is the true charge density in the wire that has to be determined selfconsistently. The true density of right–movers near the left end of the wire is then, cf. Fig. (1.7)b,

$$
\rho_+ = \frac{e(U_L - \varphi)}{2\pi\hbar v_F} ,
$$

which means that the bare charge density (152) is partially screened.

In terms of the Bosonic phase field $\vartheta$, we have by virtue of the relations (102)

$$
\rho_+ = \frac{1}{2\sqrt{\pi}} \left( \frac{\partial \vartheta}{\partial x} - \frac{1}{v_F} \frac{\partial \vartheta}{\partial t} \right) ,
$$

where we have omitted the prime on $\vartheta$ as in all equations following (102). We remark that in the relation

$$
\frac{1}{\sqrt{\pi}} \frac{\partial \vartheta}{\partial t} = -v_F(\rho_+ - \rho_-)
$$

(156)
Figure 1.7. Density of right-movers in a quantum wire attached at the left end to a reservoir with voltage $U_L$ against the gate electrode. The equilibrium density is represented as light-grey area and the nonequilibrium excess density as dark-grey area.

a) Sketch of the situation in the absence of Coulomb effects and b) with interaction taking the shift $e\varphi$ of the band bottom into account.
the Fermi velocity \( v_F \) is not altered by the Coulomb interaction such as the velocity of charge density waves, because the microscopic expression of the particle current operator in terms of Fermi operators is independent of the interaction in the absence of a vector potential [28]. For the electric potential \( \varphi \) we obtain from Eqs. (153) and (102)

\[
e\varphi = \sqrt{\pi \hbar v_F} \left( \frac{1}{g^2} - 1 \right) \frac{\partial \vartheta}{\partial x},
\]

which can be combined with the relations (154) and (155) to give for \( x \) near the left end of the wire

\[
\frac{1}{g^2} \frac{\partial \vartheta}{\partial x} - \frac{1}{v_F} \frac{\partial \vartheta}{\partial t} = \frac{eU_L}{\sqrt{\pi \hbar v_F}}.
\]

Similar considerations hold for the density of left–movers near the right end of the wire attached to an electrode at potential \( U_R \). Strictly speaking, all of the above considerations hold for the average densities of right– and left–movers. Hence, we find that the coupling to reservoirs can be described in the Bosonized TL model in terms of radiative boundary conditions for the phase field [17]

\[
\frac{1}{g^2} \frac{\partial \vartheta}{\partial x} - \frac{1}{v_F} \frac{\partial \vartheta}{\partial t} \langle \vartheta(x, t) \rangle_{x=-\frac{L}{2}} = \frac{eU_L}{\sqrt{\pi \hbar v_F}},
\]

\[
\frac{1}{g^2} \frac{\partial \vartheta}{\partial x} + \frac{1}{v_F} \frac{\partial \vartheta}{\partial t} \langle \vartheta(x, t) \rangle_{x=\frac{L}{2}} = \frac{eU_R}{\sqrt{\pi \hbar v_F}}.
\]

If we impose these conditions at a point near the reservoirs, they will be seen to be obeyed at any point in the impurity–free clean sections at the ends of the wire.

### 3.5. IMPURITY POTENTIAL

Nontrivial dc transport properties of the quantum wire do not arise from the Coulomb interaction and the resulting non–Fermi liquid behavior alone but only in connection with impurities causing backscattering of electrons. A nonmagnetic impurity at position \( x_0 \) couples to the charge density and gives rise to an energy

\[
H_W = \int dx W(x-x_0)\rho(x),
\]

where \( W(x) \) is the impurity potential. When we want to add this coupling term to the TL model, we have to note that the perturbation affects all states, also those far from the Fermi points, and we should again
start from the underlying physical model with a single branch. Applying essentially the same line of reasoning used to include electron–electron interactions in the TL model, we obtain for the case of an interaction with an impurity at position \( x_0 \)

\[
H_W = \frac{W_0}{\sqrt{\pi}} \frac{\partial \vartheta}{\partial x} \bigg|_{x=x_0} + \frac{W_{2k_F} k_F}{\pi} \cos \left[ 2k_F x_0 + 2\sqrt{\pi} \vartheta(x_0) \right].
\] (161)

Of course, this form results when we insert the representation (110) of the electronic density into Eq. (160) and then replace the Fourier coefficients \( W_q \) of the impurity potential by constants for \( q \approx 0 \) and \( |q| \approx 2k_F \), respectively. Since an elimination of states far from the Fermi points is necessary before we can rewrite the Hamiltonian in terms of the Fermi operators \( c_{p,k}, c_{p,k}^\dagger \) of the TL model, the coefficients \( W_0 \) and \( W_{2k_F} \) must be interpreted as effective parameters that are not directly related to Fourier coefficients of the microscopic interaction potential. In addition, the elimination of states will generate higher order processes involving several electrons with momentum transfer \( 4k_F, 6k_F \), and so on. It turns out that the \( 2k_F \)--processes dominate at low temperatures [18].

Although an impurity is typically laterally displaced from the center of the wire, the potential \( W(x) \) will essentially have the same behavior as the electron–electron potential \( U(x) \) discussed previously. Thus, from the bare Fourier components we would conclude \( W_{2k_F} \ll W_0 \). But, as we shall see, the forward scattering \( W_0 \), which scatters a \( p \)--mover into another state of the same branch, has no effect on the transport properties of the wire, while even a small backscattering term \( W_{2k_F} \) has a dramatic effect at low energy scales [18].

To show that forward scattering is unimportant, we write the TL Hamiltonian in presence of an impurity in the form

\[
H = \frac{\hbar v_F}{2} \int dx \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right] + \frac{U_0}{2\pi} \int dx \left( \frac{\partial \vartheta}{\partial x} \right)^2 
\] (162)

\[
+ \frac{W_0}{\sqrt{\pi}} \frac{\partial \vartheta}{\partial x} \bigg|_{x=x_0} + \frac{W_{2k_F} k_F}{\pi} \cos \left[ 2k_F x_0 + 2\sqrt{\pi} \vartheta(x_0) \right],
\]

which includes the Hamiltonian (95) of the noninteracting model, the interaction (128), and the impurity term (161). Now, the unitary transformation

\[
U = \exp \left[ -i\sqrt{\pi} \int dx \alpha(x) \phi(x) \right] \] (163)

shifts the charge density, since

\[
U \frac{\partial \vartheta}{\partial x} U^{-1} = \frac{\partial \vartheta}{\partial x} + \sqrt{\pi} \alpha(x). \] (164)
This is readily shown by considering the auxiliary function

\[ F(s) = e^{isA} \frac{\partial}{\partial x} e^{-isA}, \tag{165} \]

where \( A = \sqrt{\pi} \int dx \alpha(x)\phi(x) \) is the exponent of \( U \). One then finds

\[ \frac{\partial}{\partial s} F(s) = e^{isA} i \left[ A, \frac{\partial}{\partial x} \right] e^{-isA} - e^{-isA} = \sqrt{\pi} \alpha(x), \tag{166} \]

where the commutator is evaluated by means of Eq. (60). Since \( F(0) = \frac{\partial}{\partial x} \), we find \( F(1) = \frac{\partial}{\partial x} + \sqrt{\pi} \alpha(x) \) which is just the relation (164).

Based on the same commutator, one also finds

\[ U e^{2i\sqrt{\pi} \vartheta(x)} U^{-1} = e^{2i\sqrt{\pi} \vartheta(x) + i\eta(x)}, \tag{167} \]

where

\[ \eta(x) = \pi \int dy \, \text{sign}(x - y) \alpha(y). \tag{168} \]

To see this, we write the left hand side of Eq. (167) as \( e^{iA} e^{iB} e^{-iA} \) with the operator \( A \) introduced previously and \( B = 2\sqrt{\pi} \vartheta(x) \). Then, using twice the relation (83), we find

\[ e^{iA} e^{iB} e^{-iA} = e^{iA} e^{i(B-A)} e^{-\frac{1}{2}[A,B]} - e^{iB-[A,B]} = e^{iB-[A,B]}, \tag{169} \]

which gives the transformation (167).

With the help of the relations (164) and (167), the Hamiltonian (162) is transformed into

\[ U H U^{-1} = \frac{\hbar v_F}{2} \int dx \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{g^2} \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right] \]

\[ + \frac{W_0}{\sqrt{\pi}} \frac{\partial}{\partial x} \bigg|_{x=x_0} + \frac{\sqrt{\pi} \hbar v_F}{g^2} \int dx \alpha(x) \frac{\partial \vartheta}{\partial x} \]

\[ + \frac{W_2k_F}{\pi} \cos [2k_Fx_0 + 2\sqrt{\pi} \vartheta(x_0) + \eta(x_0)] \tag{170} \]

where we have omitted terms that depend only on \( \alpha(x) \) but not on the phase fields. Further, we have introduced the interaction parameter (130). Now, with the choice

\[ \alpha(x) = -\frac{g^2 W_0}{\pi \hbar v_F} \delta(x - x_0) \tag{171} \]
the terms in the second line of Eq. (170) cancel, and we are left with the Hamiltonian

\[ H' = \frac{\hbar v_F}{2} \int dx \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{g^2} \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right] \]

\[ + \frac{W_{2k_F} k_F}{\pi} \cos \left[ 2k_F x_0 + 2\sqrt{\pi} \vartheta(x_0) \right], \]

which contains only a backscattering term. Note that for an impurity with charge \( eQ \) giving rise to the Coulomb potential \( W(x) = QU(x) \), the quantity

\[ \alpha(x) = -\frac{g^2 U_0}{\pi \hbar v_F} Q \delta(x - x_0) = -(1 - g^2)Q \delta(x - x_0) \]

is just the screening charge density (138) caused by the static impurity charge. The unitary transformation thus removes the screening cloud, which is the main effect of the forward scattering term. Below we will determine the current–voltage relation of a quantum wire in presence of a single impurity at position \( x = 0 \). This study will be based on the action

\[ S = \frac{\hbar}{2g} \int dt \int dx \left[ \frac{1}{v} \left( \frac{\partial \vartheta}{\partial t} \right)^2 - v \left( \frac{\partial \vartheta}{\partial x} \right)^2 \right] \]

\[ - \lambda \int dt \cos [2\sqrt{\pi} \vartheta(x = 0, t)] \]

associated with the Hamiltonian (172), where \( \lambda \) characterizes the impurity strength.

### 3.6. INTERACTING ELECTRONS WITH SPIN

Here we briefly summarize the changes necessary to include the electron spin. For noninteracting electrons we found that the action of spinful electrons can be split into a charge and a spin contribution. Since the Coulomb interaction couples only to the charge density, we might argue that the charge part of the action is modified by the interaction in the same way as for spinless electrons while the spin part remains unchanged. However, this would ignore the fact that the Coulomb interaction must be introduced in the underlying physical model and the transcription to TL Fermions can only be made after an elimination of high energy excitations. This has consequences also for the spin density
waves, in particular, the spin wave velocity $v_\sigma$ becomes smaller than the Fermi velocity $v_F$ ([29]). The low energy physics is then governed by the action

$$S = \frac{\hbar}{2g} \int dt \int dx \left[ \frac{1}{v} \left( \frac{\partial \theta_\rho}{\partial t} \right)^2 - v \left( \frac{\partial \theta_\rho}{\partial x} \right)^2 \right]$$

$$+ \frac{\hbar}{2} \int dt \int dx \left[ \frac{1}{v_\sigma} \left( \frac{\partial \theta_\sigma}{\partial t} \right)^2 - v_\sigma \left( \frac{\partial \theta_\sigma}{\partial x} \right)^2 \right].$$

(175)

The TL model is thus characterized by three parameters $v$, $v_\sigma$, and $g$. There is no coupling constant $g_\sigma$ for the spin sector, which can be traced back to spin rotation invariance [8, 15]. A detailed discussion of these issues would go beyond the scope of this article and we refer to the literature cited. As in the noninteracting case, the action splits into a charge and a spin part. However, the difference between the charge and spin wave velocities in an interacting wire has important consequences and leads to the notable phenomenon of spin–charge separation. When an electron is injected into a wire it causes a charge and a spin pulse propagating with different velocities.

In case an impurity is added at position $x = 0$ to the model we see from the $2k_F$–part of the electron density in the spinful case (124) that the important backscattering term now has the form

$$S_\lambda = -\lambda \int dt \cos \left[ \sqrt{2\pi} \theta_\rho(x = 0,t) \right] \cos \left[ \sqrt{2\pi} \theta_\sigma(x = 0,t) \right].$$

(176)

Hence, the impurity couples the charge and spin sectors making the theory of dirty quantum wires in the absence of a spin–polarizing magnetic field more involved.

4. CURRENT–VOLTAGE RELATION OF A QUANTUM WIRE

In this section we apply the theory developed so far and determine the current in a quantum wire with an impurity as a function of the applied voltage and the temperature. Rather than giving an overview of the results available in the literature, we treat a special case in some detail to illustrate how the formalism explained in the previous sections can be employed to obtain concrete results. With this background readers should then be prepared to embark on reading the recent original literature on the subject.
4.1. PARTICULAR SOLUTION AND FOUR–TERMINAL VOLTAGE

We study a single channel quantum wire with an impurity at position $x = 0$. The conductor is attached to reservoirs at voltages $U_L$ and $U_R$ relative to the gate electrode screening the wire as sketched in Fig. (1.8). The applied voltage

$$U = U_L - U_R$$  \hfill (177)

will then drive a current $I$ through the wire. For simplicity, we shall restrict ourselves to the case of spinless electrons. We can then base the consideration on the action functional (174) and the boundary conditions (159).

As discussed previously, the phase field $\theta(x, t)$ describes also the mean particle densities (100) due to a shift introduced in Eq. (101). To take care of the boundary conditions, we look for a particular field $\theta(x, t)$ that satisfies the wave equation in the clean parts of the wire and the boundary conditions. This solution is of the form of the shift in Eq. (101)

$$\theta(x, t) = \frac{g^2 e}{2\sqrt{\pi}hv_F} [(U_L + U_R)x - V|x|] - \frac{e}{2\sqrt{\pi}\hbar}(U_L - U_R - V)t, \hfill (178)$$

where $V$ is an arbitrary parameter. We now split the phase field $\theta(x, t)$ into

$$\varphi(x, t) = \theta(x, t) + \varphi(x, t). \hfill (179)$$

The deviation $\varphi(x, t)$ from the particular solution (178) will then obey equilibrium boundary conditions. Further, we fix the parameter $V$ by the requirement that in the steady nonequilibrium state

$$\frac{\partial}{\partial t} \langle \varphi(x, t) \rangle = 0. \hfill (180)$$
Now, the average current $I$ may be written as

$$I = e v_F \langle \rho_+ - \rho_- \rangle = -\frac{e}{\sqrt{\pi}} \frac{\partial}{\partial t} \langle \vartheta \rangle, \quad (181)$$

where we have used Eq. (102) to obtain the second equality. Hence, the condition (180) means that the average current is determined solely by the particular solution (178), and we have

$$I = \frac{e^2}{\hbar}(U - V). \quad (182)$$

Since in a steady state the current $I$ is independent of $x$, we may impose the condition (180) fixing the voltage $V$ at any point $x$.

The particular solution (178) not only determines the average current but also the average charge density in the nonequilibrium quantum wire. Since $\langle \varphi \rangle$ obeys equilibrium boundary conditions as well as the condition (180), it does not contribute to the average density (110)

$$\langle \rho \rangle = \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial x} \langle \vartheta \rangle, \quad (183)$$

where we have omitted the $2k_F$-component which gives an additional oscillatory contribution near the impurity. This Friedel oscillation component is not seen in a density smoothed over length scales of order $\lambda_F = 2\pi/k_F$. From the particular solution (178) we obtain for the average density

$$\langle \rho \rangle = \frac{g^2 e}{2\pi \hbar v_F} \frac{(U_L + U_R)}{2} - \frac{g^2 e V}{2\pi \hbar v_F} \text{sign}(x). \quad (184)$$

The first term just describes the change (151) of the average electronic density as a consequence of the average shift $e(U_L + U_R)/2$ of the Fermi energy. This term is absent if the voltage $U$ is applied asymmetrically, i.e., for $U_L = -U_R = U/2$. The second term gives an asymmetric component of the charge density in presence of an impurity. The density drop

$$\Delta \rho = \frac{g^2 e V}{\pi \hbar v_F} \quad (185)$$

across the impurity site is associated with a difference

$$\Delta \mu = g^2 e V \quad (186)$$

of the effective chemical potential on both sides of the impurity. Furthermore, the drop of the charge density across the impurity site is also associated with a change of the electric potential (153) by

$$\Delta \varphi = \frac{e}{g^2} \left( \frac{1}{g^2} - 1 \right) \Delta \rho = (1 - g^2) \hbar v_F. \quad (187)$$
In a hypothetical ideal measurement of the voltage drop across the impurity site one would observe the difference of the electrochemical potential [16]
\[ \Delta \varphi + \frac{1}{e} \Delta \mu = V. \] (188)
Hence, the parameter $V$ introduced above coincides with the average four-terminal voltage $V$, which is the part of the applied voltage $U$ dropping across the scatterer.\(^5\)

In view of Eq. (182) the determination of the current–voltage relation corresponds to a calculation of the four–terminal voltage. Two limiting cases are evident from physical grounds. In the absence of a backscatterer ($\lambda \to 0$) we have $V = 0$ and obtain from Eq. (182)
\[ I = G_0 U, \] (189)
where $G_0 = e^2/h$ is the conductance of a clean wire. This is the same result as obtained previously for noninteracting electrons. Hence, for a clean quantum wire with adiabatic contacts to the reservoirs the interaction has no effect on the conductance [30, 31, 32, 17]. On the other hand, for a very strong backscatterer ($\lambda \to \infty$) we have $V = U$ and the current $I$ vanishes. In the remainder we shall discuss how we get from one limit to the other.

4.2. PATH INTEGRAL ON THE KELDYSH CONTOUR

To treat the nonequilibrium quantum wire with the action functional (174) quantum mechanically, we have to evaluate a Feynman path integral on the Keldysh contour. For an introduction to the Keldysh technique we refer to the review article [33], however, the basic idea can be understood in the following way. Assume that at time $t_0$ the system is described by the density matrix $W(t_0)$ and let $H$ be the Hamiltonian including the coupling to the reservoirs. The density matrix at a later time $t_f$ is then given by
\[ W(t_f) = e^{-\frac{i}{\hbar} H(t_f-t_0)} W(t_0) e^{\frac{i}{\hbar} H(t_f-t_0)}. \] (190)
Each of the two time evolution operators $e^{\pm \frac{i}{\hbar} H(t_f-t_0)}$ may be written as a Feynman path integral. Since we are interested in steady state properties independent of the initial state $W(t_0)$, we take the limit $t_0 \to -\infty$. The

\(^5\)The readers should be aware of the fact that in some early treatments of transport properties of the TL model the discrimination between $U$ and $V$ was not made.
trace over such a time propagated operator leads to a path integral of the form

\[ Z_0 = \int \mathcal{D}[\vartheta] e^{i \hbar S[\vartheta]}, \]

where \( S[\vartheta] \) is the action functional (174) with the time integration \( \int dt \) running along the Keldysh contour depicted in Fig. (1.9). In order to employ this path integral for the calculation of expectation values, we first decompose the phase field according to Eq. (179). The action (174) then takes the form

\[
S = \frac{\hbar}{2g} \int dt \int dx \left[ \frac{1}{v} \left( \frac{\partial \varphi}{\partial t} \right)^2 - v \left( \frac{\partial \varphi}{\partial x} \right)^2 \right] - \frac{eV}{\sqrt{\pi}} \int dt \varphi(0,t) - \lambda \int dt \cos \left[ 2\sqrt{\pi} \varphi(0,t) - \frac{e}{\hbar} (U - V)t \right],
\]

where we have omitted terms independent of \( \varphi(x,t) \). Further, we have taken into account that

\[ \int dt \frac{\partial \varphi}{\partial t} = 0 \]

for a time integral along the Keldysh contour and

\[ \int dx \text{sign}(x) \frac{\partial}{\partial x} \varphi(x,t) = -2\varphi(0,t). \]

For the phase field \( \varphi(x,t) \), which obeys equilibrium boundary conditions, the nonequilibrium situation becomes apparent in two modifications of the action (192). The term in the second line comes from the voltage drop across the impurity site. It describes the potential energy

\[ \int dx \left[ -\frac{V}{2} \text{sign}(x) \right] e^{i \sqrt{\pi} \frac{\partial \varphi(x,t)}{\partial x}} = \frac{eV}{\sqrt{\pi}} \varphi(0,t) \]

of a charge density fluctuation \( \langle e/\sqrt{\pi} \rangle \frac{\partial \varphi}{\partial x} \) in presence of an electrochemical potential \(- (V/2) \text{sign}(x)\). The remainder \( U - V \) of the applied voltage
$U$ shows up in the third line of Eq. (192) as a Josephson–type phase shift in the pinning potential caused by the impurity.

Instead of the path integral (191) we now study the functional

$$Z[\eta] = \int \mathcal{D}[\varphi] \ e^{i \int \eta(t) \ H(0,t)} ,$$

where $S[\varphi]$ is the action (192), and where we have introduced an auxiliary field $\eta(t)$ on the Keldysh contour. $Z[\eta]$ is a generating functional for expectation values of $\frac{\partial}{\partial \epsilon} \varphi(0,t)$. In particular, the condition (180) is now equivalent to

$$\left. \frac{\delta Z[\eta]}{\delta \eta(t)} \right|_{\eta=0} = 0 ,$$

which needs to be evaluated to determine the four–terminal voltage $V$.

### 4.3. COULOMB GAS REPRESENTATION

In the sequel we outline one of the methods available to determine the current–voltage relation. While for the one impurity problem considered here, an approach based on the thermodynamic Bethe ansatz is most powerful [34], we present here a technique which remains useful also for multi impurity problems. First, we split the Keldysh contour explicitly into the two branches and denote the phase field on the branch from $-\infty$ to $t_f$ by $\varphi(x,t)$ and the field on the branch from $t_f$ back to $-\infty$ by $\varphi'(x,t)$. Further, we split the action (192) into two terms

$$S_0 = \frac{\hbar}{2g} \int_{-\infty}^{t_f} dt \int dx \left[ \frac{1}{v} \left( \frac{\partial \varphi}{\partial t} \right)^2 - v \left( \frac{\partial \varphi}{\partial x} \right)^2 - \frac{1}{v} \left( \frac{\partial \varphi'}{\partial t} \right)^2 + v \left( \frac{\partial \varphi'}{\partial x} \right)^2 \right]$$

$$- \frac{eV}{\sqrt{\pi}} \int_{-\infty}^{t_f} dt \ [\varphi(0,t) - \varphi'(0,t)]$$

and

$$S_\lambda = -\lambda \int_{-\infty}^{t_f} dt \ \left\{ \cos \left[ 2 \sqrt{\pi} \varphi(0,t) - \frac{e}{h} (U - V) t \right] \right.$$

$$- \left. \cos \left[ 2 \sqrt{\pi} \varphi'(0,t) - \frac{e}{h} (U - V) t \right] \right\} .$$

With the help of the trigonometric relation

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

(200)
the action $S_{\lambda}$ may be written as

$$
S_{\lambda} = 2\lambda \int_{-\infty}^{t_f} dt \cos A(t) \sin B(t),
$$

(201)

where

$$
A(t) = \sqrt{\pi} [\varphi(0, t) + \varphi'(0, t)] - \frac{e}{\hbar} (U - V) t + \delta
$$

(202)

$$
B(t) = \sqrt{\pi} [\varphi(0, t) - \varphi'(0, t)].
$$

From Eqs. (199) and (200) we obtain $\delta = -\frac{\pi}{2}$, however, the precise value of this phase must be irrelevant, since we can always add a constant phase to the particular solution (178), e.g., by replacing $t$ by $t - t_0$. Such an additional phase of $\theta(x, t)$ leads to a shift of $\delta$.

Using Eq. (201) we find by expanding in powers of $\lambda$

$$
e^{\frac{i}{\hbar} S_{\lambda}} = 1 + \sum_{n=1}^{\infty} \left(\frac{2i\lambda}{\hbar}\right)^n \int D_n t \prod_{j=1}^{n} \cos A(t_j) \sin B(t_j),
$$

(203)

where we have introduced the abbreviation

$$
\int D_n t = \int_{-\infty}^{t_f} dt_n \int_{-\infty}^{t_{n-1}} dt_{n-1} \ldots \int_{-\infty}^{t_2} dt_1.
$$

(204)

Next, we write the trigonometric functions as

$$
\cos A(t_j) = \frac{1}{2} \sum_{u_j = \pm} e^{iu_j A(t_j)},
$$

(205)

$$
\sin B(t_j) = \frac{1}{2i} \sum_{v_j = \pm} v_j e^{iv_j B(t_j)},
$$

which gives

$$
e^{\frac{i}{\hbar} S_{\lambda}} = 1 + \sum_{n=1}^{\infty} \sum_{\{u_j, v_j\}} \left(\prod_{j=1}^{n} \frac{\lambda v_j}{2\hbar}\right) \int D_n t \ e^{\frac{i}{\hbar} S_n},
$$

(206)

where

$$
\frac{1}{\hbar} S_n = \sum_{j=1}^{n} [u_j A(t_j) + v_j B(t_j)]
$$

(207)

is linear in the phase fields $\varphi, \varphi'$ by virtue of Eq. (202). The benefit of this expansion is that now the path integral (196) is Gaussian order by order.
Therefore, we can integrate out the \( \varphi \) and \( \varphi' \) fields. Essentially, the calculation of these path integrals amounts to an explicit computation of the fields minimizing the action.

On a formal level we may consider the auxiliary variables \( u_j, v_j \) as charges on the time axis that are coupled to a harmonic string described by the phase fields. The integration over the fields \( \varphi, \varphi' \) then corresponds to the elimination of a harmonic bath in the context of dissipative quantum mechanics. This problem is well studied in the literature [35, 36, 37]. It would go beyond the scope of this article to present these methods explicitly here. Once the Gaussian integrals over the phase fields are carried out, the sum over the charges \( v_j \) can be done straightforwardly, and the generating functional (196) is obtained in the form

\[
Z[\eta] = \exp \left\{ -\int \! dt \int \! dt' \eta(t) \tilde{C}(t-t') \eta(t') - i \frac{e}{\hbar} V \int \! dt \eta(t) \right\} \\
\times \left( 1 + \sum_{m=1}^{\infty} Z_m[\eta] \right),
\]

where

\[
Z_m[\eta] = \left( \frac{i \lambda}{\hbar} \right)^{2m} \int \! D_{2m} t \sum_{\{u_j\}} \exp \left( \sum_{j>k=1}^{2m} u_j u_k C(t_j - t_k) \right) \\
+ \sum_{j=1}^{2m} u_j \left[ \int \! dt \eta(t) \tilde{C}(t-t_j) - i \frac{e}{\hbar} (U - V + gV) t_j \right] \\
\times \sin \left[ \pi g \eta(t_{2m}) \right] \prod_{j=1}^{2m-1} \sin \left( \pi g \left[ \eta(t_j) + \sum_{k=j+1}^{2m} u_k \right] \right) 
\]

Here the sine functions arise from the sum over the charges \( v_j \) by means of Eq. (205). The function

\[
C(t) = 2g \ln \left[ \frac{\beta \Delta}{\pi} \sinh \left( \frac{\pi |t|}{\hbar \beta} \right) \right] 
\]

describes an effective interaction between the charges \( u_j \) whereby \( \beta = 1/k_B T \) is the inverse temperature of the quantum wire and \( \Delta \) is the cutoff energy of the harmonic string. The temperature emerges from the asymptotic conditions on the fluctuations of the phase field for \( t \to -\infty \). To simplify notation we have chosen the same auxiliary field \( \eta(t) \) on both branches of the Keldysh contour so that the second term in the exponent
of (196) reads \(\sqrt{\pi} \int_{-\infty}^{t_f} dt' \eta(t') \frac{\partial}{\partial t'} [\varphi(0, t') + \varphi'(0, t')]\). Furthermore, we have taken the limit \(t_f \to \infty\).

In the series (208) only even terms in \(\lambda\) survive, because \(C(t)\) is a long range interaction which suppresses all terms that do not satisfy the charge neutrality condition

\[
\sum_{j=1}^{n} u_j = 0. \tag{211}
\]

Since the charges \(u_j = \pm 1\), this condition can only hold for \(n = 2m\).

The constraint (211) is indicated as a prime at the sum over the \(u_j\) in Eq. (209). Because of this condition the phase \(\delta\) in Eq. (202) drops out. The representation (208) of the generating functional is known as the Coulomb gas representation of the problem, since some terms allude to the partition function of one–dimensional charges interacting with the “Coulomb potential” (210). Note that in view of the factors \(\sin[\pi g\eta(t_{2m})]\) the generating functional (208) obeys the normalization \(Z[\eta = 0] = 0\).

The four–terminal voltage \(V\) can now be determined from the condition (197). A nonvanishing contribution of \(Z_m[\eta]\) only arises if the variational derivative acts upon \(\sin[\pi g\eta(t_{2m})]\). Introducing the time–difference variables

\[
\tau_m = t_{m+1} - t_m
\]

we find

\[
\frac{e}{\hbar} V = K \left( \frac{e}{\hbar} [U - V + gV] \right), \tag{213}
\]

where

\[
K(\Omega) = \pi \Im \sum_{m=1}^{\infty} (-1)^m \left( \frac{\lambda}{\hbar} \right)^{2m} \int_{0}^{\infty} d\tau_1 \cdots \int_{0}^{\infty} d\tau_{2m-1} \int_{0}^{\infty} d\tau
\]

\[
\times \sum_{\{u_j\}'} \exp \left[ \sum_{j>k=1}^{2m} u_j u_k C \left( \sum_{l=k}^{j-1} \tau_l \right) \right] \prod_{j=1}^{m-1} \sin(\pi g p_j) e^{-i p_j \Omega \tau_j}, \tag{214}
\]

Here

\[
p_j = \sum_{k=j+1}^{2m} u_k = - \sum_{k=1}^{j} u_k \tag{215}
\]

is the accumulated charge in the time interval \(t > t_j\). Sums of the form (214) with an interaction (210) are again familiar from dissipative quantum mechanics [38].
4.4. EXACT SOLUTION FOR $G = \frac{1}{2}$

The explicit evaluation of the sum (214) is greatly simplified in the special case $g = \frac{1}{2}$, since the sine–functions then suppress many terms. The so–called “collapsed blip approximation” [38] of the Coulomb gas problem becomes exact for $g = \frac{1}{2}$ and $K(\Omega)$ can easily be calculated explicitly. One finds

$$K(\Omega) = 2\frac{\lambda_B}{\hbar} \text{Im} \psi \left( \frac{1}{2} + \frac{\beta [\lambda_B + i\hbar \Omega]}{2\pi} \right), \quad (216)$$

where $\psi(z)$ is the digamma function and

$$\lambda_B = \pi \frac{\lambda^2}{\Delta} \quad (217)$$

is a renormalized energy scale of the impurity. The four–terminal voltage $V$ then follows from Eq. (213), which for $g = \frac{1}{2}$ reads

$$\frac{e}{\hbar} V = K \left( \frac{e}{\hbar} [U - V/2] \right). \quad (218)$$

It can be easily seen that the solution is of the form

$$\frac{eV}{\lambda_B} = f \left( \frac{eU}{\lambda_B}, \frac{k_BT}{\lambda_B} \right). \quad (219)$$

This shows that the solution exhibits scaling. When the energies $eU$ and $k_BT$ are measured in units of the renormalized impurity energy scale $\lambda_B$, we find the same four–terminal voltage (in units of $\lambda_B$) for any impurity and hence also the same form of the current–voltage relation (182).

In Fig. (1.10) we depict the current–voltage curve for various temperatures. We see that the interaction has a dramatic effect on the $I$–$V$–characteristics in presence of a scatterer. While in the noninteracting model the conductance is essentially independent of $k_BT$ and $eU$ for small energy scales, there is a strong suppression of the conductance in the interacting system for energies below $\lambda_B$. At zero temperature

$$\frac{I}{I_0} = \frac{1}{24} \left( \frac{eU}{\lambda_B} \right)^2 \quad \text{for} \quad eU \ll \lambda_B \quad (220)$$

and

$$\frac{I}{I_0} = 1 - \pi \frac{\lambda_B}{eU} \quad \text{for} \quad eU \gg \lambda_B, \quad (221)$$

where $I_0 = (e^2/h)U$. Hence, the differential conductance shows a perfect zero bias anomaly and vanishes $\sim U^2$ for $U \to 0$. On the other hand, the
conductance approaches the value $e^2/h$ of a clean wire for large voltages. In practice, this latter limit will not be reached, since for large $U$ the low energy description in terms of the TL model breaks down. On the other hand for $T > 0$ the linear conductance for $U \to 0$ reads

$$G(T) = \frac{e^2}{h} \frac{\Theta - \psi'(\frac{1}{2} + \frac{1}{\Theta})}{\Theta + \psi'(\frac{1}{2} + \frac{1}{\Theta})},$$

(222)

where $\Theta = 2\pi k_B T / \lambda_B$. Hence, the conductance vanishes $\sim T^2$ for $k_B T \ll \lambda_B$. We mention that the exact solution for $g = \frac{1}{2}$ was first obtained by re–Fermionization techniques [17].

4.5. CONCLUDING REMARKS

The explicit results for the current–voltage characteristics derived in the preceding section show that for a spin–polarized quantum wire the crossover from the weak impurity problem at higher temperatures or larger applied voltages to the strong coupling problem at low temperatures and small voltages can be solved exactly. The scaling function can also be obtained for arbitrary $g < 1$ [34]. Mostly, one has to be content with more limited results. In the spinful case the current is known only asymptotically for large and small energy scales [18]. Several topics are
still under investigation. For instance, in the presence of many impurities additional features such as resonant tunneling may arise. Another area of active research is the current noise in one-dimensional conductors.

We have focused here on the physics of screened single channel quantum wires at low energy scales where the finite range of the interaction is unimportant. Many results can in fact be extended to finite range interactions. However, in the absence of a gate, the long range nature of the Coulomb interaction requires special attention [39]. There are many similarities between the transport properties of quantum wires and edge currents in fractional quantum Hall bars [40]. Since in these latter devices right- and left-moving branches are spatially separated, they are described by a chiral TL model [41, 42, 43] with somewhat different transport properties. The methods presented here are also useful to model the low energy electronic properties of carbon nanotubes [44, 45]. Again, due to features of the bandstructure, some differences arise leading to four channels where one channel is charged as in a semiconductor quantum wire. We hope that this article will facilitate the study of these and many other exciting new results on transport properties of one-dimensional Fermionic systems.

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