Rate of Convergence to Separable Solutions of the Fast Diffusion Equation

Marek Fila
Department of Applied Mathematics and Statistics, Comenius University
84248 Bratislava, Slovakia

and

Michael Winkler
Institut für Mathematik, Universität Paderborn
33098 Paderborn, Germany

Abstract
We study the asymptotic behaviour near extinction of positive solutions of the Cauchy problem for the fast diffusion equation with a subcritical exponent. We show that separable solutions are stable in some suitable sense by finding a class of functions which belong to their domain of attraction. For solutions in this class we establish optimal rates of convergence to separable solutions.

1 Introduction

We consider the Cauchy problem for the fast diffusion equation,
\[
\begin{cases}
  u_\tau = \Delta(u^m), & x \in \mathbb{R}^n, \tau \in (0, T), \\
  u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^n,
\end{cases}
\]
(1.1)
where \( n \geq 3, T > 0 \) and \( 0 < m < 1 \). It is known that for \( m < m_c := (n-2)/n \) all solutions with initial data satisfying
\[
u_0(x) = O \left( |x|^{-\frac{2}{1-m}} \right) \quad \text{as} \quad |x| \to \infty,
\]
extinguish in finite time. We shall consider solutions which vanish at \( \tau = T \) and study their behaviour near \( \tau = T \).

The function
\[
u(x, \tau) := ((1-m)(T-\tau))^{\frac{1}{1-m}} \varphi_{\frac{1}{m}}(x)
\]
(1.2)
is a solution of the fast diffusion equation \( u_\tau = \Delta(u^m) \) if \( \varphi \) satisfies
\[
\Delta \varphi + \varphi^p = 0, \quad x \in \mathbb{R}^n, \quad p := \frac{1}{m}.
\]
(1.3)
We call a nontrivial solution of the form (1.2) separable. We shall show that separable solutions are stable in a suitable sense if

\[ n > 10, \quad 0 < m < \frac{(n - 2)(n - 10)}{(n - 2)^2 - 4n + 8\sqrt{n - 1}}. \]

We also find optimal rates of convergence to separable solutions.

To formulate our results in more detail, it is convenient to introduce the following change of variables:

\[ v(x, t) := ((1 - m)(T - \tau))^{-\frac{m}{1-m}} u^m(x, \tau), \quad t := -\frac{1}{1-m} \ln \frac{T - \tau}{T}. \]

If \( u \) is a solution of (1.1) with extinction at \( \tau = T \) then \( v \) satisfies

\[
\begin{cases}
  (v^p)_t = \Delta v + v^p, & x \in \mathbb{R}^n, \ t > 0, \quad p = \frac{1}{m}, \\
  v(x, 0) = v_0(x), & x \in \mathbb{R}^n,
\end{cases}
\]

where

\[ v_0(x) := ((1 - m)T)^{-\frac{m}{1-m}} u^m_0(x). \]

The behaviour of \( u \) as \( \tau \to T \) corresponds to the behaviour of \( v \) as \( t \to \infty \). Hence, asymptotically separable extinction of \( u \) is equivalent to convergence of \( v \) to a steady state as \( t \to \infty \).

Concerning the existence of positive solutions of (1.3), it is well known that the Sobolev exponent

\[ p_S := \begin{cases} 
  \frac{n + 2}{n - 2} & \text{if } n > 2, \\
  \infty & \text{if } n \leq 2,
\end{cases} \]

plays a crucial role. Namely, there is a family of positive radial solutions of (1.3) if and only if \( p \geq p_S \). We denote the solution by \( \varphi = \varphi_\alpha(r) \), \( r = |x| \), \( \alpha > 0 \), where \( \varphi_\alpha(r) \) satisfies

\[
\begin{cases}
  (\varphi_\alpha)_{rr} + \frac{n - 1}{r}(\varphi_\alpha)_r + \varphi_\alpha^p = 0, & r > 0, \\
  \varphi_\alpha(0) = \alpha, \quad (\varphi_\alpha)_r(0) = 0.
\end{cases}
\]

For each \( \alpha > 0 \), the solution \( \varphi_\alpha \) is decreasing in \( |x| \) and satisfies \( \varphi_\alpha(|x|) \to 0 \) as \( |x| \to \infty \).

For the structure of the set of steady states, there is another critical exponent

\[ p_c := \begin{cases} 
  \frac{(n - 2)^2 - 4n + 8\sqrt{n - 1}}{(n - 2)(n - 10)} & \text{if } n > 10, \\
  \infty & \text{if } n \leq 10.
\end{cases} \]

It is known that for \( p_S \leq p < p_c \), any positive steady state intersects with other positive steady states (see [23]). For \( p \geq p_c \), Wang [23] showed that the family of steady states \( \{ \varphi_\alpha; \alpha \in \mathbb{R} \} \) is ordered, that is, \( \varphi_\alpha \) is increasing in \( \alpha \) for each \( x \). Moreover,

\[ \lim_{\alpha \to 0} \varphi_\alpha(|x|) = 0, \quad \lim_{\alpha \to \infty} \varphi_\alpha(|x|) = \varphi_\infty(|x|). \]
where $\varphi_\infty$ is a singular steady state given by

$$\varphi_\infty(|x|) := L|x|^{-\nu}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

with

$$\nu := \frac{2}{p - 1}, \quad L := \left\{\nu (n - 2 - \nu)\right\}^{1/(p-1)}.$$  \hfill (1.6)

Each positive steady state satisfies

$$\varphi_\alpha(|x|) = L|x|^{-\nu} - a_\alpha|x|^{-\nu-\lambda_1} + o(|x|^{-\nu-\lambda_1}) \quad \text{as } |x| \to \infty,$$  \hfill (1.7)

where $\lambda_1$ is a positive constant given by

$$\lambda_1 = \lambda_1(n, p) := \frac{n - 2 - 2\nu - \sqrt{(n - 2 - 2\nu)^2 - 8(n - 2 - \nu)^2}}{2},$$

and $a_\alpha > 0$ is such that

$$a_\alpha > a_\beta \quad \text{if } 0 < \alpha < \beta.$$  \hfill (1.8)

(see \cite{13}).

Our main results are contained in the following two theorems:

**Theorem 1.1** Let $n > 10$, $p > p_c$, $\alpha > 0$, and assume that $v_0$ is continuous in $\mathbb{R}^n$ and satisfies

$$0 \leq v_0(x) \leq L|x|^{-\nu}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and

$$|v_0(x) - \varphi_\alpha(|x|)| \leq b|x|^{-\gamma}, \quad x \in \mathbb{R}^n, \quad |x| > 1,$$  \hfill (1.10)

with some $b > 0$ and

$$\gamma \in \left(\nu + \lambda_1, \frac{n - 2}{2}\right).$$  \hfill (1.11)

Then there exists $C > 0$ such that the solution $v$ of (1.4) has the property that

$$|v(x, t) - \varphi_\alpha(|x|)| \leq Ce^{-\kappa(\gamma)t}, \quad x \in \mathbb{R}^n, \quad t \geq 0,$$  \hfill (1.12)

where the positive number $\kappa(\gamma)$ is given by

$$\kappa(\gamma) := \frac{\gamma(n - 2 - \gamma)}{pL^{p-1}} - 1 = \frac{\gamma(n - 2 - \gamma)}{(\nu + 2)(n - 2 - \nu)} - 1.$$  \hfill (1.13)

The convergence rate provided by Theorem 1.1 is in fact optimal for stabilization from above or from below:

**Theorem 1.2** Let $n > 10$, $p > p_c$, $\alpha > 0$ and $b > 0$, and assume that $\gamma$ and $\kappa(\gamma)$ are as in (1.11) and (1.13), respectively. Moreover, let $v$ denote the solution of (1.4) corresponding to some continuous $v_0$.

(i) If

$$v_0(x) \geq \varphi_\alpha(|x|) + b(|x| + 1)^{-\gamma}, \quad x \in \mathbb{R}^n,$$  \hfill (1.14)
then there exists $C > 0$ such that
\[
v(0,t) - \varphi_\alpha(0) \geq C e^{-\kappa(\gamma)t}, \quad t \geq 0.
\] (1.15)

(ii) Under the assumption that
\[
v_0(x) \leq \varphi_\alpha(|x|) - b(|x| + 1)^{-\gamma}, \quad x \in \mathbb{R}^n,
\] (1.16)
one can find $C > 0$ such that
\[
\varphi_\alpha(0) - v(0,t) \geq C e^{-\kappa(\gamma)t}, \quad t \geq 0.
\] (1.17)

Theorems 1.1 and 1.2 imply that there is a continuum of rates of convergence of solutions of (1.4) to $\varphi_\alpha$, $\alpha > 0$, since the range of $\kappa$ is
\[
\left(0, \frac{(n-2)^2}{4pL^{p-1}} - 1\right).
\]

Notice that $\kappa(\gamma) \to 0$ as $\gamma \to \nu + \lambda_1$.

Asymptotically separable extinction of solutions of (1.1) has only been known to occur for $m = (n-2)/(n+2) = 1/p_S$, $n > 2$. This was predicted in [15] and then rigorously established in [12], [7], Theorem 7.10 in [22]. The exponent $m = 1/p_S$ is the unique value of $m$ for which the second-kind selfsimilar solution is separable, see [15], [16], [19]. The rate of convergence to a separable solution is not known in this case.

Depending on the decay rate of the initial function $u_0$, other types of asymptotic behaviour near extinction may occur, such as convergence to Barenblatt profiles (see [3], [4], [5], [8], [9], [10], for example) or convergence to selfsimilar solutions of the second type (see [12], [22]).

For a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ all bounded positive solutions of $u_\tau = \Delta(u^m)$ with the homogeneous Dirichlet boundary condition and $(n-2)/(n+2) < m < 1$ extinguish in finite time and they approach separable solutions, see [11], [2], [6], [17], [21].

But little is known about the convergence rate. As far as we know, only upper bounds for the decay rates of the entropy and of a weighted $L^2$-norm of the relative distance from the separable solution are given in [6] for $m$ sufficiently close to 1.

Let us also compare Theorems 1.1 and 1.2 with corresponding results from [11], [14] on convergence to steady states for the problem
\[
\begin{cases}
v_t = \Delta v + v^p, & x \in \mathbb{R}^n, \quad t > 0, \quad p > p_c, \\
v(x,0) = v_0(x), & x \in \mathbb{R}^n.
\end{cases}
\] (1.18)

For both problems, the steady states are the same and we obtain a continuum of convergence rates which depend explicitly on the tail of initial functions. But for (1.4) the rates are exponential while for (1.18) they are algebraic, see [11], [14].

Similarly as for (1.18), the steady states $\varphi_\alpha$ are unstable from below and from above for (1.4) when $p_S \leq p < p_c$. 

4
Proposition 1.3 Let $n > 2$, $p_S \leq p < p_c$, $\alpha > 0$, and assume that $v_0$ is bounded and continuous in $\mathbb{R}^n$.

(i) If
\[
    v_0(x) \geq \varphi_{\alpha}(|x|), \quad x \in \mathbb{R}^n, \quad v_0 \neq \varphi_{\alpha},
\]
then
\[
    \|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to \infty \quad \text{as} \quad t \to \infty.
\]

(ii) If
\[
    0 \leq v_0(x) \leq \varphi_{\alpha}(|x|), \quad x \in \mathbb{R}^n, \quad v_0 \neq \varphi_{\alpha},
\]
then there is $T \in (0, \infty]$ such that
\[
    \|v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as} \quad t \to T.
\]

The paper is organised as follows. In Section 2 we collect some preliminaries. In Section 3 we prove Theorem 1.1 for radially symmetric solutions with $v_0 \geq \varphi_{\alpha}$. In Section 4 we give a corresponding bound for $v_0 \leq \varphi_{\alpha}$ and we complete the proof of Theorem 1.1. Sections 5 and 6 are devoted to the proof of Theorem 1.2 and Section 7 to Proposition 1.3.

2 Preliminaries

2.1 An ODE lemma

The following statement plays a key role in our analysis. It describes the spatial profile of separated solutions to the formal linearization of (1.4) around a given steady state $\varphi_{\alpha}$.

Lemma 2.1 Let $\alpha > 0$ and $\kappa$ be positive and such that
\[
    \kappa < \kappa_0(p) := \frac{(n-2)^2}{4pL^{p-1}} - 1. \quad (2.1)
\]
Then the solution $f = f_{\alpha,\kappa}$ of the initial-value problem
\[
    \begin{cases}
        f_{rr} + \frac{n-1}{r}f_r + (\kappa + 1)p\varphi_{\alpha}^{p-1}f = 0, & r > 0, \\
        f(0) = 1, \quad f_r(0) = 0,
    \end{cases} \quad (2.2)
\]
is positive on $[0, \infty)$, and there exist $c > 0$ and $C > 0$ such that
\[
    cr^{-\gamma} \leq f(r) \leq Cr^{-\gamma} \quad \text{for all} \quad r > 1, \quad (2.3)
\]
where $\gamma = \gamma(\kappa)$ is the positive number given by
\[
    \gamma(\kappa) := \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - (\kappa + 1)pL^{p-1}}. \quad (2.4)
\]
Proof. We first note that as an evident consequence of (2.1) and the fact that $n \geq 3$, the number $\gamma = \gamma(\kappa)$ given by (2.4) indeed is real and positive. Since $f(0) = 1$, we know that

$$r_* := \sup \left\{ r > 0 \mid f > 0 \text{ on } [0, r] \right\}$$

is a well-defined element of $(0, \infty)$, and for $\vartheta > 0$ we let

$$h(r) \equiv h_\vartheta(r) := r^\vartheta f(r), \quad r \in [0, r_*).$$

Then

$$h(0) = 0 \quad \text{and} \quad h > 0 \text{ on } (0, r_*),$$

and since

$$f_r(r) = r^{-\vartheta} h_r(r) - \vartheta r^{-\vartheta-1} h(r), \quad r \in (0, r_*),$$

and

$$f_{rr}(r) = r^{-\vartheta} h_{rr}(r) - 2\vartheta r^{-\vartheta-1} h_r(r) + \vartheta(\vartheta + 1) r^{-\vartheta-2} h(r), \quad r \in (0, r_*),$$

from (2.2) we obtain that

$$0 = r^{-\vartheta} h_{rr} - 2\vartheta r^{-\vartheta-1} h_r + \vartheta(\vartheta + 1) r^{-\vartheta-2} h + \frac{n - 1}{r} \left\{ r^{-\vartheta} h_r - \vartheta r^{-\vartheta-1} h \right\} + (\kappa + 1) p\varphi_\alpha^{p-1} r^{-\vartheta} h$$

$$= r^{-\vartheta} \left\{ h_{rr} + \frac{n - 1 - 2\vartheta}{r} h_r + \frac{\vartheta(\vartheta + 2 - n) + (\kappa + 1) p\varphi_\alpha^{p-1} r^2}{r^2} h \right\}$$

for all $r \in (0, r_*).$ We first apply this to $\vartheta := \gamma$ and $h := h_\gamma$ and use that

$$\varphi_\alpha(r) \leq L r^{-\nu}$$

for all $r > 0$, which by positivity of $h$ on $(0, r_*)$ allows us to conclude that

$$0 \leq h_{rr} + \frac{n - 1 - 2\gamma}{r} h_r + \frac{\gamma(\gamma + 2 - n) + (\kappa + 1) pL^{p-1}}{r^2} h$$

for all $r \in (0, r_*).$ (2.7)

But the definition (2.4) entails that $\gamma$ coincides with the smaller root of the equation

$$\gamma(\gamma + 2 - n) + (\kappa + 1) pL^{p-1} = 0,$$

whence (2.7) actually says that

$$0 \leq h_{rr} + \frac{n - 1 - 2\gamma}{r} h_r = r^{-n-1+2\gamma} \left( r^{n-1-2\gamma} h_r \right)_r$$

for all $r \in (0, r_*).$ Now according to (2.5) it is clearly possible to fix $r_1 \in (0, r_*)$ small enough such that $r_1 < 1$ and $h_r(r_1) \geq 0$. Therefore, integrating (2.9) over $(r_1, r)$ for arbitrary $r \in (r_1, r_*)$ yields

$$r^{n-1-2\gamma} h_r(r) \geq r_1^{n-1-2\gamma} h_r(r_1) \geq 0 \quad \text{for all } r \in (r_1, r_*).$$
so that
\[ h(r) \geq c_1 \quad \text{for all } r \in (r_1, r_*), \]
where \( c_1 := h(r_1) \) is positive thanks to (2.5). In particular, this ensures that necessarily \( r_* = \infty \), and moreover we obtain that
\[ f(r) = r^{-\gamma} h(r) \geq c_1 r^{-\gamma} \quad \text{for all } r > r_1, \]
which implies the left inequality in (2.3), because \( r_1 < 1 \).

Our derivation of the upper estimate for \( f \) in (2.3) will proceed in two steps. First, since \( \gamma < (n - 2)/2 \) by (2.4) and (2.1), we can pick \( \delta > 0 \) such that
\[ \delta < n - 2 - 2\gamma \quad \text{(2.10)} \]
and \( \delta \leq \lambda_1 \). We then fix \( \varepsilon > 0 \) small such that
\[ \gamma \varepsilon := \frac{n - 2}{2} - \sqrt{\frac{(n - 2)^2}{4} - (\kappa + 1)(pL^{p-1} - \varepsilon)} \quad \text{(2.11)} \]
satisfies
\[ \gamma \varepsilon > \gamma - \delta, \quad \text{(2.12)} \]
which can clearly be achieved by a continuity argument. Now as a consequence of (1.7) we can find \( c_2 > 0 \) such that
\[ \varphi_\alpha(r) \geq Lr^{-\nu} - c_2 r^{-\nu - \lambda_1} \quad \text{for all } r > 0, \quad \text{(2.13)} \]
whence in particular \( \varphi_\alpha^{p-1}(r)^2 \to L^{p-1} \) as \( r \to \infty \). Accordingly, there exists \( r_2 > 0 \) such that
\[ p\varphi_\alpha^{p-1}(r)^2 \geq pL^{p-1} - \varepsilon \quad \text{for all } r \geq r_2, \]
so that (2.6) applied to \( \vartheta := \gamma \varepsilon \) shows that \( h := h_{\gamma \varepsilon} \) satisfies
\[ 0 \geq h_{rr} + \frac{n - 2 - 2\gamma \varepsilon}{r} h_r + \frac{\gamma \varepsilon (\gamma \varepsilon + 2 - n) + (\kappa + 1)(pL^{p-1} - \varepsilon)}{r^2} h \quad \text{for all } r \geq r_2. \]
Due to (2.11), the zero-order term again vanishes, and hence we have
\[ 0 \geq h_{rr} + \frac{n - 1 - 2\gamma \varepsilon}{r^2} h_r = r^{n+1+2\gamma \varepsilon} (r^{n-1-2\gamma \varepsilon} h_r)_r \quad \text{for all } r \geq r_2, \]
on successive integration implying that
\[ r^{n-1-2\gamma \varepsilon} h_r(r) \leq c_3 := r_{2}^{n-1-2\gamma \varepsilon} h_r(r_2) \quad \text{for all } r \geq r_2 \]
and
\[ h(r) \leq h(r_2) + c_3 \int_{r_2}^{r} \rho^{1-n+2\gamma \varepsilon} d\rho \quad \text{for all } r \geq r_2. \]
Since from (2.11) we know that $\gamma_\epsilon < (n - 2)/2$ and thus $1 - n + 2\gamma_\epsilon < -1$, from this and 
the definition of $h = h_\gamma$, we infer that
\[
f(r) = r^{-\gamma_\epsilon}h(r) \leq c_4 r^{-\gamma_\epsilon} \quad \text{for all } r \geq r_2
\]
with
\[
c_4 := h(r_2) + \frac{c_3 r_2^{2-n+2\gamma_\epsilon}}{n-2-2\gamma_\epsilon}.
\]

In order to finally derive the second inequality in (2.3), from this, we fix $c_5 > 0$ and $r_3 > r_2$ large enough fulfilling
\[
(1 - \delta)^p-1 \geq 1 - c_5 \delta \quad \text{for all } \delta \in [0, 1/2] \quad \text{and} \quad \frac{c_2 r_3^{-\lambda_1}}{L} \leq \frac{1}{2}.
\]

Then (2.13) says that for all $r \geq r_3$ we can estimate
\[
p^{p-1}(r)r^2 \geq p^{p-1}(Lr^{-\nu} - c_2 r^{-\lambda_1})^p \geq p L^{p-1} \left( 1 - \frac{c_2 r^{-\lambda_1}}{L} \right) \geq p L^{p-1} - \frac{c_2 r^{-\lambda_1}}{L}.
\]
with an evident definition of $c_6$. Hence, returning to our original choice $h = h_\gamma$, from (2.6) and (2.8) we obtain that
\[
0 \geq h_{rr} + r^{-n+1+2\gamma} \int_r \frac{\rho^{n-3-\gamma-\epsilon-\delta}}{h} \quad \text{for all } r \geq r_3.
\]
Here we use (2.14) and the fact that $\delta \leq \lambda_1$ in estimating
\[
c_6r^{-2-\lambda_1}h(r) = c_6r^{-2-\lambda_1} + \gamma f(r) \leq c_4c_6r^{-2-\lambda_1+\gamma-\epsilon} \leq c_7 r^{-2-\delta+\gamma-\epsilon} \quad \text{for all } r \geq r_3
\]
with $c_7 := c_4c_6 r_3^{-(\lambda_1-\delta)}$, so that an integration in (2.15) shows that if we abbreviate
\[
c_8 := r_3^{n-1-2\gamma}h_{r}(r_3),
\]
then
\[
r^{n-1-2\gamma}h_{r}(r) \leq c_8 + c_7 \int_{r_3}^{r} \rho^{n-3-\gamma-\epsilon-\delta} d\rho \quad \text{for all } r \geq r_3.
\]

Since the inequality $\gamma_\epsilon < \gamma$ along with (2.10) guarantees that
\[
n - 2 - \gamma - \gamma_\epsilon - \delta > n - 3 - 2\gamma - \delta > 0,
\]
we have
\[
\int_{r_3}^{r} \rho^{n-3-\gamma-\epsilon-\delta} d\rho \leq \frac{r^{n-2-\gamma-\epsilon-\delta}}{n - 2 - \gamma - \gamma_\epsilon - \delta} \quad \text{for all } r \geq r_3,
\]
so that (2.16) implies that
\[
h_{r}(r) \leq c_8 r^{-n+1+2\gamma} + c_9 r^{-1+\gamma-\epsilon-\delta} \quad \text{for all } r \geq r_3, \quad c_9 := \frac{c_7}{n - 2 - \gamma - \gamma_\epsilon - \delta}.
\]
In view of our restriction (2.12) on $\gamma_\varepsilon$, from this we obtain
\[
h(r) \leq h(r_3) + \frac{c_8}{n-2-2\gamma} + \frac{c_9}{\gamma_\varepsilon - \gamma + \delta}
\] for all $r \geq r_3$.

By definition of $h = h_\gamma$ and the positivity of $f$ on $[0, r_3]$, this finally completes the proof of (2.3).

2.2 Two auxiliary asymptotic estimates

The following two auxiliary lemmata will be used both in Lemma 3.4 and in Lemma 6.1 in order to provide appropriate control of certain higher order expressions arising during linearization.

**Lemma 2.2** Let $\alpha > 0$, $\beta > \alpha$, $\mu > 0$ and $\kappa \in (0, \kappa_0(p))$, where $\kappa_0(p)$ is as defined in (2.1). Then for all $A \geq 0$ and $B \geq 0$ there exist $r_0 > 0$ and $c > 0$ such that with $f_{\mu,\kappa}$ as in (2.2) we have
\[
\left\{ \varphi_\beta(r) - B f_{\mu,\kappa}(r) \right\}^{p-1} - \left\{ \varphi_\alpha(r) + A f_{\mu,\kappa}(r) \right\}^{p-1} \geq cr^{-2-\lambda_1}
\] for all $r > r_0$. (2.17)

**Proof.** Since $\beta > \alpha$ and hence $a_\beta < a_\alpha$ by (1.3), we can fix some small $\eta > 0$ such that
\[
c_1 := (p-1-\eta)(a_\alpha - \eta) - (p-1+\eta)(a_\beta + \eta) > 0.
\] (2.18)

Then in view of (1.7) we can find $r_1 > 1$ fulfilling
\[
\varphi_\alpha(r) \leq L r^{-\nu} - \left( a_\alpha - \frac{\eta}{2} \right) r^{-\nu - \lambda_1}
\] for all $r > r_1$ (2.19)

and
\[
\varphi_\beta(r) \geq L r^{-\nu} - \left( a_\beta + \frac{\eta}{2} \right) r^{-\nu - \lambda_1}
\] for all $r > r_1$. (2.20)

Now from Lemma 2.1 we know that there exists $c_2 > 0$ such that $f = f_{\mu,\kappa}$ satisfies
\[
f(r) \leq c_2 r^{-\gamma}
\] for all $r > 1$ with $\gamma = \gamma(\kappa)$ given by (2.4). As $\gamma > \nu + \lambda_1$, we can therefore choose $r_2 > r_1$ in such a way that
\[
\max\{A, B\} f(r) \leq \frac{\eta}{2} r^{-\nu - \lambda_1}
\] for all $r > r_2$. Together with (2.19) and (2.20), this yields the inequalities
\[
\varphi_\alpha(r) + A f(r) \leq L r^{-\nu} - (a_\alpha - \eta) r^{-\nu - \lambda_1}
\] for all $r > r_2$ (2.21)

and
\[
\varphi_\beta(r) + B f(r) \geq L r^{-\nu} - (a_\beta + \eta) r^{-\nu - \lambda_1}
\] for all $r > r_2$. (2.22)
We next take \( z_1 > 0 \) small enough fulfilling
\[
1 - (p - 1 + \eta)z \leq (1 - z)^p \leq 1 - (p - 1 - \eta)z \quad \text{for all } z \in [0, z_1], \tag{2.23}
\]
and then fix \( r_3 > r_2 \) such that
\[
\frac{a_\alpha - \eta}{L} r_3^{-\lambda_1} \leq z_1,
\]
which by (2.18) implies that also
\[
\frac{a_\beta + \eta}{L} r_3^{-\lambda_1} \leq z_1.
\]
Hence, (2.21) and the second inequality in (2.23) show that
\[
\left\{ \psi_\alpha(r) + Af(r) \right\}^{p-1} \leq \left( Lr^{-\nu} \right)^{p-1} \left\{ 1 - \frac{a_\alpha - \eta}{L} r_3^{-\lambda_1} \right\}^{p-1}
\]
\[
\leq \left( Lr^{-\nu} \right)^{p-1} \left\{ 1 - (p - 1 - \eta) \frac{a_\alpha - \eta}{L} r_3^{-\lambda_1} \right\}
\]
\[
= L^{p-1} r^{-2} - (p - 1 - \eta)(a_\alpha - \eta)L^{p-1} r^{-2-\lambda_1} \tag{2.24}
\]
for all \( r > r_3 \), whereas similarly (2.22) and the first inequality in (2.23) entail that for any such \( r \) we have
\[
\left\{ \psi_\beta(r) - Bf(r) \right\}^{p-1} \geq \left( Lr^{-\nu} \right)^{p-1} \left\{ 1 - (p - 1 + \eta) \frac{a_\beta + \eta}{L} r_3^{-\lambda_1} \right\}
\]
\[
= L^{p-1} r^{-2} - (p - 1 + \eta)(a_\beta + \eta)L^{p-1} r^{-2-\lambda_1}. \tag{2.25}
\]
In light of (2.18), the desired estimate immediately results from (2.24) and (2.25).

**Lemma 2.3** Let \( \alpha > 0, \mu > 0 \) and \( \kappa \in (0, \kappa_0(p)) \) with \( \kappa_0(p) \) from (2.1). Then there exists \( C > 0 \) such that
\[
\varphi^{p-2}_\alpha(r)f_{\mu,\kappa}(r) \leq C(r + 1)^{\nu-\gamma-2} \quad \text{for all } r \geq 0, \tag{2.26}
\]
where \( \gamma = \gamma(\kappa) \) is as in (2.4).

**Proof.** We apply (1.7) and (2.3) to find positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1 r^{-\nu} \leq \varphi_\alpha(r) \leq Lr^{-\nu} \quad \text{for all } r > 1, \tag{2.27}
\]
and such that \( f := f_{\mu,\kappa} \) satisfies
\[
f(r) \leq c_2 r^{-\gamma} \quad \text{for all } r > 1. \tag{2.28}
\]
Writing \( c_3 := c_2 \max\{c_1^{p-2}, L^{p-2}\} \), we therefore see that
\[
\varphi^{p-2}_\alpha(r)f(r) \leq c_3 r^{\nu-\gamma-2} \leq c_2 2^{\nu-\gamma-2}(r + 1)^{\nu-\gamma-2} \quad \text{for all } r > 1,
\]
because \( \gamma > \nu + \lambda_1 \) implies that \( \nu - \gamma - 2 < 0 \). Since for \( r \in [0, 1] \), (2.26) is obvious from the positivity of \( \varphi_\alpha \) and the boundedness of both \( \varphi_\alpha \) and \( f \), the proof is complete. \( \square \)
3 Convergence from above: upper bound

In our first estimate of solutions from above, in Lemma 3.2 below, we shall apply a comparison argument involving comparison functions which monotonically decrease with time. The initial data of the latter will be constructed separately in the following lemma.

**Lemma 3.1** Assume that \( \alpha > 0 \) and \( \kappa \in (0, \kappa_0(p)) \) with \( \kappa_0(p) \) as in (2.1), and let \( f_{\alpha,\kappa} \) denote the corresponding solution of (2.2). Then there exists \( A_0 = A_0(\alpha, \kappa) > 0 \) with the property that given any \( A > A_0 \) one can find \( r_A > 1 \) such that for each \( \epsilon \in (0, 1) \), the number

\[
  r_{A\epsilon} := \sup \left\{ \tilde{r} > 0 \left| \varphi_\alpha(r) + Af_{\alpha,\kappa}(r) > L(r + \epsilon)^{-\nu} \right. \text{ for all } r \in (0, \tilde{r}) \right\}
\]

is well-defined with

\[
  1 \leq r_{A\epsilon} \leq r_A,
\]

and

\[
  v_{0A\epsilon}(r) := \begin{cases} 
  L(r + \epsilon)^{-\nu} & \text{if } r \in (0, r_{A\epsilon}), \\
  \varphi_\alpha(r) + Af_{\alpha,\kappa}(r) & \text{if } r \geq r_{A\epsilon},
  \end{cases}
\]

determines a positive function \( v_{0A\epsilon} \in W^{1,\infty}_{\text{loc}}((0, \infty)) \cap C^2((0, \infty) \setminus \{r_{A\epsilon}\}) \) which satisfies

\[
  (v_{0A\epsilon})_{rr} + \frac{n-1}{r}(v_{0A\epsilon})_r + v_{0A\epsilon}^p \leq 0 \quad \text{for all } r \in (0, \infty) \setminus \{r_{A\epsilon}\}
\]

as well as

\[
  \lim \inf_{r \nearrow r_{A\epsilon}} (v_{0A\epsilon})_r \geq \lim \sup_{r \searrow r_{A\epsilon}} (v_{0A\epsilon})_r.
\]

**Proof.** According to (1.7), we can fix \( c_1 > 0 \) such that

\[
  \varphi_\alpha(r) \geq Lr^{-\nu} - c_1 r^{-\nu - \lambda_1} \quad \text{for all } r > 0.
\]

Since \( \kappa > 0 \), we can then take \( z_1 \in (0, 1) \) such that

\[
  (1 + z)^p \leq 1 + (\kappa + 1)pz \quad \text{for all } z \in [0, z_1],
\]

and pick \( r_1 > 1 \) large fulfilling

\[
  \frac{2c_1}{L} r_1^{-\lambda_1} \leq z_1.
\]

We thereupon choose \( A_0 > 0 \) large enough such that with \( f := f_{\alpha,\kappa} \) we have

\[
  \varphi_\alpha(r) + Af(r_1) > Lr_1^{-\nu} \quad \text{for all } A > A_0,
\]

and let \( A > A_0 \) and \( \epsilon \in (0, 1) \) be given. To see that then the set on the right-hand side of (3.1) is nonempty and bounded, we first note that by (3.9) we have

\[
  \varphi_\alpha(r) + Af(r_1) > L(r_1 + \epsilon)^{-\nu}.
\]
Moreover, again by (1.7) and by (2.3) there exist $c_2 > 0$ and $c_3 > 0$ such that
\[ \varphi_\alpha(r) \leq L r^{-\nu} - c_2 r^{-\nu - \lambda_1} \quad \text{for all } r > 1 \]
and
\[ f(r) \leq c_3 r^{-\gamma(\kappa)} \quad \text{for all } r > 1 \]
with $\gamma(\kappa)$ as in (2.4). Since the positivity of $\kappa$ implies that
\[ \gamma(\kappa) > \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - pL^{p-1}} = \nu + \lambda_1, \]
for some large $r_2(A)$ we thus obtain
\[ \varphi_\alpha(r) + A f(r) \leq L r^{-\nu} - \frac{c_2}{2} r^{-\nu - \lambda_1} \quad \text{for all } r > r_2(A). \] (3.10)

On the other hand, by convexity of $[0, \infty) \ni z \mapsto (1 + z)^{-\nu}$, we know that
\[ L(r + \varepsilon)^{-\nu} = L r^{-\nu} \left(1 + \frac{\varepsilon}{r}\right)^{-\nu} \geq L r^{-\nu} - \nu \varepsilon L r^{-\nu - 1} \quad \text{for all } r > 0. \] (3.11)
Since it can easily be checked that $\lambda_1 > 1$, combining (3.10) with (3.11) we obtain $r_3(A) > 0$ such that
\[ \varphi_\alpha(r) + A f(r) < L(r + \varepsilon)^{-\nu} \quad \text{for all } r > r_3(A). \]

Having thereby shown that $r_\varepsilon$ is well-defined and satisfies
\[ 1 < r_1 < r_\varepsilon < r_3(A), \] (3.12)
we let $v_{0\varepsilon}$ be given by (3.3) and proceed to verify (3.4). To this end, for small $r$ we use the definition (1.3) of $L$ in estimating
\[ (v_{0\varepsilon})_{rr} + \frac{r - 1}{r}(v_{0\varepsilon})_r + v_{0\varepsilon}^p = -\nu(n - 1)L(r + \varepsilon)^{-\nu - 1} v_{0\varepsilon}^p \left(1 - \frac{1}{r + \varepsilon}\right) < 0 \]
for all $r < r_\varepsilon$.

In the corresponding outer region, we first observe that from the definition of $r_\varepsilon$ it follows that
\[ \varphi_\alpha(r) + A f(r) \leq L(r + \varepsilon)^{-\nu} \quad \text{for all } r > r_\varepsilon, \]
so that according to (3.6),
\[ Af(r) \leq L(r + \varepsilon)^{-\nu} - \left(L r^{-\nu} - c_1 r^{-\nu - \lambda_1}\right) \leq c_1 r^{-\nu - \lambda_1} \quad \text{for all } r > r_\varepsilon. \] (3.13)

In conjunction with (3.5) and (3.12), (3.6) furthermore guarantees that
\[ \varphi_\alpha(r) \geq L r^{-\nu} \left(1 - \frac{c_1}{L} r^{-\lambda_1}\right) \geq L r^{-\nu} \left(1 - \frac{c_1}{L} r_1^{-\lambda_1}\right) \geq L r^{-\nu} \left(1 - \frac{z_1}{2}\right) = \frac{L}{2} r^{-\nu} \]
for all $r > r_A$, because $z_1 < 1$. Therefore, (3.13) and again (3.8) and (3.6) yield
\[ Af(r) < \frac{2c_1}{L}r^{-\lambda_1} \leq z_1 \quad \text{for all } r > r_A, \]
and hence (3.7) applies to show that the nonlinear term in (3.4) can be estimated according to
\[ v^p_{0,A} = \varphi^p_\alpha(r) \left( 1 + \frac{Af(r)}{\varphi^p_\alpha(r)} \right)^p \leq \varphi^p_\alpha(r) \left( 1 + (\kappa + 1)p \frac{Af(r)}{\varphi^p_\alpha(r)} \right) \]
\[ = \varphi^p_\alpha(r) + (\kappa + 1)pf_\alpha^{-1}(r)f(r) \quad \text{for all } r > r_A. \]
Since $\varphi_\alpha$ satisfies (1.5), by (2.2) we have
\[ (v^p_{0,A})_r + \frac{n-1}{r}(v_{0,A})_r + v^p_{0,A} = (\varphi^p)_r + Af_r + \frac{n-1}{r}(\varphi_\alpha)_r + A\frac{n-1}{r}f_r + v^p_{0,A} \]
\[ \leq (\varphi^p)_r + \frac{n-1}{r}(\varphi_\alpha)_r + \varphi^p + A\left\{f_r + \frac{n-1}{r}f_r + (\kappa + 1)\varphi_\alpha^{-1}f\right\} = 0 \]
for all $r > r_A$, which completes the proof of (3.4).

To conclude, it only remains to observe that if we let $r_A := r_\ell(A)$ then (3.12) implies (3.2), and that the claimed regularity properties of $v_{0,A}$ and (3.5) are immediate consequences of the smoothness of $\varphi_\alpha$ and the definition of $r_A$. □

We subsequently consider the radial version of (1.4), that is, we investigate nonnegative solutions of
\[ \left\{ \begin{array}{lcl} (v^p)_t &=& v_{rr} + \frac{n-1}{r}v_r + v^p, & r > 0, \ t > 0, \\ v(r,0) &=& \varphi_\alpha(r), & r \geq 0, \end{array} \right. \]
(3.14)
and to this end we introduce the operator $\mathcal{P}$ defined by
\[ \mathcal{P}v := (v^p)_t - v_{rr} - \frac{n-1}{r}v_r - v^p. \]
(3.15)
Then given radial initial data above $\varphi_\alpha$ but suitably close to $\varphi_\alpha$ asymptotically, we may compare the corresponding solution $v$ with certain solutions of (3.14), emanating from appropriate initial data taken from Lemma 3.1 to show that the deviation $v - \varphi_\alpha$ essentially maintains its spatial decay throughout evolution. We can moreover make sure that $v$ approaches $\varphi_\alpha$ in the large time limit, yet without any information on the rate of convergence.

**Lemma 3.2** Let $\alpha > 0$, and assume that $v_0$ satisfies (1.9) and
\[ \varphi_\alpha(r) \leq v_0(r) \leq \varphi_\alpha(r) + br^{-\gamma} \quad \text{for all } r > 1 \]
(3.16)
with some $b > 0$ and $\gamma \in (\nu + \lambda_1, (n-2)/2)$. Then for the solution $v$ of (3.14) we have
\[ \sup_{r \geq 0} |v(r,t) - \varphi_\alpha(r)| \to 0 \quad \text{as } t \to \infty. \]
(3.17)
Moreover, there exists $C > 0$ such that
\[ v(r,t) \leq \varphi_\alpha(r) + C(r + 1)^{-\gamma} \quad \text{for all } r \geq 0 \text{ and } t \geq 0. \]
(3.18)
PROOF. It can easily be checked that since \( \gamma \in (\nu + \lambda_1, (n - 2)/2) \), the number \( \kappa \) introduced in (1.13) is positive and satisfies (2.3), and that with \( \gamma(\kappa) \) as in (2.4) we have \( \gamma = \gamma(\kappa) \). Therefore, Lemma 2.1 applies to yield \( c_1 > 0 \) such that the solution \( f = f_{\alpha, \kappa} \) of (2.2) satisfies

\[
f(r) \geq c_1 r^{-\gamma} \quad \text{for all } r > 1.
\]

(3.19)

Taking \( b \) and \( A_0 \) from (3.16) and Lemma 3.1 respectively, we now fix \( A > A_0 \) such that

\[
A \geq \frac{b}{c_1},
\]

(3.20)

and let \( r_A > 1 \) be as provided by Lemma 3.1. Then since \( v_0(r) < Lr^{-\nu} \) for all \( r > 0 \) by (1.9), the function \( \chi \) given by

\[
\chi(r) := \left( \frac{L}{v_0(r)} \right)^{\frac{\nu - 1}{2}} - r, \quad r \in [0, r_A],
\]

is positive, and hence \( \varepsilon_0 := \min_{r \in [0, r_A]} \chi(r) \) satisfies \( \varepsilon_0 > 0 \). This enables us to finally choose \( \varepsilon \in (0,1) \) such that \( \varepsilon < \varepsilon_0 \), and let \( r_{A\varepsilon} \in (1, r_A) \) and \( v_{0A\varepsilon} \) be as given by Lemma 3.1. Then since \( r_{A\varepsilon} > 1 \), (3.16), (3.19) and (3.20) imply that

\[
v_0(r) - v_{0A\varepsilon}(r) \leq (\varphi_\alpha(r) + br^{-\gamma}) - (\varphi_\alpha(r) + Af(r)) \leq br^{-\gamma} - Ac_1r^{-\gamma} \leq 0
\]

for all \( r \geq r_{A\varepsilon} \), whereas the inequality \( r_{A\varepsilon} < r_A \) in combination with our choice of \( \chi \) and \( \varepsilon \) ensures that

\[
v_0(r) = L(r + \chi(r))^{-\nu} \leq L(r + \varepsilon_0)^{-\nu} \leq L(r + \varepsilon)^{-\nu} = v_{0A\varepsilon}(r) \quad \text{for all } r < r_{A\varepsilon}.
\]

This means that if we let \( \overline{v} \) denote the solution of

\[
\begin{align*}
(\overline{v})_r & = \frac{\nu - 1}{r} \overline{v} + \varphi_\alpha(r), \quad r > 0, \ t \in (0, T), \\
\overline{v}(r, 0) & = v_{0A\varepsilon}(r), \quad r > 0,
\end{align*}
\]

(3.21)

defined up to its maximal existence time \( T \in (0, \infty] \), then

\[
v_0(r) \leq v_{0A\varepsilon}(r) = \overline{v}(r, 0) \quad \text{for all } r \geq 0.
\]

(3.22)

But the properties (3.4) and (3.5) entail that \([0, \infty)^2 \ni (r, t) \mapsto v_{0A\varepsilon}(r)\) is a supersolution of (3.14) in the natural weak sense, which implies (20) that the solution \( \overline{v} \) of (3.21) satisfies

\[
\overline{v}_t(r, t) \leq 0 \quad \text{for all } r \geq 0 \text{ and } t \in (0, T).
\]

(3.23)

Since clearly \( \overline{v}(r, t) \geq \varphi_\alpha(r) \) for all \( r \geq 0 \) and \( t \in (0, T) \) by comparison, this entails that actually \( T = \infty \), and that hence

\[
\overline{v}(r, t) \searrow v_\infty(r) \quad \text{as } t \to \infty
\]

(3.24)
with some limit function $v_\infty$ fulfilling $\varphi_\alpha(r) \leq v_\infty(r) \leq v_{0,A}\epsilon(r)$ for all $r \geq 0$. By a straightforward parabolic regularity argument (13) based on the two-sided bound $\varphi_\alpha(r) \leq \pi(r,t) \leq v_{0,A}\epsilon(r)$, $(r,t) \in [0,\infty)^2$, the convergence in (3.23) can be seen to take place in $C^0([0,\infty)) \cap C^2_{loc}([0,\infty))$, which implies that $v_\infty$ is a stationary solution of (3.14). This means that with $\alpha' := v_\infty(0) \geq \varphi_\alpha(0) = \alpha$ we must have $v_\infty \equiv \varphi_\alpha'$, so that for the verification of (3.17) it remains to be shown that $\alpha' \leq \alpha$.

Indeed, if we had $\alpha' > \alpha$ then by (1.8) we would have $a_\alpha > a_{\alpha'}$ and hence could pick $a$ and $a'$ such that $a_{\alpha'} < a' < a < a_\alpha$. By (1.7) we could thus find $r_1 > 0$ fulfilling

$$\varphi_\alpha'(r) \geq Lr^{-\nu} - a'r^{-\nu-\lambda_1} \quad \text{for all } r > r_1$$

and

$$\varphi_\alpha(r) \leq Lr^{-\nu} - ar^{-\nu-\lambda_1} \quad \text{for all } r > r_1,$$

whereas Lemma 2.4 in view of the fact that $\gamma > \nu + \lambda_1$ would allow us to fix $r_2 > 0$ satisfying

$$Af(r) < (a - a')r^{-\nu-\lambda_1} \quad \text{for all } r > r_2.$$  

With $r_3 := \max\{r_{A\epsilon}, r_1, r_2\}$ we would thus arrive at the absurd conclusion that

$$Lr^{-\nu} - a'r^{-\nu-\lambda_1} \leq \varphi_\alpha'(r) \leq v_{0,A}\epsilon(r) = \varphi_\alpha(r) + Af(r)$$

$$< Lr^{-\nu} - ar^{-\nu-1} + (a - a')r^{-\nu-\lambda_1}$$

$$= Lr^{-\nu} - a'r^{-\nu-\lambda_1} \quad \text{for all } r > r_3,$$

from which we infer that in fact $\alpha' = \alpha$ and that thus (3.17) is valid. Finally, (3.18) immediately results from (3.23), (3.24) and the upper estimate for $f$ in (2.3).

Another comparison argument shows that under the hypotheses of the last lemma, the solution will furthermore eventually lie below any of the equilibria which are larger than the asymptotic profile.

**Lemma 3.3** Let $v_0$ be such that (1.9) holds, and such that (3.16) is valid with some $\alpha > 0$, $b > 0$ and $\gamma \in (\nu + \lambda_1, (n-2)/2)$. Then for any $\alpha' > \alpha$ one can find $t_0 \geq 0$ such that the solution $v$ of (3.14) satisfies

$$v(r,t) \leq \varphi_\alpha'(r) \quad \text{for all } r \geq 0 \text{ and } t \geq t_0.$$  

**PROOF.** In accordance with Lemma 3.2 let us fix $c_1 > 0$ such that

$$v(r,t) \leq \varphi_\alpha(r) + c_1(r+1)^{-\gamma} \quad \text{for all } r \geq 0 \text{ and } t \geq 0.$$  

Since $\alpha' > \alpha$, by (1.8) we can moreover pick positive numbers $a$ and $a'$ such that $a_{\alpha'} < a' < a < a_\alpha$, and thereupon use (1.7) to find $r_1 > 0$ such that

$$\varphi_\alpha'(r) \geq Lr^{-\nu} - a'r^{-\nu-\lambda_1} \quad \text{for all } r > r_1.$$
as well as
\[ \varphi_\alpha(r) \leq L r^{-\nu} - a r^{-\nu-\lambda_1} \quad \text{for all } r > r_1. \]

This shows that if we abbreviate \( c_2 := a - a' \) then
\[ \varphi_{\alpha'}(r) - \varphi_\alpha(r) \geq c_2 r^{-\nu-\lambda_1} \quad \text{for all } r > r_1. \quad (3.27) \]

Now thanks to the fact that \( \gamma > \nu + \lambda_1 \), choosing
\[ r_2 := \max \left\{ r_1, \left( \frac{c_1}{c_2} \right)^{\frac{1}{\gamma-\nu-\lambda_1}} \right\} \]
we see that (3.26) and (3.27) imply the inequality
\[ v(r, t) \leq \varphi_\alpha'(r) - c_2 r^{-\nu-\lambda_1} \left( 1 - \frac{c_1}{c_2} r^{-(\gamma-\nu-\lambda_1)} \right) \]
\[ \leq \varphi_{\alpha'}(r) - c_2 r^{-\nu-\lambda_1} \left( 1 - \frac{c_1}{c_2} r_2^{-(\gamma-\nu-\lambda_1)} \right) \]
\[ = \varphi_{\alpha'}(r) \quad \text{for all } r > r_2 \text{ and } t > 0. \quad (3.28) \]

Next, using that \( \varphi_{\alpha'}(r) > \varphi_\alpha(r) \) for all \( r \geq 0 \) we can find \( c_3 > 0 \) fulfilling
\[ \varphi_{\alpha'}(r) - \varphi_\alpha(r) \geq c_3 \quad \text{for all } r \in [0, r_2], \quad (3.29) \]
whereas the convergence statement in Lemma 3.2 provides \( t_0 \geq 0 \) such that
\[ v(r, t) \leq \varphi_\alpha(r) + c_3 \quad \text{for all } r \geq 0 \text{ and } t \geq t_0. \quad (3.30) \]

Combining (3.29) and (3.30) we thus obtain that
\[ v(r, t) \leq \varphi_{\alpha'}(r) \quad \text{for all } r \in [0, r_2] \text{ and } t \geq t_0, \]
which together with (3.28) establishes (3.25). \( \square \)

In a third comparison procedure, we can finally establish a quantitative upper estimate of the form asserted in Theorem 1.1. For the first time we shall use here comparison functions which deviate from \( \varphi_\alpha \) in a separated manner.

**Lemma 3.4** Assume that \( v_0 \) satisfies (1.9), and that there exist \( \alpha > 0, b > 0 \) and \( \gamma \in (\nu + \lambda_1, (n-2)/2) \) such that (3.16) holds. Then there exists \( C > 0 \) such that the solution \( v \) of (3.14) satisfies
\[ |v(r, t) - \varphi_\alpha(r)| \leq C e^{-\kappa(\gamma)t} \quad \text{for all } r \geq 0 \text{ and } t \geq 0 \quad (3.31) \]
with \( \kappa(\gamma) \) as given by (1.13).
Proof. We fix any $\beta > \alpha + 1$ and let $f := f_{\beta, \kappa}$ denote the corresponding solution of (2.2). Then according to (2.3) we can find $c_1 > 0$ and $c_2 > 0$ such that
\[ c_1 r^{-\gamma} \leq f(r) \leq c_2 r^{-\gamma} \quad \text{for all } r > 1, \]
whence in particular by (3.18) and the positivity of $f$ we can pick $A > 0$ such that
\[ v(r, t) \leq \varphi_\alpha(r) + Af(r) \quad \text{for all } r \geq 0 \text{ and } t \geq 0. \]
Next, since $\beta - 1 > \alpha$ we may apply Lemma 2.2 with $B := 0$ to find $r_1 > 1$ and $c_3 > 0$ satisfying
\[ \varphi_{\beta-1}^p(r) - \left( \varphi_\alpha(r) + Af(r) \right)^p \geq c_3 r^{-2-\lambda_1} \quad \text{for all } r > r_1, \]
whereas Lemma 2.3 provides $c_4 > 0$ such that
\[ \varphi_\alpha^{p-2}(r) f(r) \leq c_4 (r + 1)^{p-2-\gamma} \quad \text{for all } r \geq 0, \]
which clearly entails that
\[ \varphi_\alpha^{p-2}(r) f(r) \leq c_4 r^{p-2-\gamma} \quad \text{for all } r > 0, \]
because $\gamma + 2 > \nu$.

Let us now pick $c_5 > 0$ and then $c_6 > 0$ such that
\[ \varphi_\alpha(r) \geq c_5 r^{-\nu} \quad \text{for all } r > 1, \]
and such that with $z_1 := c_2 A/c_5$ we have
\[ (1 + z)^p \leq 1 + pz + c_6 z^2 \quad \text{for all } z \in [0, z_1]. \]
Then since
\[ \frac{Af(r)}{\varphi_\alpha(r)} \leq \frac{c_2 A}{c_5} r^{-(\gamma - \nu)} \leq \frac{c_2 A}{c_5} = z_1 \quad \text{for all } r > 1 \]
by (3.32) and (3.37), it follows from (3.34) and (3.35) that
\[ \frac{\kappa p \left\{ \varphi_{\beta}^{p-1}(r) - \left( \varphi_\alpha(r) + Af(r) \right)^{p-1} \right\}}{c_6 A \varphi_\alpha^{p-2}(r) f(r)} \geq \frac{\kappa pc_3}{c_4 c_6 A} r^{\gamma - \nu - \lambda_1} \quad \text{for all } r > r_1. \]
As $\gamma > \nu + \lambda_1$, we can thus choose $r_2 > r_1$ such that
\[ \kappa p \left\{ \varphi_{\beta}^{p-1}(r) - \left( \varphi_\alpha(r) + Af(r) \right)^{p-1} \right\} f(r) \geq c_6 A \varphi_\alpha^{p-2}(r) f^2(r) \quad \text{for all } r > r_2. \]
With this value of $r_2$ fixed, by (1.7) we can easily find $z_2 > 0$ such that
\[ \frac{\varphi_\beta(r)}{\varphi_\alpha(r)} \leq z_2 \quad \text{for all } r \in [0, r_2] \]
and thereupon let \( c_7 > 0 \) be large enough satisfying
\[
(1 + z)^p \leq 1 + pz + c_7 z^2 \quad \text{for all } z \in [0, z_2],
\] (3.42)
Recalling (3.35), we then take \( c_8 > 0 \) such that
\[
c_7 \varphi_\alpha^{p-2}(r)f(r) \leq c_8 \quad \text{for all } r \in [0, r_2],
\] (3.43)
and use that \( \varphi_\beta > \varphi_{\beta-1} \) and \( f > 0 \) on \([0, \infty)\) to obtain \( c_9 > 0 \) fulfilling
\[
p \left[ \varphi_\beta^{p-1}(r) - \varphi_{\beta-1}^{p-1}(r) \right] f(r) \geq c_9 \quad \text{for all } r \in [0, r_2].
\] (3.44)
Fixing \( \delta \in (0, 1) \) suitably small such that
\[
c_8 \delta \leq c_9,
\] (3.45)
by an argument involving continuous dependence for the initial-value problem (1.5) we are now able to find some \( \alpha' \in (\alpha, \beta - 1) \) sufficiently close to \( \alpha \) such that
\[
\varphi_{\alpha'}(r) - \varphi_\alpha(r) \leq \delta \quad \text{for all } r \in [0, r_2].
\] (3.46)
Finally, Lemma 3.3 allows us to choose \( t_0 > 0 \) large enough such that
\[
v(r, t) \leq \varphi_{\alpha'}(r) \quad \text{for all } r \in [0, r_2] \text{ and } t \geq t_0.
\] (3.47)
We proceed to define a comparison function \( \varpi \) on \([0, \infty) \times (t_0, \infty)\) by letting
\[
\varpi(r, t) := \min \left\{ \varphi_{\alpha'}(r), \varphi_\alpha(r) + f(r)g(t) \right\} \quad \text{for } r \geq 0 \text{ and } t \geq t_0,
\]
where \( g(t) := Ae^{-\kappa(t-t_0)} \) for \( t \geq t_0 \). Then inside the set
\[
Q := \left\{ (r, t) \in (0, \infty) \times (t_0, \infty) \mid f(r)g(t) < \varphi_{\alpha'}(r) \right\},
\]
we compute
\[
\overline{p}(r, t) = \frac{p^{p-1}r_{tt} - \frac{n-1}{r}r_{tr} - \varpi}{\varpi} - \frac{n-1}{r} \varpi - \varpi^p
\]
\[
= -\kappa p \left[ \varphi_\alpha(r) - f(r)g(t) \right]^{p-1} f(r)g(t) - \varphi_\alpha r(r) - \frac{n-1}{r} \varphi_\alpha r(r)
\]
\[
- g(t) \left[ f_{rr}(r) + \frac{n-1}{r} f_r(r) \right] - \left[ \varphi_\alpha(r) + f(r)g(t) \right]^p
\]
\[
= g(t) \left\{ - f_{rr}(r) - \frac{n-1}{r} f_r(r) - \kappa p \left[ \varphi_\alpha(r) + f(r)g(t) \right]^{p-1} f(r)
\]
\[
- \frac{1}{g(t)} \varphi_\alpha^p(r) \left[ \left( \frac{1 + f(r)g(t)}{\varphi_\alpha(r)} \right)^p - 1 \right] \right\} \quad \text{for all } (r, t) \in Q,
\]
whence (2.2) shows that

\[
\frac{1}{g(t)} P(r, t) = (\kappa + 1)p\varphi_\beta^{p-1}(r)f(r) - \kappa p\left[\varphi_\alpha(r) + f(r)g(t)\right]^{p-1} f(r)
\]

\[
- \frac{1}{g(t)} \varphi_\alpha(r) \left[ \left(1 + \frac{f(r)g(t)}{\varphi_\alpha(r)}\right)^p - 1 \right] \quad \text{for all } (r, t) \in Q. \quad (3.48)
\]

Hence, if \((r, t) \in Q\) is such that \(r > r_2\), then by (3.39), (3.38) and the fact that \(g \leq A\) we see that

\[
\frac{1}{g(t)} P(r, t) \geq (\kappa + 1)p\varphi_\beta^{p-1}(r)f(r) - \kappa p\left[\varphi_\alpha(r) + Af(r)\right]^{p-1} f(r)
\]

\[
- \frac{1}{g(t)} \varphi_\alpha(r) \left\{ p\frac{f(r)g(t)}{\varphi_\alpha(r)} + c_0 \frac{f^2(r)g^2(t)}{\varphi_\alpha^2(r)} \right\}
\]

\[
= \kappa p \left\{ \varphi_\beta^{p-1}(r) - \left[\varphi_\alpha(r) + Af(r)\right]^{p-1} \right\} f(r)
\]

\[
+ p\left( \varphi_\beta^{p-1}(r) - \varphi_\alpha^{p-1}(r) \right) f(r) - c_0 \varphi_\alpha^{p-2}(r) f^2(r)g(t).
\]

Again since \(g \leq A\), and since \(\varphi_\beta \geq \varphi_\alpha\), from (3.40) we therefore obtain that

\[
P(r, t) \geq 0 \quad \text{whenever } (r, t) \in Q \text{ is such that } r > r_2. \quad (3.49)
\]

On the other hand, if \((r, t) \in Q\) satisfies \(r \leq r_2\), then in particular

\[
f(r)g(t) < \varphi_\alpha'(r) - \varphi_\alpha(r), \quad (3.50)
\]

so that (3.41) and the fact that \(\alpha' < \beta\) imply that

\[
\frac{f(r)g(t)}{\varphi_\alpha(r)} \leq \frac{\varphi_\beta(r) - \varphi_\alpha(r)}{\varphi_\alpha(r)} \leq c_2.
\]

We may thus invoke (3.42) and once more rely on (3.50) to conclude from (3.48) that for such \((r, t)\) we have

\[
\frac{1}{g(t)} P(r, t) \geq (\kappa + 1)p\varphi_\beta^{p-1}(r)f(r) - \kappa p\left[\varphi_\alpha(r) + f(r)g(t)\right]^{p-1} f(r)
\]

\[
- \frac{\varphi_\alpha^2(r)}{g(t)} \left\{ p\frac{f(r)g(t)}{\varphi_\alpha(r)} - c_7 \frac{f^2(r)g^2(t)}{\varphi_\alpha^2(r)} \right\}
\]

\[
= \kappa p \left\{ \varphi_\beta^{p-1}(r) - \left[\varphi_\alpha(r) + f(r)g(t)\right]^{p-1} \right\} f(r)
\]

\[
+ p\left( \varphi_\beta^{p-1}(r) - \varphi_\alpha^{p-1}(r) \right) f(r) - c_7 \varphi_\alpha^{p-2}(r) f^2(r)g(t)
\]

\[
\geq \kappa p \left\{ \varphi_\beta^{p-1}(r) - \varphi_\alpha^{p-1}(r) \right\} f(r)
\]

\[
+ p\left( \varphi_\beta^{p-1}(r) - \varphi_\alpha^{p-1}(r) \right) f(r) - c_7 \varphi_\alpha^{p-2}(r) \left[\varphi_\alpha'(r) - \varphi_\alpha(r)\right] f(r).
\]
Since $\alpha' < \beta$ and $\alpha \leq \beta - 1$, in light of (3.43), (3.44), (3.46), (3.45) and the fact that $\delta < 1$, this yields

$$\frac{1}{g(t)} g(r) \geq \phi_{\beta}^{-1}(r) + \phi_{\beta}^{-1}(r) - b(r + 1)^{-\gamma} \geq \gamma \phi_{\alpha}^{-1}(r) - c_{1} r^{-\gamma} \geq c_{1} (r + 1)^{-\gamma}$$

with some $c_{1} > 0$. Since $f$ is positive on $[0, 1]$, we can thus pick $c_{2} > 0$ such that

$$f(r) \geq c_{2} (r + 1)^{-\gamma} \quad \text{for all } r \geq 0,$$

by parabolic comparison we infer that $v(r, t) \leq \phi_{\alpha}(r)$ for all $r \geq 0$ and $t \geq t_{0}$, which evidently yields (3.31) because of the boundedness of $f$.

4 Convergence from below: upper bound for the rate

For initial data below $\phi_{\alpha}$, a quantitative convergence result can be derived by using an argument which is based on a single comparison procedure, and which is thus somewhat simpler than the reasoning in the previous section.

**Lemma 4.1** Assume that $\alpha > 0$, and that $v_{0}$ is nonnegative such that

$$\phi_{\alpha}(r) \geq v_{0}(r) \geq \phi_{\alpha}(r) - b(r + 1)^{-\gamma} \quad \text{for all } r \geq 0 \quad (4.1)$$

with some $b > 0$ and $\gamma \in (\nu + \lambda_{1}, (n - 2)/2)$. Then there is $C > 0$ such that the solution $v$ of (3.14) satisfies

$$|v(r, t) - \phi_{\alpha}(r)| \leq Ce^{-\kappa(\gamma)t} \quad \text{for all } t \geq 0,$$

with $\kappa(\gamma)$ as in (1.13).

**Proof.** As $\phi_{\alpha}$ is an equilibrium of (3.14), the first inequality in (4.1) along with a parabolic comparison shows that

$$v(r, t) \leq \phi_{\alpha}(r) \quad \text{for all } r \geq 0 \text{ and } t \geq 0,$$

so that we only need to establish a lower bound for $v$. To this end, we let $f = f_{\alpha, \kappa}$ be as given by Lemma 2.1 so that from (2.3) we know that

$$f(r) \geq c_{1} r^{-\gamma} \geq c_{1} (r + 1)^{-\gamma} \quad \text{for all } r > 1$$

with some $c_{1} > 0$. Since $f$ is positive on $[0, 1]$, we can thus pick $c_{2} > 0$ such that

$$f(r) \geq c_{2} (r + 1)^{-\gamma} \quad \text{for all } r \geq 0,$$

and

$$f(r) \geq c_{2} (r + 1)^{-\gamma} \quad \text{for all } r \geq 0,$$
and let \( A := b/c_2 \). Then again writing \( g(t) := Ae^{-\kappa t}, \ t \geq 0, \) with \( \kappa := \kappa(\gamma) \), we see that

\[
\varpi(r, t) := \max \left\{ 0, \varphi_\alpha(r) - f(r)g(t) \right\}, \quad r \geq 0, \ t \geq 0,
\]
satisfies \( \varpi(r, 0) \leq v_0(r) \) for all \( r \geq 0 \). This is obvious whenever \( \varpi(r, 0) = 0 \), while otherwise \( (4.3) \) and \( (4.1) \) assert that

\[
\varpi(r, 0) = \varphi_\alpha(r) - Af(r) \leq \varphi_\alpha(r) - Ac_2(r + 1)^{-\gamma} = \varphi_\alpha(r) - b(r + 1)^{-\gamma} \leq v_0(r).
\]

In order to show that \( \varpi \) is a subsolution of \( (3.14) \) in \( (0, \infty)^2 \), we evidently only need to consider points \( (r, t) \) where \( \varpi(r, t) > 0 \), at which we compute

\[
\mathcal{P}\varpi(r, t) = \kappa p \left\{ \varphi_\alpha(r) - f(r)g(t) \right\}^{p-1} f(r)g(t) - (\varphi_\alpha)_{rr}(r) - \frac{n-1}{r} (\varphi_\alpha)_r(r) - g(t) \left\{ f_{rr}(r) + \frac{n-1}{r} f_r(r) \right\} - \left\{ \varphi_\alpha(r) - f(r)g(t) \right\}^p.
\]

By monotonicity and convexity, respectively, we see that at any such point we have

\[
\left\{ \varphi_\alpha(r) - f(r)g(t) \right\}^{p-1} \leq \varphi_\alpha^{p-1}(r)
\]

and

\[
\left\{ \varphi_\alpha(r) - f(r)g(t) \right\}^p \geq \varphi_\alpha^p(r) + p \varphi_\alpha^{p-1}(r)f(r)g(t),
\]

so that since \( \varphi_\alpha \) is a solution of \( (1.5) \), at all those \((r, t)\) we obtain

\[
\mathcal{P}\varpi(r, t) \leq g(t) \left\{ \kappa p \varphi_\alpha^{p-1}(r)f(r) - f_{rr}(r) - \frac{n-1}{r} f_r(r) + p \varphi_\alpha^{p-1}(r)f(r) \right\} = 0
\]

according to \( (2.2) \). Using the comparison principle, we thus conclude that \( \varpi(r, t) \leq v(r, t) \) for all \( r \geq 0 \) and \( t \geq 0 \), whence

\[
\varphi_\alpha(r) - v(r,t) \leq \varphi_\alpha(r) - \varpi(r,t) \leq f(r)g(t) \quad \text{for all } r \geq 0 \text{ and } t \geq 0.
\]

Since \( f \) is bounded, by definition of \( g \) this yields the desired inequality and thereby proves \( (4.2) \).

Now a straightforward combination of Lemma \( 4.1 \) and Lemma \( 3.4 \) provides the claimed upper estimate on the convergence rate.

**Proof of Theorem 1.1.** If \( v_0 \) is as in Theorem 1.1 then one can find functions \( \varpi_0, \varphi_0 \),

\[
\varpi_0(|x|) \leq \min \{ v_0(x), \varphi_\alpha(|x|) \}, \quad x \in \mathbb{R}^n,
\]

and

\[
\max \{ v_0(x), \varphi_\alpha(|x|) \} \leq \varphi_0(|x|) \leq L|x|^{-\nu}, \quad x \in \mathbb{R}^n, \ x \neq 0,
\]

such that \( \varpi_0, \varphi_0 \) satisfy the assumptions of Lemma \( 4.1 \) Lemma \( 3.4 \) respectively. By comparison, these two lemmata yield then the claim. \( \square \)
5 Convergence from above: lower bound for the rate

In this section we shall prove the first statement in Theorem 1.2. For this purpose, we employ comparison functions similar to those used in Lemma 3.4 and Lemma 4.1 to establish the following.

**Lemma 5.1** Let \( \alpha > 0 \), and assume that \( v_0 \) satisfies (1.9) as well as

\[
v_0(r) \geq \varphi_\alpha(r) + b(r + 1)^{-\gamma} \quad \text{for all } r \geq 0
\]  

(5.1)

with some \( b > 0 \) and \( \gamma \in (\nu + \lambda_1, (n - 2)/2) \). Then there exists \( c > 0 \) such that the solution \( v \) of (3.14) satisfies

\[
v(0, t) - \varphi_\alpha(0) \geq ce^{-\kappa(\gamma)t} \quad \text{for all } t \geq 0,
\]

(5.2)

where \( \kappa(\gamma) \) is as in (1.13).

**Proof.** We let \( f := f_{\alpha, \kappa} \) be as given by Lemma 2.1, and then obtain from (2.3), (5.1) and the boundedness of \( f \) that there exists \( A > 0 \) such that

\[
v_0(r) \geq \varphi_\alpha(r) + Af(r) \quad \text{for all } r \geq 0.
\]

This means that \( v \) initially dominates the function \( \underline{v} \) defined on \([0, \infty) \) by setting

\[
\underline{v}(r, t) := \varphi_\alpha(r) + f(r)g(t) \quad \text{for } r \geq 0 \text{ and } t \geq 0,
\]

with \( g(t) := Ae^{-\kappa t} \) for \( t \geq 0 \) and \( \kappa := \kappa(\gamma) \). Then

\[
P_{\underline{v}}(r, t) = -\kappa p\{\varphi_\alpha(r) + f(r)g(t)\}^{p-1} f(r)g(t) - (\varphi_\alpha)^{r}(r) - \frac{n-1}{r} (\varphi_\alpha)_r(r) - g(t)\left\{ f^{rr}(r) + \frac{n-1}{r} f_r(r) \right\} - \left\{ \varphi_\alpha(r) + f(r)g(t) \right\}^p
\]

(5.3)

for all \( r > 0 \) and \( t > 0 \), where by monotonicity and convexity of \([0, \infty) \ni z \mapsto z^{p-1} \) we can estimate

\[
\left\{ \varphi_\alpha(r) + f(r)g(t) \right\}^{p-1} \geq \varphi_\alpha^{p-1}(r)
\]

and

\[
\left\{ \varphi_\alpha(r) + f(r)g(t) \right\}^p \geq \varphi_\alpha^p(r) + p\varphi_\alpha^{p-1}(r)f(r)g(t)
\]

for all \( r > 0 \) and \( t > 0 \). Since \( \varphi_\alpha \) satisfies (1.9), (5.3) implies that

\[
P_{\underline{v}}(r, t) \leq g(t)\left\{ -\kappa p\varphi_\alpha^{p-1}(r)f(r) - f^{rr}(r) - \frac{n-1}{r} f_r(r) - p\varphi_\alpha^{p-1}(r)f(r) \right\} = 0
\]

for all \( r > 0 \) and \( t > 0 \) due to (2.2). The comparison principle thus shows that \( \underline{v}(r, t) \leq v(r, t) \) for all \( r \geq 0 \) and \( t \geq 0 \), whence in particular

\[
v(0, t) \geq \underline{v}(0, t) = \varphi_\alpha(0) + Ae^{-\kappa t} \quad \text{for all } t \geq 0,
\]

for all \( t \geq 0 \).
because \( f(0) = 1 \).

From this we immediately obtain the first conclusion of Theorem 1.2.

**Proof of Theorem 1.2 (i).** If \( v_0 \) is as in Theorem 1.2 (i) then one can find a function \( v_0 \leq v_0 \) satisfying the assumptions of Lemma 5.1. By comparison, Lemma 5.1 yields then the claim.

---

6 Convergence from below: lower bound for the rate

We shall next verify that the convergence statement in Theorem 1.1 indeed yields the precise convergence rate also for solutions approaching their limit from below. Paralleling Lemma 5.1, the following lemma provides a technical preparation for this.

**Lemma 6.1** Let \( \alpha > 0 \) and \( v_0 \) be nonnegative and such that

\[
v_0(r) \leq \varphi_\alpha(r) - b(r + 1)^{-\gamma} \quad \text{for all } r \geq 0 \tag{6.1}
\]

with some \( b > 0 \) and \( \gamma \in (\nu + \lambda_1, (n - 2)/2) \). Then there exists \( c > 0 \) such that for the solution \( v \) of (3.14) we have

\[
\varphi_\alpha(0) - v(0, t) \geq ce^{-\kappa(t)} \quad \text{for all } t \geq 0, \tag{6.2}
\]

where \( \kappa(\gamma) \) is as in (1.13).

**Proof.** Let us pick \( \beta \in (0, \alpha) \) and take \( f := f_{\beta, \kappa} \) from Lemma 2.1. Then Lemma 2.2 yields \( r_1 \geq 1 \) and \( c_1 > 0 \) such that

\[
\left\{ \varphi_\alpha(r) - f(r) \right\}^{p-1} - \varphi_\beta^{p-1}(r) \geq c_1 r^{-2-\lambda_1} \quad \text{for all } r > r_1, \tag{6.3}
\]

and from Lemma 2.3 we obtain \( c_2 > 0 \) such that

\[
\varphi_\alpha^{p-2}(r) f(r) \leq c_2 (r + 1)^{\nu - 2 - \gamma} \quad \text{for all } r > 0. \tag{6.4}
\]

Moreover, since \( f \) is bounded and \( \varphi_\alpha \) is positive with

\[
\frac{f(r)}{\varphi_\alpha(r)} \leq c_3 r^{-\gamma+\nu} \quad \text{for all } r > 1
\]

and some \( c_3 > 0 \) by (2.3) and (1.7), due to the fact that \( \gamma > \nu \) we can fix \( c_4 > 0 \) satisfying

\[
\frac{f(r)}{\varphi_\alpha(r)} \leq c_4 \quad \text{for all } r \geq 0. \tag{6.5}
\]
Similarly, using (2.3) and the boundedness of \( f \) we can find \( c_5 > 0 \) such that
\[
(r + 1)^\gamma f(r) \leq c_5 \quad \text{for all } r \geq 0.
\tag{6.6}
\]
We now pick \( c_6 > 0 \) fulfilling
\[
(1 - z)^p \leq 1 - pz + c_6 z^2 \quad \text{for all } z \in [0, 1/2],
\tag{6.7}
\]
and finally choose \( A \in (0, 1] \) such that
\[
A \leq \min \left\{ \frac{1}{2c_4}, b, \frac{\kappa pc_1}{\kappa p c_6} \right\},
\tag{6.8}
\]
as well as
\[
c_7 := \inf_{r \in (0, r_1)} \left\{ \left\{ \phi_\alpha(r) - Af(r) \right\}^{p-1} - \phi_\beta^{p-1}(r) \right\} > 0.
\tag{6.9}
\]
With these choices, we again define \( g(t) := Ae^{-\kappa t} \) for \( t \geq 0 \) where \( \kappa := \kappa(\gamma) \). Then we set
\[
\phi(r, t) := \phi_\alpha(r) - f(r) g(t) \quad \text{for } r > 0 \text{ and } t \geq 0,
\]
and first observe that as a consequence of (6.5) and (6.8) we have
\[
\frac{f(r) g(t)}{\phi_\alpha(r)} \leq \frac{Af(r)}{\phi_\alpha(r)} \leq c_4 A \leq \frac{1}{2} \quad \text{for all } r > 0 \text{ and } t > 0,
\tag{6.10}
\]
whence in particular
\[
\phi(r, t) \geq \frac{1}{2} \phi_\alpha(r) \quad \text{for all } r > 0 \text{ and } t > 0.
\]
Furthermore, (6.10) allows us to apply (6.10) in estimating
\[
\phi(r, t) = \left\{ \phi_\alpha(r) - f(r) g(t) \right\}^p \leq \phi_\alpha^p(r) - p \phi_\alpha^{p-1}(r) f(r) g(t) + c_6 \phi_\alpha^{p-2}(r) f^2(r) g^2(t),
\]
so that using the equilibrium property of \( \phi_\alpha \) and (2.2) we find that
\[
\mathcal{P}(\phi)(r, t) = \kappa p \left\{ \phi_\alpha(r) - f(r) g(t) \right\}^{p-1} f(r) g(t) \left\{ \phi_\alpha(r) + \frac{n-1}{r} \phi_\alpha(r) \right\} - g(t) \left\{ f_{rr}(r) + \frac{n-1}{r} f_{r}(r) \right\} - \phi(r, t)
\geq \kappa p \phi_\alpha^{p-1}(r) f(r) g(t) - c_6 \phi_\alpha^{p-2}(r) f^2(r) g^2(t)
+ \left( p \phi_\alpha^{p-1}(r) f(r) g(t) \right) - c_6 \phi_\alpha^{p-2}(r) f^2(r) g^2(t)
= f(r) g(t) \left\{ \kappa p \left( \phi_\alpha(r) - f(r) g(t) \right)^{p-1} - \phi_\beta^{p-1}(r) \right\}
+ p \left[ \phi_\alpha^{p-1}(r) - \phi_\beta^{p-1}(r) \right] - c_6 \phi_\alpha^{p-2}(r) f(r) g(t)
\tag{6.11}
\]
for all \( r > 0 \) and \( t > 0 \). Here
\[
\phi_\alpha^{p-1}(r) - \phi_\beta^{p-1}(r) > 0 \quad \text{for all } r > 0,
\]
because \( \alpha > \beta \). Moreover, since \( g(t) \leq A \) for all \( t \geq 0 \), we see that (6.4) implies
\[
c_6 \varphi^\alpha_{\alpha} - f(r) g(t) \leq c_6 A \varphi^\alpha_{\alpha} - f(r) \leq c_2 c_6 A (r + 1)^{-2 + \nu - \gamma} \quad \text{for all } r > 0, \tag{6.12}
\]
and that
\[
\kappa p \left[ \left( \varphi_{\alpha}(r) - f(r) g(t) \right)^{p-1} - \varphi^\beta_{\beta}(r) \right] \geq \kappa p \left[ \left( \varphi_{\alpha}(r) - A f(r) \right)^{p-1} - \varphi^\beta_{\beta}(r) \right] \tag{6.13}
\]
for all \( r > 0 \). In particular, using (6.3) and (6.8) along with the fact that \( A \leq 1 \), we obtain from (6.12) and (6.13) that
\[
kappa p \left[ \left( \varphi_{\alpha}(r) - f(r) g(t) \right)^{p-1} - \varphi^\beta_{\beta}(r) \right] c_6 \varphi^\alpha_{\alpha} - f(r) g(t) \geq \kappa p c_7 \left( \varphi_{\alpha}(r) - c_5 A (r + 1)^{-\gamma} \right) \geq 1 \tag{6.14}
\]
for all \( r \geq r_1 \) and \( t > 0 \), because \( \gamma > \nu + \lambda_1 \). On the other hand, for small \( r \) we apply (6.9) and (6.8) to find that
\[
kappa p \left[ \left( \varphi_{\alpha}(r) - f(r) g(t) \right)^{p-1} - \varphi^\beta_{\beta}(r) \right] c_6 \varphi^\alpha_{\alpha} - f(r) g(t) \geq \kappa p c_7 \left( \varphi_{\alpha}(r) - c_5 A (r + 1)^{-\gamma} \right) \geq 1 \tag{6.15}
\]
for all \( r \in (0, r_1) \) and \( t > 0 \). From (6.11), (6.14) and (6.15) we thus infer that
\[
P \varphi(r, t) \geq 0 \quad \text{for all } r > 0 \text{ and } t > 0,
\]
so that since for any \( r \geq 0 \) we have
\[
\varphi(r, 0) = \varphi_{\alpha}(r) - A f(r) \geq \varphi_{\alpha}(r) - c_5 A (r + 1)^{-\gamma} \geq \varphi_{\alpha}(r) - b(r + 1)^{-\gamma} \geq v_0(r)
\]
according to (6.6), (6.8) and (6.1), by comparison we conclude that \( \varphi(r, t) \geq v(r, t) \) for all \( r \geq 0 \) and \( t \geq 0 \). In particular, this implies that
\[
\varphi_{\alpha}(0) - v(0, t) \geq \varphi_{\alpha}(0) - \varphi(0, t) = A e^{-\kappa t} \quad \text{for all } t \geq 0,
\]
which proves (6.2).

We can thereby complete the proof of Theorem 1.2.

**Proof of Theorem 1.2 (ii).** If \( v_0 \) is as in Theorem 1.2 (ii) then one can find a function \( \varphi_0 \geq v_0 \) satisfying the assumptions of Lemma 6.1. By comparison, the claim follows from Lemma 6.1. 

\[
\square
\]
7 Instability of the steady states when $p < p_c$

Proof of Proposition 1.3. The proof is based on intersection properties of the steady states similarly as the proof of Theorem 1.14 in [13]. We recall that for $p_S \leq p < p_c$ any two steady states intersect, see [23].

To prove (i) we may assume without loss of generality that $v_0(x) > \varphi_\alpha(|x|)$ for $x \in \mathbb{R}^n$. Then we choose $\varepsilon > 0$ small enough such that

$$v_*(|x|) := \max\{\varphi_\alpha(|x|), \varphi_\alpha + \varepsilon(|x|)\} \leq v_0(x), \quad x \in \mathbb{R}^n.$$ 

Then the solution $v$ of (1.4) with the initial condition $v(0) = v_*(|x|)$ satisfies $v_t \geq 0$ in $\mathbb{R}^n \times (0, t_{\text{max}})$ where $t_{\text{max}} \in (0, \infty)$ is the maximal existence time. In fact, $t_{\text{max}} = \infty$ for every solution of (1.4) with a bounded initial function $v_0$ because $\|v_0\|_{L^\infty(\mathbb{R}^n)} e^{t/p}$ is a supersolution.

Suppose the increasing function $\underline{v}(0, t) = \|v(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}$ has a finite limit as $t \to \infty$. Then $\underline{v}(\cdot, t)$ converges to a steady state that is bigger than $\varphi_\alpha$ which is a contradiction.

To prove (ii) we assume that $v_0(x) < \varphi_\alpha(|x|)$ for $x \in \mathbb{R}^n$, and choose $\varepsilon > 0$ small enough such that

$$v^*(|x|) := \min\{\varphi_\alpha(|x|), \varphi_\alpha - \varepsilon(|x|)\} \geq v_0(x), \quad x \in \mathbb{R}^n.$$ 

Then the solution $\overline{v}$ of (1.4) with the initial condition $\overline{v}(0) = v^*(|x|)$ satisfies $\overline{v}_t \leq 0$ in $\mathbb{R}^n \times (0, T)$ where $T \in (0, \infty)$ is the maximal time such that $\overline{v} > 0$ in $\mathbb{R}^n \times (0, T)$.

If the decreasing function $\overline{v}(0, t) = \|v(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}$ had a positive limit as $t \to T$ then $\overline{v}(\cdot, t)$ would converge to a positive steady state smaller than $\varphi_\alpha$ which is a contradiction. \qed

Acknowledgments. The first author was supported in part by the Slovak Research and Development Agency under the contract No. APVV-0134-10 and by the VEGA grant 1/0711/12.

References

[1] G. Akagi and R. Kajikiya, Stability analysis of asymptotic profiles for sign-changing solutions to fast diffusion equations, Manuscripta Math., 141 (2013), 559–587.

[2] J. G. Berryman and C. J. Holland, Stability of the separable solution for fast diffusion, Arch. Rat. Mech. Anal., 74 (1980), 379–388.

[3] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, Asymptotics of the fast diffusion equation via entropy estimates, Arch. Rat. Mech. Anal., 191 (2009), 347–385.

[4] M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities, Proc. Nat. Acad. Sciences, 107 (2010), 16459–16464.
[5] M. Bonforte, G. Grillo and J. L. Vázquez, *Special fast diffusion with slow asymptotics. Entropy method and flow on a Riemannian manifold*, Arch. Rat. Mech. Anal., 196 (2010), 631–680.

[6] M. Bonforte, G. Grillo and J. L. Vázquez, *Behaviour near extinction for the Fast Diffusion Equation on bounded domains*, J. Math. Pures Appl., 97 (2012), 1–38.

[7] M. del Pino and M. Sáez, *On the extinction profile for solutions of $u_t = \Delta u^{(N-2)/(N+2)}$*, Indiana Univ. Math. J., 50 (2001), 611–628.

[8] M. Fila, J. R. King and M. Winkler, *Rate of convergence to Barenblatt profiles for the fast diffusion equation with a critical exponent*, J. London Math. Soc., to appear.

[9] M. Fila, J. L. Vázquez and M. Winkler, *A continuum of extinction rates for the fast diffusion equation*, Comm. Pure Appl. Anal., 10 (2011), 1129–1147.

[10] M. Fila, J. L. Vázquez, M. Winkler and E. Yanagida, *Rate of convergence to Barenblatt profiles for the fast diffusion equation*, Arch. Rat. Mech. Anal., 204 (2012), 599–625.

[11] M. Fila, M. Winkler and E. Yanagida, *Convergence rate for a parabolic equation with supercritical nonlinearity*, J. Dynam. Differential Equations, 17 (2005), 249–269.

[12] V. A. Galaktionov and L. A. Peletier, *Asymptotic behaviour near finite-time extinction for the fast diffusion equation*, Arch. Rat. Mech. Anal., 139 (1997), 83–98.

[13] C. Gui, W.-M. Ni, and X. Wang, *On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$*, Comm. Pure Appl. Math., 45 (1992), 1153–1181.

[14] M. Hoshino and Y. Yanagida, *Sharp estimates of the convergence rate for a semilinear parabolic equation with supercritical nonlinearity*, Nonlin. Anal. TMA, 69 (2008), 3136–3152.

[15] J. R. King, *Self-similar behaviour for the equation of fast nonlinear diffusion*, Phil. Trans. Roy. Soc. Lond. A, 343 (1993), 337–375.

[16] J. R. King, *Asymptotic analysis of extinction behaviour in fast nonlinear diffusion*, J. Eng. Math., 66 (2010), 65–86.

[17] Y. C. Kwong, *Asymptotic behavior of a plasma type equation with finite extinction*, Arch. Rat. Mech. Anal., 104 (1988), 277–294.

[18] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, “Linear and Quasilinear Equations of Parabolic Type”, Amer. Math. Soc., Providence, RI, 1968.

[19] M. A. Peletier and H. Zhang, *Self-similar solutions of a fast diffusion equation that do not conserve mass*, Diff. Int. Equations, 8 (1995), 2045–2064.
[20] P. Quittner and Ph. Souplet, “Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States”, Birkhäuser Advanced Texts, Birkhäuser, Basel, 2007.

[21] G. Savaré and V. Vespri, *The asymptotic profile of solutions of a class of doubly nonlinear equations*, Nonlin. Anal. TMA, **22** (1994), 1553–1565.

[22] J. L. Vázquez, “Smoothing and Decay Estimates for Nonlinear Diffusion Equations”, Oxford Lecture Notes in Maths. and its Applications, vol. 33, Oxford University Press, Oxford, 2006.

[23] X. Wang, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc., **337** (1993), 549–590.