NONINTEGRABILITY OF A HALPHEN SYSTEM

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We study the Halphen system with real variables and real constants. We show that in the case where at least one constant is nonzero, this system does not admit any first integral that can be described by formal power series. It hence follows that analytic first integrals do not exist. Furthermore, we prove that first integrals of the Darboux type also do not exist.

Keywords: Halphen system, analytic first integral

1. Introduction to the problem

We consider the system

\[
\begin{align*}
\dot{x}_1 &= F_1(x_1, x_2, x_3) = x_2x_3 - x_1(x_2 + x_3) - \alpha_1^2(x_1 - x_2)(x_3 - x_1), \\
\dot{x}_2 &= F_2(x_1, x_2, x_3) = x_3x_1 - x_2(x_3 + x_1) - \alpha_2^2(x_2 - x_3)(x_1 - x_2), \\
\dot{x}_3 &= F_3(x_1, x_2, x_3) = x_1x_2 - x_3(x_1 + x_2) - \alpha_3^2(x_3 - x_1)(x_2 - x_3),
\end{align*}
\]

where \(x_1, x_2,\) and \(x_3\) are real variables and \(\alpha_1, \alpha_2,\) and \(\alpha_3\) are real constants. We call system (1) the second Halphen system (Halphen himself called it the second system [1]) because system (1) with \(\alpha_1 = \alpha_2 = \alpha_3 = 0\) becomes the so-called classical Halphen system. The classical Halphen system is a famous model (see, e.g., [1]–[3]), which first appeared in Darboux’s work [1] and was later solved by Halphen [3]. One of the circumstances making this system famous is that this system, as was shown, is equivalent to the Einstein field equations for a diagonal self-dual Bianchi-IX metric with a Euclidean signature (see [2], [4]). The classical Halphen system also arises in similarity reductions of associativity equations on a three-dimensional Frobenius manifold [5].

From the standpoint of integrability, the classical Halphen system has been intensively studied using different theories. One of the main results in this direction is that system (1) with \(\alpha_1 = \alpha_2 = \alpha_3 = 0\) can be explicitly integrated because we can express its general solution in terms of elliptic integrals (see [3], [6], [7]), but the first integrals are not global and are multivalued nonalgebraic functions (see [8]). Other results that we mention are in [9], where the so-called Darboux polynomials were used to prove that system (1) with \(\alpha_1 = \alpha_2 = \alpha_3 = 0\) does not admit a nonconstant algebraic first integral, and finally in [10], where a complete characterization of the formal and analytic first integrals was provided.

Our first aim in this paper is to show the nonexistence of first integrals of system (1) that can be described by formal series. We restrict ourself to the case where \((\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}.\) The case

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Prepared from an English manuscript submitted by the authors; for the Russian version, see Teoreticheskaya i Matematicheskaya Fizika, Vol. 181, No. 2, pp. 296–311, November, 2014. Original article submitted May 5, 2014.
where $\alpha_1 = \alpha_2 = \alpha_3 = 0$ was completely studied in [10]. In [11], with another generalization of the classical Halphen system, the nonexistence of Darboux first integrals was proved under some restrictive conditions on the parameters $\alpha_1$, $\alpha_2$, and $\alpha_3$. Here, we can characterize the nonexistence of Darboux first integrals for the generalized Halphen system given in (1) for all values of $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ (see [12] for other applications of the Darboux integrability).

One of the ideas that we use to characterize the existence of formal first integrals is as follows. The three planes

$$H_1 := x_1 - x_2 = 0, \quad H_2 := x_1 - x_3 = 0, \quad H_3 := x_2 - x_3 = 0$$

are invariant under the flows of system (1), and if $f := f(x_1, x_2, x_3)$ is a first integral of system (1), then the restriction of $f$ to $H_i = 0$ for each $i = 1, 2, 3$ is also a first integral of system (1). Hence, the method for proving our results consists of completely studying the integrability of the reduced systems on each $H_i = 0$ to obtain exact information on the integrals of the whole system (1).

The main results in this paper are as follows.

A formal first integral $f = f(x_1, x_2, x_3)$ of system (1) is a nonconstant formal power series in the variables $x_1$, $x_2$, and $x_3$ such that

$$\sum_{i=1}^{3} \frac{\partial f}{\partial x_i} F_i(x_1, x_2, x_3) = 0.$$

**Theorem 1.** For any $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, system (1) does not admit any formal first integral.

Here an analytic first integral of system (1) is a nonconstant analytic function that is constant over the trajectories of system (1). We obtain the following result as a corollary of Theorem 1.

**Theorem 2.** For any $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, system (1) does not admit any analytic first integral.

A rational first integral $f = f(x_1, x_2, x_3)$ of system (1) is a nonconstant rational function that is constant over solutions of system (1).

**Theorem 3.** For any $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, system (1) does not admit any rational first integral.

Finally, we study the Darboux first integrals of system (1) (see below for the definition).

**Theorem 4.** For any $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, system (1) does not admit any Darboux first integral.

This paper is organized as follows. In Sec. 2, we present some auxiliary results needed for proving Theorems 1 and 2. In Sec. 3, we prove Theorems 1 and 2. In Sec. 4, we prove Theorem 3. Finally, in Sec. 5, we prove Theorem 4.

2. **Auxiliary results**

We note that system (1) is invariant under the changes

$$(x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3) \rightarrow (x_2, x_3, x_1, \alpha_2, \alpha_3, \alpha_1),$$

$$(x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3) \rightarrow (x_3, x_1, x_2, \alpha_3, \alpha_1, \alpha_2).$$

(3)

The first auxiliary result can be easily proved using Newton's binomial formula, and we omit its proof (see [10] for a proof).
Lemma 1. Let $f = f(x_1, x_2, x_3)$ be a formal power series such that at $x_l = x_j$, $l, j \in \{1, 2, 3\}$, $l \neq j$, we have $f(x_1, x_2, x_3)|_{x_l = x_j} = \bar{f}$, where $\bar{f}$ is a formal power series in the variables $x_j, x_k$, $k \in \{1, 2, 3\}$, $k \neq j$ and $k \neq l$. Then there exists a formal series $g = g(x_1, x_2, x_3)$ such that $f = \bar{f} + (x_l - x_j)g$.

We recall the definition of a Darboux polynomial for system (1) with a cofactor $K$. We say that $f = f(x_1, x_2, x_3)$ is a Darboux polynomial of system (1) if it satisfies

$$\frac{\partial f}{\partial x_1} \dot{x}_1 + \frac{\partial f}{\partial x_2} \dot{x}_2 + \frac{\partial f}{\partial x_3} \dot{x}_3 = K f.$$  \hspace{1cm} (4)

Furthermore, because the polynomials in the right-hand side of (1) have degree two, the cofactor $K$ has a degree at most one. We write it as

$$K = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3.$$  

We note that $f = 0$ is an invariant algebraic surface for the flow of system (1), and a polynomial first integral of system (1) is a Darboux polynomial with a zero cofactor.

We recall that if there exist invariant planes under the flows of system (1) and a Darboux polynomial of system (1) with a cofactor $K$, then the restriction of $f$ to each of the invariant planes is a Darboux polynomial of system (1) restricted to each of the planes and with cofactors being the restriction of $K$ to each of the planes. We note that for real polynomial differential systems such as system (1), when we seek their Darboux first integrals, we use complex Darboux polynomials and complex exponential factors in the general case because these objects appear in pairs (of them and their conjugates), which forces the Darboux first integral to become real whenever it exists.

The following two lemmas are well-known.

Lemma 2. For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, any Darboux polynomial $f$ of system (1) has a cofactor of the form $K = c_1 x_1 + c_2 x_2 + c_3 x_3$.

Lemma 3. For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, if we write $f$ as a sum of its homogeneous parts, $f = f_1 + \cdots + f_n$, then $f$ is a Darboux polynomial of system (1) with a cofactor $K$ if and only if $f_i$ is a Darboux polynomial of system (1) with a cofactor $K$ for all $i = 1, \ldots, n$.

Lemma 4. Let $F$ be an analytic function and $F = \sum_i F_i$ be its decomposition into homogeneous polynomials of degree $i$. Then $F$ is an analytic first integral of homogeneous polynomial differential system (1) if and only if $F_i$ is a homogeneous polynomial first integral of system (1) for all $i$.

The proofs of Lemmas 2–4 use the homogeneity of system (1). Their proofs are well known (see, e.g., [13] for the first two and [14] for the third).

As usual, $\mathbb{N}$ denotes the set of positive integers.

Lemma 5. If we decompose the polynomial $f$ into its irreducible factors in $\mathbb{C}[x_1, x_2, x_3]$ as $\prod_{j=1}^s f_j^{n_j}$, with $n_j \in \mathbb{N} \cup \{0\}$, then $f$ is a Darboux polynomial if and only if every $f_j$ is a Darboux polynomial. Moreover, if $K$ and $K_j$ are the cofactors of $f$ and $f_j$, then $K = \sum_{j=1}^s n_j K_j$.

Lemma 5 was proved, for instance, in [13].

The following statement is important for investigating the rational integrability of polynomial systems. It was proved in [15].
Proposition 1. The existence of a rational first integral for polynomial differential system (1) implies either the existence of a polynomial first integral (and hence a Darboux polynomial with a zero cofactor) or the existence of two coprime Darboux polynomials with the same nonzero cofactor.

An exponential factor $F$ of polynomial differential system (1) is a function $F = e^{f/g} \notin \mathbb{C}[x_1, x_2, x_3]$ with coprime $f, g \in \mathbb{C}[x_1, x_2, x_3]$ that satisfies
\[ \frac{\partial F}{\partial x_1} \dot{x}_1 + \frac{\partial F}{\partial x_2} \dot{x}_2 + \frac{\partial F}{\partial x_3} \dot{x}_3 = LF \]  
(5)
for some polynomial $L$ of degree one.

The following result is well known. Its proof and geometric meaning were given in [13] and [16].

Proposition 2. The following statements hold:
1. If $E = e^{g_0/g_1}$ is an exponential factor for polynomial system (1) and $g_1$ is not a constant polynomial, then $g_1 = 0$ is an invariant algebraic curve.
2. Eventually, $e^{g_0}$ can be exponential factors, coming from the multiplicity of the infinite invariant straight line.

The following result given in [16] characterizes the algebraic multiplicity of an invariant algebraic surface using the number of exponential factors of system (1) associated with this invariant algebraic surface.

Theorem 5. Given an irreducible invariant algebraic surface $g_1 = 0$ of degree $m$ of system (1), it has the algebraic multiplicity $k$ if and only if the vector field associated with system (1) has $k-1$ exponential factors of the form $e^{g_{0,i}/g_1}$, where $g_{0,i}$ is a polynomial of degree at most $im$ and $g_{0,i}$ and $g_1$ are coprime for $i = 1, \ldots, k-1$.

In view of Theorem 5, if we prove that $e^{g_0/g_1}$ is not an exponential factor with a degree $g_0$ not exceeding the degree $g_1$, then there are no exponential factors associated with the invariant algebraic surface $g_1 = 0$.

A first integral $G$ of system (1) is of the Darboux type if it has the form
\[ G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \]  
(6)
where $f_1, \ldots, f_p$ are Darboux polynomials, $F_1, \ldots, F_q$ are exponential factors, and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$.

We need the following result, whose proof was given in [13].

Theorem 6. Let system (1) admit $p$ Darboux polynomials with cofactors $K_i$, and $q$ exponential factors $F_j$ with cofactors $L_j$. Then there exists $\lambda_j, \mu_j \in \mathbb{C}$, $j = 1, \ldots, q$, not all zero such that
\[ \sum_{i=1}^{q} \lambda_i K_i + \sum_{i=1}^{q} \mu_i L_i = 0 \]
if and only if the function $G$ given in (6) (of the Darboux type) is a first integral of system (1).

As already noted in the introduction, the planes $x_1 = x_2, x_1 = x_3,$ and $x_2 = x_3$ are invariant under flows of (1). Therefore, if $f$ is a formal first integral of system (1), then
\[ f_1(x_2, x_3) = f(x_2, x_2, x_3), \]
\[ f_2(x_2, x_3) = f(x_3, x_2, x_3), \]
\[ f_3(x_1, x_3) = f(x_1, x_3, x_3) \]  
(7)
are formal first integrals of system (1) restricted to the respective planes $x_1 = x_2, x_1 = x_3,$ and $x_2 = x_3.
Proposition 3. For \((\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\), if \(f\) is a formal first integral of system (1), then
\[
f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)g,
\]
where \(c_0\) is some constant and \(g := g(x_1, x_2, x_3)\) is a formal power series.

Proof. Let \(f\) be a formal first integral of system (1). We first prove that \(f_1 = c_0\), where \(f_1\) is given in (7). Indeed, \(f_1\) satisfies
\[
-x_2^2 \frac{\partial f_1}{\partial x_2} + \left(x_2^2 - 2x_2x_3 + \alpha_3^2(x_2 - x_3)^2\right) \frac{\partial f_1}{\partial x_3} = 0.
\]
We now introduce the linear change of variables
\[
y_2 = x_2, \quad z_2 = x_2 - x_3.
\]
In these new variables, we have \(f_1(x_2, x_3) = h(y_2, z_2)\), which satisfies
\[
-y_2^2 \frac{\partial h}{\partial y_2} - (2y_2 + \alpha_3^2 z_2)z_2 \frac{\partial h}{\partial z_2} = 0. \tag{9}
\]
We show that \(h = c_0\). For this, we write \(h\) as a power series in \(y_2\) and \(z_2\),
\[
h = \sum_{k, l \geq 0} h_{k, l} y_2^k z_2^l.
\]
Hence, requiring that \(h\) satisfies (9), we obtain
\[
0 = \sum_{k, l \geq 0} (k + 2l) h_{k, l} y_2^{k+1} z_2^l + \alpha_3^2 \sum_{k, l \geq 0} l h_{k, l} y_2^k z_2^{l+1} = \sum_{k, l \geq 0} \left((k + 2l - 1) h_{k-1, l} + \alpha_3^2(l - 1) h_{k, l-1}\right) y_2^k z_2^l, \tag{10}
\]
where \(h_{m, n} = 0\) for \(m < 0\) or \(n < 0\). Computing the different degrees in \(y_2\) and \(z_2\) in (10), we now obtain
\[
(k + 2l - 1) h_{k-1, l} + \alpha_3^2(l - 1) h_{k, l-1} = 0, \quad k, l \geq 0. \tag{11}
\]
We claim that
\[
h_{k, l} = 0 \quad \text{for } k, l \geq 0 \text{ if } (k, l) \neq (0, 0). \tag{12}
\]
We prove this statement by induction on \(l\). For \(l = 0\), Eq. (11) implies that \((k - 1) h_{k-1, 0} = 0\) for all \(k \geq 0\), which yields \(h_{k, 0} = 0\) for all \(k > 0\) and concludes the proof of (12) for \(l = 0\). We now suppose that (12) holds for \(l = 0, \ldots, m - 1\) (with \(m \geq 1\)) and prove that it holds for \(l = m\). By the induction hypothesis, \(h_{k, m}\) satisfies \((k + 2m - 1) h_{k-1, m} = 0\) for \(k \geq 0\), which yields \(h_{k, m} = 0\) for \(k \geq 0\). Then (12) is proved for \(l = m\), and (12) holds for all \(l \geq 0\) by induction. From (12), we then obtain \(h = h_{0, 0} = c_0\) and consequently \(f_1(x_2, x_3) = c_0\). Using Lemma 1 with \(x_l = x_1\) and \(x_j = x_2\), we obtain
\[
f = c_0 + (x_1 - x_2) g_0 \tag{13}
\]
for some formal power series \(g_0 := g_0(x_1, x_2, x_3)\).
For $f_2$, repeating the arguments that we applied for $f_1$, we now find that there exists a constant $c_1$ and a formal power series $g_1 = g_1(x_1, x_2, x_3)$ such that
\[ f = c_1 + (x_1 - x_3)g_1. \] (14)

Further, repeating the same arguments for $f_3$, we find that there exists a constant $c_2$ and a formal power series $g_2 = g_2(x_1, x_2, x_3)$ such that
\[ f = c_2 + (x_2 - x_3)g_2. \] (15)

Substituting $x_1 = x_2 = x_3 = 0$ in Eqs. (13)–(15), we now obtain $c_0 = c_1 = c_2$. From (13)–(15), we also obtain
\[ (x_1 - x_2)g_0 = (x_1 - x_3)g_1 = (x_2 - x_3)g_2, \]
which clearly implies that there exists a formal power series $g := g(x_1, x_2, x_3)$ such that
\[ g_0 = (x_1 - x_3)(x_2 - x_3)g, \quad g_1 = (x_1 - x_2)(x_2 - x_3)g, \quad g_2 = (x_1 - x_2)(x_1 - x_3)g. \] (16)

The proposition now follows from (13) and the first relation in (16).

3. Proof of Theorems 1 and 2

**Proof of Theorem 1.** Let $f$ be any formal first integral of system (1). By Proposition 3, we know that $f$ can be written as
\[ f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)g \] (17)
for some constant $c_0$ and some formal power series $g := g(x_1, x_2, x_3)$. Requiring that $f$ be a first integral of system (1) and simplifying by $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, we find that $g$ must satisfy
\[ \frac{dg}{dt} = ((2 - 2\alpha_1^2 + \alpha_2^2 + \alpha_3^2)x_1 + (2 + \alpha_1^2 - 2\alpha_2^2 + \alpha_3^2)x_2 + (2 + \alpha_1^2 + \alpha_2^2 - 2\alpha_3^2)x_3)g, \] (18)
where the derivative is evaluated along a solution of system (1). We prove that $g = 0$. For this, we proceed by reduction to the absurd: we suppose $g \neq 0$ and reach a contradiction. We consider two different cases.

**Case 1.** In the case where $g$ is not divisible by $x_1 - x_2$, using Lemma 1 with $x_i = x_1$ and $x_j = x_2$, we can write $g$ as $g = g_0 + (x_1 - x_2)g_1$, where $g_0 := g_0(x_2, x_3) \neq 0$ and $g_1 := g_1(x_1, x_2, x_3)$ are formal power series. Then $g_0$ satisfies Eq. (18) restricted to $x_1 = x_2$. Again introducing change of variables (8), we find that in these new variables, if $g_0(x_2, x_3) = h_0(y_2, z_2)$, then $h_0$ satisfies
\[ -y_2^2 \frac{\partial h_0}{\partial y_2} - (2y_2 + \alpha_3^2 z_2)z_2 \frac{\partial h_0}{\partial z_2} = (6y_2 - (2 + \alpha_1^2 + \alpha_2^2 - 2\alpha_3^2)z_2)h_0. \] (19)

We now write
\[ h_0 = \sum_{j \geq 0} h_{0,j}z_2^j, \quad h_{0,j} = h_{0,j}(y_2), \]
where $h_{0,j}$ is a formal numeric series for each $j$. We prove that
\[ h_{0,j} = 0 \quad \text{for} \quad j \geq 0. \] (20)

Clearly, $h_{0,0}$ satisfies Eq. (19) restricted to the manifold $z_2 = 0$, i.e.,
\[ -y_2^2 \frac{dh_{0,0}}{dy_2} = 6y_2h_{0,0}, \]

1388
solving which we obtain \( h_{0,0} = c / y_2^6, \ c \in \mathbb{C} \). Because \( h_{0,0} \) is a formal series in \( y_2 \), we have \( c = 0 \) and hence \( h_{0,0} = 0 \), which proves (20) for \( j = 0 \).

We now suppose that (20) holds for \( j = 0, \ldots, m - 1 \) with \( m \geq 1 \) and prove it for \( k = m \). Clearly, by the induction hypothesis, we have

\[
h_0 = \sum_{j \geq 0} h_{0, j+m} z_2^j + m,
\]

and then from (19), after dividing by \( z_2^m \), we obtain

\[
- y_2^2 \sum_{j \geq 0} \frac{d h_{0, j+m}}{d y_2} z_2^j - (2 y_2 + \alpha_3^2 z_2) \sum_{j \geq 0} (j + m) h_{0, j+m} z_2^j =
\]

\[
= (6 y_2 - (2 + \alpha_1^2 + \alpha_2^2 - 2 \alpha_3^2) z_2) \sum_{j \geq 0} h_{0, j+m} z_2^j.
\]

(21)

Evaluating this equation at \( z_2 = 0 \), we then obtain

\[
- y_2^2 \frac{d h_{0,m}}{d y_2} = y_2 (6 + 2m) h_{0,m},
\]

whose solution is \( h_{0,m} = c_m / y_2^{6+2m}, \ c_m \in \mathbb{C} \). Because \( h_{0,m} \) is a formal series in \( y_2 \), we have \( c_m = 0 \) and hence \( h_{0,m} = 0 \), which proves (20) for \( j = m \). By the induction hypothesis, (20) then holds for all \( j \geq 0 \), and we obtain \( h_0 = 0 \) from (20). Hence \( g_0 = 0 \), in contradiction with the fact that \( g \) is not divisible by \( x_1 - x_2 \).

**Case 2.** In the case where \( g \) is divisible by \( x_1 - x_2 \), \( g = (x_1 - x_2)^j h \) with \( j \geq 1 \) and \( h \neq 0 \) and \( h := h(x_1, x_2, x_3) \) is a formal power series that is not divisible by \( x_1 - x_2 \) and after dividing by \( (x_1 - x_2)^j \), we find that the series satisfies the equation

\[
\frac{d h}{d t} = [(2 + \alpha_1^2 + \alpha_3^2 - \alpha_1^2 (2 + j)) x_1 + (2 + \alpha_1^2 + \alpha_3^2 - \alpha_1^2 (2 + j)) x_2 +
\]

\[
+ (2 + \alpha_1^2 + \alpha_3^2) (1 + j) - 2 \alpha_3^2) x_3] h,
\]

where the derivative of \( h \) is evaluated along a solution of system (1). Applying the arguments to \( h \) similar to those used for \( g \) in Case 1, we conclude that \( h = 0 \), thus obtaining a contradiction.

Hence, \( g = 0 \), and the proof of the theorem follows from (17) and the definition of a formal first integral.

**Proof of Theorem 2.** By Theorem 1, Halphen system (1) has no polynomial first integrals. The proof of Theorem 2 now follows immediately from Lemma 4.

### 4. Proof of Theorem 3

We recall that the equation defining a Darboux polynomial is given in (4) and that in view of Lemma 2, we can take

\[
K = c_1 x_1 + c_2 x_2 + c_3 x_3, \ (c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}.
\]

We prove Theorem 3 using Proposition 1, Theorem 1, and the following result.
Theorem 7. For \((\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\), every Darboux polynomial of system (1) has the form

\[
f = cH_1^{n_1}H_2^{n_2}H_3^{n_3} = c(x_1 - x_2)^{n_1}(x_1 - x_3)^{n_2}(x_2 - x_3)^{n_3},
\]

where \(c\) is some constant and \(n_1, n_2, n_3\) are nonnegative integers. Furthermore, the cofactor of \(f\) is

\[
K = n_1(-2x_3 - \alpha_1^2(x_3 - x_1) + \alpha_2^2(x_2 - x_3)) + \\
+ n_2(-2x_2 + \alpha_1^2(x_1 - x_2) + \alpha_3^2(x_3 - x_2)) + \\
+ n_3(-x_1 - \alpha_2^2(x_1 - x_2) + \alpha_3^2(x_3 - x_1)).
\] (22)

Our main objective in this section is to prove Theorem 7 because, as becomes clear later, it easily implies the proof of Theorem 3. For this, we study the Darboux polynomials of system (1) of degree one and of a degree greater than one. We do this in two separate propositions.

Proposition 4. For \((\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\), the unique homogeneous Darboux polynomials of system (1) of degree one are

\[
H_1 := x_1 - x_2, \quad H_2 := x_1 - x_3, \quad H_3 := x_2 - x_3
\]

with the respective cofactors

\[
K_1 = -2x_3 - \alpha_1^2(x_3 - x_1) + \alpha_2^2(x_2 - x_3), \\
K_2 = -2x_2 - \alpha_2^2(x_2 - x_3) + \alpha_1^2(x_1 - x_2), \\
K_3 = -2x_1 - \alpha_2^2(x_1 - x_2) + \alpha_3^2(x_3 - x_1).
\]

Proof. Let \((\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\), and let \(f\) be a homogeneous Darboux polynomial of system (1) of degree one, i.e., \(f = b_1x_1 + b_2x_2 + b_3x_3\). Then the result easily follows because \(f\) must satisfy (4).

Proposition 5. For \((\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}\), let \(f\) be an irreducible homogeneous Darboux polynomial of system (1) with a degree at least two and the cofactor \(K\) as in Lemma 2. Then \(K = 0\).

To prove this proposition, we show that each of the coefficients \(c_1, c_2,\) and \(c_3\) in the definition of \(K\) given in Lemma 2 is zero for any Darboux polynomial of system (1) of a degree greater than or equal to two. For this, we need the following preliminary result, which describes the Darboux polynomials and their cofactors of system (1) restricted to each plane \(H_j, j = 1, 2, 3\), defined in (2).

Proposition 6. Let \(\bar{f} = \bar{f}(x_2, x_3)\) be a homogeneous Darboux polynomial of a degree \(n\) of system (1) restricted to \(x_1 = x_2\) with the cofactor \(K = (c_1 + c_2)x_2 + c_3x_3\), where \((c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}\). Then \(\bar{f} \neq 0\) has the form

\[
\bar{f} = c_0x_2^{n-l_2-l_3}(x_2 - x_3)^{l_2}(x_2 + \alpha_3^2(x_2 - x_3))^{l_3},
\]

where

\[
c_0 \in \mathbb{C} \setminus \{0\}, \quad l_2, l_3, l \in \mathbb{N} \cup \{0\}, \quad l_2 + l_3 \leq n,
\]

and, moreover,

\[
c_3 = \alpha_3^2(l_2 + l_3), \quad c_1 + c_2 + c_3 = -n - l_2.
\]
We seek Darboux polynomials of this system. Let \( f \) be a homogeneous Darboux polynomial of system (1) restricted to \( x_1 = x_2 \) with the cofactor \( K = (c_1 + c_2)x_2 + c_3x_3 \), where \((c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}\). For system (1) restricted to \( x_1 = x_2 \), we obtain
\[
\dot{x}_2 = -x_2^2, \quad \dot{x}_3 = x_2^2 - 2x_2x_3 + \alpha_3^2(x_2 - x_3)^2. \tag{23}
\]
An easy computation shows that \( x_2, x_2 - x_3, \) and \( x_2 + \alpha_3^2(x_2 - x_3) \) (for \( \alpha_3 \neq 0 \)) are the unique Darboux polynomials of degree one.

We introduce the change of variables \( y_2 = x_2 \) and \( z_2 = x_2 - x_3 \). System (23) then becomes
\[
\dot{y}_2 = -y_2^2, \quad \dot{z}_2 = -z_2(2y_2 + \alpha_3^2z_2). \tag{24}
\]
We seek Darboux polynomials of this system. Let \( f = f(y_2, z_2) \) a Darboux polynomial with the cofactor \( K = d_2y_2 + d_3z_2 \), i.e., it satisfies
\[
- y_2^2 \frac{\partial f}{\partial y_2} - z_2(2y_2 + \alpha_3^2z_2) \frac{\partial f}{\partial z_2} = (d_2y_2 + d_3z_2)f. \tag{25}
\]
If we set
\[
\left. \begin{array}{ll}
u_2 = \frac{y_2^2}{z_2}, \\
w_2 = y_2,
\end{array} \right\}
\]
then we obtain \( u_2' = \alpha_3^2w_2^2 \) and \( w_2' = -w_2^2 \). Let \( H_2 = u_2 + \alpha_3^2w_2 \). We then have \( H_2' = 0 \) and \( w_2' = -w_2^2 \). Therefore, we rewrite \( g = g(H_2, w_2) = f \left( w_2, \frac{w_2^2}{H_2 - \alpha_3^2w_2} \right) \), and it follows from (25) that
\[
-w_2^2 \frac{dg}{dw_2} = \left( d_2w_2 + \frac{d_3w_2^3}{H_2 - \alpha_3^2w_2} \right)g.
\]
Solving this, we obtain
\[
g = K(H_2)w_2^{-d_2}(H_2 - \alpha_3^2w_2)^{d_3/\alpha_3^2},
\]
where \( K \) is a function of \( H_2 \). Hence,
\[
f = K \left( \frac{y_2(y_2 + \alpha_3^2z_2)}{z_2} \right) \frac{w_2^{d_3/\alpha_3^2}y_2^{2d_3/\alpha_3^2}z_2^{-d_2}}{z_2^{d_3/\alpha_3^2}y_2^{2d_3/\alpha_3^2}z_2^{-d_2}}.
\]
Because \( f \) must be a homogeneous polynomial, we have
\[
f = c_0y_2^{l_1}z_2^{l_2}(y_2 + \alpha_3^2z_2)^{l_3}, \quad c_0 \in \mathbb{C} \setminus \{0\}, \quad l_1 + l_2 + l_3 = n \geq 2. \tag{26}
\]
Hence,
\[
d_3 = -\alpha_3^2(l_2 + l_3), \quad d_2 = -n - l_2, \quad l_1 = n - l_2 - l_3.
\]
Returning to \( \bar{f} \) and its cofactor, we obtain
\[
c_3 = \alpha_3^2(l_2 + l_3), \quad c_1 + c_2 + c_3 = -n - l_2.
\]
This completes the proof.

The following two propositions can be proved similarly.
Proposition 7. Let \( \bar{f} = \bar{f}(x_2, x_3) \) be a homogeneous Darboux polynomial of degree \( n \) of system (1) restricted to \( x_1 = x_3 \) with the cofactor \( K = c_2 x_2 + (c_1 + c_3) x_3 \), where \((c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\} \). Then for \( \bar{f} \neq 0 \), we have

\[
\bar{f} = d_0 x_3^{n-j_2-j_3}(x_3 - x_2)^{j_2}(x_3 + \alpha_2^2(x_3 - x_2))^{j_3}, \quad d_0 \in \mathbb{C} \setminus \{0\},
\]

and

\[
c_2 = \alpha_2^2(j_2 + j_3), \quad c_1 + c_2 + c_3 = -n - j_2.
\]

Proposition 8. Let \( \bar{f} = \bar{f}(x_1, x_3) \) be a homogeneous Darboux polynomial of degree \( n \) of system (1) restricted to \( x_2 = x_3 \) with the cofactor \( K = c_1 x_1 + (c_2 + c_3) x_3 \), where \((c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\} \). Then

\[
\bar{f} = e_0 x_3^{n-m_2-m_3}(x_3 - x_1)^{m_2}(x_3 + \alpha_2^2(x_3 - x_1))^{m_3}, \quad e_0 \in \mathbb{C} \setminus \{0\},
\]

and

\[
c_1 = \alpha_1^2(m_2 + m_3), \quad c_1 + c_2 + c_3 = -n - m_2.
\]

Proof of Proposition 5. Let \((\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}\), and let \( f \) be an irreducible Darboux polynomial of system (1) of degree \( n \geq 2 \) with the cofactor \( K = c_1 x_1 + c_2 x_2 + c_3 x_3 \), where \((c_1, c_2, c_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}\). Because \( f \) is irreducible, it is clear that \( f_1 \neq 0, f_2 \neq 0, \) and \( f_3 \neq 0 \) (we recall that (7) defines \( f_i \) for \( i = 1, 2, 3 \)); otherwise, \( f \) would be divisible by \( x_1 - x_2 \) or by \( x_1 - x_3 \) or \( x_2 - x_3 \), and we would thus obtain a contradiction.

Furthermore, because \( f_1, f_2, \) and \( f_3 \) are homogeneous Darboux polynomials of system (1) respectively restricted to \( x_1 = x_2, x_1 = x_3, \) and \( x_2 = x_3 \), we can apply Proposition 6 with \( \bar{f} = f_1 \), Proposition 7 with \( \bar{f} = f_2 \), and Proposition 8 with \( \bar{f} = f_3 \).

It follows from Propositions 6–8 that

\[
c_1 + c_2 + c_3 = -n - l_2 = -n - j_2 = -n - m_2,
\]

i.e., \( l_2 = j_2 = m_2 \). Furthermore, we have

\[
c_1 + c_2 + c_3 = -n - l_2 = \alpha_1^2(l_2 + m_3) + \alpha_2^2(l_2 + j_3) + \alpha_3^2(l_2 + l_3).
\]  

(27)

We note that the right-hand side of (27) is always nonnegative while the quantity \(-n - l_2\) is always nonpositive. It hence follows that these expressions must be zero, \( n = l_2 = 0 \), in contradiction with \( n \geq 2 \). The proposition is proved.

Proof of Theorem 7. If \( f \) has degree one, then the proof follows directly from Proposition 4. We now assume that system (1) has an irreducible Darboux polynomial of a degree at least two with the cofactor \( K \neq 0 \) given as in Lemma 2. From Lemma 3, we can then conclude that \( f \) is a homogeneous irreducible Darboux polynomial of degree at least two and the cofactor \( K \neq 0 \). We then obtain a contradiction from Proposition 5. Therefore, all Darboux polynomials with a cofactor \( K \neq 0 \) given as in Lemma 2 must come from Darboux polynomials of degree one, i.e., \( H_1, H_2, \) or \( H_3 \). Furthermore, it follows from Theorem 1 that all Darboux polynomials with a zero cofactor, i.e., all polynomial first integrals, must be constants. Hence, applying Lemma 5, we obtain the proof of Theorem 7.
Proof of Theorem 3. By Theorem 7, it follows that every Darboux polynomial of system (1) has the form
\[ f = cH_1^{m_1}H_2^{m_2}H_3^{m_3} = c(x_1 - x_2)^{n_1}(x_1 - x_3)^{n_2}(x_2 - x_3)^{n_3}, \]
where \( c \) is some constant and \( n_1, n_2, \) and \( n_3 \) are nonnegative integers. Furthermore, the cofactor of \( f \) is given in (22).

From Proposition 1 and Theorem 1, we find that the existence of a nonconstant rational first integral implies the existence of two coprime Darboux polynomials with the same nonzero cofactor. Hence, the first integral must have the form
\[ \frac{R}{S} = \frac{c_0H_1^{m_1}H_2^{m_2}H_3^{m_3}}{c_1(cH_1^{m_1}H_2^{m_2}H_3^{m_3})} \]
with at least one nonzero \( m_i \) and \( n_i \), and the cofactors of \( R \) and \( S \) must be equal. According to (22), the equality of the cofactors of \( R \) and \( S \) then implies that
\[
(m_1 - n_1)(-2x_3 - a_1^2(x_3 - x_1) + a_2^2(x_2 - x_3)) + \\
+ (m_2 - n_2)(-2x_2 + a_1^2(x_1 - x_2) + a_3^2(x_3 - x_2)) + \\
+ (m_3 - n_3)(-2x_1 - a_2^2(x_1 - x_2) + a_3^2(x_3 - x_1)) = 0.
\]

Hence, because the \( a_i \) are real, we obtain \( m_i = n_i \) for \( i = 1, 2, 3 \), which contradicts the fact that \( R \) and \( S \) are coprime. The theorem is thus proved.

5. Proof of Theorem 4

We recall that the equation defining the exponential factor \( F = e^{h/g} \) with the cofactor \( L \) for system (1) is
\[ \dot{x}_1 \frac{\partial h}{\partial x_1} + \dot{x}_2 \frac{\partial h}{\partial x_2} + \dot{x}_3 \frac{\partial h}{\partial x_3} = L, \]  \hspace{1cm} (28)
where we simplify the common factor \( F \) and
\[ L = b_0 + b_1x_1 + b_2x_2 + b_3x_3. \]  \hspace{1cm} (29)

According to Propositions 2 and 3 and Theorems 1 and 7, if system (1) has exponential factors, then they must have the form \( e^{h/H_1^{n_1}H_2^{n_2}H_3^{n_3}} \), where \( h \in \mathbb{C}[x_1, x_2, x_3], \) \( n_1, n_2, n_3 \in \mathbb{N} \cup \{0\} \), and the degree of \( h \) does not exceed \( n_1 + n_2 + n_3 \).

Proposition 9. For \( (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \), system (1) does not admit exponential factors.

Proof. We start by showing that there are no exponential factors of the form \( e^h \). Applying (28) with \( g = 1 \), we obtain
\[ \dot{x}_1 \frac{\partial h}{\partial x_1} + \dot{x}_2 \frac{\partial h}{\partial x_2} + \dot{x}_3 \frac{\partial h}{\partial x_3} = L, \]  \hspace{1cm} (30)
where \( L \) is given in (29). Taking \( x_1 = x_2 = x_3 = 0 \) in (30), we obtain \( b_0 = 0 \). Setting \( x_1 = x_2 = 0 \) in (30), we now obtain
\[ \alpha_3^2x_3^2 \left. \frac{\partial h}{\partial x_3} \right|_{x_1=x_2=0} = b_3x_3. \]  \hspace{1cm} (31)
This equation implies that \( b_3 = 0 \). Analogously, setting \( x_1 = x_3 = 0 \) in (30), we obtain \( b_2 = 0 \), and setting \( x_2 = x_3 = 0 \) in (30), we obtain \( b_1 = 0 \). Therefore, \( L = 0 \), and from (30), we find that \( h \) is a polynomial first integral of system (1), which contradicts Theorem 1.
We now suppose that $e^{h/H_1}H_2^{n_2}H_3^{n_3}$ is an exponential factor of system (1), where $n_1$, $n_2$, and $n_3$ are nonnegative integers with at least one of them positive, $h$ is coprime to $H_1$, $H_2$, and $H_3$, and the degree of $h$ is at most $n_1 + n_2 + n_3$. Then $h$ satisfies

$$\dot{x}_1 \frac{\partial h}{\partial x_1} + \dot{x}_2 \frac{\partial h}{\partial x_2} + \dot{x}_3 \frac{\partial h}{\partial x_3} - \left( \frac{\dot{H}_1}{H_1} n_1 + \frac{\dot{H}_2}{H_2} n_2 + \frac{\dot{H}_3}{H_3} n_3 \right) h = LH_1^{n_1}H_2^{n_2}H_3^{n_3}. \quad (32)$$

Without loss of generality, we can assume that $n_1 > 0$. Taking $H_1 = 0$ in (32) and letting $h_1$ denote the restriction of $h$ to $H_1 = 0$, we conclude that $h_1$ satisfies

$$-x_2^2 \frac{\partial h_1}{\partial x_2} + \left( x_2^2 - 2x_2x_3 + \alpha_3^2(x_2 - x_3)^2 \right) \frac{\partial h_1}{\partial x_3} =$$

$$= \left[ n_1 \left( (\alpha_1^2 + \alpha_2^2)(x_2 - x_3) - 2x_3 \right) + (n_2 + n_3)(\alpha_3^2x_3 - (\alpha_2^2 + 2)x_2) \right] h_1. \quad (33)$$

Because $h$ is coprime with $H_1$ by hypothesis, we have $h_1 \neq 0$. Furthermore, from (33), $h_1$ is a Darboux polynomial of system (1) restricted to $x_1 = x_2$ with the cofactor

$$K = n_1 \left( (\alpha_1^2 + \alpha_2^2)(x_2 - x_3) - 2x_3 \right) + (n_2 + n_3)(\alpha_3^2x_3 - (\alpha_2^2 + 2)x_2).$$

In view of Lemma 3, we can assume that $h_1$ is homogeneous, and we know that $n \leq n_1 + n_2 + n_3$. In view of Proposition 6, we must then have

$$\alpha_3^2(l_3 + l_2) = -(2 + \alpha_1^2 + \alpha_2^2)n_1 + \alpha_3^2(n_2 + n_3), \quad c_1 + c_2 + c_3 = -n - l_2. \quad (34)$$

Computing $c_1 + c_2 + c_3$ from (33), we obtain $c_1 + c_2 + c_3 = -2(n_1 + n_2 + n_3)$. Hence,

$$n + l_2 = 2(n_1 + n_2 + n_3).$$

Because $n \leq n_1 + n_2 + n_3$ and $l_2 \leq n$, we have $l_2 = n = n_1 + n_2 + n_3$ and $l_1 = l_3 = 0$. The first identity in (34) then becomes

$$\alpha_3^2n_1 = -(2 + \alpha_1^2 + \alpha_2^2)n_1,$$

which is impossible because $n_1 > 0$ and the right-hand side is hence negative while the left-hand side is not. This completes the proof of the proposition.

**Proof of Theorem 4.** From Propositions 2 and 9 and Theorems 1 and 7, if system (1) has a Darboux first integral $G$, then $G = cH_1^{\lambda_1}H_2^{\lambda_2}H_3^{\lambda_3}$, where $c, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$. Because $G$ is a first integral, we must have

$$\lambda_1 (-2x_3 - \alpha_1^2(x_3 - x_1) + \alpha_2^2(x_2 - x_3)) +$$

$$+ \lambda_2 (-2x_2 + \alpha_1^2(x_1 - x_2) + \alpha_3^2(x_3 - x_2)) +$$

$$+ \lambda_3 (-2x_1 - \alpha_2^2(x_1 - x_2) + \alpha_3^2(x_3 - x_1)) = 0.$$

Hence, because the $\alpha_i$ are real for $i = 1, 2, 3$, we obtain $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which completes the proof of the theorem.

**Acknowledgments.** This work is supported in part by MINECO/FEDER (Grant No. MTM2008–03437, J. L.), CIRIT (Grant No. 2014SGR–568, J. L.), ICREA Academia (Grant Nos. FP7-PEOPLE-2012-IRSES 316338 and 318999, J. L.), FEDER (Grant No. FEDER-UNAB10-4E-378, J. L.), the Center for Mathematical Analysis, Geometry, and Dynamical Systems (C. V.), and Fundação para a Ciência e a Tecnologia (Program POCTI/FEDER, Program POSI, and Grant No. SFRH/BPD/14404/2003, C. V.).
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