RICCI SOLITONS ON LOW-DIMENSIONAL GENERALIZED SYMMETRIC SPACES

GIOVANNI CALVARUSO AND E. ROSADO

Abstract. We consider three- and four-dimensional pseudo-Riemannian generalized symmetric spaces, whose invariant metrics were explicitly described in [15]. While four-dimensional pseudo-Riemannian generalized symmetric spaces of types $A$, $C$ and $D$ are algebraic Ricci solitons, the ones of type $B$ are not so. The Ricci soliton equation for their metrics yields a system of partial differential equations. Solving such system, we prove that almost all the four-dimensional pseudo-Riemannian generalized symmetric spaces of type $B$ are Ricci solitons. These examples show some deep differences arising for the Ricci soliton equation between the Riemannian and the pseudo-Riemannian cases, as any homogeneous Riemannian Ricci soliton is algebraic [21]. We also investigate three-dimensional generalized symmetric spaces of any signature and prove that they are Ricci solitons.

1. Introduction

Generalized symmetric spaces are a natural generalization of symmetric spaces. Since their introduction, the geometry of generalized symmetric spaces has been intensively studied by several authors. Finite order automorphisms of semisimple Lie algebras and Riemannian manifolds with geodesic symmetries of order 3 were studied respectively in [22] and [19]. In [24], O. Kowalski undertook a study of generalized symmetric spaces without using neither topological invariants nor advanced algebra. Homogeneous structures of generalized symmetric Riemannian spaces were studied in [18]. S. Terzić classified generalized symmetric spaces defined as quotients of compact simple Lie groups, describing explicitly their real cohomology algebras [27] and calculating their real Pontryagin characteristic classes [28]. Formality of all generalized symmetric spaces was proved by D. Kotschick and S. Terzić in [24].

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Černý and Kowalski \cite{15} completely classified pseudo-Riemannian generalized symmetric spaces of dimension \( \leq 4 \). In dimension \( n = 2 \) they are necessarily symmetric. In dimension \( n = 3 \) the proper examples may be described as \( \mathbb{R}^3 \) endowed with a special metric, with all possible signatures. In dimension \( n = 4 \), a proper generalized symmetric space may be identified with \( \mathbb{R}^4 \) endowed with a special metric of four different types, called in \cite{15} types \( A, B, C \) and \( D \). The metrics of type \( A \) are either Riemannian or of neutral signature \((2, 2)\); metrics of type \( B \) and \( D \) always have signature \((2, 2)\); for type \( C \) (which was proved in \cite{17} to be indeed symmetric), the metric is Lorentzian. All these spaces are reductive homogeneous.

Many aspects of the geometry of four-dimensional pseudo-Riemannian generalized symmetric spaces have been investigated: Kähler and para-Kähler structures \cite{5}, harmonicity properties of vector fields \cite{6}, curvature properties \cite{8}, parallel degenerate distributions \cite{12}, homogeneous geodesics \cite{16}, parallel hypersurfaces \cite{17}. Four-dimensional pseudo-Riemannian generalized symmetric spaces of type \( A, C \) and \( D \) are algebraic Ricci solitons, whereas those of type \( B \) are never algebraic Ricci solitons \cite{1}.

A \textit{Ricci soliton} is a pseudo-Riemannian manifold \((M, g)\), together with a smooth vector field \( X \), such that
\begin{equation}
\mathcal{L}_X g + \varrho = \lambda g,
\end{equation}
where \( \mathcal{L}_X \) and \( \varrho \) respectively denote the Lie derivative in the direction of \( X \) and the Ricci tensor and \( \lambda \) is a real number. A Ricci soliton is said to be either \textit{shrinking}, \textit{steady} or \textit{expanding}, according to whether \( \lambda > 0 \), \( \lambda = 0 \) or \( \lambda < 0 \), respectively. When \( X \) is the gradient of some smooth function \( f : M \to \mathbb{R} \), the metric \( g \) is said to be a \textit{gradient Ricci soliton}.

Complete Ricci solitons with respect to some complete vector field correspond to the self-similar solutions of the Ricci flow they generate. As such, they play an essential role in understanding the singularities of the Ricci flow. We may refer to the recent survey \cite{14} for more information and further references on Ricci solitons. Pseudo-Riemannian Ricci solitons have been recently studied by several authors, some examples may be found in \cite{1,3,7,9,11,13,26} and references therein.

If we start from the explicit description of a pseudo-Riemannian metric \( g \) with respect to some (local) coordinates, the Ricci soliton equation (1.1) leads to a system of partial differential equations. However, in the case of homogeneous pseudo-Riemannian metrics, the first approach in the study of the Ricci soliton equation (1.1) is usually algebraic. A \textit{homogeneous Ricci soliton} is a homogeneous space \( M = G/H \), together with a \( G \)-invariant metric \( g \), for which equation (1.1) holds. An \textit{invariant Ricci soliton} is a homogeneous one, such that equation (1.1) holds for a \( G \)-invariant vector field.
Algebraic Ricci solitons were introduced by Lauret [25] for Riemannian manifolds and successively extended to pseudo-Riemannian settings [26]. Consider a homogeneous (reductive) pseudo-Riemannian manifold \((M = G/H, g)\) and the corresponding reductive decomposition \(g = m \oplus h\) of the Lie algebra \(g\) of \(G\). The metric \(g\) is said to be an algebraic Ricci soliton if there exists some derivation \(D \in \text{Der}(g)\), such that

\[
\text{Ric} = c \text{Id} + \text{pr} \circ D,
\]

where \(\text{Ric}\) denotes the Ricci operator of \(m\), \(\text{pr}: g \to m\) the projection and \(c\) is a real number. An algebraic Ricci soliton on a solvable Lie group is called a solvsoliton.

Any algebraic Ricci soliton metric \(g\) is also a Ricci soliton \([25, 26]\), satisfying (1.1) with \(\lambda = c\). We emphasize the fact that in Riemannian settings this algebraic approach is essentially all that is needed. In fact, homogeneous Riemannian Ricci solitons are indeed algebraic, with respect to some suitable transitive group \(G\) of isometries [21]. Therefore, the algebraic approach to the study of Ricci solitons exhausts all possibilities in the Riemannian case. On the other hand, in pseudo-Riemannian settings, neither algebraic nor invariant Ricci solitons are generally enough to determine all homogeneous Ricci solitons. Finally, we observe that also in the case of an algebraic Ricci soliton metric, this information does not codify all possible remarkable properties of this solution. In fact, given an algebraic Ricci soliton, we do not know whether it is also invariant or a gradient one.

In this paper we shall complete the study of the Ricci soliton equation (1.1) on low-dimensional pseudo-Riemannian generalized symmetric spaces, considering the three-dimensional examples and the four-dimensional pseudo-Riemannian generalized symmetric spaces of type \(B\). Three-dimensional examples, of any signature, turn out to be algebraic Ricci solitons. Retrieving them as explicit solutions of equation (1.1), we shall prove that they are not gradient. Four-dimensional generalized symmetric spaces of type \(B\) will provide a new family of examples of non-algebraic homogeneous pseudo-Riemannian Ricci solitons.

The paper is organized in the following way. In Section 2 we shall report the description in a set of (global) coordinates of three-dimensional Lorentzian generalized symmetric spaces and four-dimensional pseudo-Riemannian generalized symmetric spaces of type \(B\), and we shall compute all the curvature information with respect to the corresponding basis of coordinate vector fields. In Sections 3 and 4 we shall introduce and solve the systems of PDEs that express the Ricci soliton equation (1.1) in these global coordinates.
2. **Generalized Symmetric spaces**

Let \((M, g)\) be a pseudo-Riemannian manifold. An \(s\)-structure on \(M\) is a family of isometries \(\{s_p \mid p \in M\}\) (called symmetries) of \((M, g)\), such that each \(s_p\) has \(p\) as an isolated fixed point. The order of an \(s\)-structure is the least integer \(k\) such that \((s_p)^k \equiv \text{Id}\) for all \(p \in M\) (if such an integer does not exist, the \(s\)-structure is said to be of infinite order).

If, for any pair of points \(p, q \in M\),

- (i) the mapping \((p, q) \to s_p(q)\) is smooth, and
- (ii) \(s_p \circ s_q = s_{\tilde{q}} \circ s_p\), where \(\tilde{q} = s_p(q)\),

then the \(s\)-structure is said to be regular. Each symmetric space admits a regular \(s\)-structure, given by the family of its involutive geodesic symmetries. More in general, a pseudo-Riemannian manifold admitting at least one regular \(s\)-structure is called a generalized symmetric space. The infimum of all integers \(k(\geq 2)\) such that \((M, g)\) admits an \(s\)-structure of order \(k\), is called the order of the generalized symmetric space.

Following [15], any proper (that is, non-symmetric) three-dimensional generalized symmetric space \((M, g)\) is of order 4. Moreover, it is given by the space \(\mathbb{R}^3(x, y, t)\) with the pseudo-Riemannian metric

\[
(2.1) \quad g = \varepsilon (e^{2t}dx^2 + e^{-2t}dy^2) + \mu dt^2,
\]

where \(\varepsilon = \pm 1\) and \(\mu \neq 0\) is a real constant. With respect to the notations introduced in [15], in equation \(2.1\) we used \(\mu\) instead of \(\lambda\), which we reserved for the Ricci soliton equation \((1.1)\). Depending on the values of \(\varepsilon\) and \(\mu\), these metrics can attain any possible signature: \((3, 0), (0, 3), (2, 1), (1, 2)\).

As proved in [15], a generalized symmetric space \((M, g)\) of type \(B\) is of order 3 and is given by the space \(\mathbb{R}^4(x, y, u, v)\) with the pseudo-Riemannian metric

\[
(2.2) \quad g = \mu (dx^2 + dy^2 + dx dy) + e^{-y} (2dx + dy) dv + e^{-x} (dx + 2dy) du,
\]

where \(\mu\) is a real constant. All these metrics have neutral signature \((2, 2)\).

3. **Ricci solitons on 3D generalized symmetric spaces**

Starting from the description of the invariant metrics of three-dimensional generalized symmetric spaces given in \(2.1\), we can explicitly calculate their Levi-Civita connection and curvature. We use the coordinates \((x_1, x_2, x_3) = (x, y, t)\) and the corresponding basis of coordinate vector fields \(\{\partial_1, \partial_2, \partial_3\} = \)
\{\partial/\partial x, \partial/\partial y, \partial/\partial t\}$. Then, the Levi-Civita connection $\nabla$ is completely determined by the following non-vanishing components:

\begin{equation}
\begin{aligned}
\nabla_{\partial_1} \partial_1 &= -\frac{\varepsilon}{\mu} e^{2x_3} \partial_3, \\
\nabla_{\partial_2} \partial_2 &= \frac{\varepsilon}{\mu} e^{-2x_3} \partial_3, \\
\nabla_{\partial_3} \partial_3 &= -\partial_2.
\end{aligned}
\end{equation}

We can now describe the Riemann-Christoffel curvature tensor $R$ of $(M, g)$ with respect to $\{\partial_i\}$, computing $R(\partial_i, \partial_j) = \nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i}$ for all indices $i, j$. Denoting by $R_{ij}$ the matrix describing $R(\partial_i, \partial_j)$ with respect to the basis $\{\partial_1, \partial_2, \partial_3\}$ of coordinate vector fields, we have

\begin{align*}
R_{12} &= \begin{pmatrix} 0 & \frac{\varepsilon}{\mu} e^{-2x_3} & 0 \\ -\frac{\varepsilon}{\mu} e^{2x_3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
R_{13} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ \frac{\varepsilon}{\mu} e^{2x_3} & 0 & 0 \end{pmatrix}, \\
R_{23} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & \frac{\varepsilon}{\mu} e^{2x_3} & 0 \end{pmatrix}.
\end{align*}

Consequently, the Ricci tensor $\varrho(X, Y) = \text{tr}(Z \mapsto R(Z, X)Y)$ is then described with respect to the basis of coordinate vector fields by the matrix

\begin{equation}
\varrho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{equation}

In particular, we observe that Ricci tensor is always degenerate.

With respect to the coordinate basis $\{\partial_1, \partial_2, \partial_3\}$, let now $X = X^i \partial_i$ denote an arbitrary vector field on $(M, g)$, where $X^i = X^i(x_1, x_2, x_3)$, $i = 1, 2, 3$, are arbitrary smooth functions. We calculate $L_X g$ and obtain that the metric $g$, together with the smooth vector field $X$, is a solution of the Ricci soliton equation (1.1) if and only if the following system of 6 PDEs is satisfied:

\begin{equation}
\begin{aligned}
\varepsilon e^{2x_3} (2 \partial_1 X_1 + 2X_3 - \lambda) &= 0, \\
\varepsilon e^{2x_3} \partial_2 X_1 + \varepsilon e^{-2x_3} \partial_1 X_2 &= 0, \\
\varepsilon e^{2x_3} \partial_3 X_1 + \mu \partial_1 X_3 &= 0, \\
\varepsilon e^{-2x_3} (2 \partial_2 X_2 - 2X_3 - \lambda) &= 0, \\
\varepsilon e^{-2x_3} \partial_3 X_2 + \mu \partial_2 X_3 &= 0, \\
2 \mu \partial_3 X_3 - 2 - \lambda \mu &= 0.
\end{aligned}
\end{equation}
We first integrate the last equation in (3.3) and we get
\[ X_3 (x_1, x_2, x_3) = \frac{x_3}{\mu} + \frac{1}{2} x_3 \lambda + F_3 (x_1, x_2), \]
where \( F_3 \) is an arbitrary smooth function. Substituting the above expression of \( X_3 \) into the third and fifth equations of (3.3), they respectively become
\[
\begin{align*}
&\varepsilon e^{2x_3} \partial_1 X_1 + \mu \partial_1 F_3 (x_1, x_2) = 0, \\
&\varepsilon e^{-2x_3} \partial_2 X_2 + \mu \partial_2 F_3 (x_1, x_2) = 0,
\end{align*}
\]
which, integrated, yield
\[
\begin{align*}
X_1 (x_1, x_2, x_3) &= \frac{\mu}{2} e^{-2x_3} \partial_1 F_3 (x_1, x_2) + F_1 (x_1, x_2), \\
X_2 (x_1, x_2, x_3) &= -\frac{\mu}{2} e^{2x_3} \partial_2 F_3 (x_1, x_2) + F_2 (x_1, x_2),
\end{align*}
\]
for some smooth functions \( F_1, F_2 \). Substituting the above expressions of \( X_1 \) and \( X_2 \) into the second equation of (3.3), it now gives
\[
\varepsilon e^{2x_3} \partial_2 F_1 (x_1, x_2) + \varepsilon e^{-2x_3} \partial_1 F_2 (x_1, x_2) = 0.
\]
Since the above equation (3.4) must hold for all values of \( x_3 \), it yields \( \partial_2 F_1 = \partial_1 F_2 = 0 \), that is,
\[
F_1 (x_1, x_2) = G_1 (x_1), \quad F_2 (x_1, x_2) = G_2 (x_2),
\]
for some smooth functions \( G_1, G_2 \). System (3.3) then reduces to its first and fourth equations. In particular, the first equation in (3.3) now reads
\[
\varepsilon e^{2x_3} \left( 2 \partial_1 G_1 (x_1) + \left( \frac{2}{\mu} + \lambda \right) x_3 + 2 F_3 (x_1, x_2) - \varepsilon \lambda \right) + \mu \partial_1^2 F_3 (x_1, x_2) = 0.
\]
Since the above equation (3.5) holds for all values of \( x_3 \), in particular it implies that \( \partial_1^2 F_3 = 0 \), from which, by integration, we obtain
\[
F_3 (x_1, x_2) = P_3 (x_2) x_1 + Q_3 (x_2),
\]
for some smooth functions \( P_3, Q_3 \). We substitute into (3.5) and it becomes
\[
\varepsilon e^{2x_3} \left( 2 \partial_1 G_1 (x_1) + \left( \frac{2}{\mu} + \lambda \right) x_3 + 2 P_3 (x_2) x_1 + 2 Q_3 (x_2) - \lambda \right) = 0.
\]
Again, this equation must hold for all values of \( x_3 \) and \( x_2 \). Therefore, it implies at once that \( \lambda = -\frac{2}{\mu} \) and that \( P_3 (x_2) = B_3, Q_3 (x_2) = A_3 \), for some real constants \( A_3, B_3 \). Thus, the above equation now reduces to
\[
2 \varepsilon e^{2x_3} \left( \partial_1 G_1 (x_1) + B_3 x_1 + A_3 + \frac{1}{\mu} \right) = 0
\]
and so, integrating we find
\[
G_1 (x_1) = -\frac{1}{2} B_3 x_1^2 - \left( A_3 + \frac{1}{\mu} \right) x_1 + A_1,
\]
where $A_1$ is a real constant. We are then left with the fourth equation in (3.3), which now reads

$$2 \varepsilon e^{-2x_3} \left( \partial_2 G_2(x_2) - B_3 x_1 - A_3 + \frac{1}{\mu} \right) = 0.$$ 

Since the above equation must hold for all values of $x_1$ and $x_2$, we then necessarily have $B_3 = 0$ and, by integration,

$$G_2(x_2) = \left( A_3 - \frac{1}{\mu} \right) x_2 + A_2,$$

where $A_2$ is a real constant. All equations in system (3.3) are now satisfied. Therefore, the arbitrary invariant metric $g$ of a three-dimensional generalized symmetric space is a Ricci soliton. More precisely, replacing the explicit expressions of the functions we found above into $X_1, X_2, X_3$, we conclude that, with respect to coordinates $(x_1, x_2, x_3) = (x, y, t)$, the Ricci soliton equation (1.1) holds for the metric $g$ together with the vector field

$$(3.6) \quad X = \left( A_1 - \left( A_3 + \frac{1}{\mu} \right) x_1 \right) \partial_1 + \left( A_2 + \left( A_3 - \frac{1}{\mu} \right) x_2 \right) \partial_2 + A_3 \partial_3,$$

where $A_1, A_2, A_3$ are real constants.

In order to check the above result, we compute $L_X g$ for the vector field $X$ described in (3.6) and compare it with $-\varrho - \frac{2}{\mu} g$, which can be calculated at once from (2.1) and (3.2). We explicitly find, with respect to the basis $\{\partial_1, \partial_2, \partial_3\}$:

$$L_X g = \begin{pmatrix} -\frac{2\varepsilon}{\mu} e^{2x_3} & 0 & 0 \\ 0 & -\frac{2\varepsilon}{\mu} e^{-2x_3} & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\varrho - \frac{2}{\mu} g.$$

We now prove that the above Ricci soliton is not a gradient one, that is, there no exists a smooth function $f(x_1, x_2, x_3)$, such that $X = \nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i} \partial x_j$. In fact, suppose that such a function exists. Then, by (3.6) we have

$$\varepsilon e^{-2x_3} \partial_1 f = A_1 - \left( A_3 + \frac{1}{\mu} \right) x_1,$$

$$\varepsilon e^{2x_3} \partial_2 f = A_2 + \left( A_3 - \frac{1}{\mu} \right) x_2,$$

$$\frac{1}{\mu} \partial_3 f = A_3.$$

We integrate the third equation of (3.7), obtaining

$$f(x_1, x_2, x_3) = \mu A_3 x_3 + W(x_1, x_2),$$
where \( W \) is a smooth function. We substitute into the second equation of (3.7) and we get
\[
\varepsilon e^{x_3} \partial_2 W(x_1, x_2) = A_2 + \left( A_3 - \frac{1}{\mu} \right) x_2.
\]
Since the above equation must hold for all values of \( x_3 \), it implies that \( \partial_2 W(x_1, x_2) = 0 \) and \( A_2 + \left( A_3 - \frac{1}{\mu} \right) x_2 = 0 \), for all \( x_2 \). Therefore, \( A_2 = 0 \) and \( A_3 = \frac{1}{\mu} \) and integrating \( \partial_2 W = 0 \) we have
\[
W(x_1, x_2) = H(x_1),
\]
for a smooth function \( H \). Finally, we substitute the above into the first equation of (3.7), which now reads
\[
\varepsilon e^{-x_3} \partial_1 H(x_1) = A_1 - \frac{2}{\mu} x_1.
\]
Again, the above equation must be satisfied for all values of \( x_3 \). So, \( \partial_1 H(x_1) = 0 \) (whence, \( H(x_1) = R_1 \) is a real constant) and we are left with
\[
A_1 - \frac{2}{\mu} x_1 = 0,
\]
for all values of \( x_1 \), which cannot occur. Therefore, \( g \) is never a gradient Ricci soliton. This fact is not surprising, since the existence of a gradient non-steady Ricci soliton has some strong consequences on the geometry of the manifold [4].

The above results are summarized in the following.

**Theorem 1.** Every three-dimensional proper generalized symmetric space is a Ricci soliton. More precisely, the arbitrary metric \( g \) described by (2.1) for \( \varepsilon = \pm 1 \) and \( \mu \neq 0 \), satisfies the Ricci soliton equation (1.1) taking
\[
X = \left( A_1 - \left( A_3 + \frac{1}{\mu} \right) x \right) \frac{\partial}{\partial x} + \left( A_2 + \left( A_3 - \frac{1}{\mu} \right) y \right) \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial t},
\]
where \( A_1, A_2, A_3 \) are real constants. This Ricci soliton is not a gradient one.

4. **Ricci solitons on 4D generalized symmetric spaces of type B**

We described the general pseudo-Riemannian metric of a four-dimensional generalized symmetric space of type \( B \) in (2.2). Proceeding similarly to the three-dimensional case treated in the previous section, we first compute its Levi-Civita connection and curvature. We consider global coordinates \((x_1, x_2, x_3, x_4) = (x, y, u, v)\). With respect to the basis of coordinate vector
fields \(\{\partial_1, \partial_2, \partial_3, \partial_4\}\), where \(\partial_i = \partial/\partial x_i, \, 1 \leq i \leq 4\), the Levi-Civita connection \(\nabla\) is completely determined by the following non-vanishing components:

\[
\begin{align*}
\nabla_{\partial_1} \partial_1 &= \frac{1}{3} (\partial_1 - 2\partial_2) + \frac{1}{3} \mu (2e^{x_1} \partial_3 - e^{x_2} \partial_4), \\
\nabla_{\partial_1} \partial_2 &= -\frac{1}{3} (\partial_1 + \partial_2) + \frac{1}{3} \mu (e^{x_1} \partial_3 + e^{x_2} \partial_4), \\
\nabla_{\partial_1} \partial_3 &= -\frac{1}{3} e^{-x_1} (2e^{x_1} \partial_3 - e^{x_2} \partial_4), \\
\nabla_{\partial_1} \partial_4 &= \frac{1}{3} e^{-x_2} (2e^{x_1} \partial_3 - e^{x_2} \partial_4), \\
\nabla_{\partial_2} \partial_2 &= -\frac{1}{3} (2\partial_1 - \partial_2) - \frac{1}{3} \mu (e^{x_1} \partial_3 - 2e^{x_2} \partial_4), \\
\nabla_{\partial_2} \partial_3 &= -\frac{1}{3} e^{-x_1} (e^{x_1} \partial_3 - 2e^{x_2} \partial_4), \\
\nabla_{\partial_2} \partial_4 &= \frac{1}{3} e^{-x_2} (e^{x_1} \partial_3 - 2e^{x_2} \partial_4).
\end{align*}
\]

(4.1)

Consequently, we can then determine the Riemann-Christoffel curvature tensor \(R\) of \(g\) with respect to \(\{\partial_i\}\). Denoting by \(R_{ij}\) the matrix describing \(R(\partial_i, \partial_j)\) with respect to the basis \(\{\partial_1, \partial_2, \partial_3, \partial_4\}\), of coordinate vector fields, we have \(R_{14} = R_{23} = R_{34} = 0\) and

\[
R_{12} = \frac{1}{3} \begin{pmatrix}
-1 & -2 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2e^{x_1-x_2} \\
0 & 0 & -2e^{-x_1+x_2} & -1
\end{pmatrix},
\]

\[
R_{13} = \frac{1}{3} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
-e^{-x_1+x_2} & -2e^{-x_1+x_2} & 0 & 0
\end{pmatrix},
\]

\[
R_{24} = \frac{1}{3} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-2e^{x_1-x_2} & -e^{x_1-x_2} & 0 & 0 \\
1 & 2 & 0 & 0
\end{pmatrix}.
\]

The Ricci tensor, \(g(X, Y) = \text{tr}(Z \mapsto R(Z, X)Y)\), is then described with respect to the basis of coordinate vector fields by the matrix

\[
g = \begin{pmatrix}
-\frac{1}{3} & -\frac{2}{3} & 0 & 0 \\
-\frac{2}{3} & -\frac{4}{3} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

and so, is always degenerate.

Let now \(X = X^i \partial_i\) denote an arbitrary vector field on \((M, g)\), where \(X^i = X^i(x_1, x_2, x_3, x_4), \, 1 \leq i \leq 4\), are arbitrary smooth functions. We compute \(L_X g\) and find that the metric \(g\), together with the smooth vector field \(X\), is a solution of the Ricci soliton equation (4.1) if and only if the
following system of 10 PDEs is satisfied:

\[
\begin{align*}
(i) \quad & \mu \partial_1 (2X^1 + X^2) + e^{-x_1} \partial_1 X^3 + 2e^{-x_2} \partial_1 X^4 - \frac{4}{3} \lambda \mu = 0, \\
(ii) \quad & \mu \partial_1 (X^1 + 2X^2) + \mu \partial_2 (2X^1 + X^2) \\
& + e^{-x_2} (\partial_1 + 2\partial_2) X^4 + e^{-x_1} (2\partial_1 + \partial_2) X^3 - \frac{4}{3} \lambda \mu = 0, \\
(iii) \quad & e^{-x_1} \partial_1 (X^1 + 2X^2) + \mu \partial_3 (2X^1 + X^2) \\
& + \partial_3 (e^{-x_1} X^3 + 2e^{-x_2} X^4) - e^{-x_1} X^1 - \lambda e^{-x_1} = 0, \\
(iv) \quad & e^{-x_2} \partial_1 (2X^1 + X^2) + \mu \partial_4 (2X^1 + X^2) \\
& + \partial_4 (e^{-x_1} X^3 + 2e^{-x_2} X^4) - 2e^{-x_2} (X^2 + \lambda) = 0, \\
(v) \quad & \mu \partial_2 (X^1 + 2X^2) + 2e^{-x_1} \partial_2 X^3 + e^{-x_2} \partial_2 X^4 - \frac{4}{3} \lambda \mu = 0, \\
(vi) \quad & e^{-x_1} \partial_2 (X^1 + 2X^2) + \mu \partial_3 (X^1 + 2X^2) \\
& + \partial_3 (2e^{-x_1} X^3 + e^{-x_2} X^4) - 2e^{-x_1} (X^1 + \lambda) = 0, \\
(vii) \quad & e^{-x_2} \partial_2 (2X^1 + X^2) + \mu \partial_4 (X^1 + 2X^2) \\
& + \partial_4 (2e^{-x_1} X^3 + e^{-x_2} X^4) - e^{-x_2} (X^2 + \lambda) = 0, \\
(viii) \quad & e^{-x_1} \partial_3 (X^1 + 2X^2) = 0, \\
(ix) \quad & e^{-x_2} \partial_3 (2X^1 + X^2) + e^{-x_1} \partial_4 (X^1 + 2X^2) = 0, \\
(x) \quad & e^{-x_2} \partial_4 (2X^1 + X^2) = 0.
\end{align*}
\]

Integrating equations (viii) and (x) in (4.3) we find that there exist some smooth functions \(Y^1(x_1, x_2, x_3)\) and \(Y^2(x_1, x_2, x_3)\), such that

\[
\begin{align*}
(4.4) \quad & X^1 (x_1, x_2, x_3, x_4) + 2X^2 (x_1, x_2, x_3, x_4) = Y^1 (x_1, x_2, x_4), \\
(4.5) \quad & 2X^1 (x_1, x_2, x_3, x_4) + X^2 (x_1, x_2, x_3, x_4) = Y^2 (x_1, x_2, x_3).
\end{align*}
\]

Observe that \(\partial_3 Y^2\) does not depend on \(x_4\), while \(\partial_1 Y^1\) does not depend on \(x_3\). Consequently, taking into account (4.4) and (4.5), equation (ix) in (4.3) implies that there exists some function \(A(x_1, x_2)\), such that

\[
\frac{1}{2} e^{-x_1} \partial_4 Y^1 (x_1, x_2, x_4) = -\frac{1}{2} e^{-x_2} \partial_3 Y^2 (x_1, x_2, x_3) = A(x_1, x_2).
\]

By integrating the above equations with respect to \(Y^1(x_1, x_2, x_3)\) and \(Y^2(x_1, x_2, x_4)\) respectively, we obtain

\[
\begin{align*}
(4.6) \quad & Y^1 (x_1, x_2, x_4) = 2 e^{x_1} x_4 A(x_1, x_2) + B^1 (x_1, x_2), \\
(4.7) \quad & Y^2 (x_1, x_2, x_3) = -2 e^{x_2} x_3 A(x_1, x_2) + B^2 (x_1, x_2),
\end{align*}
\]

for some smooth functions \(B^i(x_1, x_2), \ i = 1, 2\). By (4.4)-(4.7), we now have

\[
\begin{align*}
(4.8) \quad & X^1 = -\frac{2}{3} (e^{x_1} x_4 + 2e^{x_2} x_3) A(x_1, x_2) - \frac{1}{3} B^1 (x_1, x_2) + \frac{2}{3} B^2 (x_1, x_2), \\
(4.9) \quad & X^2 = \frac{2}{3} (2e^{x_1} x_4 + e^{x_2} x_3) A(x_1, x_2) + \frac{2}{3} B^1 (x_1, x_2) - \frac{1}{3} B^2 (x_1, x_2).
\end{align*}
\]
Using (4.8) and (4.9), equation (iii) in (4.3) becomes

\[
0 = \frac{1}{2} e^{-x_1} \partial_3 X_3 + e^{-x_2} \partial_3 X_4 + \partial_1 A(x_1, x_2) x_4 + \frac{1}{2} e^{-x_1} \partial_1 B^1(x_1, x_2) + \frac{1}{2} e^{-x_1} \partial_1 B^1(x_1, x_2) + \frac{1}{2} e^{-x_1} \partial_1 B^1(x_1, x_2) + \frac{1}{2} e^{-x_1} (B^1(x_1, x_2) - 2B^2(x_1, x_2)) - \frac{1}{2} \lambda e^{-x_1}.
\]

Next, we substitute from (4.8), (4.9), (4.10) and (4.11) into equation (iv) of (4.3) it gives

We integrate the above equation with respect to \(X^4(x_1, x_2, x_3, x_4)\) and we obtain

\[
X^4 = - \left( \frac{1}{3} x_3 e^{-x_1} + \frac{1}{2} x_3 e^{-x_2} - \mu \right) e^{2x_2} x_3 A(x_1, x_2) - e^{x_2} \partial_1 A(x_1, x_2) x_3 x_4 - \frac{1}{2} \partial_1 B^1(x_1, x_2) e^{-x_1 + x_2} x_3 + \frac{1}{2} \lambda e^{-x_1 + x_2} x_3 - \frac{1}{2} e^{-x_1 + x_2} X^3 + Y^4(x_1, x_2, x_4),
\]

for some smooth function \(Y^4(x_1, x_2, x_4)\). Substituting from (4.8), (4.9) and (4.10) into equation (vi) of (4.3) it gives

\[
0 = \frac{3}{2} e^{-x_1} \partial_3 X^3 + \frac{1}{2} e^{-x_1} (2 \partial_2 B^1(x_1, x_2) - \partial_1 B^1(x_1, x_2)) - \frac{1}{2} (\partial_1 A(x_1, x_2) - 2 \partial_2 A(x_1, x_2)) x_4 + \frac{1}{2} e^{-x_1} (B^1(x_1, x_2) - 2B^2(x_1, x_2)) + A(x_1, x_2) x_3 e^{-x_1 + x_2} + \frac{1}{2} \mu A(x_1, x_2) e^{x_2} x_3 - \frac{1}{2} \lambda e^{-x_1}.
\]

We then integrate the above equation with respect to \(X^3(x_1, x_2, x_3, x_4)\) and we get

\[
X^3 = \frac{1}{2} \left( \partial_1 B^1(x_1, x_2) - 2 \partial_2 B^1(x_1, x_2) \right) x_3 + \frac{3}{2} e^{x_1} \partial_1 A(x_1, x_2) - 2 \partial_2 A(x_1, x_2) x_3 x_4 + \frac{1}{2} \left( 2B^2(x_1, x_2) - B^1(x_1, x_2) \right) x_3 - \frac{1}{3} A(x_1, x_2) e^{x_2} x_3^2 - \frac{2}{3} A(x_1, x_2) e^{x_2} x_3 + \lambda x_3 + Y^3(x_1, x_2, x_4).
\]

Next, we substitute from (4.8), (4.9), (4.10) and (4.11) into equation (iv) of (4.3) and we find

\[
-2 \left( \partial_1 A(x_1, x_2) + A(x_1, x_2) \right) x_3 + e^{-x_2} \partial_1 Y^4(x_1, x_2, x_4) + \frac{1}{2} e^{-x_2} \partial_1 B_2(x_1, x_2) - \frac{1}{2} A(x_1, x_2) x_4 e^{-x_1 - x_2} + \frac{1}{3} e^{-x_2} (B_2(x_1, x_2) - 2B^1(x_1, x_2)) - \lambda e^{-x_2} = 0.
\]
The above equation must be satisfied for the values of $x_3$. Therefore, it yields

\begin{align}
(4.12) \quad & \partial_1 A(x_1, x_2) + A(x_1, x_2) = 0, \\
(4.13) \quad & e^{-x_2} \partial_1 Y^4(x_1, x_2, x_4) + \frac{1}{2} e^{-x_2} \partial_1 B_2(x_1, x_2) - \frac{4}{3} A(x_1, x_2) x_4 e^{x_1-x_2}
+ \frac{1}{3} e^{-x_2} (B_2(x_1, x_2) - 2B_1(x_1, x_2)) - \lambda e^{-x_2} = 0.
\end{align}

By integrating (4.12) with respect to $A(x_1, x_2)$, we find

\begin{equation}
A(x_1, x_2) = C(x_2)e^{-x_1},
\end{equation}

for some smooth function $C(x_2)$. We then substitute (4.14) into (4.13) and integrate it with respect to $Y^4(x_1, x_2, x_4)$, obtaining

\begin{align}
(4.15) \quad & Y^4(x_1, x_2, x_4) = \left(-\frac{1}{2} \partial_1 B^2(x_1, x_2) + \frac{1}{3} (2B_1^1(x_1, x_2) - B^2_2(x_1, x_2))
+ \lambda \right) x_4 + \frac{2}{3} C(x_2) x_4^2 + B^4_1(x_1, x_2),
\end{align}

for some smooth function $B^4_1(x_1, x_2)$.

We now use (4.8)–(4.11), (4.14) and (4.15) into equation (vii) of (4.3), so that it yields

\begin{align*}
\frac{2}{3} e^{-x_1} \partial_1 Y^3(x_1, x_2, x_4) + \frac{1}{3} e^{-x_2} \partial_2 B_2(x_1, x_2) - \partial_1 B^2_2(x_1, x_2))
- 2 e^{-x_1} (C'(x_2) + C(x_2)) x_3 + \mu C(x_2) = 0,
\end{align*}

or equivalently,

\begin{align*}
C'(x_2) + C(x_2) &= 0, \\
3 e^{-x_1} \partial_4 Y^3(x_1, x_2, x_4) + e^{-x_2} (2\partial_2 B_2(x_1, x_2) - \partial_1 B^2_2(x_1, x_2)) + \frac{4}{3} \mu C(x_2) &= 0.
\end{align*}

By integrating the above equations with respect to $C(x_2)$ and $Y^3(x_1, x_2, x_4)$ respectively, we obtain $C(x_2) = C_2 e^{-x_2}$ and

\begin{align}
(4.16) \quad & Y^3(x_1, x_2, x_4) = \frac{1}{3} e^{-x_2} \left( \partial_1 B^2_2(x_1, x_2) - 2\partial_2 B^2_2(x_1, x_2) - 4\mu C_2 \right) x_4
+ B^3_2(x_1, x_2),
\end{align}

for some smooth function $B^3_2(x_1, x_2)$.

Taking (4.8)–(4.11) and (4.14)–(4.16) into account, equation (v) of (4.3) becomes

\begin{align}
(4.17) \quad & - \frac{2}{3} C_2 e^{-x_2} (2e^{-x_2} x_4 + 2e^{-x_1} x_3) x_4 - e^{-x_1} (\partial_2^2 B^1_1(x_1, x_2))
\int - \frac{2}{3} \left( -2\partial_2 B^2_2(x_1, x_2) + \partial_1 B^1_1(x_1, x_2) \right) + \frac{4}{3} \partial_2 B^1_1(x_1, x_2)
\int - \frac{1}{3} \mu C_2 \right) x_3
\int - e^{-x_2} \left( \partial_2^2 B^2_2(x_1, x_2) - \frac{2}{3} \partial_2 B^1_1(x_1, x_2) + \frac{2}{3} \partial_1 B^2_2(x_1, x_2)
\int - \partial_2 B^2_2(x_1, x_2) - \frac{2}{3} \mu e^{x_2} C_2 e^{-x_2} \right) x_4 + \frac{2}{3} e^{-x_1} \partial_2 B^3_2(x_1, x_2) - \frac{4}{3} 
\int + e^{-x_2} \partial_2 B^1_1(x_1, x_2) - \frac{1}{2} e^{-x_1} B^3_2(x_1, x_2) + \mu \left( \partial_2 B^1_1(x_1, x_2) - \lambda \right) = 0.
\end{align}
The above equation has to be satisfied for every value of $x_3$ and $x_4$. Hence, we necessarily have $C_2 = 0$.

Thus, we are left with equations (i), (ii) and (v) of [14.3]. Taking into account [14.8]–[14.11], [14.14]–[14.16] and $C(x_2) = 0$, these equations are rewritten as follows:

(4.18) \[ e^{-x_1} \left( \partial_1^3 B_1(x_1, x_2) - \partial_1 B_1(x_1, x_2) + \frac{2}{3} \partial_2 B_1(x_1, x_2) \right) \]
\[ - \frac{2}{3} \partial_1 B_2(x_1, x_2) \] $x_3 + e^{-x_2} \left( \partial_1^2 B_2(x_1, x_2) + \frac{2}{3} \partial_2 B_2(x_1, x_2) \right) \]
\[ + \frac{1}{3} \partial_1 B_2(x_1, x_2) - \frac{4}{3} \partial_1 B_1(x_1, x_2) \right) \] $x_4 - 2e^{-x_2} \partial_1 B_4(x_1, x_2) \]
\[ - e^{-x_1} B_3(x_1, x_2) - \mu \partial_1 B_2(x_1, x_2) + \frac{4}{3} \pm \lambda \mu = 0, \]

(4.19) \[ e^{-x_1} \left( \partial_2 \partial_1 B_1(x_1, x_2) + \frac{2}{3} \partial_1 B_1(x_1, x_2) - \frac{2}{3} \partial_1 B_2(x_1, x_2) \right) \]
\[ - \frac{1}{3} \partial_2 B_2(x_1, x_2) \] $x_3 + e^{-x_2} \left( \partial_2 \partial_1 B_2(x_1, x_2) + \frac{2}{3} \partial_2 B_2(x_1, x_2) \right) \]
\[ - \frac{1}{3} \partial_1 B_1(x_1, x_2) - \frac{2}{3} \partial_2 B_1(x_1, x_2) \] $x_4 - \frac{1}{2} e^{-x_2} \partial_1 B_4(x_1, x_2) \]
\[ - e^{-x_2} \partial_2 B_4(x_1, x_2) - \frac{3}{4} e^{-x_1} \partial_1 B_3(x_1, x_2) + \frac{1}{4} e^{-x_1} B_3(x_1, x_2) \]
\[ + \frac{2}{3} \pm \frac{1}{2} \mu \left( \partial_1 B_4(x_1, x_2) + \partial_2 B_2(x_1, x_2) - \lambda \right) = 0, \]

(4.20) \[ e^{-x_1} \left( \partial_2^3 B_1(x_1, x_2) + \frac{2}{3} \partial_1 B_1(x_1, x_2) - \frac{2}{3} \partial_2 B_2(x_1, x_2) \right) \]
\[ + \frac{1}{3} \partial_2 B_2(x_1, x_2) \] $x_3 + e^{-x_2} \left( \partial_2^2 B_2(x_1, x_2) + \frac{2}{3} \partial_1 B_2(x_1, x_2) \right) \]
\[ - \frac{2}{3} \partial_2 B_1(x_1, x_2) - \frac{2}{3} \partial_2 B_2(x_1, x_2) \] $x_4 - \frac{3}{2} e^{-x_1} \partial_2 B_3(x_1, x_2) \]
\[ - e^{-x_2} \partial_2 B_4(x_1, x_2) + \frac{1}{2} e^{-x_1} B_3(x_1, x_2) + \frac{4}{3} \]
\[ - \mu \left( \partial_1 B_4(x_1, x_2) - \lambda \right) = 0. \]
Since equations (4.18), (4.19), (4.20) must hold for all values of $x_3$ and $x_4$, they are equivalent to the following system of equations:

\[
\begin{align*}
\text{(i)} & \quad \partial_1^2 B^1(x_1, x_2) - \partial_1 B^1(x_1, x_2) + \frac{2}{3} \partial_2 B^1(x_1, x_2) - \frac{2}{3} \partial_1 B^2(x_1, x_2) = 0, \\
\text{(ii)} & \quad \partial_2 \partial_1 B^1(x_1, x_2) + \frac{2}{3} \partial_1 B^1(x_1, x_2) - \frac{2}{3} \partial_1 B^2(x_1, x_2) = 0, \\
\text{(iii)} & \quad \partial_2^2 B^1(x_1, x_2) + \frac{2}{3} \partial_1 B^1(x_1, x_2) + \frac{1}{3} \partial_2 B^1(x_1, x_2) - \frac{4}{3} \partial_2 B^2(x_1, x_2) = 0, \\
\text{(iv)} & \quad \partial_1^2 B^2(x_1, x_2) + \frac{2}{3} \partial_2 B^2(x_1, x_2) + \frac{1}{3} \partial_1 B^2(x_1, x_2) - \frac{4}{3} \partial_1 B^1(x_1, x_2) = 0, \\
\text{(v)} & \quad \partial_2 \partial_1 B^2(x_1, x_2) + \frac{2}{3} \partial_2 B^2(x_1, x_2) - \frac{2}{3} \partial_1 B^1(x_1, x_2) = 0, \\
\text{(vi)} & \quad \partial_2^2 B^2(x_1, x_2) + \frac{2}{3} \partial_2 B^2(x_1, x_2) - \frac{2}{3} \partial_2 B^1(x_1, x_2) = 0, \\
\text{(vii)} & \quad -2e^{-x_2} \partial_1 B^4(x_1, x_2) - e^{-x_1} B^3(x_1, x_2) + \frac{1}{3} - \mu (\partial_1 B^2(x_1, x_2) - \lambda) = 0, \\
\text{(viii)} & \quad e^{-x_2} \left( \frac{1}{2} \partial_1 B^4(x_1, x_2) + \partial_2 B^1(x_1, x_2) \right) + e^{-x_1} \left( \frac{1}{2} \partial_1 B^3(x_1, x_2) - \frac{1}{3} B^3(x_1, x_2) \right) - \frac{2}{3} \\
& \quad + \frac{1}{3} \mu (\partial_1 B^1(x_1, x_2) + \partial_2 B^2(x_1, x_2) - \lambda) = 0, \\
\text{(ix)} & \quad e^{-x_1} \left( \frac{2}{3} \partial_2 B^3(x_1, x_2) - \frac{1}{3} B^3(x_1, x_2) \right) - \frac{2}{3} \\
& \quad + e^{-x_2} \partial_2 B^4(x_1, x_2) + \mu (\partial_2 B^1(x_1, x_2) - \lambda) = 0, \\
\end{align*}
\]

(4.21)

In discussing the solutions of system (4.21), we shall treat separately the cases $\mu \neq 0$ and $\mu = 0$.

4.1. **Case** $\mu \neq 0$. In this case we shall provide an explicit solution. In fact, suppose that there exist some smooth functions $P^i(x_1), Q^i(x_2)$ such that

\[
B^i(x_1, x_2) = P^i(x_1) + Q^i(x_2), \quad i = 1, 2.
\]

(4.22)

Taking into account (4.22), from the equation (i) in (4.21) we deduce that there exists some real constant $J^1$, such that

\[
\frac{2}{3} \frac{d}{dx_2} Q^1(x_2) = \frac{2}{3} \frac{d}{dx_1} P^2(x_1) - \frac{d^2}{dx_2^2} P^1(x_1) + \frac{d}{dx_1} P^1(x_1) = \frac{2}{3} J^1.
\]
Therefore,

\[ Q^1(x_2) = J^1 x_2 + H^1, \]

where \( H^1 \) is a real constant. Replacing the above expressions of \( Q(x_2) \) and of \( B^i(x, x_2), i = 1, 2 \), equation (iii) of (4.21) becomes

\[
0 = -\frac{2}{3} \frac{d}{dx_1} P^1(x_1) + \frac{4}{3} \frac{d}{dx_2} Q^2(x_2) - \frac{1}{3} J^1,
\]

which in particular yields \( \frac{d^2}{dx_2^2} Q^2(x_2) = \frac{d^2}{dx_1^2} P^1(x_1) = 0 \). Hence, integrating we get

\[ Q^2(x_2) = J^2 x_2 + H^2 \text{ and } P^1(x_1) = R^1 x_1 + S^1, \]

for some real constants \( J^2, H^2, R^1 \) and \( S^1 \). Then, equations (i)-(vi) in (4.21) now give

\[
\begin{cases}
3R^1 - J^1 + \frac{d}{dx_1} P^2(x_1) = 0, \\
2R^1 - J^2 - 2 \frac{d}{dx} P^2(x_1) = 0, \\
2R^1 - 4J^2 + J^1 = 0, \\
3 \frac{d^2}{dx_1^2} P^2(x_1) + \frac{d}{dx_1} P^2(x_1) - 4R^1 + 2J^2 = 0, \\
2J^1 - 2J^2 + R^1 = 0, \\
J^1 + 3J^2 - 2 \frac{d}{dx_1} P^2(x_1) = 0,
\end{cases}
\]

which easily yield \( R^1 = J^2 = J^1 = 0 \) and \( P^2(x_1) = S^2 \) for some real constant \( S^2 \). Therefore, \( B^i(x_1, x_2) = S^i + H^i, i = 1, 2 \), where \( S^i, H^i \) are real constants. Setting \( S^i + H^i = W^i \), we then have

\[ B^i(x_1, x_2) = W^i, \quad i = 1, 2, \]

and system (4.21) reduces to:

\[
\begin{cases}
2e^{-x_2} \partial_1 B^4(x_1, x_2) + e^{-x_1} B^3(x_1, x_2) - \lambda \mu - \frac{4}{3} = 0, \\
e^{-x_1} \left( \frac{3}{2} \partial_1 B^3(x_1, x_2) - \frac{1}{2} B^3(x_1, x_2) \right) + e^{-x_2} \left( \partial_1 B^4(x_1, x_2) + 2 \partial_2 B^4(x_1, x_2) \right) - \lambda \mu - \frac{4}{3} = 0, \\
e^{-x_1} \left( \frac{3}{2} \partial_2 B^3(x_1, x_2) - \frac{1}{2} B^3(x_1, x_2) \right) + e^{-x_2} \partial_2 B^4(x_1, x_2) - \lambda \mu - \frac{4}{3} = 0.
\end{cases}
\]  

(4.23)

From the first equation of (4.23) we get

\[ B^3(x_1, x_2) = \left( \lambda \mu + \frac{4}{3} \right) e^{x_1} - 2e^{x_1-x_2} \partial_1 B^4(x_1, x_2). \]

Using the above expression for \( B^3(x_1, x_2) \) and assuming \( B^4(x_1, x_2) = P^4(x_1) + Q^4(x_2) \), from the second equation in (4.23) we deduce

\[ Q^4(x_2) = W^4 \text{ and } P^4(x_1) = R^4 e^{-\frac{4}{3}x_1} + S^4, \]
where $W^4$, $R^4$ and $S^4$ are real constants. We then substitute the expressions for $B^i(x_1, x_2)$, $i = 3, 4$ in the last equation of (4.23) and we obtain $R^4 = 0$ and $\lambda = -\frac{4}{3\mu}$, where we clearly see that the condition $\mu \neq 0$ is needed for such a kind of solution.

All equations in (4.21) are then satisfied. We now replace the functions we found by integration into $X^i(x_1, x_2, x_3, x_4)$. With respect to global coordinates $(x_1, x_2, x_3, x_4)$, we then get a solution $X(x_1, x_2, x_3, x_4) = X^i \partial_i$ of the Ricci soliton equation (1.1), explicitly given by

\begin{equation}
X^1(x_1, x_2, x_3, x_4) = -\frac{1}{3}W^1 + \frac{2}{3}W^2,
X^2(x_1, x_2, x_3, x_4) = \frac{2}{3}W^1 - \frac{1}{3}W^2,
X^3(x_1, x_2, x_3, x_4) = (\frac{1}{3}W^1 + \frac{2}{3}W^2)x_3 - \frac{4}{3\mu}x_3,
X^4(x_1, x_2, x_3, x_4) = (\frac{2}{3}W^1 - \frac{1}{3}W^2)x_4 - \frac{4}{3\mu}x_4 + W^3,
\end{equation}

for some real constants $W^1, W^2, W^4$ and $W^3 = S^4 + W^4$.

As a check, by (4.24) and computing $\mathcal{L}_X g$, we explicitly find, with respect to the basis $(\partial_i)_{1 \leq i \leq 4}$:

\[
\mathcal{L}_X g = \begin{pmatrix}
0 & 0 & -\frac{2}{3\mu}e^{-x_1} & -\frac{4}{3\mu}e^{-x_2} \\
0 & 0 & -\frac{4}{3\mu}e^{-x_1} & -\frac{2}{3\mu}\lambda e^{-x_2} \\
-\frac{2}{3\mu}e^{-x_1} & -\frac{4}{3\mu}e^{-x_1} & 0 & 0 \\
-\frac{4}{3\mu}e^{-x_2} & -\frac{2}{3\mu}e^{-x_2} & 0 & 0
\end{pmatrix} = -g - \frac{4}{3\mu}g.
\]

Therefore, the arbitrary invariant metric $g$ for $\mu \neq 0$ of a four-dimensional generalized symmetric space of type $B$ is a Ricci soliton. More precisely, replacing the explicit expressions of the functions we found above into $X_1, X_2, X_3, X_4$, we conclude that, with respect to coordinates $(x_1, x_2, x_3, x_4) = (x, y, u, v)$, the Ricci soliton equation (1.1) holds for the metric $g$ with $\mu \neq 0$, together with the vector field

\begin{equation}
X = \frac{1}{3} (W^1 + 2W^2) \partial_1 + \frac{1}{3} (2W^1 - W^2) \partial_2 - \frac{1}{3} (W^1 - 2W^2)x_3 \partial_3 \\
- \frac{4}{3\mu}x_3 \partial_3 + \frac{1}{3} (2W^1 - W^2)x_4 - \frac{4}{3\mu}x_4 + W^3 \partial_4,
\end{equation}

where $W^1 W^2$ and $W^3$ are real constants.

We now check that the above Ricci soliton is not a gradient one, that is, there no exists a smooth function $f(x_1, x_2, x_3, x_4)$, such that $X = \text{grad}(f) = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i} \partial_j$. In fact, suppose that such a function exists. Then, by (4.25)
we have
\[-2e^{x_1} \partial_3 f + 4e^{x_2} \partial_4 f + W^1 - 2W^2 = 0,\]
\[4e^{x_1} \partial_3 f - 2e^{x_2} \partial_4 f - 2W^1 + W^2 = 0,\]
\[2e^{x_1} (-\partial_1 f + 2\partial_2 f - \mu (2e^{x_1} \partial_3 f - e^{x_2} \partial_4 f)) + (W^1 - 2W^2 + \frac{1}{\mu})x_3 = 0,\]
\[2e^{x_2} (\partial_2 f - 2\partial_1 f - \mu (e^{x_1} \partial_3 f - 2e^{x_2} \partial_4 f)) - \left(2W^1 - W^2 - \frac{2}{\mu}\right)x_4 - 3W^3 = 0.\]

From the first equation of (4.26) we have
\[\partial_3 f = 2e^{x_2-x_1} \partial_4 f + e^{-x_1} \left(\frac{1}{2}W^1 - W^2\right),\]
and substituting the above expression into the second equation of (4.26), we get
\[\partial_4 f = \frac{1}{2}e^{-x_1} W^2.\]
Therefore, \(\partial_3 f = \frac{1}{2}e^{-x_1} W^1\), which by integration yields
\[f(x_1, x_2, x_3, x_4) = \frac{1}{2}e^{-x_1} W^1 x_3 + h(x_1, x_2, x_4),\]
where \(h(x_1, x_2, x_4)\) is a smooth function. We substitute the expression for \(f\) into \(\partial_4 f = \frac{1}{2}e^{-x_1} W^2\) and we get
\[\partial_1 h(x_1, x_2, x_4) = \frac{1}{2}e^{-x_2} W^2.\]

By integrating the above equation, we obtain
\[h(x_1, x_2, x_4) = \frac{1}{2}e^{-x_1} W^2 x_4 + q(x_1, x_2),\]
for a smooth function \(q\). Therefore,
\[f(x_1, x_2, x_3, x_4) = \frac{1}{2}e^{-x_1} W^1 x_3 + \frac{1}{2}e^{-x_2} W^2 x_4 + q(x_1, x_2).\]

We substitute the expressions for \(\partial_4 f\) and \(\partial_3 f\) into the third and fourth equations of (4.26) and we get
\[(2W^1 - 2W^2 + \frac{1}{\mu})x_3 - 2e^{x_1-x_2} W^2 x_4 + 2e^{x_1} \left(-\partial_1 q(x_1, x_2) + 2\partial_2 q(x_1, x_2) - \mu \left(W^1 - \frac{1}{2} W^2\right)\right) = 0,\]
\[2e^{x_1-x_2} W^1 x_3 + 2 \left(W^1 - W^2 - \frac{2}{\mu}\right)x_4 + 3W^3 - 2e^{x_2} \left(2\partial_1 q(x_1, x_2) - \partial_2 q(x_1, x_2) - \mu \left(\frac{1}{2} W^1 - W^2\right)\right) = 0.\]

Since the above equations must hold for all values of \(x_3\) and \(x_4\), we easily get \(W^1 = W^2 = 0\). Hence, the above equations reduce to
\[\frac{2}{\mu} x_3 + 2e^{x_1} \left(-\partial_1 q(x_1, x_2) + 2\partial_2 q(x_1, x_2)\right) = 0,\]
\[-\frac{4}{\mu} x_4 + 3W^3 - 2e^{x_2} \left(2\partial_1 q(x_1, x_2) - \partial_2 q(x_1, x_2)\right) = 0.
Clearly, the above equations cannot hold for all values of \( x_3 \) and \( x_4 \). Therefore, the solution of (1.1) we found for \( g \) when \( \mu \neq 0 \) is not a gradient Ricci soliton.

### 4.2. Case \( \mu = 0 \)

We integrate equations (iii) and (v) of (4.21) with respect to \( B^3(x_1, x_2) \) and \( B^4(x_1, x_2) \) respectively and we find

\[
B^3(x_1, x_2) = e^{x_1} \left( \frac{4}{3} - 2e^{-x_2}\partial_1 B^4(x_1, x_2) \right),
\]

\[
B^4(x_1, x_2) = F(x_1)G(x_2),
\]

for some smooth functions \( F(x_1) \) and \( G(x_2) \) satisfying

\[
F''(x_1) = \frac{1}{4} c F(x_1) - \frac{1}{3} F'(x_1),
\]

\[
G'(x_2) = \frac{1}{2} c G(x_2),
\]

where \( c \) is a real constant. By integrating (4.29) and (4.30) we obtain:

\[
F(x_1) = K_1 e^{\left(-\frac{1}{6} + \frac{1}{6} \sqrt{1+12c}\right)x_1} + K_2 e^{\left(-\frac{1}{6} - \frac{1}{6} \sqrt{1+12c}\right)x_1},
\]

\[
G(x_2) = H e^{\frac{1}{2} c x_2},
\]

for some constants \( K_i, i = 1, 2, \) and \( H \). We substitute in (4.28) the expressions (4.31) and (4.32) and we get

\[
B^4(x_1, x_2) = \left( K_1 e^{\left(-\frac{1}{6} + \frac{1}{6} \sqrt{1+12c}\right)x_1} + K_2 e^{\left(-\frac{1}{6} - \frac{1}{6} \sqrt{1+12c}\right)x_1} \right) H e^{\frac{1}{2} c x_2}.
\]

Next, we substitute (4.27) and (4.33) in equation (ix) of (4.21) and we obtain

\[
0 = H \left[ K_1 \left( \frac{3c}{4} + \left( \frac{1}{6} - \frac{1}{2} \right) \sqrt{1+12c} - \frac{1}{6} \right) e^{\frac{1}{2} \sqrt{1+12c}} \right.
\]

\[
+ K_2 \left( \frac{3c}{4} + \left( \frac{1}{2} - \frac{1}{6} \right) \sqrt{1+12c} - \frac{1}{6} \right) e^{-\frac{1}{2} \sqrt{1+12c}} \right] e^{\frac{1}{2} c x_2} - \frac{u}{2} - 2,
\]

which clearly cannot hold for all values of \( x_1 \) and \( x_2 \). Therefore, the case when \( \mu = 0 \) does not correspond to a Ricci soliton.

The above results are summarized in the following.

**Theorem 2.** A four-dimensional proper generalized symmetric space of type B (with the invariant metric \( g \) as described in (2.2)) is a Ricci soliton if and only if \( \mu \neq 0 \). More precisely, for \( \mu = 0 \), the metric \( g \) described in (2.2) does not satisfy the Ricci soliton equation (1.1). For \( \mu \neq 0 \), the Ricci soliton equation (1.1) holds for the metric \( g \) described in (2.2) together with a vector field

\[
X = (-W^1 + 2W^2) \left( \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} \right) + (2W^1 - W^2) \left( \frac{\partial}{\partial y} + v \frac{\partial}{\partial v} \right)
\]

\[
- \frac{4}{\mu} \left( u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) + W^3 \frac{\partial}{\partial v}.
\]
where $W^1, W^2, W^3$ are real constants. This Ricci soliton is not a gradient one.

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Giovanni Calvaruso: Dipartimento di Matematica e Fisica “E. De Giorgi”, Università del Salento, Prov. Lecce-Arnesano, 73100 Lecce, Italy., E. Rosado: Department of Applied Mathematics, Escuela Técnica Superior de Arquitectura, Universidad Politécnica de Madrid, Avda. Juan de Herrera 4, 28040 Madrid, Spain. E-mail address: eugenia.rosado@upm.es