1. Introduction

Let $\Gamma$ denote a congruence subgroup of $\text{GL}_n(\mathbb{Z})$ of level $N$. If $p$ is prime and $F$ is a field of characteristic $p$, then the cohomology groups $H^d(\Gamma, F)$ admit a natural action by a commutative ring $T$ of Hecke operators $T_{\ell,k}$ for $1 \leq k \leq n$ and for primes $\ell$ not dividing $Np$. Let $[c] \in H^d(\Gamma, F)$ denote an eigenclass for the action of $T$ with eigenvalues $a(\ell,k) \in F$. Conjecture B of [Ash92] predicts that, associated to $[c]$, there exists a continuous semisimple Galois representation $\rho: G_\mathbb{Q} \to \text{GL}_n(F)$ which is unramified outside $Np$ such that

$$\sum (-1)^k \ell^k (k-1)/2 a(\ell,k) X^k = \det(I - \rho(\text{Frob}_\ell)^{-1} X).$$

Let $\omega: G_\mathbb{Q} \to F^\times$ denote the mod-$p$ cyclotomic character. Our first theorem is the following.

**Theorem 1.1.** Fix an integer $d$. Then Conjecture B of [Ash92] is true for sufficiently large $n$. More precisely, associated to any eigenclass $[c]$ there exists a character $\chi$ of conductor $N$ and a Galois representation $\rho = \chi \otimes (1 \oplus \omega \oplus \omega^2 \oplus \ldots \oplus \omega^{n-1})$, such that $\rho(\text{Frob}_\ell)$ satisfies $\star$.

**Remark 1.2.** It follows from the argument that $n \geq 2d + 6$ will suffice.

This theorem should be interpreted as saying that the action of Hecke on stable cohomology is trivial, which is indeed how we shall prove this theorem. Note that if $[c]$ comes from characteristic zero, then the result follows from a theorem of Borel [Bor74], which shows in particular that the only rational cohomology in low degrees arises from the trivial automorphic representation.

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Example 1.3. A special case of a construction due to Soulé [Sou79] implies that $K_{22}(\mathbb{Z})$ contains an element of order 691. Associated to $[c]$ via the Hurewicz map is a stable class $[c]$ in $H_{22}(GL_n(\mathbb{Z}), F_{691})$ for all sufficiently large $n$. Our theorem implies that the class $[c]$ is associated to the representation $\rho := 1 \oplus \omega \oplus \ldots \oplus \omega^{n-1}$ via $\star$. On the other hand, the existence of $[c]$ corresponds — implicitly — to the existence of a non-semisimple Galois representation $\varrho$ with $\varrho^{ss} = 1 \oplus \omega^{11}$ such that the extension class in $\text{Ext}^1(\omega^{11}, 1)$ is unramified everywhere. Is there a generalization of Ash’s conjectures which predicts the existence of a non-semisimple Galois representations associated to Eisenstein Hecke eigenclasses?

Remark 1.4. That Theorem 1.1 (or something similar) might be true was suggested by Akshay Venkatesh in discussions with the first author. (See [Ash12] for a discussion of this conjecture and some partial results.)

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2. Arithmetic Manifolds

Torsion free arithmetic groups are well known to act freely and properly discontinuously on their associated symmetric spaces. This allows one to translate questions concerning the cohomology of arithmetic groups into questions about cohomology of the associated arithmetic quotients. While one can study such spaces independent of any adelic framework, it is more natural from the perspective of Hecke operators to work in this generality, and this is the approach we adopt in this paper.

2.1. Cohomology of arithmetic quotients. Let $K_\infty$ denote a fixed maximal compact subgroup of $GL_n(\mathbb{R})$, and let $K^0_\infty$ denote the connected component containing the identity. One has isomorphisms $K_\infty \simeq O(n)$ and $K^0_\infty \simeq SO(n)$. Let $\mathbb{A}$ denote the adeles of $\mathbb{Q}$. For any finite set of places $S$, let $\mathbb{A}^S$ denote the adeles with the components at the places $v \mid S$ missing, so (for example) $\mathbb{A}^\infty$ denotes the finite adeles. Fix a compact open subgroup $K_\ell$ of $GL_n(\mathbb{A}^{\ell, \infty})$, let $K_\ell$ denote a compact open subgroup of $GL_n(\mathbb{Z}_\ell)$, and let $K = K_\ell K_\ell$. Let

$$Y(K) = GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}) / K^0_\infty K_\ell K_\ell$$

denote the corresponding arithmetic quotient.

Assume that $F$ is a finite field of characteristic $p \neq \ell$. We now let $\Pi_{n, \ell}$ denote the direct limit

$$\Pi_{n, \ell} = \lim_{K_\ell \rightarrow K_\ell} H^d(Y(K), F).$$

For $K_\ell$ sufficiently small, the quotient $Y(K)$ is a manifold consisting of a finite number of connected components, all of which are $K(\pi, 1)$ spaces. In particular, if $K_\ell$ is the level $\ell^k$ congruence subgroup, and $K_\ell$ is the open subgroup of tame level $N$, then writing the associated space as $Y(K, \ell^k)$ there is an isomorphism

$$H^d(Y(K, \ell^k), F) \simeq \bigoplus_A H^d(\Gamma(\ell^k), F),$$

Indeed, $K_{22}(\mathbb{Z}) \simeq \mathbb{Z}/691\mathbb{Z}$.
where \( A := \mathbb{Q}^\times \backslash \mathbb{A}^\times / \det(K) \) is a ray class group of conductor \( N \) times a power of \( \ell \), and \( \Gamma \) is the corresponding classical congruence subgroup of level \( N \).

The module \( \Pi_{n,\ell} \) is endowed tautologically with an action of \( \text{GL}_n(\mathbb{Q}_\ell) \) which is \emph{admissible}; that is, letting \( G(\ell^k) \) denote the full congruence subgroup of \( \text{GL}_n(\mathbb{Z}_\ell) \) of level \( \ell^k \), we have that
\[
\dim F \Pi_{n,\ell}^G(\ell^k) < \infty
\]
for any \( k \). This property records nothing more than the finite dimensionality of the cohomology groups \( H^d(Y(K,\ell^k),F) \).

### 3. Stability for \( \ell \neq p \)

We use the following two key inputs to prove our result. The first is as follows:

**Proposition 3.1.** There is an isomorphism \( \Pi_{n,\ell} \rightarrow \Pi_{n+1,\ell} \) for all \( n \geq 2d + 6 \).

**Proof.** For any fixed \( \ell \)-power congruence subgroup, this follows immediately from the main result of Charney [Cha84] (in particular, §5.4 Example (iv), p.2118). The theorem then follow by taking direct limits. \( \square \)

The second key result is the following:

**Proposition 3.2.** Let \( \hat{V} \) be an irreducible admissible infinite dimensional representation of \( \text{GL}_n(\mathbb{Q}_\ell) \) in characteristic zero. Then the Gelfand–Kirillov dimension of \( \hat{V} \) is at least \( n - 1 \).

**Proof.** The Gelfand–Kirillov dimension of \( \hat{V} \) can be interpreted in two ways. On the one hand, it is that value of \( d \geq 0 \) for which \( \dim V^G(\ell^k) \asymp \ell^{dk} \); on the other hand, if we pull-back the character of Harish–Chandra character to a neighbourhood of 0 in \( g \), the Lie algebra of \( \text{GL}_n(\mathbb{Q}_\ell) \), via the exponential map, then a result of Howe and Harish–Chandra shows that the resulting function \( \chi \) admits an expansion
\[
\chi = \sum_{O} c_O \hat{\mu}_O,
\]
where the sum ranges over the nilpotent \( \text{GL}_n(\mathbb{Q}_\ell) \)-orbits in \( g' \), and \( \hat{\mu}_O \) denotes the Fourier transform of a suitably normalized \( \text{GL}_n(\mathbb{Q}_\ell) \)-invariant measure on \( O \) (see e.g. the introduction of [GS05], whose notation we are following here, for a discussion of these ideas); the Gelfand–Kirillov dimension of \( \hat{V} \) is then also equal \( \frac{1}{2} \max_{O \mid c_O \neq 0} \dim O \) (as one sees by pairing the characteristic function of \( G(\ell^k) \) — pulled back to \( g \) — against \( \chi \)). Since \( \hat{V} \) has Gelfand–Kirillov dimension 0 if and only if it is finite dimensional, and since the minimal non-zero nilpotent orbit in \( g \) has dimension \( 2(n - 1) \) (see §1.3 p. 459 and Table 1 on p. 460 of [Smi83]), the proposition follows. \( \square \)

We deduce from these results the following.

**Lemma 3.3.** For \( n \geq 2d + 6 \), the action of \( \text{GL}_n(\mathbb{Q}_\ell) \) on \( \Pi_{n,\ell} \) is via the determinant.

**Proof.** By Proposition 3.1, it suffices to prove the result for sufficiently large \( n \). Let \( V \) be an irreducible sub-quotient of \( \Pi_{n,\ell} \). Since the cohomology of pro-\( \ell \) groups vanishes in characteristic \( p \neq \ell \), we deduce that, for all \( k \),
\[
\dim V^G(\ell^k) \leq \dim \Pi_{n,\ell}^G(\ell^k) = \dim \Pi_{m,\ell}^G(\ell^k),
\]
where \( m = 2d + 6 \) is fixed, and where the equality follows from Proposition \[3.2\]. Yet, for a fixed \( m \), there is the trivial inequality relating the growth of cohomology to the growth of the index (up to a constant) and thus

\[
\dim \Pi_{m,\ell}^{G(\ell)} \ll [\text{GL}_m(\mathbb{Z}_\ell) : G(\ell)] \ll \ell^{km^2}.
\]

We deduce a corresponding bound for the invariants of \( V \). It follows from the main theorem of Vignéras ([Vig01], p.182) that any irreducible admissible irreducible representation \( V \) of \( \text{GL}_n(\mathbb{Q}_\ell) \) over \( \mathbb{F} \) lifts to an irreducible representation \( \hat{V} \) in characteristic 0. The bound on invariant growth then implies that the Gelfand–Kirillov dimension of \( \hat{V} \) is at most \( m^2 \). By Proposition \[3.2\] this is a contradiction for sufficiently large \( n \) unless \( \hat{V} \) and thus \( V \) is one dimensional. It follows that every irreducible constituent of \( \Pi_{n,\ell} \) is a character. Since the action of \( \text{GL}_n(\mathbb{Q}_\ell) \) on the extension of any two characters still acts through the determinant, the result follows for \( \Pi_{n,\ell} \).

\[ \square \]

4. Hecke Operators

Let \( g \in \text{GL}_n(\mathbb{A}_\infty) \) be invertible. Associated to \( g \) one has the Hecke operator \( T(g) \), defined by considering the composition:

\[
H^\bullet(Y(K), \mathbb{F}) \to H^\bullet(Y(gKg^{-1} \cap K), \mathbb{F}) \to H^\bullet(Y(K \cap g^{-1}Kg), \mathbb{F}) \to H^\bullet(Y(K), \mathbb{F}),
\]

the first map coming from the obvious inclusion, the final coming from corestriction map. The Hecke operators preserve \( H^\bullet(Y(K), \mathbb{F}) \), but not necessarily the cohomology of the connected components. Indeed, the action on the component group is via the determinant map on \( \mathbb{G}(\mathbb{A}_\infty) \) and the natural action of \( \mathbb{A}_\infty^\times \) on \( A \). The Hecke operator \( T_{\ell,k} \) is defined by taking \( g \) to be the diagonal matrix consisting of \( k \) copies of \( \ell \) and \( n-k \) copies of \( 1 \). The algebra \( \mathbb{T} \) of endomorphisms generated by \( T_{\ell,k} \) on \( H^\bullet(Y(K), \mathbb{F}) \) for \( \ell \) prime to the level of \( K \) generates a commutative algebra. For any such \( T = T(g) \), let \( \langle T \rangle \) denote the isomorphism of \( H^\bullet(Y(K), \mathbb{F}) \) which acts by permuting the components according to the image of \( \det(g) \) in \( A \).

There are many definitions of the notion of “Eisenstein” in many different contexts. For our purposes, the following very restrictive definition is appropriate:

**Definition 4.1.** A cohomology class \( [c] \in H^d(Y(K), \mathbb{F}) \) is Eisenstein if \( T[c] = \langle T \rangle \deg(T)[c] \) for any Hecke operator \( T \). A maximal ideal \( m \) of \( \mathbb{T} \) is Eisenstein if and only if \( m \) contains \( T - \deg(T) \) for all \( T \) with \( \langle T \rangle = 1 \) in \( A \).

If \( [c] \in H^d(Y(K), \mathbb{F}) \) is a Hecke eigenclass for which \( T[c] = \deg(T)[c] \) for all \( \langle T \rangle = 1 \), then \( [c] \) is necessarily Eisenstein, and moreover \( T[c] = \chi(\langle T \rangle) \deg(T)[c] \) for some character \( \chi : A \to \mathbb{F}_\infty^\times \) of \( A \). By class field theory, \( \chi \) corresponds to a finite order character of \( G_\mathbb{Q} \) of conductor dividing \( N \). A easy computation of \( \deg(T_{\ell,k}) \) then implies that \( * \) will be satisfied with \( \rho \) equals \( \chi \otimes (1+\omega+\ldots+\omega^{n-1}) \) if and only if \( [c] \) is Eisenstein.

4.1. **Proof of Theorem [1.1]** Given any eigenclass \( [c] \in H^d(Y(K), \mathbb{F}) \), consider its image \( [\iota(c)] \) in \( \Pi_{n,\ell} \). If this image is non-zero, we can determine the eigenvalues of \( [c] \) by determining those of \( [\iota(c)] \). Since the \( \text{GL}_n(\mathbb{Q}_\ell) \)-action on \( \Pi_{n,\ell} \) factors through \( \det \), the action of any \( g \in \text{GL}_n(\mathbb{Q}_\ell) \) on \( [\iota(c)] \), as well as the action of the \( \deg(T(g)) \) representatives in the double coset decomposition of \( g \), is via \( \langle T \rangle \); hence \( T[\iota(c)] - \langle T \rangle \deg(T)[\iota(c)] \) is zero in \( \Pi_{n,\ell} \). It follows that either \( [c] \) is Eisenstein or it lies in the kernel of the map \( H^d(Y(K), \mathbb{F}) \to \Pi_{n,\ell} \). Hence Theorem [1.1] follows by induction from the following Lemma.
Lemma 4.2. Let \( \mathfrak{m} \) be a maximal ideal of \( T \) which is not Eisenstein, and suppose that \( H^i(Y(L), F)_{\mathfrak{m}} = 0 \) for all \( i < d \) and \( L = K^\ell L_\ell \) for all compact normal open subgroups \( L_\ell \subset \text{GL}_n(\mathbb{Z}_\ell) \). If \([c]\) lies in the kernel of the map \( H_d(Y(K), F) \to \Pi_{n,\ell} \), then \([c]\) is Eisenstein.

Proof. By assumption, the class \([c]\) lies in the kernel of the map

\[
H^d(Y(K), F) \to H^d(Y(L), F)
\]

for some \( L \) as above. It suffices to show that this kernel vanishes after completion at any non-Eisenstein prime \( \mathfrak{m} \). By assumption, the cohomology of \( Y(K) \) localized at \( \mathfrak{m} \) vanishes in degree less than \( d \). Hence localizing the Hochschild–Serre spectral sequence at \( \mathfrak{m} \), we obtain an inflation-restriction sequence: and thus the set of classes in \( H_d(Y(K), F)_{\mathfrak{m}} \) which are annihilated under the level raising map is isomorphic to a finite number of copies of the group \( H_d(K/L, F)_{\mathfrak{m}} \). Let \( g \in \text{GL}_n(\mathbb{A}) \). There is a canonical isomorphism

\[
(gKg^{-1} \cap K)/(gLg^{-1} \cap L) \simeq K/L.
\]

Hence we obtain a commutative diagram:

\[
\begin{array}{cccc}
H^d(K/L, F) & \longrightarrow & H^d(K/L, F) & \longrightarrow & H^d(K/L, F) \\
\downarrow & & \downarrow & & \downarrow \\
H^d(Y(K), F) & \longrightarrow & H^d(Y(gKg^{-1} \cap K), F) & \longrightarrow & H^d(Y(K \cap g^{-1} \cap Kg), F) \\
\end{array}
\]

It follows that the action of \( T = T(g) \) is given by multiplication by \( \det(T) \) composed with the permutation \( \langle T \rangle = \det(g) \in A \) of components. This action is Eisenstein, and hence the kernel vanishes for any non-Eisenstein prime \( \mathfrak{m} \). \( \Box \)

This completes the proof of Theorem 1.1.

Remark 4.3. By the universal coefficient theorem, one has

\[
\Pi_{n,\ell}^\vee := \text{Hom}(\Pi_{n,\ell}, F) = \varprojlim H^d(Y(K), F).
\]

Suppose that one defines

\[
\Pi_{n,\ell}^\vee(\mathbb{Z}_\ell) = \varprojlim H^d(Y(K), \mathbb{Z}_\ell).
\]

Then it follows from Lemma 3.3 for \( d \) and \( d + 1 \) that the action of \( \text{GL}_n(\mathbb{Q}_\ell) \) on \( \Pi_{n,\ell}^\vee(\mathbb{Z}_\ell) \) is via the determinant for sufficiently large \( n \).

Remark 4.4. One can ask whether a Hecke operator \( T \) with \( \langle T \rangle \) trivial in \( A \) acts via the degree on the entire cohomology group \( H^d(\Gamma, \mathbb{Z}) \). Our argument shows that, for such \( T \), the image of \( T - \deg(T) \) on \( H^d(\Gamma, \mathbb{Z}) \) is — in the notation of \( \text{[CV12]} \) — congruence; i.e., it lies in the kernel of the map \( H^d(\Gamma, \mathbb{Z}) \to H^d(\Gamma(M), \mathbb{Z}) \) for some \( M \).

Remark 4.5. The main theorem and its proof remain valid, and essentially unchanged, if one replaces \( \text{GL}_n(\mathbb{Z}) \) by \( \text{GL}_n(\mathcal{O}_F) \) for any number field \( F \).
5. Stability for $\ell = p$

Although not necessary for the proof of Theorem 1.1, it is of interest to ask whether the methods of this paper can be extended to $\ell = p$. One obvious obstruction is that the na"ive notion of stability fails, even for $d = 1$: by the congruence subgroup property [BLS64, Men65] one easily sees that, for $n \geq 3$,

$$H_1(\text{SL}(n,p),\mathbb{Z}) \simeq \text{SL}(n,p)/\text{SL}(n,p^2) \simeq (\mathbb{Z}/p\mathbb{Z})^{n^2-1},$$

which clearly do not stabilize. On the other hand, these classes arise for purely local reasons, namely, as the pullback of classes from the homology of the $p$-congruence subgroup of the congruence completion $\text{SL}_n(\mathbb{Z}_p)$; this latter homology is evidently insensitive to the finer arithmetic properties of $\text{SL}_n(\mathbb{Z})$. It is natural, then, to excise the cohomology arising for “local” reasons and consider what remains. The analogue of the modules $\Pi_{n,\ell}$ defined above are the completed cohomology groups discussed in [CE09, CE12, Eme06], whose definition for a finite field $\mathbb{F}$ we recall below. Since we have no need for Hecke operators, we may replace $\text{GL}_n$ by $\text{SL}_n$ and work in the context of group cohomology (although what we say works equally well — with appropriate adjustments — for $\text{GL}_n$ and Betti cohomology). We fix, once and for all, a tame level $N$. Let $\Gamma(p^k)$ denote the principal congruence subgroup of $\text{SL}_n(\mathbb{Z})$ of level $Np^k$.

**Definition 5.1.** The completed cohomology groups $\tilde{H}^d_n$ are defined as follows [CE12]:

$$\tilde{H}^d_n := \lim \rightarrow H^d(\Gamma(p^k),\mathbb{F})$$

Although this definition is formally the same as $\Pi_{n,\ell}$, the theory when $\ell = p$ is quite different, due to the non-semisimple nature of pro-$p$ groups acting on mod-$p$ vector spaces. However, we still conjecture the following:

**Conjecture 5.2** (Stability of completed cohomology). For $n$ sufficiently large, the modules $\tilde{H}^d_n$ are finite dimensional over $\mathbb{F}$ and are independent of $n$.

**Remark 5.3.** We have an isomorphism $\tilde{H}_{d,n} := \text{Hom}(\tilde{H}^d_n,\mathbb{F}) \simeq \lim \rightarrow H_d(\Gamma(p^k),\mathbb{F})$. If we define

$$\tilde{H}_{d,n}(\mathbb{Z}_p) := \lim \leftarrow H_d(\Gamma(p^k),\mathbb{Z}_p),$$

then Conjecture 5.2 also implies that $\tilde{H}_{d,n}(\mathbb{Z}_p)$ is a finite $\mathbb{Z}_p$-module for sufficiently large $n$.

**Remark 5.4.** Conjecture 5.2 is true for $d = 0$ and $d = 1$. For $d = 0$ one has $\tilde{H}^0_n = \mathbb{F}$ for all $n$, and for $d = 1$ one has $\tilde{H}^1_n = 0$ for all $n \geq 3$ by the congruence subgroup property.

In the remainder of this paper, we explain (see Theorem 5.8) why Conjecture 5.2 holds modulo an (unproven) hypothesis (see Conjecture 5.6 below).

We begin by recalling some facts concerning non-commutative Iwasawa theory. Let $G$ be the pro-$p$ subgroup of $\text{SL}_n(\mathbb{Z}_p)$, and let $G(p^k)$ denote the principal congruence subgroups of $G$. By construction, the module $\tilde{H}^d_n$ is naturally a module over the completed group ring $\Lambda = \Lambda_{\mathbb{F}_p} := \mathbb{F}_p[[G]]$. If $M$ is a $\Lambda$-module, then let $M^\vee := \text{Hom}(M,\mathbb{F}_p)$ be the dual $\Lambda$-module. By Nakayama’s Lemma, $\tilde{H}_{d,n}$ is finitely generated, which implies (by definition) that $\tilde{H}^d_n$ is cofinitely generated. The ring $\Lambda$ is Auslander regular [Ven02], which implies that there is a nice notion of dimension and co-dimension of finitely generated $\Lambda$-modules. One characterization of dimension for modules is given by the following result of Ardakov and Brown [AB06].
Proposition 5.5. If $M$ is finitely generated and $M^\vee$ is the co-finitely generated dual, then $M$ has dimension at most $m$ if and only if, as $k$ increases without bound,
$$\dim(M^\vee)^G(p^k) \ll p^{mk}.$$ The following is the natural analogue of Proposition 3.2 in this context.

Conjecture 5.6. Let $G$ be the pro-$p$ subgroup of $\text{SL}_n(\mathbb{Z}_p)$, and $M$ be a finitely generated $\mathbb{Z}_p[[G]]$-module. If $M$ is infinite dimensional over $\mathbf{F}$, then the dimension of $M$ is at least $n - 1$.

Remark 5.7. The analogue of this conjecture for $\Lambda_{\mathbb{Q}_p}$ is true by Theorem A of [AW12]. In particular, if any finitely generated pure $\Lambda_{\mathbb{F}_p}$-module is the reduction of a $p$-torsion free $\Lambda_{\mathbb{Z}_p} = \mathbb{Z}_p[[G]]$-module, then the conjecture is true. Such a lifting always exists for commutative regular local rings, but it is unclear whether one should expect it to hold for $\Lambda$.

The main goal of the remainder of the paper is to prove the following:

Theorem 5.8. Assume Conjecture 5.6. Then Conjecture 5.2 holds.

Proof. Our arguments proceed in a manner quite similar to the $\ell \neq p$ case. One missing ingredient is that Proposition 3.1 is no longer valid. We use the central stability results of Putman [Put12] as a replacement.

Let $(V_n)$ be a collection of representations of $S_n$. Recall from [Put12] that a sequence
$$\ldots \rightarrow V_{n-1} \xrightarrow{\phi_{n-1}} V_n \xrightarrow{\phi_n} V_{n+1} \rightarrow \ldots$$
is centrally stable if:

1. The $\phi_s$ are equivariant with respect to the natural inclusions on symmetric groups,
2. For each $n$, there is an isomorphism from $V_{n+1}$ to the unique largest quotient of $\text{Ind}_{S_n}^{S_{n+1}} V_n$ on which the 2-cycle $(n, n+1)$ acts trivially on $\phi_{n-1}(V_{n-1})$.
3. The natural map $V_n \rightarrow V_{n+1}$ is $\phi_n$.

Lemma 5.9. Suppose that Conjecture 5.2 holds for $j < d$. Then the modules $\widetilde{H}_{d,n}$ are centrally stable for sufficiently large $n$.

Proof. This argument is essentially taken directly from Putman [Put12], and we follow his argument closely. Putman considers a spectral sequence at a fixed level, which we may take to be $\Gamma(p^k)$. Taking the inverse limit, one obtains a corresponding spectral sequence for completed homology. In order for this sequence to degenerate at the relevant terms on page 2, it suffices to show that the appropriate $E^2_{i,j}$ are actually zero. By assumption, the modules $\widetilde{H}_{j,n}$ for $j < d$ are finite dimensional vector spaces which are independent of $n$ and have a trivial $S_n$-action. Thus it suffices (following Putman) to show that the $(n+1)$th central stability complex for the trivial sequence:
$$\mathbf{F} \rightarrow \mathbf{F} \rightarrow \mathbf{F} \rightarrow \ldots$$
of modules for the symmetric group (starting at the trivial group) is exact. This is a very special case of Proposition 6.1 of [Put12], but can be verified directly in this case. Hence one deduces — as in Putman — that the $H_{d,n}$ are centrally stable.

Corollary 5.10. Suppose that Conjecture 5.2 holds for $j < d$. Then the dimension of $\widetilde{H}_{d,n}$ as a $\Lambda_{\mathbb{F}_p}$-module is bounded independently of $n$. 


Proof. This is obvious for any fixed collection of $n$. Yet, for $n$ sufficiently large, central stability implies that the natural map
$$\text{Ind}_{S_n}^{S_{n+1}} \tilde{H}_{d,n} \to \tilde{H}_{d,n+1}$$
induced by the $n+1$ embeddings of $\text{SL}(n)$ into $\text{SL}(n+1)$ is surjective. It follows by Proposition 5.5 that the dimension of $\text{Ind}_{S_n}^{S_{n+1}} \tilde{H}_{d,n}$ and hence the quotient $\tilde{H}_{d,n+1}$ have dimension at most the dimension of $\tilde{H}_{d,n}$. □

Remark 5.11. Using Putman’s spectral sequence, one may prove unconditionally that $\dim_F H_d(\Gamma(p^k), F) \ll p^{mk}$ for some constant $m$ which does not depend on $n$ (although the implied constant does depend on $n$). This leads to an alternate proof of Corollary 5.10 via the Hochschild–Serre spectral sequence and Proposition 5.5.

Let us now complete the proof of Theorem 5.8. By assumption and induction, we may assume Conjecture 5.6 and Conjecture 5.2 for $j < d$. We deduce by the proceeding corollary that the dimension of $\tilde{H}_{d,n}$ must be zero for sufficiently large $n$, and hence $\tilde{H}_{d,n}$ is finite and the action of $\text{SL}_n(\mathbb{Z})$ on $\tilde{H}_{d,n}$ is trivial. In particular, the action of $S_n$ must also be trivial for sufficiently large $n$. By Lemma 5.9 we also deduce that the sequence $\tilde{H}_{d,n}$ is centrally stable. Yet a centrally stable sequence of trivial modules is stable, and hence $\tilde{H}_{d,n}$ stabilizes. □

5.1. Consequences for classical cohomology groups. The homology (or cohomology) of arithmetic groups with $F$ coefficients can be recovered from completed homology via the Hochschild–Serre spectral sequence [CE12]. For example, if we take our arithmetic group to be the principal congruence subgroup $\Gamma(p)$ of $\text{SL}_n(\mathbb{Z})$, then, recalling that $G(p)$ is the principal subgroup of $\text{SL}_n(\mathbb{Z}_p)$, this spectral sequence has the form
$$E^{i,j}_{2} := H^i(G(p), \tilde{H}^j) \implies H^{i+j}(\Gamma(p), F).$$
As noted above, $\tilde{H}^0 = F$, and Conjecture 5.2 implies that each $\tilde{H}^j$ is finite with trivial $G(p)$-action. Thus this conjecture implies that the cohomology group $H^d(\Gamma(p), F)$ has a filtration consisting of three types of classes: those arising via pullback from $H^d(G(p), F)$, those arising via (higher) transgressions from $H^j$ of $G(p)$ for $j < d$, and those arising from $\tilde{H}^d$, which does not depend on $n$. In this optic, the notion of representation stability developed by Church and Farb [CF11] is then seen (in this context) to have its origin in the mod-$p$ cohomology of $p$-adic Lie groups, rather than in the properties of arithmetic groups.

One interpretation of Conjecture 5.2 is that the $\ell = p$ and $\ell \neq p$ theories after completion are quite similar. The difference in phenomenology between these two cases is then a consequence of the vanishing of $H^i(G(\ell^k), F)$ when $k \geq 1$ and the characteristic $p$ of $F$ is not $\ell$.

5.2. Relation to $K$-theory. Finally, we remark that even if one knew that $\tilde{H}_d$ was finite, it is not at all apparent how one would compute it. A natural hope is that there is some interpretation of $\tilde{H}_d$ in terms of $K$-theory. For example, can $\tilde{H}_2(\mathbb{Z}_p)$ be related — via some étale chern class map — to $H^2(F, \mathbb{Z}_p(2))$? Some related speculation along these lines occurs in §8.3 of [CV12].
HECKE OPERATORS ON STABLE COHOMOLOGY

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