On Frequency Estimation for Partially Observed Processes with Small Noise in Observations.

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Abstract

We consider the problem of frequency estimation of the periodic signal multiplied by a stationary Gaussian process (Ornstein-Uhlenbeck) and observed in the presence of the white Gaussian noise. We show the consistency and asymptotic normality of the maximum likelihood estimator in the asymptotics of small noise in observations. The model of observations is a linear nonhomogeneous partially observed system and the construction and study of the estimator is essentialy based on the asymptotics of the equations of Kalman-Bucy filtration.

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1 Introduction

The problem of frequency estimation is of special interest in telecommunication theory. The shift of the frequency allows to estimate the speed of the object (Doppler effect). The estimation of the frequency of the Gaussian signals observed in the WGN like the studied in this work is of interest in statistical radio physics.
This is the second work devoted to the frequency estimation by the observations of partially observed linear nonhomogeneous system. In the first work we studied the properties of the maximum likelihood estimators (MLE) and Bayes estimators (BE) in the situation, where the noises in the state and observation equations tend to zero [2]. It was shown that these estimators are consistent and asymptotically normal. Here we study a slightly different model of observations, where the noise in the state equation does not tend to zero and we have asymptotics of small noise in the observations equation. The detailed study of dynamical systems with small perturbations (noise) can be found in [3]. The statistical problems for such models are presented in [12].

Let us remind the models of observations. Consider the following model of observations

\[ dX(t) = A \cos(2\pi \vartheta t) Y_t dt + \sigma dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T \quad (1) \]

where \( Y_t, 0 \leq t \leq T \) is some stationary Gaussian process with the spectral density function \( f(\lambda) \) and \( W_t, 0 \leq t \leq T \) is the Wiener process. We have to estimate the frequency \( \vartheta \) by the continuous time observations \( X_T = (X_t, 0 \leq t \leq T) \). We suppose that the process \( Y_t \) satisfies a linear equation. The most simple examples are given below.

**Example 1.** Ornstein - Uhlenbeck (O-U) process

\[ dY_t = -aY_t dt + b dV_t, \quad Y_0 = y_0, \quad 0 \leq t \leq T \]

has the correlation function \( R(\tau) \) and the spectral density \( f(\lambda) \) are the following

\[ R(\tau) = \frac{b^2}{2a} e^{-a|\tau|}, \quad f(\lambda) = \frac{b^2}{a^2 + 4\pi^2 \lambda^2}. \]

We suppose that \( a > 0, b > 0 \) and \( V_t \) is a Wiener process independent of \( W_t \).

**Example 2.** Suppose that the process \( Y_t \) satisfies the equation

\[ d\dot{Y}_t = -a_1 \dot{Y}_t dt - a_0 Y_t dt + b dV_t, \quad 0 \leq t \leq T, \]

where \( a_1^2 < 4a_0 \). Then it has the correlation function

\[ R(\tau) = \frac{b^2}{2a_0 a_1} e^{-a_1|\tau|} \left( \cos(\beta\tau) + \frac{\alpha}{\beta} \sin(\beta\tau) \right), \]
where \( \alpha = a_1/2 \) and \( \alpha^2 + \beta^2 = a_0 \). The spectral density is

\[
f(\lambda) = \frac{b^2}{[a_0 - 4\pi^2\lambda^2]^2 + 4\pi^2a_1^2\lambda^2}.
\]

In this work like [2] we suppose that \( Y_t \) is O-U process and we use the formalism of stochastic calculus. Note that all results presented in this work for O-U process can be directly extended on the model of Example 2.

We consider a slightly more general model, where the signal \( A \cos (2\pi \vartheta t) \) we replace by a known smooth periodic function \( f(\vartheta t) \). For simplicity we suppose that the period is equal 1. Of course, the signal \( f(\vartheta t) \) has period \( \tau = \frac{1}{\vartheta} \). Therefore we have a two-dimensional stochastic process \((X_t, Y_t, 0 \leq t \leq T)\) satisfying the differential equations

\[
\begin{align*}
\text{d}X_t &= f(\vartheta t) \, \text{d}Y_t + \sigma \, \text{d}W_t, \quad X_0 = 0, \\
\text{d}Y_t &= -aY_t \, \text{d}t + b \, \text{d}V_t, \quad Y_0 = y_0.
\end{align*}
\]

Here \( W_t, 0 \leq t \leq T \) and \( V_t, 0 \leq t \leq T \) are two independent Wiener processes. The parameters \( A, \sigma, a, b \) are supposed to be known and positive, the parameter \( \vartheta \in \Theta = (\alpha, \beta) \) (frequency) is unknown and has to be estimated by the observations \( X^T = (X_t, 0 \leq t \leq T) \).

We study the maximum likelihood estimator (MLE) \( \hat{\vartheta}_T \). Let us remind the definition of it. Introduce the conditional mathematical expectation

\[
m(\vartheta, t) = \mathbb{E}_\vartheta (Y_t | X_s, 0 \leq s \leq t), \quad 0 \leq t \leq T.
\]

The likelihood ratio function is (see [15])

\[
\begin{align*}
V(\vartheta, X^T) &= \exp \left\{ \frac{1}{\sigma^2} \int_0^T f(\vartheta t) \, m(\vartheta, t) \, \text{d}X_t \\
&\quad - \frac{1}{2\sigma^2} \int_0^T f(\vartheta t)^2 \, m(\vartheta, t)^2 \, \text{d}t \right\}, \quad \vartheta \in \Theta.
\end{align*}
\]

The MLE \( \hat{\vartheta}_T \) is defined by the relation

\[
V(\hat{\vartheta}_T, X^T) = \sup_{\vartheta \in \Theta} V(\vartheta, X^T).
\]

We are interested in the asymptotic behavior of this estimator. The asymptotics providing the consistency of estimators for this model of observations can be, for example, the following:
a) $\sigma \to 0$, $b \to 0$ and $T$ is fixed,

b) $\sigma \to 0$, $b$ and $T$ are fixed,

c) $T \to \infty$, $\sigma$ and $b$ are fixed.

Note that the asymptotic $A \to \infty$ in (1) can be reduced to case b). In all three cases this problem of parameter estimation is regular and in such situations these estimators are usually asymptotically normal with the natural normalization by the Fisher information:

$$\sqrt{I_T(\vartheta)} \left( \hat{\vartheta}_T - \vartheta \right) \Rightarrow \mathcal{N}(0,1), \quad \sqrt{I_T(\vartheta)} \left( \tilde{\vartheta}_T - \vartheta \right) \Rightarrow \mathcal{N}(0,1).$$

Here $I_T(\vartheta)$ is the Fisher information

$$I_T(\vartheta) = \frac{1}{\sigma^2} \int_0^T \left[ f(\vartheta t) \dot{m}(\vartheta, t) + tf'(\vartheta t) m(\vartheta, t) \right]^2 dt. \quad (4)$$

Here and in the sequel dot means differentiation w.r.t. $\vartheta$ and prim means differentiation w.r.t. $t$. For example, $f'(\vartheta t) = \frac{df(s)}{ds} \big|_{s=\vartheta t}$.

In the work [2] we studied the asymptotic a) with $\sigma = b = \varepsilon \to 0$. It was shown that the MLE and BE are consistent, asymptotically normal

$$\frac{\hat{\vartheta}_\varepsilon - \vartheta}{\varepsilon} \Rightarrow \mathcal{N}(0, I(\vartheta)^{-1}), \quad \frac{\tilde{\vartheta}_\varepsilon - \vartheta}{\varepsilon} \Rightarrow \mathcal{N}(0, I(\vartheta)^{-1})$$

and are asymptotically efficient.

In the present work we consider the asymptotic of the case b), i.e., we put $\sigma = \varepsilon \to 0$ and the coefficient $b > 0$ and $T > 0$ keep fixed. This model of observation has some interesting features. Let us see what happens with the Fisher information (4). The first strange result is the following limit: for all $t \in (0, T]$

$$\lim_{\varepsilon \to 0} [f(\varepsilon t) \dot{m}(\vartheta, t) + tf'(\vartheta t) m(\vartheta, t)] = 0.$$

This means that

$$\varepsilon^2 I_T(\vartheta) = \int_0^T \left( \frac{\partial}{\partial \vartheta} [f(\vartheta t) m(\vartheta, t)] \right)^2 dt \to 0$$

as $\varepsilon \to 0$. Further, we have the convergence in distribution

$$\varepsilon^{-1/2} [f(\varepsilon t) \dot{m}(\vartheta, t) + tf'(\vartheta t) m(\vartheta, t)] \Rightarrow \sqrt{b} tf'(\vartheta t) \xi_t.$$
where \( \{\xi_t, t \in (0, T]\} \) is a family of independent Gaussian random variables, 
\( \xi_t \sim \mathcal{N}(0, \frac{1}{2}) \). The limit integral of the normalized Fisher information
\( \varepsilon I_T(\vartheta) \) is equal (formally) to
\[
b \int_0^T t^2 f'(\vartheta t)^2 \xi_t^2 dt
\]
but this integral by the well known reason does not exist. It is shown that
the following limit
\[
\varepsilon I_T(\vartheta) \longrightarrow I_0(\vartheta) = \frac{b}{2} \int_0^T t^2 f'(\vartheta t)^2 dt
\]
holds.

The main result of the work is the asymptotically normality of the MLE:
\[
\frac{\vartheta - \vartheta}{\sqrt{\varepsilon}} \Longrightarrow \mathcal{N}(0, I_0(\vartheta)^{-1}).
\]
In the next section we give some auxiliary results from the Kalman filtration
and show that the limit model \( \varepsilon = 0 \) admits estimation of the parameter \( \vartheta \)
without error.

2 Auxiliary results

The process \( m(\vartheta, t), 0 \leq t \leq T \) satisfies the following equations of Kalman-
Bucy filtration \([10]\) (see details in \([15]\), Theorem 10.1)
\[
dm(\vartheta, t) = -am(\vartheta, t) dt + \frac{\gamma(\vartheta, t) f'(\vartheta t)}{\varepsilon^2} \left[dX_t - f(\vartheta t) m(\vartheta, t) dt\right],
\]
where the function \( \gamma(\vartheta, t) = \mathbb{E}_\vartheta (m(\vartheta, t) - Y_t)^2 \) is solution of the Riccati
equation
\[
\frac{\partial \gamma(\vartheta, t)}{\partial t} = -2a \gamma(\vartheta, t) - \frac{\gamma(\vartheta, t)^2 f'(\vartheta t)^2}{\varepsilon^2} + b^2, \quad \gamma(\vartheta, 0) = 0.
\]
Therefore for the derivative \( \dot{m}_t(\vartheta, t) \) we obtain
\[
d\dot{m}(\vartheta, t) = -a + \frac{\gamma(\vartheta, t) f'(\vartheta t)^2}{\varepsilon^2} \dot{m}(\vartheta, t) dt
\]
\[
- h(\vartheta, t) m(\vartheta, t) dt + \mathbb{E}_\vartheta [dX_t - f(\vartheta t) m(\vartheta, t) dt],
\]
where \( \dot{m}(\vartheta, 0) = 0 \) and we denoted

\[
\begin{align*}
g(\vartheta, t) &= \frac{\dot{\gamma}(\vartheta, t) f(\vartheta t) + t\gamma(\vartheta, t) f'(\vartheta t)}{\varepsilon^2}, \\
h(\vartheta, t) &= \frac{t\gamma(\vartheta, t) f(\vartheta t) f'(\vartheta t)}{\varepsilon^2}.
\end{align*}
\]

For the derivative \( \dot{\gamma}(\vartheta, t) \) we have the equation

\[
\frac{\partial \dot{\gamma}(\vartheta, t)}{\partial t} = -\frac{2}{\varepsilon^2} \left[ a + \frac{\gamma(\vartheta, t) f(\vartheta t)^2}{\varepsilon^2} \right] \dot{\gamma}(\vartheta, t) - \frac{2t\gamma(\vartheta, t)^2 f(\vartheta t) f'(\vartheta t)}{\varepsilon^2}, \quad \dot{\gamma}(\vartheta, 0) = 0.
\]  
(8)

These equations are obtained by the formal differentiation but this derivation can be justified by the standard methods. The both functions (Gaussian \( m(\vartheta, t) \) and deterministic \( g(\vartheta, t) \)) are infinitely differentiable.

The equations (7) and (8) are linear and their solutions can be written explicitly. Let us denote

\[
q(\vartheta, t) = a + \frac{\gamma^*(\vartheta, t) f(\vartheta t)^2}{\varepsilon}, \quad \gamma^*(\vartheta, t) = \frac{\gamma(\vartheta, t)}{\varepsilon}.
\]

Then we have

\[
\dot{m}(\vartheta, t) = -\int_0^t e^{-\int_0^s q(\vartheta, v) dv} h(\vartheta, s) m(\vartheta, s) ds + \int_0^t e^{-\int_0^s q(\vartheta, v) dv} g(\vartheta, s) [dX_s - f(\vartheta s) m(\vartheta, s) ds]
\]

and

\[
\dot{\gamma}(\vartheta, t) = -2 \int_0^t se^{-2\int_0^s q(\vartheta, v) dv} \gamma^*(\vartheta, s)^2 f(\vartheta s) f'(\vartheta s) ds.
\]  
(9)

Let us see how can be constructed a consistent estimator of \( \vartheta \) in the case of asymptotic \( \varepsilon \to 0 \) by the observations \( X^T \). The model of observations is

\[
\begin{align*}
dX_t &= f(\vartheta t) Y_t dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \\
dY_t &= -a Y_t dt + b dV_t, \quad Y_0 = y_0.
\end{align*}
\]
Suppose that \( \varepsilon = 0 \) (limit system) and construct an estimator of \( \vartheta \) without error. Hence

\[
\frac{dx_t}{dt} = f(\vartheta)Y_t, \quad x_0 = 0, \tag{10}
\]
\[dY_t = -aY_t dt + bdV_t, \quad Y_0 = y_0\]

and we have to estimate \( \vartheta \) by the observations \( x^T = (x_t, 0 \leq t \leq T) \). Let us put \( z_t = \frac{dx_t}{dt} \), then

\[
dz_t = \vartheta f'(\vartheta)Y_t dt + f(\vartheta) dY_t = [\vartheta f'(\vartheta) - af(\vartheta)]Y_t dt + bf(\vartheta) dV_t.
\]

By the Itô formula

\[
z_t^2 = 2 \int_0^t z_s dz_s + b^2 \int_0^t f(\vartheta s)^2 ds.
\]

Hence the function

\[
\Psi(t) = z_t^2 - 2 \int_0^t z_s dz_s = b^2 \int_0^t f(\vartheta s)^2 ds
\]

is deterministic and for any \( t \in (0, T] \) the observed \( \Psi(t) \) defines \( \vartheta \) without error. This means that if we have the limit model \( \text{(10)} \), then the measures corresponding to the observations are singular.

Suppose that \( f(\vartheta t) = A \cos(\vartheta t) \). Then

\[
\int_0^t f(\vartheta s)^2 ds = \frac{A^2 t}{2} + \frac{A^2}{4\vartheta} \sin(2\vartheta t)
\]

and

\[
\tilde{\Psi}(t) = \frac{\Psi(t)}{b^2} - \frac{A^2 t}{2} = \frac{A^2}{4\vartheta} \sin(2\vartheta t).
\]

If we denote

\[
\tau = \arg \inf_{t > t_0} \tilde{\Psi}(t) = 0
\]

then \( \vartheta = \frac{\pi}{2\tau} \).

Therefore if \( \varepsilon \to 0 \) then the consistent estimation is possible. Of course, we cannot differentiate the observations \( X^T \) w.r.t. \( t \) but we can do it “asymptotically” with the help of the kernel. For example, let us define

\[
\tilde{z}_t = \frac{1}{\varphi \varepsilon} \int_0^T K \left( \frac{s - t}{\varphi \varepsilon} \right) dX_s.
\]
Here the kernel \( K(\cdot) \) satisfies the usual conditions:

\[
K(u) \geq 0, \quad \int_{c_1}^{c_2} K(u) \, du = 1
\]

and \( K(u) = 0 \) for \( u \not\in (c_1, c_2) \). Moreover we suppose that \( K(u) \) is continuously differentiable.

Then for \( t \in (0, T) \) and small \( \varepsilon \) we have

\[
z_t = f(\vartheta_t) Y_t = \frac{1}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) f(\vartheta_t) Y_t \, ds.
\]

Hence we can write

\[
E_{\vartheta}(\tilde{z}_t - z_t)^2 = E_{\vartheta}\left(\frac{\varepsilon}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) dW_s\right)^2 + E_{\vartheta}\left(\frac{1}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) [f(\vartheta_t) Y_s - f(\vartheta_t) Y_t] \, ds\right)^2.
\]

Below we put \( s = t + \varphi_\varepsilon u \)

\[
E_{\vartheta}\left(\frac{\varepsilon}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) dW_s\right)^2 = \frac{\varepsilon^2}{\varphi_\varepsilon} \int_{\frac{t}{v_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}} K(u)^2 \, du
\]

and

\[
\frac{1}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) [f(\vartheta_t) Y_s - f(\vartheta_t) Y_t] \, ds
\]

\[
= \frac{1}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) [f(\vartheta_t) - f(\vartheta_t)] Y_s \, ds
\]

\[
+ f(\vartheta_t) \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) [Y_s - Y_t] \, ds
\]

\[
= \int_{\frac{\varphi_\varepsilon}{T}}^{\frac{\varphi_\varepsilon}{t}} K(u) [f(\vartheta_t u) - f(\vartheta_t)] Y_{t+\varphi_\varepsilon u} \, du
\]

\[
+ f(\vartheta_t) \int_{\frac{\varphi_\varepsilon}{T}}^{\frac{\varphi_\varepsilon}{T}} K(u) [Y_{t+\varphi_\varepsilon u} - Y_t] \, ds
\]

\[
= \varphi_\varepsilon \vartheta \int_{\frac{\varphi_\varepsilon}{T}}^{\frac{\varphi_\varepsilon}{T}} uK(u) f'(\vartheta_t) Y_{t+\varphi_\varepsilon u} \, du
\]

\[
+ \sqrt{\varphi_\varepsilon} f(\vartheta_t) \int_{\frac{\varphi_\varepsilon}{T}}^{\frac{\varphi_\varepsilon}{T}} K(u) \left[ \frac{Y_{t+\varphi_\varepsilon u} - Y_t}{\sqrt{\varphi_\varepsilon}} \right] \, ds.
\]
For the process $Y_t$ we have

$$\frac{Y_{t+\varphi u} - Y_t}{\sqrt{\varphi \varepsilon}} = -a \int_t^{t+\varphi u} Y_s ds + b \frac{V_{t+\varphi u} - V_t}{\sqrt{\varphi \varepsilon}}.$$ 

Hence

$$E_\theta \left( \frac{Y_{t+\varphi u} - Y_t}{\sqrt{\varphi \varepsilon}} \right)^2 \leq C$$

with some constant $C = C(u, t) > 0$. Therefore we obtain the following estimate for the error

$$E_\theta (\hat{z}_t - z_t)^2 \leq C_1 \varepsilon^2 + C_2 \varphi \varepsilon \leq C \varepsilon$$

if we take the optimal choice $\varphi \varepsilon = \varepsilon$. This means that

$$\hat{z}_t = f(\vartheta t) Y_t + O(\sqrt{\varepsilon}).$$

The function $\Psi(t)$ can be estimated as follows

$$\hat{\Psi}_\varepsilon (t) = \hat{z}_t^2 - 2 \sum_{k=0}^{K-1} \hat{z}_{t_k} \left[ \hat{z}_{t_{k+1}} - \hat{z}_{t_k} \right].$$

Here $t_k = \frac{kt}{K}$ and it can be shown that for a special choice $K = K(\varepsilon) \to \infty$ we obtain $\hat{\Psi}_\varepsilon (t) \to \Psi_\varepsilon (t)$. Using the standard arguments we can verify the consistency of the estimator $\vartheta^*_\varepsilon$ defined by the relation

$$\hat{\Psi}_\varepsilon (t) = b^2 \int_0^t f(\vartheta^*_s)^2 ds.$$ 

Of course, this estimator has a bad rate of convergence.

### 3 Main result

The MLE and BE we denote as $\hat{\vartheta}_\varepsilon$ and $\tilde{\vartheta}_\varepsilon$ respectively. We have the model of observations

\[
\begin{align*}
    dX_t &= f(\vartheta t) Y_t dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \\
    dY_t &= -aY_t dt + bV_t, \quad Y_0 = y_0.
\end{align*}
\]
Our goal is to estimate $\vartheta$ and to describe the properties of the estimators as $\varepsilon \to 0$. We suppose that the periodic function $f(t)$ is positive and two times continuously differentiable. We denote $f'(t)$ the derivative and put

$$\kappa = \inf_{0 \leq t \leq 1} f(t) > 0, \quad K = \sup_{0 \leq t \leq 1} f(t) < \infty. \quad (11)$$

This allows us to avoid the situation, where $Y_t$ is multiplied by 0. Recall that we suppose that $a, b > 0, y_0$ are known, the conditions (11) are fulfilled and the Fisher information is

$$I_0(\vartheta) = \frac{b}{2} \int_0^T t^2 f'(\vartheta t)^2 dt.$$  

The main result of this work is the following theorem.

**Theorem 1** The MLE $\hat{\vartheta}_\varepsilon$ is consistent and asymptotically normal

$$\frac{\hat{\vartheta}_\varepsilon - \vartheta}{\sqrt{\varepsilon}} \Rightarrow \mathcal{N}\left(0, I_0(\vartheta)^{-1}\right). \quad (12)$$

**Proof.** Let us denote $M(\vartheta, t) = f(\vartheta t) m(\vartheta, t)$, where $m(\vartheta, t)$ is solution of the equation (5). We have to study the log-likelihood ratio

$$\ln V(\vartheta, X^T) = \frac{1}{\varepsilon^2} \int_0^T M(\vartheta, t) dX_t - \frac{1}{2\varepsilon^2} \int_0^T M(\vartheta, t)^2 dt, \quad \vartheta \in \Theta = (\alpha, \beta).$$

We have the relation

$$\dot{M}(\vartheta, t) = tf'(\vartheta t) m(\vartheta, t) + f(\vartheta t) \dot{m}(\vartheta, t)$$

and we need to know the asymptotics of the random processes $m(\vartheta, t), 0 \leq t \leq T$ and $\dot{m}(\vartheta, t), 0 \leq t \leq T$ as $\varepsilon \to 0$. Introduce the function $\gamma_*(\vartheta, t) = \varepsilon^{-1} \gamma(\vartheta, t)$ and note that

$$dm(\vartheta, t) = -am(\vartheta, t) dt + \gamma_*(\vartheta, t) f(\vartheta t)d\bar{W}_t.$$ 

Here $\bar{W}_t$ is the innovation Wiener process

$$\bar{W}_t = \frac{1}{\varepsilon} [dX_t - M(\vartheta, t) dt].$$

The asymptotics of the solution of Riccati equation is given by the following lemma.
Lemma 1 For any $t_0 \in (0, T]$ we have the convergence

$$\sup_{t_0 \leq t \leq T} \left| \gamma(\vartheta, t) - \frac{b \varepsilon}{f(\vartheta t)} \right| = O(\varepsilon^2). \quad (13)$$

Proof. Recall that the function $\gamma(\vartheta, t) = E_{\vartheta} (m(\vartheta, t) - Y_t)^2$ satisfies the Riccati equation (6)

$$\frac{\partial \gamma(\vartheta, t)}{\partial t} = -2a \gamma(\vartheta, t) - \gamma^*(\vartheta, t)^2 f(\vartheta t)^2 + b^2, \quad \gamma(\vartheta, 0) = 0.$$ 

To verify the convergence (13) we introduce the equation

$$\frac{\partial \gamma^*(t)}{\partial t} = -2a \gamma^*(t) - \frac{\gamma^*(t)^2 \kappa^2}{\varepsilon^2} + b^2, \quad \gamma^*(0) = 0.$$ 

and note that by the comparison theorem for ordinary differential equations we have the relation

$$\gamma(\vartheta, t) \leq \gamma^*(t), \quad 0 \leq t \leq T.$$ 

The solution $\gamma^*(t)$ can be written explicitly

$$\gamma^*(t) = e^{-2rt} \left[ \frac{1}{\gamma^*(0) - \hat{\gamma}} + \frac{\kappa^2}{2r \varepsilon^2} \left( 1 - e^{-2rt} \right) \right]^{-1} + \hat{\gamma}.$$ 

Here we denoted

$$r = \left( a^2 + \frac{b^2 \kappa^2}{\varepsilon^2} \right)^{1/2}, \quad \hat{\gamma} = \frac{a \varepsilon^2}{\kappa^2} \left( \sqrt{1 + \frac{b^2 \kappa^2}{a^2 \varepsilon^2}} - 1 \right).$$

It is easy to see that for any $t_0 \in (0, T]$ we have the representations

$$r = \frac{b \kappa}{\varepsilon} (1 + O(\varepsilon)), \quad \hat{\gamma} = \frac{b \varepsilon}{\kappa} (1 + O(\varepsilon)),$$

$$\sup_{t_0 \leq t \leq T} \left| \gamma^*(t) - \frac{b \varepsilon}{\kappa} \right| = O(\varepsilon^2).$$

Hence for $t > t_0$ and $0 < \varepsilon \leq \varepsilon_0$ with some $\varepsilon_0 > 0$ we have

$$0 \leq \gamma^*(\vartheta, t) = \frac{\gamma(\vartheta, t)}{\varepsilon} \leq \frac{2b}{\kappa}.$$ 

Using the similar arguments we obtain the following estimate from below

$$\gamma^*(\vartheta, t) \geq \frac{b}{2K}.$$
We have the relation
\[ \gamma(\vartheta, t) - \gamma(\vartheta, t_0) + a \int_{t_0}^{t} \gamma(\vartheta, s) \, ds = - \int_{t_0}^{t} \gamma_{*}(\vartheta, s)^2 f(\vartheta s)^2 \, ds + b^2 (t - t_0), \]
where the left hand part tends to zero.

Hence, we verified (13) and can write
\[ dm(\vartheta, t) = -am(\vartheta, t) \, dt + b(1 + o(1)) \, dW_t, \quad m(\vartheta, 0) = y_0. \]

Below we will use several time the following technical elementary lemma.

**Lemma 2.** Suppose that the functions \( F(t), t \in [0, T] \) and \( G(t), t \in [0, T] \) are continuously differentiable, the function \( F(0) = 0, F(t) > 0, t \in (0, T] \) and \( \varepsilon \to 0 \), then we have the estimate
\[
N_\varepsilon(t) = \int_0^t e^{-\frac{1}{2} \int_0^s F(v) \, dv} G(s) \, ds = \varepsilon \frac{G(t)}{F(t)} (1 + O(\varepsilon))
\]
for any \( t > 0 \).

**Proof.** Let us take some (small) \( t_0 > 0 \) and \( t_1 \in (t_0, t) \) and denote
\[
\inf_{t_0 \leq s \leq T} F(s) = c_1 > 0, \quad \sup_{0 \leq s \leq T} F(s) = C_1 < \infty, \quad \sup_{0 \leq s \leq T} |G(s)| = C_2 < \infty.
\]
We have the estimate
\[
J_0 = \int_0^{t_0} e^{-\frac{1}{2} \int_0^s F(v) \, dv} |G(s)| \, ds \leq C_2 t_0 e^{-\frac{1}{2} \int_0^{t_0} F(v) \, dv} \leq C_2 t_0 e^{-\frac{(t - t_0)}{2}}.
\]
Then we can write
\[
J_1 = \int_{t_0}^{t_1} e^{-\frac{1}{2} \int_0^s F(v) \, dv} |G(s)| \, ds \leq \int_{t_0}^{t_1} e^{-\frac{(t - s)c_1}{c_1}} |G(s)| \, ds \leq \frac{C_2 \varepsilon}{c_1} e^{-\frac{(t_{t_1})}{c_1}}.
\]
Below we change the variables \( s = t - u\varepsilon, v = t - q\varepsilon \) and use the Taylor expansion
\[
J_2 = \int_{t_1}^{t} e^{-\frac{1}{2} \int_0^u F(v) \, dv} G(s) \, ds = -\varepsilon \int_{t_1}^{t} e^{-\frac{1}{2} \int_{t-u\varepsilon}^{t} F(v) \, dv} G(t - u\varepsilon) \, du
\]
\[ = \varepsilon G(t) \int_0^{t_{t_1}} \exp \left\{ -\frac{1}{\varepsilon} \int_{t-u\varepsilon}^{t} \left[ F(t) + (t - v) F'(t) \right] dv \right\} (1 + O(\varepsilon))
\]
\[ = \varepsilon G(t) \int_0^{t_{t_1}} \exp \left\{ -uF(t) + O(\varepsilon) \right\} (1 + O(\varepsilon)) = \varepsilon \frac{G(t)}{F(t)} (1 + O(\varepsilon))
\]
The asymptotics of \( \bar{M}(\vartheta, t) \) is described in the next lemma.
Lemma 3 For any \( t \in (t_0, T] \) we have the limits
\[
\lim_{\varepsilon \to 0} \left( m(\vartheta, t) - Y_t \right) = 0, \quad \lim_{\varepsilon \to 0} \left( \dot{m}(\vartheta, t) - \frac{t f'(\vartheta)}{f(\vartheta)} Y_t \right) = 0
\]
as \( \varepsilon \to 0 \) and therefore \( \dot{M}(\vartheta, t) \to 0 \).

Proof. The first convergence follows immediately from \( \mathbb{E}_\vartheta (m(\vartheta, t) - Y_t)^2 = \varepsilon \gamma_* (\vartheta, t) \to 0 \), i.e., we have the mean square convergence \( m(\vartheta, t) \to Y_t \) uniformly on \( t \in [t_0, T] \) for any \( t_0 \in (0, T] \). The derivative \( \dot{m}(\vartheta, t) \) satisfies the equation
\[
\dot{m}(\vartheta, t) = -\frac{1}{\varepsilon} \int_0^t e^{-\int_s^t q(\vartheta, v) \, dv} h_*(\vartheta, s) m(\vartheta, s) \, ds + \int_0^t e^{-\int_s^t q(\vartheta, v) \, dv} g_*(\vartheta, s) \, d\bar{W}_s,
\]
where we denoted \( g_*(\vartheta, t) = \dot{\gamma}_*(\vartheta, t) f(\vartheta t) + t \gamma_* (\vartheta, t) f'(\vartheta) \) and \( h_*(\vartheta, t) = t \gamma_* (\vartheta, t) f(\vartheta t) f'(\vartheta) \). Here \( \gamma_* (\vartheta, t) = \varepsilon^{-1} \dot{\gamma}(\vartheta, t) \). Note that for the values \( v \in [s, t] \) with \( s > t_0 \) and \( |t - s| \leq C\varepsilon \) we have
\[
q(\vartheta, v) = \frac{1}{\varepsilon} \left[ a\varepsilon + \gamma_* (\vartheta, v) f(\vartheta v)^2 \right] = \frac{b f(\vartheta t) (1 + o(1))}{\varepsilon}
\]
The derivative \( \dot{\gamma}_*(\vartheta, t) \) according to Lemma 2 and the equation (9) has the following asymptotics (below \( f_t = f(\vartheta t) \))
\[
\dot{\gamma}_*(\vartheta, t) = -2 \int_0^t s \exp \left\{ -\frac{2(t - s) b f_t}{\varepsilon} \right\} \gamma_* (\vartheta s)^2 f(\vartheta s) f'(\vartheta s) \, ds \left( 1 + O(\varepsilon) \right)
\]
\[
= -\frac{2\varepsilon}{b f(\vartheta t)} \int_0^{\frac{b t}{f_t}} e^{-2u} \gamma_* (\vartheta, s_u)^2 f(\vartheta s_u) f'(\vartheta s_u) \, du \left( 1 + O(\varepsilon) \right)
\]
\[
= -\frac{2\varepsilon \gamma_* (\vartheta, t)^2 f'(\vartheta t)}{b} \int_0^{\frac{b t}{f_t}} e^{-2u} \, du \left( 1 + O(\varepsilon) \right)
\]
\[
= -\frac{\varepsilon b f'(\vartheta t)}{f(\vartheta t)} \left( 1 + O(\varepsilon) \right),
\]
where we put \( s = s_u = t - \frac{ut}{b f_t} \) and used the Taylor formula. Further
\[
g_*(\vartheta, t) = -\frac{tb f'(\vartheta t)}{f(\vartheta t)} \left( 1 + O(\varepsilon) \right) + \frac{tb f''(\vartheta t)}{f(\vartheta t)} \left( 1 + O(\varepsilon) \right) = O(\varepsilon).
\]
This allows us to write
\[
\mathbb{E}_{\vartheta_0} \left( \int_0^t e^{-\int_0^s q(\vartheta, v) \, dv} g_*(\vartheta, s) \, d\bar{W}_s \right)^2 = O(\varepsilon^2)
\]
and

\[ \dot{m}(\vartheta, t) = -\frac{1}{\varepsilon} \int_0^t e^{-\int_0^s q(\vartheta,v)dv} s \gamma_*(\vartheta, s) f'(\vartheta s) m(\vartheta, s) ds + O(\varepsilon) \]

\[ = \frac{-t \gamma_*(\vartheta, t) f'(\vartheta t)}{\varepsilon} \int_0^t e^{-\int_s^t q(\vartheta,v)dv} m(\vartheta, s) ds + O(\varepsilon) \]

\[ = \frac{-t \gamma_*(\vartheta, t) f'(\vartheta t) Y_t}{b} + O(\sqrt{\varepsilon}) = \frac{-t b f'(\vartheta t) Y_t}{f(\vartheta t)} + O(\sqrt{\varepsilon}). \]

Here we used the relations

\[ m(\vartheta, s) = Y_s + O(\sqrt{\varepsilon}) = Y_t + O(\sqrt{\varepsilon}) \]

which can be easily verified.

Therefore we have the limits (14) and

\[ tf'(\vartheta t) m(\vartheta, t) \longrightarrow tf'(\vartheta t) Y_t, \quad f'(\vartheta t) \dot{m}(\vartheta, t) \longrightarrow -tf'(\vartheta t) Y_t, \]

\[ \dot{M}(\vartheta, t) \longrightarrow 0. \]

Hence for the Fisher information we obtain the limit

\[ \varepsilon^2 I_\varepsilon(\vartheta) = \int_0^T \left( tf'(\vartheta t) m(\vartheta, t) + f'(\vartheta t) \dot{m}(\vartheta, t) \right)^2 dt \]

\[ \longrightarrow \int_0^T \left( tf'(\vartheta t) Y_t - tf'(\vartheta t) Y_t \right)^2 dt = 0. \]

This means that we have to study the limits of the random processes

\[ r_{t,\varepsilon} = m(\vartheta, t) - Y_t, \quad k_{t,\varepsilon} = \dot{m}(\vartheta, t) + \frac{tf'(\vartheta t)}{f(\vartheta t)} Y_t. \]

Introduce the random processes

\[ \zeta_{t,\varepsilon} = \int_0^{\frac{b t}{\varepsilon}} e^{-u} dW_{t,\varepsilon}(u), \quad \xi_{t,\varepsilon} = \int_0^{\frac{b t}{\varepsilon}} e^{-u} dV_{t,\varepsilon}(u) \]

with the independent Wiener processes

\[ W_{t,\varepsilon}(u) = \sqrt{\frac{bf(\vartheta t)}{\varepsilon}} \left( W_t - \frac{u}{b} - W_t \right), \quad V_{t,\varepsilon}(u) = \sqrt{\frac{bf(\vartheta t)}{\varepsilon}} \left( V_t - \frac{u}{b} - V_t \right). \]

For example, we have \( E V_{t,\varepsilon}(u) = 0 \) and \( E V_{t,\varepsilon}(u_1) V_{t,\varepsilon}(u_2) = u_1 \wedge u_2. \)
Lemma 4 We have the representations
\[ r_{t,\varepsilon} = \left( \frac{b \varepsilon}{f(\partial t)} \right)^{1/2} [\zeta_{t,\varepsilon} - \xi_{t,\varepsilon}] (1 + o(1)), \]  
\[ k_{t,\varepsilon} = -\frac{tf'(\partial t)}{f(\partial t)} r_{t,\varepsilon} (1 + o(1)) - tf'(\partial t) \sqrt{\frac{b \varepsilon}{f(\partial t)^3}} \xi_{t,\varepsilon} (1 + o(1)). \]  

Proof. We have
\[ dr_{t,\varepsilon} = -\left[ a + \frac{\gamma_*(\partial, t) f(\partial t)^2}{\varepsilon} \right] r_{t,\varepsilon} dt + \gamma_*(\partial, t) f(\partial t) dW_t - bdV_t, \quad r_{0,\varepsilon} = 0, \]
and
\[ r_{t,\varepsilon} = \int_0^t e^{-\int_s^t q(\partial, v) dv} \left[ \gamma_*(\partial, s) f(\partial s)^2 + b^2 \right] ds = \frac{b \varepsilon}{f(\partial t)} (1 + O(\varepsilon)). \]

This process for \( t > t_0 > 0 \) has the following asymptotics
\[ r_{t,\varepsilon} = b \int_0^t e^{-\int_s^t \frac{bt}{\partial t} ds} dW_s (1 + o(1)) - b \int_0^t e^{-\int_s^t \frac{bt}{\partial t} ds} dV_s (1 + o(1)) \]
\[ = \sqrt{\frac{b \varepsilon}{f(\partial t)}} [\zeta_{t,\varepsilon} - \xi_{t,\varepsilon}] (1 + o(1)), \]
where we changed the variables \( s = t - \frac{\mu t}{bt} \). This proves the first relation (15).

For \( k_{t,\varepsilon} \) we can write
\[ k_{t,\varepsilon} = -\frac{1}{\varepsilon} \int_0^t e^{-\int_s^t q(\partial, v) dv} h_*(\partial, s) m(\partial, s) ds + \frac{tf'(\partial_0 t) Y_t}{f(\partial t)} + O(\varepsilon). \]

Recall the estimates
\[ \gamma_*(\partial, t) = \frac{tbf'(\partial t)}{f(\partial t)^2} (1 + O(\varepsilon)), \quad \gamma_*(\partial, t) = \frac{b}{f(\partial t)} (1 + O(\varepsilon)), \]
\[ h_*(\partial, t) = t\gamma_*(\partial, t) f(\partial t) f'(\partial t) = tbf'(\partial t) (1 + O(\varepsilon)). \]
This allow us to write
\[ k_t = -\frac{tb'f'(\vartheta t)}{\varepsilon} \int_0^t e^{-\int_0^s q(\vartheta, v) dv} m(\vartheta, s) \, ds + \frac{tf'(\vartheta t) Y_t}{f(\vartheta t)} + O(\varepsilon) \]
\[ = -\frac{tb'f'(\vartheta t)}{\varepsilon} \int_0^t e^{-\int_0^s q(\vartheta, v) dv} r_s \, ds + \frac{tf'(\vartheta t) Y_t}{f(\vartheta t)} \]
\[ - \frac{tb'f'(\vartheta t)}{\varepsilon} \int_0^t e^{-\int_0^{t-s} Y_s \, ds} + O(\varepsilon). \]

Consider the integral
\[ R_{\varepsilon}(t) = \frac{bf'(\vartheta t)}{\varepsilon} \int_0^t e^{-\int_0^s q(\vartheta, v) dv} [Y_s - Y_t] \, ds \]
\[ = \int_0^{\frac{tb't}{\varepsilon}} e^{-u} [Y_{t-u} - Y_t] \, du = \sqrt{\frac{b\varepsilon}{f_t}} \int_0^{\frac{tb't}{\varepsilon}} e^{-uy_{t,\varepsilon}(u)} \, du, \]
where we put \( s = t - \frac{u \varepsilon}{bf_t} \) and
\[ y_{t,\varepsilon}(u) = \sqrt{\frac{f_t}{b\varepsilon}} \left[ Y_{t-u \varepsilon} - Y_t \right] = V_{t,\varepsilon}(u) + o(\sqrt{\varepsilon}). \]

Therefore for any \( t \in [t_0, T] \) as \( \varepsilon \to 0 \) we have the convergence
\[ \sqrt{\frac{f_t}{b\varepsilon}} R_{\varepsilon}(t) \implies \xi_t \equiv \int_0^\infty e^{-u} V_t(u) \, du = \int_0^\infty e^{-u} dV_t(u) \]
and the random variables \( \xi_{t_1}, \ldots, \xi_{t_k} \) are independent for any \( 0 < t_1 < \ldots < t_k < T \). Here \( V_t(\cdot) \) is a Wiener process.

Further
\[ \frac{tb'f'(\vartheta t)}{\varepsilon} \int_0^t e^{-\int_0^s q(\vartheta, v) dv} r_s \, ds = \frac{tb'f'(\vartheta t)}{\varepsilon} \int_0^t e^{-\int_0^{t-s} Y_s \, ds} \, ds \left( 1 + o(1) \right) \]
\[ = \frac{tf'(\vartheta t)}{f(\vartheta t)} \int_0^{\frac{tb't}{\varepsilon}} e^{-u r_{t-u \varepsilon}} \, du \left( 1 + o(1) \right) = \frac{tf'(\vartheta t)}{f(\vartheta t)} r_t \left( 1 + o(1) \right). \]

Finally we obtain the second presentation (16):
\[ k_{t,\varepsilon} = -\frac{tf'(\vartheta t)}{f(\vartheta t)} r_t \left( 1 + o(1) \right) - \frac{tf'(\vartheta t)}{f(\vartheta t)} \sqrt{\frac{b\varepsilon}{f(\vartheta t)}} \xi_{t,\varepsilon} \left( 1 + o(1) \right) \]
From the representations (15) and (16) it follows that
\[ tf' (\vartheta t) r_t + f (\vartheta t) k_t = tf' (\vartheta t) \sqrt{\frac{b \varepsilon}{f (\vartheta t)}} \xi_{t, \varepsilon} (1 + o (1)) \]
and
\[ \varepsilon I_{\vartheta} = b \int_0^T t^2 f' (\vartheta t)^2 \xi_{t, \varepsilon}^2 dt + o (1) . \]

Of course, \( \xi_{t, \varepsilon}, t \in (0, T] \) has no limit process and the limit in distribution of each \( \xi_{t, \varepsilon} \) is Gaussian random variable \( \xi_t \sim \mathcal{N} (0, \frac{1}{\varepsilon}) \). The set \( \xi_t, t \in (0, T] \) is just a family of independent random variables. Let us denote
\[ J_{\varepsilon} (\vartheta) = b \int_0^T t^2 f' (\vartheta t)^2 \xi_{t, \varepsilon}^2 dt. \]

We have the following properties
\[ \lim_{\varepsilon \to 0} \mathbb{E} \vartheta J_{\varepsilon} (\vartheta) = \frac{b}{2} \int_0^T t^2 f' (\vartheta t)^2 dt \equiv I_0 (\vartheta), \quad \lim_{\varepsilon \to 0} \mathbb{E} \vartheta J_{\varepsilon} (\vartheta)^2 = I_0 (\vartheta)^2, \]
which imply that
\[ J_{\varepsilon} (\vartheta) \longrightarrow I (\vartheta). \]  

**Remark.** Note that the integral
\[ \int_0^T t^2 f' (\vartheta t)^2 \xi_t^2 dt \]
does not exist and the limit (17) can be explained as follows. The Gaussian processes \( \xi_{t, \varepsilon}, t \in [0, T], \varepsilon > 0 \) are continuous and the integral \( J_{\varepsilon} (\vartheta) \) can be well approximated by the sum
\[ S_{n, \varepsilon} = \frac{b T}{n} \sum_{j=1}^n t_{ij}^2 f' (\vartheta t_j)^2 \xi_{t_j, \varepsilon}^2, \quad t_j = \frac{j T}{n}. \]

Then we have the first limit (\( \varepsilon \to 0 \))
\[ S_{n, \varepsilon} \longrightarrow S_n = \frac{b T}{n} \sum_{j=1}^n t_{ij}^2 f' (\vartheta t_j)^2 \xi_{t_j}^2. \]
The second limit \((n \to \infty)\) by the law of large numbers is

\[
S_n \to I_0(\vartheta) = \frac{b}{2} \int_0^T t^2 f'(\vartheta t)^2 \, dt.
\]

Indeed, we have

\[
E_\vartheta S_n = \frac{bT}{2n} \sum_{j=1}^n t_{ij}^2 f'(\vartheta t_j)^2 \to \frac{b}{2} \int_0^T t^2 f'(\vartheta t)^2 \, dt,
\]

\[
E_\vartheta (S_n - E_\vartheta S_n)^2 = \frac{b^2 T^2}{n^2} \sum_{j=1}^n \sum_{i=1}^n t_{ij}^2 t_{ij}^4 f'(\vartheta t_j)^2 f'(\vartheta t_i)^2 E_\vartheta \left( \xi_{ij}^2 - 1 \right) \left( \xi_{ij}^2 - 1 \right)
\]

\[
= \frac{C b^2 T^6 K^4}{n} \to 0.
\]

Let us introduce the family of measures \(\{P^\varepsilon_\vartheta, \vartheta \in \Theta\}\), where \(P^\varepsilon_\vartheta\) is the measure induced in the space of continuous on \([0, T]\) functions by the observations \(X^T\) satisfying \((2)\) and define the normalized likelihood ratio

\[
Z_\varepsilon(u) = \frac{V(\vartheta + \sqrt{\varepsilon} u, X^T)}{V(\vartheta, X^T)}, \quad u \in U_\varepsilon = \left( \frac{\alpha - \vartheta}{\sqrt{\varepsilon}}, \frac{\beta - \vartheta}{\sqrt{\varepsilon}} \right).
\]

Recall that a statistical experiment is considered as regular in Le Cam’s sense if the corresponding family of measures \(\{P^\varepsilon_\vartheta, \vartheta \in \Theta\}\) is locally asymptotically normal (LAN) \([9], [5]\). The studied in the present work model of observations is regular in this sense.

**Lemma 5** The family of measures \(\{P^\varepsilon_\vartheta, \vartheta \in \Theta\}\) is LAN, i.e., we have the representation

\[
\ln Z_\varepsilon(u) = u \Delta_\varepsilon(\vartheta, X^T) - \frac{u^2}{2} I(\vartheta) + \rho_\varepsilon,
\]

where \(\rho_\varepsilon \to 0\),

\[
\Delta_\varepsilon(\vartheta, X^T) = \frac{1}{\sqrt{\varepsilon}} \int_0^T \left[ tf'(\vartheta t) m(\vartheta, t) - f(\vartheta t) \dot{m}(\vartheta, t) \right] \, d\bar{W}_t
\]

\[
= \sqrt{b} \int_0^T tf'(\vartheta t) \xi_{t,\varepsilon} \, d\bar{W}_t (1 + o(1)) \Rightarrow \mathcal{N}(0, I_0(\vartheta)).
\]
The asymptotic normality of \( \Delta \varepsilon (\vartheta, X_T) \) follows from the central limit theorem for stochastic integrals (see, e.g., [12], Lemma 1.8).

Let us verify the consistency of the MLE \( \hat{\vartheta}_\varepsilon \). Consider the log-likelihood ratio

\[
\varepsilon \ln \frac{V (\vartheta, X_T)}{V (\vartheta_0, X_T)} = \int_0^T \left[ M (\vartheta, t) - M (\vartheta_0, t) \right] d\tilde{W}_t - \varepsilon \int_0^T \frac{(M (\vartheta, t) - M (\vartheta_0, t))^2}{2} dt,
\]

where we denoted by \( \vartheta_0 \) the true value. We have to show that the first integral tends to zero and the second integral tends to a deterministic function \( G (\vartheta, \vartheta_0) \), which has a unique minimum and the point \( \vartheta = \vartheta_0 \).

**Lemma 6** We have the convergence

\[
\varepsilon \ln \frac{V (\vartheta, X_T)}{V (\vartheta_0, X_T)} \rightarrow -b \int_0^T \frac{f (\vartheta t) - f (\vartheta_0 t)^2}{4 f (\vartheta t)} dt \equiv -G (\vartheta, \vartheta_0) \quad (18)
\]

**Proof.** It will be convenient to work with the Kalman filter for the stochastic process \( Z_t = f (\vartheta t) Y_t \). This leads us to the system of equations

\[
\begin{align*}
    dX_t &= Z_t dt + \varepsilon dW_t, \quad X_0 = 0, \\
    dZ_t &= A (\vartheta t) Z_t dt + b f (\vartheta t) dV_t, \quad Z_0 = f (0) y_0, \quad 0 \leq t \leq T.
\end{align*}
\]
Here $A(\vartheta t) = \vartheta J_t(\vartheta t) - a$. The corresponding filtration equations are

$$
\begin{align*}
\frac{dM(\vartheta, t)}{dt} &= A(\vartheta t) M(\vartheta, t) dt + \frac{\Gamma(\vartheta, t)}{\varepsilon^2} [dX_t - M(\vartheta, t) dt], \\
\frac{\partial \Gamma(\vartheta, t)}{\partial t} &= 2A(\vartheta t) \Gamma(\vartheta, t) - \frac{\Gamma(\vartheta, t)^2}{\varepsilon^2} + b^2 f(\vartheta t)^2, \quad 0 \leq t \leq T,
\end{align*}
$$

with the initial values $M(\vartheta, 0) = f(0) y_0$ and $\Gamma(\vartheta, 0) = 0$. Here $\Gamma(\vartheta, t) = E_{\vartheta} \left( M(\vartheta, t) - f(\vartheta t) \right)$. Using the same arguments as above we obtain a similar to (13) approximation

$$
\Gamma(\vartheta, t) = \varepsilon b f(\vartheta t) (1 + O(\varepsilon)) = \varepsilon \Gamma^*(\vartheta, t).
$$

If we write the same equations for $M(\vartheta_0, t)$ and $\Gamma(\vartheta_0, t)$ and take the difference $R(t) = M(\vartheta, t) - M(\vartheta_0, t)$, then we obtain the equation for $R(t)$:

$$
\begin{align*}
\frac{dR(t)}{dt} &= B_\varepsilon(t) R(t) dt + [A(\vartheta t) - A(\vartheta_0 t)] M(\vartheta_0, t) dt + \delta(t) d\bar{W}_t,
\end{align*}
$$

where

$$
B_\varepsilon(t) = A(\vartheta t) - \frac{\Gamma^*(\vartheta, t)}{\varepsilon}, \quad \delta(t) = \Gamma^*(\vartheta, t) - \Gamma^*(\vartheta_0, t).
$$

The solution of this equation is

$$
R(t) = \int_0^t e^{\int_s^t B_\varepsilon(v) dv} \left\{ [A(\vartheta s) - A(\vartheta_0 s)] M(\vartheta_0, s) ds + \delta(s) d\bar{W}_s \right\}.
$$

For the first integral we have the asymptotics

$$
\int_0^t e^{\int_s^t B_\varepsilon(v) dv} [A(\vartheta s) - A(\vartheta_0 s)] M(\vartheta_0, s) ds
= \varepsilon [A(\vartheta t) - A(\vartheta_0 t)] \frac{\Gamma^*(\vartheta, t)}{\Gamma^*(\vartheta, t)} Y_t \left( 1 + O(\sqrt{\varepsilon}) \right).
$$

The second integral is of order $\sqrt{\varepsilon}$ because

$$
E_{\vartheta_0} \left( \int_0^t e^{\int_s^t B_\varepsilon(v) dv} \delta(s) d\bar{W}_s \right)^2
= \int_0^t e^{2\int_s^t B_\varepsilon(v) dv} \delta(s)^2 ds
= \frac{\varepsilon \delta(t)^2}{2\Gamma^*(\vartheta, t)} (1 + o(1)).
$$
Further
\[
\int_0^t e^{\int_0^t B_s(v) dW_s} \delta (s) \, dW_s = \frac{\sqrt[\varepsilon]{\delta (t)}}{\sqrt{\Gamma_* (\vartheta, t)}} \int_0^{t_* (\vartheta, t)} e^{-u} d\tilde{W}_{t, \varepsilon} (u) \, (1 + o (1))
\]
\[
= \frac{\sqrt[\varepsilon]{\delta (t)}}{\sqrt{\Gamma_* (\vartheta, t)}} \tilde{\zeta}_{t, \varepsilon} (1 + o (1)) = \frac{\sqrt{b} | f (\vartheta t) - f (\vartheta_0 t)|}{\sqrt{f (\vartheta t)}} \tilde{\zeta}_{t, \varepsilon} (1 + o (1))
\]
where \( \tilde{W}_{t, \varepsilon} (u) \) is a Wiener process and \( \tilde{\zeta}_{t, \varepsilon} \) is Gaussian random variable. As it was shown above the variables \( \tilde{\xi}_{t_1, \varepsilon}, \ldots, \tilde{\xi}_{t_k, \varepsilon} \) converge in distribution to the independent i.i.d. random variables \( \tilde{\xi}_1, \ldots, \tilde{\xi}_k, \tilde{\zeta} \sim N (0, 1) \).

All these allow us to write
\[
\int_0^T \frac{(M (\vartheta, t) - M (\vartheta_0, t))^2}{\varepsilon} \, dt = b \int_0^T \frac{| f (\vartheta t) - f (\vartheta_0 t)|^2}{f (\vartheta t)} \tilde{\zeta}_{t, \varepsilon}^2 \, dt (1 + o (1))
\]
\[
\rightarrow b \int_0^T \frac{| f (\vartheta t) - f (\vartheta_0 t)|^2}{2 f (\vartheta t)} \, dt \equiv 2 G (\vartheta, \vartheta_0)
\]
and
\[
\int_0^T [M (\vartheta, t) - M (\vartheta_0, t)] \, dW_t \longrightarrow 0.
\]
It can be shown that all estimates can be done uniformly in \( \vartheta \in \Theta \).

Note that the function \( G (\vartheta, \vartheta_0) \) has a unique minimum at the point \( \vartheta = \vartheta_0 \). Moreover, according to [9], Lemma 3.5.3 we have the estimate
\[
G (\vartheta, \vartheta_0) \geq c | \vartheta - \vartheta_0 |^2.
\]
The uniform in \( \vartheta \) convergence [18] provides us the consistency of the MLE. Recall that the MLE satisfies the equation
\[
\dot{V} (\hat{\vartheta}_\varepsilon, X_t) = \int_0^T \frac{\dot{M} (\hat{\vartheta}_\varepsilon, t)}{\varepsilon} \, dW_t - \int_0^T \frac{\dot{M} (\hat{\vartheta}_\varepsilon, t)}{\varepsilon^2} \left[ M (\hat{\vartheta}_\varepsilon, t) - M (\vartheta_0, t) \right] \, dt
\]
\[
= \int_0^T \frac{\dot{M} (\hat{\vartheta}_\varepsilon, t)}{\varepsilon} \, dW_t - \frac{\hat{\vartheta}_\varepsilon - \vartheta_0}{\varepsilon^2} \int_0^T \dot{M} (\hat{\vartheta}_\varepsilon, t) \, dt = 0.
\]
Therefore using the consistency of \( \hat{\vartheta}_\varepsilon \) we can write
\[
\frac{\hat{\vartheta}_\varepsilon - \vartheta_0}{\sqrt{\varepsilon}} = \varepsilon^{-1/2} \int_0^T \frac{\dot{M} (\vartheta_0, t)}{\varepsilon} \, dW_t (1 + o (1)).
\]
Recall that the limit [17] provides us the convergence
\[
\varepsilon^{-1/2} \int_0^T \dot{M} (\vartheta_0, t) \, dW_t \Rightarrow N (0, I_0 (\vartheta_0)), \quad \varepsilon^{-1} \int_0^T \dot{M} (\vartheta_0, t)^2 \, dt \rightarrow I_0 (\vartheta_0).
\]
Hence the asymptotic normality [12] is proved.
4 Discussions

The results on frequency estimation by the observations of stationary Gaussian process $Y_t$ satisfying linear equation in the presence of WGN are valid for much more general models of inhomogeneous processes $Y_t$ and (smooth) functions $f(\vartheta, t)$. We did not use the periodicity of $f(\vartheta, t) = f(\vartheta t)$ and the relation

$$\lim_{\varepsilon \to 0} \int_0^T \dot{M}(\vartheta, t)^2 \, dt = 0$$

holds for inhomogeneous processes $Y_t$ too. Recall that we took this model of observations just for the comparison of the properties of estimators for different limits.

Recall that if the smooth signal $f(\vartheta t)$ is observed in the WGN (say, $Y_t \equiv 1$ in [2]) and $T \to \infty$, then the rate of convergence of the MLE $\hat{\vartheta}_T$ is $T^{3/2}$ [8]:

$$T^{3/2} \left( \hat{\vartheta}_T - \vartheta \right) \Longrightarrow \mathcal{N} \left( 0, I_s(\vartheta)^{-1} \right)$$

with some $I_s(\vartheta) > 0$. In the case $\sigma = b = \varepsilon \to 0$ studied in [2] we have

$$\frac{\hat{\vartheta}_\varepsilon - \vartheta}{\varepsilon} \Longrightarrow \mathcal{N} \left( 0, I(\vartheta)^{-1} \right).$$

and the limit variance $I(\vartheta)^{-1}$ for large $T$ is of order $T^{-3}$, i.e.; if we consider the second limit $T \to \infty$, then the normalization formally can be written as follows

$$\frac{T^{3/2} \left( \hat{\vartheta}_\varepsilon - \vartheta \right)}{\varepsilon} \Longrightarrow \mathcal{N}.$$

In our case $\sigma = \varepsilon \to 0$ and $b > 0$ fixed we have

$$\frac{\hat{\vartheta}_\varepsilon - \vartheta}{\sqrt{\varepsilon b}} \Longrightarrow \mathcal{N} \left( 0, \tilde{I}(\vartheta)^{-1} \right),$$

where

$$\tilde{I}(\vartheta) = \int_0^T t^2 f'(\vartheta t)^2 \, dt.$$
Therefore if \( b \to 0 \) (second limit), then we can write formally the normalization \( \sqrt{\varepsilon b} \). If we consider now the third limit \( T \to \infty \), then we can write that

\[
\frac{T^{3/2} (\hat{\vartheta}_\varepsilon - \vartheta)}{\sqrt{\varepsilon b}} \Rightarrow \mathcal{N}.
\]

We have to note that the calculation of the MLE for the model of partially observed linear system is of extreme computational complexity because to calculate it we have to know the solutions of the filtration equations (5), (6) for all \( \vartheta \in \Theta \). To realize an estimator asymptotically equivalent to the MLE and much more easy calculated we can use the Multi-step MLE approach developed in [14], [11]. In the case of periodic signal in WGN the similar One-step MLE approach was realized in [4]. The most interesting case of frequency estimation for the model of observation (2), (3) is \( T \to \infty \) and it will be considered in our next work. For periodic diffusion processes the similar problems of frequency estimation were considered in [6], [7].

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