MINIMAL MASS BLOW-UP SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH A HARTREE NONLINEARITY

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ABSTRACT. We consider the following nonlinear Schrödinger equation with a Hartree nonlinearity:

\[ i \frac{\partial u}{\partial t} + \Delta u + |u|^\sigma u \pm \left( \frac{1}{|x|^{2\sigma}} \ast |u|^2 \right) u = 0 \]

in \( \mathbb{R}^N \). We are interested in the existence and behaviour of minimal mass blow-up solutions. Previous studies have shown the existence of minimal mass blow-up solutions with an inverse power potential and investigated the behaviour of the solution. In this paper, we investigate Hartree nonlinearity, which is a nonlinear term similar to the inverse power-type potential in terms of scaling.

1. INTRODUCTION

We consider the following nonlinear Schrödinger equation with a Hartree nonlinearity:

\[ i \frac{\partial u}{\partial t} + \Delta u + |u|^\sigma u \pm \left( \frac{1}{|x|^{2\sigma}} \ast |u|^2 \right) u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]

where \( N \in \mathbb{N} \) and \( \sigma \in \mathbb{R} \). It is well known that if

\[ 0 < \sigma < \min \left\{ \frac{N}{2}, 2 \right\}, \]

then (1) is locally well-posed in \( H^1(\mathbb{R}^N) \) from [5] Proposition 3.2.5, Proposition 3.2.9, Theorem 3.3.9, and Proposition 4.2.3. This means that for any initial value \( u_0 \in H^1(\mathbb{R}^N) \), there exists a unique maximal solution \( u \in C((T_*, T^*), H^1(\mathbb{R}^N)) \cap C^1((T_*, T^*), H^{-1}(\mathbb{R}^N)) \) for (1) with \( u(0) = u_0 \). Moreover, the mass (i.e., \( L^2 \)-norm) and energy \( E \) of the solution \( u \) are conserved by the flow, where

\[ E(u) := \frac{1}{2} \| \nabla u \|^2 - \frac{1}{2 + \frac{\sigma}{2}} \| u \|^{2 + \frac{\sigma}{2}} + \frac{1}{4} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{2\sigma}} \ast |u|^2 \right) (x)|u(x)|^2 dx. \]

Furthermore, the blow-up alternative holds:

\[ T^* < \infty \quad \text{implies} \quad \lim_{t \uparrow T^*} \| \nabla u(t) \|_2 = \infty. \]

We define \( \Sigma^k \) by

\[ \Sigma^k := \left\{ u \in H^k(\mathbb{R}^N) \mid |x|^k u \in L^2(\mathbb{R}^N) \right\}, \quad \| u \|_{x^k}^2 := \| u \|_{H^k}^2 + \| |x|^k u \|_2^2. \]

Particularly, \( \Sigma^1 \) is called the virial space. If \( u_0 \in \Sigma^1 \), then the solution \( u \) for (1) with \( u(0) = u_0 \) belongs to \( C((T_*, T^*), \Sigma^1) \) from [5] Lemma 6.5.2.

Moreover, we consider the case

\[ 0 < \sigma < \min \left\{ \frac{N}{2}, 1 \right\}. \]

If \( u_0 \in H^2(\mathbb{R}^N) \), then the solution \( u \) for (1) with \( u(0) = u_0 \) belongs to \( C((T_*, T^*), H^2(\mathbb{R}^N)) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N)) \) and \( |x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N)) \) from [5] Theorem 5.3.1. Furthermore, if \( u_0 \in \Sigma^2 \), then the solution \( u \) for (1)
with \( u(0) = u_0 \) belongs to \( C((T_*, T^*), \Sigma^2) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N)) \) and \( |x| \nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N)) \) from the same proof as in [5 Lemma 6.5.2].

1.1. Previous results. Firstly, we describe the results regarding the mass-critical problem:

(4) \[ i \frac{\partial u}{\partial t} + \Delta u + |u|^{4} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]

In particular, (1) with \( \sigma = 0 \) is reduced to (4).

It is well known ([2], [6], [18]) that there exists a unique classical solution \( Q \) for

\[ -\Delta Q + Q - |Q|^{4} Q = 0, \quad Q \in H^1(\mathbb{R}^N), \quad Q > 0, \quad Q \text{ is radial,} \]

which is called the ground state. If \( \|u\|_2 = \|Q\|_2 \) (\( \|u\|_2 < \|Q\|_2, \|u\|_2 > \|Q\|_2 \)), we say that \( u \) has the critical mass (subcritical mass, supercritical mass, respectively).

We note that \( E_{\text{crit}}(Q) = 0 \), where \( E_{\text{crit}} \) is the energy with respect to (4). Moreover, the ground state \( Q \) attains the best constant in the Gagliardo-Nirenberg inequality

\[ \|v\|_{2+\frac{4}{N}}^2 \leq \left( 1 + \frac{2}{N} \right) \left( \frac{\|v\|_2}{\|Q\|_2} \right)^{\frac{4}{N}} \|\nabla v\|_2^2 \text{ for } v \in H^1(\mathbb{R}^N). \]

Therefore, for all \( v \in H^1(\mathbb{R}^N) \),

\[ E_{\text{crit}}(v) \geq \frac{1}{2} \|\nabla v\|_2^2 \left( 1 - \left( \frac{\|v\|_2}{\|Q\|_2} \right)^{\frac{4}{N}} \right) \]

holds. This inequality and the mass and energy conservations imply that any subcritical mass solution for (4) is global and bounded in \( H^1(\mathbb{R}^N) \).

Regarding the critical mass case, we apply the pseudo-conformal transformation

\[ u(t, x) \mapsto \frac{1}{|t|^{\frac{4}{N}}} u \left( -\frac{1}{t}, \pm \frac{x}{t} \right) e^{i\frac{|u|^2}{4}} \]

to the solitary wave solution \( u(t, x) := Q(x)e^{it} \). Then we obtain

\[ S(t, x) := \frac{1}{|t|^{\frac{4}{N}}} Q \left( \frac{x}{t} \right) e^{-i\frac{|x|^2}{4}} e^{i\frac{|u|^2}{4}}, \]

which is also a solution for (4) and satisfies

\[ \|S(t)\|_2 = \|Q\|_2, \quad \|\nabla S(t)\|_2 \sim \frac{1}{|t|} (t \searrow 0). \]

Namely, \( S \) is a minimal mass blow-up solution for (4). Moreover, \( S \) is the only finite time blow-up solution for (4) with critical mass, up to the symmetries of the flow (see [12]).

Regarding the supercritical mass case, there exists a solution \( u \) for (4) such that

\[ \|\nabla u(t)\|_2 \sim \sqrt{\frac{\log \log |T^* - t|}{T^* - t}} (t \nearrow T^*) \]

(see [14], [15]).

Le Coz, Martel, and Raphaël [7] based on the methodology of [16] obtains the following results for

(5) \[ i \frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}} u \pm |u|^{p-1} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \]

In [8, 10], the methods used in [7] are improved and in [11], the result of [7] are strengthened.

**Theorem 1.1** ([7], [11]). Let \( 1 < p < 1 + \frac{4}{N}, \) and \( \pm = +. \) Then for any energy level \( E_0 \in \mathbb{R} \), there exist \( t_0 < 0 \) and a radially symmetric initial value \( u_0 \in H^1(\mathbb{R}^N) \) with

\[ \|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0 \]
such that the corresponding solution $u$ for (6) with $\pm = +$ and $u(t_0) = u_0$ blows up at $t = 0$. Moreover,

$$
\left\| u(t) - \frac{1}{\lambda(t)} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{4} \frac{|x|^2}{\lambda(t)^2} + i \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \to (0, \infty)$ and $b, \gamma : (t_0, 0) \to \mathbb{R}$ such that

$$
P(t) \to Q \quad \text{in} \quad H^1(\mathbb{R}^N),
$$

$$
\lambda(t) = C_1(\sigma) |t|^{\frac{4}{4-N(p-1)}} (1 + o(1)),
$$

$$
b(t) = C_2(\sigma) |t|^{\frac{4}{4-N(p-1)}} (1 + o(1)),
$$

$$
\gamma(t)^{-1} = O \left( |t|^{\frac{4}{4-N(p-1)}} \right)
$$

as $t \nearrow 0$.

**Theorem 1.2** (6). Let $1 < p < 1 + \frac{4}{N}$, and $\pm = -$. If an initial value has critical mass, then the corresponding solution for (5) with $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

The results show that when a small perturbation term is added to the critical problem (4), the perturbation term affects the existence of the minimal mass blow-up solution and, if it exists, its behaviour near the blow-up time.

On the basis of this result, (10) obtains the following results for

$$
i \frac{\partial u}{\partial t} + \Delta u + |u|^\frac{4}{N} u \pm \frac{1}{|x|^{2\sigma}} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.
$$

**Theorem 1.3** (10). Assume $0 < \sigma < \min \{1, \frac{N}{2} \}$. Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with

$$
\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0
$$

such that the corresponding solution $u$ for (6) with $\pm = +$ and $u(t_0) = u_0$ blows up at $t = 0$. Moreover,

$$
\left\| u(t) - \frac{1}{\lambda(t)} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{4} \frac{|x|^2}{\lambda(t)^2} + i \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \to (0, \infty)$ and $b, \gamma : (t_0, 0) \to \mathbb{R}$ such that

$$
P(t) \to Q \quad \text{in} \quad H^1(\mathbb{R}^N),
$$

$$
\lambda(t) = C_1(\sigma) |t|^{\frac{4}{4-N(p-1)}} (1 + o(1)),
$$

$$
b(t) = C_2(\sigma) |t|^{\frac{4}{4-N(p-1)}} (1 + o(1)),
$$

$$
\gamma(t)^{-1} = O \left( |t|^{\frac{4}{4-N(p-1)}} \right)
$$

as $t \nearrow 0$.

In previous results [1, 3, 4, 8], the blow-up rate of the minimal mass blow-up solutions for (4) with smooth potentials do not change. However, **Theorem 1.3** shows that the blow-up rate of the minimal mass blow solution changes when the potential has a singularity.

On the other hand, the following result holds in (6) with $\pm = -$.

**Theorem 1.4** (10). Assume $N \geq 2$ and $0 < \sigma < \min \{1, \frac{N}{2} \}$. If $u_0 \in H^1_{\text{rad}}(\mathbb{R}^N)$ such that $\|u_0\|_2 = \|Q\|_2$, the corresponding solution $u$ for (6) with $\pm = -$ and $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

Moreover, (9) obtain the following result for

$$
i \frac{\partial u}{\partial t} + \Delta u + |u|^\frac{4}{N} u \pm \frac{1}{|x|^{2\sigma}} \log |x| u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.
$$

**Theorem 1.5** (9). Assume $0 < \sigma < \min \{1, \frac{N}{2} \}$. Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with

$$
\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0
$$

such that the corresponding solution $u$ for (10) with $\pm = -$ and $u(t_0) = u_0$ blows up at $t = 0$. Moreover,

$$
\left\| u(t) - \frac{1}{\lambda(t)} P \left( t, \frac{x}{\lambda(t)} \right) e^{-i \frac{\lambda(t)}{4} \frac{|x|^2}{\lambda(t)^2} + i \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
$$

as $t \nearrow 0$. 

there exist $t_0, H_t$ such that the corresponding solution

$$P(t) \to Q \text{ in } H^1(\mathbb{R}^N), \quad \lambda(t) \approx |t|^{-\frac{4}{N-2}} \log |t| |t|^{-\frac{4}{N-2}}, \quad \gamma(t)^{-1} = O\left(|t|^{-\frac{2}{N-2}}\right)$$

as $t \nearrow 0$.

Comparing Theorem 1.6 with Theorem 1.3 we see that the strength of the singularity of the potential corresponds to the magnitude of the blow-up rate.

**Theorem 1.6 (2).** Assume $N \geq 2$ and (2). If $u_0 \in H^1_{\text{rad}}(\mathbb{R}^N)$ such that $\|u_0\|_2 = \|Q\|_2$, the corresponding solution $u$ for (1) with $\pm = +$ and $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

1.2. **Main results.** Assuming (1) with $\pm = -$ or (3) with $\pm = +$, we can show from Gagliardo–Nirenberg inequality that subcritical solution for (1) are global and bounded in $H^1(\mathbb{R}^N)$.

On the other hand, regarding critical mass in (1) with $\pm = +$, we obtain the following result:

**Theorem 1.7 (Existence of a minimal-mass blow-up solution).** Assume (3). Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with $\|u_0\|_2 = \|Q\|_2$, $E(u_0) = E_0$

such that the corresponding solution $u$ for (1) with $\pm = +$ and $u(t_0) = u_0$ blows up at $t = 0$. Moreover,

$$\left\| u(t) - \frac{1}{\lambda(t)} \frac{1}{2} P\left(t, \frac{x}{\lambda(t)}\right) e^{-\frac{1}{\lambda(t)} \frac{|x|^2}{2} + i \gamma(t)} \right\|_{L^1} \to 0 \quad (t \nearrow 0)$$

holds for some blow-up profile $P$ and $C^1$ functions $\lambda : (t_0, 0) \to (0, \infty)$ and $b, \gamma : (t_0, 0) \to \mathbb{R}$ such that

$$P(t) \to Q \text{ in } H^1(\mathbb{R}^N), \quad \lambda(t) = C_1(\sigma) |t|^{-\frac{4}{N-2}} (1 + o(1)), \quad \gamma(t)^{-1} = O\left(|t|^{-\frac{2}{N-2}}\right)$$

as $t \nearrow 0$.

On the other hand, the following result holds in (1) with $\pm = -$.

**Theorem 1.8 (Non-existence of a radial minimal-mass blow-up solution).** Assume (2). If $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u_0\|_2 = \|Q\|_2$, the corresponding solution $u$ for (1) with $\pm = -$ and $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

See Section 9 for the proof.

1.3. **Comments regarding the main results.** We compare Theorem 1.3 and Theorem 1.7. In terms of blow-up rate, these result behave very similarly. On the other hand, in the construction of the blow-up profile in Proposition 3.1 we do not need to pay attention to the singularity near the origin due to the smoothing effect of convolution.

We compare Theorem 1.2, Theorem 1.4 and Theorem 1.8. In Theorem 1.8 the radial symmetry of solution is not assumed. The reason for not assuming radial symmetry in Theorem 1.2 is that $L^p$-norms are invariant with respect to translations. On the other hand, the reason for assuming radial symmetry in Theorem 1.4 is that the inverse potential is not invariant with respect to translations. In the case of Theorem 1.8 since the energy is invariant with respect to translations, the proof can be done without assuming radial symmetry, as in the case of Theorem 1.2. This suggests that, with respect to blow-up, Hartree nonlinearity behaves in an intermediate way between inverse potentials and power nonlinearity.

2. Notations

In this section, we introduce the notation used in this paper.

Let $N := \mathbb{Z}_{\geq 1}, \ N_0 := \mathbb{Z}_{\geq 0}$. 
We define
\[(u, v)_2 := \text{Re} \int_{\mathbb{R}^N} u(x) \overline{v}(x) dx, \quad \|u\|_p := \left( \int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}}, \]
\[f(z) := |z|^\frac{\alpha}{2} z, \quad F(z) := \frac{1}{2 + \frac{\alpha}{N}} |z|^{\frac{\alpha}{2} + \frac{\lambda}{N}} \quad \text{for } z \in \mathbb{C}.\]

By identifying \(\mathbb{C}\) with \(\mathbb{R}^2\), we denote the differentials of \(f\) and \(F\) by \(df\) and \(dF\), respectively. Moreover, we define
\[G(v) := \frac{1}{4} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{2\sigma}} \star |v|^2 \right)(x)|v(x)|^2 dx, \quad g(v) := \left( \frac{1}{|x|^{2\sigma}} \star |v|^2 \right)v \quad \text{for } v \in H^1(\mathbb{R}).\]

We define
\[\Lambda := \frac{N}{2} + x \cdot \nabla, \quad L_+ := -\Delta + 1 - \left( 1 + \frac{4}{N} \right) Q_{\frac{\alpha}{2}}, \quad L_- := -\Delta + 1 - Q_{\frac{\alpha}{2}}.\]

Namely, \(\Lambda\) is the generator of \(L^2\)-scaling, and \(L_+\) and \(L_-\) come from the linearised Schrödinger operator to close \(Q\). Then
\[L_\pm = 0, \quad L_+ \Lambda Q = -2Q, \quad L_- |x|^2 Q = -4\Lambda Q, \quad L_+ \rho = |x|^2 Q\]
hold, where \(\rho \in \mathcal{S}(\mathbb{R}^N)\) is the unique radial solution for \(L_+ \rho = |x|^2 Q\). Note that there exist \(C_\alpha, \kappa_\alpha > 0\) such that
\[\left| \left( \frac{\partial}{\partial x} \right)^\alpha Q(x) \right| \leq C_\alpha Q(x), \quad \left| \left( \frac{\partial \lambda}{\partial x} \right)^\alpha \rho(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x),\]
for any multi-index \(\alpha\). Furthermore, there exists \(\mu > 0\) such that for all \(u \in H^1_{\text{rad}}(\mathbb{R}^N)\),
\[(L_+ \text{Re } u, \text{Re } u) + (L_- \text{Im } u, \text{Im } u) \geq \mu \|u\|_{H^1}^2 - \frac{1}{\mu} \left( (\text{Re } u, Q)_2^2 + (\text{Re } u, |x|^2 Q)_2^2 + (\text{Im } u, \rho)_2^2 \right)\]
eq \text{(8), see [13, 14, 16, 17]. We denote by } \mathcal{Y} \text{ the set of functions } g \in C^\infty(\mathbb{R}^N) \text{ such that}
\[\exists C_\alpha, \kappa_\alpha > 0, \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha g(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x)\]
for any multi-index \(\alpha\).

Finally, we use the notation \(\preceq\) and \(\succeq\) when the inequalities hold up to a positive constant. We also use the notation \(\approx\) when \(\preceq\) and \(\succeq\) hold. Moreover, positive constants \(C\) and \(\epsilon\) are sufficiently large and small, respectively.

3. Construction of a blow-up profile

In this section, we construct a blow-up profile \(P\) and introduce a decomposition of functions based on the methodology in [17].

Heuristically, we state the strategy. We look for a blow-up solution in the form of
\[u(t, x) = \frac{1}{\lambda(s)^{\frac{\alpha}{2}}} v(s, y) e^{-\frac{|x|^2}{4} + \gamma(s)}, \quad y = \frac{x}{\lambda(s)}, \quad \frac{ds}{dt} = -\frac{1}{\lambda(s)^2};\]
where \(v\) satisfies
\[0 = i \frac{\partial v}{\partial s} + \Delta v - v + f(v) + \lambda^\alpha \left( \frac{1}{|y|^{2\sigma}} \star |v|^2 \right) v \]
\[\text{where } \alpha = 2 - 2\sigma. \text{ Since we look for a blow-up solution, it may holds that } \lambda(s) \to 0 \text{ as } s \to \infty. \text{ Therefore, it seems that } \lambda^\alpha \left( |y|^{-2\sigma} \star |v|^2 \right) v \text{ is ignored. By ignoring } \lambda^\alpha \left( |y|^{-2\sigma} \star |v|^2 \right) v, \]
\[v(s, y) = Q(y), \quad \frac{1}{\lambda} \frac{\partial}{\partial s} + b = 1 - \frac{\partial \gamma}{\partial s} = \frac{\partial b}{\partial s} + b^2 = 0\]
is a solution of (9). Accordingly, \(v\) is expected to be close to \(Q\). We now consider the case where \(\sigma = 0\), i.e., the critical problem. Then \(\lambda^2 v\) corresponds to the linear term with the constant coefficient and can be removed by an
appropriate transformation. In other words, $\lambda^2v$ is a negligible term for the construction of minimal-mass blow-up solutions. This suggests that $\alpha = 2$ may be the threshold for ignoring the term in the context of minimal-mass blow-up. Therefore, $\lambda^\alpha (|y|^{-2\sigma} \ast |v|^2) v$ may become a non-negligible term if $\alpha < 2$, i.e., $\sigma > 0$. Also, (9) is difficult to solve explicitly. Consequently, we construct an approximate solution $P$ that is close to $Q$ and fully incorporates the effects of $\lambda^\alpha (|y|^{-2\sigma} \ast |v|^2) v$, e.g., the singularity of the origin.

For $K \in \mathbb{N}$, we define

$$
\Sigma_K := \left\{ (j, k) \in \mathbb{N}_0^2 \mid j + k \leq K \right\}.
$$

**Proposition 3.1.** Let $K \in \mathbb{N}$ be sufficiently large. Let $\lambda(s) > 0$ and $b(s) \in \mathbb{R}$ be $C^1$ functions of $s$ such that $\lambda(s) + |b(s)| \ll 1$.

(i) Existence of blow-up profile. For any $(j, k) \in \Sigma_K$, there exist $P_{j,k}^+, P_{j,k}^- \in \mathcal{Y}$, $\beta_{j,k} \in \mathbb{R}$, and $\Psi \in H^1(\mathbb{R}^N)$ such that $P$ satisfies

$$
\frac{i}{\partial s} P + \Delta P - P + f(P) + \lambda^\alpha \left( \frac{1}{|y|^{2\sigma}} \ast |P|^2 \right) P + \theta \frac{|y|^2}{4} P = \Psi,
$$

where $\alpha = 2 - 2\sigma$, and $P$ and $\theta$ are defined by

$$
P(s, y) := Q(y) + \sum_{(j,k) \in \Sigma_K} \left( b(s)^2 \lambda(s)^{(k+1)\alpha} P_{j,k}^+(y) + ib(s)^2j+1 \lambda(s)^{(k+1)\alpha} P_{j,k}^-(y) \right),
$$

$$
\theta(s) := \sum_{(j,k) \in \Sigma_K} b(s)^2j \lambda(s)^{(k+1)\alpha} \beta_{j,k}.
$$

Moreover, for some sufficiently small $\epsilon' > 0$,

$$
\left\| e^{\epsilon'|y|} \Psi \right\|_{H^1} \lesssim \lambda^\alpha \left( \frac{1}{|y|^\frac{2\sigma}{2}} \right) \left( \frac{1}{\lambda(s)} + b \right) + \left( b^2 + \lambda^\alpha \right)^{K+2}
$$

holds.

(ii) Mass and energy properties of blow-up profile. Let define

$$
P_{\lambda, b, \gamma}(s, x) := \frac{1}{\lambda(s)^\frac{2}{\sigma}} \left( \frac{x}{\lambda(s)} \right) \frac{1}{\lambda(s)^{\frac{2}{\sigma}}} e^{-i \frac{k(x)\cdot j k}{\lambda(s)^{\frac{2}{\sigma}}} + i \gamma(s)}.
$$

Then

$$
\left\| \frac{d}{ds} \left( P_{\lambda, b, \gamma} \right) \right\|_2 \lesssim \lambda^\alpha \left( \frac{1}{|y|^\frac{2\sigma}{2}} \right) \left( \frac{1}{\lambda(s)} + b \right) + \left( b^2 + \lambda^\alpha \right)^{K+2},
$$

$$
\frac{d}{ds} E(P_{\lambda, b, \gamma}) \lesssim \Lambda^\alpha \left( \frac{1}{|y|^\frac{2\sigma}{2}} \right) \left( \frac{1}{\lambda(s)} + b \right) + \left( b^2 + \lambda^\alpha \right)^{K+2}
$$

hold. Moreover,

$$
8E(P_{\lambda, b, \gamma}) - ||y|Q|_2^2 \left( \frac{b^2}{\lambda^2 \lambda^2} - \frac{2\beta}{2 - \alpha} \lambda^{-2} \right) \lesssim \lambda^\alpha \left( \frac{b^2 + \lambda^\alpha}{\lambda^2} \right)
$$

holds, where

$$
\beta := \beta_{0,0} = \frac{2\sigma \left( \frac{[y]^{2\sigma} \ast |Q|^2}{\lambda^2} \right)}{||\cdot|Q|_2^2}.
$$

**proof.** See [7, 10] for details of proofs.

We prove (i). We set

$$
Z := \sum_{(j,k) \in \Sigma_K} b^{2j} \lambda^{\alpha j} P_{j,k}^+ + i \sum_{(j,k) \in \Sigma_K} b^{2j+1} \lambda^{\alpha j} P_{j,k}^-.
$$

Then $P = Q + \lambda^\alpha Z$ holds.

$$
\Psi := \frac{i}{\partial s} P + \Delta P - P + f(P) + \lambda^\alpha \left( \frac{1}{|y|^{2\sigma}} \ast |P|^2 \right) P + \theta \frac{|y|^2}{4} P,
$$

where $P_{j,k}^+, P_{j,k}^- \in \mathcal{Y}$ and $\beta_{j,k}, c_{j,k}^+ \in \mathbb{R}$ are to be determined.
Firstly, we have
\[ i \frac{\partial P}{\partial s} = -i \sum_{(j,k) \in \Sigma_K} ((k+1)\alpha + 2j) b^{2j+1} \lambda^{(k+1)\alpha} P_{j,k}^{+} \]
\[ + i \sum_{j,k \geq 0} b^{2j+1} \lambda^{(k+1)\alpha} F_{j,k}^{\alpha P} - + \sum_{j,k \geq 0} b^{2j+1} \lambda^{(k+1)\alpha} F_{j,k}^{\alpha P} + + \Psi \frac{\partial P}{\partial s}, \]
where
\[ \Psi \frac{\partial P}{\partial s} = \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \sum_{(j,k) \in \Sigma_K} (k+1)\alpha b^{2j} \lambda^{(k+1)\alpha} (i P_{j,k}^{+} - b P_{j,k}^{-}) \]
\[ + \left( \frac{\partial b}{\partial s} + b^2 - \theta \right) \sum_{(j,k) \in \Sigma_K} b^{2j+1} \lambda^{(k+1)\alpha} (2j+1) P_{j,k}^{+} - (2j+1) b P_{j,k}^{-} \]
and for \( j, k \geq 0 \), \( F_{j,k}^{\alpha P} \) consists of \( P_{j,k}^{\pm} \) and \( \beta_{j,k} \) for \( (j', k') \in \Sigma_K \) such that \( k' \leq k-1 \) and \( j' \leq j+1 \) or \( k' \leq k \) and \( j' \leq j-1 \). Only a finite number of these functions are non-zero. In particular, \( F_{0,0}^{\alpha P} \) belongs to \( \mathcal{Y} \) and \( F_{0,0}^{\alpha P} = 0 \).

Next, we have
\[ \Delta P - P + |P|^2 P = - \sum_{(j,k) \in \Sigma_K} b^{2j} \lambda^{(k+1)\alpha} L_{j,k}^{+} + \sum_{(j,k) \in \Sigma_K} b^{2j+1} \lambda^{(k+1)\alpha} L_{j,k}^{-} \]
\[ + \sum_{j,k \geq 0} b^{2j+1} \lambda^{(k+1)\alpha} F_{j,k}^{f,\pm} + \sum_{j,k \geq 0} b^{2j+1} \lambda^{(k+1)\alpha} F_{j,k}^{f,\pm} + + \Psi f, \]
where
\[ \Psi f = f(Q + \lambda^\alpha Z) - \sum_{k=0}^{K+1} \frac{1}{k!} f(Q)(\lambda^\alpha Z, \ldots, \lambda^\alpha Z) \]
and for \( j, k \geq 0 \), \( F_{j,k}^{f,\pm} \) consists of \( Q, P_{j,k}^{\pm}, \) and \( \beta_{j,k} \) for \( (j', k') \in \Sigma_K \) such that \( k' \leq k-1 \) and \( j' \leq j \). Only a finite number of these functions are non-zero. In particular, \( F_{0,0}^{f,\pm} \) belongs to \( \mathcal{Y} \) and \( F_{0,0}^{f,\pm} = 0 \).

Next, we have
\[ \lambda^\alpha \left( \frac{1}{|y|^2 \ast |P|^2} \right) P = \sum_{j,k \geq 0} b^{2j} \lambda^{(k+1)\alpha} F_{j,k}^{\sigma,\pm} + ib^{2j+1} \lambda^{(k+1)\alpha} F_{j,k}^{\sigma,-} \]
where for \( j, k \geq 0 \), \( F_{j,k}^{\sigma,\pm} \) consists of \( Q, P_{j,k}^{\pm}, \) and \( \beta_{j,k} \) for \( (j', k') \in \Sigma_K \) such that \( k' \leq k-1 \) and \( j' \leq j \). Only a finite number of these functions are non-zero. In particular, \( F_{0,0}^{\sigma,\pm} \) belongs to \( \mathcal{Y} \) and \( F_{0,0}^{\sigma,\pm} = 0 \).

Finally, we have
\[ \theta \frac{|y|^2}{4} P = \sum_{(j,k) \in \Sigma_K} b^{2j} \lambda^{(k+1)\alpha} \beta_{j,k} \frac{|y|^2}{4} Q + \sum_{j,k \geq 0} b^{2j} \lambda^{(k+1)\alpha} F_{j,k}^{\theta,\pm} + i \sum_{j,k \geq 0} b^{2j+1} \lambda^{(k+1)\alpha} F_{j,k}^{\theta,-} \]
and for \( j, k \geq 0 \), \( F_{j,k}^{\theta,\pm} \) consists of \( Q, P_{j,k}^{\pm}, \) and \( \beta_{j,k} \) for \( (j', k') \in \Sigma_K \) such that \( k' \leq k-1 \) and \( j' \leq j \). Only a finite number of these functions are non-zero. In particular, \( F_{0,0}^{\theta,\pm} \) belongs to \( \mathcal{Y} \) and \( F_{0,0}^{\theta,\pm} = 0 \).

Here, we define
\[ F_{j,k}^{\pm} := F_{j,k}^{\alpha P} + F_{j,k}^{f,\pm} + F_{j,k}^{\sigma,\pm} + F_{j,k}^{\theta,\pm}, \]
\[ \Psi^{>K} := \sum_{(j,k) \not\in \Sigma_K} b^{2j} \lambda^{(k+1)\alpha} F_{j,k}^{+} + i \sum_{(j,k) \not\in \Sigma_K} b^{2j+1} \lambda^{(k+1)\alpha} F_{j,k}^{-}, \]
\[ \Psi := \Psi^{\alpha P} + \Psi f + \Psi^{>K} \]
Then $\Phi^{> K}$ is a finite sum and we obtain

$$
\int P + \Delta P - P + f(P) + \chi' \left( \frac{1}{|y|^{2\sigma}} \ast |P|^2 \right) P + \theta \frac{|y|^2}{4} P
$$

$$= \sum_{(j,k) \in \Sigma_K} b^{j\lambda} \lambda^{(k+1)\alpha} \left( -L_+ P_{j,k}^+ + \beta_{j,k} \frac{|y|^2}{4} Q + F_{j,k}^+ \right)
+ i \sum_{(j,k) \in \Sigma_K} b^{j+1} \lambda^{(k+1)\alpha} \left( -L_- P_{j,k}^- - ((k+1)\alpha + 2j) P_{j,k}^+ + F_{j,k}^- \right)
$$

For each $(j,k) \in \Sigma_K$, we choose recursively $P_{j,k}^\pm \in Y$ and $\beta_{j,k}, c_{j,k}^\pm \in \mathbb{R}$ that are solutions for the systems

$$(S_{j,k}) \begin{cases}
L_+ P_{j,k}^+ - F_{j,k}^+ - \beta_{j,k} \frac{|y|^2}{4} Q = 0, \\
L_- P_{j,k}^- - F_{j,k}^- + ((k+1)\alpha + 2j) P_{j,k}^+ = 0.
\end{cases}$$

Such solutions $(P_{j,k}^+, P_{j,k}^-, \beta_{j,k})$ are obtained from the later Propositions 3.2.

In the same way as [7, Proposition 2.1], we have

$$
\|e^{\epsilon' y} \Psi\|_{H^1} \lesssim \lambda^\alpha \left( \left| \frac{1}{\lambda} \frac{\partial \lambda}{\partial \sigma} + b \right| + \left| \frac{\partial b}{\partial \sigma} + b^2 - \theta \right| + (b^2 + \lambda^\alpha)^{K+2}. \right)
$$

The rest is the same as in [7, 10].

In the rest of this section, we construct solutions $(P_{j,k}^+, P_{j,k}^-, \beta_{j,k}) \in Y^2 \times \mathbb{R}$ for systems $(S_{j,k})$ in the proof of Proposition 3.1.

**Proposition 3.2.** The system $(S_{j,k})$ has a solution $(P_{j,k}^+, P_{j,k}^-, \beta_{j,k}) \in Y^2 \times \mathbb{R}$.

**proof.** We solve

$$
(S_{j,k}) \begin{cases}
L_+ P_{0,0}^+ - \beta_{0,0} \frac{|y|^2}{4} Q - (\frac{1}{|y|^{2\sigma}} \ast Q^2) Q = 0, \\
L_- P_{0,0}^- + \alpha P_{0,0}^+ = 0.
\end{cases}
$$

Firstly, we solve

$$(S_{0,0}) \begin{cases}
L_+ P_{0,0}^+ - \beta_{0,0} \frac{|y|^2}{4} Q - (\frac{1}{|y|^{2\sigma}} \ast Q^2) Q = 0, \\
L_- P_{0,0}^- + \alpha P_{0,0}^+ = 0.
\end{cases}
$$

For any $\beta_{0,0} \in \mathbb{R}$, there exists a solution $P_{0,0}^+ \in Y$. Let

$$
\beta_{0,0} := 2\sigma \left( \left( \frac{1}{|y|^{2\sigma}} \ast Q^2 \right) Q, Q \right)_2.
$$

Then since

$$
(P_{0,0}^+, Q)_2 = -\frac{1}{2} \langle L_+ P_{0,0}^+, \Lambda Q \rangle = \frac{1}{2} \left( \beta_{0,0} \frac{||Q||_2^2 - \sigma}{2} - \frac{1}{2} \left( \left( \frac{1}{|y|^{2\sigma}} \ast Q^2 \right) Q, Q \right)_2 \right) = 0,
$$

there exists a solution $P_{0,0}^- \in Y$. Here, let $H(j_0, k_0)$ denote by that

$$
\forall (j,k) \in \Sigma_K, \ k < k_0 \text{ or } (k = k_0 \text{ and } j < j_0)
\Rightarrow (S_{j,k}) \text{ has a solution } (P_{j,k}^+, P_{j,k}^-, \beta_{j,k}) \in Y^2 \times \mathbb{R}^2.
$$

From the above discuss, $H(1, 0)$ is true. If $H(j_0, k_0)$ is true, then $F_{j_0, k_0}^\pm$ is defined and belongs to $Y$. Moreover, for any $\beta_{j_0, k_0}$, there exists a solution $P_{j_0, k_0}^+$. Let be $\beta_{j_0, k_0}$ such that

$$
\left( -F_{j_0, k_0}^- + ((k_0 + 1)\alpha + 2j_0) P_{j_0, k_0}^+ - \frac{1}{|y|^{2\sigma}} \ast F_{j_0, k_0}^\sigma, Q \right) = 0.
$$
Then we obtain a solution $P^{\perp}_{\lambda_0, b_0}$.

4. Decomposition of functions

The parameters $\tilde{\lambda}, \tilde{b}, \tilde{\gamma}$ to be used for modulation are obtained by the following lemma:

**Lemma 4.1** (Decomposition). There exists $\overline{t}, \overline{C} > 0$ such that the following statement holds. Let $I$ be an interval and $\delta > 0$ be sufficiently small. We assume that $u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))$ satisfies

$$\forall \ t \in I, \ |\lambda(t)^2 u (t, \lambda(t)y) e^{i\gamma(t)} - Q|_{H^1} < \delta$$

for some functions $\lambda : I \to (0, \overline{t})$ and $\gamma : I \to \mathbb{R}$. Then there exist unique functions $\tilde{\lambda} : I \to (0, \overline{t}), \tilde{b} : I \to \mathbb{R}$, and $\tilde{\gamma} : I \to \mathbb{R}/2\pi\mathbb{Z}$ such that

$$(11) \quad u(t, x) = \frac{1}{\lambda(t)^2} (P + \tilde{\varepsilon}) \left( t, \frac{x}{\lambda(t)} \right) e^{-\frac{i}{\lambda(t)} \frac{|x|^2}{4} + i\tilde{\gamma}(t)}$$

$$\left| \frac{\tilde{\lambda}(t)}{\lambda(t)} - 1 \right| + |\tilde{b}(t)| + |\tilde{\gamma}(t) - \gamma(t)|_{\mathbb{R}/2\pi\mathbb{Z}} < \overline{C}$$

hold, where $| \cdot |_{\mathbb{R}/2\pi\mathbb{Z}}$ is defined by

$$|c|_{\mathbb{R}/2\pi\mathbb{Z}} := \inf_{m \in \mathbb{Z}} |c + 2\pi m|,$$

and that $\tilde{\varepsilon}$ satisfies the orthogonal conditions

$$(\tilde{\varepsilon}, i\Lambda P)_2 = (\tilde{\varepsilon}, |y|^2 P)_2 = (\tilde{\varepsilon}, i\rho)_2 = 0$$

on $I$. In particular, $\tilde{\lambda}, \tilde{b}, \tilde{\gamma}$ are $C^1$ functions and independent of $\lambda$ and $\gamma$.

For the proof, see [10].

5. Approximate blow-up law

In this section, we describe the initial values and the approximation functions of the parameters $\lambda$ and $b$ in the decomposition.

We expect the parameters $\lambda$ and $b$ in the decomposition to approximately satisfy

$$\frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b = \frac{\partial b}{\partial s} + b^2 - \theta = 0.$$

Therefore, the approximation functions $\lambda_{\text{app}}$ and $b_{\text{app}}$ of the parameters $\lambda$ and $b$ will be determined by the following lemma:

**Lemma 5.1.** Let

$$\lambda_{\text{app}}(s) := \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2 - \alpha}} \right)^{-\frac{2}{\alpha}} s^{-\frac{1}{\alpha}}, \quad b_{\text{app}}(s) := \frac{2}{\alpha s},$$

Then $(\lambda_{\text{app}}, b_{\text{app}})$ is a solution for

$$\frac{\partial b}{\partial s} + b^2 - \beta \lambda^\alpha = 0, \quad \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b = 0$$

in $s > 0$.

Furthermore, the following lemma determines $\lambda(s_1)$ and $b(s_1)$ for a given energy level $E_0$ and a sufficiently large $s_1$.

**Lemma 5.2.** Let define $C_0 := \frac{8E_0}{\|y|Q\|_2^2}$ and $0 < \lambda_0 \ll 1$ such that $\frac{2\beta}{2 - \alpha} + C_0 \lambda_0^{2-\alpha} > 0$. For $\lambda \in (0, \lambda_0)$, we set

$$\mathcal{F}(\lambda) := \int_{\lambda}^{\lambda_0} \frac{1}{\mu^{\frac{\alpha}{2} + 1} \sqrt{\frac{2\beta}{2 - \alpha} + C_0 \mu^{2-\alpha}}} d\mu.$$
Then for any \( s_1 \gg 1 \), there exist \( b_1, \lambda_1 > 0 \) such that
\[
\left| \frac{\lambda_1^2}{\lambda_{\text{app}}(s_1)^2} - 1 \right| + \left| \frac{b_1}{b_{\text{app}}(s_1)} - 1 \right| \lesssim s_1^{-\frac{1}{2}} + s_1^{2-\frac{\alpha}{2}}, \quad \mathcal{F}(\lambda_1) = s_1, \quad E(P_{\lambda_1, b_1, \gamma_1}) = E_0.
\]
Moreover,
\[
\left| \mathcal{F}(\lambda) - \frac{2}{\alpha \lambda^2 \sqrt{\frac{2\beta}{2-\alpha}}} \right| \lesssim \lambda^{-\frac{2}{\alpha}} + \lambda^{3-\frac{\alpha}{2}}
\]
holds.

**Proof.** For the proof, see [7, 10]. \( \square \)

### 6. Uniformity estimates for decomposition

In this section, we estimate modulation terms.

Let define
\[
C := \frac{\alpha}{4 - \alpha} \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2-\alpha}} \right)^{-\frac{2\alpha}{\beta}}.
\]
For \( t_1 < 0 \) that is sufficiently close to 0, we define
\[
s_1 := |C^{-1} t_1|^{\frac{3}{2\alpha}}.
\]
Additionally, let \( \lambda_1 \) and \( b_1 \) be given in Lemma [5.2] for \( s_1 \) and \( \gamma_1 = 0 \). Let \( u \) be the solution for (I) with \( \pm = + \) with an initial value
\[
u(t_1, x) := P_{\lambda_1, b_1, 0}(x).
\]
Then since \( u \) satisfies the assumption of Lemma [4.1] in a neighbourhood of \( t_1 \), there exists a decomposition \((\tilde{\lambda}_{t_1}, \tilde{b}_{t_1}, \tilde{\gamma}_{t_1}, \tilde{\varepsilon}_{t_1})\) such that (II) in a neighbourhood \( I \) of \( t_1 \). The rescaled time \( s_{t_1} \) is defined by
\[
s_{t_1}(t) := s_1 - \int_1^t \frac{1}{\lambda_{t_1}^2} d\tau.
\]
Then we define an inverse function \( s_{t_1}^{-1} : s_{t_1}(I) \rightarrow I \). Moreover, we define
\[
t_{t_1} := s_{t_1}^{-1}, \quad \lambda_{t_1}(s) := \tilde{\lambda}(t_{t_1}(s)), \quad b_{t_1}(s) := \tilde{b}(t_{t_1}(s)),
\]
\[
\gamma_{t_1}(s) := \tilde{\gamma}(t_{t_1}(s)), \quad \varepsilon_{t_1}(s, y) := \tilde{\varepsilon}(t_{t_1}(s), y).
\]
For the sake of clarity in notation, we often omit the subscript \( t_1 \). In particular, it should be noted that \( u \in C((T_*, T^*), \Sigma^2(\mathbb{R}^N)) \) and \( |x| \nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N)) \). Furthermore, let \( I_{t_1} \) be the maximal interval such that a decomposition as (II) is obtained and we define
\[
J_{s_1} := s(I_{t_1}).
\]
Additionally, let \( s_0 (\leq s_1) \) be sufficiently large and let
\[
s' := \max \{ s_0, \inf J_{s_1} \}.
\]
Let
\[
0 < M < \min \left\{ \frac{1}{2}, \frac{4}{\alpha - 2} \right\}
\]
and \( s_* \) be defined by
\[
s_* := \inf \{ \sigma \in (s', s_1) \mid (13) \text{ holds on } [\sigma, s_1] \},
\]
where
\[
\| e(s) \|_{H^1}^2 + b(s)^2 \| |y| e(s) \|_{2}^2 < s^{-2K}, \quad \sum_{\text{app}}(s)^{2\alpha} - 1 + \left| \frac{b(s)}{b_{\text{app}}(s)} - 1 \right| < s^{-M}.
\]
Finally, we define
\[
\text{Mod}(s) := \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \frac{\partial b}{\partial s} + b^2 - \theta, 1 - \frac{\partial \gamma}{\partial s} \right).
\]
Lemma 6.1. For $s \in (s_*, s_1)$,
\[
|\langle \varepsilon(s), Q \rangle| \lesssim s^{-(K+2)}, \quad |\text{Mod}(s)| \lesssim s^{-(K+2)}, \quad \|e^{t' y} \Psi\|_{H^1} \lesssim s^{-(K+4)}
\]
hold.

**proof.** For the proof, see [7, 10].

\[\square\]

7. Modified energy function

We proceed with a modified version of the technique presented in Le Coz, Martel, and Raphaël [7] and Raphaël and Szefelt [16]. Let $m > 0$ be sufficiently large and define
\[
H(s, \varepsilon) := \frac{1}{2} \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_2^2 - \int_{\mathbb{R}^N} (F(P + \varepsilon) - F(P) - dF(P)(\varepsilon)) \, dy
\]
\[
- \lambda \alpha (G(P + \varepsilon) - G(P) - dG(P)(\varepsilon)),
\]
\[
S(s, \varepsilon) := \frac{1}{\lambda^m} H(s, \varepsilon).
\]

**Lemma 7.1 (Estimates of $S$).** For $s \in (s_*, s_1)$,
\[
\|\varepsilon\|_{H^1}^2 + b^2 \|y\|_2^2 + O(s^{-2(K+2)}) \lesssim H(s, \varepsilon) \lesssim \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_2^2
\]
hold. Moreover,
\[
\frac{1}{\lambda^m} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_2^2 + O(s^{-2(K+2)}) \right) \lesssim S(s, \varepsilon) \lesssim \frac{1}{\lambda^m} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_2^2 \right)
\]
hold.

**proof.** Since
\[
|\langle G(P + \varepsilon) - G(P) - dG(P)(\varepsilon) \rangle| \lesssim \|\varepsilon\|_{H^1}^2,
\]
we obtain the conclusion as in [7, 10].

\[\square\]

**Lemma 7.2.** For $s \in (s_*, s_1)$,
\[
|\langle f(P + \varepsilon) - f(P), \Lambda \varepsilon \rangle_2 | \lesssim \|\varepsilon\|_{H^1}^2 + s^{-3K}, \quad |\langle g(P + \varepsilon) - g(P), \Lambda \varepsilon \rangle_2 | \lesssim \|\varepsilon\|_{H^1}^2
\]
hold.

**proof.** For $f$, see [7, 10]. Similarly, we obtain.
\[
\int_{\mathbb{R}^N} y \cdot \nabla \left( \frac{1}{4} \left( \frac{1}{|y|^{2\sigma}} \ast |P + \varepsilon|^2 \right) |P + \varepsilon|^2 - \frac{1}{4} \left( \frac{1}{|y|^{2\sigma}} \ast |P|^2 \right) |P|^2 - \frac{1}{|y|^{2\sigma}} \ast |P|^2 \right) \text{Re}(P \overline{\varepsilon}) \right) \, dy = O \left( \|\varepsilon\|_{H^1}^2 \right).
\]
On the other hand, since
\[
dg(v)(w) = \left( \frac{1}{|y|^{2\sigma}} \ast |v|^2 \right) w + 2 \left( \frac{1}{|y|^{2\sigma}} \ast \text{Re}(v \overline{\omega}) \right) v,
\]
we obtain
\[
\int_{\mathbb{R}^N} y \cdot \nabla \left( \frac{1}{4} \left( \frac{1}{|y|^{2\sigma}} \ast |P + \varepsilon|^2 \right) |P + \varepsilon|^2 - \frac{1}{4} \left( \frac{1}{|y|^{2\sigma}} \ast |P|^2 \right) |P|^2 - \frac{1}{|y|^{2\sigma}} \ast |P|^2 \right) \text{Re}(P \overline{\varepsilon}) \right) \, dy
\]
\[
= (g(P + \varepsilon) - g(P) - dg(P)(\varepsilon), y \cdot \nabla P)_2 + (g(P + \varepsilon) - g(P), \Lambda \varepsilon)_2 - \frac{N}{2} (g(P + \varepsilon) - g(P), \varepsilon)_2
\]
\[
= (g(P + \varepsilon) - g(P), \Lambda \varepsilon)_2 + O \left( \|\varepsilon\|_{H^1}^2 \right).
\]
Consequently, we have the conclusion.

\[\square\]
Lemma 7.3 (Derivative of $S$ in time). For $s \in (s_*, s_1]$,
\[
\frac{d}{ds} S(s, \varepsilon(s)) \gtrsim -b \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_{l_2}^2 \right) + O(s^{-2(K+2)})
\]
holds. Moreover,
\[
\frac{d}{ds} S(s, \varepsilon(s)) \gtrsim \frac{b}{\lambda_m} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_{l_2}^2 + O(s^{-(2K+3)}) \right)
\]
holds.

**proof.** From Lemma 7.2, we obtain the conclusion as in [7, 10].

From Lemma 7.1 and Lemma 7.3, we confirm (13) on $[s_0, s_1]$. Namely, we obtain the following result:

Lemma 7.4 (Re-estimation). For $s \in (s_*, s_1]$,
\[
\|\varepsilon(s)\|_{H^1}^2 + b(s)^2 \|y\|_{l_2}^2 \lesssim s^{-(2K+2)},
\]
(14)
\[
\left| \frac{\lambda(s)^{\frac{2}{m}}}{\lambda_{app}(s)^{\frac{2}{m}}} - 1 \right| + \left| \frac{b(s)}{b_{app}(s)} - 1 \right| \lesssim s^{-\frac{1}{2}} + s^{\frac{1}{2}}
\]
holds.

**proof.** For the proof, see [7, 10].

Lemma 7.5. If $s_0$ is sufficiently large, then $s_* = s' = s_0$.

**proof.** This result is proven from Lemma 7.2 and the definitions of $s_*$ and $s'$. See [8] for details of the proof.

Finally, we rewrite the uniform estimates obtained for the time variable $s$ in Lemma 7.4 into uniform estimates for the time variable $t$.

Lemma 7.6 (Interval). If $s_0$ is sufficiently large, then there is $t_0 < 0$ that is sufficiently close to 0 such that for $t_1 \in (t_0, 0)$,
\[
[t_0, t_1] \subset s_{t_1}^{-1}([s_0, s_1]), \quad \|CS_{t_1}(t) - \frac{4\pi}{\alpha} - |t| \| \lesssim |t|^{1 + \frac{\alpha M}{4}} \quad (t \in [t_0, t_1])
\]
holds.

**proof.** For the proof, see [7, 10].

Lemma 7.7 (Conversion of estimates). Let
\[
C_\lambda := C^{-\frac{2}{m}} \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2-\alpha}} \right)^{-\frac{2}{m}}, \quad C_b := \frac{2}{\alpha} C^{-\frac{2}{m}}.
\]
For $t \in [t_0, t_1]$,
\[
\tilde{\lambda}_{t_1}(t) = C_\lambda |t|^{\frac{2}{m}} \left( 1 + \epsilon_{\tilde{\lambda}_{t_1}}(t) \right), \quad \tilde{b}_{t_1}(t) = C_b |t|^{\frac{2}{m}} \left( 1 + \epsilon_{\tilde{b}_{t_1}}(t) \right)
\]
hold. Furthermore,
\[
\sup_{t_1 \in [t_0, t_1]} \left| \epsilon_{\tilde{\lambda}_{t_1}}(t) \right| \lesssim |t|^{\frac{\alpha M}{1+\alpha}} , \quad \sup_{t_1 \in [t_0, t_1]} \left| \epsilon_{\tilde{b}_{t_1}}(t) \right| \lesssim |t|^{\frac{\alpha M}{1+\alpha}}.
\]

**proof.** For the proof, see [7, 10].
8. Proof of Theorem 1.7

In this section, we complete the proof of Theorem 1.7. See [7, 8] for details of proof.

**proof of Theorem 1.7** Let \((t_n)_{n \in \mathbb{N}} \subset (t_0, 0)\) be a monotonically increasing sequence such that \(\lim_{n \to \infty} t_n = 0\). For each \(n \in \mathbb{N}\), \(u_n\) is the solution for (1) with \(\pm = +\) with an initial value

\[
u_n(t_n, x) := P_{\lambda_{1,n}, b_{1,n}, 0}(x)
\]

at \(t_n\), where \(b_{1,n}\) and \(\lambda_{1,n}\) are given by Lemma 5.2 for \(t_n\).

According to Lemma 4.1 with an initial value \(\tilde{\gamma}_n(t_n) = 0\), there exists a decomposition

\[
u_n(t, x) = \frac{1}{\lambda_n(t)} (P + \epsilon_n) \left( t, \frac{x}{\lambda_n(t)} \right) \epsilon_n \left( \frac{\lambda_n(t)}{\lambda_n(t)} + \epsilon_n(t) \right).
\]

Then \((\nu_n(t_0))_{n \in \mathbb{N}}\) is bounded in \(\Sigma^1\). Therefore, up to a subsequence, there exists \(\nu_\infty(t_0) \in \Sigma^1\) such that

\[
u_n(t_0) \to \nu_\infty(t_0) \text{ in } \Sigma^1, \quad \nu_n(t_0) \to \nu_\infty(t_0) \text{ in } L^2(\mathbb{R}^N) \quad (n \to \infty),
\]

see [7, 8] for details.

Let \(\nu_\infty\) be the solution for (1) with \(\pm = +\) and an initial value \(\nu_\infty(t_0)\), and let \(T^*\) be the supremum of the maximal existent interval of \(\nu_\infty\). Moreover, we define \(T := \min(\{0, T^*\})\). Then for any \(T' \in [t_0, T], [t_0, T] \subset [t_0, t_n]\) if \(n\) is sufficiently large. Then there exist \(n_0\) and \(C(T', t_0) > 0\) such that

\[
\sup_{n \geq n_0} \|\nu_n\|_{L^\infty([t_0, T'], \Sigma^1)} \leq C(T', t_0)
\]

holds. Therefore,

\[\nu_n \to \nu_\infty \quad \text{in } C([t_0, T], L^2(\mathbb{R}^N)) \quad (n \to \infty)\]

holds (see [8]). In particular, \(\nu_n(t) \to \nu_\infty(t)\) in \(\Sigma^1\) for any \(t \in [t_0, T]\). Furthermore, from the mass conservation, we have

\[
\|\nu_\infty(t)\|_2 = \|\nu_\infty(t_0)\|_2 \leq \lim_{n \to \infty} \|\nu_n(t_0)\|_2 = \lim_{n \to \infty} \|\nu_n(t_n)\|_2 = \lim_{n \to \infty} \|P(t_n)\|_2 = \|Q\|_2.
\]

Based on weak convergence in \(H^1(\mathbb{R}^N)\) and Lemma 4.1 we decompose \(\nu_\infty\) to

\[
\nu_\infty(t, x) = \frac{1}{\lambda_\infty(t)} \left( P + \tilde{\nu}_\infty \right) \left( t, \frac{x}{\lambda_\infty(t)} \right) \epsilon_\infty \left( \frac{\lambda_\infty(t)}{\lambda_\infty(t)} + \epsilon_\infty(t) \right).
\]

Furthermore, for any \(t \in [t_0, T]\), as \(n \to \infty\),

\[
\hat{\lambda}_n(t) \to \hat{\lambda}_\infty(t), \quad \hat{b}_n(t) \to \hat{b}_\infty(t), \quad \epsilon_n \to \epsilon_\infty, \quad \tilde{\nu}_n \to \tilde{\nu}_\infty \quad \text{in } \Sigma^1
\]

hold. Consequently, from the uniform estimate in Lemma 7.7 as \(n \to \infty\), we have

\[
\|\tilde{\nu}_\infty(t)\|_{H^1} \lesssim |t|^{\frac{4}{1+4\alpha}}, \quad |||y\tilde{\nu}_\infty(t)\|_2 \lesssim |t|^{\frac{2\alpha}{1+4\alpha}}, \quad |\epsilon_{\infty}(t)| \lesssim |t|^{\frac{4\alpha}{1+4\alpha}}, \quad |\tilde{\nu}_{\infty}(t)| \lesssim |t|^{\frac{4\alpha}{1+4\alpha}}.
\]

Consequently, we obtain that \(u\) converges to the blow-up profile in \(\Sigma^1\).

Finally, we check energy of \(\nu_\infty\). Since

\[
E(u_n) - E\left( P_{\lambda_n, \hat{b}_n, \tilde{\gamma}_n} \right) = \int_0^1 \left\langle E'(P_{\lambda_n, \hat{b}_n, \tilde{\gamma}_n}), \xi_{\lambda_n, \hat{b}_n, \tilde{\gamma}_n} \right\rangle \, dt
\]

and \(E'(w) = -\Delta w - |w|^{4\alpha} w - (|x|^{-2\sigma} * |w|^2) \, w\), we have

\[
E(u_n) - E\left( P_{\lambda_n, \hat{b}_n, \tilde{\gamma}_n} \right) = O \left( \frac{1}{\lambda_n^2} \|\tilde{\nu}_n\|_{H^1} \right) = O \left( |t|^{\frac{4\alpha}{1+4\alpha}} \right).
\]

Similarly, we have

\[
E(u_\infty) - E\left( P_{\lambda_\infty, \hat{b}_\infty, \tilde{\gamma}_\infty} \right) = O \left( \frac{1}{\lambda_\infty^2} \|\tilde{\nu}_\infty\|_{H^1} \right) = O \left( |t|^{\frac{4\alpha}{1+4\alpha}} \right).
\]
From the continuity of $E$, we have
\[
\lim_{n \to \infty} E \left( P_{\lambda_n, \delta_n, \gamma_n} \right) = E \left( P_{\lambda_\infty, \delta_\infty, \gamma_\infty} \right)
\]
and from the conservation of energy,
\[
E(u_n) = E(u_n(t_n)) = E \left( P_{\lambda_{1,n}, \delta_{1,n}, 0} \right) = E_0.
\]
Therefore, we have
\[
E(u_\infty) = E_0 + \alpha_{\gamma > 0}(1)
\]
and since $E(u_\infty)$ is constant for $t$, $E(u_\infty) = E_0$.  \hfill \Box

9. PROOF OF THEOREM 1.8

In this section, we describe the proof of Theorem 1.8.

Proof of Theorem 1.8 We assume that $u$ is a critical-mass radial solution for (1) with $\pm = -$ and blows up at $T^*$. Let a sequence $(t_n)_{n \in \mathbb{N}}$ be such that $t_n \to T^*$ as $n \to T^*$ and define
\[
\lambda_n := \frac{\|\nabla Q\|_2}{\|\nabla u(t_n)\|_1}, \quad v_n(x) := \lambda_n \nabla u(t_n, \lambda_n x).
\]
Then
\[
\|v_n\|_2 = \|Q\|_2, \quad \|\nabla v_n\|_2 = \|\nabla Q\|_2
\]
hold. Moreover,
\[
E_0 := E(u(t_n)) \geq E_{\text{crit}}(u(t_n)) = \frac{E_{\text{crit}}(v_n)}{\lambda_n^2}.
\]
Therefore, we obtain
\[
\limsup_{n \to \infty} E_{\text{crit}}(v_n) \leq 0.
\]
From the standard concentration argument (see [14, 17]), there exist sequences $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N$ and $(\gamma_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that
\[
v_n(\cdot - x_n)e^{i\gamma_n} \to Q \quad \text{in} \quad H^1(\mathbb{R}^N) \quad (n \to \infty).
\]
Here, since
\[
\int_{\mathbb{R}} \left( \frac{1}{|x|^{2\sigma}} \ast |v_n(\cdot - x_n)|^2 \right)(x) |v_n(x - x_n)|^2 dx = \int_{\mathbb{R}} \left( \frac{1}{|x|^{2\sigma}} \ast |v_n|_2^2 \right)(x) |v_n(x)|^2 dx,
\]
we obtain
\[
\int_{\mathbb{R}} \left( \frac{1}{|x|^{2\sigma}} \ast |u(t_n)|^2 \right)(x) |u(t_n, x)|^2 dx = \frac{1}{\lambda_n^{2\sigma}} \int_{\mathbb{R}} \left( \frac{1}{|x|^{2\sigma}} \ast |v_n(\cdot - x_n)|^2 \right)(x) |v_n(x - x_n)|^2 dx.
\]
Therefore, since $E_{\text{crit}}(u) \geq 0$ and $v_n(\cdot - x_n)e^{i\gamma_n} \to Q$ in $H^1(\mathbb{R}^N)$,
\[
E_0 = E(u(t_n)) \geq \frac{1}{4\lambda_n^{2\sigma}} \int_{\mathbb{R}} \left( \frac{1}{|x|^{2\sigma}} \ast |v_n(\cdot - x_n)|^2 \right)(x) |v_n(x - x_n)|^2 dx \to \infty \quad (n \to \infty).
\]
It is a contradiction.  \hfill \Box
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