Confinement and Strings in MQCD

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We study aspects of confinement in the M theory fivebrane version of QCD (MQCD). We show heavy quarks are confined in hadrons (which take the form of membrane-fivebrane bound states) for $N = 1$ and softly broken $N = 2$ $SU(N)$ MQCD. We explore and clarify the transition from the exotic physics of the latter to the standard physics of the former. In particular, the many strings and quark-antiquark mesons found in $N = 2$ field theory by Douglas and Shenker are reproduced. It is seen that in the $N = 1$ limit all but one such meson disappears while all of the strings survive. The strings of softly broken $N = 2$, $N = 1$, and even non-supersymmetric $SU(N)$ MQCD have a common ratio for their tensions as a function of the amount of flux they carry. We also comment on the almost BPS properties of the Douglas-Shenker strings and discuss the brane picture for monopole confinement on $N = 2$ QCD Higgs branches.
1. Introduction

Recently, many interesting results about field theory have been obtained by realizing gauge field theories on the world-volume of branes in string theory. A particularly interesting configuration was constructed in [1] to study $\mathcal{N} = 4$ three-dimensional gauge theories. The construction was generalized to study gauge theories with the same amount of supersymmetry in various dimensions in [2,3,4,6]. The four-dimensional $\mathcal{N} = 1$ case was first studied in [5] and further investigated in [6,7,8,9,10,11,12,13]. A different approach, which involves encoding gauge theories in type II geometry, has been studied in [15,16,17,18,19,20,21,22].

With this construction, we can easily realize a pure $\mathcal{N} = 2$ or $\mathcal{N} = 1$ four-dimensional gauge theory as the low energy limit of a configuration of branes in the weakly coupled Type IIA theory. A powerful method to solve aspects of these theories has been presented in [2]. If we go to the strong coupling limit of the Type IIA string theory, the configuration of branes become smooth enough to allow a semi-classical analysis in M theory. Using this method, the Seiberg-Witten curve for a large number of gauge theories was found in [2,3,23,24] (see also [25,26,27]). A different approach based on a fivebrane interpretation of some four-dimensional theories was earlier used in [28,29] to solve a very large family of models.

The brane theory (which we will call MQCD) is by no means identical to QCD. It contains, among other things, extra colored Kaluza-Klein states from the compact $x^{10}$ direction around which it is wrapped. The possibility of varying the radius $R$ of this compact direction makes MQCD a one-parameter generalization of QCD. For $\mathcal{N} = 2$ supersymmetry we may take $R$ small enough that these Kaluza-Klein states are heavy compared with the QCD scale, and all of the interesting field theory gauge couplings and BPS states are independent of $R$. This is not true for $\mathcal{N} = 1$ [30] if we want to study strings and confinement. Still, MQCD can be a useful method for extracting physics that cannot be computed or easily visualized in the context of ordinary field theory. It was shown in [30] that $\mathcal{N} = 1$ MQCD has flux tubes and undergoes spontaneous breaking of its discrete chiral symmetry. The tensions of the MQCD strings and BPS-saturated domain walls [31] were computed, and a number of interesting results (such as the fact that the MQCD string can end on a domain wall) were derived. Our point of view, following [30], is to assume that $\mathcal{N} = 1$ MQCD is in the same universality class as QCD and, therefore, has the same qualitative properties. We will extract some qualitative insights and one
quantitative formula, whose reliability we cannot prove but which we find suggestive. This will be extensively discussed in section 9.

In this paper we want to investigate various aspects of confinement in MQCD. It will be relatively straightforward, using the results in [30], to introduce heavy quarks into the theory and study the topological objects corresponding to mesons and baryons. We obtain a picture which is consistent with the standard lore of confinement in ordinary QCD. In addition we consider the possible existence of multiple stable QCD strings. In principle, QCD flux tubes can carry between 1 and $N - 1$ flux units; we will refer to a string with $k$ units of flux as a “$k$-string”. A $k$-string could be important in the dynamics of a meson built from a quark in the $k$-index antisymmetric tensor representation and a corresponding antiquark. But it is a dynamical question as to whether the $k$-string is stable against decay to $k-1$-strings. We will show that in $\mathcal{N} = 1$ MQCD (and also non-supersymmetric MQCD) the $k$-strings are all stable.

By contrast, in the $\mathcal{N} = 2$ gauge theory softly broken to $\mathcal{N} = 1$, the physics is quite different. Using the explicit solution for the $\mathcal{N} = 2$ theory [32,33,34,35,36], Douglas and Shenker [37] found an exotic spectrum in which quarks in the fundamental representation form $\lfloor (N + 1)/2 \rfloor$ distinguishable mesons. The Weyl group is broken in this theory and different color components of the quarks are bound by different strings. However, as is implicit in [37], the $N - 1$ strings found there are nothing more than a set of stable $k$-strings. We show that MQCD reproduces these result, with string tensions which agree with those found in field theory. Also, as we will show, MQCD provides a convenient picture for the transition from the softly broken $\mathcal{N} = 2$ physics to the more conventional $\mathcal{N} = 1$ expectations.

Moreover, we find also that all the fivebrane generalizations of QCD — the weakly broken $\mathcal{N} = 2$, the $\mathcal{N} = 1$ and even the non-supersymmetric proposal of [30] — exhibit a common universal ratio between the tensions of the $k$-strings. This ratio naturally agrees with the field theoretical prediction for the weakly broken $\mathcal{N} = 2$ theory [37]. Unfortunately, far from the $\mathcal{N} = 2$ limit, the string tensions are not protected from renormalization, and so the MQCD results are highly questionable. But it is possible that the ratios of tensions are weakly renormalized. The suggestion that the $k$-strings are stable may well be correct, and it is also possible that the quantitative MQCD result is fairly accurate for $\mathcal{N} = 1$ QCD or even for non-supersymmetric QCD. At the moment there is no data on ratios of string tensions with which to compare; one requires a lattice computation using a group larger than $SU(3)$. 

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The paper is organized as follows. After a rudimentary overview of confinement and heavy quarks in section 2, we explain in section 3 how heavy gauge bosons and quarks appear as membranes in $\mathcal{N} = 2$ MQCD. In section 4 we turn to $\mathcal{N} = 1$ MQCD and show that there are no membrane states corresponding to isolated heavy quarks. However, heavy quarks can connect to the MQCD strings identified by Witten [30], and M theory membranes corresponding to mesons and baryons can exist in the theory. The meson membranes bear some resemblance to the brane picture for confinement of monopoles in the Abelian Higgs model found in [38]. We discuss the $k$-strings of $\mathcal{N} = 1$ MQCD in section 5.

In section 6 we identify the flux tubes of Douglas and Shenker in MQCD, and show that their quantum numbers and tensions are given correctly. As the $\mathcal{N} = 2$ breaking parameter is taken to be large, the brane construction gives a nice physical picture for the transition from these flux tubes to those of $\mathcal{N} = 1$ MQCD. We discuss the process by which the order-$N$ different mesons decay to a single one during this transition. We comment in section 7 on the “almost BPS” properties of the flux tubes of [37], and in section 8 present a brane picture of monopole confinement along Higgs branches of non-abelian $\mathcal{N} = 2$ QCD. Finally, in section 9, we discuss our observation that the ratios of the MQCD $k$-string tensions are independent of the amount of supersymmetry. We analyze critically this and the other results obtained in order to clarify to what extent MQCD predictions can be trusted in ordinary QCD, and note the possibility of numerical tests. Section 10 contains a brief conclusion, and the appendices present some conventions and a computation in non-supersymmetric MQCD.

2. Introduction of Heavy Quarks in QCD

Confinement is usually studied by computing the potential between static sources (or equivalently the expectation value of a Wilson loop.) In M theory the easiest technique is to add dynamical but very heavy quarks to the theory.

We remind the reader that it is naively reasonable to talk about a QCD string in the context of heavy quarks. Imagine adding two heavy quarks, $U$ and $D$, with masses $m_U, m_D \gg \Lambda_{QCD}$, to a pure $SU(N)$ gauge theory. Consider the bound states of a $U$ quark and $\bar{D}$ antiquark. These states carry conserved flavor quantum numbers and so the ground state is stable. The lowest lying bound states are smaller than $\Lambda_{QCD}^{-1}$; they only sample a region where QCD is weakly coupled, and their spectrum is that of the
Hydrogen atom. Highly excited states of size $L \gg \Lambda_{QCD}^{-1}$ are subject to the linear QCD potential, whose slope is given by the QCD string tension $T \sim \Lambda_{QCD}^{2}$, and have mass of order $m_U + m_D + TL$. If the total energy of the string $TL$ is much greater than twice the mass of the lighter quark, then the string can break via quark pair production. However, this process is slow, as the energy of the string is spread out over a distance $L$ much greater than the Compton wavelength $m^{-1}$ of the quarks. Of course, it is also possible for any excited state to decay via the emission of glueballs. These are closed string loops of mass near $\Lambda_{QCD}$. Such a process is of order $1/N^2$ in the large $N$ limit, and can therefore be controlled.

Thus, as long as we consider states of size $L$ in the intermediate regime between weak coupling and heavy quark pair production,

$$1 \ll \Lambda_{QCD} L \ll \frac{\min(m_U, m_D)}{\Lambda_{QCD}}, \quad (2.1)$$

the meson can be modeled as a quark and antiquark joined by a QCD string.

3. Identification of Gauge Bosons and Quarks in $\mathcal{N} = 2$ Supersymmetry

In order to set the groundwork for our study of $\mathcal{N} = 1$ supersymmetric theories, in which quark states do not exist as independent entities, we first discuss the unconfined quarks and gauge bosons of $\mathcal{N} = 2$ supersymmetric theories. We omit most technical details (which are discussed extensively in the original papers [1,2]) and instead provide a light review useful (we hope) to the non-expert.

![Figure 1: $\mathcal{N} = 2 SU(2)$ gauge theory realized by stretching two D4 branes between two NS branes. The configuration is independent of spacetime; the coordinates $x^4, x^5, x^6$ are shown. The D4 branes are located at $v = x_4 + ix_5 = \pm \phi$; the string marked $W$ is a charged vector multiplet of mass $2\phi$. The semi-infinite D4 brane located at $v = m$ introduces a hypermultiplet $Q$ of bare mass $m$.](image-url)
A simple model studied in Ref. [39] is $SU(2)$ gauge theory with a single hypermultiplet in the doublet representation. In Type IIA string theory, the representation of the classical theory is given in figure 1. The fields on the world-volume of two Dirichelet fourbranes (D4 branes) make up a $U(2)$ gauge theory on five-dimensional Minkowski space $M^5$ [10]. We will take the coordinates of this theory to be $x^0, x^1, x^2, x^3, x^6$. When the two D4 branes are stretched between two Neveu-Schwarz fivebranes (NS branes), which fill coordinates $x^0, x^1, x^2, x^3, x^4, x^5$, this theory is constrained to exist on $M^4 \times I$, where $I$ is an interval of length $\Delta x^6$. The NS branes both cut off the volume of the gauge theory in the $x^6$ direction, making it four-dimensional in the infrared, and reduce the supersymmetry to the equivalent of $\mathcal{N} = 2$ in four dimensions. The gauge coupling of the effective four-dimensional theory on the D4 branes is $1/g^2 = \Delta x^6/g_s \ell_s$, where $g_s$ and $\ell_s$ are the Type IIA string coupling and length. The D4 branes are free to move in the two dimensions of the NS branes which are perpendicular to the D4 branes. These dimensions, $x^4$ and $x^5$, can be combined into the holomorphic coordinate $v = x^4 + ix^5$. The distance $\delta v$ between the two D4 branes is proportional to the expectation value $\phi$ of the scalar in the $\mathcal{N} = 2$ vector multiplet of the $SU(2)$ gauge theory; recall this scalar has a single complex eigenvalue. As explained in [2], the requirement of having finite energy configuration on the NS branes imposes that the sum of the positions in $v$ of all the D4 branes is zero. As a consequence, the $U(1)$ subgroup of $U(2)$ is non-dynamical.

The four gauge bosons (and their scalar partners) of the $U(2)$ theory consist of Type IIA strings which stretch between two (not necessarily different) D4 branes. In particular, the $W$ bosons of the $SU(2)$ subgroup, whose masses are classically proportional to $2\phi$, are the IIA strings of length $\delta v$ which stretch from the first D4 brane to the second. $W^+$ bosons are strings of one orientation, $W^-$ bosons have the opposite orientation.

A hypermultiplet in the doublet of $SU(2)$ may be added to the theory by attaching a semi-infinite D4 brane to the right of the righthand NS brane. (We will refer to finite D4 branes as color branes, since they carry color quantum numbers, and to the semi-infinite D4 branes as flavor branes for the analogous reason.) The position of the flavor brane in the $v$ plane is the bare mass for the hypermultiplet. In the classical theory, in which $SU(2)$ is broken to $U(1)$ by non-zero $\phi$, the quark has two color states, of charges $\pm \frac{1}{2}$ and masses $m \pm \phi$. Since the separations between the flavor brane and the two color branes are precisely $m \pm \phi$, it is natural that a quark is a IIA string lying in the NS brane and stretching from the flavor brane to one of the color branes. The string of opposite orientation is the antiquark.
If the Type IIA string coupling is large, the physics of the branes will be given in terms of the semiclassical limit of the eleven-dimensional description known as M theory. This theory exists on $M^{10} \times S^1$, where the radius $R$ of the circle in the tenth spatial direction $x^{10}$ grows with the IIA string coupling. M theory has two-dimensional membranes and the corresponding electromagnetic dual objects, fivebranes. Just as IIA strings can end on D-branes \[40\], M theory membranes can end on M theory fivebranes \[41\], with the intersection being a closed curve inside the fivebrane. The IIA string in M theory language is a membrane wrapped \textit{once} around the compact $x^{10}$ direction. The NS brane is an M theory fivebrane, while the D4 branes of the IIA string theory are M theory fivebranes wrapped \textit{once} around the compact $x^{10}$ direction.

Thus, the NS and D4 branes of the Type IIA construction outlined above are made from the same type of object, and it is therefore natural that in M theory the singular intersections between them would be smoothed out. As shown in \[2\] the construction of figure 1, which is shown embedded in the space $R^3$ made from $v$ and $x^6$, becomes a continuous six-dimensional surface filling the \textit{eight}-dimensional space consisting of spacetime $M^4$ and the coordinates $v = x^4 + i x^5$ and $t = e^{(x^6 + i x^{10})/R}$. Since the construction is translationally invariant in space-time, the six-dimensional surface factors into $M^4 \times \Sigma$, where $\Sigma$ is a two-dimensional Riemann surface embedded in the flat $v, t$ space and specified by a single complex equation in $v$ and $t$. This Riemann surface is equivalent to the Seiberg-Witten torus \[32,39\] which specifies the gauge coupling of the low-energy effective $U(1)$ gauge theory. The embedding of the surface determines the Seiberg-Witten one-form from which the masses of BPS states may be determined.

For the theory in figure 1, the Riemann surface is given by \[39\]
\begin{equation}
(1 - \frac{v}{m})t + \Lambda_2^{-2}(v^2 - \phi^2) + 1/t = 0
\end{equation}

Note that the gauge coupling has disappeared and been replaced by $\Lambda_2$. We show two renderings of the surface $\Sigma$ \[2\] in figures 2 and 3. The first rendering shows the embedding of the surface in the $v, |t|$ space. Although we cannot draw four dimensions, we note that the surface $\Sigma$ wraps around the compact direction $\arg t = x^{10}/R$. We indicate with two dark lines on $\Sigma$ the points at which $\Sigma$ intersects $x^{10} = 0$; a curve which travels on $\Sigma$ from one dark line to the next wraps once around $x^{10}$. Note that the picture (drawn for $\phi \gg \Lambda_2$) roughly resembles figure 1 and that, as required, each D4 brane has become an M theory fivebrane wrapping once around $x^{10}$. In figure 3, $\Sigma$ is considered as a double-sheeted cover.
Figure 2: The curve $\Sigma$ for $SU(2)$ with one flavor; compare with figure 1. The curve is wrapped around the compact $x^{10}$ direction, which is not shown. The intersection of the curve with $x^{10} = 0$ is indicated by the two curved dark lines; notice each tube corresponding to a D4 brane contains one such line, showing it wraps once around $x^{10}$.

of the $v$ plane, with singularities on both sheets near $v = \pm \phi$ and with a singularity on the top sheet at $v = m$. We show the top sheet in the figure. Here dashed lines indicate intersections of the surface with $x^{10} = 0$.

We now identify the $W$ bosons and quarks in this M theory picture. The Type IIA strings of figure 1 which stretch between D4 branes must now become membranes which wrap once around $x^{10}$ and which attach to the fivebrane along closed curves. It is clear from figure 3 that the curves $\gamma_m, \gamma_\phi, \gamma_{-\phi}$ drawn around the three singular points in $v$ are suitable for the ends of such membranes. A $W$ boson thus consists of a two-dimensional curve, with cylindrical topology, lying in $v, t$ but not in $\Sigma$, which has one boundary on the contour $\gamma_\phi$ and the other on $\gamma_{-\phi}$. Again we emphasize that the boundaries of the membrane do lie in $\Sigma$ though the bulk of the membrane does not. In fact the membrane will be the minimal area surface in $v, t$ with these boundaries, and will appear roughly as a cylinder of radius $2\pi R$ and length $2\phi$. The mass of the four-dimensional particle is proportional to
the area of the membrane, and will therefore be proportional to $2\phi$. Similar statements apply to the two quark states which connect $\gamma_m$ with one of the other two curves; the masses will be of order $m \pm \phi$.  

As a technical matter, we note that if the theory contained two heavy quarks, there would be singularities at $v = m_1$ and $v = m_2$, and in addition to the $W$ boson and quark states, one could consider an open membrane whose two boundaries wrap around these two singularities. This would correspond to a gauge boson of the flavor group. Since the flavor branes are semi-infinite, the flavor theory is actually five dimensional and the flavor gauge bosons do not couple dynamically to the four-dimensional theory. They instead couple as background gauge fields to the corresponding flavor currents.

A monopole in Type IIA string theory is a rectangular D2 brane with two boundaries on D4 branes and two on NS branes. In figure 1 the monopole fills the “hole” between the branes, like a soap-bubble. In M theory the monopole is a membrane stretched across an opening in the Riemann surface, such as the large hole in figure 2. Its mass is

\footnote{The details of dimension counting in this system are given in appendix A.}
proportional to the minimal area of the hole. When $\phi$ is tuned to a special value, the area of the hole shrinks to zero, corresponding to the point in the moduli space of the $\mathcal{N} = 2$ theory where the monopole is massless. Since when $\mathcal{N} = 2$ is broken to $\mathcal{N} = 1$ the vacua which survive are those with massless monopoles or dyons, we will need to discuss in more detail precisely how this occurs.

4. $\mathcal{N} = 1$ Supersymmetry and Confinement: Mesons, Baryons, and Strings

In this section we show that in the M theory representation of the quantum theory of $\mathcal{N} = 1$ supersymmetric gauge theory with heavy quarks, the quarks do not exist by themselves. However, a quark can join onto the MQCD string identified by Witten [30]. Consequently one can show that quark-antiquark states bound by a string do exist. One can also show that baryonic states exist. In addition one can discuss strings that carry more than one unit of flux, which are relevant for baryons and for dynamics of quarks in higher representations than the fundamental. Our results in this section follow directly from combining the discussion of the previous section with the results of [30].

![Figure 4: Brane configuration for the $\mathcal{N} = 1$ theory. It is obtained from figure 1 by rotating the leftmost NS brane from the $v = x^4 + ix^5$ plane into the $w = x^8 + ix^9$ plane. We show the case of $SU(N)$ (there are $N$ D4 branes combined in the central dark line) with two massive flavors (each given by a semi-infinite D4 brane on the right).](image)

The classical construction of the $\mathcal{N} = 1$ theory, using the Type IIA string theory, merely involves rotating the lefthand NS brane of figure 1 from the $v = x^4 + ix^5$ plane into the $w = x^8 + ix^9$ plane [3]. This is indicated in figure 4. (The rotated brane will be referred to as the NS’ brane.) The rotation makes it impossible for the color branes to move apart, corresponding to the absence of an adjoint scalar in the $\mathcal{N} = 1$ theory. The flavor branes can still be placed anywhere in $v$. Classically, the quarks and gauge bosons are constructed as Type IIA strings just as in figure 1.
When the $\mathcal{N} = 1$ quantum theory is studied using M theory, the physics is again described in terms of a Riemann surface $\Sigma$, which is now embedded in the flat six-dimensional space $v, w, t$, and is specified by two complex equations in these coordinates. The curve for the pure MQCD theory with gauge group $SU(N)$ is given by the equations

$$vw = \zeta; \quad v^N = \zeta^{N/2}t; \quad (4.1)$$

this has also been shown in [30,43,44]. The constant $\zeta$ essentially determines the MQCD scale $\Lambda_1$. We will return to the exact relation in the following. The addition of two flavor branes at $v = m_1$ and $v = m_2$, where $m_1, m_2 \gg \Lambda_1$, modifies the curve to [43,44]

$$vw = \zeta; \quad v^N = \zeta^{N/2}\left(1 - \frac{v}{m_1}\right)\left(1 - \frac{v}{m_2}\right)t. \quad (4.2)$$

![Figure 5: Singularities of $\Sigma$ in the $v$ plane for $SU(6)$ with two heavy quarks. The points $m_i$ correspond to positions of the flavor branes, while $v = 0$ corresponds to the position of the 6 color branes. Dashed lines correspond to the intersection of $\Sigma$ with $x^{10} = 0$. The contour $\gamma_0$ wraps 6 times around $x^{10}$, while $\gamma_{1,2}$ wrap only once.](image)

It is difficult to represent this curve in its entirety because of its embedding in six dimensions, but we may still consider $w$ and $t$ (now single-sheeted) as a function of $v$. As shown in figure 5 there are three singularities, one at $v = 0$ (which corresponds to the NS' brane) and one each at $v = m_1$ and $v = m_2$. Note that although the curves $\gamma_1$ and $\gamma_2$ wrap once around $x^{10}$, the curve $\gamma_0$ wraps $N$ times around $x^{10}$. 

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Can we construct gauge bosons or quarks? As gauge bosons are expected to be light, they need not be easy to see. However, the heavy quarks have such large masses and such a small effect on the dynamics that we should be able to construct them. Specifically, we expect to find states carrying one unit of a flavor quantum number, with mass of order $m_1$. In analogy with the $\mathcal{N} = 2$ case above, we should construct such a quark by taking a membrane with one boundary on $\gamma_1$ and closing it on a curve which wraps once in $x_{10}$. However, the only other curve of this type is $\gamma_2$, which carries a flavor quantum number. Therefore quark states with one unit of flavor do not exist in the quantum theory.

However, a membrane with a boundary on $\gamma_1$ can end on a MQCD string. To see this, consider in detail a finite two-dimensional surface given by $\{v, w, t\}(\sigma, \tau)$ (for $0 \leq \sigma, \tau \leq 1$) with the properties that at each $\tau$ it wraps once around $x_{10}$

$$v(\sigma + 1, \tau) = v(\sigma, \tau) ; w(\sigma + 1, \tau) = w(\sigma, \tau) ; t(\sigma + 1, \tau) = e^{2\pi i t(\sigma, \tau)}, \quad (4.3)$$

and that at $\tau = 0$ it intersects the curve $\gamma_1$

$$v(\sigma, 0) = \gamma_1(\sigma) = \frac{\zeta}{w(\sigma, 0)} ; t(\sigma, 0) = e^{2\pi i \sigma t(0, 0)}. \quad (4.4)$$

\textbf{Figure 6:} $\Sigma$ (the diagonal lines) is pictured, for fixed values of $|t|$ and $|v|$, in the $(\arg(v), x_{10})$ plane. A rotation of $2\pi/N$ in $v$ corresponds to shifting $\Sigma$ once around $x_{10}$. The curve $C_t$, which is closed and passes through point A, is homotopic to $C_v$, which connects points A and B on $\Sigma$. The length of $C_t$ is proportional to $R$, while $C_v$ has length of order $\sqrt{\zeta}$. 

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We may deform this curve (by moving the membrane smoothly inside of the \(v, w, t\) space) so that
\[
v(\sigma, 1) = \sqrt{\zeta t_0^{1/N}} = \frac{\zeta}{w(\sigma, 1)} ; \quad t(\sigma, 1) = e^{2\pi i \sigma} t_0. \tag{4.5}
\]
Then the curve \(C_t = \{v, w, t\}(\sigma, 1)\) intersects \(\Sigma\) (at a unique point A) at \(\sigma = 0\) and at \(\sigma = 1\); see figure 6 in which the intersection of \(\Sigma\) with \(|v| = |t_0^{1/N}|\) is shown along with \(C_t\). But as noted in [30], this curve is homotopic to the curve \(C_v\) as is obvious from the figure. We emphasize that \(C_v\) intersects \(\Sigma\) only at its endpoints and does not represent the end of a membrane. Instead, this homotopic transformation represents the opening of a hole in the membrane world-volume; the curve \(C_t\) is closed in \(v, w, t\) but \(C_v\) is not.

The curves \(C_v\) and \(C_t\) have quite different physical implications. A closed membrane wrapping once around the eleventh dimension, like that of \((4.3)\), is identified in double dimensional reduction with the elementary Type IIA string, whose tension goes like \(R\) in M theory units. This string can exist anywhere in ten-dimensional spacetime. The curve \(C_v\), by contrast, has length \(\sqrt{\zeta}\), which is related to the MQCD scale, and, having boundary on \(\Sigma\), gives an open membrane which must be localized around the fivebrane. According to Witten [30], the curve \(C_v\), when extended into a membrane by dragging it along a curve \(C\) in space, represents a MQCD string lying on the curve \(C\).

The tension of the MQCD string is proportional to the length of \(C_v\), which is a straight line in \(v, w, t\) space connecting the points \([30]\)
\[
A = (\sqrt{\zeta t_0^{1/N}}, \sqrt{\zeta t_0^{-1/N}}, t_0) \quad \text{and} \quad B = (\sqrt{\zeta t_0^{1/N} e^{-2\pi i/N}}, \sqrt{\zeta t_0^{-1/N} e^{2\pi i/N}}, t_0) \tag{4.6}
\]
Its length is
\[
\sqrt{|\Delta v|^2 + |\Delta w|^2} = 2\sqrt{\zeta} \sqrt{t_0^{2/N} + t_0^{-2/N}} \sin(\pi/N). \tag{4.7}
\]
To get the MQCD string tension we should further minimize \((4.7)\) with respect to \(t_0\). Since \(\Sigma\) has a symmetry (for very heavy quarks) under \(t \leftrightarrow 1/t\) which exchanges \(w\) and \(v\) — the reflection symmetry that exchanges the two NS branes — the minimum will be at \(t_0 = 1\), giving length
\[
2\sqrt{2\zeta} \sin(\pi/N) \tag{4.8}
\]
Multiplying by the membrane tension \((1\) in these units\) we get the MQCD string tension. If we want to match on to \(\mathcal{N} = 1\) QCD field theory expectation, where the string tension

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\[2\] In this expression we assume \(\sqrt{\zeta t_0^{1/N}} \ll m_i\) and approximate \(\Sigma\) by its form \((1.1)\).
should be of order $\Lambda^2$, we must take $\zeta \sim \Lambda^4$ \[30\]. For the MQCD string to be stable against decay to Type IIA strings, it must be that its tension $\sim \sqrt{\zeta}$ is much less than $R$ \[30\].

Thus, a quark can connect to a MQCD string, and we can form a meson by connecting the string to an antiquark at the other end. A meson $Q(x) - \bar{Q}(y)$ is thus given by a single membrane with three boundaries on $\Sigma$, one on $\gamma_1$ at the point $x$, another on $\gamma_2$ at the point $y$, and a third which opens up at $x$ during the transition from $C_t$ to $C_v$, stretches along the MQCD string $C$, and closes again via the reverse homotopy at $y$ (figure 7). This picture is quite similar to that given for abelian confinement by Greene, Morrison and Vafa \[38\] though it has significant differences also. The mass of the meson will be roughly given by the sum of the quark masses and the tension of the MQCD string times its length, in agreement with expectations, as long as the meson is neither too long nor too short. For short strings a Coulomb potential, rather than a linear potential, applies between the quarks, but this effect is not visible in the semiclassical membrane picture of a meson. Long strings can break via heavy quark pair production; this process can easily be seen in

\[3\] Actually the minimal tension is given not by $C_v$ but by the nearby line which intersects $\Sigma$ at right angles; the difference in lengths is very small, of order $\sqrt{\zeta}/R$, and can be ignored until section 9.
Figure 8: The homotopy transformation by which 5 quarks form a baryon of $SU(5)$. The notation is as in figure 6, except that here we emphasize the periodicity in arg $v$. The vertical curves $C_t(i)$ can be homotopically deformed to the horizontal curves $C_v(i)$, which can be joined together in a closed loop that can be detached from $\Sigma$.

the membrane picture, though we omit any further discussion here.

Next, we construct a baryon from $N$ massive quarks. It is convenient to add many flavor D4 branes to the theory, each one giving a singularity in $v$ around which $x^{10}$ winds once. We can attach $N$ membranes to $N$ contours surrounding these singularities, and bring them toward the origin in $v$. Following [30], and as shown in figure 8, the quarks can be brought to the curves $C_t(i)$ and then homotopically deformed to the curves $C_v(i)$; the latter can be joined together into a closed loop that can then be pulled off of the surface $\Sigma$, following which it can be shrunk to a point in the $v, w, t$ space. This means that a baryon consists of a single membrane with $N$ boundaries, each wrapping (with the same orientation) around one of the singularities at $v = m_i$, along with a single additional boundary running along the $N$ strings and ending at the vertex which joins them together.

As a final comment, we note that when some number $N_f < N_c$ quarks are taken to be light compared with the MQCD scale, the situation is topologically the same but requires physical reinterpretation. (A different phase structure emerges for $N_f \geq N_c$.) The $N_f$ light squarks $q_r$ acquire expectation values, breaking $SU(N)$ to $SU(N - N_f)$; their components are eaten by gauge bosons, except for $N_f^2$ light singlets. The heavy quarks $Q^i$ split into components which are charged under $SU(N - N_f)$ and components which are not. It is easy to see that the mesonic membranes connecting two light quarks make up the

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4 It naively appears that the mesons we have identified can decay to membranes stretching directly from $\gamma_1$ to $\gamma_2$, namely to gauge bosons of the flavor group. We have explained in section 3 why the flavor gauge bosons do not couple dynamically to the four dimensional theory.
$N_f^2$ light singlets, those connecting a light and a heavy quark are the neutral components of the heavy quarks, while those connecting two heavy quarks are quark-antiquark mesons confined by the $SU(N - N_f)$ interaction. This conforms with field theory expectations.

5. Strings with Multiple Units of Flux

One expects, in general, as many as $N - 1$ stable $k$-strings in ordinary QCD, characterized by their quantum number $k$ under the center of the gauge group, or equivalently by the number of units of flux $k$ which they carry (mod $N$). The tension as a function of $k$ is both periodic in $N$ and symmetric under $k \sim N - k$, so the strings can have as many as $[N/2]$ different tensions. The stability of the $k$-strings with $k > 1$ is of physical interest. For example, a quark in the antisymmetric representation may either connect to two 1-strings or to a single 2-string. Since the two choices are not distinguished by a quantum number, the preferred configuration is determined dynamically.

![Figure 9](image_url)

**Figure 9:** The curve $\Sigma$ is shown for $SU(6)$, with the circle $|v| = \sqrt{\zeta}$ highlighted. Intersections of $\Sigma$ with $x^{10} = 0$ are shown with dashed lines. The line $AB$, which lies outside $\Sigma$ except at its endpoints and is at a fixed value of $x^{10}$, corresponds to a string with one unit of flux. It is clear that the line $AC$ (a string with two units of flux) is energetically preferred to the line $ABC$ (two strings with one unit of flux each.)

In MQCD, the 2-string is preferred. The tension of a $k$-string is given by considering the minimal tension of a string which can connect to $k$ quarks. A membrane connected to $k$ quarks wraps $k$ times around $x^{10}$. Using the homotopy transformation of figure 6, such a membrane can be rotated into $k$ MQCD strings of one unit of flux, each of which connects two points $v_0$ and $v_0 e^{2\pi i/N}$ on $\Sigma$, or by a single $k$-string given by connecting two points $v_0$
and \( v_0 e^{2\pi ik/N} \) (figure 9). It is obvious that the latter possibility leads to the shorter curve, whose length, as in (4.6)-(4.8), is given by

\[
2\sqrt{2\zeta} \sin \frac{\pi k}{N},
\]

(5.1)

so the \( k \)-string tensions are proportional to \( \sin \frac{\pi k}{N} \). The importance of this formula and its relevance for ordinary QCD will be discussed in the coming sections.

6. Breaking \( \mathcal{N} = 2 \) Supersymmetry to \( \mathcal{N} = 1 \)

The images of strings, mesons and baryons from the previous sections are in contrast to those which emerge when pure \( \mathcal{N} = 2 \) SU(\( N \)) Yang-Mills theory is weakly broken, at least for \( N > 2 \). In this section we review the results of Douglas and Shenker [37], who analyzed this breaking in detail, generalizing the approach of [32]. We show that M theory reproduces these results, and discuss the picture that it suggests for the transition from the physics of weakly broken \( \mathcal{N} = 2 \) gauge theory to that of pure \( \mathcal{N} = 1 \) gauge theory.

6.1. Brief Review of Weakly Broken Pure \( \mathcal{N} = 2 \) Gauge Theory

The \( \mathcal{N} = 2 \) vector multiplet consists of an \( \mathcal{N} = 1 \) vector multiplet along with a chiral multiplet \( \phi \) in the adjoint representation. With addition of a mass term \( W = \mu u \), where \( u = \frac{1}{2} \text{tr} \phi^2 \), \( \mathcal{N} = 2 \) supersymmetry is broken to \( \mathcal{N} = 1 \).

In the quantum \( \mathcal{N} = 2 \) SU(\( N \)) gauge theory [32], the low energy effective theory on the Coulomb branch has gauge symmetry \( U(1)^{N-1} \). The elliptic curve for this theory has been studied by various authors [33,34,35,36], and, as discussed for SU(2) in section 3, can be identified as part of the world-volume of a fivebrane [2]. The curve \( \Sigma \) is given by

\[
t + \Lambda_2^{-2N} P_N(v) + 1/t = 0,
\]

(6.1)

where \( P_N(v) \) is a polynomial of order \( N \) and \( \Lambda_2 \) is the dynamical scale of the \( \mathcal{N} = 2 \) theory. The curve has the manifest symmetry \( t \leftrightarrow 1/t \).

The theory has \( N \) special vacua at which \( N - 1 \) mutually local monopoles or dyons become massless [32]. Only these vacua survive when \( \mathcal{N} = 2 \) supersymmetry is broken. These vacua are related by a symmetry, so without loss of generality we limit ourselves to the one with monopole states.
In the monopole variables, the mechanism of $\mathcal{N} = 2$ breaking very closely resembles the addition of Fayet-Iliopolous terms to $N - 1$ decoupled $\mathcal{N} = 2$ supersymmetric Abelian Higgs models. One can choose a basis for the $N - 1$ $U(1)$ factors in which each monopole is charged under only one magnetic photon \[33,34,37\]; that is, their magnetic charges in this basis are $(1,0,0,\cdots,0), (0,1,0,\cdots,0), (0,0,1,\cdots,0), \ldots, (0,0,0,\cdots,1)$. The superpotential for the monopoles is then
\[
W = \sum_{p=1}^{N-1} \sqrt{2} a_D^{(p)} M_p \tilde{M}_p + \mu u(a_D)
\]
where $a_D^{(p)}$ is the scalar in the vector multiplet of the $p$-th $U(1)$ factor, $(M_p, \tilde{M}_p)$ is the $p$-th monopole hypermultiplet, and
\[
u(a_D) = b \Lambda_2^2 - \sum_j c_j \Lambda_2 a_D^{(j)} + \mathcal{O}(a_D^{(i)} a_D^{(j)}) ,
\]
where $b, c_j$ are constants determined by the solution of the theory \[33,34,37\]. The linear term in $a_D^{(j)}$ would be an $(\mathcal{N} = 2)$-preserving Fayet-Iliopolous term for the $j$-th $U(1)$ factor, and the analysis would be identical to that of the Abelian Higgs model, were it not for the higher order terms in $u(a_D)$ which break $\mathcal{N} = 2$ supersymmetry. The potential energy is minimized for $\langle M_p \tilde{M}_p \rangle = c_p \mu \Lambda_2$ and $a_D^{(p)} = 0$. In each $U(1)$ factor, the non-zero monopole expectation value breaks the gauge symmetry and permits a Nielsen-Olesen string solution to the classical equations \[45\]. One finds \[37\] that the $N - 1$ strings have string tensions
\[
T(p) = 2\pi |\langle M_p \tilde{M}_p \rangle| = 4\sqrt{2}\pi |\mu \Lambda_2| \sin \frac{\pi p}{N}
\]
where $p$ runs from 1 to $N - 1$. Note the symmetry under $p \leftrightarrow N - p$. The calculation is reliable for small $\mu$ since the monopole Lagrangian is weakly coupled. (The failure of these strings to be BPS saturated will be discussed in section 7.)

To understand how these strings manifest themselves physically, it is essential to note the following. Since monopoles have condensed, the flux stemming from electrically charged states must be confined into strings. In the basis mentioned above, in which the monopole $U(1)^{N-1}$ magnetic charges are simple, a heavy quark in the fundamental representation has $N$ color states $Q^1, Q^2, Q^3, \cdots Q^N$, with electric charges $(1,0,0,\cdots,0), (-1,1,0,\cdots,0), (0,-1,1,\cdots,0), \ldots, (0,0,0,\cdots,1)$. From this it is easy to see that the strings of Douglas and Shenker are indeed distinguished by the amount of flux that they carry. The state $(Q^1 Q^2 Q^3 \cdots Q^p)$ has charge $p$ under the center of the group and must couple to strings
carrying a total of $p$ units of flux. Since this state is charged under the $p$-th $U(1)$ factor and neutral under the others, it couples only to the string labelled $p$ above, and so the $p$-th string is indeed a $p$-string.

In addition, the charges of the state $Q^p$, which carries one unit of flux, are such that it must attach to two strings, a $p$-string and a $(p - 1)$-antistring. Since this is the case, we are led to the surprising conclusion that there are actually many types of heavy quark mesons \[37\]. A (highly excited) $Q^p - \tilde{Q}_p$ state will be bound by a string of tension $T(p)$ and an antistring of tension $T(p - 1)$, so there are, in all, $\lfloor (N + 1)/2 \rfloor$ distinguishable sets of meson states made from a quark and antiquark in the fundamental representation.

\[\text{Figure 10: Standard picture for baryons in } SU(6): \text{ six quarks each with a 1-string connected by a six-string vertex.}\]

\[\text{Figure 11: Expectation for a baryon in weakly broken } \mathcal{N} = 2 \text{ } SU(6): \text{ the } k\text{-th quark is connected to a } (k - 1)\text{-string on the left and a } k\text{-string on the right.}\]

These features also affect the baryons. A common picture for a baryon in QCD is that of figure 10. Here, the expectation would more naturally be that of figure 11 \[37\].

Note that a sufficiently excited state $Q^p - \tilde{Q}_p$ is only metastable, due to pair production of $W$ bosons \[37\]. If an excited state with length $L$ is long enough that the quantity $[T(p) + T(p - 1) - T(1)]L$ is greater than twice the mass of the $W$ boson $[W_\alpha]_p^1$, then the state $(Q^p[W_\alpha]_p^1) - ([W_\alpha]_p^1\tilde{Q}_p)$ will have lower energy (for the same angular momentum) than the $Q^p - \tilde{Q}_p$ state. However, one can always find a range for $L$ in which the $Q^p - \tilde{Q}_p$ states are metastable and can be observed.

This theory differs significantly from the expectation for $\mathcal{N} = 1$ or non-supersymmetric QCD, where only one type of meson is anticipated. The key point \[37\] is that in weakly
broken $\mathcal{N} = 2$ gauge theory, the Weyl group of $SU(N)$ is spontaneously broken by the expectation value of the field $\phi$. This makes it possible for the $Q^p - \tilde{Q}_p$ bound state spectra to depend on $p$. The absence of scalar fields in $\mathcal{N} = 1$ and in non-supersymmetric QCD makes it plausible that the Weyl group is unbroken in these theories.

However, the $k$-strings of weakly broken $\mathcal{N} = 2$ QCD and those of $\mathcal{N} = 1$ MQCD share an important property: although their tensions (6.4) and (5.1) have different overall normalizations, they both satisfy the formula

$$\frac{T(k)}{T(k')} = \frac{\sin \frac{\pi k}{N}}{\sin \frac{\pi k'}{N}}.$$  

(We will see in section 9 and appendix B that this formula even applies for nonsupersymmetric MQCD.) Still, the dynamics of the Douglas-Shenker and $\mathcal{N} = 1$ MQCD strings are somewhat different, as the MQCD picture makes clear. We should also explore the physics behind the disappearance of the $[(N+1)/2]$ mesons in favor of the single meson of $\mathcal{N} = 1$ MQCD. In the remainder of this section we will discuss the transition from small to large $\mu$ in detail.

### 6.2. The M Theory Fivebrane for Broken $\mathcal{N} = 2$ Gauge Theory

We now examine the fivebrane theory as a function of $\mu$. As a starting point, we identify the properties of the fivebrane describing the $\mathcal{N} = 2$ theory as we approach the vacuum with massless monopoles. In this limit the areas of the holes in the genus $N - 1$ Riemann surface simultaneously shrink to zero, as seen in figure 12; the monopoles become massless, and the curve $\Sigma$ degenerates to a genus zero surface.

The equation for the degenerated surface is

$$t = e^{iN\sigma}; \quad 2\pi Rv = \Lambda_2 \left( t^{1/2} + t^{-1/2} \right) = 2\Lambda_2 \cos \sigma \tag{6.6}$$

where $\sigma$ is complex. The two halves of $\Sigma$ join where $\sigma$ is real, on the line shown in figure 13. The $N$ line segments touch at $t = \pm 1$, that is, at $v = 2(\Lambda_2/2\pi R)\cos(\pi k/N)$. The $N - 1$ massless monopoles are localized at these points on the fivebrane. Note that the $W$ bosons (given by membranes wrapping around two different line segments) and heavy quarks (given by membranes wrapping at one end around one line segment and at the other

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5 Here we need to be more careful with units. We use those appropriate for M theory, and put $l_P = 1$. Additional details are given in appendix A.

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Figure 12: The curve $\Sigma$ for $\mathcal{N} = 2\ SU(6)$ gauge theory near the massless monopole point. The masses of the monopoles, proportional to the area of the holes in the surface, are all becoming small.

Figure 13: The curve $\Sigma$, projected into the $v$ plane, for $SU(6)$ at the massless monopole point. The two halves of $\Sigma$ meet at $|t| = 1$ along a line. The line segments are the colored D4 branes. Massless monopoles are localized at the positions of the filled circles; the open circles are ends of D4 branes where there are no monopoles. We indicate $t$ real and positive (negative) using dashed (dotted) lines. In going between two dashed lines or two dotted lines, $\Sigma$ wraps once around $x^{10}$.

around a flavor brane singularity at $v = m$) remain massive despite the degeneration. The
presence of $2\pi R$ in equation (6.6) is simply explained by checking the mass for a $W$ boson. The $W$ boson is a membrane wrapped around $x$ and with a length along the line of order $\Lambda^2/2\pi R$. Its mass is therefore proportional to $\Lambda^2$, as expected.

What happens when the $\mathcal{N} = 2$ theory is broken by the mass $\mu$ for the adjoint chiral superfield $\phi$? The NS branes are rotated relative to one another [6,8], and the surface becomes

$$2\pi Rv = \Lambda_2(t^{\frac{1}{N}} + t^{-\frac{1}{N}}), \quad 2\pi Rw = \alpha \Lambda_2 t^{\frac{1}{N}}$$

(6.7)

We can roughly identify $|\alpha|$ with the tangent of the rotation angle, and $\mu$ with $\alpha/2\pi R$. We are not careful with overall normalizations. For $\alpha = 0$ this clearly reduces to the the $\mathcal{N} = 2$ curve since $w$ goes to zero.

For any value of $\mu$ the surface is still symmetric under the reflection which exchanges the two NS branes (the regions where $v$ or $w$ go to infinity). The symmetric point where we expect the surface to have minimal size is

$$|t|^{1/N} = t_0^{1/N} = \left| \frac{1}{(1 + |\mu R|^2)^{1/4}} \right| .$$

(6.8)

The intersection of $\Sigma$ with this hyperplane is therefore the curve

$$t = t_0 e^{2\pi i N \sigma}, \quad v = \frac{\Lambda_2}{2\pi R} \left( (1 + |\mu R|^2)^{1/4} e^{-2\pi i \sigma} + \frac{e^{2\pi i \sigma}}{(1 + |\mu R|^2)^{1/4}} \right); \quad w = \frac{\Lambda_2 \mu e^{2\pi i \sigma}}{(1 + |\mu R|^2)^{1/4}} e^{2\pi i \sigma}$$

(6.9)

which describes an ellipse in the $v$ plane with semi-axes

$$\frac{\Lambda_2}{2\pi R} ((1 + |\mu R|^2)^{1/4} \pm (1 + |\mu R|^2)^{-1/4})$$

(6.10)

and a circle in the $w$ plane with radius $\frac{\Lambda_2 \mu}{(1 + |\mu R|^2)^{1/4}}$.

In the $\mathcal{N} = 2$ limit $\alpha \to 0$, the circle in the $w$ plane, which has radius proportional to $\alpha$, shrinks to zero size while the ellipse in the $v$ plane shrinks to a line, reproducing equation (6.6).

In the $\mathcal{N} = 1$ limit $\alpha \to \infty$, the ellipse (6.9) becomes a circle, as in figure 9. To see this requires a rescaling of variables [13]. The curve (6.7) has a smooth limit provided that we rescale $t$ as follows:

$$t^{1/N} = \frac{\tilde{t}^{1/N}}{\sqrt{\alpha}} .$$

(6.11)
Matching the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) QCD scales \( \Lambda_1 \) and \( \Lambda_2 \), using
\[
\Lambda_1^{3N} = \mu^N \Lambda_2^{2N},
\]
we find in the limit the \( \mathcal{N} = 1 \) curve
\[
\sqrt{2\pi R v} = \Lambda_1^{3/2} \tilde{t}^{-1/N}, \quad \sqrt{2\pi R w} = \Lambda_1^{3/2} \tilde{t}^{1/N}.
\]

In the notations of section 4, \( \zeta = \Lambda_1^4 / 2\pi R \). As shown in [30] and reviewed in appendix A, we must fix \( R \sim 1/\Lambda_1 \), in order that \( \zeta \sim \Lambda_1^2 \) and the string tension, from equation (4.8), be proportional to \( \Lambda_1^2 \).

![Figure 14: The ellipse (6.9) in \( \Sigma \) for the weakly broken \( \mathcal{N} = 2 \) SU(6) theory. The dashed (dotted) lines correspond to the intersection of \( \Sigma \) with \( \arg t = 0 \) (\( \arg t = \pi \)).](image)

Figure 14 focuses on the relevant region of \( \Sigma \) for the weakly broken \( \mathcal{N} = 2 \) theory; the ellipse is the curve (6.9). The dashed (dotted) lines correspond to the intersection with \( \arg t = 0 \) (\( \arg t = \pi \)). Traveling from one dashed line to the next inside \( \Sigma \) involves wrapping once around \( x^{10} \). The points \( A_k \) and \( B_k \) are important in our later discussion. They are given by \( t = (-1)^k \). Notice that as \( \mu \to 0 \) they intersect at the double branch points where the massless monopoles were localized for \( \mu = 0 \).

\[6.3. \text{Physics of the Transition from } \mathcal{N} = 2 \text{ to } \mathcal{N} = 1 \text{ Gauge Theory}\]

We will now discuss how strings and mesons behave for large and small \( \mu/\Lambda_2 \), using MQCD as a guide. In particular we will see how the \( \lfloor (N+1)/2 \rfloor \) mesons of weakly broken
$\mathcal{N} = 2$ go over to a single stable meson in the $\mathcal{N} = 1$ limit. We will also study the modification of the $N − 1$ strings during the transition, and explain why (6.3) holds for all values of $\mu$.

First, we confirm that confinement of quarks does indeed occur when $\mu \neq 0$. As in the pure $\mathcal{N} = 1$ case, this follows from the fact that when $\mu$ is non-zero there are no closed curves in $\Sigma$ (except for those corresponding to flavor branes) which wrap once around $x^{10}$. From the equation (6.7) for $\Sigma$, it can be seen that any closed curve in $v$ with this property cannot close in the variable $w$ for $\mu \neq 0$. We conclude there are no heavy quarks.

However, Nielsen-Olesen strings do exist, and for small $\mu$ their properties agree with the results of Douglas and Shenker, as we will now show. As discussed in section 5, a string carrying $k$ units of flux (a $k$-string) in pure $\mathcal{N} = 1$ MQCD is specified by a finite real curve in $v, w, t$ with the following properties. It should not wrap around $x^{10}$, so that its length is much less than $R$; its endpoints should lie on $\Sigma$, so that it gives an open membrane; and any curve inside $\Sigma$ joining its endpoints should wrap $k$ times around $x^{10}$, so that this string can be attached to a state containing $k$ quarks. Loosely, in the language of figure 14, between the two endpoints there must lie $k$ dashed lines in $\Sigma$.

For the weakly broken $\mathcal{N} = 2$ theory, we must determine the curves of minimal length which satisfy these conditions. By the symmetry which exchanges the two NS branes, the endpoints of these curves must lie on the ellipse (6.9). For $k = 1$, we need two points on (6.9) which are separated by one wrapping in $x^{10}$. It is clear from figure 14 that the straight line connecting $A_1$ and $B_1$ is the shortest line available. (The line connecting $A_{N−1}$ to $B_{N−1}$ would also do.) For $\mu$ small the major axis of the ellipse is of order $\Lambda_2/R$ and the minor axis of order $\Lambda_2 \mu^2 R$, while the radius of the $w$ circle is of order $\Lambda_2 \mu$. The length of the straight line between $A_1$ and $B_1$, $\sqrt{|\Delta v|^2 + |\Delta w|^2}$, is therefore proportional to $\mu \Lambda_2 \sin(\pi/N)$ for small $\mu$.

For $k = 2$ it is clear that the shortest distance between points separated by two wrappings of $x^{10}$ is given by connecting $A_2$ and $B_2$; the length of the straight line between them is proportional to $\mu \Lambda_2 \sin(2\pi/N)$. In general the line which gives $k$ units of flux connects $A_k$ with $B_k$ and has length $\sim \sin(\pi k/N)$. The tensions of the $k$-strings thus agree with the field theory result (6.4).

Notice that the $k$-string is pinned near the point where one of the massless monopoles is localized; any attempt to move the 1-string around the ellipse, so that it connected, say, the points $B_2$ and $B_4$, would give a curve whose length would be of order $\Lambda_2/R$, which is much greater than $\Lambda \mu$. The positions of the strings are consistent with our assertion that
Figure 15: Transition from the Coulomb branch (a) through the massless monopole point (b) to $\mathcal{N} = 2$ supersymmetry breaking (c). In (a) and (b) we draw a closed curve $\gamma_k$ in $\Sigma$ which surrounds the $k$-th D4 brane and wraps once around $x^{10}$. After the transition, $\gamma_k$ does not exist. The curve $\hat{\gamma}_k$ wraps once around $x^{10}$. Its segments $A_k - B_k$ and $B_{k-1} - A_{k-1}$ lie outside $\Sigma$, at fixed values of $t$; they can join to a $k$-string and $(k-1)$-antistring, respectively.

the $k$-string is the Nielsen-Olesen string of the $k$-th monopole. We will now verify this assertion by looking at quark-antiquark mesons.

As discussed previously, far out along the Coulomb branch of the $\mathcal{N} = 2$ theory, a heavy quark $Q^k$ is constructed by attaching one boundary of a membrane to $\Sigma$ on a curve $\gamma_m$ surrounding $v = m$, and the other boundary to a curve $\gamma_k$ surrounding the branch cut at $v = \phi_k$. As we move in along the Coulomb branch toward the monopole point, keeping track of the quark $Q^k$, the curve $\gamma_k$ moves to the position indicated in figure 15. After $\mathcal{N} = 2$ supersymmetry is broken, the curve $\gamma_k$ no longer exists as a closed curve in $\Sigma$. It is useful to define a closed curve $\hat{\gamma}_k$ which consists of segments $A_{k-1} - A_k - B_k - B_{k-1} - A_{k-1}$, with the property that the segments $A_{k-1} - A_k$ and $B_k - B_{k-1}$ lie in $\Sigma$ while the segments $A_k - B_k$ and $B_{k-1} - A_{k-1}$ do not. Note that this curve wraps once around $x^{10}$. We may therefore construct a quark $Q^k$ as a membrane with one boundary on $\gamma_m$ and the other on $\hat{\gamma}_k$, but only if we join it to a $k$-string along the line $A_k - B_k$ and to a $(k-1)$-antistring along the line $B_{k-1} - A_{k-1}$. These strings can end only by attaching to an antiquark $\tilde{Q}_k$. This reproduces the results of Douglas and Shenker for mesons, when we identify the $k$-string as the string of the $k$-th monopole.

It is similarly straightforward to verify that the state $Q^1 Q^2 \cdots Q^k$, a membrane which can attach to the curve $A_0 - A_k - B_k - A_0$, connects only to the $k$-string given by $A_k - B_k$.

We can also see that sufficiently excited mesons are unstable to $W$ boson pair production. Ignoring the energetics of the process, this is easy to see topologically. Before $\mathcal{N} = 2$ is broken, the $W$ bosons $[W_{\alpha}]^k_p$ are given by attaching a membrane on the curve $\gamma_k$ at one end and on $\gamma_p$ on the other. Here, the $W$ boson can be attached at $\hat{\gamma}_k$ and $\hat{\gamma}_p$; on the one side it will connect to a $k$-string/$(k-1)$-antistring pair, while on the other it
Figure 16: The ellipse (6.9) for small $\mu$, with notation as in figure 14. A $k$-string/$(k-1)$-antistring pair is homotopic to a curve stretched along the ellipse and wrapping once around $x^{10}$. This curve can be shifted to the position of a 1-string. The intermediate steps involve energies of order $\Lambda_2$ and so this string-antistring partial annihilation process is energetically suppressed.

will connect to a $p$-string/$(p-1)$-antistring pair. In particular the state $[W_\alpha]^k_1$ can convert the $k$-string/$(k-1)$-antistring pair to a 1-string. Of course, this process can only occur if enough energy is available, and the required energy is of order the $W$ mass $m_W \sim \Lambda_2$, which is much larger than $\Lambda_1$ for small $\mu$.

There is another process, string-antistring partial annihilation, which is homotopically equivalent to this one. We illustrate in figure 16 a deformation by which a $k$-string/$(k-1)$-antistring pair annihilates to form a 1-string. The energy of this process is estimated by multiplying the length of the segment $A_{k-1}A_{k-2}$, of order $\Lambda_2/R$, by the length in space along the MQCD string which is required for the transition. Assuming on physical grounds that the required length is of order $\Lambda_1^{-1}$, and recalling that $\Lambda_1 R \sim 1$, we find that the energy of this process is again of order $\Lambda_2$. Without a detailed study of the energetics one cannot say whether $W$ boson production or string annihilation is the better language for explaining the physics; the two descriptions are complementary.

To show the baryons are correctly represented as in figure 11 is completely straightforward, and we omit a detailed discussion.
Now, what happens when $\mu$ is taken large? Since the $W$ mass is now less than $\Lambda_1$, the $W$ boson pair production process occurs rapidly, and the $k$-string/$(k - 1)$-antistring pair decays very quickly to a 1-string. A complementary description is that the string-antistring annihilation process becomes instantaneous. To see this, consider figure 16 when $\mu$ is large and the ellipse is nearly circular. The line connecting $A_{k-2}$ to $A_k$ will now be shorter than the sum of the lines $A_k-B_k$ and $A_{k-1}-B_{k-1}$, so the energy barrier to the annihilation process is essentially gone.

Thus, for large enough $\mu$, all mesons with quarks in the fundamental representation are indeed built from a single string tension, that of the 1-string. The $k$-string/$(k - 1)$-antistring pair is no longer metastable. But the $k$-strings themselves are stable. Their tensions (in M theory) still satisfy (6.5), as a consequence of the geometry of the ellipse (6.9) (figure 14) which goes smoothly over to the circle of figure 9. These strings therefore are still relevant for heavy quarks in higher representations, as discussed in section 7. In particular, for quarks in the $k$-index antisymmetric representation of $SU(N)$, which have charge $k$ under the center of $SU(N)$, a quark-antiquark meson will be bound by a $k$-string, not by $k$ 1-strings. The same relation suggests that certain excited baryons may indeed resemble the chain suggested in figure 11.

What are the key physical differences between the large and small $\mu$ limits? All of the unusual phenomena can be traced back to the breaking of the Weyl symmetry and the projection to the $U(1)^{N-1}$ subgroup (abelian projection) implemented by the expectation value for $\phi$. The breaking of the Weyl group is visible in the structure of the fivebrane. The Weyl group exchanges color indices, and so in the Type IIA brane language exchanges D4 color branes. Since in the vacuum with massless monopoles the D4 branes are lined up side by side, each one lying between two adjacent circles in figure 13, it is clear that the symmetry which would exchange them is spontaneously broken in this vacuum. The breaking is still present when $\mu$ is non-zero, as represented by the inequivalence of the points $A_k$ and $A_{k'}$.

From the positions for small $\mu$ of the curves $\hat{\gamma}_k$ (figure 15), it is evident that the quarks with different color indices are distinguished. The $\lfloor (N+1)/2 \rfloor$ different mesons result from this breaking. Only when $\mu \to \infty$ does the ellipse become a circle and the Weyl group become fully restored.

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7 However, the structure of a baryon will depend on its excitation quantum numbers; figure 10 may apply for some states, figure 11 for others, while still others may have an intermediate structure.
The effect of the abelian projection is that the Nielsen-Olesen strings are pinned at the points \( A_k \) and \( B_k \) for small \( \mu \), while as \( \mu \) becomes large they break free of these points and move easily around the ellipse. This freedom of movement corresponds to the presence of light non-abelian gauge bosons (relative to \( \Lambda_1 \)) in the theory, in contrast to the small \( \mu \) situation in which the \( W \) bosons are all heavy compared to \( \Lambda_1 \). Having \( W \) bosons much more massive than the photons of the unbroken \( U(1)^{N-1} \) gauge theory impedes the free flow of color quantum numbers, and inhibits annihilation of strings, inevitably resulting in each \( k \)-string carrying an approximately conserved quantum number. Only when the gauge bosons all are treated equally, as in the large \( \mu \) limit, is color free to flow as expected in an unbroken non-abelian gauge theory. This can be viewed as an argument against using abelian reduction techniques [46] to study confinement in QCD.

As a final comment, we note that all of the physics discussed in this section is essentially trivial in the case of \( SU(2) \) gauge theory. In this case there is only one meson and only one type of string for all values of \( \mu \), and so the discussion of Seiberg and Witten [32] is the complete story.

7. QCD Strings, Nielsen-Olesen Strings, and BPS Conditions.

A brief comment on the “almost BPS” properties of the Nielsen-Olesen strings is in order. As noted in section 6.1, the massless monopole vacuum is well-described as an \( \mathcal{N} = 2 \) supersymmetric Abelian Higgs model of \( N - 1 \) decoupled \( U(1) \) gauge theories, each with a massless charged hypermultiplet. With a properly normalized quartic potential, such as that generated by a Fayet-Iliopolous term, Abelian Higgs models have BPS saturated strings [45,47,48] in the semiclassical limit. The string tension is equal to the coefficient of the Fayet-Iliopolous term. Note that \( \mathcal{N} = 2 \) supersymmetry is preserved by a Fayet-Iliopolous term; only the \( SU(2) \) R-symmetry is broken.

As can be seen from (6.2)-(6.3), the breaking parameter \( \mu \) acts to leading order in \( 1/\Lambda_2 \) like a Fayet-Iliopolous term for each \( U(1) \) in the monopole Lagrangian. This results in \( N - 1 \) strings with tension of order \( \mu \Lambda_2 \). However, the \( 1/\Lambda_2 \) corrections break \( \mathcal{N} = 2 \) supersymmetry. A more serious obstruction, as pointed out in [30], is that we cannot expect \( \mathcal{N} = 1 \) QCD strings to be BPS saturated; it cannot be that one string satisfies \( T \geq Q_{BPS} \) while \( N \) identical strings, which must carry charge \( NQ_{BPS} \), can decay to the vacuum.
In exactly what limit does the perturbation of equation (6.2) become a Fayet-Iliopolous term? If we take \( \Lambda_2 \to \infty \) and \( \mu \to 0 \) simultaneously, holding \( \mu \Lambda_2 \) fixed, then the term linear in \( a_D^{(k)} \) survives while the non-linear terms drop out. From the point of view of the electric description of the theory, this is a rather odd scaling in which the theory is taken to ultra-strong coupling while the \( \mathcal{N} = 2 \) breaking parameter is taken to zero. This is perhaps not surprising, since on the one hand the Fayet-Iliopolous term must not break \( \mathcal{N} = 2 \) supersymmetry, and yet it must break \( SU(2)_R \).

In short, if \( \Lambda_2 \gg \sqrt{\mu \Lambda_2} \gg \mu \), then the strings should behave as though BPS saturated, carrying accidentally conserved charges and satisfying an approximate Bogomolnyi bound on their tensions. The strings’ energy scale \( \sim \sqrt{\mu \Lambda_2} \) lies between two difficulties. Above lie the irrelevant operators which know the theory is really non-abelian; these violate conservation of \( U(1)^{N-1} \) flux quantum numbers, which would otherwise serve as conserved BPS charges. Below we find the relevant but small operators which break \( \mathcal{N} = 2 \) supersymmetry and destroy the associated BPS bound. All approximate BPS properties are lost when \( \mu \sim \Lambda_2 \). These aspects of the theory are all clearly visible in the fivebrane construction discussed in section 4.

As an aside, we note that the number of Fayet-Iliopolous parameters always matches the number of \( \mathcal{N} = 2 \)-breaking terms in \( \mathcal{N} = 2 \) QCD; both are equal to the dimension of the Coulomb branch, which in turn equals the rank of the group. For example, in \( SU(N) \), the \( N-1 \) operators \( \text{tr} \phi^2, \text{tr} \phi^3, \ldots, \text{tr} \phi^N \) parameterize the Coulomb branch. We may break \( \mathcal{N} = 2 \) supersymmetry by adding the general superpotential

\[
W = \sum_{k=2}^{N} \lambda_k \text{tr} \phi^k ;
\]

The \( N-1 \) Fayet-Iliopolous terms are then linear combinations of the \( N-1 \) coefficients \( \lambda_k \Lambda_2^{k-1} \) in the limit \( \Lambda_2 \to \infty, \lambda_k \to 0 \).

8. An Additional Comment on Confinement

It is interesting to consider, in \( \mathcal{N} = 2 \) supersymmetry, what happens to monopoles when the electric non-abelian gauge group is broken along a Higgs branch. We expect the monopoles to be confined by strings carrying magnetic flux. In the Abelian Higgs model, this is well-understood in field theory \([45,47,48]\) and a brane representation (in the context of Type II compactifications) for a monopole-antimonopole pair bound by an abelian flux
the D4 color branes, as shown in figure 17. If lying in the shaded region between the color branes, and attached to the NS branes and doublet representation. The broken $SU(2)$ gauge theory, given by two finite D4 branes between two NS branes, and attach semi-infinite branes to the finite brane at the same value of $\phi$. This breaks the remaining symmetry, which means we have moved along the Higgs branch and expect the monopole to be confined. Now, if a D2 brane is attached to the NS and D4 branes, two pieces of its boundary are left open, as indicated in figure 17. This means this configuration by itself is not consistent — there are no isolated monopoles. However, to each of the open pieces of the D2 brane boundary we may sew on another D2 brane which also extends along a

![Figure 17](image)

**Figure 17:** The Type IIA brane construction for $\mathcal{N} = 2$ $SU(2)$ with two doublets. In each picture, the $SU(2)/U(1)$ monopole is shown as a D2 brane with boundary on the NS and D4 branes. On the left, $m > \phi$, while in the center and right, $m = \phi$. The low energy theory in the central picture is $U(1)$ with two massless hypermultiplets. In the righthand picture we have moved onto the Higgs branch by moving the infinite D4 brane off the NS branes. The dark lines indicate edges of the D2 brane which cannot be attached to NS or D4 branes. A D2 brane extending in a spatial direction cannot be attached to these edges.

We discuss the question semiclassically, using Type IIA string language. Consider $SU(2)$ gauge theory, given by two finite D4 branes between two NS branes, and attach one semi-infinite D4 brane to each of the NS branes, giving two hypermultiplets in the doublet representation. The broken $SU(2)$ theory has a monopole, made of a D2 brane lying in the shaded region between the color branes, and attached to the NS branes and the D4 color branes, as shown in figure 17. If $\phi = m$ then it is possible to connect the semi-infinite branes to the finite brane at the same value of $v$ and pull the resulting infinite D4 brane off of the NS branes, as shown in figure 17. This breaks the remaining $U(1)$ symmetry, which means we have moved along the Higgs branch and expect the monopole to be confined. Now, if a D2 brane is attached to the NS and D4 branes, two pieces of its boundary are left open, as indicated in figure 17. This means this configuration by itself is not consistent — there are no isolated monopoles. However, to each of the open pieces of the D2 brane boundary we may sew on another D2 brane which also extends along a
curve in space. Because of the relative orientations of the two boundaries, one of these D2 branes is a string in spacetime, while the other is an antistring; the pair together are a magnetic flux tube. The flux tube can terminate by being connected to an antimonopole (a D2 brane with orientation opposite to a monopole).

On dimensional grounds, the flux tube tension ought to be related to the square of the expectation value by which the gauge group is broken. The tension of the flux tube is apparently proportional (in this semiclassical regime) to the distance between the NS branes and the infinite D4 brane. The expectation values $\langle Q^1 \tilde{Q}_1 \rangle = -\langle Q^2 \tilde{Q}_2 \rangle$ are also proportional to this distance \[1\].

The confined monopole-antimonopole state, bound by a string-antistring pair, is thus a single continuous D2 brane. This structure is very similar to the picture proposed for the purely abelian case in \[35\] and to our earlier quark-antiquark meson.

We may also consider truly non-abelian weakly-coupled examples. A simple example is $SU(3)$ with six equally massive flavors at a point on the Coulomb branch where it is broken to $SU(2) \times U(1)$ with six massless doublets, an infrared free theory. There are monopoles in the coset $SU(3)/SU(2) \times U(1)$ which are not neutral under $SU(2)$ (as can be seen by breaking the theory slightly to $U(1) \times U(1)$.) On the Higgs branch, where doublet expectation values break $SU(2)$ completely, the monopoles are confined as described above.

Adapting this mechanism to strongly coupled theories, using the M theory fivebrane and membrane to replace the NS/D4 and D2 branes, is straightforward. Furthermore, a similar mechanism applies for confinement of quarks along monopole Higgs branches; it is closely related to the M theory picture of monopole condensation and string formation discussed in section \[3\].

9. Conjectures in Supersymmetric and Non-Supersymmetric QCD

We have seen that in the M theory construction of $\mathcal{N} = 1$ and weakly broken $\mathcal{N} = 2$ QCD, there are stable $k$-strings with tensions satisfying (6.3). Witten has proposed a minimal surface fivebrane solution to non-supersymmetric MQCD \[31\]. (Other work on non-supersymmetric QCD using branes has appeared in \[50\].) While many aspects of this solution are not understood, and although not only renormalization effects but even

\[8\] These simple statements about tensions and expectation values are actually quite naive. There are both classical and quantum mechanical subtleties, since this flux tube is not a BPS-saturated semiclassical soliton; see for example \[49\].
phase transitions may separate the semiclassical fivebrane picture from real QCD, we may still try to construct the MQCD $k$-strings as we have done in the supersymmetric case. It is straightforward to verify, as we have done in appendix B, that for all values of the parameters in Witten’s solution, the candidate $k$-strings still connect points on a curve similar to that of (6.9) (figure 14). Consequently, the formula (6.5) still applies, and the $k$-strings are stable.

Nevertheless, if we want to extract information about ordinary field theoretic QCD, we must view a formula such as (6.5) with considerable skepticism. To what extent can semiclassical results in M theory give reliable results in strongly coupled gauge theories?

The MQCD gauge theories considered in this paper arise from configurations of branes in string theory. In the weakly coupled Type IIA string theory, these configurations realize at low energy the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric field theory we are interested in, but the dynamics of the strongly interacting field theory cannot be computed there. In the strong coupling limit of Type IIA string theory, the configuration of branes becomes smooth enough to allow a semi-classical analysis using M theory on $M^{10} \times S^1$ [2,30], where $S^1$ is a circle of radius $R \gg 1$ in M theory units; in this approach much of the non-perturbative structure of the brane theory may be obtained. However, for large $R$ it is not obvious which aspects of the field theory physics are in fact reflected in the MQCD brane theory.

The difference between the $\mathcal{N} = 1$ and the $\mathcal{N} = 2$ case is simply understood [30]. The brane configuration depends on two parameters, the scale $\zeta$ and the radius of the eleventh dimension $R$. In $\mathcal{N} = 2$, $\zeta$ fixes the only parameter of the theory $\Lambda_2$ and we can vary $R$ as we like without changing any of the physics of interest. In $\mathcal{N} = 1$, the BPS saturated domain wall [31] and the MQCD string tension depend on $R$ and $\zeta$ in different ways [30]. In order to get a theory which at least resembles $\mathcal{N} = 1$ QCD, we must fix the parameters in such a way that the MQCD string has a tension of order $\Lambda_1^2$ and the domain walls of the spontaneously broken chiral symmetry have a tension of order $\Lambda_1^3$. As shown in [30], and reviewed in appendix A, this requires $R$ to be of order $1/\Lambda_1$. Unfortunately, for this choice, the Kaluza-Klein modes with momentum around $S^1$, which of course do not exist in the QCD field theory we want to study, have masses of order $1/R \sim \Lambda_1$, too low to be ignored. If we try to take $R$ small, then the tension of the MQCD string becomes very large, and the theory does not behave like QCD.

It is reasonable to believe that the qualitative properties of $\mathcal{N} = 1$ QCD (such as confinement, presence of mesons and baryons, etc.) are independent of $R$ and are correctly
described by MQCD. By contrast, quantitative predictions should only be trusted when they are protected in some way from renormalization, and therefore can be followed also to small radius. This is the case for all the computable properties in the $\mathcal{N} = 2$ case and for the domain wall tension in $\mathcal{N} = 1$, but is not the case for the QCD string tension.

There are essentially two quantitative results in this paper concerning $\mathcal{N} = 1$ gauge theories. The first is the computation of the Douglas-Shenker string tensions in the weakly broken $\mathcal{N} = 2$ theory. We find agreement with quantum field theory, as is natural, since not only is the theory nearly $\mathcal{N} = 2$ supersymmetric, but also the strings are almost BPS saturated and are described at weak coupling in the monopole variables. The second result, concerning the tensions of $k$-strings for strongly broken $\mathcal{N} = 2$ gauge theory, is much more subtle. We can trust our estimate (5.1) for the tension of a $k$-string only for large radius MQCD. To extend this result to ordinary QCD, we need to be able to follow this quantity down to small radius. Unfortunately, large renormalizations are expected, and the tensions implied by (5.1) certainly cannot be trusted.

However, the fact that all of the fivebrane generalizations of QCD — the weakly broken $\mathcal{N} = 2$, the $\mathcal{N} = 1$ and even the non-supersymmetric case — exhibit the same ratio of tensions (6.5) is remarkable. Perhaps this is merely a property of semiclassical M theory, and field theory agrees with it only when it has to, namely for weak $\mathcal{N} = 2$ breaking. But it is also possible that ratios of string tensions are rather weakly renormalized and that (6.5) is fairly accurate for $\mathcal{N} = 1$ or even for non-supersymmetric QCD.

Even if the formula (6.5) is not quantitatively accurate, there is still the qualitative question of whether the $k$-strings in various forms of QCD are stable or unstable to decay to 1-strings, that is, whether $T(k)/T(1)$ is greater than or less than $k$. M theory seems to come down squarely on the side of stability in all of these theories. In particular, even though (6.5) certainly has $1/R$ corrections, as pointed out in the footnote before (4.6), these corrections do not alter the stability properties of the $k$-strings. We should also note that other techniques, such as the strong coupling expansion, agree with M theory that $T(k)/T(1) = k$ in the large $N$ limit; the real issue in this context is the sign of the leading $1/N$ correction.

The ratios of string tensions can be studied in numerical lattice simulations of gauge theory. This would be straightforward for the non-supersymmetric theory, but could be carried out even for weakly broken $\mathcal{N} = 1$ or $\mathcal{N} = 2$ gauge theories [51]. Of course, for $SU(2)$ or $SU(3)$ there is only one string tension, so one must at least study $SU(4)$. To our knowledge this has not been done even in the non-supersymmetric case [12]. It seems to
us that the ratios of string tensions are fundamental quantities in confining gauge theories, and that it would be useful to have numerical values for a few non-trivial examples.

Why should we care whether the M theory result survives, at least qualitatively, into pure $\mathcal{N} = 1$ QCD? First, it would imply the presence of markedly different string tensions for mesons built from quarks in higher representations. Second, it would confirm our picture for the transition from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ QCD, which depended on (6.3) holding qualitatively. Third, it would give some indication as to whether M theory is a useful guide for extracting physics that ordinary field theoretic methods cannot compute.

10. Conclusion

Extending the results of [30,43,44], we have found that the MQCD picture correctly describes the standard lore for confinement in $\mathcal{N} = 1$ QCD and the field theoretical predictions for the weakly broken $\mathcal{N} = 2$ gauge theory. In $\mathcal{N} = 1$ MQCD we have studied construction of heavy quark mesons and baryons as M theory membranes, and identified the $k$-strings of MQCD (a $k$-string is a flux tube carrying $k$ units of flux, $0 < k < N$.) We have seen explicitly that the $N - 1$ strings of Douglas and Shenker go smoothly to the $k$-strings of MQCD. The metastability of a $k$-string/$(k - 1)$-antistring pair explains the presence of the many quark-antiquark mesons observed by Douglas and Shenker; this metastability is lost when $\mathcal{N} = 2$ supersymmetry is strongly broken. In addition we have discussed the scaling limit in which the Douglas-Shenker strings are BPS saturated, and have explored the brane picture for monopole confinement in $\mathcal{N} = 2$ nonabelian gauge theory, which closely resembles that of [38].

We have shown that the $\mathcal{N} = 1$ and non-supersymmetric MQCD results of [30] imply that the ratios of the MQCD $k$-string tensions are independent of the $\mathcal{N} = 2$ supersymmetry breaking parameters. Our formula predicts that $k$-strings are always stable. We have proposed that this formula might undergo relatively little renormalization and might hold, at least qualitatively, in ordinary $\mathcal{N} = 1$ and perhaps even non-supersymmetric QCD. The question of whether these conjectures are correct could be studied using lattice gauge theory, and we hope that some attention will be given to this problem.

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Appendix A. Dimensional Analysis for the M Theory Construction

We use M theory units in this paper. Quantities are measured in eleven dimensional Planck length units \( (l_P) \).

If we compactify M theory on a circle of radius \( R \), we recover the Type IIA string theory with string length \( l_s \) and coupling constant \( g_s \) given by

\[
R = g_s l_s, \quad l_P = g^{1/3} l_s. \tag{A.1}
\]

The membrane tension is \( 1/l_P^3 \). A membrane wrapped around the eleventh dimension is identified with the Type IIA string and equation (A.1) correctly gives its tension \( 2\pi R/l_P^3 = 2\pi/l_s^2 \).

In the M theory picture, quarks and W bosons are identified as membranes wrapped around \( x^{10} \). The mass of a state corresponding to a membrane with length \( v \) in the \( x_4, x_5 \) plane is therefore proportional to \( 2\pi R v/l_P^3 \). As a consequence, in the units appropriate for M theory, the \( N = 2 \) curve for an \( SU(N) \) gauge theory reads,

\[
t + 1/t + \Lambda^{-N} \prod_{i}^{N} (2\pi R v/l_P^3 - \phi_a) = 0. \tag{A.2}
\]

where \( t \) is dimensionless. In Type IIA units (see equation (A.1)) the power of \( R \) disappears and the curve assumes the form used in section 3 (see for example equation (3.1) with \( m = \infty \)).

In section 4 we rotated the \( N = 2 \) curve and found the \( N = 1 \) Riemann surface,

\[
v = \frac{\Lambda^{3/2} l_P^3}{(2\pi R)^{1/2}} t^{-1/N}, \quad w = \frac{\Lambda^{3/2} l_P^3}{(2\pi R)^{1/2}} t^{1/N}. \tag{A.3}
\]

so that \( vw = l_P^6 \Lambda^3/2\pi R \).

\( N = 1 \) MQCD has strings \( \lambda^3 \) with tension \( \sqrt{vw}/l_P^3 \) and domain walls with tension \( Rvw/l_P^6 \). If we fix the value of \( R \) to be \( 1/\Lambda_1 \) we correctly reproduce the expectations that the string tension is proportional to \( \Lambda_1^2 \) and the domain wall tension is proportional to \( \Lambda_1^3 \). We have not determined overall normalizations. The exact coefficients could be fixed by a detailed analysis of the normalization of the superpotential (gluino condensate) generated by the fivebrane theory.
Appendix B. The Non-Supersymmetric Fivebrane Solution

A non-supersymmetric fivebrane solution reproducing, in a certain limit, the bosonic $SU(N)$ Yang-Mills theory has been proposed in [30]. It is obtained by an arbitrary rotation of the NS' brane. The equation is parametrized in terms of a complex number $\lambda$ and depends on two complex four-vectors $\vec{p}, \vec{q}$ and a real constant $c$, with the constraints

$$\vec{p}^2 = \vec{q}^2 = 0, \quad -\vec{p} \cdot \vec{q} + \frac{R^2 N^2}{2} (1 - c^2) = 0. \quad (B.1)$$

The condition for unbroken supersymmetry is $\vec{p} \cdot \vec{q} = 0$. Combining $x_4, x_5, x_8, x_9$ in a real four-vector $\vec{A}$, the solution looks like (in the notation of [30])

$$\vec{A}(\lambda) = \text{Re} (\vec{p} \lambda + \vec{q} \lambda^{-1})$$

$$x^6 = -(RNc) \text{ Re } \ln \lambda$$

$$x^{10} = -RN \text{ Im } \ln \lambda \quad (B.2)$$

The curve $\Sigma$ still wraps $N$ times around $x^{10}$, and its topology has not changed, so we can try to find the MQCD string tension by the same arguments used in section 4 and 6.3. To describe $k$-strings, we need to find the real curve of minimum length connecting two points on $\Sigma$ separated by $k$ wrappings around $x^{10}$. From (B.2), the two points must be specified by $\lambda$ values which differ in phase by $e^{2\pi i k/N}$. Since $\Sigma$ is symmetric with respect to exchanging $p$ and $q$, we expect the two points to lie on the plane of symmetry. We may take $|p| = |q|$, without loss of generality, to ensure that the plane of symmetry is $x^6 = \text{Re } \ln \lambda = 0$. The two points must then be given by $\lambda = e^{i\sigma_1}, e^{i\sigma_2}$, with $\sigma_1 - \sigma_2 = 2\pi k/N$. Since these two points have the same $x^6, x^{10}$ coordinates, the length of the line connecting them is given by

$$|\vec{A}(e^{i\sigma_1}) - \vec{A}(e^{i\sigma_2})| = \left| -2 \sin \frac{\sigma_1 - \sigma_2}{2} \text{ Im } (\vec{p} e^{i\alpha} - \vec{q} e^{-i\alpha}) \right|$$

$$= \sqrt{2} \sin \frac{\pi k}{N} \left( |p|^2 + |q|^2 + 2 \text{Re } \vec{p} \cdot \vec{q} - \vec{p} \cdot \vec{q} e^{2i\alpha} - \vec{p}^* \cdot \vec{q} e^{-2i\alpha} \right)^{1/2}, \quad (B.3)$$

where $\alpha$ is $(\sigma_1 + \sigma_2)/2$. This is minimized at $\tan \alpha = -\text{Im } \vec{p} \cdot \vec{q}^*/\text{Re } \vec{p} \cdot \vec{q}^*$ for all values of $k$. We thus find the string tensions are proportional to $\sin \pi k/N$, as claimed in section 6.
References

[1] A. Hanany, E. Witten, Type IIB Superstrings, BPS Monopoles, and Three Dimensional Gauge Dynamics, [hep-th/9611230].
[2] E. Witten, Solutions Of Four-Dimensional Field Theories Via M Theory, [hep-th/9703166].
[3] O. Aharony, A. Hanany, Branes, Superpotentials and Superconformal Fixed Points, [hep-th/9704170].
[4] I. Brunner, A. Karch, Branes and Six Dimensional Fixed Points [hep-th/9705022].
[5] B. Kol, 5d Field Theories and M Theory, [hep-th/9705031].
[6] S. Elitzur, A. Giveon, D. Kutasov, Branes and N=1 Duality in String Theory, Phys. Lett. B400 (1997) 269, [hep-th/9702014].
[7] J. de Boer, K. Hori, H. Ooguri, Y. Oz, Z. Yin Branes and Mirror Symmetry in N=2 Supersymmetric Gauge Theories in Three Dimensions, [hep-th/9702154].
[8] J. L. F. Barbon, Rotated Branes and N=1 Duality, Phys.Lett. B402 (1997) 59, [hep-th/9703051].
[9] N. Evans, C. V. Johnson, A. D. Shapere, Orientifolds, Branes, and Duality of 4D Gauge Theories, [hep-th/9703210].
[10] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici, A. Schwimmer, Brane Dynamics and N=1 Supersymmetric Gauge Theory, [hep-th/9704104].
[11] J.H. Brodie, A. Hanany, Type IIA Superstrings, Chiral Symmetry, and N = 1 4D Gauge Theory Duality, [hep-th/9704043].
[12] A. Brandhuber, J. Sonnenschein, S. Theisen, S. Yankielowicz, Brane Configurations and 4D Field Theory Dualities, [hep-th/9704044].
[13] R. Tatar, Dualities in 4D Theories with Product Gauge Groups from Brane Configurations, [hep-th/9704198].
[14] A. Hanany, A. Zaffaroni, Chiral Symmetry from Type IIA Branes, [hep-th/9706047].
[15] S. Katz, C. Vafa, Geometric Engineering of N=1 Quantum Field Theories, [hep-th/9704090].
[16] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov, C. Vafa, F-theory, Geometric Engineering and N=1 Dualities, [hep-th/9612052].
[17] C. Vafa, B. Zwiebach, N=1 Dualities of SO and USp Gauge Theories and T-Duality of String Theory, [hep-th/9701013].
[18] H. Ooguri, C. Vafa, Geometry of N=1 Dualities in Four Dimensions, [hep-th/9702180].
[19] C. Ahn, K. Oh, Geometry, D-Branes and N=1 Duality in Four Dimensions I, [hep-th/9704061].
[20] C. Ahn, Geometry,D-Branes and N=1 Duality in Four Dimensions II, [hep-th/9705004].
[21] C. Ahn, R. Tatar, Geometry, D-branes and N=1 Duality in Four Dimensions with Product Gauge Group, [hep-th/9705100].
[22] C. Ahn, K. Oh, R. Tatar, Branes, Geometry and N=1 Duality with Product Gauge Groups of SO and Sp, hep-th/9707027.
[23] K. Landsteiner, E. Lopez, D. A. Lowe, N=2 Supersymmetric Gauge Theories, Branes and Orientifolds, hep-th/9705199.
[24] A. Brandhuber, J. Sonnenschein, S. Theisen, S. Yankielowicz, M Theory and Seiberg-Witten Curves: Orthogonal and Symplectic Groups, hep-th/9705232.
[25] A. Marshakov, M. Martellini, A. Morozov, Insights and Puzzles from Branes: 4d SUSY Yang-Mills from 6d Models, hep-th/9706050.
[26] A. Fayyazuddin, M. Spalinski, The Seiberg-Witten Differential From M-Theory, hep-th/9706087.
[27] C. V. Johnson, From M-theory to F-theory, with Branes, hep-th/9706153.
[28] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. Warner, Self-Dual Strings and N=2 Supersymmetric Field Theory, hep-th/9604034.
[29] S. Katz, P. Mayr, C. Vafa, Mirror symmetry and Exact Solution of 4D N=2 Gauge Theories I, hep-th/9706110 and references therein.
[30] E. Witten, Branes And The Dynamics Of QCD, hep-th/9706109.
[31] G. Dvali, M. Shifman, Domain Walls in Strongly Coupled Theories, Phys.Lett. B396 (1997) 64, hep-th/9612128. A. Kovner, M. Shifman, A. Smilga, Domain Walls in Supersymmetric Yang-Mills Theories, hep-th/9706089. B. Chibisov, M. Shifman, BPS-Saturated Walls in Supersymmetric Theories, hep-th/9706141.
[32] N. Seiberg, E. Witten, Monopole Condensation And Confinement In N = 2 Supersymmetric Yang-Mills Theory, Nucl.Phys. B426 (1994) 19, hep-th/9407087.
[33] A. Klemm, W. Lerche, S. Theisen, S. Yankielowicz, Simple Singularities and N=2 Supersymmetric Yang-Mills Theory, Phys.Lett. B344 (1995) 169, hep-th/9411048.
[34] P. C. Argyres, A. E. Faraggi, The Vacuum Structure and Spectrum of N=2 Supersymmetric SU(N) Gauge Theory, Phys. Rev. Lett. 74 (1995) 3931, hep-th/9411057.
[35] A. Hanany, Y. Oz, On the Quantum Moduli Space of Vacua of N = 2 Supersymmetric SU(Nc) Gauge Theories, Nucl. Phys. B452 (1995) 283, hep-th/9505073.
[36] P. C. Argyres, M. R. Plesser, A. Shapere, The Coulomb Phase of N=2 Supersymmetric QCD, Phys. Rev. Lett. 75 (1995) 1699, hep-th/9505100.
[37] M. R. Douglas, S. H. Shenker, Dynamics of SU(N) Supersymmetric Gauge Theory, Nucl.Phys. B447 (1995) 271, hep-th/9503163.
[38] B. R. Greene, D. R. Morrison, C. Vafa, A Geometric Realization of Confinement, Nucl.Phys. B481 (1996) 513, hep-th/9608039.
[39] N. Seiberg, E. Witten, Monopoles, Duality and Chiral Symmetry Breaking in N=2 Supersymmetric QCD, Nucl.Phys. B431 (1994) 484, hep-th/9408099.
[40] J. Polchinski, Dirichlet-Branes and Ramond-Ramond Charges Phys.Rev.Lett. 75 (1995) 4724, hep-th/9510017.
[41] A. Strominger, *Open P-Branes* Phys.Lett. B383 (1996) 44, hep-th/9512059; P. K. Townsend, *D-branes from M-branes* Phys.Lett. B373 (1996) 68, hep-th/9512062.

[42] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Superstrings In D = 10 From Supermembranes In D = 11*, Phys. Lett 191B (1987) 70; P.K. Townsend, *The Eleven-Dimensional Supermembrane Revisited*, Phys.Lett. B350 (1995) 184, hep-th/9501068; E. Witten, *String Theory Dynamics in Various Dimensions*, Nucl.Phys. B443 (1995) 85, hep-th/9503124.

[43] K. Hori, H. Ooguri, Y. Oz, *Strong Coupling Dynamics of Four-Dimensional N=1 Gauge Theories from M Theory Fivebrane*, hep-th/9706082.

[44] A. Brandhuber, N. Itzhaki, V. Kaplunovsky, J. Sonnenschein, S. Yankielowicz, *Comments on the M Theory Approach to N=1 SQCD and Brane Dynamics*, hep-th/9706127.

[45] H.B. Nielsen, P. Olesen, Nucl.Phys. B61 (1973) 45.

[46] G. ’t Hooft, *Topology of the Gauge Condition and New Confinement Phases in Non-Abelian Gauge Theories*, Nucl.Phys. B190 (1981) 455.

[47] E.B. Bogomolny, Sov.J.Nucl.Phys. 24 (1976) 449.

[48] H.J. de Vega, F.A. Schaposnik, Phys.Rev. D14 (1976) 1100; J. Edelstein, C. Nunez, F. Schaposnik, *Supersymmetry and Bogomolny Equations in the Abelian Higgs Model*, Phys.Lett. B329 (1994) 39, hep-th/9311055.

[49] A.A. Penin, V.A. Rubakov, P.G. Tinyakov, S.V. Troitskii, *What becomes of vortices in theories with flat directions*, Phys.Lett. B389 (1996) 13-17, hep-ph/9609257; O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, M.J. Strassler, *Aspects of N = 2 Supersymmetric Gauge Theories in Three Dimensions*, hep-th/9703110.

[50] A. Brandhuber, J. Sonnenschein, S. Theisen, S. Yankielowicz, *Brane Configurations and 4-D Field Theory Dualities*, hep-th/9704044; N. Evans, *Softly Broken SQCD: in the Continuum, on the Lattice, on the Brane*, hep-th/9707197.

[51] I. Montvay, *Supersymmetric Gauge Theories on the Lattice*, Nucl. Phys. B53, Proc. Suppl. (1997) 853-855, hep-lat/9607035, and references therein.

[52] See for example E. Marinari, M.L. Paciello, B. Taglienti, *The String Tension in Gauge Theories*, Int.J.Mod.Phys. A10 (1995) 4265, hep-lat/9503027.