A 3-states magnetic model of binary decisions in sociophysics

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Abstract

We study a diluted Blume-Capel model of 3-states sites as an attempt to understand how some social processes as cooperation or organization happen. For this aim we study the effect of the complex network topology on the equilibrium properties of the model, by focusing on three different substrates: random graph, Watts-Strogatz and Newman substrates. Our computer simulations are in good agreement with the corresponding analytical results.

Keywords:
Sociophysics, Three-states models, Small-World, Coalitions, Cooperation, Decision

1. Introduction

There is great interest in applying physical models of proven efficiency in the new field of sociophysics. Problems such as decision of living in a neighborhood \[1\], to go to a crowed bar \[2\] or to participate in a strike \[3\]
are some particular examples of such applications.

Physics ideas in social science have been introduced in order to study hierarchical structures by using the principle of least difficulty [4], the effect of frustration for modelling the dissemination of culture within the Axelrod model in his energy landscape theory [5], and by Galam in his models of coalitions [6]. To study the emergence of order in such a system, magnetic models of the Ising family have been used [7, 8, 9]. In particular, within the social sciences, models have been developed to describe urban segregation, language change [10], business confidence and economic opinions [11]. These studies apply either the traditional two-states Ising model to binary choice or a Potts multistates model [12] to multiple choice. Representing opinions or positioning by discrete values, society can be modeled as a system in which an analogue of the magnetic spin variable represents the state of individuals while the couplings represent their interactions [13, 14].

The Ising model is usually applied in the context of binary decisions, which can be to vote or not to vote, to buy or not to buy a certain good [15, 16], etc. The Random Field Ising Model (RFIM) was first applied to social systems for studying collective phenomena such as consensus and attitude changes in groups [17] and rational group decision making [18]. Another useful application of the RFIM in social and economics sciences is presented in [19], by taking into consideration heterogeneities and interaction in decision making, thus providing a unified framework to account for many collective socioeconomic phenomena leading to ruptures or crisis. Although in the mean-field approximation, quantitative details of the model might be sensitive to the type of the topology or the distribution of the idiosyncratic fields,
the qualitative behavior does not depend very much on them [19].

By using coupled Ising models, the interdependent binary choices under social influence for homogeneous unbiased populations have been analyzed [20]. Currently a comparison of the model with real data of the European labor market is under investigation [21].

Important results have also been obtained in [22], where real data concerning the drop of birth rates in European countries in the second half of the XXth century, the increase of cell phones in Europe in the 90s and the way clapping dies out at the end of a concert have been successfully explained by using the RFIM.

Concerning three-states models, several real examples can be given, such as the decision to vote, not to vote or to abstain, to belong to NATO, to the Warsaw Alliance or to a third part Alliance, and so on. For example, Ref. [23] studies the opinion dynamics in a three-choice system dividing the population in random groups of fixed size, demonstrating that such a system always reaches an equilibrium, and in [24] there is a discussion on the opinion formation in a voter-like model of a 3-states agents system (yes, no, undecided), defined on a regular lattice.

The review [25] provides many examples of applications of statistical physics methodology and concepts for the study of social systems, including opinion formation, cooperation, cultural dynamics, language evolution, crowd behavior or human dynamics among many others.

On the other hand, modeling of Small World and scale-free networks allowed for powerful theoretical analyses of dynamical collective social phenomena, such as opinion formation, infection, or damage propagation in social
networks, where the analysis of social structures and interactions is crucial. Both models can be combined to study dynamical processes on complex networks [8, 26, 27].

Activities like cooperation and coalition forming have been traditionally approached by game theory in politics and diplomacy [28], yet it is only recently that the concepts of Statistical Physics have been applied. In this context, various investigations have been done by representations in terms of cellular automata, [29, 30] or in terms of magnetic models on a Cayley tree, similar to what we propose in this work [31, 32]. In this article we demonstrate that starting from a rather weak assumption, stating that each individual in the ensemble is an agent who can adopt three different states, we show that it is possible to effectively model a broad range of sociological processes. We model the society as a set of agents whose interactions attempt to minimize their frustration. In order to introduce the neutrality as a possible decision option, different to the symmetric choice, we use the Blume-Capel model [33, 34]. Further, we make our model reside on an underlying complex network, which plays the role of substrate, and study the effects of its topology on the model behavior.

The Blume-Capel model has been applied in the context of urban segregation for the case of a fixed number of agents on a lattice [35] and for the case of an external reservoir of agents, thus modelling an open city [36]. Although the interpretation of the variables is different, in the sense that in the problem of opinion formation studied here the zero state corresponds to neutrality, while in the above cited references they correspond to an empty location, there are similarities among these models that will be commented
In this paper we will restrict ourselves to the mean-field case, in the sense that the agents perceive the opinion of the rest of the agents through the average opinion, thus the total “demand” becomes a public information that influences the individual agent.

The rest of the paper is organized as follows: in Section 2 we study magnetic models in which three states are allowed, in contrast with traditional Ising models; in Section 3 we study the network on which our model will be residing, with emphasis on the Annealed Network Approximation approach, which enables us to derive expressions for the order parameters on the grid substrate; Section 4 is devoted to the study of the order parameters and critical temperature of our three-states magnetic model on three different network substrates, with a special focus on the Newman substrate, which is deemed the most realistic. Finally, in Section 5 we discuss the results.

2. Three-states magnetic models

Blume, Emery and Griffiths (BEG) proposed the first version of their three-states model in [33] to explain the behavior of a $^3He - ^4He$ mixture, which has been since successfully applied to a number of different problems.

The three states BEG model has been applied in the context of neural networks [37, 38, 39] giving way to enhanced techniques for information storage. It has been studied by a number of techniques: Bethe-Peierls approximation [40, 41], real-space normalization [42] and exact recursion [43, 44, 45, 46], among many others. In this paper we will use the mean-field approximation.

In analogy with the original model, we will consider an ensemble made
up by agents of two species, neutral with spin 0 and non-neutral with spin ±1, defining thus the following two parameters:

\[ M = \frac{1}{N} \sum_{i=1}^{N} \langle S_i \rangle \]  \hspace{1cm} (1)

and

\[ x = 1 - \langle S_i^2 \rangle, \]  \hspace{1cm} (2)

where \( M \) and \( x \) represent the number of "active" agents \( (S_i = \pm 1) \) and the neutral agents, respectively, being \( S_i \) the site spin (agent state). The Hamiltonian of this system has the general form

\[ H = -J \sum_{i,j} S_i S_j - \kappa \sum_{i,j} S_i^2 S_j^2 + \Delta \sum_i S_i^2. \]  \hspace{1cm} (3)

Here \( J \) is the coupling constant, \( \kappa \) is the interaction between agents of different species and \( \Delta \) is the anisotropy term. The limit \( \Delta \to -\infty \) corresponds to the Ising case, or an absence of neutral agents, while large values of \( \Delta \) corresponds to large density of neutral agents. If the interactions between agents are equivalent, the biquadratic term can be neglected, thus obtaining a simplified version of the Blume-Emery-Griffiths model known as Blume-Capel model, described with detail in \cite{34}. We will follow this simplification and adopt the Blume-Capel model to capture the essence of competition between two processes, one favoring confrontation and another favoring neutrality. The dilution manifests itself as random coupling strengths between nodes, and has no effect in the anisotropy term, which corresponds to self-interaction. Consequently, we generate in our simulation \( J \) from a Poisson distribution, obtaining results in good agreement with the theory.
We start from the Hamiltonian in (3) with $\kappa = 0$ and notice it is convenient to work with $1 - x$ instead $x$. We then apply the Mean Field (MF) approach \[34\] by splitting the site spin in two parts, the thermal average and the fluctuation, i.e., as $S_i = \langle S_i \rangle + \delta S_i = m + \delta S_i$, obtaining thus the following MF Hamiltonian

$$
\hat{H}_{MF} = \frac{1}{2} NJ_0 m^2 - J_0 m \sum_i S_i + \Delta \sum_i S_i^2,
$$

where we introduced $J_0$, the discrete Fourier transform of $J(\vec{R})$ for wavevector $q = 0$. After rescaling by $J_0$ as

$$
\theta = \frac{k_B T}{J_0}, \delta = \frac{\Delta}{J_0},
$$

we obtain the following expressions for the partition function and the normalized free energy respectively

$$
Z = e^{-\frac{1}{2} \beta N J_0 m^2} \left[ 1 + 2 e^{-\frac{\delta}{\theta}} \cosh \left( \frac{m}{\theta} \right) \right]^N,
$$

$$
f = \frac{F}{NJ_0} = \frac{1}{2} m^2 - \theta \ln \left[ 1 + 2 e^{-\frac{\delta}{\theta}} \cosh \left( \frac{m}{\theta} \right) \right].
$$

Finally, minimizing with respect to $m$ and $x$ respectively, we get the order parameters

$$
m = \frac{2 e^{-\frac{\delta}{\theta}} \sinh \left( \frac{m}{\theta} \right)}{1 + 2 e^{-\frac{\delta}{\theta}} \cosh \left( \frac{m}{\theta} \right)}
$$

and

$$
1 - x = \frac{2 e^{-\frac{\delta}{\theta}} \cosh \left( \frac{m}{\theta} \right)}{1 + 2 e^{-\frac{\delta}{\theta}} \cosh \left( \frac{m}{\theta} \right)}.
$$

This system exhibits a critical temperature at

$$
\theta_c = \frac{2}{\exp \left( \frac{\delta}{\theta_c} \right) + 2}
$$
and a tricritical point separating first order from second order transition located at

\[ \theta = \frac{1}{3}, \delta = \frac{2}{3} \ln 2, \]  

shown in Figure 1.

3. Annealed Network Approximation

In this section and afterwards we will use the annealed MF approach by accounting for the heterogeneity of a complex network [7, 8, 26, 47, 48].

The main idea is to replace a model on a complex network by a model on a weighted fully connected graph. This approximation is exact in the limit of large systems for a Small World network [49]. Let us consider our magnetic model described by the Hamiltonian obtained in Section 2 which resides on
a graph with a given topology, that will be defined in the next sections. The
corresponding adjacency matrix $a_{ij}$ will be replaced by an effective one, thus
introducing the probability that nodes $i$ and $j$ with degrees $q_i$ and $q_j$ are
connected with weights $\frac{q_i q_j}{\langle q \rangle N}$, being $\langle q \rangle$ the average degree.

In other words, we substitute the original network with an effective sub-
strate where the sum of couplings for each node $i$ has the same value $J q_i$ as
in the original graph. In this way the fully connected graph approximates
the original complex network.

Using the Hamiltonian (3) with $\kappa = 0$

$$H = -\sum_{i,j} J_{ij} S_i S_j + \Delta \sum_i S_i^2.$$  (12)
we can write the corresponding node energy, in the Mean Field Approxima-
tion, as follows:

$$E_i = H(S_i) = -S_i \left( \sum_j J_{ij} S_j \right) + \Delta S_i^2$$

$$\approx -S_i \sum_j J_{ij} m_j + \Delta S_i^2,$$  (13)

where $J_{ij}$ are the coupling constants between node $i$ and its neighbors $j$, $m_j$
is the magnetization of node $j$ and $\Delta$ is the anisotropy term.

The Annealed Network Approximation described above gives the follow-
ing approximations for the order parameters (see the Appendix for the de-
tails)

$$m_i = \frac{2 \sinh \beta q_i M}{e^{\beta \Delta} + 2 \cosh [\beta q_i M]}$$  (14)
and
1 - x = \frac{2 \cosh \beta J q_i M}{e^{\beta \Delta} + 2 \cosh [\beta J q_i M]}.

(15)

Here J is the expected value of the coupling constants and we have introduced the weighted magnetic moment, defined as

\[ M = \sum_j q_j m_j \langle q \rangle N. \tag{16} \]

M is a solution of the following equation:

\[ M = \sum_q qP(q) \frac{2 \sinh \beta J q M}{\langle q \rangle \exp (\beta \Delta) + 2 \cosh \beta J q M} \tag{17} \]

and is obtained after substituting Eq. (14) into the definition of M and by using the fact that \( \sum_i f(q_i) = \sum_q P(q)f(q) \), being \( P(q) \) the degree distribution of the original network \[7, 8, 26\]. As can be seen from Eq.(17), the order parameter in our model is the weighted magnetic moment M.

4. Magnetic Models on Complex Networks

Complex networks represent better the topologies found in real world than regular grids. In particular, Small Worlds networks are often used to describe economic and social organizations \[50\].

To generate a complex network, Watts and Strogatz add disorder to a regular grid and thus get a Small World analytic model which combines high clustering and short paths \[51, 52\]. The rationale behind is that a pure random network ignores the clustering, whereby nodes connected to a node are often also connected between themselves. In this way, a Watts-Strogatz network reunites local properties of a regular grid with global properties of
a random network, thanks to the introduction of long range connections in an otherwise initially regular grid. In the original variation of the Watts-Strogatz network, as described in [50] and [52], the network starts from $N$ nodes located on an one-dimensional grid, with links to $z$ next neighbors. Then, a link between two nodes is selected randomly with probability $\phi$ and replaced by a shortcut from the first to a third random node. In a second variation, due to Newman [49, 53], shortcuts are just added without deleting the original links.

We postulate that our model resides in a grid described by a Small-World type network, for which we will use three different models: a random graph and the two variations of the Watts-Strogatz model described above, the original by Watts and Strogatz and the variation proposed by Newman. In this section, we will apply the annealed network approximation to a Blume-Capel system residing in these three different diluted network substrates. For that purpose, the weighted neutrality and magnetization are derived using in each case the relevant degree distribution.

### 4.1. Random Graph Substrate

The random graph is the first model utilized to study complex networks. In it, randomly chosen nodes are connected, so that in a certain way, a random graph model is complementary to a lattice type model like the ones we will study in following sections. Its degree distribution shows a rapid drop at large degrees, so that we can safely consider a cutoff of 20 for the sum obtained substituting the distribution in Eq. (17). Therefore we get the
Figure 2: (Color online). Solutions for magnetization (left) and neutrality (right) in Random Graph substrate.

following expression for the weighted magnetic moment

\[ M = \sum_{q=1}^{20} \frac{q}{\langle q \rangle} e^{-\langle q \rangle^2} \left( \frac{1}{q!} \exp(\beta \Delta) + 2 \cosh \beta J q M \right). \]  \hspace{1cm} (18)

We will see in next Section (4.2) that Watts-Strogatz model is not defined for \( q < 6 \), hence we make \( \langle q \rangle = 6 \), so that it agrees with the values used to study the other topologies. Then we have the following order parameters

\[ M = \sum_{q=1}^{20} \frac{qe^{-6q^{-1}}}{q!} \frac{2 \sinh \beta J q M}{\exp(\beta \Delta)} + 2 \cosh \beta J q M. \]  \hspace{1cm} (19)

and

\[ 1 - x = \sum_{q=1}^{20} \frac{qe^{-6q^{-1}}}{q!} \frac{2 \cosh \beta J q M}{\exp(\beta \Delta) + 2 \cosh \beta J q M}. \]  \hspace{1cm} (20)

Figure 2 shows solutions of Eq. (19), being \( J \) the expected value of a normal distribution for the quenched coupling constants. \( M = 0 \) is always a solution, and there is a limiting value of \( \Delta \) for which it is unique. Below
this value only exists the disordered state and above there are more finite solutions. A very small connectivity causes also \( M = 0 \) to be the only solution for every value of \( \Delta \) and temperature. Either very low connectivity -because there is no interaction- or high propensity to neutrality prevent collective decisions.

The critical temperature at which a non-zero value of \( M \) is possible is found graphically looking at the intersections of \( y = M \), represented by a solid line in the left of Figure 2 and the rhs of Eq. (19) to get:

\[
\frac{1}{\beta_c} = \frac{CJ}{e^{\beta_c \Delta} + 2},
\]

where \( C \) depends on the connectivity. The meaning of the existence of a critical temperature is that a very high agitation, either thermal as in the magnetic model or social, prevents the agents to settle in a fixed state. With respect to the order parameter related with neutrality, when \( \Delta = 0 \) and \( M = 0 \) all solutions of \( 1 - x \) are null at any temperature, i.e., all the agents are neutral.

4.2. Watts-Strogatz Substrate

To build their model, Watts and Strogatz start from a regular unidimensional grid in which each node is connected to \( 2z \) next neighbors. Then links are selected randomly with probability \( \phi \) and one end is reconnected to another randomly selected node.

To get the distribution of connectivity \( P(q) \), see [50], we realize that by network construction, the average connectivity is \( K = 2z \). The connectivity of a node can be written as \( q_i = z + n_i \), with \( n_i \geq 0 \). Now, \( n_i \) can be split in two parts: \( n_i^1 \leq z \) remaining links (with probability \( 1 - \phi \)) and \( n_i^2 = n_i - n_i^1 \).
links which have been reconnected to node $i$ with probability $\frac{\phi}{N}$ obtaining

$$P(n^1_i) = \binom{z}{n^1_i}(1 - \phi)^{n^1_i}\phi^{z-n^1_i}$$

(22)

and

$$P(n^2_i) = \frac{(z\phi)^{n^2_i}}{n^2_i!}\exp(-\phi z),$$

(23)

for large $N$. Finally

$$P(q, \phi, z) = \sum_{n=0}^{\min(q - z, z)} \binom{z}{n}(1 - \phi)^n\phi^{z-n}\frac{(\phi z)^{q-z-n}}{(q - z - n)!}\exp(-\phi z)$$

(24)

As this expression is valid for $z \geq 3$ (otherwise we could have finite probability for degrees $< 0$), we will take $z = 3$ in the rest of this section.

Summarizing, the degree distribution is expressed in the following way:

1. For $\phi = 0$, the distribution is $\delta(x - K)$. This case corresponds to the original grid without shortcuts, so that its distribution is a Dirac’s delta centered on the average value, $\langle q \rangle = 2z$ as all the nodes have the same connectivity.

2. For $\phi = 1$ we have a Poisson distribution $P(q) = \exp(-\langle q \rangle)\langle q \rangle^{q/2}/q!$. This is an Erdős-Renyi network, historically the first used to study complex networks. It is analytically easy to deal with, but not too representative of real networks.

3. For values of $\phi$ between 0 and 1, which better approximates real networks we have Eq. (24).

Here, $\phi$ is the reconnection probability, $z$ is the level of separation of nearest neighbors and $\langle q \rangle = 2z$ is the connectivity. Figure 3 shows the connectivity distribution for various values of neighborhood levels. We see
Figure 3: (Color online). Connectivity in Watts-Strogatz substrate for different levels of neighborhood.

that, unlike real networks, this distribution is peaked with low cutoff, with width depending only on $\phi$. But then, this model was conceived to reproduce the clustering and short path lengths observed in real networks, not their connectivity.

Observing the degree distribution in Figure 3 we see a rapid drop at large degrees, as in the random graph, therefore we use a cutoff of 20. Then the Annealed Network approximation, applied to the medium range of the reconnection probability, yields for the order parameters

\[
M = \sum_{q=3}^{20} \left\{ \frac{q}{\langle q \rangle} \cdot P(q, \phi, z) \cdot \frac{2 \sinh \beta JqM}{\exp (\beta \Delta) + 2 \cosh \beta JqM} \right\}
\]  

(25)

and

\[
1 - x = \sum_{q=3}^{20} \left\{ \frac{q}{\langle q \rangle} \cdot P(q, \phi, z) \cdot \frac{2 \cosh \beta JqM}{\exp (\beta \Delta) + 2 \cosh \beta JqM} \right\}.
\]  

(26)
As for the case of Poisson, the disordered state characterized by $M = 0$ is always a solution, and below certain value of $\Delta$ it is unique. The critical temperature is found graphically as before, looking at the intersections of $y = M$, represented by a solid line in the left of Figure 4 and the rhs of Eq. (25). In this way we get the critical value of $\beta$ for which a non-zero value of $M$ is possible:

$$\frac{C \beta_c J}{e^{\beta_c \Delta} + 2} = 1.$$  \hspace{1cm} (27)

Here $C$ is a constant which depends approximately on the reconnection probability $\phi$ by a potential law of the form

$$C = C_0 \phi^\gamma.$$  \hspace{1cm} (28)

Numerically we determine $C_0 = 5.9$ and $\gamma = 1.82$. For large $\Delta$ the exponential in Eq. (27) is too abrupt and there are no real solutions. For lower $\Delta$ there is a domain which admits real solution, which is bigger the lower is $\Delta$, so that there can be solutions of finite magnetization at low temperature. For low $\phi$, the only solution for $M$ is 0 if $J$ approaches 1.

With respect to the order parameter related with neutrality, see Figure 4 (right), when $\Delta = 0$ and $M = 0$ all solutions of $1 - x$ are null at any temperature, i.e., all the agents are neutral. Comparing this figure with Figure 2 we can see that while neutrality in the fully disordered solution is approximately the same in both models, in Watts-Strogatz the neutrality is narrower, i.e., the same neutrality corresponds to lower magnetization.

The effect of varying it and $J$ can be seen in Figure 5 which represents neutrality for a fixed $\phi$. For a fixed $\Delta$, increasing $J$ reduces the region of high neutrality. Comparing to the random graph, the addition of shortcuts
increases the importance of the strength of interactions. If the message can be conveyed to other distant groups, its repercussion is bigger and overcomes the propensity to immobility or neutrality. This influence is stronger than increasing $\Delta$ while keeping $J$ constant.

4.3. Newman Substrate

Let’s analyze now the variant proposed by Newman to the Watts-Strogatz model, as described in Section (4.2), and remember that to build this model, Newman starts from a regular unidimensional grid in which each node is connected to $2z$ nearest neighbors, then adds a link with probability $\phi$ to a randomly chosen pair of nodes.

The calculations are simpler in this case. We start initially from a degree $2z$ without shortcuts, so that we have $nz$ links. If now we add a shortcut with probability $\phi$ we will have $nz\phi$ shortcuts, $2nz\phi$ shortcut ends from which $2z\phi$
Figure 5: (Color online). Effect of varying $J$ and $\Delta$ on the neutrality, represented in $z$ axis, in the Watts-Strogatz substrate.
Figure 6: (Color online). Connectivity in Newman substrate for different levels of neighborhood.

link by average to a given node. The number $s$ of shortcuts which end in a given node follows a Poisson distribution with mean $2z\phi$:

$$P(s) = e^{-2z\phi} \frac{(2z\phi)^s}{s!}. \tag{29}$$

As the degree of a node is $q = 2z + s$ the distribution is

$$P(q, z, \phi) = e^{-2z\phi} \frac{(2z\phi)^{q-2z}}{(q-2z)!} \tag{30}$$

for $q \geq 2z$, and $P(q, z, \phi) = 0$ for $q < 2z$.

Figure 6 shows the connectivity distribution of this model for different levels of next neighbors.

Observing the rapid drop at large degrees in the distribution for $z = 3$ it is apparent that we can safely use a cutoff of 20. Then the Annealed Network approach having substituted the degree distribution Eq. (30) in Eq. (17)
yields for the order parameters

\[ M = \sum_{q=6}^{20} \left\{ \frac{q}{\langle q \rangle} \cdot P(q, z, \phi) \cdot \frac{2 \sinh \beta J q M}{\exp (\beta \Delta) + 2 \cosh \beta J q M} \right\} \quad (31) \]

and

\[ 1 - x = \sum_{q=6}^{20} \left\{ \frac{q}{\langle q \rangle} \cdot P(q, z, \phi) \cdot \frac{2 \cosh \beta J q M}{\exp (\beta \Delta) + 2 \cosh \beta J q M} \right\}. \quad (32) \]

As in former models \( M = 0 \) is always a solution, and there is a limiting value of \( \Delta \) for which it is unique, so that below this value only exists the disordered state and above there are more finite solutions.

For large \( \Delta \) the exponential in Eq. (33) is too abrupt and there is not a real value for critical temperature, implying that finite magnetization at low temperature is not possible. On the contrary, for low \( \Delta \) there is a domain which admits real solution, bigger the lower is \( \Delta \), and in this case finite magnetization at low temperature is possible. The effect of varying \( \Delta \) and \( J \) can be seen in Figure 7. Comparing this dependence with the corresponding one shown in Figure 5 one can see that both cases have an inverse dependence on the average degree, being larger in the Newman’s case. One can also observe that the increase of \( \Delta \) leads to higher values for the neutrality, in correspondence with the results in [36] where higher values of the same parameter (interpreted as urban attractiveness) lead to an unwelcoming environment as more and more agents leave.

For low \( \phi \), the only solution for \( M \) is 0 if \( J \) approaches 1. The critical temperature is found graphically as before, looking at the intersections of \( y = M \) and the rhs of Eq. (31) to get the critical value of \( \beta \) at which a non-zero value of \( M \) is possible:

\[ \frac{C \beta_c J}{e^\beta_c \Delta + 2} = 1. \quad (33) \]
Figure 7: (Color online). Effect of varying $J$, $\Delta$ and $\phi$ on the neutrality, represented in $z$ axis, in the Newman substrate.
where $C$ is a constant with an approximate linear dependence on the reconnection probability $\phi$ of the form

$$C = C_0 + \alpha \phi.$$  

and numerically we obtain $C_0 = 12.1, \alpha = 12.9$.

In order to understand the effect of the topology on the model behavior, we have performed an additional analysis for different values of the probability of connection $\phi$, shown in Figure 7. As can be seen, although the quantitative behavior vary upon the probability distribution of the shortcut connections, the qualitative behavior does not depend very much on the detail. This observation also holds when comparing the results with the Mean Field (MF) results of the Blume-Capel model on a regular lattice ($\phi \rightarrow 1$), although one observes a slight enhancement of the area of the ordered phase when decreasing the value of $\phi$.

4.3.1. Phase Diagram

In this section we will study the Newman substrate in further detail. The reason is that we think that while it seems realistic to establish new links between agents, it is not so to destroy existing ones. This could be the case of the break of diplomatic relations in case of war. But then, the war is not subject of study in this paper.

Figure 8 shows the phase diagram for $z = 3$. We can see in it three main regions. The solid line separates a disordered phase with $M = 0$ on the left from an ordered phase with $M \neq 0$ on the right. The line $\Delta = J$ separates the region of finite magnetization with an appreciable concentration of neutral agents from another in which alignment is so strong that there are almost no
neutrals in temperatures for which fluctuations are not important.

![Phase diagram in Newman substrate for $z = 3$.](image)

If $J$ is high enough, one of the options prevails over the other and the existence of neutrality is strongly influenced by the value of $\Delta$, being higher the higher the temperature. For $\Delta$ below that line, the neutral component is negligible.

For this case, a comparison with the results obtained by using the Ising model is interesting [16, 19]. For values of the parameter $\Delta$ below the dashed line shown in Fig. 8, the social influence is stronger, and the agents make a decision in accordance with the classical Ising model in absence of external field (absence of personal inclination in [16, 19]).

In our model, the effect of network topology is manifested on its effect on the connectivity of the reconnection probability. A high value of $\phi$ favors alignment, as it increases connectivity and hence the amount of interactions.
for the same $J$ so that the null magnetization phase occurs for smaller $J$.

**Region I.** $J$ small, $\Delta$ big. **Null magnetization with finite neutrality.**

In this region the competing options do not appeal the agents enough so as to produce a clear prevalence. Due to the low intensity of interaction in comparison with the propensity to remain neutral, very few agents decide to align, in such a way that magnetization is kept low because the alignment is symmetric.

In politics this is like apathy or general disappointment with the main contending options produced by these not being able to communicate, or failing to convey the right message. In short, abstention wins.

Figure 9 displays graphs of solutions for magnetization and neutrality for different values of the control parameters in this region.
Region II. $J$ big, $\Delta > J$. Finite magnetization with finite neutrality. In this region, the strength of interactions is such that the advantages of one of the contending options are clearly perceived, but however the effect of $\Delta$ favouring neutrality is stronger.

If $\Delta \gg J$ neutrality prevails over other options in equilibrium states although the magnetization is not null. People behave in a conservative way, not aligning, not making investments, not expending, but more meaningfully, not deciding. Politic parties don’t excite the society and the options voted win with notorious abstention.

Figure 10 displays graphs of solutions for magnetization and neutrality for different values of the control parameters in this region.

Region III. $J$ big, $\Delta < J$. Finite magnetization with low neutrality. This region corresponds to the clear predominance of one of the options,
whose advantages the agents perceive clearly, so that they feel motivated for a sharp alignment. The theoretical solution corresponds to high magnetization with null neutrality, with a prevailing option which depends on the initial state.

In politics, this is the situation of absolute majority of a party. Commercially, one of the options is the preferred choice by majority.

Figure 11 displays graphs of solutions for magnetization and neutrality for different values of the control parameters in this region.

5. Discussion

In this paper we have studied a 3-states magnetic model residing on three topologically different substrates, establishing sociophysical analogies for the control parameters.
Local interaction is determined by the coupling constant $J$. Socially corresponds to an exchange of information, which happens formally as advertisement, or by direct contact. In politics is related to commercial or diplomatic treatises. The anisotropy term $\Delta$, which in the Blume-Capel model is associated to the chemical potentials of transformation from one species to another, represents in our model the energetic cost to pass from neutrality to alignment or vice versa.

Global interaction, in turn, is determined by the reconnection probability $\phi$. It defines the number of linked agents and hence the connectivity. Therefore, it has only any impact in the Newman substrate, because connectivity is fixed in the others. The effect of this parameter is manifested as a decrease in neutrality, almost unseen in the Watts-Strogatz substrate. This seems to indicate that isolation favors neutrality and is appreciable in the extremes, i.e., for the finite solutions of magnetization. In sociophysics, this means that the polarization is more intense for higher connectivity, which can be interpreted perhaps paradoxically as if a better communication favors discrepancy.

Neutrality is strongly conditioned by the anisotropy constant $\Delta$. Its effect is apparent in the critical temperature and is the key control parameter in our model, as the magnetization is only relevant to distinguish between ordered and disordered states. For a small value of $\Delta$ the system tends to distribute in the two contending options, but if $\Delta$ is too high, there is no chance for consensus in any of the options, so that all the agents are indifferent (neutral). Socially, the higher interaction strength due to the bigger connectivity implies better communication and information, thus canceling
the effects of indifference towards the advantages of any of the contending options. \( \Delta \) can be regarded as opposition to the main options in a two-party politic system, or in other words, abstention with sympathy for minority options. In diplomacy it represents the advantages of not aligning with any of two conflicting options. When it comes to study customer options, it is the savings of not buying anything.

The results presented in this work can be compared with those obtained by using the RFIM \[16, 19\]. For low values of the parameter \( \Delta \) the system tends to align in a binary-like scenario, thus leading to well known MF results. It is also interesting to point out that there is a good qualitative agreement with the MF results of the Blume-Capel model on a regular lattice, although the quantitative comparison depends on the probability of connections.

The analogy between the BEG model and his variant of Blume-Capel spin-1 model with the Schelling’s model \[35\] and its generalization for an open city \[36\] permits to relate variables such as neutrality (in opinion formation) with empty location (in Schelling’s models), showing similar behaviors in the sense that higher values of the anisotropy parameter \( \Delta \) – the urban attractiveness in \[36\] – leads to higher neutrality.

With respect to the substrate, we found that in Newman substrate the value of neutrality is less than in Watts-Strogatz for the same value of \( \Delta \) and \( J \). This appears to be due to the higher connectivity, because in the Newman model links are just added without deleting previously existing ones.

For high reconnection probability, Watts-Strogatz substrate approaches the random graph, as its degree distribution tends to Poisson for reconnection probability near to 1. Logically, for low reconnection probability, Newman
and random graph models are more similar as both approach the regular grid behavior (compare Figures 2 and 4 with the different regions in the phase space of Newman substrate model). We think that the later represents better more real networks in which an agent has closer relations with its neighbors. Vicinity must be understood here as imposed by the system considered: work mates, neighboring countries, etc., extending links equally intimate with geographically far agents, so being neighbors in this concept of distance. We also feel this model is more realistic as, being normal to establish new relations, it is not so normal that these replace existing relations. Moreover, the connectivity in this model varies with the reconnection probability, which allows to study more different situations.

In the Newman substrate the critical temperature is lower than in Watts-Strogatz, while below this temperature finite solutions for magnetization are higher. The meaning of the existence of a critical temperature is that a very high agitation, either thermal as in the magnetic model, or social, prevents the agents to settle in a fixed state. In this paper we interpret the temperature as a finite probability of doing a change of state which is not energetically prescribed. In sociophysical terms, an agent will take a decision which is not the most interesting.

To summarize, if we model society as made of individuals which can adopt one of two possible options or none of these, we conclude that the higher connectivity will increase the level of alignment at the expenses of neutrality. The RFIM provides a unifying framework to account for many collective socioeconomic phenomena that lead to endogenous ruptures and crisis. Moreover, it has been quite successfully used to analyze quantitatively
real data [22]. However, we think that the three-states model considered in this work is a good approach for modeling more complex decision processes. A straightforward comparison between our model and real situations is difficult, but the World War II, the Cold War [12] or the fragmentation of the ex-Yugoslavia [54] might be good real examples of application.

In order to test our analytical results and to apply our model to more complicated and realistic situations, we have developed an agent-based computer simulation tool. The simulator is based on the Netlogo v5.0.2 and implements the Uri Wilenski algorithm for the generation of different substrates [52]. To identify small worlds, the average path length and clustering coefficient of the network are calculated and plotted. All control parameters can be set up, and both numerical and graphical outputs are available. With it, we have confirmed that the outcomes correspond to stable configurations in a broad range of the control parameters, in good agreement with the analytical results. Moreover, as a first step towards more realistic cases, we have applied our model to the study of the coalitions during the Cold War using real data obtained from The Correlates of War Project (COW) [55]. For that purpose, we chose 1955 as the oldest year, for which good commerce real data are known, and used the correspondent commerce flows between different countries to fix the coupling interactions in the context of our model. We ran simulations in Region III of the phase diagram (see Figure 8). Our preliminary results show the grouping of two big coalitions, with a good resemblance to the NATO and Warsaw Pact, as well as the group of neutral countries. More detailed work in this direction is in progress.
Acknowledgements

The authors thank H. Chamati and N. Tonchev for fruitful discussions, M. Jimenez for the preprocessing of real data, and the anonymous referees for their interesting comments and suggestions.

Appendix. Order parameters in the Annealed Network Approximation

If \( P(S_i) \) is the probability of a site having the spin \( S_i \) we have

\[
m_i = \langle S_i \rangle = \frac{\sum_{\{S\}} S_i P(S_i)}{\sum_{\{S\}} \exp(-\beta E_i)} = \frac{e^{-\beta E_+} - e^{-\beta E_-}}{e^{-\beta E_+} + e^{-\beta E_0} + e^{-\beta E_-}} =
\]

\[
\frac{\exp \left\{ \beta \left[ \sum_{i \neq j} (J_{ij} m_j) - \Delta \right] \right\} - \exp \left\{ -\beta \left[ \sum_{i \neq j} (J_{ij} m_j) + \Delta \right] \right\}}{
\exp \left\{ \beta \left[ \sum_{i \neq j} (J_{ij} m_j) - \Delta \right] \right\} + \exp \left\{ -\beta \left[ \sum_{i \neq j} (J_{ij} m_j) + \Delta \right] \right\} + 1
\]

\[
= \frac{\exp \left\{ \beta \sum_{i \neq j} (J_{ij} m_j) \right\} - \exp \left\{ -\beta \sum_{i \neq j} (J_{ij} m_j) \right\}}{\exp \left\{ \beta \sum_{i \neq j} (J_{ij} m_j) \right\} + \exp \left\{ -\beta \sum_{i \neq j} (J_{ij} m_j) \right\} + \exp(\beta \Delta)} \quad (A.1)
\]

and finally

\[
m_i = \frac{2 \sinh \left[ \beta \sum_{i \neq j} (J_{ij} m_j) \right]}{e^{\beta \Delta} + 2 \cosh \left[ \beta \sum_{i \neq j} (J_{ij} m_j) \right]} \quad (A.2)
\]

Note the formal similarity with Eq. [8]. Proceeding in the same way for
the neutrality we have

\[ 1 - x = \langle S_i^2 \rangle = \frac{\sum_{\{S\}} S_i^2 P(S_i)}{\sum_{\{S\}} \exp(-\beta E_i)} = \frac{e^{-\beta E_k} + e^{-\beta E_-}}{e^{-\beta E_k} + e^{-\beta E_0} + e^{-\beta E_-}} = \]

\[ = \exp \left\{ \beta \left[ \sum_{i \neq j} (J_{ij} m_j) - \Delta \right] \right\} + \exp \left\{ -\beta \left[ \sum_{i \neq j} (J_{ij} m_j) + \Delta \right] \right\} = \]

\[ = \frac{2 \cosh \left[ \beta \sum_{i \neq j} (J_{ij} m_j) \right]}{e^{\beta \Delta} + 2 \cosh \left[ \beta \sum_{i \neq j} (J_{ij} m_j) \right]} \quad (A.3) \]

Using now the Annealed Network Approximation described in Section 3 we get the following approximations for the order parameters

\[ m_i = \frac{2 \sinh \left[ \frac{\beta q}{q_j} \frac{\sum_j q_j m_j}{N} \right]}{e^{\beta \Delta} + 2 \cosh \left[ \frac{\beta q}{q_j} \frac{\sum_j q_j m_j}{N} \right]} = \frac{2 \sinh \beta J_{q_i} M}{e^{\beta \Delta} + 2 \cosh \left[ \beta J_{q_i} M \right]} \quad (A.4) \]

and

\[ 1 - x = \frac{2 \cosh \left[ \beta \sum_{i \neq j} (J_{ij} m_j) \right]}{e^{\beta \Delta} + 2 \cosh \left[ \beta \sum_{i \neq j} (J_{ij} m_j) \right]} = \frac{2 \cosh \beta J_{q_i} M}{e^{\beta \Delta} + 2 \cosh \left[ \beta J_{q_i} M \right]} \quad (A.5) \]
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