1. Introduction

The notion of a Massey F-product was introduced in [4] in order to give a unified description of various deformation problems in terms of an extension of the usual Massey product. The usual Massey product which arises in algebra deformation theory describes the conditions under which an infinitesimal deformation can be prolonged to a higher order or formal deformation. The Massey F-product was introduced in order to solve some more general deformation problems. Associated to each Massey F-product is a $\mathbb{Z}_2$-graded coalgebra, which depends on the problem being described. In this paper we consider some examples which arise in connection with the problem of deforming $A_\infty$ and $L_\infty$ algebras, as well as the more restricted problem of deforming associative algebras into $A_\infty$ algebras and Lie algebras into $L_\infty$ algebras. The general example of deforming $A_\infty$ and $L_\infty$ algebras turns out to give the same coalgebra structure as arises in the problem of prolonging an infinitesimal deformation of a $\mathbb{Z}_2$-graded Lie algebra into a higher order deformation. The restricted problem yields a more interesting coalgebra structure, which we will describe here.

We also consider deformations of $A_\infty$ and $L_\infty$ algebras with a base given by a $\mathbb{Z}_2$-graded commutative algebra, and prove an extension of a theorem in [4], that deformations of Lie algebras with a base are determined by triviality of Massey F-products.

In order to make the reading of this paper more self-contained, we include definitions of $A_\infty$ and $L_\infty$ algebras. $A_\infty$ algebras were first introduced by J. Stasheff in [16, 17], while the notion of an $L_\infty$ algebra first appeared in [14]. Our definitions are based on the exposition in [13], but the articles [10, 11, 12, 5] contain an introduction to infinity algebras as well.

2. Notation

In the following we shall be considering vector spaces graded by some group equipped with a $\mathbb{Z}_2$-valued inner product. For any graded vector space $V$, define maps $S : V \otimes V \to V \otimes V$ and $C : V \otimes V \otimes V \to V \otimes V \otimes V$ by $S(u \otimes v) = (-1)^{uv}(u \otimes v)$ and $C(u \otimes v \otimes w) = (-1)^{u(w+v)}u \otimes w \otimes u$, where we adopt the convention that expressions like $(-1)^{uv}$ stand for the sign determined by the inner product of the gradings of $u$ and $v$. These signs appear, for example, when considering tensor products of mappings from graded spaces.

A graded coassociative coalgebra is a graded vector space $V$ equipped with a degree zero mapping (comultiplication) $\Delta : V \to V \otimes V$ satisfying $(1 \otimes \Delta) \circ \Delta =
$(\Delta \otimes 1) \circ \Delta$. It is cocommutative if $S \circ \Delta = \Delta$. In what follows, we shall consider $\mathbb{Z}_2$-graded vector spaces, but everything said applies to the $\mathbb{Z}$-graded case as well. If $V$ is a $\mathbb{Z}_2$-graded vector space, then the tensor and exterior (co)algebras are $\mathbb{Z}_2 \times \mathbb{Z}$-bigraded algebras. We shall encounter two different inner products on $\mathbb{Z}_2 \times \mathbb{Z}$, the usual inner product given by $\langle (k, m), (l, n) \rangle = kl + mn$, and the inner product $\langle (k, m), (l, n) \rangle = (k + m)(l + n)$, which we will call the good inner product, for reasons which will appear later. (In the formulas above, the bar stands for integers mod 2.) We consider the two inner products as giving different gradings on $T(V)$. In both gradings, a homogeneous element $v$ of bidegree $(k, m)$ will be called odd if $k + m$ is an odd integer. If $v$ is an element of a $\mathbb{Z}_2 \times \mathbb{Z}$-graded space $V$ then $(-1)^v = (-1)^{|v|}$ is the sign determined by the parity of $v$.

3. $A_\infty$ Algebras

Suppose $V$ is a $\mathbb{Z}_2$-graded vector space, and consider the tensor coalgebra $T(V) = \bigoplus_{n=1}^{\infty} V^n$. $T(V)$ has a natural $\mathbb{Z}_2 \times \mathbb{Z}$-bigrading, the bidegree of $v \in V^n$ being given by $\text{bid}(v) = (|v|, n)$. An $A_\infty$ algebra is a vector space $V$ with an odd map $\mu \in \text{Hom}(T(V), V)$, in other words a sequence of maps $\mu_k : V^k \to V$, which satisfy for $n \geq 1$,

$$
\sum_{k+i=n+1 \atop 0 \leq j \leq k-1} (-1)^j \mu_k(v_1, \ldots, v_j, v_{j+1}, \ldots, v_{j+l}, v_{j+l+1}, \ldots, v_n) = 0,
$$

(1)

where $r = l(v_1 + \cdots + v_j) + j(l - 1) + (k - 1)i$. An $A_\infty$ algebra may be seen as a generalization of an associative algebra, and in particular, if $\mu$ consists only of an $\mu_2$ part, then it is an associative algebra. To explain the origin of the signs above consider the complex, $C(V) = \text{Hom}(T(V), V)$ with the usual $\mathbb{Z}_2 \times \mathbb{Z}$-grading. The bigrading on $C(V)$ is given by $\text{bid}(\varphi) = (|\varphi|, k - 1)$ for $\varphi \in C^k(V) = \text{Hom}(V^k, V)$. There is a natural isomorphism $C(V) \cong \text{Coder}(T(V))$, the $\mathbb{Z}_2 \times \mathbb{Z}$-graded coderivations of $T(V)$. Since $\text{Coder}(T(V))$ is a graded Lie algebra with respect to the usual $\mathbb{Z}_2 \times \mathbb{Z}$-grading, $C(V)$ has a Lie bracket, which is given by

$$
[\varphi, \psi](v_1, \ldots, v_n) = \sum_{0 \leq j \leq k-1} (-1)^j \varphi(v_1, \ldots, v_j, \psi(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n)
$$

$$
- (-1)^r \psi(\varphi(v_1, \ldots, v_j, v_{j+1}, \ldots, v_{j+l+1}, \ldots, v_n)).
$$

(2)

for $\varphi \in C^k(V)$, $\psi \in C^l(V)$, where $n = k + l - 1$, $r = \psi(v_1 + \cdots + v_j) + j(l - 1)$ and $s = \varphi(v_1 + \cdots + v_j) + j(k - 1)$. This bracket is called the Gerstenhaber bracket. Lie algebras with the usual $\mathbb{Z}_2 \times \mathbb{Z}$-grading have some undesirable properties, but there is a procedure for modifying the bracket which transforms a Lie algebra with the usual $\mathbb{Z}_2 \times \mathbb{Z}$-grading into a Lie algebra with the good $\mathbb{Z}_2 \times \mathbb{Z}$-grading. The modified Gerstenhaber bracket is defined by

$$
\{\varphi, \psi\} = (-1)^{(k-1)s} [\varphi, \psi].
$$

An $A_\infty$ structure is determined by an odd element $\mu$ satisfying $\{\mu, \mu\} = 0$. This definition is equivalent to the relations given by equation (1). Then $\delta(\varphi) = \{\mu, \varphi\}$ defines a differential on $\text{Hom}(T(V), V)$ and gives it a DGLA (Differential Graded Lie Algebra) structure. The property that $\delta^2 = 0$ follows from the Jacobi identity. This property is what makes the good grading behave better than the usual one.
We shall call $\mu$ the codifferential determining the $A_\infty$ structure on $V$, although it is not precisely a codifferential, since it is given by the vanishing of the modified bracket, rather than the bracket of coderivations.

4. $L_\infty$ Algebras

To define an $L_\infty$ algebra, we begin with the exterior algebra $\Lambda V$ of a $\mathbb{Z}_2$-graded vector space $V$. With respect to the usual $\mathbb{Z}_2 \times \mathbb{Z}$-grading, $\Lambda V$ is a cocommutative coalgebra. For $v_1 \wedge \cdots \wedge v_n \in \Lambda V$ and any permutation $\sigma$, $\epsilon(\sigma; v_1, \cdots, v_n)$ is defined by the equation

$$v_1 \wedge \cdots \wedge v_n = (-1)^\sigma \epsilon(\sigma; v_1, \cdots, v_n) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)},$$

(3)

where $(-1)^\sigma$ is the sign of the permutation $\sigma$. For ease of notation, $\epsilon(\sigma; v_1, \cdots, v_n)$ will be denoted by $\epsilon(\sigma)$. An $L_\infty$ algebra is given by an odd map $l \in C(V) = \text{Hom}(\Lambda V, V)$, that is, with a sequence of maps $l_k : \Lambda^k V \to V$, with $|l_k| = k$, which satisfy

$$\sum_{k+l=n+1} (-1)^{k} \epsilon(\sigma)(-1)^{(k-1)l} l_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}), v_{\sigma(k+1)} \cdots, v_{\sigma(n)}) = 0$$

(4)

for $n \geq 1$. The set of relations above will be referred to as the generalized Jacobi identity. This structure is also called a homotopy Lie algebra and may be viewed as a generalization of a Lie algebra as follows. If $l_k = 0$ for $k > 2$ then $V$ has a DGLA structure with differential $l_1$ and bracket given by $l_2$. This generalization was motivated by deformation theory [14] and has applications in quantum mechanics and string theory [20, 21].

The bigrading on $C(V)$ is given by $\text{bid}(\varphi) = (|\varphi|, k-1)$ for $\varphi \in C^k(V) = \text{Hom}(V^k, V)$. There is a natural isomorphism $C(V) \cong \text{Coder}(\Lambda V)$, equipping it with a $\mathbb{Z}_2 \times \mathbb{Z}$-graded Lie bracket defined by

$$[\varphi, \psi](v_1, \cdots, v_n) = \sum_{\sigma \in \text{Sh}(k,l-1)} (-1)^{\sigma} \epsilon(\sigma) \varphi(\psi(v_{\sigma(1)}, \cdots, v_{\sigma(l)}), v_{\sigma(l+1)} \cdots, v_{\sigma(n)})$$

$$-(-1)^{\psi+1(k-1)(l-1)} \sum_{\sigma \in \text{Sh}(k,l-1)} (-1)^{\sigma} \epsilon(\sigma) \psi \varphi(v_{\sigma(1)}, \cdots, v_{\sigma(k)}, v_{\sigma(k+1)} \cdots, v_{\sigma(n)}),$$

(5)

$\varphi \in \text{Hom}(\Lambda^k V, V), \psi \in \text{Hom}(\Lambda^l V, V)$, and $v_1, \cdots, v_n \in V$, where $k + l = n + 1$. This bracket is simply the bracket of $\varphi$ and $\psi$ as coderivations with respect to the usual $\mathbb{Z}_2 \times \mathbb{Z}$-grading. As in the $A_\infty$ algebra case, a modified bracket is defined by

$$\{\varphi, \psi\} = (-1)^{(k-1)\psi}[\varphi, \psi].$$

(6)

The modified bracket gives $C(V)$ the structure of a $\mathbb{Z}_2 \times \mathbb{Z}$-graded Lie Algebra with respect to the good inner product on $\mathbb{Z}_2 \times \mathbb{Z}$. Then equation (6) is simply the property that $\{l, l\} = 0$. As before, we define a differential on $C(V)$ by $\delta(\varphi) = \{l, \varphi\}$. Thus $C(V)$ inherits the structure of a DGLA. We shall call $l$ the codifferential determining the $L_\infty$ structure on $V$. In this fashion, we say that $A_\infty$ and $L_\infty$ structures are determined by codifferentials, although this is not precisely true.
5. Definition of Massey F-Products

Let $L$ be a DGLA with commutator $\mu: L \otimes L \to L$, $\mu(u, v) = [u, v]$, and differential $\delta: L \to L$. Denote the cohomology of $L$ with respect to $\delta$ by $H = \bigoplus_i H^i$. Let $F$ be a graded cocommutative coassociative coalgebra with comultiplication operator $\Delta: F \to F \otimes F$. Suppose also that a filtration $F_0 \subset F_1 \subset F$ of $F$ is given, such that $F_0 \subset \text{Ker} \Delta$ and $\text{im} \Delta \subset F_1 \otimes F_1$. Let $a: F_0 \to H, b: F/F_1 \to H$ be linear maps of degree 1. We say that $b$ is contained in the Massey $F$-product of $a$ if there exists a degree 1 linear mapping $\alpha: F_1 \to L$ satisfying the condition

$$\delta \circ \alpha = \mu \circ (\alpha \otimes \alpha) \circ \Delta,$$

and such that the diagrams

$$
\begin{array}{c}
\begin{array}{ccc}
F_0 & \xrightarrow{\alpha|_{F_0}} & \text{Ker} \delta \\
| & | & | \\
F_0 & \xrightarrow{a} & H \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
F & \xrightarrow{\mu \circ (\alpha \otimes \alpha) \circ \Delta} & \text{Ker} \delta \\
| & | & | \\
F/F_1 & \xrightarrow{b} & H \\
\end{array}
\end{array}
$$

are commutative, where the vertical maps labeled $\pi$ denote the projections of each space onto the quotient space.

Note that from (5) it follows that $\alpha(F_0) \subset \text{Ker} \delta$ and $\mu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset \text{Ker} \delta$, making upper horizontal maps of the diagrams well-defined (see [1]).

In the case that $F_1 = F$, $b$ does not need to be specified and $a$ is said to satisfy the condition of triviality of Massey $F$-products if such an $\alpha$ exists.

6. Graded Lie Algebra Deformations

Massey F-Products arise in the deformation theory of graded Lie algebras in the following fashion. Let $V$ be a $\mathbb{Z}_2$-graded Lie algebra over $k$ with bracket $l(v_1, v_2) = [v_1, v_2]$. A formal deformation of $V$ may be defined as a power series

$$[g, h]_t = [g, h] + \sum_{p=1}^{\infty} t^p(\gamma_p(g, h) + \theta \beta_p(g, h)), \quad (8)$$

where $t$ is an even parameter and $\theta$ is an odd parameter. In order for the new bracket to be bilinear and antisymmetric each $\gamma_i$ and $\beta_i$ must be bilinear and antisymmetric, thus $\gamma_i$ and $\beta_i$ are cochains in the cohomology of $V$. The bracket of cochains, $\varphi \in C^k(V), \psi \in C^l(V)$, is given by

$$[\varphi, \psi](v_1, \ldots, v_n) =$$

\begin{equation}
\sum_{\sigma \in \text{Sh}(l, k-1)} \epsilon(\sigma) \varphi(\psi(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)})
\end{equation}

\begin{equation}
- (-1)^r \sum_{\sigma \in \text{Sh}(k, l-1)} \epsilon(\sigma) \psi(\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}),
\end{equation}

where $r = \varphi + (k - 1)(l - 1)$ and $k + l = n + 1$. Defining a differential on $C(V)$ by $\delta \varphi = [t, \varphi]$ for any $\varphi \in C(V)$ gives a DGLA structure on $C(V)$. Using this
structure, the deformed bracket satisfies the graded Jacobi identity precisely when
\[ \delta \gamma_p = -\frac{1}{2} \sum_{i=1}^{p-1} [\gamma_i, \gamma_{p-i}] \] (10)

\[ \delta \beta_p = -\frac{1}{2} \sum_{i=1}^{p-1} [\beta_i, \gamma_{p-i}] + [\gamma_i, \beta_{p-i}] . \] (11)

For \( p = 1 \), the sums in (10) and (11) are empty so \( \gamma_1 \) and \( \beta_1 \) are cocycles representing certain cohomology classes. The question of interest is, when does a deformation (8) exist with \( \gamma_1 \in c, \beta_1 \in b \) representing two given cohomology classes, \( c \) and \( b \)? Such a deformation exists when \( a \) satisfies the condition of triviality of Massey \( F \)-products where

\[ F = \text{span}\{e^1, e^2, \ldots, f^1, f^2, \ldots\}, \]

\[ F_0 = \text{span}\{e^1, f^1\}, \]

\[ F_1 = F, \]

\[ \text{bid} e^i = (0, 0), \text{bid} f^i = (1, 0), \]

\[ \Delta e^k = -\frac{1}{2} \sum_{i=1}^{k-i} e^i \otimes e^{k-i}, \]

\[ \Delta f^k = -\frac{1}{2} \sum_{i=1}^{k-i} f^i \otimes e^{k-i} + e^i \otimes f^{k-i}, \]

\[ a(e^i) = c, a(f^i) = b. \]

Let \( G \) be the dual algebra to \( F \), with basis \( e_k, f_k \). Then \( G \) is a commutative algebra with structure given by multiplication

\[ e_k e_l = -\frac{1}{2} e_{k+l}, \quad e_k f_l = -\frac{1}{2} f_{k+l}, \quad f_k f_l = 0. \]

If we let \( t = e_1 \) and \( \theta = f_1 \) then \( e_k = (-2)^{k-1} t^k, f_l = (-2)^{l-2} t^k \theta, \) and \( \theta^2 = 0 \). In this form, \( G \) is the maximal ideal in \( k[[t, \theta]]/\theta^2 \), and \( F_0^* = G/G^2 \).

### 7. Deformations of \( A_\infty \) and \( L_\infty \) Algebras

A formal deformation of an \( L_\infty \) or \( A_\infty \) algebra \( V \) with codifferential \( d \) is given by

\[ d_t = \sum_{i=0}^{\infty} t^i (\gamma_i + \theta \beta_i) \] (12)

with the property that \( \{d_t, d_t\} = 0 \). Here \( \gamma_i, \beta_i \) are cochains in \( C(V) \), \( \theta \) is an odd parameter, \( \alpha_0 = d \) denotes the original codifferential determining the structure on \( V \), and \( \beta_0 = 0 \). The property \( \{d_t, d_t\} = 0 \) holds exactly when, for all \( p \),

\[ \delta \gamma_p = -\frac{1}{2} \sum_{i=1}^{p-1} \{\gamma_i, \gamma_{p-i}\} \] (13)

\[ \delta \beta_p = \frac{1}{2} \sum_{i=1}^{p-1} \{\beta_i, \gamma_{p-i}\} - \{\gamma_i, \beta_{p-i}\} . \] (14)
This gives us the same coalgebra structure as the graded Lie algebra case (the apparent sign differences work out because of the change in the bracket being used).

8. Lie and Associative Algebra Case

The cochains $\gamma_i$ and $\beta_i$ may be written as infinite sums $\gamma_i = \sum_{m=1}^{\infty} \phi_{i,m}$ and $\beta_i = \sum_{m=1}^{\infty} \psi_{i,m}$, where $\text{bid}(\phi_{i,m}) = (m, m - 1)$, and $\text{bid}(\psi_{i,m}) = (m - 1, m - 1)$.

In the case of deforming a Lie algebra into an $L_\infty$ algebra or an associative algebra into an $A_\infty$ algebra $\varphi_{0.j} = 0$ for $j \neq 2$, so equations (13) and (14) may be broken down into conditions on the $\varphi$ and $\psi$ as follows:

$$\delta \varphi_{p,q} = -\frac{1}{2} \sum_{i=1}^{p-1} \sum_{j=1}^{q+1} \{ \phi_{i.j}, \phi_{p-i,q+2-j} \}$$

$$\delta \psi_{p,q} = \frac{1}{2} \sum_{i=1}^{p-1} \sum_{j=1}^{q+1} \{ \phi_{i.j}, \phi_{p-i,q+2-j} \} - \{ \phi_{i,j}, \phi_{p-i,q+2-j} \} .$$

Given sequences $c_i$ and $b_i$ of cohomology classes a deformation $[12]$ with $\varphi_{1.i} \in c_i$ and $\psi_{1.i} \in b_i$ exists if and only if $a$ satisfies the condition of triviality for the following Massey $F$-product:

$$F = \text{span} \{ e^{i,j}, f^{i,j} \}_{i+1, j+2}$$

$$F_0 = \text{span} \{ e^{i,j}, f^{i,j} \}_{i \leq 1}$$

$$F_1 = F$$

$$\text{bid} e^{i,j} = (j, j-2), \text{bid} f^{i,j} = (j-1, j-2)$$

$$\Delta e^{p,q} = -\frac{1}{2} \sum_{i=1}^{p-1} \sum_{j=1}^{q+1} e^{i,j} \otimes e^{p-i,q+2-j}$$

$$\Delta f^{p,q} = -\frac{1}{2} \sum_{i=1}^{p-1} \sum_{j=1}^{q+1} f^{i,j} \otimes e^{p-i,q+2-j} + e^{i,j} \otimes f^{p-i,q+2-j}$$

$$a(e^{i,j}) = \begin{cases} 
  c_j & i = 1 \\
  0 & i < 1 
\end{cases}$$

$$a(f^{i,j}) = \begin{cases} 
  b_j & i = 1 \\
  0 & i < 1 
\end{cases}$$

The dual algebra $G$ is spanned by $\{e_{i,j}, f_{i,j}\}_{i+1, j+2}$, and the multiplication in $G$ is defined by the formulas

$$e_{i,k} e_{j,l} = -\frac{1}{2} \delta_{i+j,k+l-2}, \quad f_{i,k} e_{j,l} = -\frac{1}{2} \delta_{i+j,k+l-2}. \quad f_{i,k} f_{j,l} = 0$$

If we let $t_i = e_{1,i}, u_i = f_{1,i}$ then $G$ is the algebra generated by $\{t_i, u_i\}_{i \geq 1}$, with the relations $u_i u_j = 0$, $t_i t_j = t_k t_l$ if $i + j = k + l$, and $t_i u_j = t_k u_l$ if $i + j = k + l$.

9. Algebraic Deformation Theory

The discussion below holds for both $L_\infty$ and $A_\infty$ algebras and the term infinity algebra will be used to refer to either one. Let $S$ be a graded commutative $k$-algebra with identity, containing a distinguished maximal ideal $m \subset S$ such that $S/m \cong k$.

Let $\varepsilon : S \to S/m = k$ be the projection with $\varepsilon(1) = 1$. Let $V$ be an infinity algebra
over the field $k$, with codifferential $d$. Then $V \otimes S$ is an $S$-module and a deformation of $V$ with base $S$ is defined to be an infinity algebra structure over $S$ on $V \otimes S$, with the condition that $1 \otimes \varepsilon : V \otimes S \to V \otimes k = V$ is an infinity algebra homomorphism.

Suppose that $S$ is finite-dimensional and we have a series $\tau = \sum \tau_n$ of $S$-linear maps $\tau_n : \wedge^n(V \otimes S) \to V \otimes S$ which satisfy the condition that $1 \otimes \varepsilon$ is an infinity algebra homomorphism, that is

$$(1 \otimes \varepsilon)\tau_n(v_1 \otimes s_1, \ldots, v_n \otimes s_n) = d_n(v_1, \ldots, v_n) \varepsilon(s_1 \cdots s_n)$$

(15)

for any $v_1, \ldots, v_n \in V, s_1, \ldots, s_n \in S$. Here, by homomorphism, we are not implying that $\tau$ actually satisfies the codifferential condition. Then a linear map $\alpha : m^* \to C(V)$ can be defined by

$$\alpha(\phi)(v_1, \ldots, v_k) = (1 \otimes \phi)(\tau_k(v_1 \otimes 1, \ldots, v_k \otimes 1) - d_k(v_1, \ldots, v_k))$$

(16)

for any linear functional $\phi \in m^*$. With this definition $\tau$ and $\alpha$ determine each other. Let $F = m^*$, $\Delta$ be the comultiplication in $F$ dual to the multipication in $m$, and $F_0 = (m/m^2)^*$. $F_0$ may be considered naturally as a subset of $F$ and in this sense $F_0 \subset \text{Ker} \Delta$. We will show that if $\tau$ is a codifferential, providing $V \otimes S$ with structure of an infinity algebra, then $\delta \circ \alpha + \frac{1}{2} \mu \circ (\alpha \circ \alpha) \circ \Delta = 0$, so the map

$$a : F_0 \xrightarrow{\alpha} \text{Ker} \delta \xrightarrow{\pi} H(V)$$

(17)

is well-defined. In this case, $a$ is called the differential of the deformation $\tau$. An infinitesimal deformation of $g$ with base $S$ is defined to be any linear map $F_0 \to H(V)$. We will use proposition $\Box$ below to show

**Theorem 1.** An infinitesimal deformation $a : F_0 \to H(V)$ is the differential of some deformation with base $S$ if and only if $-\frac{1}{\lambda} a$ satisfies the condition of triviality of Massey $F$-products.

**Proposition 1.** The operator $\tau$ satisfies the infinity algebra structure equation if and only if $\alpha$ satisfies the Maurer-Cartan equation

$$\delta \circ \alpha + \frac{1}{2} \mu \circ (\alpha \circ \alpha) \circ \Delta = 0.$$

**proof:** $(L_{\infty}$ Algebra Case$)$ Let $\{m_i\}$ be a (homogeneous) basis of $m$ and let $\{m^i\}$ be the dual basis of $F$. Suppose $m_im_j = \sum_k \epsilon_{ij}^k m_k$. Then the comultiplication is given by $\Delta m^k = \sum_{i,j} (-1)^{m_im_j} \epsilon_{ij}^k (m^i \otimes m^j)$. Also for some $\beta_n^i \in C^n(V)$, we have $\tau_n(v_1 \otimes 1, \ldots, v_n \otimes 1) = l_n(v_1, \ldots, v_n) + \sum_i \beta_n^i(v_1, \ldots, v_n) \otimes m_i$. Thus for $v_1, \ldots, v_n \in V$, and $n = k + l - 1$,

$$\tau_k(\tau_l(v_1, \ldots, v_l), v_{l+1}, \ldots, v_n)$$

$$= \tau_k(l_1(v_1, \ldots, v_l) + \sum_i \beta_l^i(v_1, \ldots, v_l) \otimes m_i, v_{l+1}, \ldots, v_n)$$

$$= \tau_k(l_1(v_1, \ldots, v_l), v_{l+1}, \ldots, v_n)$$

$$+ \sum_i (-1)^{m_i(v_1, \ldots, v_l)} \tau_k(\beta_l^i(v_1, \ldots, v_l), v_{l+1}, \ldots, v_n) \otimes m_i$$

$$= l_k(l_1(v_1, \ldots, v_l), v_{l+1}, \ldots, v_n)$$

$$+ \sum_i \beta_k^i(l_1(v_1, \ldots, v_l), v_{l+1}, \ldots, v_n) \otimes m_i$$
\[
+ \sum_{i} (-1)^{m_i(v_i+1+\cdots+v_n)} l_k(\beta_i(1, \cdots, v_i), v_{i+1}, \cdots, v_n) \otimes m_i
\]
\[
+ \sum_{i} (-1)^{m_i(v_i+1+\cdots+v_n)} l_k(\beta_i'(1, \cdots, v_i), v_{i+1}, \cdots, v_n) \otimes m_r m_s
\]
\[
= l_k(l_i(v_1, \cdots, v_l), v_{l+1}, \cdots, v_n)
\]
\[
+ \sum_{i} \beta_i^k(l_i(v_1, \cdots, v_l), v_{i+1}, \cdots, v_n) \otimes m_i
\]
\[
+ \sum_{i} (-1)^{m_i(v_i+1+\cdots+v_n)} l_k(\beta_i^l(1, \cdots, v_i), v_{i+1}, \cdots, v_n) \otimes m_i
\]
\[
+ \sum_{i} (-1)^{m_i(v_i+1+\cdots+v_n)} c_{rs}^i \beta_i^k(\beta_i^l(1, \cdots, v_i), v_{i+1}, \cdots, v_n) \otimes m_i
\]

If we use the notation \( \alpha_k(m^i) = \alpha_k^i \) then
\[
\beta_i^k(v_1, \cdots, v_k) = (-1)^{m_i(k+1+v_i+\cdots+v_k)} \alpha_k^i(v_1, \cdots, v_k).
\]

Substituting this into the previous expression yields
\[
l_k(l_i(v_1, \cdots, v_l), v_{l+1}, \cdots, v_n) + \sum_i (-1)^{m_i(n+v_i+\cdots+v_n)} M_i(v_1, \cdots, v_n) \otimes m_i,
\]
where
\[
M_i(v_1, \cdots, v_n) = \alpha_k^i(l_i(v_1, \cdots, v_l), v_{l+1}, \cdots, v_n)
\]
\[
+ (-1)^{km_i} l_k(\alpha_i(1, \cdots, v_i), v_{i+1}, \cdots, v_n)
\]
\[
+ \sum_{r,s} c_{rs}^i (-1)^{m_r m_s} \alpha_k^i(\alpha_i(1, \cdots, v_i), v_{i+1}, \cdots, v_n).
\]

The requirement for \( \tau \) to be a codifferential is that \( \{ \tau, \tau \} = 0 \) or equivalently, \( \sum_{k+l=n+1} \{ \tau_k, \tau_l \} = 0 \), for all \( n \). Since \((-1)^{kl(k+1)(l+1)+(k-1)l} = (-1)^{(l-1)k} \) we have
\[
\sum_{k+l=n+1} \{ \tau_k, \tau_l \}(v_1, \cdots, v_n) = \sum_{\sigma \in \text{Sh}(l,k-1)} (-1)^{\epsilon(\sigma)} (-1)^{(k-1)l} \tau_k(\tau_l(v_{\sigma(1)}, \cdots, v_{\sigma(l)}), v_{\sigma(l+1)}, \cdots, v_{\sigma(n)})
\]
\[
+ \sum_{\sigma \in \text{Sh}(k,l-1)} (-1)^{\epsilon(\sigma)} (-1)^{(l-1)k} \tau_l(\tau_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}), v_{\sigma(k+1)}, \cdots, v_{\sigma(n)}),
\]
so
\[
\{ \tau_k, \tau_l \} + \{ \tau_l, \tau_k \} = 2 \sum_{\sigma \in \text{Sh}(l,k-1)} (-1)^{\epsilon(\sigma)} (-1)^{(k-1)l} \tau_k(\tau_l(v_{\sigma(1)}, \cdots, v_{\sigma(l)}), v_{\sigma(l+1)}, \cdots, v_{\sigma(n)}).
\]

Because of this fact, \( \tau \) gives an \( L_\infty \) algebra structure exactly when
\[
\sum_{\substack{k+l=n+1 \\sigma \in \text{Sh}(l,k-1)}} (-1)^{\epsilon(\sigma)} (-1)^{(k-1)l} \left( l_k(l_i(v_1, \cdots, v_i), v_{i+1}, \cdots, v_n)
\]
\[
+ \sum_i (-1)^{m_i(n+v_i+\cdots+v_n)} M_i(v_1, \cdots, v_n) \otimes m_i \right) = 0.
\]
Hence the proposition is proved for $L_\infty$ algebras. 

\textbf{proof:} (A_\infty Algebra Case) Let \{m_i\} be a (homogeneous) basis of $\mathfrak{m}$ and let \{m^i\} be the dual basis of $F$. Suppose that $m_i m_j = \sum_k c_{ij}^k m_k$. Then $\Delta m^k = \sum_{i+j=k} c_{ij}^k \Delta m^i \otimes m^j$. 

Since the first term is zero by the generalized Jacobi identity and \{m^i\} is a basis, this is the same as requiring 

$$
\sum_{k+l=n+1} \sigma \in \text{Sh}(k,k-1) (-1)^{\sigma} \sum_{k+l=n+1} \delta (\alpha \otimes \alpha) \circ \Delta(\mu) = 0 
$$

for all $i$ and $n$.

Looking at the terms in the Maurer-Cartan equation, 

$$
\delta \circ \alpha(m^i) = \delta \circ \alpha^i = \{l, \alpha^i\} = \sum_{k,l} \{l_k, \alpha^i_l\}.
$$

Thus 

$$
\delta \circ \alpha(m^i)(v_1, \ldots, v_n) = \sum_{k+l=n+1} \{l_k, \alpha^i_l\}(v_1, \ldots, v_n)
$$

and 

$$
\frac{1}{2} \mu \circ (\alpha \otimes \alpha) \circ \Delta(m^i) = \frac{1}{2} \sum_{r,s} (-1)^{m_r m_s} c_{rs}^i \mu \circ (\alpha \otimes \alpha)(m^r \otimes m^s)
$$

so, by an identity similar to (18), 

$$
\frac{1}{2} \mu \circ (\alpha \otimes \alpha) \circ \Delta(m^i)(v_1, \ldots, v_n)
$$

Putting the pieces together, 

$$
(\delta \circ \alpha + \frac{1}{2} \mu \circ (\alpha \otimes \alpha) \circ \Delta)(m^i)(v_1, \ldots, v_n) = \sum_{\sigma \in \text{Sh}(l,k-1)} (-1)^{\sigma} \epsilon(\sigma) M_\sigma(v_{\sigma(1)}, \ldots, v_{\sigma(n)}).
$$

Hence the proposition is proved for $L_\infty$ algebras. 

\[Q.E.D.\]
\[ \sum (-1)^{m_i m_j} \varepsilon_{ij}^k (m^i \otimes m^j). \] Also for \( v_1, \ldots, v_n \in V, \)

\[ \tau_n(v_1 \otimes 1, \ldots, v_n \otimes 1) = \mu_n(v_1, \ldots, v_n) + \sum_i \beta_n^i(v_1, \ldots, v_n) \otimes m_i, \]

for some \( \beta_n^i \in C^n(V). \)

Thus for \( v_1, \ldots, v_n \in V, \) any \( j, \) and \( k + l - 1 = n, \)

\[ \tau_n(v_1, \ldots, v_{j-1}, \tau(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \]

\[ = \tau_n(v_1, \ldots, v_j, \mu(v_{j+1}, \ldots, v_{j+l}) + \sum_i \beta_n^i(v_{j+1}, \ldots, v_{j+l}) \otimes m_i, v_{j+l+1}, \ldots, v_n) \]

\[ = \mu_k(v_1, \ldots, v_j, \mu(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \]

\[ + \sum_i \beta_k^i(v_1, \ldots, v_j, \mu(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \otimes m_i \]

\[ + \sum_i (-1)^{m_x} \mu_k(v_1, \ldots, v_j, \beta_n^i(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \otimes m_i \]

\[ + \sum_i (-1)^{m_x} \beta_k^i(v_1, \ldots, v_j, \beta_n^i(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \otimes m_i \]

\[ + \sum_{r,s} (-1)^{m_x} c_{rs} \beta_k^i(v_1, \ldots, v_j, \beta_n^i(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \otimes m_i \]

where \( x = v_{j+l+1} + \cdots + v_n. \)

As before, if we use the notation \( \alpha_k(m^i) = \alpha_k^i \) then

\[ \beta_k^i(v_1, \ldots, v_k) = (-1)^{m_i (k+1+v_{j+1}+\cdots+v_n)} \alpha_k^i(v_1, \ldots, v_k) \]

and substituting this into the previous expression yields

\[ \mu_k(v_1, \ldots, v_j, \mu(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \]

\[ + \sum_i (-1)^{m_x (n+v_{j+1}+\cdots+v_n)} M_k(v_1, \ldots, v_n) \otimes m_i, \]

where

\[ M_k(v_1, \ldots, v_n) \]

\[ = \alpha_k^i(v_1, \ldots, v_j, \mu(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \]

\[ + (-1)^{m_x (k+v_{j+1}+\cdots+v_{j-1})} \mu_k(v_1, \ldots, v_j, \alpha_k^i(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \]

\[ + \sum_{r,s} c_{rs} (-1)^{x} \alpha_k^i(v_1, \ldots, v_j, \alpha_k^i(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n), \]

where \( x = m_x (m_x+k+v_{j+1}+\cdots+v_{j-1}). \) The requirement for \( \tau \) to be a codifferential is that \( \{\tau,\tau\} = 0 \) or equivalently, \( \sum_{k+l=n+1} \{\tau_k,\tau_l\} = 0, \) for all \( n. \) By reasoning similar
to (18) this may be expressed as
\[ \sum_{1 \leq j \leq k} (-1)^x \tau_k (v_1, \ldots, v_j, \tau_l (v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) = 0, \]
where \( x = l(k + v_1 + \cdots + v_{j-1}) + j(l - 1) \), or,
\[ \sum_{k+i=n+1 \atop 1 \leq j \leq k} (-1)^x (\mu_k (v_1, \ldots, v_j, \mu_l (v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \]
\[ + \sum_i (-1)^{m_i (n+v_1+\cdots+v_n)} M_i (v_1, \ldots, v_n) \otimes m_i) = 0, \]
where \( x = l(k + v_1 + \cdots + v_{j-1}) + j(l - 1) \). The first term is zero because of the \( A_\infty \) algebra structure on \( V \) and since \( \{ m^i \} \) is a basis, this may be written as,
\[ \sum_{k+i=n+1 \atop 1 \leq j \leq k} (-1)^{(k-1)i+l(v_1+\cdots+v_{j-1})+(j-1)(l-1)} M_i (v_1, \ldots, v_n) = 0, \]
for all \( i \) and \( n \).

From the Maurer-Cartan equation we have
\[
\delta \circ \alpha (m^i) (v_1, \ldots, v_n)
\]
\[ = \sum_{k+i=n+1} \{ \mu_k, \alpha_i^l \} (v_1, \ldots, v_n) \]
\[ = \sum_{k+i=n+1 \atop 1 \leq j \leq k} (-1)^x (\mu_k (v_1, \ldots, v_j, \alpha_i^l (v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \]
\[ + (-1)^y \alpha_k^i (v_1, \ldots, v_j, \mu_l (v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n) \],
where \( x = (l+m_i)(k+v_1+\cdots+v_{j-1})+j(l-1)+m_i+1, y = m_i(k+v_1+\cdots+v_{j-1}) \), and
\[
\frac{1}{2} \nu \circ (\alpha \otimes \alpha) \circ \Delta (m^i) (v_1, \ldots, v_n)
\]
\[ = \frac{1}{2} \sum_{k+i=n+1} (-1)^{m_r (m_s +1)} c_{rs}^j \{ \alpha_k^r, \alpha_i^s \} (v_1, \ldots, v_n) \]
\[ = \sum_{k+i=n+1 \atop 1 \leq j \leq k} c_{rs}^j (-1)^{m_r m_s + k(l+m_i) + l(m_r) (v_1+\cdots+v_{j-1}) + j(l-1)} \]
\[ \times \alpha_k^r (v_1, \ldots, v_j, \alpha_i^s (v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n). \]

Together we have,
\[
(\delta \circ \alpha + \frac{1}{2} \nu \circ (\alpha \otimes \alpha) \circ \Delta) (m^i) (v_1, \ldots, v_n)
\]
\[ = \sum_{k+i=n+1 \atop 1 \leq j \leq k} (-1)^{(k-1)i+l(v_1+\cdots+v_{j-1})+(j-1)(l-1)} M_i (v_1, \ldots, v_n). \]

Thus the proposition is proved for \( A_\infty \) algebras.
10. Conclusion

When the examples of Massey F-products in [4] were gathered together, the examples which we present here were known to the authors, but it was decided that the extra definitions necessary to appreciate the deformation theory of infinity algebras merited a separate treatment. At the same time, we did not know how to extend the notion of deformation of algebras with a base to infinity algebras. When we discovered how to do this, we decided that the combination of results might be interesting, and so we have collected them in this article. We would like to take this opportunity to thank Dmitry Fuchs, who encouraged us to write this paper, and who offered some insightful remarks.

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