A level set method for shape reconstruction in seismic full waveform inversion using a linear elastic model in 2D

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Abstract. We present a novel shape reconstruction technique for seismic full waveform inversion that uses a linear elastic system in 2D for modeling wave propagation in the Earth and a level set technique for performing shape evolution. A cost functional is formulated and minimized by a gradient based descent technique with backtracking line search and an Armijo condition. The three distributed elastic parameter profiles are in our model assumed to follow a given reference profile inside an unknown region \( D \), and a different reference profile outside of \( D \). The task is to reconstruct the unknown region \( D \) from seismic data obtained at or close to the surface. We present numerical experiments that address the imaging of salt domes buried in the Earth. The results demonstrate that our new algorithm is able to reconstruct shapes with non-trivial topology from a simpler starting guess in few iterations.

1. Introduction
Seismic full waveform inversion (FWI) has become an interesting alternative to more traditional geophysical exploration techniques [12]. A sufficiently accurate model for describing seismic wave propagation in the Earth is given by a linear elastic system, which however is high-dimensional and therefore difficult to handle computationally as well as theoretically. Due to this, many practical algorithms for data inversion are based on lower-dimensional approximations such as the acoustic wave equation in time domain or a Helmholtz equation in frequency domain. However, it is not clear whether those simplified models are able to correctly represent fundamental physical processes occurring during the wave propagation, possibly leading to incorrect results or other difficulties. This is why in our study we use a linearised elastic wave model instead which is assumed to be more accurate. So far this is done in a proof-of-concept approach using a 2D setup. We anticipate, however, that the algorithm as well as general observations directly extend to a more realistic elastic 3D setup.

Regardless of the model used for wave propagation, the inverse problem of seismic FWI is highly ill-posed such that strong regularization techniques are necessary in order to achieve stable results. Usually regularization techniques aim at incorporating any available a-priori information on the desired solution into the reconstruction. In the work presented here, we use a shape-based approach for this task which is targeting situations where structural a-priori information is available from additional imaging modalities such as gravity, well-logs, controlled source electromagnetic or outcrop data which often can be carried out prior to a full-waveform seismic survey.
During the iterative non-linear inversion strategy, an initial profile is successively modified until a provided cost functional is minimized. We use a level set representation of shapes for this iteration which provides us with sufficient flexibility to adjust the interfaces during the reconstruction process to the unknown topology of the hidden objects in an efficient and automatic way. Related approaches for a simpler scalar acoustic or Helmholtz model have been presented in [3, 4, 5]. Our approach goes beyond that work in that it directly incorporates a more complex time-dependent elastic wave propagation model in the level set evolution.

2. The elastic forward model for waveform inversion

We model seismic wave propagation in a 2D Earth by a first-order symmetric hyperbolic system [7] of the following form

\[
\begin{align*}
\frac{\partial v_x}{\partial t} &= \frac{1}{\rho} \left( \frac{\partial \eta_{xx}}{\partial x} + \frac{\partial \eta_{xy}}{\partial y} + F_x \right) \\
\frac{\partial v_y}{\partial t} &= \frac{1}{\rho} \left( \frac{\partial \eta_{xy}}{\partial x} + \frac{\partial \eta_{yy}}{\partial y} + F_y \right) \\
\frac{\partial \eta_{xx}}{\partial t} &= 2\mu \frac{\partial v_x}{\partial x} \\
\frac{\partial \eta_{yy}}{\partial t} &= 2\mu \frac{\partial v_y}{\partial y} \\
\frac{\partial \eta_{xy}}{\partial t} &= \mu \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \\
\frac{\partial p}{\partial t} &= \lambda \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right)
\end{align*}
\]  

(1)

with appropriate initial and boundary conditions. It models elastic wave propagation inside a domain \( \Omega \) [7] where \( \rho \) denotes density and \( \mu \) and \( \lambda \) are two Lamé parameters. The quantity \( v = (v_x, v_y)^T \) is a vector function describing velocity, \( \eta \) is a \( 2 \times 2 \) tensor function related to the stress tensor, and \( p \) is a scalar function describing pressure. \( F_x \) and \( F_y \) are seismic source terms, \( t \) denotes time and \( x \) and \( y \) the two cartesian directions.

The unknown model parameters \( (\rho, \mu, \lambda) \) used in (1) are related to the P and S wave velocities \( V_p \) and \( V_s \) by

\[
V_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad V_s = \sqrt{\frac{\mu}{\rho}},
\]  

(2)

which gives rise to the equivalent alternative representation of model parameters as \( (\rho, V_p, V_s) \). We will use both representations in this paper. The system (1) can be written in more compact notation as

\[
\mathcal{L}(m)w = \Gamma(m) \frac{\partial w}{\partial t} + D_x \frac{\partial w}{\partial x} + D_y \frac{\partial w}{\partial y} = q
\]  

(3)

where \( m \) represents the estimated model parameters \( (\rho, \mu, \lambda) \), \( w \) the states, \( q \) the seismic source, \( \Gamma(m) \) is a diagonal matrix containing functions of \( (\rho, \mu, \lambda) \) on its diagonal, and \( D_x, D_y \) are scalar symmetric matrices.

Let us assume that we apply \( s = 1, \ldots, S \) different source patterns \( q^s \) (usually located at discrete positions \( x_s \) with a prescribed time dependence) giving rise to solutions \( w^s \) of (3). For each source, we measure these states \( w^s \) at a discrete set of \( r = 1, \ldots, R \) different receivers located at positions \( x_r \). In our case, source positions \( x_s \) and receiver positions \( x_r \) are close to the surface above the region of interest. Let us denote these measured data by \( d_{sr} = \mathcal{P}_r w^s \) where \( \mathcal{P}_r \) is the linear measurement operator at receiver with index \( r \). For example, it might record point measurements of all components of \( w^s \) at the location \( x_r \). For a given model \( m \) we can
simulate corresponding data by running our forward model with a seismic simulator producing simulated data \(d_{sr}[m]\). The mathematical task in the inverse problem will be to minimize the mismatch between predicted data \(d_{sr}[m]\) and measured data \(\hat{d}_{sr}\) in some sense. Here we use the minimization of a classical least squares data misfit for that purpose, namely

\[
\min_m E(m) = \frac{1}{2} \sum_s \sum_r \|d_{sr}[m] - \hat{d}_{sr}\|^2_2
\]

constrained by

\[
\mathcal{P}_r w^s = d_{sr}[m] \quad \text{(State-data map)}
\]
\[
\mathcal{L}(m) w^s = q^s \quad \text{(Elastic wave equation)}.
\]

This inverse problem is ill-posed and requires some form of regularization for its stable solution. We are using a model based approach for this regularization task based on a level set representation of parameters as explained in the next section.

3. The new level set shape reconstruction approach

In this work we assume that the unknown model parameters are shape-like in the sense that regions filled with constant and known parameters are embedded in a smoothly varying (also known) background. This way we reduce the number of unknowns and thereby stabilize the underlying inverse problem. Furthermore, we assume that all model parameters share the same regions and interfaces towards the smooth background profile. The main application we have in mind is the mapping of salt bodies in seismic full waveform inversion. A similar approach has been taken in [3, 4, 5], however using a scalar Helmholtz model in frequency domain instead.

In our level set approach, we assume that the parameters \(\rho\), \(\mu\) and \(\lambda\) obtain a prescribed functional profile \(m_{obj}(x)\) inside an unknown region \(D\) (which can have complicated shape) and a different functional profile \(m_{back}(x)\) outside of \(D\). As already mentioned, in this paper we assume in particular that \(m_{obj}(x)\) assumes constant values for all model parameters. This assumption will be relaxed in future work, but is a good approximation when mapping salt bodies in the geophysical prospecting of hydrocarbon reservoirs.

![Figure 1](image-url)

Figure 1. Reference models for our numerical experiments. Left: test case 1; right: test case 2. For both the upper image indicates \(V_p\), the central image \(V_s\) and the bottom image \(\rho\).
In the level set representation employed here, the unknown region $D$ is implicitly represented by a level set function $\phi$ according to

$$D = \{x | \phi(x) \geq 0\},$$
$$\Omega \setminus D = \{x | \phi(x) < 0\}.$$  

The model parameter $m$ is then parameterised as

$$m = m_{\text{obj}}(x) \quad \text{for} \quad x \in D,$$
$$m = m_{\text{back}}(x) \quad \text{for} \quad x \in \Omega \setminus D.$$  

For details of the level set technique see for example [1, 6, 10]. Using the one-dimensional Heaviside function this can alternatively be written as

$$m = m_{\text{obj}}(x)H(\phi(x)) + m_{\text{back}}(x)(1 - H(\phi(x))). \quad (5)$$  

In the numerical simulations presented further below we assume that $m_{\text{obj}}(x)$ is equal to a given constant value representing salt properties, and $m_{\text{back}}(x)$ represents a typical background profile which is assumed to approximately follow a linear dependence on depth and being horizontally invariant.

A standard way of minimizing the least squares data functional (4) in a pixel or voxel based formulation for the parameters is to employ gradient techniques by a reverse-time migration adjoint-state method [2, 8]. We follow a similar route, using however the above formulation of a symmetric hyperbolic system (3) for wave propagation, and furthermore extend the resulting expressions to our level set formulation. In this new approach, gradients need to be expressed with respect to the level set function, which is achieved for example by formally applying the chain rule to the results of the classical adjoint technique using (5).

Assume that we have computed the gradient $g_{m} = \partial E/\partial m$ following such a modification of the adjoint-state method. By formally applying the chain rule, we obtain the level set based gradient $g_{\phi} = \partial E/\partial \phi$ satisfying

$$g_{\phi} = [g_{m}^{T}(m_{\text{obj}} - m_{\text{back}})]\delta(\phi(x)) \quad (6)$$  

where $^{T}$ means ‘transpose’. Once this gradient $g_{\phi}$ has been calculated, we will follow here a steepest descent style approach (using $g_{\phi}$) on the level set function combined with a backtracking line search with Armijo condition and a narrow-band approximation to $\delta(\phi(x))$ in each iteration. The numerical implementation of this approach results in a shape evolution which, upon convergence, will deliver a shape that minimizes the least squares data misfit functional (4). For more details we refer here to [1, 9].

4. Numerical experiments

We test our algorithm by using two different reference setups displayed in Figure 1, both addressing the imaging of salt domes buried in the ground with FWI. The left hand column of the figure shows a profile where only one salt dome is included in the ground. The right hand column shows a slightly more complicated setup with two separate salt domes hidden in the ground. The top row of this figure shows the p-wave velocities $V_{p}$, the center row the s-wave velocities $V_{s}$, and the bottom row the densities $\rho$ for both cases. Notice that this figure is displayed with equal axes in x- and y-directions, which indicates the real physical dimensions of the field. All following images presented here will use different axes for x- and y-directions in order to obtain a more compact form of visualization. We use 21 equidistant seismic point sources distributed along a horizontal line close to the surface (i.e. close to the top of the
computational domain displayed in Figure 1) with time dependence being of Ricker wavelet type in order to generate our data. Receivers are located at 81 equidistant locations along a different horizontal line close to the surface measuring the time series of all components of the arriving elastic waves at the receiver locations. This means that we have only a top view on the domain of interest available for gathering data, as it is typical in seismic surveys.

As forward solver for generating synthetic data as well as for reconstruction we use the elastic wave propagation module of the k-Wave toolbox [11]. We discretize the domain of interest into a grid of size $30 \times 120$ pixels each having dimension $100\text{m} \times 100\text{m}$. Therefore the total physical size of the test domain is $3\text{km} \times 12\text{km}$. The parameter values in each region can be seen in Figure 1. In this proof-of-concept study, 5% Gaussian noise is added to the data prior to running the reconstruction algorithm, in order to avoid the inverse crime. In future work, different discretization levels for simulating data will be tested as well.

![Figure 2. Level set shape reconstruction for the first test case. Left column: reference model as on the left of Figure 1; central column: initial guess; right column: final reconstruction. The bottom image shows the evolution of the least squares data misfit (4) against iteration number.](image)

Figure 2 demonstrates the result of the first test case with just one embedded object after running the algorithm for 20 iterations. As stopping criterion for both cases we assume that the relative error for the data misfit is less than $10^{-6}$. The top row shows $V_p$, the center row $V_s$ and the bottom row $\rho$ for each case. More details are provided in the caption of the figure.

Figure 3 demonstrates the results of the second test case with the same arrangement of images as in Figure 2. In this slightly more complicated geometry, 24 iterations are needed to obtain convergence.

Figure 4 visualizes the shape evolution for the second test case in steps of 3 iterations using...
Figure 3. Level set shape reconstruction for the second test case. Left column: reference model as on the right of Figure 1; central column: initial guess; right column: final reconstruction. The bottom image shows the evolution of the least squares data misfit (4) against iteration number.

$V_p$ as reference profile. The shape evolution of the $V_s$ and $\rho$ profiles is similar since they all share the same level set function.

We observe that in both cases the algorithm is able to obtain a very good approximation of the correct shapes of salt domes in the area of interest. The obtained shapes appear slightly irregular. This could be controlled, if desired, by either adding specific regularization terms to (4) or by using Sobolev gradients for the level set update as in [9]. In deeper areas some artefacts build up in the second test case which as well might be avoidable when applying stronger regularization tools.

5. Summary and future research
We have proposed a novel shape-based reconstruction technique for seismic FWI using a linear elastic system for the forward modelling and a level set technique for shape evolution. A gradient based line search strategy provides an efficient reconstruction which converges to a shape that approximates the reference model well. Therefore, this novel technique represents a promising alternative to more traditional pixel or voxel based reconstruction strategies as long as the available a-priori information justifies the incorporated structural information.

Future work will include the development of additional regularization techniques for controlling artefacts in deeper regions and obtaining smoother shape boundaries. Also the simultaneous estimation of internal and external profiles with the correct shapes, and the separate estimation of parameter-specific region for $V_p$, $V_s$ and $\rho$, are challenging tasks to work on. In addition, more efficient minimization approaches for the cost functional (4) will be investigated as for example a nonlinear Kaczmarz type approach. This includes the investigation
Figure 4. Shape evolution for the second test case: shown is $V_p$ for different iterations.

of optimal data types, measurement geometries and suitable stopping criteria.

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