Dynamic analysis of elementary differential gear with rigid links

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Abstract. At present the machine dynamics investigations are conducted in the next directions: dynamic processes research in special machines; research of the machines which links are solids; research of mechanisms representing a combinations of a solids and elastic systems. In this paper the elementary differential gear with rigid links dynamics is investigated. A number of dynamic problems can be solved accurately enough without taking into account the loaded element elasticity. Research of dynamic processes of one degree of freedom mechanisms are quite fully published in scientific literature, which cannot be said about mechanisms that have two degrees of freedom. In this paper are compiled and solved the movement equations for differential mechanism with two degrees of freedom. Mechanism has two degrees of mobility, therefore, the position of all its links is determined by two generalized coordinates. The mechanism movement is described by two Lagrange equations of II kind. Rotation angles of the drive and driven shafts are taken as generalized coordinates. System of differential equations of the second order relativ generalized coordinates is compiled. All coefficients of this system can be calculated in advance. Joint solution of equations of the system makes it possible to determine the angular accelerations of the two main links. Equations for determining angular velocities and angular accelerations of drive and driven shafts are recieved.

Effective activity of enterprises of various industries is ensured by the efficiency of the use of equipment. Rational design and effective use of equipment are considered in the research works [1 – 7]. To modern equipment of enterprises of various industries the following requirements are placed: working speeds increase, weight reduction, use of new drive systems, kinematic parameters accuracy increase, etc. [1]. This makes it necessary to study the dynamics of machines and mechanisms. Investigations on the dynamics of the machines are conducted in the following directions: research of dynamic processes in special machines; research of machines whose links are solids; research of machines which represent a combination of solids and elastic systems [8]. Works [9 – 13] devoted to the study of static and dynamic parameters of the planetary gears of various designs. In this paper the dynamics of the differential mechanism whose links are solids is investigated. A number of dynamic problems can be solve accurately enough without taking into account the elasticity of loaded elements [9]. Notice, that research of dynamic processes of one degree of freedom mechanisms are quite fully published in scientific literature, which cannot be said about mechanisms that have two degrees of freedom.

An elementary differential gear consists of three main links D, B, T and satellite group \( \Sigma g \) (figure 1). Mechanism has two degrees of mobility, therefore, the position of all its links is determined by two
generalized coordinates. The movement of the mechanism can be described by two Lagrange equations of II kind. Following external moments applied to the mechanism: to the drive shaft – driving moment $M_D$, to the driven shaft – moment of forces of useful resistance $M_B$, and to the shaft of the brake link – torque $M_T$. When fully braked, the mechanism has one degree of freedom, in other cases – two degrees of freedom.

![Kinematic scheme of elementary differential gear](image)

**Figure 1.** Kinematic scheme of elementary differential gear

Differential equation of mechanism movement in form Lagrange equations of II kind has the form [10]:

$$\frac{d}{dt} \left( \frac{\partial E_k}{\partial \omega_v} \right) - \frac{\partial E_k}{\partial \phi_v} = Q_v,$$

where $E_k$ – kinetic energy of mechanism; $Q_v$ – generalized force; $\omega_v$ – generalized angular velocity; $\phi_v$ – generalized coordinate.

For the generalized coordinates will take the angles of rotation of the drive $D$ and driven $B$ shafts – corresponding $\phi_D$ and $\phi_B$; generalized velocities are $\omega_D$ and $\omega_B$ ($\dot{\phi}_D$ and $\dot{\phi}_B$).

So, for the considered mechanical system Lagrange equations of II kind will have the form:

$$\begin{align*}
\frac{d}{dt} \left( \frac{\partial E_k}{\partial \omega_D} \right) - \frac{\partial E_k}{\partial \phi_D} &= M_D, \\
\frac{d}{dt} \left( \frac{\partial E_k}{\partial \omega_B} \right) - \frac{\partial E_k}{\partial \phi_B} &= M_B,
\end{align*}$$

(1)
where \( M_D^п \) – reduced to link \( D \) moment of forces; \( M_B^п \) – reduced to link \( B \) moment of forces; Reduced moments are coefficients in expressions of elementary works of external forces in corresponding variations of generalized coordinates.

Will define generalized forces in coordinates \( \varphi_D \) and \( \varphi_B \). For this purpose suppose, that constraints, imposed on mechanism links movement, are ideally; links are absolutely rigid bodies.

Instantaneous power of all forces acting on the mechanism

\[
N = M_D \omega_D + M_B \omega_B + M_T \omega_T. \tag{2}
\]

Because \( M_D^п \omega_D = N = M_B^п \omega_B \), generalized force on coordinate \( \varphi_D \)

\[
M_D^п = M_D + M_T i_{TD}^B, \tag{3}
\]

and generalized force on coordinate \( \varphi_B \)

\[
M_B^п = M_B + M_T i_{TB}^D. \tag{4}
\]

In expressions (2) – (4) moments and angular velocities are algebraic values and their signs are implicitly represented.

Kinetic energy of mechanism (see figure 1) will take expression [10]

\[
T = I_D^п \frac{\omega_D^2}{2} + I_B^п \frac{\omega_B^2}{2} + I_{DB}^п \omega_D \omega_B. \tag{5}
\]

where \( I_D^п \), \( I_B^п \) and \( I_{DB}^п \) – inertia coefficients of system.

For Lagrange equations of II kind derivatives: partial derivative in kinetic energy on generalized velocity (generalized impulses)

\[
\begin{align*}
\left( \frac{\partial E_k}{\partial \omega_D} \right) &= I_B^п \omega_D + I_{DB}^п \omega_B \\
\left( \frac{\partial E_k}{\partial \omega_B} \right) &= I_B^п \omega_B + I_{DB}^п \omega_D
\end{align*} \tag{6}
\]

derivative in time by generalized impulses

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial E_k}{\partial \omega_D} \right) &= I_B^п \varepsilon_D + I_{DB}^п \varepsilon_B \\
\frac{d}{dt} \left( \frac{\partial E_k}{\partial \omega_B} \right) &= I_B^п \varepsilon_B + I_{DB}^п \varepsilon_D
\end{align*} \tag{7}
\]

As generalized coordinates \( \varphi_B \) and \( \varphi_D \) are not included into kinetic energy expression (5), second number of the left part of Lagrange equations of II kind (1) will equal to zero and equations system can be written

\[
\begin{align*}
\varepsilon_D I_D^п + \varepsilon_B I_{DB}^п &= M_D + M_T i_{TD}^B, \\
\varepsilon_B I_B^п + \varepsilon_D I_{DB}^п &= M_B + M_T i_{TB}^D.
\end{align*} \tag{8}
\]

System (8) is the system of differential equations of the second order relative generalized coordinates \( \varphi_B \) and \( \varphi_D \), which all coefficients can be calculated in advance. Solving the first equation relative to \( \varepsilon_D \) and solving the second equation relative to \( \varepsilon_B \), we obtain
\[ \epsilon_D = \frac{M_D + M_T i_D^B}{i_D^B} - \epsilon_B \frac{i_B^D}{i_B^D}; \]
\[ \epsilon_B = \frac{M_B + M_T i_B^D}{i_B^D} - \epsilon_D \frac{i_B^D}{i_B^D}. \]

Joint solution of equations of the system (9) makes it possible to determine the angular accelerations of two main links

\[ \epsilon_D = \frac{i_B^D (M_D + M_T i_D^B) - i_B^D (M_B + M_T i_B^D)}{i_B^D (i_B^D)^2}; \]
\[ \epsilon_B = \frac{i_B^D (M_B + M_T i_B^D) - i_B^D (M_D + M_T i_D^B)}{i_B^D (i_B^D)^2}. \]

The equations of the system (10) make it possible to determine the angular accelerations of the two main links of elementary differential gear. Third main link acceleration is determined from the known kinematic formulas.

To solve any dynamic problems it is necessary to determine the change laws of moments \(M_D\), \(M_B\) and \(M_T\). Moments change laws make it possible to determine change laws of the angular velocities and accelerations. When solving equations of system (10) it is necessary to consider, that moment \(M_D\) doing a positive work, moment \(M_B\) doing negative work. Braking torque work can be both positive and negative, depending on the operation mode of the planetary mechanism. For the case of steady movement of mechanism the sum of the moments of all external forces is zero, i.e.

\[ M_D + M_B + M_T = 0. \]  

If the mechanism works as gearbox,

\[ M_D i_B^D = |M_B|. \]

Because \(i_B^D > 1\), consequently \(|M_B| > |M_D|\), and

\[ M_B = M_D + M_T; \quad M_D - M_B + M_T = 0. \]  

If the mechanism works as multiplier,

\[ M_B i_B^D = |M_D|. \]

Because \(i_B^D < 1\), consequently \(|M_B| < |M_D|\), and

\[ M_D = M_B + M_T; \quad M_D - M_B - M_T = 0. \]  

On base experimental data can consider that brake torque in brake process changes very slightly for correctly calculation brakes. So, in practical calculations brake torque can be considered constant.

Dynamic processes in machine drives substantially depend on motor type that is from its speed-torque characteristic. Asynchronous slip-ring motors are often used in mining machine drives. The working section of their speed-torque characteristic can be represented by the straight line. Then, the motor shaft moment value can be determined according to angular velocity from the following formula [9]:

\[ M_1 = M_n \frac{\omega_o - \omega}{\omega_o - \omega_n}, \]
where $M_n$, $\omega_n$ – nominal values of moment and angular velocity on motor shaft; $\omega_o$ – angular velocity with no load.

Will write the expression (14) in the following form

$$M_D = M_n \omega_o - \omega \omega_o - \omega_n = a - b \omega_D,$$

where $a = M_n \omega_o / \omega_o - \omega_n$; $b = M_n \omega_o / (\omega_o - \omega_n)$.

Considering that $M_T = \text{const}$, and equation (11), will write the law of the driven shaft moment change in the following form

$$M_B = -a_1 + b_1 \omega_B,$$

where $a_1 = M_0$ – constant moment; $b_1$ – angular coefficient.

Considering equations (11) and (12), will write system of equations (10):

$$\begin{cases}
A \frac{d\omega_D}{dt} = K + a - b \omega_D; \\
A_1 \frac{d\omega_B}{dt} = K_1 + a_1 - b_1 \omega_B.
\end{cases}$$

Integrating equations (15), we get

$$\begin{cases}
t = A \int_{\omega_D_0}^{\omega_D} \frac{d\omega_D}{K + a - b \omega_D}; \\
t = A_1 \int_{\omega_B_0}^{\omega_B} \frac{d\omega_B}{K_1 + a_1 - b_1 \omega_B}.
\end{cases}$$

Solving equations (17) relative $\omega_D$ and $\omega_B$, we get

$$\begin{cases}
\omega_D = \omega_D_0 \exp \left( -\frac{b}{A} t \right) + \frac{K + a}{b} \left[ 1 - \exp \left( -\frac{b}{A_1} t \right) \right]; \\
\omega_B = \omega_B_0 \exp \left( -\frac{b_1}{A_1} t \right) + \frac{K_1 + a_1}{b_1} \left[ 1 - \exp \left( -\frac{b_1}{A_1} t \right) \right].
\end{cases}$$

Equations (18) represent the change laws of angular velocities of drive and driven shafts.
Conclusions
Equations for determining angular velocities and angular accelerations for two main shafts for elementary differential gear with rigid links are received.

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