Zero Attractors of Partition Polynomials

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Abstract

A partition polynomial is a refinement of the partition number $p(n)$ whose coefficients count some special partition statistic. Just as partition numbers have useful asymptotics so do partition polynomials. In fact, their asymptotics determine the limiting behavior of their zeros which form a network of curves inside the unit disk. An important new feature in their study requires a detailed analysis of the “root dilogarithm” given as the real part of the square root of the usual dilogarithm.

Keywords: Partition, Polynomials, Asymptotics, Dilogarithm, Lerch zeta function

2000 MSC: 11M35, 11P82, 11P55, 33B30, 30E15

1. Introduction

More than twenty years ago, Richard Stanley gave a talk on the geometry of the roots of various polynomial sequences where the roots appear to lie on complicated unknown curves in the complex plane. The examples he emphasized are influential within combinatorics and include the Bernoulli polynomials \cite{15}, chromatic polynomials \cite{29}, and $q$-Catalan numbers. The list contains a single example from the theory of integer partitions \cite[Chapter 2]{1}

\[ F_n(z) = \sum_{k=1}^{n} p_k(n) z^k, \]

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which we now call the partition polynomials. Here the number \( p_k(n) \) counts the total number of integer partitions of \( n \) with \( k \) parts. This polynomial sequence appeared already in several areas of mathematics and physics. First, E.M. Wright studied \( F_n(z) \) to develop the “Wright circle method” to produce estimates for \( F_n(z) \) when \( z > 0 \) under the notion of “weighted partitions.” This paper has been an inspiration to many works in combinatorics and number theory (see [13] for example). Second, Navez et al. discovered that \( F_n(z) \) can define a probability measure called the “Maxwell’s Demon Ensemble” which approximates a Bose gas near certain temperatures [31, 18, 24].

With Richard Stanley’s talk in mind, Boyer and Goh [7, 8] initiated a study of the roots of \( F_n(z) \). They announced the results of their study but they never published proofs. In fact they confirmed Richard Stanley’s suspicion that there are indeed curves that attract the roots of the partition polynomials.

For background, the Wright circle method produces an asymptotic estimate of \( F_n(z) \) as \( n \to \infty \), which requires the choice of a “heaviest” or “dominant” singularity of a bivariate generating function \( P(z, q) \). It turns out that for \( z > 0 \), this heaviest singularity always occurs at \( q = 1 \) but for general \( z \) in the unit disk \( \mathbb{D} \) the heaviest singularity depends on the location of \( z \). Since the asymptotics for \( F_n(z) \) depends primarily on the heaviest singularity, a Stokes phenomenon [22] appears forcing the roots fall near the Stokes/anti-Stokes lines. This is analogous to the phenomenon observed by the well studied Szego curve [28] and the Appell polynomials [6, 14].

Boyer and Goh also observed that the dilogarithm \( \text{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2 \) plays an important role in determination of the heaviest singularity. In itself, this is not unusual in integer partition theory because the dilogarithm is intricately connected to partitions ([27, 33, 25] for example) but what is unusual is that overcoming the additional challenge of allowing \( z \) be a complex number required univalent function theory including conformal properties of the dilogarithm [3, 23, 21].

With that, Boyer and Parry began the study of a similar polynomial sequence called the plane partition polynomials [12, 11, 10] whose study was suggested by Stanley in conversation in 2005. This polynomial sequence acts in the same way as Boyer and Goh predicted with the different “phases” of the asymptotic structure of \( F_n(z) \) driving the location of the roots of \( F_n(z) \). It is now clear that the partition polynomials are not an isolated case but part of an entire family of polynomial sequences in combinatorics that have this Stokes phenomenon.

Construction of a general framework for partition polynomials began
with [26] when Parry found a polynomial version of the classical Meinardus Theorem for the asymptotics of sequences whose generating functions have a special (Euler) infinite product structure. This structure is common in the classical study of integer partitions and given one knows the phase structure, one can determine the asymptotics of the polynomial sequence.

This paper finishes the framework of Boyer and Parry and confirms the conjectures of Boyer and Goh. Specifically, we will show that the same phenomenon observed by the original partition polynomials and established for the plane partition polynomials occurs for the partition polynomials for arithmetic progressions.

2. Mathematical Preliminaries

Partition polynomials have been studied in a few contexts [2, 4] and loosely define a family of polynomial generating functions that have an interpretation in integer partition theory. For our purposes, we will use the following definition:

Definition 1. The partition polynomials, $F_n(z)$, are defined as the Fourier coefficients of

$$1 + \sum_{n=1}^{\infty} F_n(z) q^n = \prod_{m=1}^{\infty} \frac{1}{(1 - zq^m)^{a_m}}$$

where the integer exponents $a_m$ are either 0 or 1.

For $S \subset \mathbb{Z}^+$, let $a_m = \chi_S(m)$, then the coefficient $[z^k]F_n(z)$ counts the number of partitions of $n$ with $k$ parts that all lie in $S$. The principal cases we study are $S = \mathbb{Z}^+$; that is, the parts have no restriction; and where the elements of $S$ satisfy a linear congruence equation modulo $p$.

We recall the notion of phase [1, Definition 1] for a sequence \{L_n(z)\} of functions on the open subset $U$ of the complex plane where $L_n(z)$ is analytic on $U$ except perhaps for a branch cut.

Definition 2. Assume $U$ has a decomposition as a finite disjoint union of open nonempty sets $R(m_1), \ldots, R(m_k)$ together with their boundaries such that

$$\Re L_{m_j}(z) > \max\{L_m(z) : m \neq m_j\}$$

for $z \in R(m_j)$. If the family of sets $R(m)$ is a maximal family with these properties, we call $R(m_j)$ a phase with phase function $L_{m_j}(z)$. 

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The term phase comes from statistical mechanics; in that context a phase function represents a metastable free energy. In our examples, $U$ will either be the complex plane $\mathbb{C}$ or the open unit disk $\mathbb{D}$. By maximality, the phase functions are not analytic continuations of each other.

Our description of the asymptotics for partition polynomials $\{F_n(z)\}$ inside the open unit disk $\mathbb{D}$ uses the notion of phase and phase functions. On $\mathbb{D}$, we find that there is a sequence $\{R_{m_j}(z)\}$ of functions and finitely many indices $m_1, m_2, \ldots, m_\ell$, say, so on each $R_{m_j}$

$$F_n(z) \sim_{\mathcal{X}} A_{n,m}(z) \exp(n^{-\beta}L_{m_j}(z)), \quad \beta > 0,$$

where $A_{n,m}(z)$ are nonzero functions on $R_{m_j}$ such that

$$\lim_{n \to \infty} n^{-\beta} \ln |A_{n,m}(z)| = 0$$

uniformly on the compact subsets of $R_{m_j}$, $L_{m_j}(z)$ is analytic on $R_{m_j}$ except perhaps for a branch cut, and $\beta = 1/2$.

Next we recall the definition of principal object of this paper.

**Definition 3.** Let $Z(P(z))$ denote the finite set of zeros of the polynomial $P(z)$. For a polynomial sequence $\{F_n(z)\}$ whose degrees go to infinity, its zero attractor $A$ is the limit of $Z(F_n(z))$ in the Hausdorff metric on the non-empty compact subsets $K$ of $\mathbb{C} \cup \{\infty\}$. 

We now discuss fully below the connection between phases and the zero attractor. For a partition polynomial sequence $\{F_n(z)\}$ on the unit disk, if its phases there are, $R_{m_1}, \ldots, R_{m_\ell}$ say, then its zero attractor $A$ is simply the union of the boundaries of the phases

$$A = \partial R_{m_1} \cup \cdots \cup \partial R_{m_\ell}$$

with perhaps further contributions from branch cuts if necessary.

Finally we mention that we continue using our notational conventions in [1]. So $\sqrt{s} = s^{\frac{1}{2}} \exp \left( \frac{\log z}{s} \right)$ with the imaginary part of the logarithm defined on $(-\pi, \pi]$. Both $[x]^{-}$ and $\overline{x}$ denote the complex conjugate of $x$. Let $\{g_n(x)\}$ a sequence of functions, we say $g_n(x) = O_V(a_n)$ where $a_n$ a sequence of complex numbers, if there exists a constant $C_V$ dependent solely on a collection of parameters $V$, such that $|g_n(x)| \leq C_V|a_n|$ as $n \to \infty$. Similarly, we say $g_n(x) = o_V(a_n)$ for every $C_V$ dependent solely on a collection of parameters $V$, $|g_n(x)| \leq C_V|a_n|$ as $n \to \infty$. Absence of any $V$ indicates that the constant is uniform.

For convenience, we use the notation $e_k(z) = \exp(2\pi iz/k)$ and $e(z) = e_1(z)$. 

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2.1. Summary of the Meinardus Polynomial Setup \[26\]

Let \(\{a_m\}\) be the exponent sequence for the generating function for the partition polynomials \(\{F_n(z)\}\). For each positive integer \(k\) and character of \(\mathbb{Z}_k\), we associate the Dirichlet series \(D_{t,k}(s)\) where \(t\) is an integer \(1 \leq t \leq k\) that indexes the character of the finite cyclic group \(\mathbb{Z}_k\) given by \(j \mapsto e_k(tj)\). We define the Dirichlet series as

\[
D_{t,k}(s) = \sum_{m=1}^{\infty} a_m \frac{e_k(mt)}{m^s}.
\]

Observe that \(D_{k,k}(s)\) is the usual Dirichlet generating function for the sequence \(\{a_m\}\). In the examples in this paper, these Dirichlet series either have an analytic continuation to the complex plane as an entire function or as a meromorphic function with a unique singularity at \(s = 1\), which will be a simple pole. Further, for each positive integer \(k\), we need to introduce two functions on the character group \(\hat{\mathbb{Z}}_k\) as well as their finite Fourier transforms given in terms of the Dirichlet series family.

**Definition 4.** Let \(b_k\) be the \(\hat{\mathbb{Z}}_k\)-Fourier transform of \(t \mapsto D_{t,k}(0)\) while \(c_k\) be the \(\hat{\mathbb{Z}}_k\)-Fourier transform of \(t \mapsto \text{Res}(D_{t,k}(s), s = 1)\); explicitly, for \(1 \leq j \leq k\),

\[
b_k(j) = \frac{1}{k} \sum_{t=1}^{k} e_k(-tj)D_{t,k}(0), \quad c_k(j) = \frac{1}{k} \sum_{t=1}^{k} e_k(-tj) \text{Res}(D_{t,k}(s), s = 1).
\]

The actual form of the asymptotics of \(\{F_n(z)\}\) is given in terms of these functions on \(\mathbb{Z}_k\) and its dual. For \(k, n \in \mathbb{Z}^+\) and \(1 \leq h < k\) relatively prime to \(k\), define \(L_{h,k}(z)\) and \(\omega_{h,k,n}(z)\):

\[
L_{h,k}(z)^2 = \frac{1}{k^2} \sum_{r=1}^{k} z^r L(z^k, 2, r/k) \text{Res}(D_{r-h,k}, s = 1) = \sum_{j=1}^{k} c_k(j) L_2(e_k(jh)z),
\]

\[
\omega_{h,k,n}(z) = e_k(-hn) \prod_{j=1}^{k} (1 - e_k(hj)z)^{-b_k(j)}, \quad \Omega_{n,k}(z) = \sum_{(h,k)=1} \omega_{h,k,n}(z)
\]

where \(L(z, s, \nu)\) is the Lerch zeta function. When the functions \(L_{h,k}(z)\) are independent of \(h\), we will suppress the \(h\) dependence in the indexing when convenient.

The asymptotic results in \[26\] depend on knowing the phases for the function sequence \(\{L_{h,k}(z)\}\). For simplicity of exposition, we assume that \(L_{h,k}(z)\) are independent of \(h\) and that the phases are \(R(m_1), \ldots, R(m_j)\). On
each phase $R(m)$ there are possibly two different regimes of the asymptotics on compact subsets $X$:

$$F_n(z) \sim_X \frac{1}{2\sqrt{\pi n^{3/4}}} \Omega_{n,m}(z) \sqrt{L_m(z)} \exp(2n^{1/2}L_m(z)), \ z \in X \setminus \{z : L_m(z) \leq 0\},$$

(3)

$$\sim_X 2\Re \left[ \frac{1}{2\sqrt{\pi n^{3/4}}} \Omega_{n,m}(z) \sqrt{L_m(z)} \exp(2n^{1/2}L_m(z)) \right], \ z \in X \cap \{z : L_m(z) \leq 0\},$$

(4)

where the second asymptotics represent the contribution of a branch cut, if needed.

2.2. General Results about Zero Attractor

We show that the zero attractor lies inside the closed unit disk $\overline{D}$ and always contains the unit circle for any family of partition polynomials if their exponent sequence $\{a_n\}$ satisfies the initial condition $a_1 = 1$ and contains infinitely many 1’s. Of course, the more detailed structure of the zero attractor depends on the moduli of $\{F_n(z)\}$ so the chief contribution to consider is $\exp(2n^{1/2}\Re L_m(z))$ in its asymptotics.

We restate Sokal’s result in language convenient for partition polynomials.

**Proposition 1.** [29] Let $\{\phi_n(z)\}$ be a sequence of analytic functions on an open connected set $U$. For a fixed $\beta > 0$, we assume that the sequence $\{|\phi_n(z)|^{n^\beta}\}$ is uniformly bounded on the compact subsets of $U$. A point $z_0 \in \mathbb{C}$ is in their zero attractor if there exists a neighborhood $V(z_0)$ of $z_0$ with the property that there cannot exist a harmonic function $v(z)$ on $V(z_0)$ satisfying

$$\liminf_{n \to \infty} n^{-\beta} \ln |\phi_n(z)| \leq v(z) \leq \limsup_{n \to \infty} n^{-\beta} \ln |\phi_n(z)|.$$

For partition polynomials, there are two choices for the scales $\beta = 1/2$ or 1. In next Proposition, we make no assumptions on the phase structures inside the unit disk.

**Proposition 2.** Let $\{F_n(z)\}$ be a sequence of partition polynomials. Then
(a) $|F_n(z)|^{1/\sqrt{n}}$ is uniformly bounded on the unit disk.
(b) Uniformly on compact subsets of $\mathbb{C} \setminus \overline{D}$:

$$\lim_{n \to \infty} \frac{1}{n} \ln |F_n(z)| = \ln |z|.$$
(c) Let \( \{a_m\}_{m=1}^{\infty} \) be an exponent sequence for the partition polynomial sequence \( \{F_n(z)\} \) with \( a_1 = 1 \). Then its zero attractor \( A \) lies inside the closed unit disk and \( \infty \notin A \).

**Proof.** (a) Since the exponents \( \{a_m\} \) form a binary 0-1 sequence, \( F_n(1) \) is bounded above by the total number \( p(n) \) of the standard partitions of \( n \). But \( p(n) \) has order \( \exp(\pi \sqrt{2n/3}) \).

(b) Assume \( |z| > 1 \). Then

\[
\frac{1}{n} \ln |F_n(z)| = \frac{1}{n} \ln |z| + \frac{1}{n} \ln \left| 1 + \left( [z^{n-1}]F_n(z) \right)/z + \cdots + ([z^1]F_n(z))/z^{n-1} \right|
\]

But

\[
\frac{1}{n} \ln \left| 1 + \left( [z^{n-1}]F_n(z) \right)/z + \cdots + ([z^1]F_n(z))/z^{n-1} \right| \leq \frac{1}{n} \ln F_n(1)
\]

as \( n \to \infty \). Hence \( \ln |F_n(z)|/n \to \ln |z| \).

(c) Let \( H(z) = \prod_{m=1}^{\infty} (1 - z^m)^{-a_{m+1}} \) so \( H(z) \) is analytic, nonvanishing in the open unit disk \( \mathbb{D} \), and \( H(0) = 1 \). By the Stabilization result in [9], we find that

\[
[z^{n-k}]F_n(z) = [z^k]H(z), \quad 0 \leq k < \lfloor n/2 \rfloor.
\]

Write \( H(z) = 1 + \sum_{m=1}^{\infty} h_m z^m \). Let \( 1 < r_0 \). For \( |z| \geq r_0 \), we find there is a cancellation of the first \( \lfloor n/2 \rfloor \) terms below to give the bound:

\[
\left| \frac{F_n(z)}{z^n} - \prod_{m=1}^{\infty} (1 - 1/z)^{-a_{m+1}} \right| \leq \sum_{\ell=\lfloor n/2 \rfloor}^{n} \left| z^{n-\ell} \right| \left| F_n(z) \right| z^{-\ell} + \sum_{\ell=\lfloor n/2 \rfloor+1}^{\infty} h_\ell |z|^{-\ell}
\]

\[
\leq r_0^{\lfloor n/2 \rfloor} F_n(1) + \sum_{\ell=\lfloor n/2 \rfloor+1}^{\infty} h_\ell r_0^{-\ell}.
\]

Hence, on \( |z| \geq r_0 > 1 \), \( F_n(z)/z^n \) converges uniformly to \( H(1/z) \). Let \( \{z_{n_k}\} \) be a convergent sequence of zeros of \( F_{n_k}(z_{n_k}) = 0 \) such that \( |z_{n_k}| \geq r_0 \) with limit \( z^* \). Then \( F_{n_k}(z_{n_k})/z_{n_k}^{n_k} = 0 \to H(1/z^*) \neq 0 \) which contradicts that \( H(1/z) \) does not vanish on \( \mathbb{D} \). On the other hand, \( |z_{n_k}| \to \infty \) forces \( H(0) = 0 \) but this contradicts \( H(1) = 0 \).

\[\square\]

**Theorem 1.** Let \( \{F_n(z)\} \) be a partition polynomial sequence whose phases are the disjoint nonempty open subsets are \( R(m_1), \ldots, R(m_\ell) \) inside \( \mathbb{D} \). Assume the functions \( \Omega_{n,m}(z) \neq 0 \) for \( 0 \neq z \in R(m_j) \).

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(a) Then any \( z_0 \in \partial R(m_j) \cap \mathbb{D} \) is in the zero attractor of \( \{F_n(z)\} \).

(b) No nonzero point in \( R(m) \) lies in the zero attractor.

(c) The unit circle lies in the zero attractor.

Proof. (a) To apply Sokal’s result (see Proposition \( \ref{prop:sokal} \)), we need to consider the limit
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \ln |F_n(z)| = 2\Re L_m(z), \quad z \in R(m),
\]
that holds uniformly on compact subsets of the phase \( R(m) \). By assumption, \( \Re L_m(z) \) is harmonic on \( R(m) \) and for \( \ell \neq k \), \( \Re L_\ell(z) \) and \( \Re L_k(z) \) are not harmonic continuations of each other. Hence, for \( z_0 \in \mathbb{D} \cap \partial R(m) \), there cannot be a harmonic function \( v(z) \) in a neighborhood of \( z_0 \) such that \( \liminf \ln |F_n(z)| \leq v(z) \leq \limsup \ln |F_n(z)| \). We conclude that \( z_0 \) lies in the zero attractor.

(b) By assumption, we know that
\[
\lim_{n \to \infty} n^{-3/4} F_n(z) \exp(-2n^{1/2}L_m(z)) = \frac{1}{2\sqrt{\pi}} \Omega_{n,m}(z) \sqrt{L_m(z)}
\]
uniformly on the compact subsets of \( R(m) \). Further, this limit is nonzero by construction. Hence \( z \) cannot lie in the zero attractor.

(c) Let \( z^* \) be a point on the unit circle. Consider the open disk \( V \) with center \( z^* \) and radius \( r > 0 \). We consider the limit
\[
\lim_{n \to \infty} \frac{1}{n} \ln |F_n(z)|, \quad z \in V.
\]

The above theorem leads to an algorithm for drawing zero attractors. Locally, the boundary of a phase is a portion of an integral curve of the differential equation:
\[
\frac{dy}{dx} = \frac{\Re[L_k'(x+iy) - L_\ell'(x+iy)]}{\Im[L_k'(x+iy) - L_\ell'(x+iy)]},
\]
where its initial condition is found by solving the appropriate single nonlinear equation with a specified radius or angle; typically on the unit circle, say \( \Re L_k(e^{it}) = \Re L_\ell(e^{it}) \).

For chromatic polynomials, the analogue of the phases is the equimodular set \( \mathcal{E} \) which can also be described by means of integral curves but of a single differential equation.
3. A New Special Function: The Root Dilogarithm

The determination of the phases for the families of partition polynomials in this paper rests on finding the bounds among the family of functions given through the dilogarithm.

Definition 5. The root dilogarithms are the functions given by

\[ f_k(z) = \frac{1}{k} \Re \left[ \sqrt{\text{Li}_2(z^k)} \right] \]  

where the square root is chosen as nonnegative and \( k \) is a positive integer.

In general, the study of \( f_k(z) \) is a lengthy tangent which we leave to the appendix. We will state here what we need and refer the reader to the appendix for their proofs. These results come in two kinds. First will be the calculus of \( f_1(z) \) on radial lines and circles.

Proposition 3. The function \( t \to f_1(it) \) is a positive, increasing, concave function of \( t \in (0, 1) \).

On circles, \( f_k(re^{it}) \) behaves in a cosine like fashion.

Proposition 4. For a fixed value of \( r \in (0, 1] \), the function \( t \mapsto f_1(re^{it}) \) is decreasing on \([0, \pi]\).

Second are root dilogarithm dominance facts that we will use extensively throughout our examples.

Theorem 2. For \( 0 < |z| \leq 1 \),

1. \( f_k(z) \leq f_k(|z|) \leq f_2(|z|) < f_1(z) \), \( k \geq 2 \), \( |\arg z| \leq \pi/3 \).
2. \( f_k(z) \leq f_k(|z|) \leq f_1(z) \), \( k \geq 3 \), \( |\arg z| \leq \pi/2 \).
3. \( f_k(z) < f_1(z) \), \( k \geq 2 \), \( 0 \leq \arg z \leq \pi/2 \).
4. \( f_k(z) < \max[f_1(z), f_2(z), f_3(z)] \), \( k \geq 4 \), \( \pi/2 \leq \arg z \leq \pi \).

4. Partition Polynomials

4.1. Partition Polynomials Weighted by Number of Parts

Consider the constant exponent sequence \( \{a_m\} \) where \( a_m = 1 \) for all \( m \) so there is no restriction on the parts of a partition. For \( k \in \mathbb{Z}^+ \) and \( t \in \hat{\mathbb{Z}}_k \), the Dirichlet series \( D_{t,k}(s) \) are given by

\[ D_{t,k}(s) = \sum_{m=1}^{\infty} \frac{e_k(mt)}{m^s} = \frac{1}{k^s} \sum_{r=1}^{k} e_k(rt)\zeta(s, r/k) = F(t/k, s) \]
where $F(\lambda, s)$ is the periodic zeta function

$$F(\lambda, s) = \sum_{m=1}^{\infty} \frac{e(m\lambda)}{m^s}.$$ 

$F(\lambda, s)$ is periodic in $\lambda$ with period 1. When $\lambda$ is not an integer, then $F(\lambda, s)$ is an entire function of $s$; otherwise, $F(\lambda, s)$ reduces to the Riemann zeta function.

**Remark 1.** In this example $D_{t,k}(s)$ reduces to the usual Dirichlet $L$-function relative to a character of $\mathbb{Z}_k$ when $(t, k) = 1$.

Since $\zeta(0, r/k) = 1/2 - r/k$, for $1 \leq r < k$,

$$D_{t,k}(0) = \sum_{r=1}^{k} \left(\frac{1}{2} - \frac{r}{k}\right) e_k(tr) = -\frac{1}{k} \sum_{r=1}^{k} re_k(tr), \quad 1 \leq t < k,$$

while $D_{k,k}(0) = -1/2$. Since $F(s, \lambda)$ has a pole at $s = 1$ when $\lambda$ is an integer, we find

$$Res(D_{t,k}, s = 1) = Res(\zeta(s), s = 1) \delta_{t,k} = \delta_{t,k}.$$ 

Hence its $\hat{\mathbb{Z}}_k$ Fourier transform is

$$c_k(j) = \frac{1}{k}, \quad 1 \leq j \leq k.$$ 

For $h$ relatively prime to $k$ such that $1 \leq h < k$, we find an explicit form for $L_{h,k}(z)$:

$$L_{h,k}(z)^2 = \sum_{j=1}^{k} c_k(j)Li_2(e_k(j)z) = \frac{1}{k} \sum_{j=1}^{k} Li_2(e_k(j)z) = \frac{1}{k^2} Li_2(z^k);$$

in particular, the functions $L_{h,k}(z)$ are independent of the choice of $h$ so we write $L_k(z)$ for them. Furthermore, each $L_k(z)$ is analytic on the unit disk except for branch cuts at the rays $\arg z = \pi j/k$, $0 \leq j < k$.

For small values of $k$, we write out the values of $b_k$:

$$b_1(1) = -1/2; \quad b_2(1) = (-1)^n/2; \quad b_3(1) = 1/6, b_3(2) = -1/6, b_3(3) = -1/2.$$
Hence we find that
\[
\omega_{1,1,n}(z) = \sqrt{1 - z}, \quad \omega_{1,2,n}(z) = (-1)^n \sqrt{1 - z}, \\
\omega_{1,3,n}(z) = e_3(-n) \frac{(1 - e_3(2)z)^{1/6}(1 - z)^{1/2}}{(1 - e_3(1)z)^{1/6}}, \\
\omega_{2,3,n}(z) = e_3(-2n) \frac{(1 - e_3(1)z)^{1/6}(1 - z)^{1/2}}{(1 - e_3(2)z)^{1/6}},
\]

We can verify directly that \(\omega_{1,3,n}(z) \neq -\omega_{2,3,n}(z)\) for \(|z| < 1\). Using Theorem 2, we obtain that there are only three phases \(R(1), R(2),\) and \(R(3)\). These phases can be more concretely written down as
\[
R(1) = \left\{ z \in \mathbb{D} : \Re \sqrt{Li_2(z)} > \frac{1}{2} \Re \sqrt{Li_2(z^2)}, \frac{1}{3} \Re \sqrt{Li_2(z^3)} \right\}, \\
R(2) = \left\{ z \in \mathbb{D} : \frac{1}{2} \Re \sqrt{Li_2(z^2)} > \frac{1}{2} \Re \sqrt{Li_2(z)}, \frac{1}{3} \Re \sqrt{Li_2(z^3)} \right\}, \\
R(3) = \left\{ z \in \mathbb{D} : \frac{1}{3} \Re \sqrt{Li_2(z^3)} > \frac{1}{2} \Re \sqrt{Li_2(z)}, \Re \sqrt{Li_2(z)} \right\}.
\]

Using Theorem 1, we record the following proposition.

**Proposition 5.** The zero attractor for the partition polynomials is
\[
S^1 \cup \partial R(1) \cup \partial R(2) \cup \partial R(3).
\]

**Remark 2.** The appendix shows that there are at most three phases; in fact, all three do occur. In the right-half plane, including the imaginary axis, \(f_2(z), f_3(z) < f_1(z)\). In the second quadrant, consider \(f_1(re^{it})\) which is decreasing to 0 while \(f_2(re^{it})\) is increasing from 0, \(t \in [\pi/2, \pi]\). Hence, for each \(0 < r \leq 1\), there is a unique value \(t_{12}(r)\) such that \(f_1(r e^{it_{12}(r)}) = f_2(r e^{it_{12}(r)})\). Similar reasoning shows \(f_2(re^{it}) = f_3(re^{it})\) has at most one solution for \(2\pi/3 \leq t \leq \pi\); in fact, \(f_3(re^{i\pi/6}) < f_2(re^{i\pi/6})\) on \((0, 1]\).

The phase \(R(1)\) includes the unit disk that lies in the right-half plane. The phases \(R(2)\) and \(R(3)\) lie in the left-hand plane since \(f_1(z) > f_k(z)\), for \(k \geq 2, 0 < |z| \leq 1,\) and \(Rz \geq 0\). For fixed \(r > 0\), \(f_1(re^{it})\) is decreasing to 0 while \(f_2(e^{it})\) is increasing from 0 on \([\pi/2, \pi]\). So \(f_1(z) = f_2(z)\) determines a level set that connects 0 to a unique point on the unit circle. See Figure 4.1.
Figure 1: Phases for weighted partition polynomials: (a) Region 1 (b) Region 2, (c) Region 3 in upper half plane.

Figure 2: (a) Level sets $f_1(z) = f_2(z)$ (solid line), $f_2(z) = f_3(z)$ (dotted line), $f_1(z) = f_3(z)$ (dash-dot lines) inside the unit disk, (b) Closeup near the triple point
4.2. Partition Polynomials Whose Parts Satisfy a Linear Congruence

For a fixed positive integer \( p \geq 2 \), consider the exponent sequence \( \{a_m\} \) whose nonzero entries satisfy \( m \equiv 1 \mod p \). The corresponding partition polynomials count partitions all of whose parts satisfy the congruence \( 1 \mod p \). For \( k \in \mathbb{Z}^+ \) and \( t \in \mathbb{Z}_k \), let

\[
D_{t,k}(s) = \sum_{m=1}^{\infty} \frac{a_m e_k(tm)}{m^s} = \sum_{\ell=0}^{\infty} \frac{e_k(t(p\ell + 1))}{(p\ell + 1)^s} = \frac{e_k(t)}{p^s} L(pt/k, s, 1/p)
\]

where \( L(\lambda, s, \nu) \) is the Lerch zeta function. We observe that if \( k \mid pt \) then \( D_{t,k}(s) \) reduces to \( e_k(t)p^{-s}\zeta(s, 1/p) \). Since \( \zeta(0, \alpha) = 1/2 - \alpha \) and

\[
L(\lambda, 0, \nu) = \frac{1}{1 - \exp(2\pi i \lambda)}, \quad \lambda \notin \mathbb{Z}.
\]

we have for \( 1 \leq t \leq k \)

\[
D_{t,k}(0) = \begin{cases}
    e_k(t)(1/2 - 1/p), & k \mid pt, \\
    e_k(t)/e_k(pt), & k \nmid pt
\end{cases}
\]

In the special case for \( p = 2 \) with \( 1 \leq t < k \) and \( t \) is relatively prime to \( k \)

\[
D_{t,k}(0) = \frac{i}{2 \sin(2\pi t/k)}
\]

and, with \( p = 2 \), \( D_{k,k}(0) = 0 \).

Recall that \( L(\lambda, s, \nu) \) is an entire function of \( s \) if and only if \( \lambda \notin \mathbb{Z} \). When \( \lambda \) is an integer, \( L(\lambda, s, \nu) \) is a meromorphic function of \( s \) with a unique singularity at \( s = 1 \) which is a simple pole with residue 1. Hence \( D_{t,k}(s) \) has a singularity if and only if \( k \mid pt \). If this is case, then the singularity is a simple pole at \( s = 1 \) with residue

\[
\text{Res}(D_{t,k}(s), s = 1) = \frac{1}{p} e_k(t).
\]

Lemma 1. For \( p \geq 3 \), the finite Fourier transform \( b_k(t) \) of \( t \mapsto D_{t,p}(0) \) is

\[
b_k(t) = \begin{cases}
    \frac{p/2 - 1}{p}, & t = 1, 2 \mid p \text{ and } 2 \mid p \\
    \frac{p-2}{2p}, & t = 1, 2 \nmid p \\
    0, & 2 \leq t \leq p
\end{cases}
\]

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Example 1. For the special case: \( p = 2 \) with small values of \( k \), we record the values of the finite Fourier transform \( b_k \):

\[
\begin{align*}
b_3(1) &= -b_3(2) = 1/3, \quad b_3(3) = 0, \\
b_4(1) &= b_4(3) = 1/4, \quad b_4(2) = b_4(4) = 0, \\
b_6(1) &= -b_6(5) = 1/3, \quad b_6(2) = b_6(3) = b_6(4) = b_6(6) = 0.
\end{align*}
\]

This allows us to write out the explicit forms for the corresponding \( \omega_{h,k,n}(z) \) functions:

\[
\begin{align*}
\omega_{1,1,n}(z) &= 1, \quad \omega_{1,2,n}(z) = (-1)^n, \\
\omega_{1,4,n}(z) &= i^{-n} \sqrt{\frac{z - i}{z + i}}, \quad \omega_{3,4,n}(z) = i^n \sqrt{\frac{z + i}{z - i}}.
\end{align*}
\]

Further we can check directly that \( \omega_{1,4,n}(z) = -\omega_{3,4,n}(z) \) only if \( z = 0 \).

Example 2. For \( p = 3 \), we have

\[
\begin{align*}
b_1(1) &= -1/2, \quad b_2(1) = 1/3, b_2(2) = -1/6, \\
b_6(1) &= 1/3, b_6(4) = -1/6, \quad b_6(2) = b_6(3) = b_6(5) = b_6(6) = 0.
\end{align*}
\]

Lemma 2. If \( (k, p) = m > 1 \), then

\[
c_k(j) = \begin{cases} 
\frac{(k,p)}{kp}, & j \equiv 1 \mod (k, p), \\
0, & j \not\equiv 1 \mod (k, p)
\end{cases}
\]

Proof. Let \( 1 \leq t \leq k \). Then \( k \mid pt \) if and only if \( k_0 \mid pt \) where \( k_0 = k/(k, p) \) and \( p_0 = p/(k, p) \). In other words, \( k \mid pt \) if and only if \( pt \in k_0 \mathbb{Z} \). In our case, \( t = vk_0, 1 \leq v \leq (k, p) \). Consider

\[
\begin{align*}
c_k(j) &= \frac{1}{k} \sum_{t=1}^{k} e_k(-jt) \text{Res}(D_{t,k}(s), s = 1) = \frac{1}{k} \sum_{t=1}^{k} e_k(-jt) \delta_{k|pt} \frac{e_k(t)}{p} \\
&= \frac{1}{k} \sum_{t=1}^{k} e_k(-jt) \delta_{k_0|pt} \frac{e_k(t)}{p} = \frac{1}{k} \sum_{v=1}^{(k,p)} e_k(-jvk_0) \frac{e_k(vk_0)}{p} \\
&= \frac{1}{kp} \sum_{v=1}^{(k,p)} e_k(vk_0(1 - j)) = \frac{1}{kp} \sum_{v=1}^{(k,p)} e_{(k,p)}(v(1 - j)) \\
&= \begin{cases} 
\frac{(k,p)}{pk}, & j \equiv 1 \mod (k, p) \\
0, & j \not\equiv 1 \mod (k, p)
\end{cases}
\]

\[\square\]
We use this Kubert-type identity in the next lemma:

\[ \sum_{m=1}^{k} Li_2(ze_k(m)) = \frac{1}{k} Li_2(z^k). \]

**Lemma 3.** Let \( 1 \leq h < k \) and \((h,k) = 1\). Then

\[ L_{h,k}(z)^2 = \frac{(k,p)^2}{k^2p} Li_2 \left( e_{(k,p)}(h) z^{k/(k,p)} \right). \]

**Proof.** Consider the following expansions

\[ L_{h,k}(z)^2 = \sum_{j=1}^{k} c_k(j) Li_2(e_k(jh)z) = \sum_{k|(j-1)} \frac{(k,p)}{kp} Li_2(e_k(jh)z) \]

\[ = \sum_{v=1}^{k/(k,p)} \frac{(k,p)}{kp} Li_2(e_k((v(k,p) + 1)h)z) \]

\[ = \frac{(k,p)}{kp} \sum_{v=1}^{k/(k,p)} Li_2(e_{k/(k,p)}(v)[e_k(h)z]) \]

\[ = \frac{(k,p)}{kp} \frac{1}{k/(k,p)} Li_2 \left( [e_k(h)z]^{k/(k,p)} \right) \]

\[ = \frac{(k,p)}{kp} \frac{1}{k/(k,p)} Li_2 \left( [e_k(hk/(k,p))] z^{k/(k,p)} \right) \]

\[ = \frac{(k,p)^2}{k^2p} Li_2 \left( e_{(k,p)}(h) z^{k/(k,p)} \right). \]

\[ \Box \]

**Remark 3.** For polynomials counting partitions whose parts are congruent to a modulo \( p \) with \((a,p) = 1\), then

\[ L_{h,k}(z)^2 = \frac{(k,p)^2}{k^2p} Li_2 \left( e_{(k,p)}(ha) z^{k/(k,p)} \right). \]

For \( p \geq 2 \), assume that the only nonzero entries of the exponent sequence \( \{a_m\} \) satisfy \( m \equiv 1 \mod p \). Calculating directly from the generating function \( G(z, q) \), we get

\[ F_\ell(e^{2\pi i/p} z) = e^{2\pi i \ell/p} F_\ell(z) \]

Since the polynomial \( F_\ell(z) \) has real coefficients, \( \overline{F_\ell(z)} = F_\ell(\overline{z}) \) as well. So their moduli \( |F_\ell(z)| \) are invariant under the action of the dihedral group.
generated by the rotation by angle $2\pi/p$ and reflection about the real axis. Note the choice of $1 \mod p$ is done for simplicity and a similar analysis can be done for $a \mod p$ together with the requirement $a_1 = 1$ although the computations are needlessly lengthy. Nonetheless, for clarity for the reminder of this section, we will still use the two variable notation for phases

$$R(h, k) = \{ z \in \mathbb{D} \setminus 0 : \Re L_{h,k}(z) \text{ is maximal} \}.$$ 

with $R(p,p) = R(1,1) = R(0,p)$ for notational uniformity. The phase functions connect to the root dilogarithm function by

$$\Re L_{h,k}(z) = \frac{1}{\sqrt{p}} f_{k/(k,p)}(e_k(h)z).$$

(6)

We need an elementary number theory lemma to handle the cyclic nature of $\Re L_{h,k}(z)$.

**Lemma 4.** For every $k, j \in \mathbb{N}$, $k/(k, j) = 1$ if and only if $k \mid j$.

**Proof.** Suppose that $k \mid j$ and of course $k \mid k$. By the definition of greatest common divisor, $k \mid (k, j)$. But $k \leq j$ so $k/(k, j) = 1$. If $k/(k, j) = 1$ then $k \mid (k, j)$. Therefore by the definition of greatest common divisor, $k \mid j$. \[\square\]

The next lemma is a consequence of Theorem 2 part (1).

**Lemma 5.** We can reduce the set of phase functions for every $z \in \mathbb{D}$, $j \geq 3$,

$$\max \{ \Re L_{h,k}(z) : k \mid p, (k, p) = 1 \} > \max \{ \Re L_{h,k}(z) : k \nmid p \}$$

In terms of phases we have

$$\mathbb{D} = \bigcup_{h \in \mathbb{Z}_p} R(h,p).$$

**Proof.** If $p$ is a prime, we observe, using Lemma 4 that if $p/(p,k) = 1$ then $k \mid p$ and either $k = 1$ or $k = p$. In the case $k = 1$, then $h = 1$ since $(h,k) = 1$; otherwise, $k = p$ and so $h \in \mathbb{Z}_p^\times$. Using the definition of $\Re L_{h,k}(z)$ in Equation 6 and applying Proposition 4, we obtain

$$\max \{ \Re L_{h,k}(z) : k \mid p, (k, h) = 1 \} = \max \{ \frac{1}{\sqrt{p}} f_1 (e_p(h)z) : 1 \leq h \leq p \}$$

$$= \frac{1}{\sqrt{p}} f_1 (e_p(h^*)z).$$
where \( h^* \) is chosen as the minimizer of \(|\arg(e_p(h^*)z)|\) over \( h \in \mathbb{Z}_p \). By Theorem 2 part (1) and the Equation 6, we next find that if \(|\arg(e_p(h^*)z)| \leq \pi/3\) and \( k \nmid p \) then

\[
\Re L_{h,k}(z) = \frac{1}{\sqrt{p}} f_{k/(k,p)}(e_k(h)z) \leq \frac{1}{\sqrt{p}} f_2(|z|) \leq \frac{1}{\sqrt{p}} f_1(e_p(h^*)z).
\]

Hence it is enough to show there exists at least one \( h \in \mathbb{Z}_p \) such that \(|\arg(e_p(h)z)| \leq \pi/p \leq \pi/3\). But this is a classic consequence of the pigeon-hole principle.

**Theorem 3.** The phase \( R(h,p) \) is an open angular wedge of angle \( 2\pi/p \) of radius 1 centered at \( \arg z = 2h\pi/p \).

**Proof.** By Lemma 5, we already know that

\[
\mathbb{D} = \bigcup_{h \in \mathbb{Z}_p} R(h,p)
\]

We also have a symmetry that \( f_1(e_p(h)z) = f_1(e_p(h-1)ze_p(1)z) \) implying that if \( z \in R(h,p) \) if and only if \( e_p(1)z \in R(h-1,p) \). Set wise we have

\[
R(n,p) = e_p(1)R(n-1,p).
\]

Hence we need only to show \( R(1,1) \) contains the wedge \(|\arg z| < \pi/p\) and because phases are disjoint and symmetric we will be done. But by Proposition 4, Equation 6 and the previous lemma we need only determine for \(|\arg z| \leq \pi/p\) that \( |\arg z| = \min_{1 \leq h \leq p} |\arg(e_p(h)z)| \).

Suppose \( |\arg e_p(n)| > \frac{2\pi}{3p} \) then by the triangle inequality \( |ze_p(n)| > \frac{\pi}{p} \) and by Proposition 4

\[
f_1(z) \geq f_1(e_{2p}(1)) > f_1(ze_p(n)).
\]

This leaves only two potential candidates which can dominate \( f_1(z) \); namely, \( f_1(e_p(-1)z) \) and \( f_1(e_p(1)z) \). So long as

\[
\frac{\pi}{p} - \arg z > \arg z, \quad \arg z + \frac{\pi}{p} > -\arg(z),
\]

then by Proposition 4, \( f_1(z) \) dominates the both of them. This concludes the proof.

Like the other examples all we need now do is apply Theorem 1. Of course, the boundaries of wedges can be written as the solution to \( z^p = -1 \) and so we conclude the example.
**Proposition 6.** Given \( p > 2 \), the zero attractor for partition polynomials corresponding to partitions into parts congruent to 1 \( \mod p \) is the unit circle and a set of \( p \) spokes.

\[
S^1 \cup \{ z \in \mathbb{D} : z^p = -1 \}
\]

### 4.3. Odd Partition Polynomials

When \( p = 2 \), we obtain a nondegenerate behavior for the odd parts polynomials. Using Remark 3, we obtain

\[
\sqrt{2} L_{h,2k-1}(z) = L_{2k-1}(z) = \frac{1}{2^{k-1}} \sqrt{Li_2(z^{2k-1})},
\]

\[
\sqrt{2} L_{h,2k}(z) = L_{2k}(z) = \frac{1}{k} \sqrt{Li_2(-z^k)}.
\]

Once again that we note the connection to the root dilogarithm function where \( \Re L_k(z) = f_k(z) \) for \( k \) odd and \( \Re L_{2k}(z) = f_1(-z^k)/k \) for even indices.

**Theorem 4.** The only nonempty phases are \( R(1) \), \( R(2) \), and \( R(4) \).

**Proof.** Inspection of \( L_k(z) \) near the points \( z = 1, -1, i \) proves that \( R(1), R(2), \) and \( R(4) \) are nonempty. For \( k > 4 \), we note that \( \Re L_k(z) \leq f_3(|z|) \) which by Theorem 2 must be bounded by either \( \Re L_1(z) = f_1(z) \) or \( \Re L_2(z) = f_1(-z) \) depending on whether \( \Re z > 0 \) or \( \Re z < 0 \). Hence \( R(k) \) is empty.

The case of \( k = 3 \) follows by the same Lemma since

\[
\Re L_3(z) = f_3(z) \leq f_3(|z|) \leq \max(\Re L_1(z), \Re L_2(z)).
\]

Using Theorem 1, we record the following proposition.

**Proposition 7.** The zero attractor for the odd parts partition polynomials is given by

\[
S^1 \cup \partial R(1) \cup \partial R(2) \cup \partial R(4).
\]

However, we can go a bit further and identify curves to the boundaries of \( R(m) \).

**Proposition 8.** On \( (0, 1] \), there is a unique solution \( \beta \) to \( f_1(i\beta) = f_2(\beta) \); further, \( 3/4 < \beta < 1 \).

**Theorem 5.** There exists exactly three phases \( R(1), R(2), \) and \( R(4) \).
1. The boundary of \( R(4) \) consists of \( \gamma, -\gamma, \overline{\gamma}, \) and \( -\overline{\gamma} \) where \( \gamma \) is the level set \( \{ z : \Re L_1(z) = \Re L_4(z); \Re z, \Im z \geq 0 \} \).

2. The boundary of \( R(1) \) is \( i[0, \beta] \cup i[-\beta, 0] \cup \gamma \cup \overline{\gamma} \).

3. The boundary of \( R(2) \) is \( i[0, \beta] \cup i[-\beta, 0] \cup -\gamma \cup -\overline{\gamma} \).

Proof. Since \( \Re L_1(-z) = \Re L_2(z) \) and \( \Re L_4(z) = \Re L_4(-z) \), we need only classify the phases on the upper quarter disk \( \mathbb{D}^{++} \). On this quarter disk, we have by Proposition 4 that \( \Re L_1(z) > \Re L_2(z) \), \( z \neq 0 \). It remains to examine \( \Re L_1(z) \) and \( \Re L_4(z) \). Consider the function

\[
h(z) = \Re L_1(z) - \Re L_4(z) = f_1(z) - \frac{1}{2} f_1(-z^2).
\]

If \( z \in \mathbb{D}^{++} \) and \( h(z) > 0 \) then \( z \in R(1) \) and if \( h(z) < 0 \) then \( z \in R(4) \). But by Proposition \( 4 \) \( h(r e^{it}) \) is decreasing.

The theorem now follows from Proposition 8 and the intermediate values theorem.

The more detailed reduction of the zero attractor now follows directly from Theorem 1.

**Theorem 6.** If \( A \) is the zero attractor of \( F_n(z) \) then

\[
A = S^1 \cup \gamma \cup -\gamma \cup \overline{\gamma} \cup -\overline{\gamma} \cup i[-\beta, \beta].
\]

5. Further Examples and Conclusion

This framework for the zeros of partition polynomials is quite deep and general. For mathematical convenience we restricted ourselves to the cases that can be fully worked out analytically, but this framework generalizes beyond just these cases.

Here are a few more examples, where we have done a numerical setup and computed potential phase functions but have yet to identify \( \partial R(m) \). In these cases, we require that the partition parts are solutions to \( x^2 \equiv 1 \mod p \); that is, they lie in the residue classes 1 or \( p - 1 \mod p \). The calculation of the corresponding phase functions follow from the same method as those for linear congruences.

Let \( (h, k) = 1 \) with \( 1 \leq h < k \). Then set

\[
L_{h,k}(z)^2 = \frac{(k,p)^2}{k^2p} \left( L_{i2}(e_{(k,p)}(h) z^{k/(k,p)}) + L_{i2}(e_{(k,p)}(-h) z^{k/(k,p)}) \right).
\]
Figure 3: Closeups of zero attractor inside unit disk: (a) Closeup: parts are congruent to 0 or 2 modulo 3; (b); parts are congruent to 0 or 2 modulo 3; (c) parts are congruent to 1 or 4 modulo 5,
Figure 4: Closeups of zero attractor inside unit disk: (a) all parts, zeros of degree 25,000; (b) parts are odd, zeros of degree $2^k$, $k = 12, \ldots, 15$; (c) parts are congruent to 1 or 2 modulo 3, degree 2500
When we let \( p = 3 \),

\[
L_{h,3k}(z) = L_k(z) = \frac{1}{k\sqrt{p}} \sqrt{2Li_2(z^k)}
\]

and for \( k \) not divisible by 3,

\[
L_{h,k}(z) = \frac{1}{k\sqrt{p}} \sqrt{\frac{1}{3}Li_2(z^{3k}) - Li_2(z^k)}.
\]

The zero attractor can be seen in Figure (4.3) where we have identified the curves which appear to be portions of the level sets \( f_1 = f_3 \) and \( f_2 = f_3 \).

For \( p = 5 \), we obtain the same kind of phenomenon except things cannot be reduced as easily. As before, the phase functions are

\[
L_{h,5k}(z) = L_k(z) = \frac{1}{k\sqrt{p}} \sqrt{2Li_2(z^k)}
\]

and for \( k \) not divisible by 5,

\[
L_{h,k}(z) = \frac{1}{k\sqrt{p}} \sqrt{\frac{1}{3}Li_2(e_5(1)z^k) + Li_2(e_5(-1)z^k)}.
\]

Figure (4.3) shows with a little bit stronger evidence that the unit disk is made up of three phases \( R(1) \), \( R(2) \), and \( R(3) \).

This framework reaches beyond the narrow scope we discussed here. After all, we first worked out the asymptotics of polynomials for plane partitions whose coefficients are indexed by the trace. The corresponding phase functions involved a cube root rather than a square root [11]. Other more interesting applications would be the polynomial analogue of Wright’s partitions into powers \( (a_m = \chi_{mk}(m) \text{ for } k \in \mathbb{N} \text{ fixed}) \) and the polynomial analogue of Cayley double partitions [19] \( (a_m = p(n) \text{ where } p(n) \text{ counts the number of partitions of } n) \).

The topological behavior of all the examples studied is also quite intriguing. It seems that every zero attractor drawn so far is a connected. Does this generalize the behavior of the eigenvalues of the truncations of infinite banded Toeplitz matrices [30]? Is it possible to classify zero attractors by their fundamental group properties or even perhaps identify each phase as the interior of a specific set of level curves using the Jordan curve theorem? These are all interesting questions which beg an answer for them.
Appendix A. Analysis of the Root Dilogarithm

For convenience, we repeat the definition of the root dilogarithm

\[ f_k(z) = \frac{1}{k} \Re \left[ \sqrt{\text{Li}_2(z^k)} \right], \quad k \in \mathbb{Z}^+, \quad (A.1) \]

where the real part of the square root is chosen as nonnegative. By construction each \( f_k(z) \) is harmonic on the sectors in the unit disk determined by the rays \( \arg z = \pi j/k, \ 0 \leq j < k \), and symmetric about the real axis. Further \( f_1(z) = 0 \) if and only if \( z \in [-1, 0] \).

From this we learn several facts about \( f_k(z) \) which derive from \( \text{Li}_2(z) \).

For handling the real component of the square root we note that for any \( z \in \mathbb{C}, \ z \neq 0 \),

\[ \sqrt{z} = \pm \frac{1}{\sqrt{2}} \left[ \sqrt{\Re(z) + |z|} + i \text{sign}(\Im(z)) \sqrt{-\Re(z) + |z|} \right]. \quad (A.2) \]

There are two special values of the polylogarithm worthy of note \( \text{Li}_2(1) = \zeta(2) = \pi^2/6 \) and \( \text{Li}_2(-1) = -\eta(2) = -\pi^2/12 \).

The goal of this appendix is to prove the facts in Section 3. We begin with computing the behavior of \( f_1(z) \) on circles.

Lemma 6. For a fixed value of \( r \in (0, 1] \), the function \( t \mapsto \arg \text{Li}_2(re^{it}) \) is increasing on \( [0, \pi] \).

Proof. Let \( g(z) \) be a univalent function on the unit disk normalized by \( g(0) = 0 \) and \( g'(0) = 1 \). Then \( g(z) \) is star-like if its derivative satisfies

\[ \frac{\partial \arg g(re^{it})}{\partial t} > 0, \]

for \( 0 < r < 1 \). By a result of Lewis [21], any polylogarithm \( \text{Li}_s(z), \ s > 0, \) is univalent and star-like so the result follows.

By equation (A.2), \( f_1(z) = 0 \) if and only if \( \Im \text{Li}_2(z) = 0 \) and \( \Re \text{Li}_2(z) \leq 0 \). Since \( \text{Li}_2(z) \) is negative on \([-1, 0]\), we find \( f_1(z) = 0 \) if and only if \( z \in [-1, 0] \).

Lemma 7. For a fixed value of \( r \in (0, 1] \), the function \( t \mapsto |\text{Li}_2(re^{it})| \) is decreasing on \([0, \pi]\).

Proof. Let \( h(z) = \sum_{n=1}^{\infty} c_n z^n \) be a power series convergent in the unit disk with \( c_n > 0 \). For fixed \( 0 < r \leq 1 \), the theorem of Fejér described in [17, p. 513] states that \( t \mapsto |h(re^{it})| \) is decreasing provided \( \Delta^4(r^n c_n) \geq 0 \) for
all $n$. For $Li_2(z)$, Fejér’s condition becomes $\Delta^4(r^n/n^2) \geq 0$ for all $n$. These inequalities hold because $a_n = r^n/(n+1)^2$ is a moment sequence for the probability measure on $[0,1]$ with density function $-(1/r)\ln(x/r)$; in fact,

$$\frac{r^n}{(n+1)^2} = -\int_0^1 \frac{x^n}{r} \log(x/r) \, dx$$

It follows that $(-1)^j \Delta^j(a_n) > 0$; in particular, the required fourth-order difference is also nonnegative.

These two lemmas combine to prove our first objective.

**Proposition 9** (Proposition 4 in main text). For a fixed value of $r \in (0,1]$, the function $t \mapsto f_1(re^{it})$ is decreasing on $[0,\pi]$.

**Proof.** We begin by writing $f_1(re^{it})$ in polar form

$$f_1(re^{it}) = |Li_2(re^{it})|^{1/2} \cos\left(\frac{1}{2} \arg Li_2(re^{it})\right).$$

Lemmas 6 and 7 show that both factors in this factorization are decreasing functions of $t$. In particular, since $\arg Li_2(re^{it})$ is an increasing function of $t$ which is 0 on the positive real axis and $\pi$ on the negative axis, the function $\cos\left(\frac{1}{2} \arg Li_2(re^{it})\right)$ is decreasing on $[0,\pi]$.

An immediate application of this proposition is to obtain the elementary bounds we use throughout the appendix.

**Lemma 8.** For $|z| \leq 1$, we have the bounds

1. $\frac{\pi^2}{12}|z| \leq |Li_2(z)| \leq \frac{\pi^2}{6}|z|,$
2. $0 \leq f_k(z) \leq \frac{\pi}{k\sqrt{6}}|z|^{k/2}$.

**Proof.** (1) Apply the maximum and minimum modulus principle to $Li_2(z)/z$. Since $Li_2(z)$ is univalent in the unit disk, it has a unique zero at $z = 0$ which is simple. The maximum occurs at $z = 1$ and minimum at $z = -1$ which is observed using Proposition 7.

(2) This follows immediately from part (1) and the fact that $Li_2(1) = \pi^2/6$. 

\[\]
Appendix A.1. Behavior of the root dilogarithm on the imaginary axis

As a heuristic, studying $f_k(z)$ on radial lines and circles is how one proves facts about phases. Proposition 4 is conclusive on $f_k(z)$’s behavior on the circles $|z| = r \leq 1$ but the behavior on radial lines is much harder. We can scrape by with knowing how $f_1(z)$ behaves on the “main” radial lines; the real and imaginary axes. On the negative axis $f_1(z) = 0$, and on the positive axis, $f_1(z) = \sqrt{Li_2(z)}$ which is routine. So we focus on the imaginary axis and prove

Proposition 10. [Proposition 3 in the main text] The function $r \rightarrow f_1(ir)$ is a positive, increasing, concave function of $r \in (0, 1)$.

We break up this theorem into several lemmas; for convenience, we let

$$ r \mapsto \theta = \arg(Li_2(ir)). $$

We will begin with an improvement to the bound of Lemma 8 when $z$ lies on the imaginary axis. Recall that the real and imaginary parts of $Li_2(ir)$ have the explicit forms [20, Chapter 5]:

$$ \Re Li_2(ir) = Li_2(-r^2)/4 \leq 0, \quad \Im Li_2(ir) = \int_0^r \tan^{-1}(y) \frac{dy}{y} \geq 0. \quad (A.3) $$

Lemma 9. For $0 < r \leq 1$, $\frac{\pi \sqrt{2}}{8} \sqrt{r} < f_1(ir)$.

Proof. To find this bound for $f_1(ir)$ on $(0, 1]$, we start by explicitly writing $f_1(ir)$ using equation (A.2)

$$ f_1(ir) = \Re \sqrt{Li_2(ir)} = \frac{1}{\sqrt{2}} \sqrt{\Re Li_2(ir) + \Im Li_2(ir)}, $$

where

$$ \Re Li_2(ir) = \frac{1}{4} Li_2(-r^2) > \frac{1}{4} Li_2(-1)r = -\frac{1}{48} \pi^2 r, \quad 0 < r < 1. $$

Here we used that $Li_2(-r^2)$ is concave so its secant line on $[0, 1]$ gives a lower bound.

Lemma 10. The function $r \mapsto \theta = \arg(Li_2(ir))$ is increasing on $[0, 1]$. 

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Proof. For convenience, write $x(r) = \Re Li_2(ir)$, $y(r) = \Im Li_2(ir)$, and $dy/dx = \tan{\theta}$. We will use the dot notation for derivative. By equation (A.3), we note that $\dot{x}$, $\ddot{x}$, and $\ddot{y}$ are negative while $\dot{y}$ is positive. Consequently, the sign of the desired derivative is positive:

$$\frac{d}{dr} \tan{\theta} = \frac{\dot{x} \ddot{y} - \ddot{x} \dot{y}}{(\dot{x})^2} > 0$$

\[\square\]

Remark 4. Since $\theta(r)$ is increasing, $\Re Li_2(ir) < 0$, and $\Im Li_2(ir) > 0$, the range of $\theta(r)$ must lie in $[\pi/2, 3\pi/2]$. By equation (A.3), $Li_2(i)$ is given explicitly by $-\pi^2/48 + IG$, where $G$ is the Catalan constant $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$. We find that $\arg Li_2(i) = \pi - \arctan(48G/\pi^2) \approx 1.79161$ so the range of $\theta(r)$ is exactly $[\pi/2, \arg Li_2(i)]$ which lies inside $[\pi/2, \pi/3]$. Further, using equation (A.2), we record the explicit form for $f_1(i)$

$$f_1(i) = \frac{1}{4\sqrt{6}}\sqrt{-\pi^2 + \sqrt{\pi^4 + 2304G^2}}$$

so we can verify $f_1(i) < f_2(1) = \pi\sqrt{6}/12$ by direct calculation.

To complete the proofs in the section, we need the bounds

$$\frac{1}{2} \leq \cos \left(\frac{\theta(r)}{2}\right) \leq \frac{\sqrt{2}}{2} \leq \sin \left(\frac{\theta(r)}{2}\right) \leq \frac{\sqrt{3}}{2}.$$ 

If needed, these bounds can be tighten by using

$$\cos(\theta(1)/2) = \sqrt{(a - \pi^2)/(2a)} \approx 0.62488,$$

$$\sin(\theta(1)/2) = \sqrt{(a + \pi^2)/(2a)} \approx 0.78071$$

with $a = \sqrt{\pi^4 + 2304G^2}$.

Lemma 11. $f_1(ir)$ is increasing on $[0,1]$.

Proof. We write out the real part of the derivative

$$\frac{d}{dr} \sqrt{Li_2(ir)} = -\frac{\ln(1 - ir)}{2r\sqrt{Li_2(ir)}}.$$ 

using the polar form of $\sqrt{Li_2(ir)} = |Li_2(ir)|^{1/2}e^{i\theta(r)/2}$ to get

$$\frac{d}{dr} \Re \sqrt{Li_2(ir)} = -\Re \left( \frac{\frac{1}{2}\ln(1 + r^2) - i \arctan(r)}{2r|Li_2(ir)|^{1/2}e^{i\theta(r)/2}} \right).$$ 

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Now its sign is the same as the sign of

$$\sin(\theta(r)/2) \arctan(r) - \frac{1}{2} \cos(\theta(r)/2) \ln(1 + r^2)$$

which is bounded below by

$$\frac{\sqrt{2}}{2} \left( \arctan(r) - \frac{1}{2} \ln(1 + r^2) \right) = \frac{\sqrt{2}}{2} \int_0^r \frac{1 - t}{1 + t^2} \, dt \geq 0$$

since \( \theta(r)/2 \) lies in the interval \([\pi/4, \pi/3]\). \( \Box \)

**Lemma 12.** For \( r \in (0, 1) \), \( f_1(ir) \) is a concave function.

**Proof.** We use polar form again this time for the second derivative of \( f_1(ir) \). For the sake of notation, we introduce \( g_1(r) \) and \( g_2(r) \) by

$$\frac{d^2}{dr^2} \Re \sqrt{L_{i2}(r)} = \Re (g_1(r) + g_2(r))$$

where

$$g_1(r) = -\frac{1}{4} \left( \frac{1}{2} \ln(1 + r^2) - i \arctan(r) \right)^2 \frac{1}{L_{i2}(r)^{1/2} r^2},$$

$$g_2(r) = \frac{i/2}{L_{i2}(r)^{1/2} (1 - ir)} + \frac{1}{2} \frac{\ln(1 + r^2) - i \arctan(r)}{L_{i2}(r)^{1/2} r^2}.$$  

Since \( \pi/2 \leq \theta(r) \leq 2\pi/3 \), we have \( \sin(3\theta(r)/2) \) is positive while \( \cos(3\theta(r)/2) \) is negative. It is enough show that the signs of \( \Re g_1(r) \) and \( \Re g_2(r) \) are negative. Now the sign of \( \Re g_1(r) \) is the same as the sign of

$$-\Re \left[ \left( \frac{1}{4} \ln(1 + r^2)^2 - i(\arctan(r))^2 \right) \left( \cos(3\theta(r)/2) - i \sin(3\theta(r)/2) \right) \right]$$

$$= \cos(3\theta(r)/2) \left\{ \left( \arctan(r) \right)^2 - \frac{1}{4} \ln(1 + r^2)^2 \right\}$$

$$- \sin(3\theta(r)/2) \ln(1 + r^2) \arctan(r)$$

which is negative since \( \sin(3\theta(r)/2) \ln(1 + r^2) \arctan(r) \) is positive and

$$(\arctan(r))^2 - \frac{1}{4} (\ln(1 + r^2))^2 = \int_0^r \int_0^r \frac{(1 - s)(1 + t)}{(1 + s^2)(1 + t^2)} \, ds \, dt > 0$$
as well. The sign of $Rg_2(r)$ is the same as the sign of
\[
\frac{1}{1 + r^2} (-r \cos(\theta(r)/2) + \sin(\theta(r)/2))
\]
\[+ \frac{1}{r} \left( \frac{1}{2} \ln(1 + r^2) \cos(\theta(r)/2) - \arctan(r) \sin(\theta(r)/2) \right)
\]
\[= \frac{1}{r} \int_0^r \frac{r \cos(\theta(r)/2) + \sin(\theta(r)/2)}{1 + r^2} \, dt + \frac{1}{r} \int_0^r \frac{t \cos(\theta(r)/2) - \sin(\theta(r)/2)}{1 + t^2} \, dt
\]
\[= \frac{\cos(\theta(r)/2)}{r} \int_0^r \left( \frac{t}{1 + t^2} - \frac{r}{1 + r^2} \right) \, dt
\]
\[+ \frac{\sin(\theta(r)/2)}{r} \int_0^r \left( \frac{1}{1 + r^2} - \frac{1}{1 + t^2} \right) \, dt
\]
which is negative since both integrals are negative while $\sin(\theta(r)/2)$ and $\cos(\theta(r)/2)$ are positive.

Although the second derivative of $f_1(r)$ changes sign, $f_2(r)$ is convex.

**Lemma 13.** $f_2(r)$ is convex.

**Proof.** The second derivative of $f_2(r)$ is
\[
\frac{-(\ln(1 - r^2))^2}{r^2 L_2(r^2)^{3/2}} + \frac{\ln(1 - r^2)}{r^2 \sqrt{L_2(r^2)}} + \frac{2}{(1 - r^2) \sqrt{L_2(r^2)}},
\]
which has the same sign as
\[h(r) = 2r^2 L_2(r^2) - (1 - r^2)(\ln(1 - r^2))^2 + (1 - r^2) \ln(1 - r^2).
\]
It is elementary that the Taylor expansion of each term above has only even powers with nonnegative coefficients starting with $r^6$.

We record that $f_k(r)$ is convex for $k \geq 2$.

**Proposition 11** (Proposition 8 in the main text). There is a unique solution $\beta$ to $f_1(\beta) = f_2(r)$ in $(0, 1]$ further, it satisfies $3/4 < \beta < 1$. For $r \in (0, \beta)$, $f_1(\beta) > f_2(r)$ while for $r \in (\beta, 1)$, $f_1(\beta) < f_2(r)$.

**Proof.** Define $h(r) = f_2(r) - f_1(\beta)$ with $\beta \in (0, 1)$. By the last two Lemmas 12 and 13 $h(r)$ must be a convex function and therefore has at most one critical point $h'(r) = 0$ in $(0, 1)$. This also implies that it can have at most two roots since if $h(x_1) = h(x_2) = h(x_3) = 0$ for some $x_1 < x_2 < x_3$ then
by Rolle’s theorem $h'(\xi_1) = h'(\xi_2) = 0$ for $\xi_1 \neq \xi_2$. We then identify that $h(0) = 0$ trivially and from the inequalities in Lemma \ref{lem:inequality} we find that

$$f_2(r) < \frac{\pi r}{2\sqrt{6}} < \frac{\pi \sqrt{2} r}{8} < f_1(ir)$$

for $0 < r < 3/4$. Further, $f_2(i) > f_1(i)$ by Remark \ref{rem:remark}. Hence, by continuity there exists $\beta \in (3/4, 1)$ such that $f_2(i\beta) = f_1(i\beta)$.

\end{proof}

\section*{Appendix A.2. Generic Dominance of $f_k(z)$ on the Unit Disk}

The last major goal of this appendix is to show $f_1(z)$, $f_2(z)$, and $f_3(z)$ dominate all the other $f_k(z)$ in the unit disk and determine where this dominance holds. In particular we need to show

\begin{theorem} \[\text{Theorem } 2 \text{ in the main text}\] For $0 < |z| \leq 1$,

1. $f_k(z) \leq f_k(|z|) \leq f_2(|z|) < f_1(z)$, $k \geq 2$, $|\arg z| \leq \pi/3$,
2. $f_k(z) \leq f_k(|z|) < f_1(z)$, $k \geq 3$, $|\arg z| \leq \pi/2$,
3. $f_k(z) < f_1(z)$, $k \geq 2$, $0 \leq \arg z \leq \pi/3$,
4. $f_k(z) < \max[f_1(z), f_2(z), f_3(z)]$, $k \geq 4$, $\pi/2 \leq \arg z \leq \pi$.

\end{theorem}

As with the previous theorems, we will prove this using a series of lemmas and propositions. Part (1) of this theorem is an immediate consequence of the next lemma.

\begin{lemma}
The following are true:

1. Let $0 < |z| < 1$. If $\Re Li_2(z) > 0$, then $f_k(z) < f_2(|z|) < f_1(z)$ for $k \geq 2$.
2. For every $r \in (0, 1)$ and $\theta \in (0, \pi/3)$, $\Re Li_2(re^{i\theta}) > 0$.

\end{lemma}

\begin{proof}
We begin with the simple bound

$$f_1(z) = \frac{1}{\sqrt{2}} \sqrt{\Re Li_2(z) + |Li_2(z)|} > \frac{1}{\sqrt{2}} \sqrt{|Li_2(z)|} \geq \frac{1}{\sqrt{2}} \sqrt{-Li_2(-1)|z|}$$

$$= \frac{1}{\sqrt{2}} \sqrt{\frac{1}{2} Li_2(1)|z|} = \frac{1}{2} \sqrt{Li_2(1)|z|} \geq \frac{1}{2} \sqrt{Li_2(|z|)} \geq f_2(z).$$

To check that $\Re Li_2(re^{i\pi/3}) > 0$, we use the identity that $\Re Li_2(re^{i\pi/3}) = \frac{1}{6} Li_2(-r^3) - \frac{1}{2} Li_2(-r)$ \cite[p. 133]{20} and Lemma \ref{lem:identity}.

\end{proof}
Part (2) of Theorem 2 is part (1) of this corollary.
For $k \geq 3$, $f_k(z) \leq f_k(|z|) < f_1(z)$, $0 \leq \arg(z) \leq \pi/2$.

Proposition 12. For $0 < r \leq 1$, we have the following bounds:
(a) $f_3(r) < f_1(ir)$, (b) $f_5(r) < f_2(re^{i3\pi/4})$, (c) $f_7(r) < f_3(ir)$.

Proof. Note that $f_2(re^{i3\pi/4}) = f_1(ir^2)/2$ and $f_3(ir) = f_1(ir)/3$. In other words, the desired inequalities reduce to the following easily to verify bounds which follow from Lemma 9 and Equation (2):

$$f_3(r) \leq \frac{\pi}{3\sqrt{6}} r^{3/2} < \frac{\pi\sqrt{2}}{8} r^{1/2} \leq f_1(ir),$$
$$f_5(r) \leq \frac{\pi}{5\sqrt{6}} r^{5/2} < \frac{\pi\sqrt{2}}{16} r \leq f_2(re^{i3\pi/4}),$$
$$f_7(r) \leq \frac{\pi}{7\sqrt{6}} r^{7/2} < \frac{\pi\sqrt{2}}{24} r^{3/2} \leq f_3(ir).$$

\qed

By symmetry we need only consider the upper unit disk. For convenience we divide the upper disk into three sectors: $\arg(z) \in [0, \pi/2]$, $\arg(z) \in [\pi/2, 3\pi/4]$, and $\arg(z) \in [3\pi/4, \pi]$.

Corollary 1. Let $0 < |z| \leq 1$.
(1) For $k \geq 3$, $f_k(z) \leq f_k(|z|) < f_1(z)$, $0 \leq \arg(z) \leq \pi/2$.
(2) For $k \geq 5$, $f_k(z) \leq f_k(|z|) < f_2(z)$, $3\pi/4 \leq \arg(z) \leq \pi$.
(3) For $k \geq 7$, $f_k(z) \leq f_k(|z|) < f_3(z)$, $\pi/2 \leq \arg(z) \leq 3\pi/4$.

Part (2) of Theorem 2 is part (1) of this corollary.
To sum up, we know that, for $0 < |z| \leq 1$, $f_k(z) < f_1(z)$ with $\arg(z) \in [0, \pi/2]$, $k \geq 3$ and $f_k(z) < \max(f_2(z), f_3(z))$ with $\arg(z) \in [\pi/2, \pi]$, $k \geq 7$. In order to complete the proof of parts (3) and (4) of Theorem 2, we must establish the five remaining cases which requires working through the remaining three special sectors.

Appendix A.3. Refining the Dominance of $f_k(z)$ on the Open Unit Disk

We now show for $k \geq 4$, $f_k(z) < \max(f_1(z), f_2(z), f_3(z))$ for $0 < |z| \leq 1$. The maximum modulus principle applies in these exceptional cases since the functions $f_k$ are harmonic on the appropriate sector $S$ in the unit disk. In particular, we need only to check dominance on the two boundary radial
lines, say arg(z) = α or β and on the boundary arc of the unit circle e^{it},
α ≤ t ≤ β. On the unit circle, Li_2(e^{it}) has an explicit representation:

$$Li_2(e^{it}) = r(t) - iCl_2(t)$$

where

$$r(t) = \frac{\pi^2}{6} - \frac{t(2\pi - t)}{4}, \quad Cl_2(t) = \int_0^t \ln[2\sin(s/2)] \, ds$$

(see [20, p. 101]). Cl_2(t) is called the Clausen integral and has special values
Cl_2(\pi/2) = -G, the Catalan constant, and Cl_2(\pi) = 0. By construction,
\Im Li_2(e^{it}) = -Cl_2(t) is a nonnegative concave decreasing function on [\pi/2, \pi]
so its secant line

$$s(t) = 2G(\pi - t)/\pi$$
gives a lower bound on [\pi/2, \pi]. The tangent line L(t) at s = \pi/2 yields an
upper bound where L(t) = s/2 + (1/2) \ln(2) - \pi/4. Given the explicit form
of the real part of the square root (equation (A.2)), we have a lower bound
for f_1(e^{it}) on [\pi/2, \pi]:

$$f_1(e^{it}) \geq \frac{1}{\sqrt{2}} \sqrt{r(t) + \sqrt{r^2(t) + s^2(t)}}.$$  

We obtain the following inequalities by direct calculation using the secant
lower bound and the exact forms of f_1(1) and f_1(i):

$$f_1(1)/4 < f_1(e^{i3\pi/4}), \quad f_1(1)/4 < f_1(e^{i2\pi/3}), \quad f_1(e^{i2\pi/3})/5 < f_1(e^{i3\pi/4})$$

(A.4)
as well as

$$f_1(e^{i2\pi/3})/5 < f_1(e^{i4\pi/5})$$

(A.5)
needed in the proofs of Propositions [15] and [17] below. Further, using the
lower bound ln[2\sin(\pi/4)] for ln[2\sin(s/2)] on [\pi/2, 2\pi/3], we obtain

$$-Cl_2(2\pi/3) = - \int_{\pi/2}^{\pi/2} \ln[2\sin(s/2)] \, ds - \int_{\pi/2}^{2\pi/3} \ln[2\sin(s/2)] \, ds$$

$$= G - \int_{\pi/2}^{2\pi/3} \ln[2\sin(s/2)] \, ds < G - \ln[2\sin(\pi/4)](2\pi/3 - \pi/2)$$

$$= G - \frac{\pi \ln 2}{12}.$$  

Substituting, we find that equation (A.5) now follows.
Note that that $f_1(e^{i2\pi/3}) < f_3(e^{i2\pi/3}) = f_1(1)/3$ by the tangent line upper bound $L(t)$ for $-C l_2(2\pi/3)$ on $[\pi/2, 2\pi/3]$. Since $f_2(e^{i2\pi/3}) = f_1(e^{i2\pi/3})/2$, we find that

$$f_2(e^{i2\pi/3}) < f_1(e^{i2\pi/3}) < f_3(e^{i2\pi/3}),$$

so all three functions $f_1(z), f_2(z),$ and $f_3(z)$ occur in describing phases.

We will also make use of the symmetry relation $f_1(e^{i(\pi+t)}) = f_1(e^{i(\pi-t)})$ on $[0, \pi]$.

**Lemma 15.** For $0 < r \leq 1$, $f_3(re^{i5\pi/6}) < f_2(re^{i5\pi/6})$.

**Proof.** We record that $f_3(re^{i5\pi/6}) = f_1(ir^3)/3$ and $f_2(re^{i3\pi/4}) = f_1(ir^2)/2$. Since $f_2(re^{it})$ is increasing for $t \in [\pi/2, \pi]$ and $f_1(ir)$ is also increasing, the result follows. 

**Appendix A.3.1. Dominance of $f_k(z)$ on the Quarter Disk $\arg(z) \in [0, \pi/2]$**

On the quarter disk, we already know $f_k(z) \leq \max (f_1(z), f_2(z))$ with $k \geq 3$. We must reduce this to show $f_2(z) < f_1(z)$. This also concludes part (3) in Theorem 2.

**Proposition 13.** For $0 < |z| \leq 1$ and $0 \leq \arg(z) \leq \pi/2$,

$$f_2(z) < f_1(z).$$

**Proof.** Since both $f_1$ and $f_2$ are harmonic on the open quarter unit disk, we can use the maximum modulus principle. On the radial boundary lines, $f_2(r) < f_1(r)$ and $0 = f_2(ir) < f_1(ir), r \in (0,1]$. On the arc $e^{it}, t \in [0, \pi/2]$, both $f_1(e^{it})$ and $f_2(e^{it})$ are decreasing functions by Proposition 4. On $t \in [0, \pi/4],

$$f_2(e^{it}) \leq f_2(1) \leq f_1(e^{i\pi/4}) < f_1(e^{it}).$$

as $f_2(1) < f_1(e^{i\pi/4})$ by Lemma 13. For $t \in [\pi/4, \pi/2]:$

$$f_2(e^{it}) \leq f_2(e^{i\pi/4}) = \frac{1}{2} f_1(i) \leq f_1(i) < f_1(e^{it}).$$

**Appendix A.3.2. Dominance of $f_k(z)$ on the sector $\arg(z) \in [3\pi/4, \pi]$**

On this sector we have for $k \geq 5,$

$$\max (f_1(z), f_2(z), f_3(z), f_4(z)) > f_k(z).$$

To refine this result, we show $f_4(z) < f_2(z)$.
Proposition 14. On the sector $\arg z \in [3\pi/4, \pi]$ and $0 < |z| \leq 1$, $f_4(z) < f_2(z)$.

Proof. The sector $S$ has boundary lines with $\alpha = 3\pi/4$ and $\beta = \pi$. When $\arg(z) = 3\pi/4$, $f_4(z) = 0$ while $f_2(re^{i3\pi/4}) = f_1(ir^2)/2 > 0$ when $0 < r \leq 1$. For $\beta = \pi$, $0 < f_4(-r) = f_4(r) < f_2(-r) = f_2(r)$, $0 < r \leq 1$. On the arc $e^{it}$, $3\pi/4 \leq t \leq \pi$, both $f_2(e^{it})$ and $f_4(e^{it})$ are increasing by Proposition 4. We check the chain of inequalities

$$f_4(e^{i5\pi/6}) = f_1(e^{i2\pi/3})/4 < f_2(e^{i3\pi/4}) = f_1(i)/2$$

by Proposition 4 and

$$f_4(1) = (1/2)(f_1(1)/2) < f_1(e^{i\pi/3})/2 = f_2(e^{i5\pi/6})$$

by Lemma 14. Thus

$$f_4(e^{it}) < f_4(1) < f_2(e^{i\pi/4}) < f_2(e^{i3\pi/4}) < f_2(e^{it}).$$

□

Appendix A.3.3. Dominance of $f_k(z)$ on the sector $\arg(z) \in [\pi/2, 3\pi/4]$

As in the previous two sections, we have to reduce the number of relevant $f_k(z)$. Things are a bit more fine tuned in this section however. There are three cases.

Proposition 15. On the sector with $\arg z \in [\pi/2, 3\pi/4]$, $f_4(z) < f_1(z)$, $z \neq 0$.

Proof. The sector $S$ has boundary lines with $\alpha = \pi/2$ and $\beta = 3\pi/4$. We need to recall Theorem 2 part (2) (which is already proven) that $f_3(r) \leq f_1(ir)$. When $\arg(z) = \pi/2$, $f_4(ir) = f_4(r) < f_3(r) \leq f_1(ir)$, $0 < r \leq 1$. When $\arg(z) = 3\pi/4$, $f_4(re^{i3\pi/4}) = 0$ while $f_1(re^{i3\pi/4}) > 0$, $0 < r \leq 1$. Hence $f_4(z) < f_1(z)$ on the radial boundary lines.

On the arc $e^{it}$, $t \in [\pi/2, 3\pi/4]$, we can use Proposition 4 to show both $f_1(e^{it})$ and $f_4(e^{it})$ are decreasing. Using the secant line bounds for $f_1(1)/4 < f_1(e^{i2\pi/3})$ and $f_1(i)/4 < f_1(e^{i3\pi/4})$, we find that the required inequalities hold on $t \in [\pi/2, 2\pi/3]$

$$f_4(e^{it}) \leq f_4(i) = f_1(i)/4 = \frac{\pi}{4\sqrt{6}} < f_1(e^{i2\pi/3}) \leq f_1(e^{it})$$

and for $t \in [2\pi/3, 3\pi/4]$

$$f_4(e^{it}) \leq f_4(e^{i2\pi/3}) = f_1(e^{i2\pi/3})/4 < f_1(i)/4 < f_1(e^{i3\pi/4}) \leq f_1(e^{it})$$
where \( f_1(e^{i2\pi/3})/4 < f_1(i)/4 \) holds since \( f_1(e^{it}) \) is decreasing and \( f_1(i)/4 < f_1(e^{i3\pi/4}) \) by equation (A.3).

\[ \square \]

**Proposition 16.** On the sector with \( \arg z \in [\pi/2, 3\pi/4], \) \( f_0(z) < f_3(z), \) \( z \neq 0. \)

**Proof.** We need to consider two sectors \( S_1 \) and \( S_2 \) so \( \arg z \in [\pi/2, 2\pi/3] \) and \([2\pi/3, 3\pi/4]\) so \( f_0(z) \) will be harmonic in each. The boundary lines of the sector \( S_1 \) are \( \alpha = \pi/2 \) and \( \beta = 2\pi/3. \) For \( \arg z = \pi/2, \) \( f_6(ir) = f_1(-r^6) = 0 \) while \( f_3(ir) = f_1(ir^3) > 0 \) for \( r \in (0, 1]. \) For \( \arg z = 2\pi/3, \)

\[
f_0(re^{i2\pi/3}) = f_1(r)/6 \quad \text{while} \quad f_3(re^{i2\pi/3}) = f_1(r^3)/3.
\]

For \( \arg z = 3\pi/4, \)

\[
f_0(re^{i3\pi/4}) = 1/6 \quad \text{and} \quad \frac{1}{3} f_1(ir^3) < f_3(r^3e^{i\pi/4}) = f_3(re^{i3\pi/4})
\]

since \( f_1(ir) \) is increasing in \( r \) by Proposition 3 and \( f_1(z) \) is decreasing in the argument of \( z \) by Proposition 4. Hence \( f_0(z) \leq f_3(z) \) are the radial lines.

On the arc for \( S_1, \) both \( f_3(e^{it}) \) and \( f_6(e^{it}) \) are increasing on \([\pi/2, 2\pi/3]\) by Proposition 4. It convenient to use two subintervals \([\pi/2, 7\pi/12]\) and \([7\pi/12, 2\pi/3]\) in this case. We need to check that

\[
f_0(e^{i7\pi/12}) < f_3(e^{i\pi/2}), \quad f_0(e^{i2\pi/3}) < f_3(e^{7\pi/12}).
\]

Now

\[
f_0(e^{i7\pi/12}) = f_1(e^{i\pi/2})/6 = f_1(i)/6 < f_1(i)/3 = f_3(e^{i\pi/2}),
\]

\[
f_0(e^{i2\pi/3}) = f_1(1)/6 < f_3(e^{i7\pi/12}) = f_1(e^{i\pi/4})/3
\]

since \( \Re L_{2}(e^{i\pi/4}) > 0 \) and Lemma 11.

On the arc \([2\pi/3, 3\pi/4]\) for \( S_2, \) both \( f_3(e^{it}) \) and \( f_6(e^{it}) \) are decreasing. The required inequality \( f_0(e^{i2\pi/3}) < f_3(e^{i3\pi/4}) \) holds because \( f_3(e^{i3\pi/4}) = f_1(e^{i\pi/4})/3; \) so this inequality reduces to the last inequality above. \( \square \)

**Proposition 17.** On the sector \( \arg z \in [\pi/2, 5\pi/8], \) \( f_5(z) < f_1(z), \) \( z \neq 0. \)

**Proof.** (a) Since \( f_4(z) < f_1(z) \) on \([\pi/2, 3\pi/4]\), it is enough to show \( f_5(z) < f_4(z). \) For \( t \in [\pi/2, 5\pi/8], \) \( f_4(re^{it}) \) is decreasing while \( f_5(re^{it}) \) is decreasing on \([\pi/2, 3\pi/5]\) and increasing on \([3\pi/5, 5\pi/8]. \) For fixed \( r \in (0, 1], \) the minimum value of \( f_4(re^{it}) \) is \( f_4(re^{i5\pi/8}) = f_1(ir^4)/4 \) while the maximum value of \( f_5(re^{it}) \) is \( f_5(ir) = f_1(ir^5)/5. \) Since \( f_1(ir) \) is an increasing function, the desired inequality holds.

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(b) We apply the maximum modulus principle on a sector $S$ with boundary radial lines $\arg(z) = \alpha$ or $\beta$ where $\alpha = 3\pi/5$ and $\beta = 3\pi/4$.

For $\arg(z) = 3\pi/5$, $f_5(z) = 0$ while $f_2(re^{i3\pi/5}) > 0$ with $r \in (0,1]$. For $\arg(z) = 4\pi/5$, we see by part (2) of Proposition II that

$$f_5(r) < f_2(re^{i3\pi/4}) < f_2(re^{i4\pi/5}).$$

On the arc $e^{it}$, $3\pi/5 \leq t \leq 4\pi/5$, both $f_2(e^{it})$ and $f_5(e^{it})$ are increasing so it is enough to check the two evaluations:

$$f_5(e^{i2\pi/3}) < f_2(e^{i3\pi/5}), \quad f_5(e^{3\pi/4}) < f_2(e^{i3\pi/4}).$$

We just saw that the last inequality automatically holds. For the remaining inequality, consider

$$f_5(e^{i2\pi/3}) = f_1(e^{i2\pi/3})/5, \quad f_2(e^{i3\pi/5}) = f_1(e^{i4\pi/5})/2$$

then we can use equation (A.5).

\[ \square \]

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