Tableau Correspondences and Representation Theory

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Abstract. We deduce decompositions of natural representations of general linear groups and symmetric groups from combinatorial bijections involving tableaux. These include some of Howe’s dualities, Gelfand models, the Schur-Weyl decomposition of tensor space, and multiplicity-free decompositions indexed by threshold partitions.

1. Introduction

Bijections in algebraic combinatorics are often manifestations of results in representation theory. For example, the Robinson-Schensted correspondence, between permutations and pairs of standard tableaux of the same shape, reflects the fact that the sum of squares of dimensions of irreducible representations of a symmetric group is its order. Sometimes combinatorial identities can be used to prove results in representation theory. For instance, the classification of the irreducible representations of symmetric groups and Young’s rule are deduced from the Robinson-Schensted-Knuth (RSK) correspondence in [11]. The dual RSK correspondence is used to determine what happens when a representation is twisted by the sign character. This article collects many more instances of this phenomenon.

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2010 Mathematics Subject Classification. 05E10, 20C30, 22E46.

Key words and phrases. Burge correspondence, Gelfand model, Schur-Weyl duality, Specht modules, RSK correspondence, tableaux, Weyl modules.
Section 2 reviews basic facts about semistandard Young tableaux and Schur functions. Section 3 reviews fundamental results in the polynomial representation theory of general linear groups. The RSK correspondence is a bijection from matrices with non-negative integer entries onto pairs of semistandard tableaux of the same shape. The dual RSK correspondence is a bijection from matrices with entries 0 or 1 onto pairs of semistandard tableaux of mutually conjugate shape. In Section 4, the RSK correspondence and its dual are shown to imply the \((\text{GL}_m, \text{GL}_n)\)-duality and the skew \((\text{GL}_m, \text{GL}_n)\)-duality theorems of Howe. The symmetry property of the RSK correspondence gives a Gelfand model for \(\text{GL}_n(\mathbb{C})\). Schützenberger’s lemma on the RSK correspondence is shown to give a refinement of the Gelfand model.

The Burge correspondence is another bijection from matrices with non-negative integer entries onto pairs of semistandard tableaux of the same shape, less well-known than the RSK correspondence. In Section 5, we show that its symmetry property gives rise to another Gelfand model for \(\text{GL}_n(\mathbb{C})\), which also has a refinement based on an adaptation of Schützenberger’s lemma to this setting. Two more correspondences of Burge give rise to multiplicity-free decompositions into representations parameterized by threshold partitions, and conjugate threshold partitions. These are discussed in Section 6.

Section 7 describes a passage from representations of \(\text{GL}_n(\mathbb{C})\) to representations of \(S_n\) via the all-ones weight space. Applying this to one side of \((\text{GL}_m, \text{GL}_n)\)-duality, we recover Schur-Weyl duality. Applying it to the Gelfand models of \(\text{GL}_n(\mathbb{C})\), we recover combinatorial Gelfand models for \(S_n\) due to Inglis, Richardson, and Saxl \([8]\). Applying it to the multiplicity-free representations of \(\text{GL}_n(\mathbb{C})\) in Section 6, we recover some other interesting multiplicity-free representations of \(S_n\), which were outlined by Bump \([3]\) using completely different methods.

Most of the results in this article have appeared in the MSc thesis of Arghya Sadhukhan \([13]\).

2. Semistandard Young Tableaux and Schur Polynomials

In this section we recall basic facts about semistandard Young tableaux and Schur polynomials. For detailed, self-contained expositions, see \([11, 12]\).

**Definition 1.** A semistandard Young tableau in \(n\) letters is a left-justified array of boxes with rows of weakly decreasing length filled with numbers between 1 and \(n\) such that

1. the numbers increase weakly from left to right along rows,
2. the numbers increase strictly from top to bottom along columns.
If a semistandard Young tableau has \( l \) rows, and \( \lambda_i \) boxes in the \( i \)-th row for \( i = 1, \ldots, l \), then the partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \) is called the shape of the tableau. The weight of a tableau is the integer vector \( (\mu_1, \ldots, \mu_n) \) where \( \mu_i \) the number of times the number \( i \) occurs. If \( t \) is a tableau, we write \( \text{wt}(t) \) for its weight. Write \( \text{Tab}_n(\lambda) \) for the set of all semistandard Young tableaux in \( n \) letters having shape \( \lambda \). The subset of \( \text{Tab}_n(\lambda) \) consisting of tableaux with weight \( \mu \) is denoted by \( \text{Tab}(\lambda, \mu) \).

**Example 2.** The tableau in 7 letters

\[
\begin{array}{cccccc}
2 & 2 & 3 & 5 & 5 \\
4 & 4 & 4 & 6 \\
5 & 7
\end{array}
\]

has shape (5, 4, 2) and \( \text{wt}(t) = (0, 2, 1, 3, 3, 1, 1) \).

Given an integer vector \( \mu = (\mu_1, \ldots, \mu_n) \) with \( n \)-coordinates, let \( x^\mu \) denote the monomial \( x_1^{\mu_1} \cdots x_n^{\mu_n} \) in \( n \) variables.

**Definition 3 (Schur Polynomial).** For each integer partition \( \lambda \), the Schur polynomial in \( n \) variables corresponding to \( \lambda \) is defined as:

\[
\begin{align*}
s_\lambda(x_1, \ldots, x_n) &= \sum_{t \in \text{Tab}_n(\lambda)} x^{\text{wt}(t)}. \\
\end{align*}
\]

It turns out that \( s_\lambda \) is a symmetric polynomial in \( n \) variables. If the partition \( \lambda \) has more than \( n \) non-zero parts, there are no tableaux in \( n \) letters of shape \( \lambda \) (because the first column of the tableau has to be strictly increasing, and has length equal the number of parts of \( \lambda \)). On the other hand, if the number of non-zero parts of \( \lambda \) is at most \( n \), then there is at least one such tableau, namely the one whose \( i \)-th row has all boxes filled with \( i \). Therefore \( s_\lambda(x_1, \ldots, x_n) = 0 \) if and only if \( \lambda \) has more than \( n \) non-zero parts.

Let \( \Lambda_n^d \) denote the set of all integer partitions of \( d \) with at most \( n \) parts. By padding a partition with 0’s, write it as an integer vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with exactly \( n \) parts. Thus

\[
\Lambda_n^d = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 + \cdots + \lambda_n = d, \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \}.
\]

**Theorem 4.** [\( \text{[11]} \text{, Theorem 5.4.3} \)] The set

\[
\{s_\lambda \mid \lambda \in \Lambda_n^d\}
\]

is a basis for the space of all symmetric polynomials of degree \( d \) in \( n \) variables with coefficients in \( \mathbb{C} \).
3. Polynomial Representations of $GL_n(\mathbb{C})$

In this section we recall basic facts about polynomial representations of $GL_n(\mathbb{C})$. Details can be found in [11, Chapter 6].

**Definition 5.** A **polynomial representation** of $GL_n(\mathbb{C})$ is a finite dimensional vector space $W$ over $\mathbb{C}$, together with a group homomorphism $\rho : GL_n(\mathbb{C}) \to GL(W)$, such that each entry of the matrix of $\rho(g)$ (with respect to any basis of $W$) is a polynomial in the entries of the matrix $g$. The polynomial representation $W$ is said to be **homogeneous of degree $d$** if these polynomials are homogeneous of degree $d$. It is said to be **irreducible** if there is no non-trivial proper subspace $W' \subset W$ that is left invariant by $\rho(g)$ for all $g \in GL_n(\mathbb{C})$. Representations $(\rho_1, W_1)$ and $(\rho_2, W_2)$ are said to be **isomorphic** if there exists a linear isomorphism $T : W_1 \to W_2$ such that $\rho_2(g) \circ T = T \circ \rho_1(g)$ for each $g \in GL_n(\mathbb{C})$.

**Example 6** (The trivial representation). The only homogeneous polynomial representations of degree 0 are where $\rho(g)$ is identically the identity operator of $W$.

**Example 7** (Direct sum). If $(\rho_1, W_1)$ and $(\rho_2, W_2)$ are polynomial representations of $GL_n(\mathbb{C})$, then $(\rho_1 \oplus \rho_2, W_1 \oplus W_2)$ is also a polynomial representation of $GL_n(\mathbb{C})$. If $\rho_1$ and $\rho_2$ are homogeneous of degree $d$, then so is $\rho_1 \oplus \rho_2$.

**Example 8** (The defining representation). The identity map $\rho : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ defines a representation of $GL_n(\mathbb{C})$ on $\mathbb{C}^n$. This representation is homogeneous of degree 1 and is called the **defining representation**.

**Example 9** (Tensor product). If $(\rho_1, W_1)$ and $(\rho_2, W_2)$ are polynomial representations of $GL_n(\mathbb{C})$, then so is $(\rho_1 \otimes \rho_2, W_1 \otimes W_2)$. If $\rho_1$ has degree $d_1$ and $\rho_2$ has degree $d_2$, then $\rho_1 \otimes \rho_2$ has degree $d_1 + d_2$. As a particular example, $\otimes^d \mathbb{C}^n$, the $d$-fold tensor power of the defining representation $\mathbb{C}^n$, is homogeneous of degree $d$ and dimension $n^d$.

**Example 10** (Symmetric and alternating tensors). The symmetric group $S_d$ acts on $\otimes^d \mathbb{C}^n$ by permuting the tensor factors. The subspace of invariant tensors is denoted by $\text{Sym}^d \mathbb{C}^n$. Since the actions of $GL_n(\mathbb{C})$ and $S_d$ on $\otimes^d \mathbb{C}^n$ commute, $\text{Sym}^d \mathbb{C}^n$ is invariant under the action of $GL_n(\mathbb{C})$, forming a homogeneous representation of degree $d$, which is known as the $d$th symmetric tensor. Let $L$ denote the subalgebra of $\bigoplus_{d=0}^{\infty} \otimes^d \mathbb{C}^n$ generated by $x \otimes x$. Let $L_d = L \cap \otimes^d \mathbb{C}^n$. Then $L_d$ is an invariant subspace of $\otimes^d \mathbb{C}^n$, and the quotient $\wedge^d \mathbb{C}^n = \otimes^d \mathbb{C}^n / L_d$ is a representation of $GL_n(\mathbb{C})$, known as the $d$th alternating tensor.
Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of $d$. Then
\[
\text{Sym}^\lambda \mathbb{C}^n = \bigotimes_{i=1}^l \text{Sym}^{\lambda_i} \mathbb{C}^n
\]
\[
\wedge^\lambda \mathbb{C}^n = \bigotimes_{i=1}^l \wedge^{\lambda_i} \mathbb{C}^n
\]
are homogeneous polynomial representations of $\text{GL}_n(\mathbb{C})$ of degree $d$.

Let $T_n \subset \text{GL}_n(\mathbb{C})$ denote the subgroup of invertible diagonal matrices. We use $\Delta(x_1, \ldots, x_n)$ to denote the diagonal matrix with entries $x_1, \ldots, x_n$.

**Definition 11 (Weight vector).** Let $(\rho, W)$ be a polynomial representation of $\text{GL}_n(\mathbb{C})$. A vector $v \in W$ is said to be a weight vector with weight $\mu = (\mu_1, \ldots, \mu_n)$ if, for all $x_1, \ldots, x_n \in \mathbb{C}^*$,
\[
\rho(\Delta(x_1, \ldots, x_n))v = x_1^{\mu_1} \cdots x_n^{\mu_n} v.
\]
The subspace of all weight vectors of weight $\mu$ is called the $\mu$-weight space of $W$ and denoted $W(\mu)$.

**Theorem 12.** [11 Theorem 6.6.5] Every polynomial representation $(\rho, W)$ of $\text{GL}_n(\mathbb{C})$ admits a basis of weight vectors.

**Example 13.** In the defining representation of $\text{GL}_n(\mathbb{C})$ (Example 8), the coordinate vector $e_i \in \mathbb{C}^n$ is a weight vector with weight $e_i$. The set $\{e_i \mid i = 1, \ldots, n\}$ is a basis of weight vectors for $\mathbb{C}^n$.

**Example 14.** In the tensor space $\otimes^d \mathbb{C}^n$ (Example 9),
\[
\{e_{i_1} \otimes \cdots \otimes e_{i_d} \mid (i_1, \ldots, i_d) \in \{1, \ldots, n\}^d\}
\]
is a basis of $\otimes^d \mathbb{C}^n$ consisting of weight vectors. The weight of $e_{i_1} \otimes \cdots \otimes e_{i_d}$ is $(\mu_1, \ldots, \mu_n)$, where $\mu_k$ is the number of $k$ for which $i_k = i$.

**Example 15.** For non-negative integers $n$ and $d$, define
\[
M(n, d) = \{(i_1, \ldots, i_d) \mid 1 \leq i_1 \leq \cdots \leq i_d \leq n\}.
\]
Given $I = (i_1, \ldots, i_d) \in M(n, d)$ the symmetric tensor $e_{i_1} \cdots e_{i_d}$ is a weight vector in $\text{Sym}^d \mathbb{C}^n$ with weight $(\mu_1, \ldots, \mu_n)$ where $\mu_i$ is the number of $k$ for which $i_k = i$. As $I$ runs over $M(n, d)$, these vectors form a basis of $\text{Sym}^d \mathbb{C}^n$.

For non-negative integers $n$ and $d$, define
\[
N(n, d) = \{(i_1, \ldots, i_d) \mid 1 \leq i_1 < \cdots < i_d \leq n\}.
\]
Given $I = (i_1, \ldots, i_d) \in N(n, d)$ the alternating tensor $e_{i_1} \wedge \cdots \wedge e_{i_d}$ is a weight vector in $\wedge^d \mathbb{C}^n$ with weight $(\mu_1, \ldots, \mu_n)$ where $\mu_i$ is the number
of $k$ for which $i_k = i$. As $I$ runs over $N(n, d)$, these vectors form a basis of $\wedge^d \mathbb{C}^n$.

Most proofs in this article will be based on finding bases of weight vectors of representations as in the examples above.

**Definition 16.** The *character* of a polynomial representation $W$ of $\text{GL}_n(\mathbb{C})$ is defined as

$$\text{ch}_W(x_1, \ldots, x_n) = \text{tr}(\rho(\Delta(x_1, \ldots, x_n)); W).$$

For any permutation $w \in S_n$, $\Delta(x_{w(1)}, \ldots, x_{w(n)})$ is conjugate to $\Delta(x_1, \ldots, x_n)$. Therefore $\text{ch}_W$ is a symmetric polynomial. If $W$ is homogeneous of degree $d$, then $\text{ch}_W$ is homogeneous of degree $d$.

**Remark 17.** Theorem 12 implies that

$$\text{ch}_W(x_1, \ldots, x_n) = \sum \dim W(\mu)x^\mu,$$

the sum being over all $n$-tuples $\mu$ of non-negative integers.

Let $\lambda'$ denote the partition *conjugate* to $\lambda$, namely,

$$\lambda'_i = \# \{ j \mid \lambda_j \geq i \} \text{ for } i = 1, \ldots, n.$$

**Theorem 18.** [π, Chapter 6] Let $\rho : \text{GL}_n(\mathbb{C}) \to \text{GL}(W)$ be a polynomial representation of $\text{GL}_n(\mathbb{C})$. Then we have:

1. For every partition $\lambda \in \Lambda^d_n$, there exists an irreducible homogeneous polynomial representation $W^n_\lambda$ of $\text{GL}_n(\mathbb{C})$ of degree $d$ which occurs in both $\text{Sym}^\lambda \mathbb{C}^n$ and $\wedge^\lambda \mathbb{C}^n$. This representation satisfies:

$$\text{ch}_{W^n_\lambda}(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n),$$

the Schur polynomial corresponding to $\lambda$, in $n$ variables.

2. Every polynomial representation $W$ of $\text{GL}_n(\mathbb{C})$ has a unique decomposition of the form:

$$W = \bigoplus_{\lambda} (W^n_\lambda)^{\oplus m_\lambda}$$

into irreducible polynomial representations (since $W$ is finite dimensional, it should be understood that $m_\lambda$ is positive for only finitely many $\lambda$).

3. Two polynomial representations of $\text{GL}_n(\mathbb{C})$ are isomorphic if and only if their characters are equal.
4. The RSK Correspondence and its Dual

Let \( \mu = (\mu_1, \ldots, \mu_m) \) and \( \nu = (\nu_1, \ldots, \nu_n) \) be non-negative integer vectors with common sum \( d \). Let \( M_{\mu \nu} \) denote the set of all \( m \times n \) matrices \( A = (a_{ij}) \) with non-negative integer entries such that
\[
\sum_j a_{ij} = \mu_i \text{ for } i = 1, \ldots, m \text{ and } \sum_i a_{ij} = \nu_j \text{ for } j = 1, \ldots, n.
\]

The RSK correspondence [9, Section 3] is an algorithmic bijection:
\[
\text{RSK} : M_{\mu \nu} \rightarrow \bigsqcup_{\lambda \vdash d} \text{Tab}(\lambda, \nu) \times \text{Tab}(\lambda, \mu).
\]

**Theorem 19.** Let \( W = \text{Sym}^d(C^m \otimes C^n) \). Viewing \( C^m \) and \( C^n \) as the defining representations of \( \text{GL}_m(C) \) and \( \text{GL}_n(C) \) respectively, the functorial nature of \( \text{Sym}^d \) implies that \( W \) is a representation of \( \text{GL}_m(C) \times \text{GL}_n(C) \). The decomposition of \( W \) into irreducible representations of \( \text{GL}_m(C) \times \text{GL}_n(C) \) is given by:
\[
W = \bigoplus_{\lambda \vdash d} W^n_{\lambda} \otimes W^m_{\lambda}.
\]

**Proof.** Let \( e_1, \ldots, e_n \) and \( f_1, \ldots, f_m \) denote the coordinate vectors in \( C^n \) and \( C^m \) respectively. Let \( \mu \) and \( \nu \) be non-negative integer vectors with common sum \( d \). Given \( A \in M_{\mu \nu} \), define a vector \( v_A \in W \) by
\[
v_A = \prod_{i=1}^n \prod_{j=1}^m (e_i \otimes f_j)^{a_{ij}}.
\]

Then \( v_A \) is an eigenvector for the action of \( \Delta(x_1, \ldots, x_n) \in T_n \), and \( \Delta(y_1, \ldots, y_m) \in T_m \) with eigenvalue \( x^\nu y^\mu \). Therefore
\[
\text{tr}(\Delta(x)\Delta(y); W) = \sum_{\nu} \sum_{\mu} \sum_{A \in M_{\mu \nu}} x^\nu y^\mu
\]
\[
= \sum_{\nu} \sum_{\mu} \sum_{\lambda} \sum_{\lambda' \in \text{Tab}(\lambda, \nu)} \sum_{\lambda'' \in \text{Tab}(\lambda, \mu)} x^{\text{wt}(\lambda')} y^{\text{wt}(\lambda'')}
\]
\[
= \sum_{\lambda} s_\lambda(x) s_\lambda(y)
\]
\[
= \sum_{\lambda} \text{ch}_{W^n_{\lambda}}(x) \text{ch}_{W^m_{\lambda}}(y)
\]
\[
= \sum_{\lambda} \text{ch}_{W^n_{\lambda} \otimes W^m_{\lambda}}(x, y).
\]
\[\Box\]
Remark 20. Theorem \[19\] is called \((GL_n, GL_m)\)-duality by Howe \[7, Theorem 2.1.2\], who gives a different proof. See also Benson and Ratcliff \[1, Theorem 4.1.1\].

Let \(\mu = (\mu_1, \ldots, \mu_m)\) and \(\nu = (\nu_1, \ldots, \nu_n)\) be non-negative integer vectors with common sum \(d\). Let \(N_{\mu\nu}\) denote the set of all \(m \times n\) matrices \(A = (a_{ij})\) with entries in \(\{0, 1\}\) such that

\[
\sum_j a_{ij} = \mu_i \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad \sum_i a_{ij} = \nu_j \quad \text{for} \quad j = 1, \ldots, n.
\]

The dual RSK correspondence \[9, Section 5\] is an algorithmic bijection:

\[
RSK^*: N_{\mu\nu} \to \bigsqcup_{\lambda \vdash d} \text{Tab}(\lambda', \nu) \times \text{Tab}(\lambda, \mu).
\]

Theorem 21. Let \(W = \wedge^d(C^m \otimes C^n)\). The decomposition of \(W\) into irreducible representations of \(GL_m(C) \times GL_n(C)\) is given by:

\[
W = \bigoplus_{\lambda \vdash d} W^m_{\lambda} \otimes W^n_{\lambda}.
\]

Proof. Given \(A\) in \(N_{\mu\nu}\) define \(v_A \in W\) by:

\[
v_A = \bigwedge_{i=1}^n \bigwedge_{j=1}^m (e_i \otimes f_j)^{a_{ij}}.
\]

Then \(v_A\) is an eigenvector for the action of \(\Delta(x_1, \ldots, x_n) \in T_n\) and \(\Delta(y_1, \ldots, y_m) \in T_m\) with eigenvalue \(x^\nu y^\mu\). Now, proceeding as in the proof of Theorem \[19\] gives the result. \(\square\)

Remark 22. Theorem \[21\] is called skew \((GL_n, GL_m)\)-duality by Howe \[7, Theorem 4.1.1\], who gives a different proof.

Knuth proved a symmetry theorem for the RSK correspondence:

\[
\text{RSK}(A) = (P, Q) \text{ if and only if } \text{RSK}(A') = (Q, P).
\]

Here \(A'\) denotes the transpose of the matrix \(A\). Therefore, if \(A\) is symmetric, then \(\text{RSK}(A)\) is of the form \((P, P)\) for some semistandard Young tableau \(P\). Let \(M^\text{sym}_{\nu\nu}\) denote the set of symmetric \(n \times n\) matrices in \(M_{\nu\nu}\). The symmetry property of the RSK correspondence implies that it induces a bijection:

\[
(1) \quad \text{RSK} : M^\text{sym}_{\nu\nu} \to \bigsqcup_{\lambda \vdash d} \text{Tab}(\lambda, \nu).
\]
Theorem 23. Let $W = \bigoplus_{k+2l=d} \text{Sym}^k(C^n) \otimes \text{Sym}^l(\wedge^2 C^n)$. This representation of $\text{GL}_n(C)$ has decomposition into irreducibles given by:

$$W = \bigoplus_{\lambda \vdash d} W^n_\lambda.$$ 

Proof. As $A$ runs over symmetric $n \times n$ matrices with non-negative integer entries summing to $d$, the vectors

$$v_A = \prod_{i=1}^n e_i^{a_{ii}} \prod_{i<j} (e_i \wedge e_j)^{a_{ij}}$$

form a basis of $W$ consisting of eigenvectors for the action of $\Delta(x_1, \ldots, x_n)$. When $A \in M_{\nu \nu}^{\text{sym}}$, the eigenvalue corresponding to $v_A$ is $x^\nu$. Now using the correspondence (1) on symmetric matrices:

$$\text{ch}_W(x_1, \ldots, x_n) = \sum_{\nu} |M_{\nu \nu}^{\text{sym}}| x^\nu$$

$$= \sum_{\nu} \sum_{\lambda} |\text{Tab}(\lambda, \nu)| x^\nu$$

$$= \sum_{\nu} s_\lambda(x)$$

$$= \sum_{\lambda} \text{ch}_{W_\lambda}(x),$$

thereby proving the result. \hfill \Box

Remark 24 (A Gelfand Model). We may say that the symmetric algebra:

$$\text{Sym}(C^n \oplus \wedge^2 C^n) := \bigoplus_{k,l \geq 0} \text{Sym}^k(C^n) \otimes \text{Sym}^l(\wedge^2 C^n)$$

is a model for polynomial representation theory of $\text{GL}_n(C)$, in the sense of Bernstein, Gelfand and Gelfand [2]: it is a completely reducible representation containing each irreducible polynomial representation of $\text{GL}_n(C)$ exactly once. For an alternate approach, see [1] Theorem 4.5.1.

The following result was proved for permutation matrices by Schützenberger [14, Theorem 4.4]. An elegant proof using the light-and-shadows version of the Robinson-Schensted correspondence was suggested (as an exercise) by Viennot [15, Proposition 4.1]. This proof can be adapted to integer matrices using Fulton’s matrix ball construction [5, Section 4.2], or the generalization of Viennot’s light-and-shadows algorithm in Prasad [11, Chapter 3].
Theorem 25 (Schützenberger’s lemma). The RSK correspondence takes symmetric matrices with trace $k$ to semistandard Young tableaux with $k$ odd columns.

The term “odd columns” alludes to the Young diagram of $\lambda$, whose rows are the parts of $\lambda$. The columns of $\lambda$ are the parts of the conjugate partition $\lambda'$. Schützenberger’s lemma gives the following refinement of Theorem 23:

Theorem 26. For all non-negative integers $k$ and $l$,
\[
\text{Sym}^k(C^n) \otimes \text{Sym}^l(\wedge^2 C^n) = \bigoplus_{\lambda \vdash k+2l \text{ with } k \text{ odd columns}} W^n_\lambda.
\]

Taking $k = 0$ gives a better-known special case (see Benson and Ratcliff [1, Theorem 4.3.1])

Theorem 27. For every positive integer $l$,
\[
\text{Sym}^l(\wedge^2 C^n) = \bigoplus_{\lambda \vdash 2l \text{ with every part appearing even number of times}} W^n_\lambda.
\]

Remark 28. The restriction of the RSK correspondence to symmetric matrices with zeros on the diagonal is the correspondence concerning graphs without loops in Burge [4, Section 2]. Theorem 27 can be deduced from it.

5. The Burge Correspondence

Burge [4] described four variants of the RSK correspondence. We begin with a correspondence which, although absent from Burge’s paper, is now commonly known as the Burge Correspondence [5, A.4.1]:

\[
\text{BUR} : M_{\mu\nu} \rightarrow \bigsqcup_{\lambda \vdash d} \text{Tab}(\lambda, \nu) \times \text{Tab}(\lambda, \mu).
\]

Indeed, this is a different bijection between the same two sets between which the RSK correspondence defines a bijection. This correspondence shares the symmetry property with the RSK correspondence:

\[
\text{BUR}(A) = (P, Q) \text{ if and only if } \text{BUR}(A') = (Q, P),
\]

and consequently induces a bijection:

\[
\text{(2) BUR} : M_{\mu\nu}^{\text{sym}} \rightarrow \bigsqcup_{\lambda \vdash d} \text{Tab}(\lambda, \mu).
\]

An important difference between the Burge correspondence and the RSK correspondence is that Theorem 25 (Schützenberger’s lemma) fails. Instead we have [5 A4.1, Exercise 18]:

\[
\text{BUR} : M_{\mu\nu}^{\text{sym}} \rightarrow \bigsqcup_{\lambda \vdash d} \text{Tab}(\lambda, \mu).
\]
Theorem 29 (Schützenberger’s lemma for the Burge correspondence). The Burge correspondence (2) takes a symmetric integer matrix with \( k \) odd entries on its diagonal to a semistandard Young tableaux with \( k \) odd rows.

Theorem 30. For non-negative integers \( k \) and \( l \), the representation:

\[
W = \wedge^k(C^n) \otimes \text{Sym}^l(\text{Sym}^2C^n)
\]

of \( \text{GL}_n(C) \) has decomposition into irreducible representations given by:

\[
W = \bigoplus_{\lambda \vdash k + 2l \text{ with } k \text{ odd rows}} W^n_{\lambda}.
\]

Proof. Given a symmetric integer matrix \( A \) with \( k \) odd diagonal entries, write \( A = A' + A'' \), where \( A' \) is a diagonal matrix with diagonal entries 0 or 1, and \( A'' \) is an integer matrix with even diagonal entries. Write \( b_{ii} = a''_{ii}/2 \) (half of the \((i, i)\)th entry of \( A'' \)) and \( b_{ij} = a''_{ij} \) for \( i \neq j \). Set

\[
v_A = \bigwedge_{a'_{ii}=1} e_i \otimes \prod_{i < j} (e_i e_j)^{b_{ij}}.
\]

As \( A \) runs over symmetric matrices with non-negative integer entries, \( k \) odd entries on the diagonal, and off-diagonal entries summing to \( 2l \), \( v_A \) forms a basis of eigenvectors for the action of \( \Delta(x_1, \ldots, x_n) \) on \( W \). Moreover, when \( A \in M^{\text{sym}}_{\mu \mu} \) then \( v_A \) has eigenvalue \( x^\mu \).

Schützenberger’s lemma for the Burge correspondence (Theorem 29) gives:

\[
\text{ch}_W(x) = \sum_{\lambda \vdash k+2l \text{ with } k \text{ odd rows}} s_\lambda(x),
\]

whence the theorem follows. \( \square \)

When restricted to matrices with even diagonal entries, the correspondence (2) becomes the correspondence in [4, Section 3] concerning graphs with loops and multiple edges. In terms of Theorem 30, this is the special case where \( k = 0 \):

Theorem 31. For every non-negative integer \( k \),

\[
\text{Sym}^k(\text{Sym}^2C^n) = \bigoplus_{\lambda \vdash 2k \text{ with all parts even}} W^n_{\lambda}.
\]

Remark 32. Theorem 31 appears in Howe [7, Theorem 3.1] with a different proof.
Remark 33 (Another Gelfand model). We get a second model (see Remark 24) for the polynomial representation theory of $GL_n(\mathbb{C})$, namely:

$$\bigoplus_{k,l \geq 0} \Lambda^k \mathbb{C}^n \otimes \text{Sym}^l(\text{Sym}^2 \mathbb{C}^n).$$

6. Two more correspondences of Burge

Besides the restrictions of RSK and BUR to symmetric matrices with zeros on the diagonal, the article of Burge [4] contains two more correspondences. Let $N^\text{sym}_{\mu \mu}$ denote matrices in $M^\text{sym}_{\mu \mu}$ with entries in \{0, 1\}. Let $N^\text{sym, tr}=0_{\mu \mu}$ denote the subset of $N^\text{sym}_{\mu \mu}$ consisting of matrices with trace zero. Given a partition $\lambda$, for each $i$ such that $\lambda_i \geq i$, let $\alpha_i = \lambda_i - i$, and $\beta_i = \lambda'_i - i$. The largest index $d$ such that $\lambda_d \geq d$ is called the Durfee rank of $\lambda$, and $(\alpha_1, \ldots, \alpha_d | \beta_1, \ldots, \beta_d)$ are called the Frobenius coordinates of $\lambda$.

Definition 34 (Threshold Partition). A partition $\lambda$ is called a threshold partition if its Frobenius coordinates satisfy $\beta_i = \alpha_i + 1$ for $i = 1, \ldots, d$, where $d$ is the Durfee rank of $\lambda$.

Let $\text{TP}(n)$ denote the set of threshold partitions of $n$. Let $\mu$ be a partition of $d$. The correspondences of Burge concerning graphs without loops or multiple edges [4 Section 4] and graphs without multiple edges [4 Section 5] are:

$$\text{BUR}_1 : N^\text{sym, tr}=0_{\mu \mu} \to \bigoplus_{\lambda \in \text{TP}(d)} \text{Tab}(\lambda, \mu)$$

$$\text{BUR}_2 : N^\text{sym}_{\mu \mu} \to \bigoplus_{\lambda' \in \text{TP}(d)} \text{Tab}(\lambda, \mu).$$

Their significance in terms of the representation theory of $GL_n(\mathbb{C})$ are given by:

Theorem 35. For every non-negative integer $d$, we have:

$$\Lambda^d(\Lambda^2 \mathbb{C}^n) \cong \bigoplus_{\lambda \in \text{TP}(d)} W^\mu_{\lambda},$$

$$\Lambda^d(\text{Sym}^2 \mathbb{C}^n) \cong \bigoplus_{\lambda' \in \text{TP}(d)} W^n_{\lambda}.$$

Proof. Given a symmetric matrix $A$ with entries in \{0, 1\}, define

$$v_A = \bigwedge_{\{(i,j)|i<j, a_{ij}=1\}} e_i \wedge e_j.$$
taking the terms in lexicographic order. As $A$ runs over symmetric matrices with entries in $\{0, 1\}$, zeros on the diagonal, and all entries summing to $d$, $v_A$ forms a basis of $\wedge^d(\wedge^2 \mathbb{C}^n)$. Moreover, if $A \in N^{\text{sym}, \text{tr}=0}$, $v_A$ is an eigenvector for the action of $\Delta(x_1, \ldots, x_n)$ with eigenvalue $x^\mu$.

Now using the bijection BUR, it is easy to see that the characters of both sides in the first identity in Theorem 35 are equal. The proof of the second identity is similar. 

7. Representations of Symmetric Groups

If $W_1$ and $W_2$ are polynomial representations of $\text{GL}_n(\mathbb{C})$ of degree $d$, and $T : W_1 \to W_2$ is a homomorphism of representations, then for each $\mu$, $T(W_1(\mu)) \subset W_2(\mu)$. Here $W_i(\mu)$ denotes the $\mu$ weight space of $W_i$ (Definition 11). The symmetric group $S_n$ may be regarded as a subgroup of $\text{GL}_n(\mathbb{C})$ via permutation matrices. Given a polynomial representation $W$ of $\text{GL}_n(\mathbb{C})$ of degree $n$, let $R_n(W) = W(1, \ldots, 1)$. Clearly, the action of $S_n$ leaves $R_n(W)$ invariant, making it a representation of the symmetric group. Moreover, for a morphism $T : W_1 \to W_2$ of polynomial representations of $\text{GL}_n(\mathbb{C})$, let $R_n(T) : R_n(W_1) \to R_n(W_2)$ denote the linear map obtained by restriction of $T$ to $R_n(W)$. We have a functor:

$$R_n : \text{Rep}^n(\text{GL}_n(\mathbb{C})) \to \text{Rep}(S_n),$$

where $\text{Rep}^n(\text{GL}_n(\mathbb{C}))$ denotes the category of homogeneous polynomial representations of $\text{GL}_n(\mathbb{C})$ of degree $n$, and $\text{Rep}(S_n)$ denotes the category of complex representations of $S_n$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$, define:

$$X_\lambda = \{(S_1, \ldots, S_l) \mid S_1 \sqcup \cdots \sqcup S_l = \{1, \ldots, n\}, |S_i| = \lambda_i\}.$$

The action of $S_n$ on $\{1, \ldots, n\}$ induces an action on $X_\lambda$.

Recall a characterization of irreducible complex representations of $S_n$:

**Theorem 36.** [11, Section 4.4] For every partition $\lambda$ of $n$, there exists a unique representation $V_\lambda$ of $S_n$ which occurs in both $K[X_\lambda]$ and $K[X_\lambda] \otimes \epsilon$. As $\lambda$ runs over all partition of $n$, $V_\lambda$ forms a set of representatives of isomorphism classes of irreducible representations of $S_n$.

**Theorem 37.** For every partition $\lambda$ of $n$, $R_n(W_\lambda) \cong V_\lambda$.

**Proof.** Every submultiset of $\{1, \ldots, n\}$ is of the form $I = \{1^{m_1}, \ldots, n^{m_n}\}$, where $m_i \geq 0$ denotes the multiplicity of $i$ in $I$. The size of $I$ is, by definition, the sum $m = m_1 + \cdots + m_n$. Then $e_I = \prod_{i=1}^n e_i^{m_i}$ may be regarded as a tensor in $\text{Sym}^m \mathbb{C}^n$. Let $J(n, m)$
denote the set of all submultisets of \( n \) of size \( m \). For \( S = (S_1, \ldots, S_l) \in \prod_{k=1}^l J(n, \lambda_k) \), define
\[
e_S = e_{S_1} \otimes \cdots \otimes e_{S_l} \in \text{Sym}^\lambda \mathbb{C}^n.
\]
Then as \( S \) runs over elements of \( \prod_{i=1}^l J(n, \lambda_k) \), \( e_S \) forms a basis of \( \text{Sym}^\lambda \mathbb{C}^n \). The basis vector \( v_S \) has weight \( (\mu_1, \ldots, \mu_n) \), where \( \mu_i \) is the sum of the multiplicities of \( i \) in \( S_1, \ldots, S_l \). Thus \( v_S \) has weight \( (1, \ldots, 1) \) if and only if the submultisets \( S_1, \ldots, S_l \) are actually subsets (each element has multiplicity 0 or 1), and moreover, form a partition of \( \{1, \ldots, n\} \). In other words, \( R_n(\text{Sym}^\lambda \mathbb{C}^n) \) is spanned by \( \{v_S \mid S \in \lambda\} \). It follows that \( R_n(\text{Sym}^\lambda \mathbb{C}^n) \cong K[X_\lambda] \). Similarly, \( R_n(\wedge^\lambda \mathbb{C}^n) \approx K[X_\lambda] \otimes \epsilon \). Since \( W_\lambda \) occurs in \( \text{Sym}^\lambda \mathbb{C}^n \) and also in \( \wedge^\lambda \mathbb{C}^n \), it follows that \( R_n(W_\lambda) \) occurs in \( K[X_\lambda] \) and \( K[X_\lambda] \otimes \epsilon \). Therefore, by Theorem \( \text{30} \), \( R_n(W_\lambda) \) would have to be isomorphic to \( V_\lambda \), unless \( R_n(W_\lambda) = 0 \). But \( \dim(R_n(W_\lambda)) \) is the coefficient of \( x_1 \cdots x_n \) in \( s_\lambda(x_1, \ldots, x_n) \), which is the number of tableaux of shape \( \lambda \) and weight \( (1, \ldots, 1) \) (the standard tableaux), which is always positive. \( \square \)

**Theorem 38 (Schur-Weyl duality).** Let \( S_m \) act on \( \otimes^m \mathbb{C}^n \) by permuting the tensor factors, and \( \text{GL}_n(\mathbb{C}) \) act on each tensor factor. Then, as a representation of \( S_m \times \text{GL}_n(\mathbb{C}) \),
\[
\otimes^m \mathbb{C}^n \cong \bigoplus_{\lambda \in \Lambda^n} V_\lambda \otimes W_\lambda.
\]

**Proof.** Apply the functor \( R_m \otimes \text{id} \) to \( (\text{GL}_n(\mathbb{C}), \text{GL}_n(\mathbb{C})) \) duality (Theorem \( \text{30} \)). The \((1^m)\)-weight space of \( \text{Sym}^m(\mathbb{C}^m \otimes \mathbb{C}^n) \) is spanned by vectors of the form \( v_A \), where each of the \( m \) rows of \( A \) sums to 1. If the 1 in the \( i \)th row occurs in the \( j \)th column, then map \( v_A \) to the vector \( e_{j_1} \otimes \cdots \otimes e_{j_m} \in \otimes^m \mathbb{C}^n \). This induces an isomorphism \( (R_m \otimes \text{id})(\mathbb{C}^m \otimes \mathbb{C}^n) \rightarrow \otimes^m \mathbb{C}^n \) of \( S_m \times \text{GL}_n(\mathbb{C}) \)-representations. \( \square \)

**Remark 39.** Schur-Weyl duality was used by Schur \( \text{12} \) to give a second proof of the classification of irreducible polynomial representations of \( \text{GL}_n(\mathbb{C}) \). It was popularized by Hermann Weyl \( \text{16} \) and is known as Schur-Weyl duality. See also \( \text{6} \) and \( \text{11} \) Section 6.4.

**Theorem 40 (Gelfand models for \( \text{Sn} \)).** Let \( M_{n,k} \) denote the set of all elements \( w \in \text{Sn} \) such that \( w = w^{-1} \) and \( w \) has \( k \) fixed points. Define a representation of \( \text{Sn} \) on \( \mathbb{C}[M_{n,k}] \) (the space of complex-valued functions on \( M_{n,k} \)) by:
\[
\rho_1(g)f(w) = (-1)^{i_1(g,w)}f(g^{-1}wg)
\]
where $i_1(g, w)$ is the number of indices $i < j$ such that $w(i) = j$ and $g(i) > g(j)$. Then

$$C[M_{n,k}] = \bigoplus_{\{\lambda-n\} \text{ with } k \text{ odd columns}} V_\lambda.$$  

In particular, if $M_n = \bigsqcup M_{n,k}$ is the set of all elements $w \in S_n$ such that $w^2 = \text{id}$, then $C[M_n]$ is a Gelfand model for $S_n$.

**Proof.** Apply the functor $R_n$ to both sides of the first identity in Theorem 26. □

**Remark 41.** The above Gelfand model for symmetric groups is well-known. For an alternative approach see [8, 10]. The decomposition in Theorem 30 also gives a Gelfand model for $S_n$; the model obtained is the twist of the model in Theorem 40 by the sign character. Note that the twist of $V_\lambda$ by the sign character is $V_\lambda'$.

**Remark 42 (Relation to induced representations).** The representation $\rho_1$ of $S_n$ on $C[M_{n,k}]$ in Theorem 10 can be viewed as an induced representation. For each positive integer $l$, let $B_{2l} \subset S_{2l}$ denote the centralizer of the involution $(1,2)(3,4) \cdots (2l-1,2l) \in S_{2l}$. Let $e_{2l}$ denote the restriction of the sign character of $S_{2l}$ to $B_{2l}$. If $n = 2l + k$, then $B_{2l} \times S_k$ is a subgroup of $S_n$. We have $\rho_1 = \text{Ind}_{B_{2l} \times S_k}^{S_n} e_{2l} \otimes 1_{S_k}$, where $1_{S_k}$ denotes the trivial character of $S_k$. Taking $k = 0$ gives:

$$\text{Ind}_{B_{2l}}^{S_{2l}} e_{2l} = \bigoplus_{\{\lambda-2l\} \text{ every part appearing even number of times}} V_\lambda.$$  

Similarly, applying the functor $R_{2l}$ to both sides of Theorem 31 gives:

$$\text{Ind}_{B_{2l}}^{S_{2l}} 1_{B_{2l}} = \bigoplus_{\{\lambda-2l\} \text{ with all parts even}} V_\lambda.$$  

Applying the functor $R_n$ to the first identity in Theorem 55 gives an interesting multiplicity-free representation of $S_n$:

**Theorem 43.** Let $n$ be an even integer, say $n = 2m$. Given pairs $(i, j)$ and $(k, l)$, let $(i', j')$ and $(k', l')$ denote their sorted versions, so that $i' < j'$ and $k' < l'$. Write $(i, j) < (k, l)$ if $(i', j') < (k', l')$ in lexicographic order. Define a representation of $S_n$ on $C[M_{n,0}]$ by:

$$\rho_2(g)f(w) = (-1)^{i_2(g,w)} f(g^{-1}wg),$$

where $i_2(g, w) = i_1(g, w) + \# \{(i, j) < (k, l) \mid i < j, k < l, w(i) = j, w(k) = l, (g(i), g(j)) > (g(k), g(l))\}.$
Then
\[ C[M_{n,0}] = \bigoplus_{\lambda \in TP(n)} V_{\lambda}. \]

**Remark 44** (Relation to induced representations). The representation \( \rho_2 \) of \( S_{2m} \) on \( M_{2m,0} \) in Theorem 43 can be viewed as an induced representation. The group \( B_{2m} \) is a semidirect product \((\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m\). The \( \eta_m \) be the character of \( B_{2m} \) whose restriction to \( S_m \) is the sign character of \( S_m \) and whose restriction to \((\mathbb{Z}/2\mathbb{Z})^m\) is the product of the non-trivial characters of each of the \( m \) factors. Then Theorem 43 can be restated as:
\[ \text{Ind}^{S_{2m}}_{B_{2m}} \eta_m = \bigoplus_{\lambda \in TP(n)} V_{\lambda}. \]

For a different approach to this result see [3], Exercise 45.6]

When \( R_n \) is applied to the second identity in Theorem 35, the representations obtained are twists by the sign character of the representations in Theorem 43.

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