Wedge disclination in the field theory of elastoplasticity

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Abstract
In this paper we study the wedge disclination within the elastoplastic defect theory. Using the stress function method we found exact analytical solutions for all characteristic fields of a straight wedge disclination in a cylinder. The elastic stress, elastic strain, elastic bend-twist, displacement and rotation have no singularities at the disclination line. We found a modified stress function for the wedge disclination.

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Disclinations are very important lattice defects analogous to dislocations. A disclination is characterized by a closure failure of the rotation for a closed circuit round the disclination line. There are wedge and twist disclinations. It seems that very strong plastic distortions are necessary in order to create disclinations in crystals. If the Frank angle (rotation failure) of the disclination is a symmetry angle of lattice, then the disclination is called a perfect disclination. Such disclinations have been introduced by Anthony [1] and deWit [2, 3]. A wedge disclination (rotation axis parallel to the disclination line) may be generated by adding or removing a wedge-shaped piece of material and re-welding the material. The smallest value of the Frank vector is $\pi/2$ in a cubic lattice and $\pi/3$ in a hexagonal lattice. If the Frank angle is not a symmetry angle of lattice, the disclination is called partial disclination. They play an important role, e.g., in building of twin boundaries (see, e.g., [4]). Disclinations correspond, in general, to Volterra’s distortions of the second kind (see also [5]). Thus, these defects are of rotational type. They are different from the so-called Frank’s (spin) disclinations which are (elementary) defects in liquid crystals.

In recent articles [6–9] we have studied screw and edge dislocations in the framework of field theory of elastoplasticity. This field theory of elastoplasticity can be considered as a gauge theory of defects where the defects cause plasticity. The corresponding gauge fields may be identified with the plastic distortion. By the help of this theory the elastic and plastic part of the total distortion can be calculated. The total distortion is defined in terms of a displacement and consists of the elastic and plastic part. In the case of dislocations (see, e.g., [9]) the elastic distortion is continuous even in the dislocation core and the plastic part becomes discontinuous. The field equation of the elastic stress in an isotropic medium is an inhomogeneous Helmholtz equation (see, e.g., [8])

$$
\left(1 - \kappa^{-2}\Delta\right) \sigma_{ij} = \sigma_{ij}^{0}, \quad \kappa^2 = \frac{2\mu}{a_1}
$$

(1)
Here $\sigma_{ij}$ is the classical stress tensor and $\mu$ is the shear modulus. The coefficient $a_1$ has the dimension of a force and $\kappa$ has the dimension of a reciprocal length. It is important to note that Eq. (1) agrees with the field equation for the elastic stress in Eringen’s nonlocal elasticity [10–12] and in gradient elasticity [13–15] where the factor $\kappa$ is called non-locality parameter or gradient coefficient. Using the inverse of Hooke’s law for $\sigma_{ij}$ and $\sigma_{ij}$, Eq. (1) implies an inhomogeneous Helmholtz equation for the elastic strain

$$\left(1 - \kappa^{-2} \Delta\right) E_{ij} = \tilde{E}_{ij}, \quad (2)$$

where $\tilde{E}_{ij}$ is the classical strain tensor. It is worth noting that Eq. (2) is analogous to an equation for the elastic strain in gradient theory used by Gutkin and Aifantis [13–15]. It is, therefore, quite reasonable to use the field theory of elastoplasticity for disclinations.

Here we consider a straight wedge disclination inside an infinitely long cylinder with outer radius $R$. Disclinations are defined by the disclination line and the Frank vector. The $z$-axis is along the disclination line and coincides with the axis of the cylinder. For a wedge disclination the Frank vector is parallel to the disclination line: $\Omega_x = \Omega_y = 0$ and $\Omega_z = \Omega$.

In absence of body forces, the force equilibrium condition can be identically satisfied by using the so-called stress function ansatz [16]. If we specialize to the plane problem, the stress function $f$ is related to the stress tensor

$$\sigma_{ij} = \begin{pmatrix} \partial_{yy}^2 f & -\partial_{xy}^2 f & 0 \\ -\partial_{xy}^2 f & \partial_{xx}^2 f & 0 \\ 0 & 0 & \nu \Delta f \end{pmatrix}. \quad (3)$$

Here $\Delta$ denotes the two-dimensional Laplacian $\partial_{xx}^2 + \partial_{yy}^2$ and $\nu$ is Poisson’s ratio. In addition, the strain is given in terms of the stress function as

$$E_{ij} = \frac{1}{2\mu} \begin{pmatrix} \partial_{yy}^2 f - \nu \Delta f & -\partial_{xy}^2 f & 0 \\ -\partial_{xy}^2 f & \partial_{xx}^2 f - \nu \Delta f & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

We use the classical stress field of a straight wedge disclination in terms of the Airy stress function

$$\sigma_{ij} = \begin{pmatrix} \partial_{yy}^2 \chi & -\partial_{xy}^2 \chi & 0 \\ -\partial_{xy}^2 \chi & \partial_{xx}^2 \chi & 0 \\ 0 & 0 & \nu \Delta \chi \end{pmatrix}. \quad (5)$$

The stress function of a “classical” wedge disclinations is given by

$$\chi = A r^2 \left\{ \ln r - \frac{1 - 4\nu}{2(1 - 2\nu)} - C \right\}, \quad A = \frac{\mu \Omega}{4\pi(1 - \nu)}, \quad (6)$$

where $r^2 = x^2 + y^2$. That stress function is chosen so that in the case of $C = 0$ it reproduces the stress of a wedge disclination given by deWit [3]. It fulfils the following inhomogeneous bipotential (or biharmonic) equation

$$\Delta \Delta \chi = 8\pi A \delta(r). \quad (7)$$

More precisely, the stress function $\chi = \frac{1}{\pi} r^2 \ln r$ is Green’s function of the two-dimensional bipotential equation. Substituting (3), (5) and (6) into (1) we get

$$\left(\Delta - \kappa^2\right) f = -\kappa^2 A r^2 \left( \ln r - \frac{1 - 4\nu}{2(1 - 2\nu)} - C \right). \quad (8)$$
Now we use the ansatz

\[ f = Ar^2 \left( \ln r - \frac{1 - 4\nu}{2(1 - 2\nu)} - C \right) + f_{(1)} \]  \hspace{1cm} (9)

and obtain

\( (\Delta - \kappa^2)f_{(1)} = -4A \left( \ln r + \frac{1}{2(1 - 2\nu)} - C \right). \)  \hspace{1cm} (10)

Its solution reads

\[ f_{(1)} = \frac{4A}{\kappa^2} \left( \ln r + \frac{1}{2(1 - 2\nu)} + K_0(\kappa r) - C \right). \]  \hspace{1cm} (11)

Finally, we find the solution of (8)

\[ f = \frac{\mu\Omega}{4\pi(1 - \nu)} \left\{ r^2 \left( \ln r - \frac{1 - 4\nu}{2(1 - 2\nu)} - C \right) + \frac{4}{\kappa^2} \left( \ln r + K_0(\kappa r) + \frac{1}{2(1 - 2\nu)} - C \right) \right\}, \]  \hspace{1cm} (12)

where the first piece is the “classical” stress function and \( K_n \) is the modified Bessel function of the second kind and \( n = 0, 1, \ldots \) denotes the order of this function.

Let us obtain the stress field of the wedge disclination in the cylinder with radius \( R \). It is convenient to use cylindrical coordinates

\[ \sigma_{rr} = \frac{1}{r} \partial_r f, \quad \sigma_{\varphi\varphi} = \partial_{\varphi}^2 f, \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\varphi\varphi}). \]  \hspace{1cm} (13)

In this way we find for the non-vanishing components

\[ \sigma_{rr} = \frac{\mu\Omega}{2\pi(1 - \nu)} \left\{ \ln r + \frac{\nu}{1 - 2\nu} + K_0(\kappa r) + \frac{1}{\kappa^2 r^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) - C \right\}, \]

\[ \sigma_{\varphi\varphi} = \frac{\mu\Omega}{2\pi(1 - \nu)} \left\{ \ln r + 1 + \frac{\nu}{1 - 2\nu} + K_0(\kappa r) - \frac{1}{\kappa^2 r^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) - C \right\}, \]

\[ \sigma_{zz} = \frac{\mu\Omega \nu}{\pi(1 - \nu)} \left\{ \ln r + \frac{1}{2(1 - 2\nu)} + K_0(\kappa r) - C \right\}. \]  \hspace{1cm} (14)

It can be seen that the boundary term \( C \) just gives a constant contribution to the stress (14). The trace of the stress tensor \( \sigma_{kk} = \sigma_{rr} + \sigma_{\varphi\varphi} + \sigma_{zz} \) produced by the wedge disclination is

\[ \sigma_{kk} = \frac{\mu\Omega (1 + \nu)}{\pi(1 - \nu)} \left\{ \ln r + \frac{1}{2(1 - 2\nu)} + K_0(\kappa r) - C \right\}. \]  \hspace{1cm} (15)

It is interesting to note that for the case when \( C = 0 \) the solution (14) agrees with the result obtained by Povstenko [12, 17] in the framework of nonlocal elasticity if we use the identification \( \kappa \equiv 1/(\tau l) \). He used the two-dimensional nonlocal kernel which is Green’s function of the two-dimensional Helmholtz equation. The constant \( C \) is determined by the “semi-classical” boundary condition on the surface \( r = R \) of the cylinder

\[ \sigma_{rr}(R) = 0, \] \hspace{1cm} (16)

which means the absence of external forces on the boundary of the cylinder. So we find for the constant

\[ C = \ln R + \frac{\nu}{1 - 2\nu} + K_0(\kappa R) + \frac{1}{\kappa^2 R^2} \left( 2 - \kappa^2 R^2 K_2(\kappa R) \right). \]  \hspace{1cm} (17)
In the limit $\kappa^{-1} \to 0$, the stress function (12) with (17) agrees with the stress function given by [18]. The constant (17) diverges in the limit $R \to \infty$. Therefore, we consider a cylinder of finite radius. Note that the cylinder size $R$ is the characteristic screening parameter which appears under the logarithm. If we use the limiting expressions for $r \to 0$,

$$K_0(\kappa R) \to -\left[\gamma + \ln \frac{\kappa R}{2}\right], \quad K_2(\kappa R) \to -\frac{1}{2} + \frac{2}{(\kappa R)^2},$$

(18)

where $\gamma$ denotes the Euler constant, we obtain

$$\sigma_{rr}(0) = \sigma_{\varphi\varphi}(0) = \frac{1}{2 \nu} \sigma_{zz}(0) = \frac{1}{2(1 + \nu)} \sigma_{kk}(0) = -\frac{\mu \Omega}{2\pi(1 - \nu)} \left\{ \ln \frac{\kappa R}{2} + \gamma - \frac{1}{2} + K_0(\kappa R) + \frac{1}{\kappa^2 R^2} \left(2 - \kappa^2 R^2 K_2(\kappa R)\right) \right\}.$$  

(19)

Consequently, the stress is finite at the disclination line in contrast to the unphysical stress in “classical” disclination theory. The graphs of the components of the stress calculated from (14) are plotted over $\kappa R$ in Fig. 1 (with the radius of the cylinder $R = 10/\kappa$). With $R = 10/\kappa$ we obtain for (19) the value

$$\sigma_{rr}(0) = \sigma_{\varphi\varphi}(0) = \frac{1}{2 \nu} \sigma_{zz}(0) = \frac{1}{2(1 + \nu)} \sigma_{kk}(0) \simeq -1.707 \frac{\mu \Omega}{2\pi(1 - \nu)}. \tag{20}$$

At the outer boundary $R$ the non-vanishing components of the stress tensor (14) are

$$\sigma_{\varphi\varphi}(R) = \frac{1}{\nu} \sigma_{zz}(R) = \frac{1}{2(1 + \nu)} \sigma_{kk}(R) = \frac{\mu \Omega}{2\pi(1 - \nu)} \left\{ 1 - \frac{4}{\kappa^2 R^2} + 2 K_2(\kappa R) \right\}. \tag{21}$$

With $R = 10/\kappa$ we obtain for (21) the value (see Fig. 1)

$$\sigma_{\varphi\varphi}(R) = \frac{1}{\nu} \sigma_{zz}(R) = \frac{1}{2(1 + \nu)} \sigma_{kk}(R) \simeq 0.960 \frac{\mu \Omega}{2\pi(1 - \nu)}. \tag{22}$$

By means of Eqs. (3) and (12) we obtain the modified stress of the wedge disclination in the cylinder with radius $R$ in Cartesian coordinates

$$\begin{align*}
\sigma_{xx} &= \frac{\mu \Omega}{2\pi(1 - \nu)} \left\{ \ln r + \frac{y^2}{r^2} + \frac{\nu}{1 - 2\nu} + K_0(\kappa R) + \frac{(x^2 - y^2)}{\kappa^2 r^4} \left(2 - \kappa^2 r^2 K_2(\kappa R)\right) - C \right\}, \\
\sigma_{yy} &= \frac{\mu \Omega}{2\pi(1 - \nu)} \left\{ \ln r + \frac{x^2}{r^2} + \frac{\nu}{1 - 2\nu} + K_0(\kappa R) - \frac{(x^2 - y^2)}{\kappa^2 r^4} \left(2 - \kappa^2 r^2 K_2(\kappa R)\right) - C \right\}, \\
\sigma_{xy} &= -\frac{\mu \Omega}{2\pi(1 - \nu)} \frac{xy}{r^2} \left\{ 1 - \frac{2}{\kappa^2 r^2} \left(2 - \kappa^2 r^2 K_2(\kappa R)\right) \right\}, \\
\sigma_{zz} &= \frac{\mu \Omega \nu}{\pi(1 - \nu)} \left\{ \ln r + \frac{1}{2(1 - 2\nu)} + K_0(\kappa R) - C \right\}. \tag{23}
\end{align*}$$

If we identify $\kappa \equiv 1/\sqrt{c}$ ($c$ is the gradient coefficient used by Gutkin and Aifantis) and put $C = 0$, the elastic stress is in agreement with the stress field given by Gutkin and Aifantis [14,15] in the framework of strain gradient elasticity by using the Fourier transform method.

Using (4) and (12) we find for the elastic strain of the straight wedge disclination

$$\begin{align*}
E_{xx} &= \frac{\Omega}{4\pi(1 - \nu)} \left\{ (1 - 2\nu) \left( \ln r + K_0(\kappa R) - C \right) + \frac{y^2}{r^2} + \frac{(x^2 - y^2)}{\kappa^2 r^4} \left(2 - \kappa^2 r^2 K_2(\kappa R)\right) \right\}, \\
E_{yy} &= \frac{\Omega}{4\pi(1 - \nu)} \left\{ (1 - 2\nu) \left( \ln r + K_0(\kappa R) - C \right) + \frac{x^2}{r^2} - \frac{(x^2 - y^2)}{\kappa^2 r^4} \left(2 - \kappa^2 r^2 K_2(\kappa R)\right) \right\}, \\
E_{xy} &= -\frac{\Omega}{4\pi(1 - \nu)} \frac{xy}{r^2} \left\{ 1 - \frac{2}{\kappa^2 r^2} \left(2 - \kappa^2 r^2 K_2(\kappa R)\right) \right\}. \tag{24}
\end{align*}$$
Figure 1: The stress components for a wedge disclination in a cylinder with radius $R = 10/\kappa$: (a) $\sigma_{rr}$, (b) $\sigma_{\phi\phi}$, are given in units of $\mu \Omega / [2\pi(1 - \nu)]$ and (c) $\sigma_{zz}$ is given in units of $\mu \Omega \nu / [\pi(1 - \nu)]$. The dashed curves represent the classical stress components.
In the case of $C = 0$ the strain (24) coincides with the result given by Gutkin and Aifantis [14, 15, 19]. The strain reads in cylindrical coordinates

$$E_{rr} = \frac{\Omega}{4\pi(1 - \nu)} \left\{ (1 - 2\nu) \left( \ln r + K_0(\kappa r) - C \right) + \frac{1}{\kappa^2 r^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right\},$$

$$E_{\varphi\varphi} = \frac{\Omega}{4\pi(1 - \nu)} \left\{ (1 - 2\nu) \left( \ln r + K_0(\kappa r) - C \right) + 1 - \frac{1}{\kappa^2 r^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right\}. \quad (25)$$

The dilatation reads

$$E_{kk} = \frac{\Omega}{2\pi(1 - \nu)} \left\{ (1 - 2\nu) \left( \ln r + K_0(\kappa r) - C \right) + \frac{1}{2} \right\}. \quad (26)$$

By the help of (18) the expressions of the strain (25) and the dilatation (26) read for the limit $r \to 0$, respectively,

$$E_{rr}(0) = E_{\varphi\varphi}(0) = -\frac{\Omega}{4\pi(1 - \nu)} \left\{ (1 - 2\nu) \left[ \ln \frac{\kappa R}{2} + \gamma - \frac{1}{2} + K_0(\kappa R) \right] \right. \left. + \frac{1}{\kappa^2 R^2} \left( 2 - \kappa^2 R^2 K_2(\kappa R) \right) \right\}, \quad (27)$$

and

$$E_{kk}(0) = -\frac{\Omega(1 - 2\nu)}{2\pi(1 - \nu)} \left\{ \ln \frac{\kappa R}{2} + \gamma - \frac{1}{2} + K_0(\kappa R) + \frac{1}{\kappa^2 R^2} \left( 2 - \kappa^2 R^2 K_2(\kappa R) \right) \right\}. \quad (28)$$

The strain and the dilatation at the cylindrical surface $r = R$ are given by

$$E_{\varphi\varphi}(R) = -\frac{1 - \nu}{\nu} E_{rr}(R) = \frac{1 - \nu}{(1 - 2\nu)} E_{kk}(R) = \frac{\Omega}{4\pi} \left\{ 1 - \frac{4}{\kappa^2 R^2} + 2K_2(\kappa R) \right\}. \quad (29)$$

In the conventional disclination theory the torsion tensor (linear version of Cartan’s torsion) is defined by

$$\alpha_{ij} = \epsilon_{jkl} \left( \partial_k \beta_{il} + \epsilon_{ilm} \varphi^*_m \right). \quad (30)$$

It is the dislocation density in the theory of disclinations (see, e.g., [2, 3, 20]). On the other hand, Anthony [1] called it the disclination torsion. $\beta_{ij}$ is the elastic distortion. The $\varphi^*_m$ was introduced by Mura [20] who called it “plastic rotation” and deWit [2, 3, 21] called this quantity “disclination loop density”. Using the elastic bend-twist tensor (see, e.g., [22])

$$k_{ij} = \partial_j \omega_i - \varphi^*_{ij}, \quad (31)$$

with the rotation vector

$$\omega_i = -\frac{1}{2} \epsilon_{ijk} \beta_{jk}, \quad (32)$$

Eq. (30) can be rewritten according to (see also [1, 21, 22])

$$\alpha_{ij} = \epsilon_{jkl} \left( \partial_k E_{il} + \epsilon_{ilm} k_{mk} \right) = \epsilon_{jkl} \partial_k E_{il} + \delta_{ij} k_{il} - k_{ji}. \quad (33)$$

Consequently, the elastic bend-twist may be determined from the condition that the dislocation density (disclination torsion) has to be zero for a straight wedge disclination

$$\alpha_{xz} = -\frac{1}{2\mu} \partial_y \Delta f - k_{xz} \equiv 0, \quad \alpha_{yz} = \frac{1}{2\mu} \partial_x \Delta f - k_{zy} \equiv 0. \quad (34)$$
Figure 2: Elastic bend-twist $k_{z\phi}$ of a wedge disclination (solid) is given in units of $\Omega/(2\pi)$. The dashed curve represents the classical solution.

So we find for the elastic bend-twist

$$
k_{zx} = -\frac{\Omega}{2\pi} \frac{y}{r^2} \left\{ 1 - \kappa r K_1(\kappa r) \right\}, \quad k_{zy} = \frac{\Omega}{2\pi} \frac{x}{r^2} \left\{ 1 - \kappa r K_1(\kappa r) \right\}, \quad (35)
$$

and in cylindrical coordinates

$$
k_{z\phi} = \frac{\Omega}{2\pi} \frac{1}{r} \left\{ 1 - \kappa r K_1(\kappa r) \right\}, \quad k_{zr} = 0. \quad (36)
$$

Obviously, the components of the elastic bend-twist (35) and (36) fulfil an inhomogeneous Helmholtz equation

$$
\left( 1 - \kappa^{-2} \Delta \right) k_{ij} = \tilde{k}_{ij}, \quad (37)
$$

where $\tilde{k}_{ij}$ denotes the classical elastic bend-twist. The elastic bend-twist $k_{z\phi}$ is plotted in Fig. 2. It has no singularity at the disclination line. It is $k_{z\phi}(r = 0) = 0$. The elastic bend-twist has a maximum of $k_{z\phi}^{\text{max}} \approx 0.399\kappa\Omega/(2\pi)$ at $r \approx 1.1/\kappa$. The boundary condition $k_{zr}(R) = 0$ is trivially satisfied.

The elastic bend-twist tensor can be decomposed according to (31) into a gradient of the rotation vector and an incompatible part. We identify the incompatible part with the disclination loop density. It is analogous to the decomposition of the elastic distortion of a dislocation into a gradient of the displacement vector and an incompatible distortion (see [6–8]). The disclination loop density turns out to be

$$
\varphi_{xx}^* = \frac{\Omega}{2\pi} \kappa^2 x K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right), \\
\varphi_{xy}^* = \frac{\Omega}{2\pi} \left\{ \kappa^2 y K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right) + \pi \delta(y) \left( 1 - \text{sign}(x) \right) \left[ 1 - \kappa r K_1(\kappa r) \right] \right\}. \quad (38)
$$

It contains the angle $\varphi$ and its form is analogous to the plastic distortion of a screw dislocation (see [8]). Here we use a single-valued discontinuous form of $\varphi$ (see [3]). Only the component $\varphi_{xy}^*$ has a $\delta$-singularity at $y = 0$ like the disclination loop density [3, 20] $\varphi_{yz}^* = (\Omega/2) \delta(y)(1 - \text{sign}(x))$. The rotation vector reads

$$
\omega_z = \frac{\Omega}{2\pi} \left\{ \varphi \left( 1 - \kappa r K_1(\kappa r) \right) + \frac{\pi}{2} \text{sign}(y) \kappa r K_1(\kappa r) \right\}. \quad (39)
$$
Figure 3: Rotation vector of a wedge disclination $\omega_z(x, y \to +0)$ (solid). The dashed curve represents the classical solution.

The far field of the rotation vector (39) agrees with deWit’s expression given in [3]. When $y \to +0$, the expression (39) is plotted in Fig. 3. One can see that the Bessel function terms which appear in (39) lead to the symmetric smoothing of the rotation vector profile, in contrast to the abrupt jump occurring in the classical solution. It is interesting to note that the size of such a transition zone is approximately $12/\kappa$ which gives the value $6/\kappa$ for the radius of the disclination core. Replacing $\Omega$ by $b$ in Eq. (39) it can be seen that the rotation of the wedge disclination $\omega_z$ has the same form as the displacement of a screw dislocation $u_z$ given in [8]. Consequently, the elastic rotation (antisymmetric part of the elastic distortion) of a wedge disclination is discontinuous due to $\varphi$ in contrast to the dislocation case where the corresponding expressions are continuous (see also [3]).

Finally, we find for the elastic distortion

\begin{align*}
\beta_{xx} &= \frac{\Omega}{4\pi(1-\nu)} \left\{ (1 - 2\nu) \left( \ln r + K_0(\kappa r) - C \right) + \frac{y^2}{r^2} + \frac{(x^2 - y^2)}{\kappa^2 r^4} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right\}, \\
\beta_{xy} &= -\frac{\Omega}{4\pi(1-\nu)} \frac{xy}{r^2} \left\{ 1 - \frac{2}{\kappa^2 r^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right\} - \frac{\Omega}{2\pi} \left\{ \varphi(1 - \kappa r K_1(\kappa r)) + \frac{\pi}{2} \text{sign}(y) \kappa r K_1(\kappa r) \right\}, \\
\beta_{yx} &= -\frac{\Omega}{4\pi(1-\nu)} \frac{xy}{r^2} \left\{ 1 - \frac{2}{\kappa^2 r^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right\} + \frac{\Omega}{2\pi} \left\{ \varphi(1 - \kappa r K_1(\kappa r)) + \frac{\pi}{2} \text{sign}(y) \kappa r K_1(\kappa r) \right\}, \\
\beta_{yy} &= \frac{\Omega}{4\pi(1-\nu)} \left\{ (1 - 2\nu) \left( \ln r + K_0(\kappa r) - C \right) + \frac{x^2}{r^2} - \frac{(x^2 - y^2)}{\kappa^2 r^4} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right\}. \tag{40}
\end{align*}

We note that the elastic distortion (40) contains the angle $\varphi$ in contrast to the dislocation case. The elastic distortion satisfies the following relations that the Burgers vector vanishes for a wedge disclination

\begin{align*}
\oint_{\gamma} (\beta_{xx} dx + \beta_{xy} dy) &= 0, \\
\oint_{\gamma} (\beta_{yx} dx + \beta_{yy} dy) &= 0. \tag{41}
\end{align*}

Here $\gamma$ denotes the Burgers circuit.
We obtain for the effective Frank vector of the wedge disclination
\[
\Omega(r) = \oint \left( k_{zz} \, dx + k_{zy} \, dy \right) = \Omega \left\{ 1 - \kappa r K_1(\kappa r) \right\}.
\]  

(42)

This expression is in agreement with the one obtained in [23]. The form of the effective Frank vector is similar to the effective Burgers vector of screw and edge dislocations (see, e.g., [6–9]). It differs appreciably from the constant value \(\Omega\) in the region from \(r = 0\) up to \(r \simeq 6/\kappa\). In fact, we find \(\Omega(0) = 0\) and \(\Omega(\infty) = \Omega\). Thus, it is suggestive to take \(r_c \simeq 6/\kappa\) as the core radius of the disclination.

The so-called disclination density tensor of a discrete disclination is defined by [1–3, 20–22]
\[
\Theta_{ij} = \epsilon_{jmn} \partial_m \varphi_{in} = -\epsilon_{jmn} \partial_m \varphi_{in}^*. 
\]  

(43)

Here the index \(i\) indicates the direction of the Frank vector, \(j\) the disclination line direction. Thus, the diagonal components of \(\Theta_{ij}\) represent wedge disclinations, the off-diagonal components twist disclinations. The disclination density tensor satisfies the continuity condition \(\partial_j \Theta_{ij} = 0\) which implies that disclinations do not end inside the body. From (35) and (38) we find for the non-vanishing component of the disclination density of a wedge disclination
\[
\Theta_{zz} = \frac{\Omega \kappa^2}{2\pi} K_0(\kappa r).
\]  

(44)

In the limit as \(\kappa^{-1} \to 0\), the result (44) converts to the classical expression \(\Theta_{zz} = \Omega \delta(r)\). It is interesting to note that (44) coincides with the expression calculated in [23]. Additionally, the disclination density (44) agrees with Eringen’s two-dimensional nonlocal kernel used in [10–12].

We use the decomposition of the elastic distortion (40) into a compatible and an incompatible distortion
\[
\beta_{ij} = \partial_j u_i + \tilde{\beta}_{ij}.
\]  

(45)

In this way we restore a modified displacement vector (see also [8])
\[
u_x = -\frac{\Omega}{2\pi} \left\{ y \left[ \varphi (1 - \kappa r K_1(\kappa r)) + \frac{\pi}{2} \text{sign}(y) \kappa r K_1(\kappa r) \right] - \frac{x}{2(1 - \nu)} \left[ (1 - 2\nu) \left( \ln r - 1 + K_0(\kappa r) - C \right) - \frac{1}{\kappa^2 \nu^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right] \right\},
\]
\[
u_y = \frac{\Omega}{2\pi} \left\{ x \left[ \varphi (1 - \kappa r K_1(\kappa r)) + \frac{\pi}{2} \text{sign}(y) \kappa r K_1(\kappa r) \right] + \frac{y}{2(1 - \nu)} \left[ (1 - 2\nu) \left( \ln r - 1 + K_0(\kappa r) - C \right) - \frac{1}{\kappa^2 \nu^2} \left( 2 - \kappa^2 r^2 K_2(\kappa r) \right) \right] \right\},
\]  

(46)

and the incompatible part
\[
\tilde{\beta}_{xx} = \frac{\Omega}{2\pi} \left\{ \kappa r K_1(\kappa r) + \kappa^2 xy K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right) \right\},
\]
\[
\tilde{\beta}_{xy} = 0,
\]
\[
\tilde{\beta}_{yx} = \frac{\Omega}{2\pi} \kappa^2 y^2 K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right),
\]
\[
\tilde{\beta}_{yy} = -\frac{\Omega}{2\pi} \kappa^2 x^2 K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right) - \pi \delta(y) \varphi \left( 1 - \text{sign}(x) \right) \left( 1 - \kappa r K_1(\kappa r) \right),
\]  

(47)

Eq. (47) fulfills \(\alpha_{ij} = \epsilon_{jkl} (\partial_k \tilde{\beta}_{il} + \epsilon_{ilm} \varphi_{mk}) = 0\) and Eqs. (46) and (47) satisfy \(\omega = \partial_j \nu_x + \tilde{\beta}_{jx}\). The \(\delta\)-term in (47) has a similar form like the plastic strain or dislocation loop density of a wedge disclination [3, 20, 21] \(E_{yy}^p = \tilde{\beta}_{yy} = -\Omega(2) \delta(y)x(1 - \text{sign}(x))\).
Using the gauge theory of defect [24], which is an ISO(3)-gauge theory ($ISO(3) = T(3) \otimes SO(3)$), we are able to decompose the incompatible distortion (47). Namely, the incompatible distortion takes the form [24, 25]

\[
\tilde{\beta}_{ij} = \phi_{ij} + \epsilon_{ikl}W_{kj}x_l,
\]

where $\phi_{ij}$ and $W_{ij}$ are the translational and rotational gauge fields, respectively. We obtain for (47) the following decomposition

\[
\begin{align*}
\tilde{\beta}_{xx} &= \phi_{xx} - yW_{zx}, \\
\tilde{\beta}_{xy} &= -yW_{zy}, \\
\tilde{\beta}_{yx} &= xW_{zx}, \\
\tilde{\beta}_{yy} &= \phi_{yy} + xW_{zy},
\end{align*}
\]

(49)

into the translational gauge field

\[
\phi_{xx} = \phi_{yy} = \frac{\Omega}{2\pi} \kappa r K_1(\kappa r),
\]

(50)

and the rotational gauge field

\[
W_{zx} \equiv -\varphi^*_{xx}, \quad W_{zy} \equiv -\varphi^*_{zy}.
\]

(51)

Therefore, the negative disclination loop density (38) is equivalent to the rotational gauge potential (51).

In conclusion, the field theory of elastoplasticity has been employed on the consideration of a straight wedge disclination. We were able to calculate the elastic and plastic fields. We found that the elastic stress, elastic strain, elastic bend-twist and disclination density are continuous and the displacement, plastic distortion, rotation and the disclination loop density of the wedge disclination are discontinuous fields. Exact analytical solutions for all characteristic field quantities of a wedge disclination in a cylinder have been reported which demonstrate the elimination of “classical” logarithmic singularities at the disclination line. All logarithmic terms are influenced by the “semi-classical” boundary term $C$ (17). In addition, the disclination core appears naturally as a result of the smoothing of the rotation vector profile. For an infinitely extended body ($C = 0$) the elastic stress of a wedge disclination calculated in the field theory of elastoplasticity agrees with the stress calculated within the theory of nonlocal elasticity and strain gradient elasticity. The reason is that in all three theories the fundamental equation for the elastic stress has the form of an inhomogeneous Helmholtz equation (see Eq. (1)). The boundary-value problem of a wedge disclination in a cylinder considered in this paper should be help in studies of mechanical behaviour of nano-objects including nanotubes and nanomembranes and of disclinated nanoparticles of cylindrical shape (nanowires). Finally, we note that one observes an interesting relation between the wedge disclination and the screw dislocation. Namely the rotation (39), elastic bend-twist (36), effective Frank vector (42) and the disclination density (44) of a wedge disclination have the same form as the displacement, elastic distortion, effective Burgers vector and the dislocation density of a screw dislocation given in [7, 8] when the Frank vector is replaced by the Burgers vector.

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