Fredholm modules over categories, Connes periodicity and classes in cyclic cohomology

Mamta Balodi∗† Abhishek Banerjee ‡§

Department of Mathematics, Indian Institute of Science, Bangalore - 560012, India.

Abstract

We replace a ring with a small $\mathcal{C}$-linear category $\mathcal{C}$, seen as a ring with several objects in the sense of Mitchell. We introduce Fredholm modules over this category and construct a Chern character taking values in the cyclic cohomology of $\mathcal{C}$. We show that this categorified Chern character is homotopy invariant and is well-behaved with respect to the periodicity operator in cyclic cohomology. For this, we also obtain a description of cocycles and coboundaries in the cyclic cohomology of $\mathcal{C}$ (and more generally, in the Hopf-cyclic cohomology of a Hopf module category) by means of DG-semicategories equipped with a trace on endomorphism spaces.

MSC(2010) Subject Classification: 18E05, 47A53, 53C99, 58B34

Keywords: Hopf-module categories, Differential graded categories, Fredholm modules, cyclic cohomology

1 Introduction

In his celebrated work [12], Connes extended differential calculus beyond the framework of manifolds to include noncommutative spaces such as that of leaves of a foliation or the orbit space of the action of a group on a manifold. For this, he began by considering Fredholm modules over an algebra $A$ which could in general be noncommutative. When $A$ is commutative, such as the space of smooth functions on a manifold $M$, examples of Fredholm modules over $A$ may be obtained by considering elliptic operators on $M$. More generally, by considering Schatten classes inside the collection of bounded operators on a Hilbert space, Connes studied the notion of $p$-summable Fredholm modules over $A$ in [12]. The Fredholm modules over $A$ lead to Chern characters taking values in the cyclic cohomology of $A$. Moreover, these cohomology classes are related by means of Connes’ periodicity operator.

In this paper, we study Fredholm modules over linear categories, along with their Chern characters taking values in cyclic cohomology. Our idea is to have a counterpart of the algebraic notion of modules over a category, a subject which has been highly developed in the literature (see, for instance, [7], [17], [35], [36], [44], [45]). A small preadditive category is treated as a ring with several objects, following an idea first advanced by Mitchell [39]. We note that there is also a well-developed study of spaces in algebraic geometry over categories (see, for instance, [16], [17], [43]). It is also important to mention here the work of Baez [2] with the category of Hilbert spaces as well as the recent work of Henriques [22], Henriques and Penneys [23] with fusion categories with potential applications to physics.

Let $\mathcal{C}$ be a small linear category. We consider pairs $(\mathcal{H}, \mathcal{F})$, where $\mathcal{H}$ is a functor

$$\mathcal{H} : \mathcal{C} \rightarrow \text{SHilb}_{\mathbb{Z}_2}$$

(1.1)
taking values in \( \mathbb{Z}_2 \)-graded separable Hilbert spaces and \( \mathcal{F} = \{ \mathcal{F}_X : \mathcal{H}(X) \to \mathcal{H}(X) \}_{X \in \text{Ob}(\mathcal{C})} \) is a family of bounded and involutive linear operators each of degree 1. When the elements of \( \mathcal{F} \) satisfy certain commutator conditions with respect to the operators \( \{ \mathcal{H}(f) \}_{f \in \text{Mor}(\mathcal{C})} \), we say that the pair \( (\mathcal{H}, \mathcal{F}) \) is a Fredholm module over the category \( \mathcal{C} \). Following the methods of Connes [12], we construct Chern characters of these Fredholm modules taking values in the cyclic cohomology of \( \mathcal{C} \) and study how they are related by means of the periodicity operator. We hope this is the first step towards a larger program which mixes together the techniques in categorical algebra with those in differential geometry.

The paper consists of two parts. In the first part, we study cyclic cohomology. We work more generally with a small linear category \( \mathcal{D}_H \) whose morphism spaces carry a well-behaved action of a Hopf algebra \( H \). In other words, \( \mathcal{D}_H \) is a small Hopf-module category (or \( H \)-category) in the sense of Cibils and Solotar [10]. We recall that in [13], Connes and Moscovici introduced Hopf-cyclic cohomology as a generalization of Lie algebra cohomology adapted to noncommutative geometry. For an \( H \)-category \( \mathcal{D}_H \), we describe the cocycles and coboundaries that determine its Hopf cyclic cohomology groups by extending Connes’ original construction of cyclic cohomology from [11] and [12] in terms of cycles and closed graded traces on differential graded algebras. An important role in our paper is played by “semicategories,” which are categories that may not contain identity maps. This notion, introduced by Mitchell [10], is precisely what we need in order to categorify non-unital algebras. We work with the Hopf cyclic cohomology groups \( HC^*_{\mathcal{H}}(\mathcal{D}_H, M) \) having coefficients in \( M \), where \( M \) is a stable anti-Yetter Drinfeld module in the sense of [13].

Let \( k \) be a field. After collecting some preliminaries in Section 2, we begin in Section 3 by considering the universal differential graded Hopf module semicategory (or DGH-semicategory) associated to the \( H \)-category \( \mathcal{D}_H \). For a DGH-semicategory \( (\mathcal{S}_H, \partial_H) \) and \( n \geq 0 \), we let an \( n \)-dimensional closed graded \((H,M)\)-trace on \( \mathcal{S}_H \) be a collection of maps

\[
\hat{\mathcal{J}}^H := \{ \hat{\mathcal{J}}^H_X : M \otimes \text{Hom}^2_{\mathcal{S}_H}(X,X) \to k \}_{X \in \text{Ob}(\mathcal{S}_H)} \tag{1.2}
\]

satisfying certain conditions (see Definition 3.6). A cycle over \( \mathcal{D}_H \) consists of a triple \( (\mathcal{S}_H, \partial_H, \hat{\mathcal{J}}^H) \) along with an \( H \)-linear semifunctor \( \rho : \mathcal{D}_H \to \mathcal{S}_H^0 \). In Theorem 3.8, we provide a description of the cocycles \( Z^*_H(\mathcal{D}_H, M) \) in Hopf cyclic cohomology in terms of characters of cycles over \( \mathcal{D}_H \). This result is an \( H \)-linear categorical version of Connes [12] Proposition 1, p. 98]. It also follows from Theorem 3.8 that there is a one-one correspondence between \( Z^*_H(\mathcal{D}_H, M) \) and the collection of \( n \)-dimensional closed graded \((H,M)\)-traces on the universal DGH-semicategory \( \Omega(\mathcal{D}_H) \) associated to \( \mathcal{D}_H \).

In Sections 4 and 5, we provide a description of the space \( B^*_H(\mathcal{D}_H, M) \) of coboundaries. Throughout, we take \( k = \mathbb{C} \). We consider families \( \eta \) of automorphisms \( \eta = \{ \eta(X) \in \text{Aut}_{\mathcal{D}_H}(X) \}_{X \in \text{Ob}(\mathcal{D}_H)} \) such that \( h(\eta(X)) = \epsilon(h)\eta(X) \) for all \( h \in H \) and \( X \in \text{Ob}(\mathcal{D}_H) \). We show that these families form a group, which we denote by \( \mathcal{U}_H(\mathcal{D}_H) \). Further, we show that the inner automorphism of \( \mathcal{D}_H \) induced by conjugating with an element \( \eta \in \mathcal{U}_H(\mathcal{D}_H) \) induces the identity functor on \( HC^*_H(\mathcal{D}_H, M) \). Using this, we obtain in Proposition 5.5 a set of sufficient conditions for the Hopf cyclic cohomology of an \( H \)-category to be zero.

We say that a cycle \( (\mathcal{S}_H, \partial_H, \hat{\mathcal{J}}^H) \) is vanishing if \( \mathcal{S}_H^0 \) is an \( H \)-category and \( \epsilon^*_H \) satisfies the assumptions in Proposition 5.5. We describe the elements of \( B^*_H(\mathcal{D}_H, M) \) in Theorem 5.10 as the characters of vanishing cycles over \( \mathcal{D}_H \). Finally, in Theorem 5.12 we use categorified cycles and vanishing cycles to construct a product in Hopf-cyclic cohomologies

\[
HC^p_H(\mathcal{D}_H, M) \otimes HC^q_H(\mathcal{D}_H', M') \to HC^{p+q}_H(\mathcal{D}_H \otimes \mathcal{D}_H', M \otimes M') \quad p, q \geq 0 \tag{1.3}
\]

where \( \mathcal{D}_H \) and \( \mathcal{D}_H' \) are \( H \)-linear categories and \( M \) and \( M' \) are stable anti-Yetter Drinfeld modules over \( H \) satisfying certain conditions.

In the second part of the paper, we study Fredholm modules and Chern classes. For this, we assume \( H = \mathbb{C} = M \) and consider a small \( \mathbb{C} \)-linear category \( \mathcal{C} \). Let \( p \geq 1 \) be an integer. We will say that a pair \( (\mathcal{H}, \mathcal{F}) \) over \( \mathcal{C} \) as in [11] is a \( p \)-summable Fredholm module if it satisfies

\[
[\mathcal{F}, f] := (\mathcal{F}_Y \circ \mathcal{H}(f) - \mathcal{H}(f) \circ \mathcal{F}_X) \in \mathcal{B}^p(\mathcal{H}(X), \mathcal{H}(Y)) \quad \tag{1.4}
\]
for any morphism \( f : X \rightarrow Y \) in \( C \) (see Definition 6.1). Here, \( B^p(B(\mathcal{H}(X), \mathcal{H}(Y))) \) is the \( p \)-th Schatten class inside the space of bounded linear operators from \( \mathcal{H}(X) \) to \( \mathcal{H}(Y) \). We mention here that in this paper, we will consider only even Fredholm modules. We hope to tackle the case of odd Fredholm modules over linear categories in a future paper 6.

Let \( H^*_c(C) := H C^*_c(C, C) \) denote the cyclic cohomology groups of \( C \). Corresponding to a \( p \)-summable Fredholm module \( (\mathcal{H}, \mathcal{F}) \) and any \( 2m \geq p - 1 \), we construct a DG-semicategory \( (\Omega(\mathcal{H}, \mathcal{F}), D, \Delta) \) along with a closed graded trace \( Tr_s = \{ Tr_s : Hom^m_{\mathcal{H}(\mathcal{H}, \mathcal{F})}(X, X) \rightarrow \mathbb{C} \}_{X \in Ob(C)} \) of dimension \( 2m \). Let \( CN^*_c(C) \) denote the cyclic nerve of \( C \) and \( CN^*_c(C) \) its linear dual. By taking the character of the cycle \( (\Omega(\mathcal{H}, \mathcal{F}), \mathcal{F}, Tr_s) \) over \( C \), we obtain \( \phi^{2m} \in CN^*_c(C) \) which is given by (see Theorem 6.3)

\[
\phi^{2m}(f^0 \otimes f^1 \otimes \cdots \otimes f^{2m}) := Tr_s(\mathcal{H}(f^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^{2m}])
\]

for any \( f^0 \otimes f^1 \otimes \cdots \otimes f^{2m} \in CN_{2m}(C) \). Then, \( \phi^{2m} \) lies in the space \( Z^2_c(C) \) of cocycles for the cyclic cohomology of \( C \). The Chern character \( ch^{2m}(\mathcal{H}, \mathcal{F}) \) of the Fredholm module \( (\mathcal{H}, \mathcal{F}) \) will be of the class of \( \phi^{2m} \) in the cyclic cohomology \( H^*_c(C) \) of \( C \).

We relate the Chern characters by means of the periodicity operator in Section 7. We know that the action of the periodicity operator \( S : H^*_c(C) \rightarrow H^{*-2}_c(C) \) is given by taking the product as in (1.3) with a certain class in the cohomology \( H^*_c(C) \). If \( (\mathcal{H}, \mathcal{F}) \) is a \( p \)-summable Fredholm module over \( C \) and \( 2m \geq p - 1 \), we show in Theorem 7.3 that

\[
S(\phi^{2m}) = -(m + 1)\phi^{2m+2} \quad \text{in} \quad H^{2m+2}_c(C)
\]

Finally, in Section 8, we describe the homotopy invariance of the Chern character. For this, we consider a family \( \{ (\rho_t, F_t) \}_{t \in [0, 1]} \) of \( p \)-summable Fredholm modules

\[
\{ \rho_t : C \rightarrow \text{SHilb}_H \}_{t \in [0, 1]} \quad F_t(X) : \rho_t(X) \rightarrow \rho_t(X)
\]

each having the same underlying Hilbert space and satisfying some conditions. Then, if \( \rho_t \) and \( F_t \) vary in a strongly continuous manner with respect to \( t \in [0, 1] \), we show in Theorem 8.7 that the \((p + 2)\)-dimensional character \( ch^{p+2}(\mathcal{H}, \mathcal{F}) \) is independent of \( t \in [0, 1] \).

**Notations:** Throughout the paper, \( H \) is a Hopf algebra over the field \( k \) of characteristic zero, with comultiplication \( \Delta \), counit \( \varepsilon \) and bijective antipode \( S \). We will use Sweedler’s notation for the coproduct \( \Delta(h) = h_1 \otimes h_2 \) and for a left \( H \)-coaction \( \rho : M \rightarrow H \otimes M \), \( \rho(m) = m_{(-1)} \otimes m_{(0)} \) (with the summation sign suppressed). The small cyclic category of Connes [11] will be denoted by \( \Lambda \). The Hochschild differential will always be denoted by \( b \) and the modified Hochschild differential (with the last face operator missing) will be denoted by \( b' \).

On any cocyclic module \( \mathcal{C} \), we will denote by \( \tau_n \) the unsigned cyclic operator on \( C^n(\mathcal{C}) \) and by \( \lambda_n \) the signed cyclic operator \( (-1)^n \tau_n \) on \( C^n(\mathcal{C}) \). The complex computing cyclic cohomology of \( \mathcal{C} \) will be denoted by \( C^*_c(\mathcal{C}) \). Accordingly, the cyclic cocycles and cyclic coboundaries will be denoted by \( Z^*_c(\mathcal{C}) \) and \( B^*_c(\mathcal{C}) \) respectively.

**Acknowledgements:** We are grateful to Gadadhar Misra for several useful discussions.

### 2 Preliminaries on \( H \)-categories and Hopf-cyclic cohomology

A small Hopf module category may be treated as a “Hopf algebra with several objects.” In this section, we will collect some preliminaries on Hopf module categories and on Hopf cyclic cohomology. We note that the Hopf cyclic cohomology introduced by Connes and Moscovici (13, 14, 15) has been developed extensively by a number of authors (see, for instance, 11, 3, 19, 20, 21, 26, 27, 28, 32, 33, 44).

**Definition 2.1.** (see Cibils and Solotar [10]) Let \( H \) be a Hopf algebra over a field \( k \). A \( k \)-linear category \( \mathcal{D}_H \) is said to be a left \( H \)-module category if
(i) $\text{Hom}_{\mathcal{D}_H}(X,Y)$ is a left $H$-module for all $X,Y \in \text{Ob}(\mathcal{D}_H)$

(ii) $h(\text{id}_X) = \varepsilon(h)\text{id}_X$ for all $X \in \text{Ob}(\mathcal{D}_H)$ and $h \in H$

(iii) the composition map is a morphism of $H$-modules, i.e., $h(gf) = (h_1g)(h_2f)$ for any $h \in H$, $f \in \text{Hom}_{\mathcal{D}_H}(X,Y)$ and $g \in \text{Hom}_{\mathcal{D}_H}(Y,Z)$.

A small left $H$-module category will be called a left $H$-category. We will denote by $\text{Cat}_H$ the category of all left $H$-categories with $H$-linear functors between them.

For more on Hopf-module categories, we refer the reader, for instance, to [4], [5], [24], [29]. Let $H$ be a small left $H$-category. We set

$$CN_n(\mathcal{D}_H) := \bigoplus \text{Hom}_{\mathcal{D}_H}(X_1,X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_2,X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_0,X_n)$$

where the direct sum runs over all $(X_0,X_1,\ldots,X_n) \in \text{Ob}(\mathcal{D}_H)^{n+1}$.

**Lemma 2.2.** Let $M$ be a right $H$-module. For each $n \geq 0$, $M \otimes CN_n(\mathcal{D}_H)$ is a right $H$-module with action determined by

$$(m \otimes f^0 \otimes \ldots \otimes f^n)h := mh_1 \otimes S(h_2)(f^0 \otimes \ldots \otimes f^n)$$

for any $m \in M$, $f^0 \otimes \ldots \otimes f^n \in CN_n(\mathcal{D}_H)$ and $h \in H$.

We now recall the notion of a stable anti-Yetter-Drinfeld module (SAYD) module from [19, Definition 2.1].

**Definition 2.3.** Let $H$ be a Hopf algebra with a bijective antipode $S$. A $k$-vector space $M$ is said to be a right-left anti-Yetter-Drinfeld module over $H$ if $M$ is a right $H$-module and a left $H$-comodule such that

$$\rho(mh) = (mh)(-1) \otimes (mh)_0 = S(h_3)m(-1) \otimes m_0 h_2$$

for all $m \in M$ and $h \in H$, where $\rho : M \rightarrow H \otimes M$, $m \mapsto m(-1) \otimes m_0$ is the coaction. Moreover, $M$ is said to be stable if $m_0 m(-1) = m$.

We now take the Hopf-cyclic cohomology $HC^n_H(\mathcal{D}_H, M)$ of an $H$-category $\mathcal{D}_H$ with coefficients in a SAYD module $M$ (see also [24]). This generalizes the construction of the Hopf-cyclic cohomology for $H$-module algebras with coefficients in an SAYD module (see [18] and also [37]). For each $n \geq 0$, we set

$$C^n(\mathcal{D}_H, M) := \text{Hom}_k(M \otimes CN_n(\mathcal{D}_H), k) \quad C^n_H(\mathcal{D}_H, M) := \text{Hom}_H(M \otimes CN_n(\mathcal{D}_H), k)$$

where $k$ is considered as a right $H$-module via the counit. It is clear from the definition that an element in $C^n_H(\mathcal{D}_H, M)$ is a $k$-linear map $\phi : M \otimes CN_n(\mathcal{D}_H) \rightarrow k$ satisfying

$$\phi(mh_1 \otimes S(h_2)(f^0 \otimes \ldots \otimes f^n)) = \varepsilon(h)\phi(m \otimes f^0 \otimes \ldots \otimes f^n)$$

We recall that a (co)simplicial module is said to be para-(co)cyclic if all the relations for a (co)cyclic module are satisfied except $\tau_n^{n+1} = \text{id}$ (see, for instance [29]). The following may be verified directly.

**Proposition 2.4.** Let $\mathcal{D}_H$ be a left $H$-category and let $M$ be a right-left SAYD module over $H$. Then,

1. we have a para-cocyclic module $C^*(\mathcal{D}_H, M) := \{C^n(\mathcal{D}_H, M)\}_{n \geq 0}$ with the following structure maps

$$(\delta_i\phi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} \phi(m \otimes f^0 \otimes \ldots \otimes f^i f^{i+1} \otimes \ldots \otimes f^n) & 0 \leq i \leq n-1 \\ \phi(m_0 \otimes (S^{-1}(m_{-1})) f^n) & i = n \end{cases}$$

$$(\sigma_i\psi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} \psi(m \otimes f^0 \otimes \ldots \otimes f^i \otimes \text{id}_{X_{i+1}} \otimes f^{i+1} \otimes \ldots \otimes f^n) & 0 \leq i \leq n-1 \\ \psi(m \otimes f^0 \otimes \ldots \otimes f^n \otimes \text{id}_{X_n}) & i = n \end{cases}$$

$$(\tau_n\varphi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \varphi(m_0 \otimes S^{-1}(m_{-1}) f^n) \otimes f^0 \otimes \ldots \otimes f^{n-1}$$

for any $\phi \in C^{n-1}(\mathcal{D}_H, M)$, $\psi \in C^{n+1}(\mathcal{D}_H, M)$, $\varphi \in C^n(\mathcal{D}_H, M)$, $m \in M$ and $f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{\mathcal{D}_H}(X_1,X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_2,X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_0,X_n)$.

2. by restricting to right $H$-linear morphisms $C^*_H(\mathcal{D}_H, M) = \text{Hom}_H(M \otimes CN_n(\mathcal{D}_H), k)$, we obtain a cocyclic module $C^*_H(\mathcal{D}_H, M) := \{C^n_H(\mathcal{D}_H, M)\}_{n \geq 0}$.
The cohomology of the cocyclic module $C_{\bullet}^H(D_H, M)$ is referred to as the Hopf-cyclic cohomology of the $H$-category $D_H$ with coefficients in the SAYD module $M$. The corresponding cohomology groups are denoted by $HC_{\bullet}^H(D_H, M)$.

**Remark 2.5.** As $k$ contains $\mathbb{Q}$, we recall that the cohomology of a cocyclic module $\mathcal{C}$ can be expressed alternatively as the cohomology of the following complex (see, for instance [34, 2.5.9]):

$$C_{\lambda}^0(\mathcal{C}) \overset{b}{\rightarrow} \ldots \overset{b}{\rightarrow} C_{\lambda}^n(\mathcal{C}) \overset{b}{\rightarrow} C_{\lambda}^{n+1}(\mathcal{C}) \overset{b}{\rightarrow} \ldots \quad (2.4)$$

where $C_{\lambda}^n(\mathcal{C}) = Ker(1 - \lambda) \subseteq C^n(\mathcal{C})$, $b = \sum_{i=0}^{n+1}(-1)^i \delta_i$ and $\lambda = (-1)^n \tau_n$. In particular, an element $\phi \in C^n_H(D_H, M)$ is a cyclic cocycle if and only if

$$b(\phi) = 0 \quad \text{and} \quad (1 - \lambda)(\phi) = 0 \quad (2.5)$$

In this paper, the cocycles and coboundaries of a cocyclic module will always refer to this complex.

**Proposition 2.6.** Let $D_H$ be a left $H$-category and let $M$ be a right-left SAYD module. Then:

1. We obtain a para-cyclic module $C_{\bullet}(D_H, M) := \{C_n(D_H, M) := M \otimes CN_n(D_H)\}_{n \geq 0}$ with the following structure maps

   $$d_i(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} m \otimes f^0 \otimes f^1 \otimes \ldots \otimes f^{i+1} \otimes \ldots \otimes f^n & 0 \leq i \leq n - 1 \\ m_{(0)} \otimes (S^{-1}(m_{(-1)})) f^n \otimes f^1 \otimes \ldots \otimes f^{n-1} & i = n \end{cases}$$

   $$s_i(m \otimes f^0 \otimes \ldots \otimes f^n) = \begin{cases} m \otimes f^0 \otimes f^1 \otimes \ldots \otimes f^i \otimes id_{X_{i+1}} \otimes f^{i+1} \otimes \ldots \otimes f^n & 0 \leq i \leq n - 1 \\ m \otimes f^0 \otimes f^1 \otimes \ldots \otimes f^n \otimes id_{X_0} & i = n \end{cases}$$

   $$t_n(m \otimes f^0 \otimes \ldots \otimes f^n) = m_{(0)} \otimes S^{-1}(m_{(-1)})) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}$$

for any $m \in M$ and $f^0 \otimes f^1 \otimes \ldots \otimes f^n \in Hom_{D_H}(X_0, X_0) \otimes Hom_{D_H}(X_2, X_1) \otimes \ldots \otimes Hom_{D_H}(X_0, X_n)$.

2. By passing to the tensor product over $H$, we obtain a cyclic module $C_{\bullet}^H(D_H, M) := \{C_n^H(D_H, M) := M \otimes_H CN_n(D_H)\}_{n \geq 0}$.

The cyclic homology groups corresponding to the cyclic module $C_{\bullet}^H(D_H, M)$ will be denoted by $HC_{\bullet}^H(D_H, M)$.

### 3 Traces, cocycles and DGH-semicategories

We continue with $D_H$ being a left $H$-category and $M$ a right-left SAYD module over $H$. Our purpose is to develop a formalism analogous to that of Connes [12] in order to interpret the cocycles $Z_{\bullet}^H(D_H, M)$ of the complex $C_{\bullet}(D_H, M)$ and its coboundaries $B_{\bullet}^H(D_H, M)$ as characters of differential graded semicategories. In this section, we will describe $Z_{\bullet}^H(D_H, M)$, for which we will need the framework of DG-semicategories. Let us first recall the notion of a semicategory introduced by Mitchell in [38] (for more on semicategories, see, for instance, [8]).

**Definition 3.1.** (see [38] Section 4) A semicategory $\mathcal{C}$ consists of a collection $Ob(\mathcal{C})$ of objects together with a set of morphisms $Hom_{\mathcal{C}}(X, Y)$ for each $X, Y \in Ob(\mathcal{C})$ and an associative composition. A semifunctor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between semicategories assigns an object $F(X) \in Ob(\mathcal{C}')$ to each $X \in Ob(\mathcal{C})$ and a morphism $F(f) \in Hom_{\mathcal{C}'}(F(X), F(Y))$ to each $f \in Hom_{\mathcal{C}}(X, Y)$ and preserves composition.

A left $H$-semicategory is a small $k$-linear semicategory $S_H$ such that

1. $Hom_{S_H}(X, Y)$ is a left $H$-module for all $X, Y \in Ob(S_H)$

2. $h(gf) = (h_1 g)(h_2 f)$ for any $h \in H$, $f \in Hom_{S_H}(X, Y)$ and $g \in Hom_{S_H}(Y, Z)$.
It is clear that any ordinary category may be treated as a semicategory. Conversely, to any \( k \)-semicategory \( \mathcal{C} \), we can associate an ordinary \( k \)-category \( \hat{\mathcal{C}} \) by setting \( \text{Ob}(\hat{\mathcal{C}}) = \text{Ob}(\mathcal{C}) \) and adjoining unit morphisms as follows:

\[
\text{Hom}_\mathcal{C}(X,Y) := \begin{cases} 
\text{Hom}_\mathcal{C}(X,X) \oplus k & \text{if } X = Y \\
\text{Hom}_\mathcal{C}(X,Y) & \text{if } X \neq Y 
\end{cases}
\]

A morphism in \( \text{Hom}_\mathcal{C}(X,Y) \) will be denoted by \( f = f + \mu \), where \( f \in \text{Hom}_\mathcal{C}(X,Y) \) and \( \mu \in k \). It is understood that \( \mu = 0 \) whenever \( X = Y \). Any semifunctor \( F : \mathcal{C} \longrightarrow \mathcal{D} \) where \( \mathcal{D} \) is an ordinary category may be extended to an ordinary functor \( \hat{F} : \hat{\mathcal{C}} \longrightarrow \mathcal{D} \). If \( \hat{\mathcal{S}}_H \) is a left \( H \)-semicategory, we note that \( \hat{\mathcal{S}}_H \) is a left \( H \)-category in the sense of Definition 2.1

**Definition 3.2.** A differential graded semicategory (DG-semicategory) \((\mathcal{S}, \hat{\partial})\) is a \( k \)-linear semicategory \( \mathcal{S} \) such that

(i) \( \text{Hom}^*_{\mathcal{S}}(X,Y) = \left( \text{Hom}^0_{\mathcal{S}}(X,Y), \hat{\partial}^n_{XY} \right)_{n \geq 0} \) is a cochain complex of \( k \)-spaces for each \( X, Y \in \text{Ob}(\mathcal{S}) \).

(ii) the composition map \( \text{Hom}^*_{\mathcal{S}}(Y,Z) \otimes \text{Hom}^*_{\mathcal{S}}(X,Y) \longrightarrow \text{Hom}^*_{\mathcal{S}}(X,Z) \) is a morphism of complexes.

Equivalently, we have \( \hat{\partial}^n_{YZ}(gf) = \hat{\partial}^n_{XY}(g)f + (-1)^{n-r}g\hat{\partial}^n_{XY}(f) \) for any \( f \in \text{Hom}_\mathcal{S}(X,Y)^r \) and \( g \in \text{Hom}_\mathcal{S}(Y,Z)^n \).

Whenever the meaning is clear from context, we will drop the subscript and simply write \( \hat{\partial}^* \) for the differential on any \( \text{Hom}^*_{\mathcal{S}}(X,Y) \).

A small DG-semicategory may be treated as a differential graded (but not necessarily unital) \( k \)-algebra with several objects. The DG-semicategories may be treated in a manner similar to DG-categories (see, for instance, [30], [31]). For instance, there is an obvious notion of DG-semifunctor between DG-semicategories.

We now construct a “universal DG-semicategory” associated to a given \( k \)-linear semicategory, similar to the construction of the universal differential graded algebra associated to a (not necessarily unital) \( k \)-algebra (see, for instance, [12], p. 315).

Let \( \Omega \mathcal{C} \) be the semicategory with \( \text{Ob}(\Omega \mathcal{C}) := \text{Ob}(\mathcal{C}) \) and \( \text{Hom}_{\Omega \mathcal{C}}(X,Y) = \bigoplus_{n \geq 0} \text{Hom}^n_{\Omega \mathcal{C}}(X,Y) \), where

\[
\text{Hom}^n_{\Omega \mathcal{C}}(X,Y) := \begin{cases} 
\text{Hom}_\mathcal{C}(X,Y) & \text{if } n = 0 \\
\bigoplus_{(X_1, \ldots, X_n) \in \text{Ob}(\mathcal{C})^n} \text{Hom}_\mathcal{C}(X_1, Y) \otimes \text{Hom}_\mathcal{C}(X_2, X_1) \otimes \cdots \otimes \text{Hom}_\mathcal{C}(X_n, X_{n-1}) & \text{if } n \geq 1 
\end{cases}
\]

(3.1)

Here the sum runs over the ordered tuples \((X_1, \ldots, X_n) \in \text{Ob}(\mathcal{C})^n\). In particular, \((\Omega \mathcal{C})^0 = \mathcal{C}\). For \( n \geq 1 \), an element of the form \( f^0 \otimes f^1 \otimes \cdots \otimes f^n \) in \( \text{Hom}^n_{\Omega \mathcal{C}}(X,Y) \) will be denoted by \( f^0 df^1 \cdots df^n = (f^0 + \mu) df^1 \cdots df^n \) and said to be homogeneous of degree \( n \). By abuse of notation, we will continue to use \( f^0 df^1 \cdots df^n \) to denote an element of \( \text{Hom}^n_{\Omega \mathcal{C}}(X,Y) \) even when \( n = 0 \). In that case, it will be understood that \( \mu = 0 \).

The composition in \( \Omega \mathcal{C} \) is determined by

\[
f^0 \circ df^1 \cdots \circ df^n = f^0 df^1 \cdots df^n \quad (df^0) \circ f^1 = d(f^0 f^1) - f^0 (df^1) \quad df^1 \circ \cdots \circ df^n = df^1 \cdots df^n \quad (3.2)
\]

In particular, it follows that

\[
((f^0 + \mu) df^1 \cdots df^i) \cdot ((g^0 + \mu') dg^1 \cdots dg^j) \\
= (f^0 + \mu) \left( df^1 \cdots df^{i-1} (f^j g^0) dg^1 \cdots dg^j + \sum_{l=1}^{i-1} (-1)^{l-1} df^1 \cdots df^l f^j (f^{l+1} g^0) \cdots dg^1 \cdots dg^j \right) \\
\quad + (-1)^i (f^0 + \mu) f^1 df^2 \cdots df^i \cdot dg^0 \cdots dg^1 \cdots \cdots + \mu' (f^0 + \mu) df^1 \cdots df^i \cdot dg^1 \cdots dg^j
\]

(3.3)
For each $X, Y \in \text{Ob}(\Omega C)$, the differential $\partial_{XY}^n : \text{Hom}_{\Omega C}^n(X, Y) \rightarrow \text{Hom}_{\Omega C}^{n+1}(X, Y)$ is determined by setting

$$\partial_{XY}^n((f^0 + \mu)df^1 \ldots df^n) := df^0df^1 \ldots df^n$$

It follows from definition that $\partial_{XY}^{n+1} \circ \partial_{XY}^n = 0$. Therefore, $\text{Hom}_{\Omega C}^n(X, Y) := (\text{Hom}_{\Omega C}^n(X, Y), \partial_{XY}^n)_{n \geq 0}$ is a cochain complex for each $X, Y \in \text{Ob}(\Omega C)$. It may also be verified that the composition in $\Omega C$ is a morphism of complexes. Thus, $\Omega C$ is a DG-semicategory.

**Proposition 3.3.** Let $C$ be a small $k$-linear semicategory. Then, the associated DG-semicategory $(\Omega C, \partial)$ is universal in the following sense: given

(i) any DG-semicategory $(S, \partial_S)$ and

(ii) a $k$-linear semifunctor $\rho : C \rightarrow S^0$,

there exists a unique DG-semifunctor $\hat{\rho} : (\Omega C, \partial) \rightarrow (S, \partial_S)$ such that the restriction of $\hat{\rho}$ to the semicategory $C$ is identical to $\rho : C \rightarrow S^0$.

**Proof.** We extend $\rho$ to obtain a DG-semifunctor $\hat{\rho} : (\Omega C, \partial) \rightarrow (S, \partial_S)$ as follows:

$$\hat{\rho}(X) := \rho(X)$$

$$\hat{\rho}((f^0 + \mu)df^1 \ldots df^n) := \rho(f^0) \circ \hat{\partial}^0(\rho(f^1)) \circ \ldots \circ \hat{\partial}^0(\rho(f^n))$$

for all $X \in \text{Ob}(\Omega C) = \text{Ob}(C)$ and $(f^0 + \mu)df^1 \ldots df^n \in \text{Hom}_{\Omega C}^n(X, Y)$, $n \geq 1$. Since each $\rho(f^i)$ is a morphism of degree $0$ in $S$, it follows from (3.2) and (3.3) that

$$\hat{\rho}((f^0 + \mu)df^1 \ldots df^n) \circ ((f^{n+1} + \mu')df^{n+2} \ldots df^m)) = \hat{\rho}((f^0 + \mu)df^1 \ldots df^n) \circ \hat{\rho}((f^{n+1} + \mu')df^{n+2} \ldots df^m)$$

It is also clear by construction that $\hat{\rho}|C = \rho$. Moreover, we have

$$\hat{\partial}^n(\hat{\rho}((f^0 + \mu)df^1 \ldots df^n)) = \hat{\partial}^n(\rho(f^0)\hat{\partial}^0(\rho(f^1)) \ldots \hat{\partial}^0(\rho(f^n))) + \mu \hat{\partial}^n(\hat{\partial}^0(\rho(f^1)) \ldots \hat{\partial}^0(\rho(f^n)))$$

$$= \hat{\partial}^0(\rho(f^0))\hat{\partial}^0(\rho(f^1)) \ldots \hat{\partial}^0(\rho(f^n)) + \rho(f^0)\hat{\partial}^n(\hat{\partial}^0(\rho(f^1)) \ldots \hat{\partial}^0(\rho(f^n)))$$

The uniqueness of $\hat{\rho}$ is also clear from (3.2) and (3.3). \hfill \Box

**Definition 3.4.** A left DGH-semicategory is a left $H$-semicategory $S_H$ equipped with a DG-semicategory $(S_H, \partial_H)$ structure such that for all $n \geq 0$:

(a) $\text{Hom}_{S_H}^n(X, Y)$ is a left $H$-module for $X, Y \in \text{Ob}(S_H)$.

(b) $\partial_H^n : \text{Hom}_{S_H}^n(X, Y) \rightarrow \text{Hom}_{S_H}^{n+1}(X, Y)$ is $H$-linear for $X, Y \in \text{Ob}(S_H)$.

We can similarly define the notion of a DGH-semifunctor between DGH-semicategories. If $(S_H, \partial_H)$ is a left DGH-semicategory, we note that $S_H^0$ is a left $H$-semicategory.

**Proposition 3.5.** Let $D_H$ be a left $H$-category. Then, the universal DGH-semicategory $(\Omega(D_H), \partial_H)$ associated to $D_H$ is a left DGH-semicategory with the $H$-action determined by

$$h \cdot ((f^0 + \mu)df^1 \ldots df^n) := (h_1f^0 + \mu \varepsilon(h_1))d(h_2f^1) \ldots d(h_{n+1}f^n)$$

for all $h \in H$ and $(f^0 + \mu)df^1 \ldots df^n \in \text{Hom}_{\Omega(D_H)}(X, Y)$.

**Proof.** This is immediate from the definitions in (3.3) and (3.4). \hfill \Box
Definition 3.6. Let \((S_H, \hat{\partial}_H)\) be a left DGH-semicategory and \(M\) be a right-left SAYD module over \(H\). A closed graded \((H,M)\)-trace of dimension \(n\) on \(S_H\) is a collection of \(k\)-linear maps

\[
\hat{\mathcal{H}} := \{ \hat{\mathcal{H}}_X^H : M \otimes \text{Hom}_{S_H}^r(X,X) \to k \}_{X \in \text{Ob}(S_H)}
\]

such that

\[
\begin{align*}
\hat{\mathcal{H}}_X(mh_1 \otimes S(h_2)f) &= \varepsilon(h) \hat{\mathcal{H}}_X(m \otimes f) \\
\hat{\mathcal{H}}_X(m \otimes \hat{\partial}_H^{n-1}(f')) &= 0 \\
\hat{\mathcal{H}}_X(m \otimes g'g) &= (-1)^{ij} \hat{\mathcal{H}}_X(m_{(0)} \otimes (S^{-1}(m_{(-1)}g)) g')
\end{align*}
\]

for all \(h \in H, m \in M, f \in \text{Hom}_{S_H}^n(X,X), f' \in \text{Hom}_{S_H}^{n-1}(X,X), g \in \text{Hom}_{S_H}^n(X,Y), g' \in \text{Hom}_{S_H}^j(Y,X)\) and \(i + j = n\).

Definition 3.7. An \(n\)-dimensional \(S_H\)-cycle with coefficients in a SAYD module \(M\) is a triple \((S_H, \hat{\partial}_H, \hat{\mathcal{H}})\) such that

(i) \((S_H, \hat{\partial}_H)\) is a left DGH-semicategory.

(ii) \(\hat{\mathcal{H}}\) is a closed graded \((H,M)\)-trace of dimension \(n\) on \(S_H\).

Let \(\mathcal{D}_H\) be a left \(H\)-category. By an \(n\)-dimensional cycle over \(\mathcal{D}_H\), we mean a tuple \((S_H, \hat{\partial}_H, \hat{\mathcal{H}}, \rho)\) such that

(i) \((S_H, \hat{\partial}_H, \hat{\mathcal{H}})\) is an \(n\)-dimensional \(S_H\)-cycle with coefficients in a SAYD module \(M\).

(ii) \(\rho : \mathcal{D}_H \to S_H^0\) is an \(H\)-linear semifunctor.

We fix a left \(H\)-category \(\mathcal{D}_H\). Given an \(n\)-dimensional cycle \((S_H, \hat{\partial}_H, \hat{\mathcal{H}}, \rho)\) over \(\mathcal{D}_H\), we define its character \(\phi \in C^n_H(\mathcal{D}_H, M)\) by setting

\[
\phi : M \otimes C_n(\mathcal{D}_H) \to k \quad \phi(m \otimes f^0 \ast \ldots \ast f^n) := \hat{\mathcal{H}}_X^H(m \otimes \rho(f^0) \hat{\partial}_H^{0}(\rho(f^1)) \ldots \hat{\partial}_H^{0}(\rho(f^n)))
\]

for \(m \in M\) and \(f^0 \ast \ldots \ast f^n \in \text{Hom}_{\mathcal{D}_H}(X_1, X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_0, X_n)\). We will often suppress the semifunctor \(\rho\) and refer to \(\phi\) simply as the character of the \(n\)-dimensional cycle \((S_H, \hat{\partial}_H, \hat{\mathcal{H}})\).

We now have a characterization of the space \(Z^n_H(\mathcal{D}_H, M)\) of \(n\)-cocycles in the Hopf-cyclic cohomology of the category \(\mathcal{D}_H\) with coefficients in the SAYD module \(M\).

Theorem 3.8. Let \(\mathcal{D}_H\) be a left \(H\)-category and \(M\) be a right-left SAYD module over \(H\). Let \(\phi \in C^n_H(\mathcal{D}_H, M)\). Then, the following conditions are equivalent:

(1) \(\phi\) is the character of an \(n\)-dimensional cycle over \(\mathcal{D}_H\), i.e., there is an \(n\)-dimensional cycle \((S_H, \hat{\partial}_H, \hat{\mathcal{H}})\) with coefficients in \(M\) and an \(H\)-linear semifunctor \(\rho : \mathcal{D}_H \to S_H^0\) such that

\[
\begin{align*}
\phi(m \otimes f^0 \ast \ldots \ast f^n) &= \hat{\mathcal{H}}_X^H((id_M \otimes \hat{\partial}) (m \otimes f^0 \ast df^1 \ldots df^n)) \\
&= \hat{\mathcal{H}}_X^H(m \otimes \rho(f^0) \hat{\partial}_H^{0}(\rho(f^1)) \ldots \hat{\partial}_H^{0}(\rho(f^n)))
\end{align*}
\]

for any \(m \in M\) and \(f^0 \ast \ldots \ast f^n \in \text{Hom}_{\mathcal{D}_H}(X_1, X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_0, X_n)\).

(2) There exists a closed graded \((H,M)\)-trace \(\mathcal{H}\) of dimension \(n\) on \((\Omega(\mathcal{D}_H), \partial_H)\) such that

\[
\phi(m \otimes f^0 \ast \ldots \ast f^n) = \mathcal{H}_X^H(m \otimes f^0 df^1 \ldots df^n)
\]

for any \(m \in M\) and \(f^0 \ast \ldots \ast f^n \in \text{Hom}_{\mathcal{D}_H}(X_1, X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_0, X_n)\).
Thus, using the condition in (3.9), we obtain
\[ b \text{Hom} \Rightarrow \]
for any \( m \in M \) and \( f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{D_H}(X_1, X) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X, X_n) \). In particular, it follows from (3.12) that

\[ \phi(m \otimes f^0 \otimes \ldots \otimes f^n) = \hat{\mathcal{F}}^H_X(m \otimes \rho(f^0)\hat{\partial}^0_H(\rho(f^1)) \ldots \hat{\partial}^0_H(\rho(f^n))) = \hat{\mathcal{F}}^H_X(m \otimes f^0 f^1 \ldots d^n) \]

(3.12)

It may be verified that the collection \( \hat{\mathcal{F}}^H \) is an \( n \)-dimensional closed graded \((H, M)\)-trace on \( \Omega(D_H) \).

(2) \( \Rightarrow \) (1). Suppose that we have a closed graded \((H, M)\)-trace \( \hat{\mathcal{F}}^H \) of dimension \( n \) on \( \Omega(D_H) \) satisfying (3.11). Then, the triple \((\Omega(D_H), \partial_H, \hat{\mathcal{F}}^H)\) forms an \( n \)-dimensional cycle over \( D_H \) with coefficients in \( M \). Further, by observing that \( \hat{\partial}^0_H(f) = df \) for any \( f \in \text{Hom}_{D_H}(X, Y) \), we get (3.10).

(1) \( \Rightarrow \) (3). Let \( (S_H, \partial_H, \hat{\mathcal{F}}^H) \) be an \( n \)-dimensional cycle over \( D_H \) with coefficients in \( M \) and \( \rho : D_H \rightarrow S^0_H \) be an \( H \)-linear semifunctor satisfying

\[ \phi(m \otimes f^0 \otimes \ldots \otimes f^n) = \hat{\mathcal{F}}^H_X(m \otimes \rho(f^0)\hat{\partial}^0_H(\rho(f^1)) \ldots \hat{\partial}^0_H(\rho(f^n))) \]

for any \( m \in M \) and \( f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X, X_n) \). For simplicity of notation, we will drop the functor \( \rho \). To show that \( \phi \) is an \( n \)-cocycle, it suffices to check that (see (2.5))

\[ b(\phi) = 0 \quad \text{and} \quad (1 - \lambda)(\phi) = 0 \]

where \( b = \sum_{i=0}^{n+1} (-1)^i \delta_i \) and \( \lambda = (-1)^n \tau_n \). For any \( p^0 \otimes \ldots \otimes p^{n+1} \in \text{Hom}_{D_H}(X_1, X_0) \otimes \text{Hom}_{D_H}(X_2, X_1) \otimes \ldots \otimes \text{Hom}_{D_H}(X, X_{n+1}) \), we have

\[
\sum_{i=0}^{n+1} (-1)^i \delta_i(\phi)(m \otimes p^0 \otimes \ldots \otimes p^{n+1}) \]

\[ = \sum_{i=0}^{n} (-1)^i \phi(m \otimes p^0 \otimes \ldots \otimes p^i \partial H^0(p^{i+1}) \otimes \ldots \otimes p^{n+1}) + (-1)^{n+1} \phi(m_{(0)} \otimes (S^{-1}(m_{(-1)}))^{n+1}) p^0 \otimes p^1 \otimes \ldots \otimes p^n \]

\[ = \hat{\mathcal{F}}^H_{X_0}(m \otimes p^0 \partial H^0(p^1) \ldots \partial H^0(p^{n+1})) + \sum_{i=1}^{n} (-1)^i \hat{\mathcal{F}}^H_{X_0}(m \otimes p^0 \partial H^0(p^1) \ldots \partial H^0(p^{i+1}) \ldots \partial H^0(p^{n+1})) + \]

\[ (-1)^{n+1} \hat{\mathcal{F}}^H_{X_{n+1}}(m_{(0)} \otimes (S^{-1}(m_{(-1)}))^{n+1}) p^0 \partial H^0(p^1) \ldots \partial H^0(p^n)) \]

Now using the equality \( \partial H^0(fg) = \partial H^0(f)g + f \partial H^0(g) \) for any \( f \) and \( g \) of degree 0, we have

\[ (p^0 \partial H^0(p^1) \ldots \partial H^0(p^{n+1})) = \sum_{i=1}^{n} (-1)^{n+1} p^0 \partial H^0(p^1) \ldots \partial H^0(p^{i+1}) \ldots \partial H^0(p^{n+1}) + (-1)^{n+1} p^0 \partial H^0(p^1) \ldots \partial H^0(p^{n+1}) \]

Thus, using the condition in (3.10), we obtain

\[ \sum_{i=0}^{n+1} (-1)^i \delta_i(\phi)(m \otimes p^0 \otimes \ldots \otimes p^{n+1}) \]

\[ = (-1)^n \hat{\mathcal{F}}^H_{X_0}(m \otimes (p^0 \partial H^0(p^1) \ldots \partial H^0(p^{n+1})) + (-1)^{n+1} \hat{\mathcal{F}}^H_{X_{n+1}}(m_{(0)} \otimes (S^{-1}(m_{(-1)}))^{n+1}) p^0 \partial H^0(p^1) \ldots \partial H^0(p^n)) = 0 \]
Next, using (3.8), (3.9), and the $H$-linearity of $\partial_H$, we have

\[
(1 - (-1)^n r_n) \phi (m \otimes f^0 \otimes \ldots \otimes f^n)
\]
\[
= \phi (m \otimes f^0 \otimes \ldots \otimes f^n) - (-1)^n \phi (m(0) \otimes S^{-1}(m(-1)) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1})
\]
\[
= \hat{T}_X^H (m \otimes f^0 \otimes \ldots \otimes f^n) - (-1)^n \hat{T}_X^H (m(0) \otimes (S^{-1}(m(-1)) f^n \otimes f^0 \otimes \ldots \otimes f^{n-1}))
\]
\[
= (-1)^n \hat{T}_X^H (m(0) \otimes (S^{-1}(m(-1)) \partial_H^n (f^n) \otimes f^0 \otimes \ldots \otimes f^{n-1}))
\]
\[
= (-1)^n \hat{T}_X^H (m(0) \otimes (S^{-1}(m(-1)) \partial_H^n (f^n) \otimes f^0 \otimes \ldots \otimes f^{n-1})) + (-1)^n \hat{T}_X^H (m(0) \otimes \partial_H^n (f^n) \otimes f^0 \otimes \ldots \otimes f^{n-1})
\]
\[
= (-1)^n \hat{T}_X^H (m(0) \otimes \partial_H^n (f^n) \otimes f^0 \otimes \ldots \otimes f^{n-1}) = 0
\]

(3) $\Rightarrow$ (2). Let $\phi \in Z^n_H (D_H, M)$. For each $X \in Ob(\Omega(D_H))$, we define an $H$-linear map $T^H_X : M \otimes Hom^n_H (\Omega(D_H)) (X, X) \to k$ given by

\[
T^H_X (m \otimes (f^0 + \mu) df^1 \ldots df^n) := \phi (m \otimes f^0 \otimes \ldots \otimes f^n)
\]

for $f^0 \otimes \ldots \otimes f^n \in Hom^1_H (X_1, X) \otimes Hom^1_H (X_2, X_1) \otimes \ldots \otimes Hom^1_H (X_n, X_n)$. We now verify that the collection \{ $T^H_X : M \otimes Hom^n_H (\Omega(D_H)) (X, X) \to k$ \}_{X \in Ob(\Omega(D_H))} is a closed graded $(H, M)$-trace on $(\Omega(D_H), \partial_H)$. For any $(p^0 + \mu) dp^1 \ldots dp^{n-1} \in Hom^{n-1}_H (\Omega(D_H)) (X, X)$, we have

\[
T^H_X (m \otimes \partial_H^{n-1} ((p^0 + \mu) dp^1 \ldots dp^{n-1})) = T^H_X (m \otimes 1 dp^0 dp^1 \ldots dp^{n-1}) = \phi (m \otimes 0 \otimes p^0 \otimes \ldots \otimes p^{n-1}) = 0
\]

This proves the condition in (3.8). Using (2.3), it is also clear that \{ $T^H_X : M \otimes Hom^n_H (\Omega(D_H)) (X, X) \to k$ \}_{X \in Ob(\Omega(D_H))} satisfies condition (3.7). Finally, for any $g' = (g^0 + \mu') dg^1 \ldots dg'^r \in Hom^r_H (Y, X)$ and $g = (g^{r+1} + \mu) dg^{r+2} \ldots dg^{n+1} \in Hom^{n-r}_H (X, Y)$, we have

\[
T^H_X (m \otimes g' g) = \sum_{j=1}^n (-1)^{r-j} T^H_X (m \otimes (g^0 + \mu') dg^1 \ldots dg^{j+1} \ldots dg^{n+1}) + (-1)^r T^H_X (m \otimes (g^0 + \mu') g^1 dg^2 \ldots dg^{n+1})
\]
\[
+ T^H_X (m \otimes \mu' dg^1 \ldots dg^{r-1} dg^r \ldots dg^{n+1})
\]
\[
= \sum_{j=1}^n (-1)^{r-j} \phi (m \otimes g^0 \otimes \ldots \otimes g^j g^{j+1} \otimes \ldots \otimes g^{n+1}) + (-1)^r \phi (m \otimes g^0 \otimes g^1 \otimes \ldots \otimes g^{n+1})
\]
\[
+ (-1)^r \mu' \phi (m \otimes g^1 \otimes g^2 \otimes \ldots \otimes g^{r-1} \otimes g^{r+2} \otimes \ldots \otimes g^{n+1})
\]
\[
= \sum_{j=0}^r (-1)^{r+j} \phi (m \otimes g^0 \otimes \ldots \otimes g^j g^{j+1} \otimes \ldots \otimes g^{n+1}) + (-1)^r \mu' \phi (m \otimes g^1 \otimes g^2 \otimes \ldots \otimes g^{n+1})
\]
\[
+ \mu \phi (m \otimes g^0 \otimes g^1 \otimes \ldots \otimes g^r \otimes g^{r+2} \otimes \ldots \otimes g^{n+1})
\]
On the other hand, we have

\[
(-1)^{(n-r)} \mathcal{T}^H_Y \left( m_{(0)} \otimes (S^{-1}(m_{-1})g) g' \right) \\
= (-1)^{(n-r)} \mathcal{T}^H_Y \left( m_{(0)} \otimes \left[ S^{-1} \left( (m_{-1})_{n-r} \right) (g^{r+1} + \mu) \right] [d \left( S^{-1} \left( (m_{-1})_{n-r} \right) g^{r+2} \right)] \ldots [d \left( S^{-1} \left( (m_{-1})_{1} \right) g^{n+1} \right)] \right) \\
\quad \otimes ((g^0 + \mu') dg^1 \ldots dg^r) \\
= (-1)^{(n-r)} \sum_{j=r+2}^{n} (-1)^{n-j+1} \mathcal{T}^H_Y \left( m_{(0)} \otimes \left[ S^{-1} \left( (m_{-1})_{n-r} \right) (g^{r+1} + \mu) \right] \ldots d \left( S^{-1} \left( (m_{-1})_{1} \right) g^{n+1} \right) g^0 \ldots dg^r \right) \\
\quad + (-1)^{(n-r)} \mathcal{T}^H_Y \left( m_{(0)} \otimes \left[ S^{-1} \left( (m_{-1})_{n-r} \right) (g^{r+1} + \mu) \right] \ldots [d \left( S^{-1} \left( (m_{-1})_{1} \right) g^{n+1} \right)] (dg^0 dg^1 \ldots dg^r) \right) \\
\quad + (-1)^{(n-r)} \mu' \mathcal{T}^H_Y \left( m_{(0)} \otimes \left[ S^{-1} \left( (m_{-1})_{n-r} \right) (g^{r+1} + \mu) \right] \ldots [d \left( S^{-1} \left( (m_{-1})_{1} \right) g^{n+1} \right)] \right) \\
\quad dg^1 \ldots dg^r \\
\]

Using repeatedly the fact that \( \phi = (-1)^{n-r} \tau_n \phi \), we get

\[
(-1)^{(n-r)} \mathcal{T}^H_Y \left( m_{(0)} \otimes (S^{-1}(m_{-1})g) g' \right) \\
= - \sum_{j=r+1}^{n} (-1)^{r+j} \phi \left( m \otimes g^0 \ldots \otimes g^r g^{j+1} \ldots \otimes g^{n+r} \right) - (-1)^{n+r+1} \phi \left( m_{(0)} \otimes \left( S^{-1}(m_{-1}) \right) g^n \right) g^0 \otimes g^1 \otimes \ldots \otimes g^n \\
\quad + (-1)^r \mu' \phi \left( m \otimes g^1 \otimes g^2 \otimes \ldots \otimes g^n \right) + \mu(\phi \left( m \otimes g^0 \otimes g^1 \otimes \ldots \otimes g^r \otimes g^{r+2} \otimes \ldots \otimes g^{n+1} \right)) \\
\]

The condition (3.9) now follows using the fact that \( b(\phi) = 0 \). This proves the result. \( \square \)

**Remark 3.9.** From the statement and proof of Theorem 3.8, it is clear that there is a one to one correspondence between \( n \)-dimensional closed graded \((H,M)\)-traces on \( \Omega(D_H) \) and \( Z^2_H(D_H,M) \).

### 4 Linearity: matrices and Hopf-cyclic cohomology

In the previous section, we described the spaces \( Z^2_H(D_H,M) \). The next aim is to find a characterization of \( B^*_H(D_H,M) \) which will be done in several steps. For this, we will show in this section that the Hopf-cyclic cohomology of an \( H \)-category \( D_H \) is the same as that of its linearization \( D_H \otimes M_r(k) \) by the algebra of \( r \times r \)-matrices. We observe that \( D_H \otimes M_r(k) \) is also a left \( H \)-category. We denote by \( \mathcal{O} \) the category whose objects are left \( H \)-categories and whose morphisms are \( H \)-linear functors.

We denote by \( \text{Vect}_k \) the category of all \( k \)-vector spaces and by \( H\text{-Mod} \) the category of all left \( H \)-modules. Let \( \text{Hom}_H(-, -) : H\text{-Mod} \to \text{Vect}_k \) be the functor that takes \( N \to \text{Hom}_H(N,k) \).

We fix \( r > 1 \). For \( 1 \leq i, j \leq r \) and \( \alpha \in k \), we let \( E_{ij}(\alpha) \) denote the elementary matrix in \( M_r(k) \) having \( \alpha \) at \((i,j)\)-th position and 0 everywhere else. We will often use \( E_{ij} \) for \( E_{ij}(1) \). For each \( 1 \leq p \leq r \), we have an inclusion \( \text{inc}_p : D_H \to D_H \otimes M_r(k) \) in \( \mathcal{O} \) which fixes the objects and \( \text{inc}_p(f) = f \otimes E_{pp} = f \otimes E_{pp}(1) \) for any morphism \( f \in D_H \).  

11
For any right-left SAYD-module $M$, the inclusion $inc_p : D_H \to D_H \otimes M_r(k)$ induces an inclusion map $(inc_p, M) : M \otimes CN_n(D_H) \to M \otimes CN_n(D_H \otimes M_r(k))$ which takes $m \otimes f^0 \otimes \ldots \otimes f^n \mapsto m \otimes (f^0 \otimes E_{p0}) \otimes \ldots \otimes (f^n \otimes E_{pn})$. This induces a morphism of Hochschild complexes $C_i(inc_p, M)^{hoc} : C_i(D_H, M)^{hoc} \to C_i(D_H \otimes M_r(k), M)^{hoc}$. Applying the functor $\text{Hom}_H(-, k)$, we obtain morphisms of Hochschild complexes $C_H(inc_p, M)^{hoc} : C_H(D_H, M)^{hoc} \to C_H(D_H \otimes M_r(k), M)^{hoc}$. We also have the induced morphism of double complexes computing cyclic homology $C_i(inc_p, M)^{cy} : C_i(D_H, M)^{cy} \to C_i(D_H \otimes M_r(k), M)^{cy}$. Applying the functor $\text{Hom}_H(-, k)$, we obtain a morphism of double complexes computing cyclic homology $C_H(inc_p, M)^{cy} : C_H(D_H, M)^{cy} \to C_H(D_H \otimes M_r(k), M)^{cy}$.

For each $n \geq 0$, there is an $H$-linear trace map $tr^M : M \otimes CN_n(D_H \otimes M_r(k)) \to M \otimes CN_n(D_H)$ given by

$$tr^M (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) := (m \otimes f^0 \otimes \ldots \otimes f^n)\text{trace}(B^0 \ldots B^n) \quad (4.1)$$

for any $m \in M$ and $(f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n) \in CN_n(D_H \otimes M_r(k))$. It may be verified easily that the trace map as in (4.1) defines a morphism $C_i(tr^M) : C_i(D_H \otimes M_r(k), M) \to C_i(D_H, M)$ of para-cyclic modules. In particular, we have an induced morphism between underlying Hochschild complexes

$$C_i(tr^M)^{hoc} : C_i(D_H \otimes M_r(k), M)^{hoc} \to C_i(D_H, M)^{hoc}$$

**Proposition 4.1.** The maps $C_i(inc_p, M)^{hoc}$ and $C_i(tr^M)^{hoc}$ are homotopy inverses of each other.

**Proof.** It may be easily verified that $C_i(tr^M)^{hoc} \circ C_i(inc_p, M)^{hoc} = id$. To show that $C_i(inc_p, M)^{hoc} \circ C_i(tr^M)^{hoc} \sim id$, we define $k$-linear maps $\{h_i : C_n(D_H \otimes M_r(k), M) \to C_{n+1}(D_H \otimes M_r(k), M)\}_{0 \leq i \leq n}$ by setting:

$$h_i (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) := m \otimes \sum_{1 \leq j,k,l,\ldots, p,q,r \leq n} (f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq})) \otimes (id_{x_{ij}} \otimes E_{ij}(1)) \otimes (f_1^0 \otimes B^0_{ij}) \otimes \ldots \otimes (f_1^n \otimes B^n_{ij})$$

for $0 \leq i < n$ and

$$h_n (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) := m \otimes \sum_{1 \leq j,k,m,\ldots, p,q,r} (f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{km})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq})) \otimes (id_{x_{ij}} \otimes E_{ij}(1))$$

We now verify that $h_n := \sum_{i=0}^{n} (-1)^i h_i$ is a pre-simplicial homotopy (see, for instance, [34] § 1.0.8]) between $C_i(inc_p, M)^{hoc} \circ C_i(tr^M)^{hoc}$ and $id_{C_i(D_H \otimes M_r(k), M)}$. For this, we need to verify the following identities:

$$d_i h_i = d_i - 1 d_i$$

for $i < i'$

$$d_i h_i = d_i - 1 h_i$$

for $0 < i \leq n$

$$d_i h_i = d_i - 1 h_i$$

for $i > i' + 1$

$$d_0 h_0 = id_{C_i(D_H \otimes M_r(k), M)^{hoc}}$$

and $d_n h_n = C_i(inc_p, M)^{hoc} \circ C_i(tr^M)^{hoc}$

where $d_i : C_{n+1}(D_H \otimes M_r(k), M) \to C_n(D_H \otimes M_r(k), M)$, $0 \leq i \leq n + 1$ are the face maps. We only verify the last one in [4.2] because the others follow similarly. Using the fact that $E_{1q}(1) E_{j1}(B_{jk}) = 0$ unless $q = j$, we have

$$d_n h_n (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n))$$

$$= d_{n+1} (m \otimes (f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq})) \otimes (id_{x_{ij}} \otimes E_{ij}(1)))$$

$$= m \otimes \sum_{1 \leq j,k,l,\ldots, p,q,r} (S^{-1}(m(-1))(id_{x_{ij}} \otimes E_{ij}(1))) (f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq}))$$

$$= m \otimes \sum_{1 \leq j,k,l,\ldots, p,q,r} (f^0 \otimes E_{j1}(1) E_{11}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq}))$$

$$= m \otimes \sum_{1 \leq j,k,l,\ldots, p,q,r} (f^0 \otimes E_{11}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq}))$$

$$= \left(m \otimes (f^0 \otimes E_{11}) \otimes \ldots \otimes (f^n \otimes E_{11})\right) \otimes (f^0 \otimes E_{j1}(B^0_{jk})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq}))$$

$$= \left(m \otimes (f^0 \otimes E_{11}) \otimes \ldots \otimes (f^n \otimes E_{11})\right) \otimes (f^0 \otimes E_{j1}(B^0_{jk})) \otimes \ldots \otimes (f^n \otimes E_{n1}(B^n_{pq}))$$

$$= \left(m \otimes (f^0 \otimes E_{11}) \otimes \ldots \otimes (f^n \otimes E_{11})\right) \otimes \ldots \otimes \left(m \otimes (f^0 \otimes E_{11}) \otimes \ldots \otimes (f^n \otimes E_{11})\right)$$

$$= \left(m \otimes (f^0 \otimes E_{11}) \otimes \ldots \otimes (f^n \otimes E_{11})\right) \otimes \ldots \otimes \left(m \otimes (f^0 \otimes E_{11}) \otimes \ldots \otimes (f^n \otimes E_{11})\right)$$

$$= \left(C_i(inc_p, M)^{hoc} \circ C_i(tr^M)^{hoc}\right) (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n))$$

$$= \left(C_i(inc_p, M)^{hoc} \circ C_i(tr^M)^{hoc}\right) (m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n))$$
This proves the result.

**Proposition 4.2.** Let \( \mathcal{D}_H \) be a left \( H \)-category and \( M \) be a right-left SAYD module. Then,

1. The morphisms

\[
\begin{align*}
HC^\ast_H(\text{inc}_1, M)^{\operatorname{hoc}} &:\ HC^\ast_H(\mathcal{D}_H \otimes M_r(k), M)^{\operatorname{hoc}} \to HC^\ast_H(\mathcal{D}_H, M)^{\operatorname{hoc}} \\
HC^\ast_H(\text{tr}^M)^{\operatorname{hoc}} &:\ HC^\ast_H(\mathcal{D}_H, M)^{\operatorname{hoc}} \to HC^\ast_H(\mathcal{D}_H \otimes M_r(k), M)^{\operatorname{hoc}}
\end{align*}
\]

induced by \( C^\ast_H(\text{inc}_1, M)^{\operatorname{hoc}} \) and \( C^\ast_H(\text{tr}^M)^{\operatorname{hoc}} \) are mutually inverse isomorphisms of Hochschild cohomologies.

2. We have isomorphisms

\[
HC^\ast_H(\mathcal{D}_H, M) \xrightarrow{\mu} HC^\ast_H(\text{inc}_1, M)
\]

**Proof.**

1. By Proposition 4.1, we know that \( C^\ast(\text{tr}^M)^{\operatorname{hoc}} \circ C^\ast(\text{inc}_1, M)^{\operatorname{hoc}} = \text{id}_{C^\ast(\mathcal{D}_H, M)^{\operatorname{hoc}}} \) and \( C^\ast(\text{inc}_1, M)^{\operatorname{hoc}} \circ C^\ast(\text{tr}^M)^{\operatorname{hoc}} \sim \text{id}_{C^\ast(\mathcal{D}_H^\ast \otimes M_r(k), M)^{\operatorname{hoc}}} \). Thus, applying the functor \( \text{Hom}_H(-, k) \), we obtain

\[
\begin{align*}
C^\ast_H(\text{inc}_1, M)^{\operatorname{hoc}} \circ C^\ast_H(\text{tr}^M)^{\operatorname{hoc}} &= \text{id}_{C^\ast_H(\mathcal{D}_H, M)^{\operatorname{hoc}}} \\
C^\ast_H(\text{tr}^M)^{\operatorname{hoc}} \circ C^\ast_H(\text{inc}_1, M)^{\operatorname{hoc}} &\sim \text{id}_{C^\ast_H(\mathcal{D}_H^\ast \otimes M_r(k), M)^{\operatorname{hoc}}}
\end{align*}
\]

Therefore, \( C^\ast_H(\text{inc}_1, M)^{\operatorname{hoc}} \) and \( C^\ast_H(\text{tr}^M)^{\operatorname{hoc}} \) are homotopy inverses of each other.

2. This follows immediately from (1) and the Hochschild to cyclic spectral sequence.

**Corollary 4.3.** For an \( n \)-cocycle \( \phi \in Z^n_H(\mathcal{D}_H, M) \), the \( n \)-cocycle \( \bar{\phi} = \text{Hom}_H(\text{tr}^M, k)(\phi) = \phi \circ \text{tr}^M \in Z^n_H(\mathcal{D}_H \otimes M_r(k), M) \) may be described as follows

\[
\bar{\phi}(m \otimes (f^0 \otimes B^0) \otimes \ldots \otimes (f^n \otimes B^n)) = \phi(m \otimes f^0 \otimes \ldots \otimes f^n) \text{trace}(B^0 \ldots B^n)
\]

## 5 Vanishing cycles on an \( H \)-category and coboundaries

From now onwards, we will always assume that \( k = \mathbb{C} \). In this section, we will describe the spaces \( B^\ast_H(\mathcal{D}_H, M) \). We will then use the formalism of categorified cycles and vanishing cycles developed in this paper to obtain a product on Hopf cyclic cohomologies of \( H \)-categories. We begin by recalling the notion of an inner automorphism of a category.

**Definition 5.1.** [see \( [22] \), p 24] Let \( \mathcal{D}_H \) be a left \( H \)-category. An automorphism \( \Phi \in \text{Hom}_{\text{Cat}_h}(\mathcal{D}_H, \mathcal{D}_H) \) is said to be inner if \( \Phi \) is isomorphic to the identity functor \( \text{id}_{\mathcal{D}_H} \). In particular, there exist isomorphisms \( \{\eta(X) : X \to \Phi(X)\}_{X \in \text{Ob}(\mathcal{D}_H)} \) such that \( \Phi(f) = \eta(Y) \circ f \circ (\eta(X))^{-1} \) for any \( f \in \text{Hom}_{\mathcal{D}_H}(X, Y) \).

We now set

\[
G(\mathcal{D}_H) := \prod_{X \in \text{Ob}(\mathcal{D}_H)} \text{Aut}_{\mathcal{D}_H}(X)
\]

By definition, an element \( \eta \in G(\mathcal{D}_H) \) corresponds to a family of automorphisms \( \{\eta(X) : X \to X\}_{X \in \text{Ob}(\mathcal{D}_H)} \). We now set

\[
\mathcal{U}_H(\mathcal{D}_H) := \{\eta \in G(\mathcal{D}_H) \mid h(\eta(X)) = \varepsilon(h)\eta(X) \text{ for every } h \in H \text{ and } X \in \text{Ob}(\mathcal{D}_H)\}
\]

**Lemma 5.2.** \( \mathcal{U}_H(\mathcal{D}_H) \) is a subgroup of \( G(\mathcal{D}_H) \).

**Proof.** The element \( e = \prod_{X \in \text{Ob}(\mathcal{D}_H)} \text{id}_X \) is the identity of the group \( G(\mathcal{D}_H) \). By definition of an \( H \)-category, we know that \( h \cdot \text{id}_X = \varepsilon(h) \cdot \text{id}_X \) for each \( X \in \text{Ob}(\mathcal{D}_H) \) and \( h \in H \). Thus, \( e \in \mathcal{U}_H(\mathcal{D}_H) \). Now, suppose that \( \eta, \eta' \in \mathcal{U}_H(\mathcal{D}_H) \). Then, for each \( X \in \text{Ob}(\mathcal{D}_H) \) and \( h \in H \),

\[
h((\eta \circ \eta')(X)) = h(\eta(X) \circ \eta'(X)) = (h_1(X)) \circ (h_2(X)) = (\varepsilon(h_1)\eta(X)) \circ (\varepsilon(h_2)\eta'(X)) = \varepsilon(h)(\eta(X) \circ \eta'(X))
\]
Hence, $\eta \circ \eta' \in \mathbb{U}_H(D_H)$. Also, $\eta^{-1} \in G(D_H)$ corresponds to a family of morphisms $\{\eta^{-1}(X) := \eta(X)^{-1} : X \to X\}_{X \in \text{Ob}\(D_H\)}$. Then, for each $h \in H$ and $X \in \text{Ob}\(D_H\)$,

$$
\varepsilon(h) \text{id}_X = h(\eta(X) \circ \eta^{-1}(X)) = (\varepsilon(h_1) \eta(X)) \circ (h_2 \eta^{-1}(X)) = \eta(X) \circ (h \eta^{-1}(X))
$$

which gives $\varepsilon(h) \eta^{-1}(X) = h \eta^{-1}(X)$. Therefore, $\eta^{-1} \in \mathbb{U}_H(D_H)$. □

**Lemma 5.3.** Let $D_H$ be a left $H$-category and let $\eta \in \mathbb{U}_H(D_H)$.

1. Consider $\Phi_\eta : D_H \to D_H$ defined by

$$
\Phi_\eta(X) = X \quad \Phi_\eta(f) := \eta(Y) \circ f \circ \eta(X)^{-1}
$$

for every $X \in \text{Ob}(D_H)$ and $f \in \text{Hom}_{D_H}(X,Y)$. Then, $\Phi_\eta : D_H \to D_H$ is an inner automorphism of $D_H$.

2. Consider $\tilde{\Phi}_\eta : D_H \otimes M_2(k) \to D_H \otimes M_2(k)$ defined by

$$
\tilde{\Phi}_\eta(X) = X \quad \tilde{\Phi}_\eta(f \otimes B) = (id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ (f \otimes B) \circ (id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22})
$$

for every $X \in \text{Ob}(D_H \otimes M_2(k))$. Then, $\tilde{\Phi}_\eta : D_H \otimes M_2(k) \to D_H \otimes M_2(k)$ is an inner automorphism.

**Proof.** 1. Using the fact that $\eta, \eta^{-1} \in \mathbb{U}_H(D_H)$, we have

$$
h(\Phi_\eta(f)) = (h_1 \eta(Y)) \circ (h_2 f) \circ (h_3 \eta(X)^{-1}) = (\varepsilon(h_1) \eta(Y)) \circ (h_2 f) \circ (\varepsilon(h_3) \eta(X)^{-1}) = \eta(Y) \circ (h_1 f) \circ (\varepsilon(h_2) \eta(X)^{-1}) = \eta(Y) \circ (h f) \circ \eta(X)^{-1}
$$

for any $h \in H$ and $f \in \text{Hom}_{D_H}(X,Y)$. By Definition 5.1, we now see that $\Phi_\eta$ is an inner automorphism.

2. Setting $\tilde{\eta}(X) : X \to X$ in $D_H \otimes M_2(k)$ as $\tilde{\eta}(X) = id_X \otimes E_{11} + \eta(X) \otimes E_{22}$, we see that

$$
\tilde{\Phi}_\eta(f \otimes B) = (id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ (f \otimes B) \circ (id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) = \tilde{\eta}(Y) \circ (f \otimes B) \circ \tilde{\eta}(X)^{-1}
$$

for any $f \otimes B \in \text{Hom}_{D_H \otimes M_2(k)}(X,Y)$. Considering the $H$-action on the category $D_H \otimes M_2(k)$, we have

$$
h(\tilde{\Phi}_\eta((f \otimes B))) = h_1(id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ h_3(id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22})
$$

$$
= (h_1 id_Y \otimes E_{11} + h_1 \eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ (h_3 \eta(X)^{-1} \otimes E_{22})
$$

$$
= \varepsilon(h_1)(id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ \varepsilon(h_3)(id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22})
$$

$$
= (id_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h(f \otimes B) \circ (id_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22})
$$

$$
= \tilde{\Phi}_\eta(h(f \otimes B))
$$

for any $h \in H$ and $f \otimes B \in \text{Hom}_{D_H \otimes M_2(k)}(X,Y)$. By Definition 5.1, we now see that $\tilde{\Phi}_\eta$ is an inner automorphism. □

For any $\eta \in \mathbb{U}_H(D_H)$, we will always denote by $\Phi_\eta$ and $\tilde{\Phi}_\eta$ the inner automorphisms defined in Lemma 5.3.

**Lemma 5.4.** Let $M$ be a right-left SAYD module over $H$. Then,

1. A semifunctor $\alpha \in \text{Hom}_{\text{Cat}_H}(D_H, D'_H)$ induces a morphism (for all $n \geq 0$)

$$
C^n_H(\alpha, M) : C^n_H(D'_H, M) = \text{Hom}_H(M \otimes C_n(D'_H), k) \to C^n_H(D_H, M) = \text{Hom}_H(M \otimes C_n(D_H), k)
$$

determined by

$$
C^n_H(\alpha, M)(\phi)(m \otimes f^0 \otimes \ldots \otimes f^n) = \phi(m \otimes \alpha(f^0) \otimes \ldots \otimes \alpha(f^n))
$$

for any $\phi \in C^n_H(D'_H, M)$, $m \in M$ and $f^0 \otimes \ldots \otimes f^n \in C_n(D'_H)$. This leads to a morphism $C^n_{H}(\alpha, M)^{cy} : C^n_{H}(D'_H, M)^{cy} \to C^n_{H}(D_H, M)^{cy}$ of double complexes and induces a functor $HC^n_{H}(\alpha, M) : \text{Cat}_H^{op} \to \text{Vect}_k$.

2. Let $\eta \in \mathbb{U}_H(D_H)$. Then, $\Phi_\eta$ induces the identity map on $HC^n_{H}(D_H, M)$. 

14
Proof. (1) Since $\phi$ and $\alpha$ are $H$-linear, the morphisms $C^*_{H}(\alpha, M)$ are well-defined and well behaved with respect to the maps appearing in the Hochschild and cyclic complexes. The result follows.

(2) Let $\eta \in \mathbb{U}_H(D_H)$ and $\Phi_\eta \in Hom_{Cat_H}(D_H, D_H)$ be the corresponding inner automorphism. By Proposition [1.2], the maps $HC^*_H(inc_1, M)$ and $HC^*_H(tr^M)$ are mutually inverse isomorphisms of Hopf-cyclic cohomology groups. Thus, we have

$$HC^*_H(inc_2, M) \circ (HC^*_H(inc_1, M)^{-1} = HC^*_H(inc_2, M) \circ HC^*_H(tr^M) = HC^*_H(tr^M \circ (inc_2, M)) = id \quad (5.3)$$

Further, we have the following commutative diagram in the category $Cat_H$:

$$\begin{array}{ccc}
D_H & \xrightarrow{inc_1} & D_H \otimes M_2(k) \\
\downarrow{id_D} & \downarrow{\phi_0} & \downarrow{\phi} \\
D_H & \xrightarrow{inc_2} & D_H \otimes M_2(k) \\
\end{array} \quad (5.4)$$

Thus, by applying the functor $HC^*_H(-, M)$ to the commutative diagram (5.4) and using (5.3), we obtain

$$HC^*_H(\Phi_\eta, M) = (HC^*_H(inc_2, M)) \circ HC^*_H(inc_1, M)^{-1} \circ HC^*_H(id_{D_H}, M) \circ (HC^*_H(inc_1, M)) \circ HC^*_H(inc_2, M)^{-1} = id_{HC^*_H(D_H, M)}$$

\[\square\]

Proposition 5.5. Let $D_H$ be a left $H$-category. Suppose that there is a semifunctor $\upsilon \in Hom_{\underline{Cat}_H}(D_H, D_H)$ and an $\eta \in \mathbb{U}_H(D_H)$ such that

1. $\upsilon(X) = X$ for all $X \in Ob(D_H)$
2. $\Phi_\eta(f \otimes E_{11} + \upsilon(f)) = \upsilon(f) \otimes E_{22}$

for all $f \in Hom_{D_H}(X, Y)$ and $X, Y \in Ob(D_H)$. Then, $HC^*_H(D_H, M) = 0$.

Proof. Let $\alpha, \alpha' \in Hom_{\underline{Cat}_H}(D_H, D_H \otimes M_2(k))$ be the semifunctors defined by

$$\alpha(X) = X \quad \alpha(f) := f \otimes E_{11} + \upsilon(f) \otimes E_{22}$$

$$\alpha'(X) := X \quad \alpha'(f) := \upsilon(f) \otimes E_{22}$$

for all $X \in Ob(D_H)$ and $f \in Hom_{D_H}(X, Y)$. Then, by assumption, $\alpha' = \Phi_\eta \circ \alpha$. Therefore, applying the functor $HC^*_H(\cdot, M)$ and using Lemma [5.3](2), we get

$$HC^*_H(\alpha', M) = HC^*_H(\alpha, M) \circ HC^*_H(\Phi_\eta, M) = HC^*_H(\alpha, M) : HC^*_H(D_H \otimes M_2(k), M) \longrightarrow HC^*_H(D_H, M) \quad (5.5)$$

Let $\phi \in Z^0_H(D_H, M)$ and $\tilde{\phi} = Hom_H(tr^M, k)(\phi) = \phi \circ tr^M \in Z^0_H(D_H \otimes M_2(k), M)$ as in Corollary [4.3]. Let $[\tilde{\phi}]$ denote the cohomology class of $\tilde{\phi}$. Then, by [5.5], we have $HC^*_H(\alpha, M)([\tilde{\phi}]) = HC^*_H(\alpha', M)([\tilde{\phi}])$, i.e.,

$$\tilde{\phi} \circ (id_M \otimes CN_n(\alpha)) + B^n_H(D_H, M) = \tilde{\phi} \circ (id_M \otimes CN_n(\alpha')) + B^n_H(D_H, M) \quad (5.6)$$

so that $\tilde{\phi} \circ (id_M \otimes CN_n(\alpha)) - \tilde{\phi} \circ (id_M \otimes CN_n(\alpha')) \in B^n_H(D_H, M)$. Applying the definition of $\tilde{\phi}$, we now have

$$\tilde{\phi} \circ (id_M \otimes CN_n(\alpha)(m \otimes f^0 \otimes \ldots \otimes f^n))$$

$$= \tilde{\phi}(m \otimes \alpha(f^0) \otimes \ldots \otimes \alpha(f^n))$$

$$= \tilde{\phi}(m \otimes (f^0 \otimes E_{11} + \upsilon(f^0) \otimes E_{22}) \otimes \ldots \otimes (f^n \otimes E_{11} + \upsilon(f^n) \otimes E_{22}))$$

$$= \tilde{\phi}(m \otimes f^0 \otimes \ldots \otimes f^n) + \tilde{\phi}(m \otimes \upsilon(f^0) \otimes \ldots \otimes \upsilon(f^n))$$

Similarly, $\tilde{\phi} \circ (id_M \otimes CN_n(\alpha'))(m \otimes f^0 \otimes \ldots \otimes f^n) = \phi(m \otimes \upsilon(f^0) \otimes \ldots \otimes \upsilon(f^n))$. Thus, $\phi = \tilde{\phi} \circ (id_M \otimes CN_n(\alpha)) - \tilde{\phi} \circ (id_M \otimes CN_n(\alpha')) \in B^n_H(D_H, M)$. This proves the result. \[\square\]
Lemma 5.9. We have

We note that the condition in Remark 5.8.

Then, we verify that the category

There exists an algebra homomorphism

Using the identification

satisfies the assumptions in Proposition 5.5.

Then, $HC^*(C) = 0$.

Remark 5.8. We note that the condition in (5.7) ensures that $\omega(1) = 1$, where $1$ is the unit element of $C$.

For any $k$-algebra $A$, we may define a $k$-linear category $A \otimes \mathcal{D}_H$ by setting $\text{Ob}(A \otimes \mathcal{D}_H) = \text{Ob}(\mathcal{D}_H)$ and $\text{Hom}_{A \otimes \mathcal{D}_H}(X, Y) = A \otimes \text{Hom}_{\mathcal{D}_H}(X, Y)$. The category $A \otimes \mathcal{D}_H$ is a left $H$-category via the action $h(a \otimes f) := a \otimes hf$ for any $h \in H$, $a \otimes f \in A \otimes \text{Hom}_{\mathcal{D}_H}(X, Y)$.

Lemma 5.9. We have $HC^*_B(C \otimes \mathcal{D}_H, M) = 0$.

Proof. We will verify that the category $C \otimes \mathcal{D}_H$ satisfies the assumptions of Proposition 5.5. Let $\omega$ and $\tilde{U}$ be as in Lemma 5.7. We now define $\nu : C \otimes \mathcal{D}_H \rightarrow C \otimes \mathcal{D}_H$ given by

for any $x \in \text{Ob}(C \otimes \mathcal{D}_H)$ and $b \otimes f \in \text{Hom}_{C \otimes \mathcal{D}_H}(X, Y)$. Since $\omega : C \rightarrow C$ is an algebra homomorphism, it follows that $\nu$ is a semifunctor. By the definition of the $H$-action on $C \otimes \mathcal{D}_H$, it is also clear that $\nu$ is $H$-linear.

Using the identification $C \otimes \mathcal{D}_H \otimes M_2(C) = M_2(C) \otimes \mathcal{D}_H$, we now define an element $\eta \in \mathbb{G}(C \otimes \mathcal{D}_H \otimes M_2(C)) = \mathbb{G}(M_2(C) \otimes \mathcal{D}_H)$ given by the family of morphisms

Since $\tilde{U}$ is a unit in $M_2(C)$, it follows that each $\eta(x)$ in (5.8) is an automorphism. Since $H$ acts trivially on $M_2(C)$, we see that $\eta \in U_H(C \otimes \mathcal{D}_H \otimes M_2(C))$. Moreover, for any $b \otimes f \in \text{Hom}_{M_2(C) \otimes \mathcal{D}_H}(X, Y) = M_2(C) \otimes \text{Hom}_{\mathcal{D}_H}(X, Y)$, we have

\[\Phi_\eta(B \otimes f) = \eta(Y) \circ (\tilde{B} \otimes f) \circ \eta(X)^{-1} = (\bigotimes \text{id}_Y) \circ (\tilde{B} \otimes f) \circ (\bigotimes \text{id}_X) = \tilde{U} \tilde{B} \tilde{U}^{-1} \otimes f = \Xi(B) \otimes f\]

Therefore, for any $b \otimes f \in C \otimes \text{Hom}_{\mathcal{D}_H}(X, Y)$, we have

$$\Phi_\eta((b \otimes f) \otimes e_{11} + \nu(b \otimes f) \otimes e_{22}) = \Phi_\eta(b \otimes f \otimes e_{11} + \omega(B) \otimes f \otimes e_{22})$$

This proves the result.

We are now ready to describe elements in the space $B_\mathcal{H}^n(D_H, M)$. 

16
Theorem 5.10. An element $\phi \in C^n_H(\mathcal{D}_H,M)$ is a coboundary iff $\phi$ is the character of an $n$-dimensional vanishing $S_H$-cycle $(S_H,\partial_H,\mathcal{T}_H,\rho)$ over $\mathcal{D}_H$.

Proof. Let $\phi$ be the character of an $n$-dimensional vanishing $S_H$-cycle $(S_H,\partial_H,\mathcal{T}_H,\rho)$. By definition, $\mathcal{T}_H$ is an $n$-dimensional closed graded $(H,M)$-trace on the $H$-semicategory $S_H$ and that $S^n_H$ is an ordinary $H$-category. We now define $\psi \in C^n_H(S^n_H,\rho)$ by setting

$$
\psi(m \otimes g^0 \otimes \ldots \otimes g^n) := \mathcal{T}_H(m \otimes g^0 \partial_H(g^1) \ldots \partial_H(g^n))
$$

for $m \in M$ and $g^0 \otimes \ldots \otimes g^n \in \text{Hom}_S(X_0,X_0) \otimes \text{Hom}_S(X_1,X_1) \otimes \ldots \otimes \text{Hom}_S(X_n,X_n)$. Then, by the implication (1) $\Rightarrow$ (3) in Theorem 5.4, we have that $\psi \in Z^n_H(S^n_H,M)$. Since $HC^n_H(S^n_H,M) = 0$, we have that $\psi = b\psi'$ for some $\psi' \in C^n_H(S^n_H,M)$.

By Lemma 5.4, the semifunctor $\rho \in \text{Hom}_{\text{Cat}_{\text{gr}}}^c(\mathcal{D}_H,S^n_H)$ induces a map $C^n_H(\rho,M) : C^n_H(S^n_H,M) \rightarrow C^n_H(\mathcal{D}_H,M)$. Setting $\psi' := C^n_H(\rho,M)(\psi')$, we have

$$
(\psi')(m \otimes p^0 \otimes \ldots \otimes p^{n-1}) = \psi'(m \otimes \rho(p^0) \otimes \ldots \otimes \rho(p^{n-1}))
$$

for any $m \in M$ and $p^0 \otimes \ldots \otimes p^{n-1} \in C^{n-1}_H(\mathcal{D}_H)$. Therefore,

$$
\phi(m \otimes f^0 \otimes \ldots \otimes f^n) = \mathcal{T}_H(m \otimes \rho(f^0) \partial_H(f^1) \ldots \partial_H(f^n)) = \psi(m \otimes (\rho(f^0) \otimes \ldots \otimes \rho(f^n)) = (b\psi')(m \otimes (\rho(f^0) \otimes \ldots \otimes \rho(f^n))
$$

for any $m \in M$ and $f^0 \otimes \ldots \otimes f^n \in \text{Hom}_{\mathcal{D}_H}(X_0,X_0) \otimes \text{Hom}_{\mathcal{D}_H}(X_1,X_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{D}_H}(X_n,X_n)$. Thus, $\phi \in B^n_H(\mathcal{D}_H,M)$.

Conversely, suppose that $\phi \in B^n_H(\mathcal{D}_H,M)$. Then, $\phi = b\psi'$ for some $\psi' \in C^n_H(\mathcal{D}_H,M)$. We now extend $\psi'$ to get an element $\psi' \in C^n_H(\mathcal{C} \otimes \mathcal{D}_H,M)$ as follows:

$$
\psi'(m \otimes (B^0 \otimes f^0) \otimes \ldots \otimes (B^{n-1} \otimes f^{n-1})) = \psi(m \otimes B^0_0 f^0 \otimes \ldots \otimes B^{n-1}_1 f^{n-1})
$$

which fixes objects and takes any morphism $f$ to $1 \otimes f$. Then, we have

$$
(C^n_H(\rho,M)(\phi'))(m \otimes f^0 \otimes \ldots \otimes f^n) = \phi'(m \otimes \rho(f^0) \otimes \ldots \otimes \rho(f^n)) = (b\psi')(m \otimes \rho(f^0) \otimes \ldots \otimes \rho(f^n)) = (b\psi')(m \otimes f^0 \otimes \ldots \otimes f^n) = \phi(m \otimes f^0 \otimes \ldots \otimes f^n)
$$

Since $\phi' \in Z^n_H(\mathcal{C} \otimes \mathcal{D}_H,M)$, the implication (3) $\Rightarrow$ (2) in Theorem 5.4 gives us a closed graded $(H,M)$-trace $\mathcal{T}_H$ of dimension $n$ on the DGH-semicategory $(\Omega(\mathcal{C} \otimes \mathcal{D}_H),\partial_H,\mathcal{T}_H,\rho)$ such that

$$
\mathcal{T}_H(m \otimes \rho(f^0) \partial_H(f^1) \ldots \partial_H(f^n)) = \phi'(m \otimes \rho(f^0) \otimes \ldots \otimes \rho(f^n)) = \phi(m \otimes f^0 \otimes \ldots \otimes f^n)
$$

for some $m \in M$. Then, by the characterization of the character $\phi$ as the $n$-dimensional closed graded $(H,M)$-trace on the $H$-semicategory $S_H$ and that $S^n_H$ is an ordinary $H$-category, we find that $\phi$ is the character associated to the cycle $\mathcal{T}_H(m \otimes \rho(f^0) \otimes \ldots \otimes \rho(f^n))$.

Hence, $\mathcal{T}_H$ is a vanishing cycle over $\mathcal{D}_H$. From this, the result follows. $\square$

For the remaining part of this section, we shall suppose that $H$ is cocommutative. If $\mathcal{D}_H, \mathcal{D}'_H$ are left $H$-categories, we observe that $\mathcal{D}_H \otimes \mathcal{D}'_H$ then becomes a left $H$-category under the diagonal action of $H$.

Let $M, M'$ be left $H$-comodules equipped respectively with coactions $\rho_M : H \otimes M$ and $\rho_{M'} : H \otimes M'$. Since $H$ is cocommutative, $M$ may be treated as a right $H$-comodule and we can form the cotensor product $M \Box H M'$ defined by the kernel

$$
M \Box H M' := \text{Ker}(M \otimes M' \xrightarrow{\rho_M \otimes id_{M'} - id_M \otimes \rho_{M'}} M \otimes H \otimes M')
$$

in $\text{Vect}_k$. It follows by Proposition 7.2.2] that the map $\rho_M \otimes id_{M'}$, gives $M \Box H M'$ a left $H$-comodule structure. We also note that $M \otimes M'$ carries a right $H$-module structure via the diagonal action.
Lemma 5.11. Let $H$ be a cocommutative Hopf algebra and $M, M'$ be right-left SAYD modules over $H$ such that $M \square_H M'$ is a right $H$-submodule of $M \otimes M'$. Then, $M \square_H M'$ is also an SAYD module over $H$.

Proof. For any $m \otimes m' \in M \square_H M'$, we have

$$(m \otimes m')h_{(-1)} \otimes ((m \otimes m')h_{(0)} = (mh_1 \otimes m'h_2)_{(-1)} \otimes (mh_1 \otimes m'h_2)_{(0)} = (mh_1)_{(-1)} \otimes (mh_1)_{(0)} \otimes m'h_2$$

On the other hand, we have

$$S(h_3)(m \otimes m')_{(-1)}h_1 \otimes (m \otimes m')_{(0)}h_2 = S(h_3)m_{(-1)}h_1 \otimes m_{(0)}h_2 \circ m'h_4$$

Since $H$ is cocommutative, we see that the two expressions are the same. This proves that $M \square_H M'$ is an anti-Yetter-Drinfeld module. We now check that it is also stable. Using the cocommutativity of $H$ and the stability of $M, M'$, we have

$$(m \otimes m')_{(0)}(m \otimes m')_{(-1)} = m_0m_1 \otimes m_2 = m_0m_1 \otimes m'm_1 = m \otimes m'$$

for any $m \otimes m' \in M \square_H M'$.

Let $(S_H, \hat{\partial}_H)$ and $(S'_H, \hat{\partial}'_H)$ be DGH-semicategories. Then, their tensor product $S_H \otimes S'_H$ is the DGH-semicategory defined by setting $Ob(S_H \otimes S'_H) = Ob(S_H) \times Ob(S'_H)$ and

$$Hom_{S_H \otimes S'_H}^0((X,X'), (Y,Y')) = \bigoplus_{i+j=n} Hom^i_{S_H}(X,Y) \otimes_k Hom^j_{S'_H}(X',Y')$$

The composition in $S_H \otimes S'_H$ is given by the rule:

$$(g \otimes g') \circ (f \otimes f') = (-1)^{deg(g')deg(f)}(gf \otimes g'f')$$

for homogeneous $f : X \to Y, g : Y \to Z$ in $S_H$ and $f' : X' \to Y', g' : Y' \to Z'$ in $S'_H$. The differential $(\partial_H \otimes \partial'_H)^n : Hom_{S_H \otimes S'_H}^n((X,X'), (Y,Y')) \to Hom_{S_H \otimes S'_H}^{n+1}((X,X'), (Y,Y'))$ is determined by

$$(\partial_H \otimes \partial'_H)^n(f_i \otimes g_j) = \partial_H(f_i) \otimes g_j + (-1)^i f_i \otimes \partial'_H(g_j)$$

for any $f_i \in Hom_{S_H}^i(X,Y)$ and $g_j \in Hom_{S'_H}^j(X',Y')$ such that $i + j = n$. Clearly, $(S_H \otimes S'_H)^0 = S_H^0 \otimes S'_H^0$.

Theorem 5.12. Let $H$ be a cocommutative Hopf algebra and $M, M'$ be right-left SAYD modules over $H$ such that $M \square_H M'$ is a right $H$-submodule of $M \otimes M'$. Let $D_H, D'_H$ be left $H$-categories. Then, we have a pairing

$$HC^p_H(D_H, M) \otimes HC^q_H(D'_H, M') \to HC^{p+q}_H(D_H \otimes D'_H, M \square_H M')$$

for $p, q \geq 0$.

Proof. Let $\phi \in Z^p_H(D_H, M)$ and $\phi' \in Z^q_H(D'_H, M)$. We may express $\phi$ and $\phi'$ respectively as the characters of $p$- and $q$-dimensional cycles $(S_H, \hat{\partial}_H, \hat{\mathcal{F}}^H, \rho)$ and $(S'_H, \hat{\partial}'_H, \hat{\mathcal{F}}'^H, \rho')$ over $D_H$ and $D'_H$ with coefficients in $M$ and $M'$ respectively. We now consider the collection $\hat{\mathcal{F}}^H \# \hat{\mathcal{F}}'^H = \{(\hat{\mathcal{F}}^H \# \hat{\mathcal{F}}'^H)_{(X,X')} : M \square_H M' \to Hom_{S_H \otimes S'_H}^{p+q}((X,X'),(X',X')) \to C)_{(X,X')} \otimes Ob(S_H \otimes S'_H) \}$ of $\mathbb{C}$-linear maps defined by

$$(\hat{\mathcal{F}}^H \# \hat{\mathcal{F}}'^H)_{(X,X')} (m \otimes m' \otimes f \otimes f') = \hat{\mathcal{F}}^H_X (m \otimes f_p) \hat{\mathcal{F}}'^H_X (m' \otimes f'_q)$$

for any $m \otimes m' \in M \square_H M'$ and $f \otimes f' = (f_i \otimes f'_j)_{i+j=p+q} \in Hom_{S_H \otimes S'_H}^{p+q}((X,X'),(X',X'))$. We will now prove that $\hat{\mathcal{F}}^H \# \hat{\mathcal{F}}'^H$ is a $(p + q)$-dimensional closed graded trace on the DGH-semicategory $S_H \otimes S'_H$ with coefficients.
Thus, we see that the category $\mathcal{S}_{\hat{H}}$ may also be easily verified that
\[(\hat{H} \# \hat{\mathcal{H}})(X, X') \left( m \otimes m' \otimes (\hat{H} \otimes \hat{\mathcal{H}})^{p+q-1} (g \otimes g') \right) = \sum_{i+j=p+q-1} \left( \hat{H} \# \hat{\mathcal{H}} \right)(X, X') \left( m \otimes m' \otimes \hat{H}(g_i) \otimes g_j + (-1)^i m \otimes m' \otimes g_i \otimes \hat{\mathcal{H}}(g_j) \right) \]
\[= \hat{\mathcal{H}}_X^\delta (m \otimes \hat{H}(g_{p-1})) \hat{\mathcal{H}}_X^\delta (m' \otimes g_q') + (-1)^p \hat{\mathcal{H}}_X^\delta (m \otimes g_q) \hat{\mathcal{H}}_X^\delta (m' \otimes \hat{H}(g_{q-1})) = 0 \]
This proves the condition in [5.3]. Next for any homogeneous $f : X \to Y$, $g : Y \to X$ in $\mathcal{S}_H$ and $f' : X' \to Y'$, $g' : Y' \to X'$ in $\mathcal{S}_H$, we have
\[(\hat{\mathcal{H}} \# \hat{\mathcal{H}})(X, X') \left( m \otimes m' \otimes (g \otimes g') (f \otimes f') \right) = (-1)^{\deg(g') \deg(f)} (\hat{\mathcal{H}} \# \hat{\mathcal{H}})(X, X') \left( m \otimes m' \otimes g(f \otimes g') \right) \]
\[= (-1)^{\deg(g') \deg(f)} (\hat{\mathcal{H}} \# \hat{\mathcal{H}})(X, X') \left( m \otimes m' \otimes (f \otimes g')(g \otimes g') \right) \]
This proves the condition in [5.4]. We may similarly verify the condition in [5.7]. Thus, we get a $(p+q)$-dimensional cycle $(\mathcal{S}_H \otimes \mathcal{S}_H', \hat{H} \otimes \hat{\mathcal{H}}, \rho \otimes \rho')$ with coefficients in $M \sqcup M'$ over the category $\mathbb{D}_H \otimes \mathbb{D}_H'$. Then, the character of this cycle, denoted by $\phi \# \# \phi' \in Z^{p+q}_H(\mathbb{D}_H \otimes \mathbb{D}_H', M \sqcup M')$, gives a well defined map $\gamma : Z^p_H(\mathbb{D}_H, \mathcal{S}_H) \to Z^q_H(\mathbb{D}_H', \mathcal{S}_H)$.

We now verify that the map $\gamma$ restricts to a pairing
\[B^p_H(\mathcal{D}_H) \otimes Z^q_H(\mathcal{D}_H', \mathcal{S}_H') \to B^{p+q}_{\hat{H}}(\mathcal{D}_H \otimes \mathcal{D}_H', \mathcal{S}_H \otimes \mathcal{S}_H') \]
For this, we let $\phi \in Z^p_H(\mathbb{D}_H, \mathcal{S}_H)$ be the character of a $p$-dimensional vanishing cycle $(\mathcal{S}_H, \hat{H}, \mathcal{F}_H, \rho)$ over $\mathbb{D}_H$. In particular, it follows from Definition 5.6 that $\mathcal{S}_H^0$ is an ordinary left $H$-category. From the implication (1) $\Rightarrow$ (2) in Theorem 5.3, it follows that we might as well take $\mathcal{S}_H^0$ to be an ordinary left $H$-category. In fact, we could assume that $\mathcal{S}_H = \emptyset \mathbb{D}_H$. Then, $\mathcal{S}_H \otimes \mathcal{S}_H$ is an ordinary left $H$-category. It suffices to show that the tuple $(S_H \otimes S_H', \hat{H} \otimes \hat{\mathcal{H}}, \mathcal{F}_H \# \hat{\mathcal{H}}, \rho \otimes \rho')$ is a vanishing cycle.

Since $(\mathcal{S}_H, \hat{H}, \mathcal{F}_H)$ is a vanishing cycle, we have an $H$-linear semifunctor $v : S_H^0 \to S_H^0$ and an $\eta \in \mathbb{U}(S_H^0 \otimes H_2(\mathcal{C}))$ satisfying the conditions in Proposition 5.5. Extending $v$, we get the the $H$-linear semifunctor $v \otimes id : S_H^0 \otimes S_H^0 \to S_H^0 \otimes S_H^0$. Identifying, $S_H^0 \otimes S_H^0 \otimes H_2(\mathcal{C}) \cong S_H^0 \otimes H_2(\mathcal{C}) \otimes S_H^0$, we obtain $\eta \in \mathbb{U}(S_H^0 \otimes H_2(\mathcal{C}) \otimes S_H^0)$ given by
\[\{\hat{\eta}(X, X') = \eta(X) \otimes id_{X'}, \in Hom_{S_H^0 \otimes H_2(\mathcal{C}) \otimes S_H^0}((X, X'), (X, X')) = Hom_{S_H^0 \otimes H_2(\mathcal{C})}(X, X) \otimes Hom_{S_H^0}(X', X') \}
\]
It may also be easily verified that
\[\Phi_{\hat{\eta}}(f \otimes f' \otimes E_{11} + (v \otimes id)(f \otimes f') \otimes E_{22}) = (v \otimes id)(f \otimes f') \otimes E_{22} \]
Thus, we see that the category $(\mathcal{S}_H \otimes \mathcal{S}_H^0)^0 = S_H^0 \otimes S_H^0$ satisfies the conditions in Proposition 5.5. Therefore, the tuple $(\mathcal{S}_H \otimes \mathcal{S}_H', \hat{H} \otimes \hat{\mathcal{H}}, \mathcal{F}_H \# \hat{\mathcal{H}}, \rho \otimes \rho')$ is a vanishing cycle. This proves the result. $

\section{Characters of Fredholm modules over categories}

In the rest of this paper, we will study Fredholm modules and Chern characters. We fix a small $\mathcal{C}$-linear category $\mathcal{C}$. Our categorified Fredholm modules will consist of functors from $\mathcal{C}$ taking values in separable Hilbert spaces. Let $S\tilde{H}ilb$ be the category whose objects are separable Hilbert spaces and whose morphisms are bounded linear maps.
Given separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, let $B(\mathcal{H}_1, \mathcal{H}_2)$ denote the space of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $B^*(\mathcal{H}_1, \mathcal{H}_2) \subseteq B(\mathcal{H}_1, \mathcal{H}_2)$ be the space of all compact operators. For any bounded operator $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, let $\mu_n(T)$ denote the $n$-th singular value of $T$. In other words, $\mu_n(T)$ is the $n$-th (arranged in decreasing order) eigenvalue of the positive operator $|T| := (T^* T)^{\frac{1}{2}}$. For $1 \leq p < \infty$, the $p$-th Schatten class is defined to be the space

$$B^p(\mathcal{H}_1, \mathcal{H}_2) := \{ T \in B(\mathcal{H}_1, \mathcal{H}_2) \mid \sum \mu_n(T)^p < \infty \}$$

Clearly, $B^p(\mathcal{H}_1, \mathcal{H}_2) \subseteq B^q(\mathcal{H}_1, \mathcal{H}_2)$ for $p \leq q$. For $p = 1$, the space $B^1(\mathcal{H}_1, \mathcal{H}_2)$ is the collection of all trace class operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $T \in B^1(\mathcal{H}_1, \mathcal{H}_2)$, we write $Tr(T) := \sum \mu_n(T)$. It is well known that

$$Tr(T_1 T_2) = Tr(T_2 T_1) \quad \forall T_1 \in B^{n_1}(\mathcal{H}, \mathcal{H}'), \, T_2 \in B^{n_2}(\mathcal{H}', \mathcal{H}), \frac{1}{n_1} + \frac{1}{n_2} = 1$$

(6.1)

We note that $B^p(\mathcal{H}_1, \mathcal{H}_2)$ is an “ideal” in the following sense: consider the functor

$$B(\mathcal{H}, \mathcal{H}) = \text{Hom}_{\mathcal{H}_1^* \mathcal{H}_1}(\mathcal{H}_1, \mathcal{H}_2) \to B(\mathcal{H}_1, \mathcal{H}_2)$$

$$B(-,-)(\mathcal{H}_1, \mathcal{H}_2) := B(\mathcal{H}_1, \mathcal{H}_2)$$

where $\mathcal{H} \to \mathcal{H}_1$ be a functor and $\mathcal{G} := \{ G_X : \mathcal{H}(X) \to \mathcal{H}(X) \}_{X \in \text{Ob}(\mathcal{C})}$ be a collection of bounded linear operators. Then, we set

$$[G,-] : B(\mathcal{H}(X), \mathcal{H}(Y)) \to B(\mathcal{H}(X), \mathcal{H}(Y))$$

$$[G,T] := G_Y \circ T - T \circ G_X \in B(\mathcal{H}(X), \mathcal{H}(Y))$$

(6.2)

for each $X, Y \in \mathcal{C}$.

**Definition 6.1.** Let $\mathcal{C}$ be a small $\mathcal{C}$-category and let $p \in [1, \infty)$. We consider a pair $(\mathcal{H}, \mathcal{F})$ as follows.

1. A functor $\mathcal{H} : \mathcal{C} \to SHilb_{\mathbb{C}}$ such that $\mathcal{H}(f) : \mathcal{H}(X) \to \mathcal{H}(Y)$ is a linear operator of degree 0 for each $f \in \text{Hom}_{\mathcal{C}}(X,Y)$.

2. A collection $\mathcal{F} := \{ F_X : \mathcal{H}(X) \to \mathcal{H}(X) \}_{X \in \text{Ob}(\mathcal{C})}$ of bounded linear operators of degree 1 such that $\mathcal{F}_X^2 = \text{id}_{\mathcal{H}(X)}$ for each $X \in \text{Ob}(\mathcal{C})$.

The pair $(\mathcal{H}, \mathcal{F})$ is said to be a $p$-summable even Fredholm module over the category $\mathcal{C}$ if every $f \in \text{Hom}_{\mathcal{C}}(X,Y)$ satisfies

$$[\mathcal{F}, f] := (\mathcal{F}_Y \circ \mathcal{H}(f) - \mathcal{H}(f) \circ \mathcal{F}_X) \in B^p(\mathcal{H}(X), \mathcal{H}(Y))$$

(6.3)

Taking $H = \mathbb{C} = M$ in Definition 6.6, we note that a closed graded trace of dimension $n$ on a DG-semicategory $(\mathcal{S}, \partial)$ is a collection of $\mathbb{C}$-linear maps $\tilde{T} : \{ T_X : \text{Hom}_S^n(X,X) \to \mathbb{C} \}_{X \in \text{Ob}(\mathcal{S})}$ satisfying the following two conditions

$$\tilde{T}_X(\partial^n f) = 0 \quad \tilde{T}_X(g g') = (-1)^{ij} \tilde{T}_Y(g' g)$$

(6.4)

for all $f \in \text{Hom}_S^{n-1}(X), \, g \in \text{Hom}_S^j(Y,X), \, g' \in \text{Hom}_S^i(X,Y)$ and $i + j = n$. Accordingly, we will consider cycles $(\mathcal{S}, \partial, \tilde{T}, \rho)$ over $\mathcal{C}$ by setting $H = \mathbb{C} = M$ in Definition 6.7.
Lemma 6.2. Let \((\mathcal{H}, \mathcal{F})\) be a pair that satisfies conditions (1) and (2) in Definition 6.1. We define a graded-semicategory \(\mathcal{V}' \mathcal{C} = \mathcal{V}'\mathcal{C}_X \mathcal{V}'\mathcal{C}_Y\) as follows: we put \(\text{Ob}(\mathcal{V}' \mathcal{C}) := \text{Ob}(\mathcal{C})\) and for any \(X, Y \in \mathcal{C}, \ j \geq 0\), we set \(\text{Hom}^j_{\mathcal{V}' \mathcal{C}}(X, Y)\) to be the linear span in \(\mathcal{B}(\mathcal{H}(X), \mathcal{H}(Y))\) of the operators
\[
\mathcal{H}(\tilde{f}^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^j]
\] (6.5)
where \(\tilde{f}^0 \otimes f^1 \otimes \ldots \otimes f^j\) is a homogeneous element of degree \(j\) in \(\text{Hom}_{\mathcal{V}' \mathcal{C}}(X, Y)\). Here, we write \(\mathcal{H}(\tilde{f}^0) = \mathcal{H}(f^0) + \mu \cdot \text{id}\), where \(f^0 = f^0 + \mu\). Using the fact that
\[
[f, f'] \mathcal{H}(f') = [f, f'\circ f'] - \mathcal{H}(f)[f, f']
\]
for composable morphisms \(f, f'\) in \(\mathcal{V}'\), we observe that \(\mathcal{V}' \mathcal{C}\) is closed under composition. We set
\[
\partial' := [\mathcal{F}, -]\mathcal{B}(\mathcal{H}(X), \mathcal{H}(Y)) \rightarrow \mathcal{B}(\mathcal{H}(X), \mathcal{H}(Y))
\]
\[
\partial'T = [\mathcal{F}, T] = \mathcal{F}_Y \circ T - (-1)^{|T|} T \circ \mathcal{F}_X
\]
We now have the following Lemma.

Lemma 6.2. Let \((\mathcal{H}, \mathcal{F})\) be a pair that satisfies conditions (1) and (2) in Definition 6.1. Then,
(a) \((\mathcal{V}' \mathcal{C}, \partial')\) is a DG-semicategory and \(\mathcal{V}' \mathcal{C}\) is an ordinary category.
(b) There is a canonical semifunctor \(\hat{\rho} : \mathcal{V}' \mathcal{C} \rightarrow \mathcal{V}' \mathcal{C}\) which is identity on objects and takes any \(f \in \text{Hom}_{\mathcal{C}}(X, Y)\) to \(\mathcal{H}(f) \in \mathcal{B}(\mathcal{H}(X), \mathcal{H}(Y))\). This extends to a unique DG-semifunctor \(\hat{\rho} : (\mathcal{V}' \mathcal{C}, \partial') \rightarrow (\mathcal{V}' \mathcal{C}, \partial')\) such that the restriction of \(\hat{\rho}'\) to \(\mathcal{C}\) is identical to \(\rho'\).
(c) Suppose that \((\mathcal{H}, \mathcal{F})\) is a p-summable Fredholm module. Choose \(n \geq p - 1\). Then, for \(X, Y \in \text{Ob}(\mathcal{C})\) and \(k \geq 0\), we have \(\text{Hom}^k_{\mathcal{V}' \mathcal{C}}(X, Y) \subseteq B^{(n+1)k}(\mathcal{H}(X), \mathcal{H}(Y))\).

Proof. (a) Since each \(\mathcal{F}_X\) is a degree 1 operator and \(\mathcal{F}_Y[f, f] = -[\mathcal{F}, f] \mathcal{F}_X\) for any \(f \in \text{Hom}_{\mathcal{C}}(X, Y)\), we have \(\partial'(\text{Hom}^j_{\mathcal{V}' \mathcal{C}}(X, Y)) \subseteq \text{Hom}^{j+1}_{\mathcal{V}' \mathcal{C}}(X, Y)\). We now check that \(\partial'^2 = 0\). For any homogeneous element \(\mathcal{H}(\tilde{f}^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^j]\) of degree \(j\), we have
\[
\partial'^2(\mathcal{H}(\tilde{f}^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^j]) = \partial'(\partial' \mathcal{H}(\tilde{f}^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^j])
\]
\[
= \partial'(\mathcal{F}_Y \circ \mathcal{H}(\tilde{f}^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^j])
\]
\[
= \mathcal{F}_Y \circ \mathcal{H}(\tilde{f}^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^j]
\]
\[
= \mathcal{F}_Y \circ \mathcal{H}(\tilde{f}^0)[\mathcal{F}, f^1][\mathcal{F}, f^2] \cdots [\mathcal{F}, f^j]
\]
\[
= 0
\]
The fact that \(\partial'\) is compatible with composition follows by direct computation. It is also easy to see that \(\mathcal{V}' \mathcal{C}\) is an ordinary category.

(b) This is immediate using the universal property in Proposition 3.3.

(c) This is a consequence of Hölder's inequality and the condition \((0, 3)\) in Definition 6.1.

For any \(Z_2\)-graded Hilbert space \(\mathcal{H}\), the grading operator on it will be denoted by \(\epsilon_{\mathcal{H}}\) or simply \(\epsilon\). For any \(T \in \mathcal{B}(\mathcal{H}(X), \mathcal{H}(Y))\) such that \(\mathcal{F}, T\) \(\in B^1(\mathcal{H}(X), \mathcal{H}(Y))\), we define
\[
\text{Tr}_s(T) := \frac{1}{2} \text{Tr}(\epsilon \mathcal{F}_Y[\mathcal{F}, T]) - \frac{1}{2} \text{Tr}(\epsilon \mathcal{F}_Y \partial'(T)) = \frac{1}{2} \text{Tr}(\epsilon \mathcal{F}_Y(\mathcal{F}_Y \circ T - (-1)^{|T|} T \circ \mathcal{F}_X))
\]

Proposition 6.3. Let \((\mathcal{H}, \mathcal{F})\) be a p-summable Fredholm module over \(\mathcal{C}\). Take \(2m \geq p - 1\). Then, the collection
\[
\hat{T}_s = \{\text{Tr}_s : \text{Hom}^m_{\mathcal{V}' \mathcal{C}}(X, X) \rightarrow \mathbb{C}\}_{X \in \text{Ob}(\mathcal{C})}
\]
defines a closed graded trace of dimension \(2m\) on \((\mathcal{V}' \mathcal{C}, \partial')\).
Proof. From Lemma 6.2(a), it is clear that for any \( T \in \text{Hom}_{\text{odd}}^2(X, Y) \), we have \([F, T] \in \text{Hom}_{\text{odd}}^{m+1}(X, X)\). Applying Lemma 6.2(c), it follows that \([F, T] \in B^{1}(\mathcal{H}(X), \mathcal{H}(X))\). Accordingly, each of the maps \( T_{n} : \text{Hom}_{\text{odd}}^{m}(X, X) \rightarrow \mathbb{C} \) is well-defined.

For \( T' \in \text{Hom}_{\text{odd}}^{m-1}(X, X) \), we notice that

\[
T_{n}(\partial^2 T') = \frac{1}{2} \text{Tr} (\epsilon F_{X}(\partial^2 T')) = 0
\]

We now consider \( T_{1} \in \text{Hom}_{\text{odd}}^{m}(X, Y) \), \( T_{2} \in \text{Hom}_{\text{odd}}^{m-1}(Y, X) \) such that \( i + j = 2m \). We notice that

\[
\epsilon F_{Y}(\partial' (T_{1})) = \partial' (T_{1}) \epsilon F_{X} \quad \epsilon F_{X}(\partial' (T_{2})) = \partial' (T_{2}) \epsilon F_{Y}
\]

(6.7)

We note that \( i \equiv j (\mod 2) \). Using (6.7) and (6.1), we now have

\[
2 \cdot T_{n}(T_{1}T_{2}) = \text{Tr} (\epsilon F_{Y}(\partial'(T_{1})T_{2})) = \text{Tr} (\epsilon F_{Y} \partial'(T_{1})T_{2}) + (-1)^{i} \text{Tr} (\epsilon F_{Y} T_{1} \partial'(T_{2}))
\]

\[
= \text{Tr} (\partial'(T_{1}) \epsilon F_{X}T_{2}) + (-1)^{i} \text{Tr} (\partial'(T_{2}) \epsilon F_{Y}T_{1})
\]

\[
= \text{Tr} (\epsilon F_{X}T_{2} \partial'(T_{1})) + (-1)^{i} \text{Tr} (\epsilon F_{X} \partial'(T_{2})T_{1})
\]

\[
= \text{Tr} (\epsilon F_{X}T_{2} \partial'(T_{1})) + (-1)^{i+1} \text{Tr} (\epsilon F_{X} \partial'(T_{2})T_{1})
\]

\[
= (-1)^{i+1} \cdot 2 \cdot T_{n}(T_{1}T_{2})
\]

Theorem 6.4.

Let \((\mathcal{H}, F)\) be a \(p\)-summable Fredholm module over \(\mathcal{C}\). Take \(2m \geq p - 1\). Then, the tuple \((\Omega\mathcal{C}, \partial', \hat{T}_{n}, \rho')\) defines a \(2m\)-dimensional cycle over \(\mathcal{C}\). Then, \(\phi^{2m} \in C^{2m}(\mathcal{C}) = C_{\mathcal{C}}^{2m}(\mathcal{C}, \mathcal{C}) = \text{Hom}(CN_{2m}(\mathcal{C}), \mathcal{C})\) defined by

\[
\phi^{2m}(f^{0} \otimes f^{1} \otimes \ldots \otimes f^{2m}) \equiv \text{Tr}_{n}(\mathcal{H}(f^{0})[\mathcal{F}, f^{1}][\mathcal{F}, f^{2}] \ldots [\mathcal{F}, f^{2m}])
\]

for any \(f^{0} \otimes f^{1} \otimes \ldots \otimes f^{2m} \in \text{Hom}_{\mathcal{C}}(X_{1}, X) \otimes \text{Hom}_{\mathcal{C}}(X_{2}, X_{1}) \otimes \ldots \otimes \text{Hom}_{\mathcal{C}}(X, X_{2m})\) is a cyclic cocycle over \(\mathcal{C}\).

Proof. It follows directly from Lemma 6.2 and Proposition 6.3 that \((\Omega\mathcal{C}, \partial', \hat{T}_{n}, \rho')\) is a \(2m\)-dimensional cycle over \(\mathcal{C}\). The rest follows by applying Theorem 3.8 with \(H = \mathbb{C} = M\).

We will refer to \(\phi^{2m}\) as the \(2m\)-dimensional character associated with the \(p\)-summable even Fredholm module \((\mathcal{H}, F)\) over the category \(\mathcal{C}\).

Remark 6.5. The appearance of only even cyclic cocycles in Theorem 6.4 is due to the following fact from [12, Lemma 2 a)]: if \(T \in B(\mathcal{H}(X), \mathcal{H}(X))\) is homogeneous of odd degree, then \(T_{n}(T) = 0\).

7 Periodicity character for Fredholm modules

We continue with \(\mathcal{C}\) being a small \(\mathcal{C}\)-category. Taking \(H = \mathbb{C} = M\), we denote the cyclic cohomology groups of \(\mathcal{C}\) by \(H^{*}_{\mathcal{C}}(\mathcal{C}) := HC^{*}_{\mathcal{C}}(\mathcal{C}, \mathbb{C})\). The cyclic complex corresponding to the cocyclic module \(\{CN^{n}(\mathcal{C}) = \text{Hom}_{\mathbb{C}}(CN_{n}(\mathcal{C}), \mathbb{C})\}_{n \geq 0}\) as in [24] will be denoted by \(C^{*}_{\mathcal{C}}(\mathcal{C})\). The cocycles of this complex will be denoted by \(Z^{*}_{\mathcal{C}}(\mathcal{C}) := Z^{*}_{\mathcal{C}}(\mathcal{C}, \mathbb{C})\) and the coboundaries by \(B^{*}_{\mathcal{C}}(\mathcal{C}) := B^{*}_{\mathcal{C}}(\mathcal{C}, \mathbb{C})\).

Let \((\mathcal{H}, F)\) be a \(p\)-summable Fredholm module over \(\mathcal{C}\). We take \(2m \geq p - 1\). Let \(\phi^{2m}\) be the \(2m\)-dimensional character associated to the Fredholm module \((\mathcal{H}, F)\). We denote by \(ch^{2m}(\mathcal{H}, F) \in H^{2m}_{\mathcal{C}}(\mathcal{C})\) the cohomology class of \(\phi^{2m}\). Since \(B^{p}(\mathcal{H}(X), \mathcal{H}(Y)) \subseteq B^{p}(\mathcal{H}(X), \mathcal{H}(Y))\) for any \(p \geq q\), the Fredholm module \((\mathcal{H}, F)\) is also \((p+2)\)-summable. Using Theorem 6.4 we then have the \((2m+2)\)-dimensional character \(\phi^{2m+2}\) associated to \((\mathcal{H}, F)\). We will show that the cyclic cocycles \(\phi^{2m}\) and \(\phi^{2m+2}\) are related to each other via the periodicity operator.


If \( \mathcal{C} \) and \( \mathcal{C}' \) are small \( \mathcal{C} \)-categories, from the proof of Theorem 5.12 it follows that there is a pairing on cyclic cocycles

\[
Z^r_{\lambda}(\mathcal{C}) \otimes Z^r_{\lambda}(\mathcal{C}') \longrightarrow Z^{r+s}_{\lambda}(\mathcal{C} \otimes \mathcal{C}')
\]

which descends to a pairing on cyclic cohomologies:

\[
H^r_{\lambda}(\mathcal{C}) \otimes H^r_{\lambda}(\mathcal{C}') \longrightarrow H^{r+s}_{\lambda}(\mathcal{C} \otimes \mathcal{C}')
\]

given by

\[
(\hat{T}_X \# \hat{T}_{X'})(f \otimes f') := \hat{T}_X(f) \hat{T}_{X'}(f')
\]

for any \( f \otimes f' = \sum_{i,r,s} (f_i \otimes f'_j) \in Hom_{\mathcal{S} \otimes \mathcal{S}'}^{r+s} ((X, X'), (X, X')) \). Here \( \phi \) and \( \phi' \) are expressed respectively as the characters of \( r \) and \( s \)-dimensional cycles \( (S, \hat{\delta}, \hat{T}_\phi, \rho) \) and \( (S', \hat{\delta}', \hat{T}_{\phi'}, \rho') \) over \( \mathcal{C} \) and \( \mathcal{C}' \). In particular, \( \phi \# \phi' \) is the character of the \((r + s)\)-dimensional cycle \( (S \otimes S', \hat{\delta} \otimes \hat{\delta}', \hat{T}_\phi \# \hat{T}_{\phi'}, \rho \otimes \rho') \) over \( \mathcal{C} \otimes \mathcal{C}' \). For a morphism \( f \) in \( \mathcal{C} \), we will often suppress the functor \( \rho \) and write the morphism \( \rho(f) \) in \( \mathcal{S} \) simply as \( f \). Similarly, when there is no danger of confusion, we will often write the morphism \( \mathcal{H}(f) \) simply as \( f \).

Now setting \( \mathcal{C}' = \mathcal{C} \) (the category with one object) and considering the cyclic cocycle \( \psi \in H^r_{\lambda}(\mathcal{C}) \) determined by \( \psi(1,1,1) = 1 \), we obtain the periodicity operator:

\[
S : Z^r_{\lambda}(\mathcal{C}) \longrightarrow Z^{r+2}_{\lambda}(\mathcal{C}) \quad S(\phi) := \phi \# \psi
\]

for any \( r \geq 0 \) and \( \phi \in Z^r_{\lambda}(\mathcal{C}) \).

**Lemma 7.1.** Let \( \psi \in Z^r_{\lambda}(\mathcal{C}) \). For any \( f^0 \otimes f^1 \otimes \ldots \otimes f^{r+2} \in C_{N+2}(\mathcal{C}) \), we have

\[
(S(\phi))(f^0 \otimes f^1 \otimes \ldots \otimes f^{r+2}) = (S(\phi))(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
= (T^\phi \# T^\psi)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
= (T^\phi \# T^\psi)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
+ T^\phi(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
+ T^\phi(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
+ T^\phi(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
= (T^\phi \# T^\psi)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
+ T^\phi(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
+ T^\phi(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
+ T^\phi(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

\[
+ T^\phi(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

The last equality follows by using the fact that \( T^\psi((1d)^2) = \psi(1,1,1,1) = 1 \).

**Proposition 7.2.** Let \( \psi \) be the character of an \( r \)-dimensional cycle \( (S, \hat{\delta}, \hat{T}_\phi, \rho) \) over \( \mathcal{C} \). Then, \( S(\phi) \) is a coboundary. In particular, we have \( S(\phi) = b\psi \), where \( \psi \in C_{N+2}(\mathcal{C}) \) is given by

\[
\psi(f^0 \otimes f^1 \otimes \ldots \otimes f^{r+1}) = \sum_{j=1}^{r+1} (-1)^{j-1} \hat{T}_\phi(f^0 \otimes f^1 \ldots \otimes f^{j-1} \hat{\delta} f^{j+1} \ldots \otimes f^{r+1})
\]

\[23\]
Proof. Again, we illustrate the case of \( r = 2 \). The general computation is similar.

\[
(b\psi)(f^0 \otimes f^1 \otimes f^2 \otimes f^{3^4}) = \psi(f^0 \otimes f^1 \otimes f^2 \otimes f^3) + \psi(f^0 \otimes f^1 \otimes f^2 \otimes f^4) + \psi(f^0 \otimes f^1 \otimes f^3 \otimes f^4) + \psi(f^0 \otimes f^1 \otimes f^4)
\]

\[
+ (f^1 \otimes f^2 \otimes f^3 \otimes f^4) + (f^1 \otimes f^2 \otimes f^4) + (f^1 \otimes f^3 \otimes f^4) + (f^1 \otimes f^4)
\]

\[
= \sum_{j=0}^{2m-1} Tr_s\left( f^0[f^1, f^2, f^3, f^4] + f^1[f^0, f^2, f^3, f^4] + f^2[f^0, f^1, f^3, f^4] + f^3[f^0, f^1, f^2, f^4] + f^4[f^0, f^1, f^2, f^3] \right)
\]

\[
= (S(\psi))(f^0 \otimes f^1 \otimes f^2 \otimes f^3 \otimes f^4)
\]

Theorem 7.3. Let \( \mathcal{C} \) be a small \( \mathcal{C} \)-category and let \( (\mathcal{H}, \mathcal{F}) \) be a \( p \)-summable even Fredholm module over \( \mathcal{C} \). Take \( 2m \geq p - 1 \). Then,

\[
S(\phi^{2m}) = -(m+1)\phi^{2m+2} \quad \text{in } H^{2m+2}_\lambda(\mathcal{C})
\]

Proof. We will show that \( S(\phi^{2m}) + (m+1)\phi^{2m+2} = b\psi \) for some \( \psi \in Z^{2m+1}_\lambda(\mathcal{C}) \). By Theorem 7.1, we know that \( \phi^{2m} \) is the character of the \( 2m \)-dimensional cycle \((\Omega^\lambda, \partial^\lambda, Tr_s, \rho^\lambda)\) over the category \( \mathcal{C} \). Applying Lemma 7.1 and using the fact that \( Tr_s(T) = 0 \) for any homogeneous \( T \) of odd degree, we have

\[
(S(\phi^{2m}))(f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+2}) = \sum_{j=0}^{2m+1} Tr_s\left( f^0[f^1, f^2, f^3, f^4] + f^1[f^0, f^2, f^3, f^4] + f^2[f^0, f^1, f^3, f^4] + f^3[f^0, f^1, f^2, f^4]ight.
\]

\[
+ f^4[f^0, f^1, f^2, f^3] \right)
\]

We now consider \( \psi = \sum_{j=0}^{2m+1} (-1)^j \psi^j \), where

\[
\psi^j(f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+1}) = Tr\left( \epsilon \mathcal{F} f^j[f^1, f^2, f^3, f^4] + \epsilon \mathcal{F} f^j[f^0, f^1, f^2, f^3]ight) + \epsilon \mathcal{F} f^j[f^0, f^1, f^2, f^3]
\]

Since \( 2m \geq p - 1 \) and \( (\mathcal{H}, \mathcal{F}) \) is a \( p \)-summable even Fredholm module over \( \mathcal{C} \), it follows that the operator

\[
\epsilon \mathcal{F} f^j[f^1, f^2, f^3, f^4] + \epsilon \mathcal{F} f^j[f^0, f^1, f^2, f^3]
\]

is trace class.

We observe that \( \tau \psi^j = \psi^{j-1} \) for \( 1 \leq j \leq 2m + 1 \) and \( \tau \psi^0 = \psi^{2m+1} \). It follows that \((1 - \lambda)(\psi) = 0\). Hence, \( \psi \in Z^{2m+1}_\lambda(\mathcal{C}) \). Using 7.1, we have

\[
(b\psi)(f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+2})
\]

\[
= \sum_{i=0}^{2m+1} (-1)^i \psi^i(f^0 \otimes \ldots \otimes f^{2i+1} \otimes \ldots \otimes f^{2m+1}) + \psi^j(f^{2m+2} f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+1})
\]

\[
= Tr\left( \epsilon \mathcal{F} f^j[f^1, f^2, f^3, f^4] + \epsilon \mathcal{F} f^j[f^0, f^1, f^2, f^3]ight) + \epsilon \mathcal{F} f^j[f^0, f^1, f^2, f^3]
\]

We now set \( \beta^j = [\mathcal{F}, f^{j-1}] \ldots [\mathcal{F}, f^{2m+2}] f^0[f^1, f^2, f^3, f^4] \). Then, we have

\[
[S(\phi^{2m}) + (m+1)\phi^{2m+2}](f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+2})
\]

\[
= Tr\left( \epsilon \mathcal{F} f^j[f^1, f^2, f^3, f^4] + \epsilon \mathcal{F} f^j[f^0, f^1, f^2, f^3]ight) + \epsilon \mathcal{F} f^j[f^0, f^1, f^2, f^3]
\]

\[
= (-1)^{j-1} f^0[f^1, f^2, f^3, f^4]
\]

\[
= (-1)^{j-1} f^0[f^1, f^2, f^3, f^4]
\]
With $\alpha^j = f^j F f^{j+1}$, we get

$$\begin{align*}
(-1)^j & T r \left( \epsilon F f^{j+1} [F, f^{j+2}] \ldots [F, f^{2m+2}] [F, f^0] [F, f^1] \ldots [F, f^{j-1}] f^j \right) \\
& = T r \left( \epsilon F f^{j+1} [F, \beta^j] f^j \right) = T r_s (\alpha^j [F, \beta^j]) = T r_s ([F, \alpha^j] \beta^j) \\
& = (-1)^{j-1} T r (\epsilon F f^{j+1} [F, f^{j+2}] \ldots [F, f^{2m+2}] [F, f^0] [F, f^1] \ldots [F, f^{j-1}] f^j)
\end{align*}$$

(7.6)

where we have used the fact that $T r_s$ is a closed graded trace and $T r_s (T) = T r (\epsilon T)$ for any operator that is trace class (see [12] Lemma 2). Thus, we have

$$(b \psi^j)(f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+2}) = - T r_s \left( [F, f^j] F f^{j+1} \beta^j \right) + T r_s ([F, \alpha^j] \beta^j) + T r_s (F f^j [F, f^{j+1}] \beta^j)$$

Since

$$F [F, f^j f^{j+1}] = F [F, f^j] f^{j+1} + F f^j [F, f^{j+1}] = - [F, f^j] F f^{j+1} + F f^j [F, f^{j+1}],$$

we obtain

$$(b \psi^j)(f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+2}) = T r_s \left( \left( F [F, f^j f^{j+1}] + [F, \alpha^j] \beta^j \right) + [F, \alpha^j] F = [F, f^j] [F, f^{j+1}] + 2 F f^j f^{j+1} \right)$$

we get

$$\begin{align*}
(b \psi)(f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+2}) &= \sum_{j=0}^{2m+1} (-1)^{j-1}(b \psi^j)(f^0 \otimes f^1 \otimes \ldots \otimes f^{2m+2}) \\
& = \sum_{j=0}^{2m+1} (-1)^{j-1} \left( 2 T r_s \left( f^j f^{j+1} \beta^j \right) + T r_s ([F, f^j] [F, f^{j+1}] \beta^j) \right) \\
& = \sum_{j=0}^{2m+1} 2 T r_s \left( f^0 [F, f^1] \ldots [F, f^{j-1}] (f^j f^{j+1}) [F, f^{j+2}] \ldots [F, f^{2m+2}] \right) \\
& + \sum_{j=0}^{2m+1} T r_s \left( f^0 [F, f^1] \ldots [F, f^{j-1}] (f^j f^{j+1}) [F, f^{j+2}] \ldots [F, f^{2m+2}] \right) \\
& = 2 \sum_{j=0}^{2m+1} 2 T r_s \left( f^0 [F, f^1] \ldots [F, f^{j-1}] (f^j f^{j+1}) [F, f^{j+2}] \ldots [F, f^{2m+2}] \right) \\
& + (2m + 2) T r \left( f^0 [F, f^1] \ldots [F, f^{2m+2}] \right)
\end{align*}$$

The result now follows by (7.4).

## 8 Homotopy invariance of the Chern character

Let $SHilb_2$ be the full subcategory of $SHilb_{D_2}$ whose objects are of the form $D = \mathcal{H} \oplus \mathcal{H}$, for some separable Hilbert space $\mathcal{H}$. If $D = \mathcal{H} \oplus \mathcal{H} \in SHilb_2$, we denote by $F(D)$ the morphism in $SHilb_2(D, D) = \mathcal{B}(\mathcal{H} \oplus \mathcal{H}, \mathcal{H} \oplus \mathcal{H})$ given by the matrix

$$(\begin{array}{cc}
0 & 1 \\
1 & 0 
\end{array})$$

swapping the two copies of $\mathcal{H}$.

**Lemma 8.1.** Let $\mathcal{C}$ be a small $\mathcal{C}$-category and $\{ \mathcal{H}_t : \mathcal{C} \to SHilb_2 \}_{t \in [0, 1]}$ be a family of functors such that for each $X \in \text{Ob}(\mathcal{C})$, we have $\mathcal{H}_t(X) = \mathcal{H}_t(X)$ for all $t, t' \in [0, 1]$. We put $\mathcal{H}(X) := \mathcal{H}_1(X)$ for all $t \in [0, 1]$. For each $f : X \to Y$ in $\mathcal{C}$, we assume that the function

$$p_f : [0, 1] \to SHilb_2 (\mathcal{H}_1(X), \mathcal{H}_1(Y)), \quad t \mapsto \mathcal{H}_t(f)$$

is strongly $C^1$. Then if $\delta_t(f) := p_f(t)$, we have

$$\delta_t(f g) = \mathcal{H}_t(f) \circ \delta_t(g) + \delta_t(f) \circ \mathcal{H}_t(g)$$

for composable morphisms $f, g$ in $\mathcal{C}$. 

25
Proof. We have
\[
\delta_t(fg) - \mathcal{H}_t(f) \circ \delta_t(g) - \delta_t(f) \circ \mathcal{H}_t(g) \\
= p'_{fg}(t) - \mathcal{H}_t(f) \circ p'_g(t) - \mathcal{H}_t(g) \\
= \lim_{s \to 0} \left( p_{fg}(t + s) - p_{fg}(t) - \mathcal{H}_t(f) \circ p_g(t + s) + \mathcal{H}_t(f) \circ p_g(t) + p_f(t + s) \circ \mathcal{H}_t(g) + p_f(t) \circ \mathcal{H}_t(g) \right) \\
= \lim_{s \to 0} \left( \mathcal{H}_{t+s}(fg) - \mathcal{H}_t(fg) + \mathcal{H}_t(f) \mathcal{H}_t(g) - \mathcal{H}_{t+s}(f) \mathcal{H}_t(g) + \mathcal{H}_t(f) \mathcal{H}_{t+s}(g) \right) \\
= \lim_{s \to 0} \left( p_f(t + s) \circ \mathcal{H}_t(g) - p_f(t) \circ \mathcal{H}_t(g) \right) \\
= p'_f(t) \lim_{s \to 0} (p_g(t + s) - p_g(t)) = 0
\]
\[\square\]

For each \(n \in \mathbb{Z}_{\geq 0}\), we now define an operator \(A : CN^n(C) \to CN^n(C)\) given by
\[A := 1 + \lambda + \lambda^2 + \ldots + \lambda^n\]
where \(\lambda\) is the (signed) cyclic operator. We observe that if \(\psi \in C^n_\lambda(C) = Ker(1 - \lambda)\), then \(A\psi = (n + 1)\psi\). From the relation
\[(1 - \lambda)(1 + 2\lambda + 3\lambda^2 + \ldots + (n + 1)\lambda^n) = A - (n + 1) \cdot 1\]
it is immediate that \(Ker(A) \subseteq Im(1 - \lambda)\). Let \(B_0 : CN^{n+1}(C) \to CN^n(C)\) be the map defined as follows:
\[(B_0\phi)(f^0 \otimes \ldots \otimes f^n) := \phi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^n) - (-1)^{n+1} \phi(f^0 \otimes \ldots \otimes f^n \otimes id_{X_0})\]
for any \(f^0 \otimes f^1 \otimes \ldots \otimes f^n \in Hom_C(X_1, X_0) \otimes Hom_C(X_2, X_1) \otimes \ldots \otimes Hom_C(X_0, X_n)\). We now set
\[B := AB_0 : CN^{n+1}(C) \to CN^n(C)\]

Lemma 8.2. We have
(1) \(bA = A b'\).
(2) \(bB + Bb = 0\).

Proof. (1) This follows from the general fact that the dual \(CN^*(C)\) of the cyclic nerve of \(C\) is a cocyclic module (see, for instance, [34 § 2.5]).
(2) For any \(f^0 \otimes f^1 \otimes \ldots \otimes f^n \in Hom_C(X_1, X_0) \otimes Hom_C(X_2, X_1) \otimes \ldots \otimes Hom_C(X_0, X_n)\) and \(\phi \in CN^n C\), we have
\[(B_0\phi)(f^0 \otimes \ldots \otimes f^n) = (b\phi)(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^n) - (-1)^{n+1}(b\phi)(f^0 \otimes \ldots \otimes f^n \otimes id_{X_0})\]
\[= \phi(f^0 \otimes \ldots \otimes f^n) + \sum_{i=0}^{n-1} (-1)^{i+1} \phi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^i f^{i+1} \otimes \ldots \otimes f^n) + (-1)^{n+1} \phi(f^n \otimes f^0 \otimes \ldots \otimes f^{n-1})\]
\[= (-1)^{n+1} \sum_{i=0}^{n-1} (-1)^i \phi(f^0 \otimes \ldots \otimes f^i f^{i+1} \otimes \ldots \otimes f^n \otimes id_{X_0})\]

On the other hand,
\[(b' B_0\phi)(f^0 \otimes \ldots \otimes f^n) = \sum_{i=0}^{n-1} (-1)^i \phi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^i f^{i+1} \otimes \ldots \otimes f^n) - (-1)^n \sum_{i=0}^{n-1} (-1)^i \phi(f^0 \otimes \ldots \otimes f^i f^{i+1} \otimes \ldots \otimes f^n \otimes id_{X_0})\]

Thus, we obtain
\[(B_0 b + b' B_0)(\phi)(f^0 \otimes \ldots \otimes f^n) = \phi(f^0 \otimes \ldots \otimes f^n) + (-1)^{n+1} \phi(f^n \otimes f^0 \otimes \ldots \otimes f^{n-1})\]

26
Therefore,
\[(B_0b + b'B_0)(\phi) = \phi - \lambda \phi\]  \hspace{1cm} (8.1)

Now, by applying the operator $A$ to both sides of (8.1), we have
\[AB_0b + A'b'B_0 = 0\]

The result now follows from part (1).

**Proposition 8.3.** The image of the map $B : CN^{n+1}(C) \to CN^n(C)$ is $C^n(C)$.

**Proof.** Let $\phi \in C^n(C)$ and let $R := \bigoplus_{X,Y \in \text{Ob}(C)} \text{Hom}(X,Y)$. Then $A$ is an algebra with multiplication given by composition wherever possible and 0 otherwise. We choose a linear map $\eta : A \to C$ such that
\[
\eta(f) = 0 \quad \text{for } f \in \text{Hom}_C(X,Y), \ X \neq Y
\]
\[
\eta(id_X) = 1 \quad \forall X \in \text{Ob}(C)
\]

We now define $\psi \in CN^{n+1}(C)$ by setting
\[
\psi(f^0 \otimes \ldots \otimes f^{n+1}) := \eta(f^0) \phi(f^1 \otimes \ldots \otimes f^{n+1}) + (-1)^n \left( \phi(f^0 \otimes f^1 \otimes \ldots \otimes f^n) - \eta(f^0) \phi(id_X \otimes f^1 \otimes \ldots \otimes f^n) \eta(f^{n+1}) \right)
\]
for any $f^0 \otimes f^1 \otimes \ldots \otimes f^{n+1} \in \text{Hom}_C(X_1, X_0) \otimes \text{Hom}_C(X_2, X_1) \otimes \ldots \otimes \text{Hom}_C(X_0, X_{n+1})$. We observe that if the tuple $(f^1, \ldots, f^{n+1})$ is not cyclically composable, i.e., $X_0 \neq X_1$, then the first term vanishes as $\eta(f^0) = 0$. Similarly, if the tuple $(f^n, \ldots, f^n)$ is not cyclically composable, i.e., $X_{n+1} \neq X_0$, then the second term vanishes. For the last term, $\eta(f^n)$ and $\eta(f^{n+1})$ will be non zero only if $X_1 = X_0$ and $X_0 = X_{n+1}$ which means that $X_{n+1} = X_1$ and the tuple $(id_{X_1}, f^1, \ldots, f^n)$ is cyclically composable.

Then, for any $g^0 \otimes g^1 \otimes \ldots \otimes g^n \in \text{Hom}_C(Y_1, Y_0) \otimes \text{Hom}_C(Y_2, Y_1) \otimes \ldots \otimes \text{Hom}_C(Y_0, Y_n)$, we have
\[
\psi(id_{Y_0} \otimes g^0 \otimes \ldots \otimes g^n) = \eta(id_{Y_0}) \phi(g^0 \otimes \ldots \otimes g^n) + (-1)^n \left( \phi(id_{Y_0} \otimes g^0 \otimes \ldots \otimes g^{n-1}) \eta(g^n) \right)
\]
\[
= \phi(g^0 \otimes \ldots \otimes g^n)
\]

Also
\[
\psi(g^0 \otimes \ldots \otimes g^n \otimes id_{Y_0}) = \eta(g^0) \phi(g^1 \otimes \ldots \otimes g^n \otimes id_{Y_0}) + (-1)^n \left( \phi(g^0 \otimes \ldots \otimes g^n) \eta(id_{Y_0}) \right)
\]
\[
= (-1)^n \phi(g^0 \otimes \ldots \otimes g^n)
\]
where the second equality follows from the fact that $\phi \in C^n_C(C)$ and that $\eta(g^0) = 0$ whenever $Y_1 \neq Y_0$. Thus,
\[
(B_0\psi)(g^0 \otimes \ldots \otimes g^n) = \psi(id_{Y_0} \otimes g^0 \otimes \ldots \otimes g^n) - (-1)^{n+1} \psi(g^0 \otimes \ldots \otimes g^n \otimes id_{Y_0})
\]
\[
= 2\phi(g^0 \otimes \ldots \otimes g^n)
\]

Since $\phi \in \text{Ker}(1 - \lambda)$, we now have $B\psi = 2A\phi = 2(n + 1)\phi$. Thus, $\phi \in \text{Im}(B)$. Conversely, let $\phi \in \text{Im}(B)$. Then, $\phi = B\psi$ for some $\psi \in CN^{n+1}(C)$. Using the fact that $(1 - \lambda)A = 0$, we have
\[
(1 - \lambda)(\phi) = (1 - \lambda)(B\psi) = ((1 - \lambda)AB_0)\psi = 0
\]
This proves the result.

**Proposition 8.4.** Let $\psi \in CN^n(C)$ be such that $b\psi \in C^{n+1}_C(C)$. Then,

1. $B\psi \in Z^{n-1}_C(C)$ i.e., $b(B\psi) = 0$ and $(1 - \lambda)(B\psi) = 0$.
2. $S(B\psi) = n(n + 1)b\psi$ in $R^{n+1}_C(C)$.

27
Proof. (1) We know that \((1 - \lambda)(B\psi) = (1 - \lambda)(AB_0)(\psi) = 0\). Further, for any \(\phi \in Ker(1 - \lambda)\), we have \(B_0\phi = 0\). Therefore, it follows that \(bB\psi = -B_b\psi = -AB_0b\psi = 0\).

(2) We have to show that \(SB\psi - n(n + 1)b\psi = b\zeta\) for some \(\zeta \in C^n(C)\). We set \(\phi = B\psi\). Then, \(\phi = B\psi\). Hence, \(\psi = n(n + 1)\psi'\) is given by \(\psi'\) is defined by \(\psi' = CN^{n}(C)\) such that \(\psi'' = \psi \in B^n(C)\) and \(\zeta = \psi' - n(n + 1)\psi'' \in C^n(C)\). This would give

\[
\psi'' = cn^{n-1}(C), \quad \text{where} \quad B\psi \in Z_{n-1}(C),
\]

We set \(\theta := B_0\psi\), \(\theta' := 1/\theta\) and \(\theta'' := \theta - \theta' \in C^n(C)\). Since \(B\psi \in Z^n(C)\), we have

\[
A\theta'' = AB_0\psi - \frac{1}{n}A\phi = B\psi - \frac{1}{n}AB\psi = B\psi - \frac{1}{n}nB\psi = 0.
\]

Since \(Ker(A) \subseteq Im(1 - \lambda)\), we have \(\theta'' = (1 - \lambda)(\psi)\) for some \(\psi_1 \in C^n(C)\). We take \(\psi'' = \psi - b\psi_1\). We now show that \((1 - \lambda)\zeta = 0\), i.e., \(\lambda(n + 1)(1 - \lambda)(\psi'')\) where \(\zeta = \psi' - n(n + 1)\psi''\). We see that

\[
(\tau_n\psi')(f_0 \otimes \cdots \otimes f^n) = \psi'(f_0 \otimes f_0 \otimes \cdots \otimes f^n) = \sum_{j=0}^{n-1} (-1)^j T (\hat{f}_0 f_1 \hat{f}_j f_1^{-1} f_1^{-1} \otimes f_1 \otimes \cdots \otimes f_0)
\]

where we have used the fact that \(T\) is a graded trace. For \(1 \leq j \leq n - 1\), we now set

\[
\omega_j := f_0(\hat{f}_0 f_1 \hat{f}_j f_1^{-1} f_j \otimes \cdots \otimes f_n) f^n
\]

Then,

\[
\hat{\omega}_j = (\hat{\omega}_0 \hat{f}_1 \hat{f}_j f_1^{-1} f_j \hat{f}_1 f_1^{-1} \otimes \cdots \otimes f_n) f^n + (-1)^{-1} f_0(\hat{\omega}_0 f_1 \hat{f}_1 f_1^{-1} f_1^{-1} \hat{f}_1 \hat{f}_1 \otimes \cdots \otimes f_n) f^n + (-1)^{-1} f_0(\hat{\omega}_0 f_1 \hat{f}_1 f_1^{-1} f_1^{-1} \hat{f}_1 \hat{f}_1 \otimes \cdots \otimes f_n) f^n + (-1)^{-1} f_0(\hat{\omega}_0 f_1 \hat{f}_1 f_1^{-1} f_1^{-1} \hat{f}_1 \hat{f}_1 \otimes \cdots \otimes f_n) f^n
\]

Thus,

\[
0 = T(\hat{\omega}_j) = \hat{T}(\hat{\omega}_0 f_1 \hat{f}_1 f_1^{-1} f_j \hat{f}_1 f_1^{-1} \otimes \cdots \otimes f_n) f^n + (-1)^{-1} \hat{T}(\hat{\omega}_0 f_1 \hat{f}_1 f_1^{-1} f_1^{-1} \hat{f}_1 \hat{f}_1 \otimes \cdots \otimes f_n) f^n + (-1)^{-1} \hat{T}(\hat{\omega}_0 f_1 \hat{f}_1 f_1^{-1} f_1^{-1} \hat{f}_1 \hat{f}_1 \otimes \cdots \otimes f_n) f^n + (-1)^{-1} \hat{T}(\hat{\omega}_0 f_1 \hat{f}_1 f_1^{-1} f_1^{-1} \hat{f}_1 \hat{f}_1 \otimes \cdots \otimes f_n) f^n
\]

Therefore,

\[
(1 - \lambda)(\psi')(f_0 \otimes \cdots \otimes f^n)
\]

Hence,

\[
(1 - \lambda)(\psi')(f_0 \otimes \cdots \otimes f^n) = (1 - \lambda)\phi(f_n f_0 \otimes f_1 \otimes \cdots \otimes f_n)
\]
Since \( b\psi \in C_{\lambda+1}(C) \), we have from (8.1) that 
\[
(1 - \lambda)(\psi) = (B_0b + b'B_0)(\psi) = b'B_0\psi = b'b\theta = b'\theta + b'\theta''.
\]
Hence, 
\[
(1 - \lambda)(\psi') = b'\theta' = \frac{1}{n} b'\phi,
\]
Since \( \phi = B\psi \in Z_{\lambda}^{-1}(C), \) \( b\phi = 0 \) and therefore
\[
(1 - \lambda)(\psi''_n)(f^0 \otimes \ldots \otimes f^n) = \frac{1}{n} (b'\phi)(f^0 \otimes \ldots \otimes f^n) = \frac{1}{n} (-1)^{n-1} \phi(f^n f^0 \otimes f^1 \otimes \ldots \otimes f^{n-1})
\]
(8.3)

The result now follows by comparing (8.2) and (8.3).

\[\square\]

**Proposition 8.5.** Let \( C \) be a small \( \mathbb{C} \)-category and \( \{ \mathscr{H}_t : C \to \text{SHilb}_2 \}_{t \in [0,1]} \) be a family of functors such that for each \( X \in \text{Ob}(C) \), we have \( \mathscr{H}_t(X) = \mathscr{H}_r(X) \) for all \( t, t' \in [0,1] \) and \( \mathscr{H}_t(f) \) is of degree zero for each \( f \in \text{Hom}_C(X,Y) \) and \( t \in [0,1] \). We put \( \mathscr{H}(X) := \mathscr{H}_t(X) \) for all \( t \in [0,1] \).

Let \( \mathcal{F} \) be the family of operators
\[
\mathcal{F} = \left\{ (F(\mathscr{H}(X))) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}_{X \in \text{Ob}(C)}
\]
(8.4)

Let \( p = 2m \) be an even integer. We assume that
1. for each \( f \in \text{Hom}_C(X,Y) \), the association \( t \mapsto [\mathcal{F}, \mathscr{H}_t(f)] \) is a continuous map
   \[
   \zeta_f : [0,1] \to \text{B}^p(\mathscr{H}(X),\mathscr{H}(Y)) \quad t \mapsto [\mathcal{F}, \mathscr{H}_t(f)]
   \]
2. for each \( f \in \text{Hom}_C(X,Y) \), the association
   \[
   p_f : [0,1] \to \text{SHilb}_2(\mathscr{H}_t(X),\mathscr{H}_t(Y)) \quad t \mapsto \mathscr{H}_t(f)
   \]
is piecewise strongly \( C^1 \).

Let \( (\mathscr{H}_t, \mathcal{F}) \) be the corresponding \( p \)-summable Fredholm modules over \( C \). Then, the class in \( H^{p+2}_\lambda(C) \) of the \( (p+2) \)-dimensional character of the Fredholm module \( (\mathscr{H}_t, \mathcal{F}) \) is independent of \( t \).

**Proof.** For any \( t \in [0,1] \), let \( \phi_i \) be the \( p \)-dimensional character of the Fredholm module \( (\mathscr{H}_t, \mathcal{F}) \). We will show that \( S(\phi_{t_1}) = S(\phi_{t_2}) \) for any \( t_1, t_2 \in [0,1] \).

By assumption, we know that there exists a finite set \( R = \{0 = r_0 < r_1 < \ldots < r_k < r_{k+1} = 1\} \subseteq [0,1] \) such that \( p_f : [0,1] \to \text{SHilb}_2(\mathscr{H}_t(X),\mathscr{H}_t(Y)) \) is continuously differentiable in each \([r_i, r_{i+1}])\). By abuse of notation, we set for each \( f \in \text{Hom}_C(X,Y) \):
\[
\delta_i(f) := p_f'(t) \in \text{SHilb}_2(\mathscr{H}_t(X),\mathscr{H}_t(Y))
\]
(8.5)

Here, it is understood that if \( t = r_i \) for some \( 1 \leq i \leq k \), we use the right hand derivative when \( r_i \) is treated as a point of \([r_i, r_{i+1})\) and the left hand derivative when \( r_i \) is treated as a point of \([r_{i-1}, r_i)\).

Using Lemma (8.1), we know that
\[
\delta_i(fg) = \mathscr{H}_t(f) \circ \delta_i(g) + \delta_i(f) \circ \mathscr{H}_t(g)
\]
(8.6)

for any \( t \in [0,1] \) and for any pair of composable morphisms \( f \) and \( g \) in \( C \).

For any \( t \in [0,1] \) and \( 1 \leq j \leq p + 1 \), we set
\[
\psi^j_t(f^0 \otimes \ldots \otimes f^{p+1}) := \text{Tr}(\epsilon\mathscr{H}_t(f^0)[\mathcal{F}, \mathscr{H}_t(f^1)] \ldots [\mathcal{F}, \mathscr{H}_t(f^{j-1})] \delta_t(f^j)[\mathcal{F}, \mathscr{H}_t(f^{j+1})] \ldots [\mathcal{F}, \mathscr{H}_t(f^{p+1})])
\]
Using the expression in (8.6) and the fact that $\epsilon\mathcal{H}(f) = \mathcal{H}(f)\epsilon$ for any morphism $f \in \mathcal{C}$, it may be easily verified that $b\psi_j^i = 0$. For example, when $j = 1$, we have (suppressing the functor $\mathcal{H}$)

$$(b\psi_j^i)(f^0 \otimes \ldots \otimes f^{p+2})$$

$$= \sum_{i=0}^{p+1} \psi_i^j(f^0 \otimes \ldots \otimes f^{i+1} \otimes \ldots \otimes f^{p+2}) + \psi_i^j(f^{p+2}f^0 \otimes f^1 \otimes \ldots \otimes f^{p+2})$$

$$= Tr(\epsilon f^0 \delta_i(f^2)[\mathcal{F}, f^3] \ldots [\mathcal{F}, f^{p+2}]) - Tr(\epsilon f^0 \delta_i(f^1f^2)[\mathcal{F}, f^3] \ldots [\mathcal{F}, f^{p+2}])$$

$$+ Tr(\epsilon f^0 \delta_i(f^1)[\mathcal{F}, f^2f^3] \ldots [\mathcal{F}, f^{p+2}]) - Tr(\epsilon f^0 \delta_i(f^1)[\mathcal{F}, f^2][\mathcal{F}, f^3f^4] \ldots [\mathcal{F}, f^{p+2}]) + \ldots$$

$$\ldots - Tr(\epsilon f^0 \delta_i(f^1)[\mathcal{F}, f^2] \ldots [\mathcal{F}, f^{p+1}f^{p+2}]) + Tr(\epsilon f^{p+2}f^0 \delta_i(f^1)[\mathcal{F}, f^2][\mathcal{F}, f^3f^4] \ldots [\mathcal{F}, f^{p+1}]) = 0$$

We then define

$$\psi_t := \sum_{j=0}^{p+1} (-1)^{j-1} \psi_t^j$$

We have $b\psi_t = 0$.

For fixed $f$, it follows from the compactness of $[0,1]$ and the assumptions (1) and (2) that the families \(\{\mathcal{H}_t(f_j)\}_{t \in [0,1]}, \{\rho_f(t)\}_{t \in [0,1]}\) and \(\{\delta_t(f)\}_{t \in [0,1]}\) are uniformly bounded. For the sake of simplicity, we assume that there is only a single point $r \in R$ such that $t_1 \leq r \leq t_2$. Then, we form $\psi \in C^{p+1}(\mathcal{C})$ by setting

$$\psi(f^0 \otimes \ldots \otimes f^{p+1}) := \int_{t_1}^{r} \psi_t(f^0 \otimes \ldots \otimes f^{p+1})dt + \int_{r}^{t_2} \psi_t(f^0 \otimes \ldots \otimes f^{p+1})dt$$

We now have

$$\psi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^p)$$

$$= \int_{t_1}^{r} \psi_t(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^p)dt + \int_{r}^{t_2} \psi_t(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^p)dt$$

$$= \int_{t_1}^{r} \left( \sum_{j=0}^{p} (-1)^j Tr(\epsilon[\mathcal{H}_t(f^0)] \ldots [\mathcal{H}_t(f-j)] \delta_t(f^j)[\mathcal{H}_t(f^j+1)] \ldots [\mathcal{H}_t(f^p)]) \right) dt$$

$$+ \int_{r}^{t_2} \left( \sum_{j=0}^{p} (-1)^j Tr(\epsilon[\mathcal{H}_t(f^0)] \ldots [\mathcal{H}_t(f-j)] \delta_t(f^j)[\mathcal{H}_t(f^j+1)] \ldots [\mathcal{H}_t(f^p)]) \right) dt$$

Let $\phi : [0,1] \rightarrow Z^{p+1} \mathcal{H}$ be the map given by $t \mapsto \phi_t$. We now claim that

$$\psi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^p) = \int_{t_1}^{r} \phi_t(t)(f^0 \otimes \ldots \otimes f^p) dt + \int_{r}^{t_2} \phi_t(t)(f^0 \otimes \ldots \otimes f^p) dt$$

Indeed, we have

$$\phi_t(t)(f^0 \otimes \ldots \otimes f^p) = \lim_{t \rightarrow s} \frac{1}{t-s} (\phi_{t+s} - \phi_t)(f^0 \otimes \ldots \otimes f^p)$$

$$= \lim_{t \rightarrow s} \left( Tr(\epsilon[\mathcal{H}_s(f^0)] \ldots [\mathcal{H}_s(f-j)] [\mathcal{F}, \mathcal{H}_s(f^j)] \ldots [\mathcal{F}, \mathcal{H}_s(f^p)]) \right) + \ldots$$

By (1), we know that the association $t \mapsto [\mathcal{F}, \mathcal{H}_t(f)]$ is a continuous map for each morphism $f \in \mathcal{C}$. Therefore, we have

$$\lim_{t \rightarrow s} \left( Tr(\epsilon[\mathcal{H}_s(f^0)] \ldots [\mathcal{H}_s(f-j)] [\mathcal{F}, \mathcal{H}_s(f^j)] \ldots [\mathcal{F}, \mathcal{H}_s(f^p)]) \right)$$

$$= (-1)^j Tr(\epsilon[\mathcal{H}_t(f^0)] \ldots [\mathcal{H}_t(f-j)] \delta_t(f^j)[\mathcal{H}_t(f^j+1)] \ldots [\mathcal{H}_t(f^p)])$$

From this, we obtain

$$\int_{t_1}^{r} \phi_t(t)(f^0 \otimes \ldots \otimes f^p) dt + \int_{r}^{t_2} \phi_t(t)(f^0 \otimes \ldots \otimes f^p) dt$$

$$= \int_{t_1}^{r} \sum_{j=0}^{p} (-1)^j Tr(\epsilon[\mathcal{H}_t(f^0)] \ldots [\mathcal{H}_t(f-j)] \delta_t(f^j)[\mathcal{H}_t(f^j+1)] \ldots [\mathcal{H}_t(f^p)]) dt$$

$$+ \int_{r}^{t_2} \sum_{j=0}^{p} (-1)^j Tr(\epsilon[\mathcal{H}_t(f^0)] \ldots [\mathcal{H}_t(f-j)] \delta_t(f^j)[\mathcal{H}_t(f^j+1)] \ldots [\mathcal{H}_t(f^p)]) dt$$

$$= \psi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^p)$$
Therefore, using assumption
\[\psi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^p) = \phi_{t_2}(f^0 \otimes \ldots \otimes f^p) - \phi_1(f^0 \otimes \ldots \otimes f^p) = \phi_{t_2}(f^0 \otimes \ldots \otimes f^p) - \phi_{t_1}(f^0 \otimes \ldots \otimes f^p)\]
Since \(\psi(f^0 \otimes \ldots \otimes f^p \otimes id_{X_0}) = 0\), we now have
\[(B_0\psi)(f^0 \otimes \ldots \otimes f^p) = \psi(id_{X_0} \otimes f^0 \otimes \ldots \otimes f^p) - \psi(f^0 \otimes \ldots \otimes f^p \otimes id_{X_0}) = (\phi_{t_2} - \phi_{t_1})(f^0 \otimes \ldots \otimes f^p)\]
Since \(b\psi = 0\), using Proposition 8.4 and the fact that \(\phi_{t_2} - \phi_{t_1} \in Ker(1 - \lambda)\), we have
\[0 = S(B\psi) = S(AB_0\psi) = (p + 1)S(\phi_{t_2} - \phi_{t_1})\]
This proves the result.

**Theorem 8.6.** Let \(\mathcal{C}\) be a small \(\mathbb{C}\)-category and \(\{\rho_t : \mathcal{C} \rightarrow SHilb_2\}_{t \in [0,1]}\) be a family of functors such that for each \(X \in \text{Ob}(\mathcal{C})\), we have \(\rho_t(X) = \rho_1(X)\) for all \(t, t' \in [0,1]\). We put \(\rho(X) := \rho_t(X)\) for all \(t \in [0,1]\). Further, for each \(t \in [0,1]\), let
\[
\mathcal{F}_t := \frac{\mathcal{P}_t(X) \otimes Q_t(X)}{0} : \rho(X) \rightarrow \rho(X) \quad (8.7)
\]
with \(\mathcal{P}_t(X) = Q_t^{-1}(X)\) to be such that \((\rho_t, \mathcal{F}_t)\) is a \(p\)-summable Fredholm module over the category \(\mathcal{C}\). We set \(\rho(X) = \rho'(X) \oplus \rho'(X) \in SHilb_2\). We further assume that for some even integer \(p\) and for any \(f \in Hom_{\mathcal{C}}(X,Y)\), we have
1. \(t \mapsto \rho_t^+(f) - Q_t\rho_t^-(f)\mathcal{P}_t\) is a continuous map from \([0,1]\) to \(\mathcal{B}^p(\rho'(X),\rho'(Y))\), where \(\rho_t^\pm\) are the two components of the morphism \(\rho_t\) of degree zero.
2. \(t \mapsto \rho_t^+(f)\) and \(t \mapsto Q_t\rho_t^-(f)\mathcal{P}_t\) are piecewise strongly \(C^1\) maps from \([0,1]\) to \(SHilb(\rho'(X),\rho'(Y))\).
Then, the \((p+2)\)-dimensional character \(ch_{F^{p+2}}(\rho_t, \mathcal{F}_t) \in H_{\mathcal{X}}^{p+2}(\mathcal{C})\) is independent of \(t \in [0,1]\).

**Proof.** For each \(t \in [0,1]\), we set \(T_t := \begin{pmatrix} 1 & 0 \\ 0 & Q_t \end{pmatrix}\). Then, \(T_t^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & P_t \end{pmatrix}\) and \(\mathcal{F}_t := T_t\mathcal{F}_t T_t^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

For each \(t \in [0,1]\), we also define a functor \(\mathcal{H}_t : \mathcal{C} \rightarrow SHilb_{\mathbb{Z}_2}\) given by
\[
\mathcal{H}_t(X) := \rho(X) \quad \mathcal{H}_t(f) := T_t\rho_t(f)T_t^{-1}
\]
Then, we have
\[
[\mathcal{F}_t, \mathcal{H}_t(f)] = \begin{pmatrix} \rho_t^+(f) - Q_t\rho_t^-(f)\mathcal{P}_t & 0 \\ 0 & Q_t\rho_t^-(f)\mathcal{P}_t - \rho_t^+(f) \end{pmatrix}
\]
Therefore, using assumption (1), we see that the map \(t \mapsto [\mathcal{F}_t, \mathcal{H}_t(f)]\) from \([0,1]\) to \(\mathcal{B}^p(\mathcal{H}_t(X), \mathcal{H}_t(Y))\) is continuous for each \(f \in Hom_{\mathcal{C}}(X,Y)\). Further,
\[
\mathcal{H}_t(f) = T_t\rho_t(f)T_t^{-1} = \begin{pmatrix} \rho_t^+(f) & 0 \\ 0 & Q_t\rho_t^-(f)\mathcal{P}_t \end{pmatrix}
\]
Therefore, by applying assumption (2), we see that the map \(t \mapsto \mathcal{H}_t(f)\) is piecewise strongly \(C^1\). Since trace is invariant under similarity, the result now follows using Proposition 8.3.\(\square\)

**Theorem 8.7.** Let \(\mathcal{C}\) be a small \(\mathbb{C}\)-category and \(\{\rho_t : \mathcal{C} \rightarrow SHilb_2\}_{t \in [0,1]}\) be a family of functors such that for each \(X \in \text{Ob}(\mathcal{C})\), we have \(\rho_t(X) = \rho_1(X)\) for all \(t, t' \in [0,1]\). We put \(\rho(X) := \rho_t(X)\) for all \(t \in [0,1]\). Further, for each \(t \in [0,1]\) and \(X \in \text{Ob}(\mathcal{C})\), let
\[
\mathcal{F}_t(X) := \begin{pmatrix} 0 & Q_t(X) \\ P_t(X) & 0 \end{pmatrix} : \rho(X) \rightarrow \rho(X)
\]
with $Q_t^{-1} = P_t$ be such that $(\rho_t, \mathcal{F}_t)$ is a $p$-summable Fredholm module over the category $\mathcal{C}$. We further assume that for some even integer $p$, we have

1. For any $f \in \text{Hom}_\mathcal{C}(X, Y)$, $t \mapsto \rho_t(f)$ is a strongly $C^1$-map from $[0, 1]$ to $\text{SHilb}_{Z^2}(\rho(X), \rho(Y))$.

2. For any $X \in \mathcal{C}$, $t \mapsto \mathcal{F}_t(X)$ is a strongly $C^1$-map from $[0, 1]$ to $\text{SHilb}_{Z^2}(\rho(X), \rho(X))$.

Then, the $(p + 2)$-dimensional character $\chi^{p+2}(\rho_t, \mathcal{F}_t) \in H^{p+2}_t(\mathcal{C})$ is independent of $t \in [0, 1]$.

Proof. By definition, $\rho_t(f) = \left( \begin{array}{cc} \rho^+(f) & 0 \\ 0 & \rho^-(f) \end{array} \right)$ and $\mathcal{F}_t(X) = \left( \begin{array}{c} 0 \\ P_t(X) \\ Q_t(X) \end{array} \right)$. As such, it is clear that a system satisfying the assumptions (1) and (2) above also satisfies the assumptions in Theorem 8.6. This proves the result.

References

[1] R. Akbarpour and M. Khalkhali, Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras, J. Reine Angew. Math. 559 (2003), 137–152.

[2] J. C. Baez, Higher-dimensional algebra. II. 2-Hilbert spaces, Adv. Math. 127 (1997), no. 2, 125–189.

[3] M. Balodi, Morita invariance of equivariant and Hopf-cyclic cohomology of module algebras over Hopf algebroids, arXiv:1804.10898 [math.QA].

[4] M. Balodi, A. Banerjee, and S. Ray, Cohomology of modules over $H$-categories and co-$H$-categories, Canad. J. Math. 72 (2020), no. 5, 1352-1385.

[5] ———, On entwined modules over linear categories and Galois extensions, Isreal J. Math. 241 (2021), 623-692.

[6] M. Balodi and A. Banerjee, Odd Fredholm modules over linear categories and cyclic cohomology, (in preparation).

[7] A. Banerjee, On differential torsion theories and rings with several objects, Canad. Math. Bull. 62 (2019), no. 4, 703–714.

[8] G. Böhm, S. Lack, and R. Street, Idempotent splittings, colimit completion, and weak aspects of the theory of monads, J. Pure Appl. Algebra 216 (2012), no. 2, 385–403.

[9] S. Caenepeel, Brauer groups, Hopf algebras and Galois theory, K-Monographs in Mathematics, vol. 4, Kluwer Academic Publishers, Dordrecht, 1998.

[10] C. Cibils and A. Solotar, Galois coverings, Morita equivalence and smash extensions of categories over a field, Doc. Math. 11 (2006), 143–159.

[11] A. Connes, Cohomologie cyclique et foncteurs Ext$, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), no. 23, 953–958.

[12] ———, Noncommutative differential geometry, Inst. Hautes Études Sci. Publ. Math. 62 (1985), 257–360.

[13] A. Connes and H. Moscovici, Hopf algebras, cyclic cohomology and the transverse index theorem, Comm. Math. Phys. 198 (1998), no. 1, 199–246.

[14] ———, Cyclic cohomology and Hopf algebras, Lett. Math. Phys. 48 (1999), no. 1, 97–108. Moshé Flato (1937–1998).

[15] ———, Cyclic Cohomology and Hopf Algebra Symmetry, Lett. Math. Phys. 52 (2000), no. 1, 1–28.

[16] P. Deligne, Catégories tannakiennes, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 111–195.

[17] S. Estrada and S. Virili, Cartesiian modules over representations of small categories, Adv. Math. 310 (2017), 557–609.

[18] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser, Hopf-cyclic cohomology and cohomology with coefficients, C. R. Math. Acad. Sci. Paris 338 (2004), no. 9, 667–672.

[19] ———, Stable anti-Yetter-Drinfeld modules, C. R. Math. Acad. Sci. Paris 338 (2004), no. 8, 587–590.

[20] M. Hassanzadeh, D. Kucerovsky, and B. Rangipour, Generalized coefficients for Hopf cyclic cohomology, SIGMA Symmetry Integrability Geom. Methods Appl. 10 (2014), Paper 093, 16.

[21] M. Hassanzadeh, M. Khalkhali, and I. Shapiro, Monoidal categories, 2-traces, and cyclic cohomology, Canad. Math. Bull. 62 (2019), no. 2, 293–312.

[22] A. G. Henriques, What Chern-Simons theory assigns to a point, Proc. Natl. Acad. Sci. USA 114 (2017), no. 51, 13418–13423.

[23] A. Henriques and D. Penneys, Bicocommutant categories from fusion categories, Selecta Math. (N.S.) 23 (2017), no. 3, 1609–1708.

[24] E. Herscovich and A. Solotar, Hochschild-Mitchell cohomology and Galois extensions, J. Pure Appl. Algebra 209 (2007), no. 1, 37–55.
M. Karoubi and O. Villamayor, *K*-théorie algébrique et *K*-théorie topologique. I, Math. Scand. 28 (1971), 265–307.

A. Kaygun, Bialgebra cyclic homology with coefficients, *K*-Theory 34 (2005), no. 2, 151–194.

———, The universal Hopf-cyclic theory, J. Noncommut. Geom. 2 (2008), no. 3, 333–351.

———, Products in Hopf-cyclic cohomology, Homology Homotopy Appl. 10 (2008), no. 2, 115–133.

A. Kaygun and M. Khalkhali, Bivariant Hopf cyclic cohomology, Comm. Algebra 38 (2010), no. 7, 2513–2537.

B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63–102.

———, On differential graded categories, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

M. Khalkhali and B. Rangipour, Cup products in Hopf-cyclic cohomology, C. R. Math. Acad. Sci. Paris 340 (2005), 9–14.

I. Kobyzev and I. Shapiro, A Categorical Approach to Cyclic Cohomology of Quasi-Hopf Algebras and Hopf Algebroids, Appl. Categ. Structures 27 (2019), no. 1, 85–109.

J.-L. Loday, Cyclic homology, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1992.

W. Lowen and M. Van den Bergh, Deformation theory of abelian categories, Trans. Amer. Math. Soc. 358 (2006), no. 12, 5441–5483.

W. Lowen, Hochschild cohomology with support, Int. Math. Res. Not. IMRN 13 (2015), 4741–4812.

R. McCarthy, The cyclic homology of an exact category, J. Pure Appl. Algebra 93 (1994), no. 3, 251–296.

B. Mitchell, The dominion of Isbell, Trans. Amer. Math. Soc. 167 (1972), 319–331.

———, Rings with several objects, Adv. Math. 8 (1972), 1–161.

———, Some applications of module theory to functor categories, Bull. Amer. Math. Soc. 84 (1978), no. 5, 867–885.

B. Rangipour, Cup products in Hopf cyclic cohomology via cyclic modules, Homology Homotopy Appl. 10 (2008), no. 2, 273–286.

H. Schubert, Categories, Springer-Verlag, New York-Heidelberg, 1972.

B. Toën and M. Vaquié, Au-dessous de Spec Z, J. K-Theory 3 (2009), no. 3, 437–500.

F. Xu, On the cohomology rings of small categories, J. Pure Appl. Algebra 212 (2008), no. 11, 2555–2569.

———, Hochschild and ordinary cohomology rings of small categories, Adv. Math. 219 (2008), no. 6, 1872–1893.