Separability Criteria from Uncertainty Relations

Otfried Gühne∗† and Maciej Lewenstein†

∗Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, A-6020 Innsbruck, Austria
†Institut für Theoretische Physik, Universität Hannover, Appelstraße 2, D-30167 Hannover, Germany

Abstract. We explain several separability criteria which rely on uncertainty relations. For the derivation of these criteria uncertainty relations in terms of variances or entropies can be used. We investigate the strength of the separability conditions for the case of two qubits and show how they can improve entanglement witnesses.

INTRODUCTION

Entanglement is one of the counterintuitive phenomena of quantum mechanics and is, despite a lot of progress in the last years, not fully understood. A state $\rho$ on a bipartite system is called separable when it can be written as a convex combination of product states, i.e., $\rho = \sum_i p_i |a_i b_i\rangle \langle a_i b_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$. A state which is not separable is called entangled. The question, whether a state $\rho$ is entangled or not, is the so-called separability problem, and no general answer to this question is known (for a review see [1]). Geometrically, the definition of separability implies that the separable states form a convex set in the (high dimensional) real vector space of all density matrices of a given system (see Fig. 1(a)).

Given a state $\rho$, what shall we measure to detect the entanglement in this state? This is a question of great importance in many experiments, since the presence of entanglement is a necessary precondition for certain tasks as quantum key distribution or teleportation. The usual tools to answer this question are entanglement witnesses (EW) [2]. An EW $W$ is a Hermitean observable $W$ with an positive expectation value on all separable states. Thus $\text{Tr}(W \rho) < 0$ implies that the state $\rho$ is entangled. An EW provides a criterion, which depends linearly on the state. Geometrically, the set where $\text{Tr}(W \rho) = 0$ is a hyperplane, separating the detected states from the non-detected ones.

Can we use nonlinear witnesses? This is a very natural question for the following reason: One might expect that one can approximate the convex set of the separable states better by using a nonlinear expression (see Fig. 1(a)). This paper deals with special types of nonlinear witnesses, and in fact, it will turn out that sometimes they can improve already known linear witnesses. The nonlinear expressions we will use are based on uncertainty relations.

UNCERTAINTY BASED CRITERIA

Let us start with criteria based on variances of observables. The variance of an observable $M$ in the state $\rho$ is given by $\delta^2(M)_{\rho} := \langle (M - \langle M \rangle_{\rho})^2 \rangle_{\rho} = \langle M^2 \rangle_{\rho} - \langle M \rangle_{\rho}^2$, where $\langle M \rangle_{\rho} = \text{Tr}(\rho M)$. If $\rho = |\psi\rangle \langle \psi|$ describes a pure state, the variance of $M$ is zero iff $|\psi\rangle$ is an eigenstate of $M$. Furthermore, the variance is concave in the state. If $\rho = \sum_k p_k \rho_k$ is a convex combination of some states $\rho_k$, then

$$\sum_i \delta^2(M_i)_{\rho} \geq \sum_k p_k \sum_i \delta^2(M_i)_{\rho_k}$$

(1)

holds. This inequality can be straightforwardly calculated [3][4], and expresses the simple physical fact that one cannot decrease the uncertainty of an observable by mixing several states. H. Hofmann and S. Takeuchi were the first who realized that this property of the variance gives rise to separability criteria, the so-called local uncertainty relations (LURs). They showed the following:
that for two observables

\[ \mathcal{L} \]

set where

\[ \mathcal{L} \] a possible nonlinear witness.

(b) A comparison between the LUR in Eq. (2), the witness resulting from the linear part of the LUR, and the PPT criterion for the family of states described in the text. In dependence on \( p \) the fraction of states which are detected via the different criteria is shown.

**Criterion 1** [4]. Let \( A_i, B_i, i = 1, \ldots, n \) be operators on Alice’s (respectively, Bob’s) space, fulfilling

\[
\sum_{i=1}^{n} \delta^2(A_i) \geq U_A \quad \text{and} \quad \sum_{i=1}^{n} \delta^2(B_i) \geq U_B.
\]

We define \( M_i := A_i \otimes I + I \otimes B_i \). Then we have for a separable \( \rho \) the inequality

\[
\sum_{i=1}^{n} \delta^2(M_i) \rho \geq U_A + U_B.
\]

These criteria have a beautiful and clear physical interpretation. The Eqs. (2) are just uncertainty relations, expressing the fact that the \( A_i \) and \( B_i \) do not share a common eigenstate. Then, Eq. (3) shows that the separable states inherit the bounds from the local uncertainty relations in Eqs. (2). In Ref. [4] these criteria were generalized in the following way:

**Criterion 2** [4]. A state \( \rho \) is entangled if there exist \( M_i \) and a constant \( C > 0 \) such that \( \sum_{i} \delta^2(M_i) \rho < C \), holds, while for all product states \( \sum_{i} \delta^2(M_i) \rho > C \) is valid.

Although this criterion looks fairly obvious, it turns out that a proper choice of the \( M_i \) guarantees to detect many entangled states. For instance, all pure bipartite entangled states and a family of bound entangled states can be detected [4]. Also, one can detect multipartite entanglement and relate the variance based criteria for finite dimensional systems with criteria for infinite dimensional systems. But these connections are beyond the scope of this paper.

A different way of using uncertainty relations to detect entanglement uses entropic uncertainty relations (EURs). Let us briefly recall what these are. If we have a non-degenerate observable \( M = \sum_{i} \mu_i |m_i\rangle \langle m_i| \), a measurement of this observable in a quantum state \( \rho \) gives rise to a probability distribution of the different outcomes:

\[
\mathcal{P}(M)_{\rho} = (p_1, \ldots, p_n); \quad p_i = \langle m_i|\rho|m_i\rangle.
\]

It is now possible to measure the uncertainty of this measurement by taking the entropy of this probability distribution, i.e., by defining \( S(M) := S(\mathcal{P}(M))_{\rho} \). The entropy \( S \) used in this definition may be the standard Shannon entropy \( S^S(\mathcal{P}) := -\sum_k p_k \ln(p_k) \), or, more generally any so-called entropic function \( S(\mathcal{P}) = \sum s(p_i) \) where \( s : [0;1] \rightarrow R \) is a concave function, may be used. An example is the Tsallis entropy \( S^T_q(\mathcal{P}) := (1 - \sum (p_k)^q)/(q - 1) \), which depends on a parameter \( q > 0 \). For \( q = 1 \) we have \( S^T_1 = S^S \). With this definition of the uncertainty of a measurement it is clear that for two observables \( M = \sum \mu_i |m_i\rangle \langle m_i| \) and \( N = \sum v_i |n_i\rangle \langle n_i| \) which do not share a common eigenstate, there must exist a strictly positive constant \( C \) such that

\[
S^S(M) + S^S(N) \geq C
\]

holds. Estimating \( C \) is not easy, however it was shown in Ref. [5] that one could take \( C = -2 \ln(\max_{i,j} \langle m_i|n_j\rangle) \).

When \( M \) is a degenerate observable the definition of \( S(M) \) in the previous way is not applicable, since the spectral decomposition is not unique in this case. However, there is a unique way in writing \( M = \sum \mu_i X_i \) where the \( \mu_i \) are

**FIGURE 1.** (a) Schematic view of the set of all states and the separable states as a convex subset. \( \mathcal{X} \) denotes a witness (i.e. the set where \( Tr(\rho \mathcal{X}) = 0 \)) and \( \mathcal{L} \) a possible nonlinear witness. (b) A comparison between the LUR in Eq. (2), the witness resulting from the linear part of the LUR, and the PPT criterion for the family of states described in the text. In dependence on \( p \) the fraction of states which are detected via the different criteria is shown.
Thus, in this case, the LUR can be viewed as a nonlinear EW which improves a linear EW.

Let us start with an investigation of the LURs. For a single-qubit system it has been shown [3] that for the Pauli matrices the uncertainty relation \( \sum_{i=x,y,z} \delta^2(\sigma_i) \geq 2 \) holds. Defining \( M_i = \sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i \) this yields the LUR \( \sum_{i=x,y,z} \delta^2(M_i) \geq 4 \). A short calculation shows that from this equation it follows that for all separable states

\[
\langle \mathbb{1} \otimes \mathbb{1} + \sigma_x \otimes \sigma_x + \sigma_y \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_y + \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z \rangle - \frac{1}{2} \sum_{i=x,y,z} \langle \sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i \rangle^2 \geq 0
\]

has to hold. This is a quite remarkable equation for the following reason. The first part, which is linear in the expectation values is known to be an entanglement witness [8]. From this witness some quadratic terms are subtracted.

Let us investigate how big the improvement is. To this aim, we look at states of the form \( |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \). Physically, these states are a mixture of a singlet state and some separable noise. The parameter \( p \) determines the fidelity of the singlet state, and the parameter \( d \) the properties of the noise. For \( d = 0 \) the noise consists of white noise. The set of matrices \( \rho(p,d) \) governs a ball in the space of all matrices. We take the value \( d = 0.2 \) and generate matrices of the form \( \rho(p,0.2) \) randomly distributed in this ball [2]. Then we investigate the separability properties of these matrices. We determine the fraction of matrices which are detected by the witness and the LUR and check whether the matrices have a positive partial transpose (PPT), which is necessary and sufficient for entanglement in this case [2, 10]. Of course, these fractions depend on the value of \( p \). The results are shown in Fig. 1(b). On can clearly see that the LUR improves the witness significantly, although it is not capable of detecting all states.

To give a simple example for Criterion 2 we take as observables projectors onto Bell states. Let us denote them by \( |BS_i\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2} \). From the fact that the Schmidt coefficient of the Bell states are \( 1/\sqrt{2} \) one can calculate that for separable states

\[
\sum_{i} \delta^2(|BS_i\rangle \langle BS_i|) \rho = 1 - \sum_{i} Tr(\rho |BS_i\rangle \langle BS_i|)^2 \geq \frac{1}{2}
\]

holds [4]. To investigate the strength and the geometrical meaning of this inequality, let us introduce the coordinates \( i = Tr(\rho \sigma_i \otimes \sigma_i) \) for \( i = x,y,z \). In these coordinates we have \( \langle BS_1| \rho |BS_1\rangle = (1 + x - y + z)/4 \), \( \langle BS_2| \rho |BS_2\rangle = (1 - x + y + z)/4 \), \( \langle BS_3| \rho |BS_3\rangle = (1 + x + y - z)/4 \), and \( \langle BS_4| \rho |BS_4\rangle = (1 - x - y - z)/4 \).
How does the state space look in these coordinates? An arbitrary density matrix has to obey $\text{Tr}(\rho \mathbb{1} |B_{S_i}\rangle \langle B_{S_i}|) \geq 0$, which leads to the four conditions $x - y + z \leq 1, -x + y + z \leq 1, x + y - z \leq 1$, and $-x - y - z \leq 1$. These conditions describe a tetrahedron in the three-dimensional space (see Fig. 2(a)). A separable state has to obey in addition $\text{Tr}(\rho \mathbb{1} |BS_i\rangle \langle BS_i|) \geq 0$ for the witnesses $\mathbb{W}^{(i)}=1/2: \mathbb{1} - |BS_i\rangle \langle BS_i|$. This results in $x + y + z \leq 1, -x - y + z \leq 1, x - y - z \leq 1$, and $-x + y - z \leq 1$, which describes an octahedron in the tetrahedron from above. Furthermore, a straightforward calculation proves that in these coordinates Eq. (10) reads

$$x^2 + y^2 + z^2 \leq 1.$$ 

This is the equation of a three-dimensional sphere. The states inside this sphere are not detected by Eq. (11). As one can see in Fig. 2(a), some states which are detected by the witnesses $\mathbb{W}^{(i)}$ escape the detection via Eq. (10).

One can improve now the detection by using Criterion 4. Indeed, if we take an observable $M = \sum_i \mu_i |BS_i\rangle \langle BS_i|$ this criterion requires for a separable state $\rho$:

$$S_q^T(M)_{\rho} \geq \frac{1 - 2^{1-q}}{q - 1}. \quad (11)$$

This criterion, depending on $q$, can again be expressed in the coordinates $x, y, z$. Note that for $q = 2$ Eq. (11) is equivalent to Eq. (10). For two other values of $q$ Eq. (11) is plotted in Fig. 1(b) and 1(d). One can see that the strength of the criterion increases with $q$. This can also be proved analytically, and one can show that in the limit $q \to \infty$ Eq. (11) is equivalent to the witnesses $\mathbb{W}^{(i)}=1/2: \mathbb{1} - |BS_i\rangle \langle BS_i|$.  

CONCLUSION

In conclusion, we have shown that separability conditions can be derived from variance based uncertainty relations as well as from entropic uncertainty relations. The investigation of the resulting criteria showed that they are powerful tools for the detection of entanglement.

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