Low autocorrelated multi-phase sequences

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The interplay between the ground state energy of the generalized Bernasconi model to multi-phase, and the minimal value of the maximal autocorrelation function, $C_{\text{max}} = \max_K |C_K|$, $K = 1, \ldots, N - 1$, is examined analytically and the main results are: (a) The minimal value of $\min_N C_{\text{max}}$ is 0.435\sqrt{N} significantly smaller than the typical value for random sequences $O(\sqrt{\log N \sqrt{N}})$. (b) $\min_N C_{\text{max}}$ over all sequences of length $N$ is obtained in an energy which is about 30% above the ground-state energy of the generalized Bernasconi model, independent of the number of phases $m$. (c) The maximal merit factor $F_{\text{max}}$ grows linearly with $m$. (d) For a given $N$, $\min_N C_{\text{max}} \sim \sqrt{N/m}$ indicating that for $m = N$, $\min_N C_{\text{max}} = 1$, i.e. a Barker code exists. The analytical results are confirmed by simulations.

In many applications of communication science \cite{1,2}, as well as a variety of other fields, it is necessary to find sequences with low autocorrelation functions. Some of these applications utilize the pulse compression feature of the low autocorrelated sequences to obtain high resolution in radars and sonars. In other applications, the shifts of such periodic sequences can be used to identify users in multi-user systems. Due to their importance, low autocorrelated sequences have evoked a wide spread interest accompanied by the development of various methods for constructing such sequences \cite{3-6}.

In order to construct a binary sequence $S = (s_1, \ldots, s_N)$ with low off peak autocorrelations, one has to define which quantity has to be minimized. The different applications divide the low autocorrelated sequences into two types in correspondence with the quantity which they minimize. The first kind of low autocorrelated sequences minimizes the Hamiltonian of the Bernasconi model \cite{3-6} which is given by:

$$H = \sum_{K=1}^{N-1} C_K^2$$

where for the case of non-periodic boundary conditions, which is at the center of our study,

$$C_K = \sum_{j=1}^{N-K} s_j s_{j+K},$$

and for the periodic case $C_K = \sum_{j=1}^{N} s_j s_{(j+K-1) \mod N + 1}$. Note that for random sequences the average value of $H$ in the non-periodic case is $N^2/2$, whereas for low autocorrelated sequences this energy is reduced by a merit factor $F > 1$ to $N^2/2F$.

However, there are applications for which the maximal off peak autocorrelation, $C_{\text{max}} = \max_K |C_K|$, has to be minimized. The second kind of low autocorrelated sequences are the solutions of this minimization problem. Note that for random sequences $C_{\text{max}}$ is typically $O(\sqrt{\log N \sqrt{N}})$. For a sequence of length $N$, the maximal possible ratio between the peak, $|C_0| = N$, and the maximal off-peak autocorrelations, $|C_K|$ with $K = 1, \ldots, N - 1$ is $N/1$. The only known binary sequences with this ratio are the Barker codes of length 2, 3, 4, 5, 7, 11 and 13. Obviously, the Barker sequences, when they exist, furnish a minimum for the two minimization problems. However, Turyn \cite{7} has shown that no other binary codes such as this exist for any length less than 144 or for odd length greater than 13. An exact solution for these two minimization problems is known only for systems (up to $N = 59$) which are small enough to permit an effective exhaustive search \cite{8,9}. Extrapolation of the ground state energies which were found for small systems using an exhaustive search indicates $F_{\text{max}} = \lim_{N \to \infty} \sqrt{E_N} = 8.5$ \cite{8}. Moreover, $F_{\text{max}}$ was conjectured by Golay \cite{10,11} to be bounded from above by 12.324.

The following questions regarding low autocorrelated sequences are still open and are at the center of our study: (a) Do the sequences that minimize $C_{\text{max}}$ minimize the energy of the Bernasconi model as well? (b) In case that the two minimization problems are not equivalent, how far are the energy values of the sequences which minimize $C_{\text{max}}$ from the values of the ground state energy? (c) Exhaustive search of sequences with length up to $N = 59$ show that the degeneracy of the ground state is of $O(1)$ and is bounded from above by 8. Is the degeneracy of the states which minimize $C_{\text{max}}$ is of $O(1)$ as well? (d) In this context, which of the two minimal quantities is easier to approach? Additional interesting questions arise when multi-phase sequences, whose terms are complex $m$-th roots of 1 for $m > 2$, are considered. For such sequences

$$s_l = \exp\left(i\frac{2\pi l}{m}\right)$$

where $l=1,\ldots,m$ and the correlations are defined as.
\[ C_K = \sum_{j=1}^{N-K} s_j s_{j+K}^*. \]  

where \( s_{j+K}^* \) is the conjugate of \( s_{j+K} \). The information about multi-phase low autocorrelated sequences is more limited, mainly since the configuration space grows like \( m^N \) which makes the exhaustive search ineffective even for very small sequences. Hence, not much is known about the influence of the number of phases both on the maximal value of \( F \) and on the minimal value of \( C_{max} \).

Is it possible to attain better solutions by increasing the number of phases?

In this study we suggest an analytical technique to calculate the quantities of interest, namely, \( \max F \) and \( \min C_{max} \) for both binary and multi-phase sequences. An obvious lower bound of \( \min_{N,F} C_{max} \) over all sequences of length \( N \) and a merit factor \( F \) is obtained by assuming that all \( |C_K| \) equal \( \sqrt{N/(2F)} \). Nevertheless it is tempting to investigate the relations between \( F \) and \( \min_{N,F} C_{max} \) and to measure more accurately the optimal value \( \min_{N,F} C_{max} \) over all sequences of length \( N \) similarly to the aforementioned upper bound given for \( F_{max} \). For our analytical study we consider the non-periodic Bernasconi model. We approximate the sequences to be random and the autocorrelation functions to be independent variables with the following Gaussian distribution

\[ P(C_K) = \frac{1}{\sqrt{2\pi(N-K)}} \exp\left(-\frac{C_K^2}{2(N-K)}\right). \]

Using Eq. (5), the probability \( P_F(C_{max}) \) that the autocorrelations of a sequence with a merit factor \( F = N^2/2H \) have an upper bound \( C_{max} \) such that \( |C_K| \leq C_{max} \) is given by

\[ P_F(C_{max}) = \int_{-C_{max}}^{C_{max}} \prod_K dC_K P(C_K) \delta\left(1 - \frac{1}{N} \sum_{K=1}^{N-1} C_K^2 - \frac{N}{2F}\right). \]

Since there are \( 2^N \) distinct binary sequences of length \( N \), it is necessary that \( P_F \geq 2^{-N} \) in order for a sequence with the corresponding features, namely a merit factor \( F \) and an upper bound \( C_{max} \), to exist. Hence, by equating \( P_F(C_{max}) \) to \( 2^{-N} \) one can find the minimal upper bound \( \min_{N,F} C_{max} \) among the upper bounds of all sequences with a merit factor \( F \). Moreover, the Gaussian distributions of \( C_K \) imply that \( \min_{N,F} C_{max} \) must be of \( O(\sqrt{N}) \) in order for \( P_F(C_{max}) \) to be equal \( 2^{-N} \). Assigning \( \min_{N,F} C_{max} = B(F)\sqrt{N} \) and inserting the integral representation of \( \delta \) function in Eq. (6), the saddle point method can be used to obtain the following set of equations

\[ - \int_0^1 \ln \text{erf}(B(F)g(y, \lambda))dy = \ln 2, \]

\[ \int_0^1 \frac{1}{2F} - \frac{1}{4\lambda^2}(\ln(1 - 2\lambda) + 2\lambda) + \frac{B(F)\sqrt{2}}{\pi} \int_0^1 \frac{\exp(-B(F)^2g(y, \lambda)^2)}{\text{erf}(B(F)g(y, \lambda))\sqrt{2g(y, \lambda)}}dy = 0. \]

where \( g(y, \lambda) = \frac{1-\lambda}{\lambda - 2-\lambda}/21(1-y) \). Solving numerically this set of equations, it turns out that a solution exists only for a bounded region of \( F \), where \( 0.215 \leq F \leq 1.123 \). These two limit values of \( F \) are alternatively obtained by taking the limits of the integration in Eq. (6) to infinity. The existence of a lower bound is a consequence of the fact that as the energy becomes smaller the allowed values of the correlations decrease and the probability for such sequences (Eq. (6)) decreases as well. As was pointed out above, when this probability drops below \( 2^{-N} \), there are no more sequences with the required energy. Similarly, for high energy values, high values of correlations are required and the corresponding probabilities become smaller again. The analysis of \( B(F) \) inside the allowed region reveals that the minimal value of \( B(F_c) = 0.435 \) is obtained for \( F_c = 8.839 \) whereas \( \max F = 12.324 \) corresponds to a higher value of \( B(F) = 1 \). Thus, the merit factor of sequences which minimizes \( C_{max} \) is around 71% lower than \( F_{max} \). Exhaustive search results in the region \( N \in [15, 28] \) indicate that the average ratio between the two merit factors, \( \ll \ll F(\min_{N,F} C_{max}) \gg /F_{max} \gg N \), is \( \sim 0.677 \) where \( \ll \gg \) denotes the average over all sequences which minimize \( C_{max} \) whereas \( < > \) denotes the average over the different sequences length \( N \). The deviation from the analytical result \( \sim 0.717 \) is attributed to finite size effects. Fig. 2 shows the behavior of \( B(F) \) derived from the numerical solution of Eqs. (6) and (7). The behavior of \( P_F(C_{max}) \) divides this graph into three regimes. Starting with the large \( F \) regime, there is only one sequence whose \( C_{max} \) equals \( B(F)\sqrt{N} \). This feature holds until \( F_c = 8.839 \) for which the minimal \( B(F) = 0.435 \) is obtained. Further decrease of \( F \) results in an increasing number of sequences with an upper bound of \( B(F)\sqrt{N} \). This increment terminates at \( F = 0.8 \) and for smaller values of \( F \) the number of sequences reduces to 1 as \( F \) approaches 0.215. In order to compare the analytical behaviour of \( B \) to that obtained from simulations, it is necessary to find a way to circumvent the mismatch between the analytical values of \( F \) and those obtained from simulations. The rescaling of \( B \) by \( F_{max} \) yields a parameter \( F/F_{max} \) whose range of values, \([0, 1]\), does not depend on a specific realization. This allows a comparison between analytical results and simulations. Results of an exhaustive search over sequences of length \( N = 32 \) show that the behavior of \( B \) as a function of \( F/F_{max} \) resembles the analytical prediction (Fig. 3).
Moreover, the existence of the three aforementioned regions of \( P_F(C_{\text{max}}) \) is confirmed by counting the number of sequences with the minimal upper bound \( B \) for each of the micro-canonical ensembles with a merit factor \( F \) and length \( N = 32 \). The histogram is demonstrated in the inset of Fig. 2 where the merit factor is rescaled to \( F/F_{\text{max}} \). In the small \( F \) regime, the average number of sequences increases with \( F \) until \( F/F_{\text{max}} \sim 0.2 \) from which the number of sequences decreases up to the plateau which starts at \( F/F_{\text{max}} \sim 0.83 \).

**FIG. 1.** \( B(F) \) vs. \( F \). This graph is composed of three regions corresponding to the three different behaviours of \( P_F(C_{\text{max}}) \). In the dashed region \( 0.215 \leq F \leq 0.8 \), the fraction of sequences \( P_F(C_{\text{max}}) \) is getting larger as \( F \) increases, while in the dotted-line region, \( 0.8 < F < F_c \), this fraction decreases to \( 2^{-N} \) as \( F \) approaches \( F_c \). For \( F > F_c \) (solid-line) the fraction of sequences with the minimal value \( B(F) \sqrt{N} \) is constant and equals \( 2^{-N} \).

**FIG. 2.** \( B vs. F/F_{\text{max}} \). The solid line represents the analytical results while the circles represent the exhaustive search results for \( N = 32 \). Note that the analytical \( F_{\text{max}} \) is 12.324 whereas for \( N = 32 \) \( F_{\text{max}} = 8 \). Inset: a histogram describing the logarithm of the number of sequences with the minimal upper bound \( B \) for each of the micro-canonical ensembles with a merit factor \( F \) as was derived from an exhaustive search for \( N = 32 \).

Now we turn to study the influence of the number of phases both on the maximal value of \( F \) and on the minimal value of \( C_{\text{max}} \). We assign \( D_K \) and \( E_K \) to be independent Gaussian variables which represent the real and the imaginary parts of \( C_K \) respectively. In case that \( m = 2^p \) where \( p \) is an integer number greater than 1, these two parts have the same probability distribution

\[
P(D_K) = \frac{1}{\sqrt{\pi (N - K)}} \exp\left(\frac{-D_K^2}{(N - K)}\right).
\]  

Under the same assumptions that have been used in the derivation of the binary case, Eq. (3) becomes

\[
P^m_F(C_{\text{max}}) = \int_{-C_{\text{max}}}^{C_{\text{max}}} \prod_K dD_K dE_K P(D_K) P(E_K)
\]

\[
d\left(\frac{1}{N} \sum_{K=1}^{N} (D_K^2 + E_K^2) - \frac{N}{2F}\right).
\]

Similarly to the binary case, if \( P^m_F(C_{\text{max}}) = m^{-N} \) then \( B(F) \sqrt{N} \) is the minimal value of \( C_{\text{max}} \) for which there is a sequence with a merit factor \( F \). The same procedure that has been used to calculate \( B(F) \) for the binary case results in the following set of equations

\[
\frac{\lambda}{2F} - \frac{1}{\lambda^2} (\ln(1 - \lambda) + \lambda)
\]

\[-2 \int_0^1 \ln \text{erf}(B(F) g(y, \lambda)) dy = \ln(m).
\]

\[
\frac{1}{2F} - \frac{1}{\lambda^2} (\ln(1 - \lambda) + \lambda) + \frac{2B(F)}{\sqrt{\pi}} \int_0^1 \frac{\exp(-B(F)^2 g(y, \lambda)^2)}{\text{erf}(B(F) g(y, \lambda))} dy = 0,
\]

where \( g(y, \lambda) = \sqrt{\frac{1 - \lambda (1 - y)}{1 - y}} \). Solving numerically these two equations, the behaviors of \( \min_N C_{\text{max}}/\sqrt{N} \) and \( F_{\text{max}} \) as a function of \( m \) are obtained. Fig. 3 shows that \( \min_N C_{\text{max}}/\sqrt{N} \) is proportional to \( 1/\sqrt{m} \) with slight deviations for small \( m \). This relation between \( B(F_c) = \min_N C_{\text{max}}/\sqrt{N} \) and \( m \) implies that such a relation holds for \( B(F_{\text{max}}) \) as well. Since the variance of \( C_K^2 \) equals \( N - K \), their typical values drop linearly with \( K \). Hence, \( H = N^2/2F \) is approximated by a sum of an algebraic series of \( N - 1 \) terms, \( C_K^2 \), with an upper bound, \( C_{\text{max}}^2 \).
of $O(N/m)$. Assuming that $C_K^2$ is homogeneously distributed between 0 to $C_{\text{max}}^2$ yields a linear increment of $F_{\text{max}}$ as a function of $m$ in agreement with the numerical solution of $F_{\text{max}}$ which is depicted in the inset of Fig. 4. The results of $F_{\text{max}}$ and $B(F_c) = \min_N C_{\text{max}}/\sqrt{N}$ as a function of $m$ show that the solutions of the two minimization problems are improved by increasing the number of phases $m$.

These results raise the question whether it is possible to increase $m$ such that $\min_N C_{\text{max}}$ becomes 1. An asymptotic expansion of Eqs. (11) and (12) reveals that for $m = N$, $\min_N C_{\text{max}} = 1$. Note that $C_{-1}$ is always 1 and therefore $\min_N C_{\text{max}} = 1$ holds for the entire regime $m > N$.

We used the simulated annealing method for sequences of lengths $N = 32$ with different number of phases $m$, to find $\min_N C_{\text{max}}$. The relatively small sequences size was chosen to enable an appropriate scan of the configuration space in reasonable computational time. The results are exhibited in Fig. 4, and support the anticipated $\sqrt{N/m}$ behavior of $\min_N C_{\text{max}}$. For $m = 32$, there is a deviation of $\min_N C_{\text{max}}$, from the analytical prediction $\min_N C_{\text{max}} = 1$, probably since the simulated annealing method yields only suboptimal solutions.

Finally, we would like to examine the simulated annealing method in light of the results of this study. Simulations show that for the same running times, the simulated annealing method yields $C_{\text{max}}$ which is closer to its minimal value than $\sum_{K=1}^{N-1} C_K^2$. This can be explained by the larger degeneracy of $C_{\text{max}}$ compared with that of $\sum_{K=1}^{N-1} C_K^2$. Moreover, it turns out that in order to minimize $C_{\text{max}}$, it is preferable to start the searching process with the minimization of the energy function, $\sum_{K=1}^{N-1} C_K^2$, and then replace it with $C_{\text{max}}$. In this way the system avoids the plateaus which characterize the landscape of $C_{\text{max}}$ in the configuration space. However, future research is necessary to find out how the results of this study can be applied to further improve the searching processes of low autocorrelated sequences.

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FIG. 3. $\min_N C_{\text{max}}/\sqrt{N}$ vs. $m$ as was obtained from the numerical solution of Eqs. (11) and (12). The solid line is the least square fit $0.54/\sqrt{m}$.

FIG. 4. Results for $\min_N C_{\text{max}}$ as a function of $m$ for sequences of length $N = 32$. The filled circles stand for the simulated annealing results. The deviation from the analytical results (solid-line) is attributed to finite size effects and to the suboptimal solution obtained in our limited running times of the simulations. Inset: Analytical results for $F_{\text{max}}$ as a function of $m$ which were obtained from the numerical solution of Eqs. (11) and (12). The solid line is the least square fit, $3.7m$.

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