Algebraic Bethe ansatz for singular solutions

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Abstract
The Bethe equations for the isotropic periodic spin-1/2 Heisenberg chain with N sites have solutions containing ±i/2 that are singular: both the corresponding energy and the algebraic Bethe ansatz vector are divergent. Such solutions must be carefully regularized. We consider a regularization involving a parameter that can be determined using a generalization of the Bethe equations. These generalized Bethe equations provide a practical way of determining which singular solutions correspond to eigenvectors of the model.

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1. Introduction
It is well known that the isotropic periodic spin-1/2 Heisenberg quantum spin chain with N sites, with Hamiltonian

\[ H = \frac{1}{4} \sum_{n=1}^{N} (\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - 1), \quad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1, \]  

(1)
can be solved by algebraic Bethe ansatz (ABA): the eigenvalues are given by

\[ E = -\frac{1}{2} \sum_{k=1}^{M} \frac{1}{\lambda_k^2 + \frac{1}{4}}, \]  

(2)
and the corresponding su(2) highest-weight eigenvectors are given by the Bethe vectors

\[ |\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1) \cdots B(\lambda_M)|0\rangle, \]  

(3)
where |0\rangle is the reference state with all spins up, \{\lambda_1, \ldots, \lambda_M\} are distinct and satisfy the Bethe equations

\[ \left(\frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}}\right)^N = \prod_{j \neq k}^{M} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \quad k = 1, \ldots, M, \]  

(4)
and \( M = 0, 1, \ldots, \frac{N}{2} \). The spin s of the state is given by \( s = \frac{N}{2} - M \). (See, for example, [1, 2].)
It is also well known that the so-called two-string \((\lambda_1, \lambda_2) = \left(\frac{1}{2}, -\frac{i}{2}\right)\) is an exact solution of the Bethe equations for \(N \geq 4\). This fact is particularly easy to see from the Bethe equations in the pole-free form
\[
\begin{align*}
\left(\lambda_1 + \frac{i}{2}\right)^N &\ (\lambda_1 - \lambda_2 - i) = \left(\lambda_1 - \frac{i}{2}\right)^N (\lambda_1 - \lambda_2 + i), \\
\left(\lambda_2 + \frac{i}{2}\right)^N &\ (\lambda_2 - \lambda_1 - i) = \left(\lambda_2 - \frac{i}{2}\right)^N (\lambda_2 - \lambda_1 + i).
\end{align*}
\]
This solution is singular, as both the corresponding energy (2) and Bethe vector (3) are divergent\(^1\. Clearly, it is necessary to regularize this solution. The naive regularization
\[
\lambda_1^{\text{naive}} = \frac{i}{2} + \epsilon, \quad \lambda_2^{\text{naive}} = -\frac{i}{2} + \epsilon
\]
gives the correct value of the energy in the \(\epsilon \to 0\) limit, namely, \(E = -1\).

What is perhaps not so well known is that this naive regularization gives a wrong result for the eigenvector\(^2\. Indeed, the vector \(\lim_{\epsilon \to 0} |\lambda_1^{\text{naive}}, \lambda_2^{\text{naive}}\rangle\) is finite, but it is not an eigenvector of the Hamiltonian! For example, in the case \(N = 4\), we easily find with Mathematica that
\[
\lim_{\epsilon \to 0} |\lambda_1^{\text{naive}}, \lambda_2^{\text{naive}}\rangle = (0, 0, 0, 2, 0, 0, -2, 0, 0, 0, 0, 0, 2, 0, 0, 0),
\]
while the correct eigenvector with \(E = -1\) and \(s = 0\) is known to be instead\(^3\)
\[
(0, 0, 0, 2, 0, 0, -2, 0, 0, -2, 0, 0, 0, 2, 0, 0, 0).
\]
We further observe that, for general values of \(N\), the correct eigenvector can be obtained within the ABA approach by introducing a suitable additional correction of order \(\epsilon^N\) to the Bethe roots\(^4\):
\[
\lambda_1 = \frac{i}{2} + \epsilon + c\epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon,
\]
where the parameter \(c\) is independent of \(\epsilon\). Returning to the example of \(N = 4\), we find
\[
\lim_{\epsilon \to 0} |\lambda_1, \lambda_2\rangle = (0, 0, 0, 2, 0, 0, -2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0).
\]
Comparing with (9), we see that the requisite value of the parameter in this case is \(c = 2i\).

In section 2, we address the question of how to determine in a systematic way the parameter \(c\) in (10), which (as we have already seen) is necessary for obtaining the correct eigenvector. Clearly, it is not a matter of simply solving the Bethe equations (4), since they are not satisfied by (10) for \(\epsilon\) finite. Indeed, we shall find that the Bethe equations themselves acquire \(\epsilon\)-dependent corrections. These ‘generalized’ Bethe equations (see equation (20) below) constitute our main new result. In section 3, we extend this approach to general singular solutions, i.e., solutions of the Bethe equations where two of the roots are \(\pm \frac{i}{2}\). Typically, there are many such solutions, but relatively few correspond to eigenvectors of the model. We find

\(^1\) While the divergence of the energy is obvious, the divergence of the Bethe vector is a consequence of our non-standard conventions, which we specify in section 2 below. In the standard conventions, the Bethe vector would instead be null.

\(^2\) Difficulties with constructing the eigenvector corresponding to the Bethe roots \(\pm \frac{i}{2}\) were already noted in [3, 4].

\(^3\) For any even \(N\), the Bethe vector corresponding to the two-string \(\pm \frac{i}{2}\) can be expressed as [5]
\[
\sum_{i=1}^{N} (-1)^i S_i S_{i+1}(0).
\]
One can easily verify that for \(N = 4\) this vector is indeed proportional to (9).

\(^4\) Such higher-order corrections of singular Bethe roots were already noted in equation (3.4) of [6] and studied further in [7].
that the generalized Bethe equations provide a practical way of determining which of the singular solutions correspond to eigenvectors. Section 4 summarizes our main conclusions.

Singular solutions do not appear in a related model, namely, the Heisenberg chain with twisted boundary conditions. A small twist angle $\phi$ then plays a similar role to our parameter $\epsilon$. This alternative approach for dealing with singular solutions is briefly considered in appendix B.3 of [8] and in section 2.1 of [9]. Since the twist breaks the $su(2)$ symmetry, the Bethe vectors are no longer highest-weight vectors. Our point of view is that the isotropic periodic Heisenberg chain for finite $N$ is a well-defined model, and therefore should be understandable independently of other models; it is only its Bethe ansatz solution that is not completely well defined.

Yet another approach for constructing the Bethe vectors corresponding to singular solutions, involving Sklyanin’s separation of variables, was carefully analyzed in [10].

2. Determining the parameter

We begin by briefly establishing our conventions. Following [1], the $R$-matrix is given by

$$ R_{a_1 a_2}^{a_3 a_4}(\lambda) = \lambda^{a_1 a_2} + i P_{a_1 a_2}, $$

where $\lambda$ and $P$ are the 4 × 4 identity and permutation matrices, respectively. However, as explained below, we choose a different normalization for the Lax operator, namely,

$$ L_{na}(\lambda) = \left[ \left( \lambda - \frac{i}{\lambda + \frac{1}{2}} \right) I_{na} + i P_{na} \right], $$

which diverges for $\lambda = -\frac{1}{2}$. As usual, the monodromy matrix is given by

$$ T_a(\lambda) = L_{Na}(\lambda) \cdots L_{1a}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, $$

and the transfer matrix is given by

$$ t(\lambda) = \text{tr}_a T_a(\lambda) = A(\lambda) + D(\lambda). $$

The reference state is denoted by $|0\rangle = (|0\rangle)^{\otimes N}$.

We next recall the action of the transfer matrix on an off-shell Bethe vector (3) [1]

$$ t(\lambda)|\lambda_1, \ldots, \lambda_M\rangle = \Lambda(\lambda)|\lambda_1, \ldots, \lambda_M\rangle + \sum_{k=1}^{M} F_k(\lambda, \{\lambda\}) B(\lambda_1) \cdots \hat{B}(\lambda_k) \cdots B(\lambda_M) B(\lambda)|0\rangle, $$

where a hat is used to denote an operator that is omitted, and

$$ \Lambda(\lambda) = \prod_{j=1}^{M} \left( \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} \right) + \prod_{j=1}^{M} \left( \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j} \right), $$

$$ F_k(\lambda, \{\lambda\}) = \frac{i}{\lambda - \lambda_k} \left[ \prod_{j \neq k} \left( \frac{\lambda_k - \lambda_j - i}{\lambda_k - \lambda_j} \right) - \left( \frac{\lambda_k - \lambda_j + i}{\lambda_k + \frac{1}{2}} \right) \right]. $$

The Bethe equations (4) are precisely the conditions $F_k(\lambda, \{\lambda\}) = 0$, which ensure that the ‘unwanted’ terms vanish, in which case the Bethe vector $|\lambda_1, \ldots, \lambda_M\rangle$ is an eigenvector of the transfer matrix $t(\lambda)$, with corresponding eigenvalue $\Lambda(\lambda)$ given by (17). In particular, for $M = 2$, the relation (16) reduces to

$$ t(\lambda)|\lambda_1, \lambda_2\rangle = \Lambda(\lambda)|\lambda_1, \lambda_2\rangle + F_1(\lambda, \{\lambda\}) B(\lambda_2) B(\lambda)|0\rangle + F_2(\lambda, \{\lambda\}) B(\lambda_1) B(\lambda)|0\rangle, $$

which holds for generic values of $\lambda, \lambda_1$ and $\lambda_2$. 

Let us now focus on the special case of the two-string solution ± 1/2. As already mentioned in the introduction, the corresponding Bethe vector |1/2, − 1/2⟩ is singular: some of its components have the form 0/0. (If we had defined the Lax operator (13) without dividing by (λ + 1/2) as in (11), then the corresponding Bethe vector would instead be null.) In particular, the creation operator B(± 1/2) is finite, but B(− 1/2) is singular.

Let us first consider the naive regularization (6). The key observation is that, for $\epsilon \to 0$, the most singular matrix elements of $B(\lambda^\text{naive}_2)$ are of order $1/\epsilon$. (See (A.5).) It follows from the off-shell relation (19) that, for $\epsilon \to 0$, the coefficients $F_1$ and $F_2$ must satisfy

$$F_1(\lambda, \{\lambda\}) \sim \epsilon^{N+1}, \quad F_2(\lambda, \{\lambda\}) \sim \epsilon,$$  (20)

in order that the Bethe vector $\lim_{\epsilon \to 0} |\lambda^\text{naive}_1, \lambda^\text{naive}_2\rangle$ be an eigenvector of the transfer matrix. However, explicit computation using (6) shows that $F_1(\lambda, \{\lambda\}) \sim \epsilon^N$ (instead of $\epsilon^{N+1}$) and $F_2(\lambda, \{\lambda\}) \sim 1$ (instead of $\epsilon$). Hence, the ‘unwanted’ terms in (19) are finite (do not vanish), which explains why the corresponding Bethe vector is not an eigenvector$^5$.

Let us therefore consider the regularization (10). The leading behavior of $B(\lambda_1)$ and $B(\lambda_2)$ as $\epsilon \to 0$ remains the same as with the naive regularization; i.e., $B(\lambda_1) \sim \lambda^N$ and $B(\lambda_2) \sim 1$. Hence, the conditions (20) must still be satisfied to ensure that the Bethe vector is an eigenvector of the transfer matrix. Explicit computation using (10) gives

$$F_1(\lambda, \{\lambda\}) = \left( c + 2i^{−(N+1)} \right) \epsilon^N + O(\epsilon^{N+1}), \quad F_2(\lambda, \{\lambda\}) = \left( \frac{2i^{−N}}{\lambda + \frac{1}{2}} \right) + O(\epsilon).$$

(21)

For even $N$, both conditions (20) can be satisfied by setting

$$c = 2i^{−Nolec}{\frac{3}{2}},$$

(22)

which reproduces our earlier result for $N = 4$ (see below equation (11)). We have also explicitly verified that, for $N = 6$, the ABA Bethe vector constructed using (10) and (22) is indeed an eigenvector of the Hamiltonian$^6$. Interestingly, the two conditions (20) cannot be simultaneously satisfied for odd $N$, implying that the two-string ± 1/2 is not a bona fide solution for odd $N$.$^7$

We note that the regularization (10) can be slightly generalized. Indeed, we can introduce a two-parameter regularization

$$\lambda_1 = \frac{i}{2} + \epsilon + c_1 \epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon + c_2 \epsilon^N.$$  (23)

The conditions (20) now imply (for even $N$) that

$$c_1 - c_2 = 2i^{−Nolec}{\frac{3}{2}}.$$  (24)

For finite $\epsilon$, the corresponding energy (2) depends only on the difference $c_1 - c_2$. If we impose the additional constraint $\lambda_1 = \lambda_2^*$ [11], then we obtain $c_1 = c_2^* = i^{−Nolec}{\frac{3}{2}}$. In short, for even $N$, a regularization of the singular solution ± 1/2 that produces the correct eigenvector in the $\epsilon \to 0$ limit, and also satisfies $\lambda_1 = \lambda_2^*$, is given by

$$\lambda_1 = \frac{i}{2} + \epsilon + i^{−Nolec}{\frac{3}{2}} \epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon - i^{−Nolec}{\frac{3}{2}} \epsilon^N.$$  (25)

5 The fact that $B(\lambda^\text{naive}_2)$ has matrix elements of order $1/\epsilon$ suggests that $|\lambda^\text{naive}_1, \lambda^\text{naive}_2\rangle \sim \frac{1}{\epsilon}$. However, as shown in the appendix, this vector is finite for $\epsilon \to 0$.

6 It was claimed in [4] that the Bethe ansatz fails for this case.

7 For $N = 5$, the Clebsch–Gordan theorem implies that there are five highest-weight eigenvectors with $s = 1/2$, and we have explicitly verified that all of these eigenvectors can be constructed with Bethe roots other than ± 1/2, thereby directly proving that the solution ± 1/2 must be discarded.
3. General singular solutions

We now consider a general singular solution of the Bethe equations, which has the form

\[ \{ \frac{i}{2}, -\frac{i}{2}, \lambda_3, \ldots, \lambda_M \} \], \hspace{1cm} (26)

where \( \lambda_3, \ldots, \lambda_M \) are distinct and are not equal to \( \pm \frac{i}{2} \). Proceeding as before, we regularize the first two roots as in equation (10). The Bethe equations (4) imply that the last \( M-2 \) roots \( \{ \lambda_3, \ldots, \lambda_M \} \) obey

\[ \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right)^{N-1} \left( \frac{\lambda_k - \frac{3i}{2}}{\lambda_k + \frac{3i}{2}} \right) = \prod_{j \neq k}^{M} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \hspace{1cm} k = 3, \ldots, M. \hspace{1cm} (27) \]

We again impose the two generalized Bethe equations

\[ F_1 (\lambda, \{ \lambda \}) \sim \epsilon^{N+1}, \hspace{1cm} F_2 (\lambda, \{ \lambda \}) \sim \epsilon, \hspace{1cm} (28) \]

where \( F_k \) is defined in (18). The equation (28) ensure that the Bethe vector corresponding to the singular solution (26), namely

\[ \lim \epsilon \to 0 |\lambda_1, \ldots, \lambda_M \rangle, \hspace{1cm} (29) \]

where \( \lambda_1, \lambda_2 \) are given by (10) and \( |\lambda_1, \ldots, \lambda_M \rangle \) is given by (3), is an eigenvector of the transfer matrix.

In other words, given a solution \( \{ \lambda_3, \ldots, \lambda_M \} \) of (27), if the equations (28) can be satisfied, then they determine the parameter \( c \) in (10), and the corresponding Bethe vector (29) is an eigenvector of the transfer matrix. We call such a singular solution ‘physical’. On the other hand, if the equations (28) cannot be satisfied, then—despite the fact that the usual Bethe equations (4), (27) are obeyed—this solution cannot be used to construct an eigenvector of the transfer matrix. We call such a singular solution ‘unphysical’. Hence, according to the previous section, all singular solutions with odd \( N \) and \( M = 2 \) are unphysical.

Equations (28) can be simplified as follows. Using (10), we find that these two equations imply

\[ c = -\frac{2}{i^{N+1}} \prod_{j=3}^{M} \frac{\lambda_j - \frac{3i}{2}}{\lambda_j + \frac{i}{2}}, \hspace{1cm} c = 2i^{N+1} \prod_{j=3}^{M} \frac{\lambda_j + \frac{3i}{2}}{\lambda_j - \frac{i}{2}}, \hspace{1cm} (30) \]

respectively. These equations in turn imply the consistency condition

\[ \prod_{j=3}^{M} \left( \frac{\lambda_j - \frac{3i}{2}}{\lambda_j + \frac{i}{2}} \right) \left( \frac{\lambda_j + \frac{3i}{2}}{\lambda_j - \frac{i}{2}} \right) = (-1)^N. \hspace{1cm} (31) \]

By forming the product of all the Bethe equations (27), we obtain the relation

\[ \prod_{k=3}^{M} \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right)^{N-1} \left( \frac{\lambda_k - \frac{3i}{2}}{\lambda_k + \frac{3i}{2}} \right) = 1, \hspace{1cm} (32) \]

using which the consistency condition (31) takes the simple form

\[ \left[ -\prod_{k=3}^{M} \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right) \right]^N = 1. \hspace{1cm} (33) \]

We remark that the condition (33) provides a practical way to select from among the many singular solutions of the Bethe equations (27) the physically relevant subset, which is
generally much smaller. For example, for \( N = 6 \) and \( M = 3 \), the Bethe equations (4), (27) have five singular solutions, of which only one is physical. Similarly, for \( N = 8 \) and \( M = 4 \), we find 21 singular solutions, of which only three are physical.8

4. Conclusion

We have seen that the ABA for the isotropic periodic Heisenberg chain must be extended for solutions of the Bethe equations containing \( \pm \frac{i}{2} \). Indeed, such singular solutions must be carefully regularized as in (10) or (23). This regularization involves a parameter that can be determined using a generalization of the Bethe equations given by (20), where \( F_k \) is defined in (18). These equations also provide a practical way of determining which singular solutions correspond to eigenvectors of the model. In particular, the solution \( \pm \frac{i}{2} \) must be excluded for odd \( N \).

It would be interesting to know whether the finite-\( \epsilon \) corrections to the energy have any physical significance. We expect that our analysis can be extended to the anisotropic case.

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Note added. After completing this work, we became aware of [13], where similar results were obtained for the solution \( \pm \frac{i}{2} \). However, our approach differs significantly from theirs.

Appendix

Here we fill in some details. It is convenient to define an unrenormalized Lax operator (as in [1]):

\[
\tilde{L}_{na}(\lambda) = \left( \lambda - \frac{i}{2} \right) I_{na} - \imath P_{na},
\]

and correspondingly

\[
\tilde{T}_a(\lambda) = \tilde{L}_{Na}(\lambda) \cdots \tilde{L}_{1a}(\lambda) = \begin{pmatrix} \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & \tilde{D}(\lambda) \end{pmatrix}.
\]

Evidently,

\[
L_{na}(\lambda) = \frac{1}{(\lambda + \frac{i}{2})} \tilde{L}_{na}(\lambda), \quad T_a(\lambda) = \frac{1}{(\lambda + \frac{i}{2})^N} \tilde{T}_a(\lambda).
\]

In particular,

\[
B(\lambda) = \frac{1}{(\lambda + \frac{i}{2})^N} \tilde{B}(\lambda).
\]

Since \( \tilde{B}(\pm \frac{i}{2}) \) are finite, it follows that \( B(\pm \frac{i}{2}) \) is also finite, and

\[
B\left( \pm \frac{i}{2} + \epsilon \right) \sim \frac{1}{\epsilon^N}
\]

plus less singular terms.

8 The number of singular states of the XXZ chain are estimated in [12].
The fact (A.5) suggests that \(|\tilde{\lambda}_1^{\text{naive}}, \tilde{\lambda}_2^{\text{naive}}\rangle = B\left(\frac{i}{2} + \epsilon\right)B\left(-\frac{i}{2} + \epsilon\right)|0\rangle\) should be similarly divergent for \(\epsilon \to 0\). However, we shall now argue that this vector is in fact finite. In view of (A.4), it suffices to show that\(^9\)

\[
\tilde{B}\left(\frac{i}{2} + \epsilon\right)\tilde{B}\left(-\frac{i}{2} + \epsilon\right)|0\rangle \sim \epsilon^N.
\]

To this end, we proceed by induction. The behavior (A.6) can be easily verified explicitly for \(N = 4\) using Mathematica. We observe from (A.2) that the monodromy matrices for \(N - 1\) and \(N\) sites are related by

\[
\tilde{L}_N^a(\lambda) = \tilde{L}_{N-1}^a(\lambda)\tilde{L}_{N-1}^a(\lambda),
\]

which implies that

\[
\begin{pmatrix}
\tilde{A}^{(N)}(\lambda) \\
\tilde{C}^{(N)}(\lambda)
\end{pmatrix}
\begin{pmatrix}
\tilde{B}^{(N)}(\lambda) \\
\tilde{D}^{(N)}(\lambda)
\end{pmatrix}
= \begin{pmatrix}
\tilde{a}_N(\lambda) & \tilde{b}_N(\lambda) \\
\tilde{c}_N(\lambda) & \tilde{d}_N(\lambda)
\end{pmatrix}
\begin{pmatrix}
\tilde{A}^{(N-1)}(\lambda) & \tilde{B}^{(N-1)}(\lambda) \\
\tilde{C}^{(N-1)}(\lambda) & \tilde{D}^{(N-1)}(\lambda)
\end{pmatrix},
\]

where

\[
\begin{align*}
\tilde{a}_N(\lambda) &= \begin{pmatrix} \lambda + \frac{i}{2} & 0 \\ 0 & \lambda - \frac{i}{2} \end{pmatrix}, \\
\tilde{b}_N(\lambda) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\tilde{c}_N(\lambda) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\tilde{d}_N(\lambda) &= \begin{pmatrix} \lambda - \frac{i}{2} & 0 \\ 0 & \lambda + \frac{i}{2} \end{pmatrix}.
\end{align*}
\]

In particular,

\[\tilde{B}^{(N)}(\lambda) = \tilde{a}_N(\lambda)\tilde{B}^{(N-1)}(\lambda) + \tilde{b}_N(\lambda)\tilde{D}^{(N-1)}(\lambda).\]

It follows that

\[
\tilde{B}^{(N)}(\lambda_1)\tilde{B}^{(N)}(\lambda_2)|0\rangle^{(N)} = [\tilde{a}_N(\lambda_1)\tilde{B}^{(N-1)}(\lambda_1) + \tilde{b}_N(\lambda_1)\tilde{D}^{(N-1)}(\lambda_1)]
\]

\[
\times [\tilde{a}_N(\lambda_2)\tilde{B}^{(N-1)}(\lambda_2) + \tilde{b}_N(\lambda_2)\tilde{D}^{(N-1)}(\lambda_2)]|0\rangle^{(N-1)}\begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

\[
= [\tilde{a}_N(\lambda_1)\tilde{a}_N(\lambda_2)\tilde{B}^{(N-1)}(\lambda_1)\tilde{B}^{(N-1)}(\lambda_2) \\
+ \tilde{a}_N(\lambda_1)\tilde{b}_N(\lambda_2)\tilde{B}^{(N-1)}(\lambda_1)\tilde{D}^{(N-1)}(\lambda_2) \\
+ \tilde{b}_N(\lambda_1)\tilde{a}_N(\lambda_2)\tilde{B}^{(N-1)}(\lambda_1)\tilde{B}^{(N-1)}(\lambda_2) \\
+ \tilde{b}_N(\lambda_1)\tilde{b}_N(\lambda_2)\tilde{D}^{(N-1)}(\lambda_1)\tilde{D}^{(N-1)}(\lambda_2)]|0\rangle^{(N-1)}\begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

for \(\lambda_1, \lambda_2\) arbitrary.

We now set \(\lambda_1 = \lambda_1^{\text{naive}} = \frac{i}{2} + \epsilon\) and \(\lambda_2 = \lambda_2^{\text{naive}} = -\frac{i}{2} + \epsilon\), and we consider the four terms on the rhs of (A.11), starting with the first: by the induction hypothesis,

\[\tilde{B}^{(N-1)}(\lambda_1)\tilde{B}^{(N-1)}(\lambda_2)|0\rangle^{(N-1)} \sim \epsilon^{N-1}.
\]

Moreover, it is easy to see that

\[
\tilde{a}_N(\lambda_1)\tilde{a}_N(\lambda_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sim \epsilon.
\]

Hence, the first term on the rhs of (A.11) is of order \(\epsilon^N\).

The fourth term on the rhs of (A.11) is zero because

\[
\tilde{b}_N(\lambda_1)\tilde{b}_N(\lambda_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.
\]

\(^9\) The result (A.6) implies, as already noted, that this vector is null in the limit \(\epsilon \to 0\).
Using the exchange relation [1]
\[
\tilde{D}(\lambda_1)\tilde{B}(\lambda_2) = \frac{\lambda_1 - \lambda_2 + i\tilde{B}(\lambda_2)\tilde{D}(\lambda_1)}{\lambda_1 - \lambda_2} - \frac{i}{\lambda_1 - \lambda_2} \tilde{B}(\lambda_1)\tilde{D}(\lambda_2)
\] (A.15)
in the third term, we see that the second and third terms on the rhs of (A.11) combine to give
\[
[[\tilde{a}_N(\lambda_1)\tilde{b}_N(\lambda_2) - \tilde{b}_N(\lambda_1)\tilde{a}_N(\lambda_2)]\tilde{B}^{(N-1)}(\lambda_1)\tilde{D}^{(N-1)}(\lambda_2)
+ 2\tilde{b}_N(\lambda_1)\tilde{a}_N(\lambda_2)\tilde{B}^{(N-1)}(\lambda_2)\tilde{D}^{(N-1)}(\lambda_1)][|0\rangle^{(N)}_{N}].
\] (A.16)
The first line of (A.16) gives a vanishing contribution because
\[
[[\tilde{a}_N(\lambda_1)\tilde{b}_N(\lambda_2) - \tilde{b}_N(\lambda_1)\tilde{a}_N(\lambda_2)]|0\rangle_{N} = 0.
\] (A.17)
The second line of (A.16) is of order \(\epsilon^N\), since
\[
\tilde{D}^{(N-1)}(\lambda_1)|0\rangle^{(N-1)}_{N} \sim \epsilon^{N-1},
\] (A.18)
and
\[
\tilde{b}_N(\lambda_1)\tilde{a}_N(\lambda_2)|0\rangle_{N} \sim \epsilon.
\] (A.19)
In short, we have shown that
\[
\tilde{B}^{(N)}(\lambda_1)\tilde{B}^{(N)}(\lambda_2)|0\rangle^{(N)}_{N} \sim \epsilon^{N},
\] (A.20)
which concludes the inductive proof of our claim (A.6).

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