An improved estimate for electric fields between two spherical insulators in three dimensions

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Abstract

In this paper, we consider a gradient estimate for a conductivity problem whose inclusions are two neighboring insulators in three dimensions. When inclusions with extreme conductivity (insulator or perfect conductor) are closely located, the gradients of solutions may become arbitrarily large in the narrow region in between inclusions as the distance between inclusions approaches zero. The estimate for gradient between insulators in three dimensions has been regarded as a challenging problem, while the optimal estimates in terms of the distance have been known for the other problems of perfectly conducting inclusions in two and higher dimensions, and insulators in two dimensions. In this paper, we establish an upper bound of gradient on the shortest line segment between two insulating unit spheres in three dimensions. It presents an improved dependency of gradient on the distance which is substantially different from the blow-up rates in the other extreme conductivity problems.

1 Introduction

Let $B_1$ and $B_2$ be two bounded simply connected domains in $\mathbb{R}^d$, $d = 2, 3$. We consider the following conductivity problem: for a given harmonic function $H$ in $\mathbb{R}^d$,

\[
\begin{cases}
\nabla \cdot \left( (k - 1)\chi_{(B_1 \cup B_2)} + \chi_{(\mathbb{R}^d \setminus (B_1 \cup B_2))} \right) \nabla u = 0 \quad \text{in } \mathbb{R}^d, \\
u(x) - H(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty.
\end{cases}
\]

where $\chi$ is the characteristic function. Thus, two inclusions $B_1$ and $B_2$ are conductors with conductivity $k \neq 1$, embedded in the background with conductivity 1. Let $\epsilon$ denote the distance between $B_1$ and $B_2$, i.e.,

$\epsilon := \text{dist}(B_1, B_2),$

and we assume that the distance $\epsilon$ is small.

The problem is to estimate $|\nabla u|$ in the narrow region in between inclusions. This was raised by Babuška in relation to the study of material failure of composites [4]. In fiber-reinforced composites which consist of stiff inclusions and the matrix, a high shear stress concentration can occur in between closely spaced neighboring inclusions. It is important to estimate the shear stress tensor $\nabla u$ in the study of material failure.

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while $u$ means the out-of-plane displacement, and $B_1$ and $B_2$ are the cross-sections of fibers.

Successful results have been achieved in all cases except three dimensional insulators which we consider in this paper. They can be divided into three cases when $k$ is stays away from 0 and $\infty$, when $k$ degenerates to either 0 (insulating) or $\infty$ (perfectly conducting) in two dimensions, and when $k = \infty$ in three and higher dimensions.

First, if $k$ stays away from 0 and $\infty$, i.e., $c_1 < k < c_2$ for some positive constants $c_1$ and $c_2$, then it was proved by Li-Vogelius [19] that $|\nabla u|$ remains bounded regardless of $\epsilon$. This result was extended to elliptic system by Li-Nirenberg [18]. The results in [19] [18] are valid in the case of multiple inclusions.

Second, if $k$ is either 0 or $\infty$ in the two dimensional problem, then $\nabla u$ may blow up as the distance $\epsilon$ tends to 0 and the generic blow-up rate of $\nabla u$ is $1/\sqrt{\epsilon}$. For two circular inclusions, it was shown in [7] that the gradient in general becomes unbounded as $\epsilon$ approaches zero and the blow-up rate is $\epsilon^{-1/2}$. A lower bound and an upper bound for the gradient were derived by Kang-Lim et.al. in [3] [2]. These bounds are valid for all $k$ including extreme values ($k = 0$ and $k = \infty$) and provide the precise dependence of $\nabla u$ on $\epsilon$, $k$ and radii of disks. In [23] [24], Yun showed that the blow-up rate is $\epsilon^{-1/2}$ for perfectly conducting and insulating inclusions of sufficiently general shape. Lim-Yun showed in [21] that if there is a small bump in between two inclusions, then the concentration of $|\nabla u|$ can be enhanced strongly. In [11], Kang-Lim-Yun established an asymptotic for the distribution $\nabla u$, when $B_1$ and $B_2$ are disks. For sufficiently general shape of inclusions, Ammari et.al. proved in [1] that $\nabla u$ has an asymptote proportional to the distribution between circular ones which osculate the inclusions at the approaching points, and Kang-Lee-Yun devised a numerically stable method to get the magnitude of $\nabla u$ in [10]. Hence, the distribution of $\nabla u$ can be described well, even though the inclusions have sufficiently general shape in $\mathbb{R}^2$.

Third, if $k = \infty$ in three dimensions and higher, Bao-Li-Yin [5] proved that the generic blow-up rate for the perfectly conducting inclusions is $|\epsilon \log \epsilon|^{-1}$ in $\mathbb{R}^3$ and $|\epsilon|^{-1}$ in higher dimensions, and they also proved in [6] that the generic rates work in the case of multiple inclusions in two and higher dimensions. Lim-Yun [20] also found the explicit dependency of $|\nabla u|$ on the radii as well as the distance $\epsilon$, when two inclusions are spheres in three and higher dimensions, see also [14] [15] [16]. Kang-Lim-Yun established an asymptotic for the distribution $\nabla u$ in [12], when $B_1$ and $B_2$ are two perfectly conducting unit spheres in $\mathbb{R}^3$.

This paper is mainly concerned with the gradient estimate for a conductivity problem in three dimensions whose conductivity $k$ degenerates to 0. The three dimensional insulating case has been regarded as a challenging problem. In [6], Bao-Li-Yin derived an upper bound of $|\nabla u|$ with order $1/\sqrt{\epsilon}$ in three dimensions. To our best knowledge, there has not been any updated or improved result yet. In this paper, we present an improved estimate for $|\nabla u|$ on the shortest line segment between two insulating unit spheres in three dimensions, and the dependency of $|\nabla u|$ on $\epsilon$ is substantially different from the blow-up rates known in the other extreme conductivity problems. Moreover, the estimate is derived in a new method not used in any other cases due to the different nature of the problem.

2 Main Result

Let $B_1$ and $B_2$ be a pair of unit spheres $\epsilon$ apart as follows:

$$B_1 = B_1 \left(-1 - \frac{\epsilon}{2}, 0, 0\right) \quad \text{and} \quad B_2 = B_1 \left(1 + \frac{\epsilon}{2}, 0, 0\right).$$
Thus, the quantity $\epsilon$ means the distance between $B_1$ and $B_2$, and the centers of $B_1$ and $B_2$ lie on the $x$ axis. For any harmonic function $H$ defined on $\mathbb{R}^3$, let $u$ be the solution to the conductivity problem whose conductivity $k$ degenerates to 0:

$$\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^3 \setminus (B_1 \cup B_2), \\
\partial_{\nu} u = 0 & \text{on } \partial (B_1 \cup B_2), \\
\ |x| \to \infty.
\end{cases}$$

(2.1)

Here, $x = (x, y, z)$ in $\mathbb{R}^3$.

**Theorem 2.1** For any harmonic function $H$ defined in $\mathbb{R}^3$, let $u$ be the solution to (2.1). Then, there is a constant $C$ independent of the distance $\epsilon > 0$ such that

$$|\nabla u(x, 0, 0)| \leq \frac{1}{\epsilon^{2 - \sqrt{2}}}$$

for $|x| < \frac{\epsilon}{2}$ and any small $\epsilon > 0$.

The proof of this theorem is presented immediately after Proposition 2.4. This theorem can be obtained from Propositions 2.2, 2.3, and 2.4.

The following proposition means that the directional derivative of $u$ in the $x$ direction is bounded, while two centers of $B_1$ and $B_2$ lie on the $x$ axis.

**Proposition 2.2** Let $u$ be the solution to (2.1) for a harmonic function $H$ in $\mathbb{R}^3$ as given in Theorem 2.1. Then, there is a constant $C$ independent of $\epsilon > 0$ such that

$$|\partial_{x} u(x, 0, 0)| \leq C$$

for $|x| < \frac{\epsilon}{2}$ and any small $\epsilon > 0$.

This proof of Proposition 2.2 is provided in Subsection 3.1.

Estimating the other directional derivatives, we use the following proposition to simplify the problem.

**Proposition 2.3** Let $H_1$ be a harmonic function in $\mathbb{R}^3$ and $H_2$ be a linear function defined for a constant $M$ as

$$H_2(x, y, z) = My$$

for any $(x, y, z) \in \mathbb{R}^3$. Let $u_1$ and $u_2$ be the solutions to (2.1) for $H = H_1$ and $H = H_2$, respectively. Then, there is a large constant $M$ independent of $\epsilon > 0$ such that

$$|\partial_{y} u_1(x, 0, 0)| \leq |\partial_{y} u_2(x, 0, 0)|$$

for $|x| < \frac{\epsilon}{2}$ and any small $\epsilon > 0$.

In Subsection 3.2, we prove Proposition 2.3.

The following proposition is an essential part of this paper that actually yields the main result.

**Proposition 2.4** Let $u$ be the solution to (2.1) for $H(x, y, z) = y$. Then, there is a constant $C$ independent of $\epsilon > 0$ such that

$$|\nabla u(x, 0, 0)| \leq \frac{1}{\epsilon^{2 - \sqrt{2}}}$$

for $|x| < \frac{\epsilon}{2}$ and any small $\epsilon > 0$.  

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In Section 4, we prove Proposition 2.4. To do so, we present Proposition 4.1 that obviously implies Proposition 2.4. Thus, Section 4 is mainly devoted to showing Proposition 4.1.

Now, we are ready to prove the main result by virtue of Propositions 2.2, 2.3 and 2.4.

**Proof of Theorem 2.1.** Propositions 2.2 means that the directional derivative \( u \) in the direction of the \( x \)-axis is bounded independently of \( \epsilon > 0 \) on the line segment between \( B_1 \) and \( B_2 \), supposed that the distance \( \epsilon \) is small enough. We consider the other directional derivatives. Propositions 2.3 and 2.4 imply that

\[
|\partial_y u(x,0,0)| \leq |\nabla u_2(x,0,0)| \leq C_2 \frac{1}{\frac{1}{2} - \frac{x^2}{2}}
\]

for \( |x| < \frac{1}{2} \) and any small \( \epsilon > 0 \), where \( u_2 \) is the solution to (2.1) for \( H = M_2 y \) for a large \( M_2 > 0 \). Similarly, we can choose a positive constant \( M_3 \) independent of \( \epsilon > 0 \) so that

\[
|\partial_z u(x,0,0)| \leq |\nabla u_3(x,0,0)| \leq C_3 \frac{1}{\frac{1}{2} - \frac{x^2}{2}}
\]

for \( |x| < \frac{1}{2} \) and any small \( \epsilon > 0 \) where \( u_3 \) is the solution to (2.1) for \( H = M_3 z \) for a large \( M_3 > 0 \). Therefore, we complete the proof. □

3 Representation of the solution \( u \)

The solution can be decomposed into three harmonic functions as

\[
u(x,y,z) = H(x,y,z) + R_{B_1}(x,y,z) + R_{B_2}(x,y,z), \tag{3.1}
\]

where the harmonic function \( R_{B_i} \) is defined in \( \mathbb{R}^2 \setminus B_i \) and satisfies the decay condition \( R_{B_i} = O\left(\frac{1}{|x|}\right) \) as \( |x| \to \infty \) for \( i = 1, 2 \). The decomposition can be derived from the representation of \( u \) as a sum of \( H \) and two single layer potentials on \( \partial D_1 \) and \( \partial D_2 \), respectively. For details, refer to the invertibility of \(-\frac{1}{2}I - K^*\) presented in Section 2 in [1], where \( K^* \) is the Neumann-Poincaré operator. Physically, \( R_{B_1} \) is the reflection of \( H + R_{B_2} \) occurring on the insulated inclusion \( B_1 \) only, and similarly \( R_{B_2} \) is the reflection of \( H + R_{B_1} \) on \( B_2 \).

In this paper two harmonic functions \( R_{B_1} \) and \( R_{B_2} \) play an important role, since they are used for proving Proposition 2.4 that is actually the main result in this paper. We study the properties of \( R_{B_1} \) and \( R_{B_2} \) in Section 4 where the proof of Proposition 2.4 is presented.

Another representation of \( u \) is also introduced in Lemma 3.1. This involves the derivations of Propositions 2.2 and 2.3. To illustrate the representation, we consider the reflection only for a single inclusion \( B_0 \) that denotes the unit sphere with center \((0,0,0)\), i.e.,

\[
B_0 = B_1(0,0,0).
\]

For any harmonic function \( h \) defined in a neighborhood containing \( B_0 \), let \( R_0(h) \) be the reflection of a given harmonic function \( h \) with respect to \( B_0 \), i.e.,

\[
\begin{align*}
\Delta R_0(h) &= 0 &\text{in } \mathbb{R}^3 \setminus B_0, \\
\partial_n(h + R_0(h)) &= 0 &\text{on } \partial B_0 \\
R_0(h)(x) &= O\left(\frac{1}{|x|^2}\right) &\text{as } |x| \to \infty.
\end{align*}
\tag{3.2}
\]
In the spherical coordinate system,

\[ R_0(h)(\rho, \theta, \phi) = \frac{1}{\rho} h \left( \frac{1}{\rho}, \theta, \phi \right) - \int_0^\rho h(s, \theta, \phi) \, ds \]  
(3.3)

for \( \rho \geq 1 \). In the Cartesian coordinate system,

\[ \partial_y R_0(h)(x, 0, 0) = \frac{1}{x^3} \partial_y h \left( \frac{1}{x}, 0, 0 \right) - \frac{1}{x} \int_0^\frac{1}{x} \partial_y h(s, 0, 0) \, ds \]  
(3.4)

for \( x \geq 1 \).

Similarly, for any harmonic function \( h \) defined in a neighborhood containing \( \overline{B_i} \), we define \( R_i(h) \) as the reflection of \( h \) with respect to \( B_i \) for \( i = 1, 2 \) as follows:

\[
\begin{align*}
\Delta R_i(h) &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_i}, \\
\partial_y (h + R_i(h)) &= 0 & \text{on } \partial B_i, \\
R_i(h)(x) &= O \left( \frac{1}{|x|^2} \right) & \text{as } |x| \to \infty.
\end{align*}
\]

**Lemma 3.1** The following sum converges to the solution \( u \) in the sense of the Sobolev space \( W^{4, \infty}(\mathbb{R}^3 \setminus (B_1 \cup B_2)) \):

\[
u(x) = H(x) + R_1(H)(x) + R_2(H)(x) + \sum_{n=1}^{\infty} (R_1 R_2)^n(H)(x) + (R_1 R_2)^n R_1(H)(x) + \sum_{n=1}^{\infty} (R_2 R_1)^n(H)(x) + (R_2 R_1)^n R_2(H)(x)
\]  
(3.5)

for any \( x \in \mathbb{R}^3 \setminus (B_1 \cup B_2) \).

**Proof.** Let \( h \) be a harmonic function defined in a neighborhood containing \( \overline{B_0} \) with the decay condition \( h(x) = O \left( \frac{1}{|x|^4} \right) \) as \( |x| \to \infty \). Then, we shall show two properties of (3.6) and (3.7) that are essential to prove the convergence of the series.

We first show that

\[
\left| \int_{\partial B_2} R_2(h) \partial_y (R_2(h)) \, dS \right| \leq \left( \frac{1}{1+\epsilon} \right)^3 \left| \int_{\partial B_1} h \partial_y h \, dS \right|.
\]  
(3.6)

Since \( B_1 \) is the unit sphere with the center \( (-1 - \frac{\epsilon}{2}, 0, 0) \), the function \( h \) can be expressed in terms of spherical harmonic functions whose center is \( (-1 - \frac{\epsilon}{2}, 0, 0) \). By the decay condition of \( h \), we have

\[
\frac{1}{(1+\epsilon)^3} \left| \int_{\partial B_1} h \partial_y h \, dS \right| \geq \left| \int_{|x-(1-\frac{\epsilon}{2},0,0)|=1+\epsilon} h \partial_y h \, dS \right| = \left| \int_{|x-(1-\frac{\epsilon}{2},0,0)|>1+\epsilon} |\nabla h|^2 \, dV \right| \geq \left| \int_{\partial B_2} h \partial_y h \, dS \right| \geq \left| \int_{\partial B_2} R_2(h) (\partial_y R_2(h)) \, dS \right|.
\]
Thus, we have (3.6).

Second, we prove that there is a constant $C_1$ such that

$$C_1 \left| \int_{\partial B_1} h \partial \nu \, dS \right|^2 \geq \| R_2(h) \|_{W^{1,\infty}(\mathbb{R} \setminus B_2)}.$$  \hspace{1cm} (3.7)

The mean value property for harmonic functions yields a positive constant $C_2$ such that

$$\left| \int_{\partial B_1} h \partial \nu \, dS \right|^2 = \left| \int_{\mathbb{R}^3 \setminus B_1} |\nabla h|^2 \, dV \right|^2 \geq C_2 \| \nabla h \|_{L^\infty((x-(1+\frac{1}{n}\epsilon,0,0)) \leq 1+\frac{1}{\epsilon})} \geq C_3 \left( \max_{|x-(1+\frac{1}{n}\epsilon,0,0)| \leq 1+\frac{1}{\epsilon}} (h(x)) - \min_{|x-(1+\frac{1}{n}\epsilon,0,0)| \leq 1+\frac{1}{\epsilon}} (h(x)) \right).$$

The positive constants $C_2$ and $C_3$ are used above regardless of choosing a harmonic function $h$ in $H^1(\mathbb{R}^3 \setminus B_1)$. Note that $R_2(h)$ can be extended as a harmonic function into the area $|x-(1+\frac{1}{n}\epsilon,0,0)| > \frac{1}{1+\epsilon}$, and that the analogous formula for $R_2(h)$ with (3.3) is valid in the extended domain. Thus, the formula for $R_2(h)$ implies

$$C_4 \left| \int_{\partial B_1} h \partial \nu \, dS \right|^2 \geq \| R_2(h) \|_{L^\infty(\mathbb{R}^3 \setminus B_2)}.$$

The distance between the boundaries of $B_2$ and the extended domain of $R_2(h)$ is at least $\frac{1}{4} \epsilon$. For any point in $\mathbb{R}^3 \setminus B_2$, the extended domain contains the open sphere whose center is the given point and the radius is $\frac{1}{4} \epsilon$. Thus, a gradient estimate for harmonic functions implies

$$C_5 \left| \int_{\partial B_1} h \partial \nu \, dS \right|^2 \geq \| \nabla R_2(h) \|_{L^\infty(\mathbb{R}^3 \setminus B_2)},$$

and moreover, the bound (3.7) for the higher order derivatives can be derived in the same way.

Now, we are ready to prove this lemma. By (3.9), we have

$$\left| \int_{\partial B_1} (R_1 R_2)^n(H) \partial \nu \left( (R_1 R_2)^n(H) \right) dS \right| + \left| \int_{\partial B_1} (R_1 R_2)^n(R_1 H) \partial \nu \left( (R_1 R_2)^n(R_1 H) \right) dS \right| + \left| \int_{\partial B_2} (R_2 R_1)^n(H) \partial \nu \left( (R_2 R_1)^n(H) \right) dS \right| + \left| \int_{\partial B_2} (R_2 R_1)^n(R_2 H) \partial \nu \left( (R_2 R_1)^n(R_2 H) \right) dS \right| \leq \frac{2}{(1+\epsilon)^{3(2n-1)}} \left| \sum_{i=1}^{2} \int_{\partial B_i} R_i(H) \partial \nu R_i(H) dS \right| \leq \frac{2}{(1+\epsilon)^{3(2n-1)}} \left| \sum_{i=1}^{2} \int_{\partial B_i} H \partial \nu H dS \right|$$
for any \( n = 1, 2, 3, \cdots \). By (3.7), we have
\[
\|(R_1 R_2)^n(H)\|_{W^{4, \infty}(\mathbb{R}^3 \setminus B_1)} + \|(R_1 R_2)^n(R_3 H)\|_{W^{4, \infty}(\mathbb{R}^3 \setminus B_1)} \\
+ \|(R_2 R_1)^n(H)\|_{W^{4, \infty}(\mathbb{R}^3 \setminus B_2)} + \|(R_2 R_1)^n(R_2 H)\|_{W^{4, \infty}(\mathbb{R}^3 \setminus B_2)} \\
\leq C_1 \frac{2}{(1 + \epsilon)^{(2n-1)}} \sum_{i=1}^{2} \int_{\partial B_i} H \partial_x H dS
\]  
(3.8)
for any \( n = 1, 2, 3, \cdots \), where \( C_1 \) is the constant in (3.7). This implies that the series in the right hand side of (3.5) are convergent in the sense of \( W^{4, \infty}(\mathbb{R}^3 \setminus (B_1 \cup B_2)) \) and thus satisfies (2.1). Hence, the series converges to the solution \( u \). \( \square \)

Propositions 2.2 and 2.3 can be derived from basic properties of the representations introduced before.

3.1 Proof of Proposition 2.2

We begin in considering the case of a single inclusion \( B_0 \). As defined before, \( R_0(h) \) is the reflection of a given harmonic function \( h \) with respect to \( B_0 \). By the equation (3.3),
\[
\partial_x R_0(h)(x, 0, 0) = -\frac{1}{x^2} \partial_x h \left( \frac{1}{x}, 0, 0 \right) \quad \text{for} \quad x > 1.
\]  
(3.9)
Here, \( \frac{1}{x} \) in the equation means the first coordinate of \( \left( \frac{1}{x}, 0, 0 \right) \) at which is the image charge of \( (x, 0, 0) \) with respect to \( B_0 \) is located.

Lemma 3.1 implies that the solution \( u \) results from the recursive reflections on two single inclusions. Dealing with the recursive reflections, we define \( r_1(x) \) and \( r_2(x) \) as the first coordinates of the image charges of \( (x, 0, 0) \) with respect to \( B_1 \) and \( B_2 \), respectively. Thus,
\[
\begin{align*}
    r_1(x) &= \frac{1}{x + 1 + \frac{\epsilon}{2}} - \left( 1 + \frac{\epsilon}{2} \right) \quad \text{for} \quad x \geq -\frac{\epsilon}{2}, \\
    r_2(x) &= 1 + \frac{\epsilon}{2} - \frac{1}{1 + \frac{\epsilon}{2} - x} \quad \text{for} \quad x \leq \frac{\epsilon}{2}.
\end{align*}
\]

For \( |x| < \frac{\epsilon}{2} \), we define two sequences \((r_{A_n})\) and \((r_{B_n})\) as follows:
\[
\begin{align*}
    r_{A2n-1} &= (r_2 r_1)^{n-1} r_2(x) \quad \text{and} \quad r_{A2n} = -(r_1 r_2)^n(x), \\
    r_{B2n-1} &= -(r_1 r_2)^{n-1} r_1(x) \quad \text{and} \quad r_{B2n} = (r_2 r_1)^n(x)
\end{align*}
\]
for \( n = 1, 2, 3, \cdots \), where \( (r_i r_j)(x) = r_i(r_j(x)) \) for \( \{i, j\} = \{1, 2\} \). Applying (3.9) to (3.5),
\[
\begin{align*}
    \partial_x u(x, 0, 0) &= \partial_x H(x, 0, 0) \\
    &+ \sum_{n=1}^{\infty} (-1)^n \left( \prod_{k=1}^{n} \left( 1 + \frac{\epsilon}{2} - r_{A_k} \right) \right)^3 \partial_x H(r_{A_n}, 0, 0) \\
    &+ \sum_{n=1}^{\infty} (-1)^n \left( \prod_{k=1}^{n} \left( 1 + \frac{\epsilon}{2} - r_{B_k} \right) \right)^3 \partial_x H(r_{B_n}, 0, 0)
\end{align*}
\]
for any \( |x| \leq \frac{\epsilon}{2} \).
Indeed, two positive sequences \((r_{A_n})\) and \((r_{B_n})\) are increasing and converge to a number that is \(\sqrt{r} + O(\epsilon)\). There are some properties that can be shown easily:

\[
\frac{\epsilon}{2} \leq r_{A_n}(x) \leq 2\sqrt{r}, \quad \frac{\epsilon}{2} \leq r_{B_n}(x) \leq 2\sqrt{r}
\]

for \(|x| \leq \frac{1}{2}\) and \(n = 0, 1, 2, 3, \ldots\), and

\[
\frac{1}{20} \sqrt{r} \leq (r_2 r_1)^n(x) \quad \text{and} \quad \frac{1}{20} \sqrt{r} \leq -(r_1 r_2)^n(x)
\]

for any \(n > \frac{1}{10\sqrt{r}}\) and \(|x| \leq \frac{1}{2}\). These properties follows immediately from Lemma 4.8 in this paper. For \(j = A\) or \(B\),

\[
\left| \sum_{n=1}^{\infty} (-1)^n \left( \prod_{k=1}^{n} \left( 1 + \frac{\epsilon}{2} - r_{jk} \right) \right)^3 \partial_x H(r_{jn}, 0, 0) \right| \leq \sum_{\tilde{n} = 1}^{\infty} \left( \prod_{k=1}^{2\tilde{n}-1} \left( 1 + \frac{\epsilon}{2} - r_{jk} \right) \right)^3 \times \left( |\partial_x H(r_{j(2\tilde{n}-1)}, 0, 0) - \partial_x H(r_{j(2\tilde{n})}, 0, 0)| + (6\sqrt{r} + O(\epsilon)) |\partial_x H(r_{j(2\tilde{n})}, 0, 0)| \right) \leq \sum_{\tilde{n}=1}^{\infty} \left( 4\sqrt{r} \| \partial_x^2 H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} + (6\sqrt{r} + O(\epsilon)) \| \partial_x H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} \right) + \sum_{\tilde{n}=1}^{\infty} 8 \left( 1 + \frac{\epsilon}{2} \frac{\sqrt{r}}{20} \right)^{6 \left( \tilde{n} - \frac{1}{20\sqrt{r}} \right)} \times \left( 4\sqrt{r} \| \partial_x^2 H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} + (6\sqrt{r} + O(\epsilon)) \| \partial_x H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} \right) \leq C \left( \| \partial_x^2 H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} + \| \partial_x H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} \right)
\]

for small \(\epsilon > 0\). Hence,

\[
\| \partial_x u(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} \leq C \left( \| \partial_x^2 H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} + \| \partial_x H(x, 0, 0) \|_{L^\infty([-\frac{1}{2}, \frac{1}{2}])} \right)
\]

for small \(\epsilon > 0\).

### 3.2 Proof of Proposition 2.3

In the following lemma, we first consider the model of a single inclusion \(B_0\) that is much simpler than our model of two inclusions. Applying the lemma to Lemma 3.1 we prove Proposition 2.3.

**Lemma 3.2** Let \(h\) be a harmonic function defined in a neighborhood containing the closure of the unit sphere \(B_0 = B_1(0, 0, 0)\), and \(R_0(h)\) be defined as (?). Suppose that

\[
\partial_y h(x, 0, 0) \geq 0 \quad \text{and} \quad \partial_x \partial_y h(x, 0, 0) \geq 0
\]

for \(0 \leq x \leq 1\). Then,

\[
\partial_y R_0(h)(x, 0, 0) \geq 0 \quad \text{and} \quad \partial_x \partial_y R_0(h)(x, 0, 0) \leq 0
\]

for \(x \geq 1\).
Proof. By (3.1) and the assumption,
\[
\partial_y R_0(h)(x, 0, 0) = \frac{1}{x^3} \partial_y h \left( \frac{1}{x}, 0, 0 \right) - \frac{1}{x} \int_0^\ell s \partial_y h \left( s, 0, 0 \right) ds 
\]
\[
= \frac{1}{x} \int_0^x s \left( 2 \partial_y h \left( \frac{1}{x}, 0, 0 \right) - \partial_y h \left( s, 0, 0 \right) \right) ds \geq 0 
\]
for \( x \geq 1 \). Thus, we have the first bound.

We can derive the second bound from (3.4) as follows:
\[
\partial_x \partial_y R_0(h)(x, 0, 0) = -\frac{2}{x^4} \partial_y h \left( \frac{1}{x}, 0, 0 \right) + \frac{1}{x^3} \int_0^\ell s \partial_y h \left( s, 0, 0 \right) ds - \frac{1}{x^3} \partial_x \partial_y h \left( \frac{1}{x}, 0, 0 \right) 
\]
\[
\leq -\frac{1}{x^2} \int_0^\ell s \left( 4 \partial_y h \left( \frac{1}{x}, 0, 0 \right) - \partial_y h \left( s, 0, 0 \right) \right) ds \leq 0. 
\]
\( \square \)

Now, we are ready to prove Proposition 2.3.

Proof of Proposition 2.3. From the definitions of \( R_1 \) and \( R_2 \),
\[
\partial_y R_i(y)(x, 0, 0) = \frac{1}{2 (1 + \frac{x}{2} - (-1)^i x)} > 0, 
\]
(3.10)
\[
(-1)^i \partial_x \partial_y R_i(y)(x, 0, 0) = \frac{3}{2 (1 + \frac{x}{2} - (-1)^i x)} > 0 
\]
(3.11)
for \(-1 - \frac{x}{2} \leq (-1)^i x \leq \frac{x}{2} \) and \( i = 1, 2 \). There is a large \( M > 0 \) such that
\[
\partial_y R_i(H)(x, 0, 0) \leq \partial_y R_i(M y)(x, 0, 0) 
\]
and
\[
(-1)^i \partial_x \partial_y R_i(H)(x, 0, 0) \leq (-1)^i \partial_x \partial_y R_i(M y)(x, 0, 0) 
\]
for \(-1 - \frac{x}{2} \leq (-1)^i x \leq \frac{x}{2} \) and \( i = 1, 2 \). By mathematical induction, Lemma 3.2 allows
the upper and lower bounds of \( \partial_y (R_1 R_2)^n(H)(x, 0, 0) \) and \( \partial_x \partial_y (R_1 R_2)^n(H)(x, 0, 0) \)
and so on such that for any \( n = 1, 2, 3, 4, \cdots, \)
\[
\partial_y (R_1 R_2)^n(H)(x, 0, 0) \leq \partial_y (R_1 R_2)(M y)(x, 0, 0), \n\]
\[
\partial_x \partial_y (R_1 R_2)^n(H)(x, 0, 0) \geq \partial_x \partial_y (R_1 R_2)(M y)(x, 0, 0) 
\]
and
\[
\partial_y (R_1 R_2)^{n - 1} R_1(H)(x, 0, 0) \leq \partial_y (R_1 R_2)^{n - 1} R_1(M y)(x, 0, 0), \n\]
\[
\partial_x \partial_y (R_1 R_2)^{n - 1} R_1(H)(x, 0, 0) \geq \partial_x \partial_y (R_1 R_2)^{n - 1} R_1(M y)(x, 0, 0) 
\]
for \(-\frac{x}{2} \leq x \leq 1 + \frac{x}{2}, \) and
\[
\partial_y (R_2 R_1)^n(H)(x, 0, 0) \leq \partial_y (R_2 R_1)(M y)(x, 0, 0), \n\]
\[
\partial_x \partial_y (R_2 R_1)^n(H)(x, 0, 0) \leq \partial_x \partial_y (R_2 R_1)(M y)(x, 0, 0) 
\]
and
\[
\partial_y (R_2 R_1)^{n - 1} R_1(H)(x, 0, 0) \leq \partial_y (R_2 R_1)^{n - 1} R_2(M y)(x, 0, 0), \n\]
\[
\partial_x \partial_y (R_2 R_1)^{n - 1} R_1(H)(x, 0, 0) \leq \partial_x \partial_y (R_2 R_1)^{n - 1} R_2(M y)(x, 0, 0) 
\]
for $-1 - \frac{\epsilon}{2} \leq x \leq \frac{\epsilon}{2}$. By Lemma 3.1, we have the upper bound of $\partial_y u_1(x, 0, 0)$

$$\partial_y u_1(x, 0, 0) \leq H(x, 0, 0) + R_1(My)(x, 0, 0) + R_2(My)(x, 0, 0) + \sum_{n=1}^{\infty} (R_1 R_2)^n (My)(x, 0, 0) + \sum_{n=1}^{\infty} (R_2 R_1)^n (My)(x, 0, 0)$$

and the lower bound is also obtained in the same way.

\[\square\]

4 Derivation of Proposition 2.4

In this section, we assume that $H(x, y, z) = y$ in $\mathbb{R}^3$ and $u$ is the solution to (2.1) for $H = y$. As defined in the decomposition (3.1), two harmonic functions $R_{B_1}$ and $R_{B_2}$ satisfies

$$u(x, y, z) = y + R_{B_1}(x, y, z) + R_{B_2}(x, y, z),$$

$R_{B_i}$ is defined in $\mathbb{R}^2 \setminus B_i$ and satisfies the decay condition $R_{B_i} = O\left(\frac{1}{|x|^2}\right)$ as $|x| \to \infty$ for $i = 1, 2$.

In this section, we shall prove the following proposition that obviously implies Proposition 2.4.

**Proposition 4.1** $\partial_y R_{B_1}(x, 0, 0)$ is decreasing in $[-\frac{\epsilon}{2}, \infty)$ and

$$0 \leq \partial_y R_{B_1} \left(-\frac{\epsilon}{2}, 0, 0\right) \gtrsim \frac{1}{\epsilon^{\frac{\sqrt{2}}{2}}}. \tag{4.1}$$

It follows from the property $\partial_y R_{B_1}(x, 0, 0) = \partial_y R_{B_2}(-x, 0, 0)$ that

$$\partial_y u(x, 0, 0) = 1 + \partial_y R_{B_1}(x, 0, 0) + \partial_y R_{B_2}(x, 0, 0) \lesssim \frac{1}{\epsilon^{\frac{\sqrt{2}}{2}}}$$

for $|x| < \frac{\epsilon}{2}$.

The proof of Proposition 4.1 is presented in Subsection 4.2 based on the lemmas in Subsection 4.1. Lemma 4.3 implies the decreasing property of $\partial_y R_{B_1}(x, 0, 0)$, and Subsection 4.2 is mainly devoted to deriving the estimate (4.1) in Proposition 4.1.

4.1 Basic Properties of $\partial_y R_{B_1}(x, 0, 0)$

We consider the behavior of $\partial_y R_{B_1}(x, 0, 0)$ to derive Proposition 4.1. In this subsection, $H(x, y, z) = y$ in $\mathbb{R}^3$ as assumed early in Section 4. For convenience, we define the function $P : [1, \infty) \to \mathbb{R}$ as

$$P(x) = \partial_y R_{B_1} \left(x - 1 - \frac{\epsilon}{2}, 0, 0\right)$$
that is a horizontal shift of $\partial_y R_{B_1}(x,0,0)$. The translation moves the left inclusion $B_1$ to $B_1(0,0,0)$ so that the domain of $P$ is the interval $[1, \infty)$ with the initial point 1. The symmetry between $\partial_y R_{B_1}(x,0,0)$ and $\partial_y R_{B_2}(x,0,0)$ yields

$$
\partial_y u(x,0,0) = \partial_y H(x,0,0) + \partial_y R_{B_1}(x,0,0) + \partial_y R_{B_2}(x,0,0)
= 1 + P\left(x + 1 + \frac{\epsilon}{2}\right) + P\left(-x + 1 + \frac{\epsilon}{2}\right)
$$

for $|x| \leq \frac{1}{\epsilon}$, since $H(x,y,z) = H(-x,y,z)$. We study the behavior of $P(x)$ especially for small $x - 1 \geq 0$ to prove the bound

$$
P(x) \lesssim \frac{1}{\epsilon^{2-2\sqrt{2}}} \quad (4.2)
$$

that means Propositions 4.1 and 2.4.

The basic properties of $P$ are introduced in this subsection. First, Lemma 4.2 provides a fundamental equation (4.5) of $P$ that yields almost all properties of $P$ including the main result in this paper. Second, Lemmas 4.3 and 4.4 describe the geometric behavior of $P$. Third, Lemma 4.5 presents an estimate for the integral value of $P$ that determines the blow-up rate of $P$ as $\epsilon$ approaches 0. Finally, based on these properties, our main result (4.2) can be obtained in Subsection 4.2 to show Proposition 4.1.

Dealing with lemmas, it is necessary to define two special points $p_1$ and $p_2$ as a pair of solutions to $2 + \epsilon - \frac{1}{x} = x$, where $p_1 < p_2$. Indeed, they are the fixed points of the composition of two Kelvin transforms with respect to $B_1(0,0,0)$ and $B_1(2+\epsilon,0,0)$, and can be calculated directly as

$$
p_i = 1 + (-1)^i \sqrt{\epsilon} + O(\epsilon) \quad (4.4)
$$

for $i = 1, 2$.

In the following lemma, we establish a fundamental equation of $P$ which is essential in deriving most properties of $P$ in this paper.

**Lemma 4.2** The function $P : [1, \infty) \to \mathbb{R}$ satisfies

$$
\frac{1}{x^3} P\left(2 + \epsilon - \frac{1}{x}\right) - \frac{1}{x} \int_0^{\frac{1}{x}} sP(2 + \epsilon - s) ds + \frac{1}{2 x^3} = P(x) \quad (4.5)
$$

for any $x \geq 1$.

**Proof.** We first consider the case of a single inclusion. We recall the property (4.6) of the reflection with respect to a single inclusion. Second, we apply (4.6) to the case of two neighboring inclusions thus to derive (4.5).

We first consider the reflection only for a single inclusion $B_0$ that is the unit sphere $B_1(0,0,0)$. For any harmonic function $h$ defined in a neighborhood containing $B_0$, the reflection $R_0(h)$ with respect to $B_0$ satisfies

$$
\partial_y R_0(h)(x,0,0) = \frac{1}{x^3} \partial_y h\left(\frac{1}{x},0,0\right) - \frac{1}{x} \int_0^{\frac{1}{x}} s\partial_y h(s,0,0) ds \quad (4.6)
$$

for $x \geq 1$, that is (3.3).

Second, we consider the solution $u$ to (3.1). It can be decomposed into three harmonic functions as $u(x,y,z) = H(x,y,z) + R_{B_1}(x,y,z) + R_{B_2}(x,y,z)$. From definition,
\( R_{B_1}(x, y, z) \) can be regarded as the reflection of \( H(x, y, z) + R_{B_2}(x, y, z) \) with respect to \( B_1 \) and \( R_{B_2}(x, y, z) = R_{B_1}(-x, y, z) \) due to the symmetric property of \( H(x, y, z) = y \). Since \( P(x) = \partial_y R_{B_1}(x - 1 - \frac{\epsilon}{2}, 0, 0) \), the equality (4.6) thus yields

\[
P(x) = \partial_y R_{B_1} \left( x - 1 - \frac{\epsilon}{2}, 0, 0 \right) \\
= \partial_y R_0 \left( H \left( x - 1 - \frac{\epsilon}{2}, y, z \right) + R_{B_2} \left( x - 1 - \frac{\epsilon}{2}, y, z \right) \right) (x, 0, 0) \\
= \partial_y R_0 \left( \beta + R_{B_1} \left( -x + 1 + \frac{\epsilon}{2}, y, z \right) \right) (x, 0, 0) \\
= \frac{1}{2x^3} + \frac{1}{2x^3} \partial_y R_{B_1} \left( -\frac{1}{x} + 1 + \frac{\epsilon}{2}, 0, 0 \right) - \frac{1}{2x^3} - \frac{1}{x} \int_0^{\frac{1}{x}} s \partial_y R_{B_1} \left( -s + 1 + \frac{\epsilon}{2}, 0, 0 \right) ds \\
= \frac{1}{2x^3} + \frac{1}{2x^3} P \left( 2 + \frac{1}{x} \right) - \frac{1}{x} \int_0^{\frac{1}{x}} s P \left( 2 + \epsilon - s \right) ds.
\]

Thus, we have this lemma. \( \square \)

Lemma describes the graph of \( P(x) \) as an application of Lemma 4.2.

**Lemma 4.3** The function \( P(x) \) satisfies

\[
P(x) > 0, \ P'(x) < 0, \ P''(x) > 0, \ P'''(x) < 0, \ P''''(x) > 0
\]

for \( x > 1 \) and

\[
\lim_{x \to \infty} P(x) = 0.
\]

**Proof.** It is obvious that \( P(x) \) is bounded on \( [1 + \frac{1}{x}, 2 + \frac{1}{x}] \), since the interval is some distance from 1. In an alternative way, the boundedness can also be induced by 3.8. We consider the limits of left- and right-hand sides of the equality (4.5) as \( x \) approaches \( \infty \). Then, we have

\[
\lim_{x \to \infty} P(x) = 0.
\]

Similarly with the previous lemma, we first study the properties of the reflection only with respect to a single inclusion. Second, we apply such properties to the case of two neighboring inclusions.

According to plan, we consider the properties 4.7, 4.8, 4.9, 4.10 and 4.11 of the reflection with respect to a single inclusion \( B_0 \) that denotes the unit sphere \( B_0(0, 0, 0) \). As defined in the previous lemma, \( R_0(h) \) denotes the reflection of a given harmonic function \( h \) with respect to \( B_0 \). Suppose that for \( n = 0, 1, 2, 3, 4 \),

\[
\partial^n_x \partial_y h(x, 0, 0) \geq 0 \text{ on } (0, 1].
\]

We shall show that for \( n = 0, 1, 2, 3, 4 \),

\[
(-1)^n \partial^n_x \partial_y R_0(h)(x, 0, 0) \geq 0 \text{ on } (0, 1].
\]

To do so, we use the equality (4.6) in the previous lemma or (4.4) so that

\[
\partial_y R_0(h)(x, 0, 0) = \frac{1}{x^3} \partial_y h \left( \frac{1}{x}, 0, 0 \right) - \frac{1}{x} \int_0^{\frac{1}{x}} s \partial_y h(s, 0, 0) ds \text{ for } x \geq 1.
\]

First, the positivity of \( \partial_y R_0(h)(x, 0, 0) \) results from the equality

\[
\partial_y R_0(h)(x, 0, 0) = \frac{1}{x^3} \partial_y h \left( \frac{1}{x}, 0, 0 \right) - \frac{1}{x} \int_0^{\frac{1}{x}} s \partial_y h(s, 0, 0) ds \\
= \frac{1}{x} \int_0^{\frac{1}{x}} s \left( 2\partial_y h \left( \frac{1}{x}, 0, 0 \right) - \partial_y h(s, 0, 0) \right) ds \geq 0 \quad (4.7)
\]
due to the increasing property of $\partial_y h(x, 0, 0) \geq 0$. Second, dealing with the decreasing property of $\partial_y R_0(h)(x, 0, 0)$, we take a derivative and then, the increasing assumption of $\partial_y h(x, 0, 0)$ yields that

$$\partial_x \partial_y R_0(h)(x, 0, 0) = -\frac{2}{x^2} \partial_y h \left(\frac{1}{x}, 0, 0\right) + \frac{1}{x^2} \int_0^\frac{1}{x} s \partial_y h(s, 0, 0) \, ds - \frac{1}{x^2} \partial_x \partial_y h \left(\frac{1}{x}, 0, 0\right) \leq -\frac{1}{x^2} \int_0^\frac{1}{x} s \left(4 \partial_y h \left(\frac{1}{x}, 0, 0\right) - \partial_y h(s, 0, 0)\right) \, ds \leq 0. \tag{4.8}$$

Thus, $\partial_y R_0(h)(x, 0, 0)$ is decreasing. Third, the concavity result can be also obtained in the same way. Thus,

$$\partial_x^2 \partial_y R_0(h)(x, 0, 0) = \frac{7}{x^3} \partial_y h \left(\frac{1}{x}, 0, 0\right) - \frac{2}{x^3} \int_0^\frac{1}{x} s \partial_y h(s, 0, 0) \, ds + \frac{7}{x^6} \partial_x \partial_y h \left(\frac{1}{x}, 0, 0\right) + \frac{1}{x^7} \partial_x^2 \partial_y h \left(\frac{1}{x}, 0, 0\right) \geq \frac{1}{x^7} \int_0^\frac{1}{x} s \left(14 \partial_y h \left(\frac{1}{x}, 0, 0\right) - 2 \partial_y h(s, 0, 0)\right) \, ds \geq 0. \tag{4.9}$$

Fourth, we have similarly

$$\partial_x^2 \partial_y R_0(h)(x, 0, 0) \leq -\frac{33}{x^6} \partial_y h \left(\frac{1}{x}, 0, 0\right) + \frac{6}{x^7} \int_0^\frac{1}{x} s \partial_y h(s, 0, 0) \, ds \leq 0. \tag{4.10}$$

and

$$\partial_x^4 \partial_y R_0(h)(x, 0, 0) \geq \frac{192}{x^8} \partial_y h \left(\frac{1}{x}, 0, 0\right) - \frac{24}{x^7} \int_0^\frac{1}{x} s \partial_y h(s, 0, 0) \, ds \geq 0. \tag{4.11}$$

Now, we are ready to prove this lemma. By Lemma 3.1,

$$u(x) = H(x) + R_1(H)(x) + R_2(H)(x) + R_3(R_1(H))(x) + R_4(R_2(H))(x) + \cdots,$$

and

$$\partial_y u(x, 0, 0) = \partial_y H(x, 0, 0) + \partial_y R_1(H)(x, 0, 0) + \partial_y R_2(H)(x, 0, 0) + \partial_y R_3(R_1(H))(x, 0, 0) + \partial_y R_4(R_2(H))(x, 0, 0) + \cdots,$$

where $R_1$ and $R_2$ are the reflections with respect to the insulated inclusions $B_1$ and $B_2$ as defined in (3.3). We apply (4.7), (4.8), (4.9), (4.10), (4.11). Since $\partial_y H(x, 0, 0) = 1$,

$$(-1)^n \partial_x^n \partial_y R_1(H)(x, 0, 0) \geq 0 \quad \text{for} \quad x > -\frac{\epsilon}{2},$$

and

$$\partial_x^n \partial_y R_2(H)(x, 0, 0) \geq 0 \quad \text{for} \quad x < \frac{\epsilon}{2},$$

for $n = 0, 1, 2, 3, 4$. In the same way, one can show by the mathematical induction that for $n = 0, 1, 2, 3, 4$,

$$(-1)^n \partial_x^n \partial_y ((R_1 R_2)^m(H))(x, 0, 0) \geq 0,$$

and

$$(-1)^n \partial_x^n \partial_y ((R_2 R_1)^m(H))(x, 0, 0) \geq 0$$

for $x > -\frac{\epsilon}{2}$, and

$$\partial_x^n \partial_y ((R_2 R_1)^m(H))(x, 0, 0) \geq 0,$$
\[ \partial^m_x \partial_y (R_2(R_1 R_2)^m(H))(x, 0, 0) \geq 0 \]

for \( x < \frac{\epsilon}{2} \) for \( m \in \mathbb{N} \). From definition,

\[ \partial_y R_{B_1}(x, 0, 0) = \partial_y R_1(H)(x, 0, 0) + \partial_y R_1(R_2(H))(x, 0, 0) + \cdots, \]

and by (3.10),

\[ (-1)^n \partial^n_x \partial_y R_1(H)(x, 0, 0) > 0. \]

For \( n = 0, 1, 2, 3, 4 \), we thus have

\[ (-1)^n \partial^n_x \partial_y R_{B_1}(x, 0, 0) > 0 \quad \text{for} \quad x > -\frac{\epsilon}{2} \]

that implies this lemma, since \( P(x + 1 + \frac{\epsilon}{2}) = \partial_y R_{B_1}(x, 0, 0) \).

Another property of \( P \) is provided by the following lemma based on the previous lemma. This property is used to reduce (4.5) into an ordinary differential equation in Lemma 4.9.

**Lemma 4.4** For \( n = 1, 2, 3, 4 \),

\[ |(x - 1)^n P^{(n-1)}(x)| \lesssim (x - 1)P\left(\frac{x - 1}{2} + 1\right) \]

for any \( x > 1 \).

**Proof.** In the previous lemma, we proved the decreasing property of \(|P^{(n-1)}|\) in \((1, \infty)\) for \( n = 1, 2, 3, 4 \). In the case of \( n = 1 \), it yields

\[ |(x - 1)P(x)| \lesssim (x - 1)P\left(\frac{x - 1}{2} + 1\right) \]

for any \( x > 1 \).

Let \( n \) be one of \( 2, 3, 4 \). By the mean value theorem, for any \( x > 1 \), there exists \( x_0 \in (x - \frac{z-1}{2^n}, x) \) such that

\[ \left| P^{(n-1)}(x_0) \right| = 2^{n-1} \left| \frac{P^{(n-2)}(x - \frac{x-1}{2^{n-1}}) - P^{(n-2)}(x)}{x - 1} \right| \lesssim \left| \frac{P^{(n-2)}(x - \frac{x-1}{2^{n-1}})}{x - 1} \right|, \]

since the value of \((-1)^{(n-2)} P^{(n-2)}\) is always positive. It follows from the decreasing property of \(|P^{(n-1)}|\) that

\[ \left| P^{(n-1)}(x) \right| \leq \left| P^{(n-1)}(x_0) \right| \lesssim \left| \frac{P^{(n-2)}(x - \frac{x-1}{2^{n-1}})}{x - 1} \right|. \]

Continuing this process, we have

\[ \left| P^{(n-1)}(x) \right| \lesssim \left| \frac{P^{(n-2)}(x - \frac{x-1}{2^{n-1}})}{x - 1} \right| \lesssim \left| \frac{P^{(n-3)}(x - \frac{x-1}{2^{n-1}})}{(x - 1)^2} \right| \lesssim \cdots \lesssim \left| \frac{P(x - \frac{x-1}{2^n})}{(x - 1)^{n-1}} \right| = \left| \frac{P(\frac{x-1}{2^n} + 1)}{(x - 1)^{n-1}} \right|. \]

\( \square \)
The fundamental equation (4.5) can be rewritten as
\[
\frac{1}{2} \frac{1}{x^2} - \int_0^x sP(2 + \epsilon - s)ds = xP(x) - \frac{1}{x^2} P\left(2 + \epsilon - \frac{1}{x}\right).
\]
The left-hand side is positive by the following lemma. Indeed, the value of the left-hand side is very important, since the blow-up rate of \(P\) is proportional to
\[
\frac{1}{\sqrt{\epsilon}} \left(\frac{1}{2} \frac{1}{x^2} - \int_0^x sP(2 + \epsilon - s)ds\right) \text{ at } x = 1 + 2\sqrt{\epsilon}.
\]
Please refer to Lemma 4.15 for the details.

Lemma 4.5
\[
\int_0^x sP(2 + \epsilon - s)ds < \frac{1}{2} \frac{1}{x^2} \tag{4.12}
\]
for any \(x \in [1, 2 + \epsilon]\).

Proof. First, we derive the inequality (4.12) on the restricted interval \([1, p_2]\). Here, \(p_2\) is the fixed point defined in (4.4). Second, the inequality on \([1, 2 + \epsilon]\) is proved by contradiction.

We prove according to plan that
\[
\int_0^x sP(2 + \epsilon - s)ds - \frac{1}{2} \frac{1}{x^2} < 0 \text{ for any } x \in [1, p_2]. \tag{4.13}
\]
It is easy to show that \(x < 2 + \epsilon - \frac{1}{x}\) and \(x \geq 1\) for any \(x \in [1, p_2]\). The decreasing property of \(P\) in Lemma 4.3 yields \(0 > P(2 + \epsilon - \frac{1}{x}) - P(x) \geq 0\).

By Lemma 4.2,
\[
0 > x \left(\frac{1}{x^2} P\left(2 + \epsilon - \frac{1}{x}\right) - P(x)\right) = \int_0^x sP(2 + \epsilon - s)ds - \frac{1}{2} \frac{1}{x^2}
\]
for any \(x \in [1, p_2]\), since \(x \geq 1\). Thus, we got the result (4.13) restricted on \([1, p_2]\).

Suppose that
\[
\int_0^{x_0} sP(2 + \epsilon - s)ds - \frac{1}{2} \frac{1}{x_0^2} \geq 0
\]
for some \(x_0 \in (1, 2 + \epsilon]\). By the mean value theorem, there exists a point \(s_0 \in (0, \frac{1}{x_0})\) such that \(P(2 + \epsilon - s_0) \geq 1\), since \(\int_0^{x_0} sds = \frac{1}{2} \frac{1}{x_0^2}\). The decreasing property \(P\) yields
\[
P(2 + \epsilon - s) \geq 1
\]
for any \(s > s_0\). For any \(x \in [1, x_0]\), \(\frac{1}{x} > \frac{1}{x_0} > s_0\) so that
\[
\int_0^x sP(2 + \epsilon - s)ds - \frac{1}{2} \frac{1}{x^2}
\]
\[
= \int_0^{x_0} sP(2 + \epsilon - s)ds - \frac{1}{2} \left(\frac{1}{x^2} - \frac{1}{x_0^2}\right) + \int_{x_0}^x sP(2 + \epsilon - s)ds - \frac{1}{2} \frac{1}{x_0^2}
\]
\[
\geq \int_{x_0}^x sds - \frac{1}{2} \left(\frac{1}{x^2} - \frac{1}{x_0^2}\right) = 0.
\]
This leads to a contradiction for the first result (4.13) in this proof. Thus, we have
\[ \int_0^\frac{1}{x} sP(2 + \epsilon - s)ds - \frac{1}{2} \frac{1}{x^2} < 0 \]
for any \( x \in (1, 2 + \epsilon) \).

\[ \square \]

**Remark 4.6** It follows from Lemma 4.5 that
\[ 0 < \frac{1}{2} - \int_0^\frac{1}{x} sP(2 + \epsilon - s)ds \]
for any \( x \in [1, 2 + \epsilon] \). Thus, we have
\[ \int_\epsilon^\frac{1}{x} P(1 + s) ds \leq 1 \]
and by the decreasing property of \( P \),
\[ \int_\epsilon^1 P(1 + s) ds \leq 3. \]

Roughly speaking, \( P \) satisfies an ordinary differential equation as will be seen in Lemma 4.9. Thus, \( P \) can be expressed as a particular solution plus a linear combination of two homogenous solutions. The following lemma provides the estimates for the coefficients in the linear combination that are essential to estimate the blow-up rate of \( P \). Please refer to Proposition 4.13 for the details.

**Lemma 4.7**
\[ 3\gamma \sqrt{\epsilon} P(1 + \gamma \sqrt{\epsilon} - \gamma^2 \epsilon) + \left( (\gamma^2 - 1)\epsilon - \gamma^3 \epsilon^2 \right) P'(1 + \gamma \sqrt{\epsilon}) > 0 \]
in \( 2 < \gamma < \frac{1}{10\sqrt{\epsilon}} \).

**Proof.** Applying Lemma 4.5 to (4.5), we have
\[
0 \leq P(x) - \frac{1}{x^3} P \left( 2 + \epsilon - \frac{1}{x} \right) \\
= \left( 1 - \frac{1}{x^3} \right) P \left( 2 + \epsilon - \frac{1}{x} \right) + \left( P(x) - P \left( 2 + \epsilon - \frac{1}{x} \right) \right). \tag{4.14}
\]
Let \( x = 1 + \gamma \sqrt{\epsilon} \), while \( 2 < \gamma < \frac{1}{10\sqrt{\epsilon}} \). Then,
\[ 1 - \frac{1}{x^3} \leq 3\gamma \sqrt{\epsilon}, \]
since \( \gamma < \frac{1}{10\sqrt{\epsilon}} \). Since \( \gamma > 2, x > 2 + \epsilon - \frac{1}{x} \). By the mean value theorem, there exists \( x_0 \in (2 + \epsilon - \frac{1}{x}, x) \) such that
\[
P(x) - P \left( 2 + \epsilon - \frac{1}{x} \right) = \left( x + \frac{1}{x} - 2 - \epsilon \right) P'(x_0) \leq \left( x + \frac{1}{x} - 2 - \epsilon \right) P'(x),
\]
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since $P'(x_0) \leq P'(x) \leq 0$ by the monotonic property of $P'$ in Lemma 4.3. Since $\gamma > 2$, we also have

$$-\left( x + \frac{1}{x} - 2 - \epsilon \right) \leq -\left( \gamma^2 - 1 \right) \epsilon - \gamma^3 \epsilon^2 \frac{1}{2}.$$  

Applying these bounds above to (4.14), we have this lemma.  

The key ingredient in the proof of the main result is the equation (4.5) the function $P(x)$ satisfies. Thus, we need to consider the relation between $x$ and $2 + \epsilon - \frac{1}{x}$ in the equation.

**Lemma 4.8** Suppose that the sequence $\{x_n\}$ is defined as

$$x_1 = 1$$

and

$$x_{n+1} = 2 + \epsilon - \frac{1}{x_n} \text{ for } n \in \mathbb{N}.$$  

Then, for any $n \leq \frac{1}{2\sqrt{\epsilon}}$,

$$x_n = 1 + (n-1)\epsilon + o_n$$

and

$$|o_n| \leq 10n^2\epsilon\sqrt{\epsilon}.$$  

**Proof.** One can show that

$$x_n = p_2 + \frac{2 + \epsilon - 2p_2}{c_0d^{n-1} + 1}$$  

where

$$p_2 = 1 + \sqrt{\epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{2}}$$

as given in (4.4),

$$c_0 = \frac{1 - p_2 + \epsilon}{1 - p_2} \quad \text{and} \quad d = \frac{p_2}{2 + \epsilon - p_2}.$$  

We estimate the quantities that composes $x_n$. Thus,

$$p_2 = 1 + \sqrt{\epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{8} + O(2\epsilon^3)},$$

$$c_0 = 1 - \sqrt{\epsilon + \frac{\epsilon}{2} + O(\epsilon\sqrt{\epsilon})} \geq 1 - \sqrt{\epsilon + \frac{\epsilon}{2}}$$

and

$$d = 1 + 2\sqrt{\epsilon + 2\epsilon + O(\epsilon\sqrt{\epsilon})} > 1 + 2\sqrt{\epsilon + 2\epsilon}$$  

for small $\epsilon > 0$. It has been proved in [20] that

$$1 + (n-1)x \leq (1 + x)^{n-1} \leq 1 + (n-1)x + (n-1)^2 x^2,$$

supposed that $x \in (0, 2)$ and $(1 + x)^{n-1} \leq 2$. Since $n \leq \frac{1}{2\sqrt{\epsilon}}$, $c_0d^{n-1}$ can be estimated as

$$1 + (2n-3)\sqrt{\epsilon} \leq c_0d^{n-1} \leq 1 + (2n-3)\sqrt{\epsilon} + 10n^2\epsilon.$$

Applying these bounds above to (4.15), we have

$$x_n = 1 + \sqrt{\epsilon + \frac{\epsilon}{2} - \sqrt{\epsilon}} \left( 1 + \left( \frac{3}{2} - n \right) \sqrt{\epsilon} \right) + o_n$$

$$= 1 + (n-1)\epsilon + o_n$$

where

$$|o_n| \leq 10n^2\epsilon\sqrt{\epsilon}.$$  

\hfill $\square$
4.2 Proof of Proposition 4.1

We first show that \( P(1 + r_0 \sqrt{\epsilon}) \lesssim \frac{1}{x^{2+\sqrt{2}}} \) for a large \( r_0 > 2 \), where \( r_0 \) is independent of \( \epsilon \). Second, a relation between the values \( P(1) \) and \( P(1 + r_0 \sqrt{\epsilon}) \) is established in (4.31), and we prove that \( P(1) \lesssim \frac{1}{x^{2+\sqrt{2}}} \). Here and throughout this paper, \( a \lesssim b \) means \( a \leq Cb \) and \( a \simeq b \) stands for \( \frac{1}{C_2}a \leq b \leq C_2a \) for some constants \( C_1 \) and \( C_2 \) independent of \( \epsilon \).

4.2.1 Estimate for \( P(1 + r_0 \sqrt{\epsilon}) \) for a large \( r_0 > 0 \)

Let \( t = x - 1 \), and let \( f \) be as

\[
f(t) = P(1 + t) = P(x)
\]

for \( t \geq 0 \).

Speaking of the scheme, the function \( f \), defined in \([0, \infty)\), satisfies an ordinary differential equation in Lemma 4.9. The function \( f \) can be decomposed into three functions in (4.19) as follows:

\[
f(t) = f_p + C_\alpha f_\alpha + C_\beta f_\beta,
\]

where \( f_p \) is a particular solution, and \( f_\alpha \) and \( f_\beta \) are two homogeneous solutions satisfying

\[
f_\alpha(t) \simeq \frac{1}{t^2 - \sqrt{2}} \text{ and } f_\beta(t) \simeq \frac{1}{t^2 + \sqrt{2}}
\]

for \( t \geq 10\sqrt{\epsilon} \). The boundedness of \( f_p \) is provided in Lemma 4.10. The boundedness of \( C_\alpha \) and the smallness of \( C_\beta \) can be derived by Lemma 4.11, and hence we can estimate \( f(r_0 \sqrt{\epsilon}) = P(1 + r_0 \sqrt{\epsilon}) \) in Proposition 4.13 and Remark 4.14.

Lemma 4.9

\[
(t^2 - \epsilon)f''(t) + 5tf'(t) + 2f(t) = -\frac{1}{(1 + t)^2} + g(t)
\]

and

\[
|g(t)| \lesssim \left| tg\left(\frac{t}{2}\right)\right|
\]

for any \( t > 10\sqrt{\epsilon} \).

Proof. By Lemma 4.2,

\[
\frac{1}{2} \frac{1}{x^2} = -\frac{1}{x^2} P\left(2 + \epsilon - \frac{1}{x}\right) + \int_0^x sP(2 + \epsilon - s)ds + xP(x)
\]

for any \( x \geq 1 \). Taking derivative, we have

\[
-\frac{1}{x^3} = -\frac{1}{x^3} P'(2 + \epsilon - \frac{1}{x}) + xP'(x) + P(x) + \frac{1}{x^3} P\left(2 + \epsilon - \frac{1}{x}\right)
\]

(4.16)

\[
= -\frac{1}{x^3} \left( P'(2 + \epsilon - \frac{1}{x}) - P'(x) \right) + \left( \frac{1}{x^3} + x \right) P'(x)
\]

\[
+ \left( 1 + \frac{1}{x^3} \right) P(x) + \frac{1}{x^3} \left( P\left(2 + \epsilon - \frac{1}{x}\right) - P(x) \right).
\]
Since \(2 + \epsilon - \frac{1}{x} = x + \left(2 + \epsilon - \frac{1}{1+t} - (1+t)\right) = x + (\epsilon - t^2) + O(t^3)\) in \(0 < t < 1\), the mean value theorem yields

\[
(t^2 - \epsilon)f'''(t) + 5tf''(t) + 2f(t) = -\frac{1}{(1+t)^3} + g(t) \quad \text{for } t \geq 10\sqrt{\epsilon}.
\]

(4.17)

for \(t > \sqrt{\epsilon}\), where the points \(t_1, t_2, t_3\) are located between \(t + \epsilon - t^2\) and \(t\), and they are depending on \(t\). Lemma 4.4 means that

\[
\left|t^n f^{(n-1)}(t)\right| \lesssim tf\left(\frac{t}{2}\right)
\]

for \(t > 0\). Thus,

\[
|g(t)| \lesssim tf\left(\frac{t}{2}\right)
\]

for any \(t \in (10\sqrt{\epsilon}, \frac{1}{2})\). \(\square\)

Now, we consider the solution to

\[
(t^2 - \epsilon)f'''(t) + 5tf''(t) + 2f(t) = -\frac{1}{(1+t)^3} + g(t) \quad \text{for } t \geq 10\sqrt{\epsilon}.
\]

We shall find three proper functions \(f_p, f_\alpha\) and \(f_\beta\) that satisfy

\[
(t^2 - \epsilon)f_\alpha''(t) + 5tf_\alpha'(t) + 2f_\alpha(t) = -\frac{1}{(1+t)^3} + g(t)
\]

(4.18)

and

\[
(t^2 - \epsilon)f_\beta''(t) + 5tf_\beta'(t) + 2f_\beta(t) = 0
\]

for \(i = \alpha, \beta\). The general solution is decomposed into the three functions as follows:

\[
f = f_p + C_\alpha f_\alpha + C_\beta f_\beta,
\]

(4.19)

where \(f_\alpha\) and \(f_\beta\) is homogeneous solutions defined as

\[
f_\alpha = t^{-2+\sqrt{2}} + t^{-2+\sqrt{2}} \sum_{n=1}^{\infty} \frac{\left(\frac{\epsilon}{t}\right)^n}{n!} \prod_{k=1}^{n} \frac{2k - \sqrt{2}(2k + 1 + \sqrt{2})}{2k(2k - 2\sqrt{2})},
\]

(4.20)

\[
f_\beta = t^{-2-\sqrt{2}} + t^{-2-\sqrt{2}} \sum_{n=1}^{\infty} \frac{\left(\frac{\epsilon}{t}\right)^n}{n!} \prod_{k=1}^{n} \frac{2k + \sqrt{2}(2k + 1 + \sqrt{2})}{2k(2k + 2\sqrt{2})}
\]

(4.21)

for \(t \geq 10\sqrt{\epsilon}\). The functions \(f_\alpha\) and \(f_\beta\) can be established by induction. To do so, we regard \(f_\alpha\) and \(f_\beta\) as the sums \(\sum_{n=0}^{\infty} f_{\alpha n}\) and \(\sum_{n=0}^{\infty} f_{\beta n}\), where \(f_{\alpha 0} = t^{-2+\sqrt{2}}\) and \(f_{\beta 0} = t^{-2-\sqrt{2}}\) are the solutions to \(t^2 f_{\alpha}'''(t) + 5tf_{\alpha}'(t) + 2f_{\alpha}(t) = 0\), and \(f_{\alpha n}\) is the
The functions $f_\alpha$ and $f_\beta$ are defined well on $[10\sqrt{\epsilon}, \infty)$, because

$$\left| \frac{f_{\alpha n}(t)}{f_{\alpha(n-1)}(t)} \right| = \left( \frac{\epsilon}{t^2} \right) \left| \frac{(2n - \sqrt{2})(2n + 1 - \sqrt{2})}{2n(2n - 2\sqrt{2})} \right| \leq 4 \left( \frac{\epsilon}{t^2} \right),$$

$$\left| \frac{f_{\beta n}(t)}{f_{\beta(n-1)}(t)} \right| = \left( \frac{\epsilon}{t^2} \right) \left| \frac{(2n + \sqrt{2})(2n + 1 + \sqrt{2})}{2n(2n + 2\sqrt{2})} \right| \leq 4 \left( \frac{\epsilon}{t^2} \right)$$

and the variable $t \geq 10\sqrt{\epsilon}$. Moreover, we have

$$f_\alpha \simeq t^{-2 + \sqrt{\epsilon}} \text{ and } f_\beta \simeq t^{-2 - \sqrt{\epsilon}}.$$  

Dealing with (4.19), we consider the contribution of $f_p$ to $f$. The boundedness of $f_p$ is derived in the following lemma.

**Lemma 4.10** There is a particular solution $f_p$ to (4.18) and a constant $C_0$ satisfying

$$|f_p(t)| \lesssim 1$$

and

$$|f_p'(t)| \lesssim \frac{1}{t}$$

for any $t > C_0\sqrt{\epsilon}$.

**Proof.** Let

$$G(t) = -\frac{1}{(1+t)^2} + g(t).$$

Then, we shall find the sequence of functions $\{f_{pn}\}$ satisfying

$$t^2 f''_{pn}(t) + 5 t f'_p(t) + 2 f_p(t) = G(t), \quad (4.22)$$

$$t^2 f''_{pn}(t) + 5 t f'_p(t) + 2 f_{pn}(t) = \epsilon f_{p(n-1)}(t) \quad (4.23)$$

for $n = 1, 2, 3, \cdots$. The sum $\sum_{n=0}^{\infty} f_{pn}$ is the desirable function $f_p$. As a matter of convenience, we simplify the result $|g(t)| \lesssim t f \left( \frac{t}{2} \right)$ in Lemma 4.9. Thus, without any loss of generality, we assume in this proof that

$$|g(t)| \leq t f \left( \frac{t}{2} \right) \quad (4.24)$$

for any $t > 10\sqrt{\epsilon}$.

Define $f_{p0}(t)$ as

$$f_{p0}(t) = \frac{1}{t^{2 + \sqrt{2}}} \int_{10\sqrt{\epsilon}}^{t} \frac{1}{w^{1 - 2\sqrt{2}}} \int_{10\sqrt{\epsilon}}^{w} G(s) s^{1 - \sqrt{2}} dsdw. \quad (4.25)$$

By Remark 4.6, $\int_0^1 f \left( \frac{s}{2} \right) ds \leq 2$. By Lemma 4.9 and (4.24),

$$|f_{p0}(t)| \leq \frac{1}{t^{2 + \sqrt{2}}} \int_{10\sqrt{\epsilon}}^{t} \frac{1}{w^{1 - 2\sqrt{2}}} \int_{10\sqrt{\epsilon}}^{w} s^{1 - \sqrt{2}} + f \left( \frac{s}{2} \right) s^{2 - \sqrt{2}} dsdw \leq \frac{1}{2} + \frac{1}{t^{2 + \sqrt{2}}} \int_{10\sqrt{\epsilon}}^{t} w^{1 + \sqrt{2}} \int_{10\sqrt{\epsilon}}^{w} f \left( \frac{s}{2} \right) dsdw \leq 2.$$
in $10\sqrt{\varepsilon} < t < 1$. Taking the derivative of (4.25), we can get similarly

$$ |f_p'(t)| \leq \frac{11}{t} $$

and by (4.22)

$$ |f_p''(t)| \leq \frac{1}{t^2}(60 + |g(t)|) $$

in $10\sqrt{\varepsilon} < t < 1$.

We also define $f_{pn}(t)$ as

$$ f_{pn}(t) = \frac{1}{\sqrt{2}} \int_{\sqrt{120\varepsilon}}^{t^2} \int_{w^2}^{w^2 + 2\sqrt{2}} \int_{\sqrt{120\varepsilon}}^{w} \epsilon f_p''(w) s^{1-\sqrt{2}} ds dw. $$

In the same way, we can prove by mathematical induction that for any $n = 1, 2, \cdots$, 

$$ |f_{pn}(t)| \leq \frac{2}{2^n}, \quad |f_p'(t)| \leq \frac{11}{2^n} $$

and

$$ |\epsilon f_p''(t)| \leq \frac{1}{2^n} \epsilon (60 + |g(t)|) \leq \frac{1}{2^n+1}(1 + |g(t)|) $$

that is the right-hand side of (4.23), while $\sqrt{120\varepsilon} \leq t < \frac{1}{2}$. Hence, the sum $\sum_{n=0}^{\infty} f_{p0}$ is well defined and is the desirable function $f_p$. □.

We shall consider the contribution of $f_\alpha$ and $f_\alpha$ to $f$ in Proposition 4.13, since the boundedness of $f_p$ was derived in the previous lemma. To do so, we need Lemmas 4.11 and 4.12.

**Lemma 4.11**

$$ 3\gamma\sqrt{\varepsilon} f (\gamma\sqrt{\varepsilon} - 2\gamma^2\epsilon) + \left((\gamma^2 - 1)\epsilon - \gamma^3\epsilon^2\right) f'(\gamma\sqrt{\varepsilon}) > 0 $$

in $2 < \gamma < \frac{1}{m\sqrt{\varepsilon}}$.

The lemma above is a rewritten version of Lemma 4.7, since $f(t) = P(1 + t)$.

We need the following lemma to help in proving Proposition 4.13.

**Lemma 4.12** Suppose that the constants $\tilde{M} > 0$, $\tilde{C}_\alpha$, $\tilde{C}_\beta > 0$ and $\tilde{C}_0 > 0$ satisfy

$$ \tilde{M} + \tilde{C}_\alpha \frac{1}{t^2-\sqrt{2}} \geq \tilde{C}_\beta \frac{1}{t^{2+\sqrt{2}}} \text{ for any } t \geq \tilde{C}_0\sqrt{\varepsilon}. $$

Then,

$$ \frac{1}{2} \left(\tilde{M} + \tilde{C}_\alpha \frac{1}{t^2-\sqrt{2}}\right) \geq \tilde{C}_\beta \frac{1}{t^{2+\sqrt{2}}} \text{ for any } t \geq 2\tilde{C}_0\sqrt{\varepsilon}. $$

**Proof.** For any $t \geq \tilde{C}_0\sqrt{\varepsilon}$,

$$ \tilde{M} t^{2-\sqrt{2}} + \tilde{C}_\alpha \geq \tilde{C}_\beta \frac{1}{t^{2+\sqrt{2}}}. $$

Let $t = s(\tilde{C}_0\sqrt{\varepsilon})$. For any $s \geq 2\frac{1}{\sqrt{\varepsilon}}$, Then,

$$ \tilde{M} t^{2-\sqrt{2}} + \tilde{C}_\alpha \geq \tilde{M} (\tilde{C}_0\sqrt{\varepsilon})^{2-\sqrt{2}} + \tilde{C}_\alpha \geq \frac{1}{(\tilde{C}_0\sqrt{\varepsilon})^{2+\sqrt{2}}} \geq 2\tilde{C}_\beta \frac{1}{t^{2+\sqrt{2}}}. $$
Using Lemmas 4.11 and 4.12, we estimate $f(t)$ in $[\gamma_0 \sqrt{\varepsilon}, \frac{1}{10}]$ in the following proposition, supposed that $\gamma_0$ is sufficiently large and independent of $\varepsilon$. Indeed, we shall prove that $P(1 + \gamma_0 \sqrt{\varepsilon}) = f(\gamma_0 \sqrt{\varepsilon})$ has the same blow-up rate as $P(1)$. In this respect, the estimate for $f(\gamma_0 \sqrt{\varepsilon})$ is meaningful.

**Proposition 4.13** There is a constant $\gamma_0$ independent of $\varepsilon$ such that

$$f(t) \lesssim f_\alpha(t) \simeq \frac{1}{t^{1+\sqrt{\varepsilon}}}$$

for $t \in [\gamma_0 \sqrt{\varepsilon}, \frac{1}{10})$.

**Proof.** We consider the decomposition of $f$ as

$$f = f_p + C_\alpha f_\alpha + C_\beta f_\beta.$$  \hspace{1cm} (4.26)

We shall use Lemma 4.11 to estimate $C_\alpha$ and $C_\beta$. The boundedness of $\|f_\beta(t)\|_{L^\infty(C_0 \sqrt{\varepsilon}, \frac{1}{10})}$ and $\|tf_\beta(t)\|_{L^\infty(C_0 \sqrt{\varepsilon}, \frac{1}{10})}$ was presented in Lemma 4.10. From the definitions (4.20) and (4.21), two functions $f_\alpha$ and $f_\beta$ are dominated by $\frac{1}{t^{1+\sqrt{\varepsilon}}}$ and $\frac{1}{t^{2+\sqrt{\varepsilon}}}$, respectively, for $t > C_0 \sqrt{\varepsilon}$, supposed that $C_0$ is sufficiently large and independent of $\varepsilon$. Note that $3 - (2 + \sqrt{2}) > 0$ and $3 - (2 + \sqrt{2}) < 0$. Applying the decomposition (4.26) to Lemma 4.11 we can find a constant $M > 0$ so that

$$M + C_\alpha \frac{1}{t^{1+\sqrt{\varepsilon}}} \gtrsim C_\beta \frac{1}{t^{2+\sqrt{\varepsilon}}}$$ \hspace{1cm} (4.27)

and

$$M + C_\alpha f_\alpha(t) \gtrsim C_\beta f_\beta(t)$$ \hspace{1cm} (4.28)

in $C_0 \sqrt{\varepsilon} < t \leq \frac{1}{10}$. The constant $M$ is provided by the boundedness of $f_\beta(t)$ and $tf_\beta(t)$. We need to consider four kinds of (4.27) dealing with four cases when $C_\alpha \geq 0$ and $C_\beta \geq 0$, when $C_\alpha \geq 0$ and $C_\beta < 0$, when $C_\alpha < 0$ and $C_\beta \geq 0$ and when $C_\alpha < 0$ and $C_\beta < 0$, respectively.

To obtain the bound in this lemma, we consider two cases when $C_\beta \geq 0$ and when $C_\beta \leq 0$, separately. First, we consider the case when $C_\beta \geq 0$. Then, $f \geq f_p + C_\alpha f_\alpha$ so that

$$C_\alpha \int_{C_0 \sqrt{\varepsilon}}^{1} f_\alpha dt \lesssim \int_{C_0 \sqrt{\varepsilon}}^{1} f - f_\beta dt \lesssim 1$$

by Lemma 4.10 or Remark 4.10. Since $\int_{C_0 \sqrt{\varepsilon}}^{1} f_\alpha dt \simeq 1$, the constant $C_\alpha \lesssim 1$. Thus, by (4.26) and (4.23), we have

$$0 \leq f \lesssim 1 + f_\alpha \lesssim f_\alpha$$

for $t \in [C_0 \sqrt{\varepsilon}, \frac{1}{10})$.

Second, we consider the case when $C_\beta \leq 0$. The decomposition (4.26) and the positivity $f > 0$ yield that $f_p + C_\alpha f_\alpha \geq -C_\beta f_\beta > 0$. Thus,

$$M + C_\alpha f_\alpha(t) \gtrsim |C_\beta| f_\beta(t).$$

Note that $f_\alpha \simeq \frac{1}{t^{1+\sqrt{\varepsilon}}}$ and $f_\beta \simeq \frac{1}{t^{2+\sqrt{\varepsilon}}}$, and the difference between the exponents so that $2 - \sqrt{2} < 2 + \sqrt{2}$. By Lemma 4.12 there is $r_0 > 0$ such that

$$\frac{1}{2} (M + C_\alpha f_\alpha(t)) \geq |C_\beta| f_\beta(t)$$ \hspace{1cm} (4.29)
for \( t \geq r_0 \sqrt{\epsilon} > C_0 \sqrt{\epsilon} \). Thus,
\[
1 \geq \int_{r_0 \sqrt{\epsilon}}^{t} f + Mt \geq C_0 \int_{r_0 \sqrt{\epsilon}}^{t} \frac{1}{2} f_\alpha dt \geq C_0.
\]
Hence, we use (4.29) to get
\[
f(t) \lesssim 1 + C_0 f_\alpha(t) \lesssim f_\alpha(t)
\]
for \( t \geq r_0 \sqrt{\epsilon} \). Hence, regardless of whether \( C_\beta \geq 0 \) or \( C_\beta \leq 0 \),
\[
f(t) \lesssim f_\alpha(t)
\]
for \( t \geq r_0 \sqrt{\epsilon} \). □

**Remark 4.14** It follows from Proposition 4.13 that
\[
P(1 + t) \lesssim \frac{1}{t^{2 + \sqrt{2}}}
\]
for \( t \geq r_0 \sqrt{\epsilon} \). Hence, we have
\[
P(1 + r_0 \sqrt{\epsilon}) \lesssim \frac{1}{\epsilon^{2 + \sqrt{2}}/2}
\]

### 4.2.2 Estimate for \( P(1) \)

Our estimate for \( P(1) \) is based on \( P(1 + r_0 \sqrt{\epsilon}) \lesssim \frac{1}{t^{2 + \sqrt{2}}} \) in Proposition 4.13 and Remark 4.14. The bound yields Lemmas 4.13 and 4.16 which imply \( P(1 + \frac{1}{100} \sqrt{\epsilon}) \lesssim \frac{1}{\epsilon^{2 + \sqrt{2}}} \).

We then establish a chain of correlations between \( P(1) \) and \( P(1 + \frac{1}{100} \sqrt{\epsilon}) \), and hence, \( P(1) \) can be estimated by Lemmas 4.13 and 4.16.

To derive Lemmas 4.13 and 4.16, we use the equality
\[
P \left( 2 + \epsilon - \frac{1}{x} \right) = x^3 P(x) + x^2 \int_0^x s P(2 + \epsilon - s) ds - \frac{1}{2} \tag{4.30}
\]
that is given in Lemma 4.2.

**Lemma 4.15**

\[
0 < \frac{1}{2} - \int_{1+\frac{1}{2} \sqrt{\epsilon}}^{2+\epsilon} (2 + \epsilon - t) P(t) dt \lesssim \frac{1}{\epsilon^{2 + \sqrt{2}}}. \tag{4.31}
\]

**Proof.** Let \( x_* = 1 + \gamma_0 \sqrt{\epsilon} \) where \( \gamma_0 \) was given in Proposition 4.13 and \( r_0 > 3 \). Since \( x_* > 2 + \epsilon - \frac{1}{x_*} \), the decreasing property of \( P \) in Lemma 4.14 implies \( P(x_*) \leq P \left( 2 + \epsilon - \frac{1}{x_*} \right) \). By Lemmas 4.3 and 4.4, \( 0 < \frac{1}{x_*^2} - \int_{0}^{x_*} s P(2 + \epsilon - s) ds = x_* \left( P(x_*) - \frac{1}{x_*} P \left( 2 + \epsilon - \frac{1}{x_*} \right) \right) \). By Proposition 4.14 and Remark 4.13
\[
0 < P(x_*) - \frac{1}{x_*^2} P \left( 2 + \epsilon - \frac{1}{x_*} \right)
\]
\[
= P(x_*) - P \left( 2 + \epsilon - \frac{1}{x_*} \right) + \left( 1 - \frac{1}{x_*} \right) P \left( 2 + \epsilon - \frac{1}{x_*} \right)
\]
\[
\lesssim \left( 1 - \frac{1}{x_*} \right) P \left( 2 + \epsilon - \frac{1}{x_*} \right) \lesssim \sqrt{\epsilon} \frac{1}{\epsilon^{2 + \sqrt{2}}} = \frac{1}{\epsilon^{2 + \sqrt{2}}}. 
\]
By Remark 4.6 and the positivity of $P$, 
\[
0 < \frac{1}{2} - \int_{1+2\sqrt{\epsilon}}^{2+\epsilon} (2 + \epsilon - t)P(t)dt \\
\leq \frac{1}{2} \frac{1}{x^2} - \int_{2+\epsilon-\frac{1}{x^2}}^{2+\epsilon} (2 + \epsilon - t)P(t)dt + r_0 \sqrt{\epsilon} \lesssim \frac{1}{e^{\frac{1}{x^2}}},
\]
since $1 + 2\sqrt{\epsilon} < 2 + \epsilon - \frac{1}{x^2}$ due to $r_0 > 3$.

\[\square\]

**Lemma 4.16**

\[P \left(1 + \frac{1}{100} \sqrt{\epsilon} \right) \lesssim \frac{1}{e^{\frac{1}{x^2}}}. \quad (4.32)\]

**Proof.** Applying $x = 1 + \frac{1}{100} \sqrt{\epsilon}$ to (4.30), we have 
\[
P \left(2 + \epsilon - \frac{1}{x^2} \right) = x^3 P(x) + x^2 \int_0^{1/x^2} sP(2 + \epsilon - s)ds - \frac{1}{2},
\]
and note that 
\[x < 2 + \epsilon - \frac{1}{x^2}.\]

It follows from (4.31) that 
\[
\frac{1}{e^{\frac{1}{x^2}}} \geq \frac{1}{2} - x^2 \int_0^{1/x^2} sP(2 + \epsilon - s)ds \\
= x^3 P(x) - P \left(2 + \epsilon - \frac{1}{x^2} \right) \\
\geq (x^3 - 1)P(x).
\]

Then, 
\[P \left(1 + \frac{1}{100} \sqrt{\epsilon} \right) \lesssim \frac{1}{e^{\frac{1}{x^2}}}. \quad \square\]

To estimate $P(1)$, we use a correlation between $P(1)$ and $P \left(1 + \frac{1}{100} \sqrt{\epsilon} \right)$ that is estimated just before. The correlation is described as a chain based on 
\[
P \left(2 + \epsilon - \frac{1}{x^2} \right) = x^3 P(x) + x^2 \int_0^{1/x^2} sP(2 + \epsilon - s)ds - \frac{1}{2}.
\]

In Lemma 4.8 the sequence $\{x_n\}$ was defined as 
\[x_1 = 1 \text{ and } x_{n+1} = 2 + \epsilon - \frac{1}{x_n} \text{ for } n \in \mathbb{N}.
\]
Then, 
\[P(x_{n+1}) = x_n^3 P(x_n) + x_n^2 \int_0^{1/x^2} sP(2 + \epsilon - s)ds - \frac{1}{2}\]
for \( n \in \mathbb{N} \). Let \( n_0 = \left\lceil \frac{1}{20\sqrt{\epsilon}} \right\rceil \). Then,

\[
P(x_{n_0 + 1}) + \sum_{n=1}^{n_0} x_{n_0}^3 x_{n+1} \cdots x_{n_0}^3 P(x_{n+1})
\]

\[
= x_{n_0}^3 P(x_{n_0}) + x_{n_0}^2 \int_0^{x_{n_0}} sP(2 + \epsilon - s)ds - \frac{1}{2} + \sum_{n=1}^{n_0} x_{n+1}^3 \cdots x_{n_0}^3 \left( x_n^3 P(x_n) + x_n^2 \int_0^{x_n} sP(2 + \epsilon - s)ds - \frac{1}{2} \right).
\]

By cancellation,

\[
x_1^3 x_2^3 \cdots x_{n_0}^3 P(1)
\]

\[
= P(x_{n_0 + 1}) + \left( \frac{1}{2} - x_{n_0}^2 \int_0^{x_{n_0}} sP(2 + \epsilon - s)ds \right) + \sum_{n=1}^{n_0} x_{n+1}^3 \cdots x_{n_0}^3 \left( \frac{1}{2} - x_n^2 \int_0^{x_n} sP(2 + \epsilon - s)ds \right).
\]

(4.34)

It follows from Lemma 4.8 that \( 1 + \frac{1}{100} \sqrt{\epsilon} \leq x_{n_0 + 1} \). Thus, by Lemma 4.16,

\[
P(x_{n_0 + 1}) \leq P\left( 1 + \frac{1}{100} \sqrt{\epsilon} \right) \lesssim \frac{1}{\epsilon^{\frac{1}{2}} - \sqrt{\epsilon}}.
\]

We use Lemma 4.8 again so that

\[
1 \leq x_1^3 x_2^3 \cdots x_{n_0}^3 \lesssim 1,
\]

and \( x_n \geq 1 \) for \( n \in \mathbb{N} \). Note that \( 1 + 2\sqrt{\epsilon} > p_2 > x_n \) for \( n = 1, \cdots, n_0 \), where \( p_2 \) is the fixed point in (4.4). By Lemma 4.31

\[
\frac{1}{\epsilon^{\frac{1}{2}} - \sqrt{\epsilon}} \geq \frac{1}{2} - x_n^2 \int_0^{x_n} sP(2 + \epsilon - s)ds
\]

for \( n = 1, \cdots, n_0 \). Therefore, the bound (4.34) is reduced into

\[
P(1) \lesssim \frac{1}{\epsilon^{\frac{1}{2}} - \sqrt{\epsilon}}.
\]

\[\square\]

References

[1] Ammari, H., Ciraolo, G., Kang, H., Lee, H., Yun, K.: Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity, Arch. Ration. Mech. An. 208, 275–304 (2013)

[2] Ammari, H., Kang, H., Lee, H., Lee, J., Lim, M.: Optimal bounds on the gradient of solutions to conductivity problems, J. Math. Pure. Appl. 88, 307–324 (2007)

[3] Ammari, H., Kang, H., Lim, M.: Gradient estimates for solutions to the conductivity problem, Math. Ann. 332(2), 277–286 (2005)
[4] Babuška, I., Andersson, B., Smith, P., Levin, K.: Damage analysis of fiber composites. I. Statistical analysis on fiber scale, Comput. Methods Appl. Mech. Engrg. 172, 27–77 (1999)

[5] Bao, E., Li, Y.Y., Yin, B.: Gradient estimates for the perfect conductivity problem, Arch. Ration. Mech. An. 193, 195-226 (2009)

[6] Bao, E., Li, Y.Y., Yin, B.: Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions, Commun. Part. Diff. Eq. 35, 1982–2006 (2010)

[7] Budiansky, B., Carrier, G. F.: High shear stresses in stiff fiber composites, J. Appl. Mech. 51, 733–735 (1984)

[8] Cheng, H., Greengard, L.: A method of images for the evaluation of electrostatic fields in systems of closely spaced conducting cylinders, SIAM J. Appl. Math., 58, 122–141 (1998)

[9] Gilbarg, D., Trudinger, N: Elliptic Partial Differential Equations of Second Order, Berlin-Heidelberg-New York: Springer-Verlag (1983)

[10] Optimal estimates and asymptotic for the stress concentration between closely located stiff inclusions, Math. Ann, on-line (2015)

[11] Kung, H., Lim, M., Yun, K.: Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities, J. Math. Pure. Appl. 99, 234–249 (2013)

[12] Kung, H., Lim, M., Yun, K.: Characterization of the electric field concentration between two adjacent spherical perfect conductors, SIAM J. Appl. Math., 74(1), 125146 (2014)

[13] Keller, J.B.: Stresses in narrow regions, Trans. ASME J. Appl. Mech., 60, 1054–1056 (1993)

[14] Lekner, J.: Analytical expression for the electric field enhancement between two closely-spaced conducting spheres, J. Electrostatics 68, 299–304 (2010)

[15] Lekner, J.: Near approach of two conducting spheres: enhancement of external electric field, J. Electrostatics 69, 559-563 (2011)

[16] Lekner, J.: Electrostatics of two charged conducting spheres, Proc. R. Soc. A, 468, 2829-2848 (2012)

[17] Li, H., Li, Y.Y., Bao, E., Yin, B.: Derivative estimates of solutions of elliptic systems in narrow regions, Quart. Appl. Math., to appear.

[18] Li, Y.Y., Nirenberg, L.: Estimates for elliptic system from composite material, Comm. Pure Appl. Math. LVI, 892–925 (2003)

[19] Li, Y.Y., Vogelius, M.: Gradient estimates for solution to divergence form elliptic equation with discontinuous coefficients, Arch. Rat. Mech. Anal. 153, 91–151 (2000)

[20] Lim, M., Yun, K.: Blow-up of electric fields between closely spaced spherical perfect conductors, Commun. Part. Diff. Eq. 34, 1287–1315 (2009)

[21] Lim, M., Yun, K.: Strong influence of a small fiber on shear stress in fiber-reinforced composites, J. Differ. Equations 250, 2402–2439 (2011)

[22] McPhedran, R.C., Poladian, L., Milton, G.W.: Asymptotic studies of closely spaced, highly conducting cylinders, Proc. R. Soc. Lond. A 415, 185–196 (1988)

[23] Yun, K.: Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape, SIAM J. Appl. Math. 67, 714–730 (2007)

[24] Yun, K.: Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross sections, J. Math. Anal. Appl. 350, 306-312 (2009)