CRITICAL POINTS FOR TWO-VIEW TRIANGULATION

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Abstract. Two-view triangulation is a problem of minimizing a quadratic polynomial under an equality constraint. We derive a polynomial that encodes the local minimizers of this problem using the theory of Lagrange multipliers. This offers a simpler derivation of the critical points that are given in Hartley-Sturm [6].

1. Introduction

Two-view triangulation is the problem of estimating a point \( x \in \mathbb{R}^3 \) from two noisy image projections; see [5, Chapter 12] for its significance in structure from motion in computer vision. Assuming a Gaussian error distribution, one way to solve the problem is to compute the maximum likelihood estimates (MLE) for the true image point correspondences. After that the point \( x \in \mathbb{R}^3 \) can be recovered via linear algebra [5]. In this paper we study the above problem of finding the MLEs. According to the discussion in [1] or [5, Chapter 12], the problem is formulated as follows.

Consider a rank two matrix \( F \in \mathbb{R}^{3 \times 3} \) which is called a fundamental matrix in multi-view geometry. This matrix \( F \) encodes a pair of projective cameras [5, Chapter 9]. Given two points \( u_1, u_2 \in \mathbb{R}^2 \) which denote the noisy image projections, we solve the problem

\[
\min_{x_1, x_2 \in \mathbb{R}^2} \|x_1 - u_1\|^2 + \|x_2 - u_2\|^2 \\
\text{subject to } \hat{x}_2^T F \hat{x}_1 = 0
\]

(1.1)

where \( \hat{x}_k := (x_k^T 1)^T \in \mathbb{R}^3 \) for \( k = 1, 2 \). The equation \( \hat{x}_2^T F \hat{x}_1 = 0 \) is called the epipolar constraint, which indicates that \( x_1 \) and \( x_2 \) are the true image projections under the projective cameras associated with \( F \). The minimizers of (1.1) are the MLEs for the true image correspondences, assuming the error is Gaussian.

In [5, Chapter 12] (or [6]) there is a technique for finding the global minimizers of (1.1) using a non-iterative approach. They use multi-view geometry to reformulate the problem (1.1) as minimizing a fraction in a single real variable say \( t \). Using the Fermat rule in elementary calculus, it turns out that the minimizers can be computed via finding the real roots of a polynomial in \( t \) of degree 6.

In this note, we view the problem (1.1) as minimizing a multivariate quadratic polynomial over one single equality constraint, and then employ the classical method of Lagrange multipliers to locate the potential local minimizers. These candidates are called critical points. For general rank two matrices \( F \) and general points \( u_1, u_2 \), there are six critical points. They can be computed via finding the roots of a polynomial of degree 6 in the Lagrange multiplier. Assuming that a global minimizer exists, the minimizer of (1.1) can be obtained from the critical points.
2. Six critical points for two-view triangulation

2.1. Reformulation of the problem. Given a fundamental matrix \( F \in \mathbb{R}^{3 \times 3} \) and \( u_1 = (u_{11} \ u_{12})^\top \), \( u_2 = (u_{21} \ u_{22})^\top \in \mathbb{R}^2 \), consider the invertible matrices \( W_1 := \begin{pmatrix} 1 & 0 & -u_{11} \\ 0 & 1 & -u_{12} \end{pmatrix} \) and \( W_2 := \begin{pmatrix} 1 & 0 & -u_{21} \\ 0 & 1 & -u_{22} \end{pmatrix} \). Note that \( \| x_k - u_k \|^2 = \| \hat{x}_k - \hat{u}_k \|^2 \). and that problem (1.1) is equivalent to the problem

\[
\begin{align*}
\min_{x_1, x_2 \in \mathbb{R}^2} & \quad \| W_1 \hat{x}_1 \|^2_2 + \| W_2 \hat{x}_2 \|^2_2 \\
\text{subject to} & \quad \hat{x}_2^\top F \hat{x}_1 = 0
\end{align*}
\]

For all \( k = 1, 2 \), the last coordinate of \( W_k \hat{x}_i \) equals one. As a result, we let \( y_k \in \mathbb{R}^2 \) be such that \( \hat{y}_k = W_k \hat{x}_k \). Then (1.1) is further equivalent to the problem

\[
\begin{align*}
\min_{y_1, y_2 \in \mathbb{R}^2} & \quad \frac{1}{2} (\| \hat{y}_1 \|^2_2 + \| \hat{y}_2 \|^2_2) \\
\text{subject to} & \quad \hat{y}_2^\top F^r \hat{y}_1 = 0
\end{align*}
\]

where \( F^r := W_2^{-\top} F W_1^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \) is another fundamental matrix.

2.2. Derivation of a six degree polynomial. Let \( G(y_1, y_2) := \frac{1}{2} (\| \hat{y}_1 \|^2_2 + \| \hat{y}_2 \|^2_2) \) and \( H(y_1, y_2) := \hat{y}_2^\top F^r \hat{y}_1 \). The Karush-Kuhn-Tucker (KKT) equation for (2.1) is \( \nabla G + \lambda \nabla H = 0 \) for some \( \lambda \in \mathbb{C} \) called the Lagrange multiplier; see any nonlinear programming text e.g. [2]. Unwinding this equation we obtain a linear system in four variables, namely,

\[
\begin{pmatrix}
1 & 0 & \lambda a & \lambda b \\
0 & 1 & \lambda d & \lambda e \\
\lambda a & \lambda d & 1 & 0 \\
\lambda b & \lambda e & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_{21} \\
y_{22} \\
y_{11} \\
y_{12}
\end{pmatrix}
= -\lambda
\begin{pmatrix}
c \\
d \\
e \\
g
\end{pmatrix}
\]

where \( y_k = (y_{k1} \ y_{k2})^\top \) for \( k = 1, 2 \), and \( \lambda \) is the Lagrange multiplier. To acquire the critical points we derive a polynomial equation in \( \lambda \). It comes from first expressing \( y_k, k = 1, 2 \), in terms of \( u_1, u_2, F \) and then substituting these expressions into the epipolar constraint \( \hat{y}_2^\top F^r \hat{y}_1 = 0 \). Let \( A_\lambda \) be the \( 4 \times 4 \) coefficient matrix of the above system. One has

\[
\det(A_\lambda) = (bd - ae)^2 \lambda^4 - (a^2 + b^2 + c^2 + d^2 + e^2) \lambda^2 + 1.
\]

Define \( p_{kl} := \det(A_\lambda) \) for \( k, l = 1, 2 \). By Cramer’s rule one has

\[
\begin{align*}
p_{21} &= \lambda(bd - ae)(eg - dh)\lambda^3 + (d^2 c + e^2 c - a d f - b e f)\lambda^2 + (a g + b h)\lambda - c \\
p_{22} &= \lambda[(bd - ae)(ah - bg)\lambda^3 + (a^2 f + b^2 f - a c d - b c e)\lambda^2 + (d g + c h)\lambda - f] \\
p_{11} &= \lambda[(bd - ae)(ce - bf)\lambda^3 + (b^2 g + e^2 g - a b h - d e h)\lambda^2 + (a c + d f)\lambda - g] \\
p_{12} &= \lambda[(bd - ae)(af - cd)\lambda^3 + (a^2 h + b^2 h - a b g - d e g)\lambda^2 + (b c + e f)\lambda - h].
\end{align*}
\]

Consider the polynomial

\[
T := -\det(A_\lambda) \hat{y}_2^\top F^r \hat{y}_1 = -p_{22}^\top F^r p_{12}
\]

where \( p_k := (p_{k1} \ p_{k2} \ \det(A_\lambda))^\top \) for \( k = 1, 2 \). Since \( \det(A_\lambda) \) is a quartic in \( \lambda \), and \( p_{kl} \) is also a quartic in \( \lambda \) for \( k, l = 1, 2 \), we know \( T \) is a polynomial in \( \lambda \) of
degree at most 8. By a careful and slightly tedious computation without using any machines, or by using the following Macaulay2 code:

```plaintext
R = QQ[a,b,c,d,e,f,g,h,i,L];
A = matrix({{1,0,L*a,L*b},{0,1,L*d,L*e},{L*a,L*d,1,0},{L*b,L*e,0,1}});
detA = det A;
p21 = det matrix({{-L*c,0,L*a,L*b},{-L*f,1,L*d,L*e},{-L*g,L*d,1,0},{-L*h,L*e,0,1}});
p22 = det matrix({{1,-L*c,L*a,L*b},{{1,-L*f,L*d,L*e},{L*a,-L*g,1,0},{L*b,-L*h,0,1}}};
p11 = det matrix({{1,0,-L*c,L*b},{0,1,-L*f,L*e},{L*a,-L*d,L*g,0},{L*b,-L*e,-L*h,1}});
p12 = det matrix({{1,0,L*a,-L*c},{0,1,L*d,-L*f},{L*a,L*d,1,-L*g},{L*b,L*e,0,-L*h}};
T = -(a*p11*p21+b*p12*p21+c*p21*detA+dp11*p22+
    e*p12*p22+f*p22*detA+g*p11*detA+h*p12+ip*detA*detA); treat
```

we know the coefficient of $\lambda^7$ is zero. The coefficient of $\lambda^8$ is

$$-(bd-ac)^2(eg-dh)(ace-abf+baf-bcd+cdb-cae)+$$

$$-(bd-ac)^2(ah-bg)(ace-abf+baf-bcd+cdb-cae)+$$

$$-(bd-ac)^2(gec-gbf+hae-hcd+ibd-iae) = (bd-ac)^2 \det(F) = 0$$

since $F$ has rank two. This implies $T$ is a polynomial in $\lambda$ of degree at most six. Here we record the explicit expression of $T$:

$$T = (bd-ac)^2(acg+dfg+bch+efh-a^2i-b^2i-d^2i-e^2i)\lambda^6 +$$

$$a^2c^2d^2\lambda^5 + c^2d^4\lambda^5 + 2abc^2d^2e\lambda^5 + b^2c^2e^2\lambda^5 + a^2c^2d^2e\lambda^5 -$$

$$2a^3c^2d^2\lambda^5 - 2abc^2d^2\lambda^5 - 2ac^2d^3e\lambda^5 - 2a^2bcef\lambda^5 - 2b^3ce\lambda^5 - 2bc^2de\lambda^5 -$$

$$2acde2\lambda^5 - 2bce^3\lambda^5 + a^3f3\lambda^5 + b^2c^2f^2\lambda^5 + b^2f^2\lambda^5 + a^2d^2f^2\lambda^5 +$$

$$2abde2\lambda^5 + b^2c^2f^2\lambda^5 + a^2b^2g^2\lambda^5 + b^2g^2\lambda^5 + 2abde\lambda^5 + 2bc^2de\lambda^5 +$$

$$2d^2c^2\lambda^5 + c^2d^2\lambda^5 - 2ab^3gh\lambda^5 - 2ab^2gh\lambda^5 - 2ab^2gh\lambda^5 -$$

$$2b^2deg\lambda^5 - 2b^2deg\lambda^5 - 2b^2deg\lambda^5 - 2ab^2gh\lambda^5 - 2abc^2d^2\lambda^5 +$$

$$2a^2d^2\lambda^5 + d^2h^2\lambda^5 + 2abde\lambda^5 + d^2e^2h^2\lambda^5 + a^3c\lambda^4 +$$

$$abc\lambda^4 - 5bcde\lambda^4 + 6ace^2g\lambda^4 + 2d^2\lambda^4 + 6b^2d^2\lambda^4 + d^2f^2\lambda^4 -$$

$$5abef\lambda^4 + de^2f\lambda^4 + a^2bch\lambda^4 + b^3ch\lambda^4 + 6bc^2d\lambda^4 - 5acde\lambda^4 +$$

$$bce^2\lambda^4 - 5abdf\lambda^4 + 6a^2efh\lambda^4 + b^2efh\lambda^4 + d^2efh\lambda^4 + a^2d^2\lambda^4 -$$

$$2a^2b^2\lambda^4 - 4b^2\lambda^4 - 2a^2d^2\lambda^4 - 4b^2d^2\lambda^4 - d^2\lambda^4 + 4abde\lambda^4 +$$

$$4a^2e^2\lambda^4 - 2b^2e^2\lambda^4 - 4e^4\lambda^4 - 2c^2d^2\lambda^4 - 2c^2d^2\lambda^4 + 4acdf\lambda^3 +$$

$$4bce\lambda^3 - 2a^2f^2\lambda^3 - 2b^2g^2\lambda^3 - 2c^2g^2\lambda^3 + 4abgh\lambda^3 +$$

$$4degh\lambda^3 - 2a^2h^2\lambda^3 -$$

$$2d^2h^2\lambda^3 - 3acg\lambda^2 - 3dfg\lambda^2 - 3bch\lambda^2 - 3efh\lambda^2 + 2a^2\lambda^2 +$$

$$2d^2\lambda^2 + 2e^2\lambda^2 + c^2\lambda + f^2\lambda + g^2\lambda + h^2\lambda - i.$$

### 2.3. The six critical points.

By solving $T = 0$ for $\lambda$, we get six (complex) solutions (counting multiplicities) for $\lambda$, say $\lambda_1, \ldots, \lambda_6$. Plugging in these six values of $\lambda$ into the linear system [2.2], solving the linear system for $y_1$ and $y_2$, and computing $x_1$ and $x_2$, one obtains the critical points for two-view triangulation. If $\det(A_{\lambda_k}) \neq 0$ for every $k = 1, \ldots, 6$ then there are precisely six critical points counting multiplicities.
For general points $u_1, u_2 \in \mathbb{R}^2$, there are six distinct critical points for two-view triangulation. The claim is false if and only if the discriminant of $T$ or the resultant of $T$ and $\det(A_\lambda)$ are zero polynomials. Instead of computing the desired discriminant and resultant which depend on $u_1, u_2$ and $F$, one can find an example of $(u_1, u_2, F)$ such that the discriminant of $T$ and the resultant of $T$ and $\det(A_\lambda)$ take a nonzero value, that is, $\det(A_\lambda) \neq 0$ for every solution $\lambda$ of $T$, and the six critical points obtained are distinct. If we consider the data $u_1 = (0\ 0)^T$, $u_2 = u_1$ and $F = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \\ 3 \end{pmatrix}$, then the polynomial $T$ becomes $-2\lambda^6 + 6\lambda^5 + 3\lambda^4 - 12\lambda^3 - 3\lambda^2 + 12\lambda - 3$, and there are six distinct complex critical points for the problem \([14]\); see Table 1.

We summarize the discussion in the following theorem.

**Theorem 2.1.** For general points $u_1, u_2 \in \mathbb{R}^2$ and fundamental matrices $F$, there are six complex critical points for the problem \([14]\).

### 3. Discussion

One can make sense of the critical points for $n$-view triangulation where $n$ is greater than two. The authors in \([5]\) (cf. \([7]\)) computed the number of critical points for 2 to 7 view triangulation are 6, 47, 148, 336, 638, 1081. Draisma et al. \([3]\) call this list of numbers the Euclidean distance degrees of the multi-view variety associated to 2 to 7 cameras. They conjecture that the general term of this sequence is

$$C(n) := \frac{9}{2} n^3 - \frac{21}{2} n^2 + 8n - 4.$$ 

One can apply the Bézout’s theorem to conclude that $C(n)$ has order $n^3$, and our paper verified $C(2) = 6$. However a proof of the above general formula is still unknown.

### References

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