Bianchi spaces and their three-dimensional isometries as $S$-expansions of two-dimensional isometries

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Abstract

In this paper we show that certain three-dimensional isometry algebras, specifically those of type I, II, III and V (according to Bianchi’s classification), can be obtained as expansions of the isometries in two dimensions. In particular, we use the so-called $S$-expansion method, which makes use of the finite Abelian semigroups, because it is the most general procedure known until now. Also, it is explicitly shown why it is impossible to obtain the algebras of type IV, VI–IX as expansions from the isometry algebras in two dimensions. All the results are checked with computer programs. This procedure shows that the problem of how to relate, by an expansion, two Lie algebras of different dimensions can be entirely solved. In particular, the procedure can be generalized to higher dimensions, which could be useful for diverse physical applications, as we discuss in our conclusions.

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1. Introduction

To find non-trivial correspondence between two different Lie algebras is an important task from a physical as well as from a geometrical point of view. On the physical side, such a correspondence would help us to understand how two different field theories with different groups of symmetry are related, or, in other words, how physical quantities calculated in these theories are related. On the geometrical side, it would help to relate characteristics
of different manifolds possessing different symmetric groups (relations between their Killing vectors, etc). In both cases we are talking about the Lie groups [1]. In [2] the relation between solvable and semisimple Lie algebras was studied via the contraction method. The contraction method has been introduced into the theory of Lie algebras by Inönü and Wigner [3], and was later generalized by Weimar-Woods [4]. The expansion methods are generalizations of the Weimar-Woods (WW) contraction method and were introduced some years ago in [5–8]. In the WW contraction method, a rescaling of several generators from the basis of a Lie algebra \( \mathcal{G} \) by a parameter is performed. The S-expansion method of [9] presents by itself a generalization of the expansion methods. This article is dedicated to further development of the S-expansion method of [9] and its application to the theory of Lie algebras.

In the expansion method [6–8], the Maurer–Cartan (MC) equations for the MC forms \( \omega^i \) on the manifold of its associated group \( \mathcal{G} \) were studied. As is known, the MC equations

\[
d d \omega^k = -\frac{1}{2} c^k_{ij} \omega^i \wedge \omega^j
\]

are an equivalent description to that given in terms of Lie brackets \([X_i, X_j] = c^k_{ij} X_k\) for the algebra \( \mathcal{G} \) with the basis of generators \( X_i \). When a rescaling by a parameter \( \lambda \) is performed for some of the group coordinates \( \phi^i \), the forms \( \omega^i(\phi, \lambda) \) can be expanded as power series in \( \lambda \). Inserting these expansions back into the original MC equations for \( \mathcal{G} \), one obtains the MC equations of a new finite-dimensional expanded Lie algebra. As shown in [5–8] this method reproduces all the WW contraction methods where the dimension is preserved (in [6] it is explicitly shown that it reproduces the Inönü–Wigner and WW contractions), but furthermore, in certain cases it allows the obtainment of higher dimensional Lie algebras.

A natural generalization of the method developed in [5–8] is the S-expansion of [9], which combines the structure constants of algebra \( \mathcal{G} \) with the inner multiplication law of an Abelian semigroup \( S \) in order to define the Lie bracket of a new S-expanded algebra. When certain conditions are met, the method permits extraction of smaller algebras which are called resonant subalgebras, and then, extraction of reduced algebras from the resonant subalgebras. This method reproduces the results of the first application of expansion methods described above (which uses a parameter \( \lambda \)) for a particular choice of the semigroups, namely, the semigroups denoted as \( S_E \) and whose definition is given in [9]. Since it reproduces the method above, it reproduces all the procedures of the WW contraction as well. This is explicitly shown in [9].

In [2, 10] it has been shown that the WW contractions do not cover all the possible contractions. In other words, the S-expansions generalize the expansions of [5–8], which in turn generalize the WW contractions, but it is not clear yet if the S-expansion exhausts all possible contractions.

In any case, these methods appeared to be powerful tools in order to find non-trivial relations between different Lie algebras. The discovery of these relations presents in itself a very interesting problem from both physical and mathematical points of view. In fact, many physical applications have been found in this context (see, for example, [6, 11–16]). The S-expansion with semigroup \( S_E^{(\omega)} \) is an example of the existence of such non-trivial relations. It is necessary to say that the major part of the nontrivial relations previously

\[\text{By non-trivial relations we mean that these mechanisms of contractions and expansions allow us to obtain some Lie algebras starting with other algebras that have completely different properties. Also, the original algebra is not necessarily (but could be in specific cases) contained as a subalgebra of the algebra obtained by these processes.}\]
uncovered between distinct Lie algebras was found by using this semigroup $S^{(n)}_E$. One of the advantages of the $S$-expansion method, which is used in some of those applications, is that if we know the invariant tensors of a certain Lie algebra, then the mechanism gives the invariant tensors for the expanded algebras, even if the last ones are not semisimple. This feature is especially useful in the construction of new Chern–Simons (CS) theories of gravity [11, 31] where the invariant tensors of the symmetry group under which the theory is invariant are fundamental ingredients of the theory. The dual formulation of the $S$-expansion procedure for a Lie algebra was also constructed in [12], which enables us to understand this procedure at the level of the Lagrangians. Another interesting application of the expansion methods is in establishing a relation between general relativity and CS gravity. As shown in [12], general relativity in five-dimensional spacetime may emerge at a special critical point from a CS action. To achieve this result, both the Lie algebra and the symmetric invariant tensor that defines the CS Lagrangian are constructed by means of the Lie algebra $S$-expansion method with the semigroup $S^{(n)}_E$. To investigate similar relationships possibly appearing in other theories we need the methodology of relating distinct Lie algebras by a suitable semigroup. This technique is developed in the present paper. The procedure for searching for the semigroup relating given Lie algebras may play a crucial role in relating different solutions for Einstein equations such as black holes, etc [32]. Possible non-trivial relations between quantities in exactly solvable models of statistical mechanics [17] could be among the other applications of the methods developed here for the construction of new semigroups.

It is the aim of this paper to show that the $S$-expansion method permits us to obtain some types of three-dimensional isometries from the two-dimensional isometries. We present a complete study on the possibility of discovering non-trivial relations between two- and three-dimensional isometry Lie algebras. Even when these isometries are well known in literature (see [18]), the non-trivial relations we find between two- and three-dimensional isometry algebras are new results. In fact, we identify the three-dimensional algebras that can be obtained from the two-dimensional algebras by means of an $S$-expansion and describe explicitly how to do it. We have constructed a systematic way of identifying the semigroups which relate two given algebras, if the semigroups exist. All the semigroups that we have found can be identified by means of a computer algorithm specially developed by us for this task with the semigroups classified in papers [19–29]. For other three-dimensional algebras we show that it is impossible to obtain them from the information of the two-dimensional algebras, and in some sense the information that they contain is intrinsic to three dimensions.

This paper is organized as follows: in section 2 we present a brief technical description of the basic procedures that we use alongside this work. In subsection 2.1 we review some aspects of the $S$-expansion procedure. In subsection 2.2 we introduce the expansions that will be applied in the next sections. In subsection 2.3 we summarize the history of enumeration and characterization of finite semigroups existing for each order $n = 1, \ldots, 9$, while in section 2.4 we briefly describe the Bianchi classification of isometries in two- and three-dimensional spaces. Section 3 contains the main proposition of the paper. In section 4 it is shown in an instructive way how some types of isometries are related to two-dimensional isometries using known semigroups and also by introducing other semigroups that have not previously been used in the applications of the $S$-expansion procedure. In subsection 4.4 we briefly summarize the results obtained by this iterative procedure. In section 5 it is shown why it is not possible to obtain, by expansions, the other three-dimensional isometries from the two-dimensional algebras. Finally in section 6 we check the results using computer programs and solve the problem entirely.
2. Technical description

2.1. The S-expansion procedure

In this section we describe certain technical details about the Abelian semigroup expansion procedure (S-expansion for short). We refer to [9] for further details.

The S-expansion method is a procedure which conceptually includes three steps:

(a) constructing a Lie algebra by means of the tensor product \( \mathcal{G} \otimes S \)
(b) extracting the resonant subalgebra
(c) performing a procedure called the 0\(_S\) reduction.

After applying all the three steps we will obtain the expanded algebras in which we are interested. Since this will be used in the main body of the paper, we will briefly describe (a) in this section while (b) and (c) will be explained in the next subsection.

Consider a Lie algebra \( \mathcal{G} \) and a finite Abelian semigroup \( S = \{\lambda_\alpha\} \). According to theorem 3.1 from [9], the tensor product

\[
\mathcal{G}_S = S \otimes \mathcal{G},
\]

is also a Lie algebra. The elements of this expanded algebra are denoted by

\[
X_{(i,\alpha)} = \lambda_\alpha \otimes X_i
\]

where the product is understood as a tensor product of the matrix representations of the generators \( X_i \) of \( \mathcal{G} \) and the elements \( \lambda_\alpha \) of the semigroup \( S \). For short, we will omit the symbol \( \otimes \) on those elements, so we will denote them by just

\[
X_{(i,\alpha)} = \lambda_\alpha X_i.
\]

The Lie product in \( \mathcal{G}_S \) is defined as

\[
\left[ X_{(i,\alpha)}, X_{(j,\beta)} \right] = \lambda_\alpha \beta \left[ X_i, X_j \right].
\]

Set (1) with composition law (3) is called a \( S \)-expanded Lie algebra.

In a nutshell, the S-expansion method can be seen as a natural generalization of the Inönü–Wigner contraction, where instead of multiplying the generators by a numerical parameter, we multiply the generators by the elements of a certain Abelian semigroup; for more details see [9].

2.2. Resonant subalgebra and 0\(_S\)-reduced algebra

As shown in [9] smaller algebras can be extracted from the above expanded algebra: the resonant subalgebra and the reduced algebra. Their existence depends on certain conditions expressed by equations (23) and (34) of [9].

In fact, the original algebra will be one of the two-dimensional isometry algebras (8)–(9), both having the subspace structure \( \mathcal{G} = V_0 \oplus V_1 \) given by

\[
[V_0, V_0] \subset V_0
\]

\[
[V_0, V_1] \subset V_1
\]

\[
[V_1, V_1] \subset V_0
\]

As mentioned in the introduction, by making use of a specific semigroup it is possible to obtain this formalism as a special case, whatever the contraction procedure (not only the Inönü–Wigner one). For example, as shown in [9] it is also possible to reproduce the contractions in the sense of WW.
where $V_0$, $V_1$ are generated by $X_2$ and $X_1$, respectively. On the other hand, the semigroups we are going to construct will possess a resonant decomposition, i.e., they must be of the form $S = S_0 \cup S_1$ where
\[
S_0 \times S_0 \subset S_0 \\
S_0 \times S_1 \subset S_1 \\
S_1 \times S_1 \subset S_0.
\] (6)

Then, according to theorem 4.2 of [9], the resonant subalgebra is of the form
\[
G_{S,R} = (S_0 \otimes V_0) \oplus (S_1 \otimes V_1).
\] (7)

Note that equation (6) is a particular case of equation (34) of ([9]).

Even a smaller algebra can be obtained when there is a zero element in the semigroup, i.e., an element $0_S \in S$ such that, for all $\lambda \in S$, $0_S \lambda = 0_S$. When this is the case, the whole $0_S \otimes G$ sector can be removed from the resonant subalgebra by imposing $0_S \otimes G = 0$ (see definition 3.3 from [9]). The resulting algebra continues to be a Lie algebra and we denote it as $G_{S,R}^{\text{red}}$.

2.3. Finite semigroup programming

The quantity of finite non-isomorphic semigroups of order $n$ is given in the following table:

| Order | $Q =$No of Semigroups | References |
|-------|------------------------|------------|
| 1     | 1                      |            |
| 2     | 4                      |            |
| 3     | 18                     |            |
| 4     | 126                    | [19]       |
| 5     | 1160                   | [20]       |
| 6     | 15 973                 | [23]       |
| 7     | 836 021                | [24]       |
| 8     | 1843 120 128           | [25]       |
| 9     | 52 989 400 714 478     | [26]       |

All the semigroups of order 4 were classified by Forsythe in [19], those of order 5 by Motzkin and Selfridge in [20], those of order 6 by Plemmons in [21–23], those of order 7 by Jürgensen and Wick in [24], and those of order 8 by Satoh, Yama and Tokizawa in [25], and the monoids and semigroups of order 9 by Distler and Kelsey in [26, 27] and by Distler and Mitchell in [28]. Also, for semigroups of order 9 the number of semigroups can be found in [29].

As shown in the table, the task of enumerating the all non-isomorphic finite semigroups of a certain order is a non-trivial problem. In fact, the number $Q$ of semigroups increases very quickly with the order of the semigroup.

In [30], a set of algorithms is given which permits us to make certain calculations with finite semigroups. The first program, gen.f, gives all the non-isomorphic semigroups of order $n$ for $n = 1, 2, \ldots, 8$. The input is the order $n$ of the semigroups we want to obtain and the output is a list of all the non-isomorphic semigroups that exist in this order.

8 The order $n = 9$ is non-trivial and the algorithms of the mentioned reference fails. This non-trivial problem was solved in 2009 by Distler, Kelsey and Mitchell. However, in this paper we will consider calculations with semigroups of order 4 at most.
In this work, the elements of the semigroup are labeled by $\lambda_\alpha$ with $\alpha = 1, \ldots, n$ and each semigroup will be denoted by $S^\alpha_{(n)}$, where the super-index $\alpha = 1, \ldots, Q$ identifies the specific semigroup of order $n$. The second program in [30], com.f, takes as the input one of the mentioned lists for a certain order, picks up just the symmetric tables and generates another list with all the Abelian semigroups. For example, for $n = 3$ the elements of the semigroup are labeled $\lambda_1$, $\lambda_2$ and $\lambda_3$ and the program com.f gives the following list of semigroups:

| $S^1_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^2_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^3_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|------|-------------|-------------|-------------|------|-------------|-------------|-------------|------|-------------|-------------|-------------|
| $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ |
| $\lambda_2$ | $\lambda_1$ | $\lambda_1$ | $\lambda_2$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_2$ | $\lambda_1$ | $\lambda_2$ |
| $\lambda_3$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ |
| $S^4_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^5_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^6_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
| $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ |
| $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ |
| $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_3$ |
| $S^7_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^8_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^9_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
| $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ |
| $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ |
| $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_3$ |
| $S^{10}_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^{11}_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $S^{12}_{(3)}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
| $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ |
| $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_1$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ | $\lambda_2$ |
| $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_3$ |

In general, then, the program com.f of [30] gives a list of tables of all the Abelian non-isomorphic semigroups of a certain order\(^9\) (up to order 8). At the end of this paper we are going to use these lists (as an input in some algorithms on which we elaborated) to check all our results on the relations between 2- and three-dimensional isometries. More specifically, in section 6 we show that all the semigroups we construct in this paper to make the relations between two- and three-dimensional isometries, are isomorphic to one of the semigroups of the lists given by [30]. This is a direct way of checking that the iterative procedure (which we shall present in sections 4, 5) to find semigroups with zero elements and resonant decompositions is working well.

2.4. Bianchi spaces

In [18] Bianchi proposed a procedure to classify the three-dimensional spaces that admit a three-dimensional isometry. He showed how to represent the generators as Killing vectors and how to get the corresponding metrics.

In this paper we study the possibility of relating, by means of expansions, two- and three-dimensional isometry algebras. The two-dimensional algebras are given by

$$[X_1, X_2] = 0$$

$$[X_1, X_2] = X_1.$$  

\(^9\) As the number of non-isomorphic semigroups increases very quickly with the order $n$ (see table (1)), the mentioned lists are very large for higher orders. Note also that the semigroups $S^3_{(3)}, S^4_{(3)}, S^5_{(3)}, S^6_{(3)}, S^7_{(3)}, S^8_{(3)}$ and $S^{11}_{(3)}, S^{12}_{(3)}$ are not given in the list for $n = 3$ because they are not Abelian (non-commutative).
The three-dimensional algebras are given in the following table,

| Group | Algebra |
|-------|---------|
| Type I | \([X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0\) |
| Type II | \([X_1, X_2] = [X_1, X_3] = 0, \ [X_2, X_3] = X_1\) |
| Type III | \([X_1, X_2] = [X_2, X_3] = 0, \ [X_1, X_3] = X_1\) |
| Type IV | \([X_1, X_2] = 0, \ [X_1, X_3] = X_1, \ [X_2, X_3] = X_1 + X_2\) |
| Type V | \([X_1, X_2] = 0, \ [X_1, X_3] = X_1, \ [X_2, X_3] = X_2\) |
| Type VI | \([X_1, X_2] = 0, \ [X_1, X_3] = X_1, \ [X_2, X_3] = hX_3, \) where \(h \neq 0, 1\) |
| Type VII | \([X_1, X_2] = 0, \ [X_1, X_3] = X_1, \ [X_2, X_3] = -X_1\) |
| Type VII | \([X_1, X_3] = 0, \ [X_1, X_3] = X_5, \ [X_2, X_3] = -X_1 + hX_5, \) where \(h \neq 0 (0 < h < 2)\) |
| Type VIII | \([X_1, X_2] = X_1, \ [X_1, X_3] = 2X_2, \ [X_2, X_3] = X_3\) |
| Type IX | \([X_1, X_2] = X_1, \ [X_2, X_3] = X_1, \ [X_1, X_3] = X_2\) |

In [18] Bianchi showed that these symmetries fixed completely the metric of three-dimensional spaces on which the isometries act transitively. The spaces (whose metrics are listed in [18]) are known as Bianchi spaces.

3. Main proposition

This whole paper is dedicated to demonstrating that the following proposition is valid.

**Proposition 1.** Some, but not all of the Bianchi algebras can be obtained as \(S\)-expansions of the two-dimensional algebras (8)–(9).

This proposition will be proven by performing the \(S\)-expansion of (8)–(9) and by applying two known procedures which allow the extraction of a smaller algebra from the expanded algebra. As mentioned in section 2.2, these are the construction of resonant subalgebra and the 0\(_1\)-reduction of the resonant subalgebra.

4. Bianchi spaces related to two-dimensional isometries

4.1. The type III algebra

**Theorem 2.** An Abelian semigroup of four elements \(S = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\) with decomposition

\[
S_0 = \{\lambda_2, \lambda_3, \lambda_4\} \\
S_1 = \{\lambda_1, \lambda_4\}
\]

compatible with the resonance condition (6) and possessing the following constraints in the multiplication table

\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & \lambda_4 & \lambda_1 & \lambda_4 \\
\lambda_2 & \lambda_4 & \lambda_4 & \lambda_4 \\
\lambda_3 & \lambda_1 & \lambda_4 & \lambda_4 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4
\end{array}
\]

produces, after 0\(_2\)-reduction of the resonant subalgebra of the \(S\)-expanded algebra of the two-dimensional isometry (9), an algebra which coincides with three-dimensional Bianchi type III isometry.

**Proof.** Let us begin with an unknown semigroup \(S = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\) where we impose the following conditions.
(i) $\lambda_4$ is a zero of the semigroup, so the multiplication table is restricted to the form

\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & & & \\
\lambda_2 & & & \\
\lambda_3 & & & \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

where the empty spaces must be filled in a way such that it is in fact an Abelian semigroup, i.e., closed, associative and commutative. In addition, in order to get a smaller algebra we demand

(ii) that it contains a decomposition like that given in (10) which is resonant, i.e., that satisfies equation (6).

Then, according to (7), the resonant subalgebra of $G_{S,R} = S \times G$ is given by

\[
G_{S,R} = (S_0 \otimes V_0) \oplus (S_1 \otimes V_1)
\]

\[
= \{ \lambda_2 X_2, \lambda_3 X_2, \lambda_4 X_2 \} \oplus \{ \lambda_1 X_1, \lambda_4 X_1 \}
\]

\[
= \{ \lambda_2 X_2, \lambda_3 X_2, \lambda_4 X_2, \lambda_1 X_1, \lambda_4 X_1 \}.
\]

Now we have to extract an even smaller algebra by means of the $S$-reduction. This is done by simply removing from (12) the elements that contain the zero element, $\lambda_4$. Therefore, the reduction of the resonant subalgebra is given by

\[
G_{\text{red}} = \{ \lambda_2 X_2, \lambda_3 X_2, \lambda_1 X_1 \}
\]

with the following commutation relations:

\[
[\lambda_2 X_2, \lambda_3 X_2] = 0
\]

\[
[\lambda_2 X_2, \lambda_1 X_1] = -\lambda_1 \lambda_2 X_1
\]

\[
[\lambda_3 X_2, \lambda_1 X_1] = -\lambda_1 \lambda_3 X_1.
\]

In order for this algebra to be closed we should choose:

(a) $\lambda_1 \lambda_2 = \lambda_1$ or $\lambda_1 \lambda_2 = \lambda_4$
(b) $\lambda_1 \lambda_3 = \lambda_1$ or $\lambda_1 \lambda_3 = \lambda_4$.

So we are led to four possibilities to construct a closed algebra:

(i) $\lambda_1 \lambda_2 = \lambda_1 \lambda_3 = \lambda_4$,
(ii) $\lambda_1 \lambda_2 = \lambda_1$ and $\lambda_1 \lambda_3 = \lambda_4$,
(iii) $\lambda_1 \lambda_2 = \lambda_4$ and $\lambda_1 \lambda_3 = \lambda_1$,
(iv) $\lambda_1 \lambda_2 = \lambda_1 \lambda_3 = \lambda_1$.

The case (i) will lead to translations in three dimensions, i.e., to the type I algebra.

We can check that the case (iv) is not useful, because in this case the multiplication law is non-associative. On the other hand, it can be seen that both (ii) and (iii) will lead to the type III algebra. In fact, in case (ii) we have

\[
[\lambda_2 X_2, \lambda_3 X_2] = 0
\]

\[
[\lambda_2 X_2, \lambda_1 X_1] = -\lambda_1 X_1
\]

\[
[\lambda_3 X_2, \lambda_1 X_1] = -\lambda_4 X_1 = 0 (\lambda_4 \text{ is a zero element})
\]

10 Note that we start with a two-dimensional algebra and we are looking for a three-dimensional one. The $S$-expanded algebra with a semigroup of order 4 is eight-dimensional and the reduced one is six-dimensional. So to obtain a three-dimensional algebra we need to extract an even smaller algebra. This is only possible by extracting a resonant subalgebra.

11 Note that we have an abuse of notation here. The last equation must be read as: the resonant subalgebra is generated by the set of generators that appear in the right-hand side.

12 This case will be analyzed later.
renaming the generators as
\[ Y_1 = \lambda_1 X_1 \]
\[ Y_2 = \lambda_2 X_2 \]
\[ Y_3 = \lambda_3 X_2 \]
and we immediately recognize the type III algebra (see table 3). In case (iii) we would have
\[ [\lambda_2 X_2, \lambda_3 X_2] = 0 \]
\[ [\lambda_2 X_2, \lambda_1 X_1] = -\lambda_4 X_1 = 0 \text{ (} \lambda_4 \text{ is a zero element)} \]
\[ [\lambda_3 X_2, \lambda_1 X_1] = -\lambda_1 X_1 \]
and we again recover the type III algebra by renaming the generators as
\[ Y_1 = \lambda_1 X_1 \]
\[ Y_2 = \lambda_2 X_2 \]
\[ Y_3 = \lambda_3 X_2 \].

We choose to study case (iii) to construct a semigroup that leads to a type III algebra, although the case (ii) may also be studied to generate other semigroups leading to the same result. So, the table describing the multiplication law of case (iii) is given by
\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\end{array}
\]
where the empty spaces must be filled in such a way that they satisfy associativity and the decomposition (10) satisfies the resonant condition (6).

**Proposition 3.** There are semigroups that fit multiplication table (11) and resonant condition (10).

We give several examples of semigroups of this type.

4.1.1. **Semigroup \( S^{(3)}_K \).** Consider the semigroup \( S^{(n)}_K \), with the multiplication law defined by
\[
\begin{align*}
\lambda_\alpha \lambda_\beta &= \lambda_{\min(\alpha, \beta)}, \quad \alpha + \beta > n \\
\lambda_\alpha \lambda_\beta &= \lambda_{\alpha+1}, \quad \alpha + \beta \leq n
\end{align*}
\]
(13)
\[ \alpha, \beta = 1, 2, \ldots, n. \]
It may be directly seen that for \( n = 3 \) the table of multiplication law
\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_4 \\
\end{array}
\]
fits with the form of table (11), where we have filled those empty spaces that appear in (11). Therefore, an expansion with the semigroup \( S^{(3)}_K \) reproduces the type III algebra after the \( 0_L \)-reduction of the resonant subalgebra.
4.1.2. Semigroup $S_{N1}$. Let us consider the following table of multiplication:

|   | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|-------------|-------------|-------------|-------------|
| $\lambda_1$ | $\lambda_4$ | $\lambda_4$ | $\lambda_3$ | $\lambda_4$ |
| $\lambda_2$ | $\lambda_4$ | $\lambda_2$ | $\lambda_4$ | $\lambda_4$ |
| $\lambda_3$ | $\lambda_1$ | $\lambda_4$ | $\lambda_3$ | $\lambda_4$ |
| $\lambda_4$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ |

Here we have filled the empty spaces of (11) in another way, leading to a new semigroup that satisfies the required properties. It can be directly shown that this product is associative, satisfies the resonant condition and fits the form of table (11). The proof is direct but a little tedious. Therefore this semigroup, $S_{N1}$, reproduces the type III algebra as well and is not isomorphic to the previous semigroup, $S_K^{(3)}$.

4.1.3. The semigroup $S_{E}^{(2)}$, another way of obtaining the type III algebra. In order to show that there are other semigroups that lead to the type III algebra we consider the semigroup $S_{E}^{(2)}$ introduced in [9] for $n = 2$. Its multiplication law is given by the following table:

|   | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|-------------|-------------|-------------|-------------|
| $\lambda_0$ | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
| $\lambda_1$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ |
| $\lambda_2$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3$ | $\lambda_3$ |
| $\lambda_3$ | $\lambda_3$ | $\lambda_3$ | $\lambda_3$ | $\lambda_3$ |

and its resonant partition is

$S_0 = \{\lambda_0, \lambda_2, \lambda_3\}$

$S_1 = \{\lambda_1, \lambda_3\}$

The $0_3$-reduction of the resonant subalgebra is given by

$G_{S,E}^{red} = \{\lambda_0X_2, \lambda_2X_2, \lambda_1X_1\}$

with commutation relations

$[\lambda_0X_2, \lambda_2X_2] = 0$

$[\lambda_0X_2, \lambda_1X_1] = -\lambda_1X_1$

$[\lambda_2X_2, \lambda_1X_1] = 0$.

Renaming the generators as

$Y_1 = \lambda_1X_1$

$Y_2 = \lambda_2X_2$

$Y_3 = \lambda_3X_2$

we obtain again the type III algebra

$[Y_1, Y_2] = [Y_2, Y_3] = 0$, $[Y_1, Y_3] = Y_1$.

4.2. The type II and V algebras

The natural question here is: is it possible to generate other type of Bianchi algebras from the isometries in two dimensions? To answer this question we continue the procedure of section 4.1, considering a semigroup $S = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where $\lambda_4$ is a zero element, but modify resonant decomposition (10). The decomposition can be chosen in another way, as can be seen in the following.
Lemma 4. A resonant decomposition
\[ S_0 = \{\lambda_2, \lambda_4\} \]
\[ S_1 = \{\lambda_1, \lambda_3, \lambda_4\} \]  
(14)
satisfying resonant condition (6) can produce, after $0_S$-reduction of the resonant subalgebra of the $S$-expanded algebra of two-dimensional isometry (9), a three-dimensional algebra.

Proof. The $0_S$-reduction of the resonant subalgebra is given by
\[ G_{S,R}^{\text{red}} = \{\lambda_2X_2, \lambda_1X_1, \lambda_3X_1\} \]
and the commutation relations are given by
\[
\begin{align*}
[\lambda_2X_2, \lambda_1X_1] &= -\lambda_1\lambda_2X_1 \\
[\lambda_2X_2, \lambda_3X_1] &= -\lambda_2\lambda_3X_1 \\
[\lambda_1X_1, \lambda_3X_1] &= 0.
\end{align*}
\]  
(15)

Resonant condition (6) guarantees that (15) is a closed algebra. □

Here we have different possibilities in order to make this algebra closed.

4.2.1. Type II algebra and the $S_{N^2}$ semigroup.

Theorem 5. An Abelian semigroup of four elements $S = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ satisfying the conditions of lemma 5 and possessing the following constraints in the multiplication table

|     | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|-----|-------------|-------------|-------------|-------------|
| $\lambda_1$ | $\lambda_3$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ |
| $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ |
| $\lambda_3$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ |
| $\lambda_4$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ | $\lambda_4$ |

produces, after $0_S$-reduction of the resonant subalgebra of the $S$-expanded algebra of the two-dimensional isometry (9), an algebra which coincides with three-dimensional Bianchi type II isometry.

Proof. To reproduce the type II algebra we must choose, for example,
\[ \lambda_1\lambda_2 = \lambda_3 \text{ and } \lambda_2\lambda_3 = \lambda_4. \]  
(17)

In this case the commutation relations (15) take the form
\[
\begin{align*}
[\lambda_2X_2, \lambda_1X_1] &= -\lambda_3X_1 \\
[\lambda_2X_2, \lambda_3X_1] &= 0 \\
[\lambda_1X_1, \lambda_3X_1] &= 0
\end{align*}
\]
and renaming the generators as
\[ Y_1 = \lambda_3X_1 \]
\[ Y_2 = \lambda_1X_1 \]
\[ Y_3 = \lambda_2X_2 \]
we obtain the type II algebra
\[ [Y_1, Y_2] = [Y_1, Y_3] = 0, \quad [Y_2, Y_3] = Y_1. \]
In order for this result to be true, however, we must provide an explicit semigroup that satisfies the conditions (17). Until now our table has had the form

|   | \(\lambda_1\) | \(\lambda_2\) | \(\lambda_3\) | \(\lambda_4\) |
|---|---|---|---|---|
| \(\lambda_1\) | \(\lambda_1\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_2\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) |

and the empty spaces must be filled in such a way that this table defines an associative, commutative product such that the decomposition (14) satisfies the resonant condition (6). □

**Proposition 6.** There are semigroups that fit multiplication table 16 and resonant condition (14).

After looking for different possibilities we have found a way of filling table (16). The proposed semigroup is

|   | \(\lambda_1\) | \(\lambda_2\) | \(\lambda_3\) | \(\lambda_4\) |
|---|---|---|---|---|
| \(\lambda_1\) | \(\lambda_1\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_2\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) | \(\lambda_4\) |

We can check that this multiplication table in fact represents an Abelian semigroup. That it is also commutative is seen from the table. The associativity is proved by a tedious but direct calculation.

Note that there may be other semigroups that can also lead to the type II algebra. These correspond to other ways of filling the empty spaces in table (16).

### 4.2.2. Type V and the \(S_{33}\) semigroup.

**Theorem 7.** An Abelian semigroup of four elements \(S = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\) satisfying the conditions of lemma 5 and possessing the following constraints in the multiplication table

|   | \(\lambda_1\) | \(\lambda_2\) | \(\lambda_3\) | \(\lambda_4\) |
|---|---|---|---|---|
| \(\lambda_1\) | \(\lambda_1\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_2\) | \(\lambda_1\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_3\) | \(\lambda_1\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) |
| \(\lambda_4\) | \(\lambda_1\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_4\) |

produces, after \(0_3\)-reduction of the resonant subalgebra of the \(S\)-expanded algebra of the two-dimensional isometry (9), an algebra which coincides with three-dimensional Bianchi type V isometry.

**Proof.** In fact, if we choose in the commutation relations (15), for example,

\[
\lambda_1 \lambda_2 = \lambda_1 \quad \text{and} \quad \lambda_2 \lambda_3 = \lambda_3
\]

in that case the commutation relations (15) take the form

\[
\begin{align*}
[\lambda_2 X_2, \lambda_1 X_1] &= -\lambda_1 X_1 \\
[\lambda_2 X_2, \lambda_3 X_1] &= -\lambda_3 X_1 \\
[\lambda_1 X_1, \lambda_3 X_1] &= 0
\end{align*}
\]

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and by renaming the generators as
\[ Y_1 = \lambda_1 X_1 \]
\[ Y_2 = \lambda_3 X_1 \]
\[ Y_3 = \lambda_2 X_2 \]
we obtain the type V algebra
\[ [Y_1, Y_2] = 0, \quad [Y_1, Y_3] = Y_1, \quad [Y_2, Y_3] = Y_2. \]

But again, in order for this result to be true we must provide an explicit semigroup that satisfies the conditions (20). Until now our table has had the form

\[
\begin{array}{cccc}
\lambda_1 & \lambda_1 & \lambda_3 & \lambda_4 \\
\lambda_2 & \lambda_1 & \lambda_3 & \lambda_4 \\
\lambda_3 & \lambda_3 & \lambda_4 & \lambda_4 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

and the empty spaces must be filled in a way that respects the required conditions. Note that there are \(4^4 = 256\) possibilities to fill this table in a closed form. This number is reduced by imposing associativity, commutativity and the resonant condition for the decomposition (14). This number can even be reduced by the associativity condition. □

**Proposition 8.** There are semigroups that fit multiplication table 19 and resonant condition (14).

After studying different possibilities we found a way of filling table (19). The proposed semigroup is

\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & \lambda_1 & \lambda_4 & \lambda_4 \\
\lambda_2 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

Again, we can check that by direct calculation that this multiplication represents an Abelian semigroup.

We point out again that there may be other semigroups that can lead to the type V algebra. Those correspond to other ways of filling the empty spaces in table (19).

### 4.3. The type I algebra

Starting from the Abelian two-dimensional algebra
\[ [X_1, X_2] = 0 \quad (21) \]
we note that it also possesses the subspace structure (5) where \(V_0 = \{X_2\}\) and \(V_1 = \{X_1\}\). So, for example, by choosing the semigroup \(S_K^{(3)}\) or \(S_{N_1}\), both having a resonant decomposition of the form
\[
S_0 = \{\lambda_2, \lambda_3, \lambda_4\} \\
S_1 = \{\lambda_1, \lambda_4\}
\]
we observe that the reduction of the resonant subalgebra
\[ G_{0,K}^{\text{red}} = \{\lambda_2 X_2, \lambda_3 X_2, \lambda_1 X_1\} \quad (23) \]
will have the following commutation relations:

\[
\begin{align*}
\{\lambda_2 X_2, \lambda_3 X_3\} &= \lambda_2 \lambda_3 \{X_2, X_3\} = 0 \\
\{\lambda_2 X_2, \lambda_1 X_1\} &= \lambda_2 \lambda_1 \{X_2, X_1\} = 0 \\
\{\lambda_3 X_2, \lambda_1 X_1\} &= \lambda_3 \lambda_1 \{X_2, X_1\} = 0
\end{align*}
\] (24)

which means that it does not matter if we use the semigroup \(S^{(3)}_K\) or \(S_{N1}\), as the result will be always an Abelian algebra in three dimensions because the original algebra is Abelian. The same result can be reached with the semigroup \(S^{(2)}_E\) whose decomposition is similar to (22).

Also, by using the semigroups \(S_{N2}, S_{N3}\) and probably others that have a resonant decomposition of the form

\[
\begin{align*}
S_0 &= \{\lambda_2, \lambda_4\} \\
S_1 &= \{\lambda_1, \lambda_3, \lambda_4\}
\end{align*}
\] (25)

we obtain a reduction of the resonant subalgebra

\[G_{0,K}^{\text{red}} = \{\lambda_2 X_2, \lambda_1 X_1, \lambda_3 X_1\}\]

whose commutation relations

\[
\begin{align*}
\{\lambda_2 X_2, \lambda_1 X_1\} &= \lambda_1 \lambda_2 \{X_2, X_1\} = 0 \\
\{\lambda_2 X_2, \lambda_3 X_1\} &= \lambda_2 \lambda_3 \{X_2, X_1\} = 0 \\
\{\lambda_1 X_1, \lambda_3 X_1\} &= \lambda_1 \lambda_3 \{X_1, X_1\} = 0
\end{align*}
\]

are again the three-dimensional Abelian algebra.

Thus, we conclude that starting from (21) an expansion with any given semigroup with a zero element and having a resonant decomposition of the form (22) or (25) will lead to the type I algebra. Moreover, this result can be generalized by the following.

**Proposition 9.** An Abelian algebra in \(d\) dimensions can be obtained as an expansion of the Abelian algebra in two-dimensions by using a semigroup with a zero element and a suitable resonant decomposition.

Note that a crucial property relating a three-dimensional algebra (of whichever type, I, II, III or V) with a two-dimensional algebra is the existence of the resonant subalgebra and the \(0_x\)-reduction. This is the only way of obtaining three generators starting from two generators.

### 4.4. Brief summary

The semigroups with which it is possible to generate the type I, II, III and V algebras starting from the two-dimensional algebra (9) appear in the following table:

| Algebra | Semigroup used |
|---------|----------------|
| Type I  | Any given semigroup with 0-element and a resonant decomposition |
| Type II | \(S_{N2}\) and probably others |
| Type III | \(S_{E}, S^{(3)}_K, S_{N1}\) and probably others |
| Type V  | \(S_{N3}\) and probably others |

(26)
where the mentioned semigroups are described in the following table

Table 4. Properties of semigroups that relate two-dimensional Lie algebras to 3-dimensional Lie algebras.

| Semigroup | Table of multiplication | Resonant decomposition | \(0_2\)-element |
|-----------|-------------------------|-----------------------|----------------|
| \(S_{E}^{(2)}\) | \(\lambda_0 \lambda_1 \lambda_2 \lambda_3\) | \(S_0 = \{\lambda_0, \lambda_2, \lambda_3\}\) | \(\lambda_3\) |
| \(S_{X}^{(3)}\) | \(\lambda_1 \lambda_2 \lambda_4 \lambda_4 \lambda_4 \lambda_4\) | \(S_0 = \{\lambda_2, \lambda_3, \lambda_4\}\) | \(\lambda_4\) |
| \(S_{N1}\) | \(\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_4 \lambda_4\) | \(S_0 = \{\lambda_2, \lambda_3, \lambda_4\}\) | \(\lambda_4\) |
| \(S_{N2}\) | \(\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_4 \lambda_4\) | \(S_0 = \{\lambda_2, \lambda_4\}\) | \(\lambda_4\) |
| \(S_{N3}\) | \(\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_4 \lambda_4\) | \(S_0 = \{\lambda_2, \lambda_4\}\) | \(\lambda_4\) |

5. The Bianchi spaces non-related with two-dimensional isometries

5.1. Type IV, VI, VII, VIII and IX algebras

Let us consider for example the type IV algebra

\[
[Y_1, Y_2] = 0, \quad \text{(27)}
\]

\[
[Y_1, Y_3] = Y_1, \quad \text{(28)}
\]

\[
[Y_2, Y_3] = Y_1 + Y_2. \quad \text{(29)}
\]

Since the \(S\)-expansion method uses an induced bracket

\[
[\lambda_{\alpha}X_i, \lambda_{\beta}X_j] = \lambda_{\alpha}\lambda_{\beta}[X_i, X_j] = \lambda_{\gamma(\alpha, \beta)}[X_i, X_j]
\]

for the expanded algebra and considering that for our original algebra \(i, j = 1, 2\) and its commutation relation is given by

\[
[X_1, X_2] = X_1,
\]

we have the fact that the first two relations (27), (28) can easily be reproduced with some semigroup product, but to reproduce (29) we need a non-zero result as the first requirement. This means that we must have a relation like

\[
[\lambda_{\alpha}X_1, \lambda_{\beta}X_2] = \lambda_{\alpha}\lambda_{\beta}[X_1, X_2] = \lambda_{\gamma(\alpha, \beta)}X_1.
\]
Here we can see that no matter which semigroup we choose, $\lambda_{y(x,\beta)}$ will always be an element of the semigroup (it is closed) and therefore we will never be able to reproduce a sum of two generators.

In this proof we need to consider the possibility of basis change for the generators on which the algebra is spanned. A change of the basis can be done with group $GL(n)$, where $n$ is dimension of the algebra. First of all, let us consider the change of basis in the three-dimensional $S$-expanded algebras. Starting from the type IV algebra we cannot obtain (by means of a basis change) the commutation relations in a form in which on the right-hand side (rhs) there is at most one term only, i.e., a monomial that is a product of a generator and a number (as happens for the type I, II, III and V algebras). Actually, Bianchi has classified the isometric spaces by the form of the rhs of the commutation relations, that is, by the number of monomials on the rhs [18]. Then, the following statement is also true: if whatever expansion of the original algebra (9) is such that their corresponding commutation relations have on the rhs at most one term only, then these expanded algebras are not isomorphic to the type IV algebra. To complete the proof we must now consider a change of basis in the initial two-dimensional algebra. Consider, for example, the change of basis $X_1' = X_1 - X_2$, and $X_2' = X_1 + X_2$ which transforms the commutation relation (9) to the equivalent relation $[X_1', X_2'] = X_1' + X_2'$. One might think that in this basis the above argument does not apply because now it is possible to obtain commutation relations, for the expanded algebra, with more than one element on their rhs. However, if one constructs all the three-dimensional expanded algebras starting with these generators none of them will give the type IV algebra. In fact, the three-dimensional algebra can be constructed with a semigroup of four elements with a zero element, $\lambda_4$, and one of the following two possible resonances:

(i) $S_0 = \{\lambda_2, \lambda_3, \lambda_4\}$, $S_1 = \{\lambda_1, \lambda_4\}$

(ii) $S_0 = \{\lambda_2, \lambda_4\}$, $S_1 = \{\lambda_1, \lambda_3, \lambda_4\}$,

leading respectively to the following reduction of the resonant subalgebra:

(i') $G_{S,R,\text{red}} = \{\lambda_2 X_2', \lambda_3 X_2', \lambda_4 X_1'\}$

(ii') $G_{S,R,\text{red}} = \{\lambda_2 X_2', \lambda_4 X_1', \lambda_3 X_1'\}$.

Note that, apart from the resonant condition, there is still some freedom to fix the semigroup multiplication. However, we can check directly that, in both the cases, there is no semigroup able to reproduce (29) from $[\lambda_\alpha X_1', \lambda_\beta X_2'] = \lambda_{y(x,\beta)} (X_1' + X_2')$, because it is not possible to close the algebra by demanding to have a sum of expanded generators on the rhs. This is, in fact, a consequence of a general feature of the $S$-expansion procedure: the $S$-expanded algebras of any two isomorphic algebras are isomorphic, if the semigroup $S$ and the resonance used are the same. Therefore, if the initial algebra (9) cannot be $S$-expanded to the type IV Bianchi algebra, none of the two-dimensional algebras isomorphic to (9) can be $S$-expanded to the type IV Bianchi algebra.

Now consider the type VI algebra

$[Y_1, Y_2] = 0,$

$[Y_1, Y_3] = Y_1,$

$[Y_2, Y_3] = hY_2, \quad h \neq 0, 1.$

Again the first two brackets could be reproduced by a certain semigroup, but for the third one we would have something like

$[\lambda_\alpha X_1, \lambda_\beta X_2] = \lambda_{y(x,\beta)} X_1$

and again, no matter which semigroup we choose, $\lambda_{y(x,\beta)}$ will always be an element of the semigroup and we will never be able to reproduce a semigroup element multiplied by a
numeral factor. As in the previous case of the type IV Bianchi algebra, a basis change can be considered. First, let us look at the three-dimensional $S$-expanded algebras of (9). The type VI Bianchi algebra cannot be transformed by any of the basis changes of the algebra to the case where on the rhs of the commutation relations we have monomials with coefficient 1. Thus, we cannot obtain the type VI Bianchi algebra from any of the algebras $S$-expanded from (9). Consider now the case of two-dimensional algebras isomorphic to (9). For example, if we define $X'_i = X_i$, and $X'_2 = hX_2$, the commutation relation (9) transforms to the equivalent relation $[X'_i, X'_j] = hX'_i$. Starting with this basis one might also think that in this basis the above argument does not apply. However, again it is straightforward to check that there exists no semigroup which leads to the type VI algebra starting with this new basis and the result holds in general for any other basis.

Note also that

$$[X_1, X_2] = X_1; \quad [X_1, X_2] = X_1 + X_2; \quad [X_1, X_2] = hX_1$$

define the same two-dimensional algebra, just in a different basis. On the other hand, the following algebras

$$[Y_1, Y_2] = 0; \quad [Y_1, Y_3] = Y_1; \quad [Y_2, Y_3] = Y_2 + Y_3$$

$$[Y_1, Y_2] = 0; \quad [Y_1, Y_3] = Y_1; \quad [Y_2, Y_3] = Y_2$$

$$[Y_1, Y_2] = 0; \quad [Y_1, Y_3] = Y_1; \quad [Y_2, Y_3] = hY_2$$

which correspond to type IV, V and VI algebras, respectively, are completely different algebras. In other words, there exists no change of basis connecting them. This is the key point of the Bianchi classification [18]. In fact, all the possible three-dimensional real algebras (9) may be divided in the sets in such a way that any algebra of the given set is isomorphic (in the sense of the basis change with group $GL(3)$) to one of the algebras in the Bianchi classification. The $S$-expanded algebras of all the algebras isomorphic to (9) may be distributed in these sets of three-dimensional algebras as well. So, if all the $S$-expansions of the algebras $\{X_1, X_2\}$, with commutation relations (9), do not belong to the set to which all the three-dimensional algebras isomorphic to the type IV or VI Bianchi algebra belong, all the $S$-expanded algebras of a two-dimensional algebra $\{X'_i, X'_j\}$ isomorphic to (9) do not belong to these sets either.

A similar argument can be used to show that type VII and type VII algebras cannot be obtained by the $S$-expansion procedure from the original algebra (9).

The algebras VIII and IX correspond to simple algebras $sl(2, R)$ and $so(3)$, respectively. It is known that they are rigid and do not admit any deformation [34], therefore they cannot be obtained as a result of WW contraction. As we are going to show now, it is not possible to obtain type VIII and IX algebras from the two-dimensional algebra (9) by means of an $S$-expansion. The following arguments are two independent proofs for the last statement.

(a) Consider, for example, the type IX Bianchi algebra,

$$[V_1, V_2] = V_3; \quad [V_2, V_3] = V_1; \quad [V_3, V_1] = V_2.$$  

A candidate for being the expanded algebra will have three commutation relations of the form

$$\lambda_{\alpha} X_{i} \lambda_{\beta} X_{j} = \lambda_{\gamma(\alpha, \beta)} [X_{i}, X_{j}]$$

where $\alpha, \beta$ takes the values 1 and 2. Therefore, in one of the three commutation relations one index will always be repeated leading to a vanishing bracket. Thus, it is impossible to generate, by means of an $S$-expansion, a three-dimensional algebra with the three brackets having a non-zero value. The same argument applies for the type VIII algebra.
(b) The other argument for the impossibility of obtaining type VIII and IX Bianchi algebras from the original algebra of two-dimensional isometry (9) is that the $S$-expansion conserves the property of solvability of the algebra [33, 35]. This statement forbids us from obtaining any simple algebra from a solvable algebra [33].

Thus, we conclude that these types of algebra cannot be obtained by an expansion of the two-dimensional isometries and are in some sense intrinsic in three dimensions.

6. Checking with computer programs

A common question when working with semigroups in section 4 is that of the existence of diverse semigroups given a multiplication table with some elements already chosen. We have, for example, a table like this one

$$
\begin{array}{c|cccc}
S & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\lambda_1 & & & & \\
\lambda_2 & & & & \\
\lambda_3 & & & & \\
\lambda_4 & & & & \\
\end{array}
$$

In principle there are 256 different symmetric matrices which fill this template, but not all of them will be semigroups because the multiplication table will not always be associative. Moreover, we have to select only those that satisfy a certain resonant condition. Finally, many of these associative tables will be isomorphic, so we only have to select those that are not.

In what follows we find all the non-isomorphic forms to fill tables (11), (16) and (19) with the mentioned conditions and show that all the semigroups given in table (26) (those semigroups that we have constructed by hand) are isomorphic to one of the semigroups given by the computer program $\text{com.f}$ of [30].

6.1. Type II

The template is:

$$
\begin{array}{c|cccc}
S & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\lambda_1 & & & & \\
\lambda_2 & & & & \\
\lambda_3 & & & & \\
\lambda_4 & & & & \\
\end{array}
$$

By using computer programs, we have found that there are two non-isomorphic ways of filling this template such that: (a) the resulting table is an Abelian semigroup and (b) the resonant decomposition is given by

$$
S_0 = \{\lambda_2, \lambda_4\}, \quad S_1 = \{\lambda_1, \lambda_3, \lambda_4\}.
$$

Those ways are:

$$
\begin{array}{c|cccc}
S^1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\lambda_1 & \lambda_4 & \lambda_3 & \lambda_4 & \\
\lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 & \\
\lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 & \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \\
\end{array}
\quad \quad
\begin{array}{c|cccc}
S^2 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \\
\lambda_2 & \lambda_3 & \lambda_4 & \lambda_4 & \\
\lambda_3 & \lambda_4 & \lambda_4 & \lambda_4 & \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \\
\end{array}
$$

(31)
Each of them is isomorphic to one of the semigroups of the list given by the program \textit{com.f} of [30] for \( n = 4 \). We give this information in the following table:

| Isomorphic to \( S_J^{10} \) | Isomorphism
|-----------------------------|-----------------------------
| \( S_J^{10} \) \( \iff \) \( S_J^{12} \) | (\( \lambda_4 \lambda_3 \lambda_1 \lambda_2 \))
| \( S_J^{12} \) \( \iff \) \( S_J^{14} \) | (\( \lambda_4 \lambda_3 \lambda_2 \lambda_1 \))

where the isomorphism denoted by \((\lambda_a, \lambda_b, \lambda_c, \lambda_d)\) means: change \( \lambda_1 \) by \( \lambda_a \), \( \lambda_2 \) by \( \lambda_b \), \( \lambda_3 \) by \( \lambda_c \) and \( \lambda_4 \) by \( \lambda_d \). The semigroups \( S_J^{10} \) and \( S_J^{12} \) of the list given by the program \textit{com.f} for \( n = 4 \) are:

\[
\begin{array}{cccccc}
S_J^{10} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_3 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_4 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\end{array}
\quad \quad
\begin{array}{cccccc}
S_J^{12} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_3 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_4 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\end{array}
\]

We can check directly that applying the isomorphism \((\lambda_4 \lambda_3 \lambda_1 \lambda_2)\) to \( S_J^{10} \) obtains \( S_J^{12} \) and applying the isomorphism \((\lambda_4 \lambda_3 \lambda_2 \lambda_1)\) to \( S_J^{12} \) obtains \( S_J^{10} \).

### 6.2. Type III

The template is:

\[
\begin{array}{cccc}
S^3 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & \lambda_4 & \lambda_1 & \lambda_4 \\
\lambda_2 & \lambda_4 & \lambda_2 & \lambda_4 \\
\lambda_3 & \lambda_4 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

We have found that there are seven non-isomorphic ways of filling this template such that: (a) the resulting table is an Abelian semigroup and (b) the resonant decomposition is given by

\[
S_0 = \{\lambda_2, \lambda_3, \lambda_4\}, \quad S_1 = \{\lambda_1, \lambda_4\}.
\]

Those ways are:

| \( S_{III}^{1} \) | \( S_{III}^{2} \) | \( S_{III}^{3} \) |
|----------------|----------------|----------------|
| \( \lambda_1 \) | \( \lambda_1 \) | \( \lambda_1 \) |
| \( \lambda_2 \) | \( \lambda_2 \) | \( \lambda_2 \) |
| \( \lambda_3 \) | \( \lambda_3 \) | \( \lambda_3 \) |
| \( \lambda_4 \) | \( \lambda_4 \) | \( \lambda_4 \) |

We can check directly that applying the isomorphism \((\lambda_4 \lambda_3 \lambda_1 \lambda_2)\) to \( S_{III}^{1} \) obtains \( S_{III}^{2} \) and applying the isomorphism \((\lambda_4 \lambda_3 \lambda_2 \lambda_1)\) to \( S_{III}^{2} \) obtains \( S_{III}^{3} \).
As before, each of these forms is isomorphic to one of the semigroups of the list given by the program \textit{com.f} of \cite{30} for \( n = 4 \). Those semigroups and the corresponding isomorphisms are given in the following table:

| Isomorphic to | Isomorphism |
|--------------|-------------|
| \( S_{1} \) | \( S_{13}^{(4)} \) |
| \( S_{2} \) | \( S_{28}^{(4)} \) |
| \( S_{3} \) | \( S_{42}^{(4)} \) |
| \( S_{4} \) | \( S_{43}^{(4)} \) |
| \( S_{5} \) | \( S_{44}^{(4)} \) |
| \( S_{6} \) | \( S_{45}^{(4)} \) |

where the semigroups \( S_{13}^{(4)}, S_{28}^{(4)}, S_{42}^{(4)}, S_{43}^{(4)}, S_{44}^{(4)}, S_{45}^{(4)} \), of the list generated by the program \textit{com.f} for \( n = 4 \), are explicitly given in the appendix.

6.3. Type V

The template is:

\[
\begin{array}{cccc}
S_{V} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & \lambda_1 & \lambda_3 & \lambda_4 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

In this case we have found that there is just one way of filling this template such that: (a) the resulting table is an Abelian semigroup and (b) the resonant decomposition is given by

\[
S_0 = \{\lambda_2, \lambda_4\}, \quad S_1 = \{\lambda_1, \lambda_3, \lambda_4\}.
\]

This is:

\[
\begin{array}{cccc}
S_{V} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_1 & \lambda_4 & \lambda_1 & \lambda_4 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_3 & \lambda_4 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

This table is isomorphic to the semigroup \( S_{42}^{(4)} \) given in the appendix. The isomorphism is given by

\[
(\lambda_4, \lambda_1, \lambda_3, \lambda_2).
\]

Note that the semigroup \( S_{42}^{(4)} \) also permits us to obtain the type III algebra. So we can ask, \textit{how can the same semigroup lead to type III and V algebras at the same time?} The reason is that this semigroup has two different resonant decompositions, \textit{30} and \textit{34}. Each of them permits us to extract different kinds of resonant subalgebras leading, after the reduction, to completely different algebras.

6.4. Isomorphisms and consistency of the procedure

In the following table we summarize our results by specifying \textit{all} the non-isomorphic semigroups that permit us to generate the type I, II, III and V algebras, starting from the two-dimensional algebra \textit{(9)}:  

\[
\begin{array}{cccc}
\text{Isomorphic to} & \text{Isomorphism} \\
S_{1} & S_{13}^{(4)} (\lambda_4, \lambda_2, \lambda_1, \lambda_3) \\
S_{2} & S_{28}^{(4)} (\lambda_4, \lambda_2, \lambda_3, \lambda_1) \\
S_{3} & S_{42}^{(4)} (\lambda_4, \lambda_1, \lambda_2, \lambda_3) \\
S_{4} & S_{43}^{(4)} (\lambda_4, \lambda_2, \lambda_4, \lambda_3) \\
S_{5} & S_{44}^{(4)} (\lambda_4, \lambda_1, \lambda_2, \lambda_3) \\
S_{6} & S_{45}^{(4)} (\lambda_4, \lambda_1, \lambda_3, \lambda_2) \\
\end{array}
\]
Table 5. Summary of permitted semigroups.

| Algebra | Semigroup used |
|---------|----------------|
| Type I  | Many semigroups (see section 4.3) |
| Type II | $S_{10}^4$, $S_{12}^4$ |
| Type III| $S_{13}^4$, $S_{28}^4$, $S_{42}^4$, $S_{43}^4$, $S_{44}^4$, $S_{45}^4$ and $S_{64}^4$ |
| Type V  | $S_{42}^4$ |

For consistency we should prove that each semigroup of the table (26) (which we have constructed by hand in section 4) is isomorphic to one of the semigroups of table 5 that we have found by using computer programs. This information is given in the following table:

| Isomorphic to | Isomorphism |
|---------------|-------------|
| $S_{N1}$     | $(\lambda_2 \lambda_1 \lambda_2 \lambda_3)$ |
| $S_{N2}$     | $(\lambda_2 \lambda_3 \lambda_2 \lambda_1)$ |
| $S_{N3}$     | $(\lambda_4 \lambda_1 \lambda_3 \lambda_2)$ |
| $S_{K}^{(2)}$| $(\lambda_4 \lambda_3 \lambda_2 \lambda_1)$ |
| $S_{K}^{(3)}$| $(\lambda_4 \lambda_2 \lambda_1 \lambda_3)$ |

7. Comments

In this work we present a complete study of the possibility of relations, by means of an expansion between the isometry algebras that act transitively in two and three dimensions. It was found that some isometries in three dimensions, specifically those of type I, II, III and V (according to Bianchi’s classification), can be obtained as expansions of the isometries in two dimensions. In general, there is more than one possibility of obtaining these results, i.e., it may happen that different semigroups will lead to the same expanded algebra. Also, it is shown that the other Bianchi type IV, VI-IX algebras cannot be obtained as an expansion from the isometry algebras in two dimensions. This means that the first isometry algebras have properties that can be obtained from isometries in two dimensions but the second set has properties that are in some sense intrinsic in three dimensions.

The results obtained in this work are interesting, because even when two- and three-dimensional isometry algebras are well known in the literature, the non-trivial relations we have found are something completely new. It could be thought of as a simple problem to perform extensions of Lie algebras by adding generators and where the original algebra is a subalgebra of the resulting algebras. This is, for example, what happens with the algebras of type III and V (see table (26) and (3)) where there are simply two ways of extending the algebra (9) to a three-dimensional one (it is easy to see that the algebra (9) is contained as a subalgebra in the algebras of types III and V of table (3)). In the present work we have obtained these results by means of the expansion procedure but, furthermore, new kinds of relations have been found. This is the case for the type II algebra (see table (26)) that was obtained as an expansion of the algebra (9) and where this original algebra is not present as a subalgebra. This is why we refer to these relations between two and three dimensions as non-trivial relations. These results are completely new and can be reached by means of the $S$-expansion method only. The reason is that the expansion method (which uses a parameter) is equivalent to an

13 This in some sense tells us that the $S$-expansion method includes not only the contraction methods: it contains also some of the extension procedures of a Lie algebra.
$S$-expansion, but using just one specific semigroup, the semigroup $S_E^{(n)}$ introduced in [9]. Therefore, the use of another semigroup will lead us to more general expansions, and this is why effectively this result can only be obtained via the $S$-expansion procedure.

It is also interesting that to address this problem we had to look for other semigroups that have not yet been used in the applications of the $S$-expansion method (the semigroups $S_{N1}, S_{N2}, S_{N3}$ and $S_E^{(3)}$). Their principal properties, such as the problem of finding a resonant decomposition, were studied for each of them. By using computing programs we have checked our results and solved the problem in a complete way.

As mentioned in the introduction, different applications in the construction of gauge theories of gravity have been found in this context and using the methodology and computer algorithms developed in the present paper it would be possible to uncover and classify all the relations of this type. On the other hand, a generalization of the results presented here can be useful for studying isometries in higher dimensions, particularly in applications related to isometries of black hole solutions. A first step in this direction is done in [32]. According to our present experience, the black hole solutions can be related via semigroups different from $S_E^{(n)}$ [32].

We conclude by remarking that what remains in common in all the physical applications mentioned above is the question: given two symmetry algebras, can they be related by means of some contraction or expansion procedure? The method presented in this paper is very instructive to answer this question. If the answer is yes, there is a way of constructing the semigroup that gives the relation (as made in section 4). On the contrary, if the answer is no then it should be shown explicitly that no expansion method exists that can reproduce this relation (as made in section 5). This mechanism can be developed to a general algorithm to study more complicated cases, where the construction of the semigroups by hand (as made here) would be impossible. General criteria and the mentioned algorithm is a work in progress (see [33]).

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Appendix

We give explicitly the multiplication tables of the semigroups that we have used in this paper and which belong to the list generated by the program com.f of [30] for $n = 4$. These semigroups are:

\[
\begin{array}{c|cccc}
S_E^{(10)} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
S_E^{(12)} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
S_E^{(13)} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\end{array}
\]
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