K-ORBIT CLOSURES ON G/B AS UNIVERSAL DEGENERACY LOCI FOR FLAGGED VECTOR BUNDLES SPLITTING AS DIRECT SUMS

BENJAMIN J. WYSER

Abstract. We use equivariant localization and divided difference operators to determine formulas for the torus-equivariant fundamental cohomology classes of K-orbit closures on the flag variety G/B for various symmetric pairs (G, K). In type A, we realize the closures of K = GL(p, C) × GL(q, C)-orbits on GL(p + q, C)/B as universal degeneracy loci for a vector bundle over a variety which is equipped with a single flag of subbundles and which splits as a direct sum of subbundles of ranks p and q. The precise description of such a degeneracy locus relies upon knowing a set-theoretic description of K-orbit closures, which we provide via a detailed combinatorial analysis of the poset of “(p, q)-clans,” which parametrize the orbit closures. We describe precisely how our formulas for the equivariant classes of K-orbit closures can be interpreted as formulas for the classes of such degeneracy loci in the Chern classes of the bundles involved. In the cases outside of type A, we suggest that the orbit closures should parametrize degeneracy loci involving a vector bundle equipped with a non-degenerate symmetric or skew-symmetric bilinear form, a single flag of subbundles which are isotropic or Lagrangian with respect to the form, and a splitting as a direct sum of subbundles with each summand satisfying some property (depending on K) with respect to the form. The precise description of such a degeneracy locus is conjectured for all cases in types B and C.

Suppose that G is a complex reductive group of classical type, and that K = G^θ is the subgroup fixed by an involution θ of G. K is referred to as a symmetric subgroup. K acts on the flag variety G/B with finitely many orbits [Mat79], and the geometry of these orbits and their closures plays an important role in the theory of Harish-Chandra modules for a certain real form of the group G. For this reason, the geometry of K-orbits and their closures have been studied extensively, primarily in representation-theoretic contexts.

In [Wys12a], the K-orbit closures for the symmetric pairs (G, K) = (GL(n, C), O(n, C)), (SL(n, C), SO(n, C)), and (SL(2n, C), Sp(2n, C)) were studied from the perspective of torus-equivariant geometry. In the current paper, we carry out a similar program of study for a number of other symmetric pairs (G, K) with G a classical reductive group. The pairs we consider are (in all cases, n = p + q):

1. (GL(n, C), GL(p, C) × GL(q, C));
2. (SO(2n + 1, C), S(O(2p, C) × O(2q + 1, C)));
3. (Sp(2n, C), Sp(2p, C) × Sp(2q, C));
4. (Sp(2n, C), GL(n, C));
5. (SO(2n, C), S(O(2p, C) × O(2q, C)));
6. (SO(2n, C), GL(n, C));
7. (SO(2n, C), S(O(2p + 1, C) × O(2q − 1, C))).

The study of [Wys12a], as well as that of this paper, was motivated by earlier work of W. Fulton [Ful92, Ful96b, Ful96a] which realized Schubert varieties as universal degeneracy...
loxi for maps of flagged vector bundles, and by connections between that work and the equivariant cohomology of the flag variety, elucidated by W. Graham in [Gra97]. $K$-orbit closures are, in a sense, generalizations of Schubert varieties, and so it is natural to try to fit these more general objects into a similar framework as that described in the aforementioned works.

To this end, in [Wys12a], the following program is carried out for the pairs $(GL(n, \mathbb{C}), O(n, \mathbb{C}))$, $(SL(n, \mathbb{C}), SO(n, \mathbb{C}))$ and $(SL(2n, \mathbb{C}), Sp(2n, \mathbb{C}))$:

1. Determine formulas for the $S$-equivariant cohomology classes of the closed $K$-orbits using equivariant localization, together with the self-intersection formula. (Here, $S$ is a maximal torus of $K$ contained in a $\theta$-stable maximal torus $T$ of $G$.)

2. Using such formulas as a starting point, describe the weak order on $K\backslash G/B$ combinatorially, and outline how divided difference calculations can give formulas for the equivariant classes of the remaining orbit closures.

3. Give a set-theoretic description of the $K$-orbit closures as sets of flags, and using this description, realize the $K$-orbit closures as universal degeneracy loci of a certain type, involving a vector bundle over a smooth complex variety $X$ equipped with a single flag of subbundles and a certain additional structure determined by $K$.

In the present paper, we carry out this program for the remaining symmetric pairs $(G, K)$ listed above. As far as step (3) above is concerned, in the cases considered in [Wys12a], the “additional structure” possessed by the vector bundle is a non-degenerate symmetric or skew-symmetric bilinear form taking values in the trivial bundle, and the associated degeneracy loci are defined by imposing rank conditions on the fibers of the various components of the flag of subbundles.

In the present paper, the relevant additional structure is a splitting of the vector bundle as a direct sum of two subbundles. Thus the degeneracy loci corresponding to the $K$-orbit closures in these cases can all be described roughly as follows: We are given a complex vector bundle $V$ over a smooth complex variety $X$, a single flag $F_\bullet$ of subbundles of $V$, and a splitting of $V$ as a direct sum $V' \oplus V''$ of subbundles. (In types $BCD$, the bundle $V$ is equipped with a symmetric or skew-symmetric bilinear form, the flag $F_\bullet$ is isotropic/Lagrangian with respect to the form, and the summands $V'$ and $V''$ are required to satisfy further properties with respect to the form.) The degeneracy loci are then defined by imposing conditions on the relative position of the fibers of the flag and the two summands. These conditions are encoded in “clans” (which are character strings consisting of $+$’s, $-$’s, and natural numbers subject to some further conditions) which parametrize the $K$-orbit closures.

The precise meaning of “relative position” and the way in which it is read off of a clan $\gamma$ is given in Section 2. It is determined by a linear algebraic description of the orbit closure $Y_\gamma$ corresponding to $\gamma$, mentioned in (3) above. While set-theoretic descriptions of the $K$-orbits themselves are known in the case $(G, K) = (GL(n, \mathbb{C}), GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$ ([Yam97]), no set-theoretic description of orbit closures for this particular symmetric pair has appeared in the literature. We give such a description as Theorem 2.5. Since the $K$-orbits in the remaining cases are closely related to those in the type $A$ case (in a sense made precise in Section 2), Theorem 2.5 suggests a naive guess at a set-theoretic description of the orbit closures in the type $BCD$ cases. Alas, this naive guess is incorrect in all cases in type $D$, but based upon experimental evidence, we conjecture that it is correct in types $BC$. (See Conjecture 2.18.)
The paper is organized as follows: In Section 1, we cover some preliminary facts on equivariant cohomology and localization, and state the general facts supporting the techniques which we apply to $K$-orbit closures. In Section 2, we discuss the parametrization of $K$-orbits in the various cases. We pay most attention to the case $(G, K) = (GL(p+q, \mathbb{C}), GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$, then describe how information on the orbits in the remaining cases can be deduced from this case. In Section 3, we give formulas for the classes of closed $K$-orbits in the various cases, using the localization techniques described in Section 1. Finally, in Section 4, we tie these formulas to Chern class formulas for degeneracy loci of the type loosely described above.

A number of the results presented herein were part of the author’s PhD thesis, written at the University of Georgia under the direction of his research advisor, William A. Graham. The author thanks Professor Graham wholeheartedly for his help in conceiving that project, as well as for his great generosity with his time and expertise throughout.

1. Preliminaries

We start by collecting the general facts that we will need to carry out our equivariant computations in individual examples. Many of the facts given in this section are stated without proof, as all of them are either standard or else covered in a fair amount of detail in [Wys12a].

1.1. Notation. Here we define some notation which will be used throughout the paper.

We denote by $I_n$ the $n \times n$ identity matrix, and by $J_n$ the $n \times n$ matrix with 1’s on the antidiagonal and 0’s elsewhere, i.e., the matrix $(e_{i,j}) = \delta_{i,n+1-j}$. If $n = p + q$, then $I_{p,q}$ will denote the $n \times n$ diagonal matrix having $p$ 1’s followed by $q$ -1’s on the diagonal. If $a + b + c = n$, then $I_{a,b,c}$ will denote the $n \times n$ diagonal matrix having $a$ 1’s, followed by $b$ -1’s, followed by $c$ 1’s, on the diagonal. $J_{n,n}$ shall denote the block matrix which has $J_n$ in the upper-right block, $-J_n$ in the lower-left block, and 0’s elsewhere. That is,

$$J_{n,n} := \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

We will use both “one-line” notation and cycle notation for permutations. When giving a permutation in one-line notation, the sequence of values will be listed with no delimiters, while for cycle notation, parentheses and commas will be used. Hopefully this will remove any possibility for confusion on the part of the reader. So, for example, the permutation $\pi \in S_4$ which sends 1 to 2, 2 to 3, 3 to 1, and 4 to 4 will be given in one-line notation as 2314 and in cycle notation as (1, 2, 3).

We will consider signed permutations of $\{1, \ldots, n\}$. These will be given in one-line notation, possibly with bars over some of the numbers to indicate negative values. For instance, $\overline{2}\overline{1}35$ denotes the signed permutation sending 1 to -2, 2 to -4, 3 to 1, 4 to 3, and 5 to -5. These signed permutations will sometimes be thought of as embedded in some larger symmetric group, either $S_{2n}$ or $S_{2n+1}$, as follows: The signed permutation $\pi$ of $\{1, \ldots, n\}$ is associated to the permutation $\sigma \in S_{2n}$ defined by

$$\sigma(i) = \begin{cases} \pi(i) & \text{if } \pi(i) > 0 \\ 2n + 1 - |\pi(i)| & \text{if } \pi(i) < 0, \end{cases}$$

with $\sigma$ in $S_{2n}$ being the disjoint union of $\pi$ in $S_n$ and $\overline{\pi}$ in $S_n$, where $\overline{\pi}$ is the signature of $\pi$. That is,

$$\sigma = \pi \cup \overline{\pi}.$$
and
\[ \sigma(2n + 1 - i) = 2n + 1 - \sigma(i) \]
for \( i = 1, \ldots, n \).

Embedding signed permutations in \( S_{2n+1} \) works the same way, with \( 2n \) replaced by \( 2n + 1 \) in the definitions above. Note that this forces \( \sigma(n + 1) = n + 1 \).

We will also deal often with flags, i.e. chains of subspaces of a given vector space \( V \). A flag
\[ \{0\} \subset F_1 \subset F_2 \subset \ldots \subset F_{n-1} \subset F_n = V \]
will often be denoted by \( F_* \). When we wish to specify the components \( F_i \) of a given flag \( F_* \) explicitly, we will typically use the shorthand notation
\[ F_* = \langle v_1, \ldots, v_n \rangle, \]
which shall mean that \( F_i \) is the linear span \( \mathbb{C} \cdot \langle v_1, \ldots, v_i \rangle \) for each \( i \).

We will always be dealing with characters of tori \( S \) (the maximal torus of \( K \)) and \( T \) (the maximal torus of \( G \)). Coordinate functions on \( s = \text{Lie}(S) \) will be denoted by capital \( Y \) variables, while those on \( t = \text{Lie}(T) \) will be denoted by capital \( X \) variables. Equivariant cohomology classes, on the other hand, will be represented by polynomials in lower-case \( x \) and \( y \) variables, where the lower-case variable \( x_i \) means \( 1 \otimes X_i \), and where the lower-case variable \( y_i \) means \( Y_i \otimes 1 \). (See Proposition 1.1.)

Unless stated otherwise, \( H^*(-) \) shall always mean cohomology with \( \mathbb{C} \)-coefficients.

Lastly, we note here once and for all that \( K \backslash G/B \) should always be taken to mean the set of \( K \)-orbits on \( G/B \), unless explicitly stated otherwise. (This as opposed to \( B \)-orbits on \( K \backslash G \), or \( B \times K \)-orbits on \( G \).)

1.2. Equivariant cohomology, the localization theorem, and classes of closed orbits. Our primary cohomology theory is equivariant cohomology with respect to the action of a maximal torus \( S \) of \( K \). The \( S \)-equivariant cohomology of an \( S \)-variety \( X \) is, by definition,
\[ H^*_S(X) := H^*((ES \times X)/S). \]
Here, \( ES \) denotes the total space of a universal principal \( S \)-bundle (a contractible space with a free \( S \)-action).

Note that \( H^*_S(X) \) is always an algebra for the ring \( \Lambda_S := H^*_S(\{\text{pt.}\}) \), the \( S \)-equivariant cohomology of a 1-point space (equipped with trivial \( S \)-action). The algebra structure is given by pullback through the constant map \( X \to \{\text{pt.}\} \).

Taking \( X \) to be the flag variety \( G/B \), we now describe \( H^*_S(X) \) explicitly. Let \( R = S(t^*) \), the \( \mathbb{C} \)-symmetric algebra on the dual to the Lie algebra \( t \) of a maximal torus \( T \) of \( G \). Let \( R' = S(s^*) \), the \( \mathbb{C} \)-symmetric algebra on the dual to the Lie algebra \( s \) of \( S \). It is a standard fact that \( R \cong \Lambda_T \), and \( R' \cong \Lambda_S \). Let \( n \) be the dimension of \( T \), and let \( r \) be the dimension of \( S \). Let \( X_1, \ldots, X_n \) denote coordinates on \( t^* \), taken as generators for the algebra \( R \). Likewise, let \( Y_1, \ldots, Y_r \) denote coordinates on \( s^* \), algebra generators for \( R' \).

Note that there is a map \( R \to R' \) induced by restriction of characters, whence \( R' \) is a module for \( R \). Note also that \( W \) acts on \( R \), since it acts naturally on the characters \( X_i \). Then it makes sense to form the tensor product \( R' \otimes_{RW} R \). As it turns out, this is the \( S \)-equivariant cohomology of \( X \).

**Proposition 1.1.** With notation as above, \( H^*_S(X) = R' \otimes_{RW} R \). Thus elements of \( H^*_S(X) \) are represented by polynomials in variables \( x_i := 1 \otimes X_i \) and \( y_i := Y_i \otimes 1 \).
A proof of the previous theorem is given in [Wys12a]. In the “equal rank” case, where
$S = T$, the statement of this theorem is the standard fact that $H^*_\mathbb{Z}(X) = R \otimes_{R^1} R$, for
which a proof can be found in [Bri98]. We remark here that in this paper, all but one of our
examples is an equal rank case, so we are usually in this more familiar situation.

Next, we have the standard localization theorem for actions of tori, which can also be found
in [Bri98]:

**Theorem 1.2.** Let $X$ be an $S$-variety, and let $i : X^S \hookrightarrow X$ be the inclusion of the $S$-fixed
locus of $X$. The pullback map of $\Lambda_S$-modules

$$i^* : H^*_S(X) \to H^*_S(X^S)$$

is an isomorphism after a localization which inverts finitely many characters of $S$. In particular,
if $H^*_S(X)$ is free over $\Lambda_S$, then $i^*$ is injective.

The last statement is what is relevant for us, since when $X$ is the flag variety, $H^*_S(X) = R' \otimes_{R^{\mathbb{Z}}} R$ is free over $R'$. Thus in the case of the flag variety, the localization theorem tells us
that any equivariant class is entirely determined by its image under $i^*$. We have the
following result on the $S$-fixed locus:

**Proposition 1.3 ([Bri99]).** If $K = G^\theta$ is a symmetric subgroup of $G$, $T$ is a $\theta$-stable maximal
torus of $G$, and $S$ is a maximal torus of $K$ contained in $T$, then $(G/B)^S = (G/B)^T$. Thus
$(G/B)^S$ is finite, and indexed by the Weyl group $W$, even in the event that $S$ is a proper
subtorus of $T$.

Thus in our setup,

$$H^*_S(X^S) \cong \bigoplus_{w \in W} \Lambda_S,$$

so that in fact a class in $H^*_S(X)$ is determined by its image under $i^*_w$ for each $w \in W$, where
here $i_w$ denotes the inclusion of the $S$-fixed point $wB$. Given a class $\beta \in H^*_S(X)$ and an
$S$-fixed point $wB$, we will typically denote the restriction $i^*_w(\beta)$ at $wB$ by $\beta|_{wB}$, or simply
by $\beta|_w$ if no confusion seems likely to arise.

The following proposition describes exactly how the restrictions maps are computed:

**Proposition 1.4.** Suppose that $\beta \in H^*_S(X)$ is represented by the polynomial $f = f(x, y)$ in
the $x_i$ and $y_i$. Then $\beta|_{wB} \in \Lambda_S$ is the polynomial $f(\rho(wX), Y)$, where $\rho$ once again
denotes restriction $t^* \to \mathfrak{s}^*$.

1.3. **Closed Orbits.** Our computations of equivariant classes start with the closed orbits.
For a given closed $K$-orbit $Y$, since $[Y]$ is completely determined by the restrictions $[Y]|_w$
for $w \in W$, the idea is to compute them and then try to “guess” a formula for $[Y]$ based on them.
For us, a “formula for $[Y]$” is a polynomial in the variables $x_i$ and $y_i$ (defined in the
statement of Proposition 1.1) which represents $[Y]$. Such a formula amounts to a particular
choice of lift of $[Y]$ from $R' \otimes_{R^{\mathbb{Z}}} R$ to $R' \otimes_{\mathbb{C}} R$.

The following proposition tells us precisely how to compute $[Y]|_w$.

**Proposition 1.5.** Let $\Phi^+$ denote a chosen positive system for $G$, such that the Borel group
$B$ corresponds to the negative roots $-\Phi^+$. Let $\Phi_K$ denote the roots of $K$. Consider the
(multi)-set of weights $\rho(w\Phi^+) \subset \mathfrak{s}^*$, where $\rho : t^* \to \mathfrak{s}^*$ denotes the restriction map. Let
\[ \mathcal{R} = \rho(w\Phi^+) - \Phi_K. \] For a closed \( K \)-orbit \( Y \), we have
\[ [Y]_w = \begin{cases} \prod_{\alpha \in \mathcal{R}} \alpha & \text{if } w \in Y, \\ 0 & \text{otherwise}. \end{cases} \]

This is proved in [Wys12a], and follows from the self-intersection formula. We remark once again that in all cases considered in this paper save one, the tori \( S \) and \( T \) coincide, and the restriction map is the identity and can thus be omitted from the notation. In such cases, \( \mathcal{R} = w\Phi^+ - \Phi_K \).

Thus one has hope of finding formulas for the classes of the closed orbits \( Y \) provided one knows how many closed orbits there are, and which \( S \)-fixed points are contained in each closed orbit. First, we have the following easy result:

**Proposition 1.6.** Suppose that \( K \) is a connected symmetric subgroup of \( G \). Then each closed \( K \)-orbit is isomorphic to the flag variety for the group \( K \).

This is [Wys12a, Proposition 1.5]. It says in particular that any closed \( K \)-orbit contains \( |W_K| \) \( S \)-fixed points. In general, however, for a given \( S \)-fixed point \( wB \), it need not be the case that the orbit \( K \cdot wB \) is closed.

To describe precisely which \( K \cdot wB \) are closed in the way which will be most useful to us in our examples, we must first define twisted involutions and the Richardson-Springer map. The references for what follows are [RS90, RS93].

First, observe that because \( T \) is a \( \theta \)-stable torus, \( N_G(T) \) is also \( \theta \)-stable, and hence there is an induced map (which we also call \( \theta \)) on \( W \).

**Definition 1.7.** A twisted involution is an element \( w \in W \) such that \( w = \theta(w)^{-1} \). We shall denote the set of twisted involutions by \( I \).

We now describe a map from \( K \setminus G/B \) to \( I \). First, define the map
\[ \tau : G \rightarrow G \]
by \( \tau(g) = g\theta(g)^{-1} \). Next, define the set
\[ V := \{ g \in G \mid g\theta(g)^{-1} \in N_G(T) \} = \tau^{-1}(N_G(T)). \]

The set \( V \) has a left \( T \)-action and a right \( K \)-action, and the orbit set \( V = T \setminus V/K \) is in bijective correspondence with \( K \setminus G/B \). (One direction of this bijection is given by \( TgK \mapsto K \cdot g^{-1}B \).) Given an element \( v = TgK \) of \( V \), we denote the corresponding \( K \)-orbit \( K \cdot g^{-1}B \) by \( O(v) \). The map
\[ \phi : V \rightarrow W \]
given by \( \phi(g) = \pi(\tau(g)) \) (where \( \pi \) is the natural projection \( N_G(T) \rightarrow W \)) is constant on \( T \times K \) orbits, so we have a map (which we also call \( \phi \)) \( \phi : V \rightarrow W \). It is easy to check that \( \phi \) actually maps \( V \) into \( I \).

**Remark 1.8.** Obviously, the map \( \phi \) can also be thought of as a map \( K \setminus G/B \rightarrow I \), defined by \( \phi(O(v)) = \phi(v) \). We will generally think of \( \phi \) in this way, and will use notation such as \( \phi(Q) \) for \( Q \in K \setminus G/B \) an orbit, without explicitly mentioning a corresponding element of \( V \).

With the map \( \phi \) defined, we now give the following characterization of the closed orbits.
Proposition 1.9 ([RS93, Proposition 1.4.2]). Let $w \in W$ be given. The $K$-orbit $Q = K \cdot wB$ is closed if and only if $\phi(Q) = 1$.

Note that if $W_K$ is the Weyl group for $K$, then $W_K$ is naturally a subgroup of $W$. This is obvious in the event that $\text{rank}(K) = \text{rank}(G)$, so that $S = T$. Then $N_K(T)$ is obviously a subgroup of $N_G(T)$, and $W_K = N_K(T)/T$ is obviously a subgroup of $W = N_G(T)/T$.

It is less obvious in the event that $S \subset T$, since it is not a priori clear that $N_K(S)$ is a subgroup of $N_G(T)$. That it is follows from that fact that $T$ can be recovered as $Z_G(S)$, the centralizer of $S$ in $G$ (see [Spr85, Bri99]). Since any element of $G$ normalizing $S$ must also normalize $Z_G(S) = T$, we have an inclusion $N_K(S) \subset N_G(T)$. This gives a map $W_K = N_K(S)/S \to N_G(T)/T = W$ defined by $nS \mapsto nT$. The kernel of this map is $\{nS | n \in N_K(S) \cap T\}$. Since $S = K \cap T$, the group $N_K(S) \cap T$ is simply $S$:

$$N_K(S) \cap T = N_K(S) \cap (T \cap K) = N_K(S) \cap S = S.$$  

Thus the kernel of the map $W_K \to W$ is $\{1\}$, and so it is an inclusion.

With this in mind, note that if $K \cdot wB$ is a closed orbit, the $S$-fixed points it contains correspond to elements of $W$ having the form $w'w$, with $w' \in W_K$ (viewed as an element of $W$ via the inclusion of Weyl groups we have just described). Thus, by Proposition 1.9, the number of closed orbits is $|N/|W_K|$, where $N$ is the number of $w \in W$ with $\phi(w) = 1$.

In particular, we have the following easy corollary of Proposition 1.9, which applies to all but one of the cases that we consider in this paper.

Corollary 1.10. Suppose that $\text{rank}(K) = \text{rank}(G)$, so that $S = T$. Then $K \cdot wB$ is closed for all $w \in W$. Thus the number of closed $K$-orbits is $|W|/|W_K|$. Each orbit $K \cdot wB$ contains the $|W_K|$ $S$-fixed points corresponding to the elements of the left coset $W_Kw$.

Proof. The equal rank condition is equivalent to the condition that $\theta$ be an inner involution ([Spr87, 1.8]). An inner involution acts trivially on $W$, meaning that $\phi(w) = 1$ for all $w \in W$. \qed

Finally, we mention another characterization of the closed orbits from [RS93], which we will make use of in the lone unequal rank case considered in this paper.

Proposition 1.11 ([RS93, Proposition 1.4.3]). For $w \in W$, the $K$-orbit $K \cdot wB$ is closed if and only if $wBw^{-1}$ is a $\theta$-stable Borel.

1.4. Other Orbits. Once one has formulas for the equivariant classes of closed $K$-orbits, one can compute formulas for classes of all remaining $K$-orbit closures using divided difference operators. This is because the closed orbits are precisely the minimal orbits with respect to the “weak order” on $K \backslash G/B$ ([RS90, Theorem 4.6]). We describe this ordering, and how divided difference operators enter the picture. Let $\alpha \in \Delta$ be a simple root, and let $P_\alpha$ be the minimal parabolic subgroup of $G$ of type $\alpha$ containing $B$. Consider the canonical map

$$\pi_\alpha : G/B \to G/P_\alpha.$$  

This is a $\mathbb{P}^1$-bundle. Letting $Q \in K \backslash G/B$ be given, consider the set $Z_\alpha(Q) := \pi_\alpha^{-1}(\pi_\alpha(Q))$. The map $\pi_\alpha$ is $K$-equivariant, so $Z_\alpha(Q)$ is $K$-stable. Assuming $K$ is connected, $Z_\alpha(Q)$ is also irreducible, so it has a dense $K$-orbit. In the event that $K$ is disconnected, one sees that the component group of $K$ acts transitively on the irreducible components of $Z_\alpha(Q)$, and from this it again follows that $Z_\alpha(Q)$ has a dense $K$-orbit.
If \( \dim(\pi_\alpha(Q)) < \dim(Q) \), then the dense orbit on \( Z_\alpha(Q) \) is \( Q \) itself. However, if \( \dim(\pi_\alpha(Q)) = \dim(Q) \), the dense \( K \)-orbit will be another orbit \( Q' \) of one dimension higher. In either event, using notation as in [MT09], we make the following definition:

**Definition 1.12.** With notation as above, \( s_\alpha \cdot Q \) shall denote the dense \( K \)-orbit on \( Z_\alpha(Q) \).

**Definition 1.13.** The partial order on \( K\backslash G/B \) generated by relations of the form \( Q < Q' \) if and only if \( Q' = s_\alpha \cdot Q \neq Q \) for some \( \alpha \in \Delta \) is referred to as the **weak order**.

Let \( Y, Y' \) denote the closures of \( Q, Q' \), respectively. Assuming that \( Q' = s_\alpha \cdot Q \), we have

\[
[Y'] = \frac{1}{d} \partial_\alpha([Y]),
\]

where \( d \) is the degree of the map \( \pi_\alpha|_Y \) over its image, and where \( \partial_\alpha \) is the “divided difference operator” or “Demazure operator” corresponding to \( \alpha \). This is an operator on \( H^*_S(X) \) defined by

\[
\partial_\alpha(f) = f - s_\alpha(f)/\alpha.
\]

The degree \( d \) mentioned above is always either 1 or 2, see [RS90] Section 4). Namely, \( s_\alpha \cdot Q \neq Q \) only in cases where \( \alpha \) is a “complex” or “non-compact imaginary” root for the orbit \( Q \). Non-compact imaginary roots come in two varieties, “type I” and “type II”, which are differentiated by whether \( Q \) is fixed by the “cross-action” of \( s_\alpha \). The cross-action of \( W \) on \( K\backslash G/B \) is defined by

\[
w \times (K \cdot gB) = K \cdot gw^{-1}B.
\]

We say that a non-compact imaginary root \( \alpha \) is of “type I” if \( s_\alpha \times Q \neq Q \), and of “type II” if \( s_\alpha \times Q = Q \). With this terminology explained, the result implicit in [RS90] Section 4] is that \( d \) is 2 whenever \( \alpha \) is a non-compact imaginary root of type II, and 1 otherwise. In the cases we consider, the weak order on \( K\backslash G/B \), as well as which roots are complex and non-compact imaginary, is understood in an explicit combinatorial way. It is also easy to compute the cross-action on \( W \) combinatorially, which makes it easy to determine which of the divided difference operators must be scaled back by a factor of \( 1/2 \) when performing the calculation. For more details, including the precise definitions of the terms “complex”, “non-compact imaginary”, etc., see [RS90] [RS93].

In [Bri01], the graph for the weak order on \( K \)-orbit closures is endowed with additional data, as follows: If \( Y' = s_\alpha \cdot Y \neq Y \), then the directed edge originating at \( Y \) and terminating at \( Y' \) is labelled by the simple root \( \alpha \), or perhaps by an index \( i \) if \( \alpha = \alpha_i \) for some predetermined ordering of the simple roots. Additionally, if the degree of \( \pi_\alpha|_Y \) is 2, then this edge is double. (In other cases, the edge is simple.) We modify this convention as follows: Rather than use simple and double edges, in our diagrams we distinguish the degree two covers by blue edges, as opposed to the usual black. (We do this simply because the example weak order graphs given in the appendix were created using GraphViz, which does not, as far as the author can ascertain, have a mechanism for creating a reasonable-looking double edge. On the other hand, coloring the edges is straightforward.)

2. Parametrization of \( K \)-Orbits by Clans

In this section, we describe parametrizations of the \( K \)-orbits and their closures in all cases. We start with the most important case, the symmetric pair \( (GL(p + q, \mathbb{C}), GL(p, \mathbb{C}) \times GL(q, \mathbb{C})) \).
2.1. \((G, K) = (GL(p + q, \mathbb{C}), GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))\) — Known Results. The majority of the results detailed in this subsection appeared for the first time in [MO90]. Proofs and further details appear in [Yam97]. The combinatorics are also given a nice exposition in [MT09], and much of our description is taken from there.

We start by defining “clans”, which parametrize the orbits.

Definition 2.1. A \((p, q)\)-clan is a string \(c_1 \ldots c_n\) of \(n = p + q\) characters, each of which is a +, a −, or a natural number, subject to the following conditions:

1. Every natural number which appears must appear exactly twice.
2. The difference in the number of + signs and the number of − signs must be \(p - q\).

(If \(q > p\), there are \(q - p\) more minus signs than plus signs.)

We consider such strings only up to an equivalence which says, essentially, that it is the positions of matching natural numbers, and not the numbers themselves, which determine the clan. For instance, the clans 1212, 2121, and 5757 are all the same, since they all have matching natural numbers in positions 1 and 3, and also in positions 2 and 4. On the other hand, 1221 is a different clan, since it has matching natural numbers in positions 1 and 4, and in positions 2 and 3.

As an example, suppose \(n = 4\), and \(p = q = 2\). Then we must consider all clans of length 4 where the number of +’s and the number of −’s is the same (since \(p - q = 0\)). There are 21 of these, and they are as follows:

\[
\begin{align*}
&+ + - -; + - + -; + - + -; - + + -; - + - +; - - + -; \\
&11 + -; 11 - +; 1 + 1 -; 1 - 1 +; 1 + - 1; 1 - 1 +; \\
&+ 11 - -; - 11 + +; + 1 - 1; - 1 + 1; + - 11; - - + 1; \\
&1122; 1212; 1221
\end{align*}
\]

We now spell out the precise correspondence between clans and \(K\)-orbits. Let \(E_p = \mathbb{C} \cdot \langle e_1, \ldots, e_p \rangle\) be the span of the first \(p\) standard basis vectors, and let \(E_q = \mathbb{C} \cdot \langle e_{p+1}, \ldots, e_n \rangle\) be the span of the last \(q\) standard basis vectors. Let \(\pi: \mathbb{C}^n \to E_p\) be the projection onto \(E_p\).

For any clan \(\gamma = c_1 \ldots c_n\), and for any \(i, j\) with \(i < j\), define the following quantities:

1. \(\gamma(i; +)\) = the total number of plus signs and pairs of equal natural numbers occurring among \(c_1 \ldots c_i\);
2. \(\gamma(i; -)\) = the total number of minus signs and pairs of equal natural numbers occurring among \(c_1 \ldots c_i\); and
3. \(\gamma(i; j)\) = the number of pairs of equal natural numbers \(c_s = c_t \in \mathbb{N}\) with \(s \leq i < j < t\).

For example, for the \((2, 2)\)-clan \(\gamma = 1 + 1 -\),

1. \(\gamma(i; +) = 0, 1, 2, 2\) for \(i = 1, 2, 3, 4\);
2. \(\gamma(i; -) = 0, 0, 1, 2\) for \(i = 1, 2, 3, 4\); and
3. \(\gamma(i; j) = 1, 0, 0, 0, 0, 0\) for \((i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\).

We refer to the numbers \(\gamma(i; +), \gamma(i; -), \) and \(\gamma(i; j)\) as the “rank numbers” for \(\gamma\).

Proposition 2.2. The rank numbers \(\gamma(i; +), \gamma(i; -), \) and \(\gamma(i; j)\) determine \(\gamma\) uniquely.
Proof. Say that $\gamma = c_1 \ldots c_n$ “has a + jump at $i$” if $\gamma(i; +) = \gamma(i - 1; +) + 1$. Likewise, say that $\gamma$ “has a $-$ jump at $i$” if $\gamma(i; -) = \gamma(i - 1; -) + 1$. From the definitions, it is clear that $\gamma$ has a $+$ jump at $i$ if and only if $c_i$ is a $+$ or the second occurrence of a natural number, and does not have a $+$ jump at $i$ if and only if $c_i$ is a $-$ or the first occurrence of a natural number. Likewise, $\gamma$ has a $-$ jump at $i$ if and only if $c_i$ is a $-$ or the second occurrence of a natural number, and does not have a $-$ jump at $i$ if and only if $c_i$ is a $+$ or the first occurrence of a natural number.

Thus the numbers $\gamma(i; +)$ and $\gamma(i; -)$, by themselves, completely determine the location of $+$’s, $-$’s, first occurrences of natural numbers, and second occurrences of natural numbers. The only choice left in constructing $\gamma$, then, is when we see the second occurrence of a natural number, which natural number is it the second occurrence of? This is determined by the numbers $\gamma(i; j)$. Let $k$ be the first index at which $\gamma$ has the second occurrence of a natural number. Supposing there is only one first occurrence to the left of position $k$, $c_k$ is determined. So suppose there is more than one first occurrence to the left of position $k$, with $i_1, \ldots, i_m$ the indices less than $k$ at which $\gamma$ has first occurrences. Consider the numbers $\gamma(i_1; k), \gamma(i_2; k), \ldots, \gamma(i_m; k)$. From the definitions, it is clear that for all $l = 1, \ldots, m$, $\gamma(i_l; k) \leq l$, and also that $\gamma(i_m; k) = m - 1$. Thus there is some first $l$ at which $\gamma(i_l; k) < l$. Then $c_k$ must be the second occurrence of $c_{i_l}$. Indeed, $c_k$ cannot be the second occurrence of any $c_{i_j}$ with $j < l$, because if it were, the pair $(c_{i_j}, c_k)$ would not be counted in the number $\gamma(i_l; k)$, so we would necessarily have $\gamma(i_l; k) < j$. On the other hand, if the second occurrence of $c_{i_l}$ were beyond position $k$, then we would necessarily have that $\gamma(i_l; k) = l$. Thus $c_k$ is determined by the numbers $\gamma(i_1; k), \ldots, \gamma(i_m; k)$.

Working in order from left to right, the remaining indices at which $\gamma$ has second occurrences can be filled in using similar logic. At each such index, we consider the indices left of $k$ at which $\gamma$ has the first occurrence of a natural number which does not yet have a mate. Applying the same argument as above will allow us to determine which of those first occurrences $c_k$ should be the mate of.

Example 2.3. As an example of the preceding proof, suppose that we are told that the $(3, 3)$ clan $\gamma$ has rank numbers

- $\gamma(i; +) = 0, 0, 1, 2, 2, 3$ for $i = 1, 2, 3, 4, 5, 6$;
- $\gamma(i; -) = 0, 0, 1, 2, 2, 3$ for $i = 1, 2, 3, 4, 5, 6$;
- $\gamma(i; j) = 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$ for $(i, j) = (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)$.

The first two sets of numbers tell us that $\gamma$ has the pattern $FFSSFS$, where $F$ represents the first occurrence of a natural number, and $S$ represents the second occurrence of a natural number. We can assign natural numbers to the positions of the $F$’s any way we’d like — they may as well be 1, 2, 3, in order. Thus $\gamma$ has the form $12SS3S$. Looking at the first $S$ in position 3, we need to determine whether $c_3 = 2$ or $c_3 = 1$. Looking at the numbers $\gamma(1; 3) = 1$ and $\gamma(2; 3) = 1$, we see that $c_3 = 2$. This forces $c_4 = 1$, and $c_6 = 3$. Thus $\gamma = 122133$.

With the rank numbers for a clan defined, we have the following theorem on $K$-orbits on $G/B$:

Theorem 2.4 (Yam97). Suppose $p + q = n$. For a $(p, q)$-clan $\gamma$, define $Q_\gamma$ to be the set of all flags $F_\gamma$ having the following three properties for all $i, j$ ($i < j$):

[Theorem statement follows here]
(1) \( \dim(F_i \cap E_p) = \gamma(i;) + \)
(2) \( \dim(F_i \cap E_q) = \gamma(i;) - \)
(3) \( \dim(\pi(F_i) + F_j) = j + \gamma(i;j) \)

For each \((p,q)\)-clan \( \gamma \), \( Q_{\gamma} \) is nonempty, stable under \( K \), and in fact is a single \( K \)-orbit on \( G/B \).

Conversely, every \( K \)-orbit on \( G/B \) is of the form \( Q_{\gamma} \) for some \((p,q)\)-clan \( \gamma \). Hence the association \( \gamma \mapsto Q_{\gamma} \) defines a bijection between the set of all \((p,q)\)-clans and the set of \( K \)-orbits on \( G/B \).

As in the statement of the above theorem, we will typically denote a clan by \( \gamma \), and the corresponding orbit by \( Q_{\gamma} \).

Next, we outline an algorithm, described in [Yam97], for producing a representative of \( Q_{\gamma} \) given the clan \( \gamma \).

First, for each pair of matching natural numbers of \( \gamma \), assign one of the numbers a “signature” of +, and the other a signature of −. Now choose a permutation \( \sigma \) of \( 1, \ldots, n \) with the following properties for all \( i = 1, \ldots, n \):

(1) \( 1 \leq \sigma(i) \leq p \) if \( c_i = + \) or if \( c_i \in \mathbb{N} \) and the signature of \( c_i \) is +.
(2) \( p + 1 \leq \sigma(i) \leq n \) if \( c_i = - \) or if \( c_i \in \mathbb{N} \) and the signature of \( c_i \) is −.

Having determined such a permutation \( \sigma \), take \( F_\bullet = \langle v_1, \ldots, v_n \rangle \) to be the flag specified as follows:

\[
v_i = \begin{cases} 
  e_{\sigma(i)} & \text{if } c_i = \pm, \\
  e_{\sigma(i)} + e_{\sigma(j)} & \text{if } c_i \in \mathbb{N}, c_i \text{ has signature } +, \text{ and } c_i = c_j, \\
  -e_{\sigma(i)} + e_{\sigma(j)} & \text{if } c_i \in \mathbb{N}, c_i \text{ has signature } -, \text{ and } c_i = c_j.
\end{cases}
\]

For example, for the orbit corresponding to the clan \( ++-+-- \), we could take \( \sigma = 1 \), which would give the standard flag \( \langle e_1, \ldots, e_6 \rangle \). For \( 1-+1 \), we could assign signatures to the 1’s as follows: \( 1_+ + 1_- \). We could then take \( \sigma \) to be the permutation 1324. This would give the flag

\[
F_\bullet = \langle e_1 + e_4, e_3, e_2, e_1 - e_4 \rangle.
\]

Relative to this parametrization, the closed orbits are those whose clans consist only of +’s and −’s. Note then that this algorithm tells us in particular how to determine an \( S \)-fixed point contained in such an orbit. Indeed, any representative determined by the algorithm above for an orbit whose clan consists only of +’s and −’s necessarily produces an \( S \)-fixed flag corresponding to a permutation which assigns to the positions of the +’s the numbers \( 1, \ldots, p \), and to the positions of the −’s the numbers \( p+1, \ldots, n \). Note that we have a choice of the permutation \( \sigma \), and each choice gives a different \( S \)-fixed point in the orbit. The different choices of \( \sigma \) all belong to the same \( W_K \)-coset (where \( W_K = S_p \times S_q \)), consistent with the description of the \( S \)-fixed points contained in each closed orbit given in Corollary 1.10.

We now give a combinatorial description of the weak ordering on the orbit set in terms of this parametrization. Let \( \gamma = c_1 \ldots c_n \). The simple root \( \alpha_i = Y_i - Y_{i+1} \) is complex for the orbit \( Q_{\gamma} \), with \( s_{\alpha_i} \cdot Q_{\gamma} \neq Q_{\gamma} \), if and only if one of the following occurs:

(1) \( c_i \) and \( c_{i+1} \) are unequal natural numbers, and the mate of \( c_i \) is to the left of the mate of \( c_{i+1} \);
(2) \( c_i \) is a sign, \( c_{i+1} \) is a natural number, and the mate of \( c_{i+1} \) is to the right of \( c_{i+1} \);
(3) \( c_i \) is a natural number, \( c_{i+1} \) is a sign, and the mate of \( c_i \) is to the left of \( c_i \).

So, for example, taking \( p = 3, q = 2 \), and letting \( i = 2 \), \( 112 + 2 \) satisfies the first condition; \( +1 - 1+ \) satisfies the second; \( 11 + -+ \) satisfies the third; and \( 1 + 122 \) satisfies none of them (since the mate of the 1 in the 3rd slot occurs to its left, rather than to its right).

On the other hand, \( \alpha_i \) is non-compact imaginary for the orbit \( Q\gamma \) if and only if \( c_i \) and \( c_{i+1} \) are opposite signs.

Furthermore, one sees that the clan \( \gamma' \) for \( s_{\alpha_i} \cdot Q\gamma \) is obtained from \( \gamma \) by interchanging \( c_i \) and \( c_{i+1} \) in the complex case, and by replacing the opposite signs in the \( c_i \) and \( c_{i+1} \) slots by a pair of equal natural numbers in the non-compact imaginary case. So, again taking \( p = 3, q = 2, i = 2 \), we have, for example,

- \( s_{\alpha_2} \cdot 112 + 2 = 121 + 2; \)
- \( s_{\alpha_2} \cdot + -11+ = +1 - 1+; \)
- \( s_{\alpha_2} \cdot 11 + -+ = 1 + 1 - +; \)
- \( s_{\alpha_2} \cdot 1 + -1+ = 1221+. \)

Finally, we describe the cross action of \( W = S_n \) on the orbits in terms of this parametrization. In fact, the action is the obvious one, given by permuting the symbols of any clan according to the underlying permutation of any \( w \in W \). (The most straightforward way to see this is to note the effect of simple transpositions on the representatives specified by the algorithm of \[Yam97\] described above.)

Thus we see that if \( \alpha_i \) is non-compact imaginary for the orbit \( Q\gamma \), then \( \gamma \) has (\( c_i, c_{i+1} \)) equal to either \( + - \) or \( - + \), and the cross-action of \( s_{\alpha_i} \) (the simple transposition \( (i, i + 1) \)) on \( \gamma \) switches these signs. In particular, \( s_{\alpha_i} \times Q\gamma \neq Q\gamma \), and so we see that all non-compact imaginary roots are of type I in this case. This means that the weak order graph consists only of black edges, and no factors of \( \frac{1}{2} \) are required in our divided difference computations.

2.2. \( GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \)-orbit Closures on \( GL(p+q, \mathbb{C})/B \) — New Results. Although Theorem 2.4 gives an explicit description of \( K \)-orbits themselves as sets of flags, for our purposes here we would like an explicit set-theoretic description of \( K \)-orbit closures, and none has been given in the literature. In this section, we prove such a description. Conveniently, it turns out that one passes from a \( K \)-orbit to its closure just as one passes from a Schubert cell to the corresponding Schubert variety: by changing the equalities in the description of the orbit to inequalities.

**Theorem 2.5.** Let \( \gamma \) be a \((p, q)\)-clan, with \( Y_\gamma = \overline{Q\gamma} \) the corresponding \( K \)-orbit closure. Then \( Y_\gamma \) is precisely the set of flags \( F_\bullet \) satisfying the following conditions for all \( i < j \):

1. \( \dim(F_i \cap E_p) \geq \gamma(i; +) \)
2. \( \dim(F_i \cap E_q) \geq \gamma(i; -) \)
3. \( \dim(\pi(F_i) + F_j) \leq j + \gamma(i; j) \)

Our proof of Theorem 2.5 is along the same lines as the proof of \[Ful97\] §10.5, Proposition 7], regarding closures of type \( A \) Schubert cells and the Bruhat order on \( S_n \). We proceed as follows:

1. Define a “combinatorial Bruhat order” \( \preceq \) on \((p, q)\)-clans, which we secretly know reflects the true geometric Bruhat order.
2. Describe the covering relations with respect to this combinatorial Bruhat order.
(3) Show that orbit closures $Y_\gamma$, $Y_\tau$ satisfy $Y_\gamma \subseteq Y_\tau$ if and only if $\gamma \leq \tau$.

One direction of (3) is easy. The other requires knowledge of the covering relations with respect to the order $\leq_s$, acquired in (2) above, together with some limiting arguments involving specific representatives of the orbits $Q_\gamma$ and $Q_\tau$ when $\tau$ covers $\gamma$ in the combinatorial order.

**Definition 2.6.** We define the **combinatorial Bruhat order** $\leq$ on $(p, q)$-clans as follows: $\gamma \leq \tau$ if and only if

1. $\gamma(i; +) \geq \tau(i; +)$ for all $i$;
2. $\gamma(i; -) \geq \tau(i; -)$ for all $i$;
3. $\gamma(i; j) \leq \tau(i; j)$ for all $i < j$.

In light of Proposition 2.2, it is clear that $\leq$ is a partial order on $(p, q)$-clans. We wish to describe the covering relations with respect to this order. We note first that clans can be thought of as involutions in $S_n$ with signed fixed points. Indeed, the positions of matching natural numbers naturally give two-cycles, while the positions of ± signs mark fixed points of the underlying involution. (So, for example, the clan $12 + 12$ corresponds to the involution $(1, 5)(2, 6) = 563412$.)

Note conversely that any involution in $S_n$ can likewise be thought of as a character string consisting of pairs of matching natural numbers and (say) dots, with the positions of matching numbers marking two-cycles, and the dots marking fixed points. (So, for example, the involution $563412 = (1, 5)(2, 6)$ corresponds to the character string $12 \cdot 12$.) We first wish to observe that when the involutions of $S_n$ are translated into such “clan-like” symbols, their Bruhat order (by which we mean the restriction of the Bruhat order on $S_n$ to the subset of involutions) is closely related to the combinatorial Bruhat order on clans defined above.

Paralleling the notation for clans, given an involution $\gamma \in S_n$, with $\gamma_1 \ldots \gamma_n$ the corresponding character string just described, define the following for any $i = 1, \ldots, n$ and for any $1 \leq s < t \leq n$:

1. $\gamma(i; \cdot) = \text{Number of dots plus twice the number of pairs of equal natural numbers occurring among } \gamma_1 \ldots \gamma_i; \text{ and}$
2. $\gamma(s; t) = \#\{\gamma_a = \gamma_b \in \mathbb{N} \mid a \leq s, b > t\}$.

We omit the easy proof of the following proposition, which is simply a reformulation of the Bruhat order on involutions which more closely mirrors our definition of the combinatorial Bruhat order on clans.

**Proposition 2.7.** For involutions $\gamma, \tau \in S_n$, $\gamma \leq \tau$ in Bruhat order if and only if

1. $\gamma(i; \cdot) \geq \tau(i; \cdot)$ for $i = 1, \ldots, n$; and
2. $\gamma(s; t) \leq \tau(s; t)$ for all $1 \leq s < t \leq n$.

**Corollary 2.8.** Suppose $\gamma$ and $\tau$ are two $(p, q)$-clans. Then $\gamma \leq \tau$ in the combinatorial Bruhat order if and only if

1. The underlying involutions of $\gamma, \tau$ are related in the Bruhat order on $S_n$, and additionally
2. $\gamma(i; +) \geq \tau(i; +)$ and $\gamma(i; -) \geq \tau(i; -)$ for all $i$.

**Proof.** Clearly, $\gamma(i; +) + \gamma(i; -) = \gamma(i; \cdot)$, and $\tau(i; +) + \tau(i; -) = \tau(i; \cdot)$. So if $\gamma \leq \tau$ in the combinatorial Bruhat order, then $\gamma(i; +) \geq \tau(i; +)$ and $\gamma(i; -) \geq \tau(i; -)$, so $\gamma(i; \cdot) \geq \tau(i; \cdot)$. Thus the underlying involutions are related in the Bruhat order by Proposition 2.7.
On the other hand, relation of the underlying involutions in the Bruhat order is not enough, since \( \gamma(i; \cdot) \geq \tau(i; \cdot) \) does not imply that \( \gamma(i; +) \geq \tau(i; +) \) and \( \gamma(i; -) \geq \tau(i; -) \). So we must insist on those conditions separately. □

Corollary 2.8 simplifies some of the arguments given in the proof of the following theorem, because in certain instances we are able to use the results of [Inc04] regarding the covering relations in the Bruhat order on ordinary involutions.

**Theorem 2.9.** Suppose that \( \gamma, \tau \) are \((p, q)\)-clans, with \( \gamma < \tau \). Then there exists a clan \( \gamma' \) such that \( \gamma < \gamma' \leq \tau \), and such that \( \gamma' \) is obtained from \( \gamma \) by a “move” of one of the following types:

1. Replace a pattern of the form \(+ -\) by \([11] \), i.e. replace a plus and minus by a pair of matching natural numbers.
2. Replace a pattern of the form \( - + \) by \([11] \).
3. Replace a pattern of the form \([11]+\) by \(1 + 1\), i.e. interchange a number and a + sign to its right if in doing so you move the number farther from its mate.
4. Replace a pattern of the form \([11]-\) by \(1 - 1\).
5. Replace a pattern of the form \(+11\) by \(1 + 1\).
6. Replace a pattern of the form \(11-1\) by \(1 - 1\).
7. Replace a pattern of the form \(1122\) by \([1212] \).
8. Replace a pattern of the form \(1122\) by \(1 + -1\).
9. Replace a pattern of the form \(1122\) by \(1 - +1\).
10. Replace a pattern of the form \(1212\) by \([1221] \).

**Proof.** We prove this by explicitly constructing \( \gamma' \) from \( \gamma \) and \( \tau \), then showing that the \( \gamma' \) so constructed satisfies \( \gamma < \gamma' \leq \tau \). (This is the clan version of what is done in [Ful97] §10.5, Lemma 11] in the case of permutations.)

In fact, when one translates the clans \( \gamma, \gamma', \) and \( \tau \) to their underlying involutions, one sees that the construction of \( \gamma' \) follows very closely the “covering moves” for involutions described in [Inc04] §3-4. In fact, we are able to apply directly the results of [Inc04] regarding these covering moves in certain cases.

The proof is by case analysis. As the statement of the theorem may indicate, there are a number of cases to consider. First, we say that clans \( \gamma = \gamma_1 \ldots \gamma_n \) and \( \tau = \tau_1 \ldots \tau_n \) differ at position \( i \) if and only if one of the following holds:

- \( \gamma_i \) and \( \tau_i \) are opposite signs;
- \( \gamma_i \) is a sign and \( \tau_i \) is a number;
- \( \gamma_i \) is a number and \( \tau_i \) is a sign; or
- \( \gamma_i \) and \( \tau_i \) are numbers whose mates are in different positions.

If \( \gamma < \tau \), then obviously there is a first position at which \( \gamma \) and \( \tau \) differ. Denote this position by \( f \). One of the following must hold:

1. \( \gamma_f \) is a + sign, and \( \tau_f \) is the first occurrence of some natural number;
2. \( \gamma_f \) is a – sign, and \( \tau_f \) is the first occurrence of some natural number; or
3. \( \gamma_f \) and \( \tau_f \) are both first occurrences of a natural number, with \( \gamma_f = \gamma_i \) and \( \tau_f = \tau_j \) for \( f < i < j \).

To see this, note first that neither \( \gamma_f \) nor \( \tau_f \) can be the second occurrence of a natural number, since if either was, a difference would’ve been detected prior to position \( f \), namely
at the position of the first occurrence of that natural number. Thus the only possibilities for 
\((\gamma_f, \tau_f)\) are \((+, -), (-, +), (+, F), (-, F), (F, +), (F, -), (F, F)\), where here \(F\) stands for
the first occurrence of a natural number. We can rule out the cases \((+, -), (-, +), (F, +)\),
and \((F, -)\), since in those cases we would not have \(\gamma < \tau\) in the combinatorial Bruhat order.
Indeed,
- \((\gamma_f, \tau_f) = (\gamma, \tau) \Rightarrow \gamma(f; -) < \gamma(f; -)\);
- \((\gamma_f, \tau_f) = (-, \tau) \Rightarrow \gamma(f; +) < \gamma(f; +)\);
- \((\gamma_f, \tau_f) = (F, +) \Rightarrow \gamma(f; +) < \gamma(f; +)\);
- \((\gamma_f, \tau_f) = (F, -) \Rightarrow \gamma(f; -) < \gamma(f; -)\).

Note also that in the case \((F, F)\), if we had \(\gamma_f = \gamma_i\) and \(\tau_f = \tau_j\) with \(j < i\),
then this would imply that \(\gamma(f; j) > \tau(f; j)\), which is again contrary to our assumption that \(\gamma < \tau\).
Thus in the case \((F, F)\), the mate for \(\tau_f\) must occur to the right of the mate for \(\gamma_f\).

We remark that the position we are denoting \(f\) (for “
first”) is the same as the “difference index” \(d_i\) defined in [Inch1, Definition 4.1]. In fact, there is no loss in generality in assuming that \(f = 1\), and we do so in what follows. This essentially allows us to ignore, over certain ranges of indices, any second occurrences of natural numbers whose first occurrences are prior to position \(f\). Since \(\gamma\) and \(\tau\) match in all such positions anyway, considering them does nothing but clutter our arguments unnecessarily. So from this point forward, we assume \(f = 1\).

We now consider the cases \((\gamma_1, \tau_1) = (+, F), (-, F), and (F, F)\) in turn.

**Case 1** \(((\gamma_1, \tau_1) = (+, F))\). Suppose that \(\gamma_1 = +\), and that \(\tau_1\) is the first occurrence of
a natural number whose second occurrence is at position \(i\): \(\tau_1 = \tau_i \in \mathbb{N}\). We claim first
that there exists \(j \in [1, i]\) such that either \(\gamma_j\) is a \(-\) sign, or \(\gamma_j\) is the first of a pair of
matching natural numbers whose second occurrence is in position at most \(i\), i.e. \(\gamma_j = \gamma_k \in \mathbb{N}\)
\(1 < j < k \leq i\). To see this, note that \(\tau_1(1; -) = \gamma_1(1; -) = 0\), and that \(\tau\) has a \(-\) jump at
position \(i\), \(\tau_i\) being the second occurrence of \(\tau_1\). Thus in order to ensure \(\gamma(i; -) \geq \tau(i; -)\), \(\gamma\)
must have a \(-\) jump in the range \([1, i]\). This can occur only at either a \(-\) sign, or the second
occurrence of a natural number.

Let \(j \in [1, i]\) be the smallest index such that \(\gamma_j\) is either a \(-\) or the first occurrence of a
natural number whose second occurrence is in position at most \(i\), with \(\gamma_j = \gamma_k \in \mathbb{N}\) for
\(1 < j < k \leq i\). We consider the two subcases separately, but first we prove the following
lemma, which is of use in both instances.

**Lemma 2.10.** For any \(l \in [1, j - 1]\), let \(\gamma_F(l)\) denote the number of first occurrences of
natural numbers among \(\gamma_1 \ldots \gamma_l\), and define \(\tau_F(l)\) similarly. Then \(\gamma_F(l) < \tau_F(l)\).

**Proof.** Note that, by our choice of \(j\), all first occurrences for \(\gamma\) in the range \([1, l]\) have their
second occurrence strictly after position \(i\). Since \(\gamma(1; i) \leq \tau(1; i)\), \(\tau\) must have at least as
many first occurrences in this range whose second occurrences are strictly after position \(i\).
And \(\tau\) has (at least) one more first occurrence, namely \(\tau_1\), whose second occurrence is at
position \(i\). \(\square\)

**Case 1.1** \((\gamma_i = -)\). This case looks like the following:

\[
\begin{array}{cccccc}
\tau: & 1 & \ldots & \ldots & 1 & \ldots \\
\gamma: & + & \ldots & - & \ldots & \ldots \\
\end{array}
\]
In this case, we claim that the clan $\gamma'$ obtained from $\gamma$ by replacing $\gamma_1$ and $\gamma_j$ by a new pair of matching natural numbers (i.e. replacing the pattern $(\gamma_f, \gamma_j) = +-$ by 11) satisfies $\gamma < \gamma' \leq \tau$. To see that $\gamma < \gamma'$, we simply note how the rank numbers for $\gamma'$ differ from those of $\gamma$. From the definitions, it is clear that the only changes in the rank numbers are

- $\gamma'(k;+) = \gamma(k;+) - 1$ for all $k \in [1, j - 1]$;
- $\gamma'(k;l) = \gamma(k;l) + 1$ for all $k < l$ with $1 \leq k < l < j$.

Thus $\gamma < \gamma'$. To see that $\gamma' \leq \tau$, having noted how the rank numbers for $\gamma'$ differ from those of $\gamma$, and knowing that $\gamma < \tau$, we need only establish the following:

- $\tau(k;+) < \gamma(k;+)$ for all $k \in [1, j - 1]$; and
- $\tau(k;l) > \gamma(k;l)$ for all $k < l$ with $1 \leq k < l < j$.

We start with the first statement. Since $\gamma(1;+) = 1$ and $\tau(1;+) = 0$, it is clear for $k = 1$, and so we consider $k \in [2, j - 1]$. Consider the following observations:

1. By our choice of $j$, $\gamma$ has no $-$ jumps in the range $[2, j - 1]$.
2. $\gamma(1; -) = \tau(1; -) = 0$. Thus, by the previous bullet, and because $\gamma < \tau$, we must have that $\tau(k;-) = \gamma(k;-)$ for all $k \in [2, j - 1]$. As a consequence, note that there can be no $-$ jumps for $\tau$ in the range $[2, j - 1]$, i.e. there can be no $k \in [2, j - 1]$ where $\tau_k = -$, or where $\tau_k$ is a second occurrence of a natural number.
3. Thus in the range $[2, j - 1]$, the only possible characters for both $\tau$ and $\gamma$ are $+$’s and first occurrences of natural numbers.

Since the number of first occurrences for $\gamma$ in the range $[1, k]$ is strictly less than the number of first occurrences for $\tau$ in the same range by Lemma 2.10, the number of $+$ signs for $\gamma$ must be strictly greater than the number of $+$ signs for $\tau$ in this range. Thus $\gamma(k;+) > \tau(k;+)$, as desired.

Now, we must see that $\tau(k;l) > \gamma(k;l)$ whenever $1 \leq k < l < j$. Let such a $k, l$ be given.

By the above observations, it is clear that any first occurrence of a natural number for $\gamma$ occurring in the range $[1, k]$ has its mate in position strictly greater than $i$ (and hence strictly greater than $l$, since $l < j \leq i$), while any first occurrence of a natural number for $\tau$ occurring in this range has its mate in position at least $j$ (hence also in a position strictly greater than $l$). Thus $\gamma(k;l)$ and $\tau(k;l)$ are simply the numbers of first occurrences for $\gamma$ and $\tau$, respectively, occurring in the range $[1, k]$. So again it follows from Lemma 2.10 that $\gamma(k;l) < \tau(k;l)$.

This completes the proof that $\gamma < \gamma' \leq \tau$.

**Case 1.2** $(\gamma_j = \gamma_k \in \mathbb{N}, 1 < j < k \leq i)$. Here, we have the following setup:

- $\tau: 1 \ldots \ldots 1 \ldots$
- $\gamma: + \ldots 1 \ldots 1 \ldots$

In this case, we claim that the $\gamma'$ obtained from $\gamma$ by interchanging $\gamma_1$ and $\gamma_j$ (replacing the substring $(\gamma_1, \gamma_j, \gamma_k) = +11$ by $1+1$) satisfies $\gamma < \gamma' \leq \tau$.

To see that $\gamma < \gamma'$, simply note how the rank numbers for $\gamma'$ and $\gamma$ differ. The only differences are

- $\gamma'(l;+) = \gamma(l;+) - 1$ for $l = 1, \ldots, j - 1$; and
\(\gamma'(l;m) = \gamma(l;m) + 1\) for pairs \(l < m\) such that \(1 \leq l < j\) and \(l < m < k\).

This shows that \(\gamma < \gamma'\). To see that \(\gamma' \leq \tau\), one shows that
- \(\tau(l;+) < \gamma(l;+)\) for \(l = 1, \ldots, j - 1\); and
- \(\tau(l;m) > \gamma(l;m)\) whenever \(1 \leq l < j\) and \(l < m < k\).

The proofs of these facts are exactly the same as those given in the previous subcase, so we do not repeat them here.

**Case 2** \(((\gamma_f, \tau_f) = (-, F))\). In this case, we are able to make the exact same arguments as in the previous case, interchanging \(-\) and \(+\) signs everywhere. The upshot is that \(\gamma'\) can be obtained from \(\gamma\) by a move of either type \(-+ \rightarrow 11\) or \(-11 \rightarrow 1-1\).

**Case 3** \(((\gamma_1, \tau_1) = (F, F))\). Now, suppose that \((\gamma_1, \tau_1) = (F, F)\), with \(\gamma_1\) and \(\tau_1\) natural numbers such that \(\gamma_1 = \gamma_j \in \mathbb{N}\) and \(\tau_1 = \gamma_i\). Recall that \(j < i\).

There are multiple subcases to consider here. To see what they are, we first note the following easy lemma.

**Lemma 2.11.** One of the following must be true:

1. There exists a pair of matching numbers \(\gamma_k = \gamma_l \in \mathbb{N}\) with \(1 < k < j < l \leq i\).
2. There exists a pair of natural numbers \(\gamma_k = \gamma_l \in \mathbb{N}\) with \(j < k < l \leq i\).
3. There exists a \(\gamma_k = \pm\) for some \(k \in [j + 1, i]\).
4. There exists a pair of natural numbers \(\gamma_k = \gamma_l \in \mathbb{N}\) with \(l > i\), and with \(k \in [j + 1, i]\).

**Proof.** Indeed, consider the possible values of \(\gamma_{j+1}, \ldots, \gamma_i\). If any is a \(\pm\), we are done, as we are in case (3). If any is a second occurrence, then its first occurrence must occur either between 1 and \(j - 1\) (case (1)), or else after \(j\) (case (2)). And if any is a first occurrence, then its second occurrence must occur either in position at most \(i\) (case (2)), or in a position strictly beyond \(i\) (case (4)). \(\square\)

We consider each of the above cases in turn.

**Case 3.1** (There exist \(\gamma_k = \gamma_l \in \mathbb{N}\) with \(1 < k < j < l \leq i\)). The picture here is as follows:

\[
\begin{array}{cccccccccc}
\tau: & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots \\
\gamma: & 1 & \cdots & 2 & \cdots & 1 & \cdots & 2 & \cdots & \cdots \\
& k & j & l & & & & & & \\
\end{array}
\]

Then choose the pair such that \(k\) is minimal. Note that the numbers \((\gamma_1, \gamma_k, \gamma_j, \gamma_l)\) form the pattern 1212. Let \(\gamma'\) be obtained from \(\gamma\) by changing this pattern to 1221 (i.e. by either interchanging \(\gamma_1\) and \(\gamma_k\), or \(\gamma_j\) and \(\gamma_l\)). Then we claim that \(\gamma < \gamma' \leq \tau\).

To see that \(\gamma < \gamma'\), we simply note that the only changes in the rank numbers as we move from \(\gamma\) to \(\gamma'\) are that \(\gamma'(s;t) = \gamma(s;t) + 1\) whenever \(1 \leq s < k\) and \(j \leq t < l\). To see that \(\gamma' \leq \tau\), then, we simply need to see that \(\gamma(s;t) \leq \tau(s;t)\) for such \(s\) and \(t\).

In fact, by Corollary 2.8 it suffices here to observe that the underlying involutions of \(\gamma'\) and \(\tau\) are suitably related in Bruhat order. That they are follows immediately from [Inc04, Corollary 4.6]. Indeed, one checks that, on the underlying involutions, the move made here is the “minimal covering transformation” of \(\gamma\) relative to \(\tau\), with \((1, k)\) being a “non-crossing ee-rise".
Case 3.2 (There is either a $+$, a $-$, or a pair of natural numbers occurring in the range $j + 1, \ldots, i$). In this case, choose $k$ to be the smallest index in the range $[j + 1, i]$ such that either $\gamma_k = +$, $\gamma_k = -$, or $\gamma_k = \gamma_l \in \mathbb{N}$ with $k < l \leq i$. We treat each of the three possibilities in turn, starting with the last.

Case 3.2.1 ($\gamma_k = \gamma_l \in \mathbb{N}$ with $j < k < l \leq i$). We have the following picture:

$$
\begin{array}{cccccccc}
\tau & : & 1 & \ldots & \ldots & \ldots & \ldots & 1 & \ldots \\
\gamma & : & 1 & \ldots & 1 & \ldots & 2 & \ldots & 2 & \ldots \\
& j & k & l
\end{array}
$$

Note that the numbers $(\gamma_1, \gamma_j, \gamma_k, \gamma_l)$ form the pattern 1122. We claim that the $\gamma'$ obtained from $\gamma$ by changing this pattern either to $1 + -1$ or $1 - +1$ satisfies $\gamma < \gamma' \leq \tau$. To see this, we first establish a few basic lemmas which will be used both in this subcase and the next.

Recall the following notation, defined in a previous case:

$$
\gamma_F(s) := \# \{ t \in [1, s] \mid \gamma_t \text{ is a first occurrence} \},
$$

and

$$
\tau_F(s) := \# \{ t \in [1, s] \mid \tau_t \text{ is a first occurrence} \}.
$$

Lemma 2.12. For any $s \in [1, k - 1]$,

$$
\gamma_F(s) \leq \tau_F(s).
$$

Proof. Since we are not in Case 3.1, any first occurrence in the range $[2, j - 1]$ has its mate in a position strictly beyond $i$. By our choice of $k$, all first occurrences in the range $[j + 1, k - 1]$ also have their second occurrences in a position strictly beyond $i$. Thus $\gamma_F(s) = \gamma(s; i) + 1$, the additional 1 being $\gamma_1$, whose second occurrence is $\gamma_j$. Likewise, $\tau(s; i) \leq \tau_F(s) + 1$, the additional 1 being $\tau_1$, whose second occurrence is $\tau_i$. Since $\gamma_F(s) + 1 = \gamma(s; i) \leq \tau(s; i) \leq \tau_F(s) + 1$, we get the desired inequality. \qed

Lemma 2.13. For any $s \in [j, k - 1]$, if $\gamma(s; -) = \tau(s; -)$, then $\gamma(s; +) > \tau(s; +)$, and if $\gamma(s; +) = \tau(s; +)$, then $\gamma(s; -) > \gamma(s; -)$.

Proof. Since $\gamma(s; \pm) \geq \tau(s; \pm)$ by definition of the combinatorial Bruhat order, the statement here amounts to the fact that we cannot have both $\gamma(s; -) = \tau(s; -)$ and $\gamma(s; +) = \tau(s; +)$ at the same time at any point over this range of indices.

Consider the possible values of characters $\gamma_t$ for $t \in [2, s]$. They are

- $+$ signs;
- $-$ signs;
- First occurrences of natural numbers whose second occurrences are beyond position $s$;
- Pairs of numbers each occurring in the range $[2, s]$;
- The lone character $\gamma_j$ (the second occurrence of $\gamma_1$).

Consider the possible values of characters $\tau_t$ for $t \in [2, s]$. They are

- $+$ signs;
- $-$ signs;
- First occurrences of natural numbers whose second occurrences are beyond position $s$;
The various “+ 1”'s in the equations involving $\gamma$ come from counting $\gamma_j$, the second occurrence of $\gamma_1$.

Combining all of the above observations, we get

$$\gamma(s;+) + \gamma(s;-) + \gamma_F(s) - 1 = \tau(s;+) + \tau(s;-) + \tau_F(s).$$

If $\gamma(s;-) = \tau(s;-)$ and $\gamma(s;+) = \tau(s;+)$, this reduces to

$$\gamma_F(s) = \tau_F(s) + 1,$$

which contradicts Lemma 2.12. Thus we cannot have both $\gamma(s;-) = \tau(s;-)$ and $\gamma(s;+) = \tau(s;+)$. $\square$

**Lemma 2.14.** Either $\gamma(s;+) > \tau(s;+)$ for all $s \in [j,k-1]$, or $\gamma(s;-) > \tau(s;-)$ for all $s \in [j,k-1]$ (or both).

**Proof.** If $\gamma(s;+) > \tau(s;+)$ for all $s \in [j,k-1]$, we are done. Otherwise, there is some

first index $s \in [j,k-1]$ for which $\gamma(s;+) = \tau(s;+)$. By Lemma 2.13 we must have $\gamma(s;-) > \tau(s;-)$. Since there are no + jumps for $\gamma$ in the range $s+1, \ldots, k-1$ (by the fact that we are not in Case 3.1 and by our choice of $k$), the equality $\gamma(t;+) = \tau(t;+)$ must hold for all $t = s+1, \ldots, k-1$ as well, and thus, again by Lemma 2.13, we have $\gamma(t;-) > \tau(t;-)$ for all $t$ in this range as well. Since there are no − jumps for $\gamma$ in the range $[j+1,s-1]$ (again, because we are not in Case 3.1 and by our choice of $k$), the strict inequality $\gamma(t;-) > \tau(t;-)$ must hold also for all $t \in [j,s-1]$. Indeed, assuming by induction that strict inequality holds at position $t$, we have

$$\gamma(t-1;-) = \gamma(t;-) > \tau(t;-) \geq \tau(t-1;-).$$

$\square$

Now, using Lemma 2.14, we are able to show that the $\gamma'$ obtained from $\gamma$ by replacing $(\gamma_1, \gamma_j, \gamma_k, \gamma_l) = 1122$ by either $1 + -1$ or $1 - +1$ satisfies $\gamma < \gamma' \leq \tau$. If $\gamma(s;+) > \tau(s;+)$ for all $s \in [j,k-1]$, then we replace the pattern by $1 + -1$, and if $\gamma(s;-) > \tau(s;-)$ for all $s \in [j,k-1]$, we replace the pattern by $1 + -1$. (If both of these are true, we can make either move.)
Assume that $\gamma(s;+) > \tau(s;+)$ for all $s \in [j, k - 1]$. Then $\gamma'$ is obtained via the move $1122 \rightarrow 1 - + 1$. Note how the rank numbers of $\gamma'$ differ from those of $\gamma$:

- $\gamma'(s;+) = \gamma(s;+) - 1$ for all $s \in [j, k - 1]$;
- $\gamma'(s;t) = \gamma(s;t) + 1$ whenever $s < t$, $f \leq s < k$, and $j \leq t < l$.

This shows that $\gamma < \gamma'$. To see that $\gamma' \leq \tau$, we must show that

- $\gamma(s;-) > \tau(s;-)$ for all $s \in [j, k - 1]$; and
- $\gamma(s;t) < \tau(s;t)$ whenever $s < t$, $f \leq s < k$, and $j \leq t < l$.

The first of these two items has already been assumed. The second follows from Corollary 2.8 and [Inc04, Corollary 4.6]. Indeed, on the level of the underlying involutions, the move from $\gamma$ to $\gamma'$ is the minimal covering transformation of $\gamma$ relative to $\tau$, with $(1,k)$ being a “crossing ef-rise”.

Now, if it is not the case that $\gamma(s;+) > \tau(s;+)$ for all $s \in [j, k - 1]$, then by Lemma 2.14 it is the case that $\gamma(s;-) > \tau(s;-)$ for all $s \in [j, k - 1]$. Thus we instead make the move $1122 \rightarrow 1 + - 1$, and repeat the above argument with signs reversed.

**Case 3.2.2 ($\gamma_k = +$).** We have the following picture:

$$
\begin{array}{cccccccc}
\tau: & 1 & \ldots & \ldots & \ldots & \ldots & 1 & \ldots \\
\gamma: & 1 & \ldots & 1 & \ldots & + & \ldots & \ldots \\
& j & & \gamma & & k & & \\
\end{array}
$$

Note that $(\gamma_1, \gamma_j, \gamma_k)$ form the pattern $11+$. We would like to obtain $\gamma'$ from $\gamma$ by converting this pattern to $1 + 1$. Alas, this does not always work. The $\gamma'$ so obtained satisfies $\gamma < \gamma' \leq \tau$ in some cases, and does not in other cases.

To see this, note how the rank numbers for $\gamma$ differ from those of $\gamma'$:

- $\gamma'(s;-) = \gamma(s;-) - 1$ for $s = j, \ldots, k - 1$.
- $\gamma'(s;t) = \gamma(s;t) + 1$ for $s < t$ with $1 \leq s$, $j \leq t < k$.

Thus it is always the case that $\gamma < \gamma'$. However, it is only true that $\gamma' \leq \tau$ if $\gamma(s;-) > \tau(s;-)$ for all $s \in [j, k - 1]$. If this holds, then we do indeed have that $\gamma' \leq \tau$. (Again, the inequalities $\gamma'(s;t) \leq \tau(s;t)$ follow from Corollary 2.8 and [Inc04, Corollary 4.6], since, on the level of the underlying involutions, the move $11+ \rightarrow 1 + 1$ is the minimal covering transformation of $\gamma$ relative to $\tau$, with $(1,k)$ being an “ef-rise”.)

We need not have $\gamma(s;-) > \tau(s;-)$ for all $s \in [j, k - 1]$, however. In the event that we do not, then we claim that one of the following must hold:

1. There exists $l \in [k + 1, i]$ with $\gamma_l = -$, or
2. There exists a pair of matching natural numbers $\gamma_l = \gamma_m \in \mathbb{N}$ with $k < l < m \leq i$.

To see this, note that there is some $s \in [j, k - 1]$ for which $\gamma(s;-) = \tau(s;-)$, and since $\gamma$ has no $-$ jumps in the range $[s, k]$ (by our choice of $k$), we also have that $\gamma(t;-) = \tau(t;-)$ for $t = s + 1, \ldots, k$. Now, $\tau$ has a $-$ jump at position $i$, since $\tau_i$ is the second occurrence of $\tau_i$. Thus to ensure that $\gamma(i;-) \geq \tau(i;-)$, $\gamma$ must have a $-$ jump somewhere in the range $[k + 1, i]$. This can occur either with a $-$ sign, or with the second occurrence of a natural number. If it occurs at the second occurrence of a natural number, the first occurrence of that number cannot be in the range $[2, j - 1]$, since we are not in Case 3.1 and it cannot be
in the range $[j+1,k]$, by our choice of $k$. Thus the first occurrence of that number must be beyond position $k$.

Now, choose $l$ to be the smallest index in the range $[k+1,i]$ such that either $\gamma_l = -$ or $\gamma_l = \gamma_m \in \mathbb{N}$ with $k < l < m \leq i$.

**Case 3.2.2.1 ($\gamma_l = -$).** The picture is

\[
\begin{align*}
\tau: & \quad 1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad 1 \quad \ldots \\
\gamma: & \quad 1 \quad \ldots \quad 1 \quad \ldots \quad + \quad \ldots \quad - \quad \ldots \quad \ldots \quad \ldots \\
& \quad j \quad \quad k \quad \quad l
\end{align*}
\]

Then $(\gamma_1, \gamma_j, \gamma_l)$ form the pattern $11-$, and we claim that the $\gamma'$ obtained from $\gamma$ by converting this pattern to $11$ satisfies $\gamma < \gamma' \leq \tau$. Note how the rank numbers for $\gamma$ and $\gamma'$ are related:

- $\gamma'(s;+) = \gamma(s;+)-1$ for $s = j, \ldots, l-1$.
- $\gamma'(s;t) = \gamma(s;t)+1$ for $f \leq s$, $j \leq t < l$.

Thus $\gamma < \gamma'$. To see that $\gamma' \leq \tau$, we need to see that

1. $\gamma(s;+) > \tau(s;+)$ for $s = j, \ldots, l-1$, and
2. $\gamma(s;t) < \tau(s;t)$ for $s < t$ with $1 \leq s$ and $j \leq t < l$.

For the first, note that we have already assumed that $\gamma(s;-) = \tau(s;-) $ for some $s \in [j+1,k]$, so by Lemma 2.14 $\gamma(s;+) > \tau(s;+)$ for all $s \in [j,k-1]$. Since $\gamma_k = +$, $\gamma(k;+) > \tau(k;+)$ as well. Now, since $\gamma(s;-) = \tau(s;-)$ and since there are no $-$ jumps in the range $[s,l]$ (by our choices of both $k$ and $l$), the equality $\gamma(t;-) = \tau(t;-)$ is maintained for all $t \in [s,l-1]$. Then by Lemma 2.13 we have $\gamma(t;+) > \tau(t;+)$ in all of these positions as well. (Note that Lemma 2.13 refers to indices in the range $[j,k-1]$, but in fact this restriction on the indices is not necessary. Indeed, the upper bound of the interval for which Lemma 2.13 holds can be taken to be any index less than $i$.)

Unlike several of our other cases, here the inequality $\gamma(s;t) < \tau(s;t)$ does not follow immediately from the results of [Inc04] since the move being described here is not a “minimal covering transformation”. Thus we must argue it directly here. So let $s < t$ be given with $1 \leq s$ and $j \leq t < l$. By our choice of $j$, $k$, $l$, etc., and by virtue of the case that we are currently in, it is clear that all first occurrences in the range $[1,s]$ either have their mate in a position at most $j$, or else strictly beyond $i$. So since $t \geq j$, we have

$$
\gamma(s;t) = \# \{ \gamma_a = \gamma_b \in \mathbb{N} \mid a \leq s, b > t \} = \# \{ \gamma_a = \gamma_b \in \mathbb{N} \mid a \leq s, b > i \} = \gamma(s;i).
$$

As for $\tau(s;t)$, since $t < l \leq i$, we have

$$
\tau(s;t) = \# \{ \tau_a = \tau_b \in \mathbb{N} \mid a \leq s, b > t \} > \# \{ \tau_a = \tau_b \in \mathbb{N} \mid a \leq s, b > i \} = \tau(s;i),
$$

since the pair $\tau_1 = \tau_i$ contributes to $\tau(s;t)$, but not to $\tau(s;i)$. Since

$$
\gamma(s;t) = \gamma(s;i) \leq \tau(s;i) < \tau(s;t),
$$

we are done.
Case 3.2.2.2 ($\gamma_l = \gamma_m$ for $k < l < m \leq i$). Here, the picture is

\[
\begin{array}{cccccccccccccccccccccc}
\tau : & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & i \\
\gamma : & 1 & \ldots & 1 & \ldots & + & \ldots & 2 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\]

\[

t_j \quad k \quad l \quad m
\]

Here, we claim that the $\gamma'$ obtained from $\gamma$ by changing the pattern $\gamma, \gamma_j, \gamma_l, \gamma_m = 1122$ to $1 - - 1$ satisfies $\gamma < \gamma' \leq \tau$. Indeed, the arguments for this are identical to those given in the previous subcase. Recall how the rank numbers for $\gamma$ and $\gamma'$ differ:

- $\gamma'(s; +) = \gamma(s; +) - 1$ for all $s \in [j, l - 1]$;
- $\gamma'(s; t) = \gamma(s; t) + 1$ whenever $s < t$, $1 \leq s < l$, and $j \leq t < m$.

Thus $\gamma < \gamma'$, and to see that $\gamma' \leq \tau$, we simply need to see that $\gamma(s; +) > \tau(s; +)$ for $s \in [j, l - 1]$, and that $\gamma(s; t) < \tau(s; t)$ for $s < t$, $1 \leq s < l$, and $j \leq t < m$. As mentioned, the arguments for these facts are identical to those given in the previous subcase, where $\gamma_l$ was equal to $-$. Since they are identical, we do not repeat them here.

In conclusion, the result of all the subcases considered here can be stated succinctly as follows: A suitable $\gamma'$ can be obtained from $\gamma$ through one of the following moves: $11+ \rightarrow 1 + 1$, $11- \rightarrow 1 - 1$, or $1122 \rightarrow 1 - 1 + 1$.

Case 3.2.3 ($\gamma_k = -$). Here, we repeat the arguments of the previous subcase, reversing all signs. The upshot is that $\gamma'$ can be obtained from $\gamma$ by one of the moves $11- \rightarrow 1 - 1$, $11+ \rightarrow 1 + 1$, or $1122 \rightarrow 1 + 1 - 1$.

Case 3.3 (None of the previous cases apply.). Then by Lemma 2.11, there exists a pair of matching natural numbers $\gamma_k = \gamma_j \in \mathbb{N}$ with $k \in [j + 1, i]$ and $l > i$. Choose the pair such that $l$ is minimal — that is, choose $l$ to be the smallest index greater than $i$ which is the second occurrence of a natural number whose first occurrence is between indices $j + 1$ and $i$. We have the following schematic:

\[
\begin{array}{cccccccccccccccccccccc}
\tau : & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & i \\
\gamma : & 1 & \ldots & 1 & \ldots & 2 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\]

\[

t_j \quad k \quad l
\]

Note that the indices $\gamma_1, \gamma_j, \gamma_k, \gamma_l$ form the pattern 1122. We claim that the $\gamma'$ obtained from $\gamma$ by converting this pattern to 1212 (i.e. by interchanging $\gamma_j$ and $\gamma_k$) satisfies $\gamma < \gamma' \leq \tau$. To see that $\gamma < \gamma'$, note how the rank numbers for $\gamma$ and $\gamma'$ are related. The differences are:

- $\gamma'(s; +) = \gamma(s; +) - 1$ for $s = j, \ldots, k - 1$;
- $\gamma'(s; -) = \gamma(s; -) - 1$ for $s = j, \ldots, k - 1$;
- $\gamma'(s; t) = \gamma(s; t) + 1$ for $f \leq s < j < t < k$;
- $\gamma'(s; t) = \gamma(s; t) + 1$ for $j \leq s < k \leq t < l$;
- $\gamma'(s; t) = \gamma(s; t) + 2$ for $j \leq s < k < t < k$.

This shows that $\gamma < \gamma'$. To see that $\gamma' \leq \tau$, we must show that

- $\gamma(s; \pm) > \tau(s; \pm)$ for $s = j, \ldots, k - 1$;
- $\gamma(s; t) > \tau(s; t)$ for $f \leq s < j \leq t < k$;
- $\gamma(s; t) < \tau(s; t)$ for $j \leq s < k \leq t < l$;
\begin{itemize}
  \item $\gamma(s; t) + 1 < \tau(s; t)$ for $j \leq s < t < k$.
\end{itemize}

The last three items above follow from Corollary 2.8 and [Inc04, Corollary 4.6]. Indeed, on the level of underlying involutions, the move from $\gamma$ to $\gamma'$ is the minimal covering transformation of $\gamma$ relative to $\tau$, with $(1, l)$ being an “ed-rise”.

So we concern ourselves only with the first item. Note that, by virtue of the case we currently have information of $\gamma$ relative to $\tau$, the move from $\gamma$ to $\gamma'$ is the minimal covering transformation of $\gamma$ relative to $\tau$, with $(1, l)$ being an “ed-rise”.

This completes the proof of Theorem 2.9.

With Theorem 2.9 in hand, we can now prove the following

**Proposition 2.15.** Let $\gamma$, $\tau$ be $(p, q)$-clans, with $Q_\gamma, Q_\tau$ the corresponding K-orbits, and $Y_\gamma, Y_\tau$ the corresponding K-orbit closures. Then $Y_\gamma \subseteq Y_\tau$ if and only if $\gamma \leq \tau$.

**Proof.** We show first that $\gamma \nleq \tau \Rightarrow Y_\gamma \nsubseteq Y_\tau$. Let $Y_\gamma$ be the purported orbit closure described in Theorem 2.9. defined by inequalities determined by the rank numbers of $\tau$. $Y_\gamma$ is a closed subvariety of $G/B$, being defined locally by the vanishing of certain minors. Indeed, conditions (1) and (2) of Theorem 2.9 amount to the vanishing of lower-left $i \times q$ minors and upper-left $i \times p$ minors, respectively, of a generic matrix whose non-specialized entries give affine coordinates on a translated big cell. Condition (3) amounts to the vanishing of certain minors of a matrix whose first $i$ columns are the first $i$ columns of this generic matrix with the lower-left $i \times p$ submatrix zeroed out, and whose last $j$ columns are the first $j$ columns of the generic matrix. Since $Q_\tau \subseteq Y_\tau$ by Theorem 2.4, we clearly have that $Y_\gamma \subseteq Y_\tau$. Thus to show that $Y_\gamma \nsubseteq Y_\tau$, it suffices to show that $Q_\gamma \cap Y_\tau = \emptyset$. This is clear from the definitions, since if $\gamma \nleq \tau$, there exists some $i$ such that $\gamma(i; +) < \tau(i; +)$, or $\gamma(i; -) < \tau(i; -)$, or some $i < j$ such that $\gamma(i; j) > \tau(i; j)$. Since any point of $Q_\gamma$ must meet the description of Theorem 2.4, it cannot possibly lie in $Y_\tau$.

The preceding argument establishes that $Y_\gamma \subseteq Y_\tau \Rightarrow \gamma \leq \tau$. We now consider the converse. Suppose that $\gamma < \tau$. It suffices to consider the cases where $\tau$ covers $\gamma$, so we may assume that $\gamma$ and $\tau$ are related by a “move” of the type described in Theorem 2.9. For each of those possible moves, we give the following sort of argument: We take representatives $F_\bullet \in Q_\tau$ and $E_\bullet \in Q_\gamma$, provided by the algorithm of [Yam97] described in Subsection 2.1. We also take, for each $t \in \mathbb{C}^*$, a matrix $k(t) \in K$, so that the flag $F_\bullet(t) := k(t) \cdot F_\bullet$ is in $Q_\gamma$ for all $t \in \mathbb{C}^*$. We then show that $\lim_{t \to 0} F_\bullet(t) = F_\bullet$. This shows that $E_\bullet$ is a limit point for $Q_\gamma$, so that it is an element of $Y_\tau$. Now, any other point $P_\bullet \in Q_\gamma$ is of the form $P_\bullet = k \cdot E_\bullet$ for some $k \in K$. Moreover, the curve $k \cdot F_\bullet(t)$ is contained in $Q_\gamma$, and tends to $P_\bullet$ as $t \to 0$. So this argument establishes that in fact $Q_\gamma \subseteq Y_\tau$, which implies that $Y_\gamma \subseteq Y_\tau$.

For ease of notation, when writing the flags $E_\bullet = \langle v_1, \ldots, v_n \rangle$ and $F_\bullet = \langle w_1, \ldots, w_n \rangle$, we may indicate only those $v_i$ and $w_j$ which differ. For instance, if $E_\bullet = \langle e_1, e_2, e_3, e_4, e_5 \rangle$ and $F_\bullet = \langle e_1, e_2 + e_3, e_4, e_5 \rangle$, then for short we will write $E_\bullet = \langle e_2 \rangle$ and $F_\bullet = \langle e_2 + e_3 \rangle$.

**Case 1** ($\pm \to 11$). Suppose that the + and − are in positions $i < j$. We may choose the representatives $E_\bullet = \langle v_1, \ldots, v_n \rangle$ and $F_\bullet = \langle w_1, \ldots, w_n \rangle$ so that $v_i = e_1$, $v_j = e_n$,
$w_i = e_1 + e_n$, $w_j = e_n$, and $v_l = w_l$ for all remaining $l$. Now, for $t \in \mathbb{C}^*$, let $k(t) \in K$ be the diagonal matrix

$$k(t) = \text{diag}(1/t, 1, 1, \ldots, 1).$$

Then

$$F_*(t) := k(t) \cdot F_* = \langle (1/t)e_1 + e_n \rangle = \langle e_1 + te_n \rangle.$$

(To obtain the last equality, we have simply scaled the $i$th vector for $F_*(t)$ by a factor of $t$.) From this, it is clear that $\lim_{t \to 0} F_*(t) = E_*$. 

**Case 2** ($-+ \to 11$). This case is extremely similar to the previous case. We omit the details.

**Case 3** ($11+ \to 1 + 1$). Suppose the $11+$ and $1 + 1$ occur in positions $i < j < k$. We may choose $E_* = \langle v_1, \ldots, v_n \rangle$ so that $v_i = e_1 + e_n$, $v_j = e_1$, and $v_k = e_2$. We write

$$E_* = \langle e_1 + e_n, e_1, e_2 \rangle = \langle e_1 + e_n, e_1, 2e_1 + e_2 + e_n \rangle,$$

where we have simply replaced $v_k$ by $v_i + v_j + v_k$, which does not change the point $E_*$. Likewise, we may choose $F_* = \langle w_1, \ldots, w_n \rangle$ so that $w_i = e_1 + e_n$, $w_j = e_2$, and $w_k = e_1$. Then

$$F_* = \langle e_1 + e_n, e_2, e_1 \rangle = \langle e_1 + e_n, e_2, 2e_1 + e_2 + e_n \rangle.$$

Finally, choose $k(t) \in K$ to be the matrix with $1$'s on the diagonal, $1/t$ in entry $(1, 2)$, and $0$'s elsewhere. Then

$$F_*(t) = k(t) \cdot F_* = \langle e_1 + e_n, (1/t)e_1 + e_2, (2 + (1/t))e_1 + e_2 + e_n \rangle = \langle e_1 + e_n, (1/t)e_1 + e_2, 2e_1 + e_2 + e_n \rangle.$$

(Note that to obtain the equality (*), we simply replace $(2 + (1/t))e_1 + e_2 + e_n$ by

$$(1 - (1/t))(2 + (1/t))e_1 + e_2 + e_n + (1/t)(1/t)e_1 + e_2 + (1/t)(e_1 + e_n),$$

which does not change the flag $F_*(t)$.)

By the last description of $F_*(t)$ given above, we see that $E_* = \lim_{t \to 0} F_*(t)$. 

**Case 4** ($+11 \to 1 + 1$). Suppose that $+11$ and $1 + 1$ occur in positions $i < j < k$. We may choose $E_* = \langle v_1, \ldots, v_n \rangle$ so that $v_i = e_1$, $v_j = e_2 + e_n$, and $v_k = e_n$. We write

$$E_* = \langle e_1, e_2 + e_n, e_n \rangle = \langle e_1, e_1 + e_2 + e_n, e_n \rangle.$$

We may choose $F_* = \langle w_1, \ldots, w_n \rangle$ so that $w_i = e_1 + e_n$, $w_j = e_2$, and $w_k = e_n$. Then

$$F_* = \langle e_1 + e_n, e_2, e_n \rangle = \langle e_1 + e_n, e_1 + e_2 + e_n, e_n \rangle.$$

Finally, choose $k(t) \in K$ to be the matrix with $1/t$ in entry $(1, 1)$, $1$'s on all of the other diagonal entries, $1 - (1/t)$ in entry $(1, 2)$, and $0$'s elsewhere. Then

$$F_*(t) = \langle (1/t)e_1 + e_n, e_1 + e_2 + e_n, e_n \rangle = \langle e_1 + te_n, e_1 + e_2 + e_n, e_n \rangle,$$

from which it is clear that $E_* = \lim_{t \to 0} F_*$. 

**Case 5** ($11- \to 1 - 1$, $-11 \to 1 - 1$). These cases are extremely similar to the previous two cases, so we omit the details.
\textbf{Case 6} (1122 → 1212). Suppose that the 1122 and 1212 patterns occur in positions $i < j < k < l$. We choose the flag $E_\bullet = \langle v_1, \ldots, v_n \rangle$ so that $v_i = e_1 + e_{n-1}$, $v_j = e_2 + e_n$, and $v_l = e_n$. We choose the flag $F_\bullet = \langle w_1, \ldots, w_n \rangle$ so that $w_i = e_1 + e_{n-1}$, $w_j = e_2 + e_n$, $w_k = e_{n-1}$, and $w_l = e_n$. Let $k(t) \in K$ be the matrix with 1’s on the diagonal, $1/t$ in position $(n-1, n)$, and 0’s elsewhere. Then
\[
F_\bullet(t) = (e_1 + e_{n-1}, e_2 + (1/t)e_{n-1} + e_n, e_{n-1}, (1/t)e_{n-1} + e_n) = \ast
\]
\[
(e_1 + e_{n-1}, e_2 + (1/t)e_{n-1} + e_n, e_{n-1}, e_n) = \ast\ast
\]
\[
(e_1 + e_{n-1}, e_2 + (1/t)e_{n-1} + e_n, e_2 + e_n, e_n) = \ast\ast\ast
\]
\[
note{\text{Note that the equality (\ast) is obtained by replacing } (1/t)e_{n-1} + e_n \text{ by } ((1/t)e_{n-1} + e_n) - \frac{1}{t}(e_{n-1}), \text{ while the equality (\ast\ast) is obtained by replacing } e_{n-1} \text{ by } (-1/t)(e_{n-1}) + (e_2 + (1/t)e_{n-1} + e_n), \text{ neither of which changes the flag } F_\bullet(t). \text{ By the last description of } F_\bullet(t) \text{ given above, we see that } F_\bullet = \lim_{t \to 0} F_\bullet(t).}
\]

\textbf{Case 7} (1122 → 1+−1). Suppose that the 1122 and 1+−1 occur in positions $i < j < k < l$. Choose the representative $E_\bullet = \langle v_1, \ldots, v_n \rangle$ so that $v_i = e_1 + e_{n-1}$, $v_j = e_1 + e_{n-1}$, $v_k = e_2 + e_n$, and $v_l = e_n$. Choose the flag $F_\bullet = \langle w_1, \ldots, w_n \rangle$ so that $w_i = e_1 + e_n$, $w_j = e_1 + e_{n-1}$, $w_k = e_2$, and $w_l = e_n$. We may rewrite $F_\bullet = \langle w_1, w_2, w_3, w_4 \rangle$ as $\langle e_1 + e_n, e_{n-1}, e_1 + e_2 + e_n, e_n \rangle$.

Let $k(t) \in K$ be the matrix whose upper-left 2 × 2 corner is
\[
\begin{pmatrix}
\frac{1}{t} & -1/t \\
1 & 0
\end{pmatrix},
\]
whose lower-right 2 × 2 corner is
\[
\begin{pmatrix}
-1/t & 1/t \\
0 & 1
\end{pmatrix},
\]
and which has 1’s in all other diagonal entries, and 0’s elsewhere. Then
\[
F_\bullet(t) = k(t) \cdot F_\bullet = ((1/t)e_1 + e_2 + (1/t)e_{n-1} + e_n, e_2 + (1/t)e_{n-1} + e_n, e_2 + e_n, (1/t)e_{n-1} + e_n) = \ast
\]
\[
((1/t)e_1 + e_2 + (1/t)e_{n-1} + e_n, e_2 + (1/t)e_{n-1} + e_n, e_2 + e_n, e_n) = \ast\ast
\]
\[
note{\text{where the equality (\ast) is obtained by replacing } (1/t)e_{n-1} + e_n \text{ by } (-1)(e_2 + (1/t)e_{n-1} + e_n).}}
\]
From the last description of $F_\bullet(t)$ above, we see that $\lim_{t \to 0} F_\bullet(t) = E_\bullet$.

\textbf{Case 8} (1122 → 1−+1). This case is extremely similar to the last one, except a bit simpler, so we omit the details.

\textbf{Case 9} (1212 → 1221). Suppose that the 1212 and 1221 occur in positions $i < j < k < l$. Choose the representative $E_\bullet = \langle v_1, \ldots, v_n \rangle$ so that $v_i = e_1 + e_{n-1}$, $v_j = e_2 + e_n$, $v_k = e_{n-1}$, and $v_l = e_n$. Choose the flag $F_\bullet = \langle w_1, \ldots, w_n \rangle$ so that $w_i = e_1 + e_n$, $w_j = e_2 + e_{n-1}$, $w_k = e_{n-1}$, and $w_l = e_n$. We may rewrite $F_\bullet = \langle w_1, w_2, w_3, w_4 \rangle$ as $\langle e_1 + e_n, e_1 + e_2 + e_{n-1} + e_n, e_{n-1}, e_n \rangle$.

Let $k(t) \in K$ be the same matrix described in the case 1122 → 1+−1. Then
\[
F_\bullet(t) = k(t) \cdot F_\bullet = ((1/t)e_1 + e_2 + (1/t)e_{n-1} + e_n, e_2 + e_n, (-1/t)e_{n-1}, (1/t)e_{n-1} + e_n) =
\]
\[(1/t)e_1 + e_2 + (1/t)e_{n-1} + e_n, e_2 + e_n, e_{n-1}, e_n) = (e_1 + e_{n-1} + t(e_2 + e_n), e_2 + e_n, e_{n-1}, e_n).\]

From the last description of $F_\bullet(t)$, we see that $\lim_{t \to 0} F_\bullet(t) = E_\bullet$. \hfill \Box

**Proof of Theorem 2.5.** Given a clan $\gamma$, let $\mathcal{Y}_\gamma$ denote the purported closure of $Q_\gamma$, described in the statement of Theorem 2.5, and let $Y_\gamma = \overline{Q_\gamma}$ be the true closure. Clearly, $Y_\gamma$ is $K$-stable, and hence is a union of $K$-orbits. Namely, it is the union of all $K$-orbits $Q_\gamma$ which are contained in $Y_\gamma$. Thus

\[Y_\gamma = \bigcup_{Q_\gamma \subseteq Y_\gamma} Q_\gamma = \bigcup_{\gamma \leq \tau} Q_\gamma.\]

Suppose that $F_\bullet \in Q_\gamma$ for some $\gamma \leq \tau$. Then, by Theorem 2.4,

- $\dim(F_i \cap E_p) = \gamma(i; +) \geq \tau(i; +)$ for $i = 1, \ldots, n$,
- $\dim(F_i \cap E_q) = \gamma(i; -) \geq \tau(i; -)$ for $i = 1, \ldots, n$,
- $\dim(\pi(F_i) + F_j) = j + \gamma(i; j) \leq j + \tau(i; j)$ for $1 \leq i < j \leq n$.

Thus $Y_\gamma \subseteq \mathcal{Y}_\tau$.

For the other inclusion, a flag $F_\bullet \in \mathcal{Y}_\tau$ is clearly in some $K$-orbit $Q_\gamma$, and since

- $\gamma(i; +) = \dim(F_i \cap E_p) \geq \tau(i; +)$ for $i = 1, \ldots, n$;
- $\gamma(i; -) = \dim(F_i \cap E_q) \geq \tau(i; -)$ for $i = 1, \ldots, n$;
- $j + \gamma(i; j) = \dim(\pi(F_i) + F_j) \leq j + \tau(i; j)$ for $1 \leq i < j \leq n$,

we have $\gamma \leq \tau$. Thus $F_\bullet \in Y_\gamma$. We conclude that $\mathcal{Y}_\tau = Y_\tau$. \hfill \Box

**2.3. Orbit Parametrizations in Other Cases.** For the other symmetric pairs $(G, K)$ considered in this paper (those in types $BCD$), it turns out that $K$ can be realized as $G \cap K'$ with $K' \cong GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \subseteq G' = GL(p + q, \mathbb{C})$ for some $p$ and $q$. The flag variety $X$ for $G$ naturally embeds in the flag variety $X'$ for $G'$, and so the intersection of a $K'$-orbit on $X'$ with $X$, if non-empty, is stable under $K$ and hence is a priori a union of $K$-orbits.

In general, such an intersection need not be a single $K$-orbit. Indeed, it could either be a single $K$-orbit or a union of 2 $K$-orbits, and this depends critically on the chosen representative of the isogeny class of $G$, which in turn can affect the connectedness of $K$. In the cases we consider, we choose $G$ (and the corresponding $K$) so that the intersection of a $K'$-orbit on $X'$ with $X$ is always a single $K$-orbit.

The upshot is that in each of our cases outside of type $A$, the set of $K$-orbits can be parametrized by a subset of the $(p, q)$-clans (for the appropriate $p, q$) possessing some additional combinatorial properties which amount to the corresponding $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-orbits on $X'$ meeting the smaller flag variety $X$ non-trivially. These combinatorial properties always involve one of the following two symmetry conditions.

**Definition 2.16.** We say that $\gamma = c_1 \ldots c_n$ is **symmetric** if the clan $c_n \ldots c_1$ obtained from $\gamma$ by reversing its characters is equal to $\gamma$ as a clan. Specifically, we require

1. If $c_i$ is a sign, then $c_{n+1-i}$ is the same sign.
2. If $c_i$ is a number, then $c_{n+1-i}$ is also a number, and if $c_{n+1-i} = c_j$, then $c_{n+1-j} = c_i$. 

Definition 2.17. We say that \( \gamma = c_1 \ldots c_n \) is skew-symmetric if the clan \( c_n \ldots c_1 \) is the “negative” of \( \gamma \), meaning it is the same clan, except with all signs changed. Specifically,

1. If \( c_i \) is a sign, then \( c_{n+1-i} \) is the opposite sign.
2. If \( c_i \) is a number, then \( c_{n+1-i} \) is also a number, and if \( c_{n+1-i} = c_j \), then \( c_{n+1-j} = c_i \).

Note that condition (2) of each of the above definitions allows for the possibility that \( c_i = c_{n+1-i} \). However, this is not necessary for a clan to be symmetric or skew-symmetric. Indeed, the \( (2,2) \)-clan 1212 is symmetric (and also skew-symmetric), since its reverse 2121 is the same clan, but there are no matching natural numbers in positions \( (i, n+1 - i) \) for any \( i \).

The \( K \)-orbits in our remaining examples are parametrized as follows. Note that the numbering here starts from (2) in order to make this list correspond to that given in the introduction, as well as in the statement of Theorem 3.7 in the form of Table 1.

1. \((SO(2n + 1, \mathbb{C}), S(O(2p, \mathbb{C}) \times O(2q + 1, \mathbb{C})))\): Symmetric \((2p, 2q + 1)-\)clans;
2. \((Sp(2n, \mathbb{C}), Sp(2p, \mathbb{C}) \times Sp(2q, \mathbb{C}))\): Symmetric \((2p, 2q)-\)clans \( \gamma = c_1 \ldots c_{2n} \) such that \( c_i \neq c_{2n+1-i} \) whenever \( c_i \in \mathbb{N} \);
3. \((SO(2n, \mathbb{C}), GL(n, \mathbb{C}))\): Skew-symmetric \((n, n)-\)clans;
4. \((SO(2n, \mathbb{C}), S(O(2p, \mathbb{C}) \times O(2q, \mathbb{C})))\): Symmetric \((2p, 2q)-\)clans;
5. \((SO(2n, \mathbb{C}), GL(n, \mathbb{C}))\): Skew-symmetric \((n, n)-\)clans \( \gamma = c_1 \ldots c_{2n} \) such that \( c_i \neq c_{2n+1-i} \) whenever \( c_i \in \mathbb{N} \), and such that \( \gamma(n; -) \) is even (i.e. among \( c_1 \ldots c_n \), the total number of – signs and pairs of equal natural numbers is even);
6. \((SO(2n, \mathbb{C}), S(O(2p + 1, \mathbb{C}) \times O(2q - 1, \mathbb{C})))\): Symmetric \((2p + 1, 2q-1)-\)clans.

These parametrizations, along with the corresponding weak orders, are given in [MO90]. We remark that no proofs of the correctness of the parametrizations are given in [MO90]. Some details are provided in [Yam97] for two of these cases. Proofs of the correctness of all of these parametrizations are given in [Wys12b, Appendix A].

We also remark that in cases (2), (5), and (7) above, the group \( K \) is disconnected, having two components in each case. Thus some of the \( K \)-orbits are disconnected, having two components. In particular, in cases (2) and (5) above, all of the closed orbits have two components. (As it turns out, in case (7), all of the closed orbits are connected.) The torus \( S \) of \( K \) (which coincides with the torus \( T \) of \( G \) in cases (2) and (5)) is contained in the identity component \( K^0 \) of \( K \), so each of the two components of each closed orbit, being stable under \( K^0 \), is stable under \( S \) and hence has an \( S \)-equivariant class. In these cases, we determine a formula for each component of each closed orbit using the techniques described in Section 1. We then add the formulas for the two components to get formulas for the \( S \)-equivariant classes of the closed \( K \)-orbits.

Since each \( K \)-orbit in all of these cases is the intersection of a \( K' \)-orbit on \( X' \) with \( X \), it is clear that the linear algebraic description of a \( K \)-orbit is the same as that given for the corresponding \( GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \)-orbit, and that we simply restrict our attention to flags meeting this description which lie in \( X \), which are those flags which are isotropic or Lagrangian with respect to the form which defines the group \( G \). (In type \( D \), we should restrict our attention to isotropic flags in the appropriate “family”, i.e. in the appropriate \( SO(2n, \mathbb{C}) \)-orbit on the variety of all isotropic flags.) To give a useful characterization of the \( K \)-orbit closures as universal degeneracy loci, we again would like to describe the orbit closures as sets of flags. The obvious hope is that each \( K \)-orbit closure is the intersection
of the corresponding $K'$-orbit closure with $X$, so that a set-theoretic description of a $K$-orbit closure would be given by Theorem 2.5 again restricting our attention to isotropic or Lagrangian flags.

Let $Q$ be a $K$-orbit, with $Q = X \cap Q'$ for $Q'$ a corresponding $K'$-orbit on $X'$. Our hope is that $\overline{Q} = X \cap \overline{Q}'$. It is clear on general topological grounds that $\overline{Q}$ is at least contained in $X \cap \overline{Q}'$. However, it is not obvious that the opposite containment should hold. Combinatorially, the possible issue can be framed as follows: The set of $K' \setminus X$ of $K$-orbits is essentially a subset of $K' \setminus X'$. There are two possible partial orders one can put on the set $K \setminus X$. On one hand, there is the Bruhat order, corresponding to containment of $K$. On the other hand, there is the order on $K \setminus X$ induced by the Bruhat order on $K' \setminus X'$. The question is then whether these two partial orders on $K \setminus X$ coincide.

This seems to be a somewhat subtle question. As explained in [MT09] (using results of [RS90]), given an explicit understanding of the weak order on $K \setminus X$, one can compute the full Bruhat order explicitly using a simple recursive algorithm. Since [MO90] describes the weak order explicitly in all of our examples, one can use a computer to calculate the Bruhat order on $K \setminus X$ and compare it to the order induced by the Bruhat order on $K' \setminus X'$, then test whether these partial orders coincide.

The results of experiments of this type lead the author to make the following conjecture.

**Conjecture 2.18.** In cases (2)-(4) above, the Bruhat order on $K$-orbits coincides with the induced Bruhat order on the appropriate set of clans. Thus for any $K$-orbit $Q = X \cap Q'$ for $Q'$ the corresponding $K'$-orbit on $X'$, we have that $\overline{Q} = X \cap \overline{Q}'$. In particular, the description of $\overline{Q}$ as a set of flags is given by Theorem 2.5, and we simply restrict our attention to the set of flags meeting this description which lie in $X$ — namely, isotropic flags in the type $B$ case, or Lagrangian flags in the type $C$ cases.

Conjecture 2.18 has been verified for each of cases (2)-(4) through rank 7.

The author has further determined by computer experiment that in fact the analogous conjecture does not hold in any of the type $D$ cases.

**Fact.** In each of our examples in type $D$, the Bruhat order on $K \setminus X$ is strictly weaker than the order induced by the Bruhat order on $K' \setminus X'$. Thus for a general $K$-orbit $Q = X \cap Q'$, $\overline{Q}$ is contained in, but is not equal to, $X \cap \overline{Q}'$.

We give the following examples:

- In the case $(G, K) = (SO(8, \mathbb{C}), GL(4, \mathbb{C}))$, the clans $1 + 12 + 2$ and $12341234$ are related in the Bruhat order on $K' \setminus X'$ (where $K' = GL(4, \mathbb{C}) \times GL(4, \mathbb{C})$), but are not related in the Bruhat order on $K \setminus X$.
- In the case $(G, K) = (SO(8, \mathbb{C}), S(O(4, \mathbb{C} \times O(4, \mathbb{C}))$, the clans $+ - 122 - +$ and $- + 122 - +$ are related in the Bruhat order on $K' \setminus X'$ (with $K' = GL(4, \mathbb{C}) \times GL(4, \mathbb{C})$), but are not related in the Bruhat order on $K \setminus X$.
- In the case $(G, K) = (SO(8, \mathbb{C}), S(O(5, \mathbb{C} \times O(3, \mathbb{C}))$, the clans $+12323$ and $+123123$ are related in the Bruhat order on $K' \setminus X'$ (with $K' = GL(5, \mathbb{C}) \times GL(3, \mathbb{C})$), but are not related in the Bruhat order on $K \setminus X$. 
3. Formulas for the Equivariant Classes of Closed Orbits

In this section, we give formulas for the equivariant classes of closed $K$-orbits in our various examples. In each case, the method of proof is as follows. Combining the results of Subsection 1.3 (namely Corollary 1.10 in all cases but one) with the information on orbit parametrizations given in Section 2, we describe the closed orbits and the torus-fixed points contained in each. Then, using Proposition 1.5, we compute the restriction of the class of each closed orbit to each of the torus-fixed points. Finally, employing Proposition 1.4, we verify that our putative formulas localize as required, which verifies their correctness.

We start by summarizing our results in tabular format, so that they are all in one place and easily accessible, as opposed to being sprinkled over several subsections. For the formulas in the table to make sense, we must first define several notations which appear in them.

**Definition 3.1.** Given any permutation $w \in S_{p+q}$, denote by $l_p(w)$ the number

$$l_p(w) := \# \{(i, j) \mid 1 \leq i < j \leq n, w(j) \leq p < w(i)\}.$$

**Definition 3.2.** Given any signed permutation $w$ of $\{1, \ldots, n\}$, define the (ordinary) permutation $|w|$ by $|w|(i) = |w(i)|$. For example $|24135| = 24135$.

**Definition 3.3.** Given any signed permutation $w$ of $\{1, \ldots, n\}$, and $p < n$, define

$$\phi_p(w) = \# \{i \in \{1, \ldots, n\} \mid w(i) < 0, |w(i)| \leq p\}.$$

For example, if $n = 5, p = 3, w = \overline{24135}$, then $\phi_3(w) = 1$.

We remark that the function $\phi_p$ does not appear in Table 1 but it is in fact referred to and used in the proof of Theorem 3.7.

**Definition 3.4.** For any signed permutation $w$ of $n$ elements, define the set

$$\text{Neg}(w) := \{i \mid w(i) < 0\}.$$

Define $\mathbb{N}$-valued functions $\psi, \sigma$ on the set of such permutations by

$$\psi(w) = \#\text{Neg}(w),$$

and

$$\sigma(w) = \sum_{i \in \text{Neg}(w)} (n - i).$$

**Definition 3.5.** Given a signed permutation $w$ of $n$ elements, denote by $\Delta_n(x, y, w)$ the polynomial

$$\Delta_n(x, y, w) := \det(c_{n+1+j-2i}),$$

where

$$c_k = e_k(x_{w^{-1}(1)}, \ldots, x_{w^{-1}(n)}) + e_k(y_1, \ldots, y_n).$$

Here, $e_k$ denotes the $k$th elementary symmetric function in the inputs, and $x_{w^{-1}(i)}$ means $x_{w^{-1}(i)}$ if $w^{-1}(i) > 0$, and $-x_{|w^{-1}(i)|}$ if $w^{-1}(i) < 0$. 


**Definition 3.6.** Given a signed permutation $w$ of $n$ elements, and any $p < n$, denote by

$$I_w := \{ i \in \{1, \ldots, n - 1\} \mid w(i) > p + 1 \}.$$ 

For each $i \in I_w$, define

$$C(i) := \#\{ j \mid i < j \leq n - 1, w(j) \leq p \}.$$ 

Finally, define an $\mathbb{N}$-valued function $\tau$ on such signed permutations by

$$\tau(w) := \sum_{i \in I_w} C(i).$$

**Theorem 3.7.** The formulas for the equivariant classes of closed $K$-orbits in all of our cases are given in Table 1 of the appendix. In all cases, $n = p + q$.

**Proof.** We give proofs of our formulas for some of the cases, omitting others which are very similar.

**Case (1).** Here, we realize $K$ as $G^\theta$ for $\theta = \text{int}(I_{p,q})$, where $\text{int}(g)$ denotes the inner automorphism “conjugation by $g$”. (Recall that the notation $I_{p,q}$ was defined in Section 1.1.) Then

$$K = \left\{ \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \in GL(n, \mathbb{C}) \mid K_{11} \in GL(p, \mathbb{C}), K_{22} \in GL(q, \mathbb{C}) \right\} \cong GL(p, \mathbb{C}) \times GL(q, \mathbb{C}).$$

We choose the standard positive system for $G$,

$$\Phi^+ = \{ X_i - X_j \mid i < j \},$$

where $X_i$ denote coordinate functions on the maximal torus $T$ of $G$. We choose $B$ to be the Borel subgroup of $G$ corresponding to the negative roots $\Phi^- = -\Phi^+$. Concretely, one can take $T$ to be the diagonal matrices, with the $X_i$ being the obvious coordinate functions, and take $B$ to be the Borel subgroup of lower-triangular matrices.

It is easy to check that $S = K \cap T$ is a maximal torus of $K$, and that $B_K = K \cap B$ is a Borel subgroup of $K$. (In fact, $S = T$, which in turns implies that any Borel subgroup of $G$ intersects $K$ in a Borel subgroup of $K$, see [RS90]. However, we generally prefer to think of $S$ as a separate torus, with separate coordinate functions $Y_i$, because this makes the notation of our proofs less confusing.) The roots of $K$ are

$$\Phi_K = \{ Y_i - Y_j \mid 1 \leq i, j \leq p \} \cup \{ Y_i - Y_j \mid p + 1 \leq i, j \leq n \}.$$ 

Let $Q = K \cdot wB$ be a closed orbit. This corresponds to a $(p, q)$-clan $\gamma$ consisting only of signs. Via the algorithm of [Yam97] (described in Subsection 2.1) used for obtaining representatives of the orbit corresponding to $\gamma$, it is easy to see that the $S$-fixed points contained in this orbit are those corresponding to permutations whose one-line notation has $\{1, \ldots, p\}$ occurring at the positions of the $+$’s, and $\{p + 1, \ldots, n\}$ occurring at the positions of the $-$’s. Thus $w$ should be taken to be any such permutation.

First, we observe that the formula given is independent of the choice of $w$. Indeed, any other $S$-fixed point $\bar{w} \in Q$ is of the form $\bar{w} = \sigma w$ for some $\sigma \in W_K = S_p \times S_q$. Since $\sigma$ preserves the sets $\{1, \ldots, p\}$ and $\{p + 1, \ldots, n\}$, we see that

$$(\sigma w)(j) \leq p < (\sigma w)(i) \iff w(j) \leq p < w(i),$$

and so \( l_p(w) = l_p(w) \). Further, the set \( \{ w^{-1}(i) \mid i \leq p \} \) (that is, the set of indices on the \( x \)'s in our proposed formula) is clearly the same as \( \{ \tilde{w}^{-1}(i) \mid i \leq p \} = \{ (w^{-1}\sigma^{-1})(i) \mid i \leq p \} \), again because \( \sigma^{-1} \) permutes those \( i \) which are less than or equal to \( p \).

With that established, we now use Proposition 1.5 to identify the restriction of \( [Q] \) at each \( S \)-fixed point. The set \( \rho(w\Phi^+) \) is

\[
\rho(\{ wa \mid a \in \Phi^+ \}) = \{ Y_{w(i)} - Y_{w(j)} \mid i < j \}.
\]

(Note that we are in an equal rank case, so \( \rho \) is simply the map sending the coordinate \( X_i \) on the torus \( t \) to the coordinate \( Y_i \) on the torus \( s = t \).) Throwing out roots of \( K \), we are left with precisely one of \( \pm(Y_i - Y_j) \) for each \( i, j \) with \( i \leq p < j \). The number of remaining roots of the form \( -(Y_i - Y_j) \) is precisely \( l_p(w) \).

Thus

\[
[Q]_w = F(Y) := (-1)^{l_p(w)} \prod_{i \leq p < j} (Y_i - Y_j).
\]

(Note that because \( l_p \) is constant on cosets \( W_Kw \), the restriction \( [Q]_w \) is actually the same at every \( S \)-fixed point \( w \in Q \).)

So for any \( u \in W \),

\[
[Q]_u = \begin{cases} 
F(Y) & \text{if } u \in Q, \\
0 & \text{otherwise.}
\end{cases}
\]

Recalling the precise definition of the restriction maps \( i_u^* \) given in Proposition 1.4, we see that we are looking for a polynomial \( p \) in the \( x \)'s and \( y \)'s such that \( p(\sigma w(Y), Y) = F(Y) \) for any \( \sigma \in W_K \), and such that \( p(w'Y, Y) = 0 \) for any \( w' \in W \) such that \( w'w^{-1} \notin W_K \).

It is straightforward to check that \( P \) has these properties. Indeed, for \( \sigma \in W_K \), we see that

\[
P(\sigma w(Y), Y) = (-1)^{l_p(w)} \prod_{i \leq p < j} (Y_{\sigma(i)} - Y_j),
\]

and since \( \sigma \) permutes \( \{1, \ldots, p\} \), this is precisely \( F(Y) \).

On the other hand, given \( w' \) with \( w'w^{-1} \notin W_K \),

\[
P(w'(Y), Y) = (-1)^{l_p(w)} \prod_{i \leq p < j} (Y_{w'w^{-1}(i)} - Y_j) = 0,
\]

since \( w'w^{-1} \), not being an element of \( W_K \), necessarily sends some \( i \leq p \) to some \( j > p \). We conclude that \( P(x, y) \) represents \([Q]\). \( \square \)

**Case (2).** We realize the odd orthogonal group \( SO(2n + 1, \mathbb{C}) \) as the subgroup of \( SL(2n + 1, \mathbb{C}) \) preserving the orthogonal form given by the antidiagonal matrix \( J = J_{2n+1} \). That is,

\[
SO(2n + 1, \mathbb{C}) = \{ g \in SL(2n + 1, \mathbb{C}) \mid gJg^t = J \}.
\]

Fix a maximal torus \( T \) of \( G \), and let \( X_i \) (\( i = 1, \ldots, n \)) denote coordinates on \( t \). The roots are

\[
\Phi = \{ \pm(X_i \pm X_j) \mid 1 \leq i < j \leq n \} \cup \{ \pm X_i \mid 1 \leq i \leq n \}.
\]

We choose the standard positive system

\[
\Phi^+ = \{ X_i \pm X_j \mid i < j \} \cup \{ X_i \mid i = 1, \ldots, n \},
\]
and take $B$ to be the Borel subgroup containing $T$ and corresponding to the negative roots. Concretely, in our chosen realization, one may take $T$ to be the diagonal elements of $G$, let $X_i$ be the usual coordinate functions, and take $B$ to be the lower-triangular elements of $G$. The Weyl group $W$ acts on the $X_i$ as the $2^n n!$ signed permutations of \{1, \ldots, n\} which change any number of signs. The $T$-fixed points of $X = G/B$ correspond to such permutations, in the usual way.

We realize $K$ as the fixed points of the involution $\text{int}(l_{p,2q+1,p})$ (cf. Section [1.1]). One checks easily that

$$K = G^\theta = \left\{ k = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix} \bigg| K_{11}, K_{13}, K_{31}, K_{33} \in \text{Mat}(p, p), \begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} \end{pmatrix} \in \text{O}(2p, \mathbb{C}) \right\}$$

$$\cong \text{SO}(2p, \mathbb{C}) \times \text{O}(2q + 1, \mathbb{C}).$$

Note that this is an equal rank case, so $S := K \cap T = T$. We label coordinates on $s$ by variables $Y_i$, so that the restriction map is given by $\rho(X_i) = Y_i$. The Weyl group $W_K$ for the Lie algebra of $K$ acts on the $Y_i$ as signed permutations of \{±1, \ldots, ±n\} which preserve the sets \{±1, \ldots, ±p\} and \{±(p + 1), \ldots, ±n\}, and which change an even number of signs on the first set.

As remarked in Section [2.3] this $K$ is disconnected. Each closed $K$-orbit, corresponding to a symmetric $(2p, 2q + 1)$-clan consisting only of +’s and −’s, is a union of $2 K^0 = \text{SO}(2p, \mathbb{C}) \times \text{SO}(2q + 1, \mathbb{C})$-orbits. These components are actually orbits of the connected symmetric subgroup $S(\text{Pin}(2p, \mathbb{C}) \times \text{Pin}(2q + 1, \mathbb{C})) \subset \text{Spin}(2n + 1, \mathbb{C})$, so all the results of Section [1.3] apply to them. We use those results to obtain formulas for the individual components, then add the appropriate formulas to get a formula for each closed $K$-orbit.

By the results of Section [1.3], each closed $K^0$-orbit contains $|W_K| = 2^{n-1} p! q!$ $S$-fixed points, and there are $|W|/|W_K| = 2^{n}$ of them. We claim that the class of the closed $K^0$-orbit $K^0 \cdot wB$ is represented by the polynomial $(-1)^{\phi_p(w) + l_p(|w|)} P(x, y)$, where

$$P(x, y) = \frac{1}{2} (x_{w^{-1}(1)} \cdots x_{w^{-1}(p)} + y_1 \cdots y_p) \prod_{i \leq p < j} (x_{w^{-1}(i)} - y_j)(x_{w^{-1}(i)} + y_j).$$

First, we verify that this formula is independent of the choice of $w$. To start, note that the function $\phi_p$ is constant modulo 2 on right cosets $W_K w$, since elements of $W_K$ permute \{1, \ldots, p\} with an even number of sign changes. Considering $l_p(|w|)$, note that if $w' = w Kw$ for $wK \in W_K$, then $|w'| = |wK||w|$, and $|wK|$ is a permutation of \{1, \ldots, n\} which acts separately on \{1, \ldots, p\} and \{p + 1, \ldots, n\}. This implies that $l_p(|w'|) = l_p(|w|)$.

Next, consider the term

$$x_{w^{-1}(1)} \cdots x_{w^{-1}(p)} + y_1 \cdots y_p.$$

Replacing $w$ by $wKw$, we get

$$x_{w^{-1}(w_K^{-1}(1))} \cdots x_{w^{-1}(w_K^{-1}(p))} + y_1 \cdots y_p = x_{w^{-1}(1)} \cdots x_{w^{-1}(p)} + y_1 \cdots y_p.$$
Applying in the type $A$ unchanged if we replace $w$ also does not depend on the choice of $w$. With that established, we now apply Proposition 1.5 to compute the restrictions $\mathcal{Q}|_w$. The number of weights of the form $\pm Y_j$ occurring with a negative sign is clearly $\phi_p(w)$. It is an easy argument to determine that the number of weights of the latter type occurring with a minus sign is congruent modulo 2 to $l_p(|w|)$. Thus for any $S$-fixed point $w \in \mathcal{Q}$, 

$$[\mathcal{Q}]_w = F(Y) := (-1)^{\phi_p(w)+l_p(|w|)}Y_1 \cdots Y_p \prod_{i \leq p < j} (Y_i + Y_j)(Y_i - Y_j).$$

Thus we must prove that 

$$P(\sigma Y, Y) = \begin{cases} F(Y) & \text{if } \sigma = w'w \text{ for some } w' \in W_K, \\ 0 & \text{if } \sigma w^{-1} \notin W_K. \end{cases}$$

First, we establish this when $Q$ is the orbit containing the $S$-fixed point corresponding to the identity. The general case follows easily. Suppose first that $w \in W_K$. Since $w$ permutes $\{1, \ldots, p\}$ with an even number of sign changes, we have 

$$Y_{w(1)} \cdots Y_{w(p)} = Y_1 \cdots Y_p.$$

Further, again because $w$ permutes $\{1, \ldots, p\}$ with an even number of sign changes, we see that 

$$\prod_{i \leq p < j} (Y_{w(i)} + Y_j)(Y_{w(i)} - Y_j) = \prod_{i \leq p < j} (Y_i + Y_j)(Y_i - Y_j).$$

This says that 

$$P(wY, Y) = Y_1 \cdots Y_p \prod_{i \leq p < j} (Y_i + Y_j)(Y_i - Y_j)$$

for $w \in W_K$.

Now, suppose $w \notin W_K$. Then one of two things is true: Either $w$ is separately a signed permutation of $\{1, \ldots, p\}$ and $\{p + 1, \ldots, n\}$, but permutes $\{1, \ldots, p\}$ with an odd number of sign changes, or $w$ is not separately a signed permutation of $\{1, \ldots, p\}$ and $\{p + 1, \ldots, n\}$, in which case $w$ sends some $i \leq p$ to $\pm j$ for some $j > p$. In the former case, we see that 

$$Y_{w(1)} \cdots Y_{w(p)} + Y_1 \cdots Y_p = 0,$$
while in the latter case, either $Y_{w(i)} + Y_j = 0$, or $Y_{w(i)} - Y_j = 0$, whence

$$\prod_{i \leq p < j} (Y_{w(i)} + Y_j)(Y_{w(i)} - Y_j) = 0.$$ 

Together, these two facts say that

$$P(wY, Y) = 0$$

whenever $w \notin W_K$. We conclude that $P(x, y)$ represents $[Q]$.

Now, suppose that $\tilde{Q}$ is another closed $K$-orbit, containing the $S$-fixed point $w \notin W_K$. All $S$-fixed points contained in $\tilde{Q}$ are then of the form $w'w$ for $w' \in W_K$. So for any $w'w \in \tilde{Q}$, we have

$$P(w'wY, Y) = \frac{1}{2}(Y_{w'(1)} \ldots Y_{w'(p)} + Y_1 \ldots Y_p) \prod_{i \leq p < j} (Y_{w'(i)} - Y_j)(Y_{w'(i)} + Y_j) =$$

$$Y_1 \ldots Y_p \prod_{i \leq p < j} (Y_i + Y_j)(Y_i - Y_j),$$

by our previous argument, since $w' \in W_K$. Noting that this is precisely what $P(w'wY, Y)$ is to be up to sign, and noting that we have corrected the sign by the appropriate factor of $(-1)^{\phi_p(w) + \ell_p(|w|)}$ in our putative formula, we see that it restricts correctly at $S$-fixed points contained in $\tilde{Q}$.

On the other hand, for any $S$-fixed point $\tilde{w}$ not contained in $\tilde{Q}$, we may write $\tilde{w} = w'w$ for $w' \notin W_K$. Then

$$P(\tilde{w}Y, Y) = P(w'wY, Y) =$$

$$\frac{1}{2}(Y_{w'(1)} \ldots Y_{w'(p)} + Y_1 \ldots Y_p) \prod_{i \leq p < j} (Y_{w'(i)} - Y_j)(Y_{w'(i)} + Y_j) = 0,$$

again by our previous argument, since $w' \notin W_K$.

With formulas for the closed $K^0$-orbits now in hand, we next identify how closed $K$-orbits split up as unions of closed $K^0$-orbits. Let $\gamma$ be a symmetric $(2p, 2q + 1)$-clan consisting only of $+$’s and $-$’s, let $Q'_{\gamma}$ be the associated $GL(2p, \mathbb{C}) \times GL(2q + 1, \mathbb{C})$-orbit on $GL(2n + 1, \mathbb{C})/B$, and let $Q_{\gamma} = X \cap Q'_{\gamma}$ be the corresponding $K$-orbit. Using the algorithm of [Yam97] described in Section 2.1 to produce $S$-fixed representatives of $Q_{\gamma}$, it is easy to see that the $S$-fixed points contained in $Q_{\gamma}$ correspond to all signed permutations of $\{1, \ldots, n\}$ which can be assigned to $\gamma$ in the following way: Considering only the first $n$ characters of $\gamma$, one assigns either $\pm i$ ($i = 1, \ldots, p$) to the positions of the $p$ plus signs, and either $\pm j$ ($j = p + 1, \ldots, n$) to the positions of the $q$ minus signs.

Now, if $w \in W$ is an $S$-fixed point, the $S$-fixed points contained in the closed orbit $K^0 \cdot wB$ are $W_Kw$, and $W_K$ consists of signed permutations which act separately on $\{1, \ldots, p\}$ and $\{p + 1, \ldots, n\}$, with an even number of sign changes on the first set. On the other hand, the above characterization of $S$-fixed points contained in a closed $K$-orbit says that they are of the form $\sigma w$, where $\sigma$ is a signed permutation which acts separately on $\{1, \ldots, p\}$ and $\{p + 1, \ldots, n\}$, changing any number of signs on either set. This implies that the closed $K$-orbit $K \cdot wB$ is the union of $K^0 \cdot wB$ and $K^0 \cdot \pi wB$, where $\pi \in W$ is the signed permutation $T2 \ldots n$. (In fact, $\pi$ could be taken to be any signed permutation which acts separately on $\{1, \ldots, p\}$ and $\{p + 1, \ldots, n\}$, and which changes an odd number of signs on the first set. This particular $\pi$ seems to the author to be the simplest such choice.)
Thus \( Q_\gamma = K^0 \cdot wB \cup K^0 \cdot \pi wB \), so \([Q_\gamma] = [K^0 \cdot wB] + [K^0 \cdot \pi wB]\). Using the formulas that we obtained above for these individual \( K^0 \)-orbits, this sum simplifies to the formula given in Table \[ when one makes the following easy observations:

1. \( \phi_p(\pi w) = \phi_p(w) + 1; \)
2. \( l_p(|w|) = l_p(|\pi w|); \)
3. \( x_{w^{-1}(1)} \cdots x_{w^{-1}(p)} = -x_{(\pi w)^{-1}(1)} \cdots x_{(\pi w)^{-1}(p)}; \)
4. \( (-1)^{\phi_p(w)} x_{w^{-1}(1)} \cdots x_{w^{-1}(p)} = x_{|w|^{-1}(1)} \cdots x_{|w|^{-1}(p)}. \)

\( \Box \)

The proofs of the correctness of the formulas in cases (3) and (5) is very similar to that for case (2), so we omit them.

**Case (4).** We realize the symplectic group case (2), so we omit them. The proofs of the correctness of the formulas in cases (3) and (5) is very similar to that for case (2), so we omit them.

Fix a maximal torus \( T \) of \( K \) we obtained above for these individual \( K^0 \)-orbits, this sum simplifies to the formula given in Table \[ when one makes the following easy observations:

1. \( \phi_p(\pi w) = \phi_p(w) + 1; \)
2. \( l_p(|w|) = l_p(|\pi w|); \)
3. \( x_{w^{-1}(1)} \cdots x_{w^{-1}(p)} = -x_{(\pi w)^{-1}(1)} \cdots x_{(\pi w)^{-1}(p)}; \)
4. \( (-1)^{\phi_p(w)} x_{w^{-1}(1)} \cdots x_{w^{-1}(p)} = x_{|w|^{-1}(1)} \cdots x_{|w|^{-1}(p)}. \)

\( \Box \)

The proofs of the correctness of the formulas in cases (3) and (5) is very similar to that for case (2), so we omit them.

**Case (4).** We realize the symplectic group \( Sp(2n, \mathbb{C}) \) as the subgroup of \( SL(2n, \mathbb{C}) \) preserving the symplectic form given by the antidiagonal matrix \( J_{n,n} \). That is,

\[
Sp(2n, \mathbb{C}) = \{ g \in SL(2n, \mathbb{C}) \mid g J_{n,n} g^t = J_{n,n} \}.
\]

Fix a maximal torus \( T \) of \( G \), and let \( X_i (i = 1, \ldots, n) \) denote coordinates on \( t \). The roots are

\[
\Phi = \{ \pm(X_i \pm X_j) \mid 1 \leq i < j \leq n \} \cup \{ \pm 2X_i \mid i = 1, \ldots, n \}.
\]

Take the positive roots to be those of the form

\[
\Phi^+ = \{ X_i \pm X_j \mid i < j \} \cup \{ 2X_i \mid i = 1, \ldots, n \},
\]

and take \( B \supset T \) to be the negative Borel. Concretely, in our realization, take \( T \) to be the diagonal elements of \( G \), with \( X_i \) the obvious coordinate functions, and take \( B \) to be the lower-triangular elements of \( G \).

As in the type \( B \) case, the Weyl group \( W \) acts on the \( X_i \) as the \( 2^n n! \) signed permutations of \( \{1, \ldots, n\} \) which change any number of signs. The \( T \)-fixed points of \( X = G/B \) once again correspond to such signed permutations.

Let \( \theta = \text{int}(i \cdot 1_{n,n}) \). Then

\[
K = G^\theta = \left\{ \begin{pmatrix} g & 0 \\ 0 & J_n (g^t)^{-1} J_n \end{pmatrix} \mid g \in GL(n, \mathbb{C}) \right\} \cong GL(n, \mathbb{C}).
\]

Let \( S \) be a maximal torus of \( K \) contained in \( T \). Once again, \( S = T \), but we again denote coordinates on \( s \) by \( Y_i \), with restriction \( t \to s \) given by \( \rho(X_i) = Y_i \).

Since \( K \) has a root system of type \( A \), the roots of \( K \) are as follows:

\[
\Phi_K = \{ \pm(Y_i - Y_j) \mid 1 \leq i < j \leq n \}.
\]

\( W_K \cong S_n \) is embedded in \( W \) as ordinary permutations of \( \{1, \ldots, n\} \), which we interpret as signed permutations of \( \{1, \ldots, n\} \) which change no signs.

Once again, we start by noting that the formula for the class of \( Q = K \cdot wB \) given in the table is independent of the choice of fixed point \( w \) representing \( Q \). First, note that because fixed points contained in \( Q \) are all (left) \( W_K \) translates of \( w \), and since elements of \( W_K \) are signed permutations which change no signs, all the fixed points in \( Q \) correspond to \( W \)-elements having the same “sign pattern”. By this, we mean that the set \( \text{Neg}(w) \) is independent of
the choice of $w$. As an example, when $n = 2$, the 4 closed $K$-orbits on $G/B$ contain $S$-fixed points $\{12, 21\}$, $\{12, 21\}$, $\{12, 2\}$, and $\{12, 2\}$.

It follows that the functions $\psi, \sigma$ are constant on $W_K w$, so that $(-1)^{\psi(w)+\sigma(w)}$ is independent of the choice of $w$. It is also easy to see that $\Delta_n(x, y, w)$ is independent of this choice. Indeed, replacing $w$ by $w'w$ for $w' \in W_K$, each $e_k$ becomes

$$e_k(x_{w^{-1}(w'w^{-1}(1))}, \ldots, x_{w^{-1}(w'w^{-1}(n))}) + e_k(y_1, \ldots, y_n).$$

Because $w'$ is just an ordinary permutation of $\{1, \ldots, n\}$, the effect is simply to permute the $x_{w^{-1}(i)}$, and because $e_k$ is invariant under permutation of the variables $x_i$, each $e_k$ is unchanged.

With this established, we now apply Proposition 1.5 to compute the restriction $[Q]_w$. Applying $w$ to positive roots of the form $2X_i$ and restricting to $s$, we get weights of the form $2Y_{w(i)} = \pm 2Y_j$. The number of such weights occurring with a minus sign is $\psi(w)$. Note that none of these weights are roots of $K$.

Applying $w$ to positive roots of the form $X_i \pm X_j$ ($i < j$) and then restricting, we get weights of the form $Y_{w(i)} \pm Y_{w(j)}$, and these two weights together are of the form $\pm Y_k \pm Y_l$, $\pm Y_k \mp Y_l$, for some $k, l$. Those of the latter form $\pm Y_k \mp Y_l$ are roots of $K$, while those of the former are not. So we eliminate the latter weights, and retain the former. The number of roots surviving which are negative (i.e. of the form $-Y_k - Y_l$) is precisely $\sigma(w)$. To see this, note that if $w(i)$ is positive, then applying $w$ to any pair of roots $X_i + X_j, X_i - X_j$ with $i < j$ and then restricting is going to necessarily give a positive root of the form $Y_k + Y_l$, where $k = w(i), l = |w(j)|$. If $w(i)$ is negative, then applying $w$ to any such pair will necessarily give a negative root of the form $-Y_k - Y_l$. For any fixed $i$, the number of pairs $\{X_i \pm X_j\}$ with $i < j$ is precisely $n - i$. So for each $i$ with $w(i)$ negative, $n - i$ negative roots occur, for a total of $\sigma(w)$ negative roots.

All of this adds up to the following conclusion: For any $S$-fixed point $w \in Q$, we have

$$[Q]_w = F(Y) := (-1)^{\psi(w)+\sigma(w)} 2^n Y_1 \cdots Y_n \prod_{i < j} (Y_i + Y_j).$$

So by the localization theorem, $[Q]$ is uniquely determined by the following property: For any $u \in W$,

$$[Q]_u = \begin{cases} F(Y) & \text{if } u = w'w \text{ for some } w' \in W_K, \\ 0 & \text{otherwise.} \end{cases}$$

Letting $P$ be the polynomial given in the table, the claim is thus that $P(uY, Y)$ is $F(Y)$ if $uw^{-1} \in W_K$, and is 0 otherwise. For any given $u \in W$, let $w' = uw^{-1}$. Noting that $\Delta_n(x, y, w) = \Delta(w^{-1}x, y, id)$, we have

$$P(uY, Y) = (-1)^{\psi(w)+\sigma(w)} \Delta_n(w'Y, Y, 1).$$

If $w' \in W_K$, then $w'$ is an ordinary permutation (i.e. a signed permutation with no sign changes), whereas if $uw^{-1} \notin W_K$, $w'$ has at least one sign change. So our claim that $P(x, y)$ represents $[Q]$ amounts to the claim that $\Delta_n(x, y, id)$ has the following two properties:

1. It is invariant under permutations of the $x_i, y_i$.
2. If $\epsilon_i = \pm 1$, then

$$\Delta_n((\epsilon_1Y_1, \ldots, \epsilon_nY_n), (Y_1, \ldots, Y_n), id)$$
is zero unless all $\epsilon_i$ are equal to 1, in which case it is equal to
\[ 2^n Y_1 \cdots Y_n \prod_{i<j} (Y_i + Y_j). \]
That $\Delta_n(x, y, id)$ has these properties is proved directly in [Ful96b, §3].

The proof of the correctness of the formulas in case (6) is very similar to that for case (4), and again uses properties of the determinants in question which are established in [Ful96b]. Because the argument is so similar, we omit the details.

**Case (7).** We realize $G = SO(2n, \mathbb{C})$ to be the subgroup of $SL(2n, \mathbb{C})$ which preserves the standard (diagonal) quadratic form on $\mathbb{C}^{2n}$ given by
\[ \langle x, y \rangle = \sum_{i=1}^{2n} x_i y_i. \]
Thus $G$ is the set of determinant 1 matrices $g$ such that $gg^t = I_{2n}$.

Let $T$ be a maximal torus of $G$, and let $X_i \in t^*$ ($i = 1, \ldots, n$) denote coordinates on $t$.

Note that given our chosen realization of $G$, the diagonal elements no longer form a maximal torus. Concretely, one may take $T \subseteq G$ to be the maximal torus of $G$ such that $t$ consists of matrices of the following form:
\[
\begin{pmatrix}
0 & a_1 & & & \\
-a_1 & 0 & & & \\
 & & \ddots & & \\
 & & & \ddots & \\
 & & & & 0 & a_n \\
 & & & & -a_n & 0
\end{pmatrix}
\]
The function $X_i$ gives $a_i$ when evaluated upon the above general element of $t$.

We take the positive roots to be
\[ \Phi^+ = \{ X_i \pm X_j \mid (i < j) \}, \]
and let $B \subseteq G$ be the Borel subgroup containing $T$ and corresponding to the negative roots.

Let $X = G/B$ be the flag variety, chosen to be one component of the variety of flags which are isotropic with respect to the form $\langle \cdot, \cdot \rangle$.

Let $\theta$ be the involution given by
\[ \theta(g) = I_{2p+1,2q-1}gI_{2p+1,2q-1}. \]
One checks easily that our chosen realization of $G$ is stable under $\theta$, that $T$ is stable under $\theta$, and that
\[ K = G^\theta = \left\{ k = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mid A \in O(2p + 1, \mathbb{C}), B \in O(2q - 1, \mathbb{C}), \det(k) = 1 \right\} \cong S(O(2p + 1, \mathbb{C}) \times O(2q - 1, \mathbb{C})). \]
Note here that we are in an unequal rank case, with \( \text{rank}(K) = n - 1 \). We take \( S \subseteq T \) to be the maximal torus of \( K \) such that \( S \) consists of matrices of the form

\[
\begin{pmatrix}
0 & a_1 \\
-a_1 & 0 \\
& & \ddots \\
& & 0 & a_p \\
& & -a_p & 0 \\
& & & & 0 & 0 \\
& & & & 0 & 0 \\
& & & & & 0 & a_{p+2} \\
& & & & & -a_{p+2} & 0 \\
& & & & & & & \ddots \\
& & & & & & & & 0 & a_n \\
& & & & & & & & -a_n & 0 \\
& & & & & & & & & & & & & & & \end{pmatrix}
\]

One checks easily that \( S \) is also stable under \( \theta \). We label coordinates on \( S \) as \( Y_1, \ldots, Y_p, Y_{p+2}, \ldots, Y_n \), with \( Y_i(s) = a_i \) when \( s \) is a matrix of the above block form. With this choice of labelling, the restriction map \( \rho: \mathfrak{t}^* \to \mathfrak{s}^* \) is given by \( \rho(Y_i) = Y_i \) for \( i \neq p + 1 \), and \( \rho(X_{p+1}) = 0 \).

The roots of \( K \) are as follows:

\[
\Phi_K = \{ \pm Y_i \mid i \neq p + 1 \} \cup \{ \pm (Y_i \pm Y_j) \mid i < j \leq p \text{ or } p + 1 < i < j \}.
\]

Just as in cases (2) and (5), here \( K \) is disconnected. However, unlike in those cases, this time the closed \( K \)-orbits are nonetheless connected. (We establish this below.) Thus it turns out that there is no need to concern ourselves with closed \( K \)-orbits versus closed \( K^0 \)-orbits, as here they coincide.

Let \( W_K \) be the Weyl group for \( \text{Lie}(K) \). \( W_K \) embeds in \( W \) as those signed permutations of \( \{1, \ldots, n\} \) which act separately on the first \( p \) elements \( \{1, \ldots, p\} \) and the last \( q - 1 \) elements \( \{p + 2, \ldots, n\} \), changing any number of signs on each set, and which either fix \( p + 1 \) or send it to its negative, whichever is necessary to guarantee that the resulting signed permutation changes an even number of signs. There are \( 2^{n-1}p!(q-1)! \) such signed permutations.

Since this is an unequal rank case, there will not be \( |W/W_K| \) closed orbits. We first use Proposition 1.11 to determine how many closed \( K^0 \)-orbits there are, and which \( S \)-fixed points they contain.

**Proposition 3.8.** Let \( wB \) be an \( S \)-fixed point, with \( w \in W \). Then \( K^0 \cdot wB \) is closed if and only if \( w(n) = \pm (p + 1) \). There are \( (n-1 \choose p) \) closed \( K^0 \)-orbits.

**Proof.** We use the characterization of closed orbits given in Proposition 1.11. Since we have chosen \( B \) to be the negative Borel, the condition that \( wBw^{-1} \) be \( \theta \)-stable is equivalent to the condition that \( w\Phi^- \) is a \( \theta \)-stable subset of \( \Phi \). One checks easily that the action of \( \theta \) on \( \Phi \) is defined by \( \theta(X_i) = X_i \) for \( i \neq p + 1 \), and \( \theta(X_{p+1}) = -X_{p+1} \). Any positive system contains, for each \( i < j \), exactly one of \( X_i + X_j \) and \( -X_i - X_j \), and exactly one of \( X_i - X_j \) and \( -X_i + X_j \). For \( i, j \neq p + 1 \), all such roots are fixed by \( \theta \). Thus for \( \theta \)-stability, it suffices to focus on roots of the form \( \pm X_i \pm X_{p+1} \), with \( i \neq p + 1 \). It is easy to check that a positive system is \( \theta \)-stable if and only if it contains either \( \{X_i - X_{p+1}, X_i + X_{p+1}\} \) or \( \{-X_i + X_{p+1}, -X_i - X_{p+1}\} \) for each \( i \neq p + 1 \).
This holds if and only if \( w(n) = \pm (p+1) \). Recall that \( w\Phi^- = \{ -wX_i \pm wX_j \mid i < j \} \). Suppose that \( w(n) = \pm (p+1) \). Let \( i \neq p+1 \) be given, with \( k = |w|^{-1}(i) \). Then \( -wX_k \pm wX_n \) is either the set \( \{ X_i + X_{p+1}, X_i - X_{p+1} \} \) or \( \{ -X_i + X_{p+1}, -X_i - X_{p+1} \} \), as required. Conversely, suppose that \( |w(n)| = j \neq p + 1 \). Let \( k = |w|^{-1}(p+1) \). Then \( -wX_k \pm wX_n \) is either the set \( \{ -X_{p+1} + X_j, -X_{p+1} - X_j \} \) or \( \{ X_{p+1} + X_j, X_{p+1} - X_j \} \), and thus \( w\Phi^- \) is not \( \theta \)-stable. This establishes the first claim.

To establish the claim on the number of closed orbits, note that any element \( u \in W \) such that \( u(n) = \pm (p+1) \) is in the same left \( W_K \)-coset as a unique element \( w \in W \) having the following properties:

1. \( w \) changes no signs.
2. \( w(n) = p + 1 \).
3. \( w^{-1}(1) < w^{-1}(2) < \ldots < w^{-1}(p) \).
4. \( w^{-1}(p + 2) < w^{-1}(p + 3) < \ldots < w^{-1}(n) \).

Recall that all elements of \( W_K \) are separately signed permutations of \( \{1, \ldots, p \} \) and \( \{ p + 2, \ldots, n \} \), which either fix \( p + 1 \) or send it to its negative so as to ensure that the entire signed permutation changes an even number of signs. Supposing that, in the one-line notation for \( u \), the values \( 1, \ldots, p \) (possibly with signs) are “scrambled”, then there is precisely one signed permutation of \( \{1, \ldots, p \} \) which will unscramble them and remove all negative signs, and likewise for the set \( \{ p + 2, \ldots, n \} \). Taking \( w' \in W_K \) to be the unique element which separately acts on \( \{1, \ldots, p \} \) and \( \{ p + 2, \ldots, n \} \) as required, we have that \( w'u = w \).

As an example, suppose that \( p = q = 3 \), and let \( u \) be the signed permutation \( 316254 \). To unscramble the \( 312 \), we must multiply on the left by \( 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1 \), and to unscramble the \( 65 \) we must multiply on the left by \( 5 \mapsto 6, 6 \mapsto 5 \). Thus we multiply \( u \) on the left by \( w' = 231465 \) to get \( w'u = w = 125364 \).

Note that a permutation \( w \) having the properties above is completely determined by the positions (in the one-line notation) of \( 1, \ldots, p \) among the first \( n - 1 \) spots, which can be chosen freely. Thus there are \( \binom{n-1}{p} \) such \( w \), and hence \( \binom{n-1}{p} \) closed \( K^0 \)-orbits, as claimed. \( \square \)

We shall refer to the special element \( w \) described in the proof of Proposition 3.8 as the “standard representative” of the orbit \( K \cdot wB \). We remark that in this case, the formula for the class of \( K \cdot wB \) given in Table 1 is not independent of the choice of \( w \) contained in the orbit. Indeed, the formula given there assumes \( w \) to be the standard representative of the orbit. In what follows, \( w \) is always assumed to be the standard representative.

It is noted in [MO90] (and proved in [Wys12b, Appendix A]) that the \( K \)-orbits in this case are precisely the intersections with \( X \) of the \( K' \)-orbits \( GL(2p + 1, \mathbb{C}) \times GL(2q - 1, \mathbb{C}) \)-orbits on \( X' = GL(2n, \mathbb{C})/B \) with \( X \), that these correspond to symmetric \( (2p + 1, 2q - 1) \)-clans, and that the \( K' \)-orbits minimal in Bruhat order among those which meet \( X \) non-trivially are those whose clans consist exclusively of \( \pm \) signs along with a single pair of matching numbers in the middle positions \( n \) and \( n + 1 \). (In the table, we have denoted this by \( (\pm, 1, 1, \pm) \).) By symmetry, such a clan is determined by the first \( n - 1 \) characters, which are comprised of \( p + \) signs and \( q - 1 - \) signs, which can occur in any positions. Clearly, there are \( \binom{n-1}{p} \) such clans. This establishes that there are the same number of closed \( K \)-orbits as there are closed \( K^0 \)-orbits, so that in fact these closed orbits coincide.
With this noted, we now prove the correctness of our formula. First, consider $\rho(w^+)$, the elements of $\mathfrak{s}^*$ obtained by first applying the standard representative $w$ to the positive roots, then restricting to $\mathfrak{s}$. They are as follows:

- $Y_i$ ($i \neq p + 1$), with multiplicity 2. (One is the restriction of $w(X_i + X_{p+1}) = X_{w(i)} + X_{p+1}$, the other the restriction of $w(X_i - X_{p+1}) = X_{w(i)} - X_{p+1}$.)
- $Y_i + Y_j$ ($i < j$, $i, j \neq p + 1$), with multiplicity 1.
- For each $i < j$ with $i, j \neq p + 1$, exactly one of $\pm(Y_i - Y_j)$, with multiplicity 1.

Removing roots of $K$, we have the following set of weights:

- $Y_i$ ($i \neq p + 1$), with multiplicity 1.
- $Y_i + Y_j$ ($i \leq p < p + 1 < j$), with multiplicity 1.
- For each $i < j$ with $i \leq p < p + 1 < j$, exactly one of $\pm(Y_i - Y_j)$, with multiplicity 1.

Recall that $w$ is an honest permutation, with no sign changes. This means that the only way to get a weight of the form $-(Y_i - Y_j)$ by the action of $w$ is to apply $w$ to some $X_k - X_l$ ($k < l$) with $w(k) > w(l)$, then restrict. (Clearly, we want $k, l \neq n$.) For this root to remain after discarding roots of $K$, it must be the case that $w(k) > p + 1$, while $w(l) \leq p$. Thus for each $k < n$ such that $w(k) > p + 1$ (this says that $k \in I_w$), we count the number of $l$ with $k < l < n - 1$ such that $w(l) \leq p$ (this says that $l \in C(k)$). Adding up the total number of such pairs as we let $k$ range over $I_w$, we arrive at $\tau(w)$. This says that the number of weights of the form $-(Y_i - Y_j)$ contained in $\rho(w^+) \setminus (\rho(w^+) \cap \Phi_K)$ is $\tau(w)$.

Now we consider the set $\rho(w'w^+) \setminus (\rho(w'w^+) \cap \Phi_K)$ with $w' \in W_K$, and compute the restriction $[Q]|_{w'w}$ at an arbitrary $S$-fixed point. Since the action of $w'$ on $t$ commutes with restriction to $\mathfrak{s}$, and since $w'$ acts on the roots of $K$ (and hence also on $\rho(\Phi) \setminus \Phi_K$), we can simply apply $w'$ to the set of weights described in the previous paragraph. We temporarily forget that some of those roots are of the form $-(Y_i - Y_j)$ ($i < j$), and add the sign of $(-1)^{\tau(w)}$ back in at the end. So consider the action of $w' \in W_K$ on the following set of weights, each with multiplicity 1:

- $Y_i$ ($i \neq p + 1$)
- $Y_i + Y_j$ ($i \leq p < p + 1 < j$)

Since $w'$ acts separately as signed permutations on $\{1, \ldots, p\}$ and $\{p + 2, \ldots, n\}$, it clearly sends the set of weights $Y_i \pm Y_j$ to itself, except possibly with some sign changes. We observe that the number of sign changes must be even. Suppose first that $w'(Y_i + Y_j)$ is a negative root. Then it is either of the form $-Y_k - Y_i$ or $-Y_k + Y_i$, with $k = |w(i)|$ and $l = |w(j)|$. In the former case, $w'(Y_i - Y_j) = -Y_k + Y_l$, also a negative root. In the latter, $w'(Y_i - Y_j) = -Y_k - Y_l$, again a negative root. Likewise, if $w'(Y_i - Y_j)$ is a negative root of the form $-Y_k - Y_l$ or $-Y_k + Y_l$, then $w'(Y_i + Y_j)$ is also a negative root, equal to $-Y_k + Y_l$ in the former case, and $-Y_k - Y_l$ in the latter. Thus the negative roots arising from the action of $w'$ on roots of the form $Y_i \pm Y_j$ occur in pairs.

Now consider roots of the form $Y_i$, $i \neq p + 1$. The action of $w'$ again preserves this set of roots, except possibly with some sign changes. The number of sign changes could be either even or odd. (Recall that $w'$ acts with any number of sign changes on $\{1, \ldots, p\}$ and $\{p + 2, \ldots, n\}$, and sends $p + 1$ either to itself or to $-(p + 1)$, whichever ensures that the total number of sign changes for $w'$ is even.)
This discussion all adds up to the following. The product of the weights \( \rho(w'w\Phi^+) \setminus (\rho(w'w\Phi^+) \cap \Phi_K) \) is
\[
[Q]_{w'w} = (-1)^{\tau(w) + \#(\text{Neg}(w') \setminus \{p+1\})} \prod_{i \neq p+1} Y_i \prod_{i \leq p < p+1 < j} (Y_i + Y_j)(Y_i - Y_j).
\]

Thus we wish to prove that the polynomial \( P(x, y) \) has the properties that \( P(\rho(w'wX), Y) \) is equal to this restriction for all \( w' \in W_K \), and that \( P(\rho(w'wX), Y) = 0 \) whenever \( w' \notin W_K \).

Consider first the action of \( w'w \) on \( P(x, y) \) for \( w' \in W_K \). Since \( w \) sends the set \( \{1, \ldots, n-1\} \) to the set \( \{1, \ldots, p, p+2, \ldots, n\} \) with no sign changes, the action of \( w'w \) on \( X_1 \ldots X_{n-1} \) is clearly to send it to \((-1)^{\#(\text{Neg}(w') \setminus \{p+1\})} \prod_{i \neq p+1} X_i \). Thus applying \( w'w \) to \((-1)^{\tau(w)}x_1 \ldots x_{n-1} \), then restricting, gives us the portion
\[
(-1)^{\tau(w) + \#(\text{Neg}(w') \setminus \{p+1\})} \prod_{i \neq p+1} Y_i
\]

of the required restriction. Now consider the action of \( w'w \) on the product
\[
\prod_{i \leq p < p+1 < j} (x_{w^{-1}(i)} + y_j)(x_{w^{-1}(i)} - y_j).
\]

We get
\[
\prod_{i \leq p < p+1 < j} (Y_{w^{-1}(i)} + Y_j)(Y_{w^{-1}(i)} - Y_j).
\]

Since \( w' \) acts as a signed permutation on \( \{1, \ldots, p\} \), this is clearly the same as
\[
\prod_{i \leq p < p+1 < j} (Y_i + Y_j)(Y_i - Y_j),
\]
giving us the remaining part of the required restriction.

Now, consider the action of \( w'w \) on \( P(x, y) \) for \( w' \notin W_K \). Suppose first that \( w'(p+1) \neq \pm(p+1) \). Then \( w'(i) = \pm(p+1) \) for some \( i \neq p+1 \). Let \( j = w^{-1}(i) \). Then the action of \( w'w \) sends \( x_j \) to \( \pm X_{p+1} \), which then restricts to zero. Now suppose that \( w'(p+1) = \pm(p+1) \). Then since \( w' \notin W_K \), \( w' \) must send some \( i \leq p \) to \( \pm j \) for some \( j > p+1 \). If it sends \( i \) to \( j \), then \( w'w \) applied to the term \( x_{w^{-1}(i)} - y_j \) is zero. If it sends \( i \) to \( -j \), then \( w'w \) applied to the term \( x_{w^{-1}(i)} + y_j \) is zero. This shows that \( P(\rho(w'wX), Y) = 0 \) for \( w' \notin W_K \), and completes the proof. \( \square \)

This concludes the proof of Theorem \[3.7\] \( \square \)

**Examples.** An example calculation for each of our cases is given in the appendix. In each case, the weak order graph for \( K \setminus G/B \) is given, as well as a table of formulas. The formulas for the closed orbits are given by Theorem \[3.7\] while the rest are computed from these using divided difference operators (scaled by a factor of \( \frac{1}{2} \) when passing through a blue edge), as explained in Section \[1.4\].
4. K-orbit Closures as Universal Degeneracy Loci

In this section, we describe one application of the formulas obtained in the previous section, realizing the K-orbit closures as universal degeneracy loci of a certain type determined by K. We describe a translation between our formulas for equivariant fundamental classes of K-orbit closures and Chern class formulas for the fundamental classes of such degeneracy loci.

4.1. Generalities. Before handling the specifics of the cases at hand, we first recall the general setup, which is described in some detail in [Wys12a]. Let $E := EG$ be a contractible space with a free action of $G$, and hence, by restriction, a free action of any subgroup of $G$ (e.g. $T$, $S$, $B$, $K$). Let $BG = E/G$ be a classifying space for $G$, and similarly define $BK = E/K$, $BB = E/B$, etc.

Although we have worked in $S$-equivariant cohomology throughout, the $S$-equivariant fundamental classes which we have computed actually live in the subring $H^*_K(G/B) \cong H^*_S(G/B)^{W_K} \subset H^*_S(G/B)$ [Bri98]. (If $K$ is disconnected, we can work in the subring $H^*_K(G/B)$, since in each case, the torus $S$ is actually a subgroup of $K^0$.) $H^*_K(G/B)$ is, by definition, $H^*(E \times^K (G/B))$, and the space $E \times^K (G/B)$ is easily seen to be isomorphic to the fiber product $BK \times_{BG} BB$.

Suppose that $X$ is a smooth complex variety, and that $V \to X$ is a complex vector bundle. If no further structure on $V$ is assumed, then for $G = GL(n, \mathbb{C})$, we have a classifying map $X \to BG$ such that $V$ is the pullback $\rho^*(V)$, where $V = E \times^G \mathbb{C}^n$ is a universal vector bundle over $BG$, with $\mathbb{C}^n$ carrying the natural action of $G$. If $V$ carries a quadratic (resp. skew) form $\gamma$ taking values in the trivial bundle, then for $G = SO(2n + 1, \mathbb{C})$ or $SO(2n, \mathbb{C})$ (resp. $G = Sp(2n, \mathbb{C})$), we have a classifying map into $BG$ for those $G$, and the universal bundle $V$ is equipped with a “universal” form of the same type, with the form $\gamma$ pulled back from this universal form.

For any closed subgroup $H$ of $G$, $BH \to BG$ is a fiber bundle with fiber isomorphic to $G/H$. A lift of the classifying map $\rho$ to $BH$ corresponds to a reduction of structure group to $H$ of the bundle $V$. Such a reduction of structure group can often be seen to amount to some additional structure on $V$. For instance, in type $A$, reduction of the structure group of $V$ from $GL(n, \mathbb{C})$ to the Borel subgroup $B$ of lower-triangular matrices is well-known to be equivalent to $V$ being equipped with a complete flag of subbundles. In the other types, this flag is isotropic or Lagrangian with respect to the bilinear form $\gamma$.

We will be concerned with structures on $V$ which amount to a reduction of structure group to $K$. Such a reduction gives us a lift of the classifying map $\rho$ to $BK$. Suppose that we know what this structure is, and that $V$ possesses this structure, along with a single flag of subbundles $E_\bullet$. (In types $BCD$, $V$ additionally carries a form, and the flag is isotropic with respect to the form.) Then we have two separate lifts of $\rho$, one to $BK$, and one to $BB$. Taken together, these two lifts give us a map

$$X \to BK \times_{BG} BB.$$ 

Our general thought is to consider a subvariety $D$ of $X$ which is defined as a set by linear algebraic conditions imposed on fibers over points in $X$. These linear algebraic conditions mirror the linear algebraic descriptions of the $K$-orbit closures, so that $D$ is the (set-theoretic) preimage of an orbit closure $Y$ under the map $\phi$ defined above. (More precisely, if $Y$ is an orbit closure, we must pull back the isomorphic image of $E \times^K Y$ to $BK \times_{BG} BB$ via the
aforementioned isomorphism $E \times K (G/B) \cong BK \times_{BG} BB.$) We also realize various bundles on $X$ as pullbacks by $\phi$ of certain tautological bundles on the universal space, so that the Chern classes of the various bundles on $X$ are pullbacks of $S$-equivariant classes represented by the variables $x_i$ and $y_i$, or perhaps polynomials in these classes.

Assuming that
\begin{equation}
[D] = [\phi^{-1}(Y)] = \phi^*([Y]),
\end{equation}
our equivariant formula for $[Y]$ gives us, in the end, a formula for $[D]$ in terms of the Chern classes of the bundles involved. See \cite{Ful97} §B.3, Lemma 5 for a sufficient condition to guarantee that $[\phi^{-1}(Y)] = \phi^*([Y])$ for any map $\phi$ of nonsingular varieties. Philosophically, this is the case when the various structures on the bundle $V$ are in sufficiently general position with respect to one another.

With the general picture painted, we now proceed to our specific examples.

4.2. Type $A$. We start with the case $(G, K) = (GL(n, \mathbb{C}), GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$. We assume that $V \to X$ is a complex vector bundle of rank $n$ over the smooth complex variety $X$. We know that a lift of the classifying map for $V$ to $BB$ amounts to $V$ possessing a complete flag of subbundles, so we assume that $V$ comes with such a flag, denoted by $F_*$. We must determine what structure on $V$ amounts to the lift of the classifying map to $BK$.

**Proposition 4.1.** Given the aforementioned setup, the classifying map $X \to BGL(n, \mathbb{C})$ lifts to $BK$ if and only if $V$ splits as a direct sum of subbundles of ranks $p$ and $q$.

**Proof.** ($\Leftarrow$): Suppose that $V = V' \oplus V''$, with $V'$ of rank $p$, and $V''$ of rank $q$. Let $\{U_\alpha\}$ be an open cover of $X$ over which both $V'$ and $V''$ are locally trivial (say by taking the common refinement of the open covers associated to atlases of $V'$ and $V''$). Over each $U_\alpha$, we can choose a basis of sections $s_{1,\alpha}, \ldots, s_{p,\alpha}$ for $V'$ and a basis of sections $s_{p+1,\alpha}, \ldots, s_{n,\alpha}$ for $V''$, with $s_{i,\alpha}(x) = (x, e_i)$ for $x \in U_\alpha$ ($e_1, \ldots, e_n$ the standard basis for $\mathbb{C}^n$). Then $s_{i,\alpha}$ for $i = 1, \ldots, n$ are a basis of sections for $V$. The gluing data for $V'$ and $V''$ dictates that for $x \in U_\alpha \cap U_\beta$,

$$s_{i,\alpha}(x) = \sum_{j=1}^{p} \lambda_{i,j} s_{j,\beta}(x)$$

for $i = 1, \ldots, p$, and

$$s_{i,\alpha}(x) = \sum_{j=p+1}^{n} \lambda_{i,j} s_{j,\beta}(x)$$

for $i = p+1, \ldots, n$. This defines a family of transition functions for $V$, associating to $x \in U_\alpha \cap U_\beta$ the matrix $(\lambda_{i,j}) \in GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$. Thus the classifying map for $V = V' \oplus V''$ lifts to $BK$.

($\Rightarrow$): Conversely, suppose that $V$ admits a reduction of structure group to $K$. Let $\{U_\alpha, h_\alpha\}$ be an atlas for $V$ whose transition functions take values in $K$. Then there are sections $s_{1,\alpha}, \ldots, s_{p,\alpha}$ and $s_{p+1,\alpha}, \ldots, s_{n,\alpha}$ satisfying linear relations of the above form, more or less by definition. Taking the sections $s_{1,\alpha}, \ldots, s_{p,\alpha}$, together with gluing information determined by composing the transition functions $\tau_{\alpha,\beta}$ of $V$ with projection to $GL(p, \mathbb{C})$, we have the data of a rank $p$ subbundle of $V$. Likewise, taking the sections $s_{p+1,\alpha}, \ldots, s_{n,\alpha}$ together with gluing information determined by composing the $\tau_{\alpha,\beta}$ with projection to $GL(q, \mathbb{C})$, we have the data of a rank $q$ subbundle $V''$. Clearly, $V'$ and $V''$ are in direct sum, by construction. □
With this established, let $V \to X$ be a vector bundle possessing a complete flag of subbundles $F_x$ and a splitting $V = V' \oplus V''$ as a direct sum of subbundles of ranks $p$ and $q$. In light of Theorem 2.5, let $\gamma$ be a $(p,q)$-clan, and over a point $x \in X$, let us say that the flag $F_x(x)$ and the splitting $V'(x) \oplus V''(x)$ are in relative position $\gamma$ if and only if

1. $\dim(F_i(x) \cap V'(x)) \geq \gamma(i; +)$
2. $\dim(F_i(x) \cap V''(x)) \geq \gamma(i; -)$
3. $\dim(\pi(F_i(x)) + F_j(x)) \leq j + \gamma(i; j)$

for all $i = 1, \ldots, n$, and for all $j > i$. (Here, $\pi : V(x) \to V'(x)$ is the projection onto the $p$-dimensional subspace.) Then define

$D_\gamma := \{ x \in X \mid F_x(x) \text{ and } V'(x) \oplus V''(x) \text{ are in relative position } \gamma \}.$

We describe how to use our formula for the equivariant class $[Y_\gamma]$ to obtain a formula for the fundamental class $[D_\gamma]$ of this locus in terms of the Chern classes of $V'$, $V''$, and $F_i/F_{i-1}$ ($i = 1, \ldots, n$).

As described in the previous section, the splitting of $V$ as a direct sum of subbundles, together with the complete flag of subbundles, gives us a map

$X \xrightarrow{\phi} BK \times_{BG} BB.$

Our first task is to see that $D_\gamma$ is precisely $\phi^{-1}(Y_\gamma)$, where $Y_\gamma$ denotes the isomorphic image of $E \times^K Y_\gamma$ in $BK \times_{BG} BB$.

First, note that $G/K$ can naturally be identified with the space of splittings of $\mathbb{C}^n$ as a direct sum of subspaces of dimensions $p$ and $q$, respectively. Indeed, $G$ acts transitively on the space of such splittings, and $K$ is precisely the isotropy group of the “standard” splitting of $\mathbb{C}^n$ as $E_p \oplus E_q$, in the notation of Section 2.1. Now $BK$ is a $G/K$-bundle over $BG$, and a point of $BK$ lying over $eG \in BG$ should be thought of as a splitting of the fiber $\mathcal{V}_{eG}$, where $\mathcal{V} = E \times^K \mathbb{C}^n$ is the universal rank $n$ vector bundle over $BG$. Specifically, the point $eK \in BK$ over $eG \in BG$ is the splitting of $\mathcal{V}_{eG}$ as the direct sum

$\mathbb{C} \langle [e, g \cdot e_1], \ldots, [e, g \cdot e_p] \rangle \oplus \mathbb{C} \langle [e, g \cdot e_{p+1}], \ldots, [e, g \cdot e_n] \rangle.$

Note that $BK$ carries two tautological bundles, say $S'$ and $S''$, of ranks $p$ and $q$ respectively, which sum directly to $\pi_K^* \mathcal{V}$, where $\pi_K$ is the projection $BK \to BB$. The fiber of $S'$ (resp. $S''$) over a point $eK \in BK$ is the $p$-dimensional (resp. $q$-dimensional) summand of the splitting of $\mathcal{V}_{eG}$ determined by that point.

Similarly, $BB$ is a $G/B$-bundle over $BG$, with a point of $BB$ representing a complete flag on $\mathcal{V}_{eG}$. Specifically, the point $eB \in BB$ is the flag

$\langle [e, g \cdot e_1], \ldots, [e, g \cdot e_n] \rangle.$

The space $BB$ carries a complete tautological flag of subbundles of $\pi_B^* \mathcal{V}$ ($\pi_B$ the projection $BB \to BG$), say $T_x$. The fiber of $T_x$ over a point $eB \in BB$ is simply the $i$th subspace of the flag on $\mathcal{V}_{eG}$ determined by that point.

Thus a point of $BK \times_{BG} BB$ should be thought of concretely as a pair consisting of a splitting and a flag of a fiber of $\mathcal{V}$. Now let $\gamma$ be a $(p,q)$-clan, with $Y_\gamma$ the corresponding $K$-orbit closure on $G/B$. We now note that the isomorphic image of $E \times^K Y_\gamma$ is precisely the set of points consisting of splittings and flags where the flag is in position $\gamma$ relative to the splitting. Indeed, a point $[e, gB] \in E \times^K Y_\gamma$ (with the flag $gB = \langle g \cdot e_1, \ldots, g \cdot e_n \rangle$ in position $\gamma$ relative to the standard splitting of $\mathbb{C}^n$) is carried by the isomorphism $E \times^K G/B \to BK \times_{BG} BB$. 


to the point \((eK, egB)\). This point represents the standard splitting of \(V_{eG}\), along with the flag \(gB\) on \(V_{eG}\). Thus the flag is in position \(\gamma\) relative to the splitting, since \(gB \in Y_\gamma\). On the other hand, any such point in \(BK \times _{BG} BB\) is of the form \((eK, egB)\) for some \(e \in E\) and \(g \in G\), which is then carried back to the point \([e, gB] \in E \times K Y_\gamma\) by the inverse isomorphism.

Now, consider the map \(\phi\). If \(\rho\) is the classifying map \(X \to BG\) for \(V\), denote by \(\rho_K\) and \(\rho_B\) the lifts of \(\rho\) to \(BK\) and \(BB\), respectively. The subbundles \(V'\) and \(V''\) are the pullbacks \(\rho_K^* S', \rho_K^* S''\) of the tautological bundles on \(BK\) mentioned above. Likewise, the flag \(E_\bullet\) is \(\rho_B^* T_\bullet\). The map \(\phi\) sends \(x \in X\) to the pair

\[
(S'(\rho_K(x)) \oplus S''(\rho_K(x)), T_\bullet(\rho_B(x)) = (V'(x) \oplus V''(x), F_\bullet(x)).
\]

In light of this, we see that \(\phi(x) \in \tilde{Y}_\gamma\) if and only if \(F_\bullet(x)\) is in position \(\gamma\) relative to the splitting \(V'(x) \oplus V''(x)\). This says that \(\phi^{-1}(\tilde{Y}_\gamma)\) is precisely the locus \(D_\gamma\) defined above.

Thus, assuming \([\phi^{-1}(\tilde{Y}_\gamma)] = \phi^*([\tilde{Y}_\gamma])\), as we do, we have that \([D_\gamma] = \phi^*([\tilde{Y}_\gamma])\). Again we mention that this should be thought of as a genericity assumption which requires that our splitting and our flag of subbundles are in general position with respect to one another.

Our next task is to relate the classes \(x_1, \ldots, x_n, y_1, \ldots, y_n\), in terms of which we have expressed the equivariant classes of \(K\)-orbit closures, to the Chern classes of the bundles \(V', V''\), and \(F_i/F_{i-1}\) \((i = 1, \ldots, n)\) on \(X\). The space \((G/B)_K = E \times K G/B\) carries two bundles \(S'_K\) and \(S''_K\) of ranks \(p\) and \(q\), respectively. Explicitly, the bundle \(S'_K\) is \((E \times K \mathbb{C}(e_1, \ldots, e_p)) \times G/B\), while the bundle \(S''_K\) is \((E \times K \mathbb{C}(e_{p+1}, \ldots, e_n)) \times G/B\).

When pulled back to \((G/B)_S\) via the natural map \((G/B)_S \to (G/B)_K\), these two bundles split as direct sums of line bundles. \(S'_K\) splits as a direct sum of \((E \times S \mathbb{C}_{Y_1}) \times G/B\) for \(i = 1, \ldots, p\), while \(S''_K\) splits as a direct sum of \((E \times S \mathbb{C}_{Y_1}) \times G/B\) for \(i = p + 1, \ldots, n\). The classes \(y_i \in H^*_S(G/B)\) are the first Chern classes of these line bundles. (This is spelled out in detail in the proof of [Wys12a, Proposition 1.1].) So the pullbacks of the Chern classes of \(S'_K\) and \(S''_K\) are the elementary symmetric polynomials in \(y_1, \ldots, y_p\) and \(y_{p+1}, \ldots, y_n\), respectively.

Since the pullback is an injection, when we consider \(H^*_S(G/B)\) as a subring of \(H^*_S(G/B)\), the Chern classes \(c_1(S'_K), \ldots, c_p(S'_K)\) are identically \(e_1(y_1, \ldots, y_p), \ldots, e_p(y_1, \ldots, y_p)\), respectively, while the Chern classes \(c_1(S''_K), \ldots, c_q(S''_K)\) are \(e_1(y_{p+1}, \ldots, y_n), \ldots, c_q(y_{p+1}, \ldots, y_n)\). The bundles \(S'_K\) and \(S''_K\) are identified with the bundles \(S'\) and \(S''\) on the isomorphic space \(BK \times _{BG} BB\), and as we have noted, the latter two bundles pull back to \(V'\) and \(V''\), respectively. Thus pulling back elementary symmetric polynomials in the \(y_i\) to \(X\) gives us the Chern classes of the bundles \(V'\) and \(V''\).

Now, consider the classes \(x_i\). \(G/B\) has a tautological flag of bundles \(T_\bullet\). Each bundle in this flag is \(K\)-equivariant, so that we get a flag of bundles \((T_\bullet)_K = E \times K T_\bullet\) on \((G/B)_K\). This flag pulls back to a tautological flag \((T_\bullet)_S\) on \((G/B)_S\) whose subquotients \((T_i)_S/(T_{i-1})_S\) are the line bundles \(E \times S \mathbb{C}_{X_1}\). Recall that the classes \(x_i\) are precisely the first Chern classes of the latter line bundles. The bundles \((T_\bullet)_K\) match up with the bundles \(T_\bullet\) on \(BK \times _{BG} BB\) via our isomorphism, and as we have noted, the latter bundles pull back to the flag \(F_\bullet\) of bundles on \(X\). Thus when we pull back to \(X\), the class \(x_i\) is sent to \(c_1(F_i/F_{i-1})\) for \(i = 1, \ldots, n\).

As an illustration, suppose we have a smooth complex variety \(X\) and a rank 4 vector bundle \(V \to X\). Suppose that \(V\) splits as a direct sum of rank 2 subbundles \((V = V' \oplus V'')\), and suppose further that \(V\) is equipped with a complete flag of subbundles \((F_1 \subset F_2 \subset F_3 \subset V)\). Let \(z_1, z_2, z_3, z_4\) be \(c_1(V'), c_2(V'), c_1(V''), c_2(V'')\), respectively. Let \(x_i = c_1(F_i/F_{i-1})\) for
Moreover, as has been mentioned, in each of our cases outside of type A skew-symmetric.

Thus given such a setup, using arguments as in the previous subsection, we can see that
certain types of degeneracy loci with respect to these various structures on the bundle V
are parametrized by the K′-orbit closures, and that our equivariant formulas for the orbit
 closures imply Chern class formulas for the classes of such loci. Again, the x-variables play
the role of the first Chern classes of the subquotients of the flag of subbundles, while the
elementary symmetric polynomials in the y-variables play the role of the Chern classes of
the two summands.
We note that a precise set-theoretic description of the degeneracy loci so parametrized by $K$-orbit closures in these cases depends upon knowing a set-theoretic description of the orbit closures, which we have only conjectured in the type $B$ and $C$ cases, and which we don’t even offer a conjecture for in any of the type $D$ cases. Presuming Conjecture 2.18 is correct, then the degeneracy loci for the type $B$ and $C$ cases are described set-theoretically just as those in type $A$ are, with regard to the relative position of the flag and the splitting, and we simply assume the further structures on the bundle to be in place. In the type $D$ case, it’s less clear what degeneracy loci are being parametrized. Some of the orbit closures in these cases are described set-theoretically just as in the type $A$ case, so that some of degeneracy loci in question are just as in the type $A$ case. However, as we have noted, other orbit closures are not described so simply, being only contained in the variety described by the type $A$ conditions. Thus degeneracy loci corresponding to such orbit closures are at least contained in the degeneracy loci described by the type $A$ conditions, but there are additional conditions that one must impose. It is not at all clear to the author what these conditions should be.
| Case | $(G, K)$ | Parameters for Closed Orbits | Formula for $K \cdot wB$ |
|------|------------|-------------------------------|-------------------------|
| 1    | $(GL(n, \mathbb{C}), GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$ | $(\pm)p,q$ | $(-1)^b(w) \prod_{i \leq p < j} (x_{w^{-1}(i)} - y_j)$ |
| 2    | $(SO(2n + 1, \mathbb{C}), S(O(2p, \mathbb{C}) \times O(2q + 1, \mathbb{C})))$ | Symmetric $(\pm)_{2p,2q+1}$ | $(-1)^b(|w|) x_{|w|^{-1}(1)} \cdots x_{|w|^{-1}(p)} \prod_{i \leq p < j} (x_{w^{-1}(i)} - y_j)(x_{w^{-1}(i)} + y_j)$ |
| 3    | $(Sp(2n, \mathbb{C}), Sp(2p, \mathbb{C}) \times Sp(2q, \mathbb{C}))$ | Symmetric $(\pm)_{2p,2q}$ | $(-1)^b(|w|) \prod_{i \leq p < j} (x_{w^{-1}(i)} - y_j)(x_{w^{-1}(i)} + y_j)$ |
| 4    | $(Sp(2n, \mathbb{C}), GL(n, \mathbb{C}))$ | Skew-symmetric $(\pm)_{n,n}$ | $(-1)^{\psi(w)+\sigma(w)} \Delta_n(x, y, w)$ |
| 5    | $(SO(2n, \mathbb{C}), S(O(2p, \mathbb{C}) \times O(2q, \mathbb{C})))$ | Symmetric $(\pm)_{2p,2q}$ | $(-1)^b(|w|) \prod_{i \leq p < j} (x_{w^{-1}(i)} - y_j)(x_{w^{-1}(i)} + y_j)$ |
| 6    | $(SO(2n, \mathbb{C}), GL(n, \mathbb{C}))$ | Skew-symmetric $(\pm)_{n,n}$ | $(-1)^{\sigma(w)} \frac{1}{2} n^{-1} \Delta_{n-1}(x, y, w)$ |
| 7    | $(SO(2n, \mathbb{C}), S(O(2p + 1, \mathbb{C}) \times O(2q - 1, \mathbb{C})))$ | Symmetric $(\pm, 1, 1, \pm)_{2p+1,2q-1}$ | $(-1)^{\tau(w)} x_{1} \cdots x_{n-1} \prod_{i \leq p < p+1 < j} (x_{w^{-1}(i)} - y_j)(x_{w^{-1}(i)} + y_j)$ |

Table 1. Formulas for equivariant classes of closed $K$-orbits
Appendix: Weak Order Graphs and Tables of Formulas in Examples

Figure 1. \((GL(4, \mathbb{C}), GL(2, \mathbb{C}) \times GL(2, \mathbb{C}))\)

References

[Bri98] Michel Brion. Equivariant cohomology and equivariant intersection theory. In Representation theories and algebraic geometry (Montreal, PQ, 1997), volume 514 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., pages 1–37. Kluwer Acad. Publ., Dordrecht, 1998. Notes by Alvaro Rittatore.

[Bri99] M. Brion. Rational smoothness and fixed points of torus actions. Transform. Groups, 4(2-3):127–156, 1999. Dedicated to the memory of Claude Chevalley.

[Bri01] Michel Brion. On orbit closures of spherical subgroups in flag varieties. Comment. Math. Helv., 76(2):263–299, 2001.

[Ful92] William Fulton. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. Duke Math. J., 65(3):381–420, 1992.

[Ful96a] William Fulton. Determinantal formulas for orthogonal and symplectic degeneracy loci. J. Differential Geom., 43(2):276–290, 1996.

[Ful96b] William Fulton. Schubert varieties in flag bundles for the classical groups. In Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), volume 9 of Israel Math. Conf. Proc., pages 241–262, Ramat Gan, 1996. Bar-Ilan Univ.

[Ful97] William Fulton. Young tableaux, volume 35 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[Gra97] William Graham. The class of the diagonal in flag bundles. J. Differential Geom., 45(3):471–487, 1997.

[Inc04] Federico Incitti. The Bruhat order on the involutions of the symmetric group. J. Algebraic Combin., 20(3):243–261, 2004.

[Mat79] Toshihiko Matsuki. The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. J. Math. Soc. Japan, 31(2):331–357, 1979.
Figure 2. \((SO(7, \mathbb{C}), S(O(4, \mathbb{C}) \times O(3, \mathbb{C}))\)

---

[MO90] Toshihiko Matsuki and Toshio Ōshima. Embeddings of discrete series into principal series. In *The orbit method in representation theory (Copenhagen, 1988)*, volume 82 of *Progr. Math.*, pages 147–175. Birkhäuser Boston, Boston, MA, 1990.

[MT09] William M. McGovern and Peter E. Trapa. Pattern avoidance and smoothness of closures for orbits of a symmetric subgroup in the flag variety. *J. Algebra*, 322(8):2713–2730, 2009.

[RS90] R. W. Richardson and T. A. Springer. The Bruhat order on symmetric varieties. *Geom. Dedicata*, 35(1-3):389–436, 1990.

[RS93] R. W. Richardson and T. A. Springer. Combinatorics and geometry of \(K\)-orbits on the flag manifold. In *Linear algebraic groups and their representations (Los Angeles, CA, 1992)*, volume 153 of *Contemp. Math.*, pages 109–142. Amer. Math. Soc., Providence, RI, 1993.

[Spr85] T. A. Springer. Some results on algebraic groups with involutions. In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, volume 6 of *Adv. Stud. Pure Math.*, pages 525–543. North-Holland, Amsterdam, 1985.

[Spr87] T. A. Springer. The classification of involutions of simple algebraic groups. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(3):655–670, 1987.
Figure 3. \((Sp(6, \mathbb{C}), Sp(4, \mathbb{C}) \times Sp(2, \mathbb{C}))\)

\[
\begin{array}{c}
(1,2,+,+,1,2) \\
\downarrow \\
(1,+,2,1,+,2) \\
\downarrow \\
(+,1,2,1,2,+) \\
\downarrow \\
(+,1,1,2,2,+) \\
\downarrow \\
(+,+,-,-,+,+) \\
\end{array}
\]

Figure 4. \((Sp(4, \mathbb{C}), GL(2, \mathbb{C}))\)

\[
\begin{array}{c}
(1,2,2,1) \\
\downarrow \\
(1,+,1,1) \\
\downarrow \\
(+,1,1,+) \\
\downarrow \\
(+,+,+,+) \\
\end{array}
\]

[Wys12] Benjamin J. Wyser. K-orbit closures on \(G/B\) as universal degeneracy loci for flagged vector bundles with symmetric or skew-symmetric bilinear form. ArXiv e-prints, June 2012. Submitted. arXiv:1206.6907
[Wys12b] Benjamin J. Wyser. Symmetric subgroup orbit closures on flag varieties: Their equivariant geometry, combinatorics, and connections with degeneracy loci. PhD thesis, University of Georgia, 2012.

[Yam97] Atsuko Yamamoto. Orbits in the flag variety and images of the moment map for classical groups. I. Represent. Theory, 1:329–404 (electronic), 1997.
Figure 7. \((SO(6,\mathbb{C}), S(O(3,\mathbb{C}) \times O(3,\mathbb{C}))\)

Table 2. Formulas for \((GL(4,\mathbb{C}), GL(2,\mathbb{C}) \times GL(2,\mathbb{C}))\)

| (2, 2)-clan \(\gamma\) | Formula for \([Y_\gamma]\) |
|------------------------|--------------------------|
| ++ + -                 | \((x_1 - y_3)(x_1 - y_4)(x_2 - y_3)(x_2 - y_4)\) |
| + - + -                | \(-(x_1 - y_3)(x_1 - y_4)(x_3 - y_3)(x_3 - y_4)\) |
| + - - +                | \((x_1 - y_3)(x_1 - y_4)(x_4 - y_3)(x_4 - y_4)\) |
| + + + -                | \((x_2 - y_3)(x_2 - y_4)(x_3 - y_3)(x_3 - y_4)\) |
| + + - +                | \(-(x_2 - y_3)(x_2 - y_4)(x_4 - y_3)(x_4 - y_4)\) |
| + - + +                | \((x_3 - y_3)(x_3 - y_4)(x_4 - y_3)(x_4 - y_4)\) |
| 11 + -                 | \((x_1 - y_3)(x_1 - y_4)(x_2 + x_3 - y_3 - y_4)\) |
| 11 + 1                 | \((x_1 - y_3)(x_1 - y_4)(x_1 + x_2 - y_3 - y_4)\) |
| 11 - +                 | \((x_1 - y_3)(x_1 - y_4)(x_3 + x_4 - y_3 - y_4)\) |
| 11 - 1                 | \((x_2 - y_3)(x_2 - y_4)(x_3 + x_4 - y_3 - y_4)\) |
| -1 + 1                 | \((x_2 - y_3)(x_2 - y_4)(x_2 + x_3 - y_3 - y_4)\) |
| 1212                   | \((x_1 + x_2 - y_3 - y_4)(x_3 + x_4 - y_3 - y_4)\) |
| 1211                   | \((x_4 - y_3)(x_4 - y_4)\) |
| 1221                   | \((x_4 - y_3)(x_4 - y_4)\) |
| 1222                   | \((x_3 + x_4 - y_3 - y_4)\) |
| 1221                   | \((x_1 + x_2 - y_3 - y_4)(x_3 + x_4 - y_3 - y_4)\) |
| 1211                   | \((x_4 - y_3)(x_4 - y_4)\) |
| 1122                   | \((x_1 + x_2 - y_3 - y_4)(x_3 + x_4 - y_3 - y_4)\) |
| 1121                   | \((x_4 - y_3)(x_4 - y_4)\) |
| 1211                   | \((x_4 - y_3)(x_4 - y_4)\) |
| 1221                   | \((x_1 + x_2 - y_3 - y_4)(x_3 + x_4 - y_3 - y_4)\) |
| 1211                   | \((x_4 - y_3)(x_4 - y_4)\) |
Table 3. Formulas for \((SO(7, \mathbb{C}), S(O(4, \mathbb{C}) \times O(3, \mathbb{C}))\)

| Symmetric (4,3)-clan \(\gamma\) | Formula for \([Y_{\gamma}]\) |
|----------------------------------|----------------------------------|
| \(+ + - - + +\)                 | \(x_1 x_2 (x_1 - y_3)(x_1 + y_3)(x_2 - y_3)(x_2 + y_3)\) |
| \(+ - + - + +\)                 | \(-x_1 x_3 (x_1 - y_3)(x_1 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| \(- + + - + +\)                 | \(-x_2 x_3 (x_2 - y_3)(x_2 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| \(+11 - 22+\)                   | \(x_1 (x_1 - y_3)(x_1 + y_3)(x_2^2 + x_2 x_3 + x_3^2 - y_3^2)\) |
| \(11 + - 22\)                   | \(-x_3 (x_3 - y_3)(x_3 + y_3)(x_1^2 + x_1 x_2 + x_2^2 - y_3^2)\) |
| \(+ - 1 + 1 -\)                 | \(-x_1 (x_1 - y_3)(x_1 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| \(- + 1 + 1 +\)                 | \(x_2 (x_2 - y_3)(x_2 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| \(1 + 1 - 2 + 2\)               | \(x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2 x_3^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_2 x_3 - x_1^2 y_3^2 - x_2^2 y_3^2 - x_3^2 y_3^2 + y_3^2\) |
| \(12 - 12+\)                    | \(2x_1 x_2 (x_1 - y_3)(x_1 + y_3)\) |
| \(+1 - + - 1+\)                 | \(-x_1 (x_1 - y_3)(x_1 + y_3)(x_2 + x_3)\) |
| \(113 + 322\)                   | \(-(x_3 - y_3)(x_3 + y_3)(x_1^2 + x_1 x_2 + x_2^2 - y_3^2)\) |
| \(-1 + + 1-\)                   | \((x_2 - y_3)(x_2 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| \(1 + 2 - 1 + 2\)               | \(2x_1 x_2 + 2x_1 x_3\) |
| \(+12 - 21+\)                   | \(x_1 (x_1 - y_3)(x_1 + y_3)\) |
| \(1 + + + - 1\)                 | \((x_1 + x_2)(x_3 - y_3)(x_3 + y_3)\) |
| \(12 + + 12\)                   | \(2x_1^2 + 2x_1 x_2 + 2x_1 x_3\) |
| \(1 + 2 - 2 + 1\)               | \(x_1^2 + x_1 x_2 + x_2^2 - y_3^2\) |
| \(132 - 132\)                   | \(2x_1 (x_1 + x_2)\) |
| \(311 + 223\)                   | \(x_1 x_2 + x_1 x_3 + x_2 x_3 + y_3^2\) |
| \(12 - + 21\)                   | \(x_1 + x_2 + x_3\) |
| \(123 + 312\)                   | \(2x_1\) |
| \(312 + 123\)                   | \(2(x_1 + x_2)\) |
| \(123 + 321\)                   | \(1\) |

Table 4. Formulas for \((Sp(6, \mathbb{C}), Sp(4, \mathbb{C}) \times Sp(2, \mathbb{C}))\)

| Symmetric (4,2)-clan \(\gamma\) | Formula for \([Y_{\gamma}]\) |
|----------------------------------|----------------------------------|
| \(+ + - - + +\)                 | \((x_1 - y_3)(x_1 + y_3)(x_2 - y_3)(x_2 + y_3)\) |
| \(+ - + - + +\)                 | \(- (x_1 - y_3)(x_1 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| \(- + + - + +\)                 | \((- (x_2 - y_3)(x_2 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| \(+1122+\)                      | \((x_1 - y_3)(x_1 + y_3)(x_2 + x_3)\) |
| \(11 + 22\)                     | \(- (x_3 - y_3)(x_3 + y_3)(x_1 + x_2)\) |
| \(1 + 12 + 2\)                  | \(x_1 x_2 + x_1 x_3 + x_2 x_3 + y_3^2\) |
| \(+1212+\)                      | \((x_1 - y_3)(x_1 + y_3)\) |
| \(1 + 21 + 2\)                  | \(x_1 + x_2\) |
| \(12 + + 12\)                   | \(1\) |
Table 5. Formulas for \((Sp(4, \mathbb{C}), GL(2, \mathbb{C}))\)

| Skew-symmetric \((2, 2)\)-clan \(\gamma\) | Formula for \([Y_\gamma]\) |
|-------------------------------------------|-----------------|
| + + --                                   | \((x_1 + x_2 + y_1 + y_2)(x_1 x_2 + y_1 y_2)\) |
| + -- +                                   | \(-(x_1 - x_2 + y_1 + y_2)(-x_1 x_2 + y_1 y_2)\) |
| - + ++                                   | \(-(x_1 + x_2 + y_1 + y_2)(-x_1 x_2 + y_1 y_2)\) |
| - -- ++                                   | \(-(x_1 - x_2 + y_1 + y_2)(x_1 x_2 + y_1 y_2)\) |
| +11--                                    | \((x_1 + y_1)(x_1 + y_2)\) |
| 1122                                     | \(2(x_1 x_2 - y_1 y_2)\) |
| -11+                                      | \((x_1 - y_1)(x_1 - y_2)\) |
| 1 + -1                                     | \(x_1 + x_2 + y_1 + y_2\) |
| 1212                                     | \(2x_1\) |
| 1 -- +1                                    | \(x_1 + x_2 - y_1 - y_2\) |
| 1221                                     | \(1\) |

Table 6. Formulas for \((SO(6, \mathbb{C}), S(O(4, \mathbb{C}) \times O(2, \mathbb{C}))\))

| Symmetric \((4, 2)\)-clan \(\gamma\) | Formula for \([Y_\gamma]\) |
|---------------------------------------|-----------------|
| + + -- +                              | \((x_1 - y_3)(x_1 + y_3)(x_2 - y_3)(x_2 + y_3)\) |
| + -- + --                              | \(-(x_1 - y_3)(x_1 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| - -- + +                              | \(-(x_2 - y_3)(x_2 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| +1122++                                 | \((x_2 - y_3)(x_2 + y_3)(x_3 - y_3)(x_3 + y_3)\) |
| +1212++                                 | \((x_1 - y_3)(x_1 + y_3)(x_2 - y_3)\) |
| 11 + +22                                | \(-(x_2 - y_3)(x_3 + y_3)(x_1 + x_2)\) |
| 1 + 12 + 2                              | \(x_1 x_2 + x_1 x_3 + x_2 x_3 + y_3^2\) |
| +1221+                                  | \((x_1 - y_3)(x_1 + y_3)\) |
| 1 + 21 + 2                              | \(x_1 x_2 - x_1 x_3 - x_2 x_3 + y_3^2\) |
| 12 + +12                                | \(x_1\) |
| 1 + 22 + 1                              | \(x_1 + x_2\) |
| 12 + +21                                | \(1\) |

Table 7. Formulas for \((SO(6, \mathbb{C}), GL(3, \mathbb{C}))\)

| Skew-symmetric \((3, 3)\)-clan \(\gamma\) | Formula for \([Y_\gamma]\) |
|-------------------------------------------|-----------------|
| + + + --                                 | \(\frac{1}{3} \Delta_2(x, y, id)\) |
| - + + +--+                               | \(-\frac{1}{3} \Delta_2(x, y, (1 2 3))\) |
| - + + --                                 | \(\frac{1}{3} \Delta_2(x, y, (1 2 3))\) |
| + -- + +--+                               | \(-\frac{1}{3} \Delta_2(x, y, (1 2 3))\) |
| +1212--                                  | \(\frac{1}{3}(x_1 x_2 - x_3 + x_1 y_1 + x_1 y_2 + y_1 y_2 + x_1 y_3 + y_1 y_3 + y_2 y_3)\) |
| -1122+                                   | \(\frac{1}{3}(x_1 x_2 - x_3 - x_1 y_1 - x_1 y_2 + y_1 y_2 - x_1 y_3 + y_1 y_3 + y_2 y_3)\) |
| 11 + +22                                 | \(\frac{1}{3}(x_1 x_2 - x_3 - x_1 y_1 + x_1 y_2 + y_1 y_2 + x_1 y_3 + y_1 y_3 + y_2 y_3)\) |
| 1 + 21 --                                 | \(\frac{1}{3}(x_1 + x_2 - x_3 + y_1 + y_2 + y_3)\) |
| 1 -- 12 +                                 | \(\frac{1}{3}(x_1 + x_2 + x_3 - y_1 - y_2 - y_3)\) |
| 12 + +12                                 | \(1\) |
Table 8. Formulas for \((SO(6, \mathbb{C}), S(O(3, \mathbb{C}) \times O(3, \mathbb{C}))\)

| Symmetric (3, 3)-clan \(\gamma\) | Formula for \([Y_\gamma]\) |
|---------------------------------|----------------------------------|
| + − 11 − +                      | \(x_1x_2(x_1 - y_3)(x_1 + y_3)\) |
| − + 11 + −                      | \(-x_1x_2(x_2 - y_3)(x_2 + y_3)\) |
| 112233                          | \(x_1x_2(x_1 + x_2)\)           |
| +1 − −1+                        | \(x_1(x_1 - y_3)(x_1 + y_3)\)   |
| −1 + +1−                        | \(-x_1(x_2^2 + x_2x_3 + x_3^2 - y_3^2)\) |
| 121323                          | \(x_1(x_1 + x_2 + x_3)\)        |
| 123123                          | \(x_1(x_1 + x_2 - x_3)\)        |
| 1 + − − +1                      | \(x_1^2 + x_1x_2 + x_2^2 - y_3^2\) |
| 1 − + + −1                      | \(x_1x_2 - x_3^2 + y_3^2\)      |
| 122331                          | \(x_1 + x_2 + x_3\)             |
| 123312                          | \(2x_1\)                        |
| 123231                          | \(x_1 + x_2 - x_3\)             |
| 123321                          | 1                                |