FLUID STRUCTURE INTERACTION PROBLEM WITH CHANGING THICKNESS BEAM AND SLIGHTLY COMPRESSIBLE FLUID

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ABSTRACT. In this work, we consider the dynamical response of a non-linear beam with changing thickness, perturbed in both vertical and axial directions, interacting with a Darcy flow. We explore this fluid-structure interaction problem where the fluid is assumed to be slightly compressible. In an appropriate Sobolev norm, we build an energy functional for the displacement field of the beam and the gradient pressure of the fluid flow. We show that for a class of boundary conditions the energy functional is bounded by the flux of mass through the inlet boundary.

1. Introduction. The dynamics of non-linear thin structures interacting with fluid flows attracted the attention of many researchers in recent years (see e.g. Refs. [4], [8], [9], [11], [25] and references therein). This problem is inherent to a number of topics in bio-medicine, engineering, geo-sciences, etc. It was observed for many years that ignoring non-linearity in solid part can lead to significant errors in forecasting the dynamics of elastic structures (see Refs. [24], [20]). We want specifically refer on an excellent paper by H. Koch, and I. Lasiecka [17] in which a comprehensive review of Hadamard well-posedness of corresponding IBVP with different homogeneous (zero) boundary conditions and RHS in the system of the equation is presented. These are particular important in exploring the stability of the solid structure subjected to the loads generated by the fluid flow. In turn, the fluid domain is under the impact of the solid body as a moving boundary. The mutual interaction of fluid-structure system is a difficult problem and is far from being fully understood. Biggest progress so far made for solids with non-changing thickness, and incompressible fluid. Fluid-plate interaction between elastic plates and viscous incompressible fluids (described by Navier Stokes or Stokes equations) have also been recently studied in great detail (see e.g. [4], [6], [7], [9], [10], [11], [21] and references therein). Existence of the weak solutions for the coupled systems has been proven in [9], [10]. Existence of solutions to the free boundary fluid-structure interaction system was proven for Navier-Stokes in the fluid part and wave equation for solid part defined in two different but adjacent domains in [18]. This type of model was first introduced in [5] for incompressible fluid, and in [3] for compressible fluid. Long term dynamics of the solutions has been studied in [4], and strong stability estimates have been derived under additional assumptions on the data [6]. In these studies, a more simple plate model is used compared to the von Karman
model; namely, either in-plane displacements ([6], [7], [9]) or transverse displace-
ments ([4], [10], [11]) of the plate are disregarded. Moreover, the fluid domain is in
most cases subjected to small deformations and treated as a pure Eulerian system.
In [18] most current review of the results was presented in big detail. The biggest
progress so far is achieved in numerical studies and is based on the machinery de-
veloped to simulate moving boundaries (see e.g. Refs. [2], [23], [22]) and references
therein). In the majorities of these studies,

- thickness of the plate is assumed to be constant. However, in the present
  work, we consider a non-linear beam model derived in [13], where the change
  in the thickness of the beam is taken into account by introducing it into
  the momentum of inertia under the incompressibility constraint of the solid
  material. So, even if we use a one dimensional structure for the solid part
  of our model, considering the change in the thickness of this structure makes
  this study quite interesting.

- either in-plane displacements (Refs. [6], [7], [9]) or transverse displacement
  (Refs. [4], [10], [11]) of the plate is neglected relative to the other. In our
  non-linear beam model however, longitudinal and transverse displacements of
  the neutral axis are fully coupled as a system of equations.

- the fluid domain is in most cases subjected to small deformations (and treated
  as a pure Eulerian system) and, to our knowledge, no fluid has been considered
  to be compressible. The fluid flow in the present work is described by the
  linear Darcy’s momentum equation but the fluid is considered to be (slightly)
  compressible.

In this work we primarily investigate mathematical aspects of the dynamics of
a one dimensional non-linear beam interacting with a two dimensional potential
flow. Having in mind a class of geofluidics problems, we consider the elastic beam
as the top boundary of a porous media domain, saturated with one phase slightly
compressible fluid. In the current paper we assume that the fluid-structure interac-
tion occurs directly on the boundary between beam and fluid domain. The porous
media is represented only as the moving domain of the liquid flow. Although this
an idealization of complex processes, it will provide important understanding of the
impact of parameters on the dynamics of the coupled system.

This article is partially motivated by our previous works Refs.[16], [15], [13], [14],
and can be considered as extension of the analysis of two coupled models derived in
[14] for fluid part and [13] for solid part. Energy estimates were derived for a non-
linear beam (with constant thickness [15] and changing thickness[13]) perturbed
in both the transverse and axial directions and interacting with an incompressible
potential flow. In the present work considering a Darcy type potential flow of the
compressible fluid makes the problem more difficult in a theoretical and practical
point of view. Considering a non-linear Forchheimer model [1] instead of Darcy
would make the problem even more interesting and challenging but we leave this
case to upcoming research.

In this work, we introduce the fluid-beam coupled model and investigate the
stability of this system with respect to the inflow (accessible) boundary condition.
The difficulty here is that the fluid domain changes as the beam deforms which
requires an arbitrary Lagrangian-Eulerian (ALE) formulation for the fluid part.
The system is coupled on the top boundary through the pressure exerted from the
fluid to the beam. We construct an energy balance equation which relates energy
functionals corresponding to both beam and fluid domains with the work done by the external loads on the beam.

Firstly, in Subsection 3.1 we investigate stability of the trivial solution of coupled system without any a priory constraints on the coupled fluid-structure system - this results in Theorem 3.2. Although the obtained result is generic, it does not lead to strong stability of trivial solution because the estimate for energy functional depends on time and may be effective only for vanishing in time input boundary data.

To eliminate this deficiency we impose an extra constraint on the thickness of the beam and geometry of the moving boundary in the Subsection 3.2. In depth analysis of the energy balance equation obtained in Theorem 3.2 results in Theorem 3.5, which states that under some constraints, the coupled fluid-beam energy functional is bound by the flux of mass through the accessible boundary, plus the initial data.

2. System of non-linear beam equations and energy equation. Consider a beam of length $L$ clamped at the end points subjected to some external forces. We enhance the Euler-Bernoulli beam model by considering the axial displacement in addition to the transversal displacement and also by relaxing the simplifying assumption of constant thickness. The following is the system of non-linear equations for this modified beam model (see [13])

$$
\begin{align*}
\ddot{u} + K_1 u_t - D_1 \left[ u_x + \frac{1}{2}(w_x)^2 \right]_x &= q_1, \quad (1) \\
\ddot{w} + K_1 w_t - D_1 \left[ w_x \left( u_x + \frac{1}{2}(w_x)^2 \right) \right]_x + D_2 \left[ w_{xx} \delta^3 \right]_{xx} &= q_2. \quad (2)
\end{align*}
$$

Here,

$$
\begin{align*}
u &= u(x, t) \text{ and } w = w(x, t)
\end{align*}
$$

are the axial and transversal displacements of a generic point on the beam neutral axis,

$$
\begin{align*}
K_1 &= C_1/\rho A, \quad D_1 = E/\rho, \quad D_2 = E I_0/\rho A
\end{align*}
$$

are all constant physical parameters of the system where $C_1$ is the damping coefficient, $\rho$ is the density of the beam, $A$ is the area of the beam cross section, $E$ is the Young’s modulus, and $I_0$ is the momentum of inertia in the undeformed configuration,

$$
\begin{align*}
q_1 &= q_1(x, t) = f_1/\rho A \text{ and } q_2 = q_2(x, t) = f_2/\rho A,
\end{align*}
$$

where $f_1 = f_1(x, t)$ and $f_2 = f_2(x, t)$, respectively, are the external axial and transversal applied loads to the beam,

$$
\begin{align*}
\delta = \delta(x, t) = \frac{1}{\sqrt{1 + 2 \left[ u_x + \frac{1}{2}(w_x)^2 \right]}}
\end{align*}
$$

is the relative thickness term which is defined as the ratio of the distance of a point from the neutral axis in the deformed configuration to the distance in the undeformed configuration \(^1\).

\(^1\)See [13] for derivation of $\delta$. 
Since the beam is considered to be clamped at the end points, the system (1)-(2) satisfies the following boundary conditions

\[ u(0,t) = u(L,t) = 0, \forall t \geq 0, \]  
\[ w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0, \forall t \geq 0. \]  

For the further analysis, we will make some physical assumptions on the relative thickness term \( \delta \).

**Constraint 1.** We will assume that the relative thickness is not vanishing in time, and the rate of the changes of weighted relative thickness of the beam, \( \delta \), satisfies the following. For some \( b_0 \) and \( \tilde{C} \), which will be selected later depending on the parameters of the problem, exist constants \( 0 < b < b_0 \) and \( 0 < C < \tilde{C} \) such that

\[ (e^{-bK_1t}\delta^3)_t \leq Ce^{-bK_1t}\delta^3, \forall t, \]  
and there exists positive constant \( C \) such that

\[ C \leq \delta, \forall t. \]  

Before we develop a beam-fluid interaction problem, analyzing the non-linear beam system (1)-(2) yields the following energy-balance equation (see [13] for details)

\[
\frac{d}{dt} \int_0^L \left( \frac{a}{2} (u_t + K_1 u)^2 + \frac{a}{4} (w_t + K_1 w)^2 + \frac{1-a}{2} u_t^2 + \left( \frac{1}{2} - \frac{a}{4} \right) w_t^2 \right) dx + \\
+ \frac{D_1}{2} \left( u_x + \frac{1}{2} (w_x)^2 \right)^2 + \frac{D_2}{2} \left( w_{xx} \delta^3 \right) dx + \\
+ \int_0^L \left( (1-a)K_1 u_t^2 + \left( 1 - \frac{a}{2} \right) K_1 w_t^2 + aK_1D_1 \left[ u_x + \frac{1}{2} (w_x)^2 \right]^2 + \\
+ \frac{D_2}{2} \left( aK_1 \delta^3 - (\delta^3)_t \right) \right) dx = \int_0^L q_1 (aK_1 u + u_t) + q_2 \left( \frac{aK_1 w}{2} + w_t \right) dx.
\]

where \( a \) is any real number such that \( 0 \leq a \leq 1 \). We will use this energy equation later in the beam-fluid coupled analysis.

**3. Beam-fluid interaction.** We will now formulate the fluid-beam interaction problem, where the beam is modeled as the top boundary of the 2-dimensional porous media. The beam equations are modeled in a Lagrangian reference domain \( \Gamma \). The fluid equation is modeled in an moving Euler domain \( \tilde{\Omega}(t) \), whose reference configuration is given below. Let

- \( \Omega \) be the reference/undeformed configuration of the porous media domain :
  \[ \Omega := [0,L] \times [0,H], \]
- \( \Gamma \) be the reference/undeformed top boundary: \( \Gamma = [0,L] \times \{H\}, \)
- \( \Gamma_1 \) be the inlet region (left): \( \Gamma_1 = \{0\} \times [0,H], \)
- \( \Gamma_2 \) be the outlet region (right): \( \Gamma_2 = \{L\} \times [0,H], \)
- \( \Gamma_3 \) be the impermeable bottom boundary: \( \Gamma_3 = [0,L] \times \{0\}. \)
The top boundary $\bar{\Gamma}(t)$ is the only moving boundary which deforms according to the the beam displacements. By continuity constraint it satisfies the following

$$\bar{\Gamma}(t) = \{(\bar{x}, \bar{y}) = (x + u(x,t), H + w(x,t)) \mid (x,H) \in \Gamma\}.$$  \hfill (8)

where $u$, and $w$ are the axial and transversal displacements of a point on the beam reference domain. The moving domain $\bar{\Omega}(t)$ is then the region $\bar{x} = (\bar{x}, \bar{y})$ enclosed by $\bar{\Gamma}(t)$, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$.

**Note 1.** The boundaries $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are fixed in time because the beam is clamped on its boundary $\partial \Gamma$ (conditions (3)-(4)).

For modeling the balance of fluid momentum through the porous media $\bar{\Omega}(t)$, Darcy equation is used

$$v_f = -\Pi \nabla P,$$  \hfill (9)

where $v_f(\bar{x},t)$ is the pore velocity, $P(\bar{x},t)$ is the pressure, and $\Pi$ is equal to the ratio between the the permeability of the porous rock $k$ and the fluid viscosity $\mu$. Here the porous medium is considered to be homogeneous, thus $\Pi$ is a constant.

In this work, we assume that the fluid is slightly compressible and satisfies the following equation of state

$$\frac{1}{\rho_f} \frac{d\rho_f}{dp} = \frac{1}{\gamma}, \text{ or } \rho_f(P) = \rho_{f_0} e^{\frac{p-P_0}{\gamma}},$$  \hfill (10)

where $\rho_f(\bar{x},t)$ is the density of the fluid, $\rho_{f_0} = \rho_f(P_0)$ is the density of the fluid at the reference pressure $P_0$, and $\frac{1}{\gamma}$ is the fluid compressibility. For simplicity we assume $P_0 = 0$.

Using continuity equation

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot (\rho_f v_f).$$  \hfill (11)

together with the state equation (10), we can easily obtain the following relation for the pressure

$$\frac{\partial P}{\partial t} + v_f \cdot \nabla P = -\gamma \nabla \cdot v_f \text{ in } \bar{\Omega}(t).$$  \hfill (12)

Finally, we consider the following boundary conditions for the porous medium

$$\begin{cases} Q := \rho_f v_f \cdot n = \psi(y,t) & \text{ on } \Gamma_1, \\ \rho_f = \rho_{f_0} \text{ (or } P = 0) & \text{ on } \Gamma_2, \\ \nabla P \cdot n = 0 & \text{ on } \Gamma_3, \end{cases}$$  \hfill (13)

where $Q$ is the flux of mass through the boundary. On the top boundary the coupled system satisfies

- the **continuity constraint**

$$\langle u_t(x,t), w_t(x,t) \rangle \cdot n(\bar{x}, \bar{y}) = v_f(\bar{x}, \bar{y}) \cdot n(\bar{x}, \bar{y}),$$  \hfill (14)

where $(\bar{x}, \bar{y}) = (\bar{x}(x,t), \bar{y}(x,t))$ as defined in (8) and $n$ is the unit normal vector of $\bar{\Gamma}(t)$,

- and the **momentum balance constraint**

$$\rho h \int_{\Gamma} < q_1, q_2 > \cdot \omega(x,t) \, dx = \int_{\bar{\Gamma}(t)} P n \cdot \omega(\bar{x}, \bar{y}, t) \, d\bar{s},$$  \hfill (15)

for any test function $\omega$. Here, $\rho h < q_1, q_2 >$ is the force field ($\rho$ and $h$ are the density and initial thickness of the beam), and $d\bar{s}$ is the infinitesimal arc length on $\bar{\Gamma}(t)$.
Assuming that the classical solution of the coupled system exists for all time, we will now analyze the dynamics of the porous media-beam coupled system:

3.1. **Estimate of Type 1: No a priori constraint on the deformation and compressibility** \((a = 0)\). Choosing the test function \(\omega = \langle u_t, w_t \rangle\) in the *momentum balance constraint* (15) yields

\[
\int_\Gamma q_1 u_t + q_2 w_t \, dx = \frac{1}{\rho h} \int_{\Gamma(t)} P < u_t, w_t > \, d\mathbf{s}.
\]

Using continuity constraint (14) on the RHS of the equation above, and substituting this result in the RHS of the beam energy equation (7), with \(a = 0\), we obtain the following energy equation for the porous media-beam coupled system

\[
\frac{d\tilde{F}(t)}{dt} + \tilde{G}(t) = \frac{1}{\rho h} \int_{\Gamma(t)} P \mathbf{v}_f \cdot \mathbf{n} \, d\mathbf{s},
\]

where

\[
\tilde{F}(t) := \int_\Gamma \left( \frac{1}{2} (u^2 + w^2) + \frac{D_1}{2} \left[ u_x + \frac{1}{2} (w_x)^2 \right]^2 + \frac{D_2}{2} w_{xx} \delta^3 \right) dx \geq 0,
\]

and

\[
\tilde{G}(t) = \int_\Gamma K_1 (u^2 + w^2) - \frac{D_2}{2} w_{xx} (\delta^3)_t \, dx \geq 0.
\]

Using Green’s Theorem, Darcy’s equation (9), and the boundary conditions (13), the RHS of the equation (17) can be rewritten as

\[
\int_{\Gamma(t)} P \mathbf{v}_f \cdot \mathbf{n} \, d\mathbf{s} = \int_{\Omega(t)} \nabla \cdot (P \mathbf{v}_f) \, d\mathbf{A} - \int_{\Gamma(t) \cup \Gamma_3} P \mathbf{v}_f \cdot \mathbf{n} \, d\mathbf{s}
\]

\[
- \int_{\Omega(t)} \Pi \nabla |P|^2 \, d\mathbf{A} + \int_{\Omega(t)} P \nabla \cdot \mathbf{v}_f \, d\mathbf{A} + \int_{\Gamma_1} P \mathbf{v}_f \cdot \mathbf{n} \, d\mathbf{s}.
\]

In the following, second term on the RHS of the last equation above will be analyzed.

**Lemma 3.1.** For any \(k \geq 1\), the following relation holds

\[
\int_{\Omega(t)} P^{k-1} \nabla \cdot \mathbf{v}_f \, d\mathbf{A} = \frac{1}{k \gamma} \left[ \int_{\Omega(t)} P^k \nabla \cdot \mathbf{v}_f \, d\mathbf{A} - \frac{d}{dt} \int_{\Omega(t)} P^k \, d\mathbf{A} + \int_{\Gamma_1} P^k \mathbf{v}_f \cdot \mathbf{n} \, d\mathbf{s} \right].
\]

**Proof.** From the equation (12), we have

\[
\int_{\Omega(t)} P^{k-1} \nabla \cdot \mathbf{v}_f \, d\mathbf{A} = - \int_{\Omega(t)} \frac{P^{k-1}}{\gamma} \left( \frac{\partial P}{\partial t} + \mathbf{v}_f \cdot \nabla P \right) \, d\mathbf{A}
\]

\[
= - \frac{1}{k \gamma} \int_{\Omega(t)} \frac{\partial P^k}{\partial t} \, d\mathbf{A} - \frac{1}{k \gamma} \int_{\Omega(t)} \mathbf{v}_f \cdot \nabla P^k \, d\mathbf{A}.
\]

Partial derivative with respect to time in the integral term \(\int_{\Omega(t)} \frac{\partial P^k}{\partial t} \, d\mathbf{A}\) can be taken out of the integral as a total derivative in the following way

\[
\int_{\Omega(t)} \frac{\partial P^k}{\partial t} \, d\mathbf{A} = \frac{d}{dt} \int_{\Omega(t)} P^k \, d\mathbf{A} - \int_{\Gamma(t)} P^k \mathbf{v}_f \cdot \mathbf{n} \, d\mathbf{s}.
\]
which is an extension of Leibniz’s formula for moving domain and can be found in [12]. Using Green’s Theorem in the second integral term on the RHS of the equation above yields the following
\[
\int_{\Omega(t)} \frac{\partial P_k}{\partial t} \, d\tilde{A} = \frac{d}{dt} \int_{\Omega(t)} P_k \, d\tilde{A} - \int_{\Omega(t)} \nabla (P_k \cdot \vec{v}_f) \, d\tilde{A} + \int_{\Gamma_1} P_k \vec{v}_f \cdot \vec{n} \, d\tilde{s} \\
= \frac{d}{dt} \int_{\Omega(t)} P_k \, d\tilde{A} - \int_{\Omega(t)} (\vec{v}_f \cdot \nabla P_k + P_k \nabla \cdot \vec{v}_f) \, d\tilde{A} - \int_{\Gamma_1} P_k \vec{v}_f \cdot \vec{n} \, d\tilde{s},
\]
(24)

Then the equation (21) can be easily obtained by substituting (24) into (22).

Using the result stated in Lemma 3.1 recursively yields the following Proposition.

**Proposition 1.**
\[
\int_{\Omega(t)} P \nabla \cdot \vec{v}_f \, d\tilde{A} = -\gamma \frac{d}{dt} \int_{\Omega(t)} \left( e^\frac{\rho}{\gamma} - \frac{P}{\gamma} - 1 \right) \, d\tilde{A} + \gamma \int_{\Gamma_1} \left( e^\frac{\rho}{\gamma} - \frac{P}{\gamma} - 1 \right) \vec{v}_f \cdot \vec{n} \, d\tilde{s}.
\]
(25)

**Proof.** Using Lemma 3.1, it is straightforward to show by induction that
\[
\int_{\Omega(t)} P \nabla \cdot \vec{v}_f \, d\tilde{A} = \lim_{N \to \infty} \left[ \int_{\Omega(t)} \frac{P^N}{N! \gamma^{N-1}} \nabla \cdot \vec{v}_f \, d\tilde{A} + \frac{d}{dt} \int_{\Omega(t)} \left( \sum_{k=2}^{N} \frac{P^k}{k! \gamma^{k-1}} \right) \, d\tilde{A} + \int_{\Gamma_1} \left( \sum_{k=2}^{N} \frac{P^k}{k! \gamma^{k-1}} \right) \vec{v}_f \cdot \vec{n} \, d\tilde{s} \right],
\]
(26)

where the limit
\[
\lim_{N \to \infty} \int_{\Omega(t)} \frac{P^N}{N! \gamma^{N-1}} \nabla \cdot \vec{v}_f \, d\tilde{A} = \gamma \int_{\Omega(t)} \left[ \lim_{N \to \infty} \frac{1}{N!} \left( \frac{P}{\gamma} \right)^N \nabla \cdot \vec{v}_f \, d\tilde{A} = 0, \quad (27)
\]
for the classical solution of the coupled system. The infinite sum
\[
\sum_{k=2}^{\infty} \frac{P^k}{k! \gamma^{k-1}} = \gamma \sum_{k=2}^{\infty} \frac{1}{k!} \left( \frac{P}{\gamma} \right)^k = \gamma \left( e^\frac{\rho}{\gamma} - \frac{P}{\gamma} - 1 \right).
\]
(28)

Combining the two equations (17) and (20), using Proposition 1 and the inlet boundary condition in (13), we can obtain the following result.

**Theorem 3.2.** [Energy Balance for the Coupled System (case \(a = 0\))] The following energy equation holds for all time \(t > 0\),
\[
\frac{d}{dt} \left[ \tilde{F}(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho f_0}{P_{f_0}} - \frac{P}{\gamma} - 1 \right) \, d\tilde{A} \right] + \tilde{G}(t) + \frac{1}{\rho h} \int_{\Omega(t)} \Pi \nabla P^2 \, d\tilde{A} \\
= \frac{\gamma}{\rho h \rho f_0} \int_{\Gamma_1} (\psi - \psi_0) \, d\tilde{s}, \quad (29)
\]
or integrating in time

\[
\tilde{F}(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho f}{\rho f_0} - \frac{P}{\gamma} - 1 \right) d\tilde{A} + \int_0^t \tilde{G}(\tau) d\tau + \frac{1}{\rho h} \int_{\Omega(\tau)} \Pi |\nabla P(\cdot, \tau)|^2 d\tilde{A} d\tau = \frac{\gamma}{\rho h} \rho f_0 \int_{\Omega(t)} \int_{\Gamma_1} \left( \psi - \psi_0 \right) d\tilde{s} d\tau + C_0,
\]

(30)

where \( \psi_0 := \psi|_{\rho_f = \rho f_0} \) and \( C_0 := \tilde{F}(0) + \gamma \int_{\Omega(t)} \left( \frac{\rho f}{\rho f_0} - \frac{P}{\gamma} - 1 \right) d\tilde{A} \) if \( t = 0 \geq 0 \) is a constant which depends on the initial data.

From Theorem 3.2, one can get an estimate for the energy functional

\[
\tilde{F}(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho f}{\rho f_0} - \frac{P}{\gamma} - 1 \right) d\tilde{A}
\]

for all time. As time increases, RHS of the equation (30) can increase in time although \( \rho_f \) on the inlet is bounded uniformly. In order to obtain more refined estimate, we will impose a priori condition on the domain deformation and on the fluid density.

3.2. Estimate of Type 2: Under constraints on the deformation \((a > 0)\).

If we consider \( 0 < a < 1 \) in the beam energy equation (7), by following the same steps as in the previous section 3.1, we can obtain an energy equation analogous to (29)

\[
\frac{d}{dt} \left[ F(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho f}{\rho f_0} - \frac{P}{\gamma} - 1 \right) d\tilde{A} \right] + G_1(t) + \frac{1}{\rho h} \int_{\Omega(t)} \Pi |\nabla P|^2 d\tilde{A} +
\]

\[
+ \frac{D_2}{2} \int_{\Gamma} w_x^2 \left[ aK_1 \delta^3 - (\delta^3)_t \right] dx = \frac{\gamma}{\rho h} \rho f_0 \int_{\Gamma_1} (\psi - \psi_0) d\tilde{s} + \frac{aK_1}{\rho h} \int_{\Gamma(t)} P \langle u, w \rangle \cdot n d\tilde{s},
\]

(31)

where

\[
F(t) := \int_{\Gamma} \left[ \frac{a}{2} (u_t + K_1 u)^2 + \frac{a}{4} (w_t + K_1 w)^2 + \frac{1-a}{2} u_t^2 + \left( \frac{1}{2} - \frac{a}{4} \right) w_t^2 + \right.
\]

\[
\left. + \frac{D_1}{2} \left[ u_x + \frac{1}{2} (w_x)^2 \right]^2 + \frac{D_2}{2} w_{xx}^2 \delta^3 \right] dx,
\]

(32)

and

\[
G_1(t) := \int_{\Gamma} (1-a)K_1 u_t^2 + (1-a)K_1 w_t^2 + aK_1 D_1 \left[ u_x + \frac{1}{2} (w_x)^2 \right] dx.
\]

(33)

In the following, the integral term \( \frac{aK_1}{\rho h} \int_{\Gamma(t)} P \langle u, w \rangle \cdot n d\tilde{s} \) on the RHS of the equation (31) will be analyzed. But in order to deal with this term, we will assume that the following constraint holds.

**Constraint 2.** There exists a constant \( M_1 > 0 \) such that

\[
\int_{\Gamma(t)} f_w d\tilde{s} \leq M_1 \int_{\Gamma} f_w dx,
\]

(34)

for any nonnegative weight function \( f_w \).
Lemma 3.3. Define \( <P_1, P_2> := P_n \). If the constraint 2 is satisfied, then the following estimates hold for all time \( t \geq 0 \)

\[
\int_{\Gamma(t)} P_1 u \, d\bar{s} \leq \frac{1}{4\epsilon_1} \|P_1\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_1 M_1 C_{P_1}^2 \|u_x + \frac{1}{2}(w_x)^2\|_{L^2(\Gamma)}^2 + \frac{C_{P_1}^2}{2} \|P_1\|_{L^1(\bar{\Gamma}(t))} \|w_{xx}\|_{L^2(\Gamma)}^2,
\]

\[
\int_{\Gamma(t)} P_2 w \, d\bar{s} \leq \frac{1}{4\epsilon_2} \|P_2\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_2 M_1 C_{P_1}^4 \|w_{xx}\|_{L^2(\Gamma)}^2,
\]

for any positive constants \( \epsilon_1, \epsilon_2 \), and poincare constant \( C_{P_1} > 0 \).

Proof. Define

\[
g(x) := \int_0^x \frac{1}{2} w_x^2 \, dx,
\]

then

\[
\int_{\Gamma(t)} P_1 (u + g - g) \, d\bar{s} = \int_{\Gamma(t)} P_1 (u + g) \, d\bar{s} - \int_{\Gamma(t)} P_1 g \, d\bar{s}.
\]

Using Cauchy’s inequality, constraint 2 and Poincare inequality, respectively, for the first integral term on the RHS of the equation above, we can obtain the following

\[
\int_{\Gamma(t)} P_1 (u + g) \, d\bar{s} \leq \frac{1}{4\epsilon_1} \|P_1\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_1 \|u + g\|_{L^2(\Gamma)}^2
\]

\[
\leq \frac{1}{4\epsilon_1} \|P_1\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_1 M_1 \|u\|_{L^2(\Gamma)}^2
\]

\[
\leq \frac{1}{4\epsilon_1} \|P_1\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_1 M_1 C_{P_1}^2 \|(u + g)x\|_{L^2(\Gamma)}^2
\]

\[
= \frac{1}{4\epsilon_1} \|P_1\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_1 M_1 C_{P_1}^2 \|u_x + \frac{1}{2}(w_x)^2\|_{L^2(\Gamma)}^2,
\]

for any positive constant \( \epsilon_1 \). Here, \( C_{P_1} \) is the Poincare constant.

Now consider the second integral term on RHS of the equation (38)

\[
- \int_{\Gamma(t)} P_1 g \, d\bar{s} = - \int_{\Gamma(t)} P_1 \left[ \int_0^x \frac{1}{2} w_x^2 \, dx \right] \, d\bar{s} \leq \int_{\Gamma(t)} |P_1| \left[ \int_{\Gamma} \frac{1}{2} w_x^2 \, dx \right] \, d\bar{s}
\]

\[
\leq \frac{1}{2} \|P_1\|_{L^1(\bar{\Gamma}(t))} \|w_x\|_{L^2(\Gamma)}^2
\]

\[
\leq \frac{C_{P_1}^2}{2} \|P_1\|_{L^1(\bar{\Gamma}(t))} \|w_{xx}\|_{L^2(\Gamma)}^2,
\]

where the last inequality follows from Poincare inequality. Finally, combining (38), (39) and (40) yields the estimate (35).

To obtain the estimate (36), we use again Cauchy’s inequality, constraint 2 and Poincare inequality, respectively

\[
\int_{\Gamma(t)} P_2 w \, d\bar{s} \leq \int_{\Gamma(t)} \left( \frac{1}{4\epsilon_2} P_2^2 + \epsilon_2 w^2 \right) \, d\bar{s} \leq \frac{1}{4\epsilon_2} \|P_2\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_2 M_1 \|w\|_{L^2(\Gamma)}^2
\]

\[
\leq \frac{1}{4\epsilon_2} \|P_2\|_{L^2(\bar{\Gamma}(t))}^2 + \epsilon_2 M_1 C_{P_1}^4 \|w_{xx}\|_{L^2(\Gamma)}^2,
\]

for any positive constant \( \epsilon_2 \), and Poincare constant \( C_{P_1} \). \( \Box \)
Now, we will use Lemma 3.3 in equation (31) for the last term in the RHS with
\[ \epsilon_1 := \frac{D_1 \rho h}{2C_p^2 M_1}, \] (42)
which leads to
\[ \frac{aK_1}{\rho h} \int_{\Gamma(t)} P \langle u, w \rangle \cdot n \, d\bar{s} \leq \frac{aK_1 C_p^2 M_1}{\rho h} \left\| \frac{P_1}{\rho h} \right\|_{L^2(\Gamma(t))}^2 + \frac{aK_1 D_1}{2} \left\| u_x + \frac{1}{2} (w_x)^2 \right\|_{L^2(\Gamma)}^2 \]
\[ + \frac{aK_1 C_p^2}{2} \left\| P_1 \right\|_{L^1(\Gamma(t))} \left\| w_{xx} \right\|_{L^2(\Gamma)}^2 \]
\[ + \frac{aK_1}{\rho h} \frac{1}{4\epsilon_2} \left\| P_2 \right\|_{L^2(\Gamma(t))}^2 + \frac{aK_1}{\rho h} \epsilon_2 M_1 C_p^4 \left\| w_{xx} \right\|_{L^2(\Gamma)}^2. \] (43)

Using (43) in the equation (31), rearranging some of the terms, and defining the functional \( G_2 \) as
\[ G_2(t) := G_1(t) - \frac{aK_1 D_1}{2} \left\| u_x + \frac{1}{2} (w_x)^2 \right\|_{L^2(\Gamma)}^2 \]
\[ = \int_{\Gamma} \left( 1 - a \right) K_1 u^2_x + \left( 1 - \frac{a}{2} \right) K_1 w^2_x + \frac{aK_1 D_1}{2} \left[ u_x + \frac{1}{2} (w_x)^2 \right]^2 \, dx \geq 0, \] (44)
yields the following inequality
\[ \frac{d}{dt} \left[ F(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho t}{\rho_f} - \frac{P}{\gamma} - 1 \right) \, d\bar{A} \right] + G_2(t) + \frac{1}{\rho h} \int_{\Omega(t)} \Pi |\nabla P|^2 \, d\bar{A} + \]
\[ + \int_{\Gamma} \left( \frac{D_2}{2} \left[ aK_1 \delta^3 - \delta^3 \right] - aK_1 \frac{C_p^2}{2\rho h} \left\| P_1 \right\|_{L^1(\Gamma(t))} + aK_1 \epsilon_2 M_1 C_p^4 \right) \left| w_{xx} \right|^2 \, dx \]
\[ \leq \frac{\gamma}{\rho h \rho_f} \int_{\Gamma_1} \left( \psi - \psi_0 \right) \, d\bar{s} + \frac{aK_1}{\rho h} \int_{\Gamma(t)} \frac{C_p^2 M_1}{2D_1} P_2^2 + \frac{1}{4\epsilon_2} P_2^2 \, d\bar{s}. \] (45)

Let \( M_2 := \max \left\{ \frac{C_p^2 M_1}{2D_1}, \frac{1}{4\epsilon_2} \right\} \). By Trace Theorem the following inequality for the second integral term on the RHS of the inequality (45) is obtained
\[ \int_{\Gamma(t)} \frac{C_p^2 M_1}{2D_1} P_2^2 + \frac{1}{4\epsilon_2} P_2^2 \, d\bar{s} \leq M_2 \int_{\Gamma(t)} P^2 \, d\bar{s} \leq C_T M_2 \int_{\Omega(t)} |\nabla P|^2 \, d\bar{A}, \] (46)
where \( C_T > 0 \) is the Trace theorem constant. Then using the inequality above (46) and Poincare-Sobolev inequality, (45) can be written as
\[ \frac{d}{dt} \left[ F(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho t}{\rho_f} - \frac{P}{\gamma} - 1 \right) \, d\bar{A} \right] + G_2(t) + M_3 \int_{\Omega(t)} P^2 \, d\bar{A} + \]
\[ + \int_{\Gamma} \left( \frac{D_2}{2} \left[ aK_1 \delta^3 - \delta^3 \right] - aK_1 \frac{C_p^2}{2\rho h} \left\| P_1 \right\|_{L^1(\Gamma(t))} + aK_1 \epsilon_2 M_1 C_p^4 \right) \left| w_{xx} \right|^2 \, dx \]
\[ \leq \frac{\gamma}{\rho h \rho_f} \int_{\Gamma_1} \left( \psi - \psi_0 \right) \, d\bar{s}, \] (47)
where \( M_3 := \frac{1}{\rho h} (\Pi - aK_1 C_T M_2) \).
Note 2. Choose $a$ such that $0 < a < \min \left\{ \frac{\Pi}{\kappa_3 c_2}, 1 \right\}$, which implies that $M_3 > 0$.

Now define the following functions

$$I(t) := F(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho f}{\rho f_0} - \frac{P}{\gamma} - 1 \right) dA,$$

$$\tilde{I}(t) := \tilde{F}(t) + \frac{\gamma}{\rho h} \int_{\Omega(t)} \left( \frac{\rho f}{\rho f_0} - \frac{P}{\gamma} - 1 \right) dA. \quad \text{(48)}$$

Using the constraint Poincare-Sobolev inequality, and Theorem 3.2, we can estimate the term $\|P_1\|_{L^1(\tilde{\Gamma}(t))}$ as follows

$$\|P_1\|_{L^1(\tilde{\Gamma}(t))} \leq \|P\|_{L^1(\tilde{\Gamma}(t))} \leq C^2_p \frac{\rho h}{\Pi} \left( \frac{\gamma}{\rho h \rho f_0} \int_{\Gamma_1} (\psi - \psi_0) \, ds - \tilde{\mathcal{T}}(t) - \tilde{G}(t) \right). \quad \text{(50)}$$

Then we can rewrite the inequality (47) as

$$\mathcal{T}'(t) + G_2(t) + M_3 \int_{\Omega(t)} P^2 \, dA + M_4 \int_{\Omega(t)} (M_5 - \frac{\gamma}{\rho h \rho f_0} \int_{\Gamma_1} (\psi - \psi_0) \, ds + \tilde{\mathcal{T}}(t) + \tilde{G}(t)) |w_{xx}|^2 \, dx$$

$$\leq \frac{\gamma}{\rho h \rho f_0} \int_{\Gamma_1} (\psi - \psi_0) \, ds, \quad \text{(51)}$$

where

$$M_4 := a K_1 \frac{C_p^4}{2 \Pi}, \quad \text{and} \quad M_5 := M_5(t) := \frac{1}{M_4} \left( \frac{D_2}{2} [a K_1 \delta^3 - (\delta^3)_{\lambda}] - a K_1 \frac{\epsilon_2 M_4 C_m^4}{\rho h} \right). \quad \text{(52)}$$

Note 3. Select in constraint 1 constants $b_0 < a$ and $\tilde{C} < (a - b_0)$, then it is not difficult to show that bracket $[a K_1 \delta^3 - (\delta^3)_{\lambda}] \geq (a - b_0 - \tilde{C}) \delta^3$. Choose $\epsilon_2$ such that

$$0 < \epsilon_2 < \frac{D_2 \rho h}{2 a K_1 C_p^4 M_1} (a - b_0 - \tilde{C}) \delta^3,$$

Existence of such $t$ independent $\epsilon$ follow from assumption (6). Latter implies that $M_5(t) \geq C_{M_5} > 0$ for all $t$, for some constant $C_{M_5}$.

Define

$$G_3(t) := G_2(t) + M_3 \int_{\Omega(t)} P^2 \, dA + M_4 \left( M_5 - \frac{\gamma}{\rho h \rho f_0} \int_{\Gamma_1} (\psi - \psi_0) \, ds + \tilde{G}(t) \right) \int_{\Gamma} |w_{xx}|^2 \, dx,$$

and rewrite the inequality (51) as

$$\mathcal{T}'(t) + G_3(t) + M_4 \tilde{\mathcal{T}}(t) \int_{\Gamma} |w_{xx}|^2 \, dx \leq \frac{\gamma}{\rho h \rho f_0} \int_{\Gamma_1} (\psi - \psi_0) \, ds. \quad \text{(54)}$$

In the following we assume that the following constraint holds.

**Constraint 3.** The fluid density satisfies the following constraint

$$\frac{\rho f}{\rho f_0} \leq \left( \frac{c}{\gamma} P^2 + \frac{P}{\gamma} + 1 \right), \quad \text{(55)}$$

for all $t$ and for some positive constant $c$. 


The above constraint reflects the fact that the considered fluid is slightly compressible. It should be noticed that the state equation (10) is valid only for $|P| \leq C\gamma$, for some small constant $C$, otherwise the density saturates to a minimum value $\rho_{\text{min}}$ for $P < -C\gamma$ or a maximum value $\rho_{\text{max}}$ for $P > C\gamma$ (see [19]). In our analysis however we don’t consider these upper and lower bounds, resulting in a density which grows exponentially with the pressure. In order to prove the following result, we assume boundedness of the density growth with a quadratic function of the pressure $P$. Then the constraint 3 is needed, for mathematical reasons, to satisfy the physics of the problem.

Now we can state the following lemma.

**Lemma 3.4.** Let constraints 1 and 3 are satisfied. Assume that there that

$$C_{M_5} > \frac{\gamma}{\rho_0\rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s} \quad \forall \ t \geq 0,$$

and

$$C_{M_5} > \limsup_{t \to \infty} \frac{\gamma}{\rho_0\rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s}.$$  

where $C_{M_5}$ is an upper bound for $M_5(t)$. Then, there exist constants $c_1, c_2, c \geq 0$ depending on the physical parameters of the porous medium - beam coupled system such that

$$M_4 \int_{\Gamma} |w_{xx}|^2 \, dx \leq 2c_1 \bar{F}(t) \quad \forall \ t \geq 0,$$

$$\bar{I}(t) \leq c_2 G_3(t) \quad \forall \ t \geq 0,$$

and

$$I(t) + c_1 \bar{I}(t)^2 \leq c \left[ G_3(t) + G_3^2(t) \right] \quad \forall \ t \geq 0.$$

**Proof.** Assuming constraint 1 holds, the inequality (58) is obvious.

For the inequality (59), observe that each integral term in $\bar{I}(t)$ has an immediate analogous term in $G_3(t)$ with respect to some constant except the following terms

- $\frac{D_2}{2} \int_{\Gamma} \delta^3 w_{xx}^2 \, dx$: This term can be compared with the term

$$M_4 \left( M_5 - \frac{\gamma}{\rho_0\rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s} + \tilde{G}(t) \right) \int_{\Gamma} |w_{xx}|^2 \, dx.$$

Since $\delta^3$ is a non-increasing function (constraint 1), it is bounded above for all time. On the other hand, by the assumption (56), $M_5 - \frac{\gamma}{\rho_0\rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s}$ is bounded below by a positive constant for all time. Therefore, these two terms can be compared with respect to some constant.

- $\gamma \int_{\Omega(t)} \left( \frac{\rho_f}{\rho_0} - \frac{P}{\gamma} - 1 \right) \, d\bar{A}$: This term can be compared with the term

$$M_3 \int_{\Omega(t)} P^2 \, d\bar{A}$$

in $G_3$, if the constraint 3 holds.

Similarly for the inequality (60), observe that each term in $I(t)$ can be compared with a term in $G_3(t)$ by the help of the previous result (inequality (59)) except the following term

$$\frac{a}{2} \|u_1 + K_1 u_1\|_{L^2(\Gamma)}^2.$$
But, from classical Poincare and Holder inequalities, the following estimate can be obtained

\[
\| u_t + K_1 u \|_{L^2(\Gamma)}^2 = \left\| u_t + K_1 u + K_1 \int_0^x \frac{w_x^2}{2} \, dx_1 - K_1 \int_0^x \frac{w_x^2}{2} \, dx_1 \right\|_{L^2(\Gamma)}^2 \\
\leq 2 \| u_t \|_{L^2(\Gamma)} + 2C_P^2 K_1^2 \left( \| u_x + \frac{1}{2} w_x \|_{L^2(\Gamma)} + \frac{K_1^2 L^2}{2} \| w_x \|_{L^4}^4 \right) \\
\leq 2 \| u_t \|_{L^2(\Gamma)} + 2C_P^2 K_1^2 \left( \| u_x + \frac{1}{2} w_x \|_{L^2(\Gamma)} + \frac{C_P^2 K_1^2 L^2}{2} \| w_{xx} \|_{L^2(\Gamma)} \right),
\]

(61)

where \( C_P \) is the Poincare constant. The RHS of the inequality above can now be compared with \( G_3(t) + G_3^2(t) \).

**Note 4.** Under assumptions 56 and 57 on the boundary Data in the previous lemma 3.4 there exist constant \( C > 0 \) such that

\[
\min \left\{ C_{M_5}, \frac{1}{4C_1C_2} \right\} - \frac{\gamma}{\rho h \rho_0} \int_{\Gamma_1} (\psi - \psi_0) \, ds \geq C \quad \forall \, t \geq 0,
\]

(62)

where \( C_{M_5} \) is an upper bound for \( M_5(t) \) and \( c_1 \) and \( c_2 \) are the constants defined in Lemma 3.4

**Note 5.** Define \( \Phi(\xi) := 4c \left( \xi + \xi^2 \right) \) for any functional \( \xi \), and observe that

\[
c \left[ G_3 + G_3^2 \right] \leq 4c \left[ \frac{G_3}{2} + \left( \frac{G_3}{2} \right)^2 \right] = \Phi \left( \frac{G_3}{2} \right).
\]

(63)

Then the inequality (60) can be rewritten as

\[
\mathcal{I}(t) + c_1 \tilde{I}(t)^2 \leq \Phi \left( \frac{G_3}{2} \right), \quad \forall \, t \geq 0.
\]

(64)

**Theorem 3.5 (Boundedness of Energy Functional for Coupled System-II).** Let constraints 1, 2 and 3 are satisfied. Assume that assumptions 56 and 57 on the boundary Data in the lemma 3.4 are satisfied.

Let

\[
B_{in}(t) := \max \left\{ 0, \max_{\tau \in [0,t]} \left\{ \frac{\gamma}{\rho h \rho_0} \int_{\Gamma_1} (\psi(\tau) - \psi_0(\tau)) \, ds \right\} \right\},
\]

(65)

be a non-decreasing function, then

\[
\mathcal{I}(t) + c_1 \tilde{I}(t)^2 \leq \Phi(B_{in}(t)) + I(0) \quad \forall \, t \geq 0.
\]

(66)

**Proof.** Assume that the inequality (66) is not true. Then there exist a sufficiently small interval \([t_1, t_2]\) and \( \delta > 0 \), such that

\[
\mathcal{I}(t_1) + c_1 \tilde{I}(t_1)^2 = \Phi(B_{in}(t_1)) + I(0),
\]

(67)

\[
\mathcal{I}(t_2) + c_1 \tilde{I}(t_2)^2 = \Phi(B_{in}(t_2)) + I(0) + \delta,
\]

(68)

\[
\mathcal{I}(t) + c_1 \tilde{I}(t)^2 \geq \Phi(B_{in}(t)) + I(0) \quad \forall \, t \in [t_1, t_2].
\]

(69)

Let \( t \in [t_1, t_2] \), then either \( \tilde{I}'(t) \geq 0 \) or \( \tilde{I}'(t) < 0 \):
• Case 1. If $\ddot{\mathcal{I}}(t) \geq 0$, then drop $\ddot{\mathcal{I}}(t)$ in the inequality (54) to obtain

$$\mathcal{I}'(t) \leq -G_3(t) + \frac{\gamma}{\rho h \rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s}.$$  

(70)

Since we have the following relation from Theorem 3.2 and Lemma 3.4

$$\left[\mathcal{I}(t) + c_1 \ddot{\mathcal{I}}(t)^2\right]' \leq \left[1 - 2c_1 c_2 \frac{\gamma}{\rho h \rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s}\right] G_3(t) + \frac{\gamma}{\rho h \rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s}$$

we can conclude that,

$$\left[\mathcal{I}(t) + c_1 \ddot{\mathcal{I}}(t)^2\right]' \leq -\left[1 - 2c_1 c_2 B_{in}(t) G_3(t) + B_{in}(t)\right] \leq -\frac{1}{2} G_3(t) + B_{in}(t).$$  

(72)

The last inequality comes from condition (62).

• Case 2. If $\ddot{\mathcal{I}}(t) < 0$, then by Lemma 3.4

$$M_4 \ddot{\mathcal{I}}(t) \int_\Gamma |w_{xx}|^2 \, dx \geq 2c_1 \ddot{\mathcal{I}}(t) \ddot{F}(t) \geq 2c_1 \ddot{\mathcal{I}}(t) \ddot{\mathcal{I}}(t) = c_1 \left[\ddot{\mathcal{I}}(t)^2\right]'$$  

(73)

From inequality (54) it then follows

$$\left[\mathcal{I}(t) + c_1 \ddot{\mathcal{I}}(t)^2\right]' \leq -G_3(t) + \frac{\gamma}{\rho h \rho_f} \int_{\Gamma_1} (\psi - \psi_0) \, d\bar{s} \leq -G_3(t) + B_{in}(t).$$  

(74)

From both cases, that is from the inequalities (72) and (74), we can conclude that for any $t \in [t_1, t_2]

$$\left[\mathcal{I}(t) + c_1 \ddot{\mathcal{I}}(t)^2\right]' \leq -\frac{1}{2} G_3(t) + B_{in}(t) \leq -\Phi^{-1} \left(\mathcal{I}(t) + c_1 \ddot{\mathcal{I}}(t)^2\right) + B_{in}(t),$$  

(75)

where the latter inequality comes from the relation (64) and the fact that $\Phi^{-1}(\cdot)$ is an increasing function. Now use the relation (69) and increasing monotonicity of $\Phi^{-1}$ to rewrite the inequality above (75) for the interval $[t_1, t_2]$ as

$$\left[\mathcal{I}(t) + c_1 \ddot{\mathcal{I}}(t)^2\right]' \leq -\Phi^{-1} (\Phi(B_{in}(t)) + I(0)) + B_{in}(t) \leq -\Phi^{-1} (\Phi(B_{in}(t))) + B_{in}(t) = -B_{in}(t) + B_{in}(t) = 0.$$  

(76)

But then

$$\mathcal{I}(t_2) + c_1 \ddot{\mathcal{I}}(t_2)^2 \leq \mathcal{I}(t_1) + c_1 \ddot{\mathcal{I}}(t_1)^2 = \Phi(B_{in}(t_1)) + I(0) < \Phi(B_{in}(t_2)) + I(0) + \delta = \mathcal{I}(t_2) + c_1 \ddot{\mathcal{I}}(t_2)^2,$$  

(77)

which leads to contradiction.
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