ON THE LOCALIZATION FORMULA IN EQUIVARIANT COHOMOLOGY

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ABSTRACT. We give a generalization of the Atiyah-Bott-Berline-Vergne localization theorem for the equivariant cohomology of a torus action. We replace the manifold having a torus action by an equivariant map of manifolds having a compact connected Lie group action. This provides a systematic method for calculating the Gysin homomorphism in ordinary cohomology of an equivariant map. As an example, we recover a formula of Akyildiz-Carrell for the Gysin homomorphism of flag manifolds.

Suppose $M$ is a compact oriented manifold on which a torus $T$ acts. The Atiyah-Bott-Berline-Vergne localization formula calculates the integral of an equivariant cohomology class on $M$ in terms of an integral over the fixed point set $M^T$. This formula has found many applications, for example, in analysis, topology, symplectic geometry, and algebraic geometry (see [2], [8], [10], [14]). Similar, but not entirely analogous, formulas exist in $K$-theory ([3]), cobordism theory ([13]), and algebraic geometry ([9]).

Taking cues from the work of Atiyah and Segal in $K$-theory [3], we state and prove a localization formula for a compact connected Lie group action in terms of the fixed point set of a conjugacy class in the group. As an application, the formula can be used to calculate the Gysin homomorphism in ordinary cohomology of an equivariant map. For a compact connected Lie group $G$ with maximal torus $T$ and a closed subgroup $H$ containing $T$, we work out as an example the Gysin homomorphism of the canonical projection $f : G/T \to G/H$, a formula first obtained by Akyildiz and Carrell [1].

The application to the Gysin map in this article complements that of [14]. The previous article [14] shows how to use the ABBV localization formula to calculate the Gysin map of a fiber bundle. This article shows how to use the relative localization formula to calculate the Gysin map of an equivariant map.

We thank Michel Brion for many helpful discussions.

1. BOREL-TYPE LOCALIZATION FORMULA FOR A CONJUGACY CLASS

Suppose a compact connected Lie group $G$ acts on a manifold $M$. For $g \in G$, define $M^g$ to be the fixed point set of $g$:

$$M^g = \{ x \in M \mid g \cdot x = x \}.$$
The set $M^g$ is not $G$-invariant. The $G$-invariant subset it generates is
\[ \cup_{h \in G} h \cdot (M^g) = \cup_{h \in G} M^{gh^{-1}} = \cup_{k \in C(g)} M^k \]
where $C(g)$ is the conjugacy class of $g$. This suggests that for every conjugacy class $C$ in $G$, we consider the set $M^C$ of elements of $M$ that are fixed by at least one element of the conjugacy class $C$:
\[ M^C = \cup_{k \in C} M^k. \]
Then $M^C$ is a closed $G$-subset of $M$ ([2], footnote 1, p. 532); however it may not be always smooth. From now on we make the assumption that $M^C$ is smooth.

Suppose $C = C(g)$ is the conjugacy class of an element $g$ in $G$. Let $T$ be a maximal torus of $T$ containing $g$. Then we have the following inclusions of fixed-point sets:
\[ M^G \subset M^T \subset M^g \subset M^C. \]

Remark 1.1. If $T$ is a maximal torus in the compact connected Lie group $G$ and $\dim T = \ell$, then
\[ H^*(BG) = H^*(BT)^W_G = \mathbb{Q}[u_1, \ldots, u_\ell]^{W_G}, \]
where $W_G$ is the Weyl group of $T$ in $G$. Thus, $H^*(BG)$ is an integral domain. Let $Q$ be its field of fractions. For any $H^*(BG)$-module $V$, we define the localization of $V$ with respect to the zero ideal in $H^*(BG)$ to be
\[ \hat{V} := V \otimes_{H^*(BG)} Q. \]
It is easily verified that $V$ is $H^*(BG)$-torsion if and only if $\hat{V} = 0$. For a $G$-manifold $M$, we call $\hat{H}_G^*(M)$ the localized equivariant cohomology of $M$.

Lemma 1.2. Let $M$ be a $G$-manifold and $T$ a maximal torus of $G$. If $H^*_T(M)$ is $H^*(BT)$-torsion, then $H^*_G(M)$ is $H^*(BG)$-torsion.

Proof. Recall that $H^*_G(M)$ is the subring of $H^*_T(M)$ consisting of the $W_G$-invariant elements. Since $H^*_T(M)$ is $H^*(BT)$-torsion, there is $a \in H^*(BT)$ such that $a \cdot 1_{H^*_T(M)} = 0$. Consider the average of $a$ over the Weyl group $W_G$ of $T$ in $G$,
\[ \tilde{a} = \frac{1}{|W_G|} (a + \omega_1 a + \cdots + \omega_\ell a) \in H^*(BG). \]
Under $\psi$, the element $\tilde{a} \cdot 1_{H^*_G(M)}$ goes to
\[ \frac{1}{|W_G|} (\omega_1 a + \cdots + \omega_\ell a)1_{H^*_G(M)}. \]
But $(\omega_j a)1_{H^*_G(M)} = \omega_j (a1_{H^*_G(M)}) = 0$ for any $j$. Thus $\tilde{a} \cdot 1_{H^*_G(M)} = 0$ in $H^*_G(M)$. \hfill \square

Proposition 1.3. Let $G$ be a compact connected Lie group acting on a compact manifold $M$, and let $C$ be a conjugacy class in $G$. If $U \subset M - M^C$ is an open $G$-subset, then the equivariant cohomology $H^*_G(U)$ is $H^*(BG)$-torsion.

Proof. It follows from ([1]) that $U \subset M - M^C \subset M - M^T$. Since the inclusion map $U \to M - M^T$ is $T$-equivariant, and $H^*_T(M - M^T)$ is $H^*(BT)$-torsion by ([1], Th. 11.4.1), $H^*_T(U)$ is also $H^*(BT)$-torsion. By Lemma 1.2, $H^*_G(U)$ is $H^*(BG)$-torsion. \hfill \square

In the rest of this section, “torsion” will mean $H^*(BG)$-torsion.
Theorem 1.4 (Borel-type localization formula for a conjugacy class). Let $G$ be a compact connected Lie group acting on a compact manifold $M$, and $C$ a conjugacy class in $G$. Then the inclusion $i : M^C \to M$ induces an isomorphism in localized equivariant cohomology

$$i^* : \hat{H}^*_G(M) \to \hat{H}^*_G(M^C).$$

Proof. Let $U$ be a $G$-invariant tubular neighborhood of $M^C$. Then $\{U, M - M^C\}$ is a $G$-invariant open cover of $M$. Moreover, $H^*_G(U) \simeq H^*_G(M)$ because $U$ has the $G$-homotopy type of $M^C$.

By Prop. 1.3, $H^*_G(M - M^C)$ and $H^*_G(U \cap (M - M^C))$ are torsion. Then in the localized equivariant Mayer-Vietoris sequence

$$\cdots \to \hat{H}^{*-1}_G(U \cap (M - M^C)) \to \hat{H}^*_G(M - M^C) \oplus \hat{H}^*_G(U) \to \hat{H}^*_G(U \cap (M - M^C)) \to \cdots,$$

all the terms except $\hat{H}^*_G(M)$ and $\hat{H}^*_G(U)$ are zero. It follows that

$$\hat{H}^*_G(M) \to \hat{H}^*_G(U) \simeq \hat{H}^*_G(M^C)$$

is an isomorphism of $H^*(BG)$-modules.

When the group is a torus $T$, a conjugacy class $C$ consist of a single element $t \in T$. If $t$ is generator, then the fixed point set of $t$ is the same as the fixed point set of the whole group $T$: $M^C = M^t = M^T$. In this case $M^C$ is smooth. Thus Borel’s localization theorem follows from Theorem 1.4 by taking the conjugacy class $C = \{t\}$ in $T$.

2. The equivariant Euler class

Suppose a compact connected Lie group $G$ acts on a smooth compact manifold $M$. Let $C$ be a conjugacy class in $G$, and $M^C$ as before. From now on we assume that $M^C$ is smooth with oriented normal bundle. Denote by $i : M^C \to M$ the inclusion map and by $e_M \in \hat{H}^*_G(M^C)$ the equivariant Euler class of the normal bundle of $M^C$ in $M$.

Proposition 2.1. Let $M$ be a compact connected oriented $G$-manifold. Then the equivariant Euler class $e_M$ of the normal bundle of $M^C$ in $M$ is invertible in $\hat{H}^*_G(M^C)$.

Proof. Fix a $G$-invariant Riemannian metric on $M$. Then the normal bundle $\nu \to M^C$ is a $G$-equivariant vector bundle. Let $\nu_0$ be the normal bundle minus the zero section. Since $\nu_0$ is equivariantly isomorphic to an open set in $M - M^C$, $\hat{H}^*_G(\nu_0)$ vanishes by Prop. 1.3. From the Gysin long exact sequence in localized equivariant cohomology

$$\cdots \to \hat{H}^*_G(\nu_0) \to \hat{H}^*_G(M^C) \times e_M \hat{H}_G(M^C) \to \hat{H}^*_G(\nu_0) \to \cdots$$

it follows that multiplication by the equivariant Euler class gives an automorphism of $\hat{H}^*_G(M^C)$. Thus $e_M$ has an inverse in the ring $\hat{H}^*_G(M^C)$. \hfill $\Box$

Recall that the inclusion map $i : M^C \to M$ satisfies the identity

$$i^* i_*(x) = xe_M, \quad x \in \hat{H}^*_G(M).$$
in equivariant cohomology. In the localized equivariant cohomology \( \hat{H}_G^*(M^C) \),
\[
\frac{i^*i_*}{e_M} x = \frac{i^*x}{e_M} e_M = i^*x.
\]
By Theorem 1.4 \( i^* \) is an isomorphism. Hence,
\[
(2)
\]
for \( a \in \hat{H}_G^*(M) \).

3. Relative localization formula

Let \( N \) be a \( G \)-manifold, \( e_N \) the equivariant Euler class of the normal bundle of \( N^C \), and \( f : M \rightarrow N \) a \( G \)-equivariant map. There is a commutative diagram of maps
\[
\begin{array}{ccc}
M^C & \xrightarrow{i_M} & M \\
\downarrow{f^C} & & \downarrow{f} \\
N^C & \xrightarrow{i_N} & N,
\end{array}
\]
(3)
where \( i_M \) and \( i_N \) are inclusion maps and \( f^C \) is the restriction of \( f \) to \( M^C \). Let
\[
(f_G)_* : \hat{H}_G^*(M) \rightarrow \hat{H}_G^*(N), \quad f^C_* : \hat{H}_G^*(M^C) \rightarrow \hat{H}_G^*(N^C)
\]
be the push-forward maps in localized equivariant cohomology.

**Theorem 3.1** (Relative localization formula). Let \( M \) and \( N \) be compact oriented manifolds on which a compact connected Lie group \( G \) acts, and \( f : M \rightarrow N \) a \( G \)-equivariant map. For \( a \in \hat{H}_G^*(M) \),
\[
(f_G)_* a = (i_N^*)^{-1} f^C_* \left( \frac{(f^C)_* e_N}{i^*_M e_M} i^*_M a \right)
\]
where the push-forward and restriction maps are in localized equivariant cohomology.

**Proof.** The commutative diagram (3), induces a commutative diagram in localized equivariant cohomology
\[
\begin{array}{ccc}
\hat{H}_G^*(M^C) & \xrightarrow{i^*_M} & \hat{H}_G^*(M) \\
\downarrow{f^C_*} & & \downarrow{(f_G)_*} \\
\hat{H}_G^*(N^C) & \xrightarrow{i^*_N} & \hat{H}_G^*(N),
\end{array}
\]
(4)
By eq. (2) and the commutativity of the diagram (4),
\[
(f_G)_* a = (f_G)_* i^*_M \left( \frac{1}{i^*_M e_M} i^*_M a \right)
\]
\[
= i^*_N f^C_* \left( \frac{1}{e_M} i^*_M a \right).
\]
Hence,
\[ i_N^*(f_G)_* a = i_N^* i_N_* f^*_C \left( \frac{1}{e_M} i_M^* a \right) = e_N f_C^* \left( \frac{1}{e_M} i_M^* a \right) = (f_C)_* \left( \frac{(f_C)^* e_N}{e_M} i_M^* a \right) \text{ (projection formula)}. \]

By Theorem 1.4, \( i_N^* \) is an isomorphism in localized equivariant cohomology,
\[ (f_G)_* a = (i_N^*)^{-1} (f_C)_* \left( \frac{(f_C)^* e_N}{e_M} i_M^* a \right). \]

\[ \square \]

If in Theorem 3.1 we take the group \( G \) to be a torus \( T \) and the conjugacy class \( C \) to be the conjugacy class of a generator \( t \) for \( T \), then \( M_C = M_T = M_T \) and Theorem 3.1 specializes to the following formula of Lian, Liu, and Yau [12].

**Corollary 3.2 (Relative localization formula for a torus action).** Let \( M \) and \( N \) be manifolds on which a torus \( T \) acts, and \( f : M \to N \) a \( T \)-equivariant map with compact oriented fibers. For \( a \in H^*_T(M) \),
\[ (f_T)_* a = (i_N^*)^{-1} (f_T)_* \left( \frac{(f_T)^* e_N}{e_M} i_M^* a \right), \]
where the push-forward and restriction maps are in localized equivariant cohomology.

When \( N \) is a single point, Cor. 3.2 reduces to the Atiyah-Bott-Berline-Vergne localization formula.

4. Applications to the Gysin homomorphism in ordinary cohomology

Let \( G \) be a compact connected Lie group acting on a manifold \( M \). Denote by \( M_G \) the homotopy quotient of \( M \) by \( G \), and by \( M^G \) the fixed point set of the action of \( G \) on \( M \). Let \( h_M : M \to M_G \) be the inclusion of \( M \) as a fiber of the bundle \( M_G \to BG \) and \( i_M : M^G \to M \) the inclusion of the fixed point set \( M^G \) in \( M \). The map \( h_M \) induces a homomorphism in cohomology
\[ h_M^* : H^*_G(M) \to H^*(M). \]
The inclusion \( i_M \) induces a homomorphism in equivariant cohomology
\[ i_M^* : H^*_G(M) \to H^*_G(M^G). \]

A cohomology class \( a \in H^*(M) \) is said to have an equivariant extension \( \tilde{a} \in H^*_G(M) \) under the \( G \) action if under the restriction map \( h_M^* : H^*_G(M) \to H^*(M) \), the equivariant class \( \tilde{a} \) restricts to \( a \).

Suppose \( f : M \to N \) is a \( G \)-equivariant map of compact oriented \( G \)-manifolds. In this section we show that if a class in \( H^*(M) \) has an equivariant extension, then its image under the Gysin map \( f_* : H^*(M) \to H^*(N) \) in ordinary cohomology can be computed from the relative localization formulas (Cor. 3.2 or Th. 3.1).

We consider first the case of an action by a torus \( T \). Let \( f_T : M_T \to N_T \) be the induced map of homotopy quotients and \( \tilde{f_T} : M^T \to N^T \) the induced map of fixed
point sets. As before, $e_M$ denotes the equivariant Euler class of the normal bundle of the fixed point set $M^T$ in $M$.

**Proposition 4.1.** Let $f : M \to N$ be a $T$-equivariant map of compact oriented $T$-manifolds. If a cohomology class $a \in H^*(M)$ has an equivariant extension $\tilde{a} \in H^*_T(M)$, then its image under the Gysin map $f_* : H^*(M) \to H^*(N)$ is,

1) in terms of equivariant integration over $M$:

$$f_* a = h_N^* f_T^* \tilde{a},$$

2) in terms of equivariant integration over the fixed point set $M^T$:

$$f_* a = h_N^* (i_M^*)^{-1} (f_C)^* \left( \frac{e_N^* e_M}{e_M} i_M^* \tilde{a} \right).$$

**Proof.** The inclusions $h_M : M \to M_T$ and $h_N : N \to N_T$ fit into a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{h_M} & M_T \\
\downarrow f & & \downarrow f_T \\
N & \xrightarrow{h_N} & N_T
\end{array}$$

This diagram is Cartesian in the sense that $M$ is the inverse image of $N$ under $f_T$. Hence, the push-pull formula $f_* h_M^* = h_N^* f_T^*$ holds. Then

$$f_* a = f_* h_M^* \tilde{a} = h_N^* f_T^* \tilde{a}.$$

2) follows from 1) and the relative localization formula for a torus action (Cor. 3.2). □

Using the relative localization formula for a conjugacy class, one obtains analogously a push-forward formula in terms of the fixed point sets of a conjugacy class. Now $h_M$ and $i_M$ are the inclusion maps

$$h_M : M \to M_G, \quad i_M : M^C \to M,$$

$e_M$ is the equivariant Euler class of the normal bundle of $M^C$ in $M$, and $f^C : M^C \to N^C$ is the induced map on the fixed point sets of the conjugacy class $C$.

**Proposition 4.2.** Let $f : M \to N$ be a $G$-equivariant map of compact oriented $G$-manifolds. Assume that the fixed point sets $M^C$ and $N^C$ are smooth with oriented normal bundle. For a class $a \in H^*(M)$ that has an equivariant extension $\tilde{a} \in H^*_G(M)$,

$$f_* a = h_N^* (i_N^*)^{-1} (f_C)^* \left( \frac{e_N^* e_M}{e_M} i_M^* \tilde{a} \right).$$

**5. Example: the Gysin homomorphism of flag manifolds**

Let $G$ be a compact connected Lie group with maximal torus $T$, and $H$ a closed subgroup of $G$ containing $T$. In [1] Akyildiz and Carrell compute the Gysin homomorphism for the canonical projection $f : G/T \to G/H$. In this section we deduce the formula of Akyildiz and Carrell from the relative localization formula in equivariant cohomology.

Let $N_G(T)$ be the normalizer of the torus $T$ in the group $G$. The Weyl group $W_G$ of $T$ in $G$ is $W_G = N_G(T)/T$. We use the same letter $w$ to denote an element
of the Weyl group $W_G$ and a lift of the element to the normalizer $N_G(T)$. The Weyl group $W_G$ acts on $G/T$ by
\[(gT)w = gwT \quad \text{for } gT \in G/T \text{ and } w \in W_G.\]
This induces an action of $W_G$ on the cohomology ring $H^\ast(G/T)$.

We may also consider the Weyl group $W_H$ of $T$ in $H$. By restriction the Weyl group $W_H$ acts on $G/T$ and on $H^\ast(G/T)$.

To each character $\gamma$ of $T$ with representation space $C_\gamma$, one associates a complex line bundle $L_\gamma := G \times_T C_\gamma$ over $G/T$. Fix a set $\Delta^+(H)$ of positive roots for $T$ in $H$, and extend $\Delta^+(H)$ to a set $\Delta^+$ of positive roots for $T$ in $G$.

**Theorem 5.1** ([1]). The Gysin homomorphism $f_* : H^\ast(G/T) \to H^\ast(G/H)$ is given by, for $a \in H^\ast(G/T)$,
\[f_*a = \sum_{w \in W_H} (-1)^w w \cdot a \prod_{\alpha \in \Delta^+(H)} c_1(L_\alpha).\]

**Remark 5.2.** There are two other ways to obtain this formula. First, using representation theory, Brion [6] proves a push-forward formula for flag bundles that includes Th. 5.1 as a special case. Secondly, since $G/T \to G/H$ is a fiber bundle with equivariantly formal fibers, the method of [14] using the ABBV localization theorem also applies.

To deduce Th. 5.1 from Prop. 4.1 we need to recall a few facts about the cohomology and equivariant cohomology of $G/T$ and $G/H$ (see [14]).

**Cohomology ring of $BT$.** Let $ET \to BT$ be the universal principal $T$-bundle. To each character $\gamma$ of $T$, one associates a complex line bundle $S_\gamma$ over $BT$ and a complex line bundle $L_\gamma$ over $G/T$:
\[S_\gamma := ET \times_T C_\gamma, \quad L_\gamma := G \times_T C_\gamma.\]
For definiteness, fix a basis $\chi_1, \ldots, \chi_\ell$ for the character group $\hat{T}$, where we write the characters additively, and set
\[u_i = c_1(S_{\chi_i}) \in H^2(BT), \quad z_i = c_1(L_{\chi_i}) \in H^2(G/T).\]
Let $R = \text{Sym}(\hat{T})$ be the symmetric algebra over $\mathbb{Q}$ generated by $\hat{T}$. The map $\gamma \mapsto c_1(S_\gamma)$ induces an isomorphism
\[R = \text{Sym}(\hat{T}) \to H^\ast(BT) = \mathbb{Q}[u_1, \ldots, u_\ell].\]
The map $\gamma \mapsto c_1(L_\gamma)$ induces an isomorphism
\[R = \text{Sym}(\hat{T}) \to \mathbb{Q}[z_1, \ldots, z_\ell].\]
The Weyl groups $W_G$ and $W_H$ act on the characters of $T$ and hence on $R$: for $w \in W_G$ and $\gamma \in \hat{T}$,
\[(w \cdot \gamma)(t) = \gamma(w^{-1}tw).\]
Cohomology rings of flag manifolds. The cohomology rings of $G/T$ and $G/H$ are described in [5]:

$$H^*(G/T) \simeq \frac{R}{(R^W_G)} \simeq \frac{\mathbb{Q}[z_1, \ldots, z_l]}{(R^W_G)},$$

$$H^*(G/H) \simeq \frac{R^W_H}{(R^W_G)} \simeq \frac{\mathbb{Q}[z_1, \ldots, z_l]^W_H}{(R^W_G)},$$

where $(R^W_G)$ denotes the ideal generated by the $W_G$-invariant homogeneous polynomials of positive degree.

The torus $T$ acts on $G/T$ and $G/H$ by left multiplication. For each character $\chi$ of $T$, let $K_\chi := (L_\chi)_T$ be the homotopy quotient of the bundle $L_\chi$ by the torus $T$. Then $K_\chi$ is a complex line bundle over $(G/T)_T$. Their equivariant cohomology rings are (see [14])

$$H^*_T(G/T) = \frac{\mathbb{Q}[u_1, \ldots, u_l, y_1, \ldots, y_l]}{J},$$

$$H^*_T(G/H) = \frac{\mathbb{Q}[u_1, \ldots, u_l] \otimes (\mathbb{Q}[y_1, \ldots, y_l]^W_H)}{J},$$

where $y_i = c_1(K_\chi_i) \in H^*_T(G/T)$ and $J$ denotes the ideal generated by $q(y) - q(u)$ for $q \in R^W_G$.

Fixed point sets. The fixed point sets of the $T$-action on $G/T$ and on $G/H$ are the Weyl group $W_G$ and the coset space $W_G/W_H$ respectively. Since these are finite sets of points,

$$H^*_T(W_G) = \bigoplus_{w \in W_G} H^*_T(\{w\}) \simeq \bigoplus_{w \in W_G} R,$$

$$H^*_T(W_G/W_H) = \bigoplus_{w \in W_H \in W_G/W_H} R.$$

Thus, we may view an element of $H^*_T(W_G)$ as a function from $W_G$ to $R$, and an element of $H^*_T(W_G/W_H)$ as a function from $W_G/W_H$ to $R$.

Let $h_M : M \to M_T$ be the inclusion of $M$ as a fiber in the fiber bundle $M_T \to BT$ and $i_M : M^T \to M$ the inclusion of the fixed point set $M^T$ in $M$. Note that $i_M$ is $T$-equivariant and induces a homomorphism in $T$-equivariant cohomology, $i^*_M : H^*_T(M) \to H^*_T(M^T)$. In order to apply Prop. 4.1 we need to know how to calculate the restriction maps

$$h^*_M : H^*_T(M) \to H^*(M) \quad \text{and} \quad i^*_M : H^*_T(M) \to H^*_T(M^T)$$

as well as the equivariant Euler class $e_M$ of the normal bundle to the fixed point set $M^T$, for $M = G/T$ and $G/H$. This is done in [14].

Restriction and equivariant Euler class formulas for $G/T$. Since $h^*_M : H^*_T(M) \to H^*(M)$ is the restriction to a fiber of the bundle $M_T \to BT$, and the bundle $K_\chi_i = (L_\chi_i)_T$ on $M_T$ pulls back to $L_\chi_i$ on $M$,

$$h^*_M(u_i) = 0, \quad h^*_M(y_i) = h^*_M(c_1(K_\chi_i)) = c_1(L_\chi_i) = z_i.$$

Let $i_w : \{w\} \to G/T$ be the inclusion of the fixed point $w \in W_G$ and

$$i^*_w : H^*_T(G/T) \to H^*_T(\{w\}) = R$$
the induced map in equivariant cohomology. By (14), for \( p(y) \in \tilde{H}^*_G(G/T), \)
(6) \[ i^*_w u_i = u_i, \quad i^*_w p(y) = w \cdot p(u), \quad i^*_w c_1(K_\gamma) = w \cdot c_1(S_\gamma). \]
Thus, the restriction of \( p(y) \) to the fixed point set \( W_G \) is the function \( i^*_M p(y) : W_G \to R \) whose value at \( w \in W_G \) is
(7) \[ (i^*_M p(y))(w) = w \cdot p(u). \]

The equivariant Euler class of the normal bundle to the fixed point set \( W_G \) assigns to each \( w \in W_G \) the equivariant Euler class of the normal bundle \( \nu_w \) at \( w \); thus, it is also a function \( e_M : W_G \to R \). By (14),
(8) \[ e_M(w) = e_T(\nu_w) = w \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right) = (-1)^w \prod_{\alpha \in \Delta^+} c_1(S_\alpha). \]

**Restriction and equivariant Euler class formulas for** \( G/H \). For the manifold \( M = G/H \), the formulas for the restriction maps \( h^*_N \) and \( i^*_N \) are the same as in (5) and (6), except that now the polynomial \( p(y) \) must be \( W_H \)-invariant. In particular,
(9) \[ h^*_N(\nu_w) = 0, \quad h^*_N p(y) = p(z), \quad h^*_N(c_1(K_\gamma)) = c_1(L_\gamma), \]
and
(10) \[ (i^*_N p(y))(wW_H) = w \cdot p(u). \]
If \( \gamma_1, \ldots, \gamma_m \) are characters of \( T \) such that \( p(c_1(K_{\gamma_1}), \ldots, c_1(K_{\gamma_m})) \) is invariant under the Weyl group \( W_H \), then
(11) \[ (i^*_N p(c_1(K_{\gamma_1}), \ldots, c_1(K_{\gamma_m}))(wW_H) = w \cdot p(c_1(S_{\gamma_1}), \ldots, c_1(S_{\gamma_m})). \]

The equivariant Euler class of the normal bundle of the fixed point set \( W_G/W_H \) is the function \( e_N : W_G/W_H \to R \) given by
(12) \[ e_N(wW_H) = w \cdot \left( \prod_{\alpha \in \Delta^+ - \Delta^+(H)} c_1(S_\alpha) \right). \]

**Proof of Th. 5.1** With \( M = G/T \) and \( N = G/H \) in Prop. 4.1, let
\[ p(z) \in H^*(G/T) = \mathbb{Q}[z_1, \ldots, z_d]/(R^W_G). \]
It is the image of \( p(y) \in \tilde{H}^*_T(G/T) \) under the restriction map \( h^*_M : H^*_T(G/T) \to H^*_G(G/T) \). By Prop. 4.1
(13) \[ f_* p(z) = f_* h^*_M p(y) = h^*_N f_{T*} p(y) \]
and
\[ f_{T*} p(y) = (i^*_N)^{-1}(f^T)_* \left( \frac{e_N}{e_M} i^*_M p(y) \right). \]
By Eq. (7), (8), and (12), for \( w \in W_G \),
\[ (i^*_M p(y))(w) = i^*_w p(y) = w \cdot p(u), \]
and
\[
\left( \frac{(f^T)^*e_N}{e_M} \right)(w) = \frac{e_N(wW_H)}{e_M(w)} = w \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right)
\]
\[
= \frac{1}{w \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right)}.
\]

To simplify the notation, define temporarily the function \( k : W_G \to R \)
by
\[
k(w) = w \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right).
\]
Then
\[
(14) \quad f_T^*p(y) = (i_N^*)^{-1}(f^T)_*(k).
\]

Now \((f^T)_*(k) \in H^*_T(W_G/W_H)\) is the function: \( W_G/W_H \to R \) whose value at the point \( wW_H \) is obtained by summing \( k \) over the fiber of \( f^T : W_G \to W_G/W_H \) above \( wW_H \). Hence,
\[
((f^T)_*k)(wW_H) = \sum_{wv \in wW_H} wv \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right)
\]
\[
= w \cdot \sum_{v \in W_H} v \cdot \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right).
\]

By (11), the inverse image of this expression under \( i_N^* \) is
\[
(15) \quad (i_N^*)^{-1}(f^T)_*k = \sum_{v \in W_H} v \cdot \left( \prod_{\alpha \in \Delta^+} c_1(K_\alpha) \right).
\]

Finally, combining (13), (14), (15) and (9),
\[
f_*p(z) = h_N^*(f^T)_*p(y) = \sum_{v \in W_H} v \cdot \left( \prod_{\alpha \in \Delta^+} c_1(L_\alpha) \right).
\]

\[\Box\]

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