SINGULAR UNITARITY IN “QUANTIZATION COMMUTES WITH REDUCTION”

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Abstract. Let $M$ be a connected compact quantizable Kähler manifold equipped with a Hamiltonian action of a connected compact Lie group $G$. Let $M//G = \phi^{-1}(0)/G = M_0$ be the symplectic quotient at value 0 of the moment map $\phi$. The space $M_0$ may in general not be smooth. It is known that, as vector spaces, there is a natural isomorphism between the quantum Hilbert space over $M_0$ and the $G$-invariant subspace of the quantum Hilbert space over $M$. In this paper, without any regularity assumption on the quotient $M_0$, we discuss the relation between the inner products of these two quantum Hilbert spaces under the above natural isomorphism; we establish asymptotic unitarity to leading order in Planck’s constant of a modified map of the above isomorphism under a “metaplectic correction” of the two quantum Hilbert spaces.

1. Introduction

Let $M$ be an integral connected compact Kähler manifold with symplectic form $\omega$. Then $M$ is quantizable, i.e., there is a Hermitian holomorphic line bundle $L$ over $M$ with connection whose curvature is $-i\omega$. We consider the $k$th tensor power $L^\otimes k$ of $L$. The Hermitian structure on $L$ induces a Hermitian structure on $L^\otimes k$. The Hermitian structure on $L^\otimes k$ naturally equips the space of holomorphic sections of $L^\otimes k$ over $M$ with an inner product. For each $k$, the quantum Hilbert space $\mathcal{H}(M, L^\otimes k)$ is the space of holomorphic sections of $L^\otimes k$ over $M$ with the inner product. This is the first “reducing” and then “quantizing” Hilbert space. The first “quantizing” and then “reducing” quantum Hilbert space is the $G$-invariant subspace $\mathcal{H}(M, L^\otimes k)_G$ of $\mathcal{H}(M, L^\otimes k)$. By Guillemin and Sternberg ([6]), there is a natural invertible linear map $A_k$ between $\mathcal{H}(M, L^\otimes k)_G$ and $\mathcal{H}(M_0, (L^\otimes 0)_0)$. Let us call this linear map the Guillemin-Sternberg map. For quantum mechanics, the

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inner products of the quantum Hilbert spaces are also important. A few authors have observed that the Guillemin-Sternberg map is not unitary, and, it does not become asymptotically unitary as \( k \to \infty \). Moreover, they identified the volume of the \( G \)-orbits in the zero level set as an obstruction to asymptotic unitarity. We refer to the work of Flude [5], Paoletti [14], Ma-Zhang [12] and [13], Charles [3] and Hall-Kirwin [9]. Flude was the first who gave a formal computation of the leading-order term of the asymptotic density function (the function which relates the norm of an invariant holomorphic section upstairs and the norm of the descended section downstairs) and who obtained the non-unitarity result. Paoletti proved this result in his study of the asymptotic expansion of the Szegő kernels upstairs and downstairs (using microlocal analysis [1, 2]). Ma and Zhang obtained this result (Theorem 0.10 for \( E = \mathbb{C} \) in [13]) on their way of studying the asymptotic expansion of the \( G \)-invariant Bergman kernel of the spin\(^c\) Dirac operator associated with vector bundles on a symplectic manifold. Charles obtained this result in his study of (invariant) Toeplitz operators on \( M \) and on the symplectic quotient \( M_0 \) (for torus actions) by looking at the relations he obtained of the (principal) symbols of the Toeplitz operators on \( M \) and of the Toeplitz operators on \( M_0 \).

In the recent study of the inner products of the quantum Hilbert spaces by Hall and Kirwin [9], they proved again non-unitarity by writing down an exact expression for the norm of an invariant holomorphic section upstairs as an integral over \( M_0 \) and by estimating the leading term of the asymptotic behavior of the density function. Moreover, for this “free action” case, they obtained asymptotic unitarity results for a modified quantization procedure. More precisely, they took the tensor products of the line bundles \( L^\otimes k \)'s with the square root of the canonical bundle of \( M \) (assuming it exists), called the metaplectic correction, and they showed that a new defined Guillemin-Sternberg type map \( B_k \) between the new quantum Hilbert spaces is invertible for all sufficiently large \( k \), and that this map is asymptotically unitary to leading order as \( k \to \infty \).

In general, the action of \( G \) on \( \phi^{-1}(0) \) may not be free. Consequently, the quotient \( M_0 \) may not be smooth. By [15] and by [16], \( M_0 \) is in general a stratified Kähler space, with the stratification being given by orbit types of the action. When there is only one orbit type, \( M_0 \) is still a smooth Kähler manifold. In this general case when the action of \( G \) on \( \phi^{-1}(0) \) may not be free, the Hermitian line bundle \( L^\otimes k \) descends to a Hermitian V-line bundle \((L^\otimes k)_0\) over \( M_0 \). Let \( \mathcal{H}(M_0,(L^\otimes k)_0) \) still be the space of holomorphic sections of the V-line bundle \((L^\otimes k)_0\) over \( M_0 \) with the induced inner product. By Sjamaar (see Theorem 6), there is a natural linear isomorphism \( A'_k \) between \( \mathcal{H}(M,L^\otimes k)^G \) and \( \mathcal{H}(M_0,(L^\otimes k)_0) \).

When the action of \( G \) on \( \phi^{-1}(0) \) is not free, the volume of the \( G \)-orbits in \( \phi^{-1}(0) \) is of course less “uniform”. One guesses by the above authors’ results that \( A'_k \) would not be unitary or asymptotically unitary after suitable quantum norms are defined. In this paper, we drop the assumption that the action of \( G \) on \( \phi^{-1}(0) \) is free. We give a formula on the relation of the quantum norm of an invariant holomorphic section upstairs and the quantum norm of the descended section downstairs under the map \( A'_k \) and we give an asymptotic formula of this to leading order as \( k \to \infty \). We see that \( A'_k \) is not unitary and it is not asymptotically unitary. We still consider the “metaplectic correction”. We give a description on how the square root of the canonical bundle of \( M \) descends to \( M_0 \), we show the existence of a family of modified isomorphisms \( B'_k \) for sufficiently large \( k \) between the new quantum Hilbert
spaces, and we establish asymptotic unitarity to leading order term for the maps $B'_k$.

There are two main problems that need to be addressed in this new study. One is that we find a suitable way to descend the half form bundle of $M$ to the stratified quotient $M_0$. Another problem has to do with a large piece of the manifold $M$, the semistable set $M^{ss}$, which is open dense and connected in $M$. By Theorem 6, $\mathcal{H}(M_0, (L^\otimes k)_0) \simeq \mathcal{H}(M^{ss}, L^\otimes k)^G$. The holomorphic action of $G$ can be analytically extended to a $G_C$-action, where $G_C$ is the complexification of $G$. If $G$ acts freely on $\phi^{-1}(0)$, $M^{ss}$ consists of free $G$-orbits and it consists of complex $G_C$-orbits each of which intersects $\phi^{-1}(0)$ at one $G$-orbit. In the general case, $M^{ss}$ may contain complex $G_C$-orbits which do not intersect $\phi^{-1}(0)$ but contain those $G_C$-orbits which intersect $\phi^{-1}(0)$ in their closures. We will analyze the structure of these complex orbits and study their contribution to the quantum norms.

Our main results are Theorem 7, Theorem 9, Theorem 11, Theorem 12 (and the corollaries of Theorems 11 and 12 corollaries 2 and 3), and Theorem 14.

We will use three different notations interchangeably for the symplectic quotient at $0$, $M_0$, $M/G$, and $M^{ss}/G_C$, depending on the context.

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2. Reduction of Kähler manifolds

In this section, we will recall some main results obtained by R. Sjamaar in [10] on general Kähler quotients. This will help us to understand our space $M_0$ as well as its relation with $M$. One may see the difference between the case when the action of $G$ on $\phi^{-1}(0)$ is free and the case when this assumption is removed. This section also serves as a preparation for the tools needed in the subsequent sections.

To understand stratified Kähler spaces and their quantizations, we also refer to the work of Huebschmann, [7] and [8].

Let $(M, \omega, J, B = \omega(\cdot, J \cdot))$ be a connected compact Kähler manifold with symplectic form $\omega$, compatible complex structure $J$ and Riemannian metric $B$. Let $G$ be a connected compact Lie group acting holomorphically on $M$. Assume that the $G$ action is Hamiltonian with an equivariant moment map $\phi$. Assume $a$ is a value of $\phi$. Then the quotient $M_a = \phi^{-1}(G \cdot a)/G$ is called the symplectic quotient or the reduced space at the coadjoint orbit $G \cdot a$. Let us restrict attention to the value $a = 0$. By [13], the quotient $M_0$ is a connected compact stratified symplectic space with a connected open dense stratum. If $M_0$ has only one stratum, then it is a smooth symplectic manifold. We will see that $M_0$ also admits an analytic structure such that $M_0$ is a stratified Kähler space.

Since $G$ acts holomorphically, the action can be analytically continued to a holomorphic action of the “complexified” group $G_C$ on $M$. The Lie algebra $\mathfrak{g}_C$ of $G_C$
is the complexification of \( \mathfrak{g} \). The Cartan decomposition gives a diffeomorphism 
\[ G_C \simeq \exp(i\mathfrak{g})G. \]
For \( \xi \in \mathfrak{g} \), let \( X^\xi \) be the infinitesimal vector field on \( M \) generated by \( \xi \). Then \( X^{i\xi} = JX^\xi \) is the infinitesimal vector field generated by \( i\xi \).

Define a point \( m \) in \( M \) to be \textbf{(analytically) semistable} if the closure of the 
\( G_C \)-orbit through \( m \) intersects the zero level set \( \phi^{-1}(0) \). Let \( M^{ss} \) be the set of
semistable points in \( M \). The point \( m \) is called \textbf{(analytically) stable} if the closure of
the \( G_C \)-orbit through \( m \) intersects the zero level set \( \phi^{-1}(0) \) at a point where \( d\phi \)
is surjective. Let \( M^s \) be the set of stable points in \( M \). When the action of \( G \) on
\( \phi^{-1}(0) \) is free or locally free, \( d\phi \) is surjective at any point of \( \phi^{-1}(0) \). In this case,
\( M^{ss} \) coincides with \( M^s \).

Assuming there is a \( G \)-invariant inner product on \( \mathfrak{g} \), by Lemma 6.6 in [10], the
gradient vector field of \( \|\phi\|^2 \) is given by
\[ \text{grad}((\|\phi\|^2))(m) = 2JX^{\phi(m)}(m), \]
where we have identified \( \phi(m) \in \mathfrak{g}^* \) with a vector in \( \mathfrak{g} \) using the inner product,
and where \( X^{\phi(m)}(m) \) is the vector field on \( M \) induced by \( \phi(m) \), evaluated at the
point \( m \). So \( \text{grad}((\|\phi\|^2))(m) \) is tangent to the \( G_C \)-orbits. Let \( F_1 \) be the flow of
\( -\text{grad}((\|\phi\|^2)) \). Kirwan has proved that \( M^{ss} \) is the set of points \( m \in M \) such that
the path \( F_1(m) \) has a limit point in \( \phi^{-1}(0) \) ([10]). By [11] or by [17], the limit map
\( F_\infty(m) \) gives an equivariant deformation retraction from \( M^{ss} \) onto \( \phi^{-1}(0) \).

2.1. The holomorphic slice theorem. In order to describe the complex analytic
structure on \( M_0 \), let us first recall the holomorphic slice theorem due to R. Sjamaar.
The results on the orbit structure of \( M^{ss} \) and on the stratified Kähler structure of
\( M_0 \) are due to this theorem.

**Theorem 1.** \((\text{Holomorphic slice theorem}) \ [10]\) Let \( M \) be a Kähler manifold and
let \( G_C \) act holomorphically on \( M \). Assume the action of the compact real form \( G \) is Hamiltonian.
Let \( m \) be any point in \( M \) such that the \( G \)-orbit through \( m \) is isotropic.
Then there exists a \textbf{holomorphic slice} at \( m \) for the \( G_C \)-action.

If \( X \) is a complex space and \( G_C \) a reductive complex Lie group acting holomorphically on \( X \), we have the following definition of a \textbf{holomorphic slice}.

**Definition 1.** A \textbf{holomorphic slice} at \( x \) for the \( G_C \) action is a locally closed analytic subspace \( D \) of \( X \) with the following properties:
1. \( x \in D \);
2. \( G_C D \) of \( D \) is open in \( X \);
3. \( D \) is invariant under the action of the stabilizer \( (G_C)_x \);
4. the natural \( G_C \)-equivariant map from the associated bundle \( G_C \times (G_C)_x \rightarrow D \) into \( X \),
which sends an equivalence class \([g, y]\) to the point \( gy \), is an analytic isomorphism
onto \( G_C D \).

2.2. Kähler reduction. For the orbit structure of \( M^{ss} \), one may see Proposition
2.4 in [10]. We list a few of them which are more relevant to us.

**Proposition 1.** In the following, “closed” means “closed in \( M^{ss} \)” and “closure”
means “closure in \( M^{ss} \).”
1. The semistable set \( M^{ss} \) is the smallest \( G_C \)-invariant open subset of \( M \) containing \( \phi^{-1}(0) \), and its complement in \( M \) is a complex-analytic subset;
2. A \( G_C \)-orbit in \( M^{ss} \) is closed if and only if it intersects \( \phi^{-1}(0) \);
3. The closure of every \( G_C \)-orbit in \( M^{ss} \) contains exactly one closed \( G_C \)-orbit.
We call two semistable points \( x \) and \( y \) related if the closures in \( M^{ss} \) of the orbits \( G_Cx \) and \( GCy \) intersect. This relation is an equivalence relation. Let \( M^{ss}/G_C \) be the quotient space and let \( \pi_C : M^{ss} \to M^{ss}/G_C \) be the quotient map.

**Theorem 2.** ([16]) The inclusion \( \phi^{-1}(0) \subset M^{ss} \) induces a homeomorphism \( M_0 = \phi^{-1}(0)/G \to M^{ss}/G_C \).

We say that a subset \( A \) of \( M^{ss} \) is saturated with respect to \( \pi_C \) if \( \pi_C^{-1} \pi_C(A) = A \).

**Proposition 2.** ([16]) At every point of \( \phi^{-1}(0) \), there exists a holomorphic slice \( D \) such that the set \( \pi_C^{-1} \pi_C(A) = A \).

We identify the spaces \( M^{ss}/G_C \) and \( M_0 \). We furnish \( M_0 \) with a complex-analytic structure such that the quotient map \( \pi_C \) is holomorphic. We define a function \( f \) defined on an open subset \( O \) of \( M_0 \) to be holomorphic if the pullback of \( f \) to \( \pi_C^{-1}(O) \) is holomorphic. Let \( \mathcal{O}_{M_0} \) be the sheaf of holomorphic functions on \( M_0 \).

**Theorem 3.** ([16]) The ringed space \( (M_0, \mathcal{O}_{M_0}) \) is an analytic space.

The following theorem describes the property of the stable set \( M^s \subset M^{ss} \). If \( 0 \) is a regular value of \( \phi \), then \( M^s = M^{ss} \). In general, if \( M^s \neq \emptyset \), then \( M^s \) is open and dense in \( M^{ss} \).

**Theorem 4.** ([16]) If \( x \in M \) is stable, then the orbit \( G_Cx \) is closed in \( M^{ss} \). Let \( Z \) be the set of points \( m \in \phi^{-1}(0) \) with the property that \( d\phi_m \) is surjective. Then the stable set \( M^s \) is equal to \( F^{-1}_\infty(Z) \). Every fiber of \( \pi_C|_{M^s} \) consists of a single orbit.

By this theorem, we see that if a \( G \)-orbit \( O = G \cdot x \) in \( \phi^{-1}(0) \) has the dimension of \( G \), then only one complex orbit \( G_C \cdot x = G_C \cdot O \) flows to \( O \) under the gradient flow of \(-\|\phi\|^2\).

The stratification of \( M_0 \) as a stratified symplectic space is given by orbit types. Let \( p \in M_0 \), and let \( x \in \pi^{-1}(p) \), where \( \pi : \phi^{-1}(0) \to M_0 \) is the quotient map. Let \( (H) \) be a conjugacy class of closed subgroups of \( G \). Then \( p \) is said to be of orbit type \( (H) \) if the stabilizer of \( x \) is conjugate to \( H \). By [13], the set of all points of orbit type \( (H) \) in \( M_0 \) is a symplectic manifold.

We can similarly define \( G_C \)-orbit types. We can show that if \( x \in \phi^{-1}(0) \), then the complex stabilizer \((G_C)_x\) is equal to the compactification \((G_x)_C\) of the compact stabilizer \( G_x \) (see Proposition 1.6 in [16]). By Proposition 1 the fiber \( \pi_C^{-1}(p) \) contains a unique closed \( G_C \)-orbit \( G_Cx \). Let us say \( p \) is of \( G_C \)-orbit type \( (H_C) \) if the stabilizer \((G_C)_x\) is conjugate to \( H_C \) in \( G_C \).

**Theorem 5.** ([16]) The stratification of \( M_0 \) by \( G \)-orbit types is identical to the stratification by \( G_C \)-orbit types. Each stratum \( S \) is a complex manifold and its closure is a complex-analytic subvariety of \( M_0 \). The reduced symplectic form on \( S \) is a Kähler form.

3. Quantization of Kähler Manifolds

Let \( M \) a connected compact Kähler manifold as in the last section. Assume that the Kähler form \( \omega \) is integral, i.e., the cohomology class \([\omega/2\pi]\) is an integral cohomology class. Then \( M \) is quantizable, i.e., there is a Hermitian line bundle \( L \) with compatible connection \( \nabla \) such that its curvature is \(-i\omega \). The \( k \)-th tensor
power $L^\otimes k$ of $L$ is a Hermitian line bundle over $M$ with induced Hermitian structure from $L$. For each $k$, $L^\otimes k$ may be given the structure of a holomorphic line bundle. For each fixed $k$, the quantum Hilbert space is the space of holomorphic sections of $L^\otimes k$ over $M$, denoted $\mathcal{H}(M, L^\otimes k)$. Let $\epsilon_\omega = \omega^n/n$ be the Liouville volume form on $M$. Then the inner product on $\mathcal{H}(M, L^\otimes k)$ is usually defined to be

$$< s_1, s_2 > = (k/2\pi)^{n/2} \int_M (s_1, s_2) \epsilon_\omega,$$

where $(s_1, s_2)$ is the pointwise Hermitian structure on $L^\otimes k$.

In this paper, we will study quantizable Kähler manifolds with a holomorphic Hamiltonian Lie group action. The symplectic quotient at value 0 may not be smooth. To adapt to this situation, we will give two definitions of the inner product on $\mathcal{H}(M, L^\otimes k)$, respectively in Definitions 4 and 5 of Section 13.

4. QUANTUM REDUCTION

Let $M$ be a connected compact quantizable Kähler manifold. Let $G$ be a connected compact Lie group acting on $M$ holomorphically and in a Hamiltonian fashion with moment map $\phi$. The $G$ action lifts to a holomorphic action on the line bundle $L$ preserving the Hermitian structure. Both the $G$ action on $M$ and on $L$ can be analytically continued to holomorphic $G_\mathbb{C}$ actions. The $G$ action on $L$ induces $G$ actions on $\mathcal{H}(M, L^\otimes k)$. Infinitesimally, the action is given by

$$Q_\xi s = \nabla^{(k)}_{\chi\xi} s - ik\phi_\xi s, \quad \text{for} \ \xi \in \mathfrak{g},$$

where $\nabla^{(k)}$ is the induced connection on $L^\otimes k$, and $\phi_\xi$ is the "$\xi$-moment map component", i.e., $\phi_\xi = < \phi, \xi >$. The reduction at quantum level amounts to taking $G$-invariant holomorphic sections, i.e., taking $\mathcal{H}(M, L^\otimes k)^G$.

5. QUANTIZATION AFTER REDUCTION

Let $M$ be a connected compact quantizable Kähler manifold equipped with a holomorphic Hamiltonian action of a connected compact Lie group $G$. Let $\phi$ be the moment map. Let $L_0 = L|_{\phi^{-1}(0)}/G$. Then $L_0$ is a $V$-line bundle over $M_0$, i.e., each point in $M_0$ has an open neighborhood $O$ which is the quotient of a space $\tilde{O}$ by a finite group $\Gamma$ such that $L_0|_O$ is the quotient by $\Gamma$ of a $\Gamma$-equivariant line bundle over $\tilde{O}$. As an analytic space, $L_0$ can be identified with the quotient $L|_M^{ss}/G_\mathbb{C}$. A holomorphic section of $L$ defined over a $G_\mathbb{C}$-invariant open set is $G$-invariant if and only if it is $G_\mathbb{C}$-invariant. Let $L'$ be the sheaf of holomorphic sections of $L$ and define a sheaf $L'$ on $M_0$, the sheaf of invariant sections, by letting $L'(O) = L(\pi^{-1}_C(O))^{G_\mathbb{C}}$ for each open set $O$ of $M_0$. Then we have

**Proposition 3.** ([16]) The sheaf $L'$ is (the sheaf of sections of) the holomorphic V-line bundle $L_0$ over $M_0 = M^{ss}/G_\mathbb{C}$.

We take the space of holomorphic sections $\mathcal{H}(M_0, L_0)$ of the $V$-line bundle $L_0$ as the quantization of the reduced space $M_0$.

If we replace $L$ by $L^\otimes k$, we have the quantum spaces $\mathcal{H}(M_0, (L^\otimes k)_0)$.

Since the action of $G$ preserves the Hermitian structure on $L^\otimes k$, the Hermitian structure on $L^\otimes k$ descends to a Hermitian structure on $(L^\otimes k)_0$. Let $s \in \mathcal{H}(M, L^\otimes k)^G$. Then by restricting $s$ to $\phi^{-1}(0)$ and by letting it descend to $M_0$, we
get an element of $\mathcal{H}(M_0, (L^\otimes k)_0)$. Let us call this linear map $A'_k$. So, if $x \in \phi^{-1}(0)$, then we have
\[ |s|^2(x) = |A'_ks|^2(\ldots). \]

We still need to define an inner product on $\mathcal{H}(M_0, (L^\otimes k)_0)$. Denote
\[ Z_{(H)} = \{ m \in \phi^{-1}(0) : \text{the stabilizer group of } m \text{ is conjugate to } H \subset G \}, \]
and
\[ S(H) = Z_{(H)}/G. \]

As remarked by Sjamaar (see Remark 3.9 in [16], each $S(H)$ has finite symplectic volume. For a fixed stratum $S$ of $M_0$, let $d_S$ be the complex dimension of $S$, and let $\omega_S$ be the volume form on $S$, where $\omega_S$ is the reduced symplectic form on $S$.

Let $s'_1, s'_2 \in \mathcal{H}(M_0, (L^\otimes k)_0)$, and, let $(s'_1, s'_2)$ be the pointwise Hermitian inner product on $(L^\otimes k)_0$ inherited from the one on $L^\otimes k$. Since there is an open dense connected stratum, say $S^O$, in $M_0$, which has full measure, we give the first definition of an inner product on $\mathcal{H}(M_0, (L^\otimes k)_0)$:
\[ <s'_1, s'_2>_{(1)} = \frac{k}{2\pi} d_{S^O}/2 \int_{S^O} (s'_1, s'_2) \epsilon_{\omega_{S^O}}. \]

The following second definition of an inner product on $\mathcal{H}(M_0, (L^\otimes k)_0)$ takes into account all the strata of $M_0$:
\[ <s'_1, s'_2>_{(2)} = \sum_{S(H)} \frac{k}{2\pi} d_{S(H)}/2 \int_{S(H)} (s'_1, s'_2) \epsilon_{\omega_{S(H)}}. \]

If $S$ is a single point, then the above integral of $(s'_1, s'_2)$ over $S$ is just the value of $(s'_1, s'_2)$ over this point.

6. The linear space isomorphism

In the last section, we defined a linear map $A'_k$ from $\mathcal{H}(M, L^\otimes k)^G$ to $\mathcal{H}(M_0, (L^\otimes k)_0)^G$. We have

**Theorem 6.** ([16]) Under our hypotheses, the quotient map $\pi_S : M^{ss} \to M_0$ and the inclusion $M^{ss} \subset M$ induce isomorphisms $\mathcal{H}(M_0, (L^\otimes k)_0) \simeq \mathcal{H}(M^{ss}, L^\otimes k)^G \simeq \mathcal{H}(M, L^\otimes k)^G$.

By Proposition 3, we have the isomorphism $\mathcal{H}(M_0, (L^\otimes k)_0) \simeq \mathcal{H}(M^{ss}, L^\otimes k)^G$. The isomorphism $\mathcal{H}(M^{ss}, L^\otimes k)^G \simeq \mathcal{H}(M, L^\otimes k)^G$ is based on the observation that the norm of an invariant holomorphic section $s$ of $L^\otimes k$ is increasing along the trajectories of $-\text{grad}(\|s\|^2)$. It follows that if $s$ is defined on $M^{ss}$, then $<s, s>$ is bounded on $M$. By Riemann’s Extension Theorem, $s$ extends to a $G$-invariant holomorphic section on $M$. See [16] for details.

From this theorem, we can deduce that a point $x \in M$ is semistable if there exists an invariant global holomorphic section $s \in \mathcal{H}(M, L^\otimes k)^G$ for some $l$ such that $s(x) \neq 0$ (see [16]). So the set of unsemistable points is contained in the 0 set of $s$, therefore it has complex codimension at least one.
7. Half form bundles on $M$

Let $K = \bigwedge^n (T^{1,0}M)^*$ be the canonical bundle of $M$. A smooth section of $K$ is called an $(n,0)$-form. We know that the first Chern class of $K$ is $-c_1(M)$. Assume $c_1(M)/2$ is integral. Then the square root $\sqrt{K}$ of the bundle $K$ exists. We fix a choice of $\sqrt{K}$. The group $G$ acts on sections of $K$. Infinitesimally, a Lie algebra element $\xi \in \mathfrak{g}$ acts on $(n,0)$-forms by taking the Lie derivative $L_{X\xi}$ of the form. This induces an action of $\mathfrak{g}$ on half forms by $2(L_{X\xi} \mu) = L_{X\xi}(\mu^2)$, where $\mu \in \sqrt{K}$. Since $G$ acts holomorphically on $M$, we can check that $\mathfrak{g}$ preserves the space of holomorphic sections of $K$ and of $\sqrt{K}$.

Let us define a Hermitian structure on $\Gamma(M, \sqrt{K})$, where $\Gamma(M, \sqrt{K})$ is the space of smooth sections of $\sqrt{K}$. Let $\mu, \nu \in \Gamma(M, \sqrt{K})$ be half forms, then $\mu^2 \wedge \overline{\nu}^2 \in \Gamma(\bigwedge^{2n} T^* (M))$. The volume form $\epsilon_\omega$ is a global trivializing section of $\bigwedge^{2n} T^* (M)$. So there is a function, denoted $(\mu, \nu)$, such that

$$\mu^2 \wedge \overline{\nu}^2 = (\mu, \nu)^2 \epsilon_\omega.$$ (5)

The function $(\mu, \nu)$ is defined to be the pointwise inner product of $\mu$ and $\nu$.

We use this to define a Hermitian form on $\Gamma(M, L^{\otimes k} \otimes \sqrt{K})$. Let $t_1, t_2 \in \Gamma(M, L^{\otimes k} \otimes \sqrt{K})$ which are locally represented by $t_j (x) = s_j (x) \mu_j (x)$, we define

$$(t_1, t_2)(x) = (s_1(x), s_2(x))(\mu_1, \mu_2)(x).$$ (6)

8. The “push down” of the half form bundle $\sqrt{K}$ to the reduced space $M_0$

8.1. When the $G$ action on $\phi^{-1}(0)$ is free. Before we come to the general case, let us recall first the procedure given by Hall and Kirwin of pushing down a half form bundle $\sqrt{K}$ of $M$ to a half form bundle $\sqrt{K}$ on $M//G$ in the case the action of $G$ on $\phi^{-1}(0)$ is free.

Let $\alpha$ be a $G_\mathfrak{C}$-invariant $(n,0)$-form on $M$. Hall and Kirwin obtained an $(n-d,0)$-form $(d$ is the dimension of $G$) $\hat{\beta}$ on $M//G$ in the following way. Choose a $G$-invariant inner product on $\mathfrak{g}$ normalized so that the volume of $G$ with respect to the associated Haar measure is $1$. Fix an orthonormal basis $\xi_1, \xi_2, \ldots, \xi_d$ of the Lie algebra $\mathfrak{g}$. Let $X_{\xi_1}, X_{\xi_2}, \ldots, X_{\xi_d}$ be the vector fields they generate on $M$. For any $x \in M^s$, define

$$\beta = i(\bigwedge_j X_{\xi_j}) \alpha.$$ $\beta$ is basic with respect to the projection map $\pi_\mathfrak{C}$. So $\beta = \pi_\mathfrak{C}(\hat{\beta})$, where $\hat{\beta}$ is an $(n-d,0)$-form on $M//G$. Let $\mathfrak{B}$ be the map

$$\mathfrak{B}(\alpha) = \hat{\beta}.$$ Consequently, one can construct the inverse map of this push down map. Given an $(n-d,0)$-form $\hat{\beta}$ on $M//G$. The pull back $\beta = \pi_\mathfrak{C}(\hat{\beta})$ is a $G_\mathfrak{C}$-invariant $(n-d,0)$-form on $M^s$. One can construct a $G_\mathfrak{C}$-invariant $(n,0)$-form $\alpha$ on $M^s$ from $\beta$. Given a local frame $X_{\xi_1}, X_{\xi_2}, \ldots, X_{\xi_d}, Y_1, \ldots, Y_{n-d}$ for $T_x M^s$, set

$$\alpha(X_{\xi_1}, X_{\xi_2}, \ldots, X_{\xi_d}, Y_1, \ldots, Y_{n-d}) = \pi_\mathfrak{C}(\hat{\beta}(Y_1, \ldots, Y_{n-d})).$$
and define \( \alpha \) on any other frame by \( GL(n, \mathbb{C}) \)-equivariance and the requirement that \( \alpha \) be an \((n, 0)\)-form. Every other frame is equivalent to a linear combination of frames which are \( GL(n, \mathbb{C}) \)-equivalent to one of the form \( W_1, W_2, \ldots, W_d, Y_1, \ldots, Y_{n-d} \) where \( W_j = X^{\xi_j} \) or \( JX^{\xi_j} \).

Assume that the \( g \) action on \( \sqrt{K} \) exponentiates to a \( G \) action and it is compatible with the \( G \) action on \( K \). It can be shown that the \( G \) action on \( \sqrt{K} \) can be analytically continued to a \( G_C \) action. Define a line bundle \( \sqrt{K} \) over \( M//G \) whose fiber is the equivalence class of \( \sqrt{K} \) under the \( G_C \) action. For a \( G_C \)-invariant smooth section \( \mu \in \Gamma(M, \sqrt{K})^{G_C} \), we define the map

\[
B : \Gamma(M, \sqrt{K})^{G_C} \to \Gamma(M//G, \sqrt{K})
\]

by \((B\mu)^2 = \mathfrak{B}(\mu^2)\).

Since for an \((n, 0)\)-form \( \alpha \), contracting with \( \bigwedge \pi_+ X^{\xi} \) is the same as contracting with \( \bigwedge \pi_+ X^{\xi} \), where \( \pi_+ X^{\xi} = \frac{i}{2}(X^{\xi} - iJX^{\xi}) \), and the vector fields \( \pi_+ X^{\xi} \) are holomorphic, \( \alpha \) is locally holomorphic if and only if \( \mathfrak{B}(\alpha) \) is locally holomorphic; and, \( \mu \) is locally holomorphic if and only if \( B(\mu) \) is locally holomorphic.

### 8.2. When the \( G \) action on \( \phi^{-1}(0) \) is not necessarily free.

Now, we come to the general case.

**Lemma 1.** Let \( \alpha \in \Gamma(M, K)^{G_C} \). Then, \( \alpha \) descends to a smooth \((d_S, 0)\)-form \( \hat{\beta}|_S \) on each smooth stratum \( S \) of \( M_0 \) of complex dimension \( d_S \). If \( \alpha \) is holomorphic, then each \( \hat{\beta}|_S \) is holomorphic.

**Proof.** Let \( Z(H) \) and \( S(H) \) be as in (1) and (2). Take the complex submanifold \( G_C \cdot Z(H) \). Let

\[
\alpha |_s = \alpha|_{G_C \cdot Z(H)}.
\]

Then \( \alpha |_s \) is a \( G_C \)-invariant \((m, 0)\)-form on \( G_C \cdot Z(H) \), assuming \( m \) is the complex dimension of \( G_C \cdot Z(H) \).

Case 1. Assume \( H = G \). Then \( G_C \cdot Z_G = Z_G = Z_G//G = S_G \). We define

\[
\hat{\beta}|_{S_G} = \alpha|_{Z_G}.
\]

Case 2. Assume \( H \neq G \). Assume we have chosen a normalized \( G \)-invariant inner product on \( g \). Let \( \xi_1, \ldots, \xi_h \) be an orthonormal basis of \( \mathfrak{h} = \text{Lie}(H) \), expand it to an orthonormal basis of \( g = \text{Lie}(G) \) by joining \( \xi_{h+1}, \ldots, \xi_d \). At each point \( x \) of \( Z(H) \) with stabilizer group \( H \), we define

\[
\beta|_x = i( \bigwedge_{j=h+1, \ldots, d} X^{\xi_j}) \alpha|_x.
\]

We contract the form \( \alpha|_x \) similarly at the points of \( Z(H) \) with stabilizers conjugate to \( H \). So, along \( Z(H) \), we have a new form \( \beta|_x \). Let

\[
\beta|_x = \beta|_{Z(H)}.
\]

This restriction “cuts off” the \( JX^{Ad(G)}\xi_j \), \( j = h + 1, \ldots, d \) directions which are normal to \( Z(H) \) in \( G_C \cdot Z(H) \). Now, \( \beta|_x \) is a smooth \( G \)-invariant \((m - d_G//H)\)-form defined on \( Z(H) \). By the above contraction and by \( G \)-invariance of the form \( \alpha \),
clearly, $\beta_m = \pi^*(\hat{\beta})$, where $\hat{\beta}$ is a $(m - d_{G/H}, 0)$-form on $S_{\langle H \rangle}$, $\pi : Z_{\langle H \rangle} \to S_{\langle H \rangle}$ is the quotient map.

By the above construction, if $\alpha$ is holomorphic, then $\hat{\beta}|_S$ is holomorphic (see the reason we mentioned in Section 8.1).

Next, we will use the holomorphic slice theorem to see how the forms $\hat{\beta}|_S$’s are related.

**Lemma 2.** The forms $\hat{\beta}|_S$’s in Lemma 1 satisfy: if $S \subset \tilde{S}$, then $\hat{\beta}|_S$ is obtained from $\hat{\beta}|_{S'}$ by degenerating some directions.

**Proof.** Let $x_0 \in Z_{\langle H \rangle}$ be a point with stabilizer group $H$. Take a saturated open neighborhood $U = G_{\mathbb{C}}D = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} D$ of $x_0$ (see Proposition 2), where $H_{\mathbb{C}} = (H)_{\mathbb{C}}$ is the complex stabilizer group of $x_0$ which is the complexification of $H$. Split $D = D_1 \times D_2$, where $D_1$ is the fixed complex subspace of the $H$ action and therefore the $H_{\mathbb{C}}$ action. So $U = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times D_2)$. The set

$$U/H = (D_1 \times D_2)/H_{\mathbb{C}} = D_1 \times D_2/\!/H_{\mathbb{C}}$$

is a neighborhood of $[x_0]$ in $M_0$.

The set $U$ is $G$-equivariantly diffeomorphic to $G \times_H (\sqrt{-1}m \times D_1 \times D_2)$, where $m$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. We pull back (or restrict) the symplectic form $\omega$ on $M$ to $U$. The group $H_{\mathbb{C}}$ acts on $D_2$ holomorphically. Assume $\phi|_1$ is the moment map for the $H$-action on $D_2$ with respect to the restricted Kähler form. Then,

$$U \cap \phi^{-1}(0) = G \times_H (D_1 \times \phi^{-1}(0)).$$

Denote

$$Z_{\langle H' \rangle} = \{m \in \phi^{-1}(0) \subset D_2 : \text{the compact stabilizer group of } m \text{ is conjugate to } H' \subset H\},$$

and recall (1) for $Z_{\langle H \rangle}$. We have

$$U \cap Z_{\langle H' \rangle} = G \times_H (D_1 \times Z_{\langle H' \rangle}),$$

in particular, $U \cap Z_{\langle H \rangle} = G \times_H (D_1 \times 0)$, where 0 is in the closure of $Z_{\langle H' \rangle}$. While

$$G_{\mathbb{C}} \cdot (U \cap Z_{\langle H' \rangle}) = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times H_{\mathbb{C}}Z_{\langle H' \rangle}),$$

and

$$G_{\mathbb{C}} \cdot (U \cap Z_{\langle H \rangle}) = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times 0).$$

The quotients are

$$(U \cap Z_{\langle H' \rangle})/G = D_1 \times Z_{\langle H' \rangle}/H,$$

which is the same as

$$G_{\mathbb{C}} \cdot (U \cap Z_{\langle H' \rangle})/G_{\mathbb{C}} = D_1 \times (H_{\mathbb{C}}Z_{\langle H' \rangle})/H_{\mathbb{C}},$$

and

$$(U \cap Z_{\langle H \rangle})/G = D_1 \times 0,$$

which is the same as $G_{\mathbb{C}} \cdot (U \cap Z_{\langle H \rangle})/G_{\mathbb{C}}$.

Now, restricting to the open set $U$, resp., $U//G_{\mathbb{C}}$, the relation between $Z_{\langle H \rangle}$ and $Z_{\langle H' \rangle}$, resp., the relation between $S_{\langle H \rangle}$ and $S_{\langle H' \rangle}$ is clear. In the open set $U$, do the specified restricting, contracting, restricting again, and pushing down of the form $\alpha|_1$ as in the proof of Lemma 1, we see that, if we use local coordinates, and if $\hat{\beta}|_{S_{\langle H' \rangle}} = g(w_1, ..., w_k) dw_1 \wedge \ldots \wedge dw_k$, then $\hat{\beta}|_{S_{\langle H \rangle}} = g(w_1, ..., w_i, 0, ..., 0) dw_1 \wedge \ldots \wedge dw_i$, where $\{i_1, ..., i_j\} \subset \{1, ..., k\}$. \qed
Let us simply use $\hat{\beta}$ to denote this family of forms on $M_0$ we got. It has different dimensions on different dimensional strata.

Let us denote the above push down map by

$$\mathfrak{B}': \mathfrak{B}'(\alpha) = \hat{\beta}.$$

**Remark 1.** Let us use local coordinates on $U = G_C \times_H (D_1 \times D_2)$ based at a point $x$ with stabilizer group $H$ to see the push down map described in the proof of Lemma [1]. Let $z_0$ be the coordinate along the $G_C$-orbit direction, and let $(z_1, z_2) \in D_1 \times D_2$ be the coordinate in the transversal direction. Then, for instance, we may write a $G_C$-invariant $(n,0)$-form $\alpha = f(z_0, z_1, z_2) dz_0 \wedge dz_1 \wedge dz_2$ locally, where $f$ is a $G_C$-invariant function. Restricting $\alpha$ to $G_C \cdot (U \cap Z(H))$, we get $\alpha|_1 = f(z_0, z_1, 0) dz_0 \wedge dz_1$. The contraction gives $\beta|_1 = f(z_0, z_1, 0) dz_1$ (up to a sign), and the restriction of $\beta|_1$ to $U \cap Z(H)$ gives $\beta|_1 = f(x_0, z_1, 0) dz_1$. By the $G$-invariance of $\alpha$, $\beta|_1 = \pi^*(\hat{\beta}|_1)$, where $\hat{\beta}|_1 = f([x], z_1, 0) dz_1$ is a local form on $D_1$ which is a neighborhood of $[x]$ in $S(H)$. Notice that, conversely, if we have such a $(dS, 0)$-form $\hat{\beta}$ on $S(H)$, we can lift it to a $(dS + dG/H, 0)$-form on $G_C \cdot Z(H)$ by using $G_C$-invariance and by “growing back” the coordinate $z_0$.

**Remark 2.** If $G$ acts freely on $\phi^{-1}(0)$, then $\phi^{-1}(0) = Z_1$, where $1 \in G$ is the identity element. By the holomorphic slice theorem, a saturated neighborhood of each point $x \in \phi^{-1}(0)$ is biholomorphic to $U = G_C \times D$. So $U \cap \phi^{-1}(0) = G \times D$, and $D = (U \cap \phi^{-1}(0))/G = U/G \times D$ is biholomorphic to a neighborhood of $[x]$ in $M_0$. In our point of view, we restrict a $G_C$-invariant $(n,0)$-form $\alpha$ to $U$, then we contract the form at the points in $U \cap \phi^{-1}(0)$ with the generating vector fields of the free $G$-action, then we restrict the resulting form to $U \cap \phi^{-1}(0)$ and push it down to $D$ by the quotient map $\pi: U \cap \phi^{-1}(0) \to D$. One may see this in local coordinates as we did in the last remark. In the point of view of Hall and Kirwin, they contract the form $\alpha$ at the points in $U$ with the generating vector fields of the free $G$-action ($G$ acts freely on $U$), and then push down the resulting form to $D$ by the quotient map $\pi_C: U \to U/G \times D$. We see that the results are the same. Their pulling back of an $(n', 0)$-form (let $n' = \dim(D)$) on $D$ to a $G_C$-invariant $(n,0)$-form to $U$ is just by using the $G_C$-action and by “adding” the $G_C$-orbit direction.

We define $\hat{K}$ to be $K||G_C = K/|\phi^{-1}(0)/G|$, and we define $\sqrt{K}$ to be $\sqrt{K}/|G_C = \sqrt{K}/|\phi^{-1}(0)/G|$. Sections of $\hat{K}$ over $M//G$ are “stratified forms” $\hat{\beta}$ whose restriction to each stratum $S$ of complex dimension $d_S$ is a smooth $(dS, 0)$-form. If $O$ is a small open set in $M//G$, a section of $\hat{K}$ over $O$ looks like $\hat{f}_i dw_{i_1} \wedge dw_{i_2} \wedge \ldots \wedge dw_{i_r}$ on the open dense stratum and looks like $\hat{f}_j dw_{i_1} \wedge \ldots \wedge dw_{i_j}$ for some subset $\{i_1, \ldots, i_j\}$ of $\{1, \ldots, r\}$ on other strata, where $\pi_C^*(\hat{f})$ is a $G_C$-invariant function on $\pi_C^*(O)$.

We defined the map $\mathfrak{B}': \Gamma(M, K)^{G_C} \to \Gamma(M//G, \hat{K})$. Using this map, we define a linear map

$$B': \Gamma(M, \sqrt{K})^{G_C} \to \Gamma(M//G, \sqrt{K})$$

such that

$$(B'\mu)^2 = \mathfrak{B}'(\mu^2).$$

Using the map $A'_k$ and the map $B'$, for each $k$, we define a linear map

$$B'_k : \Gamma(M, L^0 \otimes \sqrt{K})^G \to \Gamma(M//G, (L^0 \otimes \sqrt{K}),$$
unique up to an overall sign, such that
\[ B'_k(s \otimes \mu) = A'_k(s) \otimes B'(\mu) \]
where \( s \in \Gamma(L^\otimes k)^G \) and \( \mu \in \Gamma(\sqrt{K})^G \).

Next, we give an argument of the facts
\[ \mathcal{H}(M//G, \tilde{K}) = \mathcal{B}'(\mathcal{H}(M^{ss}, K)^{Gc}), \]
and
\[ \mathcal{H}(M//G, \sqrt{K}) = B'(\mathcal{H}(M^{ss}, \sqrt{K})^{Gc}). \]

A holomorphic section of \( K \) (or of \( \sqrt{K} \)) defined over a \( Gc \)-invariant open set is \( G \)-invariant if and only if it is \( Gc \)-invariant. Let \( K \) (or \( \sqrt{K} \)) be the sheaf of holomorphic sections of \( K \) (or of \( \sqrt{K} \)), and, we define a sheaf \( K' \) (or \( \sqrt{K'} \)) on \( M_0 \), by letting \( K'(O) = \mathcal{B}'(K(\pi^{-1}_C(O))^{Gc}) \) (or by letting \( \sqrt{K'}(O) = B'(\sqrt{K}(\pi^{-1}_C(O))^{Gc}) \)) for each open set \( O \) of \( M_0 \). Using our results above and combining the argument of the proof of Proposition 3, we have the following:

**Proposition 4.** The sheaf \( K' \) (or \( \sqrt{K'} \)) is the sheaf of holomorphic sections of the stratified-line bundle \( \tilde{K} \) (or \( \sqrt{K} \)) over \( M_0 = M^{ss}/G \).

So we have proved the following

**Theorem 7.** Let \( \sqrt{K} = \sqrt{K}/Gc \). There exists a linear map \( B' : \Gamma(M, \sqrt{K})^{Gc} \to \Gamma(M//G, \sqrt{K}), \) unique up to an overall sign, such that for each \( \mu \in \Gamma(M, \sqrt{K})^{Gc}, \) \( B'(\mu) \) is a “stratified form” on \( M//G \) such that for each stratum \( S \) of \( M//G \) with complex dimension \( d_S \), \( (B'(\mu))^{2}|_S \) is a \((d_S, 0)\)-form on \( S \), and these forms are related by suitable degenerating of directions from higher dimensional strata to lower dimensional strata. Moreover, \( \mathcal{H}(M//G, \sqrt{K}) = B'(\mathcal{H}(M^{ss}, \sqrt{K})^{Gc}). \)

Consequently, for each \( k \), there exists a linear map \( B'_k : \Gamma(M, L^\otimes k \otimes \sqrt{K})^{Gc} \to \Gamma(M//G, (L^\otimes k)^G \otimes \sqrt{K}), \) unique up to an overall sign, such that
\[ B'_k(s \otimes \mu) = A'_k(s) \otimes B'(\mu) \]
for \( s \in \Gamma(M, L^\otimes k)^{Gc} \) and \( \mu \in \Gamma(M, \sqrt{K})^{Gc} \), and such that \( \mathcal{H}(M//G, (L^\otimes k)^G \otimes \sqrt{K}) = B'_k(\mathcal{H}(M^{ss}, L^\otimes k \otimes \sqrt{K})^{Gc}). \)

We end this section by giving the definition of a pointwise Hermitian structure on \( \Gamma(M//G, \sqrt{K}). \) Let \( \mu', \nu' \in \Gamma(M//G, \sqrt{K}). \) We define a Hermitian structure on \( \Gamma(M//G, \sqrt{K}) \) stratum-wise as
\[ (\mu')^2 \wedge (\nu')^2|_S = (\mu', \nu')^2|_S \epsilon_\omega|_S, \]
where \( \epsilon_\omega|_S \) is the volume form on the stratum \( S \) of \( M//G \).

9. **MODIFIED LINEAR SPACE ISOMORPHISM**

The following theorem gives the growth of the pointwise norm square of a \( G \)-invariant holomorphic section and a modified \( G \)-invariant holomorphic section along the gradient curves of the moment map components. We need this theorem to prove Theorem 3 and we will need this theorem in the subsequent sections.
Theorem 8. Let \( s \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \) and let \( r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \). Let \( y_0 \in M \), and let \( H \) be its stabilizer group. Let \( h = \text{Lie}(H) \). Let \( m \) be the orthogonal complement of \( h \) in \( g = \text{Lie}(G) \) (assuming we have chosen a \( G \)-invariant metric on \( g \)). Then for \( 0 \neq \xi \in m \), we have

\[
(a) \ |s|^2(e^{i\xi} \cdot y_0) = |s|^2(y_0)e^{\exp\{- \int_0^1 2k\phi_\xi(e^{it\xi} \cdot y_0)dt\}}, \\
(b) \ |r|^2(e^{i\xi} \cdot y_0) = |r|^2(y_0)e^{\exp\{- \int_0^1 (2k\phi_\xi(e^{it\xi} \cdot y_0) + \frac{\partial f_{\xi}}{\partial \xi}(y_0!)(e^{it\xi} \cdot y_0))dt\}}.
\]

If we let \( f(\xi, y_0) = \frac{1}{2} \int_0^1 \phi_\xi(e^{it\xi} \cdot y_0)dt \), then as a function of \( \xi \in m \), \( f(\xi, y_0) \) achieves its unique minimum at \( \xi = 0 \). The Hessian of \( f(\xi, y_0) \) at \( \xi = 0 \) is given by

\[
D_\xi D_\xi f(\xi, y_0)|_{\xi = 0} = 2B_{y_0}(J\xi_1, J\xi_2), \quad \xi_1, \xi_2 \in m.
\]

Proof. See the proof of Theorem 4.1 in \([9]\). Modify the proof by noticing the following: for \( y_0 \in M \), if \( H \) is the stabilizer group of \( y_0 \), and if \( 0 \neq \xi \in h = \text{Lie}(H) \), then \( X\xi(y_0) = 0 \), so \( J\xi\xi(y_0) = 0 \) as well, therefore \( e^{i\xi} \cdot y_0 = y_0 \).

Using (b) of the above theorem, we obtain the following modified linear space isomorphism:

Theorem 9. For \( k \) sufficiently large, the map

\[
B'_k : \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \rightarrow \mathcal{H}(M//G, (L^{\otimes k})_0 \otimes \sqrt{K})
\]

is bijective.

Proof. We use a similar argument as used by Guillemin and Sternberg in \([6]\), by Sjamaar in \([16]\), and by Hall and Kirwin in \([9]\) (the proof of Theorem 3.2 in \([9]\)).

By Theorems \([6]\) and \([7]\), elements in \( \mathcal{H}(M//G, (L^{\otimes k})_0 \otimes \sqrt{K}) \) lift to elements in \( \mathcal{H}(M^{ss}, L^{\otimes k} \otimes \sqrt{K})^G \).

The map is injective because two holomorphic sections which agree on the semistable set, which is open and dense in \( M \), must be equal.

Let \( \hat{r} \in \mathcal{H}(M//G, (L^{\otimes k})_0 \otimes \sqrt{K}) \), and let \( r \in \mathcal{H}(M^{ss}, L^{\otimes k} \otimes \sqrt{K})^G \) be its lift. We only need to show that \( |r|^2 \) remains bounded as we approach the unsemistable set (which is of complex codimension at least one), the Riemann Extension Theorem will imply that \( r \) extends holomorphically to all of \( M \).

By Theorem \([8]\) (b), for \( y_0 \in M^{ss} \) with stabilizer group \( H \), and for \( \xi \in m \), we have

\[
\frac{d}{dt}|r|^2(e^{it\xi} \cdot y_0) = |r|^2(e^{it\xi} \cdot y_0)(-2k\phi_\xi(e^{it\xi} \cdot y_0) - \frac{\partial f_{\xi}}{\partial \xi}(y_0!)(e^{it\xi} \cdot y_0)).
\]

Notice that, for \( \xi \in h = \text{Lie}(H) \), \( |r|^2(e^{it\xi} \cdot y_0) = |r|^2(y_0) \), and so \( \frac{d}{dt}|r|^2(e^{it\xi} \cdot y_0) = 0 \).

By compactness of \( M \) and by compactness of the set \( \{ \xi \in g : |\xi| = 1 \} \), \( \frac{\partial f_{\xi}}{\partial \xi}(y_0!) \) is bounded uniformly for all \( \xi \in g \) with \( |\xi| = 1 \) and at all points in \( M \).

By the monotonicity of \( \phi_\xi(e^{it\xi} \cdot y_0) \) in \( t \) for \( \xi \in m \), and by the above fact about \( \xi \in h \), we see that for all sufficiently large \( k \), \( \frac{d}{dt}|r|^2(e^{it\xi} \cdot y_0) \leq 0 \) for all \( y_0 \in M^{ss} \), all \( \xi \in g \) with \( |\xi| = 1 \), and all \( t \geq 1 \). It follows that the \( r \) obtained extends holomorphically to all of \( M \). 

\[\Box\]

10. THE POINTWISE NORMS OF THE MODIFIED SECTIONS

Theorem 10. Suppose \( r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \). Let \( x_0 \in \phi^{-1}(0) \) be a point with stabilizer group \( H \). So \( [x_0] \in S(H) = Z(H)/G \). Then,
Lemma 1). Case 1. Assume $H$ and $S\xi$ on $L$ where $d$. Proof. prove Theorem 10. Let $\sqrt{\text{Lemma 3}}$. Let $\text{Lemma 4}$. Let $\alpha$ and let $\alpha_0 = |\alpha|$. If $h \neq g$, we expand $\xi_1, \ldots, \xi_h$ to an orthonormal basis of $G$ by joining $\xi_{h+1}, \ldots, \xi_d$. Then, the function $\sqrt{\det_{j,k=h+1,\ldots,d}(B(X^{Ad(g)\xi_j}, X^{Ad(g)\xi_k}))_{g,x}}$ is a constant along the $G$ orbit $G \cdot x$, and $\text{vol}(G \cdot x) = \sqrt{\det_{j,k=h+1,\ldots,d}(B(X^{\xi_j}, X^{\xi_k}))}$. Then $\text{vol}(G \cdot x) = \sqrt{\det_{j,k=h+1,\ldots,d}(B(X^{\xi_j}, X^{\xi_k}))}$. Let $\omega = \omega|G \cdot Z(\xi_1), \ldots, \xi_d \rangle$, and let $(\epsilon_\omega) = \frac{\omega}{(d_{S(\xi_1)} + d_{G/H})!}$. Then $i(\bigwedge_j Z^j) \cdot \bigwedge_k Z^k(\epsilon_\omega)|x_0|Z_{(\xi_1), \ldots, \xi_d}$.

The proof of Lemma 4 uses the result of Lemma 3. Now, we use Lemma 4 to prove Theorem 11.

Proof. Near $x_0$, we can write $r = s\mu$, where $s$ is a local $G$-invariant holomorphic section of $L^{\otimes k}$ and $\mu$ is a local $G$-invariant holomorphic section of $\sqrt{K}$. Let $\alpha = \mu^2$, and let $\alpha_0 = |\alpha|$. Then, by Lemma 3 and by Lemma 2, we have

$$(*): \quad \pi^*(\langle B^j(\mu, B^j(\mu)\dot{\omega}|_{\text{S(\xi_1)}, \ldots, \xi_d}(x_0) \rangle) = \pi^*(\langle B^j(\alpha)|\langle x_0 | \wedge B^j(\dot{\alpha})|\langle x_0 | S_{(\xi_1), \ldots, \xi_d} \rangle$$

Case 1. Assume $H = G$. Then, by construction of the map $\mathfrak{B}$ (see the proof of Lemma 1),

$$(*): \quad (\alpha_0 \wedge \bar{\alpha}_0)(x_0)|_{Z_G} \quad \text{by Lemma 5}\quad = (\mu, \mu)^2(\epsilon_\omega)|\langle x_0 | \wedge (\mu, \mu)^2(\epsilon_\omega)|\langle x_0 | Z_G = (\mu, \mu)^2 \omega_{(d_{S(\xi_1)} + d_{G/H})!}(x_0)|Z_G.$$

Case 2. Assume $H \neq G$. Then, by construction of the map $\mathfrak{B}'$,

$$(*): \quad (\alpha_0 \wedge \bar{\alpha}_0)(x_0)|_{Z_G} \quad \text{by Lemma 5}\quad = (\mu, \mu)^2(\epsilon_\omega)|\langle x_0 | \wedge (\mu, \mu)^2(\epsilon_\omega)|\langle x_0 | Z_G = (\mu, \mu)^2 \omega_{(d_{S(\xi_1)} + d_{G/H})!}(x_0)|Z_G.$$
is connected. Since there is an open dense connected stratum $Z_{\phi}$ in the norms of the sections in the quantum spaces. The main result of this section is Theorem 11.

In this section, we compute

\[ \int_Q J_p f \, d\text{vol}(Q) = \int_R \text{dvol}(R)(y) \int_{p^{-1}(y)} (f|_{p^{-1}(y)}) \, d\text{vol}(p^{-1}(y)), \]

where the Jacobian is $J_p := \sqrt{\text{det}(p_* \circ p_*^{\text{adj}})}$.

For instance, consider $G_C \cdot Z(H)$ or $G_C \cdot S_i$ occurring in Lemma 5 or in Lemma 2 of Section 11.2. Let $m$ be the orthogonal complement of $h = \text{Lie}(H)$ or of $h'$ in $g$. Denote $Z(H)$ or $S_i$ simply as $S$. Let $\Lambda : m \times S \to G_C \cdot S$ be the diffeomorphism $\Lambda(\xi, u) = e^{Ad(g)}\xi \cdot u$, if $u$ has infinitesimal isotropy Lie algebra $Ad(g)h$ or $Ad(g)h'$. The volume element of $G_C \cdot S$ inherited from $M$ at a point $(\xi, g \cdot u)$, where $u$ has isotropy Lie algebra $h$ or $h'$, decomposes as

\[ \Lambda^*(\text{dvol}(G_C \cdot S))(\xi, g \cdot u) = \tau(\xi, u) \text{dvol}(m) \wedge \text{dvol}(S)_{g \cdot u} \]

for some $G$-invariant smooth Jacobian function $\tau$, where $\text{dvol}(m)$ is the Lebesgue measure on $m$.

11.2. Norms of sections in the quantum spaces. In this section, we compute the norms of the sections in the quantum spaces. The main result of this section is Theorem 11.

Let $Z(H)$ be as in 5. Then

\[ M^{ss} = \bigcup_{(H)} F_\infty^{-1}(Z(H)), \]

where $F_\infty$ is the limit map of the flow $F_t$ of the gradient of $-\|\phi\|^2$. By dividing further into connected components for each $Z(H)$, we assume that each $Z(H)$ is connected. Since there is an open dense connected stratum $Z(H)$ (for some $H$) in $\phi^{-1}(0)$ (15), there is an open dense connected set $F_\infty^{-1}(Z(H))$ in $M^{ss}$. We will compute the integral of the pointwise norm square of the sections over each $F_\infty^{-1}(Z(H))$ with $H$ varying. This integral will relate to the integral over $S(H)$ (see 2 for the definition of $S(H)$) of the pointwise norm square of the descended section. In particular, if $Z(H)$ or $S(H)$ is a single point, then the integral on it is regarded.
as the pointwise norm square of the sections over this point.

Now, let us take a look at $F_{∞}^{-1}(Z(H))$. By 2. and 3. of Proposition 1, $G_{C} \cdot Z(H) \subseteq F_{∞}^{-1}(Z(H))$. By the holomorphic slice theorem or by Theorem 4, we have

**Lemma 6.** If $dφ_{x}$ is surjective for all $x \in Z(H)$ (H is necessarily finite), then $F_{∞}^{-1}(Z(H)) = G_{C} \cdot Z(H)$.

Now, we assume that $F_{∞}^{-1}(Z(H))$ is strictly larger than $G_{C} \cdot Z(H)$. Then $F_{∞}^{-1}(Z(H))$ contains complex orbits whose closures contain the complex orbits in $G_{C} \cdot Z(H)$. We decompose the set $F_{∞}^{-1}(Z(H)) - G_{C} \cdot Z(H)$ into a disjoint union of connected $G_{C}$-invariant complex submanifolds each of which has the same infinitesimal compact orbit type, say $(h')$. Let $M_{(h')}^{(H)}$ denote one of these invariant complex submanifold.

**Lemma 7.** Assume that $F_{∞}^{-1}(Z(H))$ is strictly larger than $G_{C} \cdot Z(H)$. We decompose $F_{∞}^{-1}(Z(H)) - G_{C} \cdot Z(H) = \bigcup_{y'} M_{(y')}^{(H)}$, where $\bigcup_{y'} M_{(y')}^{(H)}$ is a disjoint union with each $M_{(y')}^{(H)}$ being a connected $G_{C}$-invariant complex submanifold of a certain infinitesimal compact orbit type $(h')$. Then $0 \notin φ(M_{(y')}^{(H)})$ and $\dim(h') < \dim(H)$. Therefore, for any possible $H$, if $m$ is the orthogonal complement of $h'$ in $g$, then $\dim(m) > 0$.

**Proof.** By 2. of Proposition 1, the complex orbits in $M_{(h')}^{(H)}$ do not intersect $φ^{-1}(0)$. So $0 \notin φ(M_{(h')}^{(H)})$.

By Theorem 1, a neighborhood $U$ of $x \in Z(H)$ in $M$ is $G$-equivariantly biholomorphic to $G_{C} \times_{H_{C}} D$. Split $D = D_{1} \times D_{2}$, where $D_{1}$ is the complex subspace fixed by $H$ and $H_{C}$. Let $φ_{1}$ be the moment map of the $H$ action on $D_{2}$ with respect to the restricted Kähler form. Then $U \cap φ^{-1}(0) = G \times_{H} (D_{1} \times φ^{-1}_{1}(0))$, $U \cap Z(H) = G \times_{H} (D_{1} \times 0)$, and $U \cap G_{C} \cdot Z(H) = G_{C} \times_{H_{C}} (D_{1} \times 0)$. By the assumption, $D_{2} \neq \emptyset$ and $M_{(h')}^{(H)} \cap D_{2} \neq \emptyset$. The set $M_{(h')}^{(H)} \cap D_{2}$ is an $H$-invariant subset of $D_{2}$ consisting of points with isotropy Lie algebra $h' \subset h = Lie(H)$ (A group $H'$ such that $Lie(H') = h'$ is a subgroup of $H$ since any point in $D_{2}$ has isotropy group a subgroup of $H$).

Since $φ(M_{(h')}^{(H)})$ does not intersect 0, $φ(M_{(h')}^{(H)} \cap D_{2})$ does not intersect 0. One only needs to argue when $H$ is not connected and when $\dim(H') = \dim(H)$ and exclude this possibility by using the fact that a finite group action does not contribute to the moment map $φ_{1}$.

By definition of the moment map, for $x \in M$ with isotropy Lie algebra $h'$, the image of $dφ_{x} : T_{x}M → g^{*}$ is the annihilator in $g^{*}$ of $h'$. So the image $φ(M_{(h')}^{(H)})$ intersects with a closed positive Weyl chamber at a certain dimension. This image may lie on one or more than one open faces of the moment polytope $Δ$ of $φ$. These faces form a connected set since we took $M_{(h')}^{(H)}$ to be connected. For a non-abelian Lie group action, the moment polytope is defined to be the intersection of the image of the moment map with a fixed closed positive Weyl chamber. The faces of the moment polytope are caused by symplectic submanifolds with different isotropy groups. One should distinguish the faces of the moment polytope with the faces of the Weyl chamber.
Lemma 8. Assume \( \phi(M^{(H')}_0) \) only lies on one open face \( F_0 \) of \( \Delta \). Then \( \dim(F_0) > 0 \). Let \( 0 \neq a_0 \in \phi(M^{(H')}_0) \subset F_0 \) be a value. Then, we can write \( M^{(H')}_0 = G_C \cdot S_0 \), where \( S_0 \subseteq S_{(b')} = \{ x \in \phi^{-1}(G \cdot a_0) : x \) has isotropy Lie algebra type \((b')\} \) and \( S_0 \) is \( G \)-invariant.

Proof. Since \( F_\infty(M^{(H')}_0) \subset \phi^{-1}(0) \), there are points in \( \phi(M^{(H')}_0) \) arbitrarily near 0. Since \( \phi(M^{(H')}_0) \) is connected, \( \dim(F_0) > 0 \).

Since \( S_0 \subset M^{(H')}_0 \) and since \( M^{(H')}_0 \) is \( G_C \cdot S_0 \subset M^{(H')}_0 \) is \( G \)-invariant, we have \( G_C \cdot S_0 \subset M^{(H')}_0 \).

Conversely, if \( x \in M^{(H')}_0 \), then \( \phi(x) \in G \cdot F_0 \). Without loss of generality, we assume the isotropy Lie algebra of \( x \) is \( \mathfrak{h}' \) and \( \phi(x) = b \in F_0 \). If \( b = a_0 \), then \( x \in S_{(b')} \).

If \( b \neq a_0 \), then \( x \) can be reached by the flow line of \( JX^\xi \) from a point in \( \phi^{-1}(a_0) \), where \( \xi \) is a vector in \( F_0 \) pointing from \( a_0 \) to \( b \). So \( x \in G_C \cdot S_{(b')} \).

Corollary 1. Let \( a \) be any point on the face \( F_0 \), and let \( S \subseteq \{ x \in \phi^{-1}(G \cdot a) : x \) has isotropy Lie algebra type \((b')\} \). Then \( G_C \cdot S_0 = G_C \cdot S = M^{(H')}_0 \).

Proof. We have \( F_0 \subset m \), where \( m \) is the orthogonal complement of \( \mathfrak{h}' \) in \( \mathfrak{g} \) which is identified with the annihilator of \( \mathfrak{h}' \) in \( \mathfrak{g}^* \). The image \( \phi(G_C \cdot S_0) \) must cover the face \( F_0 \). So \( a \in \phi(M^{(H')}_0) \).

Lemma 9. Assume \( \phi(M^{(H')}_0) \) lies on more than one faces of \( \Delta \). Let \( F_1, ..., F_p \) be the ones whose closures contain 0. Let \( 0 \neq a_i \in F_i \), \( i = 1, ..., p \), and let

\[ S_i \subseteq \{ x \in \phi^{-1}(G \cdot a_i) : x \) has isotropy Lie algebra type \((b')\} \].

If \( F_k \) is in the closure of \( F_i \), then \( G_C \cdot S_k \subset G_C \cdot S_i \). Moreover, we can write \( M^{(H')}_0 = \bigcup_{i \in I}(G_C \cdot S_i) \), where \( I \) is the subset of \( \{1, ..., p\} \) such that \( F_i \in I \) are the top dimensional faces among the \( F_i \)'s.

Proof. We have \( F_k \subset F_i \subset m \), where \( m \) is the orthogonal complement of \( \mathfrak{h}' \) in \( \mathfrak{g} \) which is identified with the annihilator of \( \mathfrak{h}' \) in \( \mathfrak{g}^* \). Since the points in \( S_k \) have isotropy Lie algebra \((b')\), the \( G_C \) action (or the \( i(m) \) action) will take the points in \( S_k \) out and emerge them into \( G_C \cdot S_i \). Or, equivalently, the moment map value increases along the flow lines of \( JX^\xi \), where \( \xi \in m \) is orthogonal to \( F_k \). This proves \( G_C \cdot S_k \subset G_C \cdot S_i \).

So, using Lemma 8 \( \bigcup_{i \in I}(G_C \cdot S_i) = \bigcup_{i=1}^p(G_C \cdot S_i) \subset M^{(H')}_0 \). If \( \phi(M^{(H')}_0) \) lies on another face \( F_{p+1} \) whose closure does not contain 0, then \( G_C \cdot S_{p+1} \) (where \( S_{p+1} \) is taken similarly as the \( S_i \)'s) should emerge into \( \bigcup_{i=1}^p G_C \cdot S_i \) to converge to \( \phi^{-1}(0) \).

This proves \( M^{(H')}_0 = \bigcup_{i \in I}(G_C \cdot S_i) \).

By this lemma, if two faces \( F_i \) and \( F_j \) where \( i, j \in I \) contain a one dimensional face \( F_k \) in their common closure, then \( G_C \cdot S_i \cap G_C \cdot S_j = G_C \cdot S_k \).

Remark 3. In the above lemma, generally we cannot get all \( G_C \cdot S_i \) from \( G_C \cdot S_k \) by the flow lines of \( JX^\xi \), where \( \xi \in (m) \). Some orbits in \( G_C \cdot S_i \) may converge to more singular orbits in \( \phi^{-1}(G \cdot F_k) \).

Since 0 is in the closure of each \( F_i \), \( \dim(F_i) > 0 \) for each \( i = 1, ..., p \).

So we have proved...
Lemma 10. We can decompose $F^{-1}_\infty(Z_{(H)})$ into a (finite) disjoint union $F^{-1}_\infty(Z_{(H)}) = G_C \cdot Z_{(H)} \cup_{b'} M^{(H)}_{(b')}$, where $\cup_{b'} M^{(H)}_{(b')} = \emptyset$, or, each $M^{(H)}_{(b')}$ can be written as in Lemma 8 or in Lemma 9. In the case, for any $i = 0, 1, \ldots, p$, we may choose $a'_i \neq 0$ on $F_i$ different from $a_i$ and choose $S'_i \subseteq \{ x \in \phi^{-1}(G \cdot a'_i) : x \text{ has isotropy Lie algebra type } (b') \}$ and we have $G_C \cdot S_i = G_C \cdot S'_i$.

Definition 2. Let $n_{(H)}$ be the complex dimension of $G_C \cdot Z_{(H)}$, and let $n'_{(b')}$ be the complex dimension of $M^{(H)}_{(b')}$. Take $s \in \mathcal{H}(M, L^\otimes k)^G$. Define

$$I^Z_{k} = (k/2\pi)^{n_{(H)}/2} \int_{G_C \cdot Z_{(H)}} |s|^2 dvol(G_C \cdot Z_{(H)}),$$
and,

$$I^Z_{k} = \sum_{b'} (k/2\pi)^{n'_{(b')} / 2} \int_{M^{(H)}_{(b')}} |s|^2 \cdot dvol(M^{(H)}_{(b')}).$$

Define

$$\int_{F^{-1}_\infty(Z_{(H)})} |s|^2 dvol(F^{-1}_\infty(Z_{(H)})) = I^Z_{k} + I^Z_{k}.$$

Lemma 11. Let $s \in \mathcal{H}(M, L^\otimes k)^G$. Then

(a). $I^Z_{k} = (k/2\pi)^{d_{(H)/2}} \int_{S_{(H)}} |A'_k s|^2([x]) \epsilon_{Z_{(H)}}([x])$, where

$I^Z_{k}([x]) = 1$, if $H = G$; and

$I^Z_{k}([x]) = \text{vol}(G \cdot x)(k/2\pi)^{d_{(H)/2}} \int_{m} \tau(\xi, x) \epsilon_{Z_{(H)}}([x])$, if $H \neq G$. Here, $m$ denotes the orthogonal complement of $\mathfrak{h} = \text{Lie}(H)$ in $\mathfrak{g}$, and $x$ is a point with stabilizer group $H$.

(b). $I^Z_{k} = 0,$

or

$I^Z_{k} = \sum_{b'} (k/2\pi)^{n'_{(b')} / 2} \sum_{i} \pm \int_{S_{i}} |s|^2 (g \cdot u) dvol(S_i)$

$$\int_{m'} \tau(\zeta, u) \epsilon_{Z_{(H)}}([x]) dvol(m'),$$

where the second sum is over some subset of indices of $i$ occurring in Lemma 8 or in Lemma 9. $m'$ is the orthogonal complement of $b'$ in $g$, and the points $u \in S_{i}$ are of isotropy Lie algebra $b'$.

Proof. We will drop the subscripts and superscripts in $I^Z_{k}$ and $I^Z_{k}$ and simply write $I$ and $II$. 

(a). If $H = G$, then $G_C \cdot Z_{G} = Z_{G} = S_{G}$. So

$I = (k/2\pi)^{n_{(H)/2}} \int_{G_C \cdot Z_{G}} |s|^2 dvol(G_C \cdot Z_{G}) = (k/2\pi)^{n_{(H)/2}} \int_{Z_{G}} |s|^2 (x) dvol(Z_{G})$.
where $x$ is any point in $Z(H)$ with stabilizer group $gHg^{-1}$ (for some $g$).

By $G$-invariance of the function $\tau$, and by $G$-equivariance of the moment map $\phi$, we have

$$\int_m \tau(\xi, g^{-1}x') \exp\{ -2k \int_0^1 \phi_{Ad(g)}(e^{itAd(g)\xi} \cdot x') \} d\text{vol}(m),$$

where $x = g^{-1}x'$ has stabilizer $H$.

Using the fact that $d\text{vol}(Z(H)) = d\text{vol}(G \cdot x) \wedge \pi^* d\text{vol}(S(H))$, the integral

$$I = (k/2\pi)^{n(H)/2} \int_{S(H)} |A_k s|^2([x]) d\text{vol}(S(H)) d\text{vol}(G \cdot x) (k/2\pi)^{d_G/H}/2 \int_m \tau(\xi, x) \exp\{ -2k \int_0^1 \phi_{Ad(g)}(e^{it\xi} \cdot x) \} d\text{vol}(m)$$

$$= (k/2\pi)^{d_{S(H)}/2} \int_{S(H)} |A_k s|^2([x]) I_k^S([x]) \epsilon_\omega_{S(H)},$$

where

$$I_k^S([x]) = vol(G \cdot x)(k/2\pi)^{d_G/H}/2 \int_m \tau(\xi, x) \exp\{ -2k \int_0^1 \phi_{Ad(g)}(e^{it\xi} \cdot x) \} d\text{vol}(m),$$

with $x$ being taken as a point (on the orbit $G \cdot x$) with stabilizer group exactly $H$.

(b). If $\bigcup_{b'} M_{(b')} = \emptyset$, then $II = 0$.

Otherwise, let us only consider one summand for the first summation in $II$. The others follow similarly. So, we assume

$$II = (k/2\pi)^{n(H)/2} \int_{M_{(b')}^{(H)}} |s|^2 d\text{vol}(M_{(b')}^{(H)}).$$

By Lemma\(\ref{lem9}\) or Lemma\(\ref{lem10}\) we can compute this integral over one set $G_C \cdot S_0$, or we can compute it over a finite union $G_C \cdot S_{i \in I}$ and possibly subtract some integrals over some mutual intersections which have similar forms (if a mutual intersection has less dimension, then we do not subtract). So we only need to write one such integral in the stated form.

Using the coarea formula, the formula \(\ref{eq9}\) and Theorem\(\ref{thm8}\)(a) on the space $G_C \cdot S$, we have

$$\int_{G_C \cdot S_i} |s|^2 d\text{vol}(G_C \cdot S_i) = \int_{S_i} |s|^2(u') d\text{vol}(S_i)$$

$$\int_m \tau(\xi, g^{-1}u') \exp\{ -2k \int_0^1 \phi_{Ad(g)}(e^{itAd(g)\xi} \cdot u') \} d\text{vol}(m'),$$
Lemma 9, where the second sum is over some subset of indices of \(i\) is the isotropy Lie algebra \(h\).

For the same reason as in (a), we have

\[
\int_{m'} \tau(\zeta, g^{-1} u') \exp\{-2k \int_0^1 \phi_{Ad(g)}(e^{it\text{Ad}(g)\zeta} \cdot u') \} dvol(m') = \int_{m'} \tau(\zeta, u) \exp\{-2k \int_0^1 \phi_{\zeta}(e^{it\zeta} \cdot u) \} dvol(m'),
\]

where \(u = g^{-1} u'\) has isotropy Lie algebra \(h'\).

\[\square\]

**Definition 3.** We use the same notations as those in Definition 2. Take \(r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G\). Define

\[
\tilde{I}^Z_k(H) = (k/2\pi)^{n(H)/2} \int_{G \cdot Z(H)} |r|^2 dvol(G \cdot Z(H)),
\]

and,

\[
\tilde{\Pi}^Z_k(H) = \sum_{h'} (k/2\pi)^{n(H')/2} \int_{M(H') \setminus M(H')} |r|^2 dvol(M(H')).
\]

Define

\[
\tilde{I}^Z_k(Z(H)) = \int_{F^{-1}(Z(H))} |r|^2 dvol(F^{-1}(Z(H)))
\]

\[= \tilde{I}^Z_k(H) + \tilde{\Pi}^Z_k(H).
\]

**Lemma 12.** Let \(r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G\). Then

(a). \(\tilde{I}^Z_k(H) = (k/2\pi)^{d_{S(H)}} J_k^{S(H)} (|x|) J_k^{S(H)} (|x|) \in \omega_{S(H)}\), where

\[
J_k^{S(H)} (|x|) = 1, \text{ if } H = G; \text{ and } J_k^{S(H)} (|x|) = (k/2\pi)^{d_{G/H}} 2^{d_{G/H}}.
\]

\[
\int_m \tau(\xi, x) \exp\{-2k \int_0^1 \phi_{\zeta}(e^{it\zeta} \cdot x) + \frac{\mathcal{L}_{\xi} \epsilon \epsilon \omega}{2\epsilon} (e^{it\zeta} \cdot x) \} dvol(m),
\]

if \(H \neq G\). Here, \(m\) denotes the orthogonal complement of \(h = \text{Lie}(H)\) in \(g\), and \(x\) is a point with stabilizer group \(H\).

(b).

\[
\tilde{\Pi}^Z_k(H) = 0,
\]

or

\[
\tilde{\Pi}^Z_k(H) = \sum_{h'} (k/2\pi)^{n(H')/2} \left( \sum_i \int_{S_i} |r|^2 (g \cdot u) dvol(S_i) \right)
\]

\[
\int_{m'} \tau(\zeta, u) \exp\{-2k \int_0^1 \phi_{\zeta}(e^{it\zeta} \cdot u) + \frac{\mathcal{L}_{x} \epsilon \epsilon \omega}{2\epsilon} (e^{it\zeta} \cdot u) \} dvol(m'),
\]

where the second sum is over some subset of indices of \(i\) occurring in Lemma 3 or Lemma 4. \(m'\) is the orthogonal complement of \(h'\) in \(g\), and the points \(u \in S_i\) are of isotropy Lie algebra \(h'\).
Proof. (a). The proof is similar to the proof of (a) of Lemma 11, but we will use Theorem 10. We will drop the subscript and superscript in $\tilde{I}_k^{Z(H)}$ and simply write $\tilde{I}$.

If $H = G$, then $G_C \cdot Z_G = Z_G = S_G$. Then

$$\tilde{I} = (k/2\pi)^{n_G/2} \int_{G \cdot Z_G} |r|^2 d\text{vol}(G_C \cdot Z_G)$$

$$= (k/2\pi)^{n_G/2} \int_{Z_G} |r|^2(x) d\text{vol}(Z_G) = (k/2\pi)^{d_{G/H}/2} \int_{S_G} |B'_k r|^2([x]) \epsilon_{\omega_S G}$$

by Theorem 10.

If $H \neq G$, by the coarea formula, the formula (9), Theorem 8 (b), and by a $G$-invariance argument as in the proof of Lemma 11, we have

$$\tilde{I} = (k/2\pi)^{n_{(H)}}/2 \int_{Z(H)} |r|^2(g \cdot x) d\text{vol}(Z(H))$$

$$\int_m \tau(\xi, x) \exp\left\{ - \int_{\gamma_{\xi}} (2k\phi_\xi + \frac{L_{JX} \epsilon_{\omega}}{2\epsilon_{\omega}}) \right\} d\text{vol}(m).$$

By Theorem 10

$$\tilde{I} = (k/2\pi)^{d_{S(H)}}/2 \int_{S(H)} |B'_k r|_2^2([x]) d\text{vol}(S(H)) 2^{d_{G/H}/2}(k/2\pi)^{d_{G/H}/2}$$

$$\int_m \tau(\xi, x) \exp\left\{ - \int_{\gamma_{\xi}} (2k\phi_\xi + \frac{L_{JX} \epsilon_{\omega}}{2\epsilon_{\omega}}) \right\} d\text{vol}(m)$$

$$= (k/2\pi)^{d_{S(H)}}/2 \int_{S(H)} |B'_k r|_2^2([x]) J_{k}^{S(H)}([x]) \epsilon_{\omega_{S(H)}},$$

where

$$J_{k}^{S(H)}([x]) = (k/2\pi)^{d_{G/H}}/2 2^{d_{G/H}/2} \int_m \tau(\xi, x) \exp\left\{ - \int_{\gamma_{\xi}} (2k\phi_\xi + \frac{L_{JX} \epsilon_{\omega}}{2\epsilon_{\omega}}) \right\} d\text{vol}(m).$$

(b). Similar to the proof of (b) of Lemma 11. We omit it. \hfill \Box

Now, we come to our main result of this section:

**Theorem 11.** (a). Let $s \in \mathcal{H}(M, L^\otimes k)$. Then

$$\sum_{Z(H)} \int_{F^{-1}_{\omega}(Z(H))} |s|^2 d\text{vol}(F_{\omega}^{-1}(Z(H)))$$

$$= \sum_{S(H)} (k/2\pi)^{d_{S(H)}}/2 \int_{S(H)} |A'_k s|_2^2([x]) J_{k}^{S(H)}([x]) \epsilon_{\omega_{S(H)}} + \sum_{Z(H)} I_{k}^{Z(H)},$$

where $J_{k}^{S(H)}([x])$ is as in Lemma 11 (a), and each $I_{k}^{Z(H)}$ is as in Lemma 11 (b).

In particular, the above is true for each individual summand with respect to $(H)$.
(b). Let \( r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \). Then

\[
\sum_{Z(H)} \int_{F_{\infty}^{-1}(Z(H))} |r|^2 \text{dvol}(F_{\infty}^{-1}(Z(H)))
\]

\[
= \sum_{\mathcal{S}(H)} (k/2\pi)^{d_{\mathcal{S}(H)}/2} \int_{\mathcal{S}(H)} |B_k|^2([x])J^H_k([x])\epsilon_{\mathcal{S}(H)} + \sum_{\mathcal{Z}(H)} \hat{\Pi}_k^H,
\]

where \( J^H_k([x]) \) is as in Lemma \ref{lemma:J} (a), and each \( \hat{\Pi}_k^H \) is as in Lemma \ref{lemma:Pi} (b).

In particular, the above is true for each individual summand with respect to \((H)\).

Proof. Lemmas \ref{lemma:I} and \ref{lemma:J} proved the statements for the individual summands. The statements for the sums follow from these lemmas by taking the sum of the individual terms. \( \square \)

The asymptotic properties of \( I^H_k([x]) \), of \( J^H_k([x]) \), of \( \Pi_k^H \) and of \( \hat{\Pi}_k^H \) will be studied in the next section (see Theorem \ref{thm:asymptotics}).

12. ASYMMETRICITIES

Our main result of this section is

**Theorem 12.** (a). The densities \( I^H_k \) and \( J^H_k \) for \( H \neq G \) satisfy

\[
\lim_{k \to \infty} I^H_k([x]) = 2^{-d_G/2} \text{vol}(G \cdot x),
\]

and

\[
\lim_{k \to \infty} J^H_k([x]) = 1.
\]

The limits are uniform for \( [x] \in Z(H)/G \).

(b). If \( \Pi_k^H \neq 0 \) and \( \hat{\Pi}_k^H \neq 0 \), then they satisfy

\[
\lim_{k \to \infty} \Pi_k^H = 0,
\]

and

\[
\lim_{k \to \infty} \hat{\Pi}_k^H = 0.
\]

The proof of this theorem will be in Section 12.3.

12.1. Growth estimates. In Lemmas \ref{lemma:I} and \ref{lemma:J} in the expressions of \( I^H_k \), or of \( J^H_k \) (for \( H \neq G \)), or in the summands of \( \Pi_k^H \) or \( \hat{\Pi}_k^H \), we had the following types of integrals

\[
\int_m \tau(\xi, x) \exp\{-2k \int_{\gamma_\xi} \phi_\xi\} \text{dvol}(m)
\]

and

\[
\int_m \tau(\xi, x) \exp\{- \int_{\gamma_\xi} (2k\phi_\xi + \frac{L_{\lambda(x)} e_\omega}{2e_\omega})\} \text{dvol}(m),
\]

where \( x \in S \) with \( S = Z(H) \) or \( S = S_i \) for some \( S_i \) as in Lemma \ref{lemma:S} or in Lemma \ref{lemma:Si} \( \xi \in m, \) and \( \gamma_\xi = e^{it\xi} \times x, \) \( t \in [0, 1] \).
Remark 4. In this and the next subsections, for simplicity, we will only use $m$ to denote the orthogonal complement of $\mathfrak{h}$ or of $\mathfrak{h}'$ in $\mathfrak{g}$ as we did in Formula (9).

Theorem 13. Consider $G\subset S$, where $S = Z(H)$ or $S = S_i$ for an $S_i$ as in Lemma 8 or in Lemma 4. There exist constants $b$, and $D > 0$ such that for all $[x] \in S/G$ (the integral is a function of $[x]$), and for all $R$ and $k$ sufficiently large,

$$\int_{m-B_R(0)} \tau(\xi,x) \exp\{-2k \int_{\gamma_\xi} \phi_\xi d\omega(m)\} \leq be^{-RDk},$$

where $m$ is the orthogonal complement of $\mathfrak{h}$ or of $\mathfrak{h}'$ in $\mathfrak{g}$, and $B_R(0)$ is a ball in $m$ of radius $R$ centered at 0.

Since we can find a uniform bound for $-\frac{\mathcal{L}_{\xi}^0}{2\omega}$ on $M$, the above inequality is also true for the integral $\int_{m-B_R(0)} \tau(\xi,x) \exp\{-\int_{\gamma_\xi} (2k\phi_\xi + \frac{\mathcal{L}_{\xi}^0}{2\omega})\} d\omega(m)$.

The proof of this theorem relies on the following two lemmas.

Lemma 13. Consider $G\subset S$, where $S = Z(H)$ or $S = S_i$ for an $S_i$ as in Lemma 8 or in Lemma 8. For any $t_0 > 0$, there exists $C > 0$ such that for all $t > t_0$,

$$\exp\{-\int_{\gamma_\xi} 2k\phi_\xi\} \leq e^{-2ktC}$$

uniformly on $S/G$, where $\hat{\xi} \in m$ with $|\hat{\xi}| = 1$.

Proof. By definition, $\int_{\gamma_\xi} \phi_\xi = \int_0^1 < \phi(e^{it\hat{\xi}} \cdot x), t\hat{\xi} > d\tau = t \int_0^1 \phi_\xi(e^{it\hat{\xi}} \cdot x) d\tau$ when $t$ is sufficiently large. We prove the lemma for the case $S = Z(H)$. The argument applies to other cases. Since $f_t$ is $G$-invariant, on each $G$-orbit, we only need to consider a particular point $x$ which has isotropy Lie algebra exactly $\mathfrak{h}$. So we take $S^\mathfrak{h} \subset S$ to be the set of such points. First, fix $\hat{\xi} \in m$ with $|\hat{\xi}| = 1$, and consider the $\hat{\xi}$-moment map $\phi_{\hat{\xi}}$. Then, $\phi_{\hat{\xi}}(S^\mathfrak{h}) = \text{constant}$. For $x \in S^\mathfrak{h}$, $f_t(\hat{\xi},x)$ for any $t > 0$ is strictly increasing since $e^{it\hat{\xi}} \cdot x$ is the gradient line of $\phi_{\hat{\xi}}$ and $JX\hat{\xi}(x) \neq 0$. If $S^\mathfrak{h}$ is compact, we can find a positive lower bound $C_{\hat{\xi}}$ for $f_t(\hat{\xi},x)$ for all points in $S^\mathfrak{h}$ and for all $t > t_0$ for any chosen $t_0 > 0$. If $S^\mathfrak{h}$ is not compact, we do the following. Consider a nearby regular value $a > 0$ of $\phi_{\hat{\xi}}$. For $y \in \phi_{\hat{\xi}}^{-1}(a)$, consider the function $f'_t(\hat{\xi},y) = \int_{-\epsilon}^{1-\epsilon} \phi_{\hat{\xi}}(e^{it\hat{\xi}} \cdot y) d\tau$, where $\epsilon$ is a small number. By choosing $\epsilon$ properly, for each $x$ in $S^\mathfrak{h}$, there exists $y \in \phi_{\hat{\xi}}^{-1}(a)$, such that $f'_t(\hat{\xi},y) = f_t(\hat{\xi},x)$ (since the $x$'s are not fixed by the circle action generated by $\hat{\xi}$, this can be achieved). We choose the positive minimum of $f'_t$ on its compact domain $\phi_{\hat{\xi}}^{-1}(a)$ as $C_{\hat{\xi}}$ (the positivity of $C_{\hat{\xi}}$ is due to $a$ is a regular value). So, for each $\xi \in m$ with $|\xi| = 1$, there exists $C_{\xi} > 0$, such that $f_t(\hat{\xi},x) = \int_0^1 \phi_{\hat{\xi}}(e^{it\hat{\xi}} \cdot x) d\tau \geq C_{\hat{\xi}}$ for all $[x] \in S/G$. By the compactness of the set $\{\xi \in m, |\xi| = 1\}$, and by continuous dependence of $f_t$ on $\xi$, we can find a positive constant $C$ such that $f_t \geq C$ uniformly for all $[x] \in S/G$ and for all $\xi \in m$ with $|\xi| = 1$. 
For the proof of other $S'$s, we replace the above $S^b$ by $S^b \cap \phi^{-1}(a_i)$ (recall that $0 \neq a_i \in F_i$) so that $\phi_{\xi}(S^b \cap \phi^{-1}(a_i)) = \text{constant}$, noticing the fact that $S^b \cap \phi^{-1}(a_i)$ has all the representatives of $S/G$.

\begin{lemma}
\label{lem:14}
Consider $G_C \cdot S$, where $S = Z_{(H)}$ or $S = S_i$ for an $S_i$ as in Lemma \ref{lem:8} or in Lemma \ref{lem:2}. There exist constants $a$ and $b > 0$ such that for all $t > 0$
\[ \tau(t, x) \leq bt^{-m}e^{at} \]
uniformly on $S/G$, where $\xi \in \mathfrak{m}$ with $|\xi| = 1$, and $m$ is the dimension of $\mathfrak{m}$.
\end{lemma}

\begin{proof}
The manifold $G_C \cdot S$ is a complex submanifold of $M$. Since $M$ can be embedded into projective spaces, $G_C \cdot S$ is a complex submanifold of projective spaces. The proof of Lemma 5.7 in \cite{9} applies. (The proof of Lemma 5.7 in \cite{9} does not need the domain of $\hat{\tau}$ to be compact, but it uses the fact that the domain of $\hat{\tau}$ is compact.)
\end{proof}

Once we have the above two lemmas, using polar coordinates, we can prove Theorem \ref{thm:13}. One may refer to the proof of Theorem 5.5 in \cite{9}.

12.2. Approximation.

\begin{lemma}
\label{lem:15}
The function $\tau(x, e^t \cdot x)$ equals $\text{vol}(G \cdot x)$ on $S$, where $S = Z_{(H)}$ or $S = S_i$ for an $S_i$ as in Lemma \ref{lem:8} or in Lemma \ref{lem:2}.
\end{lemma}

\begin{proof}
We prove the lemma for the case $S = S_i$ for some $i$. The proof for the other $S'$s is similar. Consider the complex submanifold $G_C \cdot S$. We take $S^b \subset S$, the set of points with isotropy Lie algebra exactly $\mathfrak{h}'$. Let $\hat{S} = S^b \cap \phi^{-1}(a_i)$. Then $\hat{S}$ contains all the representatives of $S/G$. Since $\tau(x, e^t \cdot x)$ is $G$-invariant, we only need to consider the value $\tau(0, x)$ with $x \in \hat{S}$. So we only consider Formula \ref{eq:9} on $e^t \cdot \hat{S}$. Consider the submanifold $e^t \cdot \hat{S}$. At each point $x \in \hat{S}$, the $B$-orthogonal complement of $T_x \hat{S}$ in $T_x(e^t \cdot \hat{S})$ is exactly the linear span of the vectors $JX^\xi$ with $\xi \in \mathfrak{m}$: for any $JX^\xi$ with $\xi \in \mathfrak{m}$ and any vector $v \in T_x \hat{S}$, we have $B(JX^\xi, v)_x = \omega(v, X^\xi)_x = v(\phi_{\mathfrak{h}})_x = 0$ since $\phi_{\mathfrak{h}}$ takes constant value on $\hat{S}$. So $B$ is block diagonalizable at $x$ on the submanifold $e^t \cdot \hat{S}$, and
\[ \text{dvol}(e^t \cdot \hat{S})_x = \sqrt{\det B(JX^\xi, JX^\xi)}_x \text{dvol}(\mathfrak{m}) \wedge \text{dvol}(\hat{S})_x \]
with $\xi_i, \xi_j \in \mathfrak{m}$. By Lemma \ref{lem:8},
\[ \sqrt{\det B(JX^\xi, JX^\xi)}_x = \text{vol}(G \cdot x). \]
\end{proof}

The result of the above lemma will be used in the proof of the following lemma.

\begin{lemma}
\label{lem:16}
Consider $G_C \cdot S$, where $S = Z_{(H)}$ or $S = S_i$ for an $S_i$ as in Lemma \ref{lem:8} or in Lemma \ref{lem:2}. Define
\[ I_k R(x) = (k/2\pi)^{m/2} \int_{B_R(0)} \tau(x, e^{-kf(\xi, x)} \text{dvol}(\mathfrak{m}), \]
where $f(\xi, x) = 2 \int_0^1 \phi_{\mathfrak{h}}(e^{it\xi}, x) dt$ at a point $x \in S$ with isotropy Lie algebra $\mathfrak{h}$ or $\mathfrak{h}'$, $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ or of $\mathfrak{h}'$ in $\mathfrak{g}$ and $m = \text{dim}(\mathfrak{m})$.

Then there exists some $R > 0$ such that
\[ \lim_{k \to \infty} |I_k R(x)| - 2^{-m/2} = 0 \]
uniformly on $S/G$.
\end{lemma}
Proof. We will prove the lemma for the case $S = Z(H)$. The other cases follow similarly. We refer to the proof of Lemma 5.10 in [9]. By Theorem 8, the function $f(\xi, x)$ is a $G$-invariant Morse-Bott function on $G_C \cdot Z(H)$ with $0 \times Z(H)$ being a minimum. By the Morse-Bott lemma, for each point $x \in Z(H)$, there exists a neighborhood of this point on which $f(\xi, x)$ can be written as a quadratic function. If $Z(H)$ is compact, we can choose the smallest positive radius of the (finitely many) neighborhoods as $R$. If $Z(H)$ is not compact, note that if $Z(K)$ is in the closure of $Z(H)$, then (up to conjugacy) the orthogonal complement $m'$ of $\text{Lie}(K)$ is a linear subspace of the orthogonal complement $m$ of $\text{Lie}(H)$. Because of this property, for the function $f(\xi, x)$ on $G_C \cdot Z(K)$, we may assume that the neighborhoods of the points $x \in Z(K)$ overlap the strata $Z(H)$'s whose closures contain $Z(K)$. So, we can use the compactness of $\phi^{-1}(0)$ to have finitely many neighborhoods, and therefore to choose the smallest $R$ for all the strata $Z(H) \subset \phi^{-1}(0)$. Once $R$ is chosen, on each $G_C \cdot Z(H)$, follow the arguments of the proof of Lemma 5.10 in [9]. In the proof of Lemma 5.10 in [9], there are some estimates on the bounds of the absolute value of some continuous functions of $x \in Z(H)$ which involve certain integrals of the derivative of $\tau(\xi, x)$ in the direction of $x$ (the constants $Q_1$ and $Q_2$). If $Z(H)$ is not compact, the formula on $\tau$ in (9) of Section 11.1 should continuously transform from higher dimensional strata $Z(H)$ of $\phi^{-1}(0)$ to lower dimensional ones. This should allow us to extend continuously the above continuous functions to the closure of $Z(H)$ in $\phi^{-1}(0)$ and take the maximal of the absolute values. (The constant $Q_3$ in the proof of Lemma 5.10 in [9], is $2^m/2$ in our case.) □

Lemma 17. Consider $G_C \cdot S$, where $S = Z(H)$ or $S = S_i$ for an $S_i$ as in Lemma 5.10 or in Lemma 8. Define

$$J_{k, R}([x]) = (k/2\pi)^{m/2}2^{m/2}\int_{B_R(0)} \tau(\xi, x)e^{-kf(\xi, x)}e^{\int_{\gamma_{\zeta}} \frac{\xi J X_{\xi}}{2\epsilon} d\omega} d\text{vol}(m).$$

Then, there exists $R > 0$ such that

$$\lim_{k \to \infty} |J_{k, R}([x]) - 1| = 0$$

uniformly on $S/G$.

Proof. In the proof of Lemma 10 replace $\tau(\xi, x)$ by $\tau(\xi, x)e^{\int_{\gamma_{\zeta}} \frac{\xi J X_{\xi}}{2\epsilon} d\omega}$, just to notice that the exponent is 0 when $\xi = 0$. □

12.3. Proof of Theorem 12

Proof. (a). We write $I_k^{S_1(H)}$ as the sum of an integral over $B_R(0)$ and an integral over the complement of $B_R(0)$. The result follows from Lemma 10 and Theorem 13.

The proof for $J_k^{S_1(H)}$ is similar but using Lemma 17 and Theorem 13.

(b). We assume that $II_k^{Z(H)} \neq 0$ and $II_k^{Z(H)} \neq 0$.

Now we prove $\lim_{k \to \infty} II_k^Z = 0$. Since we have a finite summation in the expression of $II_k^Z$, we only need to prove that each summand goes to 0 when $k \to \infty$. We will simply write $G_C \cdot S_i$ as $G_C \cdot S$. By Theorem 13 and Lemma 10 there exists $K_0 > 0$, such that when $k > K_0$,

$$\int_{m'} \tau(\xi, u)e^{\int_{\gamma_{\zeta}} \phi_\xi} d\text{vol}(m') \leq be^{-RDk} + 2(k/2\pi)^{-m'/2}2^{-m'/2}$$
Therefore \( \lim S \) with \( S \) a flow lines of the vector fields \( JX^\xi \), where \( \xi \in (m') \). We use Theorem 8 (a) to express \( |s|^2(u) \) in terms of \( |s'|^2(u') \) and we use the arguments in the proof of Lemma 13 to find a constant \( C' > 0 \) such that \( |s|^2(u) \leq |s'(u')e^{-kC'} | \) for all \( u' \in S' \). Now, since \( M \) is compact, \( |s|^2(u) \) is bounded. The volume of \( S \) is also bounded. So \( \int_S |s|^2(u) d\text{vol}(S) \leq C''e^{-kC'} \) for some constant \( C'' \).

So, for each summand in \( II_k^{H(Z)} \), there exist \( K_0 > 0 \) and constants \( C, C', b, b', R, D \) with \( C', R \) and \( D \) positive such that when \( k > K_0 \), the summand

\[
\leq (k/2\pi)^{n(H)/2}C e^{-kC'}(be^{-RDk} + (k/2\pi)^{-m'/2}b').
\]

Therefore \( \lim_{k\to\infty} II_k^{H(Z)} = 0 \).

The proof for the statement about \( II_k^{\tilde{H}(Z)} \) is similar. \( \square \)

13. ASYMPTOTIC UNITARITY

Now, it comes to the definition of the inner products on \( \mathcal{H}(M, L^0k)^G \) and on \( \mathcal{H}(M, L^0k \otimes \sqrt{K})^G \). Recall that \( M^{ss} \) is open and dense in \( M \), and

\[
M^{ss} = \bigcup_{(H)} F_{\infty}^{-1}(Z(H)).
\]

There is an open and dense set \( F_{\infty}^{-1}(Z(H)) \) for some \( H \) in \( M^{ss} \), and, correspondingly, there is an open and dense stratum \( S(H) \) in \( M//G \). Let us denote the open dense piece \( F_{\infty}^{-1}(Z(H)) \) as \( F_{\infty}^{-1}(Z^0) \), and denote the corresponding open and dense stratum \( S(H) \) of \( M//G \) as \( S^0 \).

**Definition 4.** Let \( s_1, s_2 \in \mathcal{H}(M, L^0k)^G \) and let \( r_1, r_2 \in \mathcal{H}(M, L^0k \otimes \sqrt{K})^G \). We define

\[
<s_1, s_2>(1) = \int_M (s_1, s_2) d\text{vol}(M) = \int_{F_{\infty}^{-1}(Z^0)} (s_1, s_2) d\text{vol}(F_{\infty}^{-1}(Z^0)),
\]

and we define

\[
<r_1, r_2>(1) = \int_M (r_1, r_2) d\text{vol}(M) = \int_{F_{\infty}^{-1}(Z^0)} (r_1, r_2) d\text{vol}(F_{\infty}^{-1}(Z^0)).
\]

By Theorems 11 and 12 we have

**Corollary 2.** Let \( s \in \mathcal{H}(M, L^0k)^G \), and let \( r \in \mathcal{H}(M, L^0k \otimes \sqrt{K})^G \). Then,

\[
|s|^2_{(1)} = \int_M |s|^2 d\text{vol}(M) = (k/2\pi)^{d_{S^0}/2} \int_{S^0} |A_k s|^2([x]) I_k^{S^0}([x]) e_{S^0} + II_k^{Z^0},
\]

where, \( I_k^{S^0}([x]) = 1 \) or \( \lim_{k\to\infty} I_k^{S^0}([x]) = 2^{-d_{S^0}/2} \text{vol}(G \cdot x) \) uniformly for \( x \in S^0 \) for some \( H \neq G \), and, \( II_k^{Z^0} = 0 \) or \( \lim_{k\to\infty} II_k^{Z^0} = 0 \);

\[
|r|^2_{(1)} = \int_M |r|^2 d\text{vol}(M) = (k/2\pi)^{d_{S^0}/2} \int_{S^0} |B_k r|^2([x]) J_k^{S^0}([x]) e_{S^0} + II_k^{Z^0},
\]
where, \( J_k^{S^G}(\{x\}) = 1 \) or \( \lim_{k \to \infty} J_k^{S^G}(\{x\}) = 1 \) uniformly for \( \{x\} \in S^G \), and, \( \tilde{I}_k^{Z^G} = 0 \) or \( \lim_{k \to \infty} \tilde{I}_k^{Z^G} = 0 \).

The following definition modifies the usual definition of quantum norms, but it takes into account all the strata. Physical interpretations of this definition would be desirable.

**Definition 5.** Let \( s_1, s_2 \in \mathcal{H}(M, L^{\otimes k})^G \) and let \( r_1, r_2 \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \). We define

\[
\langle s_1, s_2 \rangle_{(2)} = \int_M (s_1, s_2) \, d\text{vol}(M) = \sum_{Z(H)} \int_{Z^{-1}(Z(H))} (s_1, s_2) \, d\text{vol}(F^{-1}_{\infty}(Z(H))),
\]

and we define

\[
\langle r_1, r_2 \rangle_{(2)} = \int_M (r_1, r_2) \, d\text{vol}(M) = \sum_{Z(H)} \int_{Z^{-1}(Z(H))} (r_1, r_2) \, d\text{vol}(F^{-1}_{\infty}(Z(H))).
\]

Again, by Theorems 11 and 12 we have

**Corollary 3.** Let \( s \in \mathcal{H}(M, L^{\otimes k})^G \), and let \( r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \). Then,

\[
||s||_{(2)}^2 = \int_M |s|^2 \, d\text{vol}(M)
\]

\[
= \sum_{S(H)} (k/2\pi)^{d_{S(H)}/2} \int_{S(H)} |A_k^s|^2([x]) I_k^{S(H)}([x]) d\omega_{S(H)} + \sum_{Z(H)} \tilde{I}_k^{Z(H)},
\]

where, \( I_k^{S(H)} = 1 \) or \( \lim_{k \to \infty} I_k^{S(H)}([x]) = 2^{-d_{G/H}/2} \text{vol}(G \cdot x) \) uniformly for \( [x] \in S(H) \) with \( H \neq G \), and, \( \tilde{I}_k^{Z(H)} = 0 \) or \( \lim_{k \to \infty} \tilde{I}_k^{Z(H)} = 0 \);

\[
||r||_{(2)}^2 = \int_M |r|^2 \, d\text{vol}(M)
\]

\[
= \sum_{S(H)} (k/2\pi)^{d_{S(H)}/2} \int_{S(H)} |B_k^r|^2([x]) J_k^{S(H)}([x]) d\omega_{S(H)} + \sum_{Z(H)} \tilde{I}_k^{Z(H)},
\]

where, \( J_k^{S(H)} = 1 \) or \( \lim_{k \to \infty} J_k^{S(H)}([x]) = 1 \) uniformly for \( [x] \in S(H) \), and, \( \tilde{I}_k^{Z(H)} = 0 \) or \( \lim_{k \to \infty} \tilde{I}_k^{Z(H)} = 0 \).

For both Definitions 4 and 5 we have the following asymptotic unitarity for the maps \( B_k^l \).

**Theorem 14.** The maps \( B_k^l \) are asymptotically unitary, in the sense that

\[
\lim_{k \to \infty} ||B_k^l B_k^* - I|| = \lim_{k \to \infty} ||B_k^l B_k^* - I|| = 0,
\]

where \( ||\cdot|| \) refers to the operator norm.

**Proof.** We use Theorem 9 the definitions in (3) and in (11) of Section 5, and we use the above results in Corollaries 2 and 3 of Theorems 11 and 12. For the case of the quantum norms in Definition 5 we also use the fact that there are finitely many strata. One may refer to [9], the proof of Theorem 5.2 for the asymptotic unitarity of the maps \( B_k^l \).
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