Strong Comparison Principles for Some Nonlinear Degenerate Elliptic Equations

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Dedicated to the memory of Xiaqi Ding

Abstract

In this paper, we obtain the strong comparison principle and Hopf Lemma for locally Lipschitz viscosity solutions to a class of nonlinear degenerate elliptic operators of the form $\nabla^2 \psi + L(x, \nabla \psi)$, including the conformal hessian operator.

Key words: Hopf Lemma; Strong Comparison Principle; Degenerate Ellipticity; Conformal invariance.

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1 Introduction

In this paper, we establish the strong comparison principle and Hopf Lemma for locally Lipschitz viscosity solutions to a class of nonlinear degenerate elliptic operators.

For a positive integer $n \geq 2$, let $\Omega$ be an open connected bounded subset of $\mathbb{R}^n$, the $n$-dimensional euclidean space. For any $C^2$ function $u$ in $\Omega$, we consider a symmetric matrix function

$$F[u] := \nabla^2 u + L(\cdot, \nabla u),$$

where $L \in C_{loc}^{0,1}(\Omega \times \mathbb{R}^n)$, is in $\mathcal{S}^{n \times n}$, the set of all $n \times n$ real symmetric matrices.

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One such matrix operator is the conformal hessian operator (see e.g. [21], [27] and the references therein), that is,

\[ A[u] = \nabla^2 u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 I, \]

where \( I \) denotes the \( n \times n \) identity matrix, and for \( p, q \in \mathbb{R}^n \), \( p \otimes q \) denotes the \( n \times n \) matrix with entries \( (p \otimes q)_{ij} = p_i q_j, i, j = 1, \ldots, n \). Some comparison principles for this matrix operator have been studied in [22]-[25]. Comparison principles for other classes of (degenerate) elliptic operators are available in the literature. See [1]-[5], [7]-[20], [26] and the references therein.

Let \( U \) be an open subset of \( \mathcal{S}^{n \times n} \), satisfying

\[ 0 \in \partial U, \quad U + \mathcal{P} \subset U, \quad tU \subset U, \quad \forall \ t > 0, \tag{2} \]

where \( \mathcal{P} \) is the set of all non-negative matrices. Furthermore, in order to conclude that the strong comparison principle holds, we assume Condition \( U_\nu \), as introduced in [25], for some unit vector \( \nu \) in \( \mathbb{R}^n \): there exists \( \mu = \mu(\nu) > 0 \) such that

\[ U + C_\mu(\nu) \subset U. \tag{3} \]

Here \( C_\mu(\nu) := \{ t(\nu \otimes \nu + A) : A \in \mathcal{S}^{n \times n}, \|A\| < \mu, t > 0 \} \). Some counter examples for the strong maximum principle were given in [24] to show that the condition (3) cannot be simply dropped.

**Remark 1.1.** If \( U \) satisfies (3),

\[ \text{diag}\{1,0,\ldots,0\} \in U, \]

and

\[ O^t U O \subset U, \quad \forall \ O \in O(n), \]

where \( O(n) \) denotes the set of \( n \times n \) orthogonal matrices, then it is easy to see that \( U \) satisfies (3).

Let \( u, v \in C^{0,1}_{\text{loc}}(\Omega) \). We say that

\[ F[u] \in \mathcal{S}^{n \times n} \setminus U \quad (F[v] \in \bar{U}), \quad \text{in} \ \Omega \]

in the viscosity sense, if for any \( x_0 \in \Omega, \varphi \in C^2(\Omega), (\varphi - u)(x_0) = 0 \) \( ((\varphi - v)(x_0) = 0) \)

and

\[ u - \varphi \geq 0 \quad (v - \varphi \leq 0), \quad \text{near} \ x_0, \]

there holds

\[ F[\varphi](x_0) \in \mathcal{S}^{n \times n} \setminus U \quad (F[\varphi](x_0) \in \bar{U}). \]

We have the following strong comparison principle and Hopf Lemma.
Theorem 1.2. (Strong Comparison Principle) Let $\Omega$ be an open connected subset of $\mathbb{R}^n$, $n \geq 2$, $U$ be an open subset of $S^{n \times n}$, satisfying (2) and Condition $U_\nu$ for every unit vector $\nu$ in $\mathbb{R}^n$, and $F$ be of the form (4) with $L \in C^0_{0,1}(\Omega \times \mathbb{R}^n)$. Assume that $u, v \in C^0_{0,1}(\Omega)$ satisfy (4) in the viscosity sense, $u \geq v$ in $\Omega$. Then either $u > v$ in $\Omega$ or $u \equiv v$ in $\Omega$.

Theorem 1.3. (Hopf Lemma) Let $\Omega$ be an open connected subset of $\mathbb{R}^n$, $n \geq 2$, $\partial \Omega$ be $C^2$ near a point $\hat{x} \in \partial \Omega$, and $U$ be an open subset of $S^{n \times n}$, satisfying (2) and Condition $U_\nu$ for $\nu = \nu(\hat{x})$, the interior unit normal of $\partial \Omega$ at $\hat{x}$, and $F$ be of the form (4) with $L \in C^0_{0,1}(\Omega \times \mathbb{R}^n)$. Assume that $u, v \in C^0_{0,1}(\Omega \cup \{\hat{x}\})$ satisfy (4) in the viscosity sense, $u > v$ in $\Omega$ and $u(\hat{x}) = v(\hat{x})$. Then we have
\[
\liminf_{s \to 0^+} \frac{(u-v)(\hat{x} + s\nu(\hat{x}))}{s} > 0.
\]

Remark 1.4. If $u$ and $v \in C^2$, then Theorems 1.2 and 1.3 were proved in [25].

2 Proof of Theorem 1.2

Proof of Theorem 1.2. We argue by contradiction. Suppose the conclusion is false. Since $u - v \in C^0_{0,1}(\Omega)$ is non-negative, the set $\{x \in \Omega : u = v\}$ is closed. Then there exists an open ball $B(x_0, R) \subset \subset \Omega$ centered at $x_0 \in \Omega$ with radius $R > 0$ such that
\[
\begin{cases}
\ u - v > 0, & \text{in } B(x_0, R) \setminus \{\hat{x}\}, \\
\ u(\hat{x}) - v(\hat{x}) = 0, & \hat{x} \in \partial B(x_0, R).
\end{cases}
\]

We make use of the standard comparison function
\[
h(x) := e^{-\alpha|x-x_0|^2} - e^{-\alpha R^2}, \quad \forall \alpha > 0, x \in \Omega. \tag{5}
\]
For $i, j = 1, \ldots, n$, we have
\[
h_i(x) = \frac{\partial}{\partial x_i} h(x) = -2\alpha(x_i - (x_0)_i)e^{-\alpha|x-x_0|^2}, \tag{6}
\]
and
\[
h_{ij}(x) = \frac{\partial^2}{\partial x_i \partial x_j} h(x) = 4\alpha^2 e^{-\alpha|x-x_0|^2} \left[ (x_i - (x_0)_i)(x_j - (x_0)_j) - \frac{1}{2\alpha} \delta_{ij} \right]. \tag{7}
\]
Choose $0 < R' < \frac{R}{2}$ such that $B(\hat{x}, R') \subset \Omega$. For any $\delta \in (0, R')$, we have that for any $x \in B(\hat{x}, \delta)$,

$$-1 \leq h(x) \leq 1, \quad |\nabla h(x)| + |\nabla^2 h(x)| \leq C$$

for some $C > 0$ independent of $\delta$ and $\alpha$.

It follows that, for any $0 < \hat{\varepsilon} < \frac{\min(B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{1}{2} \delta)) \cap B(x_0, R)}{2}$,

$$u - v - \hat{\varepsilon}h > 0, \quad \text{on } B(\hat{x}) \setminus B(\hat{x}, \frac{1}{2} \delta), \quad (u - v - \hat{\varepsilon}h)(\hat{x}) = 0. \quad (9)$$

Indeed, by (8) and the fact that $h(\hat{x}) = 0$ outside $\overline{B(x_0, R)}$, for any $x \in (B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{1}{2} \delta)) \setminus B(x_0, R)$,

$$(u - v)(x) \geq 0 > \hat{\varepsilon}h(x);$$

and for any $x \in (B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{1}{2} \delta)) \cap B(x_0, R)$,

$$(u - v)(x) \geq \frac{\min(B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{1}{2} \delta)) \cap \overline{B(x_0, R)}}{2} (u - v) > \hat{\varepsilon} \geq \hat{\varepsilon}h(x).$$

For any $\varepsilon > 0$, we define the $\varepsilon$-lower and upper envelope of $u$ and $v$ as

$$u_\varepsilon(x) := \min_{y \in B(x_0, R) \cup B(\hat{x}, R')} \{u(y) + \frac{1}{\varepsilon} |x - y|^2\}, \quad \forall x \in B(x_0, R) \cup B(\hat{x}, R'),$$

and

$$v_\varepsilon(x) := \max_{y \in B(x_0, R) \cup B(\hat{x}, R')} \{v(y) - \frac{1}{\varepsilon} |x - y|^2\}, \quad \forall x \in B(x_0, R) \cup B(\hat{x}, R'),$$

respectively.

Then we conclude that there exists $\varepsilon_0 = \varepsilon_0(\delta, \alpha, \hat{\varepsilon})$ such that for $0 < \varepsilon < \varepsilon_0$,

$$\min_{B(\hat{x}, \delta)} (u_\varepsilon - v_\varepsilon - \hat{\varepsilon}h) \leq 0, \quad u_\varepsilon - v_\varepsilon - \hat{\varepsilon}h > 0 \text{ on } B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{1}{2} \delta). \quad (10)$$

Indeed, the first part of (10) follows from the definitions of $u_\varepsilon$ and $v_\varepsilon$, and the fact that $h(\hat{x}) = 0; (u_\varepsilon - v_\varepsilon - \hat{\varepsilon}h)(\hat{x}) \leq (u - v)(\hat{x}) = 0$. Now we prove the second part of (10). By theorem 5.1 (a) in [4], we have that

$$u_\varepsilon - v_\varepsilon \uparrow u - v \quad \text{uniformly on } B(x_0, R) \cup B(\hat{x}, R'), \quad \text{as } \varepsilon \to 0.
It follows that for any $M > 0$, there exists $\epsilon_0(M) > 0$ such that

$$(u_\epsilon - \nu - \hat{\epsilon} h)(x) > \min_{B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{\delta}{2})} (u - v - \hat{\epsilon} h) - M$$

for any $0 < \epsilon < \epsilon_0$ and any $x \in \overline{B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{\delta}{2})}$. Then by taking $0 < M < \frac{1}{2} \min_{B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{\delta}{2})} (u - v - \hat{\epsilon} h)$, (10) is obtained.

It follows from (10) that there exists $\bar{\eta} = \bar{\eta}(\delta, \alpha, \hat{\epsilon}) > 0$ such that for any $\eta \in (0, \bar{\eta})$, there exists $\tau = \tau(\epsilon, \eta, \delta, \alpha, \hat{\epsilon}) \in \mathbb{R}$ such that

$$\min_{B(\hat{x}, \delta)} (u_\epsilon - v - \hat{\epsilon} h - \tau) = -\eta, \quad u_\epsilon - v - \hat{\epsilon} h - \tau > 0 \text{ on } \overline{B(\hat{x}, \delta) \setminus B(\hat{x}, \frac{1}{2} \delta)}. \quad (11)$$

Let

$$\xi_\epsilon := u_\epsilon - v - \hat{\epsilon} h - \tau,$$

and $\Gamma_{\xi^-}$ denote the convex envelope of $\xi^- := \min\{\xi_\epsilon, 0\}$ on $\overline{B(\hat{x}, \delta)}$. Then by (20) in [24] and [8], we have

$$\nabla^2 \xi_\epsilon \leq \frac{4}{\epsilon} I + C \hat{\epsilon} I \quad \text{a.e. in } B(\hat{x}, \frac{1}{2} \delta).$$

And by lemma 3.5 in [6], we have

$$\int_{\{\xi_\epsilon = \Gamma_{\xi^-}\}} \det(\nabla^2 \Gamma_{\xi^-}) > 0,$$

which implies that the Lebesgue measure of $\{\xi_\epsilon = \Gamma_{\xi^-}\}$ is positive. Then there exists $x_{\epsilon, \eta} \in \{\xi_\epsilon = \Gamma_{\xi^-}\} \cap \overline{B(\hat{x}, \frac{1}{2} \delta)}$ such that both of $v^\epsilon$ and $u_\epsilon$ are punctually second order differentiable at $x_{\epsilon, \eta}$,

$$0 > \xi_\epsilon(x_{\epsilon, \eta}) \geq -\eta, \quad (12)$$

$$|\nabla \xi_\epsilon(x_{\epsilon, \eta})| \leq C \eta, \quad (13)$$

and

$$\nabla^2 \xi_\epsilon(x_{\epsilon, \eta}) = \nabla^2 (u_\epsilon - v - \hat{\epsilon} h)(x_{\epsilon, \eta}) \geq 0. \quad (14)$$

For $x_{\epsilon, \eta} \in \Omega$, by the definitions of $u_\epsilon$ and $v^\epsilon$, there exist $(x_{\epsilon, \eta})_*$ and $(x_{\epsilon, \eta})^* \in \Omega$ such that

$$u_\epsilon(x_{\epsilon, \eta}) = u((x_{\epsilon, \eta})_*) + \frac{1}{\epsilon} |(x_{\epsilon, \eta})_* - x_{\epsilon, \eta}|^2,$$

and

$$v^\epsilon(x_{\epsilon, \eta}) = v((x_{\epsilon, \eta})^*) - \frac{1}{\epsilon} |(x_{\epsilon, \eta})^* - x_{\epsilon, \eta}|^2.$$
Since $u$ and $v \in C_{\text{loc}}^{0,1}(\Omega)$, by (2.6) and (2.7) in [23], we have

$$|(x,\eta)_* - x,\eta| + |(x,\eta)^* - x,\eta| \leq C_1\epsilon,$$

and

$$|\nabla u_e(x,\eta)| + |\nabla v^e(x,\eta)| \leq C_2,$$

where $C_1$ and $C_2$ are two universal positive constant independent of $\epsilon$ and $\eta$.

Since $u_\epsilon$ is punctually second order differentiable at $x,\eta$, we have

$$u_\epsilon(x,\eta + z) \geq u_\epsilon(x,\eta) + \nabla u_\epsilon(x,\eta) \cdot z + \frac{1}{2}z^T \nabla^2 u_\epsilon(x,\eta)z + o(|z|^2), \quad \text{as } z \to 0. \quad (17)$$

By the definition of $u_\epsilon$, we have

$$u_\epsilon(x,\eta + z) \leq u((x,\eta)_* + z) + \frac{1}{\epsilon}|(x,\eta)_* - x,\eta|^2,$$

and therefore, in view of (17),

$$u((x,\eta)_* + z) \geq u_\epsilon(x,\eta + z) - \frac{1}{\epsilon}|(x,\eta)_* - x,\eta|^2$$

$$\geq P_\epsilon((x,\eta)_* + z) + o(|z|^2), \quad \text{as } z \to 0,$$

where $P_\epsilon$ is a quadratic polynomial with

$$P_\epsilon((x,\eta)_*) = u_\epsilon(x,\eta) - \frac{1}{\epsilon}|(x,\eta)_* - x,\eta|^2 = u((x,\eta)_*),$$

$$\nabla P_\epsilon((x,\eta)_*) = \nabla u_\epsilon(x,\eta),$$

$$\nabla^2 P_\epsilon((x,\eta)_*) = \nabla^2 u_\epsilon(x,\eta).$$

Since $u$ satisfies (4) in the viscosity sense, we thus have

$$\nabla^2 u_\epsilon(x,\eta) + L((x,\eta)_*, \nabla u_\epsilon(x,\eta)) = F[P_\epsilon((x,\eta)_*)] \in S_n^* \setminus U. \quad (18)$$

On the other hand, in view of (15), (16) and the fact that $L \in C_{\text{loc}}^{0,1}(\Omega \times \mathbb{R}^n)$,

$$L(x,\eta, \nabla u_\epsilon(x,\eta)) - L((x,\eta)_*, \nabla u_\epsilon(x,\eta)) \leq C|x,\eta - (x,\eta)_*|I \leq a_1 \epsilon I, \quad (19)$$

where $C$ and $a_1 > 0$ are universal constants.

It follows from (2), (18) and (19) that

$$F[u_\epsilon(x,\eta)] - a_1 \epsilon I \in S_n^* \setminus U. \quad (20)$$
Analogously, we can obtain
\[ F[v^\varepsilon](x_{\epsilon,\eta}) + a_2 \epsilon I \in U \]
for some universal constants \(a_2 \geq 0\).

By (13), (14), (16) and the fact that \(L \in C_{loc}^{0,1}(\Omega \times \mathbb{R}^n)\),
\[
F[u_\epsilon](x_{\epsilon,\eta}) \geq \nabla^2 (v^\varepsilon + \hat{\epsilon} h)(x_{\epsilon,\eta}) + L(x_{\epsilon,\eta}, \nabla u_\epsilon(x_{\epsilon,\eta})) = F[v^\varepsilon + \hat{\epsilon} h](x_{\epsilon,\eta}) + L(x_{\epsilon,\eta}, \nabla v^\varepsilon(x_{\epsilon,\eta})) \geq F[v^\varepsilon + \hat{\epsilon} h](x_{\epsilon,\eta}) - C|\nabla(u_\epsilon - v^\varepsilon)(x_{\epsilon,\eta})| \]
\[
\geq F[v^\varepsilon + \hat{\epsilon} h](x_{\epsilon,\eta}) - C(\eta + \hat{\epsilon}|\nabla h(x_{\epsilon,\eta})|)I.
\]
(21)

By (8), (16) and the fact that \(L \in C_{loc}^{0,1}(\Omega \times \mathbb{R}^n)\), we have
\[
F[v^\varepsilon + \hat{\epsilon} h](x_{\epsilon,\eta}) = F[v^\varepsilon](x_{\epsilon,\eta}) + \hat{\epsilon} \nabla^2 h(x_{\epsilon,\eta}) + L(x_{\epsilon,\eta}, \nabla (v^\varepsilon + \hat{\epsilon} h)(x_{\epsilon,\eta})) - L(x_{\epsilon,\eta}, \nabla v^\varepsilon(x_{\epsilon,\eta})) \]
\[
\geq F[v^\varepsilon](x_{\epsilon,\eta}) + \hat{\epsilon} \nabla^2 h(x_{\epsilon,\eta}) - C|\nabla h(x_{\epsilon,\eta})|I.
\]
(22)

Then by (6), (7) and the fact \(|x_{\epsilon,\eta} - x_0| < 2R|
\[
\nabla^2 h(x_{\epsilon,\eta}) - C|\nabla h(x_{\epsilon,\eta})|I \]
\[
= 4 \alpha^2 e^{-\alpha|x_{\epsilon,\eta} - x_0|^2} \left[ (x_{\epsilon,\eta} - x_0) \otimes (x_{\epsilon,\eta} - x_0) - \frac{1}{2\alpha} I - \frac{C}{4\alpha} |x_{\epsilon,\eta} - x_0| I \right] \]
\[
\geq 4 \alpha^2 e^{-\alpha|x_{\epsilon,\eta} - x_0|^2} \left[ (x_{\epsilon,\eta} - x_0) \otimes (x_{\epsilon,\eta} - x_0) - \frac{C}{\alpha} I \right] \]
\[
\geq 4 \alpha^2 e^{-\alpha|x_{\epsilon,\eta} - x_0|^2} \left[ (\hat{x} - x_0) \otimes (\hat{x} - x_0) - C \delta R I - \frac{C}{\alpha} I \right] \]
\[
= 4R^2 \alpha^2 e^{-\alpha|x_{\epsilon,\eta} - x_0|^2} \left[ \left( \frac{\hat{x} - x_0}{R} \right) \otimes \left( \frac{\hat{x} - x_0}{R} \right) - C \delta I - \frac{C}{\alpha} I \right] \]
\[
\geq 4R^2 \alpha^2 e^{-4R^2 \alpha} \left[ \left( \frac{\hat{x} - x_0}{R} \right) \otimes \left( \frac{\hat{x} - x_0}{R} \right) - C \delta I - \frac{C}{\alpha} I \right].
\]
(23)

Inserting (23) into (22), we have
\[
F[v^\varepsilon + \hat{\epsilon} h](x_{\epsilon,\eta}) \geq F[v^\varepsilon](x_{\epsilon,\eta}) + 4R^2 \hat{\epsilon} \alpha e^{-4R^2 \alpha} \left[ \left( \frac{\hat{x} - x_0}{R} \right) \otimes \left( \frac{\hat{x} - x_0}{R} \right) - C \delta I - \frac{C}{\alpha} I \right].
\]
(24)
It follows from (21) and (24) that
\[
F[u_\epsilon](x_\epsilon, \eta) - a_1\epsilon I \\
\geq F[v^\epsilon](x_\epsilon, \eta) + a_2\epsilon I \\
+ 4R^2\hat{\epsilon}\alpha^2 e^{-4R^2\alpha} \left[ \left( \frac{\hat{x} - x_0}{R} \right) \otimes \left( \frac{\hat{x} - x_0}{R} \right) \right] - C\delta I - \frac{C}{\alpha} I - C \hat{\epsilon} \alpha^2 (\epsilon + \eta) I.
\]

(25)

We can firstly fix the value of small \( \delta > 0 \) and a large \( \alpha > 1 \), then fix the value of small \( \hat{\epsilon} > 0 \), and lastly fix the value of small \( \epsilon \) and \( \eta > 0 \) such that
\[
\|C\delta I + \frac{C}{\alpha} I + C \hat{\epsilon} \alpha^2 (\epsilon + \eta) I\| < \frac{1}{2} \mu \left( \frac{\hat{x} - x_0}{R} \right),
\]
where \( \mu \) is obtained from condition (3).

Therefore, by (3) and (25), we have that
\[
F[u_\epsilon](x_\epsilon, \eta) - a_1\epsilon I \in U,
\]
which is a contradiction with (20). Theorem 1.2 is proved.

3 Proof of Theorem 1.3

Proof of Theorem 1.3. Since \( \partial \Omega \) is \( C^2 \) near \( \hat{x} \), there exists an open ball \( B(x_0, R) \subset \Omega \) such that \( B(x_0, R) \cap \partial \Omega = \{ \hat{x} \} \) and
\[
\begin{cases}
  u - v > 0, & \text{in } B(x_0, R) \setminus \{ \hat{x} \}, \\
  u(\hat{x}) - v(\hat{x}) = 0.
\end{cases}
\]

Let \( h \) be defined as in (5). We work in the domain
\[
A_\delta := B(\hat{x}, \delta) \cap B(x_0, R).
\]

It is easy to see that
\[
u - v \geq \hat{\epsilon} h, \quad \text{on } \partial A_\delta
\]
for any \( 0 < \delta < \frac{R}{2} \) and \( 0 < \hat{\epsilon} < \min_{\partial B(\hat{x}, \delta) \cap B(x_0, R)} (u - v) \).
We claim that for $\varepsilon$ small enough,
\[ u - v \geq \hat{\varepsilon}h, \quad \text{on } \overline{A_\delta}. \]

Once the claim is proved, then we have that
\[
\liminf_{s \to 0^+} \frac{(u-v)(\hat{x} + s \nu(\hat{x}))}{s} \geq \hat{\varepsilon} \liminf_{s \to 0^+} \frac{h(\hat{x} + s \nu(\hat{x}))}{s} = 2\alpha R e^{-\alpha R^2} > 0.
\]

Therefore, in order to finish the proof of Theorem 1.3 we only need to prove the above claim. Suppose the contrary, that is,
\[
\zeta = \zeta(\hat{\varepsilon}, \alpha, \delta) := \min_{\overline{A_\delta}} (u - v - \hat{\varepsilon}h) < 0.
\]

It follows that
\[
\min_{\overline{A_\delta}} (u - v - \hat{\varepsilon}h - \zeta) = 0, \quad u - v - \hat{\varepsilon}h - \zeta \geq -\zeta > 0 \text{ on } \partial A_\delta.
\]

Now we can follow the argument as in the proof of Theorem 1.2 to get a contradiction. Theorem 1.3 is proved.

\[ \square \]

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