Constrained fitting of three-point functions

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We determine matrix elements for $B \rightarrow D$ semileptonic decay. The use of the constrained fitting method and multiple smearings for both two- and three-point correlators allows an improved calculation of the form factors.

1. Constrained curve fitting

The aim of this contribution is to illustrate the method of constrained curve fitting by application to three-point functions describing the semileptonic decay of heavy-light mesons. For a discussion of the method see \cite{1}. The constraint method offers the chance to include knowledge about the fit parameters and generally results in very stable fits. This allows for much more ambitious fits than previous methods and the results are less prone to systematic errors from fitting.

We apply the method to the subset ($\beta = 5.7$, $a m_Q = 2, 4, 8$) of the data analysed in \cite{2}, where NRQCD is used for the heavy quark. The lattice size is $12^3 \times 24$. The smearings used are a Gaussian with a radius of $a r_Q = 2.0$ for the heavy and local for the light quark (referred to as smearing 1) and $a r_{q/Q} = 3.0$ for both (smearing '2). All combinations of these smearings at source and sink were used. It has been demonstrated that these are not the optimal smearings \cite{3}, in particular for nonvanishing momenta, which leaves room for improvement of the results presented here.

2. Form factors

We aim at extracting the form factors $h^+(\omega)$ from the matrix elements

$$\langle B | V_0 | D \rangle = F_1(q^2) (p_B + p_D)_0 = \sqrt{m_B m_D} h^+(\omega) (v_B + v_D)_0$$

for the elastic case $p_B^2 = p_D^2$ with degenerate masses $m_B = m_D$. $V_0$ is the leading order heavy-heavy current for NRQCD \cite{4}.

First we extract the form factors for each value of $q^2$. We fit the two-point functions parametrised by

$$G_{2}^{ij} = \sum_{k=1}^{n} A_{i}^{j} A_{k}^{j} \exp(-E_{k}(t-t_{0}))$$

simultaneously with the three-point functions using the ansatz

$$G_{3}^{ij} = \sum_{k,l=1}^{n} A_{i}^{j} J_{kl} A_{l}^{j} e^{-E_{k}(t-t_{0})} e^{-E_{l}(T-t)}$$

with current matrix elements $J_{kl}$, where $i,j$ label the smearings and $k,l$ the energy levels. We set $J_{kl} = J_{lk}$, and $J_{11}$ is $h^+(\omega)$. The distance between quark source and sink is 12. Instead of the energies we in fact used as parameters the logs of energy differences as in \cite{1}. Our standard priors are $0.7(\ast/2)$ for the ground state energy\cite{3}, and $0.6(\ast/2)$ for the energy differences between\textsuperscript{\ast} i.e. our prior favours $\ln 0.7 \pm \ln 2$ for the fit parameter $\ln E_{g.s.}$.
the excitations. The first is known from previous fits: the excitation energy priors are appropriate to typical radial excitations at this lattice spacing. For the amplitudes we took as priors 1.0(±1.5) and 0.5(±1.5) for the ground states, −0.4(±2) for \( A^2 \) and 0.1(±2) for the rest. These are based on experience from previous fits but allow for plenty of variation (the typical errors on the amplitudes from the fits are 0.01...0.05). The priors for the \( J_{kl} \) are 1(±0.5) for \( J_{kk} \) and 0.1(±0.5) for \( J_{kl} \) (\( k \neq l \)).

![Figure 1](image1)

Figure 1. Three-point functions and 6-exp fit results \( (p^2 = 1, q^2 = 2, m_Q = 2) \); the gray boxes are (displaced) enlargements of the functions underneath.

The minimisation was done using the Levenberg-Marquardt algorithm. The value of \( \chi^2_{\text{aug}} \) for the reliable fits (i.e. using a sufficient number of exponentials) is generally about 1. The contribution of the two-point functions is very small (as the examples in [1]), the priors contribute only marginally. Quoted errors are extracted from the curvature matrix. Figure 1 shows a typical fit result for the three-point functions. To shed light on the effect of the priors we repeated the calculation with priors away from the expectations and allowing for smaller deviations. We find that the data strongly constrain the lowest lying current matrix elements \( J_{11} \) and \( J_{12} \), but have little information about the higher ones. This is expected because off-diagonal matrix elements should be, and are, small, because they represent the overlap (as \( q^2 \to 0 \)) of orthogonal states.

The dependence of the lowest currents on the number of exponentials in the fit is demonstrated in Fig. 2. Stability sets in for six exponentials.

![Figure 2](image2)

Figure 2. The current matrix elements for the lowest lying states vs. the number of exponentials used in the fit \( (p^2 = 1, q^2 = 2, m_Q = 2) \)

3. Kinetic masses

To convert these form factors into \( h^+(\omega) \) one needs the kinetic masses. They are calculated from the energy splittings \( \Delta E \) between the \( p^2 = 1 \) and \( p^2 = 0 \) states

\[
m = (p^2 - (\Delta E)^2)/(2\Delta E).
\]
Our method permits us to extract them directly from simultaneous fits of the respective smeared-smeared two-point functions. This is an example where the usage of the classical fitting method could be misleading if one considers the two-point functions separately, because the two-point functions seem to plateau much earlier than their ratio.

The values for the kinetic masses for the three heavy quark masses investigated are listed in the following table:

| $m_Q$ | $\Delta E$ | $m_{\text{kin}}$ |
|-------|------------|-----------------|
| 2     | 0.058(1)   | 2.9(2)          |
| 4     | 0.029(3)   | 4.7(5)          |
| 8     | 0.019(3)   | 7.3 (1.3)       |

This much simpler fit shows stable results already using three exponentials (see Fig. 3).

In the future we plan to fit simultaneously all the two- and three-point functions at different momenta and the kinetic mass will then come directly from the same fits that give the current matrix elements.

![Figure 3. Kinetic mass as function of the number of exponentials in the fit; the horizontal lines are the perturbative results from 2](image)

4. Isgur-Wise function

Now we can plot the form factors as function of $\omega = v \cdot v'$, the Isgur-Wise function (Fig. 4). The errors are half of those from using only one smearing. Each dataset extrapolates to the point (1, 1) as it has to for NRQCD. There seems to be a tendency of increasing slope with the heavy quark mass which has to be explored further.

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![Figure 4. Form factors as functions of $\omega$](image)

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