ON AUTOMORPHISMS OF DANIELEWSKI SURFACES

ANTHONY J. CRACHIOLA

Abstract. We develop techniques for computing the AK invariant of a domain with arbitrary characteristic. We use these techniques to describe for any field \( k \) the automorphism group of \( k[X, Y, Z]/(X^nY - Z^2 - h(X)Z) \), where \( h(0) \neq 0 \) and \( n \geq 2 \), as well as the isomorphism classes of these algebras.

1. Introduction

All rings in this paper are commutative with identity. Throughout this paper, let \( k \) denote field of arbitrary characteristic, and let \( k^* = k \setminus \{0\} \). For a ring \( A \), let \( A^{[n]} \) denote the polynomial ring in \( n \) indeterminates over \( A \). Let \( \mathbb{C} \) denote the field of complex numbers.

One fundamental algebraic problem is to describe the automorphism group of a given algebra. From the perspective of algebraic geometry, this means describing automorphisms of a given affine variety. The Jung-van der Kulk [J, K] theorem provides the answer for the polynomial ring \( k^{[2]} \), i.e. the affine plane \( k^2 \), but for higher dimensional affine spaces the problem is still open. For other affine varieties there is no general approach to solving the problem.

Let \( A \) be an algebra with characteristic zero. A new tool appeared in the 1990s when Leonid Makar-Limanov [M1] introduced the AK invariant (more commonly known now as the Makar-Limanov invariant) of \( A \) as the intersection of the kernels of all locally nilpotent derivations on \( A \). Each automorphism of \( A \) restricts to an automorphism of the subalgebra \( AK(A) \), making this invariant useful in describing the automorphism group of \( A \). As one demonstration Makar-Limanov has computed the automorphism group of a surface \( x^n y = P(z) \) over \( \mathbb{C} \) [M2]. The successful application of the AK invariant to this and other algebro-geometric problems, such as the linearization conjecture for \( \mathbb{C}^* \)-actions on \( \mathbb{C}^3 \) [KKMR], has contributed to its current popular status in algebraic geometry.

The AK invariant as defined by Makar-Limanov loses its potency for algebras with prime characteristic \( p \) because the kernel of each derivation becomes much larger, containing the \( p \)th power of every element. To the author’s knowledge, while the AK invariant is still bearing fruit, all the research is being conducted under the restriction of zero characteristic. Now, in the characteristic zero arena locally nilpotent derivations on an algebra \( A \) are interchangeable with algebraic additive group actions on \( \text{Spec}(A) \), and unlike derivations these actions maintain

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their attractive properties for prime characteristic algebras. We can use this point of view to redefine the AK invariant for rings with arbitrary characteristic.

In the first part of this paper we explain how to use the AK invariant for domains of arbitrary characteristic. While complex algebraic geometry can utilize the topological properties of the complex numbers, the techniques in this paper rely only on algebraic structures and do not even require algebraic closure of the ground field. In fact, the results in this paper are valid over any field of any characteristic. We next compute the AK invariant of the algebra \( R = k[X, Y, Z]/(X^nY - Z^2 - h(X)Z) \), where \( n \geq 2 \) and \( h(0) \neq 0 \), and use it to describe the automorphism group of \( R \).

Over \( \mathbb{C} \) the algebra \( R \) is the coordinate ring of the surface \( x^ny = z^2 + h(x)z \). This is a generalization of the celebrated Danielewski surface which plays a role in the cancellation problem for affine varieties. Here is the one form of the problem. If \( V \) and \( W \) are affine varieties over \( k \), does \( V \times k^n \cong W \times k^n \) imply \( V \cong W \)? If \( \dim(V) = \dim(W) = 1 \) the answer is affirmative. This was shown algebraically by Shreeram Abhyankar, Paul Eakin, and William Heinzer [AEH]. (As a side remark, there is a new proof of this fact which employs the AK invariant [CM].) For an algebraist the problem is reformulated as follows. If \( A \) and \( B \) are \( k \)-algebras, does \( A^{[n]} \cong B^{[n]} \) imply \( A \cong B \)? Mel Hochster published the first counterexample [H] in 1972, the same year of the Abhyankar-Eakin-Heinzer paper, using 4-dimensional algebras over the field of real numbers. For a counterexample over an algebraically closed field the world waited until 1989. It is due to Wlodzimierz Danielewski [D] who never published the result. For a published treatment on the Danielewski surfaces, refer to the paper of Karl-Heinz Fieseler [F] which gives a classification of normal surfaces equipped with a nontrivial \( C^+ \)-action. Here is Danielewski’s original counterexample. Let \( V \) and \( W \) be surfaces over \( \mathbb{C} \) given by \( xy = z^2 + z \) and \( x^2y = z^2 + z \), respectively. Danielewski showed geometrically that the cylinders \( V \times \mathbb{C} \) and \( W \times \mathbb{C} \) are isomorphic while \( V \) and \( W \) are not. To explain the isomorphism of cylinders, Danielewski showed that \( V \) and \( W \) are total spaces of some principal \( C^+ \)-bundles over a line with a double point \( \mathbb{C} \), and that each of these total spaces is isomorphic to \( V \times \mathbb{C} W \) which is a trivial bundle over each \( V \) and \( W \). To distinguish \( V \) and \( W \) Danielewski used the first homology group at infinity. The AK invariant can also tell them apart.

Now let \( V_i \) be the surface in \( \mathbb{C}^3 \) given by \( x^ny = z^2 + h_i(x)z \), \( i = 1, 2 \), where \( h_i(0) \neq 0 \). In the same unpublished paper [D] Danielewski conjectured that \( V_1 \cong V_2 \) if and only if \( n_1 = n_2 \) and \( h_2(x) = \lambda h_1(\mu x) \) for some \( \lambda, \mu \in \mathbb{C}^\times \). Jörn Winkelmann [W] proved this conjecture and also that, as with Danielewski’s original example, the cylinders over \( V_1 \) and \( V_2 \) are isomorphic for any \( n_1, n_2 \) and any \( h_1(x), h_2(x) \). At the end of our paper we shall revisit this connection to cancellation as an application of the AK invariant. We describe the isomorphism classes of \( x^ny = z^2 + h(x)z \) over any field \( k \) and show how an isomorphism of cylinders can be explained algebraically.

Danielewski surfaces continue to be a source of interest for current research. In addition to this paper, see for instance [Du] [FM] [SY].

The main inspiration for this paper is the paper [M2] of Leonid Makar-Limanov in which similar results are achieved on the surface \( x^ny = P(z) \) over \( \mathbb{C} \).
2. Methods

Exponential maps, the AK invariant, and locally finite iterative higher derivations. Let $A$ be a $k$-algebra. Suppose $\varphi : A \rightarrow A^{[1]}$ is a $k$-algebra homomorphism. We write $\varphi = \varphi_U : A \rightarrow A[U]$ if we wish to emphasize an indeterminate $U$.

We say that $\varphi$ is an exponential map on $A$ if it satisfies the following two additional properties.

(i) $\varepsilon_0 \varphi_U$ is the identity on $A$, where $\varepsilon_0 : A[U] \rightarrow A$ is evaluation at $U = 0$.

(ii) $\varphi S \varphi_U = \varphi S_U$, where $\varphi S$ is extended by $\varphi S(U) = U$ to a homomorphism $A[U] \rightarrow A[S, U]$.

(When $A$ is the coordinate ring of an affine variety $\text{Spec}(A)$ over $k$, the exponential maps on $A$ correspond to algebraic actions of the additive group $k^+$ on $\text{Spec}(A)$ [9.5].)

Given an exponential map $\varphi : A \rightarrow A[U]$, set $\varphi(U) = U$ to obtain an automorphism of $A[U]$ with inverse $\varphi_{-U}$. Consider the map $\varepsilon_1 : A \rightarrow A$, where $\varepsilon_1 : A[U] \rightarrow A$ is evaluation at $U = 1$. One can check that $\varepsilon_1 \varphi$ is an automorphism of $A$ with inverse $\varepsilon_1 \varphi_{-U}$.

Define

$$A^\varphi = \{ a \in A \mid \varphi(a) = a \},$$

a subalgebra of $A$ called the ring of $\varphi$-invariants. Let $\text{EXP}(A)$ denote the set of all exponential maps on $A$. We define the AK invariant, or ring of absolute constants of $A$ as

$$\text{AK}(A) = \bigcap_{\varphi \in \text{EXP}(A)} A^\varphi.$$

This is a subalgebra of $A$ which is preserved by isomorphism. Indeed, any isomorphism $f : A \rightarrow B$ of $k$-algebras restricts to an isomorphism $f : \text{AK}(A) \rightarrow \text{AK}(B)$.

To understand this, observe that if $\varphi \in \text{EXP}(A)$ then $f \varphi f^{-1} \in \text{EXP}(B)$. Remark that $\text{AK}(A) = A$ if and only if the only exponential map on $A$ is the standard inclusion $\varphi(a) = a$ for all $a \in A$.

It is often helpful to view a given $\varphi \in \text{EXP}(A)$ as a sequence in the following way. For each $a \in A$ and each natural number $n$, let $D^n(a)$ denote the $U^n$-coefficient of $\varphi(a)$. Let $D = \{D^0, D^1, D^2, \ldots \}$. To say that $\varphi$ is a $k$-algebra homomorphism is equivalent to saying that the sequence $\{D^i(a)\}$ has finitely many nonzero elements for each $a \in A$, that $D^n : A \rightarrow A$ is $k$-linear for each natural number $n$, and that the Leibniz rule

$$D^n(ab) = \sum_{i+j=n} D^i(a)D^j(b)$$

holds for all natural numbers $n$ and all $a, b \in A$. The above properties (i) and (ii) of the exponential map $\varphi$ translate into the following properties of $D$.

(i') $D^0$ is the identity map.

(ii') (iterative property) For all natural numbers $i, j$,

$$D^iD^j = \binom{i+j}{i} D^{i+j}.$$

Due to all of these properties, the collection $D$ is called a locally finite iterative higher derivation on $A$. More generally, a higher derivation on $A$ is a collection $D = \{D^i\}$ of $k$-linear maps on $A$ such that $D^0$ is the identity and the above Leibniz rule holds. The notion of higher derivations is due to H. Hasse and F.K. Schmidt [15].
When the characteristic of $A$ is zero, each $D^i$ is determined by $D^1$, which is a locally nilpotent derivation on $A$. In this case, $\varphi = \exp(UD^1) = \sum_{j=0}^{\infty} \frac{1}{j!}(UD^1)^j$ and $A^\varphi$ is the kernel of $D^1$. So we retrieve the original characteristic zero definition of $\text{AK}(A)$ given by L. Makar-Limanov as the intersection of the kernels of locally nilpotent derivations on $A$.

The above discussion of exponential maps, locally finite iterative higher derivations, and the AK invariant makes sense more generally for any (not necessarily commutative) ring. However, we will not require this generality.

**Degree functions and related lemmas.** Given an exponential map $\varphi : A \rightarrow A[U]$ on a domain $A$ over $k$, we can define the $\varphi$-degree of an element $a \in A$ by $\deg_\varphi(a) = \deg_U(\varphi(a))$ (where $\deg_U(0) = -\infty$). Note that $A^\varphi$ consists of all elements of $A$ with non-positive $\varphi$-degree. The function $\deg_\varphi$ is a degree function on $A$ because it satisfies the following two properties for all $a, b \in A$.

(i) $\deg_\varphi(ab) = \deg_\varphi(a) + \deg_\varphi(b)$.

(ii) $\deg_\varphi(a + b) \leq \max\{\deg_\varphi(a), \deg_\varphi(b)\}$.

Equipped with these notions, we now collect some useful facts.

**Lemma 2.1.** Let $\varphi$ be an exponential map on a domain $A$ over $k$. Let $D = \{D^i\}$ be the locally finite iterative higher derivation associated to $\varphi$.

(a) If $a, b \in A$ such that $ab \in A^\varphi \setminus 0$, then $a, b \in A^\varphi$. In other words, $A^\varphi$ is factorially closed in $A$.

(b) $A^\varphi$ is algebraically closed in $A$.

(c) For each $a \in A$, $\deg_\varphi(D^i(a)) \leq \deg_\varphi(a) - i$. In particular, if $a \in A \setminus 0$ and $n = \deg_\varphi(a)$, then $D^n(a) \in A^\varphi$.

**Proof.** (a): We have $0 = \deg_\varphi(ab) = \deg_\varphi(a) + \deg_\varphi(b)$, which implies that $\deg_\varphi(a) = \deg_\varphi(b) = 0$.

(b): If $a \in A \setminus 0$ and $c_0 a^n + \cdots + c_1 a + c_0 = 0$ is a polynomial relation with minimal possible degree $n \geq 1$, where each $c_i \in A^\varphi$ with $c_0 \neq 0$, then $a(c_0 a^{n-1} + \cdots + c_1) = -c_0 \in A^\varphi \setminus 0$. By part (a), $a \in A^\varphi$.

(c): Use the iterative property of $D$ to check that $D^i(D^i(a)) = 0$ whenever $j > \deg_\varphi(a) - i$. \hfill $\Box$

**Lemma 2.2.** Let $\varphi$ be a nontrivial exponential map (i.e., not the standard inclusion) on a domain $A$ over $k$ with $\text{char}(k) = p \geq 0$. Let $x \in A$ have minimal positive $\varphi$-degree $n$.

(a) $D^i(x) \in A^\varphi$ for each $i \geq 1$. Moreover, $D^i(x) = 0$ whenever $i > 1$ is not a power of $p$.

(b) If $a \in A \setminus 0$, then $n$ divides $\deg_\varphi(a)$.

(c) Let $c = D^\varphi(x)$. Then $A$ is a subalgebra of $A^\varphi[c^{-1}][x]$, where $A^\varphi[c^{-1}] = \text{Frac}(A^\varphi)$ is the localization of $A^\varphi$ at $c$.

(d) Let $\text{trdeg}_k$ denote transcendence degree over $k$. If $\text{trdeg}_k(A)$ is finite, then $\text{trdeg}_k(A^\varphi) = \text{trdeg}_k(A) - 1$.

**Proof.** In proving parts (a) and (b) we will utilize the following fact. If $p$ is prime and $i = p^iq$ for some natural numbers $i, j, q$, then $\binom{i}{p^j} \equiv q \pmod{p}$ [21, Lemma 5.1].

(a): By part (c) of Lemma 2.1, $D^i(x) \in A^\varphi$ for all $i \geq 1$. If $p = 0$ then $n = 1$, for given any element in $A \setminus A^\varphi$ we can find an element with $\varphi$-degree 1 by applying the locally nilpotent derivation $D^1$ sufficiently many times. In this case, the second
statement is immediate. Suppose now that \( p \) is prime and that \( i > 1 \) is not a power of \( p \), say \( i = p^j q \), where \( j \) is a nonnegative integer and \( q \geq 2 \) is an integer not divisible by \( p \). Then \( D^{i-p^j q}(x) \in A^x \) and
\[
0 = D^{p^j q} D^{i-p^j q}(x) = \left( \frac{i}{p^j q} \right) D^i(x) = q D^i(x).
\]
We can divide by \( q \) to conclude that \( D^i(x) = 0 \).

(b): Again if \( p = 0 \) then \( n = 1 \) and the claim is obvious. Assume that \( p \) is prime. By part (a) we have \( n = p^m \) for some integer \( m \geq 0 \). If \( m = 0 \), the claim is immediate. Assume that \( m > 0 \). Let \( d = \deg \varphi(a) \). Suppose that \( p \) does not divide \( d \). By part (c) of Lemma 2.1, \( \deg \varphi(D^{d-1}(a)) \leq 1 \). Now, \( D^1 D^{d-1}(a) = d D^d(a) \neq 0 \). So \( \deg \varphi(D^{d-1}(a)) = 1 < n \), contradicting the minimality of \( n \). Hence we can write \( d = p^k d_1 \) with \( k \geq 1 \) and \( d_1 \) not divisible by \( p \). Making a similar computation, \( D^{p^k} D^{d-p^k}(a) = d_1 D^d(a) \neq 0 \). This implies that \( \deg \varphi(D^{d-p^k}(a)) = p^k \). Since \( n = p^m \) is minimal, we must have \( k \geq m \), and so \( n \) divides \( d \).

(c): Let \( a \in A \setminus \{0\} \). By part (b) we can write \( \deg \varphi(a) = ln \) for some natural number \( l \).

If \( l = 0 \) then \( a \in A^x \) and we are done. We use induction on \( l > 0 \). Elements \( c^i a \) and \( D^{in}(a)x^i \) both have \( \varphi \)-degree \( ln \). Let us check that \( D^{in}(c^i a) = D^{in}(D^{in}(a)x^i) \). First, \( D^{in}(c^i a) = c^i D^{in}(a) \) by the Leibniz rule and because \( c^i \) is \( \varphi \)-invariant. Secondly, since \( D^{in}(x^i) = D^n(x^i) = c^i \) and \( D^{in}(a) \) is \( \varphi \)-invariant, we see that \( D^{in}(D^{in}(a)x^i) = c^i D^{in}(a) \) as well. (Remark: Though the equality \( D^{in}(x^i) = D^n(x^i) \) does follow from the Leibniz rule, it may be more immediately observed as follows. \( D^n(x^i) \) is the leading \( U \)-coefficient of \( \varphi(x) \), and \( \varphi \) is a homomorphism. Hence the leading \( U \)-coefficient of \( \varphi(x^i) \) is also that of \( \varphi(x^i) \).) Therefore, the element \( y = c^i a - D^{in}(a)x^i \) has \( \varphi \)-degree less than \( ln \) and hence less than or equal to \( (l-1)n \). By the inductive hypothesis, \( y \in A^x[c^{-1}][x] \). So \( a = c^{-l}(y + D^{in}(a)x^i) \in A^x[c^{-1}][x] \).

(d): This is immediate from part (c), together with part (b) of Lemma 2.1 which states that \( A^x \) is algebraically closed in \( A \).

Weights. We often produce a degree function on a domain \( A \) by assigning degree values to some specified generators. On a product of generators the degree is then defined by the above property (i) of degree functions. Each element of \( A \) can be expressed as a summation of linearly independent terms, each of which is a product of generators. The degree of such an expression is then defined to be the highest degree occurring among the terms. This is the case with the usual degree functions on polynomials, which are defined by assigning values to the indeterminates. In this situation we say that the degree function is obtained by assigning weights to some generators. We will use the idea of weights repeatedly in proving the results of this paper.

Homogenization of an exponential map. Let \( A \) be a domain over \( k \). Let \( Z \) denote the integers. Suppose that \( A \) has a \( Z \)-filtration \( \{ A_n \} \). This means that \( A \) is the union of linear subspaces \( A_n \) with these properties.

(i) \( A_i \subseteq A_j \) whenever \( i \leq j \).
(ii) \( A_i : A_j \subseteq A_{i+j} \) for all \( i, j \in Z \).
(iii) \( \bigcap_{n \in Z} A_n = 0 \).

Additionally, suppose that
\[
(A_i \setminus A_{i-1}) \cdot (A_j \setminus A_{j-1}) \subseteq A_{i+j} \setminus A_{i+j-1}
\]
for all $i, j \in \mathbb{Z}$. This will be the case if the filtration is induced by a degree function. Suppose also that $\chi$ is a set of generators for $A$ over $K$ with the following property: if $a \in A_i \setminus A_{i-1}$ then we can write $a = \sum c_i x^i$, a summation of monomials $c_i x^i$ built from $\chi$ which are all contained in $A_i$. This is not an unreasonable property. It merely asserts some homogeneity on the generating set $\chi$.

Given $a \in A \setminus 0$ there exists $i \in \mathbb{Z}$ for which $a \in A_i \setminus A_{i-1}$. Write

$$\overline{a} = a + A_{i-1} \in A_i/A_{i-1},$$

the top part of $a$. We can construct a graded $K$-algebra

$$\text{gr}(A) = \bigoplus_{n \in \mathbb{Z}} A_n/A_{n-1}.$$

Addition on $\text{gr}(A)$ is given by its vector space structure. Given $\overline{a} = a + A_{i-1}$ and $\overline{b} = b + A_{j-1}$, define $\overline{a} \overline{b} = ab + A_{i+j-1}$. Note that $\overline{a} \overline{b} = \overline{ab}$. Extend this multiplication to all of $\text{gr}(A)$ by the distributive law. By our assumption on the filtration, $\text{gr}(A)$ is a domain. Also, $\text{gr}(A)$ is generated by the top parts of the elements of $\chi$. Therefore, if $\chi$ is a finite set then $\text{gr}(A)$ is an affine domain.

Let $\text{grdeg}$ be the degree function induced by the grading on $\text{gr}(A)$. By assigning a weight to an indeterminate $U$ — call the weight $\text{grdeg}(U)$, we can extend the grading on $\text{gr}(A)$ to $\text{gr}(A)[U]$. Given an exponential map $\varphi : A \to A[U]$ on $A$, the goal is to obtain an exponential map $\overline{\varphi}$ on $\text{gr}(A)$. For $a \in A$, let $\text{grdeg}(a)$ denote $\text{grdeg}(\overline{a})$. Note that $\text{grdeg}(\overline{a}) = i$ if and only if $a \in A_i \setminus A_{i-1}$. Consequently, $\text{grdeg}$ can also be viewed as a degree function on $A$ and on $A[U]$ once the value of $\text{grdeg}(U)$ is determined.

Define

\begin{equation}
(\ast) \quad \text{grdeg}(U) = \min \left\{ \frac{\text{grdeg}(x) - \text{grdeg}(D_i(x))}{i} \mid x \in \chi, i \in \mathbb{Z}^+ \right\}.
\end{equation}

Let us assume now that $\text{grdeg}(U)$ does exist, i.e. is a rational number. This will indeed occur whenever $\chi$ is a finite set, as will be the case with the Danielewski surfaces. If $x \in \chi$ and $n$ is a natural number, then $\text{grdeg}(D^n(x)U^n) \leq \text{grdeg}(x)$ by our choice of $\text{grdeg}(U)$. From this it follows by straightforward calculation that $\text{grdeg}(D^n(a)U^n) \leq \text{grdeg}(a)$ for all $a \in A$ and all natural numbers $n$. (Here we use the homogeneity assumption imposed on $\chi$.) The reader can easily work out the details or refer to [C]. Note that this inequality is sharp since

$$\text{grdeg}(U) = \frac{1}{n} (\text{grdeg}(x) - \text{grdeg}(D^n(x)))$$

for some $x \in \chi$ and some positive integer $n$ (and also since $D^n(a) = a$ for all $a \in A$).

For $a \in A$, let

$$S(a) = \{ n \mid \text{grdeg}(D^n(a)) + n \text{grdeg}(U) = \text{grdeg}(a) \}.$$

Define

$$\overline{\varphi}(\overline{a}) = \sum_{n \in S(a)} D^n(a)U^n$$

and extend this linearly to define $\overline{\varphi} : \text{gr}(A) \to \text{gr}(A)[U]$, the homogenization or top part of $\varphi$. One can verify that $\overline{\varphi}$ is an exponential map on $\text{gr}(A)$. Refer to [DHM] for the case $A = K^{[n]}$. The proof of the general case is symbolically identical. Let $\overline{A^\varphi}$ denote the domain generated by the top parts of all elements in $A^\varphi$. The end result is
Theorem 2.3 (H. Derksen, O. Hadas, L. Makar-Limanov [DHM]). Let \( A \) be a domain over \( k \) with \( \mathbb{Z} \)-filtration \( \{ A_n \} \) such that \( (A_i \setminus A_{i-1}) \cdot (A_j \setminus A_{j-1}) \subseteq A_{i+j} \setminus A_{i+j-1} \) for all \( i, j \in \mathbb{Z} \). Let \( \varphi \) be a nontrivial exponential map on \( A \). Assume that \( \text{grdeg}(U) \) exists as defined above. Then \( \varphi \) as defined above is a nontrivial exponential map on \( \text{gr}(A) \). Moreover, \( \overline{A^\varphi} \) is contained in \( \text{gr}(A)\overline{\varphi} \).

An important special case of homogenization is when \( A \) itself is graded. Then we can filter \( A \) so that \( \text{gr}(A) \) is canonically isomorphic to \( A \) itself, and we can choose \( \chi \) to be a set of homogeneous generators of \( A \). In this case the top part of \( \varphi \) is a nontrivial exponential map on \( A \) (assuming \( \text{grdeg}(U) \) exists).

Example 2.4. Let \( A = k[X,Y] \), where \( \text{char}(k) = p \), prime. Define \( \varphi \in \text{EXP}(A) \) by \( \varphi(X) = X \) and \( \varphi(Y) = Y + U + XU^p \). We can grade \( A \) by assigning weights \( \text{grdeg}(X) = \alpha \) and \( \text{grdeg}(Y) = \beta \) (with \( \text{grdeg}(\lambda) = 0 \) for all \( \lambda \in k^* \), and \( \text{grdeg}(0) = -\infty \)). Since \( \text{grdeg}(D^i) = -\infty \) for all \( i \geq 1 \), \( X \) will not contribute to the value of \( \text{grdeg}(U) \). Therefore,

\[
\text{grdeg}(U) = \min \left\{ \frac{\text{grdeg}(Y) - \text{grdeg}(1)}{1}, \frac{\text{grdeg}(Y) - \text{grdeg}(X)}{p} \right\} = \min \left\{ \beta, \frac{\beta - \alpha}{p} \right\}
\]

In any case, \( \overline{\varphi}(X) = X \). If \( \beta < \frac{1}{p}(\beta - \alpha) \) then \( \text{grdeg}(U) = \beta \) and \( \overline{\varphi}(Y) = Y + U \). If \( \beta = \frac{1}{p}(\beta - \alpha) \) then \( \text{grdeg}(U) = \beta \) and \( \overline{\varphi}(Y) = \varphi(Y) \). If \( \beta > \frac{1}{p}(\beta - \alpha) \) then \( \text{grdeg}(U) = \frac{1}{p}(\beta - \alpha) \) and \( \overline{\varphi}(Y) = Y + XU^p \).

3. Exponential maps of \( x^n y = z^2 + h(x)z \)

Let \( k \) be a field with characteristic \( p \geq 0 \). Let

\[
R = k[X,Y,Z]/(X^nY - Z^2 - h(X)Z),
\]

where \( n \geq 2 \) and \( h(X) \in k[X] \) with \( h(0) \neq 0 \). Assume that \( \text{deg}_X(h(X)) < n \). (If \( d = \text{deg}_X(h(X)) \geq n \) then we can replace \( Y \) by \( Y + h_0X^{d-n}Z \), where \( h_0 \) is the leading coefficient of \( h(X) \), and replace \( h(X) \) by \( h(X) - h_0X^d \). Iterating this process finitely many times, we can replace \( h(X) \) by a polynomial with \( X \)-degree smaller than \( n \).) Let \( x, y, z \in R \) denote the cosets of \( X, Y, Z \), respectively. To study the exponential maps of \( R \), we will use filtrations and homogenization of exponential maps.

Theorem 3.1. If \( \varphi : R \to R[U] \) is a nontrivial exponential map on \( R \), then \( R^2 = k[x] \). Moreover, \( z \) has minimal positive \( \varphi \)-degree and \( D^j(z) \) is divisible by \( x^n \) for each \( i \geq 1 \), so that

\[
\varphi(z) = z + x^n f_1(x)U + \sum_{i=1}^j x^n f_{pi}(x)U^{pi}
\]

for some polynomials \( f_i \) and some \( j \geq 1 \). In fact, any choice of \( j, f_1, f_p, \ldots, f_{pi} \) in this formula determines a nontrivial exponential map on \( R \).

Corollary 3.2. \( \text{AK}(R) = k[x] \).
Proof of Theorem 3.1. One easily verifies the final sentence of the theorem by applying \( \varphi \) to the relation \( x^n y = z^2 + h(x)z \) (with \( \varphi(x) = x \)), solving for \( \varphi(y) \in R \), and checking the exponential properties on the generators \( x, y, z \).

Suppose \( \varphi : R \to R[U] \) is a nontrivial exponential map on \( R \). We know that \( R^x \) is a subalgebra of \( R \) with transcendence degree 1 over \( k \). So in order to show that \( R^x = k[x] \) it suffices to show that \( x \in R^x \).

View \( R \) as a subalgebra of \( k[x,x^{-1},z] \) with \( y = x^{-n}(z^2 + h(x)z) \). Introduce a degree function \( w_f \) given by the weights \( w_1(x) = 0 \), \( w_1(y) = 2 \), and \( w_1(z) = 1 \), with \( w_1(\lambda) = 0 \) for all \( \lambda \in k^\times \) and \( w_1(0) = -\infty \), and consider the \( \mathbb{Z} \)-filtration \( \{R_i\} \) on \( R \) induced by these weights, namely \( R_i = \{ r \in R \mid w_1(r) \leq i \} \). Observe that \( \varphi = \varphi^{w_1} \). The graded domain \( \text{gr}(R) \) which corresponds to \( w_1 \) is generated by \( \varpi, \vartheta, \varphi \) and subject to the relation \( \varpi^i \vartheta = \varpi^2 \). Writing \( x, y, z \) in place of \( \varpi, \vartheta, \varphi \), respectively, we have

\[
\text{gr}(R) = k[x, y, z]/(x^2 y - z^2).
\]

Sublemma 3.3. \( R^x \subset k[x, z] \).

Proof. Suppose not. Let \( f \in R^x \) such that \( f \notin k[x, z] \). Since \( z^2 = x^2 y - h(x)z \) we can write \( f = f_1(x, y) + z f_2(x, y) \) for some polynomials \( f_1, f_2 \). Now \( \varphi \) is either \( f_1(x, y) \) or \( z f_2(x, y) \), since \( f_1(x, y) \) has even weight while \( z f_2(x, y) \) has odd weight. Therefore, \( \varphi = y^i z^j g(x) \), where \( i \) is a positive integer, \( j \) is 0 or 1, and \( g \) is some polynomial. (The number \( i \) cannot be 0 because \( y \) carries the heaviest weight, and our assumption on \( f \) is that some term of either \( f_1(x, y) \) or \( z f_2(x, y) \) must involve \( y \).) Since \( R \) is finitely generated by \( \chi = \{x, y, z\} \), it is clear that the value \( \text{grdeg}(U) \) exists as defined by formula (\ref{formula}). By Theorem 2.3 the map \( \varphi \) induces a nontrivial exponential map \( \varphi \) on \( \text{gr}(R) \) with \( \varphi \in \text{gr}(R) \varphi \). Since all factors of \( \varphi \) belong to \( \text{gr}(R) \varphi \), it follows that \( y \in \text{gr}(R) \varphi \). Since \( \text{gr}(R) \varphi \) has transcendence degree 1 over \( k \), we must have \( \text{gr}(R) \varphi = k[y] \).

Suppose \( r \) is an arbitrary element of \( R \). Just as with the element \( f \) above we have \( \varpi = \varpi^i \varphi = z^j g(x) \) for some natural numbers \( i, j \) and some polynomial \( g \). Introduce a new grading on \( \text{gr}(R) \) by the weights \( w_2(x) = -1 \), \( w_2(y) = n \), and \( w_2(z) = 0 \). This gives us a graded domain \( \text{gr}(\text{gr}(R)) \) which is naturally isomorphic to \( \text{gr}(R) \).

Let us write \( \text{gr}(R) \) in place of \( \text{gr}(\text{gr}(R)) \), and let us continue to write \( x, y, z \) in place of \( \varpi, \vartheta, \varphi \). Under these new weights the top part of element \( \varpi \) is \( \varpi = \lambda x^i y^j z^k \) for some natural numbers \( i, j, k \) and some \( \lambda \in k \). The effect of imposing \( w_2 \) on \( \text{gr}(R) \) is to refine the top parts that were obtained via \( w_1 \) in such a way that the top part of every element from \( R \) is a monomial in \( x, y, z \). The reader should be mindful that now on the elements \( x, y, z \) there are two weights: the primary weights given by \( w_1 \) and the secondary weights given by \( w_2 \). By Theorem 2.3 we obtain a new exponential map \( \varphi \) on \( \text{gr}(R) \), a “refinement” of \( \varphi \). But let us avoid this double-bar notation and use \( \varphi \) to denote the homogenization of \( \varphi \) under the primary and secondary weights \( w_1 \) and \( w_2 \).

Let \( a = \text{deg}_{\varphi}(x) \) and \( b = \text{deg}_{\varphi}(z) \). Let \( D = \{ D^m \} \) be the locally finite iterative higher derivation associated to \( \varphi \). Since \( \text{deg}_{\varphi}(y) = 0 \), the relation \( x^n y = z^2 \) indicates that \( a n = 2 b \). Applying \( \varphi \) to this relation and examining the highest power of \( U \) which appears (that being \( U^{an} = U^{2b} \)), we see that

\[
(D^a(x))^n y = (D^b(z))^2.
\]

Now \( D^a(x), D^b(z) \in \text{gr}(R) \varphi = k[y] \). Also, both \( D^a(x), D^b(z) \) are top parts of elements of \( R \) by the way in which \( \varphi \) is defined. Thus each side of the above
equation is a monomial in $y$. If $n$ is even, then the left side has odd $y$-degree while the right side has even $y$-degree, bringing us to a contradiction.

Assume that $n$ is odd. We will now argue that $x$ must have minimal positive $\tau$-degree, and then we will bring this to a contradiction. The $\tau$-degree of $x$ is $a$. To show that no element has positive $\tau$-degree smaller than $a$, we will show that $D^l$ is identically zero for $1 \leq l < a$. Because $\text{gr}(\mathbb{R})$ is generated by $x, y, z$, it suffices to check $D^l$ on these elements. Of course $D^l(y) = 0$ for all $l \geq 1$. It remains to study $x$ and $z$.

Write $D^a(x) = \lambda y^i$ for some $\lambda \in \mathbb{k}^*$ and some natural number $i$. From the above equation we see that $D^b(z) = \lambda^{n/2} y^{(i+1)/2}$. Also, since $an = 2b$ we know that $2$ divides $a$ and $n$ divides $b$. Write $a = 2k$. Then $b = nk$. By the homogeneity of $\tau$ under both weights $w_1$ and $w_2$, we know that

$$w_\tau (x) = w_\tau (D^a(x)U^n)$$

for $\tau = 1, 2$. This can be rewritten as

$$w_\tau (x) = iw_\tau (y) + 2kw_\tau (U)$$

for $\tau = 1, 2$, where $w_\tau (U)$ represents the value grdeg(U) for the appropriate grading.

Recall that $w_1(x) = 0$ and $w_1(y) = 2$. So with $\tau = 1$ the above equations indicate that

$$w_1(U) = -\frac{i}{k}.$$ 

Recall that $w_2(x) = -1$ and $w_2(y) = n$, and so with $\tau = 2$ we obtain

$$w_2(U) = -\frac{in + 1}{2k}.$$ 

Suppose now that $D^1(x) \neq 0$ for some positive integer $l$. The homogeneity of $\tau$ again means that

$$w_\tau (x) = w_\tau (D^1(x)U^1)$$

for $\tau = 1, 2$. Also, $D^l(x)$ is a monomial, because by the definition of $\tau$ it is the top part of some element from $\mathbb{R}$. Write $D^l(x) = \mu x^\alpha y^\beta z^\gamma$ for some $\mu \in \mathbb{k}^*$ and some natural numbers $\alpha, \beta, \gamma$. In fact, $\alpha$ and $\gamma$ must be zero by part (c) of Lemma 2.1, which states that $\text{deg}(D^l(x)) \leq 2k - l$. We can therefore write

$$w_\tau (x) = \beta w_\tau (y) + lw_\tau (U)$$

for $\tau = 1, 2$, which in turn becomes the system

$$0 = 2\beta - \frac{il}{k},$$

$$-1 = n\beta - \frac{inl + l}{2k}.$$ 

We solve this system for $l$ to obtain $l = 2k$. Thus $D^l(x) = 0$ when $1 \leq l < 2k$.

We now argue similarly with $z$ in place of $x$. Suppose that $D^l(z) \neq 0$ for some positive integer $l$. Again $D^l(z)$ is a monomial by the homogeneity of $\tau$. We can write $D^l(z) = \mu x^\alpha y^\beta$ for some $\mu \in \mathbb{k}^*$ and some natural numbers $\alpha, \beta$. ($z$ cannot appear as a factor in $D^l(z)$ again by part (c) of Lemma 2.1). Applying $w_1$ and $w_2$ to this presentation of $D^l(z)$ yields the system

$$1 = 2\beta - il,$$

$$0 = -\alpha + n\beta - \frac{inl + l}{2k}.$$
We solve this system for \( l \) to obtain \( l = k(n - 2\alpha) \). Therefore \( l \) must be an odd multiple of \( k \). In particular, we see that \( D^l(z) = 0 \) for \( 1 \leq l < k \) and \( k < l \leq 2k \).

Let us briefly consider two cases. First, suppose \( \text{char}(k) = p \neq 2 \). We already saw that \( D^l(x) = 0 \) for \( 1 \leq l < 2k \), and so applying \( D^k \) to the relation \( x^a y = z^2 \) yields \( 0 = 2zD^k(z) \). Hence \( D^k(z) = 0 \). We now see that \( D^l(x), D^l(y), \) and \( D^l(z) \) are identically zero for \( 1 \leq l < 2k \). Thus \( D^l \) is identically zero in that range. As previously discussed, this means \( x \) must be an element of minimal positive \( \wp \)-degree when \( p \neq 2 \). We now obtain the same conclusion for \( p = 2 \). Suppose that there does exist an element in \( \text{gr}(R) \) of minimal positive \( \wp \)-degree smaller than \( 2k \). By our analysis of \( D^l(x) \) and \( D^l(z) \), that element necessarily must have \( \wp \)-degree \( k \). By part (a) of Lemma 2.1, \( k \) must be a power of 2. Also recall that \( n \) is odd. Consequently, \( \binom{n}{k} \equiv n \equiv 1 \pmod{2} \). (We used this fact about binomial coefficients previously.

Refer to the beginning of the proof of Lemma 2.1.) Now, since \( n - 1 \) is even we know that \( D^{(n-1)k}(z) = 0 \). Thus

\[
0 = D^kD^{(n-1)k}(z) = \binom{n}{k}D^{nk}(z) = D^{nk}(z) \neq 0.
\]

With this contradiction, we now conclude that \( x \) must be an element of minimal positive \( \wp \)-degree \( 2k \) for arbitrary characteristic \( p \).

By part (b) of Lemma 2.2, we then see that \( 2k \) must divide the \( \wp \)-degree of \( z \), namely \( nk \). But this implies that \( 2 \) divides \( n \), contradicting our assumption that \( n \) is odd. This contradiction finishes the proof of the sublemma. \( \square \)

Let us continue with the proof of Theorem 3.1. Suppose now that \( f \in R^\wp \) but \( f \notin k[x] \). Since \( z^2 = x^a y - h(x)z \) and \( R^\wp \subset k[x, z] \), we can write \( f = f_1(x) + zf_2(x) \) for some polynomials \( f_1, f_2 \) with \( f_2 \neq 0 \). Once again we consider \( \text{gr}(R) \) induced by \( w_1 \) and the nontrivial exponential map \( \wp \) induced by \( \varphi \). Since \( w_1(x) = 0 \) and \( w_1(z) = 1 \), we have \( \wp = zf_2(x) \). Since \( \text{gr}(R)^\wp \) is factorially closed and \( \wp \in \text{gr}(R)^\wp \), we must have \( z \in \text{gr}(R)^\wp \). Thus \( x^a y = z^2 \in \text{gr}(R)^\wp \), and this implies that \( x, y \in \text{gr}(R)^\wp \). But this means \( \wp \) is trivial, a contradiction. Therefore \( R^\wp \) is contained in \( k[x] \). Since \( R^\wp \) is algebraically closed in \( R \), we see that \( x \in R^\wp \) and \( R^\wp = k[x] \).

Now let us check that \( z \) is an element of minimal positive \( \varphi \)-degree. Let \( s \in R \) have minimal positive degree. By part (c) of Lemma 2.2 there exists \( c \in k[x] \) such that \( R \subseteq k[x][c^{-1}][s] \). So \( R \subseteq k(x)[s] \) and in particular \( z \in k(x)[s] \). On the other hand, viewing \( y = x^{-n}(z^2 + h(x)z) \) we know that \( R \subseteq k(x)[z] \), and thus \( s \in k(x)[z] \).

This implies that \( z = as + b \) for some \( a, b \in k(x) \), and since \( \deg_z(x) = 0 \) we have \( \deg_z(z) = \deg_z(s) \).

Since \( z \) has minimal positive \( \varphi \)-degree, \( D^i(z) \in k[x] \) for all \( i \geq 1 \) by part (a) of Lemma 2.2. If \( k \geq 1 \), then

\[
x^nD^k(y) = \sum_{i=0}^{k} D^i(z)D^{k-i}(z) + h(x)D^k(z)
= 2zD^k(z) + \left( \sum_{i=1}^{k-1} D^i(z)D^{k-i}(z) + h(x)D^k(z) \right).
\]

So the right hand side of the above equation is a linear (or possibly constant) polynomial in \( z \) with coefficients in \( k[x] \), and both of these coefficients must be divisible by \( x^n \). Recall that \( h(0) \neq 0 \) by assumption, so \( x \) does not divide \( h(x) \). By induction on \( k \) we see that each \( D^k(z) \) is divisible by \( x^n \). We have checked for each
is a field with characteristic $p$.

By part (a) of Lemma 2.2, $f_i = 0$ whenever $i \geq 2$ is not a power of $p$. Thus we obtain the formula for $\varphi(z)$ given in the statement of the theorem. \hfill $\square$

Among the exponential maps on $R$ as described by Theorem 3.1 we have those given by $x \mapsto x$ and $z \mapsto z + x^nf(x)U$. When char$(k) = 0$ these are all the exponential maps on $R$. When char$(k)$ is prime, we can rewrite the formula for $\varphi(z)$ in the statement of Theorem 3.1 as

$$z \mapsto z + x^ngcd(f_i)P(x,U)$$

for a polynomial $P(x,U)$ which can be viewed as a new indeterminate $V$. Of course this change of variables will take several different exponential maps to the same new map.

4. Automorphisms of $x^ny = z^2 + h(x)z$

As in the previous section, let $R = k[X,Y,Z]/(X^nY - Z^2 - h(X)Z)$, where $k$ is a field with characteristic $p \geq 0$, $n \geq 2$, and $h(X) \in k[X]$ with $h(0) \neq 0$ and $\deg_X(h(X)) < n$. Let $x,y,z \in R$ denote the cosets of $X,Y,Z$, respectively. The objective of this section is to describe the group Aut$(R)$ of $k$-algebra automorphisms of $R$ using the results of the previous section. Let us begin with

**Lemma 4.1.** Let $\alpha \in$ Aut$(R)$. Then $\alpha(x) = \mu x$ for some $\mu \in k^*$ such that $h(\mu x) = h(x)$, and either

(a) $\alpha(z) = z + f(x)$ for some $f \in k[x]$ with $f(x) \equiv 0 \pmod{x^n}$, or

(b) $\alpha(z) = -z + f(x)$ for some $f \in k[x]$ with $f(x) \equiv -h(x) \pmod{x^n}$.

**Proof.** If $\varphi = \sum_i U^i D^i$ is an exponential map on $R$, then $\alpha^{-1} \varphi \alpha = \sum_i U^i \alpha^{-1} D^i \alpha$ is again an exponential map on $R$. Note that if $r \in R$, then the $(\alpha^{-1} \varphi \alpha)$-degree of $r$ is equal to the $\varphi$-degree of $\alpha(r)$. From this it follows that $\alpha(z)$ must be an element of minimal positive $\varphi$-degree. Indeed, if $s \in R$ has lower positive $\varphi$-degree than that of $\alpha(z)$, then

$$\deg_{\alpha^{-1} \varphi \alpha}(\alpha^{-1}(s)) = \deg_{\varphi}(s) < \deg_{\varphi}(\alpha(z)) = \deg_{\alpha^{-1} \varphi \alpha}(z).$$

By Theorem 3.1 $z$ has minimal positive $(\alpha^{-1} \varphi \alpha)$-degree. This must mean that $\deg_{\alpha^{-1} \varphi \alpha}(\alpha^{-1}(s)) \leq 0$, i.e. $\alpha^{-1}(s)$ is invariant under $\alpha^{-1} \varphi \alpha$. But then $\alpha^{-1}(s) \in k[x]$. We know that $\alpha$ restricts to an automorphism of $AK(R) = k[x]$. Thus $s \in k[x] = R^\wedge$, contradictory to our choice of $s$.

Let us fix $\varphi = \sum_i U^i D^i$ to be the exponential map on $R$ given by $\varphi(z) = z + x^nu$. Since $z$ and $\alpha(z)$ are both elements of minimal positive $\varphi$-degree 1, we have $\alpha(z) = \lambda z + f$ for some $\lambda, f \in k[x, x^{-n}]$ (refer to part (c) of Lemma 2.2). Actually, we must have $\lambda, f \in k[x]$, since the relation $y = x^{-n}(z^2 + h(x)z)$ in $R$ does not allow for negative powers of $x$ to appear in the linear polynomial $\lambda z + f$. Moreover, $\lambda \in k^*$ because $\alpha$ is invertible.
Since $\alpha$ restricts to an automorphism of $k[x]$, we have $\alpha(x) = \mu x + c$ for some $\mu \in k^*$ and some $c \in k$. Now $\deg_{\alpha^{-1}\varphi}(z) = \deg_{\varphi}(\alpha(z)) = 1$, and
\[
(\alpha^{-1}D^1\alpha)(z) = \alpha^{-1}D^1(\lambda z + f) = \alpha^{-1}(\lambda x^n) = \lambda \mu^{-n}(x - c)^n.
\]
At the same time, $(\alpha^{-1}D^1\alpha)(z)$ is divisible by $x^n$ by Theorem 3.1. Hence $c = 0$ and $\alpha(x) = \mu x$.

We are now finished exploiting $\varphi$. It remains to study how the relation $x^n y = z^2 + h(x)z$ imposes the remaining information in the statement of the lemma. Applying $\alpha$ to that relation we obtain
\[
\mu^n x^n\alpha(y) = \alpha(z)^2 + h(\mu x)\alpha(z) = \lambda^2(z^2 + h(x)z) + g(x, z),
\]
where
\[
g(x, z) = (2\lambda f(x) + \lambda h(\mu x) - \lambda^2 h(x))z + (f(x)^2 + h(\mu x)f(x)).
\]
So
\[
\alpha(y) = \lambda^2 \mu^{-n} x^{-n}(z^2 + h(x)z) + \mu^{-n} x^{-n} g(x, z).
\]
Hence $x^{-n}g(x, z) \in R$. Now $g(x, z)$ is linear as a polynomial in $z$ with coefficients in $k[x]$, so $x^{-n}g(x, z)$ cannot have negative $x$-degree. (Again, remember that negative powers of $x$ can only appear in $R$ when an expression involves $z^2$ or higher powers of $z$.) Thus $x^n$ must divide $g(x, z)$, and this means that $x^n$ must divide each coefficient of $g(x, z)$ in $k[x]$:
\[
\begin{align*}
(f(x)^2 + h(\mu x)f(x)) & \equiv 0 \pmod{x^n}, \\
(2\lambda f(x) + \lambda h(\mu x) - \lambda^2 h(x)) & \equiv 0 \pmod{x^n}.
\end{align*}
\]
To restate equation (4.1), we know that $x^n$ divides $f(x)(f(x) + h(\mu x))$. In the following cases we will demonstrate that $x^n$ must divide either $f(x)$ or $f(x) + h(\mu x)$. This piece of information allows us to complete the lemma.

Case (a). Suppose $f(0) = 0$. Then
\[
f(0) + h(\mu \cdot 0) = h(0) \neq 0,
\]
so $x$ does not divide $f(x) + h(\mu x)$, and according to equation (4.1) $x^n$ must divide $f(x)$. By equation (4.2),
\[
\lambda h(\mu x) - \lambda^2 h(x) \equiv 0 \pmod{x^n}.
\]
Since $\deg_x(h) < n$, we must have
\[
\lambda h(\mu x) - \lambda^2 h(x) = 0.
\]
Setting $x = 0$ in this equation, we see that $\lambda = 1$, and hence $h(\mu x) = h(x)$. We now have $\alpha(z) = z + f(x)$ and the conditions of the lemma are satisfied.

Case (b). Suppose $f(0) \neq 0$. Then $x$ does not divide $f(x)$, and so $x^n$ must divide $f(x) + h(\mu x)$ by equation (4.1). Substituting
\[
f(x) \equiv -h(\mu x) \pmod{x^n}
\]
in equation 4.2, we obtain
\[-\lambda h(\mu x) - \lambda^2 h(x) \equiv 0 \pmod{x^n}.
\]
So
\[-\lambda h(\mu x) - \lambda^2 h(x) = 0
\]
since \(\deg_x(h) < n\). Setting \(x = 0\) in this equation yields \(\lambda = -1\), and then \(h(\mu x) = h(x)\). So \(f(x) \equiv -h(x) \pmod{x^n}\) and \(\alpha(z) = -z + f(x)\) as in part (b) of the lemma.

We are now in position to prove

**Theorem 4.2.** The group \(\text{Aut}(R)\) is generated by

(a) the automorphisms \(E_f\) given by
\[
E_f(x) = x,
E_f(y) = y + 2f(x)z + x^n f(x)^2 + f(x)h(x),
E_f(z) = z + x^n f(x),
\]
where \(f \in k^{[1]}\),
(b) the automorphism \(T\) given by
\[
T(x) = x, \\
T(y) = y, \\
T(z) = -z - h(x),
\]
(c) and, if \(h(x) = h_1(x^m)\) for some \(h_1 \in k^{[1]}\) and some \(m \in \mathbb{N}\), the linear automorphisms \(L_\mu\) given by
\[
L_\mu(x) = \mu x, \\
L_\mu(y) = \mu^{-n} y, \\
L_\mu(z) = z,
\]
where \(\mu \in k\) such that \(\mu^m = 1\).

**Proof.** The map \(E_f\) in (a) is obtained by evaluating \(U = 1\) in the exponential map given by \(\varphi(z) = z + x^n f(x)U\). It is easy to check that all of the maps in (b) and (c) are indeed automorphisms. The conditions \(h(x) = h_1(x^m)\) and \(\mu^m = 1\) in (c) describe the only possible way that \(h(\mu x) = h(x)\) for \(\mu \neq 1\), as in Lemma 4.1. If \(m = 1\) then (c) describes only the identity automorphism, and if \(m = 0\) then \(h\) is constant and \(\mu\) can be any element of \(k^*\). Automorphisms of the forms \(L_\mu E_f\) and \(L_\mu T E_f\) cover all possible maps described in Lemma 4.1. □

The set \(L\) of automorphisms \(L_\mu\) in (c) is an abelian subgroup of \(\text{Aut}(R)\). Let \(H\) be the subgroup generated by \(L\) and \(T\). \(H\) is an abelian group, the internal direct product \((T) \times L\). Let \(N\) be the set of automorphisms \(E_f\) in (a). \(N\) is a normal subgroup of \(\text{Aut}(R)\), and \(\text{Aut}(R)\) is the semi-direct product \(N \rtimes H\).

\(N\) is isomorphic to \(k^{[1]}\) as an additive group, and \((T)\) is cyclic of order 2. Turning to the conditions of (c), we see that \(L\) is trivial if \(m = 1\) and isomorphic to \(k^*\) if \(m = 0\). Otherwise \(L\) is a cyclic group of order dividing \(m\). (If \(k\) is algebraically closed then \(L\) has order \(m\).) Let \(C_k\) denote the cyclic group of order \(k\). To summarize:

**Corollary 4.3.** \(\text{Aut}(R) \cong k^{[1]} \rtimes H\), where \(H\) is as follows.
1. If $h$ is a constant polynomial, then $H \cong C_2 \times \mathbb{k}^*$.

2. If $h(x) = h_1(x^m)$ for some $m > 1$, then $H \cong C_2 \times C_k$ for some factor $k$ of $m$.

3. Otherwise (for a “typical” polynomial $h$), $H \cong C_2$.

5. REMARKS ON THE CANCELLATION PROBLEM

For $i = 1, 2$ let $R_i = \mathbb{k}[X, Y, Z]/(X^n Y - h_i(X)Z)$, where $n_i \geq 2$ and $h_i(0) \neq 0$. As mentioned in the introduction, these algebras (when $\mathbb{k} = \mathbb{C}$) are known to be a class of counterexamples to the cancellation problem. That is, $R_1^{[1]} \cong R_2^{[1]}$, while in general $R_1 \ncong R_2$. This has been explained geometrically [D, W], but let us briefly provide an algebraic explanation. To show the isomorphism of polynomial rings, we can try the following approach. Embed $R_1$ in $R_2[T]$, and then find an exponential map $\phi : R_2[T] \to R_2[T][U]$ with ring of invariants $R_1$ and such that $\phi(s) = s + U$ for some $s$. This will imply that $R_2[T] = R_1[s]$ by part (c) of Lemma 2.2 (because, in the notation of that lemma, we will have $c = 1$). The element $s$ is commonly called a slice.

Here are the formulae for a special case. Assume that $n_1 < n_2 \leq 2n_1$ and that $h_1(X) = h_2(X) = 1$. $\mathbb{k}$ can be any field. Let $x_i, y_i, z_i$ denote the cosets of $X, Y, Z$ in $R_i$, respectively. So for $i = 1, 2$ we have algebras $R_i$ generated by $x_i, y_i, z_i$ subject to the relations

\begin{align*}
(5.1) & \quad x_1^{n_1}y_1 = z_1^2 + z_1, \\
(5.2) & \quad x_2^{n_2}y_2 = z_2^2 + z_2.
\end{align*}

Embed $R_1$ in $R_2$ by sending $x_1$ to $x_2$, $z_1$ to $z_2$, and $y_1$ to $x_2^{n_2-n_1}y_2$. Let $\tilde{R}_1$ denote the image of $R_1$ in $R_2$. By Theorem 3.1 we have an exponential map on $\tilde{R}_1$ defined by sending $x_2$ to $x_2$ and $z_2$ to $z_2 + x_2^{n_1}T$, where $T$ is the indeterminate which parameterizes the exponential map. Of course exponential maps are injective, and the composition of the embedding $R_1 \hookrightarrow \tilde{R}_1$ with the exponential map on $\tilde{R}_1$ gives us an embedding of $R_1$ in $R_2[T]$ given by

\begin{align*}
x_1 & \mapsto x_2, \\
z_1 & \mapsto z_2 + x_2^{n_1}T, \\
y_1 & \mapsto x_2^{n_2-n_1}y_2 + (2z_2 + 1)T + x_2^{n_1}T^2.
\end{align*}

Let us identify $R_1$ with its image under this embedding, yielding $x_1 = x_2 = x$ and

\begin{align*}
(5.3) & \quad z_1 = z_2 + x^nT, \\
(5.4) & \quad y_1 = x^{n_2-n_1}y_2 + (2z_1 + 1)T - x^{n_1}T^2.
\end{align*}

Relations (5.1), (5.2), (5.3), and (5.4) completely describe $R_2[T]$. In fact, relations (5.2) and (5.3) are unnecessary since from (5.1) and (5.4) we recover the relation

\[ x^{n_2}y_2 = (z_1 - x^nT)^2 + (z_1 - x^nT). \]

This means $R_2[T]$ is generated by $x, y_1, y_2, z_1, T$ and subject to the relations (5.1) and (5.4). Define $\phi : R_2[T] \to R_2[T][U]$ by $\phi(x) = x$, $\phi(y_1) = y_1$, and $\phi(z_1) = z_1$, with

\begin{align*}
\phi(y_2) & = y_2 + (2z_1 - 2x^nT + 1)U + x^{n_2}U^2, \\
\phi(T) & = T - x^{n_2-n_1}U.
\end{align*}
One can easily observe that $\varphi$ is an exponential map on $R_2[T]$ with ring of invariants $R_1$ by checking the exponential properties on the generators. Moreover one can verify that $\varphi(s) = s + U$, where
\[
s = -4x^{3n_1-n_2}T^3 + 3x^{2n_1-n_2}(2z_1 + 1)T^2 + 4x^{n_1}y_2T + y_2(2z_1 + 1).
\]
Consequently, $R_2[T] = R_1[s]$ by part (c) of Lemma 2.2. Because $T$ is also an element of minimal positive $\varphi$-degree 1, that same lemma leads us to expect that $s$ should be a linear polynomial in $T$ with coefficients in $R_1[x^{n_1-n_2}]$, and indeed one can check that
\[
s = -x^{n_1-n_2}(T - y_1(2z_1 + 1)).
\]
To conclude, let us prove the following theorem which shows that two Danielewski surfaces are in general not isomorphic.

**Theorem 5.1.** Let $k$ be a field. For $i = 1, 2$ let $R_i = k[X, Y, Z]/(X^n_iY - h_i(X)Z)$, where $n_i \geq 2$ and $h_i(0) \neq 0$. Let $x_i, y_i, z_i$ denote the cosets of $X, Y, Z$ in $R_i$, respectively. Then $R_1 \cong R_2$ if and only if $n_1 = n_2$ and $h_2(x) = \eta h_1(\mu x)$ for some $\eta, \mu \in k^\ast$.

**Proof.** ($\Rightarrow$) The map given by $x_1 \mapsto \mu x_2$, $y_1 \mapsto \eta^{-1} \mu^{-n} y_2$, $z_1 \mapsto \eta^{-1} z_2$ clearly defines an isomorphism of $R_1$ onto $R_2$, where $n = n_1 = n_2$.

($\Leftarrow$) We can again assume that $\deg_{x_i}(h_i(x_i)) < n_i$, $i = 1, 2$. We proceed as in Lemma 4.1 and the reader should refer to the arguments made there. Suppose $\alpha : R_1 \to R_2$ is an isomorphism. If $\varphi \in \mathrm{EXP}(R_2)$, then $\alpha^{-1} \varphi \alpha \in \mathrm{EXP}(R_1)$. Consequently, as in the proof of Lemma 4.1 we can conclude that $\alpha(x_1) = \mu x_2$ and $\alpha(z_1) = \lambda z_2 + f(x_2)$ for some $\lambda, \mu \in k^\ast$ and some polynomial $f_2$. Applying $\alpha$ to the relation $x_1^{n_1} y_1 = z_1^n + h_1(x_1) z_1$ we see that $x_2^{n_1} \alpha(y_1)$ is a polynomial in $x_2$ and $z_2$ with $z_2$-degree 2. The relation $y_2 = x_2^{n_2}(z_2^2 + h_2(x_2)z_2)$ on $R_2$ will not allow us to divide by $x_2^{n_1}$ unless $n_1 \leq n_2$. Repeating this analysis with $\alpha^{-1}$ we also must have $n_2 \leq n_1$, so that $n_1 = n_2 = n$. Next, just as in the proof of Lemma 4.1 we obtain
\[
f(x_2)^2 + h_1(\mu x_2) f(x_2) \equiv 0 \pmod{x_2^n},
\]
\[
2\lambda f(x_2) + \lambda h_1(\mu x_2) - \lambda^2 h_2(x_2) \equiv 0 \pmod{x_2^n}.
\]
Recall now that $\deg_{x_i}(h_i(x_i)) < n$ for $i = 1, 2$. Continuing as with Lemma 4.1 we consider two possibilities. If $f(0) = 0$ then we find that $h_2(x_2) = \lambda^{-1} h_1(\mu x_2)$, and the conditions of the theorem are satisfied with $\eta = \lambda^{-1}$. If $f(0) \neq 0$ we conclude that $h_2(x_2) = -\lambda^{-1} h_1(\mu x_2)$, and again we are done with $\eta = -\lambda^{-1}$. \hfill $\square$

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**Department of Mathematics, Wayne State University, Detroit, MI 48202, USA**

**E-mail address:** crach@math.wayne.edu