Waves and instabilities in dissipative rotating superfluid neutron stars

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Accepted 2007 December 4. Received 2007 December 4; in original form 2007 June 6

ABSTRACT

We discuss wave propagation in rotating superfluid neutron star cores, taking into account the vortex-mediated mutual friction force. For models where the two fluids corotate in the unperturbed state, our analysis clarifies the role of chemical coupling and entrainment for sound and inertial waves. We also investigate the mutual friction damping, providing results that demonstrate the well-known fact that sound waves propagating along a vortex array are undamped. We show that the same is not true for inertial waves, which are damped by the mutual friction regardless of the propagation direction. We then include the vortex tension, which arises due to local vortex curvature. Focusing on purely transverse inertial waves, we derive the small correction that the tension induces in the wave frequency. Finally, we allow for a relative linear flow in the background (along the rotation axis). In this case we show how the mutual friction coupling may induce an instability in the inertial waves. We discuss the critical flow required for the instability to be present, its physical interpretation and the possible relevance it may have for neutron star physics.

Key words: hydrodynamics – instabilities – stars: neutron.

1 INTRODUCTION

We do not (yet) have a direct way to probe the interior of a neutron star. There is some hope that this may change with the advent of gravitational wave astronomy, but our current understanding relies on indirect inferences from electromagnetic data. In some cases we have upper limits on the temperature, which can be combined with an estimated age to yield information about neutron star cooling. This in turn depends on the interior physics, e.g. whether the core contains superfluid components or not (Page et al. 2004). Other evidence comes from the way that the neutron star interacts with its environment, e.g. how the magnetosphere affects the spin-down of an isolated radio pulsar. It is, however, much harder to link this information to the properties of the interior. It is also difficult to draw definite conclusions about the nature of the core fluid from bulk quantities such as mass and radius.

The currently most potent tests of the theoretical possibilities are provided by observed crust oscillations in the tails of magnetar giant flares (Strohmayer & Watts 2006), and glitches in the spin-down of radio pulsars (Lyne, Shemar & Smith 2000). It seems plausible that the crust motion in the magnetar events will to some extent depend on the core physics, e.g. whether the magnetic field penetrates the core or not (Glampedakis, Samuelsson & Andersson 2006; Levin 2007). Meanwhile, the glitches remain the strongest indication that the core contains partially decoupled superfluid components that may (probably following the onset of some instability) transfer angular momentum to the crust. In order to improve our models of these events we need to understand the dynamics of large-scale superfluid systems. One key question that must be addressed if we want to be able to compare our models to real data concerns how energy is dissipated in the system. Consider the (relatively simple!) case where the internal fluid is a mixture of superconducting protons, electrons and superfluid neutrons. The equations of motion for such multiconstituent fluids have been formulated (Prix 2004) and constrained to a three fluid model (Andersson & Comer 2006). The three fluids are neutrons, entropy and charged particles. It has recently been argued that for this system there are 19, more or less unknown, dissipation coefficients (Andersson & Comer 2006). This problem is clearly much more intricate than the standard single-fluid case, where one need only worry about shear and bulk viscosities.

A natural way to gain insight into the nature of a fluid system is to carry out a local analysis of wave propagation. Such a plane-wave study should provide a better understanding of energy dissipation in the system, and could perhaps also help constrain the different parameters. Given our current understanding it is natural to divide this effort into a number of steps. This first study is focused on the two-constituent model which applies in the low-temperature limit. In this limit, we know from results for superfluid helium that the main dissipation mechanism is due to the presence of rotational vortices. The vortices induce a mutual friction between the two constituents. We have recently discussed the associated force for neutron stars (Andersson, Sidery & Comer 2006), including the important effect of entrainment. The entrainment is due to the...
The amplitudes $\bar{v}_i$ and $\bar{\rho}$ will be taken to be constant throughout the analysis. With these assumptions, the continuity equation becomes
\[ \omega \bar{\rho} + \rho k_j \bar{v}^j = 0, \tag{5} \]
where $\rho$ now represents the background density.

To perturb the equations of motion we assume a one-parameter equation of state. Representing the equation of state by an energy functional $E = E(\rho)$ we then have
\[ \bar{\mu} = \frac{\delta E}{\delta \rho} \longrightarrow \delta \bar{\mu} = \frac{\delta \bar{E}}{\delta \rho} = c_s^2 \frac{1}{\rho} \delta \bar{\rho}, \tag{6} \]
where the sound speed is defined as
\[ c_s^2 = \frac{\rho}{\rho} \frac{\partial E}{\partial \rho}. \tag{7} \]
It now follows that
\[ i \omega \bar{v}_i + 2 \epsilon_{ijk} \Omega^k \bar{v}^j + i k_i c_s^2 \frac{1}{\rho} \bar{\rho} = 0. \tag{8} \]
Substituting the continuity equations (5) into (8) we finally find
\[ i \omega \bar{v}_i + 2 \epsilon_{ijk} \Omega^k \bar{v}^j - i c_s^2 k_j \bar{v}^j \omega s = 0. \tag{9} \]

2.2 Finding the dispersion relation
The derivation of the dispersion relation is straightforward. Since it is preferable to work with scalar equations, we first contract equation (9) with $k^i$. Rearranging the result we have
\[ i(k^i \bar{v}_i) \left( \omega - c_s^2 k^2 \right) - 2 \Omega^j \epsilon_{ijk} \bar{v}^k = 0. \tag{10} \]
Contracting equation (9) with $\Omega^i \epsilon^{ijk} k_j$ then leads to
\[ i \omega \left( \epsilon_{ijk} k^j \bar{v}^i \right) + 2 \Omega^j \epsilon_{ijk} \bar{v}^k = 0 \tag{11} \]
Finally, contracting equation (9) with $\Omega_i^j$ gives
\[ \bar{v}_j \Omega^j = \frac{c_s^2}{\omega} \left( k_j \Omega^j \right) \left( \bar{v}_j k^j \right). \tag{12} \]
Defining the angle $\theta$ such that $k_j \Omega^j = k \Omega \cos \theta$ we can now use equations (11) and (12) to substitute for $\epsilon_{ijk} k^j \bar{v}^k$ in equation (10). Thus we get
\[ (\bar{v}_j k^j) \left[ \left( \omega - c_s^2 k^2 \right) + \frac{2 \Omega^2}{\omega} \left( 1 - \cos^2 \theta \frac{k^2 c_s^2}{\omega^2} \right) \right] = 0. \tag{13} \]
Provided that the wave is not purely transverse, in which case we would have $\bar{v}_j k^j = 0$, we arrive at the dispersion relation
\[ \omega = c_s^2 \left( c_s^2 + 4 \Omega^2 \right) + 4 \Omega^2 \cos^2 \theta k^2 c_s^2 = 0. \tag{14} \]
In the limit of slow rotation, this quartic in $\omega$ has approximate solutions,
\[ \omega \approx \pm c_s k \left( 1 + \frac{\Omega^2}{c_s^2 k^2} \sin^2 \theta \right) \approx \pm c_s k, \tag{15} \]
\[ \omega \approx \pm 2 \Omega \cos \theta. \tag{16} \]
These are the well-known results for sound waves and inertial waves, respectively.

Before we proceed, we need to consider whether the system may admit purely transverse waves. This is important since equation (14) does not apply when $\bar{v}_j k^j = 0$. In this case, we first of all see from equation (12) that we must also have $\Omega^j$ parallel to the wavevector.
\( k' \). The corresponding dispersion relation follows easily by taking the cross product of the Euler equation (9) with \( \Omega' \). This leads to

\[
\left( \omega - 4 \frac{\Omega'}{\omega} \right) \bar{v}_i = 0. \tag{17}
\]

Hence we have \( \omega = \pm 2\Omega \). In other words, we obtain the \( \theta \to 0 \) limit for the inertial waves. Thus we learn that, in this limit, the inertial waves are purely transverse. By returning to the perturbed Euler equations, and representing the solution in a Cartesian coordinate system where the \( z \)-axis is aligned with the wavevector, we see that we must have \( \bar{v}_i = \pm i \bar{v}_y \). This solution represents a helical wave, and obviously satisfies \( \bar{v}_j \bar{v}'_j = 0 \).

3 THE TWO-FLUID PROBLEM

We want to extend the plane-wave analysis to the two-fluid problem. In principle, we then expect a doubling of the number of solutions. In addition, we are interested in the new features that become relevant when we are dealing with a multifluid situation. Our focus will be on the entrainment, which represents a non-dissipative coupling between the fluids, and the mutual friction, which (in the most often considered form) represents dissipation due to electrons scattering off of the neutron vortices.

3.1 Formulation

As we want to account for vortex effects it is natural to consider the equations of motion in a rotating frame. We will assume that the background equilibrium is such that the two fluids rotate together. This situation, which would correspond to the two fluids being in chemical equilibrium, is slightly simplified which keeps the analysis manageable. Having said that, it is worth emphasizing that we also need to understand what happens when there is a velocity difference in the background. Since relative rotation is required for the standard explanation of pulsar glitches, one would in principle expect the situation if one considers the dissipative scattering of electrons off the magnetic field of the vortex core (in an otherwise non-magnetic neutron star core). As discussed by, for example, Prix (2004) and Andersson & Comer (2006), the required equations of motion can be written as (again, in a frame rotating uniformly with angular velocity \( \Omega' \))

\[
\left( \frac{\partial}{\partial t} + v_i \nabla \right) \tilde{p}_i^s + \epsilon_{ikl} w_{jk}^s \nabla_i v_k^s = \nabla_i (\Phi + \tilde{\mu}_s) + 2\epsilon_{ijk} \Omega' v_k^s = f_i^s, \tag{18}
\]

where

\[
\tilde{p}_i^s = v_i^s + \epsilon_{ikl} w_{jk}^s = \frac{1}{m_s} p_i^s, \tag{19}
\]

and \( p_i^s \) represents the momentum. The constituent indices \( x \) and \( y \) (\( x \neq y \)) label the fluids, and will be either \( n \) or \( p \) in the following. The former represents the superfluid neutrons and the latter a charge neutral conglomerate of protons and electrons. The relative velocity is denoted by \( w_{ij}^s = v_i^s - v_j^s \). Mass conservation requires that

\[
\frac{\partial \rho_s}{\partial t} + \nabla_j (\rho_s v_j^s) = 0. \tag{20}
\]

The force \( f_i^s \) on the right-hand side of equation (18) represents a dynamical coupling between the two fluids. Here we will focus on the vortex-mediated mutual friction. Then, the form of the force follows from balancing a resistive drag force (per unit length),

\[
f_i^s = \mathcal{R} (v_i^p - v_i^n), \tag{21}
\]

acting on a neutron vortex (assumed to move with velocity \( v_i^p \)) and the standard Magnus force due to the neutron superfluid flowing past the vortex, see Andersson et al. (2006). Assuming that the vortex array is straight, this leads to

\[
f_i^s = \frac{n_s}{\rho_s} B^2 \epsilon_{ijk} k_i w_{jk}^s + \frac{n_s}{\rho_s} B \epsilon_{ijk} k_i \epsilon_{klm} k_j w_{lm}^s. \tag{22}
\]

In this expression a hat represents a unit vector, and \( n_s \) is the vortex number density per unit area. At the macroscopic level we have (Andersson et al. 2007)

\[
n_s \kappa_i = \epsilon_{ijk} \nabla_i \hat{p}_s^k. \tag{23}
\]

For a straight vortex array, representing bulk rotation, we would have

\[
n_s \kappa_i = 2 \Omega_i^p + 2 \tilde{\epsilon}_s (\Omega_i^p - \Omega_i^n), \tag{24}
\]

For our chosen background configuration the two fluids rotate together, so the second term in the above expression vanishes.

Relating back to the resistive drag (equation 21), the dimensionless coefficients \( B \) and \( B' \) follow from

\[
B = \frac{\mathcal{R}}{1 + \mathcal{R}s}, \quad \text{and} \quad B' = \frac{\mathcal{R}^2}{1 + \mathcal{R}s}, \tag{25}
\]

where \( \mathcal{R} = \mathcal{R}/\rho_s \kappa \). In our various examples, we will focus on the weak drag problem. That is, we assume that \( \mathcal{R} \) is small, and as a result we can neglect the \( B' \sim B^2 \) terms. This would be the situation if one considers the dissipative scattering of electrons off the magnetic field of the vortex core (in an otherwise non-magnetic neutron star core). As discussed by, for example, Andersson et al. (2006), one would then typically have \( \mathcal{R} \approx 10^{-4} \). It has also been suggested that the weak drag assumption is relevant when the core protons form a type I superconductor (Sedrakian 2005). On the other hand, it may well be that it is the strong drag regime that is relevant. Strong coupling between the neutron and proton fluids is thought to originate from the interaction of vortices with the much more numerous magnetic flux tubes in a type II superconductor, see e.g. Ruderman, Zhu & Chen (1998) and Link (2003). In this picture, each vortex is ‘pinned’ to a large number of flux tubes. As a consequence it tends to be forced to move with the proton fluid. Provided that the flux tubes move with the protons, this situation can also be represented by a force of form equation (22). One would still expect to have \( B' \ll 1 \), but the \( B' \sim 1 \) contribution can no longer be neglected.

Later, when discussing the nature of the various waves in the two-fluid system, we will focus on the weak drag case. This is simply because it allows us to omit the first term in equation (22). As already indicated, this may not be a valid representation of a ‘realistic’ neutron star core, although there are good reasons to believe that there will exist regions in the star where the model applies. Basically, we are keen to simplify the analysis. Moreover, we do not expect the results to be qualitatively different in the strong drag regime.\(^1\)

\(^1\) In fact, a more recent analysis (Glampedakis, Andersson & Jones 2007) in the context of neutron star free precession has shown that this expectation is correct.
3.2 Plane waves

We now consider small perturbations away from a background where the two fluids are at rest in the rotating frame. That is, we use

$$\frac{\partial \delta \rho_x}{\partial t} + \nabla \cdot (\rho_x \delta v_x) = 0.$$  \hspace{1cm} (26)

Making also the Cowling approximation ($\delta \Phi = 0$), the perturbed equations of motion become

$$\frac{\partial \delta v_x}{\partial t} + \epsilon \delta u_{yz} = \nabla \cdot \delta \rho_x + 2 \epsilon j_{ijk} \Omega^k / \delta v_x = \delta f_x,$$  \hspace{1cm} (27)

where

$$\delta f_x = \frac{\partial}{\partial t} \left[ n_x B \epsilon_{ijk} \delta u_{iky}^x \right] + \frac{\partial}{\partial t} \left[ n_x B \epsilon_{ijk} \delta \rho_x \right] / \Omega, \delta u_{yz}.$$  \hspace{1cm} (28)

It should be noted that setting the background rotation rates equal greatly reduces the complexity of the perturbed mutual friction force.

We next assume that the perturbations can be represented by plane waves. That is, we have

$$\delta \rho_x = \rho_x e^{i \omega t + k_x x + k_y y + k_z z},$$

$$\delta v_x = v_x e^{i \omega t + k_x x + k_y y + k_z z},$$  \hspace{1cm} (29)

where $v_x$ and $\rho_x$ are assumed to be constant (varying on length-scales much longer than the wavelength). The continuum equations then become

$$k_x v_x = \frac{\rho_x}{\rho_x} \omega.$$ \hspace{1cm} (30)

Finally, to make progress we need a representation of the equation of state. In this formalism (Prix 2004; Andersson & Comer 2006), we need to provide an energy functional $E$ from which the chemical potentials follow according to

$$\delta \rho_x = \frac{\partial E}{\partial \rho_x |_{\rho_x, \rho_y}}.$$ \hspace{1cm} (31)

The entrainment $\epsilon_x$ is similarly determined as

$$\epsilon_x = \frac{2 \alpha}{\rho_x}, \text{ where } \alpha = \frac{\partial E}{\partial u_{yz}^x |_{\rho_x, \rho_y}}.$$ \hspace{1cm} (32)

The basic idea is that if the system is isotropic, then the energy functional can be constructed from the various scalars that can be formed from the dynamical variables $\rho_x$ and $v_x$. This means that one would generally expect to have

$$\delta \rho_x = \frac{\partial \delta \rho_x}{\partial \rho_x} \delta \rho_x + \frac{\partial \delta \rho_x}{\partial \rho_y} \delta \rho_y + \frac{\partial \delta \rho_x}{\partial u_{yz}^x} \delta u_{yz}^x.$$ \hspace{1cm} (33)

In the present case, where the two fluids move together in the background, the last term will vanish since $\delta u_{yz}^x = 2 u_{yz}^x \delta u_{yz}^x = 0$. Hence, we can use

$$\delta \rho_x = \frac{\partial \delta \rho_x}{\partial \rho_x} \delta \rho_x + \frac{\partial \delta \rho_x}{\partial \rho_y} \delta \rho_y + \delta \rho_x \delta \rho_y + \delta \rho_x \delta \rho_y.$$ \hspace{1cm} (34)

Later, when we allow for relative flow in the background, this form for the perturbed chemical potential will no longer be generally valid. In order to simplify the analysis, we will use it nevertheless. In practice, this amounts to assuming that the equation of state is ‘separable’ in the sense that it can be written as

$$E = f(\rho_x, \rho_y) + g(u_{yz}^x).$$ \hspace{1cm} (35)

Under the above assumptions, the momentum equations become

$$i \omega \left( \tilde{v}_x^i + \epsilon \tilde{w}_{yx}^i \right) - \delta \rho_x \left( \tilde{\rho}_{xx} \tilde{v}_x + \tilde{\rho}_{xy} \tilde{v}_y + 2 \epsilon j_{ij} \Omega^k \right) = \frac{\partial}{\partial \rho_x} \left[ B \epsilon_{ijk} \Omega^k e^{ikm} \Omega^l \tilde{w}_{mx}^y \right].$$ \hspace{1cm} (36)

After using the perturbed continuity equation (30) and rearranging we arrive at

$$\tilde{v}_m^x \left[ i \omega (1 - \epsilon_x) \delta_{ij}^m + 2 \epsilon j_{ij} \Omega^l / \Omega^k \right] - \frac{\partial}{\partial \rho_x} \left[ B \epsilon_{ijk} \Omega^k e^{ikm} \Omega^l \right] - \frac{\partial}{\partial \rho_x} \left[ B \epsilon_{ijk} \Omega^k e^{ikm} \Omega^l \right]
+ \tilde{v}_m^x \left[ i \omega \epsilon_{ij}^m + 2 \epsilon j_{ij} \Omega^l / \Omega^k \right]
+ \tilde{v}_m^x \left[ i \omega \epsilon_{ij}^m + 2 \epsilon j_{ij} \Omega^l / \Omega^k \right]
= 0.$$ \hspace{1cm} (37)

Let us now introduce the speeds of sound as (Andersson & Comer 2001)

$$c_s^2 = \frac{\rho_x}{\rho_x} \tilde{\rho}_{xx},$$ \hspace{1cm} (38)

and represent the ‘chemical coupling’ by

$$C_x = \rho_x \tilde{\rho}_{xy} = \rho_x \tilde{\rho}_{yx}.$$ \hspace{1cm} (39)

Since the partial derivatives commute we have

$$C_p = \frac{\rho_x}{\rho_x} c_p.$$ \hspace{1cm} (40)

Then we arrive at

$$\tilde{v}_m^x \left[ i \omega (1 - \epsilon_x) \delta_{ij}^m + 2 \epsilon j_{ij} \Omega^l / \Omega^k \right] - \frac{\partial}{\partial \rho_x} \left[ B \epsilon_{ijk} \Omega^k e^{ikm} \Omega^l \right]
+ \tilde{v}_m^x \left[ i \omega \epsilon_{ij}^m + 2 \epsilon j_{ij} \Omega^l / \Omega^k \right]
+ \tilde{v}_m^x \left[ i \omega \epsilon_{ij}^m + 2 \epsilon j_{ij} \Omega^l / \Omega^k \right]
= 0.$$ \hspace{1cm} (41)

The dispersion relation for wave propagation in the general two-fluid system is encoded in this equation. It is, in principle, straightforward to derive it using the same strategy as in the single-fluid problem (cf. Section 2.2). However, in reality it is still a messy problem. We have six scalar equations and the dispersion relation can be written in terms of the determinant of a $6 \times 6$ matrix. An alternative is to take some further steps. This calculation is provided in Appendix A, where we introduce useful matrix notation to make the equations more compact. The final result is a $2 \times 2$ matrix problem, represented by equation (A20), for which the determinant can be easily evaluated. This leads to the general dispersion relation, including entrainment, rotation and mutual friction.

In order to understand the role of the different parameters in the problem, we will now consider various particular cases of this general result.

4 ILLUSTRATIVE EXAMPLES

In the previous section we wrote down all the relations we need to derive the general dispersion relation for the two-fluid problem, and the derivation was completed in Appendix A. It is clear that, since the generic dispersion relation is a high-order polynomial in $\omega$, this problem is very rich. In order to understand the solutions it is useful to consider a sequence of increasingly complex model situations, represented by different forms for the energy functional $E$ (the ‘equation of state’). This will give us a feeling for how the various parameters in the problem affect the wave propagation.

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4.1 No rotation, coupling or friction

It is natural to start with the very simplest case, with the two fluids completely decoupled. This model corresponds to an equation of state of form

\[ E = f(n_a) + g(n_p). \] (42)

This leads to \( \mu_{xy} = 0 \) and \( \epsilon_x = 0 \), the two fluids are not coupled either chemically or by entrainment. If we also assume that there is no background rotation or friction in the system, i.e. let \( \Omega = \mathcal{B} = \mathcal{B}' = 0 \), then the dispersion relation follows from the determinant (cf. equation A20)

\[ \left| \omega I - \frac{k^2}{\omega} \hat{\mu} \right| = 0, \] (43)

where \( I \) is the identity matrix and

\[ \hat{\mu} = \begin{pmatrix} C_a^2 & C_p \\ C_a & C_p^2 \end{pmatrix}. \] (44)

With \( C_a = 0 \), the determinant expands to

\[ \omega^2 \left( 1 - \frac{k^2}{\omega^2} c_a^2 \right) \left( 1 - \frac{k^2}{\omega^2} c_p^2 \right) = 0, \] (45)

which has the non-trivial solutions

\[ \omega^2 = k^2 c_a^2 \] and \( \omega^2 = k^2 c_p^2. \) (46)

Hence, we have the anticipated result that the system only supports sound waves,

\[ \omega = \pm k c_a, \]
\[ \omega = \pm k c_p. \] (47)

It is also easy to show that these waves are longitudinal, as one would expect.

4.2 Including entrainment

We can now investigate how various coupling mechanisms modify these waves. Let us first consider the entrainment. Then we need an equation of state that depends on the relative velocity. Thus, we assume that

\[ E = f(n_a) + g(n_p) + h \left( w_{xy}^2 \right). \] (48)

This is obviously not the general case, but since we want to be able to analyse the problem analytically it is natural to restrict ourselves to this class of ‘separable’ models. It is straightforward to study more generic situations numerically, but the results should not differ qualitatively from the ones we discuss here.

Still assuming that \( \Omega = \mathcal{B} = \mathcal{B}' = \mu_{xy} = 0 \) we obtain the dispersion relation from

\[ \omega \xi - \frac{k^2}{\omega} \hat{\mu} = 0, \] (49)

where

\[ \xi = \begin{pmatrix} 1 - \epsilon_a & \epsilon_a \\ \epsilon_p & 1 - \epsilon_p \end{pmatrix}. \] (50)

As we still have \( C_a = 0 \), we can expand this to get

\[ \omega^2 \xi - \omega^2 \left( (1 - \epsilon_a) k^2 c_a^2 + (1 - \epsilon_p) k^2 c_p^2 \right) + k^4 c_a^2 c_p^2 = 0, \] (51)

where \( \xi = 1 - \epsilon_p - \epsilon_a. \) Solving for \( \omega^2 \) we have

\[ \omega^2 = \frac{1}{2 \xi} \left( (1 - \epsilon_a) k^2 c_a^2 + (1 - \epsilon_p) k^2 c_p^2 \right) \]
\[ \pm \frac{k^2}{2 \xi} \left( \left( (1 - \epsilon_a) c_a^2 + (1 - \epsilon_p) c_p^2 \right)^2 - 4 \xi c_a^2 c_p^2 \right)^{1/2}. \] (52)

To make further progress it is useful to assume that the entrainment is a small effect and use Taylor expansion in \( \epsilon_x. \) To linear order in entrainment, the frequencies are given by

\[ \omega = \pm \left( 1 + \frac{1}{2} \epsilon_x \right) k c_n, \] (53)
\[ \omega = \pm \left( 1 + \frac{1}{2} \epsilon_p \right) k c_p. \] (54)

This illustrates how the sound waves are affected by a weak entrainment coupling. At this level there appears to be no interaction between the two wave speeds. This would be a higher order effect for this equation of state.

4.3 Chemical coupling

Let us consider the other way that the two fluids in a non-rotating system may be coupled. In order to see what effect chemical coupling has on the waves, we consider an equation of state of the form

\[ E = f(n_a, n_p). \] (55)

The key difference is that the chemical potential of one fluid can now be affected by the population density of the other constituent, i.e. we have \( C_a \neq 0. \) The relevant dispersion relation still follows from equation (49), which now expands to give

\[ \omega^4 - \omega^2 k^2 (c_a^2 + c_p^2) + k^4 \left( c_a^2 c_p^2 - \frac{\rho_p}{\rho_a} c_n^2 \right) = 0. \] (56)

Taking \( C_a^2 \) small and solving for \( \omega, \) we get either

\[ \omega = \pm k c_n \left[ 1 + \frac{\rho_p}{2 \rho_a k^2 c_n^2 (c_a^2 - c_p^2)} \right] \] (57)

or

\[ \omega = \pm k c_p \left[ 1 + \frac{\rho_p}{2 \rho_a k^2 c_p^2 (c_a^2 - c_p^2)} \right]. \] (58)

These are still modified sound waves associated with each constituent.

4.4 Slow rotation

Let us now move on to the slow-rotation problem. In addition to the sound waves, we expect to find inertial modes. Since we are assuming that the two fluids corotate in the background the inertial waves are likely to be degenerate. To keep the problem simple, we assume that there is no chemical coupling. In practice, we again let the equation of state be of the form (42). Taking \( \mathcal{B} = \mathcal{B}' = \epsilon_x = \mu_{xy} = 0 \) the dispersion relation follows from

\[ \omega^4 - \omega^2 \frac{k^2}{\omega} - \omega \frac{\Omega^2}{\omega^2} \left( 4 \omega^2 - 4 k^2 \cos^2 \theta \hat{\mu} \right) = 0. \] (59)

As before, \( \theta \) is the angle between \( k_i \) and \( \Omega \), such that \( k_i \Omega_j = k_i \Omega_j \cos \theta. \) The determinant expands to give

\[ \omega^2 \left( 1 - \frac{k^2}{\omega^2} c_a^2 - \frac{\Omega^2}{\omega^2} + \frac{4 k^2}{\omega^2} \cos^2 \theta c_a^2 \right) \]
\[ \times \left( 1 - \frac{k^2}{\omega^2} c_p^2 - \frac{\Omega^2}{\omega^2} + \frac{4 \Omega^2}{\omega^2} \cos^2 \theta c_p^2 \right) = 0. \] (60)

There are clearly two decoupled cases. The solutions are found from

\[ \omega^4 - \omega^2 \left( k^2 c_a^2 + 4 \Omega^2 \right) + 4 \Omega^2 k^2 \cos^2 \theta c_a^2 = 0. \] (61)
If, for simplicity, we assume slow rotation, the solutions to equation (61) are
\[
\omega \approx \pm k c_s \left( 1 + \frac{2 \Omega^2}{k^2 c_s^2} \sin^2 \theta \right),
\]  
(62)
\[
\omega \approx \pm 2 \Omega \cos \theta.
\]
(63)
The first pair of solutions (equation 62) represents sound waves with a correction due to the slow rotation. The second pair (equation 63) is the (in this case degenerate) inertial waves. The form of the solutions is exactly as one would expect from the single-fluid problem (cf. Section 2.2).

4.5 Mutual friction

The simple cases we have considered so far provide an insight into the different classes of waves that are present in the rotating two-fluid problem. We now want to develop an understanding of how these waves are affected by the mutual friction. To do this, it is natural to assume that the induced damping is weak. This will be the case in both the weak and the strong drag limits, i.e. when \( \mathcal{R} \gg 1 \) or \( \mathcal{R} \ll 1 \). For simplicity, we will focus on the weak drag case. Then we also expect to have \( B' \ll B \) which allows us to simplify the algebra somewhat. In other words, we assume that the mutual friction is sufficiently weak that we can include it as a perturbation of the solutions we found previously. The frequency \( \omega \) can then be replaced by \( \omega_0 + \delta \omega \), where \( \omega_0 \) represents one of the undamped solutions from Section 4.4 and \( \delta \omega \) is a correction of the order of \( B \).

In order to be consistent we cannot consider the effects of mutual friction without at the same time accounting for rotation. Without rotation there would be no neutron vortices in the background and hence no mutual friction. We therefore consider the situation where both \( B \) and \( \Omega \) can be assumed small (in a suitable sense). To make the analysis tractable we assume that \( \epsilon_s = \mu_{xy} = 0 \). Strictly speaking, it is not consistent to neglect the entrainment here. It may play a central role in generating the mutual friction since it is the entrained protons flowing around a neutron vortex that generates the main component of the vortex magnetic field (Andersson et al. 2006). Hence, if we neglect the entrainment then we should not have the mutual friction (in this form) either. Of course, the two contributions have completely different effects on the dynamics. As long as we are mainly interested in the dissipation, the assumptions we make here should be acceptable.

The equation that we need to solve can be written as (cf. equation A20)
\[
\left| a_1 \right| = \frac{i k^2}{\omega} \mu + \frac{2B}{\rho_p \Omega} \left[ \Omega^2 - \left( \frac{k \mu}{\omega^2} \right) \right] - \frac{\Omega^2}{\omega^2 - 2i \omega B \Omega (1 + \rho_p / \rho_n)} \times \left\{ 8 \omega \Omega + \frac{4B \rho_o}{\rho_p} \left[ \frac{k \mu}{\omega^2 \Omega^2} \left( i 4 \omega \mu + 8 \frac{B \Omega}{\rho_p} \rho_o \right) \right] \right\} = 0,
\]
(64)
where
\[
\rho_0 = \left( \begin{array}{cc} \rho_p & \rho_p \\ -\rho_n & \rho_n \end{array} \right),
\]
(65)
and
\[
\rho_0^{zero} = \left( \begin{array}{cc} \rho_n & \rho_p \\ \rho_p & \rho_n \end{array} \right).
\]
(66)

Before substituting the appropriate solutions for \( \omega_0 \) in equation (64) we note that the off-diagonal elements in the above matrix are proportional to \( B \). This means that, to first order in \( B \), the dispersion relation follows from the vanishing of the diagonal elements. Hence, we need to solve either
\[
\omega_0 \left( 1 + \frac{\delta \omega}{\omega_0} \right) - \frac{i k^2}{\omega_0} \left( 1 - \frac{\delta \omega}{\omega_0} \right) c_n^2 + 2 \frac{B}{\Omega} \Omega^2 - \left( \frac{k \mu}{\omega_0} \right) c_n^2 \right]
\]
\[
- \frac{\Omega^2}{\omega_0} \left[ 1 - 2 \frac{\delta \omega}{\omega_0} + 4 \Omega \left( \frac{1 + \rho_p}{\rho_p} \right) \right] \times \left\{ 4i \omega \left( 1 + \frac{\delta \omega}{\omega_0} \right) + 8 \frac{B \Omega}{\rho_p} \rho_n \right\} = 0
\]
(67)
or
\[
\omega_0 \left( 1 + \frac{\delta \omega}{\omega_0} \right) - \frac{i k^2}{\omega_0} \left( 1 - \frac{\delta \omega}{\omega_0} \right) c_n^2 + 2 \frac{B \rho_n}{\rho_p} \Omega^2 - \left( \frac{k \mu}{\omega_0} \right) c_n^2 \right]
\]
\[
- \frac{\Omega^2}{\omega_0} \left[ 1 - 2 \frac{\delta \omega}{\omega_0} + 4 \Omega \left( \frac{1 + \rho_p}{\rho_p} \right) \right] \times \left\{ 4i \omega \left( 1 + \frac{\delta \omega}{\omega_0} \right) + 8 \frac{B \Omega}{\rho_p} \rho_n \right\} = 0
\]
(68)
Linearizing equation (67) we find that the mutual friction correction to the waves associated with the neutron fluid is given by
\[
\delta \omega = 2i B \Omega \left( \frac{\omega_n^2 + 4 \Omega^2}{\omega_n^2} \left( \omega_n^2 - k^2 \cos^2 \theta c_n^2 \right) \right). \]
(69)

Let us first consider the inertial waves, i.e. take \( \omega_n^2 = 4 \Omega^2 \cos^2 \theta \). To first order in \( \Omega \) this leads to
\[
\delta \omega = iB \Omega \left( 1 + \cos^2 \theta \right).
\]
(70)
The correction to the sound waves follows by taking \( \omega_n^2 = k^2 \ c_n^2 + 4 \Omega^2 \sin^2 \theta \). This leads to
\[
\delta \omega = iB \Omega \sin^2 \theta.
\]
(71)

To find the other set of solutions we linearize equation (68) and solve for \( \delta \omega \). This leads to
\[
\delta \omega = 2i B \Omega \rho_p \left( \frac{\omega_n^2 + 4 \Omega^2}{\omega_n^2 + k^2 \omega_n^2 c_n^2 + 4 \Omega^2 \omega_n^2 - 12 k^2 \Omega^2 \cos^2 \theta c_n^2} \right). \]
(72)
For the inertial waves we again use \( \omega_n^2 = 4 \Omega^2 \cos^2 \theta \) and find that
\[
\delta \omega = iB \Omega \rho_p \left( 1 + \cos^2 \theta \right). \]
(73)
Finally, using the proton sound wave solution \( \omega_n^2 = k^2 c_p^2 + 4 \Omega^2 \sin^2 \theta \) we have
\[
\delta \omega = iB \Omega \rho_p \sin^2 \theta. \]
(74)

To summarize the results, we now have two sets of sound waves that are damped by mutual friction. Their frequencies are
\[
\omega = \pm \left( k c_n + 2 \Omega^2 \sin^2 \theta \right) + iB \Omega \sin^2 \theta \]
(75)
and
\[ \omega = \pm (k \epsilon_p + 2 \Omega \sin^2 \theta) + iB \Omega \frac{\rho_0}{\rho_p} \sin^2 \theta. \]  
(76)

There are also two sets of inertial waves. In the undamped case their frequencies are degenerate, but they become distinct when we account for the mutual friction. These solutions are
\[ \omega = \pm 2 \Omega \cos \theta + iB \Omega (1 + \cos^2 \theta) \]  
(77)
and
\[ \omega = \pm 2 \Omega \cos \theta + iB \Omega \frac{\rho_0}{\rho_p} (1 + \cos^2 \theta). \]  
(78)

From these results we learn the following. First of all, equations (75) and (76) show that there will be no dissipation of sound waves that travel along the axis of rotation (\( \theta = 0 \)). This is natural since the sound waves are longitudinal and the mutual friction only affects motion orthogonal to the vortex array. It is interesting to contrast this with the result for the inertial waves. From equations (77) and (78) we see that these are always damped. In fact, the effect of mutual friction is maximal when the wave travels along the vortex array. This result is easy to understand from the discussion of the single-fluid problem in Section 2.2. Since the inertial waves generally have a component that is orthogonal to the vortex array it is natural that they experience damping due to mutual friction regardless of the direction of propagation.

5 Including Vortex Tension

Up to this point we have implicitly assumed that the vortices can be considered straight. In effect, we have ignored the tension that arises because of vortex curvature. This tends to be a small effect, so one would not expect our results to change much if we account for it. However, it turns out that the vortex tension is important for the instability that we will discuss in the next section. In particular, it determines the critical wavelength at which the instability sets in. Hence, it is useful to extend our discussion in such a way that the tension of the neutron vortex array is accounted for. Our discussion is modelled on Hall’s analysis of the corresponding problem in superfluid helium (Hall 1958). By redoing his calculation within our formulation we will show how entrainment affects these waves. For simplicity, we ignore the mutual friction in this section. The derivation of the tension term is provided in Appendix B. Including the relevant contribution, the equations of motion in a rotating frame are (as earlier, we ignore the gravitational potential)
\[ \frac{\partial}{\partial t} \left( \partial_i v_i \nabla_j \right) \overset{\rho_0}{\rho} = \nabla_i \mu + 2 \epsilon_{ijk} \Omega \ff v_k. \]  
(79)

As discussed in Appendix B, we have (recall equation 23)
\[ n_i \kappa_i = \epsilon_{ijk} \nabla_j \overset{\rho_0}{\rho} = \epsilon_{ijk} \nabla_j (v_k + \epsilon_n w_k \rho_0), \]  
(81)
while
\[ v = \frac{1 - \epsilon_p}{1 - \epsilon - \epsilon_p} = \frac{1 - \epsilon - \epsilon_p}{4 \pi} \log \left( \frac{b}{\delta_0} \right), \]  
(82)
see Andersson et al. (2007) for a detailed discussion.

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We have already worked out most of the terms that we need to discuss perturbations of these equations. The only new piece is the tension contribution. If we consider the same background configuration as in the previous sections, then the two fluids rotate uniformly at the same rate and we have \( n_i \kappa_i = 2 \Omega \) in the background. We then only need to work out
\[ \delta f^{\text{tension}} = \delta \left[ \nabla_i (\kappa \psi) \right] = \nabla \delta \left[ \frac{1}{2} (n_i \kappa_i) \nabla \psi + 2 \Omega \nabla \delta \psi \right]. \]  
(83)

The first term is easily worked out from equation (81). The definition also leads to
\[ \delta \kappa_i = \frac{1}{2 \Omega} \left( \epsilon_{ijk} \nabla_j \delta \overset{\rho}{\rho_0} = \frac{1}{2} \Omega \epsilon_{ijk} \nabla_j \delta \psi \right). \]  
(84)

If the background configuration is uniformly rotating, we find that
\[ \delta f^{\text{tension}} = \delta \psi \nabla \left( \epsilon_{ilm} \nabla^m \delta \overset{\rho_0}{\rho} = \frac{1}{2} \Omega \epsilon_{ilm} \nabla^m \delta \psi \right). \]  
(85)

As in the previous sections we now make the plane-wave ansatz, i.e. we assume that \( \delta \overset{\rho}{\rho_0} = \exp(i(\omega t + k \cdot x)) \). Then
\[ \delta f^{\text{tension}} = -\nu k_z \left[ \epsilon_{ijk} \delta \overset{\rho_0}{\rho} - \epsilon_{ilm} \Omega^l \delta \overset{\rho_0}{\rho} \right], \]  
(86)
where we have defined \( k_z = k \cdot \hat{\Omega} \).

Since our main interest is to see how the vortex tension affects the various waves that we have discussed previously, it is useful to make a further simplification at this point. We will concentrate on waves that propagate along the axis of rotation. Then \( k_z = |k| \) and since \( k \) is parallel to \( \hat{\Omega} \) the last term in equation (86) vanishes. Hence, we have
\[ \delta f^{\text{tension}} = -\nu k_z \epsilon_{ijk} \delta \overset{\rho_0}{\rho} = \nu k_z^2 \epsilon_{ijk} \kappa^j \left[ \delta \overset{\rho_0}{\rho} + \epsilon_n (\psi - \overline{\psi}) \right]. \]  
(87)

This expression shows that the tension has no effect on longitudinal waves that travel along the rotation axis. In other words, the sound waves are unaffected by the inclusion of the tension. The same is not true for the inertial waves.

Combining the above results with results from the previous sections we arrive at the perturbed equations of motion:
\[ i \omega \left[ \delta \overset{\rho_0}{\rho} + \epsilon_n (\psi - \overline{\psi}) \right] = k_i \mu_i \rho_0 + 2 \epsilon_{ijk} \Omega^j \delta \overset{\rho_0}{\rho}, \]
\[ = -\nu k_z^2 \epsilon_{ijk} \delta \overset{\rho_0}{\rho} \left[ \delta \overset{\rho_0}{\rho} + \epsilon_n (\psi - \overline{\psi}) \right], \]  
(88)
and
\[ i \omega \left[ \delta \overset{\rho_0}{\rho} + \epsilon_n (\psi - \overline{\psi}) \right] = k_i \rho_0 \mu_i \rho_0 + 2 \epsilon_{ijk} \Omega^j \delta \overset{\rho_0}{\rho} = 0. \]  
(89)

While we could work out the dispersion relation for generic waves in this system, we have chosen not to do this. The reason is very simple. As already mentioned, when the wavenumber is aligned with the rotation axis, as in the above equations, then the sound waves are unaffected by the tension. Given this, it is natural to simplify the analysis by focusing on purely transverse inertial waves. For transverse waves we have \( \delta \overset{\rho_0}{\rho} = 0 \) which leads to \( \chi = 0 \) by virtue of the continuity equations. Hence the perturbation equations can be written as
\[ i \omega \delta \overset{\rho_0}{\rho} = -2 \Omega \epsilon_{ijk} \hat{k}^l \delta \overset{\rho_0}{\rho} = -\nu k_z^2 \epsilon_{ijk} \kappa^j \delta \overset{\rho_0}{\rho}, \]  
(90)
and
\[ i \omega \delta \overset{\rho_0}{\rho} = -2 \Omega \epsilon_{ijk} \hat{k}^l \delta \overset{\rho_0}{\rho}. \]  
(91)

To derive the dispersion relation we first take the cross product of each equation with \( \hat{k} \). This leads to the relations
\[ i \omega \delta \overset{\rho_0}{\rho} = 2 \Omega \delta \overset{\rho_0}{\rho} + \nu k_z^2 \delta \overset{\rho_0}{\rho}, \]  
(92)
and
\[ i \omega \delta \overset{\rho_0}{\rho} = 2 \Omega \delta \overset{\rho_0}{\rho}. \]  
(93)
Recalling the definition of the momenta \( \bar{p}_j \), we can solve equations (90) and (91) for \( \varepsilon_{ijkl}\bar{k}_i\bar{v}_l \). Inserting the results in equations (92) and (93) we have

\[
\begin{align*}
\left\{ 2\Omega + (1 - \varepsilon_n)\bar{v}\bar{k}_j^2 \right\}^2 & - \omega^2 (1 - \varepsilon_n) \left[ 2\Omega + (1 - \varepsilon_n)\bar{v}\bar{k}_j^2 \right] \\
- \omega^2 (1 - \varepsilon_n) & \left[ 1 - \varepsilon_n - \frac{\varepsilon_ne\bar{v}\omega^2\bar{k}_j^2}{2\Omega} - \frac{\varepsilon_ne\bar{v}\omega^2}{2\Omega} \right] \varepsilon_i^j
\end{align*}
\]

\[
= \left\{ \varepsilon_n^2 (1 - \varepsilon_n) \left[ 1 - \varepsilon_n - \frac{\varepsilon_ne\bar{v}\omega^2\bar{k}_j^2}{2\Omega} - \frac{\varepsilon_ne\bar{v}\omega^2}{2\Omega} \right] \\
+ \frac{\varepsilon_n (1 - \varepsilon_n)\bar{v}\omega^2}{2\Omega} \left[ 2\Omega + (1 - \varepsilon_n)\bar{v}\bar{k}_j^2 \right] \varepsilon_i^j \right\} \varepsilon_i^j
\]

and

\[
4\Omega^2 - (1 - \varepsilon_p)\omega^2 \varepsilon_i^j \varepsilon_i^j = \omega^2 \varepsilon_p^2 \varepsilon_i^j \varepsilon_i^j .
\]

From these two relations we see that the required dispersion relation is

\[
[4\Omega^2 - (1 - \varepsilon_p)\omega^2 \varepsilon_i^j \varepsilon_i^j ] \left[ 2\Omega + (1 - \varepsilon_n)\bar{v}\bar{k}_j^2 \right]^2
- \omega^2 (1 - \varepsilon_n) \left[ 1 - \varepsilon_n - \frac{\varepsilon_ne\bar{v}\omega^2\bar{k}_j^2}{2\Omega} - \frac{\varepsilon_ne\bar{v}\omega^2}{2\Omega} \right] \varepsilon_i^j
\]

\[
= \omega^2 \varepsilon_p \left( \varepsilon_n^2 (1 - \varepsilon_n) \left[ 1 - \varepsilon_n - \frac{\varepsilon_ne\bar{v}\omega^2\bar{k}_j^2}{2\Omega} - \frac{\varepsilon_ne\bar{v}\omega^2}{2\Omega} \right] \\
+ \frac{\varepsilon_n (1 - \varepsilon_n)\bar{v}\omega^2}{2\Omega} \left[ 2\Omega + (1 - \varepsilon_n)\bar{v}\bar{k}_j^2 \right] \varepsilon_i^j \right) \varepsilon_i^j .
\]

In principle, it is straightforward to write down the solutions to this equation. After all, it is just a quadratic in \( \omega^2 \). Of course, the final expressions will be so complicated that we learn very little from them. Let us instead focus on two limiting cases. First of all, we see that if we neglect the entrainment we have

\[
(4\Omega^2 - \omega^2) \left[ 2\Omega + \bar{v}\bar{k}_j^2 \right]^2 - \omega^2 = 0.
\]

The solutions are obviously

\[
\omega = \pm \left( 2\Omega + \bar{v}\bar{k}_j^2 \right)
\]

and

\[
\omega = \pm 2\Omega .
\]

The first solution represents the neutron inertial waves, and the second corresponds to the inertial waves in the proton fluid. As one might expect, the former are affected by the neutron vortex tension while the latter are not. These waves are analogous to those found by Hall (1958) in the case of superfluid helium. Of course, our calculation adds to the standard analysis for helium by accounting for the entrainment. To get a first idea of how it affects the inertial waves, we can include entrainment as a small correction to the above solutions. We then find that, to linear order in \( \varepsilon_n \), we have

\[
\omega = \pm \left( 2\Omega + \bar{v}\bar{k}_j^2 + 2\varepsilon_n\omega \right)
\]

and

\[
\omega = \pm 2\varepsilon_n\Omega .
\]

It should, of course, be emphasized here that there is no physical reason why the entrainment parameters should be small. We have made this assumption in order to simplify the result.

To summarize, we have shown how the tension of the neutron vortex array provides a small correction to the inertial waves in the neutron fluid. We have demonstrated that this remains true when the entrainment is considered weak and the calculation is carried out to linear order. The full solution to the problem, obtained from equation (96), is likely to exhibit a more complex structure. This could easily be investigated via numerical solutions of equation (96) for some suitable model equation of state. At this point we are, however, not going to discuss this further. Instead, we will consider the effect of introducing a relative flow in the background configuration.

### 6 Instability of the Vortex Array

So far we have assumed that the two fluids rotate together in the background configuration. This is a natural assumption given that dissipation will tend to damp any relative motion. However, there are situations where one may be interested in dynamics that takes place on a time-scale shorter than that associated with dissipation. Then one can relax the conditions of both chemical and dynamical equilibrium. In particular, one can allow for relative motion in the background configuration used in the plane-wave analysis. The question is whether a relative background flow alters the solutions we have discussed in an interesting way. This turns out to be the case. In fact, when a relative flow is introduced the vortex array may become unstable. This instability is well known for helium, and is often referred to as the Donnelly–Glaberson instability (Glaberson et al. 1974). Since the two-fluid model for a superfluid neutron star core is completely analogous to the standard formulation for superfluid helium, it should be no surprise that this instability is relevant also for neutron stars (Peralta et al. 2005, 2006). In this section we derive the critical relative velocity for this vortex instability, and discuss its interpretation.

In order to keep the analysis tractable we extend the case discussed in the previous section. That is, we focus on purely transverse waves for which the wavevector \( \bar{k} \) is aligned with the rotation \( \Omega \). In addition, we assume that it is sufficient to consider the dynamics of one of the fluids. In practice, we consider the protons as ‘clamped’ and ignore their contribution entirely. This set-up is analogous to that discussed by Glaberson et al. (1974) for helium. In their case, the assumption can to some extent be justified since the ‘normal’ fluid is viscous. In our case, this would also be true, since our ‘proton’ fluid accounts for the electron component, which will be affected by viscosity. It is not clear, however, that the viscous time-scale is short enough that the clamping assumption is truly justified. This is an important caveat, but we do not believe that relaxing this assumption would alter the results in a significant way. For simplicity, we have also chosen to neglect the entrainment.

We focus on the perturbed neutron equation in the case when there is a relative flow in the background. To facilitate the analysis we assume that this background flow is aligned with both the wavevector and the rotation axis. Representing the background flow by \( V_n\bar{k} \), the perturbation equation can be written as

\[
i (\omega + V_n\bar{k}) \varepsilon_i^j + 2\Omega\varepsilon_{ijkl}\bar{k}_i\bar{v}_l = \delta \bar{f}_j .
\]

The force on the right-hand side has three contributions. The contribution from the vortex tension remains unchanged from the previous section. We also need the mutual friction force. Under the present assumptions, and if we neglect \( B^2 \) (which is valid in the weak drag regime\(^3\)), we get from equation (22),

\[
\delta \bar{f}_j^m = B\varepsilon_{ijkl}\epsilon^{lkm} [i\bar{\Omega}V^m_{\alpha}\epsilon^{nq}_{\beta}\bar{v}_n^\alpha + i\bar{\Omega}^l\epsilon_{lpq}\bar{v}_l^p V^m_{\alpha} + 2\Omega^l\bar{\Omega}^m\bar{v}_l^m] .
\]

\(^3\) See Glampedakis et al. (2007) for a discussion of the strong drag problem.
Finally, we also want to account for the contribution to the mutual friction from the self-induced flow. This is small, but it is natural to include it if we are keeping the vortex tension. As long as the background flow is uniform, this term can be written as [cf. equation (36) in Andersson et al. (2007)]

\[ f_i^{\text{ind}} = -v_B n_i \epsilon_{ijk} k^i \nabla k^j. \]  

(104)

Perturbing this we arrive at the contribution

\[ \delta f_i^{\text{ind}} = v B(\vec{\Omega} \cdot \vec{k}) n_i \epsilon_{ijk} \epsilon^{lmn} k_m \nu_k n_l. \]  

(105)

Putting all this together, we consider an equation of form

\[ i(\omega + V_n k_z) n_i + 2\nu \epsilon_{ijk} k^i \partial_j n_l = \frac{-(v k_z^2 - iB V_n k_z) \epsilon_{ijk} k^i n_k - B (2\Omega + v k_z^2) n_l}{\Omega_1}. \]  

(106)

Taking the cross product of this equation with \( \vec{k} \) and combining the two equations we find that the dispersion relation is simply

\[ [\omega + V_n k_z, -iB (2\Omega + v k_z^2)]^\dagger = (2\Omega + v k_z^2) \mp V_n k_z^2. \]  

(107)

That is, the inertial waves in this system have frequency

\[ \omega + V_n k_z = \pm (2\Omega + v k_z^2) + iB (2\Omega + v k_z^2) \mp V_n k_z. \]  

(108)

Given that our assumed time dependence is exp (i\( t \)) this expression shows that the solution corresponding to the upper sign will be exponentially growing (\( \omega \) has a negative imaginary part) when

\[ V_n > \frac{2\Omega}{k_z^2} + v k_z. \]  

(109)

In other words, for any given wavevector \( k_z \), there exists a critical relative flow above which the wave is unstable. Of course, we see from (109) that the critical flow must be large both in the limits of large and small \( k_z \). If we are interested in the critical flow at which the instability first sets in, then we simply need to find the minimum of the function on the right-hand side of (109). Thus we need

\[ \frac{-2\Omega}{k_z^2} + v = 0 \quad \rightarrow \quad k_z = \sqrt{\frac{2\Omega}{v}}. \]  

(110)

Inserting this in the expression for the critical flow we see that the system will have unstable waves when

\[ V_n > V_c = 2\sqrt{2\Omega v}. \]  

(111)

This is exactly the condition derived by Glaberson et al. (1974) for the helium problem.

Even though we have not attempted the general problem, without assuming that the protons are clamped, we have released some of the other assumptions. In particular, one does not have to assume that the waves are purely transverse. The interested reader can find a more general discussion in Appendix C.

Let us now see if we can understand the nature of this instability better. To do this it is helpful to consider the phase velocity of the waves. Recall that in the present problem set-up, a constant phase would mean that

\[ \text{Re} \, \omega t + k_z z = \text{constant} \quad \rightarrow \quad \sigma_p = -\frac{\text{Re} \, \omega}{k_z}, \]  

(112)

where \( \sigma_p \) is the phase velocity. Hence the phase velocity of the inertial waves is

\[ \sigma_p = V_n \pm \frac{1}{k_z} (2\Omega + v k_z^2). \]  

(113)

Comparing this to the condition for the critical velocity we immediately see that the instability sets in through the waves that propagate in the direction opposite to the background flow (for a suitably small \( V_n \)). The critical point is simply identified with \( \sigma_p = 0 \). The interpretation of this condition is that a wave that is originally seen as travelling downwards (relative to \( V_n \)) is dragged upwards by the flow and becomes unstable when its direction of propagation changes (according to a fixed observer). This condition is typical for a two-stream instability. We have previously considered this class of instabilities for neutron stars, see Andersson et al. (2004) for a discussion and a list of relevant references to the plasma physics literature. A two-stream instability typically requires two identifiable flows and some coupling between them. In our previous discussion, we focused on chemical coupling and the role of entrainment. We now see that the instability can also be caused by the mutual friction.

This possibility is particularly interesting since the instability may be intimately linked to the formation of vortex loops and superfluid turbulence (Peralta et al. 2005, 2006; Andersson et al. 2007). In fact, the present analysis provides an important complement to our previous discussion of the turbulence problem.

It is obviously necessary to ask whether this instability is likely to operate in neutron stars. For this to be the case, one would require the critical wavelength to be much smaller that (say) the size of the star. Otherwise, the plane-wave analysis does not apply. From Andersson et al. (2007) we know that

\[ v = \frac{\kappa}{4\pi} \log \left( \frac{b}{d_o} \right), \]  

(114)

where \( \kappa \approx 2 \times 10^{-3} \text{ cm}^2 \text{ s}^{-1} \) and \( \log (b/d_o) \approx 20 \). Thus, we see that the critical wavenumber at which the instability first appears is

\[ k_z \approx 250 \left( \frac{\Omega}{100 \text{ s}^{-1}} \right)^{1/2} \text{ cm}^{-1}, \]  

(115)

corresponding to a wavelength

\[ \lambda = \frac{2\pi}{k_z \approx 2.5 \times 10^{-2} \left( \frac{\Omega}{100 \text{ s}^{-1}} \right)^{-1/2} \text{ cm}. \]  

(116)

If we compare this to the typical intervortex spacing

\[ b \approx 3.4 \times 10^{-3} \left( \frac{\Omega}{100 \text{ s}^{-1}} \right)^{-1/2} \text{ cm}, \]  

(117)

we conclude that one may well expect waves with a wide range of wavelengths to be unstable in a typical neutron star. This is an interesting possibility, and it would be exciting to consider various scenarios where the instability may operate.

## 7 TURBULENT MUTUAL FRICTION

The presence of an instability in the vortex array will lead to oscillations in the vortices, triggering reconnections and the formation of vortex loops with a range of different sizes (Andersson et al. 2007). This behaviour is very similar to the standard cascade seen in normal fluid turbulence. If a turbulent tangle is present, then our analysis is no longer valid. After all, the form we are using for the mutual friction force is based on the assumption that the vortex array is (loosely) straight. One of the outstanding issues in superfluid helium research concerns the nature of the force in the turbulent case. While some sort of consensus has been reached in the case of isotropic turbulence, problems with relative flow and rotation are still far from understood. Yet this is the problem that we need to solve in order to model neutron stars. Our system is rotating, and if it becomes turbulent then any tangle that develops should be polarized.

In absence of a clear strategy for developing a model for the mutual friction force in the case of polarized turbulence, we have previously proposed a phenomenological prescription (Andersson
It is of the order of $2i\beta \approx 1$, so the critical velocity is approximately given by

$$V_n \approx \frac{6n^2}{\chi B} \kappa k^c.$$  

For typical parameters, this velocity would be vastly greater than the critical velocity at which the instability sets in. In fact, it may well be the case that one cannot reach such large relative flows in a realistic neutron star. Nevertheless, the result is conceptually interesting. One should also keep in mind that $B$ is of the order of unity in superfluid helium (Donnelly 1991), so this upper cut-off for the vortex instability may not be out of reach in that context.

**8 BRIEF SUMMARY**

We have analysed the wave propagation in a rotating superfluid neutron star core, taking into account the standard mutual friction force. Our plane-wave analysis has added to previous discussions of this problem in a number of important ways. First of all, for models where the two background fluids corotate, we have clarified the role of chemical coupling and entrainment for both sound and inertial waves. Secondly, we have considered the mutual friction damping, demonstrating the well-known fact that sound waves propagating along a vortex array are undamped. We have also shown that the same is not true for inertial waves, which are damped by the mutual friction regardless of the propagation direction. We have accounted for the relatively small contribution of the vortex tension, which arises due to local vortex curvature. Focusing on purely transverse inertial waves, we derived the correction that the tension induces in the wave frequency.

The most exciting result of our investigation concerns the presence of an instability associated with the inertial waves. The instability requires a linear relative flow in the background. We analysed the particular case when this flow is aligned with the rotation axis. This led to a demonstration that the mutual friction coupling induces an instability once the relative flow has reached a critical level. This instability is well known from the analogous problem for superfluid helium, and hence our result should not come as a great surprise. Nevertheless, the possibility that this instability may operate in neutron stars has only recently been appreciated (Peralta et al. 2005, 2006; Andersson et al. 2007). We have argued that the instability belongs to the general class of two-stream instabilities. This interpretation is (we believe) new, and adds insight also into the helium problem.

If this instability operates in a neutron star, it is likely to lead to the formation of a vortex tangle and a state of superfluid turbulence. The impact of this on, for example, glitch recovery is not yet understood. Nevertheless, it is clear that much of our current ‘understanding’ (which tends to be based on the assumption of a locally straight vortex array) may have to be revised. In view of this, the results we have presented here are exciting. Having said that, it is clear that there are a number of difficult issues that need to be addressed if we really want to understand this problem. Our analysis was based on a number of simplifying assumptions. In particular, we assumed that the proton fluid was clamped. It would be relevant to try to consider the general problem. One would certainly want to account for the entrainment, which will alter the critical velocity for the onset of the instability etc. It would also be relevant to try to quantify the damping (and possibly stabilizing role) of shear viscosity, which should be important for short-wavelength oscillations. We also need to consider various astrophysical scenarios for which the instability may be relevant. If it is the case that the key features required are a straight vortex array and some imposed relative flow, then the instability could be relevant in a number of situations. The most obvious possibilities would be (i) neutron star free precession where the neutrons and protons essentially rotate with respect to different axes (in the simplest model), see the recent discussion by Glampedakis et al. (2007), (ii) neutron star spin-down which (in a non-magnetic star) is facilitated by a viscous Ekman layer at the base of the crust inducing a global flow in the charged component and (iii) global mode oscillations, where the length-scale of the mode is vastly larger than the typical length-scale of the instability. These are all interesting problems, well worth giving further attention.
ACKNOWLEDGMENTS

This work was supported by PPARC/STFC through grant numbers PPA/G/S/2002/00038 and PP/E001025/1. NA also acknowledges support from PPARC via Senior Research Fellowship no PP/C505791/1. GLC acknowledges partial support from NSF via grant number PHY-0457072.

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APPENDIX A: THE GENERAL DISPERSION RELATION

In this appendix we provide the derivation of the general dispersion relation for the two-fluid problem, including entrainment, rotation and mutual friction. The starting point is equation (41) in the main text, i.e.

\[ \mathbf{B}^m \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] + \frac{\kappa}{k^2} \mathbf{m} = 0. \]  

(A1)

The dispersion relation for wave propagation is encoded in this equation.

As in the single-fluid case, it is preferable to work with scalar equations. Hence, we contract equation (A1) with \( \mathbf{B} \). This gives

\[ \mathbf{B}^m \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] + \frac{\kappa}{k^2} \mathbf{m} = 0. \]  

(A2)

We obtain a second scalar equation by contracting equation (A1) with \( \Omega \), \( \epsilon \), \( k \).

\[ \mathbf{B}^m \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] + \frac{\kappa}{k^2} \mathbf{m} = 0. \]  

(A3)

A third equation follows from the contraction of equation (A1) with \( \Omega \).

\[ \mathbf{B}^m \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] \left[ \text{det} \frac{\delta}{\delta \mathbf{B}} \right] + \frac{\kappa}{k^2} \mathbf{m} = 0. \]  

(A4)

To facilitate the construction of the general dispersion relation, it is useful to introduce some further notation. If we define the matrices

\[ \xi = \frac{1}{ \epsilon_n - \epsilon_p } \]  

(A5)

\[ \mu = \left( \begin{array}{cc} \epsilon_n & 1 - \epsilon_p \\ \epsilon_p & 1 - \epsilon_n \\ \end{array} \right) \]  

(A6)

\[ v_{\Omega} = \left( \begin{array}{cc} \Omega & \epsilon_j \epsilon_k^* \\ \Omega & \epsilon_j \epsilon_k^* \\ \end{array} \right) \]  

(A7)

\[ v = \left( \begin{array}{cc} k \epsilon_j \epsilon_k^* \\ k \epsilon_j \epsilon_k^* \\ \end{array} \right) \]  

(A8)

then equation (A4) can be compactly written as

\[ \frac{\omega^2}{k^2} v_{\Omega} = \mu v. \]  

(A9)

As a note for future reference, we will also use

\[ \xi^{-1} = \frac{1}{\epsilon} \left( \begin{array}{cc} 1 - \epsilon_p & -\epsilon_p \\ -\epsilon_p & 1 - \epsilon_n \\ \end{array} \right) \]  

(A10)

\[ \xi = \text{det} \xi = 1 - \epsilon_n - \epsilon_p. \]  

(A11)

\[ v = \left( \begin{array}{cc} \Omega \epsilon_j \epsilon_k^* \\ \Omega \epsilon_j \epsilon_k^* \\ \end{array} \right) \]  

(A12)

\[ \rho_0 = \left( \begin{array}{cc} \rho_p & -\rho_p \\ -\rho_p & \rho_p \\ \end{array} \right) \]  

(A13)

We can now rewrite equations (A2) and (A3) as

\[ \left( \begin{array}{cc} \Omega \epsilon_j \epsilon_k^* - \frac{1}{\epsilon} \mu & + \frac{B}{\rho_p} \rho_0 \\ \Omega \epsilon_j \epsilon_k^* - \frac{1}{\epsilon} \mu & \rho_0 \rho_0 \end{array} \right) v = \left( \begin{array}{cc} 2I - \frac{B}{\rho_p} \rho_0 \end{array} \right) v. \]  

(A14)
and
\[
\left( i\omega \xi + \frac{2B \Omega}{\rho_p} \rho_0 \right) v_i + \left( 2I - \frac{2B'}{\rho_p} \rho_0 \right) \Omega^2 v_k = 0.
\]

Assuming that \(1 - \epsilon_n - \epsilon_p \neq 0\), and substituting equation (A9) into (A14) and (A15) we get
\[
\left( i\omega \xi - \frac{k^2}{\omega} \mu + \frac{2B}{\rho_p} \rho_0 \left( \Omega^2 \left( \frac{k_i \Omega_j^2}{\omega^2} - \xi^2 \xi - \mu \right) \right) \right) v_k = 0.
\]

We now define
\[
\rho_0^{\text{zero}} = \left( \rho_0, \rho_0 \right),
\]
then that \(\rho_0^{\text{zero}} \rho_0 = 0\). Inverting the matrix on the left-hand side of equation (A17), and using the result in equation (A18) we obtain
\[
\left\{ \left( i\omega \xi - \frac{k^2}{\omega} \mu + \frac{2B}{\rho_p} \rho_0 \left( \Omega^2 \left( \frac{k_i \Omega_j^2}{\omega^2} - \xi^2 \xi - \mu \right) \right) \right) \right\} v_k = 0.
\]

By using \(\rho_0 \rho_0^{\text{zero}} = 0\) this expands to give the final result
\[
v_k \left( \frac{\rho_0}{\rho_p} \right) \partial^2 \xi + \frac{2B}{\rho_p} \rho_0 \left( \Omega^2 \left( \frac{k_i \Omega_j^2}{\omega^2} - \xi^2 \xi - \mu \right) \right) = 0.
\]

In order for us to have \(v_i \neq 0\) the determinant of the matrix in the curly brackets must vanish. This condition provides the dispersion relation for waves in the two-fluid system. The final equation is undoubtedly very complex, but this should not come as a great surprise. We are including quite a lot of physics in this general model.

As in the single-fluid problem, our analysis does not apply to waves that are purely transverse. Such waves are, however, not very likely unless we align the wavevector with the rotation axis. In the general case, we see from equations (A16) and (A17) that we can have purely transverse waves (for which \(v_i = 0\) only if also \(v_e = 0\) or if
\[
det \left( i\omega \xi + \frac{B \Omega}{\rho_p} \right) = 0
\]
and
\[
det \left( 2I - \frac{B'}{\rho_p} \rho_0 \right) = 0.
\]
These conditions lead to
\[
\omega^2 \left( 1 - \epsilon_n - \epsilon_p \right) - i\omega \xi \left( 1 + \frac{\rho_n}{\rho_p} \right) = 0
\]
and
\[
4 - 2B' \left( 1 + \frac{\rho_n}{\rho_p} \right) = 0.
\]
Since these conditions are extremely restrictive a purely transverse wave is unlikely. This suggests that purely transverse waves are not possible unless the wave is aligned with the rotation in such a way that \(v_i = 0\). Although somewhat contrived, this particular case is interesting and we discuss it in more detail in Sections 5 and 6.

**APPENDIX B: THE VORTEX TENSION**

In this appendix we provide the argument that leads to the form for the neutron vortex tension that is used in the main body of the paper. The calculation is based on the intuitive reasoning of, in particular, Hall (1958). It is important in the sense that it demonstrates how the entrainment parameters enter in the vortex tension.

The starting point is the conservation of vorticity. Defining the macroscopic vorticity as
\[
\omega' = n \kappa' = \frac{1}{m_n} \epsilon^{ijk} \nabla_j p^j_n,
\]
where the neutron momentum is
\[
p^j_n = m_n \left[ v^j_n + e_n \left( v^j_p - v^j_n \right) \right],
\]
we have
\[
\frac{D}{Dt} \int v_n \kappa_n \, dV + \int S_m n \kappa_n v^j_n \, dS = 0.
\]

Here it is assumed that the vortices move collectively with velocity \(v_n^j\). Using the divergence theorem, we must have
\[
\partial_t \omega + \nabla \cdot \left( \omega v_n^j \right) = 0.
\]

Now note that \(\nabla \cdot \omega = 0\) and \(\omega \nabla v_n^j = 0\). The first statement is trivial given the definition of the vorticity. The second should be true provided that there is no motion along the vortices themselves. This way the above conservation law can be recast as
\[
\partial_t \omega_n + \nabla \cdot \left( \epsilon_{ijk} \kappa_n v^j_n \right) = 0.
\]

This leads to
\[
\epsilon_{ijk} \kappa_n \left( p_n^k - \epsilon^{klm} v^l_n \epsilon_{nmo} \nabla^m p_n^l \right) = 0,
\]
which then requires that
\[
\partial_t p_n^k - \epsilon^{klm} v^l_n \epsilon_{nmo} \nabla^m p_n^l = \nabla^k \Psi,
\]
where \(\Psi\) is some scalar potential.

Let us now, for simplicity, assume that the only force that acts on the vortex is the Magnus force. Then we must have
\[
v^j_n = v^j_n + v^j_{n,\text{ind}}.
\]
The first term represents the smooth irrotational flow past the vortex, due to for instance the presence of all other vortices. The second term represents the self-induced flow that arises when the vortex is curved [see the appendix of Andersson et al. (2007)]. This term can be written as

$$v_{ind}^n = \frac{1 - \varepsilon_p}{1 - \varepsilon_n} - v_{eijk} \hat{k} \times \nabla \hat{k} = v_{eijk} \hat{k} \times \nabla \hat{k}. \quad (B9)$$

In order to use this vortex velocity in the equation of motion (B7), we note that

$$e^{kin} (v_t^p + v_{ind}^n) \kappa_n = e^{kin} v_n^0 \kappa_n + \bar{v} \kappa \nabla \hat{k}. \quad (B10)$$

Then we need

$$m_n v_n^0 \bar{v} \kappa_n = v_n^0 \nabla \hat{k}^2 - v_n^0 \nabla \hat{k}^2 + m_n v \kappa \nabla \hat{k}. \quad \text{(B11)}$$

Use this in equation (B7) to get

$$\partial_t \bar{p}^n + v_n^0 \nabla \bar{p}^n - v_n^0 \nabla \bar{p}^n = \nabla \hat{k} \Psi + m_n v \kappa \nabla \hat{k}. \quad \text{(B12)}$$

Finally use the definition of the momentum to get

$$\partial_t (v_n^0) = (v_n^0 + e_{mun} \bar{v}^m) + e_{mun} \bar{v} \kappa \nabla \hat{k}, \quad \text{(B13)}$$

where we recall that the velocity difference is $w_n^m = v_n^m - v_n^m$. This is the equation of motion for the superfluid neutrons, with the contribution from the vortex tension accounted for. The scalar potential $\chi$ can be easily interpreted as the sum of the chemical and gravitational potentials to arrive at the standard form for this term. This way we arrive at equation (79) in the main text.

APPENDIX C: THE VORTEX INSTABILITY IN A MORE GENERAL CONTEXT

In this appendix we provide a slightly more general derivation of the vortex two-stream instability that was discussed in Section 6. While we still assume that the proton fluid is clamped and neglect entrainment, we initially relax the assumption that the wavevector is aligned with the rotation axis. We also do not assume that the waves are purely transverse. The results obtained in Section 6 follow in the appropriate limits, and the more complicated calculation that we outline here shows how the instability threshold can be derived under less constrained conditions.

In the general case, the plane-wave equation for the neutron fluid can be written as (cf. equation 22)

$$i \omega \bar{v}^n + ik_\perp \mu_n + 2e_{ijk} \Omega / \bar{v}^k = \delta f^n, \quad \text{(C1)}$$

where $\omega = \omega + V_{\perp \perp} k_j$ and we have used $\delta \mu_n = \mu_n e^{i (\omega + V_{\perp \perp} k_j)}$. The force $\delta f^n$ is made up of three contributions. The first is the mutual friction for a straight vortex array, and it leads to

$$\delta f^n = B e_{ijk} e^{kin} \left[ i \hat{k} \cdot V_n \left( e^{i \bar{v}^m k \bar{v}^n} - \hat{k} \cdot \bar{v}^m \bar{v}^n \right) + i \hat{k} \cdot \bar{v}^m \bar{v}^n V_n \right] + 2 \Omega \cdot \hat{k} \bar{v}^n. \quad \text{(C2)}$$

Next we have the contribution from the self-induced flow, which accounts for the vortex curvature. As long as the background flow is uniform, we can perturb (104) to get the contribution

$$\delta f_{ind} = vB \hat{k} \bar{v}^m e^{i \bar{v}^m \bar{v}^n} - \hat{k} \cdot \bar{v}^m \bar{v}^n \hat{k} \bar{v}^n. \quad \text{(C3)}$$

Finally, we have the vortex tension which is given by equation (86), i.e.

$$\delta f_{tension} = -v \left( \hat{k} \cdot \bar{v}^m \right) \left[ e_{ijn} k^i \bar{v}^m - \hat{k} \cdot \left( \bar{v}^n \hat{k} \bar{v}^n \right) \right]. \quad \text{(C4)}$$

Putting all the pieces together and rearranging, the perturbed momentum equation can be written as

$$\left\{ \partial_t - B \left[ 2 \Omega + v \left( k_j \hat{k}_j \right)^2 \right] \right\} \bar{v}^n$$

$$+ \left[ \bar{\mu}_n + 2v B \left( \hat{k} \bar{v}^m \right) \right] \left( \hat{k} \bar{v}^n \right) k_j - 2e_{ijk} \Omega / \bar{v}^k$$

$$= \left\{ \left[ B V_n^m - B \left( V_n^m \hat{k} \right) \hat{k}_j - iv \left( k_j \hat{k}_j \right) \right] e^{i \bar{v}^m k \bar{v}^n} - 2i B \Omega \left( \hat{k} \bar{v}^n \right) \right\} \hat{k}_j$$

$$- (D + \bar{v}^m \hat{k} \bar{v}^n) V_n^m + \left[ B \left( V_n^m \hat{k} \right) \hat{k}_j + iv \left( k_j \hat{k}_j \right) \right] e_{ijm} k^i \bar{v}^n. \quad \text{(C5)}$$

Here the chemical potential perturbation is (in the clamped case we are also assuming that the proton density variation vanishes) given by

$$\mu_n = \frac{\partial \mu_n}{\partial \rho_n} \bar{\rho}_n. \quad \text{(C6)}$$

Since the continuity equation gives

$$i \omega \bar{\rho}_n + \bar{\rho}_n \left( k_j \bar{v}^j \right) = 0, \quad \text{(C7)}$$

we get, using the standard definition of the sound speed from equation (38),

$$\bar{\rho}_n = -e^{2i \bar{v}^m} \left( k_j \bar{v}^j \right). \quad \text{(C8)}$$

The trick now is to form different scalar equations from equation (C5). By taking the scalar product with $\hat{k}$ we get

$$\hat{k}_j \left( \hat{k} \cdot \bar{v}^n \right) = B \left( V_n^m \hat{k} \right) \hat{k}_j$$

$$+ i \bar{v}^m B \left( \hat{k} \bar{v}^m \right) \hat{k}_j$$

$$+ \left[ i \left( 2 \Omega + v k^2 \right) - B \left( k_n V_n^m \right) \right] W = B \left( e^{i \bar{v}^m k \bar{v}^n} \right) \hat{k}_j. \quad \text{(C9)}$$

In writing down this expression we have decomposed the wavevector into a piece along the rotation axis and a piece orthogonal to it, i.e. we are using

$$k' = k'_{\parallel} + k'_{\perp}, \quad \text{where} \quad k'_{\parallel} \Omega = 0. \quad \text{(C10)}$$

We have also defined the scalar quantity $W = e_{ijk} \Omega / \bar{v}^n$.

If we take the scalar product of equation (C5) with $\hat{k}$ we find another scalar relation:

$$\hat{k}_j \left( \hat{k} \cdot \bar{v}^n \right) - \frac{k^2}{\omega} \left( k_j \bar{v}^j \right) + B \left( V_n^m \hat{k} \right) W = B \left( e^{i \bar{v}^m k \bar{v}^n} \right) \hat{k}_j. \quad \text{(C11)}$$

From the combination (C9) - $k_{\perp} \cdot (C11)$ we then get

$$\hat{k}_j \left( \hat{k} \cdot \bar{v}^n \right) - \frac{k^2}{\omega} \left( k_j \bar{v}^j \right) = B \left( V_n^m \hat{k} \right) \hat{k}_j$$

$$+ i \bar{v}^m B \left( \hat{k} \bar{v}^m \right) \hat{k}_j$$

$$+ \left[ i \left( 2 \Omega + v k^2 \right) + B \left( k_n V_n^m \right) \right] W = 0. \quad \text{(C12)}$$

By taking the cross product between $k'$ and equation (C5) we get another useful relation. After some work it can be written as

$$i k \left( 2 \Omega + v k^2 \right) + B \left( V_n^m \hat{k} \right) \hat{k}_j$$

$$= \left[ B \left( V_n^m \hat{k} \right) \hat{k}_j + iv \left( k_j \hat{k}_j \right) \right]$$

$$= 2i \left( k_j \hat{k}_j \right) V_n^m - (\hat{k}_j - B \left( 2 \Omega + v k^2 \right)) \bar{v}^n$$

$$+ \left[ \left[ B V_n^m - B \left( V_n^m \hat{k} \right) \hat{k}_j - iv \left( k_j \hat{k}_j \right) \right] e^{i \bar{v}^m k \bar{v}^n} - 2i B \Omega \left( \hat{k} \bar{v}^n \right) \hat{k}_j \right] e_{ijm} k^i \bar{v}^n$$

$$- B W \bar{v}^m \hat{k} \bar{v}^n. \quad \text{(C13)}$$

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Taking the scalar product of equation (C13) with $\hat{\Omega}'$ we arrive at
\begin{align}
\left[ ik_z \left( 2\Omega + vk_z^2 \right) + Bk_z \left( \hat{\Omega} \cdot \hat{v}_n \right) \right] (\hat{\Omega}' \cdot \hat{v}_n) \\
- \left[ i \left( 2\Omega + vk_z^2 \right) + Bk_z \left( \hat{\Omega} \cdot V_n \right) \right] (k_z \hat{v}_n) \\
+ \left[ \hat{\omega} - iB \left( 2\Omega + vk_z^2 \right) + B\hat{\Omega} \right] W = 0.
\end{align}
(C14)

Here we have defined yet another scalar, $\hat{\Omega} = \epsilon_{ijk} \hat{\Omega}' \cdot V_n^k$.

We now have three equations, (C9), (C12) and (C14), for four unknown scalar quantities. To solve the general problem we need another relation. Although this relation can be obtained in a few further simplifications. To discuss these examples we first note that $k = k \cos \theta$, where $\theta$ is the angle between the wavevector and the rotation axis. It is then straightforward to verify that we retain the solution from Section 5 in the case when $\theta = B = 0$. We obviously also get the neutron sound waves. If we focus our attention on the possible vortex instability, then it would be natural to first relax the assumption that $\theta$ vanishes. Doing this, but still leaving $B = 0$ and in addition assuming slow rotation and weak tension) we find the leading order wave solutions:
\begin{align}
\hat{\omega}^2 = \pm \epsilon_0 \left[ 1 + \left( \Omega \sin^2 \theta \right) \left( 2\Omega + \nu k^2 \cos^2 \theta \right) \right],
\end{align}
(C20)

for sound waves, and
\begin{align}
\hat{\omega}^2 = \pm \cos \theta (2\Omega + \nu k^2)^{1/2} (2\Omega + \nu k^2 \cos^2 \theta)^{1/2},
\end{align}
(C21)

for inertial waves. If we linearize the dispersion relation in $B$, and assume that the waves take the form $\hat{\omega} = \omega_0 + B\hat{\delta} \omega$, then we find that the mutual friction induced frequency correction follows from
\begin{align}
\hat{\delta} \hat{\omega} = i \frac{\omega_0 - kV_s \cos \theta}{\omega_0} \Omega \sin^2 \theta,
\end{align}
(C22)

for sound waves and
\begin{align}
\hat{\delta} \hat{\omega} = i \frac{\omega_0 - kV_s \cos \theta}{\omega_0} \Omega (1 + \cos^2 \theta) + \nu k^2 \cos^2 \theta,
\end{align}
(C23)

for inertial waves. Recalling that the waves are unstable if the imaginary part is negative, we see that (assuming that $k \cos \theta \geq 0$) the solutions for which $\hat{\omega}_0 < 0$ are always stable. In contrast, the $\hat{\omega}_0 > 0$ solutions become unstable at the critical velocity
\begin{align}
V_{c} = \pm \frac{\omega_0}{k \cos \theta}.
\end{align}
(C24)

As one might have guessed, the onset of the instability depends on the projection of the wavevector along the relative flow. For the sound waves we thus find that the critical flow is
\begin{align}
V_{csound} \approx \frac{c_0}{\cos \theta}.
\end{align}
(C25)

Since the superfluidity is likely broken before the wave propagation reaches the speed of sound, this indicates that these waves are always stable in a real system. Again the inertial waves are different. We find that
\begin{align}
V_{cinertial} \approx \frac{1}{k} (2\Omega + \nu k^2)^{1/2} (2\Omega + \nu k^2 \cos^2 \theta)^{1/2}.
\end{align}
(C26)

According to this criterion, the instability actually sets in at a lower relative velocity when the wavevector is not aligned with the rotation axis. Of course, in reality one may expect the tension term to be small compared to the rotation term. Then the difference between the above result and the aligned case discussed in Section 6 may only be significant at extremely short wavelengths.

\footnote{We did not assume alignment from the beginning since we wanted to outline how the general problem would be solved.}

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