Approximation Algorithm for N-distance Minimal Vertex Cover Problem

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Abstract—Evolution of large scale networks demand for efficient way of communication in the networks. One way to propagate information in the network is to find vertex cover. In this paper we describe a variant of vertex cover problem naming it N-distance Vertex Minimal Cover(N-MVC) Problem to optimize information propagation throughout the network. A minimum subset of vertices of a unweighted and undirected graph $G = (V, E)$ is called N-MVC if $\forall v \in V$, $v$ is at distance $\leq N$ from at least one of the the vertices in N-MVC. In the following paper, this problem is defined, formulated and an approximation algorithm is proposed with discussion on its correctness and upper bound.

Index Terms—Minimal Vertex Cover, Approximation, N-Trail, N-distance, Maximal Matching, Graph Reduction, Extended Graph

I. INTRODUCTION

Network such as Internet, human body, malicious botnet, mobile networks are part of everyday life. Networks in physical world are mapped to graphs in computer world. A graph $G(V, E)$ is set of nodes called vertices(V) and connections between these nodes called Edges(E). Essence of network is communication between nodes, in case of large networks it is a big challenge to perform this task efficiently. Primarily there are two objectives, first information should reach to each node of graph and second is to perform this task efficiently with minimum resources and given constraints. Given unlimited resources and no constraints, information can propagate to each node but this is not the case in real life scenarios. In practice, there is need to select some of the nodes which can propagate given information to other nodes. Example of such scenarios can be found in social networks[7], P2P botnets and similar densely connected graphs of various networks. As a specific example, consider the case where one has to select the minimal set of influential nodes in a social network such that some critical information is propagated to all nodes in the network in a finite number of hops.

One solution of this problem is to determine an approximate Minimal Vertex Cover(MVC) of the network and use the nodes in MVC for propagating information. Due to the property of MVC, the information will be propagated to all nodes in the network in a single hop. But in practice, the cardinality of MVC in huge networks will be very large. Hence, the resources used to propagate information using a single hop from vertices of MVC will be very high. A smaller set of nodes can be used to propagate the information in multiple hops, to reduce the number of resources used. The challenge is to find a set of nodes given a constraint of the maximum hops($N$), before which information has to be propagated to all nodes in the network. We propose a solution to estimate this smaller set of nodes and term it as approximated N-distance minimum vertex cover (N-MVC), where $N$ is the maximum number of hops within which information has to reach all nodes in the network. If the network is static and don’t change over time then once computed nodes in N-MVC require propagation capability, all other node may behave as sink nodes (don’t propagate the information) whereas in dynamic networks where nodes or connections are changed over time, N-MVC needs to be recomputed as soon as new any change happens. We can say $N$ is capacity of nodes to propagate the information. We are considering homogeneous situation where $N$ is same for all nodes. In this paper, we discuss the problem statement, present the approximation algorithm, correctness and discuss upper bound on the solution.

One of the similar type of problem is k-path Vertex Cover[2] and discussed[4][5] thoroughly. k-path Vertex Cover ensures existence of special(having some defined properties) nodes in any path of k vertices in the network. Variants of Vertex cover problem are part of research in domains related to secure communication in sensor networks[6], topology analysis of malicious bots network and security and resources optimization in various types of network.

This paper is organized into 4 sections. Section II defines the problem statement, Section III describe the algorithm in detail, Section IV proves correctness of the algorithm using contradiction, Section V discusses the upper bound for the solution with reference to $N$. The paper ends with summary and concluding remarks in Section VI.

II. N-DISTANCE MINIMAL VERTEX COVER PROBLEM

Given an unweighted and undirected graph $G(V, E)$, we define the following.

**Vertex Cover (VC):** The Vertex cover of Graph $G(V, E)$ is a subset of vertices $S \subseteq V(G)$ such that every edge has atleast one endpoint in $S$, that is $(u, v) \in E(G) \implies u \in S \lor v \in S$. Alternatively, vertex cover of a given Graph $G(V, E)$ is a set $S$ of vertices such that any vertex of the graph either $\in S$ or
which is original MVCP, it is implied that N-MVCP is NP-Complete.

III. APPROXIMATION ALGORITHM TO FIND N-DISTANCE MINIMAL VERTEX COVER

Before discussing the approximation algorithm, we will revisit definitions and concepts of graph theory being used.

Walk: Given a graph G = (V, E), a walk W(v₀, vₙ) joining v₀ and vₙ is defined as an alternating sequence of vertices and edges of G

W(v₀, vₙ) = v₀, e₁, v₁, e₂, v₂, ..., eₙ, vₙ

such that eᵢ = (vᵢ₋₁, vᵢ), 1 ≤ i ≤ n. The length of a walk W denoted by ℓ(W) is the number of edges in W.

Trail: A walk W(v₀, vₙ) is called a trail if all edges in the walk are different.

N-Trail: A walk W(v₀, vₙ) is called a N-Trail if there are N in the walk and all are different.

Degree(v, E): Number of edges of E incident to the vertex v.

In this paper Degree(v, E) is denoted as deg(v, E).

For a given G(V, E) and value of N (where G has at least one N-path as described in Algorithm [I] and N ≥ 2 because for N = 1 algorithm reduces to original vertex cover problem), the approximation algorithm is described as Algorithm [II] in subsection III-C which calls Algorithm [I] (III-A) and Algorithm [II-B] as subalgorithms.

A. Finding N – Trail (Trail of length N

Algorithm 1 Algorithm to Find N – Trail

Input: G(V, E) where V = {v₁, v₂, ..., vₙ}, E = {e₁, e₂, ...}

Output: Trail Eₙ such that ℓ(Eₙ) = N

1: Eₙ ← φ
2: Eₙ ← Eₙ ∪ eᵢ where eᵢ = {vⱼ, vₖ} s.t. eᵢ ∈ E & deg(vⱼ, E) ≥ 2
3: repeat
4: E' = φ
5: E' ← eᵢ, s.t. eᵢ = {vⱼ, vₖ} & eᵢ ∈ E
6: Eₙ ← Eₙ ∪ eⱼ where eⱼ = {vₖ, vₘ} s.t. eⱼ ∈ E' & deg(vₘ, E') ≥ 2 if |Eₙ| < N
7: vₖ ← vₘ
8: until |Eₙ| ≠ N
9: return Eₙ

1) Compute degree of each vertex vᵢ ∈ V and store it.
2) Randomly pick an edge eᵢ = {vⱼ, vₖ} s.t. degree of vₖ is at least 2. Add eᵢ to Eₙ.
3) Pick the vertex vₖ which has degree at least 2. Find all edges E’ s.t. each edge in E’ has vₖ as one end point.
4) As we know one end point of eⱼ is vₖ and let us say another end point is vₘ. Now pick any edge eⱼ ∈ E’ s.t. degree(vₘ) ≥ 2 and add it to Eₙ. Now repeat step 3 with vₖ ← vₘ until size of Eₙ ≠ N
5) We define endpoints of Eₙ as vertices {vₓ, vᵧ} s.t. deg(vₓ) and deg(vᵧ) = 1 considering only edges eᵢ ∈ Eₙ
Algorithm 2 Algorithm for Graph Reduction

Input: \( G(V, E) \) and \( \text{distance}(e) = 1 \ \forall e \in E \)

Output: Reduced Graph \( G'(V, E') \)

1: \( G'(V, E') \leftarrow G(V, E) \)
2: loop:
3: \( \text{while } N \geq 2 \text{ do} \)
4: \( \text{if } N - \text{Trail} \text{ not exists in } G' \text{ then} \)
5: \( N \leftarrow N - 1 \)
6: go to loop
7: \( \text{end if} \)
8: \( V' \leftarrow \phi, V'' \leftarrow \phi, E' \leftarrow \phi, E'' \leftarrow \phi, E''' \leftarrow \phi \)
9: \( \text{Pick a } N - \text{Trail } E_N \text{ from Algorithm 1 with endpoints } (v_0, v_N) \text{ and edges } (E_N) \in E \)
10: \( V' \leftarrow \{v_i\} \ \text{s.t. } \text{either } d(v_0, v_i) = N \text{ or } (2 < d(v_0, v_i) < N \text{ and } \deg(v_i, E) = 1) \forall v_i \in V' \)
11: \( V'' \leftarrow \{v_j\} \ \text{s.t. } \text{either } d(v_j, v_N) = N \text{ or } (2 < d(v_j, v_N) < N \text{ and } \deg(v_j, E) = 1) \forall v_j \in V'' \)
12: \( E' \leftarrow \{e = (v_0, v_i)\} \forall v_i \in V' \)
13: \( E'' \leftarrow \{e = (v_j, v_N)\} \forall v_j \in V'' \)
14: \( E''' \leftarrow \{e_1, e_2, \dots, e_N\} \text{ are edges of } k - \text{Trail from } v_0 \text{ to } v_N \forall v_i \in V', e_1 \in E \& k \leq N \}
15: \( E'''' \leftarrow \{e_2, e_3, \dots, e_N\} \text{ are edges of } k - \text{Trail from } v_j \text{ to } v_N \forall v_j \in V''', e_j \in E \backslash E'''' \& k \leq N \}
16: \( G'(V, E') \leftarrow G'(V, E' \cup E' \cup E'' \cup E'''' \backslash E''''') \)
17: \( \text{end while} \)
18: return \( G'(V, E') \)

B. Graph Reduction Algorithm

Initialization: Initialize \( G' \leftarrow G \) and Compute degree of each vertex of \( G' \) and store it. Assign each edge of \( G'(V, E) \) 1 unit of distance.

1) Pick a Trail of \( N \) connected edges (using Algorithms discussed in subsection III-A) \{\( e_1, e_2, \ldots, e_N \)\} connecting \( N + 1 \) vertices \{\( v_0, v_2, \ldots, v_N \)\} from \( G \) s.t. \( e_1 = \{v_0, v_1\} \) and we define endpoints of \( N \) connected edges as \( \{v_0, v_N\} \).

2) Find set of vertices \( V' \) and \( V'' \), which are either \( N \) (where \( N \geq 2 \)) distance away from \( \{v_0 \) or \( v_N\) \) or \( < N \) distance away from \( \{v_0 \) or \( v_N\) \) and having degree one, s.t. each vertex from \( V' \) or \( V'' \) is connected to \( v_1 \) or \( v_{N+1} \) respectively by a trail.

3) For each vertex \( v_i \) from \( V' \), add an edge between \( v_0 \) and \( v_i \) and mark edges connecting \( v_0 \) and \( v_i \) for deletion and For each vertex \( v_j \) from \( V'' \) add an edge between \( v_j \) and \( v_N \) and mark edges connecting \( v_j \) and \( v_N \) for deletion. Now delete all the edges marked for deletion and recompute the degree and update the degree table w.r.t edges with distance assigned (\( e \in E \)).

4) Repeat step 1,2,3 with new Trail of length \( N \) using algorithm 1.s.t. edges (Trail) \( \in E \).

Note: We will not use newly added edges (not assigned any distance) of \( G \) to form \( N - \text{Trail} \) in Algorithm 1.

5) if no \( N - \text{Trail} \) is found then go to step 1 with \( N \leftarrow N - 1 \) (\( N \geq 2 \))

C. Approximation algorithm for N-distance Minimal Vertex Cover Problem (NMVCP)

Approximation algorithm is application of graph reduction and then Approx-Vertex-Cover[3] algorithm subsequently.

Algorithm 3 Approximation Algorithm for NMVCP

Input: \( G(V, E) \) s.t. \( V = \{v_1, v_2, \ldots\} \) \( E = \{e_1, e_2, \ldots\\)\)

Output: Approximated Solution of NMVCP for \( G(V, E) \)

1: \( G'(V, E') \leftarrow \text{Graph Reduction Algorithm 2 with input } G(V, E) \)
2: \( \text{NMVC} \leftarrow \text{Approx-Vertex-Cover}[3] \text{ Algorithm 4 with input } G'(V, E') \)
3: return NMVC

Algorithm 4 Approx-Vertex-Cover(G) [3]

Input: \( G(V, E) \)

Output: Approx-Vertex-Cover solution \( AVC \)

1: \( AVC \leftarrow \phi \)
2: \( \text{while } E \neq \phi \text{ do} \)
3: \( \text{pick any } (u, v) \in E \)
4: \( AVC \leftarrow AVC \cup \{u, v\} \)
5: \( \text{delete all edges incident to either } u \text{ or } v \)
6: \( \text{end while} \)
7: return \( AVC \)

D. Example

As algorithm described in subsection III-C (Algorithm 2) to find \( N\)-distance vertex cover, first step is to apply graph reduction algorithm to the given graph. If we consider the graph shown in Fig. 1 as input then step 1 of Algorithm 2 for graph reduction (Algorithm 2) with \( N = 3 \) will proceed as follows:

Initialization: \( G' \leftarrow G \) and each edge is assigned a distance of 1 unit and compute degree of each vertex.

1) From algorithm 1 we get \( N = 3 \)-path as described in subsection III-A. Suppose we get set of three edges \( \{e_{v_1-v_2}, e_{v_2-v_3}, e_{v_3-v_4}\} \) as one of the \( N\)-Trail. End point of this \( N\)-Trail are \( \{v_1, v_4\} \).

2) a) Vertices at \( N (= 3) \) distance away from \( v_1 \) are \( \{v_4, v_6, v_{11}, v_{12}\} \)
\( \text{Vertices at } < N (= 3) \text{ but } \geq 2 \text{ distance from } v_1 \text{ and with degree 1 are } \{v_{10}\} \)
\( \text{Therefore } V' = \{v_4, v_6, v_{10}, v_{11}, v_{12}\} \)

b) Vertices at \( N (= 3) \) distance away from \( v_4 \) are \( \{v_1, v_7, v_8, v_9, v_{10}\} \)
\( \text{Vertices at } < N (= 3) \geq 2 \text{ distance from } v_4 \text{ and with degree 1 are } \{v_9\} \)
\( \text{Therefore } V'' = \{v_1, v_7, v_8, v_9, v_{10}\} \)

3) a) Add an edge \( e_{v_1-v_7} \in V' \) and mark connecting edges (of corresponding Trail) for deletion and add an edge \( e_{v_1-v_7} \in V'' \) and mark connecting edges for
Now if we search for a new N(=3)-Trail in reduced graph in Fig. 3, there are no such edges. There is no N(=2)-path also, therefore only edges left are \(e_{v_4-v_5}, e_{v_1-v_11}, e_{v_1-v_12}, e_{v_9-v_11}, e_{v_2-v_10}\) with \(N(=1)\) which is not considered in algorithm.

After graph reduction process \(G'\) is the graph as shown in Fig. 3 but without distances assigned to edges. Now as described in step 2 of algorithm 3 Approx-Vertex-Cover( algorithm 4) for \(G'\) will return the solution NMVC.

**Approx-Vertex-Cover Algorithm on \(G'\)**

1) Select an edge randomly from \(G'\) and add its endpoints to the solution AVC and remove all edges connected to the endpoints of this edge. Let's pick the edge \(e_{v_1-v_4}\) then all edges connected to \(v_1\) or \(v_4\) will be removed and we can see only one edge will remain after this process \(e_{v_11-v_13}\)

2) Now we need to pick another edge randomly but as we know only one edge is there in the graph therefore we need to pick \(e_{v_11-v_13}\) and add its endpoints to the solution AVC

3) Now no edges is remaining in graph therefore solution AVC= \(\{v_1, v_4, v_{11}, v_{13}\}\) which is NMVC in algorithm 3

Here we can see optimal solution is for NMVC is \(\{v_1, v_4, v_{11}\}\) but approx-vertex-cover algorithm outputs approximate solution with 1 extra vertex.

**IV. PROOF OF CORRECTNESS OF ALGORITHM**

We will prove the correctness of proposed algorithm using contradiction. By graph reduction described in algorithm 2 we are reducing \(G(V, E) \rightarrow G'(V, E')\) and solving \(G'\) for Vertex-Cover using Approx-Vertex-Cover algorithm which will provide solution for N-distance vertex cover for \(G\). By reduction algorithm any edge \(e \notin E\) is either an edge from original graph \(e \in E\) or added by graph reduction algorithm.

Let's assume there is an edge \(e \in E\) which is not covered by N-distance vertex cover solution(NMVC) given by the proposed algorithm. It means both the endpoint of \(e\) are \(> N\) distance from each vertex in NMVC.

From reduction algorithm we can say either \(e \in E'\) or \(e\) is removed during reduction \(G \rightarrow G'\) if \(e \in E\) then \(e\) has to be covered by approx-vertex-cover algorithm for \(G' (V, E')\) because of correctness of Approx-Vertex-Cover algorithm(contradiction to assumption).

If \(e\) is removed during graph reduction algorithm then by properties of reduction algorithm

1) An edge is only removed when it’s endpoints are \(\leq N\) distance from one of the the endpoints of \(E_N\).

2) During reduction all the edges of \(E_N\) are removed but one new edge is added between endpoints of \(E_N\)

From subsection step 3, \(e\) could be an edge connecting \(v_0\) and \(v_j\) or connecting \(v_j\) and \(v_N\). When edges connecting these vertices will be removed, one edge connecting endpoint vertices will be added. So, one of these endpoints (of newly added edge) has to be in solution (by Approx-Vertex-Cover Algorithm property) and \(e\) is \(\leq N\) distance from both endpoints. Therefore \(e\) is \(\leq N\) distance from one of the vertex in solution which is contradiction to the assumption.

**V. DISCUSSION ON UPPER BOUND ON SOLUTION**

Given graph instance \((I) \ G(V, E)\), solution for N-distance vertex cover depends on reduction \(G \rightarrow G'\). From Approx-Vertex-Cover upper bound on solution for \(G(V, E)\):

\[
\text{OPT}(I) \geq |M| \quad (1)
\]

\[
A(I) = 2|M| \leq 2 \text{OPT}(I) \quad (2)
\]

where \(M\) is maximal matching for \(G\) and \(|M|\) is size of maximal matching. OPT(I) is the optimal solution for the given instance \(I\).

After Graph Reduction Algorithm performed \(G \rightarrow G'\), new
The above construction guarantees that in any instance \( (\tilde{G}) \) is \( G' \) ( \( V, E' \) ). To get final solution, Approx-Vertex-Cover algorithm is applied on \( G' \) in algorithm [3]. Therefore we can say

\[
OPT(\tilde{G}) \geq |M'| \tag{3}
\]

\[
A(\tilde{G}) = 2|M'| \leq 2OPT(\tilde{G}) \tag{4}
\]

where \( M' \) is maximal matching for \( G' \) and \( |M'| \) is size of maximal matching. \( OPT(\tilde{G}) \) is the optimal solution for the given instance \( \tilde{G} \).

We know each edge of maximal matching in \( M \) is added by removal of exactly one edge of \( G \). As we discussed each edge of maximal matching in \( M \) is added by removal of exactly one edge of \( G \). Therefore, the construction of \( M'' \) we can write

\[
|M''| \geq |M'| \tag{5}
\]

\[
A(\tilde{G}) = 2|M''| \leq 2OPT(\tilde{G}) \tag{6}
\]

where \( M'' \) is maximal matching for \( G'' \) and \( |M''| \) is size of maximal matching. \( OPT(\tilde{G}) \) is the optimal solution for the given instance \( \tilde{G} \).

As we discussed each edge of maximal matching in \( M'' \) is replacement of a \( N \)-Trail in \( G'' \). Therefore, from the construction of \( M'' \) we can write

\[
|M''| \geq |M'| \times N/2 \tag{7}
\]

\[
|M'| \leq 2/N \times |M''| \tag{8}
\]

\(( N \leftarrow N + 1 \) if \( N \) is odd \)

from equations 4 and 8

\[
A(\tilde{G}) = 2|M''| \leq 2 \times (2/N) \times |M''| \tag{9}
\]

From equations 6 and 9

\[
A(\tilde{G}) \leq (2/N) \times 2|M''| \leq (4/N)OPT(\tilde{G}) \tag{10}
\]

from equation 10 we can say for a given graph \( G \) if we extend \( G \) to \( G' \) and reduce \( G \) to \( G' \) then solution to \( N \)-distance vertex cover problem for \( G \) is the solution from Approx-Vertex-Cover algorithm for \( G' \) which have upper bound w.r.t \( G'' \), which is extended version of \( G \).

Utility of Extended Graph: We constructed extended graph to discuss upper bound on solution but there are utilities of extended version of a graph. As we discussed extending a graph ensures the traversal of exact \( N \)-Trail which inherently suggest a process to maximize a network’s capability to propagate the information. Fig. 3 shows the extended graph where dotted edges are extended edges and diamond shaped nodes are extended nodes. In a network scenario if more no. of nodes are to be added then these extended nodes are the possible places where one should add the new nodes without worrying about propagating the information to the new nodes. These new nodes could be sink nodes which don’t require capability to propagate information.

VI. Summary

This paper presents a variant of vertex cover problem called \( N \)-distance vertex cover problem which addresses challenges in networks to propagate information efficiently. We proposed a solution by approximation algorithm using graph reduction and Approx-Vertex-Cover algorithm. Correctness of proposed solution is proved using contradiction and upper bound is also discussed by construction of extended graph.

References

[1] Karp, R. M. (1972). Reducibility among combinatorial problems (pp. 85-103). Springer US.

[2] Brear, Botjan, et al. "Minimum k-path vertex cover." Discrete Applied Mathematics 159.12 (2011): 1189-1195.

[3] Chakrabarti, Amit (Winter 2005). "Approximation Algorithms: Vertex Cover" (PDF). Computer Science 105. Dartmouth College.

[4] Li, Xiaosong, Zhao Zhang, and Xiaohui Huang. "Approximation algorithms for minimum (weight) connected k-path vertex cover." Discrete Applied Mathematics (2015).

[5] Li, Xiaosong, Zhao Zhang, and Xiaohui Huang. "Approximation algorithms for minimum (weight) connected k-path vertex cover." Combinatorial Optimization and Applications: 8th International Conference, COCOA (2014)
[6] Novotn, Marin. "Design and analysis of a generalized canvas protocol."
Information Security Theory and Practices. Security and Privacy of Pervasive Systems and Smart Devices. Springer Berlin Heidelberg, 2010.
106-121.

[7] Chen, Ning. "On the approximability of influence in social networks."
SIAM Journal on Discrete Mathematics 23.3 (2009): 1400-1415.