The Origin of the Dynamical Quantum Non-locality

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Non-locality is one of the hallmarks of quantum mechanics and is responsible for paradigmatic features such as entanglement and the Aharonov-Bohm effect. Non-locality comes in two “flavours”: a kinematic non-locality—arising from the structure of the Hilbert space—and a dynamical non-locality—arising from the quantum equations of motion. Despite of its main role in quantum information processing, kinematic non-locality is unable to induce any change in the probability distributions, so that the “action-at-a-distance” cannot manifest. Conversely, dynamical non-locality does create explicit changes in probability, though in a “causality-preserving” manner. Recently, the origin of the kinematic non-locality was related to the uncertainty principle, here we trace the origin of dynamical non-locality to the superposition principle. This relation adds to the more fundamental understanding of the nature of quantum dynamics and allows us to establish and identify how the uncertainty and the superposition principles determine the non-local character of the outcome of a quantum measurement. Thus, dynamical non-locality emerges as the responsible of the breakdown of the dynamical classical realism and therefore, as key feature in the classical-quantum transition. Most importantly, being based on group theoretical and path integral formulations, our formulation admits immediate generalizations and extensions to, e.g., quantum field theory.

Since its formulation in the first decades of the XXth century, quantum mechanics, the physics theory of the atomic and molecular phenomena, has proved to be one the of the most successful theories ever conceived. In these more that hundred years, quantum mechanics has not only provided a complete picture of the atomic and the subatomic realms, but also has been the fulcrum for the development of our more complete understanding of Nature, the standard model of elementary particles. Despite of its success in the long way till now, it is fair to say that our understanding of the quantum theory is inabsolutus.

Undoubtedly, non-locality is the most intriguing and barely understood phenomenon of the quantum theory. However, the great interest in developing the field of quantum information processing has provided some physical insights into the kinetic flavour of the non-locality. In particular, it has been related to the Hilbert space structure, formulated in terms of the Bell inequalities and experimentally verified. In the modern formulation of the quantum theory, the kinematic character has been translated in terms of entropic measures and has been even extended to more non-local and involved theories than quantum mechanics. By contrast, our understanding of the dynamical character seems to have been frozen in time since its original formulation by Aharonov and Bohm. Despite the fact that the dynamical character has also been tested experimentally, in terms of understanding, it has been at the shadow of the kinematic non-locality.

To introduce the dynamical character of the quantum theory, we take the discussion where the last significant progress was made by Aharonov et al., more than four decades ago, in terms of the non-local equation of motion of the modular variables. In doing so, let us consider, e.g., the displacement operator defined as \( \hat{D} = e^{\hat{p}q/L} \), where \( \hat{p} \) denotes the momentum operator, \( L \) is the length of the displacement and \( h \) stands for the reduced Planck’s constant, \( h = \hbar/2\pi \). When acting on a wave function \( \psi(q) \), the displacement operators leads to \( \psi(q+L) \). This operator can be decomposed into its Hermitian (in principle measurable) components \( \cos(\hat{p}L/h) \) and \( \sin(\hat{p}L/h) \) or re-expressed in terms of the modular momentum, \( \hat{p} \) mod \( 2\pi L \). For simplicity, below we focus on the displacement operator itself.

Under the action of a one-dimensional standard Hamiltonian \( \hat{H} = \hat{p}^2/2m + V(q) \), where \( m \) is the mass of the particle and \( V(q) \) is the potential energy, the displacement operator evolves “classically” according to

\[
\frac{d e^{i\hat{p}q/L}}{dt} = -\frac{i}{\hbar} \frac{dV(q)}{dq} e^{i\hat{p}q/L},
\]

while quantum mechanically according to

\[
\frac{d e^{i\hat{p}q/L}}{dt} = -\frac{i}{\hbar} [V(q+L) - V(q)] e^{-i\hat{p}q/L}.
\]

The direct comparison of the classical and quantum equations of motion, clearly, expresses the non-local character of the quantum evolution provided by the fact that it depends on the potential at two different places, \( V(q+L) \) and \( V(q) \). This result, known as the scalar Aharonov-Bohm effect, suggests that the quantum dynamics is fundamentally non-local, which is in sharp contrast with the local evolution of the wave function (or probability amplitude) governed by the Schrödinger equation.

As it was pointed out by Popescu, the non-local character of its quantum dynamics is not the only interesting feature about the displacement operator, it is also the
only operator that is able to encode the key information of quantum interference experiments: the relative phase between interfering wave packets\(^5\). This fact is of foundational relevance because it would provide insights into the design of purely deterministic quantum experiments to unravel the intricate and non-local character of the quantum interference\(^5,6\). Hence, understanding the non-local time evolution of the displacement operator would shed new light on the non-local character of the quantum interference.

But why is the displacement operator so special? Why has the dynamical non-locality been entirely formulated in terms of modular variables? Could it be observed in other operators? What is the origin of the dynamical non-locality? Having a locally time evolving wave function, how could we understand the non-local character of the quantum dynamics in the Copenhagen interpretation? Is the quantum dynamics fundamentally non-local as suggested by the Aharonov-Bohm effect or is it effectively non-local? These are just a few of the foundational questions our current comprehension of the quantum dynamics has no answer for but our formulation below adds to.

In order to appreciate the crucial role of the displacement operator and of the modular variables, we start by making a connection with the formulation of quantum mechanics developed by Weyl back in 1927\(^9\). In doing so, we need to realize that a physical system with \( f \) degree of freedoms could be described in terms of its state or in terms of its observables and the constraints on them\(^20\), i.e., in terms of the algebra of observables. In general, the observables of a system are functions of the position of the system \( q \) and its momentum \( p \), which turns out to be operators in quantum mechanics. If we were about to define a quantum system in terms of an algebra generated by the operators \( \hat{q} \) and \( \hat{p} \), we need to consider the Heisenberg commutation relations

\[
\hat{q} \hat{p} - \hat{p} \hat{q} = i\hbar, \quad \hat{q} \hat{q} = 0, \quad \hat{p} \hat{p} = 0. \tag{3}
\]

The vector space generated by the operators \( \hat{q} \) and \( \hat{p} \) with the commutation relations in equation (3) define what is known as the Heisenberg Lie algebra\(^20\) (see Section M1 for more details on this formulation).

Weyl noted that the commutation relations in equation (3) imply that \( \hat{q} \) and \( \hat{p} \) cannot be given a finite norm\(^19,20\). Since the measurement of physical observables made by a measurement device is expected to be bounded, \( \hat{q} \) and \( \hat{p} \) are unphysical choices to construct the algebra of quantum observables. Thus, Weyl suggested to consider instead the polynomial algebra generated by the bounded functions \( \hat{P}_u \) and \( \hat{Q}_v \) with \( u, v \in \mathbb{R}^d \). In shorthand notation, the algebra of the quantum observables is generated by the operator

\[
\hat{T}(u, v) = e^{i\phi(u, v)} \hat{P}_u \hat{Q}_v, \tag{4}
\]

where \( \phi(u, v) = v \cdot u / \hbar^2 \). In physical terms, that \( \hat{T}(u, v) \) is the generator means that any observable can be expressed in terms of it, in the same way as any vector in space can be expressed in terms of the unitary vectors along the \( x \), \( y \) and \( z \) axes.

For convenience, we assume that the system we want to characterize has only one degree of freedom, \( f = 1 \), and take then the double Fourier transform of \( \hat{T}(u, v) \)

\[
\hat{d}(p, q) = \frac{1}{2\pi\hbar} \int dudv \exp \left[ \frac{i}{\hbar} (up + vq) \right] \hat{T}(-u, -v), \tag{5}
\]

where \( p, q \in \mathbb{R} \) and \( \text{Tr}(\hat{d}) = 1 \). By means of this transformation, we have associated an operator \( \hat{d}(p, q) \) to each point \( r = (p, q) \) of the phase space. Since \( \hat{T} \) is unitary, then one can easily show that \( \hat{d}(p, q) \) is a Hermitian operator and therefore, a legitimate observable. Being \( \hat{d}(p, q) \) the Fourier transform of the generator of the algebra \( \hat{T}(u, v) \), we can also expressed any operator associated to the physical system in terms of \( \hat{d}(p, q) \). In particular, if we represent the state of a quantum system by means of the density operator \( \hat{\rho} \), then we have that

\[
\hat{\rho} = \int dr \rho_W(r) \hat{d}(r), \tag{6}
\]

where \( \rho_W(r) \) can be interpreted as the coefficients of the expansion of \( \hat{\rho} \) in terms of \( \hat{d}(r) \). The “generalized” displacement operator \( \hat{d}(p, q) \) is then the most fundamental basis in quantum mechanics and therefore, it is expected that all its intricate features are encoded in it. Since \( \hat{d}(p, q) \) is constructed with the “modular” momentum \( \hat{P}_u \) and the “modular” position \( \hat{Q}_v \) variables, we can now understand why the modular variables are so special.

Under the action of the Hamiltonian \( \hat{H} \), the state of the system at time \( t \) is given by \( \hat{\rho}(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}^\dagger(t) \), where \( \hat{U}(t) = \exp(-i\hat{H}t/\hbar) \) is the unitary time-evolution operator and \( \hat{U}^\dagger(t) \) denotes its adjoint. In terms of the generator \( \hat{d}(r) \) we have that

\[
\hat{\rho}(t) = \int dr'' \rho_W(r'', t) \hat{d}(r''), \tag{7}
\]

where

\[
\rho_W(r'', t) = \int dr' G_W(r'', r'; t; 0) \rho_W(r', 0) \tag{8}
\]

are just the expansion coefficients at time \( t \) (see the Supplementary Information for details). The function \( G_W(r'', t; r'; 0) \) transforms the expansion coefficients at \( t = 0 \) into the expansion coefficients at the later time \( t \). Thus, the non-local or local character of the quantum dynamics is fundamentally encoded in the time evolution of \( \rho_W(r'', t) \) and, more concretely, in the function \( G_W(r'', t; r'; 0) \).

If we invert the equation (7) in favour of \( \rho_W(r'', t) \) [see equation (A5) below], it turns out that the expansion coefficients are nothing but the Wigner function associated to the state \( \hat{\rho} \). Alternately, note that up to a constant
factor, the Wigner function can be interpreted as the expectation value of the generator of the algebra of quantum observables $\hat{d}(r)$. In the past, the Wigner function already offered a unique scenario for the study of quantum dynamical features such as decoherence and “scars” of classical chaotic systems. In particular, Zurek showed that the phase space structures of the Wigner function associated with sub-Planck scales ($\ll \hbar$), originated from interference processes, are not only physically meaningful but are also related to fundamental aspects of the classical-quantum transition. The Wigner function, and in particular its propagator (see below), also allowed for the discovery of scars that, in contrast to scars in wave functions, are not subject to the uncertainty relation and therefore need not show any smearing. These scars in the propagator constitute important exceptions from a continuous convergence in the classical limit.

Our observation above not only explains the privileged role of the Wigner function in the description of interference processes in quantum mechanics, but also enables the entire field of quantum homodyne tomography (experimental reconstruction of the Wigner function) for the study of fundamental phenomena such as the dynamical non-locality. Thus suggesting the possibility of performing measurements of the dynamical breakdown of the classical realism (violations of Bell’s inequalities due to the time evolution). Moreover, this not only suggests that in order to understand the dynamical non-locality we need to understand the time evolution of the Wigner function, but also offers the possibility of formulating the dynamical non-locality in different contexts and in different theories (we shall return to this point below).

If $\rho_W(r^n, t)$ is the Wigner function, then $G_W(r^n, t; r', 0)$ is its propagator. In the classical limit, the propagator of the Wigner function has a clear and well defined analog—the Frobenius-Perron or Livonille propagator, which corresponds to a Dirac delta function along the solution of the classical equations of motion. By focusing on the discussion of the Wigner propagator, we isolate the dynamics features from the state of the system, and we study then the dynamical non-locality in terms of a dynamical object instead of doing so in a dynamical equation of motion. Undoubtedly, this is a great advantage in order to compare quantum and classical dynamics on the same ground.

In close analogy to Feynman’s path-integrals, the propagator of the Wigner function can be expressed in terms of Marinov’s path-integrals as

$$G_W(r^n', r') = \frac{1}{\hbar} \int D^2r D^2\tilde{r} \exp \left(\frac{i}{\hbar} S[\{r\}, \{\tilde{r}\}, t] \right),$$

where $D^2r$ and $D^2\tilde{r}$ denote a summation over all possible paths with initial point at $r(0) = r'$ and time evolved in the presence of the fluctuations $\tilde{r}$ (see the Supplementary Information for details). If we assume the same standard Hamiltonian as above, $\hat{H}(p, q) = \frac{\hbar}{2} p^2 + V(q)$, then the action in equation (A17) is given by

$$S[\{r\}, \{\tilde{r}\}, t] = \int_0^t dt' \left[ V(q + \tilde{q}/2) - V(q - \tilde{q}/2) - \tilde{q} \frac{dV(q)}{dq} \right],$$

where $q$ and $\tilde{q}$ are functions of time. A crucial remark follows at this point: The propagator of the Wigner function contains the non-local terms $V(q + \tilde{q}/2) - V(q - \tilde{q}/2)$ of the quantum evolution in equation (2) and associated to the potential effect as well as the local term $-\tilde{q} dV(q)/dq$ of the classical evolution in equation (1) and associated to the classical force. Thus, the dynamical quantum non-locality is imprinted in the very generator of the quantum dynamics and is not an obscure curiosity of a particular operator.

Non-locality and non-linearity—For the case of a harmonic oscillator of natural frequency $\omega$, the potential energy is $V_{osc} = \frac{1}{2} m \omega^2 q^2$, so that $V_{osc}(q + \tilde{q}/2) - V_{osc}(q - \tilde{q}/2) = m \omega^2 \tilde{q}q$ and $\tilde{q} dV(q)/dq = m \omega^2 \tilde{q}q$. Thus, for a harmonic oscillator the quantum and the classical dynamics coincide, this turns out to lead to the fact that the resulting $G_W(r^n, r')$ is a structureless delta function along the classical trajectory. Thus, deviations from the classical evolution are only expected for non-linear interactions: a very old and known fact; however, what equation (A16) is really expressing is that the dynamical non-local character of quantum mechanics, and therefore the breakdown of the classical realism, is provided by the degree of non-linearity of the system!

Non-linear interactions are everywhere in Nature, from atoms and elementary particles all the way up to stars and galaxies. There are four fundamental interactions in Nature: the weak and the strong interactions (dominant at sub-atomic scales), the electromagnetic interactions (dominant at atomic and molecular scales) and the gravitational interaction (dominant at cosmological scales). All of them, with no exception, lead to non-linear dynamics. Thus, it should be no surprising that there is a dynamical non-local aspect to every interaction in Nature, a non-local aspect that remains vastly unexplored and unexploited.

In order to explore our ideas in a concrete case, let us consider a diatomic molecule and assume that the interaction between the atoms is described by means of the Morse potential, $V(q) = D_0 [1 - e^{\alpha(q-q_0)}]^2$, where $q$ is the distance between atoms, $q_0$ is the equilibrium distance, $D_0$ is the well depth (deep of the potential defined relative to the dissociated atoms) and $\alpha$ controls the width of the potential. This potential energy describes a highly non-linear interaction and it is an accurate model, e.g., of the nitrogen molecule $N_2$. For small distances from $q_0$, $V(q)$ can be taken in the harmonic approximation, $V(q) \approx \frac{1}{2} \omega_M^2 (q - q_0)^2$, where $\omega_M = \sqrt{2 D_0 / m}$ and $m$ denotes the reduced mass of the two atoms.

In Fig. 4, we have depicted the Wigner propagator $G_W(r^n, r_0)$ as a function of $r^n = (q^n, p^n)$ for the Morse...
interaction potential (left panel) and for its harmonic version (right panel) for a given initial condition \( r_0 = (0, 0.1) \) at approximately one quarter of the period of the underlying classical orbit. The fact that the propagator of the Morse potential exhibits a highly structured pattern is an unambiguous hallmark of a non-local, in position and momentum, dynamics because it means that at time \( t \), the propagation of a single point depends on different regions of phase space. By contrast, the propagator of the harmonic version consists of a structureless Dirac delta function, in position and momentum, and therefore can unambiguously associated to a local dynamics.

Non-locality and the uncertainty principle—Recently, it was suggested that the uncertainty principle determines the non-locality of quantum mechanics\(^8\); however, this claim can only be applied in the kinematic sense. In order to explore the effect of the non-locality, kinematic as well as dynamical, in the measurement of a given observable \( O \), we calculate the expectation value of \( O \) in terms of the Wigner function, i.e., \( \langle \hat{O}(t) \rangle = \int dr'O_W(r', t)\rho_W(r', 0) \) or

\[
\langle \hat{O}(t) \rangle = \int dr'O_W(r') \int dr'G_W(r', t; r', 0)\rho_W(r', 0),
\]

where we have made use of equation (8). Hence, the non-local character of a quantum measurement is determined by the dynamical non-locality provided by \( G_W(r', t; r', 0) \) and by the kinematic non-locality provided by \( \rho_W(r', 0) \). Not being a state, it is clear that the propagator of the Wigner function is not restricted by the uncertainty principle\(^{22,24,28}\) Note, e.g., that at \( t = 0 \), \( G_W(r'', 0; r', 0) = \delta(r'' - r') \) and that for the harmonic case discuss above, \( G_W(r'', t; r', 0) = \delta[r'' - r^{cl}(r', t)] \), being \( r^{cl}(r', t) \) the classical trajectory with initial condition \( r' \). Thus, it is evident that the uncertainty principle influences the outcome of the measurement only throughout \( \rho_W(r', 0) \), in a kinematic sense.

Non-locality and the superposition principle—At this point, a natural question is, if the Wigner propagator is not restricted by the uncertainty principle, is there any relation between the latter and the non-locality encoded in the former? The answer is no, the dynamical non-locality is generated by the superposition principle, a dynamical principle. As a motivation for the discussion on this respect, note that for the quantum harmonic description of the diatomic molecule above, even though the dynamics are local, no dynamical non-locality at all, its canonical conjugate variables \( \hat{p} \) and \( \hat{q} \) still do not commute. This means that the observables are still restricted by the uncertainty principle encoded in the initial state of the system \( \rho_W(r', 0) \).

We could argue in several ways about the connection between the dynamical non-locality and the superposition principle. However, in its simplest formation, the origin of this connection can be understood directly from the path integral expression in equation (A17). There we see that the propagator is constructed from the interference of multiples trajectories. To pursue in this direction while keeping our discussion as simple as possible, we resort here to the semiclassical description of the Wigner propagator\(^{24,27,28}\) (see section M3 below) and sketch the general case in the Supplementary Information. By semiclassical description we mean that the path integral expression we mean that the path integral expression in equation (A17) is evaluated along the leading contributing trajectories\(^{24,27,28}\). By doing so, the following simple and appealing scenario naturally emerges\(^{24,27,28,30}\); every single point \( r' \) in phase space is propagated by all the trajectory pairs \( r_{j,\pm} \) for which \( r' \)
When we refer to propagate a point $r$ for the mass $m\sqrt{\lambda}$ the limit at which the number of bounded states, are dealing with, the classical limit can be interpreted as of a general theory of open quantum systems in phase for the classical-quantum transition because of the lack of the reduced Planck constant. We follow this approach equation (A16) changes dramatically at scales of the order of the potential in such way that the action $S$ considering large masses or the values of the parameters of expected to be absent. This limit can be reached by con-

| $\hbar$ | $m_{12}$ | $M_{q}$ |
|-------|---------|---------|

in Fig. 4. Hence, the interference of all possible paths gen-

Non-locality and the classical-quantum transition—In the classical limit $\hbar \to 0$, dynamical non-locality is expected to be absent. This limit can be reached by con-

Although, as we mentioned, in the presence of decoherence a formal characterization of the Wigner propagator is not available yet, we can certainly anticipate that decoherence will suppress the dynamical non-

Discussion—In Wheeler’s words\textsuperscript{11} “Quantum physics [...] has explained the structure of the atoms and molecules, the thermonuclear burning that lights the stars, [...] and the comings and goings of particles from neutri-

is the midpoint between their initial conditions $r_{j+}^\prime$ and $r_{j-}^\prime$ (see Figs. 2.a and 2.b). The trajectories $r_{j+}^\prime$ and $r_{j-}^\prime$ evolve in time under the action of the associated classical equation of motion, and at time $t$ they reach the endpoints $r_{j+}^\prime\prime$ and $r_{j-}^\prime\prime$. The contribution from each pair to the evolution of $r'$ is then located in the midpoint between $r_{j+}^\prime\prime$ and $r_{j-}^\prime\prime$. Note that in general, the classical propagation of the initial midpoint $\bar{r}_j$, between $r_{j+}^\prime\prime$ and $r_{j-}^\prime\prime$, does not correspond to the midpoint between the endpoints $r_{j+}^\prime\prime$ and $r_{j-}^\prime\prime$ (see Fig. 2.b). This fact plays a determinant role in the discussion below.

We could think of the propagation in the following way: When we refer to propagate a point $r'$, we indeed refer to propagate the probability of finding the system at that particular point. However, since the probability is constructed from the probability amplitude, the probability of finding the system at $r'$ will depend on how the probability amplitude of finding the system at other points interfere at $r'$. Thus, during the course of the time evolution, the interference between the probability amplitude of finding the system at, e.g., $r_{j+}^\prime\prime$ and $r_{j-}^\prime\prime$ will be located, as in the case of a cat state, at the midpoint between them, which of course, does not follow any classical trajectory. This connection between the interference of the probability amplitude and non-locality is the first element in constructing a “Copenhagen interpretation” for the quantum dynamics.

For harmonic systems (see Fig. 2.a), the contribution from every pair is located exactly over the classical solution $r^{cl}(r', t)$ of the equation of motion with initial condition $r'$. In contrast, for non-linear systems (see Fig. 2.b), the contribution could be located anywhere. These contributions outside $r^{cl}(r', t)$ makes the propagator to be non-local and differ from a structureless point at $r^{cl}(r', t)$ as observe in Fig. 4. Although in the Supplementary Information we discuss the general case, in Fig. 2.c, we show that the semiclassical picture is enough to reproduce the non-local character of the quantum propagator depicted in Fig. 4. Hence, the interference of all possible paths generate a complex pattern which, unambiguously, expresses the non-local character of the quantum evolution.

For the particular case of the diatomic molecule we are dealing with, the classical limit can be interpreted as the limit at which the number of bounded states, $\Lambda = \sqrt{2mD_{q}/\hbar}$, is large. So, we take, e.g., a small value for the mass $m$ and increase it systematically in order to see how the dynamical non-locality just disappears. In Fig. 3, we have depicted the Wigner propagator for this situation at times equivalent to a quarter of the natural period, $2\pi/\omega_{M}$, in each case. As we increase the mass, we see how the Wigner propagator has less and less structure, a feature related to the vanishing of dynamical the non-locality in the classical limit.

FIG. 3. Disappearance of the dynamical non-locality in the classical limit. We present the Wigner propagator of the Morse oscillator with the parameters used in Fig. 4, but four different values of the mass: (a) $m = 0.25$, (b) $m = 1$, (c) $m = 2$ and (d) $m = 10$. It is clear how the propagator looses its oscillatory structure, and therefore its non-locality character, when entering into the classical regime (larger masses).
Balancing the glory of quantum achievements, we have the shame of not knowing “how come.” Why does the quantum exist?” Although, Wheeler’s words represent the concerns of generations of scientists, they do not make any reference of the eeriness of the quantum dynamics. In this direction, Aharonov et al.’s efforts\textsuperscript{2,3,5,6,15,17} are not only the first but also the only stones, in more than four decades, in the conceptual construction of the underlying physical picture of the quantum dynamics. The connection presented here between the superposition principle and the dynamical non-locality, based on the solid grounds of the group theoretical and path integrals formulations, contributes to the development of that conceptual construction.

We consider that the fact that the interference of local evolutions accounts for the non-local character of the dynamics (cf. Fig. 2 and the Supplementary Information) has the power of suggesting a reinterpretation of the non-local character of the quantum dynamics itself, and straightforwardly explains why quantum dynamics preserves causality. Hence, before a deeper understanding of the kinematic non-locality is developed, it differs conceptually from dynamical non-locality by the fact that the latter is an effective non-locality. Below we discuss some of the consequences and questions arisen by our formulation.

Quantum Chaos and Dynamical Non-Locality—We consider that this new way of looking at quantum evolution could, e.g., finally explain how chaos is suppressed in the quantum realm. At the quantum level, dynamical non-locality requires that points in phase space located at regular or chaotic regions are propagated taking into account contributions by trajectory pairs formed of regular-regular, regular-chaotic or chaotic-chaotic initial conditions. This generates a dynamics neither completely regular nor completely chaotic. When approaching the classical limit, only contributions pertaining to the same region, regular or chaotic, contribute and finally in the classical realm, the pair of trajectories coalesces into a single point, defining so the nature of the evolution: chaotic or regular.

Dynamical Non-locality, Hilbert Space Structures and Extensions—We consider that one of the reasons of why our understanding of dynamical non-locality is at the shadow of our understanding of kinematic non-locality is the unfounded believe that this is a curiosity of the modular variables. After the treatment presented here, the formulation of the dynamical non-locality in terms of Hilbert space structures, as well as its generalization to other theories, is straightforward. In equation (4) we write the general displacement operator as $\hat{T}(u, v) = e^{i\phi(u,v)} \hat{P}_u \hat{Q}_v$, which is nothing but the generator of the algebra of observables. It is also the generator of the quantum dynamical group of quantum mechanics, namely, the Weyl-Heisenberg group. Thus, the extension to, e.g., finite dimensional Hilbert spaces (as in the case of spin systems) involves the generators of Pauli’s group\textsuperscript{25}, while the generators of Poincaré’s group are involved in the extension to quantum field theory\textsuperscript{25}.

Dynamical Non-locality and the Aharonov-Bohm Effect—In the seminal works of Aharonov and Bohm\textsuperscript{2,3}, their effect was completely understood in terms of the exchange of modular momentum\textsuperscript{5,6,15,17}. Recently, based on the non-local time-evolution of the modular momentum (cf. equation 2), Aharonov offered an appealing interpretation of quantum interference in the Heisenberg picture\textsuperscript{18}. In the light of our formulation, the dynamics inside the region with no field is affected by the region with non-vanishing field, not non-locally, but via the interference between the local dynamics in each region. This is quite similar to the case of the chaotic and regular contributions discussed above and can be extended to the case of tunnelling\textsuperscript{24,28}.

Dynamical Non-locality and Entanglement—Entanglement between quantum systems appears after they interact. Since their interaction will induce a dynamical situation, which will be described by the corresponding Wigner propagator, it will not be surprising if a relationship between entanglement and dynamical non-locality could be established.

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Methods

\textbf{M1: Group Theoretical Formulation of Quantum Mechanics}—In classical mechanics, the state of an $f$-dimensional system is described by its position $\mathbf{q}$ and its momentum $\mathbf{p}$, i.e., in classical mechanics the state of a system is a point in phase space. In a more general context, a classical system is defined by the algebra of observables $\mathcal{A}$, which in the case of a compact phase space can be obtained in terms of the sup-norm closure of the polynomial algebra generated by $\mathbf{q}$ and $\mathbf{p}$. If we were about to define a quantum system in terms of an algebra generated by the operators $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$, we need to consider the Heisenberg commutation relations in equation (3). The vector space, over the field of complex numbers, generated by the operators $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ with the Lie products in equation (3) define the so-called Heisenberg Lie algebra\textsuperscript{20}.

In 1927, Weyl noted that the Lie products in equation (3) imply that $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ cannot be self-adjoint elements of a $C^*$-algebra because they cannot be given a finite norm\textsuperscript{19,20}. The need of a $C^*$ algebra is motivated by the von Neumann theorem which states that all the regular representation of a $C^*$ algebra are unitarily equivalent. This independence of the representation is certainly a desirable feature in any physics theory. Hence, Weyl suggested to consider instead the polynomial algebra generated by the bounded (normal) functions of $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$

$$\hat{P}_u = e^{iu \hat{\mathbf{p}}/\hbar}, \quad \hat{Q}_v = e^{iv \hat{\mathbf{q}}/\hbar}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$  

(11)
satisfying \( \hat{Q}_v \hat{P}_w = \hat{P}_w \hat{Q}_v e^{-i \nu w / \hbar} \), \( \hat{Q}_v \hat{Q}_v' = \hat{Q}_{v+v'} \) and \( \hat{P}_w \hat{P}_w' = \hat{P}_{w+w'} \) which can be compactly written, as we did in equation (4), as \( \hat{T}_{(w,v)}(q,p) = e^{i (u \cdot v - \nu / \hbar)} \hat{P}_w \hat{Q}_v \). This leads to the Heisenberg-Weyl algebra, which supports the Heisenberg-Weyl group \( \mathbb{U} \).19,20

\[ M_2: \text{The Phase-Space Formulation of Quantum Mechanics—The fact that the } T_{(w,v)}(q,p) \text{’s, or equivalently the } \hat{d}(p,q) \text{’s, are the generator of the algebra of observables, means that every element of the Heisenberg-Weyl group can be written as a combination of them, i.e., } \hat{\rho} = \int \hat{d}(p,q) \hat{d}(p,q) \text{, where } \rho_W(p,q) \text{ is the Weyl symbol of } \hat{\rho}. \text{ By inverting this relation, we obtain} \]

\[ \rho_W(p,q) = \frac{1}{2\pi \hbar} \text{Tr}\left[\hat{\rho} \hat{d}(p,q)\right], \]

so that we can interpret the Weyl symbol of an operator, as the coefficients of the expansion of that operator onto the generators of the algebra of observables.

Although all the irreducible representations of the Heisenberg-Weyl group, e.g., the Schrödinger or the phase-space representations, are equivalent, the description above unambiguously shows the privileged character of the phase-space representation. In harmonic analysis and group theoretical applications beyond quantum mechanics, the phase space representation also enjoys a privileged position.20

\[ M_3: \text{Semiclassical Approximation of Quantum Dynamics—The semiclassical Wigner propagator}^{24,27,28} \text{ is given by} \]

\[ G_W(r''',t';r'',0) = \frac{4^d}{h^d} \sum_j 2 \cos \left( \frac{1}{2} s^V_j (r''',r'',t) - \nu_j \frac{\pi}{2} \right) \left( \frac{\det |M_{j+k} - M_{j-k}|}{|S_j^V(r''',r'',t)|} \right)^{1/2}, \]

where \( S_j^V(r'',r',t) = \int_0^t ds \left[ \hat{f}_j(s) \wedge \hat{r}_j(s) - H_{+j}(r_{j+}) + H_{-j}(r_{j-}) \right] \), \( \hat{f}_j(s) \equiv \{ r_{j+}(s) - r_{j-}(s) \} / 2 \) and \( \hat{r}_j(s) \equiv r_{j+}(s) - r_{j-}(s) \). \( M_{j+k} \) denotes the stability matrix along the trajectory \( r_{j+}, \nu_j \) is the Maslov’s index associate to the trajectory pair \( j \) and \( \wedge \) stands for the symplectic product. As illustrated by Fig. (2), the sum is over all the trajectories \( r_{j+} \) for which \( r'' \) is the midpoint between their initial conditions. See the Supplementary Information for more details.

\[ 1 \quad \text{Einstein, A., Podolsky, B. & Rosen, N. Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777–780 (1935).} \]
\[ 2 \quad \text{Aharonov, Y. & Bohm, D. Significance of electromagnetic potentials in the quantum theory. Phys. Rev. 115, 485–491 (1959).} \]
\[ 3 \quad \text{Aharonov, Y. & Bohm, D. Further considerations on electromagnetic potentials in the quantum theory. Phys. Rev. 123, 1511–1524 (1961).} \]
\[ 4 \quad \text{Bell, J. S. On the problem of hidden variables in quantum mechanics. Rev. Mod. Phys. 38, 447–452 (1966).} \]
\[ 5 \quad \text{Popescu, S. Dynamical quantum non-locality. Nature Phys. 6, 151 (2010).} \]
\[ 6 \quad \text{Tollaksen, J. Dynamical quantum non-locality. AIP Conf. Proc. 1327, 269 (2011).} \]
\[ 7 \quad \text{Popescu, S. & Rohrlich, D. Quantum nonlocality as an axiom. Found. Phys. 24, 379–385 (1994).} \]
\[ 8 \quad \text{Oppenheim, J. & Wehner, S. The uncertainty principle determines the nonlocality of quantum mechanics. Science 330, 1072 (2010).} \]
\[ 9 \quad \text{Gat, O. Quantum dynamics and breakdown of classical realism in nonlinear oscillators. J. Phys. A: Math. and Theor. 40, F911.} \]
\[ 10 \quad \text{Feynman, R. P. & Hibbs, A. R. Quantum physics and path integrals (McGraw-Hill, New York, 1965).} \]
\[ 11 \quad \text{Wheeler, J. A. ‘A Practical tool,’ but Puzzling Too. The New York Times, December 12, (2000).} \]
\[ 12 \quad \text{Zurek, W. H. Decoherence, einselection, and the quantum origins of the classical. Rev. Mod. Phys. 54, 685 (2003).} \]
\[ 13 \quad \text{Buhrman, H., Cleve, R., Massar, S. & de Wolf, R. Nonlocality and communication complexity. Rev. Mod. Phys. 82, 665–698 (2010).} \]
\[ 14 \quad \text{Aspect, A., Grangier, P. & Roger, G. Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A new violation of Bell’s inequalities. Phys. Rev. Lett. 49, 91–94 (1982).} \]
\[ 15 \quad \text{Aharonov, Y., Pendleton, H. & Petersen, A. Modular variables in quantum theory. Int. J. Theor. Phys. 2, 213–230.} \]
\[ 16 \quad \text{Osakabe, N. et al. Experimental confirmation of Aharonov-Bohm effect using a toroidal magnetic field confined by a superconductor. Phys. Rev. A 34, 815–822 (1986).} \]
\[ 17 \quad \text{Tollaksen, J., Aharonov, Y., Casher, A., Kaufherr, T. & Nussino, S. Quantum interference experiments, modular variables and weak measurements. New J. Phys. 12, 013023 (2010).} \]
\[ 18 \quad \text{Aharonov, Y. On the Aharonov-Bohm Effect and Why Heisenberg Captures Nonlocality Better Than Schrödinger. ArXiv e-prints (2013). 1303.2137.} \]
\[ 19 \quad \text{Weyl, H. Quantummechanik und gruppentheorie. Z. Phys. 46 (1927).} \]
\[ 20 \quad \text{Strocchi, F. An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians: Lecture Notes. Advanced series in mathematical physics (World Scientific Publishing Company Incorporated, 2005).} \]
\[ 21 \quad \text{Zurek, W. H. Sub-Planck structure in phase space and its relevance for quantum decoherence. Nature 412, 712 (2001).} \]
\[ 22 \quad \text{Dittrich, T. & Pachón, L. A. Time-domain scars: resolving the spectral form factor in phase space. Phys. Rev. Lett. 102, 150401 (2009).} \]
\[ 23 \quad \text{Heller, E. J. Bound-state eigenfunctions of classically chaotic Hamiltonian systems: Scars of periodic orbits. Phys. Rev. Lett. 53, 1515–1518 (1984).} \]
\[ 24 \quad \text{Pachón, L. A. Coherence and Decoherence in the Semiclassical propagation of the Wigner function. Ph.D. thesis, Universidad Nacional de Colombia (2010).} \]
\[ 25 \quad \text{Duque, S., Botero, J. D. & Pachon, L. A. Dynamical quantum non-locality in finite dim. Hilbert spaces and quantum field theory In preparation. (2013).} \]
\[ 26 \quad \text{Marinov, M. S. A new type of phase-space integrals. Phys. Rev. Lett. A 153, 5 (1991).} \]
\[ 27 \quad \text{Dittrich, T., Viviescas, C. & Sandoval, L. Semiclassical propagator of the Wigner function. Phys. Rev. Lett. 96, 070403 (2006).} \]
\[ 28 \quad \text{Dittrich, T., Gómez, E. A. & Pachón, L. A. Semiclassical propagation of Wigner functions. J. Chem. Phys. 132,} \]
Appendix A: Quantum Dynamics in Phase Space

Before discussing the phase space formulation of quantum mechanics, we briefly review the standard formulation in terms of the density operator \( \hat{\rho} \). For an isolated system described by a time-independent Hamiltonian \( \hat{H} \), the time evolution of an initial density operator \( \hat{\rho}(0) \) is determined by the Landau-von Neumann equation, \( \frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}] \), whose solution can be determined through the unitary time-evolution operator

\[
U(t) = \exp\left(-\frac{i}{\hbar} \hat{H}t\right)
\]

and its adjoint operator \( U^\dagger(t) \) by means of the relation

\[
\hat{\rho}(t) = U(t)\hat{\rho}(0)U^\dagger(t).
\]

In position representation, this expression turns into

\[
\rho(q_+^n, q_-^n, t) = \int dq_+^0 dq_-^0 J(q_+^n, q_-^n, t; q_+^0, q_-^0, 0) \rho(q_+^0, q_-^0, 0),
\]

where

\[
J(q_+^n, q_-^n, t; q_+^0, q_-^0) = U(q_+^0, q_+^n, t)U^*(q_-^0, q_-^n, t),
\]

with \( U(q_+^0, q_+^n, t) = \langle q_+^n | U(t) | q_+^0 \rangle \) and \( \rho_0(q_+, q_-) = \langle q_+ | \rho_0 | q_- \rangle \).

Our goal below is to use the generator of the algebra of observables to translate the expression (A1) into the phase space language. In doing so, we discuss first, in detail, the composition law of the generators and how to construct the expansion coefficients for the case of a product of operators.

1. Phase Space Formulation of Quantum Mechanics

In section Methods we already introduced the standard formulation of quantum mechanics developed by Weyl and Wigner and Moyal. Having in mind its relation with dynamical-non-locality, and the fact that we want to study the product of three operators \( U(t)\hat{\rho}(0)U^\dagger(t) \), we review now the interference of generators \( \hat{d}(\mathbf{r}) \) in the product of operators.

In doing so, we recall that the Weyl symbol \( \hat{O}_W(\mathbf{r}) \) of an operator \( \hat{O} \) is defined by

\[
\hat{O}_W(\mathbf{p}, \mathbf{q}) = \text{Tr} \left[ \hat{O}\hat{d}(\mathbf{r}) \right],
\]

214102 (2010).
29 Pachón, L. A., Ingold, G.-L. & Dittrich, T. Nonclassical phase-space trajectories for the damped harmonic quantum oscillator. *Chem. Phys.* **375**, 209 (2010).
30 de M. Rios, P. P. & Ozorio de Almeida, A. M. On the propagation of semiclassical Wigner functions. *J. Phys. A* **35**, 2609 (2002).
31 Wigner, E. P. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.* **40**, 749 (1932).
32 Moyal, J. E. Quantum mechanics as a statistical theory. *Proc. Camb. Phil. Soc.* **45**, 99 (1949).
33 Note that in equation (12) of the main text, we define the Weyl symbol with an additional scale factor \((2\pi\hbar)^{-1}\), this is just to be consistent with the fact that the Wigner function is defined as the Weyl symbol of \( \hat{\rho}/2\pi\hbar \).
34 van Vleck, m. H. The correspondence principle in the statistical interpretation of quantum mechanics. *Proc. Natl. Acad. Sci. U.S.A.* **14**, 178 (1928).
35 Marinov, M. S. Path integrals in quantum theory: an outlook of basic concepts. *Phys. Rep.* **160**, 1 (1986).
36 Brack, M. & Bhaduri, R. *Semiclassical Physics*. Frontiers in Physics (Addison-Wesley, Advanced Book Program, 1997).
37 The discussion for the case of mixed states, \( \hat{\rho} = \sum_n \langle \psi_n | \psi_n \rangle \) follows immediately by applying the considerations below to each component \( |\psi_n\rangle \langle\psi_n| \).
38 Botero, B., Juan David & Pachon, L. A. *Dynamical quantum non-locality in finite dim. Hilbert spaces* In preparation. (2013).
39 Duque, S. & Pachon, L. A. *Dynamical quantum non-locality in quantum field theory* In preparation. (2013).
and discuss first the product of two general operators $\hat{A}_1$ and $\hat{A}_2$. Thus,

$$A_{12}(r) = \operatorname{Tr} \left[ \hat{A}_1 \hat{A}_2 \hat{d}(r) \right] = \operatorname{Tr} \left[ \int dr_1 A_1(r_1) \hat{d}(r_1) \int dr_2 A_2(r_2) \hat{d}(r_2) \hat{d}(r) \right] = \int dr_1 A_1(r_1) \operatorname{Tr} \left[ \hat{d}(r_1) \hat{d}(r_2) \hat{d}(r) \right]. \tag{A6}$$

Before we proceed further, we need to calculate the composition law of generators $\hat{d}(r)$. For the case of two generators, we have

$$\hat{d}(r_2) \hat{d}(r) = \frac{2}{\pi \hbar} \int d\tilde{r} \hat{d}(\tilde{r}) \exp \left[ i \frac{\Delta_3(\tilde{r}, r, r_2)}{\hbar} \right], \tag{A8}$$

where $\Delta_3(\tilde{r}, r, r') = \frac{1}{2} (\tilde{r} \wedge r + r \wedge r' + r' \wedge \tilde{r})$ being $r_i \wedge r_j$ the symplectic product $r_i^T \mathbf{J} r_j$ with $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ standing for the symplectic matrix. This is a known result for those familiar with the phase space formulation of quantum mechanics and, it is crucial for the general discussion of dynamical non-locality below. The composition law in equation (A8) means that the composition of two generators of the algebra of quantum observables consists of the superposition of an infinite number of generators. This interference, phase weighted by the symplectic area $\Delta_3(\tilde{r}, r, r')$, is the first step towards interpreting dynamical non-locality as the interference of local evolutions.

Applying equation (A8) once more yields to the composition law of three generators, which allows us to express the product of two operators in phase space as

$$A_{12}(r) = \frac{1}{(\pi \hbar)^2} \int dr_1 \int dr_2 A_1(r_1) A_2(r_2) \exp \left[ \frac{i}{\hbar} \Delta_3(r_2, r_1, r) \right]. \tag{A9}$$

The interpretation of this expression is quite similar to the one for the case of the composition of two generators, i.e., when projecting onto the basic basis of quantum mechanics, the product of two observables turns out to lead to a “coherent superposition” of their respective Weyl symbols.

Since we after the Weyl symbol of $\hat{\rho}(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}^\dagger(t)$, we make use of the composition of two operators successively and get

$$\rho_W(r'', t) = \int dr' G_W(r'', r') \rho_W(r', 0), \tag{A10}$$

where

$$G_W(r'', t; r', 0) \equiv \int \frac{d\tilde{r}}{(2\pi \hbar)^2} \exp \left[ \frac{i}{\hbar} \tilde{r} \wedge (r'' - r') \right] U_W \left( \frac{r'' + r' + \tilde{r}}{2}, t \right) U_W \left( \frac{r'' + r' - \tilde{r}}{2}, t \right) \tag{A11}$$

is the propagator of the Wigner function and $U_W(r)$ is the Weyl symbol of the unitary-time evolution operator. In shorthand notation,

$$G_W(r'', t; r', 0) \equiv \frac{1}{(2\pi \hbar)^2} \int d\tilde{r} e^{i \tilde{r} \wedge (r'' - r') / \hbar} U_W (\tilde{r}, t) U_W^* (\tilde{r}, t), \tag{A12}$$

with $\tilde{r}_\pm = (r'' + r' \pm \tilde{r})/2$. Based on the discuss above, the origin of the dynamical non-locality encoded in the propagator Wigner start being completely apparent at this point.

Before discussing the properties and construction of the propagator of the Wigner function, it is illustrative to compare, on a qualitatively level, the unitary-time evolution operator and the propagator of the Wigner function. Fig. 4 shows this comparison and how the unitary-time evolution operator fails in expressing, qualitatively, the locality of the evolution in the harmonic case.

### 2. Properties of the Wigner Propagator

Due to the close relation between the Weyl propagator and the Wigner propagator, it is not surprising that the properties of the Wigner propagator depend directly on the properties of the time-evolution operator $\hat{U}$. In particular, the anti-unitarity of $\hat{U}$ is translated to the Weyl propagator as $U_W^*(r, t) = U_W(r, -t) = U_W^{-1}(r, t)$. 
FIG. 4. Comparison the Unitary-Time Evolution Propagator and the Propagator of the Wigner Function. We calculate unitary-time evolution operator for the Morse potential discussed in the Fig. 1 of the main text, for the non-linear and the linear approximation, and compare them with the propagator of the Wigner function. In the upper panels we depict the real part of the $U(q_\prime'\prime, q_\prime', t)$ and in the lower panels we reproduced the Fig. 1 of the main text. The inability of the unitary-time evolution operator for qualitatively expressing the locality of the evolution in the harmonic case is evident. However, note that the propagator of the harmonic oscillator, up to a scaling of the parameters, is the kernel of the fractional Fourier transform, which is the responsible for rigid rotations in phase space, thus locality is indeed encoded in it, but in a obscure way.

- Since at $t = 0$, $\hat{U}(0) = \hat{1}$, then $U_W(r, 0) = 1$. Hence, it follows immediately that

$$G(r''', 0; r', 0) = \delta(r'' - r'),$$

which implies that the Wigner propagator is not restricted by the uncertainty principle. This fact allows for a clear and conceptually simple study of the quantum-classical transition.

- From the composition law for the unitary time-evolution operator we can show that

$$G_W(r''', t; r', 0) = \int d^2 r''' G_W(r''', t; r''', t''') G_W(r''', t'''; r', 0), \quad (A13)$$

i.e., the propagator satisfies a Chapmann-Kolmogorov type equation.

- From (A12), from the anti-unitarity of $\hat{U}$ and assuming that the quantum system is homogeneous in time, we have that

$$G_W(r''', t; r', 0) = G_W(r', -t; r''', 0) = G_W(r', 0; r''', t).$$

In this way, for autonomous Hamiltonian systems, the Wigner propagator induces a dynamical group parameterized by $t$. Other properties of $G_W(r''', t; r', 0)$ are

- Since the Wigner function is real, then $G(r''', 0; r', 0) \in \Re$.

- The propagator of the Wigner function is an orthogonal operator, i.e.,

$$\int d^2 r''' G(r'', t; r', 0) G(r''', t; r', 0) = \delta(r''' - r').$$
3. Wigner Propagator from Path Integral in Phase Space

In order to derive a path integral expression for the Wigner propagator, we follow the work by Marinov, and divide the time interval \((t', t'')\), à la Feynman, in \(N\) small time steps \(\Delta t = (t'' - t')/N\). For small \(\Delta t\), the Weyl transform of the unitary evolution operator reads \(\mathcal{U}_W(r, \Delta t) \sim \exp \left[ -\frac{i}{\hbar} \mathcal{W}_W(r) \Delta t \right]\), where \(\mathcal{W}_W(r)\) denotes the Weyl transform of the Hamiltonian operator. By replacing this expression in equation (A12), we obtain that for short times the Wigner propagator is given by

\[
G_W(r_n, r_{n-1}) = \frac{1}{(2\pi\hbar)^j} \int d^2f_r_n \exp \left( \frac{i}{\hbar} \phi_n \right),
\]

(A14)

where we have defined \(\phi_n = \Delta r_n \land \tilde{r}_n + \left[ H_W \left( \tilde{r}_n + \frac{\nu}{2} \right) - H_W \left( \tilde{r}_n - \frac{\nu}{2} \right) \right] \Delta t\), with \(\Delta r_n \equiv r_n - r_{n-1}\) and \(\tilde{r}_n \equiv \frac{r_n + r_{n-1}}{2}\).

Since the propagator satisfies a Chapman-Kolmogorov type equation [see equation (A13)], we can derive the propagator for finite times,

\[
G_W(r, r_0) = \lim_{N \to \infty} \left[ \prod_{n=1}^{N-1} \int d^2f_{r_n} \right] \exp \left( \frac{i}{\hbar} \sum_{n=1}^{N} \phi_n \right).
\]

(A15)

In the continuous limit, the phase in equation (A15) acquires the form of an integral functional action

\[
\sum_{n=1}^{N} \phi_n \to S\{\{r\}, \{\tilde{r}\}, t\} \equiv \int_0^t \left[ \tilde{r} \land \nabla + H_W \left( r + \frac{1}{2} \tilde{r} \right) - H_W \left( r - \frac{1}{2} \tilde{r} \right) \right] dt',
\]

(A16)

where \(r(t')\) is a trajectory in phase space with initial point \(r(0) = r'\) and final point \(r(t) = r''\), \(\tilde{r} = d\tilde{r}/dt\) and \(\tilde{r}\) can be considered as a fluctuation without restrictions around \(r(t')\). So, finally we have that the Wigner propagator can be written as,

\[
G_W(r'', r') = \frac{1}{(2\pi\hbar)^j} \int D^2f_r \int D^2f_{\tilde{r}} \exp \left( \frac{i}{\hbar} S\{\{r\}, \{\tilde{r}\}, t\} \right),
\]

(A17)

where \(D^2f_r\) and \(D^2f_{\tilde{r}}\) denote each one a set of infinity measures in phase space.

4. From Phase-Space Path-Integrals to Semiclassical Wigner Propagator

In this section we evaluate the path-integral expression for the Wigner propagator given in equation (A17) by making use of the stationary-phase approximation (see e.g., Refs. 34–36). In order to calculate the extremal trajectories maximizing the action \(S\{\{r\}, \{\tilde{r}\}, t\}\),

\[
\frac{\partial S}{\partial r} = 0, \quad \frac{\partial S}{\partial \tilde{r}} = 0,
\]

(A18)

we calculate the derivatives of \(S\{\{r\}, \{\tilde{r}\}, t\}\) in the discrete-time version in equation (A14). After taking the continuous limit and defining \(r_{\pm} = r \pm \frac{\tilde{r}}{2}\), we get that the action is maximized by the trajectories \(r_{\pm}\) satisfying

\[
\dot{r}_{\pm} = J\nabla H_W(\tilde{r}_{\pm}),
\]

(A19)

which means that \(r_{\pm}\), not only determine the propagation in phase space, but also are solutions of the classical equation of motion. In this case, the path-integral expressions are replaced by summation over these trajectories and weighted by the second derivatives of the action along these trajectories. This second-derivatives matrix can be related to the difference of the stability matrices of \(r_{+}\) and \(r_{-}\),

\[
\det \left( \begin{array}{cc} \frac{\partial^2 S\{\{r\}, \{\tilde{r}\}, t\}}{\partial q_j^2} & \frac{\partial^2 S\{\{r\}, \{\tilde{r}\}, t\}}{\partial q_j \partial \tilde{r}_j} \\ \frac{\partial^2 S\{\{r\}, \{\tilde{r}\}, t\}}{\partial q_j \partial \tilde{r}_j} & \frac{\partial^2 S\{\{r\}, \{\tilde{r}\}, t\}}{\partial \tilde{r}_j^2} \end{array} \right) = \frac{1}{4J} \det \left( \begin{array}{cc} \frac{\partial q_j''}{\partial q_j'} & \frac{\partial q_j''}{\partial \tilde{r}_j'} \\ \frac{\partial q_j''}{\partial \tilde{r}_j'} & \frac{\partial \tilde{r}_j''}{\partial \tilde{r}_j'} \end{array} \right) = \frac{1}{4J} \det(M_+ - M_-)
\]

(A20)

Since the summation over trajectories contains terms \(j_+ j_-\) and \(j_- j_+\), we can guarantee that the propagator is real. By taking into account the last arguments, we show that the propagator takes the form

\[
G_W(r'', t; r', 0) = \frac{4J}{\hbar^J} \sum_j 2 \cos \left( \frac{1}{2} S_j^V(\nu, r', t) - \nu \frac{\pi}{2} \right) \left| \det (M_{j_+} - M_{j_-}) \right|^{1/2},
\]

(A21)
with $S_j^{\Psi}(r'', r', t)$ given by

$$
S_j^{\Psi}(r'', r', t) = (\tilde{r}_j^+ - \tilde{r}_j^-) \wedge (r'' - r') + S_{j+} - S_{j-}
$$

$$
= \int_0^t ds \left[ \tilde{r}_j(s) \wedge r_j(s) - H_{j+}(r_j^+(s)) + H_{j-}(r_j^-(s)) \right],
$$

(A22)

with $\tilde{r}_j(s) \equiv [r_{j+}(s) + r_{j-}(s)]/2$ and $\tilde{r}_j(s) \equiv r_{j+}(s) - r_{j-}(s)$. Besides the two Hamiltonian terms, it includes the symplectic area enclosed between the two trajectory sections and the vectors $r'_j - r'_{j-}$ and $r''_j - r''_{j-}$. This result form a path integral formulation coincides with the seminal result derived in Ref. 27.

Appendix B: Probability Amplitude Interference and Dynamical Non-Locality

Based on the local evolution of the wave function and its interpretation as a probability amplitude (Copenhagen interpretation), we offered some general arguments about the emergence of the dynamical non-locality as the interference of the local evolution of the probability amplitude. We further discuss, in detail, the emergence of the dynamical non-locality based on the semiclassical approximation of the propagator de the Wigner function. Here we want to support this discussion by considings the general case.

Let us consider the case of a pure state $|\psi\rangle$ and express the density operator as $\hat{\rho} = |\psi\rangle\langle\psi|^{37}$. Hence, based on equation (A2) above, it is clear that $\hat{U}$ and $\hat{U}^\dagger$ are responsible for the evolution of $|\psi\rangle$ and $\langle\psi|$, respectively. Since the dynamics of the wave function are local, this simply means that $\hat{U}$ and $\hat{U}^\dagger$ drive local dynamics by themselves, but when expanded onto the generators of the algebra of observables [see equation (A12) above], their combined action (interference) leads to the non-local dynamics encoded in the propagator of the Wigner function!

Although the argument above is general enough, it is only by the construction of the path integral formulation of the Wigner propagator in equation (A17), that one can verify that this interference process actually yields to the non-local and local terms discussed in the context of modular variables [see equations (1,2) and (10) in the main text]. Thus, only at this point it is clear that the non-locality of the quantum dynamics expressed in the modular variables is the result of the interference of the local dynamics of the probability amplitude in the direct (where $|\psi\rangle$ lives) and in the dual (where $\langle\psi|$ lives) Hilbert space of the physical system under consideration.

By reconstructing the non-local character of the quantum dynamics in terms of the interference of local dynamics (the dynamics of the probability amplitude), we consider that our formulation and results reconcile the dynamical non-locality of the Aharonov-Bohm effect with the fundamentally local action of the electromagnetic field. As such, we consider that our formulation allows for the development of new and different ways of exploring the fundamental aspect of the quantum dynamics. We consider also that is a first step towards extending the kinematic Copenhagen interpretation to the dynamical case.

In the preface of his 1931 *Gruppentheorie una Quantenmechanik* Weyl stated that “The importance of the standpoint afforded by the theory of groups for the discovery of the general laws of quantum theory has of late become more and more apparent.” We follow Weyl in his appreciation and report soon on the study of dynamical non-locality in finite dimensional Hilbert spaces and on its extension to quantum field theory.