SOLVING A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH INTERVAL COEFFICIENTS

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Abstract. In this study, we consider a system of homogeneous linear differential equations, the coefficients and initial values of which are constant intervals. We apply the approach that treats an interval problem as a set of real (classical) problems. In previous studies, a system of linear differential equations with real coefficients, but with interval forcing terms and interval initial values was investigated. It was shown that the value of the solution at each time instant forms a convex polygon in the coordinate plane. The motivating question of the present study is to investigate whether the same statement remains true, when the coefficients are intervals. Numerical experiments show that the answer is negative. Namely, at a fixed time, the region formed by the solution’s value is not necessarily a polygon. Moreover, this region can be non-convex.

The solution, defined in this study, is compared with the Hukuhara-differentiable solution, and its advantages are exhibited. First, under the proposed concept, the solution always exists and is unique. Second, this solution concept does not require a set-valued, or interval-valued derivative. Third, the concept is successful because it seeks a solution from a wider class of set-valued functions.

1. Introduction.

1.1. Preliminaries: The classical problem. Consider Initial Value Problem (IVP) for 2 × 2 system of homogeneous linear differential equations with constant coefficients:

\[
\begin{align*}
  x' &= ax + by \\
  y' &= cx + dy
\end{align*}
\]

under initial conditions

\[
\begin{align*}
  x(0) &= u \\
  y(0) &= v
\end{align*}
\]

where \(a, b, c, d, u, v\) are given real numbers; \(x(t), y(t)\) are unknown real functions.

In vector-matrix form, the above problem can be written as

\[
\begin{align*}
  z' &= Mz \\
  z(0) &= s
\end{align*}
\]

where \(z = \begin{bmatrix} x \\ y \end{bmatrix}\), \(M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) and \(s = \begin{bmatrix} u \\ v \end{bmatrix}\).
Similar to the one-dimensional case, the solution can be expressed as

$$z(t) = e^{Mt} s$$

Denote $$r = \frac{a + d}{2}$$, $$\varepsilon = \frac{a - d}{2}$$ and $$\delta = \sqrt{\left(\frac{a - d}{2}\right)^2 + bc} = \sqrt{\varepsilon^2 + bc}$$. If $$\delta \neq 0$$, then

$$e^{Mt} = e^{rt} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

where

$$w_{11} = \cosh \delta t + \varepsilon \frac{\sinh \delta t}{\delta}; \quad w_{12} = b \frac{\sinh \delta t}{\delta};$$

$$w_{21} = c \frac{\sinh \delta t}{\delta}; \quad w_{22} = \cosh (\delta t) - \varepsilon \frac{\sinh \delta t}{\delta}.$$ 

Therefore, the solution of (1)-(2) is

$$x_{abcduv}(t) = e^{rt} \left[ \left( \cosh \delta t + \varepsilon \frac{\sinh \delta t}{\delta} \right) u + b \frac{\sinh \delta t}{\delta} v \right]$$

$$y_{abcduv}(t) = e^{rt} \left[ c \frac{\sinh \delta t}{\delta} u + \left( \cosh \delta t - \varepsilon \frac{\sinh \delta t}{\delta} \right) v \right] \quad (4)$$

provided that $$\delta \neq 0$$.

If $$\delta = 0$$, then

$$x_{abcduv}(t) = e^{rt} \left[ (1 + \varepsilon t) u + btv \right]$$

$$y_{abcduv}(t) = e^{rt} \left[ ctu + (1 - \varepsilon t) v \right] \quad (5)$$

In the case of $$\left(\frac{a - d}{2}\right)^2 + bc < 0$$, the value of $$\delta$$ becomes a purely imaginary complex number. In this case, the solution (4) can be rewritten in terms of real functions as follows:

$$x_{abcduv}(t) = e^{rt} \left[ \left( \cos \delta t + \varepsilon \frac{\sin \delta t}{\delta} \right) u + b \frac{\sin \delta t}{\delta} v \right]$$

$$y_{abcduv}(t) = e^{rt} \left[ c \frac{\sin \delta t}{\delta} u + \left( \cos \delta t - \varepsilon \frac{\sin \delta t}{\delta} \right) v \right] \quad (6)$$

(i.e. usual cos and sin arise instead of cosh and sinh, respectively, in (4)), where $$\delta = \sqrt{-bc - \varepsilon^2}$$.

### 1.2. System of linear differential equations with interval coefficients.

In real-life problems, in general, the values of input parameters $$a, b, c, d, u$$ and $$v$$ are not known exactly, i.e. they are uncertain. Let these parameters be modeled by intervals $$A = [a, \overline{a}], B = [b, \overline{b}], C = [c, \overline{c}], D = [d, \overline{d}], U = [u, \overline{u}]$$ and $$V = [v, \overline{v}]$$, respectively (Hereinafter, the term interval will mean closed interval). Then, the following two questions arise:

(i) How to formulate the problem in the uncertain environment?

(ii) How to solve it?

Mainly, there are three different approaches.

#### 1.2.1. First approach: Interval analysis.

The followers of this approach argue as follows. Since the initial values of the solution’s components $$x$$ and $$y$$ are intervals, their values at a time instant $$t$$ (i.e. $$x(t)$$ and $$y(t)$$) also are intervals, say $$X(t) = [\underline{x}(t), \overline{x}(t)]$$ and $$Y(t) = [\underline{y}(t), \overline{y}(t)]$$. Then, according to the followers, the problem with interval uncertainties should be formulated by substituting the real values with the corresponding interval values. As a result, the interval IVP is expressed as follows:

$$\begin{cases} X'(t) = AX + BY \\ Y'(t) = CX + DY \end{cases} \quad (7)$$
with initial conditions

\[
\begin{align*}
X(0) &= U \\
Y(0) &= V 
\end{align*}
\] (8)

Or, in explicit form:

\[
\begin{align*}
[x(t), \pi(t)]' &= [a, \bar{a}] [x(t), \pi(t)] + [b, \bar{b}] [y(t), \bar{y}(t)] \\
y(t), \bar{y}(t)' &= [c, \bar{c}] [x(t), \pi(t)] + [d, \bar{d}] [y(t), \bar{y}(t)] 
\end{align*}
\] (9)

and

\[
\begin{align*}
[x(0), \pi(0)] &= [u, \bar{u}] \\
y(0), \bar{y}(0) &= [v, \bar{v}] 
\end{align*}
\] (10)

The approach in question considers the arithmetic operations in the above equations be interval operations ([9, 12]), and the derivative be an interval derivative (Hukuhara derivative, generalized Hukuhara derivative, etc.) ([3, 11, 15]). The approach has some inconveniences arising from the interval-valued analysis ([5, 6]). But the main difficulty is that, in the frame of this approach, the existence and uniqueness of solution is not guaranteed.

1.2.2. Second approach: Set-valued analysis. This approach can be explained as follows. Thinking geometrically, and considering \(x\) and \(y\)-components in couple, the initial values constitute a rectangle \(S = [u, \bar{u}] \times [v, \bar{v}]\) (i.e. a region, if speak generally) in the \(xy\)-coordinate system. Consequently, at a time instant \(t\), the solution's value \(Z(t)\) is also a region (set). Therefore, in the opinion of the followers of the approach under consideration, the solution \(Z\) is a set-valued function. In the result, the uncertain problem is formulated as a set-valued problem:

\[
\begin{align*}
Z' &= MZ \\
Z(0) &= S 
\end{align*}
\] (11)

where \(M\) is a matrix with interval entries.

Set-valued analysis is required to tackle with this problem ([1, 2, 10, 14]). But, unfortunately, the achievements in this field are modest: (i) The theoretical studies cover mainly compact convex sets; (ii) Arithmetic operations can be accomplished confidently only on a few type of simple sets, such as balls and polygons.

1.2.3. Third approach: Bunch of real solutions. This approach considers the solution of the interval problem as a bunch (set) of real solutions (or, as a bundle of trajectories, or as a trajectory tube) ([4, 8, 13]). We propose a similar approach. In our approach, we reason as follows. An interval is a set of real numbers. In similar way, it is natural to consider an interval problem as a set of real (classical) problems ([5, 6, 7]). Then, we can define the solution as follows.

**Definition 1.** Consider Interval IVP (7)-(8). We interpret it as a set of real (classical) IVPs (1)-(2), where \(a \in [\bar{a}, a], b \in [\bar{b}, b], c \in [\bar{c}, c], d \in [\bar{d}, d], u \in [\bar{u}, u]\) and \(v \in [\bar{v}, v]\). Each IVP (1)-(2) has a unique solution \((x_{abcd}(t), y_{abcd}(t))\), which is a vector-function. The set (bunch) \(Z\) of all these vector-functions we define as the solution of the Interval IVP:

\[
Z = \{(x_{abcd}(\cdot), y_{abcd}(\cdot)) \mid (x_{abcd}(\cdot), y_{abcd}(\cdot)) \text{ is the unique solution of IVP (1)-(2)}, \\
a \in [\bar{a}, a], b \in [\bar{b}, b], c \in [\bar{c}, c], d \in [\bar{d}, d], u \in [\bar{u}, u], v \in [\bar{v}, v]\}
\] (12)
where $x_{abcd}()$ and $y_{abcd}()$ are determined by one of the formulas (4), (5), or (6), depending on the sign of $\varepsilon^2 + bc$. Also we define the value of solution at a time instant $t$ as

$$Z(t) = \{(x(t), y(t)) \mid (x(\cdot), y(\cdot)) \in Z\} \quad (13)$$

It should be noted that $Z(t)$ is the reachable (or attainable) set $[4, 8]$.

The proposed approach has some evident advantages. First, it does not require defining a set-valued derivative, or an interval derivative. Second, under the developed approach, the solution exists and it is unique, as in the classical case (Note that this fact follows immediately from our definition for solution and from that each real IVP (1)-(2) has a unique solution).

Remark 1. Consider IVP with interval uncertainties formulated in the form (11). When we apply our approach to the problem (11), we interpret it as a set of real problems (3). Since the real problems (3) and (1)-(2) are equivalent, the problems (with interval uncertainties) (11) and (7)-(8) are equivalent, under our approach.

2. Numerical method. In the numerical method, proposed in the present study, we use our previous result from [6] for a homogeneous system of Linear Differential Equations (LDEs) with real coefficients but with interval initial values. In [6], we have shown that, at a time $t$, the value of solution constitutes a filled parallelogram in the coordinate plane. Also, we have given an algorithm to determine the parallelogram. The algorithm requires to solve 3 classical IVPs of form (1)-(2).

In the interval problem, which is under consideration in the present study, there are 4 coefficients ($a$, $b$, $c$ and $d$), which are given by intervals. For each of these four intervals we use a grid with $n$ equidistant nodes. Then, we have $n^4$ quadruples such as $(a_i, b_j, c_k, d_l)$. For each quadruple $(a_i, b_j, c_k, d_l)$, we can compute the solution of system of LDEs with initial values $[u_i, \varpi]$ and $[v_i, \varpi]$ ([6]). At a fixed time instant $t$, the solution's value forms a filled parallelogram, which can be determined as follows. The initial values constitutes a rectangle $S = [u_i, \varpi] \times [v_i, \varpi]$. The corners of this rectangle are determined by vectors $s_1 = (u_i, v_i), s_2 = (u_i, \varpi), s_3 = (\varpi, v_i)$ and $s_4 = (\varpi, \varpi)$. The linear transform $z = L(s) = e^{Mts}$ maps the rectangle $S$ to a parallelogram $P$ with corners $z_1 = e^{Mts_1} = (x_{a_i,b_j}, y_{a_i,b_j})$, $z_2 = e^{Mts_2} = (x_{a_i,b_j}, y_{a_i,b_j})$, $z_3 = e^{Mts_3} = (x_{a_i,b_j}, y_{a_i,b_j})$ and $z_4 = e^{Mts_4} = (x_{a_i,b_j}, y_{a_i,b_j})$, where $x_{abcd}(t)$ and $y_{abcd}(t)$ are calculated by the appropriate one of formulas (4)-(6). (Note that $z_4 = z_1 + z_3 - z_2$). The union of these $n^4$ parallelograms gives an approximate value of the solution of the Interval IVP. Thus, the computational complexity of the algorithm is $O(n^4)$ in terms of the number of classical IVPs (1)-(2), which are solved analytically by (4)-(6).

3. Examples. In this section, we give two numerical examples, which are useful for comparison of the proposed method with methods based on set-valued analysis or interval analysis.

Example 1. Consider Interval IVP (7)-(8) with $A = [2.75, 3.25], B = [-1.25, -0.75], C = [3.75, 4.25], D = [-2.25, -1.75], U = [-0.25, 0.25]$ and $V = [-1.0, 1.0]$.

We solve the problem by implementing the proposed numerical method. In calculations we approximate each interval, which represents a coefficient, with 11
equidistant nodes. The obtained numerical solution is depicted in Fig. 1. As it can be seen from the figure, the value of the solution at a fixed time instant forms a region in the coordinate plane. It can be noted that this region is not convex at $t = 0$, $t = 0.2$, $t = 0.4$ and $t = 0.6$. Since the set-valued analysis is developed mainly for compact convex sets, Example 1 justifies that the proposed method covers a wider class of problems.

**Figure 1.** The values of numerical solution of Example 1 at different time instants: $t = 0$ (upper left quarter), $t = 0.2$ (upper right quarter), $t = 0.4$ (lower left quarter), and $t = 0.6$ (lower right quarter).

**Example 2.** Consider Interval IVP (7)-(8) with $A = [2.7, 3.3]$, $B = [0.9, 1.1]$, $C = [1.8, 2.2]$, $D = [0.9, 1.1]$, $U = [2.0, 3.0]$ and $V = [0.0, 2.0]$.

The solution, obtained by the proposed numerical method, is depicted in Fig. 2 by continuous lines.

Based on the considered example, let us compare our proposed method with a method, based on interval analysis. For clarity, let us consider Hukuhara-differentiable solution.

Since all coefficients and initial values lie in the non-negative semi-axis, from (9)-(10), we have:

\[
\begin{align*}
\dot{x}(t) &= a x(t) + b y(t) \\
\dot{y}(t) &= c x(t) + d y(t) \\
x(0) &= u \\
y(0) &= v
\end{align*}
\]

The solution of the system is $\bar{x}(t) = x_{abcduv}(t)$, $\bar{y}(t) = y_{abcduv}(t)$, where $x_{abcduv}(t)$ and $y_{abcduv}(t)$ are calculated by (4). These functions are presented in Fig. 3.
Figure 2. Solutions of Example 2, obtained by two methods, at $t = 0$ (upper left quarter), $t = 0.2$ (upper right quarter), $t = 0.4$ (lower left quarter), and $t = 0.6$ (lower right quarter). The continuous lines represent the numerical solution, obtained by the proposed method, while the dashed lines represent the Hukuhara-differentiable solution.

Figure 3. The Hukuhara-differentiable solution $X(t) = [\underline{x}(t), \overline{x}(t)]$ and $Y(t) = [\underline{y}(t), \overline{y}(t)]$ for Example 2. At the left half, the lower and upper lines represent $\underline{x}(t)$ and $\overline{x}(t)$, respectively. The lines at the right half represent $\underline{y}(t)$ and $\overline{y}(t)$.

It can be checked that $X(t) = [\underline{x}(t), \overline{x}(t)]$ and $Y(t) = [\underline{y}(t), \overline{y}(t)]$ determine a Hukuhara-differentiable solution. Since $X(t)$ and $Y(t)$ are independent, in the coordinate plane, the solution's value at a time $t$ constitutes a rectangle $X(t) \times Y(t) = [\underline{x}(t), \overline{x}(t)] \times [\underline{y}(t), \overline{y}(t)]$. In Fig. 2, the boundaries of these rectangles are depicted by dashed lines.
From Fig. 2 it can be seen that the regions by our method completely lie inside the rectangles, representing the Hukuhara-differentiable solution. Therefore, we can state that our proposed method describes the uncertainties more meticulously.

4. **Comparison with Hukuhara-differentiable solution.** In this section, we show that the observations made in Example 2 are not a coincidence. Namely, when the parameters are on the non-negative semi-axis, a Hukuhara-differentiable solution exists, and it bounds the solution by our approach. In order to prove this statement we involve some notations and establish a property of the solution of IVP (1)-(2).

Let \( M \) and \( K \) be real matrices of the same size. Hereinafter, \( M \geq K \) means \( m_{ij} \geq k_{ij} \) for all \( i, j \). In particular, \( M \geq O \) (where \( O \) is the zero matrix) means \( m_{ij} \geq 0 \) for all \( i, j \). Since vectors can be identified with \( n \times 1 \) matrices, the same denotation is used also for them.

It can be easily seen that

(a) If \( M \geq O, K \geq O \) and \( M \geq K \), then \( M^2 \geq K^2 \); 
(b) If \( M \geq O, K \geq O, P \geq O \) and \( M \geq K \), then \( MP \geq KP \).

**Lemma 1.** Consider IVP (1)-(2) with non-negative parameters: \( a \geq 0, b \geq 0, c \geq 0, d \geq 0, u \geq 0 \) and \( v \geq 0 \). If a parameter \( (a, b, c, d, u \) or \( v \)) increases, then the solution's components \( x(t) \) and \( y(t) \) also increase, or remain the same.

We give an informal proof which can easily be formalized.

If a coefficient \((a, b, c \) or \( d \)) increases, then the matrix \( M \geq O \) (where \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) also increases. Under the considered conditions, from the above properties \((a)\) and \((b)\) it follows that \( M^2, M^3, \ldots \) “increase” (or, remain the same). Consequently, 

\[
e^M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n \quad \text{and} \quad e^{Mt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n \quad (t \geq 0) \quad \text{“increase”, and all these matrices are “non-negative”}.\]

In particular, \( e^{Mt} \geq O \). Since the initial values are non-negative, the solution \( z = e^{Mt} s \) also “increases”. As a result, if a coefficient increases, then \( x(t) \) and \( y(t) \) also increase (or remain the same).

If the increasing parameter is an initial value \((u \) or \( v \)), then the vector \( s \) increases, while the matrix \( e^{Mt} \geq O \) remains unchanged. Consequently, the solution \( z = e^{Mt} s \) increases.

**Remark 2.** Under the conditions of Lemma 1, \( u \geq 0 \) and \( v \geq 0 \), i.e. \( s \geq O \). By proof of the lemma, \( e^{Mt} \geq O \). Then, \( z = e^{Mt} s \geq O \). Consequently, \( x(t) = x_{abcduv}(t) \geq 0 \) and \( y(t) = y_{abcduv}(t) \geq 0 \).

Below, for an interval \( A = [a, \bar{a}] \), we use the denotation \( A \geq 0 \), if \( a \geq 0 \).

**Lemma 2.** If \( A \geq 0, B \geq 0, C \geq 0, D \geq 0, U \geq 0 \) and \( V \geq 0 \), then Interval IVP (7)-(8) has a Hukuhara-differentiable solution \((X(t), Y(t))\), given by formulas \( X(t) = [x(t), \overline{x}(t)] = [x_{abcduv}(t), x_{abcduv}(t)] \) and \( Y(t) = [y(t), \overline{y}(t)] = [y_{abcduv}(t), y_{abcduv}(t)] \), where \( x_{abcduv}(t) \) and \( y_{abcduv}(t) \) are determined by (4).

**Proof.** We perform the proof in 3 steps.

(1) First, we must be sure that the components \( x(t), \overline{x}(t), y(t) \) and \( \overline{y}(t) \), indicated in the lemma, satisfy the system, corresponding to the Hukuhara-differentiable solution. Under the considered conditions, all coefficients are positive and \([x(t), \overline{x}(t)]') = [x'(t), \overline{x}'(t)] \) and \([y(t), \overline{y}(t)]') = [y'(t), \overline{y}'(t)] \). Then, the system (9)-(10) is led to (14),
the solution of which is $x(t) = x_{abcduv}(t)$, $\overline{x}(t) = x_{abcduv}(t)$ and $y(t) = y_{abcduv}(t)$, $\overline{y}(t) = y_{abcduv}(t)$.

(2) Second, we check that $X(t)$ and $Y(t)$ are proper intervals, i.e. the lower bounds of these intervals do not exceed the upper bounds. By Lemma 1, $x(t)$ and $y(t)$ are non-decreasing functions of parameters. Consequently, $x_{abcduv}(t) \leq x_{abcduv}(t)$ and $y_{abcduv}(t) \leq y_{abcduv}(t)$.

(3) Finally, we check that $X'(t) = [x'(t), \overline{x'}(t)]$ and $Y'(t) = [y'(t), \overline{y'}(t)]$ are proper intervals. By (14), $x_{abcduv}(t)$, $y_{abcduv}(t)$, $x_{abcduv}(t)$ and $y_{abcduv}(t)$ satisfy the equations

\[
\begin{align*}
\frac{dx_{abcduv}(t)}{dt} &= \alpha x_{abcduv}(t) + \beta y_{abcduv}(t) \\
\frac{dy_{abcduv}(t)}{dt} &= \alpha x_{abcduv}(t) + \beta y_{abcduv}(t)
\end{align*}
\]

If we subtract the above equations, and take into account that $x_{abcduv}(t)$ and $y_{abcduv}(t)$ are non-negative (by Remark 2), then we have:

\[
\frac{dx_{abcduv}(t) - x_{abcduv}(t)}{dt} = \alpha x_{abcduv}(t) - \alpha x_{abcduv}(t) + \beta y_{abcduv}(t) - \beta y_{abcduv}(t) \geq \alpha (x_{abcduv}(t) - x_{abcduv}(t)) + \beta (y_{abcduv}(t) - y_{abcduv}(t)) \geq 0
\]

Therefore, $x_{abcduv}(t) \leq x_{abcduv}(t)$. Similarly, $y_{abcduv}(t) \leq y_{abcduv}(t)$. 

Now, we can summarize the above derivations to establish the main result of this section.

**Theorem.** Consider Interval IVP (7)-(8). If $A \geq 0$, $B \geq 0$, $C \geq 0$, $D \geq 0$, $U \geq 0$ and $V \geq 0$, then there exists a Hukuhara-differentiable solution, and it bounds the solution in the sense of Definition 1, i.e. $Z(t) \subseteq [x_{abcduv}(t), x_{abcduv}(t)] \times [y_{abcduv}(t), y_{abcduv}(t)].$

**Proof.** By Lemma 1,

\[
\begin{align*}
 x_{abcduv}(t) &\leq x_{abcduv}(t) \leq x_{abcduv}(t) \\
y_{abcduv}(t) &\leq y_{abcduv}(t) \leq y_{abcduv}(t)
\end{align*}
\]

Then, the proof follows immediately from Definition 1 and Lemmas 1 and 2. 

**Remark 3.** In the case when the coefficients are non-negative, but the initial values are non-positive ($U \leq 0$ and $V \leq 0$), the theorem remains in force but with the difference that the Hukuhara-differentiable solution is $X(t) = [x_{abcduv}(t), x_{abcduv}(t)]$ and $Y(t) = [y_{abcduv}(t), y_{abcduv}(t)].$

5. **Conclusion.** In this study, we consider a system of homogeneous linear differential equations, where the coefficients and initial values are constant intervals. Usually, the researchers investigate the problem by using the techniques of set-valued analysis, or interval analysis. Set-valued analysis is developed mainly for compact convex sets. Our first numerical example shows that this class of sets is not sufficient for the problem under consideration. Our second example demonstrates that, in general, the interval-analysis-based methods describe the uncertainties roughly. Thus, the provided examples show that the method, proposed in the present study, has essential advantages. In comparison with the existing approaches, the main advantage of the proposed approach is that the existence and uniqueness of solution is guaranteed, as in the classical case.
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