THE RAMSEY THEORY OF THE UNIVERSAL HOMOGENEOUS TRIANGLE-FREE GRAPH PART II: EXACT BIG RAMSEY DEGREES

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Abstract. Building on work in [3], for each finite triangle-free graph \( G \), we determine the equivalence relation on the copies of \( G \) inside the universal homogeneous triangle-free graph, \( \mathcal{H}_3 \), with the smallest number of equivalence classes so that each one of the classes persists in every isomorphic subcopy of \( \mathcal{H}_3 \). This characterizes the exact big Ramsey degrees of \( \mathcal{H}_3 \). It follows that the triangle-free Henson graph is a big Ramsey structure.

1. Overview

This paper is a sequel to [3], in which the author proved that the triangle-free Henson graph has finite big Ramsey degrees. The original hope for that paper was to find the exact degrees, and we conjectured that the bounds found were optimal. In this paper, we show that while those bounds were correct for singletons, edges, and non-edges, more generally they were not exact. However, the structure of the coding trees developed in that paper did achieve the best bounds known so far, and a small but significant modification of those coding trees in this paper will enable us to prove the exact bounds. In Section 4, we improve the results of Section 7 in [3] and then apply theorems from [3] to obtain better upper bounds. In Section 5, we prove that these bounds are exact. This is the first result on exact big Ramsey degrees for structures with forbidden irreducible substructures.

2. Introduction

The universal homogeneous triangle-free graph, denoted by \( \mathcal{H}_3 \) and also known as the triangle-free Henson graph, is the Fraïssé limit of the class of finite triangle-free graphs, \( \mathcal{G}_3 \). We say that \( \mathcal{H}_3 \) has finite big Ramsey degrees if for each finite triangle-free graph \( G \in \mathcal{G}_3 \), there is a positive integer \( T \) such that

\[ \mathcal{H}_3 \rightarrow (\mathcal{H}_3)^G_k \]

holds for any \( k \geq 1 \). When such a \( T \) exists, we let \( T(G, \mathcal{H}_3) \) denote the least such \( T \), and call it the big Ramsey degree of \( G \) in \( \mathcal{H}_3 \), using the terminology in [7].

The reader interested in a broader understanding of the area is referred to the papers [3] and [2], where extensive background on big Ramsey degrees of the Henson graphs, including results known at the time, is provided. An expository article [4].

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on forcing and the method of coding trees also provides a general overview of this area. Here, we shall provide a minimal overview of the problem, an update on currently known results, and the results in this paper.

It is important to distinguish between proving that big Ramsey degrees are finite (that is, finding upper bounds) and characterizing the actual numbers $T(G, \mathcal{H}_3)$ via some structures from which they can be calculated. In the first case, we say that $\mathcal{H}_3$ has finite big Ramsey degrees and in the second case, we say that exact big Ramsey degrees are characterized. When exact big Ramsey degrees are characterized using some extra structure (in our case, some structure implicit in coding trees), then the results of Zucker in [12] regarding big Ramsey structures and universal completion flows in topological dynamics apply. This is one reason why characterizing exact big Ramsey degrees is of much current interest.

In many cases where exact big Ramsey degrees have been characterized, this has been done via finding canonical partitions via some sort of tree structures. (A new method using category theory has recently been successfully developed by Mašulović in [10] and Barbosa in [1].) In the terminology of [9], a partition $P_0, \ldots, P_{T-1}$ of the copies of a structure $A$ in an infinite structure $\mathcal{S}$ is canonical if the following holds: For each coloring $f : (\mathcal{S}_A) \to k$, where $k \geq 2$, there is an isomorphic substructure $\mathcal{S}'$ of $\mathcal{S}$ such that for each $n < T$, $f$ takes one color in $(\mathcal{S}'_A) \cap P_n$; moreover, for each isomorphic substructure $\mathcal{S}'$ of $\mathcal{S}$, $(\mathcal{S}'_A) \cap P_n$ is non-empty. This latter property is called persistence.

In [3], we developed the notion of strong coding tree and of incremental strict similarity type and proved that the number of incremental strict similarity types of antichains coding $G$ is an upper bound for $T(G, \mathcal{H}_3)$. In this paper, we shall refine work from Sections 7–9 of [3] to prove better upper bounds. Then we shall use ideas from the proof of Theorem 4.1 in [4] and build some new methods for triangle-free graphs to prove that these better upper bounds are exact.

The characterization of the big Ramsey degrees of the triangle-free Henson graph is via the notion of essential pair similarity. Throughout, we work with an enumerated copy of $\mathcal{H}_3$, and its induced coding tree $\mathcal{S}$. An essential linked pair is a pair of nodes $s, t \in \mathcal{S}$ such that $s$ and $t$ both code an edge with a common vertex of $\mathcal{H}_3$, but the least vertex of $\mathcal{H}_3$ with which $s$ codes an edge differs from the least vertex of $\mathcal{H}_3$ with which $s$ codes an edge. Two trees with coding nodes $A$ and $B$ are essential pair similar (or epsimilar) if $A$ and $B$ are strongly similar, and additionally the strong similarity map from $A$ to $B$ preserves the order in which new essential linked pairs appear. Given a finite triangle-free graph $G$, we let $\text{Sim}^\text{ep}(G)$ be a set of representatives from among the ep-similarity types of antichains $A$ (of coding nodes) representing $G$ such that each pair of coding nodes $e^A_m, e^A_n$ in $A$ ($m < n$) coding a non-edge between their represented vertices is linked (meaning they code an edge with a common vertex in $\mathcal{H}_3$).

**Theorem 5.12** Let $G$ be a finite triangle-free graph and let $h$ be a coloring of all copies of $G$ inside $\mathcal{H}_3$. Then there is a subgraph $\mathcal{H}$ of $\mathcal{H}_3$ which isomorphic to $\mathcal{H}_3$ in which for each $A \in \text{Sim}^\text{ep}(G)$, all copies of $G$ represented by an antichain which is ep-similar to $A$ have the same color. Moreover, each ep-similarity type in $\text{Sim}^\text{ep}(G)$ persists in the coding tree induced by $\mathcal{H}$.

This characterizes the exact big Ramsey degrees of finite triangle-free graphs in the triangle-free Henson graph.
Theorem 5.13  Given a finite triangle-free graph $G$, the big Ramsey degree of $G$ in the triangle-free graph is exactly the number of essential pair similarity types of strongly skew antichains coding $G$:

\[ T(G, \mathcal{H}_3) = |\text{Sim}^{ep}(G)|. \]

Some related recent results deserve mention. Zucker proved in [13] that Fraïssé classes with finitely many binary relations and finitely many forbidden irreducible substructures have finite big Ramsey degrees. This generalized the work of the author in [3] and [2] showing that all $K_n$-free Henson graphs have finite big Ramsey degrees. Very recently, Hubička has found the first non-forcing proof that $\mathcal{H}_3$ has finite big Ramsey degrees in [6]. In that paper, he used the Carlson-Simpson theorem to prove that the universal homogeneous partial order has finite big Ramsey degrees, and by similar methods was able to recover bounds for the big Ramsey degrees of $\mathcal{H}_3$. The proofs in [13] and [6] that upper bounds exist are quite a bit shorter than those of the author in [3] and [2]. This comes at the expense of looser upper bounds. Part of the motivation of the extreme structure of the coding trees of the author was to construct a space of coding trees which themselves recover indivisibility results of Komjáth and Rödl for $\mathcal{H}_3$ [8] and of El-Zahar and Sauer for the rest of the $K_n$-free Henson graphs [5]. The other part of the motivation was to prove exact big Ramsey degrees. This paper shows that adding a small but important requirement to the coding trees in [3] results in the exact big Ramsey degrees for $\mathcal{H}_3$. Something similar ought to be possible for other binary relational structures with some forbidden irreducible substructures.

3. Review

In this section, we review some concepts from [3] to ease the reading of this paper. The reader familiar with that paper can skip this section.

3.1. The strong triangle-free coding tree $\mathcal{S}$. The set $2^{<\omega}$ is the collection of all finite length sequences of 0’s and 1’s. We let $0^{<\omega}$ denote $\{0\}^{<\omega}$, the collection of all finite sequences of 0’s. Given $s \in 2^{<\omega}$, we let $|s|$ denote the length of $s$. The meet of two nodes $s, t \in 2^{<\omega}$, denoted $s \land t$, is the longest member $u \in 2^{<\omega}$ which is an initial segment of both $s$ and $t$. In particular, if $s \subseteq t$ then $s \land t = s$. A set of nodes $A \subseteq 2^{<\omega}$ is closed under meets if $s \land t$ is in $A$, for each pair $s, t \in A$. Given $A \subseteq 2^{<\omega}$, we let $\text{cl}(A)$ denote the set $\{s \land t : s, t \in A\}$ and call this the meet-closure of $A$. Since $s$ and $t$ are allowed to be equal in the definition of $\text{cl}(A)$, the meet-closure of $A$ contains $A$. We adhere to the following definition of tree, which is standard for Ramsey theory on trees.

**Definition 3.1.** A subset $T \subseteq 2^{<\omega}$ is a tree if $T$ is closed under meets and for each pair $s, t \in T$ with $|s| \leq |t|$, $t \upharpoonright |s|$ is also in $T$.

Graphs can be coded via nodes in $2^{<\omega}$ with the edge relation coded via passing number. Given two vertices $v, w$ in some graph $G$, two nodes $s, t \in 2^{<\omega}$ represent $v$ and $w$ if, assuming $|s| < |t|$, then $v$ and $w$ have an edge between them if and only if $t(|s|) = 1$. The number $t(|s|)$ is called the passing number of $t$ at $s$ (see [11] for the first usage of this terminology). The following appears as Definition 3.1 in [3].

**Definition 3.2** (Tree with coding nodes). A tree with coding nodes is a structure $(T, N; \subseteq, <, c^T)$ in the language of $\mathcal{L} = \{\subseteq, <, c\}$, where $\subseteq$ and $<$ are binary relation symbols and $c^T$ is a unary function symbol, satisfying the following: $T$ is a subset
of $2^{<\omega}$ satisfying that $(T, \subseteq)$ is a tree, $N \leq \omega$ and $<$ is the usual linear order on $N$, and $c^T : N \to T$ is an injective function such that $m < n < N$ implies $|c^T(m)| < |c^T(n)|$.

We often denote $c^T(n)$ by $c^T_n$, especially when working with more than one tree at the same time. The following is Definition 3.3 in [3].

**Definition 3.3.** A graph $G$ with vertices enumerated as $\langle v_n : n < N \rangle$ is represented by a tree $T$ with coding nodes $\langle c^T_n : n < N \rangle$ if and only if for each pair $i < n < N$, $v_n E v_i \iff c^T_i(l_i) = 1$. We will often simply say that $T$ codes $G$.

The triangle-free Henson graph $\mathcal{H}_3$ can be represented by a tree with coding nodes.

**Construction of the strong triangle-free coding tree $S$.** Let $\mathcal{H}_3$ be a Henson graph with universe $\langle v_n : n < \omega \rangle$ labeled in order-type $\omega$. Assume that this representation of $\mathcal{H}_3$ has the following properties:

1. For each $n < \omega$, $v_n E v_{n+1}$.
2. For each $n < \omega$, for all $i < 2n$, $v_{2n+1} E v_i$ if and only if $i = 2n$.

Let $S$ be the coding tree for $\mathcal{H}_3$ constructed as follows: Let $c^S_0$ be the empty sequence; this coding node represents the vertex $v_0$. In general, given $n > 0$ and supposing $c^S_m$ is defined for all $m < n$, take $c^S_n$ to be the (unique) node in $2^n$ such that for all $m < n$, $c^S_m$ has passing number 1 at $c^S_n$ if and only if $v_n E v_m$. The $m$-th level of $S$ consists of all nodes $s \in 2^m$ for which there exists an $n \geq m$ such that $s \subseteq c^S_n$.

The reader familiar with the paper [3] will notice that this construction of $S$ slightly differs from the presentation of Example 3.15 there. In [2], we streamlined the presentation of $S$ and of strong coding trees, and that is reflected here. Requirement (i) here is the same as in [3], and requirement (ii) here is the part of (ii) in [3] corresponding to $F_{3i+j} = \emptyset$ for all $i < \omega$ and $j \in \{0, 2\}$. The rest of requirements (ii) and (iii) from [3] were formulated to ensure that the coding nodes would be dense in the tree and represent the Henson graph. This is taken care of by enumerating a copy of the Henson graph and using it to define the corresponding coding tree (see [2]).

**Remark 3.4.** Our requirement (i) ensures that all coding nodes in $S$ (besides $c^S_0$) do not split in $S$. This, in addition to using skew trees (see Definition 3.11), had the effect of recovering directly, from our Ramsey theory of strong coding trees in [3], the result of Komjáth and Rödl [S] that $\mathcal{H}_3$ is indivisible.

### 3.2. Ramsey theorem finite trees with the Strict Parallel 1’s Criterion.

In this subsection, we review the Ramsey theorem from [3], which aids in providing upper bounds for the big Ramsey degrees in the triangle-free Henson graph.

Recalling Definition 3.12, let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c^T_n : n < N \rangle$, where $N \leq \omega$, and let $\ell^T_n$ denote $|c^T_n|$. Let $\hat{T}$ denote $\{ t \mid n : t \in T$ and $n \leq |t| \}$, the tree of all initial segments of members of $T$. A node $s \in T$ is called a splitting node if both $s \prec 0$ and $s \prec 1$ are in $\hat{T}$. Given $t \in T$, the level of $T$ of length $|t|$ is the set of all $s \in T$ such that $|s| = |t|$. $T$ is skew if each level of $T$ has exactly one of either a coding node or a splitting node. A skew tree $T$ is strongly skew if additionally for each splitting node $s \in T$, every $t \in T$ such that $|t| > |s|$ and $t \not\supseteq s$ also satisfies $t(|s|) = 0$. Given a strongly skew coding tree $T \subseteq S$, let $\langle d^T_m : m < M \rangle$ enumerate
the coding and splitting nodes of $T$ in increasing order of length; the nodes $d^T_{n}$ are called the critical nodes of $T$. Let $m_n$ denote the integer such that $d^T_{m_n} = c_n$.

The 0-th interval of $T$ is the set of those nodes in $T$ with lengths in $[0, l^T_0]$, and for $0 < n < N$, the $n$-th interval of $T$ is the set of those nodes in $T$ with lengths in $(l^T_{n-1}, l^T_n]$.

The lexicographic order on $2^{<\omega}$ between two nodes $s, t \in 2^{<\omega}$, with neither extending the other, is defined by $s <_{\text{lex}} t$ if and only if $s \supseteq (s \land t)^<0$ and $t \supseteq (s \land t)^<1$. It is important to note that if $T$ is a strongly skew subset of $\mathbb{S}$, then each node $s$ at the level of a coding node $c_n$ in $T$ has exactly one immediate extension in $\hat{T}$. For two nodes $s, t$ with $|s| < |t|$, the number $t(|s|)$ is called the passing number of $t$ at $s$. The following appears as Definition 4.9 in [3], augmenting Sauer’s Definition 3.1 in [11] to the setting of trees with coding nodes.

**Definition 3.5.** Let $S, T$ be strongly skew meet-closed subsets of $\mathbb{S}$. The function $f : S \rightarrow T$ is a strong similarity of $S$ to $T$ if for all nodes $s, t, u, v \in S$, the following hold:

1. $f$ is a bijection.
2. $f$ preserves lexicographic order: $s <_{\text{lex}} t$ if and only if $f(s) <_{\text{lex}} f(t)$.
3. $f$ preserves initial segments: $s \land t \subseteq u \land v$ if and only if $f(s) \land f(t) \subseteq f(u) \land f(v)$.
4. $f$ preserves meets: $f(s \land t) = f(s) \land f(t)$.
5. $f$ preserves relative lengths: $|s \land t| < |u \land v|$ if and only if $|f(s) \land f(t)| < |f(u) \land f(v)|$.
6. $f$ preserves coding nodes: $f$ maps the set of coding nodes in $S$ onto the set of coding nodes in $T$.
7. $f$ preserves passing numbers at coding nodes: If $c$ is a coding node in $S$ and $u$ is a node in $S$ with $|u| > |c|$, then $f(u)(|f(c)|) = u(|c|)$; in words, the passing number of $f(u)$ at $f(c)$ equals the passing number of $u$ at $c$.

We are going to make one terminology shift in this paper to ease descriptions of a property central to the exact big Ramsey degrees of triangle-free graphs.

**Definition 3.6 (Linked Pairs).** We shall call a pair of nodes $\{s, t\} \subseteq \mathbb{S}$ linked if and only if there is some $\ell < \min(|s|, |t|)$ such that $s(\ell) = t(\ell) = 1$. We say that $s$ and $t$ are linked at level $\ell$ if and only if $s(\ell) = t(\ell) = 1$. In the special case that $\{s, t\}$ is a linked set such that $s \land t$ is not in $0^{<\omega}$, then we say that $s$ and $t$ are base-linked.

Given a subtree $A \subseteq \mathbb{S}$ and a node $s \in A$, let $\text{BL}_A(s)$ denote the collection of all nodes $t \in A \mid (|s| + 1)$ which are base-linked with $s$.

A level subset $X$ of $\mathbb{S}$ is pairwise linked if and only if each pair of nodes $\{s, t\} \subseteq X$ is linked. Given a subtree $A \subseteq \mathbb{S}$ and we say that a level subset $X \subseteq A$ is a maximal pairwise linked set in $A$, or is maximally linked, if and only if $X$ is pairwise linked, and for any $t \in A \mid \ell_X$ not in $X$, the set $X \cup \{t\}$ is not pairwise linked.

**Remark 3.7.** What we are calling linked in this paper is exactly what we called parallel 1’s in [3] and a pre-3-clique in [2]. It will be much less cumbersome to use this new terminology here.

Note that a pair of nodes $s$ and $t$ are linked at level $\ell$ if and only if for any $m$ and $n$ such that the coding nodes $c_m$ and $c_n$ in $\mathbb{S}$ extend $s$ and $t$, respectively, the vertices $v_m$ and $v_n$ in the enumerated Henson graph $\mathcal{H}_3$ both have an edge with the vertex $v_{\ell}$. The first instances where pairs of nodes code an edge with a
common vertex in $\mathcal{H}_3$ will turn out to be central to the exact big Ramsey degrees of triangle-free graphs.

Given a subset $A \subseteq S$ and $\ell < \omega$, define

$$A_{\ell,1} = \{s \upharpoonright (\ell + 1) : s \in A, \ |s| \geq \ell + 1, \text{ and } s(\ell) = 1\},$$

the level set of nodes in $s \in A \upharpoonright (\ell + 1)$ such that $s$ codes an edge with the vertex $v_{\ell}$ of $\mathcal{H}_3$.

**Definition 3.8.** A level set $X$ is **mutually linked** at $\ell$ if for each $t \in X$, $t(\ell) = 1$. We say that $\ell$ is a **minimal level** of a new mutually linked set in $A$ if the set $A_{\ell,1}$ has at least two distinct members, and for each $\ell' < \ell$, the set $\{s \in A_{\ell,1} : s(\ell') = 1\}$ has cardinality strictly smaller than $|A_{\ell,1}|$. In this case, we call $A_{\ell,1}$ a **new mutually linked set** in $A$.

**Definition 3.9.** Given a minimal level $\ell$ of a new mutually linked set in $A$, we say that $A_{\ell,1}$ is **witnessed by the coding node** $s_n$ in $A$ if $s_i(|s_n|) = 1$ for each $i \in I_{\ell}^A$, and either $|s_n| \leq \ell$ or else both $|s_n| > \ell$ and $A$ has no splitting nodes and no coding nodes of length in $[\ell, |s_n|]$. The following is **Definition 4.1** in $[3]$.

**Definition 3.10** (Parallel 1’s Criterion). Let $T \subseteq 2^{<\omega}$ be a strongly skew tree with coding nodes $(c_n : n < N)$, where $N < \omega$. We say that $T$ satisfies the **Parallel 1’s Criterion** if the following hold: Given any set of two or more nodes $\{t_i : i < \check{i}\} \subseteq T$ and some $\ell$ such that $t_i \upharpoonright (\ell + 1), i < \check{i}$, are all distinct, and $t_{\check{i}}(\ell) = 1$ for all $i < \check{i}$,

1. There is a coding node $c_n$ in $T$ such that for all $i < \check{i}$, $|c_n| < |t_i|$ and $t_i(|c_n|) = 1$; we say that $c_n$ **witnesses** that $\{t_i : i < \check{i}\}$ is mutually linked.
2. Letting $\ell'$ be least such that $t_i(\ell') = 1$ for all $i < \check{i}$, and letting $n$ be least such that $c_n$ witnesses that $\{t_i : i < \check{i}\}$ is mutually linked, then $T$ has no splitting nodes and no coding nodes of lengths strictly between $\ell'$ and $|c_n|$.

Strong coding trees in $S$ were defined in Subsection 4.3 of $[3]$. That definition was streamlined in the more general paper $[2]$ for all $k$-clique-free Henson graphs. We paraphrase here the essentials of that definition.

**Definition 3.11** (Strong Coding Tree). A **strong coding tree** is a strongly skew coding subtree $T$ of $S$ such that

1. The coding nodes in $T$ are dense in $T$ and represent a copy of $\mathcal{H}_3$ in the same order as $S$;
2. $T$ satisfies the Parallel 1’s Criterion.
3. For each $n < \omega$, there is a one-to-one correspondence between the nodes in $T \upharpoonright ([\ell_n^T + 1] \setminus \{0(\ell_n^T + 1)\}$ and the 1-types over the graph represented by the coding nodes $\{c_i^T : i \leq n\}$.

We also require that the splitting nodes in a strong coding tree $T$ between levels with coding nodes split in reverse lexicographic order. However, that property is not essential to the proofs, but rather serves to make all strong coding trees strongly similar to each other, a property which is important for topological Ramsey space theory.

The following is **Definition 6.1** in $[3]$. 
Definition 3.12 (Strict Parallel 1’s Criterion). A subtree $A$ of a strong coding tree satisfies the Strict Parallel 1’s Criterion if $A$ for each $\ell$ which is the minimal level of a new mutually linked set in $A$,

1. The critical node in $A$ with minimal length greater than or equal to $\ell$ is a coding node in $A$, say $c$;
2. There are no terminal nodes in $A$ in the interval $[\ell, |c|)$ ($c$ can be terminal in $A$);
3. $A_{\ell,1} = \{ t \upharpoonright (\ell + 1) : t \in A_{|c|,1} \}$; that is, $c$ witnesses the mutually linked set $A_{\ell,1}$.

Note that the Strict Parallel 1’s Criterion implies the Parallel 1’s Criterion.

The following is Theorem 6.3 in [3]; we modify the presentation, in order to avoid unnecessary definitions in this paper.

Theorem 3.13 (Ramsey Theorem for finite trees with Strict Parallel 1’s Criterion). Let $T$ be a strong coding tree and let $A$ be a finite subtree of $T$ satisfying the Strict Parallel 1’s Criterion. Then for any coloring of all strongly similar copies of $A$ in $T$ into finitely many colors, there is a strong coding subtree $S \subseteq T$ such that each $B \subseteq S$ satisfying the Strict Parallel 1’s Criterion and strongly similar to $A$ has the same color.

4. Improved upper bounds

In Theorem 8.9 of [3], we proved that for each finite antichain $A$ of coding nodes, given any coloring of the strict similarity copies of $A$ inside a strong coding tree, there is a strong coding subtree $S$ in which all strictly similar copies of $A$ have the same color. By taking an antichain of coding nodes representing $H_3$ inside an incremental strong coding tree, we obtained very good bounds for the big Ramsey degrees, which we conjectured to be the exact bounds.

It turns out that those bounds are not exact for most finite triangle-free graphs (they are exact for singletons, edges, and non-edges). However, incremental trees were structurally on the right track. In this section, we fine-tune that approach to obtain better upper bounds, which will be proved to be optimal in the next section. Subsection 4.1 contains the refined version of strict similarity, which we call essential pair similarity (Definition 4.1), which will yield the exact big Ramsey degrees for triangle-free graphs. There, we give an overview of the main theorems of this section, Theorems 4.3 and 4.5 improving the upper bounds for big Ramsey degrees. Subsections 4.2 and 4.3 provide the details on how to refine work in Sections 7–9 of [3] to produce those two theorems.

4.1. Essential pair similarity and improved upper bounds. The canonical partitions proved for the triangle-free Henson graph in this paper have a simple description, given here. The reader convinced of the ability of the methods in [3] to produce Theorems 4.3 and 4.5 below may enjoy reading this subsection and then skipping to Section 5. For the unconvinced reader, Subsection 4.2 provides the details for how to obtain these two new theorems from a small but important refinement of the work in [3].

We begin with some terminology. By an antichain of coding nodes in $S$, or simply an antichain, we mean a set of coding nodes $A \subseteq S$ such that no node in $A$ extends any other node in $A$. Since we will be working within strong coding trees, all our
antichains will have meet closures which are skew. If $A$ is an antichain, then the
tree induced by $A$ is the set
\[(3) \{ s \upharpoonright |u| : s \in A \text{ and } u \in \text{cl}(A) \}.\]
Given a finite antichain $A$, let $\ell_A$ denote the maximum length of the nodes in $A$. For a level set $X$, we let $\ell_X$ denote the length of the nodes in $X$.

Recall Definitions 3.6, 3.8 and 3.9. In terms of graphs, $s$ and $t$ are base-linked if for each coding node $c_m \geq s$ and for each coding node $c_n \geq t$, the least $i$ such that $v_m E v_i$ equals the least $i$ such that $v_n E v_i$. This means that $s$ and $t$ code first edges with a common vertex in $\mathcal{H}_3$. Given a subset $A \subseteq S$, we say that $A$ has a new linked pair at level $\ell$ if and only if there is a pair of nodes $\{s,t\} \subseteq A \upharpoonright (\ell + 1)$ such that $s(\ell) = t(\ell) = 1$, $s(\ell')$ and $t(\ell')$ are never both 1 for any $\ell' < \ell$, and for all $u \in (A \upharpoonright (\ell + 1)) \setminus \{s,t\}$, $u(\ell) = 0$. We call $\{s,t\}$ an essential pair for level $\ell$. Notice that for a base-linked pair, the minimal $\ell$ for which $s(\ell) = t(\ell) = 1$ satisfies $\ell \leq |s \wedge t|$: so by definition, an essential pair is not base-linked. Thus, linked pairs are either base-linked or essential, and never both.

**Definition 4.1 (Essential Pair Similarity).** Suppose $S$ and $T$ are meet-closed sets, and let $\langle k_i : i < M \rangle$ and $\langle \ell_i : i < N \rangle$ enumerate the levels of new essential linked pairs in $S$ and $T$, respectively. (This excludes levels of new mutually linked sets of size greater than two.) We say that a map $f : S \rightarrow T$ is an essential pair similarity map (or ep-similarity) if and only if $f$ is a strong similarity map and additionally the following hold: $M = N$, and for each $i < M$, letting $d$ be the critical node in $S$ with least length greater than $k_i$, if $\{s_0,s_1\} \subseteq S \upharpoonright |d|$ is the essential pair for level $k_i$, then $\{f(s_0),f(s_1)\}$ is the essential pair in $T \upharpoonright |f(d)|$ for level $\ell_i$.

Given finite antichains of coding nodes $A,B$ in a strong coding tree, we say that $A$ and $B$ are essential pair similar (or ep-similar) if and only if $A$ and $B$ are strongly similar, and the strong similarity map from $\text{cl}(A)$ to $\text{cl}(B)$ is an essential pair similarity. When $A$ and $B$ are ep-similar, we write $A \overset{ep}{\sim} B$.

**Remark 4.2.** Strict similarity was the structural property characterizing our upper bounds for big Ramsey degrees in [3]. As we are not using strict similarity directly in this paper, we refer the reader to Definition 8.3 in [3]. We point out that strict similarity implies ep-similarity, and not vice versa. That is, ep-similarity is a coarser equivalence relation than strict similarity. The improvement of our upper bounds in this paper is due to constructing a coding tree $S$ which represents a copy of the triangle-free Henson graph and in which every two ep-similar antichains are actually strictly similar. This will clear away all superfluous strict similarity types, leaving us with an exact characterization of the big Ramsey degrees.

By applying Theorem 3.18 finitely many times, using the methods in [3] from Sections 7–8 but substituting a canonically linked subtree $S$ (see Definition 4.8) in place of the incremental strong coding subtree in Lemma 7.5, we arrive at the following improvement of Theorem 8.9 from [3].

**Theorem 4.3 (Ramsey Theorem for Essential Pair Similar Antichains).** Given a strong coding tree $T$ and a finite triangle-free graph $G$, suppose $h$ colors all antichains $A$ of coding nodes in $T$ representing a copy of $G$ into finitely colors. Then there is a canonically linked coding tree $S \subseteq T$ such that all ep-similar antichains of coding nodes in $S$ have the same $h$-color.
Definition 4.4 (Sim$^{ep}(G)$). Given a finite triangle-free graph $G$, let Sim$^{ep}(G)$ denote a set of representatives from the different ep-similar equivalence classes of strongly skew antichains $A$ representing $G$ with the property that any coding node $c_n^A$ in $A$ with passing number 0 at another coding node $c_m^A$ (where $m < n$) is linked with $c_m^A$.

The work of Section 9 of [3] with a small but important modification made in Subsection 4.3 yields the following theorem.

Theorem 4.5 (Improved Upper Bounds). Let $G$ be a finite triangle-free graph, and let $f$ color all the copies of $G$ in $H_3$ into finitely many colors. Then there is a subgraph $H'$ of $H_3$, which is isomorphic to $H_3$, such that $f$ takes no more than $|\text{Sim}^{ep}(G)|$-many colors in $H'$. Hence,

\[(4) \quad T(G, H_3) \leq |\text{Sim}^{ep}(G)|.\]

We will prove that $T(G, H_3) = |\text{Sim}^{ep}(G)|$ in Section 4.

4.2. Canonically linked coding trees. In this subsection, we improve the main result of Section 7 in [3]. We will show in Lemma 4.11 that in any strong coding tree $T$, there is a canonically linked (see Definition 4.3) coding subtree $S$ and a subset $W$ of witnessing coding nodes with the following property: Given any antichain $A$ of coding nodes in $S$, there is a set of coding nodes $W_A$ in $W$ so that $A \cup W_A$ has the Strict Parallel 1’s Criterion. The canonical linked-ness of $S$ serves to get rid of most the superfluous strict similarity types which remained in our upper bounds in [3]: All ep-similar antichains in a canonically linked coding tree are strictly similar.

The remaining few superfluous strict similarity types will be eradicated by our construction of the antichain $\mathbb{D}$ in Lemma 4.12.

For two subsets $X, Y$ of some level set $Z$, we say that $X$ is lexicographically less than $Y$, and write $X <_{\text{lex}} Y$, if and only if, letting $\langle x_i : i < m \rangle$ and $\langle y_i : i < n \rangle$ be the lexicographically increasing enumerations of $X$ and $Y$, respectively, then either (a) $x_i <_{\text{lex}} y_i$ for the $i$ least such that $x_i \neq y_i$, or (b) $\langle x_i : i < m \rangle$ is an initial segment of $\langle y_i : i < n \rangle$. The following is Definition 7.1 in [3], where it was called Incremental Parallel 1’s.

Definition 4.6 (Incremental Linked Sets). Let $Z$ be a finite subtree of a strong coding tree $T$, and let $\langle \ell_j : j < \tilde{j} \rangle$ list in increasing order the minimal lengths of new mutually linked sets in $Z$. We say that $Z$ has incremental linked sets if the following holds. For each $j < \tilde{j}$ for which

\[(5) \quad Z_{\ell_j, 1} := \{ z \mid (\ell_{j+1})_z = z \in Z, |z| > \ell_j, \text{ and } z(\ell_j) = 1 \}\]

has size at least three, letting $m$ denote the length of the longest critical node in $Z$ below $\ell_j$, for each proper subset $Y \subseteq Z_{\ell_j, 1}$ of cardinality at least two, there is a $j' < j$ such that $\ell_{j'} > m$, $Y_{\ell_{j'}, 1} := \{ y \mid (\ell_{j'} + 1)_y = y \in Y \text{ and } y(\ell_{j'}) = 1 \}$ has the same size as $Y$, and $Y_{\ell_{j'}, 1} = Z_{\ell_{j'}, 1}$.

We shall say that an infinite tree $S$ has incremental linked sets if for each $\ell < \omega$, the initial subtree $S \upharpoonright \ell$ of $S$ has incremental linked sets.

We shall use canonical completions to construct an incremental coding tree with a minimal number of ep-similarity types.

Definition 4.7 (Canonical Completion of a Linked Pair). Suppose $A$ is a subtree of a strong coding tree $T$ and suppose $X = A \upharpoonright (\ell + 1)$ has a linked pair at $\ell$. We
call a level set $Y$ end-extending $X$ in $A$ a canonical completion of $X$ if and only if there is no splitting or coding node in $A$ in the interval $[\ell, \ell_Y]$ and the following hold:

Let $\langle M_i : i < \tilde i \rangle$ enumerate all maximal pairwise linked sets in $X$. List all subsets of size three from all $M_i$, $i < \tilde i$, in lexicographic order as $\langle P_{3,j} : j < k_3 \rangle$. Then list in lexicographic order all subsets of size four from all $M_i$, $i < \tilde i$, as $\langle P_{4,j} : j < k_4 \rangle$, etc., until all the sets $M_i$ appear as $P_{|M_i|, j}$ for some $j < k_{|M_i|}$. Let $\tilde m = \max(|M_i| : i < \tilde i)$. Then for each $3 \leq i \leq \tilde m$ and $j < k_i$, there is a level $\ell' \in (\ell, \ell_Y)$ such that $Y_{\ell',1}$ is mutually linked set end-extending $P_{i,j}$. Moreover, letting $\ell_{i,j}$ be the least level above $\ell'$ where $Y_{\ell',1}$ is a mutual linked set end-extending $P_{i,j}$, the sequence $\langle \ell_{3,j} : j < k_3 \rangle \ldots \langle \ell_{\tilde m,j} : j < k_{\tilde m} \rangle$ is an increasing sequence.

Thus, a canonical completion incrementally adds linked sets of the same size in lexicographic order, and then repeats this process for sets of the next largest size until it completes this process up to a new mutually linked set for each new maximal pairwise linked set. One can think of this as supersaturating the tree with all new linked sets in a canonical manner which will not negatively affect branching capabilities. By this, we mean that whenever $X$ is a pairwise linked set, given any $s$ and $t$ in $X$ and any coding nodes $c_m, c_n$ ($m < n$) extending $s, t$, respectively, then $c_m$ must have passing number 0 at $c_n$. Thus, adding new mutually linked sets among a pairwise linked set does not affect the ability of the tree to code a copy of $\mathcal{H}_3$. Note that if a linked pair is not included in any larger pairwise linked set, then that pair is its own canonical completion; no other linked pairs need be added.

The following refines the notion of incremental linked sets. This is the fundamental notion behind the canonical partitions, which provide exact big Ramsey degrees.

**Definition 4.8** (Canonically Linked). Let $A$ be a subtree of a strong coding tree $T$, and let $\langle c_n^A : n < N \rangle$ enumerate the coding nodes in $A$. We say that $A$ is canonically linked if for each $n < N$, the following holds. Let $\ell_s = 0$ if $n = 0$; otherwise, let $\ell_s = \ell_{s-1}^A + 1$.

1. For each splitting node $s$ in $A$ in the interval $[\ell_s, \ell_n^A)$, the minimal new mutually linked set in $A$ above $s$ is a pair of nodes $s_0, s_1$ extending $s^\sim 0, s^\sim 1$, respectively, such that $s_0(\ell) = s_1(\ell) = 1$ for some $\ell$. Moreover, above this $\ell$, there is a canonical completion before any new splitting node or any other new linked pairs occur.

2. Once we have performed the canonical completion on the maximal splitting node in $A$ below $c_n^A$, let $\langle Q_q : q < \hat q \rangle$ enumerate in lexicographic order all pairs of nodes in $A_{\ell_{\hat q}^A-1}$. Add a linked pair for $Q_0$ and then perform the canonical completion. Then add a linked pair for $Q_1$ and then perform the canonical completion. And so on until $Q_{\hat q-1}$ has been taken care of. After this, extend to the level of $c_n^A$.

In particular, whenever a new linked pair occurs, then the canonical completion occurs before any other critical node or other new linked pair occurs.

**Observation 4.9.** Any subtree of a canonically linked coding tree is again canonically linked. Moreover, for any two antichains $A, B$ of coding nodes in a canonically linked coding tree, $A$ and $B$ are strictly similar if and only if $A \preceq B$. 
Remark 4.10. A canonically linked tree cannot be a strong coding tree in the sense of the definition given in [3]. This is because we stipulated that strong coding trees have the property that taking leftmost extensions never adds a new linked set, whereas in canonically linked trees, any new maximal pairwise linked set $X$ will be followed by a new mutually linked set end-extending $X$. However, this does not affect the availability of passing numbers needed to construct subcopies of $\mathcal{H}_3$, for if $X$ is pairwise linked, then whenever one node in $X$ is extended to a coding node $c$, any other node extending a node in $X$ must have passing number 0 at $c$. Thus, adding the mutually linked set end-extending $X$ does not affect the ability to extend to a subtree coding a copy of $\mathcal{H}_3$.

In retrospect, we could have used maximally linked trees from the outset in [3], where instead of adding the canonical completion to each new linked pair, we simply add a mutually linked set immediately above each new maximal pairwise linked set. All proofs in that paper could be modified to hold using such coding trees. However, the way we defined strong coding trees, it is possible that within a given strong coding tree $T$, new linked pairs might occur before we could add a mutually linked set. Thus, for the sake of logic, we shall use what has been proved there, rather than rehashing all those proofs or asking the reader to believe without proof that the work in [3] holds if we replace strong coding trees with canonically linked or maximally linked coding trees.

By a canonically linked coding tree, we mean a strongly skew canonically linked coding subtree $S$ of $\mathbb{S}$ satisfying (1) and (3) of Definition 3.11. Every strong coding tree contains a canonically linked coding subtree, as we shall show below. The following is a revised version of Definition 7.4 in [3]. Given a node $w \in 2^{<\omega}$, we let $w^\downarrow$ denote the maximal initial segment of $w$ which is a sequence of 0’s. We say that $W \subseteq T$ is a set of witnessing coding nodes for a canonically linked coding tree $S \subseteq T$ if and only if each new mutually linked set $X \subseteq S$ is witnessed by a coding node $w \in W$ such that $|w^\downarrow|$ is less than the level of $X$ and $w$ is linked with no member of $S$.

The next Lemma says that inside any strong coding tree $T$, we can construct a canonically linked coding subtree and a set of canonical witnessing coding nodes. This improves Lemma 7.5 in [3].

Lemma 4.11 (Canonically linked coding tree). Let $T$ be a strong coding tree. Then there is a canonically linked coding tree $S \subseteq T$ and a set of witnessing coding nodes $W \subseteq T$ such that each new mutually linked set in $S$ is witnessed in $T$ by a coding node in $W$.

Proof. Let $\langle d^T_m : m < \omega \rangle$ denote the critical nodes in $T$ in order of increasing length. Let $\langle m_n : n < \omega \rangle$ denote the indices such that $d^T_{m_n} = c^T_n$, so that the $m_n$-th critical node in $T$ is the $n$-th coding node in $T$, and let $T(m)$ denote the level set $T \upharpoonright |d^T_m|$. We shall construct a canonically linked subtree $S$ of $T$ so that $S$ is strongly similar to $T$. We will let $d^S_m$ denote the $m$-th critical node of $S$, and $S(m) = S \upharpoonright |d^S_m|$. Since $T$ is a strongly skew tree coding $\mathcal{H}_3$, $d^S_0$ and $d^T_0$ are splitting nodes of $T$ which are members of $0^{<\omega}$ and $d^T_0 = c^T_0$ is a coding node (hence not in $0^{<\omega}$). Moreover, $\bigcup_{m < 3} T(m)$ has no essential linked pairs, so it is canonically linked. Thus, we let $d^S_m = d^T_m$, $S(m) = T(m)$, and $W_m = \emptyset$, for all $m < 3$.

Given $m \geq 3$, suppose we have chosen $S(k)$ for all $k < m$ so that $\bigcup_{k < m} S(k)$ is canonically linked and strongly similar to $\bigcup_{k < m} T(k)$. Moreover, suppose we have
chosen a level set extension $S(m-1)^+ \subseteq T$ of $S(m-1)$ which is a canonical completion of $S(m-1)$ and a set of coding nodes $W_m \subseteq T$ witnessing each new linked set in the canonical completion. We also suppose that $S(m-1)^+$ has no predetermined new linked sets in $T$. (This was called no predetermined new parallel 1’s in [3].) It means that it is possible to extend the level set $S(m-1)^+$ in $T$ without adding any new linked sets.) Let $T^+(m-1)$ denote the set of immediate successors $T(m-1)$ in $\tilde{T}$, and let $f : T(m-1)^+ \to S(m-1)^+$ be the lexicographic-preserving bijection between these two level sets. Let $t_*$ be the node in $T(m-1)^+$ such that $t_* \subseteq d_{m*}^T$, and let $s_* = f(t_*)$.

Suppose first that $d_{m*}^T$ is a splitting node. Take $d_{m*}^S$ to be any splitting node in $T$ extending $s_*$, and extend all other nodes in $S(m-1)^+$ along the leftmost paths in $T$ to the same length as $d_{m*}^S$. These nodes comprise the level set $S(m)$. Now take the set $\text{BL}_S(d_{m*}^S)$ of all nodes in $S(m)$ which are base-linked with $d_{m*}^S$. Let $Y$ be a level set in $T$ end-extending $\text{BL}_S(d_{m*}^S)$ such that $Y$ is a canonical completion of $\text{BL}_S(d_{m*}^S)$. As the canonical completion $Y$ is being constructed, take $W_m$ to be a set of witnessing coding nodes in $T$, similar to the construction in Lemma 7.5 in [3]. Finally, extend all nodes in $S(m) \setminus \text{BL}_S(d_{m*}^S)$ leftmost in $T$ to nodes of the length in $Y$, and let $S(m)^+$ be the union of $Y$ along with these nodes. Then $S(m)^+$ has no predetermined new linked sets, and all of its linked sets occur either in $S(m)$ or $Y$.

Now suppose that $d_{m*}^T$ is a coding node. Let $\ell$ denote $|d_{m*}^T|$, and recall that $T_{\ell,1}$ denotes the collection of nodes in $T \upharpoonright (\ell + 1)$ which have passing number 1 at $d_{m*}^T$. Let $Y$ denote the set of nodes in $T(m-1)^+$ which extend to a node in $T_{\ell,1}$, and let $Z = f[Y]$. Let $\langle P_j : j < \beta \rangle$ enumerate in lexicographic order the collection of pairs of members in $Z$.

Extend the nodes in $P_0$ to a linked pair in $T$ of the same length, say $P_0'$. Let $X_0$ be the union of $P_0'$ along with leftmost extensions in $T$ of all other nodes in $S(m-1)^+$ to the same length. Look at all the maximal pairwise linked sets in $X_0$, and take $Y_0$ to be a level set end-extending $X_0$ in $T$ such that $Y_0$ is a canonical completion of $X_0$. As this canonical completion is being formed, add new witnessing coding nodes into the set $W_m$ (similar to the construction in Lemma 7.5 in [3]). In general, for $j < \beta - 1$, given $Y_j$, let $P_{j+1}$ be the set of nodes in $Y_j$ extending the nodes in $P_{j+1}$. Let $X_{j+1}$ be an end-extension of $Y_j$ in $T$ which adds a linked pair above $P_{j+1}$. Then perform the canonical completion of $X_{j+1}$ to obtain a level set extension of $Y_{j+1}$ in $T$ while adding coding nodes to the set $W_m$ to witness each new linked set. At the end of this process, we have a level set $Y_{\beta-1}$. By Lemma 4.18 in [3], we can extend $Y_{\beta-1}$ to a level set $S(m)$ in $T$ so that the coding node in $S(m)$ extends $s_*$, and the lexicographic-preserving map from $T(m)$ to $S(m)$ preserves the passing types at the coding node in this level. It follows that $\bigcup_{k \leq m} T(k)$ is strongly similar to $\bigcup_{k \leq m} S(k)$.

To finish, let $S = \bigcup_{m < \omega} S(m)$ and let $W = \bigcup_{m < \omega} W_m$. Then $S$ is strongly similar to $T$ (and hence, represents a copy of $\mathcal{H}_3$), $S$ is canonically linked, and all new linked sets in $S$ are witnessed by coding nodes in $W$. \qed

It follows from the construction in the previous lemma that for each antichain $A \subseteq S$, there is a set of coding nodes $W_A \subseteq W$ such that $A \cup W_A$ satisfies the Strict Parallel 1’s Criterion. In particular, we can choose $W_A$ so that $A \cup W_A$ will be an envelope of $A$ (in the terminology of Section 8 of [3]).
Observe that for a canonically linked coding tree $S$, whenever $Y, Y'$ are level sets in $S$ with $Y'$ end-extending $Y$, then $Y'$ has no new mutually linked sets over $Y$ if and only if $Y'$ has no new linked pairs. Thus, inside $S$, the notion of strict similarity (Definition 8.3 in [3]) reduces simply to essential pair similarity. By replacing the uses of Lemma 7.5 of [4] with Lemma 4.14 in the proof of Theorem 8.9 in [3], we obtain Theorem 4.3.

4.3. Improved antichain of coding nodes $D$ representing $\mathcal{H}_3$. In Lemma 9.1 in [3], we showed that within any strong coding tree, there is an antichain $D$ of coding nodes which represent a copy of the triangle-free Henson graph. For the proof of Theorem 4.3 we will need to make a slight modification to this construction in order to sweep away the remaining superfluous ep-similarity types. We do this by linking any two coding nodes in the antichain where the longer one has passing number 0 at the shorter coding node.

**Lemma 4.12.** Let $S$ be a canonically incremental coding tree. Then there is an infinite antichain of coding nodes $D \subseteq S$ which code a copy of $\mathcal{H}_3$ in exactly the same way that $S$ does with the following additional property: Whenever $m < n$ and $c^D_n$ has passing type 0 at $c^D_m$, then $c^D_m$ and $c^D_n$ are linked.

**Proof.** We will construct an antichain of coding nodes $D \subseteq S$ which codes a copy of $\mathcal{H}_3$ in the same order as $S$. It is important to notice that, while taking leftmost extensions in $S$ can add new mutually linked sets (indeed this is the point of being canonically linked), it will never add a new essential linked pair. Thus, leftmost extensions of any unlinked pair in $S$ yield another unlinked pair in $S$.

We use the notation $\langle c^D_m : n < \omega \rangle$ to denote the $n$-th coding node of $S$, and $\ell^D_n$ to denote $|c^D_m|$. We let $\langle d^D_m : m < \omega \rangle$ denote the critical nodes (coding and splitting) of $S$, and $m_n$ denote the index such that $d^S_m$ equals the coding node $d^D_m$. Likewise, we will use $c^D_m$ to denote the $n$-th coding node in $D$, and $\ell^D_n$ to denote its length. The set of nodes in $D \setminus \{c^D_m\}$ of length $\ell^D_n$ shall be indexed as $\{d_s : s \in S \setminus \ell^S_n\}$. We will construct $D$ so that for each $n$, the node of length $\ell^D_n + 1$ which is going to be extended to the next coding node $c^D_n$ will split at a level lower than any of the other nodes of length $\ell^D_{n+1}$ split in $D$.

Define $d^D_0 = d^S_0$, the root of $S$, and let $D(0) = \{d^D_0\}$, the 0-th level of $D$. As the node $d^S_0$ splits in $S$, so also the node $d^D_0$ will split in $D$. Let $Y_0$ denote the set of the two immediate successors to $d^D_0$ in $S$.

For the induction step, suppose $m \geq 1$ and we have constructed $\bigcup_{k<m} D(k) \subseteq S$ so that it is ep-similar to $\bigcup_{k<m} S(k)$. Let $n$ be the index of the longest coding node in $S$ in $\bigcup_{k<m} S(k)$. We have three cases:

**Case I.** $d^S_m$ is a splitting node and $d^S_{m-1}$ is the coding node $c^S_n$.

Define $X$ to be the set of immediate successors in $\hat{S}$ of the level set $S(m-1)$, and define $Y$ to be the set of immediate successors in $\hat{S}$ of the level set $D(m-1) \setminus \{c^D_m\}$, respectively. Let $\psi : X \to Y$ be the lexicographic preserving bijection. Define $s_*$ to be the node in $X$ which extends to the splitting node $d^S_m$, and let $t_* = \psi(s_*)$. Let $x_*$ be the node in $X$ which extends to the next coding node in $S$, and let $y_*$ denote $\psi(x_*)$. Note that $s_*$ and $x_*$ are distinct, because every splitting node in $S$ has an extension with passing number 1 at the next coding node, while every node
which is base-linked with \(x_\ast\) must have passing number 0 at the next coding node, so does not extend to a splitting node in this interval.

First extend \(y_\ast\) to a splitting node \(y'_\ast\) in \(S\). Then let \(Y'\) be the level set of nodes of length \(|y'_\ast| + 1\) consisting of \(y'_\ast\) and \(y'_\ast − 1\) as well as leftmost extensions in \(S\) of the nodes in \(Y \setminus \{y_\ast\}\) to the length \(|y'_\ast| + 1\). After this, extend the node in \(Y'\) extending \(t_\ast\) to a splitting node in \(S\), and label this splitting node \(d_m\). Then let \(D(m)\) consist of the node \(d_m\) along with leftmost extensions of the nodes in \(Y' \setminus \{d_m\}\) in \(S\). Note that \(D(m)\) has one more node than \(S(m)\), precisely the node extending \(y'_\ast − 1\); label this node \(e_{n+1}\). This is the node that will be extended to the coding node \(c_{n+1}\). This construction adds no new linked pairs over \(D \cup (\ell^D + 1)\).

Case II. \(d_m^S\) and \(d_{m-1}^S\) are both splitting nodes.

Let \(e\) denote the node in \(D(m-1)\) extending \(e_{n+1}\). Define \(X\) to be the set of immediate successors in \(\hat{S}\) of the level set \(S(m-1)\), and define \(Y\) to be the set of immediate successors in \(\hat{S}\) of the level set \(D(m-1) \setminus \{e\}\), respectively. Let \(\psi : X \rightarrow Y\) be the lexicographic preserving bijection. As in Case I, let \(s_\ast\) be the node in \(X\) which extends to the splitting node \(d_m\), and let \(t_\ast = \psi(s_\ast)\). Then extend \(t_\ast\) to a splitting node in \(S\) and label it \(d_m^S\). Let \(D(m)\) be the collection of the leftmost extensions in \(S\) of the nodes in \(Y \setminus \{e\}\) along with \(d_m^S\).

Case III. \(d_m^S\) is a coding node.

In this case, \(d_m^S = c_{n+1}^S\), and \(d_{m-1}^S\) is a splitting node. Let \(X\) denote the set of immediate successors of \(S(m-1)\) in \(\hat{S}\). Let \(e\) denote the node in \(D(m-1)\) extending \(e_{n+1}\), and let \(Y\) denote the set of immediate successors of \(D(m-1) \setminus \{e\}\) in \(\hat{S}\). Let \(\psi\) be the lexicographic preserving bijection from \(X\) to \(Y\).

As a preparatory step, let \(c\) be a coding node in \(S\) extending \(e\) of long enough length that there is a level set extension \(Y'\) of \(Y\) in \(\hat{S}\) of length \(|c|\) such that the lexicographic preserving map from \(S(m)\) to \(Y'\) preserves passing numbers at the coding node at these levels. Since \(S\) is canonically linked, this automatically is inherited by \(Y'\); that is, \(Y'\) is canonically linked.

Now, we will extend \(c\) along with the nodes in \(Y'\) to construct \(D(m)\) so that for each non-coding node \(t \in D(m)\) with passing number 0 at \(c_m\), \(t\) is linked with \(c_m').\) Let \(Y'_0\) denote those nodes in \(Y'\) which have passing number 0 at \(c\), and let \(\{y_j : j < \hat{j}\}\) be the enumeration of \(Y'_0\) in lexicographic order. Take \(z_0\) extending \(y_0\) and \(u_0\) extending \(c\) in \(S\) so that \(z_0\) and \(u_0\) are linked. Given \(z_j\) and \(u_j\), where \(j < \hat{j} - 1\), take \(z_{j+1}\) extending \(y_{j+1}\) and \(u_{j+1}\) extending \(u\) in \(S\) so that \(z_{j+1}\) and \(u_{j+1}\) are linked. After this process is complete, let \(Y''\) be the level set of the leftmost extensions of the nodes \(\{z_j : j < \hat{j}\}\) to the length of \(z_{\hat{j}-1}\). Then extend \(Y'' \cup \{u_{\hat{j}-1}\}\) to a level set \(D(m)\) so that the node in \(D(m)\) extending \(u_{\hat{j}-1}\) is a coding node, label it \(c_{n+1}^S\), and the lexicographic preserving map from \(S(m)\) to \(D(m)\) \(\{c_{n+1}^S\}\) preserves passing numbers.

This concludes the construction of \(D\) satisfying the Lemma.

Recall Definition 14 where \(\text{Sim}^{ep}(G)\) was defined. Notice that for any antichain \(A \subseteq D\), \(A\) is ep-similar to a representative in \(\text{Sim}^{ep}(G)\). Replacing the uses of Theorem 8.9 and Lemma 9.1 of [3] with applications of Theorem 4.3 and Lemma 4.12 in the proof of Theorem 9.2 in [3] yields our Theorem 4.5.
In the next section, we will prove that each of the ep-similarity types in Sim^{ep}(G) persist in any subcopy of H_{3} contained in the one coded by D. It will follow that the big Ramsey degree T(G, H_{3}) is exactly the cardinality of Sim^{ep}(G).

5. Canonical Partitions

In this section, we prove that for a given finite triangle-free graph G, each of the types in Sim^{ep}(G) persists in every subcopy of H_{3}. This produces canonical partitions of the copies of G in H_{3}, characterizing the exact big Ramsey degree of G in H_{3}.

Fix a canonically linked coding tree S (recall Lemma 4.11) and an antichain of coding nodes D \subseteq S such that D represents a copy of H_{3}. (The construction of such a D is done in Lemma 9.1 of [3].)

**Theorem 5.1 (Persistence).** Let D be any subset of D representing a copy of H_{3}. Given any antichain of coding nodes A \subseteq S, there is an essential pair similarity embedding of A into D. It follows that every essential pair similarity type of an antichain in S persists in D.

**Proof.** Let C denote \{c_{n} : n < \omega\}, the set of all coding nodes in S. Fix an antichain of coding nodes D \subseteq D representing H_{3}. Without loss of generality, we may assume that D represents H_{3} in the same order that S does. Let \{c_{n}^{D} : n < \omega\} enumerate the coding nodes in D, in order of increasing length. Then the map \varphi : C \to D via \varphi(c_{n}) = c_{n}^{D} is passing number preserving, meaning that whenever m < n, then

\[(6) \quad \varphi(c_{n})(|\varphi(c_{m})|) = c_{n}(|c_{m}|).
\]

Define

\[(7) \quad \overline{D} = \{c_{n}^{D} \upharpoonright |c_{m}^{D}| : m \leq n < \omega\}.
\]

Then \overline{D} is a union of level sets, but not meet-closed. We extend the map \varphi to a map \overline{\varphi} : \overline{S} \to \overline{D} as follows: Given s \in S, let n be least such that c_{n} \supseteq s, and let m be the integer such that |s| = |c_{n}|, and define \overline{\varphi}(s) = \varphi(c_{n}) \upharpoonright |\varphi(c_{m})|; that is,

\[\overline{\varphi}(s) = c_{n}^{D} \upharpoonright |c_{m}^{D}|.
\]

Notice that \overline{\varphi} is again passing number preserving: Given s, t \in \overline{S} with |s| = |c_{m}| < |t|, and given n least such that c_{n} \supseteq t, we have

\[(8) \quad \overline{\varphi}(t)(|\overline{\varphi}(s)|) = \varphi(c_{n})(|\varphi(c_{m})|) = c_{n}^{D}(|c_{m}^{D}|) = c_{n}(|c_{m}|) = t(|s|).
\]

**Observation 5.2.** If m < n and c_{n}(|c_{m}|) = 1, then c_{n}^{D}(|c_{m}^{D}|) = 1; hence c_{n}^{D} and c_{m}^{D} have no parallel 1’s.

In what follows, for s \in S, we let \widehat{s} denote the cone of all s’ \in S extending s. A subset X \subseteq \widehat{s} is cofinal in \widehat{s} if for each s’ \supseteq s in S, there is an x \in X such that x \supseteq s’. For t \in \overline{D}, t denotes the set of all t’ \in \overline{D} extending t. Since \overline{\varphi} is a map from C onto D, it follows that for a subset L \subseteq \overline{D}, \varphi^{-1}[L] is the set of coding nodes c \in C such that \varphi(c) \in L. We work with subsets L of \overline{D} rather than just of D because we shall be interested in cones above members of \overline{D}, and allowing this flexible notation will reduce the need for extra symbols throughout.

**Definition 5.3.** Let L be a subset of \overline{D}. Given s \in S, we say that L is s-large if and only if \varphi^{-1}[L] \cap \widehat{s} is cofinal in \widehat{s}. We say that L is large if and only if there is some s \in S for which L is s-large.

We say that L is 0-large if and only if L is s-large for some s \in 0^{<\omega}. Call s’ a 0-extension of s if and only if \supseteq s and for each |s| \leq i < |s’|, s’(i) = 0. We say that L is s-0-large if and only if L is s’-large for some 0-extension s’ of s.
The next observation follows immediately from the definitions.

**Observation 5.4.** If \( L \subseteq \overline{D} \) is \( s \)-large, then \( L \) is \( s' \)-large for every \( s' \) extending \( s \). In particular, \( L \) is \( s' \)-large for each \( s' \) which 0-extends \( s \), which implies that \( L \) is \( s \)-0-large.

The following series of lemmas will aid in building an \( \epsilon \)-similarity copy of a given antichain from \( S \) inside of \( D \).

**Lemma 5.5.** Suppose \( t \) is in \( \overline{D} \) and \( \hat{t} \) is 0-large. Then \( \varphi^{-1}[\hat{t}] \) contains a copy of \( \mathcal{H}_3 \), and \( t \) is in \( 0^\omega \).

**Proof.** Suppose \( \hat{t} \) is 0-large. Then there is some \( s \in 0^\omega \) such that \( \varphi^{-1}[\hat{t}] \cap \hat{s} \) is cofinal in \( \hat{s} \). Since \( s \) is in \( 0^\omega \), the set of coding nodes in \( \hat{s} \) represents a graph which contains a copy of \( \mathcal{H}_3 \). In particular, \( \varphi^{-1}[\hat{t}] \cap \hat{s} \) being a collection of coding nodes which is cofinal in \( \hat{s} \) implies that this set contains coding nodes representing a copy of \( \mathcal{H}_3 \). Since \( \varphi \) is passing number preserving, it follows that \( \hat{t} \) contains a copy of \( \mathcal{H}_3 \). This would be impossible if \( t \) were not a sequence of 0’s. Therefore, \( t \in 0^\omega \). \( \square \)

**Lemma 5.6.** If \( L \subseteq \overline{D} \) is \( s \)-large and \( L = \bigcup_{i < n} L_i \) is a partition of \( L \) into finitely many pieces, then there is an \( i < n \) such that \( L_i \) is \( s \)-0-large.

**Proof.** Suppose that no \( L_i \) is \( s \)-0-large. Then there is an \( s_0 \) which 0-extends \( s \) such that \( \varphi^{-1}[L_0] \cap \hat{s_0} = \emptyset \). Given \( i < n - 1 \) and \( s_i \), a 0-extension of \( s \), since \( L_{i+1} \) is not \( s \)-0-large, there is some \( s_{i+1} \) which 0-extends \( s_i \) such that \( \varphi^{-1}[L_{i+1}] \cap \hat{s}_{i+1} = \emptyset \). In the end, we obtain \( s_{n-1} \in S \) which 0-extends \( s \) such that for all \( i < n \), \( \varphi^{-1}[L_i] \cap \hat{s}_{n-1} = \emptyset \). This contradicts that \( L \) is \( s \)-large. \( \square \)

**Lemma 5.7.** Suppose \( t \) is in \( \overline{D} \) \( |c^D_n| \) and \( \hat{t} \) is \( s \)-large. Let \( n \geq m \) be given satisfying \( |c^D_n| > |s| \), and let \( t \geq |c^D_n| + 1 \) be given. For \( i < 2 \), let

\[
J_i = \bigcup \{ \hat{u} : u \in \hat{t} \cap \ell \text{ and } u(|c^D_n|) = i \}.
\]

Then for each \( i < 2 \), \( J_i \) is large. Moreover, \( J_0 \) is \( s \)-0-large.

**Proof.** Let \( i < 2 \) be fixed, and suppose towards a contradiction that \( J_i \) is not large. Fix any \( s_0 \) satisfying \( |s_0| > |c^D_n| \), \( s_0(|c^D_n|) = i \), and \( |\varphi(s_0)| > \ell \). Since \( J_i \) is not large, there is some \( s_1 \geq s_0 \) such that \( \varphi^{-1}[J_i] \cap \hat{s_1} = \emptyset \). Fix some coding node \( c_k \in \varphi^{-1}[\hat{t}] \cap \hat{s_1} \). Such a coding node exists since \( \varphi^{-1}[\hat{t}] \) is a cofinal subset of \( C \cap \hat{s_1} \). Note that \( c_k \geq s_0 \) implies \( c_k(|c^D_n|) = i \), and \( c_k \in \varphi^{-1}[\hat{t}] \) implies that \( c^D_k = \varphi(c_k) \geq t \). Therefore, \( c^D_k \) is a member of \( J_i \). Hence, \( c^D_k \) is in \( \varphi^{-1}[J_i] \cap \hat{s}_1 \), contradicting that this set is empty. Thus, \( J_i \) must be large.

Now suppose that \( J_0 \) is not \( s \)-0-large. Similar to the above argument, take any \( s_0 \) which 0-extensions \( s \) such that \( |s_0| > |c^D_n| \), \( s_0(|c^D_n|) = 0 \), and \( |\varphi(s_0)| > \ell \). Since \( J_0 \) is not \( s \)-0-large, there is some 0-extension \( s_1 \) of \( s_0 \) such that \( \varphi^{-1}[J_0] \cap \hat{s}_1 = \emptyset \).

Now take some \( c_k \in \varphi^{-1}[\hat{t}] \cap \hat{s}_1 \). This time, \( c_k(|c^D_n|) = 0 \), since \( c_k \geq s_0 \), and again, \( c_k \in \varphi^{-1}[\hat{t}] \) implies that \( c^D_k = \varphi(c_k) \geq t \). Therefore, \( c^D_k \) is a member of \( J_0 \), a contradiction. Thus, \( J_0 \) must be \( s \)-0-large. \( \square \)

**Lemma 5.8.** Suppose that \( \hat{t} \) is \( s \)-0-large and \( |t| < |\varphi(s)| \). Then for each \( \ell > |\varphi(s)| \), there is an extension \( u \supseteq t \) in \( \overline{D} \) with \( |u| = \ell \) such that \( \hat{u} \) is \( s \)-0-large.

**Proof.** Since \( \hat{t} \) is \( s \)-0-large, there is some 0-extension \( s_0 \) of \( s \) such that \( \hat{t} \) is \( s_0 \)-large. Let \( n \) be any index such that \( \ell > |c^D_n| > |t| \). Letting \( L_{\ell}(\hat{u} : u \in \hat{t} \cap \ell \text{ and } u(|c^D_n|) = \)
and hence, \( j \) such that \( |u| = s_0 \)-large. Since \( s_0 \) is a 0-extension of \( t \), \( \hat{u} \) is again \( s \)-large.

We shall say that a pair of nodes \( s, t \) is **unlinked** if it is not linked; that is, if there is no \( \ell \) such that \( s(\ell) = t(\ell) = 1 \).

**Lemma 5.9.** Suppose that \( t_0, t_1 \in D \) are unlinked, and assume also that \( |t_0| = |t_1| \). Given \( s_0, s_1 \in S \) of the same length such that for each \( i < 2 \), \( \hat{t}_i \) is \( s_i \)-large, then

(a) \( s_0 \) and \( s_1 \) are unlinked; and

(b) For each \( \ell > |t_1| \) there are \( u_i \in \hat{t}_1 \mid \ell \) such that \( u_0 \) and \( u_1 \) are unlinked, and there are 0-extensions \( x_i \supseteq s_i \) such that \( \hat{u}_i \) is \( x_i \)-large. It follows that \( x_0 \) and \( x_1 \) are unlinked.

**Proof.** Let \( j \) be the integer such that \( |s_0| = |s_1| = |c_j| \), and let \( k \) be the integer such that \( |t_0| = |t_1| = |c_k| \). By Lemma 5.8, we may assume that each \( |t_i| \geq |\hat{\phi}(s_i)| \), and hence, \( j \leq k \). Let \( \ell > |c_k| \) be given. Since for each \( i < 2 \), \( \hat{t}_i \) is \( s_i \)-large, it follows that \( \bigcup \{ \hat{u} : u \in \hat{t}_1 \mid \ell \} \) is also \( s_0 \)-large. By Lemma 5.6 we can fix some \( u_i \in \hat{t}_i \mid \ell \) such that \( \hat{u}_i \) is \( s_i \)-large. Thus, there is an \( x_i \) which 0-extends \( s_i \) such that \( \hat{u}_i \) is \( x_i \)-large. By 0-extending one of the \( x_i \)'s if necessary, we may assume that \( |x_0| = |x_1| \). Take coding nodes \( c_n \in \varphi^{-1}[\hat{u}_1] \cap \hat{t}_i \); without loss of generality, say \( n_0 < n_1 \).

Since each \( c_m^D \supseteq t_i \) and the pair \( t_0, t_1 \) is unlinked, for each \( m < k \), at least one of \( c_{n_0}^D (|c_{m_0}^D|) \) and \( c_{n_1}^D (|c_{m_1}^D|) \) equals zero. Then since \( \varphi \) is passing number preserving and \( j \leq k \), we have that \( c_{n_0} \) and \( c_{n_1} \) are unlinked below \( |c_j| \). Since for each \( i < 2 \), \( c_n \) extends \( s_i \), it follows that \( s_0 \) and \( s_1 \) are unlinked. (This uses the fact that every level of \( S \) has a coding node.) Thus, (a) holds.

To finish proving (b), since \( x_0 \) and \( x_1 \) are unlinked at any \( m \) in the interval \( [j, |x_1|] \), and since by (a), they are unlinked at any \( m < j \), it follows that \( x_0 \) and \( x_1 \) are unlinked. Furthermore, \( \varphi^{-1}[\hat{u}_1] \cap \hat{t}_i \) is cofinal in \( \hat{t}_i \). Therefore, we can choose the coding nodes \( c_n \supseteq x_i \) to have the additional property that \( c_n (|c_{m_0}^D|) = 1 \). Since \( \varphi \) is passing number preserving, it also holds that \( c_{n_0}^D (|c_{m_0}^D|) = 1 \). Since \( c_{n_1}^D = \varphi(c_{n_1}^D) \supseteq u_i \), it must be the case that \( u_0 \) and \( u_1 \) are unlinked. Hence, (b) holds.

**Lemma 5.10.** Suppose that

- (1) \( t_0, t_1 \in D \) are of the same length and are unlinked.
- (2) \( s_0, s_1 \in S \) are of the same length.
- (3) \( t_0 \) is \( s_0 \)-large.
- (4) \( u_1 \supseteq t_1 \) and satisfies \( \hat{u}_1 \) is \( x_1 \)-large, for some \( x_1 \supseteq s_1 \).

Then

(a) \( s_0 \) and \( s_1 \) are unlinked; and

(b) There is some \( u_0 \in \hat{t}_0 \mid |u_1| \) and a 0-extension \( x_0 \supseteq s_0 \) such that \( \hat{u}_0 \) is \( x_0 \)-large, \( u_0 \) and \( u_1 \) are unlinked, and \( x_0 \) and \( x_1 \) are unlinked.

**Proof.** Let \( j \) be the integer such that \( |s_0| = |s_1| = |c_j| \), and let \( k \) be the integer such that \( |t_0| = |t_1| = |c_k| \). Then \( |t_0| \geq |\hat{\phi}(s_0)| \) implies that \( j \leq k \). Since \( t_0 \) is \( s_0 \)-large, letting \( \ell = |u_1| \), it follows that \( \bigcup \{ \hat{u} : u \in \hat{t}_0 \mid \ell \} \) is also \( s_0 \)-large. By Lemma 5.6, we can fix some \( u_0 \in \hat{t}_0 \mid \ell \) such that \( \hat{u}_0 \) is \( s_0 \)-large. Thus, there is an \( x_0 \) which 0-extends \( s_0 \) such that \( \hat{u}_0 \) is \( x_0 \)-large. By 0-extending one of the \( x_i \)'s if
necessary, we may assume that $|x_0| = |x_1|$. Take coding nodes $c_n \in \varphi^{-1}[\hat{u}_i] \cap \hat{x}_i$; without loss of generality, say $n_0 < n_1$.

Since each $c_n^D \supseteq t_i$ and the pair $t_0, t_1$ is unlinked, for each $m < k$, at least one of $c_n^D([c_n^D])$ and $c_m^D([c_m^D])$ equals zero. Then since $\varphi$ is passing number preserving and $j \leq k$, we have that $c_{n_0}$ and $c_{n_1}$ are unlinked below $[c_j]$. Since each $c_n$ extends $s_i$, it follows that $s_0$ and $s_1$ are unlinked. (This uses the fact that every level of $S$ has a coding node.) Thus, (a) holds.

We took $u_0 \in \hat{t}_0$ and $x_0$ to be a 0-extension of $s_0$ so that $\hat{u}_0$ is $x_0$-large. So to finish proving (b), we just need to show that $u_0$ and $u_1$ are unlinked. It suffices to show that there are coding nodes $c_n^D \supseteq u_i$ with $|c_n^D| < |c_{m}^D|$ and $c_n^D([c_n^D]) = 1$. Since $x_0$ is a 0-extension of $s_0$, and since by part (a), $s_0$ and $s_1$ are unlinked, it follows that $x_0$ and $x_1$ are unlinked. Furthermore, each $\varphi^{-1}[\hat{u}_i] \cap \hat{x}_i$ is cofinal in $\hat{x}_i$. Therefore, we can choose the coding nodes $c_n \supseteq x_i$ to have the additional property that $c_n^D([c_n^D]) = 1$. Since $\varphi$ is passing number preserving, it also holds that $c_n^D([c_n^D]) = 1$. Since $c_n^D = \varphi(c_n^D) \supseteq u_i$, it must be the case that $u_0$ and $u_1$ are unlinked. By the same reasoning as for $s_0$ and $s_1$, it follows that $x_0$ and $x_1$ are unlinked. Hence, (b) holds. □

The next lemma follows from Lemma 5.9 and the fact that $\mathbb{D}$ is canonically linked.

**Lemma 5.11.** Suppose $X = \{s_i : i < p\}$ is a level set in $\mathbb{S}$ and $Y = \{t_i : i < p\}$ is a level set in $\mathbb{D}$ such that

1. For each $i < p$, $\hat{t}_i$ is $s_i$-0-large.
2. For each $i < j < p$, $t_i$ and $t_j$ are linked if and only if $s_i$ and $s_j$ are linked.

Then for each $\ell > |t_0|$, for each $i < p$ there is some $u_i \supseteq t_i$ of length $\ell$ and there is some $x_i$ which 0-extensions $s_i$ such that each $\hat{u}_i$ is $x_i$-large, and all $x_i$ have the same length. Moreover, the set $\{u_i : i < p\}$ has no new linked pairs over $Y$.

**Proof.** Let $X = \{s_i : i < p\}$ be a level set in $\mathbb{S}$ and let $Y = \{t_i : i < p\}$ be a level set in $\mathbb{D}$ with $|\varphi(s_0)| \leq |t_0|$ satisfying assumptions (1) and (2). Let $j \leq k$ be given such that each $|s_i| = |c_j|$ and each $|t_i| = |c_k^D|$. Since $\mathbb{D}$ is canonically linked and $Y$ is a subset of $\mathbb{D}$, it follows that $Y$ is canonically linked.

By Lemma 5.9 there are $x_i$ 0-extensions $s_i$, all of the same length, and there are $u_i \supseteq t_i$ all of length $\ell$ such that each $\hat{u}_i$ is $x_i$-large and each pair $\{u_i, u_j\}$ is linked only if $\{t_i, t_j\}$ is linked. Thus, the set $Y' = \{u_i : i < p\}$ has no new linked sets over $Y$. (This follows from $\mathbb{D}$ being incrementally linked: For $Y'$ has no new linked sets over $Y$ if and only if $Y'$ has no new linked pairs.) Since $X' = \{x_i : i < p\}$ is a level set of 0-extensions of $X$, it has no new linked sets over $X$. □

For level sets $X$ and $Y$ with the same cardinality, we say that $X$ and $Y$ have the **same linked pairs** if and only if for all $i < j < p$, $\{s_i, s_j\}$ is linked iff $\{t_i, t_j\}$ is linked, where $\{s_i : i < p\}$ and $\{t_i : i < p\}$ are the lexicographically increasing enumerations of $X$ and $Y$, respectively.

Let $A$ be an antichain of coding nodes in $S$. Let $W_A$ be a minimal subset of $W$ such that each new essential linked pair in $A$ is witnessed by a coding node in $W_A$, and let $B$ denote the meet-closure of $A \cup W_A$. By our construction of $W$, we may assume that each new essential linked pair $\{s, t\}$ in $A$ is witnessed by the coding node $c$ of least length in $B$ above the minimal level $\ell$ such that $s(\ell) = t(\ell) = 1$.

Moreover, this coding node $c$ is the minimal critical node in $B$ above $\ell$ and forms
no linked pair with any other member of $B$, and $A$ has no other new linked sets in the interval $[|\ell|,|c_s|]$. ($B$ can be thought of as a minimalistic kind of envelope for $A$. The witnessing coding nodes in $W_A$ are best thought of as place holders to keep track of levels where new essential linked pairs appear.) Let $\langle b_i : i < N \rangle$, where $N \leq \omega$, enumerate the nodes in $B$ in order of increasing length.

We will be using the map $\bar{\varphi}$ to construct an ep-similarity map $f$ of $B$ as a subset of $S$ into $\Bar{D}$ as follows: For $k > 0$, let $M_k = |b_{k-1}| + 1$. Define $\Bar{D}$ to be the tree of all initial segments of members of $D$; thus, $\Bar{D} = \{ u \upharpoonright \ell : u \in D$ and $\ell \leq |u| \}$. For $k < \omega$ we will recursively define meet-closed sets $T_k \subseteq \Bar{D}$, $N_k < \omega$, a level set $\{ s_t : t \in T_k \cap N_k \} \subseteq S$, and maps $f_k$ and $\psi_k$ such that the following hold:

1. $f_k$ is an ep-similarity embedding of $\{ b_i : i < k \}$ onto a subset $\{ t_i : i < k \} \subseteq T_k$.
2. $N_k = \max\{ |t| : t \in T_k \}$.
3. All maximal nodes of $T_k$ are either in $T_k \cap N_k$, or else in the range of $f_k$.
4. For each $t \in T_k \cap N_k$, $t$ is $s_t$-large.
5. $\psi_k$ is a $\prec$ and passing type preserving bijection of $B \upharpoonright M_k$ to $T_k \cap N_k$.
6. $B \upharpoonright M_k$, $T_k \upharpoonright N_k$, and $\{ s_t : t \in T_k \cap N_k \}$ all have the same linked pairs; that is, the pair $\{ y, z \} \subseteq B \upharpoonright M_k$ is linked if and only if $\{ \psi_k(y), \psi_k(z) \}$ is linked if and only if $\{ s_{\psi_k(y)}, s_{\psi_k(z)} \}$ is linked.
7. $T_{k-1} \subseteq T_k$, $f_{k-1} \subseteq f_k$, and $N_{k-1} < N_k$.

The idea behind $T_k$ is that it will contain an ep-similarity image of $\{ b_i : i < k \} \cup (B \upharpoonright M_k)$, the nodes in the image of $B \upharpoonright M_k$ being the ones we need to continue extending in order to build an ep-similarity from $B$ into $\Bar{D}$.

To begin, let $r$ denote the root of $cl(D)$ and assume, without loss of generality, that $|r| \geq 1$. Then $\Bar{r} = \Bar{D}$, which is $0$-large, and $|r| < |c_0|^D$. By Lemma 5.6, there is a $t_{-1} \in D \upharpoonright |c_0|^D$ such that $\Bar{t}_{-1}$ is $0$-large. By Lemma 5.5, $t_{-1}$ is in $0^{<\omega}$. Let $s_{-1} \in S$ be a node of minimum length such that $\varphi^{-1}[\Bar{t}_{-1}] \cap s_{-1}$ is cofinal in $s_{-1}$. Note that $s_{-1}$ must be in $0^{<\omega}$. Define $T_{-1} = \{ t_{-1} \}$, $f_0 = \emptyset$, $N_{-1} = |t_{-1}|$ (which equals $|c_0|^D$), and letting $b_{-1} = c_0$, let $M_{-1} = 0$ and $\psi_{-1}(b_{-1}) = t_{-1}$ (noting that $c_0$ is an initial segment of all nodes in $B$).

Assume now that $k \geq 0$ and (1)-(7) hold for all $k' < k$. We have two cases: Either $b_k$ is a splitting node or else it is a coding node.

**Case I**. $b_k$ is a splitting node.

Let $t$ denote $\psi_k(b_k \upharpoonright M_k)$. Then by (5), $t$ is a member of $T_k \cap N_k$, and by (4), $\Bar{t}$ is $s_t$-large. Fix a coding node $c_j \in \varphi^{-1}[\Bar{t}] \cap C$. The purpose of $\varphi(c_j)$ (which we recall, is exactly $c_j^D$) is just to find a level of $\Bar{D}$ where we can find two distinct nodes which extend $t$, so that $t$ will be extended to a splitting node. For $i \in \{0,1\}$, let $\ell = |c_j^D|_{T_i}$, and let

\[
J_i = \bigcup \{ \Bar{u} : u \in \Bar{t} \upharpoonright \ell$ and $u(|\varphi(c_j)|) = i \}.
\]

Then by Lemma 5.7, both $J_i$ are large, and moreover, $J_0$ is $s_t$-$0$-large. Therefore, by Lemma 5.6, for $i \in \{0,1\}$, there is $u_i \in \Bar{t} \upharpoonright \ell$ and an extension $s_i$ of $s_t$ such that $\Bar{u}_i$ is $s_i$-large; moreover, this lemma ensures that we may take $s_0$ to be a $0$-extension of $s_t$. If $s_t$ is in $0^{<\omega}$, then so also $s_0$ is in $0^{<\omega}$. Note that $u_0$ and $u_1$ are incomparable, since they have different passing numbers at $\varphi(c_j)$. Moreover, their meet, $u_0 \land u_1$, extends $t$. Define $N_{k+1} = \ell$, which is $|u_i|$ for both $i \in \{0,1\}$. 

For all other \( y \in (T_k \upharpoonright N_k) \setminus \{ t \} \), by (4) we know that \( \hat{t} \) is \( s_y \)-large. Again by Lemma 5.6 there is a \( y' \in \hat{g} \upharpoonright N_{k+1} \) such that \( y' \) is \( s_y \)-0-large. Therefore, there is a 0-extension \( s_{y'} \supseteq s_y \) such that \( y' \) is \( s_{y'} \)-large. Note that if \( s_y \) is in \( 0^{<\omega} \), then so also \( s_{y'} \) is in \( 0^{<\omega} \). By 0-extending some of the nodes if necessary, we may assume that \( s_0, s_1 \) and all \( s_{y'} \) have the same length.

Define \( t_k = u_0 \land u_1 \), and extend the map \( f_k \) by letting \( f_{k+1}(b_k) = t_k \). Then \( f_k \) is an essential pair strict similarity embedding of \( \{ b_i : i \leq k \} \) onto \( \{ t_i : i \leq k \} \) so (1) holds. Define

\[
T_{k+1} = T_k \cup \{ u_0, u_1, d_k \} \cup \{ y' : y \in (T \upharpoonright N_k) \setminus \{ t \} \}.
\]

Letting \( s_{u_i} \) denote \( s_i \) for \( i \in \{ 0, 1 \} \), we have the level set \( \{ s_z : z \in T_{k+1} \upharpoonright N_{k+1} \} \) with the following properties: For each \( z \in T_{k+1} \upharpoonright N_{k+1} \), \( \hat{z} \) is \( s_z \)-large, so (4) holds. By Lemma 5.10 this set \( \{ s_z : z \in T_{k+1} \upharpoonright N_{k+1} \} \) and \( T_{k+1} \upharpoonright N_{k+1} \) have the same linked pairs, precisely because they each have no new linked pairs. Since \( b_k \) is a splitting node, \( B \upharpoonright M_{k+1} \) has no new linked pairs over \( B \upharpoonright M_k \). Thus, (5) holds.

Define \( \psi_{k+1} \) on \( D \cap (b_k \upharpoonright M_{k+1}) \) to be the unique \( \prec \)-preserving map onto \( \{ u_0, u_1 \} \); then (6) holds. Properties (2), (3), and (7) hold by our construction. This completes Case I.

Case II. \( b_k \) is a coding node.

Let \( t \) denote \( \psi_k(b_k \upharpoonright M_k) \in T_k \upharpoonright N_k \). Choose a coding node \( c_p \supseteq s_t \) in \( S \) such that \( |c_p| > |s_t| \) for all \( t \in T_k \upharpoonright N_k \), and \( |\varphi(c_p)| > t \) for all \( t \in T_k \upharpoonright N_k \). Define \( t_k = \varphi(c_p) \). Extend \( f_k \) by defining \( f_{k+1}(b_k) = t_k \), and let \( N_{k+1} = [c_p^{D+1}] \). For each \( i \in \{ k, k+1 \} \), let \( E_i = (A \upharpoonright M_i) \setminus \{ b_i \upharpoonright M_i \} \). Note that for each \( s \in E_k \), there is a unique \( e' \in E_{k+1} \) such that \( e' \triangleright e \).

Fix any \( e \in E_k \) and let \( t := \psi_k(e) \in T_k \upharpoonright N_k \). Let \( i \) denote the passing number of \( e' \) at \( b_k \). By Lemma 5.7

\[
J_i := \bigcup \{ \hat{u} : u \in \hat{t} \upharpoonright N_{k+1} \text{ and } u(|t_k|) = i \}
\]

is large, and is \( s_t \)-0-large if \( i = 0 \). Thus, by Lemma 5.6 there is some \( t' \in \hat{t} \upharpoonright N_{k+1} \) such that \( \hat{t}' \) is large, and is \( s_t \)-0-large if \( i = 0 \). If \( i = 0 \), let \( s_{t'} \) be a 0-extension of \( s_t \) of length large enough that \( \hat{t}' \) is \( s_{t'} \)-large. If \( i = 1 \), let \( s_{t'} \) be an extension of \( s_t \) such that \( \hat{t}' \) is \( s_{t'} \)-large. Thus, (4) will hold, given our definition of \( T_{k+1} \) below. Define \( \psi_{k+1}(e') = t' \), and note that \( \psi_{k+1}(e')(|t_k|) = e'(|b_k|) \). As \( \psi_{k+1} \) is \( \prec \)-preserving, (5) holds.

Let

\[
T_{k+1} = T_k \cup \{ b_k \} \cup \{ \psi_{k+1}(e') : e' \in E_{k+1} \}.
\]

By 0-extending some of the nodes \( s_{t'} \) if necessary, we may assume that the nodes in the set \( \{ s_{t'} : t' \in T_{k+1} \upharpoonright N_{k+1} \} \) all have the same length. By the induction hypothesis (6) for \( T_k \upharpoonright N_k \) and \( \{ s_t : t \in T_k \upharpoonright N_k \} \) and Lemmas 5.9 and 6.10 for each pair \( t, u \in T_k \upharpoonright N_k \) the pair of nodes \( s_{t'}, s_{u'} \) is linked if and only \( t', u' \) are linked. Rewriting, we have that \( T_{k+1} \upharpoonright N_{k+1} \) and \( \{ s_t : t \in T_{k+1} \upharpoonright N_{k+1} \} \) have the same linked pairs. Furthermore, any pair \( t', u' \in T_{k+1} \upharpoonright N_{k+1} \) are linked only if their \( \psi_{k+1} \)-preimages in \( B \upharpoonright M_{k+1} \) are linked. Thus, (6) holds. By our construction, (1), (2), (3), and (7) hold as well.

This concludes the construction in Case II.
Finally, let $f = \bigcup_k f_k$. Then $f$ is a strong similarity map from $B$ to $f[B]$: $f$ preserves the $\prec$-order and the splitting and coding node order of $B$. Moreover, whenever $b_k$ is a coding node, the construction of $\psi_{k+1}$ ensures that for any $k < n < N$ such that $b_n$ is a coding node in $B$, $f_{n+1}(b_n)$ has the same passing number at $f_k(b_k)$ as the passing number of $b_n$ at $b_k$. So $f$ is passing type preserving. Further, property (6) along with the properties of $\Psi$ ensures that $f$ is an ep-similarity map from $B$ to $f[B]$. It follows that $f \upharpoonright A$ is an ep-similarity from $A$ to $f[A]$. \hfill $\square$

**Theorem 5.12 (Canonical Partitions).** Let $G$ be a finite triangle-free graph and let $h$ be a coloring of all copies of $G$ inside $\mathcal{H}_3$. Then there is a subgraph $\mathcal{H}$ of $\mathcal{H}_3$ which isomorphic to $\mathcal{H}_3$ in which for each $A \in \text{Sim}^{ep}(G)$, all copies of $G$ in $\text{Sim}^{ep}(A)$ have the same color. Moreover, each $A \in \text{Sim}^{ep}(G)$ persists in the coding tree induced by $\mathcal{H}$.

**Proof.** Let $G \in \mathcal{G}_3$ be given, and fix a finite coloring $h$ of the copies of $G$ in $\mathcal{H}_3$. Let $S$ be a coding tree representing $H_3$. By the Partition Theorem 4.5, there is an antichain $D$ of a canonically linked subtree $S \subseteq S$ such that all sets of coding nodes $A$ in $D$ representing $G$ with the same essential pair strict similarity type have the same color.

By Theorem 5.1, every ep-similarity type in $\text{Sim}^{ep}(G)$ persists in the coding tree by any isomorphic subgraph of $\mathcal{H}_3$. Therefore, $\{\text{Sim}^{ep}(A) : A \in \text{Sim}^{ep}(G)\}$ is a canonical partition. Hence the big Ramsey degree $T(G, \mathcal{H}_3)$ is exactly the cardinality of the set $\text{Sim}^{ep}(G)$. \hfill $\square$

**Corollary 5.13.** Given a finite triangle-free graph $G$, the big Ramsey degree of $G$ in the triangle-free graph is exactly the number of essential pair similarity types of strongly skew antichains coding $G$:

$$T(G, \mathcal{H}_3) = |\text{Sim}^{ep}(G)|.$$ 

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