A SPECTRAL SZEGŐ THEOREM ON THE REAL LINE

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Abstract. We characterize even measures $\mu = w \, dx + \mu_s$ on the real line $\mathbb{R}$ with finite entropy integral $\int_{\mathbb{R}} \frac{\log w(t)}{1+t^2} \, dt > -\infty$ in terms of $2 \times 2$ Hamiltonians generated by $\mu$ in the sense of the inverse spectral theory. As a corollary, we obtain criterion for spectral measure of Krein string to have converging logarithmic integral.

1. Introduction

Each probability measure $\mu$ supported on an infinite subset of the unit circle $T = \{ z : |z| = 1 \}$ of the complex plane, $\mathbb{C}$, gives rise to the infinite family $\{ \Phi_n \}_{n \geq 0}$ of monic polynomials orthogonal with respect to $\mu$. For integer $n \geq 0$, the polynomial $\Phi_n$ has degree $n$, unit coefficient in front of $z^n$, and $(\Phi_n, \Phi_k)_{L^2(\mu)} = 0$ for all $k \neq n$. The polynomials $\{ \Phi_n \}_{n \geq 0}$ satisfy the recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n \Phi_{n}(\bar{z}),$$

$$\Phi_0 = 1, \quad (1.1)$$

where $\{ \Phi_n^* \}$ are the “reversed” polynomials defined by $\Phi_n^*(z) = z^n \Phi_n(1/z)$. Recurrence coefficients $\{\alpha_n\}$ are completely determined by $\mu$ and we have $|\alpha_n| < 1$ for every $n \geq 0$. Given any sequence of complex numbers $\{\alpha_n\}$ with $|\alpha_n| < 1$, one can find the unique probability measure $\mu$ on $T$ such that $\{\alpha_n\}$ is the sequence of the recurrence coefficients of $\mu$, see [22], [24].

Szegő Theorem. Let $\mu = w \, dm + \mu_s$ be a probability measure on $T$ with density $w$ and a singular part $\mu_s$ with respect to the Lebesgue measure $m$ on $T$. The following assertions are equivalent:

(a) the set $\text{span}\{z^n, \ n \geq 0\}$ of analytic polynomials is not dense in $L^2(\mu)$;
(b) the entropy of $\mu$ is finite: $\int_T \log w \, dm > -\infty$;
(c) the recurrence coefficients $\{\alpha_n\}$ of $\mu$ satisfy $\sum_{n \geq 0} |\alpha_n|^2 < \infty$.

We refer the reader to [22], [23] for the historical account and an extended version of this result. Independent contributions to different aspects of its proof were done by Szegő, Verblunsky, and Kolmogorov. A partial counterpart of Szegő theorem for measures supported on the real line, $\mathbb{R}$, is due to Krein [17] and Wiener [24] (see also Section 4.2 in [3] or Theorem A.6 in [6] for modern expositions). Denote by $\Pi(\mathbb{R})$ the class of all Radon measures on $\mathbb{R}$ such that $\int_{\mathbb{R}} \frac{dm(t)}{1+t^2} < \infty$.

Krein–Wiener Theorem. Let $\mu = w \, dx + \mu_s$ be a measure in $\Pi(\mathbb{R})$ where $w$ is the density with respect to the Lebesgue measure $dx$ on $\mathbb{R}$ and $\mu_s$ is the singular part. The following assertions are equivalent:

(a) the set of functions whose Fourier transform is smooth and compactly supported on $[0, +\infty)$ is not dense in $L^2(\mu)$;
(b) the entropy of $\mu$ is finite: $\int_T \frac{\log w(t)}{1+t^2} \, dt > -\infty$.

Szegő and Krein-Wiener theorems have probabilistic interpretation. Roughly, it says that a stationary Gaussian sequence/process with the spectral measure $\mu$ is non-deterministic if and only if the entropy of $\mu$ is finite, see, e.g., Section II.2 in [12] or survey [2] for more details.

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The aim of this paper is to complement assertions (a), (b) in Krein–Wiener theorem with a necessary and sufficient condition similar to condition (c) in Szegő theorem. Instead of recurrence relation \( \Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n\Phi_n^*(z) \), we will consider canonical Hamiltonian system \( J\Lambda' = z\mathcal{H}M \) which naturally appears from \( \mu \) via Krein–de Branges spectral theory.

Consider the Cauchy problem for a canonical Hamiltonian system on the half-axis \( \mathbb{R}_+ = [0, +\infty) \),

\[
J\Lambda'(t, z) = z\mathcal{H}(t)M(t, z), \quad M(0, z) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad t \geq 0, \quad z \in \mathbb{C}. \tag{1.2}
\]

Here \( J = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \), the derivative of \( M \) with respect to \( t \), the Hamiltonian \( \mathcal{H} \) is the mapping taking numbers \( t \in \mathbb{R}_+ \) into positive semi-definite matrices, the entries of \( \mathcal{H} \) are real measurable functions on \( \mathbb{R}_+ \) absolutely integrable on compact subsets of \( \mathbb{R}_+ \). In addition, we assume that the trace of \( \mathcal{H} \) is not negative on any set of positive Lebesgue measure. The Hamiltonian \( \mathcal{H} \) is called singular if \( \int_0^\infty \text{trace} \mathcal{H}(t) \, dt = +\infty \). We say that \( \mathcal{H} \) is nontrivial if there is no subset \( E \subset \mathbb{R}_+ \) of full Lebesgue measure such that \( \mathcal{H} = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \) on \( E \) or \( \mathcal{H} = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \) on \( E \).

Let \( \mathcal{H} \) be a singular nontrivial Hamiltonian on \( \mathbb{R}_+ \), and let \( M \) be the solution of (1.2). Fix a parameter \( \omega \in \mathbb{R} \cup \{ \infty \} \) and define the Weyl-Titchmarsh function \( m \) of (1.2) on \( \mathbb{C} \setminus \mathbb{R} \) by

\[
m(z) = \lim_{t \to +\infty} \frac{\omega \Phi^+(t, z) + \Phi^-(t, z)}{\omega \Theta^+(t, z) + \Theta^-(t, z)}, \quad M(t, z) = \left( \begin{array}{cc} \Theta^+(t, z) & \Phi^+(t, z) \\ \Phi^-(t, z) & -\Theta^-(t, z) \end{array} \right). \tag{1.3}
\]

The fraction \( \frac{\omega c_1 + \omega c_2}{\omega c_3 + \omega c_4} \) for non-zero numbers \( c_1, c_3 \) is interpreted as \( \frac{\omega}{\omega} \). For the Weyl-Titchmarsh theory of canonical Hamiltonian systems see [11] or Section 8 in [21]. Theorem 2.1 in [11] implies that the denominator of the fraction in (1.3) is nonzero for large \( t \geq 0 \), the function \( m \) does not depend on the choice of the parameter \( \omega \), and Im \( m(z) > 0 \) for \( z \in \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \). Hence, there exists a measure \( \mu \in \Pi(\mathbb{R}) \), and numbers \( a \in \mathbb{R}, b \geq 0 \), such that

\[
m(z) = \frac{1}{\pi} \int_\mathbb{R} \left( \frac{1}{x - z} - \frac{x}{1 + x^2} \right) \, d\mu(x) + bz + a, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.4}
\]

The measure \( \mu \) in (1.4) is called the spectral measure of system (1.2). Two singular nontrivial Hamiltonians \( \mathcal{H}_1, \mathcal{H}_2 \) on \( \mathbb{R}_+ \) are called equivalent if there exists an increasing absolutely continuous function \( \eta \) on \( \mathbb{R}_+, \eta(0) = 0, \lim_{t \to \infty} \eta(t) = \infty \), such that \( \mathcal{H}_2(t) = \eta'(t)\mathcal{H}_1(\eta(t)) \) for Lebesgue almost all \( t \in \mathbb{R}_+ \). It is easy to check that equivalent Hamiltonians have equal Weyl-Titchmarsh functions, see [28]. The following theorem is central to Krein–de Branges inverse spectral theory [13, 4].

**De Branges Theorem.** For every analytic function \( m \) in \( \mathbb{C}^+ \) with positive imaginary part, there exists a singular nontrivial Hamiltonian \( \mathcal{H} \) on \( \mathbb{R}_+ \) such that \( m \) is the Weyl-Titchmarsh function (1.3) for \( \mathcal{H} \). Moreover, any two singular nontrivial Hamiltonians \( \mathcal{H}_1, \mathcal{H}_2 \) on \( \mathbb{R}_+ \) generated by \( m \) are equivalent.

See [21, 27] for proofs to this theorem. A measure \( \mu \) on \( \mathbb{R} \) is called even if \( \mu(I) = \mu(-I) \) for every interval \( I \subset \mathbb{R}_+ \). It is well-known that a Hamiltonian \( \mathcal{H} \) has the diagonal form \( \mathcal{H} = \text{diag}(h_1, h_2) \) almost everywhere on \( \mathbb{R}_+ \) if and only if its spectral measure \( \mu \) is even and \( a = 0 \) in (1.4), see Lemma 2.2 below. Here \( \text{diag}(c_1, c_2) = \left( \begin{array}{c} c_1 & 0 \\ 0 & c_2 \end{array} \right) \) for \( c_1, c_2 \in \mathbb{R}_+ \).

Szegő class \( \text{Sz}(\mathbb{R}) \) on the real line \( \mathbb{R} \) consists of measures \( \mu \in \Pi(\mathbb{R}) \) that satisfy equivalent assertions (a), (b) in Krein–Wiener theorem. Given a measure \( \mu = w \, dx + \mu_s \) in \( \text{Sz}(\mathbb{R}) \), define its normalized entropy by

\[
\mathcal{K}(\mu) = \log \frac{1}{\pi} \int_\mathbb{R} \frac{d\mu(x)}{1 + x^2} - \frac{1}{\pi} \int_\mathbb{R} \frac{\log w(x)}{1 + x^2} \, dx.
\]

By Jensen inequality, we have \( \mathcal{K}(\mu) \geq 0 \), and, moreover, \( \mathcal{K}(\mu) = 0 \) if and only if \( \mu \) is a non-zero scalar multiple of the Lebesgue measure on \( \mathbb{R} \).
We say that a measure $\mu \in \Pi(\mathbb{R})$ generates a Hamiltonian $\mathcal{H}$ if the Weyl-Titchmarsh function \(1.3\) of $\mathcal{H}$ has the form $m : z \mapsto \frac{1}{\pi} \int_\mathbb{R} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x)$. To every $\mathcal{H}$ with $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$ we associate the sequence of points $\{\eta_n\}$ by

$$
\eta_n = \min \left\{ t \geq 0 : \int_0^t \sqrt{\det \mathcal{H}(s)} \, ds = n \right\}, \quad n \geq 0. \quad (1.5)
$$

Our main result is the following theorem.

**Theorem 1.** An even measure $\mu \in \Pi(\mathbb{R})$ belongs to the Szegő class $\Sigma(\mathbb{R})$ if and only if some (and then every) Hamiltonian $\mathcal{H} = \text{diag}(h_1, h_2)$ generated by $\mu$ is such that $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$ and

$$
\tilde{\mathcal{K}}(\mathcal{H}) = \sum_{n=0}^{+\infty} \left( \int_{\eta_n}^{\eta_{n+2}} h_1(s) \, ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s) \, ds - 4 \right) < \infty, \quad (1.6)
$$

where $\{\eta_n\}$ are given by \(1.5\). Moreover, we have $\tilde{\mathcal{K}}(\mathcal{H}) \leq c \mathcal{K}(\mu)e^{c\mathcal{K}(\mu)}$ and $\mathcal{K}(\mu) \leq c \tilde{\mathcal{K}}(\mathcal{H})e^{c\tilde{\mathcal{K}}(\mathcal{H})}$ for an absolute constant $c$.

By definition, the terms in \(1.6\) are nonnegative:

$$
\int_{\eta_n}^{\eta_{n+2}} h_1(s) \, ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s) \, ds - 4 \geq \left( \int_{\eta_n}^{\eta_{n+2}} \sqrt{\det \mathcal{H}(s)} \, ds \right)^2 - 4 = 0,
$$

and the sum in \(1.6\) equals zero if and only if $\mathcal{H}$ is a constant Hamiltonian. Note that the spectral measure $\mu$ of a constant diagonal Hamiltonian $\mathcal{H}$ with $\det \mathcal{H} \neq 0$ is a scalar multiple of the Lebesgue measure on $\mathbb{R}$, in particular, we have $\mathcal{K}(\mu) = 0$ in this case.

Diagonal canonical Hamiltonian systems are closely related to the differential equation of a vibrating string:

$$
- \frac{d}{dM(t)} \frac{d}{dt} \left( y(t, z) \right) = z y(t, z), \quad t \in [0, L), \quad z \in \mathbb{C}. \quad (1.7)
$$

Here $0 < L \leq +\infty$ is the length of the string, $M : (-\infty, L) \rightarrow \mathbb{R}_+$ is an arbitrary non-decreasing and right-continuous function (mass distribution) that satisfies $M(t) = 0$ for $t < 0$. If $M$ is smooth and strictly increasing on $\mathbb{R}_+$, then equation \(1.7\) takes the form $-y'' = zM'y$.

In this paper, we consider $L$ and $M$ that satisfy the following conditions:

$$
L + \lim_{t \rightarrow L} M(t) = \infty \quad \text{and} \quad \lim_{t \rightarrow L} M(t) > 0, \quad (1.8)
$$

where the last bound means that $M$ is not identically equal to zero. If \(1.8\) holds, we say that $M$ and $L$ form $[M, L]$ pair. To every $[M, L]$ pair one can associate a string and Weyl-Titchmarsh function $q$ with spectral measure $\sigma$ supported on the positive half-axis $\mathbb{R}_+$. We discuss these objects in more detail in Section 6. Theorem 1 can be reformulated for Krein strings as follows.

**Theorem 2.** Let $[M, L]$ satisfy \(1.8\) and $\sigma = v \, dx + \sigma_s$ be the spectral measure of the corresponding string. Then, we have $\int_0^{+\infty} \frac{\log v(x)}{(1+x)^2} \, dx > -\infty$ if and only if $\sqrt{M'} \notin L^1(\mathbb{R}_+)$ and

$$
\tilde{\mathcal{K}}[M, L] = \sum_{n=0}^{+\infty} \left( \left( t_{n+2} - t_n \right) M(t_{n+2}) - M(t_n) - 4 \right) < \infty, \quad (1.9)
$$

where $t_n = \min \{ t \geq 0 : n = \int_0^t \sqrt{M'(s)} \, ds \}$.

Condition \(1.8\) guarantees that the string $[M, L]$ has the unique spectral measure. It does not restrict the generality of Theorem 2 if \(1.8\) is violated, then either $M = 0$ and $\int_0^{+\infty} \frac{\log v(x)}{(1+x)^2} \, dx = -\infty$ because $v = 0$, or $L + \lim_{t \rightarrow L} M(t) < \infty$ in which case the Weyl-Titchmarsh function is meromorphic.
and real-valued on $\mathbb{R}$, so $v(x) = 0$ again and the logarithmic integral diverges. More details on Theorem 2 can be found in Section 6.

**Historical remarks.** Except for Krein–Wiener theorem, all previously known results on Szegő theorem in the continuous setting were proved for the so-called Krein systems, i.e., differential systems that appear as a result of “orthogonalization process with continuous parameter” invented by Krein in [19]. Krein systems with locally summable coefficients can be reduced to the canonical Hamiltonian systems with absolutely continuous Hamiltonians $\mathcal{H}$ (see, e.g., [1] for this reduction in the diagonal case). The class of Hamiltonians considered in Theorem 1 is considerably wider. Krein himself formulated a restricted version of Szegő theorem for Krein systems in [19]. In [5], the second author of this paper characterized Krein systems with coefficients from a Stummel class whose spectral measures belong to $\text{Sz}(\mathbb{R})$. In [20], Nazarov, Peherstorfer, Volberg, and Yuditskii for a closely related subject of sum rules for Jacobi matrices. See also the work [20] by Nazarov, Peherstorfer, Volberg, and Yuditskii for a closely related subject of sum rules for Jacobi matrices.

**The structure of the paper.** We start with studying the basic properties of entropy function for diagonal canonical systems in Section 2. Section 3 contains the proof of upper and lower bounds for the entropy. Theorem 1 is proved in the fourth section. The new functional class which appears in the proof of Theorem 1 is studied in Section 5. We consider Krein strings and prove Theorem 2 in Section 6. The paper end with appendix which contains some auxiliary results.

**Notation.** In the text, we use the following standard notation. Given set $E \subset \mathbb{R}$ with positive Lebesgue measure $|E| > 0$ and nonnegative $f \in L^1(E)$, we denote $(f)_E = \frac{1}{|E|} \int_E f \, dx$. Suppose $a \in \mathbb{R}$, $l > 0$, then $I_{a,l} = [a, a+l)$. The symbols $C, c$ denote absolute constants which can change the value from formula to formula. For two non-negative functions $f_1$, $f_2$, we write $f_1 \lesssim f_2$ if there is an absolute constant $C$ such that $f_1 \leq C f_2$ for all values of the arguments of $f_1$, $f_2$. We define $\gtrsim$ similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously. Given a set $E \subset \mathbb{R}$, $\chi_E$ stands for the characteristic function of $E$. The norm of the space $L^p(\mathbb{R}^+)$ is denoted by $\| \cdot \|_p$. The space $L^1_{\text{loc}}(\mathbb{R}^+)$ consists of functions that are absolutely integrable on compact subsets of $\mathbb{R}^+$. Symbol $[x]$ stands for the integer part of a real number $x$.

2. Entropy function of a canonical Hamiltonian system

In this section we introduce the entropy function of a diagonal canonical Hamiltonian system and show that it has a number of remarkable properties.

Let $\mathcal{H} = \text{diag}(h_1, h_2)$ be a singular nontrivial diagonal Hamiltonian on $\mathbb{R}^+$, and let $m$, $\mu$ be its Weyl-Titchmarsh function and the spectral measure, so that

$$\text{Im} \, m(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\text{Im} \, z}{|x-z|^2} d\mu(x) + b \, \text{Im} \, z, \quad z \in \mathbb{C}^+. \quad (2.1)$$

For every $r \geq 0$ define $\mathcal{H}_r$ to be the Hamiltonian on $\mathbb{R}^+$ taking $x$ into $\mathcal{H}(x + r)$. Let $m_r$, $\mu_r$, $b_r$ denote the Weyl-Titchmarsh function, the spectral measure, and the coefficient in (1.1) of system (1.2) for $\mathcal{H} = \mathcal{H}_r$. Each time we work with these objects later in the text we assume that $\mathcal{H}_r$ is nontrivial. Define

$$\mathcal{J}_{\mathcal{H}}(r) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\mu_r(x)}{1 + x^2} + b_r = -i m_r(i), \quad \mathcal{J}_{\mathcal{H}}(r) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log w_r(x)}{1 + x^2} \, dx, \quad (2.2)$$

4
where $w_r$ is the density of the absolutely continuous part of $\mu_r = w_r \, dx + \mu_{r,s}$. The second identity above follows from the fact that $\mu$ is even, hence $m$ takes imaginary values on imaginary axis. If $\mu_r \notin \mathcal{Sz}(\mathbb{R})$, we put $J_3(r) = -\infty$. Define the entropy function of $H$ by

$$K_3(r) = \log J_3(r) - J_3(r), \quad r \geq 0.$$  

Note again that Jensen inequality and an estimate $b_r \geq 0$ give

$$K_3(r) \geq 0. \quad (2.3)$$

For the “dual” Hamiltonian $H^d = J^* H = \text{diag}(H_2, H_1)$ we denote the corresponding objects by $J_3^d$, $m_r^d$, $\mu_r^d$, $b_r^d$, $w_r^d$, $J_3^d$, $J_3^d$ and $K_3^d$. Note that a Hamiltonian $H$ is singular and nontrivial if and only if $H^d$ is singular and nontrivial. We also will need the Hamiltonian

$$\hat{H}_r(t) = \begin{cases} H(t), & t \in [0, r), \\ \text{diag}(J_3^{-1}(r), J_3(r)), & t \in [r, +\infty), \end{cases} \quad (2.4)$$

which plays the role of “Bernstein-Szegő approximation” to $H$. From formula (2.2) we see that the Hamiltonian $\hat{H}_r$ is correctly defined and nontrivial if and only if $m_r(i) \neq 0$, that is, $H_r$ is nontrivial. Later we will use notation $\hat{\mu}_r$ for the spectral measure generated by $H_r$.

An analytic function $f$ in the upper half-plane $C^+ = \{z \in C : \text{Im} \, z > 0\}$ is said to have bounded type if $f = \frac{f_1}{f_2}$ for some bounded analytic functions $f_1, f_2$ in $C^+$, where $f_2$ is not identically zero. Denote by $N(C^+)$ the class of all functions of bounded type in $C^+$. For every function $f \in N(C^+)$ we have

$$\int_{\mathbb{R}} \frac{\log |f(x)|}{1 + x^2} \, dx < \infty, \quad (2.5)$$

see, e.g., Theorem 9 in [4]. The mean type of a function $f \in N(C^+)$ is defined by

$$\text{type}^+(f) = \limsup_{y \to +\infty} \frac{\log |f(iy)|}{y}.$$  

The upper limit above is finite for every nonzero function $f \in N(C^+)$ by Theorem 10 in [4]. A remarkable fact of the spectral theory of canonical Hamiltonian systems is that for every $t \geq 0$ the entries of solution $M(t, z)$ to Cauchy problem (1.2) are entire functions in $z$ of bounded type in $C^+$ and their mean type in $C^+$ equals

$$\xi_3(t) = \int_0^t \sqrt{\text{det} H(s)} \, ds. \quad (2.6)$$

This formula has been found by Krein [18] in the setting of the string equation and then proved in full generality by de Branges, see Theorem X in [3]. A short proof of (2.6) is in Section 6 of [21]. As a consequence, we have the following result.

**Proposition 2.1.** Let $H$ be a Hamiltonian on $\mathbb{R}_+$ and let entire function $f(z)$ be one of the entries $\{\Theta^+(t, z), \Phi^+(t, z)\}$ of the matrix $M$ in (1.3). Then, if $f$ in not equal to zero identically, we have

$$\frac{1}{\pi} \int_{\mathbb{R}} \log |f(x)| \frac{\text{Im} z}{|x - z|^2} \, dx = \log |f(z)| - \xi_3(t) \text{Im} z \quad (2.7)$$

for every $z \in C_+$.  

**Proof.** Take $f$ as one of $\{\Theta^\pm\}$ and denote by $\Theta = \begin{pmatrix} \Theta^+ \\ -\Theta^- \end{pmatrix}$ the first column of $M$ in (1.2). For every $t > 0$, we take inner product of (1.2) with $\Theta$ and integrate to get the following well-known identity:

$$\text{Im}(\Theta^+(t, z)\overline{\Theta^-(t, z)}) = \text{Im} \, z \int_0^t \langle H(s)\Theta(s, z), \Theta(s, z) \rangle_{C^2} \, ds, \quad z \in C, \quad (2.8)$$
where the inner product in $\mathbb{C}^2$ is given by $\langle (c_1, c_2), (d_1, d_2) \rangle_{\mathbb{C}^2} = c_1 \overline{d_1} + c_2 \overline{d_2}$. This identity implies that either $f$ is identically zero or $f(z) \neq 0$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Function $f \in \mathcal{N}(\mathbb{C}^+)$, it is smooth on $\mathbb{R}$, and has no zeros in $\mathbb{C}^+$. So, there exists an outer function $F$ on $\mathbb{C}^+$ such that $f(z) = e^{-i\xi \gamma(r)z} F(z)$, $z \in \mathbb{C}^+$, see Theorem 9 in [1]. Now (2.7) follows from the mean value theorem for the harmonic function $\log|F|$. The proof for $\Phi^\pm$ is similar. □

**Proposition 2.2.** Let $f$ be an analytic function in $\mathbb{C}^+$ such that $\Im f(z) > 0$ for all $z \in \mathbb{C}^+$. Then for almost all $x \in \mathbb{R}$ there exists finite non-tangential limit $f(x) = \lim_{|z-x| \to 0} f(z)$ and

$$\frac{1}{\pi} \int_{\mathbb{R}} \log|f(x)| \frac{\Im z}{|x-z|^2} \, dx = \log|f(z)|$$

for every $z \in \mathbb{C}^+$, where integral in the left hand side converges absolutely.

**Proof.** Combine Corollary 4.8 in Section 4 with Exercise 13 in Section 7 of Chapter II in [10]. □

For every $\varphi \in [0, \pi)$, set $e_\varphi = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$. An open interval $I \subset \mathbb{R}_+$ is called indivisible for $\mathcal{H}$ of type $\varphi$ if there is a function $h$ on $I$ such that $\mathcal{H}(x) = h(x) e_{\varphi} e_\varphi^\top$ for almost all $x \in I$, and $I$ is the maximal open interval having this property. Note that a Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$ is nontrivial if $(0, +\infty)$ is not an indivisible interval of type $\varphi = 0$ or $\varphi = \pi/2$ for $\mathcal{H}$.

The following four lemmas are known. We give their proofs in Appendix for the reader’s convenience.

**Lemma 2.1.** Let $\mathcal{H}$ be a Hamiltonian on $\mathbb{R}_+$ such that $(0, \ell)$ is indivisible interval of type $\varphi \in [0, \pi)$ for $\mathcal{H}$. Then the solution $M$ of (1.2) has the form $M(t, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z J \int_0^t \mathcal{H}(\tau) d\tau$ for every $t \in [0, \ell]$. In particular, for $\mathcal{H} = \text{diag}(h_1, h_2)$ and $t \in [0, \ell]$ we have

$$M(t, z) = \begin{cases} \begin{pmatrix} 1/h_1(s) ds & 1 \\ 0 & 1 \end{pmatrix} & \text{if } \varphi = 0, \\ \begin{pmatrix} -z J h_2(s) ds & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \varphi = \pi/2 \end{cases}.$$  

**Lemma 2.2.** Let $\mathcal{H}$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$, and let $m$ be its Weyl-Titchmarsh function (1.3). Then, $\mathcal{H}$ is diagonal if and only if the measure $\mu$ is even and $a = 0$ in the Herglotz representation (1.4) of $m$.  

**Lemma 2.3.** Let $\mathcal{H}$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$ and let $m$ be its Weyl-Titchmarsh function. Then, we have $b > 0$ in the Herglotz representation (1.4) of $m$ if and only if $(0, \varepsilon)$ is indivisible interval for $\mathcal{H}$ of type $\pi/2$ for some $\varepsilon > 0$. Moreover, we have $b = \int_0^\varepsilon \delta_\mathcal{H}(t) (\mu_\varepsilon, 0) dt$ in the latter case.

**Lemma 2.4.** Let $\mathcal{H} = \text{diag}(a_1, a_2)$ be the constant Hamiltonian on $\mathbb{R}_+$ generated by positive numbers $a_1, a_2$. Then for all $r > 0$ we have $w_r = \sqrt{a_2/a_1}$ on $\mathbb{R}$ and

$$\log J_{3\mathcal{H}}(r) = J_{3\mathcal{H}}(r) = \log \sqrt{a_2/a_1}.$$  

The following lemma is crucial for our paper.

**Lemma 2.5.** Let $\mathcal{H} = \text{diag}(h_1, h_2)$ be a singular nontrivial Hamiltonian on $\mathbb{R}_+$ and let $\mu$ be the spectral measure of system (1.2) generated by $\mathcal{H}$. Assume that $\mu \in \text{Sz}(\mathbb{R})$. Then for every $r > 0$ we have

(a) $\mu_r \in \text{Sz}(\mathbb{R})$ and $\mu_r^d \in \text{Sz}(\mathbb{R})$,
(b) $J_{3\mathcal{H}}(r) = J_{3\mathcal{H}}(0) - 2 \xi_{3\mathcal{H}}(r) + 2 \log |\Theta^+(r, i) + i J_{3\mathcal{H}}(r) \Theta^-(r, i)|$,
(c) $J_{3\mathcal{H}}(r) = 1/J_{3\mathcal{H}d}(r)$,
Proof. Take $r \geq 0$ and consider solutions

$$M(t, z) = \begin{pmatrix} \Theta^+(t, z) & \Phi^+(t, z) \\ \Theta^-(t, z) & \Phi^-(t, z) \end{pmatrix}, \quad M_r(t, z) = \begin{pmatrix} \Theta^+_r(t, z) & \Phi^+_r(t, z) \\ \Theta^-_r(t, z) & \Phi^-_r(t, z) \end{pmatrix}, \tag{2.10}$$

of Cauchy problem (1.2) for the Hamiltonians $\mathcal{H}$ and $\mathcal{H}_r : x \mapsto \mathcal{H}(r + x)$, respectively. We have

$$M_0(t, z) = M_r(t - r, z)M_0(r, z), \quad t \geq r, \quad z \in \mathbb{C}. \tag{2.11}$$

Indeed, the right hand side of the above equality satisfies equation $JM' = z\mathcal{H}M$ on $[r, \infty)$ and coincides with $M_0(t, z)$ at $t = r$. Multiplying matrices in (2.11) and using (1.3) with $\omega = 0$, we obtain

$$m_0(z) = \lim_{t \to +\infty} \frac{\Theta^-_r(t - r, z)\Phi^+(r, z) + \Phi^-_r(t - r, z)\Phi^-(r, z)}{\Theta^+_r(t - r, z)\Theta^-(r, z) + \Phi^+_r(t - r, z)\Phi^-(r, z)}.$$ \tag{2.12}

Suppose there is $c > 0$ such that $(c, +\infty)$ is the indivisible interval of type $\pi/2$ for $\mathcal{H}$. Then from Lemma 2.1 and formula (2.12) we see that $m_0(z) = \frac{\Phi^-(c, z)}{\Theta^-_c(c, z)}$ for all $z \in \mathbb{C}^+$. Since functions $\Phi^-, \Theta^-$ are real on the real axis, this implies that $\mu$ is a discrete measure concentrated at zeros of entire function $z \mapsto \Theta^-(c, z)$. In particular, we cannot have $\mu \in S_z(\mathbb{R})$. A similar argument applies in the case where $(c, +\infty)$ is the indivisible interval of type 0 for some $c > 0$. It follows that the Hamiltonian $\mathcal{H}_r$ is nontrivial for every $r \geq 0$, in particular, its Weyl-Titchmarsh function $m_r$ is correctly defined and nonzero. Using (2.12) and (1.3) with $\omega = 0$ for $m_r$, we get the relation

$$m_0(z) = \frac{\Phi^+(r, z) + m_r(z)\Phi^-(r, z)}{\Theta^+(r, z) + m_r(z)\Theta^-(r, z)}, \quad z \in \mathbb{C}^+, \quad r \geq 0. \tag{2.13}$$

Hence,

$$\begin{align*}
\text{Im } m_0(z) &= \text{Im}(\Phi^+(r, z)\Theta^+(r, z) + m_r(z)^2\Phi^-(r, z)\Theta^-(r, z)) \\
&\quad \cdot |\Theta^+(r, z) + m_r(z)\Theta^-(r, z)|^2 \\
&\quad + \text{Im}(m_r(z)(\Theta^+(r, z)\Phi^-(r, z) - \Theta^-(r, z)\Phi^+(r, z))) \\
&\quad \cdot |\Theta^+(r, z) + m_r(z)\Theta^-(r, z)|^2.
\end{align*}$$

Since the analytic function $m_r$ has positive imaginary part in $\mathbb{C}^+$ for every $r \geq 0$, we can take non-tangential limit as $z \to x$ in this formula for almost all $x \in \mathbb{R}$, see Proposition 2.2. The real analytic functions $\Theta^\pm, \Phi^\pm$ satisfy

$$\Theta^+(r, z)\Phi^-(r, z) - \Theta^-(r, z)\Phi^+(r, z) = \det M_0(r, z) = 1$$

for all $r \geq 0, z \in \mathbb{C}$, hence we obtain

$$w_0(x) = \text{Im } m_0(x) = \frac{\text{Im } m_r(x)}{|F_r(x)|^2} = \frac{w_r(x)}{|F_r(x)|^2}, \tag{2.14}$$

for almost all $x \in \mathbb{R}$, where $F_r : z \mapsto \Theta^+(r, z) + m_r(z)\Theta^-(r, z)$ is the analytic function in $\mathbb{C}^+$ and $F_r(x), x \in \mathbb{R}$, are the non-tangential boundary values of $F_r$. Denote the first column of the matrix-function $M$ in (2.10) by $\Theta = \begin{pmatrix} \Theta^+ \\ \Theta^- \end{pmatrix}$. Assume for a moment that $(0, r)$ is not an indivisible interval of type $\pi/2$ for $\mathcal{H}$. Then formula (2.8) implies that $\Theta^-(r, z) \neq 0$ for every $z \notin \mathbb{R}$, and, moreover, $\text{Im } \frac{\Theta^+(r, z)}{\Theta^-(r, z)} > 0$ for $z \in \mathbb{C}^+$. Thus, the function $\log |F_r|$ can be represented in the form

$$\log |F_r(z)| = \log |\Theta^-(r, z)| + \log \left| m_r(z) + \frac{\Theta^+(r, z)}{\Theta^-(r, z)} \right|, \quad z \in \mathbb{C}^+. $$
Since the functions $m_r, \frac{\Theta^+(r, z)}{\Theta^-(r, z)}$ have positive imaginary parts in $\mathbb{C}^+$ and $\Theta^-(\mathbb{C}^+)\in N(\mathbb{C}^+)$, we have $|\log |F_r(x)||dx \in \Pi(\mathbb{R})$, and, moreover,

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{\log |F_r(x)|}{1 + x^2} dx = \log |F_r(i)| - \xi \mathcal{H}(r),$$

by Proposition 2.1 and Proposition 2.2. In particular, the measure $\mu_r$ belongs to the Szegő class $Sz(\mathbb{R})$. Taking logarithms in (2.14) and integrating with $\frac{1}{1 + x^2}$, we obtain assertion (b):

$$j_{\mathcal{H}}(r) = j_{\mathcal{H}}(0) - 2 \xi \mathcal{H}(r) + 2 \log |F_r(i)|.$$  \hspace{1cm} (2.15)

Let us now prove (b) in the case where $\mathcal{H}$ has an indivisible interval $(0, \varepsilon)$ of type $\pi/2$ for some $\varepsilon > 0$ and $r \leq \varepsilon$. In that situation, we can use Lemma 2.1 to show that $F_r(z) = 1$ for all $z$, hence $w_0 = w_r$ on $\mathbb{R}$ by (2.14), yielding $j_{\mathcal{H}}(r) = j_{\mathcal{H}}(0)$ for $r \in [0, \varepsilon]$. Since $\xi \mathcal{H} = 0$ on $[0, \varepsilon]$ by definition, this gives us relation (b) in full generality.

Next, the solution $M^d(r, z)$ of the canonical Hamiltonian system generated by the dual Hamiltonian $\mathcal{H}^d$ has the form

$$M^d(r, z) = J^* M(r, z) J = \begin{pmatrix} \Phi^- (r, z) & -\Theta^- (r, z) \\ -\Phi^+ (r, z) & \Theta^+ (r, z) \end{pmatrix}.$$  \hspace{1cm} (2.16)

Note that $\mathcal{H}^d, \mathcal{H}^d_r$ are singular nontrivial Hamiltonians because $\mathcal{H}, \mathcal{H}_r$ are singular and nontrivial. Using formula (2.13) with $\omega = \infty$, we see that $m_r^d (z) = -\lim_{t \to -\infty} \frac{\Theta^+(r, z)}{\Theta^-(r, z)} = -\frac{m_r(z)}{m_r(\infty)}$ for all $r \geq 0$ and all $z \in \mathbb{C}^+$. Taking the non-tangential values of imaginary parts gives $w_r^d (x) = \frac{\text{Im} m_r (x)}{|m_r (x)|^2} = \frac{w_r(x)}{|m_r(x)|^2}$. This formula and Proposition 2.2 imply $\mu_r^d \in Sz(\mathbb{R})$ thus completing the proof of (a). Since the measures $\mu_r, \mu_r^d$ are even, we have

$$j_{\mathcal{H}^d}(r) = \text{Im} m_r^d (i) = \frac{1}{\text{Im} m_r (i)} = \frac{1}{j_{\mathcal{H}}(r)},$$

as claimed in (c). Next, using the formula $w_r^d (x) = \frac{w_r(x)}{|m_r(x)|^2}$, $x \in \mathbb{R}$, the mean value formula in Proposition 2.2, formula (2.17), and identity $m_r(i) = \text{Im} m_r (i)$, we obtain assertion (d):

$$\mathcal{K}_{\mathcal{H}^d}(r) = \log j_{\mathcal{H}^d}(r) - j_{\mathcal{H}}(r) + \log |m_r (i)|^2 = -\log |F_r (i)| - 2 \log |F_r (i)| + 2 \log j_{\mathcal{H}}(r) = K_{\mathcal{H}}(r).$$

Finally, consider the Hamiltonian $\hat{\mathcal{H}}_r$ introduced in (2.21). Since $\mathcal{H}_r$ is nontrivial, we have $j_{\mathcal{H}}(r) \neq 0$ and hence $\hat{\mathcal{H}}_r$ is defined correctly. By definition and Lemma 2.4 we have $j_{\hat{\mathcal{H}}_r}(r) = j_{\mathcal{H}}(r), \ j_{\hat{\mathcal{H}}_r}(r) = \log j_{\mathcal{H}}(r),$ and $F_r(i) = F_r(i)$ for the corresponding function $\hat{F}_r$. The proof of Lemma 2.4 shows that $\hat{m}_t$ is a constant function for each $t \geq r$. Using this and the fact that $\Phi^\pm, \Theta^\pm \in N(\mathbb{C}^+)$, from (2.13) we obtain $\hat{\mu}_r \in Sz(\mathbb{R})$. Comparing the right hand sides of formula (2.13) for $m_0$ and $\hat{m}_0$ at $z = i$, we get $j_{\hat{\mathcal{H}}_r}(0) = j_{\mathcal{H}}(0)$. Hence, relation (2.15) for $\hat{H}_r$ can be written in the form

$$j_{\hat{\mathcal{H}}_r}(r) = j_{\hat{\mathcal{H}}_r}(0) - 2 \xi \mathcal{H}(r) + 2 \log |F_r (i)| = j_{\hat{\mathcal{H}}_r}(0) - j_{\mathcal{H}}(0) + j_{\mathcal{H}}(r).$$

On the other hand, we have $\log j_{\mathcal{H}}(r) = j_{\hat{\mathcal{H}}_r}(r)$ and $j_{\hat{\mathcal{H}}_r}(0) = j_{\mathcal{H}}(0).$ This yields assertion (e):

$$\mathcal{K}_{\mathcal{H}}(r) = \log j_{\mathcal{H}}(r) - j_{\mathcal{H}}(r) = j_{\hat{\mathcal{H}}_r}(r) - j_{\mathcal{H}}(r) = j_{\hat{\mathcal{H}}_r}(0) - j_{\mathcal{H}}(0) = j_{\hat{\mathcal{H}}_r}(0) - \log j_{\mathcal{H}}(0) + j_{\mathcal{H}}(0) \mathcal{K}_{\mathcal{H}}(0) + 0.$$  \hspace{1cm} (2.18)

The lemma is proved. \hspace{1cm} \Box
Lemma 2.6. Let \( l > 0 \) and \( \mathcal{H} \) be a singular Hamiltonian on \( \mathbb{R}_+ \) satisfying \( \mathcal{H}(t) = \text{diag}(a_1, a_2) \) for all \( t \in [l, +\infty) \) where \( a_1, a_2 \) are positive parameters. Then its spectral measure \( \mu \) belongs to the Szegő class \( \text{Sz}(\mathbb{R}) \).

Proof. Formula (2.14) for \( r = \ell \) says that the absolutely continuous part of \( \mu \) coincides with \( \frac{|w_r(x)|^2}{F_r(x)} \). Since \( \mathcal{H}_r = \text{diag}(a_1, a_2) \) on \( \mathbb{R}_+ \), we have \( w_r(x) = \sqrt{a_2/a_1} \) for all \( x \in \mathbb{R} \) by Lemma 2.4. It remains to use Proposition 2.7 for the function \( F_r \neq 0 \) of class \( N(\mathbb{C}^+) \). \( \square \)

Lemma 2.7. Let \( \mathcal{H} = \text{diag}(h_1, h_2) \) be a singular nontrivial Hamiltonian on \( \mathbb{R}_+ \) whose spectral measure belongs to the Szegő class \( \text{Sz}(\mathbb{R}) \). Then the functions \( J_{3\ell}(r), J_{3\ell}(r) \) are absolutely continuous and

\[
J_{3\ell}'(r) = 2J_{3\ell}(r)h_1(r) - 2\xi_{3\ell}(r),
\]

\[
J_{3\ell}'(r) = -J_{3\ell}(r)h_1(r) - \frac{h_2(r)}{J_{3\ell}(r)} + 2\xi_{3\ell}(r),
\]

for almost all \( r \geq 0 \).

Proof. At first, assume additionally that \( h_1, h_2 \) belong to \( C^1(\mathbb{R}_+) \), the space of continuously differentiable functions on \((0, +\infty)\) whose derivatives have a finite limit at 0. Then the entries of the the solution \( M'(\cdot, i) \) of (1.12) at \( z = \ell \) belong to the space \( C^1(\mathbb{R}_+) \) as well. From formula (2.13) and identity \( m_r(i) = iJ_{3\ell}(r), r \geq 0 \), we also have \( J_{3\ell} \in C^1(\mathbb{R}_+) \). Assertion (b) of Lemma 2.5 says that

\[
J_{3\ell}(r) = J_{3\ell}(0) - 2\xi_{3\ell}(r) + 2 \log |\Theta^+(r, i) + iJ_{3\ell}(r)\Theta^-(r, i)|, \quad r \geq 0.
\]

Differentiating the above formula with respect to \( r \) at \( r = 0 \) and using the equation

\[
\begin{pmatrix}
\Theta^+(r, i)' & \Phi^+(r, i)'
\Theta^-(r, i)' & \Phi^-(r, i)'
\end{pmatrix}
\bigg|_{r=0} = M'(0, i) = iJ^*\mathcal{H}(0)M(0, i) = \begin{pmatrix} 0 & i\bar{h}_2(0) \\ -ih_1(0) & 0 \end{pmatrix},
\]

we obtain

\[
J_{3\ell}'(0) = -2\xi_{3\ell}'(0) + 2 \text{ Re} \left( \frac{\Theta^+(r, i)' + iJ_{3\ell}'(r)\Theta^-(r, i) + iJ_{3\ell}(r)\Theta^+(r, i)}{\Theta^+(r, i)' + iJ_{3\ell}(r)\Theta^-(r, i)} \right)
\bigg|_{r=0}
\]

\[
= -2\xi_{3\ell}'(0) + 2J_{3\ell}(0)h_1(0).
\]

For \( r > 0 \) we have

\[
J_{3\ell}'(r) = J_{3\ell}'(0) - 2\xi_{3\ell}'(0) + 2J_{3\ell}(0)h_1(r) = -2\xi_{3\ell}'(r) + 2J_{3\ell}(r)h_1(r).
\]

Thus, relation (2.15) holds in the case when \( h_1, h_2 \in C^1(\mathbb{R}_+) \). Now let \( \mathcal{H} = \text{diag}(h_1, h_2) \) be an arbitrary singular nontrivial Hamiltonian on \( \mathbb{R}_+ \) with spectral measure in \( \text{Sz}(\mathbb{R}) \). By Lemma 2.5 the functions \( J_{3\ell}(r), J_{3\ell}(r) \) are correctly defined on \( \mathbb{R}_+ \). Find a sequence of positive smooth functions \( \{h_{1,n}\}, \{h_{2,n}\} \) such that

\[
\lim_{n \to \infty} \int_0^T |h_j(s) - h_{j,n}(s)| \, ds = 0
\]

for every \( T > 0 \) and \( j = 1, 2 \). Solutions of the equations \( JM'_n = i\mathcal{H}_n M_n, M_n(0, i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), generated by the Hamiltonians \( \mathcal{H}_n = \text{diag}(h_{1,n}, h_{2,n}) \) will then converge uniformly on compact subsets of \( \mathbb{R}_+ \) to the solution \( M(\cdot, i) \) of the equation \( JM' = i\mathcal{H} M, M(0, i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). From formulas
Lemma 2.8. Let \( \mathcal{H}(r) = \lim_{n \to \infty} (3 \mathcal{H}_n(r) - 3 \mathcal{H}_n(0)) \) which is immediate from Lemma 2.5. Relation (2.19) follows by adding (2.18) written for \( r \leftarrow \pi/R \), for all \( r \in (0, R) \) and (2.20) we see that continuous functions \( I \) written for \( \mathcal{H} \) and \( \mathcal{H}_d = \text{diag}(h_2, h_1) \) and using identity
\[
\mathcal{K}_\mathcal{H} = -(3 \mathcal{H} + 3 \mathcal{H}_d)/2
\]
which is immediate from Lemma 2.5 (c), (d).

\[\square\]

**Lemma 2.8.** Let \( l > 0 \) and \( \mathcal{H} \) be a singular Hamiltonian on \( \mathbb{R}_+ \) satisfying \( \mathcal{H}(t) = \text{diag}(a_1, a_2) \) for all \( t \in [l, +\infty) \) where \( a_1, a_2 \) are positive parameters. Then, for every \( r \geq 0 \) we have
\[
e^{-\frac{1}{2} \mathcal{H}(r)-\mathcal{K}(r)} - \mathcal{K}_d(r) = \int_{a_1}^{\infty} e^{-\frac{1}{2} \mathcal{H}(s)-\mathcal{K}(s)} ds,
\]
(2.22)
\[
e^{-\frac{1}{2} \mathcal{H}_d(r)-\mathcal{K}(r)} = \int_{a_2}^{\infty} e^{-\frac{1}{2} \mathcal{H}(s)-\mathcal{K}(s)} ds.
\]
(2.23)

**Proof.** The right hand side of (2.22) at \( r_0 \geq l \) is equal to
\[
a_1 e^{-\mathcal{K}(r_0)} \left( a_1 \int_{a_1}^{\infty} e^{-\frac{1}{2} \mathcal{H}_d(r)-\mathcal{K}(r)} ds = \sqrt{\frac{a_1}{a_2}} \cdot e^{-\mathcal{K}(r_0)} \cdot \frac{1}{2} \mathcal{H}_d(r_0).
\]
Substituting \( \mathcal{H}_d(r_0) = \log \sqrt{\frac{a_2}{a_1}} \), \( \mathcal{H}_d(r_0) = \log \sqrt{\frac{a_1}{a_2}} \) into the formula above, we see that (2.22) holds for all \( r \geq l \). Next, differentiating the left hand side of (2.22) and using Lemma 2.5 and Lemma 2.7, we obtain
\[
- \left( \frac{3 \mathcal{H}(r)}{2} + \mathcal{K}(r) \right) e^{-\frac{1}{2} \mathcal{H}(r)-\mathcal{K}(r)} = -h_1(r) \mathcal{H}(r) e^{-\frac{1}{2} \mathcal{H}(r)-\mathcal{K}(r)}
\]
\[
= -h_1(r) e^{\frac{1}{2} \log \mathcal{H}_d(r) + \frac{1}{2} \mathcal{K}(r)-\mathcal{K}(r)}
\]
\[
= -h_1(r) e^{\frac{1}{2} \log \mathcal{H}_d(r) + \frac{1}{2} (\log \mathcal{H}_d(r)+h_2(0)-h_2(r))-\mathcal{K}(r)}
\]
\[
= -h_1(r) e^{-\frac{1}{2} \mathcal{H}_d(r)-\mathcal{K}(r)}.
\]
This agrees with the derivative of the right hand side of (2.22) for almost all \( r \geq 0 \). It follows that (2.22) holds for all \( r \geq 0 \). Formula (2.23) can be proved in a similar way. \[\square\]

3. Some estimates of the entropy function

In this section we consider Hamiltonians \( \mathcal{H} \) such that \( \det \mathcal{H} = 1 \) almost everywhere on \( \mathbb{R}_+ \). In the notations of Section 2, we have \( \mathcal{K}(\mu) = \mathcal{K}(0) \) for such Hamiltonians. Indeed, the coefficient \( b_0 \) in (2.2) is non-zero if and only if there exists \( \varepsilon > 0 \) such that \( (0, \varepsilon) \) is the indivisible interval of type \( \pi/2 \) for \( \mathcal{H}_0 = \mathcal{H} \), see Lemma 2.3. The latter never happens for Hamiltonians \( \mathcal{H} \) with \( \det \mathcal{H} = 1 \) almost everywhere on \( \mathbb{R}_+ \).
3.1. A lower bound for the entropy. We first obtain a local estimate for the entropy $\mathcal{K}(\mu) = \mathcal{K}_H(0)$ in terms of $H$ and then use assertion (e) of Lemma 2.5 to improve it.

**Lemma 3.1.** Let $h > 0$ be a function on $\mathbb{R}_+$ such that $h, 1/h \in L^1_{\text{loc}}(\mathbb{R}_+)$ and assume that $h$ equals to some positive constant on $[\ell, +\infty)$ for some $\ell > 0$. Then, for the Hamiltonian $H = \text{diag}(h, 1/h)$, we have

$$e^{\frac{1}{2}H\mathcal{K}(0)} \geq \int_0^\infty \sqrt{a(t)} \cdot te^{-t}dt,$$

where $a(t) = \frac{1}{\tau} \int_0^\tau h(s) \, ds \cdot \frac{1}{h(s)} \int_0^t \frac{1}{h(s)} \, ds$ for $t > 0$.

**Proof.** Using Lemma 2.8 twice, we get

$$e^{-\frac{1}{2}H\mathcal{K}(0)} = \int_0^\infty h(s) e^{-\frac{1}{2}H\mathcal{K}(s) - s} \, ds$$

$$= \int_0^\infty h(s) \left( \int_s^\infty \frac{1}{h(\tau)} e^{-\frac{1}{2}H\mathcal{K}(\tau) e^{s-\tau} d\tau} \right) e^{-s} \, ds$$

$$= \int_0^\infty \frac{1}{h(\tau)} e^{-\frac{1}{2}H\mathcal{K}(\tau)} \left( \int_0^\tau h(s) \, ds \right) e^{-\tau} \, d\tau. \quad (3.1)$$

Analogous formula holds for $\mathcal{J}_{H^d}$:

$$e^{-\frac{1}{2}H\mathcal{K}(0)} = \int_0^\infty h(\tau) e^{-\frac{1}{2}H\mathcal{K}(\tau) - \tau} \left( \int_0^{\tau} \frac{1}{h(s)} \, ds \right) e^{-\tau} \, d\tau. \quad (3.2)$$

We have $2\mathcal{K}_H(r) = -\mathcal{J}_{H^d}(r) - \mathcal{J}_{H^d}(r)$ for all $r \geq 0$ (see (2.21)). We also have $\mathcal{K}_H \geq 0$ on $\mathbb{R}_+$ (check, e.g., (2.23)). Multiplying formulas (3.1), (3.2) and using Cauchy-Schwarz inequality, we obtain

$$e^{\frac{1}{2}H\mathcal{K}(0)} \geq \int_0^\infty e^{\frac{1}{2}H\mathcal{K}(\tau)} e^{-\tau} \sqrt{\int_0^\tau h(s) \, ds \int_0^{\tau} \frac{1}{h(s)} \, ds \, d\tau} \geq \int_0^\infty \sqrt{a(t)} \cdot te^{-t}dt,$$

as required. □

**Remark.** We can write $a(t) = \langle h \rangle_{[0, \ell]} \langle 1/h \rangle_{[0, \ell]}$ and $a(t) \geq 1$, as follows from Cauchy-Schwarz inequality.

This lemma and additivity of the entropy $\mathcal{K}_H$ imply the following estimate.

**Proposition 3.1.** Let $h \geq 0$ be a function on $\mathbb{R}_+$ such that $h, 1/h \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $H = \text{diag}(h, 1/h)$. Then, there exists a sequence of numbers $\{t_n\}$ such that $t_n \in [3, 4]$ and

$$\sum_{n \geq 0} \left( \frac{1}{t_n} \int_{\frac{4n}{3}}^{\frac{4n+t_n}{3}} h(s) \, ds \cdot \frac{1}{t_n} \int_{\frac{4n}{3}}^{\frac{4n+t_n}{3}} \frac{ds}{h(s)} - 1 \right) \leq e^{10\mathcal{K}_H(0)} - 1.$$

**Proof.** Iteratively applying assertion (e) of Lemma 2.5, we can find a sequence of Hamiltonians $H_{(n)} = \text{diag}(h_n, 1/h_n)$ such that $H_{(n)}(x) = H(4n + x)$ for $x \in [0, 4]$, $H_{(n+1)}(x) = \text{diag}(a_n, 1/a_n)$ for almost all $x > 4$ and some constant $a_n > 0$, and

$$\mathcal{K}_H(0) \geq \sum_{n \geq 0} \mathcal{K}_{H_{(n)}}(0). \quad (3.3)$$

Take $n \geq 0$ and apply Lemma 3.1 for the Hamiltonian $H_{(n)}$. Making note of

$$\int_0^\infty te^{-t}dt = 1$$

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and applying Jensen inequality, we get
\[ \mathcal{K}_\mathcal{H}(0) \geq \int_0^\infty \log a_n(t) \cdot te^{-t} dt, \]
where \( a_n(t) = \frac{1}{t} \int_{4n}^{4n+t} h(s) ds \cdot \frac{1}{t} \int_{4n+t}^{4n+2t} h(s) ds \) for \( t \in [0, 4] \) and \( a_n(t) \geq 1 \) for all \( t > 0 \). Since \( \int_0^t te^{-t} dt \geq 0.1 \) for \( I = [3, 4] \), we have \( 10 \mathcal{K}_\mathcal{H}(0) \geq \min_{t \in I} \log a_n(t) \). Define \( t_n \) to be a point in \( I \) such that \( a_n(t_n) = \min_{t \in I} a_n(t) \). Since \( e^{x+y} - 1 \geq e^x - 1 + e^y - 1 \) for all \( x, y \geq 0 \), we notice that (3.3) implies
\[ e^{10 \mathcal{K}_\mathcal{H}(0)} - 1 \geq \sum_{n \geq 0} \left( e^{10 \mathcal{K}_\mathcal{H}(0)} - 1 \right) \geq \sum_{n \geq 0} (a_n(t_n) - 1) = \sum_{n \geq 0} \left( \frac{1}{t_n} \int_{4n}^{4n+t_n} h(s) ds \cdot \frac{1}{t_n} \int_{4n}^{4n+t_n} \frac{1}{h(s)} ds - 1 \right), \]
which is the desired estimate. □

3.2. An upper bound for the entropy.

**Proposition 3.2.** Let \( h \) be a function as in Lemma 3.1 and let \( \mathcal{H} = \text{diag}(h, 1/h) \) be the corresponding Hamiltonian. Then,
\[ \mathcal{K}_\mathcal{H}(0) \leq \int_0^\infty (\kappa(s) + \kappa_d(s) - 2) ds, \]
where \( \kappa(r) = \frac{1}{h(r)} \int_r^\infty h(s)e^{-s} ds \) and \( \kappa_d(r) = h(r) \int_r^\infty \frac{1}{h(s)} e^{-s} ds \) for \( r \geq 0 \).

**Proof.** Consider the functions
\[ u(r) = \int_r^\infty \frac{1}{h(s)} e^{-\frac{3\mathcal{H}(s)}{2} - s} ds, \quad u_d(r) = \int_r^\infty h(s) e^{-\frac{3\mathcal{H}(s)}{2} - s} ds, \]
defined on \( \mathbb{R}_+ \). By Lemma 2.8, we have
\[ e^{-\frac{3\mathcal{H}(r)}{2}} = \left( \int_r^\infty h(s) e^{-\frac{3\mathcal{H}(s)}{2}} e^{-s} ds \right)^2 \leq \left( \int_r^\infty h(s) e^{-s} ds \right) \left( \int_r^\infty h(s) e^{-\frac{3\mathcal{H}(s)}{2}} e^{-s} ds \right) = h(r) e^{\kappa(r)} u_d(r). \]
Dividing by \( he^r \), we obtain \( -u'(r) \leq \kappa(r) u_d(r) \) for almost all \( r \geq 0 \). Analogously, we have \( -u_d'(r) \leq \kappa_d(r) u(r), r \geq 0 \) for the function \( u_d \). It follows that
\[ 0 \leq -(u^2 + u_d^2)'(r) \leq 2(\kappa(r) + \kappa_d(r)) u(r) \leq (\kappa(r) + \kappa_d(r)) (u^2 + u_d^2)(r), \]
for almost all \( r \geq 0 \). Thus, we have
\[ -\frac{\partial}{\partial r} \log(u^2(r) + u_d^2(r)) \leq \kappa(r) + \kappa_d(r). \]
Taking into account that \( u(r) = u_d(r) = e^{-r} \) for \( r \geq \ell \) by (3.1), we get
\[ u^2(0) + u_d^2(0) \leq (u^2(\ell) + u_d^2(\ell)) e^{\int_0^\ell (\kappa(s) + \kappa_d(s)) ds} = 2e^{\int_0^\infty (\kappa(s) + \kappa_d(s) - 2) ds}. \]
On the other hand, we have
\[ u(0) = \int_0^\infty \frac{1}{J_\mathcal{H}(s)} e^{\mathcal{K}_\mathcal{H}(s) - s} ds, \quad u_d(0) = \int_0^\infty J_\mathcal{H}(s) h(s) e^{\mathcal{K}_\mathcal{H}(s) - s} ds. \]
by assertions (c), (d) of Lemma 2.5. From (2.19) for $h_1 = h = 1/h_2$ we now get

$$u(0) + u_d(0) = -\int_0^\infty \mathcal{X}_h(s) e^{\mathcal{X}_h(s) - s} ds + 2\int_0^\infty e^{\mathcal{X}_h(s) - s} ds$$

$$= e^{\mathcal{X}_h(0)} + \int_0^\infty e^{\mathcal{X}_h(s) - s} ds$$

$$\geq e^{\mathcal{X}_h(0)} + 1 \geq 2e^{\mathcal{X}_h(0)/2},$$

using integration by parts and the fact that $\mathcal{X}_h(s) \geq 0$ for all $s$. Last estimate and (3.4) imply

$$e^{\mathcal{X}_h(0)} \leq \left(\frac{u(0) + u_d(0)}{2}\right)^2 \leq \frac{u^2(0) + u_d^2(0)}{2} \leq e^{\int_0^\infty (\kappa(s) + \kappa_d(s) - 2) ds}.$$  

Taking the logarithms, we arrive to the statement of the proposition. \hfill \Box

4. PROOF OF THEOREM 1

The classical Muckenhoupt class $A_2(\mathbb{R})$ is defined as the set of measurable functions $h \geq 0$ on $\mathbb{R}$ with finite characteristic

$$[h]_2 \equiv \sup_{I \subset \mathbb{R}} \langle h \rangle_I (h^{-1})_I,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$. Recall that $I_{x,y}$ denotes $[x, x+y]$ for $x, y \in \mathbb{R}$. For a function $h \geq 0$ on $\mathbb{R}_+$ and a sequence $\alpha = \{\alpha_n\}$ of positive numbers, put

$$[h, \alpha] = \sum_{n=0}^\infty \left( \langle h \rangle_{I_{n,\alpha_n}} (h^{-1})_{I_{n,\alpha_n}} - 1 \right).$$

(4.1)

Each term in the sum above is nonnegative, hence $[h, \alpha] \in \mathbb{R}_+ \cup \{+\infty\}$ is correctly defined. Denote by 2 the constant sequence 2, 2, \ldots indexed by non-negative integers.

**Definition.** Let $A_2(\mathbb{R}_+, \ell^1)$ be the set of functions $h \geq 0$ on $\mathbb{R}_+$ such that the characteristic $[h]_{2, \ell^1} = [h, 2]$ is finite.

Note that $[h]_{2, \ell^1} = 0$ if and only if the function $h$ is constant. Next, for a function $h \geq 0$ on $\mathbb{R}_+$ define

$$[h]_{\text{int}} = \int_0^\infty (\kappa(s) + \kappa_d(s) - 2) ds,$$

(4.2)

where $\kappa(r) = \frac{1}{h(r)} \int_r^\infty h(s) e^{-s} ds$ and $\kappa_d(r) = h(r) \int_r^\infty \frac{1}{h(s)} e^{-s} ds$ for $r \geq 0$. Since $h \geq 0$ on $\mathbb{R}_+$, we have $\frac{h(s)}{h(r)} + \frac{h(r)}{h(s)} \geq 2$, hence the quantity $[h]_{\text{int}} \in \mathbb{R}_+ \cup \{+\infty\}$ is correctly defined.

**Proposition 4.1.** Let $h \geq 0$ be a measurable function on $\mathbb{R}_+$. Assume that $[h, \alpha]$ is finite for a sequence $\alpha = \{\alpha_n\}$ where $\alpha_n \in [3,4], \forall n \in \mathbb{Z}^+$. Then $h \in A_2(\mathbb{R}_+, \ell^1)$ and, moreover, we have $[h]_{2, \ell^1} \leq c[h, \alpha]$ with absolute constant $c$.

**Proposition 4.2.** There exists an absolute constant $c$ such that $[h]_{\text{int}} \leq c[h]_{2, \ell^1} e^{[h]_{2, \ell^1}}$ for every function $h \in A_2(\mathbb{R}_+, \ell^1)$.

Propositions 4.1, 4.2 will be proved in the next section. Later, in the proof of the theorem, we will need the following lemma.

**Lemma 4.1.** Let $\mathcal{H}$ be singular diagonal Hamiltonians on $\mathbb{R}_+$ such that $\mathcal{H}(x) = \mathcal{H}(x)$ for every $k \geq 0$ and all $x \in [0,k]$. Suppose that the spectral measure of $\mathcal{H}(x)$ belongs to $Sz(\mathbb{R})$ for every $k \geq 0$ and $\sup_{k \geq 0} \mathcal{X}_h(0) < \infty$. Then, the spectral measure of $\mathcal{H}$ belongs to $Sz(\mathbb{R})$.

$$\mathcal{X}_h(0) \leq \limsup_{k \to \infty} \mathcal{X}_h(0).$$
Proof. Let \( \mathcal{H} \) be a singular Hamiltonian on \( \mathbb{R}_- \) and let \( m \) be its Weyl-Titchmarsh function. As usual, denote by \( \Theta^\pm, \Phi^\pm \) the corresponding entries of the solution \( M \) of Cauchy problem (1.2). Then, by the nesting circles analysis (see page 42 in Section 8 of [21] or page 475 in Section 7 of [11]), we have

\[
\left| m(z) - \frac{\Phi^-(k, z)}{\Theta^-(k, z)} \right| \leq \frac{1}{\text{Im}(\Theta^+(k, z)\Theta^-(k, z))}, \quad z \in \mathbb{C}^+, \quad k \geq 0, \tag{4.3}
\]

where the right hand side tends to zero as \( k \to +\infty \) uniformly on compacts in \( \mathbb{C}^+ \). Let \( m_k \) be the Weyl-Titchmarsh function of the Hamiltonian \( \mathcal{H}_k \). Since \( \mathcal{H}_k \) coincides with \( \mathcal{H} \) on \( [0, k] \), we have estimate (4.3) with \( m \) replaced by \( m_k \) and the same right hand side. The triangle inequality now implies that \( m - m_k \) tends to zero uniformly on compact subsets of \( \mathbb{C}^+ \).

Let us consider the measures \( \tilde{\mu}, \tilde{\mu}_k \) supported on the unit circle \( \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \) whose Poisson extensions to the open unit disk \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \) coincide with positive harmonic functions \( \text{Im} m(\omega), \text{Im} m_{(\omega)}(\omega) \) in \( \mathbb{D} \), respectively, where \( \omega : w \mapsto \frac{i(z-w)}{1-w\bar{w}} \) is the conformal mapping from \( \mathbb{D} \) onto \( \mathbb{C}^+ \). Since the difference \( m - m_k \) tends to zero uniformly on compacts in \( \mathbb{C}^+ \), the measures \( \tilde{\mu}_k \) converge weakly to the measure \( \tilde{\mu} \). Recall that the relative entropy of two positive finite measures \( \nu_1, \nu_2 \) on \( \mathbb{T} \) is defined by

\[
S(\nu_1 | \nu_2) = \begin{cases} 
-\infty & \text{if } \nu_1 \text{ is not } \nu_2 \text{ a.c.,} \\
-\int_\mathbb{T} \log \left( \frac{d\nu_1}{d\nu_2} \right) d\nu_1 & \text{if } \nu_1 \text{ is } \nu_2 \text{ a.c..}
\end{cases}
\]

It is known (see Section 2.2.3 in [22]) that the relative entropy is weakly upper-semicontinuous, which means \( \limsup_{k \to +\infty} S(\nu_1 | \nu_{2,k}) \leq S(\nu_1 | \nu_2) \) for every sequence of finite measures \( \nu_{2,k} \) on \( \mathbb{T} \) converging weakly to a measure \( \nu_2 \). This implies that \( \tilde{\mu} \) belongs to the Szegő class on \( \mathbb{T} \) and

\[
-\infty < \limsup_{k \to +\infty} \int_\mathbb{T} \log \tilde{\omega}_k(\xi) \, dm(\xi) \leq \int_\mathbb{T} \log \tilde{\omega}(\xi) \, dm(\xi), \tag{4.4}
\]

where \( m \) is the Lebesgue measure on \( \mathbb{T} \) normalized by \( m(\mathbb{T}) = 1 \), and \( \tilde{\omega}, \tilde{\omega}_k \) are the densities on \( \tilde{\mu}, \tilde{\mu}_k \) with respect to \( m \). Changing variables in (4.4), we see that the spectral measure of \( \tilde{\mathcal{H}} \) lies in the class \( \text{Sz}(\mathbb{R}) \), and, moreover,

\[
\limsup_{k \to +\infty} J\mathcal{H}_k(0) \leq J\mathcal{H}(0).
\]

From the relation \( \lim_{k \to +\infty} m_k(i) = m(i) \) we get \( J\mathcal{H}(0) = \lim_{k \to +\infty} J\mathcal{H}_k(0) \). The lemma now follows.

The next result establishes the key two-sided estimates for a special class of Hamiltonians.

Lemma 4.2. Let \( h \) be a function as in Lemma 3.1 and let \( \mathcal{H} = \text{diag}(h, 1/h) \). Then, we have \( \mathcal{K}(0) \leq c\mathcal{K}(\mathcal{H})e^{c\mathcal{K}(\mathcal{H})} \) and \( \mathcal{K}(\mathcal{H}) \leq c\mathcal{K}(0)e^{c\mathcal{K}(0)} \) for an absolute constant \( c \).

Proof. By Lemma 2.6 the spectral measure of \( \mathcal{H} \) belongs to \( \text{Sz}(\mathbb{R}) \). From Proposition 3.2 we know that \( \mathcal{K}(0) \leq [h]_{\text{int}} \). Proposition 4.2 implies \([h]_{\text{int}} \leq c[h]_{2,\ell_1} e^{c[h]_{2,\ell_1}} \) with \([h]_{2,\ell_1} = \mathcal{K}(\mathcal{H}) \). Combining these estimates, we obtain inequality \( \mathcal{K}(0) \leq c\mathcal{K}(\mathcal{H})e^{c\mathcal{K}(\mathcal{H})} \). To prove the second inequality, observe that Proposition 3.1 when applied to \( \mathcal{H} \), provides a sequence \( \{t_n\} \subset [3, 4] \) such that

\[
\sum_{n \geq 0} \left( \frac{1}{t_n} \int_{4n}^{4n+t_n} h(s) \, ds \cdot \frac{1}{t_n} \int_{4n}^{4n+t_n} \frac{ds}{h(s)} - 1 \right) \leq e^{100\mathcal{K}(0)} - 1.
\]

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The same proposition applied to three “translated” Hamiltonians $\mathcal{H}_k : x \mapsto \mathcal{H}(x + k), k = 1, 2, 3,$ gives

$$\sum_{n \geq 0} \left( \frac{1}{t_n^{(k)}} \int_{4n}^{4n+t_n^{(k)}} h(s + k) \, ds \cdot \frac{1}{t_n^{(k)}} \int_{4n}^{4n+t_n^{(k)}} \frac{ds}{h(s + k) - 1} \right) \leq e^{10X\mathcal{H}_k(0)} - 1.$$  

for three new sequences $\{t_n^{(k)}\} \subset [3, 4]$ where $k = 1, 2, 3$. Summing up the above four formulas, we obtain $[h, \alpha] \leq e^{10X\mathcal{H}_k(0)} - 1 + \sum_{k=1}^{3} e^{10X\mathcal{H}_k(0)} - 1$ for the sequence $\alpha = \{\alpha_n\}$ defined by $\alpha_n = t_n, \alpha_{n+k} = t_n^{(k)}, n \geq 0, k = 1, 2, 3$. By Lemma 2.5(e), we have $\mathcal{K}_{\mathcal{H}_k(0)} \leq \mathcal{K}_{\mathcal{H}(0)}$, hence $[h, \alpha] \leq 4(e^{10X\mathcal{H}_k(0)} - 1) \leq c\mathcal{K}_{\mathcal{H}(0)}e^{10X\mathcal{H}_k(0)}$. Proposition 4.1 says that $[h]_{2,\ell^1} \leq c[h,\alpha]$ for an absolute constant $c$. By definition, we have $\mathcal{K}(\mathcal{H}) = [h]_{2,\ell^1}$, hence $\mathcal{K}(\mathcal{H}) \leq c\mathcal{K}_{\mathcal{H}(0)}e^{10X\mathcal{H}_k(0)}$. \hfill $\square$

In the next lemma, we will show that the condition that the determinant equals to one can be dropped.

**Lemma 4.3.** Let $\mathcal{H} = \text{diag}(h_1, h_2)$ be a singular Hamiltonian on $\mathbb{R}_+$ such that $h_1$, $h_2$ are equal to positive constants on $[\ell, +\infty)$ for some $\ell \geq 0$. Then, we have $\mathcal{K}(\mathcal{H}) \leq c\mathcal{K}_{\mathcal{H}(0)}e^{c\mathcal{K}_{\mathcal{H}(0)}}$ and $\mathcal{K}_{\mathcal{H}(0)} \leq c\mathcal{K}(\mathcal{H})e^{c\mathcal{K}(\mathcal{H})}$ with an absolute constant $c$.

**Proof.** For every $\varepsilon > 0$ define $\mathcal{H}(\varepsilon) : t \mapsto \mathcal{H}(t) + \varepsilon \chi_{[0,\ell]}(t)I_2, t \in \mathbb{R}_+$, where $I_2 = (1 0 \ 0 1)$ is the $2 \times 2$ identity matrix and $\chi_{[0,\ell]}$ denotes the characteristic function of $[0, \ell]$. Set $\xi_\varepsilon = \xi_{\mathcal{H}(\varepsilon)},$ and let $\eta_\varepsilon$ denote the inverse function to $\xi_\varepsilon,$ so that $\eta_\varepsilon(\xi_\varepsilon(t)) = t$ for all $t \geq 0$. Since $\xi_{\mathcal{H}(\varepsilon)}$ maps $\mathbb{R}_+$ onto $\mathbb{R}_+$, the function $\eta_\varepsilon$ is defined correctly. Moreover, we have $\det \xi_{\mathcal{H}(\varepsilon)} > 0$ almost everywhere on $\mathbb{R}_+$, hence $\xi_{\mathcal{H}(\varepsilon)}$ is absolutely continuous on $\mathbb{R}_+$ and we can define the Hamiltonian $\mathcal{H}(\varepsilon) : t \mapsto \eta_\varepsilon(t)\mathcal{H}(\varepsilon)(\eta_\varepsilon(t))$. By construction, $\eta_\varepsilon(t) = 1/\sqrt{\det \mathcal{H}(\varepsilon)(\eta_\varepsilon(t))}$ almost everywhere on $\mathbb{R}_+$, so the Hamiltonian $\mathcal{H}(\varepsilon)$ has determinant equal to one almost everywhere on $\mathbb{R}_+$. By Lemma 2.6, the spectral measures $\mu$, $\mu(\varepsilon)$, $\widetilde{\mu}(\varepsilon)$ of $\mathcal{H}$, $\mathcal{H}(\varepsilon)$, $\mathcal{H}(\varepsilon)$, respectively, belong to $\text{Sz}(\mathbb{R})$. By Lemma 1.2

$$\mathcal{K}(\mathcal{H}(\varepsilon)) \leq c\mathcal{K}_{\mathcal{H}(\varepsilon)(0)}e^{c\mathcal{K}_{\mathcal{H}(\varepsilon)(0)}}, \quad \mathcal{K}_{\mathcal{H}(\varepsilon)(0)} \leq c\mathcal{K}(\mathcal{H}(\varepsilon))e^{c\mathcal{K}(\mathcal{H}(\varepsilon))}, \quad \text{(4.5)}$$

for an absolute constant $c$. Let $h_{1,\varepsilon}, h_{2,\varepsilon}, h_\varepsilon$ be defined by $\mathcal{H}(\varepsilon) = \text{diag}(h_{1,\varepsilon}, h_{2,\varepsilon}), \mathcal{H}(\varepsilon) = \text{diag}(h_\varepsilon, 1/h_\varepsilon)$. Then, for every $t \geq 0$, we have

$$\int_{\eta_\varepsilon(t)}^{\eta_\varepsilon(t+2)} h_{1,\varepsilon}(s) \, ds \cdot \int_{\eta_\varepsilon(t)}^{\eta_\varepsilon(t+2)} h_{2,\varepsilon}(s) \, ds = \int_{t}^{t+2} h_\varepsilon(s) \, ds \cdot \int_{t}^{t+2} \frac{1}{h_\varepsilon(s)} \, ds,$$

by a change of variables. This shows that $\mathcal{K}(\mathcal{H}(\varepsilon)) = \mathcal{K}(\mathcal{H}(\varepsilon))$. It is also not difficult to see that the spectral measures $\mu(\varepsilon), \widetilde{\mu}(\varepsilon)$ of $\mathcal{H}(\varepsilon)$, $\mathcal{H}(\varepsilon)$ coincide. Indeed, solutions $M(\varepsilon)$, $\widetilde{M}(\varepsilon)$ of Cauchy problem (1.2) for $\mathcal{H}(\varepsilon)$, $\mathcal{H}(\varepsilon)$ satisfy $\widetilde{M}(\varepsilon)(x) = M(\varepsilon)(\eta_\varepsilon(x)), x \in \mathbb{R}_+$. Hence the limit in the right hand side of (2.1) defines the same harmonic function for $\mathcal{H}(\varepsilon)$ and $\mathcal{H}(\varepsilon)$. Thus, from (1.5) we get

$$\mathcal{K}(\mathcal{H}(\varepsilon)) \leq c\mathcal{K}_{\mathcal{H}(\varepsilon)(0)}e^{c\mathcal{K}_{\mathcal{H}(\varepsilon)(0)}}, \quad \mathcal{K}_{\mathcal{H}(\varepsilon)(0)} \leq c\mathcal{K}(\mathcal{H}(\varepsilon))e^{c\mathcal{K}(\mathcal{H}(\varepsilon))}, \quad \text{(4.6)}$$

for every $\varepsilon > 0$. Next, by construction, we have $\xi_{\mathcal{H}(\varepsilon)}(t) > \xi_{\mathcal{H}}(t)$ for all $t > 0$ and $\varepsilon > 0$. Moreover, the difference $\xi_{\mathcal{H}(\varepsilon)} - \xi_{\mathcal{H}}$ tends to zero uniformly on $\mathbb{R}_+$ as $\varepsilon$ tends to zero. Hence $\eta_\varepsilon(t) < \eta(t)$ for all $t > 0, \varepsilon > 0$ and $\eta(t) - \eta_\varepsilon(t)$ tends to zero for each $t \in \mathbb{R}_+$ as $\varepsilon$ tends to zero. Since $\mathcal{H}, \mathcal{H}(\varepsilon)$ are
constant on $[\ell, +\infty)$, we have

$$0 = \int_{\eta_n}^{\eta_{n+2}} h_1(s) \, ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s) \, ds - 4,$$

for all $n \geq n_0$ and all sufficiently small $\varepsilon > 0$, where $n_0$ can be chosen independently of $\varepsilon$. Hence, the sums in (1.6) which define $\tilde{K}(\mathcal{H})$, $\tilde{K}(\mathcal{H}(\varepsilon))$ contain at most $n_0$ nonzero terms for small $\varepsilon > 0$. It follows that $\lim_{\varepsilon \to 0} \tilde{K}(\mathcal{H}(\varepsilon)) = \tilde{K}(\mathcal{H})$. It remains to show that $\lim_{\varepsilon \to 0} \mathcal{K}(\mathcal{H}(\varepsilon))(0) = \mathcal{K}(\mathcal{H})(0)$. To do that, one can use formula (2.13) with $r = \ell$ for $\mathcal{H}$ and $\mathcal{H}(\varepsilon)$. Since the matrix norm of $\mathcal{H} - \mathcal{H}(\varepsilon)$ tends to zero uniformly on $[0, \ell]$ and $\mathcal{H} = \mathcal{H}(\varepsilon)$ on $[\ell, +\infty)$, we have

$$\mathcal{J}_{\mathcal{H}}(\ell) = \mathcal{J}_{\mathcal{H}(\varepsilon)}(\ell), \quad \lim_{\varepsilon \to 0} \mathcal{J}_{\mathcal{H}(\varepsilon)}(\ell) = \mathcal{J}_{\mathcal{H}}(\ell), \quad \lim_{\varepsilon \to 0} |F_{\ell, \varepsilon}(i)| = |F_{\ell}(i)|. \quad (4.7)$$

To show that the last equality holds, we notice that the Hamiltonians $\mathcal{H}_{\ell}$ and $\mathcal{H}_{\mathcal{H}(\varepsilon)}(\cdot + \ell)$ coincide on $\mathbb{R}_+$ and thus have the same Weyl-Titchmarsh functions which we denote by $m_\ell$. Hence, the corresponding functions $F_{\ell, \varepsilon} : z \mapsto \Theta^+(l, z) + m_\ell(z)\Theta^-(l, z)$ tend to $F_\ell$ uniformly on compact subsets of $\mathbb{C}_+$ as $\varepsilon \to 0$. From (1.7) and Lemma 2.3 (b) for $r = \ell$, we get $\lim_{\varepsilon \to 0} \mathcal{J}_{\mathcal{H}(\varepsilon)}(0) = \mathcal{J}_{\mathcal{H}}(0)$. Using again formula (2.13) with $r = \ell$, we obtain $\lim_{\varepsilon \to 0} \mathcal{J}_{\mathcal{H}(\varepsilon)}(0) = \mathcal{J}_{\mathcal{H}}(0)$. This completes the proof of the lemma.

Now we are ready to prove Theorem 1.

**Proof of Theorem 1** Let $\mathcal{H}$ be a nontrivial singular diagonal Hamiltonian on $\mathbb{R}_+$ such that its spectral measure $\mu$ lies in the class $\mathcal{S}_z(\mathbb{R})$ and $b = 0$ in the Herglotz representation (1.4) of its Weyl-Titchmarsh function $m$. Note that we have $\mathcal{K}(\mu) = \mathcal{K}(\mu)(0)$ and no positive $\varepsilon$ exists such that $(0, \varepsilon)$ is the indivisible interval for $\mathcal{H}$ of type $\pi/2$, see Lemma 2.3. Consider the family of Bernstein-Szegő Hamiltonians $\tilde{\mathcal{H}}_r = \text{diag}(\tilde{h}_{1r}, \tilde{h}_{2r}), r \geq 0$, generated by $\mathcal{H}$ (see (2.3) for their definition). By Lemma 2.3 the spectral measure $\tilde{\mu}_r$ of $\tilde{\mathcal{H}}_r$ belongs to $\mathcal{S}_z(\mathbb{R})$ for every $r \geq 0$. Since the Hamiltonians $\tilde{\mathcal{H}}_r$ have no indivisible intervals $(0, \varepsilon)$ of type $\pi/2$, we have $\mathcal{K}(\tilde{\mu}_r) = \mathcal{K}(\tilde{\mu}_r)(0)$. From Lemma 2.3 (e) we now get $\mathcal{K}(\tilde{\mu}_r) \leq \mathcal{K}(\mu)$. Let us first show that $\sqrt{\det \mathcal{H}} \notin L^1([0, +\infty))$. Since $2\sqrt{\det \mathcal{H}} \leq \text{trace } \mathcal{H}$, the function $\sqrt{\det \mathcal{H}}$ is integrable on compact subsets of $\mathbb{R}_+$. Suppose that $\sqrt{\det \mathcal{H}} \in L^1([0, +\infty))$. Then the function $\xi_{\mathcal{H}}$ in (2.6) is bounded, hence there exists $n_0 \geq 0$ and $r_0 \geq n_{\mathcal{H}} \geq 0$, such that for every $r \geq r_0$ the last nonzero term in the sum defining $\tilde{\mathcal{K}}(\tilde{\mathcal{H}}_r)$ equals

$$c_{r, n_0} = \int_{n_{\mathcal{H}}}^{n_{\mathcal{H}}+2} h_{1r}(s) \, ds \cdot \int_{n_{\mathcal{H}}}^{n_{\mathcal{H}}+2} h_{2r}(s) \, ds - 4,$$

where $n_{\mathcal{H}} = \min\{t \geq 0 : \xi_{\mathcal{H}}(t) = n_0\}$, and $\tilde{n}_{n_{\mathcal{H}}+2}(r) = \min\{t \geq 0 : \xi_{\mathcal{H}}(t) = n_0 + 2\}$ increases infinitely with $r$. By Lemma 2.3 and Lemma 2.3 (e), we have $c_{r, n_0} \leq \tilde{\mathcal{K}}(\tilde{\mathcal{H}}_r) \leq c\mathcal{K}(\tilde{\mu}_r)e^{c\mathcal{K}(\mu)} \leq c\mathcal{K}(\mu)e^{c\mathcal{K}(\mu)}$ for every $r$. From $\text{trace } \mathcal{H} \notin L^1([0, +\infty))$ (recall that the Hamiltonian $\mathcal{H}$ is singular) and the uniform boundedness of $c_{r, n_0}, r \geq r_0$, we get

$$\int_{n_{\mathcal{H}}}^{\infty} h_1(s) \, ds \int_{n_{\mathcal{H}}}^{\infty} h_2(s) \, ds \leq \limsup_{r \to \infty} c_{r, n_0} + 4 < \infty, \quad \int_{0}^{\infty} (h_1(s) + h_2(s)) \, ds = \infty,$$

which implies that either $\int_{n_{\mathcal{H}}}^{\infty} h_1(s) \, ds = 0$ or $\int_{n_{\mathcal{H}}}^{\infty} h_2(s) \, ds = 0$. We see that either $h_1 = 0$ or $h_2 = 0$ almost everywhere on $[r_0, +\infty)$ and the Hamiltonian $\mathcal{H}_{r_0}$ is trivial. The first part of the proof of
Lemma 2.5 shows that this is not the case, hence $\int_0^\infty \sqrt{\det \mathcal{H}(s)} \, ds = +\infty$\footnote{There is a different way to prove this fact. One needs to check that the supremum of the function $\xi_{2k}(r)$ in (2.6) determines the exponential type of the measure $\mu$ and then apply Krein-Wiener completeness theorem. See Section 6 in [21].} and the function $\eta_x$ in the statement of Theorem 1 is correctly defined on $\mathbb{R}_+$. For every $r \geq \eta_2$ the first $[\xi_{2k}(r)] - 2$ terms defining $\tilde{K}(\mathcal{H})$ and $\tilde{K}(\mathcal{H}_r)$ in (1.6) are identical. Hence,

$$\tilde{K}(\mathcal{H}) \leq \limsup_{r \to \infty} \tilde{K}(\mathcal{H}_r) \leq \limsup_{r \to \infty} c\mathcal{K}(\mu) e^{c\mathcal{K}(\mu)} \leq c\mathcal{K}(\mu) e^{c\mathcal{K}(\mu)},$$

where the second and the third inequalities follow from Lemma 4.3 and Lemma 2.5 (e), respectively.

Conversely, suppose that $\mathcal{H} = \text{diag}(h_1, h_2)$ is a singular Hamiltonian on $\mathbb{R}_+$, $\sqrt{\det \mathcal{H}} \notin L^1(\mathbb{R}_+)$, and the sum defining $\tilde{K}(\mathcal{H})$ in (1.6) converges. For every integer $k \geq 0$, fix some positive constants $a_{1k}, a_{2k}$ to be specified later, and consider

$$\tilde{K}(\mathcal{H}_k)(t) = \text{diag}(h_{1k}, h_{2k}) = \begin{cases} \mathcal{H}(t) \quad &\text{if } t \in [0, \eta_{k+2}], \\ \text{diag}(a_{1k}, a_{2k}) \quad &\text{if } t \in (\eta_{k+2}, +\infty). \end{cases}$$

For every $t > 0$, set $\tilde{\eta}_t = \min\{s \geq 0 : \xi_{3k}(k)(s) = t\}$, where $\xi_{3k}(k)(s) = \int_0^s \sqrt{\det \mathcal{H}_k(\tau)} \, d\tau$. Then we have $\tilde{\eta}_t = \eta_t$ for every $t \in [0, \eta_{k+2}]$. By construction,

$$\tilde{K}(\mathcal{H}_k) = \sum_{n=0}^k \left( \int_{\eta_n}^{\eta_{n+2}} h_1(s) \, ds \cdot \int_{\eta_n}^{\eta_{n+2}} h_2(s) \, ds - 4 \right)$$

$$+ \int_{\eta_{k+1}}^{\eta_{k+3}} h_{1k}(s) \, ds \cdot \int_{\eta_{k+1}}^{\eta_{k+3}} h_{2k}(s) \, ds - 4.$$

Indeed, $\tilde{K}(\mathcal{H})$ is constant on $[\eta_{k+2}, +\infty) = [\tilde{\eta}_{k+2}, +\infty)$ and $\mathcal{H} = \tilde{K}(\mathcal{H})$ on $[0, \eta_{n+2}]$, hence the terms with indexes $n \geq k + 1$ in formula (1.6) for $\tilde{K}(\mathcal{H})$ vanish, while the terms with indexes $n \leq k$ coincide with the corresponding terms in (1.6) for the Hamiltonian $\mathcal{H}$. Since $\tilde{K}(\mathcal{H}_k) = \text{diag}(a_{1k}, a_{2k})$ on $[\eta_{k+2}, +\infty)$, we have

$$\int_{\eta_{k+1}}^{\eta_{k+3}} h_{1k} \, ds \cdot \int_{\eta_{k+1}}^{\eta_{k+3}} h_{2k} \, ds = \frac{3}{2} \left( \int_{\eta_{n+1}}^{\eta_{n+2}} h_j \, ds + a_{jk}(\tilde{\eta}_{k+3} - \eta_{k+2}) \right).$$

A short calculation gives $\eta_{k+3} - \eta_{k+2} = 1/\sqrt{a_{1k}a_{2k}}$. Thus, we have

$$\int_{\eta_{k+1}}^{\eta_{k+3}} h_{1k} \, ds \cdot \int_{\eta_{k+1}}^{\eta_{k+3}} h_{2k} \, ds = \left( x_1 + \sqrt{\frac{a_{1k}}{a_{2k}}} \right) \left( x_2 + \sqrt{\frac{a_{2k}}{a_{1k}}} \right),$$

where $x_j = \int_{\eta_{k+1}}^{\eta_{k+2}} h_j \, ds$ for $j = 1, 2$. Denoting $y_j = \int_{\eta_{k+1}}^{\eta_{k+2}} h_j \, ds$, $j = 1, 2$, we get

$$\left( x_1 + \sqrt{\frac{a_{1k}}{a_{2k}}} \right) \left( x_2 + \sqrt{\frac{a_{2k}}{a_{1k}}} \right) \leq (x_1 + y_1)(x_2 + y_2) = \int_{\eta_{k+1}}^{\eta_{k+3}} h_1 \, ds \cdot \int_{\eta_{k+1}}^{\eta_{k+3}} h_2 \, ds,$$
By Lemma 2.6 the spectral measure of the Hamiltonian $\tilde{H}_k$ belongs to $Sz(\mathbb{R})$ for every $k$. From Lemma 4.1, Lemma 4.3, and (4.10) we obtain $\mu \in Sz(\mathbb{R})$ and

$$
\mathcal{K}(\mu) \leq \limsup_{k \to \infty} \mathcal{K}_{\tilde{H}_k}(0) \leq c \limsup_{r \to \infty} \mathcal{K}(\tilde{H}_k)e^{\mathcal{H}(\tilde{H}_k)} \leq c \mathcal{K}(\mathcal{H}) e^{\mathcal{H}(\mathcal{H})},
$$

with an absolute constant $c$. The theorem is proved. \hfill \Box

5. Functions with summable fixed-scale Muckenhoupt characteristic

In this section, we study functions from the class $A_2(\mathbb{R}^+, \ell^1)$ defined in Section 4 and prove Propositions 4.1, 4.2.

Lemma 5.1. Let $I = I^- \cup I^+$ be a splitting of an interval $I \subset \mathbb{R}$ into the union of two disjoint subintervals $I^\pm$. Let $h > 0$ be a function on $I$ such that $h/1 \in L^1(I)$, and let $\eta = \langle h \rangle_I / \langle h \rangle_{I^-} - 1$. Assume that $|I^-|/|I| \geq \frac{1}{4}$, then

$$
\frac{\langle h \rangle_{I^-}}{\langle h \rangle_{I^-}} - 1 \lesssim \sqrt{\eta} (1 + \eta), \quad \frac{\langle h \rangle_{I^+}}{\langle h \rangle_{I^+}} - 1 \lesssim \min(1, \sqrt{\eta}),
$$

(5.1)

and, moreover,

$$
\langle h \rangle_{I^-} (1/\langle h \rangle_{I^-}) - 1 \lesssim \eta.
$$

(5.2)

Proof. The number $\eta$ and all bounds are invariant with respect to multiplying $h$ with a positive constant, thus we can assume that $\langle h \rangle_I = 1$. Next, put $v = |I^-|/|I|$, $a^\pm = \langle h \rangle_{I^\pm}$, $b^\pm = \langle h^{-1} \rangle_{I^\pm}$. We have

$$
va^- + (1 - v)a^+ = 1, \quad vb^+ + (1 - v)b^- = \langle h^{-1} \rangle_{I^+} = 1 + \eta, \quad a^\pm b^\pm \geq 1.
$$

(5.3)

Adding the first two estimates and using the bounds $1/a^\pm \leq b^\pm$, one gets $v(a^- + 1/a^-) + (1 - v)(a^+ + 1/a^+) \leq 2 + \eta$. Since $x + 1/x \geq 2$ for all $x > 0$, this yields $v(a^- + 1/a^-) \leq 2v + \eta$. Dividing by $2v$, we get the inequality

$$
\frac{1}{2} \left( a^- + \frac{1}{a^-} \right) \leq 1 + \frac{\eta}{2v}.
$$

(5.4)

It can be rewritten in the form $(1/a^- - 1)^2 \leq \eta/(va^-)$. Since $v \in \left[ \frac{1}{4}, 1 \right]$ and $1/a^- \lesssim (1 + \eta)$ by (5.2), this gives the first bound in (5.1). To get the second bound in (5.1), rewrite (5.4) in the form $(a^+ - 1)^2 \leq a^- \eta/v$ and use the fact that $va^- \leq 1$. Thus,

$$
|a^+ - 1| \leq \sqrt{\frac{\eta}{v}}, \quad |a^- - 1| \leq 1 + \frac{1}{v}^{-1},
$$

which implies the second inequality in (5.1). Next, let us prove (5.2). Since $a^\pm b^\pm \geq 2$, we get $v(a^- + b^-) \leq 2v + \eta$ by summing up the first two identities in (5.3). Hence $\sqrt{a^- b^-} \leq 1 + \eta/(2v)$ and $a^- b^- \leq 1 + \eta/v + \eta^2/(4v^2)$. This gives the inequality $\langle h \rangle_{I^-} (1/\langle h \rangle_{I^-}) - 1 \lesssim \eta$ in the case where $\eta \leq v$. For $\eta \geq v$ we can use (5.3) to get $a^- \leq 1/v \leq 5$ and $b^- \leq 5(1 + \eta)$. This gives $\langle h \rangle_{I^-} (1/\langle h \rangle_{I^-}) - 1 \leq 25(1 + \eta) - 1 \lesssim \eta$ since $\eta \geq 1/5$. \hfill \Box

Proof of Proposition 4.1. Apply Lemma 5.1 to the function $h$ and the intervals $I = I_{n, \alpha_n}$, $I_- = [n, n + 2]$, $n \geq 0$. Since $\{\alpha_n\} \subset [3, 4]$, this will give the estimate $[h]_{2, \ell^1} \leq c[h, \alpha]$ with an absolute constant $c$. \hfill \Box

Lemma 5.2. For $h \in A_2(\mathbb{R}^+, \ell^1)$, define $Q_n = \langle h \rangle_{I_{n,2}} (\langle h^{-1} \rangle_{I_{n,2}} - 1$ and $f_n = \langle h \rangle_{I_{n,1}}$. Then,

$$
(1 + Q_n)^{-1} \lesssim \frac{f_{n+1}}{f_n} \lesssim 1 + Q_n,
$$

(5.5)
\[ \left| \frac{f_{n+1}}{f_n} - 1 \right| \leq c\sqrt{Q_n}, \text{ if } Q_n \leq 1. \]  

(5.6)

Moreover, we have \( \|\tilde{h} + \tilde{h}^{-1} - 2\|_1 \lesssim [h]_{2,\ell_1} = \sum_{n=0}^{\infty} Q_n \) for the function \( \tilde{h} \) defined by

\[ \tilde{h}(x) = h(x)/\langle h \rangle_{I_{n,1}}, \quad x \in I_{n,1}, \quad n \in \mathbb{Z}^+. \]  

(5.7)

**Proof.** Represent \( f_{n+1}/f_n \) in the form

\[ \frac{f_{n+1}}{f_n} = \frac{\langle h \rangle_{I_{n+1,1}}}{\langle h \rangle_{I_{n,2}}} \frac{\langle h \rangle_{I_{n,2}}}{\langle h \rangle_{I_{n,1}}} \cdot \]  

(5.8)

We write

\[ \frac{1}{2} \leq \frac{\langle h \rangle_{I_{n,2}}}{\langle h \rangle_{I_{n,1}}} \leq 1 + c\sqrt{Q_n(Q_n + 1)} \lesssim 1 + Q_n, \]  

(5.9)

where the first inequality is immediate and the second one follows from the first estimate in (5.1). Similarly, we get

\[ \frac{1}{2} \leq \frac{\langle h \rangle_{I_{n+1,1}}}{\langle h \rangle_{I_{n+1,2}}} \leq 1 + c\sqrt{Q_n(Q_n + 1)} \lesssim 1 + Q_n. \]

and

\[ (1 + Q_n)^{-1} \lesssim \frac{\langle h \rangle_{I_{n+1,1}}}{\langle h \rangle_{I_{n+1,2}}} \leq 2. \]  

(5.10)

It is now sufficient to multiply (5.10) with (5.9) and substitute into (5.8) to get (5.5). Take \( n \geq 0 \) such that \( Q_n \leq 1 \). By Lemma 5.1, we have

\[ \left| \frac{\langle h \rangle_{I_{n,2}}}{\langle h \rangle_{I_{n,1}}} - 1 \right| \lesssim \sqrt{Q_n}, \quad \left| \frac{\langle h \rangle_{I_{n+1,1}}}{\langle h \rangle_{I_{n+1,2}}} - 1 \right| \lesssim \sqrt{Q_n}. \]  

(5.11)

Substituting these bounds into (5.8) gives (5.6). Finally, observe that for every \( n \geq 0 \) we have

\[ \langle h \rangle_{I_{n,1}} \langle h^{-1} \rangle_{I_{n,1}} - 1 \lesssim Q_n \]  

(5.12)

we complete the proof of the lemma. \( \square \)

**Remark.** Notice that (5.5) and (5.6) imply

\[ |\log(f_{n+1}/f_n)| \lesssim \begin{cases} \sqrt{Q_n}, & Q_n < 2, \\ \log Q_n, & Q_n \geq 2. \end{cases} \]  

(5.13)

**Proof of Proposition 4.2** Define \( \tilde{h} \) as in (5.7) and consider the function \( f_1 = (\tilde{h} - 1)\chi_{[\tilde{h}^{-1} < h < \frac{1}{2}]} \). For shorthand, denote \( P = [h]_{2,\ell_1} = \sum_{n=0}^{\infty} Q_n \), where \( Q_n \) is defined in the previous lemma. Since the function \( \tilde{h} + \tilde{h}^{-1} - 2 \in L^1(\mathbb{R}^+) \), we have \( f_1 \in L^2(\mathbb{R}^+) \) and \( \|f_1\|_2^2 \lesssim P \). Indeed, this follows from the fact that \( x + x^{-1} - 2 \sim (x - 1)^2 \) for \( x \in [\frac{1}{2}, 3] \) and the estimate \( \|\tilde{h} + \tilde{h}^{-1} - 2\|_1 \lesssim P \) in Lemma 5.2.

Similarly, the function \( f_2 = (\tilde{h} - 1)\chi_{[\tilde{h}^{-1} < h < \frac{1}{2}]} \) belongs to \( L^1(\mathbb{R}^+) \) and \( \|f_2\|_1 \lesssim P \). Thus, we see that \( \tilde{h} \) can be represented in the form \( \tilde{h} = f_0 + f_1 + f_2 \), where \( f_0 = 1, f_1 \in L^2(\mathbb{R}^+), \quad f_2 \in L^1(\mathbb{R}^+), \) and \( \|f_1\|_2^2 + \|f_2\|_1 \lesssim P \). Function \( \tilde{h}^{-1} \) admits similar representation \( \tilde{h}^{-1} = f_0 + f_1 + f_2 \), where \( f_0 = 1, f_1 = -f_1 \) and \( f_2 \in L^1(\mathbb{R}^+) \) is such that \( \|f_2\|_1 \lesssim P \). Notice that we have got \( f_1 = -f_1 \) from

\[ \frac{\chi_{[\tilde{h} - 1 < h < 1/2]}}{h} = \frac{\chi_{[\tilde{h}^{-1} - 1 < h < 1/2]}}{1 + f_1} = \chi_{[\tilde{h}^{-1} < h < 1/2]}(1 - f_1 + O(f_1^2)). \]
and \( \hat{f}_2 \in L^1(\mathbb{R}^+) \) because \( \hat{f}_2 = \chi_{[\tilde{n}-1]<1/2}O(f_1^2) + \chi_{[\tilde{n}-1]>1/2}(\tilde{n}^{-1} - 1) \in L^1(\mathbb{R}^+) \).

Let \( g_0 \) be the function on \( \mathbb{R}^+ \) such that \( g_0 = \log f_n \) on each \( I_{n,1} \), then \( h = e^{g_0}h \) on \( \mathbb{R}^+ \). Define also the function \( g : x \mapsto g_0(x) - g_0(0) \) on \( \mathbb{R}^+ \). Then, for \( \kappa \) and \( \kappa_d \) from Proposition 3.2 we have

\[
\kappa = \sum_{0 \leq k, j \leq 2} a_{kj}, \quad a_{kj} : x \mapsto \int_x^\infty \hat{f}_k(x) f_j(x) e^{g(\xi) - g(x)} + x - \xi \, d\xi,
\]

\[
\kappa_d = \sum_{0 \leq k, j \leq 2} a_{d, kj}, \quad a_{d, kj} : x \mapsto \int_x^\infty f_k(x) \hat{f}_j(x) e^{g(x) - g(\xi)} + x - \xi \, d\xi.
\]

We will need some estimates for the function \( g \). Let \( Q_j, f_j \) be defined as in Lemma 5.2 and let \( v_n = \log (f_n/f_{n-1}), n \in \mathbb{N}, v_0 = 0 \). Observe that \( g(x) = \sum_{n=0}^{[x]} v_n \) on \( \mathbb{R}^+ \) by construction. Here, as usual, \([x]\) stands for the integer part of a number \( x \in \mathbb{R}^+ \). We can estimate

\[
\|\{v_n\}\|_2^2 = \sum_{n: Q_{n-1} < 2} v_n^2 + \sum_{n: Q_{n-1} > 2} v_n^2 \lesssim \sum_{n: Q_{n-1} < 2} Q_n + \sum_{n: Q_{n-1} > 2} \log^2 Q_n \lesssim P,
\]

where we used (5.12) and the trivial bound: \( \log^2 Q \lesssim Q \) which holds for all \( Q > 2 \). Bound (5.12) also yields

\[
\|\{v_n\}\|_\infty \lesssim \log(2 + P).
\]

For \( x < y \), we can apply (5.12) to write

\[
|g(x) - g(y)| \leq \left| \sum_{j=|x|}^{[y]} v_j \right| \leq \sum_{j=|x|, Q_{j-1} < 2} [v_j] + \sum_{j=|x|, Q_{j-1} \geq 2} [v_j] \lesssim \sum_{j=|x|, Q_{j-1} < 2} \sqrt{|Q_{j-1}|} + \sum_{j=|x|, Q_{j-1} \geq 2} \log Q_{j-1}
\]

\[
\lesssim \left( (|x - y| + 1) \sum_{j \geq 0} Q_j \right)^{1/2} + \sum_{j \geq 0} Q_j \lesssim \sqrt{(|x - y| + 1)} P + P.
\]

It follows that there is an absolute constant \( C \) such that for all \( x, y \in \mathbb{R}^+ \) we have

\[
|g(x) - g(y)| \leq \frac{1}{2} |x - y| + C(1 + P).
\]

Now, for indexes \( k, j \) such that \( k + j \geq 2 \), we can use (5.16) and the Young inequality for convolutions to estimate

\[
\|a_d, kj\|_1 \lesssim e^{CP} \int_0^\infty \int_0^\infty |f_k(x)| \chi_{\mathbb{R}^+}(\xi - x) e^{-(\xi - x)/2} \, d\xi \, d\xi \lesssim e^{CP} \|f_k\|_{L_p} \cdot \|\chi_{\mathbb{R}^+} e^{-x} \|_{j, k} \cdot \|\hat{f}_j\|_{L_p} \lesssim P e^{CP},
\]

where \( p_0 = +\infty, p_1 = 2, p_2 = 1 \), and the parameter \( r_{k, j} \) is chosen so that \( \frac{1}{p_k} + \frac{1}{r_{k, j}} + \frac{1}{p_j} = 2 \). The estimate on \( a_{kj} \) for \( k + j \geq 2 \) is similar. To prove that \( \kappa + \kappa_d - 2 \in L^1(\mathbb{R}^+) \), it remains to estimate
the $L^1(\mathbb{R}^+)$–norms of functions

\[ a_{00} + a_{d,00} - 2 = 2 \int_x^\infty e^{x-\xi} (\cosh G(x, \xi) - 1) \, d\xi, \]
\[ a_{01} + a_{d,01} = 2 \int_x^\infty \tilde{f}_1(\xi) e^{x-\xi} \sinh G(x, \xi) \, d\xi, \]
\[ a_{10} + a_{d,10} = 2 \int_x^\infty f_1(x) e^{x-\xi} \sinh G(x, \xi) \, d\xi, \]

where $G(x, \xi) = g(x) - g(\xi)$. Let us define the function $\tilde{g}$ on $[-1, \infty)$ to be continuous, linear on $I_{j,1}$ for each $j \geq -1$, and so that $\tilde{g}(-1) = 0$, $\tilde{g}(j) = \sum_{n=0}^j |v_n|$ for $j \geq 0$. Clearly, $\tilde{g}$ is non-decreasing on $[-1, \infty)$. Put $\tilde{G}(x, \xi) = \tilde{g}(\xi + 1) - \tilde{g}(x - 1)$ for every $0 < x < \xi$. Then $|G(x, \xi)| \leq \tilde{G}(x, \xi)$ and so $\cosh G(x, \xi) \leq \cosh \tilde{G}(x, \xi)$. By construction and (5.13), we have

\[ \|\tilde{g}'\|_2^2 \leq \sum_{n \geq 0} |v_n|^2 \lesssim P. \]  

(5.17)

The bound (5.13) also implies

\[ \|\tilde{G}(x, x)\|_2^2 \leq \|\{v_n\}\|_2^2 \lesssim P. \]  

(5.18)

The estimate (5.14) gives

\[ \|\tilde{G}(x, x)\|_\infty \lesssim \sup_{n \geq 0} |v_n| \lesssim \log(2 + P) \]  

(5.19)

and argument given in (5.15) yields

\[ \tilde{G}(x, \xi) \lesssim \sqrt{|x - \xi| + 1} + P, \quad \tilde{G}(x, \xi) \leq \frac{1}{2} |x - \xi| + C(1 + P) \]  

(5.20)

for all $x < \xi$. Integrate by parts to get

\[ \|a_{00} + a_{d,00} - 2\|_1 \leq 2 \int_0^\infty \int_x^\infty e^{x-\xi} (\cosh \tilde{G}(x, \xi) - 1) \, d\xi \, dx \]
\[ \leq 2 \int_0^\infty \int_x^\infty \tilde{g}'(\xi + 1) e^{x-\xi} \sinh \tilde{G}(x, \xi) \, d\xi \, dx + 2R_1, \]

where $R_1 = \int_0^\infty (\cosh \tilde{G}(x, x) - 1) \, dx$. Using the inequality $\cosh t - 1 \leq t^2 e^{|t|}$, we obtain $R_1 \leq \|\tilde{G}(x, x)\|_2^2 \exp(\|\tilde{G}(x, x)\|_\infty) \lesssim P e^{CP}$ by (5.18) and (5.19). To estimate the double integral, let us change the order of integration and integrate by parts once again:

\[ \int_0^\infty \tilde{g}'(\xi + 1) \int_0^\xi e^{x-\xi} \sinh \tilde{G}(x, \xi) \, dx \, d\xi = \int_0^\infty \tilde{g}'(\xi + 1) \int_0^\xi \tilde{g}'(x - 1) e^{x-\xi} \cosh \tilde{G}(x, \xi) \, dx \, d\xi + R_2, \]  

(5.21)

where $R_2 = \int_0^\infty \tilde{g}'(\xi + 1) (\sinh \tilde{G}(0, \xi) - e^{-\xi} \sinh \tilde{G}(0, \xi)) \, d\xi \leq \int_0^\infty \tilde{g}'(\xi + 1) \sinh \tilde{G}(\xi, \xi) \, d\xi$ because $\tilde{g}' \geq 0$. Let us estimate the integral first using the second bound in (5.20)

\[ \int_0^\infty \tilde{g}'(\xi + 1) \int_0^\xi \tilde{g}'(x - 1) e^{x-\xi} \cosh \tilde{G}(x, \xi) \, dx \, d\xi \lesssim e^{CP} \int_0^\infty \tilde{g}'(\xi + 1) \int_0^\xi \tilde{g}'(x - 1) e^{(x-\xi)/2} \, dx \, d\xi \]
\[ \lesssim e^{CP} \|\tilde{g}'\|_2^2 \lesssim P e^{CP}, \]

as follows from Young’s inequality for convolution and (5.17). We are left with estimating $R_2$. Using inequality $|\sinh t| \leq |t| e^{|t|}$ we obtain

\[ \int_0^\infty \tilde{g}'(\xi + 1) \sinh \tilde{G}(\xi, \xi) \, d\xi \leq \|\tilde{g}'(\xi + 1)\|_2 \cdot \|\tilde{G}(\xi, \xi)\|_2 \exp(\|\tilde{G}(\xi, \xi)\|_\infty) \lesssim P e^{CP}. \]
Collecting the bounds, we get \( \|a_{00} + a_{d,00} - 2\|_1 \lesssim P e^{C_P} \). It remains to bound the \( L^1(\mathbb{R}_+) \)-norms of \( a_{01} + a_{d,01} \) and \( a_{10} + a_{d,10} \). First, we write
\[
\|a_{01} + a_{d,01}\|_1 \leq 2 \int_0^\infty |f_1(\xi)| \int_0^{\xi} e^{x-\xi} \sinh \tilde{G}(x,\xi) \, dx \, d\xi \lesssim P e^{C_P}
\]
since the integral has the form similar to the left hand side in (5.21) and the estimates for (5.21) can be repeated. Finally,
\[
\|a_{10} + a_{d,10}\|_1 \leq 2 \int_0^\infty \int_x^\infty |f_1(x)| e^{x-\xi} \sinh \tilde{G}(x,\xi) \, d\xi \, dx
\]
\[
\lesssim 2 \int_0^\infty |f_1(x)| \sinh \tilde{G}(x,x) \, dx + 2 \int_0^\infty \int_x^\infty |f_1(x)| \tilde{g}'(\xi+1) e^{x-\xi} \cosh \tilde{G}(x,\xi) \, d\xi \, dx,
\]
where the first term can be estimated similarly to \( R_2 \), while the second one is dominated by \( C e^{C_P} \|f_1\|_2 \cdot ||\tilde{g}'(t-1)||_2 \lesssim P e^{C_P} \). Thus, we see that \( \kappa + \kappa_d - 2 \) belongs to \( L^1(\mathbb{R}_+) \) and \( |h| \text{loc} \lesssim P e^{C_P} \) with an absolute constant \( c \).

\[\Box\]

6. Krein strings and proof of Theorem 2

In this section, we introduce the spectral measure for Krein string and show how Theorem 1 and some results obtained in [14] imply Theorem 2. Let \( 0 < L \ll \infty \). Recall that \( M \) and \( L \) form \([M,L]\) pair if (1.8) holds, i.e., \( L + \lim_{t \to L} M(t) = \infty \) and \( \lim_{t \to L} M(t) > 0 \). Define the Lebesgue–Stieltjes measure \( m \) by \( m[0,t] = M(t) \). Next, define the increasing function \( N : t \mapsto t + M(t) \) on \([0,L]\) and let \( n \) denote the corresponding measure, \( n[0,t] = N(t) \) for \( t \geq 0 \). Define also the function \( N^{-1} \) on \( \mathbb{R}_+ \) by \( N^{-1} : y \mapsto \inf \{ t \geq 0 : N(t) \geq y \} \). The set under the last infimum is non-empty for every \( y \geq 0 \) because of the assumptions we made on \( M \) and \( L \). Using the fact that \( N \) is strictly increasing, one can show that \( N^{-1} \) is continuous on \( \mathbb{R}_+ \), and we have \( N^{-1}(N(t)) = t \) for every \( t \in [0,L] \). Let \( M' \) be the density of the absolutely continuous part of \( m \), so that \( m = M'(t) \, dt + m_s \). Denote by \( E_s \) the support of the singular part \( m_s \) of the measure \( m \). Define two functions on \( \mathbb{R}_+ \),
\[
h_1(x) = \begin{cases} 
0, & \text{if } N^{-1}(x) \in E_s, \\
\frac{1}{1 + M'(N^{-1}(x))}, & \text{otherwise},
\end{cases}
\]
and
\[
h_2(x) = \begin{cases} 
1, & \text{if } N^{-1}(x) \in E_s, \\
\frac{M'(N^{-1}(x))}{1 + M'(N^{-1}(x))}, & \text{otherwise}.
\end{cases}
\]

The proof of Lemma 6.1 below shows that functions \( h_1, h_2 \) defined by different representatives of the function \( M' \) differ on a set of zero Lebesgue measure. Notice that \( h_1, h_2 \) are non-negative Lebesgue measurable functions and we have \( h_1(x) + h_2(x) = 1 \) for all \( x \in \mathbb{R}_+ \). We are going to prove the following result from [14], pp. 1527–1528.

**Lemma 6.1.** Formulas (6.1), (6.2) establish the bijection \([M,L] \mapsto \text{diag}(h_1, h_2)\) between \([M,L]\) pairs and nontrivial diagonal Hamiltonians \( \mathcal{H} = \text{diag}(h_1, h_2) \) with unit trace almost everywhere on \( \mathbb{R}_+ \).

**Proof.** Fix any pair \([M,L]\) and consider the corresponding function \( N^{-1} \) and the measure \( n \). For every function \( f \in L^1(\mathbb{R}_+; n) \) we have \( f(N^{-1}(x)) \in L^1_{\text{loc}}(\mathbb{R}_+) \), and, moreover,
\[
\int_{[0,L]} f(t) \, dn(t) = \int_{\mathbb{R}_+} f(N^{-1}(x)) \, dx, \tag{6.3}
\]
if $f$ is compactly supported in $[0, L)$. This result is known as the change of variables in the Lebesgue–Stieltjes integral (see, e.g., Exercise 5 in Section III.13 of [7]) but we give its proof for completeness. Without loss of generality we can assume that $f \geq 0$. Then (see, e.g., [14], Proposition 6.24), we have

$$
\int_{[0, L)} f(t) \, d\nu(t) = \int_{\mathbb{R}_+} \Lambda_1(\lambda) \, d\lambda, \quad \int_{[0, L)} f(N^{-1}(x)) \, dx = \int_{\mathbb{R}_+} \Lambda_2(\lambda) \, d\lambda,
$$

where $\Lambda_1(\lambda) = n\{t : f(t) > \lambda\}$ and $\Lambda_2(\lambda) = |\{x : f(N^{-1}(x)) > \lambda\}|$. For all $0 \leq a < b$ we have

$$
n((a, b)) = N(b-) - N(a) = |(N(a), N(b-))|,
$$

where $N(b-)$ denotes the left limit of $N$ at the point $b$. In fact, $(N(a), N(b-))$ is preimage of $(a, b)$ under the continuous map $N^{-1}$. Thus, the preimage under $N^{-1}$ of any open cover $\cup(a_j, b_j)$ for $n$-measurable set $E$ will be an open cover for the set $\{x : N^{-1}(x) \in E\}$. Conversely, every open cover $\cup_j(c_j, d_j)$ for $\{x : N^{-1}(x) \in E\}$ is the preimage of some open cover for $E$. Indeed, for each $j$ we get $(c_j, d_j) = (N(a_j), N(b_j))$, where $a_j$ and $b_j$ are points of continuity for $N$ (to see this, note that the preimage of $n$’s atom under $N^{-1}$ is a closed segment). For every regular measure $\nu$ we have

$$
\nu(E) = \inf\left\{\sum_j \nu(I_j), \ E \subset \cup I_j, \ \{I_j\} \text{ are disjoint open intervals} \right\},
$$

see, e.g., Lemma 1.17 in [9]. From (6.4) and (6.3) we now get $\Lambda_1(\lambda) = \Lambda_2(\lambda)$ and, consequently, relation (6.3) follows. Next, take a number $y \geq 0$. Since $h_1(x) = 0$ for all $x$ such that $N^{-1}(x) \in E_\epsilon$, we have

$$
\chi_{[0, y]}(x)h_1(x) = f_y(N^{-1}(x)), \quad x \in [0, L),
$$

where $f_y : t \mapsto \frac{\chi_{[0, N^{-1}(y) \cap E_\epsilon]}(t)}{1 + M'(t)}$ is the compactly supported function from $L^1([0, L), n)$. Applying formula (6.3) to the function $f_y$, we get

$$
\int_0^y h_1(x) \, dx = \int_{[0, L)} \frac{\chi_{[0, N^{-1}(y) \cap E_\epsilon]}(t)}{1 + M'(t)} \, d\nu(t) = \int_{[0, N^{-1}(y) \cap E_\epsilon]} M'(t) \, dt = N^{-1}(y), \quad (6.6)
$$

where we used the fact that the singular part of $n$ is supported on $E_\epsilon$ and the absolutely continuous part of $n$ has density $M' + 1$ with respect to the Lebesgue measure on $[0, L)$. If $y$ is a point of growth for the function $N^{-1}$ (that is, there is no open interval $I$ containing $y$ such that $N^{-1}$ is constant on $I$), we have $\chi_{[0, y]}(x) = \chi_{[0, N^{-1}(y)]}(N^{-1}(x))$ for all $x \geq 0$, hence we can apply (6.3) to get

$$
\int_0^y h_2(x) \, dx = \int_{[0, N^{-1}(y) \cap E_\epsilon]} \frac{M'(t)(1 + M'(t))}{1 + M'(t)} \, dt + \int_{[0, N^{-1}(y) \cap E_\epsilon]} \, dm_x = m[0, N^{-1}(y)], \quad (6.7)
$$

From here we see that $h_1, h_2$ define $M, L$ uniquely, in particular, these functions, as elements of $L^1_{\text{loc}}(\mathbb{R}_+)$, do not depend on the choice of the representative of $M'$. Moreover, we cannot have $h_1 = 0$ or $h_2 = 0$ almost everywhere on $\mathbb{R}_+$ for any $M, L$ satisfying (1.8). Hence, $[M, L] \mapsto \text{diag}(h_1, h_2)$ is the injective mapping from a set of pairs $[M, L]$ to nontrivial diagonal Hamiltonians with unit trace. Now take a nontrivial Hamiltonian $	ext{diag}(h_1, h_2)$ with unit trace almost everywhere on $\mathbb{R}_+$, and consider the function

$$
\Psi : y \mapsto \int_0^y h_1(x) \, dx.
$$

Put $L = \sup_{y \geq 0} \Psi(y)$. Note that $|\Psi(y_1) - \Psi(y_2)| \leq |y_1 - y_2|$ for all $y_1, y_2$ in $\mathbb{R}_+$, hence there exists a measure $m$ on $[0, L)$ such that $\Psi(y) = \inf\{x \geq 0 : x + M(x) \geq y\}$ for every $y \geq 0$, where $M(x) = m[0, x]$. Using (6.6) and (6.7), it is easy to check that formulas (6.1), (6.2) for $[M, L]$ generate the singular Hamiltonian $\mathcal{H} = \text{diag}(h_1, h_2)$ and it is nontrivial. The lemma is proved. □
For any pair \([M, L]\), one can define the Krein string as the differential operator \([8][13]\). In \([14]\), the authors considered two functions \(\varphi(x, z)\) and \(\psi(x, z)\) that satisfy

\[
\varphi(x, z) = 1 - z \int_{[0, x]} (x - s) \varphi(s, z) \, dm(s), \quad x \in [0, L),
\]

\[
\psi(x, z) = x - z \int_{[0, x]} (x - s) \psi(s, z) \, dm(s), \quad x \in [0, L).
\]

These functions are uniquely determined by the string \([M, L]\) and they define the principal Weyl-Titchmarsh function \(q\) of \([M, L]\) by

\[
q(z) = \lim_{x \to L} \frac{\psi(x, z)}{\varphi(x, z)}, \quad z \in \mathbb{C} \setminus [0, \infty),
\]

see formula (2.21) in \([14]\). This function \(q\) has the unique integral representation

\[
q(z) = b + \int_{\mathbb{R}_+} \frac{d\sigma(x)}{x - z},
\]

where \(b \geq 0\) and \(\sigma\), the spectral measure of the string \([M, L]\), is a measure on \(\mathbb{R}_+ = [0, +\infty)\) satisfying condition

\[
\int_{\mathbb{R}_+} \frac{d\sigma(x)}{1 + x} < \infty.
\]

The authors of \([14]\) established, among other things, connection between \(q\) and the Weyl-Titchmarsh function of a canonical system. It is worth to mention that the definition of the Weyl-Titchmarsh function \(m\) we used in \([13]\) was taken from \([21]\). The authors of \([11], [14]\) deal with the canonical system written differently, i.e., they write the Cauchy problem

\[
W'(t, z)J = zW(t, z)H(t), \quad W(0, z) = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), \quad t \in \mathbb{R}_+, \quad z \in \mathbb{C},
\]

and define the Weyl-Titchmarsh function \(Q^+\) for \(z \in \mathbb{C} \setminus \mathbb{R}\) by

\[
Q^+(z) = \lim_{t \to +\infty} \frac{w_{11}(t, z)\tilde{\omega} + w_{12}(t, z)}{w_{21}(t, z)\tilde{\omega} + w_{22}(t, z)}, \quad W(t, z) = \begin{pmatrix} w_{11}(t, z) & w_{12}(t, z) \\ w_{21}(t, z) & w_{22}(t, z) \end{pmatrix}. \quad (6.8)
\]

It is not difficult to see that \(W(t, z) = M(t, -z)^\top\) for the solution \(M\) of \([1, 2]\). If we let \(\sigma_1 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\) and denote by \(M_{\sigma_1}\) the solution of Cauchy problem \(JM_{\sigma_1} = z\mathcal{H}_{\sigma_1}M_{\sigma_1}\), \(M_{\sigma_1}(0, z) = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\) for the dual Hamiltonian \(\mathcal{H}^d = \mathcal{H}_{\sigma_1} = \sigma_1\mathcal{H}\sigma_1\), then the function \(m_{\sigma_1}\) from formula \([1, 3]\) for \(\mathcal{H}_{\sigma_1}\) will coincide with the function \(Q^+\) in \([6, 8]\) for \(\mathcal{H}\) and \(\tilde{\omega} = 1/\omega\). Indeed, we have

\[
M_{\sigma_1}(t, z) = \sigma_1 M(t, -z)\sigma_1 = \sigma_1 W(t, z)^\top \sigma_1 = \begin{pmatrix} w_{22}(t, z) & w_{12}(t, z) \\ w_{21}(t, z) & w_{11}(t, z) \end{pmatrix}. \quad (6.9)
\]

We will need the following lemma from \([14]\).

**Lemma 6.2.** Suppose \([M, L] \mapsto \text{diag}(h_1, h_2)\) is the bijection given by \([6, 1]\) and \([6, 2]\), \(q\) is the Weyl-Titchmarsh function for the string given by \([M, L]\), and \(m, m_{\sigma_1}\) are the Weyl-Titchmarsh functions for \(\text{diag}(h_1, h_2)\) and \(\text{diag}(h_2, h_1)\), respectively. Then, we have

\[
q(z)^2 = m_{\sigma_1}(z) = -m^{-1}(z), \quad z \in \mathbb{C}^+.
\]

**Proof.** In \([14]\), formula (4.20), it is proved that

\[
Q^+(z) = q(z)^2, \quad z \in \mathbb{C}^+,
\]

where \(Q^+\) is defined in \([6, 8]\) and \(\mathcal{H}\) is obtained from \([M, L]\) by bijection discussed in Lemma \([6, 1]\).

On the other hand, \(Q^+(z) = m_{\sigma_1}(z) = m^{-1}(-z) = -m^{-1}(z)\), where the first equality follows from
It remains to use Theorem 2.

**Proof.** For given $m, \mu_{\sigma_1}$ define the Hamiltonians $H$ and $H_d = H_{\sigma_1} = \sigma_1 H_{\sigma_1}$ on $\mathbb{R}_+$. Let $m_{\sigma_1}, \mu_{\sigma_1} = w_{\sigma_1} dx + \mu_{\sigma_1,s}$ be the Weyl-Titchmarsh function and the spectral measure of $H_d$. Recall that $\sigma = \nu dx + \sigma_s$ for spectral measure of the string. In (6.10), taking the nontangential limits of $\text{Im}(m_{\sigma_1}(z))$ and $\text{Im}(zq(z^2))$ as $z \to x$, we get $w_{\sigma_1}(x)$ and $xv(x^2)$ for almost all $x \in \mathbb{R}_+$, respectively. Thus, $w_{\sigma_1}(x) = xv(x^2)$ for almost every $x \geq 0$, and, since $\mu_{\sigma_1}$ is even by Lemma 2.2, we get

$$
\int_0^\infty \log w_{\sigma_1}(x) \frac{dx}{1 + x^2} = 2 \int_0^\infty \log \frac{x}{x^2 + 1} dx + 2 \int_0^\infty \frac{v(x^2)}{x^2 + 1} dx = \int_0^\infty \frac{\log v(x)}{\sqrt{x+1}} dx,
$$

where we used the fact that $\int_0^\infty \log \frac{x}{1 + x^2} dx = \int_{-\infty}^\infty \frac{y}{e^y + e^{-y}} dy = 0$. This implies that $\int_0^\infty \frac{\log v(x)}{\sqrt{x+1}} dx$ is finite if and only if $\mu_{\sigma_1} \in \Sigma(\mathbb{R})$. On the other hand, formula (6.3) and the definition of $h_1, h_2$ imply

$$
\int_0^y \sqrt{h_1(x)h_2(x)} dx = \int_{[0,N(-1)(\eta_n)]^1 \mapsto [0,N(-1)(\eta_n+1)]^1} \sqrt{M'(t)} dt = \int_0^y \sqrt{M'(t)} dt
$$

if $y$ is a point of growth of the function $N(-1)$. For every $n \geq 1$ the points $\{\eta_n\}$ defined in (1.3) are the points of growth for $N(-1)$. Indeed, this is clear from the formula (6.6) that was proved for all $y \geq 0$. Hence we have $t_n = N(-1)(\eta_n)$ for all $n \geq 0$. It follows that

$$
t_{n+2} - t_n = N(-1)(\eta_{n+2}) - N(-1)(\eta_n) = \int_{\eta_n}^{\eta_{n+2}} h_1(x) dx,
$$

where we used (6.6) again. We also have

$$
M(t_{n+2}) - M(t_n) = m(t_n, t_{n+2}) = m(N(-1)(\eta_n), N(-1)(\eta_{n+2}) = \int_{\eta_n}^{\eta_{n+2}} h_2(x) dx,
$$

by the definition of $M$ and (6.7). Thus, $\mathcal{K}[M, L] = \mathcal{K}(H) = \mathcal{K}(H_{\sigma_1})$ and $\sqrt{\det H} \in L^1(\mathbb{R}_+)$ if and only if $\sqrt{M'} \in L^1(\mathbb{R}_+)$. Now the result follows from Theorem 1. □

**Remark.** If $[M, L] \mapsto \text{diag}(h_1, h_2)$, then the string $[M_d, L_d]$ for which $[M_d, L_d] \mapsto \text{diag}(h_2, h_1)$ is called the dual string. One can easily see that $\mathcal{K}[M, L] = \mathcal{K}[M_d, L_d]$ so the logarithmic integral for the string converges if and only if it converges for the dual string.

We give two applications of Theorem 2.

**Proposition 6.1.** Suppose that the mass distribution $M$ of a string $[M, \infty]$ satisfies $M' = 1$ almost everywhere on $\mathbb{R}_+$. Let $m, \sigma$ be the singular measure on $\mathbb{R}_+$ such that $M(t) = t + m[0, t]$ for all $t \geq 0$. Then we have

$$
\int_0^\infty \frac{\log v(x)}{\sqrt{x+1}} dx > -\infty
$$

for the spectral measure $\sigma = \nu dx + \sigma_s$ of $[M, \infty]$ if and only if $m(\mathbb{R}_+) < \infty$.

**Proof.** For given $M$, we have $t_n = n$ and $M(t_{n+2}) - M(t_n) = 2 + m, n, n + 2]$, hence

$$
\mathcal{K}[M, \infty] = \sum_{n \geq 0} (2 \cdot (2 + m, n, n + 2] - 4) = 2 \sum_{n \geq 0} m, n, n + 2].
$$

It remains to use Theorem 2. □

The next result shows that logarithmic integral can converge even if $m(\mathbb{R}_+) = \infty$. 

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Proposition 6.2. There exists a string \([M, L]\) with \(L < \infty\) and \(m_s[0, L] = +\infty\) such that
\[
\int_0^\infty \frac{\log v(x)}{\sqrt{x(x+1)}} \, dx > -\infty
\]
for its spectral measure \(\sigma = v \, dx + \sigma_s\).

Proof. Consider any sequence \(\{\varepsilon_n\} \subset (-1, 1)\), and define \(\delta_n = \prod_{j=0}^n (1 + \varepsilon_j)\), \(t_0 = 0\), \(t_n = \sum_{j=0}^{n-1} \delta_j\) for integer \(n \geq 0\), and let \(L = \sup_{n \geq 0} t_n\). Consider the function
\[
M'(t) = M_n = (\delta_n)^{-2}, \quad t \in [t_n, t_{n+1}], \quad n \geq 0.
\]
Define the measure \(m\) by \(m = M' \, dt + m_s\), where \(m_s\) is some singular measure, and let \(M(t) = m[0, t]\) for \(t \geq 0\). Then, the condition \((1.2)\) for \([M, L]\) is satisfied if and only if
\[
\left\{ (\delta_n + \delta_{n+1}) \left( \frac{1}{\delta_n} + \frac{1}{\delta_{n+1}} \right) - 4 \right\} \in \ell^1
\]
and
\[
\left\{ (\delta_n + \delta_{n+1}) (\Delta m_s)_n \right\} \in \ell^1,
\]
where \((\Delta m_s)_n = m_s(t_n, t_{n+2}]\) for \(n \geq 0\). Condition \((6.12)\) is satisfied if and only if
\[
(1 + \varepsilon_n) + (1 + \varepsilon_n)^{-1} - 2 \in \ell^1,
\]
or, equivalently, \(\{\varepsilon_n\} \in \ell^2\). If we choose \(\varepsilon_n = -(n+1)^{-\alpha}, \alpha \in (2, 1)\), then \(\sum_{n=1}^\infty (t_{n+2} - t_n) < \infty\) and we have \(L < \infty\). Condition \((6.13)\) in that case can be satisfied even if \(\sum_{n} (\Delta m_s)_n\) diverges, that is, \(m_s[0, L] = \infty\). For instance, we can take a singular measure \(m_s\) such that \((\Delta m_s)_n = 1\) for all integers \(n \geq 0\). □

7. Appendix

Proof of Lemma 2.1. Differentiate the function \(M : r \mapsto \{(0, 0)\} - zJ \int_0^r \mathcal{H}(\tau) \, d\tau\) and use the fact that the solution to Cauchy problem \((1.2)\) is unique. □

Proof of Lemma 2.2. Put \(\sigma_1 = (0, 0)\) and \(M_{\sigma_1} = \sigma_1 M_{\sigma_1}\), where \(M\) is the solution of \((1.2)\). Using identity \(\sigma_1 \mathcal{H} \sigma_1 = J^* \mathcal{H} J = \mathcal{H}_d\) and \(J \sigma_1 = -\sigma_1 J\), it is easy to check that \(J M' \sigma_1 = -z \mathcal{H}_d M_{\sigma_1}\). It follows that \(M_{\sigma_1}(t, z) = M'(t, -z)\) for all \(t \geq 0, z \in \mathbb{C}\). Using \((2.10)\), we get
\[
\begin{pmatrix}
\Phi^-(t, z) \\
\Theta^+(t, z)
\end{pmatrix} = \begin{pmatrix}
\Phi^-(t, -z) \\
\Theta^+(t, -z)
\end{pmatrix},
\]
for all \(t \geq 0\) and \(z \in \mathbb{C}\). From \((1.3)\), one has \(m(z) = -m(-z)\) for \(z \in \mathbb{C} \setminus \mathbb{R}\), hence
\[
\frac{1}{\pi} \int_{\mathbb{R}^+} \frac{\text{Im} z}{|x-z|^2} \, d\mu(x) + b \text{Im} z = \frac{1}{\pi} \int_{\mathbb{R}^+} \frac{\text{Im} z}{|x+z|^2} \, d\mu(x) + b \text{Im} z, \quad z \in \mathbb{C}^+.
\]
This implies that \(\mu\) is even. Using \(m(i+1) = -m(-i-1)\), we conclude that \(a = 0\).

Conversely, suppose that \(\mu\) is even and \(a = 0\). The approximation procedure in Section 9 of \([21]\) gives a sequence of even measures \(\mu_N\) supported at finitely many points such that the corresponding Hamiltonians, \(\mathcal{H}_N\), constructed in Theorem 7 of \([21]\) are diagonal and \(\lim_{N \to \infty} \| \int_0^t (\mathcal{H}_N(s) - \mathcal{H}(s)) \, ds \| = 0\) for every \(t \geq 0\). It follows that \(\mathcal{H}\) is diagonal, as required. □

Proof of Lemma 2.3. Let \(\mathcal{H}\) be a singular nontrivial Hamiltonian on \(\mathbb{R}_+\) such that \((0, \varepsilon)\) is the indivisible interval of type \(\pi/2\) for some \(\varepsilon > 0\). Then, for all \(z \in \mathbb{C}^+\), we have
\[
m(z) = \frac{\Phi^+(\varepsilon, z) + m_\varepsilon(z) \Phi^-(\varepsilon, z)}{\Theta^+(\varepsilon, z) + m_\varepsilon(z) \Theta^-(\varepsilon, z)} = z \int_0^\varepsilon \langle \mathcal{H}(t) (0, 1) \rangle \, dt + m_\varepsilon(z), \quad (7.1)
\]
by formula \((2.13)\) for \(r = \varepsilon\) and Lemma 2.1. So, we have \(b \geq \int_0^\varepsilon \langle \mathcal{H}(t) (0, 1) \rangle \, dt\) in this situation.
 Conversely, assume that $b > 0$ in (1.3). Consider a Hamiltonian $\mathcal{H}(b)$ whose Weyl-Titchmarsh function $m_{\mathcal{H}(b)}$ coincides with $m - bz$. Define

$$\tilde{\mathcal{H}}(x) = \begin{cases} \text{diag}(0, 1), & x \in [0, b], \\ \mathcal{H}(b)(x-b), & x > b. \end{cases}$$

Let $m_{\tilde{\mathcal{H}}}$ denote the Weyl-Titchmarsh function of $\tilde{\mathcal{H}}$. Then, a variant of (1.1) for $\tilde{\mathcal{H}}$, $\varepsilon = b$, gives

$$m_{\tilde{\mathcal{H}}} = bz + m_{\mathcal{H}(b)} = bz + m - bz = m.$$ 

Thus, the Weyl-Titchmarsh functions of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ coincide. It follows from de Branges theorem formulated in the Introduction that the Hamiltonians $\mathcal{H}$, $\tilde{\mathcal{H}}$ are equivalent. Hence, there is an absolutely continuous strictly increasing function $\eta \geq 0$ such that $\tilde{\mathcal{H}}(t) = \eta'(t)\mathcal{H}(\eta(t))$ almost everywhere on $\mathbb{R}_+$. In particular, the interval $(0, \eta(b))$ is indivisible of type $\pi/2$ for $\mathcal{H}$. It follows that for $\varepsilon = \eta(b)$ we have

$$b = \int_0^b \text{trace } \tilde{\mathcal{H}}(t) \, dt = \int_0^{\eta(b)} \text{trace } \mathcal{H}(s) \, ds = \int_0^\varepsilon \langle \tilde{\mathcal{H}}(s), (\frac{d}{ds}) \rangle \, ds,$$

completing the proof of the lemma. $\square$

**Proof of Lemma 2.4** The matrix-function

$$M(t, z) = \begin{pmatrix} \cos(t\sqrt{a_1a_2z}) & \sqrt{a_2/a_1} \sin(t\sqrt{a_1a_2z}) \\ -\sqrt{a_1/a_2} \sin(t\sqrt{a_1a_2z}) & \cos(t\sqrt{a_1a_2z}) \end{pmatrix}$$

solves Cauchy problem (1.2) for $\mathcal{H} = \text{diag}(a_1, a_2)$. It follows from (1.3) that the Weyl-Titchmarsh function of $\mathcal{H}$ is given by $m(z) = i\sqrt{a_2/a_1}$ for all $z \in \mathbb{C}^+$. Taking imaginary part, we get $w_r(x) = \sqrt{a_2/a_1}$, $x \in \mathbb{R}$, and $\log J_{\mathcal{H}}(r) = J_{\mathcal{H}}(r) = \log \sqrt{a_2/a_1}$ for all $r \geq 0$, as required. $\square$

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