PRODUCT THEOREM ON DELTA INVARIANTS VIA ADDING A GENERAL BOUNDARY

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ABSTRACT. It’s well-known that adding a general boundary would create K-stability. As an application, we reprove product theorem for delta invariants of Fano varieties.

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1. INTRODUCTION

It’s well known that a K-unstable Fano manifold cannot admit Kähler-Einstein (KE) metrics. However, it can admit conic KE metrics along some smooth divisors. This inspires the concept of twisted K-stability. In [Der16], Dervan introduced the concept of twisted K-stability and later it is algebraically reformulated using twisted generalized Futaki invariants in [BLZ19], which indicates that K-unstable Fano manifolds can be twisted K-stable. For example, if $X$ is a given K-unstable Fano manifold and $H \in |-lK_X|$ is a general smooth divisor on $X$, where $-lK_X$ is a very ample line bundle, then the pair $(X, \frac{1-\beta}{\beta}H)$ is uniformly K-stable for sufficiently small $0 < \beta \ll 1$. Thus there exists a conic KE metric $w(\beta)$ such that the following equation holds:

$$\text{Ric}(w(\beta)) = \beta w(\beta) + (1 - \beta)[H].$$

The following theorem proved by [BL18] are natural products of this phenomenon, that is, adding a general boundary would create K-stability.

**Theorem 1.1.** ([BL18 Section 7]) Let $(X, \Delta)$ be a log Fano pair, then the following statements are equivalent.

1. $\delta(X, \Delta) \geq \mu$ for some positive $0 < \mu \leq 1$,
2. For any rational $0 < \epsilon < \mu$, there exists an element $D \in |-K_X - \Delta|_Q$ such that the pair $(X, \Delta + (1 - \epsilon)D)$ is K-semistable,
3. For any rational $0 < \epsilon < \mu$, there exists an element $D \in |-K_X - \Delta|_Q$ such that the pair $(X, \Delta + (1 - \epsilon)D)$ is uniformly K-stable.

As an application of this principle, we reprove the following product theorem for delta invariants of Fano varieties, which is originally proved by [Zhu20].
Theorem 1.2. Let \((X_i, \Delta_i), i = 1, 2\), be two log Fano pairs.
(1) Both pairs are K-semistable if and only if \((X_1 \times X_2, p_1^*\Delta_1 + p_2^*\Delta_2)\) is K-semistable,
(2) If one of the two pairs is K-unstable, then
\[\delta(X_1 \times X_2, p_1^*\Delta_1 + p_2^*\Delta_2) = \min\{\delta(X_1, \Delta_1), \delta(X_2, \Delta_2)\}\]
(3) In general, we have following inequality,
\[\delta(X_1 \times X_2, p_1^*\Delta_1 + p_2^*\Delta_2) \geq \min\{\delta(X_1, \Delta_1), \delta(X_2, \Delta_2)\}, \frac{1}{2}\].

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2. Preliminaries

In this section, we present some necessary preliminaries. Throughout the note we work over complex number field \(\mathbb{C}\). A log pair \((X, \Delta)\) consists of a normal projective variety \(X\) and an effective \(\mathbb{Q}\)-divisor \(\Delta\) on \(X\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. We say a log pair \((X, \Delta)\) is log Fano if the pair admits klt singularities and \(-K_X - \Delta\) is ample. The \(\mathbb{Q}\)-linear system \(|-K_X - \Delta|\) is defined as follows:
\[|-K_X - \Delta| := \{D \geq 0 | D \sim_\mathbb{Q} -K_X - \Delta\}\].

By the works of \([FO18, BJ20]\), we can give the following definition for delta invariants of log Fano pairs.

Definition 2.1. Let \((X, \Delta)\) be a log Fano pair. We define
\[\delta_m(X, \Delta) := \inf_E \frac{A_{X, \Delta}(E)}{S_m(E)} \quad \text{and} \quad \delta(X, \Delta) := \inf_E \frac{A_{X, \Delta}(E)}{S_{X, \Delta}(E)}\]
where \(E\) runs over all prime divisors over \(X\). Here
\[A_{X, \Delta}(E) = \text{ord}_E(K_Y - f^*(K_X + \Delta)) + 1\]
for some log resolution \(f : Y \rightarrow X\) such that \(E \subset Y\); and
\[S_m(E) = \sup \text{ord}_E(D_m),\]
where \(D_m\) is of the form \[\sum_{j=0}^{m} \sum_{s_j = 0} \text{dim} H^0(X, -m(K_X + \Delta))\] and \(\{s_j\}\) is a complete basis of the vector space \(H^0(X, -m(K_X + \Delta))\); and
\[S_{X, \Delta}(E) = \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^\infty \text{vol}(-f^*(K_X + \Delta) - tE)dt.\]

The following result is now well-known.

Theorem 2.2. \([FO18, BJ20]\) Let \((X, \Delta)\) be a log Fano pair, then
(1) \(\lim_{m \rightarrow \infty} \delta_m(X, \Delta) = \delta(X, \Delta)\),
(2) \((X, \Delta)\) is K-semistable if and only if \(\delta(X, \Delta) \geq 1\),
(3) \((X, \Delta)\) is uniformly K-stable if and only if \(\delta(X, \Delta) > 1\).

One can just put this theorem as the definition of K-semistability (resp. uniform K-stability) of \((X, \Delta)\).
In this section, we will prove Theorem 1.2. We first show the following result on K-stability of product varieties.

**Theorem 3.1.** Let \((X_i, \Delta_i), i = 1, 2\), be two log Fano pairs, then \((X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2)\) is K-semistable if and only if both pairs are K-semistable.

**Proof.** The only if direction is easy. Suppose \((X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2)\) is K-semistable. For any \(m\)-basis type divisor \(D_m^{(i)}\) for \((X_i, \Delta_i)\), we see that \(p_1^* D_m^{(1)} + p_2^* D_m^{(2)}\) is an \(m\)-basis type divisor for \((X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2)\). Denote \(\delta_m := \delta_m(X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2)\), then the pair \((X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2 + \delta_m(p_1^* D_m^{(1)} + p_2^* D_m^{(2)}))\) is log canonical, so are \((X_1, \Delta_1 + \delta_m D_m^{(1)}))\) and \((X_2, \Delta_2 + \delta_m D_m^{(2)})\). Thus \(\delta_m(X_1, \Delta_1) \geq \delta_m,\) which implies that both \((X_i, \Delta_i), i = 1, 2,\) are K-semistable.

For the converse direction, we just apply Theorem 1.1 to the case \(\mu = 1\). For any fixed rational \(0 \leq \epsilon < 1\), there exist \(D_i \in |-K_{X_i} - \Delta_i|_{\mathbb{Q}}, i = 1, 2\) such that the pairs \((X_1, \Delta_1 + (1 - \epsilon)D_1) \text{ and } (X_2, \Delta_2 + (1 - \epsilon)D_2)\) are both uniformly K-stable by Theorem 1.1 (3). By [LTW19], one can construct KE metrics on the pairs \((X_i, \Delta_i + (1 - \epsilon)D_i), i = 1, 2\). Therefore, by taking product, the pair \((X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2 + (1 - \epsilon)(p_1^* D_1 + p_2^* D_2))\) also admits a KE metric, hence is K-semistable (even K-polystable) by [Ber16]. Again by Theorem 1.1 (2), we see the pair \((X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2)\) is K-semistable. The proof is finished. \(\square\)

If we assume one of the two pairs is K-unstable, then we have following more precise result on delta invariant of the product variety.

**Theorem 3.2.** Let \((X_i, \Delta_i), i = 1, 2\), be two log Fano pairs. Suppose one of the two pairs is K-unstable, then

\[
\delta(X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2) = \min\{\delta(X_1, \Delta_1), \delta(X_2, \Delta_2)\}.
\]

**Proof.** We assume \(\delta(X_1, \Delta_1) \leq \delta(X_2, \Delta_2)\). We first show

\[
\delta(X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2) \geq \delta(X_1, \Delta_1).
\]

We apply Theorem 1.1 (3). For any rational \(0 < \epsilon < \delta(X_1, \Delta_1)\), one can find an element \(D_1 \in |-K_{X_1} - \Delta_1|_{\mathbb{Q}}\) and an element \(D_2 \in |-K_{X_2} - \Delta_2|_{\mathbb{Q}}\) such that both \((X_i, \Delta_i + (1 - \epsilon)D_i)\) are uniformly K-stable. By Theorem 3.1, the pair

\[
(X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2 + (1 - \epsilon)(p_1^* D_1 + p_2^* D_2))
\]

is K-semistable. By Theorem 1.1 (2) we see that

\[
\delta(X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2) \geq \delta(X_1, \Delta_1).
\]

We next show \(\delta(X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2) \leq \delta(X_1, \Delta_1)\). It suffices to show that

\[
\delta_m(X_1 \times X_2, p_1^* \Delta_1 + p_2^* \Delta_2) \leq \delta_m(X_1, \Delta_1).
\]
Similar to the proof of Theorem 3.1, we arbitrarily choose $m$-basis type divisors $D_m(i)$ for $(X_i, \Delta_i)$, then $p_1^*D_m^{(1)} + p_2^*D_m^{(2)}$ is an $m$-basis type divisor for $(X_1 \times X_2, p_1^*\Delta_1 + p_2^*\Delta_2)$. Denote $\delta_m := \delta_m(X_1 \times X_2, p_1^*\Delta_1 + p_2^*\Delta_2)$, we see the pair

$$(X_1 \times X_2, p_1^*\Delta_1 + p_2^*\Delta_2 + \delta_m(p_1^*D_m^{(1)} + p_2^*D_m^{(2)}))$$

is log canonical, so are the pairs $(X_i, \Delta_i + \delta_m D_m^{(i)}), i = 1, 2$. Therefore we have

$$\delta_m(X_1, \Delta_1) \geq \delta_m.$$ 

Taking the limit we have

$$\delta(X_1, \Delta_1) \geq \delta(X_1 \times X_2, p_1^*\Delta_1 + p_2^*\Delta_2).$$

The proof is finished.

\[\square\]

**Proof of Theorem 1.2.** The proof is just a combination of Theorem 3.1 and Theorem 3.2. \[\square\]

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