Maximal planar scale-free Sierpinski networks with small-world effect and power law strength-degree correlation

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Abstract – Many real networks share three generic properties: they are scale-free, display a small-world effect, and show a power law strength-degree correlation. In this paper, we propose a type of deterministically growing networks called Sierpinski networks, which are induced by the famous Sierpinski fractals and constructed in a simple iterative way. We derive analytical expressions for degree distribution, strength distribution, clustering coefficient, and strength-degree correlation, which agree well with the characterizations of various real-life networks. Moreover, we show that the introduced Sierpinski networks are maximal planar graphs.

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Introduction. – In the last few years, researches on complex networks have become a focus of attention for the scientific community [1–5]. One of the main reasons behind the popularity of complex networks is their flexibility and generality for representing real systems in nature and society. Researchers have done a lot of empirical studies, uncovering that various real-life networks sharing some generic properties: power law degree distribution [6], small-world effect including small average path length (APL) and high clustering coefficient [7]. Recently, many authors have described some real-world systems in terms of weighted-networks, where an interesting empirical phenomenon has been observed that there exists a power law scaling relation between the strength \(s\) and degree \(k\) of nodes, i.e. \(s \sim k^\beta\) with \(\beta > 1\) [8–11].

With the intention of studying the above properties of real-world systems, a wide variety of models has been proposed [1–4]. Watts and Strogatz, in their pioneering paper, introduced the famous small-world network model (WS model) [7], which exhibits small APL and high clustering coefficient. Another well-known model is Barabási and Albert’s scale-free network model (BA model) [6], which has a degree distribution of power law form. However, in these two elegant models, scale-free feature and high clustering are exclusive. Driven by the two seminal papers [6,7], a considerable number of other models have been developed that may represent processes more realistically taking place in real-world networks [12–18]. Very recently, Barrat, Barthélémy, and Vespignani have introduced a model (BBV) for the growth of weighted networks [19,20], which is the first weighted-network model that yields a scale-free behavior for strength and degree distributions. Enlightened by BBV’s remarkable work, various weighted-network models have been proposed to explain the properties found in real systems [21–27]. These models may give some insight into the realities. Particularly, some of them present all the three above-mentioned characteristics such as power law degree distribution, small-world effect, and power law strength-degree correlation [21–23]. Although great progresses have been made in the research of network topology, modeling complex networks with general structural properties is still of current interest.

On the other hand, fractals are an important tool for the investigation of physical phenomena [28]. They were used to describe physical characteristics of things in nature and life systems such as clouds, trees, mountains, rivers, coastlines, waves on a lake, bronchi, and the human circulatory system, to mention but a few. A vast literature...
Fig. 1: The first two stages in the construction of two fractals from the family of Sierpinski gaskets.

Fig. 2: Illustration of a deterministic network in relation to $SG_3(2)$.

on the theory and application of fractals has appeared. Among many deterministic and statistical fractals, the Sierpinski gasket [29,30] is one of the earliest deterministic fractals; it has provided a rich source for examples of fractal behavior [28,31,32]. Our initial physical motivation for this work lies in the use of the Sierpinski gasket as models for complex networks.

In this letter, based on the well-known Sierpinski family fractals, we introduce a class of deterministic networks, named Sierpinski networks. We propose a minimal iterative algorithm for constructing the networks and studying their structural properties. The networks are maximal planar graphs, show scale-free distributions of degree and strength, exhibit small-world effect, and display power law strength-degree correlation, which may provide valuable insights into the real-life systems.

The network derived from Sierpinski gasket. – We first introduce a family of fractals, called Sierpinski fractals, by generalizing the construction of the Sierpinski gasket. The classical Sierpinski gasket, shown in fig. 1(a), is constructed as follows [29]. We start with an equilateral triangle, and we denote this initial configuration by generation $t = 0$. Then in the first generation $t = 1$, the three sides of the equilateral triangle are bisected and the central triangle removed. This forms three copies of the original triangle, and the procedure is repeated indefinitely for all the new copies. In the limit of infinite $t$ generations, we obtain the well-known Sierpinski gasket denoted by $SG_2(t)$. Another fractal based on the equilateral triangle can be obtained if we perform a trisection of its sides and remove the three resultant down-pointing triangles, six copies of it are formed. When building the network, it is equivalent that for every group of three new added nodes, six new small equilateral triangles are generated, each of which may create three nodes in the next generation. According to this, we can introduce a general algorithm to create the corresponding network, denoted by $F(t)$ after $t$ generation evolutions.

The iterative algorithm for the maximal planar network. – In the construction process of $SG_3(t)$, for each equilateral triangle at arbitrary generation, once we perform a trisection of its sides and remove the three resultant down-pointing triangles, six copies of it are formed. When building the network, it is equivalent that for every group of three new added nodes, six new small equilateral triangles are generated, each of which may create three nodes in the next generation. According to this, we can introduce a general algorithm to create the corresponding network, denoted by $F(t)$ after $t$ generation evolutions.

The iterative algorithm for the network is as follows: For $t = 0$, $F(0)$ consists of three nodes forming a triangle. Then, we add three nodes into the original triangle. These three new nodes are linked to each other shaping a new triangle, and both ends of each edge of the new triangle are connected to a node of the original triangle. Thus we get $F(1)$, see fig. 3. For $t \geq 1$, $F(t)$ is obtained from $F(t-1)$. For each of the existing triangles of $F(t-1)$ that is not composed of three simultaneously emerging nodes and has never generated a node before, we call it an active triangle. We replace each of the existing active triangles of $F(t-1)$ by the connected cluster on the right hand of fig. 3 to obtain $F(t)$. The growing process is repeated until the network reaches a desired order (node number of network).
Next we compute the order and size (number of all edges) of the network \( F(t) \). Let \( L_v(t) \), \( L_e(t) \) and \( L_\Delta(t) \) be the number of vertices, edges and active triangles created at step \( t \), respectively. By construction (see also fig. 3), each active triangle in \( F(t-1) \) will be replaced by six active triangles in \( F(t) \). Thus, it is not difficult to find the following relation: \( L_\Delta(t) = 6L_\Delta(t-1) \). Since \( L_\Delta(0) = 1 \), we have \( L_\Delta(t) = 6^t \).

Note that each active triangle in \( F(t-1) \) will lead to an addition of three new nodes and nine new edges at step \( t \), then one can easily obtain the following relations: \( L_v(t) = 3L_\Delta(t-1) = 3 \cdot 6^{t-1} \), and \( L_e(t) = 9L_\Delta(t-1) = 9 \cdot 6^{t-1} \) for arbitrary \( t > 0 \). From these results, we can compute the order and size of the network. The total number of vertices \( N_t \) and edges \( E_t \) present at step \( t \) is

\[
N_t = \sum_{t_1=0}^{t} L_v(t_1) = 3 \cdot 6^t + 12
\]

and

\[
E_t = \sum_{t_1=0}^{t} L_e(t_1) = \frac{9 \cdot 6^t + 6}{5},
\]

respectively. So for large \( t \), the average degree \( \overline{k}_t = \frac{2E_t}{N_t} \) is approximately 6, which shows the network is sparse as most real systems.

From eqs. (1) and (2), we have \( E_t = 3N_t - 6 \). In addition, by the very construction of the network, it is obvious that arbitrary two edges in the network never cross each other. Thus our network is a maximal planar network (or graph) [35], which is similar to some previously studied networks [36–39].

### Relevant characteristics of the network.

Now we study the statistical properties of the network, in terms of degree distribution, clustering coefficient, average path length, and strength distribution.

**Degree distribution.** When a new node \( i \) is added to the graph at step \( t_1 \), it has a degree of 4. Let \( L_\Delta(i,t) \) be the number of active triangles at step \( t \) that will create new nodes connected to node \( i \) at step \( t + 1 \). Then at step \( t_1 \), \( L_\Delta(i,t_1) = 3 \). From the iterative generation process of the network, one can see that at any step each two new neighbors of \( i \) generate three new active triangles involving \( i \), and one of its existing active triangle is deactivated simultaneously. We define \( k_i(t) \) as the degree of node \( i \) at time \( t \), then the relation between \( k_i(t) \) and \( L_\Delta(i,t) \) satisfies

\[
L_\Delta(i,t) = k_i(t) - 1.
\]

Now we compute \( L_\Delta(i,t) \). By construction, \( L_\Delta(i,t) = 3L_\Delta(i,t-1) \). Considering the initial condition \( L_\Delta(i,t_1) = 3 \), we can derive \( L_\Delta(i,t) = 3^t + 1 \). Then at time \( t \), the degree of vertex \( i \) becomes

\[
k_i(t) = 3^{t-1} + 1.
\]

It should be mentioned that the initial three vertices created at step 0 have a little different evolution process from other ones. We can easily obtain: \( L_\Delta(0,t) = 3^t \) and \( k_i(t) = 3^t + 1 \). So at step \( t \), the initial three vertices have the same degrees as those born at step 1.

Equation (4) shows that the degree spectrum of the network is discrete. It follows that the cumulative degree distribution [3] is given by

\[
P_{\text{cum}}(k) = \sum_{\tau \leq k} \frac{L_v(\tau)}{N_t} = \frac{3 \cdot 6^t + 12}{3 \cdot 6^t + 12}.
\]

Substituting for \( t_1 \) in this expression using \( t_1 = t + 1 - \frac{\ln(1-1)}{\ln 6} \) gives

\[
P_{\text{cum}}(k) = \frac{18 \cdot 6^t \cdot (k-1)^{-\left(\ln 6/\ln 3\right)} + 12}{3 \cdot 6^t + 12}.
\]

When \( t \) is large enough, one can obtain

\[
P_{\text{cum}}(k) = 6 \cdot (k-1)^{-\left[1 + (\ln 2/\ln 3)\right]}.
\]

So the degree distribution follows a power law form with the exponent \( \gamma_k = 2 + \frac{\ln 2}{\ln 3} \).

**Clustering coefficient.** The clustering coefficient [7] \( C_i \) of node \( i \) is defined as the ratio between the number of edges \( e_i \) that actually exist among the \( k_i \) neighbors of node \( i \) and its maximum possible value, \( k_i(k_i-1)/2 \), i.e., \( C_i = 2e_i/[k_i(k_i-1)] \). The clustering coefficient of the whole network is the average of \( C_i \)’s over all nodes in the network.

For our network, the analytical expression of clustering coefficient \( C(k) \) for a single node with degree \( k \) can be derived exactly. When a node enters the system, both \( k_i \) and \( e_i \) are 4. In the following iterations, each of its active triangles increases both \( k_i \) and \( e_i \) by 2 and 3, respectively. Thus, \( e_i \) equals \( 4 + \frac{3}{2}(k_i - 4) \) for all nodes at all steps. So one can see that there exists a one-to-one correspondence between the degree of a node and its clustering. For a node of degree \( k \), we have

\[
C(k) = \frac{2e}{k(k-1)} = \frac{2[4 + \frac{3}{2}(k-4)]}{k(k-1)} = \frac{4}{k} - \frac{1}{k-1}.
\]

In the limit of large \( k \), \( C(k) \) is inversely proportional to degree \( k \). The same scaling of \( C(k) \sim k^{-1} \) has also been observed in several real-life networks [40].
Using eq. (8), we can obtain the clustering $C_t$ of the network at step $t$:

$$C_t = \sum_{r=0}^{t} \left[ \frac{L_r(r)}{N_t} \left( \frac{4}{D_r} - \frac{1}{D_r - 1} \right) \right],$$  

(9)

where the sum runs over all the nodes and $D_r$ is the degree of the nodes created at step $r$, which is given by eq. (4). In the infinite network order limit ($N_t \to \infty$), eq. (9) converges to a nonzero value $C = 0.598$, as shown in fig. 4. Therefore, the average clustering coefficient of the network is very high.

**Average path length.** Shortest paths play an important role both in the transport and communication within a network and in the characterization of the internal structure of the network. We represent all the shortest path lengths of $F(t)$ as a matrix in which the entry $d_{ij}$ is the geodesic path from node $i$ to node $j$, where geodesic path is one of the paths connecting two nodes with minimum length. The maximum value of $d_{ij}$ is called the diameter of the network. A measure of the typical separation between two nodes in $F(t)$ is given by the average path length $d_t$, also known as characteristic path length, defined as the mean of geodesic lengths over all couples of nodes.

In fig. 5, we report the dependence relation of APL $d_t$ on network size $N_t$. From fig. 5, one can see that the average path length grows logarithmically with increasing order of the network. This logarithmic scaling of $d_t$ with network order $N_t$, together with the large clustering coefficient obtained in the preceding subsection, shows that the considered graph has a small-world effect.

**Strength distribution and strength-degree correlation.** Strength usually represents resources or substances allocated to each node, such as wealth of individuals in financial contact networks [41], the number of passengers in airports in world-wide airport networks [42], the throughput of power stations in electric power grids [43], and so on. In our model, the strength of a node is defined as the area of the removed triangle it corresponds to [44]. For uniformity, let the initial three nodes born at step 0 have the same strength as those created at step 1.

We assume that the area of the initial equilateral triangle of the Sierpinski gasket is $A$. By the very construction of the network, all simultaneously emerging nodes have the same strength, because their corresponding triangles have identical area. It is easy to find that each removed triangle covers the portion $\mu = \frac{1}{9}$ of one removed triangle in the preceding generation. After $t$ iterations, all nodes which are generated at a certain step $t_i = 1, 2, \ldots, t$ have the strength $s_i(t)$:

$$s_i(t) = \mu^{t_i} \cdot A,$$

(10)

from which we have

$$s_i(t) = \mu^{t_i - t} \cdot \triangle,$$

(11)

where $\triangle$ is the area $s_i(t)$ of a triangle removed at step $t$.

Equations (4) and (11) yield a power law correlation between strength and degree of a node:

$$s_i(t) = \mu \cdot (k_i - 1)^{3 \ln(1/\mu)/\ln 3} = \frac{\Delta}{9} (k_i - 1)^2,$$

(12)

which implies when $k$ is large enough, $s \sim k^2$. This non-trivial power law scaling between strength of a node and its degree has been empirically observed in a variety of real networks, such as the airport networks, the shareholder networks, and the Internet.

Analogously to the computation of the degree distribution, one can find that the strength distribution is also scale-free with the exponent $\gamma_s$ as

$$\gamma_s = \frac{3}{2} + \frac{\ln 2}{2 \ln 3}.$$

(13)

As known to us all, for weighted networks with non-linear strength-degree correlation $s \sim k^\beta$, if their distributions of degree and strength behave as power laws,
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\[ P(k) \sim k^{-\gamma_k} \quad \text{and} \quad P(s) \sim s^{-\gamma_s}, \]

then there is a general relation between \( \gamma_k \) and \( \gamma_s \) as \([19,20]\)

\[ \gamma_s = \frac{\gamma_k}{\beta} + \frac{\beta - 1}{\beta}. \quad (14) \]

We have shown that in our model \( \gamma_k = 2 + \frac{\ln 2}{m^3} \) and \( \beta = 2 \). According to eq. (14), the exponent of strength distribution is \( \gamma_s = \frac{2}{\beta} + \frac{\ln 2}{m^3} \), giving the same \( \gamma_s \) value as obtained in the direct calculation of the strength distribution, see eq. (13).

**Conclusion.**– Deterministic model makes it easier to gain a visual understanding of how do different nodes relate to each other forming complex networks [45–50]. On the basis of Sierpinski fractals, we have proposed and studied a kind of deterministic networks. According to the network construction processes we have presented an algorithm to generate the networks, based on which we have obtained the analytical results for degree distribution, clustering coefficient, strength distribution, as well as strength-degree correlation. We have shown that the networks have three important properties: power law degree correlations, small-world effect, and power law strength-degree correlation. Moreover, using the idea presented, one can also establish random networks, which display similar features as their deterministic counterparts studied here. As the classical Sierpinski gaskets are important for the understanding of geometrical fractals in real systems, we believe that our research could be useful in the understanding and modeling of real-world networks.

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