Reply to Comment [arXiv:1708.09341] by Birse and McGovern on “Nucleon spin-averaged forward virtual Compton tensor at large $Q^2$”

RICHARD J. HILL\textsuperscript{(a)} AND GIL PAZ\textsuperscript{(b)}

\textsuperscript{(a)} Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA, and
Fermilab, Batavia, IL 60510, USA

\textsuperscript{(b)} Department of Physics and Astronomy
Wayne State University, Detroit, Michigan 48201, USA

Abstract

We reply to misleading claims made in a comment to our work by Birse and McGovern in arXiv:1708.09341.
1 Review

In Ref. [1] we recently computed, for the first time, the leading $1/Q^2$ behavior at large $Q^2$ of the quantity $W_1(0, Q^2)$ appearing in the spin-averaged forward virtual Compton tensor. This function plays an important role in the two-photon exchange contribution to the muonic hydrogen Lamb shift. This contribution provides the dominant theoretical uncertainty in the determination of the proton charge radius, and of the Rydberg constant, one of our most precisely known fundamental constants.

The OPE evaluation of $W_1(0, Q^2)$ has a long history, and our work corrects the decades old result of Collins [2]. In that work, an incorrect quark charge factor led to an overestimate of the spin-0 contribution by an order of magnitude. In reality, the leading OPE evaluation of $W_1(0, Q^2)$ is dominated by the spin-2 contribution, which was not considered in Ref. [2].

With a correct determination of $W_1(0, Q^2)$ at large $Q^2$ in hand, we ended our paper with a discussion of the implications for the muonic hydrogen Lamb shift. As summarized by Fig. 9 in Ref. [1], we presented two results: the first, based on Fig. 8 in Ref. [1], was an interpolation using the OPE evaluation at large $Q^2$ and only the constraints [3] of NRQED at low $Q^2$. The second result, based on Fig. 11 in Ref. [1], was an interpolation using the same OPE evaluation at large $Q^2$, and including chiral lagrangian constraints (in fact, the evaluation of Birse and McGovern in Ref. [4]) to extend the low $Q^2$ range.

2 Remarks

In an arXiv posting, arXiv:1708.09341, Birse and McGovern (BM) make several misleading claims related to our work. We reply with the following remarks.

(i) In a preface to their discussion, BM claim that the OPE computation [1] validates their computation [4]. It is difficult to interpret this statement. In fact, the central value for the coefficient of $1/Q^2$ that was deduced in Ref. [4] differs by a large factor, $\sim 3 - 4$, from the correct result, as BM themselves admit. The disagreement can be mitigated by accounting for uncertainties in the treatment of higher order chiral corrections; surprisingly, the authors later criticize precisely this method for estimating uncertainties in the evaluation of the Lamb shift (see point (iv) below). Regardless of the numerical comparison, the claimed relation between the low-energy quantity proportional to $M_4^{\beta/\beta}$ of Ref. [4], and the perturbative OPE expression, given by quark and gluon matrix elements, has not been justified to be a valid QCD relation.

(ii) The bulk of the analysis in the comment by BM concerns a spurious “apples to oranges” comparison of our determination in Fig. 9 that used only NRQED without chiral lagrangian constraints, to their determination that used chiral lagrangian constraints. If they had instead employed an “apples to apples” comparison of our determination in Fig. 9 that used chiral lagrangian constraints from Fig. 11, then agreement is obtained at a level consistent with remaining analysis differences. This can be seen from Fig. 9 itself comparing “[3]” (Ref. [4]) and “Fig. 11”.

1
(iii) In their comment, BM also discuss an alternative utilization of our OPE result at large $Q^2$, that would use only the NRQED expansion at small $Q^2$. By first subtracting a combination of elastic form factors whose Taylor expansion cancels the dominant part of the $\mathcal{O}(Q^2)$ contributions to $W_1(0, Q^2)$, the resulting interpolation is argued to have smaller uncertainty. We point out however that the uncertainties arising from moments of the elastic form factors, such as $r_M$, are not avoided by this procedure, but rather shuffled into a different part of the calculation.

Let us remark on the subtraction of a term involving elastic form factors from the total $W_1(0, Q^2)$. When constrained by experimental data, the Lamb shift contribution from any such term has a controlled uncertainty, owing to the existence of elastic form factor data throughout the entire $Q^2$ range of interest. For example, while the magnetic radius may induce a large uncertainty in the small-$Q^2$ Taylor expansion of $W_1(0, Q^2)_{\text{SIFF}}$, this uncertainty remains controlled when applied to the interpolation for the Lamb shift involving complete form factors.

However, there is not a unique combination of elastic form factors whose Taylor expansion cancels a chosen part of the $\mathcal{O}(Q^2)$ expansion of $W_1(0, Q^2)$. This ambiguity manifests itself in the “Born subtraction" performed by BM in Ref. [4], using third order expressions for the elastic form factors from Ref. [5]. At any given order in the chiral expansion (e.g. third order), different Born subtractions with the above-mentioned properties may be constructed, which differ only by higher order terms than have been calculated. Choosing one amongst these possibilities amounts to an implicit scheme or model dependence. Phrased equivalently, the apparent uncertainty reduction in $W_1(0, Q^2) - W_1(0, Q^2)^{\text{Born}}$ is accompanied by an enlarged uncertainty in $W_1(0, Q^2)^{\text{Born}}$.

(iv) Beyond the issue of Born subtraction, uncertainties from inputs and formalism should be accounted for in the computation of $W_1(0, Q^2) - W_1(0, Q^2)^{\text{SIFF}}$ using chiral lagrangian analysis. We did not attempt a formal quantification of these uncertainties in the appendix of Ref. [1], but simply displayed a comparison between different orders in chiral power counting. Although BM appear to take issue with this identification of uncertainty, taking the difference between successive orders in any power counting scheme is a standard method for error estimation.

References

[1] R. J. Hill and G. Paz, Phys. Rev. D 95, no. 9, 094017 (2017).

1 We remark that the word “Born” used in this context in the literature refers to a conventional subtraction. It does not have the usual technical meaning, i.e., as the leading expression for a physical quantity obtained by expressing the Hamiltonian in terms of an unperturbed problem plus a perturbation. The label “proton pole” or “elastic” used in the literature is also conventional: the very need for a subtraction implies that $\text{Im} W_1(\nu, Q^2)$ does not uniquely determine a contribution from proton intermediate states in the dispersion relation for $W_1(\nu, Q^2)$. The most common convention coincides with a form factor insertion ansatz, for which we use the more descriptive Sticking In Form Factors (SIFF) label.
[2] J. C. Collins, Nucl. Phys. B 149, 90 (1979) Erratum: [Nucl. Phys. B 153, 546 (1979)] Erratum: [Nucl. Phys. B 915, 392 (2017)].

[3] R. J. Hill and G. Paz, Phys. Rev. Lett. 107, 160402 (2011).

[4] M. C. Birse and J. A. McGovern, Eur. Phys. J. A 48, 120 (2012).

[5] V. Bernard, N. Kaiser and U. G. Meissner, Int. J. Mod. Phys. E 4, 193 (1995).