ON THE KNOT FLOER FILTRATION OF THE CONCORDANCE GROUP

STEPHEN HANCOCK, JENNIFER HOM, AND MICHAEL NEWMAN

Abstract. The knot Floer complex together with the associated concordance invariant $\varepsilon$ can be used to define a filtration on the smooth concordance group. We show that the indexing set of this filtration contains $\mathbb{N} \times \mathbb{Z}$ as an ordered subset.

1. Introduction

Two knots in $S^3$ are called concordant if they cobound a smooth, properly embedded cylinder in $S^3 \times [0,1]$. The set of knots in $S^3$, modulo concordance, forms an abelian group, the concordance group, denoted $\mathcal{C}$, where the operation is induced by connected sum. If a knot is concordant to the unknot, then we say that it is slice. The inverse of a knot $K$ is given by $-K$, the reverse of the mirror of $K$. It is straightforward to show that $K_1$ and $K_2$ are concordant if and only if $K_1 \# -K_2$ is slice.

A powerful tool for understanding knots is the knot Floer complex, defined by Ozsváth and Szabó [OS04], and independently Rasmussen [Ras03]. To a knot $K$, they associate a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex, denoted $\mathcal{C}F(K)$, whose filtered chain homotopy type is an invariant of $K$. Associated to the complex $\mathcal{C}F(K)$ is a $\{-1,0,1\}$-valued concordance invariant $\varepsilon(K)$ defined in [Hom11].

The set of such filtered chain complexes forms a monoid under the operation of tensor product, and modulo an equivalence relation defined in terms of $\varepsilon$, this monoid can be made into a group, denoted $\mathcal{F}$.

The advantage of this approach is that there is a homomorphism from

$$\mathcal{C} \to \mathcal{F},$$

defined by $[K] \mapsto [\mathcal{C}F(K)]$. Moreover, the group $\mathcal{F}$ has a rich algebraic structure coming from a total ordering. This ordering gives a filtration on $\mathcal{F}$ that can be pulled back to a filtration on $\mathcal{C}$, called the knot Floer filtration. While the indexing set of the knot Floer filtration is largely unknown, our main theorem gives a lower bound on the complexity of this indexing set.

Theorem 1. The indexing set of the knot Floer filtration contains a subset that is order isomorphic to $\mathbb{N} \times \mathbb{Z}$. Specifically, we can index the filtration by

$$S = \{(i,j) \mid (i,j) \geq (0,0)\},$$

where $S \subset \mathbb{N} \times \mathbb{Z}$ inherits the lexicographical ordering. Furthermore, each successive quotient is infinite, i.e., for $(i,j), (i',j') \in S$ and $(i,j) < (i',j')$, we have that

$$\mathbb{Z} \subset \mathcal{F}_{(i',j')}/\mathcal{F}_{(i,j)}.$$

Filtrations have been shown to be an effective tool for studying the concordance group. For example, Cochran, Orr, and Teichner [COT03] define the $n$-solvable filtration

$$\cdots \subset \mathcal{F}_{-(n+1)} \subset \mathcal{F}_{-n} \subset \mathcal{F}_{-n} \subset \cdots \subset \mathcal{F}_{-1} \subset \mathcal{F}_{-0.5} \subset \mathcal{F}_0 \subset \mathcal{C},$$

a filtration indexed by negative half integers. (Note that we adopt the convention that an indexing set for a filtration $\mathcal{F}$ is an ordered set $S$ with the property that for $a,b \in S$, $a < b$ implies that
It was shown by Cochran, Harvey, and Leidy [CHL09] that the quotient $\mathcal{F}_{-n}/\mathcal{F}_{-n,5}$ is of infinite rank for each non-negative integer $n$. Recent work of Cochran, Harvey, and Horn [CHH12] defines the bipolar filtration, again indexed by the negative natural numbers, and they also show that each successive quotient is of infinite rank. Our approach to filtering the concordance group utilizes a finer indexing set than the filtrations of [COT03] and [CHH12].

The proof of our result requires the computation of a large family of knot Floer complexes, modulo “$\varepsilon$-equivalence”. While computing the knot Floer complex in general is difficult, we use two properties of knot Floer homology that give us a large class of knots for which the computation simplifies drastically.

Recall that an $L$-space is a rational homology $S^3$ for which $\text{rk} \hat{HF}(Y) = |H_1(Y; \mathbb{Z})|$, so named because this class of 3-manifold includes lens spaces. The first property that we use pertains to a family of knots called $L$-space knots, that is, knots which admit a positive $L$-space surgery. It is well-known that positive torus knots admit positive lens space surgeries and thus are $L$-space knots. Ozsváth and Szabó [OS05, Theorem 1.2] show that the knot Floer complex of an $L$-space knot $K$ is completely determined by the Alexander polynomial of $K$. Moreover, Hedden [Hed09] proves that sufficiently large cables of $L$-space knots are again $L$-space knots. Thus, to understand the knot Floer complexes of torus knots and appropriate cables, it is sufficient to know the knot’s Alexander polynomial. It is well-known that the Alexander polynomial of $T_{p,q}$, the $(p,q)$-torus knot, is

$$\Delta_{T_{p,q}}(t) = \frac{(tpq - 1)(t - 1)}{(tp - 1)(tq - 1)},$$

and that the Alexander polynomial of $K_{p,q}$, the $(p,q)$-cable of $K$, is

$$\Delta_{K_{p,q}}(t) = \Delta_K(t^p) \cdot \Delta_{T_{p,q}}(t),$$

where $p$ denotes the longitudinal winding and $q$ the meridional winding.

The second useful property concerns the behavior of these invariants under basic topological operations. Let $CFK^\infty(K)^*$ denote the dual of $CFK^\infty(K)$; we give the precise definition of the dual complex in Section 2. Ozsváth and Szabó [OS04] show that

$$CFK^\infty(K_1 \# K_2) \simeq CFK^\infty(K_1) \otimes CFK^\infty(K_2)$$

and that

$$CFK^\infty(-K) \simeq CFK^\infty(K)^*,$$

allowing us to compute $CFK^\infty$ for linear combinations of $L$-space knots and their inverses.

With these techniques, we are able to realize a large class of $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complexes (up to $\varepsilon$-equivalence), and by studying the structure of the group $\mathcal{F}$, we can understand where in the filtration these knots lie.

The order type in Theorem 1 is almost certainly not a complete description of the indexing set of $\mathcal{F}$. One reason for this is that we limited ourselves to linear combinations of a small class of $L$-space knots for computational reasons. Moreover, to achieve our result, we needed only to consider connected sums of at most two $L$-space knots, and the $L$-space knots in question were always cables of torus knots. Further work suggests that with linear combinations of iterated torus knots, a richer order type is possible. An interesting question to consider is whether linear combinations of non-$L$-space knots would further enlarge the order type.

The results of [Hom11] defined various numerical concordance invariants associated to $CFK^\infty(K)$ that, in a sense, are a refinement of the Ozsváth-Szabó $\tau$ invariant [OS03]. This paper studies such invariants in more depth, giving a better understanding of the relationship between these invariants and the structure of the concordance group.
Organization. We begin in Section 2 with the necessary background on knot Floer homology, totally ordered groups, and $L$-space knots, including definitions of the invariant $\varepsilon$ and the group $F$. We proceed to prove algebraic results about $F$ (Sections 3 and 4) and the existence of a certain family of elements in $F$ (Section 5) through direct computation. In Section 6, we find knots that allow us to apply our preceding lemmas to understand the order type of $F$, which leads to the proof of Theorem 1. We work with coefficients in $F = \mathbb{Z}/2\mathbb{Z}$ throughout.

Acknowledgments. The ideas for this paper began during the Summer 2011 Topology REU at Columbia University, which was partially funded by NSF grant DMS-0739392. The second author was partially supported by NSF grant DMS-1307879. The authors would like to thank the referee for many helpful suggestions.

2. Background

2.1. The knot Floer complex and concordance. We begin with the necessary background on knot Floer homology, as defined in [OS04] and [Ras03]. To a knot $K$, we associate a $\mathbb{Z} \oplus \mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{F}[U, U^{-1}]$, where $U$ is a formal variable. The $\mathbb{Z}$-grading is called the Maslov, or homological, grading. We denote this complex by $CF^\infty(K)$, and the filtered chain homotopy type of $CF^\infty(K)$ is an invariant of the knot $K$. The ordering on $\mathbb{Z} \oplus \mathbb{Z}$ is given by $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$.

The differential, $\partial$, decreases the homological grading by one and respects the $\mathbb{Z}$-grading.

Multiplication by $U$ shifts the $\mathbb{Z}$-grading by two and decreases the $\mathbb{Z} \oplus \mathbb{Z}$-filtration by $(1, 1)$. Connected sum of knots corresponds to tensor product of their respective chain complexes. That is,

$$CF^\infty(K_1 \# K_2) \simeq CF^\infty(K_1) \otimes_{\mathbb{F}[U, U^{-1}]} CF^\infty(K_2).$$

Taking the reverse of the mirror image of a knot corresponds to taking the dual of its knot Floer complex. That is,

$$CF^\infty(-K) \simeq CF^\infty(K)^*,$$

where $CF^\infty(K)^*$ denotes the dual of $CF^\infty(K)$, i.e., $\text{Hom}_{\mathbb{F}[U, U^{-1}]}(CF^\infty(K), \mathbb{F}[U, U^{-1}])$. The complex $CF^\infty(K)$ is filtered chain homotopic to the complex obtained by interchanging $i$ and $j$.

A basis $\{x_k\}$ over $\mathbb{F}[U, U^{-1}]$ for a filtered chain complex $C$ is a filtered basis if $\{U^n \cdot x_k \mid U^n \cdot x_k \in C_{i,j}, n \in \mathbb{Z}\}$ is a basis over $\mathbb{F}$ for the subcomplex $C_{i,j}$ for all $i, j \in \mathbb{Z}$, where $C_{i,j}$ denotes the $(i, j)$th-filtered subcomplex. In this paper, we will often perform a filtered change of basis, producing a new filtered basis from an old one. Given a filtered basis $\{x_k\}$, we can produce a new filtered basis $\{x'_k\}$, where

$$x'_k = \begin{cases} x_k + x_\ell & \text{if } k = n \\ x_k & \text{otherwise} \end{cases}$$

for some $n$ and $\ell$ such that the filtration level of $x_\ell$ is less than or equal to that of $x_n$. In other words, one may replace a basis element with itself plus elements of lesser or equal filtration level. We will often omit the prime from the new basis and denote this change of basis by

$$x_n \to x_n + x_\ell, \quad x_k \to x_k, \quad k \neq n.$$
the differential respects the $\mathbb{Z} \oplus \mathbb{Z}$-filtration, the arrows will necessarily point (non-strictly) to the left and down. Up to filtered chain homotopy, one may assume that the differential will strictly decrease the filtration [Ras03, Lemma 4.5], and indeed, that will be the case for all of the complexes we consider. Moreover, for all of the complexes under consideration in this paper, there exists a basis where each arrow connects elements of the same $U$-degree. This is because the knots in this paper are all linear combinations of $L$-space knots, which are described in Section 2.3. Thus, it is sufficient to consider a single copy of each generator, rather than all of the $U$-translates. At times, it will be convenient to consider only the part of $\partial$ that preserves the $j$- or $i$-filtration level. We use $\partial^{\text{horz}}$ and $\partial^{\text{vert}}$, respectively, to denote these.

The subquotient of $CFK^\infty(K)$ consisting of the $i = 0$ column yields the complex $\widehat{CF}(S^3)$, and so the homology of the $i = 0$ column (or in fact, any column, up to a grading shift) is isomorphic to $\mathbb{F}$. Similarly, the homology of any row is also isomorphic to $\mathbb{F}$.

The picture for $CFK^\infty(K)^*$ is closely related to the picture for $CFK^\infty(K)$; one simply reverses the direction of each arrow, as well as both filtrations. (In practice, this may be accomplished by turning the page upside down and reversing the directions of all of the arrows.)

A basis $\{x_i\}$ over $\mathbb{F}[U, U^{-1}]$ for $CFK^\infty(K)$ is called vertically simplified if for each basis element $x_i$, exactly one of the following holds:

- There is a unique incoming vertical arrow into $x_i$.
- There is a unique outgoing vertical arrow from $x_i$.
- There are no vertical arrows entering or leaving $x_i$.

Note that since the homology of a column is $\mathbb{F}$, there is a unique basis element of a vertically simplified basis with no incoming or outgoing vertical arrows, called the vertically distinguished element. The analogous definition can be made for a horizontally simplified basis. By [LOT08, Proposition 11.52], one may always choose a basis which is vertically simplified, or if one prefers, horizontally simplified.

Given a vertically simplified basis, consider the subquotient complex associated to the $i = 0$ column. The $j$-coordinate of the vertically distinguished element in this column is a concordance invariant, defined by Ozsváth and Szabó in [OS03] and denoted $\tau(K)$.

While it remains unknown whether a simultaneously vertically and horizontally simplified basis always exists in general, we are able to find such a basis for the complexes under consideration in this paper. Moreover, one may always find a horizontally simplified basis where one of the basis elements, say $x_0$, is the distinguished element of some vertically simplified basis [Hom12, Lemmas 3.2 and 3.3]. The $\{-1, 0, 1\}$-valued concordance invariant $\epsilon$ can be defined in terms of such a basis.

**Definition 2.1.** The invariant $\epsilon(K)$ is defined in terms of the above basis for $CFK^\infty(K)$ as follows:

1. $\epsilon(K) = 1$ if there is a unique incoming horizontal arrow into $x_0$.
2. $\epsilon(K) = -1$ if there is a unique outgoing horizontal arrow from $x_0$.
3. $\epsilon(K) = 0$ if there are no horizontal arrows entering or leaving $x_0$.

To emphasize that $\epsilon$ is in fact an invariant of a bifiltered chain complex, we may at times write $\epsilon(CFK^\infty(K))$, rather than $\epsilon(K)$. Alternatively, the invariant $\epsilon$ can be defined in terms of the (non-)vanishing of certain cobordism maps on $\widehat{HF}$, as in [Hom11, Definition 3.1].

**Proposition 2.2 ([Hom12, Proposition 3.6]).** The following are properties of $\epsilon(K)$:

1. If $K$ is smoothly slice, then $\epsilon(K) = 0$.
2. $\epsilon(-K) = -\epsilon(K)$.
3. (a) If $\epsilon(K) = \epsilon(K')$, then $\epsilon(K \# K') = \epsilon(K) = \epsilon(K')$. 


(b) If \( \varepsilon(K) = 0 \), then \( \varepsilon(K \# K') = \varepsilon(K') \).

Notice that if \( K_1 \) and \( K_2 \) are concordant, then \( \varepsilon(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*) = 0 \), motivating the following definition.

**Definition 2.3.** Two bifiltered chain complexes \( C_1 \) and \( C_2 \) are \( \varepsilon \)-equivalent, denoted \( \sim \varepsilon \), if
\[
\varepsilon(C_1 \otimes C_2^*) = 0.
\]

Recall that the concordance group \( \mathcal{C} \) is obtained as a quotient of the monoid of knots under connected sum by the equivalence relation of concordance. In a similar manner, chain complexes under tensor product form a monoid, and using the idea of \( \varepsilon \)-equivalence, we can obtain a group.

Consider the monoid \((M, \otimes_{\mathbb{F}[U,U^{-1}]} )\) of bifiltered chain complexes up to filtered chain homotopy such that:
- The underlying module is a free \( \mathbb{F}[U, U^{-1}] \) module.
- The total homology of the complex is isomorphic to \( \mathbb{F}[U, U^{-1}] \).
- The vertical homology of the complex is isomorphic to \( \mathbb{F}[U, U^{-1}] \).
- The complex obtained by interchanging \( i \) and \( j \) is filtered chain homotopic to the original complex.

**Definition 2.4.** The group \( \mathcal{F}_{\text{alg}} \) is defined to be
\[
\mathcal{F}_{\text{alg}} = \left( M, \otimes \right)/\sim \varepsilon.
\]

We denote the group operation of \( \mathcal{F}_{\text{alg}} \) by + and the identity by 0. We may also consider the subgroup of \( \mathcal{F}_{\text{alg}} \) generated by complexes that are realized as \( CFK^\infty(K) \) for some knot \( K \subset S^3 \).

**Definition 2.5.** The group \( \mathcal{F} \) is
\[
\mathcal{F} = \left( \{CFK^\infty(K) \mid K \subset S^3 \}, \otimes \right)/\sim \varepsilon.
\]

Clearly \( \mathcal{F} \subseteq \mathcal{F}_{\text{alg}} \) since \( \{CFK^\infty(K) \} \subseteq M \). It is known that \( \{CFK^\infty(K) \} \neq M \), since there does not exist an \( L \)-space knot \( K \) with \( a_1(K) > 1 \) [Ras04, Theorem 2.3]; see Section 2.4 below for the definition of \( a_1 \). However, it is an open question whether \( \mathcal{F} = \mathcal{F}_{\text{alg}} \).

It is clear from the definition of \( \mathcal{F} \) and properties of \( CFK^\infty \) that we obtain a group homomorphism
\[
\mathcal{C} \to \mathcal{F}
\]
by sending \([K]\) to \([CFK^\infty(K)\] \]. Calling this map \( \phi \), notice that \( \mathcal{F} \cong \mathcal{C}/\ker \phi = \mathcal{C}/\{[K] \mid \varepsilon(K) = 0\} \).

For ease of notation, we write
\[
[K]
\]
to denote \([CFK^\infty(K)\] \]. Note that \([-K] = [K] \) and \([\text{unknot}] = 0\).

One of the advantages of this approach is that the group \( \mathcal{F} \) has a rich algebraic structure. In particular, \( \mathcal{F} \) is totally ordered, with the ordering given by
\[
[CFK^\infty(K_1)] > [CFK^\infty(K_2)] \iff \varepsilon(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*) = 1.
\]

By considering the behavior of \( \varepsilon \) under connected sum, it follows that this total ordering is well-defined.
2.2. Totally ordered groups. Two totally ordered sets $S_1$ and $S_2$ are order isomorphic if there exists a bijection $S_1 \to S_2$ such that both the bijection and its inverse are order-preserving. The order equivalence class of $S$ is called the order type of $S$.

Given a totally ordered abelian group $G$, one can naturally define a notion of absolute value, i.e., for any $g \in G$,

$$|g| = \begin{cases} g & \text{if } g \geq \text{id}_G \\ -g & \text{otherwise.} \end{cases}$$

Two elements $g$ and $h$ of a totally ordered abelian group $G$ are said to be Archimedean equivalent, denoted $\sim_{Ar}$, if there exist $m, n \in \mathbb{N}$ such that

$$m \cdot |g| > |h| \quad \text{and} \quad n \cdot |h| > |g|.$$ 

The set of Archimedean equivalence classes of $G$ inherits an ordering from the group, and the order type of this set is called the coarse order type of the group.

Let $[g]_{Ar}$ denote the Archimedean equivalence class of $g$. If $[h]_{Ar} < [g]_{Ar}$, then $n \cdot |h| < |g|$ for all $n \in \mathbb{N}$, and we write

$$|h| \ll |g|.$$ (If one restricts oneself to only positive elements in the group, then the absolute value signs may be omitted.) For positive $[K], [J] \in F$, note that $[K] \gg [J] \iff \varepsilon(K \# -nJ) = 1$ for all $n \in \mathbb{N}$.

A totally ordered group inherits a natural filtration, with the indexing set given by the coarse order type of the group. Given an Archimedean equivalence class, choose a representative $g$, and consider the subgroup

$$H_g = \{ h \in G \mid [h]_{Ar} \leq [g]_{Ar} \}.$$ 

Indeed, it follows from the definition of Archimedean equivalence that the set $H_g$ is closed under the group operation, and it is clear that if $h \in H_g$, then the inverse of $h$ is as well. The filtration now also follows from the definition of Archimedean equivalence, since $[g_1]_{Ar} < [g_2]_{Ar}$ implies that $H_{g_1} \subset H_{g_2}$ and that $\mathbb{Z} \subset H_{g_2}/H_{g_1}$, generated by $g_2$.

Applying these tools to the group $F$, we obtain a filtration on $F$, which we may pull back to give a filtration on $C$. The effectiveness of this approach is largely determined by the coarse order type of $F$. For instance, the second author showed [Hom11, Proposition 4.8] that the coarse order type of $F$ contains $\omega$ as an ordered subset, with $F_n := \phi^{-1}[H_{[T_n,n+1]}]$ giving a filtration on $C$ indexed by $\mathbb{N}$. This is precisely the reversed order type of the $n$-solvable and bipolar filtrations. Our goal is to achieve an indexing set with finer order type.

2.3. L-space knots. One of our main tools for computing the knot Floer complex of large families of knots concerns L-space knots. Recall that an L-space $Y$ is a rational homology sphere for which $\text{rk} \hat{H}^\infty(Y) = |H_1(Y; \mathbb{Z})|$, and that an L-space knot is a knot on which some positive surgery is an L-space. In [OS05, Theorem 1.2], Ozsváth and Szabó show that if a knot $K$ admits a positive L-space surgery, then its knot Floer complex is completely determined by the Alexander polynomial of $K$. In particular, if $K$ is an L-space knot, then the Alexander polynomial of $K$ is of the form

$$\Delta_K(t) = \sum_{i=0}^{2m} (-1)^i t^{n_i}$$

for a positive integer $m$ and some strictly increasing sequence of $n_i \in \mathbb{Z}_{\geq 0}$ satisfying the symmetry requirement that

$$n_i + n_{2m-i} = 2g(K),$$

where $g(K)$ is the genus of $K$ and we have normalized the Alexander polynomial to have a constant term and no negative exponents, i.e., $n_0 = 0$. 
The sequence of \( n_i \) determines the knot Floer complex of \( K \). A filtered basis over \( \mathbb{F}[U, U^{-1}] \) for \( CFK^\infty(K) \) is given by \( \{ x_i \} \), \( i = 0, \ldots, 2m \), with the following differentials:

\[
\partial x_i = \begin{cases} 
  x_{i-1} + x_{i+1} & \text{if } i \text{ odd} \\
  0 & \text{if } i \text{ even}, 
\end{cases}
\]

where the arrow from \( x_i \) to \( x_{i-1} \) is horizontal of length \( n_i - n_{i-1} \), and the arrow from \( x_i \) to \( x_{i+1} \) is vertical of length \( n_{i+1} - n_i \). See Figure 1 for an example.

**Figure 1.** A basis for \( CFK^\infty(T_{3,4}) \), where \( T_{3,4} \) denotes the \((3, 4)\)-torus knot. The Alexander polynomial of \( T_{3,4} \) is \( \Delta_{T_{3,4}}(t) = 1 - t + t^3 - t^5 + t^6 \).

In the next section, we introduce special notation for denoting certain families of bifiltered chain complexes, with the above result taking the form

\[(2.1) \quad [K] = [(n_i - n_{i-1})^m_{i=1}] \]

in that notation. Notice that in the \((i, j)\)-plane, this complex has the appearance of a “staircase”. (In fact, having such a staircase complex is a necessary and sufficient condition for being an \( L \)-space knot [OS05].) Such complexes will play a key role in this paper.

### 2.4. Notation for chain complex classes

In this section, we define some notation that will be useful to describe the complexes of interest.

**Definition 2.6.** A bifiltered complex \( C \) that represents an element in \( F_{\text{alg}} \) is of type \((a_1, \ldots, a_m)\) if it is doubly-filtered chain homotopy equivalent to a direct sum

\[ C_{\text{red}} \oplus C_A, \]

where \( C_A \) is acyclic, \( C_{\text{red}} \) has no acyclic summands, and \( C_{\text{red}} \) admits a simultaneously vertically and horizontally simplified basis \( \{ x_0, \ldots, x_{2m} \} \) with the following vertical and horizontal differentials:

\[
\partial_{\text{horz}} x_i = \begin{cases} 
  x_{i-1} & \text{if } a_i > 0 \\
  x_{i+1} & \text{if } a_i < 0 \\
  0 & \text{otherwise}
\end{cases}
\quad \text{and} \quad
\partial_{\text{vert}} x_i = \begin{cases} 
  x_{i+1} & \text{if } a_{i+1} > 0 \\
  x_{i-1} & \text{if } a_i < 0 \\
  0 & \text{otherwise},
\end{cases}
\]

where the arrow between \( x_i \) and \( x_{i-1} \) is of length \( |a_i| \) and \( a_i = a_{2m+1-i} \) for \( i = m + 1, \ldots, 2m \).

**Remark 2.7.** In the above definition, the differentials on \( C_{\text{red}} \) necessarily imply that \( C_A \) is acyclic and that \( C_{\text{red}} \) has no acyclic summands, because of the symmetry and rank properties of \( C \).
The sequence \( a_1, \ldots, a_{2m} \) indicates the signed lengths of the arrows encountered as we trace \( C_{\text{red}} \) from the vertically distinguished element to the horizontally distinguished element along horizontal and vertical arrows. For \( i \) odd, the arrow between \( x_i \) and \( x_{i-1} \) is horizontal, and \( a_i \) is positive if it is outgoing from \( x_i \), otherwise negative. For \( i \) even, the arrow between \( x_i \) and \( x_{i-1} \) is vertical, and \( a_i \) is positive if it is incoming to \( x_i \), otherwise negative. The sign convention is chosen such that \( L \)-space knots will have all positive \( a_i \). For example, the complex in Figure 1 is of type \((1, 2)\).

Note that if \( C \) is a representative of the type \((a_1, \ldots, a_m)\), denoted \( C \in (a_1, \ldots, a_m) \), then \( C \) must have at least \( 2m + 1 \) basis elements. If \( C \) has exactly \( 2m + 1 \) basis elements (i.e., \( C \simeq C_{\text{red}} \)), then we say that \( C \) is reduced. Given \( C_1 \in (a_1, \ldots, a_m) \) with basis \( \{x_i\}, i = 0, \ldots, 2m \), we know that \( C_1 \) must be the reduced representative, and we assume the \( x_i \) are labeled in the order described in Definition 2.6. Moreover, if \( C_2 \in (b_1, \ldots, b_n) \) with basis \( \{y_j\}, j = 0, \ldots, 2n \), then \( C_1 \otimes C_2 \) naturally has (unsimplified) basis \( \{x_i y_j\} \), where \( x_i y_j = x_i \otimes y_j \).

The complex \( C_1 \) is a staircase complex if all \( a_i \) are positive, or if all \( a_i \) are negative. By Lemma 2.8 below, a staircase complex is \( \varepsilon \)-equivalent to a complex with no diagonal arrows, namely the reduced representative of \((a_1, \ldots, a_m)\) with exactly \( 2m + 1 \) generators. If each \( a_i > 0 \), i.e., \( C_1 \) is a staircase complex, we have

\[
\partial x_i = \begin{cases} x_{i-1} + x_{i+1} & \text{if } i \text{ odd} \\ 0 & \text{if } i \text{ even} \end{cases} \quad \text{and} \quad \partial x_i^* = \begin{cases} x_{i+1}^* + x_{i-1}^* & \text{if } i \text{ even, } i \neq m \pm m \\ 0 & \text{if } i \text{ odd} \\ x_{i+1}^* & \text{if } i = m \pm m. \end{cases}
\]

Using \( \text{fl}(x_i) \) to denote the filtration level of \( x_i \), we have \( \text{fl}(x_0) = (0, \tau(C_1)) \). It is further clear that

\[
\text{fl}(x_i) = \begin{cases} \text{fl}(x_{i-1}) + (a_i, 0) & \text{if } i \text{ odd} \\ \text{fl}(x_{i-1}) - (0, a_i) & \text{if } i \text{ even} \end{cases} \quad \text{and} \quad \text{fl}(x_i^*) = -\text{fl}(x_i).
\]

By definition, the operation of tensor product on chain complexes gives us

\[
\partial(x_i y_j) = \partial(x_i) y_j + x_i \partial(y_j) \quad \text{and} \quad \text{fl}(x_i y_j) = \text{fl}(x_i) + \text{fl}(y_j).
\]

Notice \( \text{fl}(x_i^*) = - (0, \tau(C_1)) = (0, \tau(C_1^*)) \) and \( \text{fl}(x_0 y_0) = (0, \tau(C_1)) + (0, \tau(C_2)) = (0, \tau(C_1 \otimes C_2)) \).

The following lemma shows that the type of a complex determines an \( \varepsilon \)-equivalence class. That is, if \( C \) is a representative of the type \( T \) and the \( \varepsilon \)-equivalence class \([C]\), then \( T \subseteq [C] \).

**Lemma 2.8.** If two complexes \( C_1 \) and \( C_2 \) are both of type \((a_1, \ldots, a_m)\), then the complexes are \( \varepsilon \)-equivalent.

**Proof.** Without loss of generality, we may assume that \( a_1 \) is positive. Let \( \{x_0, \ldots, x_{2m}\} \) be a basis as in Definition 2.6 for the reduced summand of \( C_1 \) and similarly \( \{y_0, \ldots, y_{2m}\} \) a basis for the reduced summand of \( C_2^* \), where \( C_2^* \) denotes the dual of \( C_2 \); that is, if \( \{z_0, \ldots, z_{2m}\} \) is a basis for the reduced summand of \( C_2 \) as in Definition 2.6, then \( y_i = z_i^* \). In particular, there is a horizontal arrow of length \( a_1 \) from \( x_1 \) to \( x_0 \) and a horizontal arrow of length \( a_1 \) from \( y_0 \) to \( y_1 \). More generally, if there is a vertical (respectively horizontal) arrow from \( x_i \) to \( x_j \), then there is a vertical (respectively horizontal) arrow from \( y_j \) to \( y_i \). (Note the order of the subscripts on \( x \) and \( y \).)

We need to show that \( \varepsilon(C_1 \otimes C_2^*) = 0 \). In light of [Hom11, Definition 3.1], it is sufficient to show that there exists a class in \( C_1 \otimes C_2^* \) that is non-zero in the homology of \( C \{\max(i, j - \tau) = 0\} \) and \( C \{\min(i, j - \tau) = 0\} \). See [Hom11, Section 3] for the definition of these complexes. We claim that

\[
u = \sum_{i=0}^{2m} x_i y_i \]
is such an element. Note that each term in the above sum is in the same filtration level.

We first consider the horizontal homology. We will show that \( u \) is in the kernel of \( \partial_{\text{horz}} \). Suppose that \( x_jy_i \) appears in \( \partial_{\text{horz}}u \). Then either there is a horizontal arrow from \( x_i \) to \( x_j \) or from \( y_j \) to \( y_i \), and \( j = i \pm 1 \). But as noted above, there is a horizontal arrow from \( x_i \) to \( x_j \) exactly when there is a horizontal arrow from \( y_j \) to \( y_i \). In particular, \( x_jy_i \) appears in \( \partial_{\text{horz}}u \) exactly twice: once from \( \partial_{\text{horz}}x_iy_i \) and once from \( \partial_{\text{horz}}x_jy_j \). In this situation, we also have that \( \partial_{\text{horz}}x_iy_j = x_iy_i + x_jy_j \).

We now show that \( u \) is not in the image of \( \partial_{\text{horz}} \). From the last sentence of the preceding paragraph, we have that for each \( i = 0, \ldots, m - 1 \), the sum \( x_{2i}y_{2i} + x_{2i+1}y_{2i+1} \) is in the image of \( \partial_{\text{horz}} \). It follows that

\[
\sum_{i=0}^{2m-1} x_iy_i = u - x_{2m}y_{2m}
\]

is in the image of \( \partial_{\text{horz}} \). Since \( x_{2m} \) and \( y_{2m} \) were each the distinguished horizontal element of their respective bases, the element \( x_{2m}y_{2m} \) is not in the image of \( \partial_{\text{horz}} \), so neither is \( u \).

Similarly, it follows that \( u \) also generates the vertical homology (where the role of the element \( x_{2m}y_{2m} \) is now played by \( x_0y_0 \)). Moreover, similar arguments show that \( u \) is non-zero in \( H_*(C_{\{\max(i,j-\tau) = 0\}}) \) and \( H_*(C_{\{\min(i,j-\tau) = 0\}}) \). We conclude that \( \varepsilon(C_1 \otimes C_2^*) = 0 \), implying that \( C_1 \sim \varepsilon C_2 \).

It follows that for \( C \) of type \( (a_1, \ldots, a_m) \), we may denote the element \([C]\) of \( \mathcal{F}_{\text{alg}} \) by

\[
[a_1, \ldots, a_m].
\]

Note that \(-[a_1, \ldots, a_m] = [-a_1, \ldots, -a_m]\) and \([\ ] = 0\). We will sometimes use nested iterators to write our sequences. For instance, given sequences \((a_{i,j})_{j=1}^{n_i}\) indexed by \( j \) for \( i = 1, \ldots, m \), we can form the sequence

\[
((a_{i,j})_{j=1}^{n_i})_{i=1}^m = (a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}, \ldots, a_{m,1}, a_{m,2}, \ldots, a_{m,n_m}).
\]

We will also write \((a_j)_{j=1}^m\) to denote the sequence given by \( a_1, \ldots, a_n \) repeated \( m \) times, i.e.,

\[
((a_j)_{j=1}^m)_{i=1}^m = (a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n, \ldots, a_1, a_2, \ldots, a_n).
\]

Note that not every sequence of integers \( a_1, \ldots, a_m \) corresponds to an element \([a_1, \ldots, a_m] \) of \( \mathcal{F}_{\text{alg}} \). For example, \([1, -2]\) does not admit a chain complex representative, as there is no collection of diagonal arrows that makes \( \partial^2 = 0 \). See Figure 2 for two examples.

At times, it will be convenient to have some \( a_i \) equal to zero. To this end, we make the formal identifications:

- \([a_1, \ldots, a_m] = [a_1, \ldots, a_m, 0]\).
- \([a_1, \ldots, a_j, a_{j+1}, \ldots, a_m] = [a_1, \ldots, a_j - k, k, \ldots, a_m] \) for \( \min(0, a_j) \leq k \leq \max(0, a_j) \).

In particular, this allows (2.1) to remain valid even when the sequence of \( n_i \) defining \( \Delta_K(t) \) is only non-strictly increasing. It follows that given \([a_1, \ldots, a_m] \in \mathcal{F}_{\text{alg}} \) with \( a_i = 0 \), we may use \([\ldots, a_{i-1}, 0, a_{i+1}, \ldots] = [\ldots, a_{i-1} + a_{i+1}, \ldots] \) to remove the zero entry.

Finally, note that the concordance invariants \( \tau(K) \) of [OS03] and \( \varepsilon(K), a_1(K), \) and \( a_2(K) \) of [Hom11] are well-defined invariants of elements in \( \mathcal{F}_{\text{alg}} \). For \([C] = [a_1, \ldots, a_m] \in \mathcal{F}_{\text{alg}} \), we have

\[
\tau(C) = \sum_{i=1}^m a_i \quad \text{and} \quad \varepsilon(C) = \begin{cases} 
\text{sgn}(a_1) & \text{if } m > 0 \\
0 & \text{otherwise},
\end{cases}
\]
and if $\varepsilon(C) = 1$,

$$a_1(C) = a_1 \quad \text{and} \quad a_2(C) = \begin{cases} a_2 & \text{if } a_2 > 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

3. Tensor Products of Staircase Complexes

For the calculations that follow in Sections 3 and 4, we suppress the $U$-translates from this picture, which is always possible for the complexes under consideration here, given an appropriate choice of basis. That is, the complexes of interest are all of the form $C = C' \otimes_F \mathbb{F}[U, U^{-1}]$, where $C'$ is a doubly filtered, finitely-generated complex over $\mathbb{F}$. Thus $C_1 \otimes_{\mathbb{F}[U, U^{-1}]} C_2 = (C'_1 \otimes C'_2) \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}]$.

In this section, we prove two lemmas on the group operation of $\mathcal{F}_{\text{alg}}$. Our approach is to take the tensor product of two complexes, both reduced with $\varepsilon = 1$, then vertically and horizontally simplify the basis of the product to determine the reduced representative of its $\varepsilon$-equivalence class.

**Lemma 3.1.** Let $a_i, b_j > 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. If $m$ is even and $\max\{a_i \mid i \text{ odd}\} \leq b_j \leq \min\{a_i \mid i \text{ even}\}$, then

$$[a_1, a_2, \ldots, a_m] + [b_1, b_2, \ldots, b_n] = [a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n].$$

**Proof.** For $C_1 \in \langle a_1, \ldots, a_m \rangle$ with basis $\{x_i\}$, $i = 0, \ldots, 2m$, and $C_2 \in \langle b_1, \ldots, b_n \rangle$ with basis $\{y_j\}$, $j = 0, \ldots, 2n$, we prove that $C_1 \otimes C_2$ is of type $T = (a_1, \ldots, a_m, b_1, \ldots, b_n)$. Define the sets

$$S = \{x_iy_0 \mid i < m\} \cup \{x_my_j \mid 0 \leq j \leq 2n\} \cup \{x_iy_{2n} \mid i > m\}$$

$$B_{i,j} = \begin{cases} \{x_iy_j, x_{i-1}y_j, x_{i+1}y_j, x_{i-1}y_{j+1}\} & i, j \text{ odd, } i < m \\ \{x_iy_j, x_{i+1}y_j, x_{i-1}y_{j+1}, x_{i+1}y_{j-1}\} & i, j \text{ odd, } i > m. \end{cases}$$

Supposing $m$ is even, $\bigcup_{i \leq m, j} B_{i,j} = \{x_iy_j \mid i < m, j > 0\}$ and $\bigcup_{i > m, j} B_{i,j} = \{x_iy_j \mid i > m, j < 2n\}$. Therefore, $\{x_iy_j\} = S \cup \bigcup_{i \leq m, j} B_{i,j}$. Note that $S$ is the elements $\{x_iy_0 \mid i < m\} \cup \{x_my_j \mid j \leq n\}$ along with their reflection about the diagonal. Given the differentials on these elements,

$$\partial(x_iy_0) = \begin{cases} x_{i-1}y_0 + x_{i+1}y_0 & i \text{ odd} \\ 0 & i \text{ even} \end{cases} \quad \text{and} \quad \partial(x_my_j) = \begin{cases} x_my_{j-1} + x_my_{j+1} & j \text{ odd} \\ 0 & j \text{ even} \end{cases}$$

![Figure 2](image-url). Left, a basis for $CFK^\infty(5_2)$, showing $[CFK^\infty(5_2)] = [1]$. Right, a basis for a reduced representative of $[3, -1, -2, 2]$. 


and their filtration levels,
\[
\text{fl}(x_iy_0) = \begin{cases} 
\text{fl}(x_{i-1}y_0) + (a_i, 0) & \text{if odd} \\
\text{fl}(x_{i-1}y_0) - (0, a_i) & \text{if even}
\end{cases} \quad \text{and} \quad \text{fl}(x_my_j) = \begin{cases} 
\text{fl}(x_{m}y_{j-1}) + (b_j, 0) & \text{if odd} \\
\text{fl}(x_{m}y_{j-1}) - (0, b_j) & \text{if even}
\end{cases}
\]

it is clear that \( S \) spans a subcomplex isomorphic to the reduced representative of type \( T \). We show that each of the remaining \( mn \) sets of elements \( B_{i,j} \) forms an acyclic summand under a filtered change of basis. These summands look like boxes—see Figure 3 for an example. Since \( B_{i,j} \to B_{2m-i,2n-j} \) when \( x_iy_j \to x_{2m-i}y_{2n-j} \), the proofs for \( i < m \) and \( i > m \) are the same under the transformation \( x_iy_j \to x_{2m-i}y_{2n-j} \), \( a_i \to a_{2m-i} \), and \( b_j \to b_{2n-1-j} \) by the diagonal symmetry of \( C_1 \otimes C_2 \). It therefore suffices to redefine the basis for \( i = 0, \ldots, m-1 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{\( C_1 \otimes C_2 \) with, left, basis \( \{x_iy_j\} \) and, right, simplified basis \( S \cup B'_{1,1} \cup B'_{3,1} \), where \( C_1 \in (1,3) \) has basis \( \{x_0, x_1, x_2, x_3, x_4\} \) and \( C_2 \in (2) \) has basis \( \{y_0, y_1, y_2\} \).
\end{figure}

For all odd \( i < m \) and all odd \( j \), produce the new basis
\begin{equation}
(3.1) \quad x_{i-1}y_j = x_{i-1}y_j + x_iy_{j-1}, \quad x_{i}y_{j+1} = x_{i}y_{j+1} + x_{i+1}y_j, \quad x_{i-1}y_{j+1} = x_{i-1}y_{j+1} + x_{i+1}y_{j-1}.
\end{equation}

It follows that \( B_{i,j} \to B'_{i,j} \), where \( B'_{i,j} = \{x_iy_j, x_{i-1}y_j + x_iy_{j-1}, x_{i}y_{j+1} + x_{i+1}y_j, x_{i-1}y_{j+1} + x_{i+1}y_{j-1}\} \).

First, we check that our chosen basis respects the filtration. By \((2.3, 2.5)\) with \( i \) and \( j \) odd,
\begin{align*}
\text{fl}(x_{i-1}y_j) &= \text{fl}(x_i) - (a_i, 0) + \text{fl}(y_{j-1}) + (b_j, 0) \\
&= \text{fl}(x_{i}y_{j-1}) + (b_j - a_i, 0) \\
\text{fl}(x_iy_{j+1}) &= \text{fl}(x_{i+1}) + (0, a_{i+1}) + \text{fl}(y_j) - (0, b_{j+1}) \\
&= \text{fl}(x_{i+1}y_j) + (0, a_{i+1} - b_{j+1}) \\
\text{fl}(x_{i-1}y_{j+1}) &= \text{fl}(x_{i+1}) + (-a_i, a_{i+1}) + \text{fl}(y_{j-1}) + (b_j, -b_{j+1}) \\
&= \text{fl}(x_{i+1}y_{j-1}) + (b_j - a_i, a_{i+1} - b_{j+1}).
\end{align*}

We require \( \text{fl}(x_{i-1}y_j) \geq \text{fl}(x_iy_{j-1}) \), \( \text{fl}(x_iy_{j+1}) \geq \text{fl}(x_{i+1}y_j) \), and \( \text{fl}(x_{i-1}y_{j+1}) \geq \text{fl}(x_{i+1}y_{j-1}) \). By \((3.2)\), this is equivalent to \( a_i \leq b_j \) and \( b_{j+1} \leq a_{i+1} \) for \( i \) and \( j \) odd, which in turn is equivalent to \( a_i \leq b_j \leq a_{i+1} \) for \( i \) odd and \( j = 1, \ldots, n \), which is true by hypothesis.
Second, we check that the differential on $B_{i,j}'$ gives an acyclic summand. Using (2.2, 2.4) with $i$ and $j$ odd and $\mathbb{Z}/2\mathbb{Z}$ coefficients,

$$
\partial(x_i y_j) = x_{i-1} y_j + x_{i+1} y_j + x_j y_{j-1} + x_j y_{j+1} \\
= (x_{i-1} y_j + x_i y_{j-1}) + (x_i y_{j+1} + x_{i+1} y_j)
$$

$$
\partial(x_{i-1} y_j + x_i y_{j-1}) = (x_{i-1} y_{j-1} + x_{i-1} y_{j+1}) + (x_{i-1} y_{j-1} + x_{i+1} y_{j-1})
$$

$$
\partial(x_i y_{j+1} + x_{i+1} y_j) = (x_i y_{j+1} + x_{i+1} y_{j+1}) + (x_{i+1} y_{j-1} + x_{i+1} y_{j+1})
$$

$$
\partial(x_{i-1} y_j + x_{i+1} y_{j-1}) = 0.
$$

That is, both $(x_{i-1} y_j + x_i y_{j-1})$ and $(x_i y_{j+1} + x_{i+1} y_j)$ have one incoming arrow from $x_i y_j$ and one outgoing arrow to $(x_{i-1} y_j + x_{i+1} y_{j-1})$, joining the former to the latter. As there are no outgoing arrows from $S$ to $\{x_i y_j\} \setminus S$ or $B_{i,j}'$ to $\{x_i y_j\} \setminus B_{i,j}$, neither are there incoming arrows to $B_{i,j}'$. We conclude that each $B_{i,j}'$ is a basis for an acyclic subcomplex that splits off as a direct summand. \(\Box\)

Two examples that follow directly from inductive application of Lemma 3.1 are $n[a] = [(a)^n]$ (using the relation $2[a] = [a, a]$) and $n[a_1, a_2] = [(a_1, a_2)^n]$ if $a_1 \leq a_2$. As a point of contrast, we note without proof that $n[a_1, a_2] = \left[\sum_{\ell} c_i \right]_{\ell=1}^m$ with $(c_i)_{\ell=1}^m = (a_1, a_2, a_1, a_1)^n$ if $a_1 \geq a_2$.

**Lemma 3.2.** Let $a > 0$ and $c, d, e, p, q_\ell \geq 0$ for $\ell = 1, \ldots, r$. If $\min \{q_\ell\} \geq p$ and $\max \{d_\ell\} \leq c$, then

$[1, a]^p, 1, a + c] + [(1, a)^{q_\ell}, 1, a + d_\ell]_{\ell=1}^r = [(1, a)^p, 1, a + c, (1, a)^q, 1, a + d_\ell]_{\ell=1}^r$.

**Proof.** Set $m = 2p + 2$ and $n_\ell = 2 \sum_{k=1}^\ell q_k + 2\ell$, where $q_\ell = q_{2\ell + 1 - \ell}$. For $C_1 \in ((1, a)^p, 1, a + c)$ with basis $\{x_i\}, i = 0, \ldots, 2m$, and $C_2 \in ((1, a)^q, 1, a + d_\ell)_{\ell=1}^r$ with basis $\{y_j\}, j = 0, \ldots, 2n_\ell$, we prove that $C_1 \otimes C_2$ is of type $T = ((1, a)^p, 1, a + c, (1, a)^q, 1, a + d_\ell)_{\ell=1}^r$. Define the sets

$S = \{x_i y_j \mid i < m\} \cup \{x_i y_{j+2} \mid i > m\} \cup \{x_i y_{2n_\ell} \mid i > m\}$

$B_{i,j} = \begin{cases} 
\{x_i y_j, x_{i-1} y_j, x_i y_{j+1}, x_{i-1} y_{j+1}\} & i, j \text{ odd, } \text{ } i < m, j \notin \{n_\ell - 1 \mid \ell \leq r\} \\
\{x_i y_j, x_{i+1} y_j, x_i y_{j-1}, x_{i+1} y_{j-1}\} & i, j \text{ odd, } \text{ } i > m, j \notin \{n_\ell + 1 \mid \ell \geq r\}
\end{cases}$

$R_j = \begin{cases} 
\{x_i y_j, x_{i-1} y_j \mid i < m\} & j = n_\ell - 1, \ell \leq r \\
\{x_i y_j, x_{i+1} y_j \mid i > m\} & j = n_\ell + 1, \ell \geq r
\end{cases}$

Reasoning as in the preceding proof, $\{x_i y_j\} = S \cup \bigcup_{i,j} B_{i,j} \cup \bigcup_j R_j$, and $S$ spans a subcomplex isomorphic to the reduced representative of type $T$. We show that each of the $m(n_r - 2r)$ sets of elements $B_{i,j}$ and $2r$ sets of elements $R_j$ forms an acyclic summand under a filtered change of basis. The former summands look like boxes and the latter look like rectilinear polygons with $4p + 4$ sides—see Figure 4 for an example. Since $B_{i,j} \rightarrow B_{2m-i, 2n_r-j}$ and $R_j \rightarrow R_{2n_r-j}$ when $x_i y_j \rightarrow x_{2m-i} y_{2n_r-j}$, it suffices to redefine the basis for $i = 0, \ldots, m - 1$.

Let $(a_i)_{i=1}^m = ((1, a)^p, 1, a + c)$ and $(b_j)_{j=1}^{n_r} = ((1, a)^q, 1, a + d_\ell)_{\ell=1}^r$. For $i$ and $j$ odd, $i < m$, and $j \notin \{n_\ell - 1 \mid \ell \leq r\}$, we have $\max \{a_i\} = \min \{b_j\} = 1$ and $\max \{b_{j+1}\} = \min \{a_{i+1}\} = a$. Hence by (3.2), the basis change (3.1) is valid for each $B_{i,j}$. Recall that $B_{i,j}'$ is a basis for a box subcomplex.

Define the basis element sums

$$
u_{i,j} = \sum_{k=-1}^{m-i} x_{i+k} y_{j-k} \text{ and } \quad u_{i,j} = x_i y_j + x_{m} y_{j+i-m}.$$

Supposing $q_\ell \geq p$ for each $\ell$, we may produce the new basis

$$
x_i y_j \rightarrow u_{i,j}, \quad i \neq 0, \quad x_{i+1} y_{j+1} \rightarrow v_{i,j+1}, \quad x_0 y_j \rightarrow u_{0,j} + v_{-1,j+1}.$$
for all \(i < m\) and all \(j \in \{n_{\ell} - 1 | \ell \leq r\}\). Note here that \(u_{0,j} + v_{-1,j+1} = \sum_{k=0}^{m-1} x_k y_{j-k}\). It follows that \(R_j \to R'_j\), where \(R'_j = \{u_{i,j}, v_{i,j+1}, u_{0,j} + v_{-1,j+1} | i < m\}\)ANONTHE KNOT FLOER FILTRATION OF THE CONCORDANCE GROUPON THE KNOT FLOER FILTRATION OF THE CONCORDANCE GROUPON THE KNOT FLOER FILTRATION OF THE CONCORDANCE GROUP

First, we check that our basis respects the filtration. Let \(0 < k < m - i\). By (2.3, 2.5) with \(i\) even and \(j = n_{\ell} - 1\),

\[
\begin{align*}
\text{fil}(x_i y_j) &= (\text{fil}(x_{i-1}) - (0, a)) + (\text{fil}(y_{j+1}) + (0, a + d_{\ell})) \\
&= (\text{fil}(x_{i-1} y_{j+1}) + (0, d_{\ell}), \ i \neq 0 \\
\text{fil}(x_{i+k} y_{j-k}) &= \begin{cases} \\
(\text{fil}(x_{i+k-1}) + (1, 0)) + (\text{fil}(y_{j-k+1}) - (1, 0)) & k \text{ odd} \\
(\text{fil}(x_{i+k-1}) - (0, a)) + (\text{fil}(y_{j-k+1}) + (0, a)) & k \text{ even} \\
\end{cases} \\
&= (\text{fil}(x_{i+k-1} y_{j-k+1}) \\
\text{fil}(x_m y_{j+i-m}) &= (\text{fil}(x_{m-1}) - (0, a + c)) + (\text{fil}(y_{j+i-m+1}) + (0, a)) \\
&= (\text{fil}(x_{m-1} y_{j+i-m+1}) - (0, c), \ i \neq 0
\end{align*}
\]

and \(\text{fil}(x_m y_j) = \text{fil}(x_{m-1} y_{j+1}) - (0, c - d_{\ell})\). By (2.3, 2.5) with \(i\) odd and \(j = n_{\ell} - 1\),

\[
\begin{align*}
\text{fil}(x_i y_j) &= (\text{fil}(x_{i-1}) + (1, 0)) + (\text{fil}(y_{j+1}) + (0, a + d_{\ell})) \\
&= (\text{fil}(x_{i-1} y_{j+1}) + (1, a + d_{\ell}) \\
\text{fil}(x_{i+k} y_{j-k}) &= \begin{cases} \\
(\text{fil}(x_{i+k-1}) - (0, a)) + (\text{fil}(y_{j-k+1}) - (1, 0)) & k \text{ odd} \\
(\text{fil}(x_{i+k-1}) + (1, 0)) + (\text{fil}(y_{j-k+1}) + (0, a)) & k \text{ even} \\
\end{cases} \\
&= (\text{fil}(x_{i+k-1} y_{j-k+1}) - (-1)^k (1, a) \\
\text{fil}(x_m y_{j+i-m}) &= (\text{fil}(x_{m-1}) - (0, a + c)) + (\text{fil}(y_{j+i-m+1}) - (1, 0)) \\
&= (\text{fil}(x_{m-1} y_{j+i-m+1}) - (1, a + c)
\end{align*}
\]
Letting $e_i = i \mod 2$, it follows that

$$
fi(x_i y_j) = \begin{cases} 
fi(x_i y_j - k) + e_{ik}(1, a) \\
fi(x_{i-1} y_{j+1}) + (0, d_i) + e_i(1, a) & i \neq 0 \\
fi(x_m y_{j-i-m}) + (0, c) + e_i(1, a) & i \neq 0 
\end{cases}
$$

(3.6)

$$
fi(x_{i+1} y_{j+1}) = fi(x_m y_{j+i-i-m}) + (0, c - d_i).
$$

We require $fi(x_{i+1} y_{j+1}) \geq fi(x_m y_{j+i-i-m})$ and $fi(x_{i+1} y_{j+1})$ for each $k$ summed over in $u_{i,j}, v_{i,j}$ and $u_{i,j+1, y_{i,j}}$. By (3.6), this is equivalent to $0 \leq d_i \leq c$, which is true by hypothesis.

Second, we check that the differential on $R'_j$ gives an acyclic summand. Using (2.2, 2.4) with $i$ even and $j$ odd,

$$
\partial u_{i,j} = \partial \sum_{k=-1}^{m-1} \sum_{k \text{ odd}} x_{i+k} y_{j-k} = \partial \sum_{k=-1}^{m-1} (x_{i+k} y_{j-k} + x_{i+k+1} y_{j-k-1})
$$

$$
= \sum_{k=-1}^{m-1} \left( (x_{i+k} y_{j-k} + x_{i+k} y_{j-k}) \right) + \sum_{k=-1}^{m-1} \sum_{k \text{ odd}} 2 x_{i+k+1} y_{j-k}
$$

$$
= \sum_{k=-1}^{m-1} (x_{i+2} y_{j+1} + x_{m} y_{j+m-i-m-1}) + \sum_{k=-1}^{m-1} \sum_{k \text{ odd}} 2 x_{i+k+1} y_{j-k}
$$

$$
\partial v_{i,j+1} = \partial (x_i y_{j+1} + x_m y_{j+i-i-m}) = 0.
$$

Note that we first broke up the sum $u_{i,j}$ into terms with strictly odd or even subscripts to allow application of (2.2). Using (2.2, 2.4) with $i$ odd and $j$ odd,

$$
\partial u_{i,j} = \partial \sum_{k=-1}^{m-1} \sum_{k \text{ even}} x_{i+k} y_{j-k} = \partial \sum_{k=-1}^{m-1} x_{i+k} y_{j-k}
$$

$$
= \sum_{k=0}^{m-1} (x_{i+k-1} y_{j-k} + x_{i+k+1} y_{j-k} + x_{i+k} y_{j-k+1} + x_{i+k} y_{j-k}) + 0
$$

$$
= \sum_{k=0}^{m-1} (x_{i+k-1} y_{j-k} + x_{i+k+1} y_{j-k} + x_{i+k} y_{j-k+1} + x_{i+k} y_{j-k})
$$

$$
= -(x_{i-2} y_{j+1} + x_m y_{j+i-i-m}) + \sum_{k=-1}^{m-1} x_{i-1+k} y_{j-k} - \sum_{k=-1}^{m-1} x_{i+1+k} y_{j-k}
$$

$$
= v_{i-2,j+1} + u_{i-1,j} + u_{i+1,j}.
$$
\[
\partial v_{i,j+1} = \partial (x_i y_{j+1} + x_m y_{j+1+i-m}) \\
= (x_{i-1} y_{j+1} + x_{i+1} y_{j+1}) + (x_m y_{j+1+i-m-1} + x_m y_{j+1+i-m+1}) \\
= (x_{i-1} y_{j+1} + x_m y_{j+1+i-1-m}) + (x_i y_{j+1+i} + x_m y_{j+1+i+1-m}) \\
= v_{i-1,j+1} + v_{i+1,j+1}.
\]

Abbreviating \( u_i = u_{i,j} \) and \( v_i = v_{i,j+1} \) for convenience, we have shown that

\[
(3.7) \quad \partial u_i = \begin{cases} 
  v_{i-2} + u_{i-1} + u_{i+1} & \text{i odd} \\
  v_{i-2} & \text{i even}
\end{cases}
\]

\[
\partial v_i = \begin{cases} 
  v_{i-1} + v_{i+1} & \text{i odd} \\
  0 & \text{i even}.
\end{cases}
\]

Notice \( u_m = v_{m-1} \), \( v_m = 0 \), and \( \partial(v_{-1} + u_0) = v_0 \). That is, \( R_j \) is a basis for two staircases, with arrows joining the endpoints \( \{ v_{-1} + u_0, u_{m-1} \} \) to \( \{ v_0, v_{m-1} \} \). (Note the diagonal arrows from \( u_i \) to \( v_{i-2} \) for \( i > 1 \) ensure \( \partial^2 = 0 \).) As there are no outgoing arrows from \( S \) to \( \{ x_i y_j \} \backslash B_{i,j} \), neither are there incoming arrows to \( B'_{i,j} \) or \( R' \). We conclude that each \( B'_{i,j} \) and \( R' \) is a basis for an acyclic subcomplex that splits off as a direct summand. \( \square \)

4. Ordering of Staircase Complex Classes

We now study the ordering on \( \mathcal{F}_{alg} \). We take the tensor product of a complex with \( n \) times the dual of another, both reduced with \( \varepsilon = 1 \), then partially vertically and horizontally simplify the basis of the product to determine its \( \varepsilon \) value.

**Lemma 4.1** ([Hom11, Lemmas 6.3 and 6.4]). Let \( a, b > 0 \). If \( b_1 > a_1 \) or if \( b_1 = a_1 \) and \( b_2 < a_2 \), then

\[
[a_1, a_2, \ldots, a_m] \gg [b_1, b_2, \ldots, b_n].
\]

**Lemma 4.2.** Let \( a, c > 0 \) and \( d, p, q \geq 0 \). If \( q > p \) or if \( q = p \) and \( d < c \), then

\[
[(1, a)^p 1, a + c] \gg [(1, a)^q 1, a + d].
\]

**Proof.** Set \( m = 2p + 2 \) and \( n = 2q + 2 \). For \( C \in ((1, a)^p 1, a + c) \) with basis \( \{ x_i \} \), \( i = 0, \ldots, 2m \), and \( D_r \in ((1, a)^q 1, a + d)^r \) with basis \( \{ y_j \} \), \( j = 0, \ldots, 2nr \); we prove \( \varepsilon(C \otimes rD^r_1) = 1 \) for all \( r \in \mathbb{N} \). By inductive use of Lemma 3.2, \( rD_1 \sim \varepsilon D_r \), and so we may work with the simpler complexes \( D_r \), i.e., we show \( \varepsilon(C \otimes D^*_r) = 1 \).

By [Hom11, Section 3], Definition 2.1 (1) is equivalent to finding a basis with element \( u_0 \) that is the distinguished element of some vertically simplified basis for \( C \otimes D^*_r \) and in the image of the horizontal differential. Define the basis element sums

\[
(4.1) \quad u_i = \sum_{k=0}^{m-i} x_{i+k} y_k^*.
\]

Supposing \( q \geq p \), we may produce the new basis \( x_i y_0^* \rightarrow u_i \) for all \( i < m \). Clearly \( u_0 = (x_0 y_0^*)^\ast \) has no incoming vertical arrows since \( x_0 y_0^* \) has none, so it suffices to show \( \partial^\vert u_0 = 0 \) and \( \partial^\text{horz} u_1 = u_0 \). That is, we need only partially simplify the basis.

First, we check that \( x_i y_0^* \rightarrow u_i \) respects the filtration. Let \( 0 < k < m - i \). By (2.3, 2.5) with \( i \) even,

\[
\begin{align*}
\hat{f}(x_{i+k} y_k^*) &= \begin{cases} 
  (\hat{f}(x_{i+k-1}) + (1, 0)) + (\hat{f}(y_k^*) - (1, 0)) & k \text{ odd} \\
  (\hat{f}(x_{i+k-1}) - (0, a)) + (\hat{f}(y_k^*) + (0, a)) & k \text{ even}
\end{cases} \\
\hat{f}(x_m y_{m-i}^*) &= \begin{cases} 
  (\hat{f}(x_{m-1}) - (0, a + c)) + (\hat{f}(y_{m-i}^*) + (0, a)) & \text{if } q = p \\
  \hat{f}(x_{m-1} y_{m-i}^*) - (0, c), & i \neq 0
\end{cases}
\end{align*}
\]
and \( \text{fl}(x_m y_m^*) = \text{fl}(x_{m-1} y_{m-1}^*) - (0, c - d) \) if \( q = p \). For \( i \) odd,
\[
\text{fl}(x_{i+k} y_k^*) = \begin{cases} 
\text{fl}(x_{i+k-1}) - (0, a) + \text{fl}(y_{k-1}^*) - (1, 0) & \text{if } k \text{ odd} \\
\text{fl}(x_{i+k}) + (1, 0) + \text{fl}(y_{k-1}^*) + (0, a) & \text{if } k \text{ even}
\end{cases}
\]
\[
= \text{fl}(x_{i+k-1} y_{k-1}^*) + (-1)^k (1, a)
\]
\[
\text{fl}(x_m y_m^*) = \text{fl}(x_{m-1}) - (0, a + c) + \text{fl}(y_{m-1}^*) - (1, 0)
\]
\[
= \text{fl}(x_{m-1} y_{m-1}^*) - (1, a + c).
\]

Letting \( e_i = i \mod 2 \), it follows that
\[
(4.2) \quad \text{fl}(x_i y_i^*) = \begin{cases} 
\text{fl}(x_{i+k} y_k^*) + e_{ik} (1, a) & i \neq 0 \text{ if } q = p \\
\text{fl}(x_{m-1} y_{m-1}^*) + (0, c - d) & i = 0, q = p.
\end{cases}
\]

We require \( \text{fl}(x_i y_i^*) \geq \text{fl}(x_{i+k} y_k^*) \) for each \( k \) summed over in \( u_i \). By (4.2), this is equivalent to the condition that \( c \geq 0 \) and that \( d \leq c \) if \( q = p \), which is true by hypothesis.

Second, we check differentials. Using (2.2, 2.4) with \( i \) even,
\[
\partial u_i = \partial \sum_{k=0}^{m-i} x_{i+k} y_k^* = \partial (x_i y_i^*) + \partial \sum_{k=1}^{m-i-1} (x_{i+k} y_k^* + x_{i+k+1} y_{k+1}^*)
\]
\[
= x_i y_i^* + \sum_{k=1}^{m-i-1} \left( (x_{i+k-1} y_{k-1}^* + x_{i+k+1} y_k^*) + (x_{i+k+1} y_{k+2}^* + x_{i+k+1} y_k^*) \right)
\]
\[
= x_i y_i^* + \sum_{k=1}^{m-i-1} (x_{i+k-1} y_{k-1}^* + x_{i+k+1} y_{k+2}^*) + \sum_{k=1}^{m-i-1} 2x_{i+k+1} y_k^*
\]
\[
= x_m y_m^* + \sum_{k=1}^{m-i-1} 2x_{i+k-1} y_{k-1}^* + \sum_{k=1}^{m-i-1} 2x_{i+k+1} y_k^*
\]
\[
= x_m y_m^* + \sum_{k=0}^{m-i-1} x_{i+k} y_k^*
\]

For \( i \) odd,
\[
\partial u_i = \partial \sum_{k=0}^{m-i} x_{i+k} y_k^* = \partial (x_i y_i^*) + \partial \sum_{k=1}^{m-i-1} x_{i+k} y_k^* + \partial \sum_{k=1}^{m-i} x_{i+k} y_k^*
\]
\[
= \sum_{k=0}^{m-i-1} (x_{i+k-1} y_{k-1}^* + x_{i+k+1} y_{k+1}^*) + \sum_{k=2}^{m-i} x_{i+k} y_k^* + 0
\]
\[
= \sum_{k=0}^{m-i-1} (x_{i+k-1} y_{k-1}^* + x_{i+k+1} y_{k+1}^*) + x_i y_i^* + \sum_{k=2}^{m-i} (x_{i+k+1} y_k^* + x_{i+k} y_{k-1}^*)
\]
\[
= -x_m y_{m-i+1} + \sum_{k=0}^{m-i-1} x_{i+1} y_{k+1}^* + \sum_{k=0}^{m-i-1} x_{i+1} y_{k}^* + x_i y_i^* + \sum_{k=2}^{m-i} (x_{i+k+1} y_k^* + x_{i+k} y_{k-1}^*)
\]
\[
= x_m y_{m-i+1} + u_{i-1} + u_{i+1}.
\]
We have shown that

\[ \partial u_i = \begin{cases} x_m y_m^{*i+1} + u_{i-1} + u_{i+1} & \text{if } i \text{ odd} \\ x_m y_m^{*i+1} & \text{if } i \text{ even}. \end{cases} \]

Now suppose that \( c > 0 \) and that \( d < c \) if \( q = p \). Letting \( \tau = \tau(C \otimes D^*_r) \), note that \( \text{fl}(u_0) = (0, \tau) \), \( \text{fl}(u_1) = (1, \tau) \), and \( \text{fl}(u_2) = (1, \tau - a) \) if \( p > 0 \) or \((1, \tau - a - c) \) if \( p = 0 \). Using (4.2), also note that \( \text{fl}(x_m y_m^*) = (0, \tau - c) \) if \( q > p \) or \((0, \tau - c + d) \) if \( q = p \). By (4.3), we have \( \partial u_0 = x_m y_m^{*+1} \).

Notice that the \( i \)-coordinate of \( u_0 \) is 0, while the \( i \)-coordinate of \( x_m y_m^{*+1} \) is less than 0 (because \( x_m y_m^* \) is at \( i = 0 \)); thus \( \partial_{\text{vert}} u_0 = 0 \). Again by (4.3), we have \( \partial u_1 = x_m y_m^* + u_0 + u_2 \). Notice that the \( j \)-coordinate of \( u_1 \) and \( u_0 \) is \( \tau \), while the \( j \)-coordinate of both \( u_2 \) and \( x_m y_m^* \) is less than \( \tau \); thus \( \partial_{\text{horz}} u_1 = u_0 \). \( \square \)

It is easily seen that \( \{ u_i \mid i < m \} \cup \{ x_m y_j^* \mid j \leq nr \} \) with its reflection in the above proof spans a subcomplex isomorphic to the reduced complex \( P_r \in ((1, a)^p, 1, a + c, ((-1, -a)^q, -1, -a - d)^r) \).

This set belongs to a fully vertically and horizontally simplified basis for \( C \otimes D^*_r \), from which one obtains \([C] - r[D_1] = [P_r] \), which we leave as an exercise for the dedicated reader.

5. Floer Complexes of Selected \( L \)-space Knots

In this section, we find formulas for the Alexander polynomials of certain cables of torus knots. The knots we consider are all \( L \)-space knots, and thus the computations of the Alexander polynomials in fact gives us the knot Floer complexes of these knots.

This first lemma involves taking a well-known formula for the Alexander polynomial of iterated torus knots and grouping the terms in order to simply the expression, for example by noticing telescoping sums.

**Lemma 5.1.** The Alexander polynomial of the \((p, pm(m - 1) + 1)\)-cable of the \((m, m + 1)\)-torus knot is

\[
\Delta_{T_{m,m+1;pm(m-1)+1}}(t) = \sum_{i=0}^{pm(m-1)} t^{ip} - t \left( \sum_{i=0}^{p-1} i(pm^2 - pm + 1) \right) \left( \sum_{j=0}^{m-2} j^{pm} \left( \sum_{k=0}^{m-1} k^{kp} + \sum_{k=j+1}^{m-1} k^{kp-1} \right) \right).
\]

Similarly, the Alexander polynomial of the \((p, pm(m - 1) - 1)\)-cable of the \((m, m + 1)\)-torus knot is

\[
\Delta_{T_{m,m+1;p,pm(m-1)-1}}(t) = -t \sum_{i=0}^{pm(m-1)-2} t^{ip} + \left( \sum_{i=0}^{p-1} i(pm^2 - pm - 1) \right) \left( \sum_{j=0}^{m-2} j^{pm} \left( \sum_{k=0}^{m-1} k^{kp} + \sum_{k=j+1}^{m-1} k^{kp+1} \right) \right).
\]

**Proof.** We know that

\[
\Delta_{T_{m,m+1;pm(m-1)+1}}(t) = \frac{(t^{pm(m+1)} - 1)(t^p - 1)}{(t^{pm} - 1)(t^{pm+1} - 1)} \cdot \frac{(t^{pm(m-1)+1} - 1)(t - 1)}{(t^p - 1)(t^{pm(m-1)+1} - 1)}.
\]

Let

\[
P(t) = \sum_{i=0}^{pm(m-1)} t^{ip} - t \left( \sum_{i=0}^{p-1} i(pm^2 - pm + 1) \right) \left( \sum_{j=0}^{m-2} j^{pm} \left( \sum_{k=0}^{m-1} k^{kp} + \sum_{k=j+1}^{m-1} k^{kp-1} \right) \right).
\]

To prove the lemma, we will show that

\[
(t^{pm} - 1)(t^{pm+1} - 1)(t^p - 1)(t^{pm(m-1)+1} - 1) \cdot P(t) = (t^{pm(m+1)} - 1)(t^p - 1)(t^{pm(m-1)+1} - 1)(t - 1).
\]
We first consider

\[(5.2)\]
\[
(t^{pm} - 1)(t^{p(m+1)} - 1)(t^p - 1)(t^{pm(m-1)+1} - 1) \sum_{i=0}^{pm(m-1)} t^i.
\]

Note the telescoping sum

\[
(t^p - 1) \sum_{i=0}^{pm(m-1)} t^i = t^{p^2 m^2 - p^2 m + p} - 1.
\]

Hence (5.2) is equal to

\[(5.3)\]
\[
(t^{pm} - 1)(t^{p(m+1)} - 1)(t^{pm(m-1)+1} - 1)(t^{p^2 m^2 - p^2 m + p} - 1).
\]

Next, we consider

\[(5.4)\]
\[
(t^{pm} - 1)(t^{p(m+1)} - 1)(t^p - 1) \cdot (t^{pm(m-1)+1} - 1)(-t) \left( \sum_{i=0}^{p-1} t^{i(pm^2-pm+1)} \right) \left( \sum_{j=0}^{m-2} \sum_{k=0}^{j} t^{jp} + \sum_{k=j+1}^{m-1} t^{kp-1} \right) .
\]

We first notice that

\[(5.5)\]
\[
(t^{pm(m-1)+1} - 1)(-t) \left( \sum_{i=0}^{p-1} t^{i(pm^2-pm+1)} \right) = -t \cdot (t^{p^2 m^2 - p^2 m + p} - 1).
\]

Grouping the remaining factors yields

\[
(t^{pm} - 1)(t^{p(m+1)} - 1)(t^p - 1) \left( \sum_{j=0}^{m-2} \sum_{k=0}^{j} t^{jp} + \sum_{k=j+1}^{m-1} t^{kp-1} \right)
\]

\[
= (t^{pm} - 1)(t^{p(m+1)} - 1) \left( \sum_{j=0}^{m-2} \sum_{k=0}^{j} t^{jp+p-1} + t^{pm-1} - t^{jp+p-1} \right)
\]

\[
= (t^{pm} - 1)(t^{p(m+1)} - 1) \left( \sum_{j=0}^{m-2} \sum_{k=0}^{j} t^{jp-1} + t^{jp+p-1}(t-1) \right)
\]

\[
= (t^{pm} - 1)(t^{p(m+1)} - 1) \left( (t^{pm-1} - 1) \sum_{j=0}^{m-2} j^{pm} + t^{p-1}(t-1) \sum_{j=0}^{m-2} t^{jp(m+1)} \right)
\]

\[(5.5)\]
\[
= (t^{p(m+1)} - 1)(t^{pm-1} - 1)(t^{pm^2-pm+1} - 1) + (t^{pm} - 1)t^{p-1}(t-1)t^{p^2-p-1}.
\]

We have shown that the lefthand side of (5.1) is equal to

\[(5.3) + (5.4) \cdot (5.5),\]

where we have used telescoping sums to eliminate the summation from each of those expressions. It is straightforward to verify that this is equal to the righthand side of (5.1).

The calculation for $\Delta T_{m,m+1;p,pm(m-1)-1}(t)$ follows in a similar manner. \qed

The following lemma is essentially a restatement of Lemma 5.1 in our language for the $\varepsilon$-equivalence classes of staircase complexes, i.e., knot Floer complexes of $L$-space knots.
Lemma 5.2. Let $2g = m(m - 1)$ and
\begin{align}
(x_s)_{s=1}^{2gp} &= (((i, p - i), (i - 1, p - i + 1))_{j=1}^{m-1})_{i=1}^{p} \\
(y_s)_{s=1}^{2gp} &= (((p - i, i), (p - i + 1, i - 1))_{j=1}^{m-1})_{i=1}^{p}.
\end{align}
Then for positive $p$ and $m$,
\begin{align}
[T_{m,m+1,p,pm(m-1)+1}] &= [(x_s)_{s=1}^{2gp}] \quad \text{and} \quad [T_{m,m+1,p,pm(m-1)-1}] = [(y_s)_{s=1}^{2gp}].
\end{align}

Proof. We know that $CFK^\infty(T_{m,m+1,p,pm(m-1)+1})$ are staircases. We translate backwards from the sequences of step lengths to the corresponding summations. It will be helpful to rewrite the sequences as
\begin{align}
(x_{s+1}, x_{s+2})_{s=0}^{2gp-1} &= (((i + 1, p - i - 1), (i, p - i))_{k=0}^{m-1})_{j=0}^{m-2} \quad \text{for each } s, \\
(y_{s+1}, y_{s+2})_{s=0}^{2gp-1} &= (((p - i - 1, i + 1), (p - i, i))_{k=0}^{m-1})_{j=0}^{m-2} \quad \text{for each } s.
\end{align}

and to note that each integer $s$ can be written uniquely as
\begin{align}
(s) &= i(m - 1)m + jm + k \quad \text{where } 0 \leq j < m - 1 \text{ and } 0 \leq k < m.
\end{align}

Let $n_0 = 0$, and inductively define $n_i = n_{i-1} + x_i$ for $i > 0$. More generally, it follows that $n_i = n_j + \sum_{k=j+1}^i x_k$ for $i > j$. We claim that $\sum t^{n_i}(-1)^{i}t^{p_i}$ equals $\Delta_{T_{m,m+1,p,pm(m-1)+1}}(t)$, as given by the first polynomial in the statement of Lemma 5.1. Suppose that $n_{2s} = sp$. For $i$, $j$, and $k$ given as in (5.8), i.e., $i = \lfloor s/(m-1)m \rfloor$, $j = \lfloor s/m \rfloor - i(m-1)$, and $k = s - i(m-1)m - jm$, it is clear from (5.7) that
\begin{align}
n_{2s+1} &= n_{2s} + x_{2s+1} = \begin{cases} sp + i + 1 + p - i - 1 & k \leq j \\ sp + i + p - i & k > j \end{cases} = (s + 1)p.
\end{align}

It follows that $n_{2s} = sp$ for each $s$. That is,
\begin{align}
\sum_{s=0}^{2gp} t^{n_{2s}} &= \sum_{s=0}^{2gp} t^{sp} = \sum_{i=0}^{pm(m-1)} t^{sp},
\end{align}
which are all of the terms with positive coefficient. Next, we have from (5.7) that
\begin{align}
n_{2s+1} &= n_{2s} + x_{2s+1} = \begin{cases} sp + i + 1 & k \leq j \\ sp + i & k > j \end{cases}.
\end{align}

It follows that
\begin{align}
\sum_{s=0}^{2gp-1} t^{n_{2s+1}} = \sum_{s=0}^{2gp-1} t^{sp+i+1} + \sum_{k=j}^{2gp-1} t^{sp+i} = \sum_{i=0}^{p-1} \sum_{j=0}^{m-2} \sum_{k=j+1}^{j} t^{sp+i},
\end{align}
where $s = i(m - 1)m + jm + k$, which are all of the terms with negative coefficient. It is straightforward to verify that $\sum t^{4gp(-1)^{i}t^{p_i}} = \sum t^{2gp} t^{n_{2s}} - \sum t^{2gp-1} t^{n_{2s+1}}$ is equal to $\Delta_{T_{m,m+1,p,pm(m-1)+1}}(t)$.

Now let $n_0 = 0$, and inductively define $n_i = n_{i-1} + y_i$ for $i > 0$ so that $n_i = n_j + \sum_{k=j+1}^i y_k$ for $i > j$. Note that the sequence for $[T_{m,m+1,p,pm(m-1)-1}]$ begins with $y_2$, not $y_1$. We claim that $\sum_{i=1}^{4gp}(-1)^{i}t^{p_i}$ equals $\Delta_{T_{m,m+1,p,pm(m-1)-1}}(t)$, as given by the second polynomial in the statement.
of Lemma 5.1, up to a monomial factor. We find \( \sum_{i=0}^{4p} (-1)^i t^{n_i} \) in a similar manner as before. The terms with positive coefficient \( \sum_{s=0}^{2gp} t^{m_{2s}} \) are the same, and the terms with negative coefficient are

\[
\sum_{s=0}^{2gp-1} t^{m_{2s+1}} = \sum_{i=0}^{p-1} m - 2 \sum_{j=0}^{j} t^{xp+p-1} + \sum_{i=0}^{p-1} m - 2 \sum_{k=0}^{m-1} t^{xp+p-1},
\]

where \( s = i(m-1)m + jm + k \). It is straightforward to verify that \( \sum_{i=1}^{4p-1} (-1)^i t^{n_i} = \sum_{s=1}^{2gp-1} t^{m_{2s}} - \sum_{s=0}^{2gp-1} t^{m_{2s+1}} \) is equal to \( -t^{p-1} \Delta T_{m,m+1,p,p,m(m-1)-1}(t) \). \( \square \)

The corollary below follows by taking \( p = 1 \) (cf. [HLR12, Proposition 6.1]).

**Corollary 5.3.** Let

\[
(t_s)_{s=1}^{2m-2} = (j, m - j)_{j=1}^{m-1}.
\]

Then for positive \( m \),

\[
[T_{m,m+1}] = [(t_s)_{s=1}^{m-1}].
\]

**Proof.** Take \( p = 1 \) in \( (x_s) \), since \( T_{m,m+1,1,m(m-1)+1} = T_{m,m+1} \). Then \( [(x_s)_{s=1}^{2gp}] \) simplifies as

\[
[(x_s)_{s=1}^{2gp}] = \left[ \left( (1,0), (0,1)^{m-2}, m/2 \right)_{j=1}^{(m-1)/2} \right] = \left[ \left( j, m - j \right)_{j=1}^{(m-1)/2} \right]
\]

for \( m \) odd, and similarly as \( [(x_s)_{s=1}^{2gp}] = \left[ \left( j, m - j \right)_{j=1}^{(m-2)/2}, m/2 \right] \) for \( m \) even. It follows that \( [(x_s)_{s=1}^{2gp}] = [(t_s)_{s=1}^{m-1}]. \)

Note that \( (x_s) \) and \( (y_s) \) have 0 entries where the terms with positive coefficient cancel those with negative coefficient (see proof of Lemma 5.2). This occurs when \( i = 1 \) or \( p \) in \( (5.6) \). We can therefore simplify the complexes \( [(x_s)_{s=1}^{2gp}] \) and \( [(y_s)_{s=2}^{2gp}] \) as follows:

\[
[(x_s)_{s=1}^{2gp}] = \left[ \left( (1,0), (0,1)^{m-2}, m/2 \right)_{j=1}^{(m-1)/2} \right] = \left[ \left( j, m - j \right)_{j=1}^{(m-1)/2} \right],
\]

changing index by letting \( j \rightarrow j + 1 \), we can write

\[
[(x_s)_{s=1}^{2gp}] = \left[ \left( (1,0), (0,1)^{m-2}, m/2 \right)_{j=1}^{(m-1)/2} \right] = \left[ \left( j, m - j \right)_{j=1}^{(m-1)/2} \right],
\]

6. **Archimedean Equivalence Classes of \( F \)**

We can now combine the results of Sections 3, 4, and 5 to find new Archimedean equivalence classes in \( F \). In the following lemma, we define the knots that will be at the heart of Theorem 1.
Lemma 6.1. Set $p = i + 1$ and $m = |j| + 3$ for $i, j \in \mathbb{Z}$. Define

$$K_{(i,j)} = \begin{cases} T_{m,m+1;p,pm(m-1)+1} \# -T_{pm,pm+1} & i > 0, j \geq 0 \\ T_{m,m+1;p,pm(m-1)+1} \# -T_{m,m+1;p,pm(m-1)-1} & i > 0, j < 0 \\ T_{m,m+1} \# -T_{2,3[m/2],2[m/2]+1} & i = 0, j \geq 0 \end{cases}$$

Then $[K_{(0,0)}] = 0$ and for $(i, j) > (0, 0)$,

$$[K_{(i,j)}] \sim \text{Ar} \begin{cases} [1, i, 1, 2i + 1 + j(i + 1)] & j \geq 0 \\ [(1, i) - j, 1, i, 2i + 1] & j \leq 0. \end{cases}$$

Proof. Making inductive use of Lemma 3.1 on $[(t_s)_{s=1}^{m-1}]$ of Corollary 5.3, we obtain

$$(6.1) \quad [T_{m,m+1}] = \sum_{k=1}^{n} [k, m - k] + [(t_s)_{s=2n+1}^{m-1}]$$

for any $0 \leq n < [m/2]$. Using (5.10), we can apply Lemma 3.1 to $[(x_s)^{2p}_{s=1}]$ and $[(y_s)^{2p}_{s=2}]$ of Lemma 5.2 to obtain

$$(6.2) \quad [T_{m,m+1;p,pm(m-1)+1}] = [(x_s)^{4g}_{s=1} + [(x_s)^{2p}_{s=4g+1}]$$

$$[T_{m,m+1;p,pm(m-1)-1}] = [(y_s)^{4g+1}_{s=2} + [(y_s)^{2p}_{s=4g+2}],$$

where

$$[(x_s)^{4g}_{s=1}] = [(1, p - 1)^{k}, 1, p(m - k) - 1)]_{k=0}^{m-2}$$

$$[(y_s)^{4g+1}_{s=2}] = [(1, p - 1)^{k}, 1, p(m - k) - 1)]_{k=0}^{m-3}, (1, p - 1)^{m-2}, 1, 2p - 2].$$

In addition, inductive use of Lemma 3.2 gives

$$(6.3) \quad [(x_s)^{4g}_{s=1}] = \sum_{k=0}^{m-2} [(1, p - 1)^{k}, 1, p(m - k) - 1]$$

$$[(y_s)^{4g+1}_{s=2}] = \sum_{k=0}^{m-3} [(1, p - 1)^{k}, 1, p(m - k) - 1] + [(1, p - 1)^{m-2}, 1, 2p - 2].$$

We use the class decompositions (6.1, 6.2, 6.3) along with the sequence definitions (5.6, 5.9) in the following calculations.

Case 1: $i > 0, j \geq 0$. We have $[K_{(i,j)}] = [T_{m,m+1;p,pm(m-1)+1}] - [T_{pm,pm+1}]$. Hence $[K_{(i,j)}]$ is given by

$$\sum_{k=0}^{m-2} [(1, p - 1)^{k}, 1, p(m - k) - 1] + [(x_s)^{2p}_{s=4g+1}] - [1, pm - 1] - [(t_s)_{s=3}^{pm-1}]$$

$$= \sum_{k=1}^{m-2} [(1, p - 1)^{k}, 1, p(m - k) - 1] + [2, (x_s)^{2p}_{s=4g+2}] - [2, (t_s)_{s=4}^{pm-1}],$$

where the $[1, pm - 1]$ term cancels. By Lemmas 4.1 and 4.2, $[1, p - 1, 1, p(m - 1) - 1] \gg$ the other remaining terms so that $[K_{(i,j)}] \sim \text{Ar} [1, p - 1, 1, p(m - 1) - 1]$. 


**Case 2:** $i > 0$, $j < 0$. We have $[K_{(i,j)}] = [T_{m,m+1,p,m(m-1)+1}] - [T_{m,m+1,p,m(m-1)-1}]$. Hence $[K_{(i,j)}] = [(1,1)^{m-2},1,3] - [(1,1)^{m-2},1,2] \sim_{Ar} [(1,1)^{m-2},1,3]$, and for $i \geq 2$, $[K_{(i,j)}]$ is given by

$$\sum_{k=0}^{m-2} ((1,p-1)^k,1,p(m-k)-1) + [(s)_{s=3}^2g+1] - \sum_{k=0}^{m-3} ((1,p-1)^k,1,p(m-k)-1)$$

$$- ((1,p-1)^{m-2},1,2p-2) - [(y)^{2pg}_{s=4g+2}]$$

$$= [(1,p-1)^{m-2},1,2p-1] + [2,(s)_{s=4g+2}^2g] - [(1,p-1)^{m-2},1,2p-2] - [2,(y)^{2pg}_{s=4g+2}]$$

where each $[(1,p-1)^k,1,p(m-k)-1]$ term for $0 \leq k \leq m - 3$ cancels. By Lemmas 4.1 and 4.2, $[(1,p-1)^{m-2},1,2p-1] \gg j$ the other remaining terms so that $[K_{(i,j)}] \sim_{Ar} [(1,p-1)^{m-2},1,2p-1]$.

**Case 3:** $i = 0$, $j > 0$. Letting $n = [m/2]$ and noting $[1,0,1,1+j] = [2,1+j]$, we can rephrase the lemma for this case as

$$K_{(0,2n-\lfloor -j/2 \rfloor)} = T_{2n,2n+1} \# - T_{2,3;3,2n+1} \implies [K_{(0,j)}] \sim_{Ar} [2,1+j].$$

For $j$ odd, we have $[K_{(0,j)}] = [T_{2n,2n+1}] - [T_{2,3;3,2n+1}]$. Hence $[K_{(0,1)}] = [1,3] + [2] - [1,3] = [2] \sim_{Ar} [2,2] = [2,2]$, and for $j \geq 3$, $[K_{(0,j)}]$ is given by

$$\sum_{k=1}^{2} [(k,2n-k) + [(t)_{s=5}^{2n-1}] - [1,2n-1] - [(s)_{s=5}^{2n}]]$$

$$= [2,2n-2] + [3,(t)_{s=7}^{2n-1}] - [2,n-2,(s)_{s=7}^{2n}]$$

where the $[1,2n-1]$ term cancels. By Lemma 4.1, $[2,2n-2] \gg j$ the other remaining terms so that $[K_{(0,j)}] \sim_{Ar} [2,2n-2] = [2,m-2]$. For $j$ even, we similarly find $[K_{(0,j)}] \sim_{Ar} [2,2n-3] = [2,m-2]$.

The result for $(i,j) > (0,0)$ follows by substituting $(i+1)$ for $p$ and $(|j| + 3)$ for $m$ in each case, and $[K_{(0,0)}] = [T_{3,4}] - [T_{2,3;2,3}] = [1,2] - [1,2] = 0$. \hfill \Box

We now conclude with the proof of Theorem 1.

**Proof of Theorem 1.** For $i < i'$, we have that $[K_{(i,j)}] \ll [K_{(i',j')}]$ by Lemma 4.1. For $j < j'$, we have that $[K_{(i,j)}] \ll [K_{(i',j')}]$ by Lemma 4.1 ($i = 0$) and Lemma 4.2 ($i > 0$). Thus,

$$(i,j) < (i',j') \implies [K_{(i,j)}] \ll [K_{(i',j')}].$$

It follows that $\{H_{[K_{(i,j)}]} \mid (i,j) \in S\}$ is a filtration on $\mathcal{F}$ with

$$H_{[K_{(i,j)}]} \subset H_{[K_{(i',j')}]} \quad \text{if} \quad (i,j) < (i',j')$$

and $\mathbb{Z} \subset H_{[K_{(i',j')}]}/H_{K_{(i,j)}}$, generated by $[K_{(i',j')}]$. Recall the map $\phi : \mathcal{C} \to \mathcal{F}$. Letting

$$\mathcal{F}_{(i,j)} := \phi^{-1}[H_{[K_{(i,j)}]}],$$

we pull back to a filtration $\{\mathcal{F}_{(i,j)} \mid (i,j) \in S\}$ on $\mathcal{C}$ with $\mathbb{Z} \subset \mathcal{F}_{(i',j')} / \mathcal{F}_{(i,j)}$ for $(i,j) < (i',j')$. \hfill \Box

**References**

[CHH12] Tim D. Cochran, Shelly Harvey, and Peter Horn, *Filtering smooth concordance classes of topologically slice knots*, preprint (2012), arXiv:1201.6283v1.

[CHL09] Tim D. Cochran, Shelly Harvey, and Constance Leidy, *Knot concordance and higher-order Blanchfield duality*, Geom. Topol. 13 (2009), no. 3, 1419–1482.

[COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and $L^2$-signatures*, Ann. of Math. (2) 157 (2003), no. 2, 433–519.

[Hed09] Matthew Hedden, *On knot Floer homology and cabling II*, Int. Math. Res. Not. IMRN (2009), no. 12, 2248–2274.
[HLR12] Matthew Hedden, Charles Livingston, and Daniel Ruberman, \textit{Topologically slice knots with nontrivial Alexander polynomial}, Adv. Math. \textbf{231} (2012), no. 2, 913–939.

[Hom11] Jennifer Hom, \textit{The knot Floer complex and the smooth concordance group}, preprint (2011), to appear in Comment. Math. Helv., arXiv:1111.6635v1.

[Hom12] \textit{Bordered Heegaard Floer homology and the tau-invariant of cable knots}, preprint (2012), to appear in Journal of Topology, arXiv:1202.1463v1.

[LOT08] Robert Lipshitz, Peter Ozsváth, and Dylan Thurston, \textit{Bordered Heegaard Floer homology: Invariance and pairing}, preprint (2008), arXiv:0810.0687v4.

[OS03] Peter Ozsváth and Zoltán Szabó, \textit{Knot Floer homology and the four-ball genus}, Geom. Topol. \textbf{7} (2003), 615–639.

[OS04] \textit{Holomorphic disks and knot invariants}, Adv. Math. \textbf{186} (2004), no. 1, 58–116.

[OS05] \textit{On knot Floer homology and lens space surgeries}, Topology \textbf{44} (2005), no. 6, 1281–1300.

[Ras03] Jacob Rasmussen, \textit{Floer homology and knot complements}, Ph.D. thesis, Harvard University, 2003, arXiv:math/0306378v1.

[Ras04] \textit{Lens space surgeries and a conjecture of Goda and Teragaito}, Geom. Topol. \textbf{8} (2004), 1013–1031.

\textit{E-mail address}: ssh2127@columbia.edu

\textbf{Department of Mathematics, Columbia University, New York, NY 10027}
\textit{E-mail address}: hom@math.columbia.edu

\textbf{Department of Mathematics, University of Michigan, Ann Arbor, MI 48109}
\textit{E-mail address}: mgnewman@umich.edu