Scalar multi-wormholes

A I Egorov, P E Kashargin and Sergey V Sushkov

Institute of Physics, Kazan Federal University, Kremlevskaya 16a, Kazan 420008, Russia

E-mail: sergey_sushkov@mail.ru

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Abstract
In 1921 Bach and Weyl derived the method of superposition to construct new axially symmetric vacuum solutions of general relativity. In this paper we extend the Bach–Weyl approach to non-vacuum configurations with massless scalar fields. Considering a phantom scalar field with the negative kinetic energy, we construct a multi-wormhole solution describing an axially symmetric superposition of \(N\) wormholes. The solution found is static, everywhere regular and has no event horizons. These features drastically tell the multi-wormhole configuration from other axially symmetric vacuum solutions which inevitably contain gravitationally inert singular structures, such as ‘struts’ and ‘membranes’, that keep the two bodies apart making a stable configuration. However, the multi-wormholes are static without any singular struts. Instead, the stationarity of the multi-wormhole configuration is provided by the phantom scalar field with the negative kinetic energy. Anther unusual property is that the multi-wormhole spacetime has a complicated topological structure. Namely, in the spacetime there exist \(2^N\) asymptotically flat regions connected by throats.

Keywords: wormholes, phantom scalar field, Bach–Weyl approach

(Some figures may appear in colour only in the online journal)

1. Introduction

The two-body (and, more generally, \(N\)-body) problem is one of the most important and interesting to be solved in any theory of gravity. It is well known that in Newtonian gravity this problem is well-defined and, in the two-body case \(N = 2\), has a simple exact treatment of the solution. In general relativity the \(N\)-body problem has been investigated from the early days of Einstein’s gravitation theory [1–5]. However, because of conceptual and technical difficulties the motion of the \(N\) bodies cannot be solved exactly in the context of general
relativity, even when $N = 2$. Hence analysis of a two-body system (e.g. binary pulsars) necessarily involves resorting to approximation methods such as a post-Newtonian expansion (see, e.g., [6, 7] for a review and references to the early literature).

Besides developing approximate methods for solving the $N$-body problem, essential efforts had been undertaken to search and study exact solutions of the Einstein equations which could be interpreted as those describing multi-particle configurations in general relativity. In the early days of general relativity Weyl [8] and Levi-Civita [9] demonstrated that static axially symmetric gravitational fields can be described by the following metric (now known as the Weyl metric):

$$d\sigma^2 = -e^{2\lambda}dr^2 + e^{-2\lambda}[e^{2\nu}(d\rho^2 + dz^2) + \rho^2d\varphi^2],$$

where the functions $\lambda$ and $\nu$ depend on coordinates $\rho$ and $z$ only. In vacuum, when $R_{\mu\nu} = 0$, these functions satisfy, respectively, the Laplace equation

$$\Delta \lambda = \lambda_{,\rho\rho} + \lambda_{,zz} + \frac{\lambda_{,\rho}}{\rho} = 0,$$

and two partial differential equations

$$\nu_{,\rho} = \rho(\lambda_{,\rho}^2 - \lambda_{,z}^2),$$

$$\nu_{,z} = 2\rho\lambda_{,\rho}\lambda_{,z},$$

where a comma denotes a partial derivative, e.g. $\lambda_{,\rho} = \partial \lambda / \partial \rho$, $\lambda_{,\rho\rho} = \partial^2 \lambda / \partial \rho^2$, etc. Since (2) is linear and homogeneous, the superposition of any integrals is again an integral of that equation. This fact and the particular form of equations (3), (4) allow us to find explicitly solutions that represent the superposition of two or more axially symmetric bodies of equal or different shapes. In their pioneering work [10], Bach and Weyl obtained an axially symmetric vacuum solution interpreted as equilibrium configurations of a pair of Schwarzschild black holes. Since then, the fruitful Bach–Weyl method has been used to construct a number of exact axially symmetric vacuum solutions [11–18]. It is worth noticing that all solutions, representing the superposition of two or more axially symmetric bodies, contain gravitationally inert singular structures, ‘struts’ and ‘membranes’, that keep the two bodies apart making a stable configuration. This is an expected ‘price to pay’ for the stationarity [10, 19–23].

The advances of the Bach–Weyl approach in constructing exact axially symmetric solutions are based on the specific algebraic form of the vacuum field equations (2)–(4). However, in this paper we demonstrate that this approach can be easily extended to gravitating systems with a massless scalar field $\phi$. Although such systems are not vacuum ones, we show that new axially symmetric solutions with the scalar field can be found as a superposition of known spherically symmetric configurations. More specifically, we construct and analyze a multi-wormhole configuration with the phantom scalar field that possesses the negative kinetic energy.

Wormholes are usually defined as topological handles in spacetime linking widely separated regions of a single Universe, or ‘bridges’ joining two different spacetimes [24]. As is well-known [25–27], traversable wormholes in general relativity can exist only if their throats contain exotic matter which possesses a negative pressure and violates the null energy condition. The search for realistic physical models providing the wormhole existence represents an important direction in wormhole physics. In general relativity there are models of wormholes supported by matter with exotic equations of state such as phantom energy [28, 29], a Chaplygin gas [30], and tachyon matter [31]. In the context of modified gravity
wormhole geometries can be theoretically constructed without the presence of exotic matter \[32\]. Actually, numerous examples of wormhole solutions have been found in various modifications of general relativity such as scalar–tensor theories of gravity, brane theories, semiclassical gravity, theories with nonminimal coupling, multi-dimensional theories of gravity, etc.

The paper is organized as follows. In section 2 we consider the theory of gravity with a massless scalar field and derive field equations for a static axially symmetric configuration. In section 3 we review solutions describing spherically symmetric scalar wormholes. In section 4 we construct multi-wormhole solutions. First, we consider specific properties of a single-wormhole axially symmetric configuration. Then, using the superposition method, we discuss in detail the procedure of constructing the two-wormhole solution, and, more generally, the \(N\)-wormhole one. In the concluding section we summarize results and give some remarks concerning the regularity of the solutions and the violation of the null energy condition.

Supplement. After publishing this work in the e-print service arXiv.org (see \[33\]), we received very interesting and important comments from Gérard Clément. In particular, he drew our attention to his papers and some other related works concerning multi-wormhole configurations which we missed in our investigation. Here we briefly discuss these works. For the first time the technique of linear superposition of two or more aligned wormholes was applied some time ago by Clément \[34, 35\] to generate axisymmetric multi-wormhole solutions to five- and higher-dimensional general relativity from the Chodos–Detweiler spherically symmetric wormhole solution \[36\]. A similar construction was applied in \[37\] to the Bronnikov wormhole solution to four-dimensional Einstein–Maxwell theory with a massless phantom scalar field, generalized in \[38\] to wormhole solutions with NUT charge to the same theory \[39\]. Recently, the construction of stationary axisymmetric multi-wormhole solutions given in \[37\] was revised in \[40\], and it was concluded that strut singularities always appear in such constructions. In this paper we also use the Bach–Weyl technique of superposition and independently construct an axisymmetric multi-wormhole configuration in the case of Ellis wormhole solution to the gravitating phantom scalar field theory. It is necessary to emphasize that the spacetime of multi-wormholes is everywhere regular. Therefore, the strut singularities are absent in this case.

2. Field equations

The theory of gravity with a massless scalar field \(\phi\) is described by the action

\[
S = \int d^4x \sqrt{-g} \left[ R - \epsilon g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right],
\]

where \(g_{\mu\nu}\) is a metric, \(g = \det(g_{\mu\nu})\), and \(R\) is the scalar curvature. The parameter \(\epsilon\) equals \(\pm 1\).

In the case \(\epsilon = 1\) we have a canonical scalar field with the positive kinetic term, and the case \(\epsilon = -1\) describes a \textit{phantom} scalar field with the negative kinetic energy. It is well-known that wormholes in the theory (5) exist only if the scalar field is phantom, and so hereinafter we will assume that \(\epsilon = -1\).

Varying the action (5) with respect to \(g_{\mu\nu}\) and \(\phi\) leads to the field equations

\[
R_{\mu\nu} = -\phi \partial_\mu \partial_\nu \phi, \tag{6a}
\]

\[
\Box \phi = 0. \tag{6b}
\]
Let us search for static axially symmetric solutions of equation (6). In this case a gravitational field is described by the canonical Weyl metric (1), and the scalar field $\phi$ depends on coordinates $\rho$ and $z$ only. Now the field equation (6) can be represented in the following form

$$\Delta \lambda = 0, \quad \Delta \phi = 0, \quad (7a)$$

$$\nu_{,\rho} = \rho \left[ \lambda_{,\rho}^2 - \lambda_{,z}^2 - \frac{1}{2} (\phi_{,\rho}^2 - \phi_{,z}^2) \right], \quad (7b)$$

$$\nu_{,z} = \rho \left[ 2 \lambda_{,\rho} \lambda_{,z} - \phi_{,\rho} \phi_{,z} \right]. \quad (7d)$$

Here it is necessary to emphasize that the functions $\lambda(\rho, z)$ and $\phi(\rho, z)$ satisfy the Laplace equations (7a), (7b), where $\Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2}$ is the Laplace operator in cylindrical coordinates. Since (7a), (7b) are linear and homogeneous, superpositions of any integrals are again integrals of those equations. Then, since $\nu_{,\rho} = \nu_{,z}$, equations (7c), (7d) yield

$$2 \lambda_{,z} \Delta \lambda + \phi_{,z} \Delta \phi = 0. \quad (7d)$$

The latter is fulfilled due to (7a) and (7b), and so equations (7a), (7b) are integrability conditions for equations (7c), (7d). Therefore, provided solutions $\lambda(\rho, z)$ and $\phi(\rho, z)$ of equations (7a), (7b) are given, equations (7c), (7d) can be integrated in terms of the line integral

$$\nu(\rho, z) = \int_C \rho \left[ \lambda_{,\rho}^2 - \lambda_{,z}^2 - \frac{1}{2} (\phi_{,\rho}^2 - \phi_{,z}^2) \right] d\rho + \rho \left[ 2 \lambda_{,\rho} \lambda_{,z} - \phi_{,\rho} \phi_{,z} \right] dz, \quad (8)$$

where $C$ is an arbitrary path connecting some fixed point $(\rho_0, z_0)$ with the point $(\rho, z)$. In practice, a choice of $(\rho_0, z_0)$ is determined by the boundary condition $\nu(\rho_0, z_0) = \nu_0$.

3. Spherically symmetric wormhole solution

Wormholes are usually defined as topological handles in spacetime linking widely separated regions of a single Universe, or ‘bridges’ joining two different spacetimes [24, 25]. As is well-known [26, 27], in the framework of general relativity they can exist only if their throats contain an exotic matter which possesses a negative pressure and violates the null energy condition. The simplest model which provides the necessary conditions for existence of wormholes is the theory of gravity (5) with the phantom scalar field. Static spherically symmetric wormholes in the model (5) were obtained by Ellis [41] and Bronnikov [39]. Adapting their result (see [42]), one can write down a wormhole solution as follows

$$ds^2 = -e^{2\lambda(r)} dt^2 + e^{-2\lambda(r)} [dr^2 + (r^2 + a^2)(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (9)$$

$$\phi(r) = \frac{\sqrt{2(m^2 + a^2)}}{m} \lambda(r), \quad (10)$$

where

$$\lambda(r) = \frac{m}{a} \arctan(r/a), \quad (11)$$

$r \in (-\infty, \infty)$, and $m$ and $a$ are two free parameters. In the case $m = 0$ one has $\lambda(r) = 0$, and the solution (9), (10) takes the especially simple form:
The latter describes a massless wormhole connecting two asymptotically flat regions at $r \to \pm \infty$. The throat of the wormhole is located at $r = 0$ and has the radius $a$. The scalar field $\phi$ smoothly varies between two asymptotical values $\pm \pi/\sqrt{2}$.

Hereafter, for simplicity, we will assume that $m = 0$ and hence $\lambda = 0$.

4. Multi-wormhole solution

4.1. Single wormhole

Let us consider an axially symmetric form of the spherically symmetric wormhole solution given in the previous section. For this aim we rewrite the solution (12)–(13) in cylindrical coordinates $(\rho, z)$ carrying out the coordinate transformation

\[ ds^2 = -dt^2 + dr^2 + (r^2 + a^2)(d\theta^2 + \sin^2 \theta d\varphi^2), \]

\[ \phi(r) = \sqrt{2} \arctan(r/a). \]
The corresponding Jacobian reads

\[
J = \frac{D(\rho, z)}{D(r, \theta)} = -\frac{r^2 + a^2 \cos^2 \theta}{\sqrt{r^2 + a^2}}.
\]  

It is necessary to stress that the Jacobian \(J\) turns out to be zero at \(r = 0, \theta = \frac{\pi}{2}\), hence the transformation (14) is degenerated at this point.

Recall that in the wormhole spacetime the proper radial coordinate \(r\) runs from \(-\infty\) to \(+\infty\), while the azimuthal coordinate \(\theta\) runs from 0 to \(\pi\). Therefore, the domain of spherical coordinates \((r, \theta)\) can be represented as the strip \(S = (-\infty, \infty) \times [0, \pi]\) on the plane \(\mathbb{R} \times \mathbb{R}\) (see figure 1). Note that the wormhole throat is represented as the segment \(T = \{r = 0, \theta \in [0, \pi]\}\) on \(S\). The regions with \(r \to \pm \infty\) on the strip \(S\) correspond to two asymptotically flat regions of the wormhole spacetime. The corresponding domain of cylindrical coordinates \((\rho, z)\) is found from (14) as the half-plane \(D = [0, \infty) \times (-\infty, \infty)\). Correspondingly, the wormhole throat is represented as the segment \(T_a = \{\rho \in [0, a], z = 0\}\) on \(D\), where \(a\) is the radius of the throat. Note also that the asymptotics \(r \to \pm \infty\) in cylindrical coordinates take the form \((\rho^2 + z^2)^{1/2} \to \infty\).

The transformation (14) determines the mapping \(S \to D\). It is necessary to emphasize that it maps two different points of \(S\) into a single point of \(D\). Namely, \((r, \theta) \to (\rho, z)\) and \((-r, \pi - \theta) \to (\rho, z)\). Therefore, the mapping \(S \to D\) is not bijective. To construct a bijective mapping, we first consider two half-strips, \(S_0 = (-\infty, 0) \times [0, \pi]\) and \(S_+ = (0, \infty) \times [0, \pi]\), which represent, respectively, ‘lower’ \((r < 0)\) and ‘upper’ \((r > 0)\) halves of the wormhole spacetime. The edge of both half-strips is the segment \(T\), which, in turn, corresponds to the wormhole’s throat. Additionally, we introduce two identical copies of the half-plane \(D\) denoted as \(D_\pm\); the segments \(T_{a\pm}\) on \(D_\pm\) correspond to the wormhole’s throat. Now, each of the mapping \(S_\pm \to D_\pm\) is bijective. The whole strip \(S\) is restored by means of the gluing of two half-strips \(S_\pm\) along their common edge \(T\). This procedure corresponds to the topological gluing of two half-planes \(D_\pm\) by means of identifying corresponding points of

![Figure 2. The manifold \(D_2\) consists of two regions \(D_\pm\), which are glued together by the two-sided segments \(T_{a\pm}\).](image-url)
the segments $T_{a \pm}$. As the result of this topological gluing, one obtains a new manifold denoted as $\bar{T}_2$, consisting of two sheets $\bar{T}_{\pm}$ with the common segment $\bar{a}$. It is worth noticing that the manifold $\bar{T}_2$ is homeomorphic to the self-crossing two-sheeted Riemann surface. The sheets $\bar{T}_{\pm}$ of the manifold $\bar{T}_2$ contain asymptotically flat regions $(\rho^2 + z^2)^{1/2} \to \infty$ connecting by the throat $\bar{T}$. Now, the mapping $S \to \bar{T}_2$ is a bijection. Schematically, the mapping $\bar{T}_{\pm} \to \bar{T}_2$ is shown in figure 1. Respectively, in figure 2 we illustrate the procedure of constructing the manifold $\bar{T}_2$ by means of topological gluing.

The transformation being inverse to (14) reads

$$r = \pm R(\rho, z),$$

$$\theta = \arccos(z/R(\rho, z)),$$

where

$$R(\rho, z) = \frac{1}{\sqrt{2}} \left[ (\rho^2 + z^2 - a^2) + \sqrt{(\rho^2 + z^2 - a^2)^2 + 4a^2z^2} \right]^{1/2}. \tag{17}$$

Performing the transformation (16), we obtain the single-wormhole metric (12) in Weyl coordinates:

$$ds^2_\pm = -dr^2 + e^{2\nu_\pm}[d\rho^2 + dz^2] + \rho^2 d\varphi^2, \tag{18}$$

where

$$\nu_\pm(\rho, z) = \frac{1}{2} \ln \left[ \frac{R^2(\rho, z)(R^2(\rho, z) + a^2)}{R^4(\rho, z) + a^2z^2} \right]. \tag{19}$$

and the scalar field (13) now reads

$$\phi_\pm(\rho, z) = \pm \sqrt{2} \arctan(R(\rho, z)/a). \tag{20}$$

The signs $\pm$ in equations (16)–(20) are pointing out that the metric $ds^2_\pm$, as well as the functions $\phi_\pm(\rho, z)$ and $\nu_\pm(\rho, z)$ are defined on the respective charts $\bar{T}_\pm$, i.e. one should choose the plus sign if $(\rho, z) \in \bar{T}_+(r > 0)$, and minus if $(\rho, z) \in \bar{T}_-(r < 0)$. The complete solutions defined on the whole manifold $\bar{T}_2$ are constituted from separate $\phi_\pm(\rho, z)$ and $\nu_\pm(\rho, z)$. In particular, the scalar field $\phi(\rho, z)$ on $\bar{T}_2$ is defined as follows:
\[ \phi(r, z) = \begin{cases} \phi_-(r, z) & \text{if } (r, z) \in \mathcal{D}_- \\ \phi_+(r, z) & \text{if } (r, z) \in \mathcal{D}_+ \end{cases} \] (21)

Separately graphs of \( \phi_{\pm}(r, z) \) are shown in figure 3. It is seen that \( \phi_{\pm}(r, z) \) are not smooth at the segments \( T_{a,\pm} \), i.e. at the throat. However, the complete solution \( \phi(r, z) \), shown in figure 4, is single-valued, smooth and differentiable everywhere on \( \mathcal{D}_2 \). Actually, it is obvious that the scalar field, given in the spherical coordinates as \( \phi(r) = \sqrt{2} \arctan(\frac{r}{a}) \) (see equation (13)), is smooth and differentiable for all values of \( r \) including the throat \( r = 0 \). Therefore, the scalar field (21) obtained from (13) as the result of coordinate transformations is also smooth and differentiable everywhere on \( \mathcal{D}_2 \).

The metric function \( \nu_{\pm}(r, z) \) given by equation (19) has been obtained straightforwardly through the coordinate transformation (14) applied to the solution (12). On the other hand, \( \nu(r, z) \) could be found directly as the line integral (8). Since \( \lambda(r) = 0 \), equation (8) reads

\[ \nu(r, z) = \int_{C} \left[ -\frac{1}{2} \rho \left( \phi_{\rho}^2 - \phi_{z}^2 \right) d\rho - \rho \phi_{\rho} \phi_{z} dz \right] \] (22)

where \( C \) is an arbitrary path connecting some initial point \((\rho_0, z_0)\) and any point \((\rho, z)\) in \( \mathcal{D}_2 \). Note that if \((\rho_0, z_0)\) and \((\rho, z)\) are lying in the different charts \( \mathcal{D}_- \) and \( \mathcal{D}_+ \), then the path \( C \) should go through the segment \([\rho \in [0, a], z = 0]\), which corresponds to the wormhole’s throat. The coordinates of the initial point \((\rho_0, z_0)\) are determined by the boundary conditions for function \( \nu \). Imposing the following boundary conditions

\[ \lim_{r \to \infty, z \to \infty} \nu(r, z) = 0, \quad (r, z) \in \mathcal{D}_+, \]

substituting \( \phi \) from equation (21), and integrating, we obtain the expression (19). Finally, the complete solution \( \nu(r, z) \) defined on \( \mathcal{D}_2 \) is constituted from the separate \( \nu_{\pm}(r, z) \) as follows

\[ \nu(r, z) = \begin{cases} \nu_-(r, z) & \text{if } (r, z) \in \mathcal{D}_- \\ \nu_+(r, z) & \text{if } (r, z) \in \mathcal{D}_+ \end{cases} \] (23)

Figure 4. The scalar field \( \phi(r, z) \) on the manifold \( \mathcal{D}_2 \) corresponding to the single-wormhole spacetime. The segment \( T = \{ \rho \in [0, a], z = 0 \} \) corresponds to the wormhole’s throat with the radius \( a = 5 \).
Since \( \nu(r, z) \) and \( \nu_z(r, z) \) are given by the same expression (19), a graph of \( \nu(\rho, z) \) is represented by two identical sheets topologically glued along the cuts on \( \nu_\pm(\rho, z) \) with \( \rho \in [0, a] \) and \( z = 0 \).

4.2. Two wormholes

Now, let us construct a two-wormhole solution. For this aim we remind ourselves that any superposition \( c_1 \phi_1 + c_2 \phi_2 \) of two particular solutions \( \phi_1 \) and \( \phi_2 \) is a solution of the Laplace equation. Moreover, because of the axial symmetry, if \( \phi(\rho, z) \) is a solution, then \( \phi(\rho, z + \text{const}) \) is also a solution of \( \Delta \phi = 0 \), where \( \Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} \) is the Laplace operator in cylindrical coordinates. Using these properties, we may construct new axially symmetric solutions as a superposition of single-wormhole solutions (20). In particular, the superposition of two solutions reads

\[
\phi_{12}(\rho, z) = \frac{1}{2} \left[ \phi_1(\rho, z - z_i; a_i) + \phi_2(\rho, z - z_2; a_2) \right],
\]

with

\[
\phi_i(\rho, z - z_i; a_i) = \epsilon_i \sqrt{2} \arctan \left( \frac{R(\rho, z - z_i)}{a_i} \right).
\]

where \( i = 1, 2 \), and indices \( \epsilon_i \) take two values, + or −, i.e. \( \epsilon_i = \pm \). Each particular solution \( \phi_i(\rho, z - z_i; a_i) \) describes a single wormhole with the throat of the radius \( a_i \) located at \( z_i \). In case \( z_1 = z_2 = 0 \) and \( a_1 = a_2 \), equation (24) reproduces the single-wormhole solution (20). As a result, the superposition (24) gives us four new solutions \( \phi_{++}, \phi_{+-}, \phi_{-+}, \) and \( \phi_{--} \), which could be interpreted as the scalar field in the wormhole spacetime with two throats of radii \( a_1 \) and \( a_2 \) located at \( z_1 \) and \( z_2 \), respectively.
Note that the domain of \[ f(z,1,2) \] is the same as that of the single-wormhole solutions \[ f_{\pm} \] on \[ \Delta \] = \([0, \infty) \times (-\infty, \infty)\). The segments \[ T_a = \{ \rho \in [0, a_i], z = z_i \} (i = 1, 2) \] corresponding to two throats with radii \( a_i \) located at \( z_i \).

Figure 6. The manifold \( \mathcal{D}_4 \) consists of four identical copies of the half-planes \( \mathcal{D}_{1/2} \) \((a_i = \pm)\), which are glued together by the two-sided segments \( T_i = \{ \rho \in [0, a_i], z = z_i \} (i = 1, 2) \) corresponding to two throats with radii \( a_i \) located at \( z_i \).

Figure 7. The plot represents separate solutions \( \phi_{1/2} (\rho, z) \) on half-planes \( \mathcal{D}_{1/2} \). Here \( a_1 = 1, a_2 = 2, \) and \( z_1 = 1, z_2 = -1 \).

Note that the domain of \( \phi_{1/2} (\rho, z) \) is the same as that of the single-wormhole solutions \( \phi_{a_i} (\rho, z - z_i; a_i) \), i.e. the half-plane \( \mathcal{D} = [0, \infty) \times (-\infty, \infty) \). The segments \( T_a = \{ \rho \in [0, a_i], z = z_i \} (i = 1, 2) \) on \( \mathcal{D} \) represent two throats with radii \( a_i \) located at \( z_i \), respectively. Graphs of \( \phi_{1/2} (\rho, z) \) are shown separately in figure 7. It is seen that each of \( \phi_{1/2} (\rho, z) \) is not smooth at \( T_a \). To construct a smooth solution, we consider four identical copies of the half-plane \( \mathcal{D} \), denoted as \( \mathcal{D}_{1/2} \). On each copy \( \mathcal{D}_{1/2} \) we choose two segments \( T_a \) \((i = 1, 2)\), and then constitute a new manifold gluing the half-planes \( \mathcal{D}_{1/2} \) along \( T_a \). The gluing procedure is the following: \( \mathcal{D}_{1/2} \) and \( \mathcal{D}_{1/2} \) are glued along the throat \( T_a \), and \( \mathcal{D}_{1/2} \) and
\( \Delta \) are glued along the throat \( a_{\lambda} \). The gluing procedure is schematically shown in figure 6.

As a result of this topological gluing, we obtain a new manifold denoted as \( \Delta_4 \), consisting of four sheets \( \Delta_1, \Delta_2 \). Note that each of \( \Delta_1, \Delta_2 \) contains an asymptotically flat region \( r^+ \to \infty \), \( z^2 = 1, 2 \), and so \( \Delta_4 \) represents the spacetime with four asymptotically flat regions connecting by the throats \( a_{\lambda} \). It is also worth noticing that the resulting manifold \( \Delta_4 \) is homeomorphic to a self-crossing four-sheeted Riemann surface.

The complete solution \( \phi(\rho, z) \), defined on the whole manifold \( \Delta_4 \), is constituted from separate \( \phi_{\lambda\in\ell_2}(\rho, z) \) as follows:

\[
\phi(\rho, z) = \bigcup_{\ell_1, \ell_2} \{ \phi_{\lambda\in\ell_2}(\rho, z) \text{ if } (\rho, z) \in D_{\ell_1, \ell_2} \}.
\]  

(26)

To argue that \( \phi(\rho, z) \) is a smooth function on \( \Delta_4 \), we note that the single-wormhole solutions \( \phi_{\lambda}(\rho, z - z_{\lambda}; a_{\lambda}) \) and \( \phi(\rho, z - z_{\lambda}; a_{\lambda}) \) are smoothly matched at the corresponding throat \( T_{a_{\lambda}} \). A plot of \( \phi(\rho, z) \) is shown in figure 8.

The function \( \nu(\rho, z) \) could be found as the line integral

\[
\nu(\rho, z) = \int_{C} \left[ -\frac{1}{2} \rho (\phi^2_{\rho} - \phi^2_{z}) d\rho - \rho \phi_{\rho} \phi_{\rho} dz \right],
\]

where \( C \) is an arbitrary path connecting some initial point \( (\rho_0, z_0) \) and any point \( (\rho, z) \) in \( \Delta_4 \). Note that if \( (\rho_0, z_0) \) and \( (\rho, z) \) are lying in different charts \( D_{\ell_1, \ell_2} \), then the path \( C \) should pass through some throats \( T_{a_{\lambda}} \). Coordinates of the initial point \( (\rho_0, z_0) \) are determined by boundary conditions for the function \( \nu \). Imposing the following boundary conditions

\[
\lim_{\rho \to +\infty, z \to -\infty} \nu(\rho, z) = 0, \quad (\rho, z) \in D_{++},
\]

substituting \( \phi \) from equation (26), and integrating, we can obtain the set of functions \( \nu_{\lambda\in\ell_2}(\rho, z) \) defined on \( D_{\ell_1, \ell_2} \), respectively. Finally, the complete solution \( \nu(\rho, z) \) defined on \( \Delta_4 \) is constituted from the separate \( \nu_{\lambda\in\ell_2}(\rho, z) \) as follows.

Figure 8. The scalar field \( \phi(\rho, z) \) on the manifold \( \Delta_2 \) which possesses 2\( ^2 \) asymptotically flat regions and corresponds to the two-wormhole spacetime. Segments \( T_{a_{\lambda}} = [0, a_{\lambda}], z_{\lambda} \) (\( i = 1, 2 \)) correspond to the wormhole’s throats with the radii \( a_1 = 1, a_2 = 2 \), and the positions \( z_1 = 1, z_2 = -1 \).
4.3. \(N\) wormholes

A multi-wormhole solution could be constructed analogously to the two-wormhole one. For this aim we consider \(2^N\) different superpositions of \(N\) single-wormhole solutions:

\[
\phi_{\epsilon_1\epsilon_2\ldots\epsilon_N}(\rho, z) = \frac{1}{N} \sum_{i=1}^{N} \phi_i(\rho, z - z_i; a_i),
\]

where \(\phi_i(\rho, z - z_i; a_i)\) is given by equation (25), \(i = 1, 2, \ldots, N\), and \(\epsilon_i\) is an index which stands on the \(i\)th position and takes two values, + or −, i.e., \(\epsilon_i = \pm\). Each of \(\phi_{\epsilon_1\epsilon_2\ldots\epsilon_N}(\rho, z)\) is defined on the half-plane \(D = [0, \infty) \times (-\infty, \infty)\). Consider \(2^N\) identical copies of \(D\), denoted as \(D_{\epsilon_1\epsilon_2\ldots\epsilon_N}\), and then construct a new manifold \(\mathcal{D}_{2^N}\) gluing the half-planes \(D_{\epsilon_1\epsilon_2\ldots\epsilon_N}\). The gluing procedure is the following: the charts \(D_{\epsilon_1\epsilon_2\ldots\epsilon_N}\) and \(D_{\epsilon_1\epsilon_2\ldots\epsilon_N}\) are glued along the \(i\)th segment \(\{\rho \in [0, a_{i+1}], z = z_i\}\).

As a result of this topological gluing, we obtain a new manifold \(\mathcal{D}_{2^N}\), consisting of \(2^N\) sheets \(D_{\epsilon_1\epsilon_2\ldots\epsilon_N}\). Each of \(D_{\epsilon_1\epsilon_2\ldots\epsilon_N}\) contains an asymptotically flat region \((\rho^2 + z^2)^{1/2} \to \infty\), and so \(\mathcal{D}_{2^N}\) represents the spacetime with \(2^N\) asymptotically flat regions connecting by the throats \(T_{\epsilon_i}\). It is also worth noticing that the resulting manifold \(\mathcal{D}_{2^N}\) is homeomorphic to a self-crossing \(2^N\)-sheeted Riemann surface.

The complete solution \(\phi(\rho, z)\), defined on the whole manifold \(\mathcal{D}_{2^N}\), is constituted from separate \(\phi_{\epsilon_1\epsilon_2\ldots\epsilon_N}(\rho, z)\) as follows:

\[
\nu(\rho, z) = \bigcup_{\epsilon_1\epsilon_2} \{\nu_{\epsilon_1\epsilon_2}(\rho, z) \text{ if } (\rho, z) \in D_{\epsilon_1\epsilon_2}\}. \tag{27}
\]
\[ \phi(\rho, z) = \bigcup_{\epsilon_1, \ldots, \epsilon_N} \{ \phi_{\epsilon_1 \epsilon_2 \cdots \epsilon_N}(\rho, z) \text{ if } (\rho, z) \in D_{\epsilon_1 \epsilon_2 \cdots \epsilon_N} \}. \]

(29)

Stress that the scalar field \( \phi(\rho, z) \) is a smooth function on \( \mathcal{D}_{2N} \), because the separate single-wormhole solutions \( \phi_{\epsilon}(\rho, z - z_0; a_i) \) and \( \phi_{\epsilon}(\rho, z - z_i; a_i) \) are smoothly matched at the corresponding throat \( \mathcal{T}_n = \{ \rho \in [0, z_0], z = z_i \} \).

The metric function \( \nu(\rho, z) \) is found as the line integral

\[ \nu(\rho, z) = \int_{C} \left[ -\frac{1}{2} \rho (\phi_{\rho}^2 - \phi_{z}^2) d\rho - \rho \phi_{\rho} \phi_{z} dz \right], \]

where \( \phi \) is given by equation (29), and \( C \) is an arbitrary path connecting some initial point \( (\rho_0, z_0) \) and any point \( (\rho, z) \) in \( \mathcal{D}_{2N} \).

As an illustration of a multi-wormhole configuration with \( N > 2 \), in figure 9 we depict the scalar field in the wormhole spacetime \( \mathcal{D}_{2N} \) with \( 2^N \) asymptotically flat regions connecting by the three throats.

### 5. Regularity of multi-wormhole configurations

A spacetime with a given metric is everywhere regular, i.e. has no singularities, if and only if the corresponding Kretchmann scalar \( K = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \) is also regular. Let us consider the Ricci and Kretchmann scalars, \( R \) and \( K \), computed for the metric (1). Taking into account the assumption \( \lambda = 0 \) and using the relations (7c) and (7d), we find

\[ R = -e^{-2\nu} [\phi_{\rho}^2 + \phi_{z}^2], \]

(30)

\[ K = 3e^{-4\nu} (\phi_{\rho}^2 + \phi_{z}^2)^2. \]

(31)

One can see that \( K = 3R^2 \), and so, in order to characterize the regularity of multi-wormhole configurations found above, it is enough to discuss the Ricci scalar \( R \) given by equation (30). Above we have shown that the scalar field \( \phi(\rho, z) \), generally given by equation (29), is single-valued, smooth and differentiable everywhere in the multi-wormhole spacetime. Therefore, the derivatives \( \phi_{\rho} \) and \( \phi_{z} \) are everywhere finite (one can show that they are also smooth and differentiable). The metric function \( \nu(\rho, z) \) is found as the line integral (22) so that the value \( e^{-2\nu} \) is everywhere finite. As the result, the Ricci scalar \( R \) is also everywhere finite, and ultimately the resulting multi-wormhole configuration is regular and has no singular structures.

### 6. Conclusions

Many years ago, in 1921 Bach and Weyl [10] derived the method of superposition to construct new axially symmetric vacuum solutions of general relativity and obtained the famous solution interpreted as equilibrium configurations of a pair of Schwarzschild black holes. In this paper we have extended the Bach–Weyl approach to non-vacuum configurations with massless scalar fields. Considering phantom scalar fields with the negative kinetic energy, we have constructed multi-wormhole solutions describing an axially symmetric superposition of \( N \) wormholes. Let us enumerate basic properties of the obtained solutions.

- The most unusual property is that the multi-wormhole spacetime has a complicated topological structure. Namely, in the spacetime there exist \( 2^N \) asymptotically flat regions.
connected by throats, so that the resulting spacetime manifold is homeomorphic to a self-crossing $2^N$-sheeted Riemann surface.

- The spacetime of multi-wormholes is everywhere regular and has no event horizons. This feature drastically tells the multi-wormhole configuration from other axially symmetric vacuum solutions. As is known, all static solutions, representing the superposition of two or more axially symmetric bodies, inevitably contain gravitationally inert singular structures, ‘struts’ and ‘membranes’, that keep the two bodies apart making a stable configuration. This is an expected ‘price to pay’ for the stationarity. However, the multi-wormholes are static without any singular struts. Instead, the stationarity of the multi-wormhole configuration is provided by the phantom scalar field with the negative kinetic energy. Phantom scalars represent exotic matter violating the null energy condition. Therefore, now an ‘exoticism’ of the matter source supporting the multi-wormholes is a price for the stationarity.

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