1. Introduction. We consider a viscous, compressible one layer shallow water model under discontinuous stochastic forcing. In the deterministic setting, the two-dimensional velocity $v$ and the height $h$ of the fluid are given by the following coupled system of PDEs:

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + g\nabla h + f k \times v = F \quad \text{in } \mathcal{M} \times (0, T) \quad (1a)$$

$$\frac{\partial h}{\partial t} - \delta \Delta h + \nabla \cdot (hv) = 0, \quad \text{in } \mathcal{M} \times (0, T), \quad (1b)$$

in a two-dimensional domain $\mathcal{M}$, with prescribed initial conditions and, say, Dirichlet boundary conditions. Here, $v = (u, v)$, where $u := u(x, y, t)$ is the velocity of the water in the $x$ direction and $v := v(x, y, t)$ is the velocity of the water in the $y$ direction; $h := h(x, y, t)$ is the depth of the water, but we will assume that $h = H + \hat{h}$ where $H > 0$ is the average depth of the water, a constant, and $\hat{h}$ is the amount by which the height of the water deviates from its average depth. In $(1) \nu > 0$ is the constant of viscosity, $\delta$ is a viscosity-like positive constant, $g$ is the gravitational constant and $f$ is the Coriolis parameter, which is assumed to be constant. We use the vector notation $k \times v$, which in 2D means $k \times v := (-v, u)$ when $v = (u, v)$. The term $F := F(x, y, t)$ in the conservation of momentum equation $(1a)$ is an external forcing term corresponding to, e.g., bursts of surface winds. See, e.g., [39] for a realistic example of the forcing term $F$; see also [8] and [38] for other models.

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forcing terms, and who include a term involving 1\( /h \) in the momentum equation \( (1a) \). We use the model \( (1) \) without the additional term involving 1\( /h \), which is most similar to the one used by [35] in the deterministic setting and [26] in the stochastic setting, for the convenience in our analysis. In the deterministic setting, a global solution to the system \( (1) \) exists for small initial data. For general initial data, \( (1) \) admits a unique solution for only a short period of time.

This article builds upon the paper [26] of the second two authors, which considered the stochastic shallow water system driven by Wiener processes:

\[
dv + (-\nu \Delta v + (v \cdot \nabla) v + g \nabla h + f k \times v) dt = F dt + \sigma_1(v, h) dW_1 \quad \text{in } \mathcal{M} \times (0, T) \tag{2a}
\]

\[
dh + (-\delta \Delta h + \nabla \cdot (hv)) dt = \sigma_2(v, h) dW_2 \quad \text{in } \mathcal{M} \times (0, T). \tag{2b}
\]

The system \( (2) \) builds upon \( (1) \) with the addition of external stochastic forcing driven by Wiener processes \( W_1 \) and \( W_2 \). The additional noise terms correspond to the effects of wind that randomly perturbs the surface of the water. Because of the randomness in the Wiener processes \( W_1 \) and \( W_2 \), the solution \( (v, h) \) to \( (2) \) must also be regarded as a stochastic process. Wiener processes are widely used in SPDEs to incorporate random fluctuations in PDEs that are continuous in time. Due to the continuity, \( (2) \) can only describe the behavior of the water under continuous random fluctuations in the wind. In the present article we extend \( (2) \) by incorporating additional external stochastic forcing terms driven by a canonical noise process with jump discontinuities known as a Lévy process. We consider the system

\[
dv + (-\nu \Delta v + (v \cdot \nabla) v + g \nabla h + f k \times v) dt = F dt + \sigma_1(v, h) dW_1 + \int_{E_0} \mathcal{X}_1(v, h, z) d\hat{\pi}_1(t, z), \quad \text{in } \mathcal{M} \times (0, T), \tag{3a}
\]

\[
dh + (-\delta \Delta h + \nabla \cdot (hv)) dt + \int_{E_0} \mathcal{X}_2(v, h, z) d\hat{\pi}_2(t, z), \quad \text{in } \mathcal{M} \times (0, T), \tag{3b}
\]

where \( E \) is the Hilbert space, minus the zero vector, where the Lévy process takes its values and \( E_0 \) is the open unit ball in \( E \). The system \( (3) \) is supplemented with initial conditions:

\[
v(0) = v_0(x, y) \quad \text{in } \mathcal{M}, \tag{3c}
\]

\[
h(0) = h_0(x, y) > 0 \quad \text{in } \mathcal{M}, \tag{3d}
\]

and we assume, for instance, the Dirichlet boundary conditions:

\[
v = 0 \quad \text{on } \partial \mathcal{M} \times (0, T), \tag{3e}
\]

\[
h = 0 \quad \text{on } \partial \mathcal{M} \times (0, T). \tag{3f}
\]

The Lévy noise, which has jump discontinuities, corresponds to random bursts of wind. Both the times and intensities at which the wind bursts occur are random. The Lévy noise is specified by two cylindrical Wiener processes \( W_i \), \( i = 1, 2 \), Poisson random measures \( \pi_i \), \( i = 1, 2 \) and compensated Poisson random measures \( \tilde{\pi}_i \), \( i = 1, 2 \). The Poisson random measures \( \pi_i \), \( i = 1, 2 \), represent jumps of the Lévy process of large size, i.e., bounded away from 0 by a fixed constant, while the compensated Poisson random measures \( \tilde{\pi}_i \), \( i = 1, 2 \), represent the jumps of the Lévy process of arbitrarily small size. The noise processes will be defined in precise terms in Section
2.2. In (3) we will impose certain typical Lipschitz and growth conditions on the given functions \( \sigma_1, \sigma_2, K_1 \) and \( K_2 \) (see Subsection 2.3).

Our main results are Theorems 2.7 and 2.8, which establish local existence and uniqueness of solutions to the system (3) for two different types of solutions. In Theorem 2.7 we establish local existence of solutions to (3) on some unknown probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and with respect to some unknown noise processes \(\tilde{W}_i, \tilde{\pi}_i, i = 1, 2\). This type of solution is known as a probabilistically weak or martingale solution (see Definition 2.4). Martingale solutions to other SPDEs in fluid dynamics driven by Lévy noise have been obtained in, e.g., [29] and [30]. In Theorem 2.8 we establish local existence of solutions to (3) with respect to given noise processes on a given probability space. This type of local solution is known as a probabilistically strong or pathwise solution (see Definition 2.5). See, e.g., [5] and [19] for additional settings in fluid dynamics where pathwise solutions to SPDEs driven by Lévy noise have been obtained. In addition, we obtain in Theorems 2.7 and 2.8 maximal local solutions that cannot be properly extended in time because of a blow up in the \( H^1_0(M) \)-norm of the solution (see (32)).

This article is similar to [26] in certain aspects. As in [26] we use a Galerkin method and compactness methods to establish local existence of martingale solutions to a truncated version of (3). We then establish pathwise uniqueness and use a well-known argument due to Gyöngy and Krylov that generalizes the Yamada-Watanabe lemma to infinite dimensions and implies the existence of a local pathwise solution. The implementation of this strategy in the current article differs from [26] due to the discontinuity of the Lévy noise. For instance, [26] obtains a continuous local solution in time using classical compact embedding results, due to [36] and [20], of fractional Sobolev spaces into spaces of continuous functions in time. This tool is not available in the present context because of the jump discontinuities of the Lévy noise. The Lévy noise is merely càdlàg in time, i.e., right-continuous with left-limits. So, we expect the local solution \((v, h)\) to (3) to be càdlàg as well. Instead of using the compact embedding results as in [26], we obtain a solution \((v, h)\) with càdlàg sample paths in the Lévy noise case using probabilistic methods in a similar way to [29]. Specifically, we verify the Aldous condition of [1], which is similar to a compactness criterion of [36], and obtain tightness of the laws of the Galerkin approximations on an appropriate space of càdlàg functions endowed with the Skorokhod topology.

Stochastic fluid models driven by Lévy noise have been considered in several additional settings. The Navier-Stokes equations with Lévy noise have been considered in 2-D by [17] and in 3-D by [29]. The Euler equations with Lévy noise have been considered by [11]. Concerning the numerical approximation of stochastic differential equations driven by a Lévy noise, see e.g. [23]. A general treatment of stochastic hydrodynamical systems with Lévy jump noise is given in [3], which overlaps with the current work without either paper being contained in the other. Both the current article and [3] show the existence of a maximal local pathwise solution. In addition to the examples mentioned in [3], their abstract setting for the hydrodynamical system also includes the one layer shallow water system (1) although this is not said explicitly. The main difference between [3] and the current article is in the Lévy noise; the abstract stochastic evolution equation considered in [3] contains only the pure jump noise without Wiener part, whereas we investigate the full Lévy noise. In addition, all the examples from fluid mechanics in [3] correspond to the classical situation where the nonlinear term enjoys the well-known
cancellation property \((F(u), u) = 0\), which does not hold in any model of shallow water equations. Treatment of a fluid mechanics system without the cancellation property was the motivation for the article [26] dealing with the Wiener noise case as well as this article.

Our implementation of the strategy outlined above, which is based on [15], is most similar to [11, 29, 30]. More precisely, in the theory of nonlinear evolutionary partial differential equations, when the \(L^p\) bounds on the time approximation solutions are obtained, the Galerkin approximation scheme gives us the estimates on the time derivatives and we can apply the classical compactness results such as those of Aubin-Lions or Arzela-Ascoli in order to pass to the limit for the nonlinear term. However, in the case of stochastic partial differential equations, those results are useless due to the lack of differentiability in time of a solution. We utilize a different compactness result based on fractional Sobolev spaces that enable us to pass to the limit for the nonlinear term. We also verify the Aldous condition (see [1] and below) to obtain a compactness property of the laws of the approximations. We employ commonly used tools from stochastic analysis, such as Itô’s formula and the Burkholder-Davis-Gundy inequality, to make estimates in our Galerkin scheme. A martingale solution is also obtained during the process of passing to the limit along with shifting the original underlying probability space and noise processes. While existence of martingale solutions to certain stochastic hydrodynamical systems driven by Lévy noise has also been studied in [29, 30], we are further able to establish pathwise uniqueness and hence existence of local pathwise solutions. This article contributes to existing works on SPDEs with Lévy noise an improvement on the càdlàg regularity of the solution by extending an argument of [14] from the Wiener noise case to the Lévy noise case (see Subsection 6.1). In Section 5 we show that verifying the Aldous condition yields a candidate solution \((v, h)\) that, a priori, is a càdlàg function of time into the space \((L^2(\mathcal{M}))^2 \times L^2(\mathcal{M})\) almost surely. In order to establish pathwise uniqueness and to obtain a maximal local solution to (3) we establish the stronger property that \((v, h)\) is a càdlàg function of time into the space \((H^1_0(\mathcal{M}))^2 \times H^1_0(\mathcal{M})\) almost surely.

The article is organized as follows: In Section 2 we recall background information from the PDE and probabilistic frameworks and set the notations. In Subsection 2.1 we recall the relevant function spaces where the solution \((v, h)\) to equation (3) will take values. In Subsection 2.2 we define the notion of a Lévy process in an infinite dimensional Hilbert space. We also briefly recall the definitions of the stochastic integrals appearing on the right-hand sides of (3). In Subsection 2.3 we make precise the notions of local martingale solutions and local pathwise solutions to (3). With the background information in hand, we make formal a priori estimates for a smooth solution to (3) in Section 3. This will illustrate the need to truncate the original system (3) in a Galerkin method and suggest a truncation to use. In Section 4 we introduce a modified, truncated version of (3) that will be solved using a stochastic Galerkin method in subsequent sections. We define the Galerkin approximations to the modified system in Subsection 4.1 and show that the Galerkin approximations are bounded in various spaces in Subsections 4.2 and 4.3. We establish compactness properties of the Galerkin approximations in Section 5 by verifying the Aldous condition. We pass to the limit in Section 6 and obtain a global solution to the truncated version of (3) on a new probability space (i.e., a martingale solution). In Section 7 we establish pathwise uniqueness of global solutions to the truncated equation and pathwise uniqueness of local solutions to the original system.
2.1. Function spaces. We apply a well-known argument based on a result of Gyöngy and Krylov to obtain global solutions to the truncated system on the original probability space (i.e., a pathwise solution) in Section 8. Finally, we remove the truncation and obtain maximal local pathwise solutions to the original system (3) in Section 9. Several tools from probability theory related to weak convergence and tightness of probability measures are used in Sections 5, 6 and 8. In particular, we use the Aldous condition for tightness of probability measures on the space of càdlàg functions from an interval $[0,T]$ to a Hilbert space. The main tools concerning weak convergence, tightness and the Skorokhod topology on the space of càdlàg functions are collected in Appendix A.

Concerning the terminology, note that in this article and in earlier articles, we use the terminology of martingale and pathwise solutions as in e.g. [30], [19], where other authors use the terms weak and strong stochastic solutions, as in e.g. [5]. This terminology of martingale and pathwise solutions is preferred by those closer to partial differential equations, since the terms of weak and strong solutions have been used for a long time for parabolic equations like the Navier-Stokes equations, where weak solutions refer to solutions in $L^\infty(L^2)$ and strong solutions refer to solutions in $L^\infty(H^1)$.

2. Analytic tools. In this section, we collect and define the deterministic and stochastic tools needed throughout this article. Additionally, we carefully define the types of solutions we are seeking.

\subsection{2.1. Function spaces.} We will work in the spaces $H = H_1 \times H_2$, $V = V_1 \times V_2$ and $D(-\Delta) := D(A_1) \times D(A_2)$, where

\begin{align*}
H_1 &:= L^2(\mathcal{M})^2, \quad V_1 := (H^1_0(\mathcal{M}))^2, \quad D(A_1) := V_1 \cap (H^2(\mathcal{M}))^2 \\
H_2 &:= L^2(\mathcal{M}), \quad V_2 := H^1_0(\mathcal{M}), \quad D(A_2) := V_2 \cap H^2(\mathcal{M}).
\end{align*}

On $H_2$ and $H_1$, we will use the typical $L^2$--inner product and norm denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively, while on $V_1$ and $V_2$, we will use $(\cdot, \cdot)$ and $||\cdot||$, which are the usual $L^2$--inner product and norm of the gradients.

We also consider the fractional powers of the operator $(-\Delta)$ with the boundary conditions (3c) and (3d). Classically, there exists an orthonormal basis $\{\psi_k\}_{k \geq 1}$ of $H$ with an unbounded increasing sequence of eigenvalues $\{\lambda_k\}_{k \geq 1}$ such that $-\Delta \psi_k = \lambda_k \psi_k$. We have $D(-\Delta) = D(A_1) \times D(A_2) = V \cap (H^2(\mathcal{M}))^3$ and for $\alpha \geq 0$ we define:

\begin{equation}
D((-\Delta)\alpha) = \left\{ u \in H : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < \infty \right\},
\end{equation}

endowed with the Hilbertian norm

\begin{equation}
|u|_\alpha := |(-\Delta)\alpha u| = \left( \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 \right)^{1/2}.
\end{equation}

Here, $u = \sum_{k=1}^{\infty} u_k \psi_k$ with $|u|^2 = \sum_{k=1}^{\infty} |u_k|^2 < \infty$.

For the Galerkin scheme below, we introduce the finite dimensional spaces $H_n = \text{span}\{\psi_1, \ldots, \psi_n\}$ and let $P_n, Q_n = I - P_n$ be the projection operators in $H$ onto $H_n$ and onto its orthogonal complement. By abuse of notation we will use also the operator $P_0$ to denote $P_0 \psi = P_0(\psi, 0)$ and $P_0h = P_0(0, h)$. We have the generalized and reverse Poincaré inequalities:

\begin{equation}
|P_n u|_{\alpha_2} \leq \lambda_n^{\alpha_2 - \alpha_1} |P_n u|_{\alpha_1} \quad \text{and} \quad |Q_n u|_{\alpha_1} \leq \frac{1}{\lambda_n^{\alpha_2 - \alpha_1}} |Q_n u|_{\alpha_2},
\end{equation}

where $\lambda_n$ are the eigenvalues of $H(-\Delta)$.\]
which hold for any $\alpha_1 < \alpha_2$ and all $u \in H$ and $n \geq 1$.

2.2. Stochastic preliminaries. In order to make sense of the stochastic terms in (3a) and (3b), we first recall the definitions and some properties of Hilbert space-valued Wiener processes and Lévy processes. In this section we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We begin by recalling Hilbert space-valued Lévy processes.

Definition 2.1. Let $U$ be a Hilbert space. A stochastic process $(L_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $(U, \mathcal{B}(U))$ is called a Lévy process if

i) $L_0 = 0$ a.s.,

ii) $L$ has independent increments, i.e., for every $0 = t_0 < t_1 < \cdots < t_n$ the random variables $L_{t_1} - L_{t_0}, L_{t_2} - L_{t_1}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent,

iii) $L$ has stationary increments, i.e., $L_t - L_s$ has the same distribution as $L_{t-s}$ for all $0 \leq s < t$,

iv) $(L_t)_{t \geq 0}$ is stochastically continuous, i.e., $\forall \epsilon > 0, \lim_{s \to t} \mathbb{P}(|L_s - L_t| > \epsilon) = 0$.

We will usually just use the letter $L$ when we speak about a Lévy process $(L_t)_{t \geq 0}$. Every Lévy process admits a càdlàg modification by, e.g., Theorem 4.3 in [32]. More precisely, for every Lévy process $L$ there exists a càdlàg process $\tilde{L}$ such that $\mathbb{P}[L_t = \tilde{L}_t] = 1$ for every $t \geq 0$ (i.e., $\tilde{L}$ is a modification of $L$) and such that, $\mathbb{P}$-a.s., the function $t \mapsto \tilde{L}_t$ is right continuous at every $t \geq 0$ and has left-hand limits at every $t > 0$ (i.e., $\tilde{L}$ is càdlàg). We will only consider càdlàg Lévy processes. Almost surely, a càdlàg Lévy process $L$ has at most countably many jumps on any interval $[0, T]$. This is because, for each positive integer $n$, a càdlàg function can only have finitely many jumps of size exceeding $1/n$ on $[0, T]$.

Let $L$ be a càdlàg Lévy process with values in a Hilbert space $U$. The jump of $L$ at time $t > 0$ is denoted by $\Delta L_t = L_t - L_{t-}$. Let $A \in \mathcal{B}(U)$ and define

$$\pi(t, A) := \pi(t, A, \omega) := \sum_{\substack{s \geq 0 \leq t \\ \Delta L_s(\omega) \neq 0}} \chi_A(\Delta L_s(\omega)).$$

So, $\pi(t, A)$ is the number of jumps of $L$ that occur before or at time $t$ and fall in the set $A$. More generally, for $\Gamma \in \mathcal{B}(\mathbb{R}^+ \times U)$, we define

$$\pi(\Gamma) := \sum_{\substack{s \geq 0 \\ \Delta L_s \neq 0}} \delta_{(s, \Delta L_s)}(\Gamma).$$

Equation (7) defines a random measure $\pi$ that agrees with the quantity $\pi(t, A)$ defined in (7) when $\Gamma = [0, t] \times A$. We call $\pi$ the jump measure of $(L_t)_{t \geq 0}$. It is well-known that $\pi$ is a Poisson random measure (see, e.g., [32]). Each Lévy process $L$ gives rise to a positive Borel measure $\nu$ on $U \setminus \{0\}$ defined by the property that $\nu(A)$ is the expected rate of jumps of $L$ that lie in $A$, for every $A \in \mathcal{B}(U \setminus \{0\})$, i.e.,

$$\nu(A) := \frac{1}{t} \cdot \mathbb{E}[\pi((0, t] \times A)].$$

The fact that the right-hand side of (8) does not depend on the value of $t > 0$ follows from the fact that $\pi$ is a Poisson random measure. We call $\nu$ the Lévy measure of $L$. Since $L$ is càdlàg $\mathbb{P}$-a.s., we have $\nu(A) < \infty$ when $0 \not\in \overline{A}$. Indeed, when $0 \not\in \overline{A}$, the nonnegative integer-valued process $(\pi((0, t] \times A))_{t \geq 0}$ is a Poisson process and its rate, which is finite, is $\nu(A)$. In particular, $\nu$ is a $\sigma$-finite measure.
Let \( \gamma \) be i.i.d. real-valued Brownian motions and Brownian motion. They can be constructed as follows. Let \( t \in [0, T] \) be continuous almost surely. Wiener processes are infinite dimensional generalizations of compound Poisson process, which we recall below.

Brownian motions in each direction scaled by \( \gamma \). In the coordinates of the orthonormal basis, \( \gamma \) is a Wiener process, i.e., \( \mathbb{P}\text{-a.s. in the space } C([0, T]; U) \) (see, e.g., Theorem 4.3 in [12]). It is simple to check that the process in (9) converges to a Wiener process, i.e., \( W_t := \sum_{n=1}^{\infty} \gamma_n \beta_n(t) u_n \). (9)

In the coordinates of the orthonormal basis, \( W_t \) evolves according to independent Brownian motions in each direction scaled by \( \gamma_n \). The scaling factors \( \{\gamma_n\}_{n=1}^{\infty} \) are required in order to ensure that the sum in (9) converges to a \( U \)-valued random variable. One can show that the sum in (9) converges \( \mathbb{P}\text{-a.s. in the space } C([0, T]; U) \) (see, e.g., Theorem 4.3 in [12]). It is simple to check that the process \( W_t \) defined in (9) is a Wiener process, i.e., \( W_t \) satisfies the conditions in Definition 2.1 and \( W_t \) is continuous almost surely. Conversely, every Wiener process is of the form in (9). Let \( W_t \) be a \( U \)-valued Wiener process, expressed in the form (9), and let \( Q \) denote the unique bounded linear operator on \( U \) with eigenvectors \( \{\gamma_n\}_{n=1}^{\infty} \) and eigenvalues \( \{\gamma_n^2\}_{n=1}^{\infty} \). Then \( Q \) is positive and of trace class; we call it the covariance operator of \( W_t \). The space \( \mathfrak{U} := Q^{1/2}(U) \) equipped with the inner product

\[
\langle x, y \rangle_{\mathfrak{U}} := \langle Q^{-1/2}x, Q^{-1/2}y \rangle_U,
\]

where \( Q^{-1/2} \) is the pseudoinverse of \( Q^{1/2} \), plays an important role in constructing the stochastic integral with respect to \( W_t \). We call \( \mathfrak{U} \) the reproducing kernel Hilbert space of \( W_t \). Let \( X \) and \( \mathfrak{U} \) be Hilbert spaces. We denote by \( \mathcal{L}(\mathfrak{U}, X) \) the space of bounded linear operators from \( \mathfrak{U} \) to \( X \). We denote by

\[
L_2(\mathfrak{U}, X) := \{R \in \mathcal{L}(\mathfrak{U}, X) : \sum_{k=1}^{\infty} |R e_k|^2_X < \infty \}
\]

the set of Hilbert-Schmidt operators from \( \mathfrak{U} \) to \( X \). This space \( L_2(\mathfrak{U}, X) \) is a Hilbert space endowed with the following inner product and norm

\[
\langle R, S \rangle_{L_2(\mathfrak{U}, X)} = \sum_{k=1}^{\infty} \langle R e_k, S e_k \rangle_X \quad \text{and} \quad \|R\|_{L_2(\mathfrak{U}, X)}^2 = \sum_{k=1}^{\infty} |R e_k|^2_X.
\]

The processes that we integrate with respect to the Wiener process \( W_t \) take values in the space \( L_2(\mathfrak{U}, X) \). In the context of SPDEs one often has a particular function in mind for the multiplicative Wiener noise, i.e., a specific choice of \( \sigma_1 \) and \( \sigma_2 \) in (3a) and (3b) in the present case. As these functions are \( L_2(\mathfrak{U}, X) \)-valued, this also amounts to a specific choice of the Hilbert space \( \mathfrak{U} \). Using a cylindrical Wiener process construction, it is possible to define a Wiener process \( W_t \) on some larger Hilbert space \( U_1 \) such that \( \mathfrak{U} \) is the reproducing kernel Hilbert space of \( W_t \). Any real, separable Hilbert space \( U_1 \) that contains \( \mathfrak{U} \) with a Hilbert-Schmidt embedding will do and the resulting stochastic integral does not depend on the space \( U_1 \) or the choice of the Hilbert-Schmidt embedding of \( \mathfrak{U} \) into \( U_1 \). We refer the reader to [12] for the details of the cylindrical Wiener process construction.

The fundamental example of a Lévy process with jump discontinuities is the compound Poisson process. A \( U \)-valued process \( P \) is a compound Poisson process.
if and only if there exists a finite positive Borel measure \( \mu \) on \( U \setminus \{0\} \), a Poisson process \( \Pi \) with intensity \( \mu(U) \) and i.i.d. \( U \)-valued random variables \( (Z_j)_{j=1}^\infty \) that are independent of \( \Pi \) such that
\[
P(t) = \sum_{j=1}^{\Pi(t)} Z_j \quad \text{for all } t \geq 0.
\]

Thus, \( P \) has jumps at the same times as \( \Pi \) and the value of the \( j \)th jump is \( Z_j \). We refer to [32] for a treatment of compound Poisson processes. When \( \mathbb{E}|P(t)| < \infty \), which occurs if and only if \( \int_U |y| \mu(y) < \infty \), we define the compensated compound Poisson process \( \hat{P}_t := P_t - \mathbb{E}[P_t] \).

The structure of a general Lévy process is described by the Lévy-Khinchin decomposition (see, e.g., Theorem 4.23 in [32]). This result says that every \( U \)-valued Lévy process \( L \) can be decomposed as a sum
\[
L_t = at + W_t + P^0_t + \sum_{n=1}^\infty \hat{P}^n_t,
\]
where \( a \in U \) is a fixed vector, \( W \) is a Wiener process, \( P^0 \) is a compound Poisson process, \( \{\hat{P}^n\}_{n=1}^\infty \) are compound Poisson processes and all of the process on the right-hand side of (13) are independent. Noise driven by a Lévy process \( L \) is incorporated in the main equations (3a) and (3b) using three notions of stochastic integration, one for each of the processes \( W \), \( P^0 \) and \( \sum_{n=1}^\infty \hat{P}^n \) appearing in the decomposition (13). Before discussing stochastic integration we first review the notions of filtrations and predictability.

An increasing family of \( \sigma \)-subfields \( (\mathcal{F}_t)_{t \geq 0} \) of \( \mathcal{F} \) is called a filtration. We call \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) a filtered probability space.

**Definition 2.2.** Let \( (\mathcal{F}_t)_{t \geq 0} \) be a filtration on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( T > 0 \). We denote by \( \mathcal{P}_{[0,T]} \) the \( \sigma \)-field of subsets of \( \Omega \times [0,T] \) generated by sets of the form
\[
A \times (s, t], \quad \text{where } A \in \mathcal{F}_s \text{ and } 0 \leq s \leq t \leq T.
\]
We call \( \mathcal{P}_{[0,T]} \) the predictable \( \sigma \)-field. We say that a stochastic process \( (X_t)_{t \in [0,T]} \) is predictable if and only if it is \( \mathcal{P}_{[0,T]} \)-measurable as a function of both \( \omega \) and \( t \).

**Definition 2.3.** Let \( L \) be a \( U \)-valued Lévy process defined on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \). We say that \( L \) is an \( \mathcal{F}_t \)-Lévy process if \( L \) is adapted to the filtration \( (\mathcal{F}_t)_{t \geq 0} \) and
\[
L_t - L_s \quad \text{is independent of } \mathcal{F}_s \text{ for all } t \geq s \geq 0.
\]
If \( L \) is also a Wiener process, then we will simply say that \( L \) is an \( \mathcal{F}_t \)-Wiener process.

We now recall the notions of stochastic integration that will be used here. Let \( X \) be a real, separable Hilbert space. First, we consider a Wiener process \( W \) on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) with reproducing kernel Hilbert space \( \mathcal{H} \). For use in stochastic integration we assume that \( W \) is an \( \mathcal{F}_t \)-Wiener process. The space of integrands for stochastic integration with respect to \( W \) is
\[
\mathbb{L}^2_{\mathcal{H},T}(X) := L^2(\Omega \times [0,T], \mathcal{P}_{[0,T]}, d\mathbb{P} \otimes dt; L_2(\mathcal{H}, X)),
\]
i.e., \( \mathbb{L}^2_{\mathcal{H},T}(X) \) is the space of predictable, square-integrable, \( L_2(\mathcal{H}, X) \)-valued functions on \( \Omega \times [0,T] \). The main facts about stochastic integration with respect to
$W$ are listed below. For every $\Psi \in L^2_{\Omega,T}(X)$ the stochastic integral $I^W_t(\Psi) := \int_0^t \Psi(s)dW(s)$ is a continuous $X$-valued $L^2$-martingale. By the Burkholder-Davis-Gundy (BDG) inequality (see, e.g., [32]), for every $p \in [1, \infty)$ there exists a constant $C_p > 0$ such that for every $\Psi \in L^2_{\Omega,T}(X)$ and every $F_t$-stopping time $\tau$ we have

$$\mathbb{E}\left[ \sup_{t \in [0,\tau]} \left| \int_0^t \Psi(s)dW(s) \right|^p_X \right] \leq C_p \mathbb{E}\left( \int_0^\tau \|\Psi(s)\|^2_{L^2_{\Omega,T}} ds \right)^{p/2}. \quad (16)$$

Second, we consider stochastic integration with respect to the jump measure $\pi$ of a $U$-valued Lévy process $L$ (see equation (7)). We assume that $L$ is an $F_t$-Lévy process. This implies that for each set $A \in \mathcal{B}(U \setminus \{0\})$ the Poisson process $(\pi((0,t] \times A))_{t \geq 0}$ is an $\mathcal{F}_t$-Lévy process. This says that $\pi$ is the Poisson random measure induced by a stationary $\mathcal{F}_t$-Poisson point process, namely the jumps of $L$. We introduce the notation $E := U \setminus \{0\}$ and the spaces

$$\mathbb{P}_{\nu,T}^q(X) := L^q(\Omega \times [0,T] \times E, \mathbb{P}_{[0,T]} \otimes \mathcal{B}(E), d\mathbb{P} \otimes dt \otimes d\nu; X), \quad (17)$$

for $q = 1, 2$. Below we gather basic facts from [24] about integration of functions in these spaces with respect to $\pi$. The main fact is that we are able to integrate functions $f \in \mathbb{P}_{\nu,T}^1(X)$ with respect to the measure $\pi$ for $\mathbb{P}$-a.e. fixed $\omega \in \Omega$. To be precise, for every $f \in \mathbb{P}_{\nu,T}^1(X)$ the following statements hold:

i) $\mathbb{E} \int_{[0,t]} \int_E \left| f(s,z) \right|_X d\pi_t(s,z) = \mathbb{E} \int_0^t \int_E \left| f(s,z) \right|_X d\nu ds < \infty$ for every $t \in [0,T]$.

ii) For each $t \in [0,T]$ the $X$-valued integral $\int_{[0,t]} \int_E f(s,z)d\pi_t(s,z)$ exists a.s. and is equal to the absolutely convergent sum $\sum_{s \in [0,t]} f(s,\Delta L_s)$.

iii) For each $t \in [0,T]$ we have $\mathbb{E} \int_{[0,t]} \int_E f(s,z)d\pi_t(s,z) = \mathbb{E} \int_0^t \int_E f(s,z)d\nu ds$.

For $f \in \mathbb{P}_{\nu,T}^2(X)$ and a set $A \in \mathcal{B}(E)$ with $0 \notin \mathbb{A}$ we have $f \cdot \chi_A \in \mathbb{P}_{\nu,T}^1(X)$ because $\nu(A) < \infty$. Note that the stochastic integral

$$\int_{[0,t]} \int_A f(s,z)d\pi_t(s,z) = \sum_{s \in [0,t]} f(s,\Delta L_s)\chi_A(\Delta L_s) \quad (18)$$

is a sum of finitely many vectors in $X$, $\mathbb{P}$-a.s. Stochastic integration with respect to the compound Poisson process $P_0$ in the Lévy-Khinchin decomposition of $L$, i.e. equation (13), can be described in this manner by taking $A := \{y \in U : |y|_U \geq 1\}$.

Third, we consider stochastic integration with respect to the compensated Poisson random measure $\tilde{\pi}$, which is formally given by the rule $d\tilde{\pi} = d\pi - d\nu \otimes dt$. For $f \in \mathbb{P}_{\nu,T}^1(X) \cap \mathbb{P}_{\nu,T}^2(X)$ we define

$$I^\pi_t(f) := \int_0^t \int_E f(s,z)d\tilde{\pi}_t(s,z) := \int_{[0,t]} \int_E f(s,z)d\pi_t(s,z) - \int_0^t \int_E f(s,z)d\nu ds. \quad (19)$$

It is well-known that $(I^\pi_t(f))_{t \in [0,T]}$ is a purely discontinuous $X$-valued $L^2$-martingale (see, e.g., page 62 of [24] in the case where $X$ is finite-dimensional or Theorem 4.2 and Proposition 4.10 of [34] in the case where $X$ is infinite-dimensional). We also have the isometric formula

$$\mathbb{E} \left| I^\pi_t(f) \right|^2_X = \mathbb{E} \int_0^t \int_E \left| f(s,z) \right|^2_X d\nu ds, \quad \text{for all } t \in [0,T]. \quad (20)$$
Since \( \mathbb{F}^{1,2}_{\nu,T}(X) \cap \mathbb{F}^{2}_{\nu,T}(X) \) is dense in \( \mathbb{F}^{2}_{\nu,T}(X) \) it follows from (20) that the map \( I_T^\pi \) extends uniquely by continuity to \( \mathbb{F}^{2}_{\nu,T}(X) \). It is clear that for every \( f \in \mathbb{F}^{2}_{\nu,T}(X) \) the process \( \{I_T^\pi(f)\}_{t \in [0,T]} \) is still a purely discontinuous \( X \)-valued \( L^2 \)-martingale and that (20) continues to hold. By the BDG inequality (see, e.g., [10]) for every \( E \)

\[
\mathbb{E} \left[ \sup_{t \in [0,\tau]} \left| \int_{(0,t]} \int_E f(s,z) d\pi(s,z) \right|^p \right] \leq C_p \mathbb{E} \left( \int_{(0,\tau]} \int_E |f(s,z)|^2 d\pi(s,z) \right)^{p/2}. \tag{21}
\]

Let \( E_0 := \{ y \in U : 0 < |y| < 1 \} \) and \( f \in \mathbb{F}^{2}_{\nu,T}(X) \). The stochastic integral \( \int_{(0,t]} \int_{E_0} f(s,z) d\pi(s,z) \) represents stochastic integration with respect to the process \( \sum_{n=1}^\infty \hat{P}_n \) in the Lévy-Khinchin decomposition (13); see [10] for details.

The noise in equations (3a) and (3b) will be driven by two independent Lévy processes \( L_1 \) and \( L_2 \) defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). The Wiener process \( W_1 \) appearing in (3a) is the Wiener part in the Lévy-Khinchin decomposition of \( L_1 \) and the Poisson random measure \( \pi_1 \) is the jump measure of \( L_1 \). Likewise, the driving noise \( W_2 \) and \( \pi_2 \) appearing in (3b) are defined based on \( L_2 \) in the same manner. For more general noise we can allow \( W_1 \) and \( W_2 \) to be \( \mathcal{F}_t \)-cylindrical Wiener processes and allow \( \pi_1 \) and \( \pi_2 \) to be the Poisson random measures induced by two stationary \( \mathcal{F}_t \)-Poisson point processes (see [24] for the definition), where all of these processes are independent. Throughout this article, we call a stochastic basis a tuple

\[ S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_1, W_2, \pi_1, \pi_2) \]

where \( (\mathcal{F}_t)_{t \geq 0} \) is a complete, right-continuous filtration such that \( W_1 \) and \( W_2 \) are independent \( \mathcal{F}_t \)-cylindrical Wiener processes, \( \pi_1 \) and \( \pi_2 \) are Poisson random measures on \( (0, \infty) \times E \) induced by independent stationary \( \mathcal{F}_t \)-Poisson point processes that are independent of \( W_1 \) and \( W_2 \). For notational convenience will assume that the cylindrical Wiener processes \( W_1 \) and \( W_2 \) have the same reproducing kernel Hilbert space \( \mathcal{H} \) and take values in the same space \( U \). We will also assume that \( \pi_1 \) and \( \pi_2 \) have the same intensity measure, which has the form \( dt \otimes d\nu \) for some \( \sigma \)-finite measure \( \nu \) on \( E \). For more details on Lévy processes the reader is referred to [24], [32] and the review article [10]; and the references therein.

2.3. Definitions of solutions. We now discuss the specific conditions that we impose on the coefficients in the system (3). The coefficients \( \sigma_1, \sigma_2, \mathcal{K}_1, \mathcal{K}_2 \) are functions

\[
\sigma_1 : [0, \infty) \times V_1 \times V_2 \rightarrow L_2(\mathcal{H}, V_1),
\]

\[
\sigma_2 : [0, \infty) \times V_1 \times V_2 \rightarrow L_2(\mathcal{H}, V_2),
\]

\[
\mathcal{K}_1 : [0, \infty) \times V_1 \times V_2 \rightarrow E_0 \rightarrow V_1,
\]

\[
\mathcal{K}_2 : [0, \infty) \times V_1 \times V_2 \rightarrow E_0 \rightarrow V_2.
\]

We will assume throughout the work that the following growth and Lipschitz conditions are satisfied

\[
\|\sigma_1(t, v, h)\|_{L^2(\mathcal{H}, V_1)}^2 + \|\sigma_2(t, v, h)\|_{L^2(\mathcal{H}, V_2)}^2 + \int_{E_0} \|\mathcal{K}_1(t, v, h, z)\|^2 d\nu(z) + \int_{E_0} \|\mathcal{K}_2(t, v, h, z)\|^2 d\nu(z) \leq C(1 + \|v\|^2 + \|h\|^2),
\]
for all \( \mathbf{v} \in V_1, h \in V_2, t \geq 0 \) and \( \omega \in \Omega \), and

\[
\begin{align*}
&\|\sigma_1(t, \mathbf{v}, 1, h) - \sigma_1(t, \mathbf{v}, 2, h)\|_{L^2(U, V_1)}^2 + \|\sigma_2(t, \mathbf{v}, 1, h) - \sigma_2(t, \mathbf{v}, h)\|_{L^2(U, V_2)}^2 \\
&\quad + \int_{E_0} \|\mathcal{K}_1(t, \mathbf{v}, 1, h, z) - \mathcal{K}_1(t, \mathbf{v}, 2, h, z)\|^2 d\nu(z) \\
&\quad + \int_{E_0} \|\mathcal{K}_2(t, \mathbf{v}, 1, h, z) - \mathcal{K}_2(t, \mathbf{v}, 2, h, z)\|^2 d\nu(z) \\
&\quad \leq C\|\mathbf{v}_1 - \mathbf{v}_2\|^2 + \|h_1 - h_2\|^2
\end{align*}
\]

(23)

for all \( \mathbf{v}_1, \mathbf{v}_2 \in V_1, h_1, h_2 \in V_2, t \geq 0 \) and \( \omega \in \Omega \).

For simplicity, we shall usually write \( \sigma_i(t, \mathbf{v}, h) = \sigma_i(\mathbf{v}, h) \) and \( \mathcal{K}_i(t, \mathbf{v}, h) = \mathcal{K}_i(\mathbf{v}, h) \) for \( i = 1, 2 \).

We assume that \( \mathcal{L}_i : H \times E \to V_i \) are measurable functions for \( i = 1, 2 \). No growth or Lipschitz assumptions are required for \( \mathcal{L}_i \). The stochastic integrals

\[
\int_0^t \int_{E \setminus E_0} \mathcal{L}_i(\mathbf{v}, h, z) d\pi_i(s, z), \quad i = 1, 2,
\]

(24)

appearing on the right-hand side of (3) are random sums of finitely many vectors in \( V_i \) almost surely. These terms capture the influences from large jumps of the Lévy noise. With \( E := U \setminus \{0\} \) and \( E_0 := \{\xi \in U : 0 < \|\xi\|_U < 1\} \), the stochastic integrals in (24) represent stochastic integration with respect to the jumps of a Lévy process outside of the unit ball. See [10] for an explanation of how the stochastic integrals (24) generalize stochastic integration with respect to the compound Poisson process \( P^0 \) in the Lévy-Khinchin decomposition (13) of a Lévy process.

We now introduce the notions of solution that will be used in this article. Let \( \mu_0 \) be a probability measure on \( V_1 \times V_2 \) satisfying the following moment condition:

\[
\int_{V_1 \times V_2} (\|\mathbf{v}\|^4 + \|h\|^4) d\mu_0(\mathbf{v}, h) < \infty.
\]

(25)

The time of existence of a local solution will be an accessible stopping time on \([0, T]\), which is a stopping time \( \tau \) that can be approximated by a localizing sequence of stopping times \( (\tau_n)_{n=1}^\infty \) such that \( \tau_n \uparrow \tau \) a.s. and such that \( \tau_n < \tau \) a.s. for every \( n \) on the event \( \{\tau < T\} \).

**Definition 2.4** (Local martingale solution). A martingale solution of (3) with initial law \( \mu_0 \) is a system \( (\tilde{S}, \tilde{V}, \tilde{h}, \tilde{\tau}) \), where

i) \( \tilde{S} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{P}, \tilde{W}_1, \tilde{W}_2, \tilde{\tau}_1, \tilde{\tau}_2) \) is a stochastic basis,

ii) \( \tilde{\tau} \) is a strictly positive accessible stopping time (i.e. \( \tilde{\tau} > 0 \) almost surely) relative to \( (\tilde{\mathcal{F}}_t)_{t \geq 0} \) with a localizing sequence \( (\tilde{\tau}_n)_{n=1}^\infty \) such that \( \tilde{V}(\cdot \wedge \tilde{\tau}_n), \tilde{h}(\cdot \wedge \tilde{\tau}_n) \) are \( \tilde{\mathcal{F}}_\tau \)-adapted processes with càdlàg paths in \( V_1 \) and \( V_2 \), respectively, for every \( n \) and

\[
\tilde{V}(t)1_{t \leq \tilde{\tau}_n} \in L^2(\tilde{\Omega}; L^2(0, T; V_1)),
\]

(26a)

\[
\tilde{h}(t)1_{t \leq \tilde{\tau}_n} \in L^2(\tilde{\Omega}; L^2(0, T; V_2)).
\]

(26b)

Furthermore, the law of \( (\tilde{V}(0), \tilde{h}(0)) \) is \( \mu_0 \), i.e., \( \mu_0(\Gamma) = \tilde{P}( (\tilde{V}(0), \tilde{h}(0)) \in \Gamma) \) for all Borel subsets \( \Gamma \subset V_1 \times V_2 \), and \( (\tilde{V}, \tilde{h}) \) must satisfy the following equation
almost surely, for every \( n \), every \( t \geq 0 \), every \( \phi \in H_1 \), and every \( \psi \in H_2 \):

\[
(v(t \wedge \tau_n), \phi) + \int_0^{t \wedge \tau_n} \left( -\nu \Delta v + (\nabla \cdot v) \nabla + g \nabla h + f k \times v - F, \phi \right) ds
\]

\[
= (v(0), \phi) + \int_0^{t \wedge \tau_n} (\phi, \sigma_1(\nu, h) dW_1) + \int_0^{t \wedge \tau_n} \left( \int_{E_0} \left( \phi, \mathcal{X}_1(\nu(s-), h(s-), z) \right) d\pi_1(s, z) \right)
\]

\[
+ \int_{(0,t\wedge\tau_n)} \left( \int_{E \setminus E_0} \left( \phi, \mathcal{L}_1(\nu(s-), h(s-), z) \right) d\pi_1(s, z) \right), \tag{27}
\]

and

\[
(h(t \wedge \tau_n), \psi) + \int_0^{t \wedge \tau_n} (\nabla \cdot (h \nu) - \delta \Delta h, \psi) ds = (h(0), \psi) +
\]

\[
\int_0^{t \wedge \tau_n} (\psi, \sigma_2(\nu, h) dW_2) + \int_0^{t \wedge \tau_n} \left( \int_{E_0} \left( \psi, \mathcal{X}_2(\nu(s-), h(s-), z) \right) d\pi_2(s, z) \right)
\]

\[
+ \int_{(0,t\wedge\tau_n)} \left( \int_{E \setminus E_0} \left( \psi, \mathcal{L}_2(\nu(s-), h(s-), z) \right) d\pi_2(s, z) \right). \tag{28}
\]

In a martingale solution the stochastic basis is not given in advance, rather, it is an unknown in the problem. That is the reason that the initial condition cannot be specified as a random variable, i.e., an explicit function, and must instead be specified in law by \( \mu_0 \). We now define pathwise solutions, in which the stochastic basis is given in advance.

**Definition 2.5 (Local pathwise solutions).** Suppose that \( S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_1, W_2, \pi_1, \pi_2) \) is a stochastic basis and that \( (v_0, h_0) \) is a \( V_1 \times V_2 \)-valued, \( \mathcal{F}_0 \)-measurable random variable. The same conditions as above are applied for the coefficients \( F, \sigma_i, \mathcal{X}_i \) and \( \mathcal{L}_i \) for \( i = 1, 2 \).

i) A triplet \( (v, h, \tau) \) is a local pathwise solution to (3) if \( \tau \) is a strictly positive accessible \( \mathcal{F}_\tau \)-stopping time, \( (v, h) \) is an \( \mathcal{F}_\tau \)-adapted process with values in \( D(A_1) \times D(A_2) \) (relative to the fixed basis \( S \)) such that \( (v, h) \) is càdlàg in \( V_1 \times V_2, \mathbb{P}\text{-a.s.}, \) and equations (27) and (28) hold with \( S \) in place of \( \tilde{S} \) and \( (v, h, \tau) \) in place of \( (\nu, h, \tilde{\tau}) \).

ii) Pathwise solutions of (3) are said to be unique up to a stopping time \( \tau > 0 \) if given any pair of pathwise solutions \( (v_1, h_1, \tau) \) and \( (v_2, h_2, \tau) \) which coincide at \( t = 0 \) on the subset \( \Omega_0 \) of \( \Omega \):

\[
\Omega_0 := \{ v_1(0) = v_2(0), h_1(0) = h_2(0) \} \subset \Omega, \tag{29}
\]

it follows that

\[
\mathbb{P} (\mathbb{1}_{\Omega_0} (v_1(t) - v_2(t)) = 0, \forall t \in [0, \tau)) = 1, \tag{30}
\]

and

\[
\mathbb{P} (\mathbb{1}_{\Omega_0} (h_1(t) - h_2(t)) = 0, \forall t \in [0, \tau)) = 1. \tag{31}
\]

**Definition 2.6 (Maximal local pathwise solution).** A local pathwise solution \( (v, h, \tau) \) to equation (3) is said to be a maximal local pathwise solution if \( (v, h, \tau) \) has no proper extension by another local pathwise solution to equation (3), i.e., if for every local pathwise solution \( (\nu', h', \tau') \) to equation (3) such that \( \tau \leq \tau' \) a.s. and \( (v(t), h(t)) = (\nu'(t), h'(t)) \) a.s. for all \( t \in [0, \tau) \) it follows that \( \tau' = \tau \) a.s.
It is clear that a local pathwise solution \((v, h, \tau)\) to equation (3) that satisfies
\[
\sup_{t \in [0, \tau]} \|v(t)\| + \|h(t)\| = \infty,
\]
a.s. on the event \(\{\tau < T\}\) must be a maximal local pathwise solution.

We now state the main results in this work:

**Theorem 2.7.** We are given a probability measure \(\mu_0\) on \(V_1 \times V_2\) satisfying condition (25), \(F \in L^1(0, T; V_1)\), \(\sigma_i(v, h), \mathcal{X}_i(v, h), i = 1, 2\) satisfying the growth and Lipschitz conditions (22) and (23) and the \(\mathcal{L}_i\) as in (24). Then there exists a local martingale solution \((\mathcal{S}, \tilde{v}, \tilde{h}, \tilde{\tau})\) to (3).

**Theorem 2.8.** Assume we are working relative to a given fixed stochastic basis \(\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, W_1, W_2, \pi_1, \pi_2)\) and let \((v_0, h_0)\) be an \(\mathcal{F}_0\)-measurable \(V_1 \times V_2\)-valued random variable. Then there exists a maximal local pathwise solution \((v, h, \tau)\) to (3) with initial condition \((v_0, h_0)\) and the blow-up condition (32) is satisfied.

The strategy to prove both of these theorems is to consider a modified system with a cutoff function damping the non-linear term and to set \(\mathcal{L}_i = 0\) initially. For the modified system we will establish global existence of martingale solutions and pathwise solutions using a Galerkin approximation method. We then return to the original system (3), with \(\mathcal{L}_i\) still set to 0, by introducing a stopping time which will be proven to be positive. The terms with coefficients \(\mathcal{L}_i\) in (3) represent finitely many additional jumps, a.s., and can be interlaced into the solution at the end.

### 3. Formal a priori estimates

In this section we assume the existence of a smooth solution \((v, h)\) to the system (3) with \(\mathcal{L}_i = 0\) for \(i = 1, 2\). We make a priori estimates in order to illustrate how the cutoff function for the nonlinear term should be chosen. We use techniques in the a priori estimates that will also be applicable when making uniform estimates in the Galerkin scheme that we introduce below in Section 4.1.

**Proposition 1.** Assume that the hypotheses of Theorem 2.7 are all satisfied. Assume that \(F \in L^p(0, T; V_1)\) and
\[
E\left(\|v_0\|^p + \|h_0\|^p\right) < \infty \text{ for some } p \geq 2
\]
and also there exist a constant \(C\) such that
\[
\int_{\mathcal{E}_0} \left(\|\mathcal{X}_1(\tilde{v}, \tilde{h}, z)\|^p + \|\mathcal{X}_2(\tilde{v}, \tilde{h}, z)\|^p\right) d\nu(z) \leq C E(1 + \|\tilde{v}\|^p + \|\tilde{h}\|^p),
\]
for all \((\tilde{v}, \tilde{h}) \in V_1 \times V_2\). Assume that \((v, h)\) is a pathwise solution to (3) with \(\mathcal{L}_i = 0\) such that \((v, h)\) is càdlàg in \(H\) a.s. Then there exists a constant \(C\), depending upon only on \(T, p\) and the constant in (34), such that
\[
E\left(\sup_{s \in [0, t^*]} \|v(s)\|^p + \|h(s)\|^p\right) \leq C E(1 + \|v_0\|^p + \|h_0\|^p) + C,
\]
where the random time \(t^*\) is specified later.

**Proof.** We apply the Itô formula (see, e.g., Theorem 27.2 in [27]) to (3) using the function \(\phi: V \to \mathbb{R}\) defined by \(\phi(x) := \|x\|^p\). One can easily compute the first an
second derivatives of \( \phi \) and find that \( D\phi: V \to V^* \) and \( D^2\phi: V \to \mathcal{L}(V,V^*) \) are given by
\[
(D\phi)(x) = p|x||p^{-2}(-\Delta x, \cdot) \quad \text{and} \\
(D^2\phi)(x) = p|x||p^{-2}I + p(p-2)|x||p^{-4}(-\Delta x, \cdot)x,
\]
respectively. The Itô formula implies that
\[
d\|v(s)\|^p + p\|v(s)\|^{p-2}(v\Delta v(s), \Delta v(s))ds - p\|v(s)\|^{p-2}((v(s)\cdot \nabla)v(s), \Delta v(s))ds - p\|v(s)\|^{p-2}(f_k \times v(s), \Delta v(s))ds = -p\|v(s)\|^{p-2}(F, \Delta v(s))ds
\]
\[
- gp\|v(s)\|^{p-2}(\nabla h(s), \Delta v(s))ds
\]
\[
- p\|v(s-\cdot)|^{p-2}(\Delta v(s-\cdot), \sigma_1(v(s-\cdot), h(s-\cdot))dW_1(s))
\]
\[
+ \frac{1}{2}p\|v(s)\|^{p-2}\|\sigma_1(v(s), h(s))\|^2_{L_2(U,V_1)}ds
\]
\[
+ \frac{p(p-2)}{2}\|v(s)\|^{p-4}\|\sigma_1(v(s), h(s))\|\|\Delta v(s)\|^2_{\mathcal{D}}ds
\]
\[
+ \int_{E_0} p\|v(s-\cdot)|^{p-2}(\Delta v(s-\cdot), \mathcal{K}_1(v(s-\cdot), h(s-\cdot), z))d\pi_1(s, z)
\]
\[
+ \int_{E_0} \left[\|v(s-\cdot) + \mathcal{K}_1(v(s-\cdot), h(s-\cdot), z)\|^p - \|v(s-\cdot)\|^p
\]
\[
+ p\|v(s-\cdot)|^{p-2}(\Delta v(s-\cdot), \mathcal{K}_1(v(s-\cdot), h(s-\cdot), z))\right]d\pi_1(s, z), \quad (36)
\]
and
\[
d\|h(s)\|^p + p\delta|h(s)|^{p-2}(\Delta h(s), \Delta h(s))ds - p\|h(s)\|^{p-2}(\nabla \cdot (h(s)v(s)), \Delta h(s))ds =
\]
\[
\frac{1}{2}p\|h(s)\|^p - p\|v(s)\|^{p-2}\|\sigma_2(v(s), h(s))\|^2_{L_2(U,V_2)}ds
\]
\[
- \frac{p(p-2)}{2}\|h(s)\|^{p-4}\|\sigma_2(v(s), h(s))\|\|\Delta h(s)\|^2_{\mathcal{D}}ds
\]
\[
- p\|h(s-\cdot)|^{p-2}(\Delta h(s-\cdot), \sigma_2(v(s-\cdot), h(s-\cdot))dW_2(s))
\]
\[
+ \int_{E_0} p\|h(s-\cdot)|^{p-2}(\Delta v(s-\cdot), \mathcal{K}_2(v(s-\cdot), h(s-\cdot), z))d\pi_2(s, z)
\]
\[
+ \int_{E_0} \left[\|h(s-\cdot) + \mathcal{K}_2(v(s-\cdot), h(s-\cdot), z)\|^p - \|h(s-\cdot)\|^p
\]
\[
+ p\|h(s-\cdot)|^{p-2}(\Delta h(s-\cdot), \mathcal{K}_2(v(s-\cdot), h(s-\cdot), z))\right]d\pi_2(s, z). \quad (37)
\]
Adding (36) and (37), taking the absolute value, integrating over \([0, s]\) for \(0 \leq s \leq t \leq T\) and finally taking supremum over \([0, t]\) before taking mathematical expectation, we deduce
\[
\mathbb{E}\left(\sup_{0 \leq s \leq t} \|v(s)\|^p + \|h(s)\|^p\right) + pq\mathbb{E}\int_0^t \|v(s)\|^{p-2}|\Delta v(s)|^2 ds
\]
\[
+ p\delta\mathbb{E}\int_0^t \|h(s)\|^{p-2}|\Delta h(s)|^2 ds \leq \mathbb{E}(\|v_0\|^p + \|h_0\|^p) + \sum_{i=1}^{16} K_i. \quad (38)
\]
The terms \(K_1\) and \(K_2\) are readily estimated by simply using the Cauchy-Schwarz inequality and the Young inequality.
Young inequality in a similar way to the estimate of Ladyzhenskaya’s inequality in 2-D as follows:

\[ K_1 := p\mathbb{E} \int_0^t |v(s)|^{p-2} |(F, \Delta v(s))| ds \leq C \mathbb{E} \int_0^t |F|^p \]

\[ + \|v\|^p ds + \frac{p\nu}{8} \mathbb{E} \int_0^t |\Delta v(s)|^2 \|v(s)\|^{p-2} ds. \quad (39) \]

\[ K_2 := p\mathbb{E} \int_0^t |v(s)|^{p-2} |(f \times v(s), \Delta v(s))| ds \leq C \mathbb{E} \int_0^t |v(s)|^p ds. \quad (40) \]

The term \( K_3 \) is evaluated by first using the Cauchy-Schwarz inequality and the Young inequality in a similar way to the estimate of \( K_1 \):

\[ K_3 := p\mathbb{E} \int_0^t |v(s)|^{p-2} |(\nabla h(s), \Delta v(s))| ds \]
\[ \leq C \mathbb{E} \int_0^t |v(s)|^p + \|h(s)\|^p ds + \frac{p\nu}{8} \mathbb{E} \int_0^t |v(s)|^{p-2} |\Delta v(s)|^2 ds. \quad (41) \]

The bounds for the nonlinear terms are carried out using Agmon’s inequality, which states that \( |u|^2_{L^\infty} \leq C |u| |\Delta u| \). By using Hölder’s inequality and then Agmon’s inequality, we obtain

\[ K_4 := p\mathbb{E} \int_0^t |v(s)|^{p-2} |(\nabla \cdot (v(s) - \nabla h(s)) v(s), \Delta v(s))| ds \]
\[ \leq C \mathbb{E} \int_0^t |v(s)|^{p-2} \|v(s)\|_{L^\infty} \|v(s)\| |\Delta v(s)| ds \]
\[ \leq C \mathbb{E} \int_0^t |v(s)|^{p-2} |v(s)|^{\frac{p}{2}} |\Delta v(s)|^{\frac{p}{2}} ds \]
\[ \leq C \mathbb{E} \int_0^t |v(s)|^{p-2} \|v(s)\|^{\frac{p}{2}} |\Delta v(s)|^{\frac{p}{2}} ds \]
\[ \leq C \mathbb{E} \int_0^t |v(s)|^{p+4} ds + \frac{p\nu}{4} \int_0^t |v(s)|^{p-2} |\Delta v(s)|^2 ds. \quad (42) \]

The last line follows due to applying the Young inequality to the previous one with \( p_1 = \frac{4}{3}, p_2 = 4 \).

The term \( K_5 \) is treated as follows:

\[ K_5 := p\mathbb{E} \int_0^t |h(s)|^{p-2} |(\nabla \cdot (h(s) v(s)), \Delta h(s))| ds \]
\[ \leq p\mathbb{E} \int_0^t \left( |h(s)|^{p-2} |(\nabla h(s) \cdot v(s), \Delta h(s)) + (\nabla \cdot v(s) h(s), \Delta h(s))| \right) ds \]
\[ =: K_5^1 + K_5^2. \quad (43) \]

The bound for the term \( K_5^1 \) is derived by first applying Hölder’s inequality and then Ladyzhenskaya’s inequality in 2-D as follows:

\[ K_5^1 := p\mathbb{E} \int_0^t |h(s)|^{p-2} (\nabla h(s) \cdot v(s), \Delta h(s))| ds \]
\[ \leq C \mathbb{E} \int_0^t |\nabla h(s)|^{p-2} |v(s)|_{L^4} \|h(s)\|_{L^4} \|\Delta h(s)\| ds \]
\[ \leq C \mathbb{E} \int_0^t |h(s)|^{p-2} \|h(s)\|^{\frac{p}{2}} |\Delta h(s)|^{\frac{p}{2}} \|v(s)\| \|\Delta h(s)\| ds \]
\[ \leq C \mathbb{E} \int_0^t |h(s)|^{p-2} \|h(s)\|^{\frac{p}{2}} |\Delta h(s)|^{\frac{p}{2}} \|v(s)\| |\Delta h(s)| ds. \]
By applying the Young inequality to the last three terms with \( p_1 = 12, p_2 = 6, p_3 = \frac{4}{3} \) we obtain

\[
K_5^3 \leq CE \int_0^t \|h(s)\|^{p+4} ds + CE \int_0^t \|h(s)\|^{p-2} \|\nabla\phi(s)\|^{6} ds \\
+ \frac{p\delta}{4} E \int_0^t \|h(s)\|^{p-2} |\Delta h(s)|^2 ds.
\]  

(45)

In a similar manner, the bound for the term \( K_5^2 \) is obtained using Agmon’s inequality as follows:

\[
K_5^2 := pE \int_0^t \|((\nabla \cdot \phi(s))h(s), \Delta h(s))\| |h(s)|^{p-2} ds \\
\leq CE \int_0^t \|\phi(s)\| |h(s)| \Delta h(s) \| h(s) \|^{p-2} ds \\
\leq CE \int_0^t \|\phi(s)\| |h(s)|^{\frac{1}{2}} \Delta h(s) \| h(s) \|^{\frac{3}{2}} \Delta h(s) \| h(s) \|^{p-2} ds \\
\leq CE \int_0^t \|\phi(s)\| |h(s)|^{\frac{1}{2}} \Delta h(s) \| h(s) \|^{\frac{3}{2}} \Delta h(s) \| h(s) \|^{p-2} ds \\
\leq CE \int_0^t \|h(s)\|^{p+4} ds + CE \int_0^t \|h(s)\|^{p-2} \|\phi(s)\|^{6} ds \\
+ \frac{p\delta}{4} E \int_0^t \|h(s)\|^{p-2} |\Delta h(s)|^2 ds.
\]  

(46)

By combining the estimates (44) and (46), we obtain

\[
K_5 = pE \int_0^t \|h(s)|^{p-2} ||(\nabla \cdot (h(s)\phi(s)), h(s))|| ds \\
\leq CE \int_0^t \|h(s)\|^{p+4} ds + CE \int_0^t \|h(s)\|^{p-2} \|\phi(s)\|^{6} ds \\
+ \frac{p\delta}{2} E \int_0^t \|h(s)\|^{p-2} |\Delta h(s)|^2 ds.
\]  

(47)

The estimates for the terms \( K_6 \) and \( K_7 \) are given by using the assumptions (22) and the Young inequality. We have

\[
K_6 + K_7 := E \left( \int_0^t \|\phi(s)\|^{p-2} |\sigma_1(\phi(s), h(s))| dL_{(W_1, V_1)} ds \\
+ \int_0^t \|h(s)\|^{p-2} |\sigma_2(\phi(s), h(s))| dL_{(W_2, V_2)} ds \right) \\
\leq CE \left( \int_0^t (\|\phi(s)\|^{p-2} + \|h(s)\|^{p-2}) (1 + \|\phi(s)\|^2 + \|h(s)\|^2) ds \\
\leq C + CE \int_0^t \|\phi(s)\|^{p} + \|h(s)\|^{p} ds.
\]  

(48)
We estimate the martingale terms by utilizing the BDG inequality (see inequality (16)) and the Young inequality as follows:

\[
K_8 := \mathbb{E} \left( \sup_{0 \leq r \leq t} \left| \int_0^r |v(s)|^{p-2} (\Delta v(s), \sigma_1(v(s), h(s))-dW_1(s)) \right| \right) \\
\leq C \mathbb{E} \left( \int_0^t |v(s)|^{2(p-2)} |\sigma_1(v(s), h(s))|^2 ds \right)^{\frac{1}{2}} \\
\leq C \mathbb{E} \left( \int_0^t |v(s)|^{2(p-2)} \|\sigma_1(v(s), h(s))\|^2_{L^2(U, V_1)} \|v(s)\|^2 ds \right) \\
\leq C \mathbb{E} \left[ \sup_{s \in [0,t]} |v(s)|^2 \left( \int_0^t |v(s)|^{p-2} \|\sigma_1(v(s), h(s))\|^2_{L^2(U, V_1)} \right)^{\frac{1}{2}} \right] \\
\leq \frac{1}{4} \mathbb{E} \sup_{s \in [0,t]} |v(s)|^p + C \mathbb{E} \int_0^t |v(s)|^{p-2} \|\sigma_1(v(s), h(s))\|^2_{L^2(U, V_1)}. \quad (49)
\]

Similarly, we have

\[
K_9 := \mathbb{E} \left( \sup_{0 \leq r \leq t} \left| \int_0^r |h(s)|^{p-2} (h(s)-\sigma_2(v(s), h(s))-dW_2(s)) \right| \right) \\
\leq \frac{1}{4} \mathbb{E} \sup_{s \in [0,t]} |h(s)|^p + C \mathbb{E} \int_0^t |h(s)|^{p-2} \|\sigma_2(v(s), h(s))\|^2_{L^2(U, V_2)}. \quad (50)
\]

We combine (49), (50), and assumption (22) and arrive at

\[
K_8 + K_9 \leq \frac{1}{4} \mathbb{E} \sup_{s \in [0,t]} |h(s)|^p + \frac{1}{4} \mathbb{E} \sup_{s \in [0,t]} |v(s)|^p \\
+ C \mathbb{E} \int_0^t (|h(s)|^{p-2} + |v(s)|^{p-2})(1 + |v(s)|^2 + |h(s)|^2) ds \\
\leq C + \frac{1}{4} \mathbb{E} \sup_{s \in [0,t]} |h(s)|^p + \frac{1}{4} \mathbb{E} \sup_{s \in [0,t]} |v(s)|^p + C \mathbb{E} \int_0^t (|v(s)|^p + |h(s)|^p) ds. \quad (51)
\]

We next consider the term

\[
K_{10} = \mathbb{E} \int_0^t \int_{E_0} |v(s-)|^{p-2} (\Delta v(s-), \mathcal{A}_1(v(s-), h(s-), z)) d\pi_1(s, z),
\]

whose quadratic variation is given by

\[
[K_{10}, K_{10}]_t = \int_0^t \int_{E_0} (|v(s-)|^{p-2}((v(s-), \mathcal{A}_1(v(s-), h(s-), z)))^2 d\pi_1(s, z).
\]

For \(K_{10}\) the BDG inequality (i.e., inequality (21)) takes the form

\[
\mathbb{E} \sup_{s \in [0,t]} |K_{10}(s)| \leq \\
C \mathbb{E} \left( \int_0^t \int_{E_0} |v(s-)|^{2(p-2)}(|v(s-), \mathcal{A}_1(v(s-), h(s-), z)))^2 d\pi_1(s, z) \right)^{\frac{1}{2}}.
\]
\[
\begin{align*}
\leq C E \left( \sup_{0 \leq s \leq t} \|v(s)\|^{\frac{p}{2}} \left( \int_0^t \int_{E_0} \|v(s-)^{p-2}\| \|\mathcal{J}(v(s-), h(s-), z)\|^{2} d\pi_1(s, z) \right)^{\frac{1}{2}} \right) \\
\leq \frac{1}{4} E \sup_{s \in [0, t]} \|v(s)\|^p + C E \int_0^t \|v(s)\|^{p-2}(1 + \|v(s)\|^2 + \|h(s)\|^2) ds.
\end{align*}
\]

We obtain a similar bound for the term
\[
K_{11} = E \int_0^t \int_{E_0} \|h(s-)^{p-2}(\Delta h(s-), \mathcal{K}(v(s-), h(s-), z))d\pi_2(s, z).
\]

After combining that bound with (53), we arrive at
\[
\begin{align*}
E \sup_{s \in [0, t]} (|K_{10}(s)| + |K_{11}(s)|) &\leq \frac{1}{4} E \sup_{s \in [0, t]} (\|v(s)\|^p + \|h(s)\|^p) \\
&+ C E \int_0^t (\|v(s)\|^{p-2} + \|h(s)\|^{p-2})(1 + \|v(s)\|^2 + \|h(s)\|^2) ds \\
&\leq C + \frac{1}{4} E \sup_{s \in [0, t]} (\|v(s)\|^p + \|h(s)\|^p) + C E \int_0^t (\|v(s)\|^p + \|h(s)\|^p) ds.
\end{align*}
\]

The treatment for the last two terms,
\[
\begin{align*}
K_{12} := E \left( \int_0^t \int_{E_0} \left[ \|v(s-)^{p-2}(\Delta v(s-), \mathcal{K}(v(s-), h(s-), z))d\pi_1(s, z) \right] \right)
\end{align*}
\]

and
\[
\begin{align*}
K_{13} := E \left( \int_0^t \int_{E_0} \left[ \|h(s-)^{p-2}(\Delta h(s-), \mathcal{K}(v(s-), h(s-), z))d\pi_2(s, z) \right] \right)
\end{align*}
\]

is based on the inequality
\[
\|x + y\|^p - \|y\|^p - p\|y\|^{p-2}(x, y) \| \leq C(\|y\|^{p-2}\|x\|^2 + \|x\|^p), \quad \forall x, y \in V, \quad (55)
\]

which can be derived using the fundamental theorem of calculus. Using inequality (55) and the assumption (22), we see that
\[
\begin{align*}
E \sup_{0 \leq s \leq t} |K_{12}(s)| &\leq E \left( \int_0^t \int_{E_0} \left[ \|v(s-)^{p-2}(\Delta v(s-), \mathcal{K}(v(s-), h(s-), z))d\pi_1(s, z) \right] \right) \\
&\leq C E \int_0^t \int_{E_0} \left[ \|v(s-)^{p-2}\| \|\mathcal{K}(v(s-), h(s-), z)\|^{2} d\pi_1(z) \right] ds \\
&\leq C E \int_0^t \|v(s)\|^{p-2}(1 + \|v(s)\|^2 + \|h(s)\|^2) \\
&\quad + (1 + \|v(s)\|^p + \|h(s)\|^p) ds \\
&\leq C E \int_0^t (\|v(s)\|^p + \|h(s)\|^p) ds + C. \quad (56)
\end{align*}
\]
The last two lines follow thanks to the Young inequality and the fact that \( \|u(s-)\|_{L^2} = \|u(s)\|_{L^2} \) for a.e. \( s \). Similarly, we have
\[
E \sup_{0 \leq s \leq T} |K_{13}(s)| \leq C E \int_0^T \int_{E_0} \left[ \|h(s-)\|^{p-2}\|\mathcal{X}_2(v(s-), h(s-), z)\|^2 + \|\mathcal{X}_2(v(s-), h(s-), z)\|^p \right] dv_2(z) ds
\]
where
\[
C E \int_0^T (\|v(s)\|^p + \|h(s)\|^p) ds + C. \tag{57}
\]
Rearranging all estimates from (39) through (57) and multiplying by 2, we deduce that
\[
E \left( \sup_{0 \leq s \leq T} (\|v(s)\|^p + \|h(s)\|^p) + p_\nu \int_0^T |\Delta v(s)|^2 \|v(s)\|^{p-2} ds \right.
\]
\[
+ p_\delta \int_0^T |\Delta h(s)|^2 \|h(s)\|^{p-2} ds \bigg) \leq 2E(\|v_0\|^p + \|h_0\|^p) + C E \int_0^T \|F(s)\|^p ds + \mathcal{K}_0, \tag{58}
\]
where
\[
\mathcal{K}_0 := C E \int_0^T (\|h(s)\|^p + \|v(s)\|^p)(\|v(s)\| + \|h(s)\|)^4 ds. \tag{59}
\]
Now, we assume that \( M > 1 \) and consider the stopping time
\[
\tau = \tau_M := \inf \left\{ (\|v(s)\| + \|h(s)\|) > M \right\}. \tag{60}
\]
Since \( V \) is compactly embedded in \( H \), \( \tau \) is the time that \((v, h)\) hits an open subset of \( H \). Since \((v, h)\) is càdlàg in \( H \) and the filtration is right-continuous it follows that \( \tau \) is a stopping time (see, e.g., Proposition 4.6 in Chapter I of [33] for a proof of this fact). Replacing \( t \) by \( t \wedge \tau \) in (58) yields
\[
E \left( \sup_{0 \leq s \leq t \wedge \tau} (\|v(s)\|^p + \|h(s)\|^p) + p_\nu \int_0^{t \wedge \tau} |\Delta v(s)|^2 \|v(s)\|^{p-2} ds \right.
\]
\[
+ p_\delta \int_0^{t \wedge \tau} |\Delta h(s)|^2 \|h(s)\|^{p-2} ds \bigg) \leq 2E(\|v_0\|^p + \|h_0\|^p) + C E \int_0^{t \wedge \tau} \|F(s)\|^p ds
\]
\[
+ CM^4E \int_0^{t \wedge \tau} (\|v(s)\|^p + \|h(s)\|^p) ds. \tag{61}
\]
Now, we define
\[
\mathcal{Y}(t) := E \left( \sup_{0 \leq r \leq t \wedge \tau} (\|v(r)\|^p + \|h(r)\|^p) \right), \tag{62}
\]
and
\[
\mathcal{K}_1 = 2E(\|v_0\|^p + \|h_0\|^p) + C E \int_0^{t \wedge \tau} \|F(s)\|^p ds. \tag{63}
\]
Then (61) implies
\[
\mathcal{Y}(t) \leq \mathcal{K}_1 + CM^4 \int_0^t \mathcal{Y}(s) ds,
\]
which gives
\[ \mathcal{Y}(t) \leq K_1 e^{CM^4 t} \] (64)
by an application of the deterministic Gronwall inequality. From (61) and (64), we obtain
\[
\mathbb{E} \left( \sup_{s \in [0, t]} \| v(s) \| + \| h(s) \| + p \nu \int_0^t |\Delta v(s)|^2 \| v(s) \|^{p-2} ds \right. \\
+ p \delta \int_0^t |\Delta h(s)|^2 \| h(s) \|^{p-2} ds \left. \right) \\
\leq K_1 + K_1 e^{CM^4 t},
\] (65)
which gives (35) with \( t^* = \tau \).

The estimates of the nonlinear terms, (42) and (47) suggest how to modify the nonlinear terms in (3) using a truncation so that bounds can be established in a Galerkin scheme. Specifically, when the estimates are collected in (58) and (59) we see that truncating \( \| v \| + \| h \| \) via the stopping time \( \tau_M \) defined in (60) permits an application of Gronwall’s inequality and leads to (64). Based on the reasoning in Proposition 1, we expect that Galerkin approximations to (3) modified by a truncating \( \| v \| + \| h \| \) in the nonlinear terms will be bounded in the space \( L^p(\Omega; L^\infty(0, T; V)) \). This is true and will be established in Lemma 4.1.

4. Existence of martingale solutions. The first step to show the existence of martingale solutions of the system (3) is to consider the modified system as follows:

\[
dv + (-\nu \Delta v + \theta_R(\| v \| + \| h \|)(v \cdot \nabla)v + g \nabla h + f k \times v) dt = F dt + \sigma_1(v, h) dW_1 \\
+ \int_{\mathcal{E}_0} \mathcal{K}_1(v(t-), h(t-), z) d\pi_1(t, z)
\] (66a)

\[
dh + [\theta_R(\| v \| + \| h \|)](h \nabla - \delta \Delta h) dt \\
= \sigma_2(v, h) dW_2 + \int_{\mathcal{E}_0} \mathcal{K}_2(v(t-), h(t-), z) d\pi_2(t, z) \] (66b)

\[
v(0) = v_0, \quad h(0) = h_0 > 0.
\] (66c)

Initially we use as the initial condition an arbitrary \( \mathcal{F}_0 \)-measurable \( V \)-valued random variable \( (v_0, h_0) \) with law \( \mu \) that satisfies condition (25), i.e., \( \mathbb{E}(\| v_0 \|^4 + \| h_0 \|^4) < \infty \). Here, \( \theta_R : \mathbb{R} \to [0, 1] \) is a \( C^\infty \) cutoff function satisfying
\[
\theta_R(z) = \begin{cases} 
1 & \text{if } |z| \leq R, \\
0 & \text{if } |z| \geq 2R.
\end{cases}
\] (67)

The parameter \( R \) will be fixed until global existence and uniqueness is established for the system (66) in Proposition 8 below. Once this has been completed, we will let \( R \to \infty \) in Section 9 in order to obtain a maximal local pathwise solution to the original equation (3). We will also remove the moment condition \( \mathbb{E}(\| v_0 \|^4 + \| h_0 \|^4) < \infty \) in Section 9.

Notice that the terms \( \int_{\mathcal{E}_0} \mathcal{L}_1(v, h, z) d\pi_1(t, z) \) and \( \int_{\mathcal{E}_0} \mathcal{L}_2(v, h, z) d\pi_2(t, z) \) that appeared on the right-hand sides of equations (3a) and (3b) are not present
on the right-hand sides of equations (66a) and (66b). We will use $L^2$-martingale techniques, such as the BDG inequality, to estimate Galerkin approximations to (66). Once existence and uniqueness are established for (66), the additional jumps caused by the terms $\int_{E \setminus E_0} \mathcal{L}_1(v, h, z) d\tau_1(t, z)$ and $\int_{E \setminus E_0} \mathcal{L}_2(v, h, z) d\tau_2(t, z)$ can be accounted for in the solution using the piecing out argument (see, e.g., Section 4.2 of [7]). So, in order to establish Theorems 2.7 and 2.8 it is sufficient to establish existence and pathwise uniqueness of local solutions to the equation:

$$d\nu + (-\nu \Delta \nu + (v \cdot \nabla) \nu + g\nabla h + f \times \nu) dt = F dt + \sigma_1(\nu, h) dW_1 + \int_{E_0} \mathcal{K}_1(\nu, h, z) d\tilde{\tau}_1(t, z), \quad \text{in } \mathcal{M} \times (0, T), \quad (68a)$$

$$dh + (-\delta h + \nabla \cdot (hv)) dt = \sigma_2(\nu, h) dW_2 + \int_{E_0} \mathcal{K}_2(\nu, h, z) d\tilde{\tau}_2(t, z), \quad \text{in } \mathcal{M} \times (0, T), \quad (68b)$$

which is just the original system (3) with $\mathcal{L}_1 = \mathcal{L}_2 = 0$.

4.1. **Galerkin approximation scheme.** We prove the existence of a martingale solution to (66) by using a Galerkin approximation scheme. Considering the projection $P_n$ defined as in (6), we introduce the Galerkin approximation $(\nu^n, h^n)$ of (66), with $\nu^n$ and $h^n$ being the solution to the $P_n(V_1 \times V_2)$-valued equation

$$d\nu^n - \nu \Delta \nu^n dt + P_n[\theta_R(||\nu^n|| + ||h^n||)](\nu^n \cdot \nabla) \nu^n + g\nabla h^n + f \times \nu^n dt = P_n F dt + P_n \sigma_1(\nu^n, h^n) dW_1 + \int_{E_0} P_n \mathcal{K}_1(\nu^n(s-), h^n(s-), z) d\tilde{\tau}_1(s, z) \quad (69a)$$

$$dh^n + P_n[\theta_R(||\nu^n|| + ||h^n||)] \nabla \cdot (h^n \nu^n) - \delta h^n dt = P_n \sigma_2(\nu^n, h^n) dW_2 + \int_{E_0} P_n \mathcal{K}_2(\nu^n(s-), h^n(s-), z) d\tilde{\tau}_2(s, z) \quad (69b)$$

$$\nu^n(0) = \nu_0^n = P_n \nu_0, \quad h^n(0) = h_0^n = P_n h_0. \quad (69c)$$

Recall that, by abuse of notations, we write $P_n \nu = P_n(\nu, 0)$ and $P_n h = P_n(0, h)$. The cutoff function $\theta_R$ makes the coefficients in equations (69) and (69b) globally Lipschitz, so the global existence and uniqueness of $(\nu^n, h^n)$ is standard; see, e.g., Theorem 9.1 in [24]. Here $(\nu^n, h^n)$ are adapted processes that take values in the space $D(0, T; H_\nu)^3 \cong D(0, T; \mathbb{R}^3)$ of càdlàg $H_\nu$-valued functions on $[0, T]$ almost surely. See Appendix A for a review of the space $D(0, T; H_\nu)$. In what follows, we will denote the cutoff term $\theta_R(||\nu(s)|| + ||h(s)||)$ by $\theta_R(||U(s)||)$ to simplify the writing.

4.2. **Uniform estimates on the approximate solutions.** In this subsection we establish the main uniform estimates on the Galerkin approximations $\{(\nu^n, h^n)\}_{n=1}^\infty$.

**Lemma 4.1.** Under the same hypotheses as in Proposition 1, we obtain the following estimates independent of $n$:

$$\mathbb{E}\left( \sup_{s \in [0, T]} ||\nu^n(s)||^p + ||h^n(s)||^p \right) + \nu \mathbb{E} \int_0^t |\Delta \nu^n(s)|^2 ||\nu^n(s)||^{p-2} ds$$
\[ + \delta E \int_0^t |\Delta h^n(s)|^2 \|h^n(s)\|^p - 2 ds \leq C E(1 + \|v_0\|^p + \|h_0\|^p). \quad (70) \]

**Proof.** As in the proof of Proposition 1, we apply the Itô formula to equations (69a) and (69b) using the function \( \phi: V \to \mathbb{R} \) defined by \( \phi(x) := \|x\|^p \). We obtain two equations analogous to (36) and (37) but with \((v^n, h^n)\) in place of \((v, h)\) and with the cutoff term \( \theta_R(\|U^n\|) \) appearing as a coefficient of the non-linear terms. We add these equations and proceed as in the proof of Proposition 1. All of the linear terms are defined and estimated in a similar way to the terms \( K_1, K_2 \) and \( K_3 \) that appeared in the proof of Proposition 1. We obtain analogous estimates to (39), (40) and (41) with constants that are independent of \( n \). We now move on to the estimate of the non-linear term. Thanks to the presence of the cutoff function in front of the non-linear terms, we can obtain global bounds instead of the local bounds from Section 3. The bounds for the the nonlinear terms are derived as follows (cf. (42)):

\[ I_4 := \int_0^s |(\theta_R(\|U^n\|)(v^n \cdot \nabla)v^n, \Delta v^n)| \|v^n\|^p - 2 dt \leq C \int_0^s |v^n|_{L^\infty} |\nabla v^n| \|v^n\|^p - 2 dt \leq C \int_0^s |v^n|^\frac{p}{2} |\Delta v^n|^\frac{1}{2} \|v^n\|^p - 2 dt \leq C \int_0^s \|v^n\|^2 \|v^n\|^p - 2 dt + \frac{p\nu}{4} \int_0^s |\Delta v^n|^2 \|v^n\|^p - 2 dt \leq C \int_0^s \|v^n\|^p dt + \frac{p\nu}{4} \int_0^s |\Delta v^n|^2 \|v^n\|^p - 2 dt, \quad (71) \]

where the third and the fourth lines hold true due to the Agmon’s inequality, the definition of the cutoff function in (67) and the Young inequality. The nonlinear term that arises from the equation (69b) for \( h^n \) is split up and estimated as follows:

\[ I_5 := pE \int_0^t \|h^n(s)\|^p - 2 (P_n(\theta_R(\|U^n\|)\nabla \cdot (h^n(s)v^n(s)), h(s))ds \leq pE \int_0^t \|h^n(s)\|^p - 2 \theta_R(\|U^n\|)|\nabla h^n(s) \cdot v^n(s), h^n(s))ds + pE \int_0^t \|h^n(s)\|^p - 2 \theta_R(\|U^n\|)|(|\nabla v^n(s)\cdot h(s), h^n(s))ds := I_5^1 + I_5^2. \]

We estimate the first term as in (44) using Ladyzhenskaya’s inequality and obtain

\[ I_5^1 \leq CE \int_0^t \|h^n\|^p ds + \frac{p\delta}{4} E \int_0^t |\Delta h^n|^2 \|h^n\|^p - 2 ds \quad (72) \]

using the cutoff function. We estimate the second term \( I_5^2 \) by making use of Hölder’s inequality:

\[ I_5^2 := E \int_0^t \theta_R(\|U^n\|)|(|\nabla v^n(h^n, \Delta h^n)||h^n\|^p - 2 ds \leq CE \int_0^t \theta_R(\|U^n\|) \|h^n\|_{L^\infty} |\nabla v^n| \|\Delta h^n\|^p \|h^n\|^p - 2 ds. \quad (73) \]
Using Agmon’s inequality to control the first term and the definition of the cutoff function to control the second term of the right-hand side we obtain (cf. (46)):

\[ I_5^2 \leq C E \int_0^t \| h^n(t) \|^{p-2} | \Delta h^n(t) |^2 \| h^n(t) \|^p \, ds = C E \int_0^t \| h^n(t) \|^{p-2} | \Delta h^n(t) |^2 \| h^n(t) \|^p \, ds \] (74)

\[ \leq C E \int_0^t \| h^n \|^p \, ds + \frac{p \delta}{4} E \int_0^t | \Delta h^n(t) |^2 \| h^n(t) \|^p \, ds. \] (75)

The last line holds true thanks to the Young inequality. Combining (72) and (73), we obtain:

\[ I_5 = p E \int_0^t \| h^n(s) \|^{p-2} | (\theta(t(t) U(t)) \nabla \cdot (h^n \nabla n), \Delta h^n) | \, ds \]

\[ \leq C E \int_0^t \| h^n \|^p \, ds + \frac{p \delta}{2} E \int_0^t | \Delta h^n(t) |^2 \| h^n(t) \|^p \, dt. \] (76)

By applying the deterministic Gronwall inequality to the quantity

\[ Y(t) = E \sup_{0 \leq s \leq t} \left( \| v^n(s) \|^p + \| h^n(s) \|^p \right), \]

we obtain:

\[ E \sup_{0 \leq s \leq t} \left( \| v^n(s) \|^p + \| h^n(s) \|^p \right) \leq E \| v_0 \|^p + E \| h_0 \|^p + \int_0^t \| F \|^p \, ds + 1. \] (77)

We conclude the proof of the Lemma by combining the results in both (76) and (77).

4.3. Uniform estimates in fractional Sobolev spaces. We will show that the Galerkin approximations (69) are bounded in a fractional Sobolev space. We will first recall fractional Sobolev spaces and some continuity properties of stochastic integrals into these spaces.

**Definition 4.2.** Let \( X \) be a separable Hilbert space. Given \( p \geq 2, \alpha \in (0, 1) \), we define the fractional Sobolev space \( W^{\alpha,p}(0,T;X) \) as the Sobolev space of all \( u \in L^p(0,T;X) \) such that

\[ \int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t-s|^{1+\alpha p}} \, ds \, dt < \infty, \] (78)
endowed with the norm

$$\|u\|^p_{W^{\alpha,p}(0,T;X)} = \int_0^T |u(t)|^p_X dt + \int_0^T \int_0^T \frac{|u(t)-u(s)|^p}{|t-s|^{\alpha p}} dt ds.$$  \hspace{1cm} (79)

In Lemma 2.1 of [20], the authors showed that for every $p \in [2, \infty)$ the stochastic integral with respect to a Wiener process maps $L^p_{\nu,T}(X)$ continuously into $L^p(\Omega;W^{\alpha,p}(0,T;X))$. More precisely, there exists $C = C(\alpha,p)$ such that for any $\phi \in L^p_{\nu,T}(X)$, we have:

$$\mathbb{E}\|I^W(\phi)\|^p_{W^{\alpha,p}(0,T;X)} = \mathbb{E}\|\int_0^t \phi(s)dW(s)\|^p_{W^{\alpha,p}(0,T;X)} \leq CE \int_0^T \|\phi(t)\|^p_{L^2(\Omega;X)} dt.$$  \hspace{1cm} (80)

A similar continuity result, which we now state, holds for the stochastic integral with respect to a compensated Poisson random measure.

**Proposition 2.** For every $1 \leq p \leq 2$ any every $0 < \alpha < \frac{1}{p}$ the stochastic integral $\int^\pi$ maps $F^p_{\nu,T}(X)$ continuously into $L^p(\Omega;W^{\alpha,p}(0,T;X))$, i.e., we have:

$$\mathbb{E}\|\int^\pi (g)\|^p_{W^{\alpha,p}(0,T;X)} = \mathbb{E}\|\int_0^t g(s,z)d\pi(s,z)\|^p_{W^{\alpha,p}(0,T;X)} \leq CE \int_0^T \int_{E_0} |g(s,z)|^p d\nu(z) ds,$$  \hspace{1cm} (81)

for every $g \in F^p_{\nu,T}(X)$.

The proof of Proposition 2 is based on the BDG inequality and can be carried out in an analogous way to the proof of Lemma 2.1 in [20]. See also a similar result for stochastic convolutions in Lemma 7.4 of [6].

We now arrive at the main lemma of this section.

**Lemma 4.3.** Under the same assumptions as in Theorem 2.7, we consider the associated sequence of solutions $\{(v^n, h^n)\}_{n=1}^\infty$ of the Galerkin system (69). We assume further that $\mathbb{E}(\|v_0\|^2 + \|h_0\|^2) < \infty$. Then for every $\alpha \in (0, 1/2)$ there exist a finite number $C > 0$, independent of $n$, such that

$$\sup_{n \geq 1} \mathbb{E}\|v^n\|^2_{W^{\alpha,2}(0,T;H^1)} + \|h^n\|^2_{W^{\alpha,2}(0,T;H^2)} \leq C.$$  \hspace{1cm} (82)

**Proof.** By integrating equation (69a) from 0 to $t$, we obtain:

$$v^n(t) = v^n_0 + \int_0^t P_n F dt + \int_0^t P_n \sigma_1(v^n, h^n) dW_1 + \int_0^t P_n \mathcal{X}_1(v^n(s-), h^n(s-), z) d\tilde{\eta}_1(s, z).$$  \hspace{1cm} (83)
By using the triangle inequality, we infer

\[
\mathbb{E}\|v^n(t)\|_{W^{\alpha,2}(0,T;H_1)}^2 \leq \mathbb{E}\left|\int_0^t P_n\sigma_1(v^n, h^n) dW_1\right|^2_{W^{\alpha,2}(0,T;H_1)}
\]

\[
+ \mathbb{E}\left|\int_0^t \int_{E^0} P_n\mathcal{X}_1(v^n(s^-), h^n(s^-), z) d\tilde{\mathcal{X}}_1(s, z)\right|^2_{W^{\alpha,2}(0,T;H_1)}
\]

\[
+ \mathbb{E}\left|v^n(t) - \int_0^t P_n\sigma_1(v^n, h^n) dW_1
\right.
\]

\[
- \int_0^t \int_{E^0} P_n\mathcal{X}_1(v^n(s^-), h^n(s^-), z) d\tilde{\mathcal{X}}_1(s, z)\left|\right|^2_{W^{\alpha,2}(0,T;H_1)}
\]

\[
:= N_1(t) + N_2(t) + N_3(t).
\]

(84)

It follows from (80), (83) and assumption (22) that

\[
N_1(t) \leq \mathbb{E}\left|\int_0^t P_n\sigma_1(v^n, h^n) dW_1\right|^2_{W^{\alpha,2}(0,T;H_1)} \leq C\mathbb{E}\left|\int_0^T \left\|P_n\sigma_1(v^n, h^n)\right\|^2_{L_2(\Omega,H_1)}\right| dt
\]

\[
\leq C\mathbb{E}\left|\int_0^T (1 + \|v^n(t)\|^2 + \|h^n(t)\|^2) dt \right| \leq C\mathbb{E}\left|\sup_{t \in [0,T]} \left(\|v^n(t)\|^2 + \|h^n(t)\|^2\right)\right| + C
\]

\[
\leq C.
\]

The last line holds true thanks to Lemma 4.1. The uniform bound for \(N_2(t)\) can be obtained in a similar manner using Proposition 2 and Lemma 4.1. The bound for \(N_3(t)\) is derived as follows:

\[
\mathbb{E}\left|v^n(t) - \int_0^t P_n\sigma_1(v^n, h^n) dW_1
\right.
\]

\[
- \int_0^t \int_{E^0} P_n\mathcal{X}_1(v^n(s^-), h^n(s^-), z) d\tilde{\mathcal{X}}_1(s, z)
\]

\[
\mathbb{E}\left|v^n(t) - \int_0^t P_n\sigma_1(v^n, h^n) dW_1
\right.
\]

\[
- \int_0^t \int_{E^0} P_n\mathcal{X}_1(v^n(s^-), h^n(s^-), z) d\tilde{\mathcal{X}}_1(s, z)\left|\right|^2_{W^{\alpha,2}(0,T;H_1)}
\]

\[
\leq \mathbb{E}\left(\int_0^T \left|d_1 \int_0^t P_n\sigma_1(v^n, h^n) dW_1
\right.
\]

\[
- \int_0^t \int_{E^0} P_n\mathcal{X}_1(v^n(s^-), h^n(s^-), z) d\tilde{\mathcal{X}}_1(s, z)
\]

\[
\mathbb{E}\left(\int_0^T \left|v^n - \int_0^t P_n\sigma_1(v^n, h^n) dW_1
\right.
\]

\[
- \int_0^t \int_{E^0} P_n\mathcal{X}_1(v^n(s^-), h^n(s^-), z) d\tilde{\mathcal{X}}_1(s, z)\left|\right|^2_{W^{\alpha,2}(0,T;H_1)}
\]

\[
+ \mathbb{E}\left(\int_0^T \left|v^n - \int_0^t P_n\sigma_1(v^n, h^n) dW_1
\right.
\]

\[
- \int_0^t \int_{E^0} P_n\mathcal{X}_1(v^n(s^-), h^n(s^-), z) d\tilde{\mathcal{X}}_1(s, z)\left|\right|^2_{W^{\alpha,2}(0,T;H_1)}
\]

\[
\mathbb{E}\left(\int_0^T |\nu \Delta v^n + P_n[\theta_R(||U^n||)(v^n \cdot \nabla)v^n + g \nabla h^n + f k \times v^n]|^2 dt\right)
\]

\[
+ \int_0^T |F|^2 dt.
\]

(85)
We use the embedding $H^2 \hookrightarrow L^\infty$ to estimate the nonlinear term in $N_3(t)$ in (85) and estimate the remaining terms as in Proposition 1. We infer that
\[
N_3(t) \leq CE \left(\|v_0\|^2 + \nu \int_0^T |\Delta v^n|^2 dt + g \int_0^T \|h^n\|^2 dt + \int_0^T |v^n|^2 dt + \int_0^T |F|^2 dt \right),
\]
\[
\leq CE \left(\|v_0\|^2 + \int_0^T |\Delta v^n|^2 dt + \sup_{r \in [0,T]} (\|v^n\|^2 + \|h^n\|^2) + \int_0^T |F|^2 dt \right),
\]
\[
\leq C.
\]
The last step follows from an application of Lemmas 4.1. In a similar manner, we infer that there exists a constant $C$, which is independent of $n$, such that $E[\|h^n\|_{W^{2,2}(0,T;H_2)}^2] \leq C$. Therefore, the proof of the Lemma is complete.

5. Compactness argument. In this section we use the uniform estimates established above to show that the laws of the Galerkin solutions $(v^n, h^n)$ to equation (66) have a weakly convergent subsequence. In order to obtain a candidate solution that is càdlàg we will establish tightness of the laws of $(v^n, h^n)$ on a certain space of functions that are càdlàg in time. For a complete, separable metric space $S$ we denote by $D(0, T; S)$ the space of functions $\nu: [0, T] \to S$ that are right-continuous on $(0, T]$ and have left-limits at every point in $[0, T]$. The space $D(0, T; S)$ is endowed with the Skorokhod topology, which makes $D(0, T; S)$ separable and metrizable by a complete metric. The definition of the Skorokhod topology is recalled in Appendix Subsection A.2 along with other common notions in the theory of weak convergence of probability measures, such as tightness, Prokhorov’s Theorem and the Aldous condition, that will be used extensively in this section.

We will show that the laws of $(v^n, h^n)$ are tight as probability measures on the phase space $\mathcal{X} \times \mathcal{X}_h$, where
\[
\mathcal{X} = L^2(0, T; V_1) \cap D(0, T; H_1) \quad \text{and} \quad \mathcal{X}_h = L^2(0, T; V_2) \cap D(0, T; H_2).
\]
We denote the joint law of $(v^n, h^n)$ by $\mu^n$. So each $\mu^n$ is a probability measure on $\mathcal{X}$ with marginal distributions
\[
\mu^n_v(\cdot) = P (v^n \in \cdot) \in Pr(L^2(0, T; V_1) \cap D(0, T; H_1)),
\]
and
\[
\mu^n_h(\cdot) = P (h^n \in \cdot) \in Pr(L^2(0, T; V_2) \cap D(0, T; H_2)),
\]
where $Pr(X)$ denotes the set of probability measures on a metric space $X$ (see Appendix A). We begin by observing that $\{\mu^n\}_{n=1}^\infty$ forms a tight sequence of probability measures on $L^2(0, T; V_1) \times L^2(0, T; V_2)$.

**Proposition 3.** Suppose that $E[\|v_0\|^4 + \|h_0\|^4] < \infty$. Then the sequence $\{\mu^n\}_{n=1}^\infty$ is tight over $L^2(0, T; V_1) \times L^2(0, T; V_2)$.

It is straightforward to prove Proposition 3 by verifying the definition of tightness directly with the help of the compact embedding result
\[
L^2(0, T; D(-\Delta)) \cap W^{1/2,2}(0, T; H_i) \subset L^2(0, T; V_i),
\]
which holds for $i = 1, 2$ (see, e.g., Theorem 2.1 in [20]), Chebyshev’s inequality, Lemma 4.1 and the estimate (82). We will omit the proof of Proposition 3; see, e.g., Lemma 4.1 in [14] for full details of the argument in an analogous setting.
We now show that the laws \( \{\mu^n\}_{n=1}^\infty \) of the Galerkin approximations \((v^n, h^n)\) are tight on \( D(0, T; H_1^2) \times D(0, T; H_2) \) with the Skorokhod topology. Directly below we will do this by verifying the Aldous condition established in \cite{1} (which is stated as Lemma A.5 in Appendix A). We apply Lemma A.5 in two parts. First, we observe in Lemma 5.1 below that the laws of \((v^n(t), h^n(t))\) are tight on \( H_1 \times H_2 \) for each fixed \( t \in [0, T] \). Second, we verify the Aldous condition in Proposition 4.

**Lemma 5.1.** Suppose that \( \mathbb{E}(||v_0||^4 + ||h_0||^4) < \infty \). Then for each \( t \in [0, T] \) the laws of \((v^n(t), h^n(t))\) form a tight sequence of probability measures on \( H_1 \times H_2 \).

Since \( V \) is compactly embedded in \( H \), Lemma 5.1 is a simple consequence of Chebyshev’s inequality and Lemma 4.1. We omit the proof.

We are now ready to verify the Aldous condition and show that \( \{\mu^n\}_{n=1}^\infty \) forms a tight sequence of probability measures on \( D(0, T; H_1^2) \times D(0, T; H_2) \).

**Proposition 4.** Suppose that \( \mathbb{E}(||v_0||^4 + ||h_0||^4) < \infty \). Then the laws \( \{\mu^n\}_{n=1}^\infty \) of the Galerkin approximations \((v^n, h^n)\) form a tight sequence of probability measures on \( D(0, T; H_1^2) \times D(0, T; H_2) \), endowed with the Skorokhod topology.

**Proof.** It is sufficient to prove that the sequences \( \{\mu^n\}_{n=1}^\infty \) and \( \{\mu^n\}_{n=1}^\infty \) of marginal laws are tight on \( D(0, T; H_1^2) \) and \( D(0, T; H_2) \), respectively. Because of Lemma A.5 and Lemma 5.1 it is sufficient to verify condition (181). We begin by showing that condition (181) holds for the sequence \( \{v^n\}_{n=1}^\infty \) with \( \alpha = 2 \) and \( \beta = 1 \). For this purpose, we rewrite the Galerkin system (69a) as

\[
v^n(t) = v^n_0 + \int_0^t [\nu \Delta v^n - \theta_R(||U^n||) P_n((v^n(s) \cdot \nabla) v^n(s))] ds \\
- \int_0^t g \nabla h^n(s) - f k \times v^n(s)] ds + \int_0^t P_n F ds \\
+ \int_0^t P_n \sigma_1(v^n, h^n) dW_1 + \int_0^t \int_{E_0} P_n(\mathcal{X}_1(v^n(s-), h^n(s-), z)) d\pi_1(s, z)
\]

\[
= v^n_0 + I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t) + I_5^n(t) + I_6^n(t).
\]

Let \( \tau \) be a stopping time bounded by \( T \) almost surely and let \( t \in [0, T] \). By using Hölder’s inequality, we readily obtain the following bounds:

\[
\mathbb{E} \left( |I^n_1(\tau + t) - I^n_1(\tau)|^2 \right) = \mathbb{E} \left( \int_\tau^{\tau+t} \nu \Delta v^n(s) ds \right)^2 \leq \nu^2 t \mathbb{E} \int_\tau^{\tau+t} |\Delta v^n(s)|^2 ds \\
\leq \nu^2 t \mathbb{E} \int_0^T |\Delta v^n(s)|^2 ds.
\]  

We estimate the nonlinear term in exactly the same way as in the proof of Lemma 4.3 and obtain

\[
\mathbb{E} \left( |I^n_2(\tau + t) - I^n_2(\tau)|^2 \right) = \mathbb{E} \left( \int_\tau^{\tau+t} \theta_R(||U^n||) P_n((v^n(s) \cdot \nabla) v^n(s)) ds \right)^2 \\
\leq C t \mathbb{E} \left( \int_\tau^{\tau+t} |\Delta v^n(s)|^2 ds \right) \leq C t \mathbb{E} \left( \int_0^T |\Delta v^n(s)|^2 ds \right).
\]
Using the Cauchy-Schwarz inequality we obtain for $I^n_3$:

$$
E\left( |I^n_3(\tau + t) - I^n_3(\tau)|^2 \right) = C E \left| \int_{\tau}^{\tau + t} g \nabla h^n(s)ds \right|^2 \\
\leq t E \int_{\tau}^{\tau + t} |g \nabla h^n(s)|^2 ds \leq C g^2 t E \int_0^T \|h^n(s)\|^2 ds.
$$

(93)

Similarly, we have

$$
E\left( |I^n_1(\tau + t) - I^n_1(\tau)|^2 \right) = E \left| \int_{\tau}^{\tau + t} f k \times v^n(s)ds \right|^2 \\
\leq C t E \int_{\tau}^{\tau + t} |v^n(s)|^2 ds \leq C t E \sup_{t \in [0,T]} \|v^n(s)\|^2,
$$

(94)

and

$$
E\left( |I^n_2(\tau + t) - I^n_2(\tau)|^2 \right) = \left| \int_{\tau}^{\tau + t} P_n F(s)ds \right|^2 \leq t \int_0^T |F(s)|^2 ds.
$$

(95)

By using hypothesis (22) and Itô’s isometry, we obtain

$$
E\left( |I^n_3(\tau + t) - I^n_3(\tau)|^2 \right) = E \left( \left| \int_{\tau}^{\tau + t} \sigma_1(v^n(s), h^n(s))dW_1(s) \right|^2 \right) \\
= E \int_{\tau}^{\tau + t} \left| \sigma_1(v^n(s), h^n(s))^2 \right|_{L^2(\Omega, H_1)} ds \leq C E \int_{\tau}^{\tau + t} (1 + |v^n(s)|^2 + |h^n(s)|^2) ds \\
\leq C t E (1 + \sup_{0 \leq s \leq T} [||v^n(s)||^2 + ||h^n(s)||^2]).
$$

(96)

In the same manner, we use (22) and Itô’s isometry to obtain

$$
E\left( |I^n_1(\tau + t) - I^n_1(\tau)|^2 \right) = E \left( \left| \int_{\tau}^{\tau + t} \int_{E_0} P_n(\mathcal{L}_1(v^n(s-), h^n(s-), z))d\tilde{\nu}_1(s, z) \right|^2 \right) \\
= C E \left( \int_{\tau}^{\tau + t} \int_{E_0} |\mathcal{L}_1(v^n(s-), h^n(s-), z)|^2 d\nu(z)ds \right) \\
\leq C t E (1 + \sup_{0 \leq s \leq T} [||v^n(s)||^2 + ||h^n(s)||^2]).
$$

(97)

In summary, due to the uniform estimates in Lemmas 4.1 we conclude that there exists a constant $C > 0$, independent of $n$, such that for every stopping time $\tau_n$ bounded by $T$ we have

$$
E\left( |v^n(\tau_n + t) - v^n(\tau_n)|^2 \right) \leq C t,
$$

(98)

which is condition (181) with $\alpha = 2$ and $\beta = 1$.

Turning now to verifying condition (181) for the sequence $\{h^n\}_{n \geq 1}$, we rewrite (66b) as

$$
h^n(t) = h^n_0 - \int_0^t [\partial_t(\|U^n\|)P_n(\nabla \cdot (h^n v^n))] - \delta \Delta h^n)ds + \int_0^t \sigma_2(v^n, h^n)dW_2 \\
+ \int_0^t \int_{E_0} P_n(\mathcal{L}_2(v^n(s-), h^n(s-), z))d\tilde{\nu}_2(s, z)
$$

$$
= h^n_0 + J^n_1(t) + J^n_2(t) + J^n_3(t) + J^n_4(t).
$$

(99)
Observe that we can treat $J^1_n, J^2_n, J^3_n, J^4_n$ in a similar way to $I^2_n, I^3_n, I^6_n, I^7_n$, respectively. Therefore, we conclude that there exists a constant $C > 0$, independent of $n$, such that for every stopping time $\tau_n$ bounded by $T$ we have
\[
\mathbb{E} \left( |h^n(\tau_n + t) - h^n(\tau_n)|^2 \right) \leq Ct,
\]
which is condition (181). It now follows from Lemma A.5 that the sequence $\{\mu_n\}_{n=1}^\infty$ is tight on $D(0, T; H_1) \times D(0, T; H_2)$. \hfill $\Box$

Having shown that the sequence $\{\mu_n\}_{n=1}^\infty$ of joint laws of the Galerkin approximations $\{(v^n, h^n)\}_{n=1}^\infty$ is tight on the phase space $\mathcal{X}_\nu \times \mathcal{X}_h$, we are almost ready to extract a weakly convergent subsequence. When this is done, we would also like to carry along the noise processes $W_1, \pi_1$ and $W_2, \pi_2$ and the initial conditions in the system (68). For each positive integer $n$ let $\nu^n$ denote the joint law of the tuple
\[
(v^n_0, h^n_0, v^n, h^n, W_1, W_2, \pi_1, \pi_2).
\]
Then $\nu^n$ is a Borel probability measure on the space
\[
\mathcal{X} := V \times \mathcal{X}_\nu \times \mathcal{X}_h \times \mathcal{C}([0, T]; U) \times \mathcal{C}([0, T]; U) \times \mathcal{N}^\#_{(0, \infty) \times E} \times \mathcal{N}^\#_{(0, \infty) \times E},
\]
where $\mathcal{N}^\#_{(0, \infty) \times E}$ denotes the space of counting measures on $[0, \infty) \times E$ that are finite on bounded sets. The space $\mathcal{N}^\#_{(0, \infty) \times E}$ is endowed with the topology of weak-$\#$ convergence, which is weak convergence of measures using test functions that are bounded, continuous and have bounded support. For our purposes it is enough to point out that $\mathcal{N}^\#_{(0, \infty) \times E}$ is separable and metrizable by a complete metric (see, e.g., Proposition 9.1.IV. in [13]). One reason that this observation is useful is that every Borel probability measure on a complete, separable metric space is tight (see, e.g., Theorem 1.3 in [4]). Therefore, the laws of $\pi_1$ and $\pi_2$ are tight on $\mathcal{N}^\#_{(0, \infty) \times E}$. Since $\mathcal{C}([0, T]; U)$ is separable and complete, the laws of $W_1$ and $W_2$ are tight on $\mathcal{C}([0, T]; U)$. Since the projected initial conditions $\{(v^n_0, h^n_0)\}_{n=1}^\infty$ converge $\mathbb{P}$-a.s. to the given initial condition $(v_0, h_0)$ in the space $V$, it follows that the laws of the initial conditions $\{(v^n_0, h^n_0)\}_{n=1}^\infty$ are tight on $V$. So, we have shown that the sequences of laws of each coordinate in the tuple in (101) are tight on their respective spaces in (102). It now follows easily from the definition of tightness that the laws $\{\nu^n\}_{n=1}^\infty$ of the entire tuples in (101) form a tight sequence of probability measures on the space $\mathcal{X}$.

6. Passage to the limit. In the previous section, we have shown that the sequence of measures $\{\nu^n\}_{n=1}^\infty$ associated to the Galerkin sequence $\{(v^n_0, h^n_0, v^n, h^n, W_1, W_2, \pi_1, \pi_2)\}_{n=1}^\infty$ is tight on $\mathcal{X}$. Hence, in virtue of the Prokhorov Theorem (stated as Proposition 10), $\{\nu^n\}_{n=1}^\infty$ is weakly compact over $\mathcal{X}$. This implies that there exists a probability measure $\nu^\infty$ such that $\{\nu^n\}_{n=1}^\infty$ converges weakly to $\nu^\infty$. By applying the Skorokhod convergence theorem (stated in Appendix A as Theorem A.2), there exists a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, with the associated expectation denoted by $\tilde{\mathbb{E}}$, and $\mathcal{X}$-valued random variables $\{(\tilde{v}^{\nu_{nk}}_0, \tilde{h}^{\nu_{nk}}_0, \tilde{v}^{\nu_{nk}}, \tilde{h}^{\nu_{nk}}, \tilde{W}^{\nu_{nk}}_1, \tilde{W}^{\nu_{nk}}_2, \tilde{\pi}^{\nu_{nk}}_1, \tilde{\pi}^{\nu_{nk}}_2)\}_{k=1}^\infty$ and $(\tilde{v}_0, \tilde{h}_0, \tilde{v}, \tilde{h}, \tilde{W}_1, \tilde{W}_2, \tilde{\pi}_1, \tilde{\pi}_2)$, such that

i) The distribution of $(\tilde{v}^{\nu_{nk}}_0, \tilde{h}^{\nu_{nk}}_0, \tilde{v}^{\nu_{nk}}, \tilde{h}^{\nu_{nk}}, \tilde{W}^{\nu_{nk}}_1, \tilde{W}^{\nu_{nk}}_2, \tilde{\pi}^{\nu_{nk}}_1, \tilde{\pi}^{\nu_{nk}}_2)$ is $\nu^\infty_{nk}$ and the distribution of $(\tilde{v}_0, \tilde{h}_0, \tilde{v}, \tilde{h}, \tilde{W}_1, \tilde{W}_2, \tilde{\pi}_1, \tilde{\pi}_2)$ is $\nu^\infty$. 

SHALLOW WATER EQUATIONS WITH LÉVY NOISE 3793
\[ (\tilde{W}_1^{nk}(\omega), \tilde{W}_2^{nk}(\omega), \tilde{\pi}_1^{nk}(\omega), \tilde{\pi}_2^{nk}(\omega)) = (\tilde{W}_1(\omega), \tilde{W}_2(\omega), \tilde{\pi}_1(\omega), \tilde{\pi}_2(\omega)) \quad \forall \omega \in \tilde{\Omega}. \]

iii) \( \{(\tilde{\nu}_0, \tilde{\nu}_1, \tilde{\pi}_0, \tilde{\pi}_1, \tilde{\nu}_0, \tilde{\nu}_2, \tilde{\pi}_0, \tilde{\pi}_2)\}_{k=1}^{\infty} \) converges in \( \mathcal{X}, \tilde{\mathbb{P}} \)-a.s., to \((\nu_0, \nu_1, \nu_2, W_1, \tilde{\pi}_1, \tilde{\pi}_2)\), i.e.,

\[
\begin{align*}
\tilde{\nu}_0^{nk} &\to \nu_0 \quad \text{in } V_1 \quad \tilde{\mathbb{P}}\text{-a.s.}, \\
\tilde{\nu}_0^{nk} &\to \nu_0 \quad \text{in } V_2 \quad \tilde{\mathbb{P}}\text{-a.s.}, \\
\tilde{\nu}_1^{nk} &\to \nu \quad \text{in } L^2(0, T; V_1) \cap D(0, T; H_1) \quad \tilde{\mathbb{P}}\text{-a.s.}, \\
\tilde{\nu}_2^{nk} &\to \tilde{\nu} \quad \text{in } L^2(0, T; V_2) \cap D(0, T; H_2) \quad \tilde{\mathbb{P}}\text{-a.s.} \quad (103d)
\end{align*}
\]

iv) \( \tilde{\pi}_1, \tilde{\pi}_2 \) are time homogeneous Poisson random measures over \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) with intensity measure \( dt \otimes dv \).

In order to obtain a complete, right-continuous filtration \((\tilde{\mathcal{F}}_t)_{t \geq 0}\) on the new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), to which the noise processes and solutions are adapted, we define \( \tilde{\mathcal{F}}'_t \) to be the \( \sigma \)-field of subsets of \( \tilde{\mathcal{F}} \) generated by the random variables \( \tilde{W}_1(s), \tilde{W}_2(s), \tilde{\nu}(s), \tilde{\pi}_1((0, s] \times \Gamma), \) and \( \tilde{\pi}_2((0, s] \times \Gamma) \) for all \( s \leq t \) and all Borel subsets \( \Gamma \) of \( E_0 \). Then we define

\[
\mathcal{N} := \{ A \in \tilde{\mathcal{F}} \mid \tilde{\mathbb{P}}(A) = 0 \}, \\
\tilde{\mathcal{F}}_t = \sigma(\mathcal{F}'_t, \cup \mathcal{N}), \\
\tilde{\mathcal{F}} := \bigcap_{s \geq t} \tilde{\mathcal{F}}_s. \quad (104)
\]

Now we define the new stochastic basis \( \tilde{S} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{W}_1, \tilde{W}_2, \tilde{\pi}_1, \tilde{\pi}_2) \). We will show in Proposition 5 that \((\tilde{S}, \tilde{\nu}, \tilde{\pi})\) is a global martingale solution of the system (69) in the sense of definition (2.3). For each \( nk \) we define a filtration \((\tilde{\mathcal{F}}^{nk}_t)_{t \geq 0}\) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) in the same way as above but using \((\tilde{\nu}^{nk}, \tilde{h}^{nk})\) in place of \((\tilde{\nu}, \tilde{h})\). Each \((\tilde{\nu}^{nk}, \tilde{h}^{nk})\) is adapted to \((\tilde{\mathcal{F}}^{nk}_t)_{t \geq 0}\) and we have the following:

v) Each \((\tilde{\nu}_0^{nk}, \tilde{h}_0^{nk}, \tilde{\nu}_1^{nk}, \tilde{h}_1^{nk})\) satisfies

\[
d\tilde{\nu}^{nk} - \nu \Delta \tilde{\nu}^{nk} + P_{nk} \left( \theta(||\tilde{U}^{nk}||) (\tilde{\nu}^{nk} \cdot \nabla) \tilde{\nu}^{nk} + g \nabla \tilde{h}^{nk} + f k \times \tilde{\nu}^{nk} \right) dt = (105)
\]

\[
P_{nk} F dt + P_{nk} \sigma_1(\tilde{\nu}^{nk}, \tilde{h}^{nk}) dW_1 + \int_{E_0} P_{nk} \mathcal{K}_1(\tilde{\nu}^{nk}(t-), \tilde{h}^{nk}(t-), z) d\tilde{\pi}_1(t, z)
\]

and

\[
d\tilde{h}^{nk} + P_{nk} \left( - \delta \Delta \tilde{h}^{nk} + \theta(||\tilde{U}^{nk}||) \nabla \cdot (\nabla \tilde{\nu}^{nk} \tilde{h}^{nk}) \right) dt = (106)
\]

\[
P_{nk} \sigma_2(\tilde{\nu}^{nk}, \tilde{h}^{nk}) dW_1 + \int_{E_0} P_{nk} \mathcal{K}_2(\tilde{\nu}^{nk}(t-), \tilde{h}^{nk}(t-), z) d\tilde{\pi}_2(t, z).
\]

- The proofs from i) through iii) are directly inferred from Theorem A.2.
- To prove iv), we need to show
  i) \( \forall \Gamma \in B(E_0), \tilde{\pi}_i(\Gamma) \sim \text{Poisson } (\lambda(\Gamma)), i = 1, 2, \) where \( d\lambda := dt \otimes dv \).
  ii) For disjoint sets \( \Gamma_1, \Gamma_2, ..., \Gamma_m \in B(E_0), \tilde{\pi}_i(\Gamma_1), \tilde{\pi}_i(\Gamma_2), ..., \tilde{\pi}_i(\Gamma_m) \) are independent.
  Both parts are obvious as a direct consequence of i) and ii).
- The proof of v) can be obtained by adapting the argument of [2] from the Wiener case to the Lévy noise case. See also the proof of Theorem 2.3 in [30] on page 2069.
• Because of \( v \), all of the estimates that hold for \((v^n, h^n)\) also hold for \((\tilde{v}^{nk}, \tilde{h}^{nk})\) with respect to the new stochastic basis \( \tilde{S} \). In particular, the conclusion of Lemma 4.1 holds for \((\tilde{v}^{nk}, \tilde{h}^{nk})\) and

\[
\begin{align*}
&i) \quad \tilde{v}^{nk} \text{ belong to a bounded subset of } L^2(\tilde{\Omega}, L^\infty(0, T, V_1)) \cap L^2(\tilde{\Omega}, L^2(0, T, D(A_1))), \\
&ii) \quad \tilde{h}^{nk} \text{ belong to a bounded subset of } L^2(\tilde{\Omega}, L^\infty(0, T, V_2)) \cap L^2(\tilde{\Omega}, L^2(0, T, D(A_2))).
\end{align*}
\]

By the Banach-Alaoglu theorem, we infer that there exist \( \tilde{v}, \tilde{h} \) in those spaces such that:

\[
\begin{align*}
&i) \quad \tilde{v}^{nk} \rightharpoonup \tilde{v} \text{ weak-* in } L^2(\tilde{\Omega}, L^\infty(0, T, V_1)) \text{ and weakly in } L^2(\tilde{\Omega}, L^2(0, T, D(A_1))), \\
&ii) \quad \tilde{h}^{nk} \rightharpoonup \tilde{h} \text{ weak-* in } L^2(\tilde{\Omega}, L^\infty(0, T, V_2)) \text{ and weakly in } L^2(\tilde{\Omega}, L^2(0, T, D(A_2))).
\end{align*}
\]

Our task now is to show that \( \tilde{v}, \tilde{h} \) are solutions of the system (66). Specifically, \((\tilde{v}, \tilde{h})\) will be a global martingale solution of the system (66) with respect to the stochastic basis \( \tilde{S} \), which was obtained by applying the modified version of the Skorokhod convergence theorem. Note that \((\tilde{v}_0, \tilde{h}_0)\), which will serve as the initial condition for \((\tilde{v}, \tilde{h})\) as a solution to (66), has the same law as the original initial condition \((v_0, h_0)\) but is defined on the new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\).

**Proposition 5.** Suppose that \( \mathbb{E} (\|v_0\|^4 + \|h_0\|^4) < \infty \). Then \((\tilde{S}, \tilde{v}, \tilde{h})\) is a global martingale solution to the modified system (66).

**Proof.** From Lemma 4.1 we obtain the following estimates

\[
\sup_{k \in \mathbb{N}} \mathbb{E} \left( \int_0^T \|\tilde{v}^{nk}\|^2 dt \right)^2 \leq C \sup_{k \in \mathbb{N}} \mathbb{E} \sup_{t \in [0, T]} \|\tilde{v}^{nk}\|^4 < \infty, \quad \text{and} \quad (107)
\]

\[
\sup_{k \in \mathbb{N}} \mathbb{E} \left( \int_0^T \|\tilde{h}^{nk}\|^2 dt \right)^2 \leq C \sup_{k \in \mathbb{N}} \mathbb{E} \sup_{t \in [0, T]} \|\tilde{h}^{nk}\|^4 < \infty.
\]

By applying the Vitali Convergence Theorem, we infer that

\[
\tilde{v}^{nk} \to \tilde{v} \text{ in } L^2(\tilde{\Omega}, L^\infty(0, T, V_1)) \text{ and } \tilde{h}^{nk} \to \tilde{h} \text{ in } L^2(\tilde{\Omega}, L^2(0, T, V_2)).
\]

This implies that for \( \tilde{\mathbb{P}} \text{-a.s and for a.e } t \in [0, T] \), the following convergence holds along a further subsequence (which we continue to denote in the same way)

\[
\|\tilde{v}^{nk} - \tilde{v}\| \to 0 \text{ and } \|\tilde{h}^{nk} - \tilde{h}\| \to 0.
\]

Fix \( \phi \in D(A_1) \) and take the inner product in \( H_1 \) with \( \phi \) on both sides of equation (105) and then let \( k \to \infty \) term-by-term. At the same time we take the inner product in \( H_2 \) with a fixed \( \psi \in D(A_2) \) on both sides of equation (106) and let \( k \to \infty \) term-by-term. We begin with the initial conditions. Thanks to (103a), we obtain

\[
(\tilde{v}_0^{nk}, \phi) \to (\tilde{v}_0, \phi) \quad \text{and} \quad (\tilde{h}_0^{nk}, \psi) \to (\tilde{h}_0, \psi)
\]

By applying the Vitali Convergence Theorem, we obtain:

\[
\lim_{k \to \infty} \mathbb{E}_0[|\tilde{v}_0^{nk} - \tilde{v}_0, \phi|^2] = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathbb{E}_0[|\tilde{h}_0^{nk} - \tilde{h}_0, \psi|^2] = 0.
\]

Hence,

\[
\lim_{k \to \infty} \left| (\tilde{v}_0^{nk} - \tilde{v}_0, \phi) \right|_{L^2(\tilde{\Omega} \times [0, T])}^2 = 0 \quad \text{and} \quad \lim_{k \to \infty} \left| (\tilde{h}_0^{nk} - \tilde{h}_0, \psi) \right|_{L^2(\tilde{\Omega} \times [0, T])}^2 = 0.
\]

(110)
Using Hölder’s inequality and (108) it is also easy to infer that
\[ \lim_{k \to \infty} \| (\tilde{\nu}^{nk} - \tilde{\nu}, \phi) \|_{L^2(\Omega \times [0,T])}^2 = 0, \quad \text{and} \quad \lim_{k \to \infty} \| (\tilde{h}^{nk} - \tilde{h}, \psi) \|_{L^2(\Omega \times [0,T])}^2 = 0. \]

Using Hölder’s inequality and (108) it is also easy to infer that
\[ \lim_{k \to \infty} \left| \left| \nu \Delta \tilde{\nu}^{nk} - \nu \Delta \tilde{\nu}, \phi \right| \right|^2_{L^2(\Omega \times [0,T])} = \lim_{k \to \infty} \left| \left| (\delta \Delta \tilde{h}^{nk} - \delta \Delta \tilde{h}, \psi) \right| \right|^2_{L^2(\Omega \times [0,T])} = 0. \]

Next, we consider the nonlinear term:
\[
0 \leq \left| \int_0^t \left( \theta_R((\tilde{U}^{nk}) ||) P_{nk} (\tilde{\nu}^{nk} \cdot \nabla) \tilde{\nu}^{nk} - \theta_R((\tilde{U} ||)) (\nabla \tilde{\nu} \cdot \tilde{\nu}, \phi) \right) ds \right|
\]
\[
\leq \left| \int_0^t \left( \theta_R((\tilde{U}^{nk}) ||) P_{nk} (\tilde{\nu}^{nk} \cdot \nabla) \tilde{\nu}^{nk} - \theta_R((\tilde{U} ||)) (P_{nk} (\tilde{\nu} \cdot \nabla) \tilde{\nu} + Q_{nk} (\tilde{\nu} \cdot \nabla) \tilde{\nu}, \phi) \right) ds \right|
\]
\[
\leq \left| \int_0^t \theta_R((\tilde{U}^{nk}) ||) (P_{nk} (\tilde{\nu}^{nk} \cdot \nabla) \tilde{\nu}^{nk} - P_{nk} (\tilde{\nu} \cdot \nabla) \tilde{\nu}, \phi) ds \right|
\]
\[
+ \left| \int_0^t \theta_R((\tilde{U} ||)) (Q_{nk} (\tilde{\nu} \cdot \nabla) \tilde{\nu}, \phi) ds \right|
\]
\[
:= M_1 + M_2 + M_3.
\]

For $M_1$ we have
\[
M_1 := \left| \int_0^t \theta_R((\tilde{U}^{nk}) ||) (\tilde{\nu}^{nk} \cdot \nabla) \tilde{\nu}^{nk} - (\tilde{\nu} \cdot \nabla) \tilde{\nu}, P_{nk} \phi) ds \right|
\]
\[
\leq \int_0^t \left| (\tilde{\nu}^{nk} - \tilde{\nu}) \nabla \cdot \tilde{\nu}^{nk} - \tilde{\nu} \cdot \nabla (\tilde{\nu} - \tilde{\nu}^{nk}), P_{nk} \phi) \right| ds
\]
\[
\leq \int_0^t \left| \tilde{\nu}^{nk} - \tilde{\nu} \right| \left| \tilde{\nu}^{nk} \right| \left| \phi \right|_{L^\infty} + \left| \tilde{\nu} \right| \left| \tilde{\nu}^{nk} - \tilde{\nu} \right| \left| \phi \right|_{L^\infty} ds
\]
\[
\leq C \left| \phi \right|_{D(A_1)} \left( \int_0^T \left| \tilde{\nu}^{nk} - \tilde{\nu} \right|^2 \right)^{\frac{1}{2}} \left( \int_0^T \left| \tilde{\nu}^{nk} \right|^2 + \left| \tilde{\nu} \right|^2 \right)^{\frac{1}{2}}.
\]
So, $M_1 \to 0$, $\bar{\nu}$-a.s. for all $t \in [0,T]$ because $\tilde{\nu}^{nk} \to \tilde{\nu}$ in $L^2(0,T;V_1)$, $\bar{\nu}$-a.s., along a subsequence. Next, since $\theta_R$ is Lipschitz and by using the embedding $H^2 \hookrightarrow L^\infty$ in 2-D, we obtain
\[
M_2 := \left| \int_0^t \left( \theta_R((\tilde{U}^{nk}) ||) - \theta_R((\tilde{U} ||)) \right) (P_{nk} (\tilde{\nu} \cdot \nabla) \tilde{\nu}, \phi) ds \right|
\]
\[
\leq C \left| \phi \right|_{L^\infty} \int_0^t \left( \left| \tilde{\nu}^{nk} - \tilde{\nu} \right| + \left| \tilde{h}^{nk} - \tilde{h} \right| \right) \left| \tilde{\nu} \right| \left| \phi \right| ds
\]
\[
\leq C \left| \phi \right|_{D(A_1)} \int_0^t \left( \left| \tilde{\nu}^{nk} - \tilde{\nu} \right| + \left| \tilde{h}^{nk} - \tilde{h} \right| \right) \left| \tilde{\nu} \right|^2 ds
\]
\[
\leq C \left| \phi \right|_{D(A_1)} \left( \left| \tilde{\nu} \right|_{L^2(0,T;V_1)} \left( \left| \tilde{\nu}^{nk} - \tilde{\nu} \right|^2 + \left| \tilde{h}^{nk} - \tilde{h} \right|^2 \right) + \left| \tilde{h}^{nk} - \tilde{h} \right|^2_{L^2(0,T;V_2)} \right). \quad (111)
\]
We have $M_2 \to 0$, $\bar{\mathbb{P}}$-a.s., for all $t \in [0, T]$ because $\vec{v}_{nk}^{\ast} \to \vec{v}$ in $L^2(0, T; V_1)$, $\bar{\mathbb{P}}$-a.s., along a subsequence and because $\vec{v} \in L^4(0, T; V_1)$, $\bar{\mathbb{P}}$-a.s. Next, for $M_3$ we have

\[
M_3 := \left| \int_0^t (\theta_R(||\vec{U}||)Q_{nk}(\vec{v} \cdot \nabla \vec{v}, \phi))ds \right| \leq \int_0^t (|\theta_R(||\vec{U}||)(\vec{v} \cdot \nabla \vec{v})Q_{nk, \phi})|ds \\
\leq C \int_0^t |\vec{v}|_{L^4} \left| \theta_R(||\vec{U}||)\nabla \vec{v} \right| |Q_{nk, \phi}|_{L^4} ds \leq C \int_0^t |\vec{v}| |Q_{nk, \phi}| ds \\
\leq C \frac{1}{\lambda_{nk}} |\phi|_{D(A_1)} \left( \int_0^T ||\vec{v}||^2 ds \right)^{\frac{1}{2}},
\]

which tends to zero $\bar{\mathbb{P}}$-a.s. for all $t \in [0, T]$ because $\lambda_{nk} \to \infty$. It is direct to deduce the following bounds thanks to Lemma 4.1:

\[
\mathbb{E} \left| \int_0^t (\theta_R(||\vec{U}_{nk}||)P_{nk}(\vec{v}_{nk} \cdot \nabla \vec{v}_{nk}, \phi))ds \right|^4 \leq C \mathbb{E} \int_0^t |(\theta_R(||\vec{U}_{nk}||)(\vec{v}_{nk} \cdot \nabla)\vec{v}_{nk}, P_{nk, \phi})|^4 ds \\
\leq C \mathbb{E} \int_0^t \left| \theta_R(||\vec{U}_{nk}||)\nabla \vec{v}_{nk} \right|^4 |\vec{v}_{nk}|^4_{L^1} |P_{nk, \phi}|^4_{L^\infty} ds \\
\leq C |\phi|^4_{D(A_1)} \mathbb{E} \int_0^T ||\vec{v}_{nk}||^4 ds \\
< C < \infty,
\]

where the constant $C$ does not depend on $k$. And

\[
\int_0^T \mathbb{E} \left| \int_0^t (\theta_R(||\vec{U}_{nk}||)P_{nk}(\vec{v}_{nk} \cdot \nabla)(\vec{v}_{nk}, \phi))ds \right|^4 dt \leq CT ||\phi||^4_{D(A_1)} \mathbb{E} \int_0^T ||\vec{v}_{nk}||^4 ds \\
< C < \infty.
\]

By utilizing the Vitali Convergence Theorem, we infer that

\[
\left| \int_0^t (\theta_R(||\vec{U}_{nk}||)P_{nk}(\vec{v}_{nk} \cdot \nabla \vec{v}_{nk} - \theta_R(||\vec{U}||)(\vec{v} \cdot \nabla)\vec{v}, \phi))ds \right|^2_{L^2(\vec{v} \times [0, T])} = 0. \tag{115}
\]

Next, we consider

\[
0 \leq \left| \int_0^t (\theta_R(||\vec{U}_{nk}||)P_{nk} \nabla \cdot (\vec{v}_{nk} h_{nk}) - \theta_R(||\vec{U}||)\nabla \cdot (\vec{v} h), \psi)ds \right| \\
\leq \left| \int_0^t (\theta_R(||\vec{U}_{nk}||)P_{nk} (\nabla \cdot \vec{v}_{nk} h_{nk} + \nabla h_{nk} \cdot \vec{v}_{nk}) - \theta_R(||\vec{U}||)(\vec{h} \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \vec{h}), \psi)ds \right| \\
\leq \left| \int_0^t (\theta_R(||\vec{U}_{nk}||)P_{nk} (\nabla \cdot \vec{v}_{nk} h_{nk}) - \theta_R(||\vec{U}_{nk}||)P_{nk} (\nabla \cdot \vec{h}), \psi)ds \right| + \left| \int_0^t (\theta_R(||\vec{U}||)Q_{nk} (\nabla \cdot \vec{h}), \psi)ds \right| \\
+ \left| \int_0^t (\theta_R(||\vec{U}||)Q_{nk} (\nabla \vec{h} \cdot \vec{v}), \psi)ds \right| \\
+ \left| \int_0^t (\theta_R(||\vec{U}_{nk}||)P_{nk} (\nabla h_{nk} \cdot \vec{v}_{nk}) - \theta_R(||\vec{U}_{nk}||)P_{nk} (\nabla \vec{h} \cdot \vec{v}), \psi)ds \right|.
\]
+ \left| \int_0^t \left( \theta_R(\|\tilde{U}_{\text{nk}}\|)P_{nk}(\nabla \tilde{h} \cdot \tilde{\nu}) - \theta_R(\|\tilde{U}_{\text{nk}}\|)P_{nk}(\nabla \tilde{h} \cdot \tilde{\nu}), \psi \right) ds \right| \\
+ \left| \int_0^t \left( \theta_R(\|\tilde{U}_{\text{nk}}\|)Q_{nk}(\nabla \tilde{h} \cdot \tilde{\nu}), \psi \right) ds \right| \\
:= \sum_{i=1}^6 N_i. \tag{116}

For $N_1$, we have

$N_1 := \int_0^t \left( \theta_R(\|\tilde{U}_{\text{nk}}\|)P_{nk}(\nabla \tilde{v} \cdot \tilde{\nu}_{\text{nk}}) - \theta_R(\|\tilde{U}_{\text{nk}}\|)P_{nk}(\nabla \tilde{v} \tilde{\nu}), \psi \right) ds$

\[
\leq \left| \int_0^t (\nabla \cdot (\tilde{v} \cdot \tilde{\nu}) \tilde{\nu}_{\text{nk}} + \nabla \cdot \tilde{\nu}(\tilde{\nu}_{\text{nk}} - \tilde{\nu}), \psi) ds \right| \\
\leq \int_0^t \left( |\nabla \cdot (\tilde{v} \cdot \tilde{\nu})| \|\tilde{\nu}_{\text{nk}}| + |\nabla \tilde{\nu}| \|\tilde{\nu}_{\text{nk}} - \tilde{\nu}\| \right)_{L^\infty} ds \\
\leq C |\psi|_{D(A_2)} \int_0^t \|\tilde{v} \cdot \tilde{\nu}\| \|\tilde{\nu}_{\text{nk}}\| + \|\nabla \tilde{\nu}\| \|\tilde{\nu}_{\text{nk}} - \tilde{\nu}\| ds \\
\leq C |\psi|_{D(A_2)} \left[ \left( \int_0^t \|\tilde{v} \cdot \tilde{\nu}\|^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\tilde{\nu}_{\text{nk}}\|^2 ds \right)^{\frac{1}{2}} \right] (117)
\]

which converges to 0 as $k \to \infty$ for all $t \in [0, T]$ because $\tilde{v} \cdot \tilde{\nu}_{\text{nk}} \to \tilde{v}$ in $L^2(0, T; V_1)$ and $\tilde{\nu}_{\text{nk}} \to \tilde{\nu}$ in $L^2(0, T; V_2) \mathbb{P}$-a.s.

Next, we consider

$N_2 := \int_0^t \left( \theta_R(\|\tilde{U}_{\text{nk}}\|) - \theta_R(\|\tilde{U}_{\text{nk}}\|)P_{nk}(\nabla \cdot \tilde{\nu}), \psi \right) ds$

\[
\leq C |\psi|_{D(A_2)} \int_0^t (\|\tilde{v} \cdot \tilde{\nu}\| + \|\tilde{\nu}_{\text{nk}} - \tilde{\nu}\|) \|\nabla \tilde{\nu}\| \|\tilde{\nu}_{\text{nk}}\| ds \\
\leq C |\psi|_{D(A_2)} \left( \left( \int_0^t \|\tilde{v} \cdot \tilde{\nu}\|^2 + \|\tilde{\nu}_{\text{nk}} - \tilde{\nu}\|^2 \right)^{\frac{1}{2}} \left( \int_0^t \|\tilde{\nu}\|^4 + \|\tilde{\nu}_{\text{nk}}\|^4 ds \right)^{\frac{1}{2}} \right) \\
\leq C |\psi|_{D(A_2)} \|\tilde{v}\|_{L^4(0, T; V_1)} \|\tilde{\nu}\|_{L^4(0, T; V_1)} \|\tilde{\nu}_{\text{nk}} - \tilde{\nu}\|_{L^2(0, T; V_1)} + \|\tilde{\nu}_{\text{nk}} - \tilde{\nu}\|_{L^2(0, T; V_2)}, \tag{118}
\]

which tends to 0 as $k \to \infty$ by the same reasoning as the term $N_1$.

The term $N_3$ is bounded as follows:

$N_3 := \int_0^t \left( \theta_R(\|\tilde{U}\|)Q_{nk}(\nabla \cdot \tilde{\nu}h), \psi \right) ds \leq \int_0^t \left( \theta_R(\|\tilde{U}\|) \|\nabla \tilde{\nu}\| \|\tilde{h}\|_{L^4} |Q_{nk} \psi|_{L^4} \right) ds$

\[
\leq C \int_0^t \|\tilde{h}\|_{L^4} |Q_{nk} \psi| ds \leq \frac{C}{\lambda_{nk}^{1/2}} |\psi|_{D(A_2)} \left( \int_0^t \|\tilde{h}\|^2 ds \right)^{\frac{1}{2}} \\
\to 0 \text{ as } k \to \infty \text{ because } \lambda_{nk} \to \infty. \tag{119}
\]

The last line follows thanks to the definition of the cutoff in (67) and by the reverse Poincaré inequality in (6).
Thanks to Lemma 4.1, we obtain:

\[
\mathbb{E} \left| \int_0^t (\theta_R(\|\tilde{U}^{n_k}\|)P_{n_k} \nabla \cdot (\tilde{v}^{n_k} \tilde{h}^{n_k}), \psi) ds \right|^4 \\
= \mathbb{E} \left| \int_0^t (\theta_R(\|\tilde{U}^{n_k}\|)(\nabla \cdot \tilde{v}^{n_k} \tilde{h}^{n_k} + \nabla \tilde{h}^{n_k} \cdot \tilde{v}^{n_k}), \psi) ds \right|^4 \\
\leq C |\psi|^4 \mathbb{E} \left( \sup_{t \in [0, T]} \|\tilde{v}^{n_k}\|^4 + \|\tilde{h}^{n_k}\|^4 \right) \leq C < \infty,
\]

where \(C\) does not depend on \(k\). And

\[
\int_0^T \mathbb{E} \left| \int_0^t (\theta_R(\|\tilde{U}^{n_k}\|)P_{n_k} \nabla \cdot (\tilde{v}^{n_k} \tilde{h}^{n_k}), \psi) ds \right|^4 \\
\leq C |\psi|^4 \mathbb{E} \left( \sup_{t \in [0, T]} \|\tilde{v}^{n_k}\|^4 + \|\tilde{h}^{n_k}\|^4 \right) \leq C < \infty,
\]

where the constant \(C\) does not depend on \(k\).

By making use of the Vitali Convergence Theorem, we infer that the following convergence holds

\[
\lim_{k \to \infty} \left| \int_0^t (P_{n_k} [\theta_R(\|\tilde{U}^{n_k}\|)\nabla \cdot (\tilde{v}^{n_k} \tilde{h}^{n_k})] - \theta_R(\|\tilde{U}\|)\nabla \cdot (\tilde{h} \tilde{v}), \psi) ds \right|_{L^2(\tilde{U} \times [0, T])} = 0.
\]

The reader is referred to [26] to see a detailed proof of the following results:

\[
\lim_{k \to \infty} \left| (P_{n_k} \sigma_1(\tilde{v}^{n_k}(s), \tilde{h}^{n_k}(s))d\tilde{W}_1 - \sigma_1(\tilde{v}(s), \tilde{h}(s))d\tilde{W}_1, \phi) \right|_{L^2(\tilde{U} \times [0, T])} = 0, \quad (123)
\]

\[
\lim_{k \to \infty} \left| (P_{n_k} \sigma_2(\tilde{v}^{n_k}(s), \tilde{h}^{n_k}(s))d\tilde{W}_2 - \sigma_2(\tilde{v}(s), \tilde{h}(s))d\tilde{W}_2, \phi) \right|_{L^2(\tilde{U} \times [0, T])} = 0. \quad (124)
\]

The last two stochastic terms are treated differently

\[
\int_0^t \int_{E_0} \left| (P_{n_k} \mathcal{A}_1(\tilde{v}^{n_k}(s-), \tilde{h}^{n_k}(s-), z) - \mathcal{A}_1(\tilde{v}(s-), \tilde{h}(s-), z), \phi) \right|^2 d\nu(z) ds \\
\leq \int_0^t \int_{E_0} \left| (P_{n_k} \mathcal{A}_1(\tilde{v}^{n_k}(s-), \tilde{h}^{n_k}(s-), z) - P_{n_k} \mathcal{A}_1(\tilde{v}(s-), \tilde{h}(s-), z), \phi) \right|^2 d\nu(z) ds \\
+ \int_0^t \int_{E_0} \left| (Q_{n_k} \mathcal{A}_1(\tilde{v}(s-), \tilde{h}(s-), z), \phi) \right|^2 d\nu(z) ds \\
\leq |\phi|^2 \int_0^t \|\tilde{v}^{n_k}(s) - \tilde{v}(s)\|^2 + \|\tilde{h}^{n_k}(s) - \tilde{h}(s)\|^2 ds \\
+ \frac{C}{\lambda_{n_k}} \int_0^t \int_{E_0} \|Q_{n_k} \mathcal{A}_1(\tilde{v}(s-), \tilde{h}(s-), z)\|^2 d\nu(z) ds \\
\leq |\phi|^2 \int_0^T (\|\tilde{v}^{n_k}(s) - \tilde{v}(s)\|^2 + \|\tilde{h}^{n_k}(s) - \tilde{h}(s)\|^2 + \frac{C}{\lambda_{n_k}} (1 + \|\tilde{v}\|^2 + \|\tilde{h}\|^2)) ds \to 0.
\]
The last line follows thanks to the assumption (22), (23) and (109). Therefore, we conclude that

\[
P_{n_k} \mathcal{X}_1(\mathbf{v}^n_k(s-), h^n_k(s-), z) \text{ converges to } \mathcal{X}_1(\mathbf{v}(s-), h(s-), z)
\]  

(126)
a.s in \(L^2([0, T] \times E_0, dt \otimes d\nu, V_1)\). It is straightforward to obtain the next bound by making use of the assumption (22) and the uniform in Lemma 4.1:

\[
\mathbb{E} \int_0^t \int_{E_0} |(P_{n_k} \mathcal{X}_1(\mathbf{v}^n_k(s-), h^n_k(s-), z), \phi)|^4 d\nu(z) ds
\] 
\[
\leq C |\phi|^4 \mathbb{E} \int_0^t \int_{E_0} ||\mathcal{X}_1(\mathbf{v}^n_k(s-), h^n_k(s-), z)||^4 d\nu(z) ds
\] 
\[
\leq C |\phi|^4 \mathbb{E} \int_0^t (1 + ||\mathbf{v}^n_k||^4 + ||h^n_k||^4) ds
\] 
\[
\leq C |\phi|^4 \mathbb{E} \left( \sup_{t \in [0, T]} \left( ||\mathbf{v}^n_k||^4 + ||h^n_k||^4 \right) \right) ds + C < C < \infty. 
\]  

(127)

By applying the Vitali Convergence Theorem, we infer that

\[
\lim_{k \to \infty} \mathbb{E} \int_0^t \int_{E_0} |(P_{n_k} \mathcal{X}_1(\mathbf{v}^n_k(s-), h^n_k(s-), z) - \mathcal{X}_1(\mathbf{v}(s-), h(s-), z), \phi)|^2 d\nu(z) ds = 0. 
\]  

(128)

By making use of the Itô isometry, we deduce that

\[
\lim_{k \to \infty} \mathbb{E} \left| \int_0^t \int_{E_0} (P_{n_k} \mathcal{X}_1(\mathbf{v}^n_k(s-), h^n_k(s-), z) - \mathcal{X}_1(\mathbf{v}(s-), h(s-), z), \phi) d\mathbf{\tilde{u}}_1(s, z) \right|^2
\] 
\[
= \lim_{k \to \infty} \mathbb{E} \int_0^t \int_{E_0} |(P_{n_k} \mathcal{X}_1(\mathbf{v}^n_k(s-), h^n_k(s-), z) - \mathcal{X}_1(\mathbf{v}(s-), h(s-), z), \phi)|^2 d\nu(z) ds = 0. 
\]  

(129)

for each \(t \in [0, T]\). Note, in addition, that the expectations above are bounded by a constant that is independent of \(k\) by Lemma 4.1. Thus, along with the dominated convergence theorem, we infer that

\[
\lim_{k \to \infty} \left\| \int_0^t \int_{E_0} (P_{n_k} \mathcal{X}_1(\mathbf{v}^n_k(s-), h^n_k(s-), z) - \mathcal{X}_1(\mathbf{v}(s-), h(s-), z), \phi) d\mathbf{\tilde{u}}_1(s, z) \right\|_{L^2(\Omega \times [0, T])} = 0.
\]  

(130)

Similarly, we obtain:

\[
\lim_{k \to \infty} \left\| \int_0^t \int_{E_0} (P_{n_k} \mathcal{X}_2(\mathbf{v}^n_k(s-), h^n_k(s-), z) - \mathcal{X}_2(\mathbf{v}^n_k(s-), h^n_k(s-), z), \psi) d\mathbf{\tilde{u}}_2(s, z) \right\|_{L^2(\Omega \times [0, T])} = 0.
\]  

(131)
Gathering all the relations from \((110)\) through \((131)\), we see for any \(\phi \in D(A_1)\), \(\psi \in D(A_2)\) and a measurable set \(R \subset \bar{\Omega} \times [0, T]\) we have

\[
\mathbb{E} \int_0^T \chi_R(\bar{v}(t), \phi)dt - \mathbb{E} \int_0^T \chi_R \int_0^t (\nu \Delta \bar{v}(s), \phi)dsdt
+ \mathbb{E} \int_0^T \chi_R \int_0^t (|\theta_R(\|\bar{U}\|)|\bar{v} \cdot \nabla \bar{v} + g \nabla \bar{h} + f \mathbf{k} \times \bar{v}), \phi)dsdt
= \mathbb{E} \int_0^T \chi_R(\bar{v}(0), \phi)ds + \mathbb{E} \int_0^T \chi_R \int_0^t (\sigma_1(\bar{v}(s), \bar{h}(s)), \phi)d\bar{W}_1(s)dt
+ \mathbb{E} \int_0^T \chi_R \int_0^t \int_{E_0} (\mathcal{X}_1(\bar{v}(s-), \bar{h}(s-), z), \phi)d\bar{\pi}_1(s, z)dt \quad (132)
\]

and

\[
\mathbb{E} \int_0^T \chi_R(\bar{h}(t), \psi)ds - \mathbb{E} \int_0^T \chi_R \int_0^t (\delta \Delta \bar{h}, \psi)dsdt
+ \mathbb{E} \int_0^T \chi_R \int_0^t (\theta_R(\|\bar{U}\|)\nabla \cdot (\bar{v} \nabla \bar{h})), \psi)dsdt
= \mathbb{E} \int_0^T \chi_R(\bar{h}(0), \psi)ds + \mathbb{E} \int_0^T \chi_R \int_0^t (\sigma_2(\bar{v}(s), \bar{h}(s)), \psi)d\bar{W}_2(s)dt
+ \mathbb{E} \int_0^T \chi_R \int_0^t \int_{E_0} (\mathcal{X}_2(\bar{v}(s-), \bar{h}(s-), z), \psi)d\bar{\pi}_2(s, z)dt \quad (133)
\]

Since \(D(A_1) \times D(A_2)\) is dense in \(H_1 \times H_2\) we find that for every \(\phi \in H_1\) and \(\psi \in H_2\) we have

\[
(\bar{v}(t), \phi) - \int_0^t (\nu \Delta \bar{v}(s), \phi)ds + \int_0^t (|\theta_R(\|\bar{U}\|)|\bar{v} \cdot \nabla \bar{v} + g \nabla \bar{h} + f \mathbf{k} \times \bar{v}), \phi)ds
= (\bar{v}(0), \phi)ds + \int_0^t (\sigma_1(\bar{v}(s), \bar{h}(s)), \phi)d\bar{W}_1(s)
+ \int_0^t \int_{E_0} (\mathcal{X}_1(\bar{v}(s-), \bar{h}(s-), z), \phi)d\bar{\pi}_1(s, z) \quad (134)
\]

for \(d\bar{\mathbb{P}} \otimes dt\)-a.e. \((\bar{\omega}, t) \in \bar{\Omega} \times [0, T]\) and

\[
(\bar{h}(t), \psi) - \int_0^t (\delta \Delta \bar{h}, \psi)ds + \int_0^t (\theta_R(\|\bar{U}\|)\nabla \cdot (\bar{v} \nabla \bar{h})), \psi)ds
= (\bar{h}(0), \psi)ds + \int_0^t (\sigma_2(\bar{v}(s), \bar{h}(s)), \psi)d\bar{W}_2(s)
+ \int_0^t \int_{E_0} (\mathcal{X}_2(\bar{v}(s-), \bar{h}(s-), z), \psi)d\bar{\pi}_2(s, z) \quad (135)
\]

for \(d\bar{\mathbb{P}} \otimes dt\)-a.e. \((\bar{\omega}, t) \in \bar{\Omega} \times [0, T]\). Since \((\bar{v}, \bar{h})\) is càdlàg in the space \(H_1 \times H_2\), \(\bar{\mathbb{P}}\)-a.s., it is not hard to show (using Fubini’s theorem) that equations \((134)\) and \((135)\) hold simultaneously \(\bar{\mathbb{P}}\)-a.s. for all \(t \in [0, T]\). This shows that \((\bar{S}, \bar{v}, \bar{h})\) is a global martingale solution to the modified system \((66)\). □
6.1. Improved regularity of the solution in time. Above we established existence of a global martingale solution \((\tilde{S}, \tilde{v}, \tilde{h})\) to the modified system (66) that has càdlàg paths in the space \(H\), \(\bar{P}\)-a.s. In this subsection we show that \((\tilde{v}, \tilde{h})\) is also càdlàg in the space \(V\). This improved regularity in time will be used in Section 7 to show that \(\sup_{t \geq 0} ||\tilde{v}(t)||\) is an \(\mathcal{F}\)-measurable random variable. We will also use the improved regularity in time in Section 9 to obtain a maximal local pathwise solution to (66).

Proposition 6. Let \((\tilde{S}, \tilde{v}, \tilde{h})\) be a global martingale solution to the modified system (66). Then the sample paths of \((\tilde{v}, \tilde{h})\) are càdlàg in the space \(V\), \(\bar{P}\)-a.s.

Proof. We present an argument in the Lévy noise setting based on an analogous result in Section 7.3 of [14] in the Wiener noise setting. Consider the linear, additive noise SDEs:

\[
dZ_v - \nu \Delta Z_v dt = F dt + \sigma_1(\tilde{v}, \tilde{h})d\tilde{W}_1 + \int_{E_0} \mathcal{A}_1(\tilde{v}, \tilde{h})d\tilde{\pi}_1, \tag{136a}
\]

\[
dZ_h - \delta \Delta Z_h = \sigma_2(\tilde{v}, \tilde{h})d\tilde{W}_2 + \int_{E_0} \mathcal{A}_2(\tilde{v}, \tilde{h})d\tilde{\pi}_2, \tag{136b}
\]

with \((Z_v(0), Z_h(0)) = (\tilde{v}(0), \tilde{h}(0))\). One can establish existence of a solution \((Z_v, Z_h)\) to (136) by implementing a Galerkin method in a similar way to Section 4. The Galerkin approximations \(\{(Z^n_v, Z^n_h)\}_{n=1}^\infty\) to (136) have càdlàg sample paths in the space \(V\), \(\bar{P}\)-a.s. Furthermore, in a similar manner to the proof of Proposition 4.1, one can show that the sequence \(\{(Z^n_v, Z^n_h)\}_{n=1}^\infty\) is bounded in the space \(L^2(\bar{\Omega}, L^\infty(0, T; V)) \cap L^2(\bar{\Omega}; L^2(0, T; D(A)))\).

In contrast to the Galerkin scheme for (66) considered in Subsection 4.1, there are no nonlinear terms in (136). As a result, the approximations \(\{(Z^n_v, Z^n_h)\}_{n=1}^\infty\) converge in a much stronger sense than before. Indeed, using the Itô formula in a similar way to the proof of Proposition 4.1 it is straightforward to show that the sequence \(\{(Z^n_v, Z^n_h)\}_{n=1}^\infty\) is Cauchy in the space \(L^2(\bar{\Omega}, L^\infty(0, T; V))\). If \((Z_v, Z_h)\) denotes the limit, then one can easily show that \((Z_v, Z_h)\) solves equation (136). Moreover, \((Z_v, Z_h)\) has càdlàg sample paths in the space \(V\), \(\bar{P}\)-a.s., because \((Z^n_v, Z^n_h) \rightarrow (Z_v, Z_h)\) in \(V\) uniformly on \([0, T]\), \(\bar{P}\)-a.s., and because a uniform limit of càdlàg functions is càdlàg.

Now consider the process \((\tilde{v}, \tilde{h}) := (\tilde{v}, \tilde{h}) - (Z_v, Z_h)\), which solves the random PDEs

\[
d\tilde{v} + (-\nu \Delta \tilde{v} + \theta_R(||\tilde{U} + Z||)(\tilde{v} + Z_v) \cdot \nabla)(\tilde{v} + Z_v) + g\nabla(\tilde{h} + Z_h) + f\kappa \times (\tilde{v} + Z_v))dt = 0 \tag{137a}
\]

\[
d\tilde{h} + (-\delta \Delta \tilde{h} + \theta_R(||\tilde{U} + Z||)\nabla \cdot ((\tilde{h} + Z_h)(\tilde{v} + Z_v)))dt = 0, \tag{137b}
\]

with \((\tilde{v}(0), \tilde{h}(0)) = 0\), where we denote \(||\tilde{U} + Z|| := ||\tilde{v} + Z_v|| + ||\tilde{h} + Z_h||\) (which is equal to \(||\tilde{U}||\)). We have \((\tilde{v}, \tilde{h}) \in L^2(\bar{\Omega}, L^\infty(0, T; V)) \cap L^2(\bar{\Omega}; L^2(0, T; D(A)))\) because \((\tilde{v}, \tilde{h})\) and \((Z_v, Z_h)\) both belong to this space. As a result, we easily see that

\[
-\nu \Delta \tilde{v}, \quad -\theta_R(||\tilde{U} + Z||)(\tilde{v} + Z_v) \cdot \nabla)(\tilde{v} + Z_v), \quad g\nabla(\tilde{h} + Z_h), \quad f\kappa \times (\tilde{v} + Z_v) \quad \text{belong to } L^2(\bar{\Omega}; L^2(0, T; H_1)) \quad \text{and}
\]

\[
-\delta \Delta \tilde{h}, \quad -\theta_R(||\tilde{U} + Z||)\nabla \cdot ((\tilde{h} + Z_h)(\tilde{v} + Z_v)) \quad \text{in } L^2(\bar{\Omega}; L^2(0, T; H_2)).
\]
Therefore, we have
\[ \frac{d}{dt} \tilde{v} \in L^2(\tilde{\Omega}; L^2(0, T; H_1)) \quad \text{and} \quad \tilde{v} \in L^2(\tilde{\Omega}; L^2(0, T; D(A_1))), \]
so interpolation (cf. Lemma 1.2 in Chapter 3 of [37]) yields \( \tilde{v} \in C([0, T]; V_1), \) \( \tilde{\mathbb{P}} \)-a.s. Similarly, we deduce that \( h \in C([0, T]; V_2), \) \( \tilde{\mathbb{P}} \)-a.s. Finally, the fact that \((Z_{\nu}, Z_{h}) \in \mathcal{D}(0, T; V), \) \( \tilde{\mathbb{P}} \)-a.s., implies that \((\tilde{v}, \tilde{h}) \in \mathcal{D}(0, T; V), \) \( \tilde{\mathbb{P}} \)-a.s. \( \square \)

7. Global pathwise uniqueness. This section is devoted to establishing pathwise uniqueness for the modified system (66). Recall from Subsection 2.3 that pathwise uniqueness is said to hold for the system (66) if for every pair \((S, v, h_1)\) and \((S, v, h_2)\) of global martingale solutions of the system (66) relative to the same stochastic basis \( \mathcal{S} := (\Omega, \mathcal{F}, (\mathcal{F}_t))_{t \geq 0}, \mathbb{P}, W_1, W_2, \pi_1, \pi_2 \) we have

\[ v_1(t) = v_2(t) \quad \forall t \geq 0 \quad \text{and} \quad h_1(t) = h_2(t) \quad \forall t \geq 0, \]

\( \mathbb{P} \)-a.s. on the event \( \{v_1(0) = v_2(0), h_1(0) = h_2(0)\} \)

**Proposition 7.** There exists a constant \( C > 0 \) such that for any global martingale solutions \((S, v, h_1)\) and \((S, v, h_2)\) of the system (66) relative to the same stochastic basis \( \mathcal{S} := (\Omega, \mathcal{F}, (\mathcal{F}_t))_{t \geq 0}, \mathbb{P}, W_1, W_2, \pi_1, \pi_2 \) and every event \( A \in \mathcal{F}_0 \) we have

\[ \mathbb{E}[1_A \left( \sup_{s \in [0,T]} \|v_1(s) - v_2(s)\|^2 + \sup_{s \in [0,T]} \|h_1(s) - h_2(s)\|^2 \right)] \leq C \mathbb{E}[1_A \|v_1(0) - v_2(0)\|^2 + \|h_1(0) - h_2(0)\|^2]. \] (138)

**Proof.** Let \((S, v, h_1)\) and \((S, v, h_2)\) be two global martingale solutions of the system (66) relative to the same stochastic basis \( \mathcal{S} := (\Omega, \mathcal{F}, (\mathcal{F}_t))_{t \geq 0}, \mathbb{P}, W_1, W_2, \pi_1, \pi_2 \). Let \( v := v_1 - v_2, \tilde{v} := 1_A v \) and \( h := h_1 - h_2, \tilde{h} := 1_A h \). Substituting both \( v_1, v_2 \) and \( h_1, h_2 \) into the system (66) and taking the difference between these equations, we arrive at the following equations:

\[ dv + [-v \Delta v + f_k \times v + g \nabla h + \theta_R(\|U_1\|)(v_1 \cdot \nabla)v_1 - \theta_R(\|U_2\|)(v_2 \cdot \nabla)v_2]ds \]
\[ = [\sigma_1(v_1, h_1) - \sigma_1(v_2, h_2)]dW_1 \]
\[ + \int_{E_0} [\mathcal{X}_1(v_1(s-), h_1(s-), z) - \mathcal{X}_1(v_2(s-), h_2(s-), z)]d\tilde{h}_1(s, z), \] (139)

\[ dh + [-\delta \Delta h + \theta_R(\|U_1\|)\nabla \cdot (v_1h_1) - \theta_R(\|U_2\|)\nabla \cdot (v_2h_2)]ds \]
\[ = [\sigma_2(v_1, h_1) - \sigma_2(v_2, h_2)]dW_2 \]
\[ + \int_{E_0} [\mathcal{X}_2(v_1(s-), h_1(s-), z) - \mathcal{X}_2(v_2(s-), h_2(s-), z)]d\tilde{h}_2(s, z) \] (140)

For the sake of simplicity, we write \( \theta_{R,1} \) to mean \( \theta_R(\|U_1\|) \) and \( \theta_{R,2} \) to mean \( \theta_R(\|U_2\|) \).

Applying the Itô formula with the map \( u \rightarrow \|u\|^2 \) in (139) and (140) and adding the corresponding relations together yields

\[ d\|v\|^2 + d\|h\|^2 + 2\nu |\Delta v|^2 ds + 2\delta |\Delta h|^2 ds = 2(f_k \times v, \Delta v)ds + 2(g \nabla h, \Delta v)ds \]
\[ + 2(\theta_{R,1}(v_1 \cdot \nabla)v_1 - \theta_{R,2}(v_2 \cdot \nabla)v_2, \Delta v)ds \]
\[ + 2(\theta_{R,1} \nabla \cdot (h_1v_1) - \theta_{R,2} \nabla \cdot (h_2v_2), \Delta h)ds \]
\[ + \|\nabla \sigma_1(v_1, h_1) - \nabla \sigma_1(v_2, h_2)\|_{L^2(\Omega, H_1)\otimes H_1}^2 ds \]
\[ + \|\nabla \sigma_2(v_1, h_1) - \nabla \sigma_2(v_2, h_2)\|_{L^2(\Omega, H_2)\otimes H_2}^2 ds \]
$$-2[(\sigma_1(v_1, h_1) - \sigma_1(v_2, h_2))dW_1, \Delta v) - 2[(\sigma_2(v_1, h_1) - \sigma_2(v_2, h_2))dW_2, \Delta h)$$
$$-2 \int_{E_0} ([\mathcal{X}_1(v_1(s^-), h_1(s^-), z) - \mathcal{X}_1(v_2(s^-), h_2(s^-), z)], \Delta v(s^-))d\bar{\pi}_1(s, z)$$
$$-2 \int_{E_0} ([\mathcal{X}_2(v_1(s^-), h_1(s^-), z) - \mathcal{X}_2(v_2(s^-), h_2(s^-), z)], \Delta h(s^-))d\bar{\pi}_2(s, z)$$
$$+ \int_{E_0} \|v(s^-) - \mathcal{X}_1(v_1(s^-), h_1(s^-), z) - \mathcal{X}_1(v_2(s^-), h_2(s^-), z)||^2 - \|v(s^-)||^2$$
$$-2(\mathcal{X}_1(v_1(s^-), h_1(s^-), z) - \mathcal{X}_1(v_2(s^-), h_2(s^-), z), \Delta v(s^-))d\pi_1(s, z)$$
$$+ \int_{E_0} \|h(s^-) - \mathcal{X}_2(v_1(s^-), h_1(s^-), z) - \mathcal{X}_2(v_2(s^-), h_2(s^-), z)||^2 - \|h(s^-)||^2$$
$$-2(\mathcal{X}_2(v_1(s^-), h_1(s^-), z) - \mathcal{X}_2(v_2(s^-), h_2(s^-), z), \Delta h(s^-))d\pi_2(s, z). \quad (141)$$

For an $\mathcal{F}_t$-stopping time $\tau^n$ (to be defined below) we integrate (141) in time over $[0, t \wedge \tau^n], 0 \leq t \leq T$ and multiply the resulting expression by $1_A$. Finally taking the expected value of the supremum in $s \in [0, t \wedge \tau^n]$ yields

$$\mathbb{E}1_A \sup_{s \in [0, t \wedge \tau^n]} (\|\bar{v}\|^2 + \|\bar{h}\|^2) + 2\nu \mathbb{E}1_A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds + 2\delta \mathbb{E}1_A \int_0^{t \wedge \tau^n} |\Delta \bar{h}|^2 ds \leq \mathbb{E}[1_A(||\bar{v}(0)||^2 + ||\bar{h}(0)||^2)] + \sum_{i=1}^{10} K_i(t). \quad (142)$$

$K_1, K_2$ are bounded by simply using the Cauchy Schwarz and the Poincaré inequalities,

$$K_1 := \mathbb{E}1_A \int_0^{t \wedge \tau^n} |(f k \times \bar{v}, \Delta \bar{v})|ds \leq C \mathbb{E}1_A \int_0^{t \wedge \tau^n} \|\bar{v}\|^2 ds + \nu \mathbb{E}1_A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds. \quad (143)$$

$$K_2 := \mathbb{E}1_A \int_0^{t \wedge \tau^n} |(\nabla \bar{h}, \Delta \bar{v})|ds \leq \frac{\nu}{4} \mathbb{E}1_A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds + C \mathbb{E}1_A \int_0^{t \wedge \tau^n} ||\bar{h}\|^2 ds. \quad (144)$$

We estimate $K_3$ by splitting it as follows:

$$K_3 := \mathbb{E}1_A \int_0^{t \wedge \tau^n} |(\theta_{R,1}(v_1 \cdot \nabla)v_1 - \theta_{R,2}(v_2 \cdot \nabla)v_2, \Delta \bar{v})|ds$$
$$\leq \mathbb{E}1_A \int_0^{t \wedge \tau^n} |(\theta_{R,1}(v_1 \cdot \nabla)v_1 - \theta_{R,2}(v_1 \cdot \nabla)v_1, \Delta \bar{v})|ds$$
$$+ \mathbb{E}1_A \int_0^{t \wedge \tau^n} |((v_1 \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2, \Delta \bar{v})|ds$$
$$:= K_3^1 + K_3^2. \quad (145)$$

The estimate for $K_3^1$ is derived as follows, by using the fact that the cutoff function defined in (67) is Lipschitz:

$$K_3^1 := \mathbb{E}1_A \int_0^{t \wedge \tau^n} (\theta_{R,1}(v_1 \cdot \nabla)v_1 - \theta_{R,2}(v_1 \cdot \nabla)v_1, \Delta \bar{v})|ds$$
We derive the bound for the next nonlinear term as follows:

\[ \leq C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |\theta_{R,1} - \theta_{R,2}| |((v_1 \cdot \nabla)v_1, \Delta \bar{v})| ds \]

\[ \leq C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} (\|\bar{v}\| + \|\bar{h}\|)|v_1|_{L^4}|\nabla v_1|_{L^4} |\Delta \bar{v}| ds \]

\[ \leq C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} (\|\bar{v}\| + \|\bar{h}\|)(\|v_1\| |\Delta v_1| |\Delta \bar{v}|) ds, \quad (146) \]

where we have used the embedding \( H^1 \hookrightarrow L^4 \) in 2-D to obtain the right-hand side. We now use the Young inequality to deduce

\[ K_3^1 \leq \frac{\nu}{4} \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds + C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} (\|\bar{v}\|^2 + \|\bar{h}\|^2)(\|v_1\|^2 |\Delta v_1|^2) ds. \quad (147) \]

We next give the bound for the term \( K_3^2 \) as follows:

\[
K_3^2 := \mathbb{E}_1 A \int_0^{t \wedge \tau^n} ((v_1 \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2, \Delta \bar{v})| ds \\
\leq C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |(\bar{v} \cdot \nabla)v_1 + (v_2 \cdot \nabla)\bar{v}, \Delta \bar{v})| ds \\
\leq C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} (\|\bar{v}\|_{L^\infty} \|v_1\| + |v_2|_{L^4} |\nabla \bar{v}|_{L^4}) |\Delta \bar{v}| ds \\
\leq C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 \|v_1\|^2 \|\bar{v}\|^2 ds \\
\leq \frac{\nu}{4} \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds \\
\quad + C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} \|\bar{v}\|^2 (\|v_1\|^2 |\Delta v_1|^2 + \|v_2\|^2 |\Delta v_2|^2) ds. \quad (148) \]

The last line holds true thanks to the Young inequality and the Poincaré inequality. By collecting (146) and (148), we obtain:

\[ K_3 \leq \frac{\nu}{2} \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds \\
\quad + C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} (\|\bar{v}\|^2 + \|\bar{h}\|^2)(\|v_1\|^2 |\Delta v_1|^2 + \|v_2\|^2 |\Delta v_2|^2) ds. \quad (149) \]

We derive the bound for the next nonlinear term as follows:

\[
K_4 := \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |(\theta_{R,1} \nabla \cdot (h_1 v_1) - \theta_{R,2} \nabla \cdot (h_2 v_2), \Delta \bar{h})| ds \\
\leq C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |(\theta_{R,1} \nabla \cdot (h_1 v_1) - \theta_{R,2} \nabla \cdot (h_1 v_1), \Delta \bar{h})| ds \\
\quad + C \mathbb{E}_1 A \int_0^{t \wedge \tau^n} |(\nabla \cdot (h_1 v_1) - \nabla \cdot (h_2 v_2), \Delta \bar{h})| ds \\
:= K_4^1 + K_4^2. \quad (150) \]
The bound for $K_4^1$ is given by using the fact that the cutoff function defined in (67) is Lipschitz as follows:

$$K_4^1 = E_1 A \int_0^{t \wedge \tau^n} |(\theta_{R,1} \nabla \cdot (h_1 v_1) - \theta_{R,2} \nabla \cdot (h_1 v_1), \Delta \tilde{h})| ds$$

$$\leq C E_1 A \int_0^{t \wedge \tau^n} |\theta_{R,1} - \theta_{R,2}|(|\nabla \cdot (v_1 h_1), \Delta \tilde{h})| ds$$

$$\leq C E_1 A \int_0^{t \wedge \tau^n} (||\tilde{v}|| + ||\tilde{h}||)((|\nabla h_1 v_1, \Delta \tilde{h}|) + (|\nabla \cdot v_1 h_1, \Delta \tilde{h}|)) ds$$

$$\leq C E_1 A \int_0^{t \wedge \tau^n} (||\tilde{v}|| + ||\tilde{h}||)(|\nabla v_1| L_\infty + ||v_1|| L_\infty) |\Delta \tilde{h}| ds$$

$$\leq C E_1 A \int_0^{t \wedge \tau^n} (||\tilde{v}|| + ||\tilde{h}||)(|\nabla v_1| |\Delta v_1| + ||v_1|| |\Delta h_1|) |\Delta \tilde{h}| ds$$

$$\leq \frac{\delta}{2} E_1 A \int_0^{t \wedge \tau^n} |\Delta \tilde{h}|^2 ds$$

$$+ C E_1 A \int_0^{t \wedge \tau^n} (||\tilde{v}||^2 + ||\tilde{h}||^2)(|\nabla v_1|^2 |\Delta v_1|^2 + ||v_1||^2 |\Delta h_1|^2) ds, \quad (151)$$

where we use the embedding $H^2 \hookrightarrow L_\infty$ in 2-D.

Next, the bound for the term $K_4^2$ is deduced as follows:

$$K_4^2 := E_1 A \int_0^{t \wedge \tau^n} |(\nabla \cdot (h_1 v_1) - \nabla \cdot (h_2 v_2), \Delta \tilde{h})| ds$$

$$\leq C E_1 A \int_0^{t \wedge \tau^n} |(|\nabla h_1 (v_1 - v_2) + \nabla (h_1 - h_2) v_2, \Delta \tilde{h})| ds$$

$$+ C E_1 A \int_0^{t \wedge \tau^n} |(|\nabla \cdot v_1 (h_1 - h_2) + \nabla \cdot (v_1 - v_2) h_2, \Delta \tilde{h})| ds$$

$$= C E_1 A \int_0^{t \wedge \tau^n} |(|\nabla h_1 \tilde{v} + \nabla h v_2, \Delta \tilde{h})| + |(|\nabla \cdot v_1 \tilde{h} + h_2 \nabla \tilde{v}, \Delta \tilde{h})| ds$$

$$\leq C E_1 A \int_0^{t \wedge \tau^n} (||\nabla h_1||_{L^4} ||v_1||_{L^4} + ||v_2||_{L^\infty} ||\tilde{h}|| + ||\nabla v_1||_{L^4} ||\tilde{h}||_{L^4} + ||h_2||_{L^\infty} ||\tilde{v}||) |\Delta \tilde{h}| ds$$

$$\leq C E_1 A \int_0^{t \wedge \tau^n} (||\Delta h_1|| ||\tilde{v}|| + ||\Delta v_2|| ||\tilde{h}|| + ||\Delta v_1|| ||\tilde{h}|| + ||h_2|| ||\tilde{v}||) |\Delta \tilde{h}| ds$$

$$\leq \frac{\delta}{2} E_1 A \int_0^{t \wedge \tau^n} |\Delta \tilde{h}|^2 ds$$

$$+ C E_1 A \int_0^{t \wedge \tau^n} (||\tilde{v}||^2 + ||\tilde{h}||^2)(|\Delta h_1|^2 + |\Delta v_2|^2 + |\Delta v_1|^2 + ||\tilde{v}||^2) ds. \quad (152)$$

From (151) and (152), we see

$$K_4 \leq \delta E_1 A \int_0^{t \wedge \tau^n} |\Delta \tilde{h}|^2 ds$$

$$+ C E_1 A \int_0^{t \wedge \tau^n} (||\tilde{v}||^2 + ||\tilde{h}||^2)(|\Delta h_1|^2 + |\Delta v_2|^2 + |\Delta v_1|^2 + ||\tilde{v}||^2) ds.$$
By making use of the Lipschitz assumption (23), we easily obtain the following bounds:

\[ K_5 + K_6 := \mathbb{E}_A \int_0^{t \wedge \tau^n} \| \nabla \sigma_1(v_1, h_1) - \nabla \sigma_1(v_2, h_2) \|_{L^2(\Omega, H_1)}^2 ds \]

\[ + \| \nabla \sigma_2(v_1, h_1) - \nabla \sigma_2(v_2, h_2) \|_{L^2(\Omega, H_2)}^2 ds \leq C \mathbb{E}_A \int_0^{t \wedge \tau^n} (\| \bar{v} \|^2 + \| \bar{h} \|^2) ds. \quad (153) \]

By using integration by parts and the BDG inequality we obtain

\[ K_7 := \mathbb{E}_A \left( \sup_{r \in [0, t \wedge \tau^n]} \left| \int_0^r \left( \nabla \left[ \sigma_1(v_1, h_1) - \sigma_1(v_2, h_2) \right] \right) dW_1 \right| \right) \]

\[ \leq \frac{1}{8} \mathbb{E}_A \sup_{t \in [0, t \wedge \tau^n]} (\| \bar{v} \|^2 + \| \bar{h} \|^2) + C \mathbb{E}_A \int_0^{t \wedge \tau^n} (\| \bar{v} \|^2 + \| \bar{h} \|^2) ds. \quad (154) \]

The estimate of \( K_8 \), which is analogous to \( K_7 \) but with \( \sigma_2 \) in place of \( \sigma_1 \), is similar. By invoking the BDG inequality along with the assumption (23), we deduce the following bound:

\[ K_9 = \mathbb{E}_A \sup_{r \in [0, t \wedge \tau^n]} \left| \int_0^r \int_{E_0} \left[ \nabla \left( \mathcal{K}_1(v_1(s-), h_1(s-), z) - \mathcal{K}_1(v_2(s-), h_2(s-), z) \right), \nabla v(s-) \right] \right| d\pi_1(s, z) \]

\[ \leq C \mathbb{E}_A \left( \int_0^{t \wedge \tau^n} \int_{E_0} \left( \nabla \left( \mathcal{K}_1(v_1(s-), h_1(s-), z) - \mathcal{K}_1(v_2(s-), h_2(s-), z) \right), \nabla v(s-) \right)^2 d\pi_1(s, z) \right)^{\frac{1}{2}} \]

\[ = C \mathbb{E}_A \left( \int_0^{t \wedge \tau^n} \int_{E_0} \left( \nabla \left( \mathcal{K}_1(v_1(s-), h_1(s-), z) - \mathcal{K}_1(v_2(s-), h_2(s-), z) \right), \nabla v(s-) \right)^2 d\pi_1(s, z) \right)^{\frac{1}{2}} \]

\[ \leq C \mathbb{E}_A \sup_{s \in [0, t \wedge \tau^n]} \| \nabla v(s-) \|^2 \left[ \int_0^{t \wedge \tau^n} \int_{E_0} \| \mathcal{K}_1(v_1, h_1, z) - \mathcal{K}_1(v_2, h_2, z) \|^2 d\pi_1(s, z) \right]^{\frac{1}{2}} \]

\[ \leq \frac{1}{4} \mathbb{E}_A \sup_{s \in [0, t \wedge \tau^n]} \| \nabla v(s-) \|^2 + C \mathbb{E}_A \int_0^{t \wedge \tau^n} \| \nabla v(s-) \|^2 + \| \bar{h}(s-) \|^2 ds \]

\[ \leq \frac{1}{4} \mathbb{E}_A \sup_{s \in [0, t \wedge \tau^n]} \| \nabla v(s-) \|^2 + C \mathbb{E}_A \int_0^{t \wedge \tau^n} \| \nabla v(s) \|^2 + \| \bar{h}(s) \|^2 ds. \quad (155) \]

Similarly,

\[ K_{10} = \mathbb{E}_A \sup_{r \in [0, t \wedge \tau^n]} \left| \int_0^r \int_{E_0} \left[ \nabla \left( \mathcal{K}_2(v_1(s-), h_1(s-), z) - \mathcal{K}_2(v_2(s-), h_2(s-), z) \right), \nabla h(s-) \right] \right| d\pi_2(s, z) \]
Classically, we have the elementary equality
\[ |x + h|^2 - |x|^2 - 2(x, h) = |h|^2 \]  
for all \( x, h \in H_1 \). By applying the above equality and integration by parts, we readily obtain bounds for \( K_{11} \) and \( K_{12} \) as follows:

\[ K_{11} := E_1A \int_0^{t \wedge \tau^n} \left( \|v(s) - \mathcal{X}_1^1(s, z) - \mathcal{X}_2^2(s, z)\|^2 
- \|v(s)\|^2 - 2(\nabla \mathcal{X}_1^1(s, z) - \nabla \mathcal{X}_2^2(s, z), \nabla v(s)) \right) d\tau_1(s, z) \]

\[ = E_1A \int_0^{t \wedge \tau^n} \left( \|\mathcal{X}_1^1(s, z) - \mathcal{X}_2^2(s, z)\|^2 \right) d\tau_1(s, z) \]

\[ \leq C E_1A \int_0^{t \wedge \tau^n} (\|\bar{v}(s)\|^2 + \|\bar{h}(s)\|^2) ds. \]  

The last line is true thanks to the assumption (23). The bound for \( K_{12} \), which is analogous to \( K_{11} \), but with \( \mathcal{X}_1^1 \) and \( \tau_1 \) replaced by \( \mathcal{X}_2^2 \) and \( \tau_2 \), is the same.

Collecting all the estimates from (144) through (158) and then multiplying by 2, we obtain:

\[ E_1A \sup_{s \in [0, t \wedge \tau^n]} (\|\bar{v}\|^2 + \|\bar{h}\|^2) + 2\nu E_1A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds + 2\delta E_1A \int_0^{t \wedge \tau^n} |\Delta \bar{h}|^2 ds \]

\[ \leq 2E[1_A(\|\bar{v}(0)\|^2 + \|\bar{h}(0)\|^2)] + C E_1A \int_0^{t \wedge \tau^n} (\|\bar{v}\|^2 + \|\bar{h}\|^2)(1 + |\Delta h_1|^2 + |\Delta h_2|^2 + |\Delta v_1|^2) \]

\[ + |\Delta v_2|^2 + |h|^2 |\Delta v|^2 + \|v_1\|^2 |\Delta h_1|^2 + \|v_1\|^2 |\Delta v_1|^2 + \|v_2\|^2 |\Delta v_2|^2) ds. \]

We are now ready to specify the stopping time \( \tau_n \). Define

\[ S := 1 + |\Delta h_1|^2 + |\Delta h_2|^2 + |\Delta v_1|^2 + |\Delta v_2|^2 + \|h_1\|^2 |\Delta v_1|^2 \]

\[ + |\Delta v_1|^2 |\Delta h_1| + \|v_1\|^2 |\Delta v_1|^2 + \|v_2\|^2 |\Delta v_2|^2 \]

and let

\[ \tau_n := \inf\{t > 0 : \int_0^t S(s) ds > n\}. \]

By applying the stochastic Gronwall inequality (see, e.g., [21]) to

\[ X := 1_A \sup_{s \in [0, t \wedge \tau_n]} |\bar{v}|^2 + \|\bar{h}\|^2 \]

we obtain

\[ E_1A \sup_{s \in [0, t \wedge \tau_n]} (\|\bar{v}\|^2 + \|\bar{h}\|^2) + 2\nu E_1A \int_0^{t \wedge \tau^n} |\Delta \bar{v}|^2 ds \]

\[ + 2\delta E_1A \int_0^{t \wedge \tau^n} |\Delta \bar{h}|^2 ds \leq C E_1A(\|\bar{v}(0)\|^2 + \|\bar{h}(0)\|^2). \]
By using Lemma 4.1 with $p = 4$, we obtain that $\lim_{n \to \infty} \tau^n = \infty$. Thus, we have shown that

$$E_1 A \left( \sup_{t \in [0,T]} (\|\tilde{v}\|^2 + \|\tilde{h}\|^2) \right) \leq C E [1_A (\|\tilde{v}(0)\|^2 + \|\tilde{h}(0)\|^2)],$$

(159)

which is (138).

**Remark 1.** By taking $A := \Omega_0 := \{v_1(0) = v_2(0), h_1(0) = h_2(0)\}$ in inequation (138) we find that

$$P(1_{\Omega_0} (v_1(t) - v_2(t)) = 0; \forall t \geq 0) = 1,$$

(160)

and

$$P(1_{\Omega_0} (h_1(t) - h_2(t)) = 0; \forall t \geq 0) = 1.$$  

(161)

In other words, $v_1$ and $v_2$ are indistinguishable on $\Omega_0$ and so are $h_1$ and $h_2$. This proves global pathwise uniqueness for (66). In addition, once we establish existence of global pathwise solutions to (66) below in Proposition 8, we will obtain Lipschitz continuous dependence on initial data for (66) from the space $L^2(\Omega; V)$ into $L^2(\Omega; L^\infty(0,T; V))$ by taking $A := \Omega$ in inequation (138).

**Remark 2.** In a similar way to Proposition 7, one can show that local pathwise uniqueness holds for equation (68), which has no truncation. To be precise, if $(v_1, h_1)$ and $(v_2, h_2)$ are local martingale solutions to equation (68) with respect to the same stochastic basis $S$ and up to the same stopping time $\tau$, then we have

$$v_1(t) = v_2(t) \ \forall t \in [0, \tau], \ \text{and} \ h_1(t) = h_2(t) \ \forall t \in [0, \tau)$$

P-a.s. on the event $\{v_2(0) = v_2(0), h_1(0) = h_2(0)\}$.

The only change to the argument in Proposition 7 is replacing the deterministic interval $[0, T]$ by the random time interval $[0, \tau]$.

8. **Existence of global pathwise solution.** In this section we return to the original stochastic basis $S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_1, W_2, \pi_1, \pi_2)$ and establish the existence of a global pathwise solution to the modified system (66). In finite dimensions, the Yamada-Watanabe theorem says that existence of martingale solutions together with pathwise uniqueness imply the existence of a pathwise solution. We follow this principle in infinite dimensions in Proposition 8 using an argument based on a result of Gyöngyi and Krylov in [22]. One could also invoke a general version of the Yamada-Watanabe theorem established in Theorem 3.14 of [25], which applies to SPDEs in infinite dimensions driven by Lévy noise. In pursuit of the Gyöngyi-Krylov theorem (which is stated in Appendix A as Theorem 9), we come back to SPDEs in infinite dimensions driven by Lévy noise. In the sequel, we establish existence of global pathwise solutions relative to the original stochastic basis. We have considered the $\mathcal{X}$-valued random variables $\{(v_0^n, h_0^n, v^n, h^n, W_1, W_2, \pi_1, \pi_2)\}_{n=1}^{\infty}$ in Section 6. We now consider the double subsequence

$$\{(v_0^m, v_0^n, h_0^m, h_0^n, v^m, v^n, h^m, h^n, W_1, W_2, \pi_1, \pi_2)\}_{n,m=1}^{\infty}$$

(162)

of random variables, which take their values in the extended phase space

$$\mathcal{Y} := V_1 \times V_2 \times V_2 \times \mathcal{X}_v \times \mathcal{X}_v \times \mathcal{X}_h \times \mathcal{X}_h \times C([0,T]; U)) \times C([-\infty, 0; U)) \times \mathcal{N}_{(0,\infty) \times E} \times \mathcal{N}_{(0,\infty) \times E},$$

(163)

with $\mathcal{N}_{(0,\infty) \times E}$ still defined as in (102). We denote by $\{\zeta^{n,m}\}_{n,m=1}^{\infty}$ the sequence of joint laws of the $\mathcal{Y}$-valued random variables in (162).
Lemma 8.1. The sequence \( \{\zeta^{n,m}\}_{n,m=1}^{\infty} \) is tight over \( \mathcal{Y} \) and hence weakly compact over \( \mathcal{Y} \).

Proof. The proof is identical to the proof of Lemma 4.5 in [26] so we omit it. \( \square \)

Proposition 8. There exists a global pathwise solution of the system (66).

Proof. By Lemma 8.1 and the Prokhorov theorem, we infer that the sequence \( \{\zeta^{n,m}\}_{n,m=1}^{\infty} \) is weakly compact over the phase space \( \mathcal{Y} \). Therefore, we can deduce the existence of a subsequence \( \{\zeta^{n_k,m_k}\}_{k=1}^{\infty} \) that converges weakly to a probability measure \( \zeta \) on \( \mathcal{Y} \). After applying the Skorokhod convergence theorem (Theorem A.2), we infer the existence of a new underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a sequence of \( \mathcal{Y} \)-valued random variables \((\bar{V}^{m_k}_0, \bar{V}^{n_k}_0, \bar{h}^{m_k}_0, \bar{v}^{m_k}_0, \bar{v}^{n_k}_0, \bar{h}^{m_k}_0, \bar{h}^{n_k}_0, \bar{W}_1, \bar{W}_2, \bar{\pi}_1, \bar{\pi}_2)\) and \((\tilde{v}_0, \tilde{h}_0, \tilde{b}, \tilde{v}_h, \tilde{w}_1, \tilde{w}_2, \tilde{\pi}_1, \tilde{\pi}_2)\) such that

1. The law of \((\bar{V}^{m_k}_0, \bar{h}^{m_k}_0, \bar{v}^{m_k}_0, \bar{h}^{n_k}_0, \bar{W}_1, \bar{W}_2, \bar{\pi}_1, \bar{\pi}_2)\) converges to \((\bar{V}_0, \bar{h}_0, \bar{v}_h, \bar{W}_1, \bar{W}_2, \bar{\pi}_1, \bar{\pi}_2)\) almost surely in the topology of \( \mathcal{X} = V_1 \times V_2 \times \mathcal{X}_h \times (C([0,T];\mathbb{U}))^2 \times (\mathcal{N}^{\#}) \times E \).

2. The law of \((\bar{V}^{n_k}_0, \bar{h}^{n_k}_0, \bar{v}^{n_k}_0, \bar{h}^{m_k}_0, \bar{W}_1, \bar{W}_2, \bar{\pi}_1, \bar{\pi}_2)\) converges to \((\bar{v}_0, \bar{h}_0, \bar{v}_h, \bar{W}_1, \bar{W}_2, \bar{\pi}_1, \bar{\pi}_2)\) almost surely in the topology of \( \mathcal{X} \).

By the same argument in Section 6, we see that both \((\bar{v}_0, \bar{h}_0, \bar{v}_h, \bar{W}_1, \bar{W}_2, \bar{\pi}_1, \bar{\pi}_2)\) and \((\bar{v}_0, \bar{h}_0, \bar{v}_h, \bar{W}_1, \bar{W}_2, \bar{\pi}_1, \bar{\pi}_2)\) are global martingale solutions of the truncated system (66). It can be easily shown that both of them are identical at time \( t = 0 \) a.s. Hence, by utilizing the result of Section 7, we obtain \((\bar{v}_0, \bar{h}_0, \bar{v}_h) = (v_0, h_0, v, h)\) in \( V_1 \times V_2 \times \mathcal{X}_h \times \mathcal{X}_h = \mathcal{X}_{v,h} \). In other words,

\[
\zeta\left(\{(x,y) \in \mathcal{X} \times \mathcal{X}_h : x = y\}\right) = \mathbb{P}\left(\bar{v}_0, \bar{h}_0, \bar{v}_h = (v_0, h_0, v, h) \text{ in } \mathcal{X}_{v,h}\right) = 1. \tag{164}
\]

This implies that, by Proposition 9, the original sequence \((v^n(0), h^n(0), v^n, h^n)\) defined on the original probability space \((\Omega, \mathcal{F}, \mathbb{P})\) converges in probability to an element \((v(0), h(0), v, h)\) of \( \mathcal{X}_{v,h} \). Along a subsequence, we further infer that convergence holds almost surely in the topology \( \mathcal{X}_{v,h} \). More precisely, \( \mathbb{P}\)-a.s., the following results hold:

\[
v_0^n \to v(0) \quad \text{in } V_1 \quad \text{and} \quad h_0^n \to h(0) \quad \text{in } V_2 \tag{165}
\]

\[
v^n \to v \quad \text{in probability as } L^2(0,T;V_1) \cap D(0,T;H_1)\text{-valued random variables,} \tag{166}
\]

\[
h^n \to h \quad \text{in probability as } L^2(0,T;V_2) \cap D(0,T;H_2)\text{-valued random variables.} \tag{167}
\]

By the identical argument in Section 6, we obtain that \((v, h)\) is a global pathwise solution of the equation (66) with respect to the original stochastic basis \( \mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t))_{t \geq 0}, \mathbb{P}; W_1, W_2, \pi_1, \pi_2)\). \( \square \)
9. **Existence of solutions for the original system.** The argument in Section 8 show that for every \((v_0, h_0) \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}; V)\) there exists a unique global pathwise solution \((v, h)\) to the modified system (66) with respect to the original stochastic basis \(S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_1, W_2, \pi_1, \pi_2).\) Our aim in this section is to remove the cutoff function \(\theta_R\) from (66) and obtain a maximal local pathwise solution to equation (68), which is the original system (3) with \(\mathcal{Z}_1 = \mathcal{Z}_2 = 0.\) In this section we must pay attention to the cutoff parameter \(R\) introduced in (67), which was fixed throughout the previous sections. For emphasis, we will denote by \((v_R, h_R)\) the unique global pathwise solution to the modified system (66) for the cutoff parameter \(R.\) It is clear from the definition of \(\theta_R\) in (67) that \((v_R, h_R)\) is a local pathwise solution to (68) up to the stopping time \(\tau_R\) defined by

\[
\tau_R := \inf_{t \geq 0} \{|v_R(t)| + |h_R(t)| > R\}.
\]  

(168)

Note that since \((v_R, h_R)\) is càdlàg in the space \(V_1 \times V_2\) almost surely (by Proposition 6) and adapted to the right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\), \(\tau_R\) is indeed an \(\mathcal{F}_t\)-stopping time (see, e.g., Proposition 4.6 in Chapter I of [33]). We will obtain a maximal local solution to equation (68) by showing that the local solutions \((v_R, h_R, \tau_R)\) extend each other as \(R\) increases. The existence time of the maximal pathwise solution will then be the stopping time

\[
\tau := \sup_{R \geq 1} \tau_R.
\]  

(169)

We begin with a lemma to show that \(\mathbb{P}[\tau > 0] = 1.\)

**Lemma 9.1.** For each \(R > 0\) we have \(\tau_R > 0\) a.s. on the event \(\{|v_0| + |h_0| \leq \frac{R}{2}\}\). Thus, \(\mathbb{P}[\tau > 0] = 1.\)

**Proof.** We need to show that

\[
\mathbb{P}\left[\tau_R = 0, |v_0| + |h_0| \leq \frac{R}{2}\right] = 0.
\]  

(170)

Since

\[
\mathbb{P}[\tau = 0] = \mathbb{P}\left[\bigcup_{R \geq 1} \{\tau = 0, |v_0| + |h_0| \leq \frac{R}{2}\}\right] \leq \sum_{R \geq 1} \mathbb{P}[\tau_R = 0, |v_0| + |h_0| \leq \frac{R}{2}],
\]

equation (170) will imply that \(\mathbb{P}[\tau = 0] = 0.\) In order to establish (170) we use Chebyshev's inequality and other simple estimates to obtain:

\[
\mathbb{P}(\tau_R = 0, |v_0| + |h_0| \leq \frac{R}{2}) = \mathbb{P}(\cap_{R \geq 0}\{\tau_R < \epsilon\}, |v_0| + |h_0| \leq \frac{R}{2})
\]

\[
= \lim_{\epsilon \to 0} \mathbb{P}(\{\tau_R < \epsilon\}, |v_0| + |h_0| \leq \frac{R}{2})
\]

\[
\leq \lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0, \epsilon]} (|v_R(t)| + |h_R(t)|) > R, |v_0| + |h_0| \leq \frac{R}{2}\right)
\]

\[
\leq \lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0, \epsilon]} |v_R(t) - v_0| \geq \frac{R}{4}\right) + \lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0, \epsilon]} |h_R(t) - h_0| \geq \frac{R}{4}\right)
\]

\[
\leq \lim_{\epsilon \to 0} 16 \frac{1}{R^2} \mathbb{E}\left(\sup_{t \in [0, \epsilon]} |v_R(t) - v_0|^2 + \sup_{t \in [0, \epsilon]} |h_R(t) - h_0|^2\right).
\]
Since \((v_R, h_R) \in L^2(\Omega; L^\infty(0, T; V))\) and since \((v_R, h_R)\) is right-continuous at time 0 in the \(V\)-norm, \(\mathbb{P}\)-a.s., by Proposition 6, we deduce that the last line above is zero using the dominated convergence theorem. 

Now we establish the aforementioned extension property of the local solutions \((v_R, h_R, \tau_R)\) to (68). We do this in three steps in Lemma 9.2. First, we show that the stopping times \(\{\tau_R\}_{R=1}^\infty\) increase monotonically for indices above the random index \(R_0 := 2(\|v_0\| + \|h_0\|)\). Second, we show that for indices \(R_1 < R_2\) above \(R_0\), the trajectories of \((v_R, h_{R1})\) and \((v_{R2}, h_{R2})\) agree on the interval \([0, \tau_{R1}]\). Third, we show that on the event that \(\tau < T\), i.e., when the maximal solution is not global, the sequence \(\{\tau_R\}_{R=1}^\infty\) of stopping times does not stabilize (i.e., become constant). The third property implies that, when \(\tau < T\), every local solution \((v_R, h_R, \tau_R)\) of (68) can be extended to a local solution of (68) on a strictly larger time interval \([0, \tau_{R+1}]\). This will be used to show that the norm of the maximal solution becomes unbounded as \(t \uparrow \tau\). We now establish the steps in the extension argument.

Lemma 9.2. In the setup above, the following statements hold:

i) for every positive integer \(R_1\), the sequence \(\{\tau_R\}_{R=R_1}^\infty\) increases monotonically on the event \(\{\|v_0\| + \|h_0\| \leq R_1/2\}\);

ii) for every positive integer \(R_1\) we have \(v_{R_1}(t) = v_{R_2}(t)\) and \(h_{R_1}(t) = h_{R_2}(t)\) for all \(t \in [0, \tau_{R_1}]\), a.s. on the event \(\{\|v_0\| + \|h_0\| \leq R_1/2\}\);

iii) \(\mathbb{P}\left[\bigcup_{R=1}^\infty \{\tau_R = \tau\} \cap \{\|v_0\| + \|h_0\| \leq R/2\} \cap \{\tau < T\}\right] = 0\).

Proof. i) For positive integers \(R_1 < R_2\) is is clear that \((v_{R_1}, h_{R_1}, \tau_{R_1})\) is a local solution to (68) with cutoff parameter \(R_2\). Therefore, Proposition 7 implies that \(\mathbb{P}\)-a.s. on the event \(\{\|v_0\| + \|h_0\| \leq R_1/2\}\) we have \(v_{R_1}(t) = v_{R_2}(t)\) and \(h_{R_1}(t) = h_{R_2}(t)\) for all \(t \in [0, \tau_{R_1}]\) (see Remark 2). Almost surely on the event \(\{\|v_0\| + \|h_0\| \leq R_1/2\}\) we have

\[
R_2 > R_1 \geq \|v_{R_1}(t)\| + \|h_{R_1}(t)\| = \|v_{R_2}(t)\| + \|h_{R_2}(t)\|
\]

for all \(t \in [0, \tau_{R_1}]\). This implies that \(t \leq \tau_{R_2}\) for all \(t \in [0, \tau_{R_1}]\), so \(\tau_{R_1} \leq \tau_{R_2}\) a.s. on the event \(\{\|v_0\| + \|h_0\| \leq R_1/2\}\).

ii) We saw directly above that \(R_1 < R_2\) implies that \(\mathbb{P}\)-a.s., \((v_{R_1}, h_{R_1})\) and \((v_{R_2}, h_{R_2})\) coincide for all \(t \in [0, \tau_{R_1}]\) a.s. on the event \(\{\|v_0\| + \|h_0\| \leq R_1/2\}\). Here we just need to show that equality holds at time \(\tau_{R_1}\). Since both \((v_{R_1}, h_{R_1})\) and \((v_{R_2}, h_{R_2})\) solve (68) locally with respect to the same noise \(W_1, W_2, \pi_1\) and \(\pi_2\), they have jumps at the same times. If \(\tau_{R_1}\) is not the time of a jump, then \((v_{R_1}, h_{R_1})\) and \((v_{R_2}, h_{R_2})\) are both continuous at time \(\tau_{R_1}\), so they agree at time \(\tau_{R_1}\) because

\[
v_{R_1}(\tau_{R_1}^{-}) = v_{R_2}(\tau_{R_1}^{-}) \quad \text{and} \quad h_{R_1}(\tau_{R_1}^{-}) = h_{R_2}(\tau_{R_1}^{-}).
\]

(171)

On the other hand, if the noise experiences a jump at time \(\tau_{R_1}\), then the value of the jump for both \((v_{R_1}, h_{R_1})\) and \((v_{R_2}, h_{R_2})\) will be the same. Indeed, by (171) the common value of the jump will be

\[
(\mathcal{X}_1(v_{R_1}(\tau_{R_1}^{-}), h_{R_1}(\tau_{R_1}^{-}), \Delta L_1(\tau_{R_1}^{-})), \mathcal{X}_2(v_{R_1}(\tau_{R_1}^{-}), h_{R_1}(\tau_{R_1}^{-}), \Delta L_2(\tau_{R_1}^{-}))),
\]

where \(L_1\) and \(L_2\) are the Lévy processes driving the noise (see Subsection 2.2). This shows that \((v_{R_1}, h_{R_1})\) and \((v_{R_2}, h_{R_2})\) coincide on the closed interval \([0, \tau_{R_1}]\), a.s. on the event \(\{\|v_0\| + \|h_0\| \leq R_1/2\}\).
iii) It suffices to show that \( \mathbb{P}[\tau_R = \tau, \|v_0\| + \|h_0\| \leq R/2, \tau < T] = 0 \) for each fixed positive integer \( R \). For \( R' \geq R \) we have \( \tau_R \leq \tau_{R'} \) a.s. on the event \( \{\|v_0\| + \|h_0\| \leq R/2\} \) by part i). This means that 
\[
\mathbb{P}[\tau_R = \tau, \|v_0\| + \|h_0\| \leq R/2, \tau < T] = \mathbb{P} \left[ \bigcap_{R' = R}^{\infty} \{\tau_R = \tau_{R'}, \|v_0\| + \|h_0\| \leq R/2, \tau < T\} \right].
\]
(172)
Consider the random positive integer \( R_0 := 1 + (\lceil \|v_R(\tau_R)\| + \|h_R(\tau_R)\| \rceil) \vee R \), where \( \lceil x \rceil \) denotes the minimal positive integer that is greater than or equal to a nonnegative number \( x \). Almost surely on the event \( \{\|v_0\| + \|h_0\| \leq R/2\} \) we have 
\[
\sup_{t \in [0, \tau_R]} \|v_R'(t)\| + \|h_R'(t)\| = \sup_{t \in [0, \tau_R]} \|v_R(t)\| + \|h_R(t)\| < R_0 \tag{173}
\]
for all \( R' > R \). Since \((v_R', h_R')\) is right-continuous in the space \( V_1 \times V_2 \) a.s. it follows that 
\[
R' \leq \sup_{t \in [0, \tau_{R'}]} \|v_R'(t)\| + \|h_R'(t)\|. \tag{174}
\]
On the event \( \bigcap_{R=1}^{\infty} \{\tau_R = \tau_{R'}, \|v_0\| + \|h_0\| \leq R/2, \tau_R < T\} \), both (173) and (174) hold for all \( R' > R \) and \( \tau_R = \tau_{R'} \). This implies that \( R' < R_0 \) for arbitrarily large \( R' \), which cannot happen. Therefore, the right-hand side of (172) is zero. 

Now we construct the maximal local solution to equation (68). Let \( v_0: \Omega \to V_1 \) and \( h_0: \Omega \to V_2 \) be \( \mathcal{F}_0 \)-measurable and define \( \{(v_R, h_R)\}_{R=1}^{\infty} \) to be the sequence of global pathwise solutions to the modified system (66) with cutoff parameters \( R \) and initial conditions \((v_R(0), h_R(0)) := (v_0, h_0)1_{\{\|v_0\| + \|h_0\| \leq R/2\}} \). For each \( R \) define \( \tau_R \) by (168) and define \( \tau \) by (169). Part iii) of Lemma 9.2 implies that \( \tau_R < \tau \) a.s. for every \( R \) on the event \( \{\tau < T\} \). So, \( \tau \) is a positive accessible stopping time on \( [0, T] \) that is localized by the sequence \( (\tau_R)_{R=1}^{\infty} \). Now, for \( \omega \in \Omega \) and \( t \in [0, \tau(\omega)) \) define \( v(\omega, t) := v_R(\omega, t) \) for any positive integer \( R \) such that \( \|v_0(\omega)\| + \|h_0(\omega)\| \leq R/2 \) and \( t < \tau_R(\omega) \). Lemma 9.1 shows that, a.s., \( v \) is a well-defined \( V_1 \)-valued function on \( [0, \tau] \) that is càdlàg in the \( V_1 \)-norm. We define a \( V_2 \)-valued process \( h \) on \( [0, \tau] \) in the same manner and \( h \) is càdlàg in the \( V_2 \)-norm. It is clear that the pair \((v, h)\) is a local solution to the original system (68) up to time \( \tau \). We claim that \((v, h, \tau)\) is maximal in the sense that its norm in the space \( V_1 \times V_2 \) becomes unbounded as \( t \uparrow \tau \). Part iii) of Lemma 9.1 shows that a.s. on the event \( \{\tau < T\} \) we have either \( \tau_R < \tau \) or \( \|v_0\| + \|h_0\| \leq R/2 \) for every positive integer \( R \). In particular, we have \( \tau_R < \tau \) all but finitely often a.s. on \( \{\tau < T\} \). So, \( \mathbb{P} \)-a.s. on the event \( \{\tau < T\} \) we have that \( \|v_R(\tau_R)\| + \|h_R(\tau_R)\| \geq R \) for every positive integer \( R \). When \( \tau < T \) we conclude that 
\[
\sup_{t \in (0, \tau]} \|v_R(t)\| + \|h_R(t)\| = \infty \quad \mathbb{P}\text{-a.s. on } \{\tau < T\}. \tag{175}
\]
This shows that \((v, h, \tau)\) is indeed a maximal local solution to (68) satisfying the blow-up condition (32). The local pathwise solution \((v, h, \tau)\) to (68) is unique by Remark 2 and it extends every local solution to (68).

By combining the reasoning in Section 6 with Lemma 168, we find that \((\tilde{S}, \tilde{v}, \tilde{h}, \tau)\) is a local martingale solution of the system (68). The local existence of a maximal pathwise solution has been established directly above. As mentioned in Section 4,
there is a standard procedure — known both as “piecing out” and “interlacing” — for obtaining the unique maximal local pathwise solutions to the original system (3), in which \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) appear, from maximal local pathwise solutions to equation (68) (which has \( \mathcal{L}_1 = \mathcal{L}_2 = 0 \)). See, e.g., Subsection 4.2 of [7] for details of this procedure. Therefore, both Theorems 2.7 and 2.8 are proved.

Appendix A. Compactness and tightness. We recall in this section the notion of weak convergence of a sequence of probability measures on a complete, separable metric space \((E, d)\). In Subsection A.1 we gather results about weak convergence that are key to our analysis. In Subsection A.2 we introduce the notion of tightness for a family of probability measures and recall its connection to weak convergence. We then specialize to our primary case of interest: where \((E, d)\) is the space of càdlàg functions on \([0, T]\) taking values in some complete, separable metric space. We recall the Skorokhod topology on the space of càdlàg functions and state the Aldous condition for tightness of a sequence of càdlàg processes in Lemma A.5.

A.1. Weak convergence. Let \((E, d)\) be a complete, separable metric space and let \(\mathcal{B}(E)\) denote its Borel \(\sigma\)-algebra. Let \(\mathcal{C}_b(E)\) be the set of all real-valued, continuous, bounded functions on \(E\), and let \(\mathcal{P}(E)\) be the set of all probability measures on \((E, \mathcal{B}(E))\).

**Definition A.1.** A sequence \(\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(E)\) is said to converge weakly to a probability measure \(\mu\) if

\[
\int f \, d\mu_n \to \int f \, d\mu \quad \forall f \in \mathcal{C}_b(E).
\]  

(176)

In order to pass to the limit in Section 6 we apply the Skorokhod convergence theorem to the weakly convergent sequence \(\{\nu_0, h_0^n, v^n, h^n, W_1, W_2, \pi_1, \pi_2\}_{k=1}^\infty\) and obtain almost sure convergence on a new probability space. We invoke a modified version of the Skorokhod convergence theorem, which is stated next, that permits the noise in the resulting sequence \(\{\nu_0^n, h_0^n, v^n, h^n, W_1^n, W_2^n, \pi_1^n, \pi_2^n\}_{k=1}^\infty\) on the new probability space to remain constant in \(k\). This is essential for passing to the limit in the stochastic integral terms involving \(\hat{\pi}_1\) and \(\hat{\pi}_2\) (almost sure convergence of \(\{\tilde{\pi}_1^n\}_{k=1}^\infty\) in the weak-\# topology on \(\mathcal{N}_{\mathbb{R}_+}^{\# \times E}\) is too limited for this purpose because of the small class of test functions for the weak-\# topology). For a proof of the following modified version of the Skorokhod convergence theorem see [6].

**Theorem A.2** (Skorokhod). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(E_1\) and \(E_2\) be two separable metric spaces. Let \(\chi_n : \Omega \to E_1 \times E_2, n \in \mathbb{N}\), be a family of random variables, whose laws are weakly convergent on \(E_1 \times E_2\). Let \(p_1 : E_1 \times E_2 \to E_1\) be the natural projection onto \(E_1\), i.e., \(p_1(e_1, e_2) = e_1\) for every \((e_1, e_2) \in E_1 \times E_2\). Assume that \(p_1(\chi_n)\) has the same law for every \(n \in \mathbb{N}\).

Then there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), a sequence \(\{\tilde{\chi}_n : n \in \mathbb{N}\}\) of \(E_1 \times E_2\)-valued random variables and on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and a random variable \(\chi_*\) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) such that

i) \(\tilde{\chi}_n\) has the same law as \(\chi_n\) for every \(n \in \mathbb{N}\),

ii) \(\tilde{\chi}_n \to \chi_*\) in \(E_1 \times E_2, \tilde{\mathbb{P}}\)-a.s. and

iii) \(p_1(\tilde{\chi}_n(\tilde{\omega})) = p_1(\tilde{\chi}_n(\tilde{\omega}))\) for every \(\tilde{\omega} \in \tilde{\Omega}\).
In Section 8 we use a characterization of convergence in probability from [22], which we recall here for convenience. Suppose that \( \{Y_n\}_{n \geq 0} \) is a sequence of \( E \)-valued random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( \{\mu_{m,n}\}_{m,n \geq 0} \) be the collection of joint laws of \( \{Y_n\}_{n \geq 0} \), i.e.

\[
\mu_{m,n}(\Gamma) := \mathbb{P}((Y_m, Y_n) \in \Gamma), \quad \forall \Gamma \in \mathcal{B}(E \times E).
\]  

The result characterizes convergence of probability for the sequence \( \{Y_n\}_{n \geq 0} \) in terms of weak convergence along subsequences of \( \{\mu_{m,n}\}_{m,n \geq 0} \).

**Proposition 9** (Gyöngy-Krylov Theorem). A sequence of \( E \)-valued random variables \( \{Y_n\}_{n \geq 0} \) converges in probability if and only if for every subsequence of joint probability laws, \( \{\mu_{m,k,n}\}_{k \geq 0} \), there exists a further subsequence that converges weakly to a probability measure \( \mu \) such that

\[
\mu(\{(x, y) \in E \times E : x = y\}) = 1.
\]

**A.2. Tightness and the Skorokhod topology.** We recall the notion of tightness in this section along with the Skorokhod topology.

**Definition A.3.** A set \( \Pi \) of Borel probability measures on a metric space \( (E, d) \) is said to be tight if for every \( \epsilon > 0 \) there exists a compact set \( K \subseteq E \) such that \( \mu(K^c) < \epsilon \) for every \( \mu \in \Pi \).

Tightness is a compactness property of sets of probability measures in the topology of weak convergence on \( \text{Pr}(E) \). The exact relationship between tightness and weak convergence is described by the following theorem due to Prokhorov:

**Proposition 10** (Prokhorov’s Theorem). Let \( (E, d) \) be a complete, separable metric space. Then a set \( \Pi \subset \text{Pr}(E) \) is weakly compact if and only if it is tight.

A proof of Theorem 10 can be found in, e.g., Theorems 5.1 and 5.2 in [4].

We now turn to the specific case where \( (E, d) \) is the space of càdlàg functions in time. Let \( (\mathbb{S}, \rho) \) be a separable and complete metric space. Let \( \mathcal{D}(0, T; \mathbb{S}) \) denote the set of \( \mathbb{S} \)-valued càdlàg functions defined on \([0, T]\), i.e., the functions that are right-continuous and have a left-hand limit at every \( t \in [0, T] \). This space is endowed with the Skorokhod topology. We now briefly describe the main facts about the Skorokhod topology that will be used here. More detailed treatments can be found in, e.g., [4, 18]. A sequence \( \{u_n\}_{n=1}^{\infty} \subset \mathcal{D}(0, T; \mathbb{S}) \) converges to a function \( u \in \mathcal{D}(0, T; \mathbb{S}) \) in the Skorokhod topology if and only if there exists a sequence \( \{\lambda_n\}_{n=1}^{\infty} \) of increasing homeomorphisms of \([0, T]\) such that \( \{\lambda_n\}_{n=1}^{\infty} \) converges to the identity function uniformly on \([0, T]\) and \( u_n \circ \lambda_n \) tends to \( u \) uniformly on \([0, T]\). The Skorokhod topology is metrizable by a complete metric, for instance, the metric \( \vartheta_T \) defined by

\[
\vartheta_T(u, v) := \inf_{\lambda \in \sigma_T} \left[ \sup_{t \in [0, T]} \rho(u(t), v(\lambda(t))) + \sup_{t \in [0, T]} |t - \lambda(t)| + \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right],
\]

where \( \sigma_T \) is the set of all increasing homeomorphisms on \([0, T]\). When equipped with the metric \( \vartheta_T \), \( \mathcal{D}(0, T; \mathbb{S}) \) becomes a separable, complete metric space. Because of Prokhorov’s theorem (Proposition 10) it is useful to have a sufficient condition for tightness of a family of probability measures on \( \mathcal{D}(0, T; \mathbb{S}) \). We will use the following condition related to tightness that was introduced by Aldous in [1].
Definition A.4. A sequence \( \{X_n\}_{n=1}^{\infty} \) of \( \mathcal{D}(0, T; \mathbb{S}) \)-valued random variables is said to satisfy the Aldous condition if and only if every sequence \( \{\tau_n\}_{n=1}^{\infty} \) of \( (\mathcal{F}_t)_{t \geq 0} \)-stopping times with \( \tau_n \leq T \) we have
\[
\sup_{n \geq 1} \sup_{0 \leq t \leq \delta} \mathbb{P}\{\rho(X_{\tau_n}(t), X_{\tau_n}(\tau_n)) \geq \eta\} \leq \epsilon. \tag{180}
\]

We can easily formulate a sufficient condition for (180) using Markov’s inequality. Suppose that there exist constants \( \alpha, \beta, C > 0 \) such that for every sequence \( \{\tau_n\}_{n=1}^{\infty} \) of \( \mathcal{F}_t \)-stopping times with \( \tau_n \leq T \) we have
\[
\sup_{n \geq 1} \mathbb{E}\left(\rho(X_{\tau_n}(t), X_{\tau_n}(\tau_n))^\alpha\right) \leq Ct^\beta \tag{181}
\]
for every \( t \geq 0 \). Then \( \{X_n\}_{n=1}^{\infty} \) satisfies the Aldous condition (180). See Theorem 13.2 of [36] for a compactness result in the deterministic setting that is analogous to condition (181).

Now we state the sufficient condition for tightness of probability measures on \( \mathcal{D}(0, T; \mathbb{S}) \) that we use in Section 5.

Lemma A.5. Let \((\mathbb{S}, \rho)\) be a complete, separable metric space and let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of \( \mathcal{D}(0, T; \mathbb{S}) \)-valued random variables. If for every \( t \in [0, T] \) the laws of \( \{X_n(t)\}_{n=1}^{\infty} \) are tight on \( \mathbb{S} \) and if condition (181) holds, then the laws of \( \{X_n\}_{n=1}^{\infty} \) are tight on \( \mathcal{D}(0, T; \mathbb{S}) \).

For a proof of Lemma A.5 see, e.g., Theorem 3.2 in [28].

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SHALLOW WATER EQUATIONS WITH LÉVY NOISE

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