A uniform reconstruction formula in integral geometry

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Received 1 December 2011, in final form 23 April 2012
Published 16 May 2012
Online at stacks.iop.org/IP/28/065014

Abstract
A new method for analytic inversion of Radon-type integral transforms is proposed.

1. Introduction
We present a uniform reconstruction method for a class of geometric integral transforms for submanifolds of co-dimension 1. The reconstruction does not include summation of an infinite series and looks like a standard inversion of the Radon transform. We specify this method for classical and new acquisition geometries. The condition of regularity is necessary for an inversion operator to be bounded in a Sobolev space scale, but it is not sufficient. The existence of an exact reconstruction formula depends on vanishing of some singular integrals of rational forms on a sphere. In section 8, we discuss reconstruction for families of spheres. This subject has been in the focus of recent research; see surveys of related results in [13, 12, 15].

2. Geometry and integrals
Let $X$ and $\Sigma$ be smooth $n$-dimensional manifolds where $n > 1$, $Z$ be a smooth closed hypersurface in $X \times \Sigma$ and $p : Z \rightarrow X$, $\pi : Z \rightarrow \Sigma$ be natural projections. We suppose that there exists a real smooth function $\Phi_1$ in $X \times \Sigma$ (called the generating function) such that $Z = \{(x, \sigma) ; \Phi_1(x, \sigma) = 0\}$ and $d_x \Phi \neq 0$ on $Z$. Suppose that

(i) The map $\pi$ has rank $n$ and the mapping $P : N^*(Z) \rightarrow T^*(X)$ is a local diffeomorphism.
Here, $N^*(Z)$ denotes the co-normal bundle of $Z$ and $P(x, \sigma; v_x, v_\sigma) = (x, v_x) \in T^*(X)$. It follows that the set $Z(\sigma) = \pi^{-1}(\sigma) = \{x; \Phi(x, \sigma) = 0\}$ is for any $\sigma \in \Sigma$ a smooth hypersurface in $X$, and for any point $x \in X$ and for any tangent hyperplane $h \subset T_x(X)$, there is a locally unique hypersurface $Z(\sigma)$ through $x$ tangent to $h$.

Proposition 2.1. For an arbitrary generating function $\Phi$ property (i) is equivalent to the condition $\det(d_x \Phi, d_\sigma \Phi, \Psi) \neq 0$, where $\Psi(x, t; \sigma, \tau) = t \tau \Phi(x, \sigma)$, $t, \tau \in \mathbb{R}$, $t \tau > 0$, for any local coordinate system $x_1, \ldots, x_n$ in $X$ and any local coordinate system $\sigma_1, \ldots, \sigma_m$ in $\Sigma$. 

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For a proof, see [18], proposition 1.1.

**Definition.** We call a generating function \( \Phi \) regular if it satisfies conditions (i) and (ii) there are no conjugate points, that is, the equations \( \Phi (x, \sigma) = \Phi (y, \sigma) = 0 \) and \( d_x \Phi (x, \sigma) = d_y \Phi (y, \sigma) \) are fulfilled for no \( x \neq y \in X, \sigma \in \Sigma \).

We assume further that \( X \) is an open set in a Euclidean space \( E^n \); let \( dV \) be the volume form and \( dS \) be a hypersurface element in \( E^n \). Consider the integral

\[
M_{\Phi} f (\sigma) = \int \delta (\Phi (x, \sigma)) f dV = c = \int f q
\]

for an arbitrary continuous function \( f \) compactly supported in \( X \). The quotient \( q = dV/d\Phi \) denotes an arbitrary \( n - 1 \) form \( q \) such that \( d\Phi \wedge q = dV \). It is defined up to a term \( h d\Phi \) where \( h \) is a continuous function. An orientation of a hypersurface \( Z (\sigma) \) is defined by means of the form \( d\Phi \) and the integral of the form \( f q \) over \( Z (\sigma) \) is uniquely defined. We call the operator \( M_{\Phi} \) Funk–Radon transform generated by \( \Phi \). This transform can be written in terms of Euclidean integrals as follows:

\[
M_{\Phi} f (\sigma) = \int_{Z(\sigma)} \frac{f dS}{|\nabla_{\sigma} \Phi (x, \sigma)|}, \quad (1)
\]

where \( \nabla a \) is the gradient of a function \( a \) in \( E^n \). The function \( |\nabla_{\sigma} \Phi| \) does not vanish because of (i). Suppose that the gradient factorizes through \( X \) and \( \Sigma \), that is, \( |\nabla_{\sigma} \Phi (x, \sigma)| = m(x) \mu (\sigma) \) for some positive continuous functions \( m \) in \( X \) and \( \mu \) in \( \Sigma \). Then data of the Funk transform are equivalent to data of Euclidean hypersurface integrals

\[
R f (\sigma) = \int_{Z(\sigma)} f dS, \quad \sigma \in \Sigma
\]

since \( R f (\sigma) = \mu (\sigma) M_{\Phi} (mf) (\sigma) \). The reconstruction problem of a function \( f \) from integrals \( R f \) is then reduced to inversion of the operator \( M_{\Phi} \).

We say that a generating function \( \Phi \) is resolved if \( \Sigma = \mathbb{R} \times S^{n-1} \), and \( \Phi (x; \lambda, \omega) = \theta (x, \omega) - \lambda, \lambda \in \mathbb{R}, \omega \in S^{n-1} \) for a smooth function \( \theta \) on \( X \times S^{n-1} \), where \( S^{n-1} \) denotes the unit sphere in \( E^n \). (Here and later we replace the notation \( \sigma \) by \( (\lambda, \omega) \).) Note that the map \( p : Z \to X \) is always proper for a resolved generating function. This property guarantees that the functions \( M_{\Phi} f \) and \( R f \) have compact support in \( \Sigma \). The operator \( M_{\Phi} \) fulfills the range conditions similar to that of the Radon transform.

**Proposition 2.2.** Let \( \Phi = \theta - \lambda \) be a resolved regular generating function and \( \theta (x, \omega) \) be a polynomial function of \( \omega \) of order \( m \). Then for an arbitrary integrable function \( f \) in \( X \) with compact support and for an arbitrary polynomial \( p (\lambda) \) of order \( k \), the integral

\[
\int p (\lambda) M_{\Phi} f (\lambda, \omega) d\lambda
\]

is a polynomial of \( \omega \) of order \( \leq mk \).

**Proof.** We have

\[
\int p (\lambda) M_{\Phi} f (\lambda, \omega) d\lambda = \int p (\lambda) \int_{\theta = \lambda} \frac{f dV}{d\theta} d\lambda = \int_{X} p (\theta (x, \omega)) f (x) dV,
\]

where \( p (\theta (x, \omega)) \) is a polynomial of \( \omega \) of order \( \leq mk \). \( \Box \)
3. Main theorem

For a real smooth function \( f \) in a manifold \( X \) and a natural \( n \), we consider singular integrals
\[
I_{n \pm} (\rho) = \int_X \frac{\rho}{(f \pm i \epsilon)^n},
\]
for a smooth density \( \rho \) with compact support. If \( df \neq 0 \) on the zero set of \( f \), then the limits exist and the functionals \( I_{n \pm} \) are generalized functions in \( X \). The functional
\[
(P) \int_X \frac{\rho}{f^n} = \text{Re} I_{n+} (\rho) = \text{Re} I_{n-} (\hat{\rho})
\]
is called a principal value integral. For a resolved regular generating function \( \Phi = \theta - \lambda \), we define the function on \( X \times X \setminus \{ \text{diag} \} \)
\[
\Theta_n (x, y) = \int_{S^{n-1}} \frac{d\omega}{\theta(x, \omega) - \theta(y, \omega)},
\]
where \( d\omega \) is the Euclidean volume form on \( S^{n-1} \). The singular integral converges since by (ii) the \( d\omega (\theta(x, \omega) - \theta(y, \omega)) \neq 0 \) as \( \theta(x, \omega) - \theta(y, \omega) = 0 \).

**Theorem 3.1.** Let \( \Phi = \theta - \lambda \) be a regular resolved generating function in \( X \times \Sigma \) and \( f \in L_2(X) \) be an arbitrary function with compact support. If \( n \) is even and \( \text{Re} \Theta_n (x, y) = 0 \) for any \( x \neq y \in X \), a reconstruction from data of \( M_\Phi f \) is given by the formula:
\[
f(x) = \frac{1}{(2\pi i)^n} D_n (x) \int_{\Sigma} M_\Phi f(\lambda, \omega) \frac{d\lambda d\omega}{\theta(x, \omega) - \lambda}.
\]
(2)

If \( n \) is odd and \( \text{Im} \Theta_n (x, y) = 0 \) for \( x \neq y \), the function can be reconstructed by
\[
f(x) = \frac{1}{2(2\pi i)^{n-1} D_n (x)} \int_{\Sigma} \left[ \frac{\partial^{n-1}}{\partial \lambda^{n-1}} M_\Phi f(\lambda, \omega) \right]_{\lambda = \theta(x, \omega)} d\omega
\]
(3)

where
\[
D_n (x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \frac{d\omega}{||\nabla \theta(x, \omega)||^n}, \quad \nabla = \nabla_x.
\]

Integrals (2) and (3) converge in mean on any compact set in \( X \).

**Remark 1.** A more invariant form of (2) or (3) is the reconstruction of the form \( f \) \( dV \):
\[
f \ dV = \frac{1}{(2\pi i)^n} D_n \int \ldots d\omega.
\]
The quotient \( dV/D_n \) is invariant under scale transformations in \( E^n \).

**Remark 2.** Beylkin studied ‘the generalized Radon transform’ \([8]\), which coincides with the operator \( M_\Phi \) in the Euclidean space. He constructed a Fourier integral operator parametrix for this operator and reduced inversion of this operator to solution of a Fredholm equation.

**Lemma 3.2.** The integral transform
\[
I_n f(x) = (P) \int_{S^{n-1}} \int_\Sigma M_\Phi f(\lambda, \omega) \frac{d\lambda d\omega}{\Phi^n(\lambda; x, \omega)}
\]
for even \( n \)
\[
= \frac{1}{(2\pi i)^{n-1} D_n (x)} \int_{S^{n-1}} \left[ \frac{\partial}{\partial \lambda} \right]^{n-1} M_\Phi f(\lambda, \omega) \bigg|_{\lambda = \theta(x, \omega)} d\omega,
\]
for odd \( n \)
is a continuous operator \( L_2(X)_{\text{comp}} \to L_2(X)_{\text{loc}}. \)
Proof of lemma. We can write for even \( n \)

\[
I_n f(x) = - \int_{\Phi(x)} M_\phi f(\lambda, \omega) \frac{d\mu}{d\lambda} \frac{d\omega}{d\lambda} (P) \int_R (\mu - \lambda)^{-n} M_\phi f(\lambda, \omega) \ d\lambda
\]

since \( d_\mu \Phi(x; \mu, \omega) = -d_\mu \Phi(x; \mu, \omega) = 0 \) implies \( \mu - \lambda = \Phi(x; \lambda, \omega) \). It follows that

\[ I_n = M_\phi^* \Lambda_n M_\phi \]

where \( \Lambda_n \) is a convolution operator in \( R \) with the principal value kernel \( \lambda^{-n} \) and \( M_\phi^* \) is the back-projection operator as in \([18]\) with \( d\Sigma = d\lambda \ d\omega \). The map \( p : Z \to X \) is proper since \( \Phi \) is a resolved generating function. Therefore by \([18]\), corollary 3.3, the operator \( M_\phi \) is bounded in the spaces \( H^n(X) \to H^{(n-1)/2}(\Sigma) \), where \( K \) is an arbitrary compact set in \( X \) and \( \Lambda = \pi^{-1}(p(K)) \) is a compact in \( \Sigma \). The operator \( \Lambda_n \) is a PDO of order \( n - 1 \) and generates a bounded operator \( H^{(n-1)/2}(\Sigma) \to H^{(n-1)/2}(\Sigma) \). By \([18]\), proposition 3.1, the operator \( M_\phi^* \) is continuous in the spaces \( H^{(n-1)/2}(\Sigma) \to H^{0}_{\text{loc}}(X) \). Finally, \( I_n \) is continuous as an operator \( H^0_{\text{loc}}(X) \to H^0_{\text{comp}}(X) \).

In the case of odd \( n \), a similar factorization holds for \( I_n \) with \( \Lambda_n = (\partial/\partial \lambda)^{n-1} \) which leads to the same conclusion. \( \square \)

Proof of theorem. For even \( n \) and an arbitrary \( x \in X \) and a function \( f \) that vanishes in a neighborhood of \( x \), we calculate

\[
I_n f(x) = (P) \int_{\Sigma} M_\phi f(\lambda, \omega) \ d\lambda \ d\omega = \int_{\Lambda_n} d\omega (P) \int_{\Phi(x; \lambda)} f(y) q(\theta(x, \omega) - \lambda) \frac{d\lambda}{d\theta(x, \omega)}
\]

Here, the relation \( d\lambda = d\theta \) holds in \( Z \) and the equation \( d\theta \land q = dV \) is fulfilled in \( X \) by definition. Thus the function \( \Theta_n \) is the off-diagonal kernel of the operator \( I_n \). It vanishes by the assumption. Therefore \( \Theta_n(x, y) \) is supported in the diagonal and according to lemma 3.2, we have \( \Theta_n(x, y) = a_n(x) \delta_n(y) \) for a locally bounded function \( a_n \) in \( X \).

If \( n \) is odd, then we have

\[
I_n f(x) = \int_{\Lambda_n} d\omega \int_R f(y) \frac{dV}{(\theta(x, \omega) - \lambda)^{n-1}}
\]

that is, \( \pi^{-1}(n-1)! \int \Theta_n(x, y) f(y) dV \)

Next we calculate the function \( a_n \). Choose a smooth function \( e_0 \) of one variable with support in \([-1, 1]\) such that \( e_0(0) = 1 \) and set \( e_\varepsilon(x) = e_0(|x|^2/\varepsilon^2) \) for \( x \in R^n \) and any \( \varepsilon > 0 \).

Take a point \( x_0 \in X \) and show that

\[
\text{Re} \int_X dV \int_{\Lambda_n} f(y) \frac{dV}{(\theta(x, \omega) - \lambda(x_0))^{n-1}} \rightarrow a_n(x_0)
\]
for $n$ even and

$$
\frac{(n-1)!}{\pi} \text{Im} \int_X dV \int_{S^{n-1}} e_\epsilon (x-x_0) \, d\omega \frac{\epsilon}{(\theta (x, \omega) - \theta (x_0, \omega) - i0)^n} \to a_n (x_0)
$$

for $n$ odd as $\epsilon \to 0$. We can change order of integrals and integrate first over $X$.

**Lemma 3.3.** If $n$ is even, we have for any $x_0 \in X$, arbitrary $\omega \in S^{n-1}$ and small $\epsilon$

$$
a_n (x_0, \omega) \equiv \text{Re} \int_X e_\epsilon (x-x_0) \, dV \int_{S^{n-1}} e_\epsilon (x-x_0) \, d\omega \frac{(-1)^{n/2-1} \pi^{(n+1)/2}}{\Gamma ((n+1)/2)} \frac{1}{|\nabla \theta (x_0, \omega)|^{n+\epsilon}} + o (1),
$$

where $|o(1)| \leq C \epsilon^{1/2} \log 1/\epsilon$ where $C$ does not depend on $\omega$. For odd $n$, we have

$$
a_n (x_0, \omega) \equiv \frac{(n-1)!}{\pi} \text{Im} \int_X (\theta (x, \omega) - \theta (x_0, \omega) - i0)^n d\omega
\int_{S^{n-1}} \frac{d\omega}{|\nabla \theta (x_0, \omega)|^n} = \frac{(-1)^{(n-1)/2} (2\pi i)^n}{(n-1)!} D_n (x_0) + o (\epsilon).
$$

Taking the limit and integrating (4) over $S^{n-1}$ yields for even $n$ the equation

$$
a_n (x_0) = \lim_{\epsilon \to 0} \int_X a_n (x_0, \omega) \, d\omega = (-1)^{n/2-1} \pi^{(n+1)/2} \int \frac{d\omega}{|\nabla \theta (x_0, \omega)|^n} \int_{S^{n-1}} \frac{d\omega}{|\nabla \theta (x_0, \omega)|^n}
$$

which implies (2). For odd $n$, we obtain

$$
a_n (x_0) = \lim_{\epsilon \to 0} \int_X a_n (x_0, \omega) \, d\omega = 2 (2\pi i)^{n-1} \frac{1}{|S^{n-1}|} \int \frac{d\omega}{|\nabla \theta (x_0, \omega)|^n} = 2 (2\pi i)^{n-1} D_n (x_0),
$$

which yields (3). This completes the proof of theorem 3.1.

**Proof of lemma.** We show first that the $\theta$ can be replaced by a linear function. Choose a Euclidean coordinate system $x_1, \ldots, x_n$ in $E$ such that $\partial \theta (x_0, \omega) / \partial x_1 = |\nabla \theta|, \partial \theta (x_0, \omega) / \partial x_2 = \ldots = \partial \theta (x_0, \omega) / \partial x_n = 0$; we have then $dV = dx \equiv dx_1, \ldots, dx_n$. If $n$ is even, we integrate by parts $n$ times with respect to $x_1$:

$$
\text{Re} \int \frac{e_\epsilon (x-x_0) \, dV}{(\theta (x, \omega) - \theta (x_0, \omega) - i0)^n} = \frac{1}{(n-1)!} \text{Re} \int \frac{d_1 (x) \, dx}{(\theta (x, \omega) - \theta (x_0, \omega) - i0)^{n-1}}
$$

$$
\cdots = \frac{1}{(n-1)!} \int \log |\theta (x, \omega) - \theta (x_0, \omega)| d_n (x) \, dx
$$

$$
d_1 = \frac{\partial}{\partial x_1} \frac{e_\epsilon (x-x_0)}{|\nabla \theta (x_1)|} = \frac{\partial e_\epsilon}{\partial x_1} \frac{1}{|\nabla \theta (x_1)|} - \frac{\partial}{\partial x_1} \frac{|\nabla \theta (x_1)|}{|\nabla \theta (x_1)|} \frac{e_\epsilon}{|\nabla \theta (x_1)|^2}
$$

$$
\cdots
$$

$$
d_n = \frac{\partial}{\partial x_1} d_{n-1} (x) = \frac{\partial}{\partial x_1} \frac{1}{|\nabla \theta (x_1)|} \frac{e_\epsilon}{|\nabla \theta (x_1)|} + \cdots,
$$

where omitted terms only include derivatives of $e_\epsilon$ of order $n$. Changing the variables $x = \epsilon y$, we obtain

$$
d_n (y) = \epsilon^{-n} \frac{\partial^n e_\epsilon}{\partial y^n} \frac{1}{|\nabla \theta (\epsilon y)|^n} + o (\epsilon^{1-n}), \quad dx = \epsilon^n dy.
$$

By Lagrange’s theorem we can write

$$
\theta (\epsilon y, \omega) - \theta (0, \omega) = \epsilon \rho_\epsilon (y), \quad \rho_\epsilon (y) = \int_0^1 (y, \nabla \theta (\epsilon y)) \, dt.
$$

$$
\theta (\epsilon y, \omega) - \theta (0, \omega) = \epsilon \rho_\epsilon (y), \quad \rho_\epsilon (y) = \int_0^1 (y, \nabla \theta (\epsilon y)) \, dt.
$$
This yields
\[
\int \log |\theta (x, \omega) - \theta (x_0, \omega)| d_{\nu}(x) \, dx = \int \left( \log \varepsilon \right) d_{\nu}(x) \, dx + \int \log |\rho_\varepsilon(y)| |d_{\nu}(x)\, dx.
\]
The first integral vanishes since \(d_{\nu}\) is equal to \(x_1\)-derivatives of a function with compact support. It follows that the left-hand side of (5) equals
\[
\frac{1}{(n-1)!} \int \log |\rho_\varepsilon(y)| |d_{\nu}(x)\, dx = \int \log |\rho_\varepsilon(y)| \frac{1}{|\nabla \theta (x_0, \omega)|^n} \frac{\partial^n e}{\partial y^n} \, dy + O(\varepsilon)
\]
since the logarithmic factor is absolutely integrable. By (7) we have \(C^1\)-convergence \(\rho_\varepsilon \to \langle y, \nabla \theta (0) \rangle\) as \(\varepsilon \to 0\) in a neighborhood of the origin. This implies the inequality
\[
-\int_{|y| \leq 1, |\rho_\varepsilon(y)| \leq \varepsilon} \log |y| \, dy + \int_{|y| \leq 1, |\rho_\varepsilon(y)| \leq \varepsilon} \log |\rho_\varepsilon(y)| \, dy \leq C \varepsilon \log |\delta|,
\]
where \(\delta, 0 < \delta \leq 1\), is arbitrary and \(C\) does not depend on \(\varepsilon\) and \(\delta\). On the other hand, \(\log |\rho_\varepsilon(y)| \to \log |y|\) everywhere as \(\varepsilon \to 0\). Therefore,
\[
\int \log |\rho_\varepsilon(y)| \frac{\partial^n e}{\partial y^n} \, dy \to \int \log |y| \frac{\partial^n e}{\partial y^n} \, dy
\]
and
\[
\text{Re} \int \frac{e_\varepsilon (x - x_0) \, dV}{(\theta (x, \omega) - \theta (x_0, \omega) - i0)^n} \to \frac{1}{(n-1)!} \frac{1}{|\nabla \theta (x_0, \omega)|^n} \int \log |y| \frac{\partial^n e}{\partial y^n} \, dy.
\]
More detailed arguments show that the difference is equal to \(O(e^{1/2} \log \varepsilon)\). The same is true for the linear function \(\theta (x, \omega) = x_1\), that is,
\[
\int_X \frac{e_\varepsilon (x - x_0) \, dV}{(\theta (x, \omega) - \theta (x_0, \omega) - i0)^n} \to \frac{1}{(n-1)!} \frac{1}{|\nabla \theta (x_0, \omega)|^n} \int \text{Re} \, e(y) \, dy, \quad \varepsilon \to 0.
\]
Calculate the integral on the right-hand side by partial integration:
\[
\text{Re} \int_{\mathbb{R}^n} \frac{e \, dy}{(y_1 - i0)^n} = \frac{1}{n-1} \text{Re} \int_{\mathbb{R}^n} \frac{dy}{(y_1 - i0)^{n-1}} \frac{\partial e}{\partial y_1}
\]
\[
= \frac{1}{n-1} \text{Re} \int_{\mathbb{S}^{n-1}} (\cos \omega_1 - i0)^{2-n} \, d\omega \int_0^\infty \frac{\partial e_0}{\partial r^2} \, dr^2
\]
\[
= -\frac{1}{n-1} \text{Re} \int_{\mathbb{S}^{n-1}} (\cos \omega_1 - i0)^{2-n} \, d\omega
\]
\[
= \frac{|\mathbb{S}^{n-2}|}{n-1} \text{Re} \int_0^{\pi} \sin^{n-2} \omega_1 (\cos \omega_1 - i0)^{2-n} \, d\omega_1,
\]
where \(y = r \cos \omega_1\) since \(\frac{\partial e}{\partial y_1} = 2y_1\frac{\partial e}{\partial r} \quad y_1 - i0 = (\cos \omega_1 - i0) \, r, \quad \int_0^\infty \partial e_0/\partial r^2 \, dr^2 = -e(0) = -1.
\]
By substituting \(s = \cos^2 \omega_1\), we obtain
\[
\text{Re} \int_0^{\pi} \left( \frac{\sin \omega_1}{\cos \omega_1 + i0} \right)^{2-n}\, d\omega_1 = \frac{1}{2} B \left( \frac{n-1}{2}, \frac{3-n}{2} \right) = (-1)^{n/2} \frac{\pi}{2}.
\]
We use the formula for the Beta-function extended for all complex (non-negative integer) values of arguments. The exponent \(\lambda = 1/2 - n/2\) is a regular point, and we can use a classical formula. The right-hand side of (8) equals
\[
(-1)^{n/2-1} \frac{\pi|\mathbb{S}^{n-2}|}{2(n-1)} = (-1)^{n/2-1} \frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)},
\]
which yields (6).
In the case of odd \( n \), we integrate by parts as in the previous case and obtain

\[
\text{Im} \int \frac{e_\varepsilon (x - x_0)}{(\theta (x, \omega) - \theta (x_0, \omega) - i 0)^n} \, dV = \frac{\pi}{(n - 1)!} \int_{\theta(x_0, \omega) > \theta(x, \omega)} d_n(x) \, dx.
\]

Taking into account (6) and convergence \( \rho_\varepsilon \to \gamma_1 \), the limit of the right-hand side is

\[
\frac{\pi}{(n - 1)! |\nabla \theta (x_0, \omega)|^n} \int_{y_1 > 0} \left| \frac{\partial^n e}{\partial y_1^n} \right| dy.
\]

Integrating by parts backward gives the equation

\[
\int y_1 > 0 \frac{\partial^n e}{\partial y_1^n} dy = - \int y_1 = 0 \frac{\partial^{n-1} e}{\partial y_1^{n-1}} dy^{-} = -2^{m-1} (n - 2)!! |S^{n-2}| \int_0^\infty \frac{\partial^m e_0 (s)}{\partial s^m} s^{m-1} ds,
\]

where \( y' = (y_2, \ldots, y_m) \), \( m = (n - 1)/2 \), \( s = r^2 \). Here, we apply the formula

\[
\left. \frac{\partial^{n-1} e(y)}{\partial y_1^{n-1}} \right|_{y_1 = 0} = 2^m (n - 2)!! \frac{\partial^m e_0 (s)}{\partial s^m}.
\]

Integrating by parts \( m - 1 \) times in the interior integral, we obtain the quantity

\[
\int_0^1 \frac{\partial^m e_0 (s)}{\partial s^m} s^{m-1} ds = (-1)^{m-1} (m - 1)! \int_0^1 \frac{\partial e_0}{\partial s} ds = (-1)^m (m - 1)!!.
\]

This implies that (9) equals

\[
(-1)^m 2^{m-1} \pi (m - 1)! (n - 2)!! |S^{n-2}| \frac{1}{(n - 1)! |\nabla \theta (x_0, \omega)|^n} = 2 (2\pi i)^{m-1} \frac{1}{(n - 1)! |S^{n-2}| |\nabla \theta (x_0, \omega)|^n}.
\]

For odd \( n \) we have

\[
a_n (x_0, \omega) = \frac{(n - 1)!}{\pi} \lim_{\varepsilon \to 0} \int_{|X|} \frac{e_\varepsilon (x - x_0)}{(\theta (x, \omega) - \theta (x_0, \omega) - i 0)^n} \, dx = \frac{2 (2\pi i)^{n-1}}{|S^{n-1}| |\nabla \theta (x_0, \omega)|^n}.
\]

Integrating over \( S^{n-1} \) we obtain

\[
a_n (x_0) = \int a_n (x_0, \omega) \, d\omega = \frac{2 (2\pi i)^{n-1}}{|S^{n-1}|} \int \frac{d\omega}{|\nabla \theta (x_0, \omega)|^n}
\]

and (3) follows.

\[
\square
\]

4. Integrals of rational trigonometric functions

We focus now on the conditions of theorem 3.1 and show that the condition (ii) can be weakened in the case \( n = 2k \). A function of the form

\[
t(\varphi) = \sum_{j=0}^k a_j \cos j\varphi + b_j \sin j\varphi
\]

is called the trigonometric polynomial of degree \( k \) if \( a_j \neq 0 \) or \( b_k \neq 0 \). Any trigonometric polynomial is \( 2\pi \)-periodic, is well defined and holomorphic on the cylinder \( \mathbb{C}/2\pi \mathbb{Z} \). It always has \( 2k \) zeros in the cylinder. If the polynomial is real, then the number of real zeros is even.

**Lemma 4.1.** Let \( t(\varphi) \) and \( s(\varphi) \) be real trigonometric polynomials such that \( \deg s < \deg t \) and all the roots of \( t \) are real. Then for \( r = s/t \) and arbitrary natural \( n \),

\[
(P) \int_0^{2\pi} r^n (\varphi) \, d\varphi = \frac{1}{2} \int_0^{2\pi} (r (\varphi) + i 0)^n \, d\varphi + \frac{1}{2} \int_0^{2\pi} (r (\varphi) - i 0)^n \, d\varphi = 0.
\]

(10)
Proof. Suppose first that all roots of \( t \) are simple. Let \( \alpha_1 < \alpha_2 < \ldots < \alpha_m \) be all roots of \( \partial t / \partial \psi \) on the circle \( \mathbb{R} / 2 \pi \mathbb{Z} \). Let \( \varepsilon_k = \text{sgn} \partial t / \partial \psi \) on the interval \((\alpha_k, \alpha_k + 1) \) for \( k = 1, \ldots, m \), where \( \alpha_{m+1} = \alpha_1 \). The function \( r(\xi) \) is meromorphic for \( \xi = \psi + i \tau \in \mathbb{C} / 2 \pi \mathbb{Z} \) and has no poles in the half-cylinder \( \{ \tau > 0 \} \) because of the assumption. We have for \( k = 1, \ldots, m \),

\[
\int_{a_k}^{a_{k+1}} (r(\psi) + \varepsilon_k i 0)^n \, d\psi = \int_{a_k}^{a_{k+1}} r^n (\psi + i 0) \, d\psi.
\]

After summation we obtain

\[
\sum_k \int_{a_k}^{a_{k+1}} (r(\psi) + \varepsilon_k i 0)^n \, d\psi = \sum_k \int_{a_k}^{a_{k+1}} r^n (\psi + i 0) \, d\psi = \int_0^{2\pi} r^n (\psi + i 0) \, d\psi
\]

(11)

The mean of the left-hand sides is equal to the left-hand side of (10). Show that the right-hand sides of (11) vanish. Replace the form \( r^n (\psi + i 0) \, d\psi \) by \( r^n (\xi) \, d\xi \) for \( \xi = \psi \pm i \eta \) for an arbitrary \( \eta > 0 \) without changing the integrals on the right-hand side. We have \( |r(\xi)| \to 0 \) as \( \eta \to \infty \), hence the right-hand side of (11) vanishes. In the general case, we can approximate the polynomial \( r \) with real roots by polynomials \( \tilde{r} \) with real simple roots. Equation (10) holds for \( \tilde{r} = s/\lambda \); hence, it is true for \( r = s/\lambda \).

Corollary 4.2. Suppose that \( \dim X = 2 \) and \( \Phi = \theta - \lambda \) is a generating function in \( X \times \Sigma \) satisfying conditions (i) and (iii); \( \theta(x, \omega) - \theta(y, \omega) \) is for any \( x, y \in X, x \neq y \), a trigonometric polynomial in \( \omega \) of positive degree with only real zeros (occasionally multiple). Then equation (2) holds for an arbitrary \( f \in L_2(X) \) with compact support.

Proof. By lemma 4.1 we have \( \text{Re} \Theta_2 (x, y) = 0 \) and the arguments of theorem 3.1 can be applied to this generating function.

Proposition 4.3. Let \( v \in \mathbb{R}^2 \) and \( a \in \mathbb{R} \) be such that \( |a| < |v| \). Then for arbitrary even \( n \geq 2 \),

\[
\text{Re} \int_{S^{n-1}} \frac{d\omega}{(|\omega, v| - a - i 0)^n} = 0
\]

and for arbitrary odd \( n \geq 3 \),

\[
\text{Im} \int_{S^{n-1}} \frac{d\omega}{(|\omega, v| - a - i 0)^n} = 0.
\]

Proof. We may assume that \( |v| = 1 \). For even \( n \), we have

\[
\text{Re} \int \frac{d\omega}{(|\omega, v| - a - i 0)^n} = \text{Re} \int \frac{d\omega}{(\cos \varphi - a - i 0)^n},
\]

where \( \varphi \) is the spherical distance between \( \omega \) and \( v \). We have \( d\omega = \sin^{n-2} \varphi \, d\varphi \, d\omega' \) where \( d\omega' \) is the area of a unit sphere \( S^{n-2} \). Integrating over \( n - 2 \)-spheres \( \varphi = \text{const} \), we obtain

\[
\text{Re} \int \frac{d\omega}{(\cos \varphi - a - i 0)^n} = \frac{|S^{n-2}|}{2} \int_0^{2\pi} \sin^{n-2} \varphi \, d\varphi
\]

since the integrand is \( \pi \)-periodic. The right-hand side vanishes by lemma 4.1.
For odd \( n \), we have

\[
\text{Im} \int \frac{d\omega}{(\cos \varphi - a - i0)^n} = \frac{|S^{n-2}|}{2i(n-1)!} \left[ \int_0^\pi \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a - i0)^n} - \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a + i0)^n} \right]
\]

\[
= \frac{|S^{n-2}|}{2i(n-1)!} \int_{|\varphi - a| = \varepsilon} \frac{\sin^{n-2} \varphi d\varphi}{(\cos \varphi - a)^n} = \frac{\pi |S^{n-2}|}{(n-1)!} \text{res}_a (\sin^{n-2} \varphi d\varphi),
\]

where \( \alpha = \arccos a \in [0, \pi] \). Changing variable \( \xi = \cos \varphi \) and omitting the constant coefficient, we obtain the quantity

\[
\text{res}_a \left( 1 - \xi^2 \right)^m d\xi,
\]

where \( m = (n-3)/2 \). The residue is equal to zero since the numerator has order \( 2m = n-3 \). \qed

**Corollary 4.4.** For any regular resolved generating function \( \Phi = \theta - \lambda \), such that for any pair of points \( x \neq y \) in an open set \( \Omega \subset X \), we have \( \theta(x, \omega) - \theta(y, \omega) = \langle v, \omega \rangle + \alpha, |\alpha| < |v| \), formulas (2) and (3) hold for any function \( f \in L_2(X) \) with support in \( \Omega \).

### 5. Reconstruction in spaces of constant curvature

We apply the above results to recover a few known and unknown inversion formulas for geodesic integral transforms in spaces of curvature \( \kappa = 0, 1, -1 \).

**Euclidean space.** Take the generating function \( \Phi(x; \lambda, \omega) = \langle \omega, x \rangle - \lambda \in \mathbb{R}^n \times \Sigma, \Sigma = \mathbb{R} \times S^{n-1} \). We have \( |\nabla \theta| = D_\omega(x) = 1 \). Then (2) and (3) coincide with the classical John’s reconstruction in Euclidean space from data of hyperplane integrals.

**Elliptic space.** Funk [2] inspired by the seminal paper of Minkowski [1] found a reconstruction formula of an even function \( f \) on the unit sphere \( S^2 \) from its integrals over big circles. A generalization of Funk’s formula for odd \( n \) is due to Helgason [6] and for even \( n \) to Semyanistyi [4].

In both cases we can apply theorem 3.1 to a generating function \( \Phi(x; \lambda, \omega) = \langle \omega, y \rangle - \lambda \) defined in \( X \times \Sigma \), where \( X = \{ (x_0, x) \in E^{n+1}, x_0^2 + |x|^2 = 1, x_0 > 0 \}, y = x_0^{-1}x \in E^n \) and the metric in \( X \) is induced from the Euclidean metric in \( E^{n+1} \). Omitting some simple calculations, we arrive at the following.

**Theorem 5.1.** If \( n \) is even then any function \( f \in L_2(X) \) can be reconstructed from its integrals \( Rf(\sigma) \) over big spheres \( S(\sigma) = \{ x \in X, \langle \sigma, x \rangle = 0 \}, \sigma \in S^n_+ \) by

\[
f(x) = -\frac{(n-1)!}{(2\pi i)^n} (P) \int_{S^n_+} \frac{Rf(\sigma) d\sigma}{\langle \sigma, x \rangle^n},
\]

where \( S^n_+ = \{ \sigma \in \mathbb{R}^{n+1}, |\sigma| = 1, \sigma_0 \geq 0 \} \) is a hemisphere. If \( n \) is odd we have

\[
f(x) = \frac{1}{2} \frac{(n-1)!}{(2\pi i)^{n-1}} \int_{S^{n-1}_+} \delta^{(n-1)}(\langle \sigma, x \rangle) Rf(\sigma) d\sigma.
\]

Reconstructions (12) and (13) are different from that of Helgason [7] and of Rubin [10]. They are apparently not much known as the classical formulas (14) and (15) for the hyperbolic case, see below.

**Hyperbolic space.** Take the generating function \( \Phi(x; \lambda, \omega) = \theta - \lambda, \theta = -2(|x|^2 + 1)^{-1} \langle \omega, x \rangle, -1 < \lambda < 1 \) in the unit ball \( X \subset \mathbb{R}^n \). The hypersurfaces \( Z(\lambda, \omega) \) are fully geodesics for the hyperbolic metric \( d_s = 2(1 - |x|^2)^{-1} ds \). By similar calculations, we obtain

\[
f(x) = -\frac{(n-1)!}{(2\pi i)^n} (P) \int_{Q^n_+} \frac{Rf(\sigma) d\sigma}{\langle \sigma, x \rangle^n}
\]
for even \( n \) and
\[ f(x) = \frac{1}{2} (2\pi i)^{n-1} \int_{Q_n} \delta^{(n-1)}((\sigma, \lambda)) \, Rf(\sigma) \, d\sigma \]  
(15)
for odd \( n \), where \( Q_n = \{ \sigma = (\sigma_0, \sigma') \in \mathbb{R}^{n+1}; \sigma_0^2 - |\sigma'|^2 = -1, \sigma_0 \geq 0 \} \) is the dual one sheet hyperboloid.

A reconstruction in Funk’s form was done first by Radon [3] for \( n = 2 \) and Helgason [6, 7] for \( n > 2 \); formulas (14) and (15) are due to Gelfand–Graev–Vilenkin [5].

6. Equidistant spheres and horospheres in hyperbolic space

Equidistant spheres. Let \( X \) be again a unit \( n \)-dimensional ball, \( n \geq 2 \) and
\[ \Phi(x; \lambda, \omega) = \theta - \lambda, \quad \theta(x, \omega) = \frac{p - \langle \omega, x \rangle}{1 - |x|^2}, \quad \omega \in S^{n-1} \]
be a generating function where \( 0 \leq p < 1 \). For a fixed \( \omega \) and an arbitrary \( \lambda \neq 0 \) the hypersurface \( Z(\lambda, \omega) = \{ x; \, \Phi(x; \lambda, \omega) = 0 \} \) is the intersection of \( X \) and of an \( n - 1 \) sphere \( S(\lambda) \), whereas \( S(0) = \{ \langle \omega, x \rangle = p = 0 \} \) is a hyperplane; all the spheres \( S(\lambda) \) contain \( n - 2 \) sphere \( S(0) \cap dX \).

For arbitrary real \( \lambda \) and \( \mu \) the hypersurfaces \( S(\lambda) \cap X, S(\mu) \cap X \) are equidistant with respect to the hyperbolic metric. Check that \( \Phi \) fulfils the conditions of corollary 4.4 for arbitrary \( p, 0 \leq p \leq 1 \). A proof of regularity is a routine. Further we have
\[ \theta(x, \omega) - \theta(x, \omega) = -\left( \omega, \frac{x}{1 - |x|^2} - \frac{y}{1 - |y|^2} \right) + p(\frac{1}{1 - |x|^2} - \frac{1}{1 - |y|^2}) \]
and we need to prove that
\[ \left| \frac{x}{1 - |x|^2} - \frac{y}{1 - |y|^2} \right| > \left| \frac{1}{1 - |x|^2} - \frac{1}{1 - |y|^2} \right| \]  
(16)
for arbitrary \( x \neq y \in X \). Squaring both sides, we reduce (16) to the obvious inequality \( 2(1 - xy) > 2 - |x|^2 - |y|^2 \). The proof is complete and reconstructions (2) and (3) follow. The geodesic integral transform is
\[ Hf(\sigma) = \int_{Z(\sigma)} f d\sigma S, \quad \sigma = (\lambda, \omega), \]
where \( d\sigma S \) is the hyperbolic hypersurface element. The operator \( M_\sigma \) can be written in terms of the Euclidean integral transform \( Rf(\sigma) = \int_{Z(\sigma)} f dS \) since of the factorization \( |\nabla \theta(x, \omega)| = (1 - |x|^2)^{-1/2} \sqrt{4\lambda^2 - 4p\lambda + 1} \) (see (1)). On the other hand,
\[ d\sigma S = \left( \frac{2}{1 - |x|^2} \right)^{n-1} dS \]
which yields
\[ Mf(\lambda, \omega) = \frac{Rf_1(\lambda, \omega)}{\sqrt{\lambda^2 - p\lambda + 1/4}} = \frac{Hf_2(\lambda, \omega)}{\sqrt{\lambda^2 - p\lambda + 1/4}} \]
where
\[ f_1(x) = \frac{(1 - |x|^2)}{2} f(x), \quad f_2(x) = \left( \frac{1 - |x|^2}{2} \right)^n f(x). \]

Corollary 6.1. For any function \( f \) with compact support in the unit ball, a reconstruction is given for even \( n \) by
\[ f(x) = -\frac{1}{(4\pi i)^n} \int_{S^{n-1}} \int_{\mathbb{R}} \frac{Hf(\lambda, \omega) d\lambda}{\sqrt{\lambda^2 - p\lambda + 1/4}} \frac{d\omega}{\langle \omega, x \rangle - \lambda(1 - |x|^2)} \]  
(17)
and for odd $n$ by
\[
  f(x) = \frac{1}{2(4\pi i)^n} \frac{(1 - |x|^2)^{2n}}{D_n(x)} \int_{S^{n-1}} \frac{\partial^{n-1} H f(\lambda, \omega)}{\partial \lambda^{n-1}} \sqrt{\lambda^2 - p^2 + 1/4} \bigg|_{\lambda = \theta(x, \omega)} \, d\omega.
\] 

Another reconstruction for the case $n = 2$ and $p = 0$ was obtained in [19].

**Horospheres.** Taking $p = 1$ in the above formulas, we obtain the function
\[
  \theta(x, \omega) = \frac{1 - \langle \omega, x \rangle}{1 - |x|^2}
\]
which defines the family of horospheres $\theta(x, \omega) = \lambda$, $1/2 < \lambda < \infty$. Formulas (17) and (18) hold for horospheres if we substitute $\sqrt{4\lambda^2 - 4\lambda + 1} = 2\lambda - 1$ and integrate in (17) over the ray $(1/2, \infty)$ in the interior integral.

Reconstruction formulas (of a different form) for the horospherical transform are contained in Gelfand–Graev–Vilenkin [5].

**7. Isofocal hyperboloids**

The equation $\lambda = |x| + \varepsilon x_1$, $\varepsilon > 1$ defines a sheet of the rotation hyperboloid $H$
\[
  \left( \alpha x_1 - \varepsilon \lambda \right)^2 - x_2^2 - \cdots - x_n^2 = \frac{\lambda^2}{\alpha^2}, \quad \alpha = \sqrt{\varepsilon^2 - 1},
\]
with a focus at the origin. The function $\Phi(x; \lambda, \omega) = \theta(x, \omega) - \lambda$, $\theta(x, \varphi) = |x| + \varepsilon \langle x, \omega \rangle$ generates the family of all rotations of $H$ in $\mathbb{R}^n \setminus \{0\}$ about the origin. The function
\[
  \theta(x, \omega) - \theta(y, \omega) = \varepsilon \langle \omega, x - y \rangle + |x| - |y|, \quad \omega \in S^{n-1}
\]
satisfies the conditions of proposition 4.3 since $||x| - |y|| < \varepsilon |x - y|$ for any $x, y \in \mathbb{R}^n$, $x \neq y$.

Therefore, theorem 3.1 holds for this family. We have
\[
  |\nabla \theta(x, \omega)|^2 = 1 + \varepsilon^2 + 2\varepsilon |x|^{-1} \langle \omega, x \rangle
\]
and
\[
  D_n = \int_{S^{n-1}} \frac{d\omega}{(1 + \varepsilon^2 + 2\varepsilon |x|^{-1} \langle \omega, x \rangle)^{n/2}} = |S^{n-2}| \int_0^\pi \sin^{n-2} \varphi \, d\varphi/(1 + \varepsilon^2 - 2\varepsilon \cos \varphi)^{n/2},
\]
where $\varphi$ is the angle between $\omega$ and $|x|^{-1}x$. This integral does not depend on $x$.

**Corollary 7.1.** For any smooth function $f$ with compact support in $\mathbb{R}^n$ and even $n$, the following equation holds
\[
  f(x) = -\frac{1}{(2\pi i)^n D_n} \int_{S^{n-1}} \int_{\mathbb{R}^n} \frac{M_\Phi f(\lambda, \omega) \, d\lambda \, d\omega}{(|x| + \varepsilon \langle \omega, x \rangle - \lambda)^n}, \quad n \text{ even}
\]
\[
  f(x) = \frac{1}{2(2\pi i)^n D_n} \int_{S^{n-1}} \frac{\partial^{n-1} M_\Phi f(\lambda, \omega)}{\partial \lambda^{n-1}} \bigg|_{\lambda = |x| + \varepsilon \langle \xi(x, \omega) \rangle} \, d\omega, \quad n \text{ odd}.
\]

**8. Photoacoustic geometries**

Consider a resolved generating function
\[
  \Phi(x; \lambda, \omega) = |x - \xi(\omega)|^2 - \lambda, \quad \omega \in S^{n-1},
\]
where \( \xi : S^{n-1} \to \mathbb{R}^n \) is a smooth map. We call the image \( \text{C} \) of \( \xi \) central set. Any hypersurface \( Z(\lambda, \omega) = \{ \Phi(\cdot; \lambda, \omega) = 0 \} \) is a sphere of radius \( \sqrt{\lambda} \) with the center \( \xi(\omega) \in \text{C} \) and by (1)

\[
M_{\Phi} f(\lambda, \omega) = \frac{R f(\lambda, \omega)}{2\sqrt{\lambda}},
\]

where \( R f(\lambda, \omega) \) is the Euclidean integral over this sphere. Inversion of the operator \( M_{\Phi} \) implies inversion of the spherical integral transform \( R \) for the given central surface \( \text{C} \) (and vice versa). This subject is of special interest in view of application to the photoacoustic (thermoacoustic) tomography. Inversion formulas for a function supported in a half-space with the hyperplane central set was found by Fawcett [9]. In [17] a reconstruction was done by reduction to the Radon transform. For a spherical central surface Finch with coauthors [11, 13] found a reconstruction formula of types (20)–(21) in the physical domain for arbitrary dimension. Another reconstruction formula was proposed by Kunyanski [14]; it is similar to (20) and (21) after a simplification. An inversion for the spherical mean and for cylinder mean operators in three-dimensional space was constructed by Xu and Wang [12]. Kunyanski [15] constructed inversion for polyhedral center sets with special symmetries.

**Ellipsoids.** We show that for an arbitrary ellipsoid or elliptical cylinder \( \text{C} \) in \( \mathbb{R}^n \) as a central set a simple reconstruction follows from theorem 3.1. Independently Natterer [16] found an explicit inversion for the case \( n = 3 \). It looks different from (21).

Set \( \xi(\omega) = (a_1 \omega_1, \ldots, a_n \omega_n) \) where \( a_1, \ldots, a_n \) are positive constants. The central hypersurface \( \{ x = \xi(\omega), \omega \in S^{n-1} \} \) is the boundary of an ellipsoid \( E_\omega \) with half-axes \( a_1, \ldots, a_n \). Then

\[
\theta(x, \omega) - \theta(y, \omega) = 2\langle \xi(\omega), y - x \rangle + |x|^2 - |y|^2 = 2\langle \omega, z \rangle + |x|^2 - |y|^2,
\]

where \( z = (a_1(y_1 - x_1), \ldots, a_n(y_n - x_n)) \). The following inequality holds

\[
||x|^2 - |y|^2| \leq \sum (y_i - x_i)(y_i + x_i) \leq \sum |a_i(y_i - x_i)| \sum |a_i^{-1}(y_i + x_i)| \leq \|z\| \|w\|,
\]

where \( w = (a_1^{-1}(y_1 + x_1), \ldots, a_n^{-1}(y_n + x_n)) \). Suppose that \( x, y \in E_\omega \) and \( x \neq y \); then the point \((x + y)/2\) belongs to the interior of \( E_\omega \) which implies \( \|w\| < 2 \). It follows that the right-hand side is strictly bounded by \( 2\|z\| \). By proposition 4.3, theorem 3.1 holds for any \( n \geq 2 \). It follows that any function \( f \) supported in the closed ellipsoid \( E_\omega \) can be reconstructed by the formula

\[
f(x) = \frac{1}{(2\pi i)^n D_n(x)} \int_{S^{n-1}} (P) \int_\mathbb{R} \frac{R f(\rho^2, \omega)}{|x - \xi(\omega)|^2 - \rho^2} \, d\rho \, d\omega,
\]

for even \( n \) where we did the substitution \( \lambda = \rho^2 \), and by

\[
f(x) = \frac{1}{4(2\pi i)^{n+1} D_n(x)} \int_{S^{n-1}} \left( \frac{\partial}{\partial \rho^2} \right)^{n-1} \left( \frac{|x - \xi(\omega)|^2 - \rho^2}{\rho} \right)^{n-1} d\omega,
\]

for odd \( n \), where

\[
D_n(x) = \frac{1}{2^n|S^{n-1}|} \int \frac{d\omega}{|x - \xi(\omega)|^n}.
\]

**Elliptic cylinders.** If a central set \( \text{C} \) is unbounded we cannot apply the same method since it cannot be regularly parametrized. However, we can write a reconstruction formula for any closed cylinder \( \text{E} \) with an elliptic base. Indeed, \( \text{E} \) is a union of the family of ellipsoids \( E_\omega \) as several half-axes, say \( a_1, \ldots, a_p \), tend to infinity, with \( a_{p+1}, \ldots, a_n \) being fixed. One can come to limits in (20) and in (21). We omit details.
Algebraic plane curves. In the case $n = 2$, there are more geometries which allow exact reconstruction formulas. We call a curve $C \subset \mathbb{R}^2$ trigonometric of degree $k$ if it is given by a parametric equation

$$x_1 = \xi_1(\varphi), \quad x_2 = \xi_2(\varphi), \quad \varphi \in S^1,$$

where $\xi_1$ and $\xi_2$ are real trigonometric polynomials of degree $k$. A trigonometric curve is always a component of a real algebraic curve. A point $x \in \mathbb{R}^2$ is called hyperbolic with respect to a trigonometric curve $C$ if any straight line $L$ through $x$ meets the curve at $2k$ different points. It is easy to see that the set $H$ of all hyperbolic points is always open and convex. We call a curve $C$ hyperbolic if the set $H$ of hyperbolic points is not empty. Introduce a Euclidean structure in $\mathbb{R}^2$ and consider a function

$$\theta(x, \varphi) = |x - \xi(\varphi)|^2, \quad \xi(\varphi) = (\xi_1(\varphi), \xi_2(\varphi)), \quad 0 \leq \varphi < 2\pi, \quad x = (x_1, x_2).$$

**Proposition 8.1.** Let $H$ be the set of hyperbolic points with respect to a trigonometric curve $C$ of degree $k$. For arbitrary points $x, y \in H$, $x \neq y$, all roots of the polynomial $\theta(x, \varphi) - \theta(y, \varphi)$ (of order $k$) are real.

**Proof.** We have

$$\theta(x, \varphi) - \theta(y, \varphi) = |x - \xi(\varphi)|^2 - |y - \xi(\varphi)|^2 = |x|^2 - |y|^2 - 2 \langle x - y, \xi(\varphi) \rangle = 2 \langle x - y, \xi(\varphi) - s \rangle,$$

where $s = (x + y)/2$. This point is contained in $H$ since $H$ is convex. Therefore, the line $L = \{z = s + rv, r \in \mathbb{R}\}$ has $2k$ common points $\xi(\varphi_1), \ldots, \xi(\varphi_{2k})$ with $C$ for arbitrary vector $v \neq 0$. If $v$ is orthogonal to $x - y$, then the right-hand side of (23) vanishes. The corresponding angles $\varphi_1, \ldots, \varphi_{2k}$ are real roots of the polynomial $\theta(x, \varphi) - \theta(y, \varphi)$. \qed

The family of circles centered at the curve $C$ is generated by the function $\Phi(x; \lambda, \varphi) = \theta(x, \varphi) - \lambda$. Applying lemma 4.1 we get
Corollary 8.2. Let $C$ be a hyperbolic trigonometric curve. Reconstruction (2) holds for the family of circles centered at $C$, arbitrary function $f$ supported in the set $H$ of hyperbolic points.

There is a large variety of trigonometric curves $C$ with non-empty hyperbolic sets.

Example 1. Let $\xi_1(\phi) = 2\cos 2\phi - \cos \phi$, $\xi_2(\phi) = 2\sin 2\phi + \sin \phi$. The curve $C$ is shown in figure 1.

The hyperbolic set $H$ is the triangle in the middle, $k = 2$.

Example 2. A hyperbolic ‘square’ set is defined by the trigonometric curve $\xi_1(\phi) = 2\cos 3\phi + \cos \phi$, $\xi_2(\phi) = 2\sin 3\phi - \sin \phi$, $k = 3$; see figure 2.
Example 3. A ‘pentagon’ is the hyperbolic set of the curve \( \xi_1(\varphi) = 5 \cos 4\varphi + 4 \cos \varphi \), \( \xi_2(\varphi) = 5 \sin 4\varphi - 4 \sin \varphi \), \( k = 4 \), see figure 3, and so on.

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