On spherically symmetric solutions of the Einstein-Euler-de Sitter equations

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Abstract

We construct spherically symmetric solutions to the Einstein-Euler equations, which contains a positive cosmological constant, say, the Einstein-Euler-de Sitter equations. We assume a realistic barotropic equation of state. Equilibria of the spherically symmetric Einstein-Euler-de Sitter equations are given by the Tolman-Oppenheimer-Volkoff-de Sitter equation. We can construct solutions near time periodic linearized solutions around the equilibrium. The Cauchy problem around the equilibrium can be solved. This work can be considered as a trial of the generalization of the previous work on the problem without cosmological constants.

Key Words and Phrases. Einstein equations, Cosmological constant, Spherically symmetric solutions, Vacuum boundary, Nash-Moser theorem

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1 Introduction

We consider the Einstein-de Sitter equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (g^{\alpha\beta} R_{\alpha\beta}) - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

for the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (c^2 \rho + P) U^\mu U^\nu - P g_{\mu\nu}. $$

Here $R_{\mu\nu}$ is the Ricci tensor associated with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

$G$ is the gravitational constant, $c$ the speed of light, $\rho$ the mass density, $P$ the pressure and $U^\mu$ is the 4-dimensional velocity. $\Lambda$ is the cosmological constant which is supposed to be positive in this article. See [11] §111.

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Spherically symmetric solutions for the problem with $\Lambda = 0$ was investigated in [4] and the aim of this article is to describe the similar results on the problem with positive cosmological constants.

In this article we suppose that the pressure $P$ is a given function of the density $\rho$ and pose the following

**Assumption.** $P$ is an analytic function of $\rho > 0$ such that $0 < P, 0 < dP/d\rho < c^2$ for $\rho > 0$, and $P \to 0$ as $\rho \to +0$. Moreover there are positive constants $A, \gamma$ and an analytic function $\Omega$ on a neighborhood of $[0, +\infty]$ such that $\Omega(0) = 1$ and

$$P = A\rho^\gamma \Omega(A\rho^\gamma/c^2).$$

We assume that $1 < \gamma < 2$ and $\frac{1}{\gamma - 1}$ is an integer.

We are keeping in mind the equation of state for neutron stars. See [4], [6, p. 188].

We consider spherically symmetric metrics of the form

$$ds^2 = e^{2F(t,r)c^2dt^2} - e^{2H(t,r)}dr^2 - R(t,r)^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

We suppose that the system of coordinates is co-moving, that is,

$$U^0 = e^{-F}, \quad U^1 = U^2 = U^3 = 0$$

for $x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi$. Then the equations turn out to be

$$e^{-F}\frac{\partial R}{\partial t} = V \quad (1a)$$

$$e^{-F}\frac{\partial \rho}{\partial t} = -(\rho + P/c^2)(\frac{V'}{R'} + \frac{2V}{R}) \quad (1b)$$

$$e^{-F}\frac{\partial V}{\partial t} = -GR\left(\frac{m}{R^3} + \frac{4\pi P}{c^2}ight) + \frac{c^2\Lambda}{3}R +$$

$$-\left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2R} - \frac{\Lambda}{3}R^2\right)\frac{P'}{R'(\rho + P/c^2)} \quad (1c)$$

$$e^{-F}\frac{\partial m}{\partial t} = -\frac{4\pi}{c^2}R^2PV \quad (1d)$$

Here $X'$ stands for $\partial X/\partial r$.

The coefficients of the metric are given by

$$P' + F'(c^2\rho + P) = 0$$

and

$$e^{2H} = \left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2R} - \frac{\Lambda}{3}R^2\right)^{-1}(R')^2.$$
In order to specify the function \( F \), we introduce the state variable \( u \) by
\[
u = \int_0^\rho \frac{dP}{\rho + P/c^2},
\]
and fix the idea by putting
\[
e^F = \sqrt{\kappa} e^{-u/c^2}
\]
with a positive constant \( \kappa \) specified in the next Section. We note that there are analytic functions \( \Omega_u, \Omega_\rho, \Omega_P \) on a neighborhood of \([0, +\infty[\) such that \( \Omega_u(0) = \Omega_\rho(0) = \Omega_P(0) = 1 \) and
\[
u = \frac{\gamma A}{\gamma - 1} \rho^{\gamma - 1} \Omega_u(A \rho^{\gamma - 1}/c^2),
\]
\[
\rho = A_1 u^{\frac{1}{\gamma - 1}} \Omega_\rho(u/c^2),
\]
\[
P = A A_1^\gamma u^{\gamma - 1} \Omega_P(u/c^2).
\]
Here \( A_1 := \left(\frac{\gamma - 1}{\gamma A}\right)^{\frac{1}{\gamma - 1}} \). See [5].

We put
\[
m = 4\pi \int_0^\rho \rho R^2 R' dr,
\]
supposing that \( \rho \) is continuous at \( r = 0 \). The coordinate \( r \) can be changed to \( m \), supposing that \( \rho > 0 \), and the equations are reduced to
\[
e^{-F} \left( \frac{\partial R}{\partial t} \right)_m = (1 + \frac{P}{c^2 \rho}) V,
\]
\[
e^{-F} \left( \frac{\partial V}{\partial t} \right)_m = \frac{4\pi}{c^2} R^2 PV \frac{\partial V}{\partial m} - GR \left( \frac{m}{R^3} + \frac{4\pi P}{c^2} \right) + \frac{c^2 A}{3} \Lambda R + \frac{2Gm}{c^2 R} - \frac{A}{3} R^2 \left( 1 + \frac{P}{c^2 \rho} \right)^{-1} 4\pi R^2 \frac{\partial R}{\partial m}.
\]
Here \( (\partial/\partial t)_m \) means the differentiation with respect to \( t \) keeping \( m \) constant.

We will change the coordinate \( m \) to \( r \) later through a fixed equilibrium, and we shall construct solutions near the equilibrium.

## 2 Equilibrium

Let us consider solutions independent of \( t \), that is, \( F = F(r), H = H(r), \rho = \rho(r), V \equiv 0, R \equiv r \). The equations are reduced to the Tolman-Oppenheimer-Volkoff-de Sitter equation
\[
\frac{dm}{dr} = 4\pi r^2 \rho,
\]
\[
\frac{dP}{dr} = -(\rho + P/c^2) \frac{G \left( m + \frac{4\pi r^3}{c^2} P \right) - \frac{c^2 A}{3} r^3}{r^2 \left( 1 - \frac{2Gm}{c^2 r} - \frac{A}{3} r^2 \right)}.
\]
This equation was analyzed in [5]. Let us summarize the results.

For arbitrary positive central density $\rho_c$ there exists a unique solution germ $(m(r), P(r)), 0 < r \ll 1$, such that

$$m = \frac{4\pi}{3} \rho_c r^3 + [r^2]_2 r, \quad \text{(4a)}$$

$$P = P_c - (\rho_c + P_c/e^2) \left( \frac{4\pi G}{3} (\rho_c + 3P_c/e^2) - \frac{c^2 \Lambda}{3} \right) \frac{r^2}{2} + [r^2]_2. \quad \text{(4b)}$$

Here $[X]_Q$ denotes a convergent power series of the form $\sum_{k \geq Q} a_k X^k$.

We denote

$$\kappa(r, m) := 1 - \frac{2Gm}{c^2P} - \frac{\Lambda}{3} r^2,$$

$$Q(r, m, P) := G \left( m + \frac{4\pi r^3}{c^2} P \right) - \frac{c^2 \Lambda}{3} r^3,$$

we concentrate ourselves to solutions satisfying $\kappa(r, m(r)) > 0$. Moreover a solution $(m(r), P(r)), 0 < r < r_+$, of (3a)(3b) is said to be monotone-short if $r_+ < \infty$, $dP/dr < 0$ for $0 < r < r_+$, that is, $Q(r, m(r), P(r)) > 0$, and $P \to 0$ as $r \to r_+ - 0$ and if

$$\kappa_+ := \lim_{r \to r_+ - 0} \kappa(r, m(r)) = 1 - \frac{2Gm_+}{c^2r_+} - \frac{\Lambda}{3} r_+^2$$

and

$$Q_+ := \lim_{r \to r_+ - 0} Q(r, m(r), P(r)) = Gm_+ - \frac{c^2 \Lambda}{3} r_+^3$$

are positive. Here

$$m_+ := \lim_{r \to r_+ - 0} m(r) = 4\pi \int_0^{r_+} \rho(r)r^2 dr.$$

We suppose that there is a monotone-short solution $(\bar{m}(r), \bar{P}(r)), 0 < r < r_+$, satisfying (4a)(4b), and fix it hereafter.

As for sufficient conditions for the existence of monotone-short prolongations, see [5]. Anyway, the associated function $u = \bar{u}(r)$ turns out to be analytic on a neighborhood of $[0, r_+]$ and

$$\bar{u}(r) = \frac{Q_+}{r_+^2 \kappa_+} (r_+ - r) + [r_+ - r]_2$$

as $r \to r_+ - 0$. See [5 Theorem 4].

3 Equations for the small perturbation from the equilibrium

Using the fixed equilibrium $m = \bar{m}(r)$, we take the variable $r$ given by its inverse function. We are going to a solution near equilibrium of the form

$$R = r(1 + y), \quad V = rv.$$
Here $y, v$ are small unknowns. The equations turn out to be

$$e^{-F} \frac{\partial y}{\partial t} = \left(1 + \frac{P}{c^2 \rho}\right) v,$$

$$e^{-F} \frac{\partial v}{\partial t} = \left(1 + y\right)^2 \left(1 + y\right)^2 \frac{P}{c^2 \rho} v \frac{\partial (rv)}{\partial r} +$$
$$-G(1 + y) \left(\frac{m}{r^3(1 + y)^3} + \frac{4\pi}{c^2} P\right) + \frac{c^2 \Lambda}{3} (1 + y) +$$
$$- \left(1 + \frac{r^2 v^2}{c^2} - \frac{2Gm}{c^2 r(1 + y)} - \frac{\Lambda}{3} r^2 (1 + y)^2\right) \times$$
$$\left(1 + \frac{P}{c^2 \rho}\right)^{-1} (1 + y)^2 \frac{\partial P}{\partial r}.$$

(5b)

Here $m = \bar{m}(r)$ is a given function and $\rho, P$ are considered as given functions of $r$ and the unknowns $y, z := r \partial y / \partial r$ as follows:

$$\rho = \bar{\rho}(r)(1 + y)^{-2}(1 + y + z)^{-1},$$
$$P = \bar{P}(r)(1 - \Gamma(\bar{u}(r))(3y + z) - \Phi(\bar{u}(r), y, z)).$$

Here

$$\Gamma := \frac{\rho}{P} \frac{dP}{d\rho}$$

and $\Phi(u, y, z)$ is an analytic function of the form $\sum_{k_0 \geq 0, k_1 + k_2 \geq 2} u^{k_0} y^{k_1} z^{k_2}$. We shall denote such a function by $[u; y, z]_{0;2}$ hereafter. Moreover we shall use

$$1 + \frac{P}{c^2 \rho} = \left(1 + \frac{P}{c^2 \rho}\right) \left(1 - \frac{\bar{P}}{c^2 \rho} \left(1 + \frac{P}{c^2 \rho}\right)^{-1} (\Gamma - 1)(3y + z) +
$$
$$+ [\bar{u}(r); y, z]_{0;2}\right).$$

4 Analysis of the linearized equation

Let us linearize (5a) (5b):

$$e^{-\bar{F}} \frac{\partial y}{\partial t} = \left(1 + \frac{P}{c^2 \rho}\right) v,$$

(6a)

$$e^{-\bar{F}} \frac{\partial v}{\partial t} = E_2 y'' + E_1 y' + E_0 y,$$

(6b)
where $y'' = \partial^2 y / \partial r^2$, $y' = \partial y / \partial r$ and

$$E_2 = e^{-2\bar{H}(\rho + P/c^2)} P \Gamma,$$

$$E_1 = \frac{d}{dr} \left( \bar{H} + \bar{F} - \log(1 + P/c^2 \rho) + \log(P \Gamma r^4) \right),$$

$$E_0 = \frac{4\pi G}{c^2} \frac{3(\Gamma - 1)P +}{1 + P/c^2 \rho} +$$

$$+ \left( -1 - 3\bar{H} e^{-2\bar{H}} + 3(\Gamma - 1)e^{-2\bar{H}}(1 + P/c^2)^{-1} \right)(\rho + P/c^2)^{-1} \frac{1}{r} \frac{d}{dr} P +$$

$$+ 3e^{-2\bar{H}}(\rho + P/c^2)^{-1} \frac{d}{dr} \frac{r}{P \Gamma} +$$

$$+ \Lambda \left( c^2 + r \frac{d}{dr} \right).$$

Here $\bar{X}, \bar{XXX}$ denote the evaluations along the fixed equilibrium. Putting

$$\mathcal{L} y := -e^{-2\bar{F}}(1 + P/c^2 \rho)(E_2 y'' + E_1 y' + E_0 y),$$

we get the linearized wave equation

$$\frac{\partial^2 y}{\partial t^2} + \mathcal{L} y = 0.$$ 

We can rewrite $\mathcal{L}$ in the formally self-adjoint form

$$\mathcal{L} y = -\frac{1}{b} \frac{d}{dr} a \frac{dy}{dr} + Q y,$$

where

$$a = e^{\bar{H} + \bar{F}} \frac{P \Gamma r^4}{1 + P/c^2 \rho},$$

$$b = e^{3\bar{H} - \bar{F}} \frac{r^4 \bar{F}}{1 + P/c^2 \rho},$$

$$Q = -e^{2\bar{F}} \frac{1 + P/c^2 \rho}{b} E_0.$$ 

It is easy to see that $Q$ is bounded on $0 \leq r \leq r_+$. Therefore [4, Proposition 7] is still valid:

**Proposition 1** The operator $\mathcal{S}_0, D(\mathcal{S}_0) = C^\infty_0(0, r_+), \mathcal{S}_0 y = \mathcal{L} y$ in the Hilbert space $L^2((0, r_+); b(r) dr)$ admits the Friedrichs extension $\mathcal{S}$, a self adjoint operator, whose spectrum consists of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_\nu < \cdots \to +\infty$.

As in [4], we introduce the new variable $x$ instead of $r$ defined by

$$x := \frac{\tan^2 \theta}{1 + \tan^2 \theta} \quad \text{with} \quad \theta := \frac{\pi}{2\xi_+} \int_0^r \sqrt{\frac{\rho}{P \Gamma}} - \bar{F} + \bar{H} dr.$$
Here
\[ \xi_+ := \int_0^{r_+} \sqrt{\frac{\rho}{1 + \frac{\tilde{P}}{\Gamma}}} e^{-\tilde{P} + \tilde{H}} \, dr. \]

Then there are positive constants \( C_0, C_1 \) such that
\[ r = C_0 \sqrt{x}(1 + [x]) \quad \text{as} \quad x \to 0, \]
\[ r_+ - r = C_1 (1 - x)(1 + [1 - x]) \quad \text{as} \quad x \to 1. \]

Using this variable, we can write the operator \( \mathcal{L} \) as
\[ (\xi_+/\pi)^2 \mathcal{L} y = -x(1 - x) \frac{d^2 y}{dx^2} - \left( \frac{5}{2} (1 - x) - \frac{N}{2} x \right) \frac{dy}{dx} + L_1(x) x(1 - x) \frac{dy}{dx} + L_0(x) y, \]
where \( L_1(x), L_0(x) \) are analytic functions on a neighborhood of \([0, 1]\), and
\[ N := \frac{2\gamma}{\gamma - 1}, \]
which is supposed to be an even integer. Since the discussion is quite parallel to that of [4, §5], we omit the details. We may assume that \( \xi_+/\pi = 1 \) without loss of generality, by changing the scale of \( t \).

5 Rewriting (5a) (5b) using \( \mathcal{L} \)

Let us go back to the equations (5a) (5b). We shall use the analysis of \( \partial P/\partial r \) given in [4, (6.2)]:
\[ -\frac{1}{r \rho} \frac{\partial P}{\partial r} = -\frac{1}{r \rho} \frac{d\tilde{P}}{dr} + (1 + \partial_z \Phi / T) \frac{1}{r \rho} \frac{\partial P}{\partial r} + (3y + z) + \tilde{P} \cdot [Q0] + \frac{1}{r \rho} \frac{d\tilde{P}}{dr} \cdot [Q1], \]
where \([Q0], [Q1]\) are given by [4, (6.3a)]. [4, (6.3b)].

We put
\[ \text{the right-hand side of (5b)} = [R2] + [R1] + [W] \frac{\Lambda}{3}, \]
where
\[ [R2] := \frac{(1 + y)^2 P}{c^2 \tilde{\rho}} v(v + w) \quad \text{with} \quad w = \frac{\partial v}{\partial r}; \]
\[ [W] := c^2 (1 + y) - r^2 (1 + y)^{4(1 + P/c^2 \rho)} \left( -\frac{1}{r \tilde{\rho}} \frac{\partial P}{\partial r} \right). \]

We put
\[ [R1] = [R3] + [R4] + [R5] + [R6] + [R7]. \]
as in [4]. (The symbols Φ, γ of [4] are replaced by Φ, Γ in this article.) But the analysis of [W] is new: We put

\[ [W] = c^2 + \frac{r \, d\bar{u}}{dr} + [W1] + [W2] + [W3] + [W4], \]

\[ [W1] := c^2 y - r^2 (1 + y)^2 (1 + P/c^2 \rho)^{-1} \left( \frac{1}{r \rho} \, \frac{dP}{dr} \right) - r \, \frac{d\bar{u}}{dr} \]

\[ = [W1L] + [W1Q], \]

\[ [W1L] := c^2 y - 4r \frac{d\bar{u}}{dr} y + r \left( \frac{P/c^2}{\rho + P/c^2} \right) (\Gamma - 1)(3y + z) \, \frac{d\bar{u}}{dr}, \]

\[ [W2] := -r^2 (1 + y)^4 (1 + P/c^2 \rho)^{-1} (1 + \partial_z \Phi/\Gamma) \frac{1}{r \rho} \, \frac{\partial \rho}{\partial r} \left( 3y + z \right), \]

\[ [W3] := -r^2 (1 + y)^4 (1 + P/c^2 \rho)^{-1} \frac{\dot{P}}{r \rho} [Q0], \]

\[ [W4] := -r^2 (1 + y)^4 (1 + P/c^2 \rho)^{-1} \frac{1}{r \rho} \, \frac{d\bar{P}}{dr} [Q1]. \]

Then it follows from (3b) that

\[ [R1] + [W1] \Lambda^3 = [R3L] + [R3Q] + [R4L] + [R4Q] + [R5] + [R6] + [R7] + \]

\[ + ([W1L] + [W1Q] + [W2] + [W3] + [W4]) \frac{\Lambda}{3}. \]

Let us define

\[ 1 + G_1 = (1 + \partial_z \Phi/\Gamma) \left( 1 + \frac{r^2 v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} - r^2 (1 + y)^2 \frac{\Lambda}{3} \right) \times \]

\[ \left( 1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} \right)^{-1} \frac{1 + P/c^2 \rho}{1 + P/c^2 \rho} (1 + y)^2. \]

Then we have

\[ -e^{-2F} (1 + P/c^2 \rho)^{-1} Ly = [R3L] + [R4L] + [W1L] \frac{\Lambda}{3} + \]

\[ + \frac{1}{1 + G_1} \left( [R5] + [W2] \frac{\Lambda}{3} \right). \]

Putting

\[ G_2 := G_1 ([R3L] + [R4L] + [W1L] \frac{\Lambda}{3}) + \]

\[ - ([R3Q] + [R4Q] + [R6] + [R7] + [R2]) + \]

\[ - ([W1Q] + [W3] + [W4]) \frac{\Lambda}{3}, \]

\[ H_2 := e^{F} G_2, \]

\[ H_1 := e^{F-2F} (1 + P/c^2 \rho)^{-1} (1 + G_1), \]
we can write
e^F \times (\text{the right-hand side of (5b)}) = -H_1LY - H_2.

Using these analysis, we claim as [2, Proposition 11]:

**Lemma 1** we have

\[(\partial_z H_1)LY + \partial_z H_2 \equiv 0 \mod (1 - x)\]

as \(x \to 1\).

Here “\(Q_1 \equiv Q_0 \mod (1 - x)\)” means that there exists an analytic function \(\omega(x, y, z, v, w(:= rv')), y', y''\) such that \(Q_1 = Q_0 + (1 - x)\omega\).

Proof is similar to that of [2, Proposition 11]. We see

\[(\partial_z H_1)LY + \partial_z H_2 \equiv e^F[S]\]

and we have to show \([S] \equiv 0\), where

\[S := (\partial_z G_1)\left(e^{-2L}(1 + P/c^2\rho)^{-1}LY + [R3L] + [R4L] + [W1L] \Lambda/3\right) +
+ G_1\partial_z([R3L] + [R4L] + [W1L] \Lambda)+
- \partial_z\left([R3Q] + [R4Q] + [R6] + [R7] + [R2] + ([W1Q] + [W3] + [W4]) \Lambda/3\right).\]

But we have

\[S \equiv -\partial_z G_1\left([R5] + [W2] \Lambda/3\right) - \partial_z\left([R7] + [W4] \Lambda/3\right),\]

since \(\partial_z[R3L], \partial_z[R4L], \partial_z[R3Q], \partial_z[R4Q], \partial_z[R6], \partial_z[R7], \partial_z[W1L], \partial_z[W1Q], \partial_z[W3]\) are all \(\equiv 0\) clearly. By a tedious calculation, we get

\[-\partial_z G_1\left([R5] + [W2] \Lambda/3\right) \equiv \partial_z\left([R7] + [W4] \Lambda/3\right)\]

\[\equiv -\partial_z^2\Phi\left(1 + \frac{r^2v^2}{c^2} - \frac{2Gm}{c^2r(1 + y)} - r^2(1 + y)^2\Lambda\right)\left(1 + y\right)^2\frac{dP}{dr}\left(3y + z\right),\]

so that \([S] \equiv 0\). This completes the proof.

Putting

\[J := e^F(1 + P/c^2\rho),\]

we rewrite the system of equations (5a)-(5b) as

\[
\frac{\partial y}{\partial t} - Jv = 0, \tag{7a}
\]

\[
\frac{\partial v}{\partial t} + H_1LY + H_2 = 0. \tag{7b}
\]

9
6 Main results

Let us fix a time periodic solution of the linearized equation:
\[ Y_1 = \sin(\sqrt{\lambda} t + \Theta_0) \psi(x), \]
where \( \lambda \) is a positive eigenvalue of the operator \( \mathcal{T} \) and \( \psi \) is an associated eigenfunction. We seek a solution of the form
\[ y = \varepsilon(Y_1 + \hat{y}), \quad v = \varepsilon(V_1 + \hat{v}), \]
where
\[ V_1 = e^{-F} (1 + \frac{P}{c^2 \rho})^{-1} \frac{\partial Y_1}{\partial t}. \]
Then we have the equation
\[ P(\vec{w}) = \varepsilon \vec{c}, \]
with
\[ \vec{w} = \begin{bmatrix} \hat{y} \\ \hat{v} \end{bmatrix}. \] (8)

The Fréchet derivative of the nonlinear operator \( P \):
\[ DP(\vec{w}) \hat{h} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad \text{with} \quad \hat{h} = \begin{bmatrix} h \\ k \end{bmatrix} \]
is given by
\[
F_1 = \frac{\partial h}{\partial t} - Jk + (\partial_y J)v + (\partial_z J)vr \frac{\partial}{\partial r} h
\]
\[
F_2 = \frac{\partial k}{\partial t} + H_1 \mathcal{L}h + ((\partial_z H_1) \mathcal{L}y + \partial_z H_2) \frac{\partial}{\partial r} h + (\partial_x H_1) \mathcal{L}y + \partial_x H_2 + \partial_w H_2 r \frac{\partial}{\partial r} k.
\]

Thanks to Lemma 1 and considerations as [4] we can claim that there are analytic functions \( a_{01}, a_{00}, a_{11}, a_{10}, a_{21}, a_{20} \) of \( x, y, \partial_x y, \partial^2_x y, v, \partial_x v \) such that
\[
F_1 = \frac{\partial h}{\partial t} - Jk + \left( a_{01} x (1 - x) \frac{\partial}{\partial x} + a_{00} \right) h,
\]
\[
F_2 = \frac{\partial k}{\partial t} + H_1 \mathcal{L}h + \left( a_{11} x (1 - x) \frac{\partial}{\partial x} + a_{10} \right) h + \left( a_{21} x (1 - x) \frac{\partial}{\partial x} + a_{20} \right) k.
\]

Thus we can apply the Nash-Moser(-Hamilton) theorem to get

**Theorem 1** Given \( T > 0 \), there is a positive number \( \varepsilon_0 \) such that, for \( |\varepsilon| \leq \varepsilon_0 \), there is a solution \( \vec{w} \in C^\infty([0,T] \times [0,1]) \) of (8) such that
\[
\sup_{j+k \leq n} \left\| \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^k \vec{w} \right\|_{L^\infty([0,T] \times [0,1])} \leq C(n)|\varepsilon|,
\]
and hence a solution \((y, v)\) of (5a)(5b) of the form \( y = \varepsilon Y_1 + O(\varepsilon^2) \).
Note that
\[ R(t, r_+) = r_+(1 + \varepsilon \sin(\sqrt{\lambda}t + \Theta_0) + O(\varepsilon^2)), \]
provided that \( \psi \) has been normalized as \( \psi(x = 1) = 1 \), and that the density distribution enjoys the ‘physical vacuum boundary’ condition:
\[
\rho(t, r) = \begin{cases} 
C(t)(r_+ - r) \frac{1}{r_+} (1 + O(r_+ - r)) & (0 \leq r < r_+) \\
0 & (r_+ \leq r)
\end{cases}
\]
with a smooth function \( C(t) \) of \( t \) such that
\[
C(t) = \left( \frac{\gamma - 1}{A\gamma} \frac{Q_+}{r_+^{\gamma-1}} \right)^{\frac{1}{\gamma-1}} + O(\varepsilon).
\]

Also we can consider the Cauchy problem
\[
\frac{\partial y}{\partial t} - Jv = 0, \quad \frac{\partial v}{\partial t} + H_1 Ly + H_2 = 0,
\]
\[
y\bigg|_{t=0} = \psi_0(x), \quad v\bigg|_{t=0} = \psi_1(x).
\]

Then we have

**Theorem 2** Given \( T > 0 \), there exists a small positive \( \delta \) such that if \( \psi_0, \psi_1 \in C^\infty([0,1]) \) satisfy
\[
\max_{k \geq K} \left\{ \left\| \left( \frac{d}{dx} \right)^k \psi_0 \right\|_{L^\infty}, \left\| \left( \frac{d}{dx} \right)^k \psi_1 \right\|_{L^\infty} \right\} \leq \delta,
\]
then there exists a unique solution \( (y, v) \) of the Cauchy problem in \( C^\infty([0,T] \times [0,1]) \). Here \( K \) is sufficiently large number.

### 7 Metric in the exterior domain

Let us consider the moving solutions constructed in the preceding section, which are defined on \( 0 \leq t \leq T, 0 < r \leq r_+ \). We discuss on the extension of the metric onto the exterior vacuum region \( r > r_+ \). Keeping in mind the Birkhoff’s theorem, we try to patch the Schwarzschild-de Sitter metric
\[
ds^2 = \kappa^2 c^2 \left( dt^2 \right)^2 - \frac{1}{\kappa^2} \left( dR^2 \right)^2 - \left( R^2 \right)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\]
from the exterior region. Here \( t^2 = t^2(t, r), R^2 = R^2(t, r) \) are smooth functions of \( 0 \leq t \leq T, r_+ \leq r \leq r_+ + \delta \), \( \delta \) being a small positive number, and
\[
\kappa^2 = 1 - \frac{2Gm_+}{c^2 R^2} - \frac{\Lambda}{3} (R^2)^2.
\]
The patched metric is

\[ ds^2 = g_{00}c^2dt^2 + 2g_{01}cdtdr + g_{11}dr^2 + g_{22}(d\theta^2 + \sin^2 \theta d\phi^2), \]

where

\[
\begin{align*}
g_{00} &= \begin{cases} 
e^{-2u/c^2} & (r \leq r_+) \\
\kappa^2(\partial_t t^t)^2 - \frac{1}{c^2 \kappa^2} (\partial_t R^t)^2 & (r_+ < r) 
\end{cases} \\
g_{01} &= \begin{cases} 0 & (r \leq r_+) \\
c\kappa^2(\partial_t t^t)(\partial_t t^t) - \frac{1}{c^2 \kappa^2} (\partial_t R^t)(\partial_t R^t) & (r_+ < r) 
\end{cases} \\
g_{11} &= \begin{cases} -e^{2H} = -\left(1 + \frac{V^2}{c^2} - \frac{2Gm}{c^2 R} - \frac{\Lambda}{3} R^2\right)^{-1} (\partial_t R)^2 & (r \leq r_+) \\
c^2\kappa^2(\partial_t t^t)^2 - \frac{1}{\kappa^2} (\partial_t R^t)^2 & (r_+ < r) 
\end{cases} \\
g_{22} &= \begin{cases} -R^2 & (r \leq r_+) \\
-\left(R^2\right)^2 & (r_+ < r). 
\end{cases}
\end{align*}
\]

We require that \( R = R^t \) and \( \partial_r R = \partial_r R^t \) along \( r = r_+ \). It is necessary for that \( g_{22} \) is of class \( C^1 \). Moreover, by the same way as [5, Supplementary Remark 4], we see that \( \partial_t t^t, \partial_t t^r, \partial_t^2 t^t, \partial_t^2 R^t \) at \( r = r_+ + 0 \) are uniquely determined so that \( g_{\mu\nu} \) are of class \( C^1 \) across \( r = r_+ \).

By a tedious calculation we have

\[
\frac{\partial^2 R^t}{\partial r^2} \bigg|_{r_+ + 0} - \frac{\partial^2 R^t}{\partial r^2} \bigg|_{r_+ - 0} = A \left( \frac{\partial R^t}{\partial r} \right)^2,
\]

where

\[
A = -\frac{V^2}{c^2} \left( \left( \frac{Gm_+}{c^2 R^2} - \frac{\Lambda}{3} R + \frac{1}{\sqrt{\kappa_+} c^2} \frac{1}{\partial t} \right) \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-2} \right) \bigg|_{r_+ - 0}.
\]

Since

\[
\left( \frac{Gm_+}{c^2 R^2} - \frac{\Lambda}{3} R \right) \left( 1 + \frac{V^2}{c^2} - \frac{2Gm_+}{c^2 R} - \frac{\Lambda}{3} R^2 \right)^{-2} \bigg|_{r = r_+ - 0} \equiv \frac{Q_+}{c^2 r_+^2 \kappa_+} \neq 0,
\]

we see that \( \partial^2 R^t/\partial r^2 \equiv \partial^2 R/\partial s^2 \) if and only if \( V \equiv 0 \) at \( r = r_+ \), which is the case if the solution under consideration is an equilibrium.

**Appendix**
Let us describe the abstract theorem we have used. This has been established by the author through [2], [3], [4], therefore the proof is not repeated here.

First of all we introduce the following classes of functions: Let us denote by $\mathfrak{A}([0, 1])$ the set of all functions defined and analytic on a neighborhood of the interval $[0, 1]$, by $\mathfrak{A}_q([0, 1], [0]^p)$ the set of all functions $f$ defined and analytic on a neighborhood of $[0, 1] \times \{0\} \times \cdots \times \{0\} \in \mathbb{R}^{1+p}$ such that

$$f(x, y_1, \cdots, y_p) = \sum_{k_1 + \cdots + k_p \geq q} a_{k_1 \cdots k_p}(x) y_1^{k_1} \cdots y_p^{k_p}.$$ 

The set of equations under consideration is

$$\frac{\partial y}{\partial t} - J(x, y, z) v = 0, \quad (9a)$$
$$\frac{\partial v}{\partial t} + H_1(x, y, z, v) Ly + H_2(x, y, z, v, w) = 0 \quad (9b)$$

with

$$\mathcal{L} = -x(1 - x) \frac{d^2}{dx^2} - \left( \frac{N_0}{2} (1 - x) - \frac{N_1}{2} x \right) \frac{d}{dx} + L_1(x)x(1 - x) \frac{d}{dx} + L_0(x). \quad (10)$$

Here we denote $z = \frac{\partial y}{\partial x}, w = \frac{\partial v}{\partial x}$. We assume:

**(B0):** The parameters $N_0, N_1$ are supposed to be greater than 4.

The coefficients $J, H_1, H_2, L_1, L_0$ are supposed to be of class $\mathfrak{A}_0([0, 1], [0]^2), \mathfrak{A}_0([0, 1], [0]^3), \mathfrak{A}_2([0, 1], [0]^4), \mathfrak{A}([0, 1]), \mathfrak{A}([0, 1]),$ respectively, and their domains are supposed to include $U_0 \times U \times U, U_0 \times U \times U \times U, U_0 \times U \times U \times U \times U, U_0, U_0, U_0, U_0, U_0, U_0, U_0, U_0$, respectively, where $U_0$ is a neighborhood of $[0, 1]$ and $U$ is a neighborhood of 0.

We suppose the following assumptions:

**(B1):** $H_1(x, 0, 0, 0) = J(x, 0, 0)^{-1}$ and there is a constant $C > 1$ such that

$$\frac{1}{C} < J(x, 0, 0) < C$$

for $\forall x \in U_0$.

**(B2):** We have

$$\partial_x J \equiv 0, \quad (\partial_x H_1) Ly + \partial_x H_2 \equiv 0, \quad \partial_w H_2 \equiv 0$$

as $x \to 0$ and as $x \to 1$. 

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Here the meaning of ‘≡ 0 as \( x \to x_0 \)’ is as follows:

Let us denote by \( \mathcal{A}([x_0] \times [0,1]) \) the set of all functions defined and analytic on a neighborhood of \( (x_0,0,\cdots,0) \in \mathbb{R}^{1+q} \). A function \( f \) in \( \mathcal{A}([x_0] \times [0,1]) \) is said to satisfy \( f \equiv 0 \) as \( x \to x_0 \) iff \( f(x,y_1,\cdots,y_p) = 0 \) for \( \forall y_1,\cdots,y_p \), that is, there is a function \( \Omega \) in \( \mathcal{A}([x_0] \times [0,1]) \) such that

\[
f(x,y_1,\cdots,y_p) = (x-x_0)\Omega(x,y_1,\cdots,y_p).
\]

In the assumption (B2), the functions under consideration are regarded as functions of \( x,y,Dy,D^2y,v,Dv \). Here and hereafter \( D \) stands for \( \partial/\partial x \).

Let us fix \( T > 0 \) arbitrarily, and fix functions \( y^*,v^* \in C^\infty([0,T] \times [0,1]) \) such that all \( y^*,z^* = \partial y^*/\partial x, v^*,w^* = \partial v^*/\partial x \) are confined in \( U \) for \( 0 \leq \forall t \leq T \). We seek a solution \( y,v \in C^\infty([0,T] \times [0,1]) \) of (9a), (9b) of the form

\[
y = y^* + \tilde{y}, \quad v = v^* + \tilde{v},
\]

which satisfies

\[
\tilde{y}|_{t=0} = 0, \quad \tilde{v}|_{t=0} = 0.
\]

The conclusion is: There is a small positive number \( \delta(T) \) and a large number \( \mathcal{R} \) such that, if

\[
\max_{j+k \leq \mathcal{R}} \|\partial_j^t \partial_k^x (y^*,v^*)\|_{L^\infty} \leq \delta(T),
\]

then there exists a solution \( (y,v) \) of (9a), (9b), (11), (12).

In fact the equations for \( \tilde{u} = (\tilde{y},\tilde{v})^T \) turns out to be

\[
\begin{align*}
\frac{\partial \tilde{y}}{\partial t} - J\tilde{v} - (\Delta J)v^* &= c_1, \quad (14a) \\
\frac{\partial \tilde{v}}{\partial t} + H_1 \Delta \tilde{y} + (\Delta H_1)\Delta y^* + \Delta H_2 &= c_2, \quad (14b)
\end{align*}
\]

where

\[
\begin{align*}
J &= J(x,y^*+\tilde{y},z^*+\tilde{z}) \quad \text{with} \quad \tilde{z} = \frac{\partial \tilde{y}}{\partial x}, \quad (15a) \\
\Delta J &= J(x,y^*+\tilde{y},z^*+\tilde{z}) - J(x,y^*,z^*), \quad (15b) \\
c_1 &= -\frac{\partial y^*}{\partial t} + J(x,y^*,z^*)v^*, \quad (15c) \\
H_1 &= H_1(x,y^*+\tilde{y},z^*+\tilde{z},v^*+\tilde{v},w^*+\tilde{w}) \quad \text{with} \quad \tilde{w} = \frac{\partial \tilde{v}}{\partial x}, \quad (15d) \\
\Delta H_1 &= H_1(x,y^*+\tilde{y},z^*+\tilde{z},v^*+\tilde{v}) - H_1(x,y^*,z^*,v^*), \quad (15e) \\
\Delta H_2 &= H_2(x,y^*+\tilde{y},z^*+\tilde{z},v^*+\tilde{v},w^*+\tilde{w}) + \quad \text{H}_2(x,y^*,z^*,v^*,w^*), \quad (15f) \\
c_2 &= -\frac{\partial v^*}{\partial t} - H_1(x,y^*,z^*,v^*)\Delta y^* - H_2(x,y^*,z^*,v^*,w^*). \quad (15g)
\end{align*}
\]
We write the equations $(14a)(14b)$ as $\Psi(\vec{w}) = \vec{c}$, where $\vec{c} = (c_1, c_2)^T$. The domain of the nonlinear mapping $\Psi$ is $U$, the set of all functions $\vec{w} = (\tilde{y}, \tilde{v})^T \in \vec{E}_0$ such that
\begin{equation}
|\tilde{y}| + |D\tilde{y}| + |\tilde{v}| + |D\tilde{v}| < \epsilon_0.
\end{equation}
Here $\epsilon_0$ is small so that $(16)$ implies $y, z, v, w \in U$, and $\vec{E} = \mathcal{E} \times \mathcal{E}$, $\mathcal{E} = C^\infty([0, T] \times [0, 1])$, $\vec{E}_0 = \{(\phi, \psi) \in \vec{E} | \phi|_{t=0} = \psi|_{t=0} = 0\}$. The Fréchet derivative $D\Psi$ of $\Psi$ is
\begin{equation}
D\Psi(\vec{w}) \vec{h} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{for} \quad \vec{h} = \begin{bmatrix} h \\ k \end{bmatrix},
\end{equation}
where
\begin{align}
F_1 &= \frac{\partial h}{\partial t} - Jk - \left((\partial_y J)v + (\partial_z J)v \frac{\partial}{\partial x}\right)h, \quad (18a) \\
F_2 &= \frac{\partial k}{\partial t} + H_1 \mathcal{L}h + \\
&\quad + \left((\partial_y H_1)\mathcal{L}y + \partial_y H_2 + ((\partial_z H_1)\mathcal{L}y + \partial_z H_2) \frac{\partial}{\partial x}\right)h + \\
&\quad + \left((\partial_v H_1)\mathcal{L}y + \partial_v H_2 + \partial_w H_2 \cdot \frac{\partial}{\partial x}\right)k. \quad (18b)
\end{align}
Thanks to (B2) we see that there are $a_{\mu\nu} \in \mathbb{A}([0, 1], [0]^5), \mu = 0, 1, 2, \nu = 1, 0$, such that
\begin{align}
F_1 &= \frac{\partial h}{\partial t} - Jk - (a_{01}x(1-x)D + a_{00})h, \quad (19a) \\
F_2 &= \frac{\partial k}{\partial t} + H_1 \mathcal{L}y + \\
&\quad + (a_{11}x(1-x)D + a_{10})h + (a_{21}x(1-x)D + a_{20})k, \quad (19b)
\end{align}
where $a_{\mu\nu}$ are analytic functions of $x, y, Dy, D^2y, v, Dv$. Under this situation, we can apply the Nash-Moser(-Hamilton) theorem.

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