Gravitational Field Equation and the Structure of Black Holes

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Abstract
The problem posed by Einstein in 1913 related to the inclusion of the energy-momentum tensor of the gravitational field in the equation of the same gravitational field has been solved. At the same time, it was possible to preserve the covariance of the equation. In small fields, the new gravitational field equation successfully transforms into the Einstein equation. The new equation implies a local conservation law. The numerical solution of the equation is given. The numerical solution for $g_{00}$ is given. This parameter is strictly positive. Within the region of large fields the value quickly drops to almost zero. The region where extremely small values of $g_{00}$ can be found corresponds to the interval $0 < r < 3.4 r_g$. This interval is close to the observed radius of neutron stars. It follows that neutron stars and "black holes" are of the same nature. There are no singularities in the solutions of the equation.

Keywords: General relativity, Gravity field equation, Einstein equation, Ricci tensor, Strict solutions.

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1. Introduction

1.1 Brief historical information

In 1913 Einstein put forward the basic principles of his relativistic theory of gravity (Einstein and Grossmann, 1913). [1] At the same time Einstein presented the gravity field equation using the following formulation:

\[ D_{\mu \nu} = \kappa (T_{\mu \nu} + f_{\mu \nu}) , \]

(1)

where \( \kappa \) is constant and \( D_{\mu \nu} \) is a second rank covariant tensor formed from the derivatives of the fundamental tensor \( g_{\mu \nu} \). In this equation, the energy-momentum tensor of the gravity field \( f_{\mu \nu} \) and the similar tensor of matter \( T_{\mu \nu} \) are equally significant sources of the gravity field. In this regard Einstein made the following important remark.

“The exclusive position of the energy of the gravitational field in comparison with all other types of energy would lead to unacceptable consequences” (Einstein and Grossmann, 1913). [1]

Einstein was aided by the following equation

\[ R_{\mu \nu} = 0 , \]

(2)

where \( R_{\mu \nu} \) is Ricci tensor which uncovers the correct value of the rotation of the perihelion of Mercury (Einstein, 1915c) [4]. This result showed that taking the (low) energy density of the gravity field (1) into account is not essential.

Immediately after that, in his next paper (Einstein A. 1915d) [5], Einstein completes the work by adding a value to the equation which makes it one which satisfies the law on the conservation of matter but does not the system of matter and gravitational fields. The conservation law is preserved but it is true only for the closed systems. Now the Einstein equation was as follows:

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \kappa T_{\mu \nu} \]

(3)

Note that this equation leads only to the law of conservation of matter, but not to the system of matter plus the gravitational field (Landau and Lifshitz, 1975) [6].

1.2 Introduction of the field energy-momentum tensor into the equation. Formally from equation (1), we obtain the following modernized equation of a gravitational field:

\[ G_{\mu \nu} = \kappa (T_{\mu \nu} + f_{\mu \nu}) . \]

(4)

It should be noted that this equation guarantees the fulfillment of the law on the conservation of matter and gravitational fields. Indeed, due to the covariant divergence of the Einstein tensor \( G^\mu_{\nu;\mu} = 0 \), the law on the conservation of both energy and momentum are preserved:

\[ T^\mu_{\nu;\mu} + f^\mu_{\nu;\mu} = 0 , \]

The same problem remains. We have to demonstrate the connection between \( f_{\mu \nu} \) and gravitational fields. It must be proven that \( f_{\mu \nu} \) is a tensor.

By simplifying equation (4) we obtain the following relation:

\[ R = -\kappa (T + f) . \]

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2 Hereinafter the modern designations are used.

3 This is the value \(-\frac{1}{2} g_{\mu \nu} T\) or \(-\frac{1}{2} g_{\mu \nu} R\) for equivalent Einstein equations, where the scalars \( T \) and \( R \) are traces of their relevant tensors.
Now field equation (4) can be written in the following form:

\[ R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T + f_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f \right), \quad (4') \]

2. Development

2.1 Einstein’s division of the Ricci tensor

Einstein considered the possibility of simplifying of the gravity field equations (Einstein, 1915a, b), [3, 7] For this the Ricci tensor is represented as the sum of two covariant tensors as shown below:

\[
R_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu};
\]

\[ A_{\mu\nu} = \frac{\partial \Gamma_{\alpha\beta}^\gamma}{\partial x_{\alpha}} \Gamma_{\gamma\alpha\beta} - \Gamma_{\mu\beta}^\gamma \Gamma_{\gamma\alpha} ; \]

\[ B_{\mu\nu} = -\frac{\partial \Gamma_{\alpha\mu}^\beta}{\partial x_{\nu}} + \Gamma_{\nu\mu}^\alpha \Gamma_{\alpha\beta}^\gamma. \]

2.2 Equation covariance

We use the formula for the coordinate transformation of Christoffel symbols (equation (85.15), Landau and Lifshitz, 1975) [6]

\[
\Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\xi}^\gamma \frac{\partial x^\xi}{\partial x^\gamma} \frac{\partial x^\mu}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\nu}. \]

Let us put in this expression \( \iota = \alpha \) and \( \gamma = \xi \), then the formula will be simplified:

\[ \Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\xi}^\gamma \frac{\partial x^\xi}{\partial x^\gamma}. \]

Hence, \( \Gamma_{\mu\nu}^\alpha \) is a covariant 4-vector which generates the tensor

\[ D \frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x_{\nu}} = \frac{\partial \Gamma_{\mu\alpha}^\alpha}{\partial x_{\nu}} \Gamma_{\alpha\beta}^\beta - \Gamma_{\mu\beta}^\gamma \Gamma_{\gamma\alpha} \theta. \]

i.e. with precision up to the value \( B_{\mu\nu} \). As \( R_{\mu\nu} \) is a tensor, then from \( R_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu} \) it follows that \( A_{\mu\nu} \) is also a tensor value.

2.3 Gravity field energy-momentum tensor

If \( R_{\mu\nu} \) is represented by the equation:

\[ R_{\mu\nu} = \frac{\partial f_{\mu\nu}}{\partial x_{\alpha}} - \Gamma_{\mu\beta}^\gamma \Gamma_{\gamma\alpha} \theta - \frac{\partial \ln \sqrt{-g}}{\partial x_{\mu}} - \frac{\partial \ln \sqrt{-g}}{\partial x_{\nu}} + \Gamma_{\mu\nu}^\alpha \frac{\partial \ln \sqrt{-g}}{\partial x_{\alpha}}, \]

it can thus be seen that in the inertial system the last two terms (i.e. \( B_{\mu\nu} \)) become zero. The same can be said about the gravitational field - it disappears together with its energy-momentum tensor in the inertial coordinate system, since in the inertial coordinate system the determinant is \( g = -1 \). Disappearance in an inertial system is an exceptional property of gravitational fields. Thus, it can be assumed that the tensor \( B_{\mu\nu} \) and the sum of the tensors related to the gravity field are proportional; and the proportionality coefficient is as follows:

\[ B_{\mu\nu} = \kappa \left( f_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f \right). \]

Whence \( B = B_{\mu}^\mu = -\kappa f \). Simplifying this equation in an analogous manner as the Einstein equation, we obtain the equivalent equation:

\[ f_{\mu\nu} = \frac{1}{\kappa} \left( B_{\mu\nu} - \frac{1}{2} g_{\mu\nu} B \right). \]

As \( B_{\mu\nu} \) is a tensor, \( f_{\mu\nu} \) is a tensor and along with equation (4) become covariant.

Taking equation (5) into account, from equation (4), an equation is obtained which does not contain the components of the tensor \( f_{\mu\nu} \):

\[ \frac{\partial \Gamma_{\mu\alpha}^\alpha}{\partial x_{\alpha}} = \Gamma_{\mu\beta}^\gamma \Gamma_{\gamma\alpha} \theta = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \]

Now the number of the unknowns is equal to the number of the variables, which leads to the uniqueness of the solution to this equation.

2.4 Asymptotic properties of solutions of the gravitational field equation

If to enter the additional condition \( \sqrt{-g} = 1 \) to the equation resulting from equation (4''), we come to the little-known Einstein equation (5'):
\[
\begin{align*}
\left\{ \frac{\partial \Gamma_{\mu\nu}^{\alpha}}{\partial x^\alpha} - \Gamma_{\mu\beta}^{\alpha} T^\beta_{\nu} &= \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \\
\sqrt{-g} &= 1.
\end{align*}
\]

It was exactly in this form that Schwarzschild solved the equation for empty space (Schwarzschild, 1916). [8]

It follows from the above that if field \( g_{\mu\nu} \) is weak, then \( \sqrt{-g} \to 1 \) and the Einstein equation solutions are asymptotically equal to the solutions of equation (4').

2.5 Solutions of gravitational field equation.

For a comparison, the most suitable problem is the problem of finding of the gravitational field of a point mass. We are looking for solutions as follows:

\[
d s^2 = s(r) dt^2 - p(r) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2.
\]

From equation (6), we find the components of the mixed tensor up to a factor equal to:

\[
A_0^0 = \frac{s''}{s'} - \frac{p'}{p} - \frac{s'}{s'}.
\]

\[
A_1^1 = \left( \frac{p'}{p} \right)' - \frac{2}{r^2} - \frac{s' s'}{4 s^2}.
\]

where prime denotes the derivative with respect to \( r \). If we equate these components to zero, we obtain a system of ordinary differential equations. Eliminating quantity \( \frac{p'}{p} \) we obtain the following equation:

\[
\left( \frac{s''}{s'} \right)' - \left( \frac{s'}{s} \right)' - \frac{s' s'}{2 s^2} = \frac{4}{r^2}.
\]

This equation was solved numerically. The vicinity of the point \( r = r_g = 1 \) was of fundamental importance. The simple substitution \( s = e^{\nu} \) helped to calculate \( s \) in this area. In contrast to the Schwarzschild solution, the value of \( s \) at this point turned out to be an extremely small positive value. Moreover, in the area of small positive values \( 0 < s = g_{00} \ll 1 \), lying in the interval \( 0 < r < 3.4 r_g \) (see Figure 1).

Figure 1. Dependence of \( g_{00} \) ratio \( r/r_g \). Schwarzschild solutions of the Einstein equation (marked by the dotted line) and the numerical solution of the exact gravity field equation (the solid line).

Source: Author.

3. Discussions

Now Einstein’s idea (Einstein and Grossmann 1913) [1] about the inclusion of the gravitational field energy tensor into the right gravitational field equation can be considered as having been realized. In this case, the equation remains covariant and the number of the unknowns does not increase, while the value \( g_{00} \) stays positive in interval \( 0 < r < \infty \).

The figure shows the dependence of the solution of equation (6) \( g_{00} = s(r) \) on the distance to the coordinate’s origin—in comparison with the Schwarzschild solution \( g_{00} = 1 - \frac{1}{r} \). In accordance with the above asymptotic behavior of the solution, with decreasing field strength, the numerical solution quickly approaches the Schwarzschild solution. Notably, \( s \) is a smooth non-negative function on the interval \( r > 0 \). This changes our ideas regarding the magnitude and nature of the gravitational fields of heavy compact objects. This solution shows that the equations satisfy the V. A. Fock’s requirement (Fock, 1964), [9] namely the requirement of the absence of regions, in which the velocity of light is equal to zero, i.e. \( g_{00} > 0 \) for \( r > 0 \). There are no anomalies on the Schwarzschild radius. In general, the picture has become full-fledged physical but by no means simple. Both light and particles fall on such an object in a finite period of
time. However, their return trip is also not an impossibility.

4. Conclusions

The radius of compact heavy stars \(3.4 \, \tau_g\) is close to the estimated radius of neutron stars. This allows neutron stars and heavier “black holes” to be considered objects of the same nature.

Initial results on this topic were published in [10].

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