Quenched lattice calculation of the $B \to D\ell\nu$ decay rate

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We calculate, in the continuum limit of quenched lattice QCD, the form factor that enters in the decay rate of the semileptonic decay $B \to D\ell\nu$. Making use of the step scaling method (SSM), previously introduced to handle two scale problems in lattice QCD, and of flavour twisted boundary conditions we extract $G^{B\to D}(w)$ at finite momentum transfer and at the physical values of the heavy quark masses. Our results can be used in order to extract the CKM matrix element $V_{cb}$ by the experimental decay rate without model dependent extrapolations.

I. INTRODUCTION

The central goal of flavour physics is the determination of the Cabibbo–Kobayashi–Maskawa [1, 2] quark mixing matrix. A set of redundant and precise measurements can also provide informations about possible new physics [3]. In turn, precise measurements need an accurate theoretical determination of hadron matrix elements of the weak currents. Non perturbative tools and in particular lattice QCD can eventually provide the required precision. In this letter we address the determination at finite momentum transfer of the form factor that enters in the semileptonic heavy meson decay rate. In the infinite quark mass limit it is parametrized by the Isgur-Wise function [4]. In particular, we calculate the matrix elements entering the determination of $V_{cb}$. The calculation that we present misses the effects of dynamical fermions and is not the final one. Nevertheless, it accomplishes for the first time the calculation at finite momentum transfer and at the physical values of the heavy quark masses, allowing to compare experimental data without additional model dependent extrapolations.

We use a method [5] already applied successfully to the determination of heavy quark masses and decay constants [6, 7, 8], called the step scaling method (SSM), and special boundary conditions, called flavour twisted [9], that shift by an arbitrary amount the discretized set of lattice momenta (see also [10, 11]). The step scaling method allows to reconcile large quark masses with adequate lattice resolution and large physical volumes; the flavour twisted boundary conditions allow to perform a calculation at non zero momentum transfer with good accuracy. We present the main results of our computation and stress their phenomenological implications; a more detailed discussion on technical aspects will be presented elsewhere [12].

II. FORM FACTORS AND DECAY RATE

The semileptonic decay of a pseudoscalar meson into another pseudoscalar meson is mediated by the vector part of the weak $V-A$ current. The corresponding matrix element can be parametrized in terms of two form factors. Among possible parametrizations we chose the following one

$$\frac{(M_f | V^\mu | M_i)}{\sqrt{M_i M_f}} = (v_i + v_f)^\mu h_+ + (v_i - v_f)^\mu h_-$$

where $M_i, f$ and $v_i, f = p_i, f / M_i f$ are the meson masses and 4-velocities. The form factors depend upon the masses of the initial and final particles and upon $w \equiv v_f \cdot v_i$

$$h_\pm \equiv h_\pm^{i \to f}(w) \equiv h_\pm(w, M_i, M_f)$$

$$1 \leq w \leq (M_i^2 + M_f^2)/(2 M_i M_f)$$

In the case where $M_i$ is the $B$ meson mass and $M_f$ is the $D$ meson mass the maximum value of $w$ is around 1.6. In the infinite mass limit the form factors become

$$h_+^{i \to f}(w) = \xi(w) \hspace{1cm} M_f, M_i \to \infty$$

$$h_-^{i \to f}(w) = 0$$

where $\xi(w)$ is the universal Isgur-Wise function; the conservation of the vector current implies $\xi(1) = 1$. In what follows we will find deviations from such a limit that will allow a precise discussion on the onset of the HQET regime [12].

The differential decay rate for the process $B \to D\ell\nu$, in the case of massless leptons, is given by

$$\frac{d\Gamma^{B\to D\ell\nu}}{dw} = |V_{cb}|^2 \frac{G_F^2}{48\pi^3} (M_B + M_D)^2 M_D^3 (w^2 - 1)^{3/2} \left[ G^{B\to D}(w) \right]^2$$
The phase space factor \((w^2 - 1)^{3/2}\) in the decay rate makes its experimental determination harder as \(w\) approaches 1 and the value at zero recoil is obtained from an extrapolation. The function \(\mathcal{G}^{B-D}(w)\),

\[
G^{i-f}(w) = h^{i-f}_+ (w) - \frac{M_f - M_i}{M_f + M_i} h^{i-f}_- (w)
\]

is needed in order to extract \(V_{cb}\) by the measurement of the decay rate. Previous lattice calculations by the Fermilab collaboration \([13, 14]\) quote the value of \(\mathcal{G}^{B-D}(w)\) only at zero recoil where it can be extracted with good statistical accuracy by using the so called "double ratio" technique. In the following we calculate \(\mathcal{G}^{B-D}(w)\) in the range \(1 \leq w \leq 1.2\) that includes values of \(w\) where experimental data are available.

### III. LATTICE OBSERVABLES

We have carried out the calculation within the \(O(a)\) improved Schrödinger Functional formalism \([13, 16]\) with \(T = 2L\) and vanishing background fields. In order to fix the notations, we introduce the following lattice operators

\[
O_{rs} = \frac{a^6}{L^3} \sum_{y,z} \zeta_r(y) \gamma_5 \zeta_s(z)
\]

\[
O'_{rs} = \frac{a^6}{L^3} \sum_{y,z} \zeta'_r(y) \gamma_5 \zeta'_s(z)
\]

\[
A^0_{rs}(x) = \bar{\psi}_r(x) \gamma_5 \gamma^0 \psi_s(x), \quad P_{rs}(x) = \bar{\psi}_r(x) \gamma_5 \psi_s(x)
\]

\[
V^u_{rs}(x) = \bar{\psi}_r(x) \gamma^u \psi_s(x), \quad T^{\mu\nu}_{rs}(x) = \bar{\psi}_r(x) \gamma^\mu \gamma^\nu \psi_s(x)
\]

\[
A^0_{rs}(x) = A^0_{rs}(x) + ac_A \frac{\partial_0 + \partial^*_0}{2} P_{rs}(x)
\]

\[
V^u_{rs}(x) = V^u_{rs}(x) + ac_V \frac{\partial_u + \partial^*_u}{2} T^{\mu\nu}_{rs}(x)
\]

where \(r\) and \(s\) are flavour indexes while \(\zeta\) and \(\zeta'\) are boundary fields at \(x_0 = 0\) and \(x_0 = T\) respectively. The improvement coefficients \(c_A\) and \(c_V\) have been taken from refs. \([17, 18, 19]\). We have calculated the following correlation functions

\[
\mathcal{F}_{i-f}^u(x_0; \mathbf{p}_i, \mathbf{p}_f) = \frac{a^3}{2} \sum_x \langle O_{li} V^u_{lf}(x) O'_{fl} \rangle
\]

\[
f_i^A(x_0, \mathbf{p}_r) = -\sum_x \langle O_{lf} A^0_{lf}(x) \rangle
\]

where \(i\) and \(f\) refer to the heavy flavours while \(l\) to the light one. The external momenta have been set by using flavour twisted b.c. for the heavy flavours; in particular we have used

\[
\psi_{i,f}(x + 1L) = e^{i\theta_i,f} \psi_{i,f}(x)
\]

leading to

\[
p_1 = \frac{\theta_i,f}{L} + \frac{2\pi k_1}{L}, \quad k_1 \in \mathbb{N}
\]

and ordinary periodic b.c. in the other spatial directions and for the light quarks. We work in the Lorentz frame in which the parent particle is at rest (\(p_1 = 0\)). In this frame \(w\) is obtained from the ratio between the energy and the mass of the final particle \(w = E_f/M_f\).

By assuming ground state dominance and by relying on the conservation of the vector current, the matrix elements of \(V^\mu\) can be extracted by considering the ratio

\[
\langle V^\mu \rangle_{D_1} = \frac{\langle \mathcal{M}_f | V^\mu | \mathcal{M}_i \rangle_{D_1}}{\sqrt{\mathcal{F}_{i-f}^u(T/2; 0, \mathbf{p}_r) \mathcal{F}_{f-i}^u(T/2; \mathbf{p}_i, \mathbf{p}_f)}}
\]

(1)

An alternative definition of the matrix elements (D2), which reduces to the previous one (D1) in the limits of infinite volume and zero lattice spacing, can be obtained by considering

\[
\langle V^\mu \rangle_{D_2} = \frac{\langle \mathcal{M}_f | V^\mu | \mathcal{M}_i \rangle_{D_2}}{\sqrt{\mathcal{F}_{i-f}^u(T/2; 0, \mathbf{p}_r) \mathcal{F}_{f-i}^u(T/2; 0, \mathbf{p}_i) \mathcal{F}_{f-i}^u(T/2; 0, \mathbf{p}_i)}}
\]

(2)

In eqs. (1) and (2) the renormalization factors \(Z_V\) and \(Z_A\) cancel in the ratios together with factors containing the improvement coefficients \(b_V\) and \(b_A\).
FIG. 1: Step scaling functions $\sigma^{i-D}(w; L_0, L_1)$, lower plot, and $\sigma^{i-D}(w; L_1, L_2)$, upper plot, as functions of the inverse of the RGI heavy quark mass $m_i$ of the parent meson measured in GeV. The black vertical lines represent the physical values of $m_c$ and $m_b$. The data correspond to the definition D1.

By introducing the following ratio

$$x_f = \frac{\mathcal{F}_{i-f}^{3}(T/2; 0, p_T)}{\mathcal{F}_{i-f}^{3}(T/2; 0, p_T)}$$

we can define $w$, as well as $E_f$ and $M_f$, entirely in terms of three point correlation functions. This definition of $w$ is noisier than the one that can be obtained in terms of ratios of two point correlation functions; however it leads to exact vector current conservation when $M_f = M_i$ and reduces the final statistical error on the form factors.

The two definitions of the matrix elements lead to two definitions of $G^{i-f}$

$$G^{i-f} = \frac{\sqrt{\langle V^0 | V^{1-i} | M_f \rangle}}{\langle M_f | V^1 | M_f \rangle} \left\{ 1 + \frac{w r - 1}{r \sqrt{w^2 - 1}} \frac{\langle V^1 | V^{0-i} | M_f \rangle}{\langle V^0 | V^{1-i} | M_f \rangle} \right\}$$

$$r = \frac{M_f}{M_i}$$

The last equation is not defined at $w = 1$; this is due to the second term in parenthesis that we extrapolate at zero recoil before calculating $G^{i-f}(w = 1)$.

IV. THE STEP SCALING METHOD

The SSM has been introduced to cope with two-scale problems in lattice QCD. In the calculation of heavy-light meson properties the two scales are the mass of the heavy quarks ($b, c$) and the mass of the light quarks ($u, d, s$). Here we consider the form factor $G^{i-f}$ as a function of $w$, the volume, $L^3$, and identify heavy meson states by the corresponding RGI quark masses that in the infinite volume limit lead to the physical meson spectrum [7]; the RGI quark masses are measured by the lattice version of the PCAC relation and are not affected by finite volume effects (see [12] for further details).

First we compute the observable $G^{B-D}(w; L_0)$ on a small volume, $L_0$, which is chosen to accommodate the dynamics of the $b$-quark. As in our previous work we fixed $L_0 = 0.4$ fm. Then we evaluate a first effect of finite volume by evolving the volume from $L_0$ to $L_1 = 0.8$ fm and computing the ratio

$$\sigma^{i-D}(w; L_0, L_1) = \frac{G^{i-D}(w; L_1)}{G^{i-D}(w; L_0)}$$

The crucial point is that the step scaling functions are calculated by simulating heavy quark masses $m_i$ smaller than the $b$-quark mass. The physical value $\sigma^{B-D}(w; L)$ is obtained by a smooth extrapolation in $1/m_i$ that relies on the HQET expectations and upon the general idea that finite volume effects, measured by the $\sigma$’s, are almost insensitive to the high energy scale. The final result is obtained by further evolving the volume from $L_1$ to $L_2 = 1.2$ fm, according to

$$G^{B-D}(w; L_2) = G^{B-D}(w; L_0) \sigma^{B-D}(w; L_0, L_1) \sigma^{B-D}(w; L_1, L_2)$$

Physical values require also usual continuum and chiral extrapolations.

V. RESULTS

In figure 4 we can test the validity of the SSM. The step scaling functions are almost insensitive to the heavy quark mass $m_i$ of the parent meson for values larger than $m_c$. Moreover, finite volume effects are already very small at $L = 0.8$ fm, in particular at zero recoil.

We present results already extrapolated to the chiral and continuum limits. We have simulated three different lattices for $L_0$ and two for the other volumes (see table 1).
observing small $O(a)$ effects on the form factor and on the step scaling functions. Further details on the continuum extrapolations will be given in ref. [12]. Concerning the chiral behaviour, our results do not show any sizable dependence upon the light quark mass, as shown in figure 2 for $G^{B \rightarrow D}(w; L)$. The same feature is observed for all the combinations of heavy quark masses and on all the volumes that we have simulated. Nevertheless we make a linear extrapolation to reach the chiral limit; the resulting error largely accounts for the systematics due to these extrapolations.

As discussed in sec. III we used two definitions of the form factor that, at finite volume, differ by the finite volume effects. A check of the convergence of our SSM can be obtained by comparing these two definitions on the smallest volume (figure 3 upper plot) and on the largest one (figure 3 lower plot). We see that, while the small volume results differ, the final ones converge to a common value giving us confidence of a correct accounting of finite volume effects.

TABLE II: Final result. Form factor $G^{B \rightarrow D}(w)$ in the continuum and infinite volume limits. As a comparison we quote also the results of previous lattice calculations by the Fermilab collaboration.

| $w$  | $G^{B \rightarrow D}(w)$ | $N_f$ | reference  |
|------|----------------|------|-----------|
| 1.00 | 1.026(17)      | 0    | this work |
| 1.03 | 1.001(19)      | 0    | this work |
| 1.05 | 0.987(15)      | 0    | this work |
| 1.10 | 0.943(11)      | 0    | this work |
| 1.20 | 0.853(21)      | 0    | this work |

| $w$  | $G^{B \rightarrow D}(w)$ | $N_f$ | reference  |
|------|----------------|------|-----------|
| 1.00 | 1.058(20)      | 0    | [13]      |
| 1.00 | 1.074(24)      | 2+1  | [14]      |

FIG. 3: Comparison of the two definitions of $G^{B \rightarrow D}(w; L)$ at $L_0 = 0.4$ fm (upper plot) and at $L_2 = 1.2$ fm (lower plot).

VI. CONCLUDING REMARKS

In table II we quote our final results obtained by averaging over the two definitions and by combining in quadrature statistical errors with the systematic ones due to the small residual dispersion between $D1$ and $D2$. As a comparison we show previous lattice results obtained by the Fermilab collaboration at zero recoil.

The existence of predictions up to $w \simeq 1.2$ and physical $b$ and $c$ quark masses allows a direct comparison with experimental data, as shown in figure 4. The comparison has been done by extracting the value of $V_{cb}$ by the ratio of the experimental and lattice data at $w = 1.2$; as an indication, we get $V_{cb} = 3.84(9)(42) \times 10^{-2}$, where the first error is from our lattice result, $G^{B \rightarrow D}(w = 1.2) = 0.853(21)$, and the second from the experimental decay rate, $|V_{cb}|G^{B \rightarrow D}(w = 1.2) = 0.0327(35)$, as deduced from the plots of refs. [20, 21, 22].

The extension of this calculation to the unquenched case does not present problems of principle. The recursive matching process can be extended to the sea quark masses that, alternatively, can be kept to their physical values if the Schrödinger Functional formalism is used. Moreover, flavour twisted boundary conditions can be used for heavy valence quarks also in the $N_f = 3$ unquenched theory. The real case will further differ by the heavy flavour determinants that can be computed perturbatively.

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