Parametric Lyapunov exponents

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Abstract
In an algebraic family of rational maps of \( \mathbb{P}^1 \), we show that, for almost every parameter for the trace of the bifurcation current of a marked critical value, the critical value is Collet–Eckmann. This extends previous results of Graczyk and Świątek in the unicritical family, using Makarov theorem. Our methods are based instead on ideas of laminar currents theory.

1. Introduction
Let \( \Lambda \) be a smooth complex quasi-projective variety and \( f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1 \) an algebraic family of rational maps of degree \( d \geq 2 \); \( f \) is a morphism and for each \( (\lambda, z) \), \( f(\lambda, z) = (\lambda, f_\lambda(z)) \) where \( f_\lambda \) is a rational map of \( \mathbb{P}^1 \) of degree \( d \). Let also \( a \) be a marked point, that is, a rational function \( a : \Lambda \to \mathbb{P}^1 \). A particularly interesting case is when \( a \) is a marked critical point. A fundamental notion in complex dynamics is the notion of stability: the point \( a \) is stable at some parameter \( \lambda_0 \) if the sequence \( \lambda \mapsto (f^n_\lambda(a(\lambda)))_n \) is normal in some neighborhood of \( \lambda_0 \).

The bifurcation locus of \( a \) is then the set of unstable parameters.

One can give a measurable sense to bifurcation using the bifurcation (or activity) current of the pair \((f, a)\). It is the closed positive (1,1)-current \( T_{f,a} := (\Pi_\Lambda)_* (\hat{T} \wedge [\Gamma_a]) \), where \( \hat{T} \) is the fibered Green current of the family \( f \), \( \Gamma_a \) is the graph of \( a \) and \( \Pi_\Lambda : \Lambda \times \mathbb{P}^1 \to \Lambda \) is the canonical projection. Roughly speaking, \( \hat{T} \) is a 2-form with measure coefficients and, by duality between 2-forms and functions on a Riemann surface, the slice of \( \hat{T} \) above each parameter \( \lambda \) induces the measure of maximal entropy of the map \( f_\lambda \) (the support of this measure is the Julia set of \( f_\lambda \)). Then \( \hat{T} \wedge [\Gamma_a] \) detects the non-stable intersection between the graph of \( a \) and the fibered Julia set. We then project on \( \Lambda \) to measure bifurcation on the parameter space. Indeed, this current is exactly supported by the bifurcation locus of the marked point \( a \), see, for example, [8]. When \( \dim(\Lambda) = 1 \), then \( T_{f,a} \) is a measure that we simply denote by \( \mu_{f,a} \).

In that sense, the bifurcation current is a parametric analogous of the Green current of an endomorphism of \( \mathbb{P}^k \) which measures the dynamical unstability. As such, it is interesting to develop an ergodic theory for the bifurcation currents. This is what we did in [5] where we defined a notion of parametric entropy and proved, for example, that in a one-dimensional family, the measure \( \mu_{f,a} \) is a measure of maximal entropy. Pursuing our study, in the present article, we address the notion of parametric Lyapunov exponent.

An historically important example is the Mandelbrot set in the unicritical family: \( f_\lambda(z) = z^d + \lambda \) with \( \lambda \in \mathbb{C} \) and \( a(\lambda) := \lambda \). In this case, the bifurcation measure \( \mu_{f,a} \) is the harmonic measure \( \mu_{M_d} \) of the degree \( d \) Mandelbrot set \( M_d \). In this context, Graczyk and Świątek [11] described the dynamics of a typical parameter.

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**Theorem 1** (Graczyk–Świątek). In the unicritical family of degree $d$, for $\mu_{M_d}$-almost every parameter $\lambda \in \mathbb{C}$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log |(f^n_\lambda)'(\lambda)| = \log d.$$ 

As the measure $\mu_{M_d}$ has Hausdorff dimension 1, this result may be reinterpreted as a parametric Ruelle (in)equality ‘the Lyapunov exponent of $\mu_{M_d}$ is equal to $\log d = h_{\text{bif}}(f,a)/\dim \mu_{f,a}$’. Note also that the theorem implies that, in the unicritical family, for $\mu_{M_d}$-almost every parameter $\lambda \in \mathbb{C}$, the corresponding map $f_\lambda$ is Collet–Eckmann: the critical value $\lambda$ satisfies

$$\liminf_{n \to \infty} \frac{1}{n} \log |(f^n_\lambda)'(\lambda)| > 0.$$ 

This weaker result was already obtained by Graczyk and Świątek in [10] and independently by Smirnov in [13].

Here, we generalize partially Theorem 1 to the case of any pair $(f,a)$. For $\lambda \in \Lambda$, denote $\text{Crit}(f_\lambda) := \{z \in \mathbb{P}^1, f'_\lambda(z) = 0\}$. Denote $\omega_\lambda$ a Kähler form on $\Lambda$ so that $T_{f,a} \cdot \omega_\lambda_{\dim(\Lambda)-1}$ is the trace measure of $T_{f,a}$. When $\dim(\Lambda) = 1$, for a measure $\mu$ on an open set $U \subset \Lambda$, we define $D_{\mu}^\ast$, the upper packing dimension of $\mu$ in $U$ as

$$D_{\mu}^\ast(\mu) := \sup \{ \phi^\ast(\lambda) \leq 2; \lambda \in U \},$$

where, for $\lambda \in U$,

$$\phi^\ast(\lambda) := \limsup_{r \to 0} \frac{\log \mu(B(\lambda,r))}{\log r}.$$

We have the inequality $D^\ast(\mu) \leq 2$ since the upper packing dimension of a measure is less than the dimension of the ambient space [9, Proposition 10.4]. We prove the following, where $f^\#$ is the spherical derivative.

**Main Theorem.** Let $f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$ parametrized by a quasi-projective curve $\Lambda$ and let $a : \Lambda \to \mathbb{P}^1$ be a rational function for which there exists $\lambda_0 \in \Lambda$ such that $\{f^n_{a_\lambda}(a(\lambda_0)), n \in \mathbb{N}\} \cap \text{Crit}(f_{a_\lambda}) = \emptyset$. Then, for any subset $U \subset \Lambda$, we have

$$\mu_{f,a} - \text{a.e } \lambda \in U, \liminf_{n \to \infty} \frac{1}{n} \log (f^n_\lambda)^\#(a(\lambda)) \geq \frac{\log d}{D_{\mu_{f,a}}^\ast(\mu_{f,a})} \geq \frac{1}{2} \log d.$$ 

Using Fubini theorem, we deduce the corollary when $\dim(\Lambda) \geq 1$

**Corollary 2.** Let $f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$ parametrized by a quasi-projective curve $\Lambda$ and let $a : \Lambda \to \mathbb{P}^1$ be a rational function for which there exists $\lambda_0 \in \Lambda$ such that $\{f^n_{a_\lambda}(a(\lambda_0)), n \in \mathbb{N}\} \cap \text{Crit}(f_{a_\lambda}) = \emptyset$. Then, for almost every parameter $\lambda \in \Lambda$ with respect to the trace measure of $T_{f,a}$, we have

$$\liminf_{n \to \infty} \frac{1}{n} \log (f^n_\lambda)^\#(a(\lambda)) \geq \frac{1}{2} \log d.$$ 

A particularly interesting case is when $a = c$ is a marked critical value which is not stably precritical. Then, the corollary means that, for almost every parameter $\lambda$ with respect to the trace measure of $T_{f,c}$, $c(\lambda)$ is Collet–Eckmann. Hence, the large-scale condition of [2] is generic for the trace measure of $T_{f,c}$.

Let us say a few words about the strategy of the proof. First, the proof of Graczyk and Świątek relies deeply on the fact that $\mu_{M_d}$ is the harmonic measure of a fully connected compact
set of the complex plane, on profound results of Makarov on the harmonic measures of such compact sets \[12\] and on the specific combinatorial structure of the unicritical family \([10, 13]\) also use this combinatorial structure). As such, it cannot be used for arbitrary families of rational maps.

Instead, when \(\dim \Lambda = 1\), we construct here many disks in the graph of \(f^n(a)\) (which is an analytic set of dimension 1 in a 2-dimensional space) using classical ideas of the theory of laminar currents \([3, 6]\). We then use those disks to bound the parametric Lyapunov exponent (see Theorem 3). Finally, we use a transversality argument to bound the dynamical Lyapunov exponent.

Nevertheless, for the unicritical family, we do not recover the optimal bound of Graczyk and Świątek since we do not know whether \(D^*(\mu_{M_d}) = 1\) (Makarov theorem tells us that \(D_*(\mu_{M_d}) = 1 \leq D^*(\mu_{M_d})\)). Still, we show that the bound in Theorem 1 is sharp, in general, by considering a constant family of Lattès maps with a moving marked point. Similarly, we construct (constant) families where \(D^*(\mu_{f,a}) < 2\).

2. Bounding the parametric exponent on a set of full measure

In this section, \(\Lambda\) is a smooth quasi-projective curve and \(f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1\) an algebraic family of rational maps of degree \(d \geq 2\) endowed with a marked point \(a : \Lambda \to \mathbb{P}^1\). The reason we need the algebraic hypothesis is so that the volume of the graph of \(\lambda \mapsto f^n(a(\lambda))\) is \(O(d^n)\) (for example, this is not true if one considers the family \(z^2 + e^{\lambda}\) for the marked critical point \(\lambda \to 0\)). Let \(\omega_{\mathbb{P}^1}\) denote the Fubini–Study form on \(\mathbb{P}^1\) and \(\omega_{\Lambda}\) a volume form on \(\Lambda\). Denote by \(\Pi_{\mathbb{P}^1} : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1\) and \(\Pi_{\Lambda} : \Lambda \times \mathbb{P}^1 \to \Lambda\) the canonical projections. Let \(\hat{\omega}_{\mathbb{P}^1} := \Pi_{\mathbb{P}^1}^*(\omega_{\mathbb{P}^1})\) and \(\hat{\omega}_{\Lambda} := \Pi_{\Lambda}^*(\omega_{\Lambda})\). Let \(\mu_{f,a}\) be the bifurcation (or activity) measure of \((f,a)\):

\[
\mu_{f,a} := (\Pi_{\Lambda})_* \left( [\Gamma_a] \wedge \hat{T} \right).
\]

Let \(U \subset \Lambda\). Recall that \(\hat{T} = \lim d^{-n}(f^n)^*(\hat{\omega}_{\mathbb{P}^1}) = \hat{\omega}_{\mathbb{P}^1} + dd^c g\) where \(g\) is a \(\alpha\)-Hölder function on \(U\) (\(\alpha\) a priori depends on \(U\)) and the \(dd^c\) is the complex Laplacian taken in the sense of currents (or distributions). We are interested in the \(\mu_{f,a}\)-a.e value of the parametric Lyapunov exponent defined by

\[
\liminf_{n \to \infty} \frac{1}{n} \log \left\| \frac{d}{d\lambda} (f^n(\lambda, a(\lambda))) \right\|.
\]

Here, the norm is computed with respect to the spherical distance on \(\mathbb{P}^1\), but, as any equivalent metric will give the same result, the exponent can be computed in some finite charts. The purpose of this section is to prove

**Theorem 3.** The parametric Lyapunov exponent satisfies

\[
\mu_{f,a} - \text{a.e.} \lambda \in U, \quad \liminf_{n \to \infty} \frac{1}{n} \log \left\| \frac{d}{d\lambda} (f^n(\lambda, a(\lambda))) \right\| \geq \frac{\log d}{D_U^*(\mu_{f,a})} \geq \frac{\log d}{2}.
\]

The rest of the section is devoted to the proof of the theorem. Observe that it is enough to restrict to the case where \(U\) is a disk relatively compact in \(\Lambda\) such that \(\mu_{f,a}(U) > 0\). To simplify the notations, we write \(D^*\) instead of \(D_U^*(\mu_{f,a})\).

2.1. Constructing disks in \(f^n(\Gamma_a)\)

Let \(\varepsilon > 0\). We fix \(0 < \beta < \log d\).
By definition of $D^*$, we can find $B \subset U$ such that $\mu_{f,a}(U\setminus B) = 0$ and

$$\forall \lambda \in B, \limsup_{r \to 0} \frac{\log \mu_{f,a}(B(\lambda,r))}{\log r} \leq D^*.$$ 

In particular,

$$\forall \lambda \in B, \exists r_0, \forall r \leq r_0, \mu_{f,a}(B(\lambda,r)) \geq r^{D^*+\beta}.$$ 

Let

$$B_\ell := \left\{ \lambda \in B, \forall r < \frac{1}{\ell}, \mu_{f,a}(B(\lambda,r)) \geq r^{D^*+\beta} \right\}.$$ 

In particular, $\cup_\ell B_\ell = B$ and the union is increasing so that we fix $\ell_0$ large enough so that $\mu_{f,a}(B_{\ell_0}) \geq \mu_{f,a}(U) - \varepsilon/2$. Note that $\ell_0$ depends only on $\varepsilon$.

We construct disks of size $e^{-\beta n}$ in $f^n(\Gamma_a)$. We will use classical idea of the theory of laminar currents.

We let $C$ be a finite cover of $\mathbb{P}^1$ given by charts $C$ where $C$ is the unit square in $\mathbb{C}$ centered at 0. We also let $V$ be an open neighborhood of $\overline{U}$ which can be taken to be a square in $\mathbb{C}$ of size 1.

We can assume that $\mu_{f,a}(V) = [f^n(\Gamma_a)] \cap \hat{T}(V \times \mathbb{P}^1) \leq 1$ (up to restricting $U$ and $V$). We subdivide $C$ and $V$ into squares of size $e^{-\beta n}$ which gives us a subdivision of $C \times V$ into (4 dimensional) cubes of size $e^{-\beta n}$, we denote by $\mathcal{P}$ this tiling. Let $0 < \eta < 1$. For $P \in \mathcal{P}$ of center $c(P)$, let $P^n$ be the image of $P$ by the homotopy of ratio $\eta$ and center $c(P)$. Let $\mathcal{P}^0$ denote the union of the $P^n$. For $z \in P$, let $\mathcal{P}_z := z - c(P) + P$ denote the translation of $\mathcal{P}$ by the vector $z - c(P)$. Finally, let $\mathcal{P}_z^0$ denote the union of all the homothetics of elements of $\mathcal{P}_z$. Recall the following result [6, Lemme 4.5].

**Lemma 4.** With the above notations, fix $P \in \mathcal{P}$. Then, there exists $z \in P$ such that

$$\frac{[f^n(\Gamma_a)]}{d^n} \cap \hat{T}(\mathcal{P}_z \setminus \mathcal{P}_z^0) \leq 2(1 - \eta^4).$$

We take

$$\eta = \left(1 - \exp \left(\frac{-\beta n \alpha}{4}\right)\right)^\frac{1}{4}$$

so that $2(1 - \eta^4) \leq 2 \exp(-\beta n \alpha/4)$ (recall that $\alpha$ is the H"older exponent of the quasi-potential $g$ of $\hat{T}$). So we translate $C$ and $V$ by the $z$ given by the above lemma. Since $\mathcal{P}$ does not move much as $\text{diam}(P) \leq \exp(-\beta n)$, this gives us new $C$ and $V$ that we still denote $C$ and $V$ since the collection of the new $C$ still covers $\mathbb{P}^1$ and $U \subset V$ still holds.

We now construct disks in $f^n(\Gamma_a) \cap \Lambda \times C$. Let $\chi$ denote the Euler characteristic. Then, $\chi(f^n(\Gamma_a)) \geq \chi(\Gamma_a) =: \chi_0$ as the Euler characteristic increases by direct image $(f^n(\Gamma_a)$ and $\Gamma_a$ are seen as Riemann surfaces over $\Lambda$). Let $R_n$ denote the number of ramifications of $(\Pi_{\mathbb{P}^1})|_{f^n(\Gamma_a)}$ and let $d_n$ the topological degree of $(\Pi_{\mathbb{P}^1})|_{f^n(\Gamma_a)}$. Then $d_n \leq d^n \times d'$ where $d'$ is the topological degree of $a$. By Riemann–Hurwitz, we have $\chi(f^n(\Gamma_a)) = d_n \chi(\mathbb{P}^1) - R_n$ so $R_n \leq d d^n$ for some constant $c$ that does not depend on $n$ nor $\beta$.

Consider the set of connected components of all the preimages of $(\Pi_{\mathbb{P}^1})|_{f^n(\Gamma_a)}^{-1}(S)$ where $S$ belongs to the above tiling of $C$ into squares of size $e^{-\beta n}$. We call island such a connected component $I$ for which $(\Pi_{\mathbb{P}^1})|_{f^n(\Gamma_a)}$ is a biholomorphism from $I$ to $S$. In particular, the sum of the degrees of the projection $(\Pi_{\mathbb{P}^1})|_{f^n(\Gamma_a)}$ restricted to each connected component which is not an island is $\leq c d^n$. Let us also remove the islands whose area is $\geq \frac{n}{\ell_0}$. As $f^n(\Gamma_a)$ has area $\leq \text{area}(\Gamma_a) d^n$, we have removed at most $\frac{c \ell_0 d^n}{\pi}$ (taking a larger $c$ if necessary). Let us denote
by $\mathcal{I}_n$, the union of all the other islands, which are those we call good disks. Let $B_n := |\mathcal{I}_n|/d^n$, then

**Lemma 5.** With the above notations, there exist $n_1 \in \mathbb{N}$ and a constant $K(\varepsilon)$ such that

$$\forall n \geq n_1, \int_{V \cap C} \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge \hat{T} \leq K(\varepsilon)e^{-\beta_n}.$$

**Proof.** First, observe that there exist $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$

$$\left< \frac{f^n(\Gamma_a)}{d^n} - B_n, \hat{\omega}_B |_C + \hat{\omega}_A |_V \right> \leq \left< \frac{f^n(\Gamma_a)}{d^n} - B_n, \hat{\omega}_B |_C \right> + \left< \frac{f^n(\Gamma_a)}{d^n}, \hat{\omega}_A |_V \right> \leq \frac{c d^n e^{-2\beta n}}{d^n} + \frac{c e^2 e^{-2\beta n} d^n}{\pi d^n} + \frac{1}{d^n} \leq \frac{3 c e^2 e^{-2\beta n}}{\pi}, \quad (1)$$

where we used that $\beta \ll \log d$ and that $f^n(\Gamma_a)$ is a graph (hence the area of the projection on the first coordinate is the area of $V$). We now follow ideas of Dujardin [6]. Take a smooth cut-off function $\Psi$ which is equal to 1 on $P^0$ and near $\partial P$ for every $P \in \mathcal{P}$ and such that there exists a constant $K$ independent of $n$ satisfying

$$\|\Psi\|_{C^2} \leq K \left( \frac{1}{(1 - \eta)e^{-\beta_n}} \right)^2 \leq \frac{K e^{2\beta_n}}{\left( 1 - \left( 1 - e^{-\beta_n} \right)^{1/4} \right)^2} \approx \frac{K e^{2\beta_n}}{\left( 1 - 1 + \frac{1}{4} e^{-\beta_n} \right)^2} \leq 20 K e^{\beta_n(2 + \frac{1}{4})} \quad (2)$$

for $n \geq n_0$. Writing as above $\hat{T} = \hat{\omega}_B + dd^c g$ gives

$$\int_{V \cap C} \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge \hat{T} \leq \int_{P \cap \mathcal{I}_n} \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge \hat{T} + \int \Psi \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge \hat{T} \wedge dd^c g \leq 2 e^{-\beta_n} + \int \Psi \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge \hat{\omega}_B \wedge dd^c g$$

$$\leq 2 e^{-\beta_n} + 3 c e^2 e^{-2\beta n} + \int \Psi \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge dd^c g,$$

where we used Lemma 4 and the bound (1). For the last term, by Stokes ($c(P)$ denotes the center of the cube $P$):

$$\int \Psi \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge dd^c g = \sum_{P \in \mathcal{P}} \int_P \Psi \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge dd^c g$$

$$= \sum_{P \in \mathcal{P}} \int_P \left( g - g(c(P)) \right) \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge dd^c \Psi$$
\[
\leq \sum_{P \in \mathcal{P}} \int_P |g - g(c(P))| \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge 20Ke^{\beta n(2+\frac{4}{v})}(\hat{\omega}_{\mathbb{P}^1}|_C + \hat{\omega}_\Lambda|_V).
\]

Now, \(|g - g(c(P))| \leq ce^{-\beta an}\) since \(g\) is \(\alpha\)-Hölder (we can take the same \(c\) than in (1) up to increasing it) so that

\[
\int \Psi \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge dd^c g \leq ce^{-\beta an}20Ke^{\beta n(2+\frac{4}{v})} \int \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge (\hat{\omega}_{\mathbb{P}^1}|_C + \hat{\omega}_\Lambda|_V)
\]

\[
\leq 20Ke^{2\beta n}e^{-\beta \frac{\alpha}{4}} \frac{3c^2\ell_0^2 e^{-2\beta n}}{\pi}
\]

\[
\leq K'(\varepsilon)e^{-\beta \frac{\alpha}{4}},
\]

where we used (1), \(K'(\varepsilon)\) is a large enough constant and \(n \geq n_0\). Combining all the above gives a rank \(n_1 \geq n_0\) and a constant \(K(\varepsilon)\) such that for \(n \geq n_1\):

\[
\int_{V \times C} \left( \frac{f^n(\Gamma_a)}{d^n} - B_n \right) \wedge \hat{\Delta} \leq 2e^{-\frac{\beta an}{4}} + \frac{3c^2\ell_0^2 e^{-2\beta n}}{\pi} + K'(\varepsilon)e^{-\beta \frac{\alpha}{4}} \leq K(\varepsilon)e^{-\beta \frac{\alpha}{4}}.
\]

Taking a finite cover of \(\mathbb{P}^1\), we have the above estimate on \(V \times \mathbb{P}^1\).

### 2.2. Using the above disks to bound the Lyapunov exponent

We first show that we can find an arbitrary large set in \(U\) of parameters \(\lambda\) for which the pointwise dimension of \(\mu_{f,a}\) is controlled by \(D^\ast\) and for which all the corresponding points \((\lambda, f^n(\Lambda(\lambda)))\) belong to a good disk constructed above. Then, using Koebe’s distortion theorem, we bound from below the parametric Lyapunov exponents.

**Lemma 6.** With the above notations, there exist a set \(W \subset U\), integers \(n_2 \in \mathbb{N}\) and \(\ell_0 \in \mathbb{N}\) such that

- \(\mu_{f,a}(U \setminus W) \leq \varepsilon\),
- \(\forall \lambda \in W, \forall n \geq n_2, f^n(\lambda, z)\) belongs to a good disk \(D \subset f^n(\Gamma_a)\) and \(f^n(\lambda, z) \notin \mathcal{P}\setminus\mathcal{P}'\),
- \(\forall \lambda \in W, \forall r < \frac{1}{\ell_0}, \mu_{f,a}(B(\lambda, r)) \geq r^{D^\ast + \beta}\).

**Proof.** Take \(n \geq n_2 \geq n_1\) such that

\[
\sum_{C \in \mathcal{C}} \sum_{n \geq n_2} (2 + K(\varepsilon))e^{-\frac{\alpha n}{4}} \leq \frac{\varepsilon}{2},
\]

where \(\mathcal{C}\) denotes the finite cover of \(\mathbb{P}^1\) defined in the previous section and \(K(\varepsilon)\) is the constant given by Lemma 5 (we can take the same constant \(K(\varepsilon)\) for every \(C\)). Denote

\[
\hat{A} := \{(\lambda, z) \in (U \times \mathbb{P}^1) \cap \Gamma_a, \forall n \geq n_2, f^n(\lambda, z)\) belongs to a good disk \(D \subset f^n(\Gamma_a)\) and \(f^n(\lambda, z) \notin \mathcal{P}\setminus\mathcal{P}'\}\}
\]

and

\[
\hat{A}_n := \{(\lambda, z) \in (U \times \mathbb{P}^1) \cap \Gamma_a, f^n(\lambda, z)\) belongs to a good disk \(D \subset f^n(\Gamma_a)\) and \(f^n(\lambda, z) \notin \mathcal{P}\setminus\mathcal{P}'\}\}
\]
so that \( \mathring{A} = \cap_{n \geq n_2} \mathring{A}_n \). Then, using \( f^* \mathring{T} = d\mathring{T} \) and Lemmas 4 and 5:

\[
\begin{align*}
[\Gamma_a] \land \mathring{T} (\mathring{A} \cap \Pi_{\Lambda}^{-1} U) &= [\Gamma_a] \land \mathring{T} \left( \left( \cup_{n \geq n_2} \mathring{A}_n \right) \cap \Pi_{\Lambda}^{-1} U \right) \\
&\leq \sum_{n \geq n_2} \frac{[f^n(\Gamma_a)]}{d^n} \land \mathring{T} \left( f^n(\mathring{A}_n) \cap \Pi_{\Lambda}^{-1} U \right) \\
&\leq \sum_{C \in C} \sum_{n \geq n_2} \frac{[f^n(\Gamma_a)]}{d^n} \land \mathring{T} \left( f^n(\mathring{A}_n) \cap (U \times C) \right) \\
&\leq \sum_{C \in C} \sum_{n \geq n_2} \frac{[f^n(\Gamma_a)]}{d^n} \land \mathring{T} (\mathcal{P} \setminus \mathcal{P}^n) + K(\varepsilon) e^{-\beta a n/4} \\
&\leq \sum_{C \in C} \sum_{n \geq n_2} 2e^{-\beta a n/4} + K(\varepsilon) e^{-\beta a n/4} \leq \frac{\varepsilon}{2}.
\end{align*}
\]

Hence \( \mathring{T} \land [\Gamma_a](\mathring{A}) \geq \mu_{f,a}(U) - \varepsilon/2 \). Now, recall that

\[
B_{\ell_0} = \left\{ \lambda \in B, \forall r < \frac{1}{\ell_0}, \mu_{f,a}(B(\lambda, r)) \geq r^{D_\beta+\beta} \right\}
\]

satisfies \( \mu_{f,a}(B_{\ell_0}) \geq \mu_{f,a}(U) - \varepsilon/2 \). Then, the set \( W := B_{\ell_0} \cap \Pi_{\Lambda}(\mathring{A}) \) satisfies

\[
\mu_{f,a}(W) \geq \mu_{f,a}(U) - \varepsilon
\]

since \( \mu_{f,a}(\Pi_{\Lambda}(\mathring{A})) = \mathring{T} \land [\Gamma_a](\Pi_{\Lambda}^{-1} \Pi_{\Lambda}(\mathring{A})) \geq \mathring{T} \land [\Gamma_a](\mathring{A}) \geq \mu_{f,a}(U) - \varepsilon/2 \). This proves the lemma. \( \square \)

Let \( W \) be given by the above lemma and pick \( \lambda \in W \). Let \( n \geq n_2 \), by definition, there exists a good disk \( D \subset f^n(\Gamma_a) \) above a square \( S \) of size \( e^{-\beta a n} \) in the chart \( C \) such that \( f^n(\lambda, a(\lambda)) \in D \). As \( f^n(\lambda, a(\lambda)) \notin \mathcal{P} \setminus \mathcal{P}^n \), then \( \Pi_{\mathcal{P}^1}(f^n(\lambda, a(\lambda))) \in S^0 \) (the homothetic of \( S \) of ratio \( \eta \) with respect to its center). Define

\[
\eta' := \frac{1 + \eta}{2}.
\]

Let \( \Delta := \Pi_{\mathcal{P}^1}^{-1}(S^0') \cap D \) and let \( \Delta_n \subset \Gamma_a \) be the preimage of \( \Delta \) by \( f^n \) (\( f^n \) is injective on \( \Gamma_a \)).

**Lemma 7.** With the above notations, there exists an integer \( n_3 \geq n_2 \) such that

\[
\forall n \geq n_3, \int \left( 1_{B(\lambda, 1/\ell_0)} \circ \Pi_{\Lambda} \right) \mathring{T} \land [\Delta_n] \leq \frac{200Ke^{\beta a n} e^{2\beta n}}{d^n}.
\]

**Proof.** Let \( \psi \) be a smooth cut-off function on \( \mathbb{P}^1 \) which is equal to 1 on \( S^n' \) and 0 near \( \partial S \) and \( \varphi \) be a smooth cut-off function on \( \Lambda \) which is equal to 1 on \( B(\lambda, 1/\ell_0) \) and 0 near \( \partial B(\lambda, 1/\ell_0) \) so that \( (K \) is a universal constant)

- \( d\psi \land d^c \psi \leq K((1 - \eta')e^{-\beta n})^{-2}\omega_{\mathbb{P}^1} \) and \( d\varphi \land d^c \varphi \leq K((1 - \eta')e^{-\beta n})^{-2}\omega_{\mathbb{P}^1} \leq 0 \).
- \( d\varphi \land d^c \varphi \leq K(\ell_0)^{-2}\omega_{\Lambda} \) and \( d\varphi \land d^c \varphi \leq K\ell_0^{-2}\omega_{\Lambda} \leq 0 \).


Then, by Stokes
\[
\int_{\Delta \cap \Pi_{\lambda}^{-1}(B(\lambda, \frac{1}{2T}))} \hat{T} \leq \int \Pi_{\varphi_1}^*(\psi)\Pi_{\lambda}^*(\varphi)[D] \wedge \hat{T} \\
\leq \int \Pi_{\varphi_1}^*(\psi)\Pi_{\lambda}^*(\varphi)[D] \wedge \hat{\omega}_{\varphi_1} + \int \Pi_{\varphi_1}^*(\psi)\Pi_{\lambda}^*(\varphi)[D] \wedge dd^c g \\
\leq \int \Pi_{\varphi_1}^*(\psi)[D] \wedge \hat{\omega}_{\varphi_1} + \int g[D] \wedge dd^c (\Pi_{\varphi_1}^*(\psi)\Pi_{\lambda}^*(\varphi)) \\
\leq 1 + \int g[D] \wedge dd^c (\Pi_{\varphi_1}^*(\psi)\Pi_{\lambda}^*(\varphi)).
\]

Now,
\[
dd^c (\Pi_{\varphi_1}^*(\psi)\Pi_{\lambda}^*(\varphi)) = \Pi_{\lambda}^*(\varphi)dd^c (\Pi_{\varphi_1}^*(\psi)) + \Pi_{\varphi_1}^*(\psi)dd^c (\Pi_{\lambda}^*(\varphi)) + \Pi_{\lambda}^*(\varphi)\Pi_{\varphi_1}^*(d\psi \wedge d^c \varphi) \\
+ \Pi_{\varphi_1}^*(\psi)\Pi_{\lambda}^*(d\varphi \wedge d^c \varphi).
\]

So we have the bound
\[
\int_{\Delta \cap \Pi_{\lambda}^{-1}(B(\lambda, \frac{1}{2T}))} \hat{T} \leq 1 + 2\|g\|_{\infty} \int_{\Pi_{\lambda}^{-1}(B(\lambda, \frac{1}{2T}))} [D] \wedge K((1 - \eta')e^{-\beta n})^{-2}\hat{\omega}_{\varphi_1} \\
+ 2\|g\|_{\infty} \int_{\Pi_{\lambda}^{-1}(B(\lambda, \frac{1}{2T}))} [D] \wedge K\ell_{\omega}^{-2}\hat{\omega}_{\lambda} \\
\leq 1 + 2\|g\|_{\infty} K(4((1 - \eta')e^{-\beta n})^{-2} + 4) \\
\leq 200Ke^{\frac{3\alpha}{2}}e^{2\beta n},
\]

where we used the computations in (2) and assume \( n \geq n_3 \geq n_2 \). In particular, using that \( f^n(\Delta_n) = \Delta \) and the fact that \( f^n \) is injective on \( \Gamma_\alpha \) gives:
\[
\int (1_{B(\lambda, \frac{1}{2T})} \circ \Pi_{\lambda}) \hat{T} \wedge [\Delta] = \frac{1}{d^n} \int (1_{B(\lambda, \frac{1}{2T})} \circ \Pi_{\lambda}) \hat{T} \wedge [\Delta] \leq \frac{200Ke^{\frac{3\alpha}{2}}e^{2\beta n}}{d^n}.
\]

**Lemma 8.** For \( \lambda \in W \), we have that
\[
\liminf_{n \to \infty} \frac{1}{n} \log \left\| \frac{d}{d\lambda} (f^n(\lambda, a(\lambda))) \right\| \geq \frac{\log d - \frac{\beta_0}{2} - 2\beta}{D^* + \beta} - \frac{\beta \alpha}{4} - \beta.
\]

**Proof.** Let \( r(\lambda) \) be the largest \( r > 0 \) such that \( B(\lambda, r) \subset \Pi_{\lambda}(\Delta_n) \). Recall that the area of the good disks we consider is \( \leq \pi/\ell_{\varphi}^2 \) so that \( r(\lambda) \leq \frac{r}{4} \).

Since \( \Pi_{\varphi_1}(f^n(\lambda, a(\lambda))) \in S'^\prime \), there exists a disk \( D_0 \) of radius \( (\eta' - \eta)e^{-\beta n} \) centered at \( \Pi_{\varphi_1}(f^n(\lambda, a(\lambda))) \) and contained in \( S'^\prime \). The holomorphic map
\[
h : \lambda' \mapsto \Pi_{\varphi_1}(f^n(\lambda', a(\lambda')))
\]
is injective on \( \Pi_{\lambda}(\Delta_n) \) \((\lambda' \mapsto (\lambda', a(\lambda'))\) is injective, \( f^n \) is injective on \( \Gamma_\alpha \) and \( \Pi_{\varphi_1} \) is injective on \( D \) since \( D \) is a graph). Koebes \( \frac{1}{4} \) Theorem implies that \( h^{-1}(D_0) \) contains a disks of center \( \lambda \) and radius
\[
\frac{|(h^{-1})'(\Pi_{\varphi_1}(f^n(\lambda, a(\lambda))))|}{4} (\eta' - \eta)e^{-\beta n} \geq \frac{|(h^{-1})'(\Pi_{\varphi_1}(f^n(\lambda, a(\lambda))))|e^{-\beta n}}{32}. 
\]
By definition of \( r(\lambda) \), we have

\[
\frac{e^{-\frac{\beta \alpha n}{4}} e^{-\beta n}}{32 |h'(\lambda)|} \geq r(\lambda) \geq e^{-\frac{\beta \alpha n}{4}} e^{-\beta n}.
\]

So, by Lemma 7 and the definition of \( W \):

\[
e^{-\frac{\beta \alpha n}{4}} e^{-\beta n} \leq r(\lambda) \leq (\mu_{f, a}(B(\lambda, r(\lambda))))^{\frac{1}{n+\beta}} \leq \left( \frac{200K e^{\frac{\beta \alpha n}{2}} e^{2\beta n}}{d^n} \right)^{\frac{1}{n+\beta}}.
\]

In other words,

\[
e^{-\frac{\beta \alpha n}{4}} e^{-\beta n} \left( \frac{d^n}{200K e^{\frac{\beta \alpha n}{2}} e^{2\beta n}} \right)^{\frac{1}{n+\beta}} \leq |h'(\lambda)|.
\]

(3)

By the chain rule,

\[
|h'(\lambda)| = \left\| D\Pi_{\mathcal{P}_1} (f^n(\lambda, a(\lambda))) \circ \frac{d}{d\lambda}(f^n(\lambda, a(\lambda))) \right\| \leq \left\| \frac{d}{d\lambda}(f^n(\lambda, a(\lambda))) \right\|
\]

since projection are 1-Lipschitz. Then, taking the logarithm in (3), dividing by \( n \) and letting \( n \to \infty \) gives

\[
\liminf_{n \to \infty} \frac{1}{n} \log \left( \left\| \frac{d}{d\lambda}(f^n(\lambda, a(\lambda))) \right\| \right) \geq \frac{\log d - \frac{\beta \alpha}{2} - 2\beta}{D^* + \beta} - \frac{\beta \alpha}{4} - \beta,
\]

as required. \( \square \)

Now, the proof of Theorem 3 is complete by taking \( \beta \to 0 \) and \( \varepsilon \to 0 \) in Lemmas 6 and 8.

REMARK. As observed by the referee, using the technics of this section with the introduction of the upper packing dimension allows to improve the results of [4, Section 4] and [7, Theorem 5.1] in dimension 2. Indeed, in both cases, one can improve the inequalities \( \gg \log d/2 \) by \( \log d/D^* \) where \( D^* \) is the upper packing dimension of \( T \wedge [D] \) where \( T \) is the Green current of the endomorphism and \( D \) an appropriate disk.

3. The proof of the Main Theorem

3.1. Comparing parameter and dynamical growth

Here, we prove the following, relying on ideas of [1, 2].

PROPOSITION 9. Let \( f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1 \) be an analytic family of degree \( d \) rational maps. Assume that, for some parameter \( \lambda_0 \), there exists \( \delta > 0 \) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \log \left( \left\| \frac{d}{d\lambda}(f^n(\lambda, a(\lambda)))_{\lambda = \lambda_0} \right\| \right) \geq \delta > 0.
\]

Assume in addition that \( f^k_{\lambda_0} (a(\lambda_0)) \notin \text{Crit}(f_{\lambda_0}) \) for all \( k \geq 0 \). Then we have

\[
\liminf_{n \to \infty} \frac{1}{n} \log (f^n_{\lambda_0})^\# (a(\lambda_0)) \geq \delta.
\]
For the sake of simplicity, we set \( a_n(\lambda) := f^n(\lambda) \) so that \( a_0(\lambda) = a(\lambda) \). We also let \( \bar{f} := \partial_\lambda f(\lambda, \cdot)|_{\lambda = \lambda_0} \) and \( \bar{a} := \partial_\lambda a(\lambda_0) \). The following is [2, Lemma 4.4].

**LEMMA 10.** Pick any parameter \( \lambda_0 \) and any integer \( n \geq 1 \). As soon as we have that \( f_{\lambda_0}^k(\lambda_0) \neq 0 \) for all \( 0 \leq k \leq n \), the following holds:

\[
\partial_\lambda a_n(\lambda_0) = (f_{\lambda_0}^n)'(\lambda_0) \cdot \left( \partial_\lambda a(\lambda_0) + \sum_{k=0}^{n-1} \frac{\bar{f}(a_k(\lambda_0))}{(f_{\lambda_0}^{k+1})'(\lambda_0))} \right).
\]

**Proof of Proposition 9.** Since the coordinate on \( \Lambda \) is a local coordinate and the metric on \( \mathbb{P}^1 \) is the one induced by the spherical distance:

\[
\| \partial_\lambda a_n(\lambda_0) \| \leq \frac{|\partial_\lambda a_n(\lambda_0)|}{1 + |a_n(\lambda_0)|^2} \| f_{\lambda_0}^n(a(\lambda_0)) |(1 + |a(\lambda_0)|^2) \|
\]

\[
\times \left( \frac{|\partial_\lambda a(\lambda_0)|}{1 + |a(\lambda_0)|^2} + \sum_{k=0}^{n-1} \frac{|\bar{f}(a_k(\lambda_0))| |(1 + |a_k+1(\lambda_0)|^2)|}{(f_{\lambda_0}^{k+1})'(\lambda_0)) (1 + |a_k+1(\lambda_0)|^2)(1 + |a(\lambda_0)|^2)} \right).
\]

Hence

\[
\| \partial_\lambda a_n(\lambda_0) \| \leq (f_{\lambda_0}^n)'(\lambda_0)) \left( \| \partial_\lambda a(\lambda_0) \| + \sum_{k=0}^{n-1} \frac{|\bar{f}(a_k(\lambda_0))|}{(f_{\lambda_0}^{k+1})'(\lambda_0)) \right).
\]

We first prove \( \gamma := \liminf_{n \to \infty} \frac{1}{n} \log(f_{\lambda_0}^n)'(\lambda_0)) > -\infty \) by contradiction. If not, take \( M \gg 1 \) and let \( n_0 \) be the first integer such that \( (f_{\lambda_0}^{n_0})'(\lambda_0)) \leq e^{-n_0M} \). Taking \( M \) larger will only increase \( n_0 \) so, by hypothesis, we can assume \( \| \partial_\lambda a_{n_0}(\lambda_0) \| \geq \exp(n_0\delta/2) \). Then, (4) gives:

\[
e^{n_0\delta/2} \leq (f_{\lambda_0}^{n_0})'(\lambda_0)) \leq C_1 \left( (f_{\lambda_0}^{n_0})'(\lambda_0)) \cdot \left( 1 + \sum_{k=0}^{n_0-1} \frac{1}{(f_{\lambda_0}^{k+1})'(\lambda_0))} \right)
\]

\[
\leq C_1 e^{-n_0M} \left( 1 + \sum_{k=1}^{n_0} e^{kM} \right) \leq \frac{2C_1 e^M}{e^M - 1},
\]

which is impossible, so \( \gamma > -\infty \).

We now prove similarly that \( \gamma > 0 \). Assume by contradiction that \( \gamma \leq 0 \) and fix \( 0 < \epsilon < \delta/3 \) and let \( n_0 \geq 1 \) be such that \( \frac{1}{n} \log(f_{\lambda_0}^n)'(\lambda_0)) \geq \gamma - \epsilon \) for all \( n \geq n_0 \). Set

\[
C_2 := \max \left\{ 1, \max_{k \leq n_0} \frac{1}{(f_{\lambda_0}^{k+1})'(\lambda_0))} \right\} < +\infty.
\]
Taking $n_1 \geq n_0$ large enough, we can assume that for all $n \geq n_1$,

(1) $\frac{1}{n} \log \| \partial_\lambda a_n(\lambda_0) \| \geq \delta - \varepsilon$, and

(2) $\frac{1}{n} \log (3nC_1C_2/(\exp(\gamma + \varepsilon) - 1)) \leq \varepsilon$.

We apply again (4): for all $n \geq n_1$, we have

$$\| \partial_\lambda a_n(\lambda_0) \| \leq C_1 \cdot (f^{\#}_{\lambda_0})(a(\lambda_0)) \cdot \left( 1 + n_0C_2 + \sum_{k=n_0+1}^{n-1} \exp((k+1)(\gamma + \varepsilon)) \right)$$

$$\leq (f^{\#}_{\lambda_0})(a(\lambda_0)) \cdot (3nC_1C_2) \cdot \left( \frac{\exp((n+1)(\gamma + \varepsilon))}{\exp(-\gamma + \varepsilon) - 1} \right).$$

By the choice of $n_1$, for all $n \geq n_1$ this gives

$$\delta - \varepsilon \leq \frac{1}{n} \log (f^{\#}_{\lambda_0})(a(\lambda_0)) + \frac{1}{n} \log \left( \frac{3nC_1C_2}{\exp(-\gamma + \varepsilon) - 1} \right) + \frac{n+1}{n}(-\gamma + \varepsilon)$$

$$\leq \frac{1}{n} \log (f^{\#}_{\lambda_0})(a(\lambda_0)) + \frac{n+1}{n}(-\gamma + 2\varepsilon).$$

Taking the $\liminf$ as $n \to \infty$ yields $\delta - \varepsilon \leq -\gamma + 2\varepsilon$, whence $\delta \leq 3\varepsilon$. This is a contradiction. We thus have proved that $\gamma > 0$.

To conclude, we have to prove $\gamma \geq \delta$. Using again (4), we have

$$\varepsilon_n := \frac{1}{n} \log \| \partial_\lambda a_n(\lambda_0) \| - \frac{1}{n} \log (f^{\#}_{\lambda_0})(a(\lambda_0)) \leq \frac{1}{n} \log C_1 \left[ 1 + \sum_{k=0}^{n-1} \frac{1}{(f^{\#}_{\lambda_0})(a(\lambda_0))} \right].$$

Now, $\limsup_n \varepsilon_n \leq 0$ since, as $\gamma > 0$, the series $\sum_{k=0}^{+\infty} \frac{1}{(f^{\#}_{\lambda_0})(a(\lambda_0))}$ is absolutely convergent. □

3.2. **Proof of the Main Theorem**

The case when $\dim \Lambda = 1$ is just the combination of Theorem 3 and Proposition 9.

We now assume $\dim \Lambda > 1$. Let $\iota : \Lambda \to \mathbb{P}^N$ be an embedding of $\Lambda$ into a complex projective space, let $k := \dim \Lambda < N$ and let $X$ be the intersection of the closure $\Lambda$ be the closure of $\Lambda$ in $\mathbb{P}^N$ with the hyperplane at infinity $H_\infty := \{Z_N = 0\}$ in a given system of homogeneous coordinates $[Z_0 : \cdots : Z_N]$ on $\mathbb{P}^N$. Let $Y$ be a linear subspace of $H_\infty$ of dimension $N - k$ so that $Y \cap X$ is a finite subspace and let $W$ be the collection of all linear subspaces of $\mathbb{P}^N$ of dimension $N - k + 1$ which intersect $H_\infty$ along $Y$. For any $W \in W$, let

$$\Lambda_W := \Lambda \cap W.$$ The variety $\Lambda_W$ is a quasi-projective curve. Let $f_W$ be the restriction of the family $f$ to a family parametrized by $\Lambda_W$ and let $\mu_W$ be the slice of $T_{f,a}$ along $\Lambda_W$, that is, $\mu_W = T_{f,a} \wedge [\Lambda_W]$. According to Theorem 3, for any $W$, and for $\mu_W$-almost every $\lambda \in W$, we have

$$\liminf_{n \to \infty} \frac{1}{n} \log \left\| \frac{d}{d\lambda}(f^n(\lambda, a(\lambda))) \right\| \geq \frac{\log d}{2}.$$ By hypothesis, the set of parameters $\lambda \in \Lambda$ such that there exists $k \geq 0$ with $f^k_\lambda(a(\lambda)) \in \text{Crit}(f_\lambda)$ is a pluripolar subset of $\Lambda$. In particular, for Lebesgue almost every $W$, it intersects $W$ along a pluripolar set. As $\mu_W$ has continuous potentials, it does not give mass to pluripolar sets and Proposition 9 implies that

$$\liminf_{n \to \infty} \frac{1}{n} \log (f^{\#}_\lambda)(a(\lambda)) \geq \frac{\log d}{2},$$

for $\mu_W$-almost every $\lambda \in W$, and for almost every $W$. The conclusion follows by Fubini Theorem.
3.3. Sharpness of the bound and examples

We now prove that the bound from below of the Main Theorem is sharp in the following simple situation. Take a constant family
\[ f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \]
where \( f_0 \) is a Lattès map of degree \( d \) and take \( a : \Lambda \to \Lambda \) be the marked point defined by \( a(\lambda) = \lambda \). Then, one has \( \mu_{\text{bif}, a} = \mu_{f_0} \) where \( \mu_{f_0} \) is the maximal entropy measure of \( f_0 \). It is well known that \( \mu_{f_0} \) is absolutely continuous with respect to the Lebesgue measure (so \( D^*_{\Lambda} = 2 \) for any non empty open set \( U \subset \Lambda = \mathbb{P}^1 \)) and its Lyapunov exponent is \( \log d/2 \) [14]. This means, in particular, that Theorem 1 is sharp here, since for \( \mu_{f_0} \) for everywhere \( \lambda \) in \( U \)
\[ \limsup_{n} \frac{1}{n} \log(f^n_\lambda) \#(a(\lambda)) \leq \frac{1}{D^*_{\Lambda}} \log d. \]

In the same spirit, consider the family
\[ f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{C} \times \mathbb{P}^1 \]
and take \( a : \Lambda \to \Lambda \) be the marked point defined by \( a(\lambda) = \lambda \). Then \( \mu_{\text{bif}, a} = \lambda_{\mathbb{S}^1} \) is the Lebesgue measure on the unit disk so that \( D^*_{\mathbb{C}}(\lambda_{\mathbb{S}^1}) = 1 < 2 \).

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