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Group Classification of Generalised Eikonal Equations

Abstract

A new approach to the problem of group classification is applied to the class of first-order non-linear equations of the form \( u_a u_a = F(t, u, u_t) \). It allowed complete solution of the group classification problem for a class of equations for functions depending on multiple independent variables, where highest derivatives enter non-linearly. Equivalence groups of the class under consideration and algebraic properties of the symmetry algebra are studied.

The class of equations considered presents generalisation of the eikonal and Hamilton-Jacobi equations. The paper contains the list of all non-equivalent equations from this class with symmetry extensions, and proofs of such non-equivalence.

New first order non-linear equations possessing wide symmetry groups were constructed.

1. Introduction. The problem of group classification of differential equations is determination of all particular equations from the class possessing wider symmetry when the class in general. Mathematical models, and in particular, differential equations of mathematical physics, often contain parameters (numerical or functional), that are determined experimentally. For this reason the parameters are not fixed exactly. Theoretical selection of mathematical models can be done on the basis of the symmetry principle, that is to select among given values of parameters ones that lead to a mathematical model with given properties, or with the widest possible symmetry group [1].

We considered a class of mathematical models given by a rather general set of non-linear first-order equations that are invariant under the Euclid group – the group of space rotations and translations. We did not fix any symmetry with respect to time and dependent variables. The set of mathematical models considered incorporates models with relativistic and Galilei symmetries that are well-known and widely used in physics, geometric optics and other applications.

The basics for the group classification were constructed by S. Lie himself [2] in his classification of integrable ordinary differential equations.
The start of systematic study of similar problems for partial differential equations is linked to papers by L.Ovsyannikov (see [1] and references wherein) and by his collaborators. In particular, Ovsyannikov introduced such important concepts as arbitrary element, kernel of principal groups, equivalence group and a number of others.

At present there are quite a few papers on group classification of important classes of models in mechanics, physics, biology and other sciences (see e.g. [3]–[18], and these present only a small part of existing papers!).

Until recently the principal method for solving of group classification problems for partial differential equations was direct integration of determining equations for a Lie symmetry operator with sorting out all possible cases of integration. Application only of the above method substantially narrowed the scope of solvable group classification problems, as determining equations may be extremely difficult to solve, if it is possible at all. Most of the group classification problems solved in the literature are problems for classes of equations with arbitrary functions on only one variable each. Arbitrary functions with multiple variables result in determining equations being partial differential equations, rather than ordinary. It is obvious that this complicates integration considerably.

Recently new approaches appeared having allowed solution of group classification problems that could not be solved by means of the methods known before. Let us mention in more detail a few papers whose ideas are the closest to this paper.

I.S. Akhatov, R.K. Gazizov and N.K. Ibragimov [15], developing the approach by L.Ovsyannikov, systematically employed the equivalence group for simplification of classifying conditions.

R. Zhdanov and V. Lahno [16] suggested a new approach to group classification of PDEs that presents a generalisation of S. Lie’s approach to group classification of ODEs [2] (see also [17]). This approach is based on classification of low-dimensional abstract algebras and works best for equations with two independent variables. It allows avoiding direct solving of cumbersome determining equations and finding complete solutions for nonlinear group classification problems for parameter functions depending on many variables. Determining equations to be solved are partial differential equations with respect to the parameter functions. Further development for this approach presented in this paper is applied for equations with arbitrary number of independent variables. R. Popovych and R. Cherniha [18] employed for classification of systems of coupled non-linear Laplace equa-
tions a combined method in which the number of cases that are considered during integration of determining equations, is essentially decreased at the expense of the study of possible structure of the symmetry algebra. There are a number of other papers suggesting new approaches to group classification problems.

In the present paper the combined method is used for group classification of generalised eikonal and Hamilton-Jacobi equations. The results of the paper show that this method that includes investigation of determining equations, of equivalence transformation groups and of the symmetry algebra structures is an efficient tool for finding equations from a certain class with symmetry extensions. However, the method is not strictly algorithmic, and the steps to be used would be different for different classes of equations.

We succeeded to give a full description of subclasses for the class considered where symmetry extension arises compared to the general symmetry group for the class in total. We expect that the method suggested would allow solving other group classification problems, especially for higher order equations and for systems of PDEs, and extending the variety of non-linear equations with wide symmetry that could be used for construction of mathematical models in physics and other natural sciences.

2. Formulation of the Problem. Let us consider the equation of the form

\[ u_a u_a = F(t, u, u_t) \]  \hspace{1cm} (1)

for a real function \( u = u(t, x) \) of \( n + 1 \) independent variables \( t = x_0 \) and \( x = (x_1, x_2, \ldots, x_n), n \geq 2 \). Here and below a lower index of a function will designate differentiation with respect to the corresponding variable. The indices \( a \) and \( b \) take values from 1 to \( n \). The summation over repeated indices is implied unless stated otherwise. The class of equations (1) is the general class of non-linear first-order equations invariant under the Euclid group – the group of space rotations and translations. It is remarkable also as it includes all the following well-known equations:

\[ F = 2mu_t \] is the Hamilton-Jacobi equation for a free particle;
\[ F = u_t^2 - 1 \] is the relativistic Hamilton equation;
\[ F = u_t^2 \] is the eikonal equation.

The symmetry properties of these equations and relations between them were investigated in \cite{19, 20, 21}. Equations of the form \( u_t^2 = f(u)u_a u_a \) that
are equivalent to the eikonal equation, were used in [22] for studying of solutions of non-linear wave equations.

We perform group classification for equations of the form (1) by an arbitrary element — a smooth function $F = F(t, u, u_t) \neq 0$. (Here and below smoothness means continuous differentiability.)

3. Remarks on Formulation of Classification Problems and Schemes for their Solving. We would like to explain our choice of the class we consider for group classification. This choice was governed by the following rules:

1. A class has to generalise physically interesting equations.

2. A problem has to be symmetric by itself: a class under consideration should have a wide equivalence group.

3. A class has to preserve at least some of important symmetry properties of equations being generalized.

The second rule was the basis of the approach suggested in [16, 17]. Both second and third rules were the reason why we included dependence on the variable $t$ for the arbitrary function $F$. That preserved the possibility of symmetries interchanging $t$ and $u$ that the listed well-known equations have. Introduction of $t$ substantially increases the equivalence group, and despite most resulting classes with symmetry extensions could be represented by $F = F(u, u_t)$, or $F$ does not depend on $t$, the equivalence transformations employed in the classification process belonged to the wide equivalence group corresponding to the class with $F = F(t, u, u_t)$.

The general outline of our classification could be presented as follows.

1. Find determining equations and the kernel of main groups.

2. Find the equivalence group.

3. Integrate a part of the system of determining equations and obtain a possible form for invariance algebra operators.

4. Classify possible symmetry algebras that are lowest-order extensions of the main algebra and that possess a certain structure identified as a separate case at the step 3. Distinguish the corresponding non-equivalent cases.

5. Impose conditions for the invariance algebras having larger dimension or different structure than those considered at the step 4, find new cases. The requirement of a different structure than was considered before allowed to study remaining cases by means of the standard approach.
4. Kernel of Main Groups and Equivalence Group. Let an infinitesimal operator
\[ Q = \xi^0(t, x, u)\partial_t + \xi^a(t, x, u)\partial_a + \eta(t, x, u)\partial_u \]
generate a one-parameter symmetry group of local transformation for the equation (1). Then from the infinitesimal invariance criterion \([1, 23]\) after transition to the manifold given by the equation (1) in the prolonged space, and splitting by non-related variables we derive the following determining equations for the coefficients of the operator \(Q\):

\[ \xi^b_a + \xi^a_b = 0, \quad \xi^a_a = \xi^b_b, \quad a \neq b \]  
(2)

(here there is no summation by \(a\) and \(b\)),

\[ (\xi^a_t + \xi^a_u u_t)F_{u_t} - 2\xi^a_u F + 2\eta_a - 2u_t\xi^0_a = 0, \]  
(3)

\[ \xi^0 F_t + \eta F_u + (\eta_t + (\eta_u - \xi^0_t)u_t - \xi^0_u u_t^2)F_{u_t} = 2(\eta_u - \xi^1_u - \xi^0_u u_t)F. \]  
(4)

If the function \(F\) is not fixed, then splitting in (3) and (4) by the “variables” \(F, F_t, F_u, F_{u_t}, u_t\), we obtain that \(\eta = \xi^0 = 0, \xi^1_t = 0, \xi^a = \xi^a_u = 0\), whence taking into account (2) we come to the following statement.

Statement. The kernel of the main groups for the equation (1) is the Euclid group \(E(n)\), the algebra of which is \(A^\ker = e(n) = \langle \partial_a, J_{ab} = x_a\partial_b - x_b\partial_a \rangle\).

The equivalence group for the equation (1) coincides with the group generated by the set of one-parameter groups of local symmetries of the system

\[ u_a u_a = F, \quad w = u_t, \quad F_a = 0, \]  
(5)

whose infinitesimal operators have the form

\[ \hat{Q} = \xi^0(t, x, u)\partial_t + \xi^a(t, x, u)\partial_a + \eta(t, x, u)\partial_u + \theta(t, x, w)\partial_w + \chi(t, x, w, F)\partial_F. \]

From the infinitesimal invariance criterion for the system (5) after splitting with respect to non-related variables we obtain determining equations for coefficients of the operator \(\hat{Q}\), from which it follows that an infinitesimal operator of any one-parameter equivalence group for the equation (1) is a linear combination of operators

\[ \partial_a, \quad J_{ab}, \quad x_a\partial_a - 2F\partial_F, \quad \xi\partial_t + \eta\partial_u + 2(\eta_u - \xi_u u_t)F\partial_F, \]  
(6)
where \( \xi \) and \( \eta \) are arbitrary smooth functions of variables \( t \) and \( u \). Thus, equivalence transformations, that act non-trivially on the parameter \( F \), have the form
\[
\tilde{t} = \zeta(t, u), \quad \tilde{u} = \varphi(t, u), \quad \tilde{x} = \delta x,
\]
\[
\tilde{F} = \delta^{-2}(\varphi_u - \zeta_u \tilde{u}_t)^2 F = \delta^{-2} \left( \frac{\zeta t \varphi_u - \zeta_u \varphi_t}{\zeta_t + \zeta_u u_t} \right)^2 F,
\]
(7)
where \( \delta \) is a non-zero constant, \( \zeta \) and \( \varphi \) are arbitrary smooth functions of variables \( t \) and \( u \), for which \( \zeta_t \varphi_u - \zeta_u \varphi_t \neq 0 \).

If we limit the class of equations (1), having imposed an additional condition \( F_t = 0 \) (this subclass is separated in the process of classification), then for calculation of the corresponding equivalence group it is necessary to add this condition to the system (5). As a result, the equivalence group is narrowed: an infinitesimal operator of any one-parameter equivalence group for the equation (1) with \( F = F(u, u_t) \) is a linear combination of the operator \( t \partial_t \) and of operators (6), where the functions \( \xi \) and \( \eta \) now depend only on the variable \( u \), and therefore equivalence transformations that act non-trivially on the parameter \( F \) have the form
\[
\tilde{t} = \hat{\delta} t + \zeta(u), \quad \tilde{u} = \varphi(u), \quad \tilde{x} = \delta x,
\]
\[
\tilde{F} = \hat{\delta}^{-2}(\varphi_u - \zeta_u \tilde{u}_t)^2 F = \delta^{-2} \left( \frac{\hat{\delta} \varphi_u}{\delta + \zeta_u u_t} \right)^2 F,
\]
(8)
where \( \delta, \hat{\delta} \) are non-zero constants, \( \zeta \) and \( \varphi \) are arbitrary smooth functions of the variable \( u \), and \( \varphi_u \neq 0 \).

Further limitation of the class of equations (1) may lead to appearance of equivalence transformations of the form (7), that are different from (8) (see the proof).

5. Result of Classification. All possible cases of extension of the maximal invariance algebra in the Lie sense for the equation (1) up to equivalence transformations (7) are exhausted by the cases listed in the Table 1.

In the Table 1 \( F = F(t, u, u_t), f = f(u, u_t), h = h(u_t) \) are arbitrary smooth functions of their arguments, \( \delta \) is constant, \( \varepsilon_1, \varepsilon_2 = \pm 1, (\varepsilon_1, \varepsilon_2) \neq (-1, -1) \), \( J_{ab} = x_a \partial_b - x_b \partial_a, D = t \partial_t + u \partial_u + x_a \partial_a \). In the case 7 \( g_{\mu\nu} \) is metric tensor of Minkowsky space \( \mathbb{R}_{1,1} \), i.e. \( g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, g_{\mu\nu} = 0, \mu \neq \nu; c^\mu, b^{\mu\nu}, d, a^\nu, \eta \) are arbitrary smooth functions of the variable \( u \) (for the set of such operators to form an algebra, it is necessary to
require that these functions were infinitely differentiable or real analytical); the indices \( \mu, \nu \) and \( \varkappa \) take values from 0 to 3.

Let us note that the known Hamilton-Jacobi and eikonal equations, and the relativistic Hamilton equation are selected in the family of equations (II) as representatives of the classes of equivalent equations having the widest symmetry.

### Table 1

| \( F \)                                      | Basis Symmetry Operators                                      |
|----------------------------------------------|--------------------------------------------------------------|
| 0 \( F(t, u, u_t) \)                        | \( \partial_t, J_{ab} \)                                    |
| 1 \( e^{\delta f(u, u_t)}, \delta \in \{0; 1\} \) | \( \partial_t, J_{ab}, 2\partial_t - \delta x_a \partial_a \) |
| 2 \( e^u h(u_t) \)                         | \( \partial_t, J_{ab}, \partial_t, 2\partial_u - x_a \partial_a \) |
| 3 \( |u|^{2-\delta} h(u_t), \delta \neq 2 \) | \( \partial_t, J_{ab}, \partial_t, 2\partial_t + 2u \partial_u + \delta x_a \partial_a \) |
| 4 \( h(u_t) \)                             | \( \partial_t, J_{ab}, D \)                                  |
| 5 \( |u|^\beta, \beta \neq 0, 1, 2 \)       | \( \partial_t, J_{ab}, \partial_t, D, (\beta - 2)x_a \partial_a - 2u \partial_u \) |
| 6 \( u_t^2 \)                              | \( 2g_{\mu \nu}e^u(x_\mu x_\nu \partial_\varkappa - c^\varkappa(u)g_{\mu \nu}x_\mu x_\nu \partial_\varkappa + g_{\mu \nu}b^{\mu \nu}(u)x_\mu \partial_\varkappa + \delta x_\mu \partial_\varkappa + \eta(u)\partial_a \) |
| 7 \( \varepsilon_2 u_t^2 + \varepsilon_1 \) | \( \partial_t, J_{ab}, \partial_t, D, J_{ua} = u \partial_a + \varepsilon_1 x_a \partial_u, J_{ta} = t \partial_a + \varepsilon_2 x_a \partial_t, J_{ut} = u \partial_t - \varepsilon_1 \varepsilon_2 t \partial_u, K_a = 2x_a D - s^2 \partial_a, K_t = 2u D + \varepsilon_1 s^2 \partial_u, K_t = 2t D + \varepsilon_2 s^2 \partial_t, \) |
| 8 \( \varepsilon_2 e^u u_t^2 + \varepsilon_1 \) | \( \partial_t, J_{ab}, \partial_t, t \partial_t + 2u \partial_u, (t^2 - 4\varepsilon_1 \varepsilon_2 e^u) \partial_t + 4t \partial_u \) |
| 9 \( \cos^{-2} u u_t^2 + 1 \)             | \( \partial_t, J_{ab}, \partial_t, \cos t \tan u \partial_t - \sin t \partial_u, \sin t \tan u \partial_t + \cos t \partial_u \) |
| 10 \( \pm(\cos^{-2} u u_t^2 - 1) \)       | \( \partial_t, J_{ab}, \partial_t, \cosh t \tan u \partial_t + \sinh t \partial_u, \sinh t \tan u \partial_t + \cosh t \partial_u \) |
| 12 \( \cosh^{-2} u u_t^2 + 1 \)           | \( \partial_t, J_{ab}, \partial_t, \cosh t \tanh u \partial_t - \sinh t \partial_u, \sinh t \tanh u \partial_t - \cosh t \partial_u \) |

### 6. Proof

The complete proof is very cumbersome. We omit technical calculations, and some interim results are given as lemmas, and without proofs.

Let us denote the maximal invariance algebra in the Lie sense for the equation (II) as \( A^{\text{max}} \). In investigation of determining equations (2)–(4) two essentially different cases arise: \( F_{u_t u_{tt}} \neq 0 \) and \( F_{u_t u_{tt}} = 0 \).

If \( F_{u_t u_{tt}} \neq 0 \), then it follows from (9) and (4), that \( \eta_0 = \xi_1^0 = \xi_1^a = \xi_a = 0, \xi_1^0 = 0 \), thus \( A^{\text{max}} = A^{\text{ker}} + A^{\text{ext}} \), and \( A^{\text{ext}} \subseteq \langle \delta x_a \partial_a + \xi_1^0(t, u) \partial_t + \eta(t, u) \partial_u \rangle \), where \( \delta \) is an arbitrary constant, \( \xi \) and \( \eta \) are arbitrary smooth functions of the variables \( t \) and \( u \). \( A^{\text{ker}} \) ia an ideal, and \( A^{\text{ext}} \) is a subalgebra.
of the algebra $A^{\max}$, and $\forall Q \in A^{\text{ext}} (Q \neq 0): (\xi^0, \eta) \neq (0, 0)$. The dimension of $A^{\text{ext}}$ determines dimension of an extension of the algebra $A^{\max}$.

If $\dim A^{\text{ext}} > 0$, then we may choose any non-zero operator $Q$ from $A^{\text{ext}}$. By equivalence transformations (7) it is always possible to reduce it to the form

$$Q = 2\partial_t - \delta x_a \partial_a,$$

where $\delta \in \{0; 1\}$, after that, having solved equations (4) with $\xi^0 = 2$, $\xi^a = -\delta x_a$, $\eta = 0$ with respect to $F$, we obtain the first case of extension of the algebra $A^{\max}$.

Let $\dim A^{\text{ext}} > 1$. Then $A^{\text{ext}}$ has to contain an operator $Q$ with $\delta = 0$. By equivalence transformations (7) we can reduce the operator $Q$ to the operator $\partial_t$. In the following we will assume that $\partial_t \in A^{\text{ext}}$, whence $F_t = 0$, i.e. $F = F(u, u_t)$, and we will say that there is an additional extension of symmetry, if for such $F$ the relation $\dim A^{\max} > \dim A^{\ker} + 1$ is satisfied.

Let us assume that $\dim A^{\text{ext}} = 2$, and

$$A^{\text{ext}} = \langle Q^1 = \partial_t, \ Q^2 = \delta x_a \partial_a + \xi^0(t, u)\partial_t + \eta(t, u)\partial_a \rangle \quad \text{and} \quad \eta \neq 0.$$  

If the algebra $A^{\text{ext}}$ is commutative, then $\xi^0_t = 0, \eta_t = 0$, and by means of the transformation (8) the operator $Q^2$ may be reduced to the form

$$\tilde{Q}^2 = 2\partial_u - \tilde{\delta} x_a \partial_a,$$

where $\tilde{\delta} \in \{0; 1\}$ (the operator $Q^1$ does not change in this case). Let us substitute the coefficients of the operator $\tilde{Q}^2$ into the equation (4) and integrate it with respect to $F$: $F = e^{\tilde{\delta} u} h(u_t)$, where $h$ is a smooth function of the variable $u_t$. If $\tilde{\delta} = 1$, and $h$ is not fixed, then the algebra $A^{\text{ext}}$ is truly two-dimensional (the second case of an extension).

If the algebra $A^{\text{ext}}$ is non-commutative, we can assume, that $[Q^1, Q^2] = 2Q^1$, whence $\xi^0_t = 2, \eta_t = 0$, and by the transformation (8) the operator $Q^2$ can be reduced to the form $\tilde{Q}^2 = 2t\partial_t + 2u\partial_u + \delta x_a \partial_a$. Let us substitute coefficients of the operator $\tilde{Q}^2$ to the equation (4) and integrate it with respect to $F$: $F = |u|^{2-\delta} h(u_t)$, where $h$ is a smooth function of the variable $u_t$. If $\delta \neq 2$, and $h$ is not specified, then the algebra $A^{\text{ext}}$ is truly two-dimensional (the third case of an extension).

Let us assume that $\dim A^{\text{ext}} = 3$, and

$$A^{\text{ext}} = \langle Q^1 = \delta_i x_a \partial_a + \xi^{0i}(t, u)\partial_t + \eta^i(t, u)\partial_u, \ i = 1, 2, \ Q^3 = \partial_t \rangle,$$

the functions $\eta^1$ and $\eta^2$ are linearly independent, and the equation $\eta^1 \cdot (4.2) - \eta^2 \cdot (4.1)$ is an identity with respect to $F$ (here (4.2) is the equation, obtained from (4) by substitution of coefficients of the operator $Q^i$). By virtue of the latter condition the coefficients of the operators $Q^1$ and $Q^2$
have to satisfy the following equations:

\[
\eta_1^1 \eta_2^2 = \eta_2^2 \eta_1^1, \quad \eta_1^1 (\eta_u^2 - \delta_2) = \eta_2^2 (\eta_u^1 - \delta_1),
\]
\[
\eta_1^1 (\eta_u^2 - \xi_0^2) = \eta_2^2 (\eta_u^1 - \xi_0^1), \quad \eta_1^1 \xi_0^2 = \eta_2^2 \xi_0^1,
\]

from which it follows that \((\delta_1, \delta_2) \neq (0, 0), \eta_i^1 = 0, \xi_0^i = \delta_i, i = 1, 2, \eta_1^1 = (\delta_1 - \delta_2 \varphi)/\varphi_u, \eta_2^2 = \varphi \eta_1^1\) for a certain function \(\varphi = \varphi(u) \neq \text{const}\).

Let us substitute, if necessary, the operators \(Q_1^1\) and \(Q_2^2\) by their linear combinations \(\tilde{Q}_1^1\) and \(\tilde{Q}_2^2\) so \(\delta_1 = 1, \delta_2 = 0\). By the transformation (8) the operators \(\tilde{Q}_1^1\) and \(\tilde{Q}_2^2\) are reduced to the form \(\tilde{Q}_1^1 = t \partial_t + u \partial_u + \delta x_a \partial_a, \tilde{Q}_2^2 = \partial_u\). After substitution of the coefficients of the operators \(\tilde{Q}_1^1\) and \(\tilde{Q}_2^2\) into the equation (4) we obtain one equation for the function \(F\):

\[
F_u^2 = 0,
\]
whence \(F = h(u_t)\) where \(h\) is a smooth function of the variable \(u_t\). If \(h\) is not specified then the algebra \(A^\text{ext}\) is truly three-dimensional (the fourth case of extension).

**Lemma 1.** In all other cases when \(\dim A^\text{ext} > 1\), the function \(F\) satisfies the equation

\[
(Au_t^2 + Bu_t + C)F_{u_t} = (2Au_t + D)F,
\]

where \(A, B, C\) and \(D\) are arbitrary smooth functions of the variable \(u\). The transformation (8) is an equivalence transformation at the set of equations of the form (9).

When we integrate the equations (9), four non-equivalent cases arise where \(F_{u_t u_t u_t} \neq 0\).

1. \(F = \alpha(u) e^{u_t}\) where \(\alpha > 0\). In this case an additional extension of the symmetry is possible only for \(\alpha = A|u|^{\nu} e^{\mu u}\), where \(A, \nu, \mu = \text{const}, A > 0\).

By means of scale transformations by variables \(x_a\) we can get \(A = 1\). If \(\nu = 0\), then we can put \(\mu = 0\) (if it is not so, it is sufficient to perform the transformation \(\tilde{t} = e^{\mu t}/\mu, \tilde{u} = e^{\mu t}(u + 2t - 2/\mu)\); here and in the following variables \(x_a\) are not transformed if it is not specifically mentioned). Thus we obtain the fifth case of an extension. For \(\mu \nu \neq 0\) an additional symmetry extension will arise only for \(\nu = 2\), but then it is possible to put \(\mu = 0\) (the respective equivalence transformation is \(\tilde{t} = e^{\mu t}/\mu, \tilde{u} = e^{\mu t}u\)). If \(\nu \neq 0\) and \(\mu = 0\), then the equation (11) has the same symmetry algebra as in the more general third case of an extension.

2. \(F = \alpha(u)|u_t + \delta|^{\beta(u)}\), where \(\delta \in \{0; 1\}, \alpha > 0\). For additional symmetry extension it is necessary \(\beta = \text{const}\) (and \(\beta \notin \{0; 1; 2\}\), or otherwise.
Then any equation (I) with $\delta = 0$ is equivalent to the same equation in which in addition $\alpha = 1$ (the sixth case of an extension). The equation (II) with $\delta \neq 0$ is also reduced to the same case of extension, if $(\alpha' \alpha)' = 0$. Really, then $\alpha = A e^{\mu u}$ and we can put $A = 1$ (due to scal transformations by the variables $x_a$), $\mu = 0$ and $\delta = 0$ (the respective equivalence transformation when $\mu \neq 0$ has the form $\tilde{t} = e^{\mu \delta t/\beta}(\beta - 2)/\mu$, and when $\mu = 0$ it has the form $\tilde{t} = t$, $\tilde{u} = u + \delta t$). If $\delta \neq 0$ and $(\alpha' \alpha)' \neq 0$, then we will not obtain new cases of symmetry extension.

3. $F = \alpha(u) e^{\beta(u) \tan^{-1} u}$, where $\alpha > 0$, $\beta \neq 0$. An additional extension of the symmetry exists only on the conditions $\beta = \text{const}$ and $\alpha$ being an exponential or a power function, but it is not wider than extensions for more general cases 2–4 from the Table 1, and for this reason there is not need to list this case separately.

4. $F = \alpha(u) u_t^2 e^{u_t}$, where $\alpha > 0$. The equation with such second part is equivalent to the equation with $F = e^{u_t}$. The equivalence is determined by the transformation (of the type (7)) $\tilde{t} = u$, $\tilde{u} = t + \int \ln \alpha(u) du$.

We completed consideration of the case $F_{u_t u_t} \neq 0$.

Let in the following $F_{u_t u_t u_t} = 0$, whence

$$F = A(t, u) u_t^2 + B(t, u) u_t + C(t, u), \quad (10)$$

where $A, B, C$ are smooth functions of the variables $t$ and $u$. Splitting the equations (3) and (4) by the variable $u_t$, in addition to (2) we obtain the following system of determining equations:

$$\xi_0^a = A\xi_t^a - \frac{1}{2} B\xi_u^a, \quad \eta_a = C\xi_u^a - \frac{1}{2} B\xi_t^a, \quad (11)$$

$$A_t \xi_0^a + A_u \eta + B\xi_0^u = 2A(\xi_t^0 - \xi_1^0),$$

$$C_t \xi_0^a + C_u \eta + B\eta_t = 2C(\eta_u - \xi_1^0),$$

$$B_t \xi_0^a + B_u \eta + 2A\eta_t + 2C\xi_0^u = B(\eta_u + \xi_t^0 - 2\xi_1^0). \quad (12)$$

The equivalence group of the set of equations (II) with second parts quadratic by $u_t$ coincides with the general equivalence group of the equations (II). By means of the transformation (7) the coefficients $A, B$ and $C$
are changed as follows:
\[
\delta^2 \tilde{A} = A\zeta_t^2 - B\zeta_t\zeta_u + C\zeta_u^2,
\]
\[
\delta^2 \tilde{B} = B(\zeta_t\varphi_u + \zeta_u\varphi_t) - 2A\zeta_t\varphi_t - 2C\zeta_u\varphi_u,
\]
\[
\delta^2 \tilde{C} = A\varphi_t^2 - B\varphi_t\varphi_u + C\varphi_u^2,
\]
\[(13)\]

The condition \(B^2 - 4AC = 0\) is invariant with respect to transformations of the type \((13)\), that are generated by equivalence transformations \((7)\), and for this reason it is a classifying condition. If \(B^2 - 4AC = 0\), we can put \(A = 1\), \(B = 0\), whence \(C = 0\) the seventh case of an extension. In the following \(B^2 - 4AC \neq 0\).

**Lemma 2.** On the condition \(B^2 - 4AC \neq 0\) it follows from the equations \((2)\) and \((11)\) that coefficients of any symmetry operator of the equation \((1)\) have the following form:
\[
\xi^a = 2\gamma_b x_b x_a - \gamma_a x_b x_b + \sigma_{ab} x_b + \beta x_a + \alpha^a,
\]
\[
\xi^0 = \frac{1}{2}(A\beta - \frac{1}{2}B\beta_u)x_a x_a + (A\alpha^a - \frac{1}{2}B\alpha^a_u)x_a + \alpha^0,
\]
\[
\eta = \frac{1}{2}(C\beta - \frac{1}{2}B\beta_t)x_a x_a + (C\alpha^a - \frac{1}{2}B\alpha^a_t)x_a + \alpha^4,
\]
\[(14)\]
where \(\gamma_a\), \(\sigma_{ab}\) are constants, \(\beta\), \(\alpha^0\), \(\alpha^a\), \(\alpha^4\) are smooth functions of the variables \(t\) and \(u\).

Let us substitute the expressions \((14)\) into the system \((12)\) and split it by the variables \(x_a\). As a result we obtain \(n + 2\) systems of the same structure:
\[
H^1 A_t + H^2 A_u + H^1_u B = 2(H^1_t - \lambda)A,
\]
\[
H^1 C_t + H^2 C_u + H^2_t B = 2(H^2_u - \lambda)C,
\]
\[
H^1 B_t + H^2 B_u + 2H^2_t A + 2H^1_u C = (H^1_t + H^2_u - 2\lambda)B,
\]
\[(15)\]
where \(H^1\), \(H^2\) and \(\lambda\) take the following values:
\[
H^1 = A\beta - \frac{1}{2}B\beta_u, \quad H^2 = C\beta_u - \frac{1}{2}B\beta_t, \quad \lambda = 0;
\]
\[
H^1 = A\alpha^a - \frac{1}{2}B\alpha^a_u, \quad H^2 = C\alpha^a_u - \frac{1}{2}B\alpha^a_t, \quad \lambda = 2\gamma_a;
\]
\[
H^1 = \alpha^0, \quad H^2 = \alpha^4, \quad \lambda = \beta.
\]
\[(16)\]
Investigating the systems of determining equations \((15)\), \((16)\), we obtain the following statements.

**Lemma 3.** If \(\text{dim} A^\text{max} > \text{dim} A^\text{ker}\), then the equation \((1)\), \((10)\) is equivalent to the same equation in which in addition \(A = e^{\delta t} A(u), B = e^{\delta t} B(u)\)
and \( C = e^{\delta t} \hat{C}(u) \), and for which \( A_{\text{max}} \ni 2\partial_t - \delta x_a \partial_a \) (with functions \( \hat{A}(u) \), \( \hat{B}(u) \), \( \hat{C}(u) \) being not specified, \( A_{\text{max}} = \langle \partial_a, J_{ab}, 2\partial_t - \delta x_a \partial_a \rangle \), and it is the first case of an extension).

**Lemma 4.** If \( \dim A_{\text{max}} > \dim A_{\text{ker}} + 1 \), then the equation (I), (10) is equivalent to the same equation in which in addition \( A_t = 0, B_t = 0 \) and \( C_t = 0 \).

It follows from the Lemmas 3 and 4 that for completion of the classification it is necessary to investigate the equation (I) with the second parts of the form \( F = A(u)u_t^2 + B(u)u_t + C(u) \), where \( A, B \) and \( C \) are smooth functions of the variable \( u \), \( B^2 - 4AC \neq 0 \). The equivalence group of the set of these equations coincides with the general equivalence group of the equations (I), whose second parts do not depend on \( t \). Under the action of the transformation (S) the coefficients \( A, B \) and \( C \) change as follows:

\[
\delta^2 \tilde{A} = A\tilde{\delta}^2 - B\tilde{\delta} \zeta_u + C\zeta_u^2, \quad \delta^2 \tilde{B} = B\tilde{\delta} \varphi_u - 2C\zeta_u \varphi_u, \quad \delta^2 \tilde{C} = C\varphi_u^2,
\]

and we can assume additionally that \( B = 0, C = \varepsilon_1 = \pm 1 \). Thus in the following \( F = A(u)u_t^2 + \varepsilon_1 \) where \( A \neq 0 \).

The condition \( A_u = 0 \) gives the eighth case of a symmetry extension.

The case when \( \exists \mu = \text{const}: (u + \mu)A_u + 2A = 0 \), is reduced to the case \( A_u = 0 \) by means of the equivalence transformation \( \tilde{t} = (u + \mu) \sinh t, \quad \tilde{u} = (u + \mu) \cosh t \), if \( \varepsilon_1 A < 0 \), or \( \tilde{t} = (u + \mu) \sin t, \quad \tilde{u} = (u + \mu) \cos t \), if \( \varepsilon_1 A > 0 \).

The case when \( \exists \mu, \nu = \text{const} (\nu \neq 0, 2): (u + \mu)A_u + \nu A = 0 \), is reduced to a more general case 3 from the Table 1 by means of the equivalence transformation \( \tilde{t} = t, \quad \tilde{u} = |u + \mu|^{1-\nu/2} \).

The condition \( (A_u/A)_u = 0, A_u \neq 0 \) gives the ninth case of a symmetry extension.

Let the function \( A \) satisfy the equation \( A_u/A = \nu A + \mu \) for some non-zero constants \( \mu \) and \( \nu \). Any solution of this equation depending on values of the constants \( \mu \) and \( \nu \) is equivalent (by translations and scale transformations by the variable \( u \)) to one of the following functions:

\[
\frac{\varepsilon_2}{\cosh^2 u}, \quad \frac{\varepsilon_2}{\sinh^2 u}, \quad \frac{\varepsilon_2}{\cos^2 u},
\]

where \( \varepsilon_2 = \pm 1, (\varepsilon_1, \varepsilon_2) \neq (-1, -1) \). Let \( \varepsilon_0 = 1 \) for the first two functions, and \( \varepsilon_0 = -1 \) for the third function.
Lemma 5. The equation (1) with the second parts $A(u)u_t^2 + \varepsilon_1$ and $\tilde{A}(u)u_t^2 + \tilde{\varepsilon}_1$, where $A$ and $\tilde{A}$ are chosen from the set of functions (17), are equivalent if and only if $\varepsilon_0\tilde{\varepsilon}_0 = \varepsilon_1\tilde{\varepsilon}_1 = \varepsilon_2\tilde{\varepsilon}_2$.

By virtue of Lemma 5 the cases 10–12 of the Table 1 exhaust all possible non-equivalent equations from this class.

For any other functions $A = A(u)$ there would be no extension of the symmetry for the equation (1).

Non-equivalence of the all cases of extensions adduced in the Table 1, where it was not proved directly by means of application of equivalence transformations, follows explicitly from non-isomorphness of the respective maximal symmetry algebras, in particular, from the fact they have different dimensions.

7. Conclusion. We presented a complete solution of group classification problem for a class of PDEs with derivatives entering nonlinearly. New invariant equations obtained in this paper are very interesting also from the point of view of finding new conditional symmetries [21] for higher order equations, if these new equations are used as differential constraints. In particular, this method works for nonlinear wave equations, and that will be the subject of further papers.

[1] Ovsjannikov L V 1982 Group Analysis of Differential Equations (New York: Academic Press)

[2] Lie S 1891 Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen (Leipzig: B.G. Teubner)

[3] Akhatov I S, Gazizov R K and Ibragimov N K 1987 Proc. Acad. Sci. USSR 293 1033–5

[4] Dorodnitsyn V A 1982 Zhum. Vych. Maternal Matem. Fiziki 22 1393–400

[5] Oron A and Rosenau P 1986 Phys. Lett. A 118 172-6

[6] Edwards M P 1994 Phys. Lett. A 190 149-54

[7] Gandarias M L 1996 J. Phys. A: Math. Gen. 29 607–33

[8] Olver P J and Heredero R H 1996 J. Math. Phys. 37 6419–38
[9] Ibragimov N K, Torrisi M and Valenti A 1991 *J. Math. Phys.* **32** 2988–95

[10] Ibragimov N K and Torrisi M 1992 *J. Math. Phys.* **33** 3931–7

[11] Torrisi M, Tracina R and Valenti A 1996 *J. Math. Phys.* **37** 4758–67

[12] Torrisi M and Tracina R 1998 *Int. J. Nonlinear Mech.* **33** 473–87

[13] Serov M I and Cherniha R M 1997 *Ukrain. Math. J.* **49** 1262–70

[14] Nikitin A G and Wiltshire R J 2000 *Proc. of Institute of Mathematics of NAS of Ukraine* vol 30 part 1 (Kyiv: Institute of Mathematics of NAS of Ukraine) 47–59.

[15] Akhatov I S, Gazizov R K and Ibragimov N K 1989 *Sovremennye Problemy Matematiki. Novejshie Dostizheniya* vol 34 (Moscow: Nauka) pp 3–83

[16] Zhdanov R Z and Lahno V I 1999 *J. Phys. A: Math. Gen.* **32** 7405–18

[17] Zhdanov R Z and Roman O V 2000 *Rep. Math. Phys.* **45** no 2 273–91

[18] Popovych R and Cherniha R 2001 *Proc. of Institute of mathematics of NAS of Ukraine* (Kyiv: Institute of Mathematics of NAS of Ukraine), to be published

[19] Boyer C P and Penafiel M N 1976 *Nuovo cim. B* **31**, N 1 195–210

[20] Fushchich W I and Shtelen W M 1982 *Lett. nuovo cim.* **34**, N 16 498.

[21] Fushchych W I, Shtelen W M and Serov N I 1989 *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics* (Kiev: Naukova Dumka)

Fushchych W I, Shtelen W M and Serov N I 1992 *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics* (Dordrecht: Kluwer Academic Publisher) (English transl.)

[22] Fushchych W I and Serov N I 1991 *Ukrain. Math. J.* **43** 394–399

[23] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer-Verlag)