Choiceless large cardinals and set-theoretic potentialism

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We define a potentialist system of ZF-structures, i.e., a collection of possible worlds in the language of ZF connected by a binary accessibility relation, achieving a potentialist account of the full background set-theoretic universe \(V\). The definition involves Berkeley cardinals, the strongest known large cardinal axioms, inconsistent with the Axiom of Choice. In fact, as background theory we assume just ZF. It turns out that the propositional modal assertions which are valid at every world of our system are exactly those in the modal theory \(S4.2\). Moreover, we characterize the worlds satisfying the potentialist maximality principle, and thus the modal theory \(S5\), both for assertions in the language of ZF and for assertions in the full potentialist language.

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1 Introduction

In the current scenario of set theory, we are faced with a conflict between large cardinal axioms and the Axiom of Choice (AC). In fact, there is a whole new hierarchy of ZF large cardinals—the Berkeley hierarchy—which contradict AC and lie beyond the Kunen inconsistency of Reinhardt cardinals. Such “choiceless” large cardinals have been recently introduced in [1] and the investigation of their consistency is very involved in the present main foundational questions concerning the universe of set theory. In this article, we will combine the choiceless large cardinals with a potentialist perspective, i.e., one in which the universe of set theory reveals gradually, and never completely, as we progressively take under consideration new fragments of it: indeed, we can actually think of how we access higher and higher parts of the set-theoretic universe by considering stronger and stronger large cardinals.

Recent works of the second author focus on the idea of set-theoretic potentialism and the analysis of the modal principles validated by specific potentialist systems. The general definition is stated below.

Definition 1.1 A potentialist system is a collection \(\mathcal{W}\) of structures in a common language \(L\) called “worlds”, equipped with a binary accessibility relation \(R\), such that:

1. \(R\) is reflexive and transitive;
2. whenever \(MN\), then \(M\) is—or embeds to—a substructure of \(N\).

So, a potentialist system is a Kripke model of \(L\)-structures for some language \(L\). In order to study how truth of an assertion \(\phi\) propagates through the worlds of \(\mathcal{W}\), one adds to the basic language \(L\) the modal operators \(\Box\) and \(\Diamond\), expressing, respectively, the notions of possibility and necessity:

1. \(\Diamond \phi\) holds at a world \(M\) (i.e., “\(\phi\) is possible over \(M\)”) if \(\phi\) holds at some world \(N\) such that \(MN\);
2. \(\Box \phi\) holds at a world \(M\) (i.e., “\(\phi\) is necessary over \(M\)”) if \(\phi\) holds at all worlds \(N\) such that \(MN\).

Now one can ask which propositional modal assertions are valid in the whole system \(\mathcal{W}\) (i.e., hold in every world of \(\mathcal{W}\)); the point is that determining the modal validities of a potentialist system gives a precise account of how its worlds interact with respect to their respective truths.
Let us turn to our particular case, whose hallmark is to combine choiceless large cardinals with the potentialist ideas. Indeed, we consider the concept of set-theoretic potentialism that arises from elementary embeddings of a transitive set into itself, where we view $M$ as accessing $N \supseteq M$ whenever the restriction to $M$ of any elementary embedding $j : N \to N$ yields an elementary embedding $j' : M \to M$. Such a definition of the accessibility relation results in an interesting case as in the context of Berkeley cardinals, one can arrange non-trivial elementary embeddings fixing any desired set. The key point is that every given set is definable in some big transitive set and if there is a Berkeley cardinal $\delta$ then, by definition, any transitive set $M$ containing $\delta$ as a member admits non-trivial elementary embeddings $j : M \to M$, whose critical points are in fact cofinal in $\delta$.

**Definition 1.2** A cardinal $\delta$ is a Berkeley cardinal if for every transitive set $M$ such that $\delta \in M$, and for every ordinal $\eta < \delta$, there exists a non-trivial elementary embedding $j : M \to M$ with $\eta < \text{crit}(j) < \delta$.\(^1\)

It turns out that the set-theoretic universe $V$ equals the union of the worlds of our potentialist system, and given any world $M$ and any set $a$, there is a world $N$ accessed by $M$ such that $a \in N$. Thus, truth in $V$ is approximated by truth in our worlds: we can assert any property concerning any set of $V$ from any of the worlds of our system by using the diamond operator, and we can progressively move from any world to the wider perspective of another world which is “closer” to $V$ in that it contains additional sets and is capable to satisfy additional properties about them. The primary goal here will be to provide a definite account and determine the valid modal principles of this kind of set-theoretic potentialism; but let us mention that as a further perspective, maybe one could use such a multiverse setting to investigate further the fundamental question of the consistency of the choiceless large cardinals.

For the basics of set-theoretic potentialism and the various potentialist systems analyzed so far we refer to [5]. For more on the choiceless large cardinals cf. [1].

## 2 A potentialist account of the set-theoretic universe $V$

We start with a preliminary lemma motivating the definition of the accessibility relation we shall consider.

**Lemma 2.1** For every transitive set $M$, for every set $a$, there exists a transitive set $N \supseteq M$ with $a \in N$ such that every elementary embedding $j : N \to N$ lifts some elementary embedding $j' : M \to M$.

**Proof.** Let $M$ be a transitive set and let $a$ be any set. Let $N = \text{tr cl}((M, a))$. Then, $N$ is a transitive set such that $M, a \in N$ and $M$ is definable (without parameters) in $N$: in fact, $N$ is a transitive set with a unique $\epsilon$-maximal element, the pair $\langle M, a \rangle$, and so $M$ and $a$ are both definable in $N$. Thus, $M \subseteq N$ and every $j : N \to N$ fixes $M$, which implies $j' \restriction M \subseteq M$. Therefore $j \restriction M : M \to M$, and so actually $j$ lifts $j' = j \restriction M : M \to M$. \hfill $\Box$

**Definition 2.2** Let $\delta$ be a Berkeley cardinal.

1. Let $\mathcal{M}_\delta = \{ M : M \text{ is transitive and } \delta \in M \}$.

2. Let $\mathcal{R}$ be the binary relation on $\mathcal{M}_\delta$ defined as follows: for $M, N \in \mathcal{M}_\delta$,

   $M \mathcal{R} N$ iff $M \subseteq N$ and every elementary embedding $j : N \to N$ lifts some elementary embedding $j' : M \to M$; i.e., $M \mathcal{R} N$ iff $M \subseteq N$ and for every elementary embedding $j : N \to N$, $j \restriction M : M \to M$.

Note that the assumption that there is a Berkeley cardinal $\delta$ and the choice of $\mathcal{M}_\delta$ as collection of worlds ensure that every world $M$ admits non-trivial elementary embeddings $j : M \to M$, so $\mathcal{R}$ is not merely reduced to the subset relation. It is trivial that $\mathcal{R}$ is reflexive; also, $\mathcal{R}$ is transitive: in fact, if $j : H \to H$ lifts $j' : N \to N$, which in turn lifts $j'' : M \to M$, then $j$ lifts $j''$, as $j \restriction M = (j \restriction N)\restriction M = j' \restriction M = j''$. Therefore, $\langle \mathcal{M}_\delta, \mathcal{R} \rangle$ is a potentialist system of $\text{ZF}$-structures. Since $V_\alpha \in \mathcal{M}_\delta$ for any $\alpha > \delta$, we have that $V = \bigcup_{M \in \mathcal{M}_\delta} M$. Moreover, we show that $\mathcal{M}_\delta$ provides a potentialist account of the set-theoretic universe $V$, meaning that every world in $\mathcal{M}_\delta$ is a substructure of $V$ and for every $M \in \mathcal{M}_\delta$ and every set $a$ there is a world $N \in \mathcal{M}_\delta$ accessed by $M$ such that $a \in N$.

\(^1\) As Berkeley cardinals are incompatible with $\text{AC}$, their definition is stated in just $\text{ZF}$ and in such choiceless context, being non-trivial means $j$ is not the identity on the ordinals.
Lemma 2.3. \( \mathcal{M} \) provides a potentialist account of the universe \( V \).

Proof. First, for every \( M \in \mathcal{M} \), \( (M, \in) \) is a substructure of \( (V, \in) \) (i.e., \( M \subseteq V \) and \( \in^M = \in^V | M \)). Further, if \( M \in \mathcal{M} \) and \( a \in V \), then by Lemma 2.1 there exists a transitive set \( N \) such that \( \{M, a\} \in N \) and for every elementary embedding \( j : N \rightarrow N, j|M : M \rightarrow M \). Since \( \delta \in M \subseteq N \), we have that \( N \in \mathcal{M} \); since \( a \in N \) and \( N \) is accessed by \( M \), we are done. \( \square \)

Remark 2.4. Notice that:

1. By Lemma 2.1, for every \( M \in \mathcal{M} \) there exist cofinally many \( N \in \mathcal{M} \) which are accessed by \( M \), meaning that such \( N \) can accommodate any given set.
2. In particular, for every \( M \in \mathcal{M} \) and every ordinal \( \alpha \), there is a transitive set \( N \) containing \( V_\alpha \) such that every elementary embedding \( j : N \rightarrow N \) lifts some elementary embedding \( \tilde{j} : M \rightarrow M \); i.e., \( M \) accesses some \( N \) that includes any desired rank initial segment \( V_\alpha \).

For any assertion \( \phi \) in the language of \( ZF \), the potentialist translation \( \phi^\circ \) is the assertion in the potentialist language \( ZF^\circ \) (which augments the language of \( ZF \) with the modal operators \( \bigdiamond \) and \( \blacksquare \)) achieved by replacing every instance of \( \exists x \) with \( \bigdiamond \exists x \) and every instance of \( \forall x \) with \( \blacksquare \forall x \). As an immediate corollary of Lemma 2.3, we get that truth in \( V \) is equivalent to potentialist truth at the worlds of \( \mathcal{M} \).

Corollary 2.5. For any \( ZF \)-formula \( \phi \) and for any \( a_0, \ldots, a_n \in V \), we have: \( V \models \phi(a_0, \ldots, a_n) \) iff \( M \models \phi^\circ(a_0, \ldots, a_n) \), for any \( M \in \mathcal{M} \) in which \( a_0, \ldots, a_n \) exist.\(^2\)

Let us now state what it precisely means for a modal assertion to be valid with respect to our potentialist system.

Definition 2.6. A modal assertion \( \phi(p_0, \ldots, p_n) \) in the language of propositional modal logic is valid at a world \( M \) in \( \mathcal{M} \) for a certain class of assertions, if all the resulting substitution instances \( \phi(\psi_0, \ldots, \psi_n) \), where assertion \( \psi_i \) from the allowed class is substituted for the propositional variable \( p_i \), are true at \( M \). \( \phi \) is valid in \( \mathcal{M} \) if it is valid at every world of \( \mathcal{M} \).

Of course, the main question arising here is the following:

Question. What is the modal logic of \( \mathcal{M} \)? That is, which are the modal principles valid in \( \mathcal{M} \)?

3 The modal logic of \( \mathcal{M} \)

In this section we provide lower and upper bounds on the modal validities of \( \mathcal{M} \), and finally prove that the modal logic of \( \mathcal{M} \) is exactly \( S4.2 \).

Definition 3.1. The modal theory \( S4 \) is obtained from the following axioms by closing under modus ponens and necessitation.

\[(K) \quad \blacksquare(\phi \rightarrow \psi) \rightarrow (\blacksquare\phi \rightarrow \blacksquare\psi)\]

\[(\text{Dual}) \quad \neg\bigdiamond\phi \leftrightarrow \boxdot\neg\phi\]

\[(S) \quad \bigdiamond\phi \rightarrow \phi\]

\[(4) \quad \bigdiamond\phi \rightarrow \bigdiamond\bigdiamond\phi\]

Theorem 3.2. The modal theory \( S4 \) is valid at every world of \( \mathcal{M} \).\(^3,4\)

Proof. Let \( M \in \mathcal{M} \).

\[(K) \quad \text{Suppose } \blacksquare(\phi \rightarrow \psi) \text{ and } \bigdiamond\phi \text{ hold in } M. \text{ Then, } \phi \rightarrow \psi \text{ and } \phi \text{ hold in any world } N \text{ accessed by } M. \]

Therefore, by modus ponens, \( \psi \) holds in any such \( N \), i.e., \( \bigdiamond\psi \) holds in \( M \).

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\(^2\) The symbol \( \models_{\mathcal{M}} \) is defined recursively as usual; details can be found in [5, § 1].

\(^3\) In fact, every potentialist system validates \( S4 \).

\(^4\) Unless otherwise specified, the validities hold for all assertions in \( ZF^\circ \), with parameters.
(Dual) Immediate.

(5) Follows immediately from the fact that every world accesses itself.

(4) If \( \varphi \) holds in any world \( N \) accessed by \( M \), then so does \( \Box \varphi \), as any world \( H \) accessed by \( N \) is also accessed by \( M \). \( \square \)

Definition 3.3 The modal theory \( S4.2 \) is obtained from \( S4 \) by adding the axiom \( (.2) \Diamond \Box \varphi \rightarrow \Box \Diamond \varphi \).

Theorem 3.4 The modal theory \( S4.2 \) is valid at every world of \( M_3 \).

Proof. \( (.2) \). Let \( M \in M_3 \). Assume \( \Diamond \Box \varphi \) holds in \( M \), i.e., there exists \( N \in M_3 \) such that \( MRN \) and \( \Box \varphi \) holds in \( N \). Let \( H \in M_3 \) be such that \( MRH \). We need to show that \( \Diamond \varphi \) holds in \( H \). Note that there exists a transitive set \( K \) such that \( \langle N, H \rangle \) is definable in \( K \). Since \( K \) is transitive and \( N, H \subseteq K \), we have that \( \delta \in K \) and so \( K \in M_3 \). Now, take any non-trivial elementary embedding \( j : K \rightarrow K \). Then, \( j(N) = N \) and \( j(H) = H \). So, \( j|N : N \rightarrow N \) and \( j|H : H \rightarrow H \). Therefore, \( N \) and \( H \) both access \( K \). Since \( K \) is accessed by \( N, K \) satisfies \( \varphi \); but then, since \( K \) is accessed by \( H \) and \( \varphi \) holds in \( K \), \( \Diamond \varphi \) holds in \( H \). \( \square \)

Theorem 3.4, whose proof actually shows our potentialist system is “convergent”—or “locally directed”\(^5\)—and therefore validates \( (.2) \), establishes a first significant lower-bound result. In order to provide upper bounds on the validities of \( M_3 \), and then determine the exact set of modal principles valid through the whole system, we recall the definitions of switches, buttons and dials, specific kinds of control statements first introduced in [6]; in particular, we will be interested in finding assertions satisfying such definitions which also have the property of being independent.

Definition 3.5 An assertion \( s \) is a switch if both \( \Diamond s \) and \( \Diamond \neg s \) are true at every world, i.e., both \( \Box \Diamond s \) and \( \Box \Diamond \neg s \) hold. \( s \) is a switch at a particular world \( M \) if \( \Diamond s \) and \( \Diamond \neg s \) are true at all the worlds accessed by \( M \).

Definition 3.6 A button is a statement \( b \) such that \( \Diamond \Box b \) is true at every world, i.e., \( \Box \Diamond \Box b \) holds. The button is pushed at a world if \( \Box b \) holds at that world, and otherwise unpushed.

Definition 3.7 A (possibly infinite) list of statements \( d_0, d_1, d_2, \ldots \) is a dial if every world satisfies exactly one of the statements \( d_i \) and every world can access another world with any prescribed dial value. If a world satisfies \( d_i \), then we say that the dial value is \( i \) in that world.

Definition 3.8 A family of switches is independent if one can always flip the truth values of any finitely many of the switches so as to realize any desired finite pattern of truth.

Definition 3.9 A family of buttons and switches is independent if there is a world at which the buttons are unpushed, and every world \( M \) accesses a world \( N \) in which any additional button may be pushed without pushing any other as-yet unpushed button from the family, while also setting any finitely many of the switches so as to have any desired pattern in \( N \); and similarly with dials.

Definition 3.10 The modal theory \( S5 \) is obtained from \( S4 \) by adding the axiom \( (.5) \Diamond \Box \varphi \rightarrow \varphi \), which we call potentialist maximality principle (MP).

The following theorem summarizes some key results—first proved in [6, Theorem 6], and developed further in [4, Theorems 10 & 13]—we shall use.

Theorem 3.11 The following hold.

1. If \( W \) is a Kripke model and a world \( M \in W \) admits arbitrarily large finite collections of independent switches, then the propositional modal assertions valid at \( M \) are contained in the modal theory \( S5 \). In particular, if the switches work throughout \( W \), then the validities of every world of \( W \), and so the validities of \( W \), are contained within \( S5 \).

2. A Kripke model \( W \) admits arbitrarily large finite families of independent switches if and only if it admits arbitrarily large finite dials.

\(^5\) In fact, it shows that whenever \( MRN \) and \( MRH \) then there exists \( K \) such that \( NRK \) and \( HRK \), and so, that \( M_3 \) has amalgamation.

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3. If $W$ is a Kripke model that admits arbitrarily large finite families of independent buttons and switches, or independent buttons independent of a dial, then the propositional modal validities of $W$ are contained in $S4.2$. The validities of any particular world in which the buttons are not yet pushed are contained in $S4.2$, and in any case, are contained in $S5$.

Theorem 3.12 The propositional modal validities of $M_δ$ are contained in the modal theory $S5$.

Proof. It suffices to show that $M_δ$ admits arbitrarily large finite dials. We shall show that it admits in fact an infinite dial (notice that from an infinite dial, we can construct finite dials of any given size by keeping any desired finite number of dial statements and adding the statement that none of them holds). For $i < ω$, let $d_i$ be the assertion that the height of the ordinals is $λ + i$, where $λ$ is a limit ordinal or zero. These statements are expressible in the language of $ZF$ (without parameters or modal vocabulary), correctly interpreted inside any transitive set, and so inside any world $M \in M_δ$. Let us show that they form a dial. First, since any ordinal is uniquely expressed as $λ + i$ for some limit ordinal $λ$ or zero and some finite $i < ω$, every world in $M_δ$ satisfies exactly one of the statements $d_i$. It remains to prove that every world can access another world with any desired dial value. Let $M \in M_δ$ and fix $i < ω$. Let $δ$ be the least such that $M \subseteq V_δ$. Let $N = \text{trcl}(M) \cup (δ + ω + i)$. Then $N \in M_δ$ and $M$ is definable in $N$, which implies $M \cap N$, and the dial value in $N$ is $i$.

Theorem 3.13 $M_δ$ satisfies exactly $S4.2$, i.e., the modal logic of $M_δ$ is $S4.2$.

Proof. We show that $M_δ$ admits arbitrarily large finite families of independent buttons independent of a dial, which implies that the modal validities of $M_δ$ are contained in, and hence by Theorem 3.4 equal to, $S4.2$. For $i < ω$, let $d_i$ be the assertion that the height of the ordinals is $λ + i$, where $λ$ is a limit ordinal or zero; we already showed that these statements form a dial. For $1 < m < ω$, let $b_m$ be the assertion that the set $m \cdot N = \{m \cdot k : k < ω\}$ exists; let us show that these statements are independent buttons independent of the above dial. Since for every $m < ω$, $m > 1$, if the assertion $b_m$ is true in some transitive set then it will continue to be true in any larger transitive set, each $b_m$ is a button. These buttons are independent because every world $M$ accesses a world $N$ in which any additional button $b_m$ may be pushed without pushing any other as-yet unpushed button from the family. Finally, the above buttons and dial values can be controlled independently of each other.6

By Theorem 3.12, $S5$ is a definite upper bound on the validities of $M_δ$. An interesting point would therefore be to determine which worlds of $M_δ$ realize the maximum set of validities. In other words:

Question. Which worlds of $M_δ$ satisfy the potentialist maximality principle?

4 The worlds satisfying MP

We now give a characterization of the worlds of $M_δ$ satisfying $S5$. Depending on the language we consider, we get different criteria. The following concepts are involved.

Definition 4.1 An ordinal $δ$ is $Σ_n$-correct if $V_δ \preceq_Σ V$, meaning that $V_δ$ and $V$ agree on the truth of $Σ_n$ formulas with parameters from $V_δ$.

Definition 4.2 A cardinal $δ$ is correct if it is $Σ_n$-correct for every $n$, i.e., if it realizes the scheme $V_δ \preceq V$.7

Theorem 4.3 The following are equivalent.

1. The potentialist maximality principle holds in a world $M \in M_δ$ for assertions in the language of $ZF$ with parameters from $M$.
2. $M = V_δ$ for some $Σ_2$-correct cardinal $δ > δ$.

Proof. Let $M \in M_δ$.

(1 $\implies$ 2). Assume $M$ satisfies $(5) ∨ □φ → φ$ for assertions in the language of $ZF$ with parameters from $M$. We show that $M$ must be a $V_δ$ with $δ$ limit. First, note that the statement $Y = P(X)$ is modally expressible for $X, Y$

6 Notice that we can ensure $M$ computes the sets $b_m$ properly by assuming $V_ω \subseteq M$.

7 Note that this concept is not expressible as a single assertion in the language of set theory, although it can be expressed as a scheme of statements.
as the claim that it is impossible for there to be a subset of $X$ that is not in $Y$ (i.e., $\neg \exists b \subseteq X (b \notin Y)$) and an element of $Y$ that is not a subset of $X$ (i.e., $\neg \exists b \in Y (b \notin X)$); and similarly the statement $Y = V_\alpha$ is modally expressible as a claim about $Y$ and $\alpha$ as: necessarily $Y = V_\alpha$. If it is, then it remains true in all larger transitive sets; and if $Y$ is not actually $V_\alpha$, then there will be a larger transitive set in which this is revealed. Therefore, for all $a \in M$, the existence of the power set of $a$ is possibly necessary, and it follows that $M$ thinks $P(a)$ exists, and that $M$ computes the power sets correctly; moreover, $M$ computes $V_\alpha$ correctly for any ordinal $\alpha \in M$, since the existence of $V_\alpha$ is possibly necessary. Also, for any set $a$, it is possibly necessary that $a \in V_\alpha$ for some ordinal $\alpha$, and so this is already true in $M$. Now, suppose $\varphi(\vec{a})$ is a $\Sigma_2$ statement true in $V$, with $\vec{a} \in M$. This is witnessed by the existence of some ordinal $\alpha$ for which $V_\alpha$ satisfies $\psi(\vec{a})$ for some assertion $\psi$. So, it is possibly necessary the statement that there is an ordinal $\alpha$ for which $V_\alpha$ exists and satisfies $\psi(\vec{a})$. Thus, by (5), this statement must be true in $M$, and so $\varphi(\vec{a})$ is true in $M$. Therefore, $\theta$ is $\Sigma_2$-correct.

Theorem 4.4 The following are equivalent.

1. The potentialist maximality principle holds in a world $M \in M_\delta$ for assertions in the potentialist language $ZF^\diamond$ with parameters from $M$.

2. $M = V_\theta$ for some correct cardinal $\theta > \delta$.

Proof. Let $M \in M_\delta$.

(1 $\implies$ 2). Assume $M$ satisfies (5) $\diamond \varphi \rightarrow \varphi$ for assertions in the potentialist language $ZF^\diamond$ with parameters from $M$. Then by Theorem 4.3, $M = V_\theta$ for $\theta \Sigma_2$-correct. By the potentialist translation, truth in $V$ is expressible as $\varphi^\diamond$-truth in $M$. Thus, $M$ can express the statement for any given $\alpha$ in $M$ that there is a $\Sigma_n$-correct cardinal above $\alpha$. For each $n$, this statement is a button in $ZF^\diamond$ using parameter $\alpha$. So $M$ is a limit of $\Sigma_n$-correct cardinals, and therefore $M$ is fully correct.

(2 $\implies$ 1). Assume $M = V_\theta$ for $\theta > \delta$ a correct cardinal. Suppose $M$ satisfies $\diamond \varphi(\vec{a})$, where $\vec{a} \in M$ and $\varphi$ is $ZF^\diamond$ assertion. Then, there exists $N \in M_\delta$ accessed by $M$ that satisfies $\Box \varphi(\vec{a})$. Since the existence of such a set $N$ and the potentialist semantics are expressible in the language of set theory, it follows from $M \prec V$ that there is such a set inside $M$. So, there exists $m \in M$ such that $m$ satisfies $\varphi(\vec{a})$. Without loss of generality, there exists $m = V_\alpha$, like this (inside $M$), and so the smallest one is definable and so it accesses $M$. Since the statement that $V_\alpha$ satisfies $\Box \varphi(\vec{a})$ is a $\Pi_1$ statement, it is absolute between $M$ and $V$ by the $\Sigma_2$-correctness of $M$, and so, it holds in $V$. Thus, $\varphi(\vec{a})$ holds in $M$, which therefore satisfies (5).

Remark 4.5 Note that one can view Theorem 4.4 as a $ZF$ theorem scheme asserting the equivalence of two schemes; or, one could view it as a theorem of Gödel-Bernays set theory augmented with the assumption that there is a predicate for first-order set-theoretic truth (that theory is provable, for example, in Kelley-Morse set theory). 8

5 Consistency of MP and conclusive remarks

Theorems 4.3 & 4.4 characterize the worlds of $M_\delta$ satisfying MP, respectively, for assertions in the language of set theory and for assertions in the full potentialist language, with parameters. Let us remark that there exist indeed such worlds in $M_\delta$. In fact, for the first case, note that by the Lévy-Montague reflection theorem (which is a $ZF$ result), the class of all $\Sigma_2$-correct cardinals is closed and unbounded in the ordinals. For the second case, observe that although the existence of a correct cardinal is not provable in $ZF$, it is relatively consistent with $ZF$ (cf. [3]); so, it is relatively consistent with $ZF$ that there exist worlds in $M_\delta$ satisfying MP for assertions in the potentialist language $ZF^\diamond$. In conclusion, let us point out that there is an affinity of the characterizations of the modal validities and maximality principle instances occurring in $M_\delta$ and the corresponding analysis for transitive-set potentialism in [5, § 3.5].

8 In fact, $\Sigma_n$-correct cardinals are unbounded in the ordinals, as a consequence of the Lévy-Montague reflection theorem.

9 Cf. [2].
Recall that the definition of our potentialist system leverages on the assumption that there is a Berkeley cardinal \( \delta \); one may ask if there is any world in \( M_\delta \) which satisfies that there exists a Berkeley cardinal, and the answer is yes: in fact, the property of being a Berkeley cardinal is \( \Pi_2 \), so for any \( \Sigma_2 \)-correct cardinal \( \vartheta \), \( V_\vartheta \) correctly recognizes the Berkeley cardinals below \( \vartheta \). In other words, inside all the worlds of \( M_\delta \) satisfying MP, \( \delta \) itself is still a Berkeley cardinal; but these are not the only worlds in \( M_\delta \) recognizing \( \delta \) is Berkeley: in fact, as noted in [1], if \( \delta \) is a Berkeley cardinal then for all limit ordinals \( \lambda > \delta \), \( V_\lambda \) thinks that \( \delta \) is Berkeley.

Finally, since every world \( M \) in \( M_\delta \) can access a \( V_\lambda \) with \( \lambda \) limit, the assertion \( \varphi_{BC} \) that there exists a Berkeley cardinal is possible over any \( M \), i.e., \( \diamond \varphi_{BC} \) holds at every world.

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