Statistical errors in Monte Carlo-based inference for random elements

Yasutaka Shimizu
Department of Applied Mathematics, Waseda University; JST CREST.
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Abstract
Monte Carlo simulation is useful to compute or estimate expected functionals of random elements if those random samples are possible to be generated from the true distribution. However, when the distribution has some unknown parameters, the samples must be generated from an estimated distribution with the parameters replaced by some estimators, which causes a statistical error in Monte Carlo estimation. This paper considers such a statistical error and investigates the asymptotic distributions of Monte Carlo-based estimators when the random elements are not only the real valued, but also functional valued random variables. We also investigate expected functionals for semimartingales in details. The consideration indicates that the Monte Carlo estimation can get worse when a semimartingale has a jump part with unremovable unknown parameters.

Key words: Expected functionals; Monte Carlo estimator; asymptotic distribution, diffusion processes; semimartingales.

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1 Introduction
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{X}$ be a metric space with a norm $|| \cdot ||$. Consider a $\mathcal{X}$-valued random element $X^\theta$ with an unknown parameter $\theta \in \Theta \subset \mathbb{R}^p$, and put $P_\theta := \mathbb{P} \circ (X^\theta)^{-1}$, the distribution of $X^\theta$. Suppose that there exists the true value $\theta_0 \in \Theta$, and we are interested in the inference for the following expected functional of $X^\theta$:

$$H(\theta) = \mathbb{E} \left[ h(X^\theta, \theta) \right] = \int_{\mathcal{X}} h(x, \theta) P_\theta (dx),$$

where $h : \mathcal{X} \times \Theta \rightarrow \mathbb{R}$, and consider the estimation problem of $H(\theta_0) = \mathbb{E} \left[ h(X^{\theta_0}, \theta_0) \right]$.

When the functional $H$ is not explicit, the Monte Carlo simulation is useful to compute it by samples generated from the true distribution $P_{\theta_0}$, the Monte Carlo techniques are well-established.

*E-mail: shimizu@waseda.jp
under many methodologies and investigations; see, e.g., Robert and Casella [5]. However, most of those researches are based on the discussion on which the parameter \( \vartheta_0 \) is given. In practice, the Monte Carlo samples are given under the estimated distribution such as \( P_{\hat{\vartheta}} \), where \( \hat{\vartheta} \) is an estimator of \( \vartheta_0 \) that is consist of realizations (data). Then the estimated value of \( H(\vartheta_0) \) by Monte Carlo samples from \( P_{\hat{\vartheta}} \) has a statistical error.

Assume that a realization of \( X_{\vartheta_0} \) from \( P_{\vartheta_0} \), say \( X_{\vartheta_0,n} \), is given, where \( n \) is supposed to be a parameter on which the sample size depends: for example, when we observe \( n \)-samples of i.i.d. variables \( \{X_k\}_{k=1}^n \), it is regarded as \( X_{\vartheta_0,n} = (X_1,X_2,\ldots,X_n) \), so \( n \) represents the number of samples; when \( X_{\vartheta_0} \) is a stochastic process \( X = (X_t)_{t \geq 0} \), \( X_{\vartheta_0,n} \) can be a time-continuous observation in \([0,n]\)-time interval: \( X_{\vartheta_0,n} = (X_t)_{t \in [0,n]} \), or a discrete samples such as \( X_{\vartheta_0,n} = (X_{0_1},X_{1_1},\ldots,X_{n_1}) \), among others. We assume that a “good” estimator of \( \vartheta_0 \) are given based on the observations \( X_{\vartheta_0,n} \), say

\[
\hat{\vartheta}_n := \hat{\vartheta}(X_{\vartheta_0,n}) .
\]

We shall estimate \( H(\vartheta_0) \) as follows.

**Definition 1.1** (Monte Carlo estimator). Let \( \hat{\vartheta}_n \) be a consistent estimator of \( \vartheta_0 \): \( \hat{\vartheta}_n \overset{p}{\to} \vartheta_0 \) \((n \to \infty)\). Then a **Monte Carlo estimator** of \( H(\vartheta_0) \) is given by

\[
\hat{H}^*(\hat{\vartheta}_n) := \mathbb{E} \left[ h(X_{\hat{\vartheta}_n}, \hat{\vartheta}_n) \mid X_{\vartheta_0,n} \right] = \lim_{B \to \infty} \frac{1}{B} \sum_{k=1}^{B} h(X_{\hat{\vartheta}_n}^{k}, \hat{\vartheta}_n) , \quad a.s.,
\]

where \( X_{\hat{\vartheta}_n}^{k} \) and \( X_{\vartheta_0,n}^{k} \) \((k = 1,2,\ldots,B)\) are i.i.d. Monte Carlo samples from the estimated distribution \( P_{\hat{\vartheta}_n} \), which are independent of the data \( X_{\vartheta_0,n} \).

Since the above (conditional) expectation is computable without numerical error by letting \( B \to \infty \), we are interested in the statistical error between \( \hat{H}^*(\hat{\vartheta}_n) \) and \( H(\vartheta_0) \) in this paper.

Similar analysis of such a statistical error may be rather investigated in the context of the bootstrapping than that of the Monte Carlo estimation since random sampling from the estimated distribution \( P_{\hat{\vartheta}_n} \) is the same procedure as the parametric bootstrap. However, in the context of bootstrapping, the interest is the approximation of distributions of statistics, in our notation, an ‘asymptotical equivalence’ between the law \( \mathcal{L}(H(\hat{\vartheta}_n) - H(\vartheta_0)) \) and the conditional law \( \mathcal{L}(\hat{H}^*(\hat{\vartheta}_n) - H(\vartheta_0) \mid X_{\vartheta_0,n}) \), where \( \hat{\vartheta}_n^* \) is a statistic based on bootstrap samples, is a main concern; see, e.g., Mammen [4], Theorem 1 and the references therein. However, we are here interested in the ‘direct’ error of \( \hat{H}^*(\hat{\vartheta}_n) \) (not \( \hat{H}^*(\hat{\vartheta}_n^*) \)) to the true \( H(\vartheta_0) \). As far as our knowledge is concerned, there is no such an asymptotic analysis in the context of Monte Carlo estimation.

In this paper, we investigate such an error in details when \( X_{\vartheta} \) is a general random element. More precisely, we will find the asymptotic distribution of

\[
\gamma_n^{-1}(\hat{H}^*(\hat{\vartheta}_n) - H(\vartheta_0))
\]

if \( \gamma_n^{-1}(\hat{\vartheta} - \vartheta_0) \overset{d}{\to} Z \) as \( n \to \infty \) for some random variable \( Z \), when the random element \( X_{\vartheta} \) values in a generic metric space \( (\mathcal{X}, \| \cdot \|) \).
The cases where \( \mathcal{X} = \mathbb{R}^d \) and \( \mathcal{X} \) is a functional space are discussed separately since the sufficient conditions are given essentially in different forms. The former case is described in terms of the derivative of the density of the distribution with respect to \( \vartheta \), but the latter is done with a kind of derivative process of \( X^\vartheta \) with respect to \( \vartheta \). The case where \( X^\vartheta \) is a semimartingale values in \( \mathbb{D} \)-space is important, and our investigation indicates that an ‘error’ of (2) may get worse than the case where \( X^\vartheta \) values in \( \mathbb{C} \)-space, where (2) can be asymptotically normal.

The paper is organized as follows: In Section 2, we shall give the asymptotic distribution of the Monte Carlo estimator in general formulation, and in later sections, we will consider more specific cases: In Section 3, we treat the case where \( X^\vartheta \) is \( \mathbb{R}^d \)-valued random variable. In Section 4, we consider the case where \( X^\vartheta \) is a functional valued, and a sufficient condition to ensure the asymptotic normality of the Monte Carlo estimator are given in terms of the norm of the functional space \( \mathcal{X} \). We shall check the condition in each specific form of the functional. Section 5 is devoted to the case where \( X^\vartheta \) is described by the stochastic differential equations. The situation varies very much when \( X^\vartheta \) does not have a jump in the path (\( \mathcal{X} \) is a \( \mathbb{C} \)-space), and when \( X^\vartheta \) does (\( \mathcal{X} \) is a \( \mathbb{D} \)-space). The result indicates that the Monte Carlo estimator may not be valid in the case where \( X^\vartheta \) is a jump process.

Throughout the paper, we use the following notation.

- \( A \lesssim B \) means that there exists a universal constant \( c > 0 \) such that \( A \leq c \cdot B \).
- Denote by \( N_d(0, \Sigma) \) a \( d \)-dim Gaussian variable (distribution) with mean 0 and variance-covariance matrix \( \Sigma \). We omit the index \( d = 1 \).
- For a function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) and \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), denote by
  \[
  \nabla_x f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \right)^\top,
  \]
  and \( \nabla^k_x = \nabla_x \otimes \nabla_x^{k-1} \), \( (k = 2, 3, \ldots) \), forms a multilinear form.
- For a function \( f : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R} \) and an integer \( k \), denote by
  \[
  \dot{f}(x, \vartheta) = \nabla_{\vartheta} f(x, \vartheta); \quad f^{(k)}(x, \vartheta) = \nabla^k_x f(x, \vartheta).
  \]
  Note that \( \nabla^k_x f \) is a \( k \)-th order tensor.
- For a \( k \)-th order tensor \( x = (x_{i_1,i_2,\ldots,i_k})_{i_1,i_2,\ldots,i_k=1,d} \in \mathbb{R}^d \otimes \cdots \otimes \mathbb{R}^d \), denote by
  \[
  |x| = \sqrt{\sum_{i_1=1}^d \cdots \sum_{i_k=1}^d x_{i_1,i_2,\ldots,i_k}^2}.
  \]
- For a \( \mathcal{X} \)-valued random element \( X \), \( \|X\|_{L^p} = (\mathbb{E}\|X\|^p)^{1/p} \) for \( p > 0 \), where \( \| \cdot \| \) a norm on \( \mathcal{X} \), and write \( X \in L^p \) if \( \|X\|_{L^p} < \infty \).
2 Asymptotic distribution of Monte Carlo estimators

2.1 Main theorem

We assume that some estimator of \( \vartheta_0 \), say \( \hat{\vartheta}_n \), is given in a suitable way. We shall investigates a condition under which \( \hat{H}^* (\hat{\vartheta}_n) \) defined by (1) is asymptotically normal.

We make the following conditions.

A1. For any \( \vartheta' \in \Theta \), \( \nabla_{\vartheta} \mathbb{E}[h(X^{\vartheta'}, \vartheta)] = \mathbb{E}[h(X^{\vartheta'}, \vartheta)] \).

A2. The function \( \vartheta \mapsto \mathbb{E}[h(X^{\vartheta}, \vartheta_0)] \) is continuous.

A3. There exists a diagonal matrix \( \Gamma_n = \text{diag}(\gamma_n^{(1)}, \ldots, \gamma_n^{(p)}) \) with \( \gamma_n^{(k)} > 0 \) and \( \gamma_{ns} := \max_{1 \leq k \leq p} \gamma_n^{(k)} \downarrow 0 \) \( (n \to \infty) \) such that the estimator \( \hat{\vartheta}_n \) satisfies

\[
\Gamma_n^{-1} (\hat{\vartheta}_n - \vartheta_0) \xrightarrow{d} Z; \quad \gamma_{ns}^{-1} (\hat{\vartheta}_n - \vartheta_0) \xrightarrow{d} Z^*,
\]

as \( n \to \infty \), for a \( p \)-dim random variable \( Z \) and \( Z^* \).

A4(q). There exists a \( \mathcal{F}^p \)-valued random element \( Y^{\vartheta} \) such that \( Y^{\vartheta} \in L^q \) for \( q > 0 \) and

\[
\|X^{\vartheta} + u - X^{\vartheta} - u \; Y^{\vartheta}\|_{L^q} = o(|u|), \quad |u| \to 0,
\]

uniformly in \( \vartheta \in \Theta \).

Remark 2.1. As for the condition A3 it is usually holds that \( \gamma_n^{(k)} = 1/\sqrt{n} \) for all \( k \) in IID-cases, but there are some examples that the rates of convergence are different among parameters; e.g., for a sequence such that \( T_n/n \to 0 \) as \( n \to \infty \) and constants \( \sigma_1^2, \sigma_2^2 \neq 0, \)

\[
\text{diag}(\sqrt{T_n/n}, \sqrt{n})(\hat{\vartheta}_n^{(1)} - \vartheta_0^{(1)}, \hat{\vartheta}_n^{(2)} - \vartheta_0^{(2)})^\top \xrightarrow{d} N_2(0, \text{diag}(\sigma_1^2, \sigma_2^2)) = Z.
\]

In such a case, \( \gamma_{ns} = 1/\sqrt{T_n} \) and we have

\[
\sqrt{T_n}(\hat{\vartheta}_n^{(1)} - \vartheta_0^{(1)}, \hat{\vartheta}_n^{(2)} - \vartheta_0^{(2)})^\top \xrightarrow{d} (N(0, \sigma_1^2), 0)^\top = Z^*,
\]

which is a degenerate random variable; see also Examples 5.3 and 5.1.

Remark 2.2. In the condition A4(q), the random element \( Y^{\vartheta} \) is interpreted as the first derivative of \( X^{\vartheta} \) with respect to \( \vartheta \) in \( L^q \)-sense. This condition is useful when \( \mathcal{F} \) is a functional space such that \( X^{\vartheta} \) is a stochastic process as is shown in Section 4.

The following two results are the basis of our discussion.

Theorem 2.1. Suppose that A1–A3 hold true, and that there exists a constant vector \( C_{\vartheta_0} \in \mathbb{R}^p \) such that, for \( \gamma_{ns} := \max_{1 \leq k \leq p} \gamma_n^{(k)} \),

\[
\gamma_{ns}^{-1} \mathbb{E} \left[ h(X^{\vartheta'}, \vartheta_0) - h(X^{\vartheta_0}, \vartheta_0) \right]_{\vartheta = \hat{\vartheta}_n} = C_{\vartheta_0}^\top \gamma_{ns}^{-1} (\hat{\vartheta}_n - \vartheta_0) + o_p(1), \tag{3}
\]

as \( n \to \infty \). Then it holds that

\[
\gamma_{ns}^{-1} [\hat{H}^* (\hat{\vartheta}_n) - H(\vartheta_0)] \xrightarrow{d} (\mathbb{E}[h(X^{\vartheta_0}, \vartheta_0)] + C_{\vartheta_0})^\top Z^*, \quad n \to \infty.
\]
Proof. Let $X^{\vartheta_0} \sim P_{\vartheta_0}$, which is independent of the data $X^{\vartheta_0,n}$. Then we have that
\[
\hat{H}^*(\vartheta_n) - H(\vartheta_0) = \mathbb{E}\left[h(X^{\vartheta_0}, \vartheta_n) - h(X^{\vartheta_0}, \vartheta_0)\right]_{X^{\vartheta_0,n}} = \mathbb{E}\left[h(X^{\vartheta_0}, \vartheta_0) - h(X^{\vartheta_0}, \vartheta_0)\right]_{X^{\vartheta_0,n}} + \mathbb{E}\left|h(X^{\vartheta_0}, \vartheta_0) - h(X^{\vartheta_0}, \vartheta_0)\right|_{\vartheta = \vartheta_0}.
\]
Then, under A2, the continuous mapping theorem yields the consequence. □

Theorem 2.2. Suppose the assumptions A3 and A4(q) for a constant $q > 1$, and that there exists a $\mathbb{R}^p$-valued random variable $G_{\vartheta_0} \in L^1$ such that, for each $u \in \mathbb{R}$ with $\vartheta_0 + u \in \Theta$,\n\[
\|h(X^{\vartheta_0+u}, \vartheta) - h(X^{\vartheta_0}, \vartheta) - u^\top G_{\vartheta_0}\| \lesssim \|X^{\vartheta_0+u} - X^{\vartheta_0} - u^\top Y_{\vartheta_0}\|_{L^q} + ru, \quad (4)
\]
where $r_u = o(|u|)$ as $|u| \to 0$. Then the equality (3) holds true with $C_{\vartheta_0} = \mathbb{E}[G_{\vartheta_0}]$. Proof. The assumption A4(q) with $q > 1$ implies that\n\[
\|h(X^{\vartheta_0+u}, \vartheta) - h(X^{\vartheta_0}, \vartheta) - u^\top G_{\vartheta_0}\| = o(|u|), \quad |u| \to 0.
\]
Then it follows that\n\[
\mathbb{E}[h(X^{\vartheta_0+u}, \vartheta_0) - h(X^{\vartheta_0}, \vartheta_0)] = \mathbb{E}[G_{\vartheta_0}]^\top u + o(|u|), \quad |u| \to 0.
\]
Putting $u = \hat{\vartheta}_n - \vartheta_0$ and multiplying $\gamma_n^{-1}$ in both sides, we obtain that\n\[
\gamma_n^{-1} \mathbb{E}\left[h(X^{\vartheta_0}, \vartheta_0) - h(X^{\vartheta_0}, \vartheta_0)\right]_{\vartheta = \vartheta_0} = \mathbb{E}[G_{\vartheta_0}]^\top \gamma_n^{-1}(\hat{\vartheta}_n - \vartheta_0) + o_p(|\gamma_n^{-1}(\hat{\vartheta}_n - \vartheta_0)|).
\]
The last term converges to zero in probability under A\text{3}. This ends the proof. □

In some simple examples, the condition (3) in Theorem 2.1 can be checked by a direct computation. Actually, we can give an easy-to-check condition for (3) when $X^\vartheta$ is a $\mathbb{R}^d$-valued random variable with a probability density. When it is not possible, especially in the case where $X^\vartheta$ is a stochastic process, Theorem 2.2 will be useful to check the condition (3).

3 Euclidian-valued random variables

Let $\mathcal{X} = \mathbb{R}^d$, and let $X^\vartheta$ be a random with the probability density $f(\cdot; \vartheta)$:
\[
\mathbb{P}(X^\vartheta \in A) = \int_A f(x, \vartheta) \, dx, \quad A \subset \mathbb{R}^d.
\]
We shall investigate the condition (3) in Theorem 2.1 directly. The following theorem is an immediate consequence of Taylor’s formula.
**Theorem 3.1.** Suppose that the probability density of \( f : \mathcal{X} \times \Theta \to \mathbb{R} \) is twice differentiable with respect to \( \vartheta \in \Theta \) with \( \int_{\mathcal{X}} h(x, \vartheta_0) \hat{f}(x, \vartheta_0) \, dx < \infty \). Moreover, suppose \( A3 \) and that it holds for the second derivative of \( f \) in \( \vartheta \), say \( \hat{f} \), that

\[
\sup_{\vartheta \in \Theta} \left| \int_{\mathcal{X}} h(x, \vartheta_0) \hat{f}(x, \vartheta) \, dx \right| < \infty. \tag{5}
\]

Then the condition \( 3 \) holds true with \( C_{\vartheta_0} = \int_{\mathcal{X}} h(x, \vartheta_0) \hat{f}(x, \vartheta_0) \, dx \).

**Proof.** For \( \vartheta \in \Theta \) and \( u \in \mathbb{R}^p \) with \( \vartheta + u \in \Theta \), it follows by Taylor’s formula that

\[
\mathbb{E} \left[ h(X^{\vartheta+u}, \vartheta) - h(X^{\vartheta}, \vartheta) \right] = \int_{\mathcal{X}} h(x, \vartheta) \left[ f(x, \vartheta + u) - f(x, \vartheta) \right] \, dx
\]

\[
= \int_{\mathcal{X}} h(x, \vartheta) \left[ u^\top \hat{f}(x, \vartheta) + u^\top \hat{f}(x, \vartheta_0)u \right] \, dx,
\]

where \( \vartheta^u := \vartheta + \eta u \) for some \( \eta \in [0, 1] \). Hence, putting \( \vartheta = \vartheta_0 \) and \( u = \hat{\vartheta}_n - \vartheta_0 \) and multiplying \( \gamma_n^{-1} \) in both sides, we have that

\[
\gamma_n^{-1} \mathbb{E} \left[ h(X^{\vartheta_0}, \vartheta_0) - h(X^{\hat{\vartheta}_n}, \vartheta_0) \right] = \left( \int_{\mathcal{X}} h(x, \vartheta_0) \hat{f}(x, \vartheta_0) \, dx \right)^\top \cdot \gamma_n^{-1} (\hat{\vartheta}_n - \vartheta_0)
\]

\[
+ O_p \left( \| \hat{\vartheta}_n - \vartheta_0 \| \right), \quad n \to \infty.
\]

This completes the proof. \( \square \)

**Example 3.1.** Let \( X^{\vartheta} \) be a real valued random variable with \( \Theta \subset \mathbb{R} \), and assume that we have a set of IID samples \( X^{\vartheta_0,n} = (X_1, \ldots, X_n) \) with the probability density \( f(x, \vartheta) \) with the Fisher information \( I_0 := -\int_{\mathcal{X}} f'(x, \vartheta) f(x, \vartheta) \, dx < \infty \) for any \( \vartheta \in \Theta \). Suppose the regularities such that the maximum likelihood estimator \( \hat{\vartheta}_n \) is asymptotically normal:

\[
\sqrt{n} \left( \hat{\vartheta}_n - \vartheta_0 \right) \xrightarrow{d} N(0, I_{\vartheta_0}^{-1}), \quad n \to \infty.
\]

Moreover, suppose that \( h : \mathbb{R} \times \Theta \to \mathbb{R} \) be a function such that all the conditions in Theorems 2.1 and 3.1 and that \( C_{\vartheta} = \int_{\mathcal{X}} h(x, \vartheta) \hat{f}(x, \vartheta) \, dx \) is continuous in \( \vartheta \). Then, as \( n \to \infty \),

\[
\sqrt{n} \left( \hat{H}^*(\hat{\vartheta}_n) - H(\vartheta_0) \right) \xrightarrow{d} N(0, C_{\vartheta_0}^2 I_{\vartheta_0}^{-1}).
\]

Therefore an \( \alpha \)-confidence interval for \( H(\vartheta_0) \) via the Monte Carlo estimator \( \hat{H}^*(\hat{\vartheta}_n) \) is given by

\[
\left[ \hat{H}^*(\hat{\vartheta}_n) - \frac{z_{\alpha/2}}{\sqrt{n}} C_{\hat{\vartheta}_n} I_{\hat{\vartheta}_n}^{-1/2}, \hat{H}^*(\hat{\vartheta}_n) + \frac{z_{\alpha/2}}{\sqrt{n}} C_{\hat{\vartheta}_n} I_{\hat{\vartheta}_n}^{-1/2} \right],
\]

where \( z_{\alpha} \) is the upper \( \alpha \) percentile for \( N(0,1) \).
Example 3.2. Let us consider more concrete example than the previous. Let $X^\vartheta = X_{\alpha,\beta}$ be a random variable that has a gamma distribution with probability density

$$
\gamma(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} I_{\{x > 0\}}, \quad \vartheta = (\alpha, \beta) \in \Theta.
$$

Assuming that $\alpha > 1$, we shall estimate $H(\vartheta) = \mathbb{E}[h(X)]$ for a smooth function $h \in C^\infty(\mathbb{R})$ by Monte Carlo simulations based on $n$-IID samples: $X_1, \ldots, X_n$. We suppose that, for the $n$-th order Taylor expansion of $h$ around $x = 0$, say $h_n(x) := \sum_{k=1}^n \frac{h^{(k)}(0)}{k!} x^k$, satisfies

$$
sup_{x \in \mathbb{R}} |h_n(x) - h(x)| \rightarrow 0 \quad n \rightarrow \infty.
$$

The maximum likelihood estimator $\hat{\vartheta}_n$ is given by the solution to $\sum_{i=1}^n \nabla_{\vartheta} \log \gamma(X_i; \hat{\vartheta}_n) = 0$, and it satisfies that

$$
\sqrt{n}(\hat{\vartheta}_n - \vartheta) \overset{d}{\rightarrow} N_2(0, I^{-1}(\vartheta)), \quad n \rightarrow \infty,
$$

where $I(\vartheta)$ is the Fisher information matrix given by

$$
I(\vartheta) = \begin{pmatrix}
\psi'(\vartheta) & 1/eta \\
1/\beta & \alpha/\beta^2
\end{pmatrix},
$$

and $\psi$ is the digamma function with the derivative $\psi'$. By a direct calculation, we have $\nabla_{\alpha} \gamma(x; \vartheta) = (\nabla_{\alpha} \gamma, \nabla_{\beta} \gamma)$:

$$
\nabla_{\alpha} \gamma(x; \vartheta) = \left(\frac{\log \beta}{\Gamma(\alpha)} - \psi(\alpha)\right) \gamma(x; \alpha, \beta) + \beta \gamma(x; \alpha - 1, \beta)
$$

$$
\nabla_{\beta} \gamma(x; \vartheta) = \frac{1}{\beta} \gamma(x; \alpha, \beta) - \frac{\alpha}{\beta} \gamma(x; \alpha + 1, \beta).
$$

Therefore it follows from the dominated convergence theorem that

$$
C_\vartheta = \int_0^\infty h(x) \gamma(x; \vartheta) \, dx = \sum_{k=1}^\infty \frac{h^{(k)}(0)}{k!} \int_0^\infty x^k \gamma(x; \vartheta) \, dx
$$

$$
= \sum_{k=1}^\infty \frac{h^{(k)}(0)}{k!} \left(\frac{\log \beta}{\Gamma(\alpha)} - \psi(\alpha)\right) \mathbb{E}[X^k_{\alpha,\beta}] + \beta \mathbb{E}[X^k_{\alpha-1,\beta}]
$$

$$
\quad + \frac{1}{\beta} \mathbb{E}[X^k_{\alpha,\beta}] - \frac{\alpha}{\beta} \mathbb{E}[X^k_{\alpha+1,\beta}],
$$

which is computable by the formula

$$
\mathbb{E}[X^k_{\alpha,\beta}] = \frac{\alpha(\alpha + 1) \ldots (\alpha + k - 1)}{\beta^k}.
$$

If $\Theta$ is bounded, then all the conditions in Theorem 3.1 are satisfied, and hence we have the asymptotic distribution of the Monte Carlo estimator $\hat{H}^\star(\hat{\vartheta}_n)$:

$$
\sqrt{n}(\hat{H}^\star(\hat{\vartheta}_n) - H(\vartheta)) \overset{d}{\rightarrow} N\left(0, \frac{\alpha}{\beta}\mathbb{E}[\vartheta I^{-1}(\vartheta) C_\vartheta]\right), \quad n \rightarrow \infty.
$$
4 Expected functionals for stochastic processes

In this section, we consider the case where $\mathcal{X}$ is a functional space on a compact set $K \subset \mathbb{R}$: e.g., $C(K)$, $D(K)$, with the sup norm

$$\|x\| = \sup_{t \in K} |x_t|, \quad x = (x_t)_{t \in K} \in \mathcal{X}.$$ 

Without loss of generality, we assume that $K = [0, 1]$ for simplicity of notation, so we consider the case where $X^\vartheta$ is a continuous time stochastic process on $[0, 1]$.

4.1 Functionals of expected integrals

In this section, we are interested in the expected integral-type functionals

$$H(\vartheta) = \mathbb{E}\left[ \int_0^1 V_{\vartheta}(X^\vartheta_t, t) \, dt \right],$$

for a function $V : \mathbb{R}^d \times [0, 1] \to \mathbb{R}$. This is the case where $H(\vartheta) = \mathbb{E}[h(X^\vartheta, \vartheta)]$ with

$$h(x, \vartheta) = \int_0^1 V_{\vartheta}(x_t, t) \, dt, \quad x \in \mathcal{X}.$$ 

The marginal distribution of a stochastic process $X^\vartheta$ is generally not explicit and the expectation $\mathbb{E}[V_{\vartheta}(X^\vartheta_t, t)]$ is not clear. In such a case, Theorem 2.2 can be useful to the analysis since the assumption A4$(p)$ can be confirmed.

Example 4.1. Suppose that $X^\vartheta$ satisfies the following stochastic differential equation:

$$X^\vartheta_t = x(\vartheta) + \int_0^t a(X^\vartheta_s, \vartheta) \, ds + \int_0^t b(X^\vartheta_s, \vartheta) \, dW_s,$$

where $W$ is a Wiener process and $a, b$ are functions with some “good” regularities and $\vartheta \in \mathbb{R}^p$ is the unknown parameter. According to Section 5 under some regularities, the derivative process $Y^\vartheta = (Y^\vartheta_t)_{t \in [0, 1]}$ is given as follows.

$$Y^\vartheta_t = \dot{x}(\vartheta) + \int_0^t A(X_s, Y_s, \vartheta) \, ds + \int_0^t B(X_s, Y_s, \vartheta) \, dW_s,$$

where $\dot{x}(\vartheta) = \nabla_{\vartheta} x(\vartheta)$; $A, B$ are functions on $\mathbb{R} \times \mathbb{R}^p \times \mathbb{R}$ of the form

$$A(x, y, \vartheta) = \nabla_{x, a}(x, \vartheta) y + \dot{a}(x, \vartheta); \quad B(x, y, \vartheta) = \nabla_{x, b}(x, \vartheta) y + \dot{b}(x, \vartheta).$$

This $Y^\vartheta$ can satisfy that,

$$\mathbb{E}\|X^{\vartheta+u} - X^\vartheta - u^\top Y^\vartheta\|_p \lesssim |u|^{2p},$$

for each $u \in \mathbb{R}^p$ and any $p \geq 2$, which implies A4$(p)$. 


Theorem 4.1. Suppose that there exists an integer $n \geq 1$ and $\vartheta \in \Theta$ such that $V^{(n)}_{\vartheta}(x,t) := \nabla_x V_{\vartheta}(x,t)$ is Lipschitz continuous in $x$ uniformly in $t \in [0,1]$:

$$\sup_{t \in [0,1]} |V^{(n)}_{\vartheta}(x,t) - V^{(n)}_{\vartheta}(y,t)| \lesssim |x-y|, \quad x, y \in \mathbb{R}. $$

Moreover, suppose $A4(q)$ for some $q \geq 2n$, and that

$$\sup_{t \in [0,1]} |V^{(k)}_{\vartheta}(X^{\vartheta}_t, t)| \in L^r, \quad k = 1, \ldots, n,$n

for some $r > 1$ with $1/r + 1/q = 1$. Then the condition (41) holds true with

$$G_{\vartheta} = \int_0^1 V^{(1)}_{\vartheta}(X^{\vartheta}_t, t) Y^\vartheta_t \, dt.$$

Proof. We shall check the condition (41) in Theorem 2.2. In the proof, we consider only the case $d = 1$ for the simplicity of notation. The general case can be shown similarly.

Let

$$R^{\vartheta, u}_t := X^{\vartheta+u}_t - X^{\vartheta}_t - Y^\vartheta_t, \quad u \in \mathbb{R}. $$

We note that $||R^{\vartheta, u}_t||_{L^d} \lesssim |u|^q$ by $A4(q)$. It follows by Taylor’s formula that

$$\mathbb{E} \left[ h(X^{\vartheta+u}_t, \vartheta) - h(X^{\vartheta}_t, \vartheta) - u \vartheta \int_0^1 V^{(1)}_{\vartheta}(X^{\vartheta}_t, t) Y^\vartheta_t \, dt \right]$$

$$= \mathbb{E} \left[ \int_0^1 \left( V_{\vartheta}(X^{\vartheta+u}_t, t) - V_{\vartheta}(X^{\vartheta}_t, t) - u \vartheta V^{(1)}_{\vartheta}(X^{\vartheta}_t, t) Y^\vartheta_t \right) \, dt \right]$$

$$= \mathbb{E} \left[ \int_0^1 V^{(1)}_{\vartheta}(X^{\vartheta}_t, t) R^{\vartheta, u}_t \, dt \right] + \sum_{k=2}^{n-1} \frac{1}{k!} \mathbb{E} \left[ \int_0^1 V^{(k)}_{\vartheta}(X^{\vartheta}_t, t) (R^{\vartheta, u}_t + u \vartheta Y^\vartheta_t)^k \, dt \right]$$

$$+ \frac{1}{n!} \mathbb{E} \left[ \int_0^1 V^{(n)}_{\vartheta}(X^{\vartheta}_t, t) (R^{\vartheta, u}_t + u \vartheta Y^\vartheta_t)^n \, dt \right],$$

where $\tilde{X}^\vartheta = X^{\vartheta+u}_t - X^{\vartheta}_t$ for some random number $\eta^\vartheta_t \in [0,1]$.

Firstly, it follows from Hölder’s inequality that, for $q, r > 1$ with $1/q + 1/r = 1$,

$$\mathbb{E} \left[ \int_0^1 V^{(1)}_{\vartheta}(X^{\vartheta}_t, t) R^{\vartheta, u}_t \, dt \right] \lesssim \sup_{t \in [0,1]} V^{(1)}_{\vartheta}(X^{\vartheta}_t, t) \left\| R^{\vartheta, u}_t \right\|_{L^q}$$

Secondly, noticing that

$$\left\| \tilde{X}^{\vartheta, u} - X^{\vartheta} \right\|_{L^2} \leq \left\| R^{\vartheta, u}_t \right\|_{L^\infty} + |u| \left\| Y^\vartheta \right\|_{L^2} = O(|u|), \quad |u| \to 0,$$

we see that

$$\mathbb{E} \left[ \int_0^1 V^{(n)}_{\vartheta}(\tilde{X}^{\vartheta}_t, t) (R^{\vartheta, u}_t + u \vartheta Y^\vartheta_t)^n \, dt \right] - \mathbb{E} \left[ \int_0^1 V^{(n)}_{\vartheta}(X^{\vartheta}_t, t) (R^{\vartheta, u}_t + u \vartheta Y^\vartheta_t)^n \, dt \right]$$
$$\lesssim 2^{n-1} \int_0^1 \mathbb{E} \left[ |\tilde{X}_t^u - X_t^u| \right] \left\{ |R_t^{\theta,u}| + |u|^n |Y_t^{\theta}|^n \right\} \, dt$$
$$\lesssim \|X_t^{\theta,u} - X_t^\theta\|_{L^2} \left\{ \|R_t^\theta\|_{L^{2n}}^n + |u|^n \|Y_t^\theta\|^n_{L^{2n}} \right\} = o(|u|^n), \quad \text{as } |u| \to 0.$$  

Finally, by the Schwartz inequality, it is easy to see that, for each $k = 2, \ldots, n$,

$$\left| \mathbb{E} \int_0^1 V^{(k)}_\theta(X_t^\theta, t)(R_t^{\theta,u} + u^\top Y_t^\theta)^k \right|$$
$$\leq \int_0^1 \mathbb{E} \left[ |V^{(k)}_\theta(X_t^\theta, t)| \cdot \|R_t^{\theta,u} + u^\top Y_t^\theta\|^k \right] \, dt$$
$$\leq 2^{k-1} \int_0^1 \|V^{(k)}_\theta(X_t^\theta, t)\|_{L^q} \cdot \left\{ \|R_t^{\theta,u}\|_{L^q}^k + |u|^k \|Y_t^\theta\|^k_{L^q} \right\}$$
$$= o(|u|^k),$$

where $s > 1$ with $1/s + k/q = 1$. Note that such $s > 1$ exists under our assumption since $(1 - k/q)^{-1} \geq n/(n - 1) > 1$ when $q \geq 2n$. As a result, we have that

$$\left| \mathbb{E} \left[ h(X_t^{\theta+u, \vartheta}) - h(X_t^\theta, \vartheta) - u^\top \int_0^1 V^{(1)}_\vartheta(X_t^\theta, t)Y_t^\theta \, dt \right] \right| \lesssim \|R_t^{\theta,u}\|_{L^q} + o(|u|^2),$$

which implies the condition (4) in Theorem 2.2 with $G_\vartheta = \int_0^1 V^{(1)}_\vartheta(X_t^\theta, t)Y_t^\theta \, dt$. Therefore the proof is completed. $\square$

**Remark 4.1.** If the function $V_\vartheta$ is a “good” function such that a “lower” derivative is Lipschitz continuous, then Theorem 4.1 requires just a “small” $q \geq 2$ for which $A(q)\varphi$ holds true. The more violent the function $V$ is, the more the integrability condition becomes stronger.

**Example 4.2.** Consider a 1-dim (ergodic) diffusion process $X_t^\theta = (X_t)_{t \geq 0}$: for a constant $x > 0$,

$$X_t^{\vartheta_0} = x + \int_0^t a(X_s^{\vartheta_0}, \vartheta_0) \, ds + \int_0^t b(X_s^{\vartheta_0}) \, dW_s,$$

where $\vartheta_0 \in \mathbb{R}$ is unknown, consider the estimation of

$$H(\vartheta_0) = \int_0^T e^{-rt} U(X_t^{\vartheta_0}) \, dt,$$

for a constant $r > 0$ and a function $U \in C(\mathbb{R})$, which is the case where $V_\vartheta(x, t) = e^{-rt} U(x) \mathbf{1}_{[0,T]}(t)$; see also Example 3.3 for practical applications of this example.

Assume that we have a continuous data $\{X_t\}_{t \in [0,T]}$, and consider the long term asymptotics: $T \to \infty$. Then, under some regularities, the maximum likelihood estimator of $\vartheta$, say $\hat{\vartheta}_T$, satisfies that

$$\sqrt{T}(\hat{\vartheta}_T - \vartheta_0) \xrightarrow{d} N(0, I^{-1}(\vartheta_0)), \quad T \to \infty,$$

where $I(\vartheta) = \int_\mathbb{R} \frac{a^2(x, \vartheta)}{b^2(x)} \pi(dx)$ for a stationary distribution $\pi$, and it can be estimated by, e.g.,

$$\hat{I}_T(\vartheta) = \frac{1}{T} \int_0^T \frac{\hat{a}^2(X_t, \vartheta)}{b^2(X_t)} \, dt \xrightarrow{p} I(\vartheta), \quad T \to \infty,$$
uniformly in $\vartheta \in \Theta$; see, e.g., Kutoyants \cite{Kutoyants2010}. Therefore, considering the derivative process $Y^\vartheta$ given in Example 4.1, we have that

$$
\sqrt{T}(\hat{H}^*(\hat{\vartheta}_T) - H(\vartheta_0)) \overset{d}{\to} N(0, C^2_{\vartheta_0} I(\vartheta_0)^{-1}), \quad T \to \infty,
$$

where

$$
C_{\vartheta} = \mathbb{E} \left[ \int_0^T e^{-\alpha \nabla x U(X_{\vartheta}^t)} Y_{\vartheta}^t \, dt \right].
$$

Therefore we can obtain the $\alpha$-confidence interval

$$
\left[ \hat{H}^*(\hat{\vartheta}_T) - \frac{z_{\alpha/2}}{\sqrt{T}} C_{\vartheta_0} \hat{I}_T(\hat{\vartheta}_T)^{-1} / 2, \hat{H}^*(\hat{\vartheta}_T) + \frac{z_{\alpha/2}}{\sqrt{T}} C_{\vartheta_0} \hat{I}_T(\hat{\vartheta}_T)^{-1} / 2 \right].
$$

However, we must note that we may sometimes need Monte Carlo simulation to compute $C_{\vartheta}$ again.

### 4.2 Functionals of integrated processes

Let us consider the following quantity: for a function $\varphi_{\vartheta} : \mathbb{R} \to \mathbb{R}$ and $T \in (0, 1]$,

$$
H(\vartheta) = \mathbb{E} \left[ \varphi_{\vartheta} \left( \frac{1}{T} \int_0^T X_{\vartheta}^t \, dt \right) \right].
$$

We use the following notation for simplicity:

$$
X_s = \frac{1}{T} \int_0^T X_t \, dt,
$$

for a process $X = (X_t)_{t \in [0,1]}$. Then we have the following theorem.

**Theorem 4.2.** Suppose that there exists an integer $n \geq 1$ and $\vartheta \in \Theta$ such that $\varphi_{\vartheta}^{(n)}(x)$ is Lipschitz continuous:

$$
|\varphi_{\vartheta}^{(n)}(x) - \varphi_{\vartheta}^{(n)}(y)| \lesssim |x - y|, \quad x, y \in \mathbb{R}.
$$

Moreover, suppose $A4(q)$ for some $q \geq 2n$, and that

$$
\varphi_{\vartheta}^{(k)} (X_{\vartheta}^t) \in L^r, \quad k = 1, \ldots, n,
$$

for the constant $r > 1$ with $1/r + 1/q = 1$. Then the condition (4) holds true with

$$
G_{\vartheta} = \varphi_{\vartheta}^{(1)} (X_{\vartheta}^t) Y_{\vartheta}^t.
$$

**Proof.** It follows by Jensen’s inequality that

$$
|X_s^{\vartheta} + u - X_s^{\vartheta} - u Y_s^{\vartheta}| \leq \frac{1}{T} \int_0^T |X_t^{\vartheta} + u - X_t^{\vartheta} - u Y_t^{\vartheta}| \, dt \leq \|X^{\vartheta} + u - X^{\vartheta} - u Y^{\vartheta}\|,
$$

with probability one. Hence $Y_s^{\vartheta} = \frac{1}{T} \int_0^T Y_t^{\vartheta} \, dt$ is the derivative of $X^{\vartheta}$ w.r.t. $\vartheta$. 


We can take the same argument as in the Theorem 4.1, we use Taylor’s formula and Hölder’s inequality to obtain that
\[
\left| \mathbb{E}[h(X^0 + u) - h(X^0) - \varphi^{(1)}(X^0)u^\top Y^0] \right|
\leq \mathbb{E}\left| \varphi^{(1)}(X^0)(X^0 + u - X^0)^\top Y^0 \right| + \sum_{k=1}^{n-1} \frac{1}{k!} \mathbb{E}\left| \varphi^{(k)}(X^0)(X^0 + u - X^0)^k \right|
+ \frac{1}{n!} \mathbb{E}\left| \varphi^{(k)}(X^0)(X^0 + u - X^0)^n \right|.
\]
Then the same argument as in the proof of Theorem 4.1 enables us to check the condition (4) in Theorem 2.2.

\[ \square \]

**Example 4.3.** When \( X^0 \) is a stock price, the price of *Asian call option* for \( X^0 \) with maturity \( T \) and the strike price \( K \) is given by
\[
\mathcal{C}_{T,K} = \mathbb{E} \left[ \max \left\{ \frac{1}{T} \int_0^T X_t^{\vartheta_0} \, dt - K, 0 \right\} \right]
\]
where \( \delta > 0 \) is an interest rate, and \( \mathbb{E} \) is usually taken as an expectation with respect to the risk neutral probability. This is approximated as
\[
H_{\vartheta}(\vartheta_0) := \mathbb{E} \left[ \varphi_{\vartheta} \left( \frac{1}{T} \int_0^T X_t^{\vartheta_0} \, dt \right) \right]
\]
by a function \( \varphi_{\vartheta}(x) \in C^\infty(\mathbb{R}) \) such that
\[
\sup_x |\varphi_{\vartheta}(x) - \max\{x - K, 0\}| \to 0, \quad \vartheta \to 0.
\]
For example, we can take a function \( \varphi_{\vartheta}(x) = 2^{-1}(\sqrt{(x - K)^2 + \vartheta^2} + x - K) \). Then it follows by the dominated convergence theorem that \( H_{\vartheta}(\vartheta) \to \mathcal{C}_{T,K} \) as \( \vartheta \to 0 \) if \( X^0 \in L^1 \).

Assume that a suitable estimator of \( \vartheta_0 \in \mathbb{R}^p \) is obtained: e.g.,
\[
\sqrt{T} \left( \hat{\vartheta}_T - \vartheta_0 \right) \overset{d}{\to} N(0, \Sigma), \quad T \to \infty,
\]
for a positive definite matrix \( \Sigma \in \mathbb{R}^p \otimes \mathbb{R}^p \). Then we can apply Theorem 4.2 to \( H_{\vartheta}(\vartheta) \), and we have
\[
\sqrt{T} \left( \hat{H}_{\vartheta}(\hat{\vartheta}_T) - H_{\vartheta}(\vartheta_0) \right) \overset{d}{\to} N(0, C_{\vartheta_0} \Sigma C_{\vartheta_0}),
\]
where
\[
C_{\vartheta} = \lim_{\vartheta \to 0} \mathbb{E} \left[ \frac{Y_{\vartheta}^2}{2} \left( \frac{(X_\vartheta^K - K)}{(X_{\vartheta}^0 - K)^2 + \vartheta^2} + 1 \right) \right] = \mathbb{E} \left[ \frac{Y_{\vartheta}^2}{2} \left\{ \text{sgn}(X_{\vartheta}^0 - K) + 1 \right\} \right],
\]
with \( \text{sgn}(z) = 1_{\{z > 0\}} - 1_{\{z < 0\}} \). Note this quantity would be computed by Monte Carlo simulation in practice with \( \vartheta_0 \) replaced by \( \hat{\vartheta}_n \). We will discuss when the condition A4(\( q \)) holds true when \( S^\vartheta \) is a semimartingale with jumps in Section 5.
Remark 4.2. According to the proof of Theorem 4.2, we can consider more general functional for $X^\theta$ under some smoothness conditions for $\varphi_\theta$. That is, suppose that there exists a $\mathbb{R}^p$-valued random variable $Y^\theta$ such that the following inequality holds true:

$$|X_{t}^{\varphi + u} - X_{t}^{\varphi} - u^\top Y^\theta| \lesssim \|X^{\varphi + u} - X^{\varphi} - u^\top Y^\theta\| + |u|^{1+\delta} \text{ a.s.},$$

for $\delta > 0$ and the derivative $Y^\theta$. Then the same proof as that of Theorem 4.2 works with

$$G_\varphi = \varphi_\theta^1 (X_t^{\varphi}) Y^\theta.$$

For example, let

$$X_{t}^{\varphi} = \int_{0}^{T} U(X_{t}^{\varphi}) \, dt$$

for $T > 0$ and a function $U \in C^2(\mathbb{R})$ with bounded derivatives. Then we find that

$$\tilde{Y}^\theta = \int_{0}^{T} U^{(1)}(X_{t}^{\varphi}) Y_t^{\varphi} \, dt,$$

since it follows that

$$|X_{t}^{\varphi + u} - X_{t}^{\varphi} - u^\top Y^\theta| \leq \int_{0}^{T} |U(X_{t}^{\varphi + u}) - U(X_{t}^{\varphi}) - u^\top U^{(1)}(X_{t}^{\varphi}) Y_t^{\varphi}| \, dt$$

$$\lesssim \int_{0}^{T} |U^{(1)}(X_{t}^{\varphi})(X_{t}^{\varphi + u} - X_{t}^{\varphi} - u^\top Y_t^{\varphi})| \, dt + |u|^2$$

$$\lesssim \|X^{\varphi + u} - X^{\varphi} - u^\top Y^\theta\| + |u|^2.$$

This argument can include Theorem 4.1.

5 Expected functionals of semimartingales

5.1 Stochastic differential equations with jumps

On a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, consider a 1-dim stochastic process $X = (X_t)_{t \in [0,T]}$ that satisfies the following stochastic differential equation (SDE) with a multidimensional parameter $\varphi \in \Theta \subset \mathbb{R}^p$:

$$X^\theta_t = x(\varphi) + \int_{0}^{t} a(X^\theta_s, \varphi) \, ds + \int_{0}^{t} b(X^\theta_s, \varphi) \, dW_s + \int_{0}^{t} \int_{E} c(X^\theta_s, z, \varphi) N(dE, dz),$$

(6)

where $E = \mathbb{R} \setminus \{0\}$; $x : \Theta \to \mathbb{R}$; $a : \mathbb{R} \times \Theta \to \mathbb{R}$, $b : \mathbb{R} \times \Theta \to \mathbb{R}$ and $c : \mathbb{R} \times E \times \Theta \to \mathbb{R}$; $W$ is a $\mathbb{F}$-Wiener process. Moreover, $\tilde{N}(dr, dz) := N(dr, dz) - v(z) \, dz \, dr$, which is the compensated Poisson random measure, where $N$ is a Poisson random measure associated with a $\mathbb{F}$-Lévy process, say $Z = (Z_t)_{t \geq 0}$ with the Lévy density $v$:

$$N(A \times [0, t]) = \sum_{s \leq t} 1_{\{\Delta Z_s \in A\}}, \quad A \subset E,$$
and \( \mathbb{E}[N(dr, dz)] = \nu(z) \, dz \, dr \).

In the sequel, we assume that \( \nu \) is essentially known: some cases can be rewritten into a model for a known \( \nu \) even if \( \nu \) has some unknown parameters; see Remark 5.1 below. However, if it is not the case, the situation may be totally different from ours, and the argument in this section could not work anymore; see Remark 5.2.

**Remark 5.1.** Some cases where the Lévy density \( \nu \) depends on an unknown parameter, say \( \nu_\vartheta \), can be rewritten into the form of (5) with a known Lévy process by changing the coefficients \( a \) and \( c \). For example, consider the following SDE:

\[
dX_t = a(X_t) \, dt + b(X_t) \, dW_t + \int_E c(X_t, z) \, N_\vartheta(dr, dz),
\]

where \( N_\vartheta \) is the Poisson random measure associated with a compound Poisson process of the form \( Z_t^\vartheta = \sum_{i=1}^N U_i^\vartheta \) such that \( N \) is the Poisson process with the intensity \( \lambda_0 \) and \( U_i^\vartheta \)'s are IID sequence with the probability density \( f_\vartheta \) with \( \mathbb{E}[U_i^\vartheta] = \eta \) and \( \text{Var}(U_i^\vartheta) = \xi^2 \). Suppose that \( \lambda_0 \) is known, but \( \vartheta = (\eta, \xi) \) is unknown. In this case, we can rewrite \( Z^\vartheta \) (or \( Z(\eta, \xi) \)) as

\[
Z_t^{(\eta, \xi)} = \sum_{i=1}^N (\xi U_i^{(0,1)} + \eta) = \int_0^t \int_E (\xi z + \eta) N_{(0,1)}(dr, dz),
\]

where \( N_{(0,1)} \) is the Poisson random measure associated with \( Z^{(0,1)} \). Then the SDE (7) is written as

\[
dX_t = a(X_t) \, dt + b(X_t) \, dW_t + \int_E c(X_t, \xi z + \eta) N_0(dr, dz)
= \left[ a(X_t) + \lambda_0 \int_E c(X_t, \xi z + \eta) f_{(0,1)}(z) \, dz \right] \, dt + b(X_t) \, dW_t
+ \int_E c(X_t - \xi z + \eta) N_{(0,1)}(dr, dz).
\]

See also Example 5.2.

The semimartingale \( X^\vartheta \) in (6) is a \( \mathcal{X} = \mathcal{D}([0, T]) \)-valued random element. In the sequel, we consider a metric space \((\mathcal{X}, \| \cdot \|)\) with the sup norm:

\[
\|X^\vartheta\|_T := \sup_{t \in [0, T]} |X_t^\vartheta|.
\]

We make some assumptions.

**B1.** For each \( x, z \in \mathbb{R} \),

\[
|a(x, \vartheta)| + |b(x, \vartheta)| \lesssim 1 + |x|; \quad |c(x, z, \vartheta)| \lesssim |z|(1 + |x|),
\]

uniformly in \( \vartheta \in \Theta \).

**B2.** The functions \( a, b \) and \( c \) are twice differentiable in \( x \), and that the derivatives \( \nabla_x^k a \) and \( \nabla_x^k b \) \((k = 1, 2)\) are uniformly bounded. Moreover \( \|\nabla_x c(x, z, \vartheta)\| \lesssim |z| \).
Lemma 5.1. For any $B_1$,

$$|a(x, \theta)| + |b(x, \theta)| \lesssim 1 + |x|; \quad |c(x, z, \theta)| \lesssim |z|(1 + |x|),$$

uniformly in $\theta \in \Theta$.

B4. For any $p > 0$, $\int_{|z| > 1} z^p \nu(z) \, dz < \infty$.

B5. For any $p > 0$ and $T > 0$, $\|X\|^p_T \lesssim \infty$.

5.2 Derivative processes

Let $Y^\theta = (Y_t^\theta)_{t \geq 0}$ be a $p$-dim stochastic process satisfying the following SDE: $Y_0^\theta = \dot{x}(\theta)$,

$$dY_t^\theta = A(X_t^\theta, Y_t^\theta, \theta) \, dt + B(X_t^\theta, Y_t^\theta, \theta) \, dW_t + \int_E C(X_t^\theta, Y_t^\theta, z, \theta) \tilde{N}(dt, dz), \tag{8}$$

for each $\theta \in \Theta$, where

$$A(x, y, \theta) = \nabla_x a(x, \theta) y + \dot{a}(x, \theta);$$

$$B(x, y, \theta) = \nabla_x b(x, \theta) y + \dot{b}(x, \theta);$$

$$C(x, y; z, \theta) = \nabla_x c(x, z, \theta) y + \dot{c}(x, z, \theta).$$

In this section, we will show that the above $Y^\theta = (Y_t^\theta)_{t \geq 0}$ can be the derivative process of $X^\theta$ with respect to $\theta$ in the $L^q$-sense. For that purpose, we shall give some preliminary lemmas.

Lemma 5.1. Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of polynomial growth. Then, under H[S] it holds for $p = 2^m (m \in \mathbb{N})$ that

$$E \left[ \int_0^T \int_E g(X_{s-}, z) \tilde{N}(ds, dz) \right]^p \lesssim E \left[ \int_0^T \int_E |g(X_{s-}, z)|^p \nu(z) \, dz \, ds \right].$$

Proof. See Shimizu and Yoshida [6], Lemma 4.1. \qed

Lemma 5.2. Suppose the assumptions H[I]–H[S] and that $\dot{x}(\theta)$ is uniformly bounded on $\Theta$. Then it follows for any $T > 0$, $p \geq 2$ and $u \in \mathbb{R}^p$ with $\theta + u \in \Theta$ that

$$E \|X^\theta + u - X^\theta\|^p_T \lesssim |u|^p.$$

Proof. It follows by Jensen’s inequality that

$$|X_t^{\theta+u} - X_t^\theta|^p \lesssim |x(\theta + u) - x(\theta)|^p + T^{p-1} \int_0^T |\tilde{A}_s(u, \theta)|^p \, ds + \int_0^T |\tilde{B}_t(u, \theta)|^p \, dW_t$$

$$+ \left| \int_0^T \int_E \tilde{C}_s(u, z, \theta) \tilde{N}(ds, dz) \right|^p, \tag{9}$$

with

$$\tilde{A}_t(u, \theta) := a(X_t^{\theta+u}, \theta + u) - a(X_t^\theta, \theta);$$
Finally Gronwall’s inequality completes the proof. It follows by the mean value theorem and the assumptions B1 – B3 that
\[ |\theta| \leq |u| \] by the mean value theorem, Lemma 5.1 and Burkholder-Davis-Gundy’s inequality yield that
\[ E \|X^{\theta+u} - X^{\theta}\|_T^p \leq |u|^p + \int_0^T E \left[ |\tilde{A}_t(u, \theta)|^p + |\tilde{B}_t(u, \theta)|^p \right] ds + \mathbb{E} \left[ \int_0^T \int_E |\tilde{C}_t(u, z, \theta)|^p v(z) dz ds \right] \]
It follows by the mean value theorem and the assumptions B1 – B3 that
\[ |\tilde{A}_t(u, \theta)|^p = |\nabla_s a(X^*, \theta^*) (X_t^{\theta+u} - X_t^{\theta}) + \dot{a}(X^*, \theta^*)^\top u|^p \leq |X_t^{\theta+u} - X_t^{\theta}|^p + (1 + \|X_t^{\theta}\|)^p |u|^p \]
Hence it follows from B1 – B3 that
\[ E |\tilde{A}_t(u, \theta)|^p \leq E |X_t^{\theta+u} - X_t^{\theta}|^p + |u|^p \]
Similarly, we also have that
\[ E |\tilde{B}_t(u, \theta)|^p \leq E |X_t^{\theta+u} - X_t^{\theta}|^p + |u|^p; \]
\[ E |\tilde{C}_t(u, z, \theta)|^p \leq |z|^p \left( E |X_t^{\theta+u} - X_t^{\theta}|^p + |u|^p \right). \]
Hence, the assumption B4 yields that
\[ E \|X^{\theta+u} - X^\theta\|_T^p \leq |u|^p + \int_0^T E \|X^{\theta+u} - X^\theta\|_t^p dt. \]
Finally Gronwall’s inequality completes the proof.

The next theorem is the consequence of this section.

**Theorem 5.1.** Suppose the assumptions B1 – B5. Moreover, suppose that the initial value \( x(\theta) = X^\theta_0 \) is twice differentiable with the bounded derivatives, and that the solution \( Y^\theta \) to (8) satisfies that \( \|Y^\theta\|_T < \infty \) for any \( T > 0 \). Then, for any \( p \geq 2 \), there exists a positive constant \( C_p \) depending on \( p \) such that
\[ E \left[ X^{\theta+u} - X^\theta - u^\top Y^\theta \right]_T^p \leq C_p |u|^{2p}, \quad h \in \mathbb{R}^p. \]

**Proof.** First, we shall consider the case where \( p = 2^m \) (\( m \in \mathbb{N} \)). Applying Jensen’s inequality to the \( dr \)-integral part, we see that
\[ |X_t^{\theta+u} - X_t^\theta - u^\top Y_t^\theta|^p \leq |x(\theta + u) - x(\theta) - u^\top \dot{x}(\theta)|^p + t^{p-1} \int_0^t |\tilde{A}_s(u, \theta)|^p ds \]
\[ + \left| \int_0^t \tilde{B}_s(u, \theta) dW_s \right|^p + \left| \int_0^t \int_E \tilde{C}_{s-}(u, z, \theta) \tilde{N}(ds, dz) \right|^p, \quad (10) \]
where
\[
\begin{align*}
\tilde{A}_t(u, \vartheta) & := a(X_t^{\vartheta + u}, \vartheta + u) - a(X_t^\vartheta, \vartheta) - u^\top [\nabla_s a(X_s^\vartheta, \vartheta) Y_t^\vartheta + \dot{a}(X_t^\vartheta, \vartheta)]; \\
\tilde{B}_t(u, \vartheta) & := b(X_t^{\vartheta + u}, \vartheta + u) - b(X_t^\vartheta, \vartheta) - u^\top [\nabla_s b(X_s^\vartheta, \vartheta) Y_t^\vartheta + \dot{b}(X_t^\vartheta, \vartheta)]; \\
\tilde{C}_t(u, z, \vartheta) & := c(X_t^{\vartheta + u}, z, \vartheta + u) - c(X_t^\vartheta, z, \vartheta) - u^\top [\nabla_s c(X_s^\vartheta, z, \vartheta) Y_t^\vartheta + \dot{c}(X_t^\vartheta, z, \vartheta)].
\end{align*}
\]

Take \( \sup_{t \in [0, T]} \) and the expectation \( E \) on both sides to obtain that
\[
E\|X_t^{\vartheta + u} - X_t^\vartheta - u^\top Y_t^\vartheta\|^p_T \lesssim |u|^{2p} + \int_0^T E[\tilde{A}_t(u, \vartheta)]^p \, dt + E \left\| \int_0^T \tilde{B}_t(u, \vartheta) \, dW_t \right\|^p_T \\
+ E \left\| \int_0^T \int_E \tilde{C}_t(u, z, \vartheta) \tilde{N}(ds, dz) \right\|^p_T.
\]

Using Burkholder-Davis-Gundy’s inequality and Lemma\[5.1\] we have that
\[
E\|X_t^{\vartheta + u} - X_t^\vartheta - u^\top Y_t^\vartheta\|^p_T \lesssim |u|^{2p} + \int_0^T E[\tilde{A}_t(u, \vartheta)]^p \, dt + \int_0^T E\left[|\tilde{B}_t(u, \vartheta)|^p + |\tilde{C}_t(u, \vartheta)|^p\right] \, ds \\
+ E \left\| \int_0^T \int_E |\tilde{C}_t(u, z, \vartheta)|^p \tilde{N}(ds, dz) \right\|^p_T
\]
\[
\lesssim |u|^{2p} + \int_0^T E \left[|\tilde{A}_t(u, \vartheta)|^p + |\tilde{B}_t(u, \vartheta)|^p\right] \, ds
\]
\[
+ E \left\| \int_0^T \int_E |\tilde{C}_t(u, z, \vartheta)|^p \tilde{N}(ds, dz) \right\|^p_T.
\]

According to the assumptions B\[1\], B\[2\] and Taylor’s formula, we have, e.g.,
\[
\begin{align*}
\tilde{A}_t(u, \vartheta) &= \nabla_s a(X_t^\vartheta)(X_t^{\vartheta + u} - X_t^\vartheta) + u^\top \dot{a}(X_t^\vartheta, \vartheta) + \frac{1}{2} \left([X_t^{\vartheta + u} - X_t^\vartheta]^2 \nabla_s + u^\top \nabla \vartheta \right)^2 a(X_t^\vartheta, \vartheta^*) \\
&- u^\top \left[\nabla_s a(X_s^\vartheta, \vartheta) Y_t^\vartheta + \dot{a}(X_t^\vartheta, \vartheta)\right],
\end{align*}
\]

where \( X^* \) is a random variable between \( X_t^{\vartheta + u} \) and \( X_t^\vartheta \), \( \vartheta^* \in [\vartheta, \vartheta + u] \). Since the second derivatives are bounded, and from B\[3\] we have that
\[
|\tilde{A}_t(u, \vartheta)|^p \lesssim |X_t^{\vartheta + u} - X_t^\vartheta - u^\top Y_t^\vartheta|^p + |u|^{2p} + |X_t^{\vartheta + u} - X_t^\vartheta|^2 + |u|^p|X_t^{\vartheta + u} - X_t^\vartheta|^p
\]

Similarly, we also have that
\[
|\tilde{B}_t(u, \vartheta)|^p \lesssim |X_t^{\vartheta + u} - X_t^\vartheta - u^\top Y_t^\vartheta|^p + |u|^{2p} + |X_t^{\vartheta + u} - X_t^\vartheta|^2 + |u|^p|X_t^{\vartheta + u} - X_t^\vartheta|^p;
\]
\[
|\tilde{C}_t(u, z, \vartheta)|^p \lesssim |z|^p \left(|X_t^{\vartheta + u} - X_t^\vartheta - u^\top Y_t^\vartheta|^p + |u|^{2p} + |X_t^{\vartheta + u} - X_t^\vartheta|^2 + |u|^p|X_t^{\vartheta + u} - X_t^\vartheta|^p\right).
\]

Hence, under B\[4\] it follows from Lemma\[5.2\] that
\[
E\|X_t^{\vartheta + u} - X_t^\vartheta - u^\top Y_t^\vartheta\|^p_T \lesssim |u|^{2p} + \int_0^T E\|X_t^{\vartheta + u} - X_t^\vartheta - u^\top Y_t^\vartheta\|^p_T \, dr.
\]
and Gronwall’s inequality yields the consequence.

For any $p \geq 2$, we write $p$ by the binomial expansion as $p = \sum_{k=1}^{m} \delta_k 2^k$, where $m$ is an integer and $\delta_k = 0$ or 1. Note that we have already proved the consequence for $p$ with $m = 1$ and $\delta_k = 0, 1$. Then the Cauchy-Schwarz inequality yields that, for $q = \sum_{k=2}^{m} 2^k \delta_{k-1}$,

$$
\mathbb{E}[|X^{\vartheta+u} - X^{\vartheta} - u^T Y^{\vartheta}|^p_T] 
\leq \sqrt{\mathbb{E}[|X^{\vartheta+u} - X^{\vartheta} - u^T Y^{\vartheta}|^{2p+1}_T]} \sqrt{\mathbb{E}[|X^{\vartheta+u} - X^{\vartheta} - u^T Y^{\vartheta}|^{\delta_k 2^k}_T]}
\leq \sqrt{C_2 \delta_k |u|^2 2^{2m+1} \delta_n} \sqrt{C_q |u|^{2q}}
\lesssim |u|^{2m+1} \delta_n + q = |u|^{2p}.
$$

This completes the proof. \(\square\)

**Remark 5.2.** If the random measure $\mathcal{N}$ essentially includes unknown parameters, then the derivative process in the sense of $L^3$ can not exist. To see this, consider a simple case where $X^{\vartheta}$ is a Poisson process with the (unknown) intensity $\vartheta: X^{\vartheta} \sim Po(\vartheta t)$, which is not the case described in Remark 5.1. In this case, we can not compute the expectation $\mathbb{E}[|X^{\vartheta+u} - X^{\vartheta}|^p_T]$ since we do not know the joint distribution of $(X^{\vartheta+u}, X^{\vartheta})$. This consideration indicates that Monte Carlo estimator can get worse when $X^{\vartheta}$ has an unknown jump part.

**Example 5.1** (Lévy processes). Consider a 1-dim Lévy process $X^{\vartheta}$ starting at $x > 0$:

$$
X^{\vartheta}_t = x + \mu t + \sigma W_t + \eta S_t,
$$

where $S$ is a known Lévy process with $\mathbb{E}[S_1] = 1$ and $\eta \neq 0$. We set $\vartheta = (\mu, \sigma, \eta) \in \Theta \subset \mathbb{R}^3$. Then this is the case of (10) with

$$
a(x, \vartheta) = \mu + \eta, \quad b(x, \vartheta) = \sigma, \quad c(x, z, \vartheta) = \eta z, \quad X_0 = x,
$$

Hence the derivative process $Y^{\vartheta}$ is the 3-dim Lévy process of the form

$$
Y^{\vartheta}_t = (t, W_t, S_t)
$$

**Example 5.2.** Consider an O-U process $X = (X_t)_{t \geq 0}$ written by

$$
\text{d}X^{\vartheta}_t = -\mu X^{\vartheta}_t \text{d}t + \sigma \text{d}W_t + \text{d}Z^{\eta}_t, \quad X_0 = x \quad \text{(const.)} 
$$

(11)

where $\vartheta = (\mu, \sigma, \eta)$, $W$ is a Wiener process, and $Z^{\eta}$ is a compound Poisson process with the known intensity and the mean of the jumps is $\eta$. Then the SDE (11) is rewritten by

$$
X^{\vartheta}_t = x + \int_0^t (-\mu X^{\vartheta}_s + \eta) \text{d}s + \sigma W_t + \int_0^t \int_E (z + \eta) \tilde{N} (\text{d}t, \text{d}z),
$$

where $\tilde{N}$ is a compensated Poisson process.
where \( \tilde{N} \) is the compensated Poisson random measure associated to \( Z^0 (\eta = 0) \); see Remark 5.1. Then the derivative process \( Y^\vartheta = (Y^1_t, Y^2_t, Y^3_t)_{t \geq 0} \) satisfies the following SDE:

\[
Y^1_t = \int_0^t (\mu Y^3_s - X^\vartheta_s) \, ds; \quad Y^2_t = -\mu \int_0^t Y^2_s \, ds + W_t; \\
Y^3_t = \int_0^t (1 - \mu Y^3_s) \, ds + Z^0_t,
\]

since \( \int_{E} \check{\nu}_0(z) \, dz = 0 \). The equation for \( Y^1 \) is an ordinary differential equation for almost all \( \omega \in \Omega \), and the equations for \( Y^2 \) and \( Y^3 \) are O-U type SDEs. Therefore we can solve these equations explicitly as follows:

\[
Y^1_t = -\int_0^t X^\vartheta_s e^{-\mu(t-s)} \, ds, \quad Y^2_t = \int_0^t e^{-\mu(t-s)} \, dW_s, \\
Y^3_t = \frac{1}{\mu} (1 - e^{-\mu t}) + \int_0^t e^{-\mu(t-s)} \, dZ^0_s,
\]

and

\[
X^\vartheta_t = xe^{-\mu t} + \int_0^t e^{-\mu(t-s)}[\sigma \, dW_s + dZ^0_s]. \quad (12)
\]

### 5.3 Monte Carlo estimators for semimartingales

For each \( \vartheta \in \Theta \), let \( \varphi_\vartheta : \mathbb{R} \to \mathbb{R} \) and

\[
H(\vartheta) = \mathbb{E} \left[ \varphi_\vartheta(X^\vartheta_t) \right],
\]

where \( X^\vartheta_t \) is a \( \mathbb{R} \)-valued random functional of \( X^\vartheta \) such that the inequality

\[
|X^{\vartheta+u}_t - X^{\vartheta}_t - u^\top \tilde{Y}^\vartheta_t| \lesssim \|X^{\vartheta+u}_t - X^\vartheta_t - u^\top Y^\vartheta_t\| + |u|^{1+\delta} \quad a.s., \quad (13)
\]

holds true for some \( \tilde{Y}^\vartheta \) and \( \delta > 0 \); see Remark 4.2 for some examples. Summing up our results in Sections 2, 4 and 5 with Remark 4.2, we can immediately obtain the following result for Monte Carlo estimators of expected functionals for a semimartingale \( X^\vartheta \).

**Theorem 5.2.** Suppose the same assumptions as in Theorem 5.1. Moreover, suppose that there exists an integer \( n \geq 1 \) such that \( \varphi^{(n)}_{\vartheta_0}(x) \) is Lipschitz continuous:

\[
|\varphi^{(n)}_{\vartheta_0}(x) - \varphi^{(n)}_{\vartheta_0}(y)| \lesssim |x - y|, \quad x, y \in \mathbb{R},
\]

and that, for some constant \( r > 2 \),

\[
\varphi^{(k)}_{\vartheta_0}(X^\vartheta_t) \in L^r, \quad k = 1, \ldots, n.
\]
Furthermore assume that we have an estimator of $\hat{\vartheta}_0$ based on some observations depending on a parameter $n$, say $\hat{\vartheta}_n$, such that the assumption A[3] holds true. Then, the Monte Carlo estimator $\hat{H}^*(\hat{\vartheta}_n)$ is asymptotically normal such that

$$
\gamma_n^{-1}(\hat{H}^*(\hat{\vartheta}_n) - H(\vartheta_0)) \convergesinlaw \left( \mathbb{E} \left[ \varphi_{\hat{\vartheta}_0}(X^{\hat{\vartheta}_0}) \right] + C_{\vartheta_0} \right)^T Z^*, \quad n \to \infty,
$$

and the deterministic vector $C_{\vartheta}$ is given by

$$
C_{\vartheta} = \mathbb{E} \left[ \varphi^{(1)}(X_{\vartheta}^* \tilde{Y}_{\vartheta}) \right],
$$

where $\tilde{Y}_{\vartheta}$ is given in [13].

**Example 5.3** (Ornstein-Uhlenbeck type processes). This is the continuation of the previous Example 5.2. Let us consider the same SDE as (11), and consider the expected discounted functional: for a constant $\delta > 0$,

$$
H(\vartheta) = \mathbb{E} \left[ \int_0^T e^{-\delta t} V(X_t) \, dt \middle| X_0 = x \right],
$$

which is an important quantity in insurance and finance because such a functional can represent an option price when $X$ is a stock price; see, e.g., Karatzas and Shereve [2], or it can also represent some aggregated ‘costs’ or ‘risks’ in insurance businesses when $X$ is an asset process of the company; see, e.g., Feng and Shimizu [11]. The constant $\delta > 0$ is interpreted as a interest rate.

Here we shall consider a simple case where $V(x) = x$:

$$
H(\vartheta) = \int_0^T e^{-\delta t} \mathbb{E} [X_t] \, dt,
$$

Noticing that, from the expression (12),

$$
\mathbb{E} [X_t] = xe^{-\mu t} + \mathbb{E} \left[ \int_0^t e^{-(\mu - \eta) s} \, dZ_s \right] = xe^{-\mu t} + \frac{\eta}{\mu} (1 - e^{-\mu t}),
$$

we can compute $H(\vartheta)$ explicitly as

$$
H(\vartheta) = \frac{x}{\mu + \delta} \left( 1 - e^{-(\mu + \delta) T} \right) + \frac{\eta}{\mu} \left[ \frac{1}{\delta} (1 - e^{-\delta T}) - \frac{1}{\mu + \delta} (1 - e^{-(\mu + \delta) T}) \right].
$$

Suppose that $Z$ is a compound Poisson process, and that we have a set of discrete samples $(X_t, X_2, \ldots, X_n)$ with $t_k = kh_n$ for $h_n > 0$, and assume some asymptotic conditions to $n$ and $h_n$, e.g., $h_n \to 0$ and $nh_n^2 \to 0$. Although we omit the details of the regularity conditions here, we can construct an asymptotic normal (efficient) estimator of $\vartheta = (\mu, \sigma, \eta)$, say $\hat{\vartheta}_n$, such that

$$
\Gamma_n^{-1}(\hat{\vartheta}_n - \vartheta) \convergesinlaw N_3(0, \Sigma), \quad n \to \infty
$$

with $\Gamma_n = \text{diag}(1/\sqrt{m_n}, 1/\sqrt{m_n}, 1/\sqrt{m_n})$ and a diagonal matrix $\Sigma = \text{diag}(\Sigma_1, \Sigma_2, \Sigma_3)$; see, e.g., Shimizu and Yoshida [6]. In this case, we have $\gamma_n = 1/\sqrt{m_n}$, and Theorem 4.1 says that

$$
\sqrt{n h_n} [\hat{H}^*(\hat{\vartheta}_n) - H(\vartheta_0)] \convergesinlaw N \left( 0, C_{\vartheta_0} \text{diag}(\Sigma_1, 0, \Sigma_3) C_{\vartheta_0} \right).
$$
where

\[ C_\theta = \left( \int_0^T e^{-\delta t} \mathbb{E}[Y_t^\theta] \, dt \right) =: (C_1^\theta, C_2^\theta, C_3^\theta)^T; \]

with \( C_2^\theta = 0 \) and

\[ C_1^\theta = \frac{\eta - \mu \xi}{\mu (\delta + \mu)^2} \left[ 1 - (\mu + \delta) e^{-(\mu+\delta)T} e^{-(\mu+\delta)T} \right] + \frac{\eta}{\delta \mu^2} (1 - e^{-\delta T}) \]

\[ + \frac{\eta}{\mu^2 (\mu + \delta)} (1 - e^{-(\mu+\delta)T}); \]

\[ C_3^\theta = \frac{1}{\mu} \left[ \frac{1}{\delta} (1 - e^{-\delta T}) - \frac{1}{\mu + \delta} (1 - e^{-(\mu+\delta)T}) \right]. \]

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