THE GROTHENDIECK ALGEBRAS OF CERTAIN SMASH PRODUCT SEMISIMPLE HOPF ALGEBRAS

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ABSTRACT. Let $H$ be a semisimple Hopf algebra over an algebraically closed field $\mathbb{k}$ of characteristic $p > \dim_{\mathbb{k}}(H)^{1/2}$ and $p \nmid 2 \dim_{\mathbb{k}}(H)$. In this paper, we consider the smash product semisimple Hopf algebra $H \# \mathbb{k}G$, where $G$ is a cyclic group of order $n := 2 \dim_{\mathbb{k}}(H)$. Using irreducible representations of $H$ and those of $\mathbb{k}G$, we determine all non-isomorphic irreducible representations of $H \# \mathbb{k}G$. There is a close relationship between the Grothendieck algebra $(G_0(H \# \mathbb{k}G) \otimes_{\mathbb{k}} \mathbb{k}, \ast)$ of $H \# \mathbb{k}G$ and the Grothendieck algebra $(G_0(H) \otimes_{\mathbb{k}} \mathbb{k}, \ast)$ of $H$. To establish this connection, we endow with a new multiplication operator $\star$ on $G_0(H) \otimes_{\mathbb{k}} \mathbb{k}$ and show that the Grothendieck algebra $(G_0(H \# \mathbb{k}G) \otimes_{\mathbb{k}} \mathbb{k}, \ast)$ is isomorphic to the direct sum of $(G_0(H) \otimes_{\mathbb{k}} \mathbb{k}, \ast)^{\oplus \frac{n}{2}}$ and $(G_0(H) \otimes_{\mathbb{k}} \mathbb{k}, \ast)^{\oplus \frac{n}{2}}$.

1. Introduction

The Grothendieck rings of finite dimensional semisimple or cosemisimple Hopf algebras have been studied by Nichols and Richmond [10], Nikshych [11], Kashina [4], Yang [2, 16], etc. For a finite dimensional semisimple Hopf algebra $H$, the category $\text{Rep}(H)$ of finite dimensional representations of $H$ is a fusion category. As an important invariant of $\text{Rep}(H)$, the Grothendieck ring $G_0(H)$ of $H$ reveals the decompositions of tensor product of irreducible representations into irreducibles. Hence the Grothendieck ring $G_0(H)$ can be used to study the fusion category $\text{Rep}(H)$. For instance, the knowledge of the structure of the Grothendieck ring $G_0(H)$ allows to determine all fusion subcategories of $\text{Rep}(H)$, which correspond to the so-called based subrings of $G_0(H)$.

For a semisimple Hopf algebra $H$ with antipode $S$ over a field $\mathbb{k}$, it is known that $S^2$ is an inner automorphism of $H$ (see [6]). Here an inner automorphism is understood to be the conjugation by an invertible element of $H$. If the ground field $\mathbb{k}$ has positive characteristic $p$, whether or not $S^2$ can be given by conjugation with a group-like element is not completely solved (this problem is closely related to the Kaplansky’s fifth conjecture). However, such a Hopf algebra $H$ can be embedded into another finite dimensional Hopf algebra $H \# \mathbb{k}G$, namely, the smash product of $H$ and a group algebra $\mathbb{k}G$, in which the square of the antipode is the conjugation with a group-like element. We refer to [5, 7, 14] for such Hopf algebras and related researches.

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If \( H \) is a semisimple involutory Hopf algebra, namely, a semisimple Hopf algebra with \( S^2 = id \), the smash product Hopf algebra \( H \#_\mathbb{Z} G \) considered here is nothing but the usual tensor product Hopf algebra \( H \otimes \mathbb{Z} G \). In this case, the representations of \( H \otimes \mathbb{Z} G \) can be stemmed directly from the representations of \( H \) and those of \( \mathbb{Z} G \). Also, the Grothendieck algebra of \( H \otimes \mathbb{Z} G \) is the usual tensor product of the Grothendieck algebra of \( H \) and that of \( \mathbb{Z} G \). However, if \( H \) is not necessarily involutory (although the Kaplansky’s fifth conjecture states that a semisimple Hopf algebra is necessarily involutory), the relationship between the Grothendieck algebra of \( H \#_\mathbb{Z} G \) and that of \( H \) is not clear.

The purpose of this paper is to study representations of the smash product semisimple Hopf algebra \( H \#_\mathbb{Z} G \) and to establish a relationship between the Grothendieck algebra of \( H \#_\mathbb{Z} G \) and that of \( H \), where \( H \) is a semisimple Hopf algebra over a field \( \mathbb{K} \) of characteristic \( p > \dim_{\mathbb{K}}(H)^{1/2} \), \( G \) is a cyclic group of order \( 2 \dim_{\mathbb{K}}(H) \) and \( p \nmid 2 \dim_{\mathbb{K}}(H) \). It is worthy mentioning that such a Hopf algebra \( H \) is not necessarily involutory unless the characteristic \( p \) is larger than a certain number (see \([14, 3]\)).

The paper is organized as follows: In Section 2, we present some properties of a special element \( v \) of the semisimple Hopf algebra \( H \). Such an element is used later to describe representations of the smash product semisimple Hopf algebra \( H \#_\mathbb{Z} G \).

In Section 3, using irreducible representations of \( H \) and those of \( \mathbb{Z} G \) we are able to determine all non-isomorphic irreducible representations of \( H \#_\mathbb{Z} G \). We also describe the dual of these irreducible representations of \( H \#_\mathbb{Z} G \) in this section. In Section 4, we endow with a new multiplication operator \( \star \) on the Grothendieck algebra \( G_0(H) \otimes_{\mathbb{Z}} \mathbb{K} \) so as to obtain a new algebra \( (G_0(H) \otimes_{\mathbb{Z}} \mathbb{K}, \star) \). This algebra is nothing but the usual Grothendieck algebra \( (G_0(H) \otimes_{\mathbb{Z}} \mathbb{K}, \star) \) if \( H \) is involutory.

We show that the Grothendieck algebra \( (G_0(H \#_\mathbb{Z} G) \otimes_{\mathbb{Z}} \mathbb{K}, \star) \) is isomorphic to a direct sum of \( (G_0(H) \otimes_{\mathbb{Z}} \mathbb{K}, \star)^{\otimes 2} \) and \( (G_0(H) \otimes_{\mathbb{Z}} \mathbb{K}, \star)^{\otimes 2} \). This reveals a relationship between the Grothendieck algebra of \( H \#_\mathbb{Z} G \) and that of \( H \). Moreover, we find a fusion subcategory \( C \) of \( \text{Rep}(H \#_\mathbb{Z} G) \) whose Grothendieck algebra \( (G_0(C) \otimes_{\mathbb{Z}} \mathbb{K}, \star) \) happens to be the direct sum of \( (G_0(H) \otimes_{\mathbb{Z}} \mathbb{K}, \star) \) and \( (G_0(H) \otimes_{\mathbb{Z}} \mathbb{K}, \star) \). In view of this, the Grothendieck algebra \( (G_0(H \#_\mathbb{Z} G) \otimes_{\mathbb{Z}} \mathbb{K}, \star) \) is isomorphic to \( (G_0(C) \otimes_{\mathbb{Z}} \mathbb{K}, \star)^{\otimes 2} \).

### 2. Preliminaries

Throughout, \( H \) is a finite dimensional semisimple Hopf algebra over an algebraically closed field \( \mathbb{K} \) of positive characteristic \( p > \dim_{\mathbb{K}}(H)^{1/2} \) and \( p \nmid 2 \dim_{\mathbb{K}}(H) \). We denote \( \{ V_i \mid 0 \leq i \leq m - 1 \} \) the set of all non-isomorphic finite dimensional simple \( H \)-modules, where \( V_0 \) is fixed to be the trivial \( H \)-module \( \mathbb{K} \). The condition \( p > \dim_{\mathbb{K}}(H)^{1/2} \) is used to make sure that \( \dim_{\mathbb{K}}(V_i) \neq 0 \) in \( \mathbb{K} \) for \( 0 \leq i \leq m - 1 \). Indeed, \( p^2 > \dim_{\mathbb{K}}(H) = \sum_{i=0}^{m-1} \dim_{\mathbb{K}}(V_i)^2 \geq \dim_{\mathbb{K}}(V_i)^2 \) implies that \( p > \dim_{\mathbb{K}}(V_i) \), which turns out that \( \dim_{\mathbb{K}}(V_i) \neq 0 \) in \( \mathbb{K} \). The character afforded by simple \( H \)-module \( V_i \) is denoted by \( \chi_i \) for \( 0 \leq i \leq m - 1 \) and the character afforded by left regular module \( H \) is denoted by \( \chi_H \). We denote \( \{ e_i \mid 0 \leq i \leq m - 1 \} \) the set of
all central primitive idempotents of $H$, where the central idempotent $e_i$ acts as the identity on $V_i$ and annihilates $V_j$ for $j \neq i$.

As a Hopf algebra, $H$ has a counit $\varepsilon$, antipode $S$ and comultiplication $\Delta$, where the comultiplication $\Delta(h)$ will be written as $\Delta(h) = h^{(1)} \otimes h^{(2)}$ for $h \in H$, here we omit the summation sign. We choose a left integral $\Lambda$ in $H$ and a right integral $\lambda$ in $H^*$ such that $\lambda(\Lambda) = 1$. We denote $u := S(\Lambda^{(2)})\Lambda^{(1)}$. We refer to [8] for basic theory of Hopf algebras.

For the Hopf algebra $H$, there is a formula for $S^2$ as follows (see [15]):

$$S^2(h) = uh^{-1}$$ for $h \in H$.

For the element $u$, we have the following result (see [15, Proposition 3.3]):

**Proposition 2.1.** The element $u = S(\Lambda^{(2)})\Lambda^{(1)}$ satisfies the following properties:

1. $\varepsilon(u) = \chi_H(\Lambda^{(1)})S(\Lambda^{(2)})$.
2. $\Lambda(u) = S(u)\Lambda = 1$.
3. $\Lambda(e_i) = \dim_{k}(V_i)\chi_i(u^{-1})$.
4. $uS(u) = S(u)u = \varepsilon(\Lambda)\sum_{i=0}^{n-1} \frac{\dim_{k}(V_i)}{\lambda(e_i)} e_i$.
5. $S(u^{-1})u = uS(u^{-1})$, which is the distinguished group-like element of $H$.

We denote

$$v := \frac{u}{\sqrt{\varepsilon(\Lambda)}} \sum_{i=0}^{n-1} \frac{\sqrt{\lambda(e_i)}}{\dim_{k}(V_i)} e_i.$$  (2.1)

As we shall see, the element $v$ plays a key role in the representation theory of smash product Hopf algebras. For the element $v$, we have the following result:

**Proposition 2.2.** The element $v$ satisfies the following properties:

1. $\varepsilon(v) = 1$.
2. $S^2(h) = vh^{-1}$ for $h \in H$.
3. $v^n = uS(u^{-1})$, which is the distinguished group-like element of $H$.
4. $v^n = 1$, where $n = 2\dim_{k}(H)$.
5. $v^{-1} = S(v)$.
6. $v = 1$ if and only if $S^2 = id$.

**Proof.** (1) Applying $\varepsilon$ to both sides of the equality (2.1), we obtain that $\varepsilon(v) = 1$.

(2) Since $S^2(h) = uh^{-1}$ and the elements $u, v$ are the same up to a central invertible element $\frac{1}{\sqrt{\varepsilon(\Lambda)}} \sum_{i=0}^{n-1} \frac{\sqrt{\lambda(e_i)}}{\dim_{k}(V_i)} e_i$, it follows that $S^2(h) = vh^{-1}$ for $h \in H$. 


(3) Note that \( u^{-1}S(u^{-1}) = \frac{1}{\varepsilon(\Lambda)} \sum_{i=0}^{m-1} \frac{\lambda(e_i)}{\dim_{\mathbb{K}}(V_i)} e_i \) by Proposition 2.1 (4). It follows that

\[
u S(u^{-1}) = \frac{u^2}{\varepsilon(\Lambda)} \sum_{i=0}^{m-1} \frac{\lambda(e_i)}{\dim_{\mathbb{K}}(V_i)} e_i = v^2,
\]

which is the distinguished group-like element of \( H \) by Proposition 2.1 (5).

(4) It can be seen from Part (3) that \( v^2 \) is the distinguished group-like element of \( H \), while the order of the distinguished group-like element divides \( \dim_{\mathbb{K}}(H) \). This implies that \( v^n = (v^2)^{\frac{n}{2}} = 1 \).

(5) There is a permutation \( * \) on the index set \( \{0, 1, \ldots, m-1\} \) determined by \( i^* = j \) if the dual \( H \)-module \( V_i^\ast \) is isomorphic to \( V_j \). The permutation \( * \) satisfies that \( i^{**} = i \), \( S(e_i) = e_{i^*} \), \( \dim_{\mathbb{K}}(V_i^*) = \dim_{\mathbb{K}}(V_j) \) and \( \lambda(e_{i^*}) = \lambda(e_i) \) for \( 0 \leq i \leq m-1 \) (the last equality follows from [15, Corollary 3.4]). We have

\[
vS(v) = \frac{1}{\varepsilon(\Lambda)} u \left( \sum_{i=0}^{m-1} \frac{\sqrt{\lambda(e_i)}}{\dim_{\mathbb{K}}(V_i)} e_i \right) S(u) \left( \sum_{i=0}^{m-1} \frac{\sqrt{\lambda(e_i)}}{\dim_{\mathbb{K}}(V_i)} e_i \right)
= \frac{1}{\varepsilon(\Lambda)} u S(u) \left( \sum_{i=0}^{m-1} \frac{\sqrt{\lambda(e_i)}}{\dim_{\mathbb{K}}(V_i)} e_i \right)^2
= \frac{1}{\varepsilon(\Lambda)} u S(u) \left( \sum_{i=0}^{m-1} \frac{\lambda(e_i)}{\dim_{\mathbb{K}}(V_i)} e_i \right)
= u S(u) \left( \frac{1}{\varepsilon(\Lambda)} \sum_{i=0}^{m-1} \frac{\lambda(e_i)}{\dim_{\mathbb{K}}(V_i)} e_i \right)
= 1,
\]

where the last equality follows from Proposition 2.1 (4). We obtain that \( v^{-1} = S(v) \).

(6) If \( v = 1 \), it follows from Part (2) that \( S^2 = id \). Conversely, if \( S^2 = id \), then \( \Lambda \) is cocommutative (see [15, Proposition 3.5]) and hence \( u = \varepsilon(\Lambda) \). In this case, \( \lambda(e_i) = \frac{\dim_{\mathbb{K}}(V_i)}{\varepsilon(\Lambda)} \) by Proposition 2.1 (3). Taking it into the equality (2.1) we may see that \( v = 1 \). We complete the proof.

\[\square\]

3. Representations of smash product Hopf algebras

We denote \( n := 2 \dim_{\mathbb{K}}(H) \). Let \( G \) be a cyclic group of order \( n \) generated by \( g \). The character group \( \hat{G} \) of \( G \) is also a cyclic group of order \( n \). Let \( \psi \) be a generator of \( \hat{G} \). Then \( \hat{G} = \{ \psi^j \mid 0 \leq j \leq n-1 \} \), which is the complete set of distinct irreducible characters of simple \( \mathbb{K}G \)-modules. The simple \( \mathbb{K}G \)-module with respect to the character \( \psi^j \) is denoted by \( W_j \) for \( 0 \leq j \leq n-1 \).
Since the antipode $S$ of $H$ satisfies $S^{2n} = id$ by Radford’s formula of $S^4$ [13], the Hopf algebra $H$ is a left $\mathbb{K}G$-module algebra whose action is given by

$$g^i \mapsto h = S^{2i}(h) \text{ for } g^i \in G \text{ and } h \in H.$$  

This reduces to a Hopf algebra $H\#\mathbb{K}G$ mentioned in [14]. More precisely, the Hopf algebra $H\#\mathbb{K}G$ is the smash product of $H$ and $\mathbb{K}G$. The multiplication of $H\#\mathbb{K}G$ is given by

$$(a \# g^i)(b \# g^j) = a(g^i \mapsto b) \# g^{i+j} = aS^{2i}(b) \# g^{i+j} \text{ for } a, b \in H,$$

the identity of $H\#\mathbb{K}G$ is $1_H \# 1_{\mathbb{K}G}$. The comultiplication of $H\#\mathbb{K}G$ is given by

$$\Delta_{H\#\mathbb{K}G}(h \# g^i) = (h(1) \# g^i) \otimes (h(2) \# g^i).$$

The counit of $H\#\mathbb{K}G$ is $\varepsilon_{H\#\mathbb{K}G} = \varepsilon_H \# \varepsilon_{\mathbb{K}G}$ and the antipode of $H\#\mathbb{K}G$ is

$$S_{H\#\mathbb{K}G}(h \# g^i) = (1_H \# g^{-i})(S(h) \# 1_{\mathbb{K}G}) = S^{1-2i}(h) \# g^{-i}.$$  

Moreover, $1_H \# g$ is a group-like element of $H\#\mathbb{K}G$ that satisfies

$$(3.1) \quad S^2_{H\#\mathbb{K}G}(h \# g^i) = (1_H \# g)(h \# g^i)(1_H \# g)^{-1}.$$  

The Hopf algebra $H$ can be considered as a sub-Hopf algebra of $H\#\mathbb{K}G$ under the injective map $H \to H\#\mathbb{K}G, \ h \mapsto h \# 1_{\mathbb{K}G}$. 

Since $\Lambda$ is an integral of $H$ with $\varepsilon(\Lambda) \neq 0$ and $p \nmid n$, $\Lambda \# \frac{1}{n} \sum_{i=0}^{n-1} g^i$ is an integral of $H\#\mathbb{K}G$ with $\varepsilon_{H\#\mathbb{K}G}(\Lambda \# \frac{1}{n} \sum_{i=0}^{n-1} g^i) = \varepsilon(\Lambda) \neq 0$. Thus, $H\#\mathbb{K}G$ is a semisimple Hopf algebra over $\mathbb{K}$.

The representation theory of crossed product of an algebra with a group algebra has been studied in [9]. However, we do not take advantage of those notations and methods in [9] to describe $H\#\mathbb{K}G$-modules. Instead, since the Hopf algebra $H\#\mathbb{K}G$ is semisimple, we will determine all simple $H\#\mathbb{K}G$-modules by the study of the character of regular representation of $H\#\mathbb{K}G$.

**Lemma 3.1.** If $V$ is a finite dimensional $H$-module and $W$ is a finite dimensional $\mathbb{K}G$-module, then the vector space $V \otimes W$ is a finite dimensional $H\#\mathbb{K}G$-module, where the $H\#\mathbb{K}G$-module structure is given by

$$(3.2) \quad (h \# g^k) \cdot (v \otimes w) = (hv^n \cdot v) \otimes (g^k \cdot w) \text{ for } v \in V, w \in W.$$  

**Proof.** By Proposition 2.2 (4), we have $v^n = 1$. It follows that

$$(h \# g^n) \cdot (v \otimes w) = (hv^n \cdot v) \otimes (g^n \cdot w) = (h \cdot v) \otimes w = (h \# 1_{\mathbb{K}G}) \cdot (v \otimes w).$$

This is compatible with the equality $h \# g^n = h \# 1_{\mathbb{K}G}$. For $a, b \in H$, by $S^2(h) = hv^{-1}$ for $h \in H$, we may check that

$$((a \# g^k)(b \# g^j)) \cdot (v \otimes w) = (a \# g^k) \cdot ((b \# g^j) \cdot (v \otimes w)).$$

The proof is completed.  

**Lemma 3.2.** If $V$ is a simple $H$-module and $W$ is a simple $\mathbb{K}G$-module, then $V \otimes W$ is a simple $H\#\mathbb{K}G$-module.
Proof. Note that $H\#\mathbb{k}G$ is a semisimple Hopf algebra over an algebraically closed field $\mathbb{k}$. It is sufficient to show that $\text{End}_{H\#\mathbb{k}G}(V \otimes W) = \mathbb{k}$. Suppose that the map $\phi : V \otimes W \to V \otimes W$ is an $H\#\mathbb{k}G$-module morphism. Since $W$ is one dimensional, we fix a basis $w$ of $W$. The $H\#\mathbb{k}G$-module morphism $\phi$ induces an $H$-module morphism $\phi_0 : V \to V$ as follows: $\phi(v \otimes w) = \phi_0(v) \otimes w$ for any $v \in V$. This shows that $\phi$ is the identity map of $V \otimes W$ up to a scalar, since $V$ is simple and $\phi_0$ is the identity map of $V$ up to a scalar. \qed

Remark 3.3. For simple $H$-module $V_i$ and simple $\mathbb{k}G$-module $W_j$, it can be seen from Lemma 3.2 that $V_i \otimes W_j$ is a simple $H\#\mathbb{k}G$-module. Let $\chi_{ij}$ be the character associated to the simple $H\#\mathbb{k}G$-module $V_i \otimes W_j$. It follows from (3.2) that

$$\chi_{ij}(h \otimes g^k) = \chi_i(h^k)\psi^j(g^k) \text{ for } 0 \leq i \leq m-1, 0 \leq j \leq n-1.$$ 

Theorem 3.4. The set $\{V_i \otimes W_j \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ forms a complete set of non-isomorphic simple $H\#\mathbb{k}G$-modules.

Proof. Note that $\Lambda^1 \# \sum_{j=0}^{n-1} g^j$ is a left integral in $H\#\mathbb{k}G$ and $\lambda \# \sum_{j=0}^{n-1} \psi^j$ is a right integral in $(H\#\mathbb{k}G)^*$ satisfying $(\lambda \# \sum_{j=0}^{n-1} \psi^j)(\Lambda^1 \# \sum_{j=0}^{n-1} g^j) = 1$. By [12, Corollary 6], the characters of left regular representations of $H$ and $H\#\mathbb{k}G$ are respectively given by $\chi_H = \lambda \rightarrow u$ and $\chi_{H\#\mathbb{k}G} = (\lambda \# \sum_{j=0}^{n-1} \psi^j) \rightarrow u_{H\#\mathbb{k}G}$, where $u = S(\Lambda(2))\Lambda(1)$ and

$$u_{H\#\mathbb{k}G} = \frac{1}{n} \sum_{i=0}^{n-1} S_{H\#\mathbb{k}G}(\Lambda(2)\# g^i)(\Lambda(1)\# g^i)$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} (S^{1-2i}(\Lambda(2))\# g^{-i})(\Lambda(1)\# g^i)$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} S^{1-2i}(\Lambda(2))S^{-2i}(\Lambda(1))\# 1_{\mathbb{k}G}$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} S^{-2i}(u)\# 1_{\mathbb{k}G}$$

$$= u\# 1_{\mathbb{k}G}.$$ 

It follows that

$$\chi_{H\#\mathbb{k}G} = (\lambda \# \sum_{j=0}^{n-1} \psi^j) \rightarrow (u\# 1_{\mathbb{k}G}) = (\lambda \rightarrow u)\# \sum_{j=0}^{n-1} \psi^j = \chi_{H\#\mathbb{k}G} \sum_{j=0}^{n-1} \psi^j.$$ 

Hence,

$$(\chi_{H\#\mathbb{k}G})(h\# g^k) = \chi_H(h) \sum_{j=0}^{n-1} \psi^j(g^k) = \begin{cases} n\chi_H(h) & k = 0; \\ 0, & 1 \leq k \leq n-1. \end{cases}$$
While 
\[ \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \text{dim}_k( V_i \otimes W_j) \chi_{ij}(h \# g^k) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \text{dim}_k( V_i) \chi_j(h \# g^k) \psi^i(g^k) \]
\[ = \chi_H(h \# g^k) \sum_{j=0}^{n-1} \psi^j(g^k) \]
\[ = \begin{cases} n \chi_H(h), & k = 0; \\ 0, & 1 \leq k \leq n - 1. \end{cases} \]

We obtain that \( \chi_{H \# \mathbb{k}G} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \text{dim}_k( V_i \otimes W_j) \chi_{ij}. \) Hence, all non-isomorphic simple \( H \# \mathbb{k}G \)-modules are \( V_i \otimes W_j \) for \( 0 \leq i \leq m - 1, 0 \leq j \leq n - 1. \)

**Remark 3.5.** Note that \( \chi_{00} = \varepsilon_{H \# \mathbb{k}G}. \) Hence \( V_0 \otimes W_0 \) is the trivial \( H \# \mathbb{k}G \)-module, where \( V_0 \) is the trivial \( H \)-module and \( W_0 \) is the trivial \( \mathbb{k}G \)-module.

The dual module \( (V_i \otimes W_j)^* \) of \( V_i \otimes W_j \) can be described as follows:

**Proposition 3.6.** We have \( (V_i \otimes W_j)^* \equiv V'_i \otimes W'_j \) for \( 0 \leq i \leq m - 1, 0 \leq j \leq n - 1, \) where \( V'_i \) is the dual of \( V_i \) as an \( H \)-module and \( W'_j \) is the dual of \( W_j \) as a \( \mathbb{k}G \)-module.

**Proof.** We need to check that \( \chi_{i \otimes j} = \chi_{ij} \circ S_{H \# \mathbb{k}G} \) for \( 0 \leq i \leq m - 1, 0 \leq j \leq n - 1. \)

Note that \( S(\mathbf{v}) = \mathbf{v}^{-1} \) and \( S^{-2}(h) = \mathbf{v}^{-1} h \mathbf{v} \) for \( h \in H. \) On the one hand,
\[ \chi_{i \otimes j}(h \# g^k) = \chi_{i}(h \# g^k) \psi^j(g^k) = \chi_{i}(S(\mathbf{v}^k S(h)) \psi^j(g^k)) = \chi_{i}(S(\mathbf{v}^{-k} S(h) \psi^j(g^k)). \]

On the other hand,
\[ (\chi_{ij} \circ S_{H \# \mathbb{k}G})(h \# g^k) = \chi_{ij}(S_{H \# \mathbb{k}G}(h \# g^k) \]
\[ = \chi_{ij}(S h \# g^k) \]
\[ = \chi_{ij}(\mathbf{v}^{-k} S(h) \psi^j(g^k)) \]
\[ = \chi_{i}(\mathbf{v}^{-k} S(h) \psi^j(g^k)). \]

We conclude that \( \chi_{i \otimes j} = \chi_{ij} \circ S_{H \# \mathbb{k}G} \) for \( 0 \leq i \leq m - 1, 0 \leq j \leq n - 1. \)

4. The Grothendieck algebras of smash product Hopf algebras

In this section, we will investigate a relationship between the Grothendieck algebra of the smash product Hopf algebra \( H \# \mathbb{k}G \) and the Grothendieck algebra of \( H. \)

Recall that the Grothendieck algebra \( (G_0(H) \otimes \mathbb{Z} \mathbb{k}, \ast) \) of \( H \) is an associative algebra over \( \mathbb{k} \) with unity \( \varepsilon_{H} \) under the convolution \( \ast, \) where the convolution \( \ast \) is defined by
\[ (\chi_i \ast \chi_j)(h) = (\chi_i \otimes \chi_j)(\Delta(h)) \text{ for } h \in H. \]

We define a new multiplication operator \( \star \) on \( G_0(H) \otimes \mathbb{Z} \mathbb{k} \) by
\[ (\chi_i \star \chi_j)(h) = (\chi_i \otimes \chi_j)(\Delta(h) \Delta'(h^{-1}) h \otimes \mathbf{v}) \text{ for } h \in H. \]
Remark 4.1. If $S^2 = id$, then $\v = 1$ by Proposition 2.2 (6). In this case, the multiplication operator $\star$ is nothing but the convolution $\ast$.

Proposition 4.2. The pair $(G_0(H) \otimes \mathbb{k}, \star)$ is an associative algebra over $\mathbb{k}$ with unity $\v_H$.

Proof: We first prove that $\star$ is a multiplication operator on $G_0(H) \otimes \mathbb{k}$. That is, $\chi_i \star \chi_j \in G_0(H) \otimes \mathbb{k}$ for $0 \leq i, j \leq m-1$. Indeed, for $a, b \in H$, using $S^2(h) = vh^{-1}$ for $h \in H$, we have

$$(\chi_i \star \chi_j)(ab) = \chi_i((a_1b_1)\v^{-1}(1)\v_1)(a_2b_2)\v^{-1}(2)\v)$$

$$= \chi_i(a_1(b\v^{-1})(1)\v_1)a_2(b\v^{-1})(2)\v)$$

$$= \chi_i(a_1(\v^{-1}S^2(b))(1)\v_1)a_2(\v^{-1}S^2(b))(2)\v)$$

$$= \chi_i(a_1\v^{-1}(1)S^2(b_1)\v_1)a_2(\v^{-1}(2)S^2(b_2)\v)$$

$$= \chi_i(a_1\v^{-1}(1)b_1\v_1)a_2(\v^{-1}(2)b_2)$$

$$= \chi_i(b_1a_1\v^{-1}(1)\v_1)b_2(a_2\v^{-1}(2)\v)$$

$$= (\chi_i \star \chi_j)(ba).$$

It follows from [6] that $\chi_i \star \chi_j \in G_0(H) \otimes \mathbb{k}$ for $0 \leq i, j \leq m-1$. Since the map $H \rightarrow H \otimes H$, $h \mapsto \Delta(h)\v^{-1}(\v \otimes \v)$ is a coassociative comultiplication in $H$ for which $\v_H$ is still a counit (see [1, Eq.(12)]), the operator $\star$ dual to the coassociative comultiplication is an associative multiplication on $G_0(H) \otimes \mathbb{k}$ with unity $\v_H$. □

Next, we will use the algebras $(G_0(H) \otimes \mathbb{k}, \star)$ and $(G_0(H) \otimes \mathbb{k}, \ast)$ to describe the structure of the Grothendieck algebra $(G_0(H \# \mathbb{k}G) \otimes \mathbb{k}, \star)$ of $H \# \mathbb{k}G$. Note that $\{\chi_0, \chi_1, \cdots, \chi_{m-1}\}$ is a $\mathbb{k}$-basis of $G_0(H) \otimes \mathbb{k}$. Suppose in $(G_0(H) \otimes \mathbb{k}, \star)$ that

$$\chi_i \star \chi_j = \sum_{k=0}^{n-1} N_{ij}^k \chi_k$$

and in $(G_0(H) \otimes \mathbb{k}, \ast)$ that

$$\chi_i \ast \chi_j = \sum_{k=0}^{n-1} L_{ij}^k \chi_k,$$

where $N_{ij}^k$ and $L_{ij}^k$ are respectively the structure coefficients of the two algebras with respect to the basis $\{\chi_0, \chi_1, \cdots, \chi_{m-1}\}$. We stress that the coefficient $N_{ij}^k$ is the multiplicity of $V_k$ appeared in the decomposition of tensor product $V_i \otimes V_j$ as $H$-modules, so each $N_{ij}^k$ is indeed a nonnegative integer. For the coefficient $L_{ij}^k$, we shall see in Remark 4.4 that each $L_{ij}^k$ is an integer.

Proposition 4.3. We have the following equations in the Grothendieck algebra $(G_0(H \# \mathbb{k}G) \otimes \mathbb{k}, \star)$:

1. $\chi_{ij} = \chi_0 \star \chi_{0j} = \chi_{0j} \star \chi_{0i}$ for $0 \leq i \leq m-1, 0 \leq j \leq n-1$.
2. $\chi_{0i} \star \chi_{0j} = \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k) \chi_{kj} + \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k) \chi_{kj}$ for $0 \leq i, j \leq m-1$. 
Proof. (1) It is direct to calculate that
\[
(\chi_{i0} * \chi_{0j})(h#g^k) = \chi_{i0}(h_{(1)}#g^k)\chi_{0j}(h_{(2)}#g^k)
\]
\[
= \chi_i(h_{(1)}v^k)\psi^0(g^k)\chi_0(h_{(2)}v^k)\psi^j(g^k)
\]
\[
= \chi_i(h^k)\psi^j(g^k)
\]
\[
= \chi_{ij}(h#g^k).
\]
So we have $\chi_{i0} * \chi_{0j} = \chi_{ij}$. It is similar that $\chi_{0j} * \chi_{i0} = \chi_{ij}$.

(2) We show that the values that both sides of the desired equation taking on $h#g^l$ are the same. Note that $v^2$ is the distinguished group-like element of $H$ and $\psi^2(g) = -1$. For the case $l = 2s$, we have
\[
\sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0}(h#g^{2s}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k2s}(h#g^{2s})
\]
\[
= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k}(h v^{2s}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k}(h v^{2s})\psi^0(g^{2s})
\]
\[
= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k}(h v^{2s}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k}(h v^{2s})
\]
\[
= \sum_{k=0}^{m-1} N_{ij}^k\chi_{k}(h v^{2s}) = (\chi_i * \chi_j)(h v^{2s})
\]
\[
= \chi_i(h_{(1)}v^{2s})\chi_j(h_{(2)}v^{2s}) \quad \text{(since $v^{2s}$ is a group-like element)}
\]
\[
= \chi_{i0}(h_{(1)}#g^{2s})\chi_{0j}(h_{(2)}#g^{2s}) = (\chi_{i0} * \chi_{0j})(h#g^{2s}).
\]
For the case $l = 2s + 1$, we have
\[
\sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0}(h#g^{2s+1}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k2s+1}(h#g^{2s+1})
\]
\[
= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k}(h v^{2s+1}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k}(h v^{2s+1})\psi^0(g^{2s+1})
\]
\[
= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k}(h v^{2s+1}) - \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k}(h v^{2s+1})
\]
\[
= \sum_{k=0}^{m-1} L_{ij}^k\chi_{k}(h v^{2s+1}) = (\chi_i * \chi_j)(h v^{2s+1})
\]
\[
= \chi_i(h_{(1)}v^{2s+1})\chi_j(h_{(2)}v^{2s+1}) \quad \text{(since $v^{2s}$ is a group-like element)}
\]
\[
= \chi_{i0}(h_{(1)}#g^{2s+1})\chi_{0j}(h_{(2)}#g^{2s+1}) = (\chi_{i0} * \chi_{0j})(h#g^{2s+1}).
\]
We obtain the desired equation.

(3) Using Part (1) and Part (2) we may see that Part (3) holds.

**Remark 4.4.** It follows from Proposition 4.3 (2) that the tensor product \((V_i \otimes W_0) \otimes (V_j \otimes W_0)\) has the following decomposition as \(H \# \mathbb{Z}G\)-modules:

\[
(V_i \otimes W_0) \otimes (V_j \otimes W_0) \cong \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)(V_k \otimes W_0) \bigoplus \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)(V_k \otimes W_2).
\]

Thus, these coefficients \(\frac{1}{2}(N_{ij}^k + L_{ij}^k)\) and \(\frac{1}{2}(N_{ij}^k - L_{ij}^k)\) are both nonnegative integers.

Since all \(N_{ij}^k\) are nonnegative integers, it follows that all \(L_{ij}^k\) are integers and satisfy \(-N_{ij}^k \leq L_{ij}^k \leq N_{ij}^k\). In view of this, the multiplication operator \(\ast\) defined on the Grothendieck algebra \(G_0(H) \otimes \mathbb{Z}k\) can be defined as well on the Grothendieck ring \(G_0(H)\).

The Grothendieck algebra \((G_0(H) \# \mathbb{Z}G) \otimes \mathbb{Z}k, \ast\) is an associative unity algebra with a \(k\)-basis \(\{\chi_{ij} \mid 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}\). Denote by

\[
\theta_l = \frac{1}{n} \sum_{i=0}^{n-1} \psi(g)^{-l} \chi_{0i} \text{ for } 0 \leq l \leq n - 1.
\]

Note that \(\chi_{0i} = \psi^l\) for \(0 \leq i \leq n - 1\). Thus, \(\{\theta_l \mid 0 \leq l \leq n - 1\}\) is the set of all central primitive idempotents of the algebra \(k\tilde{G}\). Moreover, we have

\[
\chi_{0j} \ast \theta_l = \psi(g)^{2l} \theta_l \text{ and } \chi_{ij} \ast \theta_l = \chi_{i0} \ast \chi_{0j} \ast \theta_l = \psi(g)^{2l} \chi_{i0} \ast \theta_l.
\]

In particular, each \(\theta_l\) is a central idempotent of \((G_0(H \# kG) \otimes \mathbb{Z}k, \ast)\). The structure of the Grothendieck algebra \((G_0(H \# kG) \otimes \mathbb{Z}k, \ast)\) now can be described as follows:

**Theorem 4.5.** We have the following algebra isomorphisms:

(1) If \(l\) is even, then \((G_0(H \# kG) \otimes \mathbb{Z}k, \ast) \ast \theta_l \cong (G_0(H) \otimes \mathbb{Z}k, \ast).

(2) If \(l\) is odd, then \((G_0(H \# kG) \otimes \mathbb{Z}k, \ast) \ast \theta_l \cong (G_0(H) \otimes \mathbb{Z}k, \ast).

(3) We have \((G_0(H \# kG) \otimes \mathbb{Z}k, \ast) \cong (G_0(H) \otimes \mathbb{Z}k, \ast)^{\#}\).

**Proof:** (1) For the case \(l\) being even, we consider the \(k\)-linear map

\[
\phi_l : (G_0(H) \otimes \mathbb{Z}k, \ast) \rightarrow (G_0(H \# kG) \otimes \mathbb{Z}k, \ast) \ast \theta_l, \quad \chi_i \mapsto \chi_{i0} \ast \theta_l.
\]

It can be seen from (4.1) that \(\phi_l\) is bijective, and moreover, \(\chi_{i0} \ast \theta_l = \chi_{i0} \ast \theta_l\). Now

\[
\phi_l(\chi_i \ast \chi_j) = \phi_l(\sum_{k=0}^{m-1} N_{ij}^k \chi_k) = \sum_{k=0}^{m-1} N_{ij}^k \chi_{k0} \ast \theta_l
\]

\[
= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k) \chi_{k0} \ast \theta_l + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k) \chi_{k2} \ast \theta_l
\]

\[
= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k) \chi_{k0} \ast \theta_l + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k) \chi_{k2} \ast \theta_l
\]
\[
= (\chi_{i0} \ast \chi_{j0}) \ast \theta_l = (\chi_{i0} \ast \theta_l) \ast (\chi_{j0} \ast \theta_l)
= \phi_l(\chi_i) \ast \phi_l(\chi_j).
\]

This shows that \(\phi_l\) is an algebra isomorphism.

(2) For the case \(l\) being odd, we consider the \(\mathbb{k}\)-linear map
\[
\phi_l : (G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \star) \rightarrow (G_0(H \# \mathbb{k}G) \otimes_{\mathbb{Z}} \mathbb{k}, \ast) \ast \theta_l, \quad \chi \mapsto \chi_{i0} \ast \theta_l.
\]
It can be seen from (4.1) that \(\phi_l\) is injective, and moreover, \(\chi_{i0} \ast \theta_l = -\chi_{i0} \ast \theta_l\). Now
\[
\phi_l(\chi_i \ast \chi_j) = \phi_l(\sum_{k=0}^{m-1} L_{ij}^k \chi_k) = \sum_{k=0}^{m-1} L_{ij}^k \chi_{i0} \ast \theta_l
\]
\[
= \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k) \chi_{i0} \ast \theta_l - \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k) \chi_{i0} \ast \theta_l
\]
\[
= \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k) \chi_{i0} \ast \theta_l + \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k) \chi_{j0} \ast \theta_l
\]
\[
= (\chi_{i0} \ast \chi_{j0}) \ast \theta_l = (\chi_{i0} \ast \theta_l) \ast (\chi_{j0} \ast \theta_l)
= \phi_l(\chi_i) \ast \phi_l(\chi_j).
\]

Thus, \(\phi_l\) is an algebra isomorphism.

(3) Let \((G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)^{\otimes 2}\) be the direct sum of \(\frac{2}{\ast}\)-folds of \((G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)\) and \((G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)^{\otimes 2}\) the direct sum of \(\frac{2}{\ast}\)-folds of \((G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)\). Since \(\theta_0 + \theta_1 + \cdots + \theta_{n-1} = 1\), where 1 is the identity \(\chi_{00}\) of \((G_0(H \# \mathbb{k}G) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)\), using Part (1) and Part (2) we obtain the following algebra isomorphism:
\[
(G_0(H \# \mathbb{k}G) \otimes_{\mathbb{Z}} \mathbb{k}, \ast) \cong (G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)^{\otimes 2} \bigoplus (G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)^{\otimes 2}.
\]

The proof is completed. \(\square\)

**Remark 4.6.** If \(S^2 = id\), by Remark 4.1, the algebra \((G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)\) is nothing but the Grothendieck algebra \((G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)\). In this case,
\[
(G_0(H \# \mathbb{k}G) \otimes_{\mathbb{Z}} \mathbb{k}, \ast) \cong (G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}, \ast)^{\otimes 2}.
\]

Let \(C\) be the \(\mathbb{k}\)-linear subcategory of \(\text{Rep}(H \# \mathbb{k}G)\) spanned by objects
\[
\{V_i \otimes W_0, V_i \otimes W_2 \mid 0 \leq i \leq m - 1\}.
\]

Then \(C\) is closed under taking dual by Proposition 3.6. It follows from Proposition 4.3 that \(C\) is also closed under the tensor product of objects. More explicitly,
\[
(V_i \otimes W_2) \ast (V_j \otimes W_2) \cong (V_i \otimes W_0) \ast (V_j \otimes W_0)
\]
\[
\cong \bigoplus_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k)(V_k \otimes W_0) \bigoplus \bigoplus_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k)(V_k \otimes W_2),
\]
and
\[
(V_i \otimes W_0) \ast (V_j \otimes W_2) \cong (V_i \otimes W_2) \ast (V_j \otimes W_0)
\]
\[
\cong \bigoplus_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k)(V_k \otimes W_0) \bigoplus \bigoplus_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k)(V_k \otimes W_2).
\]
Hence $C$ is a fusion subcategory of $\text{Rep}(H\#\mathbb{Z}G)$. Let $(G_0(C) \otimes \mathbb{Z}, *)$ be the Grothendieck algebra of $C$. Then $\{\chi_{i0}, \chi_{i\frac{m}{2}} \mid 0 \leq i \leq m - 1\}$ forms a $\mathbb{Z}$-basis of $(G_0(C) \otimes \mathbb{Z}, *)$.

**Proposition 4.7.** We have the following algebra isomorphism:

$$(G_0(C) \otimes \mathbb{Z}, *) \cong (G_0(H) \otimes \mathbb{Z},*) \bigoplus (G_0(H) \otimes \mathbb{Z},*).$$

**Proof:** We denote $\theta = \frac{1}{2}(\chi_{00} + \chi_{0\frac{m}{2}})$. Then $1 - \theta = \frac{1}{2}(\chi_{00} - \chi_{0\frac{m}{2}})$, where $1$ is the identity $\chi_{00}$ of $(G_0(C) \otimes \mathbb{Z},*)$. Note that $\theta$ and $1 - \theta$ are both central idempotents of $(G_0(C) \otimes \mathbb{Z},*)$. In particular,

$$\chi_{i\frac{m}{2}} * \theta = \chi_{i0} * \chi_{0\frac{m}{2}} * \theta = \chi_{i0} * \theta \text{ for } 0 \leq i \leq m - 1.$$

Consider the $\mathbb{Z}$-linear map

$$\phi : (G_0(H) \otimes \mathbb{Z},*) \to (G_0(C) \otimes \mathbb{Z},*) \cdot \theta, \chi_i \mapsto \chi_{i0} * \theta.$$

It is easy to see that $\phi$ is bijective and

$$\phi(\chi_i * \chi_j) = \phi\left(\sum_{k=0}^{m-1} N_{ij}^k \chi_k\right) = \sum_{k=0}^{m-1} N_{ij}^k \chi_{i0} * \theta$$

$$= \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k) \chi_{i0} * \theta + \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k) \chi_{i0} * \theta$$

$$= \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k) \chi_{i0} * \theta + \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k) \chi_{i\frac{m}{2}} * \theta$$

$$= (\chi_{i0} * \chi_{j0}) * \theta = (\chi_{i0} * \theta) * (\chi_{j0} * \theta)$$

$$= \phi(\chi_i) * \phi(\chi_j).$$

This shows that $\phi$ is an algebra isomorphism. Consider the $\mathbb{Z}$-linear map

$$\varphi : (G_0(H) \otimes \mathbb{Z},*) \to (G_0(C) \otimes \mathbb{Z},*) \cdot (1 - \theta), \chi_i \mapsto \chi_{i0} * (1 - \theta).$$

Then $\varphi$ is bijective. Using $\chi_{i\frac{m}{2}} * (1 - \theta) = -\chi_{i0} * (1 - \theta)$ we may see that

$$\varphi(\chi_i * \chi_j) = \varphi\left(\sum_{k=0}^{m-1} L_{ij}^k \chi_k\right) = \sum_{k=0}^{m-1} L_{ij}^k \chi_{i0} * (1 - \theta)$$

$$= \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k) \chi_{i0} * (1 - \theta) - \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k) \chi_{i0} * (1 - \theta)$$

$$= \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k + L_{ij}^k) \chi_{i0} * (1 - \theta) + \sum_{k=0}^{m-1} \frac{1}{2} (N_{ij}^k - L_{ij}^k) \chi_{i\frac{m}{2}} * (1 - \theta)$$

$$= (\chi_{i0} * \chi_{j0}) * (1 - \theta) = (\chi_{i0} * (1 - \theta)) * (\chi_{j0} * (1 - \theta))$$

$$= \varphi(\chi_i) * \varphi(\chi_j).$$
Hence, \( \varphi \) is an algebra isomorphism. \( \square \)

Note that \( \theta = \theta_0 + \theta_2 + \theta_4 + \cdots + \theta_{n-2} \) and \( 1 - \theta = \theta_1 + \theta_3 + \cdots + \theta_{n-1} \). By Theorem 4.5 and Proposition 4.7, we get the following corollary:

**Corollary 4.8.** We have algebra isomorphism:

\[
(G_0(H\#kG) \otimes \mathbb{Z} \otimes \mathbb{Z}, +) \cong (G_0(C) \otimes \mathbb{Z}, *)^g.
\]

Finally, we give some remarks on the pivotal (spherical) structure of the fusion categories \( \text{Rep}(H\#kG) \) and \( C \). Since \( S^2_{H\#kG} \) is an inner automorphism of \( H\#kG \) and

\[
S^2_{H\#kG}(h\#g^i) = (1_H\#g)(h\#g^i)(1_H\#g)^{-1},
\]

where \( 1_H\#g \) is a group-like element of \( H\#kG \), the category \( \text{Rep}(H\#kG) \) is a pivotal fusion category, where the pivotal structure \( \tau \) on \( \text{Rep}(H\#kG) \) is the isomorphism of monoidal functors \( \tau_{V \otimes W} : V \otimes W \to (V \otimes W)^{**} \) natural in \( V \otimes W \). Here \( \tau_{V \otimes W}(v \otimes w) \) is defined by

\[
\tau_{V \otimes W}(v \otimes w)(f) = f(1_H\#g \cdot v \otimes w) = f(v \cdot w \otimes g \cdot v)
\]

for \( v \in V, w \in W \) and \( f \in (V \otimes W)^{**} \).

The quantum dimension of \( V \otimes W \in \text{Rep}(H\#kG) \) with respect to the pivotal structure \( \tau \) is denoted by \( \text{dim}(V \otimes W) \), which is the following composition

\[
1 \xrightarrow{\text{coev}_{(V\otimes W)^*}} (V \otimes W) \otimes (V \otimes W)^* \xrightarrow{\tau_{V \otimes W} \otimes id} (V \otimes W)^{**} \otimes (V \otimes W)^* \xrightarrow{ev_{(V\otimes W)^*}} 1,
\]

where \( 1 \) is the trivial \( H\#kG \)-module \( V_0 \otimes W_0 \). From this composition, we have

\[
\text{dim}(V \otimes W) = \chi_V(v)\chi_W(g).
\]

Especially,

\[
\text{dim}(V_i \otimes W_j) = \chi_V(v)\psi_j(g) = \sqrt{\epsilon(e_i)\lambda(e_i)}\psi_j(g).
\]

For the dual module \( (V_i \otimes W_j)^* \cong V_{i'} \otimes W_{j'} \), we have

\[
\text{dim}(V_{i'} \otimes W_{j'}) = \sqrt{\epsilon(e_i)\lambda(e_i)}\psi_j(g^{-1}) = \sqrt{\epsilon(e_i)\lambda(e_i)}\psi_j((g)^{-1}).
\]

Therefore, \( \text{dim}(V_{i'} \otimes W_{j'}) = \text{dim}(V_i \otimes W_j) \) if and only if \( j = 0 \) or \( j = \frac{m}{2} \). This means that with respect to the pivotal structure \( \tau \), the fusion category \( \text{Rep}(H\#kG) \) is pivotal but not spherical, while the the fusion subcategory \( C \) of \( \text{Rep}(H\#kG) \) spanned by objects \( \{ V_i \otimes W_0, V_i \otimes W_m | 0 \leq i \leq m-1 \} \) is both pivotal and spherical.

**Question:** Does the pivotalization of \( \text{Rep}(H) \) equal to the pivotal fusion category \( C \)?

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