INTERIOR GRADIENT ESTIMATES FOR ANISOTROPIC MEAN CURVATURE FLOW

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ABSTRACT. In this paper we consider the evolution of a graph-like hypersurface by anisotropic mean curvature flow, under some restrictions on the anisotropic area integrand. We find interior estimates (in both time and space) on the gradient of such hypersurfaces, depending only on the height of the graph and the anisotropic area integrand.

1. INTRODUCTION

Consider the evolution of a hypersurface by its mean curvature:

\( \frac{d}{dt} x(y, t) = H(y, t), \quad y \in M, \)

where \( x : M^n \times [0, T] \rightarrow \mathbb{R}^{n+1} \) is the immersion of a manifold \( M \) at each time \( t \) and \( H \) is the mean curvature vector. 

\( M \) can be written as a graph when a fixed vector \( \omega \in \mathbb{R}^{n+1} \) can be found so that for a choice of unit normal \( \nu, \langle \nu, \omega \rangle > 0 \) everywhere. Given the image \( x(y, t) \) of a point \( y \in M \), the height of \( M \) above the hyperplane defined by \( \omega \) is denoted by \( u = \langle x, \omega \rangle \), and the gradient function is given by \( v = \langle \nu, \omega \rangle^{-1} = \sqrt{1 + |Du|^2} \).

In [6], Ecker and Huisken established that when initial data is given by an entire Lipschitz graph with a linear growth bound, there is a smooth solution to (1) for all times. An important step in this proof was showing that the solutions remain graphs: this was done by showing that \( v \) is bounded above, with a constant depending on the initial Lipschitz bound. In [7], it was established that the Lipschitz bound need only be local. The estimates in this paper are intended in the spirit of the local gradient estimates of Section 2 of the latter paper.

Such gradient estimates may be found even if the initial data is not Lipschitz. In [8], Evans and Spruck showed that under mean curvature flow, surfaces that may initially be written locally as a continuous graph, become smooth for \( t > 0 \). The level set method was also used by Barles, Biton and Ley [4] to find similar gradient estimates for more a general class of equations. More recently, in [5], Colding and Minicozzi found an explicit local estimate in the form

\[ |Du(x, t)| \leq \exp \left\{ c \left( \frac{1 + t^{-1/2} \| u \|_\infty}{1} \right)^2 \right\}, \]

for solutions over a ball \( B_R(x), \ R \geq C \sqrt{t} \). Here, the constant \( c \) depends only the dimension. This estimate does not depend on an initial gradient estimate.

In this paper we find an analogous result for anisotropic mean curvature flow, under some restrictions on the anisotropy.

Such estimates are an important step in finding existence results for a variety of boundary value problems with non-smooth initial data, as in [1].

In this paper, we follow the exposition of Andrews in [1], in particular Section 8, in which the evolution equation for graph-like surfaces is derived.

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We consider surfaces $M$ with local embeddings $x = y^i e_i + u(y^1, \ldots, y^n) e_0$, and normal $\nu = Du - \phi^0 = \sum_{i=1}^n u_i \phi^i - \phi^0$, where $\{\phi^0, \phi^1, \ldots, \phi^n\}$ and $\{e_0, e_1, \ldots, e_n\}$ are dual bases for the cotangent space $V^* \cong \mathbb{R}^{n+1}$ and tangent space $V \cong \mathbb{R}^{n+1}$ respectively.

The anisotropic mean curvature flow for such a surface is given by

\[
  u_t = F D^2 F|_{Du = \phi^0} \left( \phi^i, \phi^j \right) D^2 u(e_i, e_j)
\]

(the homogeneous degree zero mobility function $m = m(\nu)$ of [1] is here taken to be identically 1).

The anisotropic area integrand $F : V^* \to \mathbb{R}$ is a positive, convex function that is $C^3$ on $V^* \setminus \{0\}$, and homogeneous of degree one, so that $F(\lambda v) = \lambda F(v)$ for $v \in V^*$ and all scalars $\lambda > 0$. The level sets of $F$ are denoted by $\Sigma_\lambda := \{v \in V^* : F(v) = \lambda\}$. We impose the condition that the convex hull of each level set, $\{v \in V^* : F(v) \leq \lambda\}$, must be uniformly convex.

**Example.** In the isotropic case, $F(\nu)^2 = \sum_{i=0}^n (v_i)^2$ and the level sets of $F$ are spheres. The coefficients are those of mean curvature: $F D^2 F|_{Du = \phi^0} \left( \phi^i, \phi^j \right) = \delta_{ij} - u_i u_j (1 + |Du|^2)^{-1}$.

We define two further conditions on $F$:

**Smallness of third derivatives condition:** Define

\[
  Q_\nu(p, q, r) := F^2(\nu) D^3 F|_{\nu} (p, q, r).
\]

This condition is satisfied when

\[
  Q_\nu(p, q, r) \leq C_1 \left[ F(\nu)^3 D^2 F|_{\nu} (p, p) D^2 F|_{\nu} (q, q) D^2 F|_{\nu} (r, r) \right]^{1/2}
\]

for all $p, q, r$ tangent to the level set $\Sigma_{F(\nu)}$ at $\nu$. Here $C_1$ is a positive constant dependent on $n$.

This $Q$ is the Cartan tensor of Bao, Chern and Shen [3], or the tensor $Q$ of [11] restricted to the tangent space of the level set. The restriction here excludes anisotropic area integrands that deviate too far from the isotropic case (such as approximations to the crystalline case). Similar restrictions on third derivatives of $F$ are made in studies of the elliptic problem (see, for example, [9]).

**Symmetry condition:** This is satisfied when

\[
  F(p + \phi^0) = F(p - \phi^0) \quad \text{for all } p = \sum_{i=1}^n p_i \phi^i.
\]

**Example.** The isotropic case clearly satisfies the second condition, and also satisfies the first with $C_1 = 0$.

We impose these conditions singly, to find two time-interior gradient estimates for periodic solutions; and jointly, to find a gradient estimate interior in both space and time. (Note that in [2], such estimates are found for periodic flows without imposing either condition.)

Let $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ be a $C^3$, bounded $|u(x, t)| \leq M$, solution to the anisotropic mean curvature flow equation (2).

**Theorem 1.** Let $u$ be periodic, so that $u(x, t) = u(x + L, t)$ for some lattice $L \in \mathbb{R}^n$. If $F$ satisfies condition [3] with $C_1^2 < 4n^{-1/2}$ then

\[
  F(Du - \phi^0) \leq \max \left\{ t^{1/2} \exp \left( \frac{Aq(|u| - 2M)^2}{4t} \right), P \right\}
\]

for $0 < t \leq T'$, where $T'$ depends on $M$, and $A, P$ and $q > 1$ depend on $F$.

**Theorem 2.** Let $u$ be periodic, so that $u(x, t) = u(x + L, t)$ for some lattice $L \in \mathbb{R}^n$. If $F$ satisfies the symmetry condition (4), then

\[
  F(Du - \phi^0) \leq \max \left\{ t \exp \left( \frac{A(|u| - 2M)^2}{2t} \right), S \right\}
\]
for $0 < t \leq T'$, where $T'$ depends on $M$. $A$ depends on $F$, and $S$ depends on $F$ and $n$.

**Theorem 3** (Interior estimate for anisotropic mean curvature flow). Let $n > 1$. If $F$ satisfies both the smallness of third derivatives condition (5) with $C_0^2 < 2/\sqrt{n}$, and the symmetry condition (4), then

$$F(Du - \phi^0) \leq \max \left\{ t^{3/2} \exp \left( \frac{Aq(|u| - 2M)^2}{4t} \right) (R^2 - 2kt - |x|^2)^{-r}, P \right\}$$

for $0 < t \leq T'$. Here, $A, P$ and $k$ depend only on $F$, while $T' > 0$, $q > 1$ and $r > 1$ depend on $F$ and $M$.

In the next section, we derive some technical results on $F$ and its derivatives. In the third section we prove Theorems 1 and 2 and in the final section we prove Theorem 3.

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## 2. Some results regarding the function $F$

The uniform convexity implies that $FD^2F|_{\nu}$ is positive definite on the tangent space of the level set $\Sigma_{F(\nu)}$. The homogeneity of $F$ leads to the disappearance of some derivatives of $F$ in radial directions:

(5a) \[ DF|_{\nu} (\nu) = F(\nu) \]

(5b) \[ D^2F|_{\nu} (\nu, \cdot) = D^2F|_{\nu} (\cdot, \nu) = 0 \]

(5c) \[ D(FD^2F)|_{\nu} (\nu, \cdot, \cdot) = 0. \]

These properties make it more convenient to work, not in the space $(TM)^*$, but rather in the tangent space to the level set $\Sigma_{F(\nu)}$. Given $\nu$ normal to $M$ at $x$, we can map $v \in V^*$ to $T_{\nu} \Sigma_{F(\nu)}$ by setting $\hat{v} := v - r(v)\nu$. The normal is not in the tangent space itself, as $DF|_{\nu} (\nu) \neq 0$, by (5a). By choosing $r(v)$ appropriately, $\hat{v}$ will be in the tangent space, with

$$0 = DF|_{\nu} (\hat{v}) = DF|_{\nu} (v - r(v)\nu) = DF|_{\nu} (v) - r(v)F(\nu),$$

where we have used (5a) in the last step. With $r(v) = DF|_{\nu} (v)/F(\nu)$ we then have

$$\hat{v} = v - \frac{DF|_{\nu} (v)}{F(\nu)} \nu,$$

which is non-zero if $v$ is not parallel to $\nu$. Consequently, (5b) implies that for non-zero $v \in (TM)^*$, $FD^2F|_{\nu} (v, v) = FD^2F|_{\nu} (\hat{v}, \hat{v}) > 0$, so we consider this as a new metric on $(TM)^*$ and write $G_{\nu}(v, w) := FD^2F|_{\nu} (v, w)$.

**Lemma 4.** For all $P_1 \geq 0$,

$$F(p - \phi^0) \geq P_1 + F(-\phi^0) \implies F(p) \geq P_1.$$

**Proof.** This is a simple consequence of convexity, as $F(p - \phi^0) \leq F(p) + F(-\phi^0)$.

**Lemma 5.** Let $\{ \phi^0, \ldots, \phi^n \}$ be a basis for $V^*$. Then for each $P > F(-\phi^0)$ there exists $A_P > 0$ such that

$$FD^2F|_{p - \phi^0} (p, p) \geq A_P$$

for all $p = \sum_{i=1}^n p_i \phi^i$ with $F(p - \phi^0) \geq P$. 

\[ \square \]
Proof. Set \( P_1 = P - F(-\phi^0) > 0 \). By Lemma 4 if \( F(p - \phi^0) \geq P \) then \( F(p) \geq P_1 \).

Define \( B(p) := F(D^2F)_{p-\phi^0}(p, p) \). Consider this for a fixed member of the level set \( p \in \Sigma_{P_1} \cap \text{span}\{\phi^1, \ldots, \phi^n\} \); as \( p \) is not parallel to \( p - \phi^0 \), \( B(p) \) is positive.

Also,

\[
\lim_{s \to \infty} B(sp) = \lim_{s \to 0} F(D^2F)_{sp-\phi^0}(sp, sp) = \lim_{s \to \infty} F(D^2F)_{sp-\phi^0}(sp - (sp - \phi^0), sp - (sp - \phi^0)) = \lim_{s \to \infty} F(D^2F)_{sp/\phi^0}(\phi^0, \phi^0) = F(D^2F)_{sp}(\phi^0, \phi^0) > 0,
\]

where in the second line, the additional terms added vanish according to (5b), and in the third line the scaling in \( s \) is allowed as \( F(D^2F) \) is homogeneous degree zero. The final inequality is because \( \phi^0 \) is not parallel to \( p \). Since \( B(sp) > 0 \) for \( 1 \leq s < \infty \), it follows that \( \inf_{s \in [1, \infty)} B(sp) = A_P > 0 \), and taking the minimum over all \( p \) in the (compact and closed) level set gives

\[
\inf_{\Sigma_{P_1} \cap \text{span}\{\phi^1, \ldots, \phi^n\}} A_P =: A_P > 0.
\]

\[\square\]

Lemma 6. If \( F \) satisfies the symmetry condition (4), then homogeneity implies that

\[
\begin{align*}
DF|_p(\phi^0) & = 0 \\
D^2F|_p(\phi^0, \phi^j) & = 0 \\
D^3F|_p(\phi^0, \phi^j, \phi^k) & = 0 \\
D^3F|_p(\phi^0, \phi^0, \phi^0) & = 0,
\end{align*}
\]

(6)

for all \( p = \sum_{i=1}^n p_i \phi^i \) and all \( j, k \neq 0 \).

Proof. The symmetry condition implies that

\[
DF|_p(\phi^0) = \lim_{s \to 0} \frac{1}{s} [F(p + s\phi^0/2) - F(p - s\phi^0/2)] = 0.
\]

The others may be proven similarly. \[\square\]

In the following lemma, we show that the symmetry condition (4) can be used in a similar way to the smallness-of-third-derivatives condition (3). We use this in the proof of Theorem 2.

Lemma 7. Suppose the symmetry condition (4) holds. For all \( \epsilon > 0 \) we can find \( S_\epsilon \) such that if \( p = \sum_{i=1}^n p_i \phi^i \) satisfies \( F(p - \phi^0) \geq S_\epsilon \), then

\[
|FD(FD^2F)|_{p-\phi^0}(p, q, \tilde{q}) \leq \epsilon \left\{ FD^2F|_{p-\phi^0}(p, p) \right\}^{1/2} FD^2F|_{p-\phi^0}(q, q)
\]

for all \( q = \sum_{i=1}^n q_i \phi^i \).

Proof. Let \( \epsilon > 0 \) be given. As (7) is unchanged under the mapping \( q \mapsto sq \), we consider only those \( q \) on a fixed level set \( \Sigma_1 \).

Our approach is to restrict \( p \) to a level set \( \Sigma_P \), for \( P > F(-\phi^0) \), and then show that under the mapping \( p \mapsto sp \), the quotient

\[
\frac{|FD(FD^2F)|_{sp-\phi^0}(sp, \tilde{q}, \tilde{q})|}{G(sp, sp)^{1/2}G(q, q)}
\]

(8)
is less than ε for large enough s. In the above expression,
\[ \hat{q} = q - \frac{DF_{sp-\phi^0}(q)}{F(sp-\phi^0)}(sp-\phi^0) = q - sr(q)p + r(q)\phi^0, \]
with \( \lim_{s \to \infty} r(q) = 0 \) and \( \lim_{s \to \infty} sr(q) = DF|_p(q)/F(p) \).

As \( s \to \infty \), the numerator of (8) is
\[ \lim_{s \to \infty} FD(FD^2F)|_{sp-\phi^0}(sp, \hat{q}, \hat{q}) \]
\[ = \lim_{s \to \infty} FD(FD^2F)|_{p-\phi^0} (\phi^0, q -sr(q)p + r(q)\phi^0, q -sr(q)p + r(q)\phi^0) \]
\[ = FD(FD^2F)|_p \left( \phi^0, q - \frac{DF|_p(q)}{F(p)}p, q - \frac{DF|_p(q)}{F(p)}p \right) \]
\[ = 0, \]
by (3).

If \( q \) is not parallel to \( p \), then the denominator is strictly positive:
\[ \lim_{s \to \infty} G_{sp-\phi^0}(sp, sp)^{1/2} G_{sp-\phi^0}(q, q) \geq \sqrt{Ap} G_p(q, q) > 0, \]
where \( Ap \) is the constant given by Lemma 5. It follows that for each such \( q \), we can find an \( S \) so that \( F(sp) \geq S \) implies (8) is less than \( \epsilon \).

In the case that \( q \) is parallel to \( p \), without loss of generality we can set \( q = p \). Multiply both numerator and denominator of (8) by \( s^2 \), so that the latter is bounded below, \( \lim_{s \to \infty} G_{sp-\phi^0}(sp, sp)^{3/2} \geq Ap^{3/2} > 0 \). The numerator is then
\[ FD(FD^2F)|_{sp-\phi^0} (sp, [s - s^2 r(p)]p + sr(p)\phi^0, [s - s^2 r(p)]p + sr(p)\phi^0) \]
and since
\[ \lim_{s \to \infty} s - s^2 r(p) = \lim_{s \to \infty} s \left[ 1 - \frac{DF|_{sp-\phi^0}(p)}{F(p - \phi^0)/s} \right] \]
\[ = \lim_{s' \to 0} \frac{1}{s'} \left[ \frac{DF|_p(p)}{F(p)} - \frac{DF|_{p-s'\phi^0}(p)}{F(p - s'\phi^0)} \right] \]
\[ = D \left( \frac{DF|_p(p)}{F(p)} \right) (\phi^0, p) \]
\[ = \frac{D^2F|_p(\phi^0, p)}{F(p)} - \frac{DF|_p(p) DF|_p(\phi^0)}{F(p)^2} \]
\[ = 0, \]
the numerator approaches \( FD(FD^2F)|_p(\phi^0, \phi^0, \phi^0) = 0 \) as \( s \to \infty \). That is, for \( q = p \) we can find an \( S \) such that \( F(sp) \geq S \) implies (8) is less that \( \epsilon \).

Since (8) is continuous in \( q \), the supremum over \( q \in \Sigma_1 \) (and hence in \( V^* \)) of these constants \( S \) is finite, and we set this to be \( S_\epsilon' \). Finally, we set \( S_\epsilon = S_\epsilon' + F(-\phi^0) \), so that whenever \( F(p - \phi^0) \geq S_\epsilon \), \( F(p) \geq S_\epsilon' \).

The final two technical lemmas are used in the proof of Theorem 3.

**Lemma 8.** Let \( \{\phi^0, \phi^1, \ldots, \phi^n\} \) be a basis for \( V^* \), where \( n > 1 \). Then there are constants \( k', k > 0 \) such that for all \( p = \sum_{i=1}^n p_i \phi_i \),
\[ k' \leq \sum_{i=1}^n FD^2F|_{p-\phi^0}(\phi^i, \phi^i) \leq k. \]
Lemma 9. If $F$ satisfies the symmetry condition (4), then there exists a constant $C_2$ depending only on $F$ such that

$$FD^2F\big|_{p\varphi^0}(p, q) \leq C_2 \frac{F(q)}{F(p - \varphi^0)}$$

for all $p = \sum_{i=1}^{n} p_i \phi_i$ and $q = \sum_{i=1}^{n} q_i \phi_i$.

Proof. This is unchanged under the mapping $q \mapsto sq$, so we may restrict $q$ to $\Sigma_1$.

Let $q$ be fixed. For any given $p \in \Sigma_1$, consider

$$\frac{F(p - \varphi^0)}{F(q)} FD^2F \big|_{p\varphi^0}(p, q)$$

under the mapping $p \mapsto sp$ as $s \to \infty$:

$$\lim_{s \to \infty} \frac{F(sp - \varphi^0)}{F(q)} FD^2F \big|_{sp\varphi^0}(sp, q)$$

$$= \lim_{s \to 0} \frac{1}{F(q)} \left[ F(p - s' \varphi^0) FD^2F \big|_{p-s'\varphi^0}(\varphi^0, q) - F^2D^2F \big|_p(\varphi^0, q) \right]$$

$$= - \frac{1}{F(q)} D \left( F^2D^2F \right) \big|_p(\varphi^0, \varphi^0, q),$$

which is bounded, as $F$ is $C^3$. Consequently,

$$\sup_{s \in [0, \infty)} \frac{F(sp - \varphi^0)}{F(q)} FD^2F \big|_{sp\varphi^0}(sp, q) \leq C(p, q) < \infty$$

for some finite $C(p, q)$. Setting $C_2 = \max_{p, q \in \Sigma_1} C(p, q)$ gives the required result. \qed
3. The Gradient Estimate for Periodic Flows

Proof of Theorem. Define $Z := F(Du - \phi^0) - \varphi(u, t)$ where $\varphi$ is a smooth positive function for $t > 0$ with $\varphi(\cdot, 0) \geq \sup_{t=0} F(Du - \phi^0)$. Later, we will choose $\varphi$ to be some inverse power of the fundamental solution to a heat equation, but we start by focusing on the first part of $Z$.

Consider the first point where $Z$ is no longer negative, so that $F = \varphi$. This point will be a spatial maximum of $Z$, due to the periodicity of $u$.

Assume that at this point, $F(Du - \phi^0) \geq P > F(-\phi^0)$.

The first derivative condition at this point is $0 = D_k Z = DF|_{\nu}(\phi^m)u_{mk} - \varphi'u_k$, where $\nu = Du - \phi^0$. That is, for all vectors $v \in \text{span}\{e_1, \ldots, e_n\}$,

\[
D^2u \left( DF|_{\nu}(\phi^m)e_m, v \right) = \varphi'Du(v).
\]

Using (5b), we can rewrite the evolution equation for $u$ in terms of the tangential (to the level set $\Sigma_{F(\nu)}$) directions $\hat{\phi}^i$,

\[
\frac{\partial F}{\partial t} = D^2 \left( DF|_{\nu}(\phi^m)e_m, \nu \right) = D^2 \left( DF|_{\nu}(\phi^m, \hat{\phi}^i)u_{ij} \right).
\]

We make use of this in finding an evolution equation for $F$:

\[
\frac{\partial F}{\partial t} = D^2 \left( DF|_{\nu}(\phi^m)e_m, \nu \right) = D^2 \left( DF|_{\nu}(\phi^m, \hat{\phi}^i)u_{ij} \right).
\]

where in the third step we have added and subtracted second derivatives of $F$. Derivatives of $\nu$ are $D_k\nu = u_{mk}\phi^m$ which we use to simplify those terms with derivatives of $\hat{\phi}^i$:

\[
D^2 \left( DF|_{\nu}(\phi^m, \hat{\phi}^i)u_{ij} \right) = D^2 \left( DF|_{\nu}(-\phi^i u'_{ij}) \right) = -\frac{DF|_{\nu}(\phi^i)}{DF(\nu)} D^2 \left( DF|_{\nu}(u_{mk}\phi^m, \hat{\phi}^i) \right) u_{mk}u_{ij}.
\]

The evolution equation is now

\[
\frac{\partial F}{\partial t} = D^2 \left( DF|_{\nu}(\phi^m)e_m, \nu \right) = D^2 \left( DF|_{\nu}(\phi^m, \hat{\phi}^i)u_{ij} \right).
\]
At a critical point of $Z$, we can use the first derivative condition (9) to simplify further. The first term of (10) becomes
\[
D^2 u (DF|\nu (\phi^k) e_k, e_m) D (FD^2 F)|\nu (\phi^m, \tilde{\phi}^i, \tilde{\phi}^j) u_{ij} = \varphi' Du (e_m) D (FD^2 F)|\nu (\phi^m, \tilde{\phi}^i, \tilde{\phi}^j) u_{ij} = \varphi' D (FD^2 F)|\nu (Du, \tilde{\phi}^i, \tilde{\phi}^j) u_{ij},
\]
while the second becomes
\[
- D F|\nu (\phi^k) D F|\nu (\phi^i) D^2 F|\nu (\phi^m, \tilde{\phi}^j) u_{mk} u_{ij} = - D^2 F|\nu (\phi^m, \tilde{\phi}^i) D u \left( D F|\nu (\phi^k) e_k, e_m \right) D^2 u (DF|\nu (\phi^i) e_i, e_j) = - \varphi^2 D^2 F|\nu (Du(e_m) \phi^m, Du(e_j) \tilde{\phi}^j) = - \varphi^2 D^2 F|\nu (Du, Du),
\]
as does the third, so the evolution equation is
\[
\frac{\partial F}{ \partial t} = \frac{\varphi'}{ \varphi} F D (FD^2 F)|\nu (Du, \tilde{\phi}^i, \tilde{\phi}^j) u_{ij} - 2 \frac{\varphi^2}{ \varphi} F D^2 F|\nu (Du, Du) + F D^2 F|\nu (\phi^i, \phi^j) D_{ij} F - \frac{1}{ \varphi} F D^2 F|\nu (\phi^i, \phi^j) F D^2 F|\nu (\phi^m, \phi^l) u_{mi} u_{lj},
\]
where we have multiplied some terms through by $1 = F/\varphi$ in order that derivatives of $F$ appear as homogeneous degree zero terms.

Derivatives of $\varphi$ are given by
\[
D \varphi = \varphi' Du \quad D_{ij} \varphi = \varphi'' u_i u_j + \varphi' u_{ij} \quad \frac{d\varphi}{dt} = \varphi' u_t + \varphi_t
\]
for $i, j \neq 0$, so an evolution equation for $\varphi$ is
\[
\frac{d\varphi}{dt} = \varphi' u_t + \varphi_t + F D^2 F|\nu (\phi^i, \phi^j) (D_{ij} \varphi - \varphi'' u_i u_j - \varphi' u_{ij}) = \varphi_t + F D^2 F|\nu (\phi^i, \phi^j) (D_{ij} \varphi - \varphi'' u_i u_j),
\]
and the entire evolution equation for $Z$, at a local maximum, is
\[
\frac{dZ}{dt} = F D^2 F|\nu (\phi^i, \phi^j) D_{ij} Z + \frac{\varphi'}{ \varphi} F D (FD^2 F)|\nu (Du, \tilde{\phi}^i, \tilde{\phi}^j) u_{ij} - 2 \frac{\varphi^2}{ \varphi} F D^2 F|\nu (Du, Du) - \frac{1}{ \varphi} F D^2 F|\nu (\phi^i, \phi^j) F D^2 F|\nu (\phi^m, \phi^l) u_{mi} u_{lj} - \varphi_t + F D^2 F|\nu (Du, Du) \varphi''.
\]

Notice that all the covectors $\phi^i$, $Du$ appear in places where replacing them by their projections in the tangent space of $\Sigma_{F|\nu}$, that is, by $\tilde{\phi}^i$ or $\tilde{Du}$, has no effect, thanks to (5b) and (5c). On the tangent space, $D^2 F$ is positive definite. Choose the basis $\{\phi^1, \ldots, \phi^n\}$ so that $G_{ij}$ is the identity at
the maximum point, $G_{\nu}^{\alpha\beta} = \delta^{\alpha\beta}$. The evolution equation for $Z$ is now

$$\frac{dZ}{dt} = G^{ij}D_{ij}Z + \frac{\varphi'}{\varphi} FD(FD^2F)\big|_\nu (\tilde{D}u, \tilde{\phi}^1, \tilde{\phi}^2)u_{ij} - 2\frac{\varphi^2}{\varphi} G(Du, Du)$$

(11)

$$- \frac{1}{\varphi} G^{ij} G^{ml} u_{mi} u_{lj} - \varphi_t + G(Du, Du)\varphi''.$$  

The Cauchy-Schwarz inequality for a positive definite matrix $B$ implies that $v^T w \leq \epsilon v^T B v + (4\epsilon)^{-1} w^T B^{-1} w$. We use this to estimate the second term of (11):

$$\frac{\varphi'}{\varphi} FD(FD^2F)\big|_\nu (\tilde{D}u, \tilde{\phi}^1, \tilde{\phi}^2)u_{ij} = \frac{\varphi'}{\varphi} \left[ DF\big|_\nu (\tilde{D}u) F D^2 F\big|_\nu (\tilde{\phi}^1, \tilde{\phi}^2) + F^2 D^3 F\big|_\nu (\tilde{D}u, \tilde{\phi}^1, \tilde{\phi}^2) \right] u_{ij}$$

$$= \frac{\varphi'}{\varphi} u_{ik} Q^{kj} u_{ij}$$

$$\leq \frac{\epsilon \varphi^2}{4\epsilon \varphi} G(Du, Du) + \frac{1}{4\epsilon \varphi} G_{\alpha\beta} Q^{\alpha ij} u_{ij} Q^{\beta kl} u_{kl},$$

where the first term of the second line is zero, as $\tilde{D}u$ is tangent to the unit ball, so $DF\big|_\nu (\tilde{D}u) = 0$. In the last line, we have used the notation for the inverse $G_{\alpha\beta} = (G^{-1})^{\alpha\beta}$.

We can use $\mathbf{[3]}$, the smallness-of-third-derivatives condition, to estimate the second term in this inequality:

$$\frac{1}{4\epsilon \varphi} G_{\alpha\beta} Q^{\alpha ij} u_{ij} Q^{\beta kl} u_{kl} = \frac{1}{4\epsilon \varphi} Q \left(G_{\alpha\beta} \tilde{\phi}^\alpha, u_{ij} \tilde{\phi}^i, \tilde{\phi}^j\right) Q \left(\tilde{\phi}^\beta, u_{kl} \tilde{\phi}^k, \tilde{\phi}^l\right)$$

$$\leq C_{1/2} \left(\frac{G(G_{\alpha\beta} \tilde{\phi}^\alpha, G_{\gamma\beta} \tilde{\phi}^\gamma) G(u_{ij} \tilde{\phi}^i, u_{mj} \tilde{\phi}^m) G(\tilde{\phi}^j, \tilde{\phi}^l)}{G(\tilde{\phi}^3, \tilde{\phi}^3) G(u_{kl} \tilde{\phi}^k, u_{pl} \tilde{\phi}^p) G(\tilde{\phi}^l, \tilde{\phi}^l)}\right)^{1/2}$$

$$= C_{1/2} \frac{2}{4\epsilon \varphi} \left(\frac{G_{\beta\beta} u_{ij} u_{ij} G^{ij} G_{\beta\beta} u_{kl} u_{kl} G_{\beta\beta}}{4\epsilon \varphi} \right)^{1/2}$$

Now we can estimate (11) from above —

$$\frac{dZ}{dt} \leq G^{ij} D_{ij} Z + \frac{1}{\varphi} \left(\frac{C_{1/2}}{4\epsilon \sqrt{n} - 1}\right) G^{ij} G^{kl} u_{ik} u_{jl}$$

$$+ \frac{\varphi^2}{\varphi} (\epsilon - 2) G(Du, Du) - \varphi_t + \varphi'' G(Du, Du).$$

The second term is zero if we choose $\epsilon = C_{1/2} \sqrt{n}/4 < 1$.

Choose $\varphi = \Phi^{-q}$ for some $q > 1$ and

(12) \[ \Phi(u, t) = \frac{1}{\sqrt{t}} \exp \left( -A \frac{(u - 2M)^2}{4t} \right), \]
which satisfies the heat equation $\Phi_t = A\Phi''$, where $A = A_P$ is the constant given by Lemma 5.

As

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} &= -q\Phi^{-q-1}\varphi' \\
\varphi'' &= q(q + 1)\Phi^{-q-2}(\varphi')^2 - q\Phi^{-q-1}\Phi'' \\
\varphi_t &= -q\Phi^{-q-1}\Phi_t,
\end{align*}
\]

the equation satisfied by $\varphi$ is $\varphi_t = A\varphi'' - A(1 + q^{-1})\varphi^2/\varphi$. If we substitute $\Phi$ and its derivatives for $\varphi$ and its derivatives, we find that

\[
\frac{\varphi'^2}{\varphi}(\epsilon - 2)G(Du, Du) = \psi_t + \varphi''G(Du, Du) = q^2\Phi^{-q-2}\varphi'^2(\epsilon - 2)G(Du, Du)
+ \left[ q(q + 1)\Phi^{-q-2}\varphi'^2 - q\Phi^{-q-1}\varphi'' \right] G(Du, Du) + Aq\Phi^{-q-1}\Phi''
= q\Phi^{-q-2}\varphi'^2[\epsilon(\epsilon - 1) + 1]G(Du, Du) + q\Phi^{-q-1}\Phi''[A - G(Du, Du)].
\]

The first term is zero if we choose $q^{-1} = 1 - \epsilon = 1 - C_1^2\sqrt{n}/4$.

As we assumed at the beginning that $F(Du - \phi^0) \geq P$, Lemma 5 implies that $G(Du, Du) \geq A_P$. As $\Phi''$ is positive for small times, for $t < T'$ we have $\frac{\partial Z}{\partial t} \leq 0$.

On the other hand, if we consider the possibility that $F(Du - \phi^0) = \varphi < P$ at this local maximum, we could replace $\varphi$ by $\sup\{\varphi, P\}$ in the definition of $Z$. In that case, the first maximum of $Z$ occurs at a point where the barrier is flat, and so the first variation is

\[0 = D_kZ = DF|_z (\phi^k)u_{mk},\]

and the evolution equation for $Z$ at the local maximum is

\[
\frac{dZ}{dt} = FD^2F|_\nu(\phi^i, \phi^j)D_{ij}Z - D^2F|_\nu(\phi^i, \phi^j) FD^2F|_\nu(\phi^m, \phi^l)u_{ml}u_{ij} \leq 0.
\]

Since $Z_t \leq 0$ at the first point where $Z = 0$, $Z \leq 0$ for all $t < T'$. The same argument works if, in the definition of $\Phi$ (12), the term $u - 2M$ is replaced by $u + 2M$. The conclusion (with $|u| - 2M$) follows.

\textit{Proof of Theorem 2.} We begin by defining $Z$ as in Theorem 1 and assume that the first non-negative value of $Z$ occurs when $F(Du - \phi^0) \geq S_\epsilon$, for some $\epsilon > 0$ to be chosen later and the corresponding $S_\epsilon$ given by Lemma 7. We then follow the earlier proof up to equation (11), the evolution equation for $Z$ at a local maximum.

This time, we choose the local coordinates $\{\phi^1, \ldots, \phi^n\}$ so that at this point $D^2u$ is diagonal. This puts the second term of (11) in a suitable form to be estimated using Lemma 7.

\[
\begin{align*}
\frac{\varphi'}{\varphi} FD^2F &\mid_{\nu}(Du, \hat{\phi}^j, \hat{\phi}^j)u_{jj} \leq \left| \frac{\varphi'}{\varphi} \sqrt{G(Du, Du)G(\hat{\phi}^j, \hat{\phi}^j)}u_{jj} \right| \\
&\leq \frac{\varphi'^2}{2\varphi}G(Du, Du) + \frac{\epsilon^2}{2\varphi} \left| G(\hat{\phi}^j, \hat{\phi}^j)u_{ij} \right|^2 \\
&\leq \frac{\varphi'^2}{2\varphi}G(Du, Du) + \frac{\epsilon^2}{2\varphi}nG^{ji}u_{jk}G^{kl}u_{li},
\end{align*}
\]

where in the last line we use trace inequality $(\text{trace } A)^2 \leq n \text{trace}(A^2)$. 

If we now choose $\epsilon = \sqrt{2/n}$, the second term of this inequality is cancelled by the fourth term of \eqref{11}. The evolution equation now becomes

$$
\frac{dZ}{dt} \leq G^{ij} D_{ij} Z - \frac{3}{2} \frac{\varphi'^2}{\varphi} G(Du, Du) - \varphi_t + G(Du, Du) \varphi''.
$$

This is negative at a local maximum if we make the same choice of barrier as before, $\varphi = \Phi^{-q}$ for $q = 2$, $\Phi$ given by \eqref{12}, with $A = A_S$, given by Lemma \[5\].

If our assumption that $F(Du - \phi^0) \geq S_\ell$ does not hold, then we can replace $\varphi$ by $\max \{S_\ell, \varphi\}$. At the local maximum, $Z_t \leq 0$ and so the conclusion follows.

**Remark.** In the last theorem, we have chosen $q = 2$ somewhat arbitrarily; in fact $q$ needs only to be strictly greater than 1, since we can set $q = (1 - n \epsilon^2/4)^{-1}$, for $\epsilon$ given by Lemma \[7\]. However, a smaller $\epsilon$ may force a larger $S_\ell$, so the optimal choice would depend on the exact form of $F$.

4. **INTERIOR ESTIMATE FOR ANISOTROPIC MEAN CURVATURE FLOW**

**Proof of Theorem**\[3\] We introduce a localising term $\eta$ into our definition of $Z$,

$$
Z := F(Du - \phi^0) - \frac{\varphi}{\eta},
$$

which is now restricted to the shrinking ball $(x, t) \in B_{\sqrt{2/n} \epsilon} \times [0, T]$, where $k$ is the constant given by Lemma \[8\]. The smooth strictly positive function $\varphi = \varphi(u, t)$ is chosen so that $Z < 0$ at the initial time, and $\eta$ is a smooth positive function chosen so that $\eta \to 0$ on the boundary of the shrinking ball.

Assume that at the first interior point where $Z = 0$, $F(Du - \phi^0) \geq P > F(\phi^0)$.

Then $F(Du) = \varphi/\eta$ and as this is a spatial maximum (since the choice of $\eta$ ensures that there are no boundary maxima) we have a first derivative condition

\[13\]$$
0 = D_k Z = DF \frac{\partial (\varphi^m) u_{mk} - D_k (\varphi/\eta)}{\partial t}.
$$

An evolution equation for $\varphi/\eta$, with second derivatives $G^{ij} D_{ij} (\varphi/\eta)$ added and subtracted, follows:

$$
\frac{d}{dt} \left( \frac{\varphi}{\eta} \right) = \frac{1}{\eta} \left( \varphi' u_t + \varphi_t \right) - \frac{\varphi}{\eta^2} \frac{d\eta}{dt} + FD^2F \left. \frac{\partial (\varphi^i, \varphi^j)}{\partial (\varphi^i, \varphi^j)} D_{ij} \left( \frac{\varphi}{\eta} \right) \right|_{\nu} \\
- FD^2F \left. \frac{\partial (\varphi^i, \varphi^j)}{\partial (\varphi^i, \varphi^j)} \left[ \frac{1}{\eta} \left( \varphi'' u_i u_j + \varphi' u_{ij} \right) - \frac{\varphi'}{\eta^2} \left( u_j D^i \eta + u_i D^j \eta \right) \right] \right|_{\nu} \\
+ 2 \frac{\varphi'}{\eta^2} D^i \eta D^j \eta - \frac{\phi'}{\eta^3} D^i \eta D_{ij} \eta
$$

$$
= G^{ij} D_{ij} \left( \frac{\varphi}{\eta} \right) + \frac{1}{\eta} \left[ \varphi_t - G(Du, Du) \varphi'' \right] - \frac{\varphi'}{\eta^2} \left( \frac{d}{dt} - G^{ij} D_{ij} \right) \eta \\
+ 2 \frac{\varphi'}{\eta^2} G(Du, D\eta) - 2 \frac{\varphi'}{\eta^3} G(D\eta, D\eta).
$$

We can incorporate the first derivative condition \eqref{13} into \eqref{10}, the evolution equation for $F$:

$$
\frac{dF}{dt} = G^{ij} D_{ij} F + D_m (\varphi/\eta) D (FD^2F) \left. \frac{\partial (\varphi^m, \varphi^i, \varphi^j)}{\partial (\varphi^m, \varphi^i, \varphi^j)} u_{ij} \right|_{\nu} \\
- 2 FD^2F \left. \frac{\partial (D(\varphi/\eta), D(\varphi/\eta))}{\partial (\varphi^i, \varphi^j)} \right|_{\nu} D^2F \left. \frac{\partial (\varphi^m, \varphi^i)}{\partial (\varphi^m, \varphi^i)} u_{mi} u_{ij} \right|_{\nu}
$$
\begin{equation}
\begin{aligned}
&= G^{ij}D_{ij}F + \frac{\eta}{\varphi} FD \left( FD^2 F \right)_{\nu} \left( D(\varphi/\eta), \tilde{\varphi}^i, \tilde{\varphi}^j \right) u_{ij} \\
&\quad - 2 \frac{\eta}{\varphi} \left[ \frac{\varphi'^2}{\eta^2} G(Du, Du) - 2 \frac{\varphi'\varphi''}{\eta} G(Du, D\eta) + \frac{\varphi^2}{\eta^2} G(D\eta, D\eta) \right] \\
&\quad - \frac{\eta}{\varphi} G^{ij}G^{ml} u_{mi} u_{lj}.
\end{aligned}
\end{equation}

Putting the last two steps together gives an evolution equation for \( Z \) at a local maximum:

\begin{equation}
\begin{aligned}
\frac{dZ}{dt} &= G^{ij}D_{ij}Z + \frac{\eta}{\varphi} FD \left( FD^2 F \right)_{\nu} \left( D(\varphi/\eta), \tilde{\varphi}^i, \tilde{\varphi}^j \right) u_{ij} \\
&\quad - \frac{\eta}{\varphi} G^{ij}G^{ml} u_{mi} u_{lj} - \frac{1}{\eta} \left[ \varphi_t - G(Du, Du)\varphi'' + 2 \frac{\varphi'^2}{\varphi} G(Du, Du) \right] \\
&\quad + \frac{\varphi}{\eta^2} \left( \frac{d}{dt} - G^{ij}D_{ij} \right) \eta + 2 \frac{\varphi' \varphi''}{\eta^2} G(Du, D\eta).
\end{aligned}
\end{equation}

The second term here may be split up into a part with \( D\varphi \) and a part with \( D\eta \):

\begin{equation}
\begin{aligned}
\frac{\eta}{\varphi} FD \left( FD^2 F \right)_{\nu} \left( \frac{\varphi'}{\eta} Du - \frac{\varphi}{\eta^2} D\eta, \tilde{\varphi}^i, \tilde{\varphi}^j \right) u_{ij} \\
&= \frac{\varphi'}{\varphi} FD \left( FD^2 F \right)_{\nu} \left( Du, \tilde{\varphi}^i, \tilde{\varphi}^j \right) u_{ij} - \frac{1}{\eta} FD \left( FD^2 F \right)_{\nu} \left( D\eta, \tilde{\varphi}^i, \tilde{\varphi}^j \right) u_{ij}.
\end{aligned}
\end{equation}

These may be individually estimated using the Cauchy-Schwarz inequality and the smallness-of-third-derivatives condition, as described in the proof of Theorem 11—

\begin{equation}
\begin{aligned}
\frac{\varphi'}{\varphi} FD \left( FD^2 F \right)_{\nu} \left( Du, \tilde{\varphi}^i, \tilde{\varphi}^j \right) u_{ij} \\
\leq \mu_1 \frac{\varphi'^2}{\varphi^3} G(Du, Du) + \frac{1}{4\mu_1} \frac{\eta}{\varphi} C_1^2 \sqrt{n} \left( G^{ij}G^{ml} u_{mi} u_{lj} \right), \\
\quad - \frac{1}{\eta} FD \left( FD^2 F \right)_{\nu} \left( D\eta, \tilde{\varphi}^i, \tilde{\varphi}^j \right) u_{ij} \\
\leq \mu_2 \frac{\varphi^2}{\eta^3} G(D\eta, D\eta) + \frac{1}{4\mu_2} \frac{\eta}{\varphi} C_1^2 \sqrt{n} \left( G^{ij}G^{kl} u_{ik} u_{jl} \right),
\end{aligned}
\end{equation}

for some \( 0 < \mu_1, \mu_2 < 1 \).

We choose the localising term \( \eta := \tilde{\eta}^r \) for some \( r > 1 \) and \( \tilde{\eta} = R^2 - 2kt - |x|^2 \). Then \( D_i \eta = r\tilde{\eta}^{r-1} D_i \tilde{\eta}, D_i \eta = r\tilde{\eta}^{r-1} D_i \tilde{\eta} + r(r-1)\tilde{\eta}^{-2} D_i \tilde{\eta}, \tilde{\eta} \), and the second-last term of the evolution equation (14) is

\begin{equation}
\begin{aligned}
\frac{\varphi}{\eta^2} \left( \frac{d}{dt} - G^{ij}D_{ij} \right) \eta &= \frac{\varphi}{\eta^2} r\tilde{\eta}^{r-1} \left[ -2k + 2 \text{trace } G - (r-1)\tilde{\eta}^{-1} G(D\tilde{\eta}, D\tilde{\eta}) \right] \\
&\leq \frac{\varphi}{\eta^2} r\tilde{\eta}^{r-2}(1 - r)G(D\tilde{\eta}, D\tilde{\eta}).
\end{aligned}
\end{equation}
As $F$ satisfies the symmetry condition (4), we may use Lemma 9 to estimate the final term of the evolution equation:

$$2\frac{\varphi'}{\eta^2}G(Du, D\eta) = 2\frac{\varphi'}{\eta^2} F D^2 F \Big|_{Du=0} (Du, D\eta)$$

$$\leq 2\frac{\varphi'}{\eta^2} C_2 F(D\eta)$$

$$= 2C_2 F(D\eta) \frac{\varphi'}{\varphi_\eta}$$

$$\leq 2C_2 C_3 r R^{2r-1} \frac{\varphi'}{\varphi_\eta},$$

where we have used that $F(D\eta) = r\eta^{-1}F(D\eta) \leq C_3 r R^{2r-1}$, for $C_3 > 0$ depending only on $F$.

The evolution equation can now be estimated from above:

$$\frac{dZ}{dt} \leq G^{ij} D_{ij} Z + \eta \varphi \left( \frac{1}{4\mu_1} C_1^2 \sqrt{n} + \frac{1}{4\mu_2} C_2^2 \sqrt{n} - 1 \right) G^{ij} G^{kl} u_{mi} u_{lj}$$

$$- \frac{1}{\eta} \left[ \varphi_t - G(Du, Du) \varphi'' + (2 - \mu_1) \frac{\varphi'}{\varphi} G(Du, Du) - 2C_2 F(D\eta) \frac{\varphi'}{\varphi} \right]$$

$$+ \frac{\varphi'}{\eta^2} r\eta^{-2} (1 - r + r\mu_2) G(D\eta, D\eta).$$

(15)

Since $C_1^2 \sqrt{n}/4 < 1/2$, we can choose $\mu_1 < 1$ and $\mu_2 < 1$ such that

$$\frac{C_1^2 \sqrt{n}}{4} \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \leq 1.$$
and so we need only to choose $q$ large enough that
\[
q \geq \frac{1}{1 - \mu_1} \left( 1 + \frac{2C_1 C_2 r R^{2r - 1} T'}{A^2 M} \right)
\]
for the last term to be negative. So, at such maxima, $Z_t \leq 0$.

At local maxima where $F(Du - \phi^0) < P$, then in the definition of $Z$ we replace $\phi/\eta$ by $\max\{\phi/\eta, P\}$, in which case the barrier is flat at the local maxima, and we again find that $Z_t \leq 0$.

In either case, the maximum principle ensures that $Z$ is never greater than zero and the conclusion follows. □

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