Nonlinear bending waves in Keplerian accretion discs

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ABSTRACT
The nonlinear dynamics of a warped accretion disc is investigated in the important case of a thin Keplerian disc with negligible viscosity and self-gravity. A one-dimensional evolutionary equation is formally derived that describes the primary nonlinear and dispersive effects on propagating bending waves other than parametric instabilities. It has the form of a derivative nonlinear Schrödinger equation with coefficients that are obtained explicitly for a particular model of a disc. The properties of this equation are analysed in some detail and illustrative numerical solutions are presented. The nonlinear and dispersive effects both depend on the compressibility of the gas through its adiabatic index $\Gamma$. In the physically realistic case $\Gamma < 3$, nonlinearity does not lead to the steepening of bending waves but instead enhances their linear dispersion. In the opposite case $\Gamma > 3$, nonlinearity leads to wave steepening and solitary waves are supported. The effects of a small effective viscosity, which may suppress parametric instabilities, are also considered. This analysis may provide a useful point of comparison between theory and numerical simulations of warped accretion discs.

Key words: accretion, accretion discs – hydrodynamics – waves.

1 INTRODUCTION
The existence of warped accretion discs, in which the orbital plane of the gas varies slowly with radius and possibly with time, is suggested by both observational evidence and theoretical reasoning. Of the observational results, perhaps the best examples are certain X-ray binary stars, including Her X-1, in which the X-ray source appears to be periodically occulted by a precessing warped disc (e.g. Gerend & Boynton 1976; Clarkson et al. 2003), and a few active galactic nuclei, including NGC 4258, in which the warped shape of the disc is revealed by maser emission (e.g. Miyoshi et al. 1993; Greenhill et al. 2003). Theoretical arguments indicate that a fluid disc will be warped by an external torque if it is misaligned with the equatorial plane of a spinning black hole or magnetized star at its centre (Bardeen & Petterson 1975; Lipunov & Shakura 1980; Lai 1999), or with the orbital plane of a binary companion, planet or other satellite (Papaloizou & Terquem 1995). Even in an initially aligned system, the disc may develop a warp through a linear instability of the coplanar state, depending on tidal, radiation or magnetic forces (Lubow 1992; Pringle 1996; Lai 1999). Interpreting the behaviour of warped discs is an important and challenging problem in astrophysical fluid dynamics.

The basic aim of theoretical approaches in this subject has been to understand how the shape of the disc evolves under the action of internal stresses and external torques, and to derive practical one-dimensional equations that describe this evolution. When the warp is sufficiently small, the governing equations are linear and can be obtained from a perturbation analysis of a flat disc (Papaloizou & Pringle 1983) and (Papaloizou & Lin 1993), derived linearized equations for warped discs, superseding the pioneering but flawed analyses of Bardeen & Petterson (1975) and Petterson (1978), and demonstrated that there are two basic dynamical regimes for warped Keplerian discs in linear theory. Let the disc have angular velocity $\Omega(r)$ and vertical scale-height $H(r)$, and let $\alpha$ denote the dimensionless effective viscosity parameter of Shakura & Sunyaev (1973). To a first approximation, if $\alpha \gtrsim H/r$ the warp satisfies a diffusion-type equation with an effective diffusion coefficient $H^2\Omega/(2\alpha)$, larger by a factor of $1/(2\alpha^2)$ than the effective kinematic viscosity of the disc. On the other hand, if $\alpha \lesssim H/r$ the warp satisfies a wave-type equation and propagates with speed $H\Omega/2$. The rapid diffusive and non-dispersive wavelike behaviours are unique to Keplerian discs as they result from a resonance between the orbital and epicyclic frequencies that couples the vertical motion associated with the warp to shearing horizontal epicyclic motions.
A possible limitation of these approaches is that the Eulerian perturbation analysis on which the linearized equations are based requires that the vertical displacement be small compared to \( H \), and is therefore formally invalid for any observable warp. It is appreciated that the linear theories may be valid for larger amplitudes but this can only be demonstrated using a Lagrangian or semi-Lagrangian method. More recently it has also become possible to test these theories and to study aspects of warped discs using numerical simulations (e.g. \( \text{Lubow } \& \text{ Ogilvie } 2000 \)). Such simulations have focused mainly on the wavelike regime \( \alpha \lesssim H/r \) and have found reasonable agreement with \( \text{Papaloizou } \& \text{ Lin } 1995 \) for small-amplitude warps. There are significant discrepancies, however, which suggest that nonlinear and dispersive effects can be important.

A particular concern has been that the shearing epicyclic motions associated with the warp in either regime are predicted to be very fast, comparable to or larger than the sound speed, for observable warps. Shocks may form if the warp has a short enough wavelength that these motions collide (Nelson & Papaloizou 1999), but the horizontal motions may also be paradoxically unstable and decay into smaller-scale inertial waves (Papaloizou & Terquem 1997; Gammie, Goodman & Ogilvie 2000). Numerical simulations indicate that the shearing motions are also damped by magnetohydrodynamic turbulence in a quasi-viscous manner (Torkelsson et al. 2003). Further studies of all these effects are needed to assess their importance.

In earlier work (Ogilvie 1999; Ogilvie 2000) I derived a fully nonlinear theory of warped discs with effective viscosity, which agrees with Papaloizou & Pringle (1983) in the appropriate limit and also bears a formal resemblance to the simplified nonlinear equations adopted by Pringle (1992). However, the theory of Ogilvie (1999) does not describe the regime of propagating bending waves in Keplerian discs with \( \alpha \lesssim H/r \). It was noted there that the nonlinear dynamics of the wavelike regime is likely to be very different, and the purpose of the present paper is to investigate this important case.

In Section 2 of this paper I review the linear theory of long-wavelength bending waves in Keplerian discs and introduce the detailed derivation that follows in Section 3. The non-linear evolutionary equation is analysed and solved numerically in Section 4 and conclusions are presented in Section 5.

### 2 LEADING-ORDER DYNAMICS

The linearized equations for long-wavelength bending waves in a thin, inviscid, Keplerian disc are (e.g. Lubow & Ogilvie 2000)

\[
\Sigma r^2 \Omega \frac{\partial W}{\partial t} = \frac{1}{r} \frac{\partial G}{\partial r},
\]

\[
\frac{\partial G}{\partial t} = \frac{1}{4} \Sigma^2 H^2 \Omega^2 \frac{\partial W}{\partial r},
\]

where \( W(r, t) \) and \( G(r, t) \) are complex variables describing the warp and the internal torque. Specifically \( W = l_x + il_y \) encodes the horizontal Cartesian components of the unit vector \( \mathbf{l} \) normal to the plane containing an annulus of the disc at radius \( r \) and time \( t \), while \( G = G_x + iG_y \) does the same for the horizontal internal torque in the disc (divided by 2\( \pi \)). Also \( \Omega(r) \) is the angular velocity, \( \Sigma(r) \) is the surface density and \( H(r) \) is an effective density scale-height, defined by

\[
\Omega = \left( \frac{GM}{r^3} \right)^{1/2}, \quad \Sigma = \int_{-\infty}^{\infty} \rho \, dz, \quad \Sigma H^2 = \int_{-\infty}^{\infty} \rho z^2 \, dz,
\]

where \( M \) is the central mass, \( \rho \) is the density and \( z \) is the vertical coordinate in the unperturbed disc. For a vertically isothermal disc \( H \) is the usual Gaussian scale-height.

When \( G \) is eliminated we obtain

\[
\frac{\partial^2 W}{\partial t^2} = \frac{GM}{4} \frac{1}{\Sigma^3^{3/2}} \left( \Sigma^2 H^2 \frac{1}{\Sigma r^{3/2}} \frac{\partial W}{\partial r} \right),
\]

which implies that linear bending waves propagate non-dispersively at speed \( H\Omega/2 \) (Papaloizou & Lin 1995). If the model of the disc is such that the product \( \Sigma H \) is independent of \( r \), we obtain the classical wave equation

\[
\frac{\partial^2 W}{\partial t^2} = \frac{\partial^2 W}{\partial x^2}
\]

in the conveniently rescaled spatial coordinate

\[
x = \int \left( \frac{GM}{4} \Sigma^2 H^2 \right)^{-1/2} \Sigma r^{3/2} \, dr = 2 \int \frac{dr}{H\Omega}.
\]

With plausible applications and numerical simulations in mind, we consider the case \( H/r = \epsilon = \text{constant} \); then \( \Sigma \propto r^{-1} \) and

\[
x = \frac{4}{3\epsilon\Omega} \propto r^{3/2}.
\]

An outwardly propagating bending wave in this theory has the simple form

\[
W = \epsilon f(x - t), \quad A = -r\Omega f(x - t),
\]
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where \( A = 2G/(\Sigma H^2r\Omega) \) is related to the radial velocity amplitude through

\[
u_r = \text{Re} \left( \frac{Az}{r} e^{-i\omega} \right).\tag{9}\]

The function \( f \) is unconstrained except by the initial conditions, and the wave propagates without change of form.

This long-wavelength theory of bending waves is subject to small corrections depending on the quantity \( kH \), where \( k \) is the radial wavenumber. Lubow & Pringle (1993) analysed the linear adiabatic wave modes in a vertically isothermal accretion disc in the case \(|kr| \ll 1\). The bending wave of interest corresponds to \( m = 1 \) and \( n = 0 \) in their notation, and its dispersion relation in a Keplerian disc can be expanded in the form

\[
\frac{\omega}{\Omega} = \pm \frac{1}{2} kH - \left( \frac{3\Gamma - 2}{8\Gamma} \right) (kH)^2 + O(kH)^3,\tag{10}\]

where \( \Gamma \) is the adiabatic index. The first term on the right-hand side corresponds to the approximation of non-dispersive propagation as obtained in the long-wavelength bending-wave theory, while the second term indicates the primary dispersive correction. The dispersion depends on the compressibility of the fluid and is a weak effect if the wavelength is long compared to \( H \).

Nonlinearity will also modify the above theory. If the nonlinearity is weak and of the same order of magnitude as the linear dispersion, we may expect a similar outwardly propagating wave solution to equation (3) to exist, except that \( f \) will no longer be an arbitrary function determined only by the initial conditions, but will evolve according to a nonlinear evolutionary equation. The situation is analogous to the classical problem of long water waves, where weak nonlinearity and dispersion combine in the Korteweg–de Vries (KdV) equation (e.g. Whitham 1974). In that case nonlinearity and dispersion have competing tendencies and solitary waves (specifically, solitons) are found. Another well known example is the nonlinear Schrödinger (NLS) equation, which arises in diverse fields including nonlinear optics. If the sign of the nonlinear term is such as to steepen waves, the NLS equation also admits solitons. The KdV and NLS equations are recognised as generic nonlinear equations that arise in many applications.

Since the bending wave propagates through an inhomogeneous medium, it is not expected in general that its evolutionary equation will have constant coefficients. However, it is found below that for the self-similar disc model in which \( H \propto r \) and \( \Sigma \propto r^{-1} \), which has reasonable scalings and is well suited for numerical simulations, the coefficients of the equation can indeed be made constant by an appropriate choice of coordinates.

3 DERIVATION OF THE EVOLUTIONARY EQUATION

3.1 Basic equations

The equations governing the dynamics of an ideal, compressible fluid are the equation of mass conservation,

\[
D\rho = -\rho \nabla \cdot \mathbf{u},\tag{11}\]

the adiabatic condition,

\[
Dp = -\Gamma p \nabla \cdot \mathbf{u},\tag{12}\]

and the equation of motion,

\[
Du = -\nabla \Phi - \frac{1}{\rho} \nabla p,\tag{13}\]

where \( \mathbf{u} \) is the velocity, \( D = \partial_t + \mathbf{u} \cdot \nabla \) is the Lagrangian time-derivative, \( p \) is the pressure and \( \Phi \) is the gravitational potential. For a polytropic gas \( \Gamma \) is a constant.

Consider a non-self-gravitating, Keplerian disc around a point mass \( M \). The time-dependent warping of the disc is best described in a nonlinear regime using the warped spherical polar coordinate system \((r, \theta, \phi)\) introduced by Ogilvie (1999). In this scheme the warped mid-plane of the disc is defined by the coordinate surface \( \theta = \pi/2 \), and the warp consists of a tilt angle \( \beta(r,t) \) together with a twist angle \( \gamma(r,t) \). The Cartesian components of the unit tilt vector are given by \( \mathbf{l} = (\sin \beta \cos \gamma, \sin \beta \sin \gamma, \cos \beta) \).

When expressed in warped coordinates, the governing equations become

\[
D\rho = -\rho \left[ \frac{1}{r^2} \partial_r (r^2 v_r) + \frac{1}{r \sin \theta} \partial_\theta (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \partial_\phi v_\phi \right],\tag{14}\]

\[
Dp = -\Gamma p \left[ \frac{1}{r^2} \partial_r (r^2 v_r) + \frac{1}{r \sin \theta} \partial_\theta (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \partial_\phi v_\phi \right],\tag{15}\]

\[
Du_r = -\frac{u_r^2}{r} - \frac{u_\theta^2}{r} = -\frac{GM}{r^2} + \frac{1}{\rho} Dp,\tag{16}\]

where \( \partial_\phi v_\phi \) is the projection of the warp.
\[ Du_\theta + \frac{u_\theta u_\phi}{r} - \frac{u_\phi}{r \sin \theta} [u_\phi \cos \theta + r(D\beta) \sin \phi - r(D\gamma) \sin \beta \cos \phi] = -\frac{1}{\rho r} \partial_\theta p, \]  
\[ Du_\phi + \frac{u_\phi u_\theta}{r} + \frac{u_\theta}{r \sin \theta} [u_\phi \cos \theta + r(D\beta) \sin \phi - r(D\gamma) \sin \beta \cos \phi] = -\frac{1}{\rho r \sin \theta} \partial_\phi p, \]  

where \((v_r, v_\theta, v_\phi)\) are the components of velocity relative to the moving coordinate system,

\[ D = \partial_t + v_r \partial_r + \frac{v_\theta}{r} \partial_\theta + \frac{v_\phi}{r \sin \theta} \partial_\phi \]  
is the Lagrangian time-derivative,

\[ D = \partial_t - [(\partial_\theta \beta) \cos \phi + (\partial_r \gamma) \sin \beta \sin \phi] \partial_\theta - [-(\partial_\theta \beta) \cos \theta \sin \phi + (\partial_r \gamma)(\cos \beta \sin \theta + \sin \beta \cos \theta \cos \phi)] \frac{1}{\sin \theta} \partial_\phi \]  
is a modified radial derivative, and \((u_r, u_\theta, u_\phi)\) are the absolute velocity components

\[ u_r = v_r, \]  
\[ u_\theta = v_\theta + r(D\beta) \cos \phi + r(D\gamma) \sin \beta \sin \phi, \]  
\[ u_\phi = v_\phi - r(D\beta) \cos \phi + r(D\gamma) \sin \beta \sin \phi. \]  

These equations involve no further approximation and are valid for arbitrary warps. As in [Ogilvie 1999], meaningful dynamical equations for \(\beta\) and \(\gamma\) can be obtained only if the disc is thin and is defined to lie close to the surface \(\theta = \pi/2\). This constraint is implied by the asymptotic analysis that follows.

### 3.2 Asymptotic expansions

We utilize the small parameter \(\delta \ll 1\), such that the angular semi-thickness of the disc is \(H/r = \epsilon = \delta^2\). The equations are to be expanded in a region of the \((r, t)\) plane that corresponds to a neighbourhood of the line \(x = t\) followed by the nominal centre of an outwardly propagating bending wave at leading order. (The problem of an inwardly propagating wave can be considered using the time-reversal symmetry of the problem.) We therefore introduce the scaled variables \(\zeta, \xi\) and \(\tau\) to resolve the vertical structure of the disc and the evolution of the wave. These are defined by

\[ \theta = \frac{\pi}{2} - \delta^2 \zeta, \quad x - t = \delta^{-1} \xi, \quad t = \delta^{-2} \tau, \]  

and the equations are to be valid where \(\zeta, \xi\) and \(\tau\) are \(O(1)\). Thus \(\zeta\) is a scaled vertical coordinate relative to the mid-plane of the thin disc, \(\xi\) is a scaled horizontal coordinate relative to the nominal centre \(x = t\) of the travelling wave, and \(\tau\) is a scaled time coordinate that follows the solution over the time-scale on which the radial location of the centre of the wave changes by a factor of order unity. The scaling of \(\xi\) means that at any one time the wave is described within a region of radial extent comparable to the geometric mean of \(r\) and \(H\) (Figure 1).

Partial derivatives transform according to

\[ \partial_r = -\delta \partial_\zeta + \delta^2 \partial_\xi, \quad \partial_\xi = \delta^{-1} \frac{2}{r \Omega} \partial_\zeta, \quad \partial_\tau = -\delta^{-2} \partial_\zeta. \]  

The radius \(r\) can be expressed in terms of \(\xi\) and \(\tau\) and expanded in the form

\[ r = 2^{-3/2} \delta^{3/2} (GM)^{1/3} \tau^{2/3} \left[ 1 + \delta \left( \frac{2\xi}{3\tau} \right) - \delta^2 \left( \frac{\xi^2}{9\tau^2} \right) + \cdots \right] = r_0(\tau) + \delta r_1(\xi, \tau) + \delta^2 r_2(\xi, \tau) + \cdots, \]  

and this allows any function of \(r\) to be expanded similarly. The Keplerian angular velocity is

\[ \Omega = (\frac{GM}{r^3})^{1/2} = \frac{4}{3 \epsilon x} = \frac{4}{3 \tau} \left[ 1 - \delta \left( \frac{\xi}{\tau} \right) + \delta^2 \left( \frac{\xi^2}{\tau^2} \right) \right] = \Omega_0(\tau) + \delta \Omega_1(\xi, \tau) + \delta^2 \Omega_2(\xi, \tau) + \cdots. \]  

We then propose the asymptotic expansions

\[ \beta = \delta^2 \left[ \delta_0(\xi, \tau) + \delta_1(\xi, \tau) + \cdots \right], \]  
\[ \gamma = \gamma_0(\xi, \tau) + \delta \gamma_1(\xi, \tau) + \cdots, \]  
\[ \rho = \rho_0(\xi, \tau) + \delta \rho_1(\xi, \tau) + \delta^2 \rho_2(\xi, \tau) + \cdots, \]  
\[ p = p_0(\xi, \tau) + \delta p_1(\xi, \tau) + \delta^2 p_2(\xi, \tau) + \cdots, \]  
\[ v_r = \delta \left[ \delta v_{r1}(\xi, \tau) + \delta^2 v_{r2}(\xi, \tau) + \cdots \right], \]  
\[ v_\theta = \delta^2 \left[ \delta v_{\theta 1}(\xi, \tau) + \delta^2 v_{\theta 2}(\xi, \tau) + \cdots \right]. \]
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Figure 1. Typical spatiotemporal domain of an outwardly propagating bending wave. The solid line corresponds to the curve $x = t$ for a disc with $H/r = \delta^2 = 0.01$. The dashed lines bound the region $|\xi| < 10$ and indicate the region that might be occupied by a wave of modest radial extent. Here $r$ is expressed in arbitrary units and $t$ in units of the orbital period at $r = 1$.

$$v_\phi = r(\Omega - D\gamma) \sin \theta + \delta \left[ \delta v_{\phi 1}(\phi, \zeta, \xi, \tau) + \delta^2 v_{\phi 2}(\phi, \zeta, \xi, \tau) + \delta^3 v_{\phi 3}(\phi, \zeta, \xi, \tau) + \cdots \right].$$

(34)

Here $\rho_0(r, \zeta)$ and $p_0(r, \zeta)$ are the density and pressure of the unperturbed disc in hydrostatic equilibrium, which satisfy

$$\partial_\zeta p_0 = -\rho_0 r^2 \Omega^2 \zeta.$$  

(35)

When expressed in terms of $\zeta$ and $\tau$, these quantities have expansions

$$\rho_0 = \rho_0(\zeta, \tau) + \delta \rho_{01}(\zeta, \xi, \tau) + \cdots,$$

(36)

$$p_0 = p_0(\zeta, \xi, \tau) + \delta p_{01}(\zeta, \xi, \tau) + \cdots,$$

(37)

and equation (35) is satisfied at every order in $\delta$, in particular

$$\partial_\zeta p_0 = -\rho_0 r^2 \Omega^2 \zeta.$$  

(38)

The scaling of $\beta$ implies that the tilt angle of the disc is $O(\delta^2)$ and therefore comparable to $H/r$, while the scaling of $v_r$ implies that the radial velocity is comparable to the sound speed. It is found below that, with this choice of scalings, the effect of nonlinearity in the solution is of comparable magnitude to the effect of linear dispersion. The indexing of the variables in expansions (28)–(34) is designed so as to give a certain order to the equations that are deduced below. The parameter $\sigma$ is unspecified and drops out of the analysis.

### 3.3 Basic structure of the disc

When these expansions are substituted into equations (14)–(18), we obtain a number of equations to be considered in turn. Equation (16) at leading order $[O(1)]$ is satisfied, as the Keplerian rotation balances the gravitational force. Equation (17) at leading order $[O(\delta^2)]$ is satisfied by virtue of the vertical hydrostatic equilibrium, equation (35).

Equation (35) can be solved if the relation between pressure and density is specified. In order to make analytical progress we consider an isothermal model for the vertical structure, such that

$$\rho_0 = \rho_m e^{-\zeta^2/2},$$

(39)

$$p_0 = p_m e^{-\zeta^2/2},$$

(40)

where $\rho_m(r)$ and $p_m(r)$ are the (scaled) density and pressure on the mid-plane. In order to satisfy equation (35), these quantities are related by

$$p_m = \rho_m r^2 \Omega^2.$$  

(41)

The surface density of the unperturbed disc is
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\[ \Sigma = \delta^{2s+2} [\Sigma_0(\xi, \tau) + \cdots], \]

with

\[ \Sigma_0 = \int_{-\infty}^{\infty} \rho_0 r_0 d\zeta = (2\pi)^{1/2} r_0 \rho_0. \]

In order that \( \Sigma \propto r^{-1} \), as adopted in Section 2, we require that \( \rho_n \propto r^{-2} \).

### 3.4 Horizontal and vertical problems

The equations derived at higher orders have a consistently repeating formal structure that merits analysis. The horizontal components of the equation of motion, equations (16) and (18) at \( O(\delta^{n+1}) \), \( n \geq 1 \), involve the unknown quantities \( v_{r_{n}} \) and \( v_{\phi_{n}} \), and have the form

\[ \Omega_0 \partial_\phi v_{r_{n}} - 2\Omega_0 v_{\phi_{n}} = F_{r_{n}}, \]

\[ \Omega_0 \partial_\phi v_{\phi_{n}} + \frac{1}{2} \Omega_0 v_{r_{n}} = F_{\phi_{n}}, \]

where the right-hand sides involve quantities known or partially known from the lower orders. These equations may be combined to give

\[ -2\Omega_0 (\partial_\phi^2 + 1) v_{\phi_{n}} = F_{h_{n}}, \]

where \( F_{h_{n}} \) is the horizontal forcing combination

\[ F_{h_{n}} = F_{r_{n}} - 2\partial_\phi F_{\phi_{n}}. \]

The linear operator \( \partial_\phi^2 + 1 \), subject to periodic boundary conditions in \( \phi \), has eigenvalues \( 1 - m^2 \) and eigenfunctions \( e^{im\phi} \), where \( m \) is any integer. The special case \( m = 1 \) is a null eigenfunction, or complementary function, corresponding to an epicyclic motion or eccentric distortion of the disc with an arbitrary dependence on \( \zeta \). The condition for equation (40) to have a solution is

\[ \int_{0}^{2\pi} e^{i\phi} F_{h_{n}} d\phi = 0, \]

i.e. that the forcing combination should contain no \( m = 1 \) component. (Since \( F_{h_{n}} \) is real there is no difference in considering \( m = 1 \) or \( m = -1 \).)

The other three governing equations give rise to a vertical problem. Equation (14) at \( O(\delta^{2s+n}) \), (15) at \( O(\delta^{2s+n+1}) \) and (17) at \( O(\delta^{2s+n+2}) \), \( n \geq 1 \), involve the unknown quantities \( \rho_{n}, p_{n} \) and \( v_{\theta_{n}} \), and have the form

\[ \Omega_0 \partial_\phi \rho_{n} - \frac{v_{\theta_{n}}}{r_0} \partial_\zeta \rho_{0} - \frac{\rho_0}{r_0} \partial_\zeta v_{\theta_{n}} = F_{\rho_{n}}, \]

\[ \Omega_0 \partial_\phi p_{n} - \frac{v_{\theta_{n}}}{r_0} \partial_\zeta p_{0} - \frac{\Gamma_0}{r_0} \partial_\zeta v_{\theta_{n}} = F_{p_{n}}, \]

\[ \Omega_0 \partial_\phi v_{\theta_{n}} - \frac{1}{\rho_0 r_0} \partial_\zeta p_{n} + \frac{\rho_0}{\rho_0 r_0} \partial_\zeta \rho_{0} = F_{\theta_{n}}. \]

The quantities \( \rho_{n} \) and \( p_{n} \) may be eliminated, using the hydrostatic condition (35), to obtain

\[ \mathcal{L} v_{\theta_{n}} = F_{v_{n}}, \]

where \( \mathcal{L} \) is a linear operator defined by

\[ \mathcal{L} v_{\theta_{n}} = -\partial_\zeta (\Gamma_0 \partial_\zeta v_{\theta_{n}}) + \rho_0 r_0^2 \Omega_0^2 (\partial_\phi^2 + 1) v_{\theta_{n}}, \]

and \( F_{v_{n}} \) is the vertical forcing combination

\[ F_{v_{n}} = r_0^3 \Omega_0^2 \zeta F_{\rho_{n}} + r_0 \partial_\zeta F_{p_{n}} + \rho_0 r_0^2 \Omega_0 \partial_\phi F_{\theta_{n}}. \]

We define the eigenvalues \( \lambda \) and eigenfunctions \( w \) of \( \mathcal{L} \) as the solutions of the equation

\[ \mathcal{L} w = \lambda \rho_0 r_0^2 \Omega_0^2 w, \]

subject to the conditions that \( w \) have periodicity \( 2\pi \) in \( \phi \) and be regular at the surfaces of the disc where the density and pressure vanish. With these boundary conditions \( \mathcal{L} \) is self-adjoint with weight function \( \rho_0 \). A vertically isothermal disc has no definite surface and the vertical boundary condition is instead that the wave energy flux tend to zero as \( |\zeta| \to \infty \). The eigenfunctions and eigenvalues are then
\[ w_{\ell,m} = e^{-im\phi} \text{He}_\ell(\zeta), \quad \lambda_{\ell,m} = \ell \Gamma - m^2 + 1, \]

where \( \ell \geq 0 \) and \( m \) are integers, and \( \text{He} \) denotes an Hermite polynomial.\(^1\) We will require only the first four, \( \text{He}_0(\zeta) = 1, \text{He}_1(\zeta) = \zeta, \text{He}_2(\zeta) = \zeta^2 - 1 \) and \( \text{He}_3(\zeta) = \zeta^3 - 3\zeta \).

The special case \( w_{0,1} = e^{-i\phi} \), \( \lambda_{0,1} = 0 \) is a null eigenfunction corresponding to a rigid tilt of the annulus at radius \( r \). For general (irrational) \( \Gamma \) this is the only null eigenfunction. The corresponding solvability condition on equation (52) is

\[ \int_{-\infty}^{\infty} \int_{0}^{2\pi} e^{i\phi} F_{\ell m} \, d\phi \, d\zeta = 0. \tag{57} \]

The forcing terms required for the development of the solution to the desired order are listed in Appendix A.

### 3.5 Development of the solution

Our objective is to determine the spatiotemporal development of the warp, and we must therefore obtain equations for \( \partial_\tau \beta_0 \) and \( \partial_\tau \gamma_0 \) in terms of \( \beta_0, \partial_\zeta \beta_0, \partial_\zeta \gamma_0, \) etc. This is done by solving the horizontal and vertical problems in turn up to the required order and extracting the relevant solvability conditions. Although straightforward in principle, this procedure involves arduous algebraic manipulations and was carried out with the aid of Mathematica.

The first horizontal problem is unforced: \( F_{\ell 1} = F_{01} = 0 \). The general solution is an epicyclic motion (or eccentric distortion) with an unknown dependence on position and time,

\[ v_{\ell 1} = U_1(\zeta, \xi, \tau) \cos \phi + V_1(\zeta, \xi, \tau) \sin \phi. \tag{58} \]

Based on our experience of the leading-order dynamics (Section\(^2\)) we anticipate that in the absence of viscosity the solution for an outwardly travelling bending wave will be \( U_1 = -r_0 \Omega_0 \beta_0 \xi \) and \( V_1 = 0 \) (cf. equations \(8\) and \(9\)). This assumption simplifies the calculation, and is verified subsequently. Then we have

\[ v_{\ell 1} = -r_0 \Omega_0 \beta_0 \xi \cos \phi, \quad v_{01} = \frac{1}{2} r_0 \Omega_0 \beta_0 \xi \sin \phi. \tag{59} \]

The first vertical problem involves non-trivial forcing, and we find

\[ \frac{F_{\ell 1}}{\rho_0 \Omega_0^2 \Omega_0} = 2 (\Gamma - 1) (\partial_\zeta \beta_0 \cos \phi + \beta_0 \partial_\zeta \gamma_0 \sin \phi) (1 - \zeta^2) - 6 \beta_0 (\partial_\zeta \beta_0 \cos 2\phi + \beta_0 \partial_\zeta \gamma_0 \sin 2\phi) \zeta, \tag{60} \]

involving two different eigenfunctions, \( w_{2,1} \) and \( w_{0,1} \), of the operator \( L \). The solvability condition is satisfied, and equation (52) has the solution

\[ \frac{v_{01}}{r_0} = \left( \frac{\Gamma - 1}{\Gamma} \right) (\partial_\zeta \beta_0 \cos \phi + \beta_0 \partial_\zeta \gamma_0 \sin \phi) (1 - \zeta^2) + \left( \frac{6}{3 - \Gamma} \right) \beta_0 (\partial_\zeta \beta_0 \cos 2\phi + \beta_0 \partial_\zeta \gamma_0 \sin 2\phi) \zeta \tag{61} \]

(We do not add any multiple of the complementary function \( w_{0,1} \), as this would amount simply to redefining the tilt \( \beta_1 \).) The quantities \( \rho_1 \) and \( p_1 \) may then be obtained from equations (59) and (60) after an integration with respect to \( \phi \), and we obtain

\[ \rho_1 = p_1^{\alpha\xi}(\zeta, \xi, \tau) + \frac{\rho_0}{\Omega_0} \left\{ \frac{1}{\Gamma} (\partial_\xi \beta_0 \sin \phi - \beta_0 \partial_\xi \gamma_0 \cos \phi) \left[ 3 - \Gamma + (\Gamma - 1) \zeta^2 \right] \zeta \right. \]

\[ + \left. \left( \frac{3}{3 - \Gamma} \right) \beta_0 (\partial_\zeta \beta_0 \sin 2\phi - \beta_0 \partial_\zeta \gamma_0 \cos 2\phi) (1 - \zeta^2) \right\}, \tag{62} \]

\[ p_1 = p_1^{\alpha\xi}(\zeta, \xi, \tau) + \frac{p_0}{\Omega_0} \left\{ \frac{1}{\Gamma} (\partial_\xi \beta_0 \sin \phi - \beta_0 \partial_\xi \gamma_0 \cos \phi) \left[ \Gamma + 1 + (\Gamma - 1) \zeta^2 \right] \zeta \right. \]

\[ + \left. \left( \frac{3}{3 - \Gamma} \right) \beta_0 (\partial_\zeta \beta_0 \sin 2\phi - \beta_0 \partial_\zeta \gamma_0 \cos 2\phi) (\Gamma - \zeta^2) \right\}, \tag{63} \]

where the axisymmetric parts are required to satisfy

\[ \partial_\zeta p_1^{\alpha\xi} = -p_1^{\alpha\xi} r_0^2 \Omega_0^2 \zeta - \rho_0 r_0^2 \Omega_0 \beta_0^2 (\partial_\xi \gamma_0) \zeta. \tag{64} \]

We may assume that \( p_1^{\alpha\xi} \) vanishes, since it would otherwise correspond to an arbitrary redefinition of the unperturbed density of the disc. Then

\[ p_1^{\alpha\xi} = \frac{\beta_0^2 (\partial_\xi \gamma_0)}{\Omega_0} \rho_0. \tag{65} \]

The second horizontal problem is also forced, and we find

\[ \frac{F_{02}}{r_0 \Omega_0} = -\frac{1}{2 \Gamma} \beta_0 \left[ \partial_\zeta \beta_0 (1 + 3 \cos 2\phi) + 3 \beta_0 \partial_\zeta \gamma_0 \sin 2\phi \right] \left[ \Gamma - 1 + (\Gamma + 1) \zeta^2 \right] - \left( \frac{12}{3 - \Gamma} \right) \beta_0^2 (\partial_\zeta \beta_0 \cos 3\phi + \beta_0 \partial_\zeta \gamma_0 \sin 3\phi) \zeta. \tag{66} \]

\(^1\) There are two definitions of Hermite polynomials. Those used here are orthogonal with respect to the weight function \( \exp(-\zeta^2/2) \).
Again, the solvability condition is satisfied. (This verifies that our choice for $U_1$ and $V_1$ was correct; if we had not assumed their form in advance, the solvability condition at this order would have implied equation (69).) Equation (60) then has the solution

$$v_{u2} = \frac{1}{4\Omega_0} r_0 \beta_0 \left[ \partial_x \beta_0 \left( 1 - \cos 2\phi \right) - \beta_0 \partial_x \gamma_0 \sin 2\phi \right] \left( \Gamma - 1 + \left( \Gamma + 1 \right) \zeta^2 \right) - \frac{3}{4(3 - \Gamma)} r_0 \beta_0^2 \left( \partial_x \beta_0 \cos 3\phi - \beta_0 \partial_x \gamma_0 \sin 3\phi \right) \zeta$$

$$- \frac{1}{2} U_2(\zeta, \xi, \tau) \sin \phi + \frac{1}{2} V_2(\zeta, \xi, \tau) \cos \phi.$$  

(67)

The last two terms are the complementary functions, presently of unknown amplitude. The quantity $v_{r2}$ may then be obtained from equation (65), and we find

$$v_{r2} = -\frac{1}{2\Omega_0} r_0 \beta_0 \left[ \partial_x \beta_0 \sin 2\phi - \beta_0 \partial_x \gamma_0 (1 + \cos 2\phi) \right] \left( \Gamma - 1 + \left( \Gamma + 1 \right) \zeta^2 \right) - \frac{3}{2(3 - \Gamma)} r_0 \beta_0^2 \left( \partial_x \beta_0 \sin 3\phi - \beta_0 \partial_x \gamma_0 \cos 3\phi \right) \zeta$$

$$- 2r_0 \beta_0^2 \left( \partial_x \gamma_0 \right) \zeta^2 + \left[ 1 - \left( \frac{3}{3 - \Gamma} \right) \beta_0^2 \right] r_0 (\partial_x \beta_0 \sin \phi - \beta_0 \partial_x \gamma_0 \cos \phi) \zeta + U_2 \cos \phi + V_2 \sin \phi.$$  

(68)

The forcing for the second vertical problem is very complicated, but fortunately it is not necessary to solve it in detail. The solvability condition alone provides an equation for the desired quantities $\partial_x \beta_0$ and $\partial_x \gamma_0$. This involves much algebra, not written out here, and simplifies to

$$\partial_x \beta_0 + i \beta_0 \partial_x \gamma_0 - \partial_x \beta_1 + i \beta_1 \partial_x \gamma_0 - \frac{1}{\Omega_0 r_0} \partial \int r_0 (U_2 + iV_2) e^{i\gamma_0} r_0 \zeta \, d\zeta = -\frac{1}{4 \Omega_0} \beta_0 - \frac{r_1}{2r_0} (\partial_x \beta_0 + i \beta_0 \partial_x \gamma_0)$$

$$- \frac{i}{2\Omega_0} \left[ \partial_x \beta_0 - \beta_0 \partial_x \gamma_0 \right]^2 + i \beta_0 \partial_x \beta_0 + 2i(\partial_x \beta_0 \partial_x \gamma_0) + i \frac{3(\Gamma - 1)}{2\Omega_0} \left[ \beta_0^2 \partial_x \beta_0 + 2 \beta_0 \partial_x \beta_0 \right]$$

$$- \frac{1}{\Omega_0} \left[ \frac{3(\Gamma - 1)}{3 - \Gamma} \beta_0^2 \partial_x \beta_0 - \frac{1}{2\Omega_0} \left( \Gamma + 3 \right) \beta_0^2 \right] \partial_x \gamma_0 + i(\partial_x \gamma_0)^2.$$  

(69)

Although this equation contains information about the evolution of $\beta_0$ and $\gamma_0$, it also involves the unknown quantities $\beta_1$, $\gamma_1$, $U_2$ and $V_2$. Further information must therefore be sought.

Fortunately it is not necessary to solve the second vertical problem to obtain $v_{u2}$, $v_{r2}$ and $p_2$ in detail. The contribution of these quantities to the third problem, which is the last problem we consider, is of the form

$$F_{t3} = \cdots + \frac{v_{u2}}{r_0} \partial_x v_{r1} - 2\partial_r \left( \frac{v_{u2}}{r_0} \partial_x v_{r1} \right).$$  

(70)

For the solvability condition we require only the $m = 1$ component of this quantity. In principle both the $m = 0$ and $m = 2$ components of $v_{u2}$ could contribute, but the $m = 2$ component gives exactly zero contribution. Now the axisymmetric part of $F_{v2}$ is

$$F_{v2}^{ax} = -r_0 \Omega_0 \beta_0 \left[ 2(\partial_x \beta_0) \partial_x \gamma_0 + \beta_0 \partial_x \xi_0 \right] \left( \frac{2\Gamma^2 + \Gamma - 2}{\Gamma} \right) \zeta + \left( \frac{(\Gamma - 1)(3\Gamma + 1)}{\Gamma} \right) \left( \zeta^3 - 3\zeta \right).$$  

(71)

involving the eigenfunctions $w_{1,0}$ and $w_{3,0}$ of $\mathcal{L}$, and so the axisymmetric part of $v_{u2}$ is

$$v_{u2}^{ax} = -\frac{r_0}{\Omega_0} \left( 2(\partial_x \beta_0) \partial_x \gamma_0 + \beta_0 \partial_x \xi_0 \right) \left( \frac{2\Gamma^2 + \Gamma - 2}{\Gamma^2 + 1} \right) \zeta + \left( \frac{(\Gamma - 1)}{\Gamma} \right) \left( \zeta^3 - 3\zeta \right).$$  

(72)

This information is sufficient to deduce the final solvability condition in the form

$$(\partial_x \beta_0 + i \beta_0 \partial_x \gamma_0 - \partial_x \beta_1 + i \beta_1 \partial_x \gamma_0) \zeta + \frac{1}{\Omega_0 r_0} \partial \int \left[ (U_2 + iV_2) e^{i\gamma_0} \right] = \cdots,$$  

(73)

where we do not write the right-hand side in full. By multiplying this equation by $r_0 \Omega_0 \zeta / \Omega_0$ and integrating with respect to $\zeta$ we obtain

$$\partial_x \beta_0 + i \beta_0 \partial_x \gamma_0 - \partial_x \beta_1 + i \beta_1 \partial_x \gamma_0 + \frac{1}{\Omega_0 r_0} \partial \int \left[ r_0 (U_2 + iV_2) e^{i\gamma_0} r_0 \zeta \, d\zeta = \frac{1}{4 \Omega_0} \beta_0 - \frac{r_1}{2r_0} (\partial_x \beta_0 + i \beta_0 \partial_x \gamma_0)$$

$$+ \frac{i}{2\Omega_0} \left( \frac{7\Gamma - 4}{\Gamma} \right) \left[ \partial_x \beta_0 - \beta_0 \partial_x \gamma_0 \right]^2 + i \beta_0 \partial_x \beta_0 + 2i(\partial_x \beta_0 \partial_x \gamma_0) + \frac{i}{2\Omega_0} \left( \frac{3}{3 - \Gamma} \right) \left[ \beta_0^2 \partial_x \beta_0 + 2 \beta_0 \partial_x \beta_0 \right]$$

$$- \frac{1}{\Omega_0} \left[ (15 + 6\Gamma - \Gamma^2) \beta_0^2 \partial_x \beta_0 - \frac{1}{2\Omega_0} \left( \Gamma + 9 \right) \beta_0^2 \partial_x \gamma_0 + (\Gamma + 9)i(\partial_x \gamma_0)^2 \right],$$  

(74)

which may be compared with equation (69). The unknown quantities $\beta_1$, $\gamma_1$, $U_2$ and $V_2$ may all be eliminated by taking the average of the two equations, yielding the desired evolutionary equation for $\beta_0$ and $\gamma_0$. This is expressed most compactly in terms of the complex tilt variable $W_0 = \beta_0 e^{i\gamma_0}$:

$$\partial_x W_0 = -\frac{3(21 - 2)}{8\Gamma} i r \partial_x W_0 + \frac{9\Gamma}{16(3 - \Gamma)} i r \left[ a |W_0|^2 \partial_x W_0 + (1 - a) W_0^* \partial_x W_0^* + b W_0^* (\partial_x W_0)^2 + (1 - b) W_0 |\partial_x W_0|^2 \right],$$  

(75)
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with dimensionless parameters
\[ a = \frac{2(\Gamma^2 + 2\Gamma + 3)}{3\Gamma(\Gamma + 1)}, \quad b = \frac{\Gamma + 6}{3\Gamma}, \quad (76) \]

\[ 1 - a = \frac{(\Gamma - 3)(\Gamma + 2)}{3\Gamma(\Gamma + 1)}, \quad 1 - b = \frac{2(\Gamma - 3)}{3\Gamma}. \quad (77) \]

We recall that the true complex tilt variable is \( W(r, t) = \delta^2 W_0(\xi, \tau) + O(\delta^3) \).

4 ANALYSIS OF THE EVOLUTIONARY EQUATION

4.1 Rescaling

Equation (75) is a complex nonlinear equation describing the spatiotemporal development of the warp. The first, linear term on the right-hand side corresponds to the linear dispersion evident in equation (10) derived from Lubow & Pringle (1993).\(^2\)

The nonlinear terms are cubic and also depend on the compressibility of the gas. The appearance of the factor \((3 - \Gamma)\) in the denominator is significant and can be traced to the involvement of the mode \(w_{1,2}\), which is coupled resonantly if \(\Gamma = 3\), even if the disc is not vertically isothermal (Ogilvie 1999). In reality \(\Gamma = 5/3\) is the case of greatest relevance. In discussing the properties of equation (75), however, it is instructive to contrast the cases of \(\Gamma < 3\) and \(\Gamma > 3\). We assume throughout that \(\Gamma \geq 1\).

There is an explicit dependence on \(\tau\) in equation (75) because the bending wave experiences a varying background as it propagates outwards through the disc. However, this dependence can be conveniently eliminated by working with the rescaled time variable
\[ T = \frac{3(3\Gamma - 2)}{16\Gamma} \tau^2. \quad (78) \]

The reason that an equation with constant coefficients can be obtained is related to the self-similarity of the disc model. The nonlinear terms can also be rescaled by defining
\[ W_0 = \left[ \frac{2(3\Gamma - 2)(3 - \Gamma)}{3\Gamma^2} \right]^{1/2} \Psi. \quad (79) \]

We then have
\[ -i\partial_T \Psi = \partial_{\xi} \Psi + s [a|\Psi|^2 \partial_{\xi} \Psi + (1 - a)\Psi^2 \partial_{\xi} \Psi^* + b\Psi^* (\partial_{\xi} \Psi)^2 + (1 - b)|\Psi|^2 \partial_{\xi} |\Psi|^2], \quad (80) \]

where
\[ s = \text{sgn}(3 - \Gamma) = \pm 1. \quad (81) \]

Equation (80) is a derivative nonlinear Schrödinger equation (DNLS).\(^3\) The irreducible dimensionless parameters of the equation, \(a\) and \(b\), satisfy \(a > 1\) and \(b > 1\) for \(1 \leq \Gamma < 3\), while \(2/3 < a < 1\) and \(1/3 < b < 1\) for \(\Gamma > 3\).

4.2 Elementary properties

Equation (80) is invariant under translations of \(T\) and \(\xi\) and also under the ‘gauge transformation’ \(\Psi \mapsto \Psi e^{i\chi}\), which corresponds to a trivial rotation of the coordinate system through an angle \(\chi\) about the z-axis. A further symmetry is \(\xi \mapsto -\xi\), which corresponds to a ‘reflection’ about the centre of the wave. The coordinate \(T\), and the equation itself, are invariant under time reversal \((\tau \mapsto -\tau)\). The trivial solution \(\Psi = 0\) corresponds to a flat disc. However, the equation is not invariant under \(\Psi \mapsto \Psi + \text{constant}\), which might appear to correspond to an additional trivial rigid tilt of the disc. The reason can be traced to equation (8) where it is seen that the amplitude \(W_0\) (or \(\Psi\)) determines the non-trivial horizontal velocities as well as the tilt, because the solution under consideration is a wave travelling in one direction.

4.3 Travelling-wave solutions

It is readily verified that solutions exist in the form of uniform travelling waves \(\Psi \propto \exp(i\omega T - ik\xi)\), where \(\omega\) and \(k\) satisfy the nonlinear dispersion relation

\[ 2 \text{ An additional factor of } 3 \text{ appears because of the relations between } \tau \text{ and } \Omega \text{ and between } x \text{ and } r. \]

\[ 3 \text{ The term DNLS refers to a wide class of equations. The present equation, in which the nonlinear terms contain second derivatives, is not the form of DNLS most commonly studied. } \]
\[ \omega = - \left( 1 + 2sb \right) \Psi^2 \]  \tag{82} 

Such a wave is linearly stable to long-wavelength disturbances if

\[ (1 + s|\Psi|^2) \left[ 1 + s(2a - 1)|\Psi|^2 \right] \geq 0, \tag{83} \]

and to short-wavelength disturbances if

\[ 4sb + (2a + b - 1)^2 |\Psi|^2 \geq 0. \tag{84} \]

In the rescaled variables \( \xi \) and \( T \) the (negative) linear dispersion coefficient is unity. Since \( 2a - 1 > 0 \) and \( b > 0 \) in practice, the waves are stable if \( \Gamma < 3 \) (i.e. \( s = +1 \)) and the nonlinear terms increase their dispersion. Therefore it may be expected that solitary waves are not supported, and this appears to be confirmed by the following analysis. However, if \( \Gamma > 3 \) (i.e. \( s = -1 \)), the waves are unstable except when \( |\Psi|^2 > 1/(2a - 1) > 1 \), and the nonlinearity counteracts the linear dispersion.

Following the usual analysis of NLS solitons (e.g. Whitham 1974), we consider more general travelling-wave solutions of the form

\[ \Psi = F(X) \exp(i\omega T), \quad X = \xi - cT, \tag{85} \]

where \( \omega \) is a frequency and \( c \) is a wave speed to be determined. Note that \( c \) is a slow speed in rescaled coordinates and is relative to the basic wave speed of \( H\Omega/2 \). Writing \( F = R \exp(i\Phi) \) with \( R \) and \( \Phi \) real, we then find

\[ (1 + sR^2)R_c' + sRR_c'' = (1 + 2sbR^2)R\Phi'' - R(c\Phi' - \omega), \tag{86} \]

\[ R \left[ 1 + s(2a - 1)R^2 \right] \Phi'' + 2 \left[ 1 + s(2a + b - 1)R^2 \right] R'\Phi' = cR'. \tag{87} \]

These equations are completely integrable. Equation (86) is linear in \( \Phi' \) and has the solution

\[ R^2\Phi' = \frac{8c}{2b} + h \left[ 1 + s(2a - 1)R^2 \right]^{-q}, \tag{88} \]

where \( h \) is an arbitrary constant and \( q = b/(2a - 1) \). We have \( 7/9 \leq q < 1 \) for \( 1 \leq \Gamma < 3 \), while \( 1 < q < 25/23 \) for \( \Gamma > 3 \). Equation (86) may then be written in the form

\[ (1 + sR^2)R_c' + sRR_c'' = -\frac{dV}{dR}, \tag{89} \]

with

\[ V(R) = \frac{c^2}{8b^2R^2} + \frac{sc}{2bR} \left[ 1 + s(2a - 1)R^2 \right]^{1-q} + \frac{h^2}{2R^2} \left[ 1 + s(2a - 1)R^2 \right]^{1-2q} - \frac{1}{2}\omega R^2, \tag{90} \]

and has the first integral

\[ \frac{1}{2} (1 + sR^2)R_c^2 + V(R) = E = \text{constant}. \tag{91} \]

There is an obvious mechanical analogy with a particle (albeit of variable inertia) moving in an effective potential.

A solitary wave would have \( R \to 0 \) and \( R' \to 0 \) as \( X \to \pm \infty \). The analogous particle would roll in a potential well between \( R = 0 \) and \( R = R_+ > 0 \) such that \( V(R_+) = V(0) \) and \( V(R) < V(0) \) for \( 0 < R < R_+ \). In this problem, as \( R \to 0 \), the effective potential diverges as \( V \sim (c + 2sbh)^2/8b^2R^2 \), so this behaviour is possible only if \( h = -sc/2b \). In this case

\[ V = \frac{c^2}{8b^2R^2} \left\{ 1 - 2 \left[ 1 + s(2a - 1)R^2 \right]^{1-q} + \left[ 1 + s(2a - 1)R^2 \right]^{1-2q} \right\} - \frac{1}{2}\omega R^2, \tag{92} \]

and \( V \) tends to a constant as \( R \to 0 \). We exclude the trivial case \( c = 0 \). It can then be shown that \( V \) is a strictly concave function of \( R^2 \) (i.e. \( d^2V/dR^2 < 0 \)) when \( \Gamma < 3 \) (i.e. \( s = +1 \) and \( q < 1 \)). Therefore it is impossible for a potential well to be formed, and we conclude that solitary waves are not supported when \( \Gamma < 3 \).

In contrast, when \( \Gamma > 3 \) (i.e. \( s = -1 \) and \( q > 1 \)), \( V \) is a strictly convex function of \( R^2 \) that diverges to \( +\infty \) as \( R^2 \) approaches \( 1/(2a - 1) \), and a potential well is always formed. The squared amplitude \( R_+^2 \) can never exceed \( 1/(2a - 1) \), so it appears that solitary waves are supported for small amplitudes \( R^2 < 1/(2a - 1) \) and stable extended wavetrains for larger amplitudes \( R^2 > 1/(2a - 1) \). An analytical approximation for the solitary waves in the case \( \Gamma > 3 \) is possible when they are of small amplitude \( R_+ \ll 1 \).

In this limit

\[ V(R) - V(0) \approx \frac{bc^2}{8} R^2 \left( R^2 - R_+^2 \right), \tag{93} \]

with

\[ R_+^2 = \frac{1}{b} \left( \frac{4\omega}{c^2} - 1 \right). \tag{94} \]

The solution of equation (81) for \( R^2 \ll 1 \) is then
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\[ R \approx R_\ast \text{sech}(kX), \quad k = \frac{1}{2} b^{1/2} c R_\ast, \quad (95) \]

and furthermore \( \Phi' \approx c/2 \). For a specified height \( (R_\ast) \) and width \((1/k)\), the frequency \( \omega \) and wave speed \( c \) are determined by these relations.

### 4.4 Numerical solutions

Equation (80) was solved numerically by applying a spatial discretization on a uniform grid, based on second-order centred differences, and solving the resulting coupled ordinary differential equations in time using a fifth-order Runge–Kutta method with adaptive stepsize. Illustrative solutions for the case \( \Gamma = 5/3 \) are shown in Fig. 2. The initial profile is a real Gaussian of relatively large amplitude. The solution of the linear Schrödinger equation, obtained when the nonlinear terms are neglected, can be written in closed form and is shown in the right-hand panels; it exhibits some linear dispersion. When the nonlinear terms are included, as shown in the left-hand panels, the wave broadens much more rapidly and there is a greater emission of short waves. (If their wavelength is comparable to or shorter than \( H \) it is likely that they will behave differently from the predictions of equation (94).) The computational domain is large enough that the boundary conditions do not affect the solution significantly for the ranges of \( \xi \) and \( T \) shown. It was also confirmed numerically that small-amplitude solitary waves in the case \( \Gamma > 3 \) propagate according to equation (94).

### 4.5 Inclusion of a small effective viscosity

It is straightforward to generalize the analysis to include a small effective viscosity. Suppose that the effective shear viscosity is given by \( \mu = \alpha p/\Omega \), where \( \alpha \) is independent of \( z \) and comparable to \( H/r \). Let \( \alpha = \delta^2 \alpha_0 \) with \( \alpha_0 = O(1) \). Then the viscosity appears in the third-order horizontal forcing terms \( F_{1,3} \) and \( F_{2,3} \), where its effect is to replace \( \partial_r \) by \( (\partial_r + \alpha_0 \Omega) \). Indeed, any process that damps the shearing epicyclic motions at a rate \( \alpha \Omega \) would have an equivalent effect. An effective bulk viscosity of comparable magnitude would not affect the analysis.

The final equation for \( W_0 \) is modified to

\[ \partial_t W_0 = \cdots - \frac{2 \alpha_0}{3 \tau} W_0, \quad (96) \]

where the dots indicate the inviscid dynamical terms found previously. The DNLS equation (80) is modified to

\[ -i \partial_t \Psi = \cdots + \frac{i \alpha_0}{3 \tau} \Psi. \quad (97) \]

When \( \Psi \) is small the effect of viscosity is to cause the solution to decay by a factor \( T^{-\alpha_0/3} \) relative to the inviscid solution. In general, however, the nonlinearity of the equation means that it is not possible to find such an integrating factor.

### 5 CONCLUSIONS

We have derived a one-dimensional equation describing the evolution of weakly nonlinear and dispersive bending waves in a thin Keplerian accretion disc with negligible viscosity and self-gravity. The nonlinear and dispersive effects both depend on the compressibility of the gas through its adiabatic exponent \( \Gamma \). Perhaps surprisingly, in the physically realistic case \( \Gamma < 3 \), nonlinearity tends to broaden the waves and enhance their linear dispersion. This is in contrast with the (hypothetical) case \( \Gamma > 3 \) in which nonlinearity counteracts linear dispersion and solitary waves are supported.

Linear theories of warped accretion discs [Papaloizou & Pringle 1983; Papaloizou & Lin 1992] previously made clear the distinction between the diffusive \( (\alpha > H/r) \) and wavelike \( (\alpha \lesssim H/r) \) regimes in Keplerian discs. Since they were based on an Eulerian perturbation analysis they could not predict the amplitude at which they would break down. In this paper it is found that nonlinear effects cause a significant modification of the bending wave (as the radial location of the wave changes by a factor of order unity) when the dimensionless amplitude of the warp \( |dW/d\ln r| \) is \( O(H/r)^{1/2} \) and the horizontal velocities are comparable to the sound speed.

Unfortunately it is difficult to explain in simple terms how these critical scalings arise or why the nonlinearity is steepening only when \( \Gamma > 3 \). The nonlinearity arises through a mode-coupling process and is complicated further by the resonant nature of the epicyclic oscillations in a Keplerian disc, arising from the coincidence of the orbital and epicyclic frequencies.

The analysis in this paper does not by any means present a complete description of the nonlinear dynamics of the wavelike regime. We have focused on the evolution of a travelling bending wave and have not considered the interaction between inward and outward waves, nor the important case of steady warps maintained by an external torque. Furthermore, Nelson & Papaloizou (1999) noted that shocks can form if the wavelength of the warp is short enough that the shearing epicyclic motions from different parts of the wave collide. This effect typically requires \( |dW/d\ln r| \) to be \( O(1) \).
Figure 2. Numerical solution of equation (80) in the case $\Gamma = 5/3$ starting from a real Gaussian initial condition. Left: solution including the nonlinear terms. Right: solution excluding the nonlinear terms. The real and imaginary parts of $\Psi$ are shown as solid and dotted lines, respectively, at times (from top to bottom) $T = 0, 1, 2$ and 4. The computational domain is much larger than shown here.
Gammie, Goodman & Ogilvie (2000) also noted that the warp could decay through a parametric instability. The critical amplitude for this effect is proportional to the effective viscosity of the disc.

It would be interesting to compare the predictions of this analysis with the results of numerical simulations of warped discs. If parametric instability sets in there should be a noticeable deviation from equation (80). The best way to make this comparison would be to set up a vertically isothermal disc with \( H \propto r \) and \( \Sigma \propto r^{-1} \), to introduce a warp and the accompanying horizontal velocities (59), and to follow the evolution of the complex tilt \( W(r,t) \). The comparison with solutions of equation (80) could then be made by transforming variables from \( W(r,t) \) to \( \Psi(\xi,T) \), or vice versa.

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APPENDIX A: FORCING TERMS

For reference, we list here the forcing terms required for the development of the solution to the desired order.

\[
F_{\nu_1} = -\frac{2\rho_0}{r_0\Omega_0} \left[ \partial_\xi v_{r_1} - (\partial_\xi \gamma_0) \partial_\phi v_{r_1} \right],
\]

\[
F_{\nu_2} = -\partial_\xi \rho_0 + \left( 1 - \frac{2v_{r_1}}{r_0\Omega_0} \right) \partial_\xi (\rho_0 + \rho_1) - \left[ \Omega_1 + \left( 1 - \frac{2v_{r_1}}{r_0\Omega_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi \rho_1 + \frac{v_{\theta_1}}{r_0} \partial_\xi (\rho_0 + \rho_1)
\]

\[
- \frac{r_1 v_{\theta_1}}{r_0^2} \partial_\phi \rho_0 + \left( \rho_0 + \rho_1 \right) \left\{ -\frac{2}{r_0\Omega_0} \left[ \partial_\xi v_{r_1} - (\partial_\xi \gamma_0) \partial_\phi v_{r_1} \right] + \frac{1}{r_0} \partial_\xi v_{\theta_1} \right\}
\]

\[
+ \rho_0 \left\{ -\frac{2}{r_0\Omega_0} \left[ \partial_\xi v_{r_2} - (\partial_\xi \gamma_0) \partial_\phi v_{r_2} \right] - (\partial_\xi \gamma_1) \partial_\phi v_{r_1} \right\}
\]
\[ F_{p1} = \Gamma p_0 \left\{ \frac{2}{r_0} \left[ \partial_\xi v_{r1} - \partial_\xi \gamma_0 \partial_\phi v_{r1} \right] \right\}, \quad \text{(A3)} \]

\[ F_{p2} = -\partial_r p_0 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi (p_{r1} + p_1) - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi p_1 + \frac{v_{r1}}{r_0} \partial_\xi (p_{r1} + p_1) - \frac{r_1 v_{r1}}{r_0^2} \partial_r p_0 + \left( \frac{r_1}{r_0} \right) \left( \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right) \partial_\phi v_{r1} + \frac{1}{r_0} \partial_\xi v_{\theta 1}, \quad \text{(A4)} \]

\[ F_{r1} = 0, \]

\[ F_{r2} = \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + 2\Omega_1 v_{r1}, \quad \text{(A5)} \]

\[ F_{r3} = -\partial_r v_{r1} + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]

\[ F_{r3} = -\partial_r v_{r1} - \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]

\[ F_{r3} = -\partial_r v_{r1} + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]

\[ F_{r3} = -\partial_r v_{r1} + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]

\[ F_{r3} = -\partial_r v_{r1} + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]

\[ F_{r3} = -\partial_r v_{r1} + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]

\[ F_{r3} = -\partial_r v_{r1} + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]

\[ F_{r3} = -\partial_r v_{r1} + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r1} + \frac{v_{r1}}{r_0} \partial_\xi v_{r2} + \left( \frac{v_{r2}}{r_0} - \frac{v_{r1} r_1}{r_0^2} \right) \partial_\xi v_{r1} - \left[ \Omega_1 + \left( 1 - \frac{2v_{r1}}{r_0} \right) \partial_\xi \gamma_0 \right] \partial_\phi v_{r2} \quad \text{r_0} \]
\[
- \left[ \Omega_2 + \frac{v_{\phi 1}}{r_0} - \partial_r \gamma_0 + \left( 1 - \frac{2v_{r 1}}{r_0 \Omega_0} \right) \partial_{\xi} \gamma_1 - \left( \frac{2v_{r 2}}{r_0 \Omega_0} - \frac{2v_{r 1} r_1}{r_0 \Omega_0} - \frac{2v_{r 1} \Omega_1}{r_0 \Omega_0} \right) \partial_{\xi} \gamma_0 \right] \partial_{\phi} v_{\phi 1}
\]

\[
+ \frac{1}{2} \Omega_1 v_{r 2} + 2 \Omega_2 v_{r 1} - \frac{v_{r 1} v_{\phi 1}}{r_0} + \frac{3}{2} \left[ \frac{\Omega_0 r_1}{r_0} v_{r 2} + \left( \frac{2 \Omega_0 r_2}{r_0} - \frac{\Omega_2}{\Omega_0} - \frac{\Omega_0 r_1^2}{r_0^2} \right) v_{r 1} \right].
\] (A12)