A note on a variance bound for the multinomial and the negative multinomial distribution

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Abstract: We prove a Chernoff-type upper variance bound for the multinomial and the negative multinomial distribution. An application is also given.

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1. Introduction

Let \( Z \) be a standard normal distribution and \( g \) be an absolutely continuous function, with a.s. derivative \( g' \). Chernoff [16] proved that \( \text{Var} \ g(Z) \leq \mathbb{E} (g'(Z))^2 \), provided that \( \mathbb{E} (g'(Z))^2 \) is finite, where the equality holds iff \( g \) is a linear polynomial; see also the previous papers by Nash [22], Brascamp and Lieb [8]. This inequality has been generalized and extended by many authors (see, e.g., [1–5, 7, 9–12, 14, 15, 17–21, 24–26]).

In discrete case, let \( X \) be an integer-valued random variable with probability mass function (pmf) \( p \) and finite mean \( \mu \) and variance \( \sigma^2 \), and consider the function \( w \) given by

\[
\sum_{j \leq x} (\mu - j)p(j) = \sigma^2 w(x)p(x) \quad \text{for all } x \in \mathbb{Z}.
\]

Then, for any suitable function \( g \) the following inequality and Stein-type covariance identity hold (see Cacoullos and Papathanasiou [11, Lemma 2.2] and [12, eq. (3.2)])

\[
\text{Var} \ g(X) \leq \sigma^2 \mathbb{E} w(X)[\Delta g(X)]^2, \quad (1)
\]

\[
\text{Cov}[X, g(X)] = \sigma^2 \mathbb{E} w(X)\Delta g(X), \quad (2)
\]

where \( \Delta \) is the forward difference operator (for the cases where \( w \) is a quadratic polynomial see also Afendras et al. [4, 5]).

Let now \( X = (X_1, \ldots, X_k) \) be a random vector with pmf supported by a “convex” set \( C^k \subseteq \mathbb{N}^k \) such that \( 0 \in C^k \) (“convex” in the sense that if \( x = (x_1, \ldots, x_k) \in C^k \) then \( \sum_{i=1}^k \{0, \ldots, x_i\} \subseteq \mathbb{N}^k \) and \( C^k \subseteq \mathbb{N}^k \)).
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(C^k). Assume that the mean \( \mu \) and the variance-covariance matrix \( \Sigma \) of \( X \) are well defined (\( \Sigma > 0 \)) and consider the vector of linear functions

\[
q(x) \equiv (q^1(x), \ldots, q^k(x))^\prime := \Sigma^{-1} x.
\]

Then the \( w \)-function of \( X \) is well defined for every \( x \in C^k \) by \( w(x) \equiv (w^1(x), \ldots, w^k(x))^\prime \) with

\[
w^i(x)p(x) = \sum_{j=0}^{x_i} \left[ \mu^i - q^i(u_i, j, v_i) \right] p(u_i, j, v_i),
\]

where \( u_i = (x_1, \ldots, x_{i-1}), \ v_i = (x_{i+1}, \ldots, x_k) \) and \( \mu^i = E q^i(X), \ i = 1, \ldots, k \) (see [13, 23]).

Cacoullos and Papathanasiou [13] extended the identity (2) as

\[
\text{Cov}[q^i(X), g(X)] = E w^i(X)g_i(X),
\]

provided that \( E|w^i(X)g_i(X)| \) and \( E|(q^i(X) - \mu^i)g(X)| \) are finite, \( i = 1, 2, \ldots, k \) [for \( g \), see Definition 1(a) below]; also, under the same conditions, they established the following inequality

\[
\text{Var} g(X) \geq E \left( w_1(X)g_1(X), \ldots, w_k(X)g_k(X) \right) \Sigma E \left( w_1(X)g_1(X), \ldots, w_k(X)g_k(X) \right)^\prime.
\]

If \( X \) is multinomial or negative multinomial distribution then the weight functions \( w^i \) are the same, say \( w \), and (6) takes the form

\[
\text{Var} g(X) \geq E \left( w(X)\nabla^i g(X) \right) \Sigma E \left( w(X)\nabla g(X) \right),
\]

where \( \nabla g \) is the discrete gradient of \( g \), see Definition 1(b) below. This note complements this lower bound with the following upper bound:

\[
\text{Var} g(X) \leq E \left( w(X)\nabla^i g(X) \right) \Sigma \nabla g(X).
\]

Notice that for the continuous case of dependent random variables the similar bound has proven only in multivariate normal distribution by Chen [15, eq. (3.1)]; also, Cacoullos [10, eq.'s (1.1), (1.4)] generalize Chen’s inequality for a vector of independent random variables, for both continuous and discrete cases (of course, Cacoullos’s results cannot be apply for the multinomial and negative multinomial distributions, since both are vectors of dependent random variables).

2. Preliminaries

The following notations will be used in the sequel.

**Definition 1.** Let \( y \in (-1, \infty) \), \( x = (x_1, \ldots, x_k)^\prime \in \mathbb{R}^k \), \( \pi = (\pi_1, \ldots, \pi_k)^\prime \in (0, 1)^k \) and \( g : \mathbb{R}^k \rightarrow \mathbb{R} \). We denote by:
(a) \( g_i(x) \equiv \Delta_i g(x) := g(x + e_i) - g(x) \), where \( e_i \) is the \( i \)-th vector of the standard orthonormal basis of \( \mathbb{R}^k \).

(b) \( \nabla^i g(x) \equiv (\nabla g(x))^i := (g_1(x), g_2(x), \ldots, g_k(x)) \).

(c) \( \pi^x := \pi_1^x \cdots \pi_k^x \).

(d) \( f(y + 1) := \frac{\Gamma(y + 1)}{x_1^{x_1} \cdots x_k^{x_k}} \), provided that \( x \in \mathbb{N}^k \) with \( \sum_{i=1}^k x_i < y + 1 \).

(e) \( x_{-k} := (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1} \).

Definition 2. We shall use the following notations:

(a) \( m_k(n, \pi) \) the \( k \)-dimensional multinomial distribution with parameters \( \pi \in (0, 1)^k \) and \( n \in \mathbb{N} \), namely with pmf \( p(x) = (\pi_i^n, x \in \mathbb{N}^k \) with \( \sum_{i=1}^k x_i \leq n \), where \( x_0 := n - \sum_{i=1}^k x_i \) and \( \pi_0 := 1 - \sum_{i=1}^k \pi_i > 0 \).

(b) \( n_{m_k}(r, \theta) \) the \( k \)-dimensional negative multinomial distribution with parameters \( \theta \in (0, 1)^k \) and \( r > 0 \), namely with pmf \( p(x) = \left( r - \sum_{i=1}^k x_i \right) \theta^x \theta^{r-1}, x \in \mathbb{N}^k \), where \( \theta_0 := 1 - \sum_{i=1}^k \theta_i > 0 \).

(c) \( p_k(x) \equiv p_{X_k}(x_k), p_{-k}(x) \equiv p_{X_{-k}}(x_{-k}) \) and \( p_{-k|k}(x) \equiv p_{X_{-k}|X_k=x_k}(x_{-k}) \) the pmf's of \( X_k, X_{-k} \) and \( X_{-k}|X_k = x_k \), say, in both cases which we study.

Remark 3. For the multinomial and negative multinomial distributions the functions \( w' \) of (4) are the same for all \( i \). Specifically, \( w'(x) = \pi_0^{-1} (n - \sum_{i=1}^k x_i) \) in \( m_k(n, \pi) \) and \( w'(x) = r^{-1} \theta_0 (r + \sum_{i=1}^k x_i) \) in \( n_{m_k}(r, \theta) \), see [13, pp. 178–179]. Note that we have corrected a minor misprint in the constant of the \( w' \) function corresponding to the negative multinomial.

If \( X \sim m_k(n, \pi) \) then \( X_k \sim m_1(n, \pi_k) \) and \( X_{-k}|X_k = x_k \sim m_{k-1}(n-x_k, \frac{1}{1-x_{-k}} \pi_{-k}) \); so we define

\[
W(x) := \frac{n - \sum_{i=1}^k x_i}{n \pi_0}, \quad W_k(x) := \frac{n - x_k}{n(1 - \pi_k)} \quad \text{and} \quad W_{-k|k}(x) := \frac{(1 - \pi_k)(n - \sum_{i=1}^k x_i)}{(n - x_k) \pi_0};
\]

noting that each function \( h \) of \( X_{-k}|X_k = x_k \) is the zero constant of \( \mathbb{R}^{k-1} \) with probability 1 \( [\text{Var} h = 0] \), so if \( x_k = n \) then \( w_{-k|k} \) is treated as zero-function. If \( X \sim n_{m_k}(r, \theta) \) then \( X_k \sim n_{m_1}(r, \theta_k) \) and \( X_{-k}|X_k = x_k \sim n_{m_{k-1}}(r + x_k, \theta_{-k}) \); so, the \( w \)-functions are defined by

\[
w(x) := \frac{\theta_0 (r + \sum_{i=1}^k x_i)}{r}, \quad w_k(x) := \frac{\theta_0 (r + x_k)}{r (\theta_0 + \theta_k)} \quad \text{and} \quad w_{-k|k}(x) := \frac{(\theta_0 + \theta_k) (r + \sum_{i=1}^k x_i)}{r + x_k}.
\]

For both cases one can easily see that

\[
p_{-k|k}(x + e_k) = p_{-k|k}(x) w_{-k|k}(x) \quad \text{and} \quad w_k(x) w_{-k|k}(x) = w(x).
\]
Lemma 4. Let \( X \sim m_k(n, \pi) \) or \( nm_k(r, \theta) \) and consider a function \( g \) such that \( E|X, g(X)| \) and \( E|X, g_i(X)| \) are finite for all \( i, j = 1, \ldots, k \). Then,

(a) the following covariance identity holds

\[
\text{Cov}
\left[
\sum_{i=1}^{k} X_i, g(X) \right] = E\left(w(X) \sum_{i=1}^{k} c_i g_i(X)\right),
\]

where \( w \) is given by (8) or (9), respectively, and \( c_i = \sum_{j=1}^{k} \sigma_{ij} \) with \( \sigma_{ij} = \text{Cov}(X_i, X_j) \);

(b) the next identity is valid (for the multinomial case only when \( X_k < n \))

\[
\Delta_k E[g(X)|X_k] = E\left[w_{-k|k}(X)\left(g_k(X) + \alpha_k \sum_{i=1}^{k-1} c_{i|k} g_i(X)\right)\right|X_k],
\]

where \( c_{i|k} = \sum_{j=1}^{k-1} \sigma_{i|jk} \) with \( \sigma_{i|jk} = \text{Cov}(X_i, X_j|X_k) \) and \( \alpha_k \equiv a(X_k) \) is \(-\frac{1-n_k}{n_k(n-x_k)} \) or \( \frac{\rho_i + \rho_k}{r_{x_k}} \).

Proof. (a) In view of Remark 3, from (5) we have that \( \text{Cov}(q^i(X), g(X)) = E(w(X)g_i(X), i = 1, \ldots, k \) By (3) we get \( X_1 = \Sigma q(X); \) so \( \sum_{i=1}^{k} X_i = \sum_{i=1}^{k} c_i q(X) \). Combining the above relations (11) follows.

(b) Writing \( \Delta_k E[g(X)|X_k] = E[g(X)|X_k] + 1 - E[g(X)|X_k] \) and using (10), it follows that

\[
\Delta_k E[g(X)|X_k] = E\left[w_{-k|k}(X)g_k(X)|X_k\right] + E\left[w_{-k|k}(X)g(X)|X_k\right] - E[g(X)|X_k]
\]

\[
= E\left[w_{-k|k}(X)g_k(X)|X_k\right] + \text{Cov}\left[w_{-k|k}(X), g(X)|X_k\right],
\]

since \( E[w_{-k|k}(X)|X_k] = 1 \) (see [13, p. 178]). In view of (8) and (9), \( w_{-k|k}(X) = \alpha_k \sum_{i=1}^{k-1} X_i + \beta_k \), where \( \beta_k \equiv \beta(X_k) \) is a constant in \( X_1, \ldots, X_{k-1} \); thus,

\[
\Delta_k E[g(X)|X_k] = E\left[w_{-k|k}(X)g_k(X)|X_k\right] + \alpha_k \text{Cov}\left[\sum_{i=1}^{k-1} X_i, g(X)|X_k\right].
\]

Finally, from the conditions on \( g \) it follows that \( E|X, g(X)|X_k| \) and \( E|X, g_i(X)|X_k| \) are finite for all \( i, j = 1, \ldots, k-1 \). Thus, applying (11) for \( X_{-k|k} \) the lemma is proved.

3. The main result

In this section we present the main result. An application in trinomial distribution is given.

Theorem 5. Let \( X \sim m_k(n, \pi) \) or \( nm_k(r, \theta) \) and consider a function \( g \) such that \( \text{Var}g(X) < \infty \). Then,

\[
\text{Var}g(X) \leq E\left[w(X)\nabla g(X)\nabla g(X)\right],
\]

where \( \Sigma \) is the variance-covariance matrix of \( X \) and \( w \) is given by (8) or (9). The equality in (13) holds iff \( g \) is a linear function with respect to \( x_1, \ldots, x_k \), i.e. of the form \( g(x) = \rho_0 + \sum_{i=1}^{k} \rho_i x_i \).
Proof. If $E[w(X)\nabla^i g(X)\nabla g(X)] = \infty$ then we have nothing to prove. Suppose that

$$E[w(X)\nabla^i g(X)\nabla g(X)] < \infty.$$  \hspace{1cm} (14)

The proof will be done by induction on $k$. For $k = 1$ (13) holds, see (1). Assuming that (13) is valid for $k - 1$ for some $k > 1$, we will prove that (13) is also valid for $k$. It is well known that

$$\text{Var} g(X) = E[\text{Var}(g(X)|X_k)] + \text{Var}[E(g(X)|X_k)].$$  \hspace{1cm} (15)

Using (1) for $X_k$ it follows that

$$\text{Var}[E(g(X)|X_k)] \leq \sigma_k^2 E w_k(X) \left( E w_{-k|k}(X) \left( g_k(X) + \alpha_k \sum_{i=1}^{k-1} c_{ijk} g_i(X) \right) \right)^2.$$  \hspace{1cm} (16)

where $\sigma_k^2 = \text{Var} X_k$. From (14) we have that the conditions of Lemma 4 are valid; noting that $w_k(X)|_{X_k = n} = 0$ with probability 1 and with help of (12) we get

$$\text{Var}[E(g(X)|X_k)] \leq \sigma_k^2 E w_k(X) \left( E w_{-k|k}(X) \left( g_k(X) + \alpha_k \sum_{i=1}^{k-1} c_{ijk} g_i(X) \right) \right)^2.$$  \hspace{1cm} (17)

Using (10),

$$\text{Var}[E(g(X)|X_k)] \leq E \left[ \sigma_k^2 w(X) \left( g_k^2(X) + 2\alpha_k \sum_{i=1}^{k-1} c_{ijk} g_i(X) g_k(X) + \left( \alpha_k \sum_{i=1}^{k-1} c_{ijk} g_i(X) \right)^2 \right) \right]^2 \leq E \left( \sum_{i=1}^{k-1} c_{ijk} g_i(X) \right)^2 \leq E \left( \sum_{i=1}^{k-1} c_{ijk} g_i(X) \right)^2.$$  \hspace{1cm} (18)

By the induction hypothesis of (13), with $k - 1$ in place of $k$, it follows that

$$\text{Var}(g(X)|X_k) \leq E \left[ w_{-k|k}(X) \nabla^i \text{Var} \left( \text{Var} g(X) \right) \nabla g(X) \right]|_{X_k}.$$  \hspace{1cm} (19)

where $\text{Var} g(X)$ is the variance-covariance matrix of $X_{-k|k}$ and $\nabla g = (g_1, \ldots, g_{k-1})^t$. Thus,

$$E[\text{Var}(g(X)|X_k)] \leq E \left[ w_{-k|k}(X) \nabla^i \text{Var} \left( \text{Var} g(X) \right) \nabla g(X) \right]|_{X_k}$$

$$= E w_{-k|k}(X) \left( \sum_{i=1}^{k-1} \sigma_{ijk} g_i^2(X) + 2 \sum_{1 \leq i < j \leq k-1} \sigma_{ijk} g_i(X) g_j(X) \right).$$  \hspace{1cm} (20)
From (15), via (18) and (20), we get
\[
\text{Var } g(X) \leq E \left[ w(X) \sigma_k^2 g_k^2(X) + \sum_{i=1}^{k-1} \left[ w(X) \sigma_k^2 \mathcal{C}_{i|j|k}^2 + w_{-i|j|k}(X) \sigma_{ij|k}^2 \right] g_i^2(X) \right] \\
+ 2 \sum_{i=1}^{k-1} \left[ w(X) \sigma_k^2 \mathcal{C}_{i|j|k} g_i(X) g_k(X) \right] \\
+ 2 \sum_{i \leq j \leq k-1} \left( w(X) \sigma_k^2 \mathcal{C}_{i|j|k} + w_{-i|j|k}(X) \sigma_{ij|k} \right) g_i(X) g_j(X).
\]

After some algebra (see [6]), (13) follows.

Consider the function \( g(x) = \rho_0 + \sum_{i=1}^{k} \rho_i x_i \). One can easily see that (13) holds as equality. Conversely, assume that (13) holds as equality. Then (16), (17) and (19) hold as equalities. From the equality in (19), under the inducional hypothesis, it follows that \( g(x) = \varrho_0(x_k) + \sum_{i=1}^{k-1} \varrho_i(x_k) x_i \). From the equality in (17) we have that the quantity \( g_k(x) + \alpha_k \sum_{i=1}^{k-1} c_{i|k} \varrho_i(x) \) is a constant in \( x_1, \ldots, x_{k-1} \). Combining the above relations it follows that the quantity \( \Delta_k \varrho_0(x_k) + \sum_{i=1}^{k-1} \Delta_k \varrho_i(x_k) x_i + \alpha_k \sum_{i=1}^{k-1} c_{i|k} \varrho_i(x_k) = \sum_{i=1}^{k-1} \Delta_k \varrho_i(x_k) x_i + h(x_k) \) is a constant in \( x_1, \ldots, x_{k-1} \). Therefore, \( \Delta_k \varrho_i(x_k) = 0 \) for all \( i = 1, \ldots, k-1 \), that is \( g_i(x_k) = \rho_i \), \( i = 1, \ldots, k-1 \), are constants. Thus, \( g(x) = \varrho_0(x_k) + \sum_{i=1}^{k-1} \rho_i x_i \). Finally, from the equality in (16) it follows that the quantity \( E(g(X)|X_k = x_k) \) is a linear function in \( x_k \). Moreover, \( E(g(X)|X_k = x_k) = E(\varrho_0(X_k) + \sum_{i=1}^{k-1} \rho_i X_i|X_k = x_k) = \varrho_0(x_k) + \sum_{i=1}^{k-1} \rho_i E(X_i|X_k = x_k) \). For both cases the quantity \( \sum_{i=1}^{k-1} \rho_i E(X_i|X_k = x_k) \) is a linear function of \( x_k \). Hence, \( \varrho_0(x_k) \) is a linear function of \( x_k \), i.e. \( \varrho_0(x_k) = \rho_0 + \rho_k x_k \), and the proof is complete.

The present technique is based, mainly, on the fact that the functions \( w^i, i = 1, 2, \ldots, k \), are the same for multinomial and negative multinomial distributions, see Remark 3. Of course, this is not true for all integer-valued multivariate distributions. Thus, in other cases, the present technique may not be applicable.

### 3.1. An application in negative trinomial distribution

Next, we give an example in the trinomial distribution, in which the exact variance is rather difficult to compute, but the upper/lower bounds can be derived.

Let \( X = (X_1, X_2)^t \sim \text{nm}_2(1, \theta = (\theta_1, \theta_2)^t) \), that is
\[
p(i, j) \equiv p_{X_1, X_2}(i, j) = \frac{(i+j)!}{i!j!} \theta_0^i \theta_1^j, \quad i, j = 0, 1, \ldots,
\]
and consider the function \( h(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \), where \( h \) is assumed to be zero if \( k = 0 \). The statistic \( T = T(X) = h(X_1) - h(X_2) \) is the unbiased estimator of \( \ln \frac{1}{\theta_1} \), since
\[
E T = E h(X_1) - E h(X_2) = -\ln \frac{\theta_0}{1-\theta_2} + \ln \frac{\theta_0}{1-\theta_1},
\]
see Afendras et al. [4, pp. 180–181]. The variance of $T$ is, clearly, quite complicated. However, the bounds of (7) and (13) can be used. Here $w(x) = \theta_0(x_1 + x_2 + 1)$, $\Delta T = (T_1, T_2) = \left(\frac{1}{x_1+1}, -\frac{1}{x_2+1}\right)$ and 
$\Sigma = \theta_0^{-2} \left( \frac{\theta_0}{\theta_1}, \frac{\theta_0}{\theta_2}, \frac{\theta_0}{\theta_1}, \frac{\theta_0}{\theta_2} \right)$.

For the Cacoullos-Papathanasiou lower bound in (7) we calculate $E w(X)T_1 = \frac{\theta_0}{1-\theta_1}$ and $E w(X)T_2 = -\frac{\theta_0}{1-\theta_1}$ (see [6]), and we get

$$\text{Var} T > \frac{(1 - \theta_1 - \theta_2)(\theta_1 + \theta_2)}{(1 - \theta_1)(1 - \theta_2)}. \tag{21}$$

Applying (13) we have that

$$\text{Var} T < \frac{\theta_0(1 - \theta_2)}{\theta_0} E_1 - 2 \frac{\theta_1 \theta_2}{\theta_0} E_{12} + \frac{\theta_2(1 - \theta_1)}{\theta_0} E_2,$$

where $E_1 = E \frac{X_1 + X_2 + 1}{(X_1+1)^2}$, $E_{1,2} = E \frac{X_1 + X_2 + 1}{(X_1+1)(X_2+1)}$ and $E_2 = E \frac{X_1}{(X_2+1)^2}$. After some algebra (see [6]) we found that $E_1 = \frac{\theta_0}{\theta_1(1-\theta_2)} \ln \frac{1-\theta_1}{\theta_0}$, $E_{1,2} = \frac{\theta_0}{\theta_2(1-\theta_1)} \ln \frac{(1-\theta_1)(1-\theta_2)}{\theta_0}$ and $E_2 = \frac{\theta_0}{\theta_2(1-\theta_1)} \ln \frac{1-\theta_1}{\theta_0}$; so,

$$\text{Var} T < -\ln[(1 - \theta_1)(1 - \theta_2)]. \tag{22}$$

Table 1 gives an idea on how the lower/upper bounds of $\text{Var} T$ in (21) and (22) behave for various $\theta_1, \theta_2$-values, noting that both bounds are symmetric to $\theta_1$ and $\theta_2$.

| $\theta_1$ | $\theta_2$ | lower bound: $-\ln[(1 - \theta_1)(1 - \theta_2)]$ | upper bound: $\frac{-(\theta_1 - \theta_2)(\theta_1 + \theta_2)}{(1 - \theta_1)(1 - \theta_2)}$ |
|------------|------------|-----------------------------------------------|-----------------------------------------------|
| 0.1        | 0.1        | 0.211                                         | 0.329                                         |
|            | 0.2        | 0.462                                         | 0.616                                         |
| 0.2        | 0.3        | 0.329                                         | 0.446                                         |
|            | 0.4        | 0.580                                         | 0.734                                         |
| 0.3        | 0.5        | 0.462                                         | 0.713                                         |
|            | 0.6        | 0.616                                         | 0.868                                         |
| 0.4        | 0.7        | 0.616                                         | 0.868                                         |
|            | 0.8        | 0.616                                         | 0.868                                         |

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