The Widom-Dyson constant for the gap probability in random matrix theory

P. Deift  
*Courant Institute of Mathematical Sciences,  
New York, NY 10003, USA*

A. Its  
*Department of Mathematical Sciences,  
Indiana University – Purdue University Indianapolis  
Indianapolis, IN 46202-3216, USA*

I. Krasovsky  
*Department of Mathematical Sciences,  
Indiana University – Purdue University Indianapolis  
Indianapolis, IN 46202-3216, USA  
and  
Department of Mathematical Sciences  
Brunel University  
Uxbridge UB83PH  
United Kingdom*

X. Zhou  
*Department of Mathematics,  
Duke University,  
Durham, NC 27708-0320*
1 Introduction

In this paper we consider an asymptotic question in the theory of the Gaussian Unitary Ensemble of random matrices [1]. In the bulk scaling limit, the probability that there are no eigenvalues in the interval $(0, 2s)$ is given by $P_s = \det(I - K_s)$, where $K_s$ is the trace-class operator with kernel

$$K_s(x, y) = \frac{\sin(x - y)}{\pi(x - y)}$$

acting on $L^2(0, 2s)$. We are interested particularly in the behavior of $P_s$ as $s \to \infty$.

In 1973, des Cloizeaux and Mehta [2] showed that as $s \to \infty$

$$\ln P_s = -\frac{s^2}{2} - \frac{1}{4} \ln s + c + o(1),$$

for some constant $c$. In 1976, Dyson [3] showed that $P_s$ in fact has a full asymptotic expansion of the form

$$\ln P_s = -\frac{s^2}{2} - \frac{1}{4} \ln s + c_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots$$

Dyson identified all the constants $c_0, a_1, a_2, \ldots$. Of particular interest is the constant $c_0$, which he found using earlier work of Widom (see [4] and below) to be

$$c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1),$$

where $\zeta(z)$ is the Riemann zeta-function.

The results in [2] and [3] were not fully rigorous. In [5], Widom gave the first rigorous proof of the leading asymptotics in [1] in the form

$$\ln P_s = -\frac{s^2}{2}(1 + o(1)).$$

In subsequent work [6, 7], which also included the multi-interval generalization, the form [2] of the full asymptotic expansion was verified rigorously, together with the correct constants $a_1, a_2, \ldots$. The expression [3] for the constant $c_0$, however, remained unproven. This was because the methods in [5, 6] and [7] naturally computed the asymptotics of $(d/ds)\ln P_s$, and the constant of integration remained undetermined.

Recently, two proofs of [3] were given independently in the literature in [8] and [9, 10]. The methods in the papers [8] and [9, 10] are very different. Our goal in this paper is to give a third proof of [3], which is closely related to the proof in [8], but as explained below, does not rely on certain a priori information. This means that our approach has the potential advantage of being applicable to other problems involving the computation of critical constants, where a priori information may not be available (see, e.g., [15]).
One way that one might try to evaluate \( c_0 \) is to express

\[
\ln P_s = \ln \det(I - K_s) = \int_0^1 \frac{d}{d\eta} \ln(I - \eta K_s) d\eta = -\int_0^1 \text{tr} \left( (I - \eta K_s)^{-1} K_s \right) d\eta \tag{5}
\]

and then evaluate \( \text{tr} \left( (I - \eta K_s)^{-1} K_s \right) \) asymptotically as \( s \to \infty \) for each fixed \( \eta \in (0, 1) \) using steepest descent methods as in \([16]\), for example. However, it turns out that the asymptotics of \( \text{tr} \left( (I - \eta K_s)^{-1} K_s \right) \) as \( s \to \infty \) have a different form for \( \eta < 1 \) and \( \eta = 1 \). This means that one must integrate the asymptotics in (5) over a boundary layer as \( \eta \to 1 \), a difficult task which we have so far been unable to perform. On the other hand, for \( 0 < \gamma < 1 \), we can indeed use (5) in the form

\[
\ln \det(I - \gamma K_s) = -\int_0^\gamma \text{tr} \left( (I - \eta K_s)^{-1} K_s \right) d\eta
\]

together with the Riemann-Hilbert/steepest-descent method to compute the asymptotics of \( \ln \det(I - \gamma K_s) \) as \( s \to \infty \), so reproducing the results in \([11, 12]\).

As mentioned above, Dyson’s computation of \( c_0 \) in \([3]\) is based on an earlier calculation of Widom \([4]\). In \([4]\), Widom considered, in particular, the Toeplitz determinant \( D_n(\alpha) \) with symbol given by the characteristic function of the interval \((\alpha, 2\pi - \alpha)\), \( 0 < \alpha < \pi \). Thus \( D_n(\alpha) = \det(M_{i-j})_{i,j=0}^{n-1} \), where \( M_k = \int_\alpha^{2\pi-\alpha} e^{-ik\theta} d\theta/2\pi \), \( k \in \mathbb{Z} \). Widom showed that for a fixed \( \alpha \) as \( n \to \infty \),

\[
\ln D_n(\alpha) = n^2 \ln \cos \frac{\alpha}{2} - \frac{1}{4} \ln \left( n \sin \frac{\alpha}{2} \right) + c_0 + o(1), \tag{6}
\]

where \( c_0 \) is the constant \([3]\). What Dyson noted was that for a fixed \( s > 0 \),

\[
\lim_{n \to \infty} D_n \left( \frac{2s}{n} \right) = \det(I - K_s) = P_s \tag{7}
\]

and hence, if the error term \( o(1) \) in (6) was uniform as \( n \to \infty, \alpha \to 0, \alpha n \to \infty \), one could conclude from (6,7) that \( c_0 \) in (2) is indeed given by (3). The main technical result in this paper, as in [8], is the proof that the error term \( o(1) \) is of the form \( O(1/(n \sin^2(\alpha/2))) \), which gives the desired uniformity.

Whereas \( P_s \) is the gap probability for the Gaussian Unitary Ensemble in the bulk scaling limit, we note that \( D_n(\alpha) \) is the gap probability for the Circular Unitary Ensemble \([1]\). Formula (7) is the scaling limit for this probability, and the fact that the limit also gives \( P_s \) is a well-known universality property.

In \([8]\), the author uses steepest descent methods to show that for \( \varepsilon > 0 \) fixed, there exists a (large) positive constant \( s_0 \) such that

\[
\frac{d}{d\alpha} \ln D_n(\alpha) = -n^2 \tan \frac{\alpha}{2} - \frac{1}{8} \cot \frac{\alpha}{2} + O \left( \frac{1}{n \sin^2(\alpha/2)} \right) \tag{8}
\]
for all \( n > s_0 \) and \( \frac{2s_0}{n} \leq \alpha \leq \pi - \varepsilon \). Integrating (8) over \((\alpha, \alpha_0)\), \(2s_0/n \leq \alpha < \alpha_0 \leq \pi - \varepsilon\), one obtains

\[
\ln D_n(\alpha) = \ln D_n(\alpha_0) + n^2 \ln \cos \frac{\alpha}{2} - \frac{1}{4} n \ln \alpha_0 - n^2 \ln \cos \frac{\alpha_0}{2} + \frac{1}{4} \ln \sin \frac{\alpha_0}{2} + O\left(\frac{1}{n \sin(\alpha/2)}\right).
\]

Using Widom’s result (6) for fixed \( \alpha_0 \), one obtains for \( \frac{2s_0}{n} \leq \alpha \leq \pi - \varepsilon \), \( n > s_0 \),

\[
\ln D_n(\alpha) = n^2 \ln \cos \frac{\alpha}{2} - \frac{1}{4} \ln \left(\frac{n \sin \alpha}{2}\right) + c_0 + O\left(\frac{1}{n \sin(\alpha/2)}\right) + \delta_n,
\]

where \( \delta_n \to 0 \) as \( n \to \infty \). For any fixed \( s > s_0 \), one sets \( \alpha = 2s/n \), and then using (7) and letting \( n \to \infty \), one obtains

\[
\ln P_s = -\frac{s^2}{2} - \frac{1}{4} \ln s + c_0 + O\left(\frac{1}{s}\right),
\]

which proves (3).

In this paper, we will derive an improved version of (9), viz.,

\[
\ln D_n(\alpha) = n^2 \ln \cos \frac{\alpha}{2} - \frac{1}{4} \ln \left(\frac{n \sin \alpha}{2}\right) + c_0 + O\left(\frac{1}{n \sin(\alpha/2)}\right),
\]

for \( \frac{2s_0}{n} \leq \alpha \leq \pi - \varepsilon \), \( n > s_0 \), where \( s_0 \) is again a (large) positive constant. Our proof of (12) is direct and does not rely on Widom’s result (6). The proof is based on the following two principles:

(i) Asymptotics of \( D_n(\alpha) \) as \( \alpha \to \pi \) and \( n \) is fixed.

(ii) Asymptotics of the solution of a regularized version of the \[7\] Riemann-Hilbert problem (see below) uniform for \( \frac{2s_0}{n} \leq \alpha \leq \pi \).

The solution of problem (i) is based in turn on the analysis of the standard multiple-integral representation for \( D_n(\alpha) \). The solution of problem (ii) is based on a mapping of the original Riemann-Hilbert problem posed on the arc \(-\alpha \leq \theta \leq \alpha\) of the unit circle to a problem on the fixed interval \([-1, 1]\). The analysis then proceeds via the steepest descent method for Riemann-Hilbert problems introduced by Deift and Zhou in \[18\] and further developed in \[19\], \[20\], and also in \[21\]. This gives an asymptotic expression for the logarithmic derivative \( (d^2/d\alpha^2) \ln D_n(\alpha) \). Formula (21) below together with its integrated version (133), plays a key role in this paper.

Note that in contrast to \[5\] and \[7\], where the analysis of the derivative \( (d/ds) \ln P_s \) fails to identify the constant \( c_0 \), we may now integrate \( (d^2/d\alpha^2) \ln D_n(\alpha) \) from \( \alpha \to \pi \), and the
limit at $\pi$ is determined by step (i). The result is the expression (12). By contrast, in [5] and [7] there is no convenient point $s_0$ from which we can integrate and then use to extract the relevant asymptotics. The key device that makes our method work is the $\Phi$-RH: In particular, we note that the 11-element in the jump matrix for the $\Phi$-RH (see (31) et seq.) is uniformly small as $n \to \infty$, for all $2s_0/n \leq \alpha \leq \pi$, $s_0 >> 1$, and for all $\lambda$ in a compact subset of $(-1,1)$. It is this uniformity in $\alpha$ as $n \to \infty$ that makes it possible to control the integration from $\alpha = \pi$ to $\alpha = 2s_0/n$.

In Section 2 we analyze step (i), and in Section 3, step (ii). Finally, in Section 4, we prove (12).

2 Step (i). Multiple integral analysis.

For the analysis of $D_n(\alpha)$ as $\alpha \to \pi$ we use the multiple integral (e.g., [17, 16])

$$D_n(\alpha) = \frac{1}{(2\pi)^n n!} \int_{C_\alpha} \cdots \int_{C_\alpha} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_n,$$

(13)

where $C_\alpha$ is the arc $\alpha \leq \theta \leq 2\pi - \alpha$ of the unit circle. The integrals are taken from $\alpha$ to $2\pi - \alpha$. Setting $\alpha = \pi - \beta$, $\beta > 0$, and

$$\theta_j = \pi + \beta x_j,$$

we rewrite (13) as follows:

$$D_n(\alpha) = \frac{1}{(2\pi)^n n!} \beta^n \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1 \leq j < k \leq n} |e^{i\beta x_j} - e^{i\beta x_k}|^2 dx_1 \cdots dx_n.$$

(14)

Observe that

$$\prod_{1 \leq j < k \leq n} |e^{i\beta x_j} - e^{i\beta x_k}|^2 = \beta^n(n-1) \left( \prod_{1 \leq j < k \leq n} |x_j - x_k|^2 + O_n(\beta^2) \right).$$

Hence we arrive at the relation

$$D_n(\alpha) = \frac{1}{(2\pi)^n n!} \beta^n \left\{ \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^2 dx_1 \cdots dx_n + O_n(\beta^2) \right\}.$$

(15)

The multiple integral in this formula can be expressed in terms of the norms of the Legendre polynomials. Indeed (see, e.g., [17])

$$\int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^2 dx_1 \cdots dx_n = n! \prod_{k=0}^{n-1} h_k,$$

(16)
where $h_n$ are the normalization constants of the monic polynomials orthogonal on the interval $[-1, 1]$ with the unit weight:

$$p_n(x) = x^n + \ldots, \quad \int_{-1}^{1} p_n(x)p_m(x)dx = h_n\delta_{nm}.$$ 

Let $P_n(x)$ denote the standard Legendre polynomials [17]. Since

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2} x^n + \ldots$$

and

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1},$$

we conclude that

$$p_n(x) = \frac{2^n(n!)^2}{(2n)!} P_n(x)$$

and

$$h_n = \int_{-1}^{1} p_n^2(x)dx = \frac{2^{2n}(n!)^4}{[(2n)!]^2} \int_{-1}^{1} P_n^2(x)dx = \frac{2^{2n}(n!)^4}{[(2n)!]^2} \frac{2}{2n+1}.$$ 

This leads us to the following representation of $D_n(\alpha)$ in the neighborhood of $\alpha = \pi$:

$$\ln D_n(\alpha) = n^2 \ln \beta - n \ln 2\pi + \ln A_n + O_n(\beta^2), \quad \alpha = \pi - \beta,$$ \hspace{1cm} (17)

where

$$A_n = \prod_{k=0}^{n-1} h_k = \prod_{k=0}^{n-1} \frac{2^{2k}(k!)^4}{(2k)!^2} \frac{2}{2k+1}. \hspace{1cm} (18)$$

For later reference, note that the asymptotic relation (17) is clearly differentiable, for fixed $n$, with respect to $\alpha$. Also, for fixed $n$, the term $O_n(\beta^2) \to 0$ as $\beta = \pi - \alpha \to 0$; no claim is made here about the behavior of $O_n(\beta^2)$ as $n \to \infty$.

Widom’s constant, $c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1)$, is generated by the quantity $A_n$. In fact, it is shown in [14], using results from classical analysis, that

$$A_n = e^{c_0} n^{-1/4} (2\pi)^{n/2} n^{-n^2(1+o(1))}, \quad n \to \infty.$$ \hspace{1cm} (19)

The appearance of the zeta function is due to the presence of the products of factorials. Indeed,

$$\ln \prod_{k=1}^{n} k! = \sum_{k=1}^{n} \ln k! = (n+1) \ln n! - \sum_{k=1}^{n} k \ln k,$$

and one can expect that the asymptotics of the sum on the r.h.s. of the last equation is related to $\zeta'(-1)$. The exact relation (see again [14]) reads as follows:

$$\sum_{k=1}^{n} k \ln k = \left(\frac{1}{2} n^2 + \frac{1}{2} n + \frac{1}{12}\right) \ln n - \frac{1}{4} n^2 + \frac{1}{12} - \zeta'(-1) + o(1).$$

Applying this formula and the asymptotics of the Gamma-function to (18) yields (19).
3 Step 2. Riemann-Hilbert analysis.

Denote the complement of $C_{\alpha}$ in the unit circle by $\Gamma_{\alpha} = \{-\alpha < \theta < \alpha\}$ traversed counterclockwise (see Figure 1). Let $m(z) \equiv m(z; n, \alpha)$ be the solution of the following $2 \times 2$ Riemann-Hilbert problem posed on $\Gamma_{\alpha}$:

- $m(z)$ is holomorphic for all $z \notin \Gamma_{\alpha}$
- $m(\infty) = I$
- $m_{-}(z) = m_{+}(z) \begin{pmatrix} 2 & -z^{n} \\ -z^{-n} & 0 \end{pmatrix}, \quad z \in \Gamma_{\alpha}$

Here, as usual, $m_{+}(z)$ (respectively, $m_{-}(z)$) are the $L^{2}$ boundary values of $m(z')$ as $z' \rightarrow z \in \Gamma_{\alpha}$ non-tangentially from the “+” side ($|z| < 1$) (respectively, “−” side ($|z| > 1$)). We shall refer to this Riemann-Hilbert problem as the “m-RH problem”.

**Theorem 1** Let $0 < \alpha < \pi$ and $n > 0$. Then the m-RH problem has a (unique) solution $m(z; n, \alpha)$, and the Toeplitz determinant $D_{n}(\alpha)$ is related to $m(z; n, \alpha)$ by the following differential and difference identities:

\[
\frac{D_{n+1}(\alpha)}{D_{n}(\alpha)} = m_{11}(0; n, \alpha), \quad (20)
\]

\[
\frac{d^{2}}{d\alpha^{2}} \ln D_{n}(\alpha) = -\frac{n^{2}}{\sin^{2}\alpha} [m_{12}(0; n, \alpha)]^{2}. \quad (21)
\]
Remark As a function of \( z \), \( m(z; n, \alpha) \) has a continuous extension up to the boundary \( \Gamma_\alpha \), apart from the two end points \( e^{i\alpha} \) and \( e^{-i\alpha} \), where it has logarithmic singularities. Moreover, \( m_\pm(z) \) admit analytic continuations into a neighborhood of every point \( z \) of the open arc \( \Gamma_\alpha = \Gamma_\alpha \setminus \{ e^{i\alpha}, e^{-i\alpha} \} \). Note also that \( \det m(z; n, \alpha) = 1 \) by a standard calculation. These properties of \( m(z; n, \alpha) \) are inherited by solutions of the transformed Riemann-Hilbert problems introduced below.

Theorem 1 was proved in [7] (cf. Eqs (6.14) and (6.82)) using standard techniques from the theory of integrable systems: derivation of the relevant Lax pair, identification of \( D_n(\alpha) \) as the relevant tau-function etc. The differential identity (21) will be of central importance for the analysis below.

A standard calculation shows that the \( m \)-RH problem has no solution for \( \alpha = \pi \). However, as we now demonstrate, the \( m \)-RH problem can be regularized for all \( \alpha \) in the range, including \( \alpha = \pi \), by a simple sequence of transformations.

3.1 Mapping onto a fixed interval.

For \( 0 < \alpha < \pi \), the linear-fractional transformation,

\[
\lambda = -i \cot \frac{\alpha}{2} \frac{z - 1}{z + 1}, \quad z = \frac{1 + i\lambda \tan \frac{\alpha}{2}}{1 - i\lambda \tan \frac{\alpha}{2}},
\]

maps the arc \( \Gamma_\alpha \) onto the interval \((-1, 1)\) and transforms the \( m \)-RH problem to the following Riemann-Hilbert problem posed on the interval \((-1, 1)\) traversed from \(-1\) to 1 (see Figure 2):

- \( Y(\lambda) \) is holomorphic for all \( \lambda \notin [-1, 1] \)
- \( Y(\infty) = I \)
- \( Y_- (\lambda) = Y_+ (\lambda) \left( \frac{2}{\left( \frac{1 + i\lambda \tan \frac{\alpha}{2}}{1 - i\lambda \tan \frac{\alpha}{2}} \right)^n} - \frac{\left( \frac{1 + i\lambda \tan \frac{\alpha}{2}}{1 - i\lambda \tan \frac{\alpha}{2}} \right)^n}{0} \right), \quad \lambda \in (-1, 1) \)

We shall refer to this Riemann-Hilbert problem as the “\( Y \)-RH problem”. The relation between the \( Y \)-RH problem and the original \( m \)-RH problem is given by the equation

\[
m(z; n, \alpha) = Y^{-1} (-i \cot \frac{\alpha}{2}; n, \alpha) Y(\lambda(z); n, \alpha).
\]

---

1 There are some differences from the notation in [2], namely, our contour is \( \Gamma_\alpha \) instead of \( C \) (\( \Gamma_\alpha \) rotated by \( \pi \)) in [2], and we make the following choice for the functions \( f_i, g_i \) which build up the kernel: \( f_1 = z^{n/2}, f_2 = z^{-n/2}, g_1 = z^{-n/2}/(2\pi i), g_2 = -z^{n/2}/(2\pi i) \).
The $Y$-RH problem is still irregular at $\alpha = \pi$. Indeed, the function $\phi(\lambda, \alpha) \equiv \frac{1 + i\lambda \tan(\alpha/2)}{1 - i\lambda \tan(\alpha/2)}$ is discontinuous at $(\lambda, \alpha) = (0, \pi)$. We have for the jump matrix $J_Y(\lambda, \alpha)$ of $Y(\lambda)$ as $\alpha \to \pi$:

$$J_Y(\lambda, \pi) = \begin{cases} 
\begin{pmatrix} 2 & -(-1)^n \\ (-1)^n & 0 \end{pmatrix}, & \lambda \neq 0 \\
\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, & \lambda = 0,
\end{cases}$$

which demonstrates the difficulty for odd $n$. For even $n$, however, the jump matrix $J_Y(\lambda, \pi)$ is continuous and constant throughout the whole interval $(-1, 1)$. This implies the solvability of the $Y$-RH problem at $\alpha = \pi$ for even $n$; in fact, one easily checks that

$$Y(\lambda; n = 2k, \pi) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1 - \frac{1}{2\pi i} \int_{-1}^{1} \frac{d\nu}{\nu - \lambda}}{1 - \nu} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}.$$ 

However, regardless of the parity of $n$, the convergence of $J_Y(\lambda, \alpha)$ to $J_Y(\lambda, \pi)$ is not uniform in $\lambda$, and this creates a significant difficulty in the direct analysis of the behavior of the solution $Y(\lambda; n, \alpha)$ near $\alpha = \pi$.

As we now show (see the $\Phi$-RH problem below), the $Y$-problem can be regularized by performing one more step which is familiar in the formalism of the nonlinear steepest descent method.

### 3.2 $g$-function transformation.

Following the nonlinear steepest descent method for Riemann-Hilbert problems (see, e.g., [21]), we introduce the following “$g$” function:

$$g(\lambda) \equiv \frac{1 + i\sqrt{\lambda^2 - 1} \sin \frac{\alpha}{2}}{1 + i\lambda \tan \frac{\alpha}{2}}. \quad (24)$$

This is essentially the $g$-function of section 6 of [14] written in the variable $\lambda$ (see equation (29) below). It possesses the following characteristic properties:

(a) $g(\lambda)$ is holomorphic for all $\lambda \notin [-1, 1]$. Here we fix the square root by the condition $\sqrt{\lambda^2 - 1} \sim \lambda, \quad \lambda \to \infty$. 

\[ \lambda, \quad \text{Figure 2: Contour for the } Y \text{-RH problem.} \]
(b) \( g(\lambda) \neq 0 \) for all \( \lambda \notin [-1,1] \). At the points \( \lambda = -i \cot \frac{\alpha}{2} \) (or \( z = \infty \)) and \( \lambda = i \cot \frac{\alpha}{2} \) (or \( z = 0 \)) the values of the function \( g(\lambda) \) are:
\[
g(-i \cot \frac{\alpha}{2}) = 1 \quad \text{and} \quad g(i \cot \frac{\alpha}{2}) = \cos^2 \frac{\alpha}{2} \equiv \kappa. \tag{25}
\]

(c) The boundary values of \( g_{\pm}(\lambda) \), \( \lambda \in [-1,1] \) satisfy the following equations:
\[
g_+g_- = \kappa \frac{1 - i\lambda \tan \frac{\alpha}{2}}{1 + i\lambda \tan \frac{\alpha}{2}} \tag{26}
\]
and
\[
g_+ \frac{g}{g_-} = \frac{1 - \sqrt{1 - \lambda^2} \sin \frac{\alpha}{2}}{1 + \sqrt{1 - \lambda^2} \sin \frac{\alpha}{2}} \tag{27}
\]

(d) The behavior of \( g(\lambda) \) as \( \lambda \to \infty \) is described by the asymptotic relation
\[
g(\lambda) = \cos \frac{\alpha}{2} + O \left( \frac{1}{\lambda} \right). \tag{28}
\]

It is worth noticing that
\[
g(\lambda(z)) = \frac{z + 1 + \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2z} \equiv \varphi(z). \tag{29}
\]

If one changes \( 1 \) to \(-1\) in the numerator, then \( \varphi(z) \) becomes the \( g \)-function of section 6 of [7]. The change of sign is due to the fact that the Riemann-Hilbert problem considered in [7] is defined on the arc \( C = e^{i\pi} \Gamma_\alpha \) rather than on \( \Gamma_\alpha \) (cf. footnote 1 above).

Equation (27) has an important consequence. Fix \( 0 < \delta < 1 \) and \( 0 < \alpha \leq \pi \). Then the following inequality holds:
\[
\left| \frac{g_+}{g_-} \right| \leq \varepsilon_0 < 1, \quad \lambda \in [-1 + \delta, 1 - \delta], \tag{30}
\]
for some \( \varepsilon_0 = \varepsilon_0(\delta, \alpha) > 0 \). Of course, for all \( \lambda \in (-1,1) \) and \( \alpha \in (0,\pi) \), we have
\[
\left| \frac{g_+}{g_-} \right| \leq 1.
\]

Following the steepest descent method, we transform the original Riemann-Hilbert problem by the formula
\[
Y(\lambda) \mapsto \Phi(\lambda) \equiv Y(\lambda)g^{-n\sigma_3} \kappa^{\frac{1}{2}} g^{\sigma_3}, \tag{31}
\]
where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is the third Pauli matrix. From the properties of the \( g \)-function listed above, it follows that the matrix function \( \Phi(\lambda) \equiv \Phi(\lambda; n, \alpha) \) is the solution of the following Riemann-Hilbert problem, which we shall refer to as the “\( \Phi \)-RH problem”:
• $\Phi(\lambda)$ is holomorphic for all $\lambda \notin [-1, 1]$

• $\Phi(\infty) = I$

• $\Phi_-(\lambda) = \Phi_+(\lambda) \begin{pmatrix} 2 \left[ \frac{1 - \sqrt{1 - \lambda^2 \sin^2 \theta}}{1 + \sqrt{1 - \lambda^2 \sin^2 \theta}} \right]^{n} & -1 \\ 1 & 0 \end{pmatrix}, \lambda \in (-1, 1)$

In view of (23), the original function $m(z)$ is related to the solution $\Phi(\lambda)$ by the formulae:

$$m(z; n, \alpha) = \kappa^{\frac{\sigma_3}{2}} \Phi^{-1}(-i \cot \frac{\alpha}{2}; n, \alpha) \Phi(\lambda(z); n, \alpha) \varphi^{\sigma_3}(z) \kappa^{-\frac{\sigma_3}{2}}$$

As indicated earlier, the $\Phi$-RH problem is regularized. Indeed, note first that the jump matrix for the $\Phi$-RH problem is now continuous for all $\lambda \in [-1, 1]$ and $\alpha \in [0, \pi]$ with the end point $\alpha = \pi$ included. Moreover, for all $0 \leq \alpha \leq \pi$, the $\Phi$-RH problem is (uniquely) $L^2$-solvable, by the following argument. Consider the $\Phi$-RH problem as defined on the whole real line with discontinuities at $\pm 1$ (the jump matrix outside $(-1, 1)$ is the identity). The limiting value of the jump matrix at these two points from inside the interval $(-1, 1)$ is

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

whose only eigenvalue is $1 \notin (-\infty, 0]$. By Theorem 5.16 of [25], the RH problem is $L^2$ Fredholm. Now by (the proof of) Theorem 9.3 in [26], a Fredholm RH problem with a jump matrix $v$ on $\mathbb{R}$ is $L^2$-solvable if $v + v^* \geq 0$ everywhere, and $v + v^* > 0$ on a set of positive Lebesgue measure. These conditions are clearly satisfied in our case. Therefore the $\Phi$-RH problem is $L^2$-solvable.

Theorem 1 and equation (32) yield representations of the Toeplitz determinant $D_n(\alpha)$ in terms of the solution of the $\Phi$-RH problem:

$$\frac{D_{n+1}(\alpha)}{D_n(\alpha)} = \Theta(n, \alpha) \cos^{2n} \frac{\alpha}{2},$$

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2 \alpha} \Delta(n, \alpha),$$

where

$$\Theta(n, \alpha) = \left[ \Phi^{-1}(-i \cot \frac{\alpha}{2}; n, \alpha) \Phi(i \cot \frac{\alpha}{2}; n, \alpha) \right]_{11}$$

and

$$\Delta(n, \alpha) = \left[ \Phi^{-1}(-i \cot \frac{\alpha}{2}; n, \alpha) \Phi(i \cot \frac{\alpha}{2}; n, \alpha) \right]_{12}^2.$$
3.3 Asymptotic analysis of the $\Phi$-RH problem.

By standard arguments, using inequality (30), one expects that $\Phi(\lambda)$ is approximated by the function

$$N(\lambda) = \begin{pmatrix} \beta(\lambda) + \beta^{-1}(\lambda) & \beta(\lambda) - \beta^{-1}(\lambda) \\ \frac{\beta(\lambda) - \beta^{-1}(\lambda)}{2i} & \frac{\beta(\lambda) + \beta^{-1}(\lambda)}{2i} \end{pmatrix}, \quad \beta(\lambda) = \frac{\lambda - 1}{\lambda + 1},$$

which solves the model Riemann-Hilbert problem:

- $N(\lambda)$ is holomorphic for all $\lambda \notin [-1, 1]$ 
- $N(\infty) = I$
- $N_-(\lambda) = N_+(\lambda) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \lambda \in (-1, 1)$

In order to estimate the precision of this approximation, we need to consider the $\Phi$-RH problem for $\lambda$ near $\pm 1$. The following result, which allows for complex values of $\alpha$ in a neighborhood of $\alpha = \pi$, is basic for our analysis (the need for this complex extension will be apparent towards the end of the paper, see (126) below).

**Theorem 2** Let $\delta$ be a positive number less than $1/4$. Introduce the domain

$$\Omega^{(\delta)} = \mathbb{C} \setminus (U \cup \bar{U}),$$

were $U$ ($\bar{U}$) denotes the open disk of radius $\delta$ centered at 1 (respectively, $-1$). Let also $\varepsilon$ be a positive number less than $\pi - 2$ and denote $\mathcal{D}_\varepsilon(\pi)$ the disk in the $\alpha$-plane of radius $\varepsilon$ centered at $\alpha = \pi$. Set

$$\rho = n \left| \sin \frac{\alpha}{2} \right|.$$

Then, for $\delta$ and $\varepsilon$ sufficiently small, there exists $s_0 > 0$ such that for all $\alpha \in \left[\frac{2\pi}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_\varepsilon(\pi)$, and $n \geq s_0$, the solution of the $\Phi$-RH problem exists (and is unique) and satisfies the estimate

$$\Phi(\lambda; n, \alpha) = \left( I + O \left( \frac{1}{(1 + |\lambda|)\rho} \right) \right) N(\lambda), \quad \rho \to \infty,$$

uniformly for $\lambda \in \Omega^{(\delta)}$ and $\alpha \in \left[\frac{2\pi}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_\varepsilon(\pi)$. Moreover, this estimate can be extended to a full asymptotic series in inverse powers of $\rho$; in particular, the order $\rho^{-3}$ extension of (37) reads:

$$\Phi(\lambda; n, \alpha) = (I + R_1(\lambda) + R_2(\lambda) + R_3(\lambda)) N(\lambda),$$

(38)
where

\[ R_1(\lambda) = \frac{1}{16i\rho} \left[ \frac{1}{1 - \lambda} \begin{pmatrix} -1 & i \\ 1 & 1 \end{pmatrix} + \frac{1}{1 + \lambda} \begin{pmatrix} 1 & i \\ 1 & -1 \end{pmatrix} \right], \quad \lambda \in \mathbb{C} \setminus (U \cup \tilde{U}), \tag{39} \]

\[ R_2(\lambda) = \frac{1}{2^{8}\rho^{2}} \left[ \frac{1}{1 - \lambda} \begin{pmatrix} 1 & 8i \\ -8i & 1 \end{pmatrix} + \frac{1}{1 + \lambda} \begin{pmatrix} 1 & -8i \\ 8i & 1 \end{pmatrix} \right], \quad \lambda \in \mathbb{C} \setminus (U \cup \tilde{U}), \tag{40} \]

\[ R_r(\lambda) = O \left( \frac{1}{(1 + |\lambda|)\rho^{3}} \right), \quad \rho \to \infty, \tag{41} \]

uniformly for \( \lambda \in \Omega^{(2\delta)} \) and \( \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup \mathcal{D}_\varepsilon(\pi) \).

**Remark 1** The last statement (41) means that there exist positive constants \( C \) and \( s_0 \), depending on \( \varepsilon \) and \( \delta \) only, such that

\[ |R_r(\lambda)| \leq \frac{C}{(1 + |\lambda|)\rho^{3}}, \tag{42} \]

\[ \forall \lambda \in \Omega^{(2\delta)}, \quad \forall \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup \mathcal{D}_\varepsilon(\pi), \quad \forall n : s_0 \leq n. \]

We shall also assume that \( \varepsilon \) is small enough for the inequality,

\[ \left| \sin \frac{\alpha}{2} \right| \geq \frac{1}{2} \tag{43} \]

to take place for all \( \alpha \in \mathcal{D}_\varepsilon(\pi) \), and hence

\[ \rho \geq \frac{1}{2}s_0, \tag{44} \]

for all \( \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup \mathcal{D}_\varepsilon(\pi) \) and \( s_0 \leq n \).

**Remark 2** Part of the assertion of Theorem 2 is that the solution of the \( \Phi \)-RH problem exists and is unique for all \( \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup \mathcal{D}_\varepsilon(\pi) \) and \( n \geq s_0 \) with \( s_0 \) sufficiently large. This is all we need in the analysis that follows; however, the solution of the \( \Phi \)-RH problem actually exists and is unique for all \( \alpha \in [0, \pi - \varepsilon] \cup \mathcal{D}_\varepsilon(\pi) \) and all \( n > 0 \) for some (possibly smaller) \( \varepsilon > 0 \). Indeed, by the discussion following (32), the \( \Phi \)-RH problem is solvable for all \( \alpha \in [0, \pi], n > 0 \), and also for all \( \alpha \in \mathcal{D}_\varepsilon'(\pi), 0 < n < s_0 \) for some \( \varepsilon' > 0 \) by continuity of the jump matrix at \( \alpha = \pi \). By Theorem 2, the \( \Phi \)-RH problem is solvable for all \( \alpha \in \mathcal{D}_\varepsilon(\pi), n \geq s_0 \). Thus the \( \Phi \)-RH problem is solvable for all \( n > 0 \) on \([0, \pi - \varepsilon_1] \cup \mathcal{D}_{\varepsilon_1}(\pi)\), where \( \varepsilon_1 = \min(\varepsilon, \varepsilon') \).

**Remark 3** The local analyticity of the jump matrix of the \( \Phi \)-RH problem implies that both boundary values of the function \( \Phi(\lambda) \) on \((-1, 1)\), i.e. the functions \( \Phi_{\pm}(x) \), admit the analytic continuation in the neighborhood of every point of the interval \((-1 + \delta, 1 - \delta)\).

**Proof of Theorem 2** We shall now construct parametrices in \( U \) and \( \tilde{U} \) which are solutions of the \( \Phi \)-RH problem in these neighborhoods with the condition at infinity replaced by the requirement that they match \( N(\lambda) \) at the disks’ boundaries to leading order (cf. [7]).
Consider the function
\[ f(\lambda) = \frac{1 + i(\lambda^2 - 1)^{1/2} \sin(\alpha/2)}{1 - i(\lambda^2 - 1)^{1/2} \sin(\alpha/2)} \] (45)
which is analytic and has no zeros in \( U \setminus (1 - \delta, 1] \) and \( \tilde{U} \setminus [-1, -1 + \delta) \). (Note, however, that it is singular outside of these disks at \( \lambda = -i \cot(\alpha/2) \).) The branch of the root is taken such that \( (\lambda^2 - 1)^{1/2} > 0 \) for \( \lambda > 1 \). The function (45) has the following boundary values on \( (-1, 1) \):
\[ f_+ (x) = \frac{1 - \sqrt{1 - x^2} \sin(\alpha/2)}{1 + \sqrt{1 - x^2} \sin(\alpha/2)}, \quad f_- (x) = f_+ (x)^{-1}, \quad x \in (-1, 1). \] (46)
Consider first the neighborhood \( U \). We look for a parametrix, an analytic function in \( U \setminus (1 - \delta, 1] \), satisfying the jump condition of the \( \Phi \)-RH problem on \( (1 - \delta, 1) \), of the form
\[ P(\lambda) = E(\lambda) \hat{P}(\lambda) f(\lambda)^{-\sigma \alpha n/2}, \quad \lambda \in U \setminus (1 - \delta, 1], \] (47)
where \( E(\lambda) \) is a non-zero analytic matrix-valued function in \( U \) (which therefore does not affect the jump condition) to be chosen below so that \( P \) matches \( N \) to leading order on the boundary \( \partial U \).

It is easy to verify using (46) that for \( P \) to satisfy the jump condition for the \( \Phi \)-RH problem across \( (1 - \delta, 1) \), \( \hat{P} \) must satisfy the jump relation
\[ \hat{P}_+(x) = \hat{P}_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad x \in (1 - \delta, 1). \] (48)
An appropriate matrix function satisfying this jump relation was constructed in [7] (cf. [7] (4.79), (4.871)).

For \( \lambda \in U \setminus (1 - \delta, 1] \), define the analytic function
\[ \omega(\lambda) = \frac{1}{2} \ln f(\lambda) \] (49)
\[ \equiv i \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} \sin^{2k+1} \frac{\alpha}{2} (\lambda^2 - 1)^{k+1/2}. \]

Note that
\[ \omega(\lambda) = i \sqrt{2} \sin \frac{\alpha}{2} (\lambda - 1)^{1/2} G(\lambda), \tag{50} \]
where \( G(\lambda) \) is analytic in all of \( U \), and
\[ G(\lambda) = 1 + (\lambda - 1) \left( \frac{1}{4} - \frac{2}{3} \sin^2 \frac{\alpha}{2} \right) + O((\lambda - 1)^2), \tag{51} \]
for \( \lambda \) near 1. Thus,
\[ e^{\omega(\lambda)} = f(\lambda)^{1/2}, \quad \lambda \in U \setminus (1 - \delta, 1], \tag{52} \]
\[ \omega(x)_+ = e^{i\pi} \omega(x)_-, \quad x \in (1 - \delta, 1). \tag{53} \]
Furthermore, the function \( \omega^2(\lambda) \) is analytic in all of \( U \) and
\[ \omega^2(\lambda) = 2ue^{i\pi} \sin^2 \frac{\alpha}{2} \left( 1 + \frac{u}{2} - u \sin^2 \frac{\alpha}{2} + O(u^2) \right), \quad u = \lambda - 1. \tag{54} \]
The term \( O(u^2) \) in \( (51, 54) \) is uniform for all \( 0 \leq \alpha \leq \pi \). In fact, the estimate \( (54) \) is uniform for \( \alpha \) belonging to any compact set in the complex \( \alpha \)-plane. Let us choose \( 0 < \varepsilon < \pi \). Then, for sufficiently small \( \delta \), the asymptotic relation \( (54) \) implies that
\[ |\omega(\lambda)| \geq \sqrt{\delta} \left| \sin \frac{\alpha}{2} \right|, \quad \forall \lambda \in \partial U, \quad \forall \alpha \in [0, \pi - \varepsilon] \cup D_\varepsilon(\pi). \tag{55} \]
Here \( D_\varepsilon(\pi) \) is the disk in the \( \alpha \)-plane of radius \( \varepsilon \) centered at \( \alpha = \pi \).

Introduce the new variable
\[ \zeta = e^{-i\pi} n^2 \omega^2(\lambda). \tag{56} \]
Note that the mapping \( \lambda \to \zeta \) of \( U \) is one-to-one.

From \( (54) \) and \( (55) \) it follows that for \( \delta \) and \( \varepsilon \) sufficiently small, the following inequalities hold:
\[ -\frac{3\pi}{4} \leq \arg \sqrt{\zeta} \leq \frac{3\pi}{4}, \tag{57} \]
and
\[ |\sqrt{\zeta}| \geq n\sqrt{\delta} \left| \sin \frac{\alpha}{2} \right| \equiv \rho \sqrt{\delta}, \tag{58} \]
\[ \forall \lambda \in \partial U, \quad \forall \alpha \in [0, \pi - \varepsilon] \cup D_\varepsilon(\pi). \]
Inequality \( (58) \) together with \( (44) \) imply the estimate
\[ |\sqrt{\zeta}| \geq \frac{\sqrt{\delta}}{2} \| s_0 > 1, \tag{59} \]
Inequality \( (58) \) together with \( (44) \) imply the estimate
\[ |\sqrt{\zeta}| \geq \frac{\sqrt{\delta}}{2} \| s_0 > 1, \tag{59} \]
\[ \forall \lambda \in \partial U, \quad \forall \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup \mathcal{D}_\varepsilon(\pi), \quad \frac{2}{\sqrt{\delta}} < s_0 \leq n. \]

A function \( \hat{P}(\lambda) \) analytic in \( U \setminus (1 - \delta, 1) \) and satisfying (48) is given by the following expression in terms of Hankel functions (cf. [7]) where \( \sqrt{\zeta} = e^{-i\pi/4}n\omega(\lambda) \):

\[
\hat{P}(\lambda) = \left( \frac{H_0^{(1)}(\sqrt{\zeta})}{\sqrt{\zeta} \left( H_0^{(1)} \right)'(\sqrt{\zeta})} \right) \frac{H_0^{(2)}(\sqrt{\zeta})}{\sqrt{\zeta} \left( H_0^{(2)} \right)'(\sqrt{\zeta})} \; \sqrt{\zeta} \left( H_0^{(1)} \right)'(\sqrt{\zeta}) \left( H_0^{(2)} \right)'(\sqrt{\zeta}) \right), \tag{60}
\]

Inequality (57) and estimate (59) allow us to use the standard expansion for Bessel functions and obtain the following asymptotics on the boundary \( \partial U \):

\[
\hat{P}(\lambda) = \sqrt{\frac{2}{\pi}} \zeta^{-\sigma_3/4} \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right) \left( I + \frac{i}{8\sqrt{\zeta}} \left( \begin{array}{cc} 1 & 2 \\ -2 & -1 \end{array} \right) + \frac{3}{2\zeta} \left( \begin{array}{cc} 1 & -4 \\ 4 & 1 \end{array} \right) + \hat{P}_r(\lambda) \right) \times e^{n\omega(\lambda)\sigma_3}e^{-i(\pi/4)\sigma_3}, \quad \lambda \in \partial U, \tag{61}
\]

where the remainder \( \hat{P}_r(\lambda) \) satisfies the uniform estimate

\[
|\hat{P}_r(\lambda)| < \frac{C_0}{|\zeta|^{3/2}}, \tag{62}
\]

\[ \forall \lambda \in \partial U, \quad \forall \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup \mathcal{D}_\varepsilon(\pi), \quad \frac{2}{\sqrt{\delta}} < s_0 \leq n. \]

Here \( C_0 \) is a numerical positive constant which comes from the universal asymptotic expansion of the Hankel function \( H_0^{(1)}(\sqrt{\zeta}) \) for \( |\zeta| > 1 \) and \(-3\pi/4 \leq \arg \sqrt{\zeta} \leq 3\pi/4\).

Now let us choose \( E(\lambda) \) so that \( P \) matches \( N \) on \( \partial U \) to leading order in \( \rho \), i.e., \( PN^{-1} \sim I \). Clearly, we should take

\[
E(\lambda) = N(\lambda)e^{i(\pi/4)\sigma_3} \left( \begin{array}{cc} 1 & -i \\ i & 1 \end{array} \right) \sqrt{\frac{\pi}{2}} \zeta^{\sigma_3/4}. \tag{63}
\]

It is easy to verify that \( E(\lambda) \) has no branch point or singularity at \( \lambda = 1 \). Hence \( E(\lambda) \) is analytic in \( U \).

Thus, the parametrix in the neighborhood \( U \) is given by the expression:

\[
P(\lambda) = N(\lambda)e^{i(\pi/4)\sigma_3} \sqrt{\frac{\pi}{2}} \left( \begin{array}{cc} 1 & -i \\ 1 & i \end{array} \right) \zeta^{-\sigma_3/4} \times \left( \begin{array}{cc} H_0^{(1)}(\sqrt{\zeta}) & H_0^{(2)}(\sqrt{\zeta}) \\ \sqrt{\zeta} \left( H_0^{(1)} \right)'(\sqrt{\zeta}) & \sqrt{\zeta} \left( H_0^{(2)} \right)'(\sqrt{\zeta}) \end{array} \right) f(\lambda)^{-\sigma_3 n/2}, \tag{64}
\]

where \( \zeta \) and \( f(\lambda) \) are defined by (56, 49, 45).
The construction of a parametrix in the neighborhood $\tilde{U}$ is similar. In this case, instead of (49) we set
\[
\tilde{\omega}(\lambda) = -\frac{1}{2} \ln f(\lambda),
\] (65)
which is analytic in $\tilde{U} \setminus [-1, -1 + \delta)$. Thus
\[
e^{-\tilde{\omega}(\lambda)} = f(\lambda)^{1/2}, \quad \lambda \in \tilde{U} \setminus [-1, -1 + \delta),
\] (66)
\[
\tilde{\omega}(x)_+ = e^{-i\pi \tilde{\omega}(x)}_-, \quad x \in (-1, -1 + \delta).
\] (67)

We find the same power series expansion for $\tilde{\omega}^2(\lambda)$ as (54) with $\lambda - 1$ replaced by $-\lambda - 1$.

We define the $\tilde{\zeta}$ variable for $\lambda$ in $\tilde{U}$ again by the equation
\[
\tilde{\zeta} = e^{-i\pi/2} n^2 \tilde{\omega}^2(\lambda).
\] (68)

Note that for both the images $\tilde{\zeta}(\tilde{U})$ and $\zeta(U)$, the slit for $\tilde{\zeta}$ (respectively $\zeta$) variable lies along the negative half-axis (if $\alpha$ is real; it is slightly rotated away from the negative half-axis if $\alpha$ is complex). However, the orientation is changed (see Figure 3).

With the above notation for $\tilde{\omega}$ and $\tilde{\zeta}$, the parametrix in $\tilde{U}$ matching $N(\lambda)$ to leading order at $\partial\tilde{U}$ is given by the following expression:
\[
\tilde{P}(\lambda) = N(\lambda) e^{-i(\pi/4)\sigma_3} \sqrt{\pi} \frac{1}{2^{3/2}} \left( \begin{array}{cc} i & 1 \\ -i & 1 \end{array} \right) \tilde{\zeta}^{-\sigma_3/4} \times \\
\sigma_1 \left( \begin{array}{cc} H_0^{(1)}(\sqrt{\tilde{\zeta}}) & H_0^{(2)}(\sqrt{\tilde{\zeta}}) \\ \sqrt{\tilde{\zeta}} (H_0^{(1)})' & \sqrt{\tilde{\zeta}} (H_0^{(2)})' \end{array} \right) \sigma_1 f(\lambda)^{-\sigma_3 n/2}, \quad \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
\] (69)

where $\sqrt{\tilde{\zeta}} = e^{-i\pi/2} n \tilde{\omega}(\lambda)$.

Following the steepest descent method, we now formulate a RH-problem for the function
\[
R(\lambda) = \begin{cases} 
\Phi(\lambda) N(\lambda)^{-1}, & \lambda \in \mathbb{C} \setminus (U \cup \tilde{U} \cup (-1, 1)) \\
\Phi(\lambda) P(\lambda)^{-1}, & \lambda \in U \setminus (1 - \delta, 1] \\
\Phi(\lambda) \tilde{P}(\lambda)^{-1}, & \lambda \in \tilde{U} \setminus [-1, -1 + \delta).
\end{cases}
\] (70)

By construction, the function $R(\lambda)$ has no jumps across $(1 - \delta, 1) \cup (-1, -1 + \delta)$. Moreover, since apriori $R(\lambda)$ can have no stronger than logarithmic singularities at the points $\pm 1$, the function $R(\lambda)$ is in fact analytic in the union of the discs $U \cup \tilde{U}$. It solves the following RH-problem on the contour $\Sigma = \partial U \cup \partial \tilde{U} \cup (-1 + \delta, 1 - \delta)$ (see Figure 4):

- $R(\lambda)$ is holomorphic for all $\lambda \notin \Sigma$
- $R(\infty) = I$
$R_+(\lambda) = R_-(\lambda)\Lambda(\lambda), \quad \lambda \in \Sigma^{(0)} \equiv \Sigma \setminus \{1 - \delta, -1 + \delta\}$, where
\[
\Lambda(x) = N_+(x) \begin{pmatrix} 1 & -2f_+^n(x) \\ 0 & 1 \end{pmatrix} N_+(x)^{-1}, \quad x \in (-1 + \delta, 1 - \delta),
\]
\[
\Lambda(\lambda) = P(\lambda)N(\lambda)^{-1}, \quad \lambda \in \partial U \setminus \{1 - \delta\},
\]
\[
\Lambda(\lambda) = \tilde{P}(\lambda)N(\lambda)^{-1}, \quad \lambda \in \partial \tilde{U} \setminus \{-1 + \delta\},
\]

Observe that for all $-1 \leq x \leq 1$ and all $0 \leq \alpha \leq \pi$, we have
\[
0 \leq f_+(x) \leq e^{-\sqrt{1-x^2} \sin \alpha}. 
\]
Moreover, for sufficiently small $\varepsilon$ there exists a positive constant $C_\delta$, depending on $\delta$ only, such that
\[
|f_+(x)| \leq e^{-C_\delta},
\]
for all $-1 + \delta \leq x \leq 1 - \delta$ and all $\alpha \in D_\varepsilon(\pi)$. Combining the two estimates above, we conclude that the jump matrix on $[-1 + \delta, 1 - \delta]$ is of order
\[
I + O(\exp(-C_{\delta,\varepsilon})),
\]
where $C_{\delta,\varepsilon}$ is a positive constant which only depends on $\delta$ and $\varepsilon$. This estimate is uniform in
\[
x \in [-1 + \delta, 1 - \delta], \quad \alpha \in [0, \pi - \varepsilon] \cup D_\varepsilon(\pi).
\]

Using \eqref{61}, we obtain the following asymptotic expansion in inverse powers of $\sqrt{\zeta}$ for the jump matrix on $\partial U$:
\[
P(\lambda)N(\lambda)^{-1} = I + \Lambda_1 + \Lambda_2 + \Lambda_r,
\]
\[
\Lambda_1 = \frac{i}{16\sqrt{\zeta}} \begin{pmatrix} 3\beta^2 - \beta^{-2} & i(3\beta^2 + \beta^{-2}) \\ i(3\beta^2 + \beta^{-2}) & -(3\beta^2 - \beta^{-2}) \end{pmatrix}, \quad \Lambda_2 = \frac{3}{2\sqrt{\zeta}} \begin{pmatrix} 1 & -4i \\ 4i & 1 \end{pmatrix},
\]
\[
\lambda \in \partial U,
\]
where $\beta(\lambda)$ is defined in \eqref{56}. Since the matrix functions $N(\lambda)$ and $N^{-1}(\lambda)$ are uniformly bounded on $\partial U$, we conclude from \eqref{61} and \eqref{62} that the error term $\Lambda_r(\lambda)$ in \eqref{74} satisfies the uniform estimate,
\[
|\Lambda_r(\lambda)| < \frac{C_\delta}{|\zeta|^{3/2}},
\]

Figure 4: Contour $\Sigma$ for the $R$-RH problem.
∀\(\lambda \in \partial U\), \(\forall \alpha \in \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_\varepsilon(\pi)\), \(\frac{2}{\sqrt{\delta}} < s_0 \leq n\).

Here \(C_\delta\) is a positive constant depending on \(\delta\) only. The jump matrix on \(\partial \tilde{U}\) is given by the similar representation with the matrices \(\Lambda\) defined as follows:

\[
\begin{align*}
\Lambda_1 &= \frac{i}{16 \sqrt{\zeta}} \begin{pmatrix} -(3\beta^2 - \beta^2) & i(3\beta^2 + \beta^2) \\ i(3\beta^2 + \beta^2) & 3\beta^2 - \beta^2 \end{pmatrix}, \\
\Lambda_2 &= \frac{3}{2\sqrt{\zeta}} \begin{pmatrix} 1 & 4i \\ -4i & 1 \end{pmatrix},
\end{align*}
\]

\(\lambda \in \partial \tilde{U}\).

Let us summarize the above calculation.

**Proposition 1** The jump matrix \(\Lambda(\lambda)\) of the \(R\)-RH problem possesses the following properties:

1. For sufficiently small \(\delta\) and \(\varepsilon\), the function \(\Lambda\) satisfies the estimates:

\[
|I - \Lambda(\lambda)| \leq \frac{C_\delta}{\rho}, \quad \lambda \in (\partial U \setminus \{1 - \delta\}) \cup (\partial \tilde{U} \setminus \{-1 + \delta\}),
\]

and

\[
|I - \Lambda(x)| \leq \widetilde{C}_{\delta, \varepsilon} \exp(-C_{\delta, \varepsilon} \rho), \quad x \in (-1 + \delta, 1 - \delta),
\]

\(\forall \alpha \in \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_\varepsilon(\pi), \quad s_0 \leq n\).

Here \(\rho = n|\sin(\alpha/2)|\), and \(C_\delta, \widetilde{C}_{\delta, \varepsilon}, \) and \(C_{\delta, \varepsilon}\) are positive constants depending on the indicated quantities only. The number \(s_0\) is any positive number satisfying the inequality \(s_0 > 2/\sqrt{\delta}\). Moreover,

\[
\rho \geq \frac{s_0}{2},
\]

\(\forall \alpha \in \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_\varepsilon(\pi), \quad s_0 \leq n\).

2. The estimate (77) can be extended to the asymptotic series

\[
\Lambda(\lambda) = I + \sum_{j=1}^{k-1} \Lambda_j(\lambda) + \Lambda_r^{(k)}(\lambda), \quad \lambda \in (\partial U \setminus \{1 - \delta\}) \cup (\partial \tilde{U} \setminus \{-1 + \delta\}),
\]

where the terms \(\Lambda_j\) of expansion (80) and the error term \(\Lambda_r^{(k)}(\lambda)\) satisfy the uniform estimates:

\[
|\Lambda_j(\lambda)| \leq \frac{C^{(j)}_{\delta}}{\rho^j}, \quad |\Lambda_r^{(k)}(\lambda)| \leq \frac{C^{(k)}_{\delta}}{\rho^k},
\]

(81)
∀α ∈ \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup D_\varepsilon(\pi), \quad s_0 \leq n.

The positive constants \(C^{(j)}_\delta\), \(j = 1,...,k\) are the functions of \(\delta\) only. The first two terms of the expansion (80), i.e. the functions \(\Lambda_1\) and \(\Lambda_2\) are given by equations (74) if \(\lambda \in \partial U \setminus \{1 - \delta\}\), and by equations (76) if \(\lambda \in \partial \tilde{U} \setminus \{-1 + \delta\}\).

3. Let \(\Lambda_u\), \(\Lambda_d\), and \(\Lambda_l\) denote the limits of \(\Lambda(\lambda)\) as \(\lambda\) approaches the node point \(1 - \delta\) from the above, from the below, and from the left along \(\Sigma\), respectively. Then these limits exist, and the following cyclic equation holds:

\[\Lambda_d \Lambda_l \Lambda_u^{-1} = I.\] (82)

A similar relation (with \(\Lambda_l\) replaced by \(\Lambda_r\)) holds at the node point \(-1 + \delta\).

4. The matrix function \(\Lambda(\lambda)\) admits an analytic continuation into a neighborhood of any point of the interval \((-1 + \delta, 1 - \delta)\). Moreover, this analytic continuation preserves the estimate (78) with constants \(\tilde{C}_{\delta,\varepsilon}, C_{\delta,\varepsilon}\) possibly somewhat modified.

The only statements which need comments are the statements #3 and #4. These statements follow directly from the explicit formulae (71 - 73) for the jump matrix \(\Lambda(\lambda)\).

**Corollary 1** The following inequalities hold:

\[||I - \Lambda||_{L^2(\Sigma) \cap L^\infty(\Sigma)} \leq \frac{C^{(1)}_{\delta,\varepsilon}}{\rho},\] (83)

\[||I + \sum_{j=1}^{k-1} \Lambda_j - \Lambda||_{L^2(\Sigma) \cap L^\infty(\Sigma)} \leq \frac{C^{(k)}_{\delta,\varepsilon}}{\rho^k},\] (84)

\[\forall \alpha \in \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup D_\varepsilon(\pi), \quad s_0 \leq n,\]

where we set \(\Lambda_j \equiv 0\) for \(\lambda \in (-1 + \delta, 1 - \delta)\).

By standard arguments of the \(L^2\) RH theory (see e.g. [7, 22]), the inequality (83) implies the solvability of the R-RH problem for sufficiently large \(s_0\). Moreover, let \(\Omega_k, k = 1, 2, 3\) denote the connected components of the set \(\mathbb{C} \setminus \Sigma\). Then, due to the cyclic relation (82), the restriction \(R|_{\Omega_k}(\lambda)\) is continuous in \(\Omega_k\) for each \(k\) (see, e.g., [23]).

To complete the proof of the theorem, we need to show that the solution \(R(\lambda)\) of the R-RH problem satisfies the estimates indicated in (83).
Lemma 1 For sufficiently small $\delta$ and $\varepsilon$, and for every $k$, the function $R(\lambda)$ admits the asymptotic representation,

$$R(\lambda) = I + \sum_{j=1}^{k-1} R_j(\lambda) + R^{(k)}(\lambda),$$

(85)

where

$$R_j(\lambda) = O\left(\frac{1}{(1 + |\lambda|)^\rho}\right), \quad R^{(k)}(\lambda) = O\left(\frac{1}{(1 + |\lambda|)^\rho^k}\right), \quad \rho \equiv n|\sin \frac{\alpha}{2}| \to \infty,$$

(86)

uniformly for all $\lambda \in \Omega^{(25)}$ and $\alpha \in \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_e(\pi)$. As in the Remark 1 to Theorem 2, the latter statement means that there exist positive constants $C$ and $s_0$ such that

$$|R_j(\lambda)| \leq \frac{C}{(1 + |\lambda|)^\rho}, \quad |R^{(k)}(\lambda)| \leq \frac{C}{(1 + |\lambda|)^\rho^k},$$

(87)

$$\forall \lambda \in \Omega^{(25)}, \quad \forall \alpha \in \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_e(\pi), \quad \forall n : \ s_0 \leq n.$$

The functions $R_j(\lambda)$ are constructed by induction as follows:

$$R_1(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} \Lambda_1(s) \frac{ds}{s - \lambda}, \quad R_2(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} (R_1(s)\Lambda_1(s) + \Lambda_2(s)) \frac{ds}{s - \lambda},$$

(88)

$$\ldots, \quad R_{k-1}(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} \sum_{j=1}^{k-1} R_{k-1-j,-}(s)\Lambda_j(s) \frac{ds}{s - \lambda}, \quad R_0 \equiv I.$$

(89)

Remark 4 We also assume (cf. Remark 1 to Theorem 2) that $\varepsilon$ is small enough so that $\rho \geq s_0/2$ for all $\alpha \in \left[\frac{2s_0}{n}, \pi - \varepsilon\right] \cup \mathcal{D}_e(\pi)$ and $n \geq s_0$.

The proof of the lemma is essentially a combination of the arguments from [21] and [27]. We consider in detail the case of $k = 3$, which is all that is needed below, but the argument extends in an obvious way to any $k = 1, 2, \ldots$. The details are left to the interested reader.

Write the jump condition for $R(\lambda)$ in the form

$$R_{0+} + R_{1+} + R_{2+} + R_{r+} = (R_{0-} + R_{1-} + R_{2-} + R_{r-})(I + \Lambda_1 + \Lambda_2 + \Lambda_r).$$

(90)

Here $\Lambda_1, \Lambda_2$ are given by (74, 76) on $\partial U, \partial \tilde{U}$, and we set $\Lambda_1 = \Lambda_2 = 0$ on $(-1 + \delta, 1 - \delta)$. Thus $\Lambda_r = O(1/\rho^3)$ on $\partial U \cup \partial \tilde{U}$ (this error term arises from the Bessel asymptotics: see (81)), and $\Lambda_r = O(e^{-C_3\rho})$ on $(-1 + \delta, 1 - \delta)$. We now show that we can define $R_1$ and $R_2$ so that they are of order $1/\rho$ and $1/\rho^2$, respectively. We then show that the remainder $R_r$ is of order $1/\rho^3$. Set

$$R_0 = I.$$
We define \( R_j \) by collecting in (90) the terms that we want to be of the same order. First,

\[
R_1^+(\lambda) = R_1^-(\lambda) + \Lambda_1(\lambda), \quad \lambda \in \Sigma.
\]

(91)

We are looking for a function \( R_1(\lambda) \), which is holomorphic outside \( \Sigma \), satisfying \( R_1(\lambda) = O(1/\rho) \), \( \lambda \to \infty \), and the above jump condition. The solution to this RH-problem is given by the Sokhotsky-Plemelj formula,

\[
R(\lambda) = C(\Lambda_1),
\]

(92)

where

\[
C(f) = \frac{1}{2\pi i} \int_{\Sigma} f(s) \frac{ds}{s - \lambda}
\]

is the Cauchy operator on \( \Sigma \). The condition \( \Lambda_1(\lambda) = O(1/\rho) \), \( \lambda \in \Sigma, \rho \to \infty \) (uniform in \( \alpha \)), implies that there exist \( c, d_0, s_0 > 0 \) such that

\[
|R_1(\lambda)| \leq \frac{c}{(1 + |\lambda|)\rho}, \quad n \geq s_0,
\]

(93)

uniformly in \( \alpha \in [\frac{2n}{n}, \pi - \varepsilon] \cup D_\varepsilon(\pi) \) and \( \lambda \) satisfying \( \text{dist}(\lambda, \Sigma) \geq d_0 \). Actually, this estimate is uniform for all \( \lambda \in \mathbb{C} \setminus \Sigma \) up to \( \Sigma \). This can be shown either by direct calculation (see below) or by suitably deforming the contour \( \Sigma \). Indeed, since

\[
R_1(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} \Lambda_1(s) \frac{ds}{s - \lambda},
\]

(94)

the estimate (93) holds for \( \lambda \) up to the interval \( (-1 + \delta', 1 - \delta') \), for any \( \delta' > \delta \). Since \( \Lambda_1(\lambda) \) is analytic in the neighborhood of \( \partial U \cup \partial \tilde{U} \) (as, actually, are \( \Lambda_j(\lambda) \) for all \( j \)), the contour of integration in (94) can be deformed so that the estimate holds up to \( \partial U \cup \partial \tilde{U} \) as well. It also should be observed that, by the same deformation of the contour of integration in (94), one obtains analytic continuations of both the functions \( R_1^+(\lambda) \) and \( R_1^-(\lambda) \) in the neighborhood of the contour \( \partial U \cup \partial \tilde{U} \). Moreover, this analytic continuation preserves the estimate (93).

Now define \( R_2(\lambda) \) by the jump condition

\[
R_2^+(\lambda) = R_2^-\left(\lambda\right) + R_1^-(\lambda)\Lambda_1(\lambda) + \Lambda_2(\lambda), \quad \lambda \in \Sigma,
\]

(95)

together with the requirement of analyticity for \( \lambda \in \mathbb{C} \setminus \Sigma \), and the condition \( R_2(\lambda) = o(1) \) for \( \lambda \to \infty \). The solution to this RH-problem is

\[
R_2(\lambda) = C(R_1^- \Lambda_1 + \Lambda_2) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} \left( R_1^-(s)\Lambda_1(s) + \Lambda_2(s) \right) \frac{ds}{s - \lambda}, \quad \lambda \in \mathbb{C} \setminus (\partial U \cup \partial \tilde{U}).
\]

(96)

Using (93), the estimates \( \Lambda_j = O(1/\rho^j) \), and the analyticity of \( R_1^- \) and \( \Lambda_j \) in the neighborhood of \( \partial U \cup \partial \tilde{U} \), we obtain in the same way as for \( R_1 \): for some \( c > 0 \)

\[
|R_2(\lambda)| \leq \frac{c}{(1 + |\lambda|)\rho^2}, \quad \lambda \in \mathbb{C} \setminus \Sigma, \quad n \geq s_0
\]

(97)
with the same uniformity and analyticity properties in \( \alpha \) and \( \lambda \). Below in the proof, the same symbol \( c \) will stand for various constants independent of \( \alpha \), \( \lambda \), and \( n \).

Now from (90,91,95) we obtain

\[
R_r(\lambda) = M(\lambda) + R_r(\lambda)\Lambda(\lambda), \quad \lambda \in \Sigma, \tag{98}
\]

where

\[
M \equiv R_2 - \Lambda_1 + (R_1 + R_2)\Lambda_2 + (I + R_1 + R_2)\Lambda_r.
\]

**Remark** In the terminology of [24], equation (98) is an inhomogeneous RH-problem of type 2.

Since \( R_r = R - I - R_1 - R_2 \), the matrix function \( R_r(\lambda) \) is holomorphic outside \( \Sigma \) and satisfies the condition \( R_r(\lambda) = o(1) \) as \( \lambda \to \infty \). Therefore,

\[
R_r(\lambda) = C(M) + C(R_r(\lambda - I)), \quad \lambda \in \mathbb{C} \setminus \Sigma. \tag{99}
\]

(It is worth mentioning that, by virtue of property \# 3 of the jump matrix \( \Lambda(\lambda) \) formulated in proposition 1, equation (99) is consistent with the absence of the singularities of the function \( R_r(\lambda) \) at the node points \( 1 - \delta \) and \(-1 + \delta \).) Equation (99), in turn, implies that

\[
R_r(\lambda) = C_-(M) + C_-(R_r(\lambda - I)), \quad \lambda \in \Sigma, \tag{100}
\]

where \( C_-(f) = \lim_{\lambda' \to \lambda} C(f) \), as \( \lambda' \) approaches a point \( \lambda \in \Sigma \) from the “−” side of \( \Sigma \). Now defining the operator

\[
C_\Lambda(f) \equiv C_-(f(\Lambda - I)),
\]

we represent (100) in the form

\[
(I - C_\Lambda)(R_r) = C_-(M). \tag{101}
\]

Because of the \( L^\infty \) part of the estimate (84), and the fact that \( C_- \) is a bounded operator from \( L^2(\Sigma) \) to \( L^2(\Sigma) \), it follows that the operator norm \( ||C_\Lambda|| = O(1/\rho) \), and hence \( I - C_\Lambda \) is invertible by Neumann series for \( s_0 \) (and therefore \( \rho \)) sufficiently large. Thus (101) gives

\[
R_r = (I - C_\Lambda)^{-1}(C_-(M)). \tag{102}
\]

Moreover, using the \( L^2 \) part of the estimate (84), we conclude that \( ||C_-(M)||_{L^2(\Sigma)} = O(\rho^{-3}) \). Together with (102), this yields the uniform estimate,

\[
||R_r||_{L^2(\Sigma)} \leq \frac{c}{\rho^3}, \tag{103}
\]

\[
\forall \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup D_\varepsilon(\pi), \quad n \geq s_0.
\]

Combining the estimate (103) with equation (99), we can complete the proof of the lemma as follows.
First, assuming that \( \text{dist}(\lambda, \Sigma) \geq d_0 \), we immediately arrive at the estimate
\[
|C(M)(\lambda)| \leq \frac{c}{(1 + |\lambda|)\rho^3}, \quad n \geq s_0,
\]
for the first term in the r.h.s. of (99), and the estimate
\[
|C(R_r - (\Lambda - I))(\lambda)| \leq \frac{c}{(1 + |\lambda|)\rho^4}, \quad n \geq s_0,
\]
for the second term. Both the estimates are uniform in \( \alpha \in \left[ \frac{2s_0}{n}, \pi - \varepsilon \right] \cup D_\varepsilon(\pi) \). Together they yield the estimate
\[
|R_r(\lambda)| \leq \frac{c}{(1 + |\lambda|)\rho^3}, \quad n \geq s_0,
\]
uniformly in \( \alpha \in \left[ \frac{2s}{n}, \pi - \varepsilon \right] \cup D_\varepsilon(\pi) \) and \( \lambda \) satisfying \( \text{dist}(\lambda, \Sigma) \geq d_0 \).

Second, we observe that the matrix \( \Lambda_r(\lambda) \) coincides with the matrix \( \Lambda(\lambda) - I \) on the interval \((-1 + \delta, 1 - \delta)\). Hence, by property \# 4 of the matrix function \( \Lambda(\lambda) \) (see proposition 1), the matrix function \( \Lambda_r(\lambda) \) admits an analytic continuation in the neighborhood of any point of the interval \((-1 + \delta, 1 - \delta)\), and this continuation preserves the estimate, \( \Lambda_r = O(e^{-C\varepsilon\rho}) \).

This means that, by bending the segment \((-1 + \delta, 1 - \delta)\) of the contour \( \Sigma \) we can extend \( \lambda \) in the estimate (104) up to the interval \((-1 + 2\delta, 1 - 2\delta)\). Using property \# 4 of the jump matrix \( \Lambda(\lambda) \) one more time, we can rewrite the second term in the r.h.s. of equation (99) as
\[
C(R_r - (\Lambda - I))(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} R_r(s)(\Lambda(s) - I) \frac{ds}{s - \lambda} + \\
\frac{1}{2\pi i} \int_{\gamma(d)} R_r(s)(\Lambda(s) - I) \frac{ds}{s - \lambda},
\]
if \( \lambda \) lies above the interval \((-1 + 2\delta, 1 - 2\delta)\), and as
\[
C(R_r - (\Lambda - I))(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} R_r(s)(\Lambda(s) - I) \frac{ds}{s - \lambda} + \\
\frac{1}{2\pi i} \int_{\gamma(u)} (R_r(s) - M(s))(I - \Lambda^{-1}(s)) \frac{ds}{s - \lambda},
\]
if \( \lambda \) lies below the interval \((-1 + 2\delta, 1 - 2\delta)\). Here, the contours \( \gamma^{(d)} \) and \( \gamma^{(u)} \) are the slight deformations of the segment \((-1 + \delta, 1 - \delta)\) down and up, respectively. Using, in representations (107) and (108), the estimate (105) for \( R_r(\lambda) \), we extend the variable \( \lambda \) in the estimate (105) up to the interval \((-1 + 2\delta, 1 - 2\delta)\).

The above extensions of the estimates (104) and (105) mean, in particular, that they both, and hence the estimate (106), are valid for all \( \lambda \in \Omega^{(2\delta)} \). The proof of the lemma is completed. \( \square \)
We now derive explicit formulae for the terms $R_1(\lambda)$ and $R_2(\lambda)$ of the expansion \(85\). By Lemma 1,

\[
R_1(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} \frac{\Lambda_1(x)dx}{x - \lambda}, \quad R_2(\lambda) = \frac{1}{2\pi i} \int_{\partial U \cup \partial \tilde{U}} \frac{R_1(x)\Lambda_1(x) + \Lambda_2(x)}{x - \lambda}dx \quad (109)
\]

As noted in [27], we can also obtain the expressions for $R_j(\lambda)$ in the following way. It is not difficult to check that $\Lambda_1(\lambda)$ and $\Lambda_2(\lambda)$ are analytic in $(U \setminus \{1\}) \cup (\tilde{U} \setminus \{-1\})$ with the simple poles at $\pm 1$. We have

\[
\Lambda_1(\lambda) = \frac{A^{(1)}}{\lambda - 1} + O(1), \quad \text{as } \lambda \to 1, \quad \Lambda_1(\lambda) = \frac{B^{(1)}}{\lambda + 1} + O(1), \quad \text{as } \lambda \to -1, \quad (110)
\]

where the constant matrices $A^{(1)}$ and $B^{(1)}$ are obtained by expanding $\omega(\lambda)$ and $\beta(\lambda)$ in \(74, 76\) at $\lambda = \pm 1$. It is easy to verify directly that the Riemann-Hilbert problem for $R_1(\lambda)$ has the solution:

\[
R_1(\lambda) = \begin{cases} 
\frac{A^{(1)}}{\lambda - 1} + \frac{B^{(1)}}{\lambda + 1}, & \text{for } \lambda \in \mathbb{C} \setminus (U \cup \tilde{U}) \\
\frac{A^{(1)}}{\lambda - 1} + \frac{B^{(1)}}{\lambda + 1} - \Lambda_1(\lambda), & \text{for } \lambda \in U \cup \tilde{U}.
\end{cases} \quad (111)
\]

Using the series \(54\) and the expansion of $\beta(\lambda)$ at $\pm 1$, it is not difficult to obtain the singular and constant term in the Laurent expansion of $\Lambda_1(\lambda)$. By the first formula in \(111\), we obtain (using the singular term) the expression \(39\).

Similarly we may calculate the singular term in the expansion of $\Lambda_2(\lambda)$ at $\pm 1$, and use the second formula in \(111\) to evaluate $R_1(\pm 1)$ (note that the formula \(39\) is valid only outside $U \cup \tilde{U}$). It is then easy to compute the integral for $R_2$ in \(109\) and obtain \(40\). This completes the proof of the theorem.$\blacksquare$.

Now we give some remarks and corollaries of Theorem 2.

**Remark 5** Estimate \(37\) and formula \(36\) imply that

\[
\Theta(n, \alpha) \sim \cos \frac{\alpha}{2}, \quad \Delta(n, \alpha) \sim \sin^2 \frac{\alpha}{2},
\]

and using either \(83\) or \(41\) we recover the master term of Widom’s asymptotics \(4\) (cf. also \(7\)),

\[
\ln D_n(\alpha) \sim n^2 \ln \cos \frac{\alpha}{2}, \quad n \to \infty.
\]

**Corollary 2** The function $\Delta(n, \alpha)$ admits the asymptotic expansion

\[
\Delta(n, \alpha) = \sin^2 \frac{\alpha}{2} - \frac{\cos^2(\alpha/2)}{4n^2} + O\left(\frac{1}{\rho^3}\right) \sin^2 \alpha, \quad \rho \to \infty, \quad (112)
\]
which is uniform for \( \alpha \in \left[ \frac{2s_0}{n}, \pi \right] \).

**Remark 6** The statement,

\[
\Delta_r(n, \alpha) = O \left( \frac{1}{\rho^3} \right) \sin^2 \alpha, \quad \rho \to \infty
\]

uniformly for \( \alpha \in \left[ \frac{2s_0}{n}, \pi \right] \), means that there exist positive constants \( C \) and \( s_0 \), such that

\[
|\Delta_r(n, \alpha)| \leq \frac{C}{\rho^3} \sin^2 \alpha,
\]

\( \forall \alpha \in \left[ \frac{2s_0}{n}, \pi \right], \) and \( s_0 \leq n \).

**Proof of Corollary 2.** To calculate \( \Delta \) we need the asymptotics of \( \Phi(\lambda) \) outside the neighborhoods \( U \) and \( \tilde{U} \). By (38) these are given by the expression:

\[
\Phi(\lambda) = (I + R_1 + R_2 + R_r^{(3)})N, \quad \lambda \in \Omega^{(24)}
\]

where \( R_r^{(3)} \) is estimated by (87) for \( k = 3 \). In particular, the estimate (87) becomes

\[
O(\rho^{-3}) \sin \frac{\alpha}{2} \quad \text{if} \quad \lambda = \pm i \cot \frac{\alpha}{2}.
\]

Similarly,

\[
R_j(\pm i \cot \alpha/2) = O(\rho^{-j}) \sin \frac{\alpha}{2}, \quad j = 1, 2, \ldots
\]

Since

\[
N(\pm i \cot \alpha/2) = \begin{pmatrix} \cos(\alpha/4) & \pm \sin(\alpha/4) \\ \mp \sin(\alpha/4) & \cos(\alpha/4) \end{pmatrix},
\]

we have \( N(-i \cot \alpha/2)^{-1} = N(i \cot \alpha/2) = O(1) \) and \( [N(-i \cot \alpha/2)^{-1}N(i \cot \alpha/2)]_{12} = \sin \alpha/2 \). Definition (35) and equations (114,115,116) then imply

\[
\Delta(n, \alpha) = \left[ N(-i \cot \alpha/2)^{-1} \left( I + (O(\rho^{-1}) + O(\rho^{-2}) + O(\rho^{-3})) \sin \frac{\alpha}{2} \right) \right. \times \\
\left( I + (O(\rho^{-1}) + O(\rho^{-2}) + O(\rho^{-3})) \sin \frac{\alpha}{2} \right) N(i \cot \alpha/2) \right]_{12}^2 =
\]

\[
= \Delta_0(\alpha) + \frac{1}{n} \Delta_1(\alpha) + \frac{1}{n^2} \Delta_2(\alpha) + \frac{f(\alpha, n)}{\rho^3} \sin^2(\alpha/2),
\]

where

\[
\Delta_0(\alpha) = \sin^2 \frac{\alpha}{2},
\]

and \( f(\alpha, n) \) is uniformly bounded for \( \alpha \in [2s_0/n, \pi - \varepsilon] \cup D_\varepsilon(\pi) \), and \( s_0 \leq n \). Note that to write (118) we used the fact that \( \det R(\lambda) = 1 \).
In order to determine the terms $\Delta_1(\alpha)$ and $\Delta_2(\alpha)$ in this equation we need $R_{1,2}(\pm i \cot \alpha/2)$. These values we obtain from (39-40):

$$R_1(\pm i \cot \alpha/2) = \pm \frac{1}{8n} \begin{pmatrix} -\cos(\alpha/2) & \pm \sin(\alpha/2) \\ \pm \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix},$$

$$R_2(\pm i \cot \alpha/2) = \pm \frac{1}{2^{\sharp}n\rho} \begin{pmatrix} \pm \sin(\alpha/2) & -8 \cos(\alpha/2) \\ 8 \cos(\alpha/2) & \pm \sin(\alpha/2) \end{pmatrix}.$$

As $\det \Phi(\lambda) = 1$, the inverse $\Phi^{-1}$ is easy to compute, and after a simple computation we arrive at the equations,

$$\Delta_1(\alpha) = 0, \quad \Delta_2(\alpha) = -\frac{1}{4} \cos^2(\alpha/2).$$

Now it only remains to show that the function $f(\alpha, n)$ in (119) satisfies the estimate

$$|f(\alpha, n)| \leq C \cos^2(\alpha/2),$$

for $\alpha \in [2s_0/n, \pi]$, and $s_0 \leq n$. In fact, since the uniform boundedness of $f(\alpha, n)$ on the sets indicated has already been established, it is enough to show that estimate (124) holds for all $\alpha \in D_{\varepsilon/4}(\pi)$.

Observe that, for fixed $n$, the quantity $\Delta(n, \alpha)$ is an analytic function at $\alpha = \pi$ with $\Delta(n, \pi) = 1$ and $(d/d\alpha)\Delta(n, \pi) = 0$ (see (17) and (34)) so that we can write down the Taylor series for $\Delta$ in $\beta = \pi - \alpha$:

$$\Delta(\alpha) = 1 + (\alpha - \pi)^2 a_2 + \cdots.$$

As follows from equation (119), $f(\alpha, n)$ is a holomorphic function of $\alpha$ in $D_{\varepsilon}(\pi)$. Using a representation of $f(\alpha, n)$ by a Cauchy integral, we obtain:

$$f(\alpha, n) = \frac{1}{2\pi i} \int_{\partial D_{\varepsilon/2}(\pi)} \frac{f(\tilde{\alpha}, n)}{\tilde{\alpha} - \alpha} d\tilde{\alpha} + \frac{(\alpha - \pi)}{2\pi i} \int_{\partial D_{\varepsilon/2}(\pi)} \frac{f(\tilde{\alpha}, n)}{(\tilde{\alpha} - \pi)^2} d\tilde{\alpha} + \frac{(\alpha - \pi)^2}{2\pi i} \int_{\partial D_{\varepsilon/2}(\pi)} \frac{f(\tilde{\alpha}, n)}{(\tilde{\alpha} - \pi)^3(\tilde{\alpha} - \alpha)} d\tilde{\alpha}, \quad |\pi - \alpha| < \varepsilon/4.$$

At the same time, from (119), (125), and (123) it follows that the Taylor series of $f(\alpha, n)$ at $\alpha = \pi$ has the form,

$$f(\alpha, n) = (\alpha - \pi)^2 a_2 + \cdots$$

Therefore, the first two integrals in the r.h.s. of (126) must be zero, and the third one, by virtue of the uniform boundedness of $f(\alpha, n)$ for all $\alpha \in D_{\varepsilon}(\pi)$ and all $n \geq s_0$, yields the estimate (124) for all $\alpha \in D_{\varepsilon/4}(\pi)$ and all $n \geq s_0$. The proof of the corollary is completed.

**Remark 7** Here is an alternative derivation of the leading terms in formula (112).
We start with equations (119) and (120). The issue is the exact evaluation of the quantities \( \Delta_1(\alpha) \) and \( \Delta_2(\alpha) \). This can be done with the help of the relevant (integrable) differential system associated in the standard way with the original \( m \)-RH problem. Indeed, it is shown in [7] that the Toeplitz determinant \( D_n(\alpha) \), considered as the function of the variable 

\[
t = e^{-2i\alpha}
\]

is the \( \tau \)-function for the Painlevé VI equation characterized by the parameters

\[
\theta_\infty = -\theta_0 = n, \quad \theta_1 = \theta_t = 0,
\]

where we use the \( \theta \)-notations of Jimbo, see [28]. According to [28], this means that the quantity

\[
\eta(t) \equiv t(t - 1) \frac{d}{dt} \ln D_n
\]

satisfies the following nonlinear differential equation (the \( \tau \)-form of Painlevé VI):

\[
\left( \frac{d\eta}{dt} - \frac{n^2}{4} \right) \left( t(t - 1) \frac{d^2\eta}{dt^2} \right)^2 + \left[ 2 \left( \frac{d\eta}{dt} - \frac{n^2}{4} \right) \left( t \frac{d\eta}{dt} - \eta \right) - \left( \frac{d\eta}{dt} \right)^2 + \frac{n^2}{2} \frac{d\eta}{dt} \right]^2 = \left( \frac{d\eta}{dt} \right)^4 .
\]

(128)

The functions \( \Delta(n, \alpha) \) and \( \eta(t) \equiv \eta(n, t) \) are related by the equation

\[
\Delta = \frac{1 - t}{n^2} \frac{d\eta}{dt} + \frac{1}{n^2} \eta,
\]

(129)

and we may anticipate an expansion for \( \eta \) similar to (112). Indeed we expect

\[
\eta(t) \equiv \eta(n, t) = n^2 \eta_0(t) + n \eta_1(t) + \eta_2(t) + O \left( \frac{1}{\rho} \right), \quad \rho \to \infty,
\]

(130)

where

\[
\eta_0(t) = \frac{1}{4} (1 - \sqrt{t})^2.
\]

(131)

A substitution of the asymptotics (130) into the equation (128) gives us, after a straightforward calculation, the following formulae for the coefficient functions \( \eta_1(t) \) and \( \eta_2(t) \):

\[
\eta_1(t) \equiv 0, \quad \eta_2(t) = -\frac{1}{16} (1 + \sqrt{t})^2.
\]

(132)

These equations together with (129) lead immediately to the leading terms in the formula (112).

It also should be noticed that the differentiability of the asymptotics (119) follow from its uniformity in the disk \( D_\varepsilon(\pi) \).
4 Asymptotic evaluation of $D_n(\alpha)$. Proof of estimate (12).

The asymptotic evaluation of the Toeplitz determinant $D_n(\alpha)$ is based on the integration of the differential identity (34) from $\alpha$ to $\alpha_0$ (which is close to $\pi$ from below). We have:

$$(\alpha_0 - \alpha)(\ln D_n)'(\alpha_0) - \ln D_n(\alpha_0) + \ln D_n(\alpha) = -n^2 \int_{\alpha}^{\alpha_0} d\theta \int_{\theta}^{\alpha_0} \frac{\Delta(\phi)}{\sin^2 \phi} d\phi. \quad (133)$$

Fix $n$ and set $\alpha_0 = \pi - \beta$. Substituting for $\ln D_n(\pi - \beta)$ the expansion (17), and for $\Delta(\phi)$ the asymptotics (112), and after taking the limit $\beta \to 0$, we immediately obtain (12) with the remainder $O(1/\{n \sin(\alpha/2)\})$ uniformly for $\frac{2n}{\pi} \leq \alpha \leq \pi - \varepsilon$, $n \geq s_0$, $\varepsilon > 0$.

Acknowledgements.

Percy Deift was supported in part by NSF grants # DMS-0296084 and # DMS 0500923. Alexander Its was supported in part by NSF grants # DMS-0099812 and # DMS-0401009. Xin Zhou was supported in part by NSF grant # DMS-0071398.

References

[1] M. L. Mehta: Random matrices. San Diego: Academic 1990

[2] J. des Cloizeaux and M. L. Mehta, Asymptotic behavior of spacing distributions for the eigenvalues of random matrices, J. Math. Phys. 14, 1648–1650 (1973)

[3] F. Dyson, Fredholm determinants and inverse scattering problems. Commun. Math. Phys. 47, 171–183 (1976)

[4] H. Widom, The strong Szegö limit theorem for circular arcs. Indiana Univ. Math. J. 21, 277–283 (1971)

[5] H. Widom, The asymptotics of a continuous analogue of orthogonal polynomials. J. Approx. Th. 77, 51–64 (1994)

[6] H. Widom, Asymptotics for the Fredholm determinant of the sine kernel on a union of intervals, Comm. Math. Phys. 171, 159–180 (1995)

[7] P. Deift, A. Its, and X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics. Ann. Math 146, 149–235 (1997)
[8] I. V. Krasovsky, Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle. *Int. Math. Res. Not.* **2004**, 1249–1272 (2004)

[9] T. Ehrhardt, Dyson’s constant in the asymptotics of the Fredholm determinant of the sine kernel. arXiv.org:math.FA/0401205

[10] E. L. Basor, T. Ehrhardt, On the asymptotics of certain Wiener-Hopf-plus-Hankel determinants. arXiv.org:math.FA/0502039

[11] E. L. Basor, C. A. Tracy, Some problems associated with the asymptotics of \( \tau \)-functions. Surikagaku (Mathematical Sciences) **30**, no. 3, 71–76 (1992) [English translation appears in RIMS-845 preprint]

[12] A. M. Budylin, V. S. Buslaev, Quasiclassical asymptotics of the resolvent of an integral convolution operator with a sine kernel on a finite interval. (Russian) Algebra i Analiz **7**, no. 6, 79–103 (1995); translation in St. Petersburg Math. J. **7**, no. 6, 925–942 (1996)

[13] J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, *J. Amer. Math. Soc.* **12**, No. 4, 1119–1178 (1999)

[14] P. A. Deift, Integrable systems and combinatorial theory, *Notices of the Ammer. Math. Soc.*, **47** (6), 631–640 (2000)

[15] P. Deift, A. Its, I. Krasovsky, Asymptotics of the Airy-kernel determinant. In preparation.

[16] P. Deift, Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Math. 1998

[17] G. Szegő, Orthogonal polynomials. AMS Colloquium Publ. **23**. New York: AMS 1959

[18] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problem. *Ann. Math.* **137**, 295–368 (1993)

[19] P. Deift and X. Zhou, Asymptotics for the Painlev II equation. *Comm. Pure Appl. Math.* **48**, 277–337 (1995)

[20] P. Deift, S. Venakides, and X. Zhou, New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems. *Int. Math. Res. Not.* **1997**, 286–299 (1997).

[21] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, X. Zhou, Strong asymptotics for orthogonal polynomials with respect to exponential weights. *Commun. Pure Appl.Math.* **52**, 1491–1552 (1999)
[22] P. Deift and X. Zhou, A priori $L^p$ estimates for solutions of Riemann-Hilbert problems, *Int. Math. Res. Noties*, **40**, 2121–2154 (2002).

[23] R. Beals, P. Deift, C. Tomei, Direct and inverse scattering on the line. Mathematical Surveys and Monographs, **28**. AMS, Providence, RI, 1988.

[24] P. Deift and X. Zhou, Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. *Comm. Pure Appl. Math.* **56**, 1029–1077 (2003).

[25] G. S. Litvinchuk and I. M. Spitkovskii, Factorization of measurable matrix functions, Birkhäuser, 1987

[26] X. Zhou, The Riemann-Hilbert problem and inverse scattering, *SIAM J. Math. Anal.* **20**, No. 4, 966–986 (1989)

[27] A. B. J. Kuijlaars, K. T-R McLaughlin, W. Van Assche, M. Vanlessen, The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1,1]$. *Adv. Math.* **188**, 337–398 (2004)

[28] M. Jimbo, Monodromy problem and the boundary condition for some Painleve equations, Publ. RIMS, Kyoto University 18, 1137-1161 (1982)