Internal structure of Einstein-Yang-Mills-Dilaton black holes

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We study the interior structure of the Einstein-Yang-Mills-Dilaton black holes as a function of the dilaton coupling constant $\gamma \in [0, 1]$. For $\gamma \neq 0$ the solutions have no internal Cauchy horizons and the field amplitudes follow a power law behavior near the singularity. As $\gamma$ decreases, the solutions develop more and more oscillation cycles in the interior region, whose number becomes infinite in the limit $\gamma \to 0$.

I. The much discussed mass inflation scenario [1] provides a picture of what happens to the unstable Cauchy horizons of the Kerr-Newman family when the back reaction of the blue-shifted perturbations is taken into account. In this scenario the blue-shifted influx, due to the unavoidable gravitational wave tail of a realistic gravitational collapse, causes the local mass function to blow up exponentially near the Cauchy horizon. Thereby, a singularity is produced along the Cauchy horizon, which leads to infinite tidal forces on a free-falling object. This singularity is, however, lightlike and mild, accompanied with a divergence of the conformal curvature, but with bounded shear and expansion. The genericity of the mass inflation picture has been established in a series of analytical and numerical studies of a class of vacuum and electrovacuum solutions [2].

What really happens very close to the singularity along the horizon cannot be decided by classical physics. This is presumably not even a physically relevant question, because the black hole will have evaporated or merged with other black holes in a cosmological big crunch. It may, nevertheless, be of some interest to study the interior black hole solutions for classical field–theoretical matter models which are important in high-energy particle physics.

Such investigations were initiated in ref. [3], where it was shown that for pure Yang-Mills fields a new type of infinitely oscillating behavior with exponentially growing amplitude is developed. It is natural to ask, whether this cyclic repetition with associated mass inflation is generic when other matter fields are included. In [4], [5] it was shown that the interior solution changes qualitatively when a Higgs field is included.

In the present contribution we discuss the interior structure of black holes for the Einstein-Yang-Mills-Dilaton (EYMD) system, considering the dilaton coupling constant, $\gamma$, as a bifurcation parameter of the associated dynamical system. It turns out that for $\gamma = 1$ the behavior is drastically different from that of the EYM system ($\gamma = 0$), in that there are no oscillations; this phenomenon has also been observed in [3]. All fields evolve quite monotonically when the radial Schwarzschild coordinate $r$ (internal time) approaches 0. This has prompted us to study the change of the phase portrait as $\gamma$ decreases from 1 to 0. Our analytical and numerical results show quite convincingly, that with decreasing $\gamma$ more and more cycles are developing, and that the limit $\gamma \to 0$ is quite singular. For $\gamma \neq 0$ the interior solutions have no Cauchy horizons. We believe that this is true quite generically for a large class of nonlinear field–theoretical matter models. Further work on this issue is in progress.
II. The black hole solutions we shall be considering arise within the context of the SU(2) EYMD theory, whose action reads in standard notations

\[ S = \int \left( -\frac{1}{4} R + \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi + \frac{1}{2} e^{2\gamma \Phi} \text{tr} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} \, d^4 x. \] (1)

The dilaton coupling \( \gamma \) is assumed to have values in the interval \([0, 1]\). We are especially interested in the limiting behavior for \( \gamma \to 0 \). When \( \gamma \) vanishes, the dilaton decouples, in which case one can put \( \Phi = 0 \). In the static spherically symmetric case the metric is given by

\[ ds^2 = S^2 N dt^2 - \frac{dr^2}{N} - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \] (2)

and the gauge potential \( A \) for a purely magnetic YM field can be parametrized as

\[ A = w(r)(-T_2 \, d\theta + T_1 \sin \theta \, d\phi) + T_3 \cos \theta \, d\phi, \] (3)

where the group generators \( T_a \) are chosen as \( \tau^a/2i \) (\( \tau^a \) = Pauli matrices). The functions \( w, S, N \) and the dilaton \( \Phi \) depend only on \( r \). It will be sometimes convenient to express \( N \) in terms of the mass function \( m \) as \( N = 1 - 2m/r \). The field equations, corresponding to (1) read

\[ (rN)' + r^2 N \Phi'^2 + U = 1, \]

\[ \left( S N r^2 \Phi' \right)' = \gamma SU, \]

\[ r^2 \left( N S e^{2\gamma \Phi} w' \right)' = S e^{2\gamma \Phi} w(w^2 - 1), \]

\[ S' = S \left( r \phi'^2 + 2 e^{2\gamma \Phi} w'^2 / r \right), \] (4)

where \( U = 2 e^{2\gamma \Phi} (N w'^2 + (w^2 - 1)^2 / 2r^2) \). These equations are invariant under the scale transformations

\[ r \mapsto e^\lambda r, \quad N \mapsto N, \quad S \mapsto e^{-\lambda} S, \quad \Phi \mapsto \Phi + \frac{\lambda}{\gamma}, \quad w \mapsto w. \] (5)

The function \( S \) can be eliminated from the system (4). When \( N, \Phi \) and \( w \) are known, \( S \) can be expressed as

\[ S = \exp \left( -\int_r^\infty (r \Phi'^2 + \frac{2}{r} e^{2\gamma \Phi} w'^2) \, dr \right). \] (6)

For any \( \gamma \in [0, 1] \), equations (4) are known to possess a family of black hole solutions [7], [8]. Close to the event horizon they behave as follows:

\[ N = (1 - V_h)x + O(x^2), \quad \Phi = \Phi_h + \frac{\gamma V_h}{(1 - V_h)} x + O(x^2), \quad w = w_h + \frac{w_h(w_h^2 - 1)}{(1 - V_h)} x + O(x^2), \] (7)

where \( x = (r - r_h)/r_h, \quad V_h = e^{2\gamma \Phi_h}(w_h^2 - 1)/r_h^2 \), and the scaling symmetry (8) can be used to set \( \Phi_h = 0 \). Note that at the event horizon one has \( N'(r_h) > 0 \). The asymptotic behavior at infinity is described by

\[ N = 1 - \frac{2M}{r} + \frac{D^2}{r^2} + O\left(\frac{1}{r^3}\right), \quad \Phi = \Phi_\infty - \frac{D}{r} + O\left(\frac{1}{r^2}\right), \quad w = \pm \left( 1 - \frac{\ell}{r} \right) + O\left(\frac{1}{r^2}\right), \] (8)
where $M$ is the ADM mass and $D$ is the dilaton charge. The black hole solutions with these asymptotics are numerically known in the whole interval $r_h \leq r < \infty$ [7], [8]. Their behavior is qualitatively similar for all values of $\gamma$: the functions $N$ and $\Phi$ are monotone in the exterior region, interpolating between the boundary values given by Eqs. (7), (8), whereas $w$ oscillates within finite bounds. The solutions form a 2-parameter family labeled by $r_h$, and $n$, the number of nodes of $w$ in $[r_h, \infty)$. In what follows we shall consider the extension of these solutions into the interior region of the black hole, $0 \leq r < r_h$. It will turn out that the behavior of the solutions in the interior region changes dramatically with varying $\gamma$.

III. When extending the black hole solutions into the interior region, the following phenomenon is observed: For $\gamma = 0$ (the dilaton is absent) the interior solution is characterized by violent oscillations of the metric coefficients and the gauge field strength [3]. The amplitude of these oscillations tends to infinity as the system approaches the singularity. For $\gamma = 1$ the behavior is rather different, without any peculiar variations (see Fig.1). It is interesting to study more closely what happens when $\gamma$ decreases from 1 to 0. Our analytical and numerical investigations lead to the following qualitative picture: For small values of $\gamma$ the solution in the vicinity of the horizon follows closely that for $\gamma = 0$. The smaller the $\gamma$, the more oscillation cycles the solution exhibits. However, as long as $\gamma \neq 0$, the functions assume a power law behavior in the vicinity of $r = 0$:

$$N \propto -N_1 r^{-(1+p^2)}, \quad \Phi \propto \Phi_1 + p \ln(r), \quad w \propto w_0 - br^{2(1-\gamma p)},$$

with $b, N_1, p > 0$, and the values of $p$ are restricted by

$$\sqrt{\gamma^2 + 1 - \gamma} < p < \frac{1}{\gamma}.$$  

In order to gain some qualitative understanding of this phenomenon, it is helpful to introduce the new variables

$$x = w' e^{\gamma \Phi}, \quad y = -\frac{(w^2 - 1)^2}{r^2 N} e^{2\gamma \Phi}, \quad z = r \Phi'.$$
Figure 2: On the left: solutions of the dynamical system (13) for $\gamma = 0$ and $\gamma = 0.00005$. For $\gamma = 0$ the trajectory is bound to stay in the $z = 0$ plane always spiralling around the center, whereas for $\gamma = 0.00005$ it leaves the plane and converges to the $z$-axis. On the right: the solutions for $\gamma = 0.005$ and $\gamma = 0.020$. For $\gamma = 0.005$, one additional oscillation cycle is performed.

In addition, the following approximations in the differential equations are numerically justified for the interior solutions:

- $w \simeq \text{const.}$ (but $w'$ is kept),
- $e^{2\gamma \Phi} (w^2 - 1)^2 \gg r^2$,
- the term $w(w^2 - 1)/r^2$ in Eqs. (4) can be neglected.

This yields, as an approximate first integral of the field equations,

$$SNe^{2\gamma \Phi} w' \simeq \text{const.}$$

(12)

Eqs. (4) can then be truncated to the following dynamical system:

$$\dot{x} = x(y + \gamma z - 1),$$

$$\dot{y} = -y(2x^2 - y - 1 + z^2 + 2\gamma z),$$

$$\dot{z} = -\gamma(2x^2 - y) + yz,$$

(13)

where a dot stands for differentiation with respect to $t = -\ln(r)$. We are interested in the behavior of the solutions as $t \to \infty$. First of all, we note that the planes $\{x = 0\}$ and $\{y = 0\}$ are invariant sets and hence divide the phase space into four regions. Since $N$ is negative and the equations are symmetric under $x \mapsto -x$, we restrict ourselves to the invariant region $x > 0$, $y > 0$. The critical points in this region are:

1. $x = \alpha \beta$, $y = \alpha \beta$, $z = -\alpha \gamma$, where $\alpha = 1/(1 + 2\gamma^2)$ and $\beta = 1 + \gamma^2$.
2. $x = 0$, $y = 0$, $z = p$, where $p$ is arbitrary.
The eigenvalues corresponding to the first critical point are \( \lambda_1 = \alpha \beta, \lambda_2, \lambda_3 = \alpha \beta (1 \pm i \sqrt{15 + 32 \gamma}) / 2 \), hence it is an attractive center in the limit \( t \to -\infty \). The solutions starting at this point spiral outwards when \( t \) increases. For \( \gamma = 0 \) one has \( z \equiv 0 \) and Eqs. (13) reduce to the 2-dimensional system

\[
\dot{x} = x(y - 1),
\]

\[
\dot{y} = y(-2x^2 + y + 1).
\]

(14)

Now, for this system one can show that the spiraling motion around the focal point can never stop. The trajectories are bound to spiral around the center \((1,1)\) for all \( t \). At the same time, they have to remain in the region \( x > 0 \) and \( y > 0 \). As a result, after each revolution the trajectories come closer and closer to the second critical point, \((0,0)\), which corresponds to the spacetime singularity, but can never be reached. This explains qualitatively the unbounded oscillations for the pure EYM case [3]. It is interesting to observe that the critical point around which the spiralling occurs has no physical meaning by itself. Indeed, it can only be reached for \( t \to -\infty \), which corresponds to \( r \to +\infty \), contradicting the assumption \( r < r_h \).

On the other hand, for \( \gamma \neq 0 \) the solutions can no longer stay in the plane \( \{z = 0\} \) for all \( t \). It turns out that the appearance of the additional degree of freedom completely changes the behavior of the system. When \( \gamma \) is small the trajectory can stay for a long time in the vicinity of the plane spiralling outwards. The smaller \( \gamma \), the smaller \( \dot{z} \) and more revolutions are executed by the trajectory. As long as \( p = z \) is small, the second critical point is repulsive, since the corresponding eigenvalues are \( \lambda_1 = 1 - \gamma p, \lambda_2 = p^2 + 2\gamma p - 1 \), and \( \lambda_3 = 0 \). For small \( p \) one has \( \lambda_1 > 0 > \lambda_2 \). However, when \( p = z \) becomes large enough to fulfill the condition (10), one will have \( \lambda_1 > 0, \lambda_2 > 0 \), and so the critical point \((0,0,p)\) will become attractive. One can also show that it is stable, despite the zero eigenvalue. As a result, having performed only a finite number of oscillations, the trajectory will eventually be attracted and end up at the singularity. The linearized solution near the critical point reproduces the power law behavior given by Eq. (9).

IV. The interior solutions described above do not have inner horizons. It is natural to ask whether this property persists for all values of \( n \) and \( r_h \). For \( \gamma = 0 \) some of the solutions are known to display Cauchy horizons. Specifically, for each \( n > 1 \) there is one distinguished black hole solution with radius \( r_h(n) \), which has an inner horizon at some \( r = r_-(n) \) [3]. It turns out that for \( \gamma \neq 0 \) this can never happen. The proof of this statement is fairly simple and goes as follows: When \( \gamma \neq 0 \), the scaling symmetry (3) implies the existence of the conserved Noether current

\[
J = r^2 \left( \frac{1}{2} SN' + NS' - NS \frac{\Phi'}{\gamma} \right).
\]

(15)

The conservation condition \( J' = 0 \) can be straightforwardly verified with the help of Eqs. (4). Taking the limit \( r \to r_h \), we obtain

\[
J = \frac{r_h^2}{2} S_h N'(r_h) > 0,
\]

(16)

since \( S(r) \) is everywhere positive, and \( N' > 0 \) at the event horizon. Now, assuming that an inner horizon exists at some \( r_- < r_h \), such that \( N(r_-) = 0, N'(r_-) < 0 \), and taking
the limit \( r \to r_\ast \), we arrive at

\[
J = \frac{r^2}{2} S(r_\ast) N'(r_\ast) < 0,
\]  

which contradicts (16). The addition of a dilaton field thus eliminates the Cauchy horizons. We believe that the same is true for a large class of non-linear matter models.

A more detailed account of this work is given in the diploma thesis by OS [9]. MSV thanks the organizers of “The Internal Structure of Black Holes and Spacetime Singularities” workshop at Haifa for an interesting meeting and the generous hospitality.

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