Real and virtual photons in an external constant electromagnetic field of most general form

Anatoly E. Shabad\textsuperscript{1} and Vladimir V. Usov\textsuperscript{2}

\textsuperscript{1} P.N. Lebedev Physics Institute, Moscow 117924, Russia

\textsuperscript{2} Center for Astrophysics, Weizmann Institute, Rehovot 76100, Israel

Abstract

The photon behavior in an arbitrary superposition of constant magnetic and electric fields is considered on most general grounds basing on the first principles like Lorentz- gauge- charge- and parity-invariance. We make model- and approximation-independent, but still rather informative, statements about the behavior that the requirement of causal propagation prescribes to massive and massless branches of dispersion curves, and describe the way the eigenmodes are polarized. We find, as a consequence of Hermiticity in the transparency domain, that adding a smaller electric field to a strong magnetic field in parallel to the latter causes enhancement of birefringence. We find the magnetic field produced by a point electric charge far from it – a manifestation of magneto-electric phenomenon. We establish degeneracies of the polarization tensor that – under special kinematical conditions – occur due to space-time symmetries of the vacuum left after the external field is imposed.

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I. INTRODUCTION

Consideration of a field theory in an external field background is a longstanding challenging physical problem, that attracts attention owing to many other reasons apart from the fact that the astrophysical reality or laser and accelerator technology do propose electromagnetic fields, so strong as to make the nonlinearity of the theory of electromagnetic interaction actual. The cases of external fields of comparatively simple configurations that admit analytical solution to the Dirac equation allow one to go beyond the perturbation theory by including fields of arbitrary strength and thereby consider extreme conditions, under which it is uncertain whether well-probed theories continue to be correct and where even most firmly established fundamental laws may be questioned. For this reason we believe that when approaching such problems it is important to be aware of the status of various encountered facts by clearly distinguishing, which of them reflect basic principles and which depend on an approximation or a model exploited in establishing them.

In this paper we concentrate on an important problem belonging to this class, namely the behavior of real and virtual electromagnetic excitations of the vacuum filled with constant and homogeneous electric ($E$) and magnetic ($B$) fields that are superposed in such a way that the both field invariants $\mathfrak{F} = (B^2 - E^2)/2$ and $\mathfrak{G} = (B \cdot E)$ are different from zero. This makes the most general case of a constant in space and in time electromagnetic field, such that neither electric nor magnetic component can be eliminated from it by a Lorentz transformation, but a special reference frame always exists, where they are parallel or antiparallel. We shall avoid calling all these excitations photons, even when they are real and not virtual, because photons only make a massless subclass of all possible real excitations of the vacuum possessing the same quantum numbers as photons and supplying poles to the same photon Green function.

In principle, the excitations are completely described using the effective Lagrangian, which, when differentiated upon the fields, generates all two- and many-excitation vertices. However, as far as calculations are concerned, the use of the effective Lagrangian is efficient only if its dependence on the space- and time-derivatives of the field strengths is not taken into account. The results obtained in this way relate to the vanishing excitation momentum components $k_\mu \to 0$, $\mu = 1, 2, 3, 0$, and correspond, if extended beyond this infrared limit, to dispersion curves that are straight lines passing through the origin in the plane of the
natural kinematical variables (we refer to the special frame) \( k_0^2 - k_3^2, k^2 \), where \( k_0 \) is the energy of the excitation, and \( k_3 \) and \( k_\perp \) are its momentum projections onto the common direction (chosen as axis 3) of \( E \) and \( B \) and onto the transverse plane, respectively. All massive excitations, i.e. the ones with nonzero rest energy \( k_0|_{k=0} \neq 0 \), are lost in this limit. Within this framework the problem of photon propagation in a constant background, where \( \mathcal{F} \) and \( \mathcal{G} \) are both nonzero, was first studied by Plebański [1], who appealed to a rather general nonlinear Lagrangian independent of the field-derivatives. Among other results, he wrote certain conditions, which the Lagrangian should satisfy lest it might contradict the causal propagation of small-amplitude electromagnetic waves. More recently, Novello et al. [2] observed an interesting possibility that in the infrared limit the constant background field may be geometrized to be represented by an equivalent metric. The present authors, too, paid attention to the infrared limit by proving recently [3] the convexity property of the Lagrangian as a function of the both variables \( \mathcal{F} \) and \( \mathcal{G} \) in the point \( \mathcal{G} = 0 \), basing on causality and unitarity requirements.

To adequately consider the linearized problem of small vacuum excitations the knowledge of the infrared limit is not sufficient, and one is obliged to appeal to the second-rank polarization tensor as a function of arbitrary 4-momentum \( k_\mu \) not restricted to any mass shell, i.e. with arbitrary virtuality \( k^2 \). (To consider the problem beyond the linear approximation higher-rank polarization tensors should be taken also at arbitrary values of all the four momentum components). The needed polarization operator was first studied by Batalin and Shabad [4] (see also the book [5]), who found the general covariant structure and eigenvector expansion (the diagonal form) of the polarization operator and photon Green function in the constant field with the both invariants \( \mathcal{F} \) and \( \mathcal{G} \), taken nonzero simultaneously, that follows exclusively from the Lorentz- gauge- and charge-invariance and parity conservation of Quantum Electrodynamics (QED). They also calculated the polarization operator as an electron-positron loop in the external field of arbitrary strength. Such one-loop calculations were repeated by Bayer et al. and Urrutia [6]. The latter author also studied in more detail the useful further approximation of small external field and zero virtuality, and observed some features, which, as a matter of fact, are independent of this approximation, as well as of the one-loop approximation itself. One-loop polarization operator was revisited in [7] and, under simplifying kinematical conditions, was separately calculated in [8].

The ardor of investigators towards the study of light propagation in the field that contains
electric component in any Lorentz frame was damped by the fact that within the perturbation theory such field is, according to Schwinger, unstable with regard to spontaneous electron-positron pair production. (It is seen from that the imaginary part of the effective Lagrangian on constant fields calculated within one- and two-loop accuracy is nonzero if either \( \Phi < 0, \Sigma = 0 \) or \( \Phi \leq 0, \Sigma \neq 0 \)). A special theory for handling such fields was developed in \([11]\). After exploited in one-loop calculations of the polarization operator in \([12]\) it indicated, with the help of the general analysis in \([13]\), that C-invariance (the Furry theorem) is violated, while PT-invariance (the Onsager theorem) is preserved, hence there is no CPT. Actually, a matrix (namely, \( \Psi_{\mu\nu}^{(5)} \)) in \([12], [13]\)) appears in the decomposition of the one-loop polarization operator, which is odd in the external field and hence incompatible with the C-invariance, but compatible with the Onsager theorem (see Refs. \([13], [5]\) for its formulation fit for the present use), because it is antisymmetric with respect to the indices. On the other hand, the matrix \( \Psi_{\mu\nu}^{(7)} \), also odd in the external field, but symmetric under the permutation of its indices, that might contradict also the Onsager theorem, is not involved. Once the external field instability that contradicts CPT-invariance and is itself a violation of general principles depends on the perturbation theory and is unknown beyond it, it is ignored in the analysis of the present paper based exclusively on these principles.

During the years that followed most efforts were devoted to important special case of one-invariant field with \( \Phi = 0 \) and \( \Phi \geq 0 \) that corresponds to a purely magnetic field in a special Lorentz frame. We shall refer to it as "magnetic-like". The general analysis of Ref. \([4]\) is also valid in this case, whereas the corresponding one-loop polarization operator is contained in the formulae of that work as a simple limit \( \Phi = 0 \), analyzed specifically a bit later in \([14]\). This limiting polarization operator was recalculated separately by Tsai \([15]\). It must be pointed that the one-loop polarization operator in a magnetic-like field for vanishing virtuality, \( k^2 = 0 \), had been known earlier, after the important papers by Adler et al. \([16]\). The simplification \( k^2 = 0 \) is sufficient for considering small dispersion and has been permanently playing a significant role in astrophysical applications. It does not serve, however, the case when large deviations from the vacuum dispersion law take place, as is the case when cyclotron resonances of the vacuum polarization at the thresholds of creations of free or mutually bound electron-positron pairs are exploited to produce the photon capture by a strong magnetic field of pulsars. Besides, the assumption that \( k^2 = 0 \) completely excludes massive states and, moreover, the whole of the one of the three
polarization modes that cannot carry massless excitations.

The reason why the magnetic field attracted so much attention was, of course, the discovery of extremely strong magnetic fields (up to $\sim 10^{14} - 10^{15}$ G) in the vicinity of many compact astronomical objects (soft gamma-ray repeaters, anomalous X-ray pulsars, and some radio pulsars) identified with rotating neutron stars [21]. Still stronger magnetic fields ($B \sim 10^{16} - 10^{17}$ G) were predicted to exist at the surface of cosmological gamma-ray bursters if they are rotation-powered neutron stars similar to radio pulsars [22]. The remarkable feature of these magnetic field strengths is that they are higher than the characteristic field value $B_0 = m_e^2 c^3 / e \hbar \simeq 4.4 \times 10^{13}$ G above which the nonlinearity of QED becomes actual, where $m_e$ and $e$ are the electron mass and its charge, respectively. [Henceforth, we set $\hbar = c = 1$ and refer to the Heaviside-Lorentz system of units.] The compact astronomical objects identified with strongly magnetized neutron stars are powerful sources of electromagnetic radiation, and therefore, propagation of photons in a strong magnetic field is one of the central problems in the theory of these objects. Correspondingly, these problems have been extensively studied aiming at applications to the pulsar physics (for a review, see [23]).

In parallel, some more academic features of nonlinear electrodynamics in a magnetic field were clarified, such as the linear growth of dielectric constant with the magnetic field [24], dimensional reduction of the Coulomb field of a point source [25], and the upper bound to the magnetic field due to positronium collapse [26]. Also the notion of anomalous magnetic moment of the photon was introduced [27], especially interesting in the large-field limit owing to the above linear growth [28].

In the meanwhile little attention has been paid to the admixture of electric field. One of the reasons was that although the electric field is generated along the magnetic field lines in the magnetospheres of rotating, strongly magnetized neutron stars [29], the component $E_\parallel$ in the vicinity of all compact astronomical objects mentioned above is sufficiently small ($E_\parallel / B \ll 1$) [29] as compared to the magnetic field. Hence, at the first sight, $E_\parallel$ could result in only minor corrections [30]. However, it may be not the case at least for some processes that are forbidden without the electric field ($E_\parallel = 0$) and might be allowed at $E_\parallel \neq 0$. Splitting of photons in a strong magnetic field ($\gamma + B \rightarrow \gamma' + \gamma'' + B$) is one of the candidates to be such a process. The point is that in a magnetic-like field splitting of one photon mode is allowed, while splitting of the other is strictly forbidden [16, 31]. As far as we are aware, the polarization selection rules for photon splitting have never been
satisfactorily considered in the case of $E_{\parallel} \neq 0$. Also it is not clear beforehand how will the processes that depend on the resonance behave under inclusion of even small electric field.

On the other hand, extremely strong electric fields with the strength as high as $\sim (10 \div 10^2)E_0$ are predicted to exist at the surface of bare strange stars that are entirely made of deconfined quarks, where $E_0 = m_e^2/e \simeq 1.3 \times 10^{16}$ V/cm is the characteristic electric field value \cite{32}. These electric fields are directed perpendicular to the stellar surface and prevent ultra-relativistic electrons of quark matter from their escape to infinity. The surface magnetic fields are expected to be more or less the same for neutron and strange stars (from $\lesssim 10^9$ G to $\sim 10^{15}$ G or even higher), and therefore, at the surface of strange stars the ratio $E_{\parallel}/B$ may be both $\ll 1$ and $\gg 1$, i.e., it may be practically arbitrary.

We, therefore, feel that the time has come to renew the study of the general combination of constant electric and magnetic field as a strong external background to smaller electromagnetic fields on the base of the recent-years experience acquired when considering the magnetic-like field.

In this paper we are elaborating consequences of the general structure of the polarization operator, established in \cite{4}, for the propagation and polarization of photons and massive vector excitations of the vacuum in the presence of a constant background field with the both field invariants different from zero. Also some observations are made depending on the one-loop approximation. These are exiled to Appendix 2.

In Section II we review properties of the polarization operator and of its three excitation eigenmodes that follow from its diagonal representation, derived in \cite{4} from the fundamental principles in an approximation- and model-independent way. We present a kinematical orthogonal basis and the decomposition of the eigenvectors of the polarization operator over it. The coefficients in this decomposition are expressed in terms of four linear invariant combinations, called $\Lambda_{1,2,4,5}$, of the polarization operator components. We also present the simpler form of this decomposition valid in the limit, where the mixing between the basic vectors is small, like for small admixture of an electric field to a large magnetic one, and under special kinematical conditions. We establish that under this admixture the distance between dispersion curves of two modes that can carry massless excitations (photons) increases in the transparency domain as a consequence of Hermiticity, thus leading to increasing the birefringence and strengthening Adler’s kinematical bans \cite{16} for photon splitting in a magnetic field. We discuss how Adler’s CP-selection rules are modified in the case of the general field
under consideration. We find in the infrared limit the functions $\Lambda_{1,2,3,4}$, on which the three polarization operator eigenvalues and eigenvectors depend in an irrational way, in terms of the field derivatives of the effective action functional defined on constant background field to see explicitly that the eigenvalues disappear in the zero point of the excitation 4-momentum as a consequence of gauge invariance.

In Section III this property is used to ground the statement that there always exist massless excitations to be identified with photons present in two polarization modes, whereas any number of massive branches may be present in all the three modes. The corresponding excitations have the same quantum numbers as photons and supply poles to the same propagator. To avoid a possible misunderstanding it is worth stressing that we do not keep to the definition of the photon mass as its 4-momentum squared $k^2 \neq 0$, used by some authors. In our opinion, such word usage is counterproductive, because, if followed, it would make us recognize a photon in a standard isotropic medium with a constant refraction index as massive. This would make the notion of the photon mass indiscriminate. On the contrary, for us, the mass of an excitation is its rest energy, i.e. the value of its frequency (energy) when its spatial momentum is zero. Then, the excitations, whose dispersion curves include the origin in the momentum space, are referred to as photons.

By restricting the group velocity of an excitation below the speed of light we establish that in the special frame each dispersion curve is limited from above in the plane $(\sqrt{k_0^2 - k_3^2}, |\mathbf{k}_\perp|)$ by a straight line that crosses the dispersion curve at $|\mathbf{k}_\perp| = 0$ and is inclined to the coordinate axes at the angle of 45° (see Fig.1). Massless branches are restricted to the exterior of the light cone $\sqrt{k_0^2 - k_3^2} \leq |\mathbf{k}_\perp|$, whereas massive ones may cross it. This situation is similar to the magnetic field alone [3], because the general field also specializes only one direction in the space, namely the common direction of the magnetic and electric fields in the special reference frame.

In Section IV we describe, in the special frame, polarizations of electric and magnetic fields of the eigenmodes and find, specially, that the electric fields of modes 2 and 3, which are responsible for massless excitations, lie in a common plane spanned by two 3-vectors, one of which is orthogonal to the plane, where the external fields and the propagation momentum $\mathbf{k}$ lie, while the other belongs to that plane and makes a universal angle that depends only on momentum components with the external fields. Electric field of mode 1 is polarized along $\mathbf{k}_\perp$. In the same section we find the large-distance behavior of the magnetic field produced
by a point electric charge placed in the background electromagnetic field.

In Section V we establish relations to be obeyed by the four invariant combinations of the polarization operator components $\Lambda_{1,2,4,5}$ under special kinematical conditions that let the symmetry of the external field under rotations around its direction in the special frame and under Lorentz boost along this direction manifest itself as degeneracies of the polarization tensor, i.e. coincidences between pairs of its eigenvalues.

Appendix 1 is technical. In Appendix 2 we present the one-loop approximation for the function $\Lambda_3$ in the limit of small electric admixture to the external magnetic field, $\mathcal{G} \to 0$, responsible for mixing eigenmodes in this limit. It has cyclotron resonances starting with the second threshold of electron-positron pair creation by a photon in a magnetic field. Therefore, the mixing does not affect the photon capture effect at the first threshold important for radiation formation in the pulsar magnetosphere.

II. POLARIZATION OPERATOR, ITS EIGENVECTORS AND EIGENVALUES

Before starting, technical conventions are in order. There are two field invariants $\mathfrak{F} = \frac{1}{4} F_{\rho\sigma} F_{\rho\sigma}$ and $\mathfrak{G} = \frac{1}{4} F_{\rho\sigma} \tilde{F}_{\rho\sigma}$ of the background fields and two Lorentz-scalar combinations $k F^2 k$ and $k \tilde{F}^2 k$ of the background field strength and momentum $k_\mu$ of the elementary excitation, subject to the relation

$$\frac{k \tilde{F}^2 k}{2 \mathfrak{F}} - k^2 = \frac{k F^2 k}{2 \mathfrak{F}}. \quad (1)$$

The dual field tensor is defined as $\tilde{F}_{\rho\sigma} = \frac{i}{2} \epsilon_{\rho\sigma\lambda\nu} F_{\lambda\nu}$, where the completely antisymmetric unit tensor is fixed in such a way that $\epsilon_{1234} = 1$. We use the notations $(\tilde{F}k)_\mu \equiv \tilde{F}_{\mu\tau} k_\tau$, $(Fk)_\mu \equiv F_{\mu\tau} k_\tau$, $F^2_{\mu\nu} \equiv F_{\mu\tau} F_{\tau\nu}$, $(F^2 k)_\mu \equiv F^2_{\mu\tau} k_\tau$, $k F^2 k \equiv k_\mu F^2_{\mu\tau} k_\tau$, $k^2 \equiv k^2 + k^2_4 = k^2 - k^2_0$ and are working in Euclidian metrics with the results analytically continued to Minkowsky space, hence we do not distinguish co- and contravariant indices. The scalar variables

$$\mathcal{B} = \sqrt{\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}}, \quad \mathcal{E} = \sqrt{-\mathfrak{F} + \sqrt{\mathfrak{F}^2 + \mathfrak{G}^2}} \quad (2)$$

make the meaning, respectively, of the magnetic, $\mathcal{B} = B = |B|$, and electric fields, $\mathcal{E} = E = |E|$, in the (special) Lorentz frame, where $B$ and $E$ are directed along the same axis chosen as axis 3 in what follows. The designation " $\iff$ " will establish correspondence between quantities relating to the general Lorentz frame and the values these take in the special
Referring to the fact that in the special frame the momentum-containing invariants become

\[ k \tilde{F}^2 k = B^2(k_3^2 - k_0^2) - E^2k_\perp^2, \quad kF^2 k = -B^2k_\perp^2 + E^2(k_3^2 - k_0^2), \]

with the two-dimensional vector \( k_\perp \) being the momentum projection onto the plane orthogonal to the common direction \( 3 \) of the electric and magnetic field we shall use the equivalence relations

\[ \frac{k^2B^2 + kF^2k}{B^2 + E^2} \Leftrightarrow k_3^2 - k_0^2, \quad \frac{k^2E^2 - kF^2k}{B^2 + E^2} \Leftrightarrow k_\perp^2 \]

throughout the paper.

Polarization operator \( \Pi_{\mu\nu}(x, y) \) is responsible for small perturbations above the constant-field background. It follows from the translation- Lorentz-, gauge-, PT- and charge-invariance [4, 5] that its Fourier transform can be presented in a diagonal form

\[ \Pi_{\mu\tau}(k, p) = \delta(k - p)\Pi_{\mu\tau}(k), \quad \Pi_{\mu\tau}(k) = \sum_{a=1}^{3} \kappa_a^2 \frac{\varphi^{(a)}_\mu \varphi^{(a)}_\tau}{(\varphi^{(a)})^2}, \]

where \( \varphi^{(a)}_\tau \) are its eigenvectors

\[ \Pi_{\mu\tau} \varphi^{(a)}_\tau = \kappa^2_a \varphi^{(a)}_\mu, \quad a = 1, 2, 3, 4, \]

while the eigenvalues \( \kappa_a \) are scalar functions of \( F, G, kF^2 k \) and \( k\tilde{F}^2 k \).

The fourth eigenvector is trivial, \( \varphi^{(4)}_\mu = k_\mu \), so the fourth eigenvalue vanishes, \( \kappa_4 = 0 \), as a consequence of the 4-transverseness of the polarization operator, \( \Pi_{\mu\tau} k_\tau = 0 \). All eigenvectors are mutually orthogonal, \( \varphi^{(a)}_\mu \varphi^{(b)}_\nu \sim \delta_{ab} \), this means that the first three ones are 4-transversal, \( \varphi^{(a)}_\mu k_\mu = 0 \).

In the special case, where the second field invariant disappears, \( G = 0 \), the three meaningful eigenvectors \( \varphi^{(1,2,3)}_\mu \) are known [4, 5, 14] in the universal final form:

\[ \varphi^{(1)}_\mu \big|_{G=0} = (F^2 k)_\mu k^2 - k_\mu (kF^2 k), \quad \varphi^{(2)}_\mu \big|_{G=0} = (\tilde{F} k)_\mu, \quad \varphi^{(3)}_\mu \big|_{G=0} = (F k)_\mu. \]

This case implies that in the special frame only magnetic, when \( \mathfrak{F} > 0 \), or only electric, when \( \mathfrak{F} < 0 \), field exists. In the limit \( k^2 = 0 \) modes 2, 3 correspond to Adler’s [16] \( \perp \) - and \( \parallel \)-modes, respectively, whereas mode 1 becomes pure gauge. Vectors [17] may be used as a convenient orthogonal basis also when no external field is present. But when \( \mathfrak{G} \neq 0 \), they no
longer diagonalize the polarization operator already because the vectors \((\tilde{F} k)_\mu\) and \((F k)_\mu\) stop being mutually orthogonal, since their scalar product \(-k \tilde{F} F k = \mathcal{G} k^2\) is now nonzero.

When \(\mathcal{G} \neq 0\), the first eigenvector is expressed in terms of the fields by the same formula as in (7):

\[
\gamma^{(1)}_\mu = (F^2 k)_\mu k^2 - k_\mu (k F^2 k), \quad (\gamma^{(1)} \tilde{F} k) = (\gamma^{(1)} F k) = 0,
\]

\[
(\gamma^{(1)})^2 = k^2 (k^2 \mathcal{E}^2 - k F^2 k)(k^2 \mathcal{B}^2 + k F^2 k) \iff k^2 (B^2 + E^2)^2 k^2 (k^2_3 - k^2_0) \quad (8)
\]

and the first eigenvalue is given by the formula

\[
\nu_1 = \frac{k^2 (B^2 + \mathcal{E}^2)}{k^2 \mathcal{B}^2 + k F^2 k} \Lambda_1 \iff \frac{k^2}{k^2_3 - k^2_0} \Lambda_1, \quad (9)
\]

where the scalar function of the fields and momentum \(\Lambda_1\) here, as well as other \(\Lambda\)'s below, is a linear superposition of the polarization tensor components \(\Pi_{\mu\nu}\). The other two eigenvectors are the linear combinations

\[
\gamma^{(2,3)}_\mu = -2 \Lambda_3 c^-\mu + \left[ \Lambda_2 - \Lambda_4 \pm \sqrt{(\Lambda_2 - \Lambda_4)^2 + 4 \Lambda^2_3} \right] c^+\mu \quad (10)
\]

(where the square root is understood algebraically: \(\sqrt{Z^2} = Z\), and not \(|Z|\)) of two orthonormalized vectors:

\[
c^-\mu = \frac{B(F k)_\mu + \mathcal{E}(\tilde{F} k)_\mu}{(B^2 + \mathcal{E}^2)^{1/2}(k^2 \mathcal{E}^2 - k F^2 k)^{1/2}} \iff \frac{B(F k)_\mu + E(\tilde{F} k)_\mu}{(B^2 + E^2)|k_\perp|},
\]

\[
c^+\mu = i \frac{\mathcal{E}(F k)_\mu - B(\tilde{F} k)_\mu}{(B^2 + \mathcal{E}^2)^{1/2}(k^2 \mathcal{E}^2 + k F^2 k)^{1/2}} \iff \frac{E(F k)_\mu - B(\tilde{F} k)_\mu}{(B^2 + E^2)(k^2_0 - k^2_3)^{1/2}},
\]

\[
(c^+ c^-) = (c^\pm \gamma^{(1)}) = (c^\pm k) = 0, \quad (c^\pm)^2 = 1, \quad (11)
\]

thereby of the former basic vectors \((\tilde{F} k)_\mu\) and \((F k)_\mu\), too. The corresponding two eigenvalues are

\[
\nu_2,3 = \frac{1}{2} \left[ -(\Lambda_2 + \Lambda_4) \pm \sqrt{(\Lambda_2 - \Lambda_4)^2 + 4 \Lambda^2_3} \right]. \quad (12)
\]

The scalar coefficients in the linear combination \((10)\) cannot be expressed in a universal way in terms of the field and momentum, but are irrational functions of the polarization tensor components. The reason is that the polarization operator is a linear combination of four independent matrices with four scalar coefficients, whereas there may be only three eigenvalues in accordance with three polarization degrees of freedom of a vector field. (When
= 0, the number of independent matrices reduces to three). The orthogonality \( b(2)b(3) = 0 \) is explicit in (10). The Lorentz-invariant coefficients \( \Lambda_{1,2,3,4} \) are functions of the background fields and momenta. Expressions for them as simple linear superpositions of the components of \( \Pi_{\mu\nu} \),

\[
\begin{align*}
\Lambda_1 &= \frac{(kF^2)_{\mu} \Pi_{\mu\nu}(F^2k)_{\nu}}{(B^2 + \mathcal{E}^2)(k^2\mathcal{E}^2 - kF^2k)}, \\
\Lambda_2 &= -c^\mu_{\mu\nu}c^\nu_{\nu}, \\
\Lambda_3 &= -c^\mu_{\mu\nu}c^\nu_{\nu}^+, \\
\Lambda_4 &= -c^\mu_{\mu\nu}c^\nu_{\nu}^+. 
\end{align*}
\]

are obtained in Appendix 1 from a less transparent representation to be found in [4, 5]; their calculations in one-loop approximation of QED are given in [4, 5].

The transparency domain of momenta is such a region where absorption is absent. The electron-positron pair production by a photon is an example of absorption. The region, where it is kinematically allowed, is not the transparency domain. The absence of absorption of small perturbation of the background field is reflected in the property of Hermiticity of the matrix \( \Pi_{\mu\nu} \). It is symmetric when the charge conjugation invariance holds (no charge-asymmetric plasma background, no spontaneous pair creation). Hence, in the transparency region all the components of \( \Pi_{\mu\nu} \) are real in the case under consideration, once the charge conjugation invariance is assumed. Then, all \( \Lambda \)'s defined by (13) are also real there, except the region \( k^2_3 - k^2_0 > 0 \) (or, in invariant terms, \( k^2B^2 + kF^2k > 0 \)) wherein \( \Lambda_3 \) becomes imaginary due to (11). (We shall see later that dispersion curves cannot get into this region without violating the stability). In this exceptional region the quantity under the square root in (10), (12) stops being manifestly positive. Nevertheless, it should remain nonnegative, since eigenvalues of a Hermitian matrix should be real.

The dispersion equations that define the mass shells of the three eigenmodes are

\[
\kappa_a(kF^2k, kF^2k, \mathcal{F}, \mathcal{G}^2) = k^2, \quad a = 1, 2, 3. \tag{14}
\]

We have explicitly indicated here that the eigenvalues should be even functions of the pseudoscalar \( \mathcal{G} \).

When, due to a certain reason, \( \Lambda_3 \) is small as compared to \( |\Lambda_2 - \Lambda_4| \), the small mixing of eigenmodes is obtained by expanding (10) in powers of \( \Lambda_3 / |\Lambda_2 - \Lambda_4| \). In this way we get, with the linear accuracy in \( \Lambda_3 \), after normalizing out the common factors 2 and \( 2\Lambda_3/(\Lambda_2 - \Lambda_4) \)

\[
\begin{align*}
\hat{b}^{(2)}_{\mu} &= -\Lambda_3 c^\mu_{\mu} + (\Lambda_2 - \Lambda_4)c^\mu_{\mu}^+, \\
\hat{b}^{(3)}_{\mu} &= (\Lambda_2 - \Lambda_4)c^\mu_{\mu}^- + \Lambda_3 c^\mu_{\mu}^+. 
\end{align*}
\]

Such situation occurs, first of all, when the electric field is small as compared to the magnetic
one, which we shall discuss now, and also for two cases of special kinematical conditions considered in Section V.

In the limiting regime of small $\mathcal{G} \to 0$, one has $\mathcal{E} \approx \mathcal{G}/\mathcal{B}$. So $\mathcal{E}$ is a pseudoscalar, hence $c^\mu_-$ is a vector, and $c^\mu_+$ a pseudovector, the same as $(Fk)_\mu$ and $(\tilde{F}k)_\mu$, respectively, are. Then, from (10) it follows that $\Lambda_3$ is a pseudoscalar vanishing linearly: $\Lambda_3 \sim \mathcal{G} \approx (\mathcal{E}\mathcal{B}) \to 0$. (This fact is also in agreement with the infrared limit (20) below, since $\mathcal{L}$ depends on $\mathcal{G}^2$, and with the one-loop result in [4, 5]). The eigenvector $\hat{\psi}^{(2)}_\mu$ given by eq. (10) with the upper sign in front of the square root becomes in this limit $c^\mu_+ \sim \tilde{F}k_\mu$, as prescribed by (7). On the contrary, the eigenvector $\hat{\psi}^{(3)}_\mu$ given by eq. (10) with the lower sign becomes $c^\mu_- \sim Fk_\mu$, because the coefficient in front of $c^\mu_+$ in (10) decreases as $\Lambda^2_3$. The coefficient $\Lambda_3$ becomes responsible for mixing eigenmodes, characteristic of an external magnetic field due to the perturbation caused by electric field. (See Appendix 2 for the linearly vanishing $\mathcal{G} \to 0$ limit of $\Lambda_3$ as calculated within one-loop approximation of quantum electrodynamics.)

Bearing in mind that $\Lambda_{2,4}$ are scalars and may, thus, contain the pseudoscalar $\mathcal{G}$ only in even powers, from (12) it may be concluded, prior to any dynamical calculations, that - as long as external electric field is small as compared to magnetic one - the birefringence, inherent to the problem of the light propagation when, in the special Lorentz frame, a single - magnetic, $\mathcal{G} > 0$, or electric, $\mathcal{G} < 0$, field is present, is enhanced as soon as the other field is added in parallel to the already existing one:

$$\left| \sqrt{2} - \sqrt{3} \right| = \sqrt{(\Lambda_2 - \Lambda_4)^2 + 4\Lambda^2_3} \geq |\Lambda_2 - \Lambda_4|,$$

i.e. the dispersion curves of modes 2, 3 tend to repulse from each other [33], as far as they lie (we shall discuss later why they are to) in the domain $k^2_0 > k^2_3$, where $\Lambda_3$ is real. Thereby, Adler’s kinematical selection rules [16] that ban some processes of one photon splitting into two in a magnetic field are strengthened if an electric field is added. As for his CP-selection rules [34], those now should be applied to the eigenwaves, given as (10), and read as follows: among the three $\gamma$-states involved into the reaction $\gamma \to \gamma\gamma$ there may be only two or none of mode-2 states, since $\hat{\psi}^{(2)}_\mu$ is a pseudovector, while $\hat{\psi}^{(1,3)}_\mu$ are vectors. However, any state, prepared as an eigenstate in the magnetic field alone may decay into two like states under the perturbation caused by the electric field disregarding the initial CP-bans, since the electric field introduces the pseudoscalar $\mathcal{G}$.

The infrared limit, $k_\mu \to 0$, of the polarization operator is important. To get it, it
is sufficient to have at one’s disposal only the effective Lagrangian \( \mathcal{L}(\mathcal{F}, \mathcal{G}) \), from where the dependance on the time- and space-derivatives of the field \( F_{\mu\nu} \) is disregarded \[35\]. In the limit of vanishing momenta the invariant coefficients \( \Lambda_{1,2,3,4} \) are quadratic functions of \( k_\mu \) expressed in terms of the (momentum-independent) derivatives \( \mathcal{L}_{\mathcal{F}} = \partial \mathcal{L} / \partial \mathcal{F}, \mathcal{L}_{\mathcal{F}\mathcal{G}} = \partial^2 \mathcal{L} / \partial \mathcal{F}^2, \mathcal{L}_{\mathcal{G}\mathcal{G}} = \partial^2 \mathcal{L} / \partial \mathcal{G}^2 \) as follows

\[
\Lambda_1|_{k_\mu \to 0} = (k_3^2 - k_0^2)\mathcal{L}_{\mathcal{F}},
\]

\[
\Lambda_2|_{k_\mu \to 0} = -k^2\mathcal{L}_{\mathcal{F}} - k_1^2(B^2\mathcal{L}_{\mathcal{G}\mathcal{G}} + E^2\mathcal{L}_{\mathcal{G}\mathcal{G}} + 2\mathcal{G}\mathcal{L}_{\mathcal{F}\mathcal{G}}),
\]

\[
\Lambda_4|_{k_\mu \to 0} = -(k_2^2 - k_0^2)(E^2\mathcal{L}_{\mathcal{G}\mathcal{G}} + B^2\mathcal{L}_{\mathcal{F}\mathcal{G}} - 2\mathcal{G}\mathcal{L}_{\mathcal{F}\mathcal{G}}),
\]

\[
\Lambda_3|_{k_\mu \to 0} = (k_1^2)^{\frac{1}{2}}(k_0^2 - k_3^2)^{\frac{1}{2}}\{\mathcal{L}_{\mathcal{F}}(B^2 + E^2) - (\mathcal{L}_{\mathcal{G}\mathcal{G}} + \mathcal{L}_{\mathcal{F}\mathcal{G}})\mathcal{G}\}. \tag{20}
\]

We have written these formulae referring to the special frame. The equivalence relations \[4\] allow to immediately restore their invariant form valid in any frame. Eqs. \[17\] – \[20\] are obtained using the definition of the polarization operator components as the second derivatives with respect to vector-potentials components (see, e.g. \[3\]). Insofar as one is interested in the quantities \( \Lambda_i|_{k_\mu \to 0} \) up to one-loop accuracy one should either take the Heisenberg-Euler expression for \( \mathcal{L} \) here or pass to the infrared limit in the expressions for \( \Lambda_i \) calculated within one-loop approximation in \[4\]. The two-loop approximation for \( \mathcal{L} \) is also available \[10\].

From eqs. \[17\] – \[20\] the vanishing of the eigenvalues in the zero-momentum point

\[
\kappa_a|_{k=0} = 0, \quad a = 1, 2, 3 \tag{21}
\]

follows. This property is, in the end, a consequence of the gauge invariance that requires that the effective Lagrangian should depend only on the field strengths, and not potentials.

### III. DISPERSION CURVES

In the special frame dispersion equations \[14\] can be represented in the form

\[
\kappa_a(k_0^2 - k_3^2, k_1^2, B^2, E^2) = k_1^2 + k_3^2 - k_0^2, \quad a = 1, 2, 3 \tag{22}
\]
and their solutions that express the energy $k_0$ of the elementary excitation of a given mode $a$ in terms of its spatial momentum components $k_3, k_\perp$ have the following general structure, provided, in the end, by the invariance of the external field under rotation around axis 3 and the Lorentz boost along this axis,

$$k_0^2 = k_3^2 + f_a(k_\perp^2), \quad a = 1, 2, 3,$$

where the dispersion functions $f_a(k_\perp^2)$ certainly depend also on the external fields.

The causality principle requires that the modulus of the group velocity, calculated on each mass shell (23), be less than or equal to the speed of light in the free vacuum $c = 1$:

$$|v_{gr}|^2 = \left( \frac{\partial k_0}{\partial k_3} \right)^2 + \left( \frac{\partial k_0}{\partial k_\perp} \right)^2 = \frac{k_3^2}{k_0^2} + \left| \frac{k_\perp}{k_0} \cdot f_a' \right|^2 = \frac{k_3^2 + k_\perp^2 \cdot (f_a')^2}{k_3^2 + f_a(k_\perp^2)} \leq 1,$$

where $f_a' = df_a(k_\perp^2)/dk_\perp^2$. This imposes the obligatory condition on the form and location of the dispersion curves (23), i.e. on the function $f_a(k_\perp^2)$ (remind that $k_3^2 + f_a(k_\perp^2) \geq 0$ due to (23)):

$$k_\perp^2 \left( \frac{df_a(k_\perp^2)}{dk_\perp^2} \right)^2 \leq f_a(k_\perp^2).$$

This inequality requires first of all that $f_a(k_\perp^2) \geq 0$, hence no branch of any dispersion curve may get into the region $k_0^2 - k_3^2 < 0$, where $\Lambda_3$, eq. (13), becomes imaginary. If it might, the photon energy $k_0$ would be imaginary within the momentum interval $0 < k_3^2 < -f_a(k_\perp^2)$, corresponding to the vacuum excitation exponentially growing in time. This sort of ghost would signal the instability of the vacuum with a background field. Inequality (25) further requires that

$$\frac{df_a^2(k_\perp^2)}{dk_\perp^2} \leq 1, \quad \text{or} \quad f_a^2(k_\perp^2) \leq f_a^2(0) + k_\perp,$$

where $m = f^{1/2}(0)$ is the rest energy (mass) of the elementary excitation: $m^2 = (k_0^2 - k_3^2)|_{k_\perp=0} = k_0^2|_{k_3=k_\perp=0}$. The inequality

$$(k_0^2 - k_3^2)^{1/2} \leq m + k_\perp$$

that follows from (26) and (23) is an obligatory restriction imposed by causality principle on the dislocation of dispersion curves in the presence of constant magnetic and electric fields. In the empty space the restriction that appears in the similar way is $k_0 \leq k_0|_{k=0} + |k|$. 

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It is certainly obeyed by the free massive particle: \( k_0 = (k^2 + m^2)^{1/2} \leq m + |k| \), where \( m = (k_0)|_{k=0} \).

The gauge invariance property (21) implies via equation (14) that for each mode there always exists a dispersion curve with \( m^2 = f_a(0) = 0 \), which passes through the origin in the \((k_0^2 - k_\parallel^2, k_\perp^2)\)-plane. It is such branch that is to be called a photon, since it is massless in the sense that the energy \( k_0 \) turns to zero for the excitation at rest, \( k_3 = k_\perp = 0 \) (although, generally, \( k^2 \neq 0 \) where \( k \neq 0 \)). Other branches for each polarization mode \( a \) may also appear provided that a dynamical model includes a massive excitation of the vacuum with quantum numbers of a photon, for instance the positronium atom \([5, 17–19]\) or a massive (pseudo)scalar particle (axion) in a gauge-invariant interaction with the electromagnetic field \([36]\). Note that while the number of polarization modes of a vector particle is three – in correspondence with the dimension of the space and, hence, with the number of degrees of freedom, – dispersion curve for each of the three modes may have any number of branches, e.g. an infinite number of excited positronium branches. The energy on a dispersion curve should be real, since the dispersion equation (14) supplies poles to the photon propagator (defined up to arbitrary longitudinal part \( \sim k_\mu k_\nu \)), and these should not get into complex plane. If the state is unstable and should therefore decay, its energy must have an imaginary part, indeed, but in this case the pole is located on a nonphysical sheet of the complex plane, whose presence must be provided by branching points in the polarization operator to be introduced within an approximation where the state is expected to be unstable. An example of such situation is given by the cyclotron resonance approximation of the polarization operator in a magnetic field. The corresponding dispersion equations are cubic with respect to energy squared. Out of its three solutions, one corresponds to a stable state and, therefore, is real, whereas the other two mutually complex conjugated branches responsible for the photon decay/capture to electron-positron pairs belong to nonphysical sheets of the complex energy plane. Neither of these solutions can be disregarded. As applied to unstable branches our appeal to the group velocity may be reasonable only provided that the absorption is small, i.e. the imaginary part is much less than the real one.

On the other hand, the number of massless modes is, as a matter of fact, not three,
but only two, as it should be for a photon. The point is that the massless branch of the dispersion curve for mode 1 does not correspond to any real elementary excitation, except for two special cases. It follows from (9) and (17) that
\[ \kappa_1 \bigg|_{k_\mu=0} = k^2 \frac{\partial \Omega(\vec{\alpha}, \vec{\sigma})}{\partial \vec{\alpha}}. \] (29)
Hence the light cone \( k^2 = 0 \) is a guaranteed solution to the dispersion equation (14) for mode 1 in the vicinity of the origin \( k_\mu = 0 \). Can there exist massless excitations in mode 1 other than \( k^2 = 0 \)? The answer is "no", because from (29) it follows that \( k^2 = 0 \) is the only possibility for a dispersion curve of mode 1, as it approaches the origin \( k_\mu = 0 \).
Now, from eq. (8) it is seen, that both electric and magnetic fields in mode 1 disappear at \( k^2 = 0 \), since on this mass shell the elementary excitation is pure gauge, unless either \( k^2 \mathcal{E}^2 - kF^2 k \sim k_\perp^2 = 0 \) or \( k^2 \mathcal{B}^2 + kF^2 k \sim k_3^2 - k_0^2 = 0 \), in which cases the common factor \( k^2 \) can be normalized out from \( b^{(1)}_\mu \). These exceptional cases propose kinematical conditions for degeneracy of the polarization tensor to be discussed in Section V. For the first of them, the one of parallel propagation, \( k_\perp^2 = 0 \), the mode-1 photon is actual, while, on the contrary, mode-2 photon no longer exists, the mode-2 excitation becoming massive, as it is argued below in Section V. The overall number of massless degrees of freedom, therefore, is again two. Note that although \( k_\perp = 0 \) may seem to be an isolated point, as a matter of fact this is not the case: every nonparallel propagation \( k_\perp \neq 0 \) reduces to perpendicular propagation \( k_3 = 0 \) by a Lorentz boost along axis 3, which does not lead us out of the special frame. In the second exceptional case, \( k_3^2 - k_0^2 = 0 \), again the mode-1 photon is actual, but the mode-3 photon does not exist according to Section V. So the number of massless degrees of freedom is two in this case, too.

We concluded above in this Section that the causality requires that in the plane \((\sqrt{k_0^2 - k_3^2}, k_\perp)\) the photon dispersion curves \((m = 0)\) are located outside or coincide with the light cone: \( k^2 \geq 0 \). (Remind that the light cone \( k^2 = k^2 - k_0^2 = 0 \) is the mass shell of a photon in the vacuum without an external field.) However, unlike the case, indicated below eq. (25), a violation of this ban would not lead to a complex-energy ghost or directly signalize the vacuum instability, but would mean a presence of superluminal wave, known as tachyon. On the other hand, massive branches of the dispersion curves as restricted by the condition (27) with \( m = f^{1/2}(0) \neq 0 \) may well cross the light cone and pass to its exterior. They may even quasi-intercept with the massless (photon) branches or with branches pos-
sessing different m. The quasi-interceptions, – i.e. the would-be interception of dispersion curves of two states taken as independent within a certain approximation,– would result in the mutual repulsion of the dispersion curves leading to formation of mixed states, polaritons, an example of which is given by a photo-positronium - a mixed state between a photon and the electron-positron bound state created by it in a strong magnetic field [5, 17–19]. This situation is illustrated by Fig.1.

![Disposition of dispersion curves](image)

**FIG. 1:** Disposition of dispersion curves. The lower bold solid line is the massless (photon) dispersion curve restricted from above by the light cone $k^2 = 0$, presented by the thin solid line. The upper solid bold line, representing a massive branch, cannot pass higher than the straight dotted line, originating from its crossing with the vertical axis according eq. (27). The dashed lines show a quasi-interception.

The refraction index squared $n_a^2$ is defined for photons of mode $a$ on the mass shell [23] as

$$n_a^2 \equiv \frac{|k|^2}{k_0^2} = 1 + \frac{k_\perp^2 - f_a(k_\perp^2)}{k_0^2}.$$  

(30)

It follows from (26) with $m = f^{1/2}(0) = 0$ that the refraction index is greater than unity - the statement common in standard optics of media (this is not, certainly, true for (massive) plasmon branches). Consequently, the modulus of the phase velocity in each mode $v_a^{ph} = (k_0/|k|^2)k$ equal to $1/n_a$ is, *for the photon proper*, also smaller than the velocity of light in
the vacuum $c = 1$. This is not the case for a massive -- e.g. positronium -- branch of the photon dispersion curve, where $|\nu^{ph}_a| > 1$ without any importance for causality.

Now that we established that for photons one has $k^2 \geq k^2_0$, or $k^2 \geq 0$, we see from the dispersion equation (14) that the eigenvalues $\kappa_a$ are nonnegative in the momentum region, where the photon dispersion curves lie, i.e. the polarization operator is nonnegatively defined matrix there.

IV. ELECTROMAGNETIC FIELDS OF SMALL PERTURBATIONS OF THE BACKGROUND FIELD

A. Polarization of eigenmodes

In the special frame some peculiarities can be revealed about orientations of electric and magnetic fields in the virtual or real eigenmodes, formed out of eigenvectors $b^{(a)}_l$ as of 4-potentials according to the standard rules $e^{(a)}_m = i(k_4 b^{(a)}_m - k_m b^{(a)}_4)$, $h^{(a)}_m = i\varepsilon_{mn} b^{(a)}_n k_l$. To this end let us write down the eigenvector $b^{(1)}_\nu$ (31) (after normalizing it) and the basic vectors $c^\pm_\nu$ (31), in the special frame:

$$
\frac{b^{(1)}_\nu}{\sqrt{|(b^{(1)}_\nu)^2|}} = \frac{\sqrt{k_0^2 - k_3^2}}{|k^2|} \begin{pmatrix}
\frac{k_1}{|k_\perp|} \\
\frac{k_3}{|k_\perp|} \\
-\frac{k_0}{|k_\perp|}
\end{pmatrix}_\nu, \\
c^-_\nu = \begin{pmatrix}
\frac{|k_\perp \times e|}{|k_\perp|} \\
0 \\
0
\end{pmatrix}_\nu, \\
c^+_\nu = \begin{pmatrix}
0 \\
\frac{-k_0}{\sqrt{k_0^2 - k_3^2}} \\
\frac{k_3}{\sqrt{k_0^2 - k_3^2}}
\end{pmatrix}_\nu. \tag{31}
$$

The upper positions in every column are occupied by two-component vectors in the perpendicular plane $(1,2)$, next go the third and fourth components. The normalizing factor is $|(b^{(1)}_\nu)^2| = (E^2 + B^2)(k_0^2 - k_3^2)|k_\perp|^2$. Here $e$ is the unit vector along axis 3 and $[k_\perp \times e]$ stands for the vector product: $[k_\perp \times e]_m = \varepsilon_{mn} (k_\perp)_n e_s$. The difference $k_0^2 - k_3^2$ is understood to be nonnegative for real excitations, but may be imaginary for virtual ones.

From the normalized expression in (31) for $b^{(1)}_\nu$ we find for the electric and magnetic fields in mode 1:

$$
e^{(1)} = k_0 \frac{k_\perp}{|k_\perp|} \sqrt{\frac{|k^2|}{k_0^2 - k_3^2}}, \quad h^{(1)} = [k_\perp \times e] \frac{k_3}{|k_\perp|} \sqrt{\frac{|k^2|}{k_0^2 - k_3^2}}. \tag{32}
$$

Naturally, the electric and magnetic fields in mode 1 are oriented in the same way as in the single-invariant external field: they both lie in the plane, orthogonal to the external
fields, they are mutually orthogonal; besides, the magnetic field is transverse, \((\mathbf{h}^{(1)} \mathbf{k}) = 0\), while the electric field, generally, is not: \((\mathbf{e}^{(1)} \mathbf{k}) \neq 0\); it is transverse for the special case of propagation along the external fields \(k_\perp = 0\). It is seen again that a massless excitation is possible in mode 1 as long as it propagates along the external field, otherwise the fields vanish if \(k^2 = 0\), unless \(k_\perp = 0\), when the square root turns to unity.

The electric and magnetic fields carried by the vectors \((\tilde{\mathbf{F}}k)_\mu\) and \((Fk)_\mu\), respectively, are

\[
\mathbf{e}^{(F)} = -k_\perp Bk_3 + [k_\perp \times \mathbf{e}]k_4 E - \mathbf{B}(k_4^2 + k_3^2), \quad \mathbf{h}^{(F)} = -k_\perp E k_3 - [k_\perp \times \mathbf{e}]H k_0 + \epsilon E k_\perp^2, \tag{33}
\]

\[
\mathbf{e}^{(F)} = k_\perp E k_3 + [k_\perp \times \mathbf{e}]k_4 B + \mathbf{E}(k_4^2 + k_3^2), \quad \mathbf{h}^{(F)} = -k_\perp B k_3 + [k_\perp \times \mathbf{e}]E k_0 + \epsilon B k_\perp^2. \tag{34}
\]

The electric fields carried by the vectors \(c_\mu^+\) and \(c_\mu^-\), respectively, are

\[
\mathbf{e}^+ = \{k_\perp k_3 + \epsilon(k_3^2 - k_0^2)\} \frac{((\mathcal{B}^2 + \mathcal{E}^2))}{(k^2 \mathcal{B}^2 + k^2 \mathcal{E}^2)} = k_\perp \frac{k_0}{\sqrt{k_3^2 - k_0^2}} + \epsilon \sqrt{k_3^2 - k_0^2}, \tag{35}
\]

\[
\mathbf{e}^- = [k_\perp \times \mathbf{e}]k_0 \sqrt{((\mathcal{B}^2 + \mathcal{E}^2))} = [k_\perp \times \mathbf{e}]k_0 \frac{k_3}{|k_\perp|}, \quad \mathbf{e}^- \mathbf{k} = 0, \tag{36}
\]

while their magnetic fields are

\[
\mathbf{h}^+ = [k_\perp \times \mathbf{e}]k_0 \frac{k_0}{\sqrt{k_3^2 - k_0^2}}, \quad \mathbf{h}^- = -k_\perp \frac{k_3}{k_\perp} + \epsilon k_\perp, \quad \mathbf{h}^+ \mathbf{k} = (\mathbf{h}^+ \mathbf{h}^-) = (\mathbf{h}^\pm \mathbf{e}^\pm) = 0. \tag{37}
\]

The orthogonality of the electric fields \((\mathbf{e}^+ \mathbf{e}^-) = (\mathbf{e}^{(1)} \mathbf{e}^-) = 0\), seen in eqs. \(32\), \(35\), \(36\), originates from the fact that, in the special frame, the time-component of one of their mutually orthogonal \((c^+ c^- = 0)\), and 4-transversal \((c^+ k = 0)\) vector-potentials \(c_\mu^\pm\) disappears: \(c_\mu^- = 0\). The 3-vector \(\mathbf{e}^-\) is directed along the axis (call it axis \(1\), orthogonal to the plane, where the external fields and the propagation vector \(\mathbf{k}\) lie (the plane \((3,2)\)). The vector \(\mathbf{e}^+\) lies in that plane. It makes the universal angle \(\alpha = \arctan(k_3 k_\perp/(k_0^2 - k_3^2))\) with the direction of the external fields \(3\). Also the magnetic field \(\mathbf{h}^-\) lies in the plane spanned by the external fields and the propagation direction, while \(\mathbf{h}^+\) is orthogonal to this plane.

The electric fields in the eigenmodes 2 and 3, \(\mathbf{e}^{(2,3)}\), lie both in the common plane spanned by the vectors \(\mathbf{e}^\pm\):

\[
\mathbf{e}^{(2,3)}_\mu = -2\Lambda_3 \mathbf{e}_\mu^- + \left[\Lambda_2 - \Lambda_4 \pm \sqrt{(\Lambda_2 - \Lambda_4)^2 + 4\Lambda_3^2}\right] \mathbf{e}_\mu^+. \tag{38}
\]
These are not, generally, mutually orthogonal, since \( e^\pm \) are not unit-length vectors. Also the magnetic fields of the modes 2, 3 lie in the common plane spanned by the two vectors \( h^\pm \) and are linearly combined of them with the same coefficients as in (38). It can be checked that the electric and magnetic fields in each mode are mutually orthogonal, of course: \( (h^{(2,3)} e^{(2,3)}) = 0. \)

In the special case of only one invariant different from zero, \( \Lambda_3 \sim \mathcal{S} \approx (E/B) \to 0 \), the eigenvectors \( b^\pm_{(2,3)} \) are the same as \( c^\pm_\mu \) owing to (10), hence \( e^\pm \), eqs. (35), (36), and \( h^\pm \), eq. (37), become the electric and magnetic fields of the corresponding eigenmodes, coinciding with their expressions known from Refs. [4, 14] with the particular property, that the electric field of mode 2 lies in the plane \( (3,2) \), while that of mode 3 is orthogonal to this plane, known for the special case of zero virtuality, \( k^2 = 0 \), from Ref. [16].

B. Magnetoelectric effect

In the magnetic-like field it is known that virtual photons of mode 2 are carriers of electrostatic [3, 25] forces, whereas those of modes 1 and 3 are responsible for magnetostatic interaction [3]. These statements follow from the representation (31) for the basic vectors, that become eigenvectors in that special case, and from the diagonal representation of the Green function (28) that allows to write electric and magnetic fields created by various (static included) configurations of small – as compared to the background field – charges and currents. The mixing of the basic vectors (11) in eigenmodes 2 and 3 for the general external field with \( \mathcal{S} \neq 0 \) makes these statements no longer true in what concerns these modes, mode 3 remaining as it was. Moreover, thanks to the mixing, a static electric charge, if placed in the external field with the both invariants different from zero, gives rise not only to an electric field, as usual, but also to a magnetic field of its own, like a magnetic charge or moment. Also stationary currents produce some electric admixture to their customary magnetic fields.

Here we consider this analogue to the magneto-electric effect known in crystals [37] using the field of a point-like static charge \( q \) taken at rest in the special frame as an example. We set the 4-current corresponding to this source in the coordinate space \( x \) as \( j_\mu(x) = q\delta_{\mu 0}\delta^3(x) \), where \( \delta_{\mu 0} \) is the Kronecker symbol and \( \delta^3(x) \) is the Dirac delta-function. Integrating this current with the Green function (28) we obtain for the vector-potential produced by the
point charge (see [25] for a more detailed explanation if needed)

\[ A_\mu(x) = \frac{q}{(2\pi)^3} \int D_{\mu 0}(0, k) \exp(-ikx) d^3k. \]  

(39)

Here the argument 0 of the Green function stands for \( k_0 \). Among the basic vectors (31) there is only one whose fourth component remains nonzero in the static limit \( k_0 = 0 \). It is \( c_0^+ \). It participates in the eigenvectors \( \hat{\nu}^{(2,3)}_\mu \) in accord with (10). Hence only these two eigenvectors will remain in the decomposition (28) of \( D \) after it is substituted into (39). On the other hand the contribution of the basic vector \( c_0^- \) may only supply the spacial components to the vector-potential (39), whereas \( c_0^+ \) cannot. Bearing in mind that eqs. (31) imply in the static limit, \( k_0 = 0 \), that

\[ c_i^+ = c_0^+ = 0, \quad c_0^- = 1, \quad c_i^- = \frac{[k_\perp \times \epsilon_i]}{|k_\perp|}, \quad i = 1, 2, \quad c_3^\pm = 0 \]  

(40)

the spacial part of the latter is

\[ A_i(x) = \frac{q}{(2\pi)^3} \int \sum_{a=2,3} \frac{\hat{\nu}^{(a)}_\mu \hat{\nu}^{(a)}_0}{(\nabla^{(a)})^2} \exp(-ikx) d^3k \]  

\[ = \frac{q}{(2\pi)^3} \int c_i^- \frac{\Lambda_3 \exp(-ikx) d^3k}{(k^2 - \xi_2)(k^2 - \xi_3)}, \quad i = 1, 2 \]

\[ A_3(x) = 0. \]  

(41)

To find the large distance behavior of this field note that it is determined by the limit \( k = 0 \) in the pre-exponential factor in the integrand. Therefore, we set \( \xi_{2,3} = 0 \) and use eq. (20) for \( \Lambda_3 \)

\[ \Lambda_3|_{k_0 \to 0} = i|k_\perp|k_3 \mathcal{M}, \]  

\[ \mathcal{M} = \mathcal{S}\mathcal{F}(B^2 + E^2) - (\mathcal{S}\mathcal{F} + \mathcal{S}\mathcal{F}) \mathcal{G}. \]  

(42)

Then

\[ A(x_ \perp, x_3)|_{|x| \to \infty} \simeq \frac{q\mathcal{M}}{(2\pi)^3} \int \frac{k_3[k_\perp \times \epsilon]}{k^4} \frac{\exp(-ikx) d^3k}{k^4}. \]  

(43)

This vector in the two-dimensional plane orthogonal to the external fields is directed as \([x_\perp \times \epsilon]\), since the coordinate vector \( x_\perp \) in that plane fixes the only direction on which the integral may depend. The length of (43)

\[ |A|_{|x| \to \infty} \simeq \frac{q\mathcal{M}}{8\pi} \frac{1}{|x|}. \]  

(44)

decreases via the Coulomb law with the radial distance \(|x|\) from the charge. The vector potential and the magnetic field carried by it at large distances are

\[ A(x_ \perp, x_3)|_{|x| \to \infty} \simeq \frac{[x_\perp \times \epsilon]}{|x_\perp|} \frac{q\mathcal{M}}{8\pi} \frac{1}{|x|}, \]  

\[ h_3 = \frac{q\mathcal{M}}{8\pi} \left( \frac{1}{|x||x_\perp|} - \frac{|x_\perp|}{|x|^3} \right), \quad h_\perp = \frac{q\mathcal{M}}{8\pi} \frac{x_\perp \times x_3}{|x_\perp| |x|^3}. \]  

(45)
The last two equations together make a magnetic field oriented along the radius-vector (when \( q \) is positive) in the upper half-plane \( x_3 > 0 \) and opposite to the radius-vector in the lower half-plane:

\[
\mathbf{h} = \frac{x \, qM}{|x|} \frac{1}{8\pi} \frac{x_3}{|x|^2 |x_\perp|}.
\]  

(46)

The magnetic lines of force make a pencil of straight lines passing through the origin where the charge is located and go radially from/to the charge with their density being the cotangent of the observation angle increasing towards the axis 3, where it is singular. The magneton \( qM/(8\pi) \) is proportional to the pseudoscalar \( \mathcal{G} \); in perturbation theory it has the fine structure constant \( \alpha \) as its overall factor. The result \((46)\) is approximation-independent, but holds true only as long as the magnetic field produced by the charge may be considered as a small perturbation of the background field.

It is worth noting that analogous magneto-electric phenomenon should be present in a plasma with external magnetic field. The reason is again in the mixing – after plasma is added – of basic vectors \([13]\) (see also \([5]\)), which are electric and magnetic carriers in the magnetized vacuum alone. If the plasma is charge-symmetric, \( e.g. \) consists of equal numbers of positively and negatively charged otherwise identical particles, say electrons and positrons, the vector \( b^{(1)}_\mu \big|_{\mathcal{G} = 0} \) from \((7)\) linearly combines with the pseudovector \( b^{(2)}_\mu \big|_{\mathcal{G} = 0} = (\tilde{F} k)_\mu \), to become an eigenvector, unlike the situation considered above, while \( b^{(3)}_\mu \big|_{\mathcal{G} = 0} = (F k)_\mu \) remains an eigenvector. The pseudoscalar, needed for this combination, is built of the same pseudovector contracted with the vector of 4-velocity of the plasma. It plays the role of \( \mathcal{G} \).

For a more general charge-non-symmetric case, say, electron gas or a gas of ionized atoms, the situation is even more rich, because all the three basic vectors mix.

V. DEGENERACIES

We have to consider the degeneracies of the polarization matrix taking place for two special kinematical conditions owing to the symmetries of the external field.

For the real or virtual excitations directed parallel to the external field in the special frame the polarization operator is symmetric under spacial rotations around the common field direction 3, since the external field is invariant under them, while the excitation does not introduce an additional anisotropy in the perpendicular plane owing to the relation

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$\mathbf{k}_\perp = \mathbf{0}$. The symmetry of the polarization operator should manifest itself as a degeneracy that implies that two out of its three eigenvalues should coincide, and their corresponding eigenvectors should transform through one another under the symmetry transformations, while remaining eigenvectors. By inspecting (31) we see that $b^{(1)}_\nu$ and $c^-_\mu$ do possess this mutual property, when $\mathbf{k}_\perp = \mathbf{0}$, but $c^+_\mu$ does not (recall that $|k^2| = k_0^2 - k_3^2$, once $\mathbf{k}_\perp = \mathbf{0}$). Hence the latter cannot be admixed to $c^-_\mu$ in (10), and we conclude that, the same as in the one-invariant external field,

$$\Lambda_3 = 0, \quad \text{and} \quad \varkappa_1 = \varkappa_3, \quad \text{when} \quad k_\perp^2 = 0. \quad (47)$$

(The condition $k_\perp^2 = 0$ in the general frame should be replaced by $k^2E^2 - kF^2k = 0$). Then eqs. (10), (12) imply that

$$\Lambda_1 = -\Lambda_2, \quad \text{when} \quad k_\perp^2 = 0. \quad (48)$$

Now from (32)–(37) we see that, when $k_\perp^2 = 0$, modes 1 and 3 carry mutually perpendicular and equal in magnitude transverse electric fields $e^{(1)}$ and $e^{(3)} = e^-$ polarized in the plane orthogonal to their propagation direction and to the external fields. The same is true for the magnetic fields in these modes. Therefore, in this special case the mode 1 does correspond to a real photon, as explained in Section III. Simultaneously, mode-2 becomes a purely longitudinal wave that may exist only as long as $k_3^2 - k_0^2 \neq 0$, since (35) otherwise disappears, i.e. it may only be massive. The conclusion is that the number of massless modes remains two, as expected.

Another degeneracy of polarization operator is provided by the kinematical situation $k_0^2 - k_3^2 = 0$. This condition is invariant under Lorentz boosts along the common direction of the fields in the special frame. (Its invariant equivalent is $k^2B^2 + kF^2k = 0$). On the other hand, the external field in the special frame is also invariant under this transformation: it does not lead out of this frame, since the constant electric and magnetic fields are not transformed by it. Hence, the polarization operator should be also invariant, which implies that some two of its eigenvalues must coincide, while the corresponding eigenvectors are transformed through one another by the Lorentz rotation in the $(3,0)$ hyperplane. This is the case for the vectors $b^{(1)}_\nu$ and $c^+_\nu$ in (31), since in the limit under consideration the 2-vector in the upper row of the former is negligible as compared to the other two components, and $|\mathbf{k}_\perp| = \sqrt{|k^2|}$, so that $b^{(1)}_\nu$ matches $c^+_\nu$ as a Lorentz boost partner. On the contrary, the
vector \( c_\nu^- \) does not transform through any of them, because it is Lorentz-boost-invariant. We conclude that \( c_\nu^- \) and \( c_\nu^+ \) can no longer mix together to form eigenvectors, but should be eigenvectors separately. The mixture becomes impossible if and only if \( \Lambda_3 \) disappears from (10), (12). Then, according to (17), \( b_\nu^{(2)} \) becomes \( \sim c_\nu^+ \), hence the degeneracy is expressed as the relation

\[ \kappa_1 = \kappa_2, \quad \text{when } k_0^2 - k_3^2 = 0, \]  

accompanied by the relations

\[ \Lambda_3 = 0, \quad -\Lambda_4 (k_3^2 - k_0^2) = k_2^2 \Lambda_1, \quad \text{when } k_0^2 - k_3^2 = 0. \]  

Now we use the fact that the electric fields in eigenmodes are defined up to a common factor to renormalize them all according to \( \tilde{\mathbf{e}} = \mathbf{e} (k_0^2 - k_3^2)^{1/2} \). Then, under the special kinematic condition under consideration \( k_0^2 - k_3^2 = 0 \), the electric fields in modes 1 and 2 (see eqs. (32), (35)) are finite and equal in length \( |\tilde{\mathbf{e}}(1)| = |\tilde{\mathbf{e}}^+| = k_\perp k_3 \), while that in mode 3 (36) disappears, \( \tilde{\mathbf{e}}^- = 0 \). Consequently this degree of freedom is impossible. As pointed in Section III, in the exceptional point \( k_0^2 - k_3^2 = 0 \), the dispersion law \( k^2 = 0 \) may correspond to an actual mode-1 photon. In view of (49) it must be accompanied by a mode-2 partner.

For \( k^2 = 0 \) the electric fields in modes 1 and 2, besides the fact that they are equal in size, become polarized in transverse directions, \( (\tilde{\mathbf{e}}^+ \tilde{\mathbf{e}}^{(1)}) = |k_\perp| k_3 k_0 \sqrt{k^2} = 0 \). Therefore, we have again two photon degrees of freedom.

The symmetry relations (47)–(50) are confirmed in the infrared limit by eqs. (17) – (20) and – for any value \( k_\mu \) of the momentum – by the one-loop calculations in QED of [4]. In a theory with the dual invariance, which is not QED, another degeneracy is possible in the one-invariant case \( \mathcal{G} = 0 \) [3] that equates the eigenvalues \( \kappa_2 \) and \( \kappa_3 \), since under the continuous duality transformation the vector \( b_\mu^{(2)} \big|_{\varphi=0} = (\tilde{F}k)_\mu \) and the pseudovector \( b_\mu^{(3)} \big|_{\varphi=0} = (Fk)_\mu \) transform through each other.

VI. CONCLUSIONS

In this paper we studied on the most general basis properties of small perturbations of the vacuum, filled with a constant and homogeneous background electromagnetic field with its both invariants different from zero. To this end the eigenvector decomposition of
the polarization operator with contribution of three modes was exploited. We saw how the eigenvectors characteristic of the one-invariant (magnetic in a special frame) background field are linearly combined with the help of dynamics-dependent coefficients to form eigenvectors of the general problem under investigation.

Among the vacuum perturbations special attention was payed to the sourceless excitations that supply poles to the photon propagator and satisfy three different dispersion equations. These may be either massive or massless. In the latter case they are called photons. The massless excitations belong only to two modes, in accordance with two polarization degrees of freedom of a gauge vector particle, the photon. Massive excitations belong to all the three modes, since a massive vector field has three degrees of freedom. These may have unrestricted number of branches in each mode depending on the properties of the corresponding dispersion equation. We described admitted disposition of various dispersion curves (in the appropriate momentum plane) as it is restricted by the causal propagation requirement. The eigenmodes are plane-polarized, and the orientations of their electric and magnetic fields with respect to propagation direction and the direction of the background field are described. We dwelled on the impact the admixture of an electric field to a magnetic background may have on the selection rules for photon splitting. We noted that such admixture results in a larger separation between two different dispersion curves enhancing the birefringence.

Among possible perturbations of the background caused by small sources we especially considered the magnetic (part of the) field produced by a point static electric charge and found its behavior far from the source.

We also established coincidences between eigenvalues of the polarization operator (degeneracies) that, under special relations between momenta, reflect the residual rotational and Lorentz symmetries of the vacuum left after the background field is imposed.

All the results are approximation-independent, except for the statement, based on the one-loop calculations, that the mixing between modes is not resonant in the limit of small electric field at the first threshold of electron-positron pair creation by a photon.
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Appendix 1

Suppose, the polarization operator $\Pi_{\mu\nu}$ is known in components. Then the invariant functions $\Lambda_i$, $i = 1, 2, 3, 4$ involved in its eigenvalues and eigenvectors are given following the receipts in [4, 5] as

$$\Lambda_1 = d^{(1)}_\mu \Pi_{\mu\nu} (d^{(1)} + d^{(2)})_\nu, \quad \Lambda_2 = -c^{(1)}_\mu \Pi_{\mu\nu} c^-_\nu, \quad \Lambda_3 = -c^{(3)}_\mu \Pi_{\mu\nu} c^+_\nu, \quad \Lambda_4 = -c^{(3)}_\mu \Pi_{\mu\nu} c^+_\nu, \quad (51)$$

where $c^\pm_\nu$ are given by (11), and

$$d^{(1)}_\mu = \frac{\mathcal{E}^2 k_\mu - iB(Fk)_\mu - (F^2 k)_\mu - i\mathcal{E}(\tilde{F}k)_\mu}{2^{1/2}(B^2 + \mathcal{E}^2)^{1/2}(k^2 \mathcal{E}^2 - kF^2 k)^{1/2}} \Leftrightarrow \frac{E^2 k_\mu - iB(Fk)_\mu - (F^2 k)_\mu - iE(\tilde{F}k)_\mu}{2^{1/2}(B^2 + E^2)|k|}, \quad (52)$$

$$d^{(1)}_\mu + d^{(2)}_\mu = 2^{1/2} \frac{\mathcal{E}^2 k_\mu - (F^2 k)_\mu}{(B^2 + \mathcal{E}^2)^{1/2}(k^2 \mathcal{E}^2 - kF^2 k)^{1/2}} \Leftrightarrow 2^{1/2} \frac{E^2 k_\mu - (F^2 k)_\mu}{(B^2 + E^2)|k|}, \quad (53)$$

$$c^{(3)}_\mu = \frac{iB^2 k_\mu + \mathcal{E}(Fk)_\mu + (F^2 k)_\mu - B(\tilde{F}k)_\mu}{(B^2 + \mathcal{E}^2)^{1/2}(k^2 B^2 + kF^2 k)^{1/2}} \Leftrightarrow \frac{B^2 k_\mu + E(Fk)_\mu + (F^2 k)_\mu - B(\tilde{F}k)_\mu}{(B^2 + E^2)(k_3^2 - k_2^2)^{1/2}} \quad (54)$$

The notations used here are connected with those of Refs. [4, 5] as follows

$$c^{(1,3)}_\mu = i\sqrt{2}d^{(1,3)}_\mu, \quad c^-_\mu = i(d^{(1)}_\mu - d^{(2)}_\mu)/\sqrt{2}, \quad c^+_\mu = i(d^{(3)}_\mu - d^{(4)}_\mu)/\sqrt{2}. \quad (55)$$

Using the orthogonality of the polarization operator to vector $k_\mu$ and the orthogonality of the vector $(F^2 k)_\mu$ to the hyperplane spanned by the two vectors $(Fk)_\mu$ and $(\tilde{F}k)_\mu$, and also the diagonal representation (5), we find that many components of the vectors between which $\Pi_{\mu\nu}$ is sandwiched disappear from (51). Then we get the simpler representations (13) for the $\Lambda$’s. Equation (13) for $\Lambda_1$ agrees with eq. (9) and with the relation $\mathcal{E}_1 = \gamma^{(1)}_\mu \Pi_{\mu\nu} \gamma^{(1)}_\nu / (\gamma^{(1)})^2$ that follows from (6), taking into account the length of the eigenvector $\gamma^{(1)}_\mu$ given in (8). Equations (13) for $\Lambda_{2,3,4}$ agree with eqs. (6), (10), (12), but cannot be deduced from them, because the latter are invariant under a similarity transformation that changes the components of $\Pi_{\mu\nu}$.
The linear combinations $\Lambda_i$ of the polarization operator components are calculated in [4] in one-loop approximation, the calculational details being presented in [5] on the basis of [38]. The latter reference as well as [6, 7] contains also calculations of alternative sets of four scalar coefficient functions of an appropriate set of basic matrices in terms of which the polarization operator may be expressed. However, the set of functions $\Lambda_i$ is preferred to them all, because the eigenvalues are given the simplest in their terms.

Appendix 2

In this Appendix we write the linear in electric field correction into invariant function $\Lambda_3$ for the case, where an electric field, much smaller than the magnetic field, is added parallel to the latter (this wording refers to the special Lorentz frame). The linear part of $\Lambda_3$ defines the leading contribution into the mixing of photon eigenmodes in a magnetic field due to the perturbation introduced by the electric field. We present it here as a result of one-loop calculations of quantum electrodynamics in external magnetic field.

Using (11) and notations (7) the expansion (15) of the slightly perturbed eigenvectors over the eigenvectors in a magnetic field alone becomes

$$b^{(2)}_\mu = -\frac{(\Lambda_2 - \Lambda_4)}{(k^2 B^2 + k F^2 k)^{1/2}} b^{(2)}_\mu \Big|_{\mathcal{G} = 0} + \left(\frac{\mathcal{G}(\Lambda_2 - \Lambda_4)}{B^2 (k^2 B^2 + k F^2 k)^{1/2}} - \frac{\Lambda_3}{(k F^2 k)^{1/2}}\right) b^{(3)}_\mu \Big|_{\mathcal{G} = 0},$$

$$b^{(3)}_\mu = \frac{(\Lambda_2 - \Lambda_4)}{(k F^2 k)^{1/2}} b^{(3)}_\mu \Big|_{\mathcal{G} = 0} + \left(\frac{\mathcal{G}(\Lambda_2 - \Lambda_4)}{B^2 (k F^2 k)^{1/2}} - \frac{\Lambda_3}{(k^2 B^2 + k F^2 k)^{1/2}}\right) b^{(2)}_\mu \Big|_{\mathcal{G} = 0}. \quad (56)$$

It is understood that $\Lambda_2 - \Lambda_4$ in the right-hand side are taken at $\mathcal{G} = 0$. In writing these equations we took into account that $\Lambda_2, \Lambda_4$ are even, and $\Lambda_3$ is an odd functions of $\mathcal{G}$.

It is seen from (12) that at $\mathcal{G} = 0$ the quantity $-\Lambda_4$ is the polarization operator eigenvalue $\kappa_2$ in a magnetic field. Analogously, $-\Lambda_2 = \kappa_3$. These quantities in the one-loop approximation are known [4, 14]. Now we shall write $\Lambda_3$ in the same approximation by calculating the $\mathcal{G} = 0$ limit of the corresponding expression from [4].

$$\Lambda_3 = -\frac{\alpha}{4\pi} \frac{\mathcal{G}}{\delta} \left(\frac{-k F^2 k}{2 \delta}\right)^{1/2} \left(\frac{-k F^2 k}{2 \delta}\right)^{1/2} \int_0^\infty \frac{\tau d\tau}{\sinh^2 \tau} \int_{-1}^1 d\eta (1 - \eta)^2 \sinh^2 \tau \left(\frac{\tau (1 + \eta)}{2}\right) \times$$

$$\times \exp \left\{ \frac{k F^2 k}{2 \delta} \frac{\sinh((1 + \eta)/2)}{\sinh(ef \tau)} + \frac{k F^2 k}{2 \delta} \frac{1 - \eta^2}{4ef} \tau - \frac{m^2 \tau}{ef} \right\}. \quad (57)$$
Here $\alpha = 1/137$ is the fine structure constant, $m_e$ and $e$ are the electron mass and charge, and $f = \sqrt{2S}$. Note that this expression vanishes, indeed, either if $-\frac{kF^2}{2S} \leftrightarrow k_1^2 = 0$, or if $-\frac{kF^2}{2S} \leftrightarrow k_2^2 - k_3^2 = 0$, as it should in accordance with what the symmetry of the external field prescribes, as it was established in the body of this article, eqs. (47), (50). The integral \((57)\) has an infinite number of branching points, the same as $\Lambda_{1,2,4}$, with singular inverse square-root behavior - the cyclotronic resonances at thresholds of electron-positron pair creation by a photon \([5,14]\). However, the lowest-lying resonance is not in the point (in the variables, referring to the special frame) $k_0^2 - k_3^2 = 4m_e^2$, like in $\tau_2 = -\Lambda_4$, but in the point $k_0^2 - k_3^2 = ((m_e^2 + 2ef)^{1/2} + m_e)^2$, like in $\tau_3 = -\Lambda_2$, because this value borders the convergence domain of the $\tau$-integration in \((57)\). This means that the (small as compared with the magnetic) electric field cannot affect the phenomenon of the mode-2 photon capture with its adiabatic conversion into a free \([20]\) or bound \([17]\) electron-positron pair in the lowest Landau level.

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