GROMOV-WITTEN INVARIANTS AND RATIONAL CURVES ON GRASSMANNIANS

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ABSTRACT. We study the enumerative significance of the \(s\)-pointed genus zero Gromov-Witten invariant on a homogeneous space \(X\). For that, we give an interpretation in terms of rational curves on \(X\).

1. INTRODUCTION

Since their appearance in the algebraic context, Gromov-Witten invariants have proven to be an indispensable tool for enumerative geometry. The problem of determining the number \(N_d\) of rational curves of degree \(d\) passing through \(3d - 1\) points in general position in the complex projective plane \(\mathbb{P}^2\) was solved, by means of Gromov-Witten theory, by Kontsevich (see [12]).

Gromov-Witten invariants arose as enumerative invariants of stable maps, which had been previously introduced independently by Ruan and Tian [16] in the symplectic case, and by Kontsevich and Manin [12] in the algebraic case. Let \(X\) be a smooth projective variety over \(\mathbb{C}\), and let \(\beta\) be a curve class on \(X\). The set of isomorphism classes of pointed maps \((C, p_1, \ldots, p_s, f)\), where \(C\) is a projective nonsingular curve and \(f\) is a morphism from \(C\) to \(X\) with \(f^*([C]) = \beta\), is denoted as \(M_{g,s}(X, \beta)\). Its compactification, the moduli space \(\overline{M}_{g,s}(X, \beta)\), parameterizes stable maps. The stability condition is equivalent to finiteness of automorphisms of the map.

The purpose of our work is to study the enumerative significance of genus zero Gromov-Witten invariants with a Grassmannian target. While Gromov-Witten theory in the algebraic context in general requires the sophisticated machinery of virtual fundamental classes [15] and the invariants have no clear enumerative significance in general, in the case of genus zero with target a homogeneous variety \(X = G/P\) the moduli stack is smooth of the expected dimension and it makes sense to ask for the enumerative significance of the Gromov-Witten invariants. Given a tuple of classes of subvarieties of \(X\) of suitable dimensions, the Gromov-Witten invariant counts isomorphism classes of \((C \cong \mathbb{P}^1, p_1, \ldots, p_s, f)\) such that \(f^*([\mathbb{P}^1])\) is equal to a given curve class and \(f\) maps \(p_i\) into a general translate of the \(i\)th subvariety for all \(i\), according to Fulton and Pandharipande [10]. (The \(p_i\) are not fixed here; as explained in op. cit., page 93, there are alternative invariants in which the \(p_i\) are fixed points in \(\mathbb{P}^1\).) The purpose of this article is to rephrase
this enumerative interpretation in terms of rational curves on $X$ satisfying incidence conditions. For a large class of enumerative conditions, we are able to exclude that due to reparameterizations of the source curve a map $f$ contributes multiply to the Gromov-Witten invariant. Then the Gromov-Witten invariant simply counts rational curves in a given curve class incident to general translates of the given subvarieties. The main Theorem asserts this to be the case when $X$ is a Grassmannian variety and the subvarieties are Schubert varieties, up to a correction factor of the degree of the curve class for each Schubert variety of codimension one.

2. Preliminaries

Let $G$ be a complex simple Lie group of classical type and $P$ a maximal parabolic subgroup. The homogeneous space $X = G/P$ is a Grassmannian variety, a usual Grassmannian of subspaces of some finite-dimensional complex vector space $V$ when $G$ is of type $A$, or of subspaces isotropic for a given nondegenerate symmetric or skew-symmetric bilinear form on $V$ in the other classical Lie types. Throughout we assume that $\dim X \geq 2$.

The quantum Schubert calculus is a set of combinatorial rules that determine the genus zero three-point Gromov-Witten invariants of $X$. Quantum analogues of the classical Pieri and Giambelli formulas are given for the usual Grassmannians by Bertram [2] and for isotropic Grassmannians by Buch, Kresch, and Tamvakis [13, 14, 7, 5, 4]. In type $A$, an explicit combinatorial rule for the invariants, i.e., a quantum Littlewood-Richardson rule, is available, due to Coskun [8].

For the sake of completeness and to fix notation, let us recall some definitions. The Schubert varieties on the usual Grassmannian of $m$-planes in $V \cong \mathbb{C}^n$ are indexed by integer partitions of length at most $m$ and biggest part at most $n - m$. There are analogous descriptions in the other Lie types, based on $k$-strict partitions; the reader is referred to [4] for a detailed description and facts about Schubert varieties in the isotropic Grassmannians. Here $k$ is $n - m$ when the bilinear form on $V$ is skew symmetric and $X = IG(m, 2n)$ and in the case of a symmetric bilinear form $k$ is $n - m$, respectively $n + 1 - m$, when $X$ is $OG(m, 2n + 1)$, respectively $OG(m, 2n + 2)$; a partition is $k$-strict when there are no repeated parts $> k$. We will generally denote by $X_{\lambda}$ the Schubert variety in $X$ corresponding to a partition $\lambda$. Its codimension is $|\lambda|$, where for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ the weight is $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$. The kernel and span of a rational map are important notions, introduced and discussed by Buch in [3]. A rational map of degree $d$ to $X$ is a morphism $f : \mathbb{P}^1 \to X$ such that $f_\ast[\mathbb{P}^1]$ has degree $d$. Here degree is understood with respect to the projective embedding of $X$ corresponding to a fundamental representation of $G$ with point stabilizer $P$. In dimension 1, the unique Schubert class has degree 1 and the corresponding Schubert
variety is a line under this embedding. Since \( \dim X \geq 2 \), \( X \) contains a projective plane or a nonsingular quadric threefold, and therefore there exist rational curves of every degree \( d \geq 1 \) on \( X \).

The moduli space of stable genus zero degree \( d \) maps \( \overline{M}_{0,s}(X,d) \) is smooth (as a stack) of dimension \( \dim X + s - 3 + d \deg c_1(X) \). There are evaluation maps \( \ev_i : \overline{M}_{0,s}(X,d) \to X \). If \( \alpha_1, \ldots, \alpha_s \in A^*(X) \) are classes in the Chow (or cohomology) group of \( X \) whose codimensions sum to \( \dim \overline{M}_{0,s}(X,d) \) then there is the (non-gravitational, genus zero) Gromov-Witten invariant

\[
I_d(\alpha_1 \cdots \alpha_s) := \int_{\overline{M}_{0,s}(X,d)} \ev_1^* \alpha_1 \cup \cdots \cup \ev_s^* \alpha_s.
\]

If \( \Gamma_1, \ldots, \Gamma_s \) are subvarieties of \( X \) with \( \alpha_i \) (Poincaré dual to) the fundamental class of \( \Gamma_i \) for each \( i \), then for general \((g_1, \ldots, g_s) \in G^s \) the Gromov-Witten invariant is equal to the number of degree \( d \) maps \( \mathbb{P}^1 \to X \) sending the \( i \)-th marked point into \( g_i \Gamma_i \) for each \( i \). For proofs of these facts, see [10].

3. **Main result**

We adopt the notation of Section 2 and prove the following result.

**Theorem.** If \( X = G/P \) where \( G \) is a complex simple Lie group of classical type and \( P \) is a maximal parabolic subgroup, \( d \) and \( s \) are positive integers, and \( \Gamma_1, \ldots, \Gamma_s \) are Schubert varieties whose codimensions sum to \( \dim \overline{M}_{0,s}(X,d) \), then the Gromov-Witten invariant \( I_d([\Gamma_1] \cdots [\Gamma_s]) \) is divisible by \( d^r \), where \( r = \# \{ i : \text{codim } \Gamma_i = 1 \} \), and the quotient is equal to the number of degree \( d \) rational curves on \( X \) incident to general translates of the \( \Gamma_i \).

We note that if \( \Gamma_i = X \) for any \( i \) then the Gromov-Witten invariant is zero (fundamental class axiom) and by Lemma 14 in [10] the set of degree \( d \) rational curves on \( X \) incident to general translates of the \( \Gamma_i \) is empty. If some \( r \geq 1 \) of the \( \Gamma_i \) have codimension 1, then the Gromov-Witten invariant is equal to \( d^r \) times the \((s-r)\)-point Gromov-Witten invariant with the divisor classes omitted (divisor axiom). This implies the divisibility assertion. We first prove the enumerative claim assuming \( \text{codim } \Gamma_i \geq 2 \) for all \( i \), then we treat the case when some of the \( \Gamma_i \) are divisors.

**Lemma.** Let \( \Gamma \) be a Schubert variety in \( X \) of codimension at least 2 with Schubert cell \( \Gamma^0 \). Fix a point \( p_1 \in \mathbb{P}^1 \). Then for each \( d \geq 1 \) there exists a degree \( d \) map \( f : \mathbb{P}^1 \to X \) such that

i. \( f \) is an unramified morphism;
ii. \( f(p_1) \in \Gamma^0 \);
iii. \( f \) maps a nonzero tangent vector at \( p_1 \) to a tangent vector at \( f(p_1) \) not contained in the tangent space to \( \Gamma^0 \);
iv. \( f^{-1}(\Gamma) = \{p_1\} \).

Recall that \( \overline{M}_{0,s}(X,d) \) is irreducible [17, 11]. For a point \( x \in X \) and for each \( i \) we observe that by the group action there is a birational isomorphism
between $ev_1^{-1}(x) \times \mathbb{A}^{\dim X}$ and $\overline{M}_{0,d}(X,d)$, and hence $ev_1^{-1}(x)$ is irreducible as well. In the situation of the Lemma we therefore obtain that $ev_1^{-1}(\Gamma)$ is irreducible, by [9, Theorem 4.17].

**Proof.** It suffices to verify (ii)-(iv) for a point of $\overline{M}_{0,1}(X,d)$, since the combination of these is an open condition in $ev_1^{-1}(\Gamma)$ and (i) is satisfied on a dense open subset of $ev_1^{-1}(\Gamma) \cap M_{0,1}(X,d)$. When $d = 1$ it is clear that (ii)-(iv) may be satisfied. When $d = 2$ and $X$ is an orthogonal Grassmannian, this follows from the description in [7, Lemma 2.1, proof of Thm. 2.3 or Lemma 3.1, proof of Thm. 3.3].

Otherwise, we consider two cases. There is a critical degree $d_0$, the smallest for which two general points on $X$ are joined by a rational curve of that degree: $d_0 = \min(m, n - m)$ when $X = G(m, n)$; otherwise $d_0$ is $m$ when $X = IG(m, 2n)$ and $m$ rounded up to the next even integer for the orthogonal Grassmannians (divided by two in the case of the maximal orthogonal Grassmannians). The first case we consider is when $d \leq d_0$. Then we have the following set-up from [6, §2.2, 3.2, 4.2], [7, §1.4, 2.4, 3.4]; specifically:

- a variety $Y_d$ parameterizing pairs $(A, B)$ with $\dim A = m - d$, $\dim B = m + d$, $A \subset B$, and $B \subset A^\perp$ when $X$ is an isotropic Grassmannian;
- an incidence correspondence

$$
\begin{array}{ccc}
T_d & \rightarrow & X \\
\downarrow \pi & & \downarrow \\
Y_d & & 
\end{array}
$$

(where $T_d$ consists of triples $(A, \Sigma, B)$ with $A \subset \Sigma \subset B$ and $\Sigma$ a point of $X$);
- “modified” Schubert varieties $Y_{\lambda} \subset Y_d$ each defined as the image by $\pi$ of the preimage $T_{\lambda} \subset T_d$ of $X_{\lambda}$; and
- a result identifying the three-point genus zero Gromov-Witten invariants in degree $d$ with (in some cases up to a certain power of 2) intersection points of modified Schubert varieties in $Y_d$. (In types $B$ and $D$ when $d$ is odd there is a codimension 3 subvariety of $Y_d$ denoted $Z_d^*$ in [7] to which we need to restrict our attention.)

The fiber of $\pi$ above a general point of $Y_{\lambda}$ is a Grassmannian of particular type (e.g., the Lagrangian Grassmannian of a symplectic $2d$-space in type $C$). Letting $\lambda^+$ be the smallest partition containing $\lambda$ so that according to [6, proof of Cor. 2, 4 or 6] or [7, Lemma 1.3, 2.1 or 3.1] the map $T_{\lambda^+} \rightarrow Y_{\lambda^+}$ is generically finite, the inequality $\dim Y_{\lambda} \geq \dim Y_{\lambda^+} = \dim T_{\lambda^+}$ is enough to guarantee that the intersection of $T_{\lambda}$ with the fiber of $\pi$ above a general point of $Y_{\lambda}$ has codimension at least 2 in the fiber and is generically smooth. Choose a general point of the intersection to be $f(p_1)$; then a general rational map of degree $d$ in the fiber of $\pi$ meets requirements (ii)-(iv), since we have an explicit description of a general rational curve on the fiber of $\pi$ by [7, Prop. 1.1]. For $d > d_0$, we simply have to take a degree $d_0$ curve as
Proof of the Theorem. In [10, Lemma 14] it is proven that, in the conditions of the Theorem, the intersection $ev_1^{-1}(g_1 \Gamma_1) \cap \cdots \cap ev_s^{-1}(g_s \Gamma_s)$ is a finite set of reduced points, each corresponding to an irreducible source curve $\mathbb{P}^1$ with automorphism-free map to the target variety $X$, for general $(g_1, \ldots, g_s) \in G^s$. The number of these is the Gromov-Witten invariant.

We prove the Theorem first in the case when each of the $\Gamma_i$ has codimension at least 2. We claim, for general $(g_1, \ldots, g_s) \in G^s$, that each of these finitely many intersection points satisfies (i)-(iv) of the Lemma, with $\Gamma = g_1 \Gamma_1$. Given this, we may repeat the argument with the other $\Gamma_i$ in place of $\Gamma_1$ and find for general $(g_1, \ldots, g_s) \in G^s - 1$ that for each of the finitely many intersection points $(C \cong \mathbb{P}^1, p_1, \ldots, p_s, f)$ and each $i$, the point $p_i$ is the unique one on $C$ having image contained in $g_i \Gamma_i$.

To prove the claim, by homogeneity we may fix $g_1 = e$ and verify the conditions for general $(g_2, \ldots, g_s) \in G^{s-1}$. Let $ev_1^{-1}(\Gamma_0^0)^{*}$ denote the open subset of $ev_1^{-1}(\Gamma_0^0) \cap M_{0,s}(X, d)$ satisfying (i)-(iv) of the Lemma. Now we apply the Kleiman-Bertini theorem to the diagram

$$
\begin{array}{c}
\Gamma_1^0 \times \cdots \times \Gamma_s^0 \\
\downarrow
\end{array}
\xrightarrow{f} X^{s-1}
$$

with action of $G^{s-1}$. Then, for general $(g_2, \ldots, g_s)$ the intersection is a finite set of reduced points, and there is no contribution to the intersection from points of $ev_1^{-1}(\Gamma_0^0)$ not in $ev_1^{-1}(\Gamma_0^0)^{*}$. The claim is verified.

For the case when some of the $\Gamma_i$ are divisors, we repeat the above argument but taking $ev_1^{-1}(\Gamma_0^0)^{*}$ to be defined by only conditions (i)-(iii) of the Lemma. Then for each of the surviving intersection points, the map $f : \mathbb{P}^1 \to X$ has properties (i)-(iii) at every preimage point of $\Gamma$. It follows that there are $d$ distinct choices for a marked point on $\mathbb{P}^1$ mapping to $\Gamma_1$. Hence there is a $d^r$-to-one correspondence (with $r$ as in the statement of the Theorem) between intersection points in $M_{0,s}(X, d)$ and degree $d$ rational curves on $X$ satisfying the incidence conditions.

Remark. The three-point genus zero Gromov-Witten invariants are those that arise as structure constants in the small quantum cohomology ring; they are those given by the quantum Schubert calculus. For these, according to the Theorem, the Gromov-Witten invariant precisely counts rational curves on $X$ except in one case: when $d = 2$ and one of the $\Gamma_i$ has codimension 1, then the Gromov-Witten invariant is twice the number of conics satisfying the incidence conditions. This is only possible when $X$ is an orthogonal
Grassmannian, reflecting the presence of $q^2$ terms in the quantum Pieri formula for multiplication with the codimension 1 Schubert class (where $q$ is the formal parameter of the quantum cohomology ring).

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