Efficient Algorithms for Scheduling Moldable Tasks

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Abstract—We study the classical problem of scheduling a set $T$ of $n$ moldable tasks on $m$ processors, which arises in multiprocessor scheduling from large scale parallel computations. This problem has been extensively studied for the objective of minimizing the makespan under the assumption of monotonic tasks, that is for tasks whose workload increases when executed in parallel on multiple processors; so far, the best result of interest is a $(\frac{3}{2} + \epsilon)$-approximation algorithm with a complexity of $O(mn \log \frac{n}{\epsilon})$. Motivated by recent results, we consider the case where the speedup is linear for small parallelization. Specifically, we introduce the following new assumption: the workload of a task is constant when assigned $\leq \delta$ processors (hence its execution time decreases linearly); but when assigned $> \delta$ processors the workload increases and the execution time decreases non-linearly. In this paper, for an arbitrary value of $\delta$, we propose simple and general procedures to separately optimize two scheduling objectives: (i) minimizing the makespan, and (ii) maximizing the sum of values of tasks completed by a deadline. In particular, we propose a $\frac{3}{2}\cdot (1 + \epsilon)$-approximation algorithm with a complexity of $O(n \log \frac{n}{\epsilon})$ for the first objective, and, a $\theta(\delta)$-approximation algorithm with a complexity of $O(n)$ for the second objective. For example, $\theta(5) = 0.7500 - \max \{ \frac{3(k-1)}{4m}, \frac{3}{4m} \}$, and $\theta(27) = 0.875 - \max \{ \frac{7(k-1)}{9m}, \frac{1}{3m} \}$, where $k$ is the upper bound on parallelized execution. In large scale parallel computations such as cluster computing, we have $m \gg \delta \geq \delta$ so that $\frac{3}{2} \to 0$ and our simple algorithms can achieve very good guarantees. Furthermore, to the best of our knowledge, our work is the first that considers the second objective for moldable tasks, while it has been considered for other types of tasks.

Index Terms—Parallel Scheduling, Approximation Algorithms, Malleable.

1 INTRODUCTION

1.1 Problem Description

We consider the classical problem of scheduling a set of $n$ independent moldable tasks $T = \{T_1, T_2, \ldots, T_n\}$ on $m$ identical processors. A task $T_i$ is specified by several characteristics: (1) value (or weight) $v_{i,p}$, (2) workload $D_{i,p}$, and (3) execution time $t_{i,p}$, when allocated $p_i$ processors, where $D_{i,p} = t_{i,p} \cdot p_i$. All tasks of $T$ are available at the starting time 0. Before the execution of a moldable task $T_i$, the scheduler can determine the number of processors allocated to it; once specified and executed, the task could not be preempted and this number cannot change during the execution.

In this paper, we will separately optimize two objectives: (i) minimizing the makespan, i.e., the maximum completion time of all tasks, and (ii) maximizing the social welfare, i.e., the sum of values of tasks completed by a deadline $d$, where partial execution of a task by the deadline yields no value. Here, appropriate schedules will be proposed to specify for each executed task a time interval within which it is executed and the numbers of processors assigned to it.

1.2 Motivations

The above problems are motivated by the optimal scheduling of multiprocessor systems designed for large scale parallel computations. In terms of the way that the workload of a task changes with the number of processors assigned to it, a classic assumption is that, tasks are monotonic, i.e., a task’s workload increases with the number of processors assigned to execute it while the execution time of this task decreases. Here, the monotonicity is used to model the inefficiency introduced by the extra communication and synchronization overhead among processors when parallelizing tasks [2], [3], [4], [6], [7], [8], [9]. So, for the objective of minimizing the makespan, the most efficient algorithm is the one proposed by Mounié et al. [8] with a time complexity of $O(mn \log \frac{n}{\epsilon})$, achieving an approximation ratio $\frac{3}{2} \cdot (1 + \epsilon)$. Here, the performance metric is makespan and the approximation ratio is defined to be the ratio of the performance achieved by an algorithm to the optimal one.

Recent study of many well-known benchmarks [10] shows by experiments that, when a small set of no more than $\delta$ processors is allocated to a task, the speedup of processing workload is linearly proportional to the number of processors being utilized; as a task is executed on more and more processors, the speedup will become slower and slower, and, even negative if executed on too many processors due to the synchronization and communication overhead. The above way of speedup is formally described as follows:

Assumption 1. The speedup is linear when a task is assigned $\leq \delta$ processors; however, the workload will increase with the number of assigned processors but the execution time will decrease in a nonlinear way when assigned $> \delta$ processors but $\leq k$ processors. In other words, we have for each task $T_i \in T$ that

$$v_{i,p} = \begin{cases} v_{i,1} & \text{if } p_i \leq \delta, \\ v_{i,1} \cdot \frac{p_i}{\delta} & \text{if } \delta < p_i \leq k, \\ v_{i,1} \cdot \frac{\delta}{p_i} & \text{otherwise.} \end{cases}$$
In [10], the way of speedup above is also approximated by a unified function $t_{j,p} = \frac{D_{j,p}}{T} + (p - 1) \cdot c$ where $c$ is a small positive real number. However, in this paper, we are interested in exploring the possibility of directly utilizing Assumption 1 to propose simpler yet more efficient algorithms than existing ones [8] where the monotonic assumption is adopted.

Besides the benchmark study in [10], parallel implementations of core algorithms in many fields also show that Assumption 1 often holds. In each such field, these core algorithms are repeatedly used in extensive applications. Their parallel implementation provides the opportunity to improve the overall performance of computing systems. For example, in cryptography, the security of lattice-based cryptosystems relies on the hardness of some lattice problems to which shortest vector problem (SVP) is central. A parallel implementation of SVP-solver [11] shows that $\delta$ could be set to 64. In computer vision, the scale-invariant feature transform (SIFT) is an algorithm to detect and describe local features in images. Applications include object recognition, robotic mapping and navigation, and 3D modeling etc. A parallel SIFT implementation [12] shows that $\delta = 8$. In bioinformatics, ProTest is one of the most popular tools for parallel SIFT implementation [12] shows that $\delta$ is linear to $\frac{1}{\lambda}$, which often holds. Other examples include computational methods such as smoothed-particle hydrodynamics (SPH) [14] and constrained-random verification (CRV) [15], and, the fundamentally important programming paradigm, MapReduce, for processing big data [16], [21]. Here, SPH and CRV are widely used respectively for simulating the dynamics of continuum media, such as solid mechanics and fluid flows, and, in industry for validating hardware designs.

### 1.3 Results

In this paper, for an arbitrary value of $\delta$, we propose a general scheduling algorithm that achieves a resource utilization of $\theta(\delta)$ with a time complexity of $O(n)$. Then, we obtain the following algorithmic results:

- a $\frac{1}{\lambda t^n} \cdot (1 + \epsilon)$-approximation algorithm with a complexity of $O(n \log \frac{1}{\delta})$ to minimize the makespan;
- a $\theta(\delta)$-approximation algorithm with a complexity of $O(n)$ to maximize the social welfare, i.e., the sum of values of tasks completed by a deadline $d$.

Here, $\theta(\delta)$ mainly depends on the parameter $\delta$ in Assumption 1, and could be computed by seeking a feasible solution of some inequalities. The values of $\theta(\delta)$ when $\delta$ takes different values are illustrated in Table 1, the larger the value of $\delta$ is, the better the performance of our algorithms is. In large scale parallel computations such as supercomputing or cluster computing [21], $m \gg k \geq \delta$ and $\frac{k}{m} \rightarrow 0$. Even in the earlier modern supercomputers such as IBM BlueGene/L, there are $m = 2^{16}$ processors inside [23], [24]. In modern clusters, there are usually tens of thousands of machines, each consisting multiple processors [25]. In the benchmark study of [10] and the application examples of [11], [12], [13], [14], the value of $\delta$ ranges in [1, 100]. We set the value of the maximum parallelism bound $k$ to be 300 [10]. Then, we could obtain Table 2 to help readers better perceive the performance of the proposed algorithms of this paper.

Previously, the best result of our interest for scheduling moldable tasks to minimize the makespan is a $(\frac{2}{\delta} + \epsilon)$-approximation algorithm in [8]. In terms of approximation ratio, we could intuitively conclude from Table 2 that the performance of the algorithm proposed in this paper is better than the one in [8], in the case where $m > k$ and $\delta \geq 5$, which often holds.

In the following, we show why our algorithm is comparable with the ones for moldable tasks [2], [6], [4], [7], [8], [9]. We could use the function $t_{j,p} = \frac{D_{j,p}}{T} + (p - 1) \cdot c$ to approximately characterize the execution time for the tasks of this paper when assigned $p$ processors where $p \leq \delta$. This function models the monotonicity of tasks but could be used to approximate the first point in Assumption 1 when $c$ is a small enough positive real number. Then, if an algorithm could produce a feasible schedule for monotonic tasks, it could also produce a feasible schedule for the tasks of this paper. The main reason for this is explained as follows. Given a schedule produced for monotonic tasks, assume each tasks $T_i \in T$ to be executed in some time interval $[t_1, t_2]$ and it is assigned $p_i$ processors. In the case where $p_i \leq \delta$, even if the actual speedup of $T_i$ is linear when assigned $\leq \delta$ processors, $T_i$ could still be successfully executed in $[t_1, t_2]$ with $p_i$ processors assigned, since the actual execution time of $T_i$ is less than the simulated one computed by the above function. So, the performance of an optimal schedule for the tasks of this paper is better than the one for monotonic tasks, and, we could conclude that, if our algorithm achieves a better approximation ratio for the algorithms for moldable tasks, it will also have a better performance.

Furthermore, the social welfare maximization objective is considered by this paper for the first time, while this objective has been considered for other types of tasks [17], [18], [19], [20].

### 2 Related Work

**Related work (i).** With the Assumption 1 where $\delta = 1$, the problem of this paper has been extensively studied in the last decades to minimize the makespan. The problem is strongly NP-hard even when $m \geq 5$ [1].

The first two lines of works are based on a two-phase approach; the first phase selects the number of processors assigned to each task and the second phase goes to solve the resulting non-moldable problem. The first line of works is to select an allotment, i.e., the number of processors allotted to each task, and solve the resulting problem of scheduling rigid tasks each of which is assigned this fixed number of processors. In particular, Turek et al. and Ludwig et al. show that any $\lambda$-approximation algorithm of a time complexity $O(f(m, n))$ for the rigid scheduling problem can be adapted into a $\lambda$-approximation algorithm of a complexity $O \left( n \log^2 m + f(m, n) \right)$ for the moldable scheduling problem [2], [3], [4]; then, the strip packing algorithm in [5] could be applied to obtain a polynomial time $2$-approximation algorithm.
The range of \( \theta \) for different ranges of \( \delta \):

| \( \delta \) | \( \theta(\delta) \) | \( \delta \) | \( \theta(\delta) \) | \( \delta \) | \( \theta(\delta) \) |
|---|---|---|---|---|---|
| [5, 9] | 0.7500 - \max \left\{ \frac{3(k-1)}{4m}, \frac{3}{m} \right\} | [22, 26] | \max \left\{ \frac{0.8571 - \delta}{k(m - \frac{88.57}{m})} \right\} | 58 | \max \left\{ \frac{9(k-1)}{10m}, \frac{885.5}{m} \right\} |
| [10, 16] | 0.8000 - \max \left\{ \frac{1(k-1)}{5m}, \frac{10.8}{m} \right\} | [27, 37] | \max \left\{ \frac{7(k-1)}{8m}, \frac{143.9}{m} \right\} | [59, 74] | \max \left\{ \frac{10(k-1)}{11m}, \frac{956.8}{m} \right\} |
| [17, 21] | \max \left\{ \frac{0.8333 - \delta}{5m}, \frac{40.67}{m} \right\} | [38, 57] | \max \left\{ \frac{0.8889 - \delta}{9m}, \frac{327.6}{m} \right\} | [75, 101] | \max \left\{ \frac{0.9167 - \delta}{12m}, \frac{1646}{m} \right\} |

An example for the values of \( \theta \) for different ranges of \( \delta \):

| \( \delta \) | \( \theta(\delta) \) | \( \delta \) | \( \theta(\delta) \) | \( \delta \) | \( \theta(\delta) \) |
|---|---|---|---|---|---|
| [5, 9] | 1.333 | [38, 57] | 1.130 | [59, 74] | 1.105 |
| [10, 16] | 1.250 | 58 | 1.116 | [75, 101] | 1.096 |
| [17, 21] | 1.200 | [59, 74] | 1.105 | [102, 122] | 1.088 |

The values of \( \theta \) for different ranges of \( \delta \):

In the second line of works \([7], [8]\), the following definition was given:

**Definition 1.** Given a positive real number \( d \), we define for each task \( T_j \) its canonical number of processors \( \gamma(j, d) \) as the minimal number of processors needed to execute task \( T_j \) in time at most \( d \).

The following relationship is used in both \([7]\) and \([9]\) to design scheduling procedures:

\[
d \geq t_{j, \gamma(j,d)} \geq \frac{\gamma(j,d) - 1}{\gamma(j,d)} \cdot d.
\]

The inequality can be derived from Definition 1 and the relation that \( D_{j, \gamma(j,d)} \geq D_{j, \gamma(j,d)-1} \).

The algorithm of \([7]\) classifies the tasks of \( \mathcal{T} \) into three subsets: \( S_1, S_2, \) and \( S_3 \); each task \( T_j \) in \( S_1, S_2, \) and \( S_3 \) respectively has an execution time in \([\lambda \cdot d, d], [\frac{3}{2} \cdot \lambda \cdot d], [0, \frac{3}{2}] \) when assigned \( \gamma(j,d) \) processors. By solving a knapsack problem, Mounié found a division of \( S_1 \) such that (i) a subset of \( S_1 \) is executed on \( m \) processors in \([0, d]\) and (ii) the other tasks of \( S_1 \) and all tasks of \( S_2 \) and \( S_3 \) could be executed in \([d, (1 + \lambda \cdot d)]\). As a result, a \( \sqrt{3} \)-approximation algorithm is proposed in \([7]\) where \( \lambda = \sqrt{3} - 1 \). The algorithm of \([8]\) also needs to solve a knapsack problem via dynamic programming to classify the tasks into two subsets \( T_1 \) and \( T_2 \) producing a complexity of \( O(m) \), where the tasks respectively in \( T_1 \) and \( T_2 \) are allocated \( \gamma(i, d) \) and \( \gamma(i, \frac{d}{2}) \) processors, and then the total workload of the tasks is \( \leq m \cdot d \). Then, the total number of processors allocated to \( T_2 \) may be \( > m \) and a series of reductions to the numbers of processors assigned to tasks is taken to get a feasible schedule. Here, Mounié et al. obtained a \( \frac{3}{2} \cdot (1 + \epsilon) \)-approximation algorithm with a complexity of \( O(mn \log_\frac{\epsilon}{\epsilon}) \) in \([9]\).

The third line of work \([9]\) formulates the original problem as a linear programming problem and propose an \( (1 + \epsilon) \)-approximation algorithm with a time complexity of \( O(n) \) given a fixed number of processors. However, independent of \( \epsilon \), the actual complexity is very high and exponential in the number of processors; this makes their algorithm of little practical interest. Like \([7], [8]\), this paper is interested in efficient, low complexity heuristics with a good performance guarantee.

**Related work (ii).** We now discuss the works on scheduling other types of parallel tasks to maximize the sum of values of tasks completed by a common deadline \([17], [18], [19], [20]\) or by individual deadlines \([21], [22]\). The scheduling objective, together with the task type (e.g., monotonicity), decides the key parameters that can indicate the performance guarantee of a scheduling algorithm, and further defines the specific goals with which we can design an algorithm. For example, Jansen et al. consider rigid tasks, each requiring a fixed number of processors to be assigned, and apply the theory of knapsack problem and linear programming to propose an \( \left( \frac{1}{2} + \epsilon \right) \)-approximation algorithm. The application of knapsack problem also enables selecting a subset of tasks to maximize their social welfare where a different type of parallel tasks is considered \([17]\).

Work-preserving tasks with multiple deadlines are considered \([21], [22]\). Jain et al. and Wu et al. \([21], [22]\) consider work-preserving tasks with individual deadlines and propose an optimal greedy algorithm with an approximation ratio of \( \frac{2}{s+3} \), where \( s \) is a parameter specific to the set of tasks. The workload of a work-preserving task does not change with the number of processors assigned to a task while the monotonicity of the tasks complicates the problem of this paper, causing the value obtained from processing a unit of workload to vary with the number of assigned processors. Here, the types of tasks (e.g., monotonic, moldable, preemptive) indeed also determines the way in which we can schedule a set of tasks, and further affect the perspective from which we can take the algorithmic analysis to derive the performance guarantee. Hence, their techniques cannot be applied to our problem directly.

### 3 Scheduling for Utilization

Recall that the problem of this paper is defined in Section \([14]\) together with Assumption \([11]\) and Definition \([1]\). In this section, we consider the situation in which the number of tasks is so large that only a subset could be completed by a deadline \( d \) on \( m \) processors; the tasks of \( \mathcal{T} \) will be considered one by one and, for an arbitrary value of \( \delta \), we will propose a general schedule to achieve as a good utilization as possible.
3.1 Parameter Identification

For every task \( T_j \in T \), \( 1 \leq \gamma(j, d) < +\infty \). In order to better understand how to design a schedule that achieves a utilization approximately equal \( r \), where \( r \) is a real number in \([0,1]\), we need to classify all tasks of \( T \) into three types respectively with execution times in \([0, (1-r) \cdot d], [r \cdot d, d] \), and another range. Such classification could be achieved by leveraging the following definition, together with Assumption [1] and the monotonicity of tasks (i.e., Inequality [1]). This definition specifies a parameter \( H \) and describes the other three parameters \( \nu, x_h, \delta' \) satisfying some properties.

**Definition 2.** Given the value of \( \delta \), let \( r \) be the maximum real number in \( \{ \frac{1}{2}, \frac{3}{4}, \ldots, \frac{2}{3} \} \) such that the following conditions are satisfied,

\[
1 \leq \nu \leq H - 1 \leq \delta' \leq \delta, \quad (2a)
\]
\[
\frac{\nu}{\delta'} \cdot r \geq 1 - r, \quad (2b)
\]
\[
\frac{(\nu - 1)}{\delta'} \cdot r < 1 - r, \quad (2c)
\]
and, further, for all \( \nu \leq h \leq H - 1 \), the following conditions are satisfied,

\[
h \cdot \frac{\delta'}{\nu} \cdot x_h \leq 1, \quad (3a)
\]
\[
\max\left\{ 1 - r, \frac{h - 1}{\delta'} \right\} \cdot x_h \geq r, \quad (3b)
\]

where \( H \) is such an integer that \( r = \frac{H - 1}{H} \) and \( \delta', \nu, x_h \) are integers.

**Proposition 3.1.** There exist feasible parameters \( H, \nu, \delta', \) and \( x_h \) described in Definition [2].

**Proof.** No matter what the value of \( \delta \) is, there exist feasible parameters that satisfy Inequalities [2a]-[2c] and Inequalities [3a]-[3b], e.g., the values of these parameters are \( \nu = H - 1 = \delta' = x_2 = 1 \) and \( r = \frac{1}{2} \). Hence, the proposition holds.

Given a fixed value of \( \delta \), one can attempt to find feasible parameters in Definition [2] by exhaustive search and the particular search procedure is described in Algorithm [1]. Here, the 5 loops that respectively begin at lines 1, 2, 3, 5, and 9 lead to that Algorithm [1] has a time complexity \( O(\delta^5) \) where \( \delta \) is a constant.

3.2 Task Classification

Task classification helps us understand how to assign different tasks (in terms of their execution times) to processors efficiently. Before giving the final classification, we first study how to classify the tasks of \( T \) with \( \gamma(j, d) \) respectively in \([1, \nu - 1]\) and \([\nu, H - 1]\).

For all \( h \leq \nu - 1 \), all the tasks \( T_j \in T \) with \( \gamma(j, d) = h \) are classified as follows:

- \( R^0_h = \{ T_j | t_{j,h} \geq r \cdot d \} \).
- \( R^1_h = \{ T_j | t_{j,h} < r \cdot d \} \).

Here, \( R^0_h \cup R^1_h = \{ T_j \in T | \gamma(j, d) = h \} \) and \( R^0_h \cap R^1_h = \emptyset \).

**Lemma 3.1.** Each task in \( R^0_h \) has an execution time \( < (1 - r) \cdot d \) when assigned \( \delta' \) processors.

**Proof.** Each task \( T_j \in R^0_h \) has an execution time \( < r \cdot d \) when assigned \( \gamma(j, d) = h \) processors. Due to Assumption [1] and \( \gamma(j, d) < \delta' \leq \delta \), when assigned \( \delta' \) processors, it has an execution time \( < \frac{\gamma(j,d)}{\delta'} \cdot r \cdot d \). Since \( \gamma(j,d) < \nu \), due to Inequality [2a], the proposition holds.

For all \( \nu \leq h \leq H - 1 \), all the tasks \( T_j \in T \) with \( \gamma(j, d) = h \) are classified as follows:

- \( A_h = \{ T_j | t_{j,h} < r \cdot d, t_{j,\delta'} \geq (1 - r) \cdot d \} \).
- \( A'_h = \{ T_j | t_{j,h} \geq r \cdot d \} \).
- \( A''_h = \{ T_j | t_{j,h} < r \cdot d, t_{j,\delta'} < (1 - r) \cdot d \} \).

Here, \( A_h \cup A'_h \cup A''_h = \{ T_j \in T | \gamma(j, d) = h \} \), and, the intersection of any two of \( A_h, A'_h \) and \( A''_h \) is empty.

**Proposition 3.2.** Any \( x_h \) tasks of \( A_h \) have a total execution time in \([r \cdot d, d] \) when executed on \( \delta' \) processors.

**Proof.** By Inequality [2a], when assigned \( h \) processors, a task \( T_j \) has an execution time \( > \frac{h - 1}{H} \cdot d \); further, by the definition of \( A_h \), the execution time of \( T_j \) is in \((\frac{h - 1}{H} \cdot d, d \cdot r \cdot d)\). By Assumption [1], when assigned \( \delta' \) processors, the execution time of \( T_j \) is reduced by a factor \( \frac{\delta'}{\delta} \) and is in \((\frac{h - 1}{H} \cdot d, \frac{\delta'}{\delta} \cdot r \cdot d)\); then, a task of \( A_h \) has an execution time in \([d \cdot \min\{1 - r, \frac{h - 1}{H}\}, \frac{\delta'}{\delta} \cdot r \cdot d]\). Further, due to Inequalities [3a]-[3b], any \( x_h \) tasks have a total execution time in \([r \cdot d, d] \) and the proposition holds.

Under the above classification of tasks, different subsets of tasks are further grouped together and all tasks of \( T \) are finally classified into 3 classes according to their execution times:

- \( A' \) = the union of \( \{ T_j \in T | \gamma(j, d) \geq H \} \), \( \cup_{\nu=1}^{H-1} A'_h \), and, \( \cup_{\nu=1}^{H-1} R^0_h \).
- \( A'' \) = the union of \( \cup_{h=1}^{H-1} A''_h \) and \( \cup_{\nu=1}^{H-1} R''_h \).
- the third class of tasks is \( \cup_{h=1}^{H-1} A_h \).

Here, \( \cup_{h=1}^{H-1} A_h \cup A' \cup A'' = T \).

**Proposition 3.3.** Each task in \( A' \) has an execution time \( \geq r \cdot d \) when assigned \( \gamma(j, d) \) processors.

**Proof.** Recall the definition of \( A' \). For all tasks of \( T \) with \( \gamma(j, d) \geq H \), we have \( t_{j,\gamma(j,d)} \geq r \cdot d \) due to Inequality (1). Together with the definition of \( A'_h \) (\( \nu \leq h \leq H - 1 \)) and \( R^0_h \) (\( 1 \leq h \leq \nu - 1 \)), the proposition holds.

**Proposition 3.4.** Each task in \( A'' \) has an execution time \( \leq (1 - r) \cdot d \) when assigned \( \delta' \) processors.

**Proof.** Recall the definition of \( A'' \). Due to Lemma 3.1 and the definition of \( A''_h \) (\( \nu \leq h \leq H - 1 \)), the proposition holds.

As illustrated by the blue tasks in Figure 1 in the next subsection, the function of Proposition 3.4 is as follows. Starting from time 0, sequentially execute as many tasks of \( A'' \) as possible on \( \delta' \) processors, and, if the next task cannot be completed by time \( d \), the previous tasks will have a total execution time \( \geq r \cdot d \) and the \( \delta' \) processors have a utilization \( \geq r \). Together with Propositions 3.2 and 3.3, this enables us to propose a schedule that approximately achieves a utilization \( r \).
Algorithm 1: Parameter Computation

```plaintext
/* sequentially seek the maximum $H$, $\delta'$, and the minimum $\nu$ */
1 for $H \leftarrow \delta + 1$ to 2 do
   for $\delta' \leftarrow \delta$ to $H-1$ do
      for $\nu \leftarrow 1$ to $H-1$ do
         temp1 $\leftarrow \nu \cdot \frac{H-1}{H} + \frac{H-1}{H}$;
         temp2 $\leftarrow \frac{(\nu-1) \cdot H-1}{H}$;
         if temp1 $\geq 1 \land$ temp2 $< 1$ then
            // Inequalities (2b) and (2c) are satisfied
            flag $\leftarrow$ an array of $H - \nu$ elements whose values are 0;
            for $h \leftarrow \nu$ to $H - 1$ do
               for $x_h \leftarrow 1$ to $\frac{h}{h'}$ do
                  // the upper bound of $x_h$ is determined by Inequality (3a)
                  temp3 $\leftarrow \frac{h}{\delta} \cdot r \cdot x_h$;
                  temp4 $\leftarrow \max \{1 - r \cdot \frac{h-1}{H}, x_h\}$;
                  if (temp3 $\leq 1$) $\land$ (temp4 $\geq \frac{H-1}{H}$) then
                     // Inequalities (3a) and (3b) are satisfied, given the current value of $h$
                     flag$(H - \nu + 1) \leftarrow 1$;
                     break;
         temp3 $\leftarrow$ the sum of all the element’s values of the array flag;
         if temp3 $= H - \nu$ then
            exit Algorithm 1
```

Fig. 1. Basic Scheduling Structure in UnitAlgo

3.3 Example

Now, we illustrate how to apply the above classification of tasks to propose a scheduling algorithm by considering a special case with $\delta = 5$. After this illustration, we will give a general schedule for the case with an arbitrary value of $\delta$ in the next subsection.

Using Algorithm 1 to solve Inequalities (2a)-(2c) and (3a)-(3b), we have that $\nu = 2$, $H = 4$, $\delta' = 5$, $x_3 = 2$, and $x_2 = 3$. Here, $r = \frac{H-1}{h} = \frac{3}{2}$. In Figure 1, each green, orange, gold, and blue rectangle denotes a task in $A'$, $A_3$, $A_2$, and $A''$; they have an execution time respectively in $[\frac{3}{4} \cdot d, d]$, $[\frac{9}{20} \cdot d, \frac{9}{20} \cdot d]$, $[\frac{3}{10} \cdot d, \frac{3}{10} \cdot d]$, and $[0, \frac{d}{3}]$, according to Proposition 3.3, the proof of Proposition 3.2 and Proposition 3.4.

As illustrated in Figure 1, the way to arrange the tasks of $T$ is as follows:

Step 1. Each (green) task $T_j \in A'$ are executed on $\gamma(j, d)$ processors alone.

Step 2. In the following, we assign processors to the tasks of $A_3, A_2, A''$.

a. In the beginning, every $x_3 = 2$ (orange) tasks in $A_3$ are sequentially executed on $\delta' = 5$ processors.

b. When there is one task of $A_3$ left, assign $\delta'$ processors to it; after finishing this task, execute as many (gold) tasks of $A_2$ as possible on these $\delta'$ processors until the next task in $A_2$ could not be completed by time $d$.

c. For the remaining tasks of $A_2$, every $x_2 = 3$ (gold) tasks in $A_2$ are sequentially executed on $\delta'$ processors.

d. When there is one (gold) task left in $A_2$, assign $\delta'$ processors to it and, after finishing it, execute as many (blue) tasks in $A''$ as possible until the next task cannot be finished before time $d$.

e. Subsequently, execute as many (blue) tasks in $A''$ sequentially on every $\delta'$ new processors in time $d$ until there are less than $\delta'$ processors.

In Figure 1, the $\delta'$ processors that process both orange tasks in $A_3$ and gold tasks in $A_2$ in Step (2.b) have an execution time $\geq d - \frac{3}{10} \cdot d$, since each gold task has an execution time $\leq \frac{h}{\delta} \cdot r \cdot d = \frac{3}{10} \cdot d$, and, there exists a task that could not be completed by time $d$, where $h = 2$. With a
similar reason, the δ′ processors that process both gold tasks in A_2 and gold tasks in A″ in Step (2.d) have an execution time \( \geq \frac{d - \frac{d}{3} \cdot 2}{m} \), since each blue task has an execution time \( \leq \frac{d}{4} \).

At last, in Step (2.e), when there are not enough processors to be assigned to the blue tasks of A″, there are at most \( \delta - 1 \) processors idle. All other assigned processors have an execution time \( \geq r \cdot d \). As a result, the above schedule achieves an average utilization of at least

\[
\left( \frac{m - 2\delta' + 1}{m} \right) \cdot r \cdot d + \delta' \cdot \left( \frac{3}{10} \cdot d \right) = \frac{3}{4} - \frac{3.25}{m} \quad \text{(4)}
\]

\[
r - \frac{\delta' - 1}{m} \cdot m \cdot r - \frac{\delta'}{m} \cdot \left( r - \left( 1 - \frac{h}{\delta} \cdot r \right) \right), \quad \text{(5)}
\]

where \( h = 2 \).

### 3.4 General Schedule

In this subsection, we generalize the example in Section 3.3 to propose a schedule for an arbitrary value of \( \delta' \) and the utilization of this schedule is also a generalization of Expression 4.

The operations in different steps of Section 3.3 are generalized to obtain some basic operations that will be used repeatedly in a general schedule. In particular, by generalizing the operations in Steps (2.a) and (2.c), the basic operations that we obtain is presented in Algorithm 2. Furthermore, Algorithm 3 (Algorithm 5) corresponds to Step (2.b) and (2.d) (resp. Step (2.e)). In these algorithms, \( B_{H-1} = A_{H-1}, \ldots, B_{k} = A_{\nu}, \) and \( B_{\nu} = A^\prime \). With these basic operations, a general schedule is proposed in Algorithm 6, also referred to as UnitAlgo. As illustrated in Figure 1, Algorithm 6 also considers the tasks in \( \tilde{A}', A_{H-1}, \ldots, A_{\nu}, \tilde{A}' \) sequentially. The details of Algorithm 6 is further explained by comments when describing it, with the support of the example in Section 3.3.

**Lemma 3.2.** With a time complexity of \( O(n) \), UnitAlgo achieves a resource utilization of

\[
\theta(\delta) = \begin{cases} 
\frac{r - \frac{(k-1)}{m}}{m}, & \text{if } \nu = H - 1, \\
\frac{r - \frac{(\delta' - 1) + \sum_{k=\nu}^{H-2} \left( r \cdot \frac{h \cdot r - 1}{m} \right)}{m}}{m}, & \text{otherwise}
\end{cases}
\]

where \( \nu, \delta', r, \) and \( H \) are computed by Algorithm 7

**Proof.** UnitAlgo considers tasks one by one and thus has a time complexity of \( O(n) \). In the following, we consider the worst-case utilization achieved when Algorithm 6 exits.

When Algorithm 6 exits at line 2, all the assigned processors have a utilization \( \geq r \) by Proposition 3.2, and there are at most \( \gamma(j, d - 1) \) processors idle. Since \( \gamma(j, d - 1) \leq k \), all the \( m \) processors have a utilization no smaller than

\[
\frac{(m - k + 1) \cdot r \cdot d}{m \cdot d} = r - \frac{r \cdot (k - 1)}{m}. \quad \text{(6)}
\]

In the following, we look at the case when Algorithm 6 does not exit at line 2. As far as the for-loop that begins at line 3 is concerned, the assigned processors during the call to Algorithm 2 (line 7) have a utilization \( \geq r \cdot d \) by Proposition 3.2 where \( \nu \leq h \leq H - 1 \). Here, the action of assigning processors occurs at line 3 of Algorithm 2. Furthermore, under the assumption of this section that there are so many tasks that not all of them could be processed on \( m \) processors by time \( d \), the assigned processors during the call to Algorithm 4 (line 9) also have a utilization \( \geq r \cdot d \) where \( h = \nu - 1 \), since each task in \( A'' \) has an execution \( \leq (1 - r) \cdot d \) by Proposition 3.4 and there exists a task that could not be completed by time \( d \). Here, the action of assigning processors occurs at line 3 of Algorithm 3.

If \( \nu = H - 1 \), the call to Algorithm 3 (line 12 of Algorithm 5) will only be executed when \( h = \nu - 1 \) where \( A^\prime \) is processed; then the assigned \( \delta' \) processors have an execution time \( \geq r \cdot d \). Here, the action of assigning processors occurs at line 11 of Algorithm step2bd. Finally, Algorithm 5 will exit at line 7, 9, 12, or 17; then there are at most \( \delta' \) processors idle. As analyzed in the third paragraph of this proof, all other assigned processors have an execution time \( r \cdot d \). As a result, when Algorithm 5 exits at line 7, 9, 12, or 17, the utilization is \( \frac{(m - \delta' + 1) \cdot r \cdot d}{m \cdot d} = \frac{r - r(\delta' - 1)}{m} \). Since \( \delta' \leq k \), in the case where \( \nu = H - 1 \), the worst case happens when Algorithm 5 exits at line 2. With 6, we conclude that Lemma 3.2 holds in the case where \( \nu = H - 1 \).

If \( \nu \leq H - 2 \), for each of the \( (H - 2) \)-th, \( \nu \)-th loops, when executing line 24 of Algorithm 5 in the worst case, there are \( \delta' \) processors that have an execution time \( \geq (1 - \frac{\nu}{r}) \cdot \frac{d}{\delta} \). To achieve the above worst case, when Algorithm 5 exits, \( h = \nu \) or \( \nu = 1 \). As a result, in the worst case, when Algorithm 5 exits at line 11, 26, or 30 where \( h = \nu - 1 \) or \( \nu \), the average utilization of \( m \) processors is

\[
\frac{(m - (H - 1) \cdot \delta' - (\delta' - 1)) \cdot r \cdot d + \sum_{h=\nu}^{H-2} (\frac{h - \nu}{r} \cdot d)}{m \cdot d} = \frac{r}{m} - \frac{r(\delta' - 1)}{m} + \frac{\sum_{h=\nu}^{H-2} (\frac{h - \nu}{r} + r - 1) \delta'}{m}. \quad \text{(7)}
\]

With 6 and 7, we could conclude Lemma 3.2 holds in the case where \( \nu \leq H - 2 \).

The values of \( \theta(\delta) \) for different values of \( \delta \) are also illustrated in Table 1 where \( \delta = \delta' \).

Given a set of tasks \( T \), let \( S \) denote the subset of tasks that are selected by Algorithm 5 to be scheduled on \( m \) processors, and \( D_{\min}^m \) denote the total workload of \( S \) to be processed when the number of processors assigned to each task of \( S \) is specified by Algorithm 5. The following lemma will be used when we propose a scheduling algorithm in Section 3.1 to minimize the makespan.

**Lemma 3.3.** Given any other feasible schedule of \( S \) on \( m \) processors, \( D_{\min}^m \) is a lower bound of the total workload of \( S \) to be processed.

**Proof.** With the maximum deadline \( d \), the minimum number of processors needed is \( \gamma(j, d) \) and the minimum workload to be processed is \( D_{\gamma(j,d)} \). In Algorithm 5, for all tasks with \( \gamma(j, d) \geq H \), they are assigned \( \gamma(j, d) \) processors. For all other tasks with \( \gamma(j, d) \leq H - 1 \leq \delta' \leq \delta \), they are assigned either \( \gamma(j, d) \) or \( \delta' \) processors; since \( H - 1 \leq \delta' \leq \delta \) and due to Assumption 1, the minimum workload of each task is processed. As a result, \( D_{\min}^m \) is the minimum total workload to be processed in order to generate a feasible schedule of \( S \) and the lemma holds.
4 Optimizing Objectives

In this section, we propose algorithms to optimize two specific scheduling objectives: (i) minimizing the makespan, and (ii) maximizing the total value of tasks completed by a deadline.

4.1 Makespan

As done by Mounié et al. [8] and Nagarajan et al. [16], we propose a binary search procedure to find a feasible makespan for a set of tasks $T$, presented as Algorithm 6. In particular, at the beginning of Algorithm 6, let $U$ and $L$ denote an upper and lower bound of a feasible makespan. Then, let $M = \frac{U + L}{2}$ and Algorithm 6 repeatedly executes the following operations until $\frac{U - L}{L} \leq 1 + \epsilon$:

1. $M \leftarrow \frac{U + L}{2}$;
2. if Algorithm 5 fails to produce a feasible schedule for all the tasks of $T$ with a makespan no more than $M$, $L \leftarrow M$; otherwise, $U \leftarrow M$.

When Algorithm 5 ends, we have that (i) Algorithm 5 could produce a feasible schedule of $T$ with a makespan no greater than $U$ but greater than $L$, and (ii) $U < (1 + \epsilon) \cdot L$.

**Theorem 4.1.** The algorithm OptiMakespan gives a $\frac{1}{(2^{\delta} + 1)(1 + \epsilon)}$-approximation algorithm with a time complexity of $O(n \log(\frac{2}{\epsilon}))$ for the makespan minimization problem.

**Proof.** By abuse of notation, let $\theta = \theta(\delta)$ and it is the worst-case utilization achieved by Algorithm 5. According to Theorem 3.2, we have that Algorithm 6 ends. On one hand, Algorithm 5 fails to generate a feasible schedule for all the tasks of $T$ with a makespan $L$. This shows that, in any feasible schedule of $T$, all the workload of $T$ to be processed is greater than $\theta \cdot m \cdot L$. Let $d^*$ denote the optimal makespan of $T$ on $m$ processors and we have that $d^* \geq \frac{\theta \cdot m \cdot L}{\delta}$ if $\nu = \nu - 1$. Set $B_i = B - B_i$ for all $h \leq i \leq H$.


\begin{algorithm}[th]
\caption{Unified Generalization of Steps (2.a) and (2.c)}
\begin{algorithmic}[1]
  \While{$|B_h| \geq x_h$ do}
  \If{there are no less than $\delta'$ processors idle then}
    \State sequentially execute $x_h$ tasks in $B_h$ on $\delta'$ processors and remove these tasks from $B_h$;
  \Else
    \State exit Algorithm 5
  \EndIf
\EndWhile
\end{algorithmic}
\end{algorithm}

\begin{algorithm}[th]
\caption{Unified Generalization of Steps (2.b) and (2.d)}
\begin{algorithmic}[1]
  \If{$\nu \leq h \leq H - 1$ then}
    \State $e \leftarrow d - \frac{h + \nu}{\delta} \cdot d$;
  \ElseIf{$h = \nu - 1$ then}
    \State $e \leftarrow r \cdot d$;
  \Else
    \State $e \leftarrow \nu \cdot d$;
  \EndIf
  \If{\(\sum_{T_j \in B_h} t_j, \delta' \geq e\) then}
    \State $b \leftarrow \sum_{T_j \in B_h} t_j, \delta'$;
    \State \Comment{as illustrated in Figure 4, $b$ denotes the execution time of the orange (resp. gold) task in Step. (2.b) (resp. Step. (2.d)).}
    \State $\mathcal{B} \leftarrow \bigcup_{h = h + 1}^H B_h$;
    \State \While{$\mathcal{B} \neq \emptyset$ do}
    \State randomly choose a task from $B_h$, denoted by $T_j$;
    \If{$b + t_j, \delta' < e$ then}
      \State $\mathcal{B} \leftarrow \mathcal{B} \cup \{T_j\}$, $b \leftarrow b + t_j, \delta'$;
      \State $B_h \leftarrow B_h - \{T_j\}$; \Comment{if $t_j, \delta' \leq 1 - e$}
    \Else
      \State \Comment{break; \Comment{then, }$e \leq b < d$}
    \EndIf
    \If{there are no less than $\delta'$ processors idle then}
      \State sequentially execute the tasks of $\mathcal{B}$ on $\delta'$ processors;
      \Comment{as illustrated in Figure 4, the processors that execute the tasks from multiple subsets $B_h$, $B_{h+1}$, \ldots have an execution time $\geq d - \frac{h + \nu}{\delta} \cdot d$ if $\nu \leq h \leq H - 2$ and $\geq r \cdot d$ if $h = \nu - 1$.}
      \State set $B_i = B - B_i$ for all $h \leq i \leq H$;
    \Else
      \State exit Algorithm 5
    \EndIf
  \EndWhile
  \State break;
\end{algorithmic}
\end{algorithm}

4.2 Scheduling to Maximize the Total Value

In this subsection, we consider the objective of maximizing the sum of values of tasks completed by a deadline as studied previously for other types of tasks by Anderson et al. [17] and Jansen et al. [18], [19], [20].

In the following, we first give a generic algorithm that will define the way that the tasks are accepted to be scheduled. Then we retrospectively analyze this algorithm and define the parameters that will decide the performance of this algorithm. The main ideas of this subsection are illustrated in Fig 7.

**Analysis of a Generic Greedy Algorithm.** To make a concise algorithmic analysis, we first consider the special case of Assumption 1, where $\delta = 1$, i.e., tasks are monotonic. For a monotonic task $T_j$, its marginal value $\frac{v_j}{D_j, p_j}$ varies with
Theorem 4.2. An upper bound of the optimal social welfare for our problem is $\sum_{T_i \in S'} \frac{v_i}{(\omega \alpha)}$. 

The algorithmic framework for the second scheduling objective is illustrated in Figure 2. The algorithm consists of two main steps: 

1. **Step 1:** Assign tasks to processors based on their marginal value. 
2. **Step 2:** Process the remaining tasks in the order of their marginal value.

The algorithm uses monochromatic task sets, where each task set is assigned to a different color processor. The algorithm first initializes the sets $B_h$ and $A_h$ for all $h \leq H$, and assigns tasks to processors based on their marginal value. The algorithm then iterates over the sets $B_h$ to process tasks, ensuring that no task allocated by GenGreedyAlgo gets allocated by an inefficient algorithm. The algorithm uses three different algorithms to process different subsets of the tasks: 

- **Algorithm 1:** For each task $T_j \in A'$, assign $\gamma(j, d)$ processors to it if there are $\gamma(j, d)$ processors idle, and exit Algorithm 5 otherwise. 
- **Algorithm 2:** Sequentially execute all the remaining tasks in $A_{h+1}$, where $\delta$ processors are processing tasks with the same color (e.g., orange, gold, blue). 
- **Algorithm 3:** Process the unprocessed tasks in $B_h$ in the previous loops that begin at line 3, as illustrated in Figure 1 where $\delta$ processors are processing tasks with different colors (e.g., orange and gold, gold and blue). 

The algorithm terminates when all tasks are processed, and the optimal social welfare is achieved.
Algorithm 7: GenGreedyAlgo

Input: \( n \) tasks
Output: a feasible allocation of resources to tasks

1. initialize \( S_i = \{T_1, T_2, \ldots, T_i\} \) \((1 \leq i \leq n)\), a list
2. \( S_0 = \emptyset, i' = 0 \);
3. for \( i = 1 \) to \( n \) do
   4. if a feasible schedule of \( S_i \) is output by GS then
      5. Record this schedule by \( \text{SCHEDULE} \);
   6. else
      7. \( i' = i; \)
5. exit;

Proof. Since a task \( T_j \) has a workload of at most \( D_{j,\gamma(j,\lambda d)} \), \( \omega \cdot d \cdot m \leq \sum_{T_j \in S_i} D_{j,\gamma(j,\lambda d)} \). Each task can get a marginal value of at most \( v_j = \frac{v_j}{D_{j,\gamma(j,\lambda d)}} \) since \( D_{j,\gamma(j,\lambda d)} \) is the minimum workload to be processed for finishing \( T_j \) within a time interval \([0, d]\). Let \( \mathcal{OPT} \) denote the optimal social welfare in this problem. Then, we have \( \sum_{T_j \in S_i} v_j' \leq \mathcal{OPT} \). Together with \( \sum_{T_j \in S_i} v_j' \leq \sum_{T_j \in S_i} D_{j,\gamma(j,\lambda d)} \leq \sum_{T_j \in S_i} D_{j,\gamma(j,\lambda d)} \leq \frac{1}{\alpha} \) this lemma holds. \( \square \)

Theorem 4.2 demonstrates the main challenge in designing the algorithm GS for monotonic tasks: increasing resource utilization without decreasing efficiency of the tasks (i.e., marginal value). Indeed, increasing resource utilization may not be beneficial if it decreases the marginal value too much. For instance, even without parallelism constraint, the trivial algorithm that assigns \( m \) processors to each task achieves resource utilization 1 but may be bad in terms of social welfare. In the following, we consider the case where GS is UnitAlgo.

Specific Algorithm. Due to Assumption 1 for each task \( T_j \) that is allocated no more than \( \delta \) processors, we still have \( \alpha_j = 1 \). In UnitAlgo, a task is allocated either \( \gamma(j, d) \) or no more than \( \delta \) processors, and, in both cases, \( \alpha_j = 1 \). Hence, \( \alpha = 1 \). GenGreedyAlgo has a time complexity of \( \mathcal{O}(n) \). The following proposition follow from Theorem 4.2 and Lemma 3.2.

Proposition 4.1. GenGreedyAlgo, with GS replaced by UnitAlgo, gives an \( \theta(\delta) \)-approximation algorithm with a complexity of \( \mathcal{O}(n) \) for the social welfare maximization problem.

5 Conclusion

We study simple algorithms for scheduling a set \( \mathcal{T} \) of \( n \) moldable tasks for two objectives: (i) minimizing the makespan, and (ii) maximizing the total value of tasks completed by a deadline. Motivated by recent experimental results, we introduce a more general assumption to study this problem: the speedup of executing a task is linear when it is assigned a small set of no more than \( \delta \) processors; its workload increases with the number of assigned processor but its execution time decreases in a nonlinear way when this number is greater than \( \delta \). We propose general procedures to address the related scheduling problem. In particular, we propose a \( \theta(\delta) \)-approximation algorithm with a complexity of \( \mathcal{O}(n) \) for the objective of maximizing the sum of values of tasks completed by a deadline, and a \( \frac{1}{\theta(\delta)} \cdot (1 + \epsilon) \)-approximation algorithm with a complexity of \( \mathcal{O}(n \log \frac{1}{\delta}) \) for the objective of minimizing the makespan. For example, \( \theta(5) = 0.7500 \) and \( \theta(27) = 0.8750 \).

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