Accessing topological order in fractionalized liquids with gapped edges

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We consider manifestations of topological order in time-reversal-symmetric fractional topological liquids (TRS-FTLs), defined on planar surfaces with holes. We derive a general formula for the topological ground state degeneracy of such a TRS-FTL, which applies to cases where the edge modes on each boundary are fully gapped by backscattering terms. The degeneracy is exact in the limit of infinite system size, and is given by \( q^{N_h} \), where \( N_h \) is the number of holes and \( q \) is an integer that is determined by the topological field theory. When the degeneracy is lifted by finite-size effects, the holes realize a system of \( N_h \) coupled spin-like \( q \)-state degrees of freedom. In particular, we provide examples where “artificial” \( \mathbb{Z}_q \) quantum clock models are realized. We also investigate the possibility of measuring the topological ground state degeneracy with calorimetry, and briefly revisit the notion of topological order in \( s \)-wave BCS superconductors.

I. INTRODUCTION

The robust ground state degeneracy (GSD) that arises in topologically ordered systems has been an object of intense study over the past quarter-century. Interest in such states of matter has been motivated in large part by the desire to access quasiparticles with non-Abelian statistics, whose nontrivial braiding could be used as a platform for quantum computation. Nevertheless, to date there has been no definitive experimental proof that such non-Abelian quasiparticles exist, nor has there been any direct observation of topological GSD. There have been several theoretical proposals for the experimental detection of topological degeneracy. One set of proposals for the (putative) non-Abelian \( \nu = 5/2 \) quantum Hall state focuses on measuring the contribution of the GSD to the electronic portion of the entropy at low temperatures. Observable signatures of this contribution include the thermopower and the temperature dependence of the electrochemical potential and orbital magnetization. The thermopower has been measured on several occasions with no conclusive signatures. Abelian fractional quantum Hall (FQH) states are also topologically ordered, but the bulk GSD in these systems is only accessible on closed surfaces (e.g., the torus). This is unnatural for experiments, which are confined to finite planar systems, although a recent proposal suggests a transport measurement in a bilayer FQH system that avoids this handicap by effectively altering the topology of the system.

In this paper, we propose that time-reversal-symmetric fractional topological liquids (FTLs) may constitute a promising alternative platform for realizing the topological GSD in experimentally accessible geometries. FTLs with time-reversal symmetry (TRS) have an effective description in terms of doubled Chern-Simons (CS), or so-called BF, theories. Examples of time-reversal-symmetric FTLs with topological order include fractional quantum spin Hall systems, Kitaev’s toric code, and even the \( s \)-wave BCS superconductor. In the present work we emphasize FTLs whose edge states in planar geometries can be completely gapped without breaking TRS, which is possible when certain criteria are satisfied. In these cases, the degenerate ground state manifold is well separated from excited states and the GSD on punctured planar surfaces is accessible experimentally.

Our program for this paper is as follows. We first derive a general formula for the GSD of a doubled CS theory defined on a plane with \( N_h \) holes, in cases where all helical edge modes are gapped by backscattering terms. This topological degeneracy increases exponentially with the number of holes, and is exact in the limit where all holes are infinitely large and infinitely far apart. We then consider finite-sized systems, where the degeneracy is split exponentially by quasiparticle tunneling processes. In this setting, we argue that the holes themselves realize an effective spin-like system, whose Hilbert space consists of what was formerly the degenerate ground state manifold. We then examine calorimetry as a possible experimental probe of the degeneracy. We argue that, for suitable materials, the contribution of the GSD to the low-temperature heat capacity could be observed experimentally, even in the presence of the expected phononic and electronic backgrounds. Finally, we also briefly revisit the notion of topological order in \( s \)-wave superconductors, which was suggested by Wen and investigated in detail by Hansson et al. in Ref. 17. We argue that, for a thin-film superconductor with \( (3+1) \)-dimensional electromagnetism, there is indeed a ground state degeneracy, which is related to flux quantization. However, this degeneracy is lifted in a power-law fashion, rather than exponentially, and is therefore not topological in the canonical sense of Refs. 18,19

II. THE TOPOLOGICAL DEGENERACY

In this section we derive a formula for the ground state degeneracy of a TRS-FTL with gapped edges. We begin
A general time-reversal-symmetric doubled Chern-Simons theory in (2+1)-dimensional space and time has the form \[ \mathcal{L}_{CS} = \frac{1}{4\pi} K_{ij} \epsilon^{\mu
u\rho} a_i^\mu \partial_\nu a_j^\rho + \frac{\epsilon}{2\pi} Q_i \epsilon^{\mu
u\rho} A_\mu \partial_\nu a_i^\rho, \] (2.1a)

where \( i,j = 1, \ldots, 2N \), \( \mu, \nu, \rho = 0, 1, 2 \), and summation on repeated indices is implied. Here, the \( 2N \times 2N \) matrix \( K_{ij} \) is symmetric, invertible, and integer-valued. The fully antisymmetric Levi-Civita tensor \( \epsilon^{\mu\nu\rho} \) appears with the convention \( \epsilon^{012} = 1 \). The components \( A_\mu \) of the electromagnetic gauge potential are restricted to (2+1)-dimensional space and time, and the vector \( Q \) has integer entries that measure the charges of the various CS fields \( a_i^\mu \) in units of the electron charge \( e \). The theory contains \( N \) Kramers pairs of CS fields, which transform into one another under the operation of time-reversal. We will therefore be particularly interested in scenarios where the \( 2N \times 2N \) matrix \( K \) has the following block form, which is consistent with TRS, as was shown in Ref. \[ K = \begin{pmatrix} \kappa & \Delta \\ \Delta^T & -\kappa \end{pmatrix}, \] (2.1b)

where the \( N \times N \) matrices \( \kappa = \kappa \) and \( \Delta = -\Delta \). TRS further imposes that the charge vector possess the block form (see Ref. \[ Q = \begin{pmatrix} \theta \\ 0 \end{pmatrix}, \] (2.1c)

The theory (2.1) can also be re-expressed in terms of an equivalent BF theory by defining the linear transformation \( a_i^\mu := R_i a_i^\mu \), where

\[ R := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \] (2.2a)

with \( 1 \) the \( N \times N \) identity matrix. This linear transformation induces the \( K \)-matrix and charge vector

\[ \tilde{K} := R^{-1} K R = \begin{pmatrix} 0 & x \\ x^T & 0 \end{pmatrix}, \] (2.2b)

\[ x := \kappa - \Delta, \] (2.2c)

\[ \tilde{Q} := R^{-1} Q = \begin{pmatrix} \theta \\ 0 \end{pmatrix}. \] (2.2d)

Note that the transformation (2.2a) preserves \( \det K \) [c.f. Eq. (2.2b)].

When defined on a manifold with boundary, the CS theory (2.1a) has an associated theory of \( 2N \) chiral bosons \( \phi_i \) at the edge. In the most generic case, the boundary of the system consists of a disjoint union of an arbitrary number of edges, each with a Lagrangian density of the form (in the absence of the gauge field \( A_\mu \))

\[ \mathcal{L}_E = \frac{1}{4\pi} (K_{ij} \partial_i \phi_j \partial_j \phi_i - V_0^2 \partial_x \phi_i \partial_x \phi_i) + \mathcal{L}_T, \] (2.3)

where \( K_{ij} \) is the same \( 2N \times 2N \) matrix as before and the matrix \( V_0^2 \) encodes non-universal information specific to a particular edge. The Lagrangian density \( \mathcal{L}_T \) generically contains all inter-channel tunneling operators,

\[ \mathcal{L}_T = \sum_{T \in \mathcal{L}} U_T(x) \cos \left( T^T K \phi(x) - \zeta_T(x) \right), \] (2.4)

where \( T \) is a \( 2N \)-dimensional integer vector, \( \theta^T = (\phi_1, \ldots, \phi_{2N}) \), and \( \mathcal{L} \) is the set of all tunneling vectors \( T \) allowed by TRS and charge conservation (if it holds). The real-valued functions \( U_T(x) \) and \( \zeta_T(x) \) encode information about disorder at the edge and are further constrained to be consistent with TRS (see Ref. \[ \text{[19]} \]). When TRS is imposed, a necessary and sufficient condition for gapping out the bosonic modes in the edge theory (2.3) is the existence of \( 2N \)-dimensional vectors \( T_i \in \mathcal{L} \) satisfying \[ T_i^T \theta = 0, \quad \forall i \quad \text{(charge conservation)}, \] (2.5a)

\[ T_i^T K T_j = 0, \quad \forall i, j \quad \text{(Haldane criterion)}. \] (2.5b)

Strictly speaking, the criterion (2.5a) need not hold in a general system, such as (for example) in the case of a superconductor. In this case, one replaces charge conservation with charge conservation mod 2, so that \( T_i^T \theta \) is only constrained to be even. In the next section, we will focus on cases where the criteria (2.5) are satisfied.

B. Calculation of the degeneracy

The ground state degeneracy on the torus of a multi-component Abelian Chern-Simons theory of the form (2.1a) is known on general grounds to be given by \[ |\det K|^{1/2} \] We now present an argument that, for a doubled CS theory with a gapped edge and a \( K \)-matrix of the form (2.1b), the ground-state degeneracy of the theory on the annulus

\[ A := [0, \pi] \times S^1 \] (2.6a)

is given by the formula

\[ \text{GSD} = \sqrt{|\det K|} = \text{Pf} \left( \begin{pmatrix} \Delta & \kappa \\ -\kappa & \Delta \end{pmatrix} \right). \] (2.6b)

Note that \( |\det K| \) is the square of an integer \[ \text{[19][21]} \] so the GSD in this case is also an integer. We will now prove this result.
1. Gauge invariance in a system with gapped edges

To proceed, we rewrite the Lagrangian density \( \mathcal{L}_{CS} \), in the absence of the electromagnetic gauge potential \( A_\mu \), in terms of two separate sets of \( N \) CS fields \( \alpha^i, \beta^j \),

\[
\mathcal{L}_{CS} = \frac{\epsilon^{\mu \nu \rho}}{4\pi} \left[ \kappa_{ij} \left( \alpha^i_\mu \partial_\nu \alpha^j_\rho - \beta^i_\mu \partial_\nu \beta^j_\rho \right) + \Delta_{ij} \left( \alpha^i_\mu \partial_\nu \beta^j_\rho - \beta^i_\mu \partial_\nu \alpha^j_\rho \right) \right]. \tag{2.7}
\]

Here, \( i, j \in \{1, \cdots, N\} \), and the “new” CS fields are defined as \( \alpha^i_\mu(x, t) = \alpha^{i+1}_\mu(x, t) \) and \( \beta^j_\mu(x, t) = \beta^{j+1}_\mu(x, t) \). We define the CS action on the annulus to be

\[
S_{CS} := \int dt \int d^2 x \mathcal{L}_{CS}(x, t). \tag{2.8}
\]

Its transformation law under any local gauge transformation of the form

\[
\alpha^i_\mu \rightarrow \alpha^i_\mu + \partial_\mu \chi^i, \quad \beta^j_\mu \rightarrow \beta^j_\mu + \partial_\mu \chi^j, \tag{2.9a}
\]

where \( \chi^i, \chi^j \) are real-valued scalar fields, is

\[
S \rightarrow S + \delta S \tag{2.9b}
\]

with the boundary contribution

\[
\delta S_{CS} := \int dt \int d\partial A \frac{\epsilon^{\mu \nu \rho}}{4\pi} \left[ \kappa_{ij} \left( \chi^i_\alpha \partial_\nu \alpha^j_\rho - \chi^j_\beta \partial_\nu \beta^i_\rho \right) + \Delta_{ij} \left( \chi^i_\alpha \partial_\nu \beta^j_\rho - \chi^j_\beta \partial_\nu \alpha^i_\rho \right) \right]. \tag{2.9c}
\]

Here, the boundary \( \partial A \) of \( A \) is the disjoint union of two circles \( \partial A := S^1 \sqcup S^1 \) and \( d\partial x := \epsilon_{\mu \nu \rho} d\sigma \), with \( d\sigma \) the line element along the boundary.

There are two ways to impose gauge invariance in the doubled Chern-Simons theory \( S_{CS} \). On the one hand, if the criteria \( \text{(2.5)} \) do not hold, we must demand that there exist a gapless edge theory with the action \( S_E = -\delta S_{CS} \) on the boundary \( \partial A \) of the annulus. On the other hand, if the criteria \( \text{(2.5)} \) hold, gauge invariance on the annulus can be achieved by demanding that the anomalous term \( \delta S_{CS} = 0 \) identically. The latter option is accomplished if the following two conditions hold for all \( i = 1, \cdots, N \),

\[
\chi^i_\alpha |_{\partial A} = \chi^j_\beta |_{\partial A}, \quad \alpha^i_\mu |_{\partial A} = \beta^j_\mu |_{\partial A}. \tag{2.10}
\]

Using the above conditions, it is possible to show that Eq. \( \text{(2.6b)} \) follows in much the same way as does its counterpart on the torus, as we show in the next section. Before proceeding with the full argument, we first provide an intuitive picture of why this is, for the case where \( \Delta = 0 \) in Eq. \( \text{(2.1b)} \). In this case, Eq. \( \text{(2.7)} \) describes two decoupled CS liquids, one with \( K \)-matrix \( \kappa \) and the other with \( K \)-matrix \( -\kappa \). We can imagine that the two CS liquids live on separate copies of the annulus \( A \), which are coupled by the tunneling processes that gap out the edges. The conditions in Eq. \( \text{(2.10)} \) ensure that the two coupled annuli can be “glued” together into a single surface, on which lives a composite CS theory with a GSD given by \( |\det \kappa| \) (see Fig. 1). Remarkably, these gluing conditions are also sufficient to treat the general case, where \( \Delta \neq 0 \) (see next section).

The gluing conditions \( \text{(2.10)} \) generalize readily to the case of a system with the topology of an \( N_h \)-punctured disk. In this generalization, the boundary \( \partial A \) is the disjoint union of \( N_h + 1 \) copies of \( S^1 \) \( (\partial A = S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1) \). Since each of these edges is gapped, anomaly cancellation enforces independent gluing conditions for each copy of \( S^1 \). Using these conditions, it is possible to show that the GSD on the annulus is given by \( |\det K|^{N_h/2} \).

2. Wilson loops, large gauge transformations, and their algebras

We can now use the gluing conditions \( \text{(2.10)} \), arising as they do from the need to cancel the anomalous boundary term \( \text{(2.9c)} \), to construct Wilson loop operators, which can in turn be used to determine the dimension of the ground state subspace. To do this, we choose to work with the BF form of the CS action, defined in Eqs. \( \text{(2.2)} \). We denote the transformed set of CS fields by

\[
a^i_{\pm \mu} := a^i_\mu \pm \beta^i_\mu, \tag{2.11a}
\]

so that

\[
\mathcal{L}_{CS} = \frac{\epsilon^{\mu \nu \rho}}{4\pi} \left( \kappa_{ij} a^i_{+, \mu} \partial_\nu a^{j-}_\rho + \kappa_{ij} a^i_{-, \mu} \partial_\nu a^{j+}_\rho \right). \tag{2.11b}
\]

where the matrix \( \kappa \) is defined in Eq. \( \text{(2.2c)} \). In this new basis, the gluing conditions \( \text{(2.10)} \) become Dirich-
let boundary conditions on the \((-\) fields, \\
\chi^i_{\mid \partial A} = 0, \quad a^i_{\mid \partial A} = 0, \quad (2.12) \\
for \(i = 1, \cdots, N\). Rewriting the Lagrangian density in the 
gauge \(a^i_{\top,0} = 0\) (this can be done using a gauge transform- 
ation obeying the gluing conditions), we obtain \\
\[ 
\mathcal{L}_{CS} = \frac{1}{4\pi} \left[ x_{ij} \left( a^i_{\top,2} \partial_0 d^j_{\top,-1} - a^j_{\top,1} \partial_0 d^i_{\top,-2} \right) + x_{ij} \right] \left( a^i_{\bot,-2} \partial_0 a^j_{\bot,+1} - a^j_{\bot,-1} \partial_0 a^i_{\bot,+2} \right) 
\] \quad (2.13a) \\
supplemented by the \(2N\) constraints (\(i = 1, \cdots, N\)) \\
\[ 
\partial_1 a_{\top,-2} - \partial_2 a_{\top,-1} = 0, \quad \partial_1 a_{\bot,-2} - \partial_2 a_{\bot,-1} = 0. \quad (2.13b) 
\]
The constraints (2.13b) are met by the decompositions \\
\[ 
a_{\top,1}^i(x_1, x_2, t) = \partial_1 \chi_{\top}^i(x_1, x_2, t) + a_{\bot,1}^i(x_1, t), \quad (2.14a) 
\]
\[ 
a_{\top,2}^i(x_1, x_2, t) = \partial_2 \chi_{\top}^i(x_1, x_2, t) + a_{\bot,2}^i(x_2, t), \quad (2.14b) 
\]
of the CS fields provided \(\chi_{\top}^i(x_1, x_2, t)\) are everywhere 
smooth functions of \(x_1\) and \(x_2\), while \(a_{\bot,1}^i(x_1, t)\) and 
\(a_{\bot,2}^i(x_2, t)\) are independent of \(x_2\) and \(x_1\), respectively. 
Furthermore, the geometry of an annulus is implemented by the boundary conditions \\
\[ 
\chi^i_{\pm}(x_1, x_2 + 2\pi, t) = \chi^i_{\pm}(x_1, x_2, t) \quad (2.15a) 
\]
for the fields parametrizing the pure gauge contributions and \\
\[ 
\chi^i_{\pm}(0, x_2, t) = \chi^i_{\pm}(\pi, x_2, t) = 0, \quad (2.15b) 
\]
\[ 
a_{\pm,1}^i(0, t) = a_{\pm,1}^i(\pi, t) = 0, \quad (2.15c) 
\]
\[ 
a_{\pm,2}(x_2, t)|_{x_1 = 0} = a_{\pm,2}(x_2, t)|_{x_1 = \pi} = a_{\pm,2}(x_2, t) = 0, \quad (2.15d) 
\]
for the gluing conditions. The coordinate system employed in these definitions is depicted in Fig. 2.

The next step is to show that the barred variables de- 
couple from the remaining (pure gauge) degrees of free- 
dom. This can be done by inserting the decomposition 
(2.14) into the action and using the boundary conditions 
(2.15). We can now consider the action governing the 
barred variables alone, \\
\[ 
S_{\text{top}} = \frac{1}{2\pi} \int dt \bar{x}_j A_{\top,2} \bar{A}_{\bot,1}, \quad (2.16a) 
\]
where, for all \(i = 1, \cdots, N\), we have defined the global 
degrees of freedom \\
\[ 
A_{\top,1}^i(t) := \int_0^\pi dx_1 \bar{a}_{\bot,-1}^i(x_1, t), \quad (2.16b) 
\]
\[ 
A_{\top,2}^i(t) := \int_0^{2\pi} dx_2 \bar{a}_{\bot,+2}^i(x_2, t). \quad (2.16c) 
\]
for any \( i, j = 1, \ldots, N \). Because we require that \( \kappa \) is an integer matrix, this means that

\[
[U_1^i, U_2^j] = [U_1^i, U_1^j] = [U_2^i, U_2^j] = 0 \tag{2.21}
\]

for all \( i, j = 1, \ldots, N \). Hence, all \( U_1^i, U_2^j \) with \( i = 1, \ldots, N \) can be diagonalized simultaneously. Since any one of \( U_1^i \) and \( U_2^j \) generates a transformation that leaves the path integral invariant, the vacua of the theory must be eigenstates of any one of \( U_1^i \) and \( U_2^j \) for \( i = 1, \ldots, N \).

3. Dimension of the ground-state subspace

In order to determine the GSD of the theory, it suffices to determine the number of eigenstates of any one of \( U_1^i \) and \( U_2^j \) for \( i = 1, \ldots, N \). To do this, we follow the argument of Wesolowski et al.\textsuperscript{23} which can be adapted to our case with only minor modifications.

First, we define the eigenstates of any one of \( U_1^i \) and \( U_2^j \) for \( i = 1, \ldots, N \) by

\[
U_1^i |\Psi\rangle = e^{i\gamma_1} |\Psi\rangle, \quad U_2^j |\Psi\rangle = e^{i\gamma_2} |\Psi\rangle. \tag{2.22}
\]

Since \( A_1^i \) and \( A_2^j \) do not commute, we may choose to represent the state \(|\Psi\rangle\) in the basis for which \( A_1^i \) is diagonal by

\[
\psi(\{A_1^i\}) := \langle \{A_1^i\} |\Psi\rangle. \tag{2.23}
\]

The representation \( \psi(\{A_1^i\}) \) follows from the representation \( \psi(\{A_1^i\}) \) by a change of basis to the one in which \( A_2^j \) is diagonal. The large gauge transformations \( \text{(2.20a)} \) are represented by

\[
U_1^i := e^{2\pi \partial/\partial A_1^i}, \quad U_2^j := e^{-i \kappa_i A_1^i}, \tag{2.24}
\]

in the basis \( \text{(2.23)} \). The eigenvalue problem then becomes

\[
U_1^i \psi(\{A_1^i\}) := \psi (A_1^1, \ldots, A_1^i + 2\pi, \ldots, A_1^N) \\
= e^{i\gamma_1} \psi(\{A_1^i\}), \tag{2.25a}
\]

\[
U_2^j \psi(\{A_1^i\}) := e^{-i \kappa_j A_1^i} \psi(\{A_1^i\}) \\
= e^{i\gamma_2} \psi(\{A_1^i\}). \tag{2.25b}
\]

Equation \( \text{(2.25a)} \) implies that we can write the following series for \( \psi \):

\[
\psi(\{A_1^i\}) \equiv \psi(\{A_1^i\}) = e^{i\gamma_1 A_1^i/2\pi} \sum_n d(n) e^{i n A_1^i}, \tag{2.26}
\]

where \( n = (n_1, \ldots, n_N)^T \in \mathbb{Z}^N \), \( A_1 = (A_1^1, \ldots, A_1^N)^T \in \mathbb{R}^N \), and \( \gamma_1 = (\gamma_1^1, \ldots, \gamma_1^N)^T \in \mathbb{R}^N \).

Second, we seek the constraints on the real-valued coefficients \( d(n) \) entering the expansion \( \text{(2.26)} \) that, as we shall demonstrate, fix the dimension of the ground-state subspace. To this end, we extract from the \( N \times N \) matrix \( \kappa \) that was defined in Eq. \( \text{(2.2c)} \) the family

\[
\kappa := \begin{pmatrix} k_1^T \\ \vdots \\ k_N^T \end{pmatrix} \tag{2.27a}
\]

of \( N \) vectors from \( \mathbb{Z}^N \) and from its inverse \( \kappa^{-1} \) the family

\[
\kappa^{-1} := (\ell_1 \cdots \ell_N) \tag{2.27b}
\]

of \( N \) vectors from \( \mathbb{Q}^N \). By construction, these vectors satisfy

\[
k_1 \cdot \ell_i = \delta_{ij}. \tag{2.27c}
\]

Using these vectors, we observe that inserting the series \( \text{(2.26)} \) into the left-hand side of Eq. \( \text{(2.25b)} \) gives

\[
U_2^j \psi(\{A_1\}) = e^{i\gamma_1 A_1^i/2\pi} e^{-i k_i A_1^i} \sum_n d(n) e^{in A_1^i} \\
= e^{i\gamma_1 A_1^i/2\pi} \sum_n d(n + k_i) e^{in A_1^i} \\
= e^{i\gamma_2} \psi(\{A_1\}), \tag{2.28}
\]

which implies

\[
d(n + k_i) = e^{i\gamma_1} d(n) \tag{2.29}
\]

for all \( i = 1, \ldots, N \). The constraint \( \text{(2.29)} \) is automatically satisfied by demanding that

\[
d(n) = e^{i\gamma_2 (\kappa^{-1})^T n} d(n) \tag{2.30a}
\]

with

\[
d(n) = \hat{d}(n + k_i), \tag{2.30b}
\]

since

\[
\gamma_2 \cdot (\kappa^{-1})^T k_i = \gamma_2^i \ell_i = \gamma_2^i. \tag{2.30c}
\]

Hence, insertion of \( \text{(2.30a)} \) into the expansion \( \text{(2.26)} \) that solves the eigenvalue problem \( \text{(2.25a)} \) gives the expansion

\[
\psi(\{A_1\}) = e^{i\gamma_1 A_1^i/2\pi} \sum_n e^{i\gamma_2 (\kappa^{-1})^T n} \hat{d}(n) e^{in A_1^i} \tag{2.31}
\]

that solves the eigenvalue problem \( \text{(2.25b)} \).

Third, condition \( \text{(2.30b)} \) implies that the set of vectors \( \{n\} \) forms a lattice with basis vectors \( \{k_i\} \). The number of inequivalent points in the lattice is therefore given by

\[
r = |\det (k_1 \cdots k_N)| = |\det \kappa^T| = |\det \kappa|. \tag{2.32}
\]

This means that we can decompose any \( n \) as

\[
n = v_m + p_i k_i, \tag{2.33}
\]
where $p_i \in \mathbb{Z}$ and we have introduced $r$ linearly independent vectors $v_m$. We can therefore rewrite

$$\psi(A_1) = \sum_{m=1}^{r} d_m f_m(A_1),$$

where

$$d_m := \tilde{d}(v_m + p_i k_i) = \tilde{d}(v_m),$$

and

$$f_m(A_1) := e^{i\gamma_1 \cdot A_1/(2\pi)} \times \sum_{p_1, \ldots, p_N} e^{i\gamma_2 \cdot (\langle x \rangle^{-1})^T(v_m + p_i k_i)} e^{i(v_m + p_i k_i) \cdot A_1}. \quad (2.34c)$$

Since any $\psi(A_1)$ in the ground-state manifold can be written in this way, we have demonstrated that there are $r = |\det \varphi|$ linearly independent ground-state wavefunctions $f_m(A_1)$ in the topological Hilbert space. In other words, we have shown that

$$\text{GSD} = |\det \varphi| = \sqrt{|\det K|}, \quad (2.35)$$

with $K$ defined in Eq. (2.21b). This is precisely the result advertised in Eq. (2.6b). Note that because $\varphi$ is an integer-valued matrix, it has an integer-valued determinant. Consequently, $\sqrt{|\det K|} = |\det \varphi|$ is an integer.

An extension of this argument to the case of a plane with $N_h$ holes, along the lines discussed at the end of Sec. II B 1, yields a set of Wilson loops like those in Eqs. (2.18a) for each hole. [See, e.g., Eqs. (3.1) in the next section.] Since these sets of Wilson loops are completely independent, one obtains a degeneracy of size $|\det K|^{N_h/2}$.

### III. APPLICATIONS

With the results of Sec. II in hand, we now explore some of the consequences of Eq. (2.6b). We begin by examining the fate of the topological degeneracy in finite-sized systems, before considering the possibility of using calorimetry to detect experimental signatures of the degeneracy. We close the section by re-evaluating the proposed topological field theory for the $s$-wave BCS superconductor in light of the results of this paper.

#### A. Finite systems: clock models and beyond

On closed manifolds, the topological degeneracy is exact only in the limit of infinite system size. This is a result of the fact, pointed out by Wen and Niu, that quasiparticle tunneling events over distances of the order of the system size lift the degeneracy exponentially. This observation was also confirmed numerically for the case of the $(2+1)$-dimensional Abelian Higgs model on the torus by Vestergren et al. in Refs. 24 and 25. A similar splitting occurs for manifolds with boundary, like those studied in this work. For a planar system with many holes, each of which carries a $q$-fold topological degeneracy (where $q := \sqrt{|\det K|}$) in the limit of infinite system size, there are two kinds of tunneling events that can lift the degeneracy. These are (1) tunnelings that encircle a single hole and (2) tunnelings between boundaries. Below we argue that, in a finite-sized system with $N_h$ holes, the array of $N_h$ coupled $q$-state degrees of freedom can be modeled as a spin-like system [see Fig. 3(a)].

To see how this arises, we first note that for a system with $N_h$ holes it is possible to define a set of Wilson loops for each hole. Analogously to Eqs. (2.18a), we define

$$W_{1,j} := \exp \left( i \int_{C_{1,j}} d\ell \cdot \bar{a}_1^\dagger(x,t) \right) \quad (3.1a)$$

$$W_{2,j} := \exp \left( i \int_{C_{2,j}} d\ell \cdot \bar{a}_2^\dagger(x,t) \right), \quad (3.1b)$$

where the open curve $C_{1,j}$ connects the $j$-th hole to the...
outer boundary, and the closed curve $C_{2,j}$ encircles the $j$-th hole [see Fig. 3(b)]. Each set of operators obeys an independent copy of the algebra \( (2.18b) \). Furthermore, for any pair of holes $j$ and $k$, the Wilson loop

\[
W_{1,jk}^i := W_{1,j}^i W_{1,k}^i
\]

(3.1c)

connects these holes. More generally, any number of holes can be connected by compositions of the Wilson loops defined in Eqs. (3.1). In an infinite system, the topological protection of the degeneracy \((2.35)\) arises because the Wilson loops defined in Eqs. (2.18a) are nonlocal operators and are therefore forbidden from entering the Hamiltonian. In a finite system, however, the Wilson loops are no longer nonlocal degrees of freedom and can therefore enter the effective theory. In principle, all powers and combinations of the Wilson loops are allowed to enter the effective Hamiltonian

\[
H_{\text{eff}} := \sum_{i=1}^{N_h} \sum_{j=1}^{N_h} \left( h_{1,j} W_{1,j}^i + h_{2,j} W_{2,j}^i + \sum_{k=1}^{N_h} J_{jk} W_{1,jk}^i + \cdots \right),
\]

(3.2)

where the omitted terms include higher powers of the Wilson loops as well as all necessary Hermitian conjugates. In practice, however, all couplings in $H_{\text{eff}}$ are exponentially small in the shortest available length scale, which limits the tunneling rates. For example, $J_{jk} \propto e^{-c_{d,jk}/\xi}$, where $c$ is a constant of order one, $d_{jk}$ is the distance between holes $j$ and $k$ [see Fig. 3(a)], and $\xi$ is a length scale associated with quasiparticle tunneling.

It is interesting to note that the Hamiltonian $H_{\text{eff}}$ admits a certain amount of external control—the holes can be arranged in arbitrary ways, and the magnitudes of the couplings can be tuned by changing the length scales $R$, $d_{jk}$, and $D$. In particular, many terms in $H_{\text{eff}}$ can be tuned to zero by varying these length scales. We will make use of this freedom below.

To illustrate in what sense the effective Hamiltonian \((3.2)\) can be thought of as a spin-like system, we consider a specific class of examples. In particular, we consider the family of TRS-FTLs defined by

\[
K := \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \quad Q := \begin{pmatrix} 2 \\ 2 \end{pmatrix},
\]

(3.3)

where $q$ is an even integer. One verifies using Eq. \((2.5)\) that a single tunneling term of the form \((2.4)\) with $T = (1, -1)^T$ is sufficient to gap out the counterpropagating edge modes without breaking TRS as defined in Ref. 10.

In this case, Eq. \((2.6a)\) predicts a $q$-fold degeneracy per hole. To obtain the explicit effective Hamiltonian, we define

\[
\sigma_j := W_{1,j}, \quad \tau_j := W_{2,j},
\]

(3.4a)

whose only nonvanishing commutation relations arise from the algebra

\[
\sigma_j \tau_j = e^{-2\pi i/q} \tau_j \sigma_j.
\]

(3.4b)

One can check by writing down explicit representations of $\sigma_j$ and $\tau_j$ that they also satisfy

\[
\sigma_j^q = \tau_j^q = 1.
\]

(3.5)

For example, in the case $q = 2$ we may use Pauli matrices, e.g.,

\[
\sigma_j = \sigma_z, \quad \tau_j = \sigma_z,
\]

(3.6)

and in the case $q = 4$ we may use

\[
\sigma_j := \text{diag} \left( 1, e^{-i\pi/2}, e^{-i\pi}, e^{-i3\pi/2} \right),
\]

(3.7a)

\[
\tau_j := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

(3.7b)

For a system with $N_h$ holes of size $R$ arranged in a one-dimensional chain with lattice spacing $d$, the effective Hamiltonian in the limit $D \gg d, R$ becomes that of a one-dimensional $\mathbb{Z}_q$ quantum clock model (see Ref. 27 and references therein),

\[
H_{\text{eff}} := \sum_{i=1}^{N_h-1} J_i \left( \sigma_{i+1}^z + \text{H.c.} \right) + \sum_{i=1}^{N_h} h_i \left( \sigma_i + \text{H.c.} \right),
\]

(3.8)

where $J_i \propto e^{-c_1 d_i/\xi}$ and $h_i \propto e^{-c_2 R_i/\xi}$, with the real constants $c_1$ and $c_2$ of order unity. For simplicity, we have constrained the couplings $J_i$ and $h_i$ to be real, although their magnitude and sign is allowed to vary from hole to hole (hence the subscripts $i$). Note that in the above Hamiltonian, terms linear in $\sigma_j$ do not appear, as the associated couplings are suppressed by factors of order $e^{-c_1 D/\xi} \ll e^{-c_1 d_i/\xi}, e^{-c_2 R_i/\xi}$. Similarly, longer-range two-body terms, as well as higher powers of the $\sigma_j$ and $\tau_j$, are also omitted, as they correspond to higher-order tunneling processes.

The Hamiltonian of the clock model \((3.8)\) is invariant under the symmetry operation

\[
H_{\text{eff}} \to S H_{\text{eff}} S^{-1}
\]

(3.9a)

generated by

\[
S := \prod_{i=1}^{N_h} \tau_i^j.
\]

(3.9b)

Indeed, under the conjugation by $S$, $\tau_j^i \to \tau_j^i$ and $\sigma_j^i \to e^{-2\pi i/q} \sigma_j^i$ for all $j$. This $\mathbb{Z}_q$ symmetry can be thought of as a remnant of the $q^{N_h}$-fold topological degeneracy of the TRS-FTL, which would be present in the limit $d, R, D \to \infty$.

**B. Probing the topological degeneracy with calorimetry**

In this section, we consider experimental avenues to detect the topological degeneracy of a punctured TRS-
FTL. We focus our attention on calorimetry as a possible probe. In a sample with \(N_h\) holes, the ground state degeneracy provides a contribution \(S_{\text{GSD}} = N_h k_B \ln q\), where \(k_B\) is the Boltzmann constant and \(q = \sqrt{\det K}\), to the total entropy \(S_{\text{tot}}\). If the areal density of holes is kept fixed, then for a sample of length \(L\), we have \(S_{\text{GSD}} \sim L^2\) for the topological contribution, which is extensive. This suggests that, were a suitable material to be discovered, one might be able to detect the topological degeneracy of a punctured TRS-FTL by measuring its heat capacity. Such a measurement is feasible with current technology, as membrane-based nanocalorimeters enable the determination of heat capacities \(C_V\) in \(\mu g\) samples (and smaller), to an accuracy of \(\delta C_V/C_V \sim 10^{-4} - 10^{-5}\) down to temperatures of order \(100\) mK.

We first determine the topological contribution to the heat capacity for some particular examples. To do this, we return to the class of TRS-FTLs defined in Eq. \(3.3\). The heat capacity in this case is easiest to determine from the clock model of Eq. \(3.8\) in the paramagnetic limit \(J_i \to 0\), which is achieved for \(d \gg R\) [see Fig. 3(a)]. Setting \(h_i = h\) for convenience, we see that the clock model can be rewritten, after a change of basis, as

\[
H_{\text{eff}} = h \sum_{i=1}^{N_h} \left( \sigma_i + \sigma_i^\dagger \right) = 2h \sum_{i=1}^{N_h} \cos \left( \frac{2\pi}{q} n_i \right),
\]

where \(n_i = 0, \ldots, q-1\). Consequently the partition function is given by

\[
Z = \left( \sum_{n=0}^{q-1} e^{-2\beta h \cos(2\pi n/q)} \right)^{N_h},
\]

where \(\beta = 1/(k_B T)\) and \(T\) is the temperature. The topological heat capacity at constant volume, \(C_V^{\text{top}}\), is then determined from the partition function by standard methods. For example,

\[
C_V^{\text{top}} = N_h \frac{h^2}{k_B T^2} \left\{ \begin{array}{ll} 4 \text{sech}^2 \left( \frac{2h}{k_B T} \right), & q = 2, \\ 2 \text{sech}^2 \left( \frac{h}{k_B T} \right), & q = 4, \\ 9 \cosh \left( \frac{2h}{k_B T} \right) + \cosh \left( \frac{4h}{k_B T} \right) + 8, & q = 6, \\ 2 \cosh \left( \frac{h}{k_B T} \right) + \cosh \left( \frac{2h}{k_B T} \right), & q = 8, \end{array} \right.
\]

and so on.

To date, there has been no experimental realization of a TRS-FTL or fractional topological insulator. Since background contributions to the heat capacity are material-dependent, it is difficult to provide a precise estimate of the observable effect. However, we can nevertheless identify some constraints on the possible materials that would favor such a measurement.

To do this, let us estimate the various background contributions to the heat capacity of a TRS-FTL. First, we note that, because any TRS-FTL must have a gap \(\Delta\), the electronic contribution \(C_V^e\) to the heat capacity is

\[
C_V^e \sim \frac{\Delta}{T} e^{-\eta \Delta/ (k_B T)},
\]

where \(\eta\) is a constant of order one. The exponential suppression of \(C_V^e\) implies that this contribution is always negligible at sufficiently small temperatures.

However, one must also consider the phononic contribution, which follows a Debye power law at low temperatures. This contribution scales with the sample volume, which could be three-dimensional if the TRS-FTL is formed in a heterostructure, as is the case in quantum Hall systems. This fact, which was noted in Ref. [7], poses the greatest challenge to detecting the topological contribution to the heat capacity, which scales with the area of the two-dimensional sample. In principle, however, one may assume that the TRS-FTL lives in a strictly two-dimensional sample, or at least in a thin film. In this case, we have that the phononic contribution \(C_V^p\) to the heat capacity is

\[
C_V^p \propto k_B \left( T/T_D \right)^2,
\]

where \(T_D\) is the Debye temperature (100 K, say). We verified numerically, by simulating a square lattice of masses and springs, that the presence or absence of holes has little effect on the phonon spectrum as long as the holes are sufficiently small. We therefore expect the Debye law to hold both with and without holes, as long as one takes into account the excluded volume due to the holes.

The total heat capacity is obtained by adding the three contributions:

\[
C_V(T) = N_a \left[ C_V^{\text{top}}(T) + \nu C_V^p(T) + \frac{1}{N_a} C_V^e(T) \right],
\]

where \(N_a\) is the number of atoms in the sample and \(\nu := N_h/N_a\) determines the number of holes. The above
degeneracy in a power-law fashion, let us consider the ori-

tator can be measured by local external probes.

magnetic gauge field that is present in the superconduc-

tion. Consequently, the true electro-

field, which, in a real planar superconductor, is not con-

sidered here, but for the sake of completeness, we note that the degen-

acy in the canonical sense of Refs. 1–3. The reason for

en is not what one might call a topological degen-

ary of the two-fold degeneracy. Recall that for an annular

superconductor (a thin-film mesoscopic ring, for exam-

ple), the phase of the superconducting order parameter

winds by $2\pi$ around the hole if a flux quantum $\phi_0 = h/2e$

is trapped inside. This indicates that the electronic spec-

trum of the superconductor cannot be used to distinguish

between cases where an even ($\phi = 0 \mod \phi_0$) or odd

($\phi = 1 \mod \phi_0$) number of flux quanta penetrate the

hole. This is precisely the origin of the degeneracy. How-

ever, because the electromagnetic field also exists outside

the sample, there is an additional electromagnetic energy

coefficient associated with having a flux quantum trapped in

the hole. If we assume for simplicity that the flux is dis-

tributed uniformly over the hole (radius $R$) and does not

penetrate into the superconductor, then the energy cost

is proportional to

\[ \int_V d^3r |B|^2 = \frac{\phi_0^2}{2\pi R^2} L_z, \]

where $V$ is the interior of the cylinder in Fig. 5 and $L_z$ is

the height of the cylinder. Strictly speaking, because the

magnetic field lines must close outside the annulus, one

needs to replace $L_z$ by a length scale bounded from below by

the outer radius of the annulus. This energy cost

vanishes as $1/R$ for $R, L_z \to \infty$, which means that the

ground state degeneracy is lifted as a power law, rather than

exponentially.

The reason underlying this power-law splitting is the fact that the electromagnetic gauge field is not an emerg-

gent gauge field in the same sense as the Chern-Simons fields that are present in, say, a fractional topological insu-

lator with gapped edges. To elaborate on this distinction,

we first recall that the topological degeneracy de-

bered in Ref. 17 arises from a dynamical treatment of the

electromagnetic gauge field in $(2+1)$-dimensional space and
time. The topological sectors in which this degener-

ey is encoded reside in the Hilbert space of the elec-

magnetic gauge field, which is in turn entangled with the

C. Are superconductors topologically ordered?

In an insightful paper, it was argued by Hansson et al.
in Ref. 17 that ordinary $s$-wave BCS superconductors are
topologically ordered. In fact, it was shown that, when
the electromagnetic gauge field is treated dynamically
and confined to $(2+1)$ dimensional space and time, the
superconductor admits a description in terms of a BF
theory like the one defined in Eqs. (2.2), with

\[
\bar{K} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.
\]

(3.14)

Furthermore, it was shown that the edge states that arise
when the above theory is defined in a finite planar geometry
are generically gapped by Cooper pair creation terms.
The proposed theory is consistent with the time-reversal
symmetry of the $s$-wave superconductor and captures the
statistical phase of $\pi$ that is acquired by an electron upon
encircling a vortex. This effective theory, which is the same
as that of the $\mathbb{Z}_2$ lattice gauge theory in its de-

confined phase, predicts a four-fold GSD on the torus,
whose exponential splitting in finite systems was verified
numerically in Refs. 24 and 25.

Since the theory defined by Eq. (3.14) falls squarely
within the class of theories studied in this paper, it is
tempting to draw the conclusion that the $s$-wave super-

conductor exhibits a two-fold GSD on the annulus. Below
we argue that, while this is indeed the case, the degener-

acy is not exponential but power-law in nature, and
therefore is not what one might call a topological degen-

eracy in the canonical sense of Refs. 11–13. The reason for
this is that the topological nature of the superconductor
results from the dynamics of the electromagnetic gauge
field, which, in a real planar superconductor, is not con-

fined to the sample itself, but rather extends through all
three spatial dimensions. Consequently, the true elec-

magnetic gauge field that is present in the superconduc-
tor can be measured by local external probes.

To see how this coupling to the environment lifts the
degeneracy in a power-law fashion, let us consider the ori-
Hilbert space of the electronic degrees of freedom. Since the photonic degrees of freedom in a real annular superconductor also exist outside the sample, there is nothing to prevent the environment from fixing a topological sector. For example, the presence of an external magnetic field in the hole can privilege one topological sector over the other by fixing the flux through the hole.

It is crucial to contrast this with the case of a “true” TRS-FTL, where the Chern-Simons fields arise naturally from electron-electron interactions. In this case, the topological sectors reside in the Hilbert space of the electrons alone, and the CS fields do not exist outside the sample. Inserting an electromagnetic flux through the hole of an annular TRS-FTL switches between topological sectors, but does not betray any information about the identity of the initial or final sector. For this reason, the degeneracy of different topological sectors is completely protected from the environment in the limit of infinite system size.

IV. SUMMARY AND CONCLUSION

In this paper we have derived a general formula for the topological ground state degeneracy of a time-reversal symmetric, multi-component, Abelian Chern-Simons theory. The formula, which holds when the edge states of the theory are gapped, says that the GSD of the system on a planar surface with \( N_h \) holes is given by \(|\text{det} K|^{N_h/2}\), where \( K \) is the \( K \)-matrix. We then examined the situation where this topological degeneracy is split exponentially by finite-size effects, and found that the set of \( N_h \) holes admits a description in terms of an effective spin-like system whose couplings can be tuned by varying the sizes and arrangement of the holes. We also examined calorimetry as a means of detecting the topological degeneracy. The proposed experiment would measure the contribution of the topological degeneracy to the heat capacity at low temperatures, which we argued could be visible on top of the expected electronic and phononic backgrounds as long as the host material is sufficiently thin. Finally, in light of these results, we revisited the notion that ordinary s-wave superconductors are topologically ordered. We argued that, while thin-film superconductors do indeed possess a ground state degeneracy on punctured planar surfaces, this degeneracy is lifted in a power-law, rather than an exponential, fashion due to the (3+1)-dimensional nature of the electromagnetic gauge field.

We close by pointing out several possible extensions of this work. First, we note that our results concerning the ground state degeneracy should still apply to TRS-FTLs where the backscattering terms of Eq. (2.4) do not respect time-reversal symmetry. We could therefore also have considered in this paper fractional topological insulators whose protected edge modes are gapped by perturbations that break TRS, as is done in Refs. 33 and 34. Second, it would be interesting to determine what other kinds of “artificial” spin-like systems could be realized in TRS-FTLs with more complicated \( K \)-matrices than those in the class of Eq. (3.3). It is conceivable that remnants of the topological degeneracy may manifest themselves as exotic properties of these less conventional models. Finally, we must point out that a fractionalized two-dimensional state of matter with time-reversal symmetry has not yet been discovered experimentally, and that the search for such a state must remain a priority.

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A formula for the GSD on the cylinder of a TRS-FTL with gapped edge states was also derived in Ref. 36. However, that formula relies on detailed knowledge of the edge theory, whereas the formula derived in this paper relies only on knowledge of the bulk CS theory and the stability of the edge theory against backscattering terms. We have verified that our formula agrees with that derived in Ref. 36 for the specific examples discussed in this paper.

In the semiclassical approximation employed in Ref. 2, $\xi \sim (m^{*}\Delta)^{-1/2}$, where $m^{*}$ is the effective mass of the quasiparticle and $\Delta$ is the gap to quasiparticle excitations. If instead we used the three-dimensional Debye formula, we would have $C_{\text{ph}}^{\text{vol}} \sim T^3$, which would produce an even smaller contribution at low temperatures, so long as the sample is not too thick.