Algebraic solution of weighted minimax single-facility constrained location problems

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Abstract

We consider location problems to find the optimal sites of placement of a new facility, which minimize the maximum weighted Chebyshev or rectilinear distance to existing facilities under constraints on the feasible location domain. We examine a Chebyshev location problem in multidimensional space to represent and solve the problem in the framework of tropical (idempotent) algebra, which deals with the theory and applications of semirings and semifields with idempotent addition. The solution approach involves formulating the problem as a tropical optimization problem, introducing a parameter that represents the minimum value in the problem, and reducing the problem to a system of parametrized inequalities. The necessary and sufficient conditions for the existence of a solution to the system serve to evaluate the minimum, whereas all corresponding solutions of the system present a complete solution of the optimization problem. With this approach, we obtain a direct, exact solution represented in a compact closed form, which is appropriate for further analysis and straightforward computations with polynomial time complexity. The solution of the Chebyshev problem is then used to solve a location problem with rectilinear distance in the two-dimensional plane. The obtained solutions extend previous results on the Chebyshev and rectilinear location problems without weights.

Key-Words: tropical mathematics, idempotent semifield, constrained optimization problem, single-facility location problem.

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1 Introduction

Location problems present an important research domain in optimization, which dates back to the XVII century and originates in the influential works of P. Fermat, E. Torricelli, J. J. Sylvester, J. Steiner and A. Weber. Many results achieved in this domain are recognized as notable contributions to various fields, such as operations research, computer science and engineering.

To solve location problems, which are formulated in different settings, a variety of analytical approaches and computational techniques exists, including methods of linear and mixed-integer linear programming, methods of discrete, combinatorial and graph optimization [28, 17, 9, 7, 25]. Another approach, which finds increasing application in solving some classes of optimization problems, is to use models and methods of tropical mathematics.

Tropical (idempotent) mathematics deals with the theory and applications of semirings and semifields with idempotent addition (see, e.g., [12, 14, 13, 26]). It includes tropical optimization as a research area concerned with optimization problems that are formulated and solved in the framework of tropical mathematics. In many cases, tropical optimization problems can be solved directly in closed form under general assumptions, whereas other problems have only algorithmic solutions based on iterative numerical procedures. For a brief overview of tropical optimization problems, one can consult, e.g., [19].

As a solution framework, tropical mathematics is used in [4, 5] to handle one-dimensional minimax location problems on graphs. A similar algebraic approach based on the theory of max-separable functions is implemented in [30, 31, 15, 16, 29] to solve constrained minimax location problems. Further examples include the solutions, given in [22, 18, 23, 24, 21] in terms of idempotent algebra, to unconstrained and constrained minimax single-facility location problems with Chebyshev and rectilinear distances.

In this paper, we consider location problems to find the optimal sites of placement of a new facility, which minimize the maximum weighted Chebyshev or rectilinear distance to existing facilities under constraints on the feasible location domain. For any two vectors \( \mathbf{r} = (r_1, \ldots, r_n)^T \) and \( \mathbf{s} = (s_1, \ldots, s_n)^T \) in the real space \( \mathbb{R}^n \), the Chebyshev distance (maximum or \( l_\infty \)-metric) is given by

\[
d_\infty(\mathbf{r}, \mathbf{s}) = \max_{1 \leq i \leq n} |r_i - s_i| = \max_{1 \leq i \leq n} \max\{r_i - s_i, s_i - r_i\}.
\]

The rectilinear distance (Manhattan, rectangular, taxi-cab, city-block or \( l_1 \)-metric) is calculated as

\[
d_1(\mathbf{r}, \mathbf{s}) = \sum_{1 \leq i \leq n} |r_i - s_i| = \sum_{1 \leq i \leq n} \max\{r_i - s_i, s_i - r_i\}.
\]

Suppose that we are given \( m \) points \( \mathbf{r}_i = (r_{i1}, \ldots, r_{in})^T \in \mathbb{R}^n \), positive reals \( w_i \) (weights), \( c_i \) (upper bounds), and reals \( h_i \) (addends) for all \( i = 1, \ldots, m \).
We need to locate a new point \( \mathbf{x} = (x_1, \ldots, x_n)^T \) in a feasible location domain \( S \subset \mathbb{R}^n \) to minimize the maximum distance, in the sense of a metric \( d \), from \( \mathbf{x} \) to existing points, under upper bound constraints on these distances. The problem is formulated in the form

\[
\min \quad \max_{1 \leq i \leq m} (w_i d(\mathbf{x}, \mathbf{r}_i) + h_i);
\]
\[
\text{s. t.} \quad d(\mathbf{x}, \mathbf{r}_i) \leq c_i, \quad i = 1, \ldots, m;
\]
\[
\mathbf{x} \in S. \tag{3}
\]

We examine the problem with Chebyshev and rectilinear distances under different settings of the dimension \( n \) and of the feasible location area \( S \). In the case of Chebyshev distance, we retain the general setting of a real space of arbitrary dimension \( n \). Given real numbers \( g_{ij}, p_i \text{ and } q_i \) such that \( p_i \leq q_i \), for all \( i, j = 1, \ldots, n \), the location area is described by the set

\[
S = \{(x_1, \ldots, x_n)^T | \quad g_{ij} + x_j \leq x_i, \quad p_i \leq x_i \leq q_i, \quad 1 \leq i, j \leq n\}, \tag{4}
\]

and takes the form of the intersection, if it exists, of the half-spaces defined by the inequalities \( g_{ij} + x_j \leq x_i \) and of the hyper-rectangle defined by \( p_i \leq x_i \leq q_i \).

In the rectilinear case, we consider a more specific two-dimensional problem defined on the real plane as follows. Given real numbers \( p_1, p_2, q_1, q_2, a \text{ and } b \) such that \( p_1 \leq q_1, \quad p_2 \leq q_2, \quad a \leq b \), the location area is given by the set

\[
S = \{(x_1, x_2)^T | \quad p_1 - x_1 \leq x_2 \leq q_1 - x_1, \quad p_2 + x_2 \leq x_1 \leq q_2 + x_2, \quad a \leq x_2 \leq b\}, \tag{5}
\]

which presents the intersection of the tilted rectangle defined by the inequalities \( p_1 - x_1 \leq x_2 \leq q_1 - x_1 \) and \( p_2 + x_2 \leq x_1 \leq q_2 + x_2 \), and the horizontal strip area given by the inequality \( a \leq x_2 \leq b \).

The above-described problems and their special cases are examined in many works, which offer various solutions to the problems. First note that these problems can be formulated as linear programs, and then solved using an appropriate linear programming computational procedure such as the simplex or Karmarkar algorithm. This approach, however, provides a numerical solution, if it exists, rather than a direct, complete solution in an exact analytical form.

For the unconstrained problems with rectilinear distance and equal weights, direct explicit solutions are obtained in [8, 10] using geometric arguments. A solution for the weighted problem with rectilinear distance is given in [6], which involves decomposition into independent one-dimensional subproblems solved by reducing to equivalent network flow problems. In [22, 18, 23, 24, 21], an approach based on idempotent algebra is applied to solve unweighted unconstrained and constrained location problems. Further results
on both unweighted and weighted location problems can be found in the survey papers [11, 1, 27, 2, 3], as well as in the books [28, 17, 9, 7, 25].

In this paper, we represent and examine the location problems in the framework of tropical (idempotent) algebra. We start with the solution of a location problem with Chebyshev distance in multidimensional space. The solution approach follows the analytical technique developed in [18, 20, 24, 21], which involves formulating the problem as a tropical optimization problem, introducing a parameter that represents the minimum value in the problem, and reducing the problem to a system of parametrized inequalities. The necessary and sufficient conditions for the existence of a solution to the system serve to evaluate the minimum, whereas all corresponding solutions of the system present a complete solution of the optimization problem. With this approach, we obtain a direct, exact solution represented in a compact closed form, which is appropriate for further analysis and straightforward computations with polynomial time complexity. The solution of the Chebyshev problem is then used to solve a location problem with rectilinear distance in the two-dimensional plane.

The proposed solutions extend and further develop previous results in [18, 21] on the location problems without weights (positive equal weighted problems). These new solutions, which are given in an explicit form, can serve to supplement and complement existing methods, and be of particular interest when the application of known algorithmic solutions, for one reason or other, appears to be impractical or impossible.

2 Elements of Tropical Algebra

In this section, we present a brief introduction to tropical (idempotent) algebra to provide a formal analytical framework for the solution of the location problems in the sequel. For further details on the theory and applications of tropical mathematics, one can refer, for example, to recent works [12, 14, 13, 26].

An idempotent semifield is an algebraic system \((\mathbb{X}, \oplus, \otimes, 0, 1)\), where \(\mathbb{X}\) is a nonempty set that has distinct elements 0 (zero) and 1 (one), and is equipped with binary operations \(\oplus\) (addition) and \(\otimes\) (multiplication) such that \((\mathbb{X}, \oplus, 0)\) is a commutative idempotent monoid, \((\mathbb{X} \setminus \{0\}, \otimes, 1)\) is an Abelian group, and \(\otimes\) distributes over \(\oplus\).

In the semifield, addition is idempotent, which means that \(x \oplus x = x\) for all \(x \in \mathbb{X}\), and induces a partial order by the rule: \(x \leq y\) if and only if \(x \oplus y = y\). This order is assumed to constitute a total order on \(\mathbb{X}\). With respect to this order, the operations \(\oplus\) and \(\otimes\) are monotone, which implies that the inequality \(x \leq y\) results in \(x \oplus z \leq y \oplus z\) and \(x \otimes z \leq y \otimes z\). Furthermore, the inequalities \(x \leq x \oplus y\) and \(y \leq x \oplus y\) hold for all \(x, y \in \mathbb{X}\). Finally, the inequality \(x \oplus y \leq z\) is equivalent to the system of inequalities
Multiplication is invertible, which provides each $x \neq 0$ with its inverse $x^{-1}$ such that $x \otimes x^{-1} = 1$. Inversion is antitone to turn the inequality $x \leq y$, where $x, y \neq 0$, into $x^{-1} \geq y^{-1}$. In what follows, the multiplication sign $\otimes$ is, as usual, omitted to save writing.

The integer powers are used in the standard way to indicate iterated products: $x^0 = 1$, $x^p = xx^{p-1}$, $x^{-p} = (x^{-1})^p$ and $0^p = 0$ for all $x \in X$ and integer $p > 0$. Furthermore, the equation $x^p = a$ has the unique solution $x = a^{1/p}$ for each $a \in X$ and integer $p > 0$, which allows the powers to have rational exponents. Moreover, it is assumed that the power notation can be further extended to real exponents (e.g., by the usual extra limiting process) to have the real powers and the power rules well defined. Exponentiation is monotone, which means that the inequality $x \leq y$ yields $x^p \leq y^p$ if $p > 0$, and $x^p \geq y^p$ if $p < 0$.

An analogue of the binomial identity holds in the form $(a \oplus b)^r = a^r \oplus b^r$ for any $a, b \in X$ and nonnegative real $r$.

As an example, we consider the real semifield $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, also known as the (max, +)-algebra, where $\oplus = \max$, $\otimes = +$, $0 = -\infty$ and $1 = 0$. In this semifield, the power $x^y$ coincides with the usual arithmetic product $xy$, and the inverse $x^{-1}$ with the opposite number $-x$. The order induced by the idempotent addition corresponds to the natural linear order on $\mathbb{R}$.

The algebra of vectors and matrices over idempotent semifields is introduced in the ordinary way. The vector (matrix) operations follow the conventional rules, where the operations $\oplus$ and $\otimes$ are used instead of arithmetic addition and multiplication. In the following, all vectors are considered column vectors unless otherwise specified. A vector that has all elements equal to $0$ is the zero vector. A vector without zero elements is called regular.

A square matrix that has all entries equal to $1$ on the diagonal, and to $0$ everywhere else, is the identity matrix denoted by $I$. For any square matrix $A$ and positive integer $p$, the power notation indicates iterated matrix products $A^0 = I$, $A^p = AA^{p-1}$. For any $(n \times n)$-matrix $A = (a_{ij})$, the trace is given by

$$\text{tr} A = a_{11} \oplus \cdots \oplus a_{nn}.$$  

The properties of the scalar operations $\oplus$ and $\otimes$ with respect to the order relation $\leq$ are readily extended to the vector (matrix) operations, where the inequalities are considered componentwise. For any nonzero vector $x = (x_i)$, the multiplicative conjugate transpose is a row vector $x^- = (x_i^-)$ with elements $x_i^- = x_i^{-1}$ if $x_i \neq 0$, and $x_i^- = 0$ otherwise. For any regular vectors $x$ and $y$ such that $x \leq y$, the conjugate transposition yields $x^- \geq y^-$. We conclude the overview with two results of tropical linear algebra. First suppose that, given an $(m \times n)$-matrix $A$ and $m$-vector $d$, we need
to find all $n$-vectors $x$ that are solutions of the inequality

$$Ax \leq d.$$  \hfill (6)

**Lemma 1.** Let $A$ be a matrix with regular columns, and $d$ a regular vector. Then, all solutions of Inequality (6) are given by $x \leq (d - A)^\top$.

Furthermore, given an $(n \times n)$-matrix $A$ and $n$-vector $b$, we consider the problem to find all regular $n$-vectors $x$ that satisfy the inequality

$$Ax + b \leq x.$$  \hfill (7)

To describe a solution to the problem, we introduce a function that maps any $(n \times n)$-matrix $A$ onto the scalar

$$\text{Tr}(A) = \text{tr} A \oplus \cdots \oplus \text{tr} A^n.$$

Provided that $\text{Tr}(A) \leq 1$, the asterate operator (the Kleene star) transforms the matrix $A$ into the matrix

$$A^* = A \oplus \cdots \oplus A^n.$$

The next statement presents a solution proposed in [20] to Inequality (7).

**Theorem 2.** For any matrix $A$ and vector $b$, the following statements hold:

1. If $\text{Tr}(A) \leq 1$, then all regular solutions of Inequality (7) are given by $x = A^* u$ for any vector $u \geq b$.

2. If $\text{Tr}(A) > 1$, then there are no regular solutions.

Below, we represent the location problems under study in terms of idempotent algebra, and obtain direct, complete solutions to the problems.

### 3 Location with Chebyshev Distance

We start with a solution of the location problem defined on the $n$-dimensional vector space with Chebyshev metric (1). In the framework of $(\max, +)$-algebra, the Chebyshev distance between vectors $r = (r_i)$ and $s = (s_i)$ in $\mathbb{R}^n$ is given by

$$d_\infty(r, s) = \bigoplus_{1 \leq i \leq n} (s_i^{-1}r_i \oplus r_i^{-1}s_i) = s^\top r \oplus r^\top s.$$

The objective function in Problem (3) takes the form

$$\bigoplus_{1 \leq i \leq m} h_i(r_i^\top x \oplus x^{-1}r_i)^{\text{arc}}.$$
The feasible location area, which is defined by \( 4 \), becomes

\[
S = \{ (x_1, \ldots, x_n)^T | g_{ij}x_j \leq x_i, \ p_i \leq x_i \leq q_i, \ 1 \leq i, j \leq n \}.
\]

With the matrix and vector notation

\[
G = \begin{pmatrix}
g_{11} & \cdots & g_{1n} \\
\vdots & \ddots & \vdots \\
g_{n1} & \cdots & g_{nn}
\end{pmatrix}, \quad
p = \begin{pmatrix}
p_1 \\
\vdots \\
p_n
\end{pmatrix}, \quad
q = \begin{pmatrix}
q_1 \\
\vdots \\
q_n
\end{pmatrix},
\]

we describe the location area in vector form through the system of inequalities

\[
Gx \leq x, \quad p \leq x \leq q.
\]

After substitution of the Chebyshev metric and vector description of the location area in terms of \((\max, +)-\)algebra into Problem \( 3 \), we formulate the problem as follows:

\[
\begin{align*}
\min & \quad \bigoplus_{1 \leq i \leq m} h_i (r\oplus x\ominus r_i)^{w_i}, \\
\text{s. t.} & \quad r\oplus x\ominus r_i \leq c_i, \quad i = 1, \ldots, m; \\
& \quad Gx \leq x, \quad p \leq x \leq q.
\end{align*}
\]

(8)

Note that we assume all data involved in the formulation of Problem \( 3 \) to be real numbers, and thus none of them is equal to the tropical zero \( 0 = -\infty \). Specifically, both the known vectors \( r_i \) for all \( i = 1, \ldots, m \), and the unknown vector \( x \) are considered regular.

To solve the problem obtained, we first introduce an additional parameter to represent the minimum value of the objective function, and then reduce the problem to a parameterized system of inequalities. Subsequently, we use existence conditions for solutions of the system to evaluate the value of the parameter. Finally, all solutions of the system, which correspond to this value, serve as a complete solution to the initial optimization problem.

Let us denote the minimum value of the objective function by \( \theta \). Then, all solutions of the problem must satisfy the equation

\[
\bigoplus_{1 \leq i \leq m} h_i (r\oplus x\ominus r_i)^{w_i} = \theta.
\]

Since we assume \( \theta \) to be the minimum of the objective function, the set of solutions remains unchanged after replacing the equation by the inequality

\[
\bigoplus_{1 \leq i \leq m} h_i (r\oplus x\ominus r_i)^{w_i} \leq \theta.
\]

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Using the extremal property of idempotent addition, we replace the last inequality by an equivalent system of inequalities to describe all solutions of Problem (8) as follows:

\[
\begin{align*}
  h_i(r_i^- x \oplus x^- r_i)^{w_i} &\leq \theta, \\
  r_i^- x \oplus x^- r_i &\leq c_i, \quad i = 1, \ldots, m; \\
  Gx &\leq x, \\
  p &\leq x \leq q.
\end{align*}
\]

We use the tropical analogue of the binomial identity to replace the inequality \(h_i(r_i^- x \oplus x^- r_i)^{w_i} \leq \theta\) by the inequalities \(h_i(r_i^- x)^{w_i} \leq \theta\) and \(h_i(x^- r_i)^{w_i} \leq \theta\). Since exponentiation is monotone, these inequalities can be further rewritten by the usual power rules as \(h_i^{1/w_i} r_i^- x \leq \theta^{1/w_i}\) and \(h_i^{1/w_i} x^- r_i \leq \theta^{1/w_i}\), and then represented as the inequalities \(r_i^- x \leq \theta^{1/w_i} h_i^{-1/w_i} x\) and \(x^- r_i \leq \theta^{1/w_i} h_i^{-1/w_i}\). Application of Lemma 1 to solve the first inequality with respect to \(x\) yields \(x \leq \theta^{1/w_i} h_i^{-1/w_i} r_i\).

Furthermore, we again use Lemma 1 to solve the second inequality with respect to \(r_i\), and then multiply both sides of the result by \(\theta^{-1/w_i} h_i^{1/w_i}\) to obtain the inequality \(\theta^{-1/w_i} h_i^{1/w_i} r_i \leq x\). Finally, we combine the results into the double inequality \(\theta^{-1/w_i} h_i^{1/w_i} r_i \leq x \leq \theta^{1/w_i} h_i^{-1/w_i} r_i\).

In the same way, we replace the inequality \(r_i^- x \oplus x^- r_i \leq c_i\) by the inequalities \(x \leq c_ir_i\) and \(c_i^{-1} r_i \leq x\), and then represent them as \(c_i^{-1} r_i \leq x \leq c_ir_i\).

We now rewrite System (9) in the form

\[
\begin{align*}
  \theta^{-1/w_i} h_i^{1/w_i} r_i &\leq x \leq \theta^{1/w_i} h_i^{-1/w_i} r_i, \\
  c_i^{-1} r_i &\leq x \leq c_i r_i, \quad i = 1, \ldots, m; \\
  Gx &\leq x, \\
  p &\leq x \leq q.
\end{align*}
\]

Furthermore, we combine the left inequalities for all \(i = 1, \ldots, m\) into one, which provides a lower bound for \(x\). Next, we represent the right inequalities as \(x^- \geq \theta^{-1/w_i} h_i^{1/w_i} r_i^-\) and \(x^- \geq c_i^{-1} r_i^-\). Summing up these inequalities and conjugate-transposing the result yield an upper bound.

After adding the remaining inequalities, we have a parameterized description of all solutions in the form of the double inequality

\[
Gx \oplus \bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) r_i \oplus p \leq x
\]

\[
\leq \left( \bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) r_i^- \oplus q^\ominus \right)^\ominus.
\]
First, we assume that \( \text{Tr}(G) \leq 1 \) and apply Theorem 2 to solve the left inequality and obtain a solution represented using a vector of parameters \( u \) as follows:

\[ x = G^* u, \quad u \geq \bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})r_i \oplus p. \]

Next, we substitute \( x \) by \( G^* u \) into the right inequality, and then apply Lemma 1 to solve the obtained inequality for \( u \) in the form

\[
\begin{align*}
\bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})r_i \oplus p \leq \left( \bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})r_i \oplus q^-)G^* \right)^- \ .
\end{align*}
\]

The set of parameter vectors \( u \) defined by the obtained inequalities is nonempty if and only if the following condition holds:

\[
\bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})r_i \oplus p \leq \left( \bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})r_i \oplus q^-)G^* \right)^- .
\]

We now use the last inequality to evaluate the parameter \( \theta \). Multiplying both sides of the inequality by the conjugate transpose of the right-hand side yields the equivalent inequality

\[
\bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})r_i \oplus q^-)G^* ((\theta^{-1/w_j} h_j^{1/w_j} \oplus c_j^{-1})r_j \oplus p) \leq 1,
\]

which we further break down into the inequalities

\[
(\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})r_i \oplus q^-)G^* \oplus ((\theta^{-1/w_j} h_j^{1/w_j} \oplus c_j^{-1})r_j \oplus p) \leq 1, \quad i, j = 1, \ldots, m.
\]

After multiplying the terms on the left-hand side, we replace each inequality by four inequalities to write

\[
\begin{align*}
\theta^{-1/w_i} & h_j^{1/w_j} G^* r_j \leq 1, \\
\theta^{-1/w_i} & h_j^{1/w_j} G^* (c_j^{-1} r_j \oplus p) \leq 1, \\
\theta^{-1/w_i} & h_j^{1/w_j} (c_i^{-1} r_i \oplus q^-)G^* r_j \leq 1, \\
(c_i^{-1} r_i \oplus q^-)G^* (c_j^{-1} r_j \oplus p) \leq 1, \quad i, j = 1, \ldots, m.
\end{align*}
\]

The solution of the first three inequalities with respect to \( \theta \) yields the result

\[
\begin{align*}
(h_i^{1/w_i} h_j^{1/w_j} r_i \oplus q^- G^* r_j)^{w_i w_j} & \leq \theta, \\
h_i (r_i \oplus q^- G^* (c_j^{-1} r_j \oplus p))^{w_i} & \leq \theta, \\
h_j ((c_i^{-1} r_i \oplus q^-)G^* r_j)^{w_j} & \leq \theta, \\
(c_i^{-1} r_i \oplus q^-)G^* (c_j^{-1} r_j \oplus p) & \leq 1, \quad i, j = 1, \ldots, m.
\end{align*}
\]
By combining the inequalities for each $i$ and $j$, we obtain the system

\[
\theta \geq \bigoplus_{1 \leq i, j \leq m} \left( (h_i^{1/w_i} h_j^{1/w_j} r_i^- G^* r_j)_{w_i, w_j} \oplus h_i(r_i^- G^*(c_j^{-1} r_j + p))^{w_i} \right) \\
\oplus h_j((c_i^{-1} r_i^- \oplus q^-) G^* r_j)^{w_j} ,
\]

\[
\bigoplus_{1 \leq i, j \leq m} (c_j^{-1} r_j^- \oplus q^-) G^*(c_j^{-1} r_j + p) \leq 1.
\]

Consider the first inequality, which gives the lower bound for the parameter $\theta$. Since $\theta$ is assumed to represent the minimum in the problem, we set it to be equal to the right-hand side of the inequality.

As one can see, the second inequality serves as necessary and sufficient conditions for the constraints of the problem to be consistent.

We now summarize the result obtained in the following statement.

**Theorem 3.** Suppose that the following conditions hold:

1. $\text{Tr}(G) \leq 1$;
2. $(c_i^{-1} r_i^- \oplus q^-) G^*(c_j^{-1} r_j + p) \leq 1$ for all $i, j = 1, \ldots, m$.

Then, the minimum value in Problem (8) is equal to

\[
\theta = \bigoplus_{1 \leq i, j \leq m} \left( (h_i^{1/w_i} h_j^{1/w_j} r_i^- G^* r_j)_{w_i, w_j} \oplus h_i(r_i^- G^*(c_j^{-1} r_j + p))^{w_i} \right) \\
\oplus h_j((c_i^{-1} r_i^- \oplus q^-) G^* r_j)^{w_j} ,
\]

and all solutions are given by

\[
x = G^* u,
\]

where the vector $u$ satisfies the condition

\[
\bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) r_i \oplus p) \leq u
\]

\[
\leq \left( \bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) r_i^- \oplus q^-) G^* \right)^-. \]

It is not difficult to see that the solution given by Theorem 3 has a polynomial time complexity in the number of points $m$ and the dimension of space $n$. Clearly, the most computationally demanding part of the solution
is the calculation of the parameter $\theta$ according to (10). The evaluation of $\theta$ requires calculating the Kleene star matrix $G^*$ with the computational time, which is at most $O(n^4)$, when computed by direct matrix multiplications. Given the matrix $G^*$, each of three terms in the big brackets on the right-hand side of (10) takes time of $O(n^2)$, and thus the overall time to compute $\theta$ is no more than $O(m^2n^2)$.

Note that, in the $(\max, +)$-algebra setting, Problem (5) can be solved as a linear program using a polynomial time iterative procedure such as the Karmarkar algorithm. However, this approach can offer a numerical solution rather than a complete, direct solution in an analytical form like that provided by Theorem 3.

Finally, we represent the result of Theorem 3 in terms of conventional algebra. For the matrix $G$, we denote the entries of the matrix $G^*$ as $g^*$ and note that

$$g^*_{ij} = \begin{cases} \gamma_{ij}, & \text{if } i \neq j; \\ \max\{\gamma_{ij}, 0\}, & \text{if } i = j; \end{cases}$$

\[\gamma_{ij} = \max_{1 \leq k \leq n} \max_{1 \leq i_1, \ldots, i_{k-1} \leq n} (g^{i_0i_1} + \cdots + g^{i_{k-1}i_k}).\]

With the identity $\max(a, b) = -\min(-a, -b)$ used to save writing, we arrive at the next corollary.

**Corollary 4.** Suppose that the following conditions hold:

1. $\max_{1 \leq i_1, \ldots, i_{k-1} \leq n} (g^{i_0i_1} + \cdots + g^{i_{k-1}i_k}) \leq 0$ for all $i, k = 1, \ldots, n$;

2. $g^*_{kl} + \max_{1 \leq j \leq m} \max_{1 \leq i \leq m} (-c_j + r_{ij}, p_j) \leq \min_{1 \leq i \leq m} \max\{c_i + r_{ki}, q_k\}$ for all $k, l = 1, \ldots, n$.

Then, the minimum value in Problem (5) is equal to

$$\theta = \max_{1 \leq i, j \leq m} \max_{1 \leq k, l \leq n} \left\{ \frac{w_j h_i}{w_i + w_j} + \frac{w_i h_j}{w_i + w_j} + \frac{w_i w_j}{w_i + w_j} (g^*_{kl} - r_{ki} + r_{ij}), \right.$$ 

$$h_i + w_i (g^*_{kl} - r_{ki} + r_{ij} - c_j, g^*_{kl} - r_{ki} + p_k),$$

$$h_j + w_j (g^*_{kl} + r_{ij} - r_{ki} - c_i, g^*_{kl} + r_{ij} - q_k) \right\},$$

and all solutions $x = (x_k)$ are given by

$$x_k = \max_{1 \leq j \leq n} (g^*_{kj} + u_j), \quad k = 1, \ldots, n;$$

where the numbers $u_j$ for each $j = 1, \ldots, n$ satisfy the condition

$$\max_{1 \leq i \leq m} \left( r_{ji} + (h_i - \theta)/w_i, r_{ji} - c_i, p_j \right) \leq u_j \leq \min_{1 \leq i \leq m} \min_{1 \leq i \leq n} \left( r_{hi} + (\theta - h_i)/w_i, r_{hi} + c_i, q_i \right) - g^*_{ij}. \]
4 Location with Rectilinear Distance

We now turn to the solution of the location problem defined on the plane with rectilinear distance. To solve the problem, we extend and further develop the technique, which is proposed in [21] to solve unweighted two-dimensional rectilinear location problems. The technique involves the representation of the problem in the form of a tropical optimization problem, following by the change of variables, which reduces the optimization problem to a problem in the form of (8). Note that this technique can hardly provide solutions to the rectilinear location problems in three and more dimensions, which require different solution methods.

First, we represent the rectilinear distance between two-dimensional vectors \( \mathbf{r} = (r_1, r_2)^T \) and \( \mathbf{s} = (s_1, s_2)^T \) in terms of \((\max, +)\)-algebra and write

\[
d_1(\mathbf{r}, \mathbf{s}) = (s_1^{-1}r_1 + r_1^{-1}s_1)(s_2^{-1}r_2 + r_2^{-1}s_2).
\]

Furthermore, we describe the feasible location area given by (5) as follows

\[
S = \{(x_1, x_2)^T | p_1x_1^{-1} \leq x_2 \leq q_1x_1^{-1}, p_2x_2 \leq x_1 \leq q_2x_2, a \leq x_2 \leq b\}.
\]

Problem (3) now takes the form

\[
\min \bigoplus_{1 \leq i \leq m} h_i((r_1^{-1}x_1 \oplus x_1^{-1}r_1)(r_2^{-1}x_2 \oplus x_2^{-1}r_2))^{w_i};
\]

\[
s. t. \quad (r_1^{-1}x_1 \oplus x_1^{-1}r_1)(r_2^{-1}x_2 \oplus x_2^{-1}r_2) \leq c_i, \quad i = 1, \ldots, m;
\]

\[
p_1x_1^{-1} \leq x_2 \leq q_1x_1^{-1}, \quad p_2x_2 \leq x_1 \leq q_2x_2, \quad a \leq x_2 \leq b.
\]

As before, we assume all parameters and vectors, involved in the problem formulation, to have nonzero values in the sense of \((\max, +)\)-algebra.

The solution of Problem (11) is based on changing variables to reduce it to Problem (8), and thus to take advantage of the above-obtained results. Note that this transformation from (11) to (8) reflects the well-known relationship between the solutions of location problems on the plane with rectilinear and Chebyshev distances.

To solve Problem (11), we first introduce new vectors

\[
\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \mathbf{s}_i = \begin{pmatrix} s_{1i} \\ s_{2i} \end{pmatrix}, \quad i = 1, \ldots, m,
\]

with their elements given by the conditions

\[
y_1 = x_1x_2, \quad y_2 = x_1x_2^{-1}, \quad s_{1i} = r_1r_2, \quad s_{2i} = r_1r_2^{-1}.
\]

Clearly, the elements of the vector \( \mathbf{x} = (x_1, x_2)^T \) are related with those of \( \mathbf{y} \) by the equalities

\[
x_1 = y_1^{1/2}y_2^{1/2}, \quad x_2 = y_1^{1/2}y_2^{-1/2}.
\]
With the new notation, for all \( i = 1, \ldots, m \), we can write
\[
(r_{1i}^{-1} x_1 + x_1^{-1} r_{1i})(r_{2i}^{-1} x_2 \oplus x_2^{-1} r_{2i}) = s_{1i}^{-1} y_1 \oplus s_{2i}^{-1} y_2 \oplus s_{2i} y_2^{-1} \oplus s_{1i} y_1^{-1} = s_i^\top y \oplus y^{-s_i},
\]

It remains to rewrite the constraints, which determine the feasible location area \( S \). The first inequality \( p_1 x_1^{-1} \leq x_2 \leq q_1 x_1^{-1} \) is equivalent to \( p_1 \leq x_1 x_2 \leq q_1 \), which can be written as \( p_1 \leq y_1 \leq q_1 \). In the same way, we represent the second inequality \( p_2 x_2 \leq x_1 \leq q_2 x_2 \) as \( p_2 \leq x_1 x_2^{-1} \leq q_2 \), and then as \( p_2 \leq y_2 \leq q_2 \).

We take the last inequality \( a \leq x_2 \leq b \), and put its left-hand side in the equivalent form \( a^2 x_1 x_2^{-1} \leq x_1 x_2 \), which can be expressed as \( a^2 y_2 \leq y_1 \). The right-hand side takes the form \( x_1 x_2 \leq b^2 x_1 x_2^{-1} \), and then becomes \( b^2 y_1 \leq y_2 \).

With the vector and matrix notation
\[
p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & a^2 \\ b^{-2} & 0 \end{pmatrix},
\]
where \( 0 = -\infty \), we express the constraints in vector form as
\[
p \leq y \leq q, \quad G y \leq y.
\]

Finally, we have Problem (11) formulated in terms of \((\max, +)\)-algebra as follows
\[
\begin{align*}
\min & \quad \bigoplus_{1 \leq i \leq m} h_i(s_i^\top y \oplus y^{-s_i})^{w_i}; \\
\text{s. t.} & \quad s_i^\top y \oplus y^{-s_i} \leq c_i, \quad i = 1, \ldots, m; \\
& \quad G y \leq y, \quad p \leq y \leq q.
\end{align*}
\]

Since the problem obtained takes the form of (8), we apply Theorem 3 to derive a complete solution to Problem (12) given by the next statement.

**Theorem 5.** Let \( s_i = (s_{1i}, s_{2i})^T \), where \( s_{1i} = r_{1i} r_{2i} \) and \( s_{2i} = r_{1i}^{-1} r_{2i}^{-1} \), and suppose that \( (c_i^{-1} s_i^\top \oplus q^-) G^* (c_j^{-1} s_j^\top \oplus p) \leq 1 \) for all \( i, j = 1, \ldots, m \).

Then, the minimum value in Problem (12) is equal to
\[
\theta = \bigoplus_{1 \leq i, j \leq m} \left( (h_i^{1/w_i} h_j^{1/w_j} s_i^- G^* s_j)_{w_i w_j} \oplus h_i(s_i^- G^* (c_j^{-1} s_j^\top \oplus p))_{w_i} \right)
\]
\[
\quad \quad \oplus h_j((c_i^{-1} s_i^\top \oplus q^-) G^* s_j)_{w_j}
\]
and all solution vectors \( x = (x_1, x_2)^T \) have the elements
\[
x_1 = y_1^{1/2} y_2^{1/2}, \quad x_2 = y_1^{1/2} y_2^{-1/2},
\]

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defined by the elements of vectors $y = (y_1, y_2)^T$, which are given by

$$y = G^* u,$$

where the vector $u$ satisfies the condition

$$\bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) s_i \oplus p \leq u$$

$$\leq \left( \bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) s_i^- \oplus q^-) G^* \right).$$

In terms of ordinary arithmetic operations, the result reads as follows.

**Corollary 6.** Let $s_{1i} = r_{1i} + r_{2i}$ and $s_{2i} = r_{1i} - r_{2i}$, and suppose that the following conditions hold:

$$\max_{1 \leq j \leq m} \{ -c_j + s_{1j}, p_1, -c_j + s_{2j} + 2a, p_2 + 2a \} \leq \min_{1 \leq i \leq m} \{ c_i + s_{1i}, q_1 \},$$

$$\max_{1 \leq j \leq m} \{ -c_j + s_{1j} - 2b, p_1 - 2b, -c_j + s_{2j}, p_2 \} \leq \min_{1 \leq i \leq m} \{ c_i + s_{2i}, q_2 \}.$$

Then, the minimum value in Problem **[12]** is equal to

$$\theta = \max_{1 \leq i, j \leq m} \max \left\{ \frac{w_j h_i}{w_i + w_j} + \frac{w_i h_j}{w_i + w_j} \right\}$$

$$+ \frac{w_i w_j}{w_i + w_j} \max \{ -s_{11} + s_{1j}, 2a - s_{11} + s_{2j}, -2b - s_{21} + s_{1j}, -2b - s_{21} + s_{2j} \},$$

$$h_i + w_i \max \{ -s_{11} + s_{1j} - c_j, -s_{11} + p_1, 2a - s_{11} + s_{2j} - c_j, 2a - s_{11} + p_1, -2b - s_{21} + s_{1j} - c_j, -2b - s_{21} + s_{2j} - c_j, -2b - s_{21} + p_2 \},$$

$$h_j + w_j \max \{ s_{11} - s_{1j} - c_j, s_{1j} - q_1, 2a + s_{2j} - s_{1j} - c_j, 2a + s_{2j} - q_1, -2b + s_{1j} - s_{2j}, -2b + s_{1j} - q_2, s_{2j} - s_{2i} - c_i, s_{2j} - q_2 \},$$

and all solutions $x = (x_1, x_2)^T$ are given by

$$x_1 = (y_1 + y_2)/2, \quad x_2 = (y_1 - y_2)/2,$$

where $y_1$ and $y_2$ are calculated as

$$y_1 = \max \{ u_1, u_2 + 2a \}, \quad y_2 = \max \{ u_1 - 2b, u_2 \},$$

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with the numbers \( u_1 \) and \( u_2 \) defined by the conditions

\[
\begin{align*}
\max_{1 \leq i \leq m} & \left\{ \frac{h_i - \theta}{w_i} + s_{1i} - c_i + s_{1i}, p_1 \right\} \leq u_1 \\
& \leq \min_{1 \leq i \leq m} \left\{ \frac{\theta - h_i}{w_i} + s_{1i}, c_i + s_{1i}, q_1, \frac{\theta - h_i}{w_i} + s_{2i} - 2a, c_i + s_{2i} - 2a, q_2 \right\}, \\
\max_{1 \leq i \leq m} & \left\{ \frac{h_i - \theta}{w_i} + s_{2i} - c_i + s_{2i}, p_2 \right\} \leq u_2 \\
& \leq \min_{1 \leq i \leq m} \left\{ \frac{\theta - h_i}{w_i} + s_{1i} + 2b, c_i + s_{1i} + 2b, q_1 + 2b, \frac{\theta - h_i}{w_i} + s_{2i} + c_i + s_{2i}, q_2 \right\}.
\end{align*}
\]

5 Conclusions

The paper has examined minimax single-facility location problems in the \( n \)-dimensional vector space with Chebyshev distance and in the two-dimensional plane with rectilinear distance. The feasible location areas are given by the intersection of half-spaces defined by a set of inequality constraints.

We have started with the multidimensional problem with Chebyshev distance and general constraints. To handle the problem, we first represented it in terms of \((\max, +)\)-algebra as a tropical optimization problem. The solution approach was implemented, which introduces an additional parameter to represent the optimal value of the objective function, and then uses properties of the operations in \((\max, +)\)-algebra to reduce the optimization problem to the solution of a set of parameterized inequalities. The existence conditions for solutions of the system serve to evaluate the parameter, whereas all solutions of the system are taken as a complete solution of the optimization problem.

Using this approach, we have derived a new exact, complete solution to the multidimensional location problem with Chebyshev distance in terms of tropical mathematics, and represented the solution in the standard form. The results obtained were extended to examine the two-dimensional problem with rectilinear distance, and to provide new solutions in both tropical and conventional algebra settings. The solutions are given in a closed form, suitable for further analytical study and direct computations with low polynomial complexity in terms of both the dimension of the location space and the number of given points.

Possible lines of further research include the development of algebraic methods to solve rectilinear location problems in the three-dimensional space, as well as to solve Chebyshev and rectilinear problems with new types of constraints.
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