An overview of Manin’s conjecture for del Pezzo surfaces

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1 Introduction

A fundamental theme in mathematics is the study of integer solutions to Diophantine equations, or equivalently, the study of rational points on projective algebraic varieties. Let \( V \subset \mathbb{P}^n \) be a projective variety that is cut out by a finite system of homogeneous equations defined over \( \mathbb{Q} \). Then there are a number of basic questions that can be asked about the set \( \mathcal{V} := V \cap \mathbb{P}^n(\mathbb{Q}) \) of rational points on \( V \): when is \( \mathcal{V} \) non-empty? how large is \( \mathcal{V} \) when it is non-empty? This paper aims to survey the second question, for a large class of varieties \( V \) for which one expects \( \mathcal{V} \) to be Zariski dense in \( V \).

To make sense of this it is convenient to define the height of a projective rational point \( x = [x_0, \ldots, x_n] \in \mathbb{P}^n(\mathbb{Q}) \) to be \( H(x) := \|x\| \), for any norm \( \| \cdot \| \) on \( \mathbb{R}^{n+1} \), provided that \( x = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \) and \( \gcd(x_0, \ldots, x_n) = 1 \).

Throughout this work we shall work with the height metrized by the choice of norm \( |x| := \max_{0 \leq i \leq n} |x_i| \). Given a suitable Zariski open subset \( U \subseteq V \), the goal is then to study the quantity

\[
N_{U,H}(B) := \# \{ x \in U(\mathbb{Q}) : H(x) \leq B \},
\]

as \( B \to \infty \). It is natural to question whether the asymptotic behaviour of \( N_{U,H}(B) \) can be related to the geometry of \( V \), for suitable open subsets \( U \subseteq V \).

Around 1989 Manin initiated a program to do exactly this for varieties with ample anticanonical divisor \( [22] \). Suppose for simplicity that \( V \subset \mathbb{P}^n \) is a non-singular complete intersection, with \( V = W_1 \cap \cdots \cap W_t \) for hypersurfaces \( W_i \subset \mathbb{P}^n \) of degree \( d_i \). Since \( V \) is assumed to be Fano, its Picard group is a finitely generated free \( \mathbb{Z} \)-module, and we denote its rank by \( \rho_V \). Then in this setting the Manin conjecture takes the following shape \([1, \text{Conjecture C}']\).

**Conjecture A.** Suppose that \( d_1 + \cdots + d_t \leq n \). Then there exists a Zariski open subset \( U \subseteq V \) and a non-negative constant \( c_{V,H} \) such that

\[
N_{U,H}(B) = c_{V,H} B^{n+1-d_1-\cdots-d_t} (\log B)^{\rho_V-1} (1 + o(1)),
\]

as \( B \to \infty \).

It should be noted that there are simple heuristic arguments that support the value of the exponents of \( B \) and \( \log B \) appearing in the conjecture. The constant \( c_{V,H} \) has also received a conjectural interpretation at the hands of
Peyre [31], and this has been generalised to cover certain other cases by Batyrev and Tschinkel [2], and Salberger [35]. In fact whenever we refer to the Manin conjecture we shall henceforth mean that the value of the constant $c_{V,H}$ should agree with the prediction of Peyre et al. With this in mind, the Manin conjecture can be extended to cover certain other Fano varieties $V$ which are not necessarily complete intersections, nor non-singular. For the former one simply takes the exponent of $B$ to be the infimum of all $a/b \in \mathbb{Q}$ such that $b > 0$ and $aH + bK_V$ is linearly equivalent to an effective divisor, where $K_V \in \text{Div}(V)$ is a canonical divisor and $H \in \text{Div}(V)$ is a hyperplane section. For the latter, if $V$ has only rational double points, then one may apply the conjecture to a minimal desingularisation $\tilde{V}$ of $V$, and then employ the functoriality of heights. A discussion of these more general versions of the conjecture can be found in the survey of Tschinkel [41]. The purpose of this note is to give an overview of our progress in the case that $V$ is a suitable Fano variety of dimension 2.

A non-singular surface $S \subset \mathbb{P}^d$ of degree $d$, with very ample anticanonical divisor $-K_S$, is known as a del Pezzo surface of degree $d$. Their geometry has been expounded by Manin [30], for example. It is well-known that such surfaces $S$ arise either as the quadric Veronese embedding of a quadric in $\mathbb{P}^3$, which is a del Pezzo surface of degree 8 in $\mathbb{P}^8$ (isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$), or as the blow-up of $\mathbb{P}^2$ along $9 - d$ points in general position, in which case the degree of $S$ satisfies $3 \leq d \leq 9$. In terms of the expected asymptotic formula for $N_{U,H}(B)$ for a suitable open subset $U \subset S$, the exponent of $B$ is 1, and the exponent of $\log B$ is at most $9 - d$, since the geometric Picard group $\text{Pic}(S \otimes \mathbb{Q} \otimes \mathbb{Q})$ has rank $10 - d$. An old result of Segre ensures that the set $S(\mathbb{Q})$ of rational points on $S$ is Zariski dense as soon as it is non-empty. Moreover, when $3 \leq d \leq 8$ there are certain so-called accumulating subvarieties contained in $S$ which may dominate the behaviour of the counting function $N_{S,H}(B)$. These are the possible lines contained in $S$, that correspond to the exceptional divisors arising from the process of blowing up the projective plane along the relevant collection of points. Now it is an easy exercise to check that

$$N_{\mathbb{P}^1,H}(B) = \frac{12}{\pi^2} B^2 (1 + o(1)),$$

as $B \to \infty$, so that $N_{V,H}(B) \gg V^{1/2}$ for any geometrically integral surface $V \subset \mathbb{P}^n$ that contains a line defined over $\mathbb{Q}$. However, if $U \subset V$ is defined to be the Zariski open subset formed by deleting all of the lines from $V$ then it follows from combining an estimate of Heath-Brown [25, Theorem 6] with a birational projection argument due to Salberger [36, §8], that $N_{U,H}(B) = o_V(B^2)$.

Returning to the setting of del Pezzo surfaces $S \subset \mathbb{P}^d$ of degree $d$, it turns out that there are no exceptional divisors when $d = 9$, or when $d = 8$ and $S$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, in which case we study $N_{S,H}(B)$. When $3 \leq d \leq 7$, or when $d = 8$ and $S$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, there are a finite number of such divisors, each producing a line in $S$. In these cases we study $N_{U,H}(B)$ for the open subset $U$ formed by deleting all of the lines from $S$. We now proceed to review the progress that has been made towards the Manin conjecture for del Pezzo surfaces. In doing so we have split our discussion according to the degree of the surface, and it will become apparent that the problem of estimating $N_{U,H}(B)$ becomes harder as the degree decreases.
1.1 Del Pezzo surfaces of degree $\geq 5$

It turns out that the del Pezzo surfaces $S$ of degree $d \geq 6$ are toric, in the sense that they contain the torus $G_m^2$ of algebraic groups as a dense open subset, whose natural action on itself extends to all of $S$. Thus the Manin conjecture for such surfaces is a special case of the more general work due to Batyrev and Tschinkel [3], that establishes this conjecture for all toric varieties. One may compare this result with the work of la Bretèche [5] and Salberger [35], who both establish the conjecture for toric varieties defined over $\mathbb{Q}$, and also the work of Peyre [31], who handles a number of special cases.

For non-singular del Pezzo surfaces $S \subset \mathbb{P}^5$ of degree $5$, the situation is rather less satisfactory. In fact there are very few instances for which the Manin conjecture has been established. The most significant of these is due to la Bretèche [6], who has proved the conjecture when all of the 10 exceptional divisors are defined over $\mathbb{Q}$. In this case we say that the surface is split. Let $S_0$ be the surface obtained by blowing up $\mathbb{P}^2$ along the four points

$$p_1 = [1, 0, 0], \quad p_2 = [0, 1, 0], \quad p_3 = [0, 0, 1], \quad p_4 = [1, 1, 1],$$

and let $U_0 \subset S_0$ denote the corresponding open subset formed by deleting the lines from $S_0$. Then $\text{Pic}(S_0)$ has rank 5, since $S_0$ is split, and la Bretèche obtains the following result.

**Theorem 1.** Let $B \geq 3$. Then there exists a constant $c_0 > 0$ such that

$$N_{U_0,H} (B) = c_0 B (\log B)^4 \left( 1 + O \left( \frac{1}{\log \log B} \right) \right).$$

We shall return to the proof of this result below. The other major achievement in the setting of quintic del Pezzo surfaces is a result of la Bretèche and Fouvry [9]. Here the Manin conjecture is established for the surface obtained by blowing up $\mathbb{P}^2$ along four points in general position, two of which are defined over $\mathbb{Q}$ and two of which are conjugate over $\mathbb{Q}(i)$. In related work, Browning [12] has obtained upper bounds for $N_{U,H} (B)$ that agree with the Manin predication for several other del Pezzo surfaces of degree 5.

1.2 Del Pezzo surfaces of degree 4

A quartic del Pezzo surface $S \subset \mathbb{P}^4$, that is defined over $\mathbb{Q}$, can be recognised as the zero locus of a suitable pair of quadratic forms $Q_1, Q_2 \in \mathbb{Z}[x_0, \ldots, x_4]$. Then $S = \text{Proj}(\mathbb{Q}[x_0, \ldots, x_4]/(Q_1, Q_2))$ is the complete intersection of the hypersurfaces $Q_1 = 0$ and $Q_2 = 0$ in $\mathbb{P}^4$. When $S$ is non-singular (1.2) predicts the existence of a constant $c_{S,H} \geq 0$ such that

$$N_{U,H} (B) = c_{S,H} B (\log B)^{\rho_S - 1} (1 + o(1)),$$

as $B \to \infty$, where $\rho_S = \text{rk} \text{Pic}(S) \leq 6$ and $U \subset S$ is obtained by deleting the 16 lines from $S$. In this setting the best result available is due to Salberger. In work communicated at the conference *Higher dimensional varieties and rational points* at Budapest in 2001, he establishes the estimate $N_{U,H} (B) = O_{\varepsilon,S} (B^{1+\varepsilon})$ for any $\varepsilon > 0$, provided that the surface contains a conic defined over $\mathbb{Q}$. In fact an examination of Salberger's approach, which is based upon fibering the
surface into a family of conics, reveals that it would be straightforward to replace the factor $B^2$ by $(\log B)^4$ for a large constant $A$. It would be interesting to find examples of surfaces $S$ for which the exponent $A$ could be reduced to the expected quantity $\rho_S - 1$.

It emerges that much more can be said if one permits $S$ to contain isolated singularities. For the remainder of this section let $S \subset \mathbb{P}^4$ be a geometrically integral intersection of two quadric hypersurfaces, which has only isolated singularities and is not a cone. Then $S$ contains only rational double points (see Wall [43], for example), thereby ensuring that there exists a unique minimal desingularisation $\pi: \tilde{S} \to S$ of the surface, such that $K_{\tilde{S}} = \pi^* K_S$. In particular it follows that the asymptotic formula (1.3) is still expected to hold, with $\rho_S$ now taken to be the rank of the Picard group of $\tilde{S}$, and $U \subset S$ obtained by deleting all of the lines from $S$. The classification of such surfaces $S$ is rather classical, and can be found in the work of Hodge and Pedoe [28, Book IV, §XIII.11], for example. The notation used there is rather old-fashioned however, and makes it difficult to follow. Let $S = \text{Proj}(\mathbb{Q}[x]/(Q_1, Q_2))$ be as above. Then it turns out that up to isomorphism over $\mathbb{Q}$, there are 15 possible singularity types for $S$, each categorised by the extended Dynkin diagram. This is the Dynkin diagram that describes the intersection behaviour of the exceptional divisors and the transforms of the lines on the minimal desingularisation $\tilde{S}$ of $S$. Of course, if one is interested in a classification over the ground field $\mathbb{Q}$, then many more singularity types can occur (see Lipman [29], for example). Over $\mathbb{Q}$, Coray and Tsfasman [19, Proposition 6.1] have calculated the extended Dynkin diagrams for all of the 15 types, and this information allows us to write down a list of surfaces that typify each possibility, together with their singularity type and the number of lines that they contain. The author is grateful to Ulrich Derenthal for helping to prepare the following table, which lists examples of surfaces $S = \text{Proj}(\mathbb{Q}[x]/(Q_1, Q_2))$ that illustrate the possible types.

| type | $Q_1(x)$ | $Q_2(x)$ | # lines | singularity |
|------|----------|----------|---------|-------------|
| i    | $x_0x_1 + x_2x_3$ | $x_0x_3 + x_1x_2 + x_2x_4 + x_3x_4$ | 12 | $A_1$ |
| ii   | $x_0x_1 + x_2x_3$ | $x_0x_3 + x_1x_2 + x_2x_4 + x_3^2$ | 9 | $2A_1$ |
| iii  | $x_0x_1 + x_2^2$ | $x_0x_2 + x_1x_2 + x_3x_4$ | 8 | $2A_1$ |
| iv   | $x_0x_1 + x_2x_3$ | $x_2x_3 + x_4(x_0 + x_1 + x_2 - x_3)$ | 8 | $A_2$ |
| v    | $x_0x_1 + x_2^2$ | $x_1x_2 + x_2^2 + x_3x_4$ | 6 | $3A_1$ |
| vi   | $x_0x_1 + x_2x_3$ | $x_1x_2 + x_2^2 + x_3x_4$ | 6 | $A_1 + A_2$ |
| vii  | $x_0x_1 + x_2x_3$ | $x_1x_3 + x_2^2 + x_3^2$ | 5 | $A_3$ |
| viii | $x_0x_1 + x_2^2$ | $(x_0 + x_1)^2 + x_2x_4 + x_3^2$ | 4 | $A_3$ |
| ix   | $x_0x_1 + x_2^2$ | $x_2^2 + x_3x_4$ | 4 | $4A_1$ |
| x    | $x_0x_1 + x_2^2$ | $x_1x_2 + x_3x_4$ | 4 | $2A_1 + A_2$ |
| xi   | $x_0x_1 + x_2^2$ | $x_2^2 + x_3x_4 + x_3^2$ | 3 | $A_3 + A_3$ |
| xii  | $x_0x_1 + x_2x_3$ | $x_0x_4 + x_1x_3 + x_2^2$ | 3 | $A_4$ |
| xiii | $x_0x_1 + x_2^2$ | $x_0^2 + x_1x_4 + x_3^2$ | 2 | $D_4$ |
| xiv  | $x_0x_1 + x_2^2$ | $x_2^2 + x_3x_4$ | 2 | $2A_1 + A_3$ |
| xv   | $x_0x_1 + x_2^2$ | $x_0x_4 + x_1x_2 + x_3^2$ | 1 | $D_5$ |

Let $\tilde{S}$ denote the minimal desingularisation of any surface $S$ from the table, and let $\rho_S$ denote the rank of the Picard group of $\tilde{S}$. Then it is natural to try
and establish (1.3) for such surfaces $S$. Several of the surfaces are actually special cases of varieties for which the Manin conjecture is already known to hold. Thus we have seen above that it has been established for toric varieties, and it can be checked that the surfaces representing types $ix$, $x$, $xiv$ are all equivariant compactifications of $G_m^2$, and so are toric. Hence (1.3) holds for these particular surfaces. Similarly it has been shown by Chambert-Loir and Tschinkel [16] that the Manin conjecture is true for equivariant compactifications of vector groups. Although identifying such surfaces in the table is not entirely routine, it transpires that the $D_5$ surface representing type $xv$ is an equivariant compactification of $G_m^2$. Per Salberger has raised the question of whether there exist singular del Pezzo surfaces of degree 4 that arise as equivariant compactifications of $G_a \times G_m$, but that are not already equivariant compactifications of $G_m^2$ or $G_m^4$. This is a natural class of varieties that does not seem to have been studied yet, but for which the existing technology is likely to prove useful.

Let us consider the type $xv$ surface

$$S_1 = \{ [x_0, \ldots, x_4] \in \mathbb{P}^4 : x_0 x_1 + x_2^2 = x_0 x_4 + x_1 x_2 + x_3^2 = 0 \},$$

in more detail. Now we have already seen that (1.3) holds for $S_1$. Nonetheless, La Bretèche and Browning [7] have made an exhaustive study of $S_1$, partly in an attempt to lay down a template for the treatment of other surfaces in the table. In doing so several new features have been revealed. For $s \in \mathbb{C}$ such that $\Re(s) > 1$, let

$$Z_{U,H}(s) := \sum_{x \in U(\mathbb{Q})} H(x)^{-s}$$

(1.4)

denote the corresponding height zeta function, where $U = U_1$ denotes the open subset formed by deleting the unique line $x_0 = x_2 = x_3 = 0$ from $S_1$. The analytic properties of $Z_{U_1,H}(s)$ are intimately related to the asymptotic behaviour of the counting function $N_{U_1,H}(B)$, and it is relatively straightforward to translate between them. For $\sigma \in \mathbb{R}$, let $\mathcal{H}_\sigma$ denote the half-plane $\{ s \in \mathbb{C} : \Re(s) > \sigma \}$. Then with this notation in mind we have the following result [7, Theorem 1].

**Theorem 2.** There exists a constant $\alpha \in \mathbb{R}$, a function $F_1(s)$ that is meromorphic on $\mathcal{H}_{9/10}$ with a pole of order 6 at $s = 1$, and a function $F_2(s)$ that is holomorphic on $\mathcal{H}_{5/6}$, such that

$$Z_{U_1,H}(s) = F_1(s) + \alpha(s - 1)^{-1} + F_2(s),$$

for $s \in \mathcal{H}_1$. In particular $Z_{U_1,H}(s)$ has an analytic continuation to $\mathcal{H}_{9/10}$.

It should be highlighted that there exist remarkably precise descriptions of the terms $F_1, F_2, \alpha$ that appear in the statement of the theorem. An application of Perron’s formula enables one to deduce a corresponding asymptotic formula for $N_{U_1,H}(B)$ that verifies (1.3), with $\rho_{S_1} = 6$. Actually one is led to the much stronger statement that there exists a polynomial $f$ of degree 5 such that for any $\delta \in (0, 1/12)$ we have

$$N_{U,H}(B) = B f(\log B) + O(B^{1-\delta}),$$

(1.5)

with $U = U_1$, in which the leading coefficient of $f$ agrees with Peyre’s prediction.

No explicit use is made of the fact that $S_1$ is an equivariant compactification of $G_m^2$ in the proof of Theorem 2, and this renders the method applicable to
other surfaces in the list that are not of this type. For example, in further work la Bretèche and Browning [8] have also established the Manin conjecture for the $D_4$ surface

$$S_2 = \{ [x_0, \ldots, x_4] \in \mathbb{P}^4 : x_0x_1 + x_2^2 = x_0^2 + x_1x_4 + x_3^2 = 0 \},$$

which represents the type xiii surface in the table. This surface is not split, since it contains the pair of lines $x_1 = x_2 = x_0 \pm ix_3 = 0$, and it turns out that $\text{Pic}(S_2)$ has rank 4. In fact $S_2$ has singularity type $C_3$ over $\mathbb{Q}$, in the sense of Lipman [29, §24], which becomes a $D_4$ singularity over $\overline{\mathbb{Q}}$. Building on the techniques developed in the proof of Theorem 2, a result of the same quality is obtained for the corresponding height zeta function $Z_{U_2, H}(s)$, and this leads to an estimate of the shape (1.5) for $\delta \in (0, 3/32)$, with $U = U_2$ and $\text{deg } f = 3$.

One of the aims of this survey is to give an overview of the various ideas and techniques that have been used to study the surfaces $S_1, S_2$ above. We shall illustrate the basic method by giving a simplified analysis of a new example from the table. Let us consider the $3A_1$ surface

$$S_3 = \{ [x_0, \ldots, x_4] \in \mathbb{P}^4 : x_0x_1 + x_2^2 = x_1x_2 + x_3^2x_4 = 0 \}, \quad (1.6)$$

which represents the type v surface in the table, and is neither toric, nor an equivariant compactification of $G^2_2$. The surface has singularities at the points $[1,0,0,0,0], [0,0,0,1,0]$ and $[0,0,0,0,1]$, and contains precisely 6 lines

$$x_i = x_2 = x_j = 0, \quad x_0 + x_2 = x_1 + x_2 = x_j = 0,$$

where $i \in \{0,1\}$ and $j \in \{3,4\}$. Since $S_3$ is split, one finds that the expected exponent of $\log B$ in (1.3) is $\rho_{S_3} - 1 = 5$. We shall establish the following result.

**Theorem 3.** We have $N_{U_3, H}(B) = O(B(\log B)^5)$.

As pointed out to the author by Régis de la Bretèche, it is possible to establish a corresponding lower bound $N_{U_3, H}(B) \gg B(\log B)^5$, using little more than the most basic estimates for integers restricted to lie in fixed congruence classes. In fact, with more work, it ought even to be possible to obtain an asymptotic formula for $N_{U_3, H}(B)$. In the interests of brevity, however, we have chosen to pursue neither of these problems here.

### 1.3 Del Pezzo surfaces of degree 3

The del Pezzo surfaces $S \subset \mathbb{P}^3$ of degree 3 are readily recognised as the geometrically integral cubic surfaces in $\mathbb{P}^3$, that are not cones. Given such a surface $S$ defined over $\mathbb{Q}$, we may always find an absolutely irreducible cubic form $C(x) \in \mathbb{Z}[x_0, x_1, x_2, x_3]$ such that $S = \text{Proj}(\mathbb{Q}[x]/(C))$. Let us begin by considering the situation for non-singular cubic surfaces. In this setting $U \subset S$ is taken to be the open subset formed by deleting the famous 27 lines from $S$. Although Peyre and Tschinkel [33, 34] have provided ample numerical evidence for the validity of the Manin conjecture for non-singular cubic surfaces, we are unfortunately still rather far away from proving it. The best upper bound available is $N_{U, H}(B) = O_e S(B^{4/3+\varepsilon})$, due to Heath-Brown [24]. This applies when the surface $S$ contains 3 coplanar lines defined over $\mathbb{Q}$, and in particular to the Fermat cubic surface $x_0^3 + x_1^3 = x_2^3 + x_3^3$. The problem of proving lower bounds is somewhat easier. Under the assumption that $S$ contains a
pair of skew lines defined over $\mathbb{Q}$, Slater and Swinnerton-Dyer [39] have shown that $N_{U,H}(B) \geq S B (\log B)^{\rho s-1}$, as predicted by the Manin conjecture. This does not apply to the Fermat cubic surface, however, since the only skew lines contained in this surface are defined over $\mathbb{Q}(\sqrt{-3})$. It would be interesting to extend the work of Slater and Swinnerton-Dyer to cover such cases.

Much as in the previous section, it turns out that far better estimates are available for singular cubic surfaces. The classification of such surfaces is a well-established subject, and essentially goes back to the work of Cayley [15] and Schläfli [37] over a century ago. A contemporary classification of singular cubic surfaces, using the terminology of modern classification theory, has since been given by Bruce and Wall [14]. As in the previous section, the Manin conjecture is already known to hold for several of these surfaces by virtue of the fact that they are equivariant compactifications of $\mathbb{G}_m^2$, or toric, such as the $3\mathbb{A}_2$ surface

$$ S_4 = \{ [x_0, x_1, x_2, x_3] \in \mathbb{P}^3 : x_0^3 = x_1 x_2 x_3 \}, $$

for example. In fact a number of authors have studied this particular surface, including la Bretêche [4], Fouvry [21], and Heath-Brown and Moroz [27]. Of the asymptotic formulae obtained, the most impressive is the first, which consists of an estimate like (1.5) for any $\delta \in (0, 1/8)$, with $U = U_4 \subset S_4$ and a suitable polynomial $f$ of degree 6. The next surface to receive serious attention was the so-called Cayley cubic surface

$$ S_5 = \{ [x_0, x_1, x_2, x_3] \in \mathbb{P}^3 : x_0 x_1 x_2 + x_0 x_1 x_3 + x_0 x_2 x_3 + x_1 x_2 x_3 = 0 \}, $$

of singularity type $4\mathbb{A}_1$. This contains 9 lines, all of which are defined over $\mathbb{Q}$, and Heath-Brown [26] has shown that there exist absolute constants $A_1, A_2 > 0$ such that

$$ A_1 B (\log B)^6 \leq N_{U_5,H}(B) \leq A_2 B (\log B)^6, $$

where $U_5 \subset S_5$ is the usual open subset. An estimate of precisely the same form has also been obtained by Browning [13] for the $D_4$ surface

$$ S_6 = \{ [x_0, x_1, x_2, x_3] \in \mathbb{P}^3 : x_1 x_2 x_3 = x_0 (x_1 + x_2 + x_3)^2 \}. $$

In both cases the corresponding Picard group has rank 7, so that the exponents of $B$ and $\log B$ agree with Manin’s prediction.

The final surface to have been studied extensively is the $E_6$ cubic surface

$$ S_7 = \{ [x_0, x_1, x_2, x_3] \in \mathbb{P}^3 : x_1 x_2^3 + x_2 x_3^2 + x_3^3 = 0 \}, $$

which contains a unique line $x_2 = x_3 = 0$. Let $U_7 \subset S_7$ denote the open subset formed by deleting the line from $S_7$, and recall the notation (1.4) for the height zeta function $Z_{U_7,H}(s)$ and that of the half-plane $\mathcal{H}_s$ introduced before Theorem 2. Then recent work of la Bretêche, Browning and Derenthal [10] has succeeded in establishing the following result.

**Theorem 4.** There exists a constant $\alpha \in \mathbb{R}$, a function $F_1(s)$ that is meromorphic on $\mathcal{H}_{9/10}$ with a pole of order 7 at $s = 1$, and a function $F_2(s)$ that is holomorphic on $\mathcal{H}_{43/18}$, such that

$$ Z_{U_7,H}(s) = F_1(s) + \alpha (s-1)^{-1} + F_2(s), $$

for $s \in \mathcal{H}_1$. In particular $Z_{U_7,H}(s)$ has an analytic continuation to $\mathcal{H}_{9/10}$. 

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As in Theorem 2, the terms $F_1, F_2, \alpha$ have a very explicit description. An application of Perron’s formula now yields an asymptotic formula of the shape (1.5) for $\delta \in (0, 1/11)$, with $U = U_7$ and a suitable polynomial $f$ of degree 6. This too is in complete agreement with the Manin conjecture. It should be remarked that, in work to appear, Michael Joyce has independently established the Manin conjecture for $S_7$ in his doctoral thesis at Brown University, albeit only with a weaker error term of $O(B(\log B)^5)$.

2 Refinements of the Manin conjecture

The purpose of this section is to consider in what way one might hope to refine the conjecture of Manin. We have already seen a number of examples in which asymptotic formulae of the shape (1.5) hold, and it is very natural to suppose that this is the case for any (possibly singular) del Pezzo surface $S \subset \mathbb{P}^d$ of degree $d$, where as usual $U \subseteq S$ denotes the open subset formed by deleting any exceptional divisors from $S$, and $\rho_S$ denotes the rank of the Picard group of $S$ (possibly of $\widetilde{S}$). Let us record this formally here.

**Conjecture B.** Let $S, U, \rho_S$ be as above. Then there exists $\delta > 0$, and a polynomial $f \in \mathbb{R}[x]$ of degree $\rho_S - 1$, such that (1.5) holds.

The leading coefficient of $f$ should of course agree with the prediction of Peyre et al. It would be interesting to gain a conjectural understanding of the lower order coefficients of $f$, possibly in terms of the geometry of $S$. At this stage it seems worth drawing attention to the surprising nature of the constants $\alpha$ that appear in Theorems 2 and 4, not least because they contribute to the constant coefficient of $f$. In both cases we have $\alpha = \frac{12}{\pi^2} + \beta$, where the first term corresponds to an isolated conic in the surface, and the second is purely arithmetic in nature and takes a very complicated shape (see [7, Eq. (5.25)] and [10, Eq. (8.49)]). It arises through the error in approximating certain arithmetic quantities by real-valued continuous functions, and involves the application of results about the equidistribution of squares in fixed residue classes.

One might also ask what one expects to be the true order of magnitude of the error term in (1.5). This a question that Swinnerton-Dyer has recently addressed [40, Conjecture 2], inspired by comparisons with the explicit formulae from prime number theory.

**Conjecture C.** Let $S, U, \rho_S$ be as above. Then there exist positive constants $\theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 < \min\{\theta_2, \theta_3\}$, a polynomial $f \in \mathbb{R}[x]$ of degree $\rho_S - 1$, a constant $\gamma \in \mathbb{R}$, and a sequence of $\gamma_n \in \mathbb{C}$, such for any $\varepsilon > 0$ we have

$$N_{U,H}(B) = Bf(\log B) + \gamma B^{\theta_3} + \Re \sum \gamma_n B^{\theta_2 + it_n} + O_\varepsilon(B^{\theta_1 + \varepsilon}).$$

Here $\frac{1}{2} + it_n$ runs through a set of non-trivial zeros of the Riemann zeta function, with the $t_n$ positive and monotonic increasing, such that $\sum |\gamma_n|^2$ and $\sum t_n^{-2}$ are convergent.

In fact Swinnerton-Dyer formulates the conjecture for non-singular cubic surfaces, with $\theta_1 < \frac{1}{2} = \theta_2$ and $\gamma = 0$. There is no reason, however, to expect that it doesn’t hold more generally, and one might even suppose that the constants $\theta_2, \theta_3$ somehow relate to the nature of the surface singularities. In work
currently under preparation, la Bretèche and Swinnerton-Dyer have provided
significant evidence for this finer conjecture for the singular cubic surface (1.7).
Under the assumption of the Riemann hypothesis it is shown that the conjecture
holds for $S_4$, with $(\theta_1, \theta_2, \theta_3) = (\frac{5}{16}, \frac{2}{13}, \frac{9}{11})$ and $\gamma \neq 0$.

3 Available tools

There are a variety of tools that can be brought to bear upon the problem
of estimating the counting function (1.1) for appropriate subsets $U$ of projective
algebraic varieties. Most of these are rooted in analytic number theory.
When the dimension of the variety is large compared to its degree, the Hardy–
Littlewood circle method can often be applied successfully (see Davenport [20],
for example). When the variety has a suitable “cellular” structure, techniques
involving harmonic analysis on adelic groups can be employed (see Tschinkel
[42], for example). We shall say nothing more about these methods here, save
to observe that outside of the surfaces covered by the collective work of Batyrev,
Chambert-Loir and Tschinkel [3, 16], they do not seem capable of establishing
the Manin conjecture for all del Pezzo surfaces.

In fact we still have no clear vision of which methods are most appropri-
ate, and it is conceivable that the methods needed to handle the singular del
Pezzo surfaces of low degree are quite different from those needed to handle
the non-singular surfaces. Given our inability to prove the Manin conjecture
for a single non-singular del Pezzo surface of degree 3 or 4, we shall say no
more about them here, save to observe that the sharpest results we have are
for examples containing conic bundle structures over the ground field. Instead
we shall concentrate on the situation for singular del Pezzo surfaces of degree 3
or 4. Disappointing as it may seem, it is hard to imagine that we will see how
to prove Manin’s conjecture for all del Pezzo surfaces without first attempting
to do so for a number of very concrete representative examples. As a cursory
analysis of the proofs of Theorems 2–4 shows, the techniques that have been
successively applied so far are decidedly ad-hoc. Nonetheless there are a few
salient features that are worthy of amplification, and this will be the focus of
the two subsequent sections.

3.1 The universal torsor

Universal torsors were originally introduced by Colliot-Thélène and Sansuc [17,
18] to aid in the study of the Hasse principle and weak approximation for rational
varieties. Since their inception it is now well-recognised that they also have a
central rôle to play in proofs of the Manin conjecture for Fano varieties. Let
$S \subset \mathbb{P}^d$ be a del Pezzo surface of degree $d \in \{3, 4, 5\}$, and let $\tilde{S}$ denote the
minimal desingularisation of $S$ if it is singular, and $\tilde{S} = S$ otherwise. Let
$E_1, \ldots, E_{10-d} \in \text{Div}(\tilde{S})$ be generators for the geometric Picard group of $\tilde{S}$, and
let $E_1^\times = E_1 \setminus \{\text{zero section}\}$. Working over $\overline{\mathbb{Q}}$, “the” universal torsor of $\tilde{S}$ is
given by the action of $\mathbb{G}^{10-d}_m$ on the map

$$
\pi : E_1^\times \times \cdots \times E_{10-d}^\times \to \tilde{S}.
$$

In practice this action can be made completely explicit, thereby giving equations
for the universal torsor. A proper discussion of universal torsors would take us
too far afield at present, and the reader should consult the survey of Peyre [32] for further details, or indeed the construction of Hassett and Tschinkel [23]. The latter outlines an alternative approach to universal torsors via the Cox ring. The guiding principle behind the use of universal torsors is simply that they ought to be arithmetically simpler than the original variety. The universal torsors that we shall encounter all have embeddings as open subsets of affine varieties of higher dimension, and the general theory ensures that there is a bijection between \( U(\mathbb{Q}) \) — where \( U \subset S \) is the usual open subset formed by deleting the lines from \( S \) — and a suitable set of integral points on the corresponding universal torsor. We shall see shortly how one may often use arguments from elementary number theory to explicitly derive these bijections.

Let us begin by giving a few examples. In the proof of Theorem 1 a passage to the universal torsor is a crucial first step, and was originally carried out by Salberger in his unpublished proof of the bound \( N_{U_0(B)} = O(B(\log B)^4) \), announced in the Borel seminar at Bern in 1993. Recall the Plücker embedding

\[
\pi_i \{-i, j, k, \ell \} \rightarrow \mathbb{P}^9 \subset \mathbb{P}^9(\mathbb{Q}) \in 2\text{-dimensional linear subspaces of } \mathbb{Q}^5.
\]

Then the universal torsor \( \pi : T_0 \rightarrow S_0 \) above \( S_0 \) is a certain open subset of the affine cone over \( \mathbb{G}(2, 5) \). To count points of bounded height in \( U_0(\mathbb{Q}) \) it is then enough to count integral points \( (z_{i,j})_{1 \leq i < j \leq 5} \in \mathbb{Z}^9 \) on this cone, where \( \mathbb{Z}_n := \mathbb{Z} \setminus \{0\} \), subject to a number of side conditions. A thorough account of this particular example, and how it extends to arbitrary del Pezzo surfaces of degree 5 can be found in the work of Skorobogatov [38]. A second example is calculated by Hassett and Tschinkel [23] for the \( E_6 \) cubic surface (1.9). There it is shown that the universal torsor above \( S_7 \) has the equation

\[
\tau \xi_1^2 + \tau_2^2 \xi_2 + \tau_3^2 \xi_3 = 0,
\]

for variables \( \tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6 \). One of the variables does not explicitly appear in (3.1), and the torsor should be thought of as being embedded in \( \mathbb{A}^{10} \). The universal torsors that turn up in the proofs of Theorems 2 and 3 can also be embedded in affine space via a single equation.

We proceed to carry out explicitly the passage to the universal torsor for the \( 3\mathbb{A}_1 \) surface (1.6). We shall use \( \mathbb{N} \) to denote the set of positive integers, and for any \( n \geq 2 \) we let \( \mathbb{Z}^n \) denote the set of \emph{primitive} vectors in \( \mathbb{Z}^n \), by which we mean that the greatest common divisor of the components should be 1. We may clearly assume that \( S_3 \) is defined by the forms \( Q_1(x) = x_0x_1 - x_2^2 \) and \( Q_2(x) = x_2^2 - x_1x_2 + x_3x_4 \). Now if \( x \in U_3(\mathbb{Q}) \) is represented by the vector \( x \in \mathbb{Z}^5 \), then \( x_0 \cdots x_4 \neq 0 \) and \( H(x) = \max\{|x_0|, |x_1|, |x_3|, |x_4|\} \). Moreover, \( x_0 \) and \( x_1 \) must share the same sign. On taking \( x_0, x_1 \) to both be positive, and noting that \( x \) and \( -x \) represent the same point in \( \mathbb{P}^4 \), we deduce that

\[
N_{U_0, H}(B) = \#\{x \in Z^5 : 0 < x_0, x_1, |x_3|, |x_4| \leq B, \ Q_1(x) = Q_2(x) = 0\}.
\]

Let us begin by considering solutions \( x \in \mathbb{Z}^5 \) to the equation \( Q_1(x) = 0 \). There is a bijection between the set of integers \( x_0, x_1, x_2 \) such that \( x_0, x_1 > 0 \) and \( x_0x_1 = x_2^2 \), and the set of \( x_0, x_1, x_2 \) such that \( x_0 = z_0^2z_2, x_1 = z_1^2z_2 \) and \( x_2 = z_0z_1z_2 \), for non-zero integers \( z_0, z_1, z_2 \) such that \( z_1, z_2 > 0 \) and \( \gcd(z_0, z_1) = 1 \).
We now substitute these values into the equation $Q_2(x) = 0$, in order to obtain

$$z_0^2 z_1^2 z_2^2 - z_0 z_1^3 z_2^3 + x_3 x_4 = 0. \tag{3.2}$$

It follows from the coprimality relation $\gcd(x_0, \ldots, x_4) = 1$ that we also have $\gcd(z_0, z_1, x_3) = 1$. Now we may conclude from (3.2) that $z_0 z_1^2 z_2^2$ divides $x_3 x_4$.

Let us write $y_1 = \gcd(z_1, x_3, x_4)$ and $z_1 = y_1 y_3$, $x_4 = y_1 y_4$, with $y_1, y_3, y_4$ non-zero integers such that $y_1, y_3 > 0$ and $\gcd(y_3, y_4) = 1$. Then $z_0 y_1^2 z_2^2$ divides $y_3 y_4$. We now write $z_0 = y_0 y_3 y_4$, $y_3 = y_0 y_3 y_4$ and $y_4 = y_0 y_3 y_4$, for non-zero integers $y_0, y_0, y_3, y_4$. We therefore conclude that $y_1^2 z_2^2$ divides $y_3 y_4$, whence there exist positive integers $y_1, y_1, y_2, y_3, y_4$ and non-zero integers $y_0, y_0, y_3, y_3, y_4$ such that $y_1 = y_1 y_3, z_2 = y_2 y_3 y_4, z_3 = y_3 y_2 y_3 y_4$ and $y_4 = y_4 y_2 y_3 y_4$.

Substituting these into (3.2) yields the equation

$$y_0 y_3 y_4 - y_1 y_3 y_4 + y_3 y_4 = 0. \tag{3.3}$$

This equation gives an affine embedding of the universal torsor over $S_3$, though we shall not prove it here. Furthermore, we may combine all of the various coprimality relations above to deduce that

$$\gcd(y_1 y_3 y_4 y_2 y_3 y_2, y_1 y_3 y_2 y_3 y_3, y_1 y_2 y_3 y_4) = 1, \tag{3.4}$$

and

$$\gcd(y_0 y_3 y_4, y_1 y_3 y_4) = \gcd(y_1, y_0 y_3 y_4 y_2 y_3 y_2) = 1. \tag{3.5}$$

At this point we may summarize our argument as follows. Let $T$ denote the set of non-zero integer vectors $y = (y_1, y_0 y_3 y_4, y_3, y_2, y_4, y_3, y_2, y_3, y_4)$ such that (3.3)–(3.5) all hold, with $y_1, y_3, y_4, y_2, y_4 > 0$. Then for any $x \in Z_5$ such that $Q_1(x) = Q_2(x) = 0$ and $x_0, x_1, |x_3, |x_4| > 0$, we have shown that there exists $y \in T$ such that

$$x_0 = y_0 y_3 y_4 y_2 y_3 y_2, \quad x_1 = y_1 y_3 y_4 y_2 y_3 y_2, \quad x_2 = y_1 y_3 y_4 y_2 y_3 y_2 y_3 y_2, \quad x_3 = y_1 y_3 y_4 y_2 y_3 y_3, \quad x_4 = y_1 y_3 y_4 y_2 y_3 y_4.$$

Conversely, it is not hard to check that given any $y \in T$ the point $x$ given above will be a solution of the equations $Q_1(x) = Q_2(x) = 0$, with $x \in Z_5$ and $x_0, x_1, |x_3, |x_4| > 0$. Let us define the function $\Psi : \mathbb{R}^9 \to \mathbb{R}_{>0}$, given by

$$\Psi(y) = \max \left\{ \frac{|y_0 y_3 y_4 y_2 y_3 y_2|}{|y_1 y_3 y_4 y_2 y_3 y_2|}, \frac{|y_1 y_3 y_4 y_2 y_3 y_2|}{|y_1 y_3 y_4 y_2 y_3 y_2|}, \frac{|y_1 y_3 y_4 y_2 y_3 y_2|}{|y_1 y_3 y_4 y_2 y_3 y_2|} \right\}.$$

Then we have established the following result.

**Lemma 1.** We have $N_{U, H}(B) = \#\{y \in T : \Psi(y) \leq B\}$.

In this section we have given several examples of universal torsors, and we have ended by demonstrating how elementary number theory can sometimes be used to calculate them with very little trouble. In fact the general machinery of Colliot-Thélène–Sansuc [17, 18], or that of Hassett–Tscheinke [23], essentially provides an algorithm for calculating the universal torsor over any singular del Pezzo surface of degree 3 or 4. It should be stressed, however, that if this constitutes being given the keys to the city, it does not tell us where in the city the proof is hidden.
3.2 The next step

The purpose of this section is to overview some of the techniques that have been developed for counting integral points on the parametrization that arises out of the passage to the universal torsor, as discussed above. In the proofs of Theorems 1–4 the torsor equations all take the shape

\[ A_j + B_j + C_j = 0, \quad (1 \leq j \leq J), \]

for monomials \( A_j, B_j, C_j \) of various degrees in the appropriate variables. By fixing some of the variables at the outset, one is then left with the problem of counting integer solutions to a system of Diophantine equations, subject to certain constraints. If one is sufficiently clever about which variables to fix first, then one can sometimes be left with a quantity that we know how to estimate — and crucially — for which we can control the overall contribution from the error term when it is summed over the remaining variables.

Let us sketch this phenomenon briefly with the torsor equation (3.1) that is used in the proof of Theorem 4. It turns out that the way to proceed here is to fix all of the variables apart from \( \tau_1, \tau_2, \tau_\ell \). One may then view the equation as a congruence

\[ \tau_2^2 \xi_2 \equiv -\tau_3^3 \xi_1^2 \xi_3 \pmod{\xi_3^3 \xi_4^2 \xi_5}, \]

in order to take care of the summation over \( \tau_\ell \). This allows us to employ very standard facts about the number of integer solutions to polynomial congruences that are restricted to lie in certain regions, and this procedure yields a main term and an error term which the remaining variables need to be summed over. However, while the treatment of the main term is relatively routine, the treatment of the error term presents a much more serious obstacle. Although we do not have space to discuss it in any detail, it is here that the unexpected constant \( \alpha \) arises in Theorem 4 (and, indeed, in Theorem 2).

The sort of approach discussed above, and more generally the application of lattice methods to count solutions to ternary equations, is a very useful one. It plays a crucial role in the proof of the following result due to Heath-Brown [15, Lemma 3], which forms the next ingredient in our proof of Theorem 3.

**Lemma 2.** Let \( K_1, \ldots, K_7 \geq 1 \) be given, and let \( M \) denote the number of non-zero solutions \( m_1, \ldots, m_7 \in \mathbb{Z} \) to the equation

\[ m_1m_2 - m_3m_4m_5 + m_6m_7 = 0, \]

subject to the conditions \( K_k < |m_k| \leq 2K_k \) for \( 1 \leq k \leq 7 \), and

\[ \gcd(m_1m_2, m_3m_4m_5) = 1. \quad (3.6) \]

Then we have \( M \ll K_1K_2K_3K_4K_5 \).

For comparison, we note that it is a trivial matter to establish the bound \( M \ll \varepsilon (K_1K_2K_3K_4K_5)^{1+\varepsilon} \), using standard estimates for the divisor function.

3.3 Completion of the proof of Theorem 3

We are now ready to complete the proof of Theorem 3. We shall begin by estimating the contribution to \( N_{U_3,H}(B) \) from the values of \( y \) appearing in
Lemma 1 that are constrained to lie in a certain region. Let $Y_1, Y_3, Y_4 \geq 1$, where throughout this section $i$ denotes a generic index from the set $\{0, 1, 2, 3\}$. Then we write $N = N(Y_1, Y_3, Y_4, Y_3, Y_4, Y_3, Y_4, Y_3, Y_4)$ for the total contribution to $N_{U_3,H}(B)$ from $Y$ satisfying

$$Y_1 \leq y_1 < 2Y_1, \quad Y_3 \leq |y_3| < 2Y_3, \quad Y_4 \leq |y_4| < 2Y_4.$$  \tag{3.7}

Clearly it follows from the inequality $\Psi(y) \leq B$ that $N = 0$ unless

$$Y_0^2 Y_1^2 Y_2 Y_3 Y_4 \ll B, \quad Y_1^2 Y_3^2 Y_4 Y_5 Y_6 Y_7 Y_8 \ll B,$$  \tag{3.8}

and

$$Y_1 Y_3 Y_4 Y_5 Y_6 Y_7 Y_8 \ll B, \quad Y_1 Y_3 Y_4 Y_5 Y_6 Y_7 Y_8 \ll B.$$  \tag{3.9}

In our estimation of $N_{U_3,H}(B)$, we may clearly assume without loss of generality that

$$Y_0^2 Y_1^2 Y_2 Y_3 \ll Y_4 Y_5 Y_6 Y_7 Y_8.$$  \tag{3.10}

We proceed to show how the equation (3.3) forces certain constraints upon the choice of dyadic ranges in (3.7). There are three basic cases that can occur. Suppose first that

$$c_2 Y_0 Y_4 \leq Y_1 Y_3 Y_4,$$  \tag{3.11}

for an absolute constant $c_2 > 0$. Then it follows from (3.3) that

$$Y_3 Y_4 \ll Y_1 Y_3 Y_4 \ll Y_3 Y_4,$$  \tag{3.12}

provided that $c_2$ is chosen to be sufficiently large. Next, we suppose that

$$c_1 Y_0 Y_4 \geq Y_1 Y_3 Y_4,$$  \tag{3.13}

for an absolute constant $c_1 > 0$. Then we may deduce from (3.3) that

$$Y_3 Y_4 \ll Y_0 Y_4 \ll Y_3 Y_4,$$  \tag{3.14}

provided that $c_1$ is chosen to be sufficiently small. Let us henceforth assume that the values of $c_1, c_2$ are fixed in such a way that (3.12) holds, if (3.11) holds, and (3.14) holds, if (3.13) holds. Finally we are left with the possibility that

$$c_1 Y_0 Y_4 \leq Y_1 Y_3 Y_4 \leq c_2 Y_0 Y_4.$$  \tag{3.15}

We shall need to treat the cases (3.11), (3.13) and (3.15) separately.

We shall take $m_{i,k} = (y_1, y_{13}, y_{14}, y_{13}, y_{14}, y_{63}, y_{64})$ in our application of Lemma 2, for $(j, k) = (0, 3)$ and $(3, 0)$. In particular the coprimality relation (3.6) follows directly from (3.4) and (3.5), and we may conclude that

$$N \ll Y_1 Y_3 Y_4 Y_5 Y_6 Y_7 Y_8 \min\{Y_0 Y_4, Y_3 Y_4\},$$  \tag{3.16}

on summing over all of the available $y_2, y_2$. It remains to sum this contribution over the various dyadic intervals $Y_1, Y_3, Y_4$. Suppose for the moment that we are interested in summing over all possible dyadic intervals $X \leq |x| < 2X$, for which $|x| \leq X$. Then there are plainly $O(\log X)$ possible choices for $X$. In addition to this basic estimate, we shall make frequent use of the estimate

$$\sum_X X^\delta \ll \delta X^\delta, \quad \text{for any } \delta > 0.$$
We begin by assuming that (3.11) holds, so that (3.12) also holds. Then we may combine (3.10) with (3.12) in order to deduce that

\[
Y_{13} \ll \min \left\{ \frac{y_{04}^{-1/2}y_{14}y_{24}y_{34}^{1/2}}{y_{03}^{-1/2}y_{23}y_{33}^{1/2}}, \frac{y_{34}y_{34}}{Y_{1}Y_{14}} \right\} \ll \frac{y_{04}^{-1/4}y_{24}^{-1/2}y_{34}^{-3/4}}{y_{03}^{-1/2}y_{23}^{-1/2}}.
\]

We may now apply (3.16) to obtain

\[
\sum_{Y_{1}, Y_{23}, Y_{34}} N \ll \sum_{Y_{1}, Y_{23}, Y_{34}} Y_{1}Y_{03}Y_{04}Y_{13}Y_{14}Y_{23}Y_{24} \ll \sum_{Y_{03}, Y_{04}, Y_{33}, Y_{34}} Y_{03}^{-1/2}y_{04}^{-3/4}y_{14}^{-1/2}y_{23}^{-1/2}y_{24}^{-1/4}y_{33}^{-1/2}y_{34}^{-1/4}.
\]

But now (3.9) implies that \( Y_{14} \ll B^{1/2}/(y_{04}^{-1/2}y_{04}^{-1/2}y_{34}^{-1/2}) \), and (3.12) and (3.11) together imply that \( Y_{03} \ll Y_{33}Y_{34}/Y_{04} \). We therefore deduce that

\[
\sum_{Y_{1}, Y_{23}, Y_{34}} N \ll B^{1/2} \sum_{Y_{03}, Y_{04}, Y_{33}, Y_{34}, Y_{23}, Y_{24}} Y_{23}^{-1/2}y_{24}^{-1/2}y_{33}^{-1/2}y_{34}^{-1/4}.
\]

Finally it follows from (3.8) and (3.12) that \( Y_{33} \ll B^{1/2}/(y_{23}^{-1/2}y_{24}^{-1/2}y_{34}^{-1/2}) \), whence

\[
\sum_{Y_{1}, Y_{13}, Y_{14}} N \ll B \sum_{Y_{03}, Y_{13}, Y_{14}, Y_{23}, Y_{24}} 1 \ll B \log B,
\]

which is satisfactory for the theorem.

Next we suppose that (3.13) holds, so that (3.14) also holds. In this case it follows from (3.10), together with the inequality \( Y_{1}Y_{13}Y_{14} \ll Y_{03}Y_{04} \), that

\[
Y_{13} \ll \min \left\{ \frac{y_{04}^{-1/2}y_{14}y_{24}y_{34}^{1/2}}{y_{03}^{-1/2}y_{23}y_{33}^{1/2}}, \frac{y_{03}y_{04}}{Y_{1}Y_{14}} \right\} \ll \frac{y_{03}^{-1/4}y_{04}^{-3/4}y_{14}^{-1/2}y_{23}^{-1/2}y_{24}^{-1/4}y_{33}^{-1/2}y_{34}^{-1/4}}{y_{03}^{-1/2}y_{23}^{-1/2}y_{33}^{-1/2}}.
\]

On combining this with the inequality \( Y_{14} \ll B^{1/2}/(y_{04}^{-1/2}y_{04}^{-1/2}y_{34}^{-1/2}) \), that follows from (3.9), we may therefore deduce from (3.16) that

\[
\sum_{Y_{1}, Y_{13}, Y_{14}} N \ll \sum_{Y_{1}, Y_{13}, Y_{14}} Y_{1}Y_{13}Y_{14}Y_{23}Y_{24}Y_{33}Y_{34} \ll \sum_{Y_{03}, Y_{04}, Y_{33}, Y_{34}, Y_{14}, Y_{23}, Y_{24}} y_{03}^{-1/4}y_{04}^{-3/4}y_{14}^{-1/2}y_{23}^{-1/2}y_{24}^{-1/4}y_{33}^{-1/2}y_{34}^{-1/4}.
\]

\[14\]
Now it follows from (3.14) that \( Y_{33} \ll Y_{03}Y_{04}/Y_{34} \). We may therefore combine this with the first inequality in (3.8) to conclude that

\[
\sum_{Y_1Y_3Y_4} N \ll B^{1/2} \sum_{Y_1Y_3Y_4} Y_{03}Y_{04}Y_{23}^{1/2}Y_{24}^{1/2} \ll B(\log B)^5,
\]

which is also satisfactory for the theorem.

Finally we suppose that (3.15) holds. On combining (3.10) with the fact that \( Y_{33}Y_{34} \ll Y_{03}Y_{04} \), we obtain

\[
Y_{33} \ll \min \left\{ \frac{Y_{03}Y_{13}Y_{23}}{Y_{03}Y_{14}Y_{24}}, \frac{Y_{03}Y_{04}Y_{34}}{Y_{34}} \right\} \ll \frac{Y_{03}Y_{14}Y_{24}}{Y_{13}Y_{23}}.
\]

Summing (3.16) over \( Y_{33} \) first, with \( \min\{Y_{03}Y_{04}, Y_{33}Y_{34}\} \ll Y_{03}Y_{04}/Y_{33}Y_{34} \), we therefore obtain

\[
\sum_{Y_1Y_3Y_4} N \ll \sum_{Y_1Y_3Y_4} Y_{03}Y_{04}Y_{13}^{1/2}Y_{14}^{1/2}Y_{23}^{1/2}Y_{24}^{1/2}Y_{34}^{1/2}
\]

But then we may sum over \( Y_{03}, Y_{13} \) satisfying the inequalities in (3.8), and then \( Y_1 \) satisfying the second inequality in (3.9), in order to conclude that

\[
\sum_{Y_1Y_3Y_4} N \ll B^{1/4} \sum_{Y_1Y_3Y_4} Y_{03}Y_{04}Y_{13}^{1/2}Y_{14}^{1/2}Y_{23}^{1/2}Y_{24}^{1/2}Y_{34}^{1/2} \ll B^{1/2} \sum_{Y_1Y_04} Y_{14}Y_{24}Y_{34}^{1/2} \ll B(\log B)^5.
\]

This too is satisfactory for Theorem 3, and thereby completes its proof.

4 Open problems

We close this survey article with a list of five open problems relating to Manin’s conjecture for del Pezzo surfaces. In order to encourage activity we have deliberately selected an array of very concrete problems.

1. Establish (1.3) for a non-singular del Pezzo surface of degree 4.
   The surface \( x_0x_1 - x_2x_3 = x_0^2 + x_1^2 + x_2^2 - x_3^2 - 2x_4^2 = 0 \) has Picard group of rank 5.

2. Establish (1.3) for a non-rational del Pezzo surface.
   The surface \( x_0x_1 - x_2^2 = x_0x_2 - x_1x_2 + x_3^2 + x_4^2 = 0 \), which is isomorphic (over \( \overline{Q} \)) to the type \( III \) surface in the table, is an example of an Iskovskih surface. It is not rational over \( Q \) [19, Proposition 7.7].

3. Break the 4/3-barrier for a non-singular cubic surface.
   We have yet to prove an upper bound of the shape \( N_{U,H}(B) = O_S(B^\theta) \), with \( \theta < 4/3 \), for a single non-singular cubic surface \( S \subset \mathbb{P}^3 \). Of course the ultimate goal is to do this for every such surface, but this seems to be much harder when the surface doesn’t have a conic bundle structure over \( Q \). The surface \( x_0x_1(x_0 + x_1) = x_2x_3(x_2 + x_3) \) admits such a structure —can one break the 4/3-barrier for this example?
4. Establish the lower bound \( N_{U,H}(B) \gg B(\log B)^3 \) for the Fermat cubic. The Fermat cubic \( x_0^3 + x_1^3 = x_2^3 + x_3^3 \) has Picard group of rank 4.

5. Better bounds for del Pezzo surfaces of degree 2.

The arithmetic of non-singular del Pezzo surfaces of degree 2 is still very elusive. These surfaces take the shape \( t^2 = F(x_0, x_1, x_2) \) for a non-singular quartic form \( F \). Let \( N(F; B) \) denote the number of integers \( t, x_0, x_1, x_2 \) such that \( t^2 = F(x) \) and \( |x| \leq B \). Can one prove that we always have \( N(F; B) = O_{\varepsilon,F}(B^{2+\varepsilon}) \)? Such an estimate would be essentially best possible, as consideration of the form \( F_0(x) = x_0^4 + x_1^4 + x_2^4 \) shows. The best result in this direction is due to Broberg [11], who has established the weaker bound \( N(F; B) = O_{\varepsilon,F}(B^{9/4+\varepsilon}) \). For certain quartic forms, such as \( F_1(x) = x_0^4 + x_1^4 + x_2^2 \), the Manin conjecture implies that one ought to be able to replace the exponent \( 2 + \varepsilon \) by \( 1 + \varepsilon \). Can one prove that \( N(F_1; B) = O(B^\theta) \) for some \( \theta < 2 \)?

Acknowledgements. The author is extremely grateful to Régis de la Bretèche and Per Salberger, who have both made a number of useful comments about an earlier version of this paper.

References

[1] V.V. Batyrev and Y.I. Manin, Sur le nombre des points rationnels de hauteur bornée des variétés algébriques. Math. Ann. 286 (1990), 27–43.

[2] V.V. Batyrev and Y. Tschinkel, Tamagawa numbers of polarized algebraic varieties. Astérisque 251 (1998), 299–340.

[3] V.V. Batyrev and Y. Tschinkel, Manin’s conjecture for toric varieties. J. Alg. Geom. 7 (1998), 15–53.

[4] R. de la Bretèche, Sur le nombre de points de hauteur bornée d’une certaine surface cubique singulière. Astérisque 251 (1998), 51–77.

[5] R. de la Bretèche, Compter des points d’une variété torique. J. Number Theory 87 (2001), 315–331.

[6] R. de la Bretèche, Nombre de points de hauteur bornée sur les surfaces de del Pezzo de degré 5. Duke Math. J. 113 (2002), 421–464.

[7] R. de la Bretèche and T.D. Browning, On Manin’s conjecture for singular del Pezzo surfaces of degree four, I. Submitted, 2005.

[8] R. de la Bretèche and T.D. Browning, On Manin’s conjecture for singular del Pezzo surfaces of degree four, II. Submitted, 2005.

[9] R. de la Bretèche and É. Fouvry, L’éclaté du plan projectif en quatre points dont deux conjugués. J. Reine Angew. Math. 576 (2004), 63–122.

[10] R. de la Bretèche, T.D. Browning and U. Derenthal, On Manin’s conjecture for a certain singular cubic surface. Submitted, 2005.

[11] N. Broberg, Rational points on finite covers of \( \mathbb{P}^1 \) and \( \mathbb{P}^2 \). J. Number Theory 101 (2003), 195–207.
[12] T.D. Browning, Counting rational points on del Pezzo surfaces of degree five. Proceedings of the Bonn session in analytic number theory and diophantine equations, Bonner Math. Schriften 360, 2003.

[13] T.D. Browning, The density of rational points on a certain singular cubic surface. Submitted, 2004.

[14] J.W. Bruce and C.T.C. Wall, On the classification of cubic surfaces. J. London Math. Soc. 19 (1979), 245–256.

[15] A. Cayley, A memoir on cubic surfaces. Phil. Trans. Roy. Soc. 159 (1869), 231–326.

[16] A. Chambert-Loir and Y. Tschinkel, On the distribution of points of bounded height on equivariant compactifications of vector groups. Invent. Math. 148 (2002), 421–452.

[17] J.-L. Colliot-Thélène and J.-J. Sansuc, Torseurs sous des groupes de type multiplicatif; applications à l’étude des points rationnels de certaines variétés algébriques. C. R. Acad. Sci. Paris Sér. A-B 282 (1976), 1113–1116.

[18] J.-L. Colliot-Thélène and J.-J. Sansuc, La descente sur les variétés rationnelles. II. Duke Math. J. 54 (1987), 375–492.

[19] D.F. Coray and M.A. Tsfasman, Arithmetic on singular Del Pezzo surfaces. Proc. London Math. Soc. 57 (1988), 25–87.

[20] H. Davenport, Analytic Methods in Diophantine Equations and Diophantine Inequalities. 2nd ed., edited by T.D. Browning, Cambridge University Press, 2005.

[21] É. Fouvry, Sur la hauteur des points d’une certaine surface cubique singulière. Astérisque 251 (1998), 31–49.

[22] J. Franke, Y.I. Manin and Y. Tschinkel, Rational points of bounded height on Fano varieties. Invent. Math. 95 (1989), 421–435.

[23] B. Hassett and Y. Tschinkel, Universal torsors and Cox rings. Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 149–173, Progr. Math. 226, Birkhäuser, 2004.

[24] D.R. Heath-Brown, The density of rational points on cubic surfaces. Acta Arith. 79 (1997), 17–30.

[25] D.R. Heath-Brown, The density of rational points on curves and surfaces. Annals of Math. 155 (2002), 553–595.

[26] D.R. Heath-Brown, The density of rational points on Cayley’s cubic surface. Proceedings of the session in analytic number theory and diophantine equations, Bonner Math. Schriften 360, 2003.

[27] D.R. Heath-Brown and B.Z. Moroz, The density of rational points on the cubic surface $X_0^3 = X_1X_2X_3$. Math. Proc. Camb. Soc. 125 (1999), 385–395.
[28] W.V.D. Hodge and D. Pedoe, *Methods of algebraic geometry*. Vol. 2, Cambridge University Press, 1952.

[29] J. Lipman, Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 195–279.

[30] Y.I. Manin, *Cubic forms*. 2nd ed., North-Holland Mathematical Library **4**, North-Holland Publishing Co., 1986.

[31] E. Peyre, Hauteurs et nombres de Tamagawa sur les variétés de Fano. *Duke Math. J.* **79** (1995), 101–218.

[32] E. Peyre, Counting points on varieties using universal torsors. *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, 61–81, Progr. Math. **226**, Birkhäuser, 2004.

[33] E. Peyre and Y. Tschinkel, Tamagawa numbers of diagonal cubic surfaces, numerical evidence. *Math. Comp.* **70** (2001), 367–387.

[34] E. Peyre and Y. Tschinkel, Tamagawa numbers of diagonal cubic surfaces of higher rank. *Rational points on algebraic varieties*, 275–305, Progr. Math., **199**, Birkhäuser, 2001.

[35] P. Salberger, Tamagawa measures on universal torsors and points of bounded height on Fano varieties. *Astérisque* **251** (1998), 91–258.

[36] P. Salberger, Rational points of bounded height on projective surfaces. *Submitted*, 2005.

[37] L. Schléfli, On the distribution of surfaces of the third order into species. *Phil. Trans. Roy. Soc.* **153** (1864), 193–247.

[38] A. Skorobogatov, On a theorem of Enriques-Swinnerton-Dyer. *Ann. Fac. Sci. Toulouse Math.* **2** (1993), 429–440.

[39] J.B. Slater and P. Swinnerton-Dyer, Counting points on cubic surfaces. I. *Astérisque* **251** (1998), 1–12.

[40] P. Swinnerton-Dyer, Counting points on cubic surfaces, II. *Geometric methods in algebra and number theory*, 303–310, Progr. Math. **235**, Birkhäuser, 2005.

[41] Y. Tschinkel, Fujita’s program and rational points. *Higher dimensional varieties and rational points (Budapest, 2001)*, 283–310, Bolyai Soc. Math. Stud. **12**, Springer, 2003.

[42] Y. Tschinkel, Lectures on height zeta functions of toric varieties. *Séminaires et Congrès* **6** (2002), 227–247.

[43] C.T.C. Wall, The first canonical stratum. *J. London Math. Soc.* **21** (1980), 419–433.