ITERATED RANDOM FUNCTIONS AND SLOWLY VARYING TAILS

PIOTR DYSZEWSKI

Abstract. Consider a sequence of i.i.d. random Lipschitz maps \( \{\Psi_n\}_{n \geq 0} \). Using this sequence we can define a Markov chain via the recursive formula \( R_{n+1} = \Psi_{n+1}(R_n) \). It is a well known fact that under some mild moment assumptions this Markov chain has a unique stationary distribution. We will study the tail behaviour of this distribution in the case when \( \Psi_0(t) \approx A_0 t + B_0 \). We will show that under subexponential assumptions on the random variable \( \log^+(A_0 \lor B_0) \) the tail asymptotic in question can be described using the integrated tail function of \( \log^+(A_0 \lor B_0) \). In particular we will obtain new results for the random difference equation \( R_{n+1} = A_{n+1} R_n + B_{n+1} \).

1. Introduction

Consider a sequence of independent identically distributed (i.i.d.) random Lipschitz maps \( \{\Psi_n\}_{n \geq 0} \), where \( \Psi_n : \mathbb{R} \to \mathbb{R} \) for \( n \in \mathbb{N} \). Using this sequence we can define a Markov chain via the recursive formula
\[
R_{n+1} = \Psi_{n+1}(R_n) \quad \text{for } n \geq 0,
\]
where \( R_0 \in \mathbb{R} \) is arbitrary but independent of the sequence \( \{\Psi_n\}_{n \geq 0} \). Put \( \Psi = \Psi_0 \). We are interested in the existence and properties of the stationary distribution of the Markov chain \( \{R_n\}_{n \geq 0} \), that is the solution of the stochastic fixed point equation
\[
R \overset{d}{=} \Psi(R) \quad R \text{ independent of } \Psi,
\]
where the distribution of random variable \( R \) is the stationary distribution of the Markov chain \( \{R_n\}_{n \geq 0} \).

The main example, we have in mind, is the random difference equation, where \( \Psi \) is an affine transformation, that is \( \Psi_n(t) = A_n t + B_n \) with \( \{(A_n, B_n)\}_{n \geq 0} \) being an i.i.d. sequence of two-dimensional random vectors. Then the formula (1.1) can be written as
\[
R_{n+1} = A_{n+1} R_n + B_{n+1} \quad \text{for } n \geq 0.
\]
Put \( (A, B) = (A_0, B_0) \). It is a well known fact that if
\[
\mathbb{E}[\log |A|] < 0 \quad \text{and} \quad \mathbb{E}[\log^+ |B|] < \infty,
\]
then the Markov chain \( \{R_n\}_{n \geq 0} \) given by (1.3) has a unique stationary distribution which can be represented as the distribution of the random variable
\[
R = \sum_{n \geq 0} B_{n+1} \prod_{k=1}^{n} A_k,
\]
for details see [28]. Random variables of this form can be found in analysis of probabilistic algorithms or financial mathematics, where \( R \) would be called a perpetuity. Such random variables occur also in number theory, combinatorics, as a solution to stochastic fixed point equation
\[
R \overset{d}{=} AR + B \quad R \text{ independent of } (A, B),
\]
atomic cascades, random environment branching processes, exponential functionals of Lévy processes, Additive Increase Multiplicative Decrease algorithms [17], COGARCH processes [22], and more. A variety of examples for possible applications of $R$ can be found in [14, 15, 11].

From the application point of view, the key information is the behaviour of the tail of $R$, that is

$$
P[R > x] \quad \text{as } x \to \infty.
$$

This problem was investigated by various authors, for example by C. M. Goldie and R. Grübel [14] and in a similar setting by P. Hitczenko and J. Wesolowski [18]. The first result says that if $B$ is bounded, $P[A \in [0,1]] = 1$ and the distribution of $A$ behaves like the uniform distribution in the neighborhood of 1, then $R$ has thin tail, more precisely $\log P[R \geq x] \sim -cx \log(x)$. Recall that for two positive functions $f(\cdot)$ and $g(\cdot)$, by $f(x) \sim g(x)$ we mean that $\lim_{x \to \infty} f(x)/g(x) = 1$. In this paper we are only interested in limits as $x \to \infty$, so from now we omit the specification of the limit.

There is also the result of H. Kesten [20] and later on, in the same setting, of C. M. Goldie [13]. The essence of this result is that under Cramér’s condition, that is if $E[|A|^\alpha] = 1$ for some $\alpha > 0$ such that $E[|B|^\alpha] < \infty$, the tail of $R$ is regularly varying, i. e. $P[R > x] \sim cx^{-\alpha}$ for some positive and finite constant $c$.

Finally, the result of A. K. Grincevičius [16], which was later generalised by D. R. Grey [15], states that in the case of positive $A$ if for some $\alpha > 0$ we have $E[|A|^\alpha] < 1$ and $P[B > x] \sim x^{-\alpha} L(x)$, where $L$ is slowly varying (that is $L(cx) \sim L(x)$ for any positive $c$), then the tail of $R$ is again regularly varying, in fact $P[R > x] \sim cx^{-\alpha} L(x)$. Note that in this case the tail of perpetuity $R$ exhibits the same rate of decay as the tail of the input, that is $P[R > x] \sim cP[B > x]$.

However, in the case when $P[A > x]$ or $P[B > x]$ is a slowly varying function of $x$, up to our knowledge, little is known about the behaviour of $P[R > x]$ as $x \to \infty$. This is the problem we consider in the present paper.

The case of general fixed point equation (1.2) was studied by C. M. Goldie [13], where several particular forms of the transformation $\Psi$ were treated. Later M. Mirek [24] found the tail asymptotic of the solution of (1.2) with $\Psi$ being Lipschitz such that $\Psi(t) \approx \text{Lip}(\Psi)t$, where Lip($\Psi$) is the Lipschitz constant. The result says that if $E[\log(\text{Lip}(\Psi))] < 0$ and $E[|B|^\alpha] = 1$ for some $\alpha > 0$, then $R$ solving (1.2) exhibits regularly varying tail $P[|R| > x] \sim cx^{-\alpha}$. D. Grey [15] also treated generalized fixed point equations (1.2) in the setting introduced by A. K. Grincevičius [16].

It turns out that the assumption $E[\log(\text{Lip}(\Psi))] < 0$ is necessary for the existence of the probabilistic solutions of (1.2). For the existence and asymptotic behaviour of the invariant measure of the Markov chain (1.1) in the critical case, that is $E[\log(\text{Lip}(\Psi))] = 0$, see [1, 6, 5, 4].

This paper gives an answer to the question about asymptotic of $P[R > x]$, where $R$ solves (1.2), in the case of slowly varying input. Assuming that the Lipschitz map $\Psi$ satisfies

$$
At + B - D \leq \Psi(t) \leq At^+ + B^+ + D^+ \quad \text{for } t \in \mathbb{R},
$$

with $D$ being relatively small and $A > 0$, we will show that under subexponential assumptions on the random variable $\log(A \lor B)$ one has

$$
P[R > x] \asymp \int_{\log(x)}^\infty P[\log(A \lor B) > y] \, dy.
$$

Recall that for two positive functions $f(\cdot)$, $g(\cdot)$ by $f(x) \asymp g(x)$ we mean that $g(x) = O(f(x))$ and $f(x) = O(g(x))$. Furthermore, in our setting, the integral expression on the right hand side will be a slowly varying function of $x$. Moreover in several cases we will establish a precise tail asymptotic of $R$. In order to obtain full description of tail behaviour for the sequence $\{R_n\}_{n \geq 0}$ we will study finite time horizon. We will show that if distribution of $\log(A \lor B)$ is subexponential, then it holds true that

$$
P[R_n > x] \asymp nP[A \lor B > x],
$$

where $\{R_n\}_{n \geq 0}$ in given by (1.1).
The main result gives description of tail asymptotic of the solutions to the following stochastic fixed point equations:

\[ R \overset{d}{=} A R + B \quad R \text{ independent of } (A, B), \]

which is the random difference equation mentioned earlier. If we take \( \Psi(t) = A t^+ + B \), we obtain

\[ R \overset{d}{=} A R^+ + B \quad R \text{ independent of } (A, B). \]

This equation is closely related to the ruin probability, for details see [9]. We can also obtain a description of the solutions to

\[ R \overset{d}{=} A_1 |R| + \sqrt{D + A_2 R^2} \quad R \text{ independent of } (A_1, A_2, D) \]

where \( \mathbb{P}[D > x] = o \left( \mathbb{P}[A_1 + \sqrt{A_2} > x] \right) \). This corresponds to an autoregressive process with ARCH(1) errors, which was described by M. Borkovec and C. Klüppelberg [3]. To find the behaviour of \( \mathbb{P}[|R| > x] \) just take \( \Psi(t) = |A_1 t + \sqrt{D + A_2 (t^+)}| \).

The paper is organised as follows: In the second section we will briefly recall basic definitions and properties of subexponential distributions, after that in the third section we will present a precise statement of the result followed by some remarks and sketch of the proof. Finally, in the last fourth section, we will give the full proof of the results.

2. Subexponential Distributions

In this section we will briefly recall well known notions from the theory of heavy-tailed distributions. Next we will quote a theorem about tail behaviour of a maxima of perturbed random walk, which will be particularly useful in the proof of the main result. Firstly, for a distribution \( F \) on \( \mathbb{R} \) we define tail function \( \overline{F} \) by the formula \( \overline{F}(x) = 1 - F(x) \) for \( x \in \mathbb{R} \).

**Definition 2.1.** A distribution \( F \) on \( \mathbb{R} \) is called long-tailed if \( \overline{F}(x) > 0 \) for all \( x \in \mathbb{R} \) and for any fixed \( y \in \mathbb{R} \)

\[ \overline{F}(x + y) \sim \overline{F}(x). \]

We denote the class of long-tailed distributions by \( \mathcal{L} \).

Notice that if \( F \in \mathcal{L} \) then the function \( x \mapsto \overline{F}(\log(x)) \) is slowly varying as \( x \to \infty \). Therefore one can use Potter’s Theorem (see [2]: Theorem 1.5.6) to obtain the following corollary.

**Corollary 2.2.** If \( F \in \mathcal{L} \), then for any chosen \( \Delta > 1 \) and \( \delta > 0 \) there exists \( X = X(\Delta, \delta) \) such that

\[ \frac{\overline{F}(x)}{\overline{F}(y)} \leq \Delta e^{\delta |x-y|} \quad \text{for} \quad x, y \geq X. \]

It turns out that class \( \mathcal{L} \) is too big for our purposes. More precisely, we will need distributions satisfying some convolution properties. Recall that \( F^{*2} \) stands for the twofold convolution of the distribution \( F \).

**Definition 2.3.** A distribution \( F \) on \( \mathbb{R} \) is called subexponential if \( F \in \mathcal{L} \) and

\[ F^{*2}(x) \sim 2 F(x). \]

The class of subexponential distributions will be denoted by \( \mathcal{S} \).

Note that if \( X_1 \) and \( X_2 \) are i.i.d. with distribution \( F \in \mathcal{S} \), then by the definition above

\[ \mathbb{P}[X_1 + X_2 > x] \sim 2 \mathbb{P}[X_1 > x] \sim \mathbb{P}[X_1 \lor X_2 > x]. \]

This is a type of phenomena that we want to use in the near future. It is a well known fact that \( \mathcal{S} \subset \mathcal{L} \) and that this inclusion is proper. For examples of distributions in \( \mathcal{L} \setminus \mathcal{S} \) see [10] or [26]. The following proposition is a well known fact which will be useful thought the proofs of the results. We follow the statement presented in [12].
Proposition 2.4. Suppose that $F \in \mathcal{S}$. Let $G_1, \ldots, G_n$ be distributions such that $G_i(x) \sim c_i F(x)$ for some constants $c_i \geq 0$, $i = 1, \ldots, n$. Then
\[ G_1 \ast \ldots \ast G_n(x) \sim (c_1 + \ldots + c_n) F(x). \]
If $c_1 + \ldots + c_n > 0$, then $G_1 \ast \ldots \ast G_n \in \mathcal{S}$.

The following theorem by Z. Palmowski and B. Zwart [25] is crucial for our future purposes.

Theorem 2.5. Let $\{(X_n, Y_n)\}_{n \geq 0}$ be a sequence of i.i.d. two-dimensional random vectors such that $\mathbb{E}[X_1] < 0$ and $\mathbb{E}[X_1 \vee Y_1] < \infty$. Assume that distribution on $\mathbb{R}_+$ given by the tail function
\[ x \mapsto 1 \wedge \int_x^\infty \mathbb{P}[X_1 \vee Y_1 > y] \, dy \]
is subexponential. Then
\[ \mathbb{P} \left[ \sup_{n \geq 0} \left\{ Y_{n+1} + \sum_{j=1}^n X_j \right\} > x \right] \sim -\frac{1}{\mathbb{E}[X_1]} \int_x^\infty \mathbb{P}[X_1 \vee Y_1 > y] \, dy. \]

The $\mathbb{R}_+$ in the above theorem and for the rest of the paper stands for $[0, +\infty)$. For conditions on $F$ guaranteeing subexponentiality of distribution given by the tail function $x \mapsto 1 \wedge \int_x^\infty \mathbb{P}(y) \, dy$ see [21].

3. Main Result

In this section we will give a precise statement of the main result of the paper followed by some remarks and idea behind the proof.

3.1. Statement. Recall that we consider a Markov chain $\{R_n\}_{n \geq 0}$ given by (1.1), where for each $n \in \mathbb{N}$ the map $\Psi_n : \mathbb{R} \to \mathbb{R}$ satisfies
\[ A_n t + B_n - D_n \leq \Psi_n(t) \leq A_n t^+ + B_n^+ + D_n^+ \quad \text{for } t \in \mathbb{R} \]
for some random variables $A_n$, $B_n$ and $D_n$. We are assuming that $\{(\Psi_n, A_n, B_n, D_n)\}_{n \geq 0}$ are i.i.d., where $\Psi_n$ are Lipschitz maps with
\[ \text{Lip}(\Psi_n) = \sup_{t_1 \neq t_2} \left| \frac{\Psi_n(t_1) - \Psi_n(t_2)}{t_1 - t_2} \right|. \]
Note that (3.1) implies
\[ A_n \leq \text{Lip}(\Psi_n). \]
Put $(\Psi, A, B, D) = (\Psi_0, A_0, B_0, D_0)$. From now our standing assumptions will be
\[ A > 0 \text{ a.s., } \mathbb{E}[\log(A)] > -\infty, \quad \mathbb{E}[\log(\text{Lip}(\Psi))] < 0, \quad \mathbb{E}[\log^+ |B|] < \infty. \]
Recall that $\log^+(x) = \log(x \vee 1)$. For infinite time horizon, that is the case of the stationary distribution, we will also need to assume
\[ \mathbb{E}[\log^4(A)], \quad \mathbb{E}[\log^4(B)^4] < \infty. \]
Assume also the following tail behaviour
\[ \mathbb{P}[A \vee (B \pm D) > x] \sim \mathbb{P}[A \vee B > x], \quad \mathbb{P}[A > x, B \leq -x] = o(\mathbb{P}[A \vee B > x]). \]
Define a probability distribution $F_I$ on $\mathbb{R}_+$ via its tail function $F_I$ which is given by
\[ F_I(x) = 1 \wedge \int_x^\infty \mathbb{P}[\log(A \vee B) > y] \, dy. \]
Having that said, we are able to give a precise statement of the main result.
Theorem 3.1. Assume that conditions (3.1), (3.3), (3.4) and (3.5) are satisfied and that $F_I$ defined by (3.6) is subexponential. Then the Markov chain $\{R_n\}_{n \geq 0}$ given by (1.1) converges in distribution to a unique stationary distribution which is a unique solution of (1.2). Furthermore

$$-rac{\mathbb{P}[R > 0]}{\mathbb{E}[\log(A)]]} \leq \liminf_{x \to \infty} \frac{\mathbb{P}[R > x]}{F_I(\log(x))} \leq \limsup_{x \to \infty} \frac{\mathbb{P}[R > x]}{F_I(\log(x))} \leq -\frac{1}{\mathbb{E}[\log(A)]]}.$$ 

In particular, if $B - D > 0$ a.s., then

$$\mathbb{P}[R > x] \sim -\frac{1}{\mathbb{E}[\log(A)]]} \int_{\log(x)}^{\infty} \mathbb{P}[\log(A \vee B) > y] \, dy.$$ 

Moreover, if

- $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$ then
  $$\mathbb{P}[R > x] \sim -\frac{1}{\mathbb{E}[\log(A)]]} \int_{\log(x)}^{\infty} \mathbb{P}[\log^+(B) > y] \, dy,$$

- $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$ then
  $$\mathbb{P}[R > x] \sim \frac{\mathbb{P}[R > 0]}{\mathbb{E}[\log(A)]]} \int_{\log(x)}^{\infty} \mathbb{P}[\log(A) > y] \, dy.$$ 

Since in last two cases of the above theorem we obtain $\mathbb{P}[R > x] \sim c F_I(\log(x))$ with $F_I \in \mathcal{S} \subseteq \mathcal{L}$ and some constant $c$ we see that in each case the distribution of $R$ exhibits slowly varying tail. From the proof of the Theorem 3.1 one can see that in order to establish the lower bound in (3.7) one only uses the fact that the distribution of the random variable $A \vee B$ has a slowly varying tail. We note that by the following remark.

Remark 3.2. Assume (3.1), (3.3), (3.5) and that the function $x \mapsto \mathbb{P}[A \vee B > x]$ is slowly varying. Then

$$-rac{\mathbb{P}[R > 0]}{\mathbb{E}[\log(A)]]} \leq \liminf_{x \to \infty} \frac{\mathbb{P}[R > x]}{F_I(\log(x))}$$

where the function $F_I$ is given by (3.6). Since when $F_I \in \mathcal{L}$ it is true that $\mathbb{P}[A \vee B > x] = \mathbb{P}[\log(A \vee B) > \log(x)] = o(F_I(\log(x)))$ we can also conclude that $\mathbb{P}[A \vee B > x] = o(\mathbb{P}[R > x]).$

In order to obtain an extensive description of the Markov chain $\{R_n\}_{n \geq 0}$ given by (1.1) we will also investigate the tail behaviour of random variables $R_n$ for finite $n$. Put

$$F_I(x) = \mathbb{P}[\log(A \vee B) > x].$$

It turns out that in case of finite time horizon one can obtain result analogous to Theorem 3.1.

Theorem 3.3. Assume (3.1), (3.3), (3.5), $0 \leq n < \infty$ and that $F$ defined by (3.9) is subexponential. Assume additionally that

$$\mathbb{P}[R_0 > x] \sim w \mathbb{P}[A \vee B > x]$$

for some constant $w \geq 0$. Then

$$w + \sum_{k=0}^{n-1} \mathbb{P}[R_k > 0] \leq \liminf_{x \to \infty} \frac{\mathbb{P}[R_n > x]}{\mathbb{P}[A \vee B > x]} \leq \limsup_{x \to \infty} \frac{\mathbb{P}[R_n > x]}{\mathbb{P}[A \vee B > x]} \leq n + w.$$ 

In particular if $R_0 > 0$ a.s. and $B - D > 0$ a.s. then

$$\mathbb{P}[R_n > x] \sim (n + w) \mathbb{P}[A \vee B > x].$$

Furthermore if

- $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$ then
  $$\mathbb{P}[R_n > x] \sim (n + w) \mathbb{P}[B > x],$$

- $\mathbb{P}[A > x] = \mathbb{P}[B > x]$ then
  $$\mathbb{P}[R_n > x] \sim (n + w) \mathbb{P}[A \vee B > x].$$
• $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$ then

$$\mathbb{P}[R_n > x] \sim \mathbb{P}[A > x] \left( w + \sum_{k=0}^{n-1} \mathbb{P}[R_k > 0] \right).$$

(3.12)

Since the proof of Theorem 3.3 follows the same ideas as the proof of Theorem 3.1 analogous remark is in order.

Remark 3.4. Assume (3.1), (3.5), (3.3), $0 \leq n < \infty$, and that the function $x \mapsto \mathbb{P}[A \lor B > x]$ is slowly varying. Then

$$w + \sum_{k=0}^{n-1} \mathbb{P}[R_k > 0] \leq \liminf_{x \to \infty} \frac{\mathbb{P}[R_n > x]}{\mathbb{P}[A \lor B > x]}.$$  

3.2. Random Difference Equation. Suppose, for the rest of this section, that $\Psi(t) = At + B$ and $D = 0$. In this case the main result of this paper is closely related to Theorem 4.1 by K. Maulik and B. Zwart [23] where perpetuity $R$ is replaced by a random variable of the form $\int_0^\infty e^{\xi_s} \, ds$ where $\{\xi_s \mid s \geq 0\}$ is a Lévy process with negative drift. Note that by the strong Markov property this is a perpetuity corresponding to $A = e^{\xi_1}$ and $B = \int_0^1 e^{\xi_s} \, ds$. The theorem in question states that

$$\mathbb{P}\left[ \int_0^\infty e^{\xi_s} \, ds > x \right] \sim \frac{1}{\mathbb{E}[\xi_1]} \int_{\log(x)}^\infty \mathbb{P}[\xi_1 > y] \, dy$$

if $x \mapsto \int_x^\infty \mathbb{P}[\xi_1 > y] \, dy$ is subexponential. We see that Theorem 4.1 by K. Maulik and B. Zwart [23] is a particular case of the main result of this paper.

Theorem 3.1 is also related to results from [16, 15, 27, 19] where arising perpetuities exhibit the tail behaviour similar to the tail behaviour of the input. The first one, for example, says that

$$\frac{\mathbb{P}[R > x]}{\mathbb{P}[B > x]} \sim \frac{1}{1 - \mathbb{E}[A^\alpha]}$$

if $\mathbb{E}[A^\alpha] < 1$ and $\mathbb{P}[B > x] \sim x^{-\alpha}L(x)$ for some slowly varying function $L$ and $\alpha > 0$. We see that when $\alpha \to 0$ the constant $(1 - \mathbb{E}[A^\alpha])^{-1}$ tends to infinity. Theorem 3.1 corresponds to the case with $\alpha = 0$ and tells us what is the proper asymptotic. This also gives the reason for the blowup of the constant. By Remark 3.2 $\mathbb{P}[A \lor B > x] = o(\mathbb{P}[R > x])$ and we can write

$$\frac{\mathbb{P}[R > x]}{\mathbb{P}[B > x]} \sim \frac{\mathbb{P}[R > x]}{\mathbb{P}[A \lor B > x]} \to \infty$$

as $x \to \infty$ and the observation follows.

In the case when $B > 0$ a.s., Theorem 3.1 gives a description of the tail of $R$ in terms of the distribution of $A \lor B$, which allows us to present an example showing that in the case when $\mathbb{P}[A > x] \sim \mathbb{P}[B > x]$, the information about marginal distributions of $A$ and $B$ is not enough to determine the tail asymptotic of $R$.

Example 3.5. Fix a distribution $F$ on $\mathbb{R}_+$ and consider two types of input: First one $(A^{(1)}, B^{(1)})$: with $A^{(1)} = B^{(1)}$ with distribution $F$. Then, assuming that the assumptions are satisfied, Theorem 3.1 states in (3.8) that the corresponding perpetuity $R^{(1)}$ satisfies

$$\mathbb{P}\left[ R^{(1)} > x \right] \sim -\frac{1}{\mathbb{E}[\log(A^{(1)})]} \int_{\log(x)}^\infty \mathbb{P}\left[ \log(A^{(1)}) > y \right] \, dy.$$ 

If now we consider the second type of input, namely $(A^{(2)}, B^{(2)})$ where $A^{(2)}$, $B^{(2)}$ are independent with the same distribution $F$, Theorem 3.1 states that the corresponding perpetuity $R^{(2)}$ satisfies

$$\mathbb{P}\left[ R^{(2)} > x \right] \sim -\frac{1}{\mathbb{E}[\log(A^{(2)})]} \int_{\log(x)}^\infty \mathbb{P}\left[ \log(A^{(2)} \lor B^{(2)}) > y \right] \, dy.$$
and since $A^{(1)} \overset{d}{=} A^{(2)}$ we can write
\[ \mathbb{P} \left[ A^{(2)} \lor B^{(2)} > x \right] \sim 2 \mathbb{P} \left[ A^{(2)} > x \right] = 2 \mathbb{P} \left[ A^{(1)} > x \right] \]
and we see that
\[ \mathbb{P} \left[ R^{(2)} > x \right] \sim 2 \mathbb{P} \left[ R^{(1)} > x \right]. \]

Even though the marginal distributions of the two types of input are exactly the same, the corresponding perpetuities have different tail asymptotic.

Next example shows the importance of second condition in (3.5). This assumption is needed to ensure that the stationary distribution has right unbounded support.

**Example 3.6.** Consider the input $(A, B)$ where $B = \mathbb{1}_{[0,1]}(A) - A$. Assume that $A > 0$ and $\mathbb{E}[\log(A)] < 0$. This ensures the existence of the solution $R$ to
\[ R \overset{d}{=} AR + \mathbb{1}_{[0,1]}(A) - A \quad R \text{ independent of } A. \]
We see that $\mathbb{P}[B > 0] = \mathbb{P}[A \in [0,1]] > 0$, but the solution is bounded. Indeed, notice that $R$ also satisfies
\[ R - 1 \overset{d}{=} AR + 1 - 1 \quad R \text{ independent of } A \]
and so $R - 1$ is a perpetuity obtained from the input $(A, \mathbb{1}_{[0,1]}(A) - 1)$. Since $\mathbb{1}_{[0,1]}(A) - 1 \leq 0$ a.s., we know that $R - 1 \leq 0$ a.s. Whence we can conclude that the perpetuity $R$ obtained from the input $(A, B)$ is bounded above by 1 a.s. This is due to the fact that in this case
\[ \mathbb{P}[A > x, B \leq -x] = \mathbb{P}[A > x] = \mathbb{P}[A \lor B > x] \]
for $x > 1$.

3.3. **Idea of the proof.** Before we dive into the proof of the main result, let’s make a brief overview of the main idea. The key problem is to understand the random difference equation, i.e. the case $\Psi(t) = At + B$. For simplicity, we will focus on that case in the following discussion. The convolution property in Definition 2.3 of the subexponential distributions says that for $X_1$ and $X_2$ independent with the same distribution $F \in \mathcal{S}$ it is true that $\mathbb{P}[X_1 + X_2 > x] \sim \mathbb{P}[X_1 \lor X_2 > x]$. It turns out that the series (1.4) exhibits a similar phenomena, more precisely we are able to approximate
\[ \mathbb{P} \left[ \sum_{n \geq 0} B_{n+1} \prod_{j=1}^{n} A_j > x \right] \text{ by using } \mathbb{P} \left[ \sup_{n \geq 0} \left\{ B_{n+1} \prod_{j=1}^{n} A_j \right\} > x \right]. \]
In order to achieve that we apply technique used in [8, 7]. This technique revolves around the idea of grouping the terms of the same order and investigating the sizes of the groups. Then, after obtaining the above relation, we can interpret random variable $\sup_{n \geq 0} B_{n+1} \prod_{j=1}^{n} A_j$ as a supremum of a perturbed random walk and use the known theory, namely Theorem 2.5, to derive upper bound for the desired tail asymptotic. Next, adapting some classical techniques, used for example in [25], we get lower bound for tail asymptotic. Roughly speaking, we find relatively big subsets of $\{R > x\}$ on which we have control over the whole sequence $\left\{ B_{n+1} \prod_{j=1}^{n} A_j \right\}_{n \geq 0}$.

4. **Proof**

In this section we will prove the main result of the paper. Recall that we consider an i.i.d. sequence $\{(\Psi_n, A_n, B_n, D_n)\}_{n \geq 0}$ such that
\[ A_n t + B_n - D_n \leq \Psi_n(t) \leq A_n t^+ + B_n^+ + D_n^+ \quad \text{for } n \geq 0 \text{ and } t \in \mathbb{R}. \]
Put $(\Psi, A, B, D) = (\Psi_0, A_0, B_0, D_0)$ and let
\[ \mu = -\mathbb{E}[\log(A)]. \]
Also define
\begin{equation}
S_n = \sum_{j=1}^{n} \log(A_j) \quad \text{for } n \geq 0
\end{equation}
and
\begin{equation}
B_n = (B_n^+ + D_n^+) \lor 1, \quad B_n = B_n - D_n \quad \text{for } n \geq 0
\end{equation}
finally let \( \overline{B} = \overline{B}_0, \underline{B} = \underline{B}_0 \). Notice that (3.5) implies
\( \mathbb{P}[A \lor \underline{B} > x] \sim \mathbb{P}[A \lor B > x] \sim \mathbb{P}[A \lor \overline{B} > x] \).

For \( k < n \) put
\begin{equation}
(4.3) \quad \Psi_{k:n}(t) = \Psi_k \circ \Psi_{k+1} \circ \ldots \circ \Psi_n(t).
\end{equation}
We will use the convention that for \( k > n \) \( \Psi_k(t) = t \). For \( n \in \mathbb{N} \) we can put
\( \Psi_n(t) = A_n t + B_n \) and \( \Psi_n(t) = A_n t^+ + B_n \) and define \( \Psi_{k:n} \) and \( \Psi_{k:n} \) in the same manner as \( \Psi_{k:n} \). Notice that using this notation and the bounds on \( \Psi_n(t) \), we get
\( \Psi_n(t) \leq \Psi_{k:n}(t) \leq \overline{\Psi}_{k:n}(t) \).

In particular
\begin{equation}
(4.4) \quad \Psi_{1:n}(t) = \sum_{k=0}^{n-1} B_{k+1} \prod_{j=1}^{k} A_j + t \prod_{j=1}^{n} A_j \leq \Psi_{1:n}(t)
\end{equation}
and
\begin{equation}
(4.5) \quad \Psi_{1:n}(t) \leq \sum_{k=0}^{n-1} B_{k+1} \prod_{j=1}^{k} A_j + t^+ \prod_{j=1}^{n} A_j = \overline{\Psi}_{1:n}(t).
\end{equation}
We will use the following lemma quite often. The proof follows the idea presented in [25].

**Lemma 4.1.** Assume (3.3) and for \( \delta, K > 0 \) consider the sets
\begin{equation}
(4.6) \quad E_n = E_n(K, \delta) = \{ S_j \in (-j(\mu + \delta) - K, -j(\mu - \delta) + K), j \leq n \}
\end{equation}
and
\begin{equation}
(4.7) \quad F_n = F_n(K, \delta) = \{|B_j| \leq e^{\delta j + K}, j \leq n \}.
\end{equation}
Then the following claim holds
\begin{equation}
(4.8) \quad \forall \delta, \varepsilon > 0 \quad \exists K > 0 \quad \mathbb{P} \left[ \bigcap_{j \geq 1} (E_j \cap F_j) \right] \geq 1 - \varepsilon.
\end{equation}

**Proof.** For \( K \) large enough it is true that \( \mathbb{P}[\log |B| > K] < 1/2 \) and since for \( y \in (0, 1/2) \) it holds that \( \log(1 - y) \geq -2y \), we can write
\[
\log(\mathbb{P}[F_n]) = \sum_{j=1}^{n} \log(1 - \mathbb{P}[\log |B_j| > \delta j + K]) \geq -2 \sum_{j=1}^{n} \mathbb{P}[\log |B| > \delta j + K]
\geq -2 \sum_{j=1}^{\infty} \mathbb{P}[\delta^{-1}(\log |B| - K) > j] \geq -2\delta^{-1} \mathbb{E}[\log |B| - K]^+]
\]
and so $\mathbb{P}[F_n] \to 1$ as $K \to \infty$ uniformly with respect to $n$ since $\mathbb{E}[\log^+(B)] < \infty$. Combining this fact with the weak law of large numbers for the sequence $\{S_n\}_{n \geq 0}$ we observe that we have shown that for any $\varepsilon, \delta > 0$ we can always take $K > 0$ large enough such that

$$\forall n \quad \mathbb{P}[E_n \cap F_n] \geq 1 - \varepsilon$$

and since the sequence of sets $\{E_n \cap F_n\}_{n \geq 0}$ is decreasing in the sense of inclusion, we can conclude that

$$\mathbb{P} \left[ \bigcap_{j \geq 1} (E_j \cap F_j) \right] \geq 1 - \varepsilon$$

and hence the proof is complete. \qed

Note that the statement of the Lemma 4.1 remains true if we replace $B_j$ by $\overline{B_j}$ in the definition of the set $F_n$. The bounds on $\Psi$ imply that we can bound the solution of (1.2) by two perpetuities, namely

$$(4.9) \quad \overline{R} = \sum_{n \geq 0} B_{n+1} \prod_{k=1}^{n} A_j$$

and

$$(4.10) \quad \underline{R} = \sum_{n \geq 0} B_{n+1} \prod_{k=1}^{n} A_j.$$ 

The main idea of the proof is to approximate $\mathbb{P}[\overline{R} > e^x]$ by using $\mathbb{P}[M > x]$ where

$$M = \sup_{n \geq 0} \left\{ \log(B_{n+1}) + \sum_{j=1}^{n} \log(A_j) \right\}.$$ 

Since $\overline{B}_1 \geq 1$ we know that $M > 0$ a.s. Furthermore, we have $e^M \leq \overline{R}$ and the last series is convergent a.s by (3.3). Having introduced this notation, we are ready to prove the main theorem.

**Proof of the Theorem 3.1.** Fix large $x \in \mathbb{R}$. The proof consists of five steps.

**Step 1: Existence, representation and uniqueness of the stationary distribution.** Note that

$$R_n \overset{d}{=} \Psi_{1,n}(R_0)$$

so in order to prove that $\{R_n\}_{n \geq 0}$ converges in distribution, it is sufficient to show that $\{\Psi_{1,n}(R_0)\}_{n \geq 0}$ converges a.s. Recall that (3.1) implies

$$A_n \leq \text{Lip}(\Psi_n)$$

also, by the definition (3.2)

$$\text{Lip}(\Psi_{1,m}) \leq \prod_{j=1}^{m} \text{Lip}(\Psi_j) \quad \text{for} \ m \in \mathbb{N}.$$
For \( n \geq m \) and \( t_1, t_2 \in \mathbb{R} \) we have
\[
|\Psi_{1:n}(t_1) - \Psi_{1:m}(t_2)| \leq \text{Lip}(\Psi_{1:m})|\Psi_{m+1:n}(t_1) - t_2| \leq \text{Lip}(\Psi_{1:m})(|\Psi_{m+1:n}(t_1)| + |t_2|)
\]
\[
\leq \text{Lip}(\Psi_{1:m}) \left( \sum_{k=m}^{n-1} \left( \bar{B}_{k+1} \lor \left| \bar{B}_{k+1} \right| \right) \prod_{j=m+1}^{k} A_j + |t_1| \prod_{j=m+1}^{n} A_j + |t_2| \right)
\]
\[
\leq \text{Lip}(\Psi_{1:m}) \left( \sum_{k=m}^{n-1} \left( \bar{B}_{k+1} \lor \left| \bar{B}_{k+1} \right| \right) \prod_{j=m+1}^{k} \text{Lip}(\Psi_j) + |t_1| \prod_{j=m+1}^{n} \text{Lip}(\Psi_j) + |t_2| \prod_{j=m+1}^{m} \text{Lip}(\Psi_j) \right) \to 0.
\]

The first term tends to 0 since the series \( \sum_{k \geq 0} \left( \bar{B}_{k+1} \lor \left| \bar{B}_{k+1} \right| \right) \prod_{j=1}^{k} \text{Lip}(\Psi_j) \) is convergent, for details see [28], and the last two terms tend to 0 by the strong law of large numbers for the sequence \( \{ \log(\text{Lip}(\Psi_n)) \} \). If we take \( t_1 = t_2 = R_0 \) we see that \( \{ \Psi_{1:n}(R_0) \} \) is convergent and if we take \( t_1 = 0, t_2 = R_0 \) and \( n = m \) we see that the limit does not depend on \( R_0 \), hence the stationary distribution is unique and it is the distribution of random variable
\[
R = \lim_{n \to \infty} \Psi_{1:n}(R_0).
\]

For the rest of the proof we will assume that \( R \) is given by the limit above.

**Step 2: Upper bound in (3.7).** We claim that
\[
P[\bar{R} > e^x, M \leq \log(\varepsilon) + x] = \sqrt{\varepsilon}O(P[M > x])
\]
for \( \varepsilon \in (0, 1) \) sufficiently small. To prove (4.12), we will apply the technique from [8, 7]. For \( k \in \mathbb{Z} \) define random set of integers by
\[
\mathcal{Q}(k) := \left\{ s \in \mathbb{N} \left| \bar{B}_{s+1} \prod_{j=1}^{s} A_j \in (e^{-k}e^x, e^{-k+1}e^x) \right. \right\}.
\]

Notice that if \( M \leq \log(\varepsilon) + x \) then \( \mathcal{Q}(k) = \emptyset \) for \( k \) satisfying \( e^{-k} > \varepsilon \). The following inclusion holds
\[
\{ \bar{R} > e^x, M \leq \log(\varepsilon) + x \} \subseteq \left\{ \exists k : e^{-k} \leq \varepsilon, \# \mathcal{Q}(k) > \frac{e^k}{5k^2} \right\}.
\]

Indeed, assume that \( \bar{R} > e^x, M \leq \log(\varepsilon) + x \) and that for any \( k \) such that \( e^{-k} \leq \varepsilon \) we have \( \# \mathcal{Q}(k) \leq \frac{e^k}{5k^2} \). Since \( \mathcal{Q}(k) = \emptyset \) for \( k \) satisfying \( e^{-k} > \varepsilon \), we can write
\[
\bar{R} = \sum_{n \geq 0} \bar{B}_{n+1} \prod_{j=1}^{n} A_j = \sum_{k \in \mathbb{Z}} \sum_{s \in \mathcal{Q}(k)} \bar{B}_{s+1} \prod_{j=1}^{s} A_j
\]
\[
= \sum_{k \geq -\log(\varepsilon)} \sum_{s \in \mathcal{Q}(k)} \bar{B}_{s+1} \prod_{j=1}^{s} A_j \leq \sum_{k \geq 0} \# \mathcal{Q}(k) e^{-k+1}e^x \leq \sum_{k \geq 0} e^x \frac{e^k}{5k^2} = \frac{\pi^2e}{30}e^x < e^x.
\]

This is a contradiction. Using the inclusion (4.13) one gets instantly that
\[
\{ \bar{R} > e^x, M \leq \log(\varepsilon) + x \} \subseteq \left\{ M \leq \log(\varepsilon) + x, \exists k \geq -\log(\varepsilon), \# \mathcal{Q}(k) > \frac{e^k}{5k^2} \right\}.
\]

Let’s focus our interest on the set \( \text{RHS}(4.14) \). Define the sequence \( \tau(k) = \inf \mathcal{Q}(k) \) (we use the convention that \( \inf \emptyset = +\infty \)). On the set \( \text{RHS}(4.14) \) there exists \( k \geq -\log(\varepsilon) \) such that \( \tau(k) < \infty \)
and from the fact that \( \tau(k) \in Q(k) \) and \( \# Q(k) > \frac{4}{e^{\sigma^2}} \), we conclude that

\[
\mathcal{B}_{\tau(k)+1} \prod_{j=1}^{\tau(k)} A_j, \quad \mathcal{B}_{\tau(k)+p+1} \prod_{j=1}^{\tau(k)+p} A_j \in (e^{-k}e^x, e^{-k+1}e^x] \text{ for some } p > \frac{e^k}{10k^2} - 1.
\]

By taking \( \varepsilon > 0 \) sufficiently small we can ensure that \( \frac{e^k}{10k^2} - 1 > \frac{e^k}{10k^2} \) for \( k \geq -\log(\varepsilon) \). By dividing the two quantities above we obtain that

\[
(4.15) \quad \frac{A_{\tau(k)+1} B_{\tau(k)+1}}{A_{\tau(k)+p+1} B_{\tau(k)+p+1}} \prod_{j=1}^{\tau(k)+p} A_j \in (e^{-1}, e^1) \text{ for some } p > \frac{e^k}{10k^2}.
\]

The quotient \( \frac{A_{\tau(k)+1} B_{\tau(k)+1}}{A_{\tau(k)+p+1} B_{\tau(k)+p+1}} \) is bounded on the set \( \text{RHS}(4.14) \), because

\[
(4.16) \quad \frac{A_{\tau(k)+1} B_{\tau(k)+1}}{A_{\tau(k)+p+1} B_{\tau(k)+p+1}} \prod_{j=1}^{\tau(k)+p} A_j \leq \frac{B_{\tau(k)+2} \prod_{j=1}^{\tau(k)+1} A_j}{e^{-k}e^x} \leq \frac{e^M x}{e^{-k}e^x} \leq e^M x.
\]

Combining bounds in (4.15) and (4.16) we can conclude: on the set \( \text{RHS}(4.14) \) there exists an integer \( k \geq -\log(\varepsilon) \) for which \( \tau(k) < \infty \) and

\[
\mathcal{B}_{\tau(k)+p+1} \prod_{j=1}^{\tau(k)+p} A_j > e^{-k-1} \text{ for some } p > \frac{e^k}{10k^2}.
\]

So the following inclusion is also correct

\[
\{ \mathcal{B} > e^x, M \leq \log(\varepsilon) + x \} \subseteq \bigcup_{k \geq -\log(\varepsilon)} \bigg\{ \tau(k) < \infty, \mathcal{B}_{\tau(k)+p+1} \prod_{j=1}^{\tau(k)+p} A_j > e^{-k-1} \bigg\}.
\]

In terms of probability it yields

\[
P \left[ \mathcal{B} > e^x, M \leq \log(\varepsilon) + x \right] \leq \sum_{k \geq -\log(\varepsilon)} \sum_{p > e^k / (10k^2)} P \left[ \tau(k) < \infty, \mathcal{B}_{\tau(k)+p+1} \prod_{j=1}^{\tau(k)+p} A_j > e^{-k-1} \right]
\]

\[
= \sum_{k \geq -\log(\varepsilon)} \sum_{p > e^k / (10k^2)} P \left[ \mathcal{B}_{\tau(k)+p+1} \prod_{j=1}^{\tau(k)+p} A_j > e^{-k-1} \mid \tau(k) < \infty \right] P \left[ \tau(k) < \infty \right]
\]

by the strong Markov property of the sequence \( \{(A_n, \mathcal{B}_n)\}_{n \geq 0} \). In order to proceed any further, we need to know that the remaining terms decay sufficiently fast. First note that

\[
E \left[ \left( \log(\mathcal{B}_{p+1}) + \sum_{j=2}^{p} \log(A_j) + \mu \right)^4 \right] = \left( \frac{p-1}{2} \right)^2 \left( E [\log(A_1) + \mu] \right)^2
\]

\[
+ (p-1)E \left[ (\log(A_1) + \mu)^4 \right] + (p-1)E[\log(\mathcal{B}_1)]E[(\log(A_1) + \mu)^2]
\]

\[
+ (p-1)E \left[ (\log(\mathcal{B}_1))^2 \right] E \left[ (\log(A_1) + \mu)^2 \right] + E \left[ (\log(\mathcal{B}_1))^4 \right] \leq c_1 (p-1)^2,
\]

since \( \log(A_j) + \mu \) are independent with mean zero and they are independent of \( \log(\mathcal{B}_{p+1}) \). The constant \( c_1 \) depends on the logarithmic moments of \( A \) and \( \mathcal{B} \) and it can be computed explicitly.
from the expression above. Of course constant $c_1$ is finite since we assume (3.4) and (3.5). By the Chebyshev’s inequality we can now write

$$
P \left[ B_{p+1} \prod_{j=2}^{p} A_j > e^{-k-1} \right] = \mathbb{P} \left[ \log(B_{p+1}) + \sum_{j=2}^{p} \log(A_j + \mu) > \mu(p-1) - k - 1 \right]$$

$$\leq \mathbb{E} \left[ \left( \log(B_{p+1}) + \sum_{j=2}^{p} \log(A_j + \mu) \right)^4 \right] \leq \frac{c_1(p-1)^2}{(\mu(p-1) - k - 1)^4}.$$

Hence

$$\sum_{p=e^{k/(10k^2)}} \mathbb{P} \left[ B_{p+1} \prod_{j=2}^{p} A_j > e^{-k-1} \right] \leq \sum_{p=e^{k/(10k^2)}} \frac{c_1(p-1)^2}{(\mu(p-1) - k - 1)^4}$$

$$\leq \int_{e^{k/(10k^2)}}^{\infty} \frac{y^2}{(\mu(y - 2) - k - 1)^4} dy \leq c_2 e^{-k/2},$$

for some constant $c_2$ dependent on $\mu$ and $c_1$. If we combine everything together, we obtain for $\eta > 0$

$$\mathbb{P} \left[ R > e^x, M \leq \log(\varepsilon) + x \right] \leq c_2 \sum_{k \geq \log(\varepsilon)} \mathbb{P}[r(k) < \infty]e^{-k/2} \leq c_2 \sum_{k \geq \log(\varepsilon)} \mathbb{P}[M > x - k]e^{-k/2}$$

$$\leq c_2 \sum_{x-\eta \geq k \geq \log(\varepsilon)} \mathbb{P}[M > x - k]e^{-k/2} + c_2 \sum_{k > x-\eta} \mathbb{P}[M > x - k]e^{-k/2}$$

$$=: c_2 I_1(x) + c_2 I_2(x).$$

Now we will investigate $I_1$ and $I_2$ separately. From the Theorem 2.5 we can conclude that the distribution of the random variable $M$ belongs to the class $\mathcal{S} \subseteq \mathcal{L}$ and so we can use Potter bounds (Corollary 2.2) for $\mathbb{P}[M > t]$ to find $\eta > 0$, such that for $t, s > \eta$ we have

$$\frac{\mathbb{P}[M > t]}{\mathbb{P}[M > s]} \leq 2 \exp \left\{ \frac{|t-s|}{4} \right\}.$$

Then for $x > \eta - \log(\varepsilon)$ we have

$$\frac{I_1(x)}{\mathbb{P}[M > x]} = \sum_{x-\eta \geq k \geq \log(\varepsilon)} \frac{\mathbb{P}[M > x - k]}{\mathbb{P}[M > x]} e^{-k/2} \leq 2 \sum_{x-\eta \geq k \geq \log(\varepsilon)} e^{-k/4} \leq C \sqrt{\varepsilon}$$

and for the second term

$$I_2(x) \leq \sum_{k > x-\eta} e^{-k/2} \leq c_3 e^{-x/2} = o(\mathbb{P}[M > x])$$

for some $c_3 > 0$, since the distribution of $M$ is long-tailed. Thus claim (4.12) follows. Now we need notice that since $R \leq \frac{1}{\mu}$ we have

$$\{R > e^x\} \subseteq \{R > e^x, M \leq \log(\varepsilon) + x\} \cup \{M > \log(\varepsilon) + x\}$$

and thus using (4.12), we get

$$\frac{\mathbb{P}[R > e^x]}{\mathbb{P}[M > x]} \leq \sqrt{\varepsilon} \frac{O(\mathbb{P}[M > x])}{\mathbb{P}[M > x]} + \frac{\mathbb{P}[M > x + \log(\varepsilon)]}{\mathbb{P}[M > x]}.$$

First let $x \to \infty$ and notice that from Theorem 2.5 it follows that

$$\mathbb{P}[M > x] \sim -\frac{1}{\mathbb{E}[\log(A)\mathbb{E}[\log(B) > y]]} \int_{x}^{\infty} \mathbb{P}[\log(A \lor B) > y] dy \sim \frac{1}{\mu} F_1(x).$$

From this we can conclude that

$$\limsup_{x \to \infty} \frac{\mathbb{P}[R > e^x]}{F_1(x)} \leq C \sqrt{\varepsilon} + \frac{1}{\mu}.$$
for some finite constant $C > 0$ independent of $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary small we get the upper bound.

**Step 3: Lower bound in (3.7).** Fix $0 < \varepsilon$ and $0 < \delta < \frac{1}{2} \land 1$. For $K > 0$ consider the sets $E_n$ and $F_n$ given by (4.6) and (4.7) respectively. Choose $K > 0$ large enough for (4.8) to be satisfied. Consider also the random variables

$$ R_n^* = \lim_{N \to \infty} \Psi_n \circ \ldots \circ \Psi_N(R_0). $$

Note that $R_n^* \overset{d}{=} R$ and

$$ R = \Psi_{1:n+1}(R_{n+2}^*). $$

Finally put

$$ G_n = E_n \cap F_n \cap \left\{ A_{n+1} \lor B_{n+1} > e^{-n(\mu-\delta)+L+K+x}, B_{n+1} \geq -e^{-n(\mu-\delta)-K+x} \right\} \cap \{ R_{n+2}^* > \delta \} $$

where $L > 0$ is a constant independent of $x$ and $n$. We see that the sets $\{G_n\}_{n \geq 0}$ are disjoint if we take $L = L(K, \delta, \mu)$ sufficiently large. Moreover on the set $G_n$ we have

$$ R = \Psi_{1:n+1}(R_{n+2}^* \geq \Psi_{1:n+1}(R_{n+2}^*) = \sum_{k=0}^{n-1} B_{k+1} \prod_{j=1}^{k+1} A_j + (B_{n+1} + R_{n+2}^* A_{n+1}) \prod_{j=1}^{n} A_j $$

$$ \geq -\sum_{k=0}^{n-1} e^{2K} \prod_{j=1}^{k+1} A_j + (B_{n+1} + e^{-n(\mu-\delta)-K+x} + A_{n+1} R_{n+2}^*) \prod_{j=1}^{n} A_j $$

$$ \geq -\frac{e^{2K}}{1-e^{-\mu+2\delta}} + \delta \left( A_{n+1} \lor (B_{n+1} + e^{-n(\mu-\delta)-K+x}) \right) e^{-n(\mu-\delta)-K-x} $$

and the last inequality is valid for all $x > 0$ and all $n \in \mathbb{N}$ if $L = L(K, \delta, \mu)$ is sufficiently large. We see that $G_n \leq \{ R > \varepsilon^x \}$ and this allows us to write

$$ \mathbb{P}[R > \varepsilon^x] \geq \sum_{n \geq 0} \mathbb{P}[G_n] $$

$$ \geq (1-\varepsilon) \sum_{n \geq 0} \mathbb{P} \left[ A_{n+1} \lor B_{n+1} > e^{n(\mu+\delta)+L+x+K}, B_{n+1} \geq -e^{x+n(\mu-\delta)-K} \right] \mathbb{P} \left[ R_{n+2}^* > \delta \right] $$

$$ \geq (1-\varepsilon) \mathbb{P}[R > \delta] \sum_{n \geq 0} \left\{ \mathbb{P} \left[ A \lor B > e^{n(\mu+\delta)+L+K+x} \right] - \mathbb{P} \left[ A > e^{n(\mu+\delta)+L+K+x}, B < -e^{x+n(\mu-\delta)+K} \right] \right\} $$

$$ \sim (1-\varepsilon) \mathbb{P}[R > \delta] \int_{x}^{\infty} \frac{\mathbb{P}[\log(A \lor B) > y]}{\mu + \delta} dy. $$

This yields

$$ \lim_{x \to \infty} \frac{\mathbb{P}[R > \varepsilon^x]}{\int_{x}^{\infty} \mathbb{P}[\log(A \lor B) > y] dy} \geq \frac{(1-\varepsilon)\mathbb{P}[R > \delta]}{\mu + \delta}. $$

If we allow $\varepsilon, \delta \to 0$ we see that we have proven the lower estimate for the desired limit.

**Step 4: The case $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$.** Firstly, notice that we need only to prove the lower bound and that in this case Theorem 2.5 yields

$$ \mathbb{P}[M > x] \sim \frac{1}{\mathbb{E}[\log(A)]} \int_{x}^{\infty} \mathbb{P}^{+}[B > y] dy. $$

(4.19)
For $0 < \varepsilon$, $0 < \delta < \mu/2$ and $K > 0$ consider the sets $E_n$ and $F_n$ given by (4.6) and (4.7) respectively with $K > 0$ large enough for (4.8) to be satisfied. Finally, put

$$J_n = E_n \cap F_n \cap \left\{ B_{n+1} > e^{x^n + n(\mu + \delta) + K + L}, A_{n+1} \leq e^{n(\mu - \delta) - K + x} \right\} \cap \left\{ |R_{n+2}^*| \leq \delta^{-1} \right\}.$$ 

For some large $L > 0$ independent of $x$. We see that the sets $\{J_n\}_{n \geq 0}$ are disjoint. Moreover on the set $J_n$ we have

$$R = \Psi_{1:\varepsilon}(R_{n+2}^*) \geq \Psi_{1:\varepsilon}(R_{n+2}^*) = \sum_{k=0}^{n-1} B_{k+1} A_j + B_{n+1} n A_j + R_{n+2}^* n A_j + R_{n+2}^* n A_j \geq \frac{e^{2K}}{1 - e^{-\mu + 2\delta}} + e^{x + L - \delta^{-1} e^x} > e^x$$

and the last inequality is valid for all $x > 0$ if $L = L(K, \delta, \mu)$ is sufficiently large. Therefore $J_n \subseteq \{ R > e^x \}$ and this allows us to write

$$\mathbb{P}[R > e^x] \geq \sum_{n \geq 0} \mathbb{P}[J_n] \geq (1 - \varepsilon) \sum_{n \geq 0} \mathbb{P} \left[ B_{n+1} > e^{x^n + n(\mu + \delta) + K + L}, A_{n+1} \leq e^{n(\mu - \delta) - K + x} \right] \mathbb{P} \left[ |R_{n+2}^*| \leq \delta^{-1} \right]$$

$$\geq (1 - \varepsilon) \mathbb{P} \left[ |R| \leq \delta^{-1} \right] \sum_{n \geq 0} \left\{ \mathbb{P} \left[ B > e^{x^n + n(\mu + \delta) + K + L} \right] - \mathbb{P} \left[ A > e^{n(\mu - \delta) - K + x} \right] \right\}$$

$$\sim \frac{(1 - \varepsilon) \mathbb{P} \left[ |R| \leq \delta^{-1} \right]}{\mu + \delta} \int_x^\infty \mathbb{P} \left[ \log^+(B) > y \right] dy.$$ 

This yields

$$\liminf_{x \to \infty} \frac{\mathbb{P}[R > e^x]}{\mathbb{P} \left[ \log^+(B) > y \right] dy} \geq \frac{(1 - \varepsilon) \mathbb{P} \left[ |R| \leq \delta^{-1} \right]}{\mu + \delta}.$$ 

If we allow $\varepsilon, \delta \to 0$ we get the lower estimate.

**Step 5:** The case $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$. Notice that we only need to prove the upper estimate and that in this case

(4.20) 

$$\mathbb{P} \left[ M > x \right] \sim -\frac{1}{\mathbb{E}[\log(A)]} \int_x^\infty \mathbb{P}[\log(A) > y] dy.$$ 

Fix $\varepsilon \in (0, 1)$ and notice that since (4.12) holds we only need to focus on the set $\text{LHS}(4.21)$:

(4.21) 

$$\{ R > e^x, M > \log(\varepsilon) + x \} \subseteq \{ M \in (\log(\varepsilon) + x, x] \} \cup \{ R > e^x, M > x \}$$

and since the distribution of $M$ is long-tailed and (4.20) is valid we have

(4.22) 

$$\mathbb{P} \left[ M \in (\log(\varepsilon) + x, x] \right] = o \left( \int_x^\infty \mathbb{P}[\log(A) > y] dy \right).$$

For the other set we have

$$\{ R > e^x, M > x \} = \{ M > x \} \setminus \{ R \leq e^x, M > x \}$$

so by (4.20) now we only need to prove that

$$\liminf_{x \to \infty} \frac{\mathbb{P}[R \leq e^x, M > x]}{\mathbb{P}[\log(A) > y] dy} \geq -\frac{\mathbb{P}[R \leq 0]}{\mathbb{E}[\log(A)]}.$$
We achieve that using the same technique, but this time we consider the sets

\[ H_n = E_n \cap F'_n \cap \left\{ B_{n+1} \leq \frac{1}{2} e^{x+n(\mu-\delta)-K}, A_{n+1} > e^{n(\mu+\delta)+K+x} \right\} \cap \{ R_{n+2}^* \leq 0 \}. \]

Where

\[ F'_n = F'_n(\delta, K) = \left\{ |B_j| < e^{\delta_j+K}, j \leq n \right\}. \]

Note that (4.8) also holds true if we replace \( F_n \) by \( F'_n \). We see that the sets \( \{ H_n \}_{n \geq 0} \) are disjoint if \( x \) is sufficiently large. Moreover on the set \( H_n \) we have

\[ R = \Psi_{1:n+1}(R_{n+2}^*) \leq \Psi_{1:n+1}(R_{n+2}^*) = \sum_{j=0}^{n-1} B_{j+1} \prod_{k=1}^{j} A_k + B_{n+1} \prod_{k=1}^{n} A_k + (R_{n+2})^{+} \prod_{k=1}^{n+1} A_k \]

and the last inequality is valid for all \( x > x_0 = x_0(K, \delta, \mu) \). Therefore \( H_n \subseteq \{ R \leq e^x \}. \) Moreover on the set \( H_n \)

\[ M \geq \sum_{k=1}^{n+1} \log(A_k) > -n(\mu+\delta) - K + n(\mu+\delta) + K + x = x \]

and this proves that \( H_n \subseteq \{ R \leq e^x, M > x \} \), which allows us to write

\[ P[R \leq e^x, M > x] \geq \sum_{n \geq 0} P[H_n] \geq (1-\varepsilon) \sum_{n \geq 0} \left[ P[B_{n+1} \leq \frac{1}{2} e^{x+n(\mu-\delta)-K}, A_{n+1} > e^{n(\mu+\delta)+K+x}] \right] P[R_{n+2}^* \leq 0] \]

\[ \geq (1-\varepsilon)P[R \leq 0] \sum_{n \geq 0} \left\{ P[A > e^{x+n(\mu+\delta)+K}] - P[B > \frac{1}{2} e^{x+n(\mu-\delta)+K}] \right\} \]

\[ \sim \frac{(1-\varepsilon)P[R \leq 0]}{\mu + \delta} \int_{x}^{\infty} P[\log(A) > y] dy. \]

This yields

\[ \lim_{x \to \infty} \inf_{x} \frac{P[R \leq e^x, M > x]}{P[\log(A) > y] dy} \geq \frac{(1-\varepsilon)P[R \leq 0]}{\mu + \delta}. \]

So if we put everything together, we notice that since \( R \leq \tilde{R} \) we have

\[ \{ R > e^x \} \subseteq \{ \tilde{R} > e^x, M \leq \log(\varepsilon) + x \} \cup \{ M \in (\log(\varepsilon) + x, x) \} \cup \{ M > x \} \backslash \{ R \leq e^x, M > x \} \]

and thus

\[ P[R > e^x] \leq P[\tilde{R} > e^x, M \leq \log(\varepsilon) + x] + P[M \in (\log(\varepsilon) + x, x)] + P[M > x] - P[R \leq e^x, M > x] \]

and so using (4.12), (4.22), (4.20) and (4.23) we get

\[ \lim_{x \to \infty} \sup \frac{P[R > e^x]}{P[\log(A) > y] dy} \leq C \sqrt{\varepsilon} + 0 + \frac{1}{\mu} - \frac{(1-\varepsilon)P[R \leq 0]}{\mu + \delta}. \]

If we allow \( \varepsilon, \delta \to 0 \) we see that we achieved the desired upper bound and hence the proof is complete. \( \square \)

Now we can turn our attention to the finite time horizon.
Proof of the Theorem 3.3. The proof mimics the one of the main result. Fix \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \).

**Step 1:** Upper bound in (3.10). Notice that

\[
R_n \overset{d}{=} \Psi_{1,n}(R_0) \leq \Psi_{1,n}(R_0) = \sum_{k=0}^{n-1} B_{k+1} \prod_{j=1}^{k} A_j + R_0 \prod_{j=1}^{n} A_j \\
\leq \sum_{k=0}^{n-1} (A_{k+1} \lor \overline{B}_{k+1}) \prod_{j=1}^{k} (A_j \lor \overline{B}_j) + R_0 \prod_{j=1}^{n} (A_j \lor \overline{B}_j) \leq (n + R_0) \prod_{j=1}^{n} (A_j \lor \overline{B}_j)
\]

and so

\[
P[R_n > x] \leq \mathbb{P} \left( (n + R_0) \prod_{j=1}^{n} (A_j \lor \overline{B}_j) > x \right) \sim (n + w)P[A \lor B > x]
\]

since \( F \) is subexponential and Proposition 2.4 holds.

**Step 2:** Lower bound in (3.10). Fix \( 0 < \varepsilon \) and \( 0 < \delta < \frac{\varepsilon}{K} \land 1 \). For \( K > 0 \) consider the sets \( E_n \) and \( F_n \) given by (4.6) and (4.7) respectively. Choose \( K > 0 \) large enough for (4.8) to be satisfied and put

\[
G_k = E_k \cap F_k \cap \left\{ A_{k+1} \lor \overline{B}_{k+1} > e^{k(\mu+\delta)+L+K+x}, \overline{B}_{k+1} \geq -e^{k(\mu-\delta)-K+x} \right\} \cap \{ \Psi_{k+2,n}(R_0) > \delta \}
\]

for \( 0 \leq k \leq n - 1 \) and

\[
G_n = E_n \cap F_n \cap \left\{ R_0 > e^{n(\mu+\delta)+L+K+x} \right\}.
\]

Recall that \( \Psi_{n+1,n}(t) = t \). We see that the sets \( \{ G_n \}_{0 \leq k \leq n} \) are disjoint if we take \( L = L(K, \delta, \mu) \) large enough. Moreover on the set \( G_k \) for \( k \leq n - 1 \) we have

\[
R_n \overset{d}{=} \Psi_{1,n}(R_0) = \Psi_{1,k+1}(\Psi_{k+2,n}(R_0)) \geq \Psi_{1,k+1}(\Psi_{k+2,n}(R_0)) \\
= \sum_{l=0}^{k-1} B_{l+1} \prod_{j=1}^{l} A_j + \left( B_{k+1} + A_{k+1} \Psi_{k+2,n}(R_0) \right) \prod_{j=1}^{k} A_j \\
\geq - \sum_{l=0}^{k-1} |B_{l+1}| \prod_{j=1}^{l} A_j + \left( B_{k+1} + A_{k+1} \Psi_{k+2,n}(R_0) \right) \prod_{j=1}^{k} A_j \\
\geq - \sum_{l=0}^{k-1} |B_{l+1}| \prod_{j=1}^{l} A_j + \left( B_{k+1} + e^{k(\mu-\delta)-K+x} + A_{k+1} \Psi_{k+2,n}(R_0) \right) \prod_{j=1}^{k} A_j - e^{k(\mu-\delta)-K+x} \prod_{j=1}^{k} A_j \\
\geq - \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + \delta(A_{n+1} \lor \overline{B}_{n+1}) e^{-n(\mu+\delta)-K} - e^{\varepsilon} \geq - \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + \delta e^{x+L} - e^{\varepsilon} > e^{\varepsilon}
\]

and the last inequality is valid for all \( x > 0 \) and all \( n \in \mathbb{N} \) if \( L = L(K, \delta, \mu) \) is sufficiently large. On the set \( G_n \) we have

\[
R_n \overset{d}{=} \Psi_{1,n}(R_0) \geq \Psi_{1,n}(R_0) = \sum_{l=0}^{n} B_{l+1} \prod_{j=1}^{l} A_j + R_0 \prod_{j=1}^{n} A_j \geq - \sum_{l=0}^{k-1} |B_{l+1}| \prod_{j=1}^{l} A_j + R_0 \prod_{j=1}^{n} A_j \geq - \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + R_0 e^{-n(\mu+\delta)-K} \geq - \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + e^{x+L} > e^{\varepsilon}
\]
and again, the last inequality holds if we take $L = L(K, \delta, \mu)$ sufficiently large. Therefore $\cup_{k=0}^{n} G_k \subseteq \{ R_n > e^x \}$ and since for $0 \leq k \leq n - 1$ we have

$$\mathbb{P}[G_k] \geq (1 - \epsilon) \mathbb{P} \left[ A_{k+1} \cup B_{k+1} > e^{k(\mu+\delta)+L+x+K}, B_{k+1} \geq -e^{k(\mu-\delta)-K+x} \right] \mathbb{P}[\Psi_{k+2:n}(R_0) > \delta]$$

$$\geq (1 - \epsilon) \mathbb{P}[\Psi_{1:n-k+1}(R_0) > \delta] \left( \mathbb{P} \left[ A_{k+1} \cup B_{k+1} > e^{k(\mu+\delta)+L+x+K} \right] - \mathbb{P} \left[ A_{k+1} > e^{k(\mu+\delta)+L+x+K}, B_{k+1} \geq -e^{k(\mu-\delta)-K+x} \right] \right)$$

$$\sim (1 - \epsilon) (\mathbb{P}[\Psi_{1:n-k+1}(R_0) > \delta]) \mathbb{P}[A \lor B > e^x]$$

and

$$\mathbb{P}[G_n] \geq (1 - \epsilon) \mathbb{P} \left[ R_0 > e^{n(\mu+\delta)+L+x+K} \right] \sim w(1 - \epsilon) \mathbb{P}[A \lor B > e^x].$$

We can write

$$\mathbb{P}[R_n > e^x] \geq \sum_{k=0}^{n} \mathbb{P}[G_k] \geq (1 - \epsilon) \left( w + \sum_{k=0}^{n-1} \mathbb{P}[\Psi_{1:n-k+1}(R_0) > \delta] + o(1) \right) \mathbb{P}[A \lor B > e^x].$$

This yields

$$\liminf_{x \to \infty} \frac{\mathbb{P}[R_n > e^x]}{\mathbb{P}[A \lor B > e^x]} \geq (1 - \epsilon) \left( w + \sum_{k=0}^{n-1} \mathbb{P}[R_k > \delta] \right).$$

If we allow $\varepsilon, \delta \to 0$ we see that we have proven the lower estimate for the desired limit.

**Step 3:** The case $\mathbb{P}[A > x] = o(\mathbb{P}[B > x])$. For $0 < \varepsilon$, $0 < \delta < \mu/2$ and $K > 0$ consider the sets $E_n$ and $F_n$ given by (4.6) and (4.7) respectively with $K$ large enough for (4.8) to be satisfied. Finally put

$$J_k = E_k \cap F_k \cap \left\{ B_{k+1} > e^{x+k(\mu+\delta)+L}, A_{k+1} \leq e^{n(\mu-\delta)-K+x} \right\} \cap \left\{ |\Psi_{k+2:n}(R_0)| \leq \delta^{-1} \right\}.$$ 

for $0 \leq k \leq n - 1$ and

$$J_n = E_n \cap F_n \cap \left\{ R_0 > e^{x+n(\mu+\delta)+L} \right\}.$$ 

For some large $L > 0$ independent of $x$. We see that the sets $\{J_k\}_{0 \leq k \leq n}$ are disjoint. Moreover on the set $J_k$ for $0 \leq k \leq n - 1$ we have

$$R_n \overset{d}{=} \Psi_{1:n}(R_0) = \Psi_{1:k+1}(\Psi_{k+2:n}(R_0)) \geq \Psi_{1:k+1}(\Psi_{k+2:n}(R_0))$$

$$= \sum_{l=0}^{k-1} B_{l+1} \prod_{j=1}^{l} A_j + B_{k+1} \prod_{j=1}^{k} A_j + \Psi_{k+2:n}(R_0) A_{k+1} \prod_{j=1}^{k} A_j$$

$$\geq - \sum_{l=0}^{k-1} |B_{l+1}| \prod_{j=1}^{l} A_j + B_{k+1} \prod_{j=1}^{k} A_j - |\Psi_{k+2:n}(R_0)| A_{k+1} \prod_{j=1}^{k} A_j$$

$$\geq - \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + e^{x+L} - \frac{1}{\delta} e^x > e^x$$

and on the set $J_n$

$$R_n \overset{d}{=} \Psi_{1:n}(R_0) \geq \Psi_{1:n}(R_0) = \sum_{l=0}^{n} B_{l+1} \prod_{j=1}^{l} A_j + R_0 \prod_{j=1}^{n} A_j \geq - \sum_{l=0}^{n} |B_{l+1}| \prod_{j=1}^{l} A_j + R_0 \prod_{j=1}^{n} A_j$$

$$\geq - \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + e^{x+L} > e^x.$$
and the last inequality is valid for all $x > 0$ if $L = L(K, \delta, \mu)$ is sufficiently large. Therefore $J_n \subseteq \{ R > e^x \}$ and this allows us to write for $0 \leq k \leq n - 1$

$$\mathbb{P}[J_k] \geq (1 - \varepsilon) \mathbb{P} \left[ B_{k+1} > e^{x+k(\mu+\delta)+K+L}, \ A_{k+1} < e^{k(\mu-\delta)-3K+x} \right] \mathbb{P} \left[ \left| \Psi_{k+2; n}(R_0) \right| \leq \delta^{-1} \right]$$

$$= (1 - \varepsilon) \mathbb{P} \left[ \left| \Psi_{k+2; n}(R_0) \right| \leq \delta^{-1} \right] \left( \mathbb{P} \left[ B > e^{x+n(\mu+\delta)+K+L} \right] - \mathbb{P} \left[ A > e^{n(\mu-\delta)-K+x} \right] \right)$$

$$\sim (1 - \varepsilon) \mathbb{P} \left[ \left| \Psi_{k+2; n}(R_0) \right| \leq \delta^{-1} \right] \mathbb{P}[B > e^x]$$

and

$$\mathbb{P}[J_n] \geq (1 - \varepsilon) \mathbb{P}[R_0 > e^{x+n(\mu+\delta)+K+L}] \sim w(1 - \varepsilon) \mathbb{P}[B > e^x].$$

This allows us to write

$$\mathbb{P}[R_n > e^x] \geq \sum_{k=0}^{n} \mathbb{P}[J_k] \geq (1 - \varepsilon) \left( w + \sum_{k=0}^{n-1} \mathbb{P} \left[ \left| \Psi_{k+2; n}(R_0) \right| \leq \delta^{-1} \right] + o(1) \right) \mathbb{P}[B > e^x]$$

This yields

$$\liminf_{x \to \infty} \frac{\mathbb{P}[R > e^x]}{\mathbb{P}[\log^+(B) > x]} \geq (1 - \varepsilon) \left( w + \sum_{k=0}^{n-1} \mathbb{P} \left[ |R_k| \leq \delta^{-1} \right] \right).$$

If we allow $\varepsilon, \delta \to 0$ we get the lower estimate.

**Step 4:** The case $\mathbb{P}[B > x] = o(\mathbb{P}[A > x])$. Notice that we only need to prove the upper estimate.

Let

$$M_n = \max_{0 \leq k \leq n-1} \left\{ \log\left( B_{k+1} \right) + \sum_{j=1}^{k} \log(A_j) \right\} \lor \left\{ \log^+(R_0) + \sum_{j=1}^{n} \log(A_j) \right\}.$$ 

Next, notice that

$$\{ \Psi_{1; n}(R_0) > e^x, M_n \leq x - \log(n+1) \} = \emptyset$$

and so

$$\begin{align*}
\{ \Psi_{1; n}(R_0) > e^x \} &= \{ \Psi_{1; n}(R_0) > e^x, M_n > x - \log(n+1) \} \\
&= \{ M_n > x - \log(n+1) \} \setminus \{ \Psi_{1; n}(R_0) \leq e^x, M_n > x - \log(n+1) \}.
\end{align*}$$

Since $M_n \leq \log^+(R_0) + \sum_{k=1}^{n} \log(A_k \lor B_k)$ we can write

$$\begin{align*}
\mathbb{P}[M_n > x - \log(n+1)] &\leq \mathbb{P}[\log^+(R_0) + \sum_{k=1}^{n} \log(A_k \lor B_k) > x - \log(n+1)] \\
&\sim (w + n) \mathbb{P}[\log(A \lor B) > x]
\end{align*}$$

so we only need to prove that

$$\liminf_{x \to \infty} \frac{\mathbb{P}[\Psi_{1; n}(R_0) \leq e^x, M_n > x - \log(n+1)]}{\mathbb{P}[\log(A) > x]} \geq \sum_{k=0}^{n-1} \mathbb{P}[R_k \leq 0].$$

We achieve that using the same technique, but this time we consider the sets

$$H_k = E_k \cap F_k' \cap \left( \bigcap_{l=0}^{k-1} B_{l+1} \leq \frac{1}{2} e^{x+k(\mu-\delta)-K}, \ A_{k+1} > e^{k(\mu+\delta)+K+x} \right) \cap \{ \Psi_{k+2; n}(R_0) \leq 0 \}.$$ 

for $0 \leq k \leq n-1$. We see that the sets $\{ H_k \}_{0 \leq k \leq n-1}$ are disjoint. Moreover on the set $H_k$ we have

$$\begin{align*}
\Psi_{1; n}(R_0) &\leq \Psi_{1; k+1}(\Psi_{k+2; n}(R_0)) = \sum_{l=0}^{k-1} B_{l+1} \prod_{j=1}^{l} A_j + B_{k+1} \prod_{j=1}^{k} A_j + \Psi_{k+2; n}(R_0) \prod_{j=1}^{k+1} A_j \\
&\leq \frac{e^{2K}}{1 - e^{-\mu+2\delta}} + \frac{1}{2} e^x + 0 \leq e^x
\end{align*}$$
and the last inequality is valid for all $x$ that are large enough. Therefore $H_k \subseteq \{\Psi_{1:n}(R_0) \leq e^x\}$. Moreover on the set $H_k$

$$M_n \geq \sum_{j=1}^{k+1} \log(A_j) > -k(\mu + \delta) - K + k(\mu + \delta) + K + x = x.$$  

We see that $\bigcup_{k=0}^{n-1} H_k \subseteq \{\Psi_{1:n}(R_0) \leq e^x, M > x - \log(n + 1)\}$ and this allows us to write

$$\mathbb{P}[\Psi_{1:n}(R_0) \leq e^x, M_n > x - \log(n + 1)] \geq \sum_{k=0}^{n-1} \mathbb{P}[H_k]$$

$$\geq (1 - \varepsilon) \sum_{k=0}^{n-1} \mathbb{P} \left[ B_{k+1} \leq \frac{1}{2} e^{x+k(\mu+\delta) - K}, A_{k+1} > e^{k(\mu+\delta) + K + x} \right] \mathbb{P}[\Psi_{k+2:n}(R_0) \leq 0]$$

$$\geq (1 - \varepsilon) \sum_{k=0}^{n-1} \left( \mathbb{P} \left[ A_{k+1} > e^{k(\mu+\delta) + K + x} \right] - \mathbb{P} \left[ B_{k+1} > \frac{1}{2} e^{x+k(\mu-\delta) - K} \right] \right) \mathbb{P}[\Psi_{k+2:n}(R_0) \leq 0]$$

$$\sim (1 - \varepsilon) \mathbb{P}[A > e^x] \left( \sum_{k=0}^{n-1} \mathbb{P}[\Psi_{1:k}(R_0) \leq 0] \right).$$

This yields

$$\liminf_{x \to \infty} \frac{\mathbb{P}[\Psi_{1:n}(R_0) \leq e^x, M_n > x - \log(n)]}{\mathbb{P}[\log(A) > x]} \geq (1 - \varepsilon) \sum_{k=0}^{n-1} \mathbb{P}[R_k \leq 0].$$

Putting everything together, that is (4.24), (4.25) and the last inequality we get

$$\limsup_{x \to \infty} \frac{\mathbb{P}[R_n \leq e^x]}{\mathbb{P}[\log(A) > x]} \leq n + w - (1 - \varepsilon) \sum_{k=0}^{n-1} \mathbb{P}[R_k \leq 0].$$

If we allow $\varepsilon \to 0$ we see that we achieved the desired upper bound and hence the proof is complete in this case. \qed

**Acknowledgments**

Part of the main result of this paper was included in author’s Master’s thesis, written under the supervision of Dariusz Buraczewski at the University of Wroclaw. The author would like to thank him for hours of stimulating conversations and several helpful suggestions during the preparation of this paper. The author would like to thank also Zbigniew Palmowski and Tomasz Rolski for pointing out references.

**References**

1. M. Babillot, P. Bougerol, and L. Elie, *The random difference equation $X_n = A_nX_{n-1} - B_n$ in the critical case*, The Annals of Probability 25 (1997), no. 1, 478–493.
2. N. Bingham, C. Goldie, and J. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1989.
3. M. Borkovec and C. Klüppelberg, *The tail of the stationary distribution of an autoregressive process with arch(1) errors*, The Annals of Applied Probability 11 (2001), no. 4, 1220–1241.
4. S. Brofferio and D. Buraczewski, *On unbounded invariant measures of stochastic dynamical systems*, to appear in Annals of Probability.
5. S. Brofferio, D. Buraczewski, and E. Damek, *On the invariant measure of the random difference equation $X_n = A_nX_{n-1} + B_n$ in the critical case*, Annales de l’Institut Henri Poincare (B) Probability and Statistics 48 (2012), no. 2, 377–395.
6. D. Buraczewski, *On invariant measures of stochastic recursions in a critical case*, Annals of Applied Probability 17 (2007), no. 4, 1245–1272.
7. D. Buraczewski, E. Damek, T. Mikosch, and J. Zienkiewicz, *Large deviations for solutions to stochastic recurrence equations under Kesten’s condition*, The Annals of Probability 41 (2013), no. 4, 2755–2790.
8. D. Buraczewski, E. Damek, and J. Zienkiewicz, *Precise tail asymptotics of fixed points of the smoothing transform with general weights*, to appear in Bernoulli.
9. J. Collamore, *Random recurrence equations and ruin in a markov-dependent stochastic economic environment*, The Annals of Applied Probability 19 (2009), no. 4, 1404–1458.
10. P. Embrechts and C. Goldie, *On closure and factorization properties of subexponential and related distributions*, Journal of the Australian Mathematical Society (Series A) 29 (1980), 243–256.
11. P. Embrechts and C. Goldie, *Perpetuities and random equations*, Asymptotic Statistics (Petr Mandl and Marie Hušková, eds.), Contributions to Statistics, Physica-Verlag HD, 1994, pp. 75–86 (English).
12. S. Foss, D. Korshunov, and S. Zachary, *An introduction to heavy-tailed and subexponential distributions*, Springer, 2011.
13. C. Goldie, *Implicit renewal theory and tails of solutions of random equations*, The Annals of Applied Probability 1 (1991), no. 1, 126–166.
14. C. Goldie and R. Grübel, *Perpetuities with thin tails*, Advances in Applied Probability 28 (1996), no. 2, 463–480.
15. D. Grey, *Regular variation in the tail behaviour of solutions of random difference equations*, The Annals of Applied Probability 4 (1994), no. 1, 169–183.
16. A. Grincevičius, *One limit distribution for a random walk on the line*, Lithuanian Mathematical Journal 15 (1975), no. 4, 580–590, English translation.
17. F. Guillemin, P. Robert, and B. Zwart, *Aimed algorithms and exponential functionals*, The Annals of Applied Probability 14 (2004), no. 1, 90–117.
18. P. Hitczenko and J. Wesołowski, *Perpetuities with thin tails revisited*, The Annals of Applied Probability 19 (2009), no. 6, 2080–2101.
19. L. Jinzhu and T. Qihe, *Interplay of insurance and financial risks in a discrete-time model with strongly regular variation*, to appear in Bernoulli.
20. H. Kesten, *Random difference equations and renewal theory for products of random matrices*, Acta Mathematica 131 (1973), no. 1, 207–248.
21. C. Klüppelberg, *Subexponential distributions and integrated tails*, Journal of Applied Probability 25 (1988), no. 1, pp. 132–141 (English).
22. C. Klüppelberg, A. Lindner, and R. Maller, *Continuous time volatility modelling: Garch versus ornstein–uhlenbeck models*, From Stochastic Calculus to Mathematical Finance, Springer Berlin Heidelberg, 2006, pp. 393–419 (English).
23. K. Maulik and B. Zwart, *Tail asymptotics for exponential functionals of lévy processes*, Stochastic Processes and their Applications 116 (2006), no. 2, 156 – 177.
24. M. Mirek, *Heavy tail phenomenon and convergence to stable laws for iterated lipschitz maps*, Probability Theory and Related Fields 151 (2011), no. 3, 705–734.
25. Z. Palmowski and B. Zwart, *Tail asymptotics of the supremum of a regenerative process*, Journal of Applied Probability 44 (2007), no. 2, 349–365.
26. E. Pitman, *Subexponential distribution functions*, Journal of the Australian Mathematical Society (Series A) 29 (1980), 337–347.
27. V. Rivero, *Tail asymptotics for exponential functionals of lévy processes: The convolution equivalent case*, Annales de l’Institut Henri Poincaré, Probabilités et Statistiques 48 (2012), no. 4, 1081–1102.
28. W. Vervaat, *On a stochastic difference equation and a representation of non-negative infinitely divisible random variables*, Advances in Applied Probability (1979), 750–783.

Instytut Matematyczny, Uniwersytet Wrocławski, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: piotr.dyszewski@math.uni.wroc.pl
URL: www.math.uni.wroc.pl/~pdysz