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Short Quantum Games

by

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Abstract

In this thesis we introduce quantum refereed games, which are quantum interactive proof systems with two competing provers. We focus on a restriction of this model that we call short quantum games and we prove an upper bound and a lower bound on the expressive power of these games.

For the lower bound, we prove that every language having an ordinary quantum interactive proof system also has a short quantum game. An important part of this proof is the establishment of a quantum measurement that reliably distinguishes between quantum states chosen from disjoint convex sets.

For the upper bound, we show that certain types of quantum refereed games, including short quantum games, are decidable in deterministic exponential time by supplying a separation oracle for use with the ellipsoid method for convex feasibility.
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List of Abbreviations

What follows is a list of the complexity classes discussed in this thesis. Each entry in the list consists of the name of a complexity class, the page in which the class is first used or defined, and an informal definition of that class.

**P**, page 3. The class of languages decidable by a deterministic polynomial-time Turing machine.

**NP**, page 4. The class of languages decidable by a deterministic polynomial-time Turing machine with the help of polynomial-size “proof” strings.

**coNP**, page 3. The class of languages whose complements are in **NP**.

**PSPACE**, page 3. The class of languages decidable by a deterministic polynomial-space Turing machine. A polynomial-space Turing machine visits at most a polynomial number of cells on its tape before halting.

**EXP**, page 3. The class of languages decidable by a deterministic exponential-time Turing machine.

**NEXP**, page 3. Same as **NP** except that the “proof” strings may have exponential size.

**coNEXP**, page 3. The class of languages whose complements are in **NEXP**.

**IP**(c, s), page 5. The class of languages that have interactive proof systems with completeness error c and soundness error s.

**RG**(c, s), page 6. The class of languages that have refereed games with completeness error c and soundness error s.

**RG_1**(c, s), page 6. The class of languages that have one-round refereed games with completeness error c and soundness error s. A round consists of a message from the verifier to each of the provers in parallel followed by their responses.

**BQP**, page 14. The class of languages decidable by a polynomial-time uniform family of quantum circuits. Widely considered to be the quantum analogue of **P**.

**QIP**(c, s), page 16. The class of languages that have quantum interactive proof systems with completeness error c and soundness error s.
The class of languages that have quantum refereed games with completeness error $c$ and soundness error $s$.

The class of languages that have short quantum games with completeness error $c$ and soundness error $s$. A short quantum game is a one-round quantum refereed game in which the verifier may process the yes-prover’s response before sending a message to the no-prover.

Same as SQG$(c, s)$ except that the verifier cannot send a message to the yes-prover.

The class of languages that have double quantum interactive proof systems with completeness error $c$ and soundness error $s$. A double quantum interactive proof system is a quantum refereed game in which the verifier exchanges several messages with only the yes-prover and then several messages with only the no-prover.
Chapter 1

Introduction

In this thesis we define a new model of computation that incorporates existing models based upon the notions of competitive interaction and quantum information. We focus on a variant of this new model with a restricted protocol for interaction and we prove lower and upper bounds on the power of this restricted model. In the process of proving these bounds, we develop new computational and information-theoretic tools that may prove useful in other fields in computer science and physics.

This introductory chapter starts with an informal overview of our results in Section 1.1. We offer a survey of relevant topics from complexity theory in Section 1.2 and a review of quantum information in Section 1.3. Chapter 2 formalizes the fundamental concepts discussed in this thesis—the interested reader can find a more precise statement of the contributions of this thesis in Section 2.2.2 on page 21. Chapters 3 and 4 are devoted to proving the results stated in that section. We conclude with Chapter 5, which mentions some open questions and possible directions for future research.

1.1 Overview

It is intended that this first section provide the reader with a bird’s eye view of the direction in which this thesis is headed. For the sake of clarity, citations, technical detail, and an adequate history are absent. These gaps will be addressed in subsequent sections and in Chapter 2 as we cover the necessary background material in greater detail. On that note, we begin.

Given a new model of computation, an effective way to quantify the expressive power of that model is to compare it to other more fundamental models. These comparisons derive meaning from the fact that fundamental models of computation often capture important notions such as efficient computation in the physical world or the difficulty of solving certain computational problems. For example, it is widely believed that any problem that can be solved efficiently by a device built based upon Newtonian physics (such as a desktop computer) can also be solved by a randomized polynomial-time Turing machine and vice versa.
An *interactive proof system* is a more exotic model of computation in which a randomized polynomial-time Turing machine (a *verifier*) is endowed with the ability to interact with an entity who is computationally unbounded but not necessarily trustworthy (a *prover*). Given an input string $x$, the prover uses his unlimited computational power to attempt to convince the verifier to accept $x$, while the verifier tries to determine the validity of the prover’s argument. At the end of the interaction, the verifier accepts $x$ if he believes the prover and rejects $x$ if he does not.

A given set $L$ of strings (a *language*) is said to have an interactive proof system if there exists a verifier $V$ satisfying the following standard completeness and soundness conditions:

**Completeness.** There exists a prover $P$ that can convince $V$ to accept any string $x \in L$ with high probability.

**Soundness.** No prover can convince $V$ to accept any string $x \notin L$ except with small probability.

A *refereed game* is another model of computation that generalizes interactive proof systems in that the verifier interacts with not just one, but two provers. In this model, the provers use their unlimited computational power to compete with each other: one prover (the *yes-prover*) attempts to convince the verifier to accept $x$, while the other prover (the *no-prover*) attempts to convince the verifier to reject $x$. At the end of the interaction, the verifier decides whether to accept or reject the input $x$, effectively deciding which of the provers wins the game.

Similar to interactive proof systems, a language $L$ is said to have a refereed game if there exists a verifier $V$ satisfying the following slightly modified completeness and soundness conditions:

**Completeness.** There exists a yes-prover $Y$ that can convince $V$ to accept any string $x \in L$ with high probability, regardless of the no-prover.

**Soundness.** There exists a no-prover $N$ that can convince $V$ to reject any string $x \notin L$ with high probability, regardless of the yes-prover.

At a glance, these two models of computation may seem obscure and uninteresting. However, it can be shown that they characterize two very fundamental models of computation in the following sense:

- A language has an interactive proof system if and only if it can be decided by a polynomial-space Turing machine.

- A language has a refereed game if and only if it can be decided by a deterministic exponential-time Turing machine.

These surprising characterizations bring to light a very deep connection between exotic interaction-based models and space- and time-bounded Turing machines.
Given the recent proliferation of the theory of quantum computation, it is natural to consider the possible effects of quantum computers on models such as interactive proof systems and refereed games. How does the power of these models change if the verifier and provers are permitted to perform quantum computations and exchange quantum messages?

Perhaps the most striking contrast between the quantum and nonquantum (classical) cases is that any quantum interactive proof system can be simulated by another quantum interactive proof system in which the verifier and prover exchange only three messages. With classical interactive proof systems, all evidence suggests that such a simulation does not exist, be it with three or any other fixed number of messages. What is it about quantum information that permits this strange shortening of interactions?

In this thesis, we turn our attention to quantum refereed games, which have not been previously studied. We focus on a restricted variant of the quantum refereed game model that we call short quantum games and we prove both a lower bound and an upper bound on the expressive power of these games.

For the lower bound, we prove that any language having a quantum interactive proof system also has a quantum refereed game with the following protocol: the yes-prover sends a quantum state to the verifier, who then processes this state and forwards it to the no-prover. The no prover performs a quantum measurement on this state and sends a single classical bit of information (either a 0 or a 1) back to the verifier, who finally accepts or rejects based upon this bit. In order to prove the correctness of our short quantum game, we establish the existence of a quantum measurement that reliably distinguishes between quantum states chosen from two disjoint convex sets.

For the upper bound, we consider a slightly looser restriction of the quantum refereed game model in which the verifier exchanges several quantum messages with only the yes-prover and then exchanges several more quantum messages with only the no-prover before deciding whether to accept or reject. We show that any language having a quantum refereed game obeying this protocol can be decided by a deterministic exponential-time Turing machine. Our proof uses the ellipsoid method, which is a polynomial-time algorithm that determines the emptiness of a convex set given implicitly by a separation oracle.

1.2 Complexity Theory

In this section we review in greater detail certain concepts from complexity theory that lead to the notion of competitive interaction as a model of computation and we survey known results pertaining to that model. We assume at the onset that the reader is familiar with fundamental notions such as languages, Turing machines, polynomial-time computability, completeness, and the following fundamental complexity classes:

- The polynomial-time classes $P$, $NP$, and $coNP$;
- The exponential-time classes $EXP$, $NEXP$, and $coNEXP$; and
- The polynomial-space class $PSPACE$. 

1
Figure 1.1 depicts the known relationships among these complexity classes. In that figure, class A contains class B if A can be reached from B by following a path of only upwardly sloped edges.

We now establish some relevant notation that will be used throughout this thesis. We let $\text{poly}$ denote the set of polynomial-time computable functions $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ for which there exists a polynomial $p$ such that $f(n) \leq p(n)$ for all $n \in \mathbb{N}$. The sets $2^{-\text{poly}}$ and $\text{poly}^{-1}$ are derived from $\text{poly}$ as follows. A polynomial-time computable function $\varepsilon : \mathbb{N} \to [0, 1]$ is in $2^{-\text{poly}}$ if there exists $f \in \text{poly}$ such that $\varepsilon(n) = 2^{-f(n)}$ for all $n \in \mathbb{N}$. Similarly, $\varepsilon$ is in $\text{poly}^{-1}$ if there exists $f \in \text{poly}$ such that $\varepsilon(n) = \frac{1}{f(n)}$ for all $n \in \mathbb{N}$.

All strings are over the binary alphabet $\{0, 1\}$. We let $\{0, 1\}^n$ denote the set of strings of length $n$ and we let $\{0, 1\}^*$ denote the set of all finite strings. For any $x \in \{0, 1\}^*$, $|x|$ denotes the length of $x$.

1.2.1 Nondeterminism

The notion of nondeterminism and the discovery in the 1970’s of $\text{NP}$-complete problems [11, 25, 34] drew unprecedented attention to the field of computational complexity theory. Since then, several characterizations of $\text{NP}$ have been found [12, 11, 3] and several generalizations of nondeterminism have been explored [14, 9, 15, 18]. One of the simpler characterizations views nondeterministic computation as deterministic verification of a short proof. Specifically, a language $L \subseteq \{0, 1\}^*$ is in $\text{NP}$ if and only if there exists $p \in \text{poly}$ and a deterministic polynomial-time Turing machine $M$ such that, for all input strings $x \in \{0, 1\}^*$:

- If $x \in L$ then there exists $y \in \{0, 1\}^{p(|x|)}$ such that $M$ accepts the pair $(x, y)$.
- If $x \notin L$ then $M$ rejects the pair $(x, y)$ for all $y \in \{0, 1\}^{p(|x|)}$. 

4
The Turing machine $M$ in this characterization can be viewed as a verifier. The string $y$ submitted to $M$ can be viewed as a proof of the claim, “$x$ is in $L$.” Informally, the conditions of this characterization tell us that every $x \in L$ has a proof $y$ that can be used by the verifier to verify this claim in deterministic polynomial-time. Moreover, if $x \notin L$ then no proof could possibly convince the verifier otherwise.

### 1.2.2 Interaction

The notion of interaction was introduced in 1985 by Babai [5] and by Goldwasser, Micali, and Rackoff [18] as a generalization of nondeterminism that extends the verifier-proof analogy by allowing a two-way dialogue between the verifier and the mysterious supplier-of-proofs. Specifically, a prover is an entity with unlimited computational power whose goal is always to convince the verifier to accept the input string $x$. The verifier may ask questions of the prover, perform randomized polynomial-time computations, and ask additional questions of the prover based upon these computations and upon answers to previous questions. At some point the verifier must end the interaction and decide whether or not to accept $x$.

Such an interaction is called an interactive proof system. It follows from the fact that the verifier is restricted to polynomial-time computation that the amount and size of the messages exchanged between the verifier and prover must be polynomial in $|x|$. Because the verifier may also invoke randomization, it is plausible that his decision to accept or reject $x$ could differ between independent executions of the same interaction. Hence, an allowance is made for unlucky coin tosses that cause the verifier to erroneously accept or reject $x$ with some small probability.

In this thesis we pay considerable attention to the error probability associated with different forms of interaction. In particular, we are interested in any possible differences between the probability of a false negative (completeness) and of a false positive (soundness). We also consider a more general case in which these probabilities might even vary as a function of the input length $|x|$. However, in order to prevent the polynomial-time verifier from accessing hard-to-compute values encoded in the error probability, we restrict our attention to error probabilities that are polynomial-time computable.

All these ideas are formalized in the following definition. For any polynomial-time computable functions $c, s : \mathbb{N} \to [0, 1]$, a language $L \subseteq \{0, 1\}^*$ is said to have an interactive proof system if there exists a randomized polynomial-time verifier $V$ that satisfies the following completeness and soundness conditions for all input strings $x \in \{0, 1\}^*$:

**Completeness.** If $x \in L$ then there exists a prover $P$ that convinces $V$ to accept $x$ with probability at least $1 - c(|x|)$.

**Soundness.** If $x \notin L$ then no prover can convince $V$ to accept $x$ with probability greater than $s(|x|)$.

The functions $c$ and $s$ are called the completeness error and soundness error respectively. Finally, we let $\text{IP}(c, s)$ denote the complexity class of languages that have interactive proof systems with completeness error $c$ and soundness error $s$. 
1.2.3 Competitive Interaction

Many different models resembling competitive interaction have been studied in the context of game theory since the 1950’s, but competitive interaction was not considered as a generalization of interactive proof systems until 1990 [15]. In this generalization, the verifier interacts with not one, but two provers with unlimited computational power. As mentioned in Section 1.1 these two provers use their power to compete with each other: one prover, called the yes-prover, attempts to convince the verifier to accept the input string \(x\), while the other prover, called the no-prover, attempts to convince the verifier to reject \(x\).

As before, the verifier may perform randomized polynomial-time computations. He may also ask questions of each of the provers and base future questions upon randomized computations and answers to previous questions. At the end of the interaction, the verifier decides whether or not to accept \(x\). Such an interaction is called a refereed game because it can be viewed as a game between the two provers in which the verifier acts as a referee by ensuring that the provers obey the rules of the game and by announcing a winner at the end.

More formally, for any polynomial-time computable functions \(c, s : \mathbb{N} \rightarrow [0, 1]\), a language \(L \subseteq \{0, 1\}^*\) is said to have a refereed game if there exists a randomized polynomial-time verifier \(V\) that satisfies the following completeness and soundness conditions for all input strings \(x \in \{0, 1\}^*\):

**Completeness.** If \(x \in L\) then there exists a yes-prover \(Y\) that convinces \(V\) to accept \(x\) with probability at least \(1 - c(|x|)\), regardless of the no-prover’s strategy.

**Soundness.** If \(x \notin L\) then there exists a no-prover \(N\) that convinces \(V\) to reject \(x\) with probability at least \(1 - s(|x|)\), regardless of the yes-prover’s strategy.

As with interactive proof systems, the functions \(c\) and \(s\) are the completeness error and soundness error respectively. Finally, we let \(\text{RG}(c, s)\) denote the complexity class of languages that have refereed games with completeness error \(c\) and soundness error \(s\).

It is also of interest to consider refereed games in which the verifier exchanges just one round of messages with the provers. In particular, these one-round refereed games obey the following protocol: a message from the verifier to each of the provers, followed by their responses to the referee, followed by the referee’s decision. We let \(\text{RG}_1(c, s)\) denote the complexity class of languages that have one-round refereed games with completeness error \(c\) and soundness error \(s\).

1.2.4 Reasonable Error

In Section 1.2.2 we prevented the verifier from accessing hard-to-compute error probabilities by requiring that they be polynomial-time computable. But even interactions with polynomial-time computable error probabilities can still have undesirable properties.

Suppose, for example, that the probability of correctly accepting or rejecting an input \(x \in \{0, 1\}^*\) is exponentially close to \(\frac{1}{2}\) in \(|x|\). It is clear that such an interaction can be simulated with exponential accuracy by a verifier who ignores all provers and accepts or
rejects based solely upon the result of a coin flip. Of course, an interaction with this property
is not very interesting.

With this example in mind, we say that polynomial-time computable functions \( c, s : \mathbb{N} \to [0, 1] \) are \textit{reasonable} if there exists \( \gamma \in \text{poly}^{-1} \) such that
\[
1 - c(n) - s(n) \geq \gamma(n)
\]
for all \( n \in \mathbb{N} \). As \( c \) and \( s \) are polynomial-time computable, the verifier can compute them
and bias his final decision so that the completeness and soundness error of his biased decision
are both bounded below \( \frac{1}{2} \) by at least an inverse polynomial as desired.

Many (but not all) results concerning interaction are known to hold only when the functions \( c \) and \( s \) are reasonable. We mention the reasonability condition explicitly whenever it is required.

1.2.5 Known Results

Several inclusions follow immediately from the definitions of \( \text{IP}(c, s) \) and \( \text{RG}(c, s) \). For example, because an interactive proof system is just a refereed game without a no-prover, it is clear that \( \text{IP}(c, s) \subseteq \text{RG}(c, s) \).

Also, these classes are easily seen to be \textit{robust} with respect to error in the sense that any interaction with reasonable error can be simulated by another interaction with exponentially small error. More formally, for every \( \varepsilon \in 2^{-\text{poly}} \) and every reasonable \( c, s : \mathbb{N} \to [0, 1] \) we have
\[
\text{IP}(c, s) \subseteq \text{IP}(\varepsilon, \varepsilon) \quad \text{and} \quad \text{RG}(c, s) \subseteq \text{RG}(\varepsilon, \varepsilon).
\]
To prove these containments it suffices to note that any interaction can be repeated independently many times in succession. If the verifier for such a repeated interaction bases his decision upon a majority vote of the outcomes of each of the repetitions then it follows from Chernoff bounds that the error of the repeated interaction decreases exponentially in the number of repetitions.

Of course, sequential repetition necessarily increases the number of rounds in an interaction and so this simple error reduction technique does not apply to bounded-round interactions such as one-round refereed games. Fortunately, as we shall soon see, \( \text{RG}_{1}(c, s) \) can still be shown to be robust with respect to error.

An even stronger robustness result is known to hold for \( \text{IP}(c, s) \). In particular, an interactive proof system with reasonable error can be simulated by another interactive proof system with zero completeness error \([11] [6]\). It follows that
\[
\text{IP}(c, s) \subseteq \text{IP}(0, \varepsilon)
\]
for every \( \varepsilon \in 2^{-\text{poly}} \) and every reasonable \( c, s : \mathbb{N} \to [0, 1] \). By contrast, it is not known whether zero error (completeness or soundness) can be achieved for refereed games. In light of these robustness results, we define the following shorthand notations:

- \( \text{IP} \) is the complexity class of all languages \( L \subseteq \{0, 1\}^{*} \) such that \( L \in \text{IP}(0, \varepsilon) \) for every \( \varepsilon \in 2^{-\text{poly}} \).
• $\text{RG}$ is the complexity class of all languages $L \subseteq \{0,1\}^*$ such that $L \in \text{RG}(\varepsilon, \varepsilon)$ for every $\varepsilon \in 2^{-\text{poly}}$.

• $\text{RG}_1$ is the complexity class of all languages $L \subseteq \{0,1\}^*$ such that $L \in \text{RG}_1(\varepsilon, \varepsilon)$ for every $\varepsilon \in 2^{-\text{poly}}$.

The interactive proof system model has a rich history. However, a comprehensive survey of that history is tangential to the scope of this thesis and so we summarize only the results most relevant to our purpose.

Because interaction is a generalization of nondeterminism, it is clear that $\text{NP} \subseteq \text{IP}$. The full extent of the power of interaction was not fully known until 1990 when Lund, Fortnow, Karloff, and Nisan developed the arithmetization technique [35] that was used in Reference [42] to show

$$\text{IP} = \text{PSPACE}$$

(see also Reference [43]). This surprising characterization of polynomial-space computation has prompted further study of models of computation based upon interaction, one example of which is refereed games.

The refereed game model and several variations thereof were studied in References [38, 15, 14, 33, 16] among others. Many results concerning refereed games can be gleaned from results in game theory. For example, in game theoretic terms, refereed games correspond to two-person games of incomplete information because messages exchanged between one prover and the verifier are kept secret from the other prover. In 1992, Koller and Megiddo [33] gave a deterministic algorithm that solves two-player games of incomplete information in time polynomial in the size of an induced structure known as a game tree. The game tree induced by a refereed game on input $x \in \{0,1\}^*$ is easily shown to have size at most exponential in $|x|$, from which it follows that $\text{RG} \subseteq \text{EXP}$.

Feige and Kilian [13] used a variant of arithmetization to prove the reverse inclusion, implying

$$\text{RG} = \text{EXP}.$$  

They also proved the inclusions $\text{RG}_1(c, s) \subseteq \text{PSPACE}$ and $\text{PSPACE} \subseteq \text{RG}_1(\varepsilon, \varepsilon)$ for every $\varepsilon \in 2^{-\text{poly}}$ and every reasonable $c, s : \mathbb{N} \to [0,1]$. These inclusions imply the aforementioned robustness of one-round refereed games as well as the characterization

$$\text{RG}_1 = \text{PSPACE}.$$  

### 1.3 Quantum Information and Computation

In this section we describe the framework of quantum information. Our description is not intended to be comprehensive, but merely to refresh the reader with the aspects of quantum information that are relevant to this thesis and to establish notation.

We assume familiarity with fundamental concepts from linear algebra such as complex numbers, vectors, vector spaces, matrices, and basic matrix-related concepts such as matrix
multiplication and the trace of a matrix. Of course, familiarity with quantum information and quantum circuits is an asset.

### 1.3.1 Linear Algebra

For any positive integer \( M \), elements of the vector space \( \mathbb{C}^M \) are identified with \( M \)-dimensional column vectors in the usual way and are denoted by lowercase Roman letters such as \( u, v, w \), etc. For any vector \( v \in \mathbb{C}^M \), the conjugate-transpose of \( v \) is denoted \( v^\ast \), which is an \( M \)-dimensional row vector. For any vectors \( v, w \in \mathbb{C}^M \), the standard inner product between \( v \) and \( w \), denoted \( \langle v, w \rangle \), is given by

\[
\langle v, w \rangle = v^\ast w.
\]

The norm induced on \( \mathbb{C}^M \) by the standard inner product is the Euclidean norm. For \( v \in \mathbb{C}^M \), this norm is denoted \( \|v\| \) and given by

\[
\|v\| = \sqrt{\langle v, v \rangle}.
\]

The vector space \( \mathbb{C}^M \) endowed with the standard inner product is a Hilbert space. Moreover, every Hilbert space in this thesis is assumed to take this form for some positive integer \( M \). Hilbert spaces are denoted by uppercase script letters such as \( \mathcal{F}, \mathcal{G}, \mathcal{H} \), etc.

We note at this point that it is a common practice in quantum information to use Dirac notation to describe column and row vectors. For example, the vector \( v \in \mathcal{H} \) would be denoted \( |v\rangle \), the corresponding row vector \( v^\ast \) would be denoted \( \langle v| \), and the inner product \( \langle v, w \rangle \) would be denoted \( \langle v|w \rangle \) using this notation. However, Dirac notation is found to be cumbersome for our purposes and so we break from convention by restricting its use in the following manner: Dirac notation is used in this thesis only to describe the vector \( |0_H\rangle \in \mathcal{H} \), which always denotes the first element in the standard orthonormal basis for \( \mathcal{H} \). In other words, the vector \( |0_H\rangle \) always denotes the vector with a 1 in the first entry and all other entries equal to zero. We sometimes write \( |0\rangle \) when the Hilbert space \( \mathcal{H} \) is clear from the context.

For any Hilbert spaces \( \mathcal{F} \) and \( \mathcal{G} \) of dimensions \( M \) and \( N \) respectively, we let \( \mathbf{L}(\mathcal{F}, \mathcal{G}) \) denote the set of linear mappings from \( \mathcal{F} \) to \( \mathcal{G} \). Elements of \( \mathbf{L}(\mathcal{F}, \mathcal{G}) \) are identified with \( N \times M \) matrices in the usual way (with respect to the standard bases for \( \mathcal{F} \) and \( \mathcal{G} \)) and are denoted by uppercase Roman letters such as \( A, B, C \), etc. For any matrix \( A \in \mathbf{L}(\mathcal{F}, \mathcal{G}) \) we let \( A[i, j] \in \mathbb{C} \) denote the \([i, j]\) entry of \( A \). The spectral norm of \( A \), denoted \( \|A\| \), is given by

\[
\|A\| = \sup_{v \in \mathcal{F}\backslash\{0\}} \frac{\|Av\|}{\|v\|}.
\]

As with vectors, the conjugate-transpose of \( A \) is denoted \( A^\ast \), which is an element of \( \mathbf{L}(\mathcal{G}, \mathcal{F}) \). As a natural extension of the standard inner product for vectors, the Hilbert-Schmidt inner product between any pair of matrices \( A, B \in \mathbf{L}(\mathcal{F}, \mathcal{G}) \), denoted \( \langle A, B \rangle \), is given by

\[
\langle A, B \rangle = \text{tr}(A^\ast B).
\]
Two Hilbert spaces $\mathcal{F}_1, \mathcal{F}_2$ of dimensions $M_1, M_2$ can be combined via the Kronecker product to form a larger Hilbert space $\mathcal{F}_1 \otimes \mathcal{F}_2$ of dimension $M_1 M_2$. The Kronecker product is also defined on vectors and matrices so that for $v_1 \in \mathcal{F}_1, v_2 \in \mathcal{F}_2$ we have
\[ v_1 \otimes v_2 \in \mathcal{F}_1 \otimes \mathcal{F}_2 \]
and for $A_1 \in \mathbf{L}(\mathcal{F}_1, \mathcal{G}_1), A_2 \in \mathbf{L}(\mathcal{F}_2, \mathcal{G}_2)$ we have
\[ A_1 \otimes A_2 \in \mathbf{L}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{G}_1 \otimes \mathcal{G}_2). \]

The Kronecker product satisfies many convenient and intuitive properties that we do not list here.

We write $\mathbf{L}(\mathcal{F})$ as a shorthand notation for $\mathbf{L}(\mathcal{F}, \mathcal{F})$ and we say that a matrix $A \in \mathbf{L}(\mathcal{F})$ acts on $\mathcal{F}$. We let $I_\mathcal{F} \in \mathbf{L}(\mathcal{F})$ denote the identity matrix acting on $\mathcal{F}$, which we often abbreviate to $I$ when the Hilbert space $\mathcal{F}$ is clear from the context. Often in this thesis we multiply matrices acting on a certain Hilbert space with matrices or vectors from a larger Hilbert space. In these cases we implicitly assume that the smaller matrix is extended to the larger Hilbert space by taking the Kronecker product with the identity. For example, if $A \in \mathbf{L}(\mathcal{F}), B \in \mathbf{L}(\mathcal{F} \otimes \mathcal{G})$, and $v \in \mathcal{F} \otimes \mathcal{G}$ then $AB$ and $Av$ always mean $(A \otimes I_\mathcal{G})B$ and $(A \otimes I_\mathcal{G})v$ respectively.

The partial trace is a linear mapping $\text{tr}_\mathcal{G} : \mathbf{L}(\mathcal{F} \otimes \mathcal{G}) \to \mathbf{L}(\mathcal{F})$ defined for all $A \in \mathbf{L}(\mathcal{F}), B \in \mathbf{L}(\mathcal{G})$ as
\[ \text{tr}_\mathcal{G}(A \otimes B) = \text{tr}(B)A \]
and extending to all of $\mathbf{L}(\mathcal{F} \otimes \mathcal{G})$ by linearity. The partial trace is in some sense complimentary to the Kronecker product in that the Kronecker product combines two matrices acting on separate Hilbert spaces to form one matrix acting on one larger Hilbert space. In contrast, the partial trace takes as input a matrix acting on a larger Hilbert space and produces a matrix that acts on a smaller Hilbert space.

A matrix $A \in \mathbf{L}(\mathcal{F})$ is unitary if $A^* A = I$, Hermitian if $A = A^*$, and positive semidefinite if $v^* Av$ is a nonnegative real number for every $v \in \mathcal{F}$. We define the following subsets of $\mathbf{L}(\mathcal{F})$:

- The set $\mathbf{U}(\mathcal{F}) \subset \mathbf{L}(\mathcal{F})$ contains all unitary matrices in $\mathbf{L}(\mathcal{F})$.
- The set $\mathbf{H}(\mathcal{F}) \subset \mathbf{L}(\mathcal{F})$ contains all Hermitian matrices in $\mathbf{L}(\mathcal{F})$.
- The set $\mathbf{Pos}(\mathcal{F}) \subset \mathbf{H}(\mathcal{F})$ contains all positive semidefinite matrices in $\mathbf{L}(\mathcal{F})$.
- The set $\mathbf{D}(\mathcal{F}) \subset \mathbf{Pos}(\mathcal{F})$ contains all positive semidefinite matrices $A$ with $\text{tr}(A) = 1$.

Elements of $\mathbf{D}(\mathcal{F})$ are called density matrices and are typically denoted by lowercase Greek letters such as $\rho, \xi$, etc.
1.3.2 Qubits

A qubit is a fundamental unit of quantum information described as follows. Any collection of \( m \) qubits has a corresponding Hilbert space \( \mathcal{F} \) of dimension \( 2^m \). Any density matrix \( \rho \in \mathcal{D}(\mathcal{F}) \) completely describes some state of those \( m \) qubits. Furthermore, any physically realizable state of those \( m \) qubits is uniquely described by some density matrix, so it makes sense to refer to any such \( \rho \) as a "state" of those \( m \) qubits.

If \( \rho \in \mathcal{D}(\mathcal{F}) \) is given by \( \rho = vv^* \) for some vector \( v \in \mathcal{F} \) then \( \rho \) is called a pure state. It must be the case that \( \|v\| = 1 \) (that is, \( v \) is a unit vector) and \( \rho \) is completely described by \( v \), so it makes sense to refer to any unit vector \( v \) as a "pure state" of those \( m \) qubits. Note that the unit vector \( v \in \mathcal{F} \) describing a pure state \( vv^* \in \mathcal{D}(\mathcal{F}) \) is not unique because the vector \( u = \omega v \) satisfies \( vv^* = uu^* \) for any \( \omega \in \mathbb{C} \) with \( |\omega| = 1 \).

Let \( \mathcal{G} \) be a Hilbert space of dimension \( 2^n \) corresponding to a collection of \( n \) qubits. Then the Hilbert space corresponding to the combined collection of \( m + n \) qubits is \( \mathcal{F} \otimes \mathcal{G} \) and has dimension \( 2^{m+n} \). If \( \rho \in \mathcal{D}(\mathcal{F} \otimes \mathcal{G}) \) is any state of those \( m + n \) qubits then \( \text{tr}_\mathcal{G}(\rho) \in \mathcal{D}(\mathcal{F}) \) always describes the state of the first \( m \) qubits and \( \text{tr}_\mathcal{F}(\rho) \in \mathcal{D}(\mathcal{G}) \) always describes the state of the remaining \( n \) qubits. Although it is the case that \( \rho = \text{tr}_\mathcal{G}(\rho) \otimes \text{tr}_\mathcal{F}(\rho) \) whenever \( \text{tr}_\mathcal{G}(\rho) \) or \( \text{tr}_\mathcal{F}(\rho) \) is a pure state, this equality does not hold for every \( \rho \in \mathcal{D}(\mathcal{F} \otimes \mathcal{G}) \).

1.3.3 Quantum Circuits

The model of quantum computation that provides a foundation for quantum interaction is the quantum circuit model. All quantum circuits in this thesis are assumed to be composed of a finite number of quantum gates, each of which is chosen from some finite universal set of quantum gates. We do not discuss the details of quantum gates or universality, as those details are not required to understand the material presented in this thesis. In lieu of such a discussion, we refer the interested reader to References [2, 7, 27, 46].

For any quantum circuit \( Q \) acting on \( m \) qubits with corresponding Hilbert space \( \mathcal{F} \), there is a unitary matrix \( U \in \mathcal{U}(\mathcal{F}) \) associated with \( Q \). This matrix models the action of \( Q \) upon its \( m \) input qubits in the state \( \rho \in \mathcal{D}(\mathcal{F}) \) so that the state of those \( m \) qubits after \( Q \) is applied becomes \( U \rho U^* \in \mathcal{D}(\mathcal{F}) \). This formalism extends without complication to any positive semidefinite matrix \( \rho \in \mathcal{Pos}(\mathcal{F}) \). If \( \rho = vv^* \) for some pure state \( v \in \mathcal{F} \) then the resulting state is the pure state \( Uv \in \mathcal{F} \) and this formalism extends without complication to any nonzero vector \( v \in \mathcal{F} \).

Two additional facts concerning quantum circuits warrant attention. First, it is important to note that we lose no generality by allowing only unitary quantum circuits because any physically realizable quantum process can be simulated by unitary circuits as described in Reference [2]. We elaborate on this simulation in Section 3.1.1.

Second, the universality condition placed on our set of admissible quantum gates implies the following fact: for any unitary matrix \( V \in \mathcal{U}(\mathcal{F}) \) and any real \( \varepsilon > 0 \), there is a quantum circuit with associated unitary matrix \( U \in \mathcal{U}(\mathcal{F}) \) that satisfies

\[ \|U - V\| < \varepsilon. \]
Although the spectral norm of the difference between $U$ and $V$ is not always a convenient measure of the “distance” between two matrices, in this case it allows us a simple way to infer that any desired unitary matrix $V$ can be approximated as closely as desired by quantum circuits considered in this thesis.

1.3.4 Measurement

So far, we have discussed qubits and quantum circuits that act on qubits. It is now time to discuss measurements, which allow us to extract classical information from qubits and enable us to solve real-world problems using quantum information.

Quantum measurements have several formalizations, each differing in their simplicity and generality. Although knowledge of only the most basic notion of measurement is required throughout most of this thesis, the results of Chapter 3 make use of the extended generality offered by more complex formalizations. Hence, we introduce in this subsection the most general form of quantum measurement, since the added complication of this form is insignificant anyway.

Let $\mathcal{F}$ be a Hilbert space corresponding to some collection of qubits and let $\Gamma$ be a finite set of outcomes. A quantum measurement of those qubits with outcomes in $\Gamma$ is defined by a set of matrices

$$\{A_\tau : \tau \in \Gamma\} \subset L(\mathcal{F})$$

satisfying

$$\sum_{\tau \in \Gamma} A_\tau^* A_\tau = I_\mathcal{F}.$$ 

When such a measurement is performed on qubits in some state $\rho \in D(\mathcal{F})$, the outcome of the measurement is $\tau$ with probability $\text{tr}(A_\tau \rho A_\tau^*)$ for each $\tau \in \Gamma$. Conditioned on the outcome $\tau$, the state of the qubits becomes

$$\frac{A_\tau \rho A_\tau^*}{\text{tr}(A_\tau \rho A_\tau^*)} \in D(\mathcal{F})$$

once the measurement is complete. If $\rho = vv^*$ for some pure state $v \in \mathcal{F}$ then the outcome of the measurement is $\tau$ with probability $\|A_\tau v\|^2$ and the resulting state is the pure state

$$\frac{A_\tau v}{\|A_\tau v\|} \in \mathcal{F}.$$ 

Often, we do not care about the state of the qubits once the measurement is complete. Because the probability of outcome $\tau$ is

$$\|A_\tau v\|^2 = v^* A_\tau^* A_\tau v$$

for pure states and

$$\text{tr}(A_\tau \rho A_\tau^*) = \text{tr}(A_\tau^* A_\tau \rho) = \langle A_\tau^* A_\tau, \rho \rangle$$
for general states, it follows that the quantum measurement is completely specified in this case by the set 

\[ \{ E_\tau : \tau \in \Gamma \} \subset \text{Pos}(\mathcal{F}) \]

of positive semidefinite matrices defined by \( E_\tau = A_\tau^* A_\tau \) for each \( \tau \in \Gamma \) and hence satisfying 

\[ \sum_{\tau \in \Gamma} E_\tau = I_\mathcal{F}. \]

Any measurement expressed in this way is called a positive operator-valued measurement (POVM) for historical reasons.

It is often convenient to specify only a POVM with the understanding that we do not care about the state of the qubits once the measurement is complete—this is the formalism of quantum measurements used in Chapter 3. In all other chapters, the quantum measurements we discuss have outcomes in \( \Gamma = \{\text{accept, reject}\} \) and take the following form. For any Hilbert space \( \mathcal{F} \) corresponding to \( m \) qubits, one of those qubits is designated as the output qubit. Let \( \mathcal{F} = \mathcal{O} \otimes \mathcal{F}' \) where \( \mathcal{O} \) is a two-dimensional Hilbert space corresponding to the output qubit and \( \mathcal{F}' \) corresponds to the remaining \( m - 1 \) qubits. We fix the binary POVM 

\[ \{ \Pi_{\text{accept}}, \Pi_{\text{reject}} \} \subset \text{Pos}(\mathcal{F}) \]

so that

\[ \Pi_{\text{reject}} = |0_\mathcal{O}\rangle\langle 0_\mathcal{O}| \otimes I_{\mathcal{F}'}, \]
\[ \Pi_{\text{accept}} = I_\mathcal{F} - \Pi_{\text{reject}}. \]

This measurement is called the standard measurement of the output qubit of \( \mathcal{F} \).

### 1.3.5 Quantum Algorithms

We now describe how qubits, quantum circuits, and quantum measurements combine to form the quantum circuit model of computation. Let \( \rho \) be a quantum state and let \( Q \) be a quantum circuit with associated unitary matrix \( U \). If desired, the input state \( \rho \) may be chosen so that it uniquely encodes an input string \( x \) to some computational problem. The circuit \( Q \) is applied to \( \rho \) and the output qubit of the resulting state \( U\rho U^* \) is measured according to the standard measurement, which indicates acceptance or rejection of \( \rho \) and hence of the input string \( x \).

By definition, quantum circuits act on a fixed number of qubits. In order to use quantum circuits to decide a language \( L \subseteq \{0,1\}^* \) of arbitrarily large strings, it is typical to specify a family of quantum circuits. In this thesis, a family is a set

\[ \{Q_x : x \in \{0,1\}^*\} \]

of quantum circuits indexed by input strings. Because the quantum gates in \( Q_x \) can depend upon \( x \), there is no need to encode \( x \) in the input state \( \rho \). Instead, we may fix once and for
all a convenient pure state $|0\rangle$ upon which all quantum circuits in the family can be assumed to act.

Families of quantum circuits do not yet form a realistic model of computation because we have not restricted the amount of computation that is used to construct the circuits in a family. In order to make the model realistic, we must introduce a uniformity constraint. In particular, a family of quantum circuits is said to be polynomial-time uniformly generated if there exists a deterministic polynomial-time Turing machine that, given input $x \in \{0,1\}^*$, outputs a description of the quantum circuit $Q_x$.

For any language $L \subseteq \{0,1\}^*$, it is widely agreed that the informal statement “$L$ has an efficient solution on a quantum computer” is adequately formalized by the condition that there exist a polynomial-time uniformly generated family $\{Q_x : x \in \{0,1\}^*\}$ of quantum circuits with associated unitary matrices $\{U_x : x \in \{0,1\}^*\}$ such that, for every $x \in \{0,1\}^*$, $Q_x$ correctly accepts or rejects $|0\rangle$ according to whether or not $x \in L$ with high probability. More specifically,

$$\|\Pi_{\text{accept}} U_x |0\rangle\|_2^2 \geq \frac{2}{3} \iff x \in L,$$

$$\|\Pi_{\text{reject}} U_x |0\rangle\|_2^2 \geq \frac{2}{3} \iff x \notin L.$$ 

The class of languages with this property is denoted BQP and is considered to be the quantum analogue of the complexity class P.
Chapter 2

Preliminaries

In this chapter we provide formal definitions of the quantum interactive proof system and quantum refereed game models of computation as well as several collections of complexity classes based upon these models. We then summarize what is currently known of these models and state the contributions of this thesis.

2.1 Formalizations of Quantum Interaction

Quantum interactive proof systems were introduced by Watrous in 1999 [45] and it is that formalization of the model that we reproduce here. Although quantum refereed games had not yet been considered prior to the work of the present thesis, the formalization of that model is a straightforward extension of the quantum interactive proof system model.

Quantum interactions consist of a verifier and one or more provers. For any function \( r : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\} \), an \( r \)-round prover \( P \) is a mapping on input strings \( x \in \{0, 1\}^* \) where

\[
P(x) = (P_1, \ldots, P_r(|x|))
\]

is an \( r(|x|) \)-tuple of quantum circuits, each of which acts upon the same number of qubits. No restrictions are placed on the complexity of the prover’s circuits, which captures the notion that the prover has unlimited computational power—each of the prover’s circuits can be viewed as an arbitrary unitary operation on its input qubits.

Similarly, an \( r \)-round verifier \( V \) is a mapping on input strings \( x \in \{0, 1\}^* \) where

\[
V(x) = (V_0, \ldots, V_r(|x|))
\]

is an \( (r(|x|) + 1) \)-tuple of quantum circuits, each of which acts upon the same number of qubits. Unlike a prover, however, we require that the verifier’s circuits be generated by a polynomial-time Turing machine on input \( x \). This uniformity constraint captures the notion that the verifier’s computational power is limited and implicitly restricts the quantity \( r(|x|) \) so that \( r \in \text{poly} \) as one might expect. We often abbreviate \( r(|x|) \) to \( r \) for easier readability.
2.1.1 Quantum Interactive Proof Systems

A quantum interactive proof system has a verifier $V$ and a prover $P$. The qubits upon which each of the circuits in the prover’s $r$-tuple acts are partitioned into two sets: one set of qubits is private to the prover and the other is shared with the verifier. These shared qubits act as a quantum channel between the verifier and the prover. The Hilbert spaces corresponding to the private and shared qubits of the prover are denoted $\mathcal{P}$ and $\mathcal{M}$ respectively.

Similarly, the qubits upon which each of the circuits in the verifier’s $(r+1)$-tuple acts are partitioned into two sets: one set of qubits is private to the verifier and the other is shared with the prover. The Hilbert spaces corresponding to the private and shared qubits of the verifier are denoted $\mathcal{V}$ and $\mathcal{M}$ respectively.

For any input string $x \in \{0,1\}^*$ we create a composite circuit $(V,P)(x)$ by concatenating the circuits $V_0, P_1, V_1, \ldots, V_{r-1}, P_r, V_r$ in sequence, each circuit acting only upon the sets of qubits stipulated previously. Such a circuit is illustrated in Figure 2.1 for the case $r = 2$. The Hilbert space upon which $(V,P)(x)$ acts is denoted $S = \mathcal{P} \otimes \mathcal{M} \otimes \mathcal{V}$.

The quantum interactive proof system is implemented by applying the circuit $(V,P)(x)$ to the initial pure state $|0_S\rangle \in S$. Hence, the pure state of the system after $(V,P)(x)$ is applied is precisely $V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0_S\rangle \in S$.

Acceptance is dictated by a standard measurement of the output qubit of $S$, which is assumed to belong to the verifier. In particular, $(V,P)(x)$ accepts $x$ with probability

$$\|\Pi_{\text{accept}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0_S\rangle\|^2$$

and rejects $x$ with probability

$$\|\Pi_{\text{reject}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0_S\rangle\|^2$$

where $\Pi_{\text{accept}}, \Pi_{\text{reject}} \in \text{Pos}(S)$ are as defined in Section 1.3.4.

We now define a collection of complexity classes based upon quantum interactive proof systems. For any polynomial-time computable functions $c, s : \mathbb{N} \rightarrow [0,1]$, the class $\text{QIP}(c,s)$
consists of all languages $L \subseteq \{0, 1\}^*$ for which there exists an $r$-round verifier $V$ that satisfies the following completeness and soundness conditions:

**Completeness.** There exists an $r$-round prover $P$ such that, for all $x \in L$, $(V, P)(x)$ rejects $x$ with probability at most $c(|x|)$. In other words, there exist unitary matrices $P_1, \ldots, P_r \in \mathbb{U}(\mathcal{P} \otimes \mathcal{M})$ such that

$$\|\Pi_{\text{reject}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0_S\rangle\|^2 \leq c(|x|).$$

**Soundness.** For all $r$-round provers $P$ and all $x \notin L$, $(V, P)(x)$ accepts $x$ with probability at most $s(|x|)$. In other words, for all unitary matrices $P_1, \ldots, P_r \in \mathbb{U}(\mathcal{P} \otimes \mathcal{M})$, we have

$$\|\Pi_{\text{accept}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0_S\rangle\|^2 \leq s(|x|).$$

As with classical interactive proof systems, the functions $c$ and $s$ are called the *completeness error* and *soundness error* respectively. We often abbreviate $c(|x|)$ to $c$ and $s(|x|)$ to $s$ for easier readability.

### 2.1.2 Quantum Refereed Games

A *quantum refereed game* has a verifier $V$ and two provers $Y$ and $N$. As with quantum interactive proof systems, the qubits upon which each of the circuits in the provers’ $r$-tuples acts are partitioned into two sets: one set of qubits is private to that prover and the other is shared with the verifier. These shared qubits act as a quantum channel between the verifier and that prover.

For clarity, $Y$ is called a *yes-prover* and $N$ is called a *no-prover*. This distinction is purely a notational convenience: the Hilbert spaces corresponding to the private and shared qubits of a yes-prover are denoted $\mathcal{Y}$ and $\mathcal{M}_Y$ respectively, whereas the Hilbert spaces corresponding to the private and shared qubits of a no-prover are denoted $\mathcal{N}$ and $\mathcal{M}_N$ respectively.

In a quantum refereed game, the qubits upon which each of the circuits in the verifier’s $(r + 1)$-tuple acts are partitioned into three sets: one set of qubits is private to the verifier and the two remaining sets have corresponding Hilbert spaces $\mathcal{M}_Y$ and $\mathcal{M}_N$ and are shared with the yes- and no-provers respectively.

For any input string $x \in \{0, 1\}^*$ we create a composite circuit $(V, Y, N)(x)$ by concatenating the circuits

$$V_0, N_1, Y_1, V_1, \ldots, V_{r-1}, N_r, Y_r, V_r$$

in sequence, each circuit acting only upon the sets of qubits stipulated previously. Such a circuit is illustrated in Figure 2.2 for the case $r = 2$. The Hilbert space upon which $(V, Y, N)(x)$ acts is denoted

$$\mathcal{S} = \mathcal{Y} \otimes \mathcal{M}_Y \otimes \mathcal{V} \otimes \mathcal{M}_N \otimes \mathcal{N}.$$

Although $\mathcal{S}$ also denotes the Hilbert space for quantum interactive proof systems, any ambiguity is always resolved by context. The quantum refereed game is implemented by applying
the circuit \((V, Y, N)(x)\) to the initial pure state \(|0_S\rangle \in S\). Hence, the pure state of the system after \((V, Y, N)(x)\) is applied is precisely
\[ V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 |0_S\rangle \in S. \]

As with quantum interactive proof systems, acceptance is dictated by a standard measurement of the output qubit of \(S\), which is assumed to belong to the verifier. In particular, \((V, Y, N)(x)\) accepts \(x\) with probability
\[ \| \Pi_{\text{accept}} V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 |0_S\rangle \|_2 \]
and rejects \(x\) with probability
\[ \| \Pi_{\text{reject}} V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 |0_S\rangle \|_2. \]

We now define a collection of complexity classes based upon quantum refereed games. For any polynomial-time computable functions \(c, s : \mathbb{N} \to [0, 1]\), the class \(\text{QRG}(c, s)\) consists of all languages \(L \subseteq \{0, 1\}^*\) for which there exists an \(r\)-round verifier \(V\) that satisfies the following completeness and soundness conditions:

**Completeness.** There exists an \(r\)-round yes-prover \(Y\) such that, for all \(r\)-round no-provers \(N\) and all \(x \in L\), \((V, Y, N)(x)\) rejects \(x\) with probability at most \(c(|x|)\). In other words, there exist unitary matrices \(Y_1, \ldots, Y_r \in U(Y \otimes M_Y)\) such that
\[ \| \Pi_{\text{reject}} V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 |0_S\rangle \|_2 \leq c(|x|) \]
for all unitary matrices \(N_1, \ldots, N_r \in U(M_N \otimes N)\).

**Soundness.** There exists an \(r\)-round no-prover \(N\) such that, for all \(r\)-round yes-provers \(Y\) and all \(x \notin L\), \((V, Y, N)(x)\) accepts \(x\) with probability at most \(s(|x|)\). In other words, there exist unitary matrices \(N_1, \ldots, N_r \in U(M_N \otimes N)\) such that
\[ \| \Pi_{\text{accept}} V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 |0_S\rangle \|_2 \leq s(|x|) \]
for all unitary matrices \(Y_1, \ldots, Y_r \in U(Y \otimes M_Y)\).
As with quantum interactive proof systems, the functions $c$ and $s$ are the completeness error and soundness error respectively.

### 2.1.3 Short Quantum Games

For most of this thesis we restrict our attention to a specific class of quantum refereed games that we call short quantum games. A short quantum game has a two-round verifier $V$, a one-round yes-prover $Y$, and a one-round no-prover $N$. In these games, the composite circuit $(V, Y, N)'(x)$ is created by concatenating the circuits

$$V_0, Y_1, V_1, N_1, V_2$$

in sequence. In other words, short quantum games are one-round quantum games in which the verifier is permitted to process the yes-prover’s response before sending a message to the no-prover. Figure 2.3 illustrates the quantum circuit for a short quantum game.

For polynomial-time computable $c, s : \mathbb{N} \rightarrow [0, 1]$, we define the collection $\text{SQG}(c, s)$ of complexity classes by restricting the definition of $\text{QRG}(c, s)$ to short quantum games. We also define the collection $\text{SQG}_*(c, s)$ of complexity classes by further restricting the definition of $\text{SQG}(c, s)$ to short quantum games in which the verifier cannot send a message to the yes-prover (in other words, the verifier’s first circuit is empty, so that $V_0 = I$).

### 2.2 Remarks and Contributions

This section contains a summary of what is currently known of quantum interactive proof systems and quantum refereed games. The contributions of this thesis are stated in Section 2.2.2.
2.2.1 Known Results

Several inclusions follow immediately from the definitions in Section 2.1. For example, because a quantum interactive proof system is just a quantum refereed game without a no-prover, it is clear that

\[ \text{QIP}(c, s) \subseteq \text{QRG}(c, s). \]

Furthermore, we have

\[ \text{SQG}_*(c, s) \subseteq \text{SQG}(c, s) \subseteq \text{QRG}(c, s), \]

as any shorter quantum refereed game can be simulated by a longer quantum refereed game.

Also, it is clear that any classical interaction can be simulated by a quantum interaction in which the verifier simply measures every message he receives from any prover (thus collapsing each message to a completely classical state) and otherwise behaves in a classical manner. In other words, we have \( \text{IP}(c, s) \subseteq \text{QIP}(c, s), \text{RG}(c, s) \subseteq \text{QRG}(c, s), \) and \( \text{RG}_1(c, s) \subseteq \text{SQG}(c, s). \)

It is instructive to note that the relation \( \text{RG}_1(c, s) \subseteq \text{SQG}_*(c, s) \) is not immediately seen to hold, as \( \text{SQG}_*(c, s) \) does not permit games in which the verifier sends a message to the yes-prover.

The method of sequential repetition used in Section 1.2.5 to demonstrate the robustness of classical interaction can be applied without complication in the quantum setting. That is, for every \( \varepsilon \in 2^{-\text{poly}} \) and every reasonable \( c, s : \mathbb{N} \to [0, 1] \) we have

\[ \text{QIP}(c, s) \subseteq \text{QIP}(\varepsilon, \varepsilon) \quad \text{and} \quad \text{QRG}(c, s) \subseteq \text{QRG}(\varepsilon, \varepsilon). \]

Similar to the classical case, quantum interactive proof systems with reasonable error can be assumed to have zero completeness error \[31\], yielding the inclusion

\[ \text{QIP}(c, s) \subseteq \text{QIP}(0, \varepsilon). \]

In light of this robustness, we define the following shorthand notations:

- **QIP** is the complexity class of all languages \( L \subseteq \{0, 1\}^* \) such that \( L \in \text{QIP}(0, \varepsilon) \) for every \( \varepsilon \in 2^{-\text{poly}} \).

- **QRG** is the complexity class of all languages \( L \subseteq \{0, 1\}^* \) such that \( L \in \text{QRG}(\varepsilon, \varepsilon) \) for every \( \varepsilon \in 2^{-\text{poly}} \).

- **SQG** is the complexity class of all languages \( L \subseteq \{0, 1\}^* \) such that \( L \in \text{SQG}(\varepsilon, \varepsilon) \) for every \( \varepsilon \in 2^{-\text{poly}} \).

- **SQG\_\_** is the complexity class of all languages \( L \subseteq \{0, 1\}^* \) such that \( L \in \text{SQG}_*(\varepsilon, \varepsilon) \) for every \( \varepsilon \in 2^{-\text{poly}} \).

The well-known characterization \( \text{IP} = \text{PSPACE} \) mentioned in Section 1.2.5 implies

\[ \text{PSPACE} \subseteq \text{QIP}. \]
Watrous gave a three-message quantum interactive proof system demonstrating this containment [15]. The existence of a three-message quantum interactive proof system for PSPACE contrasts with the classical case, wherein it is widely believed that the verifier and prover must exchange a nonconstant number of messages in order to decide PSPACE (see Section 5.1.1).

This result was later extended to show that any quantum interactive proof system can be simulated by a three-message quantum interactive proof system and that these simulations are robust with respect to error [31]. This extension gives rise to a natural complete promise problem for QIP known as close-images, which can in turn be reduced to an exponential-size instance of the semidefinite programming problem [31]. As semidefinite programs can be solved in deterministic polynomial time, it follows from the QIP-completeness of close-images that

$$\text{QIP} \subseteq \text{EXP}.$$ 

In a later talk on quantum coin-flipping, Kitaev gave a semidefinite program that directly simulates many-message quantum interactive proof systems [28], thus yielding a more direct proof of the containment of QIP in EXP.

Although quantum refereed games were not considered prior to this work, it is nonetheless appropriate to mention several facts in this subsection. First, it is clear that the characterizations RG = EXP and RG_1 = PSPACE imply

$$\text{EXP} \subseteq \text{QRG} \quad \text{and} \quad \text{PSPACE} \subseteq \text{SQG}.$$ 

Also, Kitaev’s variant [28] of the semidefinite program found in Reference [31] is easily extended to yield QRG \subseteq NEXP [22]. Due to the symmetric nature of quantum refereed games, it is clear that QRG is closed under complement, from which it follows that QRG is also contained in coNEXP. In other words, we have

$$\text{QRG} \subseteq \text{NEXP} \cap \text{coNEXP}.$$ 

In Section 5.1.4 we discuss possible implications of the curious fact that QRG contains EXP and is in turn contained in NEXP \cap coNEXP. Chapter 5 also offers a diagram of relationships among the complexity classes considered in this thesis.

### 2.2.2 Contributions of this Thesis

We prove in this thesis the following two relationships among the complexity classes defined in Section 2.1:

- QIP \subseteq SQG_*(Chapter 3)
- SQG \subseteq EXP (Chapter 4)

The first result is proven in Reference [23], while the second is proven in Reference [22]. It is those proofs that we reproduce in Chapters 3 and 4.

Two of the intermediate results that are proven in order to obtain the containment QIP \subseteq SQG_* might be of independent interest and so we also list them here:
• For any two disjoint closed convex sets $A_0, A_1 \subseteq D(H)$ of quantum states, there exists a binary POVM such that, for any state $\rho \in A_0 \cup A_1$, the POVM will correctly determine whether $\rho \in A_0$ or $\rho \in A_1$ with probability proportional to the minimal trace distance between $A_0$ and $A_1$.

• $\text{SQG}(c, s) \subseteq \text{SQG}(kc, s^k) \cap \text{SQG}(c^k, ks)$ for any $c, s : \mathbb{N} \to [0, 1]$ and any $k \in \text{poly}$. A similar containment holds with $\text{SQG}_*$ in place of $\text{SQG}$.

The first of these intermediate results is a generalization of the well-known fact that two quantum states can be distinguished with probability proportional to their trace distance. It can also be viewed as a quantitative version of the well-known separation theorems in convex analysis.

The second result indicates a partial robustness of short quantum games with respect to error. In particular, it states that the completeness (soundness) error can be made exponentially small at the possible cost of soundness (completeness). Error reduction results seem to be more elusive in quantum interaction than in classical interaction and this result represents a first step toward that end for short quantum games.

The containment $\text{SQG} \subseteq \text{EXP}$ builds upon the semidefinite program for $\text{QIP}$ [31, 28]. Hence, we offer a rigorous analysis of that semidefinite program in this thesis as a precursor to our result. This precursor leads to several extensions of the containment $\text{QIP} \subseteq \text{EXP}$. For example, we show that semidefinite programming can be used to simulate a quantum interactive proof system in deterministic exponential time even if the verifier’s circuits are generated uniformly in exponential time, so long as they still act on only a polynomial number of qubits. In particular, the verifier can exchange an exponential number of messages with the prover and can use an exponential number of gates in his quantum circuits without raising the power of the model beyond $\text{EXP}$.

The containment $\text{QRG} \subseteq \text{NEXP}$ is obtained by nondeterministically guessing a yes-prover and solving the induced quantum interactive proof system using the aforementioned semidefinite program [22]. Most of the extensions of that semidefinite program also apply to this containment concerning quantum refereed games. The only exception is that we still require that the verifier exchange only a polynomial number of rounds of messages with the provers. This strange restriction is brought on by the fact that a polynomial bound on the number of qubits required by the provers in a quantum refereed game is known to hold only if a polynomial number of messages are exchanged (see Section 4.2.3).

We prove $\text{SQG} \subseteq \text{EXP}$ by a repeated use of semidefinite programming in concert with the ellipsoid method for convex feasibility. Once again, many of the aforementioned extensions also apply to this containment. For example, our method can simulate a short quantum game in deterministic exponential time even if the verifier is permitted to exchange an exponential number of messages with the yes-prover, followed by an exponential number of messages with the no-prover.
Chapter 3

A Lower Bound for Short Quantum Games

In this chapter we prove that QIP ⊆ SQG∗, which is the main result of Reference [23]. In order to prove this containment we exhibit a short quantum game that solves the QIP-complete problem CLOSE-IMAGES with completeness error $\frac{1}{3}$ and exponentially small soundness error. To show the correctness of our game, we prove an information-theoretic assertion that there exists a quantum measurement that reliably distinguishes between quantum states chosen from two disjoint convex sets. We then prove a general error reduction technique for short quantum games that allows us to produce a game for CLOSE-IMAGES in which both the completeness error and soundness error are exponentially small.

We start by defining the CLOSE-IMAGES problem in Section 3.1. We prove the quantum measurement result in Section 3.2 and use that result in Section 3.3 to yield a short quantum game for CLOSE-IMAGES. We finish the chapter with our error reduction result in Section 3.4.

3.1 The Close-Images Problem

Before we can define CLOSE-IMAGES, we must expand our repertoire of quantum formalism. In particular, we discuss the simulation of any physically realizable quantum process by means of a unitary quantum circuit in Section 3.1.1. In Section 3.1.2 we consider several different distance measures for quantum states and we settle on the trace norm as the distance measure of choice for this thesis. A formal statement of the CLOSE-IMAGES problem appears in Section 3.1.3 along with some comments concerning that problem.

3.1.1 Mixed-State Quantum Circuits

In Section 1.3.3 we described a standard model of quantum circuits in which every $m$-qubit input state is unitarily mapped to an $m$-qubit output state. We also mentioned in that section that any physically realizable quantum process can be simulated by a unitary
In particular, suppose $\mathcal{F}$ and $\mathcal{G}$ are Hilbert spaces corresponding to the $m$ input qubits and $n$ output qubits of some physical process

$$\Phi : \mathcal{D}(\mathcal{F}) \rightarrow \mathcal{D}(\mathcal{G}).$$

Then there exists a unitary matrix $U \in U(\mathcal{F} \otimes \mathcal{G} \otimes \mathcal{G}')$ satisfying

$$\Phi(\rho) = \text{tr}_{\mathcal{G} \otimes \mathcal{G}'}(U (\rho \otimes \ket{0_{\mathcal{G} \otimes \mathcal{G}'}}\bra{0_{\mathcal{G} \otimes \mathcal{G}'}}) U^*)$$

for every $\rho \in \mathcal{D}(\mathcal{F})$ where $\mathcal{G}'$ is a new Hilbert space with $\dim(\mathcal{G}') = \dim(\mathcal{G})$. In consideration with the discussion in Section 1.3.3, it follows that the matrix $U$ can be approximated as closely as desired by a quantum circuit $Q$ acting on $m + 2n$ qubits. In this construction, the vector $\ket{0_{\mathcal{G} \otimes \mathcal{G}'}}$ describes the initial pure state of the remaining $2n$ input qubits of $Q$.

Figure 3.1 illustrates such a circuit. This circuit is sometimes called the Stinespring Dilation of $\Phi$ — its existence is typically attributed to Choi [10] and a proof may be found in Kitaev, Shen, and Vyali [29]. The quantum circuits described in this subsection are called mixed-state quantum circuits in order to differentiate them from the unitary quantum circuits of Section 1.3.3.

Because quantum circuits are composed of a finite number of gates chosen from a finite set of universal gates, it is clear that mixed-state quantum circuits, like unitary circuits, can be encoded into a finite binary string $x \in \{0, 1\}^*$. It therefore makes sense, for example, to define languages over $\{0, 1\}^*$ in terms of mixed-state quantum circuits.

### 3.1.2 Distance Measures for Quantum States

Given two quantum states $\rho, \xi \in \mathcal{D}(\mathcal{F})$, how “close” are they to each other? In particular, with what reliability can these two states be distinguished by a measurement? A natural way to address the first of these questions is in the context of some norm defined on $\mathcal{L}(\mathcal{F})$. It turns out, as we shall soon see, that the second question can be answered if we consider the right norm.

It is standard to define the distance between two matrices $\rho$ and $\xi$ as the norm of the difference $\rho - \xi$. But different norms induce different distance measures, some of which
are more physically meaningful than others. For example, we used the spectral norm in Section 1.3.3 to formalize the notion that any unitary matrix can be approximated arbitrarily closely by a quantum circuit. Although the spectral norm was sufficient for that purpose, the quantity \( \| \rho - \xi \| \) is not known to have much physical meaning. Hence, all that can be said for certain is that if \( \| \rho - \xi \| \) is small then \( \rho \) and \( \xi \) must be “close” and that the two are equal if this quantity is zero.

A norm that is much more useful for quantifying the distinguishability of quantum states is the trace norm, defined for all \( A \in \mathbb{L}(\mathcal{F}) \) as

\[
\| A \|_{\text{tr}} = \text{tr} \left( \sqrt{A^* A} \right).
\]

To make sense of this definition, we point out that \( A^* A \) is positive semidefinite for every \( A \in \mathbb{L}(\mathcal{F}) \) and that, for every \( P \in \mathbb{Pos}(\mathcal{F}) \), there is a unique \( Q \in \mathbb{Pos}(\mathcal{F}) \) satisfying \( Q^2 = P \). Thus, the trace norm of an arbitrary matrix \( A \in \mathbb{L}(\mathcal{F}) \) is the trace of the unique positive semidefinite matrix \( \sqrt{A^* A} \). If \( A \in \mathbb{H}(\mathcal{F}) \) is Hermitian then \( \| A \|_{\text{tr}} \) is just the sum of the absolute values of the eigenvalues of \( A \). In comparison, the spectral norm \( \| A \| \) of a Hermitian matrix \( A \) is the maximum of the absolute values of the eigenvalues of \( A \). The trace norm and the spectral norm are dual to each other with respect to the Hilbert-Schmidt inner product, meaning that the following fact holds (see Bhatia [8]):

**Fact 3.1 (Duality of the Spectral and Trace Norms).** For every \( A \in \mathbb{L}(\mathcal{F}) \) we have

\[
\| A \| = \max \{ |\langle B, A \rangle| : B \in \mathbb{L}(\mathcal{F}), \| B \|_{\text{tr}} \leq 1 \},
\]

\[
\| A \|_{\text{tr}} = \max \{ |\langle B, A \rangle| : B \in \mathbb{L}(\mathcal{F}), \| B \| \leq 1 \}.
\]

For every \( \rho_0, \rho_1 \in \mathbb{D}(\mathcal{F}) \), the quantity \( \| \rho_0 - \rho_1 \|_{\text{tr}} \) lies in the interval \([0, 2]\). The trace norm characterizes the distinguishability of \( \rho_0 \) and \( \rho_1 \) in the following sense: there exists a binary-valued POVM such that if \( \rho \in \{ \rho_0, \rho_1 \} \) is chosen uniformly at random then the POVM correctly determines which of \( \rho_0 \) or \( \rho_1 \) was chosen with probability

\[
\frac{1}{2} + \frac{1}{4} \| \rho_0 - \rho_1 \|_{\text{tr}}.
\]

Furthermore, such a POVM is optimal in the sense that no other quantum measurement could possibly distinguish between \( \rho_0 \) and \( \rho_1 \) with a higher rate of success. It is because of this property that the trace norm is often a very convenient and satisfactory distance measure for quantum states. The quantity \( \| \rho_0 - \rho_1 \|_{\text{tr}} \) is sometimes called the trace distance between \( \rho_0 \) and \( \rho_1 \).

Incidentally, even if \( \rho \) were chosen from \( \{ \rho_0, \rho_1 \} \) according to some arbitrary and unknown distribution, it can still be shown that the same POVM will correctly distinguish between \( \rho_0 \) and \( \rho_1 \) with probability at least \( \frac{1}{2} \| \rho_0 - \rho_1 \|_{\text{tr}} \).

Besides the trace norm, several other distance measures exist for quantum states. One example of such a measure is the fidelity, which is a function

\[
F : \mathbb{Pos}(\mathcal{F}) \times \mathbb{Pos}(\mathcal{F}) \to \mathbb{R}
\]
defined by the expression
\[ F(X,Y) = \| \sqrt{X} \sqrt{Y} \|_{\text{tr}} \]
for every \( X, Y \in \text{Pos}(\mathcal{F}) \). If \( \rho \) and \( \xi \) are density matrices then \( F(\rho, \xi) \) always lies in the interval \([0, 1]\). Furthermore, \( F(\rho, \xi) = 1 \) if and only if \( \rho = \xi \) and \( F(\rho, \xi) = 0 \) if and only if \( \rho \) and \( \xi \) describe perfectly distinguishable quantum states. The fidelity and the trace norm are related by the following inequalities, which hold for every \( \rho, \xi \in \text{D}(\mathcal{F}) \) (see Fuchs and van de Graaf [17]):
\[
1 - \frac{1}{2} \| \rho - \xi \|_{\text{tr}} \leq F(\rho, \xi) \leq \sqrt{1 - \frac{1}{4} \| \rho - \xi \|_{\text{tr}}^2}.
\]
(3.1)

Although the fidelity satisfies many useful properties, we need not consider it any further because most expressions involving the fidelity can be converted into expressions involving the trace norm via (3.1) and because the trace norm adequately meets our needs in this thesis.

3.1.3 Statement of the Problem

For any mixed-state quantum circuit \( Q : \text{D}(\mathcal{F}) \to \text{D}(\mathcal{G}) \), the image of \( Q \) is the set
\[
\{ Q(\rho) : \rho \in \text{D}(\mathcal{F}) \} \subseteq \text{D}(\mathcal{G})
\]
Given mixed-state quantum circuits \( Q_0 \) and \( Q_1 \), the definition of close-images promises that the images of \( Q_0 \) and \( Q_1 \) either intersect or are disjoint.

More formally, the close-images problem—parameterized by any desired function \( \varepsilon \in 2^{-\text{poly}} \)—is defined as in Figure 3.2 This problem was implicitly shown to be \text{QIP}-complete in Reference [31]. The statement presented in Figure 3.2 is based upon the formulation found in Reference [10]. In that paper, condition 2 in the promise is stated using the fidelity instead of the trace norm. However, it is more convenient for our purposes to rephrase the problem in terms of the trace norm. That this rephrased version is equivalent to the original follows from (3.1).

3.2 Distinguishing Convex Sets of States

In Section 3.1.2 we pointed out that the trace norm is a distance measure for quantum states that characterizes the distinguishability of two states \( \rho_0, \rho_1 \in \text{D}(\mathcal{F}) \). In this section we generalize that notion from single states \( \rho_0, \rho_1 \in \text{D}(\mathcal{F}) \) to sets of states \( A_0, A_1 \subseteq \text{D}(\mathcal{F}) \). We motivate discussion of this generalization in Section 3.2.1 before we state and prove our result in Section 3.2.2.
**Problem.** CLOSE-IMAGES($\varepsilon$).

**Input.** Two mixed-state quantum circuits $Q_0, Q_1 : D(\mathcal{F}) \to D(\mathcal{G})$ acting on $m$-qubit states.

**Promise.** Exactly one of the following conditions holds:

1. There exist $m$-qubit states $\rho_0, \rho_1 \in D(\mathcal{F})$ such that
   \[ Q_0(\rho_0) = Q_1(\rho_1). \]

2. For all $m$-qubit states $\rho_0, \rho_1 \in D(\mathcal{F})$, $Q_0(\rho_0)$ and $Q_1(\rho_1)$ satisfy
   \[ \|Q_0(\rho_0) - Q_1(\rho_1)\|_{tr} > 2 - \varepsilon(m). \]

**Output.** “Accept” if condition 1 holds, “reject” if condition 2 holds.

---

**3.2.1 Motivation and Preamble**

Let $V$ be any vector space over $\mathbb{R}$ or $\mathbb{C}$ (for example, a Hilbert space $\mathcal{F}$ and the set $L(\mathcal{F})$ are vector spaces over $\mathbb{C}$). A set $C \subseteq V$ is *convex* if for every $x, y \in C$ and $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda)y \in C$. It follows from the fact that density matrices have unit trace that $D(\mathcal{F})$ is a convex subset of $L(\mathcal{F})$. Because mixed-state quantum circuits act linearly on their input qubits, it follows that the image of any mixed-state quantum circuit is convex. Hence, many results pertaining to convexity can be applied to the images of mixed-state quantum circuits.

In particular, the separation theorems of convex analysis tell us that between any two disjoint convex sets there exists a hyperplane that separates them. Typically, the separation results are stated in terms of the vector space $\mathbb{R}^n$. At first, the restriction to real numbers might seem like a problem. But fortunately, the set $H(\mathcal{F})$ of complex Hermitian matrices acting on a Hilbert space $\mathcal{F}$ of dimension $n$ is readily shown to be isomorphic to $\mathbb{R}^{n^2}$ and can therefore be regarded as a vector space over $\mathbb{R}$. Because density matrices are always positive semidefinite and hence Hermitian, it follows that the separation results apply without complication to convex sets of density matrices such as the image of a quantum circuit. The separation result of most use to us, recast in terms of $H(\mathcal{F})$, is stated as follows (see Rockafellar [39]):

**Fact 3.2 (Separation Theorem).** Let $\mathcal{A}, \mathcal{B} \subseteq H(\mathcal{F})$ be disjoint convex sets with $\mathcal{A}$ compact and $\mathcal{B}$ open. There exists $H \in H(\mathcal{F})$ and $a \in \mathbb{R}$ such that $\langle H, X \rangle \geq a > \langle H, Y \rangle$ for every $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

By choosing suitable convex sets upon which to apply the Separation Theorem (Fact
3.2), we can use the corresponding hyperplane to define a quantum measurement that distinguishes between disjoint images of two mixed-state quantum circuits. As one might expect, such a measurement is useful for solving problems such as close-images, in which disjoint images must be distinguished from overlapping images.

### 3.2.2 A Generalized Distinguishability Result

Our goal in this subsection is to solve the following generalization of the distinguishability problem. We are given $\rho \in D(\mathcal{F})$ chosen according to some arbitrary and unknown distribution from the set $\{\rho_0, \rho_1\}$. All we know about $\rho_0$ and $\rho_1$ is that $\rho_0 \in \mathcal{A}_0$ and $\rho_1 \in \mathcal{A}_1$ where $\mathcal{A}_0, \mathcal{A}_1 \subseteq D(\mathcal{F})$ are convex sets of density matrices. Our task is to determine which of $\rho_0$ or $\rho_1$ was chosen. With what probability can we correctly make this distinction?

The answer to this question depends on the minimal trace distance between $\mathcal{A}_0$ and $\mathcal{A}_1$ in much the same way as the distinguishability of $\rho_0, \rho_1 \in D(\mathcal{F})$ depends upon the trace distance between $\rho_0$ and $\rho_1$. Our formalization of this answer begins with the following theorem.

**Theorem 3.3 (Distinguishability of Sets of States).** Let $\mathcal{A}_0, \mathcal{A}_1 \subseteq D(\mathcal{F})$ be closed convex sets of density matrices and let $d$ denote the minimum of $\|\rho_0 - \rho_1\|_{tr}$ over all $\rho_0 \in \mathcal{A}_0$ and all $\rho_1 \in \mathcal{A}_1$. There exists a binary-valued POVM such that, for every pair $\rho_0 \in \mathcal{A}_0$ and $\rho_1 \in \mathcal{A}_1$, if $\rho$ is chosen uniformly at random from $\{\rho_0, \rho_1\}$ then the POVM will correctly determine which of $\rho_0$ or $\rho_1$ was chosen with probability at least $\frac{1}{2} + \frac{d}{4}$.

**Proof.** We preface this proof with some comments regarding the quantum measurement that we are to construct. The POVM that we apply to $\rho$ will have outcomes in $\Gamma = \{0, 1\}$ where outcome $0 \in \Gamma$ indicates that $\rho_0$ was chosen and outcome $1 \in \Gamma$ indicates that $\rho_1$ was chosen. In accordance with Section 1.3.4, the POVM with outcomes in $\Gamma$ will be given by the set $\{E_0, E_1\} \subset \text{Pos}(\mathcal{F})$ of positive semidefinite matrices satisfying $E_0 + E_1 = I_\mathcal{F}$.

Let $C$ denote the event that our POVM yields the correct outcome. That is, the event $C$ is said to occur if $\rho = \rho_0$ and we obtain outcome $0 \in \Gamma$ or if $\rho = \rho_1$ and we obtain outcome $1 \in \Gamma$. According to Section 1.3.4 we have

$$\Pr[C | \rho = \rho_0] = \langle E_0, \rho_0 \rangle,$$

$$\Pr[C | \rho = \rho_1] = \langle E_1, \rho_1 \rangle.$$

As $\rho$ is chosen uniformly at random, we can combine the previous two conditional probabilities to obtain

$$\Pr[C] = \frac{1}{2} \langle E_0, \rho_0 \rangle + \frac{1}{2} \langle E_1, \rho_1 \rangle.$$

By similar reasoning it follows that

$$\Pr[\neg C] = \frac{1}{2} \langle E_1, \rho_0 \rangle + \frac{1}{2} \langle E_0, \rho_1 \rangle$$

and hence

$$\Pr[C] - \Pr[\neg C] = \frac{1}{2} \langle E_0 - E_1, \rho_0 - \rho_1 \rangle.$$
This expression will be of use later in this proof.

We are now ready to begin the proof in earnest. If the minimum \( d = 0 \) then it suffices that our POVM be as good as a random coin flip. By choosing \( E_0 = E_1 = \frac{1}{2}I \) we obtain

\[
\Pr[C] = \frac{1}{2} \text{tr}(\rho_0) + \frac{1}{2} \text{tr}(\rho_1) = \frac{1}{2},
\]

which holds for every \( \rho_0, \rho_1 \in \mathcal{D}(\mathcal{F}) \) as desired. Hence, for the remainder of this proof we assume that \( d > 0 \).

We define

\[
\mathcal{A} = \mathcal{A}_0 - \mathcal{A}_1 = \{\rho_0 - \rho_1 : \rho_0 \in \mathcal{A}_0, \rho_1 \in \mathcal{A}_1\}.
\]

The set \( \mathcal{A} \subset \mathbf{H}(\mathcal{F}) \) is a closed convex set of Hermitian matrices such that \( \|X\|_{\text{tr}} \geq d \) for every \( X \in \mathcal{A} \). Let

\[
\mathcal{B} = \{Y \in \mathbf{H}(\mathcal{F}) : \|Y\|_{\text{tr}} < d\}
\]

denote the open ball of radius \( d \) with respect to the trace norm. The sets \( \mathcal{A} \) and \( \mathcal{B} \) satisfy the conditions of the Separation Theorem (Fact 3.2), and therefore there exists a hyperplane that separates them. That is, there exists a Hermitian matrix \( H \in \mathbf{H}(\mathcal{F}) \) and a real number \( a \in \mathbb{R} \) such that

\[
\langle H, X \rangle \geq a > \langle H, Y \rangle
\]

for every \( X \in \mathcal{A} \) and \( Y \in \mathcal{B} \). Note that because the ball \( \mathcal{B} \) is centred at the origin, we have \( Y \in \mathcal{B} \) if and only if \( -Y \in \mathcal{B} \) and hence \( a > \langle H, Y \rangle \) and \( a > \langle H, -Y \rangle \), from which it follows that \( a > \langle H, Y \rangle > -a \) and in particular \( a > 0 \).

We now use the Hermitian matrix \( H \) and the positive reals \( a \) and \( d \) to construct our POVM \( \{E_0, E_1\} \). Let \( K = \frac{a}{d}H \). Then \( \langle K, X \rangle \geq d \) for every \( X \in \mathcal{A} \) and \( \langle K, \frac{1}{d}Y \rangle < 1 \) for every \( Y \in \mathcal{B} \). As \( \frac{1}{d}Y \) ranges over all Hermitian matrices with trace norm smaller than 1, it follows from the Duality of the Spectral and Trace Norms (Fact 3.1) that \( \|K\| \leq 1 \).

Now let \( K_+, K_- \in \text{Pos}(\mathcal{F}) \) be the Jordan decomposition of \( K \), meaning that \( K = K_+ - K_- \) and \( K_+ \) and \( K_- \) act on orthogonal subspaces of \( \mathcal{F} \). It follows that

\[
\|K_+ + K_-\| = \|K_+ - K_-\| = \|K\| \leq 1
\]

and hence \( I - K_+ - K_- \) is positive semidefinite.

The matrices \( E_0 \) and \( E_1 \) composing our binary-valued POVM are given by

\[
E_0 = K_+ + \frac{1}{2} (I - K_+ - K_-),
\]
\[
E_1 = K_- + \frac{1}{2} (I - K_+ - K_-).
\]

Of course, \( E_0 \) and \( E_1 \) are positive semidefinite and satisfy

\[
E_0 + E_1 = I, \quad E_0 - E_1 = K.
\]
We now compute the probability with which the POVM yields the correct outcome.

\[
\Pr[C] - \Pr[\neg C] = \frac{1}{2} \langle K, \rho_0 - \rho_1 \rangle \geq \frac{d}{2}
\]

with the inequality following from the fact that \( K = \frac{d}{a} H \) and \( \rho_0 - \rho_1 \in \mathcal{A} \) for every \( \rho_0 \in \mathcal{A}_0 \) and \( \rho_1 \in \mathcal{A}_1 \). From here, it is straightforward to solve the system

\[
\begin{align*}
\Pr[C] - \Pr[\neg C] &\geq \frac{d}{2} \\
\Pr[C] + \Pr[\neg C] &\geq 1
\end{align*}
\]

and obtain \( \Pr[C] \geq 1 - \frac{d}{4} \) as desired. \( \square \)

We now use the Distinguishability of Sets of States (Theorem 3.3) to obtain a result that holds even when \( \rho \) is chosen nonuniformly from \( \{\rho_0, \rho_1\} \).

**Corollary 3.4.** Let \( \mathcal{A}_0, \mathcal{A}_1, \) and \( d \) be defined as in the statement of Theorem 3.3. The binary-valued POVM \( \{E_0, E_1\} \) from the proof of Theorem 3.3 satisfies the following property. For every pair \( \rho_0 \in \mathcal{A}_0 \) and \( \rho_1 \in \mathcal{A}_1 \), if \( \rho \) is chosen from \( \{\rho_0, \rho_1\} \) according to some arbitrary and unknown distribution then \( \{E_0, E_1\} \) will correctly determine which of \( \rho_0 \) or \( \rho_1 \) was chosen with probability at least \( \frac{d}{2} \).

**Proof.** As in the proof of Theorem 3.3, let \( C \) denote the event that \( \{E_0, E_1\} \) yields the correct outcome. We start by pointing out that any binary probability distribution over \( \{\rho_0, \rho_1\} \) can be expressed as a composition of the uniform distribution and a zero-entropy distribution. It will then suffice to show that \( \Pr[C] \geq \frac{d}{2} \) under these two distributions.

Assume for now that \( \rho_0 \) is the more likely choice—the case in which \( \rho_1 \) is the more likely choice will follow by symmetry. In other words, we assume \( \rho \) is chosen from \( \{\rho_0, \rho_1\} \) so that \( \rho = \rho_0 \) with probability \( \lambda \) for some \( \lambda \in \left[\frac{1}{2}, 1\right] \).

Consider the following composite distribution. With probability \( 2 - 2\lambda \) we choose \( \rho \) uniformly at random from \( \{\rho_0, \rho_1\} \) and with probability \( 2\lambda - 1 \) we choose \( \rho = \rho_0 \) with certainty. It follows that, under this composite distribution, \( \rho_0 \) is chosen with probability \( \lambda \) as desired.

Under the uniform distribution (\( \lambda = \frac{1}{2} \)), Theorem 3.3 tells us that

\[
\Pr[C] \geq \frac{1}{2} + \frac{d}{4} \geq \frac{d}{2}.
\]

Hence, it suffices to show that \( \Pr[C] \geq \frac{d}{4} \) under the zero-entropy distribution (\( \lambda = 1 \)). We achieve this bound using the following facts from the proof of Theorem 3.3:

\[
\begin{align*}
\Pr[C \mid \rho = \rho_0] &= \langle E_0, \rho_0 \rangle, \\
\Pr[\neg C \mid \rho = \rho_0] &= \langle E_1, \rho_0 \rangle, \\
\langle E_0 - E_1, \rho_0 - \rho_1 \rangle &\geq d, \\
\|E_0 - E_1\| &\leq 1.
\end{align*}
\]
Since \( \rho = \rho_0 \) with certainty, it follows from the first two expressions that
\[
\Pr[C] - \Pr[\neg C] = \langle E_0 - E_1, \rho_0 \rangle.
\]
The third expression implies that
\[
\langle E_0 - E_1, \rho_0 \rangle \geq d + \langle E_0 - E_1, \rho_1 \rangle.
\]
It follows from the fourth expression and from the Duality of the Spectral and Trace Norms (Fact 3.1) that
\[
|\langle E_0 - E_1, \rho_1 \rangle| \leq \|\rho_1\|_{\text{tr}} = 1
\]
and hence \( \langle E_0 - E_1, \rho_1 \rangle \geq -1 \). Combining all these inequalities, we obtain
\[
\Pr[C] - \Pr[\neg C] \geq d - 1
\]
from which it follows that \( \Pr[C] \geq \frac{d}{2} \) as desired. \( \square \)

The most important lesson to take away from the Distinguishability of Sets of States (Theorem 3.3) and Corollary 3.4 is that the POVM \( \{E_0, E_1\} \) depends only upon \( A_0 \) and \( A_1 \) and not on any particular pair of density matrices in those sets. In other words, the very same quantum measurement can be used to distinguish between every pair of density matrices chosen from those sets. This independence is critical to the correctness of our solution to CLOSE-IMAGES.

### 3.3 A Short Quantum Game for Close-Images

In this section, we prove that any language with a quantum interactive proof system also has a short quantum game by solving the QIP-complete problem CLOSE-IMAGES from Section 3.1.3. In order to prove membership in \( \text{SQG}_s(c, s) \), we must exhibit a verifier for a short quantum game that satisfies the completeness and soundness conditions stated in Section 2.1. Such a verifier receives one message from the yes-prover and then exchanges a round of messages with the no-prover before deciding whether to accept the input.

Informally, the verifier we seek obeys the following protocol: given descriptions of two mixed-state quantum circuits \( Q_0 \) and \( Q_1 \), the verifier receives states \( \rho_0 \) and \( \rho_1 \) from the yes-prover, randomly chooses \( i \in \{0, 1\} \), applies \( Q_i \) to \( \rho_i \), and forwards the result to the no-prover. The no-prover is then challenged to identify which of \( Q_0(\rho_0) \) and \( Q_1(\rho_1) \) was sent to him. If he succeeds then the verifier assumes that the no-prover can reliably distinguish between \( Q_0(\rho_0) \) and \( Q_1(\rho_1) \) and hence the images of \( Q_0 \) and \( Q_1 \) are far apart. If he fails then the verifier assumes that the no-prover cannot reliably distinguish between \( Q_0(\rho_0) \) and \( Q_1(\rho_1) \) because they are equal and hence the images of \( Q_0 \) and \( Q_1 \) intersect. This argument is formalized in the following theorem.

**Theorem 3.5.** \( \text{QIP} \subseteq \text{SQG}_s(\frac{1}{2}, \varepsilon) \) for every \( \varepsilon \in 2^{-\text{poly}} \).
1. Receive $m$-qubit registers $X_0$ and $X_1$ from the yes-prover.

2. Choose $i \in \{0, 1\}$ uniformly at random and apply $Q_i$ to register $X_i$. Let the output be contained in a register $Y$, which is then sent to the no-prover.

3. Receive a classical bit $b$ from the no-prover. Accept if $b \neq i$ and reject if $b = i$.

---

**Figure 3.3: Verifier’s protocol for Theorem 3.5**

**Proof.** Given any $\varepsilon \in 2^{-\text{poly}}$, it suffices to show that CLOSE-IMAGES$(\varepsilon)$ is in SQG$_*(\frac{1}{2}, \varepsilon)$. Let $Q_0, Q_1 : D(F) \rightarrow D(G)$ be any given mixed-state quantum circuits acting on $m$ qubits and let $A_i$ denote the image of $Q_i$ for $i \in \{0, 1\}$. The sets $A_0, A_1 \subseteq D(G)$ are closed convex sets of density operators.

Consider the verifier for a short quantum game described in Figure 3.3. If $(Q_0, Q_1)$ is a “yes” instance of CLOSE-IMAGES$(\varepsilon)$ then there exist $\rho_0, \rho_1 \in D(F)$ such that $Q_0(\rho_0) = Q_1(\rho_1)$. The strategy for the yes-prover is to prepare the registers $X_0$ and $X_1$ in the states $\rho_0$ and $\rho_1$ respectively and to send them to the verifier in step 1 of the verifier’s protocol. Because $Q_0(\rho_0) = Q_1(\rho_1)$, the state contained in the register $Y$ is independent of $i$, so the no-prover can do no better than a random guess in step 3. The verifier will therefore accept with probability at most $\frac{1}{2}$ in this case.

Let $d$ be the minimum of $\|\rho_0 - \rho_1\|_{tr}$ over all choices of $\rho_0 \in A_0$ and $\rho_1 \in A_1$. If $(Q_0, Q_1)$ is a “no” instance of CLOSE-IMAGES$(\varepsilon)$ then we are promised that $d > 2 - \varepsilon(m)$. Regardless of the state of the registers $X_0$ and $X_1$ sent to the verifier by the yes-prover, we must have that the state $\rho \in D(G)$ of the register $Y$ sent to the no-prover is in either $A_0$ or $A_1$. Furthermore, we have

$$\Pr[\rho \in A_0] = \Pr[\rho \in A_1] = \frac{1}{2}.$$ 

Hence, by the Distinguishability of Sets of States (Theorem 3.3) there exists a binary-valued POVM that correctly determines whether $\rho \in A_0$ or $\rho \in A_1$ with probability at least

$$\frac{1}{2} + \frac{d}{4} > 1 - \frac{\varepsilon(m)}{4}.$$ 

The strategy for the no-prover is to perform the quantum measurement from Theorem 3.3 and send the result to the verifier in step 3, which causes the verifier to reject with probability greater than $1 - \frac{\varepsilon(m)}{4}$. \qed

### 3.4 Error Reduction for Short Quantum Games

In this section we prove that short quantum games are at least partially robust with respect to error in the sense that the completeness error can be made exponentially small at the possible
cost of an increase in the soundness error and vice versa. Fortunately, because the short quantum game for CLOSE-IMAGES in Section 3.3 has exponentially small soundness error, any increase in that quantity can be absorbed into the arbitrarily small factor, yielding a short quantum game for close-images with exponentially small completeness and soundness error.

The error reduction technique we present in this section relies heavily upon previous results in error reduction for quantum interaction. We summarize the necessary material in Section 3.4.1 before proving our new result in Section 3.4.2.

### 3.4.1 Parallel Repetition and Transformations

In Section 2.2.1 we mentioned that any quantum refereed game with reasonable error can be simulated by another quantum refereed game with exponentially small error that repeats the initial game many times in succession and then accepts based upon the outcomes of each of the repetitions. However, we pointed out in Section 1.2.5 that sequential repetition of this form necessarily increases the number of rounds in an interaction and so this technique does not apply to bounded-round interactions such as short quantum games. In the classical case, this problem was circumvented by identifying one-round refereed games with PSPACE. Unfortunately, no analogous result is known to hold for short quantum games.

A natural approach to the task of error reduction for bounded-round interactions is to run many copies of the interaction in parallel and act as though the repetitions were sequential, basing the decision to accept accordingly. This technique, known as parallel repetition, is purely classical and has been successfully applied to classical single- and multi-prover interactive proof systems (see Raz [37] and the references therein). A potential problem with this technique is that the provers need not treat each repetition independently—they might try to correlate the parallel repetitions (or entangle them in the quantum case) in some devious way such that the completeness or soundness error does not decrease as desired.

In the quantum setting, the general case of this problem has not been completely solved. But for three-message single-prover quantum interactive proof systems with zero completeness error, Reference [31] proves that parallel repetition followed by a unanimous vote does indeed achieve the exponential reduction in soundness error that one might expect, regardless of any possible entanglement by the prover among the parallel copies.

Because we will incorporate parts of the proof of this result into our reduction, it is necessary to summarize some of the additional formalism upon which it draws. Toward that end, recall that $L(F)$ and $L(G)$ denote the sets of linear mappings acting on $F$ and $G$ respectively. As $L(F)$ and $L(G)$ are themselves vector spaces, it makes sense to consider the set $T(F, G)$ of linear mappings from $L(F)$ to $L(G)$, also known as transformations. As one might expect, the Kronecker product extends naturally to transformations.

Recall also that the spectral norm on $L(F)$ is induced by the Euclidean norm on $F$ by the relation

$$\|A\| = \sup_{v \in F \setminus \{0\}} \frac{\|Av\|}{\|v\|}$$
for every \( A \in L(F) \). We can extend the trace norm to \( T(F,G) \) in a similar way: for any \( T \in T(F,G) \) we have
\[
\| T \|_\text{tr} = \sup_{A \in L(F) \setminus \{0\}} \frac{\| T(A) \|_\text{tr}}{\| A \|_\text{tr}}.
\]
However, this extension of the trace norm does not induce an overly desirable metric on \( T(F,G) \) in part because its value can change upon taking the Kronecker product with the identity transformation. That is, there exist Hilbert spaces \( \mathcal{H} \) and transformations \( T \in T(F,G) \) with
\[
\| T \|_\text{tr} \neq \| T \otimes I_{L(H)} \|_\text{tr}
\]
where \( I_{L(H)} \in T(\mathcal{H},\mathcal{H}) \) is the identity transformation on \( L(H) \).

With this fact in mind, the diamond norm of a transformation \( T \in T(F,G) \) is defined as
\[
\| T \|_\diamond = \| T \otimes I_{L(K)} \|_\text{tr}
\]
where \( \text{dim}(K) = \text{dim}(F) \). The diamond norm satisfies several convenient properties. For one, it is robust with respect to taking the Kronecker product with the identity. Another nice property of the diamond norm is that it is multiplicative with respect to the Kronecker product. In other words,
\[
\| T_1 \otimes T_2 \|_\diamond = \| T_1 \|_\diamond \| T_2 \|_\diamond
\]
for any choice of transformations \( T_1 \) and \( T_2 \). Proofs of these and other properties of the diamond norm can be found in Kitaev, Shen, and Vyalyi [29].

Now that we have introduced the diamond norm for transformations, we are ready to discuss its relevance to quantum interaction. In what follows, the projection \( \Pi_{\text{init}} \in Pos(S) \) is defined as
\[
\Pi_{\text{init}} = |0_S\rangle \langle 0_S|
\]
where \( |0_S\rangle \) is the initial pure state of any quantum interaction. The fact upon which we base our error reduction result is stated as follows (see Reference [31, Lemma 7]):

\textbf{Fact 3.6.} Let \( V(x) = (V_0, V_1) \) be a verifier for a one-round quantum interactive proof system on input \( x \in \{0,1\}^* \) (such an interaction consists of a message from the verifier to the prover followed by the prover’s response). Let \( T \in T(M \otimes V, M) \) be a transformation defined as
\[
T(X) = \text{tr}_V ((V_0 \Pi_{\text{init}}) X (\Pi_{\text{accept}} V_1))
\]
for every \( X \in L(M \otimes V) \). The maximum probability with which any prover could convince \( V \) to accept \( x \) is precisely \( \| T \|_\diamond^2 \).

As we shall soon see, if we consider the Kronecker product of \( T \) with itself many times then the multiplicative property of the diamond norm in concert with Fact 3.6 yields an exponentially small upper bound on the soundness error of repeated one-round quantum interactions.
3.4.2 A Partial Robustness Result

In this subsection we prove that parallel repetition followed by a unanimous vote can be used to improve the error bounds for short quantum games by reducing the problem to error reduction for single-prover quantum interactive proof systems with three or fewer messages. The reduction is achieved by fixing a yes- or no-prover reduction for single-prover quantum interactive proof systems with three or fewer messages.

We are now prepared to give the main result of this section, whose proof is based upon the proof of Theorem 6 in Reference [31].

Theorem 3.7 (Partial Robustness of Short Quantum Games).

\[ \text{SQG}(c, s) \subseteq \text{SQG}(kc, s^k) \cap \text{SQG}(c^k, ks) \]

for any choice of \( c, s : \mathbb{N} \to [0, 1] \) and \( k \in \text{poly} \).

Proof. For brevity we write \( k = k(|x|) \), \( c = c(|x|) \), and \( s = s(|x|) \) where \( x \in \{0, 1\}^* \). For any matrix \( A \) and any positive integer \( n \), we write \( A^\otimes n = A \otimes \cdots \otimes A \) as shorthand for the \( n \)-fold Kronecker product of \( A \) with itself.

We first prove that \( \text{SQG}(c, s) \subseteq \text{SQG}(kc, s^k) \). Let \( L \in \text{SQG}(c, s) \) and let \( V(x) = (V_0, V_1, V_2) \) be a verifier witnessing this fact. Let \( V'(x) = (V_0^\otimes k, V_1^\otimes k, V_2^\otimes k) \) be a verifier that runs \( k \) copies of the protocol of \( V(x) \) in parallel and accepts if and only if every one of the \( k \) copies accepts. We must show that \( V'(x) \) has completeness error at most \( kc \) and soundness error at most \( s^k \).

First consider the case \( x \in L \). Let \( Y(x) = Y_1 \) be a yes-prover that convinces \( V(x) \) to accept \( x \) with probability at least \( 1 - c \). Let \( Y'(x) = Y_1^\otimes k \) be a yes-prover that runs \( k \) independent copies of the protocol of \( Y(x) \) in parallel. Then no no-prover can win any one of the \( k \) copies with probability greater than \( c \) and so by the union bound we know that the completeness error of the repeated game is at most \( kc \).

Next consider the case \( x \notin L \). Let \( N(x) = N_1 \) be a no-prover that convinces \( V(x) \) to reject \( x \) with probability at least \( 1 - s \) and let \( \Pi_{\text{init}} \) be as defined in Section 3.4.1. As earlier intimated, we may view \((V, N)(x) = (V_0, V_2 N_1 V_1)\) as a new one-round composite verifier and the yes-prover as the lone prover for some two-message quantum interactive proof system.

Define the transformation

\[ T_s(X) = \text{tr}_{Y_0 \otimes M, N \otimes N'} ((V_0 \Pi_{\text{init}}) X (\Pi_{\text{accept}} V_2 N_1 V_1)) \]

By Fact 3.6 we know that the maximum probability with which any prover could convince \((V, N)(x)\) to accept \( x \) is \( \|T_s\|_2^2 \). As \((V, N)(x)\) has soundness error at most \( s \), we have \( \|T_s\|_2^2 \leq s \).
Now let \( N'(x) = N^{\otimes k} \) be a no-prover that runs \( k \) independent copies of the protocol of \( N(x) \) in parallel. We now show that no yes-prover can win against \( N'(x) \) using verifier \( V'(x) \) with probability greater than \( s^k \). Let \( \Pi'_\text{init} = \Pi^{\otimes k}_\text{init} \) and \( \Pi'_\text{accept} = \Pi^{\otimes k}_\text{accept} \) be the projections corresponding to the initial and accepting states of the repeated game. Define the transformation

\[
T_s'(X) = \text{tr}_{(V \otimes M, N) \otimes k} \left( (V'_0 \Pi'_\text{init})X(\Pi'_\text{accept}V'_2N'_1V'_1) \right).
\]

It is clear that \( T_s' = T^{\otimes k}_s \). By Fact 3.6 and the multiplicativity of the diamond norm it follows that the maximum probability with which any prover could convince \( (V', N')(x) \) to accept \( x \) is

\[
\|T'_s\|_2^2 = \|T^{\otimes k}_s\|_2^2 = \|T_s\|_2^{2k} \leq s^k,
\]

which establishes the desired result.

Due to the symmetric nature of quantum refereed games, we can modify the above proof to show that \( \text{SQG}(c, s) \subseteq \text{SQG}(c^k, ks) \). In particular, define the verifier \( V''(x) \) so that he rejects if and only if all \( k \) copies reject. For the case \( x \notin L \), the proof that \( V''(x) \) has soundness error \( ks \) is completely symmetric to the proof that \( V'(x) \) has completeness error \( kc \).

For the case \( x \in L \), we let \( Y(x) = Y_1 \) be a yes-prover that convinces \( V(x) \) to accept with probability at least \( 1 - c \). Let \( (V, Y)(x) = (V_1Y_1V_0, V_2) \) be a new one-round composite verifier for a two-message quantum interactive proof system in which the no-prover is the lone prover. The two differences here are that the prover’s goal is now to convince \( (V, Y)(x) \) to reject instead of to accept \( x \) and that the transformation \( T_s \) is now replaced with the transformation

\[
T_c(X) = \text{tr}_{Y \otimes M, Y \otimes V} \left( (V_1Y_1V_0\Pi_\text{init})X(\Pi_\text{reject}V_2) \right).
\]

Fortunately, Fact 3.6 still applies and so the maximum probability with which any prover could convince \( (V, Y)(x) \) to reject \( x \) is precisely \( \|T'_c\|_2^2 \). That \( V''(x) \) has completeness error \( c^k \) follows as before.

Of course, the Partial Robustness of Short Quantum Games (Theorem 3.7) holds in the special case where the verifier \( V(x) = (I, V_1, V_2) \) and so we obtain

\[
\text{SQG}_s(c, s) \subseteq \text{SQG}_s(kc, s^k) \cap \text{SQG}_s(c^k, ks)
\]
as an easy corollary. A more important corollary that follows from Theorems 3.5 and 3.7 is the main result of this chapter:

**Corollary 3.8.** \( \text{QIP} \subseteq \text{SQG}_s \).

**Proof.** Given a desired error bound \( 2^{-p} \) where \( p \in \text{poly} \), choose \( \varepsilon \in 2^{-\text{poly}} \) so that \( p(n)\varepsilon(n) \leq 2^{-p(n)} \) for all \( n \in \mathbb{N} \). We have \( \text{QIP} \subseteq \text{SQG}_s(\frac{1}{2}, \varepsilon) \subseteq \text{SQG}_s(2^{-p}, 2^{-p}) \).
Chapter 4

An Upper Bound for Short Quantum Games

Given that QIP is contained in both EXP \[31\] and SQG\[\ast\] (Chapter 3), it is natural to wonder how complexity classes based upon short quantum games relate to EXP. In this chapter we prove that SQG \(\subseteq\) EXP, which is the main contribution of Reference [22].

In order to prove this containment, we build upon previously known techniques for simulating quantum interaction with classical computation. In particular, Kitaev sketched an alternate proof [28] of the containment QIP \(\subseteq\) EXP [31]. We provide the first complete formalization of that proof and offer several extensions, one of which is that QRG \(\subseteq\) NEXP.

Finally, we show that SQG \(\subseteq\) EXP by employing the accumulated results in a separation oracle for use with the ellipsoid method for convex feasibility. In fact, the containment SQG \(\subseteq\) EXP is a special case of a stronger result proven in this chapter.

4.1 The Opt Problem

In this section we define the computational problem OPT based upon some observations regarding quantum interactive proof systems. We show that OPT can be reduced to a semidefinite program and hence admits a deterministic polynomial-time solution.

4.1.1 Optimization, Transcripts, and Consistency

Let \(c, s : \mathbb{N} \rightarrow [0, 1]\), let \(L \in\) QIP\((c, s)\), let \(V(x) = (V_0, \ldots, V_r)\) be an \(r\)-round verifier witnessing this fact, and consider the following optimization problem

\[
\begin{align*}
\text{maximize} & \quad \|\Pi_{\text{accept}}V_rP_rV_{r-1}\cdots V_1P_1V_0|0\rangle\|^2 \\
\text{subject to} & \quad P_1, \ldots, P_r \in U(\mathcal{P} \otimes \mathcal{M}).
\end{align*}
\]

(4.1)

By definition, if \(x \in L\) then the optimal value of this problem is at least \(1 - c\), whereas if \(x \not\in L\) then the optimal value of this problem is at most \(s\). Hence, if \(c\) and \(s\) are reasonable then \(L\) can be decided by solving this problem.
However, the problem (4.1) in its stated form is incompatible with standard optimization algorithms. In this subsection, we define the notion of a “transcript” of a quantum interactive proof system and we identify every prover with such a transcript. In so doing, we reduce the optimization problem (4.1) to a much more manageable problem that can be solved using algorithms for semidefinite programming.

Let $P(x) = (P_1, \ldots, P_r)$ be any $r$-round prover and consider the quantum circuit $(V, P)(x)$ for some input string $x \in \{0, 1\}^*$. For each $i \in \{1, \ldots, r\}$, let $\rho_i \in D(\mathcal{M} \otimes \mathcal{V})$ denote the state of the verifier’s qubits immediately before $V_i$ is applied. The state $\rho_i$ can be viewed as a “snapshot” of the verifier’s qubits at the beginning of the $i$th round of interaction. In this sense, the states $\rho_0, \ldots, \rho_r$ indicate a complete transcript of the quantum interactive proof system. Such a transcript is illustrated in Figure 4.1 for the case $r = 2$.

What can be said about transcripts? Two observations follow immediately from the definition of a quantum interactive proof system. First, it must be the case that $\rho_0 = |0_{\mathcal{M} \otimes \mathcal{V}}\rangle \langle 0_{\mathcal{M} \otimes \mathcal{V}}|$, as the initial pure state of the entire system is always $|0_S\rangle$. Second, the probability with which $V(x)$ accepts $x$ is given by

$$\text{tr} \left( \Pi_{\text{accept}} V_r \rho_r V_r^* \Pi_{\text{accept}}^* \right) = \langle V_r^* \Pi_{\text{accept}}^* \Pi_{\text{accept}} V_r, \rho_r \rangle$$

in accordance with the rules for quantum measurement discussed in Section 1.3.4.

As a third observation, consider for each $i \in \{1, \ldots, r\}$ the states $\xi_i, \xi'_i \in D(\mathcal{V})$ of the verifier’s private qubits immediately before and after the prover’s circuit $P_i$ is applied. These states are illustrated in Figure 4.1 for the case $r = 2$.

It is clear from Figures 4.1 and 4.2 that $\xi'_i$ is obtained from $\rho_i$ by discarding the message qubits. That is,

$$\xi'_i = \text{tr}_\mathcal{M}(\rho_i).$$

Similarly, $\xi_i$ is obtained from $\rho_{i-1}$ by applying $V_{i-1}$ and discarding the message qubits. In other words,

$$\xi_i = \text{tr}_\mathcal{M}(V_{i-1} \rho_{i-1} V_{i-1}^*).$$

Finally, since the prover circuit $P_i$ cannot act on the verifier’s private qubits, it follows that $\xi_i = \xi'_i$ (this fact is also made intuitively evident in Figure 4.2). Hence, we claim that

$$\text{tr}_\mathcal{M}(\rho_i) = \text{tr}_\mathcal{M}(V_{i-1} \rho_{i-1} V_{i-1}^*) \quad \forall \ i \in \{1, \ldots, r\}.$$
As a generalization of these observations, let $\mathcal{F}$ and $\mathcal{G}$ be Hilbert spaces, let $X_1, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G})$ be positive semidefinite matrices, and let $A_0, \ldots, A_{r-1} \in \mathbb{L}((\mathcal{F} \otimes \mathcal{G})$ be arbitrary matrices. We say that the list $X_1, \ldots, X_r$ is $\mathcal{G}$-consistent with $A_0, \ldots, A_{r-1}$ if

$$
\text{tr}_{\mathcal{G}}(X_{i+1}) = \text{tr}_{\mathcal{G}}(A_i X_i A_i^*) \quad \forall \ i \in \{0, \ldots, r-1\}
$$

where $X_0 = |0_{\mathcal{F} \otimes \mathcal{G}}\rangle\langle 0_{\mathcal{F} \otimes \mathcal{G}}|$. Our goal then is to prove that a list of density matrices $\rho_1, \ldots, \rho_r$ indicates a valid transcript for a quantum interactive proof system with verifier $V(x)$ if and only if it is $\mathcal{M}$-consistent with $V_0, \ldots, V_{r-1}$.

We require some additional formalism in order to accomplish this goal. For any positive semidefinite matrix $X \in \text{Pos}(\mathcal{F})$ and any vector $v \in \mathcal{F} \otimes \mathcal{G}$, $v$ is called a purification of $X$ if $X = \text{tr}_{\mathcal{G}}(vv^*)$.

The purifications of $X$ are related to one another as indicated by the following fact (see Hughston, Jozsa, and Wootters [24]):

**Fact 4.1 (Unitary Equivalence of Purifications).** A purification $v \in \mathcal{F} \otimes \mathcal{G}$ of $X \in \text{Pos}(\mathcal{F})$ exists if and only if $\dim(\mathcal{G}) \geq \text{rank}(X)$. Moreover, purifications of $X$ are unitarily equivalent in the sense that if $u, v \in \mathcal{F} \otimes \mathcal{G}$ are both purifications of $X$ then there exists a unitary matrix $U \in \mathbb{U}(\mathcal{G})$ such that $(I_\mathcal{F} \otimes U)u = v$.

Intuitively, we make use of the Unitary Equivalence of Purifications (Fact 4.1) in the following manner. For any $i \in \{1, \ldots, r\}$, let $u, u' \in \mathcal{S}$ be purifications of the states $\xi_i, \xi'_i \in \text{D}(\mathcal{V})$ in Figure 4.2. The vectors $u$ and $u'$ can be thought of as the pure states of the entire system corresponding to the snapshots $\xi_i$ and $\xi'_i$. As $\xi_i = \xi'_i$, it follows that $u$ and $u'$ are purifications of the same state and hence by the Unitary Equivalence of Purifications (Fact 4.1) there exists a unitary matrix $P_i \in \mathbb{U}(\mathcal{P} \otimes \mathcal{M})$ such that

$$u' = (P_i \otimes I_\mathcal{V})u.$$

The unitary matrix $P_i$ indicates precisely the actions that the prover must take during the $i$th round of the interaction in order to take the pure state of the entire system from $u$ to $u'$. Given any transcript $\rho_0, \ldots, \rho_r$, we can construct in this manner unitary matrices $P_1, \ldots, P_r \in \mathbb{U}(\mathcal{P} \otimes \mathcal{M})$ corresponding to a prover $P(x)$ who gives rise to that transcript.
We now formalize this intuition. Because our result will be applied in different contexts later in this chapter, we state it in its full generality and then follow its proof with a corollary that relates it to quantum interactive proof systems.

**Lemma 4.2 (Consistency Characterization).** Let \( A_0, \ldots, A_{r-1} \in L(\mathcal{F} \otimes \mathcal{G}) \). For every Hilbert space \( \mathcal{H} \) and every \( U_1, \ldots, U_r \in U(\mathcal{G} \otimes \mathcal{H}) \) there exist \( X_1, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G}) \) that are \( \mathcal{G} \)-consistent with \( A_0, \ldots, A_{r-1} \) such that

\[
\| AU_r A_{r-1} \cdots A_1 U_1 A_0 [0_{\mathcal{F} \otimes \mathcal{G} \otimes \mathcal{H}}] \|^2 = \langle A^* A, X_r \rangle \quad \forall \, A \in L(\mathcal{F} \otimes \mathcal{G}).
\]

Conversely, if \( \dim(\mathcal{H}) \geq \dim(\mathcal{F} \otimes \mathcal{G}) \) then for every \( X_1, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G}) \) \( \mathcal{G} \)-consistent with \( A_0, \ldots, A_{r-1} \) there exist \( U_1, \ldots, U_r \in U(\mathcal{G} \otimes \mathcal{H}) \) such that (4.2) holds.

**Proof.** We start by proving the first statement. Define \( u_0, \ldots, u_r \in \mathcal{F} \otimes \mathcal{G} \otimes \mathcal{H} \) and \( X_1, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G}) \) as follows: for every \( i \in \{0, \ldots, r-1\} \), let \( u_{i+1} = U_{i+1} A_i u_i \) with \( u_0 = |0\rangle \) and let \( X_{i+1} = \text{tr}_{\mathcal{H}}(u_{i+1} u_{i+1}^*) \). Then for any \( A \in L(\mathcal{F} \otimes \mathcal{G}) \) we have

\[
\| AU_r A_{r-1} \cdots A_1 U_1 A_0 [0] \| = \| Au_r \| = \langle A^* A, u_r u_r^* \rangle = \langle A^* A, X_r \rangle.
\]

It remains only to show the \( \mathcal{G} \)-consistency of \( X_1, \ldots, X_r \) with \( A_0, \ldots, A_{r-1} \). For every \( i \in \{0, \ldots, r-1\} \) we have

\[
\text{tr}_{\mathcal{G}}(X_{i+1}) = \text{tr}_{\mathcal{G} \otimes \mathcal{H}}(u_{i+1} u_{i+1}^*) = \text{tr}_{\mathcal{G}}(U_{i+1} A_i u_i u_i^* A_i^* U_{i+1}^*) = \text{tr}_{\mathcal{G}}(A_i \text{tr}_{\mathcal{H}}(u_i u_i^*) A_i) = \text{tr}_{\mathcal{G}}(A_i X_i A_i).
\]

To prove the converse, let \( \mathcal{H} \) be a Hilbert space with \( \dim(\mathcal{H}) \geq \dim(\mathcal{F} \otimes \mathcal{G}) \) and let \( X_1, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G}) \) be \( \mathcal{G} \)-consistent with \( A_0, \ldots, A_{r-1} \). Let \( u_0, \ldots, u_r \in \mathcal{F} \otimes \mathcal{G} \otimes \mathcal{H} \) be purifications of \( X_0, \ldots, X_r \) with \( u_0 = |0\rangle \). These purifications are guaranteed to exist by the Unitary Equivalence of Purifications (Fact 4.1) because

\[
\dim(\mathcal{H}) \geq \dim(\mathcal{F} \otimes \mathcal{G}) \geq \text{rank}(X)
\]

for any \( X \in L(\mathcal{F} \otimes \mathcal{G}) \). For every \( i \in \{0, \ldots, r-1\} \), it follows that \( A_i u_i \) is a purification of \( A_i X_i A_i \). As \( \text{tr}_{\mathcal{G}}(X_{i+1}) = \text{tr}_{\mathcal{G}}(A_i X_i A_i^* A_i^* U_{i+1}) \), the Unitary Equivalence of Purifications (Fact 4.1) implies that there exists some unitary matrix \( U_{i+1} \in U(\mathcal{G} \otimes \mathcal{H}) \) such that \( u_{i+1} = U_{i+1} A_i u_i \). Again, we have

\[
\| AU_r A_{r-1} \cdots A_1 U_1 A_0 [0] \| = \| Au_r \| = \langle A^* A, u_r u_r^* \rangle = \langle A^* A, X_r \rangle
\]

for any \( A \in L(\mathcal{F} \otimes \mathcal{G}) \). \( \square \)

The Consistency Characterization (Lemma 4.2) provides us with the ability to convert from a prover \( P(x) \) to a transcript \( \rho_1, \ldots, \rho_r \) and vice versa. We use that ability in the following corollary to reformulate the optimization problem (4.1).
Corollary 4.3. Let $c,s : \mathbb{N} \to [0,1]$, let $L \in \mathrm{QIP}(c,s)$, and let $V(x) = (V_0, \ldots, V_r)$ be a verifier witnessing this fact. Consider the following optimization problem

$$\text{maximize} \quad \langle V_r^* \Pi_{\text{accept}} \Pi_{\text{accept}} V_r, \rho_r \rangle \quad (4.3)$$

subject to $\rho_1, \ldots, \rho_r \in D(\mathcal{M} \otimes \mathcal{V})$

$$\rho_1, \ldots, \rho_r \mathcal{M}\text{-consistent with } V_0, \ldots, V_{r-1}.$$ 

If $x \in L$ then the optimal value of this problem is at least $1 - c$ and if $x \not\in L$ then the optimal value of this problem is at most $s$.

Proof. If $x \in L$ then by definition there exist $P_1, \ldots, P_r \in \mathcal{U}(\mathcal{P} \otimes \mathcal{M})$ such that

$$\|\Pi_{\text{accept}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0\rangle\|^2 \geq 1 - c.$$ 

By the Consistency Characterization (Lemma 4.2) there exists a transcript $\rho_1, \ldots, \rho_r$ that is $\mathcal{M}$-consistent with $V_0, \ldots, V_{r-1}$ such that

$$\langle V_r^* \Pi_{\text{accept}} \Pi_{\text{accept}} V_r, \rho_r \rangle = \|\Pi_{\text{accept}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0\rangle\|^2,$$

from which the first claim of the corollary follows.

Now suppose $x \not\in L$ and let $\rho_1, \ldots, \rho_r$ be any transcript that is $\mathcal{M}$-consistent with $V_0, \ldots, V_{r-1}$. By the Consistency Characterization (Lemma 4.2) there exists a prover $P(x) = (P_1, \ldots, P_r)$ such that

$$\langle V_r^* \Pi_{\text{accept}} \Pi_{\text{accept}} V_r, \rho_r \rangle = \|\Pi_{\text{accept}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0\rangle\|^2.$$ 

By definition, the quantity on the right is at most $s$. \qed

As Corollary 4.3 suggests, any language in QIP can be decided by solving the optimization problem (4.3). Indeed, this entire section is dedicated to solving that problem efficiently.

Like the Consistency Characterization (Lemma 4.2), our solution to (4.3) will be used in several different contexts later in this chapter. Hence, we name the problem $\text{opt}$ and restate it in full generality in Figure 4.3. It is assumed that the real and imaginary parts of all input numbers to $\text{opt}$ are represented in binary notation.

As a final note, we observe that the optimization problem (4.3) appearing in the statement of Corollary 4.3 can be phrased as an instance of $\text{opt}$ with input matrices $V_0, \ldots, V_{r-1}, \Pi_{\text{accept}} V_r$ and a suitably small accuracy parameter $\varepsilon$ that depends only on the completeness error $c$ and soundness error $s$.

4.1.2 Semidefinite Programming

Our proof that $\text{opt}$ admits a deterministic polynomial-time solution will rely upon existing polynomial-time algorithms for semidefinite programming. Hence, we offer a brief summary of semidefinite programming in this subsection.
Problem. \textsc{opt}.

Input. Matrices $A_0, \ldots, A_r \in \mathbb{L}(\mathcal{F} \otimes \mathcal{G})$ and an accuracy parameter $\varepsilon > 0$.

Output. A list $X_1, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G})$ of positive semidefinite matrices that is $\mathcal{G}$-consistent with $A_0, \ldots, A_{r-1}$ such that

$$\langle A_r^* A_r, X_r \rangle > \langle A_r^* A_r, Z_r \rangle - \varepsilon$$

for every list $Z_1, \ldots, Z_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G})$ that is $\mathcal{G}$-consistent with $A_0, \ldots, A_{r-1}$.

---

Figure 4.3: Definition of \textsc{opt}

Semidefinite programming is derived from linear programming, which is the name given to the problem of maximizing a linear function subject to a finite number of linear constraints. Given a Hilbert space $\mathcal{F}$, a Hermitian matrix $H \in \mathbb{H}(\mathcal{F})$, matrices $A_1, \ldots, A_m \in \mathbb{L}(\mathcal{F})$, and scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, a semidefinite program over $\mathbb{C}$ has the form

$$\begin{align*}
\text{maximize} & \quad \langle H, X \rangle \\
\text{subject to} & \quad \langle A_i, X \rangle = \alpha_i \text{ for all } i \in \{1, \ldots, m\} \\
& \quad X \in \text{Pos}(\mathcal{F}).
\end{align*}$$

The feasible set $\mathcal{A} \subseteq \text{Pos}(\mathcal{F})$ is defined by

$$\mathcal{A} = \{ X \in \text{Pos}(\mathcal{F}) : \langle A_i, X \rangle = \alpha_i \text{ for all } i \in \{1, \ldots, m\} \}.$$ 

The goal is to find a matrix $X \in \mathcal{A}$ such that $\langle H, X \rangle$ is maximized over all $X \in \mathcal{A}$. Because both $H$ and $X$ are Hermitian, it follows that $\langle H, X \rangle$ is always real and so it makes sense to consider its maximal value.

The semidefinite programming problem that we use is stated in Figure 4.3. As with \textsc{opt}, it is assumed in this problem that the real and imaginary parts of all input numbers are represented in binary notation. The SDP problem can be solved in time polynomial in the bit length of the input data using interior point methods (see, for example, Nesterov and Nemirovskii [36]).

4.1.3 A Semidefinite Program for \textsc{opt}

Our goal in this subsection is to prove that \textsc{opt} can be reduced to SDP and therefore has a deterministic polynomial-time solution. We accomplish this goal by formalizing the reduction that appears implicitly in Reference [28]. The main idea is to “stack” the positive semidefinite matrices $X_1, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G})$ into one large block-diagonal matrix. This stacked matrix will serve as the variable over which SDP is to optimize.

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Problem. SDP.

Input. A Hermitian matrix $H \in \mathbf{H}(\mathcal{F})$, matrices $A_1, \ldots, A_m \in \mathbf{L}(\mathcal{F})$ and scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ defining the feasible set $\mathcal{A}$, a feasible solution $X_{\text{init}} \in \mathcal{A}$, a positive real number $b$ such that $\|X\| \leq b$ for every $X \in \mathcal{A}$, and an accuracy parameter $\varepsilon > 0$.

Output. $X \in \mathcal{A}$ such that $\langle H, X \rangle > \langle H, Z \rangle - \varepsilon$ for every $Z \in \mathcal{A}$.

Figure 4.4: Definition of SDP

Given matrices $A_0, \ldots, A_{r-1} \in \mathbf{L}(\mathcal{F} \otimes \mathcal{G})$, we construct linear equality constraints on the stacked matrix variable that characterize $\mathcal{G}$-consistency with $A_0, \ldots, A_{r-1}$. Since the equality conditions describing the $\mathcal{G}$-consistency of $X_1, \ldots, X_r$ are already linear in those matrices, this construction is largely a technical exercise that expresses those conditions in a way that is compatible with the stacked matrix variable.

Toward that end, we introduce some new notation. For any Hilbert space $\mathcal{H}$ we let $E_{i,j} \in \mathbf{L}(\mathcal{H})$ denote the matrix with all entries equal to zero except for a 1 in the $[i,j]$ entry. It follows that $B[i,j] = \langle E_{i,j}, B \rangle$ for any $B \in \mathbf{L}(\mathcal{H})$.

For any positive integer $n$ we let $\mathcal{H}^{\otimes n}$ denote the Hilbert space with dimension $n \dim(\mathcal{H})$. For any matrices $B_1, \ldots, B_n \in \mathbf{L}(\mathcal{H})$ we let $(B_1, \ldots, B_n) \in \mathbf{L}(\mathcal{H}^{\otimes n})$ denote the block-diagonal matrix

\[
\begin{pmatrix}
B_1 & 0 \\
\vdots & \ddots \\
0 & B_n
\end{pmatrix}.
\]

Letting $\mathcal{R} = (\mathcal{F} \otimes \mathcal{G})^{\otimes r+1}$, we start by describing linear equality constraints that ensure every feasible solution $X \in \mathcal{A} \subset \text{Pos}(\mathcal{R})$ is a block-diagonal matrix of the form $(X_0, \ldots, X_r)$ for some $X_0, \ldots, X_r \in \text{Pos}(\mathcal{F} \otimes \mathcal{G})$. For this task, the “brute force” method of simply forcing every off-block-diagonal entry to zero works just fine. In other words, we require that

\[
\langle E_{R}^{i,j}, X \rangle = 0
\]

for all suitably chosen $i$ and $j$. Using this same brute force technique, we set every entry of $X_0$ to indicate the matrix $\left\lvert 0 \right\rangle \langle 0 \right\rvert$.

We require additional notation before we can proceed to the $\mathcal{G}$-consistency constraints. Define

\[
\Xi_k : \mathbf{L}((\mathcal{F} \otimes \mathcal{G})^{\otimes 2}) \to \mathbf{L}(\mathcal{R})
\]

for all $k \in \{0, \ldots, r-1\}$ so that, given $C = (C_1, C_2) \in \mathbf{L}((\mathcal{F} \otimes \mathcal{G})^{\otimes 2})$, we have

\[
\Xi_k(C) = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & C_1 & \vdots \\
0 & 0 & C_2 \\
0 & \cdots & 0
\end{pmatrix} \leftarrow \text{block } k
\]

\[
\Xi_{k+1}(C) = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & C_1 & \vdots \\
0 & 0 & C_2 \\
0 & \cdots & 0
\end{pmatrix} \leftarrow \text{block } k+1
\]

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That is, \( C \) is embedded into the all-zero matrix so that if \( X = (X_0, \ldots, X_r) \) is block-diagonal then
\[
\langle \Xi_k(C), X \rangle = \langle C, (X_k, X_{k+1}) \rangle.
\]

We also define
\[
T_{i,j} : L(\mathcal{F} \otimes \mathcal{G}) \to L((\mathcal{F} \otimes \mathcal{G})^\oplus 2)
\]
for all \( i, j \in \{1, \ldots, \dim(\mathcal{F})\} \) so that, given \( A \in L(\mathcal{F} \otimes \mathcal{G}) \), we have
\[
T_{i,j}(A) = \begin{pmatrix}
A^* (E^{i,j}_\mathcal{F} \otimes I_\mathcal{G}) A \\
0 & -E^{i,j}_\mathcal{F} \otimes I_\mathcal{G}
\end{pmatrix}.
\]

We now prove the following lemma.

**Lemma 4.4 (\( \mathcal{G} \)-Consistency Constraints).** Let \( X = (X_0, \ldots, X_r) \in \text{Pos}(\mathcal{R}) \) be a block-diagonal matrix with \( X_0 = |0\rangle \langle 0| \). Then the list \( X_1, \ldots, X_r \) is \( \mathcal{G} \)-consistent with \( A_0, \ldots, A_{r-1} \) if and only if \( X \) satisfies
\[
\langle \Xi_k(T_{i,j}(A_k)), X \rangle = 0
\]
for all \( i, j \in \{1, \ldots, \dim(\mathcal{F})\} \) and all \( k \in \{0, \ldots, r-1\} \).

**Proof.** We have
\[
\langle \Xi_k(T_{i,j}(A_k)), X \rangle = \langle T_{i,j}(A_k), (X_k, X_{k+1}) \rangle
= \langle A_k^* (E^{i,j}_\mathcal{F} \otimes I_\mathcal{G}) A_k, X_k \rangle - \langle E^{i,j}_\mathcal{F} \otimes I_\mathcal{G}, X_{k+1} \rangle
= \langle E^{i,j}_\mathcal{F} \otimes I_\mathcal{G}, A_k X_k A_k^* \rangle - \langle E^{i,j}_\mathcal{F} \otimes I_\mathcal{G}, X_{k+1} \rangle
= \langle E^{i,j}_\mathcal{F}, \text{tr}_G(A_k X_k A_k^*) \rangle - \langle E^{i,j}_\mathcal{F}, \text{tr}_G(X_{k+1}) \rangle
= \text{tr}_G(A_k X_k A_k^*)[i, j] - \text{tr}_G(X_{k+1})[i, j].
\]

Of course, \( \text{tr}_G(A_k X_k A_k^*) = \text{tr}_G(X_{k+1}) \) if and only if their entrywise difference is zero, from which the lemma follows.

The \( \mathcal{G} \)-Consistency Constraints (Lemma 4.4) are based upon similar constraints found in Reference [30]. We have thus established a polynomial number of linear equality constraints that characterize \( \mathcal{G} \)-consistency. Our next task is to bound the feasible set \( \mathcal{A} \subset \text{Pos}(\mathcal{R}) \) of matrices that satisfy these constraints.

**Lemma 4.5 (\( \mathcal{G} \)-Consistency Constraint Bound).** Let \( X = (X_0, \ldots, X_r) \in \text{Pos}(\mathcal{R}) \) be a block-diagonal matrix with \( X_0 = |0\rangle \langle 0| \) such that the list \( X_1, \ldots, X_r \) is \( \mathcal{G} \)-consistent with \( A_0, \ldots, A_{r-1} \). Then
\[
\|X\| \leq \max_{i \in \{0, \ldots, r\}} \left\{ \prod_{j=0}^{i-1} \|A_j\|^2 \right\}.
\]
Proof. It is clear that \( \|X_0\| = 1 \) and that \( \|X\| \) is just the maximum of \( \|X_i\| \) over all \( i \in \{0, \ldots, r\} \). Hence, it remains only to bound \( \|X_i\| \) for \( i \geq 1 \). Let \( \mathcal{H} \) be a Hilbert space with \( \text{dim}(\mathcal{H}) = \text{dim}(\mathcal{F} \otimes \mathcal{G}) \). By the Consistency Characterization (Lemma 4.2), there exist \( U_1, \ldots, U_i \in \text{U}(\mathcal{G} \otimes \mathcal{H}) \) such that
\[
\|AU_iA_{i-1} \cdots A_1U_1A_0|0_{\mathcal{F} \otimes \mathcal{G} \otimes \mathcal{H}}\| ^2 = \langle A^* A_i, X_i \rangle \quad \forall \ A \in \mathcal{L}(\mathcal{F} \otimes \mathcal{G}).
\]
In particular,
\[
\|IU_iA_{i-1} \cdots A_1U_1A_0|0\| ^2 = \langle I^* I, X_i \rangle = \text{tr}(X_i) \geq \|X_i\|
\]
where the final inequality follows from the fact that \( X_i \) is positive semidefinite. That
\[
\|IU_iA_{i-1} \cdots A_1U_1A_0|0\| ^2 \leq \prod_{j=0}^{i-1} \|A_j\|^2
\]
follows from the fact that \( U_1, \ldots, U_i \) are unitary.

We have now developed the tools needed to prove the following theorem.

**Theorem 4.6.** \( \text{opt} \) can be solved in time polynomial in the bit length of the input data.

*Proof.* The proof is by reduction to \( \text{sdp} \). Given inputs \( A_0, \ldots, A_r \) and \( \varepsilon \) to \( \text{opt} \), we construct inputs to \( \text{sdp} \) as follows:

- The error parameter \( \varepsilon \) is passed unchanged from \( \text{opt} \) to \( \text{sdp} \).
- The objective matrix \( H \in \mathcal{H}(\mathcal{R}) \) is the block-diagonal matrix \( (0, \ldots, 0, A_r^*A_r) \).
- The linear equality constraints are the \( \mathcal{G} \)-Consistency Constraints (Lemma 4.4).
- The bound \( b \) for all feasible solutions is given by the \( \mathcal{G} \)-Consistency Constraint Bound (Lemma 4.5). Note that if \( A_0, \ldots, A_{r-1} \) are unitary then we have \( b = 1 \).
- The initial feasible solution \( X_{\text{init}} \in \mathcal{A} \) can be taken to be the block-diagonal matrix \( (X_0, \ldots, X_r) \) where \( X_{i+1} = A_iX_iA_i^* \) for every \( i \in \{0, \ldots, r-1\} \) with \( X_0 = |0\rangle \langle 0| \). This feasible solution corresponds to a prover who always acts trivially upon his qubits.

\( \square \)

### 4.2 Some Upper Bounds

In this section we use our polynomial-time solution to \( \text{opt} \) (Theorem 4.6) to prove the upper bounds \( \text{QIP} \subseteq \text{EXP} \) and \( \text{QRG} \subseteq \text{NEXP} \). Indeed, these containments are special cases of stronger results proven in this section.

Several details must be considered before we can formalize these containments. For example, it is prudent to discuss numerical error introduced by finite-precision approximations of continuous quantities. In the case of quantum refereed games, we also require a tractable bound on the number of qubits used by the provers.
4.2.1 Roundoff Error

Let \( L \in \text{QIP} \) and let \( V(x) = (V_0, \ldots, V_r) \) be a verifier witnessing this fact. In Section 4.1.1 we pointed out that \( L \) can be decided by solving \( \text{opt} \) with input matrices \( V_0, \ldots, V_{r-1}, \Pi_{\text{accept}} V_r \) and a small enough accuracy parameter \( \varepsilon \). However, it is often the case that the unitary matrices \( V_0, \ldots, V_r \in U(M \otimes \mathcal{V}) \) associated with the verifier’s quantum circuits contain entries that are complicated algebraic expressions involving irrational numbers. Because \( \text{opt} \) was defined to accept input matrices whose entries are expressed in binary notation, it follows that the best we can do is approximate \( V_0, \ldots, V_r \) with finite-precision matrices \( \tilde{V}_0, \ldots, \tilde{V}_r \in L(M \otimes \mathcal{V}) \).

Our intuition tells us that, by choosing a suitable level of precision with which to express \( \tilde{V}_0, \ldots, \tilde{V}_r \), the induced verifier \( \tilde{V}(x) \) will always have reasonable completeness error and soundness error. Moreover, we expect that matrices \( \tilde{V}_0, \ldots, \tilde{V}_r \) with the required level of precision can be computed efficiently, so that \( L \) can still be decided in exponential time.

Indeed, these intuitions are correct. This subsection is dedicated to arguing that accurate enough approximations can be computed efficiently. For simplicity, we restrict our discussion to quantum interactive proof systems, but much of the discussion in this subsection transfers to quantum refereed games without complication.

In Section 1.3.3 we stipulated that all quantum circuits in this thesis are composed of quantum gates chosen from some finite universal set. We take it as given that the unitary matrices associated with the quantum gates in this universal set can all be computed so that each entry is accurate to \( t \) bits of precision in time polynomial in the bit length of \( t \). Of course, the unitary matrix associated with any single quantum gate is readily extended to a larger Hilbert space by taking the Kronecker product with the identity matrix as usual.

Since the verifier’s quantum circuits are generated uniformly in polynomial time, it follows that each of the \( r(|x|) \) circuits is composed of at most \( g(|x|) \) quantum gates for some \( r, g \in \text{poly} \). For each \( k \in \{0, \ldots, r\} \) and \( l \in \{1, \ldots, g\} \) let \( U_{k,l} \in U(M \otimes \mathcal{V}) \) be the unitary matrix associated with the \( l \)th gate in the verifier’s \( k \)th quantum circuit, extending to \( M \otimes \mathcal{V} \) so that \( V_k = U_{k,g} \cdots U_{k,1} \).

Next, suppose that \( \tilde{U}_{k,l} \) is an approximation of \( U_{k,l} \) such that each entry of \( \tilde{U}_{k,l} \) is accurate to \( t \) bits of precision and let \( \tilde{V}_k = \tilde{U}_{k,g} \cdots \tilde{U}_{k,1} \). Finally, let \( P(x) = (P_1, \ldots, P_r) \) be any prover. The probability \( p \) with which \( (V, P)(x) \) accepts \( x \) is precisely

\[
p = \| \Pi_{\text{accept}} V_r P_r V_{r-1} \cdots V_1 P_1 V_0 |0\rangle \|^2
\]

and the probability \( \tilde{p} \) with which \( (\tilde{V}, P)(x) \) accepts \( x \) is precisely

\[
\tilde{p} = \| \Pi_{\text{accept}} \tilde{V}_r P_r \tilde{V}_{r-1} \cdots \tilde{V}_1 P_1 \tilde{V}_0 |0\rangle \|^2.
\]

Our goal is to prove an upper bound on the difference \( |p - \tilde{p}| \) in terms of \( t \).

Toward that end, let \( \delta > 0 \) and let \( A, \tilde{A} \in L(\mathcal{F}) \) be any matrices whose entrywise difference is at most \( \delta \). In other words,

\[
|A[i, j] - \tilde{A}[i, j]| < \delta
\]
for every \( i, j \in \{1, \ldots, \dim(F)\} \). It is not difficult to show that

\[
\| A - \tilde{A} \| < \dim(F)\delta.
\]

The following lemma allows us to deduce the accuracy required of our approximations.

**Lemma 4.7.** Let \( \delta > 0 \) and let \( A_1, \ldots, A_m, \tilde{A}_1, \ldots, \tilde{A}_m \) be any matrices such that the product \( A_m \cdots A_1 \) is defined and, for all \( i \in \{1, \ldots, m\} \), \( \tilde{A}_i \) has the same dimensions as \( A_i \), \( \| A_i - \tilde{A}_i \| < \delta \), and \( \| A_i \|, \| \tilde{A}_i \| \leq 1 \). Then

\[
\left| \| A_m \cdots A_1 \| - \| \tilde{A}_m \cdots \tilde{A}_1 \| \right| < m\delta
\]

and

\[
\left| \| A_m \cdots A_1 \|^2 - \| \tilde{A}_m \cdots \tilde{A}_1 \|^2 \right| < 2m\delta.
\]

**Proof.** We have

\[
\left| \| A_m \cdots A_1 \| - \| \tilde{A}_m \cdots \tilde{A}_1 \| \right| \leq \| A_m \cdots A_1 - \tilde{A}_m \cdots \tilde{A}_1 \|.
\]

By repeated application of the triangle inequality, this quantity is at most

\[
\sum_{i=1}^{m} \| A_m \cdots A_i \tilde{A}_{i-1} \cdots \tilde{A}_1 - A_m \cdots A_{i+1} \tilde{A}_i \cdots \tilde{A}_1 \|
\]

\[
= \sum_{i=1}^{m} \| A_m \cdots A_{i+1}(A_i - \tilde{A}_i) \tilde{A}_{i-1} \cdots \tilde{A}_1 \|
\]

\[
\leq \sum_{i=1}^{m} \| A_m \cdots A_{i+1} \| \| A_i - \tilde{A}_i \| \| \tilde{A}_{i-1} \cdots \tilde{A}_1 \|
\]

\[
\leq \sum_{i=1}^{m} \| A_i - \tilde{A}_i \| < m\delta.
\]

The lemma follows from the fact that

\[
\left| a^2 - b^2 \right| = \left| a - b \right| (a + b) < 2 \left| a - b \right|
\]

whenever \( a \) and \( b \) are real numbers in the interval \([0, 1]\).

Since the verifier’s quantum circuits are generated uniformly in polynomial time, it follows that they each act on at most \( q(|x|) \) qubits for some \( q \in \text{poly} \), so that \( \dim(M \otimes V) = 2^q \). As each entry of \( \tilde{U}_{k,l} \) is accurate to \( t \) bits of precision, we have

\[
\left| U_{k,l}[i,j] - \tilde{U}_{k,l}[i,j] \right| < 2^{-t}
\]

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for every $i, j \in \{1, \ldots, 2^q\}$ and hence
\[ \|U_{k,l} - \tilde{U}_{k,l}\| < 2^{q-t}. \]

It follows from Lemma 4.7 that
\[ |p - \tilde{p}| < (r + 1)g2^{q-t+1}. \]

Hence, we can compute an exponentially close approximation $\tilde{p}$ of $p$ by choosing a suitable $t \in \text{poly}$ and approximating each matrix $U_{k,l}$ to $t(|x|)$ bits of precision. In fact, this result holds even if $r$ and $g$ grow exponentially in $|x|$.

### 4.2.2 An Extension of QIP $\subseteq$ EXP

We are now ready to prove the upper bound $\text{QIP} \subseteq \text{EXP}$. Indeed, we prove that the containment holds under the following relaxations of the definition of QIP:

- The verifier may exchange an exponential number of messages with the prover.
- The verifier’s quantum circuits may contain an exponential number of gates, so long as they still act upon at most a polynomial number of qubits.
- The completeness error and soundness error may be exponentially close to $\frac{1}{2}$ in $|x|$.

We define a strong verifier to be a verifier whose quantum circuits are generated by an exponential-time Turing machine on input $x$, but they act on at most a polynomial number of qubits.

**Theorem 4.8 (An Extension of QIP $\subseteq$ EXP).** Let $c, s : \mathbb{N} \to [0, 1]$ be any polynomial-time computable functions satisfying $1 - c(n) - s(n) > 0$ for all $n \in \mathbb{N}$. Any language $L \subseteq \{0, 1\}^*$ that can be decided by a quantum interactive proof system with a strong verifier having completeness error $c$ and soundness error $s$ is in $\text{EXP}$.

**Proof.** We assume without loss of generality that $c(n), s(n) < \frac{1}{2}$ for all $n \in \mathbb{N}$, as any verifier who can compute $c(|x|)$ and $s(|x|)$ can also bias his final decision to satisfy this condition. Let $\varepsilon = \min \left\{ \frac{1}{2} - c, \frac{1}{2} - s \right\}$. It follows from the fact that $c$ and $s$ are polynomial-time computable that $\varepsilon \in 2^{-\text{poly}}$.

Figure 4.5 describes a deterministic exponential-time algorithm that decides $L$. To see that this algorithm is correct, let $p$ (respectively $\tilde{p}$) denote the maximum probability with which $V(x)$ (respectively $\tilde{V}(x)$) can be made to accept $x$. By our choice of accuracy parameter for OPT we have $|\varpi - \tilde{p}| < \frac{\varepsilon}{2}$ and by Lemma 4.7 we have
\[ |\tilde{p} - p| < 2(r + 1)\frac{\varepsilon}{4(r + 1)} = \frac{\varepsilon}{2}, \]

from which it follows that $|\varpi - p| < \varepsilon$. By definition, if $x \in L$ then $p \geq \frac{1}{2} + \varepsilon$ and hence $\varpi > \frac{1}{2}$. Conversely, if $x \notin L$ then $p \leq \frac{1}{2} - \varepsilon$ and hence $\varpi < \frac{1}{2}$. 

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1. Run the exponential-time Turing machine that generates the verifier’s quantum circuits on input string $x \in \{0, 1\}^*$. Let $V(x) = (V_0, \ldots, V_r)$ denote the unitary matrices associated with these circuits.

2. Compute an approximation $\tilde{V}(x) = (\tilde{V}_0, \ldots, \tilde{V}_r)$ of $V(x)$ satisfying
   \[ \|V_i - \tilde{V}_i\| < \frac{\varepsilon}{4(r + 1)} \]
   for every $i \in \{0, \ldots, r\}$.

3. Solve $\text{opt}$ with input matrices $\tilde{V}_0, \ldots, \tilde{V}_{r-1}, \Pi_{\text{accept}} \tilde{V}_r$ and accuracy parameter $\frac{\varepsilon}{2}$. Let $\varpi$ denote the optimal value indicated by this solution. Accept $x$ if $\varpi > \frac{1}{2}$, otherwise reject $x$.

---

**Figure 4.5:** An exponential-time algorithm for $L \in \text{QIP}$

It remains only to verify that this algorithm runs in exponential time. According to Section 4.2.1, the approximation $\tilde{V}(x)$ in step 2 can be computed in exponential time by choosing a suitable $t \in \text{poly}$ and approximating the unitary matrices associated with each of the verifier’s quantum gates to $t(|x|)$ bits of precision. As the input matrices to $\text{opt}$ in step 3 can be computed in exponential time and have at most exponential dimension, the desired result follows from the fact that $\text{opt}$ admits a polynomial-time solution (Theorem 4.6).

### 4.2.3 Bounding the Number of Prover Qubits

In this subsection we look at extending the proof of $\text{QIP} \subseteq \text{EXP}$ (Theorem 4.8) to provide an upper bound for $\text{QRG}$.

We begin by applying the Consistency Characterization (Lemma 4.2) to quantum refereed games in much the same way as it was applied to quantum interactive proof systems in Corollary 4.3 in Section 4.1.1. In particular, if the yes-prover $Y(x)$ is fixed then the combination of $V(x)$ and $Y(x)$ can be viewed as a new verifier $(V, Y)(x)$ for an ordinary quantum interactive proof system in which the no-prover is the only prover. In this case, the actions of the no-prover can be described by a transcript, just as with quantum interactive proof systems. Such a transcript is illustrated in Figure 4.6. In this case, we use the Consistency Characterization (Lemma 4.2) to convert from a no-prover $N(x)$ to a transcript $\rho_1, \ldots, \rho_r$ and vice versa.

**Corollary 4.9.** Let $c, s : \mathbb{N} \to [0, 1]$, let $L \in \text{QRG}(c, s)$, let $V(x) = (V_0, \ldots, V_r)$ be a verifier witnessing this fact, and let $Y(x) = (Y_1, \ldots, Y_r)$ be a yes-prover. Consider the following
Figure 4.6: Transcript of a two-round quantum refereed game

optimization problem

\[
\begin{align*}
\text{maximize} & \quad \langle Y_r^* V_r^* \Pi_{\text{reject}}^* \Pi_{\text{reject}} V_r Y_r, \rho_r \rangle \\
\text{subject to} & \quad \rho_1, \ldots, \rho_r \in D(\mathcal{Y} \otimes \mathcal{M}_Y \otimes \mathcal{V} \otimes \mathcal{M}_N) \\
& \quad \rho_1, \ldots, \rho_r \text{ consistent with } V_0, V_1 Y_1, \ldots, V_{r-1} Y_{r-1}.
\end{align*}
\] (4.4)

If \( x \in L \) then there exists a yes-prover \( Y(x) \) such that the optimal value of this problem is at most \( c \) and if \( x \notin L \) then for every yes-prover \( Y(x) \) the optimal value of this problem is at least \( 1 - s \).

**Proof.** If \( x \notin L \) then by definition there exist \( N_1, \ldots, N_r \in \mathcal{U}(\mathcal{M}_N \otimes \mathcal{N}) \) such that

\[
\| \Pi_{\text{reject}} V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 \|_2 \geq 1 - s
\]

for every \( Y_1, \ldots, Y_r \in \mathcal{U}(\mathcal{Y} \otimes \mathcal{M}_Y) \). By the Consistency Characterization (Lemma 4.2) there exists a transcript \( \rho_1, \ldots, \rho_r \) that is \( \mathcal{M}_N \)-consistent with \( V_0, V_1 Y_1, \ldots, V_{r-1} Y_{r-1} \) such that

\[
\langle Y_r^* V_r^* \Pi_{\text{reject}}^* \Pi_{\text{reject}} V_r Y_r, \rho_r \rangle = \| \Pi_{\text{reject}} V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 \|_2^2,
\]

from which the second claim of the corollary follows.

Now suppose \( x \in L \), let \( Y(x) = (Y_1, \ldots, Y_r) \) witness this fact, and let \( \rho_1, \ldots, \rho_r \) be any transcript that is \( \mathcal{M}_N \)-consistent with \( V_0, V_1 Y_1, \ldots, V_{r-1} Y_{r-1} \). By the Consistency Characterization (Lemma 4.2) there exists a no-prover \( N(x) = (N_1, \ldots, N_r) \) such that

\[
\langle Y_r^* V_r^* \Pi_{\text{reject}}^* \Pi_{\text{reject}} V_r Y_r, \rho_r \rangle = \| \Pi_{\text{reject}} V_r Y_r N_r V_{r-1} \cdots V_1 Y_1 N_1 V_0 \|_2^2.
\]

By definition, the quantity on the right is at most \( c \). \( \square \)

Corollary 4.9 suggests that any language in \( \text{QRG} \) can be decided by nondeterministically “guessing” the yes-prover \( Y(x) = (Y_1, \ldots, Y_r) \) and solving the optimization problem (4.4).
Given a verifier $V(x)$ and a yes-prover $Y(x)$, this problem can be phrased as an instance of $\text{OPT}$ with input matrices

$$V_0, V_1Y_1, \ldots, V_{r-1}Y_{r-1}, \Pi_{\text{reject}}V_rY_r$$

and a suitably small accuracy parameter $\varepsilon$ that depends only on the completeness error $c$ and soundness error $s$. As $\text{OPT}$ admits a deterministic polynomial-time solution (Theorem 4.6), it is tempting to conclude that any language in $\text{QRG}$ can also be decided in nondeterministic exponential time.

However, we must take care to ensure that the size of the induced instance of $\text{OPT}$ is in fact bounded by an exponential in $|x|$. The input matrices (4.5) act upon the Hilbert space $\mathcal{Y} \otimes \mathcal{M}_{Y \otimes \mathcal{V} \otimes \mathcal{M}_N}$: as the quantum circuits belonging to any prover are unbounded, it is conceivable that the yes-prover uses a superpolynomial number of private qubits. In this case, the dimension of $\mathcal{Y}$ and hence of the input matrices (4.5) to $\text{OPT}$ may be superexponential in $|x|$. In order to achieve the desired upper bound for quantum refereed games, we require a polynomial bound on the number of qubits used by the yes-prover.

As far as quantum interactive proof systems are concerned, it follows from the Consistency Characterization (Lemma 4.2) that the prover’s Hilbert space $\mathcal{P}$ need only satisfy $\dim(\mathcal{P}) \geq \dim(\mathcal{M} \otimes \mathcal{V})$. In other words, any quantum interactive proof system can be simulated by another quantum interactive proof system in which the prover uses no more qubits than the verifier. Unfortunately, this convenient bound is not known to extend to quantum interactions with multiple provers.

Fortunately, there is a looser polynomial bound that does hold for quantum interactions with multiple provers. In particular, the following fact holds (see Kobayashi and Matsumoto [32]):

**Fact 4.10.** In any quantum interaction, the number of qubits required by each of the provers is polynomial in the number of message qubits shared with the verifier and the number of rounds in the interaction.

Hence, if both the number of message qubits and the number of rounds in the interaction are polynomial then any prover in a quantum refereed game can be assumed to use a polynomial number of qubits. In particular, the Hilbert space $\mathcal{Y}$ corresponding to the yes-prover’s private qubits has dimension at most exponential in $|x|$ as desired.

## 4.2.4 An Extension of $\text{QRG} \subseteq \text{NEXP}$

In this subsection we prove the upper bound $\text{QRG} \subseteq \text{NEXP}$. Indeed, we prove that the containment holds under the following relaxations of the definition of $\text{QRG}$:

- The verifier’s quantum circuits may contain an exponential number of gates, so long as they still act upon at most a polynomial number of qubits.
- The completeness error and soundness error may be exponentially close to $\frac{1}{2}$ in $|x|$.
It is interesting to note that the containment \( \text{QRG} \subseteq \text{NEXP} \) is not known to hold for quantum refereed games with a superpolynomial number of rounds. By contrast, we showed in Theorem 4.8 that the containment \( \text{QIP} \subseteq \text{EXP} \) holds even when the number of rounds is exponential. As explained in Section 4.2.3, this strange discrepancy is brought on by the conditions of Fact 4.10.

**Theorem 4.11 (An extension of \( \text{QRG} \subseteq \text{NEXP} \)).** Let \( c, s : \mathbb{N} \rightarrow [0, 1] \) be any polynomial-time computable functions satisfying \( 1 - c(n) - s(n) > 0 \) for all \( n \in \mathbb{N} \) and let \( r \in \text{poly} \). Any language \( L \subseteq \{0, 1\}^* \) that can be decided by a quantum refereed game with a strong \( r \)-round verifier having completeness error \( c \) and soundness error \( s \) is in \( \text{NEXP} \).

**Proof.** As in the proof of Theorem 4.8, we assume without loss of generality that \( c, s < \frac{1}{2} \) and we let \( \varepsilon \in 2^{-\text{poly}} \) be defined by \( \varepsilon = \min \left\{ \frac{1}{2} - c, \frac{1}{2} - s \right\} \).

The nondeterministic step of our solution is to guess the unitary matrices \( Y(x) = (Y_1, \ldots, Y_r) \) belonging to the yes-prover and compute an approximation \( \tilde{Y}(x) = (\tilde{Y}_1, \ldots, \tilde{Y}_r) \) of \( Y(x) \) satisfying

\[
\|Y_i - \tilde{Y}_i\| < \frac{\varepsilon}{4(2r + 1)}
\]

for every \( i \in \{1, \ldots, r\} \). This step can be accomplished in many different ways. For example, any unitary matrix \( U \) is given by \( \exp(iH) \) for some Hermitian matrix \( H \). To guess the entries of \( U \) accurate to \( t \) bits of precision, we first choose a suitable \( t' > t \), guess the entries of \( H \) to \( t' \) bits of precision, and run any stable algorithm that computes the matrix exponential (see, for instance, Golub and Van Loan [20]).

The deterministic step of our solution is to run the algorithm of Figure 4.5 in Theorem 4.8 with the following changes:

- The approximation \( \tilde{V}(x) \) in step 2 must satisfy

\[
\|V_i - \tilde{V}_i\| < \frac{\varepsilon}{4(2r + 1)}
\]

for every \( i \in \{0, \ldots, r\} \).

- In step 3 we solve \( \text{OPT} \) with input matrices \( \tilde{V}_0, \tilde{V}_1\tilde{Y}_1, \ldots, \tilde{V}_{r-1}\tilde{Y}_{r-1}, \Pi_{\text{reject}}\tilde{V}_r\tilde{Y}_r \) and we reject \( x \) if \( \varpi > \frac{1}{2} \), otherwise we accept \( x \).

To see that this algorithm is correct, let \( p \) denote the maximum probability with which \( V(x) \) can be made to reject \( x \) given yes-prover \( Y(x) \). That \( |\varpi - p| < \varepsilon \) follows just as in Theorem 4.8. By definition, if \( x \in L \) then there exists a yes-prover \( Y(x) \) such that \( p \leq \frac{1}{2} - \varepsilon \) and hence \( \varpi < \frac{1}{2} \). Conversely, if \( x \notin L \) then for all yes-provers we have \( p \geq \frac{1}{2} + \varepsilon \) and hence \( \varpi > \frac{1}{2} \).

That this nondeterministic algorithm runs in exponential time follows as in the proof of Theorem 4.8. \( \square \)
4.3 The Ellipsoid Method and Short Quantum Games

In this section we prove \( \text{SQG} \subseteq \text{EXP} \). Indeed, this containment is a special case of a stronger result proven in this section.

Given a verifier for a short quantum game, we construct a convex set of matrices that is nonempty if and only if there exists a winning yes-prover for that game. The desired result is achieved by providing an algorithm that determines in exponential time whether or not this set is empty.

4.3.1 The Set of Winning Yes-Provers

The key fact that we exploit in order to put \( \text{SQG} \) inside \( \text{EXP} \) is that the no-prover does not become involved in any short quantum game until the verifier has finished exchanging messages with the yes-prover. Because of this fact, the actions of the yes-prover prior to the no-prover’s involvement can be completely described by a transcript, just as with quantum interactive proof systems. As the verifier exchanges only one message with the yes-prover, the transcript under consideration consists only of the state \( \rho_1 \in D(M_Y \otimes V \otimes M_N \otimes N) \) illustrated in Figure 4.7.

We begin by applying the Consistency Characterization (Lemma 4.2) to short quantum games in a way that identifies the set of winning yes-provers \( Y(x) = (Y_1) \) with a set of winning transcripts \( \rho_1 \).

**Corollary 4.12.** Let \( c, s : \mathbb{N} \to [0, 1] \) satisfy \( 1 - c(n) - s(n) > 0 \) for all \( n \in \mathbb{N} \), let \( L \in \text{SQG}(c, s) \), and let \( V(x) = (V_0, V_1, V_2) \) be a verifier witnessing this fact. Define \( \text{Win}(V, c) \) to be the set of all \( \rho_1 \in D(M_Y \otimes V \otimes M_N \otimes N) \) such that \( \rho_1 \) is \( M_Y \)-consistent with \( V_0 \) and

\[
\langle V_1^* N_1^* V_2^* \Pi_{\text{reject}}^* \Pi_{\text{reject}} V_2 N_1 V_1, \rho_1 \rangle \leq c \quad \forall N_1 \in U(M_N \otimes N).
\]

Then \( \text{Win}(V, c) \) is nonempty if and only if \( x \in L \).
Proof. If \( x \in L \) then by definition there exists \( Y_1 \in U(\mathcal{V} \otimes \mathcal{M}_Y) \) such that
\[
\|\Pi_{\text{reject}} V_2 N_1 Y_1 V_0 \|_2^2 \leq c
\]
for every \( N_1 \in U(\mathcal{M}_N \otimes \mathcal{N}) \). By the Consistency Characterization (Lemma 4.2) there exists a transcript \( \rho_1 \) that is \( \mathcal{M}_Y \)-consistent with \( V_0 \) such that
\[
\langle V_1^* N_1^* V_2 \Pi_{\text{reject}} \Pi_{\text{reject}} V_2 N_1 Y_1, \rho_1 \rangle = \|\Pi_{\text{reject}} V_2 N_1 Y_1 V_0 \|_2^2,
\]
from which it follows that \( \rho_1 \in \text{Win}(V, c) \).

If \( x \not\in L \) then let \( N(x) = (N_1) \) witness this fact and let \( \rho_1 \) be any transcript that is \( \mathcal{M}_Y \)-consistent with \( V_0 \). By the Consistency Characterization (Lemma 4.2) there exists a yes-prover \( Y(x) = (Y_1) \) such that
\[
\langle V_1^* N_1^* V_2 \Pi_{\text{reject}} \Pi_{\text{reject}} V_2 N_1 Y_1, \rho_1 \rangle = \|\Pi_{\text{reject}} V_2 N_1 Y_1 V_0 \|_2^2.
\]
By definition, the quantity on the right is at least \( 1 - s > c \), from which it follows that \( \rho_1 \not\in \text{Win}(V, c) \).

As Corollary 4.12 suggests, any language \( L \in \text{SQG} \) can be decided by an algorithm that decides the emptiness of the set \( \text{Win}(V, c) \). But how can the emptiness of this set be decided efficiently?

In order to answer that question, we point out that once a yes-prover has been fixed the short quantum game essentially becomes a two-message quantum interactive proof system in which the no-prover is the only prover. As demonstrated in Section 4.1, such an interaction can be decided by solving OPT. Therefore, if we are given a candidate transcript \( \rho_1 \) for some yes-prover \( Y(x) \) then we can use our solution to OPT to decide whether there exists a no-prover that wins against \( Y(x) \). If we find that no such no-prover exists then we can safely conclude that \( \rho_1 \in \text{Win}(V, c) \) and hence \( x \in L \).

Unfortunately, if we find that there does exist a no-prover that wins against \( Y(x) \) then we cannot immediately conclude that \( x \not\in L \) because it might also be the case that \( x \in L \) but \( Y(x) \) is a bad yes-prover who does not properly witness this fact.

However, all is not lost: using our solution to OPT it is easy to recover the unitary matrix \( N_1 \in U(\mathcal{M}_N \otimes \mathcal{N}) \) belonging to a no-prover who wins against \( Y(x) \). This unitary matrix satisfies
\[
\langle V_1^* N_1^* V_2 \Pi_{\text{reject}} \Pi_{\text{reject}} V_2 N_1 Y_1, \rho_1 \rangle > c,
\]
but by the definition of \( \text{Win}(V, c) \) we have
\[
\langle V_1^* N_1^* V_2 \Pi_{\text{reject}} \Pi_{\text{reject}} V_2 N_1 V_1, \rho \rangle \leq c \quad \forall \rho \in \text{Win}(V, c)
\]
and hence \( N_1 \) is in some sense a witness to the fact that \( \rho_1 \not\in \text{Win}(V, c) \). In particular, it follows from these inequalities that the matrix \( V_1^* N_1^* V_2 \Pi_{\text{reject}} \Pi_{\text{reject}} V_2 N_1 V_1 \) and the scalar \( c \) define a hyperplane that separates \( \rho_1 \) from \( \text{Win}(V, c) \).

To summarize the ideas presented thus far, we have that if a given density matrix \( \rho_1 \) is an element of \( \text{Win}(V, c) \) then this fact can be verified efficiently by solving OPT. Otherwise, if \( \rho_1 \not\in \text{Win}(V, c) \) then we can use our solution to OPT to construct a hyperplane separating \( \rho_1 \) from \( \text{Win}(V, c) \). Later we will see that these two abilities can be used to efficiently decide the emptiness of \( \text{Win}(V, c) \) via convex feasibility methods such as the ellipsoid method.
4.3.2 Double Quantum Interactive Proof Systems

Before we formalize the ideas presented in Section 4.3.1, it is instructive to note that those ideas apply to a slightly larger subclass of quantum refereed games. Suppose, for example, that $\rho$ is the state of the verifier’s qubits after exchanging not just one, but several messages with the yes-prover. Presumably, the same method can still be used to determine whether $\rho$ indicates a winning yes-prover. The only complication is that it must be possible for the yes-prover to somehow get the verifier’s qubits into the state $\rho$. Fortunately, the Consistency Characterization (Lemma 4.2) tells us precisely when this task is possible.

Of course, once $\rho$ is given and the yes-prover is fixed there is no reason to restrict the induced quantum interactive proof system to only two messages—our solution to OPT will easily handle the case in which the verifier exchanges many messages with the no-prover. The only complication here is how to generate a separating hyperplane using the output of OPT.

With that extension in mind, consider a short quantum game in which the verifier exchanges not just one message with each prover, but $r_1$ messages with the yes-prover followed by $r_2$ messages with the no-prover before making his decision. One can think of a quantum refereed game of this strange form as two consecutive quantum interactive proof systems—one with the yes-prover, then one with the no-prover. Hence, we give the name double quantum interactive proof system to quantum refereed games that obey this protocol and we let $\text{DQIP}(c, s)$ denote the complexity class of languages that have double quantum interactive proof systems with completeness error $c$ and soundness error $s$.

Like short quantum games, it is still the case with double quantum interactive proof systems that the no-prover does not become involved until the verifier has finished exchanging messages with the yes-prover. Hence, it is still the case that actions of the yes-prover can be completely described by a transcript. This time, however, the transcript consists of several states $\rho_1, \ldots, \rho_{r_1}$ instead of just a single state.

We apply the Consistency Characterization (Lemma 4.2) to double quantum interactive proof systems in much the same way as it was applied to short quantum games in Corollary 4.12. That is, we identify a set of winning yes-provers $Y(x) = (Y_1, \ldots, Y_{r_1})$ with a set of winning transcripts $\rho_1, \ldots, \rho_{r_1}$.

**Corollary 4.13.** Let $c, s : \mathbb{N} \to [0, 1]$ satisfy $1 - c(n) - s(n) > 0$ for all $n \in \mathbb{N}$, let $L \in \text{DQIP}(c, s)$, and let $V(x) = (V_0, \ldots, V_{r_1}, W_1, \ldots, W_{r_2})$ be a $(r_1 + r_2)$-round verifier witnessing this fact. Define $\text{Win}(V, c)$ to be the set of all transcripts $\rho_1, \ldots, \rho_{r_1} \in D(M_Y \otimes V \otimes M_N \otimes N)$ such that $\rho_1, \ldots, \rho_{r_1}$ is $M_Y$-consistent with $V_0, \ldots, V_{r_1-1}$ and

$$\langle D^* D, \rho_{r_1} \rangle \leq c$$

for all $D$ of the form

$$D = \Pi_{\text{reject}} W_{r_2} N_{r_2} W_{r_2-1} \cdots W_1 N_1 V_{r_1}$$

(4.6)

for some unitary matrices $N_1, \ldots, N_{r_2} \in U(M_N \otimes N)$. Then $\text{Win}(V, c)$ is nonempty if and only if $x \in L$. 55
Proof. If \( x \in L \) then by definition there exists \( Y_1, \ldots, Y_r \in U(Y \otimes \mathcal{M}_Y) \) such that
\[
\| DY_{r_1}V_{r_1-1} \cdots V_1 Y_1 V_0 | 0 \rangle \|_2 \leq c
\]
for every \( D \) of the form (4.6). By the Consistency Characterization (Lemma 4.2) there exists a transcript \( \rho_1, \ldots, \rho_r \) that is \( \mathcal{M}_Y \)-consistent with \( V_0, \ldots, V_{r-1} \) such that
\[
\langle D^* D, \rho_r \rangle = \| DY_{r_1}V_{r_1-1} \cdots V_1 Y_1 V_0 | 0 \rangle \|_2^2,
\]
from which it follows that the transcript \( \rho_1, \ldots, \rho_r \) is an element of \( \textbf{Win}(V, c) \).

If \( x \notin L \) then let \( N(x) = (N_1, \ldots, N_r) \) witness this fact, let
\[
D = \Pi_{\text{reject}} W_{r_2} N_{r_2} W_{r_2-1} \cdots W_1 N_1 V_{r_1},
\]
and let \( \rho_1, \ldots, \rho_r \) be any transcript that is \( \mathcal{M}_Y \)-consistent with \( V_0, \ldots, V_{r-1} \). By the Consistency Characterization (Lemma 4.2) there exists a yes-prover \( Y(x) = (Y_1, \ldots, Y_r) \) such that
\[
\langle D^* D, \rho_r \rangle = \| DY_{r_1}V_{r_1-1} \cdots V_1 Y_1 V_0 | 0 \rangle \|_2^2.
\]
It follows from our choice of \( D \) that the quantity on the right is at least \( 1-s > c \) and hence the transcript \( \rho_1, \ldots, \rho_r \) is not an element of \( \textbf{Win}(V, c) \). \( \square \)

Just like Corollary 4.12 Corollary 4.13 suggests that any language with a double quantum interactive proof system can be decided by ascertaining the emptiness of \( \textbf{Win}(V, c) \). Our intention is to solve this problem by extending the ideas of Section 4.3.1 to double quantum interactive proof systems. That is, given a transcript \( \rho_1, \ldots, \rho_r \) for some yes-prover \( Y(x) \), we check to see if there exists a no-prover who wins against \( Y(x) \) by solving a certain instance of \( \text{opt} \). If such a no-prover exists then we use our solution to \( \text{opt} \) to construct a hyperplane separating the transcript \( \rho_1, \ldots, \rho_r \) from the set \( \textbf{Win}(V, c) \) of winning transcripts. Given the ability to construct separating hyperplanes, we will see that the emptiness of \( \textbf{Win}(V, c) \) can be determined via the ellipsoid method.

4.3.3 The Ellipsoid Method

The problem of ascertaining the emptiness of a convex set is sometimes called convex feasibility. This problem is a special case of the convex programming problem, wherein the task is to maximize a given convex function over some convex set of feasible solutions. Convex programming is a generalization of both the linear and semidefinite programming problems mentioned in Section 4.1.2. In the case of convex feasibility, no function is given and the goal is only to determine whether or not the set of feasible solutions is empty.

The ellipsoid method is an iterative procedure that can often be used to solve the convex feasibility problem in polynomial time. Most of our discussion concerning this method is based upon material found in the book by Grötschel, Lovász, and Schrijver [21]. The ellipsoid method has a rich history that we do not survey here. The interested reader is referred to the aforementioned book for a concise account of this history up until 1988.

The method can be described informally as follows. Given a set \( A \subset \mathbb{R}^n \) such that
• \( A \) is bounded and convex; and
• if \( A \) is nonempty then \( A \) is a full-dimensional subset of \( \mathbb{R}^n \),

we wish to decide whether \( A \) is empty. One iteration of the ellipsoid method consists of the generation of a candidate element \( x \in \mathbb{R}^n \). If \( x \) is found to belong to \( A \) then the algorithm terminates, having found a certification that \( A \) is nonempty. Otherwise, the convexity of \( A \) implies that there exists a hyperplane that separates \( x \) from \( A \). This hyperplane is used by the ellipsoid method to generate a refined candidate \( x' \in \mathbb{R}^n \) during the next iteration. If an appropriate number of iterations pass without finding a vector in \( A \) then the ellipsoid method terminates with the conclusion that \( A \) must be empty.

The ellipsoid method is an oracle algorithm in the sense that it does not produce the separating hyperplanes used to refine candidates. Often, producing such a hyperplane is a complicated task that depends heavily upon the definition of \( A \). An algorithm that computes a separating hyperplane for a given candidate in this manner is called a separation oracle. The ellipsoid method guarantees that, given a polynomial-time separation oracle for \( A \), the emptiness of \( A \) can be deduced in polynomial time.

The details of the ellipsoid method are many and a proper discussion of those details would be tedious. In lieu of such a discussion, we cite an amusing analogy found in Reference [21, page 73] that effectively conveys an intuition of how the ellipsoid method works:

Recall the well-known method of catching a lion in the Sahara. It works as follows. Fence in the Sahara, and split it into two parts; check which part does not contain the lion, fence the other part in, and continue. After a finite number of steps we will have caught the lion (if there was any) because the fenced-in zone will be so small that the lion cannot move anymore. Or we realize that the fenced-in zone is so small that it cannot contain any lion.

In this analogy, the Sahara is the vector space \( \mathbb{R}^n \) and the lion is the bounded, convex, and full-dimensional set \( A \). The ellipsoid method specifies how to “fence in” some subset of \( \mathbb{R}^n \) and the separation oracle serves to split the fenced-in area and check which side contains the lion.

Based on this analogy, the necessity of the requirement that \( A \) be bounded and full-dimensional becomes clear. If the lion were unbounded then it would be impossible to fence him in. On the other hand, if the lion were not full-dimensional then he would have zero length along one axis. We could conceivably continue fencing him in along that axis \( ad infinitum \) and he would still have room to move within the fenced-in area.

Fortunately, the requirement that \( A \) be full-dimensional can often be dropped. For example, the case in which \( A \) is a full-dimensional subset of some “simple polyhedron” \( P \subset \mathbb{R}^n \) can also be handled by the ellipsoid method as described in Reference [21, Chapter 6]. Moreover, any separation oracle for \( A \) may assume without loss of generality that the input vector \( x \in \mathbb{R}^n \) is also an element of the polyhedron \( P \).
4.3.4 The Set of Winning Yes-Provers Revisited

As of now, the set \( \text{Win}(V, c) \) is defined only loosely as a set of winning transcripts for a double quantum interactive proof system. In this subsection, we provide a more precise definition of this set and show that it meets the criteria set out by the ellipsoid method.

Before we proceed, we remind the reader of some of the notation used in Section 4.1.3. In particular, \( \mathcal{H}^\oplus_n \) denotes the Hilbert space with dimension \( n \dim(\mathcal{H}) \) and \((B_1, \ldots, B_n) \in L(\mathcal{H}^\oplus_n)\) denotes the block-diagonal matrix whose blocks are the matrices \( B_1, \ldots, B_n \in L(\mathcal{H}) \).

Let \( L \subseteq \{0, 1\}^* \) be a language that has a double quantum interactive proof system with completeness error \( c \) and let \( V(x) = (V_0, \ldots, V_{r_1}, W_1, \ldots, W_{r_2}) \) be a \((r_1 + r_2)\)-round verifier witnessing this fact. We define

\[
\text{Win}(V, c) \subset H \left( (\mathcal{M}_Y \otimes \mathcal{V} \otimes \mathcal{M}_N \otimes \mathcal{N})^\oplus_{r_1+1} \right)
\]

to be the set of all block-diagonal positive semidefinite matrices \((X_0, \ldots, X_{r_1})\) such that \( X_0 = |0\rangle\langle 0| \) and the list \( X_1, \ldots, X_{r_1} \) is \( \mathcal{M}_Y \)-consistent with \( V_0, \ldots, V_{r_1-1} \) and

\[
\langle D^* D, X_{r_1} \rangle \leq c
\]

for all \( D \) of the form \((4.6)\) in Corollary 4.13.

An advantage of defining \( \text{Win}(V, c) \) in this manner is that we can leverage the results of Section 4.1 to show that \( \text{Win}(V, c) \) is a proper candidate for use with the ellipsoid method. For example, the following lemma is a straightforward consequence of the work in that section.

**Lemma 4.14.** \( \text{Win}(V, c) \) is bounded and convex.

**Proof.** Let \( \lambda \in [0, 1] \) and let \( X, Z \in \text{Win}(V, c) \) with \( X = (|0\rangle\langle 0|, X_1, \ldots, X_{r_1}) \) and \( Z = (|0\rangle\langle 0|, Z_1, \ldots, Z_{r_1}) \). That \( \lambda X + (1 - \lambda) Z \) is block-diagonal with first block equal to \(|0\rangle\langle 0|\) follows immediately. Moreover, we know that \( X \) and \( Z \) satisfy the \( \mathcal{M}_Y \)-Consistency Constraints (Lemma 4.4). Letting \((A, \alpha)\) be any one of these constraints, we have

\[
\langle A, \lambda X + (1 - \lambda) Z \rangle = \lambda \langle A, X \rangle + (1 - \lambda) \langle A, Z \rangle = \lambda \alpha + (1 - \lambda) \alpha = \alpha,
\]

from which it follows that \( \lambda X + (1 + \lambda) Z \) also satisfies the \( \mathcal{M}_Y \)-Consistency Constraints (Lemma 4.4). By similar reasoning, any matrix \( D \) for which \( \langle D^* D, X_{r_1} \rangle \leq c \) and \( \langle D^* D, Z_{r_1} \rangle \leq c \) also satisfies

\[
\langle D^* D, \lambda X_{r_1} + (1 - \lambda) Z_{r_1} \rangle \leq c,
\]

completing the proof that \( \text{Win}(V, c) \) is convex.

That \( \text{Win}(V, c) \) is bounded follows from the \( \mathcal{M}_Y \)-Consistency Constraint Bound (Lemma 4.5), which tells us that any element of \( \text{Win}(V, c) \) has spectral norm at most 1.

According to the discussion in Section 4.3.3, if we are to use the ellipsoid method to decide the emptiness of \( \text{Win}(V, c) \) then it is necessary that the set be polynomial-time isomorphic
to a full-dimensional subset of some “simple polyhedron” in \( \mathbb{R}^n \). We now argue that such an isomorphism exists.

For our purposes, a \textit{simple polyhedron} in \( \mathbb{R}^n \) is any set \( \mathcal{P} \) defined explicitly as an intersection of at most \( p(n) \) halfspaces for some fixed \( p \in \text{poly} \). For example, the \( \mathcal{M}_Y \)-Consistency Constraints define a simple polyhedron \( \mathcal{C} \) in the vector space of \( n \times n \) Hermitian matrices, which is readily identified with \( \mathbb{R}^{n^2} \) as mentioned in Section 3.2.1. Hence, it is suffices to prove the following lemma.

\textbf{Lemma 4.15.} For every \( c' > c \), \( \text{Win}(V, c') \) is a full-dimensional subset of \( \mathcal{C} \) if it is nonempty.

\textit{Proof.} By definition, every element \( X \in \text{Win}(V, c') \):

- is block-diagonal with first block equal to \( |0\rangle\langle 0| \);
- satisfies the \( \mathcal{M}_Y \)-Consistency Constraints;
- is positive semidefinite; and
- satisfies \( \langle D^*D, X \rangle \leq c < c' \) for all appropriately chosen \( D \).

The first two restrictions are precisely the definition of the simple polyhedron \( \mathcal{C} \), from which it follows that \( \text{Win}(V, c') \subseteq \mathcal{C} \). As the positive semidefinite matrices are a full-dimensional subset of the Hermitian matrices, it follows that the third restriction does not decrease the dimension of \( \text{Win}(V, c') \).

For the final restriction, a simple continuity argument shows that if \( X \) satisfies \( \langle D^*D, X \rangle < c' \) for some \( D \) then so must all \( X' \) in some neighbourhood of \( X \). Hence, this final restriction also does not decrease the dimension of \( \text{Win}(V, c') \).

Lemma 4.15 tells us that if the verifier \( V(x) \) has completeness error strictly smaller than \( c \) then \( \text{Win}(V, c) \) is a full-dimensional subset of the simple polyhedron \( \mathcal{C} \). As \( \text{Win}(V, c) \) is also bounded and convex, it follows that the ellipsoid method can decide its emptiness if provided with an efficient separation oracle.

\subsection*{4.3.5 A Separation Oracle}

We have seen that the set \( \text{Win}(V, c) \) qualifies for use with the ellipsoid method provided that the verifier \( V \) has completeness error strictly smaller than \( c \), but it remains to show that there exists an efficient separation oracle for that set. Recall from Section 4.3.3 that a separation oracle for \( \text{Win}(V, c) \) is a computational problem that takes as input a candidate Hermitian matrix \( X \) and outputs either (i) an assertion that \( X \in \text{Win}(V, c) \); or (ii) a hyperplane separating \( X \) from \( \text{Win}(V, c) \). In this subsection we formalize the statement of that problem and we provide a polynomial-time solution.

The separation oracle we seek for \( \text{Win}(V, c) \) solves the problem \( \text{sep}(V, c) \) defined in Figure 4.8 (compare with Reference 21, Theorem 3.2.1). As with SDP and OPT, we assume that the real and imaginary parts of all input numbers (including \( c \) and the matrices belonging to \( V \)) are represented in binary notation.
Problem. \( \text{sep}(V, c) \).

Input. A block-diagonal Hermitian matrix \( X \in \mathbb{H}((M_Y \otimes V \otimes M_N \otimes N)^{\oplus r_1+1}) \) satisfying the \( M_Y \)-Consistency Constraints (Lemma 4.4) with first block equal to \( |0\rangle\langle 0| \) and an accuracy parameter \( \varepsilon > 0 \).

Output. One of the following:

1. A block-diagonal Hermitian matrix \( H \in \mathbb{H}((M_Y \otimes V \otimes M_N \otimes N)^{\oplus r_1+1}) \) with \( \|H\| = 1 \) such that \( \langle H, Z \rangle < \langle H, X \rangle + \varepsilon \) for every \( Z \in \text{Win}(V, c) \).

2. An assertion that if \( \text{Win}(V, c) \) is nonempty then there exists \( X' \in \text{Win}(V, c) \) with \( \|X - X'\| < \varepsilon \).

Figure 4.8: Definition of \( \text{sep}(V, c) \)

As per the discussion in Sections 4.3.3 and 4.3.4, we assume without loss of generality that the input matrix \( X \) to our separation oracle already satisfies the \( M_Y \)-Consistency Constraints (Lemma 4.4), as these constraints define a simple polyhedron \( \mathcal{C} \) of which \( \text{Win}(V, c) \) is a full-dimensional subset.

It is instructive to note that \( \text{sep}(V, c) \) implements a weak separation oracle in the following sense. Output case 1 in Figure 4.8 allows us to reject a “good” yes-prover if it is close to a “bad” yes prover. Conversely, output case 2 implies that we may accept a “bad” yes-prover so long as it is close to a “good” yes prover. We will soon see that leeway afforded to us by the bounded-error requirement for quantum refereed games permits this convenient relaxation. This bounded-error requirement will also permit us the necessary assumption that the verifier’s completeness error be strictly less than \( c \).

The remainder of this subsection is devoted to proving the following theorem.

**Theorem 4.16.** \( \text{sep}(V, c) \) can be solved in time polynomial in the bit lengths of \( (V, c) \) and the input data.

**Proof.** Figure 4.9 describes a polynomial-time algorithm for \( \text{sep}(V, c) \). We now verify the correctness of that algorithm.

If step 2 is reached then \( X = (|0\rangle\langle 0|, X_1, \ldots, X_{r_1}) \) must be positive semidefinite where the list \( X_1, \ldots, X_{r_1} \) is \( M_Y \)-consistent with \( V_0, \ldots, V_{r_1-1} \). Existence of the unitary matrices \( Y_1, \ldots, Y_{r_1} \) then follows from the Consistency Characterization (Lemma 4.2). The matrices \( X_1, \ldots, X_{r_1} \) represent a transcript for the yes-prover defined by \( Y(x) = (Y_1, \ldots, Y_{r_1}) \).

Suppose first that the halting condition is reached in step 3. We prove that every no-prover loses to \( Y(x) \). Let \( N(x) = N_1, \ldots, N_{r_2} \) be any no-prover and let \( p \) (respectively \( \tilde{p} \)) denote the probability with which \( N(x) \) convinces \( V(x) \) to reject \( x \) given \( Y(x) \) (respectively...
1. If $X$ is not positive semidefinite then let $u$ be a unit vector satisfying $u^* X u < 0$. Halt and output case 1, returning $-u u^*$. 

2. Write $X = (\langle 0 | 0 \rangle, X_1, \ldots, X_r)$ and let $Y_1, \ldots, Y_r \in U(\mathcal{Y} \otimes \mathcal{M}_Y)$ be unitary matrices satisfying

\[ \| D Y_{r_1} V_{r_1-1} \cdots V_1 Y_1 V_0 |0\rangle \|^2 = \langle D^* D, X_{r_1} \rangle \]

for all matrices $D$ not acting on \mathcal{Y}. Let $C = Y_{r_1} V_{r_1-1} \cdots V_1 Y_1 V_0$ and compute an approximation $\tilde{C}$ of $C$ satisfying $\| C - \tilde{C} \| < \frac{\varepsilon}{4}$. 

3. Solve \textsc{opt} with input matrices $V_{r_1} \tilde{C}, W_1, \ldots, W_{r_2-1}, \Pi_{\text{reject}} W_{r_2}$ and accuracy parameter $\frac{\varepsilon}{2}$. Let $\varpi$ denote the optimal value indicated by this solution. If $\varpi \leq c$ then halt and output case 2. 

4. Otherwise, \textsc{opt} returned matrices $Z_1, \ldots, Z_{r_2} \in \text{Pos}(\mathcal{Y} \otimes \mathcal{M}_Y \otimes \mathcal{V} \otimes \mathcal{M}_N)$ consistent with $V_{r_1} \tilde{C}, W_1, \ldots, W_{r_2-1}$ such that

\[ \langle W_{r_2}^* \Pi_{\text{reject}} \Pi_{\text{reject}} W_{r_2}, Z_{r_2} \rangle = \varpi > c. \]

Let $N_1, \ldots, N_{r_2} \in U(\mathcal{M}_N \otimes \mathcal{N})$ be unitary matrices satisfying

\[ \left\| \Pi_{\text{reject}} W_{r_2} N_{r_2} W_{r_2-1} \cdots W_1 N_1 V_{r_1} \tilde{C} |0\rangle \right\|^2 = \varpi. \]

Let $D = \Pi_{\text{reject}} W_{r_2} N_{r_2} W_{r_2-1} \cdots W_1 N_1 V_{r_1}$ and compute an approximation $\tilde{D}$ of $D$ satisfying $\| D - \tilde{D} \| < \frac{\varepsilon}{8}$. Halt and output case 1, returning the block-diagonal matrix $(0, \ldots, 0, \tilde{D}^* \tilde{D})$. 

---

Figure 4.9: A polynomial-time algorithm for \textsc{sep}(V, c)
\( \tilde{Y}(x) \) so that

\[
p = \| \Pi_{\text{reject}} W_{r_2} N_{r_2} W_{r_2-1} \cdots W_1 N_1 V_1 C|0\rangle \|^2,
\]

\[
\tilde{p} = \| \Pi_{\text{reject}} W_{r_2} N_{r_2} W_{r_2-1} \cdots W_1 N_1 V_1 \tilde{C}|0\rangle \|^2.
\]

By our choice of accuracy parameter for \( \text{opt} \) we have

\[
\tilde{p} < \varpi + \varepsilon/2 \leq c + \varepsilon/2
\]

and by Lemma 4.7 with \( \| C - \tilde{C} \| < \varepsilon/4 \) we have

\[
| \tilde{p} - p | < 2\varepsilon/4 = \varepsilon/2,
\]

from which it follows that \( p < c + \varepsilon \). As \( N(x) \) was chosen arbitrarily, it follows that the transcript \( X \) is at least “close” to \( \text{Win}(V, c) \) as required by output case 2 in the definition of \( \text{sep}(V, c) \).

Next, suppose that the algorithm proceeds to step 4. We prove that the returned matrix \((0, \ldots, 0, \tilde{D}^\ast \tilde{D})\) indicates an “almost” separating hyperplane as required by output case 1 in the definition of \( \text{sep}(V, c) \).

We start by showing that \( \langle \tilde{D}^\ast \tilde{D}, X_{r_1} \rangle \) is large. Existence of the unitary matrices \( N_1, \ldots, N_{r_2} \) in step 4 follows from the Consistency Characterization (Lemma 4.2). By Lemma 4.7 with \( \| C - \tilde{C} \| < \varepsilon/4 \) and \( \| D - \tilde{D} \| < \varepsilon/8 \) we have

\[
c < \| D\tilde{C}|0\rangle \|^2 < \| \tilde{D}C|0\rangle \|^2 + 3\varepsilon/4 = \langle \tilde{D}^\ast \tilde{D}, X_{r_1} \rangle + 3\varepsilon/4. \tag{4.7}
\]

Choose any transcript \( X' \in \text{Win}(V, c) \) and write \( X' = (|0\rangle\langle 0|, X'_{1_1}, \ldots, X'_{1_{r_1}}) \). We now show that \( \langle \tilde{D}^\ast \tilde{D}, X'_{r_1} \rangle \) is small. Let \( Y'(x) = (Y'_{1_1}, \ldots, Y'_{1_{r_1}}) \) be a yes-prover giving rise to the transcript \( X' \) and let \( C' = Y'_{r_1} V_{r_1-1} \cdots V_1 Y'_1 V_0 \). We have

\[
c \geq \langle D^\ast D, X'_{r_1} \rangle = \| DC'|0\rangle \|^2 > \| \tilde{D}C'|0\rangle \|^2 - \varepsilon/4 = \langle \tilde{D}^\ast \tilde{D}, X'_{r_1} \rangle - \varepsilon/4 \tag{4.8}
\]

Combining (4.7) and (4.8) we obtain

\[
\langle \tilde{D}^\ast \tilde{D}, X'_{r_1} \rangle < \langle \tilde{D}^\ast \tilde{D}, X_{r_1} \rangle + \varepsilon
\]

as required by output case 1 in the definition of \( \text{sep}(V, c) \). The algorithm is therefore correct.

It remains only to verify that this algorithm runs in polynomial time. Using any established method for computing a unitary matrix witnessing the Unitary Equivalence of Purifications (Fact 4.1), we can approximate the matrices \( Y_1, \ldots, Y_{r_1} \) in step 2 and \( N_1, \ldots, N_{r_2} \) in step 4 to \( t \) bits of precision in time polynomial in \( t \) and the dimensions of those matrices. The desired approximations \( \tilde{C} \) and \( \tilde{D} \) can then be computed in polynomial time by choosing an appropriate \( t \) according to Section 4.2.1. As the input matrices to \( \text{opt} \) in step 3 can be computed in polynomial time and have at most polynomial dimension, the desired result follows from the fact that \( \text{opt} \) admits a polynomial-time solution (Theorem 4.6). \( \square \)
1. Run the exponential-time Turing machine that generates the verifier’s quantum circuits on input string \( x \in \{0, 1\}^* \). Let \( V(x) = (V_0, \ldots, V_{r_1}, W_1, \ldots, W_{r_2}) \) denote the unitary matrices associated with these circuits.

2. Compute an approximation \( \tilde{V}(x) = (\tilde{V}_0, \ldots, \tilde{V}_{r_1}, \tilde{W}_1, \ldots, \tilde{W}_{r_2}) \) of \( V(x) \) satisfying
\[
\|V_i - \tilde{V}_i\|, \|W_j - \tilde{W}_j\| < \frac{\varepsilon}{8(r_1 + r_2 + 1)}
\]
for every \( i \in \{0, \ldots, r_1\} \) and every \( j \in \{1, \ldots, r_2\} \).

3. Let \( c' = c + \frac{\varepsilon}{2} \) and use the ellipsoid method with an oracle for \( \text{sep}(\tilde{V}, c') \) with accuracy parameter \( \frac{\varepsilon}{4} \) to decide the emptiness of \( \text{Win}(\tilde{V}, c') \). If it is empty then reject \( x \), otherwise accept \( x \).

Figure 4.10: An exponential-time algorithm for \( L \in \text{DQIP} \)

4.3.6 At Long Last

Now that \( \text{sep}(V, c) \) has been shown to admit a polynomial-time solution (Theorem 4.16), we are finally ready to prove the upper bound \( \text{SQG} \subseteq \text{EXP} \). Indeed, we prove that the containment holds under the following relaxations of the definition of \( \text{SQG} \):

- The verifier may exchange an exponential number of messages with the yes-prover followed by an exponential number of messages with the no-prover.
- The verifier’s quantum circuits may contain an exponential number of gates, so long as they still act upon at most a polynomial number of qubits.
- The completeness error and soundness error may be exponentially close to \( \frac{1}{2} \) in \( |x| \).

Recall the definitions of a strong verifier (Section 4.2.2) and a double quantum interactive proof system (Section 4.3.2). We prove the following theorem.

**Theorem 4.17 (An extension of \( \text{SQG} \subseteq \text{EXP} \)).** Let \( c, s : \mathbb{N} \to [0, 1] \) be any polynomial-time computable functions satisfying \( 1 - c(n) - s(n) > 0 \) for all \( n \in \mathbb{N} \). Any language \( L \subseteq \{0, 1\}^* \) that can be decided by a double quantum interactive proof system with a strong verifier having completeness error \( c \) and soundness error \( s \) is in \( \text{EXP} \).

**Proof.** Let \( \varepsilon = 1 - c - s \). It follows from the fact that \( c \) and \( s \) are polynomial-time computable that \( \varepsilon \in 2^{-\text{poly}} \).

Figure 4.11 describes a deterministic exponential-time algorithm that decides \( L \). To see that this algorithm is correct, let \( \tilde{c} \) and \( \tilde{s} \) denote the completeness error and soundness error
respectively of $\tilde{V}(x)$. It follows from Lemma 4.7 with the approximations of step 2 that
\[ \tilde{c} < c + \frac{x}{4} \text{ and } \tilde{s} < s + \frac{x}{4}. \]
As
\[ 1 - c' - \tilde{s} > 1 - c - s - \frac{3}{4} \varepsilon > 0 \]
it follows from Corollary 4.12 that $\text{Win}(\tilde{V}, c')$ is nonempty if and only if $x \in L$. As $\tilde{V}(x)$ has completeness error strictly smaller than $c'$, it follows from the remarks in Section 4.3.3 that the set $\text{Win}(\tilde{V}, c')$ qualifies for use with the ellipsoid method.

It remains only to verify our choice of accuracy parameter to $\text{sep}(\tilde{V}, c')$. Suppose that the ellipsoid method found that $\text{Win}(\tilde{V}, c')$ is nonempty. Then there must exist a transcript $X$ for which the maximum probability with which $\tilde{V}(x)$ rejects $x$ is smaller than
\[ c' + \frac{x}{4} = c + \frac{3}{4} \varepsilon = 1 - s - \frac{\varepsilon}{4} < 1 - \tilde{s}. \]
As this transcript violates the soundness condition, it must be the case that $x \in L$.

Conversely, suppose the ellipsoid method found that $\text{Win}(\tilde{V}, c')$ is empty. Then for every transcript $X$ the maximum probability with which $\tilde{V}(x)$ rejects $x$ is larger than
\[ c' - \frac{x}{4} = c + \frac{x}{4} > \tilde{c}. \]
As every transcript violates the completeness condition, it must be the case that $x \notin L$.

That our algorithm runs in exponential time follows from the fact that $\text{sep}(\tilde{V}, c')$ admits a polynomial-time solution (Theorem 4.16) and from the polynomiality of the ellipsoid method. \[\square\]
Chapter 5

Conclusion

The work of this thesis initiates the study of quantum refereed games. We chose to focus on short quantum games, proving the containments $\text{QIP} \subseteq \text{SQG}_*$ in Chapter 3 and $\text{SQG} \subseteq \text{EXP}$ in Chapter 4.

Figure 5.1 summarizes some known relationships among the complexity classes considered in this thesis. In that figure, $\text{DQIP}$ denotes the class of languages with double quantum interactive proof systems as defined in Section 4.3.2 and $\text{coDQIP}$ its complement. As with Figure 1.1, a class $A$ contains class $B$ if $A$ can be reached from $B$ by following a path of only upwardly sloped edges.

5.1 Open Problems

We now discuss several open questions relating to the material covered in this thesis.

5.1.1 Parallelization

It is known that any $k$-message classical interactive proof system can be simulated by a two-message interactive proof system for any constant $k \in \mathbb{N}$ [5] [19]. The complexity class corresponding to two-message interactive proof systems is known as $\text{AM}$ and is contained in $\Pi_2^P$, the second level of the polynomial-time hierarchy.

As $\text{IP} = \text{PSPACE}$, it is widely believed that interactive proof systems with a polynomial number of messages are strictly more powerful than $k$-message interactive proof systems. In contrast, we mentioned in Section 2.2.1 that any quantum interactive proof system can be simulated by a three-message quantum interactive proof system [31].

One can also ask whether a similar parallelization result holds for refereed games. In the classical case, we mentioned in Section 1.2.5 that one-round refereed games characterize $\text{PSPACE}$ and that many-round refereed games characterize $\text{EXP}$ [13]. However, little is known about the power of refereed games intermediate between these two extremes. For example, games with a constant number of rounds may correspond to $\text{PSPACE}$, $\text{EXP}$, or some complexity class between the two.
Even less is known in the quantum case. For example, it is unclear how to solve \textsc{close-images} with a short quantum game if the verifier is not permitted to process the yes-prover’s message before sending a message to the no-prover (Section 3.3). Does this ability separate one-round quantum refereed games from short quantum games?

### 5.1.2 Parallel Repetition

Suppose we wish to reduce the error of a given interactive protocol without increasing the number of messages in that protocol. In Section 3.4.1 we described an approach to this problem called \textit{parallel repetition}. Essentially, the idea is to run many copies of the interaction in parallel and accept or reject based upon a vote of the outcomes of the individual repetitions. The hope is that Chernoff bounds can be used to prove that the error of the repeated game decreases exponentially in the number of repetitions. Of course, we must take into account the fact that the provers need not cooperate with the verifier by treating each repetition independently, and therein lies the rub.

Although parallel repetition has been successfully applied to single- and multi-prover classical interactive proof systems (see, for instance, Raz [37]), this problem has not been completely solved in the quantum setting. It is known that parallel repetition followed by a unanimous vote of the outcomes works to reduce the soundness error for three-message quantum interactive proof systems with zero completeness error [31]. We extended that result in this thesis to obtain a partial robustness result for short quantum games (Theorem 66).
However, several questions remain unanswered. Does parallel repetition work in the quantum setting if it is followed by a majority vote of the outcomes instead of a unanimous vote? Is it even possible to improve the error of $k$-round quantum refereed games for $k \geq 2$ without increasing the number of rounds?

### 5.1.3 Insight into QIP and SQG

The cumulative results of this thesis can be viewed as wedging several complexity classes between QIP and EXP—in particular,

$$\text{QIP} \subseteq \text{SQG}* \subseteq \text{SQG} \subseteq \text{DQIP} \subseteq \text{EXP}.$$ 

In a sense, it seems as though QIP is buried deeply inside EXP. Can we prove $\text{QIP} = \text{PSPACE}$?

It is clear that DQIP contains both QIP and coQIP. Does SQG also contain coQIP? Are either of QIP or SQG closed under complement?

### 5.1.4 Do Quantum Refereed Games Characterize EXP?

In this thesis we proved that $\text{QRG} \subseteq \text{NEXP}$ (Theorem 4.11). It follows immediately from the fact that $\text{QRG}$ is closed under complement that $\text{QRG} \subseteq \text{coNEXP}$. Combined with the fact that $\text{EXP} \subseteq \text{QRG}$, we have

$$\text{EXP} \subseteq \text{QRG} \subseteq \text{NEXP} \cap \text{coNEXP}.$$ 

In the realm of polynomial-time computation, problems known to be in $\text{NP} \cap \text{coNP}$ yet not known to be in $\text{P}$ are rare and often the subject of intense study. In many cases, a problem with this property is later discovered to lay in $\text{P}$ and the accompanying proof of this fact can sometimes offer new insights in complexity theory. Popular examples of this trend include the linear programming problem [26] and the primality testing problem [1]. Based upon this historical precedent and upon recent unpublished work by the author, we make the following conjecture:

**Conjecture 5.1.** $\text{QRG} = \text{EXP}$.

Aside from offering a rare quantum characterization of a classical complexity class, such a collapse would also imply that classical refereed games are polynomially equivalent in power to quantum refereed games. If true, this equivalence would be a powerful negative example of a case in which the use of quantum information offers no advantage over the use of classical information.
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