Compactification of the Action of a Point-Map on the Palm Probability of a Point Process

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Abstract

A compatible point-shift \( f \) maps, in a translation invariant way, each point of a stationary point process \( \Phi \) to some point of \( \Phi \). It is fully determined by its associated point-map, \( g^f \), which gives the image of the origin by \( f \). The initial question of this paper is whether there exist probability measures which are left invariant by the translation of \( -g^f \). The point-map probabilities of \( \Phi \) are defined from the action of the semigroup of point-map translations on the space of Palm probabilities, and more precisely from the compactification of the orbits of this semigroup action. If the point-map probability is uniquely defined, and if it satisfies certain continuity properties, it then provides a solution to the initial question. Point-map probabilities are shown to be a strict generalization of Palm probabilities: when the considered point-shift \( f \) is bijective, the point-map probability of \( \Phi \) boils down to the Palm probability of \( \Phi \). When it is not bijective, there exist cases where the point-map probability of \( \Phi \) is absolutely continuous with respect to its Palm probability, but there also exist cases where it is singular with respect to the latter. A criterion of existence of the point-map probabilities of a stationary point process is also provided. The results are illustrated by a few examples.

Key words: Point process, Stationarity, Palm probability, Point-shift, Point-map, Allocation rule, Vague topology, Mass transport principle.

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Introduction

A point-shift is a mapping which is defined on all discrete subsets $\phi$ of $\mathbb{R}^d$ and maps each point $x \in \phi$ to some point $y \in \phi$; i.e. if $f$ is a point-shift, for all discrete $\phi \subset \mathbb{R}^d$ and all $x \in \phi$,

$$f_\phi(x) := f(\phi, x) \in \phi.$$ 

A point-shift is compatible with the translations of $\mathbb{R}^d$ or simply compatible if

$$\forall t \in \mathbb{R}^d, \quad f_{\phi+t}(x + t) = f_\phi(x) + t.$$ 

A translation invariant point-shift $f$ is fully determined by its point-map $g^f = f(0)$. The point-shift $f$ is called bijective on the point process $\Phi$ if, for almost all realizations $\phi$ of the point process, $f_\phi$ is bijective.

The Palm probability of translation invariant point process $\Phi$ is often intuitively described as the law of $\Phi$ conditionally on the presence of a point at the origin. This intuitive definition was formalized by Ryll-Nardzewski [13] based on the Matthes definition of Palm probabilities (see e.g. [3]). This is the so called local interpretation of the latter. The presence of a point at the origin makes the Palm law of $\Phi$ singular with respect to (w.r.t.) the translation invariant law of $\Phi$.

This paper is focused on the point-map probability (or the $g^f$ probability) of $\Phi$, which can be intuitively described as the law of $\Phi$ conditionally on the presence of a point with $f$-pre-images of all orders at the origin.

The first aim of this paper is to make this definition rigorous. The proposed construction is based on dynamical system theory. The action of the semigroup of translations by $-g^f$ on probability distributions on counting measures having a point at the origin is considered; the $g^f$ probabilities of $\Phi$ are then defined as the $\omega$-limits of the orbit of this semigroup action on the Palm distribution of $\Phi$ (Definition 2.3). As the space of probability distributions on counting measures is not compact, the existence of $g^f$ probabilities of $\Phi$ is not granted. A necessary and sufficient conditions for the existence of $g^f$ probabilities is given in Lemma 2.4. Uniqueness is not granted either. An instance of construction of the $g^f$ probabilities of Poisson point processes where one has existence and uniqueness is given in Theorem 2.15.

It is shown in Section 2 that, when they exist, point-map probabilities generalize Palm probabilities in sense described below. Say there is evaporation when the image of $\Phi$ by $f^n$ tends to a point process with 0 intensity.
for \( n \) tending to infinity. When there is no evaporation, the \( g_f \) probabilities of \( \Phi \) are just the Palm law of \( \Phi \) w.r.t. certain translation invariant thinnings of \( \Phi \) and they are then absolutely continuous w.r.t. the Palm probability \( \mathcal{P}_0 \) of \( \Phi \); in particular, if \( f \) is bijective, then the \( g_f \) probability of \( \Phi \) exists, is uniquely defined, and coincides with \( \mathcal{P}_0 \). However, in the evaporation case, the \( g_f \) probabilities of \( \Phi \) do not admit a representation of this type and they are singular w.r.t. \( \mathcal{P}_0 \) (Theorem 2.8).

It is also shown in Theorem 2.10 that, under appropriate continuity properties on \( g_f \), a certain mixture of the \( g_f \) probabilities of \( \Phi \) is left invariant by the shift of \(-g_f\). This generalizes Mecke’s point stationarity theorem which states that if \( f \) is bijective and if \( \Phi \) is distributed according to \( \mathcal{P}_0 \), then \( \Phi - g_f \) is also distributed according to \( \mathcal{P}_0 \).

Section 1 contains the basic definitions and notation used in the paper. Section 2 gathers the results and proofs. Several examples are discussed in Section 3.

1 Preliminaries and Notation

**General Notation** Each measurable mapping \( h : (X, \mathcal{X}) \to (X', \mathcal{X}') \) between two measurable spaces induces a measurable mapping \( h_* : M(X) \to M(X') \), where \( M(X) \) is the set of all measures on \( X \): if \( \mu \) is a measure on \( (X, \mathcal{X}) \), \( h_* \mu \) is the measure on \( (X', \mathcal{X}') \) defined by

\[
h_* \mu(A) := (h_* \mu)(A) = \mu(h^{-1} A).
\]  

(1.1)

Note that if \( \mu \) is a probability measure, \( g_\mu \) is also a probability measure.

**Point Processes** Let \( \mathcal{N} = \mathcal{N}(\mathbb{R}^d) \) be the space of all locally finite counting measures (not necessarily simple) on \( \mathbb{R}^d \). One can identify each element of \( \mathcal{N} \) with the associated multi-subset of \( \mathbb{R}^d \). The notation \( \phi \) will be used to denote either the measure or the related multi-set. Let \( \mathcal{N} \) be the Borel \( \sigma \)-field with respect to the vague topology on the space of counting measures (see Subsection A.1 in appendix for more on this subject). The measurable space \((\mathcal{N}, \mathcal{N})\) is the *canonical space* of point processes.

The support of a counting measure \( \phi \) is the same set as the multi-set related to \( \phi \), but without the multiplicities, and it is denoted by \( \bar{\phi} \). The set of all counting measure supports is denoted by \( \overline{\mathcal{N}} \), i.e. \( \overline{\mathcal{N}} \) is the set of all *simple* counting measures. \( \mathcal{N} \) naturally induces a \( \sigma \)-field \( \mathcal{N} \) on \( \overline{\mathcal{N}} \).
Let $N^0$ (respectively, $\bar{N}^0$) denote the set of all elements of $N$ (respectively, $\bar{N}$) which contain the origin, i.e., for all $\phi \in N^0$ (respectively, $\phi \in \bar{N}^0$), one has $0 \in \phi$.

A point process is a couple $(\Phi, \mathbb{P})$ where $\mathbb{P}$ is a probability measure on a measurable space $(\Omega, \mathcal{F})$ and $\Phi$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathbb{N}, \mathcal{N})$. If $(\Omega, \mathcal{F}) = (\mathbb{N}, \mathcal{N})$ and $\Phi$ is the identity on $\mathbb{N}$, the point process is defined on the canonical space. Calligraphic letters $\mathcal{P}, \mathcal{Q}, \ldots$ (Blackboard bold letters $\mathbb{P}, \mathbb{Q}, \ldots$) will be used for probability measures defined on the canonical space. The canonical version of a point process $(\Phi, \mathbb{P})$ is the point process $(\text{id}, \Phi^* \mathbb{P})$ which is defined on the canonical space. Here $\text{id}$ denotes the identity on $\mathbb{N}$.

Stationary Point Process It is assumed that $(\Omega, \mathcal{F})$ is equipped with a measurable flow $\theta_t : \Omega \to \Omega$, with $t$ ranging over $\mathbb{R}^d$. A point process is compatible if

$$\Phi(\theta_t \omega, B - t) = \Phi(\omega, B), \quad \forall \omega \in \Omega, t \in \mathbb{R}^d, B \in \mathcal{B},$$

where by convention, $\Phi(\omega, B) := (\Phi(\omega))(B)$. Here $\mathcal{B}$ denotes the Borel $\sigma$-algebra on $\mathbb{R}^d$. The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is $\theta_t$-invariant if $(\theta_t)_* \mathbb{P} = \mathbb{P}$. If, for all $t \in \mathbb{R}^d$, $\mathbb{P}$ is $\theta_t$-invariant, it is called stationary. If $\mathbb{P}$ is a stationary measure and $\Phi$ is compatible, then $(\Phi, \mathbb{P})$ is stationary in the usual sense. Below, a stationary point process is a point process $(\Phi, \mathbb{P})$ such that $\Phi$ is compatible and $\mathbb{P}$ stationary.

When the point process is simple and stationary with a non-degenerate intensity, its Palm probability is a classical object in the literature. For a general (i.e. not necessarily simple) point process, its Palm probability is defined by

$$\mathbb{P}_\Phi[F] := \frac{1}{\lambda|B|} \int \sum_{t \in \Phi(\omega) \cap \tilde{B}} 1\{\theta_t \omega \in F\} \mathbb{P}[d\omega], \quad (1.2)$$

for all $F \in \mathcal{F}$, where $\tilde{B}$ is a multi-set which is equal to $B$ as a set but all its elements are of infinity multiplicity and the summations are on multi-sets; so, in the summations, each point $t \in \Phi(\omega) \cap \tilde{B}$ occurs a number of times equal to its multiplicity in $\Phi(\omega)$.

Whenever $(\Phi, \mathbb{P})$ is a point process, $\mathcal{P}$ denotes its distribution; i.e. $\mathcal{P} = \Phi_* \mathbb{P}$. If in addition, $\Phi$ is stationary and with a non degenerate intensity, the distribution of its Palm version is denoted by $\mathcal{P}_0$; i.e. $\mathcal{P}_0 = \Phi_* \mathbb{P}_\Phi$. $(\Phi, \mathbb{P}_\Phi)$
will be called the *Palm version* of \((\Phi, \mathbb{P})\). In the canonical setup, the Palm version of \((\Phi, \mathbb{P}) = (id, \mathcal{P})\) is \((\Phi, \mathcal{P}_0) = (id, \mathcal{P}_0)\).

**Compatible Point-Shifts**  A simple point-shift on \(\mathbb{N}\) is a measurable function \(f\) which is defined for all pairs \((\phi, x)\), where \(\phi \in \mathbb{N}\) and \(x \in \phi\), and satisfies the relation \(f(\phi, x) \in \phi\). If \(f\) is a simple point-shift, it extends naturally to a point-shift \(f\) defined for all pairs \((\phi, x)\) of the form \(\phi \in \mathbb{N}\) and \(x \in \phi\) by \(f(\phi, x) = f(\phi, x)\), and by carrying the multiplicity of \(x\) in \(\phi\) to the point \(f(\phi, x)\). If \(y \in \phi\) has more than one pre-image, the multiplicity of \(y\) in the image is equal to the sum of the multiplicities of all points which are mapped to \(y\).

In order to define compatible point-shifts, it is convenient to use the notion of point-map. A measurable function \(g\) from the set \(\mathbb{N}^0\) to \(\mathbb{R}^d\) is called a simple point-map if for all \(\phi \in \mathbb{N}^0\), \(g(\phi)\) belongs to \(\phi\). All simple point-maps \(g\) extend naturally to a point-map \(g\) from \(\mathbb{N}^0\) to points of \(\mathbb{R}^d\) with multiplicity defined by \(g(\phi) := g(\phi)\) and by carrying the multiplicity of \(x\) in \(\phi\) to the point \(g(\phi)\). It is easy to verify that the measurability of \(g\) implies that of \(g\).

If \(g\) is a point-map, the associated compatible point-shift, \(f_g\), is a function which is defined for all pairs \((\phi, x)\), where \(\phi \in \mathbb{N}\) and \(x \in \phi\), by \(f_g(\phi, x) = g(\theta_x \phi) + x\). It is assumed that \(f_g(\phi, x)\) has the same multiplicity as the origin in \(\theta_x \phi\). The point-shift \(f = f_g\) is compatible because

\[
 f(\theta_t(\phi, x)) = f(\theta_t \phi, x - t) = g(\theta_{x-t} \theta_t \phi) + x - t \\
g(\theta_{x-t} \phi) + x - t = f(\phi, x - t) = f(\phi, x). \tag{1.3}
\]

In the rest of this article, point-shift always means compatible point-shift and for the point-shift \(f\), we denote its point-map by \(g^f\).

It is useful to associate with the point-shift \(f(\phi, x)\) an operator \(f_\phi\) acting on the subsets of the support of \(\phi\), and such that for all \(x\) in this support, \(f_\phi \{x\} = \{f_\phi(x)\}\). In operator notation, for all multi-subsets \(\psi\) of \(\mathbb{R}^d\), such that \(\psi \subset \phi\), \(f_\phi \psi\) is the multi-set

\[
 f_\phi \psi = \{f_\phi(x); x \in \psi\}.
\]

As mentioned above, if more than one point of \(\psi\) are mapped to the same point under \(f_\phi\), then the multiplicity of the image point is the sum of the multiplicities of all of its pre-images.
To simplify notation, \( f_\phi \) will be used as a functional on either \( \mathbb{R}^d \) or counting measures; i.e. if \( y \) is a point of \( \mathbb{R}^d \), so is \( f_\phi(y) \), whereas if it is a counting measure, so is \( f_\phi(y) \). For example, if \( \psi \subset \phi \), then \( f_\phi \psi \) is a counting measure and \((f_\phi \psi)(x)\) is the multiplicity of \( x \) in \( f_\phi \psi \).

The \( n \)-th image of \( \phi \) under \( f \) is inductively defined as

\[
f^n_\phi \phi = f_\phi(f^{n-1}_\phi \phi), \quad n \geq 1,
\]
with the convention \( f^0_\phi \phi = \phi \). Let \( f^n \) be the point-shift defined by \( f^n(\phi, x) := f^n_\phi(x) \). The compatibility of \( f \) implies that of \( f^n \). For all \( m, n \in \mathbb{N} \), \( f^n \circ f^m = f^{m+n} \). The point-map associated with \( f^n \), which will be denoted by \( g^n \), satisfies

\[
g^n(\phi) = g^{n-1}(\phi) + g(\theta_{g^{n-1}(\phi)} \phi), \quad n \geq 1,
\]
with \( g^0(\phi) = 0 \). Then for all \( m, n \in \mathbb{N} \), on \( \mathbb{N}^0 \),

\[
\theta_{g^n} \circ \theta_{g^m} = \theta_{g^{m+n}}. \tag{1.4}
\]

Several examples of point-shifts are presented in Section 3.

**Mecke’s Point Stationarity Theorem** One of the motivations of this work is to extend the following proposition proved by J. Mecke in [11].

**Theorem 1.1** (Point Stationarity). Let \((\Phi, \mathbb{P})\) is a simple stationary point process and let \( f \) be a point-shift which is bijective on \( \Phi, \mathbb{P} \)-almost all realizations of the point process. Let \( g \) denote the point-map of \( f \). Then the Palm version of the point-process is invariant under the action of \( \theta_g \); i.e.

\[
\mathbb{P}_\phi = (\theta_{g(\phi)})_* \mathbb{P}_\phi, \tag{1.5}
\]
with \( \theta_{g(\phi)} \) seen as a map on \( \Omega \) defined by

\[
(\theta_{g(\phi)})(\omega) := \theta_{g(\phi(\omega))}\omega = \theta_{f_\phi(\omega)}(0)\omega.
\]

Since \( \mathbb{P}_\phi[\Phi(\{0\}) > 0] = 1 \), \( \theta_{g(\phi)} \) is \( \mathbb{P}_\phi \)-almost surely well defined.

**Semigroup Actions** Let \( X \) be a Hausdorff space. An action of \((\mathbb{N}, +)\) on \( X \) is a collection \( \pi \) of mappings \( \pi_n : X \to X, n \in \mathbb{N} \), such that for all \( x \in X \), and \( m, n \in \mathbb{N} \), \( \pi_m \circ \pi_n(x) = \pi_{m+n}(x) \). When each of the mappings \( \pi_n \) is continuous, \( \pi \) is also often referred to as a discrete time dynamical system.
On a Hausdorff space $X$, one can endow the set $X^X$ with a topology, e.g. that of pointwise convergence. The closure of the action of $\mathbb{N}$ is then the closure $\overline{\Pi}$ of the set $\Pi = \{\pi_n, n \in \mathbb{N}\} \subset X^X$ w.r.t. this topology. A classical instance (see e.g. [6]) is that where the space $X$ is compact, the mappings $\pi_n$ are all continuous, and the topology on $X^X$ is that of pointwise convergence. Then $\overline{\Pi}$ is compact.

Denote the orbit $\{x, \pi(x), \pi_2(x), \cdots\}$ of $x \in X$ by $A_x$. For all $x \in X$, the closure $\overline{A_x}$ of $A_x$ is a closed $\pi$-invariant subset of $X$. If, for all $n$, $\pi_n$ is continuous, then the restriction of $\pi$ to $\overline{A_x}$ defines a semigroup action of $\mathbb{N}$. The compactness of $\overline{A_x}$ is not granted when $X$ is non-compact. When it holds, several important structural properties follow as illustrated by the next lemmas where $X$ is a metric space with distance $d$. Let

$$\omega_x = \{y \in X \text{ s.t. } \exists n_1 < n_2 < \cdots \in \mathbb{N} \text{ with } \pi_{n_i}(x) \to y\}$$

(1.6)

denote the $\omega$-limit set of $x$.

**Lemma 1.2** (Lemma 4.2, p. 134, and p. 166 in [5]). Assume that $\pi_n$ is continuous for all $n$ and that $\overline{A_x}$ is compact. Then, for all neighborhoods $U$ of $\omega_x$, there exists an $N = N(U, x)$ such that $\pi_n(x) \in U$ for all $n \geq N$. Moreover $\omega_x$ is non empty, compact and $\pi$-invariant.

In words, under the compactness and continuity conditions, the orbit is attracted to the $\omega$-limit set.

**Lemma 1.3** (Lemma 2.9, p. 95 in [5]). If $\overline{A_x}$ is compact, then the following property holds: for all $\epsilon > 0$, there exists $N = N(\epsilon, x) \in \mathbb{N}$ such that for all $y \in \overline{A_x}$, the set $\{\pi_n(x), 0 \leq n \leq N\}$ contains a point $z$ such that $d(y, z) \leq \epsilon$. If in addition $\pi_n$ is continuous for all $n$, then the last property is equivalent to the compactness of $\overline{A_x}$.

In words, under the compactness condition, in a long enough interval, the trajectory $\pi_n(x)$ visits a neighborhood of every point of $\overline{A_x}$.

## 2 Results

### 2.1 Semigroup Actions of a Point-shift

Below, $N = N(\mathbb{R}^d)$ and $M^1(N)$ denotes the set of probability measures on $N$. Similar definitions and notation hold when $N$ is replaced by $N^0$. For all point-shifts $f$ on $N$, consider the following actions $\pi = \{\pi_n\}$ of $(N, +)$:
1 $X = \mathbb{N}$, equipped with the vague topology, and $\pi_n(\phi) = f^n_\phi \phi$, $n \in \mathbb{N}$. Here $f^n_\phi$ is seen as a map from $\mathbb{N}$ to itself, so that $f^n_\phi \phi \in \mathbb{N}$.

1* $X = M^1(\mathbb{N})$, equipped with the weak convergence of probability measures on $\mathbb{N}$, and for all $Q \in X$, $n \in \mathbb{N}$, $\pi_n(Q) = (f^n_\phi)^* Q \in M^1(\mathbb{N})$.

2 $X = \mathbb{N}^0$, also equipped with the vague topology, and for all $\phi \in \mathbb{N}^0$ and $n \in \mathbb{N}$, $\pi_n(\phi) = \theta_{g^n(\phi)}(\phi) \in \mathbb{N}^0$.

2* $X = M^1(\mathbb{N}^0)$, equipped with the same topology as $M^1(\mathbb{N})$, and for all $Q \in M^1(\mathbb{N}^0)$ and $n \in \mathbb{N}$, $\pi_n(Q) = (\theta_{g^n(\phi)})^* Q \in M^1(\mathbb{N}^0)$.

**Action 1** The point-shift $f$ is periodic on the stationary point process $(\Phi, \mathbb{P})$ if for $\Phi_* \mathbb{P}$-almost all $\phi$ and for all $x \in \phi$, the action of $f^n_\phi$ is periodic on $x$, namely if there exists integers $p = p(\phi, x)$ and $N = N(\phi, x)$ such that for all $n \geq N$, $f^n_\phi(x) = f^{n+p}_\phi(x)$. The case where $p$ is independent of $\phi$ and $x$ is known as $p$-periodicity. The special case of 1-periodicity is that where the trajectory $f^n(\phi, x)$ is stationary (in the dynamical system sense) after some steps, i.e. such that for all $n > N(\phi, x)$, $f^n(\phi, x) = f^N(\phi, x)$.

The point-shift $f$ has finite orbits on the point process $(\Phi, \mathbb{P})$ if for $\Phi_* \mathbb{P}$-almost all $\phi$, for all $x \in \phi$, $\{f^n(x)\}_{n \in \mathbb{N}}$ has finitely many different points. Since, given the realization of the point process, the point-shift is deterministic, $f$ has finite orbits on a point process if and only if it is periodic on it.

The point process $(\Phi, \mathbb{P})$ will be said to evaporate under the action of the point-shift $f$ if Action 1 converges a.s. to the null measure on $\Phi$, i.e. for $\Phi_* \mathbb{P}$-almost all $\phi$, one has

$$
\overline{f^\infty_\phi}(\phi) := \bigcap_{n=1}^{\infty} f^n_\phi(\phi) = \emptyset.
$$

Consider the following set:

$$
I := \{ \phi \in \mathbb{N}^0, \forall n \in \mathbb{N}, \exists y \in \phi \text{ s.t. } f^n_\phi(y) = 0 \}. \quad (2.1)
$$

**Lemma 2.1.** For all point-shifts $f$ and all stationary point processes $(\Phi, \mathbb{P})$, there is evaporation of $(\Phi, \mathbb{P})$ under the action of $f$ if and only if $\Phi_* \mathbb{P}_\Phi[I] = 0$. 

8
Proof. Let $P = \Phi \ast P$ and $P_0 = \Phi \ast P$. If $m(\phi, x)$ is the indicator of the fact that if $x$ has $f_\phi$-pre-images of all orders, then $m$ is a compatible marking of the point process (see [3]). Therefore if $\Psi$ denotes the sub-point process of the points with mark 1, then $(\Psi, P)$ is a stationary point process and

$$\lambda_\Psi = \lambda_\phi \mathbb{E}_\phi [m(\Phi, 0)] = \lambda_\phi P_0 [\Phi \in I]. \tag{2.2}$$

The evaporation of $(\Phi, P)$ by $f$ means $\Psi$ has zero intensity. According to (2.2) this is equivalent to $P_0 [\Phi \in I] = 0$. □

Action 2* Let $\Phi$ be a stationary point process on $\mathbb{R}^d$ and $f$ be a point-shift. Consider Action 2* when $Q = P_0$, the Palm probability of $\Phi$. It follows from the definition and from (1.2) that, for all $n \geq 1$, for all $G \in \mathcal{N}$ and for all Borel sets $B$ with positive and finite Lebesgue measure,

$$(\theta_g^n)_* P_0 [G] = \frac{1}{\lambda |B|} \int \sum_{t \in \phi \cap \sim B} 1\{\theta_g^n \circ \theta_t (\phi) \in G\} P [d\phi]. \tag{2.3}$$

In what follows, $P_{0}^{g,n}$ is a short notation for the probability on $\mathbb{N}^0$ defined in the last equations. This probability will be referred to as the $n$-th $g$-Palm probability of the point process.

From the mass transport relation [10]:

Lemma 2.2.

$$P_{0}^{g,n} [G] = \frac{1}{\lambda |B|} \int \sum_{t \in (f_\phi^n) \cap \sim B} 1\{\theta_t \phi \in G\} P [d\phi]. \tag{2.4}$$

Hence, $P_{0}^{g,n}$ is also the distribution of $\Phi$ given that the origin is in the $n$-th image process, when considering point multiplicities. It follows from the semigroup property (1.4) that

$$(\theta_g)_* P_{0}^{g,n} = P_{0}^{g,n+1}, \quad \forall n \in \mathbb{N}, \tag{2.5}$$

when letting $P_{0}^{g,0} := P_0$.

Definition 2.3 (Point-Map Probability). Let $g$ be a point-map and let $P$ be a stationary point process with Palm distribution $P_0$. Every element of the $\omega$-limit set of $P_0$ (where limits are w.r.t. the topology of the convergence
in distribution of probability measures on $\mathbb{N}^0$ — cf. Subsection A.1) under the action of $\{(\theta^n,)_n\in\mathbb{N}\}$ will be called a $g$ probability of $\mathcal{P}_0$. In particular, if the limit of the sequence $\{(\theta^n,\mathcal{P}_0)_n\in\mathbb{N}\} = \{\mathcal{P}^{g,n}_0\}_n$ exists, it is called the $g$ probability of $\mathcal{P}_0$ and denoted by $\mathcal{P}^{g}_0$.

This action of $\mathbb{N}$ on $M^1(\mathbb{N}^0)$ can also be viewed as an action of the semigroup $\theta^n$ on $M^1(\mathbb{N}^0)$ hence the title of the paper.

In words, the set of $g$ probabilities of $\mathcal{P}_0$ is the set of all accumulation points of $\mathcal{A}_{\mathcal{P}_0}$ (i.e. the elements of $\mathcal{A}_{\mathcal{P}_0}$ the neighborhoods of which contain infinitely many elements of $\mathcal{A}_{\mathcal{P}_0}$ — see the definitions in Section 1), whereas the closure $\mathcal{A}_{\mathcal{P}_0}$ of the orbit of $\mathcal{P}_0$ is the union of the orbit of $\mathcal{P}_0$ and of the set of all $g$ probabilities of $\mathcal{P}_0$.

The relatively compactness of $\mathcal{A}_{\mathcal{P}_0}$ (and the existence of $g$ probabilities) is not granted in general. The next lemmas give conditions for this relatively compactness to hold. From Lemma 4.5. in [8], one gets:

**Lemma 2.4.** A necessary and sufficient condition for the $\mathcal{A}_{\mathcal{P}_0}$ to be relatively compact in $M^1(\mathbb{N}^0(\mathbb{R}^d))$, is that for all bounded Borel subsets $B$ of $\mathbb{R}^d$,

$$\lim_{r \to \infty} \limsup_{n \to \infty} \mathcal{P}_0^{g,n}[\phi \in \mathbb{N}^0 \text{ s.t. } \phi(B) > r] = 0. \quad (2.6)$$

**Proposition 2.5.** If $f_g$ has finite orbits on the stationary point process $(\Phi, \mathcal{P})$, then the set $\mathcal{A}_{\mathcal{P}_0}$ is relatively compact.

*Proof.* For all bounded Borel subsets $B$ of $\mathbb{R}^d$ and $\phi \in \mathbb{N}^0$, let

$$R_B(\phi) := \max_{n=0}^{\infty} \{(\theta^n_\phi)(B)\}.$$ 

Since $f$ has finite orbits, the RHS is the maximum over finite number of terms and hence $R_B$ is well-defined and finite. Clearly $R_B(\phi) \geq R_B(\theta_\phi \phi)$ and therefore the distribution of the random variable $R_B$ under $\mathcal{P}^{g,n}_0$ is stochastically decreasing w.r.t. $n$. Hence

$$\lim_{r \to \infty} \limsup_{n \to \infty} \mathcal{P}_0^{g,n}[\phi \in \mathbb{N}^0 \text{ s.t. } \phi(B) > r] \\
\leq \lim_{r \to \infty} \limsup_{n \to \infty} \mathcal{P}_0^{g,n}[\phi \in \mathbb{N}^0 \text{ s.t. } R_B(\phi) > r] \\
\leq \lim_{r \to \infty} \mathcal{P}_0[\phi \in \mathbb{N}^0 \text{ s.t. } R_B(\phi) > r] = 0.$$

$\square$
2.2 Point-Map and Palm Probabilities

The following two theorems discuss the relation between Palm probabilities and point-map probabilities. For all compatible point processes \((\Psi, \mathbb{P})\) defined on \(\Omega, \mathcal{F}\) and have a positive intensity, let \(\mathbb{P}_\Psi^0\) denote the Palm probability w.r.t. \(\Psi\) on \(\Omega, \mathcal{F}\).

**Theorem 2.6.** If \(f := f_g\) is 1-periodic on \((\Phi, \mathbb{P})\), then the \(g\) probability \(\mathcal{P}_0^g\) of \(\mathcal{P}_0\) exists and is given by

\[
\mathcal{P}_0^g = \Phi_p \mathbb{P}_\Phi^0 f_{\Phi\Phi}. \tag{2.7}
\]

Furthermore \(\mathcal{P}_0^g\) is absolutely continuous with respect to \(\mathcal{P}_0\), with

\[
\frac{d\mathcal{P}_0^g}{d\mathcal{P}_0} = \frac{(f_{\Phi\Phi}) (\{0\})}{\Phi(\{0\})}. \tag{2.8}
\]

Proof. In the 1-periodic case, for all bounded Borel sets \(B\), \(f_{\Phi\Phi}^n \Phi(B) = f_{\Phi\Phi}^\infty \Phi(B)\) a.s. for \(n\) large enough, so that by letting \(n\) to infinity in (2.4), one gets that for all \(G\),

\[
\exists \lim_n \mathcal{P}_0^{g,n}[G] = \frac{1}{\lambda|B|} \int \sum_{t \in f_{\Phi\Phi}^\infty \Phi(\omega) \cap \bar{B}} 1_{\{\theta_t \phi \in G\}} \mathcal{P}[d\phi]. \tag{2.9}
\]

Since \(f_{\Phi\Phi}^\infty \Phi\) is a stationary point process with the same intensity as the original point process (because of the conservation of intensity), this proves that \(\mathbb{P}_\Phi^g\) is the distribution of \(\Phi\) with respect to the Palm distribution of \(f_{\Phi\Phi}^\infty \Phi\) indeed.

In addition,

\[
\mathbb{P}_{f_{\Phi\Phi}}[F] = \mathbb{E}_{f_{\Phi\Phi}}[1_F] = \frac{1}{\lambda|B|} \int \sum_{t \in f_{\Phi\Phi}^\infty \Phi(\omega) \cap \bar{B}} 1_{\{\theta_t \omega \in F\}} \mathbb{P}[d\omega] = \frac{1}{\lambda|B|} \int \sum_{t \in \Phi(\omega) \cap \bar{B}} f_{\Phi\Phi}^\infty \Phi(\omega, \{t\}) \frac{1_{\{\theta_t \omega \in F\}} \mathbb{P}[d\omega]}{\Phi(\omega, \{t\})} = \frac{1}{\lambda|B|} \int \sum_{t \in \Phi(\omega) \cap \bar{B}} f_{\Phi\Phi}^\infty \Phi(\theta_t \omega, \{0\}) \frac{1_{\{\theta_t \omega \in F\}} \mathbb{P}[d\omega]}{\Phi(\theta_t \omega, \{0\})} = \mathbb{E}_\Phi \left[ \frac{f_{\Phi\Phi}^\infty \Phi(\{0\})}{\Phi(\{0\})} 1_F \right],
\]
where the second equality stems from the fact that $\overline{f^{-1}} \Phi \subset \Phi$. This proves (2.8) when $F = \Phi^{-1}G$. Finally since $f$ is $1$-periodic, $\mathcal{P}_0^g$-almost surely, $g \equiv 0$ which proves that $\mathcal{P}_0^g$ is invariant under the action of $(\theta_g)_*$. □

Note that similar statements hold in the $p$-periodic case. In this case, $f^p$ is $1$-periodic on point processes $\{(\Phi, \mathcal{P}_0^g, n)\}_{n=0}^{p-1}$, and hence there exists at most $p$ point-map probabilities. Details on this fact are omitted.

The next theorem shows that when evaporation takes place, under the same existence conditions and properties as those of Theorem 2.6, the $g$ probability is of a quite different nature compared to what is given in (2.8).

**Proposition 2.7.** If $Q$ is a $g$-probability of the point process $(\Phi, \mathbb{P})$ and $I$ is defined as (2.1), then $Q[I] = 1$.

*Proof.* Since $Q$ is a $g$-probability of the point process, there exists an increasing sequence of integers, namely $\{n_i\}_{i=1}^{\infty}$, such that

$$\lim_{i \to \infty} \mathcal{P}_0^{g, n_i} = Q.$$ 

Note that □

**Theorem 2.8.** If $(\Phi, \mathbb{P})$ evaporates under the action of $f_g$, and if the $g$ probability $\mathcal{P}_0^g$ of $\mathcal{P}_0$ exists and satisfies $\mathcal{P}_0^g = (\theta_g)_* \mathcal{P}_0^g$, then $\mathcal{P}_0^g$ is singular with respect to $\mathcal{P}_0$.

*Proof.* The result is obtained when combining the results of Lemmas 2.9 and 2.1. □

Examples of cases where Theorem 2.6 holds are given in Subsection 3.5. Examples where the assumptions of Theorem 2.8 are satisfied are presented in Subsection 3.3.

### 2.3 Mecke’s Point-Stationarity Revisited

Consider the following point-map invariant measure equation

$$(\theta_g)_* Q = Q, \quad (2.10)$$

where the unknown is $Q \in M^1(\mathbb{N}^0)$. From Mecke’s point stationarity Theorem 1.1, if $f_g$ is bijective, then the Palm probability $\mathcal{P}_0$ of any simple stationary point processes solves (2.10). Equation (2.10) will be referred to as
Mecke’s invariant measure equation. The question is whether one can construct a solution of (2.10) from the Palm probability of a stationary point process when \( f_g \) is not bijective. The bijective case shows that the solution of (2.10) is not unique in general.

Let \( I \) be the set defined in (2.1).

**Lemma 2.9.** If \( Q \) is a solution of (2.10), then \( Q[I] = 1 \). In this case, \( Q \)-almost surely, there exists a bi-infinite path which passes through the origin; i.e. \( \{y_i = y_i(\phi)\}_{i \in \mathbb{Z}} \) such that \( y_0 = 0 \) and \( f_\phi(y_i) = y_{i+1} \).

**Proof.** Let \( M_n := \{\phi \in \mathbb{N}^0; f_\phi^{-n}(0) = \emptyset\} \), where \( f := f_g \). It is sufficient to show that, for all \( n > 0 \), \( Q[M_n] = 0 \). But the invariance of \( Q \) under the action of \( (\theta_g)_* \) gives

\[
Q[M_n] = (\theta_g)_* Q[M_n] = Q[(\theta_g)^{-n} M_n] = Q[\{\phi \in \mathbb{N}^0; f_\phi^{-n}(f_\phi^n(0)) = \emptyset\}] = 0.
\]

The proof of the second statement is an immediate consequence of König’s infinity lemma [9]. \( \square \)

Consider the Cesàro sums

\[
\tilde{P}_0^{g,n} := \frac{1}{n} \sum_{i=0}^{n-1} P_0^{g,i}, \quad n \in \mathbb{N}.
\] (2.11)

When the limit of \( \tilde{P}_0^{g,n} \) when \( n \) tends to infinity exists (w.r.t. the topology of \( M^1(\mathbb{N}^0) \)), let

\[
\tilde{P}_0^g := \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} P_0^{g,i}.
\] (2.12)

In general, \( \tilde{P}_0^g \) is not a \( g \) probability.

**Theorem 2.10.** Assume there exists a subsequence \( \{\tilde{P}_0^{g,n_i}\}_{i=1}^\infty \) which converges to a probability measure \( \tilde{P}_0^g \). If \( (\theta_g)_* \) is continuous at \( \tilde{P}_0^g \), then \( \tilde{P}_0^g \) solves Mecke’s Invariant Measure Equation (2.10).

**Proof.** From (2.5),

\[
(\theta_g)_* \tilde{P}_0^{g,n} - \tilde{P}_0^{g,n} = \frac{1}{n} \left( \sum_{i=0}^{n-1} (\theta_g)_* P_0^{g,i} - \sum_{i=0}^{n-1} P_0^{g,i} \right) = \frac{1}{n} \left( \sum_{i=1}^{n} P_0^{g,i} - \sum_{i=0}^{n-1} P_0^{g,i} \right) = \frac{1}{n} (P_0^{g,n} - P_0) .
\] (2.13)
Therefore, if the subsequence \( \{ \tilde{P}_{0}^{g,n_i} \}_{i=1}^{\infty} \) converges in distribution w.r.t. the vague topology to a probability measure \( \tilde{P}_{0}^{g} \), then (2.13) implies that the sequence \( \{ (\theta_g)_* \tilde{P}_{0}^{g,n_i} \}_{i=1}^{\infty} \) converges to \( \tilde{P}_{0}^{g} \) as well. Now the continuity of \( (\theta_g)_* \) at \( \tilde{P}_{0}^{g} \) implies that \( \{ (\theta_g)_* \tilde{P}_{0}^{g,n_i} \}_{i=1}^{\infty} \) converges to \( (\theta_g)_* \tilde{P}_{0}^{g} \) and therefore \( (\theta_g)_* \tilde{P}_{0}^{g} = \tilde{P}_{0}^{g} \).

Note that when the sequence \( \{ P_{0}^{g,n} \}_{n=1}^{\infty} \) converges to \( P_{0}^{g} \), then \( \{ \tilde{P}_{0}^{g,n} \}_{n=1}^{\infty} \) converges to \( P_{0}^{g} \) too, and hence Theorem 2.10 implies the invariance of the \( g \) probability \( P_{0}^{g} \) under the action of \( (\theta_g)_* \), whenever \( (\theta_g)_* \) has the required continuity.

Note that if instead of \( \{ \tilde{P}_{0}^{g,n} \}_{n=1}^{\infty} \), \( \{ P_{0}^{g,n} \}_{n=1}^{\infty} \) has convergent subsequences with different limits, i.e. if the set of \( g \) probabilities is not a singleton, then none of the \( g \) probabilities satisfies (2.10). However, it follows from Lemma 1.2 that if \( (\theta_g)_* \) is continuous, and if \( \{ P_{0}^{g,n} \}_{n=1}^{\infty} \) is relatively compact, then the set of \( g \) probabilities of \( P_{0} \) is compact, non empty and \( (\theta_g)_* \)-invariant.

The conditions listed in the last theorem are all required. For some non-trivial point-shifts, \( \{ \tilde{P}_{0}^{g,n} \}_{n=1}^{\infty} \) has no convergent subsequence (see Subsection 3.4), whereas for others, \( \{ P_{0}^{g,n} \}_{n=1}^{\infty} \) is convergent, but \( (\theta_g)_* \) is not continuous at the limit and \( P_{0}^{g} \) is not invariant under the action of \( (\theta_g)_* \) (see Subsection 3.1). The use of Cesàro limits is required too as for some point-shifts, \( \{ P_{0}^{g,n} \}_{n=1}^{\infty} \) is not convergent, whereas \( \{ \tilde{P}_{0}^{g,n} \}_{n=1}^{\infty} \) converges to a limit which satisfies (2.10) (see Subsection 3.6).

In case of existence of \( \tilde{P}_{0}^{g} \), Theorem 2.10 gives a sufficient condition for \( \tilde{P}_{0}^{g} \) to solve (2.10); however since \( \tilde{P}_{0}^{g} \) lives in the space of probability measures on counting measures, the verification of the continuity of \( (\theta_g)_* \) at \( \tilde{P}_{0}^{g} \) can be difficult. The following propositions give more handy tools to verify the continuity criterion.

**Proposition 2.11.** If \( \theta_g \) is \( \tilde{P}_{0}^{g} \)-almost surely continuous, then \( (\theta_g)_* \), is continuous at \( \tilde{P}_{0}^{g} \).

**Proof.** The proof is an immediate consequence of Proposition A.6 as the space \( \mathbf{N}(\mathbb{R}^d) \) is a Polish space. \( \square \)

**Proposition 2.12.** If \( g \) is \( \tilde{P}_{0}^{g} \)-almost surely continuous, then \( (\theta_g)_* \), is \( \tilde{P}_{0}^{g} \)-continuous.

**Proof.** One can verify that \( \theta : \mathbb{R}^d \times \mathbf{N} \to \mathbf{N} \) defined by \( \theta(t, \phi) = \theta_t \phi \) is continuous. Also \( h : \mathbf{N}^0 \to \mathbb{R}^d \times \mathbf{N} \) defined by \( h(\phi) = (g(\phi), \phi) \) is continuous.
at continuity points of the point-map of $f$ in $\mathbb{N}^0$. Hence $\theta_g = \theta \circ h$ is continuous at continuity points of $g$. □

The converse of the statement of Proposition 2.12 does not hold in general (see Subsection 3.4). Combining the last propositions and Theorem 2.10 gives:

**Corollary 2.13.** If $g$ is $\tilde{\mathcal{P}}_0^g$-almost surely continuous, then $(\theta_g)_*$ is continuous at $\tilde{\mathcal{P}}_0^g$ and hence, when it exist, $\tilde{\mathcal{P}}_0^g$ solves Mecke’s Invariant Measure Equation (2.10).

In Theorem 2.10 and the last propositions, the continuity of the mapping $(\theta_g)_*$ associated with $f$ is required at some specific point. The overall continuity of $f$, or more accurately the continuity of its point-map is a stronger requirement which does not hold for interesting cases as shown by the following proposition (see Appendix A.2 for a proof).

**Proposition 2.14.** For $d \geq 2$ there is no continuous point-map on the whole $\mathbb{N}^0$ other than the point-map of the identity point-shift; i.e. the point-map which maps all $\phi \in \mathbb{N}^0$ to the origin.

In certain cases, the existence of $\mathcal{P}_0^g$ can be established using the theory of regenerative processes [1]. This method can be used when the point process satisfies the strong Markov property such as Poisson point processes [12].

For the point-map $g$ and the stationary point process $(\Phi, \mathcal{P}) = (\text{id}, \mathcal{P})$, denote by $\{X_n\}_{n=0}^\infty = \{X_n(g, \phi)\}_{n=0}^\infty$ the successive points of the orbit of the origin by $f_g$ under $(\text{id}, \mathcal{P}_0)$: $X_0 = 0$ and for all nonnegative $n$, $X_{n+1}(\phi) = f_g(\phi, X_n(\phi))$, where $\phi$ is the realization of the point process. Using this notation, Lemma 2.2 gives $(\theta_{X_n})_* \mathcal{P}_0 = \mathcal{P}_{0,n}^g$.

Finally, for a counting measure $\phi$ (a point process $\Phi$), denote $\theta_{X_n, \phi} (\theta_{X_n, \Phi})$ by $\phi_n (\Phi_n)$ and by $\phi^r_n (\Phi^r_n)$ the restriction of $\phi_n (\Phi_n)$ to the sphere of radius $r$ around the origin. The following theorem leverages classical results in the theory of regenerative processes.

**Theorem 2.15.** If, for all $r > 0$, there exists a strictly increasing sequence of non-lattice integer-valued random variables $(\eta_i)_{i=1}^\infty$, which may depend on $r$, such that

1. $\{\eta_{i+1} - \eta_i\}_{i=1}^\infty$ is a sequence of i.i.d. random variables with finite mean,
2. The sequence $Y^i_i := (\Phi^r_{\eta^i_1}, \Phi^r_{\eta^i_2}, \ldots, \Phi^r_{\eta^i_{i+1}-1})$ is an i.i.d. sequence and $Y^i_{i+1}$ is independent of $\eta_1, \ldots, \eta_i$, then the $g$ probability $P^g_0$ exists and, for all $h \in L_\infty(P^g_0)$ and $P_0$-almost all $\phi$,

$$
\lim_{k \to \infty} \frac{1}{k} \sum_{n=0}^{k-1} h(\theta^g_n \phi) = \int_{N^0} h(\psi)P^g_0(d\psi). \quad (2.14)
$$

If in addition, for all $n$, $g$ is $P^g_{0,n}$-almost surely continuous, then $P^g_0$ is invariant under the action of $(\theta_g)_s$ and $\theta_g$ is ergodic on $(N^0, N^0, P^g_0)$.

**Proof.** In order to prove the weak convergence of $P^g_{0,n}$ to $P^g_0$, it is sufficient to show the convergence in all balls of integer radius $r$ around the origin. Note that $P^g_{0,n}$ is the distribution of $\Phi_n^r$ and hence, to prove the existence of $P^g_0$, it is sufficient to prove the convergence of the distribution of $\Phi_n^r$ for all $r \in \mathbb{N}$.

Note that the sequence $(\eta_i^i)_{i=1}^{\infty}$ forms a sequence of regenerative times for the configurations in $B_r(0)$. Since $N^0$ is metrizable (c.f. [1] Theorem B.1.2), the distribution of $\Phi_n^r$ converges to a distribution $P^g_{0,r}$ on configurations of points in $B_r(0)$ satisfying

$$
\frac{1}{E_0[\eta_2 - \eta_1]} \mathcal{E}_0 \left[ \sum_{n=0}^{n_{\eta_2} - 1} h(\Phi^r_n) \right] = \int_{N^0} h(\psi)P^g_{0,r}(d\psi). \quad (2.15)
$$

Since the distributions $\{P^g_{0,r}\}_{r=1}^\infty$ are the limits of $\{\Phi_n^r\}_{r=1}^\infty$, they satisfy the consistency condition of Kolmogorov’s extension theorem and therefore there exists a probability distribution $P^g_0$ on $N^0$ having $P^g_{0,r}$ as the distribution of its restriction to $B_r(0)$. This proves the existence of the $g$ probability.

The LHS of (2.15) can be replaced by the ergodic average for $P_0$-almost any $\phi \in N^0$ (c.f. [1] Theorem B.3.1); i.e. for all $r \in \mathbb{N}$,

$$
\lim_{k \to \infty} \frac{1}{k} \sum_{n=0}^{k-1} h(\phi^r_n) = \int_{N^0} h(\psi^r)P^g_{0,r}(d\psi^r) = \int_{N^0} h(\psi^r)P^g_0(d\psi).
$$

Finally $r$ varies in integers and hence the last equation gives (2.14), for $P_0$-almost all $\phi$.

By defining $h$ as the continuity indicator of $g$, the $P^g_{0,n}$-almost sure continuity of $g$ and (2.14) give its $P^g_0$-almost sure continuity and hence that of $(\theta_g)_s$ at $P^g_0$. Therefore $P^g_0$ is invariant under the action of $(\theta_g)_s$. Also ergodicity is clear from regeneration. \qed
The main technical difficulty for using Theorem 2.15 consists in finding an appropriate \((\eta_i)_{i=1}^\infty\) sequence. Proposition 3.1 and Proposition 3.2 leverage the strong Markov property of Poisson point processes to find appropriate \(\eta_i\) and hence prove the existence of the point-mapprobability.

3 Examples

3.1 Strip Point-Shift

The Strip Point-Shift was introduced by Ferrari, Landim and Thorisson [7]. For all points \(x = (x_1, x_2)\) in the plane, let \(St(x)\) denote the half strip \((x_1, \infty) \times [x_2 - 1, x_2 + 1]\). Then \(f_s(x)\) is the left most point of \(St(x)\). It is easy to verify that \(f_s\) is compatible and we denote its point-map by \(g_s\).

The strip point-shift is not well defined when there are more than one left most point in \(St(x)\), nor when there does not exist a point of the point process (other than \(x\)) in \(St(x)\). However it is enough to consider the strip point-shift (and all other point-shifts) on point processes for which the point-shift is almost surely well-defined. Note that such ambiguities can always be removed by fixing, in some translation invariant manner, the choice of the image and by choosing \(f(\phi, x) = x\) in the case of non-existence. By doing so one gets a point-shift defined for all \((\phi, x)\).

Properties

Let \(P_0\) be the Palm distribution of the homogeneous Poisson point process of \(\mathbb{R}^2\). It follows from results in [7] that \(P_0\) evaporates under the action of \(f_s\). It is also shown in Proposition 3.1 below that \(P_0\), admits a unique \(g_s\) probability which satisfies the continuity requirements of Theorem 2.10.

The singularity of the \(g_s\) probability and the Palm probability is illustrated by Figures 1 and 2. Figure 1 gives an (approximate) realization of the Poisson point process under the \(g_s\) probability; the origin is at the center of the figure. The points are marked as follows: a point is black if it has pre-images of all (thousand or more) orders and grey otherwise.

Figure 2 gives a realization of the Poisson point under its Palm probability. The origin is at the center. The points are marked using a grey level proportional to the age of the point (this age is \(k\) if the point has a pre-image of order \(k\) but no pre-image of order \(k + 1\).
On the need of the continuity property in Theorem 2.10  

Consider the setup of Proposition 3.1. The origin is said to admit \( x \in \phi \) as pre-image if \( f_s^n(x) = 0 \) for some \( n \geq 0 \). For all \( \phi \in \mathbb{N}^0 \) where the origin has infinitely many pre-images, change the definition of the point-map \( g_s \) as the closest point on the right half plane which has no other point of the point process in the ball of radius 1 around it. Due to evaporation, this changes the definition of \( f_s \) on a set of measure zero under \( \mathcal{P}_0^{g_s} \), for all \( n \in \mathbb{N} \), and hence, the sequence \( \{ \mathcal{P}_n^{g_s} \}_{n=1}^{\infty} \) is again converging to the same limit as that defined in the proof of Proposition 3.1. But under the action of the new \( g_s \), \( (\theta_{g_s})_* \mathcal{P}_0^{f_s} \) is not equal to \( \mathcal{P}_0^{g_s} \) due to the fact that (i) 0 has infinitely many pre-images \( \mathcal{P}_0^{g_s} \)-a.s. and (ii) there is no point of the point process in the ball of radius 1. This does not agree with the fact that in the right half plane, the distribution of \( \mathcal{P}_0^{g_s} \) is a Poisson point process. Hence, \( \mathcal{P}_0^{g_s} \) is not invariant under the action of \( (\theta_{g_s})_* \).
3.2 Directional Point-Shift

The directional point-shift was introduced in [2]. Fix a unit vector $u$ in $\mathbb{R}^d$. The directional point-map $g_u$ w.r.t. $u$, maps the origin to the nearest point in the half-plane defined by $u$, i.e. for all $\phi \in \mathbb{N}^0$,

$$g_u(\phi) := \arg\min\{|y|; y \in \phi, y \cdot u > 0\},$$  

(3.1)

where the RHS is equal to the point $y$ for which one has the minimum value of $|y|$. We denote the associated point-shift by $f_u$.

The directional point-map with deviation limit $\alpha$, $g_{u,\alpha}$, is similar, but the point $y$ is chosen in the cone with angle $2\alpha$ and central direction $u$ rather than in a half plane; i.e.

$$g_{u,\alpha}(\phi, x) := \arg\min\{|y|; y \in \phi, \frac{y}{|y|} \cdot u > \cos \alpha\}.$$

(3.2)

When $\alpha = \frac{\pi}{2}$ one has $g_{u,\alpha} = g_u$. We denote its point-shift by $f_{u,\alpha}$.
Properties  The homogeneous Poisson point process of $\mathbb{R}^2$ evaporates under the action of this point-shift, and from Proposition 3.2 below, it admits a unique $g$ probability which satisfies the continuity requirements of Theorem 2.10.

3.3  Regeneration Techniques

This subsection is focused on the existence of $\mathcal{P}_0^g$ for Poisson point processes. It is based on Theorem 2.15 and is illustrated by two examples.

Proposition 3.1. If $g_s$ is the strip point-map defined in Example 3.1, and $(\Phi, \mathcal{P})$ is a homogeneous Poisson point process in the plane with distribution $\mathcal{P}$, then the $g_s$ probability exists and is given by (2.14). In addition, for all $n$, $g_s$ is $\mathcal{P}_0^{g_{s,n}}$-almost surely continuous. Therefore the action of $(\theta_{g_s})_*$ preserves $\mathcal{P}_0^{g_s}$ and is ergodic.

Proof. Using the same notation as that of Theorem 2.15, one gets that the random vector $X_1$ depends only on the points of the point process which belong to the rectangle $S_0(\phi) = [0, x_1] \times [-1, 1]$, where $x_1$ is the first coordinate of the left most point of $\phi \cap St(0)$. It is easy to verify that $S_0(\phi)$ is a stopping set (c.f. [12]). If $S_n(\phi)$ is the rectangle which is needed to determine the image of the origin in $\theta X_n \phi$ under the action of $f_s$, then it is clear that

$$U_k = \bigcup_{n=0}^k (S_n + X_n),$$

(3.3)

where $S_n + X_n$ is the translation of the set $S_n$ by the vector $X_n$, is also a stopping set. As a consequence, given $X_0, \ldots, X_n$, the point process on the right half-plane of $X_n$ is distributed as the original Poisson point process. Let

$$p_n = \pi_1(X_{n+1} - X_n),$$

where $\pi_1$ is the projection on the first coordinate. Since $\mathcal{P}_0^{g_{s,n}}$ in the right half-plane is the distribution of a Poisson point process and the sequence $\{p_n\}_{n=1}^{\infty}$ depends only on the configuration of points in the right half-plane, $\{p_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. exponential random variables with parameter $2\lambda$ where $\lambda$ is the intensity of the point process. Also if $\eta_i$ is the integer $n$ such that, for the $i$-th time, $p_n$ is larger than $2r$, then the sequence $\{\eta_i\}_{i=1}^{\infty}$ forms a sequence of regenerative times for configuration of points in $B_r(0)$.
Combining this with the distribution of $p_n$ gives that $\{\eta_i\}_{i=1}^\infty$ satisfies the required conditions in Theorem 2.15.

Finally, the discontinuity points of $g_s$ are included in configurations with either no point in the right half-plane or at least two points on a same vertical line, and both events are of probability zero under $\mathcal{P}_0^{g_s,n}$. Therefore all conditions of Theorem 2.15 are satisfied, which proves the proposition. □

Note that the proof shows that the distribution $\mathcal{P}_0^{g_s}$ on the right half-plane is Poisson with the original intensity.

**Proposition 3.2.** Under the assumptions of Proposition 3.1, if $g_{u,\alpha}$ is the directional point-map defined in Example 3.2 with $\alpha < \pi/2$, then the probability exists and is given by (2.14). In addition, for all $n$, $g_{u,\alpha}$ is $\mathcal{P}_0^{g_{u,\alpha},n}$-almost surely continuous and hence the action of $(\theta_{g_{u,\alpha}})_*$ preserves $\mathcal{P}_0^{g_{u,\alpha}}$ and is ergodic.

**Proof.** The proof is similar to that of Proposition 3.1, but more subtle. Again we use the same notation as that of Theorem 2.15.

If $C_{u,\alpha}$ denotes the cone with angle $2\alpha$ and central direction $u$ and apex origin, $X_1(\phi)$ is the closest point of $C_{u,\alpha} \cap \phi$ to the origin (other than the origin itself). Let $C_{0,\alpha}(\phi)$ be the closed bounded cone consisting of all points of $C_{u,\alpha}(0)$ which are not farther to the origin than $X_1(\phi)$. One may verify that $C_{0,\alpha}$ is a stopping set and that $X_1$ is determined by $C_{0,\alpha}$. If $C_{n,\alpha}(\phi)$ is the closed bounded cone which is needed to determine the image of the origin in $\theta_{X_n,\phi}$ under the action of $f_{u,\alpha}$, then it is easy to verify that

$$U_k = \bigcup_{n=0}^k (C_{n,\alpha} + X_n),$$

is also a stopping set. It is a simple geometric fact that

$$U_{n-1} \cap C_{u,\pi/2-\alpha}(X_n) = \{X_n\},$$

and as a consequence, given $U_{n-1}$, the point process in $C_{u,\pi/2-\alpha} + X_n$ is distributed as the original point process. This fact together with the facts that $U_n$ is a stopping set and $C_{n,\alpha}$ has no point of the point process other than $X_n$ and $X_{n+1}$, gives that, in the $n$-th step, with probability at least $\min\{1, (\pi/2 - \alpha)/(\alpha)\}$, $X_{n+1}$ is in $C_{u,\pi/2-\alpha}(X_n)$. Let $\eta_i$ be the $i$-th time for which $X_{n+1} \in C_{u,\pi/2-\alpha}(X_n)$ and has a distance more than $2r$ from the
The edges of $C^{u,\pi/2}(X_n)$. The Poisson distribution of points in $C^{u,\pi/2}(X_n)$ gives that the random variables $\eta_{i+1} - \eta_i$ are stochastically bounded by an exponential random variable and hence they satisfy all requirements of Theorem 2.15.

The discontinuity points of $g_{u,\alpha}$ are included in those configurations of points with either no point in $C^{u,\alpha}$ or at least two points with the same distance from the origin in $C^{u,\alpha}$. Note that since $U_{n-1}$ is a stopping set and $(C^{u,\alpha} + X_n) \cap U_{n-1}$ has no point of the point process other than $X_n$, $X_{n+1}$ is distributed as in a Poisson point process in $C_n^{u,\alpha} + X_n$ given the fact that some parts contain no point. Therefore since the discontinuities of $g_{u,\alpha}$ are of probability zero under the Poisson distribution, they are of probability zero under all $\mathcal{P}_{g_{u,\alpha}}^{n}$ and hence Theorem 2.15 proves the statements of the proposition. □

The statement of Proposition 3.2 is also true in the case $\alpha = \pi/2$ and can be proved using ideas similar to these in the proof for $\alpha < \pi/2$. However the technical details of the proof in this case may hide the main idea and this case is hence ignored in the proposition.

### 3.4 Condenser and Expander Point-Shift

Assume each point $x \in \phi$ is marked with

$$m_c(x) = \#(\phi \cap B_1(x))$$

(respectively $m_e(x) = \sup\{r > 0 : \phi \cap B_r(x) = \{x\}\}$),

where $B_r(x) = \{y \in \mathbb{R}^2 : ||x - y|| < r\}$. Note that $m_c(x)$ and $m_e(x)$ are always positive. The condenser point-shift (respectively expander point-shift) acts on marked point process as follows: it goes from each point $x \in \phi$ to the closest point $y$ such that $m_c(y) \geq 2m_c(x)$ (respectively $m_e(y) \geq 2m_e(x)$).

It is easy to verify that both point-shifts are compatible and almost surely well-defined on the homogeneous Poisson point process.

**Property** Poisson point processes evaporate under the action of both point-shifts.

**An example where no $g$ probability exists** Assume $(id, \mathcal{P})$ is the Poisson point process with intensity one on $\mathbb{R}^2$ and $f_c$ is the condenser point-shift. Clearly

$$\mathcal{P}_{g_{f_c}}^{n}[\phi(B_1(0)) > 2^n] = 1.$$
Therefore the tightness criterion is not satisfied and thus there is no convergent subsequence of \( \{ P_{0,c,n} \}_{n=1}^{\infty} \).

**On the lack of converse to Proposition 2.12** The expander point-shift provides a case where \( \theta_f \) is continuous \( P_{0}^g \)-almost surely but the point-map of \( f \) is \( P_{0}^g \)-almost surely discontinuous, which proves that the converse of the statement of Proposition 2.12 does not hold in general. Consider the expander point-shift on the homogeneous Poisson point process. One can verify that \( \{ P_{0}^g,n \}_{n=1}^{\infty} \) converges to the probability measure concentrated on the counting measure \( \delta_0 \) with a single point at the origin. In this example one has \( \theta_f \) is \( P_{0}^g \)-a.s. continuous. This follows from the fact that when looking at the point process in any bounded subset of \( \mathbb{R}^d \), it will be included in some ball of radius \( r \) around the origin and therefore the configuration of points in it will be constant (only one point at the origin) after finitely many application of \( \theta_f \). But the point-map of the point-shift makes larger and larger steps and hence the sequence of \( f_\phi(0) \) under \( P_{0}^g,n \) diverges. Hence the point-map of \( f \) is almost surely not continuous at the realization \( \delta_0 \) on which \( P_{0}^g \) is concentrated.

### 3.5 Closest Hard Core Point-Shift

The image of each point \( x \) of \( \phi \) under the closest hard core point-shift \( f_h \) is the closest point \( y \) of \( \phi \) (including \( x \) itself) such that \( \phi(B_1(y)) = 1 \).

**Properties** Clearly for all positive integer \( m \), \( f_h^m(\phi, x) = f_c(\phi, x) \) and hence \( f_h \) is 1-periodic.

**Illustration of Theorem 2.6** Consider the closest hard core point-shift acting on a stationary Poisson point process of intensity one in the plane. For the simple counting measure \( \phi \), let \( HC(\phi) \) denote those points \( y \) of \( \phi \) such that \( \phi(B_1(y)) = 1 \). If \( \phi \) is chosen w.r.t. \( \mathcal{P} \), then \( HC(\phi) \) is also a stationary point process. Let \( Q_0 \) denote the Palm probability of \( HC(\phi) \). Then \( P_{0}^g \) is absolutely continuous w.r.t. \( Q_0 \) and its Radon-Nikodym derivative at each \( HC(\phi) \in \mathbb{N}^0 \) is proportional to the number of points of \( \phi \) in the Voronoi cell of the origin in \( HC(\phi) \).
3.6 Quadri-Void Grid Point-Shift

Let $\psi = \mathbb{Z}\setminus 4\mathbb{Z}$; i.e. those integers which are not multiple of 4. If $U$ is a uniform random variable in $[0,4)$, then $\psi + U$ is a stationary point process on the real line that is called the quadri-void grid below. The Palm distribution of this point process has mass of $\frac{1}{3}$ on $\theta_1\psi, \theta_2\psi$ and $\theta_3\psi$.

Let $g$ be the point-map defined by

$$g(\theta_1\psi) = 2, \ g(\theta_2\psi) = 1 \text{ and } g(\theta_3\psi) = -2.$$ 

If $f$ is the point-shift associated to $g$, then for odd values of $n > 0$, one has

$$\mathcal{P}^{g,n}_0[\phi = \theta_3\psi] = \frac{2}{3}, \quad \mathcal{P}^{g,n}_0[\phi = \theta_1\psi] = \frac{1}{3},$$

and for even values of $n > 0$, one has

$$\mathcal{P}^{g,n}_0[\phi = \theta_3\psi] = \frac{1}{3}, \quad \mathcal{P}^{g,n}_0[\phi = \theta_1\psi] = \frac{2}{3}.$$ 

Therefore $\{\mathcal{P}^{g,n}_0\}_{n=1}^{\infty}$ has two convergent subsequences with different limits, one for even and one for odd values of $n$, and none of these limits is invariant under the action of $\theta_f$. However, the sequence $\{\mathcal{P}^{g,n}_0\}_{n=1}^{\infty}$ converges to a limit $\mathcal{P}^g_0$ which is the mean of the odd and even $g$ probabilities, i.e.

$$\mathcal{P}^g_0[\phi = \theta_3\psi] = \frac{1}{2}, \quad \mathcal{P}^g_0[\phi = \theta_1\psi] = \frac{1}{2},$$

and it is invariant under the action of $\theta_f$.

A Appendix

A.1 Random Measures

This subsection summarizes the results about random measures which are used in this paper in order to have a self-contained paper. The interested reader should refer to [8]. No proofs are given.

Let $S$ be a locally compact (all points have a compact neighborhood) second countable (has a countable base) Hausdorff space. In this case, $S$ is known to be Polish, i.e. there exists some separable and complete metrization $\rho$ of $S$.
Let $B(S)$ be the Borel algebra of $S$ and $B_b(S)$ be all bounded elements of $B(S)$; i.e. all $B \in B(S)$ such that the closure of $B$ is compact. Let $M(S)$ be the class of all Radon measures on $(S,B(S))$; i.e. all measures $\mu$ such that for all $B \in B_b(S)$, $\mu B < \infty$ and let $N(S)$ be the subspace of all $\mathbb{N}$-valued measures in $M(S)$. The elements of $N(S)$ are counting measures. For all $\mu$ in $M(S)$, define

$$B_b(S)^\mu := \{B \in B_b(S); \mu(\partial B) = 0\}.$$ 

Let $C_b(S)$ (respectively $C_c(S)$) be the class of all continuous and bounded (respectively continuous and compact support) $h : S \rightarrow \mathbb{R}^+$. Let

$$\mu h := \int_S h(x)\mu(dx),$$

where the latter is equal to $\sum_{x \in \mu} h(x)$ when $\mu$ is a counting measure. Note that in the summation one takes the multiplicity of points into account. The class of all finite intersections of $M(S)$-sets (or $N(S)$-sets) of the form $\{\mu : s < \mu h < t\}$ with real $r$ and $s$ and arbitrary $h \in C_c(S)$ forms a base of a topology on $N(S)$ which is known as the vague topology. In the vague topology $N(S)$ is closed in $M(S)$ ([8], p. 94, A 7.4.) and a necessary and sufficient condition for the convergence in this topology ([8], p. 93) is:

$$\mu_n \overset{v}{\rightarrow} \mu \iff \forall h \in C_c(S), \mu_n h \rightarrow \mu h.$$ 

If one considers the subspace of all bounded measures in $N(S)$, one may replace $C_c(S)$ by $C_b(S)$ which leads to the weak topology for which

$$\mu_n \overset{w}{\rightarrow} \mu \iff \forall h \in C_b(S), \mu_n h \rightarrow \mu h.$$ 

The convergence in distribution of the random variables $\xi_1, \xi_2, \ldots$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and taking their values in $(S,B(S))$, to the random element $\xi$ is defined as follow

$$\xi_n \overset{d}{\rightarrow} \xi \iff \xi_n \mathbb{P} \overset{w}{\rightarrow} \xi \mathbb{P}.$$ 

The next lemma describes the relation between the convergences in the vague topology and the weak one.

**Lemma A.1** ([8], p.95, A 7.6.). For all bounded $\mu, \mu_1, \mu_2, \ldots \in M(S)$, one has

$$\mu_n \overset{w}{\rightarrow} \mu \iff \mu_n \overset{v}{\rightarrow} \mu$$

and $\mu_n S \rightarrow \mu S$. 

25
According to Lemma A.1, when we discuss about the convergence of probability measures, there is no difference between the vague and the weak convergence.

The following proposition is a key point in the development of the theory of random measures and random point processes ([8], p. 95 A 7.7.).

**Proposition A.2.** Both $\mathcal{M}(S)$ and $\mathcal{N}(S)$ are Polish in the vague topology. Also the subspaces of bounded measures in $\mathcal{M}(S)$ and $\mathcal{N}(S)$ are Polish in the weak topology.

Proposition A.2 allows one to define measures on $\mathcal{M}(S)$ or $\mathcal{N}(S)$ which are Polish spaces and use for them the theory available for $S$. If $\mathcal{M}$ (respectively $\mathcal{N}$) is the $\sigma$-algebra generated by the vague topology on $\mathcal{M}(S)$ (respectively $\mathcal{N}(S)$), a random measure (respectively random point process) on $S$ is simply a random element of $(\mathcal{M}(S), \mathcal{M})$ (respectively $(\mathcal{N}(S), \mathcal{N})$). Note that a random point process is a special case of a random measure.

The next theorem and lemmas give handy tools to deal with convergence in distribution of random measures on $S$.

**Theorem A.3** ([8], p.22, Theorem 4.2.). If $\mu, \mu_1, \mu_2, \ldots$ are random measures on $S$ (i.e. random elements of $(\mathcal{M}(S), \mathcal{M})$), then

$$\mu_n \xrightarrow{d} \mu \iff \mu_n h \xrightarrow{d} \mu h, \quad \forall h \in C_c(S).$$

**Lemma A.4** ([8], p.22, Lemma 4.4.). If $\mu, \mu_1, \mu_2, \ldots$ are random measures on $S$ satisfying $\mu_n \xrightarrow{d} \mu$, then $\mu_n h \xrightarrow{d} \mu h$ for every bounded measurable function $h : S \to \mathbb{R}^+$ with bounded support satisfying $\mu(D_h) = 0$ almost surely, where $D_h$ is the set of all discontinuity points of $h$. Furthermore,

$$(\mu_n B_1, \ldots, \mu_n B_k) \xrightarrow{d} (\mu B_1, \ldots, \mu B_k), \quad k \in \mathbb{N}, \quad B_1, \ldots B_k \in \mathcal{B}_b(S)^\mu.$$

**Lemma A.5** ([8], p.23, Lemma 4.5.). A sequence $\{\mu_n\}_{n=1}^\infty$ of random measures on $S$ is relatively compact w.r.t. the convergence in distribution in the vague topology if and only if

$$\lim_{t \to \infty} \limsup_{n \to \infty} \mathbb{P}[\mu_n B > t] = 0, \quad \forall B \in \mathcal{B}_b(S).$$

Denote by $P(S)$ the set of all probability measures on $S$. Clearly $P(S) \subset \mathcal{M}(S)$ and according to Lemma A.1, the weak and the vague topologies on $P(S)$ coincide.

26
Proposition A.6 ([4] p.30, Theorem 5.1.). If \( S \) and \( T \) are Polish spaces and \( h : (S, \mathcal{B}(S)) \to (T, \mathcal{B}(T)) \) is a measurable mapping, then \( h_* \) is continuous w.r.t. the weak topology at point \( \mathbb{P} \in \mathcal{P}(S) \) if \( h \) is \( \mathbb{P} \)-almost surely continuous.

Note that the version of Proposition A.6 which is in [4], is expressed for metric spaces. But, as noted in the beginning of the appendix, Polish spaces are metrizable and hence one can apply the same statement for such spaces.

A.2 Proof of Proposition 2.14

Let \( g \) be a point-map the image of which at \( \phi \in \mathbb{N}^0 \) is \( x \in \phi \), with \( x \neq 0 \). Assume there is a point \( y \in \phi \) with \( y \notin \{0,x\} \). Since \( \phi \) is a discrete subset of \( \mathbb{R}^d \) and \( d \geq 2 \) there exist curves \( \gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^d \) such that

1. \( \gamma_1(0) = \gamma_2(1) = x \) and \( \gamma_2(0) = \gamma_1(1) = y \);
2. \( \gamma_1 \) and \( \gamma_2 \) only intersect at their end-points;
3. \( \gamma_1 \) and \( \gamma_2 \) contain no point of \( \phi \) other than \( x \) and \( y \).

Now let \( \Gamma \) be a closed curve in \( \mathbb{N}^0 \) defined as

\[
\Gamma : [0,1] \to \mathbb{N}^0; \quad \Gamma(t) = (\phi \setminus \{x,y\}) \cup \{\gamma_1(t), \gamma_2(t)\}, \ t \in [0,1].
\]

The continuity of \( g \), 2. and 3. imply that for all \( t \in [0,1], \ g(\Gamma(t)) = \gamma_1(t) \). Hence \( g(\Gamma(0)) = x \) and \( g(\Gamma(1)) = y \). But it follows from 1. that \( \Gamma(0) = \Gamma(1) = \phi \), which contradicts with the fact that \( x \) and \( y \) are different points of \( \phi \). When \( \phi = \{0,x\} \), one obtains the contradiction by letting \( x \) go to infinity whereas in this situation, \( \{0,x\} \) converges to \( \{0\} \) in the vague topology.

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