Asymmetric lasing at spectral singularities

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Scattering coefficients can diverge at spectral singularities. In such situation, the stationary solution becomes a laser solution with outgoing waves only. We explore a parity-time (PT)-symmetric non-Hermitian two-arm Aharonov-Bohm interferometer consisting of three coupled resonators enclosing synthetic magnetic flux. The synthetic magnetic flux does not break the PT symmetry, which protects the symmetric transmission. The features and conditions of symmetric, asymmetric, and unidirectional lasing at spectral singularities are discussed. We elucidate that lasing affected by the interference is asymmetric; asymmetric lasing is induced by the interplay between the synthetic magnetic flux and the system’s non-Hermiticity. The product of the left and right transmissions is equal to that of the reflections. Our findings reveal that the synthetic magnetic flux affects light propagation, and the results can be applied in the design of lasing devices.

I. INTRODUCTION

The theory of parity-time (PT)-symmetric systems has been extensively investigated [11]. In experiments, the use of optical platforms is fruitful for investigating PT-symmetric non-Hermitian quantum mechanics. To construct PT-symmetric systems, passive and active devices are mostly used. In passive systems, elements have varying loss rates [16]. A passive system is equivalently described by a PT-symmetric Hamiltonian after an overall loss rate has been removed; however, mode amplification is impossible in passive systems. In active systems, the gain balances the loss to form a PT-symmetric system [17,21]. Under steady state conditions, a resonator doped with erbium ions under pumping induces a gain that balances the loss in the dissipative resonator. Active PT-symmetric devices have numerous applications such as in optical isolators [20] and single-mode PT-symmetric lasing [22,23].

The non-Hermiticity induces asymmetric and nonunitary scattering in non-Hermitian systems [24,25]. Numerous intriguing phenomena such as unidirectional reflectionless [26,27], invisible cloaking [28,29], asymmetric and robust light transport have been reported [30,31]. Spectral singularities exist in the non-Hermitian scattering system, where the scattering coefficient diverges and the completeness of eigenstates is spoiled [32,36]. At spectral singularities, coherent perfect absorption [37–39] and PT-symmetric laser absorber [40–42] have been demonstrated.

In Hermitian systems, symmetric scattering exists even in the presence of synthetic magnetic flux, where photons mimic electrons in magnetic field [43,49]. The synthetic magnetic flux induces asymmetric couplings but does not break the time-reversal symmetry. The time-reversal symmetry protects the optical reciprocity and the symmetric scattering. However, in non-Hermitian systems, the synthetic magnetic flux and the non-Hermiticity can induce novel asymmetric behaviors [50]. The synthetic magnetic flux enclosed in a loss resonator side-coupled to a resonator chain helps to create a unidirectional perfect absorber in which light can avoid occupation of the loss resonator. The proposed system has full reflectionless absorption on one direction, and full reflectionless transmission in the opposite direction [51,54]. In contrast to a perfect absorber, a PT-symmetric laser at non-Hermitian spectral singularities, where symmetrical lasing toward both sides of a scattering system occurs has been demonstrated [40,41,55]. Moreover, unidirectional lasing at unidirectional spectral singularities has been proposed [56].

In this paper, we focus on a PT-symmetric non-Hermitian system consisting of three coupled optical resonators and in which synthetic magnetic flux is enclosed. The three coupled resonators are embedded in a resonator array and collectively serve as an Aharonov-Bohm (AB) interferometer. The non-Hermiticity of the system is due to the balanced gain and loss in the resonators. Synthetic magnetic flux is induced through an asymmetric coupling between neighboring resonators. PT-symmetry holds in the presence of synthetic magnetic flux and enables symmetric light transmission even though the coupling is asymmetric. The scattering coefficients diverge at spectral singularities of the non-Hermitian interferometer, with the scattering wave function in the steady state representing lasing. The features and conditions of symmetric, asymmetric, and unidirectional lasing at spectral singularities are discussed in this paper. The synthetic magnetic flux affects the light interference and leads to asymmetric lasing in the interferometer. As an illustration, we evaluate the scattering properties of a uniformly coupled resonator system. At spectral singularities, lasing asymmetry is minimal at trivial synthetic magnetic flux and maximal at half-integer synthetic magnetic flux. Wave emission has the signature of spectral singularities in the wave packet scattering process.

The remainder of this paper is organized as follows. In Sec. II, we model the PT-symmetric coupled resonator AB interferometer. In Sec. III, we describe its spectral singularities and asymmetric lasing. In Sec. IV, we consider a uniformly coupled resonator as an illustration.
This asymmetric coupling is introduced using the optical −travel between resonators compositions (Fig. 1), where comprises the indirect path from resonator directional hopping phase, represented by −+1; the coupling between resonators 0 and ±1; the coupling strengths compose the scattering center onator array is in the linear region [18, 22]. The couplings between resonators are constant rate − comprises the direct path from resonator −1 to resonator +1 through resonator 0, the coupling strengths −1 are −κ. The loss is caused by resonator dissipation, whereas the gain is induced by pumping the ions doped in resonator −1. The gain is modeled by a phase factor corresponds to synthetic magnetic flux Φ in the coupling process. The additional directional phase factor corresponds to synthetic magnetic flux Φ that is enclosed by the three coupled resonators [58]. The synthetic magnetic flux is gauge invariant. Although the transmission and reflection coefficients change, their moduli are invariant under gauge transformation.

In the clockwise mode, photons circle in the opposite direction and all arrows in Fig. 1 are invert. The path differences in the tunneling process are opposite for the counterclockwise and clockwise modes. Consequently, opposite synthetic magnetic fluxes Φ and −Φ are induced. The synthetic magnetic flux Φ and its opposite −Φ lead to identical transmission and reflection probabilities. Therefore, the lasing at spectral singularities is identical for the counterclockwise and clockwise modes.

The equations of motion for the scattering center are identical for the counterclockwise and clockwise probabilities. Therefore, the lasing at spectral singularities is identical for the counterclockwise and clockwise modes.

The modal amplitudes of the resonators are described by coupled-mode theory [59], whereas the equation of motion for the leads is

\[ i\dot{a}_j = \omega_c a_j - \kappa a_{j-1} - \kappa a_{j+1}, \]

for the left lead \( j < -1 \) and the right lead \( j > 1 \). The equations of motion for the scattering center are

\[ i\dot{a}_{-1} = (\omega_c + i\gamma) a_{-1} - ga_0 - J e^{\mp \Phi} a_1 - \kappa a_{-2}, \]

\[ i\dot{a}_0 = \omega_c a_0 - ga_{-1} - ga_1, \]

\[ i\dot{a}_1 = (\omega_c - i\gamma) a_1 - ga_0 - J e^{-\Phi} a_{-1} - \kappa a_2. \]

The equations of motion are Schrödinger-like equations. The dynamics of photons in the synthetic magnetic flux described by these equations of motion are equivalent to those of electrons in magnetic flux in condensed matter physics.

Notably, the interferometer is \( \mathcal{PT} \)-symmetric in the presence of synthetic magnetic flux. The parity operator \( \mathcal{P} \) satisfies \( \mathcal{P} a_j^\dagger \mathcal{P}^{-1} = a_j^\dagger \) and \( \mathcal{P} a_j \mathcal{P}^{-1} = a_{-j} \). The time-reversal operator \( \mathcal{T} \) satisfies \( \mathcal{T} i \mathcal{T}^{-1} = -i \). \( \dot{a}_j \) (\( a_j \)) is the creation (annihilation) operator for the resonator \( j \).

III. SPECTRAL SINGULARITIES AND LASING

In a finite-size \( \mathcal{PT} \)-symmetric non-Hermitian system, the system might experience a phase transition from an exact \( \mathcal{PT} \)-symmetric phase to a broken-\( \mathcal{PT} \)-symmetric phase at the exceptional points. The exceptional points are points that the finite-size non-Hermitian system is defective, where coalesced eigenstates appear. Another type of singularities are spectral singularities in infinite-size non-Hermitian systems. Spectral singularities are
singular points in a non-Hermitian scattering system at which eigenstates for a continuous spectrum is incomplete [52]. The biorthonormal basis vanishes at the singularities of non-Hermitian systems [60]. In non-Hermitian optical scattering systems, coherent perfect absorption or lasing occurs at the spectral singularities of \( \mathcal{PT} \)-symmetric systems [37, 11]. The stationary scattering wave function is that of an incoming wave for a perfect absorber, where outgoing waves vanish [Fig. 2(a)]. By contrast, lasing corresponds to a stationary scattering wave function that constitutes outgoing waves only [Fig. 2(b-d)]. Unidirectional perfect absorption has been proposed at spectral singularities in dissipative systems that are non-\( \mathcal{PT} \)-symmetric [52, 53]. The \( \mathcal{PT} \)-symmetric scattering system considered in this paper is a non-Hermitian two-arm AB interferometer. We are interested in how spectral singularities and lasing are affected by the synthetic magnetic flux that causes interference between the two arms.

The dispersion relation for the coupled resonators is 
\[ \varepsilon = \omega - 2\kappa \cos k, \]
where \( k \) is the wave vector, \( -\pi \leq k \leq \pi \). The modal amplitude satisfies \( a_j = f_j e^{-i\sigma t} \). From the equations of motion Eqs. (2, 4), we have

\[ -g f_0 - J e^{i\Phi} f_1 - \kappa f_{-2} = (E - i\gamma) f_{-1}, \]  
\[ -g f_{-1} - g f_1 = E f_0, \]  
\[ -g f_0 - J e^{-i\Phi} f_{-1} - \kappa f_2 = (E + i\gamma) f_1. \]

and \( -f_{j-1} - f_{j+1} = E f_j \) for \( |j| > 1 \), where \( E = -2\kappa \cos k \). In the elastic scattering process, the wave functions \( f_j \) at steady-state are in the form of: \( f_{j} = Ae^{ijk} + Be^{-ikj} \) (\( j < 0 \)) and \( f_{j} = Ce^{ikj} + De^{-ikj} \) (\( j > 0 \)), where \( A \) and \( D \) are the amplitudes of the incoming wave, and \( B \) and \( C \) are the amplitudes of the outgoing wave.

At spectral singularities, the product of the left and right transmissions is equal to that of the reflections; in other words, \( T_L T_R = R_L R_R \) holds, where \( T \) (\( R \)) represents the transmission (reflection) probability. The subscripts indicate either the left or right input. The scattering coefficients diverge at spectral singularities for lasing and the wave function consists of outgoing waves only

\[ f_j = \begin{cases} Be^{-ikj}, & (j < 0) \\ Ce^{ikj}, & (j > 0) \end{cases}, \]

where incoming waves \( Ae^{ikj} \) (\( j < 0 \)) and \( De^{-ikj} \) (\( j > 0 \)) vanish. For a \( \mathcal{PT} \)-symmetric interferometer, the transmission or reflection probability is symmetric. Therefore, symmetric lasing of a \( \mathcal{PT} \)-symmetric interferometer requires not only symmetric reflection and transmission probabilities but also that these probabilities possess identical values, or in other words, \( T_L = T_R = R_L = R_R \) [Fig. 2(b)]; otherwise, the lasing is asymmetric [Fig. 2(c)]. \( T_L T_R = R_L R_R \) implies asymmetric lasing with contrast \( \chi = T_L / R_L = R_R / T_R \).

The enclosed synthetic magnetic flux affects the interference and spectral singularities. By substituting \( f_{-2} = Be^{ik}, f_{-1} = B, f_1 = C, \) and \( f_2 = Ce^{ik} \) into the equations of motion [Eqs. (5, 7)], and then eliminating \( f_0 \), we obtain

\[ \left( \frac{g^2}{E} - J e^{i\Phi} \right) C = \left( E + \kappa e^{ikj} - \frac{g^2}{E} - i\gamma \right) B, \]  
\[ \left( \frac{g^2}{E} - J e^{-i\Phi} \right) B = \left( E + \kappa e^{-ikj} - \frac{g^2}{E} + i\gamma \right) C. \]

where \( E = -2\kappa \cos k \). When \( B \) and \( C \) are nonvanishing, eliminating \( B \) and \( C \) from Eqs. (9, 10), we get

\[ \left( \kappa e^{-ik} + \frac{g^2}{E} \right)^2 + \gamma^2 = \left( \frac{g^2}{E} - J e^{i\Phi} \right) \left( \frac{g^2}{E} - J e^{-i\Phi} \right). \]

After simplifying this relation [Eq. (11)], we obtain the lasing conditions at spectral singularities as follows:

\[ g^2 = 2\kappa^2 \cos^2 k, \]  
\[ \cos \Phi = \frac{\gamma^2 - J^2 - \kappa^2 \kappa}{g^2}. \]

At spectral singularities, the wave function consists of outgoing waves. The modal amplitudes satisfy \( |B|^2 / |C|^2 = (\gamma + \kappa \sin k) / (\gamma - \kappa \sin k) \), which indicates that the lasing is asymmetric in the setup of the \( \mathcal{PT} \)-symmetric interferometer.

An extreme case of asymmetric lasing occurs at unidirectional spectral singularities [50]. The lasing occurs in one direction only, where \( B \) or \( C \) vanishes in the scattering wave function \( f_j \) [Fig. 2(d)]. In this case, Eqs. (9, 10) reduce to

\[ \frac{g^2}{2\kappa \cos k} + J \cos \Phi + i\sigma J \sin \Phi = 0, \]  
\[ \kappa \cos k - \frac{g^2}{2\kappa \cos k} - i(\kappa \sin k + \sigma \gamma) = 0. \]

with \( \sigma = \pm 1 \) (\( -1 \)) for \( B = 0 \) (\( C = 0 \)). Equations (14, 15) indicate that the unidirectional spectral singularities place additional constrains on the parameters of the coupled resonator system. The real and imaginary parts of the left side of Eqs. (14, 15) vanish, respectively.
Equation (14) yields that the synthetic magnetic flux must be:
\[ \Phi = n\pi, \ (n \in \mathbb{Z}). \]  
(16)

Equation (15) yields:
\[ \gamma^2 - \kappa^2 \sin^2 k; \ g^2/2 = \kappa^2 \cos^2 k. \]  
(17)

Associated Eq. (14) and Eq. (15), we notice that the coupling, the synthetic magnetic flux, and the gain or loss rate are related to the wave vector as follows:
\[ J \cos \Phi = -\kappa \cos k. \]  
(18)

Notably, \( \cos^2 \Phi = 1 \) for \( \Phi = n\pi, \ (n \in \mathbb{Z}) \). Thus, the coupling strengths satisfy \( g^2 = 2J^2 \) and the parameters take the forms of a circle and an ellipse in parameter space,
\[ J^2 + \gamma^2 = \kappa^2, \]  
(19)

and
\[ g^2/2 + \gamma^2 = \kappa^2. \]  
(20)

At unidirectional spectral singularities, the symmetric transmission is finite; the reflection is infinite for one input direction and zero for the other input direction. The stationary wave function for \( \gamma = \kappa \sin k \) is
\[ f_j = \begin{cases} e^{-ikj}, & (j < 0) \\ 0, & (j > 0) \end{cases}, \]  
(21)

and the stationary wave function for \( \gamma = -\kappa \sin k \) is
\[ f_j = \begin{cases} 0, & (j < 0) \\ e^{ikj}, & (j > 0) \end{cases}. \]  
(22)

The stationary wave function for unidirectional lasing implies that the lasing is directional, and this directionality is generated from the gain resonator as a reflection toward the lead to which it is connected. By contrast, the reflection in the other direction is perfectly absorbed by the loss resonator. The clockwise and counter clockwise modes experience opposite synthetic magnetic fluxes. From the conditions of spectral singularities, we notice that they both lase at spectral singularities for fixed device parameters.

**IV. UNIFORMLY COUPLED PT-SYMMETRIC AB INTERFEROMETER**

As an illustrative example, we investigate the AB interferometer in the case of \( J = g = \kappa \). For the left input with wave vector \( k \), the scattering wave function in the left lead is \( f_j = e^{ikj} + r_le^{-ikj} \ (j < -1) \) and the transmission wave in the right lead is \( f_j = t_le^{ikj} \ (j > 1) \). By contrast, for the right input, the input wave function in the right lead is \( f_j = e^{-ikj} + r_re^{ikj} \ (j > 1) \) and the transmission wave in the left lead is \( f_j = t_re^{-ikj} \ (j < -1) \). \( t_{l(R)} \) (\( r_{l(R)} \)) represents the transmission (reflection) coefficient for the left (right) input. The transmission and reflection probabilities are \( T_{l(R)} = |t_{l(R)}|^2 \) and \( R_{l(R)} = |r_{l(R)}|^2 \). Substituting the wave function \( f_j \) into the equations of motion, the scattering coefficients \( r_{l,R} \) and \( t_{l,R} \) are obtained as follows,

\[ t_L = \frac{i \sin k \left( 1 + e^{-i\Phi} 2 \cos k \right)}{ie^{-2ik} \sin k + (1 - \gamma^2) \cos k + \cos \Phi}, \]  
(23)

\[ r_L = \frac{\gamma \sin (2k) - (1 - \gamma^2) \cos k - \cos \Phi}{ie^{-2ik} \sin k + (1 - \gamma^2) \cos k + \cos \Phi}, \]  
(24)

\[ t_R = \frac{i \sin k \left( 1 + e^{i\Phi} 2 \cos k \right)}{ie^{-2ik} \sin k + (1 - \gamma^2) \cos k + \cos \Phi}, \]  
(25)

\[ r_R = \frac{-\gamma \sin (2k) - (1 - \gamma^2) \cos k - \cos \Phi}{ie^{-2ik} \sin k + (1 - \gamma^2) \cos k + \cos \Phi}. \]  
(26)

From these expressions of the scattering coefficients, we find that the transmission is symmetric, or in other words, \( T = T_L = T_R \). Figure [5] illustrates the transmission and reflection probabilities. In the region \( k \in (0, \pi) \),
The scattering probability is symmetric at $k = \pi/2$ for a synthetic magnetic flux of $n\pi + \pi/2 \ (n \in \mathbb{Z})$ [Fig. 3(a–c)], where the scattering spectra are symmetric about the resonance frequency; otherwise (for example, at $\Phi = 0$), the scattering spectra are asymmetric about $k = \pi/2$ [Fig. 3(d–f)]. For $\Phi = \pi$, the transmission and reflection probabilities are mirror reflection symmetric about $k = \pi/2$ at $\Phi = 0$. When $k = \pi/2$, the reflection and transmission are symmetric and $\gamma$-independent, $T_{L} = T_{R} = (\cos^{2} \Phi + 1)^{-1}$ and $R_{L} = R_{R} = (\cos^{2} \Phi + 1)^{-1} \cos^{2} \Phi$. The synthetic magnetic flux alters the reflection zeros whereas transmission zeros occur only when $\Phi = n\pi \ (n \in \mathbb{Z})$. For an input wave vector $k \neq \pi/2$, the symmetric reflection is broken in the presence of nonzero $\gamma$ even at spectral singularities because the two arms of the interferometer are imbalanced. The reflectionless transmission is unidirectional and synthetic magnetic flux dependent.

Divergence of the scattering coefficients occurs only for wave vectors $k = \pi/4$ and $3\pi/4$ [obtained from Eq. (12)]. Correspondingly, scattering coefficient divergence emerges when the gain and loss rates satisfy $\gamma^2 = 2 \pm \sqrt{2} \cos \Phi$ [obtained from Eq. (13)]. At $k = \pi/4$, the transmissions ($t_{L}$ and $t_{R}$) are larger than zero under any balanced gain and loss $\gamma$ or when the synthetic magnetic flux $\Phi$ is enclosed. Divergence of the scattering coefficients occurs at

$$\gamma^2 = \sqrt{2} \cos \Phi + 2,$$

and spectral singularities appear in the region $\sqrt{2} - \sqrt{2} \leq \gamma/\kappa \leq \sqrt{2} + \sqrt{2}$; the spectral singularity curves are shown in Fig. 4(a). At spectral singularities, the scattering center acts as a wave emitter [55] and the lasing is asymmetric. The ratio of left-going lasing to right-going lasing, which is obtained by comparing the transmission and reflection coefficients, is plotted in Fig. 4(b). A ratio lower than one indicates that the left side, which includes the gain resonator, has a higher lasing rate. Notably, the three embedded coupled resonators act as a wave emitter at spectral singularities. The lasing is asymmetric, and the contrast ratio of right-travelling wave emission to left-travelling wave emission is

$$\chi(\Phi) = \frac{\sqrt{2} \sqrt{2} \cos \Phi + 4 - 1}{\sqrt{2} \sqrt{2} \cos \Phi + 4 + 1},$$

which is obtained by comparing the transmission and reflection probabilities. The gain resonator is on the left in this system, and thus left-travelling wave emission is stronger than right-travelling wave emission and the contrast is less than 1. For example, at $\gamma = \sqrt{2}$ and $\Phi = \pi/2$, we have $T_{L}/R_{L} = 1/3$ and $T_{R}/R_{R} = 3$. Notably, $T_{L}T_{R} = R_{L}R_{R}$ holds independent of the synthetic magnetic flux at spectral singularities, indicating that the asymmetry of the lasing is input direction independent, whereas the lasing intensity is input direction dependent. The contrast $T_{L}/R_{L}$ decreases as the synthetic magnetic flux $\Phi$ increases from 0 to $\pi$; therefore, the asymmetry of the lasing increases.

A signature of spectral singularities in a wave packet scattering process is wave emission. We numerically simulate a right input wave emission process in the AB interferometer. The initial excitation is a normalized Gaussian wave packet of $|\Psi(0,j)| = (\sqrt{\pi}/\alpha)^{-1/2} \sum e^{-i(\alpha^2/2)(j-N_{c})^2} e^{i(k_{c}j)} |j\rangle$ centered at site $N_{c}$, where $j$ indicates the label of the resonator in the array and $k_{c}$ is the wave vector of the Gaussian wave packet. At spectral singularities, lasing occurs toward both sides after the wave packet reaches the embedded resonators, and forms an asymmetric plateau described by an error function [55]. The lasing wave intensity increases linearly with time, being proportional to the wave propagating velocity $2\kappa \sin k$ and the plateau heights. The time evolution of the Gaussian wave packet is $|\Psi(t,j)| = e^{-iHt}|\Psi(0,j)|$, where $H$ is Hamiltonian of the 1200-site system including both the trimer scattering center $H_{c}$ and two finite leads connected to the center. In Fig. 5(a,c,e), a Gaussian wave packet is initially centered at $N_{c} = 900$, the wave vector is $k_{c} = \pi/4$, and the packet moves from right to left at a velocity of $\sqrt{2}\kappa$. To emphasize the different asymmetries, the density plots of intensity $P(t,j)/h$ are depicted for comparison, where $P(t,j) = \langle \Psi(t,J) | \Psi(t,j) \rangle$ is the intensity of the wave packet and $h$ is the left-travelling wave plateau height in Fig. 5(b,d,f). At $t \approx 210/\kappa$, the Gaussian wave packet reaches the scattering center and the intensity begins to increase. Figure 5(b,d,f) depicts the intensity $P(t,j)$ of the lasing at $t = 500/\kappa$ and clearly displays the asymmetric plateau. The left-travelling wave for the right input and the right-travelling wave for the left input induce the same transmitted plateau height $h$, equal to $3\sqrt{\pi}/(4\alpha)$, $3\sqrt{\pi}/(4\alpha)$, and $\sqrt{\pi}/(4\alpha)$ for $\Phi = 0, \pi/2$, and $\pi$, respectively [left plateaus displayed in Fig. 5(b,d,f)].

Moreover, after the wave packet is scattered by the embedded coupled resonators, the height of the left-travelling wave is $h\gamma^{-1}(\Phi)$ and that of the right-travelling wave is $h$ for the left input. By contrast, the...
FIG. 5. Numerical simulations of lasing wave intensity for (a, b) $\Phi = 0$, (c, d) $\Phi = \pi/2$, and (e, f) $\Phi = \pi$. (a, c, e) Density plots of the intensity $P(t,j)/h$, where $h$ is the left-traveling wave height as shown in (b, d, f) on their left halves. (b, d, f) Configuration of the lasing wave intensity $P(t,j)$ at $t = 500/\kappa$. The three coupled resonators embedded in the center of the resonator array are indicated by the red triangles. A Gaussian wave packet with $\alpha = 0.02$ and a wave vector $k_c = \pi/4$ centered at $N_c = 900$ (indicated by the red arrow) is initially travelling left. The trajectory of the wave packet center is indicated by the dashed green line. An asymmetric wave is emitted after reaching the center at approximately $t = 210/\kappa$.

height of the left-traveling wave is $h$ and that of the right-travelling wave is $h\chi(\Phi)$ for the right input. The contour plots in Fig. 5(a,c,e) indicate that the asymmetry is enhanced as the synthetic magnetic flux is increased from 0 to $\pi$, where the left-travelling wave emission plateau heights are renormalized to 1.

V. CONCLUSION

A $\mathcal{PT}$-symmetric two-arm AB interferometer where three coupled resonators are embedded in a uniformly coupled resonator array is proposed in this paper. Synthetic magnetic flux is enclosed by the three coupled resonators using the path length imbalance method. The synthetic magnetic flux acts as a new degree of freedom, controlling light interference but not breaking the $\mathcal{PT}$ symmetry. In the two-arm AB interferometer, the $\mathcal{PT}$ symmetry protects the symmetric transmission; however, the reflection is asymmetric due to the non-Hermitian gain and loss. The conditions for symmetric, asymmetric, and unidirectional lasing at spectral singularities are discussed. The interplay between the synthetic magnetic flux and non-Hermiticity controls the lasing at spectral singularities; the lasing is proven asymmetric. The asymmetric lasing at spectral singularities and the reflection-less transmission both vary with the enclosed synthetic magnetic flux. We demonstrate a uniformly coupled AB interferometer as an illustrative example and show that the asymmetry of lasing is enhanced as the synthetic magnetic flux is increased. Our results could be useful in the design of optical control and lasing devices, and may facilitate the application of $\mathcal{PT}$-symmetric metamaterials.

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