Fine Resolution of $k$-transversal Cones

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**Abstract.** Tougeron’s implicit function theorem and Hensel’s lemma are well known representatives concerning $2k$-approximation combined with $k$-non-degeneracy implies existence of solutions with identity of order $k$. This note aims to extend this principle to equations $G[z] = 0$ in Banach spaces, thereby defining generalized cones spanned by submanifolds, each characterized by a certain expansion rate.

The number of manifolds in the cone, as well as their expansion rates, are recursively increased until an appropriate desingularization of the cone is built up with linearization expressed by first $k + 1$ derivatives of the singular operator at the base point. Along these lines, a well-defined submersion is constructed in the cone with uniformly bounded inverse when approaching the singularity.

The fine resolution of the cone by the manifolds represents an improvement compared to measuring the variation of the nonlinear operator exclusively by the overall behaviour of the determinant.

In case of finite dimensions, each half-cone is characterized by a constant topological degree that can be used to investigate a solution curve in general position with respect to secondary bifurcation.

The core of all considerations is given by some characteristic patterns, valid in the system of undetermined coefficients that allow for detailed analysis of the power series resulting from plugging the power series of the ansatz into the power series of the nonlinear operator.

**Keywords:** Tougeron implicit function theorem, Transversality of order $k$, Degree theory, Banach space, System of undetermined coefficients, Submersion

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1. Introduction

Given an equation \( G[x, y] = 0, \ G \in C^q(\mathbb{K}^n \times \mathbb{K}^m, \mathbb{K}^m), \ \mathbb{K} = \mathbb{R}, \mathbb{C}, \) with base solution \( G[0,0] = 0, \) which is locally embedded within a smooth approximation \( y_0(x) \) satisfying

\[
\| G[ x, y_0(x) ] \| = O\left( \| x \|^{2k+1} \right) \quad (A_0)
\]

\[
det\{ G_y[ x, y_0(x) ] \} \neq O\left( \| x \|^{k+1} \right), \quad (N_0)
\]

i.e. along \( y_0(x) \) the map value \( G[x, y_0(x)] \) varies slowly away from zero by order of \( 2k + 1, \) compared to the fast change of the \( y \)-derivative \( G_y[x, y_0(x)] \) that varies by order of \( k \) (and not slower). Under these assumptions, it is well known that further zeros \( y(x) \) of \( G[x, y] \) exist in the vicinity of the approximation \( y_0(x), \) i.e. we obtain \( G[x, y(x)] = 0 \) and identity of order \( k \) between exact solution curve \( y(x) \) and approximation \( y_0(x) \) by

\[
\| y(x) - y_0(x) \| = O\left( \| x \|^{k+1} \right). \quad (I_0)
\]

Note that the approximation condition \( (A_0) \) defines a degeneracy condition of order \( 2k, \) i.e. we require the first \( 2k \) derivatives of \( G[x, y_0(x)] \) to vanish at \( x = 0, \) whereas \( (N_0) \) defines a nondegeneracy condition of order \( k, \) finally allowing a contraction mapping argument to ascertain the existence of \( y(x) \) as well as the identity condition \( (I_0) \) of order \( k. \) The proof is essentially based on the ansatz

\[
y = y_0(x) + det\{ G_y[ x, y_0(x) ] \} \cdot \bar{y}, \quad (1.1)
\]

as well as factoring out of the adjoint matrix of the Jacobian \( G_y[\cdot] \) yielding a remainder equation appropriate for application of the implicit function theorem with respect to \( \bar{y}. \)

Formulas \( (A_0), (N_0), (I_0) \) may be summarized as follows: \( 2k \)-approximation combined with \( k \)-nondegeneracy implies \( k \)-identity between existing solutions and given approximation. A constellation of this kind occurs in several places of algebra and analysis, applied to mappings of quite different properties with respect to smoothness and type of participating spaces. Compare Newton’s Lemma [G], Tougeron’s implicit function theorem [BH], [H], [R] or Hensel’s Lemma [F]. Nevertheless, the techniques of proof are quite similar.

Now in this note, the constellation is applied to \( G[z] = 0, \ G \in C^q(B, \bar{B}) \) with \( G[0] = 0 \) and \( B, \bar{B} \) real or complex Banach spaces. The linearization \( G'[0] \in L[B, \bar{B}] \) is assumed to be bounded, but not necessarily a Fredholm operator, i.e. in general it is not supposed that reduction to finite dimensions by Lyapunov-Schmidt is possible.

When moving from the finite dimensional setting of \( (A_0), (N_0), (I_0) \) to general Banach spaces, then the determinant and the adjoint matrix of the Jacobian are no longer present in a straightforward way, implying the nondegeneracy condition \( (N_0) \) and the proof by itself to be reformulated appropriately. In some more detail, the proof will be characterized by replacing \( (1.1) \) by

\[
z = z_0(\varepsilon) + A_0 \cdot n^c \quad (1.2)
\]

with \( z_0(\varepsilon) \) an approximation of order \( 2k, \) now parametrized by the external parameter \( \varepsilon \in \mathbb{R} \) or \( \varepsilon \in \mathbb{C} \) and \( A_0 \in L[B, B] \) an \( \varepsilon \)-dependent family of linear operators acting between appropriate subspaces of \( B. \) The action of \( A_0 \) may be interpreted as a blow-up coordinate transformation of \( B \) with singularity given at \( \varepsilon = 0, \) due to \( A_0 = 0 \in L[B, B]. \)

In some more detail the subspaces are successively build up by a fine resolution of the two Banach spaces \( B \) and \( \bar{B} \) according to the following diagram.
Thus, within $B = N_0$ a filtration $B = N_0 \supset N_1 \supset \cdots \supset N_{k+1}$ is recursively set up by kernels $N_i$ of appropriate linear operators $S_1, S_2, \ldots, S_{k+1}$, where the ranges $R_1, R_2, \ldots, R_{k+1}$ of these operators are constructed in a way to ensure, step by step, an increase of the direct sum $R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1}$ within $\bar{B}$. Then, the nondegeneracy condition of the implicit function theorem is satisfied, as soon as a value of $k$ is reached, such that the direct sum agrees with $\bar{B}$ according to

$$R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1} = \bar{B},$$

(1.3)

in this way replacing the det-nondegeneracy condition ($N_0$) from above. The implicit function theorem is applied to the variable $n^c$ in (1.2), where $n^c$ is build up by the complements $N_1^c, N_2^c, \ldots$ of the kernels, i.e. $N_0 = N_1^c \oplus N_1, N_1 = N_2^c \oplus N_2, \ldots$.

If the direct sum in (1.3) does not come to an end, then the attempt was unsuccessful. For example, this situation occurs, if we try to approximate a solution curve that is completely contained in the singular locus of $G[z] = 0$.

Now, from a broader perspective, we try to construct a fine resolution with respect to an arbitrary curve $z_0(\epsilon)$ in $B$ with $z_0(0) = 0$, which is not necessarily a high order approximation of $G[z] = 0$. Geometrically, this process is characterized by giving each of the subspaces of figure 1 a precise interpretation, as depicted in figure 2.

Figure 1: Fine resolution of the Banach spaces $B$ and $\bar{B}$. 

Thus, within $B = N_0$ a filtration $B = N_0 \supset N_1 \supset \cdots \supset N_{k+1}$ is recursively set up by kernels $N_i$ of appropriate linear operators $S_1, S_2, \ldots, S_{k+1}$, where the ranges $R_1, R_2, \ldots, R_{k+1}$ of these operators are constructed in a way to ensure, step by step, an increase of the direct sum $R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1}$ within $\bar{B}$. Then, the nondegeneracy condition of the implicit function theorem is satisfied, as soon as a value of $k$ is reached, such that the direct sum agrees with $\bar{B}$ according to

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The aim is to construct in $B$ some sort of generalized cone $C_k$ around the given curve, characterized by a simple fibration along the base $z_0(\epsilon)$ with local fibers $N^c_\epsilon$, as depicted top-left in the figure. For simplicity, only a half-cone of $C_k$ is drawn.

The construction is complete, as soon as for fixed $\epsilon \neq 0$, the fibre $N^c_\epsilon$ is mapped by the leading $\epsilon$-term of the linearization onto an open neighborhood $V_\epsilon$ of $G[z_0(\epsilon)]$ in the image space $\overline{B}$, as indicated top-right. This constellation occurs, as soon as the direct sum condition (1.3) is satisfied.

Then we call $C_k$ a $k$-transversal or submersive cone.

For $\epsilon \neq 0$ fixed, the fiber $N^c_\epsilon$ is defined using a subspace $N^c \subset B$ that is restricted to an $\epsilon$-dependent neighborhood of the origin yielding $N^c_\epsilon$, as depicted bottom-right. In addition, $N^c_\epsilon$ and $N^c_\epsilon$ are further split into $k + 1$ subspaces $m^1_\epsilon, \ldots, m^{k+1}_\epsilon$ satisfying $m^1_\epsilon \oplus \cdots \oplus m^{k+1}_\epsilon = N^c$, where the subspaces are derived from the fine resolution of figure 1.

When moving $\epsilon$ towards zero, as indicated bottom from right to left, then the subspaces $m^i_\epsilon$ are slightly rotating around the center line, thereby defining smooth Banach manifolds transversally intersecting each other along the center line $z_0(\epsilon)$. The restriction of the $m^i_\epsilon$ to $N^c_\epsilon$ yields the localized manifolds as depicted top-left by orange and green surfaces. In this sense, the generalized cone $C_k$ is spanned by $k + 1$ ruled surfaces.

The importance of the smooth manifolds defined by $m^i_\epsilon$ is given by the fact that they provide us with detailed information about the speed of variation of $G$ when moving away from $z_0(\epsilon)$ with-

Figure 2: Geometrical interpretation of a fine resolution of a $k$-transversal cone.
in one of the manifolds. More precisely, the norms of partial derivatives in the directions of the smooth manifolds satisfy

\[ \| G_{m_2^k}[z_0(\varepsilon)] \| = O(|\varepsilon|^0), \ldots, \| G_{m_{k+1}^k}[z_0(\varepsilon)] \| = O(|\varepsilon|^k), \]  

(1.4)
i.e. when moving through the curve \( z_0(\varepsilon) \), thereby monitoring successively the partial derivatives in the directions of \( m_1^k, \ldots, m_{k+1}^k \), then we obtain highest values in the direction of \( m_1^k \) by order of \( |\varepsilon|^0 \) and lowest values in the direction of \( m_{k+1}^k \) by order of \( |\varepsilon|^k \). In this sense a fine resolution concerning fast and slow variation of \( G \) is constructed in the cone. The partial derivatives are explicitly given using the linear operators \( S_1, S_2, \ldots, S_{k+1} \) from figure 1.

Further, along the manifolds \( m_1^k, \ldots, m_{k+1}^k \), the cone is shrinking to zero with different orders of magnitude with respect to \( \varepsilon \), implying an ellipsoid \( N_{\varepsilon}^c \) to occur for small values of \( \varepsilon \). More precisely, along \( m_1^k, \ldots, m_{k+1}^k \), the cone is shrinking with radius of order \( |\varepsilon|^{2k+1}, \ldots, |\varepsilon|^{k+1} \), implying the product of partial derivatives from (1.4) times radius of cone to be identical of order \( O(|\varepsilon|^{2k+1}) \) in every direction of the cone, i.e. the larger the partial derivative in a certain direction, the smaller the radius of the cone in this direction. This balancing implies that the image of a fiber under \( G \) results in the open ball \( V_\varepsilon \) comprising \( G[z_0(\varepsilon)] \) in \( B \) with radius of order \( |\varepsilon|^{2k+1} \).

As already mentioned, a submersive cone satisfying the rate conditions (1.4) can typically be derived with respect to an arbitrary curve \( z_0(\varepsilon) \) in \( B \) with \( z_0(0) = 0 \). However, in the next step, let us add some approximation properties. First, if the distance of \( G[z_0(\varepsilon)] \) to the origin is greater than the radius of the ball \( V_\varepsilon \), as shown in figure 2 top-right, then obviously no element of \( N_{\varepsilon}^c \) is mapped by \( G \) to the origin of \( B \) and no solution of \( G[z] = 0 \) exists within the fiber \( N_{\varepsilon}^c \). On the other hand, if the distance of \( G[z_0(\varepsilon)] \) to the origin is smaller or equal to the radius of the ball \( V_\varepsilon \) according to \( \| G[z_0(\varepsilon)] \| = O(|\varepsilon|^{2k+1}) \), then at least one point within \( N_{\varepsilon}^c \) is mapped to the origin in \( B \), i.e. for every \( \varepsilon \neq 0 \) at least one solution of \( G[z] = 0 \) occurs in \( N_{\varepsilon}^c \).

Now, if \( C_k \) is a \( k \)-transversal cone, then by implicit function theorem, we will see that all solutions in the cone are given by a smooth manifold \( z(\varepsilon, p) \), \( \varepsilon \neq 0 \) with parameter \( p \) taken from a certain subspace \( P \) of \( B \) with dimension of \( P \) between zero and infinity. In case of \( \dim(P) = 0 \), exactly one solution curve exists in the cone. Reversely, if \( \| G[z_0(\varepsilon)] \| \neq O(|\varepsilon|^{2k+1}) \), we can be sure that the cone represents for \( \varepsilon \neq 0 \) a solution free domain in \( B \).

The construction of the cone \( C_k \) depends exactly on first \( k \) coefficients of the Taylor expansion \( z_0(\varepsilon) = \varepsilon \cdot \tilde{z}_1 + \cdots + \frac{1}{k!} \varepsilon^k \cdot \tilde{z}_k + O(|\varepsilon|^{k+1}) \) and on \( k + 1 \) first derivatives \( G_0^1, \ldots, G_{k+1}^k \) of \( G \) at \( z = 0 \). In particular, a \( k \)-transversal cone cannot be destroyed by perturbations of \( G[z] \) of order \( O(\|z\|^{k+2}) \). In this sense, the nondegeneracy condition remains valid under perturbations of order \( O(\|z\|^{k+2}) \). Further, if solutions \( z(\varepsilon, p) \) exist within the cone, then the solutions cannot be destroyed by perturbations of order \( O(\|z\|^{2k+1}) \). This corresponds to the fact that the approximation condition is of order \( 2k \). Hence, perturbations of \( G[z] \) between \( O(\|z\|^{k+2}) \) and \( O(\|z\|^{2k}) \) may destroy solutions, but not the cone.

We also note, that by a linear diffeomorphic transformation of \( N_{\varepsilon}^c \) and a near identity transformation of \( V_\varepsilon \), the ruled surfaces from figure 2 defined by \( m_1^k, \ldots, m_{k+1}^k \) may further be straightened out and \( V_\varepsilon \) may be transformed to an exact ball, i.e. by setting \( G_\varepsilon[z] := G[z_0(\varepsilon) + z] \), an RL-transformation yields the linearization

\[ J_\varepsilon \circ G_\varepsilon \circ A_\varepsilon = \varepsilon^{2k+1} \cdot L \]  

(1.5)
with an \( \varepsilon \)-independent linear mapping \( L \in GL[N^c, \overline{B}] \), the linear bijection \( A_\varepsilon \in GL[N^c, N^c] \), \( \varepsilon \neq 0 \) from (1.2), as well as a near identity transformation \( J_\varepsilon = I_\overline{B} + O(|\varepsilon|) \) of the image space \( \overline{B} \). Formula (1.5) represents some sort of normal form of a \( k \)-transversal cone. Possibly, (1.5) can be compared to general linearization techniques in Banach spaces derived in [HM], [BH].

Up to this point, all spaces are allowed to be Banach spaces of infinite dimensions. Now, when restricting to finite dimensions, then the information (1.4) about the behaviour of the linearization along \( z_0(\varepsilon) \) can directly be used to perform an \( \varepsilon \)-dependent decoupling of the system on the linear level and to calculate the overall behaviour of the determinant of the linearization with respect to the complement \( N^c \) by

\[
\det \{ G_{N^c}[ z_0(\varepsilon) ] \} = \varepsilon^\chi \cdot r(\varepsilon) \quad \text{with} \quad \chi := 1 \cdot \dim(m_2^\varepsilon) + \cdots + k \cdot \dim(m_{k+1}^\varepsilon) . \tag{1.6}
\]

Obviously, the characteristic number \( \chi \) satisfies \( \chi \geq k \), thus offering the possibility to ascertain solution curves in case of \( \|G[z_0(\varepsilon)]\| = O(|\varepsilon|^{2k+1}) \) and \( \det \{ G_{N^c}[ z_0(\varepsilon) ] \} \neq O(|\varepsilon|^{k+1}) \), which means that the determinant is allowed to vary much slower than required by standard \( (N_0) \) according to \( \det \{ G_{N^c}[ z_0(\varepsilon) ] \} \neq O(|\varepsilon|^{k+1}) \). Note that the only constellation with \( \chi = k \) occurs in case of \( m_2^\varepsilon = \cdots = m_k^\varepsilon = \{0\} \) and \( \dim(m_{k+1}^\varepsilon) = 1 \).

This is well known, that measuring the partial derivatives in complementary directions to the approximate solution \( z_0(\varepsilon) \) in terms of the determinant may not be optimal [F], [BK], [S4]. In particular, in [F] a generalization of Hensel's Lemma is obtained by resolving space appropriately under consideration of different rates of variation. Fisher also remarks that a more geometric interpretation and application of his concept would be desirable. We hope to perform a small step in this direction.

The construction of an appropriate filtration is also performed in [BK], considering equations between modules or even free abelian groupes. We are quite confident that our approach may also work in a more general setting. The techniques used, completely rely on rather basic constructions centered around chain rule and/or substitution of power series into power series with corresponding system of undetermined coefficients. In particular, the techniques apply in infinite dimensions, such as Banach spaces or possibly in cord spaces [BH] as well.

From a more functional analytical point of view, the direct sum condition (1.3) represents some sort of generalization of the \( k \)-transversality concept concerning families of Fredholm operators \( L(\lambda) \), as introduced in [E], [EL], [L1], [LM] with respect to a known trivial solution curve \( y = 0 \) of \( G[\lambda, y] = L(\lambda)y + O(\|y\|^2) \). In that context the leading term of the determinant of the Jacobian with respect to \( y \) is calculated by \( \det \{ G_y[ \lambda, 0 ] \} = \lambda^\chi \cdot r(\lambda) \), where the exponent \( \chi \) is derived by comparable methods as in the paper at hand. In [EL], the exponent \( \chi \) is introduced as \textit{generalized algebraic multiplicity of the operator family} \( L(\lambda) \). Further, the leading term of the determinant is used to calculate the behaviour of the topological degree that ascertains secondary bifurcation, as soon as sign change of the degree occurs along the trivial solution \( y = 0 \).

Obviously by (1.6), these lines of reasoning can be transferred to a \( k \)-transversal cone, supposed it contains a unique solution curve \( z(\varepsilon) \). The corresponding constellation is indicated in figure 3.
First, the topological degree of \( G \) with respect to the section \( N^c_\varepsilon \) is given by the sign of the determinant at the solution \( z(\varepsilon) \). Now, it is interesting to note that the characteristic number \( \chi \) from (1.6) is a property of the complete cone, implying \( \det(\varepsilon G_N^c[z(\varepsilon)]) = \varepsilon^k \cdot r(\varepsilon), \ r(\varepsilon) \neq 0 \) along the solution curve \( z(\varepsilon) \) as well. Hence, supposed \( \mathbb{K} = \mathbb{R} \), we obtain constant degree \( \text{deg} \equiv -1 \) or \( \text{deg} \equiv +1 \) in each half-cone.

Now, if \( \chi \) is odd, then different signs occur in the two half-cones, and if additionally the leading coefficient of \( z_0(\varepsilon) = \frac{1}{\varepsilon} e^1 \cdot \bar{z}_1 + \cdots + \frac{1}{\varepsilon^k} e^k \cdot \bar{z}_k + \cdots, \bar{z}_l \neq 0 \) satisfies \( l \ odd \), then a cylindrical construction can be set up, as indicated in the left diagram of figure 3 by blue color. The cylindrical construction is designed to connect the two half-cones (with different degrees), thus implying by homotopy invariance of the degree at least one point on the cylinder to exist with \( G = 0 \). Finally, by shrinking of the cylinder, a continuum of points with \( G = 0 \) is shown to emanate from \( z = 0 \).

In this sense a \( k \)-transversal cone is perfectly adapted to deal locally with secondary bifurcation based on topological arguments. Far reaching local and global bifurcation results can be found in [LM] and [L2] in the context of \( G[\lambda, y] = L(\lambda)y + O(||y||^2) \).

If \( l \geq 2 \) is even, as shown by the cusp curve on the right side of figure 3 (green curve), then the half cones may have different topological degrees, but the cylindrical construction fails and no results concerning secondary bifurcation are possible, at least based on standard degree theory.

Summarizing, within the Banach space \( B \), we aim to construct different cones out of the singularity at \( z = 0 \) that are characterized by some kind of homogeneity. Above all, homogeneity with respect to \( G \) expansion rates as in (1.4), yielding the possibility to establish complete linearization of \( G \) within the cone as in (1.5), as well as offering the possibility to use topological techniques concerning secondary bifurcation.

From a mathematical point of view, it is quite clear that each cone is simply blown-up to a cylinder, when approaching the singularity from the direction of the cone as \( \varepsilon \to 0 \), i.e. a certain desingularization of the singularity with respect to the cone is performed. During this process, the operator norm of the inverse has to explode. However, this explosion is uniformly bounded within the complete cone by \( O(|\varepsilon|^{-k}) \). Again, here we follow [L1].
One may hope to understand the singularity in a better way, if the conditions in some or all of the cones comprising solution curves are known. A result of this kind appears in [DR], where it is shown that all solutions of \( G[z] = 0 \) are found, supposed that only a finite number of solution curves exist with each curve satisfying the nondegeneracy condition \((1.3)\) with \( k = 1 \), i.e. every cone is spanned by exactly \( k + 1 = 2 \) ruled surfaces. However, this result only applies in case of \( G[0] \) to be a Fredholm operator. In [J], the result is partly extended to \( k = 2 \).

From a technical point of view, the paper intends to combine the well-known concept from algebra of \( 2k \)-approximation/k-nondegeneracy with the basic concept of \( k \)-transversality from functional analysis introduced in [EL].

In section 2, some basic patterns, valid within the system of undetermined coefficients are introduced and a first existence result concerning solutions of \( G[z] = 0 \) is proved and summarized in Theorem 1.

Section 3 contains the main results of the paper concerning uniqueness, perturbation, expansion rates and norm of the inverse, comprised in Theorem 2.

In section 4, we restrict to finite dimensions, possibly after performing a Lyapunov-Schmidt reduction, for dealing with secondary bifurcation from a given solution curve in general position, based on Brouwer’s degree. Section 4 also contains an example, which aims to convince the reader that formulas concerning general position might be useful. Other examples can be found in [S3].

Sections 5, 6 and 7 contain complete proofs.

The main results and proofs are not knew, see [S1]. The interpretation of some of the results may be new. In addition, the patterns concerning the system of undetermined coefficients might be of some interest.

The content was motivated by stimulating discussions with J. López-Gómez, W.-J. Beyn and in particular E. Bohl.

### 2. Patterns within the System of undetermined Coefficients

In this section, we basically focus on the description of some characteristic patterns within the system of undetermined coefficients. As an application, we present a first theorem concerning the existence of local solution curves through \( z = 0 \), parametrized by the external parameter \( \varepsilon \in \mathbb{K} \). Depending on the dimension of the smallest subspace within the filtration of \( B \) in figure 1, the solution curves will also depend on further parameters implying the existence of a smooth solution manifold of higher (or infinite) dimension.

In some more detail, an equation \( G[z] = 0, G \in C^q(B, \overline{B}) \) with \( G[0] = 0 \) and \( B, \overline{B} \) real or complex Banach spaces is considered, with the aim of finding solution curves \( G[z(\varepsilon)] = 0 \) through the origin with \( \varepsilon \in \mathbb{K} = \mathbb{R}, \mathbb{C}, |\varepsilon| \ll 1 \).

The basic technique to derive \( z(\varepsilon) \) is very simple. For \( k \geq 1 \), insert the ansatz \( z = \varepsilon \cdot z_1 + \cdots + \frac{1}{(2k+1)!} \varepsilon^{2k+1} \cdot z_{2k+1} \) into \( G \) and perform a Taylor expansion at \( \varepsilon = 0 \) up to order \( 2k + 1 \) by

\[
G\left[ \varepsilon \cdot z_1 + \cdots + \frac{\varepsilon^{2k+1}}{(2k+1)!} z_{2k+1} \right] = \sum_{i=1}^{2k+1} \frac{1}{i!} \varepsilon^i \cdot T^i[z_i, \ldots, z_1] + \varepsilon^{2k+2} \cdot r[\varepsilon, z_{2k+1}, \ldots, z_1] \tag{2.1}
\]

with smooth remainder map \( r[\cdot], q \geq 2k + 2 \) and coefficients \( T^i[z_i, \ldots, z_1], i \geq 1 \), building up the so called system of undetermined coefficients according to
\[ T^1[z_1] = G^1_0 \cdot z_1 = 0 \]
\[ T^2[z_2, z_1] = G^1_0 \cdot z_2 + G^2_0 \cdot z_1^2 = 0 \]
\[ T^3[z_3, z_2, z_1] = G^1_0 \cdot z_3 + 3G^2_0 \cdot z_1 z_2 + G^3_0 \cdot z_1^3 = 0 \]
(2.2)

Here \( G^\beta_0 \) denotes the \( \beta \)-th derivative of \( G \) at \( z = 0 \). In general, by higher order chain rule [AMR], we obtain for \( i \geq 1 \) the expression
\[ T^i[z_i, ..., z_1] = \sum_{\beta=1}^{i} G^\beta_0 \sum_{n_1+...+n_i=\beta} \frac{i!}{n_1!...n_i!} \prod_{\tau=1}^{i} \left( \frac{1}{\tau!} z_\tau \right)^{n_\tau} \in \bar{B} \]
(2.3)

with \( T^i \) depending explicitly from \( z_1, ..., z_i \) and \( G^1_0, ..., G^i_0 \). In this sense, the \( i \)-th coefficient \( T^i \) within the \( \varepsilon \)-expansion (2.1) is given by a sum composed of first \( i \) coefficients of the ansatz \( z = \varepsilon \cdot z_1 + \cdots \) and first \( i \) coefficients of the Taylor expansion of \( G \) at \( z = 0 \). Moving \( k \) towards infinity, this process may also be interpreted as plugging the \( \varepsilon \)-power series of the ansatz into the \( z \)-power series of \( G \). This viewpoint is taken in [54], where power series solutions of differential algebraic equations are investigated.

In the next step, we summarize some results concerning the system of undetermined coefficients. First, by direct inspection of (2.3), the following linearity structure with respect to \([T^{2k}, ..., T^{k+1}]\) and higher order coefficients \([z_{2k}, ..., z_{k+1}]\) is valid for \( k \geq 1 \)

\[
\begin{pmatrix}
T^{2k}[z_{2k}, ..., z_1] \\
\vdots \\
T^{k+1}[z_{k+1}, ..., z_1]
\end{pmatrix} = \Delta^k(z_{k-1}, ..., z_1) \cdot \begin{pmatrix} z_{2k} \\ \vdots \\ z_{k+1} \end{pmatrix} + I^k(z_k, ..., z_1)
\]
(2.4)

with \( \Delta^1(z_1) = G_0^1 \) in case of \( k = 1 \). Here \( \Delta^k(z_{k-1}, ..., z_1) \in L[\bar{B}, \bar{B}] \) denotes an upper triangular matrix operator with elements \( \Delta^k_{ij}(z_{k-1}, ..., z_1) \in L[\bar{B}, \bar{B}] \) and \( I^k(z_k, ..., z_1) \) an element in \( \bar{B} \), both explicitly defined in section 5.

In addition and for later use, we state the following relation for \( k \geq 1 \)

\[
\begin{pmatrix}
T^k[z_k, ..., z_1] \\
\vdots \\
T^1[z_1]
\end{pmatrix} = (\Gamma^k)^{-1} \cdot \Delta^k(z_{k-1}, ..., z_1) \cdot \Gamma^k \cdot \begin{pmatrix} z_k \\ \vdots \\ z_1 \end{pmatrix}
\]
(2.5)

with \( \Gamma^k = diag[\Gamma^k_1, ..., \Gamma^k_k] \) and \( \Gamma^k_i = \begin{pmatrix} k + i \\ i - 1 \end{pmatrix}, \ i = 1, ..., k \),

i.e. the first \( k \) coefficients \([T^1, ..., T^k]\) of the power series (2.1) can completely be expressed by the linear part \( \Delta^k \) of the next \( k \) coefficients \([T^{k+1}, ..., T^{2k}]\) from (2.4). In this sense, \( \Delta^k \) delivers some sort of recurrency structure that is playing a key role concerning the investigation of the system of undetermined coefficients.

Finally, coefficients above \( T^{2k} \) can be formulated for \( k \geq 0, l \geq 0 \) by lower ones according to
\[ T^{2k+1+\ell}(z_{2k+1+i}, \ldots, z_{2k+1+i}, \ldots, z_{k+1}, z_k, \ldots, z_1) \]

\[ = \left[ T^0_{z_0}(z_0) \ T^1_{z_1}(z_1) \ \cdots \ T^k_{z_k}(z_k, \ldots, z_1) \right] \cdot \left[ C^{2k+1} \begin{pmatrix} z_{2k+1+i} \\ \vdots \\ z_{k+1+i} \end{pmatrix} \right] + R_{2k+1+i}(z_{k+1}, \ldots, z_1) \quad (2.6) \]

with

\[ C^{2k+1} = \text{Diag}[\gamma_0^{2k+1}, \ldots, \gamma_k^{2k+1}] \quad \text{and} \quad \gamma_k^{2k+1} = \left( 2k + 1 + \ell \right) (2t)^{-1}, \quad t = 0, \ldots, k \quad (2.7) \]

Here, \( T^0_{z_0}(z_0) = G_0 \) and \( T^i_{z_i}(z_i, \ldots, z_1) \), \( i = 1, \ldots, k \) denote partial derivatives of \( k + 1 \) low order coefficients \( T^2, T^4, \ldots, T^{2k} \) with derivatives taken with respect to low z-coefficients \( z_1, \ldots, z_k \) respectively, implying a second recursion principle. Formula (2.6) may be interpreted as an extended formula by Hurwitz [Hu] expressing high order derivatives of an expansion by lower ones. For comparable results see [DL], Lemma 2.2 and [VFZ], Theorem 3.3.5, 3.6.

The proofs of (2.4), (2.5) follow by simple and elementary calculation from (2.1), omitted in the paper at hand. The proof of (2.6) can be found in section 7 formulas (7.1)-(7.6).

Next, assume the existence of \([\tilde{z}_{2k}, \ldots, \tilde{z}_1]\) with \( T^{2k}[\tilde{z}_{2k}, \ldots, \tilde{z}_1] = \cdots = T^1[\tilde{z}_1] = 0 \), implying

\[ ||G[\varepsilon \cdot \tilde{z}_1 + \cdots + \frac{1}{(2k)!} \varepsilon^{2k} \cdot \tilde{z}_{2k}]|| = O(|\varepsilon|^{2k+1}) \quad (2.8) \]

by (2.1) with \( z_{2k+1} = 0 \). Thus, the curve \( z_0(\varepsilon) = \varepsilon \cdot \tilde{z}_1 + \cdots + \frac{1}{(2k)!} \varepsilon^{2k} \cdot \tilde{z}_{2k} \) defines an approximate solution curve of \( G[z] = 0 \) of order \( 2k \), which obviously corresponds to the approximation condition (A0) of the introduction.

Now, for given \([\tilde{z}_{2k}, \ldots, \tilde{z}_1] \in B^k \), there exists a solution \([\tilde{z}_{2k}, \ldots, \tilde{z}_{k+1}] \in B^k \) of (2.4) if and only if the range \( R[\Delta^k(\tilde{z}_{k-1}, \ldots, \tilde{z}_1)] \) contains the inhomogeneity \( 1^k(\tilde{z}_{k}, \ldots, \tilde{z}_1) \). But then, the solution \([\tilde{z}_{2k}, \ldots, \tilde{z}_1] \) of \( T^{2k} = \cdots = T^1 = 0 \) is embedded within an affine subspace of solutions given by \([\tilde{z}_k, \ldots, z_1] = [\tilde{z}_k, \ldots, \tilde{z}_1] \) and

\[ \begin{pmatrix} z_{2k} \\ \vdots \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{z}_{2k} \\ \vdots \\ \tilde{z}_{k+1} \end{pmatrix} + \begin{pmatrix} b_{2k} \\ \vdots \\ b_{k+1} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} b_{2k} \\ \vdots \\ b_{k+1} \end{pmatrix} \in N[\Delta^k(\tilde{z}_{k-1}, \ldots, \tilde{z}_1)] =: N[\Delta^k]. \quad (2.9) \]

Here \( N[\Delta^k] \) denotes the nullspace of the operator in brackets. Note also, if the leading coefficients \([\tilde{z}_k, \ldots, \tilde{z}_1]\) are fixed, then all solutions of \( T^{2k} = \cdots = T^1 = 0 \) are given by (2.9).

Further, every element \([b_{2k}, \ldots, b_{k+1}]\) from \( N[\Delta^k] \) implies an approximate solution curve of order \( 2k \) within \( B \) according to

\[ z = \varepsilon \cdot \tilde{z}_1 + \cdots + \frac{1}{(k+1)!} \varepsilon^{k+1} \cdot (\tilde{z}_{k+1} + b_{k+1}) + \cdots + \frac{1}{(2k)!} \varepsilon^{2k} \cdot (\tilde{z}_{2k} + b_{2k}) \]

\[ + \frac{1}{(2k+1)!} \varepsilon^{2k+1} \cdot b_{2k+1}, \quad (2.10) \]

where the last summand of order \( 2k + 1 \) and \( b_{2k+1} \in B \) does not disturb this property and is merely added for further reasoning. Note that \( b_{2k+1} \) occurs the first time within \( T^{2k+1+\ell}[\cdot] \).

All the approximate solution curves in (2.10), obtained by variation of \([b_{2k+1}, \ldots, b_{k+1}] \in B \times N[\Delta^k], \) share the same leading coefficients up to order \( k \), defining within the Banach space \( B \) some sort of cone of approximations of order \( 2k \), as qualitatively indicated in figure 4.
The red center line $z_0(\varepsilon)$ is obtained with $[b_{2k+1}, \ldots, b_{k+1}] = 0$, whereas the blue line represents an approximation with $[b_{2k+1}, \ldots, b_{k+1}] \neq 0$.

It remains to show that within this cone, even true solutions exist, if some further structure is given to the cone that finally allows for application of the implicit function theorem with respect to $[b_{2k+1}, \ldots, b_{k+1}]$. In this sense, the initial approximation $z_0(\varepsilon)$ of order $2k$ is only needed to construct the affine subspace (2.9), in this way offering variability to the coefficients $z_{k+1} = \bar{z}_{k+1} + b_{k+1}, \ldots, z_{2k+1} = \bar{z}_{2k+1}$ within (2.10). Note also, that this construction precisely explains the drop from initial approximation of order $2k$ down to true solutions agreeing merely up to order of $k$ with the initial approximation $z_0(\varepsilon) = \varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{(2k)!}\varepsilon^{2k} \cdot \bar{z}_{2k}$.

Figure 4: Cone of approximations of order $2k$ (red, blue) and $2k+1$ (green).

Now, plugging the affine subspace (2.9), (2.10) into (2.1) and cancelling out $\varepsilon^{2k+1}$, we end up with the smooth remainder equation

$$T^{2k+1}[z_{2k+1}, \ldots, z_1] = W^{2k+1}(z_k, \ldots, z_1) \cdot \begin{pmatrix} z_{2k+1} \\ \vdots \\ z_{k+1} \end{pmatrix} + R^{2k+1}(z_k, \ldots, z_1) \in \overline{B}$$

and $H \in C^{q-2k-1}(U \times B \times N[\Delta^k], \overline{B})$, $U = \{\varepsilon \in \mathbb{K} : |\varepsilon| \ll 1\}$. Further, direct inspection of (2.3) shows the linearity property

$$T^{2k+1}[z_{2k+1}, \ldots, z_1] = W^{2k+1}(z_k, \ldots, z_1) \cdot \begin{pmatrix} z_{2k+1} \\ \vdots \\ z_{k+1} \end{pmatrix} + R^{2k+1}(z_k, \ldots, z_1) \in \overline{B}$$

with $W^{2k+1}(z_k, \ldots, z_1) \in L[B^{k+1}, \overline{B}]$ and $R^{2k+1}(z_k, \ldots, z_1) \in \overline{B}$, which are again defined explicitly in section 5. Therefore, the remainder equation (2.11) reads for $\varepsilon = 0$

$$H(0, b_{2k+1}, b_{2k}, \ldots, b_{k+1}) = T^{2k+1}[b_{2k+1}, \begin{pmatrix} \bar{z}_{2k} \\ \vdots \\ \bar{z}_{k+1} \end{pmatrix}] + \begin{pmatrix} b_{2k} \\ \vdots \\ b_{k+1} \end{pmatrix}, \bar{z}_k, \ldots, \bar{z}_1$$
\[
W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) \cdot \left\{ \begin{pmatrix} 0 \\ \vec{z}_{2k} \\ \vdots \\ \vec{z}_{k+1} \end{pmatrix} + \begin{pmatrix} b_{2k+1} \\ b_{2k} \\ \vdots \\ b_{k+1} \end{pmatrix} \right\} + R^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) = 0.
\]

Now, if the linear mapping
\[
W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) \in L[ B \times N[\Delta^k], \overline{B} ]
\]
is surjective, then the mapping can also be restricted to a direct complement \(N_c\) of its closed kernel to obtain a bijection \(W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) \in L[N_c, \overline{B}]\) by decomposition of the Banach space \(B \times N[\Delta^k]\) according to
\[
B \times N[\Delta^k] = B \times N[\Delta^k] / N[ W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) ] \oplus N[ W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) ] 
\]
\[
= : N_c \oplus N[ W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) ].
\]

Hence, for \(\epsilon = 0\), the remainder equation \(H(0, b_{2k+1}, b_{2k}, \ldots, b_{k+1}) = T^{2k+1}[\cdot] = 0\) is given by the linear equation (2.13) with corresponding affine solution subspace
\[
\begin{pmatrix} b_{2k+1} \\ \vdots \\ b_{k+1} \end{pmatrix} = \begin{pmatrix} \overline{b}_{2k+1} \\ \vdots \\ \overline{b}_{k+1} \end{pmatrix} + \begin{pmatrix} c_{2k+1} \\ \vdots \\ c_{k+1} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} c_{2k+1} \\ \vdots \\ c_{k+1} \end{pmatrix} \in N[ W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) ]
\]

and \(\overline{b} = (\overline{b}_{2k+1}, \ldots, \overline{b}_{k+1})^T = -W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1)^{-1} \cdot R^{2k+1}(\vec{z}_{2k}, \ldots, \vec{z}_1) \in N_c\). Combining (2.9) and (2.15), we obtain by (2.10) a first improvement within the cone of approximations of order \(2k\), i.e. a family of solution curves with approximation of order \(2k + 1\) parametrized by \(\epsilon\) and \(N[ W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) ]\). In figure 4, the affine solution subspace (2.15) is indicated by the green line in \(B \times N[\Delta^k]\), implying within the cone a family of approximations of order \(2k + 1\). Only one of these approximations of order \(2k + 1\) is depicted in the cone by a green line.

Now, at this point we have to decide, whether only to look for power series solutions of \(G[z] = 0\) or to look for \(C^{q-2k-1}\) smooth solution manifolds by use of the implicit function theorem.

If only power series solutions are looked for, then we are already done, because it can be shown that from now on, every equation \(T^{2k+2}, T^{2k+3}, \ldots\) assumes a structure similar to (2.13), with leading operator \(W^{2k+2}, W^{2k+3}, \ldots\) adopting the surjectivity of \(W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1)\), hence ensuring solvability of the system of undetermined coefficients up to infinity. In this manner, the family of approximations of order \(2k + 1\) is further lifted to approximations of order \(2k + 2, 2k + 3, \ldots\).

In some more detail, this aspect is treated in [S3], Corollary 2, where approximate solution curves of a \(k\)-transversal cone are lifted to arc space \(X_\infty\). We do not want to focus on power series solutions in this paper, but note that the key result needed is obviously given by formula (2.6), supplying knowledge of leading operators up to infinity.

If \(C^{q-2k-1}\) smooth solution manifolds are looked for, then suppose the complement \(N_c\) to be closed, implying by bounded inverse theorem \(W^{2k+1}(\vec{z}_k, \ldots, \vec{z}_1) \in GL[ N_c, \overline{B} ]\) and the solution
$\bar{b} \in N_c$ of $H(0, \cdot) = 0$ can uniquely be continued within $N_c$ to $\varepsilon \neq 0$ by use of the implicit function theorem.

In some more detail, for every $n : (c_{2k+1}, \ldots, c_{k+1})^T \in N/W^{2k+1}(\bar{z}_k, \ldots, \bar{z}_1) =: N$ from (2.15), there exists a locally defined function $(b_{2k+1}, \ldots, b_{k+1})/\varepsilon, n) \in N_c, |\varepsilon| < 1$, of class $C^{q-2k-1}$ with

$$G[\varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{r!} \varepsilon^k \cdot \bar{z}_k + \sum_{i=k+1}^{2k+1} \frac{1}{i!} \varepsilon^i \cdot (\bar{z}_i + b_i(\varepsilon, n)) ] = 0 \quad (2.16)$$

and $\bar{z}_{2k+1} = 0$. In addition, if $n \in N$ is restricted to a bounded domain $B_\gamma(0) = \{n \in N : ||n|| < \gamma\}, \gamma > 0$, then we obtain smoothness according to $(b_{2k+1}, \ldots, b_{k+1})/\varepsilon, n) \in C^{q-2k-1}(B_{\delta_1}(0) \times B_\gamma(0), B_{\delta_2}(\bar{b}))$ with $B_{\delta_1}(0) = \{\varepsilon \in K : |\varepsilon| < \delta_1\}$ and $B_{\delta_2}(\bar{b}) = \{n \in N_c : ||n - \bar{b}|| < \delta_2\}$ denoting sufficiently small balls in $K$ and $N_c$ around $\varepsilon = 0$ and $n = \bar{b}$ respectively. Note also that $\gamma$ is allowed to be chosen arbitrary large, but finite.

Moreover, within the open and bounded domain $B_{\delta_1}(0) \times [B_{\delta_2}(\bar{b}) \oplus B_\gamma(0)]$, the remainder equation $H(\varepsilon, b_{2k+1}, b_{2k}, \ldots, b_{k+1}) = 0$ from (2.11), (2.15) has no other solutions and summarizing the following existence and uniqueness theorem is shown.

**Theorem 1**: Assume $G \in C^q(B, \bar{B}), q \geq 2k + 2$ with $G[0] = 0$ and the existence of $[\bar{z}_{2k}, \ldots, \bar{z}_{k+1}, \bar{z}_k, \ldots, \bar{z}_1]$ satisfying approximation of order $2k$ according to

$$T^{2k} [\bar{z}_{2k}, \ldots, \bar{z}_{k+1}, \bar{z}_k, \ldots, \bar{z}_1] = \cdots = T^1 [\bar{z}_1] = 0,$$  

(A1)
as well as nondegeneracy of order $k$ by

$$R[W^{2k+1}(\bar{z}_k, \ldots, \bar{z}_1)] = \bar{B}. \quad (N_1)$$

Then, for $\varepsilon \in B_{\delta_1}(0)$ and $(b_{2k+1}, \ldots, b_{k+1}) \in B_{\delta_2}(\bar{b}) \oplus B_\gamma(0)$, all solutions of $G[z] = 0$ within the corresponding cone of approximations of order $2k$ from (2.10) are given by

$$z(\varepsilon, n) = \varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{r!} \varepsilon^k \cdot \bar{z}_k + \sum_{i=k+1}^{2k+1} \frac{1}{i!} \varepsilon^i \cdot (\bar{z}_i + b_i(\varepsilon, n)) \quad (I_1)$$

with $b_i(\varepsilon, n) \in C^{q-2k-1}(B_{\delta_1}(0) \times B_\gamma(0), B_{\delta_2}(\bar{b}))$ and $(b_{2k+1}, \ldots, b_{k+1})(0, n) = \bar{b} \in N_c$.

From $(I_1)$ we see that the solution curves agree with the initial approximation $z_0(\varepsilon) = \varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{r!} \varepsilon^k \cdot \bar{z}_k$ (center line of cone) only with respect to the $\varepsilon$-derivatives $\bar{z}_1, \ldots, \bar{z}_k$ implying identity of order $k$ with respect to $\varepsilon$, analogously to condition $(I_0)$ from the introduction.

In advance, we also note that the operator $W^{2k+1}(\bar{z}_k, \ldots, \bar{z}_1)$ in $(N_1)$ merely depends on first $k + 1$ coefficients $G_0^1, \ldots, G_k^{k+1}$ of the Taylor expansion of $G$ at $z = 0$. Again, this corresponds exactly to condition $(N_0)$ from the introduction, where first $k + 1$ derivatives of $G$ have to be calculated in the origin for deciding about the validity of the det-condition $(N_0)$.

Finally, both of the approximation conditions $(A_1)$ and $(A_0)$ depend on first $2k$ derivatives of $G$ in the origin, in this way yielding conditions of order $2k$.

**Remark**: It should be noted that instead of $T^{2k+1}[\cdot] + O(|\varepsilon|) = 0$ in (2.11), we can also work with the remainder equation

$$T^{2k} [z_{2k}, \ldots, z_{k+1}, z_k, \bar{z}_{k-1}, \ldots, \bar{z}_1] + O(|\varepsilon|) = 0,$$

implying solutions of the form

$$z(\varepsilon, n) = \varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{r!} \varepsilon^k \cdot \bar{z}_k + \sum_{i=k+1}^{2k+1} \frac{1}{i!} \varepsilon^i \cdot (\bar{z}_i + b_i(\varepsilon, n)) \quad (II_1)$$

with $b_i(\varepsilon, n) \in C^{q-2k-1}(B_{\delta_1}(0) \times B_\gamma(0), B_{\delta_2}(\bar{b}))$ and $(b_{2k+1}, \ldots, b_{k+1})(0, n) = \bar{b} \in N_c$.
\[ z(\varepsilon, n) = \varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{(k-1)!} \varepsilon^{k-1} \cdot \bar{z}_{k-1} + \frac{1}{k!} \varepsilon^k \cdot (\bar{z}_k + b_k(\varepsilon, n)) + \sum_{i=k+1}^{2k} \frac{1}{i!} \varepsilon^i \cdot (\bar{z}_i + b_i(\varepsilon, n)), \]

which represent a certain generalization of Theorem 1, because the coefficient \( z_k = \bar{z}_k + b_k(0, n) \) is now allowed to vary in the vicinity of \( \bar{z}_k \). But also when performing this extension, first a basic solution of \( T^{2k+1} = 0 \) is constructed, which is further continued to \( \varepsilon \neq 0 \) using surjectivity of the same operator \( W^{2k+1}(\bar{z}_k, \ldots, \bar{z}_1) \).

In spite of Theorem 1, a lot of questions remain open. First, the cone of approximations within the Banach space \( B \) is only poorly described by the image of the smooth operator in (2.10) mapping \( (\varepsilon, b_{2k+1}, \ldots, b_{k+1}) \in \mathcal{B}_{\delta_1}(0) \times \left[ \mathcal{B}_{\delta_2}(b) \oplus \mathcal{B}_r(0) \right] \subset \mathbb{K} \times B^{k+1} \) into \( B \). In fact, it is even not clear, whether each half-cone defines an open domain in \( B \) comprising the center line \( z_0(\varepsilon) \).

Secondly, the family \( z(\varepsilon, n) \) of solution curves obviously shows a strong redundancy due to possible reparametrizations of a solution curve, e.g. by \( \varepsilon \to \varepsilon + \varepsilon^{k+1} \), running through the same solution orbit in \( B \), but with different parametrization within the family \( z(\varepsilon, n) \). It would be desirable to have some sort of minimal parametrization that covers all solutions in the cone.

Thirdly, Theorem 1 does not at all show the iterative aspect of successively increasing \( k \) until the direct sum \( R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1} \subset \bar{B} \) satisfies \( R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1} = \bar{B} \) from (1.3), i.e. the link between the surjectivity condition \( R[W^{2k+1}(\bar{z}_k, \ldots, \bar{z}_1)] = \bar{B} \) and the fine resolution of a \( k \)-transversal cone is not yet performed.

And finally, it has not become clear that the cone with corresponding fibration can be constructed without supposing any approximation properties. In the next section, our aim is to fill these gaps. Nevertheless, we think it is worth noting Theorem 1 separately, due to its remarkable simplicity of proof.

### 3. Main Results

Now, to answer these questions, take \( k \) leading coefficients \([\bar{z}_1, \ldots, \bar{z}_k] = [0, 0, \bar{z}_1, \ldots, \bar{z}_k] \in B^k \) of an arbitrary curve \( z_0(\varepsilon) = \varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{k!} \varepsilon^k \cdot \bar{z}_k + O(|\varepsilon|^{k+1}) \) satisfying \( \bar{z}_1 \neq 0 \), \( 1 \leq l \leq k \) and define a sequence of linear operators \( S_1, S_2, \ldots, S_{k+1} \) with corresponding direct sums along the following lines of reasoning.

As preliminary, set \( N_0 := B, R_0^\perp = \bar{B} \) and \( S_1 := G_1^b \in L[N_0, R_0^\perp] \) with closed kernel \( N_1 := N[S_1] \subset N_0 \), range \( R_1 := R[S_1] \subset R_0^\perp \) and associated direct sum decompositions of \( B \) and \( \bar{B} \) according to

\[
B = N_0 = N_1^c \oplus N_1
\]

\[
\bar{B} = R_0^\perp = R_1 \oplus R_1^c
\]

(3.1)

The complementary subspaces \( N_1^c \) and \( R_1^c \) as well as \( R_1 \) by itself are assumed to be closed subspaces with continuous bijection \( S_1 \) between \( N_1^c \) and \( R_1 \) as indicated in (3.1) by arrows. In addition choose the complement \( N_1^c \) such that \( \bar{z}_1 \notin N_1^c \). Next define

\[
S_2 := 2P_{R_1^c}G_1^b\bar{z}_1N_1 \in L[N_1, R_1^c]
\]

(3.2)

with continuous projection \( P_{R_1^c} \) evaluated with respect to decomposition (3.1). Note that \( S_2 \) is mapping from the kernel \( N_1 \) of \( S_1 \) into the complement \( R_1^c \) of the image \( R_1 \) of \( S_1 \), i.e. the second operator \( S_2 \) is just creating values in the subspace \( R_1^c \) that is not reached by the first mapping. As
for $S_1$, we assume for $S_2$ a decomposition of $N_1$ and $R_1^c$ by closed subspaces using kernel $N_2 = N[S_2] \subset N_1$ and range $R_2 = R[S_2] \subset R_1^c$ according to $N_1 = N_2^c \oplus N_2$ and $R_1^c = R_2 \oplus R_2^c$. And again choose $N_2^c$ such that $z_i \notin N_2^c$.

We note in advance that if the given curve $z_0(\varepsilon)$ is an approximation of order $2k$, then the construction simplifies somewhat due to $z_i \in N_1, N_2, \ldots$, i.e. $z_i \neq 0$ is contained in every kernel $N_i$ and no care is necessary with respect to chosen complements $N_2^c$.

Continuing this way up to $S_{k+1} \in L[N_k, R_k^c]$, a sequence of successively defined closed kernels and ranges is constructed satisfying $B = N_0 \supset N_1 \supset \cdots \supset N_k \supset N_{k+1}$, as well as defining direct sum decompositions of $B$ and $\tilde{B}$ according to

$$B = N_1^c \oplus N_2^c \oplus \cdots \oplus N_{k+1}^c \oplus N_{k+1} \quad \text{with} \quad N_{k+1} = P_{k+1} \oplus \{\tilde{z}_{i,k+1}\} \quad (3.3)$$

The complete construction merely depends on first $k$ coefficients $\tilde{z}_1, \ldots, \tilde{z}_k$ of $z_0(\varepsilon)$ and on first $k + 1$ derivatives $G^1_0, \ldots, G^{k+1}_0$ of $G$ at $z = 0$. It may well happen to obtain $S_i = 0 \in L[N_{i-1}, R_{i-1}^c]$ for some $1 \leq i \leq k + 1$ with corresponding decompositions simplifying according to $N_{i-1} = \{0\} \oplus N_i$ and $R_{i-1}^c = \{0\} \oplus R_i^c$.

In case of $G^1_0 = \cdots = G^{k+1}_0 = 0$, even the case $\tilde{B} = \{0\} \oplus \cdots \oplus \{0\} \oplus R_{k+1}^c$ occurs, which simply means that up to given $k \geq 1$ no progress is made with respect to building up the direct sum $\tilde{B} = R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1}$.

Next, considering $\tilde{z}_i \notin N_i^c$, choose the projection of $\tilde{z}_i$ onto $N_{k+1}$, i.e. $\tilde{z}_{i,k+1} \neq 0$, allowing us to refine the direct sum of $B$ in (3.3) according to $N_{k+1} = P_{k+1} \oplus \{\tilde{z}_{i,k+1}\}$ with closed complement $P_{k+1} \subset N_{k+1}$, as already indicated in (3.3). Then, the cone $\mathcal{C}_k$ is defined by the map

$$Z_k(\varepsilon, n_1^c, \ldots, n_{k+1}^c, p_{k+1}) = z_0(\varepsilon) + \underbrace{[\varepsilon^{2k+1} \ldots \varepsilon^{k+1}]}_{(2k+1)!} \cdot \underbrace{\begin{pmatrix} I_B & * & \vdots & * \\ \vdots & I_B \\ n_1^c \\ \vdots \\ n_{k+1}^c + p_{k+1} \end{pmatrix}}_{= A_{\varepsilon}[A_1^c \cdots A_{k+1}^c]} \quad (3.4)$$

The curve $z_0(\varepsilon)$ defines the center line of the cone with leading term given by $\tilde{z}_i$. The second summand is defined by a linear mapping $A_{\varepsilon}$ satisfying for $\varepsilon \neq 0$

$$A_{\varepsilon} \in GL[N_1^c \times \cdots \times N_k^c \times (N_{k+1}^c \oplus P_{k+1}), N_1^c \oplus \cdots \oplus N_k^c \oplus N_{k+1}^c \oplus P_{k+1}] \ .$$

Hence, the range of $A_{\varepsilon}$ defines by (3.3) for every $\varepsilon \neq 0$ a direct complement to the center line $z_0(\varepsilon)$ implying a simple fibration with base $z_0(\varepsilon)$ in $B$, i.e. at every point of $z_0(\varepsilon)$ a direct complement to $z_0(\varepsilon)$ is attached.

The elements of the matrix operator $\bar{M}_{k+1} \in GL[B^{k+1}, B^{k+1}]$ are given by the composition of multilinear mappings, mainly derived from (3.3), where the basic structure of the bijection $\bar{M}_{k+1}$ is given by an upper tridiagonal matrix with diagonal exclusively composed of the identity map, as already indicated in (3.4). The precise definitions are constructively given in section 5.
Then, by restriction of \( n_i^c \in N_i^c \) and \( p_{k+1} \in P_{k+1} \) to arbitrary open, but bounded domains \( U_i^c \subset N_i^c \) and \( U_{k+1}^P \subset P_{k+1} \), each comprising zero, the image \( U_\varepsilon := R[A_\varepsilon(U_i^c \times \cdots \times U_k^c \times (U_{k+1}^c \oplus U_{k+1}^P))] \) is shrinking to zero as \( \varepsilon \to 0 \), thereby creating by open mapping theorem the open and cone-like neighborhood \( C_k \) of \( z_0(\varepsilon) \) in \( B \) for \( \varepsilon \neq 0 \) according to

\[
C_k = \{ z \in B : z = Z_k(\varepsilon, n^c, p), \ n^c \in U_1^c \times \cdots \times U_{k+1}^c, \ p_{k+1} \in U_{k+1}^P, \ 0 < |\varepsilon| \ll 1 \}. \quad (3.6)
\]

In some more detail, the operator \( Z_k \) delivers by upper triangularity of \( \bar{M}_{k+1} \) in (3.4) a fine structure within the cone caused by the manifolds that occur by individually applying the operator \( Z_k \) to the domains \( U_i^c \subset N_i^c, \ldots, U_{k+1}^c \subset N_{k+1}^c \) and \( U_{k+1} \subset P_{k+1} \) according to

\[
Z_k(\varepsilon, n^c_0, 0, \ldots, 0) = z_0(\varepsilon) + \frac{1}{(2k+1)!} \varepsilon^{2k+1} \cdot n^c_1 =: Z_k^1(\varepsilon, n^c_1) \\
Z_k(\varepsilon, 0, n^c_2, \ldots, 0) = z_0(\varepsilon) + \frac{1}{(2k)!} \varepsilon^{2k} \cdot n^c_2 =: Z_k^2(\varepsilon, n^c_2) \\
Z_k(\varepsilon, 0, 0, n^c_3, \ldots, 0) = z_0(\varepsilon) + \frac{1}{(2k+1)!} \varepsilon^{2k+1} \cdot n^c_3 + O(\varepsilon^{2k}) =: Z_k^3(\varepsilon, n^c_3) \\
\vdots \quad \vdots \quad \vdots \\
Z_k(\varepsilon, 0, \ldots, 0, n^c_{k+1}, 0) = z_0(\varepsilon) + \frac{1}{(k+1)!} \varepsilon^{k+1} \cdot n^c_{k+1} + O(\varepsilon^{k+2}) =: Z_k^{k+1}(\varepsilon, n^c_{k+1}) \\
Z_k(\varepsilon, 0, \ldots, 0, 0, p_{k+1}) = z_0(\varepsilon) + \frac{1}{(k+1)!} \varepsilon^{k+1} \cdot p_{k+1} + O(\varepsilon^{k+2}) =: Z_k^0(\varepsilon, p_{k+1}) \quad (3.7)
\]

By construction, for \( \varepsilon \neq 0 \) sufficiently small, the different manifolds meet transversally along the center line \( z_0(\varepsilon) \). Further, within \( Z_k^{k+1}(\varepsilon, n^c_{k+1}) \) and \( Z_k^0(\varepsilon, p_{k+1}) \), the variables \( n^c_{k+1} \) and \( p_{k+1} \) are by leading order multiplied with \( \varepsilon^{k+1} \), which denotes shrinking of these manifolds by order of \( O(\varepsilon^{k+1}) \) as \( \varepsilon \to 0 \). The remaining manifolds show faster shrinking between \( O(\varepsilon^{k+2}) \) and \( O(\varepsilon^{2k+1}) \). The higher order terms within (3.7) cause the rotation of the manifolds around \( z_0(\varepsilon) \), as indicated in figure 2.

Note also that the behaviour of the first and the second manifold \( Z_k^1(\varepsilon, n^c_1) \) and \( Z_k^2(\varepsilon, n^c_2) \) is rather simple, showing no rotation at all. Now, if \( k = 1 \), then only these two manifolds occur in the system and the complexity of the general scheme is not completely seen. In principle, the case \( k = 1 \) is treated in many papers, maybe most prominent [CR].

Now, when mapping \( C_k \) from \( B \) to \( B \) by \( G \), we will see that the map values \( G[z_0(\varepsilon)] \) of the center line are perturbed in the following way

\[
G[Z_k(\varepsilon, n^c, p_{k+1})] = G[z_0(\varepsilon)] + \varepsilon^{2k+1} \cdot \tilde{L}_{k+1} \cdot \begin{pmatrix} n^c_1 \\ \vdots \\ n^c_{k+1} + p_{k+1} \end{pmatrix} + O(\varepsilon^{2k+2}) \quad (3.8)
\]

with linear mapping \( \tilde{L}_{k+1} \in L[B^{k+1}, B] \) satisfying \( R[\tilde{L}_{k+1}] = R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1} \subset B \).

Hence, an open neighborhood of \( G[z_0(\varepsilon)] \) in \( B \), created by the leading \( \varepsilon \)-term, can only be expected to occur, if \( R_{k+1}^c = \{0\} \) is supposed in (3.3), i.e. the surjectivity of \( \tilde{L}_{k+1} \) with respect to the complement \( N^c \) is needed to obtain by leading \( \varepsilon \)-term a constellation as depicted in figure 2 top-right.
The surjectivity of \( \tilde{L}_{k+1} \) shows to be equivalent to the surjectivity of \( W^{2k+1}(\tilde{z}_k, ..., \tilde{z}_1) \) from the last section. Also note that a necessary condition for zeros of \( G \) to exist in the cone \( C_k \) obviously reads \( \| G[z_0(\varepsilon)] \| = O(|\varepsilon|^{2k+1}) \).

We call the cone \( C_k \) from (3.6) a \textit{k-transversal cone with respect to the map} \( G \), if one of the following equivalent conditions is satisfied

\[
R[\tilde{L}_{k+1} \mid N^c] = \overline{B} \iff R^c_{k+1} = \{0\} \iff \overline{B} = R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1}. \tag{N_2}
\]

The condition \((N_2)\) is now representing nondegeneracy of order \( k \). Note that \( k \)-transversality is a property defined by the image of the cone \( C_k \) under the map \( G \).

**Theorem 2**: Assume \( G \in C^q(B, \overline{B}) \), \( q \geq 2k + 2 \) with \( G[0] = 0 \) and a \( C^q \)-curve

\[
z_0(\varepsilon) = \varepsilon \cdot \tilde{z}_1 + \cdots + \frac{1}{k!} \varepsilon^k \cdot \tilde{z}_k + O(|\varepsilon|^{k+1})
\]

with \( \tilde{z}_l \neq 0, \ 1 \leq l \leq k \), to be the first coefficient different from 0. Then, the following results are valid for \( \varepsilon \in \mathbb{K} \) sufficiently small and \( k \geq 1 \).

(i) \( G[z] = 0, z \in C_k \Rightarrow \| G[z_0(\varepsilon)] \| = O(|\varepsilon|^{2k+1}) \)

Now assume \( C_k \) to be a \textit{k-transversal cone} and set \( p := p_{k+1} \).

(ii) Suppose approximation of order \( 2k \) by \( \| G[z_0(\varepsilon)] \| = O(|\varepsilon|^{2k+1}) \).

Then all solutions of \( G[z] = 0 \) in \( C_k \) are given by

\[
z(\varepsilon, p) := z_k[\varepsilon, n^c(\varepsilon, p), p] = z_0(\varepsilon) + [A_{\varepsilon}^1 \cdots A_{\varepsilon}^{k+1}] \cdot \left( \begin{array}{c} n_1^c(\varepsilon, p) \\ \vdots \\ n_{k+1}^c(\varepsilon, p) + p \end{array} \right)
\]

with \( n_i^c(\varepsilon, p) \in C^{q-2k-1}(\mathcal{B}_{\delta^i}(0) \times U_{k+1}^p, N_i^c) \), \( i = 1, \ldots, k + 1 \).

The map \( z(\varepsilon, p) \) defines a regular Banach manifold in \( B \) for \( \varepsilon \neq 0 \).

(iii) Perturbations of \( G[z] \) of order \( O(\|z\|^{2k+1}) \) cannot destroy the family \( z(\varepsilon, p) \) of solution curves from (ii), only varying \( \varepsilon \)-derivatives of \( z(\varepsilon, p) \) above the order of \( k \).

Perturbations of \( G[z] \) of order \( O(\|z\|^{k+2}) \) cannot destroy the cone \( C_k \).

(iv) There exists an \( (\varepsilon, p) \)-dependent near identity transformation \( J_{\varepsilon,p} = I_\overline{B} + O(|\varepsilon|) \) in \( \overline{B} \), linearizing the map (3.8) from the cone \( C_k \subset B \) to \( \overline{B} \) according to

\[
J_{\varepsilon,p} \circ G_{\varepsilon,p} \circ A_{\varepsilon} = \varepsilon^{2k+1} \cdot \tilde{L}_{k+1}
\]

with linear bijection \( \tilde{L}_{k+1} \in GL[N^c, \overline{B}] \), linear bijection \( A_{\varepsilon} \in GL[N^c, N^c] \) given by (3.4) and \( G_{\varepsilon,p} \) defined by \( G_{\varepsilon,p}[z] := G[z_0(\varepsilon) + z + A_{\varepsilon}^{k+1} \cdot p] \).

(v) The operator norms of partial derivatives of \( G \) along \( z_0(\varepsilon) \) with respect to the subspaces \( R[A_{\varepsilon}^i(N_i^c)] \), \( i = 1, \ldots, k + 1 \) satisfy

\[
\| G'[z_0(\varepsilon)] \mid R[A_{\varepsilon}^i] \| = O(|\varepsilon|^0), \quad \ldots, \quad \| G'[z_0(\varepsilon)] \mid R[A_{\varepsilon}^{k+1}] \| = O(|\varepsilon|^k).
\]

The linearization with respect to the complement \( N^c \) satisfies for \( \varepsilon \neq 0 \)

\[
G_{N^c}[z_0(\varepsilon)] := G'[z_0(\varepsilon)] \mid_{N^c} \in GL[N^c, \overline{B}]
\]
with operator norm of the inverse limited by
\[ \| G_{N^c}[ z_0(\varepsilon) ]^{-1} \| = O( |\varepsilon|^{-k} ) \quad \text{as} \quad \varepsilon \to 0. \]

(vi) Suppose \( \text{dim}(N^c) < \infty \). Then, the characteristic number \( \chi \geq 0 \) is well defined by
\[ \chi := 1 \cdot \text{dim}(N^c_2) + \cdots + k \cdot \text{dim}(N^c_{k+1}) \]
and at every point \( Z(\varepsilon, n^c, p) \) of the cone \( C_k \), the determinant with respect to \( N^c \) reads
\[ \det( G_{N^c}[Z(\varepsilon, n^c, p) ] ) = \varepsilon^\chi \cdot r(\varepsilon, n^c, p) \]
with smooth map \( r(\varepsilon, n^c, p) \neq 0 \).

**Remarks : 1)** Note that only part (ii) *Existence and Uniqueness* as well as part (iii) *Perturbation* assume approximation of \( z_0(\varepsilon) \) of order 2k. Part (iv) *Linearization*, part (v) *Partial Derivatives and Inverse*, as well as part (vi) *Determinant* are valid with respect to an arbitrary curve \( z_0(\varepsilon) \) through \( z = 0 \) satisfying (3.9).

2) Theorem 2 (ii) concentrates on the description of all solutions of \( G[z] = 0 \) in the cone \( C_k \) based on a certain parametrization of the cone given by (3.4). If one is interested to obtain all elements in arc space \( X_{\alpha} \) belonging to these solutions, i.e. having leading part \( z = \varepsilon \cdot \bar{z}_1 + \cdots + \frac{1}{\varepsilon^k} \varepsilon^k \cdot \bar{z}_k \), then it is necessary to solve the system of undetermined coefficients in complete generality up to infinity for fixed \( [\bar{z}_1, \ldots, \bar{z}_k] \in B^k \).

Now, as soon as a *k-transversal cone* is reached, this process can be performed, yielding for \( k \geq 1, l \geq 1 \), the following structure concerning the solutions of the system of undetermined coefficients
\[
[z_1, \ldots, z_k] = [\bar{z}_1, \ldots, \bar{z}_k] \quad \text{(3.10)}
\]

\[
z_{k+1} = \frac{P_1}{\in N^c} + q_1 \quad q_1 \in N_{k+1}
\]
\[
z_{k+2} = \frac{P_2(q_1)}{\in N^c} + q_2 \quad q_2 \in N_{k+1}
\]
\[
\vdots
\]
\[
z_{k+l} = \frac{P_l(q_1, \ldots, q_{l-1})}{\in N^c} + q_l \quad q_l \in N_{k+1}
\]

\[
z_{k+l+1} = \frac{P_{l+1}(q_1, \ldots, q_l)}{\in N^c_1 \oplus \cdots \oplus N^c_k} + n_k \quad n_k \in N_k
\]
\[
z_{k+l+2} = \frac{P_{l+2}(q_1, \ldots, q_l, n_k)}{\in N^c_1 \oplus \cdots \oplus N^c_{k-1}} + n_{k-1} \quad n_{k-1} \in N_{k-1}
\]
\[
\vdots
\]
\[
z_{k+l+k} = \frac{P_{l+k}(q_1, \ldots, q_l, n_k, \ldots, n_2)}{\in N^c_1} + n_1 \quad n_1 \in N_1
\]

with operators \( P_i(\cdot), \quad 2 \leq i \leq l+k \) defined by the composition of multilinear mappings that can effectively be constructed by the recursion in section 5. If \( B \) and \( \bar{B} \) are of finite dimensions, e.g.
$G : \mathbb{C}^n \to \mathbb{C}^m$ with $n > m$, then the $P_i(\cdot)$ can be represented by polynomials of increasing degree with respect to chosen coordinates. For more details see [S3].

Note that moving $l$ towards infinity allows to calculate $\bar{X}_\infty$, i.e. the subset of arc space $X_\infty$ with leading coefficients $[z_1, ..., z_k]$ according to

$$\bar{X}_\infty = \{ (z_i)_{i \in \mathbb{N}} : [ z_1, ..., z_k ] = [ \bar{z}_1, ..., \bar{z}_k ], z_{k+l} = P_l(q_1, ..., q_{l-1}) + q_l, q_l \in N_{k+1}, l \in \mathbb{N} \}.$$

3) In (iii), the stability of solutions $z(\varepsilon, p)$ and cone $C_k$ are treated with respect to higher order perturbations of $G[z]$. The results in (iii) are far from optimal. In some more detail, perturbations of $G[z]$ of order $\left\lfloor \frac{2k}{l} \right\rfloor + 1$, $1 \leq l \leq k$, $\bar{z}_l \neq 0$, cannot destroy the family $z(\varepsilon, p)$ of solution curves only varying $\varepsilon$-derivatives of $z(\varepsilon, p)$ above the order of $k$. The notion $|q|$ means next integer less or equal $q \in \mathbb{Q}$. This improvement is not shown in this paper, but follows from a close inspection of the recursion given in section 5. For example, if $z_1 = 0$ and $z_2 \neq 0$, i.e. $l = 2$, then $\left\lfloor \frac{2k}{2} \right\rfloor + 1 = k + 1$ and perturbations of order $k + 1$ will not destroy $z(\varepsilon, p)$, a good improvement.

Another improvement can be derived by considering the solvability condition $1^k(\bar{z}_k, ..., \bar{z}_1) \in R[\Delta^k(\bar{z}_{k-1}, ..., \bar{z}_1)]$ from (2.4) more closely. Here, the matrix operator $\Delta^k$ depends on $G^1_0, ..., G^r_0$, the inhomogeneity $1^k$ depends on $G^1_0, ..., G^k_0$, whereas the nondegeneracy condition depends on $G^1_0, ..., G^{k+1}_0$. Hence, any perturbation of order between $k + 2$ and $2k$ that does not move the inhomogeneity $1^k$ out of the range $R[\Delta^k]$, will not destroy the solutions $z(\varepsilon, p)$. In specific examples and in the context of Newton polygons, this criterion turns out to be quite useful.

Finally, note that perturbations of order between $k + 2$ and $2k$ may destroy the solutions, by destroying the approximation condition, but not the cone.

4) In (iv), the higher order terms $O(|\varepsilon|^{2k+2})$ of the cone mapping (3.8) are eliminated by an RL-transformation, where the coordinate transformation in $B$ is directly given by the linear bijection $A_\varepsilon \in GL[N^c, N^c]$ from (3.10), i.e. the linearization in (iv) is essentially obtained by the left transformation $\mathcal{J}_{\varepsilon, p}$ of the image space $\bar{B}$.

Alternatively, (3.8) may be linearized by pure right transformation along the following lines. Simply consider the equation

$$G[Z_k(\varepsilon, n^c, p)] = G[z_0(\varepsilon)] + \varepsilon^{2k+1} \cdot \hat{L}_{k+1} \cdot \varphi$$

with $\varphi \in N^c$, or equivalently by (3.8) after cancelling out $\varepsilon^{2k+1} \neq 0$

$$\hat{L}_{k+1} \cdot n^c + \hat{L}_{k+1}^{k+1} \cdot p + O(|\varepsilon|) - \hat{L}_{k+1} \cdot \varphi = 0$$

where $\hat{L}_{k+1}^{k+1} \in L[B, \bar{B}]$ denotes the last component of $\hat{L}_{k+1}$. Now, by $\hat{L}_{k+1} \in GL[N^c, B]$, the equation can uniquely be solved with respect to $n^c \in N^c$, implying linearization of the cone mapping (3.8) according to

$$G[Z_k(\varepsilon, n^c[\varepsilon, p, \varphi], p)] = G[z_0(\varepsilon)] + \varepsilon^{2k+1} \cdot \hat{L}_{k+1} \cdot \varphi$$

with new coordinates $\varphi \in N^c$. Depending from the question to be solved, one may decide, which version of linearization to be more appropriate.

5) Concerning part (v), we notice that the approximation $z_0(\varepsilon)$ can more generally be replaced by the complete cone $Z(\varepsilon, n^c, p)$, analogously to part (vi). This means that the cone is characterized by homogenous conditions with respect to derivatives. In particular, $G'[Z_k(\varepsilon, n^c, p)] \in GL[N^c, \bar{B}]$ in case of $\varepsilon \neq 0$ and the operator norm of the inverse is uniformly limited in the cone by
\[ \| G_{N^c}[Z(\varepsilon, n_{c'}, p)]^{-1} \| = O(|\varepsilon|^{-k}) \quad \text{as} \quad \varepsilon \to 0. \]

This follows straightforwardly from (3.8) and the proof in section 6.

In the context of an existing trivial solution curve \( y = 0 \) of \( G[\lambda, y] = L(\lambda) y + O(\|y\|^2) \), the notion of \( \lambda = 0 \) to be an algebraic eigenvalue of the operator family \( L(\lambda) \) of order \( k \) is introduced in [L1], if the norm of the inverse of the Jacobian with respect to \( y \) is limited according to

\[ \| G_y[\lambda, 0]^{-1} \| = O(|\lambda|^{-k}) \quad \text{as} \quad \lambda \to 0. \]

Hence by (3.11), we merely extend this concept to curves (or cones) in general position in \( B \).

6) In (vi), also the case of \( \chi = 0 \), i.e. \( B = R_1 = R[G^\circ_1] \), may well arise. Then, \( G_1^\circ \) is already surjective and we simply meet the constellation of the standard implicit function theorem. In this situation, Theorem 2 (vi) only says that the determinant with respect to the complement \( N^c \) is different from zero in the cone, where in fact, this determinant is different from zero everywhere.

In case of \( \chi \) even, the sign of the determinant is equal in both of the half-cones and zero at \( z = 0 \). If \( \chi \) is odd, then the sign of the determinant is different in the two half-cones.

In this sense, the characteristic number \( \chi \) is a property of a \textit{k-transversal cone}. In the next section, the implications of (vi) are studied in some more detail.

7) Let us interpret one of the results from the viewpoint of an operator family. Given an arbitrary curve \( z_0(\varepsilon) = \varepsilon \cdot \bar{Z}_1 + \cdots \) of the form (3.9) and perform an expansion along this curve according to

\[ G[z_0(\varepsilon) + z] = G[z_0(\varepsilon)] + G'[z_0(\varepsilon)] \cdot z + O(\|z\|^2). \]

Further, assume the \( \varepsilon \)-dependent operator family \( G'[z_0(\varepsilon)] \in L[B, \bar{B}] \) to be singular at \( \varepsilon = 0 \), i.e. \( R[G^\circ_1] \neq \bar{B} \), but regular, when disturbing to \( \varepsilon \neq 0 \). Then, one may ask, how many of the summands within the \( \varepsilon \)-expansion of the linear part

\[ G'[z_0(\varepsilon)] \cdot z = \{ G_0 + \varepsilon \cdot G_0^2 \bar{Z}_1 + \cdots \} \cdot z \]  

(3.12)

are in fact needed for ensuring regularity of \( G'[z_0(\varepsilon)] \) for \( \varepsilon \neq 0 \)? This sort of question remembers \( k \)-jet determinacy (of an operator family). Now, build up a \( k \)-transversal cone \( C_k \) according to (3.3), (3.4), (N\(_2\)) and apply the singular transformation \( z = A_{\varepsilon} \cdot n^c \) to (3.12), implying

\[ G'[z_0(\varepsilon)] \cdot A_{\varepsilon} \cdot n^c = \varepsilon^{2k+1} \cdot L_{k+1} \cdot n^c + O(\|\varepsilon\|^{2k+2}) \]

with \( L_{k+1} \in GL[N^c, \bar{B}] \). Hence, the effect of the transformation, i.e. the effect of the cone, is to collect the regularity of \( G'[z_0(\varepsilon)] \) for \( \varepsilon \neq 0 \) in one leading operator \( L_{k+1} \), where this process involves first \( k + 1 \) summands within the expansion (3.12) of the operator family \( G'[z_0(\varepsilon)] \). It is not excluded that the concept of blowing up a cone is applicable to quite general families of operators, supposed the singularity is isolated and of non-flat type.

4. Secondary Bifurcation by Degree Theory

In this section, we impose further restrictions on \( G[z] = 0 \) for obtaining further properties besides existence, uniqueness and stability of solution curves as stated in Theorem 2.

First, within the fine resolution (3.3) of cone \( C_k \), we assume \( N^c = N^c_1 \oplus N^c_2 \oplus \cdots \oplus N^c_{k+1} \) to be of finite dimension, implying by use of the characteristic number \( \chi \geq 0 \) from Theorem 2 (vi) for each half-cone constant conditions with respect to the sign of the determinant. Hence, the application of degree theory will be possible, eventually proving secondary bifurcation from \( z = 0 \).
For example, if we start with an equation \( H[u] = 0, H \in C^q(D, \mathcal{D}) \), \( D, \mathcal{D} \) Banach spaces, with \( H[0] = 0 \) and linearization \( H'[0] \in L[D, \mathcal{D}] \) a Fredholm operator (of arbitrary index), then after performing a Lyapunov–Schmidt reduction a reduced system of the form \( G[z] = 0, G \in C^q(B, \mathcal{B}) \) occurs with finite dimensions of the Banach spaces \( B, \mathcal{B} \), implying finite dimension of \( N^c \) as well.

Further, assume the curve \( z_0(\varepsilon) \) from (3.9) to be an approximation of order \( 2k \), as well as the cone \( C_k \) to be \( k \)-transversal according to a simplified decomposition (3.3) of the form

\[
\dim(N^c) < \infty
\]

\[
B = N_1^c \oplus N_2^c \oplus \cdots \oplus N_{k+1}^c \oplus N_{k+1}^c \quad \text{with} \quad N_{k+1} = \{0\} \oplus \{\bar{z}_l\} \tag{4.1}
\]

\[
\bar{B} = R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1}
\]

In particular, by \( R_{k+1} \neq \{0\}, k \) is supposed to be minimal in the sense that for smaller values of \( k \) the Banach space \( \bar{B} \) is not yet build up by the direct sum of the ranges. Moreover, the characteristic number \( \chi \) satisfies \( \chi \geq k \) by Theorem 2 (vi) under consideration of \( N_{k+1}^c \neq \{0\} \). Finally, due to \( P_{k+1} = \{0\} \), no extra parameters occur in the system.

Thus, by Theorem 2 (ii), precisely one solution curve \( z(\varepsilon) = z_0(\varepsilon) + A_\varepsilon \cdot n^c(\varepsilon), A_\varepsilon \in GL[N^c, N^c] \)

of \( G[z] = 0 \) exists in \( C_k \), implying that the topological degree with respect to the map \( \bar{G}_e[n^c] = G[z_0(\varepsilon) + n^c], n^c \in U_e, |\varepsilon| \ll 1 \), is given by

\[
deg(\bar{G}_e, U_e) = \text{sign det}(G_{N^c}[z(\varepsilon)]) \tag{4.2}
\]

under the restriction \( \mathbb{K} = \mathbb{R} \). Hence, by Theorem 2 (vi), we have constant degree \( +1 \) or \( -1 \) in each half-cone, ending up with the following result.

**Corollary 1:** Suppose \( \mathbb{K} = \mathbb{R} \) and a curve \( z_0(\varepsilon) \) with \( \bar{z}_l \neq 0, 1 \leq l \leq k \) to be an approximation of order \( 2k \), as well as the cone \( C_k \) to be \( k \)-transversal according to (4.1). Then, if both, \( l \geq 1 \) and \( \chi \geq 1 \) are **odd**, a continuum of solutions of \( G[z] = 0 \) emanates from \( z = 0 \) existing outside the cone \( C_k \).

Note that by (4.2), assumption \( \chi \) **odd** and Theorem 2 (vi), the degree changes sign when moving from one half-cone to the other, and by \( l \) **odd**, we obtain a well-defined straight movement in \( B \) in the direction of \( \bar{z}_l \). Hence, a cylindrical construction, as depicted in figure 3, can be set up in a straightforward way to ensure the continuum of solutions by homotopy invariance of the degree. In detail, a construction of this kind is performed in [L1] in the context of a known solution curve \( y = 0 \) of \( G[\lambda, y] = L(\lambda)y + O(|y|^2) \).

Compared to the results of the Madrid school, we do not think that much new is given by Corollary 1 from a theoretical point of view. On the other hand, our procedure is working with respect to solution curves in general position within Banach space \( B \), i.e. there is no need to split the Banach space \( B \) into a product space \( B \approx \Lambda \times Y \) with \( \dim(\Lambda) = 1 \) and/or to transfer the solution curve to an axis.

In addition, using the external parameter \( \varepsilon \) for parametrization of the solution curve, offers the possibility that the characteristic number \( \chi \) and the topological degree in each half-cone can be
calculated with respect to solution curves that cannot be moved to an axis. Compare a typical cusp curve, as depicted in figure 3, arising quite frequently in singularity theory.

Let us now look at a simple example for application of Corollary 1.

**Example:** Given the real polynomial

\[ G[x, y] = -xy^3 + x^5 + y^6 = 0 \]  \hspace{1cm} (4.3)

composed of monomials of order 4, 5 and 6. Then, a corresponding solution curve of the form

\[ y(x) = x^{4/3} \cdot [1 + r(x)] \]  \hspace{1cm} (4.4)

is directly obtained by plugging the ansatz \( y = x^{4/3}[1 + r] \) into (4.3), splitting off \( x^5 \) and solving the remainder with respect to the new variable \( r \) by use of the implicit function theorem.

Let us now repeat the derivation of this solution curve, using the Banach space procedure of this paper, with the aim to investigate the solution curve with respect to secondary bifurcation. First, when neglecting the highest order term \( y^6 \) in (4.3), we obtain the equation \( x(-y^3 + x^4) = 0 \), which motivates to start with the approximation

\[ z_0(\varepsilon) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon^3 \\ \varepsilon^4 \end{pmatrix} = \frac{1}{3!} \varepsilon^3 \cdot \begin{pmatrix} 3! \\ 0 \end{pmatrix} + \frac{1}{4!} \varepsilon^4 \cdot \begin{pmatrix} 0 \\ 4! \end{pmatrix} = \frac{1}{3!} \varepsilon^3 \cdot \bar{z}_3 + \frac{1}{4!} \varepsilon^4 \cdot \bar{z}_4. \]

Then, constructing fine resolution and cones by (3.3), (3.4), we reach \( k \)-transversality by straightforward calculation according to

\[
B = \mathbb{R}^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 3! \\ 0 \end{pmatrix} = \bar{z}_3
\]

\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 3! \\ 0 \end{pmatrix}
\]

\[
[0 0] \quad [0 0] \quad S_{12} = [0 c]
\]

\[
\begin{pmatrix} [0] \\ [0] \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} [0] \\ [0] \end{pmatrix} \oplus \begin{pmatrix} [0] \\ [1] \end{pmatrix}
\]

with \( c \neq 0 \). Hence, we obtain transversality of order \( k = 11 \), characteristic number given by \( \chi = 11 \) according to Theorem 2 (vi), as well as index of leading coefficient given by \( l = 3 \). Further, the behaviour of the determinant along the approximation \( z_0(\varepsilon) \) reads by Theorem 2 (vi)

\[ \text{det} \{ G_y[z_0(\varepsilon)] \} = \varepsilon^{11} \cdot s(\varepsilon), \quad s(\varepsilon) \neq 0, \]

and \( z_0(\varepsilon) \) satisfies approximation of order \( 2k \) by

\[ | G[z_0(\varepsilon)] | = | y^6 |_{y=\varepsilon^4} | = | \varepsilon^{24} | = O \left( | \varepsilon |^{2k+1} \right) \]

under consideration of \( 2k + 1 = 23 \). Thus, we can be sure by Theorem 2 (ii) that a unique solution curve \( z(\varepsilon) \) exists within the cone \( C_{11} \), in this way repeating (4.4) by external \( \varepsilon \)-parametrization. Finally, due to \( \chi = 11 \text{ odd} \) and \( l = 3 \text{ odd} \), Corollary 1 ascertains the existence of a continuum of secondary solutions emanating from \( z = 0 \) outside the cone \( C_{11} \).

Again by direct calculation, these secondary solutions are given by \( x(y) = y^3 \cdot [1 + p(y)] \) and repeating the \( \varepsilon \)-procedure from above with respect to \( x(y) \), we obtain \( k = \chi = 3 \text{ odd} \) and \( l = 1 \text{ odd} \), yielding now by Corollary 1 the solution curve \( y(x) \) from (4.4) as secondary bifurcation. In principle, each curve of a bifurcation diagram can be used as a starting point for investigation with respect to secondary bifurcation by Corollary 1.
Note also, that the Milnor number $\mu$ of the singularity (4.3) is easily calculated by the $k$ or $\chi$ values $k_1 = \chi_1 = 11$ and $k_2 = \chi_2 = 3$ of the two solution curves according to the formula

$$\mu = \chi_1 + \chi_2 - \text{ord}(G) + 1 = 11 + 3 - 4 + 1 = 11.$$ 

In general, as derived in [S3], the formula

$$\mu = \chi_1 + \cdots + \chi_\tau - \text{ord}(G) + 1$$

is valid, if all segments of the Newton polygon factorize completely with multiplicities 1. Here, $\tau$ denotes the number of different solution curves through the singularity. See also [S3] for further applications of Corollary 1 in higher dimensions, as well as applications to simple $ADE$-singularities [AGV].

Finally, we note that we could also try to calculate the generalized algebraic multiplicity $\chi$ associated to $y(x)$ from (4.4) by first moving $y(x)$ to the $x$-axis according to

$$\tilde{G}[x, \tilde{y}] := G[x, x^{4/3} \cdot [1 + r(x)] + \tilde{y}],$$

yielding the trivial solution $\tilde{y} = 0$ with respect to $\tilde{G}[x, \tilde{y}] = 0$ as desired. However, we lose smoothness according to $\tilde{G} \not\in C^2$ and the standard transformation of $y(x)$ to the $x$-axis does not work for calculation of the generalized algebraic multiplicity $\chi$.

We do not say that it is impossible to transform $y(x)$ to an axis by choosing appropriate coordinates within $B = \mathbb{R}^2$; we only say that the standard transformation does not work, supporting the idea that procedures concerning general position might be useful.
5. Construction of Fine Resolution and Cones

In this section, we show in detail the construction of the fine resolution (3.3) and the cone (3.4). The construction is kept independent of any approximation properties of the curve \( z_0(\varepsilon) \) from (3.9). In this sense, a cone-like fibration of a curve satisfying \( \tilde{z}_l \neq 0, 1 \leq l \leq k, \) is constructed.

Using \( S_1 = G_0^1 \) and \( S_2 = P_{R^2} 2G_0^2 \tilde{z}_1 |_{N_1} \) from (3.2), we start with \( k = 1 \) and corresponding decomposition

\[
\begin{align*}
B &= N_1^c \oplus N_2^c \oplus N_2 \quad \text{with} \quad N_2 = P_2 \oplus \{ \tilde{z}_{l,2} \} \\
\uparrow & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
\uparrow S_1 & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow S_2 \\
\bar{B} &= R_1 \oplus R_2 \oplus R_2^2
\end{align*}
\]

(5.1)

Then, the first cone \( C_1 \) is defined by the map

\[
Z_1(\varepsilon, n_1^c, n_2^c, p_2) := z_0(\varepsilon) + \left[ \frac{1}{3!} \varepsilon^3 \varepsilon^2 \right] \cdot \begin{bmatrix} I_B \\ I_B \end{bmatrix} \cdot \begin{pmatrix} n_1^c \\ n_2^c + p_2 \end{pmatrix}
\]

\[
= z_0(\varepsilon) + \frac{1}{2} \varepsilon^2 \cdot (n_2^c + p_2) + \frac{1}{6} \varepsilon^3 \cdot n_1^c
\]

(5.2)

where the complete construction merely depends on the first coefficient \( \tilde{z}_1 \) and the first two derivatives \( G_0^1 \) and \( G_0^2 \). In figure 5 two possible realizations of the cone \( C_1 \) are drawn.

![Figure 5](image)

Figure 5: The cone \( C_1 \) in two different realizations.

In the left diagram, both \( N_1^c \) and \( N_2^c \) are zero and the cone is exclusively build up by the subspace \( P_2 \), which is homogenously shrinking by order of \( \varepsilon^2 \) in each direction as \( \varepsilon \to 0 \). On contrary, in the right diagram \( P_2 \) equals zero and the cone is composed of \( N_1^c \) and \( N_2^c \) shrinking by order of \( \varepsilon^3 \) and \( \varepsilon^2 \) respectively.

For general \( k \geq 2 \), the decomposition of \( B, \bar{B} \) and the cone \( C_k \) are defined by the following recursion starting with \( k = 2 \). Remember, the basis for the construction is given by the system of undetermined coefficients (2.2), obtained from plugging the power series ansatz \( z = \varepsilon \cdot \tilde{z}_1 + \frac{1}{2!} \varepsilon^2 \cdot \tilde{z}_2 + \cdots \) into the power series expansion of \( G \) at \( z = 0 \). Then, the basic formula (2.3) has to be used intensively, implying complex, but constructive formulas.
First, two simple triangular schemes of the form

\[
d_{0,1} = 1 \quad | \quad c_{0,1} = 1
\]

\[n = 1 \quad d_{1,1} = 1 \quad | \quad c_{1,1} = 1\]

\[n = 1 \quad d_{2,1} = 1 \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad | \quad c_{2,1} = 1 \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix} \]

\[n = 2 \quad d_{3,1} = 1 \quad d_{3,2} = \frac{4}{3} \quad | \quad c_{3,1} = 1 \quad c_{3,2} = \frac{3}{2}\]

\[n = 2 \quad d_{4,1} = 1 \quad d_{4,2} = \frac{5}{4} \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad | \quad c_{4,1} = 1 \quad c_{4,2} = \frac{4}{2} \quad \begin{bmatrix} 1 \\ c_{4,3} = 1 \end{bmatrix}\]

\[n = 3 \quad d_{5,1} = 1 \quad d_{5,2} = \frac{6}{5} \quad d_{5,3} = \frac{6}{4} \quad | \quad c_{5,1} = 1 \quad c_{5,2} = \frac{5}{2} \quad c_{5,3} = \frac{5}{3}\]

\[n = 3 \quad d_{6,1} = 1 \quad d_{6,2} = \frac{7}{6} \quad d_{6,3} = \frac{7}{5} \quad \begin{bmatrix} 7 \\ 4 \end{bmatrix} \quad | \quad c_{6,1} = 1 \quad c_{6,2} = \frac{6}{2} \quad c_{6,3} = \frac{5}{2} \quad \begin{bmatrix} 1 \\ c_{6,4} = 1 \end{bmatrix}\]

\[\vdots \quad \vdots \quad \vdots \quad \vdots \quad | \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[l = 1 \quad l = 2 \quad l = 3 \quad \cdots \quad | \quad l = 1 \quad l = 2 \quad l = 3 \quad \cdots \]

are defined column by column according to

\[
l \geq 1 : \quad d_{2l-2,l} := \frac{2l-1}{l} \quad \quad \quad c_{2l-2,l} := 1
\]

\[
n \geq l : \quad d_{2n-1,l} := \frac{2n}{2n+1-l} \quad \quad \quad c_{2n-1,l} := c_{2n-2,l} \cdot d_{2n-2,l} \quad (5.3)
\]

\[
n \geq l : \quad d_{2n,l} := \frac{2n+1}{2n+2-l} \quad \quad \quad c_{2n,l} := c_{2n-1, l} \cdot d_{2n-1,l}
\]

with the rows of the schemes partly comprised to diagonal matrices

\[
D^{2n-1} := Diag\left[ d_{2n-1,1}, \cdots, d_{2n-1,n} \right] \in \mathbb{R}^{n,n}
\]

\[
D^{2n} := Diag\left[ d_{2n,1}, \cdots, d_{2n,n} \right] \in \mathbb{R}^{n,n}
\]

\[
C^{2n-1} := Diag\left[ c_{2n-1,1}, \cdots, c_{2n-1,n} \right] \in \mathbb{R}^{n,n}
\]

\[
C^{2n} := Diag\left[ c_{2n,1}, \cdots, c_{2n,n} \right] \in \mathbb{R}^{n,n}.
\]

Every element of the \( d \)-scheme is explicitly defined, whereas the \( c \)-scheme is iteratively defined using the \( d \)-scheme. With respect to proofs, this kind of \( c \)-scheme definition is quite convenient. Nevertheless, the \( c \)-scheme can also be expressed explicitly using binomial coefficients according to

\[
c_{2n-1, l} \cdot \binom{2(l-1)}{l-1} = \binom{2n-1}{l-1} \quad \text{and} \quad c_{2n, l} \cdot \binom{2(l-1)}{l-1} = \binom{2n}{l-1}.
\]

Then, using the abbreviations
\[ E^2 := \begin{pmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{pmatrix} = \begin{pmatrix} I_B & -S_1^{-1} P R_1 \bar{S}_2 \\ 0 & I_B \end{pmatrix} \in L[ B^2, B^2 ] \]

\[ \begin{pmatrix} a^3 & \bar{a}^3 \\ A^3 & \bar{A}^3 \end{pmatrix} = (C^3)^{-1} \cdot E^2 \cdot C^3 \in L[ B^2, B^2 ] \] \quad (5.5)

\[ [ W_A^4, \ W_2^4 ](\bar{z}_1) := [ 4G_0^2 \bar{z}_1, \ 6G_0^3 \bar{z}_1^2 ] \in L[ B, \bar{B} ] \quad \text{and} \quad M^3 := I_B \in L[ B, B ], \]

the recursion, continued with \( k = 2 \), is well defined according to the following notation

\[ \bar{S}_{k+1} := [ W_{2k-1}^{2k}, \ldots, W_2^{2k} ](\bar{z}_{k-1}, \ldots, \bar{z}_1) \cdot \begin{pmatrix} a^{2k-1} \\ \bar{a}^{2k-1} \end{pmatrix} + \frac{(2k)!}{(k!)^2} G_0^2 \bar{z}_k \in L[ B, \bar{B} ] \]

\[ S_{k+1} := P R_k \bar{S}_{k+1} \mid_{N_k} \in L[ N_k, R^c_k ] \] \quad (5.6)

with corresponding decomposition

\[ N_k = N^c_{k+1} \quad \oplus \quad N[ S_{k+1}(\bar{z}_k, \ldots, \bar{z}_1) ] =: N^c_{k+1} \oplus N_{k+1} \]

\[ R^c_k = R[ S_{k+1}(\bar{z}_k, \ldots, \bar{z}_1) ] \quad \oplus \quad R^c_{k+1} =: R_{k+1} \oplus R^c_{k+1} \]

and all subspaces assumed to closed.

Hence, setting \( S_0 := 0 \), the linear mappings \( S_i, \ i = 1, \ldots, k + 1 \), are iteratively defined in such a way that the mapping \( S_i \) acts between the kernel of the previous mapping \( N[ S_{i-1} ] = N_{i-1} \) and a complement \( R^c_{i-1} \) of the range \( R[ S_{i-1} ] = R_{i-1} \) of this mapping.

By construction, we end up with the following decompositions of \( B \) and \( \bar{B} \) at iteration step \( k \).

\[ B = N_1^c \oplus N_2^c \oplus \cdots \oplus N_{k+1}^c \oplus N_{k+1} \quad \text{with} \quad N_{k+1} = P_{k+1} \oplus \{ \bar{z}_{i,k+1} \} \]

\[ \bar{B} = R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1} \oplus R^c_{k+1} \] \quad (5.7)

Note also that with \( \bar{S}_1 := G_0^1 \) and \( \bar{S}_2 := 2G_0^2 \bar{z}_1 \), the relations

\[ S_i := P R_{i-1}^c \mid_{N_{i-1}} \in L[ N_{i-1}, R^c_{i-1} ] \quad \text{and} \quad S_i \in G L[ N^c_i, R_i ], \ i = 1, \ldots, k + 1 \]

are in general valid, where the bijectivity of \( S_i \) between the subspaces \( N^c_i \) and \( R_i \) is depicted, as usual, by arrows within (5.7).
Now, for closing the recursion (5.6) of the linear operators, the following formulas concerning \(a^{2k+1}, \overline{a}^{2k+1}\) and \(M^{2k+1}, \overline{M}_{k+1}\) have to be established according to

\[
E_{k+1,k+1} := I_B \in L[ B, B ]
\]

\[
E_{i,k+1} := -S_i^{-1} P_{R_i} \sum_{v=i+1}^{k+1} \tilde{S}_v E_{v,k+1} \in L[ B, B ], \ i = k, ..., 1
\]

\[
E^{k+1} := \begin{pmatrix}
E_{1,1} & \cdots & E_{1,k+1} \\
\vdots & \ddots & \vdots \\
E_{k+1,k+1}
\end{pmatrix} \in L[ B^{k+1}, B^{k+1} ]
\]

\[
\begin{pmatrix}
a^{2k} \\
\overline{a}^{2k}
\end{pmatrix} = (C^{2k})^{-1} \cdot E^k \cdot C^{2k} \in L[ B^k, B^k ]
\]

\[
\begin{pmatrix}
a^{2k+1} \\
\overline{a}^{2k+1}
\end{pmatrix} = (C^{2k+1})^{-1} \cdot E^{k+1} \cdot C^{2k+1} \in L[ B^{k+1}, B^{k+1} ]
\]

with

\[
a^{2k} \in L[ B^{k-1}, B ], \ \overline{a}^{2k} \in L[ B, B ], \ A^{2k} \in L[ B^{k-1}, B^{k-1} ], \ \overline{A}^{2k} \in L[ B, B^{k-1} ]
\]

\[
a^{2k+1} \in L[ B^k, B ], \ \overline{a}^{2k+1} \in L[ B, B ], \ A^{2k+1} \in L[ B^k, B^k ], \ \overline{A}^{2k+1} \in L[ B, B ]
\]

and

\[
M^{2k+1} := \begin{pmatrix}
(a^{2k} & \overline{a}^{2k}) \\
\overline{a}^{2k-1} & (A^{2k} & \overline{A}^{2k})
\end{pmatrix} \in L[ B^k, B^k ], \ (5.9)
\]

where the index \(k+1\) denotes cancelling of last row of the \((k+1) \times k\) matrix in brackets. This implies continuation of the linear operators in (5.6). Then, setting

\[
\overline{M}_{k+1} := \begin{pmatrix}
I_B \\
M^{2k+1}
\end{pmatrix},
\]

the cone \(C_k\) is given by

\[
Z_k(\varepsilon, n_i^c, ..., n_{k+1}^c, p_{k+1}) := Z_0(\varepsilon) + \begin{pmatrix}
\frac{\varepsilon^{2k+1}}{(2k+1)!} \\
\frac{\varepsilon^{k+1}}{(k+1)!}
\end{pmatrix} \cdot \overline{M}_{k+1} \begin{pmatrix}
n_i^c \\
n_{k+1}^c + p_{k+1}
\end{pmatrix}.
\]

Finally, a simple calculation shows the upper triangularity

\[
M_{i,j}^{2k+1} = 0, \ i = 2, ..., k, \ j = 1, ..., i - 1 \quad \text{and} \quad M_{i,i}^{2k+1} = I_B, \ i = 1, ..., k
\]

implying

\[
M^{2k+1} \in GL[ B^k, B^k ]
\]
with dependency of $M^{2k+1}$ from $z_1, \ldots, z_{k-1}$ and $G_0^1, \ldots, G_0^k$. In addition, we note that the iteratively defined mappings $E_{i,k+1}$ in (5.8) can also be written in an explicit way according to

$$E_{k+1,k+1} = I_B, \quad E_{k,k+1} = -S_{k+1}^{-1} P_{R_k} \tilde{S}_{k+1}$$

$$E_{i,k+1} := -S_{i}^{-1} P_{R_i} \left( I_B + \sum_{\nu=1}^{k-i} (-1)^{\nu} \sum_{i+1 \leq n_1 < \cdots < n_{\nu} \leq k} \prod_{\tau=1}^{\nu} S_{n_\tau} S_{n_\tau}^{-1} P_{R_{n_\tau}} \right) \tilde{S}_{k+1}, \; i = k-1, \ldots, 1.$$  

It remains to state formulas concerning the $W$-Operators in (5.6). In contrast to the cone $C_k$ in (5.11), the formulas of the $W$-Operators are not defined recursively, but in an explicit way, exploiting the system of undetermined coefficients in combination with (2.3).

Finally, we should not forget, to supply explicit formulas concerning section 2. In some more detail $B^k(z_{k-1}, \ldots, z_1) \in L[B^k, B^k]$ and $1^k(z_{k-1}, z_1) \in B^k$ from (2.4) as well as $W^{2k+1}(z_{k-1}, \ldots, z_1)$ and $R^{2k+1}(z_{k-1}, \ldots, z_1)$ from (2.12) are needed. Note also, that the formulas concerning $T^1[z_1]$, $T^2[z_2, z_1]$ and $T^3[z_3, z_2, z_1]$ are already stated in (2.2) and it is sufficient to supply formulas with respect to $T^{2k}[z_{2k}, \ldots, z_1]$ and $T^{2k+1}[z_{2k+1}, \ldots, z_1]$ for $k \geq 2$.

First, even components of system (2.2) are given for $k \geq 2$ by

$$T^{2k}[z_{2k}, \ldots, z_1] = \left[ W^{2k}_{2k}, \ldots, W^{2k}_k \right] (z_{k-1}, \ldots, z_1) \cdot \frac{z_{2k}}{z_k} + R^{2k}(z_{k-1}, \ldots, z_1) + \frac{(2k)!}{2(k!)^2} G_0^2 z_k^2 \in B$$

with

$$W^{2k}_\mu(z_{k-1}, \ldots, z_1) := \frac{1}{\mu!} \sum_{\beta=1}^{2k} G_0^\beta \sum_{1=n_1+\cdots+(k-1)n_{k-1}+\mu=2k} \frac{(2k)!}{n_1! \cdots n_{k-1}!} \prod_{\tau=1}^{k-1} \left( \frac{1}{\tau!} z_\tau \right) n_\tau \in L[B, B]$$

for $\mu = 2k, \ldots, k$, as well as

$$R^{2k}(z_{k-1}, \ldots, z_1) := \sum_{\beta=1}^{2k} G_0^\beta \sum_{1=n_1+\cdots+(k-1)n_{k-1}=2k} \frac{(2k)!}{n_1! \cdots n_{k-1}!} \prod_{\tau=1}^{k-1} \left( \frac{1}{\tau!} z_\tau \right) n_\tau \in B$$

closing (5.6) also with respect to the $W$-Operators.

Next, odd components of system (2.2) read for $k \geq 2$

$$T^{2k+1}[z_{2k+1}, \ldots, z_1] = \left[ W^{2k+1}_{2k+1}, \ldots, W^{2k+1}_k \right] (z_{k}, \ldots, z_1) \cdot \frac{z_{2k+1}}{z_k+1} + R^{2k+1}(z_{k}, \ldots, z_1) \in B$$

with

$$W^{2k+1}_\mu(z_{k}, \ldots, z_1) := \frac{1}{\mu!} \sum_{\beta=1}^{2k+1} G_0^\beta \sum_{1=n_1+\cdots+n_{k+1}=2k+1} \frac{(2k+1)!}{n_1! \cdots n_k!} \prod_{\tau=1}^{k+1} \left( \frac{1}{\tau!} z_\tau \right) n_\tau \in L[B, B]$$

for $\mu = 2k + 1, \ldots, k + 1$, whereas the corresponding inhomogeneity is given by
\[ R^{2k+1}(z_k, \ldots, z_1) := \sum_{\beta=1}^{2k+1} G_0^\beta \sum_{n_1+\cdots+n_k=\beta}^{(2k+1)!} \frac{(2k+1)!}{n_1! \cdots n_k!} \prod_{\tau=1}^{k} \left( \frac{1}{\tau!} z_\tau \right)^n \in \tilde{B}. \]  

(5.19)

Note that by (5.17)-(5.19), the remainder equation (2.12) from section 2 is well defined and it remains to state explicit formulas concerning the system of equations \( T^{2k} = \cdots = T^{k+1} = 0 \) from (2.4). Now, using \( \Delta^1(z_i) = G_0^1 \in L[B, \tilde{B}] \) and \( \Gamma^1(z_i) = G_0^z z_i^2 \in \tilde{B} \) in case of \( k = 1 \) (cf. section 2), as well as combining even and odd components from (5.14) and (5.17), the upper triangular operator \( \Delta^k(z_{k-1}, \ldots, z_1) \in L[B^k, \tilde{B}^k] \) with associated inhomogeneity \( I^k(z_k, \ldots, z_1) \in \tilde{B}^k \) read for \( k \geq 2 \)

\[ \Delta^k_{i_1}(z_k, \ldots, z_1) := G_0^1 \in L[B, \tilde{B}], \ i = 1, \ldots, k \]  

(5.20)

\[ \Delta^k_{j}(z_k, \ldots, z_1) := 0 \in L[B, \tilde{B}], \ i = 2, \ldots, k, \ j = 1, \ldots, i - 1 \]  

\[ \Delta^k_{i+1,k}(z_k, \ldots, z_1) := \frac{1}{(k+j)!} \sum_{\beta=1}^{k+i} G_0^\beta \sum_{n_1+\cdots+n_k+1=\beta}^{(k+j)!} \frac{k!}{n_1! \cdots n_k!} \prod_{\tau=1}^{k} \left( \frac{1}{\tau!} z_\tau \right)^n \in L[B, \tilde{B}], \ i = 2, \ldots, k, \ j = 1, \ldots, i - 1, \]  

and

\[ I^k_{i+1}(z_k, \ldots, z_1) := \sum_{\beta=1}^{k+i} G_0^\beta \sum_{n_1+\cdots+n_k=\beta}^{(k+i)!} \frac{k!}{n_1! \cdots n_k!} \prod_{\tau=1}^{k} \left( \frac{1}{\tau!} z_\tau \right)^n \in \tilde{B}, \ i = 1, \ldots, k. \]  

For later use, we also note that by (5.15) and (5.18), we obtain for \( k \geq 1 \)

\[ W^{2k+1}_{2k+1}(z_k, \ldots, z_0) = W^{2k}_{2k}(z_{k-1}, \ldots, z_0) = G_0^z \in L[B, \tilde{B}]. \]  

(5.21)

6. Proof of Main Results

In the following Lemma, the main ingredients for proving Theorem 2 from section 3 are comprised.

Lemma 1: For \( k \geq 1 \) and arbitrary \([z_k, \ldots, z_1] \in B^k\) we obtain

(i) \( N[\Delta^k(z_{k-1}, \ldots, z_1)] = R[ M^{2k+1}_{N_1 \times \cdots \times N_k} ] \)

(ii) \( W^{2k+1}(z_k, \ldots, z_1) \cdot \tilde{M}^{k+1} = [ \tilde{S}_1 | \tilde{S}_2(z_1) | \cdots | \tilde{S}_{k+1}(z_k, \ldots, z_1) ] \cdot C^{2k+1} \)

(iii) \( N[ [ \tilde{S}_1 | \tilde{S}_2(z_1) | \cdots | \tilde{S}_{k+1}(z_k, \ldots, z_1) ] | N_0 \times \cdots \times N_{k+1} ] = R[ E^{k+1}_{N_1 \times \cdots \times N_{k+1}} ] \)

(iv) \( M^{2k+1} \cdot \begin{pmatrix} d_{2k+1,1} & \cdots \\ \vdots & \ddots \end{pmatrix} = \begin{pmatrix} d_{2k+1,1} & \cdots \\ \vdots & d_{2k+1,k+1} \end{pmatrix} \cdot \begin{pmatrix} A_{2k+1}^{k+1} \\ M^{2k+1} \cdot A_{2k+1}^{k+1} \end{pmatrix} \)

(v) Assume a curve \( z_0(\varepsilon) \) as in (3.9) with \( \tilde{z}_i \neq 0, \ 1 \leq l \leq k. \) Then,

\[ \| G[z_0(\varepsilon)] \| = O(\varepsilon^{2k+1}) \Rightarrow \tilde{z}_l \in N_{k+1} \]
Proof of Theorem 2 (i) and (ii): We start with an arbitrary curve of the form

\[ z_0(\epsilon) = \epsilon \cdot \bar{z}_1 + \cdots + \frac{1}{k!} \epsilon^k \cdot \bar{z}_k + \frac{1}{(k+1)!} \epsilon^{k+1} \cdot \bar{z}_{k+1} + \cdots + \frac{1}{(2k+1)!} \epsilon^{2k+1} \cdot \bar{z}_{2k+1} + \epsilon^{2k+2} \cdot r(\epsilon) \]

with \( \bar{z}_i \neq 0 \), \( 1 \leq l \leq k \) and \( z_0(\epsilon) \) of class \( C^q \), \( q \geq 2k + 2 \) and \( k \geq 1 \).

As already mentioned, the main idea of our ansatz is given by supplementing the coefficients \( \bar{z}_{k+1}, \ldots, \bar{z}_{2k+1} \) of \( z_0(\epsilon) \) with free coefficients \( [b_{k+1}, \ldots, b_{2k+1}] \) according to

\[
G[ z_0(\epsilon) ] + \left[ \frac{\epsilon^{2k+1}}{(2k+1)!} \cdots \frac{\epsilon^{k+1}}{(k+1)!} \right] \cdot \begin{pmatrix} b_{2k+1} \\ \vdots \\ b_{k+1} \end{pmatrix}
\]

\[
= G[ \epsilon \cdot \bar{z}_1 + \cdots + \frac{1}{k!} \epsilon^k \cdot \bar{z}_k \\
+ \frac{1}{(k+1)!} \epsilon^{k+1} \cdot (\bar{z}_{k+1} + b_{k+1}) + \cdots + \frac{1}{(2k+1)!} \epsilon^{2k+1} \cdot (\bar{z}_{2k+1} + b_{2k+1}) + \epsilon^{2k+2} \cdot r(\epsilon) ]
\]

\[
= \sum_{i=1}^{k} \frac{1}{i!} \epsilon^i \cdot T^i[\bar{z}_i, \ldots, \bar{z}_1] + \sum_{i=k+1}^{2k} \frac{1}{i!} \epsilon^i \cdot T^i[\bar{z}_i + b_i, \ldots, \bar{z}_1]
\]

\[
+ \frac{1}{(2k+1)!} \epsilon^{2k+1} \cdot T^{2k+1}[\bar{z}_{2k+1} + b_{2k+1}, \ldots, \bar{z}_1] + \epsilon^{2k+2} \cdot r_1(\epsilon, b_{2k+1}, \ldots, b_{k+1})
\]

Next, restrict \( [b_{2k+1}, \ldots, b_{k+1}] \) to \( B^{k+1} \) to \( B \times N[\Delta^k] \) and perform an expansion of \( r_1(\epsilon, \cdot) \) with respect to \( [b_{2k+1}, \ldots, b_{k+1}] \) to obtain

\[
G[] = \sum_{i=1}^{k} \frac{1}{i!} \epsilon^i \cdot T^i[\bar{z}_i, \ldots, \bar{z}_1] + \left[ \frac{\epsilon^{2k}}{(2k)!} \cdots \frac{\epsilon^{k+1}}{(k+1)!} \right] \cdot \begin{pmatrix} \Delta^k \cdot \bar{z}_{2k} + b_{2k} \\ \vdots \\ \bar{z}_{k+1} + b_{k+1} \end{pmatrix}
\]

\[
+ \frac{1}{(2k+1)!} \epsilon^{2k+1} \cdot \{ W^{2k+1}(\bar{z}_k, \ldots, \bar{z}_1) \cdot \begin{pmatrix} \bar{z}_{2k+1} + b_{2k+1} \\ \vdots \\ \bar{z}_{k+1} + b_{k+1} \end{pmatrix} + R^{2k+1}(\bar{z}_k, \ldots, \bar{z}_1) \}
\]

\[
+ \epsilon^{2k+2} \cdot r_1(\epsilon, b_{2k+1}, \ldots, b_{k+1}) = G[\cdot].
\]

(6.1)
with smooth remainder operators \( r_1(\cdot) \) and \( r_2(\cdot) \) with \( r_2(\epsilon, 0, ..., 0) = 0 \). Now, with \( b_{2k+1} = \cdots = b_{k+1} = 0 \), formulas (6.1), (6.2) obviously simplify to \( G[] = G[z_0(\epsilon)] \) and we end up with

\[
G[] = G[z_0(\epsilon)] + \frac{1}{(2k+1)!} \epsilon^{2k+1} \cdot W^{2k+1}(\bar{z}_k, ..., \bar{z}_1) \cdot \begin{pmatrix}
\begin{bmatrix}
b_{2k+1} \\
\vdots \\
b_{k+1}
\end{bmatrix} \\
\begin{bmatrix}
n_0 \\
n_k
\end{bmatrix}
\end{pmatrix} + \epsilon^{2k+2} \cdot r_2(\epsilon, b_{2k+1}, ..., b_{k+1})
\]

(5.10)  

\[
\equiv \sum_{i=1}^{(2k+1)} G[z_0(\epsilon)] + \frac{1}{(2k+1)!} \epsilon^{2k+1} \cdot W^{2k+1}(\bar{z}_k, ..., \bar{z}_1) \cdot \begin{pmatrix}
\begin{bmatrix}
\bar{M}_{k+1} \\
\vdots \\
\bar{M}_1
\end{bmatrix} \\
\begin{bmatrix}
n_0 \\
n_k
\end{bmatrix}
\end{pmatrix} + \epsilon^{2k+2} \cdot r_3(\epsilon, n_0, ..., n_k)
\]

(6.3)

by invoking Lemma 1 (i), (ii), the definition (5.10) of \( \bar{M}_{k+1} \) and \([n_0, ..., n_k] \in N_0 \times \cdots \times N_k \subset B^{k+1} \).

Then, using fine resolution (5.7), the free variables \([n_0, ..., n_k]\) in (6.3) can further be restricted to \( U^{\xi}_k \times \cdots \times U^{\xi}_k \times (U^{\xi}_k + U^{\xi}_k) \subset N^{\xi}_k \times \cdots \times N^{\xi}_k \times (N^{\xi}_{k+1} + R_{k+1}) \), hence mapping the cone \( C_k \) from \( B \) to \( \bar{B} \) according to

\[
G[Z_k(\epsilon, n^{\xi}_1, ..., n^{\xi}_{k+1}, p_{k+1})] = G[z_0(\epsilon)] + \left( \frac{\epsilon^{2k+1}}{(2k+1)!} \right) \cdot \begin{pmatrix}
\begin{bmatrix}
\bar{M}_{k+1} \\
\vdots \\
\bar{M}_1
\end{bmatrix} \\
\begin{bmatrix}
n_0 \\
n_k
\end{bmatrix}
\end{pmatrix} + \epsilon^{2k+2} \cdot r_3(\epsilon, n^{\xi}_1, ..., n^{\xi}_{k+1} + p_{k+1})
\]

(5.11)

\[
= G[z_0(\epsilon)] + \epsilon^{2k+1} \cdot \bar{L}_{k+1} \cdot \begin{pmatrix}
\begin{bmatrix}
n_{k+1}^{\xi} \\
\vdots \\
n_{1}^{\xi}
\end{bmatrix} \\
\begin{bmatrix}
n_{k+1}^{\xi} + p_{k+1}
\end{bmatrix}
\end{pmatrix} + \epsilon^{2k+2} \cdot r_3(\epsilon, n^{\xi}_1, ..., n^{\xi}_{k+1} + p_{k+1})
\]

(6.4)

\[
= \sum_{i=1}^{2k} \frac{1}{i!} \epsilon^i \cdot T^i[\bar{z}_i, ..., \bar{z}_1] + \epsilon^{2k+1} \cdot \bar{L}_{k+1} \cdot \begin{pmatrix}
\begin{bmatrix}
n_{k+1}^{\xi} \\
\vdots \\
n_{1}^{\xi}
\end{bmatrix} \\
\begin{bmatrix}
n_{k+1}^{\xi} + p_{k+1}
\end{bmatrix}
\end{pmatrix} + \epsilon \cdot r_3(\epsilon, n^{\xi}_1, ..., n^{\xi}_{k+1} + p_{k+1})
\]

(6.5)

and by direct inspection of (5.6), (5.7) we obtain

\[
\bar{L}_{k+1} \in GL\left[ N^{\xi}, R_1 \oplus R_2 \oplus \cdots \oplus R_{k+1} \right].
\]

Now, (6.4) delivers (3.8) as well as Theorem 2 (i).

Concerning Theorem 2 (ii), assume \( z_0(\epsilon) \) now to be an approximation of order \( 2k \). Then, we obtain from (6.4) the blown-up remainder equation

\[
31
\]
\[ H(\varepsilon, n^c_1, \ldots, n^c_{k+1}, p_{k+1}) = \]
\[ \frac{1}{(2k+1)!} T^{2k+1}[\tilde{z}_{2k+1}, \ldots, \tilde{z}_1] + \hat{L}_{k+1} \cdot \begin{pmatrix} n^c_i \\ n^c_{k+1} \end{pmatrix} + \varepsilon \cdot r_3(\varepsilon, n^c_1, \ldots, n^c_{k+1}, p_{k+1}) = 0 \]

with \( H(\varepsilon, n^c_1, \ldots, n^c_{k+1}, p_{k+1}) \in \mathbb{C}^{q-2k-1}(B_\delta, 0) \times U_1^c \times \ldots \times U_1^c \times U^p_{k+1, \tilde{B}} \) and \( p_{k+1} \) chosen as a direct complement of \( \tilde{z}_i \), due to Lemma 1 (v) and \( N_{k+1} = P_{k+1} \oplus \{ \tilde{z}_i \} \).

Finally, suppose \( Z_k \) to define a \( k \)-transversal cone, then \( \hat{L}_{k+1} \in GL[N^c, \tilde{B}] \) by (6.5), yielding for \( \varepsilon = 0 \) the base solutions

\[ H(0, n^c_1, \ldots, n^c_{k+1}, p_{k+1}) = 0 \iff \begin{pmatrix} n^c_i \\ n^c_{k+1} \end{pmatrix} = 0 \]

and the implicit function theorem delivers locally unique and smooth continuation of these solutions to \( \varepsilon \neq 0 \), thus finishing the proof of Theorem 2 (ii), possibly after reduction of \( \delta_1 > 0 \).

**Proof of Theorem 2 (iii)**: The approximation condition of order \( 2k \) concerns the equations

\[ T^1[\tilde{z}_1] = \ldots = T^{2k}[\tilde{z}_2, \ldots, \tilde{z}_1] = 0, \]

merely depending from \( G_0^1, \ldots, G_0^{2k} \) according to (5.14) and (5.17). Hence, perturbations of \( G[z] \) of order \( O(\|z\|^{2k+1}) \) cannot destroy the approximation property of \( z_0(\varepsilon) \).

Further, by tedious, but simple inspection of the iteration from section 5, we see that the construction of a \( k \)-transversal cone \( C_k \) only is affected by \( G_0^1, \ldots, G_0^{k+1} \), completing the proof of (iii).

**Proof of Theorem 2 (iv)**: Setting \( p := p_{k+1} \), define in view of (6.4)

\[ \tilde{G}_{\varepsilon, p}[n^c] := G[Z_k(\varepsilon, n^c, p)] = G[z_0(\varepsilon) + A_\varepsilon \cdot n^c + A^{k+1}_\varepsilon \cdot p], \]

implying \( \tilde{G}_{\varepsilon, p} : N^c \to \tilde{B} \) to be a local diffeomorphism for \( \varepsilon \neq 0 \), due to supposed transversality of the cone \( Z_k \), i.e. \( \hat{L}_{k+1} \in GL[N^c, \tilde{B}] \). Now, with \( \tilde{r}_3(\varepsilon, n^c_1, \ldots, n^c_{k+1}, p) := r_3(\varepsilon, n^c_1, \ldots, n^c_{k+1} + p) \) and defining

\[ G_{\varepsilon, p}[z] := G[z_0(\varepsilon) + z + A^{k+1}_\varepsilon \cdot p] \]

\[ \mathcal{J}_{\varepsilon, p}[\tilde{b}] := \tilde{b} - G[z_0(\varepsilon) - e^{2k+1} \cdot \tilde{S}_{k+1} \cdot p - e^{2k+2} \cdot \tilde{r}_3(\varepsilon, G_{\varepsilon, p}[\tilde{b}], p)] \]

(6.7)

we obtain by direct calculation from (6.4)

\[ \mathcal{J}_{\varepsilon, p} \circ G_{\varepsilon, p} \circ A_\varepsilon \cdot n^c = e^{2k+1} \cdot \hat{L}_{k+1} \cdot n^c, \]

ending the proof of Theorem 2 (iv).

**Proof of Theorem 2 (v)**: First note by (6.4) and the upper triangularity of \( \hat{M}_{k+1} \) according to (5.10) and (5.12)

\[ A_\varepsilon \cdot \begin{pmatrix} n^c_i \\ n^c_{k+1} \end{pmatrix} = [\frac{e^{2k+1}}{(2k+1)!} \cdot I_B] \cdot [\frac{e^{2k-1}}{(2k)!} \cdot I_B] \cdot \ldots \cdot [\frac{e^{k+1}}{(k+1)!} \cdot I_B + O(|\varepsilon|)] \cdot \begin{pmatrix} n^c_i \\ n^c_{k+1} \end{pmatrix} \]

\[ = A^1_\varepsilon \cdot n^c_i + \ldots + A^{k+1}_\varepsilon \cdot n^c_{k+1} \]

(6.8)
with $A^i_k \in L[N^c_k, B]$. Then, differentiation of (6.4) with respect to $N^c_i \times \cdots \times N^c_{k+1}$ at $z_0(\varepsilon)$ implies by chain rule

$$G'[z_0(\varepsilon)] \cdot A^i_k \cdot \begin{pmatrix} n^c_i \\ n^c_{k+1} \end{pmatrix} = \varepsilon^{2k+1} \cdot \hat{L}_{k+1} \cdot \begin{pmatrix} n^c_i \\ n^c_{k+1} \end{pmatrix} + O(\varepsilon^{2k+2}) \quad (6.9)$$

and using the definition of $\hat{L}_{k+1}$ in (6.3), the equation can further be splitted according to

$$G'[z_0(\varepsilon)] \cdot \frac{\varepsilon^{2k+1}}{(2k+1)!} \cdot n^c_i = \frac{\varepsilon^{2k+1}}{(2k+1)!} \cdot \hat{S}_1 \cdot n^c_i + O(\varepsilon^{2k+2})$$

$$\ldots$$

$$\ldots$$

$$G'[z_0(\varepsilon)] \cdot \frac{\varepsilon^{k+1}}{(k+1)!} \cdot [I_B + O(\varepsilon)] \cdot n^c_{k+1} = \frac{\varepsilon^{k+1}}{(k+1)!} \cdot \hat{S}_{k+1} \cdot n^c_{k+1} + O(\varepsilon^{k+1}) \quad (6.10)$$

Finally, cancelling out for $\varepsilon \neq 0$, we obtain

$$G'[z_0(\varepsilon)] \cdot \begin{pmatrix} n^c_i \\ n^c_{k+1} \end{pmatrix} \in R[A^i_k] = \hat{S}_1 \cdot n^c_i + O(\varepsilon^1)$$

$$\ldots$$

$$\ldots$$

$$G'[z_0(\varepsilon)] \cdot \frac{1}{(k+1)!} \cdot [I_B + O(\varepsilon)] \cdot n^c_{k+1} \in R[A^i_{k+1}]$$

with operator norms satisfying

$$\| G'[z_0(\varepsilon)] \|_{R[A^i_k]} = O(\varepsilon^0), \ldots, \| G'[z_0(\varepsilon)] \|_{R[A^i_{k+1}]} = O(\varepsilon^k), \quad (6.11)$$

as desired.

In the next step, let us estimate the operator norm of the inverse of the Jacobian with respect to the complement $N^c$ along the approximation $z_0(\varepsilon)$.

First, from the settings (5.7), (5.9), it is not too difficult to see that for $\varepsilon \neq 0$ we obtain $A^i_k \in GL[N^c_k, N^c]$ and in some more detail $A^i_k \in L[N^c_k, N^c_k], A^i_k \in L[N^c_k, N^c_k]$ as well as $A^i_k \in L[N^c_k, N^c_k \oplus \cdots \oplus N^c_{k+1}]$ in case of $3 \leq i \leq k+1$.

Then, again differentiation of (6.4) with respect to $N^c$ implies at every point of the cone

$$G'[Z_k(\varepsilon, n^c, p_{k+1})] \cdot A^i_k = \varepsilon^{2k+1} \cdot [\hat{L}_{k+1} + O(\varepsilon^k)] \quad (6.12)$$

and by $\hat{L}_{k+1} \in GL[N^c, B]$ one obtains $G'[Z_k(\varepsilon, n^c, p_{k+1})] \in GL[N^c, B]$ in case of $\varepsilon \neq 0$. Now, considering (6.8), we see for $\varepsilon \neq 0$

$$\| G'[Z_k(\varepsilon, n^c, p_{k+1})] \cdot A^i_k \| = \varepsilon^{k+1} \cdot \| G'[Z_k(\varepsilon, n^c, p_{k+1})] \cdot \frac{1}{\varepsilon^{k+1}} A^i_k \| \quad (6.13)$$

$$= \varepsilon^{k+1} \cdot \| G'[Z_k(\varepsilon, n^c, p_{k+1})] \|_{N^c} \bigg\| \frac{\varepsilon^0}{\varepsilon^{k+1}} \hat{L}_{k+1} + O(\varepsilon) \bigg\| \geq \varepsilon^{2k+1} \cdot c,$n

implying $\| G[N^c[Z_k(\varepsilon, n^c, p_{k+1})]] \| \geq \| \varepsilon^k \cdot c$ with $c > 0$ due to $\hat{L}_{k+1} \neq 0$. Hence,

$$\| G[N^c[Z_k(\varepsilon, n^c, p_{k+1})]]^{-1} \| = \sup_{\| \delta \| = 1} \| G[N^c[Z_k(\varepsilon, n^c, p_{k+1})]]^{-1} \cdot \delta \|$$
\[
\begin{align*}
\sup_{\|\tilde{n}^c\| = 1} \| G_{N^c}[\cdot]^{-1} \cdot G_{N^c}[\cdot] \| \cdot \| \tilde{n}^c \| = \sup_{\|n^c\| = 1} \| \tilde{n}^c \| \leq \frac{1}{|\varepsilon|^k},
\end{align*}
\]
yielding \( \| G_{N^c}[ Z_k(\varepsilon, n^c, p_{k+1}) ]^{-1} \| = O(|\varepsilon|^{-k}) \) at every point of a \( k \)-transversal cone. In particular, we obtain along the center line \( \| G_{N^c}[ z_0(\varepsilon) ]^{-1} \| = O(|\varepsilon|^{-k}) \), as desired.

**Proof of Theorem 2 (vi):** Now, assume \( N^c \) to be of finite dimension. Then the linear mappings \( A_\varepsilon \) from (6.4), (6.8) and \( \bar{L}_{k+1} \) from (6.3) can be represented by matrices according to

\[
A_\varepsilon = \begin{pmatrix}
\frac{\varepsilon^{2k+1}}{(2k+1)!} \cdot I_{d_1} & 0 & O(|\varepsilon|^{2k}) & \cdots & O(|\varepsilon|^{k+2}) \\
\frac{\varepsilon^{2k}}{(2k)!} \cdot I_{d_2} & 0 & \ddots & \ddots & \ddots \\
\frac{\varepsilon^{2k-1}}{(2k-1)!} \cdot I_{d_3} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\frac{\varepsilon^{k+1}}{(k+1)!} \cdot I_{d_{k+1}} & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix} \in \mathbb{K}^{d,d}
\]

and

\[
\bar{L}_{k+1} = \frac{1}{(2k+1)!} \begin{pmatrix}
I_{d_1} & O(1) & \cdots & O(1) \\
I_{d_2} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
I_{d_{k+1}} & \cdots & \cdots & C^{2k+1} \in \mathbb{K}^{d,d}
\end{pmatrix}
\]

Here \( d_i := \text{dim}(N^c_i) \geq 0, d := d_1 + \cdots + d_{k+1} \geq 1 \) and canonical bases chosen in \( R_i \) according to \( R_i = R[S_i], S_i \in GL[N^c_i, R_i], i = 1, \ldots, k + 1 \). Obviously,

\[
\text{det}\{ A_\varepsilon \} = \varepsilon_c \cdot \varepsilon^{(2k+1)d_1 + (2k)d_2 + \cdots + (k+1)d_{k+1}}, \quad c_1 \neq 0
\]

\[
\text{det}\{ \varepsilon^{2k+1} \cdot [ \bar{L}_{k+1} + O(\varepsilon) ] \} = \varepsilon_c \cdot \varepsilon^{(2k+1)d} \cdot [ 1 + O(\varepsilon) ], \quad c_2 \neq 0
\]

and by determinant multiplication theorem, equation (6.12) implies

\[
\text{det}\{ G'[ Z_k(\varepsilon, n^c, p_{k+1}) ] \| N^c \}
\]

\[
= \varepsilon^{3(2k+1) \cdot (d_1 + \cdots + d_{k+1}) - (2k+1) \cdot d_1 - (2k) \cdot d_2 - \cdots - (k+1) \cdot d_{k+1} \cdot [ 1 + O(\varepsilon) ]}
\]

\[
= \varepsilon^{2k+1 \cdot (d_1 + \cdots + d_{k+1}) \cdot [ 1 + O(\varepsilon) ]} = \varepsilon^{3 \cdot [ 1 + O(\varepsilon) ]}, \quad c_3 \neq 0
\]

with \( \chi = 1 \cdot \text{dim}(N^c_0) + \cdots + k \cdot \text{dim}(N^c_{k+1}) \), finishing the proof of Theorem 2.
7. Proof of Lemma 1

Let us now turn to the proof of Lemma 1.

**Figure 6**: The way to prove Lemma 1.

First, we obtain (iii) by direct calculation using decomposition (5.7) and some essential definitions of the iteration process within section 5. Next, (i) is obtained by an induction argument with respect to \( k \), relying on the validity of (ii), (iii) and (iv). Thirdly, (ii) is proved, again by induction as well as using (iv) as principal argument. The most technical part (iv) is shown by induction on \( k \), whereas (v) is obtained by a simple calculation using (2.5) and (i).

**Proof of Lemma 1 (iii)**: Dropping arguments, we obtain by decomposition (5.7)

\[
\begin{pmatrix} S_1 & \cdots & S_k+1 \end{pmatrix} \cdot \begin{pmatrix} n_0 \\ \vdots \\ n_k \end{pmatrix} = 0 \in \tilde{B}
\]

\[\Leftrightarrow \quad P_{R_i}[S_1 | \cdots | S_{k+1}] \cdot \begin{pmatrix} n_0 \\ \vdots \\ n_k \end{pmatrix} = 0, \quad i = 1, \ldots, k+1 \quad \land \quad P_{R_{k+1}}[S_1, \cdots, S_{k+1}] \cdot \begin{pmatrix} n_0 \\ \vdots \\ n_k \end{pmatrix} = 0
\]

\[\Leftrightarrow \quad \begin{pmatrix} S_1 & P_{R_1}S_2 & \cdots & P_{R_1}S_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_k & P_{R_k}S_{k+1} \\ \tilde{S}_{k+1} \end{pmatrix} \cdot \begin{pmatrix} n_0 \\ \vdots \\ n_{k-1} \\ n_k \end{pmatrix} = 0
\]

\[\land \quad S_1 n_0 + \cdots + \tilde{S}_{k+1} n_k \in R_1 \oplus \cdots \oplus R_{k+1},
\]

where the definitions of linear \( S \)-mappings in (5.6) imply \( S_1 n_0 \in R_1, \ldots, \tilde{S}_{k+1} n_k \in R_1 \oplus \cdots \oplus R_{k+1} \) and we can restrict to the matrix operator equation. Then, by the definitions of the operators \( E_{i,j} \) in (5.8), the following equivalences result from bottom up solution of the triangular system

\[\Leftrightarrow \quad \begin{cases} n_k &= \tilde{n}_{k+1} \\ n_{k-1} &= \tilde{n}_k - S_k^{-1} P_{R_k} \tilde{S}_{k+1} \cdot \tilde{n}_{k+1} \\ \vdots & \vdots \end{cases} = \begin{bmatrix} E_{k+1,k+1} & \cdots & E_{k,0} \\ \vdots & \ddots & \vdots \\ E_{1,k} & \cdots & E_{1,0} \end{bmatrix} \cdot \begin{pmatrix} \tilde{n}_{k+1} \\ \vdots \\ \tilde{n}_k \end{pmatrix}, \quad \tilde{n}_{k+1} \in N_{k+1}
\]

\[\land \quad \tilde{n}_k \in N_k
\]
\[ \Leftrightarrow \begin{pmatrix} n_0 \\ \vdots \\ n_k \end{pmatrix} = \begin{pmatrix} E_{1,1} & \cdots & E_{1,k+1} \\ \vdots & \ddots & \vdots \\ E_{k+1,1} & \cdots & E_{k+1,k+1} \end{pmatrix} \begin{pmatrix} \bar{n}_1 \\ \vdots \\ \bar{n}_{k+1} \end{pmatrix} \in N_1 \times \cdots \times N_{k+1} \]

with the last equivalence inferred from a simple induction argument, thus finishing the proof of Lemma 1 (iii).

For later use, some easy to show preliminaries are added, based on (5.18), (5.15) and (5.3)

\[ d_{m,1} = c_{m,1} = 1, \ m \geq 0 \] (7.1)

\[ \frac{d_{2m,2+j}}{d_{2m-1,1+j}} = \frac{d_{2m+1,2+j}}{d_{2m,1+j}} = d_{2m+1,2}, \ m \geq 1, \ j = 0, \ldots, m - 1 \] (7.2)

\[ \frac{d_{2m+1,3+j}}{d_{2m-1,1+j}} = d_{2m,2} \cdot d_{2m+1,2}, \ m \geq 2, \ j = 0, \ldots, m - 2 \] (7.3)

\[ [W_{2m+1}^2, \ldots, W_{m+2}^2](z_0, \ldots, z_0) = [W_{2m}^2, \ldots, W_{m+1}^2](z_{m-1}, \ldots, z_0) \cdot D^{2m}, \ m \geq 2 \] (7.4)

\[ W_{m+1}^{2m+1}(z_0, \ldots, z_0) = [W_m^{2m}(z_{m-1}, \ldots, z_0) + \frac{(2m)!}{(m!)^2} G_0 \cdot z_m] \cdot d_{2m,m+1}, \ m \geq 2 \] (7.5)

\[ [W_{2m}^2, \ldots, W_{m+1}^2](z_{m-1}, \ldots, z_0) = [W_{2m-1}^{2m-1}, \ldots, W_m^{2m-1}](z_{m-1}, \ldots, z_0) \cdot D^{2m-1}, \ m \geq 2 \] (7.6)

Note that the basic formulas (2.6) and (2.7), coupling high and low order derivatives within the system of undetermined coefficients, are a direct consequence of (7.1)-(7.6). This follows by straightforward calculation.

**Proof of Lemma 1 (i):** Next, according to figure 6, (i) is derived from (ii), (iii) and (iv) by induction on \( k \geq 1 \). In case of \( k = 1 \), (i) is valid according to

\[ T^2[z_2, z_1] = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} = G_1 \cdot z_2 + G_0 \cdot z_1^2 = \Delta^1 \cdot z_2 + l^1(z_1) \]

\[ M^3 = l_B \quad \text{and} \quad N_1 = N[G_0^1]. \]

Now, suppose (i) with \( k \geq 1 \). We show the validity for \( k + 1 \). According to (5.14), (5.15), (5.17), (5.18), the coefficients \( T^{2k+1}, \ldots, T^{k+1} \) can be written in the following form
\[
\begin{pmatrix}
T^{2k+1} & T^{2k} & \ldots & T^{k+2} & T^{k+1}
\end{pmatrix}^T = \begin{pmatrix}
W_{2(k+1)}^2 & W_{2k+1}^2 & \ldots & \ldots & W_{k+2}^2 & W_{k+1}^2
W_{2k+1}^2 & W_{k+2}^2 & \ldots & \ldots & \ldots & \ldots
W_{2k}^2 & W_{k+2}^2 & \ldots & \ldots & \ldots & \ldots
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots
W_{k+2}^2 & W_{k+2}^2 & \ldots & \ldots & \ldots & \\vdots
\end{pmatrix}
\begin{pmatrix}
Z_{2k+1}
Z_{2k}
Z_{k+2}
Z_{k+1}
\end{pmatrix} + \tilde{R}(\bar{z}_k, \ldots \bar{z}_1).
\]

Note that the red marked elements of the matrix $\overline{W}$ denote $\Delta^k(\bar{z}_{k-1}, \ldots, \bar{z}_1)$, whereas the matrix $\overline{W}$ without last row and last column delivers $\Delta^{k+1}(\bar{z}_k, \ldots, \bar{z}_1)$.

By induction hypothesis, the kernel of $\Delta^k$ is characterized by the range of $M^{2k+1}$ and our aim is to characterize the kernel of $\Delta^{k+1}$ by $M^{2k+3}$.

In the first step, we look for elements in $N[\Delta^k] = R[M^{2k+1}]$, satisfying additionally the equation

\[
(W_{2k+1}^2 \ W_{2k}^2 \ \ldots \ W_{k+1}^2) \cdot \begin{pmatrix}
Z_{2k+1}
M^{2k+1} \cdot \begin{pmatrix}
1
\vdots
1
\end{pmatrix}
\end{pmatrix} = 0, \quad (7.7)
\]
or equivalently by setting $Z_{2k+1} = n_0 \in B$ and use of Lemma 1 (ii), (iii)

\[
(W_{2k+1}^2 \ W_{2k}^2 \ \ldots \ W_{k+1}^2) \cdot \begin{pmatrix}
I_B
M^{2k+1} \cdot \begin{pmatrix}
1
\vdots
1
\end{pmatrix}
\end{pmatrix} = \begin{pmatrix}
n_0
n_1
\vdots
n_k
\end{pmatrix} = 0
\]

\[
\iff [\bar{S}_1 | \bar{S}_2(z_1) | \ldots | \bar{S}_{k+1}(z_k, \ldots, z_1)] \cdot C^{2k+1} \cdot \begin{pmatrix}
n_0
n_1
\vdots
n_k
\end{pmatrix} = 0
\]

\[
\iff \begin{pmatrix}
n_0
n_1
\vdots
n_k
\end{pmatrix} = (C^{2k+1})^{-1} \cdot E^{k+1} \cdot \begin{pmatrix}
\bar{n}_1
\bar{n}_2
\vdots
\bar{n}_{k+1}
\end{pmatrix}, \quad \bar{n}_i \in N_j, \quad 1 \leq i \leq k + 1
\]

\[
\iff \begin{pmatrix}
n_0
n_1
\vdots
n_k
\end{pmatrix} = (C^{2k+1})^{-1} \cdot E^{k+1} \cdot C^{2k+1} \cdot \begin{pmatrix}
\bar{n}_1
\bar{n}_2
\vdots
\bar{n}_{k+1}
\end{pmatrix} = (A^{2k+1} \ \bar{a}^{2k+1}) \cdot \begin{pmatrix}
\bar{n}_1
\bar{n}_2
\vdots
\bar{n}_{k+1}
\end{pmatrix}.
\]

Hence, the elements from $N[\Delta^k]$ that can be extended to solutions of (7.7) are exactly given by
\[
\begin{pmatrix}
  z_{2k+1} \\ z_{k} \\ \vdots \\ z_{k+1}
\end{pmatrix} = 
\begin{pmatrix}
  I_B \\ M^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  a^{2k+1} \\ a^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  n_1 \\ n_2 \\ \vdots \\ n_{k+1}
\end{pmatrix} = 
\begin{pmatrix}
  I_B \\ \vdots \\ I_B
\end{pmatrix} \cdot 
\begin{pmatrix}
  n_1 \\ n_2 \\ \vdots \\ n_{k+1}
\end{pmatrix},
\]

where the product of the two upper tridiagonal matrices gives a matrix with identity operator in the diagonal. Now, setting \( n_{k+1} = 0 \) within these solutions, we obtain the kernel of the matrix operator

\[
\bar{\Delta} := 
\begin{pmatrix}
  W^{2k+1}_{2k+1} & \cdots & W^{2k+1}_{k+2} \\
  W^{2k}_{2k} & \cdots & W^{2k}_{k+2} \\
  \vdots & \ddots & \vdots \\
  W^{2k+2}_{k+2}
\end{pmatrix}
\]

according to

\[
\begin{pmatrix}
  z_{2k+1} \\ z_{k+2}
\end{pmatrix} = 
\begin{pmatrix}
  I_B \\ M^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  a^{2k+1} \\ a^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  n_1 \\ \vdots \\ n_k
\end{pmatrix}.
\] (7.8)

For further reasoning, the diagonal matrices \( D^i \) and \( C^i \) from (5.4) without the first element \( D^i_{1,1} = C^i_{1,1} = 1 \) are denoted by \( \bar{D}^i \) and \( \bar{C}^i \) respectively. Then, we look for elements in \( N[\bar{\Delta}^k] \), satisfying additionally the equation

\[
\begin{pmatrix}
  W^{2(k+1)}_{2(k+1)} & W^{2(k+1)}_{2k+1} & \cdots & W^{2(k+1)}_{k+2}
\end{pmatrix} \cdot 
\begin{pmatrix}
  I_B \\ M^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  a^{2k+1} \\ a^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  n_1 \\ \vdots \\ n_k
\end{pmatrix} = 0,
\] (7.9)

or equivalently by setting \( z_{2(k+1)} = n_0 \in B \) and use of (7.6), (5.3) with \( m = k + 1 \) as well as Lemma 1 (ii), (iv)

\[
\begin{pmatrix}
  W^{2(k+1)}_{2(k+1)} & W^{2(k+1)}_{2k+1} & \cdots & W^{2(k+1)}_{k+2}
\end{pmatrix} \cdot 
\begin{pmatrix}
  I_B \\ M^{2k+1} \cdot A^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  a^{2k+1} \\ a^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  n_1 \\ \vdots \\ n_k
\end{pmatrix} = W^{2k+1}
\] (7.6)

\[
= \begin{pmatrix}
  W^{2k+1}_{2k+1} & W^{2k+1}_{2k} & \cdots & W^{2k+1}_{k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  I_B \\ M^{2k+1} \cdot A^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  a^{2k+1} \\ a^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  n_0 \\ n_1 \\ \vdots \\ n_k
\end{pmatrix} = W^{2k+1}
\]

\[
\]

\[
\text{or (ii)}
\]

\[
\begin{pmatrix}
  I_B \\ M^{2k+1} \cdot A^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  a^{2k+1} \\ a^{2k+1}
\end{pmatrix} \cdot 
\begin{pmatrix}
  n_0 \\ n_1 \\ \vdots \\ n_k
\end{pmatrix} = W^{2k+1}
\]
\[
\left( \begin{array}{c}
I_B \\
M^2k+1, \bar{M}^{2k+1-1}, \bar{D}^{2k+1-1}
\end{array} \right)
\left( \begin{array}{c}
n_0 \\
n_k
\end{array} \right) = \left( \begin{array}{c}
\bar{S}_1 \\
\bar{S}_2(z_1) \\
\vdots \\
\bar{S}_{k+1}(z_{k+1}) \end{array} \right)
\left( \begin{array}{c}
\bar{C}^{2k+1} \\
\bar{D}^{2k+1}
\end{array} \right)
\]

Therefore,
\[
\left( \begin{array}{c}
\bar{S}_1 \\
\bar{S}_2(z_1) \\
\vdots \\
\bar{S}_{k+1}(z_{k+1}) \end{array} \right)
\left( \begin{array}{c}
\bar{C}^{2k+1} \\
\bar{D}^{2k+1}
\end{array} \right)
\left( \begin{array}{c}
\bar{I}_B \\
\bar{M}^2k+1, \bar{A}^{2k+2}
\end{array} \right)
\left( \begin{array}{c}
n_0 \\
n_k
\end{array} \right) = 0.
\]

Now, again by Lemma 1 (iii), we obtain
\[
\left( \begin{array}{c}
n_0 \\
n_k
\end{array} \right) = \left( \begin{array}{c}
\bar{C}^{2k+1} \\
\bar{D}^{2k+1}
\end{array} \right)^{-1} \cdot \bar{E}^{k+1} \cdot \left( \begin{array}{c}
\bar{\bar{n}}_1 \\
\bar{\bar{n}}_{k+1}
\end{array} \right), \quad \bar{n}_i \in \bar{N}_i, \quad 1 \leq i \leq k + 1
\]

and the elements from \( N[\bar{\bar{\Delta}}^k] \) that can be extended to \( N[\bar{\bar{\Delta}}^{k+1}] \) are exactly given by
\[
\left( \begin{array}{c}
\bar{\bar{n}}_1 \\
\bar{\bar{n}}_{k+1}
\end{array} \right) = \left( \begin{array}{c}
\bar{C}^{2k+1} \\
\bar{D}^{2k+1}
\end{array} \right) \left( \begin{array}{c}
\bar{a}^{2k+2} \\
\bar{A}^{2k+2}
\end{array} \right) \left( \begin{array}{c}
\bar{\bar{n}}_1 \\
\bar{\bar{n}}_{k+1}
\end{array} \right)
\]

thus finishing the induction.

**Proof of Lemma 1 (ii)**: In the next step, (ii) is accomplished by induction for \( k \geq 1 \) using (iv). In case of \( k = 1 \), the identity (ii) reads
\[
W^3(z_1) \cdot \left( \begin{array}{c}
I_B \\
M^3
\end{array} \right) = \left[ \bar{S}_1 \ | \ \bar{S}_2(z_1) \right] \cdot C^3,
\]

yielding by (2.2), (5.17), (5.5), (5.7) and (5.3)
\[
\left[ G_0 \ | \ 3G_0^2z_1 \right] \cdot \left( \begin{array}{c}
I_B \\
I_B
\end{array} \right) = \left[ G_0 \ | \ 2G_0^2z_1 \right] \cdot \left( \begin{array}{c}
1 \\
\frac{1}{2}
\end{array} \right),
\]

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which is true by inspection. Now, suppose identity (ii) with \( k \) replaced by \( k - 1 \geq 1 \)

\[
W^{2k-1}(z_{k-1}, \ldots, z_1) \cdot \left( I_B \begin{array}{c} I \end{array} \right) = [ \bar{S}_1 \ | \cdots | \bar{S}_k(z_{k-1}, \ldots, z_1) ] \cdot C^{2k-1}. \tag{7.10}
\]

Our aim is to show (ii) with \( k \), where the equality with respect to the first component is obviously valid under consideration of (5.21), (7.1) and it remains to show

\[
[ W^{2k+1}_{2k}, \ldots, W^{2k+1}_{k+1} ](z_k, \ldots, z_1) \cdot M^{2k+1}
\]

\[
= [ \bar{S}_2 \ | \cdots | \bar{S}_{k+1} ](z_k, \ldots, z_1) \cdot \begin{pmatrix} c_{2k+1,2} & \cdots & c_{2k+1,k+1} \\ c_{k+1,2} & \cdots & c_{k+1,k+1} \end{pmatrix}, \tag{7.11}
\]

Further, with index \( k_s \) and \( k_z \) denoting column \( k \) and row \( k \) of a matrix respectively, as well as index \( k | \) denoting a matrix without column \( k \), the left hand side of (7.11) reads

\[
\{ [ W^{2k+1}_{2k}, \ldots, W^{2k+1}_{k+1} ](z_k, \ldots, z_1) \cdot M^{2k+1}, [ W^{2k+1}_{2k}, \ldots, W^{2k+1}_{k+1} ](z_k, \ldots, z_1) \cdot M^{2k+1}_k \}
\]

\[
\overset{(5.12)}{=} \{ [ W^{2k+1}_{2k}, \ldots, W^{2k+1}_{k+2} ](z_k, \ldots, z_1) \cdot M^{2k+1}_k, \}
\]

\[
[ W^{2k}_{2k-1}, \ldots, W^{2k}_{k+1} ](z_{k-1}, \ldots, z_1) \cdot \begin{pmatrix} d_{2k,2} & \cdots & d_{2k,k} \\ d_{k+1,2} & \cdots & d_{k+1,k} \end{pmatrix}, M^{2k+1}_k, \tag{7.12}
\]

\[
[ W^{2k}_{2k-1}, \ldots, W^{2k}_{k+1} ](z_{k-1}, \ldots, z_1) \cdot \begin{pmatrix} d_{2k,2} & \cdots & d_{2k,k} \\ d_{k+1,2} & \cdots & d_{k+1,k} \end{pmatrix}, M^{2k+1}_k
\]

\[
W^2_k (z_{k-1}, \ldots, z_1) + \frac{(2k)!}{(k!)^2} G^2_0 z_k \cdot d_{2k,k+1} \}
\]

The right hand side of (7.11) can be represented by induction hypothesis (7.10) according to

\[
\{ [ \bar{S}_2 \ | \cdots | \bar{S}_k ](z_{k-1}, \ldots, z_1) \cdot \begin{pmatrix} c_{2k+1,2} & \cdots & c_{2k+1,k+1} \\ c_{2k+1,2} & \cdots & c_{2k+1,k+1} \end{pmatrix}, \bar{S}_{k+1}(z_k, \ldots, z_1) \cdot c_{2k+1,k+1} \}
\]

\[
\overset{(7.10)}{=} [ W^{2k-1}_{2k-2}, \ldots, W^{2k-1}_{k} ](z_{k-1}, \ldots, z_1) \cdot M^{2k-1} \cdot \begin{pmatrix} c_{2k-1,2} & \cdots & c_{2k-1,k+1} \\ c_{2k-1,2} & \cdots & c_{2k-1,k+1} \end{pmatrix}^{-1} \begin{pmatrix} c_{2k+1,2} & \cdots & c_{2k+1,k+1} \\ c_{2k+1,2} & \cdots & c_{2k+1,k+1} \end{pmatrix}, \tag{7.13}
\]

\[
[ W^{2k}_{2k-1}, \ldots, W^{2k}_{k} ](z_{k-1}, \ldots, z_1) \cdot \left( A^{2k-1} \begin{array}{c} M^{2k-1} \end{array} \right) \cdot c_{2k+1,k+1} + \frac{(2k)!}{(k!)^2} G^2_0 z_k \cdot c_{2k+1,k+1}
\]
Comparing (7.12) and (7.13), sufficient conditions for equality within (7.11) are given by

\[
\begin{bmatrix}
    d_{2k,2} \\
    \vdots \\
    d_{2k,k}
\end{bmatrix} \cdot M_{k,k}^{2k+1} = \begin{bmatrix}
    d_{2k-1,2} \\
    \vdots \\
    d_{2k-1,k}
\end{bmatrix} \cdot M_{k-1,k}^{2k-1} \cdot \begin{bmatrix}
    d_{2k-1,2} \\
    \vdots \\
    d_{2k-1,k}
\end{bmatrix} \cdot \begin{bmatrix}
    d_{2k,2} \\
    \vdots \\
    d_{2k,k}
\end{bmatrix}
\]

as well as

\[
\begin{bmatrix}
    d_{2k,2} \\
    \vdots \\
    d_{2k,k}
\end{bmatrix} \cdot M_{k,k}^{2k+1} = \begin{bmatrix}
    \bar{a}^{2k-1} \\
    \vdots \\
    \bar{a}^{2k-1}
\end{bmatrix} \cdot \begin{bmatrix}
    d_{2k,2} \\
    \vdots \\
    d_{2k,k}
\end{bmatrix}
\]

or combined under consideration of (iv) with \( k \) replaced by \( k - 1 \) and (5.8), (5.9)
Then by \((\text{iv})\) and for later use, we obtain by a similar calculation

\[
\begin{pmatrix}
\vdots \\
d_{2,k,2} \\
\vdots \\
d_{2,k,k}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
(a^{2k-1} & \bar{a}^{2k-1}) \\
M^{2k-1} \cdot (A^{2k-1} & \bar{A}^{2k-1})
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\vdots \\
d_{2,k,k} \\
\vdots \\
d_{2,k,k+1}
\end{pmatrix}
\begin{pmatrix}
(a^{2k} & \bar{a}^{2k}) \\
(A^{2k} & \bar{A}^{2k})
\end{pmatrix}
\begin{pmatrix}
\vdots \\
d_{2,k,2} \\
\vdots \\
d_{2,k,k+1}
\end{pmatrix}
\]

Now, using again (iv), we obtain

\[
\begin{pmatrix}
\vdots \\
d_{2,k,2} \\
\vdots \\
d_{2,k,k}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
(a^{2k-1} & \bar{a}^{2k-1}) \\
M^{2k-1} \cdot (A^{2k-1} & \bar{A}^{2k-1})
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\vdots \\
d_{2,k,k} \\
\vdots \\
d_{2,k,k+1}
\end{pmatrix}
\begin{pmatrix}
(a^{2k} & \bar{a}^{2k}) \\
(M^{2k-1} \cdot (A^{2k-1} & \bar{A}^{2k-1}))
\end{pmatrix}
\begin{pmatrix}
\vdots \\
d_{2,k,2} \\
\vdots \\
d_{2,k,k+1}
\end{pmatrix}
\]

(7.14)

Further, (5.3) and (5.8) yield

\[
\begin{pmatrix}
(a^{2k} & \bar{a}^{2k}) \\
A^{2k} & \bar{A}^{2k}
\end{pmatrix} = (C^{2k})^{-1} \cdot E^{k} \cdot C^{2k}
\]

(7.15)

\[
= (D^{2k-1})^{-1} \cdot (C^{2k-1})^{-1} \cdot E^{k} \cdot C^{2k-1} \cdot D^{2k-1} = (D^{2k-1})^{-1} \begin{pmatrix}
(a^{2k-1} & \bar{a}^{2k-1}) \\
A^{2k-1} & \bar{A}^{2k-1}
\end{pmatrix} \cdot D^{2k-1}
\]

and for later use, we obtain by a similar calculation

\[
\begin{pmatrix}
(a^{2k} & \bar{a}^{2k}) \\
A^{2k} & \bar{A}^{2k}
\end{pmatrix} = D^{2k} \cdot \begin{pmatrix}
(a^{2k+1} & \bar{a}^{2k+1}) \\
A^{2k+1} & \bar{A}^{2k+1}
\end{pmatrix} \cdot (D^{2k})^{-1}
\]

(7.16)

Then by (7.15), the first row in (7.14) is equivalent to

\[
\begin{pmatrix}
\vdots \\
d_{2,k,2} \cdot (d_{2,k-1,1})^{-1} \cdot (a^{2k-1} & \bar{a}^{2k-1}) \\
\vdots \\
d_{2,k,k}
\end{pmatrix}
\begin{pmatrix}
\vdots \\
d_{2,k,k-1,1} \\
\vdots \\
d_{2,k,k-1,k}
\end{pmatrix}
\]

= \underbrace{D^{2k-1}}_{=D^{2k-1}}
\[
\begin{pmatrix}
\alpha^{2k-1} & \alpha^{2k-1} \\
\vdots & \vdots \\
\alpha^{2k-1} & \alpha^{2k-1}
\end{pmatrix}
\begin{pmatrix}
d_{2k,2} \\
\vdots \\
d_{2k,k+1}
\end{pmatrix}
\]

\[\Leftrightarrow 0 = (a^{2k-1} \bar{a}^{2k-1}) \cdot \begin{pmatrix}
d_{2k,2} \\
\vdots \\
d_{2k,k+1}
\end{pmatrix} - \begin{pmatrix}
d_{2k,2} \\
\vdots \\
d_{2k,k+1}
\end{pmatrix}
\]

which is obviously true using (7.2) with \( j = 0, \ldots, k - 1 \). Along the same lines of reasoning, the remaining rows \( 2, \ldots, k - 1 \) in (7.14) can be treated with \( j = 1, \ldots, k - 2 \) according to

\[d_{2k,2} \cdot (d_{2k-1,1+j})^{-1} \cdot M_{jz}^{2k-1} \cdot (A^{2k-1} \bar{A}^{2k-1}) \cdot \begin{pmatrix}
d_{2k-1,1} \\
\vdots \\
d_{2k-1,k}
\end{pmatrix}
\]

\[\Leftrightarrow 0 = M_{jz}^{2k-1} \cdot (A^{2k-1} \bar{A}^{2k-1}) \cdot \begin{pmatrix}
d_{2k,2} \\
\vdots \\
d_{2k,k+1}
\end{pmatrix}
\]

in this way finishing the proof of Lemma 1 (ii).

**Proof of Lemma 1 (iv):** The proof of (iv) is accomplished independently of (i)-(iii), using again an inductive argument for \( k \geq 1 \). For \( k = 1 \), identity (iv) is obviously true by

\[M^3 =_{l_B} d_{3,2} =_{l_B} d_{3,2} \cdot \alpha^3
\]

Then, suppose (iv) with \( k \) replaced by \( k - 1 \geq 1 \) yielding

\[M^{2k-1} \cdot \begin{pmatrix}
d_{2k-1,2} \\
\vdots \\
\alpha^{2k-1}
\end{pmatrix} = \begin{pmatrix}
d_{2k-1,2} \\
\vdots \\
\alpha^{2k-1}
\end{pmatrix} \cdot \begin{pmatrix}
d_{2k-1,2} \\
\vdots \\
\alpha^{2k-1}
\end{pmatrix}
\]

whereas the identity to prove is given by

\[= M^{2k+1}
\]

\[\begin{pmatrix}
\alpha^{2k} & \alpha^{2k} \\
\vdots & \vdots \\
\alpha^{2k} & \alpha^{2k}
\end{pmatrix} \cdot \begin{pmatrix}
d_{2k+1,2} \\
\vdots \\
\alpha^{2k+1}
\end{pmatrix}
\]
\[
\begin{pmatrix}
d_{2k+1,2} \\
\vdots \\
d_{2k+1,k+1}
\end{pmatrix}
\cdot
\left(
\begin{pmatrix}
(a^{2k} & \tilde{a}^{2k}) \\
\vdots \\
M^{2k-1} \cdot A^{2k-1}
\end{pmatrix}
\cdot
\begin{pmatrix}
A^{2k} & \tilde{A}^{2k}
\end{pmatrix}
\right)
\cdot
\begin{pmatrix}
a^{2k+1} \\
\vdots \\
\tilde{a}^{2k}
\end{pmatrix}
\]
\[ (7.16) \quad (7.15) \quad \frac{d_{2k-1,1}}{=1} \cdot a^{2k} \cdot \left( \begin{array}{cccc} d_{2k-1,1} & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \cdots \\ d_{2k-1,k-1} & \cdots & \cdots & d_{2k-1,k} \end{array} \right)^{-1} \cdot \left( \begin{array}{c} d_{2k,2} \\ \vdots \\ \vdots \\ d_{2k,k} \end{array} \right) \cdot A_k^{2k+1} \cdot \left( \begin{array}{c} d_{2k,1} \\ \vdots \\ \vdots \\ d_{2k,k} \end{array} \right)^{-1} \cdot \left( \begin{array}{c} d_{2k+1,2} \\ \vdots \\ \vdots \\ d_{2k+1,k+1} \end{array} \right) \]

\[ (7.2) \quad \frac{= a^{2k} \cdot d_{2k,2} \cdot A_k^{2k+1} \cdot d_{2k+1,2}} {= d_{2k+1,3} \cdot a^{2k} \cdot A_k^{2k+1}} \]

\[ = d_{2k+1,3} \cdot (a^{2k} \cdot \bar{a}^{2k}) \cdot A_k^{2k+1}, \]

yielding the second row in (7.19). Note that the last identity follows from the fact that the \(k\)-th row of \(A_k^{2k+1}\) equals zero according to the definition in (5.8).

Finally, in case of \(k \geq 3\), the remaining rows \(3, \ldots, k\) in (7.18), (7.19) can be treated along the same lines of reasoning with \(j = 1, \ldots, k - 2\) according to

\[ (7.16) \quad (7.15) \quad \frac{M_j^{2k-1} \cdot A_k^{2k-1} \cdot \left( A_k^{2k} \right) \cdot \left( d_{2k+1,2} \\ \vdots \\ d_{2k+1,k+1} \right)} {M_j^{2k-1} \cdot \left( \begin{array}{c} d_{2k-1,2} \\ \vdots \\ \vdots \\ d_{2k-1,k} \end{array} \right) \cdot A_k^{2k} \cdot \left( \begin{array}{c} d_{2k-1,1} \\ \vdots \\ \vdots \\ d_{2k-1,k-1} \end{array} \right)^{-1} \cdot \left( \begin{array}{c} d_{2k,2} \\ \vdots \\ \vdots \\ d_{2k,k} \end{array} \right) \cdot A_k^{2k+1} \cdot \left( \begin{array}{c} d_{2k,1} \\ \vdots \\ \vdots \\ d_{2k,k} \end{array} \right)^{-1} \cdot \left( \begin{array}{c} d_{2k+1,2} \\ \vdots \\ \vdots \\ d_{2k+1,k+1} \end{array} \right)} \]

\[ (7.2) \quad \frac{M_j^{2k-1} \cdot \left( \begin{array}{c} d_{2k-1,2} \\ \vdots \\ \vdots \\ d_{2k-1,k} \end{array} \right) \cdot A_k^{2k} \cdot d_{2k,2} \cdot A_k^{2k+1} \cdot d_{2k+1,2}} {M_j^{2k-1} \cdot \left( \begin{array}{c} d_{2k-1,2} \\ \vdots \\ \vdots \\ d_{2k-1,k} \end{array} \right) \cdot A_k^{2k} \cdot A_k^{2k+1} \cdot d_{2k+1,2}} \]

\[ (7.3) \quad \frac{d_{2k+1,3+j} \cdot M_j^{2k-1} \cdot \left( \begin{array}{c} d_{2k-1,2} \\ \vdots \\ \vdots \\ d_{2k-1,k} \end{array} \right) \cdot A_k^{2k} \cdot A_k^{2k+1}} {d_{2k-1,1+j} \cdot M_j^{2k-1} \cdot \left( \begin{array}{c} d_{2k-1,2} \\ \vdots \\ \vdots \\ d_{2k-1,k} \end{array} \right) \cdot \left( A_k^{2k} \cdot \bar{A}_k^{2k} \right) \cdot A_k^{2k+1}} \]

finishing the proof of Lemma 1 (iv).
Proof of Lemma 1 (v) : We assume a curve $z_0(\varepsilon)$ as in (3.9) with $\|G[z_0(\varepsilon)]\| = O(|\varepsilon|^{2k+1})$, implying by (2.5) with $k$ replaced by $2k$

$$
\begin{pmatrix}
T^{2k} [\varepsilon_{2k}, \ldots, \varepsilon_1] \\
\vdots \\
T^1 [\varepsilon_1]
\end{pmatrix} = \Gamma^{-2k} \cdot \Delta^{2k} (\varepsilon_{2k-1}, \ldots, \varepsilon_1) \cdot \Gamma^{2k} \cdot 
\begin{pmatrix}
\varepsilon_{2k} \\
\vdots \\
\varepsilon_1
\end{pmatrix} = 
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} .
$$

Then, by Lemma 1 (i), there exists $\tilde{n}_i \in N_i, 1 \leq i \leq 2k$, satisfying

$$
\Gamma^{2k} \cdot 
\begin{pmatrix}
\varepsilon_{2k} \\
\vdots \\
\varepsilon_1
\end{pmatrix} = M^{4k+1} \cdot 
\begin{pmatrix}
\tilde{n}_1 \\
\vdots \\
\tilde{n}_{2k}
\end{pmatrix}
$$

with upper tridiagonal matrix operator $M^{4k+1}$ and $diag[M^{4k+1}] = [I_B \cdots I_B]$. Now, using $\varepsilon_1 = \cdots = \varepsilon_{i-1} = 0$, we obtain $\tilde{n}_{2k} = \cdots = \tilde{n}_{2k-i+2} = 0$ and $\Gamma^{2k} \cdot \varepsilon_i = I_B \cdot \tilde{n}_{2k-i+1} \in N_{2k-i+1} \subset N_{k+1}$ due to $l \leq k$.

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