Abstract

These lecture notes provide an introduction to cosmic inflation. In particular I will review the basic concepts of inflation, generation of density perturbations, and reheating after inflation.
1 Preface

These lecture notes are based on invited lectures at The Second Tah Poe School on Cosmology “Modern Cosmology”, Naresuan University, Phitsanulok, Thailand, April 17 -25, 2003. See the web page
http://www.tech.port.ac.uk/staffweb/gumjudpb/TPC2FIRSTpage.html for the details of this school and conference.

2 Introduction

The proposal of General Relativity by Einstein in 1915 made it possible to discuss the structure of spacetime and the evolution of the universe in terms of physical laws. In 1922 Friedmann found the existence of expanding/collapsing cosmological solutions by solving the Einstein field equations. In 1929 Hubble discovered the expansion of the universe by the observations in the redshift of galaxies, as the Einstein theory predicts. In 1946 Gamov and his collaborators showed that the universe must begin in a very hot and dense state from the theory of nucleosynthesis. They also predicted that the present universe should be filled with the microwaves with black body radiations. In 1965 Penzias and Wilson discovered microwave background radiations which coincide well with theoretical predictions by Gamov et al.. These strong observational evidences have made people believe that the universe started out from a hot and dense state, which is called as the standard big-bang model.

In standard big-bang cosmology the state of the universe is characterized by the radiation-dominated or the matter-dominated stage. These correspond to the decelerated expansion of the universe where the second derivative of the scale factor is negative. Meanwhile this decelerated expansion is not sufficient to solve a number of cosmological problems such as flatness and horizon problems which plagues in the standard big-bang scenario. In order to overcome these fundamental problems, it is required to consider an epoch of accelerated expansion in the early universe, i.e., inflation.

The basic ideas of inflation were originally proposed by Guth [1] and Sato [2] independently in 1981, which is now termed as old inflation. This corresponds to the de-Sitter inflation which makes use of the first-order transition to true vacuum. However, it has a serious shortcoming that the universe becomes inhomogeneous by the bubble collision soon after the inflation ends. The revised version was proposed by Linde [4], and Albrecht and Steinhardt [5] in 1982, which is dubbed as new inflation. This corresponds to the slow-roll inflation with the second-order transition to true vacuum. Unfortunately this scenario also suffers from a fine-tuning problem of spending enough time in false vacuum to lead to sufficient amount of inflation. In 1983 Linde [6] considered the variant version of the slow-roll inflation called chaotic inflation, in which initial conditions of scalar fields are chaotic. According to this model, our homogeneous and isotropic universe may be produced in the regions where inflation occurs sufficiently. While old and new inflation models are based on the assumption that the universe was in a state of thermal equilibrium from the beginning, chaotic inflation can occur even without such an assumption. In addition chaotic inflation can start out in the regime close to Planck density, thereby solving the problem of initial conditions.

\[ ^2 \text{In the next section I will explain how inflation solves these problems.} \]
\[ ^3 \text{Note that a specific version of the inflationary scenario—so called } R^2 \text{ inflation—was proposed by Starobinsky one year earlier [4]. Nevertheless this did not explicitly point out the virtue of inflation as in the paper of Guth.} \]
Many kinds of inflationary models have been constructed in these twenty years (see e.g., ref. [7, 8]). In particular, the recent trend is to construct consistent models of inflation based on superstring or supergravity models. See ref. [9] for a review in such a direction.

The inflationary paradigm not only provides an excellent way in solving flatness and horizon problems but also generates density perturbations as seeds for large scale structure in the universe [10]. In fact inflation provides a causal mechanism to generate nearly scale-invariant spectra of cosmological perturbations. Quantum fluctuations of the field driving inflation—called inflaton— are typically frozen by the accelerating expansion when the scales of fluctuations leave the Hubble radius. Long after the inflation ends, the scales cross inside the Hubble radius again. Thus perturbations imprinted during inflation can be the origin of large-scale structure in the universe. In fact temperature anisotropies observed by the COBE satellite in 1992 exhibit nearly scale-invariant spectra as predicted by the inflationary paradigm. Recent observations of WMAP also show strong evidence for inflation [11].

3 The standard big-bang cosmology and its problems

3.1 The standard big-bang cosmology

The standard big-bang cosmology is based on the cosmological principle [3], which means that the universe is homogeneous and isotropic on large distance. Then the metric takes the Friedmann-Robertson-Walker (FRW) form:

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - K r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \]  

(1)

Here \( a(t) \) is the scale factor with \( t \) being the cosmic time. The constant \( K \) is the spatial curvature, where positive, zero, and negative values correspond to closed, flat, and open universes, respectively.

The evolution of the universe is dependent on the material within it. This is characterized by the equation of state between the energy density \( \rho(t) \) and the pressure \( p(t) \). Typical examples are:

\[
\begin{align*}
    p & = \rho/3, & \text{radiation}, \\
    p & = 0, & \text{dust}.
\end{align*}
\]  

(2) \hspace{1cm} (3)

In order to know the dynamical evolution of the universe, it is required to solve the Einstein equations in General Relativity. The Einstein equations are expressed as [12, 13]

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}, \]  

(4)

where \( R_{\mu\nu}, R, T_{\mu\nu}, \) and \( G \) are the Ricci tensor, Ricci scalar, energy momentum tensor, gravitational constant, respectively. The Planck energy, \( m_{\text{pl}} = 1.2211 \times 10^{19} \text{ GeV} \), is related with \( G \) through the relation \( m_{\text{pl}} = (h c^5 / G)^{1/2} \). Here \( h \) and \( c \) are the Planck’s constant and the speed of light, respectively. Hereafter we use the units \( h = c = 1 \). \( \Lambda \) is a cosmological constant originally introduced by Einstein.
For the background metric (1) with a negligible cosmological constant, the Einstein equations (4) yield

\[ H^2 = \frac{8\pi}{3m_{\text{pl}}^2}\rho - \frac{K}{a^2}, \tag{5} \]

\[ \dot{\rho} + 3H(\rho + p) = 0, \tag{6} \]

where a dot denotes the derivative with respect to \( t \), and \( H \equiv \dot{a}/a \) is the Hubble expansion rate. Eqs. (5) and (6) are so called the Friedmann and Fluid equations, respectively. Combining these relations gives the following equation,

\[ \frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\text{pl}}^2}(\rho + 3p). \tag{7} \]

The Friedmann equation (5) can be rewritten as

\[ \Omega - 1 = \frac{K}{a^2H^2}, \tag{8} \]

where

\[ \Omega \equiv \frac{\rho}{\rho_c}, \quad \text{with} \quad \rho_c \equiv \frac{3H^2m_{\text{pl}}^2}{8\pi}. \tag{9} \]

Here the density parameter \( \Omega \) is the ratio of the energy density to the critical density. When the spatial geometry is flat \((K = 0)\), the solutions for Eqs. (5) and (6) are

- Radiation dominant: \( a \propto t^{1/2}, \quad \rho \propto a^{-4} \),
- Dust dominant: \( a \propto t^{2/3}, \quad \rho \propto a^{-3} \).

In these simple cases, the universe is expanding deceleratedly \((\ddot{a} < 0)\) as confirmed by Eq. (7).

### 3.2 Problems of the standard big-bang cosmology

#### 3.2.1 Flatness problem

In the standard big-bang theory with \( \ddot{a} < 0 \), the \( a^2H^2(= \dot{a}^2) \) term in Eq. (8) always decreases. This indicates that \( \Omega \) tends to shift away from unity with the expansion of the universe. However, since present observations suggest that \( \Omega \) is within the order of magnitude of one \([11]\), \( \Omega \) needs to be very close to one in the past. For example, we require \(|\Omega - 1| < O(10^{-16})\) at the epoch of nucleosynthesis \([8, 14]\) and \(|\Omega - 1| < O(10^{-64})\) at the Planck epoch \([15]\). This is an extreme finetuning of initial conditions. Unless initial conditions are chosen very accurately, the universe soon collapses, or expands quickly before the structure can be formed. This is so called the flatness problem.
3.2.2 Horizon problem

Consider a comoving wavelength, $\lambda$, and also a physical wavelength, $a\lambda$, which is inside the Hubble radius, $H^{-1}$ (i.e., $a\lambda \lesssim H^{-1}$). The standard big-bang cosmology is characterized by the cosmic evolution of $a \propto t^p$ with $0 < p < 1$. In this case the physical wavelength grows as $a\lambda \propto t^p$, whereas the Hubble radius evolves as $H^{-1} \propto t$. Therefore the physical wavelength becomes much smaller than the Hubble radius with the passage of time. This means that the region where the causality works eventually becomes the only small fraction of the Hubble radius.

To be more precise, let us first define the particle horizon $D_H(t)$ where the light travels from the beginning of the universe, $t = t_*$,

$$D_H(t) = a(t)d_H(t), \quad \text{with} \quad d_H(t) = \int_{t_*}^{t} \frac{dt}{a(t)}. \quad (12)$$

Here $d_H(t)$ corresponds to the comoving distance. Setting $t_* = 0$, we find $d_H(t) = 3t$ in the matter-dominant era. We observe photons in the Cosmic Microwave Background (CMB) which are emitted at the time of decoupling. The particle horizon at decoupling, $D_H(t_{\text{dec}}) = a(t_{\text{dec}})d_H(t_{\text{dec}})$, corresponds to the region where photons could have contacted causally at that time. The ratio of $d_H(t_{\text{dec}})$ to the particle horizon today, $d_H(t_0)$, where $t_0$ is the present time, is estimated as

$$\frac{d_H(t_{\text{dec}})}{d_H(t_0)} \approx \left( \frac{t_0}{t_{\text{dec}}} \right)^{1/3} \approx \left( \frac{10^5}{10^{10}} \right)^{1/3} \approx 10^{-2}. \quad (13)$$

This result implies that the causality regions of photons are restricted to be small. In fact the surface of the last scattering surface only corresponds to the angle of order 1°. Observationally, however, we see photons which thermalize to the same temperature in all regions in the CMB sky. This is so called the horizon problem.

3.2.3 The origin of large-scale structure in the universe

The COBE satellite observes the anisotropies in the last scattering surface, whose amplitudes are small and close to scale-invariant. These fluctuations spread so large a scale that it is practically impossible to generate them between the big bang and the time of the last scattering in the standard cosmology. This problem is almost equivalent to the horizon problem mentioned above, i.e., the standard cosmology can not provide satisfactory explanation for the origin of large-scale structure.

3.2.4 Monopole problem

According to the viewpoint of particle physics, the breaking of supersymmetry leads to the production of many unwanted relics such as monopoles, cosmic strings, and topological defects [7]. The string theories also predict supersymmetric particles such as gravitinos, Kaluza-Klein particles, and moduli fields.

If these particles exist in the early stage of the universe, the energy densities of them decrease as a matter component ($\sim a^{-3}$). Since the radiation energy density decreases as $\sim a^{-4}$ in the radiation-dominant era, these massive relics could be the dominant materials in the universe, which contradicts with observations. This problem is generally called as the monopole problem.
4 Idea of inflationary cosmology

The problems in the big-bang cosmology lie in the fact that the universe always exhibits the decelerating expansion. Let us assume the existence of a stage in the early universe with an accelerated expansion of the universe, i.e.,

\[ \ddot{a} > 0. \] (14)

From the relation (14) this corresponds to the condition

\[ \rho + 3p < 0. \] (15)

The condition (14) essentially means that \( \dot{a} (= aH) \) increases during inflation. Then the comoving Hubble radius, \( (aH)^{-1} \), decreases in the inflationary phase. This property is the key point to solve the cosmological puzzles in the standard big-bang cosmology as I will show below.

4.1 Flatness problem

Since the \( a^2 H^2 \) term in Eq. (8) increases during inflation, \( \Omega \) rapidly approaches unity. After the inflationary period ends, the evolution of the universe is followed by the conventional big-bang phase and \( |\Omega - 1| \) begins to increase. In spite of this, as long as the inflationary expansion occurs sufficiently and makes \( \Omega \) very close to one, \( \Omega \) stays of order unity even in the present epoch.

4.2 Horizon problem

Since the scale factor evolves as \( a \propto t^p \) with \( p > 1 \) during inflation, the physical wavelength, \( a\lambda \), grows faster than the Hubble radius, \( H^{-1}(\propto t) \). Therefore the physical wavelength is pushed outside the Hubble radius during inflation. This means that the region where the causality works is stretched on scales much larger than the Hubble radius, thus solving the horizon problem.

Of course the Hubble radius begins to grow faster than the physical wavelength after inflation (radiation/matter dominant era). In order to solve the horizon problem, it is required that the following condition is satisfied for the comoving particle horizon:

\[ \int_{t_{dec}}^{t} \frac{dt}{a(t)} \gg \int_{t_{dec}}^{t_0} \frac{dt}{a(t)}. \] (16)

This implies that the comoving distance that photons can travel before decoupling needs to be much larger than that after the decoupling. According to the precise calculation, it is achieved when the universe expands about \( e^{70} \) times during inflation [8, 15].

4.3 The origin of the large scale structure

The fact that the comoving Hubble radius decreases during inflation makes it possible to generate the nearly scale-invariant density perturbations on large scales. Since the scales of perturbations are within the Hubble radius in the early stage of inflation, causal physics works to generate small quantum fluctuations. After a scale is pushed outside the Hubble radius (i.e., the first horizon crossing) during inflation, the perturbations can be described by classical ones. When the inflationary period ends, the evolution of the universe is followed by the standard big-bang cosmology, and the comoving Hubble radius begins to increase. Then the scales of perturbations
cross inside the Hubble radius again (the second horizon crossing), after which causality works. The small perturbations imprinted during inflation appear as large-scale perturbations after the second horizon crossing. In this way the inflationary paradigm naturally provides a causal mechanism in generating the seeds of density perturbations observed in the CMB anisotropies.

4.4 Monopole problem

During the inflationary phase \( \rho + 3p < 0 \), the energy density of the universe decreases very slowly. For example, when the universe evolves as \( a \propto t^p \) with \( p > 1 \), we have \( H \propto t^{-1} \propto a^{-1/p} \) and \( \rho \propto a^{-2/p} \). Meanwhile the energy density of massive particles decreases much faster (\( \sim a^{-3} \)), these particles are red-shifted away during inflation, thereby solving the monopole problem.

Of course, we have to worry for the case where these unwanted particles are produced after inflation. In the process of reheating followed by inflation, the energy of the universe can be transferred to radiation or other light particles. At this stage unwanted particles must not be overproduced in order not to violate the success of the standard cosmology such as nucleosynthesis. Generally if the reheating temperature is sufficiently low, the thermal production of unwanted relics such as gravitinos can be avoided.

5 Inflationary dynamics

The scalar fields are the important ingredients in particle physics theories. Consider a homogeneous scalar field, \( \phi \), called \textit{inflaton}, whose potential energy leads to the exponential expansion of the universe. The energy density and the pressure density of the inflaton can be described, respectively, as

\[
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\]

where \( V(\phi) \) is the potential of the inflaton. Substituting Eq. (17) for Eqs. (5) and (6), we get

\[
H^2 = \frac{8\pi}{3m_{\text{pl}}^2} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right],
\]

(18)

\[
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,
\]

(19)

where \( \kappa^2 \equiv 8\pi G = 8\pi/m_{\text{pl}}^2 \), and we neglected the curvature term \( K^2/a^2 \) in Eq. (18).

During inflation, the relation (15) yields \( \dot{\phi}^2 < V(\phi) \), which indicates that the potential energy of the inflaton dominates over the kinetic energy of it. Therefore a flat potential of the inflaton is required in order to lead to sufficient amount of inflation. Imposing the slow-roll conditions: \( \frac{1}{2} \dot{\phi}^2 \ll V(\phi) \) and \( \dot{\phi} \ll 3H \phi \), Eqs. (18) and (19) are approximately given as

\[
H^2 \simeq \frac{8\pi}{3m_{\text{pl}}^2} V(\phi),
\]

(20)

\[
3H \dot{\phi} \simeq -V'(\phi).
\]

(21)
Defining the so-called slow-roll parameters
\[ \epsilon \equiv \frac{m_{\text{pl}}^2}{16\pi} \left( \frac{V'}{V} \right)^2, \quad \eta \equiv \frac{m_{\text{pl}}^2}{8\pi} \frac{V''}{V}, \] (22)
we can easily verify that the above slow-roll approximations are valid when
\[ \epsilon \ll 1, \quad |\eta| \ll 1. \] (23)

The inflationary phase ends when \( \epsilon \) and \( |\eta| \) grow of order unity. A useful quantity to describe the amount of inflation is the number of e-foldings, defined by
\[ N \equiv \ln \frac{a_f}{a_i} = \int_{t_i}^{t_f} H dt, \] (24)
where the subscripts \( i \) and \( f \) denote the quantities at the beginning and the end of the inflation, respectively.

In order to solve the flatness problem, \( \Omega \) is required to be \( |\Omega_f - 1| < 10^{-60} \) right after the end of inflation. Meanwhile the ratio \( |\Omega - 1| \) between the initial and final phase of inflation is given by
\[ \frac{|\Omega_f - 1|}{|\Omega_i - 1|} \simeq \left( \frac{a_i}{a_f} \right)^2 = e^{-2N}, \] (25)
where we used the fact that \( H \) is nearly constant during inflation. Assuming that \( |\Omega_i - 1| \) is of order unity, the number of e-foldings is required to be \( N \gtrsim 70 \) to solve the flatness problem. We have similar number of e-foldings from the requirement to solve the horizon problem.

### 6 Models of inflation

So far we did not mention the form of the inflaton potential, \( V(\phi) \). Originating from the old inflationary scenario proposed by Guth [1] and Sato [2], we now have varieties of inflationary models: \( R^2 \), new, chaotic, extended, power-law, hybrid, natural, supernatural, extranatural, eternal, D-term, F-term, brane, oscillating, trace-anomaly driven,..., etc.

These different kinds of models can be roughly classified in the following way [16]. The first class (type I) is the “large field” model, in which the initial value of the inflaton is large and it rolls down toward the potential minimum at smaller \( \phi \). Chaotic inflation [6] is one of the representative models of this class. The second class (type II) is the “small field” model, in which the inflaton field is small initially and slowly evolves toward the potential minimum at larger \( \phi \). New inflation [4, 5] and natural inflation [17] are the examples of this type. In the first model, the second derivative of the potential \( V^{(2)}(\phi) \) usually takes positive values, whereas \( V^{(2)}(\phi) \) can change its sign in the second model. The third one (type III) is the hybrid (double) inflation model [18], in which inflation typically ends by the phase transition triggered by the presence of the second scalar field (or by the second phase of inflation after the phase transition).

Let us briefly recap each type of inflationary models.
6.0.1 Chaotic inflation— an example of type I

The chaotic inflation model is described by the quadratic or quartic inflaton potential,

\[ V(\phi) = \frac{1}{2}m^2\phi^2, \quad \text{or} \quad V(\phi) = \frac{1}{4}\lambda\phi^4. \]  

\[ (26) \]

The “chaotic” means that initial conditions of inflaton are distributed chaotically. According to this scenario, the region which undergoes the sufficient amount of inflation gives rise to our universe.

Let us investigate the evolution of the universe in the case of the quadratic potential. Then Eqs. (20) and (21) read

\[ H^2 \simeq \frac{4\pi m^2\phi^2}{3m_{\text{pl}}^2}, \quad 3H\dot{\phi} + m^2\phi \simeq 0. \]  

\[ (27) \]

Combining these relations gives the following solutions:

\[ \phi \simeq \phi_i - \frac{m m_{\text{pl}}}{2\sqrt{3}\pi}t, \]  

\[ (28) \]

\[ a \simeq a_i \exp \left[ \frac{2}{3} \sqrt{\frac{m}{m_{\text{pl}}}} \left( \frac{m m_{\text{pl}}}{4\sqrt{3}\pi}t \right)^2 \right], \]  

\[ (29) \]

where \( \phi_i \) is an integration constant corresponding to the initial values of the inflaton. We find from the relation (28) that the universe expands exponentially during the initial stage of inflation. With the increase of the second term in the square bracket of Eq. (29), the expansion rate slows down. Since the slow-roll parameters are expressed as

\[ \epsilon = \eta = \frac{m^2}{4\pi\phi^2}, \]  

\[ (30) \]

the inflationary period ends around \(|\phi| \approx m_{\text{pl}}/\sqrt{4\pi}\), after which the system enters a reheating stage. The total amount of inflation is approximately expressed as

\[ N \simeq 2\pi \left( \frac{\phi_i}{m_{\text{pl}}} \right)^2 - \frac{1}{2}. \]  

\[ (31) \]

In order to lead to sufficient inflation \( N \gtrsim 70 \), we require the initial value to be \( \phi_i \gtrsim 3m_{\text{pl}} \). The inflaton mass, \( m \), is constrained by the amplitude of density perturbations observed by the COBE satellite. In order to fit observations, \( m \) is required to be

\[ m \approx 10^{-6}m_{\text{pl}}. \]  

\[ (32) \]

In the quartic potential case, the self-coupling is constrained to be \( \lambda \approx 10^{-13} \) by the similar argument.
6.0.2 Natural inflation– an example of type II

Natural inflation model [17] is characterized by Pseudo Nambu-Goldstone bosons (PNGBs) which appear when an approximate global symmetry is spontaneously broken. The PNGB potential is expressed as

$$V(\phi) = m^4 \left[ 1 + \cos\left(\frac{\phi}{f}\right) \right],$$  \hspace{1cm} (33)

where two mass scales $m$ and $f$ characterize the height and width of the potential, respectively. The typical mass scales are of order $f \sim m_{\text{pl}} \sim 10^{19}$ GeV and $m \sim m_{\text{GUT}} \sim 10^{16}$ GeV, which are expected by particle physics.

Let us consider the case where the inflaton is initially located in the region, $0 < \phi < \pi f$, and inflation occurs while the inflaton slowly evolves toward the potential minimum at $\phi = \pi f$. The slow-roll parameters are

$$\epsilon = \frac{m_{\text{pl}}^2}{16\pi f^2} \left[ \frac{\sin(\phi/f)}{1 + \cos(\phi/f)} \right]^2, \quad \eta = -\frac{m_{\text{pl}}^2}{8\pi f^2} \frac{\cos(\phi/f)}{1 + \cos(\phi/f)}. \hspace{1cm} (34)$$

Note that $\epsilon$ and $\eta$ depend on $f$, but not on $m$. Inflation starts out from the regime where $\phi$ is close to zero. The system enters a reheating stage when the inflaton begins to oscillate around $\phi = \pi f$.

One typical property in the type II model is that the second derivative of the inflaton potential can change its sign. In the natural inflation, $V^{(2)}(\phi)$ is negative when inflaton evolves in the region of $0 < \phi < \pi f/2$. This leads to the enhancement of inflaton fluctuations by spinodal (tachyonic) instability [19, 20, 21]. When the particle creation by spinodal instability is neglected, the number of $e$-foldings is expressed by [17]

$$N = \frac{16\pi f^2}{m_{\text{pl}}^2} \ln \left[ \frac{\sin(\phi_i/2f)}{\sin(\phi_f/2f)} \right], \hspace{1cm} (35)$$
where \( \phi_i \) and \( \phi_f \) are the initial and final values of the inflaton during inflation, respectively. In order to obtain the sufficient amount of e-foldings satisfying \( N \gtrsim 70 \), the initial value of the inflaton is required to be \( \phi(t_i) \lesssim 0.1m_{\text{pl}} \) for the mass scale \( f \sim m_{\text{pl}} \).

6.0.3 Hybrid (Double) inflation— an example of type III

The inflationary model building in the presence of multiple scalar fields is the recent trend from the viewpoint of particle physics [9]. Let us consider the Linde’s hybrid inflation model [18], described by

\[
V = \frac{\lambda}{4} \left( \chi^2 - \frac{M^2}{\lambda} \right)^2 + \frac{1}{2} g^2 \phi^2 \chi^2 + \frac{1}{2} m^2 \phi^2. \tag{36}
\]

When \( \phi^2 \) is large, the field tends to roll down toward the potential minimum at \( \chi = 0 \). In this case, the potential is effectively described by a single field,

\[
V \approx \frac{M^4}{4\lambda} + \frac{1}{2} m^2 \phi^2. \tag{37}
\]

Inflation does not end for the potential (37). In the presence of the \( \chi \) field, however, the mass of the field \( \chi \) becomes negative for \( \phi < \phi_c \equiv M/g \). Then the field begins to roll down to one of the true minima at \( \phi = 0 \) and \( \chi = \pm M/\sqrt{\lambda} \) (see Fig. 3). The original version of the hybrid inflation [18] corresponds to the case where the inflation soon comes to an end after the symmetry breaking (\( \phi < \phi_c \)) due to the rapid rolling of the field \( \chi \). In this case the number of e-foldings acquired in double inflation can be approximately estimated by using the potential (37):

\[
N \approx \frac{2\pi M^4}{\lambda m^2 m_{\text{pl}}^2} \ln \frac{\phi_i}{\phi_c}, \tag{38}
\]
where $\phi_i$ is the initial value of inflaton.

If the condition, $M^2 \gg \lambda M_p^2$, is satisfied, the mass of the field $\chi$ is “light” relative to the Hubble rate around $\phi = \phi_c$, thereby leading to the second stage of inflation for $\phi < \phi_c$ [22]. This corresponds to the double inflationary scenario.

7 Density perturbations from inflation

In this section we study cosmological perturbations generated from inflation. For simplicity we shall consider the single-field model of inflation. The most general form of the line element that describes scalar metric perturbations is written as [23, 24]

$$ds^2 = a^2(\tau)\left\{ - (1 + 2A)d\tau^2 + 2B_{ij} dx^i dx^j + [(1 - 2\psi)\gamma_{ij} + 2E_{ij}]dx^i dx^j \right\},$$

where $\tau \equiv \int a^{-1}dt$ is the conformal time. The scalar quantities $A$, $\psi$, $B$, and $E$ are the functions of space and time coordinates. We denote the three-dimensional covariant derivative of a function $f$ with respect to some coordinate $i$ by $f_i$. In the linear perturbation theory of cosmological perturbations, there exist some redundant degrees of freedom in the equations of motions which can be fixed without affecting the physics. It is important to choose a proper gauge condition so that it allows for an simplified mathematical analysis and an easier physical interpretation (see e.g., Ref. [25] for details). Among several gauge conditions, the longitudinal gauge defined by the gauge condition $B = E = 0$ is often used:

$$ds^2 = a^2(\tau)[-(1 + 2\Phi)d\tau^2 + (1 - 2\Psi)\gamma_{ij} dx^i dx^j]\]
= -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)\delta_{ij} dx^i dx^j. \quad (40)$$
The potentials $\Phi$ and $\Psi$ can be constructed as gauge-invariant variables, whose forms are invariant under infinitesimal coordinate transformations (see Ref. [24] for details). When the spatial part of the energy-momentum tensor is diagonal, $\delta T^i_j \sim \delta^i_j$, $\Phi$ coincides with $\Psi$.

Let us consider the linearized Einstein equation with $\Lambda = 0$:

$$\delta G^\mu_\nu = 8\pi G \delta T^\mu_\nu. \quad (41)$$

The perturbed Einstein tensor can be evaluated in the longitudinal gauge as follows

$$\delta G^0_0 = 2a^{-2} \left[-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2 \Psi \right],$$

$$\delta G^i_i = 2a^{-2}(\mathcal{H}\Phi + \Psi')_i,$$

$$\delta G^i_j = -2a^{-2} \left\{ \left[2\mathcal{H}' + \mathcal{H}^2\right]\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' + \frac{1}{2}\nabla^2 D \right\} \delta^i_j - \frac{1}{2}\gamma^{ik} D_{,kj}, \quad (42)$$

where $\mathcal{H} \equiv a'/a$ and $D \equiv \Phi - \Psi$, with a prime being a derivative with respect to $\tau$. In the case where the universe is filled with a minimally coupled scalar field, $\phi$, the energy momentum tensor takes the following form

$$T^\mu_\nu = -\left(\nabla\phi\right)^2 + \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right] \delta^\mu_\nu. \quad (43)$$

We decompose the field $\phi$ into homogenous and fluctuational parts as $\phi = \phi_0 + \delta\phi$. Then the energy momentum tensor is decomposed into background and perturbed parts as

$$T^\mu_\nu = T^\mu_\nu(0) + \delta T^\mu_\nu, \quad (44)$$

where

$$T^0_0(0) = \frac{1}{2a^2}\phi_0'^2 + V(\phi_0), \quad T^i_0(0) = 0, \quad T^i_j(0) = \left[\frac{1}{2a^2}\phi_0'^2 + V(\phi_0)\right] \delta^i_j, \quad (45)$$

and

$$\delta T^0_0 = a^{-2} \left[-\phi_0'^2\Phi + \phi_0\delta\phi' + V'(\phi)a^2\delta\phi\right],$$

$$\delta T^i_i = a^{-2}\phi_0'\delta\phi, $$

$$\delta T^i_j = a^{-2} \left[+\phi_0'^2\Phi - \phi_0'\delta\phi' + V'(\phi)a^2\delta\phi\right] \delta^i_j. \quad (46)$$

Note that the relation $\delta T^i_j \sim \delta^i_j$ holds in this case. This leads to the relation $D = \Phi - \Psi = 0$ by considering the $ij$ part in Eq. (41). Then the perturbed Einstein equation (41) can be written as

$$\nabla^2 \Phi - 3\mathcal{H}\Phi' - (\mathcal{H}' + 2\mathcal{H}^2)\Phi = \frac{\kappa^2}{2}(\phi_0'\delta\phi + V'(\phi)a^2\delta\phi), \quad (47)$$

$$\Phi' + \mathcal{H}\Phi = \frac{\kappa^2}{2}\phi_0'\delta\phi, \quad (48)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi = \frac{\kappa^2}{2}(\phi_0'\delta\phi - V'(\phi)a^2\delta\phi), \quad (49)$$

13
where $\kappa^2 \equiv 8\pi G$. We used the background equations in deriving these equations. Combining Eqs. (47)-(49) gives the equation for the gauge-invariant scalar field:

$$
\delta \phi'' + 2\mathcal{H}\delta \phi' - \nabla^2 \delta \phi + V''(\phi)a^2 \delta \phi = 4\phi'_0 \Phi' - 2V'(\phi)a^2 \Phi.
$$

(50)

Note that the effect of gravitational perturbations appears in the rhs of Eq. (50).

Eliminating the scalar field perturbation in Eqs. (47)-(49), we find the following equation for gauge-invariant cosmological perturbations:

$$
\Phi'' + 2\left(\frac{a}{\phi'_0}\right)' \Phi' - \nabla^2 \Phi + 2\phi'_0 (\mathcal{H}/\phi')' \Phi = 0.
$$

(51)

Introducing new variables

$$
u = \left(\frac{a}{\phi'_0}\right) \Phi,
\quad z = \mathcal{H}/a\phi'_0,
$$

(52)

Eq. (51) reduces to

$$
u'' - \nabla^2 \nu - \left(\frac{z''}{z}\right) \nu = 0.
$$

(53)

Making Fourier transformations, the mode function $u_k$ with a wave number $k$ satisfies

$$u''_k + (k^2 - \frac{z''}{z}) u_k = 0.
$$

(54)

In the long-wavelength limit, $k^2 \ll \frac{z''}{z}$, we find the analytic solution

$$u = z \left( c_1 + c_2 \int \frac{d\tau}{z^2} \right),
$$

(55)

where $c_1$ and $c_2$ are integration constants. Making use of Eqs. (48), (52), and (55), we have

$$\Phi = \frac{\dot{a}}{a^2} \left( c_1 - 2c_2 \int a dt \right) + 2c_2,
$$

(56)

$$\delta \phi = -\frac{\dot{\phi}_0}{a} \left( c_1 - 2c_2 \int a dt \right).
$$

(57)

The solution (56) indicates that there exists a gauge-invariant conserved quantity (comoving curvature perturbation) in the long-wavelength limit:

$$\mathcal{R} = \Phi - \frac{H^2}{H} \left( \Phi + \frac{\dot{\Phi}}{H} \right) = 2c_2,
$$

(58)

which was firstly introduced by Bardeen [26].

During inflation, the perturbations with a comoving wavenumber $k$ are pushed outside the Hubble radius by the accelerated expansion of the universe. This makes the wavenumber toward the long wavelength regions: $k^2 \ll \frac{z''}{z}$, in which case we have the analytic solutions of Eqs. (56) [56].

4Note that the comoving wavelength $\lambda$ is related with $k$ by the relation $\lambda = 2\pi/k$. [56]
and (57). Since the terms which include the $c_1$ coefficient rapidly decrease and the relation $|\dot{H}| \ll H^2$ holds during the inflationary phase, Eqs. (56) and (57) yield

$$\Phi \simeq 2c_2 \left( \frac{1}{a} \int a dt \right),$$

$$= 2c_2 \left[ H^{-1} - a^{-1} \int a(H^{-1}) dt \right] \simeq -2c_2 \dot{H}/H^2,$$  \hspace{1cm} (59)

$$\delta \phi \simeq 2c_2 \dot{\phi} a^{-1} \int a dt \simeq 2c_2 \dot{\phi} a^{-1}/H.$$  \hspace{1cm} (60)

Combining Eqs. (58) and (60), we have

$$R = H \delta \phi_k \phi_0,$$  \hspace{1cm} (61)

where we introduced the subscript “$k$” for the fluctuation, $\delta \phi$. The $H \delta \phi_k/\dot{\phi} a^{-1}$ term should be evaluated when the wavelength of the perturbation crosses the Hubble radius: $k^2 = z''/z$. Once the scales are pushed outside the Hubble radius during inflation, the evolution of perturbations can be described by Eqs. (56) and (57). Long after the inflation ends, the scale of the perturbations reenters the Hubble radius again, which generates the seeds for the galaxies and clusters. Thus the inflationary paradigm naturally explains the origin of large-scale structure.

The spectrum of the comoving curvature perturbation is defined as

$$\mathcal{P}_R = \frac{k^3}{2\pi^2} \langle |\mathcal{R}|^2 \rangle = \frac{k^3}{2\pi^2} \frac{H^2}{\delta \phi_0^2} |\delta \phi_k|^2.$$  \hspace{1cm} (62)

From Eqs. (59) and (60) we have $\Phi \simeq cH \delta \phi_k/\dot{\phi} a^{-1}$ with $c = -\dot{H}/H^2$ being the slow-roll parameter defined in eq. (22). Taking note that $|4\dot{\phi} \Phi'| \ll |2V'(\phi) a^2 \Phi|$ on super-Hubble scales and making use of the slow-roll relation $3H \dot{\phi} \simeq -V'(\phi)$, Eq. (50) can be approximately written as

$$\delta \phi''_k + 2H \delta \phi'_k + \left[ k^2 + a^2 \nu''(\phi) + 6a^2 \epsilon H^2 \right] \delta \phi_k = 0.$$  \hspace{1cm} (63)

Redefining a field perturbation, $\delta \chi_k = a^{-1} \delta \phi_k$, this equation reduces to

$$\delta \chi''_k + \left[ k^2 \frac{1}{\tau^2} \left( \nu^2 - \frac{1}{4} \right) \right] \delta \chi_k = 0 \quad \text{with} \quad \nu^2 = \frac{9}{4} + 9\epsilon - 3\eta,$$  \hspace{1cm} (64)

where we used the fact that $a''/a = a^2(2 - \epsilon) H^2 \simeq (2 + 3\epsilon)/\tau^2$. When $\nu$ is a real number, the solution for this equation can be written in terms of the Hankel functions of the first and second kind:

$$\delta \chi_k = \sqrt{-\tau} \left[ c_1(k) H_{\nu}^{(1)}(-k\tau) + c_2(k) H_{\nu}^{(2)}(-k\tau) \right].$$  \hspace{1cm} (65)

Quantum fluctuations are generated in the ultraviolet regime ($k \gg aH$), which is required to take the plane-wave form $e^{-ik\tau}/\sqrt{2k}$ for $-k\tau \gg 1$. Matching Eq. (65) with this quantum fluctuation, we have $c_1(k) = \sqrt{\pi/2} e^{i(\nu+\frac{3}{2})\frac{\pi}{2}}$ and $c_2(k) = 0$, thereby yielding

$$\delta \chi_k = \sqrt{\frac{\pi}{2}} e^{i(\nu+\frac{3}{2})\frac{\pi}{2}} \sqrt{-\tau} H_{\nu}^{(1)}(-k\tau).$$  \hspace{1cm} (66)
Since the Hankel function takes the form \( H_\nu(-k\tau \ll 1) \approx \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} 2^{\nu - \frac{3}{2}} (\Gamma(\nu)/\Gamma(3/2))(-k\tau)^{-\nu} \) on super-Hubble scales, the amplitude of the fluctuation \( \delta \phi_k \) is estimated as

\[
|\delta \phi_k| \approx \frac{H}{\sqrt{2k^3}} \left( \frac{k}{aH} \right)^{\frac{3}{2}-\nu}.
\] (67)

Substituting this relation for eq. (62), we finally get the spectrum of the curvature perturbations on super-Hubble scales:

\[
P_R = \frac{4\pi}{m_{pl}^2} \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{aH} \right)^{3-2\nu} \equiv A_R^2 \left( \frac{k}{aH} \right)^{n_R-1}.
\] (68)

Here the spectral index \( n_R \) is given as

\[
n_R - 1 = \frac{d\ln P_R}{d\ln k} = -6\epsilon + 2\eta.
\] (69)

When \( n_R = 1 \), the spectrum of the curvature perturbation is scale-invariant. Since the slow-roll parameters \( \epsilon \) and \( \eta \) are much smaller than unity [see eq. (23)], scalar perturbations generated in standard inflation are close to scale-invariant. This property is consistent with the recent observation of WMAP [11]. When \( n_R < 1 \) and \( n_R > 1 \), we call the red-tilted spectrum and the blue-tilted spectrum, respectively. For example the chaotic inflation model exhibits the red-tilted spectrum, while the hybrid inflation model shows the blue-tilted spectrum [8]. This property is useful to constrain on inflationary models from observations.

The spectrum (68) corresponds to the adiabatic perturbation generated in the single-field inflationary scenario. In the case of the multi-field inflation, this is modified due to the presence of isocurvature (entropy) perturbations [27]. In particular the curvature perturbation \( \mathcal{R} \) is no longer a conserved quantity in the multi-field context [28]. Therefore we have to estimate the curvature perturbation at the end of inflation instead of using its value at the horizon crossing. See ref. [22] for the detailed numerical analysis of correlated adiabatic and isocurvature perturbations in concrete models of double inflation.

### 8 Reheating after inflation

After the end of inflation, the universe enters a reheating stage, during which the potential energy of the inflaton is transferred to radiation and the universe is thermalized. In the original version of the reheating scenario which is now called old reheating, the decay of the inflaton is described by the perturbation theory [29]. This process is not efficient for the success of the GUT scale baryogenesis scenario. In contrast, it was later found that the existence of the nonperturbative stage called preheating can lead to the explosive particle production in the early stage of reheating [30, 31].

#### 8.1 Old reheating

The reheating stage turns on when the inflaton reaches the potential minimum and begins to oscillate. Let us first study the background evolution during reheating for the polynomial potential

\[
V(\phi) = \frac{1}{2n} m^2 m_{pl}^2 \left( \frac{\phi}{m_{pl}} \right)^{2n},
\] (70)
where \( n \) is a positive integer. Making use of the time-averaged relation

\[
\langle \dot{\phi}^2 \rangle_T = 2n \langle V(\phi) \rangle_T,
\]

the time evolution of the scale factor and the Hubble parameter are

\[
a \propto t^{(n+1)/3n}, \quad H \propto \frac{n + 1}{3n} \frac{1}{t},
\]

where we used eqs. (18) and (19). When \( n = 1 \) and \( n = 2 \), the universe evolves as matter-dominant \((a \propto t^{2/3})\) and radiation-dominant \((a \propto t^{1/2})\), respectively.

Let us consider the case of \( n = 1 \) in order to understand the basic picture of reheating. In this case, the evolution of the inflaton is described by the sinusoidal oscillation with a decreasing amplitude \( \tilde{\Phi}(t) \):

\[
\phi = \tilde{\Phi}(t) \sin mt, \quad \tilde{\Phi}(t) = \frac{m_{\text{pl}}}{\sqrt{3\pi mt}},
\]

Then the energy density of the inflaton decreases as

\[
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi) \approx \frac{1}{2} m^2 \tilde{\Phi}^2 \propto a^{-3}.
\]

The simple estimations of Eqs. (73) and (74) should be modified in the presence of the inflaton decay. Without this decay, the universe will evolve toward emptier and colder states because of the redshift of the energy density. In the old reheating scenario [29] where the single inflaton decay is considered, the inflaton \( \phi \) is coupled to a scalar field \( \chi \) and a fermion field \( \psi \) with the interacting Lagrangian,

\[
\mathcal{L}_{\text{int}} = -\sigma \phi \chi^2 - h \phi \bar{\psi} \psi,
\]

where \( \sigma \) and \( h \) are coupling constants. Note that we assume that the bare masses of \( \chi \) and \( \psi \) are much smaller than the inflaton mass \( m \) and neglect them. We include quantum correction effects to the equation of inflaton as

\[
\ddot{\phi} + 3H \dot{\phi} + \left[ m^2 + \Pi(m) \right] \phi = 0,
\]

where \( \Pi(m) \) is the polarization operator of inflaton. Although the real part of \( \Pi(m) \) is small relative to \( m^2 \), \( \Pi(m) \) acquires an imaginary part,

\[
\text{Im} \, \Pi(m) = -m \Gamma,
\]

where \( \Gamma = \Gamma(\phi \rightarrow \chi\chi) + \Gamma(\phi \rightarrow \bar{\psi}\psi) \) is the total decay rate of inflaton. The perturbative decay rates of \( \phi \rightarrow \chi \) and \( \phi \rightarrow \psi \) are expressed as [29]

\[
\Gamma(\phi \rightarrow \chi\chi) = \frac{\sigma^2}{8\pi m}, \quad \Gamma(\phi \rightarrow \bar{\psi}\psi) = \frac{h^2 m}{8\pi}.
\]

Note that these relations are valid in the case of \( \Gamma \ll m \), which implies the relations: \( \sigma^2 \ll m^2 \) and \( h^2 \ll 1 \). We cannot use Eq. (78) in the nonperturbative stage of reheating (preheating).
Combining Eqs. (76) and (77) and setting \( \phi = \tilde{\Phi}(t) \exp(it) \), we find
\[
\tilde{\Phi}(t) = \frac{m_{pl}}{\sqrt{3\pi mt}} \exp(-\Gamma t/2) .
\] (79)
The same solution can be obtained by adding the phenomenological decay term \( \Gamma \dot{\phi} \) to the equation of inflaton:
\[
\ddot{\phi} + 3H \dot{\phi} + \Gamma \dot{\phi} + m^2 \phi = 0 .
\] (80)
The relation, \( \Gamma < 3H \), holds in the initial stage as long as coupling constants \( \sigma \) and \( h \) are small. Since the Hubble parameter decreases as \( H \propto 1/t \), the fraction of produced particles to the total energy density becomes important when \( 3H \) drops less than \( \Gamma \). The energy density of the universe at this time can be estimated by setting \( \Gamma^2 = (3H)^2 = 24\pi \rho/m_{pl}^2 \), as
\[
\rho = \frac{\Gamma^2 m_{pl}^2}{24\pi} .
\] (81)
If we assume that this energy density is rapidly transferred to light particles and newly-produced particles acquire the radiation energy with temperature \( T_R \) by instant thermalization, we obtain the relation
\[
\rho = \frac{g_\ast \pi^2 T_R^4}{30} = \frac{\Gamma^2 m_{pl}^2}{24\pi} ,
\] (82)
where \( g_\ast (\gtrsim 100) \) is the effective number of degrees of freedom at \( T = T_R \). Then the reheating temperature is estimated as
\[
T_R \lesssim 0.1 \sqrt{\Gamma m_{pl}} .
\] (83)
Taking into account the relation, \( \Gamma \ll m \lesssim 10^{-6} m_{pl} \), we find
\[
T_R \ll 10^{-4} m_{pl} .
\] (84)
This value is much smaller than the GUT scale temperature \( T_{\text{GUT}} = 10^{-3} m_{pl} \), which indicates that the GUT scale baryogenesis does not work well in the old reheating scenario.

### 8.2 Preheating

In the early 1990s, it was realized that the inflaton decay may have started in a much more explosive process, called preheating \[30, 31\] before the perturbative decay (see also refs. \[32, 33, 34\]). At this stage, the fluctuations of scalar particles can grow quasi-exponentially by parametric resonance. The process of reheating consists of three stages:

(i) Preheating stage where particles are nonperturbatively produced
(ii) Perturbative stage where the inflaton decay is described by the Born process
(iii) Thermalization of produced particles

Among variant models of inflation, chaotic inflation has an advantage that the dynamics of preheating is simply analyzed. Kofman, Linde, and Starobinsky investigated the structure of
resonance in preheating both in massive and self-interacting inflationary models \[31\]. Hereafter we shall review the dynamics of preheating for the two-field model with a massive inflaton,

\[ V(\phi, \chi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \chi^2. \] (85)

We assume that the spacetime and the inflaton \(\phi\) give a classical background and the scalar field \(\chi\) is treated as a quantum field on that background. Expanding the scalar field \(\chi\) as

\[ \chi = \frac{1}{(2\pi)^{3/2}} \int (a_k \chi_k(t)e^{-i k \cdot x} + a_k^\dagger \chi_k^*(t)e^{i k \cdot x}) \, d^3 k, \] (86)

and adopting the flat Friedmann-Robertson-Walker (FRW) metric, the each Fourier component \(\chi_k(t)\) satisfies the following equation of motion,

\[ \ddot{\chi}_k + 3H \dot{\chi}_k + \left( \frac{k^2}{a^2} + g^2 \phi^2 \right) \chi_k = 0. \] (87)

Introducing a new scalar field \(X_k \equiv a^{3/2} \chi_k\), Eq. (87) yields

\[ \ddot{X}_k + \omega_k^2 X_k = 0, \] (88)

where

\[ \omega_k^2 \equiv \frac{k^2}{a^2} + g^2 \phi^2 - \frac{3}{4} \left( \frac{2 \ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right). \] (89)

We neglect the last term in Eq. (89) for analytic investigations since this term becomes gradually negligible during the reheating phase.

Then Eq. (89) is reduced to the well known Mathieu equation

\[ \frac{d^2 X_k}{dz^2} + \left( A_k - 2q \cos 2z \right) X_k = 0, \] (90)

where \(z = mt\) and

\[ A_k = 2q + \frac{k^2}{m^2 a^2}, \] (91)

\[ q = \frac{g^2 \bar{\phi}^2(t)}{4m^2}. \] (92)

The strength of the resonance in Eq. (90) depends on the variables of \(A_k\) and \(q\), which is shown in a stability-instability chart of Fig. 4. In the unstable region (the lined region in Fig. 4), \(X_k\) grows exponentially as \(X_k \propto \exp(\mu_k z)\) with the Floquet index \(\mu_k\) and particles with momentum \(k\) are produced. For small \(q\) \((< 1)\), the width of the instability band is small and few \(k\)-modes grow by this resonance. This is called the narrow resonance. On the other hand, for the large \(q\) \((\gg 1)\), the resonance can occur for a broad range of the momentum \(k\)-space. Since the growth rate of the \(\chi\)-particle becomes larger with the increase of the variable \(q\), this resonance provides more efficient particle production than in the narrow one. This is called the broad resonance \[31\].
Note that the initial amplitude of the inflaton and the coupling constant $g$ play important roles to determine whether the resonance is narrow or broad. Since the inflaton mass is constrained to be small as $m \sim 10^{-6}m_{\text{pl}}$ by COBE normalizations, the large resonance parameter, $q \gg 1$, can be easily achieved for the coupling $g \gtrsim 10^{-4}$ with the initial amplitude, $\Phi(t_i) \sim 0.2m_{\text{pl}}$. The allowed region on the Mathieu chart is determined by Eq. (91) as

$$A_k \geq 2q.$$  \hfill (93)

Hence the broadest resonance is given by the limit line $A_k = 2q$.

As is found in Eq. (91), particles with low momenta are mainly produced. Considering the nonadiabatic condition, $d\omega_k/dt \gg \omega_k^2$, for the particle production, the maximum comoving momentum is roughly estimated as [33]

$$k \lesssim \sqrt{\frac{gm\Phi}{2}}.$$  \hfill (94)

This indicates that large values of $g$ and $\Phi$ lead to the production of particles with high momenta.

When $q$ is sufficiently large initially ($q \gg 1$), the resonance of each mode occurs stochastically [33]. In this case, the frequency $\omega_k$ decreases by cosmic expansion and $\omega_k$ drastically changes within each oscillation of the inflaton field, so the phases of $\chi$ field at successive moment of $\phi = 0$ are not correlated each other. In the first stage of preheating, the $\chi$ fields cross large number of instability bands. The periods when they are in instability bands are so short that the resonance cannot occur efficiently relative to the Minkowski spacetime case. Nevertheless, the $\chi$ fluctuations can still grow quasi-exponentially. As $q$ becomes smaller, cosmic expansion slows down, and the fields stay in each resonance band for a longer time. When the variables decrease below the lower boundary of the first resonance band by the expansion of the universe, particle productions come to an end.

We have to note that there is another mechanism which terminates the resonance. Taking into account the backreaction effect of created $\chi$ particles, the equation of inflaton is modified as

$$\ddot{\phi} + 3H\dot{\phi} + (m^2 + g^2\langle \chi^2 \rangle)\phi = 0,$$  \hfill (95)

where the expectation value of $\chi^2$ is defined as

$$\langle \chi^2 \rangle \equiv \frac{1}{2\pi^2} \int k^2|\chi_k|^2 \, dk.$$  \hfill (96)

When the initial value of $q$ is large as $q \gtrsim 3000$, which corresponds to $g \gtrsim 3.0 \times 10^{-4}$, the variance $\langle \chi^2 \rangle$ grows of order $m^2/g^2$ and the backreaction onto the inflaton field cannot be ignored. This makes the oscillation of the inflaton incoherent, which finally stops the resonance. The evolution of the variance $\langle \chi^2 \rangle$ for $g = 3.0 \times 10^{-4}$ is shown in fig. 5.

The growth of the field perturbations can stimulate the enhancement of metric perturbations on sub-Hubble and even on super-Hubble scales [35]. Whether this enhancement occurs or not is sensitive to the evolution of the large-scale $\chi$ fluctuation during inflation. In the model [35], for example, strong amplification of the $\chi$ fluctuation requires a resonance parameter $q \gg 1$ at the beginning of preheating, in which case the large-scale $\chi_k$ modes are suppressed during inflation [36]. Therefore it is difficult to amplify super-Hubble metric perturbations in this model. However there exist several models of inflation where metric perturbations can be amplified during preheating [37]. This also provides an interesting possibility of overproducing primordial black holes in certain cases [38].
Figure 4: The schematic diagram of the Mathieu chart and the typical paths for three types of resonance. The lined regions denote the instability bands (zeroth, first, second, · · ·). Note that the field perturbation evolves in the region $A_k \geq 2q$. 
Figure 5: The evolution of the variance $\langle \chi^2 \rangle = \langle \chi^2 \rangle / m^2$ for $g = 3.0 \times 10^{-4}$.

9 Summary

In this lecture note I reviewed the concept of inflation, generation of density perturbations, and reheating after inflation. I explained how inflationary cosmology solves a number of cosmological problems such as flatness, horizon, and monopole problems. In addition inflation makes it possible to generate nearly scale-invariant density perturbations, which is consistent with observations. Inflation is really an efficient mechanism to solve the cosmological problems associated with standard big-bang cosmology. In addition, elementary particles can be produced during the (p)reheating stage after inflation through the decay of the inflaton.

So far there exist several cosmological scenarios that might be the alternative to inflation such as the pre-big-bang [39] and ekpyrotic/cyclic model [40]. Nevertheless, unlike the standard inflation, it is generally difficult to solve all of the major cosmological problems at the same time in these models without a fine tuning of the model parameters. It is fair to say that standard inflation with a slow-roll flat potential is the most promising scenario in the very early universe among the models proposed so far.

Nevertheless we still have unsolved problems even in the inflationary cosmology: What is the origin of the inflaton field? What is the state of the universe before inflation? The initial singularity can be avoided?

Fortunately we begin to get the high-precision observational data, which is expected to reveal the nature of inflation in details in near future. From the theoretical side, there are a lot of extensive works to try to construct viable models of inflation based on superstring and supergravity theories. It is really encouraging that we are living in an exciting era where the early universe models can be strongly constrained from the upcoming observational data.
ACKNOWLEDGEMENTS

I thank the organizers of the school, especially Burin Gumjudpai and Rachan Rangdee, for organizing nice school and conference.

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