Criterion for the functional dissipativity of second order differential operators with complex coefficients

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Abstract In the present paper we consider the Dirichlet problem for the second order differential operator \( E = \nabla (\mathcal{A} \nabla) \), where \( \mathcal{A} \) is a matrix with complex valued \( L^\infty \) entries. We introduce the concept of dissipativity of \( E \) with respect to a given function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \). Under the assumption that the \( \text{Im} \mathcal{A} \) is symmetric, we prove that the condition

\[
|s \varphi'(s)| \left| \left( \text{Im} \mathcal{A}(x) \xi, \xi \right) \right| \leq 2 \sqrt{\varphi(s)} \left| s \varphi(s) \right| \langle \text{Re} \mathcal{A}(x) \xi, \xi \rangle
\]

(for almost every \( x \in \Omega \subset \mathbb{R}^N \) and for any \( s > 0, \xi \in \mathbb{R}^N \)) is necessary and sufficient for the functional dissipativity of \( E \).

Key Words: functional dissipativity; second order differential operator with complex coefficients.

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1 Introduction

1.1 Historical background

A linear operator \( E \) defined on \( D(E) \subset L^p(\Omega) \) and with range in \( L^p(\Omega) \) is said to be \( L^p \)-dissipative if

\[
\text{Re} \int_{\Omega} \langle Eu, u \rangle |u|^{p-2} dx \leq 0
\]
for any $u \in D(E)$. Here $\Omega$ is a domain in $\mathbb{R}^N$ and the functions $u$ are complex valued.

Let $E$ be the scalar second order partial differential operator

$$Eu = \nabla (\mathcal{A} \nabla u)$$

(1)

where $\mathcal{A}$ is a square matrix whose entries are complex valued $L^\infty$-functions.

The question of determining necessary and sufficient conditions for the $L^p$-dissipativity ($1 < p < \infty$) of the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^N$ for the operator (1) was considered in our paper [6]. It is worthwhile to remark that we do not require ellipticity and we may deal with degenerating matrices.

In particular we have proved that, if $\text{Im} \mathcal{A}$ is symmetric, the algebraic condition

$$|p - 2| |\langle \text{Im} \mathcal{A}(x) \xi, \xi \rangle| \leq 2 \sqrt{p - 1} \langle \text{Re} \mathcal{A}(x) \xi, \xi \rangle$$

(2)

for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^N$ is necessary and sufficient for the $L^p$-dissipativity of the Dirichlet problem for the operator (1).

We remark that, if $\text{Im} \mathcal{A}$ is symmetric, (2) is equivalent to the condition

$$\frac{4}{pp'} \langle \text{Re} \mathcal{A}(x) \xi, \xi \rangle + \langle \text{Re} \mathcal{A}(x) \eta, \eta \rangle - 2(1 - 2/p) \langle \text{Im} \mathcal{A}(x) \xi, \eta \rangle \geq 0$$

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^N$. More generally, if the matrix $\text{Im} \mathcal{A}$ is not symmetric, the condition

$$\frac{4}{pp'} \langle \text{Re} \mathcal{A}(x) \xi, \xi \rangle + \langle \text{Re} \mathcal{A}(x) \eta, \eta \rangle + 2 \langle (p^{-1} \text{Im} \mathcal{A}(x) + p'^{-1} \text{Im} \mathcal{A}^*(x)) \xi, \eta \rangle \geq 0$$

(3)

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^N (p' = p/(p - 1))$ is only sufficient for the $L^p$-dissipativity.

Condition (2) can be used to obtain the sharp angle of dissipativity of the operator (1). To be more precise, we proved that $zE$ ($z \in \mathbb{C}$) is $L^p$-dissipative if and only if $\vartheta_- \leq \arg z \leq \vartheta_+$, where $\vartheta_-$ and $\vartheta_+$ are explicitly given (see [7]).

If $\text{Im} \mathcal{A}$ is not symmetric or the operator $E$ contains lower order terms

$$Eu = \nabla (\mathcal{A} \nabla u) + b \nabla u + \nabla (cu) + au.$$  

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If operator (4) has smooth coefficients and it is strongly elliptic, then condition (2) is necessary and sufficient for the $L^p$-quasi-dissipativity of $E$, i.e. for the $L^p$-dissipativity of $E - \omega I$, for a suitable $\omega > 0$.

We extended these results to the class of systems of partial differential operators of the form

$$Au = \partial_h(\mathcal{A}^h(x)\partial_h u)$$

where $\mathcal{A}^h$ are $m \times m$ matrices whose elements are complex valued $L^1_{\text{loc}}$ functions (see [7]). We found that the operator $A$ is $L^p$-dissipative if and only if

$$\Re \langle \mathcal{A}^h(x)\lambda, \lambda \rangle - (1 - 2/p)^2 \Re \langle \mathcal{A}^h(x)\omega, \omega \rangle (\Re \langle \lambda, \omega \rangle)^2$$

$$- (1 - 2/p) \Re \langle \mathcal{A}^h(x)\omega, \lambda \rangle - \Re \langle \mathcal{A}^h(x)\lambda, \omega \rangle \Re \langle \lambda, \omega \rangle \geq 0$$

for almost every $x \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \ldots, N$. We have determined also the angle of dissipativity for such operators.

In the particular case of positive real symmetric matrices $\mathcal{A}^h$, we proved that $A$ is $L^p$-dissipative if and only if

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_{1}^h(x) + \mu_{m}^h(x))^2 \leq \mu_{1}^h(x)\mu_{m}^h(x)$$

almost everywhere, $h = 1, \ldots, N$, where $\mu_{1}^h(x)$ and $\mu_{m}^h(x)$ are the smallest and the largest eigenvalues of the matrix $\mathcal{A}^h(x)$ respectively. These results, obtained in [7], were new even for systems of ordinary differential equations.

Peculiar results have been obtained for the system of linear elasticity (see [7, 8])

$$Eu = \Delta u + (1 - 2\nu)^{-1} \nabla \text{div} u$$

($\nu$ being the Poisson ratio, $\nu > 1$ or $\nu < 1/2$), which is not of the form (5).

In particular, for the planar elasticity, we proved (see [7]) that operator (7) is $L^p$-dissipative if and only if

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leq \frac{2(\nu - 1)(2\nu - 1)}{(3 - 4\nu)^2}.$$ 

In [8] we showed that condition (8) is necessary for the $L^p$-dissipativity of operator (7) in any dimension, even when the Poisson ratio is not constant. At the present it is not known if condition (8) is also sufficient for the $L^p$-dissipativity of elasticity operator for $N > 2$, in particular for $N = 3$. 

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Nevertheless, in the same paper, we gave a more strict explicit condition which is sufficient for the $L^p$-dissipativity of (7). Indeed we proved that if

$$
(1 - 2/p)^2 \leq \begin{cases} 
\frac{1 - 2\nu}{2(1 - \nu)} & \text{if } \nu < 1/2 \\
\frac{2(1 - \nu)}{1 - 2\nu} & \text{if } \nu > 1,
\end{cases}
$$

then the operator (7) is $L^p$-dissipative.

In [8] we gave necessary and sufficient conditions for a weighted $L^p$-negativity of the Dirichlet-Lamé operator, i.e. for the validity of the inequality

$$
\int_\Omega (\Delta u + (1 - 2\nu)^{-1} \nabla \div u) |u|^{p-2} u \frac{dx}{|x|^\alpha} \leq 0 \quad (9)
$$

under the condition that the vector $u$ is rotationally invariant, i.e. $u$ depends only on $\rho = |x|$ and $u_\rho$ is the only nonzero spherical component of $u$. Namely we showed that (9) holds for any such $u$ belonging to $(C^\infty(\mathbb{R}^N \setminus \{0\}))^N$ if and only if

$$-(p - 1)(N + p' - 2) \leq \alpha \leq N + p - 2.
$$

We have considered also the $L^p$-positivity of the fractional powers of the Laplacian $(-\Delta)^\alpha$ ($0 < \alpha < 1$) for any $p \in (1, \infty)$ (see [9, Section 7.6, pp.230–231]. Specifically we have proved that

$$
\int_{\mathbb{R}^N} \langle (-\Delta)^\alpha u, u \rangle |u|^{p-2} dx \geq \frac{2c_\alpha}{p p'} \|u|^{p/2}\|_{L^{\alpha,2}(\mathbb{R}^N)}^2, \quad (10)
$$

for any real valued $u \in C^\infty_0(\mathbb{R}^N)$, where

$$c_\alpha = -\pi^{-N/2} 4^\alpha \Gamma(\alpha + N/2)/\Gamma(-\alpha) > 0.
$$

and $\|v\|_{L^{\alpha,2}}$ is the semi-norm

$$
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x + t) - v(x)|^2 \frac{dx dt}{|t|^{N+2\alpha}} \right)^{1/2}.
$$

All these results are collected in the monograph [9] where they are considered in the more general frame of semi-bounded operators.
The $L^p$-dissipativity of the matrix operator

$$Eu = \mathcal{B}^h(x) \partial_h u + \mathcal{D}(x) u,$$

where $\mathcal{B}^h(x)$ and $\mathcal{D}(x)$ are matrices with complex valued locally integrable entries defined in the domain $\Omega$ of $\mathbb{R}^N$ and $u = (u_1, \ldots, u_m)$ ($1 \leq i, j \leq m$, $1 \leq h \leq N$), is the subject of paper [10].

We proved that, if $p \neq 2$, $E$ is $L^p$-dissipative if, and only if,

$$\mathcal{B}^h(x) = b_h(x) I \text{ a.e.}, \quad (11)$$

$b_h(x)$ being real valued locally integrable functions, and the inequality

$$\Re \langle (p^{-1} \partial_h \mathcal{B}^h(x) - \mathcal{D}(x)) \zeta, \zeta \rangle \geq 0$$

holds for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$. If $p = 2$ condition (11) is replaced by the more general requirement that the matrices $\mathcal{B}^h(x)$ are self-adjoint a.e.. On combining this with the results we have previously obtained, we deduced sufficient conditions for the $L^p$-dissipativity of certain systems of partial differential operators of the second order.

Paper [11] concerns the "complex oblique derivative" operator, i.e. the boundary operator

$$\lambda \cdot \nabla u = \frac{\partial u}{\partial x_N} + \sum_{j=1}^{N-1} a_j \frac{\partial u}{\partial x_j}, \quad (12)$$

the coefficients $a_j$ being complex valued $L^\infty$ functions defined on $\mathbb{R}^{N-1}$. We gave new necessary and, separately, sufficient conditions for the $L^p$-dissipativity of operator (12). In the case of real coefficients we provided a necessary and sufficient condition. Specifically we proved that, if $a_j$ are real valued, the operator $\lambda \cdot \nabla$ is $L^p$-dissipative if and only if there exists a real vector $\Gamma \in L^2_{\text{loc}}(\mathbb{R}^N)$ such that

$$-\partial_j a_j \delta(x_n) \leq \frac{2}{p'} (\text{div} \Gamma - |\Gamma|^2)$$

in the sense of distributions.

In the same paper we have considered also a class of integral operators which can be written as

$$\int_{\mathbb{R}^N} [u(x) - u(y)] K(dx, dy) \quad (13)$$
where the integral has to be understood as a principal value in the sense of Cauchy and the kernel $K(dx, dy)$ is a Borel positive measure defined on $\mathbb{R}^N \times \mathbb{R}^N$ satisfying certain conditions. The class of operators we considered includes the fractional powers of Laplacian $(-\Delta)\alpha$, with $0 < \alpha < 1$. We establish the $L^p$-positivity of operator (13), extending in this way (10).

We mention that Hömberg, Krumbiegel and Rehberg [20] used some of the techniques introduced in [6] to show the $L^p$-dissipativity of a certain operator connected to the problem of the existence of an optimal control for the heat equation with dynamic boundary condition.

Beyn and Otten [1, 2] considered the semilinear system

$$A\Delta v(x) + \langle Sx, \nabla v(x) \rangle + f(v(x)) = 0, \quad x \in \mathbb{R}^N,$$

where $A$ is a $m \times m$ matrix, $S$ is a $N \times N$ skew-symmetric matrix and $f$ is a sufficiently smooth vector function. Among the assumptions they made, they require the existence of a constant $\gamma_A > 0$ such that

$$|z|^2 \Re\langle w, Aw \rangle + (p - 2) \Re\langle w, z \rangle \Re\langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2$$

for any $z, w \in \mathbb{C}^m$. This condition originates from our (6).

The results of [6] allowed Nittka [24] to consider the case of partial differential operators with complex coefficients.

Ostermann and Schratz [25] obtained the stability of a numerical procedure for solving a certain evolution problem. The necessary and sufficient condition (2) show that their result does not require the contractivity of the corresponding semigroup.

Chill, Meinschmidt and Rehberg [12] used some ideas from [6] in the study of the numerical range of second order elliptic operators with mixed boundary conditions in $L^p$.

Coming back to scalar operators (1), let us consider the class of operators such that the form (3) is not merely non-negative, but strictly positive, i.e. there exists $\kappa > 0$ such that

$$\frac{4}{pp'}\langle \Re\mathcal{A}(x)\xi, \xi \rangle + \langle \Re\mathcal{A}(x)\eta, \eta \rangle + 2\langle (p^{-1}\Im\mathcal{A}(x) + p'^{-1}\Im\mathcal{A}^*(x))\xi, \eta \rangle \geq \kappa(|\xi|^2 + |\eta|^2)$$

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^N$. The class of operators (4) whose principal part satisfies (14) and which could be called $p$-strongly elliptic, was recently considered by several authors.
Carbonaro and Dragićević [4, 5] showed the validity of some so called bilinear embeddings related to boundary value problems with different boundary conditions for second order complex coefficient operators satisfying condition (14). In a series of papers [14, 15, 16, 17] Dindoš and Pipher proved several results concerning the $L^p$ solvability of the Dirichlet problem for the same class of operators.

Finally we mention that recently Maz’ya and Verbitsky [23] gave necessary and sufficient conditions for the accretivity of a second order partial differential operator $E$ containing lower order terms, in the case of Dirichlet data. We observe that the accretivity of $E$ is equivalent to the $L^2$-dissipativity of $-E$.

1.2 Functional dissipativity

A motivation for the study of $L^p$-dissipativity comes from the decrease of the norm of solutions of the Cauchy-Dirichlet problem

\[
\begin{cases}
  u' = Eu \\
  u(0) = u_0.
\end{cases}
\]  

(15)

Here $u = u(x, t)$, $x \in \Omega \subset \mathbb{R}^N$, $t > 0$ and $u(x, t) = 0$ on $\partial \Omega$ in some sense for $t > 0$. By formal arguments, we have

\[
\frac{d}{dt} \|u(\cdot, t)\|_p^p = \frac{d}{dt} \int_{\Omega} |u(x, t)|^p dx = p \Re \int_{\Omega} \langle \partial_t u, u \rangle |u|^{p-2} dx,
\]

(16)

and then the inequality

\[
\Re \int_{\Omega} \langle Eu, u \rangle |u|^{p-2} dx \leq 0.
\]

implies the decrease of the $L^p$ norm of the solution of the Cauchy-Dirichlet problem (15).

More generally, let $\Phi$ be a Young function (a convex positive function such that $\Phi(0) = 0$ and $\Phi(+\infty) = +\infty$) and consider the Orlicz space of functions $u$ for which there exists $\alpha > 0$ such that

\[
\int_{\Omega} \Phi(\alpha |u|) dx < +\infty.
\]
For the general theory of Orlicz spaces we refer to Krasnosel’skiı, Rutickiı [21] and Rao, Ren [26]. As in (16), if \(u(x, y)\) is a solution of the Cauchy-Dirichlet problem (15), we have the decrease of the integrals

\[
\int_{\Omega} \Phi(|u(x,t)|) \, dx
\]

if

\[
\Re \int_{\Omega} \langle Eu, u \rangle |u|^{-1} \Phi'(|u|) \, dx \leq 0.
\]

This implies the decrease of the Luxemburg norm in the related Orlicz space

\[
\|u(\cdot, t)\| = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \Phi(|u(x,t)|/\lambda) \, dx \leq 1 \right\}.
\]

The aim of the present paper is to find conditions for the positive function \(\varphi\) defined on \((0, +\infty)\) to satisfy the inequality

\[
\Re \int_{\Omega} \langle Eu, u \rangle \varphi(|u|) \, dx \leq 0 \quad (17)
\]

for any complex valued \(u\) in a certain class, \(\Omega\) being a domain in \(\mathbb{R}^N\). Here \(E\) is the scalar operator (1).

In integrals like (17) the combination \(\varphi(|u|)u\) in the integrand is taken to be zero where \(u\) vanishes, even if the function \(\varphi(s)\) is not defined at \(s = 0\). If \(\varphi(t) = t^{p-2} \ (p > 1)\) we recover the concept of \(L^p\)-dissipativity.

We remark that the relation between the function \(\varphi\) in (17) and \(\Phi\) is

\[
\varphi(t) = \frac{\Phi'(t)}{t} \quad \iff \quad \Phi(t) = \int_0^t s \varphi(s) \, ds \quad (18)
\]

and the convexity of \(\Phi\) is equivalent to the increase of \(s \varphi(s)\).

If (17) holds for a general \(\varphi\), we say that the operator \(E\) is functional dissipative or \(L^\Phi\)-dissipative, in analogy with the terminology used when \(\varphi(t) = t^{p-2}\).

More precisely, in the present paper we consider the partial differential operator (1) and consider the corresponding sesquilinear form

\[
\mathcal{L}(u, v) = \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla v \rangle \, dx.
\]
We look for the conditions under which the operator $E$ is $L^\Phi$-dissipative, i.e.
\[ \Re \int_\Omega \langle \mathcal{A} \nabla u, \nabla (\varphi(|u|) u) \rangle \, dx \geq 0 \]
for any $u \in \dot{H}^1(\Omega)$ such that $\varphi(|u|) u \in \dot{H}^1(\Omega)$.

In the present paper we have considered Dirichlet problem but in principle this notion could be extended to other boundary value conditions.

1.3 The main result

Let us formulate the main result of the present paper. Under the assumption that the matrix $\Im A$ is symmetric, we prove that the operator (1) is $L^\Phi$-dissipative if and only if

\[ |s \varphi'(s)| |\langle \Im \mathcal{A}(x) \xi, \xi \rangle| \leq 2 \sqrt{\varphi(s)} |s \varphi(s)|' \langle \Re \mathcal{A}(x) \xi, \xi \rangle \] (19)

for almost every $x \in \Omega$ and for any $s > 0, \xi \in \mathbb{R}^N$. The function $\varphi$ is a positive function defined on $\mathbb{R}^+$ such that $s \varphi(s)$ is strictly increasing. The precise conditions we require on the function $\varphi$ are specified later (see section 3.1). If $\Im \mathcal{A}$ is not symmetric condition (19) is only necessary for $E$ to be $L^\Phi$-dissipative.

Condition (19) is equivalent to

\[ [1 - \Lambda^2(t)] \langle \Re \mathcal{A}(x) \xi, \xi \rangle + \langle \Re \mathcal{A}(x) \eta, \eta \rangle + 2 \Lambda(t) \langle \Im \mathcal{A}(x) \xi, \eta \rangle \geq 0 \] (20)

for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^N$, where $\Lambda$ is the function defined by the relation

\[ \Lambda \left( s \sqrt{\varphi(s)} \right) = \frac{s \varphi'(s)}{s \varphi'(s) + 2 \varphi(s)} . \]

Note that if $\varphi(s) = s^{p-2}$, this function is constant and $\Lambda(t) = -(1-2/p)$, $1 - \Lambda^2(t) = 4/(pp')$. As for (3), if $\Im \mathcal{A}$ is not symmetric, the condition

\[ [1 - \Lambda^2(t)] \langle \Re \mathcal{A}(x) \xi, \xi \rangle + \langle \Re \mathcal{A}(x) \eta, \eta \rangle + [1 + \Lambda(t)] \langle \Im \mathcal{A}(x) \xi, \eta \rangle + [1 - \Lambda(t)] \langle \Im \mathcal{A}^*(x) \xi, \eta \rangle \geq 0 \] (20)

for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^N$, is only sufficient for the $L^\Phi$-dissipativity.
If the principal part of operator (4) is such that the left-hand side of (20) is not merely non negative but strictly positive, i.e.

\[ [1 - \Lambda^2(t)] \langle \Re \AA(x) \xi, \xi \rangle + \langle \Re \AA(x) \eta, \eta \rangle + [1 + \Lambda(t)] \langle \Im \AA(x) \xi, \eta \rangle + [1 - \Lambda(t)] \langle \Im \AA^*(x) \xi, \eta \rangle \geq \kappa(|\xi|^2 + |\eta|^2) \]

for a certain \( \kappa > 0 \) and for almost every \( x \in \Omega \) and for any \( t > 0, \xi, \eta \in \mathbb{R}^N \), we say that the operator \( E \) is \( \Phi \)-strongly elliptic.

### 1.4 Structure of the paper

The present paper is organized as follows.

After the short preliminary Section 2, in Section 3 we specify the class of functions \( \varphi \) we are going to consider and introduce some related functions.

Section 4 is devoted to prove a technical lemma concerning real bilinear forms, which will be used later, in the proof of the main result.

In Section 5 we give necessary and sufficient conditions for the \( L^\Phi \)-dissipativity. Specifically we prove the equivalence between the \( L^\Phi \)-dissipativity of the operator \( E \) and the positiveness of a certain form in \( \dot{H}^1(\Omega) \). We remark that a similar result holds also for second order differential operators with lower order terms, in analogy with [6, Lemma 1, p.1070]. This can be proved with the same technique, but for the sake of simplicity here we have preferred to avoid such a more general formulation. The section ends with a lemma concerning \( \Phi \)-strongly elliptic operators.

The main result concerning condition (19) is proved in Section 6. We give also some examples showing that in some cases only real nonnegative operators are \( L^\Phi \)-dissipative, while in other cases the \( L^\Phi \)-dissipativity is equivalent to the algebraic condition

\[ \lambda_0 |\langle \Im \AA(x) \xi, \xi \rangle| \leq \langle \Re \AA(x) \xi, \xi \rangle \]

for almost any \( x \in \Omega \) and for any \( \xi \in \mathbb{R}^N \), where the constant \( \lambda_0 \) is explicitly determined.

### 2 Preliminaries and notations

Let \( \Omega \) be an open set in \( \mathbb{R}^N \). As usual, by \( C_0^\infty(\Omega) \) we denote the space of complex valued \( C^\infty \) functions having compact support in \( \Omega \) and by \( \dot{H}^1(\Omega) \)
the closure of $C_0^\infty(\Omega)$ in the norm
\[ \int_{\Omega} (|u|^2 + |\nabla u|^2) dx, \]
$\nabla u$ being the gradient of the function $u$.

The inner product either in $\mathbb{C}^N$ or in $\mathbb{C}$ is denoted by $\langle \cdot, \cdot \rangle$ and the bar denotes complex conjugation.

In what follows, $\mathcal{A}$ is a $N \times N$ matrix function with complex valued entries $a_{hk} \in L^\infty(\Omega)$, $\mathcal{A}^t$ is its transposed matrix and $\mathcal{A}^*$ is its adjoint matrix, i.e. $\mathcal{A}^* = \overline{\mathcal{A}^t}$.

Let $\mathcal{L}$ be the sesquilinear form
\[ \mathcal{L}(u, v) = \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla v \rangle dx. \]

We say that the operator $E$ is $L^\Phi$-dissipative if
\[ \Re \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla (\varphi(|u|) u) \rangle dx \geq 0 \]
for any $u \in \dot{H}^1(\Omega)$ such that $\varphi(|u|) u \in \dot{H}^1(\Omega)$.

Here $\varphi$ is a positive function defined on $\mathbb{R}^+ = (0, +\infty)$. In the next section we specify the conditions we require on $\varphi$.

In the sequel we shall sometimes use the following notations. Given two functions $F$ and $G$ defined on a set $Y$, writing $|F(y)| \lesssim |G(y)|$ we mean that there exists a positive constant $C > 0$ such that $|F(y)| \leq C |G(y)|$ for any $y \in Y$. If $|F(y)| \lesssim |G(y)|$ and $|G(y)| \lesssim |F(y)|$ we shall write $F(y) \simeq G(y)$.

### 3 The function $\varphi$ and related functions

In this Section we introduce the class of function $\varphi$ with respect to which we consider the $L^\Phi$-dissipativity. We also introduce other functions related to $\varphi$ and prove some of their properties.

#### 3.1 The functions $\varphi$ and $\psi$

The positive function $\varphi$ is required to satisfy the following conditions

1. $\varphi \in C^1((0, +\infty));$
2. \((s \varphi(s))' > 0\) for any \(s > 0\);

3. the range of the strictly increasing function \(s \varphi(s)\) is \((0, +\infty)\);

4. there exist two positive constants \(C_1, C_2\) and a real number \(r > -1\) such that

\[
C_1 s^r \leq (s \varphi(s))' \leq C_2 s^r, \quad s \in (0, s_0)
\]

(23)

for a certain \(s_0 > 0\). If \(r = 0\) we require more restrictive conditions: there exists the finite limit \(\lim_{s \to 0^+} \varphi(s) = \varphi_+(0) > 0\) and \(\lim_{s \to 0^+} s \varphi'(s) = 0\).

5. There exists \(s_1 > s_0\) such that

\[
\varphi'(s) \geq 0 \text{ or } \varphi'(s) \leq 0 \quad \forall s \geq s_1.
\]

(24)

The condition 4 prescribes the behaviour of the function \(\varphi\) in a neighborhood of the origin, while 5 concerns the behaviour for large \(s\).

The function \(\varphi(s) = s^{p-2} (p > 1)\) provides an example of such a function. Other examples can be found at the end of the paper.

From condition 4 it follows that, for any \(r > -1\),

\[
\varphi(s) \simeq s^r, \quad s \in (0, s_0).
\]

(25)

Let us denote by \(t \psi(t)\) the inverse function of \(s \varphi(s)\). The functions

\[
\Phi(s) = \int_0^s \sigma \varphi(\sigma) d\sigma, \quad \Psi(s) = \int_0^s \sigma \psi(\sigma) d\sigma
\]

are conjugate Young functions.

**Lemma 1** The function \(\varphi\) satisfies conditions 1-5 if and only if the function \(\psi\) satisfies the same conditions with \(-r/(r+1)\) instead of \(r\).

**Proof.** Inequalities (23) and (25) imply

\[
\psi(t) \simeq t^{-r/(1+r)}, \quad (t \psi(t))' \simeq t^{-r/(1+r)} \quad t \in (0, t_0)
\]

for a certain \(t_0 > 0\).
Since $-r/(1 + r) > -1$, the function $\psi$ satisfy the conditions 1-4 with $-r/(1 + r)$ instead of $r$. In the particular case $r = 0$ this follows from the equality $t \psi(t) \varphi[t \psi(t)] = t$ ($t > 0$), which implies

$$\lim_{t \to 0^+} t \psi(t) = \frac{1}{\varphi_+(0)} > 0,$$

$$\lim_{t \to 0^+} t \psi'(t) = \lim_{t \to 0^+} ((t \psi(t))' - \psi(t)) = \lim_{s \to 0^+} \frac{1}{s \varphi'(s) + \varphi(s)} - \frac{1}{\varphi_+(0)} = 0.$$ 

Since $s \varphi(s) \varphi[s \varphi(s)] = s$, we find $\psi[s \varphi(s)] = 1/\varphi(s)$ and then

$$\psi'[s \varphi(s)](s \varphi(s))' = -\frac{\varphi'(s)}{\varphi^2(s)}.$$ 

(26)

Keeping in mind condition 2, we have that the function $\psi'$ satisfies condition (24) for $t$ greater than $t_1 = s_1 \varphi(s_1)$, but with an opposite sign.

The viceversa is now obvious, since $-(-r/(1 + r)/(1 - r/(1 + r)) = r$. □

### 3.2 Some auxiliary functions

The function $s \sqrt{\varphi(s)}$ is strictly increasing. Let $\zeta(t)$ be its inverse, i.e. $\zeta(t) = \left(s \sqrt{\varphi(s)}\right)^{-1}$. The range of $s \sqrt{\varphi(s)}$ is $(0, +\infty)$ and $\zeta(t)$ belongs to $C^1((0, +\infty))$.

Define

$$\Theta(t) = \zeta(t)/t; \quad \Lambda(t) = t \Theta'(t)/\Theta(t).$$

(27)

From (25) it follows that there exists a constant $K > 0$ such that

$$\zeta(t) \leq K t^{2/(2+r)}, \quad \Theta(t) \leq K t^{-r/(2+r)}, \quad t \in (0, t_0)$$

(28)

for a certain $t_0 > 0$.

We have also

$$\Theta \left(s \sqrt{\varphi(s)}\right) = 1/\sqrt{\varphi(s)}; \quad \Theta' \left(s \sqrt{\varphi(s)}\right) = \varphi'(s) \left[\frac{1}{s \varphi'(s) + 2 \varphi(s)}\right].$$

(29)

Note that condition 2 implies

$$s \varphi'(s) + 2 \varphi(s) > 0, \quad s \in (0, +\infty).$$
We can write
\[
\Lambda \left( s\sqrt{\varphi(s)} \right) = -\frac{s \varphi'(s)}{s \varphi'(s) + 2 \varphi(s)},
\]
(30)
\[
1 - \Lambda \left( s\sqrt{\varphi(s)} \right) = 2 \frac{s \varphi'(s) + \varphi(s)}{s \varphi'(s) + 2 \varphi(s)} > 0,
\]
\[
1 + \Lambda \left( s\sqrt{\varphi(s)} \right) = \frac{2 \varphi(s)}{s \varphi'(s) + 2 \varphi(s)} > 0,
\]
from which it follows
\[
1 - \Lambda^2 \left( s\sqrt{\varphi(s)} \right) = \frac{4 \varphi(s) (s \varphi'(s) + \varphi(s))}{(s \varphi'(s) + 2 \varphi(s))^2},
\]
(31)
and
\[-1 < \Lambda(t) < 1\]
(32)
for any \( t > 0 \). This, together with (28), implies
\[|\Theta'(t)| \leq \Theta(t)/t \leq K t^{-2(1+r)/(2+r)}\]
(33)
for \( t \in (0, t_0) \).

Finally we give two equalities we shall use later. The first equality in (29) can be rewritten as
\[
\Theta^2(t) \varphi'[\zeta(t)] = 1,
\]
(34)
for any \( t > 0 \), which leads to
\[2 \Theta(t) \Theta'(t) \varphi'[\zeta(t)] + \Theta^2(t) \varphi'[\zeta(t)] \zeta'(t) = 0\]
and then
\[
\Theta(t) \varphi'[\zeta(t)] \zeta'(t) + \Theta'(t) \varphi[\zeta(t)] = -\Theta'(t) \varphi[\zeta(t)] = -\Theta'(t)/\Theta^2(t).
\]
Since \( \zeta'(t) = t \Theta'(t) + \Theta(t) \) we have also
\[
\Theta(t) \varphi'[\zeta(t)] \left[ t \Theta'(t) + \Theta(t) + \Theta'(t) \varphi[\zeta(t)] = -\Theta'(t)/\Theta^2(t)\right.
\]
(35)
for any \( t > 0 \).
Lemma 2 Let $\tilde{\zeta}(t)$ the inverse function of $t \sqrt{\psi(t)}$ and define, as in (27),

$$
\tilde{\Theta}(t) = \tilde{\zeta}(t)/t; \quad \tilde{\Lambda}(t) = t \tilde{\Theta}'(t)/\tilde{\Theta}(t).
$$

We have

$$
\tilde{\Theta}(t) = \frac{1}{\Theta(t)}, \quad \tilde{\Lambda}(t) = -\Lambda(t)
$$

(36)

for any $t > 0$.

**Proof.** The function $t \psi(t)$ being the inverse of $s \varphi(s)$, we can write

$$
\varphi(s) \psi[s \varphi(s)] = 1, \forall s > 0.
$$

(37)

From this and (29) we deduce

$$
\Theta(s \sqrt{\varphi(s)}) = \frac{1}{\sqrt{\varphi(s)}} = \sqrt{\psi[s \varphi(s)]} = \sqrt{\psi(t)} = \frac{1}{\Theta(t \sqrt{\psi(t)})}
$$

where we have set $t = s \varphi(s)$. On the other hand, keeping in mind (37), we have

$$
t \sqrt{\psi(t)} = s \varphi(s) \sqrt{\psi(s \varphi(s))} = s \sqrt{\varphi(s)}.
$$

(38)

The first equality in (36) is proved and the second one follows at once. □

### 3.3 A Lemma concerning Sobolev spaces

We conclude this Section with the next Lemma which guarantees that the function $\sqrt{\varphi(|u|)} u$ belongs to the Sobolev space $H^1(\Omega)$ or $\dot{H}^1(\Omega)$.

Lemma 3 If $u \in H^1(\Omega)$ ($\dot{H}^1(\Omega)$) is such that $\varphi(|u|) u \in H^1(\Omega)$ ($\dot{H}^1(\Omega)$), then $\sqrt{\varphi(|u|)} u$ belongs to $H^1(\Omega)$ ($\dot{H}^1(\Omega)$).

**Proof.** Let us suppose $u, \varphi(|u|) u \in H^1(\Omega)$. The function $\sqrt{\varphi(|u|)} u$ belongs to $L^2(\Omega)$, because we can write $\varphi(|u|)|u|^2$ as the product of the two $L^2$ functions $\varphi(|u|)|u|$ and $|u|$.

Consider now its gradient. Suppose $r \geq 0$ and $\varphi'(s) \geq 0$, for $s \geq s_1$ (see (24)). We have

$$
\nabla(\sqrt{\varphi(|u|)} u) = \sqrt{\varphi(|u|)} \nabla u + \left(2\sqrt{\varphi(|u|)}\right)^{-1} \varphi'(|u|) \nabla(|u|) u
$$
on the set $\Omega_0 = \{ x \in \Omega \mid u(x) \neq 0 \}$.

Let us prove that this gradient belongs to $L^2(\Omega_0)$. We can write

$$
\int_{\Omega_0} |\nabla (\sqrt{\varphi(\|u\|)} u)|^2 \, dx =
\left( \int_{0<\|u\|<s_0} |\nabla (\sqrt{\varphi(\|u\|)} u)|^2 \, dx + \int_{s_0<\|u\|\leq s_1} |\nabla (\sqrt{\varphi(\|u\|)} u)|^2 \, dx \right).
$$

Observing that $\varphi(|u|) \simeq |u|^r$ and $|\varphi'(|u|)||u| = |(\varphi(|u|)|u)' - \varphi(|u|)| \lesssim |u|^r \lesssim 1$ for $|u| < s_0$ (see (23) and (25)), we find

$$
\left( \int_{0<\|u\|<s_0} |\nabla (\sqrt{\varphi(\|u\|)} u)|^2 \, dx \right)^{1/2} \leq
\left( \int_{0<\|u\|<s_0} \varphi(|u|)|\nabla u|^2 \, dx \right)^{1/2} + \left( \int_{0<\|u\|<s_0} \frac{\varphi'(|u|)^2}{\varphi(|u|)} |u|^2 |\nabla u|^2 \, dx \right)^{1/2} \lesssim
\left( \int_{0<\|u\|<s_0} |\nabla u|^2 \, dx \right)^{1/2} + \left( \int_{0<\|u\|<s_0} |\nabla u|^2 \, dx \right)^{1/2} \lesssim \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}.
$$

Concerning the set where $s_0 \leq |u| \leq s_1$ we have

$$
\left( \int_{s_0 \leq |u| \leq s_1} |\nabla (\sqrt{\varphi(\|u\|)} u)|^2 \, dx \right)^{1/2} \leq
\left( \int_{s_0 \leq |u| \leq s_1} \varphi(|u|)|\nabla u|^2 \, dx \right)^{1/2} + \left( \int_{s_0 \leq |u| \leq s_1} \frac{\varphi'(|u|)^2}{\varphi(|u|)} |u|^2 |\nabla u|^2 \, dx \right)^{1/2} \leq
\left( \max_{s \in [s_0, s_1]} \varphi(s) \int_{s_0 \leq |u| \leq s_1} |\nabla u|^2 \, dx \right)^{1/2} + \left( \max_{s \in [s_0, s_1]} \frac{\varphi'(s)^2}{\varphi(s)} \int_{s_0 \leq |u| \leq s_1} |\nabla u|^2 \, dx \right)^{1/2}
\lesssim \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}.
$$

Observe now that

$$
|\nabla (\varphi(|u|) u)|^2 = |\varphi'(|u|) \nabla (|u|) u + \varphi(|u|) \nabla u|^2 =
\varphi'(|u|)^2 |u|^2 |\nabla (|u|) u|^2 + 2 \varphi'(|u|) \varphi(|u|) \langle \nabla (|u|), \Re (\nabla u) \rangle + \varphi^2(|u|) |\nabla u|^2 =
|\varphi'(|u|)^2 |u|^2 + 2 \varphi'(|u|) \varphi(|u|) |u| |\nabla (|u|) u|^2 + \varphi^2(|u|) |\nabla u|^2
$$

(41)
on $\Omega$. Since $\phi'(s) \geq 0$ for $s \geq s_1$, each term in the last line of (41) is nonnegative in $\Omega_0$. This implies that each of these terms is integrable on the set $|u| \geq s_1$, the gradient of $\phi(|u|) u$ belonging to $L^2(\Omega)$.

By Cauchy inequality we get

$$\int_{|u| > s_1} \phi(|u|) |\nabla u|^2 \, dx \leq \left( \frac{1}{M_0} \int_{|u| > s_1} \frac{\phi'}{\phi} \phi(|u|) |\nabla u|^2 \, dx \right) \left( \frac{1}{M_0} \int_{|u| > s_1} \phi'(s)^2 |u|^2 \, dx \right) < +\infty,$$

and

$$\int_{|u| > s_1} \frac{\phi'}{\phi} \phi(|u|) |u|^2 |\nabla u|^2 \, dx \leq \frac{1}{M_0} \int_{|u| > s_1} \phi'(s)^2 |u|^2 |\nabla u|^2 \, dx < +\infty,$$

where $M_0 > 0$ is such that $\phi(s) \geq M_0$ for any $s \geq s_1$.

We have then shown that

$$\int_{|u| > s_1} |\nabla (\sqrt{\phi(|u|)} u)|^2 \, dx < +\infty \tag{42}$$

Collecting (39), (40) and (42) we get

$$\int_{\Omega_0} |\nabla (\sqrt{\phi(|u|)} u)|^2 \, dx < +\infty.$$

Suppose now that $r \geq 0$ and $\phi'(s) \leq 0$, for $s \geq s_1$. Inequalities (39) and (40) are still valid. In order to estimate $\nabla (\sqrt{\phi(|u|)} u)$ on the set where $|u| > s_1$ we proceed as follows.

Let us define $w = \phi(|u|) u$. We have $u = \psi(|w|) w$ and then $w$ and $\psi(|w|) w$ belong to $H^1(\Omega)$. On the other hand, since $|w| = \phi(|u|) |u|$ and in view of equality (38), we can write $\sqrt{\psi(|w|)} |w| = \sqrt{\phi(|u|)} |u|$. Recalling the definition of $w$, this implies

$$\sqrt{\psi(|w|)} w = \sqrt{\phi(|u|)} u \tag{43}.$$

Since $\psi'(t) \geq 0$ for $t \geq s_1 \phi(s_1)$ (see (26)), we can rewrite formula (41) replacing $\phi(|u|) u$ by $\psi(|w|) w$ and deduce - as for (42) - that

$$\int_{|u| > s_1} |\nabla (\sqrt{\phi(|u|)} u)|^2 \, dx = \int_{|u| > s_1 \phi(s_1)} |\nabla (\sqrt{\psi(|w|)} w)|^2 \, dx < +\infty.$$

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We have then proved that, if $r \geq 0$, the vector
\[
\left(2\sqrt{\varphi(|u|)}\right)^{-1} \varphi'(|u|) \nabla(|u|) u + \sqrt{\varphi(|u|)} \nabla u \chi_{\Omega_0} \quad (44)
\]
($\chi_{\Omega_0}$ is the characteristic function of $\Omega_0$) belongs to $L^2(\Omega)$. Let us show that (44) is the weak gradient of $\sqrt{\varphi(|u|)} u$. Let $\varepsilon > 0$ and define
\[
\varphi_{\varepsilon}(t) = \begin{cases} \varphi(\varepsilon) & \text{if } |t| \leq \varepsilon \\ \varphi(|t|) & \text{if } |t| > \varepsilon. \end{cases} \quad h_{\varepsilon} = \sqrt{\varphi_{\varepsilon}(|u|)} u.
\]

The function $h_{\varepsilon}$ belongs to $H^1(\Omega)$ and
\[
\nabla h_{\varepsilon} = \begin{cases} \sqrt{\varphi_{\varepsilon}} \nabla u & \text{if } |u| \leq \varepsilon \\ \nabla(\sqrt{\varphi(|u|)} u) & \text{if } |u| > \varepsilon \end{cases}
\]
almost everywhere in $\Omega$.

For any $f \in C_0^\infty(\Omega)$ and for any $j = 1, \ldots, N$ we have
\[
\int_{\Omega} h_{\varepsilon} \partial_j f \, dx = -\int_{\Omega} f \partial_j h_{\varepsilon} \, dx = -\sqrt{\varphi(\varepsilon)} \int_{|u| \leq \varepsilon} f \partial_j u \, dx - \int_{|u| > \varepsilon} f \partial_j [\sqrt{\varphi(|u|)} u] \, dx.
\]

Observing that
\[
|h_{\varepsilon}| = \sqrt{\varphi_{\varepsilon}(|u|)} |u| \leq \max \left\{ \sqrt{\varphi(\varepsilon)}, \sqrt{\varphi(|u|)} \right\} |u| \in L^2(\Omega),
\]
it follows
\[
\int_{\Omega} \sqrt{\varphi(|u|)} u \partial_j f \, dx = \lim_{\varepsilon \to 0^+} \int_{\Omega} h_{\varepsilon} \partial_j f \, dx = -\int_{\Omega_0} f \partial_j \sqrt{\varphi(|u|)} u \, dx.
\]

This means that (44) is the weak gradient of $\sqrt{\varphi(|u|)} u$ and then $\sqrt{\varphi(|u|)} u$ belongs to $H^1(\Omega)$.

If $u$ and $\varphi(|u|) u$ are in $\dot{H}^1(\Omega)$ we shall first prove that
\[
\int_{\Omega} \sqrt{\varphi(|u|)} u \partial_i f \, dx = -\int_{\Omega} f \partial_i (\sqrt{\varphi(|u|)} u) \chi_{\Omega_0} \, dx.
\]

\(18\)
for any $f \in C_0^\infty(\mathbb{R}^N)$.

We can write
\[
\int_\Omega \sqrt{\varphi(|u|)} \ u \partial_i f \ dx = \int_\Omega \varphi_\varepsilon(|u|) \ u \left[ \varphi_\varepsilon(|u|)^{-\frac{1}{2}} \partial_i f \right] \ dx
\]
\[
\int_\Omega \varphi_\varepsilon(|u|) \ u \left[ \partial_i \left( \varphi_\varepsilon(|u|)^{-\frac{1}{2}} f \right) - f \partial_i \left( \varphi_\varepsilon(|u|)^{-\frac{1}{2}} \right) \right] \ dx.
\]

Moreover, since $\varphi(|u|) u$ is in $H^1(\Omega)$,
\[
\int_\Omega \varphi_\varepsilon(|u|) u \partial_i \left( \varphi_\varepsilon(|u|)^{-\frac{1}{2}} f \right) \ dx = \int_\Omega \varphi_\varepsilon(|u|) u \partial_i (\varphi_\varepsilon(|u|)^{-\frac{1}{2}} f) \ dx + \int_\Omega \varphi(|u|) u \partial_i (\varphi_\varepsilon(|u|)^{-\frac{1}{2}} f) \ dx =
\]
\[
\frac{1}{\sqrt{\varphi(\varepsilon)}} \int_{|u|<\varepsilon} \varphi(\varepsilon) - \varphi(|u|) \ u \partial_i f \ dx - \int_\Omega \varphi_\varepsilon(|u|)^{-\frac{1}{2}} f \partial_i (\varphi(|u|) \ u) \ dx =
\]
\[
- \int_\Omega \varphi_\varepsilon(|u|)^{-\frac{1}{2}} f \partial_i (\varphi(|u|) \ u) \ dx + o(1) =
\]
\[
- \int_{|u|>\varepsilon} \varphi(|u|)^{-\frac{1}{2}} f \partial_i (\varphi(|u|) \ u) \ dx + o(1).
\]

This leads to
\[
\int_\Omega \sqrt{\varphi_\varepsilon(|u|)} \ u \partial_i f \ dx =
\]
\[
- \int_{|u|>\varepsilon} f \left[ \varphi(|u|)^{-\frac{1}{2}} \partial_i (\varphi(|u|) \ u) + \varphi(|u|) u \partial_i (\varphi(|u|)^{-\frac{1}{2}}) \right] \ dx + o(1) =
\]
\[
- \int_{|u|>\varepsilon} f \partial_i (\sqrt{\varphi(|u|)} \ u) \ dx + o(1).
\]

In view of (45), letting $\varepsilon \to 0^+$, we get (46). This means that the function $\sqrt{\varphi(|u|)} \ u$ extended by zero outside $\Omega$ belongs to $H^1(\mathbb{R}^N)$. Now we may appeal to a result proved by Deny [13] (see also Hedberg [19]) and conclude that $\sqrt{\varphi(|u|)} \ u \in H^1(\Omega)$. The proof is complete for $r \geq 0$.

If $-1 < r < 0$ we write the function $\sqrt{\varphi(|u|)} \ u$ as in (43). What we have already proved for $r \geq 0$ shows that $\sqrt{\psi(|w|)} \ w \in H^1(\Omega)$ ($H^1(\Omega)$).
4 A lemma on a bilinear form

This Section is devoted to prove a Lemma which we shall use in the main Theorem and concerns the positivity of the bilinear form $\langle \mathcal{B} \nabla v, \nabla v \rangle$ in $C^\infty_0(\Omega) \times C^\infty_0(\Omega)$, where $\mathcal{B} = \{b_{hj}\}$ is a real matrix whose elements depend on $x$ and $|v|$. We note that if $b_{hj}$ do not depend on $|v|$, the result is well known and can be obtained by standard arguments (see, e.g., [18, pp.107–108]).

**Lemma 4** Let $\mathcal{B} = \{b_{hj}\}$ a real matrix whose elements belong to $L^\infty(\Omega \times \mathbb{R}^+)$ and $b_{hj}(x,t)$ are continuous with respect to $t \in \mathbb{R}^+$. If

$$\int_{\Omega} \langle \mathcal{B}(x,|v|) \nabla v, \nabla v \rangle dx \geq 0 \quad (47)$$

for any real valued scalar function $v \in C^\infty_0(\Omega)$, then

$$\langle \mathcal{B}(x,t) \xi, \xi \rangle \geq 0 \quad (48)$$

for almost every $x \in \Omega$ and for any $t > 0$, $\xi \in \mathbb{R}^N$.

Let us assume that the matrix $\mathcal{B}$ does not depend on $x$ and suppose that (48) is false. This means that there exists $t_0 > 0$ and $\omega_0$, $|\omega_0| = 1$ such that $\langle \mathcal{B}(t_0) \omega_0, \omega_0 \rangle < 0$.

We can find two positive constants $M, \delta$ such that

$$\langle \mathcal{B}(t) \omega, \omega \rangle \leq -M, \quad \forall \ t, \omega : |t - t_0| < \delta, |\omega - \omega_0| < \delta, |\omega| = 1. \quad (49)$$

Without loss of generality we assume that the origin belongs to $\Omega$. Let now $v(x) = \beta(\varrho) \gamma(\omega)$ (where, as usual, $x = \varrho \omega$ ($\varrho > 0$, $|\omega| = 1$)), where $\beta \in C^\infty_0(\mathbb{R}^+)$ and $\gamma \in C^\infty(\Sigma)$, $\Sigma$ being the unit sphere in $\mathbb{R}^N$. Note that

$$\nabla v(x) = \dot{\beta}(\varrho) \gamma(\omega) \omega + \beta(\varrho) \varrho^{-1} \nabla_\omega \gamma(\omega)$$

where the dot and $\nabla_\omega$ denote the derivative with respect to $\varrho$ and the tangential gradient $^1$ on the unit sphere respectively.

---

$$^1$$ The tangential gradient $\nabla_\omega$ of a function $u$ can be defined as

$$\nabla_\omega u = \varrho \left( \nabla u - \frac{\partial u}{\partial \varrho} \omega \right).$$

By introducing local coordinates on the sphere of radius $\varrho$, one can verify that $\nabla_\omega$ is a tangential operator acting on $\omega$ and that it does not depend on $\varrho$. 

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Assuming that the support of $\beta$ is so small that $\text{spt} \, v \subset \Omega$, we have

\begin{equation}
0 \leq \int_{\Omega} \langle \mathcal{B}(|v|) \nabla v, \nabla v \rangle \, dx = \int_{\Omega} \langle \mathcal{B}(\beta(\gamma(\omega))) \omega, \omega \rangle \beta^2(\omega) \gamma^2(\omega) \, dx + \\
\int_{\Omega} \langle \mathcal{B}(\beta(\gamma(\omega))) + \mathcal{B}^*(\beta(\gamma(\omega))) \rangle \omega, \nabla \omega \gamma(\omega) \rangle \beta(\omega) \beta^{-1}(\omega) \gamma(\omega) \, dx + \\
\int_{\Omega} \langle \mathcal{B}(\beta(\gamma(\omega))) \nabla \omega \gamma(\omega), \nabla \omega \gamma(\omega) \rangle \beta(\omega) \gamma^{-2} \, dx,
\end{equation}

i.e.

\begin{equation}
0 \leq \int_{0}^{+\infty} \beta^2(\varrho) \varrho^{N-1} d\varrho \int_{|\omega|=1} \langle \mathcal{B}(\beta(\gamma(\omega))) \omega, \omega \rangle \gamma^2(\omega) \, d\sigma_\omega + \\
\int_{0}^{+\infty} \beta(\varrho) \beta^{-1}(\varrho) \varrho^{N-2} d\varrho \int_{|\omega|=1} \langle \mathcal{B}(\beta(\gamma(\omega))) + \mathcal{B}^*(\beta(\gamma(\omega))) \rangle \omega, \nabla \omega \gamma(\omega) \rangle \gamma(\omega) \, d\sigma_\omega + \\
\int_{0}^{+\infty} \beta^2(\varrho) \varrho^{N-3} d\varrho \int_{|\omega|=1} \langle \mathcal{B}(\beta(\gamma(\omega))) \nabla \omega \gamma(\omega), \nabla \omega \gamma(\omega) \rangle \, d\sigma_\omega.
\end{equation}

We choose now a particular sequence of test functions. Fix $0 < \varrho_1 < \varrho_2 < \varrho_3 < \varrho_4 < \text{dist} \, (0, \partial \Omega)$. The sequences $\{\beta_m(\varrho)\}$ and $\{\gamma_m(\omega)\}$ are required to satisfy the following conditions:

\begin{align*}
\begin{cases}
\beta_m \in C^\infty_0(\mathbb{R}^+), \quad \text{spt} \, \beta_m \subset (\varrho_1, \varrho_4); \\
\beta_m(\varrho) = \beta_1(\varrho), \quad \varrho \in (\varrho_1, \varrho_4) \setminus (\varrho_2, \varrho_3), \ m = 1, 2, \ldots; \\
t_0 - \delta < \beta_m(\varrho) < t_0 + \delta, \quad \varrho \in (\varrho_2, \varrho_3), \ m = 1, 2, \ldots; \\
\lim_{n \to -\infty} \int_{\varrho_2}^{\varrho_3} \dot{\beta}_m(\varrho) \varrho^{N-1} d\varrho = +\infty;
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
\gamma_m \in C^\infty(\Sigma), \ spt \, \gamma_m \subset \Sigma_{2\delta}; \\
0 \leq \gamma_m \leq 1; \quad \gamma_m(\omega) = 1, \forall \, \omega \in \Sigma_{\delta}; \\
\int_{\Sigma_{2\delta} \setminus \Sigma_{\delta}} \gamma_m^2(\omega) \, d\sigma_\omega = \mathcal{O} \left(1/\sqrt{\lambda_m}\right); \\
\int_{\Sigma_{2\delta} \setminus \Sigma_{\delta}} |\nabla \omega \gamma_m(\omega)|^2 \, d\sigma_\omega = \mathcal{O} \left(\sqrt{\lambda_m}\right).
\end{cases}
\end{align*}

Here $\Sigma_{\delta}$ is the set $\{\omega \in \Sigma \mid |\omega - \omega_0| < \delta\}$ and

\begin{equation}
\lambda_m = \int_{\varrho_2}^{\varrho_3} \dot{\beta}_m^2(\varrho) \varrho^{N-1} d\varrho.
\end{equation}
As a consequence we have also
\[
\int_{\Sigma_{28}\setminus \Sigma_{8}} |\gamma_m(\omega)| |\nabla_\omega \gamma_m(\omega)| d\sigma_\omega \leq \left( \int_{\Sigma_{28}\setminus \Sigma_{8}} \gamma_m^2(\omega) d\sigma_\omega \right)^{\frac{1}{2}} \left( \int_{\Sigma_{28}\setminus \Sigma_{8}} |\nabla_\omega \gamma_m(\omega)|^2 d\sigma_\omega \right)^{\frac{1}{2}} = \mathcal{O}(1).
\]

Inequality (50) and conditions (51), (52) imply
\[
0 \leq \int_{g_2}^{g_1} \beta_m^2(\varrho) \varrho^{N-1} d\varrho \int_{\Sigma_{8}} \langle \mathcal{B}(|\beta_m(\varrho)|) \omega, \omega \rangle d\sigma_\omega + \int_{g_2}^{g_3} \beta_1^2(\varrho) \varrho^{N-1} d\varrho \int_{\Sigma_{28}\setminus \Sigma_{8}} \langle \mathcal{B}(\beta_m(\varrho) \gamma_m(\omega)) \omega, \omega \rangle \gamma_m^2(\omega) d\sigma_\omega + \int_{(g_1,g_4)\setminus (g_2,g_3)} \beta_1^2(\varrho) \varrho^{N-1} d\varrho \int_{\Sigma_{28}\setminus \Sigma_{8}} \langle \mathcal{B}(\beta_1(\varrho)) \omega, \omega \rangle \gamma_m^2(\omega) d\sigma_\omega + \int_{g_1}^{g_4} \beta_m^2(\varrho) \varrho^{N-3} d\varrho \int_{\Sigma_{28}\setminus \Sigma_{8}} \langle \mathcal{B}(\beta_m(\varrho) \gamma_m(\omega)) \omega, \omega \rangle \nabla_\omega \gamma_m(\omega) \gamma_m(\omega) d\sigma_\omega.
\]

It is worth noting that
\[
\int_{g_2}^{g_3} \beta_m(\varrho) \beta_m(\varrho) \varrho^{N-2} d\varrho \leq \left( \int_{g_2}^{g_3} \beta_m^2(\varrho) \varrho^{N-2} d\varrho \right)^{\frac{1}{2}} \left( \int_{g_2}^{g_3} \beta_m^2(\varrho) \varrho^{N-2} d\varrho \right)^{\frac{1}{2}}.
\]

Since the sequence \( \{\beta_m\} \) is uniformly bounded, the matrix \( \mathcal{B} \) is bounded and recalling condition (49), we deduce that there exists a constant \( K \) such
that

\[
0 \leq -M |\Sigma_\delta| \lambda_m + K \left( \lambda_m \int_{\Sigma_2} \gamma_m^2(\omega) \, d\sigma_\omega + 1 + \int_{\Sigma_2} \gamma_m^2(\omega) \, d\sigma_\omega + \left(\sqrt{\lambda_m} + 1\right) \int_{\Sigma_2} |\gamma_m(\omega)||\nabla \gamma_m(\omega)| \, d\sigma_\omega + \int_{\Sigma_2} |\nabla \gamma_m(\omega)|^2 \, d\sigma_\omega \right).
\]

Dividing by \(\lambda_m\) we get

\[
0 \leq -M |\Sigma_\delta| + K \left( \int_{\Sigma_2} \gamma_m^2(\omega) \, d\sigma_\omega + \lambda_m^{-1} \left( 1 + \int_{\Sigma_2} \gamma_m^2(\omega) \, d\sigma_\omega \right) + \left(\lambda_m^{-1/2} + \lambda_m^{-1}\right) \int_{\Sigma_2} |\gamma_m(\omega)||\nabla \gamma_m(\omega)| \, d\sigma_\omega + \lambda_m^{-1} \int_{\Sigma_2} |\nabla \gamma_m(\omega)|^2 \, d\sigma_\omega \right).
\]

Letting \(m \to \infty\) and keeping in mind (52) and (53) we obtain

\[
0 \leq -M |\Sigma_\delta|
\]

and this is absurd. Inequality (48) is then proved when the matrix \(B\) does not depend on \(x\).

In the general case, suppose (47) holds and take

\[
v(x) = w((x - x_0)/\varepsilon)
\]

where \(x_0 \in \Omega\) is a fixed point, \(w \in C_0^\infty(\Omega)\) and \(\varepsilon > 0\) is sufficiently small. In this case (47) shows that

\[
0 \leq \frac{1}{\varepsilon^2} \int_\Omega \langle B(x, |w((x - x_0)/\varepsilon)|) \nabla w((x - x_0)/\varepsilon), \nabla w((x - x_0)/\varepsilon) \rangle \, dx = \varepsilon^{N-2} \int_\Omega \langle B(x_0 + \varepsilon y, |w(y)|) \nabla w(y), \nabla w(y) \rangle \, dy.
\]

Therefore

\[
\int_\Omega \langle B(x_0, |w(y)|) \nabla w(y), \nabla w(y) \rangle \, dy = \lim_{\varepsilon \to 0^+} \int_\Omega \langle B(x_0 + \varepsilon y, |w(y)|) \nabla w(y), \nabla w(y) \rangle \, dy \geq 0
\]

for almost any \(x_0 \in \Omega\). The arbitrariness of \(w \in C_0^\infty(\Omega)\) and what we have obtained for matrices not depending on \(x\) give the result.
5 The functional dissipativity

Let \( \Omega \) be a domain in \( \mathbb{R}^N \), \( \varphi \) a function satisfying the conditions 1–4 in Section 3 and \( Eu = \nabla (\mathcal{A} \nabla u) \).

The next Theorem provides a necessary and sufficient condition for the functional dissipativity of the operator \( E \).

**Lemma 5** The operator \( E \) is \( L^\Phi \)-dissipative if and only if

\[
\Re \int_{\Omega} \left[ \langle \mathcal{A} \nabla v, \nabla \varphi(\varphi(u)) \rangle + \Lambda(|v|) \langle (\mathcal{A} - \mathcal{A}^*) \nabla |v|, |v|^{-1} \nabla \nabla v \rangle + \right. \\
- \Lambda^2(|v|) \langle \mathcal{A} \nabla |v|, \nabla |v| \rangle \right] dx \geq 0, \quad \forall v \in \dot{H}^1(\Omega),
\]  

(54)

where \( \Lambda \) is given by (27). Here and in the sequel the integrand is extended by zero on the set where \( v \) vanishes.

**Proof.** Sufficiency. Suppose \( r \geq 0 \). Let \( u \in \dot{H}^1(\Omega) \) such that \( \varphi(|u|) u \in \dot{H}^1(\Omega) \) and define \( v = \sqrt{\varphi(|u|)} u \). In view of Lemma 3 we have that \( v \) belongs to \( \dot{H}^1(\Omega) \). Moreover \( u = |v|^{-1} \xi(|v|) v = \Theta(|v|) v \), \( \varphi(|u|) \bar{u} = |v| \xi(|v|)^{-1} \bar{v} = |\Theta(|v|)|^{-1} v \) (see (27)). Therefore

\[
\langle \mathcal{A} \nabla u, \nabla (\varphi(|u|) u) \rangle = \langle \mathcal{A} \nabla (\Theta(|v|) v), \nabla (\Theta(|v|)^{-1} v) \rangle = \\
\langle \Theta'(|v|) v \nabla v + \Theta(|v|) \nabla v, -\Theta'(|v|) \Theta(|v|)^{-1} v \nabla v + \Theta(|v|)^{-1} \nabla v \rangle = \\
- \Theta'(|v|)^{-1} \Theta(|v|)^{-1} v \langle \mathcal{A} \nabla v, \nabla v \rangle + \\
\Theta'(|v|) \Theta(|v|)^{-1} \langle v, \mathcal{A} \nabla v, \nabla v \rangle - \mathcal{A} \langle \mathcal{A} \nabla v, \nabla v \rangle \\
+ \Lambda(|v|) \langle \mathcal{A} \nabla v, |v|^{-1} \nabla \nabla v \rangle - \langle |v|^{-1} \nabla \nabla v, \mathcal{A} \nabla v \rangle \\
+ \langle \mathcal{A} \nabla v, \nabla v \rangle.
\]

on the set \( \{ x \in \Omega \mid u(x) \neq 0 \} = \{ x \in \Omega \mid v(x) \neq 0 \} \).

Therefore

\[
\Re \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla (\varphi(|u|) u) \rangle dx = \\
\Re \int_{\Omega} \left[ \langle \mathcal{A} \nabla v, \nabla v \rangle + \Lambda(|v|) \langle (\mathcal{A} - \mathcal{A}^*) \nabla |v|, |v|^{-1} \nabla \nabla v \rangle + \\
- \Lambda^2(|v|) \langle \mathcal{A} \nabla |v|, \nabla |v| \rangle \right] dx \geq 0
\]

because of (54), and (22) is proved.

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If $-1 < r < 0$, setting $w = \varphi(|u|) u$, i.e. $u = \psi(|w|) w$, we can write condition (22) as

$$\Re \int_{\Omega} \langle A^* \nabla w, \nabla (\psi(|w|) w) \rangle dx \geq 0$$

for any $w \in \dot{H}^1(\Omega)$ such that $\psi(|w|) w \in \dot{H}^1(\Omega)$.

Recalling Lemma 1, what we have already proved for $r \geq 0$ shows that this inequality holds if

$$\Re \int_{\Omega} \left[ \langle A^* \nabla v, \nabla v \rangle + \tilde{\Lambda}(|v|) \langle (A^* - A) \nabla |v|, |v|^{-1} \nabla \nabla v \rangle + \right.$$ \left.
\begin{align*}
-\tilde{\Lambda}^2(|v|) \langle A^* \nabla |v|, \nabla |v| \rangle \right] dx \geq 0, \quad \forall v \in \dot{H}^1(\Omega).
\end{align*}

Since $\tilde{\Lambda}(|v|) = -\Lambda(|v|)$ (see (36)), conditions (55) coincides with (54) and there is proved also for $-1 < r < 0$.

**Necessity.** Let $v \in C^1_0(\Omega)$ and define $u_\varepsilon = \Theta(g_\varepsilon) v$, where $g_\varepsilon = \sqrt{|v|^2 + \varepsilon^2}$. The function $u_\varepsilon$ and $\varphi(|u_\varepsilon|) u_\varepsilon$ belong to $C^1_0(\Omega)$ and we have

$$\langle A \nabla u_\varepsilon, \nabla (\varphi(|u_\varepsilon|) u_\varepsilon) \rangle =$$ \vspace{1em}
$$\varphi(|u_\varepsilon|) \langle A \nabla u_\varepsilon, \nabla u_\varepsilon \rangle + \varphi'(|u_\varepsilon|) \langle A \nabla u_\varepsilon, u_\varepsilon \nabla (|u_\varepsilon|) \rangle =$$ \vspace{1em}
$$\varphi[\Theta(g_\varepsilon) |v|] \langle A \Theta'(g_\varepsilon) v \nabla g_\varepsilon + \Theta(g_\varepsilon) \nabla v, \Theta'(g_\varepsilon) v \nabla g_\varepsilon + \Theta(g_\varepsilon) \nabla v \rangle +$$ \vspace{1em}
$$\varphi'[\Theta(g_\varepsilon) |v|] \times$$ \vspace{1em}
$$\langle A \Theta'(g_\varepsilon) v \nabla g_\varepsilon + \Theta(g_\varepsilon) \nabla v, \Theta(g_\varepsilon) v (\Theta'(g_\varepsilon) |v| \nabla g_\varepsilon + \Theta(g_\varepsilon) \nabla |v|) \rangle =$$ \vspace{1em}
$$\varphi[\Theta(g_\varepsilon) |v|] \left\{ [\Theta'(g_\varepsilon)]^2 |v|^2 \langle A \nabla g_\varepsilon, \nabla g_\varepsilon \rangle + \right.$$ \vspace{1em}
$$\Theta'(g_\varepsilon) \Theta(g_\varepsilon) \left[ \langle A \nabla g_\varepsilon, \nabla |v| \rangle + \langle A (\nabla |v|), \nabla g_\varepsilon \rangle \right] + \Theta^2(g_\varepsilon) \langle A \nabla v, \nabla v \rangle \right\} +$$ \vspace{1em}
$$\varphi'[\Theta(g_\varepsilon) |v|] \left\{ [\Theta(g_\varepsilon)]^3 |v|^3 \langle A \nabla g_\varepsilon, \nabla g_\varepsilon \rangle + \right.$$ \vspace{1em}
$$\Theta^2(g_\varepsilon) \Theta'(g_\varepsilon) \left[ |v|^2 \langle A \nabla g_\varepsilon, \nabla |v| \rangle + |v| \langle A (\nabla |v|), \nabla g_\varepsilon \rangle \right] +$$ \vspace{1em}
$$\Theta^3(g_\varepsilon) \langle A \nabla v, \nabla |v| \rangle \right\}.$$ \vspace{1em}

(56)
Letting $\varepsilon \to 0^+$ the right hand side tends to
\[
\varphi[\Theta(|v|)|v|\Theta^2(|v|)] \langle \mathcal{A} \nabla v, \nabla v \rangle + \\
\varphi[\Theta(|v|)|v|\Theta'(|v|)] \Theta(|v|) \langle \mathcal{A} \nabla |v|, \nabla v \rangle + \\
\Theta(|v|)\{ \varphi[\Theta(|v|)|v|\Theta'(|v|)] + \\
\varphi'[\Theta(|v|)|v|\Theta'(|v|)][\Theta(|v|)|v| + \Theta(|v|)] \} \langle \mathcal{A} (\nabla v), \nabla |v| \rangle + \\
\Theta'(|v|)|v|^2 \{ \varphi[\Theta(|v|)|v|\Theta'(|v|)] + \\
\varphi'[\Theta(|v|)|v|\Theta'(|v|)][\Theta(|v|)|v| + \Theta(|v|)] \} \langle \mathcal{A} \nabla |v|, \nabla |v| \rangle
\] (57)
on the set $\Omega_0 = \{ x \in \Omega \mid v(x) \neq 0 \}$.

In view of (34) and (35) we have
\[
\varphi[\Theta(|v|)|v|\Theta^2(|v|)] = 1, \quad \varphi[\Theta(|v|)|v|\Theta'(|v|)] \Theta(|v|) = \Theta'(|v|)/\Theta(|v|), \\
\varphi[\Theta(|v|)|v|\Theta'(|v|)] + \varphi'[\Theta(|v|)|v|\Theta'(|v|)][\Theta(|v|)|v| + \Theta(|v|)] = \\
-\Theta'(|v|)/\Theta^2(|v|).
\]

Substituting these equalities in (57) and keeping in mind (56), we see that
\[
\lim_{\varepsilon \to 0^+} \langle \mathcal{A} \nabla u_\varepsilon, \nabla (\varphi(|u_\varepsilon|) u_\varepsilon) \rangle = \\
\langle \mathcal{A} \nabla v, \nabla v \rangle + \Lambda(|v|)(\langle \mathcal{A} \nabla |v|, |v|^{-1} \nabla v \rangle - \langle \mathcal{A} (|v|^{-1} \nabla v), \nabla |v| \rangle) + \\
-\Lambda^2(|v|)\langle \mathcal{A} \nabla |v|, \nabla |v| \rangle
\]
on $\Omega_0$.

By using (25),(28) and (33) one can prove that each term in the last expression of (56) can be majorized by $L^1$ functions. Let us consider the first one: $\varphi[\Theta(g_\varepsilon)|v|[\Theta'(g_\varepsilon)]^2|v|^2 \langle \mathcal{A} \nabla g_\varepsilon, \nabla g_\varepsilon \rangle$. Observing also that $|\nabla g_\varepsilon| \leq |\nabla v|$, we get
\[
|\varphi[\Theta(g_\varepsilon)|v|[\Theta'(g_\varepsilon)]^2|v|^2 \langle \mathcal{A} \nabla g_\varepsilon, \nabla g_\varepsilon \rangle| \leq [\Theta(g_\varepsilon)|v|][\Theta'(g_\varepsilon)]^2|v|^2 |\nabla v|^2 \leq \\
\Theta^{2+r}(g_\varepsilon)|v|^r |\nabla v|^2 \leq g_\varepsilon^{-r} |v|^r |\nabla v|^2.
\] (58)

Moreover
\[
g_\varepsilon^{-r} |v|^r \leq \begin{cases} 
C, & \text{if } r > 0 \\
C|v|^r, & \text{if } r \leq 0,
\end{cases}
\]
where the constant $C$ does not depend on $\varepsilon$. Since the function $|v|^r|\nabla v|^2 \chi_{\Omega_0}$ belong to $L^1(\Omega)$ because $r > -1$ (see Langer [22, p.312]), we see that in any
case the last term in (58) can be majorized by an $L^1$ function which does not
depend on $\varepsilon$. The other terms in (56) can be estimated in a similar way.

By the Lebesgue dominated convergence theorem, we find

\[
\lim_{\varepsilon \to 0^+} \int_{\Omega} \langle \mathcal{A} \nabla u_\varepsilon, \nabla (\varphi(|u_\varepsilon|) u_\varepsilon) \rangle \, dx = 
\int_{\Omega} \left( \langle \mathcal{A} \nabla v, \nabla v \rangle + \Lambda(|v|) \left( \langle \mathcal{A} \nabla v, |v|^{-1} \nabla v \rangle - \langle \mathcal{A} (|v|^{-1} \nabla v), \nabla |v| \rangle \right) + 
- \Lambda^2(|v|) \langle \mathcal{A} |v|, \nabla |v| \rangle \right) \, dx.
\]

The left hand side being non negative (see (22)), inequality (54) holds for
any $v \in C^1_0(\Omega)$.

Let now $v \in \dot{H}^1(\Omega)$ and $v_n \in C^\infty_0(\Omega)$ such that $v_n \to v$ in $H^1$ norm. Let us show that

\[
\lim_{n \to \infty} \int_{\Omega} \left( \langle \mathcal{A} \nabla v_n, \nabla v_n \rangle + \Lambda(|v_n|) \left( \langle \mathcal{A} \nabla v_n, |v_n|^{-1} \nabla v_n \rangle + 
- \langle \mathcal{A} (|v_n|^{-1} \nabla v_n), \nabla |v_n| \rangle \right) - \Lambda^2(|v_n|) \langle \mathcal{A} |v_n|, \nabla |v_n| \rangle \right) \, dx = 
\int_{\Omega} \left( \langle \mathcal{A} \nabla v, \nabla v \rangle + \Lambda(|v|) \left( \langle \mathcal{A} \nabla v, |v|^{-1} \nabla v \rangle - \langle \mathcal{A} (|v|^{-1} \nabla v), \nabla |v| \rangle \right) + 
- \Lambda^2(|v|) \langle \mathcal{A} |v|, \nabla |v| \rangle \right) \, dx.
\]

We may assume that $v_n \to v$, $\nabla v_n \to \nabla v$ almost everywhere in $\Omega$. Denote
by $\Omega_{0n}$ and $\Omega_0$ the sets $\{ x \in \Omega \mid v_n(x) \neq 0 \}$ and $\{ x \in \Omega \mid v(x) \neq 0 \}$, respectively. As proved in [6, p.1087-1088],

\[
\chi_{\Omega_{0n}} |v_n|^{-1} \nabla v_n \to \chi_{\Omega_0} |v|^{-1} \nabla v \quad \text{a.e. in } \Omega.
\]

Because of the continuity of $\Lambda$ on $(0, \infty)$ and its boundedness (see (32)), we deduce

\[
\chi_{\Omega_{0n}} \Lambda(|v_n|) |v_n|^{-1} \nabla v_n \to \chi_{\Omega_0} \Lambda(|v|) |v|^{-1} \nabla v \quad \text{a.e. in } \Omega.
\]

The boundedness of $\Lambda$ also leads to

\[
\int_G |\Lambda(|v_n|) \langle \mathcal{A} \nabla v_n, |v_n|^{-1} \nabla v_n \rangle - \langle \mathcal{A} (|v_n|^{-1} \nabla v_n), \nabla |v_n| \rangle \rangle + 
- \Lambda^2(|v_n|) \langle \mathcal{A} |v_n|, \nabla |v_n| \rangle \rangle \, dx \lesssim \int_G |\nabla v_n|^2 \, dx
\]
for any measurable set \( G \subset \Omega \). This inequality easily implies that the sequence of functions

\[
\langle A^{\nabla} v_n, \nabla v_n \rangle + \Lambda(|v_n|) \left( \langle A^{\nabla} |v_n|, |v_n|^{-1} v_n \nabla v_n \rangle + \right.

- \langle A (|v_n|^{-1} v_n \nabla v_n), \nabla |v_n| \rangle - \Lambda^2(|v_n|) \langle A^{\nabla} |v_n|, \nabla |v_n| \rangle
\]

satisfies the conditions of the Vitali convergence Theorem (see, e.g., [3, p.71]). This establishes (59) and the result follows from (54).

The next Corollaries provide necessary and, separately, sufficient conditions for the functional dissipativity of the operator \( E \).

**Corollary 1** If the operator \( E \) is \( L^\Phi \)-dissipative, we have

\[
\langle \text{Re} \mathcal{A} (x) \xi, \xi \rangle \geq 0 \tag{60}
\]

for almost every \( x \in \Omega \) and for any \( \xi \in \mathbb{R}^N \).

**Proof.**

Given a function \( v \in C_0^1(\Omega) \), define

\[
X = \text{Re}(|v|^{-1} \nabla v), \quad Y = \text{Im}(|v|^{-1} \nabla v) \tag{61}
\]

on the set \( \{x \in \Omega \mid v(x) \neq 0\} \). As in [6, p.1074], we have

\[
\begin{align*}
\text{Re} \langle \mathcal{A} \nabla v, \nabla v \rangle &= \langle \text{Re} \mathcal{A} X, X \rangle + \langle \text{Re} \mathcal{A} Y, Y \rangle + \langle \text{Im} (\mathcal{A} - \mathcal{A}^*) X, Y \rangle, \\
\text{Re} \langle (\mathcal{A} - \mathcal{A}^*) |v|, |v|^{-1} \nabla v \rangle &= \langle \text{Im} (\mathcal{A} - \mathcal{A}^*) X, Y \rangle, \\
\text{Re} \langle \mathcal{A} \nabla |v|, \nabla |v| \rangle &= \langle \text{Re} \mathcal{A} X, X \rangle.
\end{align*}
\]

The operator being \( L^\Phi \)-dissipative, (54) holds and we can write

\[
\int_\Omega \left\{ [1 - \Lambda^2(|v|)] \langle \text{Re} \mathcal{A} X, X \rangle + \langle \text{Re} \mathcal{A} Y, Y \rangle + \\
[1 + \Lambda(|v|)] \langle \text{Im} \mathcal{A} X, Y \rangle + [1 - \Lambda(|v|)] \langle \text{Im} \mathcal{A}^* X, Y \rangle \right\} dx \geq 0.
\tag{62}
\]

Set \( v(x) = \varrho(x) e^{i\lambda \xi \cdot x} \) where \( \varrho \in C_0^\infty(\Omega) \) is a real valued function, \( \xi \in \mathbb{R}^N \) and \( \lambda \in \mathbb{R} \). Putting \( v \) in (62) we get

\[
\int_\Omega \left[ [1 - \Lambda^2(|\varrho|)] \langle \text{Re} \mathcal{A} \nabla \varrho, \nabla \varrho \rangle dx + \lambda^2 \int_\Omega \varrho^2 \langle \text{Re} \mathcal{A} \xi, \xi \rangle dx + \\
\lambda \int_\Omega \left\{ [1 + \Lambda(|\varrho|)] \langle \text{Im} \mathcal{A} \nabla \varrho, \xi \rangle + [1 - \Lambda(|\varrho|)] \langle \text{Im} \mathcal{A}^* \nabla \varrho, \xi \rangle \right\} \varrho \ dx \geq 0.
\]

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For the arbitrariness of $\lambda$ we find
\[
\int_{\Omega} \varrho^2 \langle \operatorname{Re} A, \xi \rangle \, dx \geq 0.
\]
This inequality holding for any real valued $\varrho \in C_0^\infty(\Omega)$, we obtain the result. \hfill \Box

**Corollary 2** If
\[
[1 - \Lambda^2(t)] \langle \operatorname{Re} A(x) \xi, \xi \rangle + \langle \operatorname{Re} A(x) \eta, \eta \rangle + [1 + \Lambda(t)] \langle \operatorname{Im} A(x) \xi, \eta \rangle + [1 - \Lambda(t)] \langle \operatorname{Im} A^* (x) \xi, \eta \rangle \geq 0
\]
for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^N$, the operator $E$ is $L^\Phi$-dissipative.

**Proof.** Let $v \in \hat{H}^1(\Omega)$ and define $X$ and $Y$ as in the proof of Corollary 1. Inequality (63) implies that (62) holds. As we know, this means that (54) is satisfied and the result follows from Lemma 5. \hfill \Box

**Corollary 3** If the operator $E$ has real coefficients and satisfies condition (60) for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^N$, than it is $L^\Phi$-dissipative with respect to any $\Phi$.

**Proof.** It follows immediately from Corollary 2 and (32). \hfill \Box

**Remark 1** We shall see later a class of operators for which the positiveness of polynomials (63) is also necessary for the $L^\Phi$-dissipativity. But there are no functions $\varphi$ for which the condition (63) is necessary. This is shown by the next example.

**Example 1** Consider the operator $E$ in two independent variables where the matrix of the coefficients is
\[
A = \begin{pmatrix}
1 & i\gamma \\
-i\gamma & 1
\end{pmatrix}
\]
\( \gamma \) being a real constant. The polynomial in \( \xi, \eta \) in condition (63) is given by
\[
[1 - \Lambda^2(t)]|\xi|^2 + |\eta|^2 + 2\gamma (\xi_2\eta_1 - \xi_1\eta_2).
\]
Writing this polynomial in the form
\[
(\eta_1 + \gamma \xi_2)^2 + (\eta_2 - \gamma \xi_1)^2 + [1 - \Lambda^2(t) - \gamma^2] |\xi|^2
\]
it is clear that, if \(|\gamma| > 1\), condition (63) cannot be satisfied for any \( \xi, \eta \in \mathbb{R}^N \). However, the corresponding operator is the Laplacean, which is \( L^\Phi \)-dissipative for any \( \varphi \) (see Corollary 3).

The next results concerns \( \Phi \)-strongly elliptic operators

**Lemma 6** Let \( E \) be a \( \Phi \)-strongly elliptic operator. There exists a constant \( \kappa \) such that for any complex valued \( u \in H^1(\Omega) \) such that \( \varphi(|u|) u \in H^1(\Omega) \) we have
\[
\Re \langle A \nabla u, \nabla (\varphi(|u|) u) \rangle \geq \kappa |\nabla (\sqrt{\varphi(|u|)} u)|^2
\]
augmented everywhere on the set \( \{ x \in \Omega \mid u(x) \neq 0 \} \).

**Proof.** Let us define \( v = \sqrt{\varphi(|u|)} u \). By Lemma 3, the function \( v \) belongs to \( H^1(\Omega) \). As in the proof of Lemma 5, we find
\[
\Re \langle A \nabla u, \nabla (\varphi(|u|) u) \rangle =
\Re \left[ \langle A \nabla v, \nabla v \rangle + \Lambda(|v|) \langle (A - A^*) \nabla |v|, |v|^{-1} \nabla |v| \rangle + \right.
\]
\[- \Lambda^2(|v|) \langle A \nabla |v|, \nabla |v| \rangle \]
on the set \( \{ x \in \Omega \mid u(x) \neq 0 \} = \{ x \in \Omega \mid v(x) \neq 0 \} \).

This can be written as
\[
\Re \langle A \nabla u, \nabla (\varphi(|u|) u) \rangle =
[1 - \Lambda^2(|v|)] \Re A A^* X, X \rangle + \Re A Y, Y \rangle +
[1 + \Lambda(|v|)] \Im A^* X, Y \rangle + [1 - \Lambda(|v|)] \Im A X, Y \rangle
\]
where \( X \) and \( Y \) are given by (61). Thanks to the \( \Phi \)-strong ellipticity (see (21)) we get
\[
\Re \langle A \nabla u, \nabla (\varphi(|u|) u) \rangle \geq \kappa (|X|^2 + |Y|^2) = \kappa |\nabla v|^2
\]

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and (64) is proved.

This lemma implies the next Corollary (see Dindoš and Pipher [14, Th. 2.4, pp.263–265] for a similar result in the $L^p$ case).

**Corollary 4** Let $E$ be a $\Phi$-strongly elliptic operator. There exists a constant $\kappa$ such that for any nonnegative $\chi \in L^\infty(\Omega)$ and any complex valued $u \in H^1(\Omega)$ such that $\varphi(|u|) u \in H^1(\Omega)$ we have

$$\Re \int_\Omega \langle \mathcal{A} \nabla u, \nabla (\varphi(|u|) u) \rangle \chi(x) dx \geq \kappa \int_\Omega |\nabla (\sqrt{\varphi(|u|)} u)|^2 \chi(x) dx .$$

**Proof.** It follows immediately from inequality (64).

---

6 A necessary and sufficient condition

The aim of this section is to give a necessary and sufficient condition for the $L^\Phi$-dissipativity of the operator $E$.

**Theorem 1** Let the matrix $\Im \mathcal{A}$ be symmetric, i.e. $\Im \mathcal{A}^t = \Im \mathcal{A}$. Then the operator $E$ is $L^\Phi$-dissipative if, and only if,

$$|s \varphi'(s)| \langle \Im \mathcal{A}(x) \xi, \xi \rangle \leq 2 \sqrt{\varphi(s)} \left[ |\varphi(s)|' \langle \Re \mathcal{A}(x) \xi, \xi \rangle \right] (65)$$

for almost every $x \in \Omega$ and for any $s > 0, \xi \in \mathbb{R}^N$.

**Proof.** Sufficiency. Let us prove that (65) implies inequality (63), which, for the simmetricity of $\Im \mathcal{A}$, becomes

$$[1 - \Lambda^2(t)] \langle \Re \mathcal{A}(x) \xi, \xi \rangle + \langle \Re \mathcal{A}(x) \eta, \eta \rangle + 2 \Lambda(t) \langle \Im \mathcal{A}(x) \xi, \eta \rangle \geq 0 \quad (66)$$

for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^N$.

Fix $x \in \Omega$ in such a way (65) holds, $t > 0$ and define

$$\mathcal{J}(\xi, \eta) = \langle \Re \mathcal{A}(x) \xi, \xi \rangle + \langle \Re \mathcal{A}(x) \eta, \eta \rangle + \gamma \langle \Im \mathcal{A}(x) \xi, \eta \rangle$$

where

$$\gamma = \frac{s \varphi'(s)}{\sqrt{\varphi(s)}} \left[ s \varphi(s) \right]' , \quad s = \zeta(t) ,$$

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(ζ is the function introduced in section 3.2).

Let

$$
\lambda = \min_{|\xi|^2 + |\eta|^2 = 1} \mathcal{S}(\xi, \eta).
$$

There exist \((\xi_0, \eta_0)\) such that \(|\xi_0|^2 + |\eta_0|^2 = 1\) and \(\lambda = \mathcal{S}(\xi_0, \eta_0)\). This vector satisfies the algebraic system

\[
\begin{aligned}
\Re(\mathcal{A} + \mathcal{A}^t) \xi_0 + \gamma \Im \mathcal{A} \eta_0 &= 2 \lambda \xi_0 \\
\Re(\mathcal{A} + \mathcal{A}^t) \eta_0 + \gamma \Im \mathcal{A} \xi_0 &= 2 \lambda \eta_0.
\end{aligned}
\]

This implies

$$
\Re(\mathcal{A} + \mathcal{A}^t) (\xi_0 - \eta_0) - \gamma \Im \mathcal{A} (\xi_0 - \eta_0) = 2 \lambda (\xi_0 - \eta_0).
$$

and therefore

$$
2 \left( \Re \mathcal{A} (\xi_0 - \eta_0), \xi_0 - \eta_0 \right) - \gamma \left( \Im \mathcal{A} (\xi_0 - \eta_0), \xi_0 - \eta_0 \right) = 2 \lambda |\xi_0 - \eta_0|^2. \quad (67)
$$

The left hand-side is nonnegative because of (65). If \(\lambda < 0\), (67) implies \(\xi_0 = \eta_0\). In this case

$$
\lambda = \mathcal{S}(\xi_0, \xi_0) = 2 \left( \Re \mathcal{A} (x) \xi_0, \xi_0 \right) + \gamma \left( \Im \mathcal{A} (x) \xi_0, \xi_0 \right);
$$

but this is nonnegative because of (65) and we get a contradiction. Therefore \(\lambda \geq 0\) and \(\mathcal{S}(\xi, \eta) \geq 0\), for any \(\xi, \eta \in \mathbb{R}^N\).

We have also \(\mathcal{S}(-\sqrt{1 - \Lambda^2(t)} \xi, \eta) \geq 0\), i.e.

$$
[1 - \Lambda^2(t)] \left( \Re \mathcal{A}(x) \xi, \xi \right) + \left( \Re \mathcal{A}(x) \eta, \eta \right) - \gamma \sqrt{1 - \Lambda^2(t)} \left( \Im \mathcal{A}(x) \xi, \eta \right) \geq 0.
$$

On the other hand, (30) and (31) show that

$$
\gamma \sqrt{1 - \Lambda^2(t)} = \frac{2 s \varphi'(s)}{s \varphi'(s) + 2 \varphi(s)} = -2 \Lambda(t)
$$

and then (68) coincides with (66).

Corollary 2 shows that the operator \(E\) is \(L^\Phi\)-dissipative.

**Necessity.** As in the proof of Corollary 1, the \(L^\Phi\)-dissipativity of \(E\) implies

$$
\int_\Omega \{ [1 - \Lambda^2(|v|)] \langle \Re \mathcal{A} X, X \rangle + \langle \Re \mathcal{A} Y, Y \rangle + 2 \Lambda(|v|) \langle \Im \mathcal{A} X, Y \rangle \} dx \geq 0,
$$

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for any $v \in C^1_0(\Omega)$ (see (62)). Setting $v(x) = \varrho(x) e^{i \sigma(x)}$, where $\varrho \in C^\infty_0(\Omega)$ and $\sigma \in C^\infty(\Omega)$ are real valued, we get $|v|^{-1} \nabla v = |\varrho|^{-1} \varrho \nabla \varrho + i |\varrho| \nabla \sigma$ on the set $\{ x \in \Omega \mid \varrho(x) \neq 0 \}$. It follows

$$
\int_{\Omega} \left\{ [1 - \Lambda^2(|\varrho|)] \langle \Re \mathcal{A} \nabla \varrho, \nabla \varrho \rangle + \varrho^2 \langle \Re \mathcal{A} \nabla \sigma, \nabla \sigma \rangle + 2 \Lambda(|\varrho|) \varrho \langle \Im \mathcal{A} \nabla \varrho, \nabla \varrho \rangle \right\} dx \geq 0
$$

(69)

for any real valued $\varrho \in C^\infty_0(\Omega), \sigma \in C^\infty(\Omega)$.

We choose $\sigma$ by the equality

$$
\sigma(x) = \frac{\mu}{2} \log(\varrho^2 + \varepsilon^2)
$$

where $\mu \in \mathbb{R}$ and $\varepsilon > 0$. Inequality (69) takes the form

$$
\int_{\Omega} \left\{ [1 - \Lambda^2(|\varrho|)] \langle \Re \mathcal{A} \nabla \varrho, \nabla \varrho \rangle + \mu^2 \frac{\varrho^4}{(\varrho^2 + \varepsilon)^2} \langle \Re \mathcal{A} \nabla \varrho, \nabla \varrho \rangle + 2 \mu \frac{\varrho^2}{\varrho^2 + \varepsilon} \Lambda(|\varrho|) \langle \Im \mathcal{A} \nabla \varrho, \nabla \varrho \rangle \right\} dx \geq 0.
$$

Letting $\varepsilon \to 0^+$ we find

$$
\int_{\Omega} \left\{ [1 - \Lambda^2(|\varrho|)] \langle \Re \mathcal{A} \nabla \varrho, \nabla \varrho \rangle + \mu \langle \Re \mathcal{A} \nabla \varrho, \nabla \varrho \rangle + 2 \mu \Lambda(|\varrho|) \langle \Im \mathcal{A} \nabla \varrho, \nabla \varrho \rangle \right\} dx \geq 0.
$$

This inequality holding for any real valued $\varrho \in C^\infty_0(\Omega)$, by Lemma 4 we get

$$
[1 - \Lambda^2(t)] \langle \Re \mathcal{A}(x) \xi, \xi \rangle + \mu^2 \langle \Re \mathcal{A}(x) \xi, \xi \rangle + 2 \mu \Lambda(t) \langle \Im \mathcal{A}(x) \xi, \xi \rangle \geq 0
$$

for almost every $x \in \Omega$ and for any $t > 0, \xi \in \mathbb{R}^n$. The arbitrariness of $\mu \in \mathbb{R}$ leads to

$$
\Lambda^2(t) \langle \Im \mathcal{A}(x) \xi, \xi \rangle^2 \leq [1 - \Lambda^2(t)] \langle \Re \mathcal{A}(x) \xi, \xi \rangle^2.
$$

Recalling (31) and Corollary 1, we can write

$$
|\Lambda(t)| |\langle \Im \mathcal{A}(x) \xi, \xi \rangle| \leq \sqrt{1 - \Lambda^2(t)} \langle \Re \mathcal{A}(x) \xi, \xi \rangle.
$$

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Finally, setting $s = \zeta(t)$ and keeping in mind the expressions (30) and (31), the last inequality reads as

$$\frac{|s \phi'(s)|}{s \phi'(s) + 2 \phi(s)} |\langle \text{Im} \mathcal{A}(x) \xi, \xi \rangle| \leq \frac{2 \sqrt{\phi(s) (s \phi'(s) + \phi(s))}}{s \phi'(s) + 2 \phi(s)} \langle \text{Re} \mathcal{A}(x) \xi, \xi \rangle,$$

i.e. (65).

**Remark 2** The proof of Theorem 1 shows that condition (65) holds if and only if the inequality (66) is satisfied for almost every $x \in \Omega$ and for any $t > 0, \xi, \eta \in \mathbb{R}^N$. This means that conditions (63) are necessary and sufficient for the $L^\Phi$-dissipativity for the operators considered in Theorem 1.

**Remark 3** Suppose that the condition $\text{Im} \mathcal{A} = \text{Im} \mathcal{A}^t$ is not satisfied. Arguing as in the proof of Theorem 1, one can prove that condition (65) is still necessary for the $L^\Phi$-dissipativity of the operator $E$. However in general it is not sufficient, whatever the function $\phi$ may be. This is shown by the next example.

**Example 2** Let $n = 2$, $\Omega$ a bounded domain and

$$\mathcal{A} = \begin{pmatrix} 1 & i\lambda x_1 \\ -i\lambda x_1 & 1 \end{pmatrix}$$

Since $\langle \text{Re} \mathcal{A} \xi, \xi \rangle = |\xi|^2$ and $\langle \text{Im} \mathcal{A} \xi, \xi \rangle = 0$ for any $\xi \in \mathbb{R}^N$, condition (65) is satisfied.

If the corresponding operator $Eu = \Delta u + i \lambda \partial_2 u$ is $L^\Phi$-dissipative, then

$$\text{Re} \int_{\Omega} \langle \Delta u + i \lambda \partial_2 u, u \rangle \phi(|u|) \, dx \leq 0, \quad \forall \, u \in C_0^\infty(\Omega). \quad (70)$$

Take $u(x) = \varrho(x) e^{itx_2}$, where $\varrho \in C_0^\infty(\Omega)$ is real valued and $t \in \mathbb{R}$. Since $\langle Eu, u \rangle = \varrho[\Delta \varrho + 2i t \partial_2 \varrho - t^2 \varrho + i \lambda (\partial_2 \varrho + it \varrho)]$, condition (70) implies

$$\int_{\Omega} \varrho \Delta \varrho \phi(|\varrho|) \, dx - \lambda t \int_{\Omega} \varrho^2 \phi(|\varrho|) \, dx - t^2 \int_{\Omega} \varrho \phi(|\varrho|) \, dx \leq 0 \quad (71)$$

for any $t, \lambda \in \mathbb{R}$. The function $\phi$ being positive, we can choose $\varrho$ in such a way

$$\int_{\Omega} \varrho^2 \phi(|\varrho|) \, dx > 0.$$
Taking
\[ \lambda^2 > 4 \int_\Omega \varrho \Delta \varphi(|\varrho|) \, dx \left( \int_\Omega \varrho^2 \varphi(|\varrho|) \, dx \right)^{-1}, \]
inequality (71) is impossible for all \( t \in \mathbb{R} \). Thus \( E \) is not \( L^\Phi \)-dissipative, although (65) is satisfied.

**Corollary 5** Let the matrix \( \text{Im} \, \mathcal{A} \) be symmetric, i.e. \( \text{Im} \, \mathcal{A}^t = \text{Im} \, \mathcal{A} \). If
\[ \lambda_0 = \sup_{s > 0} \frac{|s \varphi'(s)|}{2 \sqrt{\varphi(s) |s \varphi(s)|}} < +\infty, \tag{72} \]
then the operator \( E \) is \( L^\Phi \)-dissipative if, and only if,
\[ \lambda_0 \, |\langle \text{Im} \, \mathcal{A}(x) \xi, \xi \rangle| \leq \langle \text{Re} \, \mathcal{A}(x) \xi, \xi \rangle \tag{73} \]
for almost every \( x \in \Omega \) and for any \( \xi \in \mathbb{R}^N \). If \( \lambda_0 = +\infty \) the operator \( E \) is \( L^\Phi \)-dissipative if and only if \( \text{Im} \, \mathcal{A} \equiv 0 \) and condition (60) is satisfied.

**Proof.** If \( \lambda_0 < +\infty \), the result follows immediately from Theorem 1. If \( \lambda_0 = +\infty \) and the operator \( E \) is \( L^\Phi \)-dissipative, inequality (65) implies \( \langle \text{Im} \, \mathcal{A} \xi, \xi \rangle = 0, \langle \text{Re} \, \mathcal{A}(x) \xi, \xi \rangle \geq 0 \) for almost every \( x \in \Omega \) and for any \( \xi \in \mathbb{R}^N \). Therefore \( \text{Im} \, \mathcal{A} \equiv 0 \) and condition (60) is satisfied. The viceversa was proved in Corollary 3.

**Remark 4** If we use the function \( \Phi \) (see (18)), condition (65) can be written as
\[ |s \Phi''(s) - \Phi'(s)| |\langle \text{Im} \, \mathcal{A}(x) \xi, \xi \rangle| \leq 2 \sqrt{s \Phi'(s) \Phi''(s)} \langle \text{Re} \, \mathcal{A}(x) \xi, \xi \rangle \]
for almost every \( x \in \Omega \) and for any \( s > 0, \xi \in \mathbb{R}^N \). In the same way, formula (72) becomes
\[ \lambda_0 = \sup_{s > 0} \frac{|s \Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} < +\infty. \]

We end this section by some examples in which we indicate both the functions \( \Phi \) and \( \varphi \). It is easy to verify that in each example the function \( \varphi \) satisfies conditions 1-5 of section 3.1.
Example 3 If $\Phi(s) = s^p$, i.e. $\varphi(s) = p s^{p-2}$, which corresponds to $L^p$ norm, the function in (72) is constant and $\lambda_0 = |p - 2|/(2\sqrt{p - 1})$. In this way we reobtain Theorem 1 of [6, p.1079].

Example 4 Let us consider $\Phi(s) = s^p \log(s + e)$ ($p > 1$), which is the Young function corresponding to the Zygmund space $L^p \log L$. This is equivalent to say $\varphi(s) = p s^{p-2} \log(s + e) + s^{p-1} (s + e)^{-1}$. By a direct computation we find

$$
\frac{|s \Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} = \frac{p(p - 2) \log(s + e) + \frac{(2p - 1)s}{s + e} - \frac{s^2}{(s + e)^2}}{2 \sqrt{(p \log(s + e) + \frac{s}{s + e}) \left( p(p - 1) \log(s + e) + \frac{2ps}{s + e} - \frac{s^2}{(s + e)^2} \right)}}.
$$

(74)

Since

$$
\lim_{s \to 0^+} \frac{|s \Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} = \lim_{s \to +\infty} \frac{|s \Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} = \frac{|p - 2|}{2 \sqrt{p - 1}}
$$

the function is bounded. Then we have the $L^p$-dissipativity of the operator $E$ if, and only if, (73) holds, where $\lambda_0$ is the sup of the function (74) in $\mathbb{R}^+$. 

Example 5 Let us consider the function $\Phi(s) = \exp(s^p) - 1$, i.e. $\varphi(s) = p s^{p-2} \exp(s^p)$. In this case

$$
\frac{|s \Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} = \frac{|p s^p + p - 2|}{2 \sqrt{(p s^p + p - 1)}},
$$

and $\lambda_0 = +\infty$. In view of Corollary (5), the operator $E$ is $L^p$-dissipative, i.e.

$$
\Re \int_{\Omega} \langle \varphi \nabla u, \nabla [u |u|^{p-2} \exp(|u|^p)] \rangle \, dx \geq 0
$$

for any $u \in \dot{H}^1(\Omega)$ such that $|u|^p \exp(|u|^p) u \in \dot{H}^1(\Omega)$, if and only if the operator $E$ has real coefficients and condition (60) is satisfied.

Example 6 Let $\Phi(s) = s - \arctan s$, i.e. $\varphi(s) = s/(s^2 + 1)$. In this case

$$
\frac{|s \Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} = \frac{|s^2 - 1|}{2 \sqrt{2(s^2 + 1)}}
$$
and $\lambda_0 = +\infty$. As in the previous example, we have that
\[
\Re \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla \left( \frac{|u|u}{|u|^2+1} \right) \rangle \, dx \geq 0
\]
for any $u \in \dot{H}^1(\Omega)$ such that $|u|u/(|u|^2+1) \in \dot{H}^1(\Omega)$, if and only if the operator $E$ has real coefficients and condition (60) is satisfied.

**Example 7** Let $\Phi(s) = s^4/(s^2 + 1)$, i.e. $\varphi(s) = 2s^2(2 + s^2)/(s^2 + 1)^2$. In this case
\[
\frac{|s\Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} = \frac{2}{\sqrt{(s^2 + 1)(s^2 + 2)(s^4 + 3s^2 + 6)}}.
\]
This function is decreasing and $\lambda_0$ is equal to its value at 0, i.e. $\lambda_0 = 1/\sqrt{3}$.
The operator $E$ is $L^p$-dissipative, i.e.
\[
\Re \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla \left( \frac{|u|^2(2 + |u|^2)u}{(|u|^2+1)^2} \right) \rangle \, dx \geq 0
\]
for any $u \in \dot{H}^1(\Omega)$ such that $|u|^2(2 + |u|^2)u/(|u|^2+1)^2 \in \dot{H}^1(\Omega)$, if and only if
\[
|\langle \Im \mathcal{A}(x) \xi, \xi \rangle| \leq \sqrt{3} \langle \Re \mathcal{A}(x) \xi, \xi \rangle
\]
for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^N$.

**Example 8** Let $\Phi(s) = s^2(s^2 + 2)/(s^2 + 1) - 2 \log(s^2 + 1)$, i.e. $\varphi(s) = 2s^4/(s^2 + 1)^2$. In this case
\[
\frac{|s\Phi''(s) - \Phi'(s)|}{2 \sqrt{s \Phi'(s) \Phi''(s)}} = \frac{2}{\sqrt{(s^2 + 1)(s^2 + 5)}}.
\]
This function is decreasing and $\lambda_0$ is equal to its value at 0, i.e. $\lambda_0 = 2/\sqrt{5}$.
The operator $E$ is $L^p$-dissipative, i.e.
\[
\Re \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla \left( \frac{|u|^4u}{(|u|^2+1)^2} \right) \rangle \, dx \geq 0
\]
for any $u \in \dot{H}^1(\Omega)$ such that $|u|^4u/(|u|^2+1)^2 \in \dot{H}^1(\Omega)$, if and only if
\[
2 |\langle \Im \mathcal{A}(x) \xi, \xi \rangle| \leq \sqrt{5} \langle \Re \mathcal{A}(x) \xi, \xi \rangle
\]
for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^N$.

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