Fermi acceleration in time-dependent rectangular billiards due to multiple passages through resonances.

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We consider a slowly rotating rectangular billiard with moving boundaries and use canonical perturbation theory to describe the dynamics of a billiard particle. In the process of slow evolution certain resonance conditions can be satisfied. Correspondingly, phenomena of scattering on a resonance and capture into a resonance happen in the system. These phenomena lead to destruction of adiabatic invariance and to unlimited acceleration of the particle.

When one slowly changes parameters of an integrable Hamiltonian system with two degrees of freedom, two classical actions of the unperturbed system are approximately conserved as adiabatic invariants. However, if during slow evolution resonance relations between frequencies of the system are satisfied, adiabatic invariance is destroyed. The paper describes in detail how it happens in a rectangular billiard system with moving boundaries and how it can lead to unlimited acceleration (so-called Fermi acceleration) of the billiard particle.

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I. INTRODUCTION

Billiards are important models in the theory of dynamical systems and its applications (Ref. [1–3]), especially in optics and matter-wave optics (see, e.g., Ref. [4–14]). Recently billiards with varying parameters became the object of studies (see, e.g., Ref. [15–22]). One of the most interesting questions in this area of research is Fermi acceleration (Ref. [23]).

In the present paper we consider acceleration of a billiard particle due to scattering on resonances in a rectangular billiard with slowly varying parameters. We use the methods of analysis of these resonant phenomena designed for Hamiltonian systems with slow and fast variables (Ref. [24]). The methods were developed and the corresponding theorems were proved for smooth systems. However, previous studies (Ref. [14, 16, 17]) show that they can be also applied adequately for billiard systems, which possess discontinuities (impacts). In Ref. [16], acceleration of a particle due to scattering on resonances and capture into a resonance was considered mostly for the case of crossing periodically a single low-order resonance. The numerical calculations revealed very slow acceleration, and a possibility of unlimited acceleration remained an open question. Here we describe a geometric obstacle for acceleration that considerably slows it down, so that it is hard to detect the energy growth numerically. When this obstacle is not present, the particle accelerates without bounds fast enough to clearly see the energy growth in numerical calculations.

II. THE MODEL

Let us consider a particle in a rectangle 2D box rotating at a constant angular velocity \( \omega > 0 \). We assume that the rotation is slow: \( \omega \ll 1 \). The Hamiltonian of the system in the rotating coordinate frame is
FIG. 1: Phase portraits of the Hamiltonian $F_0$ in (3). a) $|a| < |b|$, the phase portrait has an oscillatory domain. b) $|a| > |b|$, there is no oscillatory domain in the phase portrait. In both plots, $a < 0$.

FIG. 2: Jumps of adiabatic invariant $I_1$. Parameters of the system: $\varepsilon = 1 \cdot 10^{-4}$, $\omega = 4.5 \cdot 10^{-5}$, $d_1 = d_{10}(1 + A_1 \cos(\varepsilon t))$, $d_2 = d_{20}(1 + A_2 \cos(\varepsilon t))$, where $A_1 = -0.1$, $A_2 = 0.3$, $d_{10} = 1$, $d_{20} = 1.4$. Inset: a single jump of the adiabatic invariant.

\[
H = \frac{1}{2}(p_1^2 + p_2^2) + \omega(p_1 q_2 - p_2 q_1) + U(q_1, q_2, \varepsilon t), \tag{1}
\]
where \( p_i, q_i \) are canonically conjugate momenta and coordinates, respectively, and \( U(q_1, q_2, \epsilon t) \) is the hard wall potential of the 2D box, and \( \epsilon \) is a small positive parameter (the time-dependence arises from motion of boundaries described below). The second term in Eq. (1) comes from switching to the rotating frame (where billiard is irrotational): it is \(-\omega L\), where \( L = -(p_1 q_2 - p_2 q_1) \) is the angular momentum of the particle about the coordinate origin [28]. The rotation is needed to couple two degrees of freedom, otherwise the system would decouple on two one-dimensional billiards.

The Hamiltonian formalism outlined below is available in Ref. [16] and is given here for self-consistency of the presentation. Using a canonical transformation we introduce new variables \((I_i, \phi_i), i = 1, 2\) such that

\[
q_i = \begin{cases} 
-d_i + 2d_i \phi_i / \pi, & \text{if } 0 < \phi_i \leq \pi, \\
3d_i - 2d_i \phi_i / \pi, & \text{if } \pi < \phi_i \leq 2\pi,
\end{cases}
\]

\[
p_i = \frac{\pi I_i}{2d_i} \text{sgn}(\sin \phi_i),
\]

where \( 2d_i \) are lengths of the sides of the 2D box. We assume that these lengths slowly depend on time: \( d_i = d_i(\epsilon t) \), where \( \epsilon \sim \omega \ll 1 \). If \( d_i \) are constant, and the box does not rotate, the new variables \((I_i, \phi_i)\) are just the action-angle variables of the system.

In the new variables the Hamiltonian of the system is:

\[
\mathcal{H} = \frac{\pi^2}{8} \left( \frac{I_1^2}{d_1^2} + \frac{I_2^2}{d_2^2} \right) - \frac{8 \omega}{\pi} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} ' \left[ \frac{(I_1 d_2^2 k_1 + I_2 d_1^2 k_2)}{d_1 d_2 k_1^2 k_2^2} \sin(k_1 \phi_1 - k_2 \phi_2) + \frac{(I_1 d_2^2 k_1 - I_2 d_1^2 k_2)}{d_1 d_2 k_1^2 k_2^2} \sin(k_1 \phi_1 + k_2 \phi_2) \right] + \epsilon [E_1(\phi_1) + E_2(\phi_2)] = H_0 + \omega H_1,
\]

where primes denote summation over odd \( k \). We use Fourier series expansion of expressions Eq. (2) for \( p_i, q_i \) (see Appendix). Terms \( E_i \) appear because the transformation (2) is non-autonomous; the explicit form of these terms is not important for our discussion.

Let \( m, n \) be relatively prime integers. Let us call the terms in (3) with \( k_1 = rm, k_2 = rn \) and integer \( r (m, n) \)-resonant, if the following resonance condition is met:
\[ \frac{mI_1}{d_1^2} - \frac{nI_2}{d_2^2} = 0. \] (4)

Physically, the resonance condition means that frequencies of oscillation of a particle between pairs of opposite walls are commensurable \((m\omega_1 = n\omega_2,\ \text{where } \omega_j = \frac{\pi p_j}{2d_j} = \frac{\pi^2 I_j}{4d_j^2}, \ j = 1, 2\) are frequencies of oscillation ”along” and ”across” the rectangular billiard, correspondingly). If the resonance condition is met in the unperturbed billiard, the particle is moving along a closed trajectory.

The resonance condition defines a corresponding resonance ray in the plane of actions:

\[ I_{2res}^{I_1} = \frac{md_2^2}{nI_1} I_1 = \alpha_{m:n} I_1, \] (5)

where \(\alpha_{m:n}\) is a slow function of time.

### III. PASSAGE THROUGH A RESONANCE

If a phase point has such values of \(I_1 = I_1^{(0)}\) and \(I_2 = I_2^{(0)}\) that in the plane of action it is located far from any low-order resonance ray, then in the process of evolution values of \(I_1\) and \(I_2\) will be approximate adiabatic invariants and the phase point will undergo small oscillations around \((I_1^{(0)}, I_2^{(0)})\).

However, as the time grows, the values of the lengths \(d_i\) change and the system may approach a resonance. There, actions are no longer well-conserved, and two phenomena can happen: scattering on resonance and capture into the resonance. They are illustrated in Fig. 2 and Fig. 3. Fig. 2 presents evolution of action \(I_1\) in a billiard with periodically oscillating walls. The action undergoes small oscillations remaining almost constant until a resonance condition is fulfilled in the system. At a resonance, it may undergo a jump, which can be seen in detail in the inset of Fig. 2. This is scattering on a resonance. Another option for a phase point is capture into the resonance shown in Fig.3. As the system approaches a resonance, the phase point can be captured in it and continue its motion in
FIG. 3: (a) Jumps of adiabatic invariant $I_1$ (due to scattering on a resonance) and a capture of the phase point into the 1:1 resonance. The captured point moves in the plane of actions with the resonant ray $I_2 = \alpha_{1:1} I_1$ along the resonant trajectory $J = \text{const}$ until it escapes from the resonance. (b) Behaviour of $I_1, I_2$ near and at the capture into the resonance 1:1 (arcs in the center of the figure correspond to the phase point captured into the resonance). (c) Behaviour of the winding number $(\omega_1/\omega_2)$ as a function of time. For the point captured into the resonance, this winding number stays approximately constant and undergoes small oscillations around the resonance value ($\omega_1/\omega_2 = 1$ in this particular case). (d) The same as in panel (c), but on a different scale. Scattering on a resonance 1:1 causes a jump in the winding number due to jumps in $I_1, I_2$ (inset shows a single jump with even greater resolution). Parameters of the system: $\varepsilon = 1 \cdot 10^{-4}$, $\omega = 4.5 \cdot 10^{-5}$, $d_1 = d_{10}(1 + A_1 \cos(\varepsilon t))$, $d_2 = d_{20}(1 + A_2 \cos(\varepsilon t))$, where $A_1 = -0.1$, $A_2 = 0.28$, $d_{10} = 1$, $d_{20} = 1.4$. 


FIG. 4: Diffusion in the action space \((I_1, I_2)\) due to periodic crossing of resonance lines (schematically). (a) Only one low-order resonance is crossed. The particle can diffuse in the action space due to kicks it receives at each passage of the resonance ray \(I_2^{\text{res}} = \alpha_{m:n} I_1(t)\) through it, but diffusion happen only along an interval of the resonance trajectory \(J = \text{const}\) within the resonance sector since all these kicks are parallel to each other and are restricted by the resonance sector. Solid lines shows the resonance rays \(I_2 = \alpha_{m:n} I_1\) for two values of the slope \(\alpha_{m:n}\), that is, \(\alpha_{\text{min}}\) and \(\alpha_{\text{max}}\). These two lines define the resonance sector. Dashed lines illustrate the resonance sector of another resonance, which our particle does not have chance to cross. (b) Resonance sectors of several low-order resonances overlap, correspondingly the particle consequently crosses several resonances. At crossing of either resonance the phase point receives a kick along the direction corresponding to the resonance. Since these directions are not parallel \((J \equiv nI_1 + mI_2\) is defined differently at different resonances), unlimited diffusion is now possible. Note that each resonance is crossed as a single resonance, 'overlapping' means that some sector in the action plane is swept by several resonance rays, but in different times.

such a way that the resonance condition would persist. This is clearly demonstrated in Fig. 3c, where a winding number \(\omega_1/\omega_2\) is shown. Capture into the 1 : 1 resonance leads to a remarkable conservation of the winding number during a long time (comparable to a slow period \(2\pi/\varepsilon\)). There are also several events of scattering on a resonance which are clearly seen in Fig.3a (as jumps in the actions similar to Fig.2), and can also be noticed in behaviour of the winding number in Figs.3c,d. Physically, captured point moves in such
FIG. 5: Diffusion in the action space. (a) Only one low-order resonance influences dynamics of the particle, crossings of the higher-order resonances nearby lead to slight perturbation of the scheme outlined in Fig. 4a. The particle is almost trapped in the resonance sector, 'transverse' diffusion due to other resonances is very slow. Time of integration $t \approx 6 \cdot 10^7$. Parameters are $A_1 = -0.02$, $A_2 = 0.05$, $\omega = 4.5 \cdot 10^{-5}$. (b) Several low-order resonances are crossed. Diffusion in action space is effective due to the zigzag process outlined in Fig. 4b. Parameters are: $A_1 = -0.1$, $A_2 = 0.35$, $\omega = 6 \cdot 10^{-5}$. Time of integration $t \approx 8 \cdot 10^7$.

a way that its trajectory remains almost closed even though geometry of the billiard is changing considerably.

Below we discuss these phenomena in a greater detail.

Consider the motion of a particle near a $(m,n)$-resonance. We make a canonical transformation $(I_1, I_2, \phi_1, \phi_2) \mapsto (R, J, \phi, \psi)$ using the generating function $W = (m\phi_1 - n\phi_2)R - (l_1\phi_1 - l_2\phi_2)J$, where $l_1, l_2$ are integers, $ml_2 - nl_1 = 1$. This means

$$J = nI_1 + mI_2, \quad R = l_2I_1 + l_1I_2 \quad (6)$$

The physical meaning of this transformation is as follows. The new phase $\phi = m\phi_1 - n\phi_2$ (the resonance phase) changes slowly near $(m : n)$-resonance, while the new phase $\psi = -l_1\phi_1 + l_2\phi_2$ changes fast. New actions $R$ and $J$ are canonically conjugate to $\phi$ and $\psi$, respectively.
FIG. 6: Diffusion in energy and in action. Left: growth of dispersion of energy of ensemble of initially isoenergetic phase points due to crossing of a single resonance as a function of time. Right: growth of dispersion of action $I_1$ of the same ensemble. In both cases, values of the magnitudes are rescaled to $\varepsilon$. Time of integration shown is approximately 100 slow periods, with $\varepsilon = 0.0001$, $\omega = 4.5 \cdot 10^{-5}$, $A_1 = -0.02$, $A_2 = 0.05$, $d_1 = 1$, $d_2 = 1.3$, number of phase points is $N = 200$.

respectively. We will see that $J$ is well conserved, while the magnitude of $R$ experiences considerable dynamics in the vicinity of the resonance. This fact is useful for the partial averaging procedure outlined below.

The resonance condition now takes the form

$$R = R_{\text{res}}(J, \varepsilon t) = \frac{nl_2d_2^2 + ml_1d_2^2}{m^2d_2^2 + n^2d_1^2}J$$

(7)

In the new variables Hamiltonian (3) is $\mathcal{H} = \mathcal{H}_0(R, J, \varepsilon t) + \omega \mathcal{H}_1(R, J, \phi, \psi, \varepsilon t)$. The canonically conjugate pairs of variables are $(R, \phi)$ and $(J, \psi)$. As noted above, it follows from the form of the Hamiltonian that $\psi$ is a fast variable, and one can average the equations of motion over it. In the averaged system, $J$ is the integral of motion and below we consider $J$ as a parameter. We use another canonical transformation $(R, \phi) \mapsto (P, \phi)$ with generating function $W_1 = (P + R_{\text{res}}(J, \varepsilon t))\phi$ to introduce new 'action' variable $P =$
the conjugate 'angle' variable is $\phi$. This transformation allows us to obtain an effective Hamiltonian in a vicinity of the resonance, since the new action $P$ measures the deviation of $R$ from its resonance value. In a small neighborhood of the resonance where $P$ is of order $\sqrt{\varepsilon}$, the Hamiltonian takes the following form:

$$
H = \Lambda(J, \varepsilon t) + F_0 + O(\varepsilon), \quad F_0 = \frac{1}{2}gP^2 + b\chi(\phi) + a\phi,
$$

where $\Lambda$ is $H_0$ on the resonance, and

$$
g = \left( \frac{\partial^2 H_0}{\partial R^2} \right)_{R=R_{res}} = \frac{\pi^2}{4} \left( \frac{k_1^2}{d_1^2} + \frac{k_2^2}{d_2^2} \right), \quad b = -2\frac{\omega d_1 d_2 J}{k_1 k_2 (k_1^2 d_2^2 + k_2^2 d_1^2)}, \quad a = \varepsilon \frac{\partial R_{res}}{\partial \varepsilon t},
$$

$$
\chi(\phi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin n\phi = \begin{cases} 
\phi - \phi^2/\pi, & \text{if } 0 < \phi < \pi, \\
\phi + \phi^2/\pi, & \text{if } -\pi < \phi \leq 0.
\end{cases}
$$

The dynamics near the resonance depends on the properties of the resonance Hamiltonian $F_0$. Consider this Hamiltonian at frozen values of $J, \varepsilon t$. Corresponding phase portraits of the resonance Hamiltonian on the $(\phi, P)$-plane are shown in Figure 1. If $|a| < |b|$, the phase portrait has a separatrix and an oscillatory domain. If $|a| > |b|$, there is no such a domain in the phase portrait. To be definite, assume $a < 0$, $b > 0$. The area $S$ of the oscillatory domain (in the case $|a| < |b|$) is a function of $J$ and $\varepsilon t$, and can be expressed as follows:

$$
S = 2\sqrt{\frac{2}{g}} \int_{\phi_1}^{\phi_2} \sqrt{F_0^s - b\chi(\phi) - a\phi} \, d\phi,
$$

where $F_0^s$ is the value of $F_0$ on the separatrix

$$
F_0^s = \frac{\pi}{4} \frac{(a + b)^2}{b},
$$

and $\phi_1, \phi_2$ are the roots of the integrand (in particular, $\phi_2$ is the coordinate of the saddle point). The explicit expression for the area can be found in the Appendix.
Now we take into account a slow evolution of parameter \( \varepsilon t \) in the system. Suppose the area of the oscillation region \( S \) grows with time. In this case, additional space appears inside the oscillatory domain. Hence, phase points can cross the separatrix and change their mode of motion from rotational to oscillatory.

As mentioned above, this phenomenon is called a capture into a resonance. Note that for a phase point captured into the resonance, in the course of its motion in the phase portrait of Hamiltonian \( F_0 \) with slowly changing parameters, the internal adiabatic invariant (the area encircled by the point’s orbit, divided by \( 2\pi \)) is well conserved (see numerical calculations in Ref. [16]).

It is important to trace the motion of the phase point in the plane of actions \((I_1, I_2)\).

Far from the resonance, a phase point undergoes small oscillations around its adiabatic position (a point \((I_{1}^{(0)}, I_{2}^{(0)})\) in the action plane).

As time goes, the resonance ray \( I_2 = \alpha_{m,n}(\varepsilon t)I_1 \) slowly sweeps a sector (let us call it the resonance sector) on this plane. This resonance ray is a line passing through the origin and slowly periodically changing its slope between the minimal and maximal values, thus sweeping the resonance sector periodically. When the resonance ray passes through our phase point and captures it, the captured point leaves a vicinity of the initial adiabatic position \( R = R_0 = l_2I_{1}^{(0)} + l_1I_{2}^{(0)} = \text{const} \), \( J = J_0 = \text{const} \) in the action space \((I_1, I_2)\) and continues its motion following approximately the resonant ray and keeping its value of \( J \) constant: \( R = R_{\text{res}}(J, \varepsilon t), J = J_0 = \text{const} \). Thus, it travels along a resonant trajectory which is a part of the line \( J = J_0 \) located within the resonant sector.

So, it leaves a vicinity of the initial point \( I_{1,2} = I_{1,2}^{0} \) const and continues its motion approximately following the corresponding resonant line. Let us denote the value of \( S \) at the moment of capture as \( S_* \). Somewhen later, the value of \( S \) starts decreasing with time. Approximately at the moment when it equals the value \( S_* \) the phase point previously captured crosses the separatrix again and leaves the oscillatory domain. This is escape from the resonance (value of \( S \) at the moment of escape is approximately equal
to $S_*$ due to conservation of the internal adiabatic invariant). If behaviour of the area $S(t)$ is monotonous as the slope $\alpha_{m,n}(t)$ is changing between $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$, e.g. it monotonously grows as $\alpha_{m,n}$ is increasing from $\alpha_{\text{min}}$ to $\alpha_{\text{max}}$ and then monotonously decreases as $\alpha_{m,n}$ is decreasing from $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$, then the escape from the resonance happens approximately near the same point $(I_1^0, I_2^0)$ where the capture happened. This is the case in our numerical examples.

Phase points that cross the resonance without being captured undergo a jump of adiabatic invariant $R$ of order $\sqrt{\epsilon}$ (and, therefore, jumps of $I_{1,2}$, since $I_1 = mR - l_1J$, $I_2 = -nR + l_2J$). As mentioned above, this phenomenon is a scattering on a resonance. Asymptotic formula for the jump in our case is (see also Ref. [24] for a general formula)

$$\Delta R = -2\sqrt{\epsilon} \int_{-\infty}^{\phi_*} \frac{b\chi'(\phi)d\phi}{[2g(\hat{h}_* - \hat{b}\chi - \hat{a}\phi)]^{1/2}},$$

where $\hat{a} = a/\epsilon$, $\hat{b} = b/\epsilon$, $\phi_*$ is the phase at which resonance crossing happen, and $\hat{h}_*$ is the value of the Hamiltonian $F_0$ at this crossing ($F_0^*$), rescaled to $\epsilon$: $\hat{h}_* = F_0^*/\epsilon$. The value of $\xi \equiv \text{Frac}[\hat{h}_*/(2\pi\hat{a}_*)]$ is uniformly distributed on the interval $(0, 1)$.

Both capture into a resonance and scattering on a resonance are probabilistic phenomena, and they are usual in systems with resonance crossings (see Ref. [24, 26, 27]).

Multiple crossings through the resonance lead to diffusion of the adiabatic invariants due to multiple scattering on the resonance. Note that this diffusion happen along the resonance trajectory if only one resonance is taken into account. We discuss this question in a more detail in the next Section.

The phase point undergoes two crossings of the resonance per slow period, and the mean value of the change in action is equal, in the first approximation, to the area $S_*$ divided by $2\pi$, with a corresponding sign depending on whether the area increases or decreases at the moment of crossing. The mean value of the total change in action after the two crossings is therefore zero. One can consider the jumps of $R$ as a random walk.
with timestep equal to the half of the slow period and a quasi-random value of the jump
is obtained from Eq. (12) with assumption of uniform distribution of $\xi$ (Ref. [16, 25, 26]).

So, the resulting dynamics in the action space is normal diffusion, at least far from
the boundaries of the resonant sector swept by the resonance ray (near the boundaries,
motion of the resonance ray slows down and the point effectively "touches" the resonance
instead of crossing it; numerically, one can see that the point is reflected from the boundary
of the sector).

The coefficient of diffusion can be found as follows. If there is no oscillatory domain
in the phase portrait of $F_0$, the mean value of the jump $\langle 12 \rangle$ is equal to zero, and the
coefficient of diffusion is equal to the expectation value of $(\Delta R)^2$, divided by half of the
slow period, similar to that being done in Ref. [26]. It can be shown that the quasirandom
variable $\xi$ is uniformly distributed on $(0, 1)$ and its values on consequent passages through
the resonance are independent. In case there is the oscillatory domain in the phase portrait
of $F_0$, the mean value of the jump $\Delta R$ is not equal to zero, and the coefficient of diffusion
can be found by calculating the expectation value of $(\Delta R^{(1)} + \Delta R^{(2)})^2$ from (12) (where
$\Delta R^{(1)}$, $\Delta R^{(2)}$ are jumps of $R$ at two consequent passages through the resonance, with
values of $\xi$ equal to $\xi_1$ and $\xi_2$, correspondingly, assuming the same uniform distribution
of the quasirandom variables $\xi_1$ and $\xi_2$), and dividing it by the slow period.

Numerical calculations are presented in Figs. 2-3. For numerical simulations we chose
parameters $d_i$ harmonically varying with time. In Figure 2 jumps of adiabatic invariant $I_1$
are shown. A single jump of the adiabatic invariant is shown in the inset of Figure 2. In
Figure 3a one can see the evolution of the adiabatic invariant due to multiple scatterings
on the resonance and one capture of the phase point into the resonance. The captured
point moves along a resonant curve until it escapes from the resonance. Figure 3b gives
evolution of $I_1, I_2$ in the vicinity of capture into resonance, Fig. 3c presents time evolution
of the winding number $\omega_1/\omega_2$. 
IV. ACCELERATION BY SEVERAL RESONANCES

Consider what is happening if parameters $d_i(t)$ are changed periodically in such a way that the particle repeatedly cross a single resonance. I.e., only one low-order resonance, say $n:m$, is crossed, while other resonances being crossed are of high order and can be neglected (While there is no precise distinction between low- and high-order resonances, in this paper we consider as high order resonances those for which $|m| + |n|$ is bigger than $\text{const}\cdot \varepsilon^{-1/6}$. This is a condition that the resonance alone does not produce any diffusion because its effect is so small that phases of subsequent passages through this resonance are strongly correlated, see also Ref. [26]). Periodic scattering on the low-order resonance leads to diffusion in action space. However, jumps of $I_1$ and $I_2$ at the passage through the resonance are linearly dependent. Indeed, since $J \equiv nI_1 + mI_2 \approx \text{const}$ at the passage through the resonance, jumps of $I_1$ and $I_2$ are dependent in such a way that in the action plane $(I_1, I_2)$ the particle is kicked along the corresponding line, $J = J_0 = \text{const}$. Since the resonance condition is $I_2 = \alpha_{m:n}(\varepsilon t)I_1$, one can easily see that the resonance ray will periodically sweep a sector on the plane of actions, and the particle will diffuse only along the part of the line $J = nI_1 + mI_2 = \text{const}$ restricted by this sector. In other words, its diffusion in action space happens along the interval. This is an obvious geometric obstacle to unlimited acceleration. Of course, taking into account that near a low-order resonance there are always high-order resonances, one gets diffusion transverse to the interval $J = J_0$, although very slow. Now, to let the energy of the particle to grow without bounds efficiently, we should allow at least two low-order resonances (say, $n, m$ and $\hat{n}, \hat{m}$), and let the corresponding resonance sectors to overlap. Then, in the first sector diffusion happens along the line $nI_1 + mI_2 = \text{const}$ and in the second sector along the line $\hat{n}I_1 + \hat{m}I_2 = \text{const}$. Zigzag-type acceleration is possible: shift of the particle in the action space due to the first resonance is not parallel to the shift of the particle due to the other resonance. Even though the particle is trapped within the union of the two resonance
sectors, it is now can drift to infinity staying within this combined sector. Schematically this mechanism is presented in Fig. 4.

Numerical examples are given in Figs 5-6. In Figure 5a one can see diffusion of the phase point in the action space restricted by a resonance sector. The particle is effectively trapped on an interval, and diffusion transverse to this interval is extremely slow. In Fig. 5b, due to subsequent crossings of several resonances, the particle can undergo unlimited diffusion. From stroboscopic shots of the position of the particle in action space shown in this Figure, peculiar properties of the motion of the particle can be recognized. There are regimes of motion where the particle diffuses along certain direction for a long time (when it is in a region of the action space swept by only one low order resonance), and there are regimes of motion, where it receives subsequent kicks in different directions (in the region of overlapping of resonance sectors).

Fig. 6a demonstrates growth of dispersion in energy of an ensemble of phase points due to multiple scattering on a single resonance. It is seen that the initial growth is linear, in accordance with the theory for the single-resonance system. However, saturation or only extremely slow diffusion is expected after the ensemble of particles is redistributed within the restricted interval shown in Fig. 5a.

Similar results can be seen in Fig. 6b, where multiple scattering on three low-order resonances happens. Since diffusion of energy is bounded from below, we expect the total energy of the ensemble will also start grow with time after considerable spreading in energy happens.

It should be noted that the reasoning in this Section is of heuristic nature. While for the partially averaged system the formula Eq. [12] for the jump of action at the resonance and the statement about conservation of $J$ are mathematically rigorous results, the using of these results in the exact two-frequency system is based on physically plausible reasoning. We use numerical simulation in order to support heuristic reasoning.
V. CONCLUSION

An important open question left in Ref. [16] concerned the possibility of unlimited acceleration of a particle in the billiard under consideration. It is well known (see, e.g. Ref. [2]) that in a similar one-dimensional problem (Ulam’s model (Ref. [2]), motion of a particle between two periodically moving walls) unlimited acceleration is impossible, provided that the motion of the walls is described by smooth enough functions. The reason is that when a particle moves fast enough with respect to the motion of the walls, the system possesses a perpetual adiabatic invariant (see Ref. [25]) that prevents the particle’s acceleration. In the two-dimensional model considered in the present paper and in Ref. [16], the adiabatic invariance is destroyed by captures into resonances and scattering on resonances. The hypothesis of Ref. [16] is that for a majority of initial conditions the velocity of a particle can reach arbitrarily large magnitudes. However, there was not clear numerical evidence for that in Ref. [16].

Here we show that there are geometric obstacles to unlimited acceleration in the case where only a single resonance influences dynamics of the particle. The particle is then trapped in the resonance sector. Overcoming this obstacle by modulating the system in such a way that several low-order resonances are crossed during a slow period of modulation, one obtains clear numerical evidence of unlimited acceleration.

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Appendix

A. Fourier series

Variables $p_i$ and $q_i$ from Eq. 2 can be expressed in terms of $(I_i, \phi_i)$ as the following Fourier series:

\[ p_i = \sum_{k=1}^{\infty} \frac{2I_i}{d_i k} \sin k\phi_i , \quad (13) \]
\[ q_i = -\sum_{k=1}^{\infty} \frac{8d_i}{\pi^2 k^2} \cos k\phi_i , \]

where primes denote summation over odd $k$.

Note that $I_i$ is related to $p_i$ in a very simple way:

\[ I_i = \frac{2d_i |p_i|}{\pi} , \quad (14) \]

as can easily be seen from the phase portrait of the one-dimensional billiard. From this expression it is easy to obtain $\frac{p^2_i}{2} = \frac{I^2\pi^2}{8d^2}$, which give us the first term of Eq. 3.

B. Area of the oscillatory domain

The explicit expression for $\phi_1$ in Eq. 10 depends on the relation between $a$ and $b$. Consider, e.g. the case $a < (2\sqrt{2} - 3)b$. One has

\[ \phi_1 = -\frac{\pi a + b}{2} \frac{b}{b} (1 + \sqrt{2}) , \quad \phi_2 = \frac{\pi a + b}{2} \frac{b}{b} . \quad (15) \]

For the area $S$ one finds:

\[ S = \sqrt{\frac{2}{g}} \frac{(a + b)^2}{2} \left( 1 + \frac{3\pi}{4} \right) \left( \frac{\pi}{b} \right)^{3/2} . \quad (16) \]
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