On Pitfalls of Identifiability in Unsupervised Learning. A Note on:
“Desiderata for Representation Learning: A Causal Perspective”

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ABSTRACT

Model identifiability is a desirable property in the context of unsupervised representation learning. In absence thereof, different models may be observationally indistinguishable while yielding representations that are nontrivially related to one another, thus making the recovery of a ground truth generative model fundamentally impossible—as often shown through suitably constructed counterexamples. In this note, we discuss one such construction, illustrating a potential failure case of an identifiability result presented in “Desiderata for Representation Learning: A Causal Perspective” by Wang & Jordan (2021) [17]. The construction is based on the theory of nonlinear independent component analysis. We comment on implications of this and other counterexamples for identifiable representation learning.

1 Introduction

One of the goals of representation learning is to infer latent structure of the data generating process starting from raw, unstructured observations. A way to formalise this is to postulate a generative model where the observations are given by a nonlinear transformation (also termed mixing function) of some latent variables of interest, and characterise under which conditions it is a priori possible to recover those ground truth variables—also termed identifiability. This has mainly been studied in the context of independent component analysis (ICA) [1, 9], under the additional assumption that the latent variables of interest are statistically independent. For the unsupervised i.i.d. setting, a negative result states that recovering the ground truth latent variables is fundamentally impossible [2, 8]: without additional auxiliary variables [10] or restrictions on the function class to which the mixing belongs [3], different models can be indistinguishable based on observations, but yield latent representations which differ in nontrivial ways.

In the related context of disentanglement, a recent paper, “Desiderata for Representation Learning: A Causal Perspective” [17], presents a result (Theorem 11) discussing identifiability of latent, not necessarily independent, factors in a fully unsupervised data-generating process. The main claim of the theorem is that, under suitable assumptions which we restate in § 2, the latent factors are identifiable up to permutation and coordinate-wise transformations.

This appears to be at odds with the aforementioned negative results for identifiability of nonlinear ICA in an unsupervised setting [8]. To illustrate this point, we provide a counterexample inspired by the theory of nonlinear ICA (§3). In particular, we leverage the indeterminacy introduced by a measure-preserving automorphism presented in [8 § 2.2], to provide an instance where two representations satisfying the assumptions of Theorem 11 in [17] are not coordinate-wise transformations and permutations of each other.

We stress that we claim no novelty regarding the construction presented in this note, which is based on theory of nonlinear ICA [8]. It should simply be thought of as exemplifying one among the many potential pitfalls that exist in identifiable representation learning. We also acknowledge personal communication with the authors of [17], who kindly discussed this matter with us and acknowledged to upload a revised version of their paper taking this problem into account.

1 Our note concerns v1 of the manuscript, Submitted on 8 Sep 2021.
2 see, e.g., [11 § 2.3] and [3 § 2.1] for formal definitions in the context of representation learning and nonlinear ICA.
2 Identifiability of representations with independent support - Theorem 11 in [17]

Theorem 2.1 (Identifiability of representations with independent support [17]). Among all compactly supported representations (i.e., the support being a closed and bounded region) that generate the same $\sigma$-algebra, the representation with independent support (2.2) (if it exists) is identifiable up to permutation and coordinate-wise bijective transformations.

Technical statement of the theorem

Given:

- observations $X \in \mathbb{R}^m$
- latent representations $Z = f(X), Z' = f'(X) \in \mathbb{R}^d$

assuming the following,

- $f, f'$ are continuous
- The sigma algebras of the latent representations are the same, i.e., $\sigma(Z) = \sigma(Z')$
- $Z, Z'$ both have compact support in $\mathbb{R}^d$
- $Z, Z'$ both satisfy the independent support condition (2.2)

we have,

$$Z_1, Z_2, \ldots, Z_d = \text{perm}(q_1(Z_1'), q_2(Z_2'), \ldots, q_d(Z_d')),$$

where $q_j$ are continuous bijective functions with a compact domain in $\mathbb{R}$.

We also restate the definition of the independent support condition, which is important in the context of Thm. 2.1

Definition 2.2 (Independent Support Condition [17]). For a representation $Z = (Z_1, Z_2, \ldots, Z_d)$, the independent support condition is satisfied if

$$\text{supp}(Z_1, Z_2, \ldots, Z_d) = \text{supp}(Z_1) \times \text{supp}(Z_2) \times \ldots \times \text{supp}(Z_d).$$

3 Proposed counterexample

We take inspiration from a setting of nonlinear ICA in two dimensions (see [8 § 2.2]) to construct a counterexample to Thm. 2.1.

- Consider latent factors $Z \sim \text{Unif }([-1, 1]^2)$.
- Consider the mapping of the latent factors to the observations, $X = AZ$ where $A \in \mathbb{R}^{2 \times 2}$ is an invertible matrix. The probability density of $X$ is supported on a compact parallelepiped in $\mathbb{R}^2$. Note that $Z = f(X)$ where $f \equiv A^{-1}$.
- We construct an alternate latent representation $Z' = h(Z)$ defined as a transformation of $Z$ as follows:

$$h(Z) = \begin{cases} Z, & ||Z|| > c \\ Z \exp(i\alpha(||Z|| - c)), & ||Z|| \leq c \end{cases}$$

where $c$ is a real number in $(0, 1)$, $i$ is the imaginary unit, and $\alpha$ is a nonzero real number.

Note that the probability density of $Z'$ is also supported on $[-1, 1]^2$. $Z' = f'(X)$, where $f' \equiv h \circ f$

Thereby, we have constructed continuous bijective mappings from random observations $X$ to two distinct latent representations $Z$ and $Z'$ supported on $[-1, 1]^2$, a compact subset of $\mathbb{R}^2$.

The indeterminacy introduced by $h$ is not trivial. In particular, it cannot be generally expressed in terms of coordinate-wise transformations and permutations as postulated by Theorem 11 (2.1). For an example choice of parameters, we include a visualisation of the nonlinear distortion between $Z$ and $Z'$ in Fig. 1.

In order to show that the construction is a valid counterexample, we need to verify that it satisfies the assumptions of Thm. 2.1.

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\[1\] Definition of $\sigma$-algebra generated by a random variable: link

\[2\] Note that a nonlinear invertible map $f$ could also be used here in place of the linear map $A$

\[3\] It can also be shown that $Z' \sim \text{Unif }([-1, 1]^2)$. 

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Figure 1: Observations and latent representations generated as described in § 3. 
Top: Observed variables; Bottom, left: Sources $Z$; Bottom, right: Sources $Z'$ generated as in (3), with parameters $a = 3.6$ and $c = 0.9$.

3.1 Verifying that the proposed counterexample satisfies the assumptions of Thm. 2.1

- **Continuity of $f$ and $f'$:** 
  $f \equiv A^{-1}$ is continuous since it is a linear map. 
  $h(\cdot)$ is continuous by definition (see also [8]). 
  $f' \equiv h \circ f$ is the composition of two functions which are continuous at all points, and hence is also continuous (see, e.g., [15, Thm. 4.7]).

- **$\sigma(Z) = \sigma(Z')$:**
  We show that $h(\cdot)$ is (continuous) bijective with continuous inverse. First, we note that this is a radius-preserving function, such that we only need to check bijectivity with domain and co-domain restricted to each fixed radius. In the case $\|Z\| > c$, $h(\cdot)$ is the identity map and hence is bijective. In the case $\|Z\| \leq c$, $h(\cdot)$ is a radius-dependent rotation and hence is also bijective, with inverse function obtained by replacing $a$ with $-a$ in eq. (3). This last observation shows that the inverse is continuous.
  As $Z$ and $Z'$ can be written as a continuous function of each other in both ways, $Z = h(Z')$ and $Z' = h^{-1}(Z)$, this implies $\sigma(Z) = \sigma(Z')$ (see [17, § 3.3]).

- **Compactness of support:** $Z, Z'$ are both supported in $[-1, 1]^2$, which is a compact region in $\mathbb{R}^2$.
- **Checking Independent Support Condition:** Both $Z$ and $Z'$ have the (hyper-)rectangular support $[-1, 1]^2$, hence satisfy the Independent Support Condition (2.2).
Hence the proposed counterexample satisfies the premises of eq. (1). However, $Z$ and $Z'$ are related by the measure-preserving automorphism $h$, which in general cannot be expressed as a permutation and coordinate-wise bijection transformation.

4 Conclusion

The construction in § 3 allows us to build a counterexample which satisfies the assumptions in Thm. 2.1, in particular the independent support condition.

Other counterexamples can be given to illustrate the identifiability issues present in nonlinear ICA and disentanglement [8,12]. While most of these concern settings where the latent variables are assumed to be independent, they also bear broader implications: as argued in [11, Appendix D], in the unsupervised, i.i.d. setting, “models with any form of unconditional prior [...] are unidentifiable”.

At the same time, while ‘ruling out’ families of counterexamples is a necessary condition for identifiability, it is not a sufficient one: for example, our recent work [3] rules out some classes of counterexamples typically used in nonlinear ICA and disentanglement by imposing constraints on the function class to which the mixing belongs. This can, however, not yet be considered as a full identifiability result but only a first step in that direction.

A different line of work tries to recover identifiability by postulating the availability of additional information, such as auxiliary variables [6,7,10,11,14], temporal or spatial structure in the data data [5] or multiple ‘views’ [4,13,19]. The above works are also limited to the assumption of (conditionally) independent latent variables; suitable notions of identifiability may however also be provided for settings where the latent variables are linked via nontrivial causal relationships, see for example [16].

The authors of [17] have meanwhile uploaded a new version of their work [18] which introduces an additional assumption to address the issue we raised. The additional assumption (see assumption 2 in Theorem 11 in the updated version [18], uploaded on the 10th of February 2022) constrains the relationship between latent factors in two alternative representations $Z$ and $Z'$. As we argued above, assumptions for identifiability are instead typically phrased as conditions on the data generating process (e.g. as constraints on the mixing function). It is currently unclear how the new assumption should be interpreted in terms of the data generating process and to what extent it entails testable implications; we believe that more work may be needed to elaborate on this.

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