Isotropic subbundles of $TM \oplus T^*M$

by

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Abstract. We define integrable, big-isotropic structures on a manifold $M$ as subbundles $E \subseteq TM \oplus T^*M$ that are isotropic with respect to the natural, neutral metric (pairing) $g$ of $TM \oplus T^*M$ and are closed by Courant brackets (this also implies that $[E, E^\perp g] \subseteq E^\perp g$). We give the interpretation of such a structure by objects of $M$, we discuss the local geometry of the structure and we give a reduction theorem.

1 Introduction

All the manifolds and mappings of this paper are assumed of the $C^\infty$ class and the following general notation is used: $M$ is an $m$-dimensional manifold, $\chi^k(M)$ is the space of $k$-vector fields, $\Omega^k(M)$ is the space of differential $k$-forms, $\Gamma$ indicates the space of global cross sections of a vector bundle, $X, Y, ...$ are either contravariant vectors or vector fields, $\alpha, \beta, ...$ are either covariant vectors or 1-forms, $d$ is the exterior differential and $L$ is the Lie derivative. The Einstein summation convention will be used whenever possible. If required by the context, a linear space $V$ shall be identified with anyone of the spaces $V \oplus 0$, $0 \oplus V$.

The vector bundle $T^{big}M = TM \oplus T^*M$ is called the big tangent bundle. It has the natural, non degenerate metric of zero signature (neutral metric)

\begin{equation}
 g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X)),
\end{equation}

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the non degenerate, skew-symmetric 2-form

\begin{equation}
\omega((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) - \beta(X))
\end{equation}

and the Courant bracket of cross sections \[4\]

\begin{equation}
[(X, \alpha), (Y, \beta)] = ([X, Y], L_X \beta - L_Y \alpha + \frac{1}{2}d(\alpha(Y) - \beta(X))).
\end{equation}

A maximal, \(g\)-isotropic subbundle \(D \subseteq T^{\text{big}}M\) is called an almost Dirac structure and, if \(\Gamma(D)\) is closed by the Courant bracket, \(D\) is an integrable or a Dirac structure. Then, the triple \((D, pr_{TM}, [\cdot, \cdot])\) is a Lie algebroid.

The almost Dirac structure \(D\) produces the generalized distribution \(\mathcal{D} = pr_{TM}D\) endowed with a leaf-wise differentiable 2-form \(\varpi\) induced by \(\omega|_D\). Conversely, \(D\) may be recovered from the pair \((\mathcal{D}, \varpi)\) by means of the formula

\begin{equation}
D = \{(X, \alpha) / X \in \mathcal{D}, \alpha|_\mathcal{D} = i(X)\varpi\}.
\end{equation}

Furthermore, by a technical computation that uses \((1.4)\), it follows that \(D\) is a Dirac structure iff \(\mathcal{D}\) is integrable and the form \(\varpi\) is closed on the leaves of \(\mathcal{D}\). Thus, a Dirac structure on \(M\) is equivalent with a generalized foliation by presymplectic leaves where the leaf-wise presymplectic form is such that the subbundle \((1.4)\) is differentiable \[4\].

While the Dirac structures were introduced as a framework for constrained dynamics, it is rather the geometry of these structures and their integrability to a conveniently equipped Lie groupoid that were the object of numerous studies.

The aim of the present paper is to understand the geometry of a \(g\)-isotropic subbundle \(E \subseteq T^{\text{big}}M\), where the maximality requirement is dropped; we call them big-isotropic structures. Then, the subbundle \(E\) must be discussed in conjunction with its \(g\)-orthogonal bundle \(E' \supseteq E\) and the corresponding objects on \(M\) will be a pair of generalized distributions \(\mathcal{E} = pr_{TM}E \subseteq \mathcal{E}' = pr_{TM}E'\) and bilinear mappings \(\varpi_x : \mathcal{E}_x \times \mathcal{E}'_x \rightarrow \mathbb{R} (\forall x \in M)\) with a skew symmetric restriction to \(\mathcal{E}_x \times \mathcal{E}_x\). Like in the Dirac case, a big-isotropic structure \(E\) will be integrable if \(\Gamma E\) is closed by Courant brackets. From the properties of the Courant bracket, one can see that if \(E\) is integrable \(\Gamma E'\) is a module over the Lie algebra \(\Gamma E\).

This definition and the relationship with the triple \((\mathcal{E}, \mathcal{E}', \varpi)\) are made precise in Section 2, where we also give several examples. In particular, \(\mathcal{E}\)
is a generalized foliation and its leaves, called the characteristic leaves of \( E \), inherit a presymplectic form.

In Section 3 we extend the construction of a local, canonical basis of a Dirac structure given in [6] to (integrable) big-isotropic structures.

In Section 4 we discuss the pullback of a big-isotropic structure by a mapping and use this operation and the canonical bases of Section 3 in order to study the structure induced on a characteristic leaf and that induced on a local transversal submanifold of the leaf. We define a property called (strong) local decomposability and extend the Dufour-Wade proof of the essential uniqueness of the transversal structure of a characteristic leaf of a Dirac structure to strongly, locally decomposable, big-isotropic structures.

Finally, in Section 5 we discuss the push-forward of a big-isotropic structure and conditions that ensure the projectability of a big-isotropic structure to the space of leaves of a foliation. The results are used in order to prove a reduction theorem of an integrable, big-isotropic structure of a manifold \( M \) to a structure of a quotient space \( N/F \) of a submanifold \( N \subseteq M \) by a foliation \( F \).

## 2 Definitions, examples, first properties

We generalize the notion of a Dirac structure by giving the following definition.

**Definition 2.1.** A \( g \)-isotropic subbundle \( E \subseteq T^{\text{big}}M \) of rank \( k \) \((0 \leq k \leq m)\) will be called a big-isotropic structure on \( M \). A big-isotropic structure \( E \) is integrable if \( \Gamma E \) is closed by the Courant bracket operation.

Let \( E' \) be the \( g \)-orthogonal subbundle \( E^\perp g \) of \( E \). The following proposition gives an important property of an integrable, big-isotropic structure.

**Proposition 2.1.** If \( E \) is an integrable, big-isotropic structure then, \( \forall (X, \alpha) \in \Gamma(E) \), \( \forall (Y, \beta) \in \Gamma(E') \), one has

\[
[(X, \alpha), (Y, \beta)] \in \Gamma(E').
\]

**Proof.** Consider also \( (Z, \gamma) \in \Gamma E \). The Courant bracket has the following property (axiom (v) of the definition of a Courant algebroid [7])

\[
X(g((Y, \beta), (Z, \gamma))) = g([(X, \alpha), (Y, \beta)], (Z, \gamma)) + g((Y, \beta), [(X, \alpha), (Z, \gamma)])
\]
In our case, since \( E \perp g E, E \perp g E' \) and \( E \) is closed by Courant brackets, the previous formula reduces to

\[
g([((X, \alpha), (Y, \beta))], (Z, \gamma)),
\]

hence, \( (2.1) \) holds.

**Remark 2.1.** The closure of \( E \) with respect to the Courant bracket is a more complex notion than the closure of a distribution \( \Delta \subseteq TM \) with respect to the Lie bracket. For instance, in the latter case, any vector field \( X \in \Delta \) is an infinitesimal automorphism of \( \Delta \) while, in the former case, if \( (X, \alpha) \in \Gamma E \) then \( X \) is an infinitesimal automorphism of \( E \) (i.e., \( (L_X Y, \mathcal{L}_X \beta) \in \Gamma E \), \( \forall (Y, \beta) \in \Gamma E \)) iff \( d\alpha(Y, Z) = 0 \) for all \( Y \in \text{pr}_{TM} E, Z \in \text{pr}_{TM} E' \). This is an easy consequence of the expression of the Courant bracket and of the isotropy of \( E \).

**Example 2.1.** For \( k = m \), the integrable, big-isotropic structures are the Dirac structures.

**Example 2.2.** Let \( M \) be a locally product manifold with the structural foliations \( \mathcal{F}_1, \mathcal{F}_2 \), i.e., each point \( x \in M \) has a neighborhood \( U \approx V_1 \times V_2 \) where \( V_a \) is a neighborhood of \( x \) in the leaf of \( \mathcal{F}_a \) through \( x \) \((a = 1, 2)\). Equivalently, \( M \) has an atlas of local coordinates of the form \((x^h, y^a)\) such that \( \mathcal{F}_1 \) has the local equations \( dy^a = 0 \) and \( \mathcal{F}_2 \) has the local equations \( dx^h = 0 \). Then \( T^{\text{big}} M \) is the direct sum of the \( g \)-orthogonal subbundles \( T \mathcal{F}_a \oplus T^* \mathcal{F}_a \) and, if \( E \) is a maximal isotropic subbundle of \( T \mathcal{F}_1 \oplus T^* \mathcal{F}_1 \), \( E \) is a big-isotropic structure on \( M \) with the orthogonal subbundle \( E' = E \oplus (T \mathcal{F}_2 \oplus T^* \mathcal{F}_2) \). Furthermore, assume \( \Gamma E \) has local bases \((Z_l, \zeta_l)\) such that

\[
Z_l = Z^h_l(x) \frac{\partial}{\partial x^h}, \quad \zeta_l = \zeta^h_l(x) dx^h
\]

and that \( E \) is Dirac along the leaves of \( \mathcal{F}_1 \). Then, using Definition 2.1 and the Courant algebroid properties of the Courant bracket \([4, 7]\), it is easy to check that the big-isotropic structure \( E \) is integrable. For instance, if \( P \in \chi^2(M), \theta \in \Omega^2(M) \) have local expressions that depend only on \((x^h)\) (which is an invariant property) and if \([P, P] = 0, d\theta = 0\), then \( \text{graph}(\mathcal{P}|_{T* \mathcal{F}_1}) \), \( \text{graph}((\theta)|_{T \mathcal{F}_1}) \) are integrable, big-isotropic structures of \( M \) of the kind described above.
Example 2.3. For any pair of vector subbundles $F \subseteq F' \subseteq TM$, $E = F \oplus \text{ann } F'$ is a big-isotropic structure on $M$ with the $g$-orthogonal bundle $E' = F' \oplus \text{ann } F$. Furthermore, $E$ is integrable iff $F$ is tangent to a foliation and $\Gamma F'$ is invariant by Lie brackets with cross sections of $F$; this means that $F'$ is a projectable distribution with respect to the foliation $F$, i.e., $F'$ is projection-related with distributions of the local spaces of leaves [12].

Example 2.4. Let $S$ be a rank $k$ subbundle of $TM$ and $\theta \in \Omega^2(M)$ a differential 2-form. Then,

$$(2.3) \quad E_\theta = \text{graph}(\flat_\theta|_S) = \{ (X, \flat_\theta X = i(X)\theta) / X \in S \} \subseteq T^\mathrm{big}M$$

is a big-isotropic structure on $M$ with the $g$-orthogonal bundle

$$(2.4) \quad E'_\theta = \{ (Y, \flat_\theta Y + \gamma) / Y \in TM, \gamma \in \text{ann } S \}.$$ 

For the integrability conditions, we compute the Courant bracket of two cross sections of $E_\theta$. With the notation of (2.3) and with $X, Y \in S$, we get

$$(2.5) \quad [(X, \flat_\theta X), (Y, \flat_\theta Y)] = ([X, Y], (L_X i(Y) - L_Y i(X))\theta + d(\theta(X, Y))) = ([X, Y], i([X, Y])\theta + i(X \wedge Y)d\theta).$$

The final result of (2.5) is in $E_\theta$ iff $S$ is involutive and

$$(2.6) \quad d\theta(X, Y, Z) = 0, \quad \forall X, Y \in \Gamma S, \forall Z \in \chi^1(M).$$

Therefore, $E_\theta$ is integrable iff $S$ is a foliation and $\theta$ satisfies (2.6) (in particular, if $\theta$ is closed).

Example 2.5. Let $S^*$ be a subbundle of rank $k$ of $T^*M$ and $P \in \chi^2(M)$ a differentiable bivector field on $M$. Then

$$(2.7) \quad E_P = \text{graph}(\sharp_P|_{S^*}) = \{ (\sharp_P\sigma = i(\sigma)P, \sigma) / \sigma \in S^* \}$$

is a big-isotropic structure on $M$ with the $g$-orthogonal bundle

$$(2.8) \quad E'_P = \{ (\sharp_P \beta + Y, \beta) / \beta \in T^*M, Y \in \text{ann } S^* \}.$$ 

For the integrability conditions we recall that $P$ defines the bracket of 1-forms:

$$(2.9) \quad \{ \alpha, \beta \}_P = L_{\sharp_P \alpha} \beta - L_{\sharp_P \beta} \alpha - d(P(\alpha, \beta)).$$
which is related to the Schouten-Nijenhuis bracket $[P, P]$ by the Gelfand-Dorfman formula \[5\]

\[
P(\{\alpha, \beta\}_P, \gamma) = \gamma([\sharp_P \alpha, \sharp_P \beta]) + \frac{1}{2}[P, P](\alpha, \beta, \gamma).
\]

With (2.9) and (2.10) we get the Courant bracket

\[
[[\sharp_P \sigma, \sigma], [\sharp_P \tau, \tau]] = ([\sharp_P \{\sigma, \tau\}_P - \frac{1}{2}i(\sigma \wedge \beta)[P, \{\sigma, \beta\}_P], \{\sigma, \beta\}_P),
\]

where $\sigma, \tau \in S^*$. The result of (2.11) is in $\Gamma E_P$ iff $S^*$ is closed by the bracket

\[\tag{2.9}\]

\[\tag{2.12}\]

$[P, P](\sigma_1, \sigma_2, \beta) = 0, \forall \sigma_1, \sigma_2 \in S^*, \forall \beta \in T^*M$.

Therefore, $E_P$ is integrable iff $S^*$ is closed by the $P$-brackets \[2.9\] and $P$ satisfies condition \[2.12\] (in particular, $P$ is a Poisson bivector field).

**Example 2.6.** The construction indicated in \[13\] for the lift of a Dirac structure of a manifold $M$ to its tangent manifold $TM$ may also be used for a big-isotropic structure $E, E'$. More exactly, if we look at the locally free sheaves $E, E'$ of germs of cross sections of $E, E'$, the formulas

\[
thg(E) = \text{span}\{(X^C, \alpha^C), (X^V, \alpha^V) / (X, \alpha) \in E\},
\]

\[
thg(E') = \text{span}\{(X^C, \alpha^C), (X^V, \alpha^V) / (X, \alpha) \in E'\},
\]

where $C, V$ denote the complete and vertical lift, respectively, define locally free sheaves of germs of cross sections of orthogonal subbundles $\nthg(E), \nthg(E') \subseteq T^{\text{big}}(TM)$. The formulas for scalar products and Courant brackets of lifts established in \[13\] show that $\nthg(E)$ is a big-isotropic structure on $TM$ with the orthogonal bundle $\nthg(E')$ and that, if $E$ is integrable, $\nthg(E)$ is integrable too.

Now, we shall look for objects of $TM$ that are equivalent with a big-isotropic structure $E$.

For the algebraic aspects, we refer to a fixed point $x \in M$ and we associate with $E$ the vector spaces

\[
E_x = \text{pr}_{T_xM} \mathcal{E}_x \subseteq E'_x = \text{pr}_{T_xM} \mathcal{E}'_x.
\]

Then, we define a bilinear mapping $\varpi_x: \mathcal{E}_x \times \mathcal{E}'_x \to \mathbb{R}$ by means of the formula

\[
\varpi_x(X, Y) = \omega((X, \alpha), (Y, \beta)) = \alpha(Y) = -\beta(X),
\]
where \((X,\alpha) \in E_x, (Y,\beta) \in E'_x\). The last equalities hold and the result is independent on the choice of \(\alpha,\beta\) because \((X,\alpha) \perp_{g} (Y,\beta)\). Of course, \(\varpi|_{E_x \times E_x}\) is skew-symmetric. Notice that \(\varpi_x\) may be identified with a mapping \(\flat_{\varpi_x} : E_x \to \mathcal{E}^*_x M\), which sends \(X\) to \(i(X)\varpi\) (with an obvious notation) and it is easy to see that \(\ker \flat_{\varpi_x} = T_x M \cap E_x\). This implies that, if \(\varpi\) is non degenerate (i.e., \(\ker \flat_{\varpi_x} = 0\)) \(E_x\) is the graph of a mapping \(\text{pr}_{T_x M} E_x \to T_x M\). Similarly, if \(E_x\) has the property \(T_x^* M \cap E_x = 0\) then \(E_x\) is the graph of a mapping \(\mathcal{E}_x \to T_x^* M\). If we are in one (and the same) of the two cases above \(\forall x \in M\), we will say that \(E\) is of the graph type.

**Example 2.7.** In Example [2.2] we have \(\mathcal{E} = \text{pr}_{TM} E, \mathcal{E}' = \mathcal{E} \oplus TF_2\) and \(\varpi\) is the extension of the 2-form of the almost Dirac structure \(E\) along \(F_1\) by the value 0 for second arguments in \(TF_2\). In Example [2.3] \(\mathcal{E} = F, \mathcal{E}' = F'\) and \(\varpi(X,Y) = 0\). In Example [2.4] \(\mathcal{E} = S, \mathcal{E}' = TM, \varpi = \theta|_{S \times TM}\) and in Example [2.5]

\[
\mathcal{E} = \text{im}(\sharp_p|S^*), \mathcal{E}' = \mathcal{E} + \text{ann} S^*, \varpi(\sharp_p \sigma, \sharp_p \beta + Y) = -P(\sigma, \beta)
\]

where \(\sigma \in S^*, Y \in \text{ann} S^*\). In the particular case of Example [2.3] where \(P\) is the Lie-Poisson bivector field of the Lie coalgebra \(G^*\) of the connected Lie group \(G\) (e.g., see \[11\]) and \(S^* = G'\) is the Lie subalgebra of the connected subgroup \(G' \subseteq G\), \(\mathcal{E}\) are the tangent spaces of the orbits of the coadjoint action of \(G'\) on \(G^*\).

**Proposition 2.2.** For any pair of planes \(\mathcal{E}_x \subseteq \mathcal{E}'_x \subseteq T_x M\) and any bilinear mapping \(\varpi_x : \mathcal{E}_x \times \mathcal{E}'_x \to \mathbb{R}\) with a skew-symmetric restriction to \(\mathcal{E}_x \times \mathcal{E}_x\) there exists a big-isotropic plane \(E_x \subseteq T_x^{\text{big}} M\) such that \((2.14), (2.15)\) are the given planes and mapping.

**Proof.** For the given planes and mapping, put

\[
E_x = \{(X,\alpha) / X \in \mathcal{E}_x \& \forall Y \in \mathcal{E}'_x, \alpha(Y) = \varpi_x(X,Y)\},
\]

\[
E'_x = \{(Y,\beta) / Y \in \mathcal{E}'_x \& \forall X \in \mathcal{E}_x, \beta(X) = -\varpi_x(X,Y)\}.
\]

Obviously, covectors \(\alpha, \beta\) as required by \((2.16)\) exist, hence, the projection on the first term of a pair defines epimorphisms \(E'_x \to E_x, E'_x \to E_x\) with the kernels \(E_x \cap T_x^* M = \text{ann} \mathcal{E}'_x, E'_x \cap T_x^* M = \text{ann} \mathcal{E}_x\), respectively. Accordingly, one has the exact sequences

\[
0 \to \text{ann} \mathcal{E}'_x \to E_x \to \mathcal{E}_x \to 0, \quad 0 \to \text{ann} \mathcal{E}_x \to E'_x \to \mathcal{E}'_x \to 0,
\]
and we get

\[(2.18) \quad \dim E_x = \dim \mathcal{E}_x + \dim \text{ann} \mathcal{E}_x', \quad \dim E'_x = \dim \mathcal{E}'_x + \dim \text{ann} \mathcal{E}_x,\]

which implies \(\dim E_x + \dim E'_x = 2m\). Thus, \(E'_x, E_x \perp g\) have the same dimension and, since by \[(2.16)\] \(E'_x \perp g E_x\), we get \(E'_x = E_x \perp g\). Furthermore, the skew-symmetry of \(\varpi\) on \(\mathcal{E}_x \times \mathcal{E}_x\) implies \(E_x \subseteq E'_x\), hence, \(E_x\) is isotropic. The fact that the given planes and mapping are associated with \(E_x, E'_x\) of \[(2.16)\] via \[(2.14), (2.15)\] is obvious.

Now, starting with the big-isotropic structure \(E\), let \(x\) vary on \(M\). Since \(E, E'\) are differentiable vector bundles, the generalized, distributions \(\mathcal{E}, \mathcal{E}'\) defined by the spaces \[(2.14)\] are differentiable. If \(\mathcal{E}\) is a regular distribution (i.e., \(\dim \mathcal{E}_x = \text{const.}\), therefore, by \[(2.18)\], \(\dim \mathcal{E}'_x = \text{const.}\) as well), the structure \(E\) will be called regular. If \(E\) is integrable then \((E, pr_{TM}, [\cdot, \cdot])\) is a Lie algebroid and, accordingly, \(\mathcal{E}\) is a generalized foliation. Proposition \[2.1\] implies that if \(X \in \Gamma(\mathcal{E}), Y \in \Gamma(\mathcal{E}')\) then \([X, Y] \in \Gamma(\mathcal{E}')\).

It is worth formalizing the status of \(E'\) as follows since this may be useful in a discussion of the integrability of \(E\) to a Lie groupoid. Let \(B \to M\) be a vector bundle endowed with an anchor (morphism) \(\rho : B \to TM\). Assume that there exists a vector subbundle \(A \subseteq B\) endowed with a Lie algebroid structure \((A, \rho|_A, [\cdot, \cdot]|_A)\) and there exists an \(\mathbb{R}\)-bilinear operation \([\cdot, \cdot] : \Gamma A \times \Gamma B \to \Gamma B\) that reduces to \([\cdot, \cdot]|_A\) for arguments in \(\Gamma A\). Then, \((B, \rho, [\cdot, \cdot])\) will be called a modular enlargement of the Lie algebroid \(A\) if, \(\forall f, h \in C^\infty(M), a \in \Gamma A, b \in \Gamma B\), the following conditions are satisfied:

1) \(\rho[a, b] = [\rho a, \rho b]\),

2) \([fa, hb] = fh[a, b] + f((\rho a)h)b - h((\rho f)a),\)

3) \([a_1, [a_2, b]] = [[a_1, a_2], b] + [a_2, [a_1, b]]\)

(the right hand side of 1) is a Lie bracket of vector fields).

With this terminology, if \(E\) is an integrable, big-isotropic structure then \(E'\) with \(\rho = pr_{TM}\) and the Courant bracket is a modular enlargement of the Lie algebroid \(E\).

We also give the following definition:
**Definition 2.2.** The generalized foliation $\mathcal{E}$ is the *characteristic foliation* of the integrable, big-isotropic structure $E$ and its leaves are the *characteristic leaves*. The generalized distribution $\mathcal{E}'$ is the *characteristic module* of $E$.

**Example 2.8.** We can extend the construction of Dirac structures from Lie algebroids \cite{[I]} as follows. Let $A \to M$ be a Lie algebroid of anchor $\rho_A : A \to TM$ and bracket $[,]_A$ and $(B, \rho, [,])$ a modular enlargement of $A$. Assume that one also has a *co-anchor* $\sigma : B \to T^*M$ such that the following properties are satisfied for all $a \in \Gamma A, b \in \Gamma B$:

1) $<\sigma a, \rho b> = - <\sigma b, \rho a>,$

2) $\sigma[a, b] = L_{\rho a}(\sigma b) - L_{\rho b}(\sigma a) + d<\sigma a, \rho b>.$

Furthermore, assume that, $\forall x \in M$, the morphism $(\rho, \sigma) : B \to T^{big}M$ satisfies the condition

$$\text{rank}(\rho, \sigma)|_{A_x} + \text{rank}(\rho, \sigma)|_{B_x} = 2m.$$ 

Then, it is easy to check that

$$E_\sigma = \{(\rho a, \sigma a) / \forall a \in A\}$$

is an integrable, big-isotropic, structure on $M$ with the orthogonal bundle

$$E'_\sigma = \{(\rho b, \sigma b) / \forall b \in B\}.$$ 

The characteristic foliation of $E_\sigma$ is $\rho(A)$ and the characteristic module is $\rho(B)$. In fact, any integrable, big-isotropic, structure $E$ is of this kind, where $A = E$, $B = E'$, the brackets are Courant brackets, the anchor is the projection on $TM$ and the co-anchor is the projection on $T^*M$.

Concerning the mapping $\varpi$, we notice that it has the following differentiability property: for any characteristic leaf $S$ of $E$ and for any differentiable vector fields $X, Y \in \chi^1(M)$, such that $X|_S \in \Gamma \mathcal{E}|_S, Y|_S \in \Gamma \mathcal{E}'|_S$, $\varpi(X,Y)$ is a differentiable function on $S$. Indeed, since $\mathcal{E}|_S$, hence $\mathcal{E}'|_S$ too, has a constant dimension, \cite[(2.17)]{[I]} produces exact sequences of differentiable, vector bundles over $S$. Using differentiable splittings of these sequences, we see that there are 1-forms $\alpha, \beta \in T^*_S M$, which are differentiable along $S$, such that $(X, \alpha) \in \Gamma E|_S, (Y, \beta) \in \Gamma E'|_S$. Accordingly, \cite[(2.15)]{[I]} shows that $\varpi(X,Y)$
is a differentiable function on $S$. This property will be called leaf-wise differentiability. If the structure $E$ is regular, the exact sequences (2.17) have differentiable splittings over $M$, and the functions $\varpi(X,Y)$ are differentiable on the whole manifold $M$.

A multilinear mapping

$$(2.19) \quad \lambda_x : \wedge^{s-1}E_x \otimes E'_x \rightarrow \mathbb{R},$$

which is defined $\forall x \in M$, has a totally skew-symmetric restriction to $\wedge^s E_x$, and is leaf-wise differentiable with respect to the characteristic foliation of $E$, will be called a truncated $s$-form on $(M, E)$. We will denote by $\Omega^s_{tr}(M, E)$ the space of truncated $s$-forms. Because of the integrability conditions of $E$, the usual formula for the evaluation of the exterior differential of a differential form makes sense for truncated $s$-forms on $(M, E)$ and for arguments in $\chi^1(M)$ that belong to $E$. Moreover, if $E$ is a regular structure, the exterior differential also makes sense if the last argument belongs to $E'$, while the other arguments belong to $E$. Whenever it makes sense, we will denote the differential of a truncated form by $d_{tr}$. In the regular case, $d_{tr}$ is a coboundary morphism

$$(2.20) \quad d_{tr} : \Omega^s_{tr}(M, E) \rightarrow \Omega^{s+1}_{tr}(M, E), \quad d^2_{tr} = 0$$

and the cohomology spaces $H^s_{tr}(M, E)$ of the cochain complex $(\Omega^s_{tr}(M, E), d_{tr})$ will be the truncated, de Rham cohomology spaces of $(M, E)$.

**Proposition 2.3.** For any integrable, big-isotropic structure $E$ one has

$$(2.21) \quad d_{tr}\varpi(X_1, X_2, X_3) = 0, \quad \forall X_a \in E, \ a = 1, 2, 3.$$

If either $E$ is an almost Dirac structure or $E$ is a regular big-isotropic structure, $E$ is integrable iff $E$ is involutive, the corresponding distribution $E'$ is invariant by Lie brackets with vector fields of $E$ and (2.21) with $X_3$ replaced by $Y \in E'$ holds.

**Proof.** For almost Dirac structures the result is known [4]. We prove the result for regular, big-isotropic structures and we will get the first assertion on the way. It was already shown that $E$ and $E'$ satisfy the required conditions for any integrable, big-isotropic structure. Now, let $X_1, X_2 \in E, Y \in E'$ be differentiable vector fields on $M$ and $\alpha_1, \alpha_2, \beta$ differentiable 1-forms such that $(X_1, \alpha_1), (X_2, \alpha_2) \in E, (Y, \beta) \in E'$. (The existence of $\alpha, \beta$ is ensured by the
regularity hypothesis.) Then, keeping in mind the $g$-orthogonality relations among these pairs and using (2.15), we get

$$(d_{tr} \omega)(X_1, X_2, Y) = (d_C \omega)((X_1, \alpha_1), (X_2, \alpha_2), (Y, \beta))$$

$$= X_1(\alpha_2(Y)) - X_2(\alpha_1(Y)) + Y(\alpha_1(X_2)) + \alpha_1([X_2, Y]) - \alpha_2([X_1, Y])$$

$$- <L_{X_1} \alpha_2 - L_{X_2} \alpha_1 - d(\alpha_1(X_2)), Y> = 0,$$

where $d_C$ is the operator defined by the usual expression of the exterior differential of a Lie algebroid with Courant brackets instead of Lie brackets.

The previous calculation makes sense and remains true in the non regular case if we also assume $Y \in \mathcal{E}$. This proves the first assertion of the proposition.

Back to the regular case, if we start with a big-isotropic structure $E$ which satisfies the required hypotheses then, for arguments as above, we get

$$d_{tr} \omega(X_1, X_2, Y) = 2g([[(X_1, \alpha_1), (X_2, \alpha_2)], (Y, \beta)])$$

$$= 2g([[(X_2, \alpha_2), (Y, \beta)], (X_1, \alpha_1))].$$

Thus, if $\omega$ is $d_{tr}$-closed, $E$ is closed by Courant brackets and $E'$ is closed by Courant brackets with cross sections of $E$.

For the non regular case, we do not get the converse result; from condition (2.21) it only follows that $[\Gamma E, \Gamma E] \in \Gamma E'$.

**Corollary 2.1.** For any big-isotropic structure $E$ and for each point $x \in M$, there exists a canonical extension of $E_x$ to an almost Dirac space $D_x(E) \subseteq T_x^{big} M$. If differentiable with respect to $x$, these spaces define a canonical almost Dirac extension $D(E)$ of $E$ and if $E$ is integrable so is $D(E)$.

**Proof.** The restriction of $\omega$ to $\mathcal{E} \times \mathcal{E}$ is a leaf-wise differentiable 2-form and we may use (1.4) and define

$$D_x(E) = \{(X, \alpha) / X \in \mathcal{E}_x & \forall Y \in \mathcal{E}_x, \alpha(Y) = \omega_x(X, Y)\}.$$ 

From (2.16), it follows that $E_x \subseteq D_x(E)$ and that we have

$$D_x(E) = \{(X, \alpha) + (0, \gamma) / (X, \alpha) \in E_x, \gamma \in \text{ann } \mathcal{E}_x\} = E_x + \text{ann } \mathcal{E}_x.$$ 

Since $E_x \cap \text{ann } \mathcal{E}_x = \text{ann } \mathcal{E}_x'$, (2.18) and (2.23) show that $\text{dim } D_x(E) = m$. Notice that $D_x(E) \subseteq E_x'$. If $D(E)$ is differentiable and $E$ is integrable, $\mathcal{E}$ is a generalized foliation and $D(E)$ is integrable because of (2.21).
For instance, in the case of Example 2.3, $D(E) = F \oplus \text{ann } F$ and, in the case of Example 2.2, we get $D(E) = E \oplus T^*F$.

**Example 2.9.** Let $M$ be a manifold endowed with a regular, involutive, $k$-dimensional subbundle $E \subseteq TM$ and with a $d_r$-closed, truncated 2-form $\omega$ of the pair $(\mathcal{E}, \mathcal{E}') = TM$. Then, the hypotheses of the regular case of Proposition 2.3 are satisfied and via (2.16), we get the corresponding integrable, big-isotropic structure $E_\omega = \text{graph } b_\omega \subseteq T^\text{big}M$ of rank $k$. Let $\tilde{\mathcal{E}}$ be a complementary subbundle of $\mathcal{E}$ ($TM = \mathcal{E} \oplus \tilde{\mathcal{E}}$) and denote by a prime and a double prime the projections of a vector on $\mathcal{E}$, $\tilde{\mathcal{E}}$, respectively. Then, we can extend $\omega$ to a 2-form $\theta \in \Omega^2(M)$ by the formula

$$\theta(Y_1, Y_2) = \omega(Y'_1, Y'_2) + \omega(Y'_1, Y''_2) - \omega(Y'_2, Y''_1),$$

and we get $E_\omega = E_\theta$, where $E_\theta$ was defined in Example 2.4. We would also like to comment on the following particular case. If $\omega|_{\mathcal{E} \times \mathcal{E}}$ is non degenerate on each leaf of $\mathcal{E}$ we will say that the structure $\text{graph } b_\omega$ is non degenerate and there exists a regular Poisson bivector field $\Pi \in \chi^2(M)$ (Schouten-Nijenhuis bracket $[\Pi, \Pi] = 0$) with the symplectic foliation $\mathcal{E}$. Furthermore, since the non-degeneracy of $\omega|_{\mathcal{E} \times \mathcal{E}}$ implies that $\ker b_\omega = 0$ and $\text{im } b_\omega \cap \text{ann } \mathcal{E} = 0$ where $b_\omega : \mathcal{E} \to \mathcal{E}' = T^*M$, we deduce that $T^*M = \text{im } b_\omega \oplus \text{ann } \mathcal{E}$, therefore, $\omega$ also defines a normal bundle $Q$ of the foliation $\mathcal{E}$ which may be seen as $Q = \mathcal{E}^\perp$. Conversely, if a regular Poisson structure $\Pi$ with symplectic foliation $\mathcal{E}$ and a normal bundle $Q$ of $\mathcal{E}$ are given, we may extend the leaf-wise symplectic form of $\Pi$ by the value zero on $Q$ to a closed, truncated form $\omega$ and we will have the corresponding, integrable, big-isotropic structure $\text{graph } b_\omega$. Thus, a non degenerate structure $\text{graph } b_\omega$ is equivalent with a regular Poisson structure together with a normal bundle of its symplectic foliation. More exactly, if the Poisson bivector field is $P$ and the normal bundle is $Q$, one has $E = \{(\sharp P \lambda, \lambda) / \lambda \in \text{ann } Q\}$ and $E' = E \oplus (Q \oplus Q^*)$.

We finish this section by indicating that, like a Dirac structure [4], an integrable, big-isotropic structure $(E, E')$ on $M$ allows for a partial Hamiltonian formalism as follows. A function $f \in C^\infty(M)$ will be called a Hamiltonian function if there exists a vector field $X_f \in \chi^1(M)$ such that $(X_f, df) \in \Gamma E$. Similarly, if $(X_f, df) \in \Gamma E'$ is a weak-Hamiltonian function. We will denote by $C^\infty_{\text{Ham}}(M)$ the set of Hamiltonian functions and by $C^\infty_{\text{wHam}}(M)$ the set of weak Hamiltonian functions. The vector field $X_f$ is a Hamiltonian, respectively weak-Hamiltonian vector field of $f$ and any $Z \in \chi^1(M)$ which is
a (weak-)Hamiltonian vector field for some $f$ is a (weak-)Hamiltonian vector field. The fields $X^f_1, X^f_2$ are Hamiltonian vector fields of the same function $f$ iff $X^f_2 - X^f_1 \in \text{ann} \, pr_{T^*M} E'$ and are weak-Hamiltonian vector fields of $f$ iff $X^f_2 - X^f_1 \in \text{ann} \, pr_{T^*M} E$. Similarly, $Z$ is Hamiltonian (weak-Hamiltonian) for two functions $f_1, f_2$ iff $df_2 - df_1 \in \text{ann} \, E'$ (respectively, $df_2 - df_1 \in \text{ann} \, \mathcal{E}$).

We will denote by $\chi_{\text{Ham}}(M), \chi_{\text{wHam}}$, respectively, the set of Hamiltonian and weak-Hamiltonian vector fields.

Furthermore, if $f \in C^\infty_{\text{Ham}}(M)$ and $h \in C^\infty_{\text{wHam}}(M)$ the following bracket is well defined

\begin{equation}
\{f, h\} = X_f h = -\varpi(X_f, X_h) = -X_h f
\end{equation}

and does not depend on the choice of the Hamiltonian vector fields of the functions $f, h$. The bracket (2.24) will be called the Poisson bracket of the two functions. Formula (1.3) shows that $\{f, h\} \in C^\infty_{\text{wHam}}(M)$ and one of its weak-Hamiltonian vector fields is $[X_f, X_h]$. Moreover, if both $f, h \in C^\infty_{\text{Ham}}(M)$, their Poisson bracket is skew symmetric and belongs to $C^\infty_{\text{Ham}}(M)$. Now, if we notice that $d_{tr} \varpi(X_f, X_h, X_l)$ makes sense $\forall f, h \in C^\infty_{\text{Ham}}(M), \forall l \in C^\infty_{\text{wHam}}(M)$ (since the functions $\varpi(X_f, X_l)$, etc. are differentiable), the computation done during the proof of Proposition 2.3 now yields $d_{tr} \varpi(X_f, X_h, X_l) = 0$. This is easily seen to be equivalent with the Leibniz property

\begin{equation}
\{f, \{h, l\}\} = \{\{f, h\}, l\} + \{h, \{f, l\}\},
\end{equation}

which restricts to the Jacobi identity on $C^\infty_{\text{Ham}}(M)$. Therefore, $C^\infty_{\text{Ham}}(M)$ with the Poisson bracket is a Lie algebra and $C^\infty_{\text{wHam}}(M)$ is a module over this Lie algebra. Using the Poisson bracket (2.24) it also follows easily that $\chi_{\text{Ham}}(M)$ is a Lie subalgebra of $\chi^1(M)$ and $\chi_{\text{wHam}}(M)$ is a module over the former.

## 3 Canonical local bases

We will discuss local properties of a $k$-dimensional, big-isotropic structure $E$ by constructing canonical, local bases in the neighborhood of a fixed point $x_0 \in M$, as constructed by Dufour and Wade in the Dirac case [6]. In what follows the notation is the same as in Section 2.

We begin with vectors $X^0_a, Y^0_a \in T_{x_0}M$ where $X^0_a, a = 1, \ldots, \dim \mathcal{E}_{x_0}$, is a basis of $\mathcal{E}_{x_0}$ and $Y^0_h, h = 1, \ldots, \dim \mathcal{E}'_{x_0} = \dim \mathcal{E}_{x_0} - \dim \mathcal{E}_{x_0}$ (2.18) $m - k$, is a
basis of a complement of $\mathcal{E}_{x_0}$ in $\mathcal{E}'_{x_0}$. Then, there exist covariant vectors $\xi_0^h, \eta_0^h \in T^*_{x_0} M$ such that $(X_0^0, \xi_0^a)$ are linearly independent elements of $E_{x_0}$ and $(X_0^0, \xi_0^a), (Y_0^0, \eta_0^h)$ is a basis of a complement of $\ker(E'_{x_0} \to \mathcal{E}'_{x_0})$. Since $\ker(E'_{x_0} \to \mathcal{E}'_{x_0}) = \text{ann } \mathcal{E}_{x_0}$, if we add a basis $(0, \xi_0^a), (s = 1, \ldots, \text{dim } \text{ann } \mathcal{E}_{x_0})$ of $\text{ann } \mathcal{E}_{x_0}$ we get a basis of $E'_{x_0}$. Moreover, since $\text{ann } \mathcal{E}_{x_0} \supseteq \text{ann } \mathcal{E}'_{x_0}$, we may ask the basis $(0, \xi_0^a)$ to consist of elements $(0, \kappa_0^u), (0, \nu_0^v), u = 1, \ldots, \text{dim } \text{ann } \mathcal{E}'_{x_0}$, $q = 1, \ldots, \text{dim } \text{ann } \mathcal{E}_{x_0} - \text{dim } \text{ann } \mathcal{E}'_{x_0} \overset{\text{2.13}}{=} m - k$, where $(0, \kappa_0^u)$ is a basis of $\text{ann } \mathcal{E}'_{x_0}$. Then, since $\ker(E_{x_0} \to \mathcal{E}_{x_0}) = \text{ann } \mathcal{E}'_{x_0}$, $(X_0^0, \xi_0^a), (0, \kappa_0^u)$ is a basis of $E_{x_0}$. Furthermore, we shall need vectors $Z_0^\sigma \in T_{x_0} M, \sigma = 1, \ldots, \text{dim } M - \text{dim } E_{x_0}' = \text{dim } \text{ann } \mathcal{E}_{x_0}'$, which are a basis of a complement of $\mathcal{E}_{x_0}'$ in $T_{x_0} M$. Notice the important fact that the indices $h, q$, on one side, and $u, \sigma$, on the other side, have the same range.

Now we shall extend the basis $B_0 = \{(X_0^0, \xi_0^a), (0, \kappa_0^u), (Y_0^0, \eta_0^h), (0, \nu_0^v)\}$ to a basis of cross sections of $E, E'$ over a neighborhood $U$ of $x_0$ in $M$; we will allow $U$ to undergo as many restrictions as needed for the correctness of the various constructions below without changing its name.

Clearly, we may assume that there exists a basis of $T_U M$ that consists of local vector fields $(X_\sigma, Y_\sigma, Z_\sigma)$ with the values $(X_0^0, Y_0^0, Z_0^\sigma)$ at $x_0$ and we denote by $(\theta^a, \phi^h, \psi^\sigma)$ the corresponding, dual, local basis of $T^*_U M$, i.e.,

\[
\theta^a(X_h) = \delta^a_b, \theta^a(Y_h) = 0, \theta^a(Z_\sigma) = 0, \\
\phi^h(X_h) = 0, \phi^h(Y_l) = \delta^l_h, \phi^h(Z_\sigma) = 0, \\
\psi^\sigma(X_h) = 0, \psi^\sigma(Y_k) = 0, \psi^\sigma(Z_\tau) = \delta^\tau_\sigma.
\]

Accordingly, an extension of the basis $B_0$ to a basis of $(E, E')$ over $U$ has an expression of the form

\[
\begin{align*}
\mathcal{X}_a &= (A^b_a X_b + A^h_a Y_h + A^\sigma_a Z_\sigma, \alpha_b^a \theta^b + \alpha_h^a \phi^h + \alpha^\sigma_a \psi^\sigma), \\
\Xi_a &= (B^b_a X_a + B^h_a Y_h + B^\sigma_a Z_\sigma, \beta_b^a \theta^b + \beta_h^a \phi^h + \beta^\sigma_a \psi^\sigma), \\
\Upsilon_h &= (C^b_h X_a + C^h_h Y_h + C^\sigma_h Z_\sigma, \gamma^b_h \theta^b + \gamma^h_h \phi^h + \gamma^\sigma_h \psi^\sigma), \\
\Theta_q &= (L^a_q X_a + L^h_q Y_h + L^\sigma_q Z_\sigma, \lambda^a_q \theta^a + \lambda^h_q \phi^h + \lambda^\sigma_q \psi^\sigma),
\end{align*}
\]

where we use the Einstein summation convention, $\mathcal{X}_a, \Xi_a$ is a basis of $E|_U$, and $\Upsilon_h, \Theta_q$ are dual bases.
\[ Y_h, \Theta_q \text{ completes the former to a basis of } E'|_U \text{ and} \]
\[ A^b_\alpha(x_0) = \delta^b_\alpha, \ A^h_\alpha(x_0) = 0, \ A^{\nu}_{\alpha}(x_0) = 0, \]
\[ B^0_u(x_0) = 0, B^h_u(x_0) = 0, B^{\nu}_{u}(x_0) = 0, \]
\[ C^0_h(x_0) = 0, C^l_h(x_0) = \delta^l_h, C^{\nu}_{h}(x_0) = 0, \]
\[ L^0_q(x_0) = 0, L^h_q(x_0) = 0, L^{\nu}_{q}(x_0) = 0. \]

\[ (3.2) \]

We shall change this basis in order to simplify the expressions \[ (3.1) \]. However, we keep denoting the elements of the new bases by the same letters as in \[ (3.1) \]. Firstly, in view of \[ (3.2) \], we may assume that the matrix \( (A^b_\alpha) \) is non degenerate on \( U \) and change the vectors \( X_a \) by the matrix \( (A^b_\alpha)^{-1} \). As a result we get a basis \[ (3.1) \] where \( A^b_\alpha = \delta^b_\alpha \). Similarly, we may get \( C^l_h = \delta^l_h \).

Then, the new basis may be changed by \( \Xi_a \mapsto \Xi_a - B^a_u X_a \) and get a new basis \[ (3.1) \] where \( B^a_u = 0 \). Similarly, we may get \( C^a_h = 0, L^a_q = 0 \), then, with the change \( \Theta_q \mapsto \Theta_q - L^h_q Y_h \), also get \( L^h_q = 0 \).

Thus, any big-isotropic structure \( (E, E') \) has local bases of the form

\[ \mathcal{X}_a = (X_a + A^h_a Y_h + A^{\nu}_{a} Z_\sigma, \alpha^h_a \theta^b + \alpha^{\nu}_{a} \phi^h + \alpha^\nu_{a} \psi^\nu), \]
\[ \Xi_a = (B^h_a Y_h + B^{\nu}_{a} Z_\sigma, \beta^h_a \theta^a + \beta^{\nu}_{a} \phi^h + \beta^\nu_{a} \psi^\nu), \]
\[ Y_h = (Y_h + C^h_\nu Z_\sigma, \gamma^h_\nu \theta^a + \gamma^{\nu}_{h} \phi^a + \gamma^\nu_{h} \psi^\nu), \]
\[ \Theta_q = (L^\nu_q Z_\sigma, \lambda^\nu_{q} \theta^a + \lambda^{h}_{q} \phi^h + \lambda^h_{q} \psi^h), \]

where \( (\mathcal{X}_a, \Xi_a) \) is a basis of \( E \) and \[ (3.2) \] holds.

Furthermore, the following \( g \)-orthogonality conditions must be satisfied:

\[ (3.4) \quad \mathcal{X}_a \perp_g \mathcal{X}_b, \mathcal{X}_a \perp_g \Xi_u, \mathcal{X}_a \perp_g \Xi_v, \mathcal{X}_a \perp_g Y_h, \]
\[ \Xi_u \perp_g Y_h, \mathcal{X}_a \perp_g \Theta_q, \Xi_u \perp_g \Theta_q. \]

In particular, the second, fifth and sixth conditions \[ (3.1) \], taken at \( x_0 \) give

\[ \beta^u_\alpha(x_0) = 0, \beta^h_\alpha(x_0) = 0, \lambda^\nu_{q}(x_0) = 0. \]

This implies that the 1-forms \( \beta^u_\alpha \psi^\nu \), on one hand, and \( \lambda^h_\nu \phi^h + \lambda^\nu_{q} \psi^\nu \), on the other hand are linearly independent at \( x_0 \) and on a neighborhood \( U \) of \( x_0 \), which may be used for further simplifications of the basis: i) we may linearly change \( \Xi_u \) by the inverse of the matrix \( (\beta^u_\alpha) \) and get a new basis \[ (3.3) \] where
\[\beta^\nu \sigma = \delta^\nu \sigma, \text{ ii) after change i), subtract } \sum \alpha^\nu \sigma \Xi^\sigma, \sum \gamma^\nu \sigma \Xi^\sigma, \sum \lambda^\nu \sigma \Xi^\sigma \text{ from } \mathcal{X}_a, \mathcal{Y}_h, \Theta_q, \text{ respectively, and get rid of the terms with } \psi^\sigma \text{ in } \mathcal{X}_a, \mathcal{Y}_h, \Theta_q, \text{ iii) change ii) reaches a situation where the forms } \lambda^\nu \sigma \phi^h \text{ are independent and we will change the } \Theta_q \text{ by the inverse of the matrix } (\lambda^\nu \sigma) \text{ and obtain } \lambda^\nu \sigma = \delta^\nu \sigma \text{ for the new basis, iv) subtract } \sum \gamma^\nu \sigma \Theta_i \text{ from } \mathcal{Y}_h \text{ and get } \gamma^\nu \sigma = 0 \text{ in the new basis, v) since change ii) alters the coefficients of } \mathcal{Y}_h \text{ in } \mathcal{Y}_h \text{ and adds terms in } \mathcal{Y}_h \text{ to } \Theta_q, \text{ we correct that by changing the new } \mathcal{Y}_h \text{ with the corresponding coefficient matrix (which is non degenerate on } \mathcal{U} \text{) and by subtracting the necessary linear combination of } \mathcal{Y}_h \text{ from } \Theta_q. \text{ The result is a local basis of } (E, E') \text{ that looks as follows}

\[
\begin{align*}
\mathcal{X}_a &= (X_a + A^h_a Y_h + A^\nu_a Z_\sigma, \alpha^\nu_h \theta^h + \alpha^\nu_a \phi^h), \\
\Xi_a &= (B^h_a Y_h + B^\nu_a Z_\sigma, \beta^\nu_a \theta^a + \beta^\nu_a \phi^h + \psi^a), \\
\mathcal{Y}_h &= (Y_h + C^\nu_a Z_\sigma, \gamma^\nu_a \theta^a), \\
\Theta_q &= (L^\nu_a Z_\sigma, \lambda^\nu_a \theta^a + \phi^q),
\end{align*}
\]

where (3.2) are still valid.

It is easy to see that, if the vector fields \(X_a, Y_h, Z_\sigma\) are fixed, there is only one basis of \((E, E')\) which is of the form (3.5). For this reason we will say that the basis (3.5) is a canonical, local basis of the big-isotropic structure \(E\).

Now, we consider the integrable case. Let \(\mathcal{U}\) be a neighborhood of the point \(x_0\) on the characteristic leaf \(S\) through \(x_0\). Then \(\dim \mathcal{E}|_{\mathcal{U}} = \text{const.}\) and, in view of (2.18), \(\dim \mathcal{E}'|_{\mathcal{U}} = \text{const.}\) too. Furthermore, on the neighborhood \(\mathcal{U}\) of \(x_0\) in \(M\) there are coordinates \((x^a, y^h, z^\sigma)\) such that \(x_0\) has the coordinates \((0, 0, 0)\), the equations of \(\mathcal{U}\) are \(y^h = 0, z^\sigma = 0, \mathcal{E}|_{\mathcal{U}} = \text{span}\{\partial/\partial x^a\}|_{\mathcal{U}}\) and \(\mathcal{E}'|_{\mathcal{U}} = \text{span}\{\partial/\partial x^a\}|_{\mathcal{U}}, \partial/\partial y^h\}|_{\mathcal{U}}\}. \text{ Indeed, we may assume that } \mathcal{U} \text{ is a tubular neighborhood of } \mathcal{U} \text{ where the tangent space of the tubular fibers at the points of } \mathcal{U} \text{ is a direct sum of a complementary space of } \mathcal{E} \text{ in } \mathcal{E}' \text{ and a complementary space of } \mathcal{E}' \text{ in } TM; \text{ then, take } x^a \text{ coordinates on } \mathcal{U} \text{ and } y^h, z^\sigma \text{ coordinates along the tubular fibers such that } \partial/\partial y^h|_{\mathcal{U}} \text{ span the chosen complement of } \mathcal{E} \text{ in } \mathcal{E}' \text{ and } \partial/\partial z^\sigma|_{\mathcal{U}} \text{ span the chosen further complement in } T_{\mathcal{U}} M. \text{ The tubular structure of the neighborhood } \mathcal{U} \text{ will be important and we will denote by } \mathcal{F} \text{ the foliation of } \mathcal{U} \text{ by the tubular fibers, for later use. It is easy to see that we can construct bases (3.1) where}

\[
(3.6) \quad X_a = \frac{\partial}{\partial x^a}, \quad Y_h = \frac{\partial}{\partial y^h} + \chi^\sigma_h \frac{\partial}{\partial z^\sigma}, \quad Z_\sigma = \frac{\partial}{\partial z^\sigma} (\chi^\sigma_h|_{\mathcal{U}} = 0).
\]
If these values are inserted in (3.5), the result takes the following form (with new coefficients):

\[
X_a = \left( \frac{\partial}{\partial x^a} + A^h_a \frac{\partial}{\partial y^h} + A''^a \frac{\partial}{\partial z^a}, \alpha_h^b dx^b + \alpha_h^u dy^h \right),
\]
\[
\Xi_u = \left( B^h_u \frac{\partial}{\partial y^h} + B''^u \frac{\partial}{\partial z^u}, \beta_h^a dx^a + \beta_h^u dy^h + dz^u \right),
\]
\[
\Upsilon_h = \left( \frac{\partial}{\partial y^h} + \gamma_h^a dx^a, \gamma_h^u dy^h \right),
\]
\[
\Theta_q = \left( L''^u \frac{\partial}{\partial z^u}, \lambda_q^a dx^a + dy^u \right),
\]

where (3.2) holds \( \forall x \in U \) and (3.4) holds on \( U \), which means that we have

\[
\alpha_h^a + \gamma_h^a = 0, \quad \lambda_q^a + A''_q = 0, \quad \beta_h^u + C''_h = 0, \quad L''_q + B''_u = 0,
\]

(3.8)

\[
\beta_a^u + A''_a^u + \alpha_h^u B_h^a + \alpha_h^u A''_a^h = 0, \quad B''_a + B''_u + \beta_h^u B_h^a + \beta_h^u B''_u = 0,
\]

\[
\alpha_h^a + \alpha_h^b A''_b + \alpha_h^u A''_u = 0.
\]

We also notice that, if we are interested in bases of \( E' \), without requiring them to be a prolongation of a basis of \( E \), we may repeat the subtraction trick between \( X_a, \Xi_u \) and \( \Theta_q \) and between \( X_a, \Xi_u \) and \( \Upsilon_h \) as well and get a basis of the form (3.7) with the supplementary conditions

\[
\alpha_h^a = 0, \quad \beta_h^u = 0, \quad A''_a^h = 0, \quad B''_u = 0;
\]

the new pairs \( X_a, \Xi_u \) may not belong to \( E \) any more and (3.8) does not hold.

It follows easily that, if the local coordinates \( (x^a, y^h, z^\sigma) \) such that

\[
y^h|_U = 0, \quad z^\sigma|_U = 0, \quad \frac{\partial}{\partial y^h}|_U \in (E'|E), \quad \frac{\partial}{\partial z^\sigma}|_U \in (TM|E')|_U
\]

are chosen, the basis of the form (3.7) where \( X_a, \Xi_u \in E \) and \( \Upsilon_h, \Theta_q \in E'|E \) is unique. A similar fact holds for the basis of \( E' \) which satisfies the conditions (3.9). Accordingly, the basis (3.7) will be called a canonical, local basis of the integrable, big-isotropic structure \( (E, E') \) and the basis where (3.9) also holds is a canonical basis of \( E' \).

**Remark 3.1.** In the Dirac case \( k = m \), the basis (3.7) reduces to

\[
X_a = \left( \frac{\partial}{\partial x^a} + A''_a \frac{\partial}{\partial z^a}, \alpha_h^a dx^a \right), \quad \Xi_u = \left( B''_u \frac{\partial}{\partial z^u}, \beta_h^a dx^a + dz^u \right),
\]
which are the formulas given in [6]. In the almost Dirac case, we have the corresponding formulas deduced from (3.5)

\[(3.12) \quad X_a = (X_a + A''^a Z_{\sigma} \alpha_b^a \theta^b), \quad \Xi_u = (B''_u Z_{\sigma} \beta^u \alpha_a \theta^a + \psi^u).\]

For (3.11) and (3.12) the conditions (3.8) reduce to

\[(3.13) \quad \alpha^b_a + \alpha^a_b = 0, \quad B''_{u\sigma} + B''_{\sigma u} = 0, \quad \beta^u_a + A''_a u = 0.\]

As an example, we use these formulas for a straightforward proof of the fact that, in the two-dimensional case \(m = 2\), any almost Dirac structure \(D\) is Dirac. Indeed, the points of \(M\) may be classified into three classes (0, 1, 2) where \(\dim(pr_{TM} D) = 0, 1, 2\), respectively. Obviously, any point of class 2 has a neighborhood \(U\) of class 2 and \(D\) is integrable on \(U\). Formulas (3.12) and (3.13) show that any point of class 1 is regular, hence, it also has a neighborhood where \(D\) is integrable. If a point of class 0 has a neighborhood \(U\) of points of class 0, \(D\) is integrable on \(U\). Finally, if a point \(x_0\) of class 0 has no such neighborhood, every neighborhood of \(x_0\) has points that are of class 2 (necessarily), hence, \(x_0\) is the limit of points that have neighborhoods where \(D\) is closed by Courant brackets and, by continuity, \(x_0\) also has a neighborhood where \(D\) is integrable. A similar analysis, where it is simpler to use Proposition 2.3 instead of the local, canonical bases shows that a big-isotropic structure \(E\) with \(k = 1, m = 2\) must be integrable.

4 Local geometry of a big-isotropic structure

We begin with a preparatory discussion of the operation of pullback of a big-isotropic structure by a mapping (see [3, 4] for the Dirac case), which is of a more general interest. Let \(f : N^n \rightarrow M^m\) be a mapping of manifolds, \(x \in N, y = f(x)\) a pair of corresponding points, and \(E\) an arbitrary vector subbundle of \(T^{big} M\). We denote by \(f_*\) the differential of \(f\) and by \(f^*\) the transposed mapping of \(f_*\). Then

\[(4.1) \quad f^*(E_y) = \{(X, f^* \alpha) / X \in T_x N, \alpha \in T^*_y M, (f_* X, \alpha) \in E_y\}\]

is the pullback of \(E\) at \(x\).
Proposition 4.1. Let $E$ be a rank $k$, big-isotropic subbundle of $T_{\text{big}}^*M$. Then $f^*(E_y)$ is isotropic in $T_{\text{big}}^*N$ and its $g_N$-orthogonal space is $(f^*(E_y))' = f^*(E'_y)$. Furthermore, if $f$ is an embedding of $N$ in $M$, if $E$ is an integrable, big-isotropic structure on $M$ and if $f^*E = \cup_{y \in M} f^*(E_y)$ is a differentiable subbundle of $T_{\text{big}}^*N$ then $f^*E$ is an integrable, big-isotropic structure on $N$. Finally, the condition

\[(4.2)\quad E_y \cap \ker f^*_y = E'_y \cap \ker f^*_y\]

characterizes the situation where the dimension of $f^*(E_y)$ is $n - m + k$.

Proof. It is easy to check that $f^*(E_y)$ is an isotropic subspace of $(T_{\text{big}}^*N, g_N)$ and we will compute its dimension. Obviously, we have

\[(4.3)\quad \dim \ker f^*_x = n - \text{rank}_x f, \quad \dim \ker f^*_y = m - \text{rank}_x f.\]

Then, define the space

\[(4.4)\quad S_y = \{(f_*X, \alpha) \in E_y / X \in T_x N, \alpha \in T^*_y M\} = E_y \cap (\text{im} f_* \oplus T^*_y M),\]

and notice that the correspondence $(f_*X, \alpha) \mapsto (X, f^*\alpha)$ produces an isomorphism

\[(4.5)\quad S_y / (S_y \cap \ker f^*_y) \approx f^*(E_y) / \ker f^*_x.\]

Therefore,

\[(4.6)\quad \dim f^*(E_y) = \dim \ker f^*_x + \dim S_y - \dim (S_y \cap \ker f^*_y)\]

\[= \dim \ker f^*_x + \dim S_y - \dim (E_y \cap \ker f^*_y)\]

(the equality $S_y \cap \ker f^*_y = E_y \cap \ker f^*_y$ follows from the definition of $S_y$ since $\ker f^*_y \subseteq \text{im} f_* \oplus T^*_y M$).

Furthermore, we notice the equality

\[(4.7)\quad \text{im} f_* \oplus T^*_y M = (\ker f^*_y)^{+g_M},\]

which follows since the left hand side is included in the right hand side and the two spaces have the same dimension by (4.3). Accordingly, we have

\[(4.8)\quad S_y = (E'_y)^{+g_M} \cap (\ker f^*_y)^{+g_M} = (E'_y + \ker f^*_y)^{+g_M},\]
whence, using the classical relation between the dimensions of the sum and intersection of two linear subspaces, we get

\[(4.9) \quad \dim S_y = 2m - \dim(E'_y + \ker f'_y) = \dim E_y - \dim \ker f'_y + \dim(E'_y \cap \ker f'_y).\]

If we combine (4.6), (4.9) and (4.3) we get

\[(4.10) \quad \dim f^*(E_y) = n - m + k + \dim(E \cap \ker f^*_y) - \dim(E \cap \ker f^*_y).\]

The same procedure with the roles of \(E\) and \(E'\) interchanged uses the space \(S'_y = E'_y \cap (\text{im} f_\ast \oplus T^*_y M)\), which (like in (4.9)) has the dimension

\[(4.11) \quad \dim S'_y = \dim E'_y - \dim \ker f'_y + \dim(E_y \cap \ker f^*_y),\]

and leads to

\[(4.12) \quad \dim f^*(E'_y) = n + m - k - \dim(E' \cap \ker f^*_y) + \dim(E \cap \ker f^*_y).\]

Hence, \(\dim f^*(E_y) + \dim f^*(E'_y) = 2n\) and, since it is easy to check the \(g_N\)-orthogonality of these two spaces, we have \(f^*(E'_y) = (f^*(E_y))^\perp g_N\).

For the second assertion of the proposition, it suffices to notice that \(X \in T_x N\) belongs to \(pr_{T_x N} f^*(E_y)\) iff \(f_\ast X \in pr_{T_y M} E_y\). If \(f\) is an embedding and if \(f^*E\) is a differentiable subbundle (in particular, \(\dim f^*(E_y) = \text{const.}\)), a field \(X \in \Gamma(pr_{T_x N}(f^*E))\) has an \(f\)-related field \(f_\ast X \in \Gamma(pr_{T_y M} E)\). Moreover, since \(\dim \ker f'_y = m - n = \text{const.}\), a cross section of \(f^*E\) must be of the form \((X, f^* \alpha)\) where \((f_\ast X, \alpha)\) is a differentiable cross section of \(E\), and the same holds for \(f^*E'\) and \(E'\). If these facts are taken into consideration, a straightforward examination of the Courant brackets shows that the integrability conditions for \(E\) imply the integrability conditions for \(f^*E\).

Finally, hypothesis (4.2) is equivalent with \(\dim(E' \cap \ker f^*_y) = \dim(E \cap \ker f^*_y)\) and, by (4.10), we have the required dimension for \(f^*(E_y)\) iff (4.2) holds.

\[\square\]

**Remark 4.1.** 1) By (4.10), \(\dim f^*(E_y) = \text{const.}\) iff

\[\dim(E'_y \cap \ker f^*_y) - \dim(E_y \cap \ker f^*_y) = \text{const.}\]

By (4.9) and (4.11), this condition is equivalent with

\[\dim S'_y - \dim S_y = 2(m - k) - (\dim(E' \cap \ker f^*_y) + \dim(E \cap \ker f^*_y)) = \text{const.}\]
2) The algebraic content of Proposition 4.1 holds for any linear mapping \( l : V \to W \) between linear spaces and any isotropic subspace \( E \subseteq W \oplus W^\ast \). The significance of the equality \( \dim f^*(E_y) = n - m + k \) is the codimensional invariance property \( n - \dim f^*(E_y) = m - \dim E_y \). Hypothesis (4.2) holds in the Dirac case \( (E = E') \) and in the case where \( f \) is a submersion \( (\ker f^* = 0) \).

**Corollary 4.1.** With the notation of Proposition 4.1 and of its proof, if \( f \) is an embedding and if \( \dim S_y = \text{const.} \), \( \dim S'_y = \text{const.} \) the pullbacks \( f^*E, f^*E' \) are differentiable.

**Proof.** By the definition of \( S_y, S'_y \) (see (4.1)), if the dimensions of these spaces do not depend on \( x \), \( S_y, S'_y \) are differentiable with respect to \( x \); then, by Remark 4.1 1) \( \dim f^*(E_y) = \text{const.} \) and \( \dim f^*(E'_y) = \text{const.} \). On the other hand, since \( f \) is an embedding, \( \ker f^*x = 0 \) and formula (4.5) yields

\[
S_y/(E_y \cap \ker f^*_y) \approx f^*(E_y), S'_y/(E'_y \cap \ker f^*_y) \approx f^*(E'_y).
\]

Formula (4.13) and the constant dimensions of the spaces therein imply the differentiability of \( f^*E, f^*E' \) and Proposition 4.1 allows us to conclude.

Now, we shall discuss some local properties of an integrable, big-isotropic structure \( E \).

**Proposition 4.2.** The pullback of an integrable, big-isotropic structure \( E \) to a characteristic leaf \( S \) is the same as the pullback of its Dirac extension \( D(E) \) and it is a presymplectic structure of \( S \).

**Proof.** Consider the neighborhoods \( U, \mathcal{U} \) where one has the canonical basis \( \mathcal{S} \cap U \) has the equations \( y^h = 0, z^\sigma = 0 \). Then, at the points of \( \mathcal{U}, \gamma \in \Omega^1(M) \) belongs to \( \text{ann}E|_\mathcal{U} \) iff \( \gamma|_\mathcal{U} = \gamma_h dy^h + \gamma_\sigma dz^\sigma \). This implies that \( i^*\gamma = 0 \) \( (i : \mathcal{U} \to M) \), whence, by (2.23), \( i^*D(E) = i^*E \). Now, (3.2) and (3.8), which hold for the coordinate values \( (x^a, 0, 0) \), give \( \beta^u_a(x^c, 0, 0) = 0, \alpha^b_a(x^c, 0, 0) = -\alpha^b_a(x^c, 0, 0) \) and, accordingly, (4.1) shows that

\[
i^*E = \{(f_a \frac{\partial}{\partial x^a}, f_a \alpha^b_a(x^c, 0, 0) dx^b) / f_a = f_a(x^c)\}.
\]

Therefore, \( i^*E \) is the presymplectic structure defined by the 2-form of components \( \alpha^b_a(x^c, 0, 0) \), which is just \( \omega|_E \times E \).

\[\]
Remark 4.2. The canonical basis (3.7) also provides a local expression of the Dirac extension \( D(E) \) over the neighborhood \( U \). Indeed, (3.7) shows that the annihilator of \( \text{span}\{\text{pr}_{TM}X_a\} \) is spanned by the 1-forms \( dy^h -! A'^{h}dx^a, dz^\sigma -! A''^{\sigma}dx^a \) and \( \text{ann} \ E \) consists of the forms

\[
\gamma = \varphi_h(dy^h - A'^{h}dx^a) + \psi_{\sigma}(dz^\sigma - A''^{\sigma}dx^a)
\]

where

\[
(4.14) \quad \varphi_h B'^{h} + \psi_{\sigma} B''^{\sigma} = 0.
\]

Therefore,

\[
(4.15) \quad D_U(E) = \{ f^a X_a + s^u \Xi_u + (0, \varphi_h(dy^h - A'^{h}dx^a))
\]

\[
+ \psi_{\sigma}(dz^\sigma - A''^{\sigma}dx^a)) \}
\]

where \( f^a, s^u, \varphi_h, \psi_{\sigma} \in C^\infty(U) \) and (4.14) holds. In particular, if the structure \( E \) is regular then \( B'^{h} \equiv 0, B''^{\sigma} \equiv 0 \) and \( D(E) \) is a differentiable, Dirac structure.

Now, at \( x_0 \), we consider the local transversal submanifold \( Q_0 \) of the characteristic leaf \( S \) given by the equations \( x^a = 0 \) and denote by \( \iota : Q_0 \to M \) the corresponding embedding. For this embedding, (3.7) shows that the spaces \( S, S' \) of Corollary 4.1 are given by

\[
S = \text{span}\{\Xi_u|_{Q_0}\}, \quad S' = \text{span}\{\Xi_u|_{Q_0}, Y_h|_{Q_0}, \Theta_q|_{Q_0}\},
\]

hence, \( S, S' \) have a constant dimension. Then, Corollary 4.1 tells us that \( Q_0 \) has an induced, integrable, big-isotropic structure \( E'^{tr}_{x_0} = \iota^*E \), which will be called the transversal structure of \( E \) at \( x_0 \).

Furthermore, with (4.4) and (3.7), it follows that \( E \cap \ker \iota^* = S \cap \ker \iota^* = 0 \) and we see that \( E'^{tr}_{x_0} \) is isomorphic with the bundle \( S|_{Q_0} \). In particular, \( \Xi_u \) (mod \( x^a = 0 \)) is a basis of the transversal structure \( E'^{tr}_{x_0} \) along \( Q_0 \). We also notice that \( E'^{tr}_{x_0} \cap (TF \oplus \text{ann} TF) = 0 \) and, in particular, \( E'^{tr}_{x_0} \) is of the graph type (it is the graph of a mapping \( \Lambda \to TQ_0 \), where \( \Lambda \) is a field of subspaces of \( T^*Q_0 \)).

Any local transversal submanifold \( \mathcal{T}_0 \) of \( U \) at \( x_0 \) inherits a transversal structure since there exists a tubular neighborhood such that \( Q_0 = \mathcal{T}_0 \). Hence, we may speak of the transversal structure in a generic way, which, however, does not mean that the transversal structures defined on different
submanifolds $T_0$ are equivalent. We shall address this question but we need more preparations first. The following considerations are inspired by the case of foliation-coupling Dirac structures [15] and may be used in a discussion of the coupling between a big-isotropic structure and a foliation, which we do not intend to develop.

Recall that $U$ has the tubular foliation $F$ of equations $x^a = \text{const.}$. Define the field of subspaces

$$H(E, F) = \{ Z \in TM / \exists \alpha \in ann T F, (Z, \alpha) \in E \}.$$  

Using the basis (3.7) we get

$$H(E, F) = \left\{ f^a pr_{TM} x_a / \sum_a f^a \alpha'_a = 0, f^a \in C^\infty(U) \right\},$$

therefore, $H(E, F) \cap T F = 0$. Generally, $H(E, F)$ may not have a constant dimension; we will say that $H(E, F)$ is the pseudo-normal bundle of $F$. $H(E, F)$ is a true normal bundle, i.e.,

$$T_U M = H(E, F) \oplus T F,$$

iff

$$\alpha'_a = 0 \quad (\Leftrightarrow \gamma_a = 0).$$

We would like to notice that condition (4.19) has other interesting interpretations too. One of them is obtained if we define the field of subspaces

$$\mathcal{H}(E', F) = \{ \theta \in T^* M / \exists Z \in T F, (Z, \theta) \in E' \},$$

which we call the pseudo-conormal bundle of $F$ modulo $E$. Using (3.7), it follows that $\theta \in \mathcal{H}(E', F)$ iff

$$\theta = \varphi_u (\beta^u_a dx^a + \beta^h u dy^h + dz^u) + \psi_h \gamma^h_a dx^a + \mu_q (\lambda^a q dx^a + dy^u).$$

Accordingly, we see that

$$T^*_U M = \mathcal{H}(E', F) + ann T F, \mathcal{H}(E', F) \cap ann T F = \text{span}\{\gamma^h_a dx^a\}.$$ 

Therefore, we have

$$T^*_U M = \mathcal{H}(E', F) \oplus ann T F.$$
iff (4.19) holds.

For another interpretation of condition (4.19) let us define the field of subspaces

\[(4.23) \quad H(E', F) = \{ Z \in TM \mid \exists \alpha \in ann T F, (Z, \alpha) \in E' \}. \]

Notice that \( H(E, F) \subseteq H(E', F) \). From (3.7), we get

\[(4.24) \quad H(E', F) = \text{span}\{ pr_{TM} x_a - \sum \alpha_q' L_q^\tau_q \frac{\partial}{\partial z^\tau_q} , pr_{TM} y_h \}. \]

Therefore, \( E'|_U = H(E', F) + T \mathcal{F} \) and \( H(E', F) \cap T \mathcal{F} \) is the (trivial) \((m-k)\)-dimensional bundle with the basis \( \{ pr_{TM} y_h \} \). Together with (4.24), this implies

\[(4.25) \quad (\text{ann } \mathcal{F})^* \approx H(E', F)/(H(E', F) \cap T \mathcal{F}). \]

On the other hand, let us recall the truncated 2-form \( \varpi \) : \( \mathcal{E} \times \mathcal{E}' \rightarrow \mathbb{R} \) and consider its second flat morphism \( \flat'_\varpi : \mathcal{E}' \rightarrow \mathcal{E}^* \) given by \( (\flat'_\varpi Y)(X) = \varpi(X, Y) \). It follows that condition (4.19) holds iff

\[(4.26) \quad H(E', \mathcal{F}) \cap T \mathcal{F} \subseteq \ker \flat'_\varpi. \]

This again is an interpretation of (4.19). Notice also that if (4.19) holds then \( \forall X \in H(E', \mathcal{F}) \) the corresponding 1-form \( \alpha \) such that \((X, \alpha) \in E'\) is uniquely defined.

The interesting fact that follows from (4.18) is that \( E|_U \) decomposes along \( \mathcal{F} \) and \( H(E, \mathcal{F}) \). Indeed, using the canonical basis (3.7) we see that (4.18) implies

\[ T^* \mathcal{F} \approx \text{ann } H(E, \mathcal{F}) = \text{span}\{ dy^h - A^h_a dx^a, dz^\sigma - A^\sigma_a dx^a \} \]

and

\[ E \cap (T \mathcal{F} \oplus T^* \mathcal{F}) = \text{span}\{ \Xi_a \}. \]

Thus, this intersection produces the transversal structures on the fibers of \( \mathcal{F} \). We also have \( H^*(E, \mathcal{F}) \approx \text{ann } T \mathcal{F} \) and

\[ E \cap (H(E, \mathcal{F}) \oplus H^*(E, \mathcal{F})) = \text{span}\{ \mathcal{X}_a \}. \]

Therefore, the following decomposition holds

\[(4.27) \quad E|_U = [E \cap (T \mathcal{F} \oplus T^* \mathcal{F})] \oplus [E \cap (H(E, \mathcal{F}) \oplus H^*(E, \mathcal{F}))]. \]

This property justifies the following definition.
Definition 4.1. An integrable, big-isotropic structure $E$ on $M$ is called \textit{locally decomposable} if each point $x \in M$ has a tubular neighborhood $U$ of the characteristic slice (a neighborhood of the characteristic leaf) $U$ through $x$, with the tubular foliation $\mathcal{F}$, where the decomposition (4.18) holds. If (4.18) holds for any tubular neighborhood, $E$ will be called \textit{strongly, locally decomposable}.

All the Dirac structures are strongly, locally decomposable [6]; a non-Dirac example follows.

Example 4.1. Take $M = \mathbb{R}^3$ with coordinates $(x, y, z)$ and

$E = \text{span}\{ (\frac{\partial}{\partial x}, 0), (0, dz) \}$.

$E$ is a regular, integrable, big-isotropic structure of dimension 2 with

$E' = E \oplus \text{span}\{ (\frac{\partial}{\partial y}, 0), (0, dy) \}$.

The given basis is canonical and local decomposability holds. New tubular coordinates are defined by

$\tilde{x} = \tilde{x}(x, y, z), \tilde{y} = \tilde{y}(x, y, z), \tilde{z} = \tilde{z}(x, y, z),$

where (see (3.10))

$\tilde{y}(x, 0, 0) = 0, \tilde{z}(x, 0, 0) = 0, \left. \frac{\partial \tilde{y}}{\partial x} \right|_{(x,0,0)} = 0, \left. \frac{\partial \tilde{z}}{\partial x} \right|_{(x,0,0)} = 0,$

$\left. \frac{\partial \tilde{z}}{\partial y} \right|_{(x,0,0)} = 0, \frac{\partial \tilde{x}}{\partial x} \neq 0, \frac{\partial \tilde{y}}{\partial y} \neq 0, \frac{\partial \tilde{z}}{\partial z} \neq 0.$

If we make this change in the generators of $E$ and produce a canonical basis out of the result we see that (4.19), hence local decomposability, also holds for the new tubular neighborhood.

The following example shows that local decomposability of an integrable, big-isotropic structure does not imply strong, local decomposability.
Example 4.2. Take $M = \mathbb{R}^5$ with the canonical coordinates $(x^1, x^2, y^1, y^2, z)$. Define

$$E = \text{span}\{\mathcal{X}_1 = (\frac{\partial}{\partial x^2}, dx^2 + dy^1), \mathcal{X}_2 = (\frac{\partial}{\partial x^2}, -dx^1 + dy^2), \Xi_1 = (0, dz)\}.\tag{4.28}$$

This is a regular, 3-dimensional, integrable, big-isotropic structure and the orthogonal bundle $E'$ has the supplementary generators

$$\mathcal{Y}_1 = (\frac{\partial}{\partial y^1}, -dx^1), \mathcal{Y}_2 = (\frac{\partial}{\partial y^2}, -dx^2), \Theta_1 = (0, dy^1), \Theta_2 = (0, dy^2).\tag{4.29}$$

The characteristic leaf through the origin is the $(x^1, x^2)$-plane, $\mathbb{R}^5$ may be seen as a tubular neighborhood of this leaf and the basis given by (4.28), (4.29) is canonical. Accordingly, the local decomposability property does not hold for the tubular foliation that consists of the family of 3-dimensional planes with coordinates $(y^1, y^2, z)$. Now, consider the following coordinate transformation

$$\tilde{x}^1 = x^1 - y^2, \tilde{x}^2 = x^2 + y^1, \tilde{y}^1 = y^1, \tilde{y}^2 = y^2, \tilde{z} = z.$$

If we express the pairs (4.28), (4.29) and then produce a canonical basis of $(E, E')$ with respect to the new coordinates we get

$$\tilde{\mathcal{X}}_1 = (\frac{\partial}{\partial \tilde{x}^1}, d\tilde{x}^2), \tilde{\mathcal{X}}_2 = (\frac{\partial}{\partial \tilde{x}^2}, -d\tilde{x}^1), \tilde{\Xi} = (0, d\tilde{z}),$$

$$\tilde{\mathcal{Y}}_1 = (\frac{\partial}{\partial \tilde{y}^1}, 0), \tilde{\mathcal{Y}}_2 = (\frac{\partial}{\partial \tilde{y}^2}, 0), \tilde{\Theta}_1 = (0, d\tilde{y}^1), \tilde{\Theta}_2 = (0, d\tilde{y}^2).$$

Thus, with respect to the tubular fibers defined by the new coordinates the local decomposability property holds.

It is known that the transversal structure of a Dirac structure is well defined up to a natural equivalence [6]. We will show that the same holds for the integrable, big-isotropic structures that are strongly, locally decomposable. The basis (3.7) also yields a transversal structure on each submanifold $Q_x$ defined by $x^a = x^a(x)$ ($x \in \mathcal{U}$). We will denote by $E_x^{tr}$ the family of transversal structures $E_x^{tr}$, $x \in \mathcal{U}$ and prove the following lemma.
Lemma 4.1. Let $E$ be a locally decomposable, integrable, big-isotropic structure on $M$, $x_0$ a point of $M$ and $U$ a tubular neighborhood of $x_0$ with the tubular foliation $\mathcal{F}$. Then, any $\mathcal{F}$-projectable vector field $Z \in H(E, \mathcal{F})$ is an infinitesimal automorphism of the transversal structure $E^{tr}$.

Proof. Since $Z$ is $\mathcal{F}$-projectable, the flow of $Z$ sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. As explained earlier, the transversal structure $E^{tr}$ on the leaves of $\mathcal{F}$ is induced by the vector bundle $S = \text{span}\{\Xi_u\}$ (see (3.7)). Equivalently, $E^{tr}$ is induced also by the bundle $\tilde{E}^{tr} = S \oplus \text{ann} T\mathcal{F}$ (obviously, the intersection of the terms is zero and $\tilde{E}^{tr}$ is an integrable, big-isotropic structure on $U$ of the same dimension $k$ like $E$). Thus, the conclusion will be obtained if we show that $Z$ is an infinitesimal automorphism of $\tilde{E}^{tr}$. If we denote

$$X_a = (V_a, \nu^a), \quad \Xi_u = (W_u, \xi^u),$$

the projectability of $Z$ is equivalent with asking $Z = \zeta^a(x^b) V_a$ and $Z$ is an infinitesimal automorphism of $\tilde{E}^{tr}$ iff

$$\langle L_{V_a} W_u, L_{V_a} \xi^u \rangle \in \Gamma \tilde{E}^{tr}. \quad (4.31)$$

The integrability of $E$ implies the existence of local functions $f^a, \varphi^u$ such that

$$\begin{align*}
[X_a, \Xi_u] &= ([V_a, W_u], L_{V_a} \xi^u - L_{W_a} \nu^a + d(\nu^a(W_u))) \\
&= ([V_a, W_u], L_{V_a} \xi^u - i(W_u)d\nu^a) = (L_{V_a} W_u, L_{V_a} \xi^u) \\
&- (0, (W_u \alpha^b_a)dx^b) = f^a X_a + \varphi^u \Xi_u,
\end{align*}$$

where we have used the local decomposability condition $\alpha^b_a = 0$. Since $[V_a, W_u]$ does not contain $\partial/\partial x^a$, we must have $f^a = 0$ and (4.31) follows. \(\square\)

Now, we can prove

Proposition 4.3. Let $E$ be a strongly, locally decomposable, integrable, big-isotropic structure on a manifold $M$. Then, the transversal structure $E^{tr}_{x_0}$ is well defined up to a structure preserving diffeomorphism.

Proof. The proof given for the Dirac structures in [6] also holds here. First we look at local transversal submanifolds $\mathcal{T}_0, \mathcal{T}_1$ of the characteristic slice $\mathcal{U}$ at $x_0 \neq x_1 \in \mathcal{U}$. Then, there exist a tubular neighborhood such that $\mathcal{T}_0, \mathcal{T}_1$ are fibers of the tubular foliation $\mathcal{F}$ and a diffeomorphism $\Phi$, which is a composition of transformations of flows of $\mathcal{F}$-projectable vector fields.
\[ Z \in H(E, \mathcal{F}), \] that sends \( T_0 \) onto \( T_1 \). By Lemma 4.1, \( \Phi \) sends the transversal structure on \( T_0 \) onto the transversal structure on \( T_1 \). Now, if we have two different local transversal submanifolds \( T_0, T'_0 \) at the same point \( x_0 \) of \( \mathcal{U} \), we take a loop of \( \mathcal{U} \) at \( x_0 \), break it into a finite number of pieces, and go from \( T_0 \) to \( T'_0 \) through intermediate transversal submanifolds defined at the breaking points. The composition of the diffeomorphisms \( \Phi \) defined as above between the intermediate transversal manifolds gives us the required equivalence of the transversal structures on \( T_0 \) and \( T'_0 \). \[ \square \]

5 Reduction of big-isotropic structures

Recently, the generalization of symplectic reduction to Dirac structures was discussed by several authors, in particular [2, 10]. In this section we discuss a reduction scheme for big-isotropic structures.

We begin by defining a push forward procedure. The notation will be similar to that of Proposition 4.1; in particular, we consider the mapping \( f : N \to M \) and the corresponding points \( y = f(x) \). But, we start with a rank \( k \) subbundle \( E \subseteq T^\text{big}N \). Then, we define the push forward of \( E \) by

\[
(5.1) \quad f_*(E_x) = \{(f_*(X), \alpha) / X \in T_xN, \alpha \in T^*_yM, (X, f^*_y\alpha) \in E_x\}.
\]

**Proposition 5.1.** If the bundle \( E \) is a big-isotropic structure then \( f_*(E_x) \) is an isotropic subspace of \( (T^\text{big}_yM, g_M) \) and its orthogonal space is \( f_*(E'_x) \). Furthermore, iff

\[
(5.2) \quad E_x \cap \ker f_{*x} = E'_x \cap \ker f_{*x}.
\]

the dimension of \( f_*(E_x) \) is \( m - n + k \).

**Proof.** The isotropy of \( f_*(E_x) \) is obvious. Then, let us define the space

\[
(5.3) \quad \Sigma = \{(X, f^*_x\alpha) \in E_x / X \in T_xN, \alpha \in T^*_yM\} = E_x \cap (T_xN \oplus \text{im} f^*_y)
\]

\[ = (E'_x + \ker f_{*x})^\perp|_{x_0}. \]

The last equality (5.3) follows from

\[
(5.4) \quad (\ker f_{*x})^\perp|_{x_0} = T_xN \oplus \text{im} f^*_y,
\]

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which holds because the right hand side is included in the left hand side and the former has the dimension required for the orthogonal space of the latter. The correspondence \((X, f^\star \alpha) \mapsto (f_* X, \alpha)\) produces an isomorphism

\[ \Sigma/(E_x \cap \ker f_{sx}) \approx f_*(E_x)/\ker f_y^\star, \]

whence

\[
\dim f_*(E_x) = \dim \ker f_y^\star + \dim \Sigma - \dim (E_x \cap \ker f_{sx})
\]

\[ \overset{4.3}{=} m - \text{rank}_x f + \dim \Sigma - 2n + \dim (E_x \cap \ker f_{sx})^{\perp gN} \]

\[ \overset{5.4}{=} m - \text{rank}_x f + \dim \Sigma - 2n + \dim (T_x N \oplus \im f_y^\star) + E'_x
\]

\[ = m - \text{rank}_x f + \dim \Sigma - 2n + \dim (T_x N \oplus \im f_y^\star)
\]

\[ + \dim E'_x - \dim ((T_x N \oplus \im f_y^\star) \cap E'_x)
\]

\[ = m + n - k + (\dim \Sigma - \dim \Sigma'), \]

where \(\Sigma'\) is the space (5.3) for \(E'\).

The same calculations with the roles of \(E\) and \(E'\) interchanged give

\[
\dim f_*(E'_x) = m - n + k - (\dim \Sigma - \dim \Sigma'),
\]

hence, \(\dim f_*(E_x) + \dim f_*(E'_x) = 2m\). Since it is trivial to check that the two spaces are \(g_M\)-orthogonal, we get the required relation \(f_*(E'_x) \perp_{g_M} f_*(E_x)\).

Furthermore, from the last expression of \(\Sigma\) in (5.3) and the similar expression of \(\Sigma'\) we get

\[
\dim \Sigma - \dim \Sigma' = \dim \Sigma^{\perp gN} - \dim \Sigma'\perp gN
\]

\[ = \dim E_x + \dim \ker f_{sx} - \dim (E_x \cap \ker f_{sx})
\]

\[ - \dim E'_x - \dim \ker f_{sx} + \dim (E'_x \cap \ker f_{sx})
\]

\[ = 2k - 2n + (\dim (E'_x \cap \ker f_{sx}) - \dim (E_x \cap \ker f_{sx})).
\]

Accordingly, we obtain

\[ \dim f_*(E_x) = m - n + k + (\dim (E'_x \cap \ker f_{sx}) - \dim (E_x \cap \ker f_{sx})), \]

which justifies the last assertion of the proposition. \(\square\)
Remark 5.1. Proposition 5.1 holds for any linear mapping \( l : V \to W \) between linear spaces and any isotropic subspace \( E \subseteq V \oplus V^* \). The significance of the equality \( \dim f^*(E_y) = m - n + k \) is the codimensional invariance property \( n - \dim E_x = m - \dim f_*(E_x) \). Hypothesis (5.2) holds in the Dirac case \( (E = E') \) and in the case where \( f \) is an immersion \((keff = 0)\). If \( f \) is a submersion and the difference between the dimensions of \( \Sigma, \Sigma' \) is constant then \( \dim f_*(E_x) \) is the same \( \forall x \in N \). However, there may not be a well defined subbundle \( f_*(E) \) because \( f^{-1}(y) \) may have more than one point.

Proposition 5.2. If \( f \) is a submersion then \( f_* f^*(E_y) = E_y \). If \( f \) is an immersion then \( f^* f_*(E_x) = E_x \).

Proof. By looking at the formulas (4.1), (5.1) we see that \((Z, \lambda) \in f_* f^*(E_y) \) iff \((Z, \lambda) = (f_* X, \alpha + \beta) \) where \((f_* X, \alpha) \in E_y \) and \( \beta \in \text{ker} f^*_y \). Since, if \( f \) is a submersion \( \text{ker} f^*_y = 0 \), we get \((Z, \lambda) \in E_y \), hence, \( f_* f^*(E_y) \subseteq E_y \). On the other hand, again since \( f \) is a submersion, any pair of \( E_y \) is of the form \((f_* X, \alpha)\), hence, of the form \((Z, \lambda) \in f_* f^*(E_y)\). Thus, \( E_y \subseteq f_* f^*(E_y) \) and the first assertion is proven. The proof of the second assertion is similar. \( \square \)

Another kind of preparation that we need concerns the notion of projectability. Let \( M \) be an \( m \)-dimensional, differentiable manifold and \( \mathcal{F} \) a foliation of \( M \) by \( p \)-dimensional leaves. In what follows, the terms projectable and foliated are synonymous and describe objects related with corresponding objects of the space of leaves via the natural projection. In particular, a vector field is foliated if it can be projected onto the space of leaves and a differential form is projectable if it is the pullback of a form on the space of leaves (such forms are called basic forms by most authors).

Definition 5.1. An arbitrary subbundle \( E \subseteq T^{big}M \) of rank \( k \) is called foliated or projectable if each point \( x \in M \) has an open neighborhood \( U \) such that the quotient manifold \( Q_U = U/(\mathcal{F}|_U) \) is endowed with a subbundle \( \Delta_U \subseteq T^{big}Q_U \) that satisfies the condition \( E|_U = \pi^*(\Delta_U) \) \((\pi : U \to U/(\mathcal{F}|_U))\).

In Definition 5.1, \( \pi \) is the natural projection and \( \pi^*(\Delta_U) \) is obtained by the pullback of \( \Delta_U \), i.e.,

\[
\pi^*(\Delta_U) = \{(X, \pi^* \alpha) / X \in T_x M, (\pi_* X, \alpha) \in \Delta_{\pi(x)} \} \quad (x \in U).
\]

Proposition 5.3. The subbundle \( E \) is foliated iff it satisfies the following two conditions: a) \( E \supseteq T\mathcal{F} \), b) each point \( x \in M \) has an open neighborhood \( U \)
such that $\Gamma E|_U$ has a basis $(X_i, \xi^i)$ ($i = 1, ..., k$) with projectable vector fields $X_i$ and projectable 1-forms $\xi^i$. Furthermore, condition b) may be replaced by b') every $Y \in \Gamma TF$ is an infinitesimal automorphism of $E$, i.e., $\forall (X, \xi) \in \Gamma E$ one has $(L_Y X, L_Y \xi) \in \Gamma E$.

Proof. If $E$ is projectable, since $(0, 0) \in \Delta_U$, formula (5.7) shows that a) holds. Then, assume that $(V_u, \lambda^u)$ ($u = 1, ..., q = m - p$) is a local basis of the cross sections of $\Delta_U$ and put $V_u = [X_u]_{TF}, \xi_u = \pi^* \lambda^u \ (X \in \chi^1(U))$, where the brackets denote equivalence classes modulo $TF$. It follows easily that $\Gamma E|_U$ has a local basis that consists of $(X_u, \xi^u)$ and of a basis of $\Gamma TF$. Hence b) also holds. Conversely, if we have the properties a), b), we can change the basis provided by b) to a basis of the form $((X_u, \xi^u), (Y_a, 0))$ where $Y_a$ is a local basis of $TF$ and $([X_u]_{TF}, \lambda^u), \xi^u = \pi^* \lambda^u$, is a basis for the local structure $\Delta_U$ required by Definition 5.1. For the last assertion, if b) holds and if we put $(X, \xi) = \sum_{i=1}^k f^i(X_i, \xi^i)$ then

$$(L_Y X, L_Y \xi) = \sum_{i=1}^k (Y f^i)(X_i, \xi^i) \in \Gamma E.$$ 

Conversely, assume that a), b') hold. For any $x \in M$, we can take a cubical, open neighborhood with $F$-adapted local coordinates $(z^a, y^u)$ (i.e., $F$ has the local equations $dy^u = 0$ and $z^a(x) = 0, y^u(x) = 0$). Then, we can take an arbitrary basis of $E$ of the form $(\partial/\partial z^a, 0), (V_u, \lambda^u)$ along the transversal slice $z^a = 0$ and move $(V_u, \lambda^u)$ along the slice $y^u = 0$ by the linear holonomy of the foliation $F$ (e.g., [9]). By hypothesis b') the flows of the tangent vectors of the leaves preserve $E$, hence, the result of the previous procedure consists of cross sections of $E$ and we get a projectable basis as required by b).

We may apply the previous general results to speak of projectable, big-isotropic structures, integrable or not, and of projectable (almost) Dirac structures. First, we notice the following corollary of Proposition 5.3.

**Corollary 5.1.** An integrable, big-isotropic structure $E$ of $M$ is foliated iff $E \supseteq TF$.

**Proof.** For a foliated subbundle $E$ we have $E \supseteq TF$ by a) of Proposition 5.3. Conversely, since $\Gamma E$ is closed by Courant brackets, with the notation of b'), Proposition 5.3, we have

$$[(Y, 0), (X, \xi)] = (L_Y X, L_Y \xi) \in \Gamma E.$$

Hence, conditions a) and b') hold and we are done.

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Proposition 5.4. For a big-isotropic, projectable subbundle $E \subseteq T^{\text{big}}M$, $\Gamma E$ is closed by Courant brackets iff the spaces $\Gamma \Delta U$ of the local projected subbundles $\Delta U$ are closed by Courant brackets.

Proof. For two differentiable, local, cross sections of $E$ of the form prescribed by (5.7) the Courant bracket has the following expression

$$[(X, \pi^*\alpha), (Y, \pi^*\beta)] = ([X, Y], \pi^*(L_{\pi^*X}\beta - L_{\pi^*Y}\alpha)$$

$$+ \frac{1}{2}d(\alpha(\pi^*Y) - \beta(\pi^*X)))$$

and it is related by (5.7) with the Courant bracket $[(\pi^*X, \alpha), (\pi^*Y, \beta)]$ on $Q_U$. Thus, if $\Gamma \Delta$ is closed by Courant brackets, the left hand side of (5.8) is in $\Gamma E$. Furthermore, the Courant bracket satisfies the property that, $\forall f \in C^\infty(M)$, one has

$$[(X, \alpha), f(Y, \beta)] = f[(X, \alpha), (Y, \beta)] + (Xf)(Y, \beta)$$

$$- g((X, \alpha), f(Y, \beta))(0, df)$$

and the isotropy of $E$ yields

$$[f(X, \pi^*\alpha), h(Y, \pi^*\beta)] = fh[(X, \pi^*\alpha), (Y, \pi^*\beta)]$$

$$+ f(Xh)(X, \pi^*\alpha) - h(Yf)(Y, \pi^*\beta),$$

where $f, h \in C^\infty(M)$ may not be projectable. Accordingly, the existence of projectable bases of $E$ shows that the bracket-closure of $\Gamma \Delta U$ implies the bracket-closure of $\Gamma E$. The converse result is a straightforward consequence of the existence of the projectable bases of $E$. \hfill \Box

Remark 5.2. From (5.9) it follows that if $E \subseteq T^{\text{big}}M$ is closed by Courant brackets then either $E$ is isotropic or $E \supseteq T^*M$.

Proposition 5.5. If $E$ is a projectable, big-isotropic structure on the foliated manifold $(M, F)$ its orthogonal bundle $E'$ is projectable as well. Moreover, the corresponding, local, projected bundles $\Delta U$ are big-isotropic on the local transversal manifolds $Q_U$, the orthogonal bundles $\Delta' U$ are the local, projected bundles of $E'$, and $E$ is integrable iff the structures $\Delta U$ are integrable.
Proof. It is easy to check conditions a) and b’) of Proposition 5.3 for \( E' \). (In particular, for b’) take \((X, \alpha) \in \Gamma E, (Y, \beta) \in \Gamma E', Z \in T\mathcal{F} \), express \( Z(g((X, \alpha), (Y, \beta))) = 0 \) and use b’) for \( E \).) For the other assertions, use Proposition 5.4.

Example 5.1. Let \( P \in \chi^2(M) \) be a bivector field on \((M, \mathcal{F})\) and define

\[
D_P = T\mathcal{F} \oplus \{(\sharp_P \alpha, \alpha) / \alpha \in \text{ann}(T\mathcal{F})\}.
\]

This is an almost Dirac structure, which does not depend on the \( T\mathcal{F} \)-component of \( P \) in the following sense. If \( \nu\mathcal{F} \) is a normal bundle of \( \mathcal{F} \), i.e., \( TM = \nu\mathcal{F} \oplus T\mathcal{F} \), and if we put

\[
P = \frac{1}{2} P^{ab} Z_a \wedge Z_b + P^{au} Z_a \wedge Y_u + \frac{1}{2} P^{uv} Y_u \wedge Y_v,
\]

where \((z^a, y^u)\) are the local coordinates used in the proof of Proposition 5.3 and \( Z_a = \partial/\partial z^a, Y_u = \partial/\partial y^u - t^a u Z_a \) (for some coefficients \( t^a u \)), \( D_P \) is spanned by \((Z_a, 0)\) and \((P^{uv} Y_v, dy^u)\). Conditions a), b’) of Proposition 5.3 show that the projectability of \( D_P \) is equivalent with the projectability of the bivector field \( P \), i.e., \( \partial P^{uv}/\partial z^a = 0 \). Furthermore, using Proposition 5.5, we see that the integrability of \( D_P \) holds iff \( [P, P]_{\text{ann}(T\mathcal{F})} = 0 \) (Schouten-Nijenhuis bracket), hence \( D_P \) is equivalent with the transversely Hamiltonian structure defined by \( P \) on \((M, \mathcal{F}) \) \[14\].

Example 5.2. In a similar way, let \( \omega \) be a foliated 2-form on \((M, \mathcal{F})\) and \( \nu\mathcal{F} \) be a chosen normal bundle. Define

\[
D_\omega = T\mathcal{F} \oplus \{(X, \flat_\omega X) / X \in \nu\mathcal{F}\}.
\]

Then \( D_\omega \) is an almost Dirac structure. If we put

\[
\omega = \frac{1}{2} \omega_{uv}(y)dy^u \wedge dy^v,
\]

we see that \( D_\omega \) is spanned by \((Z_a, 0), (Y_u, \omega_{uv}(Y_v))\), therefore, \( D_\omega \) is foliated and independent on the choice of \( \nu\mathcal{F} \). Furthermore, \( D_\omega \) is integrable iff \( \omega \) is closed, i.e., \( D_\omega \) is equivalent with a foliated presymplectic form.

Now, we will discuss the concept of reduction. The general geometric framework may be described as follows. Let \( M \) be a manifold, \( E \) a subbundle
of $T^{\text{big}}M$ and $\iota : N \hookrightarrow M$ an embedded submanifold. Assume that $N$ is $E$-proper, meaning that the pullback $\iota^* E$ is differentiable. Then, assume that $N$ is endowed with a foliation $\mathcal{F}$ such that $T\mathcal{F} \subseteq \iota^* E$, a condition that is equivalent with

$$(r) \quad \forall Z \in T\mathcal{F}, \exists \alpha \in \text{ann} TN \text{ such that } (Z, \alpha) \in E|^N$$

and will be called the reducibility condition. Then, $\iota^* E$ may be foliated. If it is so and if the quotient space $N/\mathcal{F}$ is a paracompact, Hausdorff manifold $Q$, $\iota^* E$ projects to a subbundle $E^\text{red}_N \subseteq T^{\text{big}}Q$ that has the restrictions $\Delta_U$ of Definition 5.1 over the projections $\pi(U)$. Accordingly, $\pi^* E^\text{red}_N = \iota^* E$ and Proposition 5.2 shows that we may write

$$(5.14) \quad E^\text{red}_N = \pi_* (\iota^* E).$$

The bundle $E^\text{red}_N$ will be called the reduction of $E$ via $(N, \mathcal{F})$. This is a generalization of the framework described in the paper of Stiénon-Xu [10].

Now, the following result is immediate.

**Proposition 5.6.** Let $E$ be an integrable, big-isotropic structure on the manifold $M$. Let $\iota : N \to M$ be an embedded submanifold such that the fields of subspaces $E \cap (TN \oplus T_N^* M)$, $E' \cap (TN \oplus T_N^* M)$ are of a constant dimension. Let $\mathcal{F}$ be a foliation of $N$ by the fibers of a submersion $\pi : N \to Q$ such that the natural projection

$$(5.15) \quad \text{pr}_{T\mathcal{F}} : E \cap (T\mathcal{F} \oplus \text{ann} TN) \to T\mathcal{F}$$

is surjective. Then, there exists a well defined, integrable, reduced, big-isotropic structure $E^\text{red}$ on $Q$ that satisfies the condition $(5.14)$.

**Proof.** By Corollary 4.1 $\iota^* E$ is an integrable, big-isotropic structure of $N$ and condition $(5.15)$ is equivalent with the reducibility condition $(r)$. Thus, the reduced structure $E^\text{red}$ may exist and Corollary 5.1 tells us that $E^\text{red}$ exists indeed. Then, Proposition 5.5 shows that $E^\text{red}$ is an integrable, big-isotropic structure on $Q$. \hfill \Box

**Corollary 5.2.** Let $E$ be an integrable, big-isotropic structure on the manifold $M$. Assume that the connected, Lie group $G$ acts on $M$ and the action preserves $E$ and fixes an embedded submanifold $\iota : N \to M$. Assume that the restriction of the action of $G$ to $N$ is proper and free and denote by $\mathcal{F}$ the foliation of $N$ by the orbits of $G$. Finally, assume that the reducibility condition $(r)$ holds for the infinitesimal transformations $Z$ of $G$ on $N$. Then, there...
exists a Hausdorff manifold $Q = N/G$ endowed with a reduced, integrable, big-isotropic structure $E^{\text{red}}$.

**Proof.** Under the hypotheses, condition (r) holds for any $Z \in \chi^1(N)$ and the fields of subspaces $E \cap (TN \oplus T_N^*M)$, $E' \cap (TN \oplus T_N^*M)$ have a constant dimension.

**Example 5.3.** We apply Proposition 5.6 to a Poisson structure $E_P = E'_P = \text{graph} \sharp P$, where $P \in \chi^2(M)$ is a Poisson bivector field on $M$. Assume that the submanifold $\iota^* : N \hookrightarrow M$ is such that $\dim(E_P \cap (TN \oplus T_N^*M)) = \text{const.}$ and that $N$ has a foliation $\mathcal{F}$ with the quotient manifold $Q = M/\mathcal{F}$. Then, $N$ has the Dirac structure

$$\iota^* E_P = \{(\sharp P \alpha, \iota^* \alpha) / \sharp P \alpha \in TN, \alpha \in T_N^*M\}.$$ 

Furthermore, the reducibility condition (r) is equivalent to

$$T\mathcal{F} \subseteq \sharp P(\text{ann} TN). \quad (5.16)$$ 

Therefore, if we assume that (5.16) holds, $Q$ has the reduced Dirac structure $E^{\text{red}} = \pi_* \iota^* (E_P)$, which, pointwisely, turns out to be

$$E^{\text{red}} = \{(\pi_* \sharp P \tilde{\lambda}, \lambda) / \lambda \in T^*Q, [\tilde{\lambda}]_{\text{ann} TN} = \pi^* \lambda, \sharp P \tilde{\lambda} \in TN\}. \quad (5.17)$$

The reduced structure $E^{\text{red}}$ is Dirac. If we want a Poisson reduced structure we have to add the condition $E^{\text{red}} \cap TQ = 0$, which, by (5.17), is equivalent to

$$\sharp P(\text{ann} TN) \cap TN \subseteq T\mathcal{F}.$$ 

A more general Poisson reduction scheme, where $T\mathcal{F} = V \cap TN$ for a vector bundle $V$ over $N$ that satisfies adequate conditions, was given by Marsden and Ratiu [8]. The present example corresponds to the case $V = \sharp P(\text{ann} TN)$, while, however, we do not ask $\sharp P(\text{ann} TN)$ to be a regular vector bundle.

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