PHORMA: Perfectly Hashed Order Restricted Multidimensional Arrays

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Abstract

In this paper we propose a simple and efficient strategy to obtain a data structure generator to accomplish a perfect hash of quite general order restricted multidimensional arrays named phormas. The constructor of such objects gets two parameters as input: an \( n \)-vector \( a \) of non negative integers and a boolean function \( B \) on the types of order restrictions on the coordinates of the valid \( n \)-vectors bounded by \( a \). At compiler time, the phorma constructor builds, from the pair \( a, B \), a digraph \( G(a, B) \) with a single source \( s \) and a single sink \( t \) such that the \( st \)-paths are in \( 1-1 \) correspondence with the members of the \( B \)-restricted \( a \)-bounded array \( A(a, B) \). Besides perfectly hashing \( A(a, B) \), \( G(a, B) \) is an instance of an NW-family. This permits other useful computational tasks on it.

Keywords: Hash tables, Digraphs, Implicit enumeration, Nijenhuis-Wilf combinatorial families, Constructors of objects.

1 Motivation and objective

This work introduces a new type of data structure generator named phorma, \( P = (a, B) \), which consists of a positive integer \( n \)-vector \( a \) and a boolean function \( B \) whose literals are order restrictions on the components of the \( n \)-vectors \( \alpha \) dominated by \( a \), that is \( \alpha_i \leq a_i, \ i = 1, 2, \ldots, n \). The simplest example of phorma arises in the need to store a symmetric \((p \times q)\)-matrix. In this case the phorma is \( P_{\text{sim}}^2 = (a = (p, q), B = \alpha_1 \geq \alpha_2) \). Our basic goal is to enumerate in an efficient way all the equivalence classes of indices given that the matrix is symmetric. The work that motivates phormas, and where appears its first real use is [4]. Trying to avoid duplicates in the huge set of of equivalences classes of indices of some 3-dimensional matrices, we were led to implement the phormas: \( P_{\text{sim}}^3 = (a = (p, q, r), B = (\alpha_1 \geq \alpha_2) \lor (\alpha_2 \geq \alpha_3)) \)
and $P_3^{1=2} = (a = (p, q, r), B = (\alpha_1 \geq \alpha_2))$. The first phorma arises when there are symmetries permuting arbitrarily all the three coordinates. The second, when the first and second coordinates can be interchanged, but the third is held fixed. These phormas play a crucial role in the algorithms of [4].

To better motivate the concept and to help the reader to grasp the definition of the general problem we treat, we discuss at length an example of phorma (a less trivial one) arising in packing rectangles into rectangular and $L$-shaped pieces [5]. An $L$-shaped piece is a rectangle $R$ from which we have removed a smaller rectangle $r \subseteq R$. Moreover $R$ and $r$ have a corner in common. By effecting rotations, translations and reflections we may suppose that our $L$-shaped piece has a corner in the origin and the common vertex to $r$ and $R$ is the vertex opposite to the origin in rectangle $R$. Positioned in this canonical way, the $L$-piece is represented by a quadruple of real numbers $(\alpha_1 \alpha_2 \alpha_3 \alpha_4)$, with $\alpha_1 \geq \alpha_3$ and $\alpha_2 \geq \alpha_4$, where the big rectangle $R$ has diagonal from $(0,0)$ to $(\alpha_1, \alpha_2)$ and the smaller rectangle $r$ has diagonal from $(\alpha_3, \alpha_4)$ to $(\alpha_1, \alpha_2)$. Let $a = a_1 a_2 a_3 a_4$ be a positive integer 4-vector with $a_1 \geq a_3, a_2 \geq a_4$. In [5] we need to enumerate the canonically positioned $L$-shaped pieces with integer coordinates $a \leq a$, that is, the $L$-pieces $a = \alpha_1 \alpha_2 \alpha_3 \alpha_4$ with (1) $\alpha_1 \geq \alpha_3$ and (2) $\alpha_2 \geq \alpha_4$ are dominated by $a$, $\alpha_i \leq a_i$, $i = 1, 2, 3, 4$. Symmetry considerations enable us to partition the set of $a$-bounded $L$-pieces into equivalent classes and to distinguish a set $A$ of representatives for these classes.

For our occupancy purposes in [5] the $L$-pieces $\alpha_1 \alpha_2 \alpha_3 \alpha_4$ and $\alpha_2 \alpha_1 \alpha_4 \alpha_3$ must be considered equivalent: one such $L$ piece is transformed into the other by a reflection along the line passing through the origin and having slope 1. This is simply an axis interchange. With this in mind we have the following order restrictions for a representative of an equivalence class: (3) $\alpha_1 \geq \alpha_2$, otherwise we could use $\alpha_2 \alpha_1 \alpha_4 \alpha_3$. Also, (4) $\alpha_1 = \alpha_2 \Rightarrow \alpha_3 \geq \alpha_4$, otherwise we could use $\alpha_2 \alpha_1 \alpha_4 \alpha_3$ again. In terms of occupancy, $\alpha_1 \alpha_2 \alpha_1 \alpha_4$ with $\alpha_4 < \alpha_2$, which is a degenerated $L$, can (and must) be replaced by the rectangle $\alpha_1 \alpha_2 \alpha_1 \alpha_2$. Analogously, $\alpha_1 \alpha_2 \alpha_3 \alpha_4$ with $\alpha_3 < \alpha_1$ can be replaced by $\alpha_1 \alpha_2 \alpha_1 \alpha_2$. In this way, the equivalence $\alpha_1 = \alpha_3 \Leftrightarrow \alpha_2 = \alpha_4$ holds. The equivalence is rewritten as two opposite implications in the disguised form: (5) $((\alpha_1 \neq \alpha_3) \lor (\alpha_2 = \alpha_4))$ and (6) $((\alpha_2 \neq \alpha_4) \lor (\alpha_1 = \alpha_3))$. The restrictions (1) to (6) are gathered in a boolean expression $B^L$:

$$B^L = (\alpha_1 \geq \alpha_3) \land (\alpha_2 \geq \alpha_4) \land (\alpha_1 \geq \alpha_2) \land ((\alpha_1 \neq \alpha_2) \lor (\alpha_3 \geq \alpha_4)) \land ((\alpha_1 \neq \alpha_3) \lor (\alpha_2 = \alpha_4)) \land ((\alpha_2 \neq \alpha_4) \lor (\alpha_1 = \alpha_3)).$$

In general, a phorma, or a perfectly hashed order restricted multidimensional array, is a pair $P = (a, B)$ where $a$ is an $n$-vector of positive integers and for $\alpha$ a positive integer $n$-vector dominated by $a$, $B$ is a boolean function whose literals are of type $(\alpha_i \star \alpha_j)$, where $\star \in \{\leq, \geq, <, >, =, \neq\}$. The set $A = A(P) = A(a, B)$, (of representative of the classes in the case of the
Our objective in this work is given any phorma \( P = (a, B) \) to produce a constructive bijection \( h \) between \( A = A(P) \) and \( \{0, 1, \ldots, |A| - 1\} \), so that both \( h \) and \( h^{-1} \) are efficiently computable. Such functions are called perfect hash functions [3], [2] and their usefulness is well known in computer science.

As far as we know the problem of finding perfect hash functions for these quite general multidimensional arrays have not been considered before in the literature, whence the lack of more specific references and bibliography. Our solution is based in the theory of Nijenhuis and Wilf, chapter 13 of [6]. Their NW-combinatorial families associates a digraph to a set of combinatorial objects in such a way that an object is in \( 1-1 \) correspondence with a path in the digraph. See also a more detailed account of these combinatorial families in Wilf’s book [7], available at his page in the internet. A phorma is a particular case of NW-combinatorial family, specialized in boolean order specified multidimensional arrays. Their intrinsic structure permits us to accelerate, as we show in the final section, the calculus of \( h(\alpha) \) and \( h^{-1}(w) \).

2 The \((m, n)\)-patterns

For \( m \in \{1, 2, \ldots, n\} = N \), an \((n, m)\)-pattern \( \beta = \beta_1 \beta_2 \ldots \beta_n \) is a sequence of length \( n \) in which each of the \( m \) symbols \( 1, 2, \ldots, m \) occurs at least once. Given a phorma \((a, B)\) and \( \alpha \in A = A(a, B) \) with \( m_\alpha \) distinct entries there exists a unique \((n, m_\alpha)\)-pattern, denoted by \( \beta^\alpha \), which is order compatible with \( \alpha \): for \( i \in N \), if \( \alpha_i \) is the \( k \)-th smallest entry among the ones appearing in \( \alpha \), then define \( \beta^\alpha_i = k \). As some examples, consider the phorma \((a = 7575, B_L)\), where \( B_L \) appears in the previous section. We have \( \beta^\alpha_{7412} = 4312 \), \( \beta^\alpha_{5521} = 3321 \), \( \beta^\alpha_{5533} = 2211 \), \( \beta^\alpha_{3333} = 1111 \). Let the set \( \mathcal{L} = \mathcal{L}(P) = \mathcal{L}(a, B) \) of all \((n, m)\)-patterns induced by \( \alpha \in A(a, B) \),

\[
\mathcal{L} = \mathcal{L}(P) = \mathcal{L}(a, B) = \{ \beta^\alpha \mid \alpha \in A(a, B) \} = (\beta^1, \beta^2, \ldots, \beta^q),
\]

be given by a list in lexicographical order. The list \( \mathcal{L} \) induces a partition of \( A \) in \( q \) parts: indeed, defining \( [\beta^j] = \{ \alpha \in A \mid \beta^\alpha = \beta^j \} \), we get \( A = \bigcup_{j=1}^q [\beta^j] \), with \( [\beta^j] \cap [\beta^k] = \emptyset \) if \( j \neq k \). For the phorma \( P_{7575}^L = (a = 7575, B_L) \) we get

\[
\mathcal{L}^L = \mathcal{L}(P_{7575}^L) = (1111, 2121, 2211, 3211, 3221, 3321, 4231, 4312, 4321).
\]

There are only mild restrictions on the subset \( \mathcal{L} \): its cardinality, \( q \), should be small enough in order for the \( \beta^j \)'s to be kept in core; also, \( \mathcal{L} \) should have enough structure to be effectively generated by an implicit enumeration scheme. In the implementation of a phorma \((a, B)\), the first task of the constructor of the
data structure [1] phorma (which is activated at compiler time) is to obtain the list \( \mathcal{L} \). How this is done? In many applications the dimension \( n \) of the phorma is small enough for trying all \( n^n \) sequences of length \( n \) in symbols 1, 2, ..., \( n \), choose the ones which are \((m, n)\)-patterns and test for \( B \)-satisfiability [2]. The sequences that survive are added to \( \mathcal{L} \). In the above case \( n^n = 256 \) and this simple minded approach is convenient. In some cases it is more efficient to use appropriate \( NW \)-combinatorial families [6], [7] which generate only \((m, n)\)-patterns. Here we avoid details of these specific families. In other cases, the list \( \mathcal{L} \) is obtained by implicit enumeration. In any case, testing \( B \)-satisfiability is unavoidable and is the computational bottleneck for the phorma constructor in obtaining \( \mathcal{L} \).

3 The digraphs \( H_\gamma \)'s, \( H^n \) and \( G(P) = G(a, B) \)

Throughout this work \( \gamma = \gamma_1 \gamma_2 \ldots \gamma_m \) is an strictly increasing \( m \)-sequence, \( m \leq n \), with entries in \( N \). For each \( \alpha \in A \), let \( \gamma^\alpha \) denote the strictly increasing sequence of length \( m_\alpha \) of the \( m_\alpha \) distinct entries appearing in \( \alpha_1 \alpha_2 \ldots \alpha_n \). Observe that \( \alpha \) is recoverable from (induced by) the pair \((\beta^\alpha, \gamma^\alpha)\). The \( \alpha \) so induced by \((\beta, \gamma)\), where \( \beta \) is an \((m, n)\)-pattern and \( \gamma \) is an strictly increasing \( m \)-sequence with entries in \( N \), is denoted \( \alpha^*(\beta, \gamma) \). As examples, in the phorma \( P_{7575}^L \), \( \alpha^*(3221, 457) = 7554, \alpha^*(4321, 457) = 7543, \alpha^*(4231, 4567) = 7564 \). As we shall see, the simple correspondences \( \alpha \rightarrow (\beta^\alpha, \gamma^\alpha) \) and its inverse, \( (\beta, \gamma) \rightarrow \alpha^*(\beta, \gamma) \), inducing \( \alpha \equiv (\beta, \gamma) \), are central for the efficient implementation of the hash function \( h \) and its inverse.

For \( \beta \in \mathcal{L} \) the \((a, \beta)\)-maximal increasing sequence, denoted by \( \gamma^*(a, \beta) = \gamma_1^* \gamma_2^* \ldots \gamma_m^* \), is the strictly increasing sequence of length \( m \) satisfying the following conditions: suppose that, for \( 1 \leq i \leq m \), \( i \) occurs at positions \( p_{i1}, \ldots, p_{ii} \) of \( \beta \); recall that \( a = a_1 a_2 \ldots a_n \) and define \( \gamma_i^* = \min \{ a_{p_{am}}, a_{p_{m+1}}, \ldots, a_{p_{mj}} \} \) and for \( i = m - 1, m - 2, \ldots, 1, \gamma_i^* = \min \{ a_{p_{m}}, \ldots, a_{p_{m}}, \gamma_{i+1}^* - 1 \} \). Observe that \( \gamma^*(a, \beta) \) can alternatively be defined as the lexicographically maximal \( \gamma \) such that \( \alpha^*(\beta, \gamma) \in A \).

Having constructed the list \( \mathcal{L} = \mathcal{L}(a, B) = (\beta^1, \beta^2, \ldots, \beta^n) \), the next task for the phorma constructor is to obtain a corresponding list \( \Gamma = \Gamma(a, \mathcal{L}) = (\gamma^*(a, \beta^1), \gamma^*(a, \beta^2), \ldots, \gamma^*(a, \beta^n)) \). As an example to help the understanding of how to obtain \( \Gamma \), consider its construction for the phorma \( P_{7575}^L \). We get

\[
\Gamma_{7575}^L = \Gamma(P_{7575}^L) = (5, 57, 45, 457, 457, 345, 4567, 3457, 3457).
\]

Suppose \( \gamma = \gamma_1 \gamma_2 \ldots \gamma_m \) is an increasing \( m \)-sequence with entries in the set of positive integers. We want to define a digraph \( H_\gamma \). If \( \gamma_m > m \) let \( \neg \gamma \)
denote the increasing sequence of length \(m\) satisfying \((-\gamma)_m = \gamma_m - 1\) and 
\((-\gamma)_i = \min\{(-\gamma)_{i+1} - 1, \gamma_i\}\), for \(i = 1, 2, \ldots, m - 1\). If \(\gamma_m = m\), then \(-\gamma\) does not exist. If \(\gamma \neq t\), let \(\gamma\) be the sequence of length \(m - 1\) obtained from \(\gamma\) by removing its last entry: \(\gamma = \gamma_1 \ldots \gamma_{m-1}\). If \(\gamma = t\), then \(-\gamma\) does not exist. Given \(\gamma, \tilde{\gamma} \in \Gamma\), we say that \(\gamma \leq \tilde{\gamma}\), if there is a sequence \((\gamma_i = \gamma^1, \gamma^2, \ldots, \gamma^p = \tilde{\gamma})\), with \(\gamma^i \in \Gamma\), such that, for each \(i = 1, 2, \ldots, p - 1\), either \(\gamma^{i+1} = -\gamma^i\) or else \(\gamma^{i+1} = \uparrow\gamma^i\). The relation \(\leq\) is a partial order in the set \(F_{\infty}\), of all finite increasing sequences with integer entries. For \(\gamma \in F_{\infty}\), let \(H_\gamma\) be the acyclic digraph whose vertex set is \(VC_\gamma = \{\gamma \mid \gamma \leq \gamma\}\). The empty increasing sequence is considered a member of \(F_{\infty}\). It corresponds to a terminal vertex (the unique sink), and so, is represented by \(t\). From each vertex \(\gamma \in VC_\gamma\) there are at most two outgoing edges whose heads are \(\gamma\) (if it exists) and \(\uparrow\gamma\) (if it exists). These are all the edges, what concludes the definition of \(H_\gamma\). In Fig. 1 we show all the graphs \(H_{\gamma_j}, j = 1, 2, \ldots, 9\), corresponding to \(\Gamma_{5575}^L\). Since \(\gamma^*(a, \beta^4) = \gamma^*(a, \beta^5)\) and \(\gamma^*(a, \beta^8) = \gamma^*(a, \beta^9)\) we get only seven distinct digraphs. In picturing them, the direction of the edges are implicit. They go from higher vertices to lower ones and in the case of a draw, the direction is from right to left.

The digraph \(H^a\) is defined as the \(L\)-indexed union of digraphs \(H_\gamma\)’s:

\[H^a = \bigcup\{H_{\gamma^*(a, \beta)} \mid \beta \in L\}\]
The **digraph of the phorma**, \( P = (a, B) \) is \( G(a, B) = H^a \cup \Lambda(a, B) \), where digraph \( \Lambda(a, B) \) consists of a root \( s \) linked to vertices labelled by \( \beta^j \), \( j = 1, 2, \ldots, q \). Each vertex \( \beta^j \) is of valency 2. The edge from \( s \) enters it and there is an edge from it to the vertex of \( H^a \) labelled by \( \gamma^*(a, \beta^j) = \gamma^j \). The total number of edges of \( \Lambda(a, B) \) is \( 2q \), finishing its definition. This also concludes the definition of the digraph \( G(a, B) \). In Fig. 2 we show \( G(P^{L}_{7575}) = G^{L}_{7575} \).

Observe that \( \Lambda(a, B) \) is depicted in dashed gray edges. The numbers on gray are important in the computation of \( h(\alpha) \) and are explained in the next section.

**Figure 2:** Digraph \( G^{L}_{7575} = G(7575, B^L) \) of the phorma \( P^{L}_{7575} = (7575, B^L) \)

### 4 NW-Combinatorial Families

We briefly recall the general concept of an NW-combinatorial family. An example of such an object is the digraph \( H_\gamma \). The combinatorial family that it encodes is formed by the strictly increasing \( m \)-sequences \( \gamma \) with entries in \( N \) dominated by \( \gamma, \gamma_i \leq \gamma_i, i \in N \). Also the digraph \( G(a, B) \), for any phorma \((a, B)\), is an NW-combinatorial family.
The following concept, introduced in [6], is the central tool for this work. A *Nijenhuis-Wilf combinatorial family* or *NW-family* is a digraph \( G \), whose vertex set is denoted by \( V(G) \), having the properties below:

- \( V(G) \) has a partial order (for \( x, y \in V(G) \), \( y \preceq x \) if there is a directed path from \( x \) to \( y \)) with a unique minimal element \( t \). For each \( v \in V(G) \) the set \( \{ x \in V(G) \mid x \preceq v \} \) is finite and includes \( t \).
- Every vertex \( v \), except \( t \) has a strictly positive outvalence \( \rho(v) \). For each \( v \in V(G) \), the set \( E(v) \) of outgoing edges has a local rank-label \( \ell_v \), \( 0 \leq \ell_v(e) \leq \rho(v) - 1 \), \( e \in E(v) \).

Every directed path in \( G \), starting from a vertex \( v \) and ending at \( t \) is called a *combinatorial object of order* \( v \). Thus, the set of objects of order \( v \) is identified with the vertex \( v \). Denote by \( |v| \) the cardinality of the set of objects of type \( v \), namely, the number of paths from \( v \) to \( t \). In Fig. 2, the values \( |v| \) are shown as a gray number next to vertex \( v \) and is the first of the two gray numbers in the case that \( v = \beta^j \). The significance of the second gray numbers associated with the \( \beta^j \)’s in Fig. 2 are explained in the final section. The local rank-labels of the outcoming edges at \( s \) in the NW-combinatorial family \( G(a, B) \) is given by the lexicographical order of their heads \( \beta^j \). The unique outcoming edge at \( \beta^j \) has local rank-label 0. The local rank-labels of the outcoming edges at a vertex \( \gamma \) of \( H_\gamma^* \) is 0 for the *west edge* (the one with head \( -\gamma \)), if it exists, implying 1 for the *southwest edge* (the one with head \( \uparrow \gamma \)). Of course, if a vertex has only its southwest edge, the local rank-label of this edge is 0. Even though in the drawings the edges arriving at \( t \) are not in the southwest direction (to decrease the width of the figures), all of them are considered southwest edges.

From the definitions we get immediately a recursive formula for \( |v| \): \[ |v| = \sum \{ |\text{head}(e)| \mid e \in E(G), \text{tail}(e) = v \} \]. This recursive formula follows from the fact that a path from \( v \) to \( t \) is an outgoing edge from \( v \) followed by a path representing a combinatorial object of smaller order. Therefore, the role of the graph \( G \) defining the NW-combinatorial family is to display how the combinatorial elements of the various orders are inductively formed. The usefulness of the notion of combinatorial family is that (i) a great number of usual combinatorial objects can be encoded as paths in an NW-family; (ii) the local rank-labels of the outcoming edges induce a *unique ranking* \( h \) of the combinatorial objects of order \( v \). With respect to this ranking the following four tasks become computationally simple and as cheap as they can be. The tasks are exemplified and described in terms of the paths in digraph \( G \), without mentioning the specific combinatorial families that \( G \) encodes. More details of the algorithms to perform these tasks can be found in Chapter 13 of [6].

**Task 0: counting:** What is the cardinality of the family? **Algorithm:** As we have mentioned, \( |v| = \sum \{ |\text{head}(e)| \mid e \in E(G), \text{tail}(e) = v \} \). It is then possible for the constructor of the phorma to obtain the value of each \( |v| \) by recursion and
to store it as an attribute of \( v \in V(G) \) in a pre-processing phase (compilation time). For instance, for the phorma \( P_{7575}^L \) has cardinality 190. This is the value of \( |s| \), in Fig. 2.

**Task 1: sequencing:** Given an object in the family, construct the “next” object. **Algorithm:** A path starting at \( v \) and ending in \( t \) is encoded by the sequence of label-ranks of the sequence of its edges. The next path of a given path \( \pi \) in coded form is, in coded form, the lexicographic successor of \( \pi \). In coded form the 7 paths from the vertex \( v \) of the \( NW \)-combinatorial family \( H_{\gamma^*} \) of Fig. 3 are: rank 0 \( \rightarrow 00000 \), rank 1 \( \rightarrow 01000 \), rank 2 \( \rightarrow 01100 \), rank 3 \( \rightarrow 0111 \), rank 4 \( \rightarrow 10000 \), rank 5 \( \rightarrow 1101 \), rank 6 \( \rightarrow 111 \). In Theorem 1 we shall see that these paths are in 1−1 correspondence with the sequence of \( \gamma \)'s (123, 124, 134, 234, 125, 135, 235). This is the sequence, in rank order, of all strictly increasing sequences of length 3 in \( \{1, 2, 3, 4, 5\} \) dominated by \( \gamma^* = 235 \).

![Figure 3: All paths from \( v = \gamma^* = 235 \) to \( t \) in \( H_{235} \)](image)

**Task 2: ranking (perfect hashing):** Given an object \( \omega \) in the family, find the integer \( h(\omega) \) such that \( \omega \) is the \( h(\omega) \)-th element in the order induced by task 1. **Algorithm:** Let an element-path \( \pi \) of order \( v \) of an \( NW \)-family, \( \pi = (e_1, e_2, \ldots, e_p) \) be given. The rank of \( \pi \) is defined as \( h(\pi) = \sum_{i=1}^{p} \chi(e_i) \), where \( \chi(e) = \sum \{|\text{head}(f)| \text{ with } \ell_v(f) < \ell_v(e), f \in E(v)\} \). In the \( NW \)-combinatorial family, this formula for \( \pi \) is particularly simple: the value \( h(\pi) \) is obtained as sum of the orders of the post-falls of \( \pi \) (defined in the beginning of next section). In Fig. 3, the post-falls of the paths are the white vertices.

**Task 3: unranking:** Given an object integer \( r \) construct the \( r \)-th member of the family. **Algorithm:** Given an integer \( r \), we need to construct the \( r \)-th path from \( v \) to \( t \). Consider \( \text{pred}_v(e) \) as the highest-rank edge of the set \( \{f \in E(v) \mid \ell_v(f) < \ell_v(e)\} \), and let \( |\text{head}(|\text{pred}_v(e)|)\rangle = 0 \) if this set is empty. The required \( r \)-th path’s \( \pi_r \) is generated as follows: \( \pi_r \leftarrow \emptyset \); \( r' \leftarrow 0 \); \( v' \leftarrow v \); repeat append to \( \pi_r \) the highest-rank edge \( e \) of \( E(v') \) such that \( r' + |\text{head}(|\text{pred}_v(e)|)\rangle| \leq r \); \( r' \leftarrow r' + |\text{head}(|\text{pred}_v(e)|)\rangle| \); \( v' \leftarrow \text{head}(e) \) until \( v' = t \). It should not be difficult to check this unranking algorithm in the paths of Fig. 3.

**Task 4: getting random object:** Choose an object uniformly at random from the given family. **Algorithm:** Let \( \xi \in [0, 1] \) be uniformly chosen at random; return
the \(|v| * \xi\)-th object.

5 1 − 1 Correspondences

Let \(\pi\) be a path which starts at \(\gamma^*\) and finishes at \(s\). A fall of \(\pi\) is the tail of a southwest edge, thus \(\pi\) has \(m\) falls, where \(m\) is the length of \(\gamma^*\). A post-fall in \(\pi\) is the vertex which is the head of an edge whose tail is a fall. Path \(\pi\) has at most \(m\) post-falls.

**Theorem 1** The st-paths in digraph \(H_{\gamma^*}\) are in 1 − 1 correspondence with the strictly increasing \(m\)-sequences with entries in \(N\) which are dominated by \(\gamma^*\).

**Proof:** Any such path \(\pi\) is in 1 − 1 correspondence with its sequence of falls \((\gamma^m, \ldots, \gamma^2, \gamma^1)\). Note that \(\gamma^j\) \((j = 1, 2, \ldots, m)\), is the last vertex of \(\pi\) whose defining sequence has length \(j\). Let \(\gamma^j_{\pi} = \gamma_j^\pi\), \(j = 1, 2, \ldots, m\). Clearly \(\gamma^j_{\pi} = \gamma_j^\pi\leq \gamma_j^\pi\), and \(\gamma^\pi\) is dominated by \(\gamma^*\). Reciprocally, given a \(\gamma\) dominated by \(\gamma^*\), construct a \(\pi^\gamma \equiv (\gamma^m, \ldots, \gamma^2, \gamma^1)\) so that starting from vertex \(\gamma^*\), the last vertex whose defining sequence has length \(j\) is \(\gamma^j\) defined when we impose the equality \(\gamma^j_j = \gamma_j^\gamma\). With these definitions, it follows that \(\pi^{(\gamma^*\gamma)} = \pi\), proving the Theorem.

To exemplify the above inverse constructions, consider the path \(\pi\) from \(\gamma^* = 8CFJ\) to \(t\) in \(H_{8CFJ}\) (subscript in base 20 : \(A = 10, B = 11, \ldots, J = 19\)) defined by the sequence of its falls \((8CDE, 567, 34, 3)\). Path \(\pi\) induces \(\gamma^\pi\) given by the last digits of the falls in reverse order: \(\gamma^\pi = 347E\). Reciprocally, given \(\gamma = 347E\), starting at \(8CFJ\) the last digit of the first fall of the path \(\pi^\gamma\) that we seek is the fourth digit of \(\gamma\). Thus, we must go \(J - E = 5\) steps to the left to arrive at \(8CDE\), defining the first fall of \(\pi^\gamma\). Following the southwest edge we arrive at \(8CD\). We know that the last digit of the second fall of \(\pi^\gamma\) must be \(7\) (the third digit of \(\gamma\)). Thus we must go \(D - 7 = 6\) steps to the left arriving at the second fall 567. Go southwest, arriving at 56. The last digit of the third fall is the second digit of \(\gamma\), namely 4. We must go \(6 - 4 = 2\) steps left arriving at the third fall 34. Go southwest, arriving at 3. The last digit of the fourth fall is the first digit of \(\gamma\), namely 3. We must go \(3 - 3 = 0\) steps left to get the fourth fall of \(\pi^\gamma\), namely 3. In this way, from \(\gamma\) and \(\gamma^*\) we have obtained the sequence of falls \((8CDE, 567, 34, 3)\). This sequence of falls define \(\pi^\gamma\). Clearly, \(\pi^{(\gamma^*\gamma)} = \pi\).
Theorem 2 For any phorma $P = (a, B)$, the st-paths in digraph $G(a, B)$ are in $1 - 1$ correspondence with $A(a, B)$.

Proof: Let $\pi$ be an st-path in digraph $G(a, B)$ and $(s, \beta^1, \gamma^1, \gamma^2, \ldots, \gamma^p, t)$ be the sequence of vertices in $\pi$. By Theorem 1, the subpath from $\gamma^1$ to $t$ which is in $H_{\gamma^1}$ defines a $\gamma^\pi$ dominated by $\gamma^1$. The $\alpha$ which corresponds to $\pi$ is $\alpha^*(\beta^1, \gamma^\pi)$. Reciprocally, given $\alpha \in A(a, B)$, consider the pair $(\beta^\alpha, \gamma^\alpha)$. Let $\pi'$ be the path in $H_{\gamma^*(a, \beta^\alpha)}$ from $\gamma^*(a, \beta^\alpha)$ to $t$ which corresponds to $\beta^\alpha$, given by Theorem 1. The st-path $\pi$ in $G(a, B)$ which corresponds to $\alpha$ is obtained from $\pi'$ by prefixing to it the two edges, from $s$ to $\beta^\alpha$ and from $\beta^\alpha$ to $\gamma^*(a, \beta^\alpha)$. The correspondences $\pi \mapsto \alpha$ and $\alpha \mapsto \pi$ are inverses establishing the Theorem.

6 Implementation Issues

Entering a generic phorma type boolean function $B$. A convenient way to store such boolean functions is by means of a tree $T(B)$ with three types of internal nodes: $\lor$-nodes, $\land$-nodes, $\neg$-nodes. The leaves of the tree correspond to the basic constituent boolean functions of type $\alpha_i \star \alpha_j$, where $\star \in \{\leq, \geq, <, >, =, \neq\}$. The $\neg$-nodes (negation operator) must have at most one child. Note that each subtree rooted at an internal $\diamond$-node $v$ ($\diamond \in \{\lor, \land, \neg\}$) is a boolean tree obtained by taking the $\diamond$-operation of the boolean tree(s) corresponding to the children of $v$. Given an $\alpha$ it is rather easy to decide $B$-satisfiability of $\alpha$, by evaluating from the leaves up and arriving to the root of $T(B)$.

Properly storing $H^\alpha$. Let $a^*$ be the maximum of the $a_i$’s. Consider a bidimensional array $R[0..n, 1..a^*]$, in which cell $R[m, p]$ contains the address of a simple linked list containing in $\gamma$-lexicographical order all the pairs $(\gamma, |\gamma|)$ in which $\gamma$ is a vertex of $H^\alpha$, and, as a sequence, has length $m$ and satisfies $\gamma_m = p$. The need to use the pairs $(\gamma, |\gamma|)$ become clear to efficiently
perform the rank operation, as explained below. The maximum length of the list \( R[m, p] \), denoted by \( |R[m, p]| \) is \(|L(a, B)|\), however, these lengths tends to be very small. In particular, if all the entries of \( a \) are equal, or if \( p \leq 2 \), \( |R[m, p]| = 1 \). In the example of Fig. 2, the only entry \( |R[4, 7]| = 2 \). Graph \( H^a \) is stored as a hash table \( R[0..m, 1..a^*] \) in which the pairs \( (\gamma, \gamma') \) having \( \gamma \) with the same length \( m \) and the same last element \( p \) are stored together in a \( \gamma \)-lexicographically ordered list (to resolve the conflicts). We consider that a binary search in the list \( R[m, p] \) to locate the specific pair \( (\gamma, \gamma') \) is good enough.

**Ranking in the NW-Family \( H_\gamma \).** In order to obtain the rank of \( \gamma \in H_\gamma \) of length \( m \) based in a usual pointer implementation ([2]) of \( H^a \) we may need to walk along a path \( \pi_\gamma \) of length \( a^* + m \), where \( a^* = \max \{a_i \mid i = 1, 2, \ldots, n\} \). By using the above hash table to store \( H^a \), we do the job in \( m \) steps. This is a critical speeding up improvement, since in most applications \( a^* \gg m \). Let \( h_\gamma(\gamma) \) denote the rank of \( \gamma \) in \( H_\gamma \). Let \( m \) be the length of \( \gamma \) and \( \gamma^m, \gamma^2, \gamma^1 \) the sequence of falls of \( \gamma \). We know that \( h_\gamma(\gamma) \) is the sum of the orders of the corresponding post-falls, \( \mid -\gamma^m \mid + \mid -\gamma^{m-1} \mid + \ldots + \mid -\gamma^2 \mid + \mid -\gamma^1 \mid \). In this rank formula, if \( -\gamma^j \) does not exists then \( \mid -\gamma^j \mid \) is defined as 0. Let \( -\gamma \) be denoted by \( 2-\gamma, 3-\gamma, \ldots \) be denoted by \( 3-\gamma, \) etc; also \( 0-\gamma = \gamma \). If \( \gamma_m \geq m + j \), then \( j-\gamma \) exists and is given by \( (j-\gamma)_i = \min \{\gamma_m - j - m + i, \gamma_i\} \), for \( i = 1, 2, \ldots, m \). The \( \gamma^i \)'s can be found as follows: Let \( \xi_0 = \gamma^*_m - \gamma_m \) and \( \gamma^m = \xi_{m-\gamma} \). For \( i = m - 1, m - 2, \ldots, 2, 1 \) let \( \xi_i = \gamma^i_{i+1} - \gamma_i \) and \( \gamma^i = \xi_{i-\gamma} \). Since we need only \( \mid -\gamma^m \mid, \mid -\gamma^{m-1} \mid, \ldots, \mid -\gamma^2 \mid, \mid -\gamma^1 \mid \), it is enough to store, for each vertex \( \gamma \in V(H^a) \), the pair of entries \( (\gamma, \gamma^l) \). All such pairs with \( \gamma \) of length \( m \) and \( \gamma_m = p \) are stored in the list \( R[m, p] \), ordered lexicographically by \( \gamma \). Since all \( \gamma^l \)'s in the pairs \( (\gamma, \gamma^l) \)'s stored at \( R_m(p) \) satisfy \( \gamma_m = p \), we may drop the last entry of \( \gamma \) and store the pairs \( (\gamma, \gamma^l) \).

**Ranking in the NW-Family \( G(A, L) \).** For a vertex \( v \) with at most one incoming edge \( e_v \) of an NW-family, let \( ||v|| = \chi(e_v) \). This is the case of \( s \) and of \( \beta \) in \( L(a, B) \). The value of \( ||s|| \) is zero and the values of \( ||\beta|| \)'s are pre-computed for each \( \beta \). Translating from the general recipe for ranking in an NW-family to our specific case,

\[
h(\alpha) = ||s|| + ||\beta|| + h_\gamma(a, \beta) = ||\beta|| + h_\gamma(a, \beta)(\gamma^a).\]

\( \beta^a \) is located by a binary search on \( L(a, B) \). From the pair \( (a, \beta^a) \) we compute \( \gamma^*(a, \beta^a) \) which is the entry point in \( H^a \) of the path \( \pi^a \).
7 Conclusion

We have defined a data structure generator which permits the perfect hash of order restricted multidimensional arrays $A(a, B)$. The restrictions accord a very general type of boolean functions $B$ formed by order restricting pairs of entries of the array in arbitrary ways. Our scheme is the conjunction of two ideas: (1) To make a list $L(a, B)$ of the order patterns which induce a partition of $A(a, B)$. (2) Distinct patterns which have the same increasing sequences of symbols are treated together. In consequence, an $n$-vector $\alpha \in A(a, B)$ is subdivided into two pieces of information $\beta$, the pattern associated to $\alpha$ and $\gamma$, the increasing sequence of distinct symbols appearing in $\alpha$. This encoding has the power of perfectly addressing huge arrays $A(a, B)$ by means of logarithmically smaller digraphs $G(a, B)$ ($NW$-combinatorial families). This general type of perfect hash scheme does not seem to have been treated before in the literature. In particular, its applications to database systems is a possible source of relevant applications and remains to be investigated.

References

[1] G. Booch. Object oriented design with applications. The Benjamin/Cummings Publishing Company, Inc, ISBN 0-8053-0091-0, 1991

[2] T. Cormen, C. Leiserson and R. Rivest. Introduction to Algorithms. The MIT Electrical Engineering and Computer Science Series — The MIT Press, McGraw-Hill Book Company, ISBN 0-262-03141-8, 1990.

[3] D. Knuth, The art of computer programming, vol 3. Adinson Wesley (second edition) 1975.

[4] L. Lins, S. Lins and R. Morabito. An $n$-tet graph approach to non-guillotine packings of $n$-dimensional boxes into an $n$-container. European Journal of Operations Research 141 (2002) 421-439.

[5] L. Lins, S. Lins and R. Morabito. An $L$-approach for packing $(\ell, w)$-rectangles into rectangular and $L$-shaped pieces. Submitted to the Journal Operations Research Society (August 2002).

[6] A. Nijenhuis and H. S. Wilf. Combinatorial algorithms for computers and calculators. Academic Press (second edition) 1978.

[7] H. S. Wilf. East side, west side . . . Available in PDF at the home page of the author: www.cis.upenn.edu/~wilf, 1990.