‘t Hooft model on the lattice

M. García Pérez, A. González-Arroyo*, Liam Keegan and Masanori Okawa

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* Instituto de Física Teórica UAM/CSIC
Departamento de Física Teórica, UAM

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What? 't Hooft model

This is just $d=2$ Yang-Mills theory in the large $N$ limit. Quarks only appear as sources ($N_f/N = 0$). 't Hooft Solution:

- An infinite stable spectrum of mesons with asymptotically linearly rising mass square

$$\mu_n^2 \sim n\pi^2 m_0^2$$

where $m_0^2 = g^2 N/\pi$ is the natural unit of mass ($m_0 = 1$). States have alternating even and odd parity.

- As the quark mass $m_q$ goes to zero the mass of the lowest state (pion) goes to zero as follows:

$$\mu_0^2 = \frac{2\pi}{\sqrt{3}} m_q + O(m_q^2)$$

(The order of the limits matter).

- Higher order states do not vanish in that limit. Their masses can be determined numerically by solving a 1-dim integral eq.
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Despite its attractive power very little has been done on the lattice.

- **Gross Witten 1980:**
  The partition function and rectangular Wilson loop exp. values obtained exactly for \( V = N = \infty \) U(N) theory with Wilson action:

\[
E \equiv \frac{1}{N} \langle \text{Tr}(U(P)) \rangle = 1 - \frac{1}{4b}
\]

\[
\log W_{R \times T} = \log(E)RT
\]

- **Berruto, Giusti, Hoebling and Rebbi 2002:**
  Studied \( d=2 \) U(N) and SU(N) QCD for \( N=2,3,4 \) at rather coarse lattices but made no extrapolation.

- **Kiskis, Narayanan and Neuberger 2002:**
  First attempt towards a quantitative determination of spectrum.
  They use a **reduced model** (Very similar to the present work).
How?: Our method

We make use of **volume independence**:

Matrix model at $N = \infty \equiv V = N = \infty$ LGT

Matrix model (2d version of TEK):

$$Z = \int dU_0 \, dU_1 \, \exp\{bNz \text{Tr}(U_0 U_1 U_0^\dagger U_1^\dagger) + \text{h.c.}\}$$

where $z = e^{2\pi i k/N}$ ($k$ coprime with $N$) and $b = 1/\lambda_L$

Expectation values:

$$W_{R \times T} = \frac{Z^{RT}}{N} \langle \text{Tr}(U_0^T U_1^R U_0^{-T} U_1^{-R}) \rangle$$

Finite $N$ corrections are equivalent to finite volume for $L = N$ + only partial suppression of non-planar diagrams.
Meson correlation functions

Only assumption about quarks: They live in an $N \times (l_0 N)$ lattice. Meson operators: $O_\Gamma(x) = \bar{\Psi}(x) \Gamma \Psi(x)$ with $\Gamma = I, \vec{\sigma}$.

$$C_{\Gamma\Gamma'}(t) = -\sum_x \langle O_\Gamma(0) O_\Gamma(t, \vec{x}) \rangle$$

One gets

$$C_{\Gamma\Gamma'}(t) = \sum_{p_0} e^{ip_0 t} \text{Tr}(\Gamma D^{-1}(p_0) \Gamma' D^{-1}(0))$$

$D(p_\mu)$ is fermion a operator (Wilson/overlap) of a single-site with $U_\mu \rightarrow U_\mu \otimes \Gamma^*_\mu e^{ip_\mu}$. $p_0 = 2\pi n/(NL_0)$

$D$ is an $N^2 \times N^2$ matrix

($\Gamma_0, \Gamma_1$ are 't Hooft clock matrices)
A good testing ground for methodologies for determining the meson spectrum at large $N$: The exact results are known!!

- Computationally simple

- A first step towards studying other unsolved $d = 2$ theories at large $N$ for which many conjectures and ideas are available: Veneziano limit, quarks in other representations, etc
Simulation details

We are studying the model using the twisted reduced model at various $b$:

$$b = 3, 4, 5, 6, 8, \ldots?$$

and $N$:

$$N = 31, 43, 53, \ldots?$$

for various smearings and with

- Wilson fermions  /  naive fermions  /  overlap fermions

and various analysis methods

- variational analysis  /  multiexponential fits  /  mom. space fits
Gauge field dynamics

The reduced model captures the physics of the infinite volume system.
Twisting is not essential but helps:

\[ E = 1 - \frac{1}{4b} \quad \text{for twist} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad \text{for periodic} + \mathcal{O}\left(\frac{1}{N}\right) \]
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![Effective string tension N=53 b=8](image-url)
Wilson fermions \( r=1 \)

We fix the lattice spacing in \( m_0 \) units with \( a = 1 / \sqrt{b \pi} \).

Good behaviour for small quark masses:

\[
m^2_\pi = A m_q + B m^2_q \]

with \( A = 3.16 \) for \( b = 3 \) and \( A = 3.38 \) for \( b = 4 \)

\('t\) Hooft result is \( 3.628 = 2\pi / \sqrt{3} \)

Large errors at large masses: Lattice artifacts?

![Graph showing pion mass for Wilson fermions](image-url)
Naive fermions \( r=0 \)

**Advantages**
- Corrections are only of order \( a^2 \)
- \( M_q \) corresponds to \( 1/(2\kappa) \)

**Disadvantages**
- The states are (quasi)-doubly degenerate
- The time-dep of correlators is more complex

\[ M.q \text{ corresponds to } \frac{1}{2\kappa} \]

\[ M \text{ corresponds to } \frac{1}{2\kappa} \]
Correlator in configuration space: $N = 53 \ b = 8 \ m_q = 1$ for overlap fermions
A snapshot of the quality of overlap data

Correlator in momentum space: \( N = 53 \ b = 8 \ m_q = 1 \) for overlap fermions

\[\text{Graph showing correlator behavior with parameters} \ N=53, b=8.0, M_q=1.00\]

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Correlator in momentum space (with smearing): $N = 53 \ b = 8 \ 
m_q = 1$ for overlap fermions
Summary of overlap results

SUMMARY for Overlap fermions $b=8$ $N=53$

- lowest state with variational fit
- lowest state with momentum fit
- excited state with momentum fit

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Conclusions and Outlook

- Meson mass results are in qualitative agreement with ‘t Hooft model predictions. Up to 3 excited states can be reasonably determined.
- Our meson masses are a few percent lower at equal bare quark mass. We still have to understand size of corrections: finite $N$, lattice artifacts, bare mass definition, etc.
- Results agree when applicable with Kiskis, Narayanan and Neuberger ones.

Many interesting large $N$ models are within reach