On a Stochastic Fundamental Lemma and Its Use for Data-Driven MPC

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Abstract—Data-driven Model Predictive Control (MPC) based on the fundamental lemma by Willems et al. has shown promising results for deterministic LTI systems subject to measurement noise. However, besides measurement noise, stochastic disturbances might also directly affect the dynamics. In this paper, we extend deterministic data-driven MPC towards stochastic systems. Leveraging Polynomial Chaos Expansions (PCE), we propose a novel variant of the fundamental lemma for stochastic LTI systems. This extension allows to predict future distributions of the behaviors for stochastic LTI systems based on the knowledge of previously recorded behavior realizations and based on the knowledge of the noise distribution. Finally, we propose a framework for data-driven stochastic MPC which is applicable to a large class of noise distributions with finite variance. Numerical examples illustrate the efficacy of the proposed scheme.

Keywords: Data-driven control, learning systems, model predictive control, polynomial chaos, uncertainty quantification.

I. INTRODUCTION

Recently, data-driven system representations based on the fundamental lemma by Willems et al. [1] are subject to renewed and increasing research interest. The pivotal insight of the lemma is that the trajectories of any controllable Linear Time Invariant (LTI) system can be described without explicit identification of a state-space model. Specifically, provided persistency of excitation holds, the system behaviors are contained in the column space of a Hankel matrix constructed from recorded {input-output, behavior} data. In absence of process and measurement noise, this data-driven system representation is exact. Beyond the deterministic controllable LTI setting, there are recent variants of the lemma, e.g., extensions to nonlinear systems [2], [3], to linear parameter-varying systems [4], and to linear network systems [5]. Other extensions include uncontrollable systems [6], [7], and input affine systems [8]. For recent overviews we refer to [9], [10].

Data-driven control design and system analysis with not necessarily persistently exciting data has been investigated in [11]. The exploitation for Model Predictive Control (MPC) has been popularized by [12], while earlier attempts can be found in [13]. For stability analysis of data-driven MPC see [14], while applications are discussed in [15], [16], [17], [18]. Beyond the LTI setting, [12] proposes a heuristic approach to deal with measurement noise and mild system nonlinearities by introducing slack variables and regularization in the objective function. There is also a line of research focusing on the robustness with respect to measurement noise and/or process noise. While [19], [20], [21] consider the design of robust state feedback controllers to deal with process noise, [22] uses maximum likelihood to obtain an optimal Hankel representation, and [23] views the noise entering the Hankel matrix as a problem of distributional robustness.

However to the best of the authors’ knowledge, so far there appears to be no stochastic variant of the fundamental lemma. Moreover, the problem of intrusive uncertainty propagation and quantification—i.e., not relying on sampling and scenarios—and consequently the data-driven forward propagation of stochastic uncertainties (such as process noise) through LTI dynamics represented by Hankel matrices is also open.

In the context of stochastic MPC, and uncertainty quantification in general, Polynomial Chaos Expansions (PCE) are an established method that can be applied in Markovian and non-Markovian settings. Its core idea is based on the observation that random variables can be regarded as $L^2$ functions in a probability space and hence they admit representations in appropriately chosen polynomial bases. We refer to [24] for a general introduction to PCE and to [25], [26] for recent overviews on stochastic MPC. Early works, which have popularized PCE for systems and control, include [27], [28], [29], [30]. Moreover, PCE allows computing statistical moments efficiently [31], it has been used to analyze the region of attraction of stochastic systems [32], and PCE finds application in power systems [33].

In this paper, we link the data-driven system presentation via the fundamental lemma with the PCE approach for uncertainty propagation for LTI systems subject to process noise. Our contributions are as follows: (i) we present a stochastic variant of the fundamental lemma which enables to predict and to propagate the statistical distributions of the behavior, i.e.—states, inputs, and noise—over finite horizons. (ii) we present mild conditions under which a stochastic OCP in random variables can be formulated equivalently in a finite-dimensional data-driven fashion without explicit knowledge of the system matrices. This reformulation is built upon knowledge or estimation of noise realization trajectories. Hence (iii), we also propose a strategy to estimate past noise realizations from state measurements without explicit system knowledge. Finally (iv), drawing upon simulation examples, we demonstrate the efficacy for data-driven stochastic MPC.

The remainder of the paper is as follows: Section II
A. Stochastic Optimal Control

Consider the stochastic discrete-time LTI system

$$X_{k+1} = AX_k + BU_k + W_k, \quad X_0 = X_{ini} \tag{1}$$

with state $X_k \in \mathcal{L}^2(\Omega, \mathcal{F}_k; \mu; \mathbb{R}^{n_x})$, where $\Omega$ is the sample space, $\mathcal{F}$ is a $\sigma$-algebra, $\mathbb{F} = (\mathcal{F}_k)_{k=0,\ldots,\infty}$ is a stochastic filtration, and $\mu$ is the considered probability measure. In the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mu)$, the $\sigma$-algebra contains all available historical information, or more precisely,

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_N \subseteq \mathcal{F}. \tag{2}$$

Let $\mathcal{F}$ be the smallest filtration that the stochastic process $X$ is adapted to, i.e., $\mathcal{F}_k = \sigma(X_i, i \leq k)$, where $\sigma(X_i, i \leq k)$ denotes the $\sigma$-algebra generated by $X_i, i \leq k$. Likewise, the stochastic input $U_k$ is also modelled as a stochastic process that is adapted to the filtration $\mathcal{F}$, that is, $U_k$ only depends on the realizations of $X_0, X_1, \ldots, X_k$. Note that the influence of the noise $W_k, i \leq k$ is implicitly handled via the state recursion. For more details on filtrations we refer to [34].

The process noise $W_k, k \in \mathbb{N}$ is considered as i.i.d. random variables whose underlying probability distribution $p_W : \mathbb{R}^{n_w} \to [0, 1]$ is assumed to be known. Additionally, the distribution $p_{X_{ini}} : \mathbb{R}^{n_x} \to [0, 1]$ of the initial condition $X_{ini}$ is also assumed to be known.

**Problem 1 (Stochastic OCP):** Given the initial condition $X_0$ and i.i.d. random variables $W_i, i \in \mathbb{I}_{[k,k+N-1]}$, we consider the following Optimal Control Problem (OCP) with horizon $N \in \mathbb{N}^+$,

$$\min_{X \in \mathcal{L}^2(\Omega, \mathcal{F}; \mu; \mathbb{R}^{n_x}), \quad U \in \mathcal{L}^2(\Omega, \mathcal{F}; \mu; \mathbb{R}^{n_u})} \sum_{i=k}^{k+N-1} \mathbb{E}[||X(\cdot)\|^2_Q + ||U(\cdot)\|^2_T] \tag{3a}$$

subject to for $i \in \mathbb{I}_{[k,k+N-1]}$,

$$X_{i+1} = AX_i + BU_i + W_i \tag{3b}$$

$$X_k = X_{\text{ini}}, \quad W_i = W_i \tag{3c}$$

$$\mathbb{P}(X_{i+1}(\cdot) \in \mathbb{X}) \geq 1 - \varepsilon_x, \quad \mathbb{P}(U_{i+1}(\cdot) \in \mathbb{U}) \geq 1 - \varepsilon_u \tag{3d}$$

where $Z_{i,k}, Z \in \{X, U, W\}$ denotes the random variables at $i \in \mathbb{I}_{[k,k+N-1]}$ predicted at time step $k$. Moreover $Z = [Z_{i,k}, Z_{i+1,k}, \ldots, Z_{i+N-1,k}]^T$ is the corresponding predicted trajectory.

We here consider chance constraints for the states (3d) and the inputs (3e), and the underlying sets $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ and $\mathbb{U} \subseteq \mathbb{R}^{n_u}$ are assumed to be closed. Moreover, $1 - \varepsilon_x$ and $1 - \varepsilon_u$ specify the probabilities with which the—joint in the state dimension in time—chance constraints shall be satisfied. However, it should be noticed that, depending on the considered noise distribution and constraints, the consideration of chance constraints for states and inputs may jeopardize feasibility of the OCP.

For all $\omega \in \Omega$, the realization of $W_k$ is written as $w_k = W_k(\omega)$, likewise we denote state and input realizations. Henceforth, we suppose that the system matrices $A, B$ as well as the noise realizations $w_k$ are known, while the state realizations $x_k$ and the input realizations $u_k$ are assumed to be measured, and the distributions $p_W$ and $p_{X_{ini}}$ are given.

B. Primer on Data-driven System Representation

For a specific initial condition $x_{ini} = X_{ini}(\omega)$, the realizations of $\{\text{states, inputs, noise}\}$ generated by (1) satisfy

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad x_0 = x_{ini} \tag{4}$$

The behaviors of this deterministic LTI system can be represented using data.

**Definition 1 (Persistency of excitation [1]):** Let $T, t \in \mathbb{N}^+$. A sequence of inputs $u_{[0,T]}$ is said to be persistently exciting of order $t$ if the Hankel matrix

$$H_t(u_{[0,T-1]}) = \begin{bmatrix} u_0 & u_1 & \cdots & u_{T-t} \\ u_1 & u_2 & \cdots & u_{T-t+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{t-1} & u_t & \cdots & u_{T-1} \end{bmatrix}$$

is of full row rank.

Moreover, since (4) is driven by the inputs and the process noise realizations, the extension of Willems et al. fundamental lemma to the exogenous input data $(u, w)_{[0,T-1]}$ is immediate.

**Lemma 1 (Deterministic fundamental lemma):** Let $T, t \in \mathbb{N}^+$. Consider a realization behavior $(x, u, w)_{[0,T-1]}$ of (4). If $(u, w)_{[0,T-1]}$ is persistently exciting of order $n_x + t$, then $(\tilde{x}, \tilde{u}, \tilde{w})_{[0,T-1]}$ is a realization behavior of (4) if and only if there exists a $g \in \mathbb{R}^T \setminus t + 1$ such that

$$H_t(z_{[0,T-1]})g = \tilde{z}_{[0,t-1]} \tag{5}$$

for $z \in \{x, u, w\}$.
Proof: Observe that for arbitrary $B \in \mathbb{R}^{n_x \times n_u}$, the pair $(A, [B I_{n_u}])$ is controllable. Direct application of [1] proves the assertion.

At this point, it is fair to ask for how to obtain—or how to estimate—previous realizations $\tilde{w}_{[0,T-1]}$. We postpone our answer to Section IV-C. However, even temporarily assuming exact measurements of $\tilde{w}_{[0,T-1]}$, the future realizations $\tilde{w}_{[0,T-1]}$ are not known and without the knowledge of $\tilde{w}_{[0,T-1]}$, one cannot compute future realizations $\tilde{x}_{[0,T-1]}$ or $\tilde{u}_{[0,T-1]}$. Indeed, the ambition of our subsequent discussions is twofold: (i) the development of data-driven methods to predict the future evolution of the distributions of the behaviors based on the knowledge of past realizations of behaviors and the noise distribution, and (ii) the reformulation of Problem 1 in a data-driven fashion.

III. DATA-DRIVEN SYSTEM REPRESENTATION FOR STOCHASTIC LTI SYSTEMS

A. Basics of Polynomial Chaos Expansion

Polynomial Chaos Expansion (PCE) enables to propagate uncertainties through system dynamics and thus it provides another way to describe future random variables rather than accessing their realizations. Its origins date back to Norbert Wiener [35]; for a general introduction to PCE see [24].

The core idea of PCE is that an $L^2$ random variable can be expressed in a suitable polynomial basis. To this end, consider an orthogonal polynomial basis $\{\phi_j(\omega)\}_{j=0}^{\infty}$ which spans $L^2(\Omega, F, \mu; \mathbb{R})$, i.e.,

$$\langle \phi^i, \phi^j \rangle = \int_\Omega \phi^i(\omega)\phi^j(\omega) \mu(\omega) = \delta^{ij},$$

where $\delta^{ij}$ is the Kronecker delta.

Definition 2 (Polynomial chaos expansion): The PCE of a real-valued random variable $V \in L^2(\Omega, F, \mu; \mathbb{R})$ with respect to the basis $\{\phi_j(\omega)\}_{j=0}^{\infty}$ is

$$V = \sum_{j=0}^{\infty} \psi_j \phi_j$$

where $\psi_j \in \mathbb{R}$ is called the $j$-th PCE coefficient.

We remark that upon applying PCE component-wise the $j$-th PCE coefficient of a vector-valued random variable $V \in L^2(\Omega, F, \mu; \mathbb{R}^n)$ reads

$$\psi^i = [\psi^i, \psi^{i-j} \psi^{i-j} \ldots \psi^{i,n}]^\top,$$

where $\psi^{i,j}$ is the $j$-th PCE coefficient of component $V^i$.

In numerical implementations the series has to be terminated after a finite number of terms which may lead to truncation errors. For details on truncation errors and error propagation see [36], [37]. Indeed, random variables that follow some widely used distributions admit exact finite-dimensional PCEs in suitable polynomial bases, e.g., for Gaussian random variables Hermite polynomials are chosen.

Definition 3 (Exact PCE representation): A random variable $V \in L^2(\Omega, F, \mu; \mathbb{R})$ is said to admit an exact PCE with $L + 1$ terms if

$$\Delta^L \equiv V - \sum_{j=0}^{L} \psi_j \phi_j = 0,$$

where $\Delta^L$ denotes the truncation error.

We refer to [38], [39] for details and to Table I for the usual basis choices for other distributions. When Definition 3 holds for a random variable $V$, its expected value and variance can be efficiently calculated from its PCE coefficients

$$\mathbb{E}[V] = \mathbf{v}^0 \quad \text{and} \quad \mathbb{V}[V] = \sum_{j=1}^{L} (\psi^j)^2 \langle \phi^j, \phi^j \rangle.$$

We refer to [31] for a detailed discussion.

B. A Fundamental Lemma for Stochastic LTI Systems

Replacing all random variables of (1) with their PCE expansions with respect to the basis $\{\phi_j(\omega)\}_{j=0}^{\infty}$ and performing a Galerkin projection onto the basis functions $\phi_j(\omega)$, we obtain the dynamics of the PCE coefficients with given initial condition $x_{i0}^j$ for $j \in \mathbb{N}$

$$x_{i+1}^j = Ax_i^j + Bu_i^j + w_i^j, \quad x_0^j = x_{i0}^j, \quad \forall j \in \mathbb{N}. \quad (7)$$

Since (7) is a deterministic LTI system, it allows the conceptual application of the usual LTI fundamental lemma.

Lemma 2 (Fundamental lemma for PCE coefficients): Let $T, t \in \mathbb{N}^+$. For $j \in \mathbb{N}$ and given a behavior $(x, u, w)_{[0,T-1]}$ of the PCE coefficients generated by (7), suppose $(\tilde{x}, \tilde{u}, \tilde{w})_{[0,T-1]}$ is persistently exciting of order $n_x + t$.

Then, for all $j \in \mathbb{N}$, $(\hat{x}, \hat{u}, \hat{w})_{[0,T-1]}$ is a behavior of (7) if and only if there exists a $h_j \in \mathbb{R}^{T-1+t}$ such that

$$\mathcal{H}_t(z_{[0,T-1]}^j)h_j = Z_{[0,T-1]}^j$$

for $z \in \{x, u, w\}$.

The proof follows from [1] and is thus omitted. Lemma 2 as such is straightforward, but it is not trivial to measure/estimate PCE coefficients of a stochastic LTI system. Hence the previous result is seemingly not very practical.

Consider the stochastic LTI system (1) and the corresponding behaviors of random variables, PCE coefficients (7), and the realizations (4), which are $(X, U, W)_{[0,T-1]}$, $(\hat{x}, \hat{u}, \hat{w})_{[0,T-1]}$, $j \in \mathbb{N}$, and $(x, u, w)_{[0,T-1]}$ respectively. We have the following result.

Lemma 3 (Column-space equivalence): Consider the stochastic LTI system (1) and its $L^2(\Omega, F, \mu; \mathbb{R}^n)$, $n_z \in \{n_x, n_u, n_x\}$ random-variable

| Distribution | Support | Orthogonal polynomials |
|--------------|---------|------------------------|
| Gaussian     | $(-\infty, \infty)$ | Hermite |
| Uniform      | $[a, b]$ | Legendre |
| Beta         | $[a, b]$ | Jacobi |
| Gamma        | $(0, \infty)$ | Laguerre |

TABLE I

CORRESPONDENCE OF RANDOM VARIABLES AND UNDERLYING ORTHOGONAL POLYNOMIALS.
behaviors \((X, U, W)_{[0, T-1]}\). Let the corresponding PCE coefficient trajectories \((u, w)^j_{[0, T-1]}\), \(j \in \mathbb{N}\) and the realizations \((u, w)_{[0, T-1]}\) be persistently exciting of order \(n_x + t\).

(i) Then, for all \(j \in \mathbb{N}\) and all \((z, z) \in \{(x, x), (u, u), (w, w)\}\), it holds that

\[
\text{colspan}(H_t(z^j_{[0, T-1]})) = \text{colspan}(H_t(z_{[0, T-1]})) \quad (9a)
\]

(ii) Moreover, for all \(g \in \mathbb{R}^{T-t+1}\), there exists a \(G \in L^2(\Omega, F, \mu; \mathbb{R}^{T-t+1})\) such that

\[
H_t(z_{[0, T-1]}(g) = H_t(z_{[0, T-1]}(G) \quad (9b)
\]

for all \((z, z) \in \{(x, x), (u, u), (w, w)\}\). □

**Proof:** For the sake of readability, we omit the subscript \(\cdot_{[0, T-1]}\) in the proof. The proof of (9a) in part (i) follows directly from the observation that the realizations (4) and the PCE coefficients (7) share the same system matrices \((A, [B I_{n_x}]\).

Part (ii): Considering (9b), we have

\[
H_t(Z) = H_t(\sum_{j=0}^{\infty} Z^j \phi^j) = \sum_{j=0}^{\infty} H_t(z^j \phi^j) = \sum_{j=0}^{\infty} \phi^j H_t(Z)g.
\]

Note that \(H(\cdot)\) and the summation are both linear operators, therefore the second equality holds. Moreover, the basis function \(\phi^j \in L^2(\Omega, F, \mu; \mathbb{R})\) is a scalar random variable. In case the different components of \(Z\) have different bases, one may use the union of the bases for each component. Then using the column space equivalence (9a), for all \(j \in \mathbb{N}\) and any \(g \in \mathbb{R}^{T-t+1}\), we can find a \(g^j \in \mathbb{R}^{T-t+1}\), such that \(H_t(z^j)g = H_t(z)g^j\). This leads to

\[
H_t(Z) = \sum_{j=0}^{\infty} \phi^j H_t(z^j) = \sum_{j=0}^{\infty} \phi^j g^j.
\]

The assertion follows with \(G = \sum_{j=0}^{\infty} g^j \phi^j\). □

**Corollary 1 (PCE coefficients via realizations):**
If the realizations \((u, w)_{[0, T-1]}\) are persistently exciting of order \(n_x + t\), then, for all \(j \in \mathbb{N}\), \((\tilde{x}, \tilde{u}, \tilde{w})_{[0, T-1]}\) is a PCE coefficient behavior of (7) if and only if there exists a \(g^j \in \mathbb{R}^{T-t+1}\) such that

\[
H_t(z_{[0, T-1]}(g^j) = \tilde{z}^j_{[0, t-1]}. \quad (10)
\]

holds for all \((x, \tilde{z}) \in \{(x, \tilde{x}), (u, \tilde{u}), (w, \tilde{w})\}\). □

Observe that the core difference between (10) and (5) is that the PCE coefficients \(w\) of future process noise are given, i.e., they are known. Hence, (10) allows to predict the distributions of future behaviors of stochastic LTI systems. Moreover, one can lift the results to the corresponding \(L^2\) probability space.

**Lemma 4 (Stochastic fundamental lemma):**
Consider the stochastic LTI system (1) and its \(L^2(\Omega, F, \mu; \mathbb{R}^{n_x})\), \(n_x \in \{n_x, n_u, n_w\}\) behaviors of random variables, PCE coefficients (7), and the corresponding behavior realizations (4), which are \((X, U, W)_{[0, T-1]}\), \((X,u,w)^j_{[0, T-1]}\), \(j \in \mathbb{N}\) and \((x, u, w)_{[0, T-1]}\) respectively.

(i) Let \((u, w)_{[0, T-1]}\) be persistently exciting of order \(n_x + t\). Then \((\tilde{X}, \tilde{U}, \tilde{W})_{[0, T-1]}\) is a behavior of (1) if and only if there exists \(G \in L^2(\Omega, F, \mu; \mathbb{R}^{T-t+1})\) such that

\[
H_t(z_{[0, T-1]}(G) = \tilde{Z}^j_{[0, t-1]} \quad (11a)
\]

for all \((z, \tilde{z}) \in \{(x, \tilde{x}), (u, \tilde{u}), (w, \tilde{w})\}\).

(ii) Let \((U, W)_{[0, T-1]}\) satisfy

\[
Z_{[0, T-1]} = \sum_{j=0}^{L} Z^j \phi^j, \quad Z \in \{U, W\}
\]

with \(L \in \mathbb{N}\) and all PCE trajectories \((u, w)^j_{[0, T-1]}\) with \(j \in \{0, \ldots, L\}\) are persistently exciting of order \(n_x + t\). If there exists a \(g \in \mathbb{R}^{T-t+1}\) such that, for all \(Z \in \{X, U, W\}\),

\[
H_t(z_{[0, T-1]}(G) = \tilde{Z}^j_{[0, t-1]} \phi^j, \quad \forall j \in \mathbb{N}.
\]

We obtain

\[
H_t(z_{[0, T-1]}(G) = \sum_{j=0}^{\infty} \tilde{Z}^j_{[0, t-1]} \phi^j.
\]

Hence \(G\) determines a random variable behavior \(\tilde{Z}^j_{[0, t-1]} = \sum_{j=0}^{\infty} \tilde{Z}^j_{[0, t-1]} \phi^j\) of (1).

Next we show \(\tilde{Z} \mapsto G\). For any \(\tilde{Z}_{[0, t-1]} \in L^2(\Omega, F, \mu; \mathbb{R}^{n_x})\), \(n_x \in \{n_x, n_u, n_w\}\), i.e., for any random variable behavior of (1), the corresponding PCE coefficient behaviors \(\tilde{Z}^j_{[0, t-1]}\), \(j \in \mathbb{N}\) exist. Moreover, they correspond to \(g^j \in \mathbb{R}^{T-t+1}\) such that (10) holds, cf. Corollary 1. To conclude the proof and similarly as before, construct \(G \in L^2(\Omega, F, \mu; \mathbb{R}^{T-t+1})\) via the Galerkin projection of (11a) for all \(j \in \mathbb{N}\).

We turn to part (ii) which asserts that \(g \mapsto \tilde{Z}^j_{[0, t-1]}\). With Lemma 3, part (ii), we have that for any \(g \in \mathbb{R}^{T-t+1}\) there exists \(G \in L^2(\Omega, F, \mu; \mathbb{R}^{T-t+1})\) such that (9b) holds. Part (i) of Lemma 4 gives that \(G\) determines a random variable behavior of (1) by (11a). This concludes the proof. □

**Remark 1 (Extension to the input-output case):**
The results of Lemmata 1–4 and of Corollary 1 can be generalized to stochastic LTI systems with outputs, i.e.,

\[
X_{k+1} = AX_k + BU_k + W_k, \quad X_0 = X_{ini}
\]

\[
Y_k = CX_k + DU_k.
\]
Doing so the corresponding input-output behaviors \((y, z, Z)\) should be defined in terms of \(\{(y, y', Y), (u, U), (w, W)\}\). However, due to space limitations, we do not explore the details here. □

C. Discussion

Figure 1 summarizes the results derived above and their underlying assumptions: Indeed Lemma 1 and Lemma 2 are immediate consequences of the original result by Willems et al. [1]. Observe that the assumed \(\mathcal{L}^2(\Omega, \mathcal{F}_k, \mu; \mathbb{R}^{m'})\) nature of the behaviors of (1) allows to link the random variables and their PCE coefficients dynamics (7). Moreover, note that by sampling the PCE solutions—i.e., evaluating the basis functions \(\phi^j\) for different values of \(\omega\)—one can go from PCE solutions to realizations.

Besides these technicalities, crucial aspects are as follows: First, the fundamental lemma for the realizations (Lemma 1) does not hinge on the controllability of the pair \((A, B)\). Indeed as the noise acting on the system induces excitation, and as \((A, [B I_{n_u}])\) is always controllable, one can safely drop the controllability requirement for \((A, B)\). Moreover, as the estimation of PCE coefficients from data requires rather large data sets, the immediate usability of the fundamental lemma for the PCE coefficients (Lemma 2) appears to be limited. Yet, the equivalence of column spaces obtained in Lemma 3 and Corollary 1 provides the pivotal link, i.e., one may safely use a Hankel matrix of realizations to compute PCE coefficient behaviors and trajectories.

It is worth to be remarked that persistency of excitation of the \((u, w)\) realization trajectories is a much weaker requirement than persistency of excitation of the \((u, w)\) coefficient trajectories. This is due to the fact that in many situations one will assume that the distribution of the noise is constant while the noise realization will usually be persistently exciting. Finally, the stochastic fundamental lemma (Lemma 4)—upon once more exploiting the column-space equivalence—shows that one can extend the fundamental towards the stochastic system (1).

Another crucial observation is as follows: The usual form of a fundamental lemma, i.e., the columns of the Hankel matrix constructed by past variables span the full system behavior of the LTI system. However, if the Hankel matrix is constructed from the random variables directly, this usual form of the lemma does not necessarily hold in the stochastic setting, cf. Part (ii) of Lemma 4. Specifically, notice that upon applying Galerkin projection for \(L + 1 \in \mathbb{N}\) PCE basis functions to (11a) and using the equivalence of \(\text{colsp}(\mathcal{H}_L(z_{[0,T-1]}))\) and \(\text{colsp}(\mathcal{H}_L(z_{[0,T-1]}))\) given by (9a), we obtain the system of linear equations \([I_L \otimes \mathcal{H}_L(z_{[0,T-1]})] Z_{[0,L]}\) to compute the vector \(g_{[0,L]}\), where \(\otimes\) denotes the Kronecker product and \(g_{[0,L]}\) stacks the \(g^j\) into one vector. In contrast, Galerkin projection of (11b) combined with column-space equivalence gives \([I_L \otimes \mathcal{H}_L(z_{[0,T-1]})] Z_{[0,L]}\), where \(I_L\) is the \(L \times 1\) vector of all 1.

We conclude our discussion with a simple example illustrating why Part (ii) of Lemma 4 does not admit an iff statement.

Example 1: Consider the scalar stochastic system \(X_{k+1} = X_k + U_k\) with past behaviors given by the PCEs
\[
X_0 = 0\phi^0 + 0\phi^1, \quad U_0 = 0\phi^0 + 1\phi^1,
X_1 = 0\phi^0 + 1\phi^1, \quad U_1 = 1\phi^0 + 0\phi^1,
X_2 = 1\phi^0 + 1\phi^1, \quad U_2 = 1\phi^0 + 1\phi^1.
\]

Note that the PCE coefficients of \(U_{[0,2]}\) satisfy the persistency of excitation required by Part (ii) of Lemma 4. We aim to find \(g\) in (11b) to represent \(\hat{X}_0 = 0\phi^0 + 1\phi^1, \quad \hat{U}_0 = 0\phi^0 + 1\phi^1\). We obtain (11b) as
\[
\begin{bmatrix}
X_0 & X_1 & X_2 \\
U_0 & U_1 & U_2
\end{bmatrix}
\begin{bmatrix}
g \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
\hat{X}_0 \\
\hat{U}_0
\end{bmatrix}.
\]

After Galerkin projection onto the basis functions and stacking the projected equations we obtain \(M g = c\) with
\[
M = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix} \quad c = \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

where the upper block corresponds to \(\phi^0\) and the lower one to \(\phi^1\). By the Rouché–Capelli theorem, \(M g = c\) admits a solution \(g\) if and only if \(|M|c\) has the same rank as \(M\). Observe that \(\text{rank} M = 3\) and \(\text{rank} |M|c = 4\). Thus, we conclude that (12) does not admit solutions \(g \in \mathbb{R}^4\). □

IV. DATA-DRIVEN STOCHASTIC MPC

A. Applying PCE to the Stochastic OCP

Before reformulating Problem 1 in a data-driven fashion, we recall its PCE-based reformulation. To this end, we assume that an exact PCE for the process noise is known. This is the case if we restrict the choice of \(W_i\) in Problem 1 to certain types of random variables. cf. Table I. Put differently, for known distributions of the initial condition and the noise, one will try to choose the basis such that their PCE representation are exact, e.g., they follow Table I. Moreover, for the sake of exactness of the PCE for the input \(U\), its basis is chosen accordingly.

Assumption 1: The random variables \(U\) of Problem 1 are chosen such that they admit exact PCEs with \(L + 1\) terms, i.e., they satisfy Definition 3. □

Proposition 1 (Exact uncertainty propagation via PCE): Suppose that for \(X_k\) and all i.i.d. \(W_i\), \(i \in \mathbb{N}_{[k,k+N-1]}\) in Problem 1 satisfies Definition 3 with a PCE with \(L_x\) terms and \(L_w\) terms respectively. Moreover, suppose Assumption 1 holds. Then, for any finite horizon \(N \in \mathbb{N}^+\), the optimal solutions of Problem 1—i.e., \(X^*\) and \(U^*\)—admit exact finite PCEs with \(L + 1\) terms, where \(L\) is given by
\[
L = L_x + NL_w \in \mathbb{N}^+.
\]

The proof follows ideas from [37] and is given in the appendix.
The causality/non-antipacitivity of the filtration (2) implicitly imposes an additional constraint on the PCE coefficients of the input

\[ u_{i|k}^j = 0, j \in \mathbb{I}_{[L_z+(i-k)L_w+1,L]}; i \in \mathbb{I}_{[k,N-1]} \]  

(14)

The derivation of this constraint is also given in the appendix. We remark that the causality of \( X \) trivially holds when (14) is imposed on systems dynamics (7).

**Problem 2 (Stochastic OCP in PCE coefficients):**

Consider the PCEs with \( L+1 \in \mathbb{N}^+ \) terms determined by (13) and for given PCE coefficients \( x^j_k \) and \( w^j_k \), \( j \in \mathbb{I}_{[0,L]} \), \( i \in \mathbb{I}_{[k,N-1]} \), then the PCE reformulation of Problem 1 reads

\[
\min_{x^{[0,L]}_j, w^{[0,L]}_j} \sum_{i=k}^{k+N-1} \sum_{j=0}^{L} \left( \|x^j_{i|k}\|^2_Q + \|u^j_{i|k}\|^2_R \right) \langle \phi^j, \phi^j \rangle
\]

(15a)

s.t. for \( j \in \mathbb{I}_{[0,L]} \), \( i \in \mathbb{I}_{[k,k+N-1]} \)

\[
x^j_{i+1|k} = Ax^j_{i|k} + Bu^j_{i|k} + w^j_{i|k},
\]

(15b)

\[
x^j_{i|k} = x^j_{k|k}, w^j_{i|k} = w^j_i,
\]

(15c)

\[
x^j_{i|k} + \sigma(\varepsilon_x) \sum_{j=1}^{L} (u^j_{i|k})^2 \langle \phi^j, \phi^j \rangle \in \mathbb{R},
\]

(15d)

\[
u^j_{i|k} \pm \sigma(\varepsilon_u) \sum_{j=1}^{L} (u^j_{i|k})^2 \langle \phi^j, \phi^j \rangle \in \mathbb{U},
\]

(15e)

\[
u^j_{i|k} = 0, \forall j' \in \mathbb{I}_{[L_z+(i-k)L_w+1,L]},
\]

(15f)

where \( x^j_{i|k} \), with \( z \in \{x,u,w\} \) denotes the \( j \)-th PCE coefficient of the predicted random variables at \( i \), \( z^j_{i|k} = [z^j_{i|k}, z^j_{i+1|k}, \ldots, z^j_{i+L-1|k}]^T \) is the corresponding predicted trajectory and \( z^{[0,L]} = [z^0^T, z^1^T, \ldots, z^L]^T \) represents the vectorization of \( z^j_{i|k} \).

The chance constraint reformulation from (3d) and (3e) to (15d) and (15e) with \( \sigma(\varepsilon_x) = \sqrt{(2-\varepsilon_x)/\varepsilon_x}, z \in \{x,u\} \) follows [40]. Especially, for Gaussian random variables, (3d) and (15d), respectively, (3e) and (15e) are equivalent.

**B. Data-driven Stochastic Optimal Control**

Next we give a data-driven reformulation of (15) exploiting Corollary 1.

**Problem 3 (Data-driven Stochastic OCP):** Consider an available realization trajectory \((x,u,w)_{[0,T-1]}\) of (1).

Suppose conditions of Problem 2 hold and \((u,w)_{[0,T-1]}\) is persistently exciting of order \( n_u + N \), then the data-driven reformulation of Problem 2 reads

\[
\min_{x^{[0,L]}_j, w^{[0,L]}_j} \sum_{i=k}^{k+N-1} \sum_{j=0}^{L} \left( \|x^j_{i|k}\|^2_Q + \|u^j_{i|k}\|^2_R \right) \langle \phi^j, \phi^j \rangle
\]

(16a)

s.t. for \( j \in \mathbb{I}_{[0,L]} \), \( i \in \mathbb{I}_{[k,k+N-1]} \)

\[
\mathcal{H}_N(x_{[0,T-1]}) \quad \mathcal{H}_N(u_{[0,T-1]}) \quad \mathcal{H}_N(w_{[0,T-1]}) \quad g^j = [x^j, w^j], \quad (16b)
\]

\[
x^j_{i|k} = x^j_{k|k}, w^j_{i|k} = w^j_i, \quad (15d) - (15f)
\]

where \( g^{[0,L]} = (g^0^T, g^1^T, \ldots, g^L^T)^T \) is the vectorization of \( g^j \in \mathbb{R}^{T-N+1} \).

**Remark 2 (Num. solution with the null-space method):**

Considering the fact that the Hankel matrix \( \mathcal{H}_N(w_{[0,T-1]}) \) is of full row rank and the PCE coefficients \( w^j \) are known, the null-space method can be employed to reduce the dimensionality of the decision variables, i.e., the dimensionality of \( g^{[0,L]} \) in Problem (16). To this end, choose a matrix \( M_w \in \mathbb{R}^{T-N+1 \times (T-N(n_u+1))} \) whose columns are a basis of the null space of \( \mathcal{H}_N(w_{[0,T-1]}) \), i.e., null(\( \mathcal{H}_N(w_{[0,T-1]}) \)). The core idea is to parameterize the solution of the equality constraint \( \mathcal{H}_N(w_{[0,T-1]})g^j = w^j \) as

\[
g^j = M_w h^j + \mathcal{H}_N(w_{[0,T-1]})^T w^j,
\]

(17)

where \( h^j \in \mathbb{R}^{T-N(n_u+1)} \) and \( \mathcal{H}_N(\cdot)^T \) denotes the Moore-Penrose inverse of \( \mathcal{H}_N(\cdot) \). Thus, substitution of (17) into (16b) yields the simplified and numerically favourable expression

\[
\mathcal{H}_N(x_{[0,T-1]}) \quad \mathcal{H}_N(u_{[0,T-1]}) \quad (M_w h^j + \mathcal{H}_N(w_{[0,T-1]})^T w^j) = [x^j, w^j], \quad (18)
\]

where \( j \in \mathbb{I}_{[0,L]} \), is obtained.

**Theorem 1 (Equivalence of stochastic OCPs):**

Consider the stochastic LTI system (1). Let Assumption 1 hold and consider the PCE order \( L \in \mathbb{N}^+ \) determined by
(13). Suppose that a realization trajectory \((\mathbf{x}, \mathbf{u}, \mathbf{w})_{(0,T−1)}\), with \((\mathbf{u}, \mathbf{w})_{(0,T−1)}\) persistently exciting of order \(N + n_x\), is known. Then the following holds:

(i) Problem 2 \(\Leftrightarrow\) Problem 3, i.e., for any optimal solution \((\mathbf{x}^{(0,L)}, \mathbf{u}^{(0,L)}, \mathbf{w}^{(0,L)})\) to Problem 2 there exists a \(\mathbf{g}^{(0,L),*} \in \mathbb{R}^{(T−N+1)∗(L+1)}\) such that \((\mathbf{x}^{(0,L)}, \mathbf{u}^{(0,L)}, \mathbf{w}^{(0,L)})\) is an optimal solution to Problem 3.

(ii) If, moreover, the reformulation of chance constraints (3d)–(3e) to (15d)–(15e) is exact, Problem 1 \(\Leftrightarrow\) Problem 2 \(\Leftrightarrow\) Problem 3. In particular, it holds that \(\mathbf{x}^* = \sum_{j=0}^L \mathbf{x}^j \phi_j^*, \mathbf{U} = \sum_{j=0}^L \mathbf{u}^j \phi_j^*\) is an optimal solution to Problem 1. \(\square\)

**Proof:** The equivalence Problem 2 \(\Leftrightarrow\) Problem 3 follows from Corollary 1 with \((\mathbf{u}, \mathbf{w})_{(0,T−1)}\) persistently exciting of order \(N + n_x\). The statement Problem 1 \(\Leftrightarrow\) Problem 2 follows from the fact that with sufficiently large finite \(L\) given by (13) and Assumption 1 the PCE reformulation of random variables in the Problems 2-3 is without truncation error, cf. Proposition 1. Moreover, as by assumption the reformulation of chance constraints (3d)–(3e) to (15d)–(15e) is exact, then the assertion follows. The statement Problem 3 \(\Leftrightarrow\) Problem 1 follows straightforwardly. \(\blacksquare\)

**C. Estimation of Past Process Noise Realizations**

So far we assumed knowledge of a recorded noise data \(\mathbf{w}_{(0,T−1)}\) which is unrealistic in applications. Next, we utilize the recorded states \(\mathbf{x}_{(0,T]}\) and inputs \(\mathbf{u}_{(0,T−1)}\) to estimate \(\mathbf{w}_{(0,T−1)}\).

For the sake of compact notation, we omit the subscript \((0,T−1)\). Then inspired by [9, Theorem 1], we obtain the following result.

**Proposition 2:** If \((\mathbf{u}, \mathbf{w})\) is persistently exciting of order more than 1, then

\[
\mathcal{H}_1(\mathbf{x}^+) - \mathcal{H}_1(\mathbf{w}) = 0
\]

with \(\mathbf{x}^+ \equiv \mathbf{x}_{[0,T]}\) and \(\mathbf{S} \equiv \left[\mathcal{H}_1(\mathbf{x}^T), \mathcal{H}_1(\mathbf{u})^T\right]^T\). \(\square\)

**Proof:** Since \((\mathbf{u}, \mathbf{w})\) is persistently exciting of order more than 1, then \(\mathbf{S}\) has full row rank. Hence \(\mathbf{S}^\dagger\) exists and \(\mathbf{S} \mathbf{S}^\dagger = \mathbb{I}_{n_x+n_u}\). Moreover, (4) can be reformulated as

\[
x_{k+1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} \mathbf{S} \mathbf{S}^\dagger \begin{bmatrix} \mathbf{x}_k^T, \mathbf{u}_k^T \end{bmatrix}^T + \mathbf{w}_k.
\]

We notice that \(\begin{bmatrix} \mathbf{A} \end{bmatrix} \mathbf{S} \mathbf{S}^\dagger = \mathcal{H}_1(\mathbf{x}^+) - \mathcal{H}_1(\mathbf{w})\) and obtain

\[
x_{k+1} - \mathbf{w}_k = \begin{bmatrix} \mathcal{H}_1(\mathbf{x}^+) - \mathcal{H}_1(\mathbf{w}) \end{bmatrix} \mathbf{S} \mathbf{S}^\dagger \begin{bmatrix} \mathbf{x}_k^T, \mathbf{u}_k^T \end{bmatrix}^T.
\]

In order to reconstruct the past disturbance realizations, we apply (20) to the recorded inputs and states. Thus, we get

\[
(\mathcal{H}_1(\mathbf{x}^+) - \mathcal{H}_1(\mathbf{w})) = (\mathcal{H}_1(\mathbf{x}^+) - \mathcal{H}_1(\mathbf{w})) \mathbf{S} \mathbf{S}^\dagger \mathbf{S}.
\]

which is equivalent to (19).

We note that (19) admits infinitely many solutions of \(\mathbf{w}\). Hence, we utilize the knowledge about the distribution \(p_W\) to formulate the maximum likelihood estimate

\[
\hat{\mathbf{w}} = \arg\min_{\mathbf{w}} \sum_{k=0}^{T-1} \log p_W(w_k), \text{ s.t. (19),}
\]

where \(w_k\) is the realization of \(W\) at time \(k\).

**Algorithm 1** Stochastic data-driven MPC with (16)

**Input:** \(T, N, L \in \mathbb{N}^+, p_W, k \leftarrow 0\)

**Data collection and pre-processing:**

1: Select uniformly random distributed \(\mathbf{u}_{[0,T−1]} \in \mathbb{U}^T\)
2: Apply \(\mathbf{u}_{[0,T−1]}\) to system (1), record \(\mathbf{x}_{[0,T]}\)
3: Estimate \(\mathbf{w}_{[0,T−1]}\) by (21) or (22)
4: If \((\mathbf{u}, \mathbf{w})_{(0,T−1)}\) is persistently exciting of order less than \(N + n_x\), go to Step 1, else go to Step 5
5: Construct (16) with \((\mathbf{x}, \mathbf{u}, \mathbf{w})_{[0,T−1]}\)

**MPC loop:**

1: Measure or estimate \(x_k\)
2: Solve (16)
3: Apply \(u_k = u_k^{0,*}\) to system (1)
4: \(k \leftarrow k + 1\), go to Step 1

**Remark 3 (Least-squares estimate of Gaussian noise):**

Specifically, if \(w_k \sim N(0, \sigma^2), k \in [0,T−1]\), (21) is equivalent to

\[
\hat{\mathbf{w}} = \arg\min_{\mathbf{w}} \mathbf{w}^T \mathbf{w}, \text{ s.t. (19),}
\]

which is a least-squares estimate. The solution is

\[
\mathcal{H}_1(\hat{\mathbf{w}}) = \mathcal{H}_1(\mathbf{x}^+) (\mathbf{I}_T - \mathbf{S} \mathbf{S}^\dagger),
\]

where \(\mathbf{I}_T\) is an identity matrix of size \(T\).

**Remark 4 (Estimates of uniformly distributed noise):**

For uniformly distributed \(w_k \sim U([-a, a]), k \in [0,T−1]\), the maximum likelihood estimate (21) does not admit a unique solution due to the constant probability on its support. Therefore, we apply (22) in this case.

**D. A Framework for Data-driven Stochastic MPC**

Combining the results of the previous sections, we propose the following data-driven stochastic MPC scheme based on OCP (16) as summarized in Algorithm 1. In the data collection and pre-processing phase, random inputs \(\mathbf{u}_{[0,T−1]}\) are generated to obtain \(\mathbf{x}_{[0,T]}\) and to estimate \(\mathbf{w}_{[0,T−1]}\). During the online optimization phase, we consider the case of exact state-feedback, i.e., the current state realization is available. Thus we have \(x_k^{0} = x_k, x_j^{0} = 0, j \in [1,L]\) in (16e). Then we solve the data-driven stochastic OCP (16) for \((\mathbf{u}^{(0,L),*}, \mathbf{x}^{(0,L),*}, \mathbf{g}^{(0,L),*})\). Observe that the PCE coefficient \(u_{k,j}^{0,*}\) on \(\phi_j^0 = 1\) is applied to system (1) as the MPC feedback. A detailed analysis of the closed-loop properties of the proposed data-driven stochastic MPC framework is beyond the scope of the present paper and postponed to future work. Instead we demonstrate its efficacy via examples.

**V. NUMERICAL EXAMPLES**

We consider two examples for data-driven stochastic OCPs and MPC via PCE. Moreover, the estimation (22) is employed to reconstruct the noise realizations. In the first (scalar) example, we consider Gaussian distributed noise as well as uniformly distributed noise. The second example considers discrete-time stochastic MPC for an aircraft. Furthermore, we also verify that noise estimation performs well...
in open-loop OCPs and MPC problems. To implement the numerical examples in julia, we rely on the toolboxes PolyChaos.jl [41] and JuMP.jl [42].

A. Stochastic Optimal Control with Scalar Dynamics

We consider a scalar OCP from [43] and the stochastic extension of which has been proposed in [44]. The dynamics are \( X_{k+1} = 2X_k + U_k + W_k \), where \( W_k \) is random noise with known distribution, and \( X_0 \) follows the uniform distribution \( U(0, 1) \). The matrices \( Q \) and \( R \) in (3a) are \( Q = 0 \) and \( R = 1 \) while the state chance constraint reads \( \mathbb{P}[X(\omega) \in \mathbb{X}] \geq 1 - \varepsilon_x \) with \( \mathbb{X} = [-2, 2] \) and \( \varepsilon_x = 0.2 \). We solve the scalar example as one open-loop OCP with horizon \( N = 25 \) and leave the MPC loop aside.

**Gaussian distributed noise:** Suppose no recorded noise data \( w \) is available and for all \( k \in [0, N-1] \) the disturbance is i.i.d. \( W_k \) Gaussian \( \mathcal{N}(0, 0.2^2) \) and, hence, it admits an exact PCE with \( L_w = 1 + 2 \) terms. We apply (22) with \( T = 1000 \) to reconstruct the noise realizations. Moreover, we sample the input in stochastic manner and collect 150 additional input/state measurements—this way ensuring that the data is persistently exciting.

We choose Legendre polynomials as PCE basis with \( L_x = 1 \) for \( X_0 \) and Hermite polynomials with \( L_w = 1 \) for \( W_{k+i} \) such that Definition 3 is satisfied as shown in Table I. Consequently, we obtain \( L = 26 \) from (13) and exactness of the PCEs is ensured. The solutions of the open-loop OCP (16) are depicted in Figure 2. Therein we compare the solution with estimated noise realizations to the one subject to uniform noise. Blue-solid-left y-axis: solution with exact noise measurement vs. model-based; red-dashed-right y-axis: difference of solutions noise measurement vs. estimates.

**Uniformly distributed noise:** To verify the performance of the noise estimation (22) with respect to uniformly distributed noise, we suppose \( W_k \) follows a uniform distribution \( U(-0.866, 0.866) \) with the same mean and variance as the Gaussian distribution \( \mathcal{N}(0, 0.2^2) \). Note that the uniformly distributed noise admits an exact PCE with \( L_w = 2 \) terms in the basis of Legendre polynomials and thus \( L = 26 \) as before. We also record state/input trajectories with horizon \( T = 1000 \) and reconstruct the noise realizations by (22). The solutions of the OCPs are illustrated in Figure 3. As one can see, the estimation (22) also handles uniformly distributed noise, cf. Remark 4.

B. Data-driven Stochastic MPC for an Aircraft

As a second example, we consider a LTI aircraft model taken from [45] and discretize it with sampling time \( t_s = 0.5 \) s. The system matrices are

\[
A = \begin{bmatrix}
0.239 & 0 & 0.178 & 0 \\
0.372 & 1 & 0.25 & 0 \\
0.99 & 0 & 0.139 & 0 \\
-48.9 & 64.1 & 2.4 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
-1.23 \\
-1.44 \\
-4.48 \\
-1.8
\end{bmatrix}.
\]

Gaussian process noise \( W_k \) affects the dynamics, where \( W_k, k \in \mathbb{N} \) is i.i.d. vector-valued random variables with covariance matrix \( \text{cov}(W_k, W_k) = \text{diag}([0.01, 0.01, 0.01, 4]) \). Recall that \( X^j \) denotes the \( j \)-th element of \( X \). The state \( X^1 \) is the angle of attack, \( X^2 \) is the pitch angle, \( X^3 \) is the time derivative of the pitch angle, \( X^4 \) is the altitude, and the input
$U$ is the elevator angle. The weighting matrices in the objective function are $Q = \text{diag}([1014.7, 3.2407, 5674.8, 0.3695])$ and $R = 5188.25$. A chance constraint is only imposed on the pitch angle

$$
\mathbb{P}[X^2(\omega) \in X^2] \geq 1 - \varepsilon_x,
$$

where $X^2 = [-0.349, 0.349]$ and $\varepsilon_x = 0.1$. We apply the MPC scheme from Algorithm 1 with prediction horizon $N = 10$ and exact state feedback, i.e., the realization of $X_k$ is known upon solving each OCP. Similar to before, in the data collection phase we record state/input trajectories of 1000 steps, we obtain the estimation of noise realizations by (22), and we collect additional 150 state/input measurements to construct the Hankel matrices.

To obtain an exact PCE for $W_k$, we employ the Hermite polynomials basis with $L_{w} = 4$. Hence $L = 40$ as calculated by (13) since there is no uncertainty caused by initial state, i.e., $L_x = 0$. Considering the initial condition $x_{\text{init}} = [0, 0, 0, -400]$ and sampling 50 sequences of noise realizations, the closed-loop realization trajectories of the data-driven stochastic MPC (with noise estimation) are shown in Figure 4. We compare the obtained solutions to the data-driven scheme with exact knowledge of past noise realizations. We observe the performance difference of the data-driven stochastic MPC with noise estimation is only 0.7396% in average, which implies that the noise estimation (22) does not cause much suboptimality. Moreover, for the same fixed initial condition $x_{\text{init}}$, we sample a total of 1000 sequences of noise realizations and compute the corresponding closed-loop responses of the data-driven stochastic MPC (again with noise estimation). The time evolution of the (normalized) histograms of the state realizations of past noise realizations and compute the corresponding closed-loop responses of the data-driven stochastic MPC (again with noise estimation). Figure 5, where the vertical axis refers to the probability density of $X^4$. As one can see, the proposed control scheme achieves a narrow distribution of $X^4$.

![Figure 4. 50 different closed-loop MPC realizations (with noise estimation). The red-dashed lines represent the chance constraints.](image1)

![Figure 5. Histograms of the state $X^4$ from 1000 noise realizations.](image2)

### VI. Conclusion

This paper has addressed the extension of data-driven MPC towards stochastic systems. Specifically, we have given a stochastic extension of the fundamental lemma for LTI systems. The crucial insight of our analysis is that instead of formulating a Hankel matrix in terms of random variables, it suffices to consider a Hankel matrix constructed from past behavior realizations. We have formalized this insight in terms of a result on column-space equivalence and we have leveraged it to propose a framework for data-enabled uncertainty quantification and propagation via polynomial chaos expansions. Moreover, we have shown that formulating a Hankel matrix directly in terms of random variables will not necessarily allow to characterize system behavior.

As a by-product for our analysis, we have briefly touched upon the estimation of past noise realizations from input-state data. Finally, we have shown by means of simulation examples that the proposed approach paves the road towards data-driven stochastic MPC. Future work will consider output data with measurement noise, the estimation of past noise realizations in this setting, and guaranteeing stability and recursive feasibility of the MPC loop.

### Appendix

**Proof of Proposition 1:** The PCEs for $X_k$ and i.i.d. noise $W_{k+1}$ are given by

$$X_k(\omega_k) = x_k^0 \phi_x^0 + \sum_{j=1}^{L_x} x_k^j \phi_x^j(\omega_x),$$

$$W_{k+1}(\omega_i) = w_{k+1}^0 \phi_{w_i}^0 + \sum_{j=1}^{L_{w_i}} w_{k+1}^j \phi_{w_i}^j(\omega_i),$$

where $\phi_x^0 = \phi_i^0 = 1$ and $\phi_x^j(\omega_x), j = 1, ..., L_x$ are polynomials of independent random variables $\omega_x$, $\omega_i \in I_{[0, N-1]}$. We construct a new basis with $L_x = N_{x} + 1$ terms

$$\varphi = \begin{bmatrix} 1 & \varphi_x^T(\omega) & \varphi_{w_1}^T(\omega_1) & \cdots & \varphi_{w_{N-1}}^T(\omega_{N-1}) \end{bmatrix}^T$$

(23)
with
\[
\varphi_{x,k}(\omega) = \begin{bmatrix} \phi^1(\omega) & \cdots & \phi^{L_x}(\omega) \end{bmatrix}^T,
\]
\[
\varphi_{w,k,i}(\omega_i) = \begin{bmatrix} \phi^1_{w,k,i}(\omega_i) & \cdots & \phi^{L_w}(\omega_i) \end{bmatrix}^T,
\]
where \( i \in \mathbb{I}_{[0,N-1]} \). Observe that \( X_k \) and \( W \) satisfy Definition 3 in the basis (23). Moreover, Assumption 1 implies that Definition 3 also holds for \( U \). For linear systems (3b), \( X^* \) is linked to \( W \) and \( U \) via an affine mapping. Hence the conclusion that \( X^* \) admits an exact PCE under the same basis (23), cf. [37].

**Derivation of (14):** Consider the basis (23) and the PCEs
\[
U_{i,k} = u^0_{i,k} + \sum_{j=1}^{L} u^j_{i,k} \phi^j \quad \text{with} \quad L = L_x + NL_w,
\]
we obtain
\[
\varphi_{x,k} = \begin{bmatrix} \phi^1 & \phi^2 & \cdots & \phi^{L_x} \end{bmatrix}^T,
\]
\[
\varphi_{w} = \begin{bmatrix} \phi^{p+1} & \phi^{p+2} & \cdots & \phi^{p+L_w} \end{bmatrix}^T,
\]
\[
u_{i,x,k} = \begin{bmatrix} u^1_{i,k} & u^2_{i,k} & \cdots & u^{L_x}_{i,k} \end{bmatrix}^T,
\]
\[
u_{i,w,k} = \begin{bmatrix} u^{p+1}_{i,k} & u^{p+2}_{i,k} & \cdots & u^{p+L_w}_{i,k} \end{bmatrix}^T,
\]
where \( p = L_x + jL_w \). Hence (25) can be reformulated as
\[
u^0_{i,k} = 0, \quad j \in \mathbb{I}_{[0,N-1]} \]
[32] E. Ahbe, A. Iannelli, and R. S. Smith, “Region of attraction analysis of nonlinear stochastic systems using Polynomial Chaos Expansion,” *Automatica*, vol. 122, p. 109187, 2020.

[33] T. Mühlpfordt, L. Roald, V. Hagenmeyer, T. Faulwasser, and S. Misra, “Chance-constrained AC optimal power flow: A polynomial chaos approach,” *IEEE Transactions on Power Systems*, vol. 34, no. 6, pp. 4806–4816, 2019.

[34] B. E. Fristedt and L. F. Gray, *A Modern Approach to Probability Theory*. Springer Science & Business Media, 2013.

[35] N. Wiener, “The homogeneous chaos,” *American Journal of Mathematics*, pp. 897–936, 1938.

[36] R. V. Field Jr. and M. Grigoriu, “On the accuracy of the polynomial chaos approximation,” *Probabilistic Engineering Mechanics*, vol. 19, no. 1-2, pp. 65–80, 2004.

[37] T. Mühlpfordt, R. Findeisen, V. Hagenmeyer, and T. Faulwasser, “Comments on quantifying truncation errors for polynomial chaos expansions,” *IEEE Control Systems Letters*, vol. 2, no. 1, pp. 169–174, 2018.

[38] R. Koekoek and R. F. Swarttouw, “The askey-scheme of hypergeometric orthogonal polynomials and its q-analogue,” *arXiv preprint math/9602214*, 1996.

[39] D. Xiu and G. E. Karniadakis, “The Wiener–Askey polynomial chaos for stochastic differential equations,” *SIAM Journal on Scientific Computing*, vol. 24, no. 2, pp. 619–644, 2002.

[40] M. Farina, L. Giulioni, L. Magni, and R. Scattolini, “A probabilistic approach to model predictive control,” in *2013 52nd IEEE Conference on Decision and Control (CDC)*. IEEE, 2013, pp. 7734–7739.

[41] T. Mühlpfordt, F. Zahn, V. Hagenmeyer, and T. Faulwasser, “PolyChaos.jl—a julia package for polynomial chaos in systems and control,” *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 7210–7216, 2020, 21th IFAC World Congress.

[42] I. Dunning, J. Huchette, and M. Lubin, “Jump: A modeling language for mathematical optimization,” *SIAM Review*, vol. 59, no. 2, pp. 295–320, 2017.

[43] L. Grüne, “Economic receding horizon control without terminal constraints,” *Automatica*, vol. 49, no. 3, pp. 725–734, 2013.

[44] R. Ou, M. H. Baumann, L. Grüne, and T. Faulwasser, “A simulation study on turnpikes in stochastic LQ optimal control,” *IFAC-PapersOnLine*, vol. 54, no. 3, pp. 516–521, 2021, 16th IFAC Symposium on Advanced Control of Chemical Processes ADCHEM 2021.

[45] J. M. Maciejowski, *Predictive Control with Constraints*. Pearson Education, 2002.