ON THE LOCAL MINIMIZING PROPERTY OF THE INTEGRAL
NORM OF THE CURVATURE TENSOR

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Abstract. We consider the Riemannian functional defined on the space of Riemannian
metrics with unit volume on a closed smooth manifold \( M \) given by
\[ R_p(g) := \int_M |R(g)|^p dv_g \]
where \( R(g), dv_g \) denote the Riemannian curvature and volume form. We prove that the
rank 1 symmetric spaces are stable for \( R_p \) for certain values of \( p \). It follows by standard
technique that they are local minima for this functional for those \( p \).

1. Introduction

Fix a closed smooth manifold \( M \) of dimension \( n \geq 3 \) and \( p \geq 2 \). Let \( R_p \) denote the
\( L^p \)-norm of the Riemannian curvature tensor
\[ R_p(g) = \int_M |R(g)|^p dv_g \]
where \( R(g) \) and \( dv_g \) denote corresponding Riemannian curvature and volume form. It is
a real valued function defined on the space of Riemannian metrics and remains invariant
under the action of group of diffeomorphisms of \( M \). \( R_p \) is not scale invariant unless \( p = \frac{n}{2} \).
If \( p \) is not equal to \( \frac{n}{2} \) we restrict the functional to the space of unite volume Riemannian
metrics \( M_1 \).

Convergence and collapsing of minimizing sequence of \( R_p \) have been studied by Anderson,
Yang, Gao etc. \[1,11,8\]. If \( p \leq \frac{n}{2} \) then \( \inf R_p = 0 \) for every closed manifold. Chern-
Gauss-Bonnet formula implies that Einstein metrics are minima for \( R_{\frac{n}{2}} \) in dimension 2
and 4. If \( \inf R_p \) is zero for \( p > \frac{n}{2} \) then \( M \) admits an F-structure of positive rank \[11\]. Hence
\( R_p \) has a positive global minima for \( p > \frac{n}{2} \) when the Euler characteristic of \( M \) is non-zero \[6,7\]. For a survey in this topic we refer to \[2\].

Compact irreducible symmetric spaces are critical metrics of \( R_p \). Metrics with constant
sectional curvature are local minima for \( R_p \) \[9\] for certain values of \( p \) and there are irre-
ducible symmetric spaces which are not local minima for \( R_{\frac{n}{2}} \) \[5\]. In this paper we study
stability and local minimizing property of this functional at rank 1 symmetric spaces. They
are spheres, complex, quaternionic and the Caley projective spaces (denoted by \( \mathbb{C}P^n \), \( \mathbb{H}P^n \)
and \( \mathbb{O}P^n \) respectively) and their non-compact duals. We prove the following.

Theorem. Let \((M, g)\) be a compact quotient of a rank 1 symmetric space \((\tilde{M}, \tilde{g})\). \((M, g)\) is
a local minimizer for \( R_p \) for the indicated values of \( p \).

(i) \((\tilde{M}, \tilde{g})\) is complex hyperbolic space and \( p \geq 2 \).
(ii) \((\tilde{M}, \tilde{g})\) is quaternionic projective space and \( p \geq \frac{n}{2} \).
(iii) \((\tilde{M}, \tilde{g})\) is the Caley projective plane and \( p \geq 6 \).
(iv) \((\tilde{M}, \tilde{g})\) is a compact quotient of a non-compact rank 1 symmetric space then \( p \geq \frac{n}{2} \).

Key words and phrases. Riemannian functional, critical point, stability, local minima.
\( \mathbb{C}P^{2n+1} \) is a fibre bundle over \( \mathbb{H}P^n \) with \( S^2 \) fibres \([12]\). By shrinking \( S^2 \) fibres one constructs a one parameter family of homogeneous metrics \( g(t) \) with unit volume such that the sectional curvature remains bounded. Hence \( \inf R \) is zero on \( \mathbb{C}P^{2n+1} \). The previous theorem implies that \( R \) restricted to \( g(t) \) has a local maxima at \( g(t_0) \) for some \( t_0 \). From the principle of symmetric criticality we have that \( g(t_0) \) is critical point of \( R \). For this critical metric is not a local minimizer.

On the other hand the Euler characteristic of other projective spaces are non-zero. So they do not admit F-structures. Therefore \( \inf R \) is positive for them for \( p > \frac{3}{2} \).

To prove the main theorem we study second variation of \( R \). The gradient of \( R \) is a vector field on the space of Riemannian metric. It is a 4 th order non-linear PDE in \( g \). The Hessian at a critical point of \( R \) is given by

\[
H_p(h_1, h_2) = \langle (\nabla R_p)'(h_1), h_2 \rangle \quad \forall h_1, h_2 \in S^2(T^*M)
\]

where \( S^2(T^*M) \) denotes the space of symmetric 2-tensor fields on \( M \) and \( (\nabla R_p)'(h_1) \) denotes the derivative of \( \nabla R_p \) at \( g \) along \( h_1 \). At an Einstein metric which is not a sphere \( S^2(T^*M) \) decomposes as

\[
S^2(T^*M) = \text{Im} \delta^*_g \oplus C^\infty(M).g \oplus (\delta^{-1}_g(0) \cap \text{Tr}^{-1}_g(0))
\]

(1.1)

\( \text{Im} \delta^*_g \) is the tangent space of the group of diffeomorphisms \( H \) restricted to \( \text{Im} \delta^*_g \) is zero. Define

\[
\mathcal{W} = (\delta^{-1}_g(0) \cap \text{Tr}^{-1}_g(0)) \oplus \{ fg : f \in C^\infty(M), \int_M f dv_g = 0 \}
\]

A Riemannian metric \( g \) which is not a sphere, is called stable for \( R \) if \( H_p \) restricted to unit sphere of \( \mathcal{W} \times \mathcal{W} \) has a positive lower bound i.e. the eigenvalues of \( H_p \) are bounded below by a positive constant.

**Theorem 1.** Let \( (M, g) \) be a closed manifold with the universal cover a rank 1 symmetric space. \( (M, g) \) is stable for \( R \) restricted to \( \delta^{-1}_g(0) \cap \text{tr}^{-1}(0) \) for all \( p \geq 2 \) i.e. there exists \( \epsilon > 0 \) such that for any symmetric 2-tensor \( h \) with \( \delta_g h = 0 \) and \( \text{tr}(h) = 0 \) the following holds.

\[
H_p(h, h) \geq \epsilon \| h \|^2.
\]

The gradient of \( R \) contains higher order derivatives of \( |R|^p-2R \). For \( p = 2 \), \( |R|^p-2 \) factor vanishes and then applying differential Bianchi identity one has a formula for \( \nabla R_2 \) which contains 2nd order derivatives of Ricci(\( r \)) and scalar(\( s \)) curvature only. The expression for \( \nabla R_2 \) restricted to \( M_1 \) is given by

\[
\nabla R_2(g) = 4D^*Dr + 2Dds + 4r \circ r - 4 \circ \nabla (r) - 2R + \frac{1}{2} |R|^2 g + (\frac{2}{n} - \frac{1}{2}) ||R||^2 g
\]

For notations we refer to section 2. To prove Theorem 1 first we prove the theorem for \( R_2 \) using the above formula for \( \nabla R_2 \). We use the Holonomy representation of rank 1 symmetric to express the \( H_2 \) on \( h \in \delta^{-1}_g(0) \cap \text{tr}^{-1}(0) \) in terms of rough laplacian acting on \( S^2(T^*M) \) and scalar curvature. When \( M \) is compact the expression is positive. In non-compact case we use a Böchner-Witzenböck type formula to get the stability of \( H_2 \). Since \( R \) is parallel in case of a locally symmetric space \( H_p \) restricted to \( \delta^0_g \cap \text{tr}^{-1}(0) \) is a constant multiple of \( H_2 \) and we obtain Theorem 1 for all \( p \geq 2 \).

Next using Böchner technique we express \( H_p \) restricted to the conformal variations of \( g \) in terms of Laplace-Beltrami operator acting on functions. Then estimates for first positive
eigenvalues of the Laplace-Beltrami operator in [10] gives the stability of $\mathcal{R}_p$ restricted to the the conformal variations of a rank 1 symmetric space of compact type.

**Theorem 2.** Let $(M, g)$ be a rank 1 symmetric space of compact type. $(M, g)$ is stable for $\mathcal{R}_p$ restricted to the conformal variations of $g$ for the indicated values of $p$.

(i) $(\tilde{M}, \tilde{g})$ is complex hyperbolic space and $p \geq 2$.
(ii) $(\tilde{M}, \tilde{g})$ is quaternionic projective space and $p \geq \frac{n}{2}$.
(iii) $(\tilde{M}, \tilde{g})$ is the Cayley projective plane and $p \geq 6$.

If $(M, g)$ be a compact quotient of a rank 1 symmetric space of non-compact type then $\mathcal{R}_p$ restricted to $C^\infty(M)g$ is stable for $p \geq \frac{n}{2}$ [5]. Proof of main theorem follows from Theorem 1 and 2.

Let $(M, g)$ be an closed Einstein manifold. From the formula for $\nabla \mathcal{R}_2$ mentioned before we have, $g$ is a critical for $\mathcal{R}_2$ if and only if

$$\hat{R} = \frac{|\mathcal{R}|^2}{n} g$$

If $(M, g)$ is de Rahm irreducible then Schur’s Lemma implies that $g$ is a critical metric of $\mathcal{R}_2$. We give a criterion for stability of $g$ for $\mathcal{R}_2$ in terms of the first positive eigenvalue of the Laplacian acting on functions when it is restricted to the conformal variation of $g$. The stability of $\mathcal{R}_p$ for a general quaternionic Kähler manifolds and conformally flat manifolds is still unknown.

**Acknowledgement:** The author is a post-doctoral fellow at Fourier Institute, University Joseph Fourier, Grenoble. She would like to thank Institute Fourier for supporting this work.

### 2. Notations and Second Variation of $\mathcal{R}_2$

Let $\{v_i\}$ be an orthonormal basis and $D$ be the Riemannian connection. The divergence operator $\delta_g$ acting on symmetric two tensors is given by

$$\delta_g(h)(x) = -D_{v_i} h(v_i, x) \text{ for } h \in S^2(T^*M)$$

$D^*$ and $\delta_g^*$ denote the formal adjoints of $D$ and $\delta_g$. $\hat{R}$ is a symmetric 2-tensor defined by

$$\hat{R}(x, y) = \sum R(x, v_i, v_j, v_k)R(y, v_i, v_j, v_k)$$

$\hat{R}$ is a symmetric operator on $S^2(T^*M)$ defined by

$$\hat{R}(h)(x, y) := \sum R(v_i, x, v_j, y)h(v_i, v_j).$$

Note that $\hat{R}$ ($g$) is the Ricci curvature of $g$. If $g$ is Einstein and $tr(h)$, $\delta_g h$ are zero then $tr(\hat{R}(h))$ and $\delta_g(\hat{R}(h))$ are also zero. The Kulkarni-Nomizu product of two symmetric two tensors $h_1$ and $h_2$ is defined by

$$h_1 \wedge h_2(x, y, z, w) = [h_1(x, z)h_2(y, w) + h_1(y, w)h_2(x, z) - h_1(x, w)h_2(y, z) - h_1(y, z)h_2(x, w)]$$

We define inner product on symmetric operators on bi-vectors ($S^2 \wedge^2 TM$) by

$$\langle A, B \rangle = tr(A \circ B)$$

$\{v_i \wedge v_j : j < i\}$ forms a basis of basis of $\wedge^2 TM$ and $tr$ is taken with respect to this basis. An algebraic curvature tensor $R$ can also be viewed as an element of $S^2 \wedge TM$. Note that,
\[ |R|^2 = 4t (R \circ R). \]

Let \( g_t \) be a one-parameter family of metrics with \( \frac{d}{dt}(g_t)|_{t=0} = h \). If \( T(t) \) is a tensor depending on \( g_t \), \( T'_g(h) \) denotes \( \frac{d}{dt}T(t)|_{t=0} \). \( D'_g(h) \), \( R'_g(h) \) and \( r'_g(h) \) are denoted by \( \Pi_h \), \( \bar{R} \) and \( r' \). From [4] we have the following formulae,

\[
\begin{align*}
g(\Pi_h(x,y),z) &= \frac{1}{2}\{D_xh(y,z) + D_yh(x,z) - D_zh(x,y)\} \\
\bar{R}_h(x,y,z,u) &= \frac{1}{2}\{D^2_{y,z}h(x,w) + D^2_{x,w}h(y,z) - D^2_{z,w}h(y,x) + \lambda h(R(x,y,z),w) - h(R(x,y,w),z)\} \\
r'_g(h) &= \frac{1}{2}\{D^*Dh + r \circ h + h \circ r - 2\bar{R}(h) - 2\delta g\delta g - Dd(trh)\}
\end{align*}
\]

Ricci curvature of \( \bar{R} \) is denoted by \( \bar{r} \). From the above formula we have \( r' = \bar{r} - \bar{R}(h) \).

Let \((M,g)\) be a closed Einstein manifold which is also a critical point of \( R_2 \) and \( \lambda \) be its Einstein constant. We compute each term which appear in the expression of Hessian \( H \).

**Lemma 1.** \( (D^*Dr)'(h) = -\lambda D^*Dh + D^*Dr'_h \)

**Proof.** Since \( g \) is Einstein \( Dr = 0 \). Therefore \( (D^*Dr)'(Dr) = 0 \) and we have

\[ (D^*Dr)'(h) = D^*(D'(h)r) + D^*D(r'(h)). \]

If \( T \) is a \( t \)-independent 2-tensor then,

\[ (D_xT(y,z))'(h) = -T(\Pi(x,y),z) - T(y,\Pi(x,z)) \]

When \( T = r = \lambda g \), using the formula for \( \Pi \) we have,

\[ D'(h)r = -\lambda Dh \]

Hence the lemma follows. \( \square \)

**Lemma 2.** \( (r \circ r)'(h) = 2\lambda r'_h - \lambda^2 h \)

**Proof.** Expressing \( r \circ r \) in a local co-ordinate chart we have, \( (r \circ r)_{mn} = g^{ij}r_{mi}r_{nj} \). Differentiating this with respect to \( t \),

\[
(r \circ r)'(h)_{mn} = \frac{d}{dt}(g^{ij}r_{mi}r_{nj} + g^{ij}r'_{mi}r_{nj}) + g^{ij}r_{mi}r'_{nj} = -g^{ik}g^{jl}g^{ij}r_{mi}r_{nj} + r'(h) \circ r_{mn} + r \circ r'(h)_{mn}
\]

Since \( g \) is Einstein we have, \( (r \circ r)'(h) = -\lambda^2 h + 2\lambda r'(h) \). \( \square \)

**Lemma 3.** \( (\bar{R}(r))'(h) = -\lambda \bar{R}(h) + \bar{R}'(h) + \lambda r'_h \)

**Proof.** Expressing \( \bar{R} \) in a local co-ordinate chart we have,

\[ (\bar{R}(r))_{pq} = g^{ij}g^{ij}R_{piqjh}r_{ij}. \]

Therefore,

\[
(\bar{R}(r))'(h)_{pq} = -(g^{ij}g^{ij}R_{piqjh}r_{ij} + g^{ij}g^{ij}R_{piqjh}r_{ij}) + g^{ij}g^{ij}R_{piqjh}r_{ij} + g^{ij}g^{ij}R_{piqjh}r_{ij} + g^{ij}g^{ij}R_{piqjh}r_{ij}
\]

Since \( g_{ij}(0) = \delta_j^i \) we have,

\[ (R(r))'_g(h)_{pq} = -h_{mj}R_{pmqn}r_{nj} - h_{mn}R_{piqm}r_{in} + R'_g(h)_{piqj}r_{ij} + R_{piqj}r'_g(h)_{ij} \]
Since $r = \lambda g$ we have,

$$\lambda_{mj} R_{pqmn} r_{nj} - \lambda_{mn} R_{pqim} r_{jn} = -2\lambda \overset{\circ}{\dot{R}} (h)_{pq}$$

$$R'_{g}(h)_{piqj} r_{ij} = \lambda r'_{pq} + \lambda \overset{\circ}{\dot{R}} (h)$$

$$R_{piqj} r'_{g}(h)_{ij} = \overset{\circ}{\dot{R}} (r')_{pq}$$

Hence the lemma follows. \qed

**Lemma 4.**

$$(Dds)'(h) = -\lambda Ddtrh + Ddtr(r'_{h})$$

**Proof.** Since $g$ is Einstein $(Dds)'(h) = Dds'(h)$. Next the lemma follows from 1.181 of [4]. \qed

**Lemma 5.**

$$(|R|^2)'(fg) = 4\lambda\Delta f - 2f|R|^2$$

**Proof.** The proof follows from the proof of Lemma in [9]. \qed

Define a 4 tensor $K$ by

$$K(x, y, z, w) = \frac{1}{2} \sum_{i,j} \{R(x, v_i, z, v_j) R(y, v_i, w, v_j) + R(x, v_i, w, v_j) R(y, v_i, z, v_j)\}$$

Let $r_{R\circ\bar{R}}$ denotes the ricci tensor of $R \circ \bar{R}$.

**Lemma 6.**

$$\langle (\bar{R})'(h), h \rangle = -2\langle R \circ R, h \wedge h \rangle - 2\langle K, h \otimes h \rangle + 4\langle r_{R\circ\bar{R}}, h \rangle$$

**Proof.**

$$\bar{R}_{pq} = g^{ijz}g^{jiz}g^{k1k2}R_{pi1j1k1}R_{nq2j2k2}$$

Differentiating each terms and evaluating it in an orthonormal basis $\{v_i\}$ and using

$$\langle (g^{ij})', g^{ij} \rangle = -g^{im}h_{mn}g^{nj}$$

we have,

$$(\bar{R}'_{g})(h)_{pq} = -h_{mn} (R_{pmij} R_{qni} + R_{pimj} R_{qijn} + R_{pijm} R_{qijn}) + (R'_{g}(h)_{pijk} R_{qijk} + R_{pijk}(R'_{g}(h)_{qijk})$$

By a straightforward computations we have,

$$\langle R \circ R, h \wedge h \rangle = \frac{1}{2} h_{pq} h_{mn} R_{pijk} R_{qijk}$$

$$\langle K, h \otimes h \rangle = h_{pq} h_{mn} R_{pimj} R_{qijn}$$

$$\langle r'_{R\circ\bar{R}}, h \rangle = \frac{1}{2} h_{pq} R_{pijk} \bar{R}_{qijk}$$

Hence the lemma follows. \qed
Combining Lemma 1-5 and using (2.1) we have,

\[ H(h, h) = 4(r'_h, D^* Dh + \lambda h - \hat{R}(h)) + \frac{1}{2}(\delta g h) - 4\lambda(\langle D^* Dh + \lambda h - \hat{R}(h), h \rangle (2.1)
+ 2\langle R \circ R, h \wedge h \rangle + 4\langle K, h \otimes h \rangle - 8\langle r_{R \circ R}, h \rangle + \frac{2\|R\|^2}{n}\|h\|^2
+ \frac{1}{2}(\|R\|^2 h, h) - 2\lambda(\langle Ddtr(h), h \rangle
\]

If \( \delta_g h = 0 \) and \( tr(h) = 0 \) then \( r'_h = \frac{1}{2}\{D^* Dh + 2\lambda h - 2 \hat{R}(h)\} \). Consequently we have the following.

**Theorem 3.** Let \((M, g)\) be a Einstein critical metric of \( R_2 \) with Einstein constant \( \lambda \) and \( h \) be a symmetric two tensor with \( \delta_g h = 0 \) and \( tr(h) = 0 \) then

\[ H(h, h) = 2\langle [D^* Dh - 2 \hat{R}(h), D^* Dh + \lambda h - \hat{R}(h)] + \frac{|R|^2}{n}\|h\|^2
+ \langle R \circ R, h \wedge h \rangle + 2\langle K, h \otimes h \rangle - 4\langle r_{R \circ R}, h \rangle \]

**Theorem 4.** Let \((M, g)\) be a Einstein critical metric of \( R_2 \) with Einstein constant \( \lambda \) and \( f \in C^\infty(M) \) with \( \int_M f dv_g = 0 \) then

\[ H(fg, fg) = 2(n - 1)\|\Delta f\|^2 - 8\lambda\|df\|^2 + (4 - n)|R|^2\|f\|^2 \]

Let \( \mu \) denotes the first positive eigenvalue of the Laplacian.

**Remark 1 :** Let \( \lambda > 0 \). Using Böchner technique we have,

\[ H(fg, fg) > 2(n - 5)\|f\|^2 - (n - 4)|R|^2\|f\|^2 \]

Therefore \( g \) is stable for \( R_2 \) restricted to the conformal variations if and only if

\[ \mu^2 \geq \frac{(n - 4)}{2(n - 1)}\|R\|^2 \]

**Remark 2 :** If \( \lambda \leq 0 \) then \( g \) is stable for \( R_2 \) restricted to the conformal variations if

\[ \mu^2 \geq \frac{(n - 4)}{2(n - 1)}|R|^2 \]

### 3. Stability of \( R_2 \) at Rank 1 Symmetric Spaces

We recall some basic geometric facts of rank 1 symmetric spaces. Let \((M, g)\) be a rank 1 symmetric space which is not a sphere nor a hyperbolic space. Then \( T_p M \) admits a module structure over \( \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \) for any \( p \in M \). There are linear isometries \( \{J_\alpha, \alpha = 1, 2, \ldots, \tau\} \) on \( T_p M \) with the following properties,

\[ J_\alpha^2 = Id \]

\[ J_\alpha J_\beta = -J_\beta J_\alpha \text{ for } \alpha \neq \beta \]

\[ J_\alpha J_\beta(x) \in \text{span}(J_0(x), J_1(x), \ldots, J_\tau(x)) \]

where \( J_0 \) denotes identity of \( T_p M \). \( \tau = 1, 3, 7 \) in case of \( \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \) modules respectively. Given an unit vector \( e_1 \) of \( T_p M \) one may extend it to an orthonormal frame of the form
\{e_{\alpha j}\} in a neighborhood of p where \(e_{\alpha j} = J_{\alpha} e_j\) and the curvature tensor is given by the following.

\[
R(e_{\alpha i}, e_{\beta j}, e_{\gamma k}) = 0 \text{ if } k \neq i \neq j \\
R(e_{\gamma i}, e_{\alpha i}, e_{\beta i}) = 0 \text{ if } \alpha \neq \beta \neq \gamma \\
R(e_{\alpha i}, e_{\beta j}, e_{\alpha i}, e_{\beta j}) = c \text{ when } \{\alpha, i\} \neq \{\beta, j\} \\
R(e_{\alpha i}, e_{\beta i}, e_{\alpha j}, e_{\beta j}) = 2c \text{ if } i \neq j \text{ and } \alpha \neq \beta \\
R(e_{\alpha i}, e_{\beta i}, e_{\alpha i}, e_{\beta i}) = 4c \text{ for } \alpha \neq \beta
\]

where \(c\) is a non-zero constant. Since each \(J_{\alpha}\) is an isometry we also have, \(R(x, y, J_{\alpha} z, J_{\alpha} w) = R(x, y, z, w)\) for any \(x, y, z, w \in T_p M\). Hence \(R(e_{\alpha i}, e_{\alpha j}, e_{\beta i}, e_{\beta j}) = c\).

For a symmetric 2-tensor \(h\) define \(\tilde{h}\) by \(\tilde{h}(x, y) = \sum_{\alpha \neq 0} h(J_{\alpha} x, J_{\alpha} y)\).

**Lemma 7.** Let \((M, g)\) be a rank 1 symmetric space. Then

\[
\nabla^\circ \tilde{R}(h) = 3\tilde{h} - ch + ctr(h)g
\]

**Proof.** Let \(e_1\) be a unit vector. It is sufficient to prove the above equation for \(e_1\). We extend \(e_1\) to a basis we mentioned above.

\[
\nabla^\circ \tilde{R}(h)(e_1, e_1) = R(e_1, .., e_1, ..)h(., .) \\
= \sum_i R(e_1, e_i, e_1, e_i)h(e_i, e_i) + \sum_{\alpha, i} R(e_1, e_{\alpha i}, e_1, e_{\alpha i})h(e_{\alpha i}, e_{\alpha i}) \\
= c \sum_{i \neq 1} [h(e_i, e_i) + h(e_{\alpha i}, e_{\alpha i})] + 4c \sum_{\alpha} h(e_{\alpha 1}, e_{\alpha 1}) \\
= 3\tilde{h}(e_1, e_1) + ctr(h) - ch(e_1, e_1)
\]

Since \(\nabla^\circ \tilde{R}(g) = r = \lambda g\), the Einstein constant \(\lambda = c(3\tau + n - 1)\) and scalar curvature \(s = cn(3\tau + n - 1)\).

Let \(R_1\) be an algebraic curvature tensor. Define \(\tilde{R}_1\) by

\[
\tilde{R}_1(x \wedge y) = \sum_{\gamma \neq 0, \alpha < \beta} \langle R(x, y, J_{\gamma} e_{\alpha i}, J_{\gamma} e_{\beta i}), e_{\alpha i} \wedge e_{\beta i} \rangle + \sum_{\gamma \neq 0, i < j} \langle R(x, y, J_{\gamma} e_{\alpha j}, J_{\gamma} e_{\beta j}), e_{\alpha j} \wedge e_{\beta j} \rangle
\]

By definition \(\tilde{R}(x \wedge y) = \tau R(x \wedge y)\). Since \(R(x \wedge y)\) belongs to the Lie algebra of the holonomy group of a rank 1 symmetric space the last two entries of \(R \circ R_1\) are invariant under \(J_{\gamma}\).

**Lemma 8.**

\[
R \circ R_1(x \wedge y) = cR_1(x \wedge y) + c\tilde{R}_1(x \wedge y) + c \sum_{\alpha < \beta} R_1(x, y, J_{\gamma} e_{\alpha j}, J_{\gamma} e_{\beta j}) e_{\alpha i} \wedge e_{\beta i}
\]
Proof.

\[ g(R \circ R_1(x \wedge y), e_i \wedge e_{\alpha}) = \frac{1}{2} \sum R(e_i, e_{\alpha i}, e_{\beta j}, e_{\delta k}) R_1(x, y, e_{\beta j}, e_{\delta k}) \]
\[ = \frac{1}{2} \sum R(e_i, e_{\alpha i}, J_\gamma e_{\beta j}, J_\gamma e_{\alpha j}) R_1(x, y, J_\gamma e_{\beta j}, J_\gamma e_{\alpha j}) \]
\[ = c \sum R_1(x, y, J_\gamma e_{\alpha i}, J_\gamma e_{\alpha j}) + c \sum R_1(x, y, J_\gamma e_{\beta j}, J_\gamma e_{\alpha j}) \]

Similarly,

\[ g(R \circ R_1(x \wedge y), e_i \wedge e_j) = \sum R(e_i, e_j, e_{\alpha i}, e_{\alpha j}) R_1(x, y, e_{\alpha i}, e_{\alpha j}) \]
\[ = c \sum R_1(x, y, e_{\alpha i}, e_{\alpha j}) \]

Therefore we have,

\[ R \circ R_1(x \wedge y) = c \sum \{ R_1(x, y, J_\gamma e_{\alpha i}, J_\gamma e_{\beta j}) + R_1(x, y, J_\gamma e_{\alpha j}, J_\gamma e_{\beta j}) \} e_{\alpha i} \wedge e_{\beta j} \]
\[ + c \sum R_1(x, y, J_\gamma e_{\alpha i}, J_\gamma e_{\beta j}) e_{\alpha i} \wedge e_{\beta j} \]

Hence the lemma follows. \( \square \)

If we put \( R_1 = R \) the next lemma follows.

Lemma 9.

\[ R \circ R(x \wedge y) = c(\tau + 1) R(x \wedge y) + c(\tau + 1) \sum \sum R(x, y, e_{\alpha j}, e_{\beta j}) e_{\alpha i} \wedge e_{\beta i} \]

As a consequence we have the following lemmas.

Lemma 10.

\[ |R|^2 = 2c^2 n(5\tau^2 + 3n\tau + 4\tau + n - 1) \]

Therefore,

Lemma 11.

\[ \langle R \circ R, h \wedge h \rangle = c(n + \tau + 1) \langle \hat{R}(h), h \rangle + 4c^2 n\|h\|^2 \]

Proof. Let \( \{v_i\} \) be any orthonormal basis.

\[ \sum_{i<j} \langle R(v_i \wedge v_j), h \wedge h(v_i \wedge v_j) \rangle = \sum_{i<j,k<l} 2R(v_i, v_j, v_k, v_l)[h(v_i, v_k)h(v_j, v_l) - h(v_i, v_l)h(v_j, v_k)] \]
\[ = \langle \hat{R}(h), h \rangle \]
The remaining term of $\langle R \circ R, h \wedge h \rangle$ is the following.
\[
\frac{c}{2}(\tau + 1) \sum_{a < \beta} R(J_\gamma e_{a_k}, J_\gamma e_{\beta_k}, e_{a_j}, e_{\beta_j})\{h \wedge h(J_\gamma e_{a_k} \wedge J_\gamma e_{\beta_k}), e_{a_i} \wedge e_{\beta_i}\}
\]
\[
= 3c^2 n \sum_{a < \beta} \{h \wedge h(J_\gamma e_{a_k} \wedge J_\gamma e_{\beta_k}), e_{a_i} \wedge e_{\beta_i}\}
\]
\[
= 3c^2 n \sum_{a < \beta} \{h(e_{a_i}, J_\gamma e_{a_j})h(e_{\beta_i}, J_\gamma e_{\beta_j}) - h(J_\gamma e_{a_k}, e_{\beta_i})h(e_{a_i}, J_\gamma e_{\beta_k})\}
\]
\[
= 3c^2 n(h, \tilde{h} + h)
\]
\[
= c n(\tilde{R}(h), h) + 4c^2 n|h|^2
\]

Hence the lemma follows. \(\square\)

**Lemma 12.**

\[
r_{R_0R_1}(x, x) = cr_1(x, x) + c \sum_{\gamma \neq 1} \{R_1(x, e_{\beta_i}, J_\gamma x, J_\gamma e_{\beta_i}) + \frac{c}{2} R_1(x, J_\gamma x, e_{\beta_i}, J_\gamma e_{\beta_i})\}
\]

**Proof.** From Lemma 8 we have,

\[
r_{R_0R_1}(x, x) = cr_1(x, x) + \sum_{a < \beta} R_1(x, e_{\delta_k}, J_\gamma e_{a_j}, J_\gamma e_{\beta_j})(e_{a_i} \wedge e_{\beta_i}, x \wedge e_{\delta_k})
\]

By the definition of $\tilde{R}_1$ we have,

\[
\tilde{r}_{R_1}(x, x) = \sum_{\gamma \neq 1} \tilde{R}(x, e_{\beta_i}, J_\gamma x, J_\gamma e_{\beta_i})
\]

The remaining term is

\[
c \sum_{a \wedge \beta} R_1(x, e_{\delta_k}, J_\gamma e_{a_j}, J_\gamma e_{\beta_j})(e_{a_i} \wedge e_{\beta_i}, x \wedge e_{\delta_k})
\]

\[
= c \sum_{a \wedge \beta} x_{a_i} R_1(x, e_{\beta_i}, J_\gamma e_{a_j}, J_\gamma e_{\beta_j})
\]

\[
= \frac{c}{2} \sum_2 R(x, J_\gamma x, e_{\beta_j}, J_\gamma e_{\beta_j})
\]

Hence the lemma follows. \(\square\)

Next we consider $R_1 = \tilde{R}$ and compute $\langle r_{R_0\tilde{R}}, h \rangle$.

**Lemma 13.** Let $h$ be a symmetric two tensor with $\delta_n h = 0$ and $tr(h) = 0$.

\[
\langle r_{R_0\tilde{R}}, h \rangle = \frac{c}{2} \|Dh\|^2 - c^2 \tau(\tau - 1)\|h\|^2 + \tau c^2 \|\tilde{h}, h\|^2 + 3c^2 \|\tilde{h}\|^2
\]

**Proof.** From the previous lemmas we have,

\[
\langle r_{R_0\tilde{R}}, h \rangle = c(\tilde{r}, h) + c \sum_{\gamma \neq 1} \tilde{R}(e_{\alpha p}, e_{\beta i}, J_\gamma e_{\delta q}, J_\gamma e_{\beta i})h(e_{\delta q}, e_{\alpha p})
\]

\[
+ \tilde{R}(e_{\delta q}, e_{\alpha p}, J_\gamma e_{\beta i}, J_\gamma e_{\beta i})h(e_{\delta q}, e_{\alpha p})
\]

We know that

\[
\langle \tilde{r}, h \rangle = \frac{1}{2} \|Dh\|^2 + \lambda\|h\|^2
\]
Next we compute the remaining two terms.
\[
2 \check{R}(e_{\alpha p}, e_{\beta i}, J_\gamma e_{\delta q}, J_\gamma e_{\beta i}) = D_{e_{\alpha p}, J_\gamma e_{\beta i}}^2 h(e_{\beta i}, J_\gamma e_{\delta q}) + D_{e_{\beta i}, J_\gamma e_{\delta q}}^2 h(J_\gamma e_{\beta i}, e_{\alpha p}) - D_{e_{\alpha p}, e_{\beta i}}^2 J_\gamma e_{\delta q} h(e_{\beta i}, J_\gamma e_{\delta q}) - D_{e_{\beta i}, e_{\delta q}}^2 h(J_\gamma e_{\beta i}, J_\gamma e_{\delta q}) + h(R(e_{\alpha p}, e_{\beta i}, J_\gamma e_{\delta q}), J_\gamma e_{\beta i}) - h(R(e_{\alpha p}, e_{\beta i}, J_\gamma e_{\beta i}), J_\gamma e_{\delta q})
= (T_1 + T_2 - T_3 - T_4 + T_5)(e_{\alpha p}, e_{\delta q})
\]

Notice that
\[
T_3 = \sum_i D_{e_{\alpha p}, J_\gamma e_{\delta q}}^2 h(e_{\beta i}, J_\gamma e_{\beta i}) = 0
\]
\[
T_1 = \sum_i D_{e_{\alpha p}, J_\gamma e_{\beta i}}^2 h(e_{\beta i}, J_\gamma e_{\delta q}) = - \sum_i D_{e_{\alpha p}, e_{\beta i}}^2 J_\gamma e_{\delta q} h(J_\gamma e_{\beta i}, J_\gamma e_{\delta q}) = D\delta h(h(e_{\alpha p}, e_{\delta q})
\]
Therefore \( \langle T_1, h \rangle = 2 \langle \delta h, h \rangle = 0 \). Next notice that
\[
(T_2, h) = D_{e_{\beta i}, e_{\delta q}} h(e_{\alpha p}, e_{\delta q}) h(e_{\alpha p}, e_{\delta q})
\]
Therefore,
\[
(T_2, h) + (D\delta h, h) = D_{e_{\beta i}, e_{\delta q}} h(e_{\alpha p}, e_{\delta q}) h(e_{\alpha p}, e_{\delta q}) - D_{e_{\delta q}, e_{\beta i}} h(e_{\alpha p}, e_{\delta q}) h(e_{\alpha p}, e_{\delta q})
\]
Next using Ricci identity we have,
\[
(T_2 + D\delta h)(e_{\alpha p}, e_{\delta q}) = h(R(e_{\beta i}, J_\gamma e_{\alpha p}, e_{\delta q}), J_\gamma e_{\beta i}) + h(R(e_{\beta i}, J_\gamma e_{\delta q}, J_\gamma e_{\beta i}), e_{\alpha p}) - h(R(J_\gamma e_{\beta i}, e_{\delta q}, e_{\alpha p}), J_\gamma e_{\beta i}) - h(R(J_\gamma e_{\beta i}, e_{\delta q}, J_\gamma e_{\beta i}), e_{\alpha p}) = -\check{R}(h)(e_{\alpha p}, e_{\delta q}) - \tau \lambda h(e_{\alpha p}, e_{\delta q})
\]
Therefore \( \langle T_2, h \rangle = -\check{R}(h)(h) - \tau \lambda \| h \|^2 \).
\[
2T_3 = 2D_{e_{\alpha p}, J_\gamma e_{\beta i}}^2 h(e_{\alpha p}, J_\gamma e_{\delta q}) = D_{e_{\beta i}, J_\gamma e_{\delta q}}^2 h(e_{\alpha p}, J_\gamma e_{\delta q}) - D_{J_\gamma e_{\beta i}, e_{\delta q}}^2 h(e_{\alpha p}, J_\gamma e_{\delta q})
\]
Using Ricci identity again we have,
\[
\langle T_3, h \rangle = (n + \tau + 1)\| h \|^2
\]
\[
T_5 = h(R(e_{\alpha p}, e_{\beta i}, J_\gamma e_{\delta q}), J_\gamma e_{\beta i}) - h(R(e_{\alpha p}, e_{\beta i}, J_\gamma e_{\beta i}), J_\gamma e_{\delta q}) = h(R(J_\gamma e_{\alpha p}, J_\gamma e_{\beta i}, J_\gamma e_{\delta q}), J_\gamma e_{\beta i}) - h(R(J_\gamma e_{\alpha p}, J_\gamma e_{\beta i}, J_\gamma e_{\beta i}), J_\gamma e_{\delta q}) = \check{R}(h)(J_\gamma e_{\alpha p}, J_\gamma e_{\delta q}) + \lambda h(J_\gamma e_{\alpha p}, J_\gamma e_{\delta q})
\]
Therefore \( \langle T_5, h \rangle = \check{R}(h)(h) + \langle h, h \rangle \). Similarly, let
\[
2\check{R}(e_{\delta q}, J_\gamma e_{\alpha p}, e_{\beta i}, J_\gamma e_{\beta i}) h(e_{\delta q}, e_{\alpha p}) = S_1 + S_2 + S_3 + S_4 + S_5
\]
Notice that
\[
S_3 = - \sum_{\gamma \neq 1} h(e_{\delta q}, e_{\alpha p}) D_{e_{\delta q}, e_{\beta i}}^2 h(J_\gamma e_{\alpha p}, J_\gamma e_{\beta i}) = E_2
\]
and
\[
S_4 = - \sum_{\gamma \neq 1} h(e_{\delta q}, e_{\alpha p}) D_{J_\gamma e_{\delta q}, e_{\beta i}}^2 h(e_{\delta q}, e_{\beta i}) = S_1
\]
Now

\[ S_1 = \sum_{\gamma \neq 1} h(e_{q\gamma}, e_{\alpha p}) D_{J_\gamma e_{\alpha p}, e_{\beta i}}^2 h(e_{q\delta}, J_\gamma e_{\beta i}) \]

\[ = - \sum_{\gamma \neq 1} h(J_\gamma e_{q\delta}, J_\gamma e_{\alpha p}) D_{e_{\alpha p}, e_{\beta i}}^2 h(J_\gamma e_{q\delta}, J_\gamma e_{\beta i}) \]

\[ = - \sum_{\gamma \neq 1} h(J_\gamma e_{q\delta}, J_\gamma e_{\alpha p}) D_{e_{\alpha p}, e_{\beta i}}^2 h_{\gamma}(e_{q\delta}, e_{\beta i}) \]

\[ = \sum_{\gamma \neq 1} h_{\gamma}(e_{q\delta}, e_{\alpha p}) \delta^\gamma_{\delta q} \delta_{\gamma} h_{\gamma}(e_{\alpha p}, e_{q\delta}) \]

\[ = 0 \]

\[ S_2 = \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) D_{e_{q\delta}, J_\gamma e_{\beta i}}^2 h(J_\gamma e_{\alpha p}, e_{\beta i}) \]

\[ = - \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) D_{e_{q\delta}, e_{\beta i}}^2 h(J_\gamma e_{\alpha p}, J_\gamma e_{\beta i}) \]

\[ = - \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) D_{e_{q\delta}, e_{\beta i}}^2 \tilde{h}(e_{\alpha p}, e_{q\delta}) \]

\[ = \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) \delta^\gamma_{\delta q} \tilde{h}(e_{\alpha p}, e_{q\delta}) \]

\[ S_5 = \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) [h(R(e_{q\delta}, J_\gamma e_{\alpha p}, e_{\beta i}), J_\gamma e_{\beta i}) - h(R(e_{q\delta}, J_\gamma e_{\alpha p}, J_\gamma e_{\beta i}), e_{\beta i})] \]

\[ = 2 \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) h(R(e_{q\delta}, J_\gamma e_{\alpha p}, e_{\beta i}), J_\gamma e_{\beta i}) \]

\[ = -2 \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) [h(R(J_\gamma e_{\alpha p}, e_{\beta i}, e_{q\delta}), J_\gamma e_{\beta i}) + h(R(e_{\beta i}, e_{q\delta}, J_\gamma e_{\alpha p}), J_\gamma e_{\beta i})] \]

\[ = 2 \sum_{\gamma \neq 1} h(e_{q\delta}, e_{\alpha p}) [R(e_{\alpha p}, J_\gamma e_{\beta i}, e_{q\delta}, e_{\beta i}) h(., J_\gamma e_{\beta i}) + R(e_{q\delta}, e_{\beta i}, e_{\alpha p}, .) h(J_\gamma e_{\beta i}, .)] \]

\[ = 4 \hat{R} (\tilde{h}, h) \]

Lemma 14.

\[ (\mathcal{K}, h \otimes h) = \{ n + 10(\tau + 1) \} \| \tilde{h} + h \|^2 + 4tr^2(\tilde{h} + h) \]
Proof. Let \( h_1 = h + \tilde{h} \). Since \( R \) is \( J_\gamma \) invariant in first and last two entries we have,

\[
\langle K, h \otimes h \rangle = \sum h_1(e_p, e_q)h_1(e_m, e_n)R(e_p, e_{\alpha i}, e_m, e_{\beta j})R(e_q, e_{\alpha i}, e_n, e_{\beta j})
\]

\[
= \sum h_1^2(e_p, e_p)R^2(e_p, e_{\alpha i}, e_p, e_{\alpha i}) + 2 \sum_{p \neq m} h_1(e_p, e_p)h_1(e_m, e_m)R^2(e_p, e_{\alpha p}, e_m, e_{\alpha m})
\]

\[
+ \sum_{p \neq q} h_1^2(e_p, e_q)R(e_p, e_{\alpha i}, e_p, e_{\alpha i})R(e_q, e_{\alpha i}, e_q, e_{\alpha i})
\]

\[
+ 2 \sum_{p \neq q} h_1^2(e_p, e_q)R(e_p, e_{\alpha p}, e_q, e_{\alpha q})R(e_q, e_{\alpha p}, e_q, e_{\alpha q})
\]

\[
= \{n + 10(\tau + 1)\} \|h_1\|^2 + 4tr^2(h_1)
\]

3.1. Proof of the main theorem: First we consider a symmetric two tensor \( h \) with \( \delta_g h = 0 \) and \( tr(h) = 0 \). From the previous lemmas we have,

\[
H(h, h) = 2[\|D^*Dh - \frac{3}{2} \, \overset{\circ}{\varphi}(h)\|^2 + (n + 3\tau - 3)c\|Dh\|^2 + c^2\{2n + 20 + 6\tau - \frac{1}{4}\}]\|\tilde{h}\|^2
\]

\[
+ (7n + 42 + 39\tau + \frac{1}{2})c^2\|\tilde{h}, h\|^2 + c^2\{7n\tau + 6n\tau + 14\tau^2 + 23\tau + 17 + \frac{1}{2}\}\|h\|^2
\]

If \( c > 0 \) then it is clear from the above expression that there exists \( \epsilon > 0 \) such that \( H(h, h) > \epsilon\|h\|^2 \).

Let \( c < 0 \). From [3] we have,

\[
\langle D^*Dh - \overset{\circ}{\varphi}(h) + \lambda h, h \rangle \geq 0
\]

Using this identity we have,

\[
\langle D^*Dh - 2 \, \overset{\circ}{\varphi}(h), D^*Dh + \lambda h - \overset{\circ}{\varphi}(h) \rangle
\]

\[
= \|D^*Dh - \frac{3}{2} + \lambda h\|^2 - \lambda\langle D^*Dh - \overset{\circ}{\varphi}(h), h \rangle + \lambda^2\|h\|^2 - \frac{1}{4}\| \overset{\circ}{\varphi}(h)\|^2
\]

\[
\geq -\frac{1}{4}\| \overset{\circ}{\varphi}(h)\|^2
\]

Therefore,

\[
H(h, h) \geq \frac{13}{12}c^2\| \overset{\circ}{\varphi}(h)\|^2 + 2c^2\{n + 10(\tau + 1)\}\|\tilde{h}\|^2
\]

\[
- 3(n + 3 - 5\tau)c^2\|\tilde{h}, h\|^2 + 2c^2\{3n(\tau + 1) + 5\tau^2 - 12\tau + 15 + \frac{1}{3}\}\|h\|^2
\]

\[
\geq \epsilon_1\|h\|^2
\]

As a consequence we have Theorem 1 for \( p = 2 \). Let \( H_p \) denote the Hessian of \( \mathcal{R}_p \) then from [3] (4.1) we have

\[
H_p(h, h) = \frac{p}{2}[|R|^{p-2}H(h, h) \text{ forall } h \in \delta_g(0) \cap tr^{-1}(0)]
\]

Therefore, the theorem also holds for \( \mathcal{R}_p \) for \( p > 2 \).

Next we consider the conformal variations of \( g \). In this case the previous technique does not work. \((|R|^p)'_g \) contributes to the expression of \( H_p \) restricted to the conformal case. From
Lemma 10 and [10] we have the following values of $\frac{|R|^2}{\lambda^2}$ and $\frac{\mu}{\lambda}$ for rank 1 symmetric space of compact type.

(i) $\mathbb{C}P^n$: $\tau = 1$ and let $n = 2m$. Then $\frac{|R|^2}{\lambda^2} = \frac{m}{m+1}$ and $\frac{\mu}{\lambda} = 2$.

(ii) $\mathbb{H}P^n$: $\tau = 3$ and let $n = 4m$. Then $\frac{|R|^2}{\lambda^2} = \frac{4m(5m+7)}{(m+2)^2}$ and $\frac{\mu}{\lambda} = \frac{2(m+1)}{m+2}$.

(iii) $\mathbb{O}p^2$: $\tau = 7$ and $n = 16$. $\frac{|R|^2}{\lambda^2} = \frac{416}{27}$ and $\frac{\mu}{\lambda} = \frac{4}{3}$.

Now the compact case of Theorem 2 follows from Proposition 2.1 in [9]. The non-compact case follows from the same proposition. This completes the proof of the main theorem. □

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