Resolvent estimates for non-self-adjoint operators via semi-groups

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Dedicated to V.G. Maz’ya

Abstract

We consider a non-self-adjoint $h$-pseudodifferential operator $P$ in the semi-classical limit ($h \to 0$). If $p$ is the leading symbol, then under suitable assumptions about the behaviour of $p$ at infinity, we know that the resolvent $(z - P)^{-1}$ is uniformly bounded for $z$ in any compact set not intersecting the closure of the range of $p$. Under a subellipticity condition, we show that the resolvent extends locally inside the range up to a distance $O(1)((h \ln \frac{1}{h})^{k/(k+1)})$ from certain boundary points, where $k \in \{2, 4, ...\}$. This is a slight improvement of a result by Dencker, Zworski and the author, and it has recently been obtained by W. Bordeaux Montrieux in a model situation where $k = 2$. The method of proof is different from the one of Dencker et al, and is based on estimates of an associated semi-group.

Résumé

Nous considérons un opérateur $h$-pseudodifférentiel non-autoadjoint $P$ dans la limite semi-classique ($h \to 0$). Si $p$ désigne le symbole principal, alors sous des hypothèses convenables sur le comportement de $p$ à l’infini nous savons que la résolvante $(z - P)^{-1}$ est uniformément bornée pour $z$ dans un compact qui ne rencontre pas l’adhérence

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de l'image de $p$. Sous une hypothèse des sous-ellipticité, nous montrons que la résolvante s'étend vers l'intérieur de cet image jusqu'à une distance $\mathcal{O}(1)((h \ln \frac{1}{h})^{k/(k+1)})$ de certains points du bord, où $k \in \{2, 4, \ldots\}$. Ceci est une légère amélioration d’un résultat de Dencker, Zworski et l’auteur. Cette amélioration a été obtenue récemment par W. Bordeaux Montrieux dans une situation modèle où $k = 2$. La méthode de preuve, qui est différente de celle de Dencker et al, est basée sur des estimations sur un semi-groupe microlocal associé.

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### 1. Introduction

In this paper, we are interested in bounds on the resolvent $(z - P)^{-1}$ of a non-self-adjoint $h$-pseudodifferential operator with leading symbol $p$ when $h \to 0$, for $z$ in a neighborhood of certain points on the boundary of the range of $p$. The interest in such questions arouse with that in pseudospectra of non-self-adjoint operators, see [22, 23]. Under reasonable hypothesies we know that $(z - P)^{-1}$ is uniformly bounded for $h > 0$ small enough and for $z$ in any fixed compact set in $\mathbb{C}$, disjoint from the closure of the range of $p$. On the other hand, by a quasi-mode construction of E.B. Davies [5], that was generalized by Zworski [24] by reduction to an old quasi-mode construction of Hörmander (see also [7] for a more direct approach), we also know that if $C \ni z = p(\rho)$, where $\rho$ is a point in phase space where $i^{-1}\{p, \rho\} > 0$ and $\{\cdot, \cdot\}$ denotes the Poisson bracket, then we have quasimodes for $P - z$ in the sense that there exist $u = u_h \in C_0^\infty$, normalized in $L^2$, such that the $L^2$ norm of $(P - z)u_h$ is $\mathcal{O}(h^\infty)$, implying, somewhat roughly, that the norm of the resolvent (whenever it is defined) cannot be bounded by a negative power of $h$.

A natural question is then what happens when $z$ is close to the boundary of the range of $p$. L. Boulton [1] and Davies [6] obtained some results about
this in the case of the non-self-adjoint harmonic operator on the real line. As with the quasi-mode construction this question is closely related to classical results in the general theory of linear PDE, and with N. Dencker and Zworski ([7]) we were able to find quite general results closely related to the classical topic of subellipticity for pseudodifferential operators of principal type, studied by Egorov, Hörmander and others. See [12]. This topic in turn is closely related to the oblique derivate problem and degenerate elliptic operators, where V.G. Maz’ya has made important contributions. See [16, 17].

In [7] we obtained resolvent estimates at certain boundary points, (A) under a non-trapping condition, and (B) under a stronger “subellipticity condition”.

In case (A) we could apply quite general and simple arguments related to the propagation of regularity and in case (B) we were able to adapt general Weyl-Hörmander calculus and Hörmander’s treatment of subellipticity for operators of principal type ([12]). In the first case we obtained that the resolvent extends and has temperate growth in $1/h$ in discs of radius $O(h \ln 1/h)$ centered at the appropriate boundary points, while in case (B) we got the corresponding extension up to distance $O(h^{k/(k+1)})$, where the integer $k \geq 2$ is determined by a condition of “subellipticity type”.

However, the situation near boundary points of the type (B) is more special than the general subellipticity situations considered by Egorov and Hörmander, and the purpose of the present paper is to develop such an approach by studying an associated semi-group basically as a Fourier integral operator with complex phase in the spirit of Maslov [14], Kucherenko [13], Melin-Sjöstrand [18]. (See also the more recent works by A. Menikoff-Sjöstrand [20], O. Matte [15], extending the approach of [18] to non-homogeneous cases.) Finally it turned out to be more convenient to use Bargmann-FBI transforms in the spirit of [21] and [9]. The semigroup method led to a strengthened result in case (B): The resolvent can be extended to a disc of radius $O((h \ln 1/h)^{k/(k+1)})$ around the appropriate boundary points. This improvement has been obtained recently by W. Bordeaux Montrieux [1] for the model operator $hD_x + g(x)$, when $g \in C^\infty(S^1)$ and the points of maximum or minimum are all nondegenerate. In that case $k = 2$ and Bordeaux Montrieux also constructed quasi-modes for values of the spectral parameter that are close to the boundary points.

We next state the results and outline the proof in case (B).

Let $X$ be equal to $\mathbb{R}^n$ or equal to a compact smooth manifold of dimension $n$.

In the first case, let $m \in C^\infty(\mathbb{R}^{2n}; \mathbb{R}, 1, +\infty]$ be an order function (see [8] for more details about the pseudodifferential calculus) in the sense that for
some $C_0, N_0 > 0$, 
\[ m(\rho) \leq C_0 (\rho - \mu)^{N_0} m(\mu), \quad \rho, \mu \in \mathbb{R}^{2n}, \quad (1.1) \]
where $\langle \rho - \mu \rangle = (1 + |\rho - \mu|^2)^{1/2}$. Let $P = P(x, \xi; h) \in S(m)$, meaning that $P$ is smooth in $x, \xi$ and satisfies 
\[ |\partial_{x, \xi}^\alpha P(x, \xi; h)| \leq C_\alpha m(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2n}, \alpha \in \mathbb{N}^{2n}, \quad (1.2) \]
where $C_\alpha$ is independent of $h$. We also assume that 
\[ P(x, \xi; h) \sim p_0(x, \xi) + h p_1(x, \xi) + ..., \quad \text{in } S(m), \quad (1.3) \]
and write $p = p_0$ for the principal symbol. We impose the ellipticity assumption 
\[ \exists w \in \mathbb{C}, C > 0, \text{ such that } |p(\rho) - w| \geq m(\rho)/C, \quad \forall \rho \in \mathbb{R}^{2n}. \quad (1.4) \]
In this case we let 
\[ P = P^w(x, hD_x; h) = \text{Op}(P(x, h\xi; h)) \quad (1.5) \]
be the Weyl quantization of the symbol $P(x, h\xi; h)$ that we can view as a closed unbounded operator on $L^2(\mathbb{R}^n)$.

In the second case when $X$ is compact manifold, we let $P \in S_{1,0}^m(T^*X)$ (the classical Hörmander symbol space) of order $m > 0$, meaning that 
\[ |\partial^\alpha_{x} \partial^\beta_{\xi} P(x, \xi; h)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - |\beta|}, \quad (x, \xi) \in T^*X, \quad (1.6) \]
where $C_{\alpha, \beta}$ are independent of $h$. We also assume that we have an expansion as in (1.3), now in the sense that 
\[ P(x, \xi; h) - \sum_{j=0}^{N-1} h^j p_j(x, \xi) \in h^N S_{1,0}^{m-N}(T^*X), \quad N = 1, 2, ..., \quad (1.7) \]
and we quantize the symbol $P(x, h\xi; h)$ in the standard (non-unique) way, by doing it for various local coordinates and paste the quantizations together by means of a partition of unity. In the case $m > 0$ we impose the ellipticity condition 
\[ \exists C > 0, \text{ such that } |p(x, \xi)| \geq \frac{\langle \xi \rangle^m}{C}, \quad |\xi| \geq C. \quad (1.8) \]

Let $\Sigma(p) = \overline{\Sigma^*(T^*X)}$ and let $\Sigma_\infty(p)$ be the set of accumulation points of $p(\rho_j)$ for all sequences $\rho_j \in T^*X, j = 1, 2, 3, ..$ that tend to infinity. The following theorem is a partial improvement of corresponding results in [7].
Theorem 1.1  We adopt the general assumptions above. Let \( z_0 \in \partial \Sigma(p) \setminus \Sigma_\infty(p) \) and assume that \( dp \neq 0 \) at every point of \( p^{-1}(z_0) \). Then for every such point \( \rho \) there exists \( \theta \in \mathbb{R} \) (unique up to a multiple of \( \pi \)) such that \( d(e^{-i\theta}(p-z_0)) \) is real at \( \rho \). We write \( \theta = \theta(\rho) \). Consider the following two cases:

- **(A)** For every \( \rho \in p^{-1}(z_0) \), the maximal integral curve of \( H_{\Re(e^{-i\theta(\rho)}p)} \) through the point \( \rho \) is not contained in \( p^{-1}(z_0) \).
- **(B)** There exists an integer \( k \geq 1 \) such that for every \( \rho \in p^{-1}(z_0) \), there exists \( j \in \{1, 2, \ldots, k\} \) such that \( p^*(\exp tH_p(\rho)) = \mathcal{O}(t^{j+1}) \), \( t \to 0 \), where \( a = a(\rho) \neq 0 \). Here \( p \) also denotes an almost holomorphic extension to a complex neighborhood of \( \rho \) and we put \( p^*(\mu) = p(\overline{\mu}) \). Equivalently, \( H_j(\overline{\rho})/!(j!) = a \neq 0 \).

Then, in case (A), there exists a constant \( C_0 > 0 \) such that for every constant \( C_1 > 0 \) there is a constant \( C_2 > 0 \) such that the resolvent \( (z-P)^{-1} \) is well-defined for \( |z-z_0| < C_1 h \ln \frac{1}{h}, \ h < \frac{1}{C_2} \), and satisfies the estimate

\[
\|(z-P)^{-1}\| \leq \frac{C_0}{h} \exp\left(\frac{C_0}{h} |z-z_0|\right). \tag{1.9}
\]

In case (B), there exists a constant \( C_0 > 0 \) such that for every constant \( C_1 > 0 \) there is a constant \( C_2 > 0 \) such that the resolvent \( (z-P)^{-1} \) is well-defined for \( |z-z_0| < C_1 (h \ln \frac{1}{h})^{k/(k+1)}, \ h < \frac{1}{C_2} \), and satisfies the estimate

\[
\|(z-P)^{-1}\| \leq \frac{C_0}{h^{1+\frac{k}{k+1}}} \exp\left(\frac{C_0}{h} |z-z_0|^{1\frac{1}{k+1}}\right). \tag{1.10}
\]

In [7], we obtained (1.9), (1.10) for \( z = z_0 \), implying that the resolvent exists and satisfies the same bound for \( |z-z_0| \leq h^{k/(k+1)}/\mathcal{O}(1) \) in case (B) and with \( k/(k+1) \) replaced by 1 in case (A). In case (A) we also showed that the resolvent exists with norm bounded by a negative power of \( h \) in any disc \( D(z_0, C_1 h \ln(1/h)) \). (The condition in case (B) was formulated a little differently in [7], but as we shall see later on the two conditions lead to the same microlocal models and hence they are equivalent.) Actually the proof in [7] also gives (1.9), so even if the methods of the present paper also most likely lead to that bound, we shall not elaborate the details in that case.

Let us now consider the special situation of potential interest for evolution equations, namely the case when

\[
z_0 \in i\mathbb{R}, \tag{1.11}
\]
\( \Re p(\rho) \geq 0 \) in \( \text{neigh}(p^{-1}(z_0), T^*X) \). 

(1.12)

**Theorem 1.2** We adopt the general assumptions above. Let \( z_0 \in \partial \Sigma(p) \setminus \Sigma_{\infty}(p) \) and assume (1.11), (1.12). Also assume that \( dp \neq 0 \) on \( p^{-1}(z_0) \), so that \( d \Im p \neq 0 \), \( d \Re p = 0 \) on that set. Consider the two cases of Theorem 1.1:

- **(A)** For every \( \rho \in p^{-1}(z_0) \), the maximal integral curve of \( H_{\Im p} \) through the point \( \rho \) contains a point where \( \Re p > 0 \).

- **(B)** There exists an integer \( k \geq 1 \) such that for every \( \rho \in p^{-1}(z_0) \), we have \( H_{\Im p}^j \Re p(\rho) \neq 0 \) for some \( j \in \{1, 2, \ldots, k\} \).

Then, in case (A), there exists a constant \( C_0 > 0 \) such that for every constant \( C_1 > 0 \) there is a constant \( C_2 > 0 \) such that the resolvent \( (z - P)^{-1} \) is well-defined for

\[
|\Im(z - z_0)| < \frac{1}{C_0}, \quad -\frac{1}{C_0} < \Re z < C_1 h \ln \frac{1}{h}, \quad h < \frac{1}{C_2},
\]

and satisfies the estimate

\[
\| (z - P)^{-1} \| \leq \begin{cases} 
\frac{C_0}{|\Re z|}, & \Re z \leq -h, \\
\frac{C_0}{h} \exp(\frac{C_0}{h} \Re z), & \Re z \geq -h.
\end{cases}
\] (1.13)

In case (B), there exists a constant \( C_0 > 0 \) such that for every constant \( C_1 > 0 \) there is a constant \( C_2 > 0 \) such that the resolvent \( (z - P)^{-1} \) is well-defined for

\[
|\Im(z - z_0)| < \frac{1}{C_0}, \quad -\frac{1}{C_0} < \Re z < C_1 (h \ln \frac{1}{h})^{\frac{1}{k+1}}, \quad h < \frac{1}{C_2},
\] (1.14)

and satisfies the estimate

\[
\| (z - P)^{-1} \| \leq \begin{cases} 
\frac{C_0}{|\Re z|}, & \Re z \leq -h^{\frac{1}{k+1}}, \\
\frac{C_0}{h^{\frac{1}{k+1}}} \exp(\frac{C_0}{h} (\Re z)^{\frac{1}{k+1}}), & \Re z \geq -h^{\frac{1}{k+1}}.
\end{cases}
\] (1.15)

The case (A) in the theorems is practically identical with the corresponding results in [7] and can be obtained by inspection of the proof there, and from now on we concentrate on the case (B). Away from the set \( p^{-1}(z_0) \) we can use ellipticity, so the problem is to obtain microlocal estimates near a point \( \rho \in p^{-1}(z_0) \). After a standard factorization of \( P - z \) in such a region, we can further reduce the proof of the first theorem to that of the second one.
The main (quite standard) idea of the proof of Theorem 1.2 is to study \( \exp(-tP/h) \) (microlocally) for \( 0 \leq t \ll 1 \) and to show that in this case
\[
\| \exp \left( -\frac{tP}{h} \right) \| \leq C \exp(-\frac{tk+1}{Ch}), \tag{1.16}
\]
for some constant \( C > 0 \). Noting that that implies that \( \| \exp \left( -\frac{tP}{h} \right) \| = O(h^\infty) \) for \( t \geq h^\delta \) when \( \delta(k+1) < 1 \), and using the formula
\[
(z-P)^{-1} = -\frac{1}{h} \int_0^\infty \exp\left( \frac{t}{h} (z-P) \right) dt, \tag{1.17}
\]
leads to (1.15). (This has some relation to the works of A. Cialdea and Maz’ya \([3, 4]\) where the \( L^p \) dissipativity of second order operators is characterized.)

The most direct way of studying \( \exp(-tP/h) \), or rather a microlocal version of that operator, is to view it as a Fourier integral operator with complex phase \([14, 13, 18, 15]\) of the form
\[
U(t)u(x) = \frac{1}{(2\pi h)^n} \int \int e^{\frac{i}{h}(\phi(t,x,\eta)} a(t,x,\eta; h) u(y) dyd\eta, \tag{1.18}
\]
where the phase \( \phi \) should have a non-negative imaginary part and satisfy the Hamilton-Jacobi equation:
\[
i\partial_t \phi + p(x, \partial_x \phi) = O((\Im \phi)^\infty), \quad \text{locally uniformly}, \tag{1.19}
\]
with the initial condition
\[
\phi(0,x,\eta) = x \cdot \eta. \tag{1.20}
\]
The amplitude \( a \) will be bounded with all its derivatives and has an asymptotic expansion where the terms are determined by transport equations. This can indeed be carried out in a classical manner for instance by adapting the method of \([18]\) to the case of non-homogeneous symbols following a reduction used in \([20, 15]\). It is based on making estimates on the function
\[
S_\gamma(t) = \Im(\int_0^t \xi(s) \cdot dx(s)) - \Re \xi(t) \cdot \Re x(t) + \Re \xi(0) \cdot \Re x(0)
\]
along the complex integral curves \( \gamma : [0,T] \ni s \mapsto (x(s), \xi(s)) \) of the Hamilton field of \( p \). Notice that here and already in (1.19), we need to take an almost holomorphic extension of \( p \). Using the property (B) one can show that \( \Im \phi(t,x,\eta) \geq C^{-1}t^{k+1} \) and from that we can obtain (a microlocalized version of) (1.16) quite easily.
Finally, we prefered a variant that we shall now outline: Let
\[ T u(x) = Ch^{-\frac{4n}{2n+4}} \int e^{\frac{i}{h} \phi(x,y)} u(y) dy, \]
be an FBI – or (generalized) Bargmann-Segal transform that we treat in the spirit of Fourier integral operators with complex phase as in [21]. Here \( \phi \) is holomorphic in a neighborhood of \( (x_0, y_0) \in C^n \times R^n \), and \(-\phi'_y(x_0, y_0) = \eta_0 \in R^n \), \( \Im \phi''_{y,y}(x_0, y_0) > 0 \). Let \( \kappa_t \) be the associated canonical transformation. Then microlocally, \( T \) is bounded \( L^2 \rightarrow H_{\Phi_0} = \text{Hol}(\Omega) \cap L^2(\Omega, e^{-2\Phi_0}/h dx) \) and has (microlocally) a bounded inverse, where \( \Omega \) is a small complex neighborhood of \( x_0 \) in \( C^n \). Here the weight \( \Phi_0 \) is smooth and strictly pluri-subharmonic. If \( \Lambda_{\Phi_0} : = \{ (x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}); x \in \text{neigh}(x_0) \} \), then (in the sense of germs) \( \Lambda_{\Phi_0} = \kappa_T(T^*X) \). The conjugated operator \( \widetilde{P} = TPT^{-1} \) can be defined locally modulo \( O(h^\infty) \) (see also [11]) as a bounded operator from \( H_{\Phi} \rightarrow H_{\Phi} \) provided that the weight \( \Phi \) is smooth and satisfies \( \Phi' - \Phi'_0 = O(h^\delta) \) for some \( \delta > 0 \). (In the analytic frame work this condition can be relaxed.) Egorov’s theorem applies in this situation, so the leading symbol \( \widetilde{p} \) of \( \widetilde{P} \) is given by \( \widetilde{p} \circ \kappa_T = p \). Thus (under the assumptions of Theorem [1.2]) we have \( \Re \widetilde{p}|_{\Lambda_{\Phi_0}} \geq 0 \), which in turn can be used to see that for \( 0 \leq t \leq h^\delta \), we have \( e^{-t\widetilde{p}/h} = O(1): H_{\Phi_0} \rightarrow H_{\Phi_t} \), where \( \Phi_t \leq \Phi_0 \) is determined by the real Hamilton-Jacobi problem
\[ \frac{\partial \Phi_t}{\partial t} + \Re p(x, \frac{2}{i} \frac{\partial \Phi_t}{\partial x}) = 0, \quad \Phi_{t=0} = \Phi_0. \] (1.21)

Now the bound (1.16) follows from the estimate
\[ \Phi_t \leq \Phi_0 - \frac{t^{k+1}}{C} \] (1.22)
where \( C > 0 \). An easy proof of (1.22) is to represent the I-Lagrangian manifold \( \Lambda_{\phi_t} \) as the image under \( \kappa_T \) of the I-Lagrangian manifold \( \Lambda_{G_t} = \{ \rho + i H_{G_t}(\rho); \rho \in \text{neigh}(\rho_0, T^*X) \} \), where \( H_{G_t} \) denotes the Hamilton field of \( G_t \). It turns out that the \( G_t \) are given by the real Hamilton-Jacobi problem
\[ \frac{\partial G_t}{\partial t} + \Re (p(\rho + i H_{G_t}(\rho))) = 0, \quad G_0 = 0, \] (1.23)
and there is a simple minimax type formula expressing \( \Phi_t \) in terms of \( G_t \), so it suffices to show that
\[ G_t \leq -t^{k+1}/C. \] (1.24)
This estimate is quite simple to obtain: (1.23) first implies that $G_t \leq 0$, so $(\nabla G_t)^2 = O(G_t)$. Then if we Taylor expand (1.23), we get

$$\frac{\partial G_t}{\partial t} + H_{3p}(G_t) + O(G_t) + \Re p(\rho) = 0$$

and we obtain (1.24) from a simple differential inequality and an estimate for certain integrals of $\Re p$.

The use of the representation with $G_t$ is here very much taken from the joint work [9] with B. Helffer.

In Section 5 we discuss some examples.

2 IR-manifolds close to $\mathbb{R}^{2n}$ and their FBI-representations

Much of this section is just an adaptation of the discussion in [9] with the difference that we here use the simple FBI-transforms of generalized Bargmann type from [21], rather than the more complicated variant that was necessary to treat a neighborhood of infinity in the resonance theory of [9].

We shall work locally. Let $G(y, \eta) \in C^\infty(\text{neigh } ((y_0, \eta_0), \mathbb{R}^{2n}))$ be real-valued and small in the $C^\infty$ topology. Then

$$\Lambda_G = \{(y, \eta) + iH_G(y, \eta); (y, \eta) \in \text{neigh } ((y_0, \eta_0))\}, \quad H_G = \frac{\partial G}{\partial \eta} \frac{\partial}{\partial y} - \frac{\partial G}{\partial y} \frac{\partial}{\partial \eta}$$

is an $I$-Lagrangian manifold, i.e. a Lagrangian manifold for the real symplectic form $\Im \sigma$, where $\sigma$ denotes the complex symplectic form $\sum d\tilde{\eta}_j \wedge d\tilde{y}_j$. Here, for notational reasons we reserve the notation $(y, \eta)$ for the real cotangent variables and let the tilde indicate that we take the corresponding complexified variables.

We may also represent $\Lambda_G$ by means of a nondegenerate phase function in the sense of Hörmander in the following way:

Consider

$$\psi(\tilde{y}, \eta) = -\eta \cdot \Im \tilde{y} + G(\Re \tilde{y}, \eta)$$

where $\tilde{y}$ is complex and $\eta$ real according to the convention above. Then

$$\nabla_\eta \psi(\tilde{y}, \eta) = -\Im \tilde{y} + \nabla_\eta G(\Re \tilde{y}, \eta),$$

and since $G$ is small, we see that $d\frac{\partial \psi}{\partial \eta_1}, \ldots, d\frac{\partial \psi}{\partial \eta_m}$ are linearly independent. So $\psi$ is indeed a nondegenerate phase function if we drop the classical requirement of homogeneity in the $\eta$ variables.
Let
\[ C_\psi = \{ (\tilde{y}, \eta) \in \text{neigh} ((y_0, \eta_0), \mathbb{C}^n \times \mathbb{R}^n); \nabla_\eta \psi = 0 \} \]
and consider the corresponding I-Lagrangian manifold
\[ \Lambda_\psi = \{ (\tilde{y}, 2 \frac{\partial \psi}{i \partial \tilde{y}} (\tilde{y}, \eta)); (\tilde{y}, \eta) \in C_\psi \}. \]

Here we adopt the convention that \( \frac{\partial}{\partial \tilde{y}} \) denotes the holomorphic derivative, since \( \tilde{y} \) are complex variables:
\[ \frac{\partial}{\partial \tilde{y}} = \frac{1}{2} \left( \frac{\partial}{\partial \Re \tilde{y}} + \frac{1}{i} \frac{\partial}{\partial \Im \tilde{y}} \right). \]

Let us first check that that \( \Lambda_\psi \) is I-Lagrangian, using only that \( \psi \) is a non-degenerate phase function: That \( \Lambda_\psi \) is a submanifold with the correct real dimension = 2\( n \) is classical since we can identify \( 2 \frac{\partial \psi}{i \partial \tilde{y}} \) with \( \nabla_{\Re \tilde{y}, \Im \tilde{y}} \psi \). Further,
\begin{align*}
-3(\tilde{\eta} \cdot d\tilde{y})_{|_{\Lambda_\psi}} &\simeq -3 \left( \frac{2 \partial \psi}{i \partial \tilde{y}} \cdot d\tilde{y} \right)_{|_{C_\psi}} \\
&= -\frac{1}{2i} \left( \frac{2 \partial \psi}{i \partial \tilde{y}} d\tilde{y} + \frac{2 \partial \psi}{i \partial \tilde{y}} d\tilde{y} \right)_{|_{C_\psi}} = \left( \frac{\partial \psi}{\partial \tilde{y}} d\tilde{y} + \frac{\partial \psi}{\partial \tilde{y}} d\tilde{y} \right)_{|_{C_\psi}} \\
&= d\psi_{|_{C_\psi}}
\end{align*}

which is a closed form and using that \( \Im \psi = d(\Im \tilde{\eta} \cdot d\tilde{y}) \), we get
\[-\Im \psi_{|_{\Lambda_\psi}} = 0.\]

We next check for our specific phase \( \psi \) that \( \Lambda_\psi = \Lambda_G \): If \( (\tilde{y}, \frac{2 \partial \psi}{i \partial \tilde{y}} (\tilde{y}, \eta)) \) is a general point on \( \Lambda_\psi \), then \( \Im \tilde{y} = \nabla_\eta G(\Re \tilde{y}, \eta) \) and
\begin{align*}
\frac{2 \partial \psi}{i \partial \tilde{y}} (\tilde{y}, \eta) &= \frac{2}{i} \frac{1}{2} \left( \frac{\partial}{\partial \Re \tilde{y}} + \frac{1}{i} \frac{\partial}{\partial \Im \tilde{y}} \right)(-\eta \cdot \Im \tilde{y} + \psi(\Re \tilde{y}, \eta)) \\
&= -\left( \frac{\partial}{\partial \Im \tilde{y}} + \frac{1}{i} \frac{\partial}{\partial \Re \tilde{y}} \right)(-\eta \cdot \Im \tilde{y} + \psi(\Re \tilde{y}, \eta)) \\
&= \eta - i \nabla_\eta G(\Re \tilde{y}, \eta).
\end{align*}

Hence
\[ (\tilde{y}, \frac{2 \partial \psi}{i \partial \tilde{y}} (\tilde{y}, \eta)) = (y, \eta) + \eta H_G(y, \eta), \]
if we choose \( y = \Re \tilde{y} \). \( \square \)
Now consider an FBI (or generalized Bargmann-Segal) transform

\[ Tu(x; h) = h^{-\frac{3n}{4}} \int e^{i\phi(x,y)/h} a(x, y; h) u(y) u(y) dy, \]

where \( \phi \) is holomorphic near \((x_0, y_0) \in \mathbb{C}^n \times \mathbb{R}^n\), \( \Im \phi_{y,y}'' > 0 \), \( \det \phi_{x,y}'' 
eq 0 \), \( -\frac{\partial \phi}{\partial y} = \eta_0 \in \mathbb{R}^n \), and \( a \) is holomorphic in the same neighborhood with \( a \sim a_0(x, y) + ha_1(x, y) + \ldots \) in the space of such functions with \( a_0 \neq 0 \). We can view \( T \) as a Fourier integral operator with complex phase and the associated canonical transformation is

\[ \kappa = \kappa_T : (y, -\frac{\partial \phi}{\partial y}(x, y)) \mapsto (x, \frac{\partial \phi}{\partial x}(x, y)) \]

from a complex neighborhood of \((y_0, \eta_0)\) to a complex neighborhood of \((x_0, \xi_0)\), where \( \xi_0 = \frac{\partial \phi}{\partial x}(x_0, y_0) \). Complex canonical transformations preserve the class of I-Lagrangian manifolds and (locally),

\[ \kappa(\mathbb{R}^{2n}) = \Lambda_{\Phi_0} = \{(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x)); \ x \in \text{neigh } (x_0, \mathbb{C}^n)\}, \]

where \( \Phi_0 \) is smooth and strictly plurisubharmonic. Actually,

\[ \Phi_0(x) = \sup_{y \in \mathbb{R}^n} -\Im \phi(x, y), \quad (2.1) \]

where the supremum is attained at the nondegenerate point of maximum \( y_c(x) \) \([21]\).

**Proposition 2.1** We have \( \kappa(\Lambda_G) = \Lambda_{\Phi_G} \), where

\[ \Phi_G(x) = \text{v.c.}\tilde{y},\eta - \Im \phi(x, \tilde{y}) - \eta \cdot \Im \tilde{y} + G(\Re \tilde{y}, \eta), \quad (2.2) \]

and the critical value is attained at a nondegenerate critical point. Here \( \text{v.c.}\tilde{y},(\ldots) \) means “critical value with respect to \( \tilde{y}, \eta \) of \( \ldots \).”

**Proof.** At a critical point we have

\[ \Im \tilde{y} = \nabla \eta G(\Re \tilde{y}, \eta), \]

\[ -\frac{\partial}{\partial \tilde{y}} \Im \phi(x, \tilde{y}) + \eta = 0, \]

\[ -\frac{\partial}{\partial \Re \tilde{y}} \Im \phi(x, \tilde{y}) + (\nabla \eta G)(\Re \tilde{y}, \eta) = 0. \]

If \( f(z) \) is a holomorphic function, then

\[ \frac{\partial}{\partial \Re z} \Im f = \Re \frac{\partial f}{\partial z}, \quad \frac{\partial}{\partial \Im z} \Im f = \Im \frac{\partial f}{\partial z}. \quad (2.3) \]
so the equations for our critical point become

\[ \Im \tilde{y} = \nabla_{\eta} G(\Re \tilde{y}, \eta), \]

\[ \eta = -\Re \frac{\partial \phi}{\partial \tilde{y}} (x, \tilde{y}), \]

\[ -\nabla_y G(\Re \tilde{y}, \eta) = -\Im \frac{\partial \phi}{\partial \tilde{y}}, \]

or equivalently,

\[ (\tilde{y}, -\frac{\partial \phi}{\partial \tilde{y}} (x, \tilde{y})) = (\Re \tilde{y}, \eta) + iH_G(\Re \tilde{y}, \eta), \]

which says that the critical point \((\tilde{y}, \eta)\) is determined by the condition that \(\kappa_T\) maps the point \((\tilde{y}, \tilde{\eta}) \in \Lambda_G\) to a point \((x, \xi)\), situated over \(x\). Clearly the critical point is nondegenerate. We check it when \(G = 0\): The Hessian matrix with respect to the variables \(\Re y, \Im y, \eta\) becomes

\[
\begin{pmatrix}
-\Im \phi''_{y,y} & B & 0 \\
 iB & C & -1 \\
 0 & -1 & 0
\end{pmatrix}
\]

which is nondegenerate independently of \(B, C\).

If \(\Phi(x)\) denotes the critical value in (2.2), it remains to check that \(\frac{2 \partial \phi}{i \partial x} = \xi\) where \(\xi = \frac{\partial \phi}{\partial x} (x, \tilde{y})\), \((\tilde{y}, \eta)\) denoting the critical point. However, since \(\Phi\) is a critical value, we get

\[
\frac{2 \partial \Phi}{i \partial x} = 2 \frac{\partial}{\partial x} (-\Im \phi(x, \tilde{y})) = \frac{\partial \phi}{\partial x} (x, \tilde{y}).
\]

Also notice that when \(G = 0\), the formula (2.2) produces the same function as (2.1).

Write \(\tilde{y} = y + i\theta\) and consider the function

\[ f(x; y, \eta; \theta) = -\Im \phi(x, y + i\theta) - \eta \cdot \theta, \]

which appears in (2.2).

**Proposition 2.2** \(f\) is a nondegenerate phase function with \(\theta\) as fiber variables which generates a canonical transformation which can be identified with \(\kappa_T\).
Proof.

\[ \frac{\partial}{\partial \theta} f = -\Re \frac{\partial \phi}{\partial y} (x, y + i\theta) - \eta, \]

so \( f \) is nondegenerate. The canonical relation has the graph

\[ \{ (x, \frac{\partial \phi}{\partial x}, y, \eta, \frac{\partial}{\partial y} \Im \phi(x, y + i\theta); \eta = -\Re \frac{\partial \phi}{\partial y}(x, y + i\theta)) \} = \{ (x, \frac{\partial \phi}{\partial x}(x, y + i\theta); y, -\Re \frac{\partial \phi}{\partial y}(x, y + i\theta), \Im \frac{\partial \phi}{\partial y}(x, y + i\theta), \theta) \}, \]

and up to reshuffling of the components on the preimage side and changes of signe, we recognize the graph of \( \kappa_T \).

Now we have the following easily verified fact:

**Proposition 2.3** Let \( f(x, y, \theta) \in C^\infty(\text{neigh} (x_0, y_0, \theta_0), \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N) \) be a nondegenerate phase function with \((x_0, y_0, \theta_0) \in C_\phi\), generating a canonical transformation which maps \((y_0, \eta_0) = (y_0, -\nabla_y f(x_0, y_0, \theta_0))\) to \((x_0, \nabla_x f(x_0, y_0, \theta_0))\). If \( g(y) \) is smooth near \( y_0 \) with \( \nabla g(y_0) = \eta_0 \) and

\[ h(x) = \text{v.c.}_y f(x, y, \theta) + g(y) \]

is well-defined with a nondegenerate critical point close to \((y_0, \theta_0)\) for \( x \) close to \( x_0 \), then we have the inversion formula,

\[ g(y) = \text{v.c.}_x f(x, y, \theta) + h(x), \]

for \( y \in \text{neigh}(y_0) \), where the critical point is nondegenerate and close to \((x_0, \theta_0)\).

Combining the three propositions, we get

**Proposition 2.4**

\[ G(y, \eta) = \text{v.c.}_x \Im \phi(x, y + i\theta) + \eta \cdot \theta + \Phi_G(x). \] (2.5)

If \((\tilde{\Phi}, \tilde{G})\) is a second pair of functions close to \( \Phi_0, 0 \) and related through (2.2), (2.5), then

\[ G \leq \tilde{G} \text{ iff } \Phi \leq \tilde{\Phi}. \] (2.6)

Indeed, if for instance \( \Phi \leq \tilde{\Phi} \), introduce \( \Phi_t = t\Phi + (1 - t)\tilde{\Phi} \), so that \( \partial_t \Phi_t \geq 0 \). If \( G_t \) is the corresponding critical value as in (2.5), then \( \partial_t G_t = (\partial_t \Phi_t)(x_t) \geq 0 \), where \((x_t, \theta_t)\) is the critical point.
3 Evolution equations on the transform side

Let $\tilde{P}(x,\xi;h)$ be a smooth symbol defined in $\operatorname{neigh}((x_0,\xi_0);\Lambda \Phi_0)$, with an asymptotic expansion

$$\tilde{P}(x,\xi;h) \sim \tilde{p}(x,\xi) + h\tilde{p}_1(x,\xi) + ... \text{ in } C^\infty(\operatorname{neigh}((x_0,\xi_0),\Lambda \Phi_0)).$$

By the same letter, we denote an almost holomorphic extension to a complex neighborhood of $(x_0,\xi_0)$:

$$\tilde{P}(x,\xi;h) \sim \tilde{p}(x,\xi) + h\tilde{p}_1(x,\xi) + ... \text{ in } C^\infty(\operatorname{neigh}((x_0,\xi_0),\mathbb{C}^{2n}),$$

where $\tilde{p}, \tilde{p}_j$ are smooth extensions such that

$$\overline{\partial} \tilde{p}, \overline{\partial} \tilde{p}_j = \mathcal{O}(\text{dist}((x,\xi),\Lambda \Phi_0)^\infty).$$

Then, as developed in [11] and later in [19], if $u = u_h$ is holomorphic in a neighborhood $V$ of $x_0$ and belonging to $H_{\Phi_0}(V)$ in the sense that $\|u\|_{L^2(V,e^{-2\Phi_0/h}(dx))}$ is finite and of temperate growth in $1/h$ when $h$ tends to zero, then we can define $\tilde{P}u = \tilde{P}(x,hD_x;h)u$ in any smaller neighborhood $W \Subset V$ by the formula,

$$\tilde{P}u(x) = \frac{1}{(2\pi h)^n} \int \frac{e^{i\frac{(x-y)\cdot \theta}{2}}}{\Gamma(x)} \tilde{P}(x + y_0,\theta;h)u(y)dyd\theta, \quad (3.1)$$

where $\Gamma(x)$ is a good contour (in the sense of [21]) of the form $\theta = \frac{2}{C_1} \frac{\partial \Phi_0}{\partial x} (\frac{x_0 + y_0}{2}) + \frac{1}{C_2} (x_0 - y_0), |x - y| \leq 1/C_2, C_1, C_2 > 0$. Then $\overline{\partial} \tilde{P}$ is negligible $H_{\Phi_0}(V) \rightarrow L^2_{\Phi_0}(W)$, i.e. of norm $\mathcal{O}(h^\infty)$ and modulo such negligible operators, $\tilde{P}$ is independent of the choice of good contour. By solving a $\overline{\partial}$-problem (assuming, as we may, that our neighborhoods are pseudoconvex) we can always correct $\tilde{P}$ with a negligible operator such that (after an arbitrarily small decrease of $W$) $\tilde{P} = \mathcal{O}(1) : H_{\Phi_0}(V) \rightarrow H_{\Phi_0}(W)$. Also, if $\Phi = \Phi_0 + \mathcal{O}(h \ln 1/h)$ in $C^2$, then clearly $\tilde{P} = \mathcal{O}(h^{-N_0}) : H_{\Phi}(V) \rightarrow H_{\Phi}(W)$, for some $N_0$. Using Stokes’ formula, we can show that $\tilde{P}$ will change only by a negligible term if we replace $\Phi_0$ by $\Phi$ in the definition of $\Gamma(x)$, and then it follows that $\tilde{P} = \mathcal{O}(1) : H_{\Phi}(V) \rightarrow H_{\Phi}(W)$.

Before discussing evolution equations, let us recall ([19]) that the identity operator $H_{\Phi_0}(V) \rightarrow H_{\Phi_0}(W)$ is up to a negligible operator of the form

$$Iu(x) = h^{-n} \int \int e^{-\overline{\Phi_0}(x,\overline{y})} a(x,\overline{y};h)u(y)e^{-\frac{\overline{\Phi_0}(y)}{h}} dyd\overline{y}, \quad (3.2)$$
where $\Psi_0(x,y)$, $a(x,y;h)$ are almost holomorphic on the antidiagonal $y = \overline{x}$ with $\Psi_0(x,\overline{x}) = \Phi_0(x)$, $a(x,y;h) \sim a_0(x,y) + ha_1(x,y) + \ldots$, $a_0(x,\overline{x}) \neq 0$.

More generally a pseudodifferential operator like $\tilde{P}$ takes the form

$$
\tilde{P}u(x) = h^{-n} \int e^{2\overline{\Psi_0(x,\overline{\eta})}}q(x,\overline{\eta};h)u(y)e^{-2\overline{\Phi_0(y)/h}}dyd\overline{\eta},
$$

(3.3)

and where $q_0$ denotes the first term in the asymptotic expansion of the symbol $q$. In this discussion, $\Phi_0$ can be replaced by any other smooth exponent $\Phi$ which is $O(h^\delta)$ close to $\Phi_0$ in $C^\infty$ and we make the corresponding replacement of $\Psi_0$. Also recall that because of the strict pluri-subharmonicity of $\Phi$, we have

$$
2\Re\Psi(x,\overline{y}) - \Phi(x) - \Phi(y) \asymp -|x-y|^2,
$$

(3.4)

so the uniform boundedness $H_\Phi \to H_\Phi$ follows from the domination of the modulus of the effective kernel by a Gaussian convolution kernel.

Next, consider the evolution problem

$$
(h\partial_t + \tilde{P})\tilde{U}(t) = 0, \quad \tilde{U}(0) = 1,
$$

(3.5)

where $t$ is restricted to the interval $[0,h^\delta]$ for some arbitrarily small but fixed $\delta > 0$. We review how to solve this problem approximately by a geometrical optics construction: Look for $\tilde{U}(t)$ of the form

$$
\tilde{U}(t)u(x) = h^{-n} \int e^{2\overline{\Psi_t(x,\overline{\eta})}}a_t(x,\overline{\eta};h)u(y)e^{-2\overline{\Phi_0(y)/h}}dyd\overline{\eta},
$$

(3.6)

where $\Psi_t$, $a_t$ depend smoothly on all the variables and $\Psi_{t=0} = \Psi_0$, $a_{t=0} = a_0$ in (3.3), so that $\tilde{U}(0) = 1$ up to a negligible operator.

Notice that formally $\tilde{U}(t)$ is the Fourier integral operator

$$
\tilde{U}(t)u(x) = h^{-n} \int e^{2\overline{\Psi_t(x,\overline{\eta})}}a_t(x,\overline{\eta};h)u(y)dyd\overline{\eta},
$$

(3.7)

where we choose the integration contour $\theta = \overline{\eta}$. Writing $2\overline{\Psi_t(x,\theta)} = i\phi_t(x,\theta)$ leads to more customary notation and we impose the eiconal equation

$$
i\partial_t\phi + \overline{p}(x,\phi_x'(x,\theta)) = 0.
$$

(3.8)

Of course, we are manipulating $C^\infty$ functions in the complex domain, so we cannot hope to solve the eiconal equation exactly, but we can do so to infinite order at $t = 0$, $x = \overline{\eta} = \theta$. If we put

$$\Lambda_{\phi_t(x,\theta)} = \{(x,\phi_x'(t,x,\theta))\},
$$

(3.9)
we have to $\infty$ order at $t = 0$, $\theta = x$:

$$
\Lambda_{\phi_t(\cdot, \theta)} = \exp(t\hat{H}_{1\tilde{p}})(\Lambda_{\phi_0(\cdot, \theta)}).
$$

(3.10)

With $\hat{H}_{\tilde{p}} = H_{\tilde{p}} + \overline{H}_{\tilde{p}}$ denoting the real vector field associated to the (1,0)-field $H_p$, and similarly for $\hat{H}_{1\tilde{p}}$. (We sometimes neglect the hat when integrating the Hamilton flows.) At a point where $\overline{\tilde{p}} = 0$, we have

$$
\hat{H}_{\tilde{p}} = H_{\Re\tilde{p}} = H_{\Im\tilde{p}}; \quad \hat{H}_{1\tilde{p}} = -H_{\Re\tilde{p}} = H_{\Im\tilde{p}},
$$

(3.11)

where the other fields are the Hamilton fields of $\Re\tilde{p}$, $\Im\tilde{p}$ with respect to the real symplectic forms $\Re\sigma$ and $\Im\sigma$ respectively. See [21, 19]. Thus (3.10) can be written

$$
\Lambda_{\phi_t(\cdot, \theta)} = \exp\left(tH_{-\Im\tilde{p}}\right)(\Lambda_{\phi_0(\cdot, \theta)}).
$$

(3.12)

A complex Lagrangian manifold is also an I-Lagrangian manifold (i.e. a Lagrangian manifold for $\Im\sigma$) so (3.12) can be viewed as a relation between I-Lagrangian manifolds and it defines the I-Lagrangian manifold $\Lambda_{\phi_t(\cdot, \theta)}$ in an unambiguous way, once we have fixed an almost holomorphic extension of $\tilde{p}$ and especially the real part of that function. The general form of a smooth I-Lagrangian manifold $\Lambda$, for which the $x$-space projection $\Lambda \ni (x, \xi) \mapsto x \in C^n$ is a local diffeomorphism, is locally $\Lambda = \Lambda_{\Phi}$ where $\Phi$ is real and smooth and we define

$$
\Lambda_{\Phi} = \{(x, \frac{2}{i} \frac{\partial \Phi}{\partial x}); \ x \in \Omega\}, \ \Omega \subset C^n \text{ open}.
$$

Thus, if we define

$$
\Lambda_{\Phi_t} = \exp\left(tH_{-\Im\tilde{p}}\right)(\Lambda_{\phi_0}),
$$

(3.13)

and fix the $t$-dependent constant in this definition of $\Phi_t$ by imposing the real Hamilton-Jacobi equation,

$$
\partial_t \Phi_t + \Re\tilde{p}(x, \frac{2}{i} \frac{\partial \Phi_t}{\partial x}) = 0, \ \Phi_{t=0} = \Phi_0,
$$

(3.14)

and noticing that the real part of (3.8) is a similar equation for $-\Im\phi_t$,

$$
\partial_t(-\Im\phi) + \Re\tilde{p}(x, \frac{2}{i} \frac{\partial}{\partial x}(-\Im\phi)) = 0,
$$

(3.15)
we get
\[ \Phi_t(x) + \Phi_0(\bar{\theta}) - (-3\phi_t(x, \theta)) \asymp |x - x_t(\bar{\theta})|^2, \quad (3.16) \]
where \((x_t(\theta), \xi_t(\theta)) := \exp(t H^{-3\sigma}(\bar{\theta}, \frac{2}{i} \partial \Phi_t(\bar{\theta}))).\)

Determining \(a_t\) by solving a sequence of transport equations, we arrive at the following result:

**Proposition 3.1** The operator \(\tilde{U}(t)\) constructed above is \(O(1) : H_{\Phi_0}(V) \rightarrow H_{\Phi_t}(W), (W \Subset V\) being small pseudoconvex neighborhoods of a fixed point \(x_0\)) uniformly for \(0 \leq t \leq h^\delta\) and it solves the problem (3.3) up to negligible terms. This local statement makes sense, since by (3.16) we have
\[ 2\Re \Psi_t(x, y) - \Phi_t(x) - \Phi_0(y) \asymp -|x - x_t(y)|^2. \quad (3.17) \]

Using standard arguments, we also obtain up to negligible errors
\[ h\partial_t \tilde{U}(t) + \tilde{U}(t)\tilde{P} = 0, \quad 0 \leq t \leq h^\delta. \quad (3.18) \]

Let us quickly outline an alternative approach leading to the same weights \(\Phi_t\) (cf [19]):

Consider formally:
\[ (e^{-t\tilde{P}/h} u | e^{-t\tilde{P}/h} u)_{H_{\Phi_t}} = (u_t | u_t)_{H_{\Phi_t}}, \quad u \in H_{\Phi_0}, \]
and try to choose \(\Phi_t\) so that the time derivative of this expression vanishes to leading order. We get

\[ 0 \approx h\partial_t \int u_t \overline{u}_t e^{-2\Phi_t/h} L(dx) \]
\[ = - \left( (\tilde{P} u_t | u_t)_{H_{\Phi_t}} + (u_t | \tilde{P} u_t)_{H_{\Phi_t}} + \int 2 \frac{\partial \Phi_t}{\partial t}(x)|u|^2 e^{-2\Phi_t/h} L(dx) \right). \]

Here
\[ (\tilde{P} u_t | u_t)_{H_{\Phi_t}} = \int (\tilde{p} |_{\lambda_{\Phi_t}} + O(h))|u|^2 e^{-2\Phi_t/h} L(dx), \]
and similarly for \((u_t | \tilde{P} u_t)_{H_{\Phi_t}}\), so we would like to have
\[ 0 \approx \int (2 \frac{\partial \Phi_t}{\partial t} + 2 \Re \tilde{p} |_{\lambda_{\Phi_t}} + O(h))|u|^2 e^{-2\Phi_t/h} L(dx). \]

We choose \(\Phi_t\) to be the solution of (3.14). Then the preceding discussion again shows that \(e^{-t\tilde{P}/h} = O(1) : H_{\Phi_0} \rightarrow H_{\Phi_t}.\)
Since $\Re \tilde{p}$ is constant along the integral curves of $H_{\Re \tilde{p}}^{-3\sigma}$, we see from (3.14), that the second term in (3.14) is $\geq 0$, so
\begin{equation}
\Phi_t \leq \Phi_0, \quad t \geq 0,
\end{equation}
when
\begin{equation}
\Re \tilde{p}|_{\Lambda \Phi_0} \geq 0.
\end{equation}

Recall that we limit our discussion to the interval $0 \leq t \leq h^\delta$.

The author found it simpler to get a detailed understanding by working with the corresponding functions $G_t$ in the following way:

Let $p$ be defined by $p = \tilde{p} \circ \kappa_T$ and define $G_t$ up to a $t$-dependent constant by
\begin{equation}
\Lambda_t \Phi_t = \kappa_T(\Lambda_{G_t}).
\end{equation}

Then we also have $\Lambda_{G_t} = \exp tH_p(\Lambda_0)$, where $\Lambda_0 = \mathbb{R}^{2n}$. In order to fix the $t$-dependent constant we use one of the equivalent formulae (cf. (2.2), (2.5)):
\begin{equation}
\Phi_t(x) = v.c_{\tilde{y}, \eta}(-\Im \phi(x, \tilde{y}) - \eta \cdot \Im \tilde{y} + G_t(\Re \tilde{y}, \eta)),
\end{equation}
\begin{equation}
G_t(y, \eta) = v.c_{x, \theta}(\Im \phi(x, y + i\theta) + \eta \cdot \theta + \Phi_t(x)).
\end{equation}

If $(x(t, y, \eta), \theta(t, y, \eta))$ is the critical point in the last formula, we get
\begin{equation}
\frac{\partial G_t}{\partial t}(y, \eta) = \frac{\partial \Phi_t}{\partial t}(x(t, y, \eta)) = -\Re \tilde{p}(x, \frac{2 \partial \Phi_t}{i \partial x})|_{x=x(t,y,\eta)}. \tag{3.23}
\end{equation}

As we have seen, the critical points in (3.21), (3.22) are directly related to $\kappa_T$, so (3.23) leads to
\begin{equation}
\frac{\partial G_t}{\partial t}(y, \eta) + \Re p((y, \eta) + iH_{G_t}(y, \eta)) = 0. \tag{3.24}
\end{equation}

Notice that $G_t \leq 0$ by (2.6), (3.19).

Since we consider (3.24) only when $G_t$ and its gradient are small, we can Taylor expand (3.24) and get
\begin{equation}
\frac{\partial G_t}{\partial t}(y, \eta) + \Re p(y, \eta) + \Re(iH_{G_t}p(y, \eta)) + \mathcal{O}((\nabla G_t)^2) = 0, \tag{3.25}
\end{equation}
which simplifies to
\begin{equation}
\frac{\partial G_t}{\partial t}(y, \eta) + H_{3p}G_t + \mathcal{O}((\nabla G_t)^2) = -\Re p(y, \eta). \tag{3.26}
\end{equation}
Now, $G_t \leq 0$, so $(\nabla G_t)^2 = \mathcal{O}(G_t)$ and we obtain

$$
\left( \frac{\partial}{\partial t} + H_{3p} \right) G_t + \mathcal{O}(G_t) = -\Re p, \ G_0 = 0.
$$

(3.27)

Viewing this as a differential inequality along the integral curves of $H_{3p}$, we obtain

$$
-G_t(\exp(tH_{3p})(\rho)) \approx \int_0^t \Re p(\exp sH_{3p}(\rho))ds,
$$

(3.28)

for all $\rho = (y, \eta) \in \text{neigh}(\rho_0, \mathbb{R}^{2n})$, $\rho_0 = (y_0, \eta_0)$.

Now, introduce the following assumption corresponding to the case (B) in Theorem 1.2,

$$
H^j_{3p}(\Re p)(\rho_0) \begin{cases} 
0, & j \leq k - 1, \\
> 0, & j = k 
\end{cases},
$$

(3.29)

where $k$ necessarily is even (since $\Re p \geq 0$). We will work in a sufficiently small neighborhood of $\rho_0$. Put

$$
J(t, \rho) = \int_0^t \Re p(\exp sH_{3p}(\rho))ds,
$$

(3.30)

so that $0 \leq J(t, \rho) \in C^\infty(\text{neigh}(0, \rho_0), [0, +\infty[ \times \mathbb{R}^{2n})$, and

$$
\partial_t^{j+1} J(0, \rho_0) = H^j_{3p}(\Re p)(\rho_0) \begin{cases} 
0, & j \leq k - 1, \\
> 0, & j = k 
\end{cases}.
$$

(3.31)

**Proposition 3.2** Under the above assumptions, there is a constant $C > 0$ such that

$$
J(t, \rho) \geq \frac{t^{k+1}}{C}, \ (t, \rho) \in \text{neigh}((0, \rho_0), [0, +\infty[ \times \mathbb{R}^{2n}).
$$

(3.32)

**Proof.** Assume that (3.32) does not hold. Then there is a sequence $(t_\nu, \rho_\nu) \in [0, +\infty[ \times \mathbb{R}^{2n}$ converging to $(0, \rho_0)$ such that

$$
\frac{J(t_\nu, \rho_\nu)}{t_\nu^{k+1}} \to 0,
$$

and since $J(t, \rho)$ is an increasing function of $t$, we get

$$
\sup_{0 \leq t \leq t_\nu} \frac{J(t, \rho_\nu)}{t_\nu^{k+1}} \to 0.
$$

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Introduce the Taylor expansion,

\[ J(t, \rho_\nu) = a^{(0)}_\nu + a^{(1)}_\nu t + \ldots + a^{(k+1)}_\nu t^{k+1} + \mathcal{O}(t^{k+2}), \]

and define

\[ u_\nu(s) = \frac{J(t_\nu s, \rho_\nu)}{t_\nu^{k+1}}, \quad 0 \leq s \leq 1. \]

Then, on the one hand,

\[ \sup_{0 \leq s \leq 1} u_\nu(s) \to 0, \quad \nu \to \infty, \]

and on the other hand,

\[ u_\nu(s) = \frac{a^{(0)}_\nu}{t_\nu^{k+1}} + \frac{a^{(1)}_\nu}{t_\nu^k} s + \ldots + a^{(k+1)}_\nu s^{k+1} + \mathcal{O}(t_\nu s^{k+2}), \]

so

\[ \sup_{0 \leq s \leq 1} p_\nu(s) \to 0, \quad \nu \to \infty. \]

The corresponding coefficients of \( p_\nu \) have to tend to 0, and in particular,

\[ a^{(k+1)}_\nu = \frac{1}{(k+1)!} \langle \partial_{t^{k+1}} J(0, \rho_\nu) \rangle \to 0 \]

which is in contradiction with (3.31). \( \square \)

Combining (3.28) and Proposition 3.2 we get

**Proposition 3.3** Under the assumption (3.29) there exists \( C > 0 \) such that

\[ G_t(\rho) \leq -\frac{t^{k+1}}{C}, \quad (t, \rho) \in \text{neigh } ((0, \rho_0), [0, \infty[ \times \mathbb{R}^{2n}). \quad (3.33) \]

We can now return to the evolution equation for \( \tilde{P} \) and the \( t \)-dependent weight \( \Phi_t \) in (3.14). From (3.33), (3.21), we get

**Proposition 3.4** Under the assumption (3.29), we have

\[ \Phi_t(x) \leq \Phi_0(x) - \frac{t^{k+1}}{C}, \quad (t, x) \in \text{neigh } ((0, x_0), [0, \infty[ \times \mathbb{R}^{2n}). \quad (3.34) \]
4 The resolvent estimates

Let $P$ be an $h$-pseudodifferential operator satisfying the general assumptions of the introduction.

Let $z_0 \in (\partial \Sigma(p)) \setminus \Sigma_\infty(p)$. We first treat the case of Theorem 1.2 so that,

$$z_0 \in i\mathbb{R},$$  \hfill (4.1)

$$\Re p(\rho) \geq 0 \text{ in neigh } (p^{-1}(z_0), T^*X),$$  \hfill (4.2)

$$\forall \rho \in p^{-1}(z_0), \exists j \leq k, \text{ such that } H^j_{\partial \rho} \Re p(\rho) > 0.$$  \hfill (4.3)

**Proposition 4.1** There exists $C_0 > 0$ such that $\forall C_1 > 0$, $\exists C_2 > 0$ such that we have for $z, h$ as in (1.14), $h < 1/C_2$, $u \in C_0^\infty(X)$:

$$|\Re z| \|u\| \leq C_0 \|(z - P)u\|, \text{ when } \Re z \leq -h^{k/k+1},$$  \hfill (4.4)

$$h^{k/k+1} \|u\| \leq C_0 \exp \left( \frac{C_0}{h} (\Re z)^{k+1/k} \right) \|(z - P)u\|, \text{ when } \Re z \geq -h^{k/k+1}.$$  \hfill (4.5)

**Proof.** The required estimate is easy to obtain microlocally in the region where $P - z_0$ is elliptic, so we see that it suffices to show the following statement:

For every $\rho_0 \in p^{-1}(z_0)$, there exists $\chi \in C_0^\infty(T^*X)$, equal to 1 near $\rho_0$, such that for $z, h$ as in (1.14) and letting $\chi$ also denote a corresponding $h$-pseudodifferential operator, we have

$$|\Re z| \|\chi u\| \leq C_0 \|(z - P)u\| + C_N h^N \|u\|, \text{ when } \Re z \leq -h^{k/k+1},$$  \hfill (4.6)

$$h^{k/k+1} \|\chi u\| \leq C_0 \exp \left( \frac{C_0}{h} (\Re z)^{k+1/k} \right) \|(z - P)u\| + C_N h^N \|u\|, \text{ when } \Re z \geq -h^{k/k+1},$$

where $N \in \mathbb{N}$ can be chosen arbitrarily.

When $\Re z \leq -h^{k/(k+1)}$ this is an easy consequence of the semi-classical sharp Gårding inequality (see for instance [8]), so from now on we assume that $\Re z \geq -h^{k/(k+1)}$.

If $T$ is an FBI transform and $\tilde{P}$ denotes the conjugated operator $TPT^{-1}$, it suffices to show that

$$\|u\|_{H_{\kappa_0}(V_1)} \leq h^{k/k+1} C_0 \exp \left( \frac{C_0}{h} (\Re z)^{k+1/k} \right) \|\tilde{P} - z\|_{H_{\kappa_0}(V_2)} + O(h^\infty) \|u\|_{H_{\kappa_0}(V_3)},$$

$u \in H_{\kappa_0}(V_3)$, where $V_1 \subseteq V_2 \subseteq V_3$ are neighborhoods of $x_0$, given by $(x_0, \xi_0) = \kappa_T(\rho_0) \in A_{\kappa_0}$.
From Proposition 3.4 and the fact that \( \widetilde{U}(t) : H_{\Phi_0}(V_2) \to H_{\Phi_t}(V_1) \), we see that
\[
\| \widetilde{U}(t)u \|_{H_{\Phi_0}(V_1)} \leq Ce^{-tk^{1}/C} \| u \|_{H_{\Phi_0}(V_2)}.
\]
(4.7)
Choose \( \delta > 0 \) small enough so that \( \delta(k + 1) < 1 \) and put
\[
\widetilde{R}(z) = \frac{1}{h} \int_0^{h^\delta} e^{\frac{t}{C}} \widetilde{U}(t) dt.
\]
(4.8)

We shall verify that \( \widetilde{R} \) is an approximate left inverse to \( \widetilde{P} - z \), but first we study the norm of this operator in \( H_{\Phi_0} \), starting with the estimate in \( \mathcal{L}(H_{H_{\Phi_0}(V_2)}, H_{H_{\Phi_0}(V_1)}) \):
\[
\| e^{\frac{t}{C}} \widetilde{U}(t) \| \leq C \exp \left( \frac{1}{h^\delta} (t\Re z - \frac{tk^{1}}{C}) \right)
\]
(4.9)
and notice that the right hand side is \( O(h^{\infty}) \) for \( t = h^\delta \), since \( \delta(k + 1) < 1 \) and \( h^{-1}\Re z \leq O(1) \ln \frac{1}{h} \).

We get
\[
\| \widetilde{R}(z) \| \leq \frac{C}{h} \int_0^{+\infty} e^{\frac{1}{h^\delta} (t\Re z - \frac{tk^{1}}{C})} dt = \frac{C}{h^\delta} I(\frac{C}{h^\delta} \Re z),
\]
(4.10)
where
\[
I(s) = \int_0^{+\infty} e^{st - tk^{1}} dt.
\]
(4.11)

**Lemma 4.2** We have
\[
I(s) = O(1), \text{ when } |s| \leq 1,
\]
(4.12)
\[
I(s) = \frac{O(1)}{|s|}, \text{ when } s \leq -1,
\]
(4.13)
\[
I(s) \leq O(1)s^{-\frac{k+1}{2k}} \exp \left( \frac{k}{(k+1)s^{\frac{k+1}{k}}} \right), \text{ when } s \geq 1.
\]
(4.14)

**Proof.** The first two estimates are straight forward and we concentrate on the last one, where we may also assume that \( s \gg 1 \). A computation shows that the exponent \( f_s(t) = st - tk^{1} \) on \( [0, +\infty[ \) has a unique critical point \( t = t(s) = (s/(k + 1))^{1/k} \) which is a nondegenerate maximum,
\[
f''_s(t(s)) = -k(k + 1)^{\frac{1}{k}} s^{-\frac{k+1}{k}},
\]
22
with critical value
\[ f_s(t(s)) = \frac{k}{(k + 1)^{k+1}} s^{k+1}. \]

It follows that the upper bound in (4.14) is the one we would get by applying the formal stationary phase formula.

Now
\[ f''_s(t) = -(k + 1)kt^{k-1} \leq f''_s(t(s)), \text{ for } \frac{t(s)}{2} \leq t < +\infty, \]
so \( \int t(s/2) e^{st - tk+1} dt \) satisfies the required upper bound.

On the other hand we have
\[ f_s(t(s)) - f_s(t) \geq \frac{s^{k+1}}{C}, \text{ for } 0 \leq t \leq \frac{t(s)}{2}, s \gg 1, \]
so
\[ \int_0^{t(s)/2} e^{st - tk+1} dt \leq O(1)s^{\frac{1}{k}} \exp(f_s(t(s)) - \frac{s^{k+1}}{C}), \]
and (4.14) follows. \( \square \)

Applying this to (4.10), we get

**Proposition 4.3** We have
\[ \| \tilde{R}(z) \| \leq \frac{C}{h^{k+1}}, \quad |\Re z| \leq O(1)h^{\frac{k}{k+1}}, \quad (4.15) \]
\[ \| \tilde{R}(z) \| \leq \frac{C}{|\Re z|}, \quad -1 \ll \Re z \leq -h^{\frac{k}{k+1}}, \quad (4.16) \]
\[ \| \tilde{R}(z) \| \leq \frac{C}{h^{k+1}} \exp(C_k \frac{(\Re z)^{k+1} h^{k+1}}{k^{k+1}}), \quad h^{\frac{k+1}{k+1}} \leq \Re z \ll 1. \quad (4.17) \]

From the beginning of the proof of Lemma 4.2, or more directly from (4.9), we see that
\[ \| e^{t_\pi \tilde{U}(t)} \| \leq C \exp(C_k \frac{h^{k+1}}{k}), \]
which is bounded by some negative power of \( h \), since we have imposed the restriction \( \Re z \leq O(1)(h \ln \frac{1}{h})^{\frac{k}{k+1}} \). Working locally, we then see that modulo a negligible operator,
\[ \tilde{R}(z)(\tilde{P} - z) \equiv \frac{1}{h} \int_0^{h\delta} e^{t_\pi (-h\partial_t - z)\tilde{U}(t)} dt \equiv 1, \]
23
where the last equivalence follows from an integration by parts and the fact that the integrand is negligible for $t = h^\delta$. Combining this with Proposition 4.3, we get (4.6), and this completes the proof of Proposition 4.1.

We can now finish the

**Proof of Theorem 1.2.** Using standard pseudodifferential machinery (see for instance [8]) we first notice that $P$ has discrete spectrum in a neighborhood of $z_0$ and that $P - z$ is a Fredholm operator of index 0 from $\mathcal{D}(P)$ to $L^2$ when $z$ varies in a small neighborhood of $z_0$. On the other hand, Proposition 4.1 implies that $P - z$ is injective and hence bijective for $\Re z \leq O(k^{k/(k+1)})$ and we also get the corresponding bounds on the resolvent.

**Proof of Theorem 1.1.** We may assume for simplicity that $z_0 = 0$ and consider a point $\rho_0 \in p^{-1}(0)$. After conjugation with a microlocally defined unitary Fourier integral operator, we may assume that $\rho_0 = (0, 0)$ and that $dp(\rho_0) = d\xi_n$. Then from Malgrange’s preparation theorem we get near $\rho = (0, 0), z = 0$

$$p(\rho) - z = q(x, \xi, z)(\xi_n + r(x, \xi', z)), \quad \xi' = (\xi_1, \ldots, \xi_{n-1}),$$

(4.18)

where $q, r$ are smooth and $q(0, 0, 0) \neq 0$, and as in [7], we notice that either $\Im r(x, \xi', 0) \geq 0$ in a neighborhood of $(0, 0)$ or $\Im r(x, \xi', 0) \leq 0$ in a neighborhood of $(0, 0)$. Indeed, otherwise there would exist sequences $\rho_j^+, \rho_j^-$ in $\mathbb{R}^n \times \mathbb{R}^{n-1}$, converging to $(0, 0)$ such that $\pm \Im r(\rho_j^+, 0) > 0$. It is then easy to construct a simple closed curve $\gamma_j$ in a small neighborhood of $\rho_0$, passing through the points $(\rho_j^+, 0)$, such that the image of $\gamma_j$ under the map $(x, \xi) \mapsto \xi_n + r(x, \xi', 0)$ is a simple closed curve in $\mathbb{C} \setminus \{0\}$, with winding number $\neq 0$. Then the same holds for the image of $\gamma_j$ under $p$, and we see that $\mathcal{R}(p)$ contains a full neighborhood of 0, in contradiction with the assumption that $0 = z_0 \in \partial \Sigma(p)$.

In order to fix the ideas, let us assume that $\Im r \leq 0$ near $\rho_0$ when $z = 0$, so that $\Im (i(\xi_n + r(x, \xi', 0))) \geq 0$. From (4.18), we get the pseudodifferential factorization

$$P(x, hD_x; h) - z = \frac{1}{i} Q(x, hD_x, z; h) \hat{P}(x, hD_x, z; h),$$

(4.19)

microlocally near $\rho_0$ when $z$ is close to 0. Here $Q$ and $\hat{P}$ have the leading symbols $q(x, \xi, z)$ and $i(\xi_n + r(x, \xi', z))$ respectively.

We can now obtain a microlocal apriori estimate for $\hat{P}$ as before. Let us first check that the assumption in (B) of Theorem 1.1 amounts to the statement that for $z = z_0 = 0$:

$$H^{2}_{\hat{P}} \Im \hat{p}(\rho_0) > 0$$

(4.20)
for some $j \in \{1, 2, ..., k\}$. In fact, the assumption in Theorem \[1.1\] (B) is obviously invariant under multiplication of $p$ by non-vanishing smooth factors, so we drop the hats and assume from the start that $p = \hat{p}$ and $\Im p \geq 0$. Put $\rho(t) = \exp tH_p(\rho_0)$, $r(t) = \exp tH_{\Im p}(\rho_0)$ and let $j \geq 0$ be the order of vanishing of $\Im p(r(t))$ at $t = 0$. From $\dot{\rho}(t) = H_p(\rho(t))$, $\dot{r}(t) = H_{\Im p}(r(t))$, we get

$$
\frac{d}{dt}(\rho - r) = iH_{\Im p}(r) + O(\rho - r),
$$

so

$$
\rho(t) - r(t) = \int_0^t O(\nabla \Im p(r(s)))ds.
$$

Here, if $p_2 = \frac{1}{2}(p - p^*)$ is the almost holomorphic extension of $\Im p$, we get

$$
p^*(\rho(t)) = ip_2(\rho(t)) =
\begin{align*}
&ip_2(r(t)) + i\nabla p_2(r(t)) \cdot (\rho(t) - r(t)) + O((\rho(t) - r(t))^2) = \\
&ip_2(r(t)) + i\nabla p_2(r(t)) \cdot \int_0^t O(\nabla p_2(r(s)))ds + O(1)(\int_0^t O(\nabla p_2(r(s)))ds)^2.
\end{align*}
$$

Here, $\nabla p_2(r(t)) = O(p_2(r(t))^{1/2}) = O(t^{1/2})$, so $p^*(\rho(t)) = ia(\rho_0) t^j + O(t^{j+1})$.

Then, if we conjugate with an FBI-Bargmann transform as above, we can construct an approximation $\tilde{U}(t)$ of $\exp(-t\tilde{P}/h)$, such that

$$
\|\tilde{U}(t)\| \leq C_0 e^{(C_0(t|z-z_0|-t^{k+1})/C_0)}/h,
$$

when $|z - z_0| = O((h \ln \frac{1}{n})^{k/(k+1)})$.

From this we obtain a microlocal apriori estimate for $\tilde{P}$ analogous to the one for $P - z$ in Proposition \[1.1\] and the proof can be completed in the same way as for Theorem \[1.2\].

5 Examples

Consider

$$
P = -h^2 \Delta + iV(x), \quad V \in C^\infty(X; \mathbb{R}), \quad (5.1)
$$

where either $X$ is a smooth compact manifold of dimension $n$ or $X = \mathbb{R}^n$. In the second case we assume that $p = \xi^2 + iV(x)$ belongs to a symbol space $S(m)$ where $m \geq 1$ is an order function. It is easy to give quite general sufficient condition for this to happen, let us just mention that if $V \in C^\infty_c(\mathbb{R}^2)$ then we can take $m = 1 + \xi^2$ and if $\partial^\alpha V(x) = O((1 + |x|)^2)$ for all $\alpha \in \mathbb{N}^n$ and satisfies the ellipticity condition $|V(x)| \geq C^{-1}|x|^2$ for $|x| \geq C$, for some constant $C > 0$, then we can take $m = 1 + \xi^2 + x^2$. 25
We have $\Sigma(p) = [0, \infty[iV(X)]$. When $X$ is compact then $\Sigma_{\infty}(p)$ is empty and when $X = \mathbb{R}^n$, we have $\Sigma_{\infty}(p) = [0, \infty[i\Sigma_{\infty}(V)]$, where $\Sigma_{\infty}(V)$ is the set of accumulation points at infinity of $V$.

Let $z_0 = x_0 + iy_0 \in \partial\Sigma(p) \setminus \Sigma_{\infty}(p)$.

- In the case $x_0 = 0$ we see that Theorem 1.2 (B) is applicable with $k = 2$, provided that $y_0$ is not a critical value of $V$.

- Now assume that $x_0 > 0$ and that $y_0$ is either the maximum or the minimum of $V$. In both cases, assume that $V^{-1}(y_0)$ is finite and that each element of that set is a non-degenerate maximum or minimum. Then Theorem 1.2 (B) is applicable to $\pm iP$. By allowing a more complicated behaviour of $V$ near its extreme points, we can produce examples where 1.2 (B) applies with $k > 2$.

Now, consider the non-self-adjoint harmonic oscillator

$$Q = -\frac{d^2}{dy^2} + iy^2$$

(5.2)

on the real line, studied by Boulton [2] and Davies [6]. Consider a large spectral parameter $E = i\lambda + \mu$ where $\lambda \gg 1$ and $|\mu| \ll \lambda$. The change of variables $y = \sqrt{\lambda}x$ permits us to identify $Q$ with $Q = \lambda P$, where $P = -\frac{d^2}{dx^2} + ix^2$ and $h = 1/\lambda \to 0$. Hence $Q - E = \lambda(P - (1 + i\frac{\mu}{\lambda}))$ and Theorem 1.2 (B) is applicable with $k = 2$. We conclude that $(Q - E)^{-1}$ is well-defined and of polynomial growth in $\lambda$ (which can be specified further) respectively $O(\lambda^{-1})$ when

$$\frac{\mu}{\lambda} \leq C_1(\lambda^{-1} \ln \lambda)^{\frac{3}{2}}$$

and

$$\frac{\mu}{\lambda} \leq C_1$$

for any fixed $C_1 > 0$, i.e. when

$$\mu \leq C_1 \lambda^{\frac{3}{2}}(\ln \lambda)^{\frac{3}{2}}$$

and

$$\mu \leq C_1 \lambda^{\frac{3}{2}}$$

respectively.

(5.3)

We end by making a comment about the Kramers–Fokker–Planck operator

$$P = hy \cdot \partial_x - V'(x) \cdot h\partial_y + \frac{1}{2}(y - h\partial_y) \cdot (y + h\partial_y)$$

(5.4)

on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$, where $V$ is smooth and real-valued. The associated semi-classical symbol is

$$p(x, y; \xi, \eta) = i(y \cdot \xi - V'(x) \cdot \eta) + \frac{1}{2}(y^2 + \eta^2)$$
on $\mathbb{R}^{4n}$, and we notice that $\Re p_1 \geq 0$. Under the assumption that the Hessian $V''(x)$ is bounded with all its derivatives, $|V'(x)| \geq C^{-1}$ when $|x| \geq C$ for some $C > 0$, and that $V$ is a Morse function, F. Hérau, C. Stolk and the author [10] showed among other things that the spectrum in any given strip $i[\frac{1}{C_1}, C_1] + \mathbb{R}$ is contained in a half strip

$$i\left[\frac{1}{C_1}, C_1\right] + \left[\frac{h^{2/3}}{C_2}, \infty\right]$$

(5.5)

for some $C_2 = C_2(C_1) > 0$ and that the resolvent is $O(h^{-2/3})$ in the complementary halfstrip. (We refrain from recalling more detailed statements about spectrum and absence of spectrum in the regions where $|\Im z|$ is large and small respectively.)

The proof of this result employed exponentially weighted estimates based on the fact that $H_{p_2}^2 p_1 > 0$ when $p_2 \approx 1$, $p_1 \ll 1$. This is of course reminiscent of Theorem [12] (B) with $k = 2$ or rather the corresponding result in [7], but actually more complicated since our operator is not elliptic near $\infty$ and we even have that $i\mathbb{R} \setminus \{0\}$ is not in the range of $p$ but only in $\Sigma_\infty(p)$. It seems likely that the estimates on the spectrum of the KFP-operator above can be improved so that we can replace $h$ by $h \ln(1/h)$ in the confinement (3.23) of the spectrum of $P$ in the strip $i[1/C_1, C_1] + \mathbb{R}$ and that there are similar improvements for large and small values of $|\Im z|$. This would be obtained either by a closer look at the proof in [10] or by an adaptation of the proof above when $k = 2$.

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