Many-Body Spin Hall Effect with Space Inversion Symmetry

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In contrast to the ordinary spin Hall effect (SHE), which is a single-body phenomenon caused by the spin-orbit interaction (SOI), we propose a many-body SHE induced by the dipole-dipole interaction (DDI) between particles and demonstrate it in a system of ultracold magnetic atoms. While the SOI usually requires the breaking of space inversion symmetry, the DDI preserves it. The many-body SHE can, in principle, be observed in a wide range of systems with large dipole moments and offers a powerful tool to generate spin currents, an essential ingredient in spintronics and atomtronics.

Introduction. The spin Hall effect (SHE), first proposed for electrons by Dyakonov and Perel \cite{dyakonov1971}, is a single-body phenomenon in which carriers undergo a spin-dependent force that deflects their trajectories in the perpendicular direction. This force is reminiscent of the Lorentz force but has the opposite directions for spin-up and spin-down particles. With the SHE, one can generate a nonzero spin current $j_s = j_↑ - j_↓$ at zero net particle current $j = j_↑ + j_↓$ \cite{dyakonov1971}. The SHE was first observed in GaAs \cite{roberts1990} and later in a Bose-Einstein condensate (BEC) of $^{87}$Rb under a spin-dependent artificial gauge field \cite{zhang2008} and in the propagation of light through an air-glass interface \cite{yokoyama2008, yokoyama2009}. Here, particles with (pseudo)spin-1/2 are electrons, atoms with hyperfine spins and circularly-polarized photons. The spin current generated by the SHE has found important applications in the field of spintronics for a new generation of magnetic random access memories \cite{altarelli2007, altarelli2008, altarelli2010}. Moreover, the SHE serves as an efficient mechanism to manipulate spins and circularly-polarized photons. The spin current is even in each of the DOFs and the applied electric field $\sigma_s$ change their signs, but the spin conductivity matrix $\sigma_s$ remains unchanged ($j_s = \sigma_s E$). The SHE is therefore compatible with the SIS and should appear in systems where the spin and orbital degrees of freedom (DOF) are coupled to each other in a way that preserves the SIS. To this end, the interaction Hamiltonian must be even in each of these DOFs. Moreover, for a many-body SHE we need an anisotropic interaction $H(\mathbf{s}_i, \mathbf{s}_j, \mathbf{r}_{ij})$ between particles such that the spin DOFs are coupled to vectorial orbital DOFs rather than scalar ones. These requirements are satisfied by the dipole-dipole interaction (DDI) \cite{dzyaloshinskii1957}:

$$H_{dd}(\mathbf{d}_i, \mathbf{d}_j, \mathbf{r}_{ij}) = C \frac{\mathbf{d}_i \cdot \mathbf{d}_j - 3(\mathbf{d}_i \cdot \mathbf{e}_{ij})(\mathbf{d}_j \cdot \mathbf{e}_{ij})}{4\pi r_{ij}^3}. \quad (1)$$

Here $r_{ij} = |\mathbf{r}_{ij}|$ is the distance between two dipole moments $\mathbf{d}_i$ and $\mathbf{d}_j$, $\mathbf{e}_{ij} = \mathbf{r}_{ij}/r_{ij}$, and $C = 1/\epsilon_0 (\mu_0)$ for electric (magnetic) dipoles with $\epsilon_0$ and $\mu_0$ being the vacuum permittivity and permeability, respectively. It is evident that the right-hand side of Eq. (1) is invariant upon the space inversion transformation: $\mathbf{r}_{ij} \rightarrow -\mathbf{r}_{ij}$, $\mathbf{r}_{ij} \rightarrow -\mathbf{r}_{ij}$, and $\mathbf{r}_{ij} \rightarrow -\mathbf{r}_{ij}$. It is also clear that the...
DDI couples the spin and orbital degrees of freedom in such a manner that the total angular momentum is conserved. To this end, the DDI can be decomposed as

$$H_{ij}^{δL=0} + H_{ij}^{δL=±2} [21, 22, 24],$$

where

$$H_{ij}^{δL=0} = \frac{C}{4\pi r_{ij}^3} \left( \hat{d}_{ij}^{-1}d_{ij} + \hat{d}_{ij}^{-1}d_{ij} \right), \quad (2)$$

$$H_{ij}^{δL=±2} = -\frac{3C}{8\pi r_{ij}^3} \left( \hat{d}_{ij}^{-1}d_{ij}e^{-2iϕ_{ij}} + \hat{d}_{ij}^{-1}d_{ij}e^{2iϕ_{ij}} \right). \quad (3)$$

Here, $d_{ij}^{±1} = \mp(i\hat{p}_{ij}^x + d_{ij}^y)/\sqrt{2}$ are the spherical harmonic components of the dipole-moment operator $\hat{d}_{ij}$, and $(r_{ij}, ϕ_{ij})$ are the polar coordinates of $r_{ij}$. While $H_{ij}^{δL=0}$ conserves both the spin and orbital angular momentum by two and simultaneously lowers (raises) the orbital angular momentum by the same amount so as to conserve the total angular momentum. This is the DDI-induced spin-orbit coupling that gives rise to the many-body SHE proposed in this paper.

Many-body SHE in ultracold magnetic atoms. In the following, for the sake of concreteness we consider a system of $^{52}$Cr atoms loaded into a two-dimensional (2D) optical lattice in the $xy$ plane [22, 23] (see Fig. 1). As shown below, the many-body SHE can be demonstrated in the Mott-insulator regime where atoms are strongly localized at the lattice sites, leaving only their internal degrees of freedom free. The energy degeneracy of spin states $|S = 3, m_S \rangle = 3, \cdots, -3$ of chromium atoms can be lifted by using a light-induced quadratic Zeeman shift (via the ac Stark effect) [40, 41], which preserves the SIS. In the following, we restrict our consideration to the case in which the energy shift is sufficiently large so that the spin dynamics governed by the DDI essentially involves only three spin states $|m_S = 1, 0, -1\rangle$. In a general case that involves more spin states, the system’s dynamics is more complex; however, our prediction should not change at least qualitatively if the definition of spin current is generalized to include all spin states.

If the fraction of atoms in the excited states $|m_S = ±1\rangle$ is small, the last term $d_{ij}^{±1}d_{ij}^{±1}$ in Eq. (2) becomes negligibly small compared with the other terms in the DDI. In this case, the spin dynamics of the atomic ensemble can be mapped onto the dynamics of a system of free spin-1/2 particles where the two spin states $|m_S = 1\rangle$ and $|m_S = -1\rangle$ correspond to the spin-up and spin-down states, respectively, while the ground state $|m_S = 0\rangle$ represents the vacuum for those particles. From the energy conservation, it is clear that the term $d_{ij}^{±1}d_{ij}^{±1}$ in Eq. (2) makes the state transition at site $i$ ($j$) from $|m_S = -1\rangle$ ($|m_S = 0\rangle$) to $|m_S = 0\rangle$ ($|m_S = 1\rangle$), which implies that the state $|m_S = -1\rangle$ is transferred from $i$ to $j$ followed by a spin flip $|m_S = 1\rangle \rightarrow |m_S = -1\rangle$. The second-quantized DDI Hamiltonian of the system then reads [24]

$$H_{dd} = t_0 \sum_{i,j} \sum_{σ=1,↓} \hat{b}_{ijσ}^+ \hat{b}_{ijσ} + t_2 \sum_{i,j} \sum_{σ=1,↓} \sum_{i,j} \sum_{σ=1,↓} e^{-2iϕ_{ij}} \hat{b}_{ijσ}^+ \hat{b}_{ijσ} + e^{2iϕ_{ij}} \hat{b}_{ijσ}^+ \hat{b}_{ijσ}, \quad (4)$$

where $\hat{b}_{ijσ}$ ($\hat{b}_{ijσ}^+$) is the annihilation (creation) operator of a particle with spin $σ = ↑, ↓$ at lattice site $j$. The first term on the right-hand side of Eq. (4) describes a spin-conserving tunneling, while the second term describes tunneling accompanied by a spin flip, where $t_0 = -3C\alpha^2/(4\pi)$, $t_2 = 3t_0$, and $d = g_\mu B$ with $g$ and $\mu_B$ being the Landé g-factor and the Bohr magneton, respectively. It is noteworthy that the effective Hamiltonian [44], which is obtained by neglecting the last term in Eq. (4), also preserves the SIS as the original Hamiltonian [41]. In fact, the space inversion transformation $ϕ_{ij} \rightarrow ϕ_{ij} + π$ and in turn $e^{-2iϕ_{ij}}$ invariant. Due to the translation invariance of the system, $H_{dd}$ can be expressed as

$$H_{dd} = \sum_k \left( \hat{b}_{kσ}^+ \right)^* \left( \hat{b}_{kσ}^+ \right) H \left( \hat{b}_{kσ} \right), \quad (5)$$

where $\hat{b}_{kσ}$ ($\hat{b}_{kσ}^+$) is the annihilation (creation) operator of a particle with spin $σ = ↑, ↓$ at lattice site $j$. The first term on the right-hand side of Eq. (5) describes a spin-conserving tunneling, while the second term describes tunneling accompanied by a spin flip, where $t_0 = -3C\alpha^2/(4\pi)$, $t_2 = 3t_0$, and $d = g_\mu B$ with $g$ and $\mu_B$ being the Landé g-factor and the Bohr magneton, respectively. It is noteworthy that the effective Hamiltonian [44], which is obtained by neglecting the last term in Eq. (4), also preserves the SIS as the original Hamiltonian [41]. In fact, the space inversion transformation $ϕ_{ij} \rightarrow ϕ_{ij} + π$ and in turn $e^{-2iϕ_{ij}}$ invariant. Due to the translation invariance of the system, $H_{dd}$ can be expressed as

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and \( f^{(2)}(\mathbf{k}) = N^2 \sum_{r \neq \neq} e^{-\mathbf{i} \mathbf{k} \cdot \mathbf{r}_j} e^{2i \phi_j} / \rho^3 \). The diagonalization of Eq. (5) gives

\[
H = E(0) \left( \hat{b}_{0\uparrow}^\dagger \hat{b}_{0\uparrow} + \hat{b}_{0\downarrow}^\dagger \hat{b}_{0\downarrow} + \sum_{\mathbf{k} \neq \neq} E_{\mathbf{k}}(\mathbf{k}) \hat{b}_{\mathbf{k}\uparrow}^\dagger \hat{b}_{\mathbf{k}\uparrow} + \sum_{\mathbf{k} \neq \neq} E_{\mathbf{k}}(\mathbf{k}) \hat{b}_{\mathbf{k}\downarrow}^\dagger \hat{b}_{\mathbf{k}\downarrow} \right),
\]

where \( E(0) = t_0 f^{(0)}(\mathbf{k} = 0) \), \( E_{\mathbf{k}}(\mathbf{k}) = t_0 f^{(0)}(\mathbf{k}) + t_2 f^{(2)}(\mathbf{k}) \) and \( \hat{b}_{\mathbf{k}\uparrow} = (e^{i \mathbf{k} \cdot \mathbf{r}_\uparrow} + \hat{b}_{\mathbf{k}\downarrow}) / \sqrt{2} \), \( \hat{b}_{\mathbf{k}\downarrow} = (\hat{b}_{\mathbf{k}\uparrow} + e^{-i \mathbf{k} \cdot \mathbf{r}_\downarrow}) / \sqrt{2} \) with \( \theta_k \) being the phase of \( f^{(2)}(\mathbf{k}) \). In the following, we show that the SHE emerges if an external force is applied to these spin-1/2 particles.

**Spin Hall conductivity.** Since the spin-up and spin-down particles correspond to the \( |n_S = \pm \rangle \) spin states of the atoms, a force exerted on those particles can be generated by, for example, an spatial gradient of the light intensity that induces a quadratic Zeeman shift in the energy levels. This force plays the role of an effective electric field for the spin-1/2 particles, which can be expressed in terms of an effective vector potential. As the unperturbed system is translation invariant, the effective vector potential is identical, reflecting the fact that the system preserves the SOI, here the spin components of \( \hat{A}_\mathbf{r}(\mathbf{r}, t) = \partial \hat{A}_\mathbf{r}(\mathbf{r}, t) / \partial t \). In the presence of a vector potential, the Hamiltonian \( H \) of the system is obtained through the replacement of the momentum operator \( \hat{p} \) with \( \hat{p} - \hat{A}_\mathbf{r}(\mathbf{r}, t) \) to ensure the local gauge invariance. The first-order perturbation of the Hamiltonian is given by \( \delta H = -\int d\mathbf{r} \hat{J}(\mathbf{r}) \cdot \hat{A}_\mathbf{r}(\mathbf{r}, t) \), where \( \hat{J}(\mathbf{r}) = \frac{1}{\hbar} \{ \hat{n}(\mathbf{r}), \hat{\mathbf{v}} \} \) is the paramagnetic component of the particle-current density operator. Here, the curly brackets denote the commutator and the particle-density and group-velocity operators are given by \( \hat{n}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}) \) and \( \hat{\mathbf{v}} = \frac{\delta \hat{H}}{\delta \hat{\mathbf{p}}} \), respectively. From Eq. (6), the group-velocity operator is found to be

\[
\hat{\mathbf{v}} = \frac{\hbar}{(t_2 \mathbf{g}^{(2)}(\mathbf{p}) + t_2 \mathbf{g}^{(2)}(\mathbf{p}))},
\]

where \( \mathbf{g}^{(0),(2)}(\mathbf{k}) \equiv \nabla_{\mathbf{k}} f^{(0),(2)}(\mathbf{k}) \). The total particle-current density is given by the sum of the paramagnetic and diamagnetic components, the latter of which is given by the product of the particle density and the vector potential. In the linear-response regime, the paramagnetic particle-current density in the frequency domain is given by \( j_{\mathbf{p}}^{\alpha}(\mathbf{r}, \omega) = \sum_{\beta} \int d\mathbf{r} \delta_{\mathbf{p}}^{\alpha}(\mathbf{r}, \omega) A_\alpha(\mathbf{r}, \omega) \), where \( \alpha, \beta = x, y, z \) and \( \delta_{\mathbf{p}}^{\alpha}(\mathbf{r}, \omega) \) is the paramagnetic response function. Since the total particle current should vanish at the zero-frequency limit, where the vector potential becomes time independent, leading to the vanishing electric field, the diamagnetic current can be expressed in terms of the paramagnetic response function \( k_{\alpha,\beta}(\mathbf{r}, \omega = 0) \) at zero frequency. Summing the two components, we obtain the total particle-current density as \( j_\mathbf{p}(\mathbf{r}, \omega) = \sum_{\beta} \int d\mathbf{r'} [k_{\alpha,\beta}(\mathbf{r}, \omega) - k_{\alpha,\beta}(\mathbf{r'}, \omega = 0)] \hat{A}_\beta(\mathbf{r'}, \omega) \).

As an initial state that preserves the SIS, we consider the case in which half of the spin-1/2 particles are initially prepared in the eigenstate \( |\Psi_1\rangle = |\mathbf{q}_+\rangle \) of the Hamiltonian \( \hat{H} \) with a momentum \( \mathbf{q} \neq 0 \) and the other half in the eigenstate \( |\Psi_2\rangle = | -\mathbf{q}_+\rangle \). Such an initial state can be realized, for example, by using a pair of independent counter-propagating laser beams with appropriately chosen directions and polarizations to excite the atomic ensemble. Note that in contrast to the case of the Rashba SOI, here the spin components of \( |\mathbf{q}_+\rangle \) and \( | -\mathbf{q}_+\rangle \) are identical, reflecting the fact that the system preserves the SIS. Using the linear-response theory, the paramagnetic response function is found to be

\[
(K_s)_{\alpha,\beta}(\mathbf{r}, \mathbf{r'}, \omega) = -\frac{1}{4\pi} \sum_{j=1}^{2} \sum_{n} \int \frac{\langle \psi_j | (\hat{p}_\alpha)^n(\mathbf{r}) \psi_n \rangle \langle \psi_n | (\hat{p}_\beta)^n(\mathbf{r'}) \psi_j \rangle}{E_0 - E_n - \hbar \omega + i\eta} \frac{\langle \psi_n | (\hat{p}_\alpha)^n(\mathbf{r}) \psi_j \rangle \langle \psi_j | (\hat{p}_\beta)^n(\mathbf{r'}) \psi_n \rangle}{E_0 - E_n + \hbar \omega + i\eta},
\]

where \( \eta \) is an infinitesimal positive number and the sum is taken over all eigenstates \( |\psi_n\rangle \) of Hamiltonian \( \hat{H} \). Here \( E_0 \) is the eigenenergy of \( |\Psi_1, 2\rangle \) and \( E_n \) is that of \( |\psi_n\rangle \). The matrix elements in Eq. (8) can be evaluated straightforwardly. As the unperturbed system is translation invariant, both \( (K_s)_{\alpha,\beta}(\mathbf{r}, \mathbf{r'}, \omega) \) and \( (\chi_s)_{\alpha,\beta}(\mathbf{r}, \mathbf{r'}, \omega) \) are functions of \( \mathbf{r} - \mathbf{r}' \). By making the Fourier transformation \( (\chi_s)_{\alpha,\beta}(\mathbf{k}, \omega) = \int d(\mathbf{r} - \mathbf{r'}) e^{-i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r'})} (\chi_s)_{\alpha,\beta}(\mathbf{r} - \mathbf{r'}, \omega) \) and taking the limits of \( \mathbf{k} \to 0 \) and \( \omega \to 0 \), which corresponds to a homogeneous and static driving force, we obtain
(χ_s)_{αβ}(k → 0, ω → 0) = \frac{t_0 t_2}{2π S} \sum_{p=±q} g^{(0)}_p(p) \text{Im} \left\{ e^{iθ_p} \left[ -g^{(2)}_p(p) + e^{2iθ_p} g^{(2)}_p(p)^* \right] \right\} \frac{[E_+(p) - E_-(p)]^2}{[E_+(p) - E_-(p)]^2}, \tag{9}

FIG. 2: Dependence of the spin Hall conductivity (χ_s)_{xy} on the direction of the initial excitation with wavevector q. Here q = (q, φ_q) are the polar coordinates, and (χ_s)_{xy} is shown in units of 1/[π(Nqa)^2], where a and N are the lattice constant and the number of lattice sites in a linear dimension, respectively.

where S = (Na)^2 is the area of the lattice. In the long-wavelength limit k a ≪ 1 (a is the lattice constant) and for the square lattice, f^{(0)}(k) and f^{(2)}(k) have the asymptotic forms of a^3 f^{(0)}(k)/N^2 ≈ A = 2π k a and a^3 f^{(2)}(k)/N^2 ≈ 2π i k a e^{2iφ_k}/3, where A ≈ 9.03 is a constant and φ_k is the azimuthal angle of k. Therefore, for qa ≪ 1 the spin Hall conductivity (χ_s)_{xy}(k → 0, ω → 0) reduces to

(χ_s)_{xy} = \frac{\cos(4φ_q) \cos^2(φ_q)}{π(Nqa)^2}. \tag{10}

The dependence of (χ_s)_{xy} on the direction of the initial momentum (angle φ_q) is shown in Fig. 2 demonstrating the characteristic anisotropy of the DDI [Eq. (1)] and the π-periodicity which arises from the fact that two quanta of the angular momentum are transferred between the spin and orbital degrees of freedom by the DDI.

Numerical simulation. In deriving the spin Hall conductivity analytically [Eq. (10)], we have assumed that the fraction of atomic excitations, i.e., the ratio of the number of atoms in the |m_S = ±1⟩ spin states to that of atoms in the |m_S = 0⟩ ground state, is sufficiently small so that the interaction between excitations [the last term in Eq. (2)] can be neglected. As numerically demonstrated below, this assumption can be removed without changing qualitatively the physics we are interested in.

In the numerical simulation, atoms are located at the sites of a 3 × 3 square lattice with unit filling, i.e., one atom per site. Initially, all atoms are prepared in the |m_S = 0⟩ ground state. Atoms at the left edge are then excited to the superposition state: (|m_S = 1⟩ + |m_S = −1⟩)/√2 (see Fig. 3a). An effective electric field E_{eff} for those excitations is applied in the x direction as described by an energy potential H_E = E_{eff} \sum_j x_j \hat{P}_j^0, where x_j and \hat{P}_j^0 are the x-coordinate and the projection operator onto the |m_S = 0⟩ ground state of the atom at lattice site j, respectively. The time evolution of the system is obtained by using the exact diagonalization method.

Figure 3 shows the short-time evolutions of the magnetizations M_j(x_j, y_j) = (\hat{P}_j^1 − \hat{P}_j^{-1}) of atoms at four lattice sites (x_j, y_j) = (1, 2), (1, 3), (3, 2) and (3, 3). Here, \hat{P}_j^{±1} are the projection operators onto the spin states |m_S = ±1⟩ of an atom at lattice site j. It is evident that M_j negatively increases for atoms at the bottom edge and positively increases for atoms at the top edge, while their absolute values are almost equal. These spin accumulations offer the evidence of the SHE and can directly be measured experimentally. The time scale of the spin Hall dynamics shown in Fig. 3 is given by τ = h/ε_{dd}, where ε_{dd} = C d^2/(4πa^3) is the characteristic energy scale of the DDI. Using the parameters of the ^{52}Cr, we find that τ ≈ 25 ms, which makes the many-body SHE observable in current experiments.

Concluding remarks. We have proposed the many-body SHE induced by the DDI between particles, which is in marked contrast to the ordinary SHE that is a single-body phenomenon generated by the SOI. We have demonstrated both analytically and numerically the many-body SHE in a system of ultracold magnetic chromium atoms. Moreover, the observation of many-
body SHE, in general, does not require a Bose-Einstein condensate \cite{45}.

Although the spin and orbital degrees of freedom are coupled to each other by both the DDI and SOI, the couplings themselves are totally different, leading to important differences between the SHEs caused by the two interactions. While the SOI is purely a single-body effect, the coupling induced by the DDI is of an intrinsically many-body nature. Whereas the ordinary SHE with SOI often requires SIS breaking, the many-body SHE induced by the DDI can occur in systems with SIS. Moreover, in contrast to the isotropic SOI, the characteristic anisotropy of the DDI is clearly reflected in the strong dependence of the spin Hall conductivity on the direction of the initial momentum. It is also noteworthy that the many-body SHE is fundamentally different from the magnon Hall effect \cite{46, 48} which typically arises from a nonzero Dzyaloshinskii-Moriya interaction that breaks the SIS and plays a role similar to the SOI in the ordinary SHE. The magnon Hall effect can also be observed in the case of magnetostatic spin waves \cite{49} where the restriction of the action of the magnetic DDI between spins to one spatial dimension results in the demagnetizing field that plays a role similar to the single-body SOI in the ordinary SHE. In contrast, there is no restriction on the action of the DDI that constitutes the many-body SHE which works in general even in the strongly correlated regime where the concept of quasiparticles is no longer valid.

In the case of clean systems such as ultracold atoms and molecules, the divergence of the longitudinal conductivity makes it difficult to reach a steady state. The spin Hall conductivity, however, can be obtained by extrapolating from the time evolution of magnetization at the edges of the system (see Fig. 5). In contrast, the steady state should be reached in solid-state electronic systems due to the impurity scattering, which makes the spin Hall conductivity directly measurable. In considering the effect of impurity scattering, however, it is necessary to take account of the vertex correction, which can dramatically change the spin Hall conductivity as shown for the ordinary SHE with the Rashba SOI \cite{50}. In this case, a similar definition of the spin Hall angle \cite{2} can be introduced to characterize the dimensionless magnitude of the many-body SHE.

Concerning the strength of interaction, although the magnitude of magnetic DDI is generally smaller than that of the electric DDI by a factor of $\alpha^2$, where $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ is the fine structure constant, the magnetic DDI and the electric SOI share the same scaling with respect to the fine structure constant. Therefore, the proposed many-body SHE can find wide application not only in systems whose particles possess an electric dipole moment but also in numerous space-inversion-symmetric systems with magnetic dipole moments where the SOI is negligibly small.

Using the SOI, the single-body quantum SHE characterized by a unidirectional edge spin transport, i.e., edge states with opposite spins propagating in opposite directions, has been observed in both electronic \cite{51, 52} and photonic \cite{53} materials, and it gives rise to a new and important class of materials, namely topological insulators \cite{54, 55}. Similarly, the many-body SHE can open new avenues for studies of as yet unexplored classes of many-body topological materials in which inter-particle interactions play an essential role.

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