On the geometry underlying supersymmetric flux vacua with intermediate SU(2) structure

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Abstract
We show that supersymmetric flux vacua with constant intermediate SU(2) structure are closely related to some special classes of half-flat structures. More concretely, solutions of the SUSY equations IIA possess a symplectic half-flat structure, whereas solutions of the SUSY equations IIB admit a half-flat structure which is in a certain sense near to the balanced condition. Using this result we show that compact simply connected manifolds do not admit type IIB solutions. New solutions of the SUSY equations IIA and IIB are constructed from hyperkähler 4-manifolds, special hypo 5-manifolds and six-dimensional solvmanifolds.

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1. Introduction
In [1] new supersymmetric four-dimensional Minkowski flux vacua of type II string theory with at least $N = 1$ supersymmetry (SUSY) on nilmanifolds and solvmanifolds have been found, by extending previous results by [19].

In [17, 18] it was shown that these supersymmetric conditions can be written in terms of the so-called generalized complex geometry [20, 25] and it was proved that the internal manifold has to be a (twisted) generalized Calabi–Yau manifold. An $N = 1$ supergravity vacuum implies the existence of a pair of spinors on the internal manifold. In dimension 6 the pair of spinors defines an $SU(3)$ structure, a static $SU(2)$ structure or an intermediate $SU(2)$ structure [1], which correspond respectively to the condition that the two spinors are parallel, orthogonal or between the two. These different cases are encoded in the context of the generalized geometry into an $SU(3) \times SU(3)$ structure on the bundle $TM \oplus T^*M$. The $SU(3) \times SU(3)$ structure can be encoded in a pair of compatible pure spinors, which are objects defined in generalized...
complex geometry on the generalized tangent bundle $TM \oplus T^*M$. When one of the two pure spinors is closed, the manifold is called generalized Calabi–Yau.

An interesting question is then to look for new explicit examples and a natural class is the one of the nilmanifolds, since by [5] they admit generalized Calabi–Yau structures. In [19] the authors only look for $SU(3)$ and static $SU(2)$ structures, since only these ones seemed to be compatible with the orientifold projections. But in [26] it was shown that intermediate $SU(2)$ structures are also possible if one allows a mixing of the usual $SU(2)$ structure forms under the projection conditions.

In [1] Andriot rewrote the projection conditions imposed by the orientifold for intermediate $SU(2)$ structures by introducing the ‘projection (eigen)basis’, i.e. the set of structure forms which are eigenvectors for the projection. These forms define a new $SU(2)$ structure, obtained by a rotation from the usual one and the new $SU(2)$ structure coincides (modulo a rescaling) with the one appearing with dielectric pure spinors, introduced in [22, 27] in the ADS/CFT context. Since the pure spinors become simpler to study if they are written in terms of the projection basis variables, the supersymmetry (SUSY) conditions become simple and in this way Andriot found for constant intermediate $SU(2)$ structures new four-dimensional (Minkowski) flux vacua of type II string theory with at least $N = 1$.

Inside the class of $SU(3)$ structures there is a special one which is strictly related to the construction of metrics with holonomy $G_2$. An $SU(3)$ structure defines a non-degenerate 2-form $F$, an almost-complex structure $J$, and a complex volume form $\Psi$; the $SU(3)$ structure is called half-flat if $F \wedge F$ and the real part of $\Psi$ are closed [7]. Hypersurfaces in seven-dimensional manifolds with holonomy $G_2$ have a natural half-flat structure, given by the restriction of the holonomy group representation. In [24] Hitchin showed that, starting with a half-flat manifold $(M, F, \Psi)$, if certain evolution equations have a solution coinciding with $(F, \Psi)$ at time zero then $(M, F, \Psi)$ can be embedded isometrically as a hypersurface in a manifold with holonomy contained in $G_2$.

If in addition $F$ is closed, the half-flat structure is called symplectic, and a half-flat structure with closed complex volume form $\Psi$ is known as Hermitian balanced. Nilmanifolds of dimension 6 admitting invariant symplectic, resp. Hermitian balanced, half-flat structures have been classified in [11], resp. [28]. Recently six-dimensional nilmanifolds carrying an invariant half-flat structure have been classified by Conti in [10], extending previous partial results [6, 8, 11, 28].

In this paper we show that supersymmetric flux vacua with constant intermediate $SU(2)$ structure are closely related to some special classes of half-flat structures. More concretely, in section 3 we show that solutions of the SUSY equations IIA have a symplectic half-flat structure, whereas solutions of the SUSY equations IIB admit a half-flat structure which is in a certain sense near to the Hermitian balanced condition. In particular, we prove that compact simply connected manifolds do not admit type IIB solutions. Solutions of the SUSY equations IIA and IIB are constructed from hyperkähler 4-manifolds and, more generally, from special hypo 5-manifolds, where by hypo we mean the natural $SU(2)$ structure induced on hypersurfaces in six-dimensional manifolds with holonomy $SU(3)$ given by the restriction of the holonomy group representation [12]. In the last section we consider six-dimensional solvmanifolds having both symplectic and Hermitian balanced half-flat structures, and using them we find new solutions of the SUSY equations IIA and IIB. The nilmanifolds considered in this paper have appeared previously in [19], where solutions with $SU(3)$ or static $SU(2)$ structure were found. On the other hand, on the solvmanifold of example 4.2.1, $SU(3)$ structure solutions were given in [4, 19] (see also [2]); however, to our knowledge, the solvmanifold of example 4.2.2 has not appeared previously in relation to the SUSY equations and provides a new class of solutions. In section 4 it is also proved that in general the solutions of
equations IIA or IIB are not stable by small deformations inside the class of half-flat structures (see proposition 4.1).

2. Intermediate \(SU(2)\) structures

In this section we follow the conventions of [1] and recall the four-dimensional Minkowski flux vacua conditions of type II string theory with at least \(N = 1\) supersymmetry as well as their relation to the structure group of the internal manifold.

As in [1] we consider type II supergravity (SUGRA) backgrounds, which are warped products of the Minkowski space \(\mathbb{R}^{3,1}\) and of a six-dimensional compact manifold \(M^6\). These warped products have a metric of the form

\[
d s_{(10)}^2 = e^{2A(y)} \eta_{\mu\nu} \, dx^\mu \, dx^\nu + g_{\mu\nu}(y) \, dy^\mu \, dy^\nu, \tag{1}
\]

where \(\eta\) is the diagonal Minkowski metric with signature \((3,1)\). The solutions will also have non-zero background values for some of the RR and NS fluxes. Let \(\text{vol}_{(4)}\) denote the warped four-dimensional volume form. Poincaré invariance in dimension 4 requires the fluxes living on Minkowski space to be proportional to \(\text{vol}_{(4)}\), so we will focus on non-trivial fluxes living on the internal manifold \(M^6\).

As in [19] the total internal RR field \(F\) is given by

- IIA : \(F = F_0 + F_2 + F_4 + F_6\),
- IIB : \(F = F_1 + F_3 + F_5\),

where \(F_k\) is the internal \(k\)-form RR field, and it is related to the total ten-dimensional RR field-strength \(F_{(10)}\) by

\[
F_{(10)} = F + \text{vol}_{(4)} \wedge \lambda(*F).
\]

Here \(*\) denotes the Hodge star operator on \((M^6, g)\) and \(\lambda\) is given by

\[
\lambda(A_p) = (-1)^{\frac{p(p-1)}{2}} A_p
\]

for every \(p\)-form \(A_p\).

In order to find such solutions one has to solve the equations of motion and the Bianchi identities for the fluxes; however, for the class of supergravity backgrounds we are interested in, the equations of motion for the metric and the dilaton \(\phi\) are implied by the Bianchi identities and the ten-dimensional supersymmetry conditions [26], so one can solve the latter. These conditions are the annihilation of the supersymmetry variations of the gravitino \(\psi_\mu\) and the dilatino \(\lambda\) given by (see [17])

\[
\delta \psi_\mu = D_\mu \epsilon + \frac{1}{4} H_\mu \mathcal{P} \epsilon + \frac{1}{16} e^\phi \sum_n \mathcal{H}_{2n} \gamma_\mu \mathcal{P}_n \epsilon,
\]

\[
\delta \lambda = \left( \frac{1}{2} \phi + \frac{1}{24} m \mathcal{P} \right) \epsilon + \frac{1}{8} e^\phi \sum_n (-1)^{2n(5 - 2n)} \mathcal{H}_{2n} \mathcal{P}_n \epsilon,
\]

with \(H_\mu = \frac{1}{2} H_{\mu\nu\rho} \gamma^{\nu\rho}, H\) being the NSNS flux. For IIA, \(\mathcal{P} = \gamma_{11}\) and \(\mathcal{P}_n = \gamma_n^{11}\) for \(n = 0, \ldots, 5\), while for IIB, \(\mathcal{P} = -\sigma^3, \mathcal{P}_n = \sigma^n\) for \(n = \frac{1}{2}, \frac{3}{2}\) and \(\mathcal{P}_n = i\sigma^2\) for \(n = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}\).

The ten-dimensional supersymmetry parameter \(\epsilon\) can be written as a pair \((\epsilon^1, \epsilon^2)\) of two Majorana–Weyl supersymmetry parameters and, because of the product structure of the solution (1), there should exist independent globally defined and non-vanishing spinors \(\eta^j\) on \(M^6\) such that each \(\epsilon^j\) is given as

\[
\epsilon^j = \zeta^j \otimes \sum_a f^j_a \eta^j_a + \text{c.c.}, \quad j = 1, 2,
\]
where $\xi^1$ and $\xi^2$ are the four-dimensional supersymmetry parameters.

In order to get (at least) $N = 1$ supersymmetry, the existence of (at least) a pair $(\eta^1, \eta^2)$ of globally defined and non-vanishing spinors on the internal manifold $M^6$ satisfying the SUSY conditions is required. The existence of this pair of internal spinors generically implies that the structure group of the tangent bundle over the internal manifold $M^6$ is reduced to a subgroup $G \subset SO(6)$. This is due to the fact that the spinors which are globally defined must not transform under $G$ and therefore are singlets under the $SO(6) \to G$ decomposition. The pair $(\eta^1, \eta^2)$ can be parametrized and different types of $G$-structures are defined on the internal manifold depending on the values of the parameters. $SU(3)$ and intermediate $SU(2)$ structures on 6-manifolds arise naturally in this context as we recall next.

The existence of a globally defined non-vanishing spinor $\eta_+$ on a six-dimensional manifold $M^6$ defines a reduction of the structure group of the tangent bundle over $M^6$ to $SU(3)$. Therefore, on the internal manifold we have an almost Hermitian structure $(J, g)$ and a $(3, 0)$-form $\Psi$ such that
\[
F \wedge \Psi = 0, \quad \frac{4}{i} F^3 = i \Psi \wedge \overline{\Psi} \neq 0,
\]
where $F$ is the fundamental 2-form associated with $(J, g)$. The spinor $\eta_+$ is a Weyl spinor and it is supposed to have positive chirality and unit norm. Complex conjugation acts sending $\eta_+$ to $\eta_-$. The forms $(F, \Psi)$ can be obtained as bilinears of the globally defined spinor. Indeed:
\[
F_{\mu\nu} = -i \eta^+_1 \gamma_{\mu\nu} \eta_+, \quad \Psi_{\mu\nu\rho} = -i \eta^0_- \gamma_{\mu\nu\rho} \eta_+.
\]

An $SU(2)$ structure on a six-dimensional manifold $M^6$ is defined by two orthogonal globally defined spinors $\eta_+$ and $\chi_+$, which we can suppose of unit norm, or equivalently by an almost Hermitian structure $(J, g)$, a $(1, 0)$-form $\alpha$, a real 2-form $\omega$ and a $(2, 0)$-form $\Omega$ satisfying the following conditions
\[
\omega^2 = \frac{1}{2} \Omega \wedge \overline{\Omega} \neq 0, \quad \omega \wedge \Omega = 0, \quad \Omega \wedge \Omega = 0, \quad i_\alpha \Omega = 0, \quad i_\omega \omega = 0,
\]
where by $i_\alpha$ we denote the contraction by the vector field dual to $\alpha$ and we take $\alpha$ such that $\|\alpha\|^2 = i_\alpha^\alpha = \bar{\alpha} g^{\alpha \beta} \alpha_\beta = 2$.

The forms $(\alpha, \omega, \Omega)$ are related to the globally defined spinors $(\eta_+, \chi_+)$ by the relations
\[
\alpha_\mu = \eta^+_1 \gamma_\mu \chi_+, \quad \omega_{\mu\nu} = -i \eta^1_+ \gamma_{\mu\nu} \eta_+ + i \chi^1_+ \gamma_{\mu\nu} \chi_+, \quad \Omega_{\mu\nu} = \eta^0_- \gamma_{\mu\nu} \chi_-.
\]
The spinor $\chi_+$ can be rewritten in terms of $\eta_-$ as $\chi_+ = \frac{1}{2} \alpha \eta_-$. The $SU(2)$ structure is naturally embedded in the $SU(3)$ structure defined by $\eta_+$ by
\[
F = \omega + \frac{i}{2} \alpha \wedge \overline{\alpha}, \quad \Psi = \alpha \wedge \Omega.
\]
(2)

Conversely, if one has an $SU(3)$ structure $(F, \Psi)$ on $M^6$ and a $(1, 0)$-form $\alpha$ of norm $\sqrt{2}$, then it has been proved in [1, appendix A2] that $\omega$ and $\Omega$ defined by the formulas
\[
\omega = F - \frac{i}{2} \alpha \wedge \overline{\alpha}, \quad \Omega = \frac{1}{2} i_\alpha \Psi
\]
provide an $SU(2)$ structure.

Given a pair $(\eta^1_+, \eta^2_+)$ of globally defined non-vanishing internal spinors corresponding to the internal components of the supersymmetry parameters, one can parametrize them as
\[
\eta^1_+ = a \eta_+, \quad \eta^2_+ = b (k_1 \eta_+ + k_2 \frac{1}{2} \alpha \eta_-),
\]
(3)
with $0 \leq k| \leq 1$, $k_\perp = \sqrt{1 - k_\parallel^2}$ and $a$, $b$ non-zero complex numbers such that $a = \| \eta^+_1 \|$ and $b = \| \eta^-_2 \|$. As in [1], we consider $b = \bar{a}$ so that the relative phases of the spinors are fixed by $|a|$ and $\theta$, the latter given by $e^{i\theta} = \bar{a}/a$.

Now depending on the values of $k|_1$ and $k_\perp$ one can define starting from the spinors different types of $G$ structures. Indeed, if $k_\perp = 0$, or equivalently if $\eta^+_1$ and $\eta^-_2$ are parallel, then one has an $SU(3)$ structure. If $k_\perp \neq 0$ one has an $SU(2)$ structure and in the particular case when $k_\parallel = 1$ and $k_\perp = 0$ one gets the so-called static $SU(2)$ structure. But one can consider as in [1] the intermediate case $k_\perp \neq 0$ and $k_\parallel \neq 0$. The two orthogonal spinors $\eta_+$ and $\chi_+$ define an $SU(2)$ structure $(J, g, \alpha, \omega, \Omega)$ on the internal manifold $M^6$ as above and, relating the numbers $k_\parallel$ and $k_\perp$ to the angle $\phi \in \left[0, \frac{\pi}{2}\right]$ between the spinors by

$$k_\parallel = \cos(\phi), \quad k_\perp = \sin(\phi),$$

one obtains (see [13]) the family of $SU(3)$ structures on $M^6$ given by

$$F_\phi = \cos(2\phi)\omega + \frac{1}{2}\alpha \wedge \bar{\alpha} + \sin(2\phi)\text{Re}(\Omega),$$

$$\Psi_\phi = \alpha \wedge (\sin(2\phi)\omega + \cos(2\phi)\text{Re}(\Omega) + i\text{Im}(\Omega)),$$

or equivalently the family of $SU(2)$ structures

$$\tilde{\omega}_\phi = \cos(2\phi)\omega + \sin(2\phi)\text{Re}(\Omega),$$

$$\tilde{\Omega}_\phi = -\sin(2\phi)\omega + \cos(2\phi)\text{Re}(\Omega) + i\text{Im}(\Omega).$$

**Definition 2.1.** [1] The $SU(2)$ structure on $(M^6, J, g, \alpha, \omega, \Omega)$ defined by (4) is called intermediate if $k_\parallel$ and $k_\perp$ are both different from zero. It is called static (or orthogonal) if $k_\parallel = 1$ and $k_\perp = 0$.

We recall that an $SU(3)$ structure $(F, \Psi)$ is said to be half-flat if $d(F \wedge F) = 0$ and $d(\text{Re}(\Psi)) = 0$. If in addition $dF = 0$, then the $SU(3)$ structure is said to be symplectic half-flat.

**Definition 2.2.** An $SU(3)$ structure $(F, \Psi)$ on $M^6$ is called Hermitian balanced if $d(F \wedge F) = 0$ and $d\Psi = 0$.

From now on by a symplectic half-flat, resp. Hermitian balanced, $SU(2)$ structure $(J, g, \alpha, \omega, \Omega)$ we mean that the associated $SU(3)$ structure given by (2) is symplectic half-flat, resp. Hermitian balanced.

3. SUSY equations

In this section we show that supersymmetric flux vacua with intermediate $SU(2)$ structure are closely related to the existence of special classes of half-flat structures on the internal manifold. We begin by recalling the SUSY conditions derived by Andriot in [1].

To solve the SUSY conditions, rather than using Killing spinors methods or $G$-structures tools, in [1] the formalism of generalized complex geometry [19, 20, 25] was used. In the generalized complex geometry for a $d$-dimensional manifold $M$, one considers the bundle $TM \oplus T^*M$, whose sections are generalized vectors (sums of a vector and a 1-form). The spinors on $TM \oplus T^*M$ are Majorana–Weyl Cliff$(d, d)$ spinors, and locally they can be seen as polyforms, i.e. sums of even/odd differential forms, which correspond to positive/negative chirality spinors. A Cliff$(d, d)$ spinor is pure if it is annihilated by half of the Cliff$(d, d)$ gamma matrices. Such pure spinors can be obtained also as tensor products of Cliff$(d)$ spinors.
In the supergravity context, the Cliff(6, 6) pure spinors are defined as a biprodut
\[ \Phi_\pm = \eta_+^1 \otimes \eta_-^2 = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} (\eta_+^1 \gamma_{\mu_1 \ldots \mu_k} \eta_-^2) \gamma^\mu_{1 \ldots \mu_k}. \]
The explicit expressions of the two pure spinors in terms of the forms \( \alpha, \omega, \Omega \) are then
\[ \Phi_+ = \frac{|a|^2}{8} e^{i\varphi} e^{i\omega \pi} (k_{||} e^{-i\omega} - i k_{\perp} \Omega), \]
\[ \Phi_- = -\frac{|a|^2}{8} \alpha \wedge (k_{\perp} e^{-i\omega} + i k_{||} \Omega). \]
In general, from [20] a pure spinor \( \Phi \) can be written as
\[ \Phi = \Omega_k \wedge e^{B+iK}, \]
where \( \Omega_k \) is a holomorphic \( k \)-form, and \( B \) and \( K \) are real 2-forms. The rank \( k \) of the form \( \Omega_k \) is the type of the spinor. For the intermediate SU(2) structure where both \( k_{||} \) and \( k_{\perp} \) are different from zero, from [1], the two pure spinors \( \Phi_+ \) and \( \Phi_- \) can be rewritten as
\[ \Phi_+ = \frac{|a|^2}{8} e^{i\varphi} k_{||} e^{\frac{i}{2} \omega \pi - i\frac{\omega}{2} \eta_k}, \quad \Phi_- = -\frac{|a|^2}{8} k_{\perp} \alpha \wedge e^{-i\omega + i\frac{\omega}{2} \eta_k} \Omega \]
and thus \( \Phi_+ \) and \( \Phi_- \) have respectively type 0 and 1. In the case of the SU(3) structure (limit \( k_{\perp} = 0 \)),
\[ \Phi_+ = \frac{|a|^2}{8} e^{-i\varphi} e^{iF}, \quad \Phi_- = -i \frac{|a|^2}{8} \psi \]
and the two pure spinors are of types 0 and 3, respectively. In the case of a static SU(2) structure (the other limit \( k_{||} = 0 \)) one has
\[ \Phi_+ = -i \frac{|a|^2}{8} e^{-i\varphi} \Omega \wedge e^{\frac{i}{2} \omega \pi}, \quad \Phi_- = -\frac{|a|^2}{8} \alpha \wedge e^{-i\omega} \]
and the two pure spinors are of types 2 and 1, respectively.

Two pure spinors are said to be compatible if they have three common annihilators. A pair of compatible pure spinors defines an \( SU(3) \times SU(3) \) structure on \( TM \oplus T^* M \). Depending on the relation between the spinors \( \eta_+^1, \eta_-^2 \), this translates on \( TM \) into the \( SU(3) \), static SU(2) or intermediate SU(2) structures discussed above. So the formalism of generalized complex geometry allows a unified characterization of the topological properties that an \( N = 1 \) vacuum has to satisfy to be given: it must admit an \( SU(3) \times SU(3) \) structure on \( TM \oplus T^* M \). And so to satisfy this condition, one may verify that our vacua admit a pair of compatible pure spinors.

An \( N = 1 \) vacuum should satisfy the SUSY conditions, the equations of motion and the Bianchi identities for the fluxes. From [17, 18] the SUSY equations can be then written in terms of pure spinors by
\[ (d - H \wedge)(e^{2A - \varphi} \Phi_1) = 0, \]
\[ (d - H \wedge)(e^{2A - \varphi} \text{Re}(\Phi_2)) = 0, \]
\[ (d - H \wedge)(e^{2A - \varphi} \text{Im}(\Phi_2)) = \frac{e^{4A}}{8} \lambda(F), \]
with \( \Phi_1 = \Phi_\pm, \Phi_2 = \Phi_\mp \) for IIA/IIB (upper/lower), following the conventions of [19]. The first of these equations implies that one of the two pure spinors (the one with the same parity as the RR fields) must be twisted (because of the \(-H \wedge\) conformally closed. A manifold admitting a twisted closed pure spinor is a twisted generalized Calabi–Yau (see [19, 24]) and one looks for vacua on such manifolds.
The equations of motion of the fluxes are
\[
(d + H \wedge)(e^{A} \ast F) = 0, \quad d(e^{A - 2\theta} \ast H) = \mp e^{4A} \sum F_{\mu} \wedge \ast F_{\mu + 2}.
\]
with the upper/lower sign for IIA/IIB.

The Bianchi identities (we assume no NS source) are
\[
(d - H \wedge)F = \delta(\text{source}), \quad dH = 0,
\]
where \(\delta(\text{source})\) is the charge density of the allowed sources: space-filling \(D\)-branes or orientifold planes (\(O\)-planes). In compactification to four-dimensional Minkowski, the trace of the energy–momentum tensor must be zero. Then \(O\)-planes are needed since they are the only known sources with a negative charge that can thus cancel the flux contribution to this trace. As in [1] the RR Bianchi identities are then assumed to be
\[
(d - H \wedge)F = \sum Q^{i} V^{i},
\]
where \(Q^{i}\) is the source charge and \(V^{i}\) is (up to a sign) its internal co-volume (the co-volume of the cycle wrapped by the source). The sign of the \(Q^{i}\) indicates whether the source is a \(D\)-brane \((Q^{i} > 0)\) or an \(O\)-plane \((Q^{i} < 0)\).

For intermediate \(SU(2)\) structures (for which \(\frac{k_{\perp}}{k_{\parallel}}\) is constant) in the large volume limit from the SUSY conditions one gets that the \(H\) Bianchi identities are automatically satisfied. Furthermore, for this class of compactifications, it was shown in [19] that the equations of motion for the RR fluxes are implied by the SUSY conditions. And it was shown in [26] that the equation of motion of \(H\) is implied by the SUSY conditions and the Bianchi identities. Then, in order to find a solution, having a pair of compatible pure spinors on a twisted generalized Calabi–Yau manifold with at least one \(O\)-plane, as in [1], one has to verify that the SUSY conditions and the RR Bianchi identities are satisfied.

The presence of \(O\)-planes implies that the solution has to be invariant under the action of the orientifold. As shown in [26] the first step to derive the orientifold projection on the pure spinors is to compute those for the internal SUSY parameters. This can be done starting from the orientifold plane. Starting from the projections on the internal spinors \(\epsilon^{i}\) and then reducing to the internal spinors \(\eta_{\alpha}^{i}\). In our conventions, we have
\[
\begin{align*}
O5 : \ & \sigma(\eta_{\parallel}^{1}) = \eta_{\perp}^{1}, \quad \sigma(\eta_{\perp}^{1}) = \eta_{\parallel}^{1}, \\
O6 : \ & \sigma(\eta_{\parallel}^{2}) = \eta_{\perp}^{2}, \quad \sigma(\eta_{\perp}^{2}) = \eta_{\parallel}^{2},
\end{align*}
\]
where \(\sigma\) is the target space reflection in the directions transverse to the \(O\)-plane. Using the expressions (3) for the internal spinors, one gets as in [1] the following projection conditions at the orientifold plane:
\[
\begin{align*}
O5 : \ & e^{\omega \bar{\alpha}} = \pm 1, \quad \alpha \perp O5, \\
O6 : \ & e^{\omega \bar{\alpha}} \text{free,} \quad \text{Re}(\alpha) \parallel O6, \quad \text{Im}(\alpha) \perp O6.
\end{align*}
\]
The previous conditions can be expressed on \(\alpha\) as
\[
O5 : \ \sigma(\alpha) = -\alpha, \quad O6 : \ \sigma(\alpha) = \bar{\alpha}.
\]
By [26] if the \(G\)-structures are constant (as the one which are considering), and if we work on nil/solvmanifolds (which will be our case), these conditions are valid everywhere (not only at the orientifold plane). Starting from the projections on the \(\eta_{\alpha}^{i}\), as in [1, 26], one may derive the projections of the pure spinors \(\Phi_{\pm}\) and from them those for the \(SU(2)\) structure forms. In particular one has
\[
\begin{align*}
O5 : \ & \sigma(\omega) = \left( k_{\parallel}^{2} - k_{\perp}^{2} \right) \omega + 2k_{\parallel}k_{\perp} \text{Re}(\Omega), \quad \sigma(\Omega) = -k_{\parallel}^{2} \Omega + k_{\perp}^{2} \overline{\Omega} + 2k_{\parallel}k_{\perp} \omega, \\
O6 : \ & \sigma(\omega) = \left( k_{\perp}^{2} - k_{\parallel}^{2} \right) \omega - 2k_{\parallel}k_{\perp} \text{Re}(\Omega), \quad \sigma(\Omega) = -k_{\parallel}^{2} \Omega + k_{\perp}^{2} \overline{\Omega} - 2k_{\parallel}k_{\perp} \omega.
\end{align*}
\]
By introducing as in [26]
\[ O5 : k_\parallel = \cos(\phi), \quad k_\perp = \sin(\phi), \quad 0 \leq \phi \leq \frac{\pi}{2}, \]
\[ O6 : k_\parallel = \cos\left(\phi + \frac{\pi}{2}\right) = -\sin(\phi), \quad k_\perp = \sin\left(\phi + \frac{\pi}{2}\right) = \cos(\phi), \quad -\frac{\pi}{2} \leq \phi \leq 0, \]
one gets in both cases the following formulas:
\[ \sigma(\omega) = \cos(2\phi)\omega + \sin(2\phi)\Re(\Omega), \]
\[ \sigma(\Re(\Omega)) = \sin(2\phi)\omega - \cos(2\phi)\Re(\Omega), \]
\[ \sigma(\Im(\Omega)) = -\Im(\Omega). \]

Since the previous projection conditions are not very tractable, in [1] he worked in the projection (eigen)basis
\[ \omega_\parallel = \frac{1}{2}(\omega + \sigma(\omega)), \quad \omega_\perp = \frac{1}{2}(\omega - \sigma(\omega)), \]
\[ \Re(\Omega)_\parallel = \frac{1}{2}(\Re(\Omega) + \sigma(\Re(\Omega))), \quad \Re(\Omega)_\perp = \frac{1}{2}(\Re(\Omega) - \sigma(\Re(\Omega))), \]
(5)
which can then be expressed in terms of the original SU(2)-structure as
\[ \omega_\parallel = \frac{1}{2}((1 + \cos(2\phi))\omega + \sin(2\phi)\Re(\Omega)), \]
\[ \omega_\perp = \frac{1}{2}((1 - \cos(2\phi))\omega - \sin(2\phi)\Re(\Omega)), \]
\[ \Re(\Omega)_\parallel = \frac{1}{2}((1 - \cos(2\phi))\Re(\Omega) + \sin(2\phi)\omega), \]
\[ \Re(\Omega)_\perp = \frac{1}{2}((1 + \cos(2\phi))\Re(\Omega) - \sin(2\phi)\omega). \]

As in [1] one takes \( e^A = |a|^2 = 1 \) and go to the large volume limit, i.e. \( A = 0 \) and \( e^\phi = g \), constant. This is indeed the regime in which one will look for solutions. The only remaining freedom is \( \theta \) that we do not really need to fix. Moreover, we choose to look only for intermediate SU(2) structures, i.e. with \( k_\parallel \neq 0 \) and \( k_\perp \neq 0 \) constant. Taking the coefficients constant is important because it simplifies drastically the search for solutions and the SUSY conditions are much simpler. In fact, by using the projection (eigen)basis (5) and the results in [17, 18], together with further simplications as explained in [1], one can rewrite the SUSY equations in the following form.

- **SUSY equations IIA:**

\[
\begin{align*}
    d(\Re(\alpha)) &= 0, \\
    k_\parallel H &= k_\perp d(\Im(\Omega)), \\
    d(\Re(\Omega)_\perp) &= k_\parallel k_\perp \Re(\alpha) \wedge d(\Im(\alpha)), \\
    H \wedge \Re(\alpha) &= -\frac{1}{k_\parallel} d(\Im(\alpha) \wedge \Re(\Omega)_\parallel),
\end{align*}
\]
(6)

Together with \( F_0, F_2 \) and \( F_4 \) given by
\[
\begin{align*}
    g_s * F_0 &= \frac{1}{2} k_\parallel d(\Im(\alpha)) \wedge (\Im(\Omega))^2 + \frac{1}{k_\parallel} H \wedge \Re(\alpha) \wedge \Re(\Omega)_\parallel, \\
    g_s * F_2 &= k_\parallel d(\Im(\alpha)) \wedge \Im(\Omega) + \frac{1}{k_\parallel} d(\Re(\Omega)_\parallel) \wedge \Re(\alpha), \\
    g_s * F_4 &= -k_\perp d(\Im(\alpha)).
\end{align*}
\]
\[ d(\text{Re}(\alpha)) = 0, \]
\[ d(\text{Im}(\alpha)) = 0, \]
\[ k_\parallel H = k_\perp d(\text{Im}(\Omega)), \]
\[ \text{Re}(\alpha) \wedge H = -\frac{k_\perp}{k_\parallel} \text{Im}(\alpha) \wedge d(\text{Re}(\Omega)_\perp), \]
\[ \text{Im}(\alpha) \wedge H = k_\parallel \text{Re}(\alpha) \wedge d(\text{Re}(\Omega)_\perp), \]
\[ \text{Re}(\alpha) \wedge \text{Im}(\alpha) \wedge d(\text{Re}(\Omega)_\parallel) = -H \wedge \text{Im}(\Omega), \]

together with \( F_1 \) and \( F_\parallel \) given by
\[ k_\perp e^{i\theta} g_\times \ast F_1 = H \wedge \text{Re}(\Omega)_\parallel, \]
\[ k_\parallel e^{i\theta} g_\times \ast F_\parallel = d(\text{Re}(\Omega)_\parallel). \]

Note that \( \left( \frac{1}{\cos \phi} \text{Re}(\Omega)_\perp, \frac{1}{\sin \phi} \text{Re}(\Omega)_\parallel, \text{Im}(\Omega), \alpha \right) \) define a new \( SU(2) \) structure \((\alpha, \omega', \Omega')\) on \( M \) with
\[ \omega' = \frac{1}{\sin \phi} \text{Re}(\Omega)_\parallel, \quad \Omega' = \frac{1}{\cos \phi} \text{Re}(\Omega)_\perp + i \text{Im}(\Omega), \]
and then a new \( SU(3) \) structure \((F', \Psi')\). Moreover, since we will consider only \( O6 \) planes in IIA and \( O5 \) planes in IIB, by using a local adapted basis for this \( SU(2) \)-structure (see [12]), one has that \((\hat{\alpha}, \hat{\omega}, \hat{\Omega})\) given by
\[ \hat{\alpha} = \text{Re}(\alpha) + i k_\parallel \text{Im}(\alpha), \quad \hat{\omega} = \frac{1}{k_\perp} \text{Re}(\Omega)_\perp, \quad \hat{\Omega} = \text{Im}(\Omega) - i \frac{1}{k_\parallel} \text{Re}(\Omega)_\parallel, \]
is also an \( SU(2) \) structure on \( M \) in the IIA case, and
\[ \hat{\alpha}_1 = k_\parallel \text{Re}(\alpha) + i \text{Im}(\alpha), \quad \hat{\omega} = \frac{1}{k_\perp} \text{Re}(\Omega)_\parallel, \quad \hat{\Omega} = \frac{1}{k_\parallel} \text{Re}(\Omega)_\perp + i \text{Im}(\Omega), \]
\[ \hat{\alpha}_2 = k_\parallel \text{Im}(\alpha) - i \text{Re}(\alpha), \quad \hat{\omega} = \frac{1}{k_\perp} \text{Re}(\Omega)_\parallel, \quad \hat{\Omega} = \frac{1}{k_\parallel} \text{Re}(\Omega)_\perp + i \text{Im}(\Omega) \]
are \( SU(2) \) structures on \( M \) in the IIB case. We will use these structures in the next theorems, and the corresponding \( SU(3) \) structures will be denoted by \((\hat{F}, \hat{\Psi})\). Note that the almost complex structure \( J \) and the metric \( \hat{g} \) change with respect to those given by \((\alpha, \omega', \Omega')\).

**Theorem 3.1.** Let \((M^6, J, g, \alpha, \omega, \Omega)\) be a six-dimensional manifold endowed with an \( SU(2) \) structure such that the 2-forms \( \text{Re}(\Omega)_\parallel, \text{Re}(\Omega)_\perp, \text{Im}(\Omega) \) satisfies equations (6); then, \( M^6 \) admits a symplectic half-flat \( SU(2) \) structure \((J, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})\) with \( d(\text{Re}(\hat{\alpha})) = 0 \). Conversely, if \( M^6 \) has a symplectic half-flat \( SU(2) \) structure \((J, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})\) such that \( d(\text{Re}(\hat{\alpha})) = 0 \), then the forms \( (\text{Re}(\Omega)_\parallel, \text{Re}(\Omega)_\perp, \text{Im}(\Omega), \alpha) \) defined by
\[ \frac{1}{k_\perp} \text{Re}(\Omega)_\perp = \hat{\omega}, \quad \text{Im}(\Omega) - i \frac{1}{k_\parallel} \text{Re}(\Omega)_\parallel = \hat{\Omega}, \quad \text{Re}(\alpha) + i k_\parallel \text{Im}(\alpha) = \hat{\alpha} \]
are a solution of equations (6).

**Proof.** For type IIA the 2-forms \( \frac{1}{k_\perp} \text{Re}(\Omega)_\perp, \frac{1}{k_\parallel} \text{Re}(\Omega)_\parallel, \text{Im}(\Omega) \) together with the complex 1-form \( \text{Re}(\alpha) + i k_\parallel \text{Im}(\alpha) \) define a new \( SU(2) \) structure with
\[ \hat{\omega} = \frac{1}{k_\perp} \text{Re}(\Omega)_\perp, \quad \hat{\Omega} = \text{Im}(\Omega) - i \frac{1}{k_\parallel} \text{Re}(\Omega)_\parallel, \quad \hat{\alpha} = \text{Re}(\alpha) + i k_\parallel \text{Im}(\alpha) \]
and then a new $SU(3)$ structure $(\hat{J}, \hat{F}, \hat{\Psi})$, with

$$\hat{F} = \hat{\omega} + k_1 \text{Re}(\alpha) \wedge \text{Im}(\alpha), \quad \hat{\Psi} = \hat{\alpha} \wedge \hat{\Omega}.$$

By the second equation of (6) we have

$$H = \frac{k_\perp}{k_\parallel} d(\text{Im}(\Omega)).$$

Then by the last equation of (6) we obtain

$$d(\text{Im}(\Omega)) \wedge \text{Re}(\alpha) = -d(\text{Im}(\alpha) \wedge \text{Re}(\Omega)_\parallel),$$

i.e. that the real part of the form $\hat{\Psi} = \hat{\alpha} \wedge \hat{\Omega}$ is closed.

Moreover, by

$$d(\text{Re}(\Omega)_\perp) = k_\parallel k_\perp \text{Re}(\alpha) \wedge d(\text{Im}(\alpha)),$$

it follows that

$$d(\text{Re}(\Omega)_\perp + k_\parallel k_\perp \text{Re}(\alpha) \wedge \text{Re}(\Omega)) = k_\perp d(\hat{F}) = 0,$$

and so we have a symplectic half-flat $SU(2)$ structure on $M^6$. Conversely, if $M^6$ has a symplectic half-flat $SU(2)$ structure $(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ such that $d(\text{Re}(\hat{\alpha})) = 0$, we have that the fundamental form

$$\hat{F} = \hat{\omega} + \text{Re}(\hat{\alpha}) \wedge \text{Im}(\hat{\alpha}) = \frac{1}{k_\perp} \text{Re}(\Omega)_\perp + k_\parallel \text{Re}(\alpha) \wedge \text{Im}(\alpha)$$

is closed and thus the equation

$$d(\text{Re}(\Omega)_\perp) = k_\parallel k_\perp \text{Re}(\alpha) \wedge d(\text{Im}(\alpha))$$

in (6) holds. By the closedness of the real part of the $(3,0)$-form $\hat{\Psi} = \hat{\alpha} \wedge \hat{\Omega}$ we have that also the last equation in (6) is satisfied for $H = \frac{k_\perp}{k_\parallel} d(\text{Im}(\Omega))$. □

**Theorem 3.2.** Let $(M^6, J, g, \alpha, \omega, \Omega)$ be a six-dimensional manifold endowed with an $SU(2)$ structure such that the 2-forms $\text{Re}(\Omega)_\parallel, \text{Re}(\Omega)_\perp, \text{Im}(\Omega)$ satisfy equations (7); then, $M^6$ admits two half-flat $SU(2)$ structures $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ and $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2, \hat{\omega}, \hat{\Omega})$ such that $d(\hat{\alpha}_1) = 0$ and $\hat{\alpha}_2 = k_\parallel \text{Im}(\hat{\alpha}_1) - i \frac{\text{Re}(\hat{\alpha}_1)}{k_\parallel}, \hat{\omega}, \hat{\Omega}$ is half-flat, then the forms $(\text{Re}(\Omega)_\parallel, \text{Re}(\Omega)_\perp, \text{Im}(\Omega), \alpha)$ defined by

$$\frac{1}{k_\perp} \text{Re}(\Omega)_\parallel = \hat{\omega}, \quad \frac{1}{k_\parallel} \text{Re}(\Omega)_\perp + i \text{Im}(\Omega) = \hat{\Omega}, \quad \text{Im}(\alpha) = \hat{\alpha}_1,$$

where $k_\perp = \sqrt{1 - k_\parallel^2}$, are a solution of equations (7).

**Proof.** As we already remarked previously for type IIB, the 2-forms $\frac{1}{k_\parallel} \text{Re}(\Omega)_\perp, \frac{1}{k_\perp} \text{Re}(\Omega)_\parallel, \text{Im}(\Omega), \alpha$ define a new $SU(2)$ structure with

$$\hat{\omega} = \frac{1}{k_\perp} \text{Re}(\Omega)_\parallel, \quad \hat{\Omega} = \frac{1}{k_\parallel} \text{Re}(\Omega)_\perp + i \text{Im}(\Omega), \quad \hat{\alpha}_1 = k_\parallel \text{Re}(\alpha) + i \text{Im}(\alpha)$$

and then a new $SU(3)$ structure $(\hat{J}_1, \hat{F}, \hat{\Psi}_1)$, with

$$\hat{F} = \hat{\omega} + k_1 \text{Re}(\alpha) \wedge \text{Im}(\alpha), \quad \hat{\Psi}_1 = \hat{\alpha}_1 \wedge \hat{\Omega}.$$
Suppose that the 2-forms $\Re(\Omega)_\parallel$, $\Re(\Omega)_\perp$, $\Im(\Omega)$ satisfy equations (7), then by the first five equations we have

$$d(\alpha \wedge (\Re(\Omega)_\perp + i \Im(\Omega))) = 0.$$ 

Then, $d(\Re(\bar{\alpha} \wedge \bar{\Omega})) = 0$.

Since $(\bar{\alpha}, \bar{\omega}, \bar{\Omega})$ is an $SU(2)$ structure, the condition

$$\Im(\Omega)^2 = \frac{1}{k^2_\perp} \Re(\Omega)^2_\parallel$$

is satisfied [12] and by the last equation of (7) we get

$$\Re(\alpha) \wedge \Im(\alpha) \wedge d(\Re(\Omega)_\parallel) = -\frac{k^2_\perp}{k^2_\parallel} d(\Im(\Omega)) \wedge \Im(\Omega)$$

$$= -\frac{1}{k^2_\perp k^2_\parallel} d(\Re(\Omega)_\parallel) \wedge \Re(\Omega)_\parallel.$$ 

Therefore,

$$d((k^2_\perp k^2_\parallel \Re(\alpha) \wedge \Im(\alpha) + \Re(\Omega)_\parallel)^2) = 0,$$

i.e. $d(\hat{F} \wedge \hat{F}) = 0$. Then we have a half-flat $SU(2)$ structure.

Consider

$$\hat{\alpha}_2 = k^\parallel \Im(\alpha) - i \Re(\alpha)$$

and define $\hat{\Psi}_2 = \hat{\alpha}_2 \wedge \hat{\Omega}$. We have two $SU(3)$ structures $(\hat{F}, \hat{\Psi}_1)$ and $(\hat{F}, \hat{\Psi}_2)$. Indeed,

$$\Re(\hat{\alpha}_1) \wedge \Im(\hat{\alpha}_1) = \Re(\hat{\alpha}_2) \wedge \Im(\hat{\alpha}_2) = k^\parallel \Re(\alpha) \wedge \Im(\alpha),$$

so $\hat{F}$ is the same in both cases.

Now, equation

$$\Im(\alpha) \wedge H = \frac{k^\perp}{k^\parallel} \Re(\alpha) \wedge d(\Re(\Omega)_\perp)$$

implies that $d(\Re(\hat{\Psi}_1)) = 0$, whereas equation

$$\Re(\alpha) \wedge H = -\frac{k^\perp}{k^\parallel} \Im(\alpha) \wedge d(\Re(\Omega)_\perp)$$

implies that $d(\Re(\hat{\Psi}_2)) = 0$. In conclusion we have that $(\hat{F}, \hat{\Psi}_1)$ and $(\hat{F}, \hat{\Psi}_2)$ are half-flat.

To prove the converse, we first note that the fundamental form $\hat{F}_1$ is given by

$$\hat{F}_1 = \bar{\omega} + \Re(\hat{\alpha}_1) \wedge \Im(\hat{\alpha}_1) = \frac{1}{k^\parallel} \Re(\Omega)_\parallel + k^\parallel \Re(\alpha) \wedge \Im(\alpha),$$

and therefore the closedness of the 4-form $\hat{F}_1 \wedge \hat{F}_1$ implies the last equation of (7) for $H = \frac{k^\perp}{k^\parallel} d(\Im(\Omega))$.

Let us consider the complex 3-form $\hat{\Psi}_j = \hat{\alpha}_j \wedge \hat{\Omega}$, $j = 1, 2$. Since $\Re(\hat{\Psi}_j) = \Re(\hat{\alpha}_j) \wedge \Re(\Omega) - \Im(\hat{\alpha}_j) \wedge \Im(\Omega)$, we get that

$$d(\Re(\hat{\Psi}_j)) = \frac{k^\parallel}{k^\perp} \Im(\hat{\alpha}_j) \wedge H - \frac{1}{k^\parallel} \Re(\hat{\alpha}_j) \wedge d(\Re(\Omega)_\perp).$$

For $j = 1$ we get the equation $\Im(\alpha) \wedge H = \frac{k^\parallel}{k^\perp} \Re(\alpha) \wedge d(\Re(\Omega)_\perp)$, and for $j = 2$ we get $\Re(\alpha) \wedge H = -\frac{k^\parallel}{k^\perp} \Im(\alpha) \wedge d(\Re(\Omega)_\perp)$, so equations (7) are satisfied. 

**Remark 3.3.** Note that in the conditions of theorem 3.2 we can define a 1-parametric family of half-flat $SU(2)$ structures connecting the two given structures. In fact, by considering the usual rotation

$$\hat{\alpha}_t = k^\parallel \sin t \Re(\alpha) + k^\parallel \cos t \Im(\alpha) + i \sin t \Im(\alpha) - i \cos t \Re(\alpha),$$
we have that \((\hat{F}, \hat{\alpha}, \hat{\Omega})\) is half-flat for any value of \(t\). Note that the fundamental 2-form \(\hat{F}\) does not depend on \(t\) because \(\text{Re}(\hat{\alpha}) \wedge \text{Im}(\hat{\alpha}) = k_{||} \text{Re}(\alpha) \wedge \text{Im}(\alpha)\). The almost complex structure \(\hat{J}_t\) is given with respect to the basis \(\{\text{Re}(\alpha), \text{Im}(\alpha)\}\)

\[
\hat{J}_t = \begin{pmatrix}
\sin t \cos \left(\frac{1}{k_{||}} - k_{||}\right) & k_{||} \sin^2 t + \cos^2 t \\
-\sin^2 t - k_{||} \cos^2 t & -\sin t \cos \left(\frac{1}{k_{||}} - k_{||}\right)
\end{pmatrix}.
\]

Next, using the characterization given in theorem 3.2, we show that if there is a solution of the SUSY equations IIB for any \(k_{||} \in (0, 1)\), then the manifold must be Hermitian balanced.

**Proposition 3.4.** Let \((J, g, \alpha, \omega, \Omega)\) be a half-flat \(SU(2)\) structure on a 6-manifold \(M^6\) such that \(d(\alpha) = 0\) and for each \(\lambda \in (0, 1)\) the \(SU(2)\) structure \((J_\lambda, g_\lambda, \beta_\lambda = \lambda \text{Im}(\alpha) - i \frac{\text{Re}(\alpha)}{\lambda}, \omega, \Omega)\) is half-flat. Then, \((J, g, \alpha, \omega, \Omega)\) is Hermitian balanced.

**Proof.** Let us consider the \(SU(3)\) structure \((F, \Psi)\) given by \(F = \omega + \frac{1}{2} \alpha \wedge \Omega\) and \(\Psi = \alpha \wedge \Omega\). According to definition 2.2 we have to prove that \(\text{Im}(\Psi)\) is a closed form. Let \((F, \Phi_\lambda)\) be the \(SU(3)\) structure associated with \((J_\lambda, g_\lambda, \beta_\lambda, \omega, \Omega)\). Then, the real and imaginary parts of the forms \(\Psi\) and \(\Phi_\lambda\) are given by

\[
\text{Re}(\Psi) = \text{Re}(\alpha) \wedge \text{Re}(\Omega) - \text{Im}(\alpha) \wedge \text{Im}(\Omega),
\]
\[
\text{Im}(\Psi) = \text{Re}(\alpha) \wedge \text{Im}(\Omega) + \text{Im}(\alpha) \wedge \text{Re}(\Omega),
\]
\[
\text{Re}(\Phi_\lambda) = \frac{\text{Re}(\alpha)}{\lambda} \wedge \text{Im}(\Omega) + \lambda \text{Im}(\alpha) \wedge \text{Re}(\Omega),
\]
\[
\text{Im}(\Phi_\lambda) = -\frac{\text{Re}(\alpha)}{\lambda} \wedge \text{Re}(\Omega) + \lambda \text{Im}(\alpha) \wedge \text{Im}(\Omega).\]

The limit of the 3-form \(\Phi_\lambda\) exists when \(\lambda \to 1\) and equals \(-i \Psi\). Since \(\text{Re}(\Phi_\lambda)\) is closed for any \(\lambda \in (0, 1)\), we conclude that \(\text{Im}(\Psi)\) is closed. Therefore, the \(SU(2)\) structure \((J, g, \alpha, \omega, \Omega)\) is Hermitian balanced. \(\square\)

Given a Hermitian balanced \(SU(2)\) structure \((J, g, \alpha, \omega, \Omega)\) such that \(d(\alpha) = 0\) and \(\text{Re}(\alpha) \wedge d(\text{Re}(\Omega)) = 0\), one can construct a solution of the SUSY equations IIB for any \(k_{||} \in (0, 1)\).

**Proposition 3.5.** Let \((J, g, \alpha, \omega, \Omega)\) be a Hermitian balanced \(SU(2)\) structure on a 6-manifold \(M^6\) such that \(d(\alpha) = 0\) and \(\text{Re}(\alpha) \wedge d(\text{Re}(\Omega)) = 0\); then, for each \(\lambda \in (0, 1)\) the \(SU(2)\) structure \((J_\lambda, g_\lambda, \beta_\lambda = \lambda \text{Im}(\alpha) - i \frac{\text{Re}(\alpha)}{\lambda}, \omega, i\Omega)\) is half-flat.

**Proof.** We have \(\Psi_\lambda = (\lambda \text{Im}(\alpha) - i \frac{\text{Re}(\alpha)}{\lambda}) \wedge (i \text{Re}(\Omega) - \text{Im}(\Omega))\) and thus
\[
\text{Re}(\Psi_\lambda) = -\lambda \text{Im}(\alpha) \wedge \text{Im}(\Omega) + \frac{\text{Re}(\alpha)}{\lambda} \wedge \text{Re}(\Omega),
\]
\[
\text{Im}(\Psi_\lambda) = \lambda \text{Im}(\alpha) \wedge \text{Re}(\Omega) + \frac{\text{Re}(\alpha)}{\lambda} \wedge \text{Im}(\Omega).
\]
By the assumptions on the \(SU(2)\) structure \((J, g, \alpha, \omega, \Omega)\) we get in particular that
\[
\text{Re}(\alpha) \wedge d(\text{Re}(\Omega)) = \text{Im}(\alpha) \wedge d(\text{Im}(\Omega)) = 0
\]
and therefore that \(d(\text{Re}(\Psi_\lambda)) = 0\). \(\square\)

**Example 3.6.** Let us consider a nilmanifold corresponding to the nilpotent Lie algebra \(\mathfrak{h}_5 = (0, 0, 0, 0, 12, 14 + 23)\), that is, there is a basis \(\{e^1, \ldots, e^6\}\) satisfying
\[
d e^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{12}, \quad de^6 = e^{14} + e^{23}.
\]
We consider the structure \((F, \Psi_1)\) given by the 2-form \(F = e^{13} + e^{24} - e^{56}\) and the (3,0)-form \(\Psi_1 = (e^3 + i e^4)(e^5 + i e^6)\). Although \(\Psi_1\) admits Hermitian balanced structures, the previous structure is only half-flat. In fact, \(F^2\) and \(Re(\Psi)\) are closed, but \(d(Im(\Psi)) = -e^{1234}\).

For the complex 3-form \(\Phi_\lambda = (\lambda e^3 - i \frac{\lambda}{2})(e^5 + i e^6)(e^6 + i e^5)\), a direct calculation shows that

\[
d(Re(\Phi_\lambda)) = \frac{1}{\lambda}d(e^{125} + e^{146}) + \lambda d(e^{326} - e^{345}) = \frac{2\lambda^2 - 1}{\lambda}e^{1234},
\]

which implies that \(Re(\Phi_\lambda)\) is closed only for \(\lambda = \pm \frac{1}{\sqrt{2}}\). Note that \(Im(\Phi_\lambda)\) is closed for any \(\lambda\), so \((F, \Phi_{\pm \frac{1}{\sqrt{2}}}^-)\) are Hermitian balanced and \((F, \Phi_{\mp \frac{1}{\sqrt{2}}}^+)\) provides a solution to equations (7). In fact, by theorem 3.2 we have the following explicit solution \((\alpha, Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega))\) of the SUSY equations IIB for \(k|| = k_\perp = \frac{1}{\sqrt{2}}\):

\[
Re(\alpha) = \sqrt{2}e^1, \quad Im(\alpha) = e^3, \quad Re(\Omega)_{||} = \frac{1}{\sqrt{2}}(e^{24} - e^{56}), \quad Im(\Omega) = e^{25} + e^{46}.
\]

Note that the fluxes are

\[
H = -e^{1234}, \quad e^{i\theta}g_{i} \ast F_3 = -e^{126} + e^{145} + e^{235}, \quad e^{i\theta}g_{i} \ast F_1 = e^{23456}.
\]

Now, let us start from the (Hermitian balanced) structure \(\Phi_\lambda\) for \(\lambda = -\frac{1}{\sqrt{2}}\), that is,

\[
\Phi_{\frac{1}{\sqrt{2}}} = \left( -\frac{e^3}{\sqrt{2}} + i e^4 \right)(e^5 + i e^6)(e^6 + i e^5),
\]

and consider the complex 3-form \(\Theta_\mu\) given by

\[
\Theta_\mu = \left( \sqrt{2}\mu e^1 + i \frac{e^3}{\sqrt{2}\mu} \right)(e^5 + i e^6)(e^6 + i e^5).
\]

It is easy to check that \(Re(\Theta_\mu)\) is closed for any value of \(\mu\). Note that in particular \(\Theta_{\frac{1}{\sqrt{2}}} = \Psi\), i.e. one member in the family is precisely the half-flat structure \((F, \Psi)\) given at the beginning. Again by theorem 3.2 we have the following solution \((\alpha, Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega))\) of the SUSY equations IIB for any \(k|| = k_\perp = \mu \in (0, 1)\):

\[
Re(\alpha) = \frac{1}{\sqrt{2}k_{||}}e^3, \quad Im(\alpha) = \sqrt{2}e^1, \quad Re(\Omega)_{||} = k_\perp(e^{24} - e^{56}), \quad Im(\Omega) = e^{25} + e^{46},
\]

where \(k_\perp = \sqrt{1 - k_{||}^2}\). The fluxes are

\[
H = -\frac{k_\perp}{k_{||}}e^{1234}, \quad e^{i\theta}g_{i} \ast F_3 = -e^{126} + e^{145} + e^{235}, \quad e^{i\theta}g_{i} \ast F_1 = \frac{k_\perp}{k_{||}}e^{23456}.
\]

From theorem 3.1 it follows that a compact 6-manifold \(M^6\) admitting a solution to equations (6) satisfies those topological restrictions imposed by the existence of a symplectic form; in particular, the Betti numbers \(b_2(M^6)\) and \(b_3(M^6)\) do not vanish. Next we prove that \(b_1(M^6) \geq 2\) for any compact manifold \(M^6\) admitting solution to (7). In examples 4.2.1 and 4.2.2 we show that this lower bound can be attained.

**Proposition 3.7.** Let \((M^6, J, g, \alpha, \omega, \Omega)\) be a six-dimensional compact manifold endowed with an SU(2) structure such that the 2-forms \(Re(\Omega)_{||}, Re(\Omega)_{\perp}, Im(\Omega)\) satisfy equations (7);
then, $M^6$ has the first Betti number $\geq 2$. In particular, there is no solution on compact simply connected six-dimensional manifolds.

**Proof.** From equations (7) we can prove that the closed 1-forms $\text{Re}(\alpha)$ and $\text{Im}(\alpha)$ are harmonic with respect to $g$. In fact, the 5-form $\star \text{Re}(\alpha)$ is closed because it is a (constant) multiple of $\text{Im}(\alpha) \wedge (\text{Im}(\Omega))^2$, which is closed by the last equation of (7), taking into account the value of $H$. Similarly, the 5-form $\star \text{Im}(\alpha)$ is also closed. \hfill $\square$

Let us remember that a Riemannian manifold $(N, g)$ is called hyperkähler if there are three complex structures, $I, J, K$ on $N$ satisfying the quaternion relations
\[ I^2 = J^2 = K^2 = -1, \quad IJ = K = -JI, \]
and such that $I, J, K$ are parallel. In particular, we have three Kähler forms $\omega_I, \omega_J$ and $\omega_K$ on $N$.

**Proposition 3.8.** Let $(N^4, I, J, K)$ be a compact four-dimensional hyperkähler manifold. Then, on the six-dimensional manifold $M^6 = N^4 \times \mathbb{T}^2$ there exist solutions to the SUSY equations IIA and IIB. More precisely, if $\beta^1, \beta^2$ is a basis of 1-forms on the torus $\mathbb{T}^2$, then $(\alpha, \text{Re}(\Omega)||, \text{Re}(\Omega)|\perp, \text{Im}(\Omega))$ given by
\[
\text{Re}(\alpha) = \beta^1, \quad \text{Im}(\alpha) = \frac{1}{k||} \beta^2, \quad \text{Re}(\Omega)|| = -k|| \omega_I, \quad \text{Re}(\Omega)|\perp = k|\perp \omega_K, \quad \text{Im}(\Omega) = \omega_I
\]
solve equations (6), and $(\alpha, \text{Re}(\Omega)||, \text{Re}(\Omega)|\perp, \text{Im}(\Omega))$ given by
\[
\text{Re}(\alpha) = \frac{1}{k||} \beta^1, \quad \text{Im}(\alpha) = \beta^2, \quad \text{Re}(\Omega)|| = k|| \omega_K, \quad \text{Re}(\Omega)|\perp = k|\perp \omega_I, \quad \text{Im}(\Omega) = \omega_J,
\]
are solutions to equations (7).

**Proof.** Let us consider $\hat{\alpha}_1 = \beta^1 + i \beta^2, \hat{\omega} = \omega_K$ and $\hat{\Omega} = \omega_I + i \omega_J$. Since the SU(3) structure $(\hat{\mathcal{F}} = \beta^1 + \omega_K, \hat{\Psi} = \hat{\alpha}_1 \wedge \hat{\Omega})$ is integrable, the SU(2) structure $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ is obviously symplectic half-flat. Moreover, for any $k|| \in (0, 1)$ the SU(2) structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k|| \beta^2 - i \frac{\beta^1}{k||}, \hat{\omega}, \hat{\Omega})$ is also half-flat and then the result follows from theorems 3.1 and 3.2. \hfill $\square$

Note that any compact hyperkähler surface is either a complex torus with a flat metric or a K3-surface with Calabi–Yau metric [3]. Also observe that for the solutions given in this proposition all the fluxes vanish.

Next we generalize the previous proposition by means of hypo structures on 5-manifolds. We recall that an SU(2) structure on a 5-manifold $P^5$ is an SU(2)-reduction of the principal bundle of linear frames on $P$, equivalently a triple $(\eta, \omega_1, \Phi)$, where $\eta$ is a 1-form, $\omega_1$ is a 2-form and $\Phi = \omega_2 + i \omega_3$ is a complex 2-form on $P$ such that
\[
\eta \wedge \omega_1 \wedge \omega_1 \neq 0, \quad \Phi^2 = 0, \quad \omega_1 \wedge \Phi = 0, \quad \Phi \wedge \overline{\Phi} = 2 \omega_1 \wedge \omega_1,
\]
and $\Phi$ is of type $(2, 0)$ with respect to $\omega_1$. Following [12], a SU(2) structure on a 5-manifold $P^5$ is said to be hypo if $d\omega_1 = d(\omega_2 \wedge \eta) = d(\omega_3 \wedge \eta) = 0$.

**Proposition 3.9.** Let $(P^5, \eta, \omega_1, \omega_2, \omega_3)$ be a compact five-dimensional manifold endowed with a hypo SU(2) structure such that $d\eta = 0 = d\omega_2$. Then, on the six-dimensional manifold $M^6 = P^5 \times S^1$ there exist solutions to the SUSY equations IIA and IIB. More precisely, if $\beta$ is a global non-vanishing 1-form on $S^1$ then $(\alpha, \text{Re}(\Omega)||, \text{Re}(\Omega)|\perp, \text{Im}(\Omega))$ given by
\[
\text{Re}(\alpha) = \beta, \quad \text{Im}(\alpha) = \frac{1}{k||} \eta, \quad \text{Re}(\Omega)|| = -k|| \omega_3, \quad \text{Re}(\Omega)|\perp = k|\perp \omega_1, \quad \text{Im}(\Omega) = \omega_2,
\]
solve equations (6), and \((\alpha, \Re(\Omega)_{||}, \Re(\Omega)_{\perp}, \Im(\Omega))\) given by

\[
\Re(\alpha) = \frac{1}{k_{||}} \beta, \quad \Im(\alpha) = \eta, \quad \Re(\Omega)_{||} = k_{||} \omega_3, \quad \Re(\Omega)_{\perp} = k_{\perp} \omega_3, \quad \Im(\Omega) = \omega_2,
\]

are solutions to equations (7).

**Proof.** It is clear that the SU(2) structure \((\hat{J}, \hat{g}, \hat{\omega} = \beta + i \eta, \hat{\omega} = \omega_1, \hat{\Omega} = \omega_2 + i \omega_3)\) on \(M\) is symplectic half-flat. On the other hand, the SU(2) structure \((\hat{J}_1, \hat{g}_1, \hat{\omega}_1 = \beta + i \eta, \hat{\omega}_1 = \omega_3, \hat{\Omega}_1 = \omega_1 + i \omega_2)\) on \(M\) is half-flat and, for any \(k_{||} \in (0, 1)\), the SU(2) structure \((\hat{J}_2, \hat{g}_2, \hat{\omega}_2 = k_{||} \eta - i \frac{e_1}{\eta}, \hat{\omega}_2, \hat{\Omega}_2)\) is also half-flat. Therefore, the result follows from theorems 3.1 and 3.2.

\(\square\)

It is obvious that given a compact hyperkähler 4-manifold \(N^4\) we can consider \(P^5 = N^4 \times S^1\), but there are other manifolds to which this result can be applied. For example, a nilmanifold corresponding to the Lie algebra \((0, 0, 0, 12, 13)\) with the hypo structure \(\eta = e^1\), \(\omega_1 = e^{24} - e^{35}\), \(\omega_2 = e^{25} + e^{34}\) and \(\omega_3 = e^{23} + e^{45}\). We will treat this example in more detail in section 4.1.

**4. New explicit solutions of the SUSY equations IIA and IIB**

In this section we show compact solvmanifolds admitting structures solving the SUSY equations IIA and IIB. From theorem 3.1 and proposition 3.5 we consider compact 6-solvmanifolds admitting both symplectic half-flat and Hermitian balanced \(SU(3)\) structures.

**4.1. Nilmanifolds**

Conti and Tomassini classified [12] the nilmanifolds admitting invariant symplectic half-flat structures. It turns out that the underlying nilpotent Lie algebra must be isomorphic to the Abelian Lie algebra, \(h_6 = (0, 0, 0, 0, 12, 13)\) or \((0, 0, 0, 12, 13, 23)\). Apart from the Abelian Lie algebra, only \(h_6\) admits the Hermitian balanced structure [28]. In fact, up to equivalence, there is a 1-parametric family of Hermitian balanced structures, which are described as follows (see [15] for details).

The complex equations

\[
d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} - \omega^{21}
\]

define a complex structure \(J\) on the Lie algebra \(h_6\). Any complex structure on \(h_6\) is equivalent to \(J\). With respect to \(J\), any Hermitian balanced structure is equivalent to one and only one of the form

\[
F_t = \frac{1}{2} (\alpha^{11} + \omega^{22} + t^2 \omega^{33})
\]

for some \(t \neq 0\).

Let us consider the basis of \(1\)-forms \(\{\beta^1, \ldots, \beta^6\}\) given by

\[
\beta^1 + i \beta^4 = \omega^1, \quad \beta^2 + i \beta^3 = \omega^2, \quad \beta^5 + i \beta^6 = \frac{1}{2} \omega^3.
\]

In terms of this basis, we have the structure equations

\[
d\beta^1 = d\beta^2 = d\beta^3 = d\beta^4 = 0, \quad d\beta^5 = \beta^{12}, \quad d\beta^6 = \beta^{13},
\]

and the complex structure \(J\) and the fundamental form \(F_t\) are given by

\[
J\beta^1 = -\beta^4, \quad J\beta^2 = -\beta^3, \quad J\beta^5 = -\beta^6, \quad F_t = \beta^{14} + \beta^{23} + 4t^2 \beta^{56}.
\]
Note that the associated metric is $g_t = \beta_1 \otimes \beta_1 + \cdots + \beta_4 \otimes \beta_4 + 4t^2 \beta^5 \otimes \beta^5 + 4t^2 \beta^6 \otimes \beta^6$. From now on we consider the Hermitian balanced $SU(3)$ structure $(F_t, \Psi_t)$ on $\mathfrak{h}_6$ given by (9) and

$$\Psi_t = 2t (\beta^1 + i \beta^4) \wedge (\beta^2 + i \beta^3) \wedge (\beta^5 + i \beta^6). \quad (10)$$

- **Solutions to equations IIB arising from Hermitian balanced structures on $\mathfrak{h}_6$.** For each $t \neq 0$, the structure $(F_t, \Psi_t)$ provides solutions to the SUSY equations IIB. According to theorem 3.2, we let consider the half-flat $SU(2)$ structure $(\tilde{J}_1, \tilde{g}_1, \tilde{\alpha}_1, \tilde{\omega}, \tilde{\Omega})$ given by $\tilde{\alpha}_1 = \beta^1 + i \beta^4$, $\tilde{\omega} = \beta^{25} + 4t^2 \beta^{56}$, $\tilde{\Omega} = 2t(\beta^{25} - \beta^{36}) + 2ti(\beta^{26} + \beta^{35})$.

By (8) the forms $\beta^{25} - \beta^{36}$ and $\beta^{26} + \beta^{35}$ are closed; therefore, for any $k_1 \in (0, 1)$ we conclude that the $SU(2)$ structure $(\tilde{J}_2, \tilde{g}_2, \tilde{\omega}_2 = k_1 \text{Im}(\tilde{\alpha}_1) - i \text{Re}(\tilde{\alpha}_1), \tilde{\omega}, \tilde{\Omega})$ is half-flat. Therefore, in terms of the basis $\{\beta^1, \ldots, \beta^6\}$ we get the following explicit solutions $(\alpha, \text{Re}(\Omega)_{\parallel}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ of the SUSY equations (7):

$$\text{Re}(\alpha) = \frac{1}{k_1} \beta^1, \quad \text{Im}(\alpha) = \beta^4, \quad \text{Re}(\Omega)_{\parallel} = k_1(\beta^{25} + 4t^2 \beta^{56}),$$

$$\text{Im}(\Omega) = 2t(\beta^{26} + \beta^{35}),$$

where $k_\perp = \sqrt{1 - k_1^2}$. Note that the fluxes are $H = 0 = F_1$ and $e^{i\omega} g_* \ast F_3 = 4t^2(\beta^{126} - \beta^{135})$.

- **Solutions of the SUSY equations IIA on $\mathfrak{h}_6$.** For each $t \neq 0$, we consider the $SU(2)$ structure $(\tilde{J}_1, \tilde{\Omega})$ given by

$$\tilde{\alpha} = \beta^1 + i \beta^4, \quad \tilde{\omega} = 2t \beta^{25} - 2t \beta^{36},$$

$$\tilde{\Omega} = (\beta^2 + 2i \beta^3) \wedge (-\beta^3 + 2t i \beta^6). \quad (11)$$

Since the forms $\beta^{14}$ and $\beta^{25} - \beta^{36}$ are closed, and

$$d(\text{Re}(\tilde{\alpha} \wedge \tilde{\Omega})) = \beta^1 \wedge d(\beta^{25} + 4t^2 \beta^{56}) + 2t \beta^4 \wedge d(\beta^{26} + \beta^{35}) = 0,$$

we have that the $SU(2)$ structure is symplectic half-flat for any $t \neq 0$. According to theorem 3.1, since $d(\text{Re}(\tilde{\alpha})) = d\beta^1 = 0$, the forms $(\alpha, \text{Re}(\Omega)_{\parallel}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ given by

$$\text{Re}(\alpha) = \beta^1, \quad \text{Im}(\alpha) = \frac{1}{k_1} \beta^1, \quad \text{Re}(\Omega)_{\parallel} = -2tk_\perp(\beta^{26} + \beta^{35}),$$

$$\text{Im}(\Omega) = -\beta^{23} - 4t^2 \beta^{56},$$

provide solutions to the SUSY equations (6). Note that the fluxes are $H = -4t^2 k_\perp(\beta^{126} - \beta^{135})$, $F_0 = 0$, $g_* \ast F_2 = -4t^2(\beta^{126} - \beta^{135})$ and $F_4 = 0$.

Next we show that solutions to equations IIA (resp. IIB) in general are not stable by small deformations inside the class of half-flat structures. For that, we first show explicitly that any Hermitian balanced structure $(F_t, \Psi_t)$ on $\mathfrak{h}_6$ given by (9)–(10) can be deformed into a symplectic half-flat structure (11) along a curve of half-flat structures. For each $\vartheta \in \mathbb{R}$, let us consider the $SU(3)$ structure $(F_\vartheta, \Psi_\vartheta)$ given by

$$F_\vartheta = \beta^{14} + \cos \vartheta \beta^{23} + 2t \sin \vartheta \beta^{25} - 2t \sin \vartheta \beta^{36} + 4t^2 \cos \vartheta \beta^{56},$$

and

$$\Psi_\vartheta = (\beta^1 + i \beta^4) \wedge (\beta^2 + i \cos \vartheta \beta^3 + 2t i \sin \vartheta \beta^5) \wedge (-\sin \vartheta \beta^3 + 2t \cos \vartheta \beta^5 + 2ti \beta^6).$$
A direct calculation shows that $\text{Re}(\Psi^2_\vartheta)$ is closed and $dF^\vartheta_1 = 4t^2 \cos \vartheta (\beta^{126} - \beta^{135})$, which implies that $F^\vartheta_1$ is half-flat for any $\vartheta$, and $(F^\vartheta_0, \Psi^0_\vartheta)$ is the Hermitian balanced structure given by (9)–(10), and $(F^\vartheta_5, \Psi^5_\vartheta)$ is the symplectic structure (11).

Since $F^\vartheta_1$ is symplectic if and only if $\cos \vartheta = 0$, by theorem 3.1 we have that the half-flat structure $(F^\vartheta_0, \Psi^0_\vartheta)$ does not solve equations (6) for $\vartheta \in (0, \frac{\pi}{2})$.

On the other hand, let us fix $\vartheta$ and consider the half-flat structure $(F^\vartheta_1, \Psi^1_\vartheta)$. For any $\lambda \in (0, 1)$, a direct calculation shows that the structure $(F = F^\vartheta_1, \Phi_\vartheta)$ given by

$$\Phi_\vartheta = \left( \lambda \beta^4 - i \frac{\beta^1}{\lambda} \right) \wedge (\beta^2 + i \cos \vartheta \beta^3 + 2i \sin \vartheta \beta^5) \wedge (-\sin \vartheta \beta^3 + 2t \cos \vartheta \beta^5 + 2ri \beta^6)$$

is half-flat if and only if $\sin \vartheta = 0$. From theorem 3.2 we conclude that the half-flat structure $(F^\vartheta_1, \Psi^1_\vartheta)$ does not provide a solution to equations (7) for $\vartheta \in (0, \frac{\pi}{2})$.

Therefore, we have proved the following result.

**Proposition 4.1.** The half-flat structure $(F^\vartheta_1, \Psi^1_\vartheta)$ does not solve either (6) or (7) for any $\vartheta \in (0, \frac{\pi}{2})$. Therefore, solutions to the SUSY equations IIA or IIB in general are not stable by small deformations inside the class of half-flat structures.

### 4.2. Compact solvmanifolds

In this section we describe in detail two compact solvmanifolds solving the SUSY equations IIA and IIB.

#### 4.2.1. Example

Let us consider the six-dimensional 2-step completely solvable Lie algebra $\mathfrak{a}_1 = (0, 0, 13, -14, 15, -16)$ with structure equations

$$d\beta^1 = d\beta^2 = 0, \quad d\beta^3 = \beta^{13}, \quad d\beta^4 = -\beta^{14}, \quad d\beta^5 = \beta^{15}, \quad d\beta^6 = -\beta^{16}. \quad (12)$$

The corresponding simply-connected Lie group $S_1$ is isomorphic to $\mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R}^4)$, where

$$\phi(t) = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}, \quad t \in \mathbb{R}.\]

Since $\phi(1) = \exp^{SL(4,\mathbb{R})}(\phi(0)) \in SL(4, \mathbb{Z})$, from [16, theorem 4] we have that $\Gamma = \mathbb{Z} \ltimes \mathbb{R}^4$ is a lattice in $\mathbb{R} \ltimes \mathbb{R}^4$ and therefore $\mathbb{Z} \times \Gamma = \Gamma_1$ is a lattice of $S_1$. By Hattori’s theorem [23] we have that the de Rham cohomology of the compact quotient $S_1 / \Gamma_1$ is isomorphic to the Chevalley–Eilenberg cohomology $H^*(\mathfrak{a}_1)$ of $\mathfrak{a}_1$ and thus in particular $b_1(S_1 / \Gamma_1) = 2, b_2(S_1 / \Gamma_1) = 5$ and $b_3(S_1 / \Gamma_1) = 8$.

Let us consider the almost complex structure

$$J\beta^1 = -\beta^2, \quad J\beta^3 = -\beta^5, \quad J\beta^4 = \beta^6.$$

The basis of (1,0)-forms $\omega^1 = \beta^1 + i \beta^2, \quad \omega^2 = \beta^3 + i \beta^5$ and $\omega^3 = -\beta^4 + i \beta^6$ satisfies

$$d\omega^1 = 0, \quad d\omega^2 = \frac{1}{2}\omega^{12} + \frac{1}{2}\omega^{13}, \quad d\omega^3 = -\frac{1}{2}\omega^{13} - \frac{1}{2}\omega^{12}.$$

Therefore, the almost complex structure $J$ is integrable. Since the 2-form $F = \beta^{12} + \beta^{35} - \beta^{46}$ satisfies that $F^2 = 2(\beta^{1235} - \beta^{1246} - \beta^{3546})$ is closed, we get a Hermitian balanced SU(2) structure.
• **Solutions of equations IIB arising from the Hermitian balanced structure on s$_1$.** The previous structure provides solutions to the SUSY equations IIB. Let $(\hat{J}_1, \hat{g}_1, \hat{\alpha}_1, \hat{\omega}, \hat{\Omega})$ be the half-flat SU(2) structure given by

\[
\hat{\alpha}_1 = \beta^1 + i\beta^2, \quad \hat{\omega} = \beta^{35} - \beta^{46}, \quad \hat{\Omega} = -\beta^{34} - \beta^{56} + i(\beta^{36} + \beta^{45}).
\]

It follows from (12) that the 2-forms $\beta^{34}$, $\beta^{56}$ and $\beta^{36} + \beta^{45}$ are closed, which implies that for any $k_\parallel \in (0, 1)$ the $SU(2)$ structure $(\hat{J}_2, \hat{g}_2, \hat{\alpha}_2 = k_\parallel \Im(\hat{\alpha}_1) - \frac{\Im(\hat{\alpha}_1)}{k_\parallel}, \hat{\omega}, \hat{\Omega})$ is half-flat. By theorem 3.2 we get the following solutions $(\alpha, \Re(\Omega)_||, \Re(\Omega)_\perp, \Im(\Omega))$ of the SUSY equations (7):

\[
\Re(\alpha) = \frac{1}{k_\parallel} \beta^1, \quad \Im(\alpha) = \beta^2, \quad \Re(\Omega)_|| = k_\perp (\beta^{35} - \beta^{46}),
\]

\[
\Re(\Omega)_\perp = -k_\parallel (\beta^{34} + \beta^{56}), \quad \Im(\Omega) = \beta^{36} + \beta^{45},
\]

where $k_\perp = \sqrt{1 - k_\parallel^2}$. Note that the fluxes are $H = 0 = F_1$, and $e^{\theta} g_s \ast F_3 = 2\beta^1 \wedge (\beta^{35} + \beta^{46})$.

• **Solutions of the SUSY equations IIA on s$_1$.** Let us consider the $SU(2)$ structure $(\hat{J}, \hat{g}, \hat{\alpha}, \hat{\omega}, \hat{\Omega})$ given by

\[
\hat{\alpha} = \beta^1 + i\beta^2, \quad \hat{\omega} = \beta^{34} + \beta^{56}, \quad \hat{\Omega} = (\beta^3 + i\beta^4) \wedge (\beta^5 + i\beta^6).
\]

Since the forms $\beta^{12}, \beta^{34}$ and $\beta^{56}$ are closed, and

\[
d(\Re(\hat{\alpha} \wedge \hat{\Omega})) = \beta^1 \wedge d(\beta^{35} - \beta^{46}) - \beta^2 \wedge d(\beta^{36} + \beta^{45}) = 0,
\]

we have that the SU(2) structure is symplectic half-flat. By theorem 3.1, since $d(\Re(\hat{\alpha})) = d\beta^1 = 0$, the forms $(\alpha, \Re(\Omega)_||, \Re(\Omega)_\perp, \Im(\Omega))$ given by

\[
\Re(\alpha) = \beta^1, \quad \Im(\alpha) = \frac{1}{k_\parallel} \beta^2, \quad \Re(\Omega)_|| = -k_\perp (\beta^{36} + \beta^{45}),
\]

\[
\Re(\Omega)_\perp = k_\perp (\beta^{34} + \beta^{56}), \quad \Im(\Omega) = \beta^{35} - \beta^{46}
\]

provide solutions to the SUSY equations (6). The fluxes are $H = 2\frac{k_\parallel}{k_\perp} (\beta^{135} + \beta^{146})$, $F_0 = 0$, $g_s \ast F_2 = -\frac{2}{k_\perp} (\beta^{1235} + \beta^{1246})$ and $F_4 = 0$.

As in the previous example, the particular solutions on $s_1$ to equations IIA and IIB given above are not stable by small deformations inside the class of half-flat structures. For each $\vartheta \in \mathbb{R}$, the SU(2) structure $(F^\vartheta, \Psi^\vartheta)$ given by

\[
F^\vartheta = \beta^{12} + \cos \vartheta (\beta^{34} + \beta^{56}) + \sin \vartheta (\beta^{35} - \beta^{46})
\]

and

\[
\Psi^\vartheta = (\beta^1 + i\beta^2) \wedge (\beta^3 + i\cos \vartheta \beta^4 + i\sin \vartheta \beta^5) \wedge (-\sin \vartheta \beta^4 + \cos \vartheta \beta^5 + i\beta^6)
\]

is half-flat, and for $\vartheta = 0$ (resp. $\vartheta = \frac{\pi}{2}$) we get the symplectic (resp. Hermitian balanced) half-flat structure given above. A direct calculation shows that the half-flat structure $(F^\vartheta, \Psi^\vartheta)$ does not solve either (6) or (7) for any $\vartheta \in (0, \frac{\pi}{2})$. 

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4.2.2. Example. Let us consider the solvable Lie algebra $\mathfrak{s}_2 = (0, 0, -13 - 24, -14 + 23, 15 + 26, 16 - 25)$, that is, there is a basis of 1-forms $\{\beta^1, \ldots, \beta^6\}$ satisfying

$$
\begin{align*}
\d \beta^1 &= \d \beta^2 = 0, \\
\d \beta^3 &= -\beta^{13} - \beta^{24}, \\
\d \beta^4 &= -\beta^{14} + \beta^{23}, \\
\d \beta^5 &= \beta^{15} + \beta^{26}, \\
\d \beta^6 &= \beta^{16} - \beta^{25}.
\end{align*}
$$

(13)

The existence of a lattice $\Gamma_2$ of $\mathfrak{s}_2$ of the associated simply connected solvable Lie group $S_2$ was proved in [29] (see also [14]). The de Rham cohomology of the compact quotient $S_2 / \Gamma_2$ (also known as Nakamura manifold) is not isomorphic to $H^*(\mathfrak{s}_2)$ (see [14, 29] and more recently [9, 21] for the cohomology of solvmanifolds). In particular $b_1(S_2 / \Gamma_2) = 2$, $b_2(S_2 / \Gamma_2) = 5$ and $b_3(S_2 / \Gamma_2) = 8$.

Let us consider the almost complex structure

$$
J \beta^1 = -\beta^2, \quad J \beta^3 = -\beta^4, \quad J \beta^5 = -\beta^6.
$$

The basis of (1,0)-forms $\omega^1 = \beta^1 + i \beta^2, \omega^2 = \beta^3 + i \beta^4$ and $\omega^3 = \beta^5 + i \beta^6$ satisfies

$$
d \omega^1 = 0, \quad d \omega^2 = \omega^1, \quad d \omega^3 = -\omega^1,
$$

that is, $J$ is integrable.

For each $t \in \mathbb{R} - \{0\}$, the SU(3) structure $(F_t, \Psi_t)$ given by

$$
F_t = t^2 \beta^{12} + \beta^{34} + \beta^{56}, \quad \Psi_t = t (\beta^1 + i \beta^2) \wedge (\beta^3 + i \beta^4) \wedge (\beta^5 + i \beta^6)
$$

defines a 1-parametric family of (non-equivalent) Hermitian balanced SU(3) structures on $\mathfrak{s}_2$ and thus a 1-parametric family of (non-equivalent) Hermitian balanced SU(2) structures.

Note that the associated metric is $g_t = t^2 \beta^1 \otimes \beta^1 + t^2 \beta^2 \otimes \beta^2 + \beta^3 \otimes \beta^3 + \cdots + \beta^6 \otimes \beta^6$.

- Solutions to equations IIB arising from Hermitian balanced structures on $\mathfrak{s}_2$. For each $t \neq 0$, the structure $(F_t, \Psi_t)$ provides solutions to the SUSY equations IIB. According to theorem 3.2, we consider the half-flat SU(2) structure $(\tilde{J}, \tilde{g}, \tilde{\alpha}, \tilde{\omega}, \tilde{\Omega})$ given by

$$
\tilde{\alpha} = t \beta^1 + i t \beta^2, \quad \tilde{\omega} = \beta^{34} + \beta^{56}, \quad \tilde{\Omega} = \beta^{35} - \beta^{46} + i(\beta^{36} + \beta^{45}).
$$

By (13) the forms $\beta^{35} - \beta^{46}$ and $\beta^{36} + \beta^{45}$ are closed; therefore, for any $k_{||} \in (0, 1)$ we conclude that the SU(2) structure $(\tilde{J}, \tilde{g}, \tilde{\alpha}, \tilde{\omega}, \tilde{\Omega}) = k_{||} \text{Im}(\tilde{\alpha}) - \frac{\text{Re}(\tilde{\alpha})}{k_{||}}, \tilde{\omega}, \tilde{\Omega})$ is half-flat. Therefore, in terms of the basis $\{\beta^1, \ldots, \beta^6\}$ we get the following explicit solutions $(\alpha, \text{Re}(\Omega)_{||}, \text{Re}(\Omega)_{\perp}, \text{Im}(\Omega))$ of the SUSY equations (7):

$$
\text{Re}(\alpha) = \frac{\text{Re}(\Omega)_{||}}{k_{||}}, \quad \text{Im}(\alpha) = t \beta^2, \quad \text{Re}(\Omega)_{||} = k_{||} (\beta^{34} + \beta^{56}),
$$

$$
\text{Re}(\Omega)_{\perp} = k_{||} (\beta^{35} - \beta^{46}), \quad \text{Im}(\Omega) = \beta^{36} + \beta^{45},
$$

where $k_{\perp} = \sqrt{1 - k_{||}^2}$. Note that the fluxes are $H = 0 = F_1$ and $e^{i\theta} g_t \ast F_3 = -2 \beta^1 \wedge (\beta^{34} - \beta^{56})$.

- Solutions of the SUSY equations IIA on $\mathfrak{s}_2$. For each $t \neq 0$, we consider the SU(2) structure $(\tilde{J}, \tilde{g}, \tilde{\alpha}, \tilde{\omega}, \tilde{\Omega})$ given by

$$
\tilde{\alpha} = i \beta^1 + i \beta^2, \quad \tilde{\omega} = -\beta^{36} - \beta^{45}, \quad \tilde{\Omega} = (\beta^6 + i \beta^3) \wedge (\beta^5 + i \beta^4).
$$

Since the forms $\beta^{12}$ and $\beta^{36} + \beta^{45}$ are closed, and

$$
\frac{1}{t} (\text{d}(\text{Re}(\tilde{\alpha} \wedge \tilde{\Omega}))) = \beta^1 \wedge \text{d}(\beta^{34} + \beta^{56}) + \beta^2 \wedge \text{d}(\beta^{35} - \beta^{46}) = 0,
$$
we have that the SU(2) structure is symplectic half-flat for any $t \neq 0$. According to theorem 3.1, since $d(\Re(\tilde{\alpha})) = t d\beta^1 = 0$, the forms $(\alpha, \Re(\Omega)_{||}, \Re(\Omega)_{\perp}, \Im(\Omega))$

given by

$\Re(\alpha) = t\beta^1, \quad \Im(\alpha) = \frac{t}{k_{||}}\beta^2, \quad \Re(\Omega)_{||} = -k_{||}(\beta^{35} - \beta^{46}),$

$\Re(\Omega)_{\perp} = -k_{\perp}(\beta^{16} + \beta^{45}), \quad \Im(\Omega) = -\beta^{34} - \beta^{56}$

provide solutions to the SUSY equations (6). Note that the fluxes are $H = 2\frac{k_{||}}{k_{\perp}}(\beta^{134} - \beta^{156}), F_0 = 0, g_\times * F_2 = -2\frac{1}{k_{||}}(\beta^{1234} - \beta^{1256})$ and $F_4 = 0$.

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