ASYMPTOTIC STABILITY OF TWO TYPES OF TRAVELING WAVES FOR SOME PREDATOR-PREY MODELS

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Dedicated to Professor Sze-Bi Hsu on the occasion of his 70th birthday

Abstract. This paper is concerned with the asymptotic stability of wave fronts and oscillatory waves for some predator-prey models. By spectral analysis and applying Evans function method with some numerical simulations, we show that the two types of waves with noncritical speeds are spectrally stable and nonlinearly exponentially stable in some exponentially weighted spaces.

1. Introduction. The predator-prey model with random diffusion is one of the fundamental models describing the segregation and the dynamical behavior of interacting species in mathematical ecology. A typical case is that the predator has only one prey as his food resource, in such case the existence and stability of coexistence steady state or wave fronts may correspond to the phenomena of invading of predator into a new habitation (see [20, 21, 23, 28]).

Let $u(x,t)$ and $v(x,t)$ represent the densities of prey and predator, respectively, which can be described by the following predator-prey system

$$
\begin{align*}
    u_t &= \delta u_{xx} + \rho(u) - g(u)v, \\
    v_t &= v_{xx} + \gamma vg(u) - mv,
\end{align*}
$$

(1.1)

where $\delta$ and $\gamma$ are positive constants, $\rho(u)$ is the growth rate of prey and $g(u)$ is the predation efficiency or the functional response. Here $m > 0$ means $u$ is the unique food resource of the predator $v$ and $mv$ stands for predator mortality.

The growth rate of prey $u$ is usually of logistic type, i.e. $\rho(u) = \alpha u (1 - \frac{u}{K})$, $g(u)$ in (1.1) can be selected as various types of functional responses, for instance $g(u) = u$, and the model is known as the Lotka-Volterra predator-prey model (denoted by

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steady states: $E_\beta \triangleq \text{the concentration of reactant and autocatalyst.}$ $\rho$ can be considered as a degenerate form of (1.1) with $\beta$ of epidemic model.

The PDE system (1.1) can also describe the epidemic models and the chemical reaction models with various types of reaction models with various types of $\rho(u)$ and $g(u)$, such as the following SIR type of epidemic model

$$
\begin{align*}
    u_t &= \delta u_{xx} + \Lambda - \mu u - \vartheta uv, \\
    v_t &= v_{xx} + \vartheta uv - mv,
\end{align*}
$$

where $u$ and $v$ are densities of susceptible and infective species.

The autocatalytic chemical reaction model

$$
\begin{align*}
    u_t &= \delta u_{xx} - \gamma uv^p, \\
    v_t &= v_{xx} + \gamma uv^p - mv^q,
\end{align*}
$$

can be considered as a degenerate form of (1.1) with $\rho(u) = 0$, where $u$ and $v$ are the concentration of reactant and autocatalyst.

The general predator-prey model (1.1) usually has three nonnegative constant steady states: $E_0 = (0, 0)$, semi-trivial $E_1 = (K, 0)$ and positive $E_\beta = (u_\beta, v_\beta)$ (with $\beta \triangleq \frac{\rho}{2\gamma}$). The global stability of the coexistence state $E_\beta$ of system (1.1) without diffusion has been deeply investigated by Hsu and others (see [2, 12, 13, 26]).

Let $(U_c(x-ct), V_c(x-ct))$ be the traveling wave solution of system (1.1) connecting the positive $E_\beta = (u_\beta, v_\beta)$ and the semi-trivial $E_1 = (K, 0)$ with the positive speed $c$, which describes the invading front of the predator. In moving coordinate $z \triangleq x - ct$, the wave $(U_c(z), V_c(z))$ satisfies the following ODE system with the asymptotic boundary conditions at $z = \pm \infty$:

$$
\begin{align*}
    \delta U'' + cU' + \rho(U) - g(U)V &= 0, \\
    V'' + cV' + \gamma(g(U) - \beta)V &= 0, \\
    (U, V)(-\infty) &= (u_\beta, v_\beta), \quad (U, V)(+\infty) = (K, 0),
\end{align*}
$$

with $' = \frac{d}{dz}$.

There are lots of works on the existence of traveling waves for predator-prey model (1.1), especially for models HT1-HT3. For these models, the existence of traveling waves are obtained by applying various approaches including shooting method, sub/super solutions method and Lyapunov function method, etc (see [3, 4, 5, 6, 7, 14, 17] and references therein).

The existence of traveling waves are investigated in [8, 11, 15] for the following predator-prey model with more general monotone functions $g(u), p(u)$ and $g(u)$

$$
\begin{align*}
    u_t &= \delta u_{xx} + q(u)p(u) - g(u)v, \\
    v_t &= v_{xx} + \gamma v(g(u) - \beta).
\end{align*}
$$

Hsu et al.[11] obtained the existence of traveling waves for the general predator-prey model (1.5) by applying shooting method and LaSalle’s invariance principle; Huang [15] proved the existence of traveling fronts for system (1.5) with $\delta = 0$. Fu and Tsai [8] improved the work of [11] by applying sub-super solution method and Lyapunov function method, and they proved the existence of monotone waves and oscillatory waves with noncritical speeds under the following hypotheses on monotone functions $q(u), p(u), g(u)$:

$(H_1)$ $q(u) \in C^1([0, \infty)), \quad q(0) \geq 0$, and $q(u) > 0$, $q'(u) \geq 0$ on $(0, \infty)$. 

(H2) $p(u) \in C^1([0, \infty))$, $p(K) = 0$ for the carrying capacity $K$; $p'(u) < 0$ on $(0, \infty)$.
(H3) $g(u) \in C^2([0, \infty))$, $g(0) = 0$, $g'(0) \geq 0$, $g(u_{\beta}) = \beta$ for some positive constant $u_{\beta} < K$, and $g'(u) > 0$ on $(0, \infty)$.
(H4) The function $R(u) \triangleq \frac{g(u)p(u)}{g(u)}$ satisfies $R(u) \geq R(u_{\beta})$ for $0 < u < u_{\beta}$, and $R(u) \leq R(u_{\beta})$ for $u_{\beta} < u < K$.

In this paper, we are more interested in the asymptotic stability of traveling waves connecting $E_{\beta} = (u_{\beta}, v_{\beta})$ and $E_1 = (K, 0)$ for system (1.5), which is interesting in both mathematics and mathematical ecology. For convenience of our later investigation, we restate the main existence results obtained in [8] as follows:

**Theorem 1.1.** [8] (Existence of Traveling Waves) Assume the hypotheses (H1)-(H4) hold. Then the following hold true.

(I) For each $c < c_{\min} \triangleq 2\sqrt{\gamma[g(K) - \beta]}$, there are no non-negative solutions $(U_c(z), V_c(z))$ of system (1.4).

(II) For each $c > c_{\min}$, system (1.4) admits a non-negative solution $(U_c(z), V_c(z))$ with the following properties:

(i) $0 < U(z) < K$ and $V(z) > 0$ over $\mathbb{R}$;

(ii) There exists a positive constant $\gamma^*$ such that, if $\gamma < \gamma^*$ then $(U(z), V(z))$ is monotone in $z$; if $\gamma > \gamma^*$ then $(U(z), V(z))$ tends to $(u_{\beta}, v_{\beta})$ with exponentially damped oscillations as $z \to \infty$.

(iii) $V(z) \to 0$ exponentially as $z \to +\infty$, with the exponential decay rate $\mu^*_3 = -c + \sqrt{c^2 - 4\gamma[g(K) - \beta]} < 0$.

As far as we know, there are few theoretical works on the stability of traveling waves for predator-prey models, except for some works (see [9, 30]) on the spectral stability of traveling waves with transition layers for some predator-prey systems.

As for the autocatalytic model (1.3), the nonlinear or linear asymptotic stability of traveling waves are obtained in [18, 19, 29] by applying spectral analysis and Evans function method.

In this paper, by applying spectral analysis, Evans function method and numerical simulation, we investigate the asymptotic stability of both monotone and oscillatory traveling waves for the predator-prey model (1.5).

This paper is organized as follows. In Section 2, we investigate the location of essential spectra of the linearized operator $L$ in the unweighted and some exponentially weighted spaces. In Section 3, we obtain the precise upper bound of all the unstable eigenvalues of $L$, and introduce the definition of Evans function. In Section 4, we introduce our numerical schemes for approximating monotone and oscillatory waves and show some numerical simulation on wave profiles. In Section 5, some numerical simulations on the Evans function are shown and the main stability result is stated.

2. The location of essential spectrum of the linearized operator. In the following of this paper, we focus on the stability of traveling waves of system (1.5) with noncritical speeds, and we denote the traveling wave solution by $(U_c(x - ct), V_c(x - ct))$ for any given speed $c > c_{\min} = 2\sqrt{\gamma[g(K) - \beta]}$, where $(U_c(z), V_c(z))$ is obtained in Theorem 1.1. To investigate the spectral and nonlinear asymptotic stability of $(U_c(x - ct), V_c(x - ct))$, it suffices to investigate the following IVP of
system (1.5) in the moving coordinate \( z = x - ct \)
\[
\begin{align*}
  u_t &= \delta u_{zz} + cu_z + g(u)p(u) - g(u)v, \\
  v_t &= v_{zz} + cv_z + \gamma(g(u) - \beta)v, \\
  u(0, z) &= u_0(z), \quad v(0, z) = v_0(z).
\end{align*}
\]  
(2.1)

The linearized system of (2.1) around \((U_c(z), V_c(z))\) is
\[
\begin{pmatrix}
  \phi_t \\
  \psi_t
\end{pmatrix} = \mathcal{L} \begin{pmatrix}
  \phi \\
  \psi
\end{pmatrix},
\]
(2.2)

where \(\mathcal{L} : C^2_{\text{unif}}(\mathbb{R}) \times C^2_{\text{unif}}(\mathbb{R}) \to X \triangleq C_{\text{unif}}(\mathbb{R}) \times C_{\text{unif}}(\mathbb{R})\) is
\[
\mathcal{L} = \begin{pmatrix}
  \delta \partial_{zz} + c \partial_z + (g(U)p(U))' - g'(U)V & -g(U) \\
  \gamma g'(U)V & \partial_{zz} + c \partial_z + (g(U) - \beta)
\end{pmatrix}.
\]

Denote \(\sigma_n(\mathcal{L})\) to be the set of all the isolated eigenvalues of \(\mathcal{L}\) with finite algebraic multiplicities, and define the essential spectrum of \(\mathcal{L}\) by \(\sigma_{\text{ess}}(\mathcal{L}) = \sigma(\mathcal{L}) \setminus \sigma_n(\mathcal{L})\). By applying the classical stability theory, the nonlinear exponential stability of traveling waves of system (1.4) in some appropriate spaces follows from the strong spectral stability of traveling waves, i.e. there exists \(\sigma > 0\), such that \(\text{Re}\{\sigma(L)\} < -\sigma < 0\).

Consider the eigenvalue problem of \(\mathcal{L}\), i.e.
\[
\mathcal{L} \begin{pmatrix}
  \phi \\
  \psi
\end{pmatrix} = \lambda \begin{pmatrix}
  \phi \\
  \psi
\end{pmatrix}, \quad \text{for} \quad \begin{pmatrix}
  \phi \\
  \psi
\end{pmatrix} \in X.
\]
(2.3)

Let \(Y(z) = (y_1(z), y_2(z), y_3(z), y_4(z))^T \triangleq (\phi, \psi, \phi_z, \psi_z)^T\). For each \(\lambda \in \mathbb{C}\), (2.3) can be rewritten as the following first-order ODE system
\[
Y'(z) = A(\delta, z, \lambda)Y(z), \quad Y(z) \in \mathbb{C}^4,
\]
(2.4)

with
\[
A(\delta, z, \lambda) \triangleq \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  \frac{\lambda-(q(U)p(U))'+g'(U)V}{g(U)} & \frac{g(U)}{\beta} & \frac{-c}{\beta} & 0 \\
  -\gamma g'(U)V & \lambda - \gamma (g(U) - \beta) & 0 & -c
\end{pmatrix}.
\]

Denote
\[
A^\pm(\delta, \lambda) = \lim_{z \to \pm \infty} A(\delta, z, \lambda),
\]
with
\[
A^+(\delta, \lambda) = \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  \frac{\lambda-\mu'(K)q(K)}{g(K)} & \frac{g(K)}{\beta} & \frac{-c}{\beta} & 0 \\
  0 & \lambda - \gamma(g(K) - \beta) & 0 & -c
\end{pmatrix}
\]
and
\[
A^-(\delta, \lambda) = \begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  \frac{\lambda-\beta R(u_0)}{\beta} & \frac{\beta}{\beta} & \frac{-c}{\beta} & 0 \\
  -\gamma g'(u_0)R(u_0) & \lambda & 0 & -c
\end{pmatrix},
\]
where \(R(u) \triangleq \frac{g(u)p(u)}{g'(u)}\).

Define the curves \(S^\pm\) by
\[
S^\pm = \{ \lambda \in \mathbb{C} \mid \det (A^\pm(\delta, \lambda) - i\tau I) = 0, \text{ for some } \tau \in \mathbb{R}\}.
\]
where

Using $p'(K)q(K) < 0$ and $\gamma(g(K) - \beta) > 0$ from hypotheses (H1)-(H3), we have

$$\sup_{\lambda \in S^+} \{\text{Re}\lambda\} = \gamma(g(K) - \beta) > 0. \quad (2.5)$$

Similarly,

$$S^- = \{\lambda \in \mathbb{C} | \det (A^-(\delta, \lambda) - i\tau I) = 0, \text{ for some } \tau \in \mathbb{R}\}$$

$$= \{\lambda \in \mathbb{C} | [\lambda - \beta R'(u_\beta) - i\tau(c + i\tau\delta)][\lambda - i\tau(c + i\tau)] + g'(u_\beta)R(u_\beta)\beta \gamma = 0\}$$

$$= \{\lambda \in \mathbb{C} | \lambda = -\frac{B_0(\delta, \tau) \pm \sqrt{B_0^2(\delta, \tau) - 4C_0(\delta, \tau)}}{2}, \tau \in \mathbb{R}\} \triangleq S^-_1 \cup S^-_2,$$

where

$$B_0(\delta, \tau) \triangleq -[i\tau(c + i\tau\delta) + \beta R'(u_\beta) + i\tau(c + i\tau)] = (1 + \delta)\tau^2 - \beta R'(u_\beta) - 2c\tau i,$$

$$C_0(\delta, \tau) \triangleq [i\tau(c + i\tau\delta) + \beta R'(u_\beta)][i\tau(c + i\tau)] + g'(u_\beta)R(u_\beta)\beta \gamma$$

$$= [\delta \tau^4 - (c^2 + \beta R'(u_\beta))\tau^2 + g'(u_\beta)R(u_\beta)\beta \gamma] + [c\tau\beta R'(u_\beta) - (1 + \delta)\tau^3]i.$$

**Proposition 2.1.** For each $\delta > 0$ and $c > c_{\min}$,

$$\sup_{\lambda \in S^-} \{\text{Re}\lambda\} < 0. \quad (2.6)$$

**Proof.** Let $D_0(\delta, \tau) \triangleq B_0^2(\delta, \tau) - 4C_0(\delta, \tau)$ and $\sqrt{D_0(\delta, \tau)} \triangleq a(\delta, \tau) + b(\delta, \tau)i$, where $a, b, \tau \in \mathbb{R}$ and $i$ is the imaginary unit. Then $\lambda \in S^-$ can be expressed as the explicit functions of $\tau$, i.e.

$$\lambda = \lambda_+(\delta, \tau) = -\frac{B_0(\delta, \tau) \pm \sqrt{D_0(\delta, \tau)}}{2} = \frac{(-\text{Re} B_0 \pm a) + (-\text{Im} B_0 \pm b)i}{2}.$$

It can be checked that

$$\text{Re} D_0(\delta, \tau) = (1 - \delta)^2\tau^4 + 2(1 - \delta)A_1\tau^2 + A_1^2 - 4A_2 = a^2 - b^2,$$

$$\text{Im} D_0(\delta, \tau) = 0 = 2ab,$$

with $A_1 \triangleq \beta R'(u_\beta) < 0$ and $A_2 \triangleq g'(u_\beta)R(u_\beta)\beta \gamma > 0$, which means that either $a = 0$ or $b = 0$ for given $\tau \in \mathbb{R}$ and $\delta > 0$.

If $a = 0$, we have $\text{Re} \lambda_+(\delta, \tau) = -\frac{1}{2}\text{Re} B_0(\delta, \tau) = \frac{1}{2}[-(1 + \delta)\tau^2 + A_1] \leq \frac{A_1}{2}$, then $\max\{\text{Re} \lambda_+(\delta, \tau)\} = \text{Re} \lambda_+(\delta, 0)$.

If $b = 0$, we get $\text{Im} \lambda_+(\delta, \tau) = -\frac{1}{2}\text{Im} B_0(\delta, \tau) = c\tau$, and

$$a^2(\delta, \tau) = (1 - \delta)^2\tau^4 + 2(1 - \delta)A_1\tau^2 + A_1^2 - 4A_2$$

$$= [(1 - \delta)\tau^2 + A_1]^2 - 4A_2.$$

Let $\bar{\tau} = \tau^2 \geq 0$. Due to $\sqrt{D_0} = a(\delta, \bar{\tau}) > 0$, we obtain

$$2 \text{Re} \lambda_+(\delta, \bar{\tau}) = -(1 + \delta)\bar{\tau} + A_1 + a(\delta, \bar{\tau})$$

$$< -(1 + \delta)\bar{\tau} + A_1 + [(1 - \delta)\bar{\tau} + A_1]$$

$$\leq 2A_1 < 0, \quad \forall \bar{\tau} \geq 0.$$
By (2.5) and (2.6), we have
\[ \sigma_{ess}(L) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda > 0 \} \neq \emptyset, \tag{2.7} \]
which indicates the traveling waves of system (1.4) with noncritical speeds are spectrally unstable in \( X = C_{unif}(\mathbb{R}) \times C_{unif}(\mathbb{R}) \).

In the following of this paper, we investigate the spectral stability and the non-linear exponential stability of the traveling waves in some exponentially weighted spaces.

Choosing the exponential weight function \( \omega(z) = 1 + e^{\alpha z} \) with \( \alpha > 0 \) to be determined later, and define the exponentially weighted space \( X_\alpha = \{ (u, v) \in X \mid (\omega u, \omega v) \in X \} \) of \( X \) with the norm \( \|(u, v)\|_{X_\alpha} = \|(\omega u, \omega v)\|_X \) and \( X^2_\alpha \) can be similarly defined with \( X^2 = C^2_{unif}(\mathbb{R}) \times C^2_{unif}(\mathbb{R}) \).

Let \( L_\alpha : X^2_\alpha \to X_\alpha \) be the operator \( L \) restricted in \( X^2_\alpha \), i.e.
\[ L_\alpha \left( \phi \psi \right) = L \left( \phi \psi \right), \quad \text{for} \quad \left( \phi \psi \right) \in X^2_\alpha, \]
and rewrite the eigenvalue problem
\[ L_\alpha \left( \phi \psi \right) = \lambda \left( \phi \psi \right), \quad \text{for} \quad \lambda \in \mathbb{C}, \]
as
\[ Y'(z) = A(\delta, z, \lambda) Y(z), \quad \text{for} \quad Y(z) \in Y^1_\alpha, \tag{2.8} \]
with \( Y^1_\alpha = \{ Y \mid \omega(z) Y \in [C^1_{unif}(\mathbb{R})]^4 \} \).

Let \( \tilde{Y}(z) \equiv \omega(z) Y(z) \), then (2.8) becomes
\[ \tilde{Y}'(z) = \left( A(\delta, z, \lambda) + \frac{w'(z)}{w(z)} I \right) \tilde{Y}(z) \equiv \tilde{A}(\delta, z, \lambda) \tilde{Y}(z), \quad \tilde{Y}(z) \in [C^1_{unif}(\mathbb{R})]^4. \]

Denote
\[ \tilde{A}^\pm(\delta, \lambda) = \lim_{z \to \pm \infty} \tilde{A}(\delta, z, \lambda), \]
with \( \tilde{A}^-(\delta, \lambda) = A^-(\delta, \lambda) \) and \( \tilde{A}^+(\delta, \lambda) = A^+(\delta, \lambda) + \alpha I \).

By a straightforward computation,
\[ \tilde{S}^+(\alpha) \equiv \{ \lambda \in \mathbb{C} \mid \det \left( \tilde{A}^+(\delta, \lambda) - i\tau I \right) = 0, \text{ for some } \tau \in \mathbb{R} \} \]
\[ \equiv \tilde{S}^+_1(\alpha) \cup \tilde{S}^+_2(\alpha), \]
with
\[ \begin{cases} \tilde{S}^+_1(\alpha) = \{ \lambda \in \mathbb{C} \mid \Re \lambda = -\frac{\Im^2 \lambda}{|c - 2\delta\alpha|^2} + \delta\alpha^2 - c\alpha + p'(K)q(K) \}, \\ \tilde{S}^+_2(\alpha) = \{ \lambda \in \mathbb{C} \mid \Re \lambda = -\frac{\Im^2 \lambda}{|c - 2\alpha|^2} + \alpha^2 - c\alpha + \gamma|g(K) - \beta| \}. \end{cases} \tag{2.9} \]

It is easy to see that for any \( 0 < \delta < \delta^* \), if the weight \( \alpha \) satisfies
\[ e^{-\sqrt{c^2 - 4\gamma|g(K) - \beta|}} \leq \alpha < \min \left\{ e^{+\sqrt{c^2 - 4\gamma|g(K) - \beta|}}, e^{+\sqrt{c^2 - 4\gamma p'(K)q(K)}} \right\}, \tag{2.10} \]
then
\[ \tilde{S}^+(\alpha) \subseteq \{ \Re \lambda < 0 \}, \tag{2.11} \]
where \( \delta^* = -\frac{\mu_3^- e^{-p'(K)q(K)}}{(\mu_3^+)^2} > 1 \) with \( \mu_3^- \equiv -\frac{\mu_3^+ e^{-\sqrt{c^2 - 4\gamma|g(K) - \beta|}}}{2} < 0. \)

By (2.6) and (2.11), we have proved the following lemma.
Lemma 2.1. Suppose (H1)-(H4) hold and $c > c_{\text{min}}$. If $0 < \delta < \delta^*$ and $\alpha$ satisfies (2.10), then there exists small enough $\sigma_\alpha > 0$, such that

$$\sup \{ \Re \sigma_{\text{ess}}(L_\alpha) \} \leq -\sigma_\alpha < 0. \quad (2.12)$$

In the following, we shall show the relation between the selection of $\alpha$ and the spectral gap properties of $A^\pm(\delta, \lambda)$, which will be used to define the Evans function in next section.

Define the region $\Omega \triangleq \{ \lambda \in \mathbb{C} \mid \Re \lambda > -\sigma_\alpha \}$. For each $\lambda \in \Omega$, the four eigenvalues of $A^+(\delta, \lambda)$ are

$$\begin{align*}
\mu_1^+(\delta, \lambda) &= -\frac{c-\sqrt{c^2-4[\gamma(\delta(K)-\beta)-\lambda]}}{2}, \quad \mu_2^+(\lambda) = -\frac{c+\sqrt{c^2-4[\gamma(\delta(K)-\beta)-\lambda]}}{2}, \\
\mu_3^+(\lambda) &= -\frac{c-\sqrt{c^2-4[\gamma(K)-\beta]-\lambda}}{2}, \quad \mu_4^+(\delta, \lambda) = -\frac{c+\sqrt{c^2-4[\gamma(K)-\beta]-\lambda}}{2},
\end{align*}$$

(2.13)

with associated eigenvectors

$$\begin{align*}
v_1^+(\lambda) &= (1, 0, \mu_1^+(\delta, \lambda), 0)^T, \quad i = 1, 4, \\
v_2^+(\lambda) &= (1, \nu_j^+(\lambda), \mu_j^+(\lambda), \mu_j^+(\delta, \lambda)\nu_j^+(\lambda))^T, \quad j = 2, 3,
\end{align*}$$

(2.14)

where $\nu_j^+(\lambda) = \frac{c+\delta \mu_j^+(\lambda)}{\gamma(K)}$, $\mu_j^+(\lambda)+\delta[\gamma(K)-\lambda]$.

For $\lambda = 0$, $\mu_1^+(\delta, 0)$ is monotone increasing with respect to $\delta$, and $\lim_{\delta \to +\infty} \mu_1^+(0) = 0$ and $\lim_{\delta \to 0^+} \mu_1^+(0) = -\infty$. If $0 < \delta < \delta^*$, the four distinct real eigenvalues $\mu_i^+(\delta, 0)$, $i = 1, \cdots, 4$ satisfy

$$\mu_1^+(\delta, 0) < \mu_2^+(0) < \mu_3^+(0) < 0 < \mu_4^+(\delta, 0). \quad (2.15)$$

By Theorem 1.1, it is proved that $(U(z), V(z)) \to (K, 0)$ as $z \to +\infty$ with the exponential rate $\mu_4^+(0)$.

It can be checked that for any $0 < \delta < \delta^*$, the selection of $\alpha$ in (2.10) is equivalent to the following spectral gap condition

$$\Re \mu_1^+(\delta, \lambda), \Re \mu_2^+(\lambda) < -\alpha < \Re \mu_3^+(\lambda), \Re \mu_4^+(\delta, \lambda), \text{ for } \Re \lambda > -\sigma_\alpha. \quad (2.16)$$

Let $\mu_i^-(\delta, \lambda)$, $i = 1, \cdots, 4$ are the eigenvalues of $A^-(\delta, \lambda)$ with the eigenvectors $v_i^-(\lambda)$, $i = 1, \cdots, 4$ expressed by

$$v_i^-(\lambda) = (1, \nu_i^-(\lambda), \mu_i^- (\delta, \lambda), \mu_i^- (\delta, \lambda)\nu_i^- (\lambda))^T, \quad i = 1, \cdots, 4,$$

(2.17)

where $\nu_i^- (\lambda) = \frac{c+\delta \mu_i^- (\lambda)}{\beta}$, $\mu_i^- (\delta, \lambda)-\delta[\lambda-\beta R'(u_\beta)]$.

For any $\delta > 0$, (2.6) also implies that the following spectral gap condition must hold true,

$$\Re \mu_1^-(\delta, \lambda), \Re \mu_2^- (\delta, \lambda) < 0 < \Re \mu_3^- (\delta, \lambda), \Re \mu_4^- (\delta, \lambda), \text{ for } \Re \lambda > -\sigma_\alpha. \quad (2.18)$$

In particular for $\delta = 1$ and $\lambda = 0$, the eigenvalues $\mu_i^-(1, 0)$ can be solved explicitly, i.e.

$$\begin{align*}
\mu_1^- (1, 0) &= -\frac{c-\sqrt{c^2-4E_+}}{2}, \quad \mu_2^- (1, 0) = -\frac{c+\sqrt{c^2-4E_+}}{2}, \\
\mu_3^- (1, 0) &= -\frac{c-\sqrt{c^2-4E_-}}{2}, \quad \mu_4^- (1, 0) = -\frac{c+\sqrt{c^2-4E_-}}{2},
\end{align*}$$

with

$$E_{\pm} \triangleq \beta R'(u_\beta) \pm \frac{\sqrt{[\beta R'(u_\beta)]^2-4g'(u_\beta)R(u_\beta)\beta \gamma}}{2}. $$
satisfying $E_- < E_+ < 0$ if $\gamma < \gamma^*$, and $E_\pm$ is not real if $\gamma > \gamma^*$, where $\gamma^*$ in Theorem 1.1 is the critical value determining the types of monotone or oscillatory waves.

It is worth mentioning that the precise exponential decaying rate of $(U_\varepsilon(z), V_\varepsilon(z))$ as $z \to -\infty$ is not obtained in the literature. A natural guess is that the traveling waves with noncritical speeds decay with the slower exponential rate $\mu_0^\varepsilon(\delta, 0)$ and the waves with the critical speed decay with the faster exponential rate $\mu_1^\varepsilon(\delta, 0)$, which will be verified numerically in Section 4.2.

3. The location of the unstable isolated eigenvalues. By virtue of Lemma 2.1, we shall apply Evans function method to locate the unstable eigenvalues of $L_\alpha$ in some bounded subregion of $\{\lambda \in \mathbb{C} \mid \text{Re} \lambda > -\sigma_\alpha\}$.

3.1. The upper bound of the isolated eigenvalues. The location of $\sigma_{\text{ess}}(L_\alpha)$ and the analytic semigroup theory guarantee that all the isolated eigenvalues of $L_\alpha$ must be uniformly bounded and the corresponding eigenfunctions in $X_\alpha$ must decay exponentially at both ends. In this subsection we shall apply the energy method to get the explicit bounds of all the isolated eigenvalues of $L_\alpha$, which will be useful in our later simulation on Evans function.

**Lemma 3.1.** Let $\alpha > 0$ and $\sigma_\alpha > 0$ be chosen as in Lemma 2.1, then there exist $\sigma_0 \in (0, \frac{\sigma_\alpha}{2})$ and positive constants $M_1, M_2$, such that $\sigma(L_\alpha) \cap \{\lambda \in \mathbb{C} \mid \text{Re} \lambda \geq -\sigma_0\}$ are contained in the following bounded subregion:

$$\Omega_0 = \{\mid \text{Im} \lambda \mid < M_2, -\sigma_0 < \text{Re} \lambda < M_1\}.$$  \hspace{1cm} (3.1)

**Proof.** Let $\lambda$ be an isolated eigenvalue of $L_\alpha$ with $\text{Re} \lambda \geq -\sigma_\alpha/2$ and with an eigenfunction $(\phi(z), \psi(z)) \in X_\alpha^2$. It is obvious that $(\phi(z), \psi(z)) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and there exists small enough $\sigma_0 \in (0, \sigma_\alpha/2)$ such that there are no eigenvalues of $L_\alpha$ located on $\{\text{Re} \lambda = -\sigma_0\}$. Without loss of generality, set $\|\phi(z)\|_{L_2}^2 + \|\psi(z)\|_{L_2}^2 = 1$.

Let $A \triangleq \max_{z \in \mathbb{R}} \{\|F_u(U, V)\|\}$ and $B \triangleq \max_{z \in \mathbb{R}} \{g^\prime(U)V\}$, where $F(u, v)$ and $G(u, v)$ are reaction terms of system (1.5), then

$$|F_u(U, V)| \leq A, \ |F_v(U, V)| \leq g(K), \ |G_m(U, V)| \leq \gamma B, \ |G_p(U, V)| \leq g(K) - \beta.$$  

Multiplying (2.3) and (2.3)$_2$ by conjugated functions $\overline{\phi}(z)$ and $\overline{\psi}(z)$ resp, and integrate the equations over $\mathbb{R}$, via the integration by parts, we have

$$\begin{cases} \text{Re} \lambda \|\phi\|_{L_2}^2 + \delta \|\phi_x\|_{L_2}^2 \leq A \|\phi\|_{L_2}^2 + g(K) \|\psi\|_{L_2} \|\phi\|_{L_2}, \\
\text{Im} \lambda \|\phi\|_{L_2}^2 \leq c \|\phi\|_{L_2} \|\phi_x\|_{L_2} + g(K) \|\psi\|_{L_2} \|\phi\|_{L_2}, \end{cases}$$ \hspace{1cm} (3.2)

and

$$\begin{cases} \text{Re} \lambda \|\psi\|_{L_2}^2 + \|\psi_x\|_{L_2}^2 \leq c \|\psi\|_{L_2} \|\psi_x\|_{L_2} + \gamma B \|\phi\|_{L_2} \|\psi\|_{L_2}, \\
\text{Im} \lambda \|\psi\|_{L_2}^2 \leq c \|\psi\|_{L_2} \|\psi_x\|_{L_2} + \gamma B \|\phi\|_{L_2} \|\psi\|_{L_2}. \end{cases}$$ \hspace{1cm} (3.3)

By using the Cauchy’s inequality and adding (3.2) and (3.3) together, we have

$$\begin{align*}
\text{Re} \lambda & \leq A \|\phi\|_{L_2}^2 + \gamma (g(K) - \beta) \|\psi\|_{L_2}^2 + g(K) + \gamma B, \\
\text{Im} \lambda & \leq \frac{A}{2} \left(1 + \|\phi_x\|_{L_2}^2 + \|\psi_x\|_{L_2}^2 \right) + \frac{g(K) + \gamma B}{2} \\
& \leq \frac{A}{2} \left(1 + \|\phi\|_{L_2}^2 + \gamma (g(K) - \beta) \|\psi\|_{L_2}^2 + g(K) + \gamma B \right) + \frac{g(K) + \gamma B}{2}.
\end{align*}$$
Denote
\[
\begin{align*}
M_1 &= M_{11} + \frac{g(K)+\beta}{2}, & M_{11} &= \max\{A, \gamma(g(K) - \beta)\}; \\
M_2 &= \frac{1}{2}(1 + dM_{11}) + g(K)(\frac{C}{4} + \frac{1}{2}) + \gamma B(\frac{1}{4} + \frac{1}{2}); & d &= \max\{1, \frac{1}{3}\},
\end{align*}
\]
then
\[
\text{Re}\lambda < M_1 \quad \text{and} \quad |\text{Im}\lambda| < M_2,
\]
which completes the proof.

3.2. Definition and expression of Evans function. For each \(\lambda \in \Omega_0\), the ODE system
\[
Y'(z) = A(\delta, z, \lambda)Y(z), \quad Y(z) \in \mathbb{C}^4
\]
has two families of linearly independent solutions \(\{Y_i^+(\delta, z, \lambda), i = 1, \cdots, 4\}\) satisfying the asymptotic conditions
\[
e^{-\mu_i^+(\lambda)z}Y_i^+(z, \lambda) \to v_i^+(\lambda) \quad \text{as} \ z \to +\infty, i = 1, 2, 3, 4,
\]
\[
e^{-\mu_i^-(\lambda)z}Y_i^-(z, \lambda) \to v_i^-(\lambda) \quad \text{as} \ z \to -\infty, i = 1, 2, 3, 4,
\]
where the eigenvalues \(\mu_i^\pm(\lambda)\) satisfy the (2.16) and (2.18), and the eigenvectors \(v_i^\pm(\lambda)\) are defined in (2.14) and (2.17).

Let \(S^+(z, \lambda)\) represent the 2-dimensional stable manifold spanned by \(Y_1^+(\delta, z, \lambda)\) and \(Y_2^+(\delta, z, \lambda)\); Let \(U^-(z, \lambda)\) represent the 2-dimensional unstable manifold spanned by \(Y_3^-(z, \lambda)\) and \(Y_4^-(z, \lambda)\). Obviously, \(\lambda \in \Omega_0\) is an eigenvalue of \(\mathcal{L}_\alpha\) if and only if \(U^-(z, \lambda) \cap S^+(z, \lambda)\) has a nonzero intersection. We define the Evans function of \(\mathcal{L}_\alpha\) by
\[
D(\lambda) = e^{-\int_0^1 \text{Tr} A(\delta, z, \lambda) dz}(Y_1^+(\delta, z, \lambda) \wedge Y_2^+(\delta, z, \lambda)) \wedge (Y_3^-(\delta, z, \lambda) \wedge Y_4^-(\delta, z, \lambda)),
\]
where \(\wedge\) is the wedge product.

For convenience of our numerical simulations on Evans function, we shall express \(D(\lambda)\) in the wedge space \(\Lambda^2(\mathbb{C}^4)\), and the linear system (3.5) induces the ODE system on \(\Lambda^2(\mathbb{C}^4)\) as follows
\[
W'(z) = A^{(2)}(\delta, z, \lambda)W(z), \quad W \in \Lambda^2(\mathbb{C}^4), \quad \lambda \in \Omega_0,
\]
with the 6 \times 6 matrix \(A^{(2)}(\delta, z, \lambda) : \Lambda^2(\mathbb{C}^4) \to \Lambda^2(\mathbb{C}^4)\) defined by
\[
A^{(2)}(\delta, z, \lambda) = \begin{pmatrix}
0 & 0 & 1 & -1 & 0 & 0 \\
\frac{g(U)}{\delta} & -\varepsilon & 0 & 0 & 0 & 0 \\
a_{42}(z) & 0 & -c & 0 & 0 & 1 \\
-a_{31}(z) & 0 & 0 & -\varepsilon & 0 & -1 \\
\gamma g'(U)V & 0 & 0 & 0 & -c & 0 \\
0 & \gamma g'(U) & a_{31}(z) & -a_{42}(z) & \frac{g(U)}{\delta} & -(\frac{1}{3} + 1)c
\end{pmatrix},
\]
which is analytic in \(\lambda\) and continuous in \(\delta\) and \(z\), where \(a_{31}(z) \triangleq \lambda - (\gamma g(U) - \beta)\) and \(a_{42}(z) \triangleq \lambda - \gamma g(U) - \beta\).

Let
\[
A^{(2)}_{\pm}(\delta, \lambda) = \lim_{z \to \pm\infty} A^{(2)}(\delta, z, \lambda).
\]
We denote the generalized eigenvalues of \(A^{(2)}_{\pm}(\delta, \lambda)\) by \(\sigma^{\pm}_k(\lambda)\), \(k = 1, \cdots, 6\), which are the sum of any two distinct eigenvalues (counting the algebraical multiplicity) of \(A_{\pm}(\delta, \lambda)\). Let
\[
\sigma^{\pm}_1(\lambda) \triangleq \mu^+_1(\lambda) + \mu^+_2(\lambda), \quad \sigma^{\pm}_1(\lambda) \triangleq \mu^-_1(\lambda) + \mu^-_4(\lambda).
\]

\[\text{(3.8)}\]
Obviously, \( \sigma_1^+(\lambda) \) (or \( \sigma_1^-(\lambda) \)) is a simple eigenvalue with the largest (smallest) real part and is analytic in \( \lambda \in \Omega_0 \). The simplicity of \( \sigma_1^+(\lambda) \) guarantees that there exists the analytic eigenvectors \( \eta_1^+(\lambda) \) and \( \eta_1^-(\lambda) \) corresponding to the eigenvalues \( \sigma_1^+(\lambda) \) and \( \sigma_1^-(\lambda) \), resp., i.e.

\[
\eta_1^+(\lambda) \triangleq v_1^+(\lambda) \wedge v_2^+(\lambda), \quad \eta_1^-(\lambda) \triangleq v_3^-(\lambda) \wedge v_4^-(\lambda). \quad (3.9)
\]

For system (3.7), there exists a unique solution \( W_1^+(z,\lambda) \in \Lambda^2(\mathbb{C}^4), \) which is analytic in \( \lambda \in \Omega_0 \) and has the following asymptotic behavior

\[
e^{-\sigma_1^+(\lambda)z}W_1^+(z,\lambda) \rightarrow \eta_1^+(\lambda), \quad z \rightarrow +\infty. \quad (3.10)
\]

Similarly, system (3.7) has a unique solution \( W_1^-(z,\lambda) \in \Lambda^2(\mathbb{C}^4), \) which is analytic in \( \lambda \) and satisfies

\[
e^{-\sigma_1^-(\lambda)z}W_1^-(z,\lambda) \rightarrow \eta_1^-(\lambda), \quad z \rightarrow -\infty. \quad (3.11)
\]

By virtue of (3.6)-(3.11), we can apply the abstract results in [1] and [22] to get the existence and the expression of Evans function in \( \Lambda^2(\mathbb{C}^4) \), which can be stated as follows

**Lemma 3.2.** For each \( \lambda \in \Omega_0 \), let \( W_1^+(z,\lambda) \) and \( W_1^-(z,\lambda) \) be the unique solutions of (3.7) satisfying (3.10) and (3.11) resp. Define the Evans function \( D(\lambda) \) by

\[
D(\lambda) = e^{-\int_0^z \text{Tr} A(\delta,\sigma,\lambda) ds} \left\langle W_1^+(z,\lambda), \Sigma W_1^-(z,\lambda) \right\rangle,
\]

where \( \langle u, v \rangle = u^T v \) and the \( 6 \times 6 \) matrix \( \Sigma \) is defined by

\[
\Sigma = \begin{pmatrix} 0 & \Sigma_1 \\ \Sigma_1 & 0 \end{pmatrix}, \quad \text{with} \quad \Sigma_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]

Then, \( D(\lambda) \) is analytic in \( \lambda \) and independent of \( z \). Furthermore,

(i) \( D(\lambda) \) vanishes at \( \lambda_0 \in \Omega_0 \) if and only if \( \lambda_0 \) is an eigenvalue of \( L_\alpha \) with an eigenfunction \( (\phi, \psi) \in X^2 \).

(ii) The number of the zeros (counting the algebraic multiplicities) of \( D(\lambda) \) in \( \Omega_0 \) equates the number of eigenvalues of \( L_\alpha \) (counting the algebraic multiplicities) in \( \Omega_0 \).

4. The numerical scheme of traveling waves. In this section, we shall introduce our numerical schemes on approximating the monotone and oscillatory waves with noncritical speeds for models HT1 and HT2, which will also be used in our numerical simulation on the Evans function. The selections of the appropriating schemes are important and sensitive for numerical simulations on two types of wave profiles.

4.1. The range of parameters for models HT1 and HT2. After the rescaling of variables \( x \) and \( t \), the model HT1 can be represented by

\[
\begin{align*}
u_t = \delta u_{xx} + u(1 - u) - uv, \\
v_t = v_{xx} + \gamma v(u - \beta).
\end{align*} \quad (4.1)
\]

Similarly, the model HT2 can be rewritten as

\[
\begin{align*}
u_t = \delta u_{xx} + u(1 - u) - \frac{u}{\alpha+u} v, \\
v_t = v_{xx} + \gamma v \left( \frac{u}{\alpha+u} - \beta \right).
\end{align*} \quad (4.2)
\]
Note that for model (4.1) and model (4.2), the existence and the shapes of the waves heavily depend on the selection of the parameters $\gamma, \beta$ and $\delta$. Before doing numerical simulation on these two models, we need to specify the parameter ranges to guarantee the assumptions (H1)-(H4).

For the ODE model of (4.1), $E_\beta = (\beta, 1 - \beta)$ is a stable node for $0 < \gamma < \gamma^*$ and a stable focus for $\gamma > \gamma^*$. Thus, we just consider the following cases of parameters for system (4.1):

$$0 < \delta < \delta^*, \quad \beta \in (0, 1), \quad \gamma \in (0, \gamma^*) \text{ or } \gamma \in (\gamma^*, +\infty).$$

For the ODE model of (4.2), the types of $E_\beta = (\beta h, (1 - u_\beta)(h + u_\beta))$ depend on the parameters $\beta, h \in (0, 1)$ for each fixed $\gamma > 0$ (see Fig.1). Similarly as in [25], we denote the two black curves by $h_1(\beta) \triangleq \frac{1}{\beta} - 1$ and $h_2(\beta) \triangleq 1 - \beta$. $E_\beta$ is a stable node if $(\beta, h)$ in the region $A_2$, and $E_\beta$ is a stable focus if $(\beta, h)$ in $A_3$; In the region below $h_2(\beta)$, $E_\beta$ is an unstable point. The regions $A_2$ and $A_3$ are separated by the dash curve $h^*_\gamma(\beta)$ (see Fig.1), which is one of the roots of the following quadratic equation of $h$

$$(1 + \beta)^2 \beta h^2 + [4\gamma - 2(1 + \beta)]\beta^2(1 - \beta)h + [\beta^2(1 - \beta)^2 - 4\gamma(1 - \beta)^3] = 0.$$

The region $A_3$ is vanished if $h^*_\gamma(\beta) = \frac{1 - \beta}{1 + \beta}$ with $\gamma = 0$ (see Fig.1(a)). When $\gamma$ increases, the curve $h^*_\gamma(\beta)$ and the regions $A_2$ and $A_3$ are changed as shown in Fig.1. Thus, the parameters for model (4.2) are selected as follows

$$0 < \delta < \delta^*, \quad h \in (0, 1), \quad \beta \in (1 - h, \frac{1}{h + 1}), \quad \gamma \in (0, \gamma^*) \text{ or } \gamma \in (\gamma^*, +\infty).$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The parameters $(\beta, h) \subseteq [0, 1] \times [0, 1]$ with different $\gamma$.}
\end{figure}

4.2. The numerical schemes applied to wave fronts. The traveling fronts for models (4.1) and (4.2) can be approximated by the numerical solutions of a two-point boundary value problem (BVP) in a finite interval $[-sl, sr]$ with large enough $sl$ and $sr$ ([27]), and the asymptotic behavior of the approximated wave fronts at two ends are also required to be guaranteed by the appropriate boundary conditions at $z = -sl$ and $z = sr$.

Let $Y(z) \equiv (y_1(z), y_2(z), y_3(z), y_4(z))^T = (U(z), V(z), U'(z), V'(z))^T$, the ODE system of traveling waves (1.4) can be written as

$$Y'(z) = F(Y), \quad z \in \mathbb{R},$$

\begin{equation}
(4.5)
\end{equation}
i.e. 
\[
\begin{cases}
    y'_1 = y_3, \\
    y'_2 = y_4, \\
    y'_3 = [g(y_1)y_2 - q(y_1)p(y_1) - cy_3]/\delta, \\
    y'_4 = -\gamma(g(y_1) - \beta)y_2 - cy_4,
\end{cases}
\tag{4.6}
\]
with the asymptotic boundary conditions
\[
Y(-\infty) = (u_\beta, v_\beta, 0, 0), \quad Y(+\infty) = (K, 0, 0, 0).
\tag{4.7}
\]

By Theorem 1.1, the traveling wave \((U_c(z), V_c(z))\) tends to \((K, 0)\) as \(z \to +\infty\) with the exponential rate \(\mu^+_3(0) = -c+\sqrt{c^2-4\gamma}g(K)-\beta\), i.e.
\[
K - y_1(z) \sim C_1 e^{\mu^+_3 z}, \quad y_2(z) \sim C_2 e^{\mu^+_3 z},
\]
\[
y_3(z) \sim -C_1 \mu^+_3 e^{\mu^+_3 z}, \quad y_4(z) \sim C_2 \mu^+_3 e^{\mu^+_3 z},
\tag{4.8}
\]
where \(C_1\) and \(C_2\) are positive constants.

It is natural to expect that the wave front with a noncritical speed decays with the slower exponential rate \(\mu^-_3(\delta, 0)\) as \(z \to -\infty\), even though it is hard to be proved rigorously. Our numerical simulations on traveling waves (see Fig. 2 below) can verify the expected decaying rate as \(z \to -\infty\), i.e.
\[
y_1(z) - u_\beta \sim C_3 e^{\mu^-_3 z}, \quad v_\beta - y_2(z) \sim C_4 e^{\mu^-_3 z},
\]
\[
y_3(z) \sim C_3 \mu^-_3 e^{\mu^-_3 z}, \quad y_4(z) \sim -C_4 \mu^-_3 e^{\mu^-_3 z},
\tag{4.9}
\]
where \(C_3\) and \(C_4\) are positive constants.

Let \(\hat{Y}(z)\) be the numerical solution of the following BVP:
\[
\begin{cases}
    \hat{Y}'(z) = F(\hat{Y}), \quad z \in [-sl, sr], \\
    B_1\hat{Y}(-sl) = 0, \quad B_2\hat{Y}(sr) = 0,
\end{cases}
\tag{4.10}
\]
with large enough \(sl\) and \(sr\). The boundary conditions here approximate the asymptotic behaviors (4.8) and (4.9) of the traveling fronts in the following way
\[
B_1\hat{Y}(-sl) = \frac{\hat{y}_3(-sl) - u_\beta - \mu^-_3}{\hat{y}_4(-sl) + \mu^-_3} = 0, \tag{4.11}
\]
and
\[
B_2\hat{Y}(sr) = \frac{K - \hat{y}_1(sr) + \mu^+_3}{\hat{y}_2(sr) - e^{\mu^+_3 sr}} = 0. \tag{4.12}
\]
A initial guess function \(y_{init}(z)\) used in the numerical implementation can be chosen as
\[
y_{init}(z) = \begin{cases}
    (u_\beta, v_\beta, 0, 0), & \text{if } z < 0, \\
    (K - e^{\mu^-_3 z}, e^{\mu^-_3 z}, -\mu^-_3 e^{\mu^-_3 z}, \mu^-_3 e^{\mu^-_3 z}), & \text{if } z \geq 0,
\end{cases}
\]
which is sensitive in \(z < 0\), but the flexible form of \(y_{init}(z)\) in \(z > 0\) is based on the boundary condition (4.12) or can be chosen as the constant steady state \((K, 0, 0, 0)\).

Figure 2(a) exhibits our simulations on the monotone waves \((U(z): \text{increase}; V(z): \text{decrease})\) for model (4.1) and model (4.2).

We can also verify the asymptotic behavior (4.9) by applying the numerical scheme based on the shooting method. Precisely speaking, if we assume that the exponential rate of the waves at \(z = -\infty\) is \(\mu^-_3(\delta, 0)\), we approximate the unique
solution \( Y_4(z) \) of system (4.5) with the asymptotic condition \( \lim_{z \to -\infty} e^{-\mu_4^+ z} Y_4(z) = v_4^- \), and our numerical results indicate the solution is unbounded at \( z \gg 1 \) (see Fig. 2(b)), which implies that \((U_c(z), V_c(z))\) tends to \( E_\beta \) with the exponential rate \( \mu_3(\delta, 0) \) as \( z \to -\infty \).

4.3. The numerical schemes of oscillatory waves. The oscillatory waves occur when \( \gamma > \gamma^* \) and it is obvious that the numerical scheme of the oscillatory wave profiles should be quite different from that of the monotone waves, as oscillatory wave \((U_c(z), V_c(z))\) tends to \( E_\beta \) with a complex (not a real) exponential rate as \( z \to -\infty \), thus we need to establish new schemes to approximate the oscillatory waves.

The asymptotic behavior of oscillatory waves associated with complex eigenvalues plays the key role in our numerical scheme of oscillatory waves, and we shall apply the shooting method to simulate the oscillatory waves.

Let \( \mu \triangleq \mu_3(\delta, 0) = a + bi \) and \( \mu_4(\delta, 0) = a - bi \) be the conjugated non-real eigenvalues of \( A(\delta, 0) \) with \( a > 0 \) and \( b \in \mathbb{R} \), and denote a pair of complex conjugated eigenvectors by \( V = V_1 + iV_2 \) and \( \overline{V} = V_1 - iV_2 \) for some \( V_1, V_2 \in \mathbb{R}^4 \).

Note that the solution \( Y(z) \) of system (4.6) has the following asymptotic behavior

\[
\begin{bmatrix}
y_1(z) - u_\beta \\
y_2(z) - v_\beta \\
y_3(z) \\
y_4(z)
\end{bmatrix} \sim \kappa \left( V e^{a_\beta z} + \overline{V} e^{\overline{a}_\beta z} \right), \quad \text{as } z \to -\infty, \quad (4.13)
\]

for some real parameter \( \kappa \), and we substitute

\[
\begin{align*}
V e^{a_\beta z} &= e^{az} \left[ [V_1 \cos(bz) - V_2 \sin(bz)] + i[V_2 \cos(bz) - V_1 \sin(bz)] \right], \\
\overline{V} e^{\overline{a}_\beta z} &= e^{az} \left[ [V_1 \cos(bz) - V_2 \sin(bz)] - i[V_2 \cos(bz) - V_1 \sin(bz)] \right],
\end{align*}
\]

into (4.13), i.e.

\[
\begin{align*}
U(z) &\sim u_\beta + \kappa e^{az} [V_1^{(1)} \cos(bz) - V_2^{(1)} \sin(bz)], \\
V(z) &\sim v_\beta + \kappa e^{az} [V_1^{(2)} \cos(bz) - V_2^{(2)} \sin(bz)],
\end{align*}
\]

\[
\begin{align*}
U'(z) &\sim \kappa e^{az} [V_1^{(3)} \cos(bz) - V_2^{(3)} \sin(bz)], \\
V'(z) &\sim \kappa e^{az} [V_1^{(4)} \cos(bz) - V_2^{(4)} \sin(bz)],
\end{align*}
\]

as \( z \to -\infty, \quad (4.14) \)

where \( V_i = (V_i^{(1)}, V_i^{(2)}, V_i^{(3)}, V_i^{(4)})^T, i = 1, 2. \)
Denote RHS of (4.14) by $Y^-(\kappa, z)$, which will be chosen as the initial data at $z = -sl$ for the following IVP

$$\begin{cases}
Y'(z) &= \mathcal{F}(Y), \quad z \geq -sl, \\
Y(-sl) &= Y^-(\kappa, -sl),
\end{cases} \quad (4.15)$$

with large enough $sl > 0$. We choose an appropriate parameter $\kappa$ and solve (4.15) numerically by applying shooting method to simulate the oscillatory traveling waves of system (4.5), which are illustrated in figure 3.

Figures 4 and 5 display our simulations on the oscillatory waves for models HT1 and HT2 (i.e. (4.1) and (4.2)) with different parameters.

Figure 3. Numerical oscillatory wave profiles.

Figure 4. Oscillatory traveling waves for model HT1.

5. The simulations on Evans function. In this section, by combining spectral analysis and the Evans function method with some numerical simulation, we investigate the existence and nonexistence of the isolated unstable eigenvalues (with nonnegative real parts) of $\mathcal{L}_\alpha$ in the bounded region $\Omega_0$.

5.1. The existence/nonexistence of zero eigenvalue. Obviously, zero is an eigenvalue of $\mathcal{L}$ with an eigenfunction $(U'(z), V'(z)) \in X = C_{\text{unif}}(\mathbb{R}) \times C_{\text{unif}}(\mathbb{R})$; i.e.

$$\mathcal{L} \begin{pmatrix} U'(z) \\ V'(z) \end{pmatrix} = 0.$$
In this subsection, by asymptotic analysis and numerical verification on models (4.1) and (4.2), we shall show that zero is not an eigenvalue of $L_\alpha$ with $\alpha$ satisfying (2.10).

From (4.9), we know that $U', V' \sim e^{\mu_I^+/(\delta,0)z}$ as $z \to -\infty$. However, (2.16) and (2.18) imply $U'(z), V'(z) \in U^-(z,0)$ but $U'(z), V'(z) \notin S^+(z,0)$, thus $(U'(z), V'(z)) \notin X_\alpha$, where the stable and unstable manifolds $S^+(z,0)$, $U^-(z,0)$ are defined in Section 3.2.

To prove the existence or nonexistence of the zero eigenvalue of $L_\alpha$, it is sufficient to investigate whether or not $Y^+_4(z,0) \in S^+(z,0)$, where $Y^+_4(z,0)$ is the unique solution satisfying

$$
\begin{aligned}
Y'(z) &= A(\delta, z, 0) Y, \quad z \in \mathbb{R}, \\
\lim_{z \to -\infty} Y(z)e^{-\mu_I^+/(\delta,0)z} &= v^-_4(0),
\end{aligned}
$$

(5.1)

with the eigenvalue $\mu_I^+/(\delta,0)$ and an eigenvector $v^-_4(0)$ defined in (2.17). We have to remark that it is hard to get the precise decaying or unboundedness of $Y^+_4(z,0)$ as $z \to +\infty$ to the system (5.1) with a general form. We shall investigate the asymptotic behavior of $Y^+_4(z,0)$ by solving the following initial value problem numerically with some specific $A(\delta, z, 0)$

$$
\begin{aligned}
\hat{Y}'(z) &= A(\delta, z, 0) \hat{Y}, \quad z \geq -sl, \\
\hat{Y}(-sl) &= e^{-\mu_I^+/(\delta,0)sl} v^-_4(0).
\end{aligned}
$$

(5.2)

It can be proved (see [10]) that the numerical solution $\hat{Y}(z) = (\hat{\phi}, \hat{\psi}, \hat{\phi}', \hat{\psi}')$ tends to $Y^+_4(z,0)$ for fixed $z > -sl$ as $sl \to +\infty$. Our numerical simulations indicate that $Y^+_4(z,0)$ is unbounded as $z \to +\infty$, see Figure 6 the numerical profiles of $\hat{\phi}$ and $\hat{\psi}$ for the two models (4.1) and (4.2) with some specific parameters, which implies $Y^+_4(z,0) \notin S^+(z,0)$ and $0 \notin \sigma(L_\alpha)$.

5.2. The numerical schemes of Evans function. As stated in Lemma 3.2, the Evans function is expressed as follows

$$
D(\lambda) = \langle W_1^+(0, \lambda), \Sigma W_1^-(0, \lambda) \rangle, \quad \lambda \in \Omega_0,
$$

(5.3)

where $W_1^+(z, \lambda) \in \Lambda^2(\mathbb{C}^2)$ is the unique solution of system (3.7) satisfying the asymptotic condition (3.10), i.e.

$$
\begin{aligned}
(W_1^+)'(z) &= A^{(2)}(\delta, z, \lambda) W_1^+, \quad z \in \mathbb{R}, \\
\lim_{z \to +\infty} e^{-\sigma_1^+(\lambda)z} W_1^+(z, \lambda) &= \eta_1^+(\lambda),
\end{aligned}
$$

(5.4)
and $W_1^-(z, \lambda) \in A^2(\mathbb{C}^4)$ is the unique solution of following system
\begin{equation}
\begin{cases}
(W_1^-)' = A^{(2)}(\delta, z, \lambda)W_1^-, & z \in \mathbb{R}, \\
\lim_{z \to -\infty} e^{-\sigma_1^+(\lambda)z}W_1^-(z, \lambda) = \eta_1^-(\lambda).
\end{cases} \tag{5.5}
\end{equation}

The eigenvectors $\eta_i^\pm(\lambda)$ associated with eigenvalues $\sigma_i^\pm(\lambda)$ in (5.4) and (5.5) are
\begin{align}
\eta_1^+(\lambda) &= \left(v_2^{(2)} + \mu_1^+ v_2^+(2), \mu_4^+ v_2^+(2), -\mu_1^+ v_2^+(2), 0, \mu_4^+ v_2^+(2)\right)^T, \tag{5.6}
\eta_1^-(\lambda) &= \left(v_4^{(2)} - v_3^{(2)}, \mu_4^- - \mu_3^-, \mu_4^- v_4^{(2)} - \mu_3^- v_3^{(2)}, \\
&\quad \mu_4^- v_3^{(2)} - \mu_3^- v_4^{(2)}, (\mu_4^- - \mu_3^-)v_3^{(2)} - (\mu_3^- \mu_4^-)(v_4^{(2)} - v_3^{(2)})\right)^T, \tag{5.7}
\end{align}

with the eigenvectors $v_i^\pm(\lambda) \triangleq (v_i^{\pm(1)}, v_i^{\pm(2)}, v_i^{\pm(3)}, v_i^{\pm(4)})^T, i = 1, \cdots, 4$ defined in (2.14) and (2.17).

Let $\hat{W}_1^\pm(z, \lambda) \triangleq e^{-\sigma_i^\pm(\lambda)z}W_1^\pm(z, \lambda)$, for better approximation on $W_1^\pm(0, \lambda)$, we compute $W_1^\pm(0, \lambda)$ instead, here $W_1^\pm(z, \lambda)$ can be approximated by the unique solution of the following initial value problem
\begin{equation}
\begin{cases}
(W_1^\pm)' = A^{(2)}(\delta, z, \lambda) - \sigma_i^\pm(\lambda)\mathbf{I})W_1^\pm, & z \leq sr, \\
W_1^\pm(sr, \lambda) = \eta_i^+(\lambda),
\end{cases} \tag{5.3}
\end{equation}

and
\begin{equation}
\begin{cases}
(W_1^-)' = A^{(2)}(\delta, z, \lambda) - \sigma_1^-(\lambda)\mathbf{I})W_1^-, & z \geq -sl, \\
W_1^-(sl, \lambda) = \eta_1^-(\lambda),
\end{cases} \tag{5.4}
\end{equation}

with given large enough $sl, sr$.

The Evans function $D(\lambda)$ in (5.3) can be calculated numerically by
\begin{equation}
D(\lambda) = \left\langle \hat{W}_1^+(0, \lambda), \Sigma \hat{W}_1^-(0, \lambda) \right\rangle. \tag{5.8}
\end{equation}

By Lemma 3.2, the *winding number* $wn$ is used to count the number of zeros of $D(\lambda)$ in a bounded region. By the argument principle ([24]), we calculate
\[
wn = \frac{1}{2\pi i} \oint C D'(\lambda) / D(\lambda) d\lambda,
\]
along a simple closed curve $\Gamma$ surrounding the bounded region $\Omega_0$. We choose a smooth curve $\Gamma$ perturbed from the semicircle $\tilde{\Gamma}$

$$\tilde{\Gamma} \triangleq \{ \text{Re} \lambda = -\sigma_0, |\text{Im} \lambda| \leq M \} \cup \{ \text{Re} \lambda > -\sigma_0, (\text{Re} \lambda + \sigma_0)^2 + (\text{Im} \lambda)^2 = M^2 \},$$

(5.9)

with $\sigma_0 > 0$ obtained in Lemma 3.1 such that no eigenvalues on $\Gamma$, where the radius $M \triangleq \sqrt{M_1^2 + M_2^2}$ is selected as in (3.4). By discretizing the curve $\Gamma$ with $N$ points $\lambda_k \in \Gamma$, $k = 1, \cdots, N$, we compute the Evans function $D_k \triangleq D(\lambda_k)$ at each point $\lambda_k$ and the winding number

$$wn = \frac{1}{2\pi} \sum_{k=1}^{N-1} (\text{Arg}(D_{k+1}) - \text{Arg}(D_k)),$$

(5.10)

where $\text{Arg}(D_k)$ is the argument of $D_k$.

5.3. The numerical simulations on Evans function and winding number.

Using the schemes of the Evans function and the winding number introduced in (5.8) and (5.10), we evaluate the numerical values of $D(\Gamma)$, $\text{Arg}(D(\Gamma))$ and $wn$ along $\Gamma$ to locate the isolated eigenvalues of $\mathcal{L}_\alpha$ in the bounded region surrounded by $\Gamma$.

For models HT1/HT2 presented in Section 4.1, our numerical simulations on $D(\Gamma)$ and $\text{Arg}(D(\Gamma))$ related to the monotone waves and the oscillatory waves with different parameters in (4.3)/(4.4) are illustrated in Fig.7/Fig.9 and Fig.8/Fig.10, resp. Here the smooth curve $\Gamma$ is perturbed from $\tilde{\Gamma}$ in (5.9) with $\sigma_0 = 0.001$ and $M = \sqrt{M_1^2 + M_2^2}$ selected as in (3.4).

**Figure 7.** $\text{Arg}(D(\Gamma))$ related to monotone waves for HT1.

**Figure 8.** $D(\Gamma)$ and $\text{Arg}(D(\Gamma))$ related to oscillatory waves for HT1.
Our numerical simulations on the winding number of $D(\Gamma)$ along $\Gamma$ with all above parameters are always equal to zero (see Fig. 7-10). By virtue of (2.10), Lemmas 2.1, 3.1, 3.2 and the numerical verification of nonexistence of eigenvalues of $L_\alpha$ in $\Omega_0$, we now claim the following spectral and nonlinear stability results of both monotone waves and oscillatory waves with noncritical speeds in the exponentially weighted space.

**Main Theorem.** (Spectral and nonlinear stability of the waves with noncritical speeds) Suppose (H1)-(H4) hold and $c > c_{\text{min}}$. For the model (4.1) (model (4.2) resp), if (4.3) holds ((4.4) holds resp), and $\alpha$ satisfies (2.10), then

$$\sigma(L_\alpha) \subset \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda < -\sigma_\alpha < 0 \},$$

for some $\sigma_\alpha > 0$. Thus, all the traveling wave solutions $(U_c(x - ct), V_c(x - ct))$ of system (1.4) with noncritical speeds are locally exponentially stable in $X_\alpha$. Precisely speaking, the solution $(u(z,t), v(z,t))$ of IVP (2.1) satisfies

$$\| (u(\cdot, t) - U_c(\cdot), v(\cdot, t) - V_c(\cdot)) \|_{X_\alpha} \leq C_\alpha e^{-\sigma_\alpha t} \| (u_0 - U_c, v_0 - V_c) \|_{X_\alpha}, \quad \forall t \geq 0,$$

if $\| (u_0 - U_c, v_0 - V_c) \|_{X_\alpha}$ is small enough.

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