Robust Stabilization of Uncertain Chaotic Systems via Interval Fuzzy Control

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Abstract. This paper presents a robust interval fuzzy control scheme to stabilize a class of uncertain chaotic systems with $H_\infty$ performance. First, we apply the Takagi-Sugeno (T-S) fuzzy modelling scheme to the uncertain chaotic systems to obtain the corresponding T-S interval fuzzy model. Next, the parallel distributed compensation (PDC) method is employed to derive the fuzzy controller. The quadratic stabilizability and performance problem of the interval fuzzy control system with disturbances are then solved. Both the quadratic stability condition and performance are represented by linear matrix inequality (LMI) problems. Finally, an uncertain chaotic system is utilized to demonstrate the feasibility and validity of the proposed interval fuzzy control scheme.

1. Introduction

Over the past two decades, chaotic phenomena [1] have been discussed extensively in various fields of science. The T-S fuzzy model has been shown to be effective in modeling a high nonlinearity term and complex uncertain chaotic systems [2]. For the interval fuzzy system, each local model can be described by the linear interval model.

It is known that the quadratic stabilization and $H_\infty$ performance of linear interval systems are different from those of linear time-invariant systems. Similarly, we want to point out in advance that the robust stabilization of uncertain chaotic systems via interval fuzzy control is different from that of the ordinary T-S fuzzy control system.

2. Problem statement

Consider the uncertain chaotic systems [3] described by

\[ \dot{x}(t) = f(x(t); \bar{x}) + g(u(t); \bar{x}) \cdot u + w(t) \]

(2.1)

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $w(t) \in \mathbb{R}^{m_1}$ denote the state vector, the control input, and the unknown but bounded disturbance, respectively, and $a(t) \in \mathbb{R}^n$, $b(t) \in \mathbb{R}^m$ are time-varying vectors with bounds $\underline{a}$, $\bar{a}$, $\underline{b}$ and $\bar{b}$, respectively.

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Now, we want to introduce the interval fuzzy system to model the uncertain chaotic system (2.1). Premise variables of the interval fuzzy system might be chosen by the nonlinear terms $f\left( a(t);x \right)$ and $g\left( b(t);x \right)$. We can set up the interval model as follows.

Plant Rule $k$:

IF $z_{t0}(t)$ is $M^i_k$ and ... and $z_{p0}(t)$ is $M^r_k$, THEN $\mathcal{A}(t) = A_k x(t) + B_k u(t) + w(t), k = 1 \rightarrow r$ (2.2)

where

$$A_k = \left[ a_{ij} \right] \in \mathbb{A}_k = \left[ A_k, \bar{A}_k \right] = \left[ \begin{array}{c} a_{ij} \leq a_{ij} \leq \bar{a}_{ij}, 1 \leq i, j \leq n \end{array} \right]$$ (2.3a)

$$B_k = \left[ b_{ij} \right] \in \mathbb{B}_k = \left[ B_k, \bar{B}_k \right] = \left[ \begin{array}{c} b_{ij} \leq b_{ij} \leq \bar{b}_{ij}, 1 \leq i \leq n, 1 \leq l \leq m \end{array} \right]$$ (2.3b)

are interval matrices, $M^i_k$ is the fuzzy set, $r$ is the number of IF-THEN rules, and $z_{t0}(t), L, z_{p0}(t)$ are premise variables. The overall interval fuzzy model of (2.2) is inferred as follows.

$$\mathcal{E} = \sum_{k=1}^{r} \mu_k(z_0)(A_k x + B_k u)w(t)$$ (2.4)

where $\mu_k(z_0) = \omega_k(z_0)/\sum_{k=1}^{r} \omega_k(z_0)$, $z_0 = [z_{t0}, L, z_{p0}]$, $\sum_{k=1}^{r} \omega_k(z_0) = 1$ and $\omega_k(z_0) = \prod_{i=1}^{r} M^i_k(z_0) \geq 0, k = 1, 2, K, r$, for all $t$. $M^i_k(z_0)$ is the grade of membership of $z_{j0}$ in $M^i_k$. From (2.4) we have $\sum_{k=1}^{r} \mu_k(z_0) = 1$ and $\mu_k(z_0) \geq 0$. Applying interval expressions [4] to (2.4) to obtain

$$A_k = \left[ a^0_{ij} + \Delta a_{ij} \right] = A_{0k} + \sum_{i,j=1}^{n} \{ e_i^0 \sigma^0_{ij} \} e_j^0, \left\| \sigma^0_{ij} \right\| = \left\| \Delta a_{ij} \right\| \leq \Delta \bar{a}_{ij}$$ (2.5a)

$$B_k = \left[ b^0_{ij} + \Delta b_{ij} \right] = B_{0k} + \sum_{i,j=1}^{n} \{ h_i \sigma^0_{ij} b^0_{ij} \} h_j^0, \left\| \sigma^0_{ij} b^0_{ij} \right\| = \left\| \Delta b_{ij} \right\| \leq \Delta \bar{b}_{ij}$$ (2.5b)

where $e_i \in \mathbb{R}^n$ or $h_i \in \mathbb{R}^n$ denotes the column vector with the $i$th element to be 1 and the others to be 0.

The PDC concept is utilized to design the overall fuzzy controller and the rules of the fuzzy controller are described as follows.

Controller Rule $k$:

IF $z_{t0}(t)$ is $M^i_k$ and ... and $z_{p0}(t)$ is $M^r_k$, THEN $u(t) = F_k x(t), k = 1 \rightarrow r$ (2.6)

where $F_k$ presents the state feedback gain matrices. The final composite state feedback fuzzy controller is obtained as

$$u(t) = \sum_{k=1}^{r} \mu_k F_k x(t)$$ (2.7)

Substituting (2.7) for (2.4) yields the closed-loop interval fuzzy control system

$$\mathcal{E} = \sum_{k=1}^{r} \sum_{i=1}^{r} \mu_k \mu_r \left( (A_k + B_k F_k) x \right) + w(t).$$

The $w(t)$ will weaken the control performance of the interval fuzzy system. Therefore, eliminating the effect of $w(t)$ is an important issue. Since $H_\infty$ control is a very important control design to efficiently eliminate the effect of $w(t)$ on an uncertain chaotic system, it will be employed to deal with the robust performance control in (2.4). To consider the following $H_\infty$ control performance [5]:

$$\int_0^t x^T(t) Q x(t) dt < x^T(0) P x(0) + \rho^2 \int_0^t w^T(t) w(t) dt$$ (2.9)
where $t_f$ is the terminal time of control, $\rho$ is the prescribed attenuation level, the symmetric positive-definite weighting matrix $Q$ is specified beforehand according to the design purpose, and $P$ is some symmetric positive-definite weighting matrix.

3. Fuzzy controller and $H_\infty$ performance

This section explores the fuzzy controller in (2.7) for the uncertain chaotic system in (2.1) with a guaranteed control performance $H_\infty$. We choose a Lyapunov function for the uncertain chaotic system (2.1) as

$$V(t) = x^T(t)Px(t)$$

where the weighting matrix $P = P^T > 0$ and the time derivative of $V(t)$ is

$$\dot{V}(t) = x^T(t)P\dot{x}(t) + \dot{x}^T(t)Px(t)$$

If there exist a symmetric positive-definite matrix $X = P^{-1} \in \mathbb{R}^{n \times n}$, matrices $Z_k \in \mathbb{R}^{m \times n}$, and positive real scalars $\lambda_{kj}, \lambda_{kj}, \alpha_{ij}$ and $\alpha_{il}$ $(i, j = 1, 2, K, n, l = 1, 2, K, m, k, s = 1, 2, K, r)$ satisfy the following LMIs

$$G_i = X A_i^T + A_i X + Z_i^T \bar{B}_i \bar{Z}_i + B_{i0} Z_i + \sum_{i,j=1}^{n} \lambda_{kj} \Delta \bar{a}_{ij} \Delta \bar{e}_i \Delta \bar{e}_j + \sum_{i=1}^{n} \sum_{l=1}^{n} \alpha_{ij} \Delta \bar{a}_{il} \Delta \bar{e}_i \Delta \bar{e}_l + \frac{1}{\rho^2}$$

$$U_1 = \begin{bmatrix} X & 0 & 0 & L & 4 & 0 & 0 & 0 & 4 & 3 \end{bmatrix} \in \mathbb{R}^{n \times n}^\dagger, \quad U_2 = \begin{bmatrix} Z_1^T & 0 & Z_2^T \end{bmatrix} \in \mathbb{R}^{m \times n \times n}$$

$$V_1 = diag \{ \lambda_{111}, L \} \in \mathbb{R}^{3 \times 3}, \quad V_2 = diag \{ \alpha_{411}, L \} \in \mathbb{R}^{3 \times 3}$$

Furthermore, the state feedback matrix of each rule is described by $F_i = Z_i X^{-1}$, then, the uncertain chaotic system (2.1) is quadratically stabilizable and the $H_\infty$ control performance of (2.9) is guaranteed for a prescribed $\rho^2$. One can have

$$V(t_f) - V(0) < -\int_0^{t_f} x^T(t)Qx(t)dt + \rho^2 \int_0^{t_f} w^T(t)w(t)dt$$

$$\int_0^{t_f} x^T(t)Qx(t)dt < x^T(0)P x(0) + \rho^2 \int_0^{t_f} w^T(t)w(t)dt.$$ 

Thus the $H_\infty$ control performance is achieved with a prescribed $\rho^2$.

4. Application example

Consider the chaotic Duffing dynamical system [6],

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2^3 - \delta x_2 + u$$

The membership function of $z_i(t)$ is chosen as in figure 1, then the interval fuzzy model of (4.1) is represented by the following rules.

Plant Rule 1:

IF $z_i(t)$ is $M_i$; THEN $\dot{x}_i = A_i x(t) + B_i u$ \hspace{1cm} (4.3a)

Plant Rule 2:
The rules of the fuzzy controller are designed as follows.

Controller Rule 1:

\[ \text{IF } z_0(t) \text{ is } M_1; \text{ THEN } u(t) = F_1 x(t) \]  

(4.4a)

Controller Rule 2:

\[ \text{IF } z_0(t) \text{ is about } M_1; \text{ THEN } u(t) = F_2 x(t) \]  

(4.4b)

Then the PDC fuzzy controller becomes

\[ u(t) = \left( h_1(z_0) F_1 + h_2(z_0) F_2 \right) x(t) \]  

(4.5)

Selecting the weighting matrix \( Q = 0.01 I \) and applying Theorem 1, one can easily get

\[ X = 10^1 \begin{bmatrix} 0.0383 & -0.1 \\ -0.1 & 6.4881 \end{bmatrix}, \quad \lambda_{122} = \lambda_{222} = 10^8 \cdot 2.9028, \quad F_1 = \begin{bmatrix} -389.2114 & -152.3004 \end{bmatrix}, \]

\[ F_2 = \begin{bmatrix} -386.9601 & -152.2999 \end{bmatrix} \text{ and optimal } \rho = 0.5. \] Simulation results of applying the PDC fuzzy controller (4.5) to the original Duffing system (4.1) are shown in figure 2, where the initial condition is \((-0.5, 0.8)\). Figure 3 shows that the fuzzy controller (4.5) can regulate the states to zero after 2.2 sec even though system parameters are presented by different interval numbers.

5. Conclusions
In this paper, the uncertain chaotic system has been modeled into an interval fuzzy system, where the bounded time-varying disturbance and parametric uncertainties are allowed. The quadratic stabilization and \( H_{\infty} \) performance for interval fuzzy control systems have been explored. All the sufficient conditions of the interval fuzzy control can be expressed in terms of the LMIs. The validity of the developed control scheme has been successfully verified by application to the Duffing system.

Acknowledgments
This work is supported by the National Science Council of Taiwan under grants NSC95-2221-E-006-363-MY2 and NSC95-2221-E-006-382-MY3.

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