Propagation equations for waves in moving thin films of perfect liquids with weak sources at the bottom

A Brener¹, A Yegenova², S Botayeva³

¹ Auezov University, Computer Science Department, Tauke Khan 5, Shymkent, Kazakhstan
² Akhmet Yassavi University, Department of Mathematics, Sattarkhanov Ave 29, Turkestan, Kazakhstan
³ Auezov University, Information Systems and Modeling Department, Tauke Khan 5, Shymkent, Kazakhstan

amb_52@mail.ru

Abstract. The paper deals with the problem of modelling nonlinear waves propagation in thin liquid layers in the presence of weak sources at the bottom. It is established the conditions under which the model equation has the left-hand side bearing a resemblance to the Korteweg-de-Vries equation with slowly evolved parameters, and perturbed right-hand side. At the same time the specific conditions under which the appearance of solitary waves can be nevertheless possible have been defined.

1. Introduction

Analysis of the known results [1, 2, 3] shows that mass sources in liquid films can greatly act the stability of thin waveless flows. Despite a lot of works devoted to propagation of nonlinear waves in layers and liquid films, almost always the flows with constant consumptions have been considered [1, 4]. It is explained not by lack of relevance of this problem but by great diversity of non-linearity and dispersion phenomena [5, 6] in describing flows accompanied by mass and heat transfer or phase transitions. Consequently, it is important whenever it’s possible, to identify special features intrinsic in the evolution and propagation of nonlinear waves in moving thin liquid layers with mass sources.

The most successful of the known by us attempts to solve such problems were undertaken before in the old work [4]. Since then, of course, many works in this field have been published [7, 8, 9]. However, the purpose of this short article is not to provide a detailed overview of all such works [10, 11]. Note that much attention is paid to modeling wave propagation in media with unusual properties [12, 13].

However, some aspects of the problem should be examined with the greater thoroughness because the mentioned methods rely, as a rule, on the existence of constant stationary solutions of basic non-perturbed equations. Especially, as it was shown in [4, 14], a detailed analysis becomes quite complicated even with a constant flow consumption, but with a changing bottom shape, and it turned out that soliton solutions in this case do not exist [2].

Thus, it is important to find the correct way for using perturbation methods in reference to situations where the constants couldn’t be solutions of non-perturbed problem [15, 16].
The goal of this paper is to present propagation equations for surface waves in thin layers of perfect liquids in the presence of mass sources of certain types at the bottom. Thus the submitted paper is fully devoted to establishing the characteristic types of model equations correctly describing the propagation of nonlinear waves in thin film flows in the presence of mass sources of weak intensity. It is also presented the preliminary analysis of the likely behaviour for solutions of these equations. The problems of numerical analysis in this paper are not considered.

2. Basic equations for a thin layer of perfect fluid

Let’s consider a potential flow of perfect liquid with free surface over a plate with slowly changing shape if there exists a weak mass source at the bottom. Our approach on the whole follows the scheme submitted in [2, 7].

The equation of continuity reads [7]

\[ \varphi_x + \varphi_y = 0 \]  \hspace{1cm} (1)

The boundary condition at the bottom with allowing for the bottom mass source is. The equation of continuity reads [2, 7]

\[ \varphi_y h_x + \varphi_x = q; \quad y = -h(x). \]  \hspace{1cm} (2)

The kinematical boundary condition at the free surface is

\[ \eta_t + \varphi_x \eta_x - \varphi_y = 0. \]  \hspace{1cm} (3)

The dynamical boundary condition at the free surface is

\[ \varphi_x + g \eta + \frac{1}{2} \left( \varphi_z^2 + \varphi_y^2 \right) = 0. \]  \hspace{1cm} (4)

Here \( \varphi \) is the velocity potential, \( q \) is the density of the mass stream across the solid bottom, \( y = -h(x) \) is the function for bottom shape, \( \eta(x,t) \) is the perturbed free surface of the liquid layer, \( x \) is a longitudinal coordinate, \( y \) is a normal coordinate, \( t \) is time.

The density of mass stream \( q \) depends on the actual mechanism of physical processes in the neighbourhood of the bottom. Let us consider, for example, the simplest form of the appropriate dependence that can be obtained from the condition (Figure 1)

\[ V_n = k V_t, \]  \hspace{1cm} (5)

where \( V_n \) is the normal component of liquid velocity nearby the bottom and \( V_t \) is the tangent component.

\[ \text{Figure 1. Scheme of liquid flow with mass source at the bottom} \]
This condition can be interpreted as a linear dependence of the mass source intensity on the tangent component of liquid velocity nearby the bottom. This condition has physical meaning, since for an perfect fluid the adhesion condition is not set.

Thus it can be established at

\[ \varphi_x h_y + \varphi_y = k(\varphi_x + \varphi_y) \]  \hspace{1cm} (6)

Let us consider only long-wave perturbations of the free surface supposing the wave length \( l \) much bigger than average thickness of the liquid layer \( h_y \). It seems that such an assumption is quite correct for thin layers, when the characteristic longitudinal scale prevails the others.

Thus the small parameter \( \mu \) is introduced as follows

\[ h_y^3/l^3 = \mu << 1 \]  \hspace{1cm} (7)

In addition, suppose that the perturbation amplitude is also small that is necessary in order to stay in framework of weak nonlinearity:

\[ a/h_y = \varepsilon << 1 \]  \hspace{1cm} (8)

If both of small parameters introduced above have the same order (\( \varepsilon = \mu \)), but the coefficient of mass stream \( k \) has a higher order, then following ratio is reasonable.

\[ k = k, \varepsilon \mu = k, \varepsilon^2 \]  \hspace{1cm} (9)

In framework of weak non-linearity the bottom shape is a function of the slow variable \( X = \varepsilon x \).

Let us use the dimensionless variables [7]

\[ x \rightarrow x l, \varphi \rightarrow \varphi h_y / \sqrt{gh_y}, t \rightarrow t l / \sqrt{gh_y}, y \rightarrow y h_y, \eta \rightarrow \eta a, h \rightarrow h h_y. \]  \hspace{1cm} (10)

Using dimensionless variables, system (1)-(4) takes the form

\[ \varepsilon \varphi_{xx} + \varphi_y = 0, \]  \hspace{1cm} (11)

\[ \varepsilon^2 \varphi_y h_x + \varphi_y = k, \varepsilon^2 \varphi_x, \]  \hspace{1cm} (12)

\[ \varphi_x + \eta + \frac{1}{2} \varepsilon (\varphi_{xx} + \varphi_x) = 0, \]  \hspace{1cm} (13)

\[ \eta_x + \varepsilon \varphi, \eta_y - \frac{1}{\varepsilon} \varphi_y = 0. \]  \hspace{1cm} (14)

The Taylor expansion of the velocity potential in a vicinity of the bottom gives [2]

\[ \varphi = F(x,t) + \varphi_y (y + h) + \frac{1}{2} \varphi_{yy} (y + h)^2 + \frac{1}{6} \varphi_{yyy} (y + h)^3 + \frac{1}{24} \varphi_{yyyy} (y + h)^4 + \ldots. \]  \hspace{1cm} (15)
Using (15) and after neglecting terms of higher than \( \varepsilon \) orders, (13), (14) take the forms

\[
\eta = -F_r + \varepsilon H F_{xx} - \varepsilon F_{x x}^3, \tag{16}
\]

\[
F_{rr} - HF_{xx} = \varepsilon \left( (H_z - k_z) F_r + H^2 F_{z z} - 2 F_r F_{z z} - F_r F_{z x} - \frac{1}{6} F_{xxx} H \right), \tag{17}
\]

where \( H = 1 + h \).

For more comfortable use of the methods of a secular perturbations theory it is reasonable to look for the function \( F(x,t) \) in the form

\[
F(x,t) = F\left( \frac{\theta}{\varepsilon}, X \right) + \varepsilon F\left( \frac{\theta}{\varepsilon}, X \right) + O(\varepsilon^2), \tag{18}
\]

where \( \theta \) is a special self-similar variable depending on slow coordinates \( X = \varepsilon x, \ T = \varepsilon t \). [2].

Thus the main equation for function \( F(x,t) \) reads

\[
F_{\theta \theta} \left( \theta_x^2 - H \theta_{xx} \right) = \varepsilon \left[ F_{\theta \theta} (H \theta_{xx} - \theta_{r r} + H_x - k_z) - f_{\theta \theta} (\theta_x^2 - H \theta_x^2) - 2 F_{\theta \theta} \theta_x + 
F_{\theta \theta \theta \theta} \left( H^2 \theta_x^2 \theta_x^4 - \frac{1}{6} \theta_x^4 \right) - 3 F_{\theta \theta \theta \theta} \theta_x^2 \theta_r \theta_r - \right. \tag{19}
\]

In order to satisfy this equation in a zero order the following dispersion relation should be fulfilled

\[
\theta_x^2 - H \theta_x = 0. \tag{20}
\]

For eliminating secular terms in the next order it is necessary to suppose

\[
U_x - \frac{3}{2} \frac{\theta_x \theta_r}{H} U U_x + \frac{H \theta_x}{4} \left( \theta_x^2 - \frac{1}{3} H \theta_x \right) U_{\theta \theta} = \frac{\theta_x - H \theta_{xx} + \theta_x (k_z - H_x)}{2 H \theta_x} U, \tag{21}
\]

where \( U = F_{\theta \theta} \).

As a result it can be concluded that for the accepted order of mass source intensity at the bottom its influence on non-linear waves propagation in thin liquid layer may be described by equation (21). Under choosing \( \theta_x < 0 \) and \( \theta_x > 0 \) this equation has a structure of the left side of (21) which bears a resemblance to the Korteweg-de-Vries equation (KdV) [1].
If a lower order for mass source is accepted than the structure of the evolution equation (21) will be destroyed. Indeed let it be \( k = k_1 \varepsilon \). Thus the main order instead of (20) reads

\[
F_m (\theta_\tau^2 - H \theta_\delta^2) + k_1 \theta_x F_\theta = 0.
\]  

(22)

In next orders we obtain a system of linear recurrent equations which describe decrementing or incrementing perturbations.

However in any case the complete structure of KdV equation is destroyed by the perturbation in the right-hand side, and that can be interpreted as damping influence of mass source [17, 18].

A numerical study of equation (21) requires adaptation of the methods proposed in works [19, 20], and is the subject of the current research. The results will be published in subsequent works as interesting results are obtained.

At the same time, if we consider the right-hand side of equation as a perturbing effect of the mass source, and assume, with a weak intensity of this source, a linear relationship between the source intensity and the flow rate \( U_\theta \), then the propagation equation can acquire the following structure

\[
U_x + UU_\theta + \beta U_\theta u_\theta = -c U_\theta .
\]  

(23)

Then equation (23) is reduced to the form known in the theory of solitary waves [5]

\[
U_x + (c + U)U_\theta + \beta U_\theta u_\theta = 0 .
\]  

(24)

Let us look for a solution in the form of a stationary wave

\[
U = U(\xi) = U(\theta - cX),
\]  

(25)

where \( \xi \) is a phase variable and \( c \) is a phase velocity.

After twofold integration and some transforms, equation (24) in the case \( \beta > 0 \) transforms to the form [5]

\[
3\beta \left( \frac{dU}{d\xi} \right)^2 = (U_1 - U)(U_2 - U)(U_3 - U).
\]  

(26)

Here \( U_1, U_2, U_3 \) are the constants that are expressed through control parameters of equation (23) and integration constants.

Finite solutions of equation (26) can be obtained under \( U_1 \geq U \geq U_2 > U_3 \) in the form [5]

\[
U(\xi) = (U_1 - U_3) \text{dn}^2 \left( \frac{\xi}{\sqrt{12\beta}}, \frac{U_1 - U_2}{U_1 - U_3} \right) + U_3 .
\]  

(27)

Here \( \text{dn} \) is the Jacobi elliptic function [5].

In a particular case \( U_2 = U_3 \) expression (27) is reduced to the form of classical soliton [1, 2]

\[
U(\xi) = \frac{U_1 - U_3}{\text{ch}^2 \left( \frac{\xi}{\sqrt{12\beta}} \right)} + U_3.
\]  

(28)
Structure of the propagation equation for non-linear waves may be also essentially changed under the influence of phenomena taking place at the free surface [21, 22, 23].

Namely, equation (21) after the rearrangements and transforms which pursue an aim to eliminate secular growth of perturbations can be expanded by the term describing a surface activity [24] in the following form

$$U_x - \frac{3 \theta_x \theta_t}{2H} U_U + \frac{H \theta_x}{4} \left( \theta_t^2 - \frac{1}{3} H \theta_x^2 \right) U_{\theta\theta} =$$

$$\left( \theta_t - H \theta_{xx} + \theta_x \left( k_t - H \right) \right) U + \frac{\sigma \theta_t^2}{2H \theta_x} U_{\theta\theta},$$

where $\sigma$ is the coefficient of surface activity.

A preliminary analysis of the references allows suggesting in this case the development of solitary waves, as well as the wave breaking at various ratios of control parameters [25].

3. Conclusions

From preliminary analysis carried out in the submitted work, it can be concluded that the nature of the propagation waves and evolution of nonlinear waves characteristics in systems with mass sources substantially depends on both the order of the intensity of the sources and the form of the boundary conditions. Moreover, in various situations, even the types of propagation equations can change significantly. Presented theory allows describing the main propagation characteristics of the waves in thin layers of moving perfect liquids with sources.

It is shown that the main structure of KdV equation for flows with sources is destroyed by the perturbation in the right-hand side, and that can be interpreted as damping influence of mass source. However it is established the certain conditions when under the weak intensity of sources the solitary waves with slowly changing characteristics occurrence is possible.

Thus, the main result of the work is establishing the characteristic types of equations that correctly describe the propagation of nonlinear waves in thin film flows with mass sources. Further development of the results obtained of the work will involve both a systematic asymptotic analysis of the behavior of solutions and numerical investigations. The corresponding results will become the content of subsequent articles of the authors. The authors intend also to present the development of the asymptotic methods proposed in this paper for describing nonlinear waves in thin layers of viscous fluid in the presence of mass sources.

References

[1] Karpman V I 2011 Nonlinear Waves in Dispersive Media (Oxford, New York, Toronto, Sydney: Pergamon Press)
[2] Newell A C 1985 Solitons in mathematics and physics ( Soc. for Industrial and Applied Math. Univ. of Arizona)
[3] Dodd R K, Eilbeck J C, Gibbon J D, Morris H C 1984 Solitons and Nonlinear Wave Equations (London, New York, Tokyo: Academic Press)
[4] Grimshaw R 1979 Proc.R. Soc. A 368 359
[5] Zaslavsky G M, Sagdeev R Z 1988 Introduction to nonlinear Physics (Moscow: Nauka) (In Russian)
[6] Zerroukat M 1998 Mathematical Modelling of Moving Boundaries in Phase Change Problems ACOMEN ’98 (Maastricht) pp 113-124
[7] Brener A M, Tashimov L 2001 Wave Regimes of the Vapour Film Condensation Computational Modelling of Free and Moving Boundary Problems (Southampton) pp 51-60
[8] Tseluiko D, Kalliadasis S 2011 *J. Fluid Mech.* vol 673 (Cambridge University Press) p 19
[9] Trifonov Yu 2017 *Physics of Fluid* vol 29 (American Institute of Physics) p 1
[10] Abcha N, Pelonovsky E, Mutabazi I 2018 *Nonlinear Waves and Pattern Dynamics* (Cham: Springer)
[11] Tsvelodub O Yu 2016 *J. Phys.: Conf. Ser.* 754 032020
[12] Tsvelodub O Yu, Arkhipov D G 2017 *Journal of Applied Mechanics and Technical Physics* vol 58 No 4 p 619
[13] Oron A and Gottlieb O 2002 *Phys. Fluids* vol 14 2622
[14] Tiwari N and Davis J M 2010 *Phys. Fluids* vol 22 042106
[15] Archilla J F R, Zolotaryuk Ya, Kosevich Ya, Doi Yu 2018 *CHAOS* vol 28 (AIP Publishing) 083119
[16] Shiroky I B and Gendelman O V 2018 *CHAOS* vol 28 (AIP Publishing) 023104
[17] Vorotnikov A, Starosvetsky Y, Theocharis G, Kevrekidis P G 2018 *Physica D* vol 365 p 27
[18] Champneys A, Malomed B, Yang J, Kaup D 2001 *Physica D* vol 152-153 p 340
[19] Mnebhi-Loudyi A, Boudi E M, Ouaza D 2019 *International Journal of Mechanics* vol 13 pp 52-59
[20] Hossenny C 2012 *Numerical Methods For The KdV Equation And Related Water Waves* (University of Technology, Mauritius)
[21] Zakharov V E, Gelash A A 2013 *Theor. Math. Phys.* vol 120(2) p 997
[22] Onorato M, Proment D, Toffoli O 2010 *The European Physical Journal Special Topics* vol 185(1) p 45
[23] Yazhou Shi, Xiangpeng Li, Ben-gong Zhang 2018 *Advances in Mathematical Physics* vol 2018 p 1
[24] Bassem M, Rima M, Abdelhamid T, Kechkar N, Selima E S 2017 *WSEAS Transactions on Fluid Mechanics* vol 18 pp. 220-228
[25] Iele B, Palleschi B, Gallerano F 2020 *WSEAS Transactions on Fluid Mechanics* vol 15 pp. 41-53