Characterization of equivariant maps and application to entanglement detection

Ivan Bardet, Benoît Collins, Gunjan Sapra

December 5, 2018

Abstract
We study equivariant linear maps between finite-dimensional matrix algebras, as introduced in [1]. These maps satisfy an algebraic property which makes it easy to study their positivity or $k$-positivity. They are therefore particularly suitable for applications to entanglement detection in quantum information theory. We characterize their Choi matrices and give certain properties of certain subclass of them. In particular, we focus on a subfamily that we call $(U^{a} \otimes U^{b})$-equivariant. They can be seen as both a generalization of maps invariant under unitary conjugation as studied by Bhat in [2] and as a generalization of the equivariant maps studied in [1]. Using representation theory, we fully compute them and study their graphical representation. We finally apply them to the problem of entanglement detection. We conjecture that they form a sufficient (infinite) family of positive maps to detect all entangled density matrices.

1 Introduction

Due to their crucial role in numerous tasks in quantum processing and quantum computation, it is of great importance to decide whether a certain density matrix on a bipartite system is entangled or not [3, 4]. However this problem, referred to as entanglement detection, is known to be a computationally hard one in quantum information theory [5, 6]. In the last two decades, lots of effort have been accomplished in order to determine necessary and sufficient conditions for a density matrix to be entangled. For instance, one such criterion is the $k$-extensibility hierarchy [7], which provides a sequence of tests to check that the density matrix is separable, that ultimately detects all entanglement.

Another appealing method is the positive map criterion [8], which gives an operational interpretation of the Hahn-Banach theorem applied to the convex set of separable density matrices. The Horodecki’s Theorem thus states that a density matrix $\rho$ is entangled if and only if there exists a positive map $\Phi$ such that $I \otimes \Phi(\rho)$ is not positive semi-definite, where $\Phi$ only acts on one of the two subsystems. Necessarily, this positive map is not completely positive.
The most well-known example of such map is the transpose map, and it leads to the positive partial transpose (PPT) criterion \[9\]. However, because of the complex geometrical structure of the set of separable density matrices, an infinite number of maps that one does not know how to describe efficiently would be necessary to detect all entangled states (see for instance \[10\]). The goal of this article is to propose a family of maps, with increasing complexity, which might suffice to detect any entanglement. Their main interest lies in that it is rather easy to check if they are positive or not.

Indeed, proving that a map is completely positive is easy: it is enough to check that its Choi matrix is positive semi-definite \[11\]. Such a criterion does not exist in general to check that a given map is positive, which makes it difficult to find interesting examples of positive but not completely positive maps. Choi \[12\] gave in 1973 the first example of a linear map on \(M_n(\mathbb{C})\) which is \((n-1)\)-positive but not \(n\)-positive. One decade later in 1983, Takasaki and Tomiyama \[13\] gave a method to construct any number of linear maps on finite-dimensional matrix algebras, which are \((k-1)\)-positive and not \(k\)-positive. Interestingly, all these examples fall in the class of maps introduced by Collins et al. in \[1\], called equivariant maps. In the same article, they proved that if a linear map happens to be equivariant, its \(k\)-positivity depends upon the positivity of a \(k\)-blocks submatrix of the corresponding Choi matrix. In a sense, it means that it is as easy to check that an equivariant map is \(k\)-positive, as it is to check that it is completely positive. They subsequently studied a parametric family of equivariant linear maps on \(M_3(\mathbb{C})\) with values in \(M_3(\mathbb{C}) \otimes 2\). In this article, we analyse in more depth equivariant maps from \(M_n(\mathbb{C})\) to \(M_n(\mathbb{C}) \otimes k\) for all \(k, n \geq 1\), and characterize a large class of them.

More precisely, we are concerned with two different objectives. The first one is to get a full understanding of equivariant maps. We only get sparse results in this direction. As a first insight, we give a characterization of equivariant maps in terms of their Choi matrices in Theorem \[3\]. We also define a subclass of them, the unitarily equivariant maps, which are more tracktable objects. Corollary \[3.3\] is one of the main results of this paper, where we prove that unitarily equivariant maps were the equivariance property is given in terms of a unitary representation can be reduced to (a corner of) a \((U^a \otimes U^b)\)-equivariant maps, defined in the next section. Our guess is that any equivariant maps can be described in terms of the latter.

The second objective is to fully compute a subclass of such maps, the \((U^a \otimes U^b)\)-equivariant maps, generalizing the examples in \[1\] and the characterization in \[2\] of linear maps invariant under conjugation. We then focus on the application to entanglement detection of such maps. We prove that any entangled density matrix can be detected using a positive unitarily equivariant map, but with image on an infinite dimensional Hilbert space. We conjecture that the latter requirement is not necessary and that this Hilbert space can
be taken finite dimensional. We subsequently show that this conjecture implies that any entangled density matrix can be detected by a \((U^a \otimes U^b)\)-equivariant positive map.

We note that not all known examples of positive are equivariant (see for instance [14, 15]).

This article is organized as follows: In Section 2, we introduce the equivariant maps, list some of their properties and give some examples, among which the one of Choi [12], Takasaki and Tomiyama [13]. In Section 3, we give the different characterizations mentioned above. We study the graphical representations of the Choi matrices of \((U^a \otimes U^b)\)-equivariant maps in Section 4. We focus on entanglement detection in Section 5.

2 Equivariant maps: definitions and examples

In this section, we present definitions of equivariant linear maps and give an explanation as to why it is important to study these maps.

2.1 Notations, definitions and first examples

For a positive integer \(n\), \(M_n(\mathbb{C})\) is the set of square matrices, with entries from \(\mathbb{C}\) of size \(n\), with canonical orthonormal basis \((e_{ij})_{1 \leq i,j \leq n}\). The unitary group on \(\mathbb{C}^n\) is denoted by \(U_n\). We denote by \(1_n\) the identity matrix in \(M_n(\mathbb{C})\) and by \(i_n\) the identity map acting on \(M_n(\mathbb{C})\). We write \(B_n = \sum_{i=1}^{n} |e_i \rangle \otimes |e_i \rangle\), the non-normalized maximally entangled Bell vector in \((\mathbb{C}^n \otimes \mathbb{C}^n)\). The rank-one projection on \(B_n\) is denoted by \(B_n B_n^*\). Finally, \(A^t\) and \(\text{Tr}(A)\) denote the transpose and (non-normalized) trace of a matrix \(A \in M_n(\mathbb{C})\) respectively. \(\theta_n\) denotes the transpose map \(A \mapsto A^t\) on \(M_n(\mathbb{C})\).

If \(\mathcal{H}\) and \(\mathcal{K}\) are two Hilbert spaces, \(\mathcal{B}(\mathcal{H})\) denotes the space of all bounded linear operators on \(\mathcal{H}\) and \(\mathcal{B}(\mathcal{H}, \mathcal{K})\) the space of all bounded linear maps from \(\mathcal{H}\) to \(\mathcal{K}\). A self-adjoint map \(\Phi \in \mathcal{B}(\mathcal{H}, \mathcal{K})\) is such that \(\Phi(X^*) = \Phi(X)^*\) for all \(X \in \mathcal{H}\). In the following, we will assume that all linear maps are self-adjoint.

We are now ready to give the main definitions of this article.

**Definition 2.1.** Let \(n,N \geq 2\) be two natural numbers. A self-adjoint linear map \(\Phi : M_n(\mathbb{C}) \rightarrow M_N(\mathbb{C})\) is called:

(i) Equivariant, if for every unitary matrix \(U \in M_n(\mathbb{C})\) there exists \(V = V(U) \in M_N(\mathbb{C})\) such that

\[
\Phi(UXU^*) = V(U) \Phi(X) V(U)^* \quad \forall \ X \in M_n(\mathbb{C});
\]

(ii) Unitarily equivariant, if furthermore the operator \(V(U)\) in the previous definition can be taken unitary.
(iii) \((U \otimes a \otimes U \otimes b)\)-equivariant, if there are \(a, b\) natural numbers such that 
\[ N = n^{a+b} \] and 
\[ M_N(C) \equiv M_n(C) \otimes a \otimes M_n(C) \otimes b, \] and such that for every unitary 
\( U \in M_n(C), \)
\[ \Phi(U X U^*) = (U \otimes \alpha \otimes U \otimes \beta)^* \] \quad \forall X \in M_n(C). \quad (2)

Thus, \((U \otimes a \otimes U \otimes b)\)-equivariant maps are a subfamily of unitarily equivariant maps, that is itself a subclass of equivariant maps. Besides, we prove in Corollary \[ \text{Corollary 3.3} \] that every unitarily equivariant map where 
\( U \mapsto U \) is a unitary representation can be seen as a corner of an \((U \otimes a \otimes U \otimes b)\) equivariant map. Although we were not able to rule out their existence, we do not know of any equivariant map which is not a unitarily equivariant map.

Example 2.2. Bhat characterized in \([2]\) all \((U)\)-equivariant maps (that is, for 
\(a = 0\) and \(b = 1\)) on \(B(H)\) for some Hilbert space \(H\), not necessarily finite dimensional. More precisely, he proved that a linear map 
\( \Phi \) acting on \(B(H)\) satisfies 
\[ \Phi(U X U^*) = U \Phi(X) U^* \] for all \( X \in B(H) \) iff there exist \( \alpha, \beta \in \mathbb{C} \) such that 
\[ \Phi(X) = \alpha X + \beta \text{Tr}[X]I_H. \]

Directly from this, we get that any \((U)\)-equivariant map \( \Psi \) on \(B(H)\) is of the form 
\[ \Psi(X) = \alpha \theta_n(X) + \beta \text{Tr}[X]I_H, \] where \( \theta_n \) is the transpose map. Indeed, we can check that \( \theta_n \circ \Psi \) is \((U)\)-equivariant and apply Bhat’s result. We shall similarly characterize all \((U \otimes a \otimes U \otimes b)\)-equivariant maps in Theorem \[ \text{Theorem 3.5} \] thus generalizing these two cases, when \(H\) is finite dimensional.

Establishing \(k\)-positivity of a linear map is a difficult task, even on low dimensional matrix algebras. In this regard, a criterion which is a necessary and sufficient condition for an equivariant map to be \(k\)-positive was given in \([1]\).

Theorem 2.3. \([7]\) Theorem 2.2] Let \( \Phi : M_n(C) \rightarrow M_N(C) \) be an equivariant map. Then, for \( k \leq \min\{n, N\}, \) \( \Phi \) is \(k\)-positive if and only if the block matrix 
\[ \Phi(e_{ij})_{i,j=1}^k \] is positive, where \( (e_{ij})_{1 \leq i,j \leq n} \) are the matrix units in \(M_n(C)\).

Incidentally, some well-known examples of \(k\)-positive but not completely positive linear maps are actually examples of unitarily equivariant maps, even though this was not explicitly stated when they were introduced. We recall these examples and give alternative proof of their \(k\)-positivity, based on Theorem 2.3.

(i) Every \(*\)-homomorphism or \(*\)-anti-homomorphism on a finite-dimensional matrix algebras is equivariant. Such maps are always completely positive or co-completely positive.

(ii) Choi \([12]\) Theorem 1] gave the first example of a linear map on \(M_n(C)\) which is \((n-1)\)-positive but not \(n\)-positive, given by 
\[ \Phi : M_n(C) \rightarrow M_n(C) \]
\[ A \mapsto \{(n-1)\text{Tr}(A)\}I_n - A. \]
The above map is unitarily equivariant (take $V(U) = U$). We apply Theorem 2.3 to prove that $\Phi$ is $(n - 1)$-positive. The eigenvalues of $[\Phi(e_{ij})]_{i,j=1}^{n-1}$ are 0 with multiplicity 1 and $(n - 1)$ with multiplicities $n(n - 1) - 1$, which are positive. Hence, $\Phi$ is $(n - 1)$-positive.

(iii) Tomiyama [16, Theorem 2] gave an example of a parametric family of linear maps and studied the conditions of its $k$-positivity. The map is defined by:

$$\Psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

$$A \mapsto \frac{1}{n} \text{Tr}(A) \mathbb{1}_n + (1 - \lambda) A.$$  

This map is $k$-positive for any $1 \leq k \leq n$ if and only if $0 \leq \lambda \leq 1 + \frac{1}{nk-1}$.

We give an alternative proof based on Theorem 2.3. Indeed, $\Psi$ is unitarily equivariant (take $V(U) = U$) and to find the conditions of $k$-positivity on $\Psi$, we only need to find the values of $\lambda$ such that $[\Psi(e_{ij})]_{i,j=1}^k$ is positive. It can be easily seen that $\frac{1}{n}$ and $\frac{1}{n} + (1 - \lambda)k$ are two distinct eigenvalues of $[\Psi(e_{ij})]_{i,j=1}^k$ with different multiplicities. Therefore, the map $\Psi$ is $k$-positive if and only if $\lambda \geq 0$ and $\lambda \leq 1 + \frac{1}{nk-1}$.

(iv) Collins et.al [1] gave the following family of parametric linear maps which are $(U \otimes U)$-equivariant.

Let $\alpha$ and $\beta$ be two real numbers and $n \geq 3$. Then the family of maps,

$$\Phi_{\alpha,\beta,n} : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$$

$$A \mapsto A^t \otimes \mathbb{1}_n + \mathbb{1}_n \otimes A + \text{Tr}(A)(\alpha \mathbb{1}_{n^2} + \beta B_n).$$

There are values of parameters $\alpha$ and $\beta$ for which the family of maps $\Phi_{\alpha,\beta,n}$ is positive and not completely positive, more detail can be found in [1].

2.2 Properties of equivariant linear maps

In this section, we study some basic properties of equivariant linear maps, which will help us to give a characterization of these maps.

Lemma 2.4. Let $a, b \in \mathbb{N}$. Then,

1. The set of all $(U^\otimes a \otimes U^\otimes b)$-equivariant maps is a vector space.

2. The set of all unitarily equivariant maps is not a vector space. Neither is the set of equivariant maps.

Proof. It is an easy calculation to show that the set of all $(U^\otimes a \otimes U^\otimes b)$-equivariant maps is closed under addition and scalar multiplication. Hence, this set is a vector space.

We give an example of two unitarily equivariant maps such that their sum is not even equivariant, which is enough to conclude that both sets are not vector spaces.
spaces.

Let $\Phi_1, \Phi_2 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be given by $\Phi_1 = i_n$ and $\Phi_2 = \theta_n$. It is straightforward that both maps are unitarily equivariant. We prove that the map $(\Phi_1 + \Phi_2)$ is not equivariant, that is, there exists a unitary $U \in M_2(\mathbb{C})$ such that there is no $V \in M_2(\mathbb{C})$ with

$$(\Phi_1 + \Phi_2)(UXU^*) = V(\Phi_1 + \Phi_2)(X)V^* \quad \forall \ X \in M_2(\mathbb{C}).$$

Let

$$U = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}.$$

Take $X = e_{11}$, where $e_{11}$ is a matrix unit in $M_2(\mathbb{C})$. We prove that there is no $V \in M_2(\mathbb{C})$ such that,

$$(\Phi_1 + \Phi_2)(Ue_{11}U^*) = V(\Phi_1 + \Phi_2)(e_{11})V^*.$$  \hspace{1cm} (4)

On the contrary, assume that there exists a matrix $V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \in M_2(\mathbb{C})$ such that Equation (4) holds. We have

$$V(\Phi_1 + \Phi_2)(e_{11})V^* = 2 \begin{bmatrix} |v_{11}|^2 & v_{11} \bar{v}_{21} \\ v_{21} \bar{v}_{11} & |v_{21}|^2\end{bmatrix}, \quad (\Phi_1 + \Phi_2)(Ue_{11}U^*) = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (4)

By comparing the corresponding entries of above two matrices, we get

$|v_{11}|^2 = |v_{21}|^2 = 1$ and $v_{11} \bar{v}_{21} = v_{21} \bar{v}_{11} = 0$

which is a contradiction.

**Example 2.5.** To conclude this section, we give an example of a completely positive linear map on $M_2(\mathbb{C})$ which is not equivariant. Consider the linear map

$$\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \quad A \mapsto A - A^t + \text{Tr}(A)I_2.$$  \hspace{1cm} (5)

The map $\Phi$ is completely positive, but it is not equivariant. Let

$$U = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} \text{ and } X = e_{12} + e_{21}.$$  \hspace{1cm} (5)

Then

$$\Phi(UXU^*) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } \Phi(X) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (5)

Clearly, there is no $V \in M_2(\mathbb{C})$ such that $\Phi(UXU^*) = V\Phi(X)V^*$ for the given unitary $U$ and $X = e_{12} + e_{21}$. Hence, $\Phi$ is not equivariant.
3 Characterization of equivariant maps

In this section, we work on the problem of characterizing equivariant maps defined on finite-dimensional matrix algebras. We first characterize general equivariant linear maps in terms of their Choi matrices in Theorem 3.1. Later, we prove that every unitarily equivariant map where $U \mapsto V(U)$ is a unitary representation can be realized as an $(\mathcal{U}^{\otimes a} \otimes U^{\otimes b})$-equivariant map for some values of $a$ and $b$. We then compute explicitly all $(\mathcal{U}^{\otimes a} \otimes U^{\otimes b})$-equivariant maps in Theorem 3.5.

3.1 Characterization of equivariant maps in terms of their Choi matrix

We recall that the Choi matrix $C_\Phi$ of a linear map $\Phi \in \mathcal{B}(M_n(\mathbb{C}), M_M(\mathbb{C}))$ is defined by

$$C_\Phi := (i_n \otimes \Phi)(B_{n^2})$$

where $B_{n^2}$ is the unnormalized Bell density matrix. We recall that Equation (5) defines a one-to-one correspondence between matrices $C \in M_{nN}(\mathbb{C})$ and maps $\Phi \in \mathcal{B}(M_n(\mathbb{C}), M_M(\mathbb{C}))$. The following theorem is applicable to all equivariant maps.

**Theorem 3.1.** Let $\Phi : M_n(\mathbb{C}) \to M_M(\mathbb{C})$ be a linear map. Then the following two assertions are equivalent:

(i) $\Phi$ is an equivariant map.

(ii) For all unitary matrix $U$ in $M_n(\mathbb{C})$, there exists a matrix $V \in M_M(\mathbb{C})$ such that

$$[C_\Phi, U \otimes V] = 0$$

**Proof.** To prove (i) $\implies$ (ii), we use the fact that $(\mathcal{U} \otimes U)$ commutes with $B_{n^2}$ for every unitary matrix $U \in M_n(\mathcal{U})$. We have the following

$$C_\Phi = (i_n \otimes \Phi)(B_{n^2})$$

$$= (i_n \otimes \Phi)(\mathcal{U} \otimes U)\left(\sum_{i,j} e_{ij} \otimes e_{ij}\right)(\mathcal{U} \otimes U)^*$$

$$= \sum_{i,j} U e_{ij} U^* \otimes \Phi(U e_{ij} U^*)$$

Since $\Phi$ is equivariant, for every unitary $U \in M_n(\mathbb{C})$ there exists $V \in M_M(\mathbb{C})$ such that $\Phi$ satisfies Equation (1). From Equation (6), $C_\Phi$ becomes,

$$C_\Phi = \sum_{i,j} U e_{ij} U^* \otimes V \Phi(e_{ij}) V^*$$

$$= (\mathcal{U} \otimes V)C_\Phi(\mathcal{U} \otimes V)^*.$$
Hence, $C_{\Phi}$ commutes with $(U \otimes V)$.

We now prove the converse implication (ii) $\implies$ (i).

Assume that assertion (ii) holds. Then the previous computation shows that $(U \otimes V)C_{\Phi}(U \otimes V)^* \in \mathcal{B}(M_{N_n}(\mathbb{C}), M_N(\mathbb{C}))$, we denote by $\mathcal{A}(\Phi)$ the $C^*$-algebra generated by the image of $\Phi$: it is the subalgebra of $M_N(\mathbb{C})$ spanned by the identity operator $1_N$ and elements of the form $\Phi(X)^*\Phi(Y)$ for all $X, Y \in M_n(\mathbb{C})$.

\textbf{Theorem 3.2.} Let $\Phi : M_n(\mathbb{C}) \to M_N(\mathbb{C})$ be a unitarily equivariant linear map and assume that $\mathcal{A}(\Phi) = M_N(\mathbb{C})$. Then there exists a unitary representation $U \mapsto V(U)$ of the unitary group $\mathcal{U}_n$ on $\mathbb{C}^N$ such that for all $X \in M_n(\mathbb{C})$ and all unitary $U \in \mathcal{U}_n$,

$$\Phi(UXU^*) = V(U) \Phi(X) V(U)^*.$$ 

\textit{Proof.} Let $U_1, U_2 \in \mathcal{U}_n$ be two unitaries. We have:

$$\Phi(U_1 X U_1^*) = V(U_1) \Phi(X) V(U_1)^*$$

$$\Phi(U_2 X U_2^*) = V(U_2) \Phi(X) V(U_2)^*$$

$$\Phi(U_1 U_2 X(U_1 U_2)^*) = V(U_1 U_2) \Phi(X) V(U_1 U_2)^*. \quad (7)$$

Then, considering that $\Phi(U_1 U_2 X(U_1 U_2)^*) = \Phi(U_1 (U_2 X U_2^*) U_1^*)$ and using Equation (7), it becomes

$$V(U_1 U_2) \Phi(X) V(U_1 U_2)^* = V(U_1) V(U_2) \Phi(X) V(U_2)^* V(U_1)^*.$$ 

This equation can be extended to all $\mathcal{A}(\Phi) = M_N(\mathbb{C})$ and therefore we get that for all $Y \in M_N(\mathbb{C})$,

$$V(U_1 U_2)^* V(Y U_1 U_2)^* = V(U_1) V(U_2)^* V(Y U_1 U_2)^* V(U_1)^*.$$ 

In particular, taking $Y = V(U_2)^* V(U_1)^*$, we obtain

$$V(U_1) V(U_2)^* V(U_1 U_2) = V(U_1 U_2) V(U_1) V(U_2).$$
Since $V(U_1), V(U_2)$ and $V(U_1 U_2)$ are unitaries in $M_N(\mathbb{C})$, this is possible only when $V(U_1)V(U_2)$ is a scalar multiple of $V(U_1 U_2)$. Therefore, there exists $\lambda \in \mathbb{C}$ such that

$$V(U_1)V(U_2) = \lambda V(U_1 U_2).$$

In other words, there exists $\theta(U_1, U_2) \in \mathbb{R}$ such that

$$V(U_1)V(U_2) = e^{i\theta(U_1, U_2)}V(U_1 U_2).$$

This shows that $U \mapsto V(U)$ is a projective representation of the unitary group. It is then a classical result that there exists a unitary representation $U \mapsto \tilde{V}(U)$ such that

$$W \Phi(X) W^* = \Psi(X).$$

**Proof.** This is a direct consequence of the representation theory of the unitary group. Indeed, for all irreducible representations, there exist $a, b$ natural integers such that this irreducible representation appears in the unitary representation (see [17] for instance)

$$U \mapsto U^\otimes a \otimes U^\otimes b.$$ 

It means that there exists a partial isometry $W$ from $\mathbb{C}^N$ to $\mathbb{C}^n \otimes \mathbb{C}^b$ such that for all $U \in \mathcal{U}_n$,

$$W V(U) W^* = P_\lambda U^\otimes a \otimes U^\otimes b P_\lambda,$$

where $P_\lambda$ is the orthogonal projection on some irreducible supspace of $(\mathbb{C}^n \otimes \mathbb{C}^b)$ invariant by $U^\otimes a \otimes U^\otimes b$ for all $U \in \mathcal{U}_n$. We can then define the map $\Psi : X \mapsto W \Phi(X) W^*$ and it can be readily checked that it is an $(U^\otimes a \otimes U^\otimes b)$-equivariant map. \qed

**Remark 3.4.** In the previous corollary, we assumed that $U \mapsto V(U)$ is an irreducible representation. We will see in Section 5 that this is not a restriction when considering the problem of entanglement detection.
3.3 Characterization of \((U^\otimes a \otimes U^\otimes b)\)-equivariant maps

We now deal with the question of characterizing \((U^\otimes a \otimes U^\otimes b)\)-equivariant maps. More explicitly, let \(a, b \in \mathbb{N}\) and \(n \geq 2\). Then, what are all the linear maps

\[
\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})^\otimes a \otimes M_n(\mathbb{C})^\otimes b
\]
such that for every unitary \(U \in M_n(\mathbb{C})\), \(\Phi\) satisfy Equation \((2)\). By Lemma 2.4(1) the set of all \((U^\otimes a \otimes U^\otimes b)\)-equivariant maps is a vector space. Characterizing this vector space is then equivalent to exhibiting one of its basis. In order to do that, we need some basic results from group representation theory and more precisely the Schur-Weyl duality Theorem \([18, \text{Theorem 8.2.10}]\). More detail can be found in \([18, \text{Chapters 7,8}]\).

We recall the two unitary representations \(\sigma_k\) and \(\rho_k\) on \((\mathbb{C}^n)^\otimes k\), with \(a + b = k\), of the symmetric group \(S_k\) and the unitary group \(U_n\) respectively. For all \(v_1, \ldots, v_k \in \mathbb{C}^n\), \(\pi \in S_k\) and \(U \in U_n\), they are defined as

\[
\sigma_k(\pi)v_1 \otimes \cdots \otimes v_k = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(k)},
\]

\[
\rho_k(U)v_1 \otimes \cdots \otimes v_k = Uv_1 \otimes \cdots \otimes Uv_k.
\]

We denote by \(\sigma_k(S_k)\) (resp. \(\rho_k(U_n)\)), the \(*\)-algebra generated by the representation \(\sigma_k\) (resp. \(\rho_k\)). That is,

\[
\sigma_k(S_k) = \{\sigma_k(\pi) ; \pi \in S_k\}'' = \left\{\sum_{\pi \in S_k} f(\pi)\sigma_k(\pi) \right\} ; f : S_k \rightarrow \mathbb{C},
\]

\[
\rho_k(U_n) = \{\rho_k(U) ; U \in U_n\}'',
\]

where the prime symbol denotes the commutant of a set (we will not need the explicit formula for the elements of \(\rho_k(U_n)\)). Then Schur-Weyl duality Theorem asserts that these both algebras(\(\sigma_k(S_k)\) and \(\rho_k(U_n)\)) are the commutant of each other, which by von-Neumann commutant Theorem can be rephrased as follows: for any operator \(C \in M_n(\mathbb{C})^\otimes k\),

\[
[U^\otimes k, C] = 0 \quad \forall \quad U \in U_n \quad \text{iff} \quad C \in \sigma_k(S_k).
\]

We refer to \([18, \text{Theorem 8.2.8}]\) for instance for a presentation of this Theorem. With this result in hand, we can now prove one of the main results of this paper.

**Theorem 3.5.** Let \(a, b \in \mathbb{N}\) and \(\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})^\otimes a \otimes M_n(\mathbb{C})^\otimes b\). Then the following assertions are equivalent.

(i) \(\Phi\) is \((U^\otimes a \otimes U^\otimes b)\)-equivariant.

(ii) \([C_\Phi, U^\otimes a+1 \otimes U^\otimes b] = 0\) for all \(U \in U_n\), where \(C_\Phi\) is the Choi matrix of \(\Phi\).
There exists \( f : S_{k+1} \to \mathbb{C} \) such that

\[
C_\Phi = \sum_{\pi \in S_{k+1}} f(\pi) (\theta_n^{a+1} \otimes i_n^{b}) [\sigma_{k+1}(\pi)],
\]

where \( \theta_n \) is the transpose map on \( M_n(\mathbb{C}) \).

**Proof.** The equivalence of \((i)\) and \((ii)\) is clear from Theorem 3.1.

\((ii) \Rightarrow (iii)\).

Assume that \( C_\Phi \) commutes with \( (U^{\otimes a+1} \otimes U^{\otimes b}) \) for all \( U \in \mathcal{U}_n \), that is,

\[
(U^{\otimes a+1} \otimes U^{\otimes b}) C_\Phi (U^{\otimes a+1} \otimes U^{\otimes b})^* = C_\Phi
\]

Remark that the transpose map \( \theta_n \in B(M_n(\mathbb{C})) \) is \( \mathcal{U} \)-equivariant, so that for all \( U \in \mathcal{U}_n \) and all \( C \in M_n(\mathbb{C})^{\otimes a+b+1} \),

\[
\theta_n^{a+1} \otimes i_n^{b} (U^{\otimes a+1} \otimes U^{\otimes b}) C (U^{\otimes a+1} \otimes U^{\otimes b})^* = U^{\otimes k+1} [(\theta_n^{a+1} \otimes i_n^{b}) C] U^{* \otimes k+1}.
\]

where \( k = a + b \). Applying Equations (11) and (12), we get:

\[
(\theta_n^{a+1} \otimes i_n^{b}) [C_\Phi] = U^{\otimes k+1} (\theta_n^{a+1} \otimes i_n^{b}) [C_\Phi] U^{* \otimes k+1}.
\]

Consequently, \( (\theta_n^{a+1} \otimes i_n^{b}) [C_\Phi] \in \rho_k(\mathcal{U}_n)^c \). By the Schur-Weyl duality given in Equation (9), we obtain that \( (\theta_n^{a+1} \otimes i_n^{b}) [C_\Phi] \in \sigma_{k+1}(S_{k+1}) \), which implies (iii).

\((iii) \Rightarrow (ii)\) is straightforward using the converse part of Schur-Weyl duality and the previous computation.

Clearly by point (iii) in Theorem 3.5, the set of \( (U^{\otimes a} \otimes U^{\otimes b}) \)-equivariant maps is isomorphic as a vector space to the space \( \sigma_{k+1}(S_{k+1}) \). Therefore we get the straightforward corollary:

**Corollary 3.6.** The vector space of \( (U^{\otimes a} \otimes U^{\otimes b}) \)-equivariant maps is of dimension \( (k+1)! \), where \( k = a + b \). A basis for this vector space is given by the maps \( \{ \Phi_\pi : \pi \in S_{k+1} \} \), with the corresponding Choi matrices:

\[
C_{\Phi_\pi} = \theta_n^{a+1} \otimes i_n^{b}[\sigma_{k+1}(\pi)],
\]

where the matrix \( \sigma_{k+1}(\pi) \) acts on vectors \( (v_1 \otimes \cdots \otimes v_{k+1}) \) as

\[
\sigma_{k+1}(\pi) (v_1 \otimes \cdots \otimes v_{k+1}) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(k+1)}.
\]

We discuss the graphical representation of this basis in the next section.
4 Graphical representation of the Choi matrix of \((U^\otimes a \otimes U^\otimes b)\)-equivariant maps

In this section we study the graphical representation of \((U^\otimes a \otimes U^\otimes b)\)-equivariant maps for any \(a, b \in \mathbb{N}\). This gives a visual and very convenient method to compute them. We illustrate this with the \((U \otimes U)\) equivariant maps from [1].

The following table represents the graphical representation of operators defined on Hilbert spaces, their tensor product and how the operations of transpose, multiplication of operators perform graphically. More detail about the graphical calculus can be found in [19, 20].

| Vectors/Operators | Graphical representation |
|-------------------|-------------------------|
| \(|v\rangle \in \mathbb{C}^n\) | ![Vector](image) |
| \(<v| \in (\mathbb{C}^n)^*\) | ![Inner Product](image) |
| \(T \in B(\mathcal{H})\) | ![Operator](image) |
| \(T \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)\) | ![Tensor Product Operator](image) |
| \(AB \in M_n(\mathbb{C})\) | ![Matrix Multiplication](image) |
| \(A^t \in M_n(\mathbb{C})\) | ![Transpose](image) |

Figure 1: Graphical representations of vectors and operators
Let $W_{a,b}$ denote the vector space of all $(U^\otimes a \otimes U^\otimes b)$-equivariant maps. Recall from Section 3.3 that the maps $C_{\Phi_\pi} = (\theta_n^\otimes a+1 \otimes i_n^\otimes b)(\sigma_{k+1}(\pi))$ form a basis of $W_{a,b}$ with $\pi$ running through the permutation group $S_{k+1}$, where $\sigma_{k+1}(\pi)$ is the unitary representation of the permutation group defined in Equation (14). Based on the graphical representations given in Table 1, we give the graphical representations of the $C_{\Phi_\pi}$. We first consider the case of $a = 1$ and $b = 1$ and make some observations. We discuss the general case in Theorem 4.1.

We compute the graphical representation of the Choi matrices $C_{\Phi_\pi}$ with $\pi \in S_3$, by first computing the one of $\sigma_3(\pi)$. We start with the representation of $\sigma_3(123)$ as an example. This map is defined as follows:

$$
\sigma_3(123) : (C^n)^{\otimes 3} \rightarrow (C^n)^{\otimes 3}, \\
v_1 \otimes v_2 \otimes v_3 \mapsto v_3 \otimes v_1 \otimes v_2.
$$

Its graphical representation is given on the left in Figure 2. By Equation (13), $C_{\Phi_{(123)}} = (\theta_n^\otimes 2 \otimes i_n)\sigma_3(123)$. Using the graphical representation of the transpose map, we obtain that $C_{\Phi_{(123)}}$ is represented by the right figure in Figure 2.

Figure 2: Graphical representation of $\sigma_3(123)$ (on the left) and $C_{\Phi_{(123)}}$ (on the right)

We take the following convention in order to represent the operators $\sigma_{k+1}(\pi)$: input on the left are represented by black (full) dots while output on the right are represented by white (empty) dots. The graphical representation of the operator $\sigma_{k+1}(\pi)$ then corresponds to tracing a wire from the $i$th (black) dot on the left to the $\pi^{-1}(i)$ (white) dot on the right. Using graphical computation, it can be directly check that $\sigma_{k+1}(\pi)$ and $U^\otimes k+1$ commute for all unitary operator $U \in U_n$.

Then, applying the transpose map to the first $a+1$ tensors corresponds graphically to interchange the black and the white dot at the first $a+1$ rows (starting from the highest). Again, it can be directly check from a graphical computation that $C_{\Phi_\pi}$ and $(U^\otimes a+1 \otimes U^\otimes b)$ commute. This is illustrated in the case of the permutation (123) in Figure 3.

13
We can obtain similarly the graphical representations of the Choi matrices $C_{\Phi_{\pi}}$ corresponding to all $\pi \in S_3$. There are $3! = 6$ different possibilities to trace a wire between black and white dots in a one-to-one way. Each of them corresponds to a different permutation $\pi \in S_3$. This is illustrated in Figure 4.

Figure 4 gives the basis elements of the vector space $W_{1,1}$. Theorem 4.1 generalizes this fact to any $(U^\otimes a \otimes U^\otimes b)$-equivariant map. We omit the proof, as it follows exactly the same idea as the example above.

**Theorem 4.1.** The graphical representation of the Choi matrices of the basis elements $C_{\Phi_{(\pi)}}$ of the vector space $W_{a,b}$ is given by the following rule. Being a matrix on $\mathbb{C}^n \otimes k+1$, its graphical representative has $k+1$ input (on the left) and $k+1$ output (on the right). The first $a+1$ inputs are symbolized by black (full) dots, the remaining $b$ by white (empty) dots. In the same way, the first $a+1$ outputs are symbolized by white (empty) dots, the remaining $b$ by black (full) dots. Then each element of this basis corresponds to a possible wiring between black and white dot, in a one-to-one way.
of all \((U^a \otimes U^b)\)-equivariant maps. As an illustration, we study the graphical representation of the Choi matrix of the family of linear maps \(\Phi_{\alpha,\beta,3}\) studied in [1] and given by Equation (3). It is given as follows:

\[
\begin{align*}
&\Phi_{\alpha,\beta,3} : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \\
&\quad A \mapsto A^t \otimes \mathbb{I}_n + \mathbb{I}_n \otimes A + \text{Tr}(A)(\alpha \mathbb{I}_{n^2} + \beta B_{n^2}) + \gamma (B_{n^2} \otimes A) + (\mathbb{I}_n \otimes A)B_{n^2}.
\end{align*}
\]

We leave the study of this more general map to future work.

5 Application to entanglement detection

One application of positive maps that are not completely positive is entanglement detection in quantum information theory. We recall that a density matrix in \(M_n(\mathbb{C})\) for some integer \(n \geq 0\) is a positive semi-definite operator on \(\mathbb{C}^n\) with trace equal to one. We denote by \(\mathcal{S}(n)\) the set of density matrices on \(\mathbb{C}^n\). Then a density matrix on \(\mathbb{C}^m \otimes \mathbb{C}^n\) is called separable if it is in the convex hull of product density matrices. We denote by \(\text{SEP}(m,n)\) the set of separable density matrices on \(\mathbb{C}^m \otimes \mathbb{C}^n\), that is,

\[
\text{SEP}(m,n) = \left\{ \sum_i \lambda_i \rho_A^i \otimes \rho_B^i : \rho_A^i \in \mathcal{S}(m), \rho_B^i \in \mathcal{S}(n), \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}.
\]

What we are interested in is the complement of this set in \(\mathbb{C}^m \otimes \mathbb{C}^n\). More precisely, a density matrix \(\rho \in \mathcal{S}(m \times n) \setminus \text{SEP}(m,n)\) is called entangled. The first operational characterization of entanglement/separability was proved in [8]: a density matrix \(\rho \in \mathcal{S}(m \times n)\) is entangled if and only if there exists a positive map \(\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})\) such that \((i_m \otimes \phi)(\rho)\) is not positive semi-definite. Remark that if \(\phi\) is completely positive, then \((i_m \otimes \phi)(\rho)\) is necessarily positive semi-definite. This means that positive - but not completely positive - maps lead to entanglement detectors. The most well-known example of positive
maps leading to entanglement detection is the partial transpose, and the density matrices that remain positive under its action are the so-called PPT density matrices. More generally, given a positive map \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \), we can define
\[
S_m(\phi) = \{ \rho \in S(m \times n) : (i_m \otimes \phi)(\rho) \geq 0 \}.
\]
Thus the set of PPT density matrices on \( \mathbb{C}^m \otimes \mathbb{C}^n \) is \( S_m(\theta_n) \), where \( \theta_n \) is the transposition on \( M_n(\mathbb{C}) \). Apart from \( (m,n) = (2,2) \) or \( (m,n) = (2,3) \), it is known that \( S_m(\theta_n) \) is strictly different from SEP\((m,n)\). It means that an infinite number of positive maps are needed to detect all entangled density matrices. It is thus a central question in quantum information theory to find families of maps that are sufficient for this task. The main result of this section is the following.

**Theorem 5.1.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a positive - but not completely positive - map. Then there exists a unitarily equivariant map \( \Phi : M_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \) such that:
\[
S_m(\Phi) \subset S_m(\phi).
\]
It means that all entangled density matrices that are detected by \( \phi \) are also detected by the unitarily equivariant map \( \Phi \).

We formulated the previous theorem in terms of the set \( S_m(\phi) \). As a corollary we obtain the weaker statement:

**Corollary 5.2.** Any entangled density matrix \( \rho \in S(m,n) \) can be detected thanks to a unitarily-equivariant positive map \( \Phi : M_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \).

The fact that the Hilbert space \( \mathcal{H} \) is infinite dimensional is a huge drawback in practice. To our knowledge, it is not clear if this can be improved. We shall however discuss in Section 5.2 applications of this conjecture, based on Theorem 3.2 and Corollary 3.3.

### 5.1 Proof of Theorem 5.1

**Proof.** Let \( \rho \in S(m \times n) \setminus \text{SEP}(n, m) \) be an entangled density matrix and let \( \phi \in \mathcal{B}(M_n(\mathbb{C}), M_n(\mathbb{C})) \) be a positive map such that \( (i_m \otimes \phi)(\rho) \) is not positive semi-definite, as provided by the Horodecki Theorem [2]. Define \( \mathcal{H} = L_2(\mathcal{U}_n, \mathbb{C}^n) \), that is,
\[
\mathcal{H} = \left\{ (\psi_U)_{U \in \mathcal{U}_n} : \psi_U \in \mathbb{C}^n \ \forall U \in \mathcal{U}_n, \ \int_{\mathcal{U}_n} \|\psi_U\|^2 \mu(dU) < +\infty \right\}
\]
where the integration is with respect to the Haar measure on \( \mathcal{U}_n \). We can define a map \( \Phi \in \mathcal{B}(M_n(\mathbb{C}), \mathcal{B}(\mathcal{H})) \) as
\[
X \in M_n(\mathbb{C}) \mapsto \Phi(X) = (\phi(U X U^*))_{U \in \mathcal{U}_n}.
\]
Thus for all $X \in M_n(\mathbb{C})$ and all $(\psi_U)_{U \in \mathcal{U}_n} \in \mathcal{H}$, $\Phi(X)\psi = (\phi(U X U^*)\psi)_{U \in \mathcal{U}_n}$.

Define the following unitary representation of $\mathcal{U}_n$ on $\mathcal{H}$:

$$V : W \mapsto (V(W) : \psi \in \mathcal{H} \mapsto (\psi_U)_{U \in \mathcal{U}_n}),$$

Then we check that the map $\Phi$ is unitarily equivariant with respect to $U \mapsto V(U)$. Indeed, for all $W \in \mathcal{U}_n$, $X \in M_n(\mathbb{C})$ and all $\psi \in \mathcal{H}$,

$$V(W)^* \Phi(W X W^*) V(W) \psi = V(W)^* \Phi(W X W^*) (\psi_U)_{U \in \mathcal{U}_n}$$

$$= V(W)^* (\phi(U W X (U W)^*)\psi_U)_{U \in \mathcal{U}_n}$$

$$= \Phi(X) \psi.$$

We now show that $\Phi$ “detects” $\rho$, that is, $(i_m \otimes \Phi)(\rho)$ is not positive semi-definite. Indeed, this is true as by continuity there exist a vector $\varphi \in \mathbb{C}^m \otimes \mathbb{C}^n$ and a small neighbourhood $\mathcal{U}$ of $I_n$ in $\mathcal{U}_n$ such that for all $U \in \mathcal{U}$, $(\varphi, (i_m \otimes \phi)(I_m \otimes U \rho I_m \otimes U^*))\varphi < 0$. Then defining the vector $\psi \in \mathbb{C}^m \otimes \mathcal{H}$ as:

$$\psi_U = \varphi \quad \forall U \in \mathcal{U}, \quad \psi_U = 0 \quad \text{elsewhere},$$

we get that $\langle \psi, (i_m \otimes \Phi)(\rho)\psi \rangle < 0$ which shows that $(i_m \otimes \Phi)(\rho)$ is not positive semi-definite.

Remark 5.3. In the previous proof, the construction of the Hilbert space $\mathcal{H}$ and the representation $\pi$ does not depend on $\phi$. So is also the case for the unitary representation $U \mapsto V(U)$: it means that its decomposition into irreducible representations does not depend on $\phi$, as we should expect!

5.2 Discussion on $U^{\otimes a} \otimes U^{\otimes b}$-equivariant maps

We discuss in this section the applications to entanglement detection of the following conjecture:

Conjecture 5.4. For any entangled density matrix $\rho$ on $(\mathbb{C}^m \otimes \mathbb{C}^n)$, there exists a unitarily equivariant and a positive map $\Phi \in \mathcal{B}(M_n(\mathbb{C}), M_N(\mathbb{C}))$ for some natural integer $N$ such that the map $U \mapsto V(U)$ in Equation (1) can be assumed to be a unitary representation, and such that $(i_m \otimes \Phi)(\rho)$ is not positive semi-definite.

From this conjecture, we can deduce the following result:

Theorem 5.5. Assume that Conjecture 5.4 is true. Then, for any entangled density matrix $\rho$ on $(\mathbb{C}^m \otimes \mathbb{C}^n)$, there exist two natural integer $a, b$ and a $(U^{\otimes a} \otimes U^{\otimes b})$-equivariant positive map $\Phi$ such that $(i_m \otimes \Phi)(\rho)$ is not positive semi-definite.

Before proving this theorem, let us make a few remarks.

• First, assuming the conjecture, it allows to reduce the problem of entanglement detection to the case of $(U^{\otimes a} \otimes U^{\otimes b})$ positive equivariant maps.
• We have entirely characterized the \((U^\otimes a \otimes U^\otimes b)\)-equivariant maps in Section 3.3. As they are equivariant, it is a simple computation to know which of them are positive. Partial results in this direction were made in [1] in the case \(a = b = 1\).

**Proof of Theorem 5.5.** Let \(\rho \in \mathcal{S}(m \times n) \setminus \text{SEP}(m, n)\) be an entangled density matrix and assume that Conjecture 5.4 is true: there exists a positive unitarily equivariant map \(\Phi \in B(M_n(\mathbb{C}), M_N(\mathbb{C}))\) such that \((i_m \otimes \Phi)(\rho)\) is not positive semi-definite, and such that the map \(U \mapsto V(U) \in \mathcal{U}_N\) that makes \(\Phi\) equivariant is a unitary representation. We can therefore decompose it into irreducible representations:

\[
M_N(\mathbb{C}) = \bigoplus_{\lambda} B(E_{\lambda}) \otimes i_{N_{\lambda}},
\]

where \(E_{\lambda}\) are irreducible representations of \(\mathcal{U}_n\) appearing in the unitary representation \(V\) with multiplicity \(N_{\lambda}\). We denote by \(V_{\lambda}\) the unitary representation of \(\mathcal{U}_n\) on \(E_{\lambda}\) induced by \(V\), and by \(P_{\lambda}\) the orthogonal projection on \(E_{\lambda} \otimes \mathbb{C}^{N_{\lambda}}\).

We now define the map \(\Phi_{\lambda} \in B(M_n(\mathbb{C}), B(E_{\lambda}))\) by

\[
X \in M_n(\mathbb{C}) \mapsto \Phi_{\lambda}(X) = P_{\lambda} \Phi(X) P_{\lambda}.
\]

It can be directly checked that \(\Phi_{\lambda}\) is unitarily equivariant with respect to \(U \mapsto V_{\lambda}(U) \otimes 1_{N_{\lambda}}\) for any \(\lambda\). Then, if \((i_m \otimes \Phi)(\rho)\) is not positive semi-definite, there necessarily exists a \(\lambda\) such that \((i_m \otimes P_{\lambda} \Phi P_{\lambda})(\rho)\) is not positive semi-definite.

We can then apply Corollary 3.3: there exist \(a, b\) natural integers, a partial isometric map \(W : \mathbb{C}^N \rightarrow \mathbb{C}^n \otimes \mathbb{C}^a \otimes \mathbb{C}^b\) and a \((U^\otimes a \otimes U^\otimes b)\)-equivariant map \(\Psi\) on \(M_n(\mathbb{C})\) such that for all \(X \in M_n(\mathbb{C})\),

\[
W \Phi_{\lambda}(X) W^* = \Psi(X).
\]

Then \((i_m \otimes P_{\lambda} \Psi P_{\lambda})(\rho)\) is not positive semi-definite, which concludes the proof.

\(\square\)

6 Conclusion

In this article, following [1], we studied equivariant linear maps and their application to entanglement detection. We analysed their properties and in particular, tried to understand what could be assumed of the map \(U \mapsto V(U)\) in their definition. When this map is a unitary representation, we showed that the unitarily equivariant map is “a corner” of a \((U^\otimes a \otimes U^\otimes b)\)-equivariant map. Furthermore, we fully characterized the latter and give their graphical representation. The full understanding of equivariant maps is far from complete, and this paper opens many questions that we find interesting from the point of view of operator theory.

Subsequently, we applied unitarily equivariant maps to the study of entanglement detection. We conjectured that any entangled density matrix can be
detected by a unitarily equivariant positive map where the map $U \mapsto V(U)$ is a unitary representation of the unitary group. Using representation theory, we showed that it implies that the $\left( U^a \otimes U^b \right)$-equivariant positive maps are enough to detect all entangled density matrices. In order to support our conjecture, we proved it when the image of the equivariant map is infinite dimensional.

7 Acknowledgement

Part of this work was made during IB’s visit to Kyoto University, supported by Campus France Sakura project. IB and BC were supported by the ANR project StoQ ANR-14-CE25-0003-01. BC was supported by Kakenhi 15KK0162, 17H04823, 17K18734. GS would like to acknowledge FRIENDSHIP project of Japan International Corporation Agency (JICA) for research fellowship (D-15-90284) and kakenhi 15KK0162. All authors are very grateful to Prof Hiroyuki Osaka for a careful reading and very helpful comments on a preliminary version of the paper.

References

[1] B. Collins, H. Osaka, and G. Sapra, “On a family of linear maps from $M_n(C)$ to $M_{n^2}(C)$,” Linear Algebra and its Applications, vol. 555, pp. 398–411, 2018.

[2] B. V. R. Bhat, “Linear maps respecting unitary conjugation,” Banach J. Math. Anal, vol. 5, no. 2, pp. 1–5, 2011.

[3] M. A. Nielsen and I. Chuang, “Quantum computation and quantum information,” 2002.

[4] B. M. Terhal, “Detecting quantum entanglement,” Theoretical Computer Science, vol. 287, no. 1, pp. 313–335, 2002.

[5] S. Gharibian, “Strong np-hardness of the quantum separability problem,” Quantum Info. Comput., vol. 10, pp. 343–360, Mar. 2010.

[6] L. Gurvits, “Classical deterministic complexity of Edmonds’ Problem and quantum entanglement,” in Proceedings of the thirty-fifth annual ACM symposium on Theory of computing, pp. 10–19, ACM, 2003.

[7] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, “Complete family of separability criteria,” Physical Review A, vol. 69, no. 2, p. 022308, 2004.

[8] M. Horodecki, P. Horodecki, and R. Horodecki, “Separability of mixed states: necessary and sufficient conditions,” Physics Letters A, vol. 223, no. 1, pp. 1–8, 1996.

[9] A. Peres, “Separability criterion for density matrices,” Physical Review Letters, vol. 77, no. 8, p. 1413, 1996.
[10] S.-H. Kye and H. Osaka, “Classification of bi-qutrit positive partial transpose entangled edge states by their ranks,” Journal of Mathematical Physics, vol. 53, no. 5, p. 052201, 2012.

[11] M.-D. Choi, “Completely positive linear maps on complex matrices,” Linear algebra and its applications, vol. 10, no. 3, pp. 285–290, 1975.

[12] M.-D. Choi, “Positive linear maps on $C^*$-algebras,” Canadian Math.J, vol. 24, no. 3, pp. 520–529, 1972.

[13] T. Takasaki and J. Tomiyama, “On the geometry of positive maps in matrix algebras,” Mathematische Zeitschrift, vol. 184, pp. 101–108, Mar 1983.

[14] S. J. Cho, S.-H. Kye, and S. G. Lee, “Generalized choi maps in three-dimensional matrix algebra,” Linear algebra and its applications, vol. 171, pp. 213–224, 1992.

[15] A. Müller-Hermes, “Decomposability of linear maps under tensor products,” arXiv preprint arXiv:1805.11570, 2018.

[16] J. Tomiyama, “On the geometry of positive maps in matrix algebras. II,” Linear Algebra and its Applications, vol. 69, pp. 169–177, 1985.

[17] M. R. Sepanski, Compact lie groups, vol. 235. Springer Science & Business Media, 2007.

[18] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, Representation Theory of the Symmetric Groups: The Okounkov-Vershik Approach, Character Formulas, and Partition Algebras. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2010.

[19] C. J. Wood, J. D. Biamonte, and D. G. Cory, “Tensor networks and graphical calculus for open quantum systems,” Quantum Info. Comput., vol. 15, pp. 759–811, July 2015.

[20] B. Collins and I. Nechita, “Random Quantum Channels I: Graphical Calculus and the Bell State Phenomenon,” Communications in Mathematical Physics, vol. 297, pp. 345–370, Jul 2010.

(Ivan Bardet) Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, England France
E-mail: bardetivan@gmail.com

(Benoît Collins) Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: collins@math.kyoto-u.ac.jp

(Gunjan Sapra) Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: gunjan18@math.kyoto-u.ac.jp