Abstract. We study the nodal length of random toral Laplace eigenfunctions ("arithmetic random waves") restricted to decreasing domains ("shrinking balls"), all the way down to Planck scale. We find that, up to a natural scaling, for "generic" energies the variance of the restricted nodal length obeys the same asymptotic law as the total nodal length, and these are asymptotically fully correlated. This, among other things, allows for a statistical reconstruction of the full toral length based on partial information. One of the key novel ingredients of our work, borrowing from number theory, is the use of bounds for the so-called spectral quasi-correlations, i.e., unusually small sums of lattice points lying on the same circle.

1 Introduction

1.1 Laplace eigenfunctions in Planck (microscopic) scale. Let \((M, g)\) be a compact smooth Riemannian surface, and \(\Delta\) the Laplace–Beltrami operator on \(M\). It is well-known that in this situation the spectrum of \(\Delta\) is purely discrete, i.e., there exists a non-decreasing sequence \(\{E_j\}_{j \geq 1} \subseteq \mathbb{Z}_{\geq 0}\) of eigenvalues of \(\Delta\) ("energy levels of \(M\" ) and the corresponding eigenfunctions \(\{\phi_j\}_{j \geq 1}\) such that

\[
\Delta \phi_j + E_j \cdot \phi_j = 0,
\]

and \(\{\phi_j\}_{j \geq 1}\) is an o.n.b. of \(L^2(M)\).

We are interested in the nodal line of \(\phi_j\), i.e., its zero set \(\phi_j^{-1}(0)\) in the high-energy limit \(j \to \infty\); it is generically a smooth curve [Uh]. In this general setup Yau’s conjecture asserts that there exist constants \(0 < c_M < C_M < \infty\) such that the nodal length (i.e., the length of \(\phi_j^{-1}(0)\)) satisfies

\[
c_M \cdot \sqrt{E_j} \ll \text{len}(\phi_j^{-1}(0)) \ll C_M \cdot \sqrt{E_j}.
\]

Yau’s conjecture was resolved by Donnelly–Fefferman [DoFe] for \(M\) real analytic, and the lower bound was established more recently by Logunov [Lo1, Lo2, LoMa] for the more general smooth case.
Berry’s seminal and widely believed conjecture [Be77, Be83] asserts that, at least in some generic situation, one could model the high-energy eigenfunctions $\phi_j$ on a chaotic surface $\mathcal{M}$ with random monochromatic plane waves of wavelength $\sqrt{E_j}$, i.e., an isotropic random field on $\mathbb{R}^2$ with covariance function

$$r_{RWM}(x) = J_0(\|x\|).$$

When valid, Berry’s RWM in particular implies Yau’s conjecture (1.2); it goes far beyond the macroscopic setting, i.e., that the RWM is applicable to shrinking domains. For example, it asserts that the RWM is a good model for $\phi_j^{-1}(0) \cap B_r(x)$, i.e., the nodal length lying inside a shrinking geodesic ball $B_r(x) \subseteq \mathcal{M}$, of radius slightly above the Planck scale: $r \approx C \sqrt{E}$, with $C \gg 0$ sufficiently big. In this spirit Nadirashvili’s conjecture [Na] refines upon (1.2) in that the analogous statement is to hold in the Planck (or microscopic) scale; it was in part established by Logunov [Lo1, Lo2].

To our best knowledge the only other few small-scale results all concern the mass equidistribution. These are some small-scale refinements [He-Ri1, He-Ri2, Ha1, Ha2] for Shnirelman’s Theorem [Sn, Ze, CdV] asserting the $L^2$ mass equidistribution of $\phi_j$ on a macroscopic scale for $\mathcal{M}$ chaotic, i.e., that the $L^2$ mass of $\phi_j$ on a subdomain of $\mathcal{M}$ is proportional to its area along a density 1 sequence of $\{E_j\}$. For the particular case of the standard flat torus $\mathcal{M} = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ Lester–Rudnick [LeRu] and subsequently Granville–Wigman [GW16] used the number theoretic structure of the toral eigenfunctions in order to obtain the Planck-scale mass equidistribution or slightly above it for “most” of the eigenfunctions.

### 1.2 Arithmetic random waves.

Let

$$S = \{a^2 + b^2 : a, b \in \mathbb{Z}\} \subseteq \mathbb{Z}$$

be the set of integers expressible as a sum of two squares. For $n \in S$ let

$$\mathcal{E}_n = \{\lambda \in \mathbb{Z}^2 : \|\lambda\|^2 = n\}$$

be the set of lattice points lying on the radius-$\sqrt{n}$ circle, and denote its size

$$N_n = r_2(n) = |\mathcal{E}_n|;$$

it is the number $r_2(n)$ of ways to express $n$ as the sum of two squares. It is well known that every (complex valued) Laplace eigenfunction (1.1) on the standard 2-torus $\mathcal{M} = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ is necessarily of the form

$$T_n(x) = \frac{1}{\sqrt{2N_n}} \sum_{\lambda \in \mathcal{E}_n} a_\lambda \cdot e(\langle \lambda, x \rangle)$$
for some \( n \in S \), and coefficients \( a_\lambda \in \mathbb{C} \) (with the rationale of the normalising factor \( \frac{1}{\sqrt{2N_n}} \) becoming apparent below); the corresponding eigenvalue is

\[
(1.5) \quad E_n = 4\pi^2 n,
\]

and \( T_n \) is real-valued if and only if for every \( \lambda \in \mathcal{E}_n \) we have

\[
(1.6) \quad a_{-\lambda} = \overline{a_\lambda}
\]

(the complex conjugate of \( a_\lambda \)).

For \( n \in S \) fixed, the eigenspace of all functions (1.4) satisfying

\[
\Delta T_n + E_n \cdot T_n = 0
\]

could be endowed with a Gaussian probability measure by assuming that the coefficients \( \{ a_\lambda \}_{\lambda \in \mathcal{E}_n} \) are standard (complex) Gaussian i.i.d. save for (1.6). With a slight abuse of notation, the resulting random field, also denoted \( T_n \), is the wavenumber-\( \sqrt{n} \) “arithmetic random wave” [ORW, KKW13]. Alternatively, \( T_n \) is the (unique) centered Gaussian stationary random field with the covariance function

\[
(1.7) \quad r(x - y) = r_n(x - y) = \mathbb{E}[T_n(x) \cdot T_n(y)] = \frac{1}{N_n} \sum_{\lambda \in \mathcal{E}_n} e(\langle \lambda, x - y \rangle)
\]

as \( r(0) = 1 \), the field \( T_n \) is unit variance (this is the reason we set (1.4) to normalize \( T_n \) in the first place). The (random) zero set \( T_n^{-1}(0) \) is of our fundamental interest; it is a.s. a smooth curve [RW2008], called the **nodal line**. Of our particular interest is the distribution of its (random) total length

\[
\mathcal{L}_n = \text{len}(T_n^{-1}(0)),
\]

or the length constrained inside subdomains,

\[
\mathcal{L}_{n,S} = \text{len}(T_n^{-1}(0) \cap B(s)),
\]

where \( B(s) \subseteq \mathbb{T}^2 \) is a radius-\( s \) ball shrinking at Planck scale rate \( s > n^{-1/2} \), or, more realistically, slightly above it \( s > n^{-1/2+\epsilon} \). To be able to explain the context and formulate our main results we will require some background on the arithmetic of lattice points \( \mathcal{E}_n \) lying on circles.
1.3 Some arithmetic aspects of lattice points \( E_n \). Recall that \( S \subseteq \mathbb{Z} \) as in (1.3) is the set of integers expressible as a sum of two squares, and given \( n \in S \), the number of such expressions is \( N_n = r_2(n) \). As the distribution of the above central quantities will depend on both \( N_n \) and the angular distribution of lattice points lying on the corresponding circle, here we give some necessary background. First, it is known [Lan08] that \( N_n \) grows on average as

\[
N_n \sim c_0 \cdot \sqrt{\log n}
\]

with some \( c_0 > 0 \); equivalently, as \( X \to \infty \),

\[
S(X) := |\{ n \in S : n \leq X \}| \sim c_{LR} \cdot \frac{X}{\sqrt{\log X}}
\]

where \( c_{LR} > 0 \) is the (fairly explicit) Ramanujan–Landau constant. Moreover,

\[
N_n \sim (\log n)^{(\log 2)/2+o(1)}
\]

for a density 1 sequence of numbers \( n \in S' \subseteq S \) (though we bear in mind that \( S \subseteq \mathbb{Z} \) itself is thin or density 0). To the other end, \( N_n \) is as small as \( N_p = 8 \) for an infinite sequence of primes

\[
p \equiv 1 \mod 4,
\]

and, in general, it is subject to large and erratic fluctuation satisfying for every \( \epsilon > 0 \)

\[
N_n = O(n^\epsilon).
\]

From this point on we will always work with (generic) subsequences \( \{ n \} \subseteq S' \subseteq S \) satisfying \( N_n \to \infty \).

To understand the angular distribution of the lattice points \( E_n \) we define the probability measures \( \tau_n \) on the unit circle \( S^1 \subseteq \mathbb{R}^2 \):

\[
\tau_n = \frac{1}{N_n} \sum_{\lambda \in E_n} \delta_{\lambda/\sqrt{n}}.
\]

It is known [E-H, K-F-W] that for a “generic” (density 1) sequence \( \{ n \} \subseteq S \) the lattice points \( E_n \) are equidistributed in the sense that \( \tau_n \Rightarrow \frac{\partial}{\partial x} \), i.e., weak-* convergence of probability measures to the uniform measure on \( S^1 \) parameterized as \( (\cos \theta, \sin \theta) \). To the other extreme, there exist [Cil93] (thin) “Cilleruelo” sequences \( \{ n \} \subseteq S \) such that the number of lattice points \( N_n \to \infty \) grows, though all of them are concentrated

\[
\tau_n \Rightarrow \frac{1}{4}(\delta_{\pm 1} + \delta_{\pm i})
\]
around the four points $\pm 1, \pm i$ where we are thinking of $S^1 \subseteq \mathbb{C}$ as embedded inside the complex numbers. There exist [KKW13, KW16] other attainable measures $\tau$ on $S^1$, i.e., weak-$*$ partial limits of the sequence $\{\tau_n\}_{n \in \mathbb{N}}$; by the compactness of $S^1$ the limit measure $\tau$ is automatically a probability measure; a partial classification of such $\tau$ was obtained [KW16] via their Fourier coefficients. In particular, it follows that the 4th Fourier coefficient of attainable measures is unrestricted: for every $\eta \in [-1, 1]$ there exists an attainable measure $\tau$ with $\hat{\tau}(4) = \eta$.

### 1.4 Nodal length

Recall that

$$L_n = \text{len}(T_n^{-1}(0))$$

is the total nodal length of $T_n$, and for $0 < s < 1/2$ (say),

$$L_{n,s} = \text{len}(T_n^{-1}(0) \cap B(s))$$

is the nodal length of $T_n$ restricted to a radius-$s$ ball $B(s)$, where by the stationarity of $T_n$ we may assume that $B(s)$ is centered. A straightforward computation [RW2008, Proposition 4.1] with the Kac–Rice formula was used to evaluate the expected length

$$\mathbb{E}[L_n] = \frac{1}{2\sqrt{2}} \sqrt{E_n},$$

consistent with Yau’s conjecture, and, by the stationarity of $T_n$, the more general result

$$\mathbb{E}[L_{n,s}] = \frac{1}{2\sqrt{2}} (\pi s^2) \cdot \sqrt{E_n}$$

for the restricted length also follows from the same computation.

The asymptotic behavior of the variance of $L_n$, eventually resolved by [KKW13], is of a far more delicate nature (and even more so of $L_{n,s}$); it turned out that it is intimately related to the angular distribution (1.10) of $E_n$. More precisely, it was found [KKW13] that

$$\text{(1.11)} \quad \text{Var}(L_n) = c_n \cdot \frac{E_n}{N_n^2} \cdot \left(1 + O\left(\frac{1}{N_n^{1/2}}\right)\right),$$

where the leading coefficients

$$\text{(1.12)} \quad c_n = \frac{1 + \hat{\tau}_n(4)^2}{512} \in \left[\frac{1}{512}, \frac{1}{256}\right],$$

1Originally only $o(1)$ for the error term claimed. It is easy to obtain the $O(\frac{1}{N_n^{1/2}})$ bound for the error term using Bombieri–Bourgain’s bound for the length-6 spectral correlations (see §1.5).
depending on the arithmetics of \( \mathcal{E}_n \), are bounded away from both 0 and \( \infty \). The asymptotic formula (1.11) for the nodal length variance shows that in order for \( \text{Var}(\mathcal{L}_n) \) to observe an asymptotic law it is essential to separate \( S \) into sequences \( \{ n \} \subseteq S \) such that the corresponding \( \tau_n \Rightarrow \tau \) for some (attainable) \( \tau \); in this case

\[
\text{Var}(\mathcal{L}_n) \sim c(\tau) \cdot \frac{E_n}{N_n^2}
\]

with

\[
c(\tau) = \frac{1 + \hat{\tau}(4)^2}{512}.
\]

The variance (1.11) is significantly smaller compared to the previously expected [RW2008] order of magnitude \( \approx \frac{E_n}{N_n} \); it is an arithmetic manifestation of “Berry’s cancellation” [Be02], also interpreted [MaWi1, MaWi2] as the precise vanishing of the second chaotic component in the Wiener chaos expansion of \( \mathcal{L}_n \) (or its spherical analogue). One might expect Berry’s cancellation to be a feature of the symmetries of the full torus; that this is not so follows in particular from a principal result of the present manuscript, Theorem 1.1 below (see (1.14) and §2.1 for more details).

The fine distribution of \( \mathcal{L}_n \) was also investigated [MPRW16]. For a number \( \eta \in [0, 1] \) let \( M_\eta \) be the random variable

\[
M_\eta := \frac{1}{\sqrt{1 + \eta^2}} \cdot (2 - (1 + \eta)X_1^2 - (1 - \eta)X_2^2),
\]

where \( (X_1, X_2) \) are standard Gaussian i.i.d.; for example, if \( \eta = 0 \), the distribution of \( M_\eta \) is a linear transformation of \( \chi^2 \) with 2 degrees. It was shown [MPRW16] that as \( N_n \to \infty \), the distribution law of the normalized

\[
\tilde{\mathcal{L}}_n := \frac{\mathcal{L}_n - \mathbb{E}[\mathcal{L}_n]}{\text{Var}(\mathcal{L}_n)}
\]

is asymptotic to that of \( M_{|\hat{\tau}(4)|} \), both in mean\(^2\) and almost surely. The meaning of the convergence in mean is that there exist copies of \( \tilde{\mathcal{L}}_n \) and \( M_{|\hat{\tau}(4)|} \), defined on the same probability space, such that

\[
\mathbb{E}[|\tilde{\mathcal{L}}_n - M_{|\hat{\tau}(4)|}|] \to 0
\]

as \( N_n \to \infty \).

The first principal result of this manuscript is that an analogous statement to (1.11) holds for the nodal length \( \mathcal{L}_{n,5} \) of \( T_n \) restricted to shrinking balls slightly above Planck scale, for generic sequences of energy levels \( \{ n \} \subseteq S \), and that in this regime \( \mathcal{L}_{n,5} \) are asymptotically fully correlated with \( \mathcal{L}_n \); this also implies that

\(^2\)In fact, in \( L^p \) for all \( p \in (0, 2) \).
an analogue of (1.13) holds for $\mathcal{L}_{n;s}$ (see Corollary 1.2). Below we will specify a generic arithmetic condition, sufficient for the conclusions of Theorem 1.1 to hold (see Theorem 1.5).

**Theorem 1.1.** For every $\epsilon > 0$ there exists a density-1 sequence of numbers $S' = S'(\epsilon) \subseteq S$

so that the following hold:

1. Along $n \in S'$ we have $N_n \to \infty$, and the set of accumulation points of $\{\hat{\tau}_n(4)\}_{n \in S'}$ contains the interval $[0, 1]$.
2. For $n \in S'$, uniformly for all $s > n^{-1/2+\epsilon}$ we have

\[
\text{Var}(\mathcal{L}_{n;s}) = c_n \cdot (\pi s^2)^2 \cdot \frac{E_n}{N_n^2} \left(1 + O(\epsilon \frac{1}{N_n^{1/2}})\right),
\]

where $c_n$ is given by (1.12), and the constant involved in the ‘$O$’-notation depends on $\epsilon$ only (cf. (1.11)).
3. For random variables $X, Y$ we denote as usual their correlation

\[
\text{Corr}(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}}.
\]

Then for every $\epsilon > 0$ we have that

\[
\sup_{s > n^{-1/2+\epsilon}} |\text{Corr}(\mathcal{L}_{n;s}, \mathcal{L}_n) - 1| \to 0,
\]

i.e., the nodal length $\mathcal{L}_{n;s}$ of $f_n$ restricted to a small ball is asymptotically fully correlated with the full nodal length $\mathcal{L}_n$ of $f_n$, uniformly for all $s > n^{-1/2+\epsilon}$.

Quite remarkably (and somewhat surprisingly), (1.15) shows that one may statistically reconstruct the full nodal length of $f_n$ based on its restriction to a small toral ball. Since the full nodal length $\mathcal{L}_n$ of $f_n$ obeys the limiting law (1.13), the full correlation (1.15) of the restricted nodal length $\mathcal{L}_{n;s}$ with $\mathcal{L}_n$ implies the same limiting law for $\mathcal{L}_{n;s}$ under the same conditions as Theorem 1.1. More precisely we have the following corollary:

**Corollary 1.2.** Given $\epsilon > 0$ and a (generic) sequence $S' = S'(\epsilon) \subseteq S$ as in Theorem 1.1, there exists a coupling $(M_{\hat{\tau}_n(4)}, f_n)$ (of a random variable with a random field) satisfying

\[
\sup_{s > n^{-1/2+\epsilon}} \mathbb{E} \left[ \left| \frac{\mathcal{L}_{n;s} - \mathbb{E}[\mathcal{L}_{n;s}]}{\text{Var}(\mathcal{L}_{n;s})} - M_{\hat{\tau}_n(4)} \right| \right] \to 0.
\]
1.5 Spectral (quasi)-correlations. Our next goal is to formulate a result à la Theorem 1.1 with a more explicit control over \( \{ n \} \) satisfying the conclusions of Theorem 1.1. To this end we will require some more notation. Recall that the covariance function \( r_n(x) \) of \( T_n \) is given by (1.7); of our particular interest are the moments of \( r_n \) (and related quantities), both on the whole of \( \mathbb{T}^2 \) and restricted to \( B(s) \). More precisely, for \( l \geq 1 \) we define the “full” moments of \( r_n \) as

\[
R_n(l) = \int_{\mathbb{T}^2 \times \mathbb{T}^2} r_n(x-y) dx dy = \int_{\mathbb{T}^2} r_n(x)^l dx
\]

by the stationarity, and the “restricted moments” of \( r_n \) as

\[
R_n(l; s) = \int_{B(s) \times B(s)} r_n(x-y) dx dy.
\]

An explicit computation using the orthogonality relations

\[
\int_{\mathbb{T}^2} e(\langle x, \xi \rangle) dx = \begin{cases} 1, & \xi = 0, \\ 0, & \xi \neq 0, \end{cases} \quad \xi \in \mathbb{Z}^2,
\]

relates \( R_n(l) \) to the set of length-\( l \) spectral correlations

\[
S_n(l) = \left\{ (\lambda_1, \ldots, \lambda_l) \in (E_n)^l : \sum_{i=1}^l \lambda_i = 0 \right\}.
\]

the full moments of \( r_n \) are given by

\[
R_n(l) = \frac{|S_n(l)|}{N_n^l}.
\]

For \( l = 2k \) even we further define the diagonal spectral correlations set to be

\[
D_n(l) = \{ \pi(\lambda_1, -\lambda_1, \ldots, \lambda_k, -\lambda_k) : \lambda_1, \ldots, \lambda_k \in (E_n)^k, \pi \in S_l \}
\]

the set of all possible permutations of tuples of lattice points of the form \((\lambda_1, -\lambda_1, \ldots, \lambda_k, -\lambda_k)\). It is evident that in this case

\[
D_n(l) \subseteq S_n(l),
\]

and that for every fixed \( l \) we have

\[
|D_n(l)| = \frac{(2k)!}{2^k k!} N_n^k \cdot \left( 1 + O_{N_n \to \infty} \left( \frac{1}{N_n} \right) \right);
\]

hence for every \( k \geq 1 \) we have the lower bound

\[
|S_n(2k)| \gg N_n^k.
\]
To the other end, for \( k = 1 \) we have \( S_n(2) = D_n(2) \), by the definition, whereas for \( k = 2 \) the equality

\[
D_n(4) = S_n(4)
\]

is due to an elegant (and simple) geometric observation by Zygmund [Zy] (the so-called “Zygmund’s trick”); the same observation yields

\[
S_n(6) \ll N_n^4.
\]

A key ingredient in [KKW13] was a non-trivial improvement for the latter bound,

\[
|S_n(6)| = o(N_n^4)
\]

due to Bourgain (published in [KKW13]) implying that the l.h.s. of (1.11) is asymptotic to the r.h.s. of (1.11) (though not implying the stated bound for the error term); a further improvement

\[
S_n(6) \ll N_n^{7/2}
\]

holding for the full sequence \( n \in S \) [BB15] yields the stronger form (1.11) of their result with the prescribed error term. The sharp upper bound

\[
|S_n(6)| \ll N_n^3,
\]

or, even more striking,

\[
|S_n(6)| = 3N_n^3 + O(N_n^{3-\delta})
\]

(equivalently, \(|S_n(6) \setminus D_n(6)| \ll N_n^{3-\delta}\)) holds for a density-1 subsequence \( \{n\} \subseteq S \).

For the restricted moments of particular interest for our purposes, we need to consider the “spectral quasi-correlations” (see §2.2), i.e., the defining equality

\[
\sum_{i=1}^l \lambda_i = 0 \text{ in (1.19) holding approximately: the absolute value } |\cdot| \leq K\]

where the parameter \( K \) is typically of order of magnitude \( K \approx n^{1/2-\delta} \), \( 0 < \delta < \epsilon \).

**Definition 1.3** (Quasi-correlations and \( \delta \)-separatedness.).

1. Given a number \( n \in S \), \( l \in \mathbb{Z}_{\geq 2} \), and \( 0 < K = K(n) < l \cdot \sqrt{n} \) we define the set of length-\( l \) spectral quasi-correlations

\[
C_n(l; K) = \left\{ (\lambda_1, \ldots, \lambda_l) \in \mathbb{C}_n^l : 0 < \left\| \sum_{j=1}^l \lambda_j \right\| \leq K \}.
\]

2. For \( \delta > 0 \) we say that a number \( n \subseteq S \) satisfies the \((l, \delta)\)-separatedness hypothesis \( A(n; l, \delta) \) if

\[
C_n(l; n^{1/2-\delta}) = \emptyset.
\]
For example, a number \( n \in S \) satisfies the \((2, \delta)\)-separatedness hypothesis \( A(n; 2, \delta) \) if the nearest neighbor distance in \( E_n \) grows like \( n^{1/2-\delta} \), i.e., for all \( \lambda, \lambda' \in E_n \) with \( \lambda \neq \lambda' \) we have
\[
\| \lambda - \lambda' \| > n^{1/2-\delta}.
\]

Bourgain and Rudnick [BR11, Lemma 5] proved that all but \( O(X^{1-2\delta/3}) \) numbers
\[
n \in S(X) = \{ n \in S : n \leq X \}
\]
satisfy \( A(n; 2, \delta) \) (cf. (1.8)), and more recently Granville–Wigman [GW16] refined their estimate to yield a precise asymptotic expression for the number of exceptions \( n \) to \( A(n; 2, \delta) \). More generally, for any fixed \( l \geq 2 \), the following theorem shows that for generic \( n \in S \) the assumption \( A(n; l, \delta) \) holds; it is stronger than needed for our purposes in more than one way (see §2.2).

**Theorem 1.4.** For every \( l \geq 2 \) and \( \delta > 0 \) there exist a set \( S' = S'(l; \delta) \subseteq S \) such that:

1. The set \( S' \subseteq S \) has density 1 in \( S \).
2. The set of accumulation points of \( \{ \hat{\tau}_n(4) \}_{n \in S'} \) contains the interval \([0, 1]\).
3. For every \( n \in S' \) the length-\( l \) spectral quasi-correlation set

   \[
   C_n(l; n^{1/2-\delta}) = \emptyset
   \]

   is empty, i.e., \( A(n; l, \delta) \) is satisfied.

The following result is a version of Theorem 1.1 with an explicit sufficient condition on \( n \), by virtue of Theorem 1.4.

**Theorem 1.5.** Let \( \epsilon > 0 \), \( \delta < \epsilon \), and \( S' = \{ n \} \subseteq S \) be a sequence of energies so that for all \( n \in S' \) the hypotheses \( A(n; 2, n^{1/2-\delta/2}) \) and \( A(n; 6, n^{1/2-\delta}) \) are satisfied. Then along \( S' \) both (1.14) and (1.15) of Theorem 1.1 hold.

In light of Theorem 1.4, Theorem 1.5 clearly implies the statement of Theorem 1.1, so from this point on we will only aim at proving Theorem 1.5.

**Acknowledgements.** We are indebted to Jerry Buckley, Manjunath Krishnapur, Pär Kurlberg, Zeev Rudnick and Mikhail Sodin for many stimulating and fruitful conversations, and their valuable comments on the earlier version of this manuscript. It is a pleasure to thank Andrew Granville for his inspiring ideas, especially related to some aspects involving various subjects in the geometry of numbers. Finally, we are grateful to the anonymous referee for his very thorough
reading of the manuscript, also resulting in improved readability of the present version. The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreements n° 277742 Pascal (D.M.) and n° 335141 Nodal (J.B. and I.W.),

2 Discussion

2.1 Berry’s cancellation phenomenon. It was originally found by Berry [Be02] that the nodal length variance of the RWM (see §1.1) in growing domains, e.g., the radius-$R$ Euclidean balls $B(R) \subseteq \mathbb{R}^2$, is of lower order than what was expected from the scaling considerations, i.e., of order of magnitude $\log R$, rather than $R$: “...it results from a cancellation whose meaning is still obscure...”; it was found that the leading term of the 2-point correlation function is purely oscillatory. The same phenomenon (“Berry’s Cancellation”) was rediscovered [Wi09] for random high degree spherical harmonics, and then for the arithmetic random waves [KKW13] (“Arithmetic Berry’s Cancellation”).

In [MPRW16] the Wiener chaos expansion was applied to the nodal length, and it was interpreted that Berry’s cancellation has to do with the precise vanishing of the projection of the nodal length into the 2nd chaos, with the 4th one dominating. A similar observation with the 4th chaotic projection dominating was also made for the high degree $l \to \infty$ spherical harmonics [MRW17], also for shrinking domains of Planck scale (e.g., radius $\frac{R}{l}$ spherical caps with $R = R(l) \to \infty$ arbitrarily slowly); though for the shrinking domains the 2nd chaotic projection does not vanish precisely, it is still dominated by the 4th one.

The ability to understand the asymptotic behavior of the nodal length variance for the torus [KKW13] depended on evaluating the various moments of the covariance function $r_n$ in (1.7) via the orthogonality relations (1.18) holding for the full torus (see §2.2 to follow immediately); this no longer holds for shrinking domains (or even fixed subdomains of $\mathbb{T}^2$). It was then thought that the analogous results of [KKW13] and further [MPRW16] fail decisively for domains shrinking within Planck scale rate; our principal results show that the contrary is true (Berry’s cancellation holding; the second chaos projection, though not vanishing precisely, being dominated by the 4th one; understanding the limit distribution of the nodal length) if we are willing to excise a thin set of energies $\{n\} \subseteq S$ and work slightly above the Planck scale.
2.2 Restricted moments on shrinking domains and spectral quasi-correlations. The principal result of this manuscript asserts that the nodal length distribution for arithmetic random waves (1.4) on the whole torus [KKW13, MPRW16] is, up to a normalizing factor, asymptotic to the nodal length restricted to balls shrinking slightly above Planck scale, albeit for generic energy levels \(n\) only. Let \(r_n(x, y) = r_n(x - y)\) be the covariance function (1.7) of \(T_n\); one may expand [KKW13] the nodal length variance in terms of the moments of \(r_n\) and its derivatives, and these are also intimately related to the finer aspects of its limit distribution [MPRW16]. Let us consider the 2nd moment of \(r_n\) as an illustrative example; for the unrestricted problem (i.e., the full torus) we have by the translation invariance

\[
\int_{T^2 \times T^2} r_n(x - y)^2 \, dx \, dy = \frac{1}{N_n} \sum_{\lambda, \lambda' \in \mathcal{E}_n} \int_{T^2} e(\langle x, \lambda - \lambda' \rangle) \, dx
\]

(2.1)

upon separating the diagonal, and using the orthogonality relations (1.18).

For the restricted moments, e.g.,

\[
\int_{B(s) \times B(s)} r_n(x - y)^2 \, dx \, dy,
\]

an analogue of (2.1) no longer holds, as we no longer have the precise orthogonal relations (1.18) nor the translation invariance (first equality in (2.1)). We may still separate the diagonal to write

\[
\int_{B(s) \times B(s)} r_n(x - y)^2 \, dx \, dy
\]

(2.2)

\[
= (\pi s^2)^2 \cdot \frac{1}{N_n} + \frac{1}{N_n} \sum_{\lambda \neq \lambda'} \int_{B(s) \times B(s)} e(\langle x - y, \lambda - \lambda' \rangle) \, dx \, dy
\]

\[
= (\pi s^2)^2 \cdot \frac{1}{N_n} + \frac{1}{N_n} \left| \int_{B(s)} e(\langle x, \lambda - \lambda' \rangle) \, dx \right|^2,
\]

so we no longer need to cope with the lack of translation invariance. Comparing (2.2) with (2.1) we observe that both have the same diagonal contribution, though the off-diagonal one for (2.2) might not be vanishing. We then observe that the inner integral on the r.h.s. of (2.2) is the Fourier transform of the characteristic function \(\chi_{B(s)}\) of the Euclidean unit disc \(B(s) \subseteq \mathbb{R}^2\), evaluated at \(\lambda - \lambda'\); by scaling, we have that

\[
\int_{B(s)} e(\langle x, \lambda - \lambda' \rangle) \, dx = s^2 \chi_{B(1)}(s \cdot \|\lambda - \lambda'\|) = 2\pi s^2 \cdot \frac{J_1(s\|\lambda - \lambda'\|)}{s\|\lambda - \lambda'\|}
\]
where $J_1$ is the Bessel $J$ function (cf. (A.21)). We then obtain

\begin{equation}
(2.3) \quad \int_{B(s) \times B(s)} r_n(x - y)^2 \, dx \, dy = (\pi s^2)^2 \frac{1}{N_n} + 2 \pi s^2 \sum_{\lambda \neq \lambda'} \frac{J_1(s\|\lambda - \lambda'\|)^2}{s^2 \|\lambda - \lambda'\|^2};
\end{equation}

since $J_1$ decays at infinity, it is evident that for the diagonal contribution to dominate the r.h.s. of (2.3) it is important to control the contribution of the regime $s\|\lambda - \lambda'\| \ll 1$.

Since $s$ is assumed to be above the Planck scale $s > n^{-1/2+\epsilon}$, and $N_n$ is much smaller (1.9) than any power of $n$, it is sufficient to bound the contribution of the range $\|\lambda - \lambda'\| < n^{1/2-\delta}$ for some $\delta < \epsilon$, i.e., the size of the quasi-correlation set $\mathcal{C}_n(2; n^{1/2-\delta})$. Recalling the notation (1.17) for restricted moments, the above discussion shows that, under the assumption $\mathcal{A}(n; 2, \delta)$ that $\mathcal{C}_n(2; n^{1/2-\delta}) = \emptyset$, the second moment of $r_n$, restricted to $B(s)$, is asymptotic to

\begin{equation}
(2.4) \quad \mathcal{R}_n(2; s) \sim (\pi s^2)^2 \cdot \frac{1}{N_n}.
\end{equation}

Theorem 1.4 (also the aforementioned result of Bourgain and Rudnick [BR11, Lemma 5]) shows that the hypothesis $\mathcal{A}(n; 2, \delta)$ is satisfied for a density-1 sequence $\{n\} \subseteq S$; clearly, $\mathcal{A}(n; 2, \delta)$ not allowing any quasi-correlations is far stronger than what is required for (2.4) to be satisfied, by the above. Instead, in order to yield the asymptotics (2.4) for the second restricted moment, it would be sufficient to impose that the quasi-correlation set

$$\mathcal{C}_n(l; n^{1/2-\delta}) = o(N_n)$$

is dominated by the diagonal $\mathcal{D}_n(2)$.

More generally, for the other relevant moments associated to $r_n$ (namely, higher moments of $r_n$ or its derivatives, or second moment of various derivatives of $r_n$) we need to expand the restricted moments up to an error term $o(\frac{1}{N_n^2})$. As for the nodal length computations we need to evaluate the 2nd and 4th moments and bound the 6th restricted moment

$$\mathcal{R}_n(6; s) = o\left(\frac{s^4}{N_n^2}\right)$$

(see (3.27) and (3.35) below), that naturally brings up the questions of bounding the quasi-correlation sets $\mathcal{C}_n(2; n^{1/2-\delta})$, $\mathcal{C}_n(4; n^{1/2-\delta})$, and $\mathcal{C}_n(6; n^{1/2-\delta})$ (see Lemma 3.4 and its proof below), with the diagonal contribution coming from $S_n(2) = \mathcal{D}_n(2)$, $S_n(4) = \mathcal{D}_n(4)$ or $S_n(6)$ respectively (and $S_n(6)$ being bounded by [BB15]). It then follows that the conclusions of Theorem 1.5 hold uniformly for $s > n^{-1/2+\epsilon}$ under the hypotheses $\mathcal{A}(n; 2, \delta)$, $\mathcal{A}(n; 4, \delta)$ and $\mathcal{A}(n; 6, \delta)$ for some $\delta < \epsilon$; this is somewhat
different from the assumptions made within the formulation of Theorem 1.5. It was observed [RW, Lemma 5.2] that, in fact, the hypothesis \( A(n; 2, n^{1/2-\delta}) \) implies \( A(n; 4, n^{1/2-2\delta}) \), explaining the said discrepancy:

**Lemma 2.1** ([RW, Lemma 5.2]). For \( \delta < 1/2, n \in S \) sufficiently big, if \( n \) satisfies the separatedness hypothesis \( A(n; 2, n^{1/2-\delta}) \), then \( n \) also satisfies \( A(n; 4, n^{1/2-2\delta}) \).

By the above, rather than assuming that

\[
C_n(2; n^{1/2-\delta/2}) = C_n(6; n^{1/2-\delta}) = \emptyset
\]

are empty, it would be sufficient to make the somewhat weaker assumptions

\[
|C_n(2; n^{1/2-\delta})| = o(N_n^2), \quad |C_n(4; n^{1/2-\delta})| = o(N_n^2),
\]

and

\[
|C_n(6; n^{1/2-\delta})| = o(N_n^4).
\]

It seems likely that by combining the ideas of the proof of Theorem 1.4 with some ideas in [GW16] it would be possible to shrink the balls faster: prove Theorem 1.1 for \( s > \frac{\log n^4}{\sqrt{n}} \) for generic \( n \) for some \( A \gg 0 \) sufficiently big, i.e., save a power of \( \log n \) rather than of \( n \). We believe that, in light of the results (and the techniques) presented in [GW16] it is conceivable (if not likely) that there is a phase transition: a number \( A_0 > 0 \) such that the conclusions of Theorem 1.1 hold for generic \( n \) uniformly for all \( s > (\log n)^4 \sqrt{n} \) with \( A > A_0 \), and fail for generic \( n \) for \( s = (\log n)^4 \sqrt{n}, A < A_0 \). We leave all these questions to be addressed elsewhere. Finally, we note that all the methods presented in this manuscript work (resp. uniformly) unimpaired for generic smooth (shrinking) domains (as a replacement for discs), as long as the Fourier transform of the characteristic function is (resp. uniformly) decaying at infinity.

We believe that Theorem 1.4 is of considerable independent interest. Other than the results contained in this paper, Theorem 1.4 could be used in order to establish small-scale analogues of various other recently established and forthcoming results, though slightly restricted in terms of energy levels (density-1 sequence \( S' \subseteq S \) rather than the whole of \( S \)). As a concrete application, a Planck-scale analogue of Bourgain’s de-randomisation technique [Bo, BW] could be used for counting the number of nodal domains for (deterministic) “flat” toral eigenfunctions [Sa].

2.3 Outline of the proofs of the main results. The proof of Theorem 1.5 consists of two main steps. In the first step (§3.1–§3.4) we employ the Kac–Rice formula in order to express the variance of \( \mathcal{L}_{n;2} \) in terms of an integral on \( B(s) \times B(s) \) of the 2-point correlation function and study its asymptotic behaviour.
to yield (1.14); this step is analogous to [KKW13] posing new challenges for integrating the 2-point correlation function on a restricted domain. In the second step (§3.5) the full correlation (1.15) result is established.

A significant part of the first step is done in [KKW13]: it yields a pointwise asymptotic expansion (3.26) for the 2-point correlation function, provided that $|r_n(x)|$ is bounded away from 1; this eventually reduces the question of the asymptotic behavior of $\text{Var}(\mathcal{L}_{n,x})$ to evaluating some moments of $r_n$ and its various derivatives, restricted to $B(s) \times B(s)$, provided that we avoid the “singular set”, i.e., $(x, y)$ such that $|r_n(x - y)|$ is arbitrarily close to 1. As it was mentioned in §2.2, evaluating the restricted moments is a significant challenge of a number theoretic nature; here we use the full strength of the assumptions of Theorem 1.5 on $n$.

To bound the contribution of the singular set we modify the approach in [ORW], partitioning the singular set into small cubes of side length commensurable with $\frac{1}{\sqrt{n}}$. One challenge here is that $B(s) \times B(s)$ could not be tiled by cubes; we resolve this by tiling a slightly excised set, not beyond $B(2s) \times B(2s)$, using the latter in order to bound the total measure of the singular set. We also simplify and improve our treatment of the singular set as compared to [KKW13], following some ideas from [RW2014]: we use the Lipschitz continuity property satisfied by $r_n$ to bound the total measure of the singular set, and also apply the partition into cubes on the singular set only (as opposed to the full domain of integration).

After the variance of $\mathcal{L}_{n,x}$ has successfully been analyzed (1.14), there are two ways to further proceed to establishing the full correlation result (1.15). On one hand, we may follow along the steps of [MPRW16] to evaluate the Wiener chaos expansion of $\mathcal{L}_{n,x}$; performing this we find that, under the assumptions of Theorem 1.5, the main terms of its projection $\mathcal{L}_{n,x}[4]$ onto the 4th Wiener chaos, dominating the fluctuations of $\mathcal{L}_{n,x}$, recover, up to a scaling factor, the projection $\mathcal{L}_n[4]$ of the total nodal length of $f_n$ to the 4th Wiener chaos. This, in particular, implies the full correlation result (1.15).

On the other hand, now that much computational work has already been done, we might reuse the precise information on the 2-point correlation function [KKW13] to simplify the proofs drastically by directly evaluating the correlation between $\mathcal{L}_{n,x}$ and $\mathcal{L}_n$ without decomposing them into their respective Wiener chaos components. Equivalently, we evaluate the covariance $\text{Cov}(\mathcal{L}_{n,x}, \mathcal{L}_n)$ by employing the (suitably adapted) Kac–Rice formula once again; using the group structure of the torus, this approach yields the intriguing identity

$$\text{Cov}(\mathcal{L}_{n,x}, \mathcal{L}_n) = (\pi s^2) \cdot \text{Var}(\mathcal{L}_n),$$

which, together with (1.14) and (1.11), recovers (1.15).
3 Proof of Theorem 1.5

3.1 Preliminaries. Recall that the covariance function \( r = r_n \) of \( f_n \) is given by (1.7). We further define the gradient

\[
D = D_n:1 \times 2(x) = \nabla r_n(x) = \frac{2\pi i}{N_n} \sum_{\|\lambda\|^2=n} e(\langle \lambda, x \rangle) \cdot \lambda,
\]

and the Hessian

\[
H = H_n:2 \times 2(x) = (\frac{\partial^2 r_n}{\partial x_i \partial x_j}) = -\frac{4\pi^2}{N_n} \sum_{\|\lambda\|^2=n} e(\langle \lambda, x \rangle)(\lambda^T \lambda),
\]

and the \( 2 \times 2 \) blocks (all depending on \( n \), and evaluated at \( x \in \mathbb{T}^2 \))

\[
X = -\frac{2}{E_n(1 - r_n^2)} D'D, \quad Y = -\frac{2}{E_n} \left( H + \frac{r_n}{1 - r_n} D'D \right).
\]

Finally, let \( \Omega \) be the matrix

\[
\Omega = \Omega_n:4 \times 4(x) = I + \begin{pmatrix} X & Y \\ Y & X \end{pmatrix};
\]

it is [KKW13, Equalities (24), (25)] the normalized covariance matrix of \((\nabla f_n(0), \nabla f_n(x))\) conditioned on \( f_n(0) = f_n(x) = 0 \).

In the following lemma we evaluate the variance of the restricted length \( L_{n,s} \); it is analogous to [KKW13, Proposition 3.1] and [RW2008, Proposition 5.2] which give the variance of the total length \( L_{n,s} \) except that, accordingly, the domain of integration in (3.4) is restricted to \( B(s) \times B(s) \), rather than the full \( \mathbb{T} \times \mathbb{T} \) (reducing to \( \mathbb{T} \) by stationarity). The proof of these works is sufficiently robust to cover our case unimpaired, and thereupon conveniently omitted in this manuscript.

**Lemma 3.1.** For every \( s > 0 \) we have

\[
\text{Var}(L_{n,s}) = \frac{E_n}{2} \int_{B(s) \times B(s)} \left( K_2(x - y) - \frac{1}{4} \right) dx dy,
\]

where the (normalized) 2-point correlation function is

\[
K_2(x) = K_2_n(x) = \frac{1}{2\pi \sqrt{1 - r_n(x)^2}} \cdot \mathbb{E}[\|V_1\| \cdot \|V_2\|],
\]

where \((V_1, V_2) \in \mathbb{R}^2 \times \mathbb{R}^2\) is a centered 4-variate Gaussian vector, whose covariance is given by (3.3), with \( X \) and \( Y \) given by (3.2).
3.2 Singular set. Recall that the nodal length variance (restricted to shrinking balls) is given by (3.4), where the (normalized) 2-point correlation function $K_2(x)$ is given by (3.5), and $(V_1, V_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ is centered Gaussian with covariance (3.3), with $X$ and $Y$ given by (3.2). It is possible to expand

$$
\mathbb{E}[\|V_1\| \cdot \|V_2\|]
$$

into a degree-4 Taylor polynomial as a function of the (small) entries of $X$ and $Y$ (these are the various derivatives of $r_n$), and, provided that the absolute value $|r_n(x)|$ is bounded away from 1, we may write

$$
\frac{1}{\sqrt{1 - r}} = 1 + \frac{1}{2} r^2 + \frac{3}{8} r^4 + O(r^6).
$$

These two combined yield a point-wise approximate (3.26) of $K_2(x)$, provided that $|r_n(x)|$ is bounded away from 1, a condition that is satisfied for “most” of $(x, y) \in B(s) \times B(s)$ (“nonsingular set” $(B(s) \times B(s)) \setminus B_{\text{sing}}$, see Lemma 3.3 below), and we may integrate the Taylor polynomial of $x - y$ over the nonsingular set to yield an approximation for the integral on the r.h.s. of (3.4) while bounding the contribution of the singular set.

For the singular set $B_{\text{sing}}$ (where $|r_n(x)|$ is close to 1) we only have an easy bound (3.25), and merely bounding its measure is insufficient for bounding its contribution to the integral on the r.h.s. of (3.4). We resolve this obstacle by observing that if $|r_n(x_0)|$ is close to 1, then it is so on the whole $(4d)$ cube around $x_0$ of size length commensurable to $\frac{1}{\sqrt{n}}$, by the Lipschitz property of $r_n$. This allows us to partition $B_{\text{sing}}$ into “singular” cubes of side length commensurable with $\frac{1}{\sqrt{n}}$, possibly excising $B(s) \times B(s)$, though not beyond $B(2s) \times B(2s)$. We might then bound the number of singular cubes using a simple Chebyshev’s inequality bound (3.7) via an appropriate 6th moment (it is $R_n(6; 2s)$ as $B_{\text{sing}} \subseteq B(2s) \times B(2s)$) while controlling the contribution of a single singular cube to the integral on the r.h.s. of (3.4) (as opposed to a point-wise bound).

The presented analysis is simplified and improved compared to [RW2008, KKW13] in the following ways, borrowing in particular some ideas from [RW2014]. First, only the singular set is partitioned into cubes as opposed to the whole domain of integration (e.g., $B(s) \times B(s)$ in our case), since the point-wise estimate (3.26) might be integrated on the nonsingular set to yield a precise estimate for its contribution to the integral on the r.h.s. of (3.4). The Lipschitz property of $r_n$ simplifies the partition argument of the singular set into cubes, with no need to bound the individual cosines in (1.7). Finally, working with (shrinking) subdomains $B(s)$ of the torus poses a problem while tiling the said domain $(B(s) \times B(s))$.
into cubes; since \( s > n^{-1/2 + \epsilon} \) we might still partition \( B(s) \times B(s) \) into cubes of side length commensurable to \( \frac{1}{\sqrt{n}} \) without excising the domain of integration beyond \( B(2s) \times B(2s) \). We start from the definition of the singular set.

**Definition 3.2 (Singular set).** Let \( s > n^{-1/2 + \epsilon} \) and choose

\[
F = F(n) = \frac{1}{c_0} \cdot \sqrt{n}
\]

a large integer, with \( c_0 > 0 \) a sufficiently small constant (that will be fixed throughout the rest of this manuscript).

1. A point \((x, y) \in B(s) \times B(s)\) is singular if \( |r_n(x - y)| > \frac{7}{8} \) (say).
2. Let

\[
B(s) \times B(s) \subseteq \bigcup_{i \in I} B_i \subseteq B(2s) \times B(2s)
\]

be a covering of \( B(s) \times B(s) \) by \((4d)\) cubes \( \{ B_i \} \) of side length \( \frac{1}{F} \). We say that a cube \( B_i \) is singular if it contains a singular point \( x \in B_i \).
3. Let \( I' \subseteq I \) be the collection of all indices \( i \in I \) such that \( B_i \) is singular. We define the singular set

\[
B_{\text{sing}}(s) = B_{n; \text{sing}} = \bigcup_{i \in I'} B_i \subseteq B(2s) \times B(2s)
\]

to be the union of all singular cubes.

**Lemma 3.3.** Let \( F \) be as above (3.6), with \( c_0 \) sufficiently small.

1. If \( B_i \subseteq B_{\text{sing}} \) is singular, then for all \((x, y) \in B_i\) we have \( |r_n(x - y)| > \frac{1}{2} \).
2. The measure of the singular set is bounded by

\[
\text{meas}(B_{\text{sing}}) \ll c_0 \cdot \mathcal{R}_n(6; 2s)
\]

the 6th moment (1.17) of \( r_n \) on \( B(2s) \).
3. The number \(|I'|\) of singular cubes is bounded by

\[
|I'| \ll c_0 \cdot F^4 \cdot \mathcal{R}_n(6; 2s).
\]

**Proof.** For (1) we note that \( r_n \) is Lipschitz in all the variables with constant \( \ll \sqrt{n} \). Hence if

\[
|r_n(x_0 - y_0)| > \frac{3}{4}
\]

for some \((x_0, y_0) \in B_i\), then for all \((x, y) \in B_i\) we have for some \( C, C' \) absolute constants:

\[
|r_n(x - y)| > \frac{3}{4} - C \sqrt{n} \| (x, y) - (x_0, y_0) \| \geq \frac{3}{4} - C' \sqrt{n} \cdot \frac{c_0}{\sqrt{n}} > \frac{1}{2},
\]
provided that $c_0$ was chosen sufficiently small. The estimate (3.7) now follows from the above and Chebyshev’s inequality; note that the 6th moment $\mathcal{R}_n(6; 2s)$ is over $B(2s) \times B(2s)$ rather than $B(s) \times B(s)$, in light of the fact that $B_{\text{sing}}$ might not be contained in $B(s) \times B(s)$. Finally, (3.8) follows from (3.7) bearing in mind that the measure of each singular cube in $B_{\text{sing}}$ is of order $1/F_4$. □

3.3 Moments of $r$ and its derivatives along the shrinking balls $B(s)$.

Our final ingredient for the proof of the variance part of Theorem 1.5 is evaluating certain moments of $r_n, X_n$, and $Y_n$ in $B(s)$ in Lemma 3.4. The proof of Lemma 3.4 will be given in Appendix A.

**Lemma 3.4** (Cf. [KKW13, Lemmas 4.6 and 5.4]). Let $\epsilon > 0$,

(3.9) \[ \delta < \epsilon, \]

and $S' \subseteq S$ a sequence of energy levels such that for all $n \in S'$ the hypotheses $A(n; 2, n^{1/2-\delta/2})$ and $A(n; 6, n^{1/2-\delta})$ in Definition 1.3 are satisfied. Then, for all $A > 0$ and uniformly for all $s > n^{-1/2+\epsilon}$, along $n \in S'$ the following moments of $r_n, X_n, Y_n$ observe the following asymptotics, with constants involved in the ‘$O$’-notation depending only on $A, \epsilon, \delta$:

1. \[ \int_{B(s) \times B(s)} r_n(x - y)^2 \, dx \, dy = (\pi s^2)^2 \cdot \frac{1}{N_n} \left( 1 + O\left( \frac{1}{N_n^4} \right) \right), \] (3.10)

2. \[ \int_{B(s) \times B(s)} r_n(x - y)^4 \, dx \, dy = (\pi s^2)^2 \cdot \frac{|D_n(4)|}{N_n^4} \left( 1 + O\left( \frac{1}{N_n^4} \right) \right) \]
   \[ = (\pi s^2)^2 \cdot \frac{3}{N_n^2} \left( 1 + O\left( \frac{1}{N_n} \right) \right). \] (3.11)

3. \[ \mathcal{R}_n(6; s) = \int_{B(s) \times B(s)} r_n(x - y)^6 \, dx \, dy = O\left( s^4 \cdot \frac{1}{N_n^{5/2}} \right), \] (3.12)

4. \[ \int_{B(s) \times B(s)} \text{tr} X_n(x - y) \, dx \, dy = (\pi s^2)^2 \left( - \frac{2}{N_n} - \frac{2}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right), \] (3.13)

5. \[ \int_{B(s) \times B(s)} \text{tr}(Y_n(x - y)^2) \, dx \, dy = (\pi s^2)^2 \left( \frac{4}{N_n} - \frac{4}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right), \] (3.14)

6. \[ \int_{B(s) \times B(s)} \text{tr}(X_n(x - y)Y_n(x - y)^2) \, dx \, dy = (\pi s^2)^2 \left( - \frac{4}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right), \] (3.15)
\[
\int_{B(s) \times B(s)} \text{tr}(X_n(x - y)^2) \, dx \, dy = (\pi s^2)^2 \left( \frac{8}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right),
\]
(7)

\[
\int_{B(s) \times B(s)} \text{tr}(Y_n(x - y)^4) \, dx \, dy = (\pi s^2)^2 \left( \frac{2(11 + \hat{r}_n(4)^2)}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right),
\]
(8)

\[
\int_{B(s) \times B(s)} \text{tr}(Y_n(x - y)^2)^2 \, dx \, dy = (\pi s^2)^2 \left( \frac{4(7 + \hat{r}_n(4)^2)}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right),
\]
(9)

\[
\int_{B(s) \times B(s)} \text{tr}(X_n(x-y)) \text{tr}(Y_n(x-y)^2) \, dx \, dy = (\pi s^2)^2 \left( -\frac{8}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right),
\]
(10)

\[
\int_{B(s) \times B(s)} r_n(x-y)^2 \text{tr} X_n(x-y) \, dx \, dy = (\pi s^2)^2 \left( -\frac{2}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right),
\]
(11)

\[
\int_{B(s) \times B(s)} r_n(x-y)^2 \text{tr} Y_n(x-y)^2 \, dx \, dy = (\pi s^2)^2 \left( \frac{2}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right),
\]
(12)

\[
\int_{B(s) \times B(s)} \text{tr} X_n(x-y)^3 \, dx \, dy = (\pi s^2)^2 \cdot O\left( \frac{1}{N_n^{5/2}} \right),
\]
(13)

\[
\int_{B(s) \times B(s)} \text{tr} Y_n(x-y)^6 \, dx \, dy = (\pi s^2)^2 \cdot O\left( \frac{1}{N_n^{5/2}} \right).
\]
(14)

### 3.4 Proof of the variance part (1.14) of Theorem 1.5.

**Lemma 3.5** ([KKW13], Lemma 3.2). The matrices $X_n$ and $Y_n$ are uniformly bounded (entry-wise), i.e.,

\[
X_n(x), \, Y_n(x) = O(1),
\]
(3.24)

for all $x \in \mathbb{T}^2$, where the constant involved in the ‘$O$’-notation is absolute. In particular

\[
K_2(x) = O\left( \frac{1}{\sqrt{1 - r_n(x)^2}} \right).
\]
(3.25)

**Lemma 3.6** (Cf. Lemma 3.5 and [ORW, §6.4]). Let $K_2$ be the 2-point correlation function (3.5), and $\mathcal{B}_i$, $i \in I$ ‘a singular cube. Then

\[
\int_{\mathcal{B}_i} K_2(x-y) \, dx \, dy \ll_c \frac{1}{F^3 \sqrt{n}}.
\]
For brevity of notation in what follows we will sometimes suppress the dependency of various variables on \(x\) or \(n\), e.g., \(r\) or \(r(x)\) will stand for \(r_n(x)\), and \(X\) will denote the \(2 \times 2\) matrix \(X_n(x)\) in (3.2).

**Proposition 3.7** (Cf. [KKW13, Proposition 4.5]). For every \(x \in \mathbb{T}^2\) such that \(|r_n(x)| < \frac{1}{4}\), the 2-point correlation function \(K_2(x) = K_{2,n}(x)\) satisfies the asymptotic expansion

\[
K_2(x) = \frac{1}{4} + L_2(x) + \epsilon(x),
\]

where

\[
L_2(x) = \frac{1}{8} \left( r^2 + \text{tr} X + \frac{\text{tr}(Y^2)}{4} + \frac{3}{4} r^4 - \frac{\text{tr}(XY^2)}{8} - \frac{\text{tr}(X^2)}{16} + \frac{\text{tr}(Y^4)}{128} \right)
\]

\[
+ \frac{\text{tr}(Y^2)^2}{256} - \frac{\text{tr}(X) \cdot \text{tr}(Y^2)}{16} + \frac{1}{2} r^2 \text{tr}(X) + \frac{1}{8} r^2 \text{tr}(Y^2)
\]

and

\[
|\epsilon(x)| = O(r^6 + \text{tr}(X^3) + \text{tr}(Y^6)).
\]

**Proof of the variance part** (1.14) of Theorem 1.5.

We invoke Lemma 3.1 and separate the singular contribution to write

\[
\text{Var}(\mathcal{L}_{n:s}) = \frac{E_n}{2} \int_{B(s) \times B(s)} \left( K_2(x - y) - \frac{1}{4} \right) dx dy
\]

\[
= \frac{E_n}{2} \int_{B_{\text{sing}} \cap (B(s) \times B(s))} \left( K_2(x - y) - \frac{1}{4} \right) dx dy
\]

\[
+ \frac{E_n}{2} \int_{(B(s) \times B(s)) \setminus B_{\text{sing}}} \left( K_2(x - y) - \frac{1}{4} \right) dx dy.
\]

Now we bound the contribution of the singular set (former integral on the r.h.s. of (3.29)) by

\[
\int_{B_{\text{sing}} \cap (B(s) \times B(s))} \left( K_2(x - y) - \frac{1}{4} \right) dx dy
\]

\[
= \int_{B_{\text{sing}} \cap (B(s) \times B(s))} K_2(x - y) dx dy + O(\text{meas}(B_{\text{sing}} \cap (B(s) \times B(s))))
\]

\[
\leq \int_{B_{\text{sing}}} K_2(x - y) dx dy + O(\text{meas}(B_{\text{sing}}))
\]

\[
\ll F^4 \cdot \mathcal{R}_n(6; s) \cdot \frac{1}{F^3 \sqrt{n}} + \mathcal{R}_n(6; 2s) \ll \mathcal{R}_n(6; 2s),
\]

where we employed (3.7) and (3.8) of Lemma 3.3, Lemma 3.6, and (3.6).
On the nonsingular range \((B(s) \times B(s)) \setminus B_{\text{sing}}\) (the latter integral on the r.h.s. of (3.29)) we have \(|r_n(x - y)| < \frac{3}{4}\), hence we are eligible to invoke Proposition 3.7 to write

\[
\int_{(B(s) \times B(s)) \setminus B_{\text{sing}}} \left( K_2(x - y) - \frac{1}{4} \right) dxdy
= \int_{B(s) \times B(s)} \left( L_2(x - y) + \epsilon(x - y) \right) dxdy
= \int_{B(s) \times B(s)} \left( L_2(x - y) + \epsilon(x - y) \right) dxdy + O(\text{meas}(B_{\text{sing}}))
= \int_{B(s) \times B(s)} \left( L_2(x - y) + \epsilon(x - y) \right) dxdy + O(\mathcal{R}_n(6; 2s)),
\]

by (3.24), and (3.7). Consolidating the contributions (3.30) and (3.31) of the singular and nonsingular ranges respectively to the integral in (3.29) we obtain

\[
\int_{B(s) \times B(s)} \left( K_2(x - y) - \frac{1}{4} \right) dxdy
= \int_{B(s) \times B(s)} \left( L_2(x - y) + \epsilon(x - y) \right) dxdy + O(\mathcal{R}_n(6; 2s)).
\]

By the very definition (3.27) of \(L_2\) and (3.28), the above estimate (3.32) relates the nodal length variance to evaluating moments of the encountered expressions along shrinking balls. The latter is precisely the statement of Lemma 3.4, under the hypotheses \(A(n; 2, n^{1/2 - \delta/2})\) and \(A(n; 6, n^{1/2 - \delta})\) of Theorem 1.5, that so far haven’t been exploited. Lemma 3.4 is instrumental for the precise asymptotic evaluation of the integral

\[
\int_{B(s) \times B(s)} L_2(x - y) dxdy
\]

and bounding the contribution

\[
\int_{B(s) \times B(s)} \epsilon(x - y) dxdy
\]

and \(\mathcal{R}_n(6; 2s)\) of the error terms in the following way. First,

\[
\int_{B(s) \times B(s)} \epsilon(x - y) dxdy \ll \int_{B(s) \times B(s)} (r^6 + \text{tr}(X^3) + \text{tr}(Y^6)) dxdy \ll s^4 \frac{1}{N_n^{5/2}}
\]

and

\[
\mathcal{R}_n(6; 2s) \ll s^4 \cdot \frac{1}{N_n^{5/2}}
\]
by (3.12) (applied both on $s$ and $2s$), (3.22) and (3.23). Next,

$$
\int_{B(s) \times B(s)} L_2(x - y) \ dx \ dy
$$

\begin{align*}
&= \frac{1}{8} \cdot (\pi s^2)^2 \cdot \left( \frac{1}{N_n} + \left( - \frac{2}{N_n} - \frac{2}{N_n^2} \right) \right) \\
&\quad + \frac{1}{4} \cdot \left( \frac{4}{N_n} - \frac{4}{N_n^2} \right) + \frac{3}{4} \cdot \frac{3}{N_n^2} + \frac{1}{8} \cdot \frac{4}{N_n^2} - \frac{1}{16} \cdot \frac{8}{N_n^2} \\
&\quad + \frac{1}{128} \cdot \frac{2(11 + \hat{c}_n(4)^2)}{N_n^2} + \frac{1}{256} \cdot \frac{4(7 + \hat{c}_n(4)^2)}{N_n^2} + \frac{1}{16} \cdot \frac{8}{N_n^2} \\
&\quad - \frac{1}{2} \cdot \frac{2}{N_n^2} + \frac{1}{8} \cdot \frac{8}{N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \\
&= (\pi s^2)^2 \cdot \left( \frac{1 + \hat{c}_n(4)^2}{256 \cdot N_n^2} + O\left( \frac{1}{N_n^{5/2}} \right) \right)
\end{align*}

(3.35)

by (3.10), (3.13), (3.14), (3.11), (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21), with the $\frac{1}{N_n}$ term vanishing. Substituting (3.35), (3.33) and (3.34) into (3.32), and then finally into the first equality of (3.29) yields the variance statement (1.14) of Theorem 1.5.

\[\Box\]

### 3.5 Proof of the full correlation part (1.15) of Theorem 1.5.

**Proof.** Let $\epsilon > 0$ be given; we are going to show that for every $s > 0$ we have the precise identity

$$(3.36)\quad \text{Cov}(\mathcal{L}_{n;s}, \mathcal{L}_n) = (\pi s^2) \cdot \text{Var}(\mathcal{L}_n).$$

Once (3.36) has been established, (1.15) follows at once for $n$ satisfying (1.14) (valid for a generic sequence $\{n\} \subseteq S$) and (1.11), uniformly for $s > n^{-1/2+\epsilon}$.

To show (3.36) we recall that $K_2(x) = K_{2;n}(x)$ as in (3.5) is the (normalized) 2-point correlation function, and that we have that the total nodal length variance $\text{Var}(\mathcal{L}_n)$ is given by [RW2008, KKW13]

$$(3.37)\quad \text{Var}(\mathcal{L}_n) = \frac{E_n}{2} \int_{T^2} \left( K_2(x) - \frac{1}{4} \right) dx$$

(cf. (3.4)). For the covariance we have the analogous formula

$$(3.38)\quad \text{Cov}(\mathcal{L}_{n;s}, \mathcal{L}_n) = \frac{E_n}{2} \int_{B(s) \times T^2} \left( K_2(x - y) - \frac{1}{4} \right) dx.$$
Since for every $x$ fixed, wherever $y$ varies along the torus so does $x - y$, (3.38) reads
\[
\text{Cov}(\mathcal{L}_{n,s}, \mathcal{L}_n) = \text{Vol}(B(s)) \cdot \frac{E_n}{2} \int_{\mathbb{T}^2} \left( K_2(x) - \frac{1}{4} \right) dx = (\pi s^2) \cdot \text{Var}(\mathcal{L}_n),
\]
by (3.37). This concludes the proof of (3.36), which, as mentioned above, implies (1.15).

4 Proof of Theorem 1.4: Bound for quasi-correlations

Our first goal is to state a quantitative version of Theorem 1.4 (in terms of the exceptional number of $n \in S$ not obeying the properties claimed by Theorem 1.4), also controlling the possible weak-$\ast$ partial limits of the respective $\{\tau_n\}_{n \in S}$, namely measures $\nu_s$ introduced immediately below.

Definition 4.1. For $s \in [0, \pi/4]$ let the (symmetric) probability measure $\nu_s$ be given by
\[
\int_{S^1} f \ d\nu_s := \frac{1}{8s} \sum_{k=0}^{3} \int_{s+k\pi/2}^{s+k\pi/2} f(e^{i\theta}) d\theta \quad (f \in C(S^1)).
\]

We will adopt the following conventions:

Notation 4.2. Throughout this section we will use the notation $[N] = \{1, \ldots, N\}$ for any natural number $N$ and write $n \asymp x$ to mean $n \in [x, 2x]$. The shorthand $\log_2 n := \log \log n$ will be in use and, as usual, $\Omega(n) = \sum_{p \mid n} e$ denotes the number of prime divisors of $n$ counted with multiplicity. We will say the numbers $\theta_1, \ldots, \theta_r \in \mathbb{R}$ are $\gamma$-separated when $|\theta_j - \theta_i| \geq \gamma$ for all $i \neq j$.

Theorem 4.3 is the announced quantitative version of Theorem 1.4.

Theorem 4.3. Given any $0 < \delta \leq 1$, $l \geq 2$, the following two properties hold.
(a) The exceptional set $\mathcal{R}_N(l; \delta) := \{ N \leq n \leq 2N : \mathcal{E}_n(l; n^{(1-\delta)/2}) \neq \emptyset \}$ has size at most
\[
|\mathcal{R}_N(l; \delta)| \ll \kappa^l (2L)! \ N^{1-\rho_0(\delta,l)} (\log N)^{L+1},
\]
where $\kappa > 0$ is an absolute constant, $L := 2^l$ and $\rho_0(\delta,l) = \delta/(2 \cdot 4^L(l + 1))$.

(b) For any $s \in [0, \pi/4]$ there exists a sequence of natural numbers $\{n_k\}_{k \geq 1} \subseteq S$ so that
\[
\mathcal{E}_n(l; n_k^{(1-\delta)/2}) = \emptyset
\]
and $\tau_{n_k} \Rightarrow \nu_s$.  

Proof of Theorem 1.4 assuming Theorem 4.3. Let \( \delta \in (0, 1/2) \) and \( l \geq 2 \) be given and define

\[
S'(l, \delta) = \{ n \in S : \mathbb{C}_n(l; n^{1/2-\delta}) = \emptyset \}.
\]

Part (3) of Theorem 1.4 is true by definition while part (1) follows immediately from the power saving in (4.1). Part (2) is proven as in [KKW13, Section 7.2]: Let \( s \in [0, \pi/4] \) be arbitrary and consider any sequence \((n_k)_{k \geq 1} \subseteq S'(l, \delta)\) for which \( \tau_{n_k} \Rightarrow \nu_s \). Seeing how \( \hat{\tau}_{n_k}(4) \rightarrow \hat{\nu}_s(4) \), the result follows from the continuity of the map \( s \mapsto \hat{\nu}_s(4) \) (with boundary values \( \hat{\nu}_0(4) = 1, \hat{\nu}_{\pi/4}(4) = 0 \)), or, alternatively, the explicit evaluation of

\[
\hat{\nu}_s(4) = \frac{\sin(4s)}{4s}.
\]

\( \Box \)

The rest of Section 4 is dedicated to proving Theorem 4.3.

4.1 Preliminary results. We begin with three simple estimates.

Lemma 4.4. Let \( R \geq 1 \) and suppose that the angles \( 0 < \theta_1 < \theta_2 \leq 2\pi \) are \( 1/R \)-separated. Then the lattice points contained in the sector

\[
\Gamma(R; \theta_1, \theta_2) := \{ re^{i\theta} : r \leq R, \ \theta \in (\theta_1, \theta_2) \}
\]

can be covered by two intervals, each of size at most

\[
\frac{5u}{\|A\|_2}.
\]

Proof. The number of Gaussian integers inside \( \Gamma(R; \theta_1, \theta_2) \) is bounded by the area of the covering region

\[
\tilde{\Gamma} := \{ z_1 + z_2 \in \mathbb{C} : z_1 \in \Gamma(R; \theta_1, \theta_2), \ |z_2| \leq \sqrt{2} \}.
\]

The area of \( \tilde{\Gamma} \) is at most \( c_1((\theta_2 - \theta_1)R^2 + R) \) for some absolute constant \( c_1 > 0 \) and hence the result follows. \( \Box \)

Lemma 4.5. For any \( u > 0 \) and \( A = (A_1, A_2) \in \mathbb{R}^2 \setminus \{(0, 0)\} \), the set

\[
\mathcal{A} = \{ \theta \in [0, 2\pi] : |A_1 \cos \theta + A_2 \sin \theta| < u \}
\]

can be covered by two intervals, each of size at most \( 5u/\|A\|_2 \).

Proof. When \( u/\|A\|_2 \geq 1/\sqrt{2} \), we can use the fact that \( \mathcal{A} \) is covered by the interval \([0, 2\pi]\) which has length \( 2\pi \leq 2\sqrt{2} \cdot \pi u/\|A\|_2 < 10u/\|A\|_2 \).
Otherwise we make use of the identity

\[ |r \cos(\theta - \alpha)| = |A_1 \cos \theta + A_2 \sin \theta| < u, \]

where

\[ r = (A_1^2 + A_2^2)^{1/2} = \|A\|_2, \quad \tan \alpha = \frac{A_2}{A_1} \quad \text{(taking } \alpha = \pi/2 \text{ when } A_1 = 0), \]

yielding \( \cos(\theta - \alpha)| < u/\|A\|_2 \). The result now follows from a straightforward application of the mean value theorem. \( \square \)

**Lemma 4.6.** Given any \( \alpha \geq 0 \) and \( x \geq 1 \) we have the estimate

\[
\sum_{a \in \mathbb{Z}[i] \setminus 0, |a| \leq x} \frac{1}{|a|^{2-\alpha}} \leq \kappa_2 x^\alpha \log(2 + x),
\]

for some absolute constant \( \kappa_2 > 0 \).

**Proof.** The bound is immediate for \( 1 \leq x < 2 \) (so long as \( \kappa_2 \geq 4 \)) so we may assume that \( x \geq 2 \). Covering \([1, x]\) with dyadic intervals \([D, 2D] = [2^i, 2^{i+1}]\) \((i = 0, \ldots, \lfloor \log x/\log 2 \rfloor)\), we get

\[
\sum_{a \in \mathbb{Z}[i] \setminus 0, |a| \leq x} \frac{1}{|a|^{2-\alpha}} \leq \sum_{1 \leq D \leq x} \sum_{a \in \mathbb{Z}[i], |a| \geq D} \frac{1}{D^{2-\alpha}} \ll \sum_{1 \leq D \leq x} \text{dyadic } D^\alpha \ll x^\alpha \log x. \]

To prove Theorem 4.3(b) we will require upper and lower bounds for the number of Gaussian primes in narrow sectors. These estimates, due to Kubilius, can be found in \([E-H, \text{Lemma 1}]\).

**Theorem 4.7** ([E-H], Kubilius). There exist absolute constants \( b, c, C > 0 \) satisfying the following property. Given any sufficiently large integer \( R > R_0 \) and any choice of angles

\[ 0 \leq \alpha < \beta \leq \pi/2, \quad \beta - \alpha \geq \exp(-b \sqrt{\log R}), \]

the number of Gaussian primes in the sector \( \Gamma(R, \alpha, \beta) \) is bounded from below and above by

\[
c \frac{R^2(\beta - \alpha)}{\log R} \leq \sum_{\pi \in \Gamma(R, \alpha, \beta)} 1 \leq C \frac{R^2(\beta - \alpha)}{\log R}.
\]

**Remark.** We wish to emphasize that the remainder term in \([E-H, \text{eq. 23}]\) is uniform in \( \beta - \alpha \), whence the estimates stated just above follow readily.
As an immediate consequence of Theorem 4.7 we obtain a lower bound for the number of \( k \)-almost Gaussian primes in \( \Gamma(R, 0, \beta) \) (provided \( k \) is not too large and \( \beta \) not too small).

**Lemma 4.8.** Let \( R > R_0 \) be a sufficiently large number and suppose

\[
1 \leq k \leq (\log R)^{1/4}.
\]

Then for any choice of angles \( 0 \leq \alpha_i < \beta_i \leq \pi/2 \) \((i = 1, \ldots, k)\) satisfying

\[
\frac{1}{(\log R)^2} \leq \beta_i - \alpha_i \leq \pi/2,
\]

one has the lower bound

\[
(4.2) \quad \sum_{\pi_i \in \Gamma(R, \alpha_i, \beta_i), i \leq k \atop |\pi_1 \cdots \pi_k| \leq R} 1 \geq \left( \frac{c}{4} \right)^k \frac{R^2}{(\log R)^{3\epsilon}},
\]

where \( c > 0 \) is the constant from Theorem 4.7.

**Proof.** Writing \( \Phi(R) = \exp(\sqrt{\log R}) \) and applying Theorem 4.7 one finds that

\[
\sum_{\pi_i \in \Gamma(R, \alpha_i, \beta_i), i \leq k \atop |\pi_1 \cdots \pi_k| \leq R} 1 \geq \sum_{\pi_i \in \Gamma(R, \alpha_i, \beta_i), i \leq k-1 \atop \Phi(R) \leq |\pi_1|, \ldots, |\pi_{k-1}| \leq \Phi(R)^2} \sum_{\pi_i \in \Gamma(R, \alpha_i, \beta_i)} 1 \geq \frac{cR^2}{(\log R)^2 \Phi(R)^{4(k-1)}} \sum_{\pi_i \in \Gamma(R, \alpha_i, \beta_i), i \leq k-1 \atop \Phi(R) \leq |\pi_1|, \ldots, |\pi_{k-1}| \leq \Phi(R)^2} 1.
\]

In the second inequality we used the restriction \( k \leq (\log R)^{1/4} \) to ensure that \( R/|\pi_1 \cdots \pi_{k-1}| \geq R^{1-\omega(1)} \). Invoking Theorem 4.7 once again, we get

\[
\sum_{\pi_i \in \Gamma(R, \alpha_i, \beta_i), i \leq k-1 \atop \Phi(R) \leq |\pi_1|, \ldots, |\pi_{k-1}| \leq \Phi(R)^2} 1 \geq \left( \frac{c \Phi(R)^4}{2 \log(\Phi(R)^2)(\log R)^2} \right)^{k-1},
\]

which combined with the previous estimate yields (4.2). \( \square \)

### 4.2 An upper bound for \( R_N \)

Let \( N \) be a large natural number and suppose \( n \sim N \). In order to show that \( C_n(l; n^{(1-\delta)/2}) \) is generically empty we first recall that, typically, \( n \) has \( O(\log_2 n) \) prime divisors (see [Te, Section III.3, Theorem 4]). We will need the following quantitative result of Erdős–Sárközy to bound the number of \( n \sim N \) for which \( \Omega(n) \) is unusually large.
Theorem 4.9 ([E-S, Corollary 1]). For all $N \geq 3$, $K \geq 1$ we have the estimate

$$\left| \{ n \leq N : \Omega(n) \geq K \} \right| \ll K^4 N \log N / 2^K.$$ 

4.2.1 Using the structure of $E_n$. We now turn to the study of $C_n(l; n^{(1-\delta)/2})$ and decompose

$$n = \left| \prod_{j \leq k} \pi_j \right|^2$$

into a product of Gaussian primes. Rotating each prime $\pi_j$ over an integer multiple of $\pi/2$ we may assume that $\pi_j$ lies in the first quadrant and makes an angle $0 \leq \theta_j < \pi / 2$ with the x-axis. Each lattice point $\xi \in E_n$ takes the form

$$\xi = (i)^{n^{1/2}} \prod_{j \leq k} e^{i\epsilon_j \theta_j}, \quad \epsilon_j \in \{-1, 1\}, \ j = 1, \ldots, k, \ \nu = 1, \ldots, 4.$$

As a result, each tuple $(\xi_1, \ldots, \xi_l) \in E_n^l$ corresponds to a choice of matrix $\epsilon = (\epsilon_j^{(r)})_{1 \leq r \leq l, 1 \leq j \leq k}$ with entries in $\{-1, 1\}$, and a choice of vector

$$\nu = (\nu_1, \ldots, \nu_l) \in [4]^l.$$

The representation in (4.3) is not unique: different vectors $(\epsilon_j)_{j \leq k}$ can give rise to the same lattice point $\xi \in E_n$. We refer the reader to [Cil93] for a precise description of $E_n$ and note that Theorem 4.3 only delivers an upper bound for $R_N$ which is why we allow for overcounting.

Rewriting the quasi-correlation condition. Let $y \geq 1$. For a given tuple $(\xi_1, \ldots, \xi_l) \in E_n^l$ with associated matrix $\epsilon = (\epsilon_j^{(r)})_{1 \leq r \leq l, 1 \leq j \leq k}$ and vector $\nu \in [4]^l$, we wish to express the inequalities

$$0 < \left\| \sum_{i=1}^l \xi_i \right\| / \sqrt{n} < y^{-\delta}$$

in terms of $\epsilon$ and $\nu$. Introducing vectors $\eta^+, \eta^- \in (-1, 1)^l$ and $\nu^+, \nu^- \in [4]^l$ we consider the more general conditions

$$0 < \left| \sum_{r \leq l} \eta^+_r \cos \left( \nu^+_r \pi / 2 + \sum_{j \leq k} \epsilon_j^{(r)} \theta_j \right) \right| < y^{-\delta}, \quad (+)$$

$$0 < \left| \sum_{r \leq l} \eta^-_r \sin \left( \nu^-_r \pi / 2 + \sum_{j \leq k} \epsilon_j^{(r)} \theta_j \right) \right| < y^{-\delta}, \quad (-)$$

This slight generalization will be useful in the proof of Proposition 4.10 below.
Recalling the notation $R_N(l; \delta) := \{ n \asymp N : C_n(l; n^{1-\delta/2}) \neq \emptyset \}$, set $Y = N^{1/2}$ and let $K = K(N)$ be a large integer to be chosen later. It follows from Theorem 4.9 and the preceding discussion that

$$|R_N(l; \delta)| \ll \sum_{\eta^+ \in \{-1, 1\}^l} \sum_{k < K} \sum_{\epsilon, \eta^+ , \nu^+} 1 + O\left( K^4 N \log N \right).$$

The superscript $+$ refers to the first condition in (4.4) (choosing $\eta = Y = N^{1/2}$) and the $(\pi_j)_{j \leq k}$ in the innermost sum range over Gaussian primes in the first quadrant.

### 4.2.2 Quasi-correlations of Gaussian integers.

To prove Theorem 4.3(a) we will rewrite the sum on the r.h.s. of 4.5 in terms of the quantities

$$S^\pm_\delta(y; k, \epsilon, \eta^+, \nu^+):= \sum_{a_1, \ldots, a_k \in \mathbb{Z}^2 \setminus \{(0, 0)\}} 1,$$

where, as before, the superscripts $+$ and $-$ refer to the conditions in (4.4) and the angle $\theta_j$ belongs to the Gaussian integer $a_j$ (which is not assumed to be prime). The advantage of dealing with the more general sums $S^\pm_\delta(y; k, \epsilon, \eta^+, \nu^+)$ is that we need only consider values $k \leq 2^l$ (see section 4.2.3 below for details). In the following proposition, which lies at the heart of our argument, we give an estimate for $S^\pm_\delta$ with a power saving in $y$.

**Proposition 4.10.** For any choice of parameters $y \geq 1, l \geq 2, k \leq 2^l$ and $\delta \in (0, 1)$ we have the estimates

$$\max_{\epsilon, \eta^+, \nu^+} |S^\pm_\delta(y; k, \epsilon, \eta^+, \nu^+)| \leq \kappa^k_3 (2k)! y^{2-\delta/4k} (\log(2 + 2y))^{k-1}$$

where $\kappa_3 > 0$ is an absolute constant.

**Proof.** Let $\kappa_1, \kappa_2 \geq 3$ be the constants appearing in Lemmas 4.4 and 4.6 and put $\kappa_3 = 4 \kappa_1 \kappa_2$. We will prove the estimate (4.6) by induction on $k$, the $k = 1$ case being a consequence of the lemmas in section 4.1. To see this, we note that $|S^\pm_\delta(y; 1, \epsilon, \eta^+, \nu^+)|$ is bounded by the number of Gaussian integers $|a| \leq y$ with angle $\theta$ satisfying $|r_1 \cos(\theta) + r_2 \sin(\theta)| < y^{-\delta} \leq y^{-\delta/2}$ for some $(r_1, r_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. Combining Lemmas 4.4 and 4.5, the estimate (4.6) follows readily.

Assuming (4.6) holds for all values up to $k - 1$, let us verify the bound for $S^\pm_\delta(y; k, \epsilon, \eta^+, \nu^+)$ (the treatment of $S^-_\delta$ is almost identical). A quick inspection
of $S^\epsilon_\delta$ reveals that at least one of the summation variables, say $a_k$, must be large. More precisely, we may assume that

$$|a_k| \geq y^{1/k} \geq y^{\delta/k}. \quad (4.7)$$

Applying the addition formula for cosine we may write, for each $i \leq k$,

$$\sum_{r \leq l} \eta_r^+ \cos \left( v_r^+ \pi/2 + \sum_{j \leq k} \epsilon_j^{(r)} \theta_j \right) = \left[ \sum_{r \leq l} \eta_r^+ \cos \left( v_r^+ \pi/2 + \sum_{j \neq i} \epsilon_j^{(r)} \theta_j \right) \right] \cos(\theta_i)$$

$$- \left[ \sum_{r \leq l} \eta_r^+ \epsilon_k^{(r)} \sin \left( v_r^+ \pi/2 + \sum_{j \leq k, j \neq i} \epsilon_j^{(r)} \theta_j \right) \right] \sin(\theta_i)$$

$$=: \hat{A}_i \cos(\theta_i) + \hat{B}_i \sin(\theta_i). \quad (4.8)$$

Before proceeding with the argument we record the following observation.

**Claim.** Either of the conditions $0 < |\hat{A}_i| \leq y^{-\delta}$ or $0 < |\hat{B}_i| \leq y^{-\delta}$ imply $|\prod_{j \neq i} a_j| \geq y^{\delta}$.

To prove the claim, we write $a_j = b_j + ic_j$ so that $|\sin \theta_j| = |c_j|/|a_j|$ and $|\cos \theta_j| = |b_j|/|a_j|$. Repeatedly applying the addition formulas, one finds that both $\hat{A}_i$ and $\hat{B}_i$ may be expressed in the form $d/|\prod_{j \neq i} a_j|$ for some $d \in \mathbb{Z}$ and hence the claim follows. Returning to the proof of (4.6), we set

$$\beta := \frac{k - 1}{k} \geq \frac{1}{2}$$

and consider two cases.

**Case I.** Either $|\hat{A}_k| > y^{-\delta \beta}$ or $|\hat{B}_k| > y^{-\delta \beta}$.

Let $U^+_\delta = U^+_\delta(y; k, \epsilon, \eta^+, \nu^k)$ denote the restriction of $S^+_\delta$ in which the summation variables $a_1, \ldots, a_k$ satisfy (4.7) as well as the hypothesis in Case I. Given any pair $(\hat{A}_k, \hat{B}_k)$ we can use (4.8) to rewrite the condition (4.4)(+) as

$$0 < |\hat{A}_k \cos(\theta_k) + \hat{B}_k \sin(\theta_k)| < y^{-\delta}.$$

It follows from Lemma 4.5 that $\theta_k$ must live in one of two intervals $I_k, I'_k$, each having length at most $5y^{-\delta/k}$. Recalling that $|a_k| \geq y^{\delta/k}$ and, if necessary, expanding the intervals $I_k, I'_k$ to be of length exactly $5y^{-\delta/k}$, we may now apply
Lemmas 4.4 and 4.6 to get

\[
U^\dagger(y; k, \epsilon, \eta^+, \nu^+) \leq \sum_{a_1, \ldots, a_{k-1} \in \mathbb{Z}[i]} \frac{1}{|\prod_j a_j|} \left( \frac{2y}{|\prod_j a_j|} \right)^2 (5y^{-\delta/k})
\]

(4.9)

\[
\leq 2\kappa_1 \sum_{a_1, \ldots, a_{k-1} \in \mathbb{Z}[i]} \left( \frac{2y}{|\prod_j a_j|} \right)^2 (5y^{-\delta/k})
\]

\[
\leq 40\kappa_1(k_2)^{k-1} y^{2-\delta/k} (\log(2 + 2y))^{k-1}
\]

\[
\leq \frac{1}{2}(\kappa_3)^y y^{2-\delta/4} (\log(2 + 2y))^{k-1}.
\]

In the last step we used that \( k \geq 2 \) to get the inequality

\[
40\kappa_1(k_2)^{k-1} \leq 40\kappa_1(k_2)^{k} \leq \frac{40}{9} \cdot \frac{4 \kappa_3}{4} \kappa_2 \leq \frac{\kappa_3}{2}.
\]

**Case II.** Either \( 0 < |\hat{A}_k| \leq y^{-\beta \delta} \) or \( 0 < |\hat{B}_k| \leq y^{-\beta \delta} \).

Let \( V^\dagger = V^\dagger(y; k, \epsilon, \eta^+, \nu^+) \) denote the restriction of \( S^\dagger \) according to (4.7) and the hypothesis in Case II. By the claim we have the lower bound

\[
\prod_{j=k-1} a_j \geq y^{\beta \delta}
\]

for any tuple \( a_1, \ldots, a_k \) appearing in \( V^\dagger \) and hence we may assume that one of the \( a_i \) is large, say

(4.10)

\[
|a_1| \geq y^{\delta/k}.
\]

We will consider subcases II(a) and II(b) and write \( V^\dagger(a) \) and \( V^\dagger(b) \) for the corresponding restrictions of \( V^\dagger \).

**Case II(a).** \( |a_k| \leq 2y^{1/2} \).

In this situation we apply the induction hypothesis to get

\[
V^\dagger(a) \leq \sum_{|a_k| \leq 2y^{1/2}} \sum_{a_1, \ldots, a_{k-1} \in \mathbb{Z}[i]} \frac{1}{|\prod_j a_j|} \left( \frac{2y}{|\prod_j a_j|} \right)^2 (5y^{-\delta/k})
\]

\[
\leq 2(\kappa_3)^{k-1} (2k - 2)! (\log(2 + 2y))^{k-2} \sum_{|a_k| \leq 2y^{1/2}} \left( \frac{y}{|a_k|} \right)^{2-\beta \delta/4^{k-1}}.
\]

Applying the straightforward inequality \( 4\kappa_2 \leq \kappa_3/2 \) together with Lemma 4.6 we get

\[
V^\dagger(a) \leq 2(\kappa_3)^{k-1} (2k - 2)! (\log(2 + 2y))^{k-2}
\]

\[
\times \kappa_2 y^{2-\beta \delta/4^{k-1}} \cdot (2y^{1/2})^{\beta \delta/4^{k-1}} \log(2 + 2y)
\]

(4.11)

\[
\leq \frac{1}{2}(\kappa_3)^y (2k - 2)! y^{2-\delta/4} (\log(2 + 2y))^{k-1}.
\]
**Case II(b).** $|a_1| \leq 2y^{1/2}$.

In this scenario, interchanging the roles of $a_1$ and $a_k$, we satisfy the criteria of either Case I or Case II(a) (note that (4.10) is necessary to obtain the estimates carried out in Case I). It follows that $V_\delta^+(b) \leq V_\delta^+(a) + U_\delta^+$.

Collecting the estimates from the two subcases we find that

$$V_\delta^+ \leq (k - 1)(V_\delta^+(a) + V_\delta^+(b)) \leq (k - 1)(2V_\delta^+(a) + U_\delta^+),$$

(4.12)

where the extra factor $k - 1$ compensates for the loss we incurred by fixing the index in (4.10).

To conclude the proof of Proposition 4.10 we combine (4.9), (4.11) and (4.12) to get

$$S_\delta^+ \leq k(U_\delta^+ + V_\delta^+) \leq k^2U_\delta^+ + 2k(k - 1)V_\delta^+(a) \leq (\kappa_3)^k(2k)!y^{2-\delta/4k}(\log(2 + 2y))^{k-1}.$$  

As before, the factor $k$ in the first inequality compensates for fixing the index in (4.7). \qed

### 4.2.3 Concluding the proof of Theorem 4.3(a).

**Proof.** It remains to estimate the r.h.s. of (4.5). Let $\eta^+, \nu^+, \epsilon$ be fixed and consider any ascending $k$-tuple of Gaussian primes $|\pi_1| \leq \cdots \leq |\pi_k|$ which live in the first quadrant and satisfy the condition (4.4)(+). For each $j \leq k$ the vector $(\hat{\epsilon}_j^{(r)})_{r \leq l}$ corresponds to one of the $2^l$ elements of $\{-1, 1\}^l$. We will regroup all of the $j \leq k$ which give rise to the same element of $\{-1, 1\}^l$. To this end, let $\tau_1, \ldots, \tau_L$ be a list of all the elements in $\{-1, 1\}^l$ (so that $L = 2^l$) and define

$$a_i := \prod_{(\hat{\epsilon}_j^{(r)})_{r \leq l} = \tau_i} \pi_j, \quad i = 1, \ldots, L,$$

with the convention that the empty product is 1. In this manner we obtain a map

$$\phi^+ : \{|\pi_1| \leq \cdots \leq |\pi_k|\} \mapsto \{a_1, \ldots, a_L\}.$$  

Some remarks are in order.

First, the map $\phi^+$ depends on $\eta^+, \nu^+$ and $\epsilon$. Second, $\phi^+$ is injective since we only consider $k$-tuples of primes in ascending order. Third, any tuple of Gaussian integers $a_1, \ldots, a_L$ in the image of $\phi^+$ will also satisfy the condition (4.4)(+) for some matrix $\hat{\epsilon} = (\hat{\epsilon}_j^{(r)})_{1 \leq r \leq l, 1 \leq j \leq L}$ and the same choice of $\eta^+, \nu^+$. 


In light of (4.5) and Proposition 4.10 we now find that for $K \ll \log N$,
\[
|\mathcal{R}_N(l; \delta)| \ll \sum_{\eta \in \{-1, 1\}} \sum_{k \ll K} 2^k \max_{\epsilon = \epsilon^i} |S^+_{\delta}(Y; L, \epsilon, \eta^*, \nu^*)| + O\left(\frac{N(\log N)^5}{2^{k}}\right)
\]
\[
\ll (\kappa_3)^L 8^L (2L)! N^{1-\rho(l, \delta)} (\log(2 + 2\sqrt{N}))^{L-1} 2^{KL} + \frac{N(\log N)^5}{2^{k}}
\]
with $\rho(\delta, l) = \delta/(2 \cdot 4^L)$. Noting that $\log(2 + 2\sqrt{N}) \leq \log N$ we now set $\kappa := 16\kappa_3$ and choose $2^K \asymp N^{\rho(l, \delta)}$ (so that $2^{KL} \leq 2^{L(\rho(l, \delta))}$) to get
\[
|\mathcal{R}_N(l; \delta)| \ll (16\kappa_3)^L (2L)! N^{1-\rho(l, \delta)} (\log N)^{L-1} + N^{1-\rho(l, \delta)} (\log N)^5
\]
\[
\ll \kappa^L (2L)! N^{1-\rho(l, \delta)} (\log N)^{L+1},
\]
as claimed. \qed

4.3 A sequence $(n_k)_{k \geq 1}$ for which $\tau_{n_k} \Rightarrow v_s$ Let $s \in [0, \pi/4]$ and take $N$ to be a large natural number. We choose $k$ to be the integer for which $2^k \asymp \log N$ and select from $[0, s]$ the disjoint subintervals
\[
[\alpha_j, \beta_j] := \left[\frac{s2^j}{2^k}, \frac{s2^j}{2^k} \left(1 + \frac{1}{k^2}\right)\right], \quad j = 0, \ldots, k - 1.
\]

**Lemma 4.11.** For any tuple $(\theta_0, \ldots, \theta_{k-1}) \in [\alpha_0, \beta_0] \times \cdots \times [\alpha_{k-1}, \beta_{k-1}]$ the sumset
\[
\mathcal{B}(\theta_0, \ldots, \theta_{k-1}) := \left\{ \sum_{j=0}^{k-1} \epsilon_j \theta_j : \epsilon_j \in \{-1, 1\} \text{ for all } j \right\}
\]
forms a collection of $(s/2^k)$-separated numbers. Each of the intervals
\[
\left[\frac{s(2j + 1)(1 - \frac{1}{k})}{2^k}, \frac{s(2j + 1)(1 + \frac{1}{k})}{2^k}\right], \quad (-2^{k-1} \leq j \leq 2^{k-1} - 1)
\]
contains exactly one member of $\mathcal{B}(\theta_0, \ldots, \theta_{k-1})$.

**Proof.** For each $0 \leq i \leq k - 1$ we may write
\[
\theta_i = s2^i/2^k + \gamma_i
\]
with $|\gamma_i| \leq s2^i/(k^2 2^k)$. Since the sumset $\mathcal{B}(1, 2, \ldots, 2^{k-1})$ consists of all odd integers in $[-2^k, 2^k]$, the result follows. \qed
We now introduce the set of $k$-almost Gaussian primes
\[ G_k(N; \alpha, \beta) := \left\{ a = \prod_{j \leq k} \pi_j : |a| \leq N^{1/2}, \ \pi_j \in \Gamma(N^{1/2}, \alpha_j, \beta_j) \text{ for all } j \right\}, \]
where the angles $\alpha_i, \beta_i$ are chosen as in (4.13) (and hence depend on $N$); $G_k$ gives rise to the relatively large set of rational integers
\[ A_k(N; \alpha, \beta) := \left\{ n \leq N : n = |a|^2, a \in G_k(N; \alpha, \beta) \right\}, \]
from which we will extract the sequence $(n_k)$. Indeed, since $k = O(\log_2 N)$ we may apply Lemma 4.8 to get
\[ |A_k(N; \alpha, \beta)| = |G_k(N; \alpha, \beta)| \geq \left( \frac{c}{4} \right)^{k} \frac{N}{(\log N)^3} \gg \frac{N}{\Phi(N)} \]
(recall the notation $\Phi(N) = \exp(\sqrt{\log N})$). Comparing this lower bound to the estimate (4.1) for $|R_N(l; \delta)|$ we deduce the existence of an infinite sequence $(n_k)_{k \geq 1}$ for which
\[ C_{n_k}(l; n^{(1-\delta)/2}) = \emptyset. \]

Proof of Theorem 4.3(b). Let $f : S^1 \to \mathbb{R}$ be an arbitrary continuous test function and consider any $n_k = |\prod_{i \leq r} \pi_i|^2$ belonging to the sequence introduced just above. As before we let $\theta_i$ denote the angle between $\pi_i$ and the $x$-axis and observe that
\[ \mathcal{E}_{n_k} = \left\{ e^{ix} : x \in \left( j\pi/2 + \mathcal{B}(\theta_1, \ldots, \theta_r) \right) \right\}. \]
It follows from Lemma 4.11 that the sumset $\mathcal{B}(\theta_1, \ldots, \theta_r) = \{ x_1, \ldots, x_J \}$ is evenly distributed in $[-s, s]$: we have that $x_{j+1} - x_j = 2s/J + o_p(1/J)$ for all $j \leq J - 1$. Recalling Definition 4.1 we now get
\[ (4.14) \quad \int_{S^1} f \, d\tau_{n_k} = \frac{1}{4|\mathcal{B}(\theta_1, \ldots, \theta_r)|} \sum_{j=0}^{3} \sum_{x \in \mathcal{B}(\theta_1, \ldots, \theta_r)} f(e^{j(x+\pi/2)}). \]
The r.h.s. of (4.14) represents, up to a small error, an evenly spaced Riemann sum for the integral $\frac{1}{4s} \sum_{k=0}^{3} \int_{-s+k\pi/2}^{s+k\pi/2} f$. By construction, the size of $\mathcal{E}_{n_k}$ will grow together with $n_k$ and hence $\tau_{n_k} \Rightarrow \nu_s$. \hfill \Box

Appendix A Proof of Lemma 3.4: moments of $r$ and its derivatives along the shrinking balls $B(s)$

The goal of this section is to prove Lemma 3.4. First we need to formulate the following lemma whose purpose is evaluating some summations of oscillatory integrals; it will be proven after the proof of Lemma 3.4.
Lemma A.1. (1) For every $s = s(n) > 0$ and $K = K(n) > 0$ we have the estimate

$$\sum_{\lambda \neq \lambda' \in \mathcal{E}_n} \left\| \int_{B(s)} e((\lambda - \lambda', x)) \right\|^2 \ll s^4 \cdot \left( |\mathcal{C}_n(2; K)| + \frac{N^2_n}{(Ks)^3} \right).$$

(2) For every $s = s(n) > 0$ and $K = K(n) > 0$ we have the estimate

$$\sum_{\lambda + \lambda' + \lambda'' \neq 0} \left\| \int_{B(s)} e((\lambda + \lambda' + \lambda'', x)) \right\|^2 \ll s^4 \cdot \left( |\mathcal{C}_n(4; K)| + \frac{N^4_n}{(Ks)^3} \right).$$

(3) For every $s = s(n) > 0$ and $K = K(n) > 0$ we have the estimate

$$\sum_{\lambda_1 + \ldots + \lambda_6 \neq 0} \left\| \int_{B(s)} e((\lambda_1 + \ldots + \lambda_6, x)) \right\|^2 \ll s^4 \cdot \left( |\mathcal{C}_n(6; K)| + \frac{N^6_n}{(Ks)^3} \right).$$

Proof of Lemma 3.4 assuming Lemma A.1. First we prove (3.10); we then assume that $n \in S' \subseteq S$ satisfies the hypothesis

$$\mathcal{A}(n; 2, n^{1/2-\delta}),$$

and aim at proving (3.10) for an arbitrary ball of radius satisfying

$$s > n^{-1/2+\epsilon}.$$

Using the definition (1.7) of $r_n$ and separating the diagonal contribution we have

$$r_n(x)^2 = \frac{1}{N_n} + \frac{1}{N_n^2} \sum_{\lambda \neq \lambda' \in \mathcal{E}_n} e((\lambda - \lambda', x)),$$

hence

$$\int_{B(s) \times B(s)} r_n(x - y)^2 dxdy = (\pi s^2)^2 \cdot \frac{1}{N_n} + \frac{1}{N_n^2} \sum_{\lambda \neq \lambda' \in \mathcal{E}_n} \int_{B(s) \times B(s)} e((\lambda - \lambda', x - y)) dxdy$$

$$= (\pi s^2)^2 \cdot \frac{1}{N_n} + \frac{1}{N_n^2} \sum_{\lambda \neq \lambda' \in \mathcal{E}_n} \left\| \int_{B(s)} e((\lambda - \lambda', x)) dx \right\|^2$$

$$= (\pi s^2)^2 \cdot \frac{1}{N_n} + O\left( s^4 \cdot \left( |\mathcal{C}_n(2; K)| + \frac{N^2_n}{(Ks)^3} \right) \right),$$
by Lemma A.1; we still maintain the freedom of choosing the threshold $K = K(n)$, with the help of (A.5).

For the choice

$$K = n^{1/2 - \delta}$$

we have that the quasi-correlation set

(A.8) $\mathcal{C}_n(2; K) = \emptyset$

is empty by hypothesis (A.4) we made earlier in this proof, and

(A.9) $\frac{N_n^2}{(Ks)^3} = O\left(\frac{N_n^2}{n^{\epsilon - \delta}}\right) = O\left(\frac{1}{N_n^4}\right)$

is smaller than any power $A > 0$ of $N_n$, bearing in mind (3.9) and (1.9). Upon substituting the last couple of estimates, (A.8) and (A.9) into (A.7) we then obtain the asymptotics

$$\int_{B(s) \times B(s)} r_n(x - y) dx dy = (\pi s^2)^2 \cdot \frac{1}{N_n} \left(1 + O\left(\frac{1}{N_n^A}\right)\right)$$

holding for every $A > 0$, i.e., we obtain (3.10). Along the way we also proved that, under the assumptions of Lemma 3.4, the summation

(A.10) $\sum_{\lambda \neq \lambda' \in E_n} \left| \int_{B(s)} e\left(\langle \lambda + \lambda' \rangle, x \rangle\right) dx \right|^2 < s^4 \cdot \frac{1}{N_n^4}$

is smaller than any power of $N_n^A$.

We turn now to proving (3.11). First, under the hypothesis $\mathcal{A}(n; 2, n^{1/2 - \delta}/2)$, also

(A.11) $\mathcal{A}(n; 4, n^{1/2 - \delta})$

is satisfied, thanks to Lemma 2.1. Similarly to (A.6) and (A.7), and upon recalling the definition (1.20) of the diagonal, and that the length-4 spectral correlation set $S_n(4)$ consists of the diagonal elements only (1.21), for the 4th moment we have

$$r_n(x)^4 = \frac{[D_n(4)]}{N_n^4} + \frac{1}{N_n^4} \sum_{\lambda \neq \lambda' \in E_n} e\left(\langle \lambda + \lambda' \rangle, x \rangle\right),$$

and

$$\int_{B(s) \times B(s)} r_n(x - y)^4 dx dy = (\pi s^2)^2 \cdot \frac{[D_n(4)]}{N_n^4}$$

$$+ \frac{1}{N_n^4} \sum_{\lambda + \lambda' + \lambda'' + \lambda''' \neq 0} \left| \int_{B(s)} e\left(\langle \lambda + \lambda' \rangle, x \rangle\right) dx \right|^2.$$
Hence, by invoking Lemma A.1 once again, we obtain

\[(A.12) \quad \int_{B(s) \times B(s)} r_n(x - y)^4 \, dx \, dy = (\pi s^2)^2 \cdot \frac{|D_n(4)|}{N_n^4} \cdot O\left(s^4 \cdot \left( |\mathcal{C}_n(4; K)| + \frac{N_n^4}{(Ks)^3} \right) \right),\]

where we are still free to choose the value of the parameter \(K = K(n)\).

For the choice \(K = n^{1/2-\delta}\) as above, the length-4 correlation set \(\mathcal{C}_n(4; K)\) is empty by (A.11), and

\[
\frac{N_n^4}{(Ks)^3} = O\left(\frac{N_n^4}{n^{\epsilon-\delta}}\right) = O\left(\frac{1}{N_n^4}\right)
\]

for every \(A > 0\) by (1.9), and (A.12) now reads

\[
\int_{B(s) \times B(s)} r_n(x - y)^4 \, dx \, dy = (\pi s^2)^2 \cdot \frac{|D_n(4)|}{N_n^4} \left( 1 + O\left(\frac{1}{N_n^4}\right) \right),
\]

i.e., the first estimate of (3.11). The second estimate of (3.11) follows from the first one and

\[
|D_n(4)| = 3N_n^2 + O(N_n).
\]

On the way we also proved that, under the assumptions of Lemma 3.4, the summation

\[(A.13) \quad \sum_{\lambda + \lambda' + \lambda'' + \lambda''' \neq 0} \left| \int_{B(s)} e((\lambda + \lambda' + \lambda'' + \lambda''', x)) \, dx \right|^2 \ll s^4 \cdot \frac{1}{N_n^4}\]

is smaller than any power of \(N_n^4\).

Now we turn to proving (3.12) under

\[(A.14) \quad A(n; 6, n^{1/2-\delta/2}).\]

As above, we have

\[(B.15) \quad \int_{B(s) \times B(s)} r_n(x - y)^6 \, dx \, dy \]

\[
= (\pi s^2)^2 \cdot \frac{|S_n(6)|}{|N_n|^6} + \frac{1}{N_n^4} \sum_{\lambda_1 + \cdots + \lambda_6 \neq 0} \left| \int_{B(s)} e((\lambda_1 + \cdots + \lambda_6, x)) \, dx \right|^2.
\]

For the first summand on the r.h.s. of (A.15) we use Bombieri–Bourgain’s bound (1.23), whereas for the second summand on the r.h.s. of (A.15) we invoke Lemma A.1. These yield

\[
\int_{B(s) \times B(s)} r_n(x - y)^6 \, dx \, dy = s^4 \cdot \left( O\left(\frac{1}{N_n^{5/2}}\right) + |\mathcal{C}_n(6; K)| + \frac{N_n^6}{(Ks)^3} \right).
\]
The estimate (3.12) then finally follows upon choosing \( K = n^{1/2 - \delta} \) so that

\[
\mathcal{C}_n(6; K) = \emptyset
\]

by (A.14), and

\[
\frac{N_n}{(Ks)^3} \ll \frac{N_n}{n^2} \ll \frac{1}{N_n^A}
\]

for every \( A > 0 \), by (3.9) and (1.9).

Now we turn to proving (3.13); the proof is quite similar to the proof of (3.10) except that we need to deal with the potentially blowing-up denominator in (3.2). To this end we separate the singular set \( B_{\text{sing}} \) defined in §3.2. The contribution of \( B_{\text{sing}} \) to the l.h.s. of (3.13) is

\[
\int_{(B(s) \times B(s)) \cap B_{\text{sing}}} \text{tr} X_n(x - y)dxdy \ll \text{meas}(B_{\text{sing}}) \ll \mathcal{R}_n(6; 2s) \ll s^4 \cdot \frac{1}{N_n^{3/2}},
\]

by (3.24), (3.7) and (3.12) respectively. On \( (B(s) \times B(s)) \setminus B_{\text{sing}} \) we may expand

\[
\int_{(B(s) \times B(s)) \setminus B_{\text{sing}}} \text{tr} X_n(x - y)dxdy = -2E_n \int_{(B(s) \times B(s)) \setminus B_{\text{sing}}} \text{tr}(D' \cdot D)dxdy + O \left( \sum_{\lambda \neq \lambda' \in \mathcal{E}_n} \left| \int_{B(s)} e \left( \langle \lambda - \lambda', x \rangle \right) dx \right|^2 \right) \]

(A.16)

since \( r_n \) and \( \frac{D'}{\sqrt{n}} \) are both uniformly bounded.

Now by (3.1) we have upon separating the diagonal \( \lambda = \lambda' \)

\[
\int_{B(s) \times B(s)} \text{tr}(D' \cdot D)dxdy = \int_{B(s) \times B(s)} D \cdot D'dxdy = \left( \pi s^2 \right)^2 \cdot 4\pi^2 n \cdot \frac{1}{N_n} + O \left( \frac{n}{N_n^A} \right),
\]

(A.17)

for every \( A > 0 \), by (1.5) and (A.10). For the second summand on the r.h.s. of (A.16)
we have, again separating the diagonal $\lambda + \lambda' + \lambda'' + \lambda''' = 0$,

\[
\int_{B(s) \times B(s)} r_n^2 \text{tr}(D'D)dx dy = \int_{B(s) \times B(s)} r_n^2 D'D dx dy
\]

\[
= (\pi s^2)^2 \cdot E_n \cdot \frac{1}{n^2} + O \left( E_n \sum_{\lambda + \lambda' + \lambda'' + \lambda''' \neq 0} \left| \int_{B(s)} e(\langle \lambda + \lambda' + \lambda'' + \lambda''' , x \rangle) dx \right|^2 \right)
\]

\[
= (\pi s^2)^2 \cdot E_n \cdot \frac{1}{n^2} + O \left( s^4 E_n \cdot \frac{1}{n^4} \right),
\]

by (A.13). For the latter term on the r.h.s. of (A.16) we employ (3.1), so that

\[
\int_{B(s) \times B(s)} r_n^4 \text{tr}(D'D)dx dy
\]

\[
\ll E_n \cdot \left( (\pi s^2)^2 \cdot \frac{|S_n(6)|}{|\mathcal{N}_n|^6} + \frac{1}{N_4} \sum_{\lambda_1 + \ldots + \lambda_6 \neq 0} \left| \int_{B(s)} e(\langle \lambda_1 + \ldots + \lambda_6 , x \rangle) dx \right|^2 \right)
\]

\[
= E_n \cdot \int_{B(s) \times B(s)} r_n(x - y)^6 dx dy \ll O \left( E_n s^4 \cdot \frac{1}{N_n^{5/2}} \right)
\]

by (A.15) and (3.12). The estimate (3.13) finally follows upon substituting (A.17), (A.18) and (A.19) into (A.16). The exact same argument used for (3.13) also yields (3.14).

The proofs of all the other estimates (3.15)–(3.23) also follow the same but slightly easier pattern: we bound the contribution of the singular set $(B(s) \times B(s)) \cap B_{\text{sing}}$ using the uniform boundedness of the integrand, and expand

\[
\frac{1}{1 - r^2} = 1 + O(r^2)
\]

(no need for $r^2$) on $(B(s) \times B(s)) \setminus B_{\text{sing}}$. The main contribution will always come from evaluating the appropriate moments of $r$ on the diagonal $\mathcal{D}_n(4)$, whereas the non-diagonal contribution is bounded by (A.10), (A.13); the term corresponding to $O(r^2)$ is bounded using (3.12). The precise details are omitted here (cf. [KKW13, Proofs of Lemmas 4.6, 5.4]).

**Proof of Lemma A.1.** First, we show (A.1). To this end we transform the variables $s \cdot y = x$ to write

\[
\int_{B(s)} e(\langle \lambda - \lambda' , x \rangle) dx = s^2 \int_{B(1)} e(\langle s(\lambda - \lambda') , y \rangle) dy = s^2 \tilde{\chi}(s(\lambda - \lambda')),
\]
where \( \chi = \chi_{B(1)} \) is the characteristic function of the Euclidean unit ball \( B(1) \subseteq \mathbb{R}^2 \). As \( \chi \) is rotationally invariant so is its Fourier transform, and a standard direct computation shows that

\[
(A.21) \quad \hat{\chi}(\xi) = 2\pi J_1(\|\xi\|)/\|\xi\|,
\]

where \( J_1 \) is the Bessel \( J \) function; it is well known that

\[
(A.22) \quad J_1(\psi) \ll \min\{1, \|\psi\|^{1/2}\}.
\]

Upon substituting (A.22) into (A.21), and then into (A.20), we obtain

\[
(A.23) \quad \left| \int_{B(s)} e(\langle \lambda - \lambda', x \rangle) dx \right| \ll s^2 \cdot \min\{1, \frac{1}{(s\|\lambda - \lambda'\|^3)^{3/2}}\}.
\]

We then use the inequality (A.23) to bound the summands of (A.1), separating the contribution of the range \( \|\lambda - \lambda'\| > K \) and otherwise to yield

\[
\sum_{\lambda \neq \lambda' \in \mathcal{E}_n} \left| \int_{B(s)} e(\langle \lambda - \lambda', x \rangle) dx \right|^2 \ll s^4 \left( \{ (\lambda, \lambda') : 0 < |\lambda - \lambda'| \leq K \} + \sum_{\|\lambda - \lambda'\| > K} \frac{1}{(sK)^3} \right),
\]

which implies (A.1) upon recalling the definition (1.24) of the quasi-correlation set \( \mathcal{C}(2; K) \), and trivially bounding the number of summands in the latter summation by \( N^2 \).

Now we turn to proving (A.2). Following along the lines of the above argument for (A.1) we find that the l.h.s. of (A.2) is equal to

\[
\sum_{\lambda + \lambda' + \lambda'' + \lambda''' \neq 0} \left| \int_{B(s)} e(\langle \lambda + \lambda' + \lambda'' + \lambda'''', x \rangle) dx \right|^2 = 4\pi^2 s^4 \sum_{\lambda + \lambda' + \lambda'' + \lambda ''' \neq 0} \frac{J_1(s\|\lambda + \lambda' + \lambda'' + \lambda '''\|)^2}{(s\|\lambda + \lambda' + \lambda'' + \lambda '''\|)^2},
\]

which is bounded by separating the summands \( \|\lambda + \lambda' + \lambda'' + \lambda'''\| > K \) from the other summands and using the respective inequalities from (A.22) in each of the cases. This argument yields the precise claimed inequality (A.22). A very similar argument to the above (except that we need to deal with 6-tuples of lattice points rather than 4-tuples, and also this time \( \mathcal{C}_n(6) \) replaces \( \mathcal{D}_n(4) \) as, unlike \( l = 4 \) for \( l = 6 \), \( \mathcal{C}_n(6) \) might properly contain \( \mathcal{D}_n(6) \)) also gives (A.3).
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Jacques Benatar
DEPARTMENT OF MATHEMATICS
KING’S COLLEGE LONDON
LONDON, WC2R 2LS, UK

Current address
SCHOOL OF MATHEMATICAL SCIENCES
TEL AVIV UNIVERSITY
TEL AVIV 69978, ISRAEL
email: benatar@mail.tau.ac.il
Domenico Marinucci
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROME “TOR VERGATA”
VIA DELLA RICERCA SCIENTIFICA, 1
00133 ROMA, ITALY
email: marinucc@axp.mat.uniroma.it

Igor Wigman
DEPARTMENT OF MATHEMATICS
KING’S COLLEGE LONDON
LONDON, WC2R 2LS, UK
email: igor.wigman@kcl.ac.uk

(Received October 3, 2017 and in revised form November 20, 2018)