ISOMORPHISM TESTING OF $k$-SPANNING TOURNAMENTS IS FIXED PARAMETER TRACTABLE

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ABSTRACT. An arc-colored tournament is said to be $k$-spanning for an integer $k \geq 1$ if the union of its arc-color classes of maximal valency at most $k$ is the arc set of a strongly connected digraph. It is proved that isomorphism testing of $k$-spanning tournaments is fixed-parameter tractable.

Keywords: Graph isomorphism problem, colored tournaments, fixed-parameter tractable algorithm.

MSC: 05C20, 05C60, 05C85.

1. Introduction

Over the last two decades, there has been an increased interest in algorithms testing graph isomorphism that are of fixed parameter tractable complexity (FPT algorithms). More precisely, for the graph isomorphism instances we bound some natural graph parameter by a value $k$, which defines a graph class for the input instances. An FPT isomorphism testing algorithm for input graphs from such a class is required to have running time bounded by $f(k) \cdot n^{O(1)}$, where $f(\cdot)$ is allowed to be an arbitrary function of the parameter $k$, but is independent of the input size, and $n$ is the number of vertices in the input graph.

Some well-studied parameters in this context are, for instance, the eigenvalue multiplicity of the input graphs [5], color class size of colored input hypergraphs [1], the treewidth of the input graphs [8] etc. However, for the well known (in the context) parameter $k$ bounding the valency of the input graphs, testing isomorphism in FPT remains an open problem despite some recent interesting progress [6]. A remark in the seminal paper of Babai and Luks [4, p. 8] could be useful for constructing such an algorithm, but the paper in preparation referred there was probably not published. Motivated by this open problem, in the present note we describe an FPT algorithm for testing isomorphism of colored tournaments with respect to a new parameter that plays an analogous role for arc-colored tournaments as valency for graphs.

The parameter for tournaments. Recall that a tournament is a digraph obtained by assigning a direction for each edge in an undirected complete graph without loops. The automorphism group of a tournament is of odd order and hence solvable by the Feit-Thompson theorem. Based on this fact, it was proved in [4] that the isomorphism of tournaments can be tested in quasipolynomial time (of course, this also follows from the recent result [2]). On the other hand, to date, no algorithms are known for isomorphism testing of tournaments, which are in FPT for

The third author is supported by The Israel Science Foundation (project No. 87792731).
some parameter. In the present note, we construct such an algorithm for \( k \)-spanning tournaments defined as follows. An arc-colored tournament is said to be \( k \)-spanning for an integer \( k \geq 1 \) if the union of its arc-color classes of maximal valency at most \( k \) is the arc set of a strongly connected digraph (examples of \( k \)-spanning tournaments are discussed in Subsection 2.2).

**Theorem 1.1.** Let \( k \geq 1 \). Given \( k \)-spanning colored tournaments \( X \) and \( Y \), the set \( \text{Iso}(X,Y) \) of all isomorphisms from \( X \) to \( Y \) can be found in time \( k^{O(\log k)} \text{poly}(n) \), where \( n \) is the number of vertices in \( X \).

It should be remarked that Theorem 1.1 can be extended to a wider class of colored tournaments for which the digraph formed by the arc-color classes of bounded valency is weakly connected. Besides, for \( k \) close to \( n \), our algorithm has the same complexity as the algorithm in [4]. In fact, we use the latter algorithm for \( n \leq k \) (as well as the algorithm from [9] for finding the intersection of a solvable group with the automorphism group of a hypergraph).

The text of the paper is organized as follows. Section 2 contains the necessary background about colored digraphs, including the definition and examples of \( k \)-spanning colored tournaments, wreath product of permutation groups, and some known algorithms. In Section 3, we construct an auxiliary procedure which extends an action of a permutation group from a vertex-color class to the other one or refines the coloring. In Section 4, we prove Theorem 1.1.

## 2. Preliminaries

### 2.1. Colored digraphs.

A digraph \( X \) is said to be colored if it is equipped with linear ordered partitions \( \pi \) and \( S \) of the vertex and arc sets of \( X \), respectively; the elements of \( \pi \) and \( S \) are referred to as vertex-color classes and arc-color classes. The index numbers of vertex- and arc-color classes in the corresponding linear orderings are called vertex and arc colors, respectively.

An isomorphism from a colored digraph \( X \) to a colored digraph \( X' \) is an ordinary digraph isomorphism that preserves vertex- and arc-color classes, respecting the corresponding linear orderings. The coset of all isomorphisms from \( X \) to \( X' \) is denoted by \( \text{Iso}(X,X') \). The group \( \text{Aut}(X) = \text{Iso}(X,X) \) is called the automorphism group of \( X \). If \( X \) and \( X' \) are isomorphic, then \( \text{Iso}(X,X') \) is the right coset \( \text{Aut}(X)\pi \), where \( \pi \in \text{Iso}(X,X') \); of course, \( \text{Iso}(X,X') \) is empty if \( X \) and \( X' \) are not isomorphic.

Suppose \( X \) and \( X' \) have the same vertex set. We write \( X' \succeq X \) if each vertex-color class (respectively, arc-color class) of \( X \) is a union of some vertex-color classes (respectively, arc-color classes) of \( X' \), and \( \text{Aut}(X) = \text{Aut}(X') \). As usual, \( X' \succ X \) if \( X' \succeq X \) and \( X' \neq X \).

### 2.2. \( k \)-spanning digraphs.

Let \( \Omega \) be the vertex set of the colored digraph \( X \) and \( k \geq 1 \). Denote by \( s_k \) the union of all \( s \in S \) such that

\[
|\alpha s| \leq k, \quad \alpha \in \Omega,
\]

where \( \alpha s = \{ \beta \in \Omega : (\alpha, \beta) \in s \} \). We say that \( X \) is \( k \)-spanning if the digraph with vertex set \( \Omega \) and arc set \( s_k \) is strongly connected, i.e., there exists a directed path from every vertex to any other. Two natural examples of \( k \)-spanning tournaments
are given below. In what follows, for a binary relation $s$ the minimal number $k$ satisfying Eq. (1) is called the maximal valency of $s$.

**Cayley tournaments.** Let $G$ be a finite group of odd order and $A$ be an identity-free subset of $G$ such that $|A \cap \{g, g^{-1}\}| = 1$ for every nonidentity $g \in G$. Then the Cayley digraph $X = \text{Cay}(G, A)$ is a tournament. Every partition $A = A_1 \cup \ldots \cup A_m$ induces an ordered partition of the arcs of $X$, the $i$th class of which is the arc set of the Cayley digraph $\text{Cay}(G, A_i)$, $i = 1, \ldots, m$. The resulting colored tournament (with arbitrarily chosen vertex-colored classes) is $k$-spanning if and only if the union of all $A_i$ of cardinality at most $k$ is a generating set of $G$. In the special case $|A_i| \leq k$ for all $i$, this always gives a $k$-spanning (Cayley) tournament, and even for small $k$, the number of such tournaments is exponential in $|G|$. Note that recognizing Cayley tournaments efficiently is an open problem, and polynomial-time algorithms are known only for some groups $G$ close to cyclic, see [10, 11].

**Tournaments with bounded immersion.** Recall that a tournament $Y$ is immersed in a tournament $X$ if the vertices of $Y$ are mapped to distinct vertices of $X$ and the arcs of $Y$ are mapped to directed paths joining the corresponding pairs of vertices of $X$, in such a way that these paths are pairwise arc-disjoint (see, e.g., [2]). Given a tournament $X$, denote by $k(X)$ the maximal integer $k$ such that a transitive tournament \(^1\) with $k$ vertices is immersed in $X$. We claim that if $X$ is a tournament with $k(X) \leq k$ and colored by the Weisfeiler-Leman algorithm (see notation in Subsection 2.2), then $X$ is $O(k^3)$-spanning. Indeed, for every vertex-color class $\Delta$ of $X$, the induced subdigraph $X_{\Delta}$ is a regular tournament with $k(X_{\Delta}) \leq k(X) \leq k$. By [2], this yields $|\Delta| = O(k^3)$. Thus, the maximum valency of each arc-color class of $X$ is at most $O(k^3)$. It should be noted that every $n$-vertex tournament has a transitive subtournament with at least $\log n$ vertices; in particular, $k(X) \geq \log n$. Hence, this example does not really give a fixed parameter problem. However, the algorithm of Theorem 4 is significantly faster than the $n^{\log n}$ time algorithm for general tournaments, like for example when $k(X) = O(\log n)^{O(1)}$.

2.3. **Wreath product.**

Let $X$ be a digraph with vertex set $\Gamma \sqcup \Delta$ and the intersection $D$ of the arc set of $X$ with $\Gamma \times \Delta$ is nonempty. Suppose that for every $\gamma, \gamma' \in \Gamma$, we are given a nonempty set $H_{\gamma, \gamma'}$ of bijections $\gamma D \rightarrow \gamma' D$ (see notation in Subsection 2.2) such that

$$H_{\gamma, \gamma'} H_{\gamma', \gamma''} = H_{\gamma, \gamma''},$$

where $H_{\gamma, \gamma'} \gamma''$ is the set of all compositions $\psi \circ \rho$ with $\psi \in H_{\gamma, \gamma'}$ and $\rho \in H_{\gamma', \gamma''}$. Note that the assumption implies $|\gamma D| = |\gamma' D|$ for all $\gamma, \gamma'$. For $g \in \text{Sym}(\Gamma)$, denote by $F(g)$ the set of all full systems of distinct representatives of the family \{ $H_{\gamma, \gamma'} : \gamma \in \Gamma$ \}; we have $F(g) \neq \varnothing$, because $H_{\gamma, \gamma'} \neq \varnothing$ for all $\gamma, \gamma'$. Given such a system $F = \{f_{\gamma, \gamma'} : \gamma \in \Gamma\}$, we define a permutation $f_{\Gamma} \in \text{Sym}(D)$ as follows:

$$f_{\Gamma}(\gamma, \delta) = (\gamma g, \delta f_{\gamma, \gamma'}), \quad \gamma \in \Gamma, \delta \in \gamma D.$$

**Lemma 2.1.** Let $X$ be the above digraph and $K \leq \text{Sym}(\Gamma)$. Then the set

$$W = W(X, K) = \{f_{\Gamma} : F \in F(g), \ g \in K\}$$

\(^1\)A tournament is said to be transitive if the binary relation defined by the arcs is transitive.
is a permutation group on $D$. Moreover, this group is permutationally isomorphic to $H_{\gamma,\gamma'} \wr K$ for every $\gamma \in \Gamma$.

Proof. Let $\gamma \in \Gamma$. For every $\gamma' \in \Gamma$, we fix an arbitrary $h_{\gamma,\gamma'} \in H_{\gamma,\gamma'}$. Then the mapping 
\[
\varphi : \Gamma \times \gamma D \to D, \quad (\gamma', \delta) \mapsto (\gamma', \delta^{h_{\gamma,\gamma'}})
\]
is obviously a bijection. It suffices to show that $\varphi W \varphi^{-1} = H_{\gamma,\gamma} \wr K$. To this end, let $f = f_F \in W$, where $F = \{f_{\gamma',(\gamma')^\delta} : \gamma' \in \Gamma\} \in \mathcal{F}(g)$ and $g \in K$. Then for any $(\gamma', \delta) \in \Gamma \times \gamma D$, we have
\[
(\gamma', \delta)^s f \varphi^{-1} = (\gamma', \delta^{h_{\gamma,\gamma'}}) f \varphi^{-1} = ((\gamma')^\delta, \delta^{h_{\gamma,\gamma'} f_{\gamma',(\gamma')^\delta}})^{-1}
\]
where $f = h_{\gamma,\gamma'} f_{\gamma',(\gamma')^\delta} h_{(\gamma')^\delta,\gamma'}$. In view of Eq. (2), we have $f' \in H_{\gamma,\gamma}$. Thus, $\varphi f \varphi^{-1} \in H_{\gamma,\gamma} \wr K$ and hence $\varphi W \varphi^{-1} \leq H_{\gamma,\gamma} \wr K$. The reverse inclusion is proved similarly. \qed

2.4. Algorithms.

Let $X$ be a colored digraph with vertex set $\Omega$ of cardinality $n$ and $\tau$ a linear ordered set of subsets of $\Omega$. Using the well-known 2-dimensional Weisfeiler-Leman algorithm [12], one can construct in time $\text{poly}(n)$ a colored digraph $X' = \text{WL}_2(X, \tau)$ such that every vertex-color class of $X$ and every element of $\tau$ is a union of some vertex-color classes of $X'$ and each arc-color class of $X$ is a union of some arc-color classes of $X'$. Moreover, if $\tau$ is $\text{Aut}(X)$-invariant, then
\[
X' = \text{WL}_2(X, \tau) \geq X,
\]
where the relation $\geq$ is as in Subsection 2.1.

In the following lemma, we collect some known results; the permutation groups of degree $n$ in the statement (as well throughout the rest of the paper) are given by generating sets of size $\text{poly}(n)$.

Lemma 2.2. Let $K \leq \text{Sym}(\Omega)$ be a solvable group. Then the problems

1. given a colored digraph $X$, find the group $\text{Aut}(X) \cap K$,
2. given a hypergraph $H$, find the group $\text{Aut}(H) \cap K$,
3. given colored tournaments $T$ and $T'$, find the set $\text{Iso}(T, T')$,

can be solved in time $\text{poly}(n)$, $\text{poly}(m)$, and $n^{O(\log n)}$, respectively, where $n$ is the vertex number of $X$ and $T$, and $m$ is the size of $H$.

Proof. The algorithms for Problems (1) and (3) are well-known and can be found in [4] Corollary 3.6] and in [4] Theorem 4.1, respectively. For Problem (2), one can use Miller’s algorithm [9 Section 2], which runs in polynomial time if the group $K$ is solvable (see [5]). \qed

3. Auxiliary Algorithm

In this section, we construct an algorithm to be used for the proof of Theorem 1.1. The general idea goes back to the Babai-Luks procedure of finding a canonical labeling for a bipartite graph with respect to a group action on one of the parts [4]. However, the algorithm below uses more information from the input graph which, in turn, restricts the output group. Below, the restriction of a permutation group $G \leq \text{Sym}(\Omega)$ to a $G$-invariant subset $\Delta \subseteq \Omega$ is denoted by $G|_{\Delta}$.
Algorithm AUX

**Input:** A colored digraph $X$ with two vertex-color classes $\Gamma$, $\Delta$, and also $X_\Delta$ is a tournament, a nonempty intersection $D$ of some arc-color class with $\Gamma \times \Delta$, and a group $K \supseteq \text{Aut}(X)^\Delta$.

**Output:** Either a colored digraph $X' \succ X$, or a group $L \leq \text{Sym}(D)$ and a homomorphism $\varphi : L \rightarrow \text{Sym}(\Delta)$ such that $L^\varphi \geq \text{Aut}(X)^\Delta$.

**Step 1.** For every $\gamma \in \Gamma$, we use the bijection $\psi : \gamma D \rightarrow \{\gamma\} \times \gamma D$, $\delta \mapsto (\gamma, \delta)$, to construct the tournament $X(\gamma) = (X, \gamma D)^\psi$ on $\{\gamma\} \times \gamma D$.

**Step 2.** For all $\gamma, \gamma' \in \Gamma$, find $H_{\gamma, \gamma'} = \text{Iso}(X(\gamma), X(\gamma'))$ by the algorithm from Lemma 2.2(3). Construct the partition $\pi$ of $\Gamma$ such that $\gamma$ and $\gamma'$ are in the same class of $\pi$ if and only if $H_{\gamma, \gamma'} \neq \emptyset$.

**Step 3.** If $|\pi| > 1$, then output $X' = WL_2(X, \pi)$.

**Step 4.** Construct the permutation group $W = W(X, K)$ on $D$ by Eq. (4) and the hypergraph $\mathcal{H}$ on $D$ with hyperedge set $E = \{\delta D^* \times \{\delta\} : \delta \in \Delta\}$, where $D^* = \{\delta, \gamma) : (\gamma, \delta) \in D\}$.

**Step 5.** Output the group $L = \text{Aut}(\mathcal{H}) \cap W$ found by the algorithm from Lemma 2.2(2), and the homomorphism $\varphi : L \rightarrow \text{Sym}(\Delta)$ induced by the bijection $E \rightarrow \Delta$, $\delta D^* \times \{\delta\} \mapsto \delta$.

Denote by $n = |\Delta \cup \Gamma|$ the vertex number of $X$, and by $k$ the maximal outdegree of the digraph with vertex set $\Delta \cup \Gamma$ and arc set $D$.

**Proposition 3.1.** Algorithm AUX correctly computes the stated output. Moreover, if $K$ is a solvable group, then $L$ is also solvable and the algorithm runs in time $k^{O(\log k)} \text{poly}(n)$.

**Proof.** To prove the correctness assume first that the algorithm terminates at Step 3. Obviously, $\pi$ is $\text{Aut}(X)$-invariant and hence $X' \succeq X$ by Eq. (4). Moreover, $X' \succ X$, because $|\pi| > 1$.

Now let the algorithm terminates at Step 5. Note that $\text{Aut}(X)$ acts on $D$. We claim that

$$\text{Aut}(X)^D \leq W,$$

where $W$ is the group found at Step 4. Indeed, let $a \in \text{Aut}(X)$. Since $K \supseteq \text{Aut}(X)^\Delta$, we have $\gamma^a = \gamma^k$ for some $k \in K$ and all $\gamma \in \Gamma$. Now let $(\gamma, \delta) \in D$. Then $\delta^a \in (\gamma D)^a = \gamma^a D = \gamma^k D$. At Step 4, the set $H_{\gamma, \gamma^k} = \text{Iso}(X(\gamma), X(\gamma^k))$ is nonempty and hence $\delta^a = \delta^{f_{\gamma, \gamma^k}}$ for some $f_{\gamma, \gamma^k} \in H_{\gamma, \gamma^k}$. Therefore,

$$(\gamma, \delta)^a = (\gamma^k, \delta^{f_{\gamma, \gamma^k}}).$$

This is true for all $(\gamma, \delta) \in D$. Consequently, the permutation $a \in \text{Sym}(D)$ belongs to $W$ (see Lemma 2.3). Thus, Eq. (5) holds.

By the definition of $D$, the set $E$ defined at Step 4 is $\text{Aut}(X)^D$-invariant. Therefore, $\text{Aut}(X)^D \leq \text{Aut}(\mathcal{H})$, where $\mathcal{H}$ is the hypergraph defined at the same step. In view of Eq. (5), this yields that

$$\text{Aut}(X)^D \leq W \cap \text{Aut}(\mathcal{H}) = L.$$

Thus, $\text{Aut}(X)^\Delta = (\text{Aut}(X)^D)^\varphi \leq L^\varphi$, which completes the proof of correctness.
To estimate the running time of Algorithm AUX, we note that Steps 1, 3, and 4 can obviously be implemented in time \( \text{poly}(n) \). By Lemma 2.2(3), Step 2 can be done in time \( k^{O(\log k)}m^2 + \text{poly}(m) \), where \( m = |\Gamma| \). Next, the group \( W \) is permutationally isomorphic to the wreath product \( \text{Aut}(X(\gamma)) \wr K \) for every \( \gamma \in \Gamma \) by Lemma 2.1. Note that the group \( \text{Aut}(X(\gamma)) \) is solvable because \( X(\gamma) \) is a tournament. Therefore, if \( K \) is solvable, so are \( W \) and \( L \leq W \). Besides, the hypergraph \( H \) has \( |D| = mk \) vertices, \( |\Delta| \leq mk \) hyperedges, and each hyperedge consists of at most \( m \) vertices. Therefore, the size of \( H \) is polynomial in \( n \). Thus, the group \( L = \text{Aut}(H) \cap W \) (and hence the homomorphism \( \varphi \)) can be found in time \( \text{poly}(n) \) by Lemma 2.2(2) and, consequently, Step 5 terminates within the same time bound. □

4. The proof of Theorem 1.1

We divide the proof of Theorem 1.1 into two parts. First, we prove Theorem 4.1 in which a fixed-parameter tractable algorithm for finding the automorphism group of a \( k \)-spanning colored tournament is constructed. Then we derive Theorem 1.1 from Theorem 4.1.

**Theorem 4.1.** Let \( k \geq 1 \), \( X \) a \( k \)-spanning colored tournament with \( n \) vertices, and \( m \) the minimal size of a vertex-color class of \( X \). Then one can find the group \( \text{Aut}(X) \) in time \( (m^{O(\log m)} + k^{O(\log k)}) \text{poly}(n) \).

**Proof.** In what follows, \( \Omega \) is the vertex set of \( X \). It suffices to construct a solvable group \( K \supseteq \text{Aut}(X) \) in the required time: indeed, then the group \( \text{Aut}(X) \) can be found inside \( K \) in time \( \text{poly}(n) \) by Lemma 2.2(1). To describe the algorithm constructing \( K \), we need an auxiliary lemma.

**Lemma 4.2.** Let \( \Sigma \) be a proper union of vertex-color classes of \( X \). Then there exist vertex-color classes \( \Gamma \) inside \( \Sigma \) and \( \Delta \) outside \( \Sigma \), and an arc-color class of \( X \) of maximal valency at most \( k \) whose intersection \( D \) with \( \Gamma \times \Delta \) is nonempty.

**Proof.** Since \( X \) is \( k \)-spanning, there exist \( \alpha \in \Sigma \) and \( \beta \in \Omega \setminus \Sigma \) such that the arc \((\alpha, \beta)\) belongs to some color class \( s \) of maximal valency at most \( k \). Let \( \Gamma \) and \( \Delta \) be the vertex-color classes of \( X \) containing \( \alpha \) and \( \beta \), respectively. Then \( \Gamma \subseteq \Sigma \) and \( \Delta \subseteq \Omega \setminus \Sigma \). This completes the proof with \( D = s \cap (\Gamma \times \Delta) \). □

The algorithm starts with finding a vertex-color class \( \Sigma \) of \( X \) of size \( m \), and the group \( K = \text{Aut}(X_\Sigma) \) by the algorithm from Lemma 2.2(3). Since \( X \) and hence \( X_\Sigma \) is a tournament, the group \( K \) is solvable and the cost of this step is essentially \( m^{O(\log m)} \). Next, the algorithm proceeds as follows:

- **while** \( \Sigma \neq \Omega \) **do**
  - find classes \( \Gamma, \Delta \) and a set \( D \) as in Lemma 4.2
  - apply the algorithm AUX to \( Y = X_{\Gamma \cup \Delta}, \Gamma, \Delta, D \), and the group \( K^\Gamma \);
  - if the output of AUX is a digraph \( Y' \),
    - then change \( X \) by the tournament obtained by replacing the vertex-color classes \( \Gamma \) and \( \Delta \) with those of \( Y' \); **break**;
    - // now the output of AUX is \( L \leq \text{Sym}(D) \) and \( \varphi : L \to \text{Sym}(\Delta) \)
  - **else** \( \Sigma := \Sigma \cup \Delta \) and \( K := K \times L^\varphi \);
If the above loop terminates with break, then we repeat the whole procedure with the new colored tournament $X$. Since obviously $X' \succ X$, such a repetition occurs at most $n^2$ times. Moreover, at each iteration of the loop we have $K \geq \text{Aut}(X)^{\Delta^2}$: at the zero step this is clear, and then this follows from the inclusion $L^\varphi \geq \text{Aut}(X)^{\Delta}$. Thus, at the end,

$$K \geq \text{Aut}(X).$$

Further, at each iteration of the loop the group $K$ is solvable by induction and the fact that $L^\varphi$ is solvable (see Proposition 6.3). Therefore, the cost of each repetition is at most $m^{O(\log m)} + k^{O(\log k)} \text{poly}(n)$. \hfill \Box

It is well known that the problems of finding the set of all isomorphisms and the automorphism group are polynomial-time equivalent in the class of all graphs. However, the standard proof of this equivalence does not work for $k$-spanning colored tournaments. Lemma 4.3 proved below and based on an old observation of Babai, is sufficient to deduce Theorem 1.1 from Theorem 1.3 for $m = 1$.

**Lemma 4.3.** Given $k$-spanning colored tournaments $X$ and $Y$ with $n$ vertices, one can efficiently construct a set $T = T(X,Y)$ of $n$ colored tournaments such that

(a) each tournament from $T$ has $3n + 1$ vertices, is $k'$-spanning with $k' = \max\{3,k\}$, and has a singleton vertex-color class,

(b) the set $\text{Iso}(X,Y)$ can efficiently be found inside the set $\bigcup_{T \in T} \text{Aut}(T)$.

**Proof.** Let us fix a vertex $\alpha$ of $X$. For every vertex $\beta$ of $Y$ we define a tournament $T = T_\beta$ as the disjoint union of $X$, $Y$, $X' = X$ with attached new vertex $\mu$, and the sets

(6) $$(\Omega \times \Delta) \cup (\Delta \times \Omega') \cup (\Omega' \times \Omega) \cup \{\mu\} \times (\Omega \cup \Delta \cup \Omega')$$

of new arcs, where $\Omega$, $\Delta$, and $\Omega'$ are the vertex sets of $X$, $Y$, and $X'$, respectively. The colors of vertices in $\Omega \cup \Delta \cup \Omega'$ are the same as in $X$, $Y$, and $X'$, and $\mu$ gets a new color. Thus, $T$ contains $3n + 1$ vertices and the singleton vertex-color class $\{\mu\}$.

The colors of arcs inside $X$, $Y$, and $X'$ are not changed, the arcs from the first set from Eq. (6) get a new color, and the arcs in the second set are divided into two classes one of which consists of the arcs $(\mu,\alpha)$, $(\mu,\beta)$, and $(\mu,\alpha')$, where $\alpha'$ is the copy of $\alpha$ in $X'$. Since the union of the latter class with the arc-color classes of $X$ and $Y$ of valency at most $k$ is obviously a connected relation, the colored tournament $T$ is $k'$-spanning. Thus, the set $T = \{T_\beta : \beta \in \Delta\}$ satisfies condition (a).

**Claim.** $X$ and $Y$ are isomorphic if and only if there exist $\beta \in \Delta$ and $f \in \text{Aut}(T_\beta)$ such that $\alpha^f = \beta$, $\beta^f = \alpha'$, and $(\alpha')^f = \alpha$.

**Proof.** To prove the “if” part, it suffices to show that $\Omega^f = \Delta$. Assume that $\gamma^f \notin \Delta$ for some $\gamma \in \Omega$. Since $Y$ is a tournament, $(\alpha,\gamma)$ or $(\gamma,\alpha)$ is an arc of $T_\beta$; for definiteness, assume the first. Then the arcs $(\alpha,\gamma)$ and $(\alpha',\gamma^f) = (\beta,\gamma^f)$ are in different arc-color classes of $T_\beta$. It follows that $f \notin \text{Aut}(T_\beta)$, a contradiction. Conversely, let $f_0 \in \text{Iso}(X,Y)$ and $\beta = \alpha f_0$. Then the permutation $f$ on the union $\Omega \cup \Delta \cup \Omega' \cup \{\mu\}$, defined by

$$\mu^f = \mu, \quad f^\Omega = f_0, \quad f^\Delta = f_0^{-1}, \quad f^{\Omega'} = \text{id},$$

is in $\text{Aut}(T_\beta)$, and $f \in \text{Aut}(T_\beta)$, as desired. \hfill \Box
is obviously an automorphism of \( T_\beta \).

To complete the proof, for any \( \beta \in \Delta \) denote by \( H_\beta \) the set consisting of all permutations of \( \text{Aut}(T_\beta) \) taking \( \alpha, \beta, \alpha' \) to \( \beta, \alpha, \alpha' \), respectively (note that if \( H_\beta \neq \emptyset \), then \( H_\beta \) is a coset of the point stabilizer \( \text{Aut}(T_\beta)_{\alpha, \beta, \alpha'} \)). Then by the Claim, we have

\[
\text{Iso}(X,Y) = \bigcup_{\beta \in \Delta} H_\beta.
\]

To prove that the set \( T \) satisfies condition (b), it remains to note that the index of \( \text{Aut}(T_\beta)_{\alpha, \beta, \alpha'} \) in \( \text{Aut}(T_\beta) \) is at most \( n^3 \), and hence \( H_\beta \) can be found in time \( \text{poly}(n) \) for every \( \beta \in \Delta \). \( \square \)

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