Axions, Surface States, and the Post Constraint in Electromagnetics

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Abstract. After formulating the frequency-domain Maxwell equations for a homogeneous, linear, bianisotropic material occupying a bounded region, we found that the axionic piece vanishes from both the differential equations valid in the region and the boundary conditions, thereby vindicating the Post constraint. Our analysis indicates that characteristic effects that may be observed experimentally with magnetoelectric materials are not the consequences of the axionic piece but of an admittance that describes surface states.

Keywords: admittance, axion, bianisotropic material, linear material, Post constraint, non-reciprocal biisotropy, surface state, topological insulator

1 Introduction

Do axions \cite{1} exist? We, the authors of this communication, lack the capabilities to answer that question. But we do note several negative answers to that question have emerged from experiments \cite{2-5}. Does the “axionic piece” \cite{6} of the linear constitutive dyadic exist in modern classical electromagnetic theory? Again, we are not in a position to answer this question. But one of us summarized the developments before 2004 to state that the recognizable existence of the axionic piece is ruled out by modern classical electromagnetic theory \cite{7}. The filtering out of the axionic piece (denoted by $\Psi$ in this communication) by the Maxwell equations is enshrined as the Post constraint $\Psi \equiv 0$ \cite{8,9}, having been delivered by Post \cite{10} more than half a century ago.

The validity of the Post constraint has been vigorously debated. A simple boundary-value problem has been formulated to show that $\Psi$ is measurable \cite{6}. Multipole symmetry arguments have been derived to rule against as well as for the Post constraint \cite{11-13}. Data from scattering experiments conducted on a single “Tellegen particle” were produced to show that $\Psi$ exists \cite{14}, but those data—while sufficient to establish the magnetoelectric phenomenon (which is not in doubt)—are insufficient to invalidate the Post constraint. Although suspensions of Tellegen particles have also been shown to exhibit the magnetoelectric phenomenon \cite{15, 16}, no data has been presented to invalidate the Post constraint.

There is one item of experimental evidence against the Post constraint, as recounted in great detail by Hehl \textit{et al.} \cite{17} in 2007. This evidence emerging from (i) certain assumptions of symmetry of the linear constitutive dyadic that lead to the description of the magnetoelectric properties of $\text{Cr}_2\text{O}_3$ in terms of just two scalars; (ii) DC measurements reported in 1994 to obtain the sign and the magnitude of one of the two magnetoelectric scalars as well as the magnitude of other magnetoelectric scalar of $\text{Cr}_2\text{O}_3$ on a disk-shaped sample and a rectangular-solid sample \cite{18}, respectively; (iii) the sign of the second magnetoelectric scalar.
of Cr$_2$O$_3$ measured at 10 kHz on a spherical sample reported in 1961 [19]; and (iv) the
eperimentally supported approximation that the quasistatic permeability of Cr$_2$O$_3$ is the
same as that of vacuum [17, 20], enabled them to deduce non-zero values of $\Psi$. Although
independent confirmation of this experimental evidence does not exist to our knowledge, the
veracity of measured data from careful experiments should not be doubted.

But the interpretation of measured data in order to obtain derivative quantities of greater
significance is not necessarily unimpeachable. It is our contention here that data from
electromagnetic experiments on samples of magnetoelectric materials (such as Cr$_2$O$_3$) do
not yield evidence in favor of non-zero $\Psi$. Instead, the effects attributed to a non-zero $\Psi$,
a macroscopic quantity that supposedly exists throughout a sample of a magnetoelectric
material, arise from surface states that exist due to the abrupt cessation of microscopic
morphology at the boundary of that sample.

The remainder of this communication is planned as follows: Section 2 is devoted to a
mathematical exposition of the Post constraint being a natural outcome of the application
of Maxwell equations to linear materials. As $\Psi$ does not appear in the Maxwell equations, it
is necessarily absent from the boundary conditions derived from those differential equations
[21]. Section 3 presents the macroscopic characterization of surface states in terms of a
surface charge density and a surface current density mediated by an admittance. Section 4
contains the solution of a boundary-value problem to show that certain characteristic effects
attributed to the axionic piece are the consequences of that admittance. This communication
concludes with some remarks in Sec. 5. Vectors are underlined, 3×3 dyadics [22] are double
underlined, 0 is the null vector, $I$ is the 3×3 identity dyadic, and $\mathbb{0}$ is the 3×3 null dyadic,
$\varepsilon_0$ is the permittivity of vacuum, and $\mu_0$ is the permeability of vacuum.

2 Natural emergence of the Post constraint

Suppose that all space is divided into two mutually disjoint regions $\mathcal{V}_{\text{out}}$ and $\mathcal{V}_{\text{in}}$ as well as
the surface $\mathcal{S}$ that separates $\mathcal{V}_{\text{out}}$ and $\mathcal{V}_{\text{in}}$. We formally distinguish between $\mathcal{V}_{\text{out}}$, $\mathcal{V}_{\text{in}}$, and $\mathcal{S}$.
A differential equation equation holding in $\mathcal{V}_{\text{out}}$ (or $\mathcal{V}_{\text{in}}$) would have to be solved to satisfy
boundary values of dependent variables prescribed on $\mathcal{S}$.

Suppose that the region $\mathcal{V}_{\text{out}}$ is vacuous, and the region $\mathcal{V}_{\text{in}}$ is occupied by a homoge-
neous, linear, bianisotropic material. With an $\exp(-i\omega t)$ dependence on time $t$ with angular
frequency $\omega$ and $i = \sqrt{-1}$, the frequency-domain constitutive relations of free space are
specified as

$$
\begin{align*}
\mathbf{D}(\mathbf{r}, \omega) &= \varepsilon_0 \mathbf{E}(\mathbf{r}, \omega), \\
\mathbf{H}(\mathbf{r}, \omega) &= \mu_0 \mathbf{B}(\mathbf{r}, \omega),
\end{align*}
$$

with $\nu_0 = \mu_0^{-1}$. The frequency-domain constitutive relations of the bianisotropic material
are specified as

$$
\begin{align*}
\mathbf{D}(\mathbf{r}, \omega) &= \varepsilon(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \alpha(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) + \Psi(\omega) \mathbf{B}(\mathbf{r}, \omega), \\
\mathbf{H}(\mathbf{r}, \omega) &= \beta(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \nu(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) - \Psi(\omega) \mathbf{E}(\mathbf{r}, \omega),
\end{align*}
$$

with $\nu_0 = \mu_0^{-1}$. The frequency-domain constitutive relations of the bianisotropic material
are specified as
where
\[
\text{Trace } \left[ \alpha(\omega) - \beta(\omega) \right] = 0. \tag{3}
\]

The 3x3 permittivity dyadic \( \varepsilon \), the 3x3 impermeability dyadic \( \nu \), the two 3x3 magneto-electric dyadics \( \alpha \) and \( \beta \), and the axionic piece \( \Psi \) can be combined into a single 6x6 linear constitutive dyadic \[22\], but we do not use that compressed notation here. When writing the constitutive relations (2) for \( D \) and \( H \) in terms of \( E \) and \( B \), we have kept in mind that \( E \) and \( B \) are primitive fields because of their underlying microscopic existence while \( D \) and \( H \) are induction fields that arise at the macroscopic level but do not exist at the microscopic level. The non-existence of induction fields at the microscopic level is a cornerstone of modern physics—in contrast to a heterodox formulation of Hehl and Obukhov [23] wherein the merely convenient but inessential \( D \) and \( H \) at the macroscopic level have microscopic counterparts, thereby obscuring the distinction between the microscopic and macroscopic levels.

Let us now apply the frequency-domain macroscopic Maxwell equations
\[
\begin{align*}
\nabla \cdot B(\mathbf{r}, \omega) &= 0 \\
\nabla \times E(\mathbf{r}, \omega) - i\omega B(\mathbf{r}, \omega) &= 0 \\
\nabla \cdot D(\mathbf{r}, \omega) &= \rho(\mathbf{r}, \omega) \\
\nabla \times H(\mathbf{r}, \omega) + i\omega D(\mathbf{r}, \omega) &= J(\mathbf{r}, \omega)
\end{align*}
\]
\[
\tag{4}
\]

(4)

\( \epsilon_0 \) to both \( V_{\text{out}} \) and \( V_{\text{in}} \). After substituting Eqs. (1) in Eqs. (4) we get
\[
\begin{align*}
\nabla \cdot B(\mathbf{r}, \omega) &= 0 \\
\nabla \times E(\mathbf{r}, \omega) - i\omega B(\mathbf{r}, \omega) &= 0 \\
\epsilon_0 \nabla \cdot E(\mathbf{r}, \omega) &= \rho(\mathbf{r}, \omega) \\
\nu_0 \nabla \times B(\mathbf{r}, \omega) + i\omega B(\mathbf{r}, \omega) &= J(\mathbf{r}, \omega)
\end{align*}
\]
\[
\tag{5}
\]

Likewise, after substituting Eqs. (2) in Eqs. (4) we get
\[
\begin{align*}
\nabla \cdot B(\mathbf{r}, \omega) &= 0 \\
\nabla \times E(\mathbf{r}, \omega) - i\omega B(\mathbf{r}, \omega) &= 0 \\
\nabla \cdot \left[ \frac{\varepsilon(\omega) \cdot E(\mathbf{r}, \omega) + \alpha(\omega) \cdot B(\mathbf{r}, \omega) + \Psi(\omega) B(\mathbf{r}, \omega)}{} \right] &= \rho(\mathbf{r}, \omega) \\
\n\nabla \times \left[ \frac{\beta(\omega) \cdot E(\mathbf{r}, \omega) + \nu(\omega) \cdot B(\mathbf{r}, \omega) - \Psi(\omega) E(\mathbf{r}, \omega)}{\epsilon_0} \right] + i\omega \left[ \frac{\varepsilon(\omega) \cdot E(\mathbf{r}, \omega) + \alpha(\omega) \cdot B(\mathbf{r}, \omega) + \Psi(\omega) B(\mathbf{r}, \omega)}{\nu_0} \right] &= J(\mathbf{r}, \omega)
\end{align*}
\]
\[
\tag{6}
\]
which can be first rearranged as
\[
\begin{align*}
\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) &= 0 \\
\nabla \times \mathbf{E}(\mathbf{r}, \omega) - i\omega \mathbf{B}(\mathbf{r}, \omega) &= 0 \\
\n\nabla \cdot \left[ \varepsilon(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \alpha(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) \right] + \Psi(\omega) \nabla \cdot \mathbf{B}(\mathbf{r}, \omega) &= \rho(\mathbf{r}, \omega) \\
\n\nabla \times \left[ \beta(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \mu(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) \right] - \Psi(\omega) \left[ \nabla \times \mathbf{E}(\mathbf{r}, \omega) - i\omega \mathbf{B}(\mathbf{r}, \omega) \right] \\
&\quad + i\omega \left[ \varepsilon(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \alpha(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) \right] = \mathbf{J}(\mathbf{r}, \omega)
\end{align*}
\]
and then simplified to
\[
\begin{align*}
\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) &= 0 \\
\nabla \times \mathbf{E}(\mathbf{r}, \omega) - i\omega \mathbf{B}(\mathbf{r}, \omega) &= 0 \\
\n\nabla \cdot \left[ \varepsilon(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \alpha(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) \right] &= \rho(\mathbf{r}, \omega) \\
\n\nabla \times \left[ \beta(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \mu(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) \right] \\
&\quad + i\omega \left[ \varepsilon(\omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \alpha(\omega) \cdot \mathbf{B}(\mathbf{r}, \omega) \right] = \mathbf{J}(\mathbf{r}, \omega)
\end{align*}
\]
Most importantly, \( \Psi \) does not appear in the Maxwell equations applied to \( \mathcal{V}_\text{in} \) when the merely convenient induction fields have been translated into primitive fields.

When solving an electromagnetic boundary-value problem, it is common to use the following boundary conditions derived from Eqs. (4):
\[
\begin{align*}
\hat{\mathbf{n}}(\mathbf{r}_s) \cdot \left[ \mathbf{B}_\text{out}(\mathbf{r}_s, \omega) - \mathbf{B}_\text{in}(\mathbf{r}_s, \omega) \right] &= 0 \\
\hat{\mathbf{n}}(\mathbf{r}_s) \times \left[ \mathbf{E}_\text{out}(\mathbf{r}_s, \omega) - \mathbf{E}_\text{in}(\mathbf{r}_s, \omega) \right] &= 0 \\
\hat{\mathbf{n}}(\mathbf{r}_s) \cdot \left[ \mathbf{D}_\text{out}(\mathbf{r}_s, \omega) - \mathbf{D}_\text{in}(\mathbf{r}_s, \omega) \right] &= \rho_s(\mathbf{r}_s, \omega) \\
\hat{\mathbf{n}}(\mathbf{r}_s) \times \left[ \mathbf{H}_\text{out}(\mathbf{r}_s, \omega) - \mathbf{H}_\text{in}(\mathbf{r}_s, \omega) \right] &= \mathbf{J}_s(\mathbf{r}_s, \omega)
\end{align*}
\]
with the unit vector \( \hat{\mathbf{n}}(\mathbf{r}_s) \) at \( \mathbf{r}_s \in \mathcal{S} \) pointing into \( \mathcal{V}_\text{out} \). The quantities \( \rho_s \) and \( \mathbf{J}_s \) are the surface charge density and the surface current density, respectively. The subscripts \( \text{in} \) and \( \text{out} \) indicate that the fields are being evaluated on the interior and the exterior sides of \( \mathcal{S} \). The boundary conditions (9) are derived by integrating Eqs. (4) over pillboxes and closed contours straddling \( \mathcal{S} \), as appropriate. Substituting the constitutive relations (1) and (2) in Eqs. (9) and enforcing the charge neutrality and current neutrality of \( \mathcal{S} \), we get as the boundary conditions
\[
\begin{align*}
\hat{\mathbf{n}}(\mathbf{r}_s) \cdot \left[ \mathbf{B}_\text{out}(\mathbf{r}_s, \omega) - \mathbf{B}_\text{in}(\mathbf{r}_s, \omega) \right] &= 0 \\
\hat{\mathbf{n}}(\mathbf{r}_s) \times \left[ \mathbf{E}_\text{out}(\mathbf{r}_s, \omega) - \mathbf{E}_\text{in}(\mathbf{r}_s, \omega) \right] &= 0 \\
\hat{\mathbf{n}}(\mathbf{r}_s) \cdot \left[ \varepsilon(\omega) \mathbf{E}_\text{out}(\mathbf{r}_s, \omega) - \varepsilon(\omega) \mathbf{E}_\text{in}(\mathbf{r}_s, \omega) \\
&\quad - \alpha(\omega) \cdot \mathbf{B}_\text{in}(\mathbf{r}_s, \omega) - \Psi(\omega) \mathbf{B}_\text{in}(\mathbf{r}_s, \omega) \right] = 0 \\
\hat{\mathbf{n}}(\mathbf{r}_s) \times \left[ \mu(\omega) \mathbf{B}_\text{out}(\mathbf{r}_s, \omega) - \beta(\omega) \mathbf{E}_\text{in}(\mathbf{r}_s, \omega) \\
&\quad - \mu(\omega) \cdot \mathbf{B}_\text{in}(\mathbf{r}_s, \omega) + \Psi(\omega) \mathbf{E}_\text{in}(\mathbf{r}_s, \omega) \right] = 0
\end{align*}
\]
which clearly contain $\Psi$. Thus, one could conclude that although $\Psi$ has vanished from the Maxwell equations applied to $\mathcal{V}_in$, it still makes its presence felt through the boundary conditions (10).

That conclusion would be erroneous [21]. As stated elsewhere [24], “the actual [emphasis in the original] issue is the correct physical formulation of boundary-value problems.” The boundary conditions (9) are not sacrosanct in electromagnetics, but their emergence from the Maxwell equations is. Using the same pillboxes and closed contours as in the derivation procedure for the previous paragraph, but using

$$\begin{cases} 
\nabla \cdot \mathbf{B}(r, \omega) = 0 \\
\nabla \times \mathbf{E}(r, \omega) - i\omega \mathbf{B}(r, \omega) = 0 \\
\nabla \cdot \left[ \varepsilon(r, \omega) \cdot \mathbf{E}(r, \omega) + \varepsilon_0(r, \omega) \cdot \mathbf{B}(r, \omega) \right] = \rho(r, \omega) \\
\nabla \times \left[ \beta(r, \omega) \cdot \mathbf{E}(r, \omega) + \beta_0(r, \omega) \cdot \mathbf{B}(r, \omega) \right] \\
+ i\omega \left[ \varepsilon(r, \omega) \cdot \mathbf{E}(r, \omega) + \varepsilon_0(r, \omega) \cdot \mathbf{B}(r, \omega) \right] = \mathcal{J}(r, \omega)
\end{cases}$$

(11)

with

$$\begin{align*}
\varepsilon(r, \omega) &= \begin{cases} 
\varepsilon_0 & \text{in} \\
\varepsilon & \text{out}
\end{cases}, \\
\beta(r, \omega) &= \begin{cases} 
\beta_0 & \text{in} \\
\beta & \text{out}
\end{cases}, \\
\varepsilon_0(r, \omega) &= \begin{cases} 
0 & \text{in} \\
\frac{\alpha}{\varepsilon} & \text{out}
\end{cases}, \\
\beta_0(r, \omega) &= \begin{cases} 
0 & \text{in} \\
\frac{\nu_0\beta}{\nu} & \text{out}
\end{cases},
\end{align*}$$

(12)

instead of Eqs. (4) because $\Psi$ vanishes from Eqs. (6) to deliver Eqs. (8), we get

$$\begin{align*}
\hat{n}(r_s) \cdot [\mathbf{B}_{out}(r_s, \omega) - \mathbf{B}_{in}(r_s, \omega)] &= 0 \\
\hat{n}(r_s) \times [\mathbf{E}_{out}(r_s, \omega) - \mathbf{E}_{in}(r_s, \omega)] &= 0 \\
\hat{n}(r_s) \cdot \left[ \varepsilon_0 \mathbf{E}_{out}(r_s, \omega) - \varepsilon \cdot \mathbf{E}_{in}(r_s, \omega) - \frac{\alpha}{\varepsilon}(\omega) \cdot \mathbf{B}_{in}(r_s, \omega) \right] &= 0 \\
\hat{n}(r_s) \times \left[ \nu_0 \mathbf{B}_{out}(r_s, \omega) - \beta_0 (\omega) \cdot \mathbf{E}_{in}(r_s, \omega) - \frac{\nu}{\beta}(\omega) \cdot \mathbf{B}_{in}(r_s, \omega) \right] &= 0
\end{align*}$$

as the boundary conditions. These boundary conditions do not contain $\Psi$, in contrast to the argument raised elsewhere [25] based on the applicability of the Maxwell equations on $S$ with $\Psi$ multiplied by the unit step function with zero value for $r \in \mathcal{V}_{out}$.

What is the reason for the differences between the boundary conditions (10) and (13)? Very simply, the boundary conditions (10) were derived from Eqs. (4) without considering that the redundancies possibly contained in constitutive equations could be filtered out by the Maxwell equations when applied separately to $\mathcal{V}_in$ and $\mathcal{V}_out$. When the redundant in the form of $\Psi$ was filtered out by the Maxwell equations on application to $\mathcal{V}_in$, we obtained the boundary conditions (13) [21]. Thus, $\Psi$ does not have a recognizable existence in either $\mathcal{V}_in$ or $\mathcal{V}_out$ and it does not enter the boundary conditions. Ergo, the Post constraint

$$\Psi(\omega) \equiv 0$$

(14)

emerges as a natural outcome of the application of Maxwell equations to linear materials.
Let us stress here that the standard boundary conditions in macroscopic electromagnetics were not changed—in order to exclude $\Psi$. That constitutive parameter was absent in the applicable differential equations (i.e., the Maxwell equations containing the primitive but not the induction fields) and therefore did not appear in the boundary conditions derived from those differential equations [21]. Any suggestion that the foregoing procedure “actually changes [emphasis in the original] the physical laws, namely, the Maxwell equations at the interface between the two [homogeneous] media” [26] appears strange to us because any differential equation is held to be valid in a region but not on the boundary of that region.

**Remark 1:** If $V_{in}$ were to be occupied by a homogeneous, linear, biisotropic material, then $\varepsilon = \varepsilon I$, $\nu = \nu I$, $\alpha = \alpha I = \beta$, and $\Psi = 0$. In other words, a biisotropic material must be Lorentz reciprocal and nonreciprocal biisotropy does not have a recognizable existence in electromagnetics [8].

**Remark 2:** If $V_{out}$ were to be occupied by a homogeneous, linear, bianisotropic material different from the one occupying $V_{in}$, the process of deriving Eqs. (8) and (13) shows that the Post constraint would apply to that material too.

**Remark 3:** The Post constraint applies at every frequency, positive, negative, or zero. Therefore, it also applies in the time domain.

**Remark 4:** Even though the existence of $\Psi$ may be deduced from crystallographic symmetry [11,13] or other non-electromagnetic means, it would still be filtered out by the Maxwell equations.

**Remark 5:** The so-called perfect electromagnetic conductor, characterized by $D = \Psi B$ and $H = -\Psi E$ [27,28], does not have a recognizable existence as an electromagnetic medium because it violates the Post constraint.

## 3 Surface states

The Post constraint does not negate the various manifestations of the linear magnetoelectric phenomenon as enshrined in the constitutive dyadics $\alpha$ and $\beta$, but it does constrain the constitutive characterization of that phenomenon. In contrast, as is clear from Eqs. (11)–(13), $\Psi$ does not appear in the Maxwell equations in the two regions as well as in the boundary conditions, when the constitutive relations are written for the correct induction fields ($D$ and $H$) in terms of the correct primitive fields ($E$ and $B$). So, we sought an alternative mechanism that would preserve the Post constraint but could be invoked to interpret certain characteristics of experimental measurements on magnetoelectric materials.

As in Sec. 2, let Eqs. (1) hold in $V_{out}$ while the constitutive relations

\[
\begin{align*}
D(r, \omega) &= \varepsilon(\omega) \cdot E(r, \omega) + \alpha(\omega) \cdot B(r, \omega) \\
H(r, \omega) &= \beta(\omega) \cdot E(r, \omega) + \nu(\omega) \cdot B(r, \omega)
\end{align*}
\]

along with Eqs. (3) hold in $V_{in}$. As boundary conditions, let us use Eqs. (13) without
assuming that \( S \) is charge neutral and current neutral; i.e.,

\[
\begin{align*}
\hat{n}(r_s) \cdot [B_{\text{out}}(r_s, \omega) - B_{\text{in}}(r_s, \omega)] &= 0 \\
\hat{n}(r_s) \times [E_{\text{out}}(r_s, \omega) - E_{\text{in}}(r_s, \omega)] &= 0 \\
\hat{n}(r_s) \cdot [\varepsilon_0 E_{\text{out}}(r_s, \omega) - \varepsilon(r) \cdot E_{\text{in}}(r_s, \omega) - \alpha(\omega) \cdot \beta(\omega) \cdot B_{\text{in}}(r_s, \omega)] &= \rho_s(r_s, \omega) \\
\hat{n}(r_s) \times [\nu \cdot B_{\text{out}}(r_s, \omega) - \beta(\omega) \cdot E_{\text{in}}(r_s, \omega) - \alpha(\omega) \cdot \beta(\omega) \cdot B_{\text{in}}(r_s, \omega)] &= J_s(r_s, \omega)
\end{align*}
\]

\( r_s \in S \). \quad (16)

We posit that the bianisotropic material possesses surface states due to the abrupt cessation of its microscopic morphology, and that these surface states can be described macroscopically as

\[\begin{align*}
\rho_s(r_s, \omega) &= \gamma(\omega) \hat{n}(r_s) \cdot B_{\text{in}}(r_s, \omega) \\
J_s(r_s, \omega) &= -\gamma(\omega) \hat{n}(r_s) \times E_{\text{in}}(r_s, \omega)
\end{align*}\]

\( r_s \in S \), \quad (17)

where \( \gamma \) is an admittance characteristic of the material. What may be observed experimentally with magnetoelectric materials are not the effects of a constitutive parameter \( \Psi \) that describes electromagnetic phenomena in a volume, but of an admittance \( \gamma \) that describes electromagnetic phenomena on a surface.

Remark 6: If \( \mathcal{V}_{\text{out}} \) were to be occupied by a homogeneous, linear, bianisotropic material different from the one occupying \( \mathcal{V}_{\text{in}} \), and both materials were to possess surface states, then

\[\begin{align*}
\rho_s(r_s, \omega) &= \hat{n}(r_s) \cdot [\gamma_{\text{in}}(\omega) B_{\text{in}}(r_s, \omega) - \gamma_{\text{out}}(\omega) B_{\text{out}}(r_s, \omega)] \\
J_s(r_s, \omega) &= -\hat{n}(r_s) \times [\gamma_{\text{in}}(\omega) E_{\text{in}}(r_s, \omega) - \gamma_{\text{out}}(\omega) E_{\text{out}}(r_s, \omega)]
\end{align*}\]

\( r_s \in S \), \quad (18)

because the unit normal vector at \( r_s \in S \) pointing into \( \mathcal{V}_{\text{in}} \) is \(-\hat{n}(r_s)\).

4 Illustrative Example

In order to illustrate the effects of \( \gamma \), let us consider a simple boundary-value problem that has been put forward to indicate the measurability of \( \Psi \) \([6]\). Let us actually begin with a slightly more complicated problem \([25]\) to be simplified later.

Suppose that \( \mathcal{V}_{\text{out}} \) is the half space \( z < 0 \), \( \mathcal{V}_{\text{in}} \) is the half space \( z > 0 \), and therefore \( S \) is the plane \( z = 0 \). Furthermore, let \( \varepsilon = \varepsilon \mathbf{I}, \mu = \nu \mathbf{I}, \alpha = 0, \beta = 0 \), where the dependency upon \( \omega \) has been omitted for brevity.

Let the primitive electromagnetic fields in \( \mathcal{V}_{\text{out}} \) be

\[
\begin{align*}
E(\mathbf{r}) &= \left\{ \begin{array}{l}
\left[ a_s \hat{u}_y + a_p \left( -\frac{\hat{u}_x + \hat{u}_z}{k_0} \right) \right] \exp(i\tau_0 z) \\
\quad + \left[ r_s \hat{u}_y + r_p \left( \frac{\hat{u}_x + \hat{u}_z}{k_0} \right) \right] \exp(-i\tau_0 z) \end{array} \right\} \exp(i\kappa x) \\
B(\mathbf{r}) &= \left\{ \begin{array}{l}
\left[ -a_p \hat{u}_y + a_s \left( -\frac{\hat{u}_x + \hat{u}_z}{k_0} \right) \right] \exp(i\tau_0 z) \\
\quad + \left[ -r_p \hat{u}_y + r_s \left( \frac{\hat{u}_x + \hat{u}_z}{k_0} \right) \right] \exp(-i\tau_0 z) \end{array} \right\} \exp(i\kappa x)
\end{align*}
\]

\( z < 0 \). \quad (19)
where \( k_0 = \omega \sqrt{\varepsilon_0 / \nu_0} \) and \( \tau_0 = + \sqrt{k_0^2 - \kappa^2} \). Representing a plane wave incident on \( S \), the coefficients \( a_s \) and \( a_p \) are presumed to be known. Representing the plane wave reflected into \( \mathcal{V}_{\text{out}} \), the coefficients \( r_s \) and \( r_p \) are unknown. Equations (19) satisfy the homogeneous counterparts of Eqs. (5).

The primitive electromagnetic fields in \( \mathcal{V}_{\text{in}} \) are given as

\[
\begin{align*}
E(r) &= \left\{ t_s \hat{u}_y + t_p \left( \frac{-\hat{u}_x \tau + \hat{u}_z \kappa}{k} \right) \right\} \exp(i\tau z) \exp(i\kappa x), \\
B(r) &= \frac{k}{\omega} \left\{ -t_p \hat{u}_y + t_s \left( \frac{-\hat{u}_x \tau + \hat{u}_z \kappa}{k} \right) \right\} \exp(i\tau z) \exp(i\kappa x)
\end{align*}
\]

where \( k = k_0 \sqrt{(\nu_0 / \nu)(\varepsilon / \varepsilon_0)} \), \( \tau = + \sqrt{k^2 - \kappa^2} \), and the coefficients \( t_s \) and \( t_p \) are unknown. Representing the plane wave refracted into \( \mathcal{V}_{\text{in}} \), these expressions satisfy the homogeneous counterparts of Eqs. (6) and (8).

The foregoing expressions were substituted in Eqs. (16) and (17) to determine \( r_s, r_p, t_s, \) and \( t_p \) in terms of \( a_s \) and \( a_p \). The results found are as follows:

\[
\begin{align*}
{r_s} &= a_s \left( \eta_r - \delta_r \right) \left( 1 + \eta_r \delta_r \right) - \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r^2 + a_p \frac{2\gamma \eta_0 \eta_r^2 \delta_r}{\left( \eta_r + \delta_r \right) \left( 1 + \eta_r \delta_r \right) + \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r}, \\
{r_p} &= a_p \left( \eta_r + \delta_r \right) \left( 1 + \eta_r \delta_r \right) + \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r + \frac{2\gamma \eta_0 \eta_r^2 \delta_r}{\left( \eta_r + \delta_r \right) \left( 1 + \eta_r \delta_r \right) + \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r}, \\
{t_s} &= a_s \left( \eta_r + \delta_r \right) \left( 1 + \eta_r \delta_r \right) + \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r + \frac{2\gamma \eta_0 \eta_r^2 \delta_r}{\left( \eta_r + \delta_r \right) \left( 1 + \eta_r \delta_r \right) + \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r}, \\
{t_p} &= a_p \left( \eta_r + \delta_r \right) \left( 1 + \eta_r \delta_r \right) + \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r + \frac{2\gamma \eta_0 \eta_r^2 \delta_r}{\left( \eta_r + \delta_r \right) \left( 1 + \eta_r \delta_r \right) + \left( \gamma \eta_0 \right)^2 \left( \eta_r \right)^2 \delta_r},
\end{align*}
\]

where

\[
\delta_r = \frac{\tau/k}{\tau_0/k_0}, \quad \eta_r = \sqrt{\varepsilon_0 \nu_0 / \varepsilon \nu}.
\]

We have verified that Eqs. (21)-(24) satisfy Eqs. (16) and (17); furthermore, these expressions simplify to the standard results

\[
\begin{align*}
{r_s} &= a_s \frac{\eta_r - \delta_r}{\eta_r + \delta_r}, \quad {r_p} = a_p \frac{1 - \eta_r \delta_r}{1 + \eta_r \delta_r}, \\
{t_s} &= a_s \frac{2\eta_r}{\eta_r + \delta_r}, \quad {t_p} = a_p \frac{2\eta_r - \delta_r}{1 + \eta_r \delta_r},
\end{align*}
\]

for \( \gamma = 0 \) [29].

Cross-polarized reflection in the foregoing problem has been used to adduce the invalidity of the Post constraint [25], but Eqs. (21)-(24) show that both cross-polarized reflection and refraction are due instead to the surface states as accommodated phenomenologically through \( \gamma \).

Let us now simplify the results to apply to the boundary-value problem put forward to indicate the measurability of \( \Psi \) [6]. For that problem \( \varepsilon = \varepsilon_0 \) and \( \nu = \nu_0 \). Then, \( \delta_r = \eta_r = 1 \).
and Eqs. (21)–(24) simplify to:

\[
\begin{align*}
    r_s &= -\gamma\eta_0 \frac{a_s \gamma_0 - a_p}{4 + (\gamma_0)^2}, \\
    r_p &= \gamma\eta_0 \frac{a_p \gamma_0 + a_s}{4 + (\gamma_0)^2}, \\
    t_s &= 2 \frac{2a_s + a_p \gamma_0}{4 + (\gamma_0)^2}, \\
    t_p &= 2 \frac{2a_p - a_s \gamma_0}{4 + (\gamma_0)^2}.
\end{align*}
\] (27)

As both half spaces \( z < 0 \) and \( z > 0 \) are vacuous, Eqs. (27) indicate that (i) reflection occurs solely due to non-zero \( \gamma \) and (ii) the polarization of the reflected plane wave is rotated with respect to that of the incident plane wave. Both characteristics had been attributed to a non-zero \( \Psi \) [6], but that is clearly unnecessary.

5 Concluding remarks

We formulated the frequency-domain Maxwell equations for a homogeneous, linear, bianisotropic material occupying a bounded region, as well as for the vacuous region that makes up the rest of space. The axionic piece \( \Psi \) is absent in vacuum and it disappears from the Maxwell equations in the bianisotropic material when the induction fields \( \mathbf{D} \) and \( \mathbf{H} \) are replaced by the primitive fields \( \mathbf{E} \) and \( \mathbf{B} \) by using the constitutive equations. In consequence, the axionic piece also vanishes from the boundary conditions prevalent at the interface of the two regions. The Post constraint \( \Psi \equiv 0 \) is thereby vindicated.

Even if both regions were to be occupied by dissimilar homogeneous, linear, bianisotropic materials, the derivation process indicates that the Post constraint would apply to both materials. Even though the existence of \( \Psi \) may be deduced somehow, it will be filtered out by the Maxwell equations. Furthermore, every biisotropic material must be Lorentz reciprocal and the so-called perfect electromagnetic conductor does not have a recognizable existence as an electromagnetic medium because it violates the Post constraint.

We also found that characteristic effects that may be observed experimentally with slabs of a magnetoelectric material are not the consequences of the axionic piece of the linear constitutive dyadic of that material. Instead, they are the consequences of an admittance that describes surface states arising from the abrupt cessation of the microscopic morphology of the magnetoelectric material. This admittance will also impact classical analyses [30, 31] of light scattering by topological insulators. [32]

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