DIMENSION OF BAD SETS FOR NON-UNIFORM FUCHSIAN LATTICES

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Abstract. The set of real numbers which are badly approximable by rationals admits a filtration by sets $\text{Bad}(\epsilon)$, whose dimension converges to 1 as $\epsilon$ goes to zero. D. Hensley computed the asymptotic for the dimension up to the first order in $\epsilon$, via an analogous estimate for the set of real numbers whose continued fraction has all entries uniformly bounded. We generalize this setting considering diophantine approximations by any non-uniform lattice in $\text{PSL}(2,\mathbb{R})$. In particular we give a definition of $\epsilon$-badly approximable points which naturally generalizes the case of rationals. Then we use the thermodynamic method of Ruelle and Bowen to compute the dimension of the set of such points up to the first order in $\epsilon$. Our estimates of spectral radii of transfer operators follow Hensley’s scheme, but we use Banach spaces of piecewise Lipschitz functions.

1. Introduction and main statement

From the theory of continued fractions we know that if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is any irrational number then there exist infinitely many $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ with

$$|\alpha - \frac{p}{q}| < \frac{\epsilon_0}{q^2}$$

where $\epsilon_0 := \frac{1}{\sqrt{5}}$.

The inequality above is sharp, indeed replacing $\epsilon_0$ by any $\epsilon$ with $0 < \epsilon < (\sqrt{5})^{-1}$ and considering $\alpha := (\sqrt{5} - 1)/2$ one gets only finitely many solutions $p/q \in \mathbb{Q}$. More generally, for any $\epsilon > 0$ small enough we have a non-empty set

$$\text{Bad}(\epsilon) := \left\{ \alpha \in \mathbb{R} : \left| \alpha - \frac{p}{q} \right| \geq \frac{\epsilon}{q^2} \quad \forall \ p/q \in \mathbb{Q} \right\}.$$

We have $\text{Bad}(\epsilon') \subset \text{Bad}(\epsilon)$ for $\epsilon' \geq \epsilon$. The union $\text{Bad} := \bigcup_{\epsilon > 0} \text{Bad}(\epsilon)$ is known as the set of badly approximable numbers, and has full dimension in the real line. Moreover, denoting by $\dim_H(E)$ the Hausdorff dimension of a set $E \subset \mathbb{R}$ (see §7.3), we can measure how the size of $\text{Bad}(\epsilon)$ increases when $\epsilon \to 0$. According to [He] we have

$$\dim_H (\text{Bad}(\epsilon)) = 1 - \frac{6}{\pi^2} \cdot \epsilon + o(\epsilon).$$

More precisely, in [He] it is obtained the finer asymptotic up to order $O(n^{-2})$ of $\dim_H(E_n)$, where $E_n$ denotes the set of $\alpha \in \mathbb{R}$ whose continued fraction $\alpha = a_0 + [a_1, a_2, \ldots]$ have entries $a_k \leq n$ for any $k \in \mathbb{N}^*$.

The set of badly approximable numbers and its filtration into sets $\text{Bad}(\epsilon)$ have several natural generalizations. In particular sets of badly approximable points has been considered in detail for vectors in the euclidian space, systems of linear forms, and more generally systems of affine forms. Other natural settings arise considering Klenian groups acting on the boundary of the hyperbolic space, or the set of directions on a given translation surface.
The terminology of bad sets for such examples is somehow standard, as the notation $\text{Bad}(\epsilon)$ for the sets in their filtration. Generally, bad sets have full dimension, and more precisely they are thick. For systems of affine forms this was proved in [Kl], which establishes the most general result in this direction. In [Sc] it was proved that the set of badly approximable systems of linear forms is winning for the so-called Schmidt’s game, a property which implies thickness. The full dimension result for real numbers was established by Jarník in 1929. In [BeGhSiVe], among other natural problems in diophantine approximations, it is proved that for non-elementary geometrically finite Kleinian groups the set of badly approximable points has full dimension in the limit set of the group. This generalizes a previous result in [Pat] for Fuchsian group of the first kind. Thickness of bad sets for directions on a given translation surface has been proved in [KlWe], then in [ChaCheMa] it was proved that the same set is absolute winning. For systems of affine forms this was proved in [Kl], which establishes the most general result in this direction. In [Sc] it was proved that the set of badly approximable systems of linear forms is winning for the so-called Schmidt’s game, a property which implies thickness. The full dimension result for real numbers was established by Jarník in 1929. In [BeGhSiVe], among other natural problems in diophantine approximations, it is proved that for non-elementary geometrically finite Kleinian groups the set of badly approximable points has full dimension in the limit set of the group. This generalizes a previous result in [Pat] for Fuchsian group of the first kind. Thickness of bad sets for directions on a given translation surface has been proved in [KlWe], then in [ChaCheMa] it was proved that the same set is absolute winning. On the other hand, for $\epsilon > 0$ sets $\text{Bad}(\epsilon)$ do not have full dimension, and the main problem appearing in the literature is to compute asymptotic formulas for their dimension. The prototypes for such estimates are Hensley’s Theorem mentioned above, and a previous result of Kurzweil (see [Ku]), which establishes the linear bound

$$0.25 \cdot \epsilon \leq 1 - \dim_H \left( \text{Bad}(\epsilon) \right) \leq 0.99 \cdot \epsilon$$

for badly approximable real numbers. In general, we can call Kurzweil’s bound a pair of upper and lower bounds for the difference between the dimension of the ambient space and $\dim_H \left( \text{Bad}(\epsilon) \right)$. In [We] Kurzweil’s bounds have been obtained for points in the euclidian space, and also in the hyperbolic space, for the action of some class of geometrically finite Kleinian groups. Similar bounds are obtained in [BrKl] for systems of linear forms and in [MarTrWe] for the set of directions on a given translation surface. The common aspect of all these results is that the Kurzweil’s bound that they provide is not linear, but one inequality can only be obtained for some power $\epsilon^\beta$ with $0 < \beta < 1$. Finally, in [Si], the Kurzweil’s bound for systems of linear forms is improved up to Hensley’s first order estimate. This paper is devoted to the proof of Theorem 1.3 below, which establishes Hensley’s result for the action of non-uniform lattices in $\text{SL}(2, \mathbb{R})$ on the boundary of the upper half plane. Theorem 1.3 also gives positive answer to a question in [MarTrWe] concerning badly approximable directions on a Veech surface (see Theorem 1.1 in [MarTrWe] and the comments after the statement).

1.1. Notation. Let $\text{SL}(2, \mathbb{C})$ be the group of matrices

$$(1.1) \quad G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$, where any such $G$ acts on points $z \in \mathbb{C} \cup \{\infty\}$ by

$$(1.2) \quad G \cdot z := \frac{az + b}{cz + d}.$$ 

Given $G \in \text{SL}(2, \mathbb{C})$, when referring to its representation as in Equation (1.1), denote its coefficients by $a = a(G)$, $b = b(G)$, $c = c(G)$ and $d = d(G)$. The derivative at $z \in \mathbb{C}$ of the map in Equation (1.2), which will be denoted indifferently $D_z G$ or $DG(z)$, is equal to

$$D_z G := \frac{1}{(cz + d)^2}.$$ 

The group $\text{SL}(2, \mathbb{R})$ of those matrices $G$ as in Equation (1.1) with coefficients $a, b, c, d$ in $\mathbb{R}$ acts on the upper half plane $\mathbb{H} := \{z \in \mathbb{C}; \text{Im}(z) > 0\}$ via Equation (1.2), indeed for any
such $G$ we have

$$
(1.3) \quad \text{Im}(G \cdot z) = \text{Im} \left( \frac{az + b}{cz + d} \right) = \frac{\text{Im}(z)}{|cz + d|^2}.
$$

A topology on SL(2, $\mathbb{R}$) is induced from $\mathbb{R}^4$ via the identification with the set of those $(a, b, c, d) \in \mathbb{R}^4$ which satisfy $ad - bc = 1$. The quotient $\text{PSL}(2, \mathbb{R}) := \text{SL}(2, \mathbb{R})/\{\pm \text{Id}\}$, whose elements act on $\mathbb{H}$ as in Equation (1.2), is identified with the set of orientation preserving hyperbolic isometries of $\mathbb{H}$, and inherits a topology too. A Fuchsian group is a subgroup $\Gamma < \text{PSL}(2, \mathbb{R})$ which is discrete with respect to such topology. In the following, we will often use the same name for such a group $\Gamma$ and its pre-image in $\text{SL}(2, \mathbb{R})$. The quotient space $\Gamma \backslash \mathbb{H}$ is an orbifold and it inherits from $\mathbb{H}$ the structure of Riemann surface. The area form $\text{Im}^{-2}(z) \cdot dz \wedge d\overline{z}$ corresponds to a measure $\mu$ on $\mathbb{H}$ invariant under $\text{SL}(2, \mathbb{R})$, and it induces an area form and a measure $\mu$ on $\Gamma \backslash \mathbb{H}$. We say that $\Gamma$ is a lattice if $\mu(\Gamma \backslash \mathbb{H}) < +\infty$. A lattice $\Gamma < \text{SL}(2, \mathbb{R})$ is said non-uniform if the quotient $\Gamma \backslash \mathbb{H}$ is not compact. Any non-uniform lattice $\Gamma$ has a fundamental domain $\Omega \subset \mathbb{H}$ with $\mu(\Omega) < +\infty$ which is not compact (see §2.11), moreover the intersection $\overline{\Omega} \cap \partial \mathbb{H}$ is a finite non-empty set (see §2), whose elements are called the vertices at infinity of $\Omega$. A point $z \in \partial \mathbb{H}$ is a parabolic fixed point for $\Gamma$ if there exists a parabolic element $P \in \Gamma$ with $P(z) = z$. The set of all parabolic fixed points equals the orbit under $\Gamma$ of the vertices at infinity of $\Omega$. Two parabolic fixed points $z_1$ and $z_2$ for $\Gamma$ are equivalent if $z_2 = G(z_1)$ for some $G \in \Gamma$. Any non-uniform lattice $\Gamma$ has a finite number $p \geq 1$ of equivalence classes $\{z_1, \ldots, z_p\}$ of parabolic fixed points, which are called the cusps of $\Gamma$, and correspond to the punctures of the quotient surface $\Gamma \backslash \mathbb{H}$. Equivalently a cusp corresponds to a conjugacy class $\langle P \rangle$ in $\Gamma$ of primitive parabolic elements, where $P$ is primitive if it is not a power of one other parabolic element $P' \in \Gamma$.

1.2. Main statement. Classical diophantine approximations have a well know geometric interpretation in terms of the action of the modular group $\text{SL}(2, \mathbb{Z})$ on the upper half plane $\mathbb{H}$, where $\text{SL}(2, \mathbb{Z})$ denotes the subgroup of those $G \in \text{SL}(2, \mathbb{C})$ with coefficients $a, b, c, d$ in $\mathbb{Z}$, referring to the notation established by Equation (1.1). The modular group is a non-uniform lattice, and its action on $\mathbb{H}$ induces an action on the boundary $\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$. In particular $\mathbb{Q}$ equals the orbit $\text{SL}(2, \mathbb{Z}) \cdot \infty$, which is the unique class of parabolic fixed points for $\text{SL}(2, \mathbb{Z})$. In other words, all $p/q \in \mathbb{Q} \cup \{\infty\}$ are sent to an unique cusp on the quotient space $\text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$, which is known as modular surface. In this setting, bad approximability can be translated into the condition below:

$$
\alpha \in \text{Bad}(\epsilon) \iff |\alpha - G \cdot \infty| \geq \frac{\epsilon}{c^2(G)} \quad \forall \ G \in \text{SL}(2, \mathbb{Z}) : c(G) \neq 0.
$$

In order to generalize this notion, let $\Gamma$ be a non-uniform lattice and $p \geq 1$ be the number of its cusps. Fix a family $\mathcal{S} = (A_1, \ldots, A_p)$ of elements $A_k \in \text{SL}(2, \mathbb{R})$ such that the set of points $\{z_1, \ldots, z_p\} \subset \partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$ defined by

$$
(1.4) \quad z_k = A_k \cdot \infty \quad \text{for} \quad k = 1, \ldots, p
$$

is a complete set of inequivalent parabolic fixed points for $\Gamma$ (see Figure 1). Any parabolic fixed point has the from $G \cdot z_k$ for some $G \in \Gamma$ and $k = 1, \ldots, p$ (Equation (4.4) below uses a more refined representation), and we define its denominator as

$$
D(G, z_k) := |c(GA_k)|.
$$
Figure 1. A fundamental domain for a lattice $\Gamma$ and a complete set $\{z_1, z_2, z_3\}$ of inequivalent parabolic fixed points. In general, a set of inequivalent parabolic fixed points is a proper subset of the set of vertices at infinity of the fundamental domain.

**Remark 1.1.** According to Lemma 4.7 the denominator $D(G,z_k)$ of $\zeta = G \cdot z_k$ depends only on $\zeta$, and not on the particular choice of $G \in \Gamma$ (while $k = k(\zeta) \in \{1, \ldots, p\}$ is uniquely determined by $\zeta$, so there is no ambiguity in its choice). Moreover we have $D(G,z_k) = 0$ if and only if $G \cdot z_k = \infty$. Therefore the denominators do not change replacing the set $S$ by

$$S' := (G_1A_1, \ldots, G_PA_p) \quad \text{where} \quad G_k \in \Gamma \quad \text{for} \quad k = 1, \ldots, p. \quad (1.5)$$

The only freedom left in the choice of the set $S$ is to replace it by

$$S'' := (A_1U_1, \ldots, A_pU_p) \quad \text{where} \quad U_k \in U \quad \text{for} \quad k = 1, \ldots, p, \quad (1.6)$$

where $U$ denotes the upper triangular subgroup of those $U \in \mathrm{SL}(2, \mathbb{R})$ with $c(U) = 0$, referring to the notation in Equation (1.1). In this case the new denominator change by

$$D''(G,z_k) = |a(U_k)| \cdot D(G,z_k) \quad \text{for} \quad k = 1, \ldots, p.$$ 

Below we recall Theorem 1 in § 7 of [Pat], which is a version of Dirichlet Theorem due to S. J. Patterson (see also § 1.1 in [BeGhSiVe], or Appendix § A of this paper, where we give a proof for the shake of completeness).

**Theorem** (Patterson). There exists a constant $M = M(\Gamma, S) > 0$ such that for any $Q > 0$ big enough and any $\alpha \in \mathbb{R}$ there exists $G \in \Gamma$ and $k \in \{1, \ldots, p\}$ with $D(G,z_k) \neq 0$ such that

$$|\alpha - G \cdot z_k| \leq \frac{M}{D(G,z_k) \cdot Q} \quad \text{and} \quad D(G,z_k) \leq Q.$$ 

For $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ and $S := \{\text{Id}\}$ the statement of the classical Dirichlet Theorem corresponds to $M = 1$. In the general situation, it follows that for any $\alpha \in \mathbb{R}$ there exist infinitely many $G \in \Gamma$ such that for some $k = k(G)$ in $\{1, \ldots, p\}$ we have

$$|\alpha - G \cdot z_k| \leq \frac{M}{D^2(G,z_k)}.$$
As for the classical case, for any $\Gamma$ and $S$ the constant $M(\Gamma,S)$ in the condition above cannot be replaced by an arbitrarily small constant. In other words we obtain a non-empty set via Definition 1.2 below.

**Definition 1.2.** For any fixed $\epsilon > 0$ small enough define $\text{Bad}(\Gamma,S,\epsilon)$ as the set of those $\alpha \in \mathbb{R}$ such that

$$|\alpha - G \cdot z_k| > \frac{\epsilon}{D(G,z_k)} \quad \forall \ G \in \Gamma, k = 1, \ldots, p : D(G,z_k) \neq 0.$$

**Theorem 1.3.** Let $\Gamma < \text{SL}(2,\mathbb{R})$ be a non-uniform lattice and $S = (A_1, \ldots, A_p)$ be a family of elements $A_k \in \text{SL}(2,\mathbb{R})$ as in Equation (1.4). Then there exists a strictly positive constant $\Theta = \Theta(\Gamma,S) > 0$ such that for any $\epsilon > 0$ small enough we have

$$\dim_H(\text{Bad}(\Gamma,S,\epsilon)) = 1 - \Theta \cdot \epsilon + o(\epsilon).$$

**Remark 1.4.** Equation (9.11) at the end of this paper gives the explicit form of the constant $\Theta(\Gamma,S)$, which has several invariance properties. First of all, Equation (9.11) is obtained via some choice of a finite index free subgroup $\Gamma_0 < \Gamma$, but $\Theta$ does not depend on such choice. Furthermore the set $\text{Bad}(\Gamma,S,\epsilon)$ does not change replacing $S$ by $S'$ in Equation (1.5), and thus $\Theta(\Gamma,S') = \Theta(\Gamma,S)$. On the other hand, if $S$ is replaced by $S''$ in Equation (1.6) then $\Theta$ varies accordingly. For example, if all the elements $U_k \in \mathcal{U}$ as in Equation (1.6) satisfy $U_k = U$ for some $U \in \mathcal{U}$, then we get $\Theta(\Gamma,S'') = a(U)^{-1} \cdot \Theta(\Gamma,S)$. These properties are easily derived a posteriori from Theorem 1.3, but it seems difficult to obtain them directly from Equation (9.11), since the latter contains integrals with respect to a measure whose properties are not explicit enough to pursue further analysis. Similar properties hold for the constant $M(\Gamma,S)$ in Patterson’s Theorem. Finally the set of all badly approximable points $\bigcup_{\epsilon > 0} \text{Bad}(\Gamma,S,\epsilon)$ does not depend on the choice of $S$. In [Pat] (at page 558) it is proved that such set has full dimension. Modulo the dependence on the choice of $S$, once such set is fixed, in the following we will simply write $\text{Bad}(\Gamma,\epsilon)$ instead of $\text{Bad}(\Gamma,S,\epsilon)$.

**Contents of this paper.** This paper is devoted to the proof of Theorem 1.3, which can be resumed as follows. The set $\text{Bad}(\Gamma,\epsilon)$ is approximated by a dynamical cantor set $\mathbb{E}_T$, that is the attractor of an iterated function system generated by a finite family of elements of $\Gamma$, where the number of elements in such finite family is determined by the parameter $T = \epsilon^{-1}$. Thus the dimension of $\text{Bad}(\Gamma,\epsilon)$ is approximated by the dimension of $\mathbb{E}_T$. We introduce an extra parameter $s > 0$ and define a transfer operator $L_{(s,T)}$, in terms of the dynamics generating $\mathbb{E}_T$. Such operator is quasi-compact and its maximal eigenvalue $\lambda(s,T)$ is simple, real and positive. According to a nice formula by R. Bowen, for any $T > 0$, the Hausdorff dimension $s_T$ of $\mathbb{E}_T$ is the (unique) solution of $\lambda(s_T,T) = 1$. Roughly speaking, this reduces the computation of $s_T$ to a discrete version of the implicit function theorem for the locus of zeros of the function $(s,T) \mapsto \lambda(s,T) - 1$. More precisely, the first order of $s_T$ in $T^{-1}$ is obtained by perturbative analysis of spectral radii. This material is organized as follows.

In §2 we recall that any $\Gamma$ as in Theorem 1.3 admits a finite index free subgroup $\Gamma_0$. The pairings between sides in the fundamental domain of such $\Gamma_0$ are a nice family of generators. In §3 the generators mentioned above are used to define the so-called boundary expansion by R. Bowen and C. Series, which can be interpreted as a generalization of Farey’s map. Then we define the cuspidal acceleration of the boundary expansion, which in the analogy with the classical case plays the role of the Gauss map.
In §4 we define the dynamical Cantor set $E_T$ which approximates $\text{Bad}(\Gamma, \epsilon)$. The cuspidal acceleration of the boundary expansion generates a coding for points in the boundary of the unit disc, over a countable alphabet. We reduce to a finite alphabet bounding the size of the cuspidal words arising from the coding. In order to approximate up to the first order the dimension of $\text{Bad}(\Gamma, \epsilon)$ by the dimension of $E_T$ it is necessary to choose the right way to bound the size of cuspidal words, and this is done measuring their geometric length.

In §5 we complete the description of the finite coding by cuspidal words with bounded geometric length. In particular we check the aperiodicity of the transition matrix.

In §6 we prove estimates on contraction and distortion for the iterated function system associated to the coding by cuspidal words. This is an important part of this paper, which is the basis for the functional-analytic estimates in the next sections.

In §7 we state the Ruelle-Perron-Frobenius Theorem for aperiodic sub-shifts of finite type, which says that for a general Lipschitz potential the transfer operator has an isolated, simple and positive maximal eigenvalue. Then we apply this result to the natural potential for dimension estimates, roughly speaking the logarithm of the derivative. Finally we state Bowen’s formula for the dimension of $E_T$.

In §8 we define a proper Banach space of functions on the circle and consider the action of the transfer operator $L(s,T)$ on such space. Quasi-compactness is preserved, and the maximal eigenvalue is the same $\lambda(s, T)$ as in §7. The main goal here is to have the same space to study the action of all transfer operators $L(s,T)$, in particular for $L(1,\infty)$, which arises as limit point and is not covered by Ruelle-Perron-Frobenius Theorem in §7.

In §9 we complete the perturbative analysis of the spectral radii of the family of operators $L(s,T)$. We mainly follow the scheme of Hensley’s proof in the classical case.

In §A we provide a proof of the Dirichlet-Patterson Theorem, which prescribes the right choice of denominators in the definition of $\text{Bad}(\Gamma, \epsilon)$.

In §B we resume some basic properties of spectra and spectral projectors of bounded linear operators, in particular their stability under small perturbations.

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2. Background

Besides $\mathbb{H}$, consider also the unit disc $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$ as model for the hyperbolic space, the two models being related by the map $\varphi : \mathbb{H} \to \mathbb{D}$ defined by

$$\varphi(z) := \frac{z - i}{z + i}.$$  
(2.1)

The conjugate of $\text{SL}(2, \mathbb{R})$ under $\varphi$ is the group $\text{SU}(1, 1)$ of matrices $F \in \text{GL}(2, \mathbb{C})$ with

$$F = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1.$$  
(2.2)
Given any $F \in \text{SU}(1,1)$, when referring to its representation is in Equation (2.2) as above, denote its coefficients by $\alpha = \alpha(F)$ and $\beta = \beta(F)$. In this section, we consider Fuchsian groups as discrete subgroup $\Gamma$ of $\text{PSU}(1,1) := \text{SU}(1,1)/\{\pm \text{Id}\}$, or their pre-image in $\text{SU}(1,1)$.

2.1. Dirichlet region and Siegel Theorem. Let $\Gamma$ be a Fuchsian group and $\xi_0 \in \mathbb{D}$ be a point with $F(\xi_0) \neq \xi_0$ for any $F \in \Gamma \setminus \{\text{Id}\}$. The Dirichlet region $\Omega = \Omega(\Gamma, \xi_0)$ for $\Gamma$ centered at $\xi_0$ is

$$\Omega := \{z \in \mathbb{D} : \rho(z, \xi_0) \leq \rho(z, F(\xi_0)) \forall F \in \Gamma \setminus \{\text{Id}\}\},$$

where $\rho$ denotes the hyperbolic distance on $\mathbb{D}$. One can see that $\Omega$ is a connected, hyperbolic convex and closed subset of $\mathbb{D}$ with non-empty interior $\text{Int}(\Omega)$, where the closure is taken in $\mathbb{D}$ and convexity is meant in term of hyperbolic geodesics in $\mathbb{D}$. It is a well known fact (Theorem 3.2.2 in [Ka]) that any Dirichlet region $\Omega$ for $\Gamma$ satisfies the properties below.

1. $\mathbb{D} = \bigcup_{F \in \Gamma} F(\Omega)$,
2. $\text{Int}(\Omega) \cap F(\text{Int}(\Omega)) = \emptyset$ for any $F \in \Gamma \setminus \{\text{Id}\}$.

Moreover, according to Theorem 3.5.1 in [Ka], for any Dirichlet region $\Omega$ the tessellation $\{F(\Omega) : F \in \Gamma\}$ is locally finite, that is

3. for any compact $K \subset \mathbb{D}$ we have $K \cap F(\Omega) \neq \emptyset$ only for finitely many $F \in \Gamma$.

The boundary $\partial \Omega$ is the countable union of segments $s$ of geodesic bisectors, where for any $F \in \Gamma \setminus \{\text{Id}\}$ the geodesic bisector $H = H(F, \xi_0)$ is the set of points

$$H := \{z \in \mathbb{D} : \rho(z, \xi_0) \leq \rho(F(z), F(\xi_0))\},$$

which is a geodesic for the hyperbolic metric of $\mathbb{D}$. If a geodesic segment $s \subset \partial \Omega$ contains a point $z$ fixed by an elliptic element $F \in \Gamma$ with $F^2 = \text{Id}$ then we consider it as union of two segments $s_1$ and $s_2$ with $s_1 \cap s_2 = \{z\}$. The endpoints of segments $s$ in $\partial \Omega$ are called the vertices of $\Omega$. According Point (3) above the vertices of $\Omega$ do not cumulate inside $\mathbb{D}$. Moreover (see for example § 3.5 in [Ka]) one can see that for any $z \in \partial \Omega$ there is $F \in \Gamma \setminus \{\text{Id}\}$ with $F(z) \in \partial \Omega$ and more precisely we have

4. If $z$ is a vertex of $\Omega$ then also $F(z)$ is. Hence the set of vertices of $\Omega$ is decomposed into equivalence classes under elements of $\Gamma$. Any equivalence class is finite.
5. If $s$ is a side of $\Omega$ then there exists an unique $F \in \Gamma \setminus \{\text{Id}\}$ such that $\hat{s} := F(s)$ is a side of $\Omega$. We say that $s$ and $\hat{s}$ are paired sides.

According to Siegel Theorem (for a proof see Theorem 4.1.1 in [Ka]), if $\Omega$ is any Dirichlet region for $\Gamma$ then we have the implication

6. If $\mu(\Omega) < +\infty$ then $\Omega$ has finitely many sides.

Finally, according to Theorem 3.5.4 in [Ka] the set of pairings generates $\Gamma$.

2.2. Finite index free subgroup. Let $\Gamma$ be a Fuchsian group with $\mu(\Omega) < +\infty$. According to Siegel’s Theorem any Dirichlet region $\Omega$ for $\Gamma$ has finitely many sides. Moreover, since pairings at sides generate $\Gamma$, then $\Gamma$ is finitely generated. Therefore there exists a subgroup $\Gamma_0 < \Gamma$ with finite index $[\Gamma_0 : \Gamma] < +\infty$ such that any $F \in \Gamma_0 \setminus \{\text{Id}\}$ has not finite order, and thus is not elliptic. This was proved by several authors, a more general statement corresponds to Lemma 8 in [Se]. In particular $\Gamma_0$ acts evenly on $\mathbb{D}$, that is any point $\xi \in \mathbb{D}$ has a neighborhood $U$ such that

$$G(U) \cap F(U) = \emptyset \quad \text{for any} \quad G, F \in \Gamma_0 \quad \text{with} \quad G \neq F.$$
In this case the projection map \( p : \mathbb{D} \to \mathbb{D}/\Gamma_0 \) is a covering (Lemma 11.17 in \([Fu]\)). Moreover the group \( \Gamma_0 \) is isomorphic to the deck group \( \text{Aut}(p : \mathbb{D} \to \mathbb{D}/\Gamma_0) \), that is the group of homeomorphisms \( g : \mathbb{D} \to \mathbb{D} \) with \( p \circ g = p \) (Proposition 11.37 in \([Fu]\)). Since \( \mathbb{D} \) is simply connected, then such group is isomorphic to the fundamental group \( \Pi_1(\mathbb{D}/\Gamma_0) \) of the quotient space \( \mathbb{D}/\Gamma_0 \) (Theorem 13.11 in \([Fu]\)). The latter is a Riemann surface of genus \( g \geq 0 \) with \( 1 \leq n < +\infty \) points removed, thus \( \Pi_1(\mathbb{D}/\Gamma_0) \) is a free group with \( 2g + n - 1 \) generators (§ 14 in \([Fu]\)).

2.3. Isometric circles. Consider any \( F \in SU(1,1) \) and \( \alpha = \alpha(F) \) and \( \beta = \beta(F) \) as in Equation (2.2) and assume that \( \beta \neq 0 \), then let \( \omega_F := -\alpha/\beta \) be the pole of \( F \). The isometric circle \( I_F \) of \( F \) is the set of \( \xi \in \mathbb{C} \) such that
\[
|\beta \xi + \alpha| = 1 \iff |\xi - \omega_F| = |\beta|^{-1}.
\]
Clearly \( I_F \) is the euclidean circle centered at \( \omega_F \) with euclidean radius \( \rho(F) := |\beta|^{-1} \), and according to Theorem 3.3.2 in \([Ka]\) we have
\[
F(I_F) = I_{F^{-1}} \quad \text{where} \quad \rho(F) = \rho(F^{-1}) \quad \text{and} \quad |\omega_{F^{-1}}| = |\omega_F|.
\]

Moreover \( I_F \cap \mathbb{D} \) is a geodesic of \( \mathbb{D} \) for any \( F \in SU(1,1) \), according to Theorem 3.3.3 in \([Ka]\). Finally, for any \( F \in SU(1,1) \) denote by \( U_F \) the disc in \( \mathbb{C} \) with \( \partial U_F = I_F \), that is the interior of the isometric circle \( I_F \). We have \( |D_\xi F| < 1 \) for any \( \xi \in \mathbb{C} \setminus U_F \), that is \( F \) contracts (in weak sense) the euclidean metric.

Finally fix \( F, G \in SU(1,1) \) and \( z \in \mathbb{D} \). Observe that we have \( z \in G^{-1}(U_F) \Leftrightarrow G(z) \in U_F \) and \( |D_z FG| = |D_G(z) F| \cdot |D_z G| \), therefore
\[
G^{-1}(U_F) \cap U_G \subset U_{FG} \subset U_G \cup G^{-1}(U_F).
\]
Moreover, if \( U_F \cap U_G = \emptyset \), which is equivalent to \( G^{-1}(U_F) \subset U_G \), then we have
\[
(2.3) \quad G^{-1}(U_F) \subset U_{FG} \subset U_G.
\]

2.4. Labelled ideal polygon. Let \( \Gamma \subset SU(1,1) \) be a non-uniform lattice and \( \Gamma_0 < \Gamma \) be a finite index free subgroup as in § 2.2. In particular \( \beta(F) \neq 0 \) for any \( F \in \Gamma_0 \), where we refer to the notation in Equation (2.2), so that the isometric circle \( I_F \) and the disc \( U_F \) introduced in § 2.3 are defined. Moreover the origin \( 0 \in \mathbb{D} \) is not a fixed point of any \( F \in \Gamma_0 \). According to Theorem 3.3.5 in \([Ka]\) the set
\[
(2.4) \quad \Omega := \mathbb{D} \setminus \bigcup_{F \in \Gamma_0} U_F
\]
is a Dirichlet region for \( \Gamma_0 \) centered in \( \xi = 0 \), that is \( \Omega = \Omega(\Gamma_0, 0) \). Siegel Theorem implies that \( \Omega \) has a finite number of sides, which we denote by the letter \( s \), and Point (5) in § 2.1 implies that they are an even number \( 2d \). Thus \( \Omega \) has \( 2d \) vertices, which we denote by the letter \( \xi_i \), and Point (4) in § 2.1 implies that all vertices belong to \( \partial \mathbb{D} \). Therefore any side \( s \) is a complete geodesic in \( \mathbb{D} \) and for any such \( s \) there exists an unique \( F \in \Gamma \) such that \( F(s) \) is an other side of \( \Omega \) with \( F(s) \neq s \). The sides \( s \) and \( F(s) \) are thus paired, and by Theorem 3.5.4 in \([Ka]\) the set of \( d \) pairings arising in this way generates \( \Gamma_0 \). In order to establish a convenient labelling, consider finite alphabets \( A_0 \) and \( \hat{A}_0 \), both with \( d \) elements and a map
\[
i : A_0 \cup \hat{A}_0 \to A_0 \cup \hat{A}_0 \quad \text{with} \quad \iota^2 = \text{Id} \quad \text{and} \quad \iota(A_0) = \hat{A}_0,
\]
Figure 2. A labelled ideal polygon with $\mathcal{A} = \{a, b, c, \hat{a}, \hat{b}, \hat{c}\}$.

that is an involution of $\mathcal{A}_0 \cup \hat{\mathcal{A}}_0$ which exchanges $\mathcal{A}_0$ with $\hat{\mathcal{A}}_0$. For convenience of notation, set $\mathcal{A} := \mathcal{A}_0 \cup \hat{\mathcal{A}}_0$ and for any $a \in \mathcal{A}$, denote $\hat{a} := \iota(a)$.

1. Label the sides of $\Omega$ by the letters in $\mathcal{A}$, so that for any $a \in \mathcal{A}$ the sides $s_a$ and $s_{\hat{a}}$ are those which are paired by the action of $\Gamma$.

2. For any pair of sides $s_a$ and $s_{\hat{a}}$ as above, let $F_a$ be the unique element of $\Gamma_0$ such that $F_a(s_{\hat{a}}) = s_a$.

3. For any $a \in \mathcal{A}$ we have $F_{\hat{a}} = F_a^{-1}$, and the latter form a set of generators for $\Gamma_0$.

An example of labelled ideal polygon appears in Figure 2. In the next §4 we will consider also the image of $\Omega$ in $\mathbb{H}$. In order to avoid ambiguities, call $\Omega_D := \Omega \subset \mathbb{D}$ the labelled deal polygon defined above, and

$$\Omega_H := \varphi^{-1}(\Omega_D) \subset \mathbb{H}$$

its image in $\mathbb{H}$ via the inverse of the map $\varphi : \mathbb{H} \to \mathbb{D}$ in Equation (2.1).

3. The boundary expansion

We follow the presentation in §2.4 in \cite{ArMarUl1} and of §2 in \cite{ArMarUl2}, in particular we consider the unit disc model of the hyperbolic space. Let $\Gamma < \text{PSU}(1, 1)$ be a non-uniform lattice and $\Gamma_0 < \Gamma$ be a finite index free subgroup as in §2.2 Then let $\mathcal{A}$ be a set of labels and $\Omega_D$ be a labelled ideal polygon as in §2.4.

3.1. The boundary map. We orient $\partial \mathbb{D}$ as follows: if $I \subset \partial \mathbb{D}$ is an interval with $I \neq \partial \mathbb{D}$ and $I \neq \emptyset$ which is parametrized by $(a, b) \to \partial \mathbb{D}$, $t \mapsto e^{-it}$, we say that $e^{-ib}$ is the right endpoint, or supremum, of $I$ and $e^{-ia}$ is the left endpoint, or infimum of $I$, and we write

$$\inf I := e^{-ia} \quad \text{and} \quad \sup I := e^{-ib}.$$ 

We say that the interval $I$ is right open if $\inf I \in I$ and $\sup I \not\in I$. 

For any $a \in \mathcal{A}$ the side $s_a$ divides the disc $D$ into two open connected components. Let $[a]_D \subset \partial D$ be the right open arc such that $[a]_D \cup s_a$ is the boundary of the connected component which is disjoint from the interior of $\Omega$. Set also
\[ \xi^L_a := \inf[a]_D \quad \text{and} \quad \xi^R_a := \sup[a]_D. \]
In order to take account of the cyclic order of the arcs $[a]_D$ in $\partial D$ it is convenient to consider the map $o: \mathcal{A} \to \mathbb{Z}/2d\mathbb{Z}$, where we choose any $a_0 \in \mathcal{A}$ and set $o(a_0) := 0$, and then for any other $a, b \in \mathcal{A}$ we set
\[ (3.1) \quad o(b) = o(a) + 1 \quad \text{iff} \quad \xi^R_a = \xi^L_b. \]
Elements $F \in \Gamma_0$ act as hyperbolic isometries $z \mapsto F(z)$ on $D$, and the action extends continuously to $\partial D$. Consider any $a \in \mathcal{A}$ and the corresponding element $F_a \in \Gamma_0$. Since $s_a = I_{F_a} \cap D$ and $s_a = I_{F_a} \cap D$, where $I_{F_a}$ and $I_{F_a}$ denote the isometric circles of $F_a$ and $F_a$ respectively, and since according to $\S 2.3$ we have $F_a(I_{F_a}) = I_{F_a}$, then $F_a$ sends the complement of $[a]_D$ to $[a]_D$, that is
\[ (3.2) \quad F_a(\partial D \setminus [a]_D) = [a]_D, \]
in particular we have $F_a(\xi^R_a) = \xi^L_a$ and $F_a(\xi^L_a) = \xi^R_a$. The Bowen-Series map is the map $BS: \partial D \to \partial D$, $\xi \mapsto BS(\xi)$ defined by
\[ (3.3) \quad BS(\xi) := F_a^{-1}(\xi) \quad \text{iff} \quad \xi \in [a]_D. \]

**Definition 3.1.** The boundary expansion of a point $\xi \in \partial D$ is the sequence $(a_k)_{k \in \mathbb{N}}$ which encodes the trajectory $\{BS^k(\xi), k \in \mathbb{N}\}$ of $\xi$ by its itinerary with respect to the partition into arcs $\{[a]_D, a \in \mathcal{A}\}$, that is the sequence of letters such that
\[ BS^k(\xi) \in [a_k]_D \quad \text{for any} \ k \in \mathbb{N}. \]

The boundary expansion $(a_k)_{k \in \mathbb{N}}$ of a point $\xi \in \partial D$ will be represented as an infinite word $(a_0, a_1, \ldots)$, and Equation (3.2) implies that any such word satisfies the so-called no backtracking Condition\(^1\), that is
\[ (3.4) \quad a_{k+1} \neq \hat{a}_k \quad \text{for any} \ k \in \mathbb{N}. \]
Reciprocally, finite words $(a_0, \ldots, a_n)$ which satisfy Condition (3.4) corresponds to factors of the map $BS: \partial D \to \partial D$, that is finite concatenations $F_a^{-1} \circ \cdots \circ F_a^{-1}$ arising from iterations of $BS$. We call admissible word, or simply word, any finite word $(a_0, \ldots, a_n)$ or infinite word $(a_0, a_1, \ldots)$ in the letters of $\mathcal{A}$ satisfying the no backtracking Condition (3.4). For any finite admissible finite word $(a_0, \ldots, a_n)$, we use the notation
\[ F_{a_0 \cdots a_n} := F_{a_0} \circ \cdots \circ F_{a_n} \in \Gamma_0. \]
Moreover we also define the half-closed arc $[a_0, \ldots, a_n]_D \subset \partial D$ by
\[ [a_0, \ldots, a_n]_D := [a_0]_D \cap BS^{-1}[a_1]_D \cap \cdots \cap BS^{-n}[a_n]_D. \]
From the definition we see that
\[ [a_0, \ldots, a_n]_D = F_{a_0} \circ \cdots \circ F_{a_n-1} [a_n]_D = F_{a_0} \circ \cdots \circ F_{a_n} (\partial D \setminus [\hat{a}_n]_D). \]
\(^1\)This corresponds to a property of any hyperbolic ray $\gamma: [0, +\infty), t \mapsto \gamma(t)$ in $D$ with $\gamma(0) \in \Omega$ and $\gamma(+\infty) = \xi \in \partial D$: the edges of the tessellation by copies of $\Omega$ are geodesic too, thus $\gamma$ can cross each of them at most once.
It is also evident that given two finite words \((a_0, \ldots, a_n)\) and \((b_0, \ldots, b_m)\) we have the inclusion

\[ [a_0, \ldots, a_n]_\mathbb{D} \subset [b_0, \ldots, b_m]_\mathbb{D} \]

for the corresponding arcs if and only if \(m \geq n\) and \(a_k = b_k\) for any \(k = 0, \ldots, n\). Combining the observations above we have

**Proposition 3.2** (Bowen-Series). If \((a_k)_{k \in \mathbb{N}}\) is the boundary expansion of \(\xi \in \partial \mathbb{D}\) we have

\[ \xi = \bigcap_{n \in \mathbb{N}} F_{a_0} \cdots F_{a_n}[a_{n+1}]_\mathbb{D}. \]

For any sequence \((a_k)_{k \in \mathbb{N}}\) satisfying the no-backtrack Condition \((3.4)\) we introduce the boundary point \(\xi = [a_0, a_1, \ldots]_\mathbb{D} \in \partial \mathbb{D}\) defined by

\[ [a_0, a_1, \ldots]_\mathbb{D} := \bigcap_{n \in \mathbb{N}} F_{a_0} \cdots F_{a_n}[a_{n+1}]_\mathbb{D}. \]

**Lemma 3.3.** The Bowen-Series map \(BS : \partial \mathbb{D} \to \partial \mathbb{D}\) acts as the right shift on the space of admissible infinite words \((a_n)_{n \in \mathbb{N}}\), that is

\[ BS([a_0, a_1, \ldots]_\mathbb{D}) = [a_1, a_2, \ldots]_\mathbb{D}. \]

**Proof.** Since \([a_0, a_1, \ldots]_\mathbb{D} \in [a_0]_\mathbb{D}\) then

\[ BS([a_0, a_1, \ldots]_\mathbb{D}) = F_{a_0}^{-1}([a_0, a_1, \ldots]_\mathbb{D}) = \bigcap_{n \in \mathbb{N}} F_{a_0} \cdots F_{a_n}[a_{n+1}]_\mathbb{D} = \bigcap_{n \in \mathbb{N}} F_{a_1} \cdots F_{a_n}[a_{n+1}]_\mathbb{D} = [a_1, a_2, \ldots]_\mathbb{D}. \]

\[ \square \]

**Lemma 3.4.** For any admissible word \(a_0, \ldots, a_n\) the pole of the map \(F_{a_0,\ldots,a_n}\) belongs to the interior of the disc \(U_{F_{a_0,\ldots,a_n}}\).

**Proof.** For \(n = 0\), that is words composed by only one arrow the statement is obvious. In general the Lemma follows from Equation \((2.3)\) observing that for any \(k = 0, \ldots, n - 1\) condition \(a_{k+1} \neq \overset{\frown}{a}_k\) is equivalent to \(U_{F_{a_k+1}} \cap U_{F_{a_k}^{-1}} = \emptyset\).

\[ \square \]

### 3.2. Cuspidal words.

We follow § 4.1 in [ArMarUl1] and § 2.3 in [ArMarUl2]. Consider a finite word \(W = (a_0, \ldots, a_n)\) in the alphabet \(\mathcal{A}\) which satisfies Condition \((3.4)\).

**Left cuspidal words:** We say that \(W\) is a *left cuspidal word* if \(n \geq 1\) and we have

\[ \xi_{a_0}^L := \inf[a_0]_\mathbb{D} = \inf[a_0, a_1]_\mathbb{D} = \cdots = \inf[a_0, \ldots, a_{n-1}]_\mathbb{D} = \inf[a_0, \ldots, a_n]_\mathbb{D}, \]

that is all the \(n + 1\) arcs above share \(\xi_{a_0}^L\) as common left endpoint. In this case we define its type \(\varepsilon = \varepsilon(W)\) by setting

\[ \varepsilon(W) := L. \]

**Right cuspidal words:** We say that \(W\) is a *right cuspidal word* if \(n \geq 1\) and we have

\[ \xi_{a_0}^R := \sup[a_0]_\mathbb{D} = \sup[a_0, a_1]_\mathbb{D} = \cdots = \sup[a_0, \ldots, a_{n-1}]_\mathbb{D} = \sup[a_0, \ldots, a_n]_\mathbb{D}, \]

that is all the \(n + 1\) arcs above share \(\xi_{a_0}^R\) as common right endpoint. In this case we define its type by setting

\[ \varepsilon(W) := R. \]
Cuspidal words: We say that \( W \) is a cuspidal word if either \( n \geq 1 \) and \( W \) is either left cuspidal or right cuspidal, of if \( W = (a_0) \) for some \( a_0 \in \mathcal{A} \). In this last case the type \( \epsilon(W) \) is not defined.\(^2\)

Lemma 3.5 below corresponds to Lemma 2.3 in [ArMarUl2] and Lemma 4.8 in [ArMarUl1]. The Lemma implies that for any fixed \( n \geq 1 \), any letter \( a_0 \) determines uniquely a left cuspidal word \((a_0, \ldots, a_n)\), and the same holds for right cuspidal words. Equivalently, given a vertex \( \xi \) of \( \Omega_\mathcal{D} \), there is a unique left (right) cuspidal word \((a_0, \ldots, a_n)\) with \( n \geq 1 \) such that the arc \([a_0, \ldots, a_n]D\) has \( \xi \) as left (right) endpoint. Let \( o: \mathcal{A} \to \mathbb{Z}/2d\mathbb{Z} \) be the function defined by Equation (3.1).

**Lemma 3.5.** For \( n \geq 1 \) consider a word \((a_0, \ldots, a_n)\) in the alphabet \( \mathcal{A} \) which satisfies the no-backtracking Condition (3.4). The following holds.

1. The word \((a_0, \ldots, a_n)\) is left cuspidal if and only if \( n \geq 1 \) and
   \[
o(a_{k+1}) = o(\hat{a}_k) + 1 \quad \text{for any} \quad k = 0, \ldots, n-1.
   \]
2. The word \((a_0, \ldots, a_n)\) is right cuspidal if and only if \( n \geq 1 \) and
   \[
o(a_{k+1}) = o(\hat{a}_k) - 1 \quad \text{for any} \quad k = 0, \ldots, n-1.
   \]
3. The word \((a_0, \ldots, a_n)\) is left cuspidal if and only if \( (\hat{a}_n, \ldots, \hat{a}_0) \) is right cuspidal.

Lemma 3.6 below corresponds to Lemma 2.4 in [ArMarUl2] and Lemma 4.9 in [ArMarUl1].

**Lemma 3.6.** Let \((a_0, \ldots, a_n)\) be a left cuspidal word such that \((a_0, \ldots, a_n, a_0)\) is left cuspidal too. Then
\[
F_{a_0} \circ \cdots \circ F_{a_n}
\]
is a parabolic element of \( \Gamma_0 \) whose unique fixed point is \( \xi_{a_0}^L = \xi_{a_n}^R \).

A word \((a_0, \ldots, a_n)\) as in the Lemma before with minimal length is called a left parabolic word. In the same way one defines a right parabolic word. Moreover \((a_0, \ldots, a_n)\) is left parabolic if and only if its inverse word \((\hat{a}_n, \ldots, \hat{a}_0)\) is right parabolic, and the corresponding fixed point is \( \xi_{a_0}^R = \xi_{a_n}^L \). We write simply parabolic word when left or right is not specified.

The cusps of \( \mathcal{D}/\Gamma_0 \) are in bijection with parabolic words, modulo inversion operation and cyclical permutation of the entries. We say that a sequence \((a_n)_{n \in \mathbb{N}}\) is a cuspidal sequence if any finite sub-word of the form \((a_0, \ldots, a_n)\) for \( n \in \mathbb{N} \) is a cuspidal word and that it is eventually cuspidal if there exists \( k \in \mathbb{N} \) such that \((a_{n+k})_{n \in \mathbb{N}}\) is a cuspidal sequence.

3.3. The cuspidal acceleration. Let \( W = (a_0, \ldots, a_n) \) and \( W' = (b_0, \ldots, b_m) \) be two words such that \( b_0 \neq \hat{a}_n \). Then define the word
\[
W \ast W' := (a_0, \ldots, a_n, b_0, \ldots, b_m).
\]

\(^2\)Actually this is slightly inaccurate. Indeed one can have letters \( a \in \mathcal{A} \) such that \( F_a \) is parabolic. Anyhow, this doesn’t affect the results in this paper. Indeed taking into account such words, the transition matrix in Equation (3.7) has more positive entries, and thus aperiodicity in Proposition 5.1 still holds. Moreover the pole of such \( F_a \) have modulus \(|\alpha|/|\beta| > 1 \) strictly, and thus its distance from points in \( \mathcal{D} \) is bounded from below. This implies that the estimates in Lemma 6.1 and Lemma 6.5 keep holding. All the spectral properties of transfer operators used here are based on these two properties.
Consider a sequence \((a_n)_{n \in \mathbb{N}}\) in the letters of \(\mathcal{A}\) satisfying Condition (3.4) and not eventually cuspidal. We define positive integers \(0 = n(0) < n(1) < n(2) < \ldots\) and a decomposition of such sequence into cuspidal words
\[
W_r = (a_{n(r)}, \ldots, a_{n(r+1)-1})
\]
with \(r \geq 0\) such that \((a_0, a_1, a_2, \ldots) = W_0 * W_1 * W_2 * \ldots\).

**Initial step:** Set \(n(0) := 0\). Let \(n(1) \in \mathbb{N}\) be the maximal integer \(n(1) \geq 1\) such that \((a_0, \ldots, a_{n(1)-1})\) is cuspidal, then set
\[
W_0 := (a_0, \ldots, a_{n(1)-1})
\]

**Recursive step:** Fix \(r \geq 1\) and assume that the instants \(n(r)\) and the cuspidal words \(W_0, \ldots, W_{r-1}\) are defined. Define \(n(r+1) \geq n(r) + 1\) as the maximal integer such that \([a_{n(r)}, \ldots, a_{n(r+1)-1}]\) is cuspidal, then set
\[
W_r := (a_{n(r)}, \ldots, a_{n(r+1)-1})
\]
The sequence of words \((W_r)_{r \in \mathbb{N}}\) as above is called the **cuspidal decomposition** of \((a_n)_{n \in \mathbb{N}}\).

**Remark 3.7.** Observe that if \(W_{r-1} := (a_{n(r-1)}, \ldots, a_{n(r)-1})\) and \(W_r := (a_{n(r)}, \ldots, a_{n(r+1)-1})\) are two consecutive cuspidal words in the cuspidal decomposition of a sequence \((a_n)_{n \in \mathbb{N}}\) satisfying Condition (3.4), then the word \((a_{n(r)-1}, a_{n(r)}, \ldots, a_{n(r+1)-1})\) can be cuspidal.

Let \(\mathcal{W}\) be the set of all cuspidal words. Define the **transition matrix** \(M \in \{0, 1\}^{\mathcal{W} \times \mathcal{W}}\) as the infinite matrix such that for any pair of words
\[
W' = (a_0, \ldots, a_n), W = (b_0, \ldots, b_n) \in \mathcal{W}
\]
the coefficient \(M_{W',W} \in \{0, 1\}\) in column \(W\) and row \(W'\) is defined by
\[
M_{W',W} = \begin{cases} 
0 & \text{if } b_0 = \hat{a}_n \\
0 & \text{if } W' = (a_0) \quad \text{and} \quad o(b_0) = o(\hat{a}_n) \pm 1 \\
0 & \text{if } n \geq 1 \quad \text{or} \quad \varepsilon(W) = L \quad \text{and} \quad o(b_0) = o(\hat{a}_n) + 1 \\
0 & \text{if } n \geq 1 \quad \text{or} \quad \varepsilon(W) = R \quad \text{and} \quad o(b_0) = o(\hat{a}_n) - 1 \\
1 & \text{otherwise.}
\end{cases}
\]

In other words, the entry in column \(W\) and row \(W'\) of \(M\) satisfies \(M_{W',W} = 1\) if and only if the concatenated word \(W' * W\) is admissible and \(W' * (b_0)\) is not cuspidal, where we observe that \((a_n) * W\) is allowed to be cuspidal and this asymmetry between \(W\) and \(W'\) in Equation (3.7) reflects Remark 3.7.

### 3.4. The cuspidal acceleration of the Bowen-Series map.
If \(W = (a_0, \ldots, a_n) \in \mathcal{W}\) is any a cuspidal word, consider the group element \(F_W := F_{a_0, \ldots, a_n} \in \Gamma_0\). Consider \(\xi \in \partial \mathbb{D}\) and let \((a_k)_{k \in \mathbb{N}}\) be its boundary expansion as in Definition 3.1 where \(a_k = a_k(\xi)\) for any \(k \in \mathbb{N}\). As in §3.3 for \(r \in \mathbb{N}\), let
\[
W_r(\xi) := (a_{n(r)}(\xi), \ldots, a_{n(r+1)-1}(\xi))
\]
be the \(r\)-th cuspidal word in the cuspidal decomposition of \((a_k)_{k \in \mathbb{N}}\). The cuspidal acceleration of the Bowen-Series map \(\mathbf{BS} : \partial \mathbb{D} \to \partial \mathbb{D}\) is the map \(\mathcal{F} : \partial \mathbb{D} \to \partial \mathbb{D}\) defined by
\[
\mathcal{F}(\xi) := F_W^{-1}(\xi) \quad \text{if} \quad W_0(\xi) = W.
\]
4. Dynamically defined Cantor sets

The main result proved in this section is Proposition 4.4, which reduces the proof of Theorem 1.3 to Theorem 4.3. The proof of the latter is the subject of the rest of the paper, from §4 to §9. In this §4 we define families of dynamical Cantor sets $E_T$ which approximate the bad sets $\text{Bad}(\Gamma, \epsilon)$, where $T = \epsilon^{-1}$. In the next chapters we consider transfer operators acting on the circle, thus $E_T$ are defined as subsets of $\partial \mathbb{D}$. On the other hand bad sets $\text{Bad}(\Gamma, \epsilon)$ are defined in terms of the hyperbolic geometry of $\mathbb{H}$. In this §4 we use both models of hyperbolic space, and we need to transfer to $\mathbb{H}$ the tools developed in §3. The correspondence is resumed below.

Let $\Omega_\mathbb{D}$ be a labelled ideal polygon as in §2.4 and let $\Omega_\mathbb{H}$ be its pre-image in $\mathbb{H}$ under the map $\varphi : \mathbb{H} \to \mathbb{D}$ in Equation (2.1). The vertices of $\Omega_\mathbb{H}$ are denoted by the letter $\zeta$, and for any such point there exists an unique vertex $\xi$ of $\Omega_\mathbb{D}$ with $\zeta = \varphi^{-1}(\xi)$. The sides of $\Omega_\mathbb{H}$ are denoted by the letter $e$ and labelled by letters $a \in A$, where for any such $a$ we set $e_a := \varphi^{-1}(s_a)$ (see Figure 3). This induces a boundary expansion on $\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$. More precisely, for any admissible word $(a_0, \ldots, a_n)$ we set

$$[a_0, \ldots, a_n]_\mathbb{H} := \varphi^{-1}(\langle a_0, \ldots, a_n \rangle_\mathbb{D}).$$

Moreover, if $\xi \in \partial \mathbb{D}$ has cutting sequence $(a_n)_{n \in \mathbb{N}}$, that is $\xi = [a_0, a_1, \ldots]_\mathbb{D}$, then $\alpha := \varphi^{-1}(\xi)$ has the same cutting sequence, that is we write $\alpha = [a_0, a_1, \ldots]_\mathbb{H}$, where we set

$$[a_0, a_1, \ldots]_\mathbb{H} := \varphi^{-1}([a_0, a_1, \ldots]_\mathbb{D}).$$

For any $a \in A$ we have $G_a = \varphi^{-1} \circ F_a \circ \varphi$, and we set $G_{a_0, \ldots, a_n} := G_{a_0} \circ \cdots \circ G_{a_n}$ for any admissible word $(a_0, \ldots, a_n)$. If the same admissible word has cuspidal decomposition $(W_0, \ldots, W_r)$ we write also $G_{W_0, \ldots, W_r} := G_{a_0, \ldots, a_n}$. If the admissible sequence $(a_n)_{n \in \mathbb{N}}$ has cuspidal decomposition $(W_r)_{r \in \mathbb{N}}$ then we write $[W_0, W_1, \ldots]_\mathbb{H} := [a_1, a_2, \ldots]_\mathbb{H}$. All symbolic properties developed in §3 transfer naturally to $\mathbb{H}$, for example for any $n \in \mathbb{N}$ we have

$$G_{a_0, \ldots, a_n}([a_{n+1}, a_{n+2}, \ldots]_\mathbb{H}) = [a_0, a_1, \ldots]_\mathbb{H} \quad \text{and} \quad [a_0, a_1, \ldots]_\mathbb{H} = G_{a_0, \ldots, a_{n-1}}([a_n]_\mathbb{H}) = G_{a_0, \ldots, a_n}(\partial \mathbb{H} \setminus [a_n]_\mathbb{H}).$$

Finally, in this §4 we don’t need to consider cylinders for the cuspidal acceleration of the boundary expansion introduced in §3.4, but we only point out that they have a different form from arcs $[a_0, \ldots, a_n]_\mathbb{H}$, as it is explained in §5.4.

4.1. Geometric length of cuspidal words. Let $\varphi : \mathbb{H} \to \mathbb{D}$ be the map defined in Equation (2.1). For any vertex $\zeta$ of $\Omega_\mathbb{D}$ consider $B \in \Gamma$ and $k \in \{1, \ldots, p\}$ be such that

$$\zeta = BA_k \cdot \infty \quad \text{(4.1)}$$

Observe that the condition $BA_k \cdot \infty = B' A_j \cdot \infty$ implies $j = k$ and $B' = B P$, where $P \in \Gamma$ is parabolic fixing $A_k \cdot \infty$. In particular the integer $k = k(\zeta)$ is uniquely defined. Moreover, for any pair of elements $B$ and $B' = B P$ as above, the map $z \mapsto A_k^{-1} P A_k(z)$ is an horizontal translation in $\mathbb{H}$. Therefore, if $\xi = \varphi(\zeta)$ is a vertex of $\Omega_\mathbb{D}$ and $s$ and $s'$ are geodesics in $\mathbb{D}$ having $\xi$ as common endpoint, their pre-images in $\mathbb{H}$ under $\varphi \circ B \circ A_k$ are two parallel vertical half lines whose distance does not depend on the choice of $B$ in Equation (4.1). It follows that we have a well defined positive real number

$$\Delta(s, s', \xi) := |\text{Re}(A_k^{-1} B^{-1} \varphi^{-1}(s)) - \text{Re}(A_k^{-1} B^{-1} \varphi^{-1}(s'))|.$$
The definition of the geometric length $|W|$ of a right cuspidal word $W = (a_0, a_1, a_2, a_3)$. In the left part of the figure the fundamental domain $\Omega_D$, where the arrows represent the action of the generators: the arcs $s_0 := s_{a_0}$, $s_1 := F_{a_0}(s_{a_1})$, $s_2 := F_{a_0,a_1}(s_{a_2})$ and $s_3 := F_{a_0,a_1,a_2}(s_{a_3})$ all share $\xi_W$ as common endpoint. In the right part of the picture the point $\xi_W$ is sent to $\infty$ under the map $A^{-1}B^{-1}\phi^{-1}$, and the arcs $s_0, s_1, s_2, s_3$ in $D$ are sent to parallel vertical arcs $e_i := \varphi^{-1}(s_i)$ in $\mathbb{H}$ for $i = 0, 1, 2, 3$. We have $|W| = |\text{Re}(e_3) - \text{Re}(e_0)|$, which is also the distance between the endpoints of the geodesic represented by a dotted semicircle. The preimage in $\Omega_D$ of such geodesic is also represented by a dotted arc.

Let $W = (a_0, a_1, \ldots, a_n)$ be a cuspidal word and assume that $n \geq 1$, that is the type $\varepsilon(W)$ is defined. Define $\xi_W$ as the vertex of $\Omega_D$ such that

$$
\xi_W := \xi_{a_0}^L = \inf[a_0]_D = \cdots = \inf[a_0, \ldots, a_n]_D \quad \text{if} \quad \varepsilon(W) = L
$$

and

$$
\xi_W := \xi_{a_0}^R = \sup[a_0]_D = \cdots = \sup[a_0, \ldots, a_n]_D \quad \text{if} \quad \varepsilon(W) = R.
$$

Then define the 

geometric length $|W| > 0$ of the cuspidal word $W$ by

$$
(4.2) \quad |W| := \Delta(s_{a_0}, F_{a_0,\ldots,a_{n-1}}(s_{a_n}), \xi_W)
$$

where $s_{a_0}$ and $F_{a_0,\ldots,a_{n-1}}(s_{a_n})$ are the first and the last arc associated to $W$ which have $\xi_W$ as common endpoint (see Figure 3). Otherwise, if $n = 0$, that is $W$ contains only one letter $a_0$ and thus is not right nor left cuspidal, set

$$
|W| := 0.
$$

**Definition 4.1.** Fix $T \geq 1$ and let $E_T$ be the set of points $\xi \in \partial D$ whose boundary expansion $(a_k)_{k \in \mathbb{N}}$ satisfies

$$
|W_r| \leq T
$$

for any $r \in \mathbb{N}$, where $W_r := (a_{n(r)}, \ldots, a_{n(r+1)-1})$ denotes the $r$-th element in the cuspidal decomposition of $(a_k)_{k \in \mathbb{N}}$, defined by Equation (3.8).
Proposition 4.4. There exists positive constants

Remark 4.2. The Cantor set $\mathcal{E}_T$ is the attractor of an iterated functions system generated by a finite subset of the family of maps $\{f_W : W \in \mathcal{W}\}$. In ArMarUl2 are considered Cantor sets of the same kind, where the length of allowed cuspidal words is measured differently. The notion of geometric length used here is adapted in order to have the inclusions in Proposition 4.4 below. Such sharp inclusions give equality between the first order term in $\epsilon$ of $\text{dim}_H(\text{Bad}(\Gamma, \epsilon))$ and the first order term in $T^{-1}$ of $\text{dim}_H(\mathcal{E}_T)$. The latter is provided by Theorem 4.3 below, which is proved via thermodynamic formalism.

Theorem 4.3. Let $\Theta > 0$ be the constant in Theorem 1.3. For any $T > 0$ big enough have

$$\text{dim}_H(\mathcal{E}_T) = 1 - \Theta \cdot T^{-1} + o(T^{-1}).$$

4.2. End of the proof of Theorem 1.3. Fix a constant $\mu > 0$ and for any $\epsilon > 0$ set $\epsilon' := \frac{\epsilon}{1 - 2\mu \epsilon}$. It is clear that we have

$$\left|(1 - (1 - \Theta \cdot \epsilon + o(\epsilon))) - (1 - \Theta \cdot \epsilon' + o(\epsilon'))\right| = o(\epsilon).$$

Moreover the map $\varphi : \mathbb{H} \to \mathbb{D}$ defined in Equation (2.1) obviously does not change the Hausdorff dimension, since it is smooth. Therefore Theorem 1.3 follows from Theorem 4.3 and from Proposition 4.4 below. The rest of this chapter is devoted to develop the tools to prove Proposition 4.4 whose proof is completed in §4.6.

Proposition 4.4. There exists positive constants $\epsilon_0 > 0$ and $\mu > 0$, depending only on $\Gamma$ and on $\mathcal{S}$, such that for any $\epsilon < \epsilon_0$ we have

$$\mathcal{E}_{\epsilon - 1 - 2\mu} \subset \varphi(\text{Bad}(\Gamma, \epsilon)) \subset \mathcal{E}_{\epsilon - 1}.$$

4.3. Diameter of horoballs and separation. Fix $T > 0$ and consider the horoball at infinity

$$H_T := \{z \in \mathbb{H} : \text{Im}(z) > T\}.$$

Consider $G \in \text{SL}(2, \mathbb{R})$ and its entires $c = c(G)$ and $d = d(G)$ defined by Equation (1.1). If $c(G) \neq 0$ then $G(H_T)$ is an euclidian ball tangent to the real line at $x = G \cdot \infty$. Moreover $\partial H_T = \{z = \alpha + iT : \alpha \in \mathbb{R}\}$, thus for any $z = \alpha + iT \in \partial H_T$ Equation (1.3) gives

$$\text{Im}(G \cdot z) = \frac{\text{Im}(z)}{|cz + d|^2} = \frac{T}{|cT + \text{ic} \alpha + d|^2} = \frac{T}{|c|^2 T^2 + |c \alpha + d|^2}. $$

The expression above is maximal for $\alpha = -d/c$, thus we get

$$\text{Diam}(G(H_T)) = \frac{1}{T \cdot c^2(G)}. $$

Fix a family $\mathcal{S} = \{A_1, \ldots, A_p\}$ as in Equation (1.4), so that $\{z_k = A_k \cdot \infty : k = 1, \ldots, p\}$ is a complete set of inequivalent parabolic fixed points for $\Gamma$. Recall from §1.2 that given a parabolic fixed point $G \cdot z_k$ with $G \in \Gamma$ and $k = 1, \ldots, p$, we define its denominator as $D(G, z_k) = |c(GA_k)|$. In order to simplify the notation, for any $G \in \Gamma$, any $k = 1, \ldots, p$ and any $T > 0$ define the balls

$$B_k(G, T) := G \cdot A_k(H_T).$$

Lemma 4.5. There exists $S_0 > 0$ such that for any $G, F \in \Gamma$ and $i, j \in \{1, \ldots, p\}$ with $G \cdot z_i \neq F \cdot z_j$ we have

$$|G \cdot z_i - F \cdot z_j| \geq \min \left\{ \frac{1}{S_0 \cdot D(G, z_i)^2}, \frac{1}{S_0 \cdot D(F, z_j)^2} \right\}.$$
Proof. The Lemma follows from Equation (4.3) and from Proposition A.2 observing that if \( B_1, B_2 \) are two disjoint horoballs in \( \mathbb{H} \), tangent to \( \mathbb{R} \) at \( x_1, x_2 \) respectively, then
\[
|x_1 - x_2| \geq \min_{i=1,2} \text{Diam}(B_i).
\]
\[\Box\]

4.4. Denominators and reduced form. Consider any vertex \( \zeta \) of \( \Omega_\mathbb{H} \) and let \( B \in \Gamma \) and \( k \in \{1, \ldots, p\} \) be as in Equation (4.1). Consider also any \( G \in \Gamma_0 \), then define the denominator by
\[
D(G, \zeta) := |c(GBA_k)|.
\]

Remark 4.6. Recall that if \( \Gamma \) is a Fuchsian group in \( \text{SL}(2, \mathbb{R}) \) having \( \infty \) as parabolic fixed point, then the condition \( P \cdot \infty = \infty \) implies that \( P \) is parabolic.

The denominator introduced by Equation (4.4) is well defined according to the Lemma below.

Lemma 4.7. For any \( G \in \Gamma_0 \) and any vertex \( \zeta \) of \( \Omega_\mathbb{H} \) we have the equivalence
\[
D(G, \zeta) = 0 \iff G \cdot \zeta = \infty.
\]
Moreover, for any \( G_1, G_2 \) in \( \Gamma_0 \) and any pair of vertices \( \zeta_1 \) and \( \zeta_2 \) of \( \Omega_\mathbb{H} \) we have
\[
G_1 \cdot \zeta_1 = G_2 \cdot \zeta_2 \implies D(G_1, \zeta_1) = D(G_2, \zeta_2).
\]

Proof. In order to prove the first part of the statement, let \( k \in \{1, \ldots, p\} \) and \( B \in \Gamma \) be as in Equation (4.1), so that \( \zeta = B A_k \cdot \infty \). Then just observe that we have \( G \cdot \zeta = \infty \) if and only if \( GBA_k \) is an element of \( \text{SL}(2, \mathbb{R}) \) fixing \( \infty \), that is \( c(GBA_k) = 0 \).

In order to prove the second part of the statement, observe that the assumption implies that \( \zeta_1 \) and \( \zeta_2 \) are in the same orbit under \( \Gamma \), hence there exists \( k \in \{1, \ldots, p\} \) and \( B_1, B_2 \) in \( \Gamma \) such that \( \zeta_1 = B_1 A_k \cdot \infty \) and \( \zeta_2 = B_2 A_k \cdot \infty \). Therefore we get
\[
A_k^{-1} B_2^{-1} G_1 B_1 A_k \cdot \infty = \infty,
\]
that is \( P := A_k^{-1} B_2^{-1} G_1 B_1 A_k \) is a parabolic element of \( \Gamma_k := A_k^{-1} \Gamma A_k \) fixing \( \infty \), according to Remark 4.6. Then
\[
D(G_1, \zeta_1) = |c(G_1 B_1 A_k)| = |c(G_2 B_2 A_k P)| = |c(G_2 B_2 A_k)| = D(G_2, \zeta_2).
\]
\[\Box\]

Remark 4.8. Parabolic fixed points for \( \Gamma \) have the from \( G \cdot z_k \) with \( G \in \Gamma \) and \( k = 1, \ldots, p \). The set of such points is equal to the orbit under \( \Gamma \) of the set of vertices at infinity of a Dirichlet region \( \Omega_\Gamma \) for \( \Gamma \) (Theorem 4.2.5 in [Ka]). Moreover, writing \( \Gamma \) as disjoint union of cosets \( \Gamma = \Gamma_0 T_1 \cup \cdots \cup \Gamma_0 T_d \), we get a fundamental region \( \Omega' = \Omega_1 \) for \( \Gamma \) (Theorem 3.1.2 in [Ka]). Thus the set of parabolic fixed points for \( \Gamma \) is equal to the orbit under \( \Gamma_0 \) of the vertices at infinity of \( \Omega' \), moreover the latter is obviously equal to the orbit under \( \Gamma_0 \) of the vertices at infinity of labelled polygon \( \Omega_\mathbb{H} \) (in fact, all vertices of \( \Omega_\mathbb{H} \) are at infinity). Therefore any parabolic fixed point \( G \cdot z_k \) for \( \Gamma \) is of the from \( G \cdot z_k = \Gamma_{b_0, \ldots, b_m} \cdot \zeta_0 \), where \( (b_0, \ldots, b_m) \) is an admissible word and \( \zeta_0 \) is a vertex of \( \Omega_\mathbb{H} \).
For any point $\zeta \in \partial \mathbb{H}$ which is fixed by some parabolic element of $\Gamma$ there exists an unique admissible word $b_0, \ldots, b_m$ and a vertex $\zeta_0$ of $\Omega_\mathbb{H}$ which is not an endpoint of $e_{bm}^{-1}$ such that

$$\zeta = G_{b_0, \ldots, b_m} \cdot \zeta_0.$$ 

The representation above is called the reduced form of the parabolic fixed point $\zeta$. Lemma 4.9 below establishes a geometric property of reduced words. Lemma 6.2 gives a stronger version of it for cuspidal words.

**Lemma 4.9.** There exists a constant $\kappa_1 > 0$, depending only on $\Omega_\mathbb{H}$, such that the following holds. If $\zeta = G_{b_0, \ldots, b_m} \cdot \zeta_0 \neq 0$ is a parabolic fixed point of $\Gamma$ represented in its reduced form, then we have

$$|\zeta_0 - G_{b_0, \ldots, b_m}^{-1} \cdot \infty| \geq \kappa_1,$$

that is the vertex $\zeta_0$ of $\Omega_\mathbb{H}$ and the pole of $G_{b_0, \ldots, b_m}$ stay at distance uniformly bounded from below.

**Proof.** Observe that $G_{b_0, \ldots, b_m}$ maps $\mathbb{R} \setminus \overline{[b_m]_\mathbb{H}}$ onto $[b_0, \ldots, b_m]_\mathbb{H}$. Since $\infty$ does not belong to the interior of $[b_0, \ldots, b_m]_\mathbb{H}$ then the pole of $G_{b_0, \ldots, b_m}$ belongs to the closure of $[b_m]_\mathbb{H}$. The Lemma follows because $\zeta_0$ is a vertex of $\Omega_\mathbb{H}$ different from the endpoints of $e_{bm}^{-1}$. \qed

**Lemma 4.10.** There exists a constant $\kappa_2 > 0$, depending only on $\Gamma$, such that the following holds. Let $(b_0, \ldots, b_m)$ be any non-trivial admissible word and let $\zeta_0$ be a vertex of $\Omega_\mathbb{H}$ which is not an endpoint of $e_{bm}^{-1}$, so that $G_{b_0, \ldots, b_m} \cdot \zeta_0$ is a parabolic fixed point written in its reduced form (and different from $\infty$).

1. If $\zeta_1$ is an endpoint of $e_{bm}^{-1}$ then we have

$$D(G_{b_0, \ldots, b_m}, \zeta_0) \geq \kappa_2 \cdot D(G_{b_0, \ldots, b_m}, \zeta_1).$$

2. If $b_{m+1}$ satisfies $b_{m+1} \neq b_m$ and $\zeta_2$ is not an endpoint of $e_{bm+1}^{-1}$ we have

$$D(G_{b_0, \ldots, b_m}, \zeta_0) \geq \kappa_2 \cdot D(G_{b_0, \ldots, b_m, b_{m+1}}, \zeta_2).$$

**Proof.** We first prove Part (1) of the statement. For convenience of notation set $G := G_{b_0, \ldots, b_m}$ and $\zeta := G \cdot \zeta_0$, $\zeta' := G \cdot \zeta_1$. We have $D(G, \zeta_1) = 0$ if and only if $\zeta' = \infty$, according to Lemma 4.7. In this case the statement is obvious, since by assumption $\zeta \in \mathbb{R}$ and thus $D(G, \zeta_0) > 0$. Otherwise, if $D(G, \zeta_1) \neq 0$, let $\zeta_0 = B_0 A_k \cdot \infty$ and $\zeta_1 = B_1 A_j \cdot \infty$ as in Equation (L1). In the notation of Equation (L1), let $c, d$ be the entries of $G$. In the same notation, let $a_0, c_0$ and $a_1, c_1$ be the entries of $B_0 \cdot A_k$ and $B_1 \cdot A_j$ respectively. Part (1) of the statement follows obtaining an uniform upper bound for

$$\frac{D(G_{b_0, \ldots, b_m}, \zeta_1)}{D(G_{b_0, \ldots, b_m}, \zeta_0)} = \left| \frac{ca_1 + dc_1}{ca_0 + dc_0} \right|.$$ 

Observe that we cannot have $c_0 = c_1 = 0$, because $\zeta_0 \neq \zeta_1$ and thus in particular these points cannot be both equal to $\infty$. Moreover the image under $G$ of these two points is different from $\infty$, thus condition $c = 0$ implies $c_0 \neq c_1$. Hence for $c = 0$ Part (1) follows because the ratio above equals $\left| \frac{c_1}{c_0} \right|$, which varies over a finite set of values and is therefore bounded from above. If $c, c_0, c_1 \neq 0$ then

$$\frac{ca_1 + dc_1}{ca_0 + dc_0} = \left| \frac{c_1}{c_0} \right| \cdot \frac{(a_1/c_1) - (-d/c)}{(a_0/c_0) - (-d/c)} = \frac{c_1}{c_0} \cdot \frac{\zeta_1 - (G^{-1} \cdot \infty)}{\zeta_0 - (G^{-1} \cdot \infty)}.$$
In this case Part (1) follows because \(|c_1/c_0|\) is bounded from above, and Lemma 4.9 gives and upper bound for the second factor. If \(c, c_0 \neq 0\) and \(c_1 = 0\) then 
\[
\left| \frac{ca_1 + dc_1}{ca_0 + dc_0} \right| = \left| \frac{a_1}{c_0} \right| \cdot \left| \frac{1}{(a_0/c_0) - (d/c)} \right| = \left| \frac{a_1}{c_0} \right| \cdot \left| \frac{1}{\zeta_0 - (G^{-1} \cdot \infty)} \right| \leq \left| \frac{a_1}{c_0 \cdot k_1} \right|
\]
and Part (1) follows from Lemma 4.9 observing that \(a_1/c_0\) varies over a finite set of values. Finally, if \(c, c_1 \neq 0\) and \(c_0 = 0\) then 
\[
\left| \frac{ca_1 + dc_1}{ca_0 + dc_0} \right| = \left| \frac{a_1}{a_0} - \frac{(d/c) \cdot c_1}{a_0} \right| \leq \left| \frac{a_1}{a_0} \right| + |G^{-1} \cdot \infty| \left| \frac{c_1}{a_0} \right|
\]
and Part (1) follows observing that in this case the pole \(G^{-1} \cdot \infty\) belongs to the compact interval of \(\mathbb{R}\) delimited by the two parallel vertical segments of \(\Omega_H\) (because of definition of reduced form and of Lemma 5.4). The analysis of cases is complete and Part (1) of the Lemma is proved. Part (2) follows by a similar analysis, replacing \(\zeta_1\) by 
\[
\zeta_s := G_{b_{m+1}} \cdot \zeta_2
\]
and observing that, since \(G_{b_{m+1}}\) varies in the finite set \(\{G_a : a \in \mathcal{A}\}\) then also the entries of \(X \in \text{SL}(2, \mathbb{R})\) with \(G_{b_{m+1}} \cdot \zeta_2 = X \cdot \infty\) vary in a finite set. Moreover \(G \cdot \zeta_0 \neq G \cdot \zeta_s\), since they are two different reduced forms. Finally these two points are both different from \(\infty\), and their pre-image under \(G\) are mutually different. The Lemma is proved. \qed

4.5. Best approximations and cuspidal words. Let \(S = (A_1, \ldots, A_p)\) be as in Equation (1.4), so that \(\{z_k = A_k \cdot \infty : k = 1, \ldots, p\}\) is a complete set of inequivalent parabolic fixed points for \(\Gamma\). For any \(k = 1, \ldots, p\) let \(\mu_k > 0\) be such that the primitive parabolic element \(P_k \in A_k \Gamma A_k^{-1}\) fixing \(\infty\) acts by \(P_k(z) = z + \mu_k\). Then let \(\mu = \mu(\Gamma, S) > 0\) be defined by \(\mu := \max\{\mu_1, \ldots, \mu_p\}\). Consider \(\alpha \in \mathbb{R}\) and its boundary expansion 
\[
\alpha = [a_0, a_1, \ldots]_{\mathbb{H}} = [W_0, W_1, \ldots]_{\mathbb{H}},
\]
where we consider both the expansion in the generators \(\{G_a : a \in \mathcal{A}\}\) and its decomposition into cuspidal words. For \(r \in \mathbb{N}\) let \(W_r = (a_0, a_1, \ldots, a_n, a_0^{-1})\) be the \(r\)-th cuspidal word in the above decomposition, where \(W_r = W_r(\alpha)\). When \(|W_r| > 0\) let \(\zeta_{W_r}\) be the vertex associated \(W_r\) as in § 4.4 and let \(\zeta_{W_r} := \varphi^{-1}(\xi_{W_r})\) be its preimage in \(\mathbb{H}\) under the map in Equation (2.1).

Lemma 4.11. In the notation introduced above, for any \(r \in \mathbb{N}\) such that \(|W_r| > 0\) we have 
\[
\frac{1}{|W_r| + 2\mu} \leq D(G_{W_0, \ldots, W_{r-1}, \zeta_{W_r}}) \cdot |\alpha - G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r}| \leq \frac{1}{|W_r|}.
\]

Proof. Consider \(k \in \{1, \ldots, p\}\) and \(B \in \Gamma\) such that \(\zeta_{W_r} = B A_k \cdot \infty\), as in Equation (4.1). Then consider \(T > 0\) such that the horoball 
\[
B_T := G_{W_0, \ldots, W_{r-1}} B A_k \cdot H_T
\]
is tangent at \(G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r}\) with radius \(\rho(B_T) = |\alpha - G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r}|\). Equation (4.3) and Equation (4.4) give 
\[
D(G_{W_0, \ldots, W_{r-1}, \zeta_{W_r}}) \cdot |\alpha - G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r}| = c^2(G_{W_0, \ldots, W_{r-1}} B A_k) \cdot \frac{\text{Diam}(B_T)}{2} = \frac{1}{2T}.
\]
According to the definition of $T$, the geodesic in $\mathbb{H}$ with endpoints $(G_{W_0, \ldots, W_{r-1}}BA_k)\cdot \infty$ and $(G_{W_0, \ldots, W_{r-1}}BA_k)^{-1}\cdot \alpha$ is tangent to $H_T$ (the horoball at $\infty$). The Lemma follows observing that the definition of $|W_r|$ in Equation 112 gives

$$|W_r| \leq 2T \leq |W_r| + 2\mu.$$ 

See Figure 4 for a geometric interpretation of the last inequality. The Lemma is proved. \hfill \Box

**Lemma 4.12.** There exist constants $\epsilon_0 > 0$ and $T_0 > 0$, depending only on $\Gamma$, such that for any $G \in \Gamma$ and $k = 1, \ldots, p$ with $D(G, z_k) \neq 0$ the condition

$$D(G, z_k)^2 \cdot |G \cdot z_k| < \epsilon_0$$

implies that there exists some $r \in \mathbb{N}$ such that

$$G \cdot z_k = G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r} \quad \text{and} \quad |W_r| \geq T_0.$$ 

**Proof.** According to Remark 4.8, let $\zeta_0$ be the vertex of $\Omega_\mathbb{H}$ and $(b_0, \ldots, b_m)$ be the admissible word such that the reduced form of the parabolic fixed point $G \cdot z_k$ is

$$G \cdot z_k = G_{b_0, \ldots, b_m} \cdot \zeta_0,$$

where $\zeta_0$ is not an endpoint of $e_{b_m^{-}}$ whenever $(b_0, \ldots, b_m)$ is not the empty word. Recall that $D(G, z_k) = D(G_{b_0, \ldots, b_m}, \zeta_0)$, according to Lemma 4.7. We subdivide the proof in a sequence of steps.

*Step (0)* Consider separately the case when $(b_0, \ldots, b_m)$ is the empty word. In this case, assume first as extra assumption that the first cuspidal word $W_0 = (a_0, \ldots, a_{n(1)-1})$ in the expansion of $\alpha$ satisfies $|W_0| > 0$, and that $\zeta_0 = \zeta_{W_0}$, where $\xi_{W_0}$ is the vertex of $\Omega_\mathbb{H}$ associated with...
to $W_0$ as in §4.1. Under such extra assumption, Lemma 4.11 gives

$$D(\text{Id}, \zeta_0)^2 \cdot |\alpha - \zeta_0| = D(\text{Id}, \zeta_{W_0})^2 \cdot |\alpha - \zeta_{W_0}| \geq \frac{1}{|W_0| + 2\mu}.$$ 

Otherwise, if the extra assumption is not satisfied by $\alpha$, then it is satisfied by some $\alpha'$, which also satisfies the following three conditions

\[ \zeta_0 = \zeta_{W'_0}, \quad |W'_0| \leq |W_0| + \mu, \quad |\alpha' - \zeta_0| \leq |x - \zeta_0|, \]
where $W'_0$ denotes the first cuspidal word in the boundary expansion of $\alpha'$. We have

\[
\begin{align*}
D(\text{Id}, \zeta_0)^2 \cdot |\alpha - \zeta_0| &\geq D(\text{Id}, \zeta_0)^2 \cdot |\alpha' - \zeta_0| \\
&= D(\text{Id}, \zeta_{W'_0})^2 \cdot |\alpha' - \zeta_{W'_0}| \geq \frac{1}{|W'_0| + 2\mu} \geq \frac{1}{|W_0| + 3\mu}.
\end{align*}
\]

Therefore, if $(b_0, \ldots, b_m)$ is the empty word, then the Lemma follows for constants $\epsilon_0, T_0$ such that

\[
\frac{1}{\epsilon_0} - 3\mu > 0 \quad \text{and} \quad T_0 \leq \frac{1}{2} \left( \frac{1}{\epsilon_0} - 3\mu \right).
\]

**Step (1)** Now assume that $(b_0, \ldots, b_m)$ is not the empty word. Then $G \cdot z_k$ is an interior point of the arc $[b_0, \ldots, b_m]_\mathbb{H}$, and in particular $G \cdot z_k \neq \infty$. Denote $\zeta'_1$ and $\zeta'_2$ the endpoints of $[b_0, \ldots, b_m]_\mathbb{H}$ and let $\zeta_1, \zeta_2$ be the vertices of $\Omega_\mathbb{H}$ such that $\zeta'_i = G_{b_0, \ldots, b_m} \cdot \zeta_i$ for $i = 1, 2$. Let $N \geq -1$ be maximal such that $a_n = b_n$ for any $n = 0, \ldots, N$, where the last condition is clearly empty for $N = -1$, and obviously $N \leq m$. Observe that condition $N \leq m - 1$ implies that $\alpha \not\in [b_0, \ldots, b_m]_\mathbb{H}$, and therefore

\[
|\alpha - G \cdot z_k| \geq \min_{i=1,2} |\zeta'_i - G \cdot z_k| = \min_{i=1,2} |G_{b_0, \ldots, b_m} \cdot \zeta_i - G_{b_0, \ldots, b_m} \cdot \zeta_0| \\
\geq \frac{1}{S_0} \cdot \min \left\{ \frac{1}{D(G_{b_0, \ldots, b_m}, \zeta'_1)^2}, \frac{1}{D(G_{b_0, \ldots, b_m}, \zeta'_2)^2}, \frac{1}{D(G_{b_0, \ldots, b_m}, \zeta_0)^2} \right\} \\
\geq \frac{1}{(\kappa_2)^2 \cdot S_0} \cdot \frac{1}{D(G_{b_0, \ldots, b_m}, \zeta_0)^2},
\]

where the third inequality follows from Part (1) of Lemma 4.10 and where the second inequality follows from Lemma 4.5 in terms of the constant $S_0 > 0$ in the same Lemma. Therefore we must have $N = m$, provided that $\epsilon_0$ satisfies

\[ \epsilon_0 < \left( S_0 \cdot (\kappa_2)^2 \right)^{-1}. \]

Under the same condition, a similar argument (which is left to the reader) shows that $G \cdot z_k$ must be an endpoint of $[a_0, \ldots, a_m, a_{m+1}]_\mathbb{H} \subset [a_0, \ldots, a_m]_\mathbb{H} = [b_0, \ldots, b_m]_\mathbb{H}$.

**Step (2)** We show that $G \cdot z_k = G_{b_0, \ldots, b_m} \cdot \zeta_0$ is an endpoint of $[a_0, \ldots, a_{m+2}]_\mathbb{H}$. Otherwise $G \cdot z_k$ would not belong to the closure of $[a_0, \ldots, a_{m+2}]_\mathbb{H}$ and the distance $|G \cdot z_k - \alpha|$ would be grater then the distance of $G \cdot z_k$ from the closest endpoint of $[a_0, \ldots, a_{m+2}]_\mathbb{H}$. Such closest endpoint would have reduced from $G_{b_0, \ldots, b_m, a_{m+1}} \cdot \zeta_3$, where $\zeta_3$ is a vertex of $\Omega_\mathbb{H}$ which is not an endpoint of $e_{a_{m+1}}$ (recall that $a_i = b_i$ for $i = 0, \ldots, m$). Since by definition $\alpha \in [a_0, \ldots, a_{m+2}]_\mathbb{H}$, then
Lemma 4.5 and Part (2) of Lemma 4.10 would imply
\[
|\alpha - G_{b_0,\ldots,b_m} \cdot \zeta_0| \geq |G_{b_0,\ldots,b_m,a_{m+1}} \cdot \zeta_3 - G_{b_0,\ldots,b_m} \cdot \zeta_0| \\
\geq \frac{S_0^{-1}}{\max\{D(G_{b_0,\ldots,b_m}, \zeta_0)^2, D(G_{b_0,\ldots,b_m,a_{m+1}}, \zeta_3)^2\}} \geq \frac{\kappa_2}{S_0 \cdot D(G_{b_0,\ldots,b_m}, \zeta_0)^2},
\]
which is absurd by the condition \(\epsilon_0 < (S_0 \cdot (\kappa_2)^2)^{-1}\) in Step (1).

**Step (3)** Let \(r\) be minimal such that \((a_0, \ldots, a_m)\) is an initial factor of \(W_0, \ldots, W_{r-1}\). Observe that \((a_0, \ldots, a_{m+2})\) cannot be an initial factor of \(W_0, \ldots, W_{r-1}\), otherwise the intervals \([a_0, \ldots, a_m]_H, [a_0, \ldots, a_{m+1}]_H\) and \([a_0, \ldots, a_{m+2}]_H\) would share and endpoint, and without loss of generality we would have
\[
\inf[a_0, \ldots, a_m]_H = \inf[a_0, \ldots, a_{m+1}]_H = \inf[a_0, \ldots, a_{m+2}]_H.
\]
Such shared endpoint cannot be \(G \cdot z_k\), since the latter belongs to the interior of \([a_0, \ldots, a_m]_H\) by definition of reduced form. Therefore Step (1) implies \(G \cdot z_k = \sup[a_0, \ldots, a_{m+1}]_H\); and this is absurd because \(G \cdot z_k\) must be an endpoint of \([a_0, \ldots, a_{m+2}]_H\) according to Step (1). Hence \(W_0 \ast \cdots \ast W_{r-1}\) is either equal to \((a_0, \ldots, a_m)\) or to \((a_0, \ldots, a_{m+1})\), and these two cases are treated separately in the last two steps below.

**Step (4)** Assume that \(W_0 \ast \cdots \ast W_{r-1} = (a_0, \ldots, a_m)\). We have \(|W_r| > 0\) because \((a_{m+1}, a_{m+2})\) is an initial factor of \(W_r\), and \(\zeta_0 = \zeta_{W_r}\) because \(G_{b_0,\ldots,b_m} \cdot \zeta_0\) is an endpoint of \([a_0, \ldots, a_{m+2}]_H\). Then the statement follows because Lemma 4.11 gives
\[
\epsilon_0 \geq D(G_{b_0,\ldots,b_m}, \zeta_0)^2 \cdot |\alpha - G_{b_0,\ldots,b_m} \cdot \zeta_0| \\
= D(G_{W_0,\ldots,W_{r-1}}, \zeta_{W_r})^2 \cdot |\alpha - G_{W_0,\ldots,W_{r-1}} \cdot \zeta_{W_r}| \geq \frac{1}{|W_r| + 2\mu},
\]
which implies \(|W_r| \geq (\epsilon_0)^{-1} - 2\mu\).

**Step (5)** Assume that \(W_0 \ast \cdots \ast W_{r-1} = (a_0, \ldots, a_{m+1})\). In this case, since by Step (1) and Step (2) the intervals \([a_0, \ldots, a_{m+1}]_H\) and \([a_0, \ldots, a_{m+2}]_H\) share \(G_{b_0,\ldots,b_m} \cdot \zeta_0\) as common endpoint, then the concatenated word \(W' := (a_{m+1}) \ast W_r\) is also cuspidal (this is allowed according to Remark 3.7). We must have \(|W_r| > 0\), otherwise \(G_{b_0,\ldots,b_m} \cdot \zeta_0\) would not be endpoint of \([a_0, \ldots, a_{m+3}]_H\), so that
\[
|\alpha - G_{b_0,\ldots,b_m} \cdot \zeta_0| \geq |G_{b_0,\ldots,b_m} \cdot \zeta_0 - G_{b_0,\ldots,b_m,a_{m+1},a_{m+2}} \cdot \zeta_3|,
\]
where \(G_{b_0,\ldots,b_m,a_{m+1},a_{m+2}} \cdot \zeta_3\) is the reduced form of the endpoint of \([a_0, \ldots, a_{m+3}]_H\) closest to \(G_{b_0,\ldots,b_m} \cdot \zeta_0\), and reasoning as in Step (2) we would get a contradiction (modulo replacing the constant \(\kappa_2\) by a smaller one, and extending Part (2) of Lemma 4.10 one more step, in order to compare \(D(G_{b_0,\ldots,b_m}, \zeta_0)\) and \(D(G_{b_0,\ldots,b_m,a_{m+1},a_{m+2}}, \zeta_3)\)). Since \(W'\) is cuspidal with \(|W'| > 0\) we have \(\zeta_0 = \zeta_{W'}\) and also \(\zeta_{W'} = G_{a_{m+1}} \cdot \zeta_{W_r}\), which implies
\[
G_{b_0,\ldots,b_m} \cdot \zeta_0 = G_{a_0,\ldots,a_m} \cdot G_{a_{m+1}} \cdot \zeta_{W_r} = G_{W_0,\ldots,W_{r-1}} \cdot \zeta_{W_r}.
\]
Then arguing as in the end of Step (4) one gets \(|W_r| \geq (\epsilon_0)^{-1} - 2\mu\). The Lemma is proved. \(\square\)
4.6. **Proof of Proposition 4.4.** In this section we prove Proposition 4.4. Let \( \epsilon_0 \) be a constant as in Lemma 4.12 and fix any \( \epsilon < \epsilon_0 \). For any \( \alpha \in \mathbb{R} \) consider the cuspidal acceleration of its boundary expansion, that is write \( \alpha = [W_0, W_1, \ldots]_R \).

Consider \( \alpha \in \text{Bad}(\Gamma, \epsilon) \). The definition of \( \text{Bad}(\Gamma, \epsilon) \) and Lemma 4.11 imply that for any \( r \in \mathbb{N} \) we have

\[
\epsilon \leq D(G_{W_0, \ldots, W_{r-1}}, \zeta_{W_r})^2 \cdot |\alpha - G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r}| \leq \frac{1}{|W_r|}.
\]

Thus it follows that \( \alpha \in \varphi^{-1}(\mathbb{E}_{\epsilon, -1}) \). On the other hand, consider \( \alpha \notin \text{Bad}(\Gamma, \epsilon) \) and let \( G \in \Gamma \) and \( \varepsilon_k = A_k \cdot \infty \) such that \( D(G, \varepsilon_k)^2 \cdot |\alpha - G \cdot \varepsilon_k| < \epsilon \). Lemma 4.12 implies that there exists some \( r \in \mathbb{N} \) such that \( G \cdot \varepsilon_k = G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r} \), then for such \( r \) Lemma 4.11 implies

\[
\frac{1}{|W_r|} + 2\mu \leq D(G_{W_0, \ldots, W_{r-1}}, \zeta_{W_r})^2 \cdot |\alpha - G_{W_0, \ldots, W_{r-1}} \cdot \zeta_{W_r}| < \epsilon.
\]

Thus it follows that \( \alpha \notin \varphi^{-1}(\mathbb{E}_{\epsilon, -1 - 2\mu}) \). Proposition 4.4 is proved.

\[\Box\]

5. **Subshift of finite type**

### 5.1. Transition matrix.

Recall the transition matrix \( M_{W', W} \in \{0, 1\}^{W \times W} \) defined by Equation (3.7). Fix \( T > 0 \) and let \( W_T \) be the set of cuspidal words \( W \in \mathcal{W} \) with geometric length \( |W| \leq T \). Equation (3.7) describes also the allowed transitions between the elements of \( W_T \), and we still denote the corresponding (finite) matrix by \( M_{W', W} \). According to Lemma 1.3 in [Bo2], for any \( m \geq 1 \) the entry \( M^m_{W', W} \) of the \( m \)-th power \( M^m \) of \( M \) is the number of different words \( W_0, W_1, \ldots, W_m \) in the letters of \( W_T \) with length \( m + 1 \) with

1. \( M_{W_i, W_{i+1}} = 1 \) for any \( i = 0, \ldots, m - 1 \)
2. \( W_0 = W' \) and \( W_m = W \).

Following § 1 in [Bo2] and § 1 in [ParPo] we say that the matrix \( M \) is **aperiodic** if there exists some \( m \in \mathbb{N} \) such that \( M^m_{W', W} \geq 1 \) for any \( W, W' \in \mathcal{W} \).

**Proposition 5.1.** The matrix \( M \) is aperiodic for any \( T > 0 \) big enough. More precisely, if \( T \) is big enough, we have \( M^2_{W', W} \geq 1 \) for any \( W, W' \in \mathcal{W}_T \).

**Proof.** Recall Lemma 3.5 and fix \( r \geq 1 \), then consider the map \( \chi \mapsto \phi_{(r)}(\chi) \) from \( A \) to itself, where \( \chi' = \phi_{(r)}(\chi) \) is the unique letter such that the word \( (\chi = \chi_0, \ldots, \chi_r = \chi') \) is right cuspidal. The map \( \phi_{(r)} \) is a bijection, because it is injective. Indeed if \( (a_0, \ldots, a_r) \) and \( (b_0, \ldots, b_r) \) are two right cuspidal words with \( a_r = b_r \) then Lemma 3.5 implies that \( (\tilde{a}_r, \ldots, \tilde{a}_0) \) and \( (\tilde{b}_r, \ldots, \tilde{b}_0) \) are left cuspidal with \( \tilde{a}_r = \tilde{b}_r \), thus the two words are equal, because the first letter determines left cuspidal words of fixed length, and hence \( b_0 = a_0 \).

Fix any pair of elements \( W' = (a_0, \ldots, a_n) \) and \( W = (b_0, \ldots, b_m) \) in \( \mathcal{W} \). Assume first that the alphabet \( A \) has at least 6 letters. We show that \( M^2_{W', W} \geq 1 \) showing that there exists a right cuspidal word \( X = [\chi_0, \chi_1] \in \mathcal{W} \) such that \( M_{W', X} = 1 \) and \( M_{X, W} = 1 \), where such \( X \) is determined by the choice of its first letter \( \chi_0 \). According to Equation (3.7), a sufficient (but in general not necessary) condition on \( \chi_0 \) in order to have \( M_{W', X} = 1 \) is

\[
o(\chi_0) - o(\tilde{\alpha}_n) \neq -1, 0, 1,
\]

\[\Box\]
which corresponds to 3 forbidden values for $\chi_0$. Moreover, since $(\chi_0, \chi_1)$ is right cuspidal, condition $M_{W',X} = 1$ is satisfied if and only if

$$o(b_0) - o(\tilde{\chi}_1) \neq -1, 0,$$

which corresponds to 2 forbidden values for $\chi_1$, and thus 2 forbidden values for $\chi_0$, because the map $\phi^{(1)}$ is a bijection. Therefore there are at most 5 forbidden values for $\chi_0$, and thus at least 1 possible choices, which determines a word $X \in \mathcal{W}$ as required.

Assume now that $\mathcal{A}$ has only 4 letters, denoted $a, b, \tilde{a}, \tilde{b}$. The two corresponding generators $F_a, F_b$ of $\Gamma_0$ are either both parabolic (with different fixed point in $\partial \mathbb{D}$) or both hyperbolic (with different axis which intersect transversally). In both cases we find a cuspidal word $X \in \mathcal{W}$ as required considering all possible values for the pair $(a_n, b_0)$, that is all possible values for the last letter of $W'$ and for the first of $W$. More precisely, for any value of $(a_n, b_0)$ we exhibit a concatenation of 3 cuspidal words of the form

$$W' \ast X \ast W = (\ldots, a_n) \ast X \ast (b_0, \ldots).$$

If $F_a, F_b$ are both parabolic, then there are in total 12 different type of cuspidal words: 3 of them start with the letter $a$, that is

$$\underbrace{(a, \ldots, a)}_{n \text{ times}} \ast \underbrace{(a, \tilde{b}, \ldots, a, \tilde{b})}_{n \text{ times}} \ast \underbrace{(a, \tilde{a}, \tilde{b}, \ldots, a, \tilde{b}, a)}_{n \text{ times}},$$

while the other 9 are obtained in the obvious way replacing the role of letters. For $a_n = a$ there are 4 possible values of $b_0$, and the list below gives, for each case, a $X \in \mathcal{W}$ such that the concatenation $W \ast X \ast W'$ satisfies Condition (3.7):

$$(\ldots, a) \ast (b) \ast (a, \ldots)$$

$$((\ldots, a) \ast (b, \tilde{a}, b) \ast (b, \ldots)$$

$$((\ldots, a) \ast (b, b) \ast (\tilde{a}, \ldots)$$

$$(\ldots, a) \ast (b, \tilde{a}) \ast (\tilde{b}, \ldots).$$

For the other 3 values of $a_n$ the analogous list is obtained by obvious substituions and details are left to the reader. Finally, if $F_a, F_b$ are both hyperbolic, the are only two cuspidal sequences, which are

$$a, b, \tilde{a}, \tilde{b}, a, b, \tilde{a}, \tilde{b}, \ldots$$

$$b, a, \tilde{b}, a, b, \tilde{a}, \tilde{b}, \ldots$$

and cuspidal words are any finite subword of the sequences above. For $a_n = a$ there are 4 possible values of $b_0$, and the list below gives, for each case, a $X \in \mathcal{W}$ such that the concatenation $W \ast X \ast W'$ satisfies Condition (3.7):

$$(\ldots, a) \ast (a) \ast (a, \ldots)$$

$$((\ldots, a) \ast (a, b, \tilde{a}) \ast (b, \ldots)$$

$$((\ldots, a) \ast (a, b, \tilde{a}) \ast (\tilde{a}, \ldots)$$

$$(\ldots, a) \ast (a, \tilde{b}, \tilde{a}) \ast (\tilde{b}, \ldots).$$

For the other 3 values of $a_n$ the analogous list is obtained by obvious substituions and details are left to the reader. The Proposition is proved. $\square$
5.2. The space of the sub-shift. Denote by $w = (W_r)_{r \in \mathbb{N}}$ the elements in $\mathcal{W}_T^N$, that is half-infinite sequences in the letters $W \in \mathcal{W}_T$. The shift space is

$$\Sigma := \{w = (W_r)_{r \in \mathbb{N}} : M_{W_r, W_{r+1}} = 1 \quad \forall \quad r \in \mathbb{N}\}.$$

Considering the discrete topology on $\mathcal{W}_T$ and the product topology on $\mathcal{W}_T^N$, we obtain a compact totally disconnected topological space. The shift space $\Sigma$ is a compact subset of $\mathcal{W}_T^N$. Following §1 in [Bo2] and §1 in [ParPo] we fix $\theta$ with $0 < \theta < 1$ and define a distance on $\mathcal{W}_T^N$, and thus on $\Sigma$, by setting

$$d_\theta(w, w') := \theta^N \quad \text{where} \quad N := \max\{n \in \mathbb{N} : W_r = W'_r \quad \forall \quad r = 0, \ldots, n\}$$

and where $w = (W_r)_{r \in \mathbb{N}}$ and $w' = (W'_r)_{r \in \mathbb{N}}$ are any pair of sequences in $\mathcal{W}_T^N$. A basis of open sets for the induced discrete topology is given by the set of cylinders, where for any finite sequence $W_0, \ldots, W_n$ of elements of $\mathcal{W}_T$ such that $M_{W_i, W_{i+1}} = 1$ for any $i = 0, \ldots, n - 1$ the corresponding cylinder is

$$[W_0, \ldots, W_n]_\Sigma := \{w = (W'_r)_{r \in \mathbb{N}} \in \Sigma : W'_r = W_r \quad \forall \quad r = 0, \ldots, n\}.$$

There is a natural dynamics on $\Sigma$ given by the shift map

$$\sigma : \Sigma \to \Sigma \quad ; \quad w = (W_r)_{r \in \mathbb{N}} \mapsto \sigma(w) := (W_{r+1})_{r \in \mathbb{N}}.$$

5.3. Subshift and Cantor sets in the boundary. Fix $T$ with $0 < T < \infty$ and let $\mathbb{E}_T$ be the set introduced in Definition 4.1. Consider the map

$$\Pi : \Sigma \to \partial \mathbb{D} \quad ; \quad w = (W_r)_{r \in \mathbb{N}} \mapsto \Pi(w) := [W_0 * W_1 * \ldots]_\mathbb{D}.$$

Consider the map $F : \partial \mathbb{D} \to \partial \mathbb{D}$ defined in Equation (3.9). Lemma 6.3 below gives a uniform (i.e. depending only on the geometry of $\Omega_\mathbb{D}$ and not on $T$) contraction factor $0 < \theta < 1$ for the inverse branches of $F$. For this specific parameter the distance $d_\theta(\cdot, \cdot)$ reflects some metric properties of $\mathbb{E}_T$, and in particular we get Lemma 5.2 below.

**Lemma 5.2.** The map $\Pi : \Sigma \to \partial \mathbb{D}$ is bijective onto $\mathbb{E}_T$. Moreover $\Pi$ is Lipschitz, but its inverse $\Pi^{-1}$ is just Holder continuous.

**Proof.** Part (1) of Proposition 6.9 implies directly that $\Pi$ is Lipschitz. Holder continuity for $\Pi^{-1} : \mathbb{E}_T \to \Sigma$ follows from the other estimates in Proposition 6.9. Such property of the inverse will not be used in the following, and details are left to the interested reader. \hfill \Box

The map $F$ satisfies Lemma 5.3 below, whose proof is left to the reader.

**Lemma 5.3.** We have a commutative diagram $\Pi \circ \sigma = F \circ \Pi$.

5.4. Cylinders in the Cantor set. Let $(a_0, \ldots, a_n)$ be any admissible word in the letters of $\mathcal{A}$ and recall that

$$[a_0, \ldots, a_n]_\mathbb{D} = F_{a_0, \ldots, a_n} \cdot [a_n]_\mathbb{D} = F_{a_0, \ldots, a_n} \left( \bigcup_{\chi \notin \hat{a}_n} [\chi]_\mathbb{D} \right).$$

Let $F$ be the map in Equation (3.9). Fix any cuspidal word $W = (b_0, \ldots, b_m)$ in $\mathcal{W}$, then set

$$[W]_E := \{\xi \in \partial \mathbb{D} : F(\xi) = F_{W^{-1}}(\xi)\} \in \{\xi \in \partial \mathbb{D} : W_0(\xi) = W\}.$$
that is the arc in $\partial \mathbb{D}$ corresponding to points $\xi$ whose first cuspidal word, given by Equation (3.8), satisfies $W_0(\xi) = W$. We have $[W]\not\in [b_0,\ldots,b_m]\mathbb{D}$. In order to get the expression of such arc, set $\text{Dom}(W) := \{F_{w_0}^{-1}(W_0(\xi)) : W_0(\xi) = W\}$ and observe that

$$W_0(\xi) = W \iff M_{w_0,F_{w_0}^{-1}(\xi)} = 1,$$

that is $\xi \in [W]_E$ if and only if the first letter of the cuspidal word $W_0(F_{w_0}^{-1}(\xi))$ satisfies the Conditions in Equation (3.7). Therefore $^3$

$$\text{Dom}(W) = \begin{cases} 
\bigcup_{o(\chi) - o(b_0) \neq 0,0} [\chi]_E & \text{if } |W| = 0 \\
\bigcup_{o(\chi) - o(b_m) \neq 0,1} [\chi]_E & \text{if } |W| > 0, \quad \epsilon(W) = L \\
\bigcup_{o(\chi) - o(b_{m-1}) \neq 0,1} [\chi]_E & \text{if } |W| > 0, \quad \epsilon(W) = R,
\end{cases}$$

and of course $[W]_E = F_W(\text{Dom}(W))$. Let $w := (W_1,\ldots,W_n)$ be a finite block of cuspidal words $W_k \in \mathcal{W}$ for $k = 1,\ldots,n$ such that

$$W_k = 1 \quad \text{for } k = 0,\ldots,n-1.$$  

Then define $[W_1,\ldots,W_n]_E := \Pi([W_1,\ldots,W_n]_E)$, that is

$$[W_1,\ldots,W_n]_E = \{\xi \in \partial \mathbb{D} : W_0(F_{k-1}(\xi)) = W_k \text{ for } k = 1,\ldots,n\}.$$

We see from the definition that

$$F_{W_1}(\xi) \in [W_1,\ldots,W_n]_E \iff \xi \in [W_2,\ldots,W_n]_E \cap \text{Dom}(W_1).$$

Moreover, for $k = 0,\ldots,n-1$ condition $M_{W_k,W_{k+1}} = 1$ holds if and only if the first letter of $W_{k+1}$ and the last letter of $W_k$ are as in Equation (5.1). Hence we have the equivalence

$$M_{W_k,W_{k+1}} = 1 \iff [W_{k+1}]_E \subset \text{Dom}(W_k),$$

and since $[W_{k+1}]_E = F_{W_{k+1}}(\text{Dom}(W_{k+1}))$, we get by iteration

$$[W_1,\ldots,W_n]_E = F_{W_1,\ldots,W_n}(\text{Dom}(W_n)).$$

For a finite block $w := (W_1,\ldots,W_n)$ as in Equation (5.2) we will also use the notation

$$[w]_E := [W_1,\ldots,W_n]_E \quad \text{and} \quad F_w := F_{W_1,\ldots,W_n}.$$

6. ESTIMATES ON CONTRACTION AND DISTORTION

From now on, unless explicitly stated, constants appearing in contraction and distortion estimate only depend on the geometry of the ideal polygon $\Omega_\mathbb{D}$ and not on the parameter $T > 0$ defining $\mathcal{W}_T$. Such constants are called uniform constants, without assigning a specific name to any of them. The only exception is the contraction factor $\theta$ in Lemma 6.3, which has an important role in spectral properties of transfer operators. All contraction and distortion estimates in this section are consequence of a simple geometric property of cuspidal words, which is the content of Lemma 6.1

$^3$Recall that by definition $|W| = 0$ if $W$ is composed by only one letter, otherwise its geometric length is $|W| > 0$ and $W$ has at least two letters and the symbol $\epsilon(W) \in \{L,R\}$ is defined.
6.1. Distance from the poles. For any admissible word \((a_0, \ldots, a_n)\) in the letters of \(A\) set
\[
U_{a_0, \ldots, a_n} := U_{F_{a_0}, \ldots, a_n},
\]
that is the interior of the isometric circle of the map \(F_{a_0, \ldots, a_n}\) (see §2.3). In other words we have \(|D_F a_{n-1, a_n}| \leq 1\) if and only if \(z \in \mathbb{C}\setminus U_{a_0, \ldots, a_n}\). Observe that \(\partial U_a \cap \mathbb{D} = s_a\) for any letter \(a \in A\). Moreover, according to Equation (2.3), we have
\[
\frac{F^{-1}}{a_n-1, a_n}(U_{a_n}) = U_{a_n, \ldots, a_0} \subset U_{a_0} \subset U_{U_{a_0}}.
\]

**Lemma 6.1.** Consider letters \(b_0, \ldots, b_m\) with \(m \in \mathbb{N}\) such that \((b_0, \ldots, b_m)\) is cuspidal. Let \(a_0\) be any letter such that \((b_0, \ldots, b_m, a_0)\) is not cuspidal. Then
\[
\overline{U_{b_0, \ldots, b_m}} \cap \overline{U_{a_0}} = \emptyset.
\]

**Proof.** Recall the cyclic order \(o : A \to \mathbb{Z}/2d\mathbb{Z}\) introduced in Equation (3.1). Assume first that \(m = 0\). In this case \((b_0, a_0)\) is not cuspidal if and only if we have \(|o(a_0) - o(b_0)| \geq 2\) and this last condition implies
\[
\overline{U_{b_0}} \cap \overline{U_{a_0}} = \emptyset.
\]
If \(m \geq 1\) assume without loss of generality that \((b_0, \ldots, b_m)\) is right cuspidal, the other case being the same. In particular we have \(o(b_m) = o(b_{m-1}) - 1\). Let \(\chi\) be the letter with \(o(\chi) = o(b_m) - 1\), that is the letter such that \((b_{m-1}, b_m, \chi)\) is right cuspidal, then let \(\xi \in \partial \mathbb{D}\) be the tangency point between the discs \(U_\chi\) and \(U_{b_m}\), and therefore \(U_{b_m} \cap U_\chi = \emptyset\). We have
\[
\frac{F^{-1}}{b_m}(U_{b_{m-1}}) \cap \overline{U_\chi} = \{\xi\}.
\]
Equation (2.3) gives \(\frac{F^{-1}}{b_m}(U_{b_{m-1}}) \subset U_{b_{m-1}, b_m} \subset U_{b_m}\), and therefore \(U_{b_{m-1}, b_m}\) and \(U_\chi\) are also tangent at \(\xi\). Finally the inclusion \(U_{b_{m-1}, b_m} \subset U_{b_m}\) is strict, thus we get
\[
\overline{U_{b_{m-1}, b_m}} \subset \{\xi\} \cup U_\chi \cup U_{b_m}.
\]
On the other hand \((b_{m-1}, b_m, a_0)\) is not right cuspidal, thus \(a_0 \neq b_{m-1}, \chi\), that is the disc \(U_{a_0}\) is different from \(U_\chi\) and from \(U_{b_0}\), so that we get
\[
\overline{U_{b_{m-1}, b_m}} \cap \overline{U_{a_0}} = \emptyset.
\]
Then the statement follows because \(U_{b_0, \ldots, b_m} \subset U_{b_{m-1}, b_m}\) according to Equation (2.3). □

**Lemma 6.2.** There exists a uniform positive constant \(C > 1\) such that for any finite block \(w_k := (W_1, \ldots, W_k)\) as in Equation (5.2) and any \(\xi \in \text{Dom}(W_k)\) the distance from the pole \(\omega = \omega(F_{w_k})\) of \(F_{w_k}\) satisfies
\[
C^{-1} \leq |\xi - \omega| \leq C.
\]

**Proof.** By definition of isometric circle, the pole \(\omega\) belongs to \(U_{W_1, \ldots, W_k}\), moreover Equation (2.3) implies \(U_{W_1, \ldots, W_k} \subset U_{W_k}\). Set \(W_k = (b_0, \ldots, b_m)\) and consider any \(a_0 \in A\) such that \([a_0]_E \subset \text{Dom}(W_k)\). If \(m \geq 1\) we have
\[
\overline{U_{W_k}} = U_{b_0, \ldots, b_m} \subset U_{b_{m-1}, b_m} \quad \text{and} \quad \overline{U_{b_{m-1}, b_m}} \cap \overline{U_{a_0}} = \emptyset
\]
onlyp otherwise if \(m = 0\) we have
\[
\overline{U_{W_k}} = U_{b_0} \quad \text{and} \quad \overline{U_{b_0}} \cap \overline{U_{a_0}} = \emptyset.
\]
where in both cases the intersection is empty according to Lemma 6.1. Since the sequences \((b_0, b_1, a_0)\) and \((b_0, a_0)\) as in the two cases above vary in a finite set, then there exists an uniform constant \(C > 0\) such that, for any \(w_k = (W_1, \ldots, W_k)\) as above we have
\[
\text{Dist}(U_{W_1 \ast \ldots \ast W_k}, U_{a_0}) \geq C^{-1}.
\]
The lower bound for \(|\xi - \omega|\) follows. The upper bound holds trivially because in any infinite discrete subgroup of \(\text{SU}(1, 1)\) we have \(|\omega_F| \to 1\) as \(\|F\| \to \infty\), that is the poles \(\omega_F\) of the maps \(z \mapsto F(z)\) accumulate to the unit circle. \(\square\)

6.2. Uniform contraction.

**Lemma 6.3.** For any \(T > 0\) there exists a positive constant \(\kappa = \kappa(T) > 0\) such that for any \(W \in \mathcal{W}_T\) and any \(\xi \in \partial \mathbb{D}\) we have
\[
|D_\xi F_W| \geq \kappa.
\]

**Proof.** Fix \(W_T \in \mathcal{W}_T\). In the notation of Equation (2.2), the derivative of the map \(F_W\) is
\[
|D_\xi F_W| = \frac{1}{|\beta|^2 \cdot |\xi - (-\overline{\alpha}/\beta)|^2},
\]
where in particular \(\omega_{F_W} = -\overline{\alpha}/\beta\) is the pole of \(F_W\), and \(\alpha = \alpha(F_W)\) and \(\beta = \beta(F_W)\). Finiteness of \(\mathcal{W}_T\) gives an upper bound on \(|\beta|\), thus the Lemma follows from Lemma 6.2. \(\square\)

**Lemma 6.4.** For any admissible word \((a_0, \ldots, a_n)\) and any \(z \in \mathbb{D} \setminus U_{a_n}\) we have
\[
|D_z F_{a_0, \ldots, a_n}| \leq 1.
\]

**Proof.** For \(n = 0\) the statement corresponds to the definition: we have \(|D_z F_a| \leq 1\) for any \(a \in \mathcal{A}\) and any \(z \in \mathbb{D} \setminus U_a\). For general \(n \in \mathbb{N}\), the inequality follows by induction observing that
\[
D_z F_{a_0, \ldots, a_n} = D_{F_{a_n}(z)} F_{a_0, \ldots, a_{n-1}} \cdot D_z F_{a_n}
\]
and that admissibility, that is \(a_n \neq \widehat{a_{n-1}}\), implies
\[
F_{a_n}(\mathbb{D} \setminus U_{a_n}) = \mathbb{D} \cap U_{\widehat{a_n}} \subset \mathbb{D} \setminus U_{a_{n-1}}.
\]
\(\square\)

**Lemma 6.5.** There exists an uniform constant \(0 < \theta < 1\) such that for any \(W \in \mathcal{W}\), any \(a_0 \in \mathcal{A}\) with \([a_0]_{\mathbb{D}} \subset \text{Dom}(W)\) and any \(\xi \in U_{a_0} \cap \mathbb{D}\) we have
\[
|D_\xi F_W| \leq \theta.
\]

*In particular the property above holds for any \(\xi \in \text{Dom}(W)\).*

**Proof.** Let \(\mathcal{V}_1\) be the set of words \((b_0, a_0)\) which are admissible and not cuspidal, then set
\[
\theta_1 := \sup \{|D_\xi F_{b_0}| : (b_0, a_0) \in \mathcal{V}_1, \xi \in U_{a_0}\}.
\]
We have \(0 < \theta_1 < 1\) according to Lemma 6.1 where we recall that \(|D_\xi F_{b_0}| < 1\) if and only if \(\xi \in \mathbb{C} \setminus U_{b_0}^\circ\). Similarly, let \(\mathcal{V}_2\) be the set of admissible words \((b_0, b_1, a_0)\) such that \((b_0, b_1)\) is cuspidal but \((b_0, b_1, a_0)\) is not cuspidal. By a similar argument, we have \(0 < \theta_2 < 1\), where
\[
\theta_2 := \sup \{|D_\xi F_{b_0, b_1}| : (b_0, b_1, a_0) \in \mathcal{V}_2, \xi \in U_{a_0}^\circ\}.
\]
In the general case consider \(W = (b_0, \ldots, b_m)\). If \(m = 0\) we have \(|D_\xi F_W| = |D_\xi F_{b_0}|\), which is bounded by \(\theta_1\). Otherwise, if \(m \geq 1\), the chain rule applied to \(|D_\xi F_W|\) gives the factor
\[|D_\xi F_{\phi_{m-1},\phi_m}|, \text{ which is bounded by } \theta_2, \text{ and an other factor which is } \leq 1 \text{ because of Lemma 6.4}\]

Hence in general the statement holds with \( \theta := \max\{\theta_1, \theta_2\} \).

\[\square\]

**Corollary 6.6.** Consider a block of cuspidal words \( w_k = (W_1, \ldots, W_k) \) as in Equation (5.2), for some integer \( k \geq 1 \). Then for any \( a \in \mathcal{A} \) with \( [a]_\mathcal{D} \subset \text{Dom}(W_k) \) and any \( \xi', \xi \) in \([a]_\mathcal{D}\) we have

\[|F_{w_k}(\xi') - F_{w_k}(\xi)| \leq \theta^k|\xi' - \xi|.
\]

**Proof.** Define \( \gamma : [0, 1] \to \mathbb{C} \) by \( \gamma(t) := \xi + t(\xi' - \xi) \). In particular \( \gamma(t) \in U_\alpha \) for any \( t \in [0, 1] \) and \([a]_\mathcal{D} \subset \text{Dom}(W_k) \), so that \( |D_\xi F_{W_k}| \leq \theta \), according to Lemma 6.5. If \( k \geq 2 \), for \( l = 2, \ldots, k \), let \( a_l \) be the first letter of \( W_l \). We have \([W_l]_\mathcal{D} \subset [a_l]_\mathcal{D} \), but on the other hand \([W_l]_\mathcal{D} \subset \text{Dom}(W_{l-1}) \), according to Equation (5.3). Therefore \([a_l]_\mathcal{D} \subset \text{Dom}(W_{l-1}) \). Moreover Equation (6.1) implies

\[F_{W_1,\ldots,W_k}(\gamma(t)) \in F_{W_1,\ldots,W_k}(U_\alpha) \subset U_\alpha.
\]

We get \( |D_\gamma(t)F_{W_1,\ldots,W_k}| \leq \theta^k \) factoring the derivative with the chain rule and applying Lemma 6.5 to each factor. Finally the statement follows observing that

\[|F_{w_k}(\xi') - F_{w_k}(\xi)| \leq \int_0^1 |d/dtF_{W_k}(\gamma(t))|dt = \int_0^1 |D_\gamma(t)F_{w_k}| \cdot |d\gamma(t)/dt|dt \leq \theta^k \cdot |\xi' - \xi|.
\]

\[\square\]

### 6.3. Distortion of the derivative

Referring to § 5.4, consider blocks of cuspidal words \( w_k = (W_1, \ldots, W_k) \) as in Equation (5.2), and for any such \( w_k \) set

\[(6.2) \quad \|DF_{w_k}\|_\infty := \sup_{\xi \in \text{Dom}(W_k)} |D_\xi F_{w_k}|.
\]

**Lemma 6.7.** There exists an uniform constant \( C > 0 \) such that for any \( 0 < T \leq +\infty \) and \( 0 < s \leq 1 \), any \( a \in \mathcal{A} \), \( k \in \mathbb{N} \) and any finite block \( w_k = (W_1, \ldots, W_k) \) as in Equation (5.2), the map

\[\phi : \text{Dom}(W_k) \to \mathbb{R}_+ ; \quad \phi(\xi) := s \ln |D_\xi F_{w_k}|
\]

is Lipschitz with \( \text{Lip}(\phi) < C \).

**Proof.** In the notation of Equation (2.2), let \( \alpha \) and \( \beta \) with \(|\alpha|^2 - |\beta|^2 = 1\) such that

\[F_{w_k}(\xi) = \frac{\alpha \xi + \beta}{\beta \xi + \alpha} \quad \text{and} \quad D_\xi F_{w_k} = \frac{1}{\beta^2 \cdot (\xi - \omega)^2},
\]

where \( \omega := -\alpha/\beta \) is the pole of \( F_{w_k} \). We have

\[\phi(\xi) = \log(\|D_\xi F_{w_k}\|^s) = -2s\left(\ln|\beta| + \ln|\xi - \omega|\right),
\]

where we recall that \( \beta(F) \neq 0 \) for any \( F \in \Gamma_0 \), since the latter is free and thus in particular does not have (finite order) elliptic fixed points. Therefore for any \( \xi, \xi' \in \text{Dom}(W_k) \) we have

\[|\phi(\xi') - \phi(\xi)| = 2s \left|\ln \frac{\xi' - \omega}{\xi - \omega}\right| = 2s \left|\ln\left|1 + \frac{\xi' - \xi}{\xi - \omega}\right|\right|.
\]

Observe that if \( z \in \mathbb{C} \) and \( \epsilon > 0 \) are such that \(|1 + z| > \epsilon\) then \(|\ln|1 + z|| \leq \epsilon^{-1} \cdot |z|\). Moreover Lemma 6.2 implies

\[1 + \frac{\xi' - \xi}{\xi - \omega} = \frac{|\xi' - \omega|}{|\xi - \omega|} \geq \frac{1}{C^2},
\]
where here \( C_1 > 1 \) denotes the uniform constant in Lemma 6.2. Therefore the Lemma follows because from the observation above we get
\[
|\phi(\xi') - \phi(\xi)| = 2s \left| \ln \left| 1 + \frac{\xi' - \xi}{\xi - \omega} \right| \right| \leq 2sC_1^2 \cdot \left| \frac{\xi' - \xi}{\xi - \omega} \right| \leq 2C_1^3 \cdot |\xi' - \xi|.
\]

\[\Box\]

**Corollary 6.8.** There exists an uniform constant \( C > 0 \) such that for any \( 0 < T \leq +\infty \), any \( k \in \mathbb{N} \) and any finite block \( w_k = (W_1, \ldots, W_k) \) as in Equation (5.2) the following holds.

1. For any \( \xi', \xi \in \text{Dom}(W_k) \) we have
\[
\frac{1}{C} \leq \frac{|D_{\xi'}F_{w_k}|}{|D_{\xi}F_{w_k}|} \leq C.
\]

2. For any \( 0 < s \leq 1 \) and any \( \xi', \xi \in \text{Dom}(W_k) \) we have
\[
\|D_{\xi'}F_{w_k}^s - D_{\xi}F_{w_k}^s\| \leq C \cdot \|DF_{w_k}\|_s \cdot |\xi' - \xi|.
\]

**Proof.** The first part of the statement follows considering the expression of \( F_{w_k} \) given by Equation (2.2) and observing that the ratio of the derivative at different points is bounded by Lemma 6.2. In the proof of the second part we use the same notation as in the proof of Lemma 6.7. In particular, according to the Lemma we have \( \phi(\xi') \leq \phi(\xi) + C' \cdot |\xi' - \xi| \), where \( C' \) denotes the uniform constant in Lemma 6.7. Thus we have
\[
e^{\phi(\xi')} - e^{\phi(\xi)} \leq e^{\phi(\xi)} + C' \cdot |\xi' - \xi| - e^{\phi(\xi)} \leq e^{\phi(\xi)} \left( e^{C' \cdot |\xi' - \xi|} - 1 \right)
\]
\[
\leq e^{\phi(\xi)} \cdot e^{C' \cdot |\xi' - \xi|} \cdot C' \cdot |\xi' - \xi| = \|D_{\xi'}F_{w_k}^s\| \cdot e^{C' \cdot |\xi' - \xi|} \cdot C' \cdot |\xi' - \xi|
\]
\[
\leq \|DF_{w_k}\|_s \cdot e^{C' \cdot |\xi' - \xi|} \cdot C' \cdot |\xi' - \xi| \leq C'' e^{2C'} \cdot \|DF_{w_k}\|_s \cdot |\xi' - \xi|,
\]

where the third equality holds because \( e^{R} - 1 \leq Re^R \) for any \( R \geq 0 \) and the last inequality uses that \( |\xi' - \xi| \leq 2 \) for any \( \xi, \xi' \) in \( \partial \mathbb{D} \). The statement follows because by symmetry the same bound holds for \( e^{\phi(\xi)} - e^{\phi(\xi')} \).

\[\Box\]

### 6.4. Sizes of cylinders and gaps in the Cantor set.

Fix any letter \( a_0 \in A \) and define \( \Xi^{(0)}(a_0) \) as the set whose elements are \( \xi_{a_0}^L, \xi_{a_0}^R \) and all the parabolic fixed points of the form
\[
F_{a_0,a_1,\ldots,a_{k-1}} \cdot \xi_{a_k}^L ; \quad W := (a_0, a_1, \ldots, a_{k-1}, a_k) \text{ left cuspidal} ; \quad 0 < |W| \leq T
\]
\[
F_{a_0,a_1,\ldots,a_{k-1}} \cdot \xi_{a_k}^R ; \quad W := (a_0, a_1, \ldots, a_{k-1}, a_k) \text{ right cuspidal} ; \quad 0 < |W| \leq T.
\]

Then set \( \Xi^{(0)} := \bigcup_{a_0 \in A} \Xi^{(0)}(a_0) \), where the union is not disjoint because for consecutive letters \( a, b \), that is letters with \( o(b) = o(a) + 1 \), we have \( \Xi^{(0)}(a) \cap \Xi^{(0)}(b) = \{\xi_a^R\} = \{\xi_b^L\} \). Recalling \( \S 5.4 \), we see that the elements of \( \Xi^{(0)} \) are the endpoints of intervals of the form \([W]_\Xi\), with \( W \in W_T \).

For \( n \geq 1 \) let \( \Xi^{(n)} \) be the set of parabolic fixed points of the form \( F_{W_0,\ldots,W_{n-1}} \cdot \xi \), where \( W_0, \ldots, W_{n-1} \) are words as in Equation (5.2), \( a_0 \) is a letter such that \( W_{n-1} \ast (a_0) \) is not cuspidal, and \( \xi \in \Xi^{(0)}(a_0) \). The elements of \( \Xi^{(n)} \) are the endpoints of intervals of the form \([W_0,\ldots,W_{n-1},W_n]_\Xi\), where \( w_{n+1} = (W_0, \ldots, W_{n-1}, W_n) \) is a block of words as in Equation (5.2).

The gaps of level \( n = 0 \) are the connected components of \( \partial \mathbb{D} \setminus \Xi_T \) which contain a parabolic fixed point of \( \Xi^{(0)} \). For \( n \geq 1 \) the gaps of level \( n \) are the connected components of \( \partial \mathbb{D} \setminus \Xi_T \) which contain a parabolic fixed point of \( \Xi^{(n)} \setminus \bigcup_{l=0}^{n-1} \Xi^{(l)} \). For any such endpoint \( \xi \) denote

\[30\]
\(B_\xi^{(n)}\) the corresponding gap. The complement of all gaps \(B_\xi^{(l)}\) of level \(l = 0, \ldots, n\) are closed intervals, which are in bijection with the set of finite sequences \(w_n = (W_0, \ldots, W_n)\) as in Equation (5.2). The interval corresponding to any such \(w_n\) is the smallest closed interval which contains \(E_T \cap [W_0, \ldots, W_n]_E\).

Consider words \(W_0, \ldots, W_n\) as in Equation (5.2) and an interval \(I \subset \text{Dom}(W_n)\). Parametrizing it by arc length \([0, |I|] \to I, t \mapsto \xi(t)\) we get

\[
|F_{W_0,\ldots,W_n}(I)| = \int_0^{|I|} |d/dt (F_{W_0,\ldots,W_n} \circ \xi(t))| dt = \int_0^{|I|} |D_\xi F_{W_0,\ldots,W_n}| dt,
\]

and thus

\[
(6.3) \quad |I| \cdot \inf_{\xi \in I} |D_\xi F_{W_0,\ldots,W_n}| \leq |F_{W_0,\ldots,W_n}(I)| \leq |I| \cdot \sup_{\xi \in I} |D_\xi F_{W_0,\ldots,W_n}|.
\]

Equation (6.3) refers to arc length of segments in \(\partial \mathbb{D}\). But for short arcs the arc length which is comparable with the diameter (as subsets of \(\mathbb{C}\)). For this reason the estimates in Proposition 6.9 below, whose proof uses Equation (6.3), also hold for the diameter (modulo slightly changing the constants).

**Proposition 6.9.** There is an uniform constant \(C > 0\) such that for any finite block of words \(w_k = (W_1, \ldots, W_k)\) as in Equation (5.2) the following holds.

1. The cylinder \([W_1, \ldots, W_k]_E\) has size

\[
|([W_1, \ldots, W_k]_E) < C \cdot \theta^k.
\]

2. For any \(0 \leq m \leq k\) we have

\[
|([W_1, \ldots, W_k]_E) \leq C \cdot \theta^{k-m} \cdot |([W_1, \ldots, W_m]_E)
\]

Moreover for any \(T\) such that \(0 \leq T < +\infty\) strictly there exists a constant \(\kappa = \kappa(T) > 0\) such that the following holds too.

3. We have

\[
|([W_1, \ldots, W_k]_E) \geq \kappa \cdot |([W_1, \ldots, W_{k-1}]_E).
\]

4. If \(B_\xi^{(k+1)}\) is a gap of level \(k + 1\) with \(B_\xi^{(k+1)} \subset [W_1, \ldots, W_k]_E\) we have

\[
|B_\xi^{(k+1)}| \geq \kappa \cdot |([W_1, \ldots, W_k]_E).
\]

**Proof.** Part (1) is a direct consequence of Corollary 6.6. In order to prove Part (2) set \(E := [W_{m+1}, \ldots, W_k]_E \subset \text{Dom}(W_m)\). We have

\[
|([W_1, \ldots, W_k]_E) = |F_{W_0,\ldots,W_n}(E)| \leq |E| \cdot \sup_{\xi \in E} |D_\xi F_{W_1,\ldots,W_m}|
\]

\[
\leq C \cdot \theta^{k-m} \cdot \sup_{\xi \in E} |D_\xi F_{W_1,\ldots,W_m}|
\]

\[
\leq C \cdot \theta^{k-m} \cdot C' \cdot \inf_{\xi \in \text{Dom}(W_m)} |D_\xi F_{W_1,\ldots,W_m}|
\]

\[
\leq \frac{C \cdot \theta^{k-m} \cdot C'}{|\text{Dom}(W_m)|} \cdot |\text{Dom}(W_m)| \cdot \inf_{\xi \in \text{Dom}(W_m)} |D_\xi F_{W_1,\ldots,W_m}|
\]

\[
\leq \frac{C \cdot \theta^{k-m} \cdot C'}{|\text{Dom}(W_m)|} \cdot |([W_1, \ldots, W_m]_E),
\]

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where the first and the last inequalities follow from Equation (6.3), the second from Part (1) of this Proposition and the third from Part (1) of Corollary 6.8 (where \( C' \) here denotes the constant in the Corollary). Then the required estimate follows observing that \( \lVert \text{Dom}(W_m) \rVert \) has length uniformly bounded from below, and increasing \( C \) to a proper larger constant.

In order to prove Part (4) observe that \( B_\xi^{(k+1)} = F_{W_0,\ldots,W_k}(B_\xi^{(k)}) \), where \( \xi' \in \Xi^{(k)}(a_0) \) for some letter \( a_0 \) such that \( W_k \ast (a_0) \) is not cuspidal. Set \( B := B_\xi^{(0)} \) and \( D := \text{Dom}(W_k) \), where obviously \( B \subset D \). Equation (6.3) and Part (1) of Corollary 6.8 (where \( C' \) here denotes the constant in the Corollary) give

\[
|B_\xi^{(k+1)}| \geq |B| \cdot \inf_{\xi \in B} |D_\xi F_{W_1,\ldots,W_k}| \geq \frac{|B|}{C'} \cdot \sup_{\xi \in D} |D_\xi F_{W_1,\ldots,W_k}| = \frac{|B|}{C'} \cdot |D| \cdot \sup_{\xi \in D} |D_\xi F_{W_1,\ldots,W_k}| \geq \frac{|B|}{C'} \cdot |D| \cdot |[W_1,\ldots,W_k]|. 
\]

Part (4) follows observing that \( |B| \) is bounded from below because gaps of level \( n = 0 \) are finitely many, and that \( |D| < 2\pi \). Part (3) follows with a similar argument, observing that

\[
[W_1,\ldots,W_k] = F_{W_1,\ldots,W_{k-1}}([W_k]_r),
\]

and that by finiteness, intervals \([W]_r\) with \( W \in \mathcal{W}_T \) have length bounded from below. The Proposition is proved. 

\[\square\]

7. Transfer operator and dimension

7.1. The Theorem of Ruelle-Perron-Frobenius. Let \( \sigma : \Sigma \to \Sigma \) be the shift map and \( d_\theta(\cdot,\cdot) \) be the distance on \( \Sigma \) introduced in § 5.2 where \( 0 < \theta < 1 \) is the uniform constant in Lemma 6.5. Let \( C(\Sigma) \) be the Banach space of continuous functions \( f : \Sigma \to \mathbb{C} \) with norm

\[
\|f\|_\infty := \sup_{w \in \Sigma} |f(w)|.
\]

Following § 1 in [Bo2] and § 2 in [ParPo], for a fixed \( \varphi \in C(\Sigma) \) (also said potential) we consider the transfer operator \( \mathcal{L}_\varphi : C(\Sigma) \to C(\Sigma) \), also known as Ruelle operator (see [Ru]), which takes any \( f \in C(\Sigma) \) into the function \( \mathcal{L}_\varphi f \in C(\Sigma) \) given by

\[
(\mathcal{L}_\varphi f)(w) := \sum_{\sigma(w')=w} e^{\varphi(w')} f(w').
\]

The space \( \mathcal{M}(\Sigma) \) of Borel probability measures on \( \Sigma \) is identified with the set of \( \mu \) in the dual space \( C(\Sigma)^* \) such that \( \mu(1) = 1 \) and \( \mu(f) \geq 0 \) if \( f \) is real and positive. The dual operator \( \mathcal{L}_\varphi^* : C(\Sigma)^* \to C(\Sigma)^* \) acts sends the Borel probability measure \( \mu \) to the continuous functional \( \mathcal{L}_\varphi^* \mu \) which acts on any \( f \in C(\Sigma) \) as

\[
(\mathcal{L}_\varphi^* \mu)(f) := \int (\mathcal{L}_\varphi f) \, d\mu.
\]

According to § 1 in [Bo2] and § 1 in [ParPo], for any \( f \in C(\Sigma) \) and for any \( n \in \mathbb{N} \) define

\[
\text{Var}_n(f) := \sup \{|f(w) - f(w')| : W_r = W_r' \quad \forall \quad r = 0,\ldots,n\}
\]

Let \( \text{Lip}(\Sigma, \theta) \subset C(\Sigma) \) be the subspace of Lipschitz functions \( f : \Sigma \to \mathbb{C} \) with respect to the distance \( d_\theta \), which is a Banach space for the norm

\[
\|f\|_\theta := \|f\|_\infty + \text{Lip}(f).
\]
In the notation above, \( \text{Lip}(\Sigma, \theta) \) is the space of functions \( f \) such that there exist \( C > 0 \) with
\[
\text{Var}_n(f) \leq C \cdot \theta^n \quad \forall \quad n \in \mathbb{N}.
\]
The Lipschitz constant of such \( f \) is obviously \( \text{Lip}(f) = C \). For a potential \( \varphi \in \text{Lip}(\Sigma, \theta) \), the operator \( \mathcal{L}_\varphi \) in Equation (7.1) acts as a bounded linear operator on \( \text{Lip}(\Sigma, \theta) \) (see \cite{ParPo} at page 19, or \cite{Ba} at page 30).

Fix a continuous real-valued function \( \varphi : \Sigma \to \mathbb{R} \). A probability (non necessarily \( \sigma \)-invariant) measure \( m \) on \( \Sigma \) is a Gibbs measure if there exist constants \( P = P(\varphi) \in \mathbb{R} \) and \( C = C(\varphi) > 1 \) such that for any \( w = (W_r)_{r \in \mathbb{N}} \in \Sigma \) and any \( n \in \mathbb{N} \) we have
\[
\frac{1}{C} \leq \frac{m([W_0, \ldots, W_{n-1}])}{\exp (-nP + \sum_{k=0}^{n-1} \varphi(\sigma^k(w)))} \leq C.
\]
The real constant \( P = P(\varphi) \) above is called the pressure for the function \( \varphi \). Theorem 7.1 below is known as Ruelle-Perron-Frobenius Theorem. It corresponds to Theorem 2.2 and Corollary 3.2.1 in \cite{ParPo} or Theorem 1.7 and Theorem 1.16 in \cite{Bo2} (some basic notions of spectral theory of bounded linear operators are recalled in \S 3).

**Theorem 7.1** (Ruelle-Perron-Frobenius). Let \( \sigma : \Sigma \to \Sigma \) be a subshift of finite type determined by an aperiodic matrix \( M \). For a positive real-valued potential \( \varphi \in \text{Lip}(\Sigma, \theta) \) the following holds.

1. There exists a simple real eigenvalue \( \lambda > 0 \) for \( \mathcal{L}_\varphi : C(\Sigma) \to C(\Sigma) \), which corresponds to a real eigenfunction \( h \in \text{Lip}(\Sigma, \theta) \) with \( h(w) > 0 \) strictly for any \( w \in \Sigma \) (\cite{ParPo}, Theorem 2.2-(i)).

2. The remainder of the spectrum of \( \mathcal{L}_\varphi : \text{Lip}(\Sigma, \theta) \to \text{Lip}(\Sigma, \theta) \) is contained in the ball \( B(0, \rho) \subset \mathbb{C} \) for some \( \rho < \lambda \) (\cite{ParPo}, Theorem 2.2-(ii)).

Moreover set \( \psi := \varphi - \ln h \circ \sigma + \ln h - \ln \lambda \), in terms of the eigenfunction and eigenvalue \( h \) and \( \lambda \) above, and let \( \mathcal{L}_\psi \) be the corresponding operator.

3. There exists an unique \( \sigma \)-invariant Borel probability measure \( m \) on \( \Sigma \) such that \( \mathcal{L}_\psi^*(m) = m \), moreover \( m \) is a Gibbs measure for \( \varphi \) with pressure \( P(\varphi) = \ln \lambda \) (\cite{ParPo}, Corollary 3.2.1).

Finally consider the measure \( dm(w) := h^{-1}(w)dm(w) \) with normalization \( \int_H hd\tilde{m} = 1 \).

4. We have \( \lambda^{-n} \cdot \mathcal{L}_\varphi^n f \to (\int f dm) \cdot h \) uniformly for any \( f \in C(\Sigma) \) (\cite{ParPo}, Theorem 2.2-(iv)).

### 7.2. Potential from boundary expansion.

Recall the map \( \Pi : \Sigma \to \partial \mathbb{D} \) introduced in \S 5.2. For any sequence \( w = (W_r)_{r \in \mathbb{N}} \) denote by \( W_0(w) \) the first letter of \( w \). Consider the continuous function \( \varphi_T : \Sigma \to \mathbb{R} \) defined by
\[
\varphi_T(w) := - \ln |D_{\Pi(w)}F_{W_0(w)}^{-1}| = \ln |D_{\Pi(\sigma w)}F_{W_0(w)}|,
\]
where \( \varphi_T \) depends on \( T \) because the alphabet of the sub-shift \( \sigma : \Sigma \to \Sigma \) is the set \( \mathcal{W}_T \) of cuspidal words with \( |W| \leq T \). The second equality in Equation (7.2) follows observing that we have \( D_{\xi'}F_{W_0} = (D_{\xi}F_{W_0}^{-1})^{-1} \) for any \( W_0 \in \mathcal{W} \) and \( \xi \), where \( \xi' := F_{W_0}^{-1}(\xi) \), and that \( F_{W_0(w)}^{-1}(\Pi(w)) = \Pi(\sigma(w)) \) for any \( w \in \Sigma \), according to Lemma 5.3 (and applying the resulting identity to \( \xi := \Pi(w) \)). It is practical to develop the computation in Lemma 7.2 below.
Lemma 7.2. For any $w = (W_0, W_1, \ldots) \in \Sigma$ and any $n \in \mathbb{N}^*$ let $\xi_w := \Pi(\sigma^n w)$. Then we have $\xi_w \in \text{Dom}(W_{n-1})$ and
\[
\exp \left( \sum_{k=0}^{n-1} \varphi_T(\sigma^k w) \right) = |D_{\xi_w} F_{W_0, \ldots, W_{n-1}}|.
\]

Proof. The first statement holds because for any $k = 0, \ldots, n-1$ we have obviously
\[\Pi \circ \sigma^n \circ w = [W_{n-k}, W_{n-k+1}, \ldots] \in [W_{n-k}]_E \subset \text{Dom}(W_{n-k}).\]
In particular $W_0(\Pi \circ \sigma^n \circ w) = W_{n-k}$. Hence, if $F$ is the map defined in Equation (3.9), we have
\[\xi_w = F^k(\Pi \circ \sigma^n \circ w) = F_{W_{n-1}}^{-1} \circ \cdots \circ F_{W_{n-k}}^{-1}(\Pi \circ \sigma^n \circ w)\].
Then the Lemma follows because the chain rule gives
\[
\exp \left( \sum_{k=0}^{n-1} \varphi_T(\sigma^k w) \right) = |D_{\Pi \circ \sigma^n \circ w} F_{W_{n-1}}| \cdot \cdots \cdot |D_{\Pi \circ \sigma \circ w} F_{W_0}| = \left| D_{\xi_w} F_{W_0, \ldots, W_{n-1}} \right|.
\]

Theorem [7.41] can be applied to the function $\varphi_T$, according to Lemma [7.3] below, which corresponds to Lemma 4 in [Bo1].

Lemma 7.3. For any $T$ such that $0 < T < +\infty$ strictly there exists a constant $C = C(T) > 0$ such that for any $n \in \mathbb{N}$ we have
\[\text{Var}_n(\varphi_T) \leq C \theta^n.\]

Proof. Observe that if $w = (W_0, W_1, \ldots)$ then $W_0 = W_0(w)$ and $\Pi(\sigma w) \in [W_1]_E \subset \text{Dom}(W_0)$. The Lemma follows from Lemma [6.7] (applied for blocks of length $k = 1$) and Lemma [5.2].

7.3. Dimension of Cantor set. For a fixed subset $E \subset \mathbb{C}$ and for $\rho > 0$, a $\rho$-cover of $E$ is a countable collection \( \{B_i : i \in I\} \) of balls $B_i$ with diameter $|B_i| \leq \rho$ for each $i$ such that $E \subset \bigcup_i B_i$. Such a cover exists for every $\rho > 0$. Fix $s$ with $0 \leq s \leq 1$ and define
\[H^s_\rho(E) := \inf \sum_{i \in I} |B_i|^s,\]
where the infimum is taken over all $\rho$-covers of $E$. The Hausdorff $s$-measure $H^s(E)$ of $E$ is defined by
\[H^s(E) := \lim_{\rho \to 0} H^s_\rho(E) = \sup_{\rho > 0} H^s_\rho(E).\]
The Hausdorff dimension $\text{dim}_H E$ of a set $E$ is defined by
\[\text{dim}_H(E) := \inf \{s : H^s(E) = 0\} = \sup \{s : H^s(E) = \infty\}.\]
The following Lemma is a classical fact, for a proof see Theorem 5.7 in [Mat].

Lemma 7.4. Let $\nu$ be a probability measure on $\mathbb{C}$ and let $E := \text{Supp}(\nu)$ be its support. Assume that there exists $s > 0$, $C > 1$ and $r_0 > 0$ such that for any $r < r_0$ and any $x \in E$ we have
\[\frac{1}{C} \cdot r^s \leq \nu(B(x,r)) \leq C \cdot r^s.\]
Then we have \( C^{-1} \leq H^s(E) \leq 5^s \cdot C \) and in particular \( \dim_H(E) = s \).

Consider the function \( \varphi_T \) defined by Equation (7.2). For any \( 0 \leq s \leq 1 \) consider the transfer operator \( \mathcal{L}_{s,T} : \text{Lip}(\Sigma, \theta) \rightarrow \text{Lip}(\Sigma, \theta) \) in Equation (7.1) corresponding to the potential \( s \cdot \varphi_T \), that is the operator acting on \( f \in \text{Lip}(\Sigma, \theta) \) by

\[
(\mathcal{L}_{s,T} f)(w) := \sum_{\sigma(w') = w} \frac{1}{D_{\Pi(w)} F_{W_0(w')}^{s-1}} f(w') = \sum_{\sigma(w') = w} |D_{\Pi(w)} F_{W_0(w')}|^s f(w'),
\]

where the second inequality in the definition of \( \mathcal{L}_{s,T} \) is direct consequence of the second inequality in Equation (7.2). Let \( \lambda(s,T) > 0 \) be the maximal eigenvalue of \( \mathcal{L}_{s,T} \) as in Point (1) of Theorem 7.1. Let also \( m_{s,T} \) be the Gibbs measure associated to \( \mathcal{L}_{s,T} \) as in Point (3) of Theorem 7.1. Finally consider the measure \( \nu_{s,T} := \Pi_s(m_{s,T}) \) on \( \partial \mathbb{D} \), that is \( \nu_{s,T}(I) := m_{s,T}(\Pi^{-1}(I)) \) for any Borel set \( I \subset \partial \mathbb{D} \). In particular for any cylinder \( \nu_{s,T}([W_0, \ldots, W_n]_\Sigma) = m_{s,T}([W_0, \ldots, W_n]_\Sigma) \).

The next Proposition 7.5 corresponds to Lemma 10 in [Bo1] (see also Theorem 3.1 in [Bed]).

**Proposition 7.5.** Fix \( T \) with \( 0 < T < \infty \) strictly and let \( \mathcal{L}_{s,T} \) be the operator as above.

1. The function \( s \mapsto P(s) := \log \lambda(s,T) \) is continuous and strictly decreasing monotone with \( P(0) > 0 \) and \( P(1) \leq 0 \).

2. We have \( \dim_H(\mathbb{E}_T) = s_T \) where \( s_T \) is the unique solution of

\[
P(s_T) = 0.
\]

Moreover for \( s = s_T \) the measure \( \nu_{s_T,T} \) is equivalent to the restriction to \( \mathbb{E}_T \) of the Hausdorff measure \( H^{s_T} \).

**Proof.** Part (1) of the statement corresponds to pages 20-21 in [Bo1] (see also pages 15-16 in [Bed] and page 44 in [ParPo]). Part (2) can be found in [Bo1] and [Bed], we report the proof here since it shows clearly the relation between the Gibbs property and dimension. Moreover several steps in the proof will be used in the following.

**Step (1):** following Lemma 5 in [Bo1], we show that there exists an uniform constant \( C_1 > 1 \) such that for any \( w = (W_r)_{r \in \mathbb{N}} \in \Sigma \) and any \( n \in \mathbb{N} \) we have

\[
\frac{1}{C_1} \leq \frac{|[W_0, \ldots, W_{n-1}]_\Sigma|}{\exp \left( \sum_{k=0}^{n-1} \varphi_T(\sigma^k(w)) \right)} \leq C_1.
\]

Indeed we have \( [W_0, \ldots, W_{n-1}]_\Sigma = F_{W_0, \ldots, W_{n-1}}(D) \), where \( D := \text{Dom}(W_{n-1}) \), and Equation (6.3) gives

\[
|D| \cdot \inf_{\xi \in D} |D_\xi F_{W_0, \ldots, W_{n-1}}| \leq |[W_0, \ldots, W_{n-1}]_\Sigma| \leq |D| \cdot \sup_{\xi \in D} |D_\xi F_{W_0, \ldots, W_{n-1}}|.
\]

Thus Equation (7.5) follows observing that \( \xi_w := \Pi(\sigma^n w) \in \text{Dom}(W_{n-1}) \) and combining Lemma 7.2 with Part (1) of Corollary 6.8

**Step (2):** for the specific value \( s = s_T \) we have \( P(s_T) = \ln \lambda(s_T, T) = 0 \). Moreover \( \nu_{s,T,T} := \Pi_s(m_{s,T}) \), where \( m_{s,T} \) is the Gibbs measure for the potential \( s_T \cdot \varphi_T \) with
pressure $P(s_T) = 0$, according to Point (3) of Theorem 7.1. Therefore there exists a constant $C_2 = C_2(T) > 1$ such that for any $w = (W_r)_{r \in \mathbb{R}}$ and any $n \in \mathbb{N}$ we have

$$
\frac{1}{C_2} \leq \frac{\nu(s_T, T)([W_0, \ldots, W_{n-1}]_E)}{\exp \left(s_T \cdot \sum_{k=0}^{n-1} \varphi_T(\sigma^k(w)) \right)} \leq C_2.
$$

(7.6)

Step (3): finally we show that for $s = s_T$ the measure $\nu(s_T, T)$ satisfies Equation (7.3). Fix $r > 0$ and $\xi \in E_T$ and consider the euclidian ball $B(\xi, r) \subset \mathbb{C}$. Let $w = (W_r)_{r \in \mathbb{N}} \in \Sigma$ be the sequence of cuspidal words arising from the boundary expansion of $\xi$, that is the sequence such that $\xi = [W_0, W_1, W_2, \ldots]_\mathbb{R}$. Let $n \in \mathbb{N}$ be such that

$$
|[W_0, \ldots, W_n]_E| = r < |[W_0, \ldots, W_{n-1}]_E|.
$$

The first inequality above implies $[W_0, \ldots, W_n]_E \subset B(\xi, r)$. The lower bound in Equation (7.3) follows observing that

$$
\nu(s_T, T)(B(\xi, r)) \geq \nu(s_T, T)([W_0, \ldots, W_n]_E) \geq \frac{1}{C_2} \cdot \left(\exp \left(\sum_{k=0}^{n} \varphi_T(\sigma^k(w)) \right)\right)^{s_T} \geq \frac{|[W_0, \ldots, W_n]_E|^{s_T} \cdot C_2}{C_2 \cdot (\kappa C_1)^{s_T}},
$$

where the second inequality holds by Equation (7.6), the third by Equation (7.5) and the fourth by Part (3) of Proposition 6.9 (here $\kappa$ denotes the constant in Parts (3) and (4) of the Proposition). On the other hand, let $m \leq n$ be maximal such that

$$
B(\xi, r) \cap E_T \subset [W_0, \ldots, W_m]_E.
$$

By maximality, $B(\xi, r) \cap E_T$ contains a gap $B^{(m+1)}_\xi$ of level $m+1$ and we have

$$
|B^{(m+1)}_\xi| \geq \kappa \cdot |[W_0, \ldots, W_m]_E| \geq \frac{\kappa \cdot |[W_0, \ldots, W_{n-1}]_E|}{C_3 \cdot \theta^{n-m-1}} \geq \frac{\kappa \cdot r}{C_3 \cdot \theta^{n-m-1}},
$$

where the first and second inequalities follows respectively from Part (2) and Part (2) of Proposition 6.9 (here $C_3$ denotes the constant in Parts (1) and (2) of the Proposition), and the last by definition of $n$. Since

$$
|B^{(m+1)}_\xi| \leq |B(\xi, r) \cap \partial \mathbb{D}| \leq (\pi/3) \cdot r
$$

it follows that there exists an integer $N = N(T)$ such that for any $r$ and $n$ as above we have

$$
B(\xi, r) \cap E_T \subset [W_0, \ldots, W_{n-N}]_E.
$$

Therefore the upper bound in Equation (7.3) follows by a chain of inequalities similar to those proving the lower bound. \hfill \Box

8. Transfer operator on the circle

In this section we consider parameters $s, T$ with $9/10 \leq s \leq 1$ and $0 < T \leq +\infty$ and the corresponding transfer operators $L(s, T)$ defined by Equation (8.1) below, acting on the Banach spaces $\mathcal{B}$ and $\mathcal{B}_T$ which are described in the next § 8.4. Condition $s \geq 9/10$ guarantees summability in Corollary 8.3. As in § 6 all constant are uniform, that is depend only on
Thus we have a well-defined set of cuspidal words $W$ and only if the concatenation $W \ast B$ is shorter, we define also $B$.

8.2. The transfer operator on the circle and its spectrum. Fix $a \in A$. Consider any pair of cuspidal words $W_0 = (a_0, \ldots, a_n)$ and $W_1 = (b_0, \ldots, b_k)$ with $a_0 = b_0 = a$, where in general $n \neq k$. Observe that if $W = (c_0, \ldots, c_m)$ is any other cuspidal word, then $M_{W,W_0} = M_{W,W_1}$, that is the concatenation $W \ast W_0$ is allowed by the transition matrix in Equation (3.7) if and only if the concatenation $W \ast W_1$ is allowed. In other words, for given $W, W_0 \in \mathcal{W}$ the admissibility of the concatenation $W \ast W_0$ depends only on $W$ and on the first letter of $W_0$. Thus we have a well-defined set of cuspidal words $\mathcal{W}(a) := \{W \in \mathcal{W} : M_{W,W_0} = 1 \text{ for any } W_0 = (a_0, \ldots, a_n) \in \mathcal{W} \text{ with } a_0 = a\}$. 

8.1. Banach spaces of piecewise Lipschitz functions. For a metric space $(X,d)$ let $\text{Lip}(X)$ be the space of Lipschitz functions $f : X \rightarrow \mathbb{C}$, that is the functions such that there exists some $C = C(f) > 0$ such that $|f(x) - f(y)| \leq C|y - x|$ for any $x, y \in X$. The Lipschitz constant of any such $f$ is

$$\text{Lip}(f) := \sup_{x \neq y} \frac{|f(y) - f(x)|}{|y - x|}.$$ 

If $X$ is compact, then the space $C(X)$ of continuous function $f : X \rightarrow \mathbb{C}$ with norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$ is a Banach space. Moreover $\text{Lip}(X)$ is also a Banach space, with norm $\|f\|_* := \|f\|_\infty + \text{Lip}(f)$. According to the Theorem of Ascoli-Arzela, the unitary ball for the norm $\| \cdot \|_*$ is relatively compact for the topology induced by the norm $\| \cdot \|_\infty$.

Fix $a \in A$. Any $f \in \text{Lip}([a])$ has an unique Lipschitz extension to the closure $[a]$, with the same Lipschitz constant. Hence in particular it is continuous and bounded. Consider the space of functions

$$\mathcal{B} := \{f : \partial D \rightarrow \mathbb{C} : f|_a \in \text{Lip}([a]) \land a \in A\}.$$ 

Consider the norm $\| \cdot \|_* : \mathcal{B} \rightarrow \mathbb{R}_+$ defined by $\|f\|_* := \|f\|_\infty + \text{Lip}(f)$ where $\text{Lip}(f) := \max_{a \in A} \text{Lip}(f|_a)$.

The axioms of a normed vector space are easily verified. Moreover $\mathcal{B}$ is a Banach space, because $\text{Lip}([a])$ is a Banach space for any $a \in A$. For the same reason, the unitary ball in $\mathcal{B}$ for the norm $\| \cdot \|_*$ is relatively compact for the norm $\| \cdot \|_\infty$.

Fix any $T$ with $0 < T < +\infty$ and let $\mathcal{B}_T$ be the space of functions on $\mathbb{E}_T$ which are restrictions to $\mathbb{E}_T$ of functions in $\mathcal{B}$, that is

$$\mathcal{B}_T := \{g : \mathbb{E}_T \rightarrow \mathbb{C} : \exists f \in \mathcal{B} : g = f|_{\mathbb{E}_T}\}.$$ 

For $0 < T < +\infty$ the transfer operator $L_{(s,T)}$ defined in Equation (8.1) below acts both on $\mathcal{B}$ and $\mathcal{B}_T$, while for $T = \infty$ the operator $L_{(s,\infty)}$ acts only on $\mathcal{B}$. In order to make statements shorter, we define also $\mathcal{B}_\infty := \mathcal{B}$.
Then, for fixed \( T > 0 \) set

\[ \mathcal{W}(a, T) := \{ W \in \mathcal{W}(a) : |W| \leq T \}. \]

The set \( \mathcal{W}(a, T) \) is finite for \( 0 < T < +\infty \), while for \( T = +\infty \) we have \( \mathcal{W}(a, T) = \mathcal{W}(a) \), which is an infinite countable set. Finally, for \( k \in \mathbb{D} \) define \( \mathcal{W}(k, a, T) \) as the set of finite blocks \( w_k := (W_1, \ldots, W_k) \) as in Equation (5.2) with \( W_k \in \mathcal{W}(a, T) \). In other words we have

\[ w_k \in \mathcal{W}(k, a, T) \iff [a] \subset \text{Dom}(W_k). \]

With this notation, for any \( 0 < T \leq \infty \) let \( L_{(s,T)} : \mathcal{B} \to \mathcal{B} \) and \( L_{(s,T)} : \mathcal{B}_T \to \mathcal{B}_T \) be the operators defined by

\[
(8.1) \quad L_{(s,T)} f(\xi) = \sum_{w \in \mathcal{W}(a,T)} |D_{\xi}F_W|^s \cdot f(F_W \cdot \xi) \quad \text{for} \quad \xi \in [a],
\]

where we observe that the definition makes sense also for \( f \in \mathcal{B}_T \) because for any \( a \in \mathcal{A} \) the maps \( F_W \) with \( W \in \mathcal{W}(a, T) \) leave \( \mathbb{E}_T \) invariant. For \( T = +\infty \) the sum above is over all \( W \) in the infinite set \( \mathcal{W}(a) \), and the only invariant set is the entire circle \( \partial \mathbb{D} \). The expression of the \( k \)-th iterated of \( L_{(s,T)} \) is

\[
L_{(s,T)}^k f(\xi) = \sum_{w_k \in \mathcal{W}(k, a, T)} |D_{\xi}F_{w_k}|^s \cdot f(F_{w_k} \cdot \xi) \quad \text{for} \quad \xi \in [a].
\]

For \((s, T)\) with \( 0 < s \leq 1 \) and \( 0 < T < \infty \) recall from Equation (7.4) the definition of the transfer operator \( \mathcal{L}_{(s,T)} : \text{Lip}(\Sigma, \theta) \to \text{Lip}(\Sigma, \theta) \). Let \( \lambda(s, T) \) and \( h_{(s,T)} \) be the leading eigenvalue and eigenfunction of \( \mathcal{L}_{(s,T)} \), as in Point (1) of Theorem 7.1 Consider the probability measure \( \mu_{(s,T)} = \Pi_s(\hat{m}_{(s,T)}) \), where \( \hat{m}_{(s,T)} \) is as in Point (4) of Theorem 7.1 and \( \Pi : \Sigma \to \mathbb{E}_T \) is the map in \( \S \) 5.2 According to Lemma 5.2 the map \( \Pi : \Sigma \to \mathbb{E}_T \) is continuous with continuous inverse, thus we have a bounded invertible operator:

\[
H : (C(\mathbb{E}_T), \| \cdot \|_\infty) \to (C(\Sigma), \| \cdot \|_\infty) \quad ; \quad H(f) := f \circ \Pi.
\]

Fix \( f \in C(\mathbb{E}_T) \) and \( w = (W_0, W_1, \ldots) \in \Sigma \). Let \( a \in \mathcal{A} \) such that \( \xi := \Pi(w) \in [a] \), that is the first letter of \( W_0 \). For \( W \in \mathcal{W} \) set \( w' := (W, W_0, W_1, \ldots) \) and observe that

\[
W \in \mathcal{W}(a, T) \iff \mathcal{M}_{W, W_0}(\xi) = 1 \iff \sigma(w') = w.
\]

Therefore we have

\[
(8.2) \quad (HL_{(s,T)} f)(w) = L_{(s,T)} f(\Pi(w)) = \sum_{w \in \mathcal{W}(a,T)} |D_{\Pi(w)}F_W|^s f(F_W(\Pi w))
= \sum_{\sigma(w') = w} |D_{\Pi(w)}F_{W_0(w')}|^s f(F_W(\Pi w))
= \sum_{\sigma(w') = w} |D_{\Pi(w)}F_{W_0(w')}|^s f(\Pi(w')) = \mathcal{L}_{(s,T)}(f \circ \Pi)(w) = (\mathcal{L}_{(s,T)}H f)(w),
\]

where the second to last equality holds because \( F_W^{-1}(\Pi w') = F(\Pi w') = \Pi(\sigma w') = \Pi(w) \) and therefore \( F_W(\Pi w) = \Pi(w') \). It follows that

\[
H \circ L_{(s,T)} = \mathcal{L}_{(s,T)} \circ H.
\]

Theorem 8.1 below is completed by Corollary 9.10 in the next section. We refer to \( \S \) B.3 for the terminology of quasi-compact operators.
Theorem 8.1. Fix any \((s,T)\). Both on \(B\) and on \(B_T\) the operator \(L(s,T)\) has a simple, real and positive eigenvalue \(\hat{\lambda}(s,T) > 0\), corresponding to a strictly positive function \(g(s,T) \in \Lambda\) (respectively in \(\Lambda_T\)), that is
\[
L(s,T)(g(s,T)) = \hat{\lambda}(s,T) \cdot g(s,T).
\]
Moreover for \(T < \infty\) and any \(s\) the following holds.

1. For we have
   \[
   \hat{\lambda}(s,T) = \lambda(s,T) \quad \text{and} \quad g(s,T)|\mathcal{E}_T = h(s,T) \circ \Pi^{-1}.
   \]
2. For any \(f \in B\) (respectively in \(B_T\)) we have
   \[
   \lambda(s,T)^{-n} \cdot L^n(s,T)(f) \to \left(\int fd\mu(s,T)\right) \cdot g(s,T) \quad \text{uniformly as} \quad n \to \infty.
   \]

On the other hand, for \(0 < T \leq \infty\) and for \(s = s_T\) for following holds

3. We have \(\lambda(s_T,T) = 1\), which is the maximal eigenvalue of \(L(s_T,T)\), both on \(B\) and on \(B_T\). Moreover the latter is quasi-compact with
   \[
   \rho_{ess}(L(s_T,T)) \leq \theta.
   \]

8.3. Some preliminary uniform estimates.

Lemma 8.2. There is an uniform constant \(C > 1\) such that for any cuspidal word \(W \in \mathcal{W}\) and any \(\xi \in \text{Dom}(W)\) we have
\[
\frac{1}{C} \cdot |W|^2 \leq |D_{\xi}F_W| \leq \frac{C}{|W|^2}.
\]

Proof. Observe first that if \(F \in \text{SU}(1,1)\) is parabolic and conjugated in \(\text{SL}(2,\mathbb{C})\) to a map \(z \mapsto z + \mu\) then its coefficient \(\beta = \beta(F)\) is given by \(\beta = -i(\mu/2)\), where we refer to the notation of Equation (2.2). Moreover, according to Lemma 3.5 and Lemma 3.6, any cuspidal word \(W\) can be decomposed as
\[
W = V \ast P \ast \cdots \ast P,
\]
where \(P\) is a parabolic word and where \(V\) is a cuspidal word which does not contain any parabolic word as a factor. From the definition of the geometric length in § 4 one can see that there is an uniform constant \(C_1 > 1\) such that in the decomposition above we have
\[
C_1^{-1} \leq |W| \cdot k^{-1} \leq C_1.
\]

Fix \(W \in \mathcal{W}\) and any \(\xi \in \text{Dom}(W)\). Then decompose \(W\) as above, where \(F_P\) is conjugated in \(\text{SL}(2,\mathbb{C})\) to \(z \mapsto z + \mu\). Denoting by \(\omega_k\) and \(\omega_{k+1}\) the poles of \(F^k_P\) and \(F^{k+1}_P\) respectively, we have
\[
\frac{1}{|(k+1)(\mu/2)|^2|\xi - \omega_{k+1}|^2} = |D_{\xi}F^k_P| \leq |D_{\xi}F_W| \leq |D_{\xi}F^{k+1}_P| = \frac{1}{|k(\mu/2)|^2|\xi - \omega_k|^2},
\]
where the inequalities follow from Lemma 6.4 factorizing \(|D_{\xi}F_W|\) with the chain rule. The distance from poles is bounded by Lemma 6.2. The statement follows.

For any \(0 < T \leq \infty\) and any \(a \in \mathcal{A}\) let \(\mathcal{V}(a,T) := \mathcal{W}(a) \setminus \mathcal{W}(a,T)\), that is the set of cuspidal words \(W\) with \(|W| > T\) and \([a] \subset \text{Dom}(W)\).
Corollary 8.3. There exists an uniform constant \( C > 0 \) such that for any \( s > 9/10 \) and any \( 0 < T \leq \infty \), for any \( a \in A \) and any \( \xi \in [a] \) we have

\[
\sum_{W \in W(a,T)} |D_\xi F_W|^s \leq C \quad \text{and} \quad \sum_{W \in W(a,T)} |\ln |D_\xi F_W|| \cdot |D_\xi F_W|^s \leq C.
\]

Moreover we also have

\[
\sum_{W \in V(a,T)} |D_\xi F_W|^s \leq C \cdot \left( \frac{1}{T} \right)^{2s-1}.
\]

Proof. The definition of geometric length of a cuspidal word in Equation (1.2) implies that there exist uniform constants \( N > 1 \) and \( C_1 \) such that for any \( T > 0 \) we have

\[
1 \leq \# \{ W \in \mathcal{W} : T \leq |W| \leq T + N \} \leq C_1.
\]

Then the statement follows directly from Lemma 8.2. \( \square \)

8.4. Continuity and quasi-compactness of transfer operators.

Lemma 8.4. For \( 9/10 \leq s \leq 1 \) and \( 0 < T \leq +\infty \) the operator \( L_{(s,T)} \) is a bounded linear operator both on \( B \) and \( B_T \). Moreover there exists an uniform constant \( C > 0 \) such that, on both spaces, we have

\[
\| L_{(s,T)} \|_* \leq C.
\]

Proof. Fix \( f \in B \). For any \( a \in A \) and any \( x, y \in [a] \) we have

\[
|L_{(s,T)} f(y) - L_{(s,T)} f(x)| \leq \sum_{W \in V(a,T)} |D_s F_W|^s \cdot |f(F_W y) - f(F_W x)| + f(F_W y) \cdot ||D_s F_W|^s - |D_s F_W|^s| \leq \sum_{W \in V(a,T)} |D_s F_W|^s \cdot \text{Lip}(f) \cdot |F_W(y) - F_W(x)| + \| f \|_{\infty} \cdot \| D_F W \|_{\infty}^s \cdot C_1 \cdot |y - x| \leq \left( \sum_{W \in V(a,T)} \| D_F W \|_{\infty}^s \right) \cdot (\theta \cdot \text{Lip}(f) + \| f \|_{\infty} \cdot C_1) \cdot |y - x|,
\]

where the second inequality follows from Part (2) of Corollary 6.8 applied for \( k = 1 \) (here \( C_1 \) denotes the uniform constant in the Corollary), and the third inequality follows from Corollary 6.6 also applied for \( k = 1 \). Therefore Corollary 8.3 gives

\[
\text{Lip}(L_{(s,T)} f) \leq C_2 \cdot \left( \theta \cdot \text{Lip}(f) + \| f \|_{\infty} \cdot C_1 \right) \leq C_2 \cdot (\theta + C_1) \cdot \| f \|_*,
\]

where here \( C_2 \) denotes the uniform constant in Corollary 8.3. The inequality above implies that \( L_{(s,T)} f \in B \). Moreover for any \( a \in A \) and any \( \xi \in [a] \) we have

\[
|L_{(s,T)} f(\xi)| = \sum_{W \in V(a,T)} |D_s F_W|^s \cdot |f(F_W \cdot \xi)| \leq \left( \sum_{W \in V(a,T)} \| D_F W \|_{\infty}^s \right) \cdot \| f \|_{\infty}.
\]

Thus \( \| L_{(s,T)} f(\xi) \|_{\infty} \leq C_2 \cdot \| f \|_{\infty} \) again by Corollary 8.3. All the involved constants are uniform, thus the statement follows for the operator \( L_{(s,T)} : B \to \mathcal{B} \). The same result holds for \( L_{(s,T)} : \mathcal{B}_T \to \mathcal{B}_T \), indeed the inequality above hold for any \( x, y \in \partial \mathbb{D} \), and thus in particular restricting to \( x, y \) in \( \partial \mathbb{D} \). The Lemma is proved. \( \square \)
For any $w_k \in \mathcal{W}(k,a,T)$ recall the definition of $\|DF_{w_k}\|_\infty$ in Equation (6.2). Recall also that we denote $s_T = \dim_H(\mathbb{E}_T)$, where in particular $s_T = 1$ for $T = \infty$.

**Lemma 8.5.** For any $0 < T \leq \infty$ there exists a constant $C(T) > 0$ such that for any $a \in \mathcal{A}$ and any $k \geq 2$ we have

$$\sum_{w_k \in \mathcal{W}(k,a,T)} \|DF_{w_k}\|_{s_T}^{s_T} \leq C(T).$$

**Proof.** Consider $w_k = (W_1, \ldots, W_k)$ as in Equation (5.2) and any $w \in [w_k]_\Sigma$ and observe that $\xi := \Pi(\sigma^k w) \in \text{Dom}(W_k)$ and that $\exp \left( \sum_{k=0}^{n-1} \varphi_T(\sigma^k w) \right) = |D_\xi F_{w_k}|$, according to Lemma 7.2. For $T = \infty$, which corresponds to $s_T = 1$, Equation (7.5) gives

$$\|DF_{w_k}\|_{\infty} \leq C_2 \cdot |D_\xi F_{w_k}| = C_2 \cdot \exp \left( \sum_{k=0}^{n-1} \varphi_T(\sigma^k w) \right) \leq C_2 C_1 \cdot |[w_k]|,$$

where here $C_2$ denotes the uniform constant in Corollary 6.8 and $C_1$ the uniform constant in Equation (7.5). On the other hand, any $T$ with $0 < T < \infty$ corresponds to the Gibbs measure $\nu_{(s_T, T)}$, and Equation (7.6) gives

$$\|DF_{w_k}\|_{s_T}^{s_T} \leq C_2 \cdot |D_\xi F_{w_k}|^{s_T} = C_2 \cdot \exp \left( s_T \sum_{k=0}^{n-1} \varphi_T(\sigma^k w) \right) \leq C_2 C_3 \cdot \nu_{(s_T, T)}([w_k]),$$

where here $C_3 = C_3(T)$ denotes the constant in Equation (7.6). The matrix $M^2_{W,W'}$ is positive by Proposition 5.1 thus \{\{w_k : w_k \in \mathcal{W}(k,a,T)\}\} is a partition of $\mathbb{E}_T$ for any $k \geq 2$. The Lemma follows. \[ \Box \]

**Proposition 8.6.** For any $T$ with $0 < T \leq +\infty$ there exists a constant $C(T) > 0$ such that for any $f$ either in $\mathcal{B}$ or in $\mathcal{B}_T$ and for any $k \in \mathbb{N}$ we have

$$\|L^k_{(s_T, T)} f\|_\ast \leq C(T) \cdot (\theta^k \|f\|_\ast + \|f\|_\infty).$$

**Note:** we stress that Proposition 8.6 is proved only for the parameter $s = s_T$. A more general statement can be obtained from the proof of Lemma 9.5. See also Remark 9.6.

**Proof.** Fix $f \in \mathcal{B}$ and $k \in \mathbb{N}$. For any $a \in \mathcal{A}$ and any $\xi' \in [a]$ we have

$$\left| L^k_{(s_T, T)} f(\xi') - L^k_{(s_T, T)} f(\xi) \right| \leq \sum_{w_k \in \mathcal{W}(k,a,T)} |D_{\xi'} F_{w_k}|^{s_T} \cdot |f(F_{w_k}(\xi')) - f(F_{w_k}(\xi))| + f(F_{w_k}(\xi)) \cdot |D_{\xi'} F_{w_k}|^{s_T} - |D_{\xi} F_{w_k}|^{s_T}|$$

$$\leq \sum_{w_k \in \mathcal{W}(k,a,T)} |D_{\xi'} F_{w_k}|^{s_T} \cdot \text{Lip}(f) \cdot |F_{w_k}(\xi') - F_{w_k}(\xi)| + \|f\|_\infty \cdot \|DF_{w_k}\|_{s_T}^{s_T} \cdot C_1 \cdot |\xi' - \xi| \leq$$

$$\left( \sum_{w_k \in \mathcal{W}(k,a,T)} \|DF_{w_k}\|_{s_T}^{s_T} \right) \cdot (\theta^k \cdot \text{Lip}(f) + \|f\|_\infty \cdot C_1) \cdot |\xi' - \xi|,$$

where the second inequality follows from Corollary 6.8 (here $C_1$ denotes the uniform constant in the Corollary) and the third inequality follows from Corollary 6.6. Therefore Lemma 8.5 implies

$$\text{Lip}(L^k_{(s_T, T)} f) \leq C(T) \cdot (\theta^k \cdot \text{Lip}(f) + \|f\|_\infty).$$
For any \( a \in \mathcal{A} \) and any \( \xi \in [a] \) Lemma 8.5 implies also

\[ |L_{(s,T)}^k(f)(\xi)| = \left| \sum_{w_k \in W(k,a,T)} D_{\xi} F_{w_k} |s|^* f(F_{w_k} \xi) \right| \leq C(T) \cdot \|f\|_{\infty}, \]

that is \( \|L_{(s,T)}^k(f)\|_{\infty} \leq C(T) \cdot \|f\|_{\infty} \). Then the required inequality follows redefining \( C(T) \). The statement is proved for any \( f \in \mathcal{B} \). All the inequalities above keep true restricting to points \( \xi', \xi \in [a] \cap \mathbb{E}_T \) for any \( a \in \mathcal{A} \), thus the statement holds also for any \( f \in \mathcal{B}_T \). \( \square \)

### 8.5. An invariant set of Lipschitz functions.

We follow pages 22-23 in [ParPo]. Let \( 0 < \theta < 1 \) be the uniform constant in Lemma 6.7. Denote here \( \kappa > 0 \) the uniform constant in Lemma 6.7. Then let \( C > 0 \) be an uniform constant such that

\[ \kappa + \theta \cdot C \leq C. \]

Let \( \Lambda \) be the set of real positive functions \( f : \partial \mathbb{D} \to \mathbb{R}_+ \) such that for any \( a \in \mathcal{A} \) we have

\[ f(\xi') \leq \exp \left( C \cdot |\xi' - \xi| \right) \cdot f(\xi) \quad \text{for any} \quad \xi', \xi \in [a]. \]

Let \( \Lambda_T \) be the set of functions \( g : \mathbb{E}_T \to \mathbb{R}_+ \) such that there exists some \( f \in \Lambda \) with \( g = f|_{\mathbb{E}_T} \).

**Lemma 8.7.** We have the inclusions \( \Lambda \subset \mathcal{B} \) and \( \Lambda_T \subset \mathcal{B}_T \). More precisely, any \( f \) either in \( \Lambda \) or in \( \Lambda_T \) is bounded, and in terms of the constant \( C \) introduced above it satisfies

\[ \|f\|_* \leq \|f\|_{\infty} \cdot (1 + Ce^{2C}). \]

**Proof.** Fix \( f \in \Lambda \). Since \( |x - y| \leq \text{Diam}(\mathbb{D}) = 2 \) for any \( x, y \), then Equation (8.3) implies \( \|f\|_{\infty} \leq +\infty \), that is \( f \) is bounded (but a priori continuity is not yet proved). Moreover \( (e^R - 1) \leq Re^R \) for any \( R \geq 0 \). Therefore for any \( a \in \mathcal{A} \) and any \( x, y \in [a] \) we have

\[
\begin{align*}
(f(y) - f(x)) &\leq f(x) \cdot (e^{C|y-x|} - 1) \\
&\leq \|f\|_{\infty} \cdot (e^{C|y-x|} - 1) \\
&\leq \|f\|_{\infty} \cdot e^{C|y-x|} \cdot C |y - x| \\
&\leq \|f\|_{\infty} \cdot Ce^{2C} \cdot |y - x|.
\end{align*}
\]

The last inequality holds reversing the role of \( x \) and \( y \), thus \( f \in \text{Lip}([a]) \), and a posteriori \( f \) is also continuous on \([a]\). In particular we get \( \text{Lip}(f) \leq \|f\|_{\infty} \cdot Ce^{2C} \). All the inequality above hold restricting \( x, y \) in \( \mathbb{E}_T \), thus the statement follows also for any \( f \in \Lambda_T \). \( \square \)

**Lemma 8.8.** For any \( s, T \) and any \( f \in \Lambda \) we have \( L_{(s,T)}(f) \in \Lambda \). For any \( f \in \Lambda_T \) we have also \( L_{(s,T)}(f) \in \Lambda_T \).

**Proof.** It is enough to prove the Lemma for \( f \in \Lambda \). It is clear that \( L_{(s,T)}(f) \) is real and positive if \( f \) is. Fix \( a \in \mathcal{A} \) and consider \( y, x \in [a] \). Corollary 6.6 gives

\[ f(F_{Wy}) \leq e^{\left( C \cdot \left| f(F_{Wy}) - f(F_{Wx}) \right| \right)} \cdot f(F_{Wx}) \leq e^{(\theta C \cdot |y - x|)} \cdot f(F_{Wx}). \]

Moreover Lemma 6.7 gives

\[ |D_y F_W|^* = \exp \left( s \log(|D_y F_W|) \right) \leq \exp \left( s \log(|D_x F_W|) + \kappa \cdot |y - x| \right) = \exp \left( \kappa \cdot |y - x| \right) \cdot |D_x F_W|^*. \]
Recall that $\kappa + \theta \cdot C \leq C$. The statement follows because that the estimates above give
\[
L_{(s,T)} f(y) = \sum_{W \in \mathcal{W}(a,T)} |D_y F_W|^s f(F_W y)
\]
\[
\leq \sum_{W \in \mathcal{W}(a,T)} e^{\kappa_1 |y-x|} \cdot |D_x F_W|^s \cdot e^{\theta C |y-x|} f(F_W x)
\]
\[
\leq e^{(\kappa + \theta C) |y-x|} \sum_{W \in \mathcal{W}(a,T)} |D_x F_W|^s \cdot f(F_W x) \leq e^{C |y-x|} \cdot L_{(s,T)} f(x).
\]
\[\square\]

8.6. Maximal eigenfunction for the transfer operator on the circle. Proposition 8.9 below follows [ParPo], pages 22-24. In the next §8.7 we show that $\tilde{\lambda}(s, T) = \lambda(s, T)$, that is the eigenvalue in the Proposition is the maximal eigenvalue of the operator $L_{(s,T)}$.

**Proposition 8.9.** Fix any $(s, T)$. Both on $B$ and on $B_T$ the operator $L_{(s,T)}$ has a simple, real and positive eigenvalue $\tilde{\lambda}(s, T) > 0$, corresponding to a strictly positive function $g(s,T) \in \Lambda$ (respectively in $\Lambda_T$), that is $L_{(s,T)}(g(s,T)) = \tilde{\lambda}(s, T) \cdot g(s,T)$.

**Proof.** For simplicity set $L := L_{(s,T)}$. By Lemma 6.3 we have $\|L1\|_\infty \geq c$ for some uniform constant $c > 0$. Therefore we have $\|L(f + n^{-1})\|_\infty > c/n$ for any $f$ continuous and positive and for any integer $n \geq 1$. It follows that for any integer $n \geq 1$ we have a well defined (non linear) operator $M_n$ acting on the set of positive functions $f \in B$ by
\[
M_n(f) := \frac{1}{\|L(f + n^{-1})\|_\infty} \cdot L(f + n^{-1}).
\]
Since Lemma 8.4 gives a (uniform) bound for $\|L\|_\infty$ then the operator $M_n$ is continuous in the topology of the norm $\| \cdot \|_\infty$, indeed for any $f, g$ in $\Lambda$ we have
\[
\|M_n(f) - M_n(g)\|_\infty \leq \frac{1}{\|L(f + n^{-1})\|_\infty} \left( \|L\|_\infty \cdot \|f - g\|_\infty + \|L(g + n^{-1})\|_\infty - \|L(f + n^{-1})\|_\infty \right) \leq \frac{n \cdot 2 \|L\|_\infty}{c} \cdot \|f - g\|_\infty.
\]
Let $\Lambda^{(1)} := \{f \in \Lambda : f(\xi) \leq 1 \ \forall \xi \in \partial \mathbb{D}\}$, where $\Lambda$ is the set of functions defined in §8.5. Lemma 8.8 and the definition of $M_n$ imply that $M_n(\Lambda^{(1)}) \subset \Lambda^{(1)}$. Lemma 8.7 via the Theorem of Ascoli-Arzelà, implies that $\Lambda^{(1)}$ is compact in the topology of $\| \cdot \|_\infty$. Moreover $\Lambda^{(1)}$ is also obviously convex. Therefore the Schauder-Tychonov fixed point Theorem implies that there exists $g_n \in \Lambda^{(1)}$ such that $M_n(g_n) = g_n$, that is
\[
L(g_n + n^{-1}) = \|L(g_n + n^{-1})\|_\infty \cdot g_n.
\]
Compactness implies that, modulo subsequences, there exists $g \in \Lambda^{(1)}$ and $\lambda \geq 0$ such that $\|(g_n + n^{-1}) - g\|_\infty \to 0$ and $\|L(g_n + n^{-1})\|_\infty \to \lambda$ for $n \to +\infty$, that is
\[
L(g) = \lambda \cdot g.
\]
Since \(\|M_n(g_n)\|_\infty = 1\) by construction, then \(\|g_n\|_\infty = 1\) for any \(n\), and thus we have also \(\|g\|_\infty = 1\). Then positivity of \(L\) implies \(\lambda = \|Lg\|_\infty > 0\) strictly. Fix \(a \in \mathcal{A}\) and \(\xi \in [a]\). For any \(k \in \mathbb{N}\) we have

\[
g(\xi) = \lambda^{-k} \cdot \sum_{w_k \in W(k,a,T)} \|D_\xi F_{w_k}\|^s \cdot g(F_{w_k} \cdot \xi).
\]

The square \(M^2_{W,W'}\) of the transition matrix in Equation (3.7) is positive by Proposition 5.1. Therefore, since \(\|g\|_\infty = 1\), then Equation (8.4) implies \(\|g\|_{1} \|g\|_{\infty} > 0\) strictly for any \(a \in \mathcal{A}\) (otherwise, if \(g(\xi) = 0\) for some \(\xi\), then Equation (8.4) implies that \(g(F_{w_k} \cdot \xi) = 0\) for any \(w_k \in W(k,a,T)\), and points \(F_{w_k} \cdot \xi\) become dense for \(k \to \infty\)). Since \(g \in \Lambda\), then Equation (8.3) implies that there exists \(m(s,T) > 0\) such that

\[
m(s,T) \leq g(\xi) \leq 1 \quad \text{for any} \quad \xi \in \partial \mathbb{D}.
\]

In order to prove simplicity of \(\lambda\), let \(f \in B\) real valued such that \(Lf = \lambda \cdot f\) and set \(t := \inf_{\xi \in \partial \mathbb{D}} f(\xi)/g(\xi)\). Observe that \(t \in \mathbb{R}\). Then \(h(\xi) := f(\xi) - tg(\xi) \geq 0\) for any \(\xi\). Moreover there exists \(a \in \mathcal{A}\), \(\xi_\infty \in [a]\) and a sequence \(\xi_n \in [a]\) such that for \(n \to \infty\) we have \(\xi_n \to \xi_\infty\) and \(f(\xi_n)/g(\xi_n) = t\). Then we have also \(h(\xi_n) \to 0 = h(\xi_\infty)\), because both \(g\) and \(f\) have Lipschitz extension to \([\bar{a}]\). Fix \(k \in \mathbb{N}\). Equation (8.4) implies

\[
0 = \lim_{n \to \infty} h(\xi_n) = \lambda^{-k} \cdot \sum_{w_k \in W(k,a,T)} \lim_{\xi_n \to \xi_\infty} |D_{\xi_n} F_{w_k}|^s \cdot \lim_{\xi_n \to \xi_\infty} h(F_{w_k} \xi_n)
\]

\[
= \lambda^{-k} \cdot \sum_{w_k \in W(k,a,T)} |D_{\xi_\infty} F_{w_k}|^s h(F_{w_k} \xi_\infty).
\]

The last condition implies \(h(F_{w_k} \xi_0) = 0\) for any \(w_k \in W(k,a,T)\) and any \(k \geq 1\), thus \(h(\xi) = 0\) for any \(\xi \in \partial \mathbb{D}\) by continuity and we conclude \(f = t \cdot g\). We observe that the sum above is finite for \(0 < T < \infty\), while for \(T = \infty\) the sum is infinite and one applies the Dominated Convergence Theorem to the functions

\[
\mathcal{H}_n : W(k,a,T) \to \mathbb{R}_+ ; \quad \mathcal{H}_n(w_k) := \lambda^{-k} \cdot |D_{\xi_n} F_{w_k}|^s \cdot h(F_{w_k} \xi_n).
\]

Setting \(\widehat{\mathcal{H}}(w_k) := \lambda^{-k} \cdot (\sup_{n \in \mathbb{N}} |D_{\xi_n} F_{w_k}|^s) \cdot \|h\|_\infty\) we have \(\mathcal{H}_n(w_k) \leq \widehat{\mathcal{H}}(w_k)\) for any \(n \in \mathbb{N}\) and any \(w_k \in W(k,a,T)\). Moreover, fix any \(\xi \in [a]\) and observe that for any \(w_k\) and any \(n \in \mathbb{N}\) Part (1) of Corollary 6.8 gives \(|D_{\xi_n} F_{w_k}|^s \leq C|D_{\xi_n} F_{w_k}|^s\), so that the same holds for the supremum over \(n \in \mathbb{N}\) and we get

\[
\sum_{w_k \in W(k,a,T)} \widehat{\mathcal{H}}(w_k) \leq C \cdot \lambda^{-k} \cdot \sum_{w_k \in W(k,a,T)} |D_{\xi_n} F_{w_k}|^s \cdot \|h\|_\infty \leq C \cdot \lambda^{-k} \cdot \|L^{k}(1)\|_{\infty} \cdot \|h\|_\infty.
\]

The arguments above hold for any point \(\xi \in \partial \mathbb{D}\), and thus in particular restricting to \(\xi \in \mathbb{E}_T\). Thus the statement for \(L_{(s,T)} : \mathcal{B}_T \to \mathcal{B}_T\) holds too.

\[\Box\]

8.7. End of the proof of Theorem 8.1

Point (4) of Theorem 7.1 and Equation (8.2) imply that for any \(f \in C(\mathbb{E}_T)\) and for \(n \to \infty\) we have uniform convergence

\[
\lambda(s,T)^{-n} \cdot L_{(s,T)}^n f = \lambda(s,T)^{-n} \cdot H^{-1} \mathcal{L}_{(s,T)}^n (H f) \to \left( \int (H f) d\tilde{\mu}_{(s,T)} \right) H^{-1} h_{(s,T)}
\]

\[
= \left( \int f d\mu_{(s,T)} \right) (h_{(s,T)} \circ \Pi^{-1}).
\]
Applying the last result the function \( f = g_{(s,T)} \in \Lambda_T \) from Proposition 8.9 we obtain
\[
\frac{\lambda(s,T)^n}{\lambda(s,T)^n} \cdot g_{(s,T)} \rightarrow \left( \int g_{(s,T)} d\mu(s,T) \right) (h_{(s,T)} \circ \Pi^{-1}) \quad \text{as} \quad n \to \infty.
\]

Since both \( g_{(s,T)} \) and \( h_{(s,T)} \circ \Pi^{-1} \) are positive, then \( \lambda(s,T) = \lambda(s,T) \). Moreover introducing the normalization \( \int g_{(s,T)} d\mu(s,T) = 1 \) we have also \( g_{(s,T)} = h_{(s,T)} \circ \Pi^{-1} \). Points (1) and (2) in Theorem 8.1 follow.

In particular, for \( 0 < T < \infty \) and \( s = s_T \) we have \( \lambda(s_T, T) = 1 \), according to Proposition 7.5. In order to see that
\[
\lambda := \lambda(s_T, T) \quad \text{for simplicity. According to Lemma 7.2 and Equation (7.6), there is a constant}
\]
\[ C(T) > 1 \quad \text{such that for any} \quad w_k \in W(k, a, T) \quad \text{we have}
\]
\[
C(T)^{-1} \cdot \nu(s_T,T)([w_k]) \leq |D\xi F_{w_k}|_{s_T} \leq C(T) \cdot \nu(s_T,T)([w_k]).
\]
Since the square \( M^2_{W,W} \), of the transition matrix in Equation (3.7) is positive, then it follows that \( \{ [w_k] : w_k \in W(k, a, T) \} \) is a partition of \( \mathbb{E}_T \), for any \( k \geq 2 \), and therefore we have
\[
\sum_{w_k \in W(k, a, T)} \nu(s_T,T)([w_k]) = 1.
\]
Recalling Equation (8.4), for any \( k \geq 2 \) we have
\[
g(\xi) = \lambda^{-k} \cdot \sum_{w_k \in W(k, a, T)} |D\xi F_{w_k}|_{s_T} \cdot g(F_{w_k} \cdot \xi) \geq \lambda^{-k} \cdot m(T) \cdot \sum_{w_k \in W(k, a, T)} |D\xi F_{w_k}|_{s_T}
\]
\[
\geq \frac{m(T)}{\lambda^k \cdot C(T)} \cdot \sum_{w_k \in W(k, a, T)} \nu(s_T,T)([w_k]) = \frac{m(T)}{\lambda^k \cdot C(T)},
\]
where we use that \( m(T) \leq g(\xi) \leq 1 \) for any \( \xi \) and some \( m(T) > 0 \). We have also
\[
g(\xi) = \lambda^{-k} \cdot \sum_{w_k \in W(k, a, T)} |D\xi F_{w_k}|_{s_T} \cdot g(F_{w_k} \cdot \xi) \leq \lambda^{-k} \cdot \sum_{w_k \in W(k, a, T)} |D\xi F_{w_k}|_{s_T}
\]
\[
\leq \frac{C(T)}{\lambda^k} \cdot \sum_{w_k \in W(k, a, T)} \nu(s_T,T)([w_k]) = \frac{C(T)}{\lambda^k}.
\]
Resuming, we have \( m(T) \leq g(\xi) \leq 1 \) and \( m(T) \cdot C(T)^{-1} \leq \lambda^k g(\xi) \leq C(T) \) for any \( k \geq 2 \). Hence we must have \( \lambda = 1 \).

On the other hand, Proposition 8.6 and Corollary 1 in \[\text{[Henn]}\] imply \( \rho_{ess}(L(s_T,T)) \leq \theta \). Since \( 0 < \theta < 1 \) then \( L(s_T,T) \) is quasi-compact.

Finally let \( \lambda' \) with \( |\lambda'| > \theta \) and \( f \in \mathcal{B} \) such that \( L(s_T,T)f = \lambda' \cdot f \). Since \( \mathbb{E}_T \) is invariant under all maps \( F_W \) with \( a \in \mathcal{A} \) and \( W \in \bigcup_{a \in \mathcal{A}} \mathcal{W}(a, T) \), then letting \( f_T := f|_{\mathbb{E}_T} \) we have also \( L(s_T,T)f_T = \lambda' \cdot f_T \). Moreover set \( h_T := f_T \circ \Pi = H(f_T) \) and observe that \( h_T \in \operatorname{Lip}(\Sigma, \theta) \), since \( \Pi : \Sigma \to \mathbb{E}_T \) is Lipschitz by Lemma 5.2. Equation (8.2) implies
\[
\lambda' \cdot h_T = H(\lambda' \cdot f_T) = HL(s_T,T)f_T = L(s_T,T)Hf_T = L(s_T,T)h_T.
\]
Resuming, both for \( L(s_T,T) : \mathcal{B} \to \mathcal{B} \) and for \( L(s_T,T) : \mathcal{B}_T \to \mathcal{B}_T \), any eigenvalue in the non-essential part of the spectrum is also eigenvalue of \( L(s_T,T) \). Then \( \lambda(s_T, T) = 1 \) is the maximal eigenvalue also for \( L(s_T,T) \), acting both on \( \mathcal{B} \) and \( \mathcal{B}_T \). Theorem 8.1 is proved. \( \square \)
9. Perturbative estimate of maximal eigenvalue: proof of Theorem \[4.3\]

From now on, for parameters \(s, T\) with \(9/10 < s \leq 1\) and \(0 < T \leq +\infty\), we consider only the transfer operators \(L_{(s, T)}\) on the Banach space \(\mathcal{B}\). As in \(\S 6\) all constants are uniform, unless explicitly stated.

9.1. Expansion of transfer operator in parameter \(s\). Observe that for any \(D, s\) with \(0 < D < 1\) and \(9/10 < s \leq 1\) and any \(h \in \mathbb{R}\) such that \(9/10 < s + h \leq 1\) we have

\[
D^{s+h} = D^s + \ln(D) \cdot \int_s^{s+h} D^t \, dt = D^s + h \cdot D^s \ln(D) + \ln(D) \cdot \int_s^{s+h} (D^t - D^s) \, dt,
\]

where we recall that \(d/dt(D^t) = (\ln D) \cdot D^t\). The mean value Theorem gives

\[
|D^{s+h} - (D^s + h \cdot D^s \ln(D))| \leq |h|^2 \cdot |\ln(D)|^2 \cdot D^{s/5}.
\]

For \(0 < T \leq +\infty\) consider the operator \(A_T := \mathcal{B} \to \mathcal{B}\) defined by

\[
A_Tf(x) := \sum_{W \in \mathcal{W}(a, T)} (\ln|D_x F_W|)|D_x F_W|^{s_T} f(F_W x) \quad \text{if} \quad x \in [a].
\]

For any \(W \in \mathcal{W}\) set

\[
\|\ln DF_W\|_{\infty} := \sup_{\xi \in \text{Dom}(W)} |\ln|D_\xi F_W||.
\]

Since \(s_T \to 1\) as \(T \to \infty\), let \(T_0\) be such that \(9/10 < s_T \leq 1\) for \(T_0 \leq T \leq \infty\).

**Lemma 9.1.** There exists an uniform constant \(C > 0\) such that for \(T_0 \leq T \leq \infty\) we have

\[
\|A_T\|_* \leq C.
\]

**Proof.** Fix \(T\) with \(T_0 \leq T \leq \infty\). Corollary \(\S 3.3\) gives directly an uniform upper bound for \(\|A_T\|_{\infty}\). Fix \(W \in \mathcal{W}\) and consider the function

\[
\Phi_W : \text{Dom}(W) \to \mathbb{R}_+ \quad ; \quad \Phi_W(x) := (\ln|D_x F_W|)|D_x F_W|^{s_T}.
\]

Letting \(C_1 > 0\) be the uniform constant in Part (2) of Corollary \(\S 6.8\) and \(C_2 > 0\) be the uniform constant in Lemma \(\S 6.7\), for any \(x, y\) in \(\text{Dom}(W)\) we have

\[
|\Phi_W(y) - \Phi_W(x)| \\
\leq |\ln|D_y F_W||(|D_y F_W|^{s_T} - |D_x F_W|^{s_T}) + |D_x F_W|^{s_T} (\ln|D_y F_W| - \ln|D_x F_W|) \\
\leq (C_1 \cdot \|\ln DF_W\|_{\infty} \cdot \|DF_W\|_{\infty}^{s_T} + C_2 \cdot \|DF_W\|_{\infty}^{s_T}) \cdot |y - x|.
\]

Therefore for any \(f \in \mathcal{B}\), any \(a \in \mathcal{A}\) and any \(x, y\) in \([a]\) we have

\[
|A_T f(y) - A_T f(x)| \leq \\
\sum_{W \in \mathcal{W}(a, T)} |\Phi_W(y)(f(F_W y) - f(F_W x)) + f(F_W x) \cdot (\Phi_W(y) - \Phi_W(x))| \leq \\
\sum_{W \in \mathcal{W}(a, T)} |\Phi_W(y)| \cdot \theta \cdot \text{Lip}(f) \cdot |y - x| + \|f\|_{\infty} \cdot |\Phi_W(y) - \Phi_W(x)| \leq \\
C_3 \cdot (\theta \cdot \text{Lip}(f) + \|f\|_{\infty}) \cdot |y - x|,
\]

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where the second inequality follows from the estimate above and from Corollary 6.6, and where the last inequality follows from Corollary 8.3 in terms of some uniform constant $C_3$.

The uniform bound for $\|A_T\|_s$ follows combining the estimates above.

**Lemma 9.2.** There exists an uniform constant $C > 0$ such that for any $T$ with $T_0 \leq T \leq 1$ and any $s$ with $9/10 \leq s \leq 1$ we have

$$\|L(s,T) - L(st,T)\|_s \leq C \cdot |s - st| \quad \text{and} \quad \|L(s,T) - L(st,T) - (s - st)A_T\|_s \leq C \cdot |s - st|^2.$$

**Proof.** The first bound obviously follows from the second and from Lemma 9.1, thus we only prove that $\|R\|_s \leq C \cdot |s - st|^2$, where for convenience of notation we set

$$R := L(s,T) - L(st,T) - (s - st)A_T.$$

Fix $W \in \mathcal{W}$. In the notation of the proof of Lemma 9.1, consider the function

$$\Psi_W : \text{Dom}(W) \to \mathbb{R}_+ \quad ; \quad \Psi_W(x) := |D_x F_W|^s - |D_x F_W|^{st} - (s - st)\Psi_W(x).$$

According to Equation (9.2) for any $x$ as above we have

$$|\Psi_W(x)| \leq |s - st|^2 \cdot \ln |D_x F_W|^2 \cdot |D_x F_W|^{4/5}.$$

Moreover the same argument which gives Part (1) of Corollary 8.3 implies that there exists an uniform constant $C_1 > 0$ such that for any $a \in \mathcal{A}$ and any $x \in [a]$ we have

$$\sum_{W \in \mathcal{W}(a,T)} \ln |D_x F_W|^2 \cdot |D_x F_W|^{4/5} \leq C_1.$$

It follows immediately that $\|R\|_\infty \leq C_1 \cdot |s - st|^2$. Moreover for $W \in \mathcal{W}$ and $x \in \text{Dom}(W)$ Equation (9.1) gives

$$\Psi_W(x) = (\ln |D_x F_W|) \cdot \int_{sT}^s B_W(x,t)dt,$$

where we set $B_W(x,t) := |D_x F_W|^t - |D_x F_W|^{st}$. For $s_T < t < s$ or $s < t < s_T$, the first equality in Equation (9.1) gives

$$|B_W(x,t)| \leq |t - s_T| \cdot (\ln |D_x F_W|)|D_x F_W|^{4/5} \leq |s - s_T| \cdot \ln |D_x F_W|_\infty \cdot |DF_W|^{4/5}.$$

Moreover for $x, y$ in Dom$(W)$ again the first equality in Equation (9.1) gives

$$|B_W(y,t) - B_W(x,t)| =$$

$$\left| (\ln |D_y F_W|) \int_{sT}^t |D_y F_W|^r dr - (\ln |D_x F_W|) \int_{sT}^t |D_x F_W|^r dr \right| \leq$$

$$\ln |D_y F_W| \int_{sT}^t |D_y F_W|^r - |D_x F_W|^r dr + \ln |D_y F_W| - \ln |D_x F_W| \int_{sT}^t |D_x F_W|^r dr \leq$$

$$\ln |DF_W|_\infty \cdot |t - s_T| \cdot C_2 \cdot |DF_W|^{4/5} \cdot |y - x| + C_2 \cdot |y - x| \cdot |t - s_T| \cdot |DF_W|^{4/5} \leq$$

$$C_2 \cdot |DF_W|^{4/5} \cdot (1 + \ln |DF_W|_\infty) \cdot |s - s_T| \cdot |y - x|,$$

where $C_2$ is some uniform constant and in the second inequality $|D_y F_W|^r - |D_x F_W|^r|$ is bounded by Part (2) of Corollary 6.8 while $|\ln |D_y F_W| - \ln |D_x F_W||$ is bounded by
Lemma 6.7. The bounds obtained above for \(|B_W(y, t) - B_W(x, t)|\) and for \(|B_W(x, t)|\) give

\[
|\Psi_W(y) - \Psi_W(x)| = \\
\left| (\ln |D_y F_W|) \int_{sT}^s B(y, t) dt - (\ln |D_x F_W|) \int_{sT}^s B(x, t) dt \right| \\
\left| \ln |D_y F_W| \right| \int_{sT}^s |B(y, t) - B(x, t)| dt + \left| \ln |D_y F_W| - \ln |D_x F_W| \right| \int_{sT}^s B(x, t) dt \leq \\
\| \ln DF_W \|_\infty \cdot C_2 \cdot \| DF_W \|_F^{4/5} \cdot (1 + \| \ln DF_W \|_\infty) \cdot |s - s_T|^2 \cdot |y - x| + \\
C_3 \cdot |y - x| \cdot |s - s_T|^2 \cdot \| \ln DF_W \|_\infty \cdot \| DF_W \|_F^{4/5} \leq \\
C_4 \cdot (1 + \| \ln DF_W \|_\infty)^2 \cdot \| DF_W \|_F^{4/5} \cdot |y - x| \cdot |s - s_T|^2,
\]

where \(C_4\) is some uniform constant, and where \(C_3\) is the constant in Lemma 6.7. The same argument which gives Corollary 8.3 implies that there is some uniform constant \(C_5\) such that

\[
\sum_{W \in \mathcal{W}(a, T)} |\Psi_W(y) - \Psi_W(x)| \leq C_5 \cdot |y - x| \cdot |s - s_T|^2.
\]

Finally the statement of the Lemma follows from the estimates above because \(|Rf(y) - Rf(x)|\) is bounded by

\[
\sum_{W \in \mathcal{W}(a, T)} |\Psi_W(x)| \cdot |f(F_W y) - f(F_W x)| + |f(F_W x)| \cdot |\Psi_W(y) - \Psi_W(x)| \\
\sum_{W \in \mathcal{W}(a, T)} |\Psi_W(x)| \cdot \text{Lip}(f) \cdot \theta \cdot |y - x| + \| f \|_\infty \cdot |\Psi_W(y) - \Psi_W(x)|.
\]

\(\square\)

9.2. Expansion of transfer operator in parameter \(T\). Fix \(a \in A\) and \(0 < T \leq +\infty\) and set \(\mathcal{V}(a, T) := \mathcal{V}(a) \setminus \mathcal{W}(a, T)\), as in § 8.3. Consider parameters \(s, T\) with \(9/10 < s \leq 1\) and \(0 < T \leq +\infty\) and the operator \(\Delta_{(s, T)} : B \to B\) defined by

\[
\Delta_{(s, T)} f(x) := \sum_{W \in \mathcal{V}(a, T)} |D_x F_W|^s f(F_W x) \quad \text{if} \quad x \in [a].
\]

Lemma 9.3. There exists an uniform constant \(C > 0\) such that for any \(s, T\) as above \(\Delta_{(s, T)}\) is a bounded linear operator both on \(B\) and \(B_T\), with norm

\[
\| \Delta_{(s, T)} \|_* \leq C \cdot \left( \frac{1}{T} \right)^{2s-1}.
\]

Proof. The estimate for \(\| \Delta_{(s, T)} \|_\infty\) follows directly from Corollary 8.3. Moreover Corollary 6.8 gives a bound for the difference \(\| D_y F_W |^s - D_z F_W |^s \|\) for any \(a \in A\) and any \(x, y \in [a]\). Then, given any \(f \in B\) (resp. in \(B_T\)), the difference \(\Delta_{(s, T)} f(y) - \Delta_{(s, T)} f(x)\) can be bounded by the same arguments in the other analogous estimates in this paper. \(\square\)

Lemma 9.4. There exists an uniform constant \(C > 0\) such that for any \(T > 0\) we have

\[
\| \Delta_{(1, T)} - \Delta_{(s, T)} \|_* \leq \frac{C \cdot |s - 1|}{T^{3/5}}.
\]

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Proof. Recall the first part of Equation (9.1), and for any \( W \in \mathcal{W} \) and \( x \in \text{Dom}(W) \) set
\[
B_W(x) := |D_x F_W| - |D_x F_W|^s = (\ln |D_x F_W|) \cdot \int_s^1 |D_x F_W|^t dt.
\]
We have \( B_W(x) \leq \|\ln DF_W\|_{\infty} \cdot \|DF_W\|^s \cdot |s - 1| \). Since \( s > 9/10 \) then Corollary 8.3 implies that there is some uniform constant \( C > 0 \) such that
\[
\|\Delta_{(1,T)} - \Delta_{(s,T)}\|_{\infty} \leq |s - 1| \cdot \sum_{W \in \mathcal{V}(a,T)} \|\ln DF_W\|_{\infty} \cdot \|DF_W\|^s \leq \frac{C \cdot |s - 1|}{T^{3/5}}.
\]
Moreover for any \( W \in \mathcal{W} \) and any \( x, y \in \text{Dom}(W) \) we have
\[
|B_W(y) - B_W(x)| = \left| (\ln |D_y F_W|) \int_s^1 |D_y F_W|^t dt - (\ln |D_x F_W|) \int_s^1 |D_x F_W|^t dt \right| \leq \\
\|\ln |D_y F_W|\| \int_s^1 |D_y F_W|^t - |D_x F_W|^t dt + \|\ln |D_y F_W| - \ln |D_x F_W|\| \int_s^1 |D_x F_W|^t dt \leq \\
C_1 \cdot \left( \|\ln DF_W\|_{\infty} \cdot \|DF_W\|^s + \|DF_W\|^s \right) \cdot |s - 1| \cdot |x - y| = \\
C_1 \cdot \|DF_W\|^s \cdot (1 + \|\ln DF_W\|_{\infty}) \cdot |s - 1| \cdot |x - y|,
\]
where \( C_1 \) is some uniform constant and in the second inequality \( \|D_y F_W|^t - |D_x F_W|^t\| \) is bounded by Part (2) of Corollary 6.8 while \( \|\ln |D_y F_W| - \ln |D_x F_W|\| \) is bounded by Lemma 6.7. Therefore the upper bound for \( \text{Lip}((\Delta_{(1,T)} - \Delta_{(s,T)})f) \) follows as usual considering any \( a \in \mathcal{A} \) and any \( x, y \in [a] \) and bounding the difference
\[
(\Delta_{(1,T)} - \Delta_{(s,T)})f(y) - (\Delta_{(1,T)} - \Delta_{(s,T)})f(x) = \\
\sum_{W \in \mathcal{V}(a,T)} B_W(y)(f(F_W y) - f(F_W x)) + f(F_W x) \cdot (B_W(y) - B_W(x)).
\]
\[
\square
\]

9.3. Explicit form of spectral projectors for \( T = \infty \). In the next § 9.4 we introduce spectral projectors for the maximal eigenvalue of \( L_{(s,T)} \). It is practical to have explicit representation of such spectral projectors. Fix \( s > 0 \) and set \( g := g(s,\infty) \) and \( \lambda = \lambda(s,\infty) \), where \( g(s,\infty) \) and \( \lambda(s,\infty) \) are as in Theorem 8.1. Let also \( L := L_{(s,\infty)} \).

Consider the operator \( \mathcal{N} : \mathcal{B} \rightarrow \mathcal{B} \) defined by \( \mathcal{N} f(x) := g(s) \cdot f(x) \). It is easy to see that for any \( f \in \mathcal{B} \) we have
\[
\|\mathcal{N} f\|_{\infty} \leq \|g\|_{\infty} \cdot \|f\|_{\infty} \quad \text{and} \quad \text{Lip}(\mathcal{N} f) \leq \|g\|_{\infty} \cdot \text{Lip}(f) + \|f\|_{\infty} \cdot \text{Lip}(g).
\]
Therefore we have \( \|\mathcal{N}\|_{\ast} \leq \|g\|_{\ast} \). Moreover \( \mathcal{N} \) is also invertible with \( \|\mathcal{N}^{-1}\|_{\ast} \leq \|g^{-1}\|_{\ast} \), where we observe that \( g^{-1} \in L \) for \( g \in L \) strictly positive. Define the normalized operator \( \tilde{L} : \mathcal{B} \rightarrow \mathcal{B} \) by \( \tilde{L} = \lambda^{-1} \mathcal{N}^{-1} L \mathcal{N} \). For \( a \in \mathcal{A} \) and \( W \in \mathcal{W}(a) \) it is practical to introduce
\[
N_W : \text{Dom}(W) \rightarrow \mathbb{R}_+ \quad ; \quad N_W(x) := \lambda^{-1} \cdot \frac{|D_x F_W|^s g(F_W x)}{g(x)}.
\]
so that for any \( f \in \mathcal{B}, a \in \mathcal{A} \) and \( x \in [a] \) we have
\[
\widehat{L}f(x) = \sum_{w \in \mathcal{W}(a)} N_W(x)f(F_Wx).
\]
Since \( Lg = \lambda g \) then \( \widehat{L}1 = 1 \), and this corresponds to
\[
(9.5) \quad \sum_{w \in \mathcal{W}(a)} N_W(x) = 1 \quad \text{for any } a \in \mathcal{A}, x \in [a].
\]
Setting \( \mathcal{W}(k, a) := \mathcal{W}(k, a, \infty) \) we have \( \widehat{L}^k f(x) = \sum_{w_k \in \mathcal{W}(k, a)} N_{w_k}(x)f(F_{w_k}x) \), where for any \( k \in \mathbb{N} \) and any \( w_k \in \mathcal{W}(a) \) we consider function
\[
N_{w_k}(x) := \frac{|D_xF_{w_k}|^s \cdot g(F_{w_k}x)}{g(x)}.
\]
Equation (9.5) implies \( \sum_{w_k \in \mathcal{W}(k, a)} N_{w_k}(x) = 1 \) for any \( a \in \mathcal{A} \) and any \( x \in [a] \).

**Lemma 9.5.** There exists a constant \( C = C(s) > 0 \) such that for any \( k \in \mathbb{N} \) we have
\[
\text{Lip}(\widehat{L}^k f) \leq \theta^k \cdot \text{Lip}(f) + C \cdot \|f\|_\infty.
\]

**Proof.** Observe that \( N_W(\cdot) \) is the product of 3 positive factors \( a, b, c \). In order to estimate \( |N_W(y) - N_W(x)| \) one can consider 6 terms coupled in pairs, each pair being the product of two factors evaluated in the same point, times the difference of the third factor at the two points \( x, y \). For example \( a(x)b(y)(c(y) - c(x)) \). The estimate of two of this pairs are left to the reader, and for the third one we observe that
\[
|D_yF_W|^s - |D_xF_W|^s \leq |D_xF_W|^s \cdot (e^{C_1|y-x|} - 1) \leq |D_xF_W|^s \cdot e^{C_1|y-x|}C_1|y-x|
\]
\[
\leq (|D_xF_W|^s + |D_yF_W|^s) \cdot e^{C_1|y-x|}C_1|y-x|
\]
where \( C_1 > 0 \) denotes the uniform Lipschitz constant of \( x \mapsto \ln |D_xF_W|^s \) (see Lemma 6.7). The right hand side of the last inequality is symmetric in \( x, y \), thus it follows that
\[
\left| \frac{|D_yF_W|^s g(F_Wx)}{g(x)} - \frac{|D_xF_W|^s g(F_Wx)}{g(x)} \right| \leq \frac{g(F_Wx)}{g(x)} \cdot (|D_xF_W|^s + |D_yF_W|^s) \cdot e^{C_1|y-x|}C_1|y-x|
\]
\[
\leq \left( N_W(x) + \frac{g(F_Wx)g(y)}{g(x)g(F_Wy)} \cdot N_W(y) \right) \cdot e^{C_1|y-x|}C_1|y-x| \leq C_2(N_W(x) + N_W(y)) \cdot |y-x|,
\]
for some constant \( C_2 = C_2(s) > 0 \). Thus there exists a constant \( C_3 = C_3(s) > 0 \), depending on \( s \) via \( g \), such that
\[
|N_W(y) - N_W(x)| \leq C_3(N_W(x) + N_W(y)) \cdot |y-x|.
\]
Equation (9.5) gives
\[
\widehat{L}f(y) - \widehat{L}f(x) \leq \sum_{w \in \mathcal{W}(a)} N_W(y)|f(F_Wy) - f(F_Wx)| + f(F_Wx)|N_W(y) - N_W(x)|
\]
\[
\leq \theta \cdot \text{Lip}(f) + 2C_3 \cdot \|f\|_\infty.
\]
The statement follows for \( k = 1 \), then for general \( k \) one can argue as in the proof if Proposition 2.1 in [ParPo]. □
Remark 9.6. For any \((s,T)\) it is possible to define the operator \(\hat{L}(s,T)\) which normalizes \(L(s,T)\), and Lemma 9.3 still applies, so that \(\rho_{ess} \hat{L}(s,T) \leq \theta\), according to [Henn]. Since \(\mathcal{N}\) is an isomorphism of Banach spaces, then spectrum of \(L(s,T)\) is equal to the spectrum of \(\hat{L}(s,T)\) multiplied by \(\hat{\lambda}(s,T)\), and we have \(\rho_{ess} (L(s,T)) \leq \theta \cdot \hat{\lambda}(s,T)\). Therefore quasi-compactness of \(L(s,T)\) follows from Proposition 8.3 because \(\hat{\lambda}(s,T)\) is eigenvalue of \(L(s,T)\).

Lemma 9.7. There exists a Borel probability measure \(\nu\) on \(\partial \mathbb{D}\) such that for any \(f \in \mathcal{B}\) we have

\[
\int \hat{L} f \, d\nu = \int f \, d\nu.
\]

Proof. Consider the disjoint union \(\mathbb{E}_\infty := \bigsqcup_{a \in \mathcal{A}} [a]\), that is the set of pairs \((a,x)\) with \(a \in \mathcal{A}\) and \(x \in [a]\). A set \(E \subset \mathbb{E}_\infty\) is open iff \(\iota^{-1}_a(E)\) is open in \([a]\) for any \(a \in \mathcal{A}\), where we consider the maps \(\iota_a(x) := (a,x)\), and with this topology \(\mathbb{E}_\infty\) is a compact set. More precisely \(\mathbb{E}_\infty\) is homeomorphic to \([0,1]^4\). A function \(f : \mathbb{E}_\infty \to \mathbb{C}\) is continuous if and only if \(f \circ \iota_a : [a] \to \mathbb{C}\) is continuous for any \(a \in \mathcal{A}\). Let \(\mathcal{C}\) be the Banach space of continuous functions \(f : \mathbb{E}_\infty \to \mathbb{C}\) with the sup norm \(\| \cdot \|_\infty\). For any \(a \in \mathcal{A}\) and any \(W \in \mathcal{W}(a)\) both \(F_W\) and \(|DF_W|\) admit a continuous extension to \([a]\). The same is true for the functions \(g\) and \(g^{-1}\). Therefore the operator \(\hat{L}\) acts on \(\mathcal{C}\). The condition \(\hat{L} 1 = 1\) implies that the dual operator \(\hat{L}^* : c^* \to c^*\) preserves the convex weakly compact subset \(\mathcal{M}(\mathbb{E}_\infty)\) of Borel probability measures over \(\mathbb{E}_\infty\). Thus there exists \(m \in \mathcal{M}(\mathbb{E}_\infty)\) such that \(\hat{L}^* m = m\), according to the Schauder-Tychonoff fixed point theorem. Given the inclusion maps \(p_a : [a] \to \partial \mathbb{D}\), let \(p : \mathbb{E}_\infty \to \partial \mathbb{D}\) be the unique continuous map such that \(p \circ \iota_a = p_a\) for any \(a \in \mathcal{A}\). Then consider the Borel probability measure \(\nu := p^*(m)\) over \(\partial \mathbb{D}\). For any \(f \in \mathcal{B}\) we have

\[
\int f \, d\nu = \int f \circ pdm = \int f \circ p\hat{L}^* m = \int \hat{L}(f \circ p)dm = \int (\hat{L}f) \circ pdm = \int \hat{L}fd\nu.
\]

\(\square\)

Lemma 9.8. For any \(f \in \mathcal{B}\) we have uniform convergence

\[
\hat{L}^k f \to \int f \, d\nu.
\]

Proof. Set \(f^{(k)} := \hat{L}^k f\). Lemma 9.3 and the Theorem of Ascoli-Arzelà imply that there exists \(f_\infty \in \mathcal{B}\) with \(f^{(k_n)} \to f\) uniformly as \(n \to \infty\) along some subsequence \(k_n\). Equation (9.5) implies \(\sup f^{(k+1)} \leq \sup f^{(k)}\) for any \(k \in \mathbb{N}\). Moreover \(f^{(k)}(x) \geq \inf f\) for any \(k \in \mathbb{N}\) and any \(x\), where \(\inf f \to -\infty\) strictly. Fix \(N \in \mathbb{N}\). The last two conditions imply \(\sup \hat{L}^N f_\infty = \sup f_\infty\). Indeed assume there exists \(\epsilon > 0\) with \(\sup \hat{L}^N f_\infty \leq \sup f_\infty - 3\epsilon\). We have

\[
\|f^{(k_n)} - f_\infty\|_\infty < \epsilon \quad ; \quad \|\hat{L}^N f^{(k_n)} - \hat{L}^N f_\infty\|_\infty < \epsilon \quad ; \quad |\sup f^{(k_n)} - \sup f^{(k_n+N)}| < \epsilon
\]

for any \(n\) big enough. Therefore we get an absurd because

\[
\sup \hat{L}^N f_\infty > \sup \hat{L}^N f^{(k_n)} - \epsilon = \sup f^{(k_n+N)} - \epsilon > \sup f^{(k_n)} - 2\epsilon > \sup f_\infty - 3\epsilon.
\]

Consider \(x_0\) and \(x_N\) with \(f_\infty(x_0) = \sup f_\infty = \sup \hat{L}^N f_\infty = \hat{L}^N f_\infty(x_N)\), where \(N\) is the integer fixed above. Such points exist modulo replacing \(f_\infty\) and \(\hat{L}^N f_\infty\) by \(f_\infty \circ p\) and \(\hat{L}^N f_\infty \circ p\).
where \( p : \mathbb{E}_\infty \to \partial \mathbb{D} \) is the continuous map in the proof of Lemma 9.7. We have
\[
\hat{L}^N f_\infty(x_N) = \sum_{w_k \in \mathcal{W}(k,a)} N_{w_k}(x_N) f_\infty(F_{w_k}x_N) = f_\infty(x_0).
\]
Since \( \sum_{w_k \in \mathcal{W}(k,a)} N_{w_k}(x_N) = 1 \) and any term in the sum is positive, then for any \( w_n \in \mathcal{W}(k,a) \) we have \( f_\infty(F_{w_n}x_N) = f_\infty(x_0) \), and thus continuity implies \( f_\infty(x) = f_\infty(x_0) \) for any \( x \). Finally, setting \( \nu^{(k)} := (\hat{I}^k)^* \nu \) we get
\[
f_\infty(x_0) = \int f_\infty dv = \lim_{n \to \infty} \int f^{(k_n)} dv = \lim_{n \to \infty} \int dv^{(k_n)} = \int dv.
\]
The argument can be repeated for any subsequence of \( f^{(k)} \), extracting a sub-subsequence converging to the integral. Thus the Lemma follows. \( \square \)

Let \( \nu_{(s,\infty)} := \nu \) be the probability measure as in Lemma 9.7 and Lemma 9.8. Let \( \mu_{(s,\infty)} \) be the probability measure on \( \partial \mathbb{D} \) defined by
\[
\mu_{(s,\infty)}(f) := \int f(x) g_{(s,\infty)}^{-1}(x) dv_{(s,\infty)}(x) \quad \text{for any} \quad f \in C(\partial \mathbb{D}).
\]

**Corollary 9.9.** For any \( f \in \mathcal{B} \) we have
\[
\hat{\lambda}(s,\infty)^{-k} \cdot L_{(s,\infty)}^k f \to \left( \int f dv_{(s,\infty)} \right) \cdot g_{(s,\infty)} \quad \text{uniformly as} \quad k \to \infty.
\]

**Proof.** The Corollary follows because for any \( f \in \mathcal{B} \) Lemma 9.8 implies
\[
\hat{\lambda}(s,\infty)^{-k} \cdot L_{(s,\infty)} f = \mathcal{N} \hat{I}^k \mathcal{N}^{-1} f \to \left( \int f(x) g_{(s,\infty)}^{-1}(x) dv_{(s,\infty)}(x) \right) g_{(s,\infty)}.
\]

\( \square \)

### 9.4. Spectral properties for parameters close to \((1, \infty)\). Refer to §B.1 for basic properties of spectral projectors. Recall from Theorem 8.1 that \( L_{(1,\infty)} \) is quasi-compact with isolated and simple maximal eigenvalue \( \hat{\lambda}(1,\infty) = 1 \). Hence there exists \( \epsilon_0 > 0 \) such that its spectrum can be decomposed as
\[
\text{sp}(L_{(1,\infty)}) = \Sigma \cup \{1\} \quad \text{with} \quad \Sigma \subset B(0,1-2\epsilon_0).
\]
In terms of such \( \epsilon_0 \) let \( \gamma \) and \( \gamma^* \) be the loops in \( \mathbb{C} \) defined for \( 0 \leq t < 2\pi \) by
\[
\gamma(t) := 1 + \epsilon_0 \cdot e^{it} \quad \text{and} \quad \gamma^*(t) := (1 - \epsilon_0) \cdot e^{it}.
\]
For parameters \( (s, T) \) consider the expression
\[
P_{(s,T)} := -\frac{1}{2\pi i} \int_{\gamma} (L_{(s,T)} - \xi \cdot \text{Id})^{-1} d\xi.
\]
According to §B.1 the operator \( P_{(s,T)} \) is defined if and only if \( \gamma \) is contained in the resolvent set \( \mathcal{R}(L_{(s,T)}) \), and in this case is a projection commuting with \( L_{(s,T)} \), that is \( P^2_{(s,T)} = P_{(s,T)} \) and \( L_{(s,T)} P_{(s,T)} = P_{(s,T)} L_{(s,T)} \), so that it induces an \( L_{(s,T)} \)-invariant spectral decomposition:
\[
\mathcal{B} = L_{(s,T)} \oplus V_{(s,T)} \quad \text{where} \quad V_{(s,T)} := \ker(P_{(s,T)} - \text{Id}) \quad \text{and} \quad N_{(s,T)} := \ker(P_{(s,T)}).
\]
The discussion above implies that $P_{(1,\infty)}$ is defined, and the corresponding spectral decomposition $\mathcal{B} = N_{(1,\infty)} \oplus V_{(1,\infty)}$ satisfies

$$V_{(1,\infty)} = \ker(L_{(1,\infty)} - \text{Id}) \quad \text{and} \quad \rho(L_{(1,\infty)}|_{N_{(1,\infty)}}) \leq 1 - 2\varepsilon_0.$$ 

For $(s,T)$ close to $(1,\infty)$ let $\tilde{\lambda}(s,T)$ and $g_{(s,T)} \in \Lambda$ be the simple eigenvalue and the corresponding eigenfunction as in Theorem 8.1.

**Corollary 9.10.** There exist $T_0 > 0$ and $0 < s_0 < 1$ such that the operator $L_{(s,T)} : \mathcal{B} \to \mathcal{B}$ is quasi-compact for any $(s,T) \in [s_0,1] \times [T_0,+,\infty]$. More precisely the following holds.

1. The maximal eigenvalue of $L_{(s,T)}$ is $\tilde{\lambda}(s,T)$ and we have $|\tilde{\lambda}(s,T) - 1| < \epsilon_0$.
2. The eigenspace $V_{(s,T)} = \ker((L_{(s,T)} - \tilde{\lambda}(s,T)\cdot \text{Id})$ has dimension one and is generated by $g_{(s,T)}$.
3. The restriction $L_{(s,T)} : N_{(s,T)} \to N_{(s,T)}$ satisfies
   $$\rho((L_{(s,T)}|_{N_{(s,T)}}) \leq 1 - \epsilon_0.$$ 
4. The projection given by Equation (9.8) satisfies
   $$\|P_{(s,T)} - P_{(1,\infty)}\|_* = O(\|L_{(s,T)} - L_{(1,\infty)}\|_*).$$

**Proof.** Since $L_{(s,T)} = L_{(s,\infty)} - \Delta_{(s,T)}$ for any $(s,T)$, then Lemma 9.3 and Lemma 9.2 imply that $\|L_{(s,T)} - L_{(1,\infty)}\|_*$ can be made arbitrarily small for $(s,T)$ close enough to $(1,\infty)$. The discussion in §B.2 implies that for any such $(s,T)$ both the loops $\gamma$ and $\gamma_s$ in Equation (9.1) are contained in the resolvent set $\mathcal{R}(L_{(s,T)})$. Thus $P_{(s,T)}$ is defined and Point (4) follows. Modulo taking $(s,T)$ closer to $(1,\infty)$, we have a linear isomorphism $G_{(s,T)} : \mathcal{B} \to \mathcal{B}$ such that $G_{(s,T)}(V_{(1,\infty)}) = V_{(s,T)}$ and $G_{(s,T)}(N_{(1,\infty)}) = N_{(s,T)}$, and moreover

$$G_{(s,T)} - \text{Id} = O(\|L_{(s,T)} - L_{(1,\infty)}\|_*) \quad \text{and} \quad G_{(s,T)}^{-1} - \text{Id} = O(\|L_{(s,T)} - L_{(1,\infty)}\|_*)$$

It follows that $V_{(1,\infty)}$ has dimension 1 and is invariant under $L_{(s,T)}$, so that it is generated by some $\tilde{g}_{(s,T)} \in \mathcal{B}$ with $L_{(s,T)}(\tilde{g}_{(s,T)}) = \tilde{\lambda}(s,T) \cdot \tilde{g}_{(s,T)}$ for some $\tilde{\lambda}(s,T) \in \mathbb{C}$. Applying either Point (2) of Theorem 8.1 (if $T < \infty$) or Corollary 9.9 (if $T = \infty$) to $f = \tilde{g}_{(s,T)}$ we obtain

$$\tilde{\lambda}(s,T)^n \cdot \tilde{g}_{(s,T)} \to \left(\int \tilde{g}_{(s,T)}d\mu_{(s,T)}\right)g_{(s,T)} \quad \text{as} \quad n \to \infty.$$ 

Since $\tilde{g}_{(s,T)} - g_{(1,\infty)} = O(\|P_{(s,T)} - P_{(1,\infty)}\|_*)$ and $g_{(1,\infty)}$ is strictly positive, then the two functions in the limit above are both never zero. It follows that

$$\tilde{\lambda}(s,T) = \tilde{\lambda}(s,T) \quad \text{and} \quad \tilde{g}_{(s,T)} = g_{(s,T)}.$$ 

By §B.1 we know that $\text{sp}(L_{(s,T)}|_{N_{(s,T)}})$ is contained in the interior of $\gamma_s$. For the same reason $\tilde{\lambda}(s,T)$ is the part of $\text{sp}(L_{(s,T)})$ included in the interior of $\gamma$. Finally the spectrum of $L_{(s,T)}$ does not have other components, because $\mathcal{B} = N_{(s,T)} \oplus V_{(s,T)}$. The Corollary is proved. \(\square\)

**Lemma 9.11.** For any $T$ big enough and any $f \in \mathcal{B}$ we have

$$P_{(s,T)}(f) = \left(\int f(\xi)d\mu_{(s,T)}(\xi)\right) \cdot g_{(s,T)}.$$ 

In particular the formula above holds for $(s,T) = (1,\infty)$. 

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Proof. For $T < \infty$ this follows from Point (2) in Theorem 8.1. For $T = \infty$ the statement follows from Corollary 9.9.

9.5. Expansion of maximal eigenvalue in parameters $s$ and $T$. We follow § 7 in [He]. For $(s, T)$ close to $(1, \infty)$ call $\lambda(s, T)$ the eigenvalue in Corollary 9.10 also for the value $T = \infty$. Consider the spectral decomposition
\[ \mathcal{B} = N_{(1, \infty)} \oplus V_{(1, \infty)}, \]
where $N_{(1, \infty)} := \ker(P_{(1, \infty)})$ and $V_{(1, \infty)} := \ker(\text{Id} - P_{(1, \infty)})$. Let $\delta \in \mathbb{R}$ and $u_{(1, \infty)} \in N_{(1, \infty)}$ such that we have
\[ A_{\infty}(g_{(1, \infty)}) = \delta \cdot g_{(1, \infty)} + u_{(1, \infty)}. \]
We have $\delta < 0$ strictly, indeed $|D_x F_W| \leq \theta$ for any $W \in \mathcal{W}$ and any $x \in \text{Dom}(W)$, and the expression of $A_{\infty}$ in Equation (9.3) gives
\[ \delta = \int A_{\infty}g_{(1, \infty)}d\mu_{(1, \infty)} = \sum_{a \in A, W \in W(a)} \int |D_x F_W| \cdot |D_x F_W| \cdot g_{(1, \infty)}(F_W x) d\mu_{(1, \infty)}(x). \]

Proposition 9.12. We have
\[ \lambda(s, \infty) = 1 + \delta(s - 1) + O(|s - 1|^2). \]

Proof. According to Corollary 9.10, the restriction $L_{(1, \infty)}|_{N_{(1, \infty)}} : N_{(1, \infty)} \to N_{(1, \infty)}$ has spectral radius $\rho(L_{(1, \infty)}|_{N_{(1, \infty)}}) \leq 1 - 2\varepsilon_0$, thus its resolvent $R(L_{(1, \infty)}|_{N_{(1, \infty)}})$ contains $z = 1$. Therefore consider $u_{(1, \infty)} \in N_{(1, \infty)}$ in the decomposition of $A_{\infty}(g_{(1, \infty)})$ and set
\[ v_{(1, \infty)} := ((1 - L_{(1, \infty)})|_{N_{(1, \infty)}})^{-1}(u_{(1, \infty)}) = -R(L_{(1, \infty)}|_{N_{(1, \infty)}}, 1)(u_{(1, \infty)}). \]
Since $L_{(1, \infty)}(g_{(1, \infty)}) = g_{(1, \infty)}$, the definition of $v_{(1, \infty)}$ and Lemma 9.2 give
\[ L_{(s, \infty)}(g_{(1, \infty)} + (s - 1)v_{(1, \infty)}) = \]
\[ (1 + \delta(s - 1))g_{(1, \infty)} + (s - 1)(u_{(1, \infty)} + L_{(1, \infty)}(v_{(1, \infty)})) + O(|s - 1|^2) = \]
\[ (1 + \delta(s - 1))g_{(1, \infty)} + (s - 1)v_{(1, \infty)} + O(|s - 1|^2). \]

On the other hand we have $L_{(s, \infty)} \circ P_{(s, \infty)} = P_{(s, \infty)} \circ L_{(s, \infty)}$, therefore
\[ \lambda(s, \infty)P_{(s, \infty)}(g_{(1, \infty)} + (s - 1)v_{(1, \infty)}) = \]
\[ P_{(s, \infty)}L_{(s, \infty)}(g_{(1, \infty)} + (s - 1)v_{(1, \infty)}) = \]
\[ P_{(s, \infty)}((1 + \delta(s - 1))(g_{(1, \infty)} + (s - 1)v_{(1, \infty)}) + O(|s - 1|^2)) = \]
\[ (1 + \delta(s - 1))P_{(s, \infty)}(g_{(1, \infty)} + (s - 1)v_{(1, \infty)}) + O(|s - 1|^2), \]
where the third equality follows from the expression for $L_{(s, \infty)}(g_{(1, \infty)} + (s - 1)v_{(1, \infty)})$ obtained above, the first two equalities are just standard algebra, and the forth inequality follows because $P_{(s, \infty)}$ is close to $P_{(1, \infty)}$, and thus it has norm close to $\|P_{(1, \infty)}\|_s$. Therefore
\[ \left( \lambda(s, \infty) - (1 + \delta(s - 1)) \right) \cdot P_{(s, \infty)}(g_{(1, \infty)} + (s - 1)v_{(1, \infty)}) = O(|s - 1|^2). \]

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The statement follows because \( \|g_{(1,\infty)} + (s-1)v_{(1,\infty)}\|_* \geq \|g_{(1,\infty)}\|_* - |s-1| \cdot \|v_{(1,\infty)}\|_* \) and because \( P_{(s,\infty)} \) is close to \( P_{(1,\infty)} \), which acts on \( g_{(1,\infty)} \) as the identity. \( \square \)

For \( 0 < T \leq \infty \) consider the spectral decomposition
\[
\mathcal{B} = N_{(s,T,\infty)} \oplus V_{(s,T,\infty)},
\]
where \( N_{(s,T,\infty)} := \ker(P_{(s,T,\infty)}) \) and \( V_{(s,T,\infty)} := \ker(Id - P_{(s,T,\infty)}) \). Consider \( \beta_T \in \mathbb{R} \) and \( u_{(s,T,\infty)} \in N_{(s,T,\infty)} \) such that, in the direct sum above, we have
\[
\Delta_{(s,T)}g_{(s,T,\infty)} = \beta_T \cdot g_{(s,T,\infty)} + u_{(s,T,\infty)}.
\]

The asymptotic of \( \beta_T \) as \( T \to \infty \) is determined in the next § 9.6.

**Proposition 9.13.** We have
\[
\lambda(s,T) = \lambda(s,T,\infty) - \beta_T + O(T^{-2sT-1}) \cdot (O(|sT-1|) + O(T^{-2sT-1})).
\]

**Proof.** According to Corollary 9.10, the restriction of \( L_{(s,T,\infty)} \) to the invariant space \( N_{(s,T,\infty)} \) has spectral radius \( \rho(L_{(s,T,\infty)}|N_{(s,T,\infty)}) \leq 1 - \epsilon_0 \), thus its resolvent set \( \mathcal{R}(L_{(s,T,\infty)}|N_{(s,T,\infty)}) \) contains \( z = 1 \). For any \( u \in N_{(s,T,\infty)} \) we have
\[
(L_{(s,T,\infty)} - \text{Id})u = (L_{(s,T,\infty)} - \text{Id})P_{(s,T,\infty)}u = \]
\[
((L_{(1,\infty)} - \text{Id}) + (L_{(s,T,\infty)} - L_{(1,\infty)}))(P_{(1,\infty)} + (P_{(s,T,\infty)} - P_{(1,\infty)}))u =
\]
\[
(L_{(1,\infty)} - \text{Id})P_{(1,\infty)}u + O(\|L_{(s,T,\infty)} - L_{(1,\infty)}\|_*)u.
\]

Consider the invertible bounded operator \( G : \mathcal{B} \to \mathcal{B} \) with \( GP_{(1,\infty)}G^{-1} = P_{(s,T,\infty)} \), introduced in § 9.2. We have \( u = G(v) \) for some \( v \in N_{(1,\infty)} \), and developing
\[
u = G(v) = v + (G - \text{Id})v =
\]
\[
v + O(\|P_{(1,\infty)}\|_* \cdot \|P_{(s,T,\infty)} - P_{(1,\infty)}\|_*)v = v + O(\|L_{(s,T,\infty)} - L_{(1,\infty)}\|_*)v,
\]
we get
\[
\|(L_{(s,T,\infty)} - \text{Id})u\|_* \geq \|(L_{(1,\infty)} - \text{Id})v\|_* - \epsilon\|v\|_* \geq c_0\|v\|_* \geq \epsilon_0\|u\|_*
\]
where \( \epsilon > 0 \) above can be made arbitrarily small for \( \|L_{(s,T,\infty)} - L_{(1,\infty)}\|_* \) small enough, and \( c_0 > 0 \) and \( \epsilon_0 > 0 \) are uniform constant not depending on \( T \). In conclusion we have the uniform bound (i.e. not depending on \( T \)):
\[
\|R(L_{(s,T,\infty)}|N_{(s,T,\infty)}, 1)\|_* \leq (\epsilon_0')^{-1}.
\]

Consider \( u_{(s,T,\infty)} \) in the decomposition of \( \Delta_{(s,T)}g_{(s,T,\infty)} \) and set
\[
u_{(s,T,\infty)} := ((\text{Id} - L_{(s,T,\infty)})|N_{(s,T,\infty)})^{-1}(u_{(s,T,\infty)}) = -R(L_{(s,T,\infty)}|N_{(s,T,\infty)}, 1)(u_{(s,T,\infty)}).
\]

It is practical to introduce the quantity
\[
\text{Err}(T) := (\lambda(s,T,\infty) - \beta_T) \cdot v_{(s,T,\infty)} - u_{(s,T,\infty)} - L_{(s,T,\infty)}(v_{(s,T,\infty)}) + \Delta_{(s,T)}(v_{(s,T,\infty)}).
\]

Recall that \( \Delta_{(s,T)} = O(T^{-2sT-1}) \) and observe that this implies \( \beta_T g_{(s,T,\infty)} = O(T^{-2sT-1}) \) and thus \( \beta_T = O(T^{-2sT-1}) \), since \( \|g_{(s,T,\infty)}\|_* > 0.5 \cdot \|g_{(1,\infty)}\|_* \) for \( T \) big enough. From the above uniform bound on \( \|R(L_{(s,T,\infty)}|N_{(s,T,\infty)}, 1)\|_* \) we get \( \nu_{(s,T,\infty)} = O(T^{-2sT-1}) \) also.

Therefore the definition of \( v_{(s,T,\infty)} \) and Lemma 9.3 give
\[
\text{Err}(T) = \lambda(s,T,\infty) \cdot v_{(s,T,\infty)} - u_{(s,T,\infty)} - L_{(s,T,\infty)}(v_{(s,T,\infty)}) + O(T^{-2sT-1}) =
\]
\[
(\lambda(s,T,\infty) - 1) \cdot v_{(s,T,\infty)} + O(T^{-2(2sT-1)}) = O(T^{-2sT-1}) \cdot (O(|sT-1|) + O(T^{-2sT-1})),
\]
where the last equality follows from Proposition 9.12. Since \( L_{(s,T)}(g_{(s,T)}) = g_{(s,T)} \), then the definition of \( v_{(s,\infty)} \) gives

\[
L_{(s,T)}(g_{(s,+,\infty)} - v_{(s,\infty)}) = \\
(L_{(s,\infty)} - \Delta_{(s,T)}) (g_{(s,\infty)} - v_{(s,\infty)}) = \\
(\lambda(s_T, \infty) - \beta_T) \cdot g_{(s,\infty)} - u_{(s,\infty)} - L_{(s,\infty)}(v_{(s,\infty)}) + \Delta_{(s,T)}(v_{(s,\infty)}) = \\
(\lambda(s_T, \infty) - \beta_T) \cdot (g_{(s,\infty)} - v_{(s,\infty)}) + \text{Err}(T).
\]

On the other hand we have \( L_{(s,T)} \circ P_{(s,T)} = P_{(s,T)} \circ L_{(s,T)} \), therefore standard algebra gives

\[
\lambda(s_T, T) P_{(s,T)}(g_{(s,+,\infty)} - v_{(s,\infty)}) = \\
L_{(s,T)} P_{(s,T)}(g_{(s,+,\infty)} - v_{(s,\infty)}) = \\
P_{(s,T)} L_{(s,T)}(g_{(s,+,\infty)} - v_{(s,\infty)}) = \\
P_{(s,T)} \left( (\lambda(s_T, \infty) - \beta_T) \cdot (g_{(s,\infty)} - v_{(s,T)}) + \text{Err}(T) \right) = \\
(\lambda(s_T, \infty) - \beta_T) \cdot P_{(s,T)}(g_{(s,\infty)} - v_{(s,T)}) + P_{(s,T)}(\text{Err}(T)).
\]

Since \( P_{(s,T)} \) is close to \( P_{(1,\infty)} \), then its norm admits an uniform bound which does not depend on \( T \). It follows that \( P_{(s,T)}(\text{Err}(T)) = O(\text{Err}(T)) \). Therefore

\[
\left( \lambda(s_T, T) - (\lambda(s_T, \infty) - \beta_T) \right) \cdot P_{(s,T)}(g_{(s,\infty)} - v_{(s,\infty)}) = \\
O(\text{Err}(T)) = O(T^{-2s_T}) \cdot O(|s_T - 1| + O(T^{-2s_T})).
\]

The statement follows because \( P_{(s,T)}(v_{(s,\infty)}) = O(T^{-2s_T}) \), while on the other hand \( P_{(s,T)}(g_{s,\infty}) \) is close to \( g_{(1,\infty)} \).

9.6. **Asymptotic of \( \beta_T \).** In this section we prove that there exists a strictly positive uniform constant \( \beta > 0 \) such that

\[
\beta_T = \frac{\beta}{T} + o(T^{-1}).
\]

Independent covering arguments give a fist row lower bound on \(|s_T - 1|\). For example Theorem 1.1 in [MarTrWe], translated into the setting of Fuchsian groups, gives

\[
|s_T - 1| = o(T^{-1/2}).
\]

**Lemma 9.14.** We have

\[
\beta_T = \int \Delta_{(1,T)} g_{(1,\infty)} d\mu_{(1,\infty)} + o(T^{-1}).
\]

**Proof.** Recall that \( g_{(s,\infty)} - g_{(1,\infty)} = O\left( \| P_{(s,\infty)} - P_{(1,\infty)} \| \right) \) and that

\[
P_{(s,\infty)} - P_{(1,\infty)} = O\left( \| L_{(s,\infty)} - L_{(1,\infty)} \| \right) = O(|s_T - 1|).
\]

We have the development \( \beta_T \cdot g_{(s,\infty)} = \beta_T \cdot g_{(1,\infty)} + o(T^{-1}) \), because

\[
\beta_T \cdot (g_{(1,\infty)} - g_{(s,\infty)}) = O(|s_T - 1| \cdot T^{-4/5}) = o(T^{-1}).
\]
On the other hand $\beta_T \cdot g(\ast_T, \infty) = P_{(\ast_T, \infty)} \Delta_{(\ast_T, T)}(g(\ast_T, \infty))$, moreover the three terms
\[
\Delta_{(\ast_T, T)}(g(\ast_T, \infty) - g(1, \infty)) \; ; \; (\Delta_{(1, T)} - \Delta_{(\ast_T, T)}) \; ; \; (P_{(\ast_T, \infty)} - P_{(1, \infty)}) \Delta_{(1, T)}
\]
are of the order of $|s_T - 1| \cdot T^{-3/5} = o(T^{-1})$, where in particular $\|\Delta_{(1, T)} - \Delta_{(\ast_T, T)}\|$ is bounded by Lemma 9.4. Therefore the Lemma follows comparing the development above with Lemma 3.6, for any $\beta > 0$.

There exists the limit
\[
\beta := \lim_{T \to +\infty} T \cdot \int \Delta_{(1, T)} g(1, \infty) d\mu(1, \infty).
\]
Moreover $\beta > 0$ strictly.

Proof. For simplicity, write $g := g(1, \infty)$ and $\mu := \mu(1, \infty)$. According to Lemma 3.5 and Lemma 3.6 for any $W \in \mathcal{V}(a, T)$ there exist $k \in \mathbb{N}$, a parabolic word $P$ and a cuspidal word $V \in \mathcal{W}(a)$ which is an initial factor of $P$, such that
\[
W = P^{(k)} \ast V \quad \text{where} \quad P^{(k)} := P \ast \cdots \ast P. \quad \text{(9.9)}
\]
The set $\mathcal{P}(a)$ of parabolic words $P$ as above is finite (with cardinality bounded by twice the number of vertices of $\Omega_2$). For $P \in \mathcal{P}(a)$ let $\mathcal{I}(P)$ be the set of initial factors of $P$, which is obviously a finite set with cardinality the number of letters of $P = (a_1, \ldots, a_p)$ (here the empty word counts as a factor).

Fix $P \in \mathcal{P}(a)$ and let $\xi_P$ be the vertex of $\Omega_2$ which is the parabolic fixed point of $F_P$. For $W = P^{(k)} \ast V$ the arc $[W]_E$ shrinks to $\xi_P$ as $k \to \infty$. Moreover we have $F_W(\xi) \in [W]_E$ for any $\xi \in \text{Dom}(W)$. Since $g \in \mathcal{B}$, then there exist the limits of $g(\xi)$ for $\xi \to \xi_P$ either from the left or from the right. Denote these two limits as $g(\xi_P, \varepsilon(P))$, in terms of the symbol $\varepsilon(P) \in \{L, R\}$. The discussion above implies that the limit below exists:
\[
\lim_{k \to \infty} g(F_{P^{(k)}} \ast V \cdot \xi) = g(\xi_P, \varepsilon(P)).
\]

Let $\mu > 0$ be such that $F_P$ is conjugated in $\text{SL}(2, \mathbb{C})$ to $z \mapsto 2k\mu$. Then in the notation of Equation (2.1), the entries $\alpha = \alpha(F_P)$ and $\beta = \beta(F_P)$ are given by $\alpha = 1 + ik\mu$ and $\beta = ik\mu \cdot \xi_P$. For $\xi \in \text{Dom}(P)$ standard algebra gives $-\pi/\beta = -\xi_P \cdot \left(\frac{1 + ik\mu}{ik\mu}\right)$, and thus
\[
|D_{\xi} F_P| = \frac{1}{k^2} \cdot \frac{1}{|\xi + \xi_P (1 - i(k\mu)^{-1})|^2}.
\]

For $N \in \mathbb{N}^*$ we have the identity $1 = N \cdot \sum_{k=n}^{\infty} (k(k + 1))^{-1}$. This implies that $\lim_{k \to \infty}$ with $a_k \to 1$. However, then $N \cdot \sum_{k=N}^{\infty} a_k \cdot k^{-2} \to \lambda$ as $N \to \infty$. The discussions above, and this last remark, imply that for any $P \in \mathcal{P}(a)$, any $V \in \mathcal{I}(P)$ and any $\xi \in \text{Dom}(W = P^{(k)} \ast V)$ we have
\[
\lim_{N \to \infty} N \cdot \sum_{k=N}^{\infty} |D_{F_V(\xi)} F_P| \cdot g(F_{P^{(k)}} \ast V, \xi) = \frac{g(\xi_P, \varepsilon(P))}{|F_V(\xi) + \xi_P|^2}.
\]
The limit in the statement is equal to \( \sum_{a \in A} \left( \lim_{T \to \infty} \int_{[a]} f_{a,T} d\mu \right) \), where for any fixed \( a \in A \) we consider the function \( f_{a,T} : [a] \to \mathbb{R}_+ \) defined by

\[
f_{a,T}(\xi) := T \cdot \sum_{W \in \mathcal{V}(a,T)} |D_\xi F_W| \cdot g(F_W \xi).
\]

Fix \( a \in A \) and for any \( W \in \mathcal{V}(a,T) \) consider its decomposition \( W = P^{(k)} \ast V \) as in Equation (9.9). According to their definition in Equation (4.2), the geometric lengths \( |P| \) and \( |W| \) satisfy the relation \( ||W| - k \cdot |P|| \leq C \), where \( C > 0 \) is some uniform constant. Letting \( N(P,T) \) be the integer part of \( T \cdot |P|^{-1} \), the discussion above and the estimate in Lemma 8.2 give

\[
f_{a,T}(\xi) = \sum_{P \in \mathcal{P}(a)} \sum_{V \in \mathcal{I}(P)} |P| \cdot N(P,T) \cdot \sum_{k=0}^{\infty} |D_{F_V(\xi)} F_P^k| \cdot |D_\xi F_V| \cdot g(F_P(\xi) V + O(T^{-2})
\]

\[
= \sum_{P \in \mathcal{P}(a)} \sum_{V \in \mathcal{I}(P)} |P| \cdot g(\xi_P, \varepsilon(P)) \left| F_V(\xi) + \xi_P \right| + O(T^{-1}).
\]

The expression above is integrable, because \( F_V(\xi) \in [V]_\varepsilon \) and the latter is close to \( \xi_P \), thus bounded far away from \( -\xi_P \), so that the denominator is bounded and hence integrable. The existence of the limit \( \beta \) follows. Moreover \( \beta > 0 \) strictly because \( g \) is strictly positive. □

9.7. End of the proof of Theorem 4.3 Combining Proposition 9.12 and Proposition 9.13 we get

\[
1 = \lambda(s_T, T) = \lambda(s_T, \infty) - \beta_T + O(T^{-2s_T-1}) \cdot \left( O(|s_T - 1|) + O(T^{-2s_T-1}) \right) = 1 + \delta(s_T - 1) + O(|s_T - 1|^2) - \beta_T + O(T^{-2s_T-1}) \cdot \left( O(|s_T - 1|) + O(T^{-2s_T-1}) \right),
\]

that is

\[
s_T = 1 + \frac{\beta_T}{\delta} + O(|s_T - 1|^2) + O(T^{-2s_T-1}) \cdot \left( O(|s_T - 1|) + O(T^{-2s_T-1}) \right).
\]

Recall from § 9.6 that \( |s_T - 1| = o(T^{-1/2}) \). In particular \( 2s_T - 1 > 4/5 \) and the expression in Equation (9.10) turns into

\[
s_T = 1 + \frac{\beta_T}{\delta} + o(T^{-1}) = 1 - \left( \frac{\beta}{\delta} \right) \cdot \frac{1}{T} + o(T^{-1}),
\]

where \( \beta > 0 \) is the constant in Lemma 9.15. Therefore Theorem 4.3 follows from Lemma 9.14 and Lemma 9.15. The explicit form of the constant \( \Theta \) in Theorem 4.3 (and Theorem 1.3) is

\[
\Theta = \frac{\beta}{-\delta} = \lim_{T \to \infty} T \cdot \int \Delta_{(1,T)} g_{(1,\infty)} d\mu_{(1,\infty)} - \int A_{\infty} g_{(1,\infty)} d\mu_{(1,\infty)}.
\]

The proof of Theorem 4.3 is complete. This completes the proof of Theorem 1.3 too. □
Appendix A. Proof of Dirichlet-Patterson Theorem

A.1. A family of disjoint horoballs. Let $\Gamma$ be a non-uniform lattice in $\text{SL}(2, \mathbb{R})$ with $\infty$ as parabolic fixed point. Let $P \in \Gamma$ be a primitive parabolic element with $P \cdot \infty = \infty$. It is a well-known fact that there exists a constant $c(\Gamma) > 0$ such that

$$|c(G)| \geq c(\Gamma) \quad \text{for any } G \in \Gamma \setminus \langle P \rangle. \quad (A.1)$$

From this fact it is easy to derive Lemma A.1 below.

**Lemma A.1.** Let $\Gamma$ be a Fuchsian group and $P \in \Gamma$ be a primitive parabolic element fixing $\infty$ as above. Then for all $T \geq \frac{1}{c(\Gamma)}$ and all $G_1, G_2 \in \Gamma$ the following holds.

1. If $G_2^{-1} \cdot G_1 \in \langle P \rangle$ then $G_1(H_T) = G_2(H_T)$.

2. Otherwise $G_1(H_T) \cap G_2(H_T) = \emptyset$.

For fixed $T \geq c(\Gamma)^{-1}$ the punctured disc $U := \langle P \rangle \setminus H_T$ is isometrically embedded in $\Gamma \setminus \mathbb{H}$ and gives a so-called Margulis neighborhood of the cusp $[\infty]$.

Fix a family $S = \{A_1, \ldots, A_p\}$ of elements of $\text{SL}(2, \mathbb{R})$ such that the points $z_k = A_k \cdot \infty$ with $k = 1, \ldots, p$ given by Equation (1.4) form a complete set of inequivalent parabolic fixed points for $\Gamma$. For any $k = 1, \ldots, p$ let $\Gamma_k := A_k^{-1} G A_k$ be the corresponding conjugated of $\Gamma$ in $\text{SL}(2, \mathbb{R})$. Consider any $k \in \{1, \ldots, p\}$ and observe that $G \in \Gamma$ fixes $z_k = A_k \cdot \infty$ if and only if $A_k^{-1} G A_k \in \Gamma_k$ fixes $\infty$. Thus let $P_k \in \Gamma_k$ be the primitive parabolic element which fixes $\infty$ and $T_k > 0$ be the constant, specific of $\Gamma_k$, such that $\langle P_k \rangle \setminus H_{T_k}$ is a Margulis neighborhood of the cusp $[\infty]$ in $\Gamma_k \setminus \mathbb{H}$. The map $A_k : \mathbb{H} \to \mathbb{H}$, $z \mapsto A_k \cdot z$ descends to a map

$$A_k : \Gamma_k \setminus \mathbb{H} \to \Gamma \setminus \mathbb{H}, \quad \Gamma_k z \mapsto \Gamma A_k \cdot z,$$

and a Margulis neighborhood of the cusp $[z_k]$ in $\Gamma \setminus \mathbb{H}$ is given by the image

$$U_k := A_k(\langle P_k \rangle \setminus H_{T_k}) = \langle A_k P_k A_k^{-1} \rangle \setminus (A_k H_{T_k}),$$

which is of course an isometrically embedded punctured disc. As in §4.3 for any $G \in \Gamma$, any $k = 1, \ldots, p$ and any $T > 0$ define the balls

$$B_k(G, T) := G \cdot A_k(H_T),$$

where we observe that $B_k(G, S) \subset B_k(G, T)$ if $S \geq T$.

**Proposition A.2.** There exist constants $T > 0$ and $\delta$ with $0 < \delta < 1$, depending on $\Gamma$, such that the following holds.

1. The joint family $\{B_k(G, T) : G \in \Gamma, k = 1, \ldots, p\}$ is a family of disjoint balls, that is we have $B_k(G, T) \cap B_j(F, T) \neq \emptyset \Rightarrow G = F$ and $j = k$.

2. We have

$$\mathbb{H} = \bigcup_{k=1}^{p} \bigcup_{G \in \Gamma} B_k(G, \delta \cdot T).$$
Proof. For any \( k = 1, \ldots, p \) let \( P_k \in \Gamma \) be the primitive parabolic elements and \( T_k > 0 \) be the positive real numbers introduced above. Then consider any \( T > 0 \) with

\[
T \geq \max\{T_1, \ldots, T_p\}.
\]

Since \( T \geq T_k \), then \( \langle P_k \rangle \backslash H_T \) is a Margulis neighborhood of the cusp \([\infty]\) in \( \Gamma \backslash \mathbb{H} \), and therefore \( U_k := \langle A_k P_k A_k^{-1} \rangle \backslash \langle A_k H_T \rangle \) is a Margulis neighborhood of the cusp \([z_k]\) in \( \Gamma \backslash \mathbb{H} \) for any \( k = 1, \ldots, p \). According to Lemma \( \mathbb{A}.1 \), \( \{B_k(G, T) : G \in \Gamma\} \) is a family of disjoint balls for any \( k = 1, \ldots, p \). Then the first part of the Proposition follows because the Margulis neighborhoods \( U_1, \ldots, U_p \) are mutually disjoint in \( \Gamma \backslash \mathbb{H} \).

In order to prove the second part, let \( \Omega \) be a fundamental domain for \( \Gamma \) and let \( \mathcal{B} \subset \Gamma \) be a finite set such that, for any vertex \( \xi \) of \( \Omega \), there exist unique \( G \in \mathcal{B} \) and \( k = k(G) = 1, \ldots, p \) with \( \xi = G \cdot z_k \). Clearly \( \mathcal{B} \) has as many elements as the number of vertices of \( \Omega \). Moreover \( \{G \cdot z_k(G) : G \in \mathcal{B}\} \) contains a complete set of parabolic fixed points. The set \( \Omega \backslash \bigcup_{G \in \mathcal{B}} B_k(G, T) \) is relatively compact in \( \mathbb{H} \), then if \( \delta \) is small enough we have

\[
\overline{\Omega} \backslash \bigcup_{G \in \mathcal{B}} B_k(G, \delta \cdot T),
\]

where the closure is taken in \( \mathbb{H} \). The second part of the Proposition follows because

\[
\mathbb{H} = \bigcup_{G \in \Gamma} \overline{G(\Omega)} = \bigcup_{G \in \Gamma} \bigcup_{G \in \mathcal{B}} B_k(G, \delta \cdot T) = \bigcup_{k=1}^{p} \bigcup_{G \in \Gamma} B_k(G, \delta \cdot T).
\]

\( \square \)

A.2. End of the proof of Dirichlet-Patterson Theorem. We first prove the useful Lemma below.

**Lemma A.3.** Fix real numbers \( R > 0 \) and \( \Delta \) with \( 0 < \Delta < R \). Consider the euclidian ball \( B := \{(x, y) \in \mathbb{R}^2 : x^2 + (y - R)^2 \leq R^2\} \) centered at \((0, R)\) with radius \( R \). Let \((x_0, y_0) \in B\) be such that \( y_0 \leq \Delta \). Then

\[
x_0^2 + y_0^2 \leq 2R\Delta.
\]

**Proof.** Assume without loss of generality that \((x_0, y_0)\) belongs to the intersection between the circle \( \partial B := \{(x, y) \in \mathbb{R}^2 : x^2 + (y - R)^2 = R^2\} \) and the horizontal line \( \{(x, y) \in \mathbb{R}^2 : y = \Delta\} \). Then we have

\[
\sqrt{x_0^2 + y_0^2} = 2R \sin(\theta/2),
\]

where \( \theta \) is the angle between the rays connecting the center \((0, R)\) of \( B \) with the boundary points \((0, 0)\) and \((x_0, y_0)\). The Lemma follows observing that we have \( \theta/2 = \beta \), where \( \beta \) is the angle between the two segments connecting \((0, 0)\) to \((x_0, 0)\) and \((x_0, y_0)\) respectively, which satisfies the relation

\[
\sin(\beta) := \frac{y_0}{\sqrt{x_0^2 + y_0^2}} = \frac{\Delta}{\sqrt{x_0^2 + y_0^2}}.
\]

\( \square \)

Recall the notation of the previous § \( \mathbb{A}.1 \). Let \( \Omega \) be a fundamental domain for \( \Gamma \). Let \( T > 0 \) and \( 0 < \delta < 1 \) be the constants in Proposition \( \mathbb{A}.2 \). Recall that we assume that \( \infty \) is a parabolic fixed point of \( \Gamma \), that is we set a normalization on the set \( \mathcal{S} = \{A_1, \ldots, A_p\} \).
assuming that \( A_1 = \text{Id} \). Let \( c_T > 0 \) be the constant as in Equation (A.1). In particular, since \( T \geq c_T^{-1} \), then we have either \( G(H_T) = H_T \) or \( G(H_T) \cap H_T = \emptyset \). Finally set
\[
Q_0 := \sqrt{c_T} \cdot M \quad \text{where} \quad M := \frac{1}{\delta \cdot T}.
\]
For \( G \in \Gamma \) and \( k = 1, \ldots, p \) let \( B_{G,k} := B_k(G, M) \) be the balls as in the second part of Proposition A.2. Observe that any \( B_{G,k} \) is a standard euclidian ball if and only if \( c(GA_k) \neq 0 \) and in this case, according to Equation (4.3), its diameter is given by
\[
\text{Diam}(B_{G,k}) = \frac{M}{c(GA_k)^2}.
\]
Fix any \( Q > Q_0 \) and any \( \alpha \in \mathbb{R} \), then set
\[
z_\alpha := \alpha + \frac{M}{Q^2} \cdot i \in \mathbb{H}.
\]
By the second part of Proposition A.2 there exist \( G \in \Gamma \) and \( k \in \{1, \ldots, p\} \) such that \( z_\alpha \in B_{G,k} \). We have
\[
\text{Im}(z_\alpha) = \frac{M}{Q^2} \leq \frac{M}{Q_0^2} = \frac{1}{c_T} \leq T,
\]
therefore \( z_\alpha \not\in H_T \), and the discussion above imply \( c(GA_k) \neq 0 \). Moreover by construction we have \( |c(G)| \leq Q \), indeed
\[
\frac{M}{c(GA_k)^2} = \text{Diam}(B_{G,k}) \geq \text{Im}(z_\alpha) = \frac{M}{Q^2}.
\]
The Patterson-Dirichlet Theorem follows recalling that \( B_k,G \) is tangent to the real line at \( G \cdot z_k = GA_k \cdot \infty \), so that Lemma A.3 above implies
\[
|z_\alpha - GA_k \cdot \infty|^2 \leq \frac{M}{c(GA_k)^2} \cdot \frac{M}{Q^2}
\]
and therefore
\[
|\alpha - G \cdot z_k|^2 = |\alpha - GA_k \cdot \infty|^2 = |z_\alpha - GA_k \cdot \infty|^2 - |z_\alpha - \alpha|^2 \leq \frac{M}{c(GA_k)^2} \cdot \frac{M}{Q^2} - \frac{M^2}{Q^4}
\]
\[
= \frac{M^2}{Q^2} \cdot \left( \frac{1}{c(GA_k)^2} - \frac{1}{Q^2} \right) = M^2 \cdot \frac{Q^2 - c(GA_k)^2}{Q^2} = \frac{1}{c(GA_k)^2 \cdot Q^2} \leq \frac{M^2}{c(GA_k)^2 \cdot Q^2} \quad \square
\]

Appendix B. Some basic facts on spectra and projectors

We recall some facts on spectral properties of bounded linear operator. For more details one can see for example § III.6 in [Kato].

B.1. Spectrum and spectral projectors. Let \((B, \| \cdot \|)\) be a Banach space and \( L : B \to B \) be a bounded linear operator. The resolvent set of \( L \) is the set \( \mathcal{R}(L) \) of complex numbers \( z \in \mathbb{C} \) such that \( L - z\text{Id} : B \to B \) is an invertible operator (with bounded inverse). The set \( \text{sp}(L) := \mathbb{C} \setminus \mathcal{R}(L) \) is called the spectrum of \( L \). The spectral radius \( \rho(L) \) of \( L \) is defined by
\[
\rho(L) := \sup \{|z| : z \in \text{sp}(L)\}.
\]
The spectrum \( \text{sp}(L) \) of a bounded linear operator \( L \) is always a compact subset of \( \mathbb{C} \). The resolvent set \( \mathcal{R}(L) \) is thus open and we have the resolvent map

\[
R(L, \cdot) : \mathcal{R}(L) \to \mathcal{L}(B, B) \quad ; \quad R(L, z) := (L - z\text{Id})^{-1},
\]

where \( \mathcal{L}(B, B) \) denotes the space of bounded linear maps from \( B \) to \( B \). For \( z \in \mathcal{R}(L) \) the bounded linear operator \( R(L, z) : B \to B \) is called the resolvent of \( L \). The resolvent map is holomorphic, meaning that for any \( z \in \mathcal{R}(L) \) and any \( h \in \mathbb{C} \) small enough we have

\[
R(L, z + h) = R(L, z) + h \cdot R^2(L, z) + ho(h).
\]

The facts above can be derived simply from the first resolvent equation: for any \( z, z' \in \mathcal{R}(L) \) we have

\[
R(L, z) - R(L, z') = (z - z')R(L, z)R(L, z').
\]

If \( \Sigma, \Sigma' \) are two compact subsets of \( \mathbb{C} \) with \( \Sigma \cap \Sigma' = \emptyset \) and \( \text{sp}(L) = \Sigma \cup \Sigma' \), then consider a smooth oriented loop \( \gamma \) in \( \mathcal{R} \) containing \( \Sigma \) in its interior. Then the spectral projector for \( \Sigma \) is the linear operator let \( P : B \to B \) defined by

\[
(P := \frac{-1}{2\pi i} \int_\gamma R(L, z)dz.
\]

Since \( z \mapsto R(L, z) \) is holomorphic, the map \( P : B \to B \) does not depend on the specific choice of \( \gamma \). Moreover one can verify that \( P \) is a projection, that is \( P^2 = P \), that we have \( P \circ L = L \circ P \), and that \( P \) is bounded. More precisely the continuity of the resolvent map implies \( M := \sup_{z \in \gamma} \|R(L, z)\| < +\infty \), and thus \( \|P\| \leq |\gamma| \cdot M \). Obviously \( \text{Id} - P : B \to B \) is also a projection which commutes with \( L \), with norm satisfying \( \|\text{Id} - P\| \leq 1 + \|P\| \). Thus setting \( N := \ker(P) \) and \( V := \ker(\text{Id} - P) \) we have

\[
B = N \oplus V \quad \text{with} \quad L(N) \subset N \quad \text{and} \quad L(V) \subset V.
\]

For \( z \in \mathcal{R}(L) \) the map \( R(L, z) : B \to B \) is bijective continuous and satisfies the relation \( P \circ R(L, z) = R(L, z) \circ P \). Hence the restrictions \( R(L, z)|_N : N \to N \) and \( R(L, z)|_V : V \to V \) are bijective continuous and it follows that

\[
R(L|_N, z) = R(L, z)|_N \quad \text{and} \quad R(L|_V, z) = R(L, z)|_V \quad \text{for any} \quad z \in \mathcal{R}(L).
\]

Moreover one can prove that \( \text{sp}(L|_N) = \Sigma' \) and \( \text{sp}(L|_V) = \Sigma \). More details on the can be found in \S\ III.6.4 in \([Kato]\).

**B.2. Stability of spectral decompositions.** Let \( L_0, A : B \to B \) be bounded linear operators. Assume that \( L_0 \) is invertible and that \( \|A\| \cdot \|L_0^{-1}\| < 1 \). Then \( \|A \circ L_0^{-1}\| < 1 \) and thus \(-1 \in \mathcal{R}(A \circ L_0^{-1})\), so that \((\text{Id} + A \circ L_0^{-1}) : B \to B \) is bounded invertible. It follows that \( L_0 + A \) is bounded invertible, indeed

\[
L_0 + A = (\text{Id} + A \circ L_0^{-1}) \circ L_0.
\]

Let \( L : B \to B \) be a bounded linear operator and apply the construction above to \( A := L - L_0 \). In particular, if \( z \in \mathcal{R}(L_0) \) and \( \|L - L_0\| < \|R(L_0, z)\|^{-1} \), then \( z \in \mathcal{R}(L) \) too. Moreover, if \( \gamma \) is a closed smooth path in \( \mathcal{R}(L_0) \), then continuity of \( z \mapsto R(L_0, z) \) implies

\[
M_0 := \sup_{z \in \gamma} \|R(L_0, z)\| < +\infty,
\]

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and therefore \( \gamma \subset \mathcal{R}(L_0) \cap \mathcal{R}(L) \) provided that \( \|L - L_0\| \leq M_0^{-1} \). Furthermore, for any \( z \in \mathcal{R}(L_0) \cap \mathcal{R}(L) \) we have
\[
(L_0 - z\text{Id})(L - z\text{Id})(R(L_0, z) - R(L, z)) = (L - L_0) + (L_0L - LL_0)R(L_0, z).
\]
From \( L_0L - LL_0 = L_0(L - L^2_0 + L_0) - LL_0 \) we get \( \|L_0L - LL_0\| \leq 2\|L_0\| \cdot \|L - L_0\| \), so that
\[
\|R(L_0, z) - R(L, z)\| \leq \|R(L, z)\| \cdot \|R(L_0, z)\| \cdot (1 + 2\|L_0\|) \cdot \|L - L_0\|.
\]
Therefore the stronger condition \( \|L - L_0\| < (2M_0)^{-1} \) implies that for any \( z \in \gamma \) and any \( v \neq 0 \) we have
\[
\|(L - z\text{Id})(v)\| \geq \|(L_0 - z\text{Id})(v)\| - \|(L - L_0)(v)\| \geq \frac{\|v\|}{\|R(L_0, z)\|} \cdot \|v\| - \frac{\|v\|}{2M_0} \geq \frac{\|v\|}{2M_0},
\]
that is \( \sup_{z \in \gamma} \|R(L, z)\| \leq 2M_0 \). Thus, setting \( M_1 := 2M_0^2 \cdot (1 + 2\|L_0\|) \), we get
\[
\sup_{z \in \gamma} \|R(L_0, z) - R(L, z)\| \leq M_1 \cdot \|L - L_0\|.
\]
Therefore if \( P_0 \) and \( P \) are the spectral projectors along \( \gamma \) associated respectively to \( L_0 \) and \( L \) via Equation \( B.1 \) then we have
\[
\|P - P_0\| \leq |\gamma| \cdot M_1 \cdot \|L - L_0\|.
\]
Finally, if \( \|L - L_0\| \) is small enough to have \( \|P - P_0\| < 1 \), then there exists an isomorphism \( G : B \to B \) such that \( P = GP_0G^{-1} \) (see \cite{Kato}, pages 33-34). In particular \( G \) sends the spectral decomposition \( B = N_0 \oplus V_0 \) onto the spectral decomposition \( B = N \oplus V \), thus \( \dim N = \dim N_0 \) and \( \dim V = \dim V_0 \). Moreover we have
\[
G - \text{Id} = O(|P_0| \cdot \|P - P_0\|) \quad \text{and} \quad G^{-1} - \text{Id} = O(|P_0| \cdot \|P - P_0\|).
\]
\section*{B.3. Quasi compact operators.} Recall from \cite{Henn} the notion of quasi compact operator. The essential spectral radius \( \rho_{\text{ess}}(L) \) of a bounded linear operator \( L : B \to B \) is the infimum of those \( r \geq 0 \) such that there exists subspaces \( N, V \) of \( B \) such that the following holds.
\begin{enumerate}
\item We have \( B = N \oplus V \) with \( L(N) \subset N \) and \( L(V) \subset V \).
\item We have \( 1 \leq \dim V < +\infty \) and the restriction \( L|_V \) has only eigenvalues \( \lambda \) of modulus \( |\lambda| \geq r \).
\item The subspace \( N \) is closed and the restriction has spectral radius \( \rho(L|_N) < r \).
\end{enumerate}
The operator \( L : B \to B \) is said quasi compact if \( \rho_{\text{ess}}(L) < \rho(L) \) strictly.

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