Drinfeld modular curves have many points

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Abstract

An algebraic (smooth, projective) curve over a finite field of $q$ elements can have at most $(\sqrt{q} - 1 + o(1))g$ rational points for a growing genus $g$. If $q$ is a square, then this bound is sharp. Many examples have been found of families of curves of increasing genus attaining this bound (by, amongst others, M. A. Tsfasman, S. G. Vladut, T. Zink, Y. Ihara, A. Garcia, H. Stichtenoth, N. Elkies). These are typically towers of reductions of modular curves or Drinfeld modular curves. In the former case, it has been proven by Y. Ihara that all towers of reductions of modular curves of a certain type attain this bound. In the latter case, this has been proven only for particular examples, even though, as remarked by Noam Elkies, a similar argument should work.

This text applies the argument of Ihara to the case of Drinfeld modular curves. The bulk of work in doing this is working out the theory of (reductions of) Drinfeld modular curve, as a lot of properties of these curves that are needed for Ihara’s argument to work are absent from the literature. As a consequence, the text is almost completely self-contained.

The main result can be stated as follows. Let $X$ be a smooth projective and absolutely irreducible curve over the finite field $\mathbb{F}_q$ of $q$ elements. Let $\infty$ be a point of $X$ and denote by $A$ the ring of functions on $X$ that are regular outside $\infty$. Let $n$ be a principal ideal in $A$ and $p$ a prime ideal not containing $n$. Let $q^m$ be the cardinality of the residue field of $p$. Denote by $X_0(n)$ the compactification of the Drinfeld modular variety over $A$ that classifies the Drinfeld $A$-modules of rank 2 together with a submodule isomorphic to $A/nA$. Then the reduction modulo $p$ of $X_0(n)$ is a curve of genus $g$ having at least $(q^m - 1)(g - 1)$ rational points over the quadratic extension of the residue field of $p$. 

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1. Notation

This thesis is written in the language of schemes and representable functors, so let us fix some notation and make some conventions.

All schemes are assumed to be locally noetherian. We do not distinguish between rings or algebras (both taken to be commutative and with unit element, unless otherwise stated) and affine schemes. Often we do not write \( \text{Spec}(R) \) but just \( R \) for the spec of the ring \( R \). It should be clear from the context (usually from the direction of the arrows) if we are speaking “algebra” or “scheme”.

Categories are usually not abbreviated. For example: (abelian groups). Except for the category \((\text{Sch}/S)\) of schemes over a base scheme \( S \) or \((\text{Sch})\) for \((\text{Sch}/\mathbb{Z})\).

Isomorphisms are denoted by \( \sim \to \), \( \sim \leftarrow \) and \( \cong \). Canonical isomorphisms simply by \( = \).

2. Line bundles in positive characteristic

2.1 Review

In this chapter we study additive group schemes in positive characteristic. All schemes in this chapter will be schemes over a finite field \( \mathbb{F}_q \) of characteristic \( p \). We will study additive groups equipped with not only an \( \mathbb{F}_p \)-linear structure, which they carry in a natural way, but also an \( \mathbb{F}_q \)-linear one. Let us first recall some facts.
Drinfeld modular curves have many points

2.1.1. A line bundle $G$ over $S$ is a group scheme which is Zariski locally isomorphic to the additive group $\mathbb{G}_a$ on $S$. We call $G$ together with an action of $\mathbb{F}_q$, that is a ring morphism

$$\mathbb{F}_q \to \text{End}(G)$$

into the ring of group scheme endomorphisms, an $\mathbb{F}_q$-line bundle. We will from now on assume that all line bundles are equipped with such an $\mathbb{F}_q$-structure.

This means that $G$ is not only a commutative group object but also an $\mathbb{F}_q$-vector space object in the category $(\text{Sch}/\mathbb{F}_q)$. A morphism $G \to H$ of $\mathbb{F}_q$-line bundles over $S$ is said to be $\mathbb{F}_q$-linear (or short: an $\mathbb{F}_q$-morphism) if it commutes with the action of the finite field $\mathbb{F}_q$. This is the same as saying that $G \to H$ is a morphism of functors, where $G$ and $H$ are interpreted as contravariant functors from $(\text{Sch}/S)$ to $(\text{vector spaces}/\mathbb{F}_q)$.

A trivialization provides an open affine covering of $S$ with spectra of rings $R_i$, such that $G \times_S R_i \cong R_i[X]$ where the addition on $R_i[X]$ is given by

$$R_i[X] \to R_i[X] \otimes_{R_i} R_i[X] = R_i[Y,Z] : X \mapsto Y + Z$$

and the $\mathbb{F}_q$-linear structure by

$$\mathbb{F}_q \to \text{End}(R_i[X]) : \lambda \mapsto (X \mapsto \lambda X).$$

2.1.2. An invertible sheaf on $S$ is a coherent $\mathcal{O}_S$-module sheaf $\mathcal{L}$, locally free of rank 1. It is called invertible because the dual sheaf

$$\mathcal{L}^{-1} := \text{Hom}(\mathcal{L}, \mathcal{O}_S)$$

satisfies the following canonical isomorphism:

$$\mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{O}_S$$

2.1.3. Invertible sheaves correspond to line bundles and vice versa. There is a one-one correspondence between isomorphism classes of line bundles and isomorphism classes of invertible sheaves. This can be established in two ways.

Given an invertible sheaf $\mathcal{L}$ on $S$, the contravariant functor

$$(f : T \to S) \mapsto H^0(T, f^*(\mathcal{L}))$$

of $(\text{Sch}/S)$ to (commutative groups) is representable by a line bundle $G$, we call it $G = \mathbb{G}_{a,\mathcal{L}}$.

Alternatively, we can construct the scheme $G = \mathbb{G}_{a,\mathcal{L}}$ as the affine scheme on $S$, corresponding to the symmetric algebra of the sheaf $\mathcal{L}^{-1}$ ([5] I 9.4).

The converse construction is obtained by restricting the functor of points of $G/S$ to open immersions $U \to S$. The restricted functor is representable by an invertible sheaf $\mathcal{L}$.

2.2 Homomorphisms

This section is largely based on section 1.2 of Thomas Lehmkuhl’s habilitationsschrift [12]. From here on, we suppress the $\mathbb{F}_q$ and just talk about “line bundles” whenever we mean “$\mathbb{F}_q$-line bundles”.

2.2.1. A line bundle on $S$ can be trivialized by covering $S$ with open affines. On such an affine, corresponding to a ring $R$, the restriction of the line bundle is isomorphic to $\mathbb{G}_{a,R} = R[X]$, with the usual additive group structure on $R[X]$. On this group scheme we can easily find a ring of $\mathbb{F}_q$-endomorphisms, namely by adjoining the $q$-th power Frobenius $\tau$ to the ring $R$. $\tau$ does not commute with multiplication by an element of
The resulting ring, of which the elements are endomorphisms working on the left, is the skew polynomial ring $R\{\tau\}$ with commutation rule $r^q\tau = \tau r$.

2.2.2 PROPOSITION. $R\{\tau\}$ is the full ring $\text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,R})$ of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_{a,R}$.

Proof: An $\mathbb{F}_q$-linear morphism $\mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ corresponds to a ring morphism $R[X] \leftarrow R[X]$, sending $X$ to an $\mathbb{F}_q$-linear, additive polynomial $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$. Given such an $\mathbb{F}_q$-linear $f$, we want to prove that for all indices $i$ that are not a power of $q$ the coefficient $a_i$ vanishes. This is a purely algebraic exercise. For details, see for example ([12] 1.2.1).

2.2.3 We now use this affine result to determine the abelian group $\text{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a,L}, \mathbb{G}_{a,M})$ of $\mathbb{F}_q$-morphisms of commutative group schemes. Consider the abelian group of global sections $H^0(S, \bigoplus_{n \geq 0} \mathcal{L} \otimes \mathcal{M}^{-q^n})$.

This group is canonically isomorphic to

$$\bigoplus_{n \geq 0} H^0(S, \mathcal{L} \otimes \mathcal{M}^{-q^n})$$

of which the elements are formal power series in $\tau$

$$a_0 + a_1 \tau + a_2 \tau^2 + \cdots$$

with $a_i \in H^0(S, \mathcal{M} \otimes \mathcal{L}^{-q^i})$ such that locally only a finite number of terms are non-zero. A term $a_i \tau^i$ of an element of such a power series defines an $\mathbb{F}_q$-linear morphism of invertible sheaves by composing the $i$-th iterated $q$-th power frobenius

$$\tau^i : \mathcal{L} \to \mathcal{L}^{q^i} : s \mapsto s \otimes s \cdots \otimes s$$

with multiplication by a global section:

$$a_i : \mathcal{L}^{q^i} \to \mathcal{M} : t \mapsto t \otimes a_i \in \mathcal{L}^{q^i} \otimes \mathcal{L}^{-q^i} \otimes \mathcal{M} = \mathcal{M}.$$ 

Because we demand our power series to be locally finite, we can locally sum these morphisms to $\mathbb{F}_q$-linear morphisms, and glue them to form a global linear morphism of line bundles. If $\mathcal{L} = \mathcal{M}$, this group of endomorphisms is actually a ring of endomorphisms, multiplication is given by

$$a_i \tau^i b_j \tau^j = a_i \otimes b_j \tau^{i+j}$$

Now we claim that, as in the local case, this group is the full group of $\mathbb{F}_q$-morphisms.

2.2.4 PROPOSITION. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on the $\mathbb{F}_q$-scheme $S$. We have a canonical isomorphism of groups

$$\text{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a,L}, \mathbb{G}_{a,M}) = \bigoplus_{n \geq 0} H^0(S, \mathcal{L} \otimes \mathcal{M}^{-q^n})$$

and a canonical isomorphism of rings

$$\text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}) = \bigoplus_{n \geq 0} H^0(S, \mathcal{L}^{-q^n})$$

Proof: Given an $\mathbb{F}_q$-morphism $\mathbb{G}_{a,L} \to \mathbb{G}_{a,M}$, we can write it locally (on the affines of a trivializing covering) in the desired form. Since this is done uniquely, it follows from the uniqueness on the intersections that we can glue the coefficients to form global sections over $S$ of the sheaves $\mathcal{L} \otimes \mathcal{M}^{-q^n}$. For the converse, we
have already seen that every series of sections of the desired form induces canonically an \( \mathbb{F}_q \)-morphism of line bundles.

We are especially interested in finite morphisms. We restrict our attention to connected base schemes, to make notions as rank defined globally.

2.2.5 Proposition. Given a non-trivial \( \mathbb{F}_q \)-morphism \( \xi : \mathbb{G}_{a, \mathcal{L}} \to \mathbb{G}_{a, \mathcal{M}} \) of line bundles over the connected scheme \( S \), the following are equivalent:

i) \( \xi \) is finite

ii) \( \xi \) is finite and flat, of rank \( q^n \), for some \( n \)

iii) there exists an \( n \) such that \( \xi = x_0 + x_1 \tau + x_2 \tau^2 + \cdots + x_n \) invertible, i.e., nowhere vanishing, and all the \( x_i \) with \( i > n \) are nilpotent, i.e., everywhere vanishing.

If \( \mathcal{L} = \mathcal{M} \), and if the above conditions are satisfied, then there is a unique \( \mathbb{F}_q \)-automorphism \( \sigma = \sum s_i \tau^i \) of \( \mathbb{G}_{a, \mathcal{L}} \) with \( s_0 = 1 \) such that

\[
\sigma^{-1} \circ \xi \circ \sigma = x_0 + z_1 \tau + \cdots + z_n \tau^n
\]

where \( z_n \) is invertible.

Proof. (i) \( \Rightarrow \) (iii). If \( \xi \) is finite, it is automatically flat since flatness is local and every finite \( R \)-morphism \( R[\tau] \to R[\tau] \) defines on the target the structure of a free module over the source. Because \( \xi \) is flat it has a locally constant rank. If we write \( \xi \) in the usual form

\[
\xi = x_0 + x_1 \tau + x_2 \tau^2 + \cdots
\]

this means that in every point of \( S \), some \( x_n \) is non-zero, and all higher coefficients are zero. Moreover, since \( S \) is connected, this \( n \) is a constant.

(iii) \( \Rightarrow \) (ii). True on all open affine subschemes of \( S \), hence also globally true.

(ii) \( \Rightarrow \) (i). A fortiori.

For the last claim, assume again \( S \) is a ring \( R \). Write \( \xi \) as a finite polynomial in \( \tau \)

\[
\xi = x_0 + x_1 \tau + x_2 \tau^2 + \cdots + x_n \tau^n + \cdots + x_m \tau^m
\]

with all coefficients \( x_i \) for \( n < i \leq m \) generating a nilpotent ideal \( n \) of \( R \). Let us assume \( n^{2s} = 0 \). Remark that

\[
\left( 1 + \frac{x_m}{x_n^n} \tau^{m-n} \right)^{-1} \equiv \left( 1 - \frac{x_m}{x_n^n} \tau^{m-n} \right) \mod n^{2s-1}
\]

And that the coefficients in \( \tau^i \) for \( i \geq m \) of

\[
\left( 1 + \frac{x_m}{x_n^n} \tau^{m-n} \right)^{-1} \xi \left( 1 + \frac{x_m}{x_n^n} \tau^{m-n} \right)
\]

vanish modulo \( n^{2s-1} \). So by induction on the degree of \( \xi \) modulo \( n^{2s-1} \), we find an automorphism \( \sigma \) such that

\[
\xi' = \sigma^{-1} \xi \sigma = \sum_{0}^{m'} x'_i \tau^i
\]

has coefficients \( x'_i \) for \( n < i \leq m' \) generating a nilpotent ideal \( n' \) satisfying \( n'^{2s-1} = 0 \). Now by decreasing induction on \( s \) we find the existence of a \( \sigma \) satisfying the desired property.
For the uniqueness, it is sufficient to show that \( \sigma \) must be trivial if \( \xi \) has already the desired form with zero coefficients for \( i > n \). Because \( \sigma \) must be invertible, it looks like
\[
\sigma = 1 + s_1 \tau + s_2 \tau^2 + \cdots + s_t \tau^t
\]
where all the \( s_i \) are nilpotent. Let \( m \) be the nilpotent ideal generated by them, and let \( s \) be minimal such that \( m^{2s} = 0 \). We will apply the same trick as for the existence. If \( s = 0 \), we are done. Now assume \( s > 0 \) and let \( i \) be the highest index such that \( s_i \) is non-zero modulo \( m^{2s-1} \). By evaluating the terms in \( \tau^{n+i} \) in
\[
(x_0 + \cdots + x_n \tau^n) \sigma \equiv (z_0 + \cdots + z_n \tau^n) \mod m^{2s-1}
\]
we find that \( s_i \) vanishes modulo \( m^{2s-1} \), a contradiction.

Now for a general \( S \), consider an affine covering. On every affine we find a \( \sigma \), and these glue because of the uniqueness on the intersections of the affines. Moreover, the global \( \sigma \) is unique because it is unique on affines.

2.2.6 COROLLARY. A non-trivial finite \( \mathbb{F}_q \)-morphism of line bundles over \( S \) is an epimorphism of \( S \)-schemes.

Proof. It follows from the above explicit description that it is surjective (because it is surjective on geometric points), and hence faithfully flat. We also see that it is quasi-compact. These two properties imply the result since faithfully flat and quasi-compact morphisms are descent morphisms, and therefore strict epimorphisms, hence a fortiori epimorphisms.

2.2.7. Given a finite \( \mathbb{F}_q \)-morphism \( \xi : G_{a,S} \to G_{a,#} \) of line bundles over the connected scheme \( S \), we define its height to be the smallest index \( h \) for which the coefficient \( x_h \) is non-zero. The derivative at 0 is defined to be the coefficient \( x_0 \). We use the notations \( \text{rk} \), \( \text{ht} \) and \( \partial \) for rank, height and derivative at 0 respectively:
\[
\text{rk}(x_0 + \cdots + x_n \tau^n + \text{nilpotents}) = q^n \\
\text{ht}(x_h \tau^h + x_{h+1} \tau^{h+1} + \cdots) = h \\
\partial(x_0 + x_1 \tau + \cdots) = x_0.
\]
The operators \( \text{rk} \) and \( \text{ht} \) satisfy the following relations:
\[
\text{rk}(\xi \circ \alpha) = \text{rk}(\xi) \text{rk}(\alpha) \\
\text{rk}(\alpha + \beta) \leq \max(\text{rk}(\alpha), \text{rk}(\beta)) \\
\text{ht}(\xi \circ \alpha) = \text{ht}(\xi) + \text{ht}(\alpha) \\
\text{ht}(\alpha + \beta) \geq \min(\text{ht}(\alpha), \text{ht}(\beta)).
\]

2.3 Quotients by finite flat subgroup schemes

In this section, we fix a base scheme \( S/\mathbb{F}_q \).

2.3.1. Let us first recall the definition of a quotient group scheme (see for example [15]). We say that a short sequence
\[
0 \to X \to Y \to Z \to 0
\]
of flat commutative group schemes over \( S \), where \( X \) is finite is exact if for every group scheme \( T/S \), the induced sequences
\[
0 \to \text{Hom}(T, X) \to \text{Hom}(T, Y) \to \text{Hom}(T, Z)
\]
and

\[ 0 \to \text{Hom}(Z, T) \to \text{Hom}(Y, T) \to \text{Hom}(X, T) \]

are exact sequences of (abstract) commutative groups. If this is the case, we also say that \(Z\) is the quotient of \(Y\) by \(X\). Remark that \(Z\) does not necessarily represent the functor \(T \mapsto Y(T)/X(T)\), but \(Y(T)/X(T)\) does map injectively (and functorially) into \(Z(T)\).

We call a finite flat subgroup scheme \(G \to \mathbb{G}_{a, \mathbb{F}_q}\) a finite flat \(\mathbb{F}_q\)-subgroup scheme if it has an \(\mathbb{F}_q\)-linear structure, that is inherited from the one on \(\mathbb{G}_{a, \mathbb{F}_q}\) by base change. Above a connected component of \(S\), \(G\) has a constant rank \(r\), which is always a power of \(p\), and even of \(q\) if \(G\) is an \(\mathbb{F}_q\)-subgroup scheme. Clearly the kernel of an \(\mathbb{F}_q\)-morphism of line bundles over \(S\) is a finite flat \(\mathbb{F}_q\)-subgroup scheme, but the converse is also true.

2.3.2 THEOREM. Given a finite flat \(\mathbb{F}_q\)-subgroup scheme \(G\) of \(\mathbb{G}_{a, \mathbb{F}_q}\), then there is, unique up to a unique isomorphism, an \(\mathbb{F}_q\)-morphism of line bundles

\[ \xi : \mathbb{G}_{a, \mathbb{F}_q} \to \mathbb{G}_{a, \mathbb{F}_q} \]

such that the sequence

\[ 0 \to G \to \mathbb{G}_{a, \mathbb{F}_q} \to \mathbb{G}_{a, \mathbb{F}_q} \to 0 \]

is an exact sequence of group schemes over \(S\). Moreover, if \(S\) is connected and the rank of \(G\) is denoted by \(r\), then \(\mathbb{F}_q\) is isomorphic to \(\mathbb{F}_q^{\otimes r}\) and the rank of \(\xi\) is also \(r\).

Proof. Without loss of generality, we may assume \(S\) is connected. We know that the quotient \(H = \mathbb{G}_{a, \mathbb{F}_q}/G\) exists, as a group scheme. Let us see what it looks like locally on \(S\).

On an open affine \(R\) of \(S\), over which \(\mathbb{G}_{a, \mathbb{F}_q}\) is trivial, we have

\[ \mathbb{G}_{a, \mathbb{F}_q} \times S R \cong R[X] \]

Moreover, since \(G\) is flat and \(\mathbb{F}_q\)-linear, there is a monic \(\mathbb{F}_q\)-linear polynomial \(P(X) = \sum a_i X^{q^i}\) such that

\[ G \times S R \cong R[X]/(P(X)). \]

We see that above \(R\), the morphism \(\mathbb{G}_{a, \mathbb{F}_q} \to H\) is given by

\[ R[X] \leftarrow R[X] : P(X) \leftrightarrow X \]

Since the rank of \(G\) is the degree of \(P(X)\), the claim follows from this local discussion.

3. Drinfeld modules

Fix a nonsingular projective curve with finite field of constants \(\mathbb{F}_q\) of characteristic \(p\). Choose a closed point \(\infty\) on it and let \(A\) be the coordinate ring of the affine curve obtained by removing \(\infty\). Then \(A\) is a dedekind ring but need not be a principal ideal domain. We write \(K\) for its quotient field, that is the function field of the curve. Let \(R\) be the completion of \(K\) at the place \(\infty\), and \(C\) the completion of the algebraic closure of \(R\). Then \(C\) will be algebraically closed as well. For a prime ideal \(P\) of \(A\), we write \(A \to \overline{k}(P)\) for the corresponding residue field and \(A \to \overline{k}(P)\) for a fixed algebraic closure. For every non-zero ideal \(m\) of \(A\), \(A/m\) is a finite-dimensional vector space over \(\mathbb{F}_q\). We call its dimension the degree of \(m\).

There is a well-known deep analogy of \(A\), \(K\), \(R\) and \(C\) with, respectively, \(\mathbb{Z}\), \(\mathbb{Q}\), \(\mathbb{R}\) and \(\mathbb{C}\). As usual, we will call the former the “function field setting”, and the latter the “classical setting”. Drinfeld modules (called elliptic modules by Drinfel’d in his paper [3]) are analogs over function fields of abelian varieties over
number fields. We will follow [1] in calling drinfeld modules of rank 2, roughly corresponding to elliptic curves, elliptic modules.

3.1 Drinfeld modules over $A$-schemes

Let $\gamma : S \to A$ be an $A$-scheme. Roughly, a drinfeld module over $S$ is a nontrivial algebraic action of $A$ on a line bundle over $S$.

3.1.1 Definition. A drinfeld module of rank $r > 0$ over $S$ is a pair $E = (\mathcal{L}, \phi)$ of an invertible sheaf $\mathcal{L}$ and an $\mathbb{F}_q$-linear ring morphism

$$\phi : A \to \text{End}_{\mathbb{F}_q}(G_{a,\mathcal{L}}) : a \mapsto \phi_a$$

such that for every non-zero $a \in A$ of degree $d$ we have:

i) $\partial \phi_a = \gamma^d(a)$

ii) $\phi_a$ is finite of rank $q^{rd}$.

A morphism between two drinfeld modules of rank $r$ over $S$ is a morphism of group schemes that is compatible with the actions of $A$.

3.1.2. If $T \to S$ is an $S$-scheme, then $G_{a,\mathcal{L}}(T)$ inherits functorially the $A$-action of $E$. We write $E(T)$ for the (abstract) $A$-module of $T$-valued points. Thus, we are using the symbol $E$ both for the drinfeld module itself and for the functor

$$E : (\text{Sch}/S) \to (A\text{-modules})$$

it represents.

If we fix a scheme $f : T \to S$, and restrict the functor $E$ to the category of $T$-schemes, it is represented by a new drinfeld module $E_T/T$, which is obtained from $E/S$ by base change

$$E_T = (f^* \mathcal{L}, \psi)$$

where

$$\psi : a \mapsto \psi_a = \phi_a \times_S T.$$ 

We have already seen how a single finite endomorphism of a line bundle can be normalized, now we will do the same with all endomorphisms $\phi_a$ simultaneously. The resulting object is a standard drinfeld module.

3.1.3 Definition. A standard drinfeld module of rank $r$ over $\gamma : S \to A$ is a pair $E = (\mathcal{L}, \phi)$ of an invertible sheaf $\mathcal{L}$ and an $\mathbb{F}_q$-linear ring morphism

$$\phi : A \to \text{End}_{\mathbb{F}_q}(G_{a,\mathcal{L}}) : a \mapsto \phi_a$$

of the form

$$\phi_a = \gamma^d(a) + a_1 \tau + \cdots + a_r \tau^r$$

for non-zero $a \in A$, where $d$ is the degree of $a$ and where $a_r \tau^r$ does not vanish on $S$.

Clearly a standard drinfeld module is a drinfeld module. The converse is also true in the following sense:

3.1.4 Proposition. Every drinfeld module of rank $r$ over $S$ is isomorphic to a standard drinfeld module of the same rank over $S$. Moreover, this isomorphism is unique up to a unique isomorphism.
3.2 Isogenies

Proof. Choose a non-zero \( a \in \mathbf{A} \), say of degree \( d \). Then there is, by 22.25, a unique isomorphism \( \mathbf{E} \to \mathbf{F} = (\mathcal{L}, \psi) \) such that \( \psi_a \) is in standard form. Now assume for some non-zero \( b \in \mathbf{A} \) the endomorphism \( \psi_b \) of \( \mathbb{G}_{a, \mathcal{L}} \) is not standard. This implies there exists an affine subscheme of \( S \), corresponding to a ring \( R \), such that

\[
\psi_b \otimes R = \beta_0 + \ldots + \beta_t r^i
\]

is not standard, so \( \beta_i \) is nilpotent. If we write down the restriction of \( \psi_a \) to \( R \):

\[
\psi_a \otimes R = \alpha_0 + \ldots + \alpha_r r^d
\]

where \( \alpha_r \) is a unit in \( R \), then we read from \( \psi_a \psi_b = \psi_b \psi_a \) that

\[
\beta_i = \beta_i \alpha_i q^{d-1}
\]

This contradicts that \( \beta_i \) is nilpotent.

3.2.1. Let again \( \mathbf{E} = (\mathcal{L}, \phi) \) be a drinfeld module over the \( \mathbf{A} \)-scheme \( S \). This scheme can be covered by open affines that trivialize the line bundle \( \mathcal{L} \). So we can see a drinfeld module over a scheme as a glued collection of drinfeld modules over affine schemes with trivial invertible sheaf. In standard form, such a drinfeld module of rank \( r \) over the \( \mathbf{A} \)-algebra \( \gamma^d : \mathbf{A} \to \mathbb{R} \) is just a morphism \( \psi : \mathbf{A} \to \text{End}(\mathbb{G}_{a, \mathbb{R}}) \) that maps a non-zero \( a \) to an endomorphism

\[
\psi_a = \gamma_d(a) + a_1 r + \ldots + a_r r^d
\]

where \( d \) is the degree of \( a \), the sections \( a_i \) are elements of \( R \), the leading coefficient \( a_r \) is a unit and \( r \) is just the \( q \)-th power frobenius endomorphism of \( \mathbb{G}_{a, \mathbb{R}} \).

3.2.2. A special isogeny of \( \mathbf{E} \) onto itself is multiplication by \( a \) for a non-zero \( a \in \mathbf{A} \). We denote it by \( [a] = \phi_a \). The ring of all isogenies of \( \mathbf{E} \to \mathbf{E} \) is denoted by \( \text{End}(\mathbf{E}) \). As in the classical setting, we say that \( \mathbf{E} \) has complex multiplication if the map \( \mathbf{A} \to \text{End}(\mathbf{E}) \) is not surjective. The kernel of an isogeny \( \xi : \mathbf{E} \to \mathbf{F} \) is a finite flat group scheme over \( S \). We denote the kernel of \( \xi \) by \( \mathbf{E}[\xi] \), the kernel of \( [a] \) by \( \mathbf{E}[a] \). The intersection of the kernels of \( [a] \), that is their fiber product over \( \mathbb{G}_{a, \mathcal{L}} \), for all non-zero \( a \) in some non-zero ideal \( n \) of \( \mathbf{A} \) is denoted by \( \mathbf{E}[n] \). We call \( \mathbf{E}[n] \) the \( n \)-torsion scheme of \( \mathbf{E} \).

3.2.3 PROPOSITION. Let \( n \) be a non-zero prime ideal of \( \mathbf{A} \) with prime factorization

\[
n = q_1^{n_1} q_2^{n_2} \cdots q_m^{n_m}
\]

i) \( \mathbf{E}[n] \) is a finite flat group scheme over \( S \). It is étale over \( S \) if \( n \) is disjoint from the characteristic of \( S \).

ii) \( \mathbf{E}[n] \) has rank \( |\mathbf{A}/n|^r \) over \( S \).
iii) \( \phi \) induces an action of \( A \) on \( E[n] \) which factors through \( A/n \).

iv) There is a canonical isomorphism

\[
E[n] = E[q_1^{n_1}] \times_S E[q_2^{n_2}] \times_S \cdots \times_S E[q_m^{n_m}].
\]

**Proof.** For all the claims it is sufficient to prove them over affine open subschemes.

Let \( E/R \) be a drinfeld module of rank \( r \) over the algebra \( A \to R \) with trivial line bundle \( \mathbb{G}_{a,R} = R[X] \).

Assume first that \( n \) is a principal ideal, generated by \( a \in A \) of degree \( \deg(a) = d \). We have

\[
E[n] = R[X]/(\gamma(a)X + a_1X^q + \ldots + a_{rd}X^{q^d})
\]

This is finite and flat because \( a_{rd} \) is invertible in \( R \). By the Jacobi criterion, it is étale whenever \( \gamma(a) \) is invertible. This is precisely the case when \( n \) is disjoint from the characteristic. From the above explicit discription we read that the rank of \( E[n] \) is \([A/n]^r\). The action of \( \phi \) factors through \( A/n \) because \( n \) is in the kernel of it. Because fiber products of finite flat or finite étale schemes are again finite flat, respectively finite étale, the proposition is also true for non-principal ideals \( n \). The last claim follows from the third one and the chinese remainder theorem.

We would also want to know when a finite flat sub-module is the kernel of an isogeny: when does the quotient of a drinfeld module by such a sub-module exist and is itself a drinfeld module? Lemma 2.3.2 tells us that the quotient line bundle exists, so the remaining question is if it can be equipped with a drinfeld structure. Lemma 2.3.2 tells us that the line bundle of an isogeny \( F \to E \) exists, so the remaining question is if it can be equipped with a drinfeld structure. Because the quotient map \( \xi : \mathbb{G}_{a,L} \to \mathbb{G}_{a,M} \) is an epimorphism \( 2.2.6 \), and because \( \xi \) has to be a morphism of drinfeld modules, there can only be one choice for \( \psi \). Namely, for all \( a \in A \), \( \psi_a \) is determined by

\[
\xi \circ \phi_a = \psi_a \circ \xi
\]

Because \( G \) is \( A \)-invariant, a \( \psi_a \) satisfying this relation exists, and \( \psi \) is automatically a ring morphism of \( A \) into \( \text{End}(\mathbb{G}_{a,M}) \).

First we consider an affine scheme \( S \) with trivial line bundle \( L \). This is just the additive group \( \mathbb{G}_{a,R} \) of an \( A \)-algebra \( \gamma^2 : A \to R \). Theorem 2.3.2 tells us that the line bundle \( \mathcal{M} \) will also be trivial. Therefore the quotient \( \xi : \mathbb{G}_{a,L} \to \mathbb{G}_{a,M} \) is an endomorphism of \( \mathbb{G}_{a,R} \), which has the form \( \xi = x_{h}x_{h+1} + x_{h+1}x_{h+2} + \cdots \) with non-zero \( x_h \). Now, by comparing coefficients in \( \xi \circ \phi_a = \psi_a \circ \xi \), we find a necessary and sufficient condition for \( F = (\mathcal{M}, \psi) \) to be a drinfeld module of rank \( r \), namely:

\[
\forall a \in A : \gamma^2(a^{q^h}) = \gamma^2(a).
\]

Now consider a general \( A \)-scheme \( S \). The only condition to be checked is the compatibility of the induced action on the quotient with the structure morphism \( S \to A \), this can be done locally.
Let $A \to k$ be an $A$-field of characteristic $p$, and $A \to \overline{k}$ an algebraic closure of $k$. For such base schemes we can say a bit more about torsion and isogenies.

3.2.5 Theorem. If $n = q_1^{n_1} q_2^{n_2} \cdots q_m^{n_m}$ is the factorization of $n$ in prime ideals, then the $n$-torsion scheme of the rank $r$ Drinfeld module $E/k$ is, as $A$-module

$$E[n](k) = E[q_1^{n_1}](k) \oplus \cdots \oplus E[q_m^{n_m}](k)$$

Now let $q$ be a non-zero prime ideal of $A$, then either

i) $q \neq p$ and $E[q^n](\overline{k}) \cong (A/q^n)^r$ for all $n > 0$

ii) $q = p$ and $E[q^n](\overline{k}) \cong (A/q^n)^h$ for all $n > 0$ where $0 \leq h \leq r - 1$ is a constant.

In the second case, we call an elliptic module (that is a Drinfeld module of rank 2) ordinary if $h = 1$ and super-singular if $h = 0$.

Proof. The first statement is just 3.2.3. Now let $m$ be the order of the class of $q$ in Pic($A$), this means $m$ is the smallest positive integer such that $q^m$ is a principal ideal, generated by, say, $a$. For positive integers $l$ we have

$$\phi_a^l E[q^m](\overline{k}) = E[q^m-l](\overline{k}).$$

But because of 3.2.3 we also have

$$|E[q^m](\overline{k})| = |E[a^l](\overline{k})| = q^{la}$$

where $s = \deg(\phi_a)$ if $q \neq p$ and $s = \deg(\phi_a) - \text{ht}(\phi_a)$ if $q = p$. From this already the theorem follows for $n$ divisible by $m$. Now, for general $n$ use the filtration

$$E[q^m](\overline{k}) \hookrightarrow E[q^{m+1}](\overline{k}) \cdots \hookrightarrow E[q^{m+m}](\overline{k})$$

and the action of an element in $q$, not in $q^2$. \hfill \square

4. Level structures

Drinfeld introduced level structures for Drinfeld modules in his paper [3]. Using the same idea in the classical context, a good definition of level structures for “bad primes” was then finally given. A very good book about the classical theory of moduli problems, which focuses on the bad primes using Drinfeld’s definition is “Arithmetic moduli of elliptic curves” ([10]).

4.1 $\Gamma(n)$-structures

4.1.1. Let $E/S/A = (\mathcal{L}, \phi)$ be an elliptic module over the $A$-scheme $S$. Consider a morphism of $A$-modules

$$\alpha : (A/n)^2 \to \mathbb{G}_{a,\mathcal{L}}(S)$$

Every element $x$ of $(A/n)$ defines an $S$-rational point $\alpha(x)$, and therefore also a cartier divisor $[\alpha(x)]$ on $\mathbb{G}_{a,\mathcal{L}}$. If the sum of all these divisors is the cartier divisor corresponding to the closed subscheme $E[n]$ of $\mathbb{G}_{a,\mathcal{L}}$, we say $\alpha$ is a $\Gamma(n)$-structure on $E$.

Shorter: a $\Gamma(n)$-structure on $E$ is an $A$-morphism

$$\alpha : (A/n)^2 \to \mathbb{G}_{a,\mathcal{L}}(S)$$
inducing an equality of cartier divisors

$$\sum_{x \in (\mathbb{A}/n)^2} [\alpha(x)] = \mathbb{E}[n]$$
on \mathbb{G}_{a, \varphi}.

\(\Gamma(n)\)-structures commute with base change. An elliptic module over \(S\) only allows \(\Gamma(n)\)-structures if its \(n\)-torsion points are \(S\)-rational, therefore every elliptic module can be equipped with \(\Gamma(n)\)-structures after finite faithfully flat base change. If \(n\) is disjoint from the characteristic of \(S\), \(\Gamma(n)\)-structures exist even after finite étale base change.

Let \(n = q_1^{n_1}q_2^{n_2}\cdots q_m^{n_m}\) be the factorization of \(n\) into prime ideals. Then giving a \(\Gamma(n)\)-structure is the same as giving a \(\Gamma(q_i^{n_i})\)-structure for every \(i\).

4.1.2. Now assume our base scheme is an algebraically closed \(A\)-field \(A \to \overline{k}\). If \(n\) is coprime to the characteristic of \(\overline{k}\), then a \(\Gamma(n)\)-structure on \(E/\overline{k}\) is just an isomorphism of \(A\)-modules

$$\mathbb{A}/n \sim \to \mathbb{E}[n](\overline{k}) \cong (\mathbb{A}/n)^2$$

If \(n = q^n\), where the prime \(q\) is the characteristic of \(\overline{k}\), then a \(\Gamma(n)\)-structure is a surjection of \(A\)-modules

$$(\mathbb{A}/n)^2 \to \mathbb{E}[n](\overline{k}) \cong \begin{cases} \mathbb{A}/n & \text{(ordinary \(E\))} \\ 0 & \text{(super-singular \(E\))} \end{cases}$$

4.2 \(\Gamma_1(n)\)-structures

4.2.1. A \(\Gamma_1(n)\)-structure on \(E/S\) is a morphism of \(A\)-modules

$$\alpha : \mathbb{A}/n \to \mathbb{G}_{a, \varphi}(S)$$

such that the effective cartier divisor

$$\sum_{x \in \mathbb{A}/n} [\alpha(x)]$$
is a subgroup scheme of \(E[n]\).

\(\Gamma_1(n)\)-structures commute with base change. An elliptic module that allows \(\Gamma(n)\)-structures also allows \(\Gamma_1(n)\)-structures, but not necessarily vice versa.

Just as with \(\Gamma(n)\)-structures, we can factor \(\Gamma_1(n)\)-structures: giving a \(\Gamma_1(n)\)-structure is the same as giving a \(\Gamma_1(q_i^{n_i})\)-structure for every \(i\), where \(\prod_i q_i^{n_i}\) is the prime factorization of \(n\).

4.2.2. Let \(E/\overline{k}\) be an elliptic module over an algebraically closed field \(A \to \overline{k}\). If \(n\) is disjoint from the characteristic of \(\overline{k}\), then a \(\Gamma_1(n)\)-structure is an injection of \(A\)-modules:

$$\mathbb{A}/n \hookrightarrow \mathbb{E}[n](\overline{k}) \cong (\mathbb{A}/n)^2.$$ If \(n = q^n\), where the prime \(q\) is the characteristic of \(\overline{k}\), then a \(\Gamma_1(n)\)-structure is either an isomorphism

$$\mathbb{A}/n \sim \to \mathbb{E}[n](\overline{k})$$
or the zero morphism

$$\mathbb{A}/n \to 0 \hookrightarrow \mathbb{E}[n](\overline{k})$$

where the former can occur only if \(E\) is ordinary, and the latter occurs for both ordinary and super-singular elliptic modules.

12
4.3 $\Gamma_0(n)$-structures

The definition of a $\Gamma_0(n)$-structure is a bit more technical because we have to consider $S$-rational subgroups of which the elements themselves need not be $S$-rational.

4.3.1. A $\Gamma_0(n)$-structure on $E/S$ is a finite flat subgroup scheme

$$H \subset E[n]$$

with an induced action of $A/n$, of constant rank $\#(A/n)$ over $S$, and cyclic. This last condition means there is a finite, faithfully flat base-change $T \to S$, and a point $P \in (\mathbb{G}_{a,Z} \times_S T)(T)$ such that we have on $\mathbb{G}_{a,Z} \times_S T$ an equality of cartier divisors

$$H_T = \sum_{x \in A/n} [xP].$$

We call such a point $P$ a base point.

$\Gamma_0(n)$-structures are stable under base change because finite, faithfully flat morphisms are. An elliptic module $E/S$ need not allow $\Gamma_0(n)$-structures. If $E/S$ has a $\Gamma_1(n)$-structure $\alpha$, it has an induced $\Gamma_0(n)$-structure by taking $H$ to be the effective cartesian divisor

$$\sum_{x \in A/n} [\alpha(x)].$$

Factor $n$ as $\prod q_i^{n_i}$. Given a $\Gamma_0(n)$-structure $H$, we can construct $\Gamma_0(q_i^{n_i})$-structures $H_i$ by taking the $q_i$-torsion in $H$:

$$H_i = H[q_i^{n_i}] = \bigcap_{a \in q_i^{n_i}} \ker([a] : H \to H).$$

Conversely, given $\Gamma_0(q_i^{n_i})$-structures $H_i$ for all $i$, we construct a $\Gamma_0(n)$-structure $H$ as

$$H = H_1 \times_S H_2 \times_S \cdots \subset E[q_1^{n_1}] \times_S E[q_2^{n_2}] \times_S \cdots = E[n]$$

These constructions are mutually inverse.

4.3.2. Because over an algebraically closed field all the torsion points are rational, such a base scheme is much easier to handle. Let $E/k$ be an elliptic module over the algebraically closed $A$-field $A \to k$. If $n$ is not divisible by the characteristic $q$ of $k$, a $\Gamma_0(n)$-structure is a submodule $H$ of $E[n]/k$ that is isomorphic to $A/n$. If $n = q^n$, and $E$ is ordinary, then a $\Gamma_0(n)$-structure $H$ is either

$$H = E[n]^{\text{red}}$$

or the effective cartesian divisor

$$H = \#(A/n)[0]$$

where 0 is the zero section of $\mathbb{G}_{a,Z}$. If $E$ is super-singular, the only $\Gamma_0(n)$-structure is $H = \#(A/n)[0]$.

4.4 Mixed structures

4.4.1. Combinations of the above level structures will also be used. Let $n$, $n_1$, and $n_0$ be three ideals in $A$. Giving a $\Gamma(n) \times \Gamma_1(n_1) \times \Gamma_0(n_0)$-structure on an elliptic module $E$ is the same as giving three level structures: a $\Gamma(n)$-structure, a $\Gamma_1(n_1)$-structure, and a $\Gamma_0(n_0)$-structure.

4.5 The action of $\text{GL}(2, A/n)$

4.5.1. Let us fix an ideal $n$ and an elliptic module $E/S$. 13
The group $GL(2, \mathcal{A}/n)$ acts on the set of $\Gamma(n)$-structures on $E/S$ by left multiplication: an element $m \in GL(2, \mathcal{A}/n)$ transforms a couple $(E/S, \alpha)$ of a module over $S$ and a $\Gamma(n)$-structure on $E$ as
\[(E/S, \alpha) \mapsto (E/S, \alpha \circ m)\]
and this action commutes with base change. Similarly, the group $(\mathcal{A}/n)^\times$ acts on $Y_1(n)$. Over an algebraically closed base field $k$, we can redefine a $\Gamma_0(n)$-structure as a $\Gamma_1(n)$-structure modulo the left action of $(\mathcal{A}/n)^\times$ on it.

5. Modular schemes

In this chapter we will represent functors that classify drinfeld modules of rank 2 with certain level structures by affine schemes, and briefly mention the canonical compactifications of them. We will only consider modules of rank 2, so-called elliptic modules, and as a consequence the fibers of the moduli schemes over $\mathcal{A}$ will be curves.

5.1 The three basic moduli problems

Compare to [10]. We do not consider balanced level structures because we lack a good moduli interpretation in terms of dual isogenies, when the ring $\mathcal{A}$ is not a principal ideal domain. The associated moduli problem can however be defined as a quotient of the full level moduli problem.

5.1.1. Our three main moduli problems are functors on $(\text{Sch}/\mathcal{A})$, that associate with a scheme the set of elliptic modules with certain level structure, up to isomorphy. Concretely, we define, for an ideal $n$ of $\mathcal{A}$ the full level $n$-structure moduli problem
\[Y(n) : S \mapsto Y(n)(S) = \{\text{elliptic modules over } S \text{ with } \Gamma(n)\text{-structure}\}/ \cong\]
as well as the $\gamma_1(n)$-structure moduli problem
\[Y_1(n) : S \mapsto Y_1(n)(S) = \{\text{elliptic modules over } S \text{ with } \Gamma_1(n)\text{-structure}\}/ \cong\]
and the $\gamma_0(n)$-structure moduli problem
\[Y_0(n) : S \mapsto Y_0(n)(S) = \{\text{elliptic modules over } S \text{ with } \Gamma_0(n)\text{-structure}\}/ \cong\]

5.1.2. Given an elliptic module $E/S/\mathcal{A}$, we write, by abuse of notation,
\[Y(n)(E/S) = \{\Gamma(n)\text{-structures on } E/S\}\]
This is in fact functorial by base change. In the same way we write
\[Y_1(n)(E/S) = \{\Gamma_1(n)\text{-structures on } E/S\}\]
\[Y_0(n)(E/S) = \{\Gamma_0(n)\text{-structures on } E/S\}\]
can define new functors $Y_1(n)$ and $Y_0(n)$. We call these functors on (elliptic modules) relative moduli functors, to differentiate them from the moduli functors on $(\text{Sch}/\mathcal{A})$.

5.1.3. For mixed structures we use the following notation. We simply concatenate the symbols for functors. For example, the moduli problem $\Gamma(n) \times \Gamma_0(m)$ defines a moduli functor denoted by $\overline{Y(n)Y_0(m)}$.

5.2 Relative representability

In this section we will prove the following result, which we call relative representability.
5.2.1 Theorem. Given an elliptic module $E/S/A$ and an ideal $n$, then the three functors on $(\text{Sch}/S)$

\[ T \mapsto Y(n)(E_T/T) = \{ \Gamma(n)\text{-structures on } E_T \} \]
\[ T \mapsto Y_1(n)(E_T/T) = \{ \Gamma_1(n)\text{-structures on } E_T \} \]
\[ T \mapsto Y_0(n)(E_T/T) = \{ \Gamma_0(n)\text{-structures on } E_T \} \]

are represented by finite affine schemes over $S$.

The method is the same as used in [10], in the context of elliptic curves for $\Gamma$- and $\Gamma_1$-structures. For $\Gamma_0$-structures we give a much easier proof, using faithfully flat descent.

5.2.2 Lemma. For a fixed elliptic module $E/S/A$, the functors

\[ T \mapsto \text{Hom}_A(A/n, E[n](T)) \]
\[ T \mapsto \text{Hom}_A((A/n)^2, E[n](T)) \]

are represented by the finite $S$-schemes $E[n]$ and $E[n] \times_S E[n]$ respectively.

Proof. Because the $A$-module $E[n](T)$ descends to an $A/n$-module (3.2.3), we have canonical isomorphisms:

\[ \text{Hom}_A(A/n, E[n](T)) = \text{Hom}_{A/n}(A/n, E[n](T)) = E[n](T) \]

This proves the first claim, and with this, the second result follows from the canonical isomorphism

\[ \text{Hom}_A((A/n)^2, E[n](T)) = (\text{Hom}_A(A/n, E[n](T)))^2 \]

5.2.3 Lemma. The functor

\[ T \mapsto Y_1(n)(E_T/T) \]

is represented by a closed subscheme of $E[n]$.

Proof. By definition, the considered functor is a subfunctor of the representable functor $T \mapsto \text{Hom}_A(A/n, E[n](T))$. An $A$-morphism

\[ \alpha : A/n \rightarrow E[n](T) \]

is a $\Gamma_1(n)$-structure on $E_T$ if and only if it induces an inequality of effective cartier divisors

\[ [\alpha(A/n)] \leq [E_T[n]]. \]

This condition is represented by a closed subscheme of $E[n]$ by ([10] 3.1.4).

5.2.4 Lemma. The functor

\[ T \mapsto Y(n)(E_T/T) \]

is represented by a closed subscheme of $E[n] \times_S E[n]$.

Proof. The proof is essentially the same as the previous one. Given an

\[ \alpha : (A/n)^2 \rightarrow E[n](T), \]

it is a $\Gamma(n)$-structure if and only if

\[ [\alpha((A/n)^2)] = [E_T[n]]. \]
But since both sides automatically have the same degree, it suffices to require the inequality

\[ \alpha((A/n)^2) \leq [E_T[n]]. \]

The proof for \( \Gamma_0(n) \)-structures is more subtle, we will use the technique of faithfully flat descent (8, 7 VIII). It also motivates our definition. A priori, more definitions seemed plausible, but the given definition gives rise to a relatively representable moduli problem, a necessity to obtain good moduli schemes.

5.2.5 Lemma. Let \( Z/S \) be the scheme representing the functor

\[ T \mapsto Y_1(n)(E_T/T), \]

then the functor

\[ T \mapsto Y_0(n)(E_T/T) \]

is represented by the quotient scheme \( Z/G \), where the group \( G = (A/n)^\times \) is the group of units of the ring \( A/n \), acting on \( Z \) by its action on level structures.

Proof. Given a \( \Gamma_0(n) \)-structure \( H \) on \( E_T \), we do a finite faithfully flat base-change \( T' \to T \) such that \( H_{T'} \) has a base point. This choice of base point yields a \( \Gamma_1(n) \)-structure on \( E_{T'} \), and hence a \( T' \)-valued point of \( Z \). By composition \( Z \to Z/G \) we then find a section \( T' \to Z/G \) of the quotient. Now by faithfully flat descent, this section descends to a \( T \)-valued point. Indeed, write \( T'' = T' \times_T T' \), and consider the following commutative diagrams, indexed by \( i = 1, 2 \):

\[
\begin{array}{ccccccccc}
T'' & \overset{i}{\longrightarrow} & Z_{T''} & \longrightarrow & (Z/G)_{T''} \\
\downarrow \rho_i & & \downarrow i & & \downarrow \\
T' & \longrightarrow & Z_{T'} & \longrightarrow & (Z/G)_{T'} \\
\downarrow & & \downarrow & & \downarrow \\
T & & & & (Z/G)
\end{array}
\]

The two cases correspond to the two projections \( T'' = T' \times_T T' \to T' \). For the middle composite horizontal arrow to descend, we have to show that the composite top level arrow does not depend on \( i \) either. The induced elements of \( Z(T'') \) are \( \Gamma_1(n) \)-structures on \( E_{T''} \) refining the given \( H_{T''} \) (the composite arrow \( T'' \to T \) is independent of \( i \)). Hence they differ by an automorphism in \( G = (A/n)^\times \), and project to the same element of \( (Z/G)(T'') \).

Conversely, let \( T \to Z/(A/n)^\times \) be a morphism of \( S \)-schemes. Again, a finite faithfully flat extension yields a \( T' \to Z \), namely \( T' = T \times_{Z/G} Z \). This corresponds to a \( \Gamma_1(n) \)-structure on \( E_{T'} \), and this in turn gives us a \( \Gamma_0(n) \)-structure \( H' \) over \( T' \). The group \( G \) works on \( T' \) by its action on \( Z \), and also on \( H'/T' \) by this action on \( T' \). We get a composite \( H' \to T' \to T \) which is invariant under \( G \) and hence factors over \( H = H'/G \to T \). Hence we get a cartesian square

\[
\begin{array}{ccc}
H' & \longrightarrow & T' \\
\downarrow & & \downarrow \\
H & \longrightarrow & T
\end{array}
\]

and a canonically determined element of \( Y_0(n)(E/T) \).

The above constructions are each others inverses, and we have established a canonical bijection, functorial
in $T$, 
\[ Y_0(n)(E_T/T) = \text{Hom}(T, Z/G). \]

5.2.6 THEOREM. Fix an elliptic module $E/S$. Let $Z(n)$, $Z_1(n)$ and $Z_0(n)$ be the representing schemes for the relative moduli functors corresponding to $\Gamma(n)$, $\Gamma_1(n)$ and $\Gamma_0(n)$. We have the following quotients by subgroups of $\text{GL}(2, A/n)$, acting on level structures:

\[ Z(A) = Z(n)/\left( \begin{array}{cc} * & * \\ * & * \end{array} \right) \]
\[ Z_0(n) = Z(n)/\left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \]
\[ Z_1(n) = Z(n)/\left( \begin{array}{cc} 1 & * \\ 0 & * \end{array} \right) \]

If $m \subset n$ is an inclusion of ideals, we also have:
\[ Z(n) = Z(m)/G \]
where $G$ is the kernel of the reduction map $\text{GL}(2, A/m) \to \text{GL}(2, A/n)$.

Proof. All these claims can be proven in precisely the same way as the second one, about $Z_0$ (5.2.5).

5.3 Coarse and fine moduli schemes

A pioneering book on the topic of moduli spaces is “Geometric Invariant Theory” ([13]).

5.3.1. Let $\mathcal{F}$ be a functor from $(\text{Sch}/S)$ to (sets). A coarse moduli scheme for $\mathcal{F}$ is a couple $(Y, \Phi)$ of an $S$-scheme $Y$ and a morphism of functors
\[ \Phi : \text{Hom}(Y, -) \to \mathcal{F}(-) \]

such that

i) $\Phi(k) : Y(k) \to \mathcal{F}(k)$ is a bijection for every algebraically closed field $k$.

ii) $(Y, \Phi)$ satisfies the following universal property: if $Y'$ is a scheme over $S$, and $\Phi' : \text{Hom}(Y', -) \to \mathcal{F}(-)$ a morphism of functors, there is a unique morphism of functors $f : \text{Hom}(Y, -) \to \text{Hom}(X, -)$ such that $\Phi' = f \circ \Phi$.

If, moreover, $\Phi$ is a functorial isomorphism, we call $(Y, \Phi)$ a fine moduli scheme. In this case, the scheme $Y$ represents the functor $\mathcal{F}$. Fine moduli schemes are a fortiori coarse moduli schemes. From the universal property in the definition, it follows formally that coarse moduli schemes are unique up to a unique isomorphism.

Clearly a representable functor has a fine moduli scheme. A non-representable functor does not always have a coarse moduli scheme.

5.3.2 DEFINITION. An ideal $n$ of $A$ is called admissible if its factorization into prime ideals contains at least two different prime factors.

5.3.3 THEOREM. Let $n$ be an admissible ideal in $A$ with at least two different prime factors. The moduli problem $Y(n)$ is representable by an affine $A$-scheme. We denote it by $Y(n)$. Moreover, for all ideals $m$,
\[ \overline{Y(n)}Y_0(m) \text{ and } Y(n)\overline{Y}_1(m) \text{ are represented by affine } A\text{-schemes, which we denote by } Y(n)Y_0(m) \text{ and } Y(n)\overline{Y}_1(m). \]

This means that the scheme \( Y(n) \) is equipped with a universal elliptic module and a \( \Gamma(n) \)-structure on it, such that every elliptic module with such a level structure over a scheme \( S \) is obtained from this universal module by a base change \( S \to Y(n) \).

\textbf{Proof.} \cite[3.2.6]{3.2.6} Assume \( p \) and \( q \) are different prime divisors of \( n \). We will first prove representability of \( \overline{Y(n)} \) over \( A(p) \) and of \( A(q) \), and then glue over \( A(pq) \). The result for the two mixed moduli problems follows by applying relative representability \cite[5.2.3 and 5.2.5]{5.2.3 and 5.2.5} to the universal elliptic module \( E \big/I Y(n) \).

Now let us proceed to prove that \( \overline{Y(p^n)} \), restricted to schemes over \( A(p) \) is representable. The ring \( A \) can be presented in terms of generators and relations as

\[ A = \mathbb{F}_q[a_1, a_2, \ldots a_m]/(f_1, f_2, \ldots, f_{m-1}) \]

We are going to construct an \( A \)-algebra \( B \), equipped with an elliptic module \( E = (G_\alpha, \phi) \) for the trivial line bundle on \( B \) and a \( \Gamma(n) \)-structure on it. Then we will show that it is universal for the moduli problem restricted to \( A(p) \)-schemes. We construct \( B \) in several steps, the construction is straightforward but involves a lot of generators and relations.

i) We add the coefficients \( c_{i,j} \) of a generic elliptic module \( \phi \) (in standard form) to \( A \):

\[ \phi(a_i) = \sum c_{i,j} \tau^j \]

ii) We add the relations between the \( c_{i,j} \) expressing that \( \phi \) is a ring homomorphism. Moreover we invert the coefficients of highest degree \( c_{i,2 \deg(a_i)}, \) for all \( i \).

iii) A \( \Gamma(p^n) \)-structure on \( E \) is defined by

\[ \alpha : (A/p^n)^2 \to B : x \mapsto \alpha(x), \]

and we add the images \( \alpha(x) \) to our set of generators.

iv) They must satisfy algebraic relations expressing that they are additive in \( x \), and that they are \( A \)-linear:

\[ \phi_{a_i} \alpha(x) = \alpha(a_i x) \]

\[ \alpha(\lambda x) = \lambda \alpha(x), \forall \lambda \in \mathbb{F}_q \]

Moreover, the image of \( \alpha \) satisfies

\[ \prod_{x \in (A/p^n)^2} (z - \alpha(x)) = P(z) \]

where \( P \) is the monic polynomial corresponding to the \( q \)-torsion of \( E \). We finish by adding all these relations.

It follows from the construction that every elliptic module over an \( A \)-algebra in which \( p \) is invertible can be obtained from \( E \) by a base change. Moreover, this base change is unique since our construction of \( E \) is rigid. Remark that if we allow algebras with characteristic intersecting \( p \) then we lose this uniqueness. From this universal property it follows immediately that \( B \) represents the moduli problem over \( A(p) \)-algebras and even over \( A(p) \)-schemes. Applying relative representability to the universal elliptic module \( E/I B \) shows that also \( A(n) \), with \( p \) dividing \( n \), is representable over \( A(p) \).

So assume now that \( n \) has two different prime divisors \( p \) and \( q \). We find that \( \overline{Y(n)} \) restricted to \( A(p) \)-or \( A(q) \)-schemes is represented by some affine schemes \( Y_1 \) and \( Y_2 \) respectively. Since they represent
the same functor over $A(pq)$-schemes, there is a canonical isomorphism between open subschemes $Y_1 \otimes_A A(p) \rightarrow Y_2 \otimes_A A(q)$. We find that $Y(n)$ is represented by the scheme $Y(n)$ obtained by gluing $Y_1$ and $Y_2$ by this isomorphism.

We now want to prove the existence of coarse moduli schemes for the other moduli problems we defined.

5.3.4. Let $n$ be any ideal in $A$, and fix an admissible ideal $m$ coprime to $n$. We define quotient schemes of fine moduli schemes by the finite group $G = GL(2, A/m)$, acting on $\Gamma(m)$-structures.

$$Y(n) = Y(mn)/G$$

$$Y_1(n) = Y(m)Y_1(n)/G$$

$$Y_0(n) = Y(m)Y_0(n)/G$$

These quotient schemes are independent of the choice of $m$, since the different quotient morphisms coincide on geometric points. For the same reason, these definitions are consistent with the ones already given for $Y(n)$ with admissible $n$.

5.3.5 Theorem. $Y(n)$, $Y_1(n)$ and $Y_0(n)$ are coarse moduli schemes for the moduli problems $Y(n)$, $Y_1(n)$ and $Y_0(n)$, respectively.

Proof. Let $Y$ be one of the schemes $Y(n)$, $Y_1(n)$ or $Y_0(n)$, and let $\underline{Y}$ be the corresponding moduli functor $(\underline{Y}(n), \underline{Y}_1(n) \text{ or } \underline{Y}_0(n))$. By definition

$$Y = Y'/G$$

for some finite group $G = GL(2, A/m)$ and fine moduli scheme $Y'$, representing the moduli problem obtained by adding some $\Gamma(m)$-structure to the old one. We will silently use that this is independent of the choice of $m$. We proceed in several steps.

STEP 1. We construct a morphism of functors

$$\Phi : \underline{Y} \rightarrow \text{Hom}(-, Y)$$

Fix an $A$-scheme $S$. An element of $\underline{Y}(S)$ corresponds to an elliptic module $E$ equipped with a certain level structure. After a finite faithfully flat base change $T \rightarrow S$, we can chose a $\Gamma(m)$-structure on $E_T$. This induces a point $T \rightarrow Y_T$, which by faithfully flat descent descends to a point $S \rightarrow Y$.

STEP 2. Remark that $\Phi(\overline{k})$ is a bijection, for all algebraically closed $\overline{k}$. Indeed, $Y(\overline{k}) = Y'(\overline{k})/G$, and this is precisely $\underline{Y}(\overline{k})$.

STEP 3. (compare [13] 5.2.4) There is a canonical bijection

$$\{\text{morphisms of functors } \phi : \underline{Y} \rightarrow \text{Hom}(-, S)\} = \{\text{morphisms } f : Y \rightarrow S\},$$

functorial in $S$. We can see this as follows. The fine moduli scheme $Y'$ carries a universal elliptic module $E^{\text{univ}}/Y'$ with some level structure. Forgetting the $\Gamma(m)$-structure, we find an element $P \in \underline{Y}(Y')$. Now assume given a $\phi$ as above. Applying it to $P$ yields a morphism $Y' \rightarrow S$. But by construction it is invariant under the action of $GL(2, A/m)$ and hence factors over a unique $f : Y \rightarrow S$. On the other hand, given an
f : Y → S, we can compose with \( \Phi : \underbrace{Y} \rightarrow \text{Hom}(\cdot, Y) \) to get a \( \phi : \underbrace{Y} \rightarrow \text{Hom}(\cdot, S) : \)

\[
Y \rightarrow \text{Hom}(\cdot, Y) \rightarrow \text{Hom}(\cdot, S)
\]

The correspondences \( \phi \mapsto f \) and \( f \mapsto \phi \) are each others inverses and we established the claimed functorial bijection.

**STEP 4.** Combining the previous steps we can verify that \( Y \) is a coarse moduli scheme. \( \square \)

5.3.6. To study the coarse moduli schemes, one often covers them first with a fine moduli scheme by adding, for example, some \( \Gamma(pq) \)-structure, for suitable primes \( p \) and \( q \) to the moduli problem. The new, possibly mixed, moduli functor is representable. We can study the properties of the new, fine, moduli scheme by studying the functor it represents. Then we descend to the original, coarse, moduli scheme by taking the quotient by a suitable group. This procedure can often be avoided by using the machinery of stacks as described in [1].

### 5.4 Deformation theory

In this section we will study the regularity and dimensionality of the fine and coarse moduli schemes, and of their fibers. To do so, we first study regularity in the case of fine moduli schemes. And then take quotients by suitable finite groups, acting without fix-points, to obtain results for the coarse moduli schemes. As everywhere else in this chapter, we restrict to the case of rank two modules, but the methods used in this section are easily generalized to higher rank Drinfeld modules.

5.4.1. In what follows, \( k \) is an algebraically closed field. \( k[\epsilon] \) is the \( k \)-algebra determined by the relation \( \epsilon^2 = 0 \). An algebraic structure \( \overline{A} \) over \( k[\epsilon] \) is a *lift* of the structure \( A \) over \( k \), if \( A \) is obtained from \( \overline{A} \) by base change via

\[
k[\epsilon] \rightarrow k : \epsilon \mapsto 0
\]

The lift is said to be *trivial* if it is obtained from \( A \) by base change via the inclusion \( k \hookrightarrow k[\epsilon] \).

5.4.2 **Lemma.** Assume \( k \) has the structure of an \( A \)-algebra by a non-zero morphism \( \gamma : A \rightarrow k \). Then the \( k \)-space of lifts of \( \gamma \) to \( \overline{\gamma} : A \rightarrow k[\epsilon] \) is one-dimensional over \( k \).

**Proof.** This is just a restatement of the regularity of the curve \( A \) over \( \mathbb{F}_q \). \( \square \)

5.4.3 **Proposition.** Let \( E \) be an elliptic module over \( \gamma : A \rightarrow k \). Given a fixed lift \( \overline{\gamma} : A \rightarrow k[\epsilon] \) of \( \gamma \), the \( k \)-space of lifts \( (\overline{\gamma}, \underline{E}/k[\epsilon]) \) of \( (\gamma, E/k) \) is one-dimensional.

**Proof.** See also ([11] 1.5) and ([2] 2.2). The following proof is based on the proof in [11]. It was simplified by Marius van der Put.

Remark that all line bundles on \( k[\epsilon] \) are trivial.

Let us first lift \( E \) without fixing a given lift of \( \gamma \).

The elliptic module \( E \) is given by a morphism \( \phi : A \rightarrow k\{\tau\} \) of rings. Now consider a lift

\[
\overline{\phi} = \phi + \epsilon D : A \rightarrow k[\epsilon]\{\tau\}
\]
Let $I$ be a ring homomorphism, implying that $D$ is a derivation $D : A \to M$ to the bimodule $M = k\epsilon\{\tau\}$, on which $a \otimes b \in A \otimes A$ acts as
\[
amb = \phi(a)m\phi(b) = \gamma(a)m\phi(b).
\]
Conversely, every such derivation yields a ring homomorphism $\overline{\phi} = \phi + \epsilon D$.

A lift $\overline{\phi} = \phi + \epsilon D$ is isomorphic to the trivial lift precisely when there is a fixed $m \in M$ such that for all $a$,
\[
D : A \to M : a \mapsto am - ma.
\]

Now we introduce the universal bimodule derivation ([6] §20). We write $I$ for the kernel of the multiplication map
\[
m : A \otimes_{\mathbb{F}_q} A \to A : a \otimes b \mapsto ab
\]
and equip $I$ with the obvious $A/\mathbb{F}_q$-bimodule structure. The bimodule derivation
\[
d : A \to I : a \mapsto a \otimes 1 - 1 \otimes a
\]
has the following universal property. Every derivation $D : A \to N$ of $A$ in an $A/\mathbb{F}_q$-bimodule $N$ factors in a unique way as $D = l \circ d$, where $l : I \to N$ is a bimodule homomorphism.

In our situation, we find the space of lifts of ring homomorphisms is $\text{Hom}_{A \otimes A}(I, M)$, and the subspace of lifts equivalent to the trivial one $\text{Hom}_{A \otimes A}(A \otimes A, M)$. Hence the sought deformation space $\text{Def}$ is determined by the exact sequence
\[
\text{Hom}_{A \otimes A}(A \otimes A, M) \to \text{Hom}_{A \otimes A}(I, M) \to \text{Def} \to 0
\]
Let $l : I \to A \otimes A$ be an $A \otimes A$-morphism. The composition with the multiplication map $m$ is a morphism $I \to A$. Since this morphism is zero on $I^2$, it factors over a morphism $s(l) : I/I^2 \to A$. We have constructed a map $s : \text{Hom}_{A \otimes A}(I, A \otimes A) \to \text{Hom}_{A \otimes A}(I/I^2, A)$. To calculate its kernel, note that $s(l) = 0$ is equivalent to $l(I) \in I$. The ideal $I \subset A \otimes A$ corresponds geometrically to the diagonal divisor on the surface $\text{Spec}(A \otimes A)$, hence $I$ is a projective $A \otimes A$ module of rank 1, and thus the isomorphism $\text{Hom}_{A \otimes A}(I, I) \cong A \otimes A$ holds. Therefore, $l(I) \in I$ is equivalent to the existence of a continuation of $l : I \to A \otimes A$ to an $A \otimes A$-morphism $A \otimes A \to A \otimes A$. We have proven the exactness of the sequence of $A \otimes A$-modules
\[
\text{Hom}(A \otimes A, A \otimes A) \to \text{Hom}(I, A \otimes A) \to \text{Hom}(I/I^2, A) \to 0
\]
in which the middle map is the $s$ constructed above. The $A \otimes A$-structure on $A$ is induced by the multiplication morphism $m$. Tensoring this exact sequence with $M$ yields the previous exact sequence and we find
\[
\text{Def} \cong \text{Hom}_{A \otimes A}(I/I^2, A) \otimes_{A \otimes A} M
\]
The ideal $I = \ker(m : A \otimes A \to A)$ acts as 0 on both $I/I^2$ and $A$, hence $\text{Hom}_{A \otimes A}(I/I^2, A) = \text{Hom}_A(I/I^2, A)$ which is projective of rank 1 over $A$ since it is dual to the module of relative differentials of $A$ over $\mathbb{F}_q$.

Now the following lemma allows us to conclude that $\text{Def} \cong k^2$ as $k$-vector space.

Hence, we have proven that the space of lifts of $E$ to arbitrary lifts of $\gamma$ is two-dimensional. We still have to check that it projects surjectively to the one-dimensional space of lifts of $\gamma$. This follows from the above proof, by tracking in every step what happens to the lifts of $\gamma$ induced by the lifts of $E$.

\[\Box\]

5.4.4 LEMMA. Let $\gamma : A \to k$ be an $A$-field, and $E = (G_{a,k}, \phi)$ be an elliptic module over it. The
$k \otimes_{\mathbb{F}_q} A$-module $k\{\tau\}$ given by the action

$$(\lambda \otimes a)\alpha = \lambda \alpha \phi_a \in k\{\tau\}$$

is locally free of constant rank 2.

**Proof.** First we change the base field of the curve $A$ by $\mathbb{F}_q \hookrightarrow k$. We write $A_k = A \otimes_{\mathbb{F}_q} k$ and $\gamma_k = \gamma \otimes_{\mathbb{F}_q} k$. Choose a non-constant element $a \in A$ of degree $d$. Then $A_k$ is locally free of rank $d$ over the polynomial ring $k[a] \hookrightarrow A_k$. Moreover, by using the left euclidean algorithm we can write every element of $k\{\tau\}$ uniquely as a $k$-linear combination of terms

$$\tau^n \phi_{a^m} = \tau^n \phi_{a^m}$$

where $0 \leq n < 2d$. Hence $k\{\tau\}$ is free of rank $2d$ over $k[a]$. Because $\mathbb{F}_q$ is precisely the field of constants of $A$, $A_k$ is integral and the result follows. \qed

5.4.5 PROPOSITION. Assume $n$ is an ideal of $A$ which is coprime to the characteristic $\gamma : A \to k$. Let $(\overline{\gamma}, \overline{E})$ be a lift of $(\gamma, E)$. A $\Gamma(n)$, $\Gamma_1(n)$ or $\Gamma_0(n)$-structure lifts uniquely to a similar structure on $\overline{E}$.

**Proof.** The $n$-torsion of $E$ is given by a separable, additive polynomial $f$ over $k$. This implies the coefficient of its linear term is non-zero, say $c$. Similarly, $\mathbb{E}[n]$ is given by a lift $\overline{f}$ of $f$ to $k[e]$. Now let $u \in k$ be a zero of $f$, which means it corresponds to a $k$-rational torsion point of $E$. For the lift $u + \epsilon \in k[e]$ to be a $k[e]$-rational torsion point of $\overline{E}$, it suffices that $\overline{f}(u + \epsilon v) = 0$. We calculate

$$\overline{f}(u + \epsilon v) = \overline{f}(u) + \epsilon vc$$

where $\overline{f}(u)$ is an element of the radical $k\epsilon$ of $k[e]$. Hence, since $c \neq 0$, we find a unique $v \in k$ such that $\overline{f}(u + \epsilon v) = 0$. This already proves the proposition for $\Gamma(n)$ and $\Gamma_1(n)$-structures.

Now for $\Gamma_0(n)$-structures, remark that we have just shown that both $E[n](k)$ and $E[n](k[e])$ are isomorphic to $(A/n)^2$. Therefore choosing a $\Gamma_0(n)$-structure on one of them is nothing but choosing a submodule $A/n \subset (A/n)^2$ and no base extensions are needed to make torsion rational. Now the lemma follows immediately by the lifting bijection from $E[n](k)$ to $E[n](k[e])$ we established. \qed

5.4.6 COROLLARY. For all $n$, the moduli schemes $Y(n)$, $Y_1(n)$ and $Y_0(n)$ have dimension 2 over $\mathbb{F}_q$, the field of constants of $A$.

**Proof.** Let $p$ be a prime ideal. It follows from the proof of 5.3.3 that $Y(p)$ is representable by an affine scheme $Y$ over $A(p)$. Moreover, it follows from the given construction that this affine scheme is reduced. The previous proposition calculates the tangent space at a geometric point of $Y$, and since it is everywhere 2-dimensional, $Y$ must be regular of dimension 2 over $\mathbb{F}_q$. Moreover, $Y$ must coincide with the restriction $Y(p) \otimes_A A(p)$ of the coarse moduli scheme $Y(p)$, which is therefore also 2-dimensional. All the other moduli schemes are finite covers of $Y(p)$, or finite quotients of these finite covers, and hence they are 2-dimensional. \qed

5.4.7 PROPOSITION. Let $p$ be the kernel of $\gamma$, and assume it is non-zero. Let $E$ be an elliptic module over $\gamma : A \to k$, equipped with a $\Gamma(p^*)$-structure $\alpha$. The lifted couple $(\overline{\gamma}, \overline{E})$ allows a lift $\overline{\alpha}$ of $\alpha$ if and only if it is the trivial lift of $(\gamma, E)$. In this case, the space of lifts of $\alpha$ is two-dimensional over $k$.

**Proof.** Write $n = p^s$. The $\Gamma(n)$-structure

$$\alpha : (A/n)^2 \to G_{a,k}(k) = k$$
is given by its values on a free basis of \((\mathbb{A} / n)^2\) as \(\mathbb{A} / n\)-module. We set \(u_1 = \alpha(1, 0), u_2 = \alpha(0, 1)\), and similarly \(u_1 = u_1 + ev_1 = \overline{\alpha}(1, 0)\) and \(u_2 = u_2 + ev_2 = \overline{\alpha}(0, 1)\) for the lift
\[
\overline{\alpha} : (\mathbb{A} / n)^2 \rightarrow \mathbb{G}_{a,k}[e](k) = k[e]
\]
of \(\alpha\).

We must have that the \(n\)-torsion schemes (divisors) of \(E\) and \(\overline{E}\) correspond to the polynomials
\[
f = \prod_{x \in (\mathbb{A} / n)^2} (X - \alpha(x)) \quad \text{and} \quad \overline{f} = \prod_{x \in (\mathbb{A} / n)^2} (X - \overline{\alpha}(x))
\]
Because \(n\) is a power of the characteristic, the kernel of \(\alpha\) is non-trivial and therefore contains a submodule acting as \(\mathbb{F}_q\). Now using the formula
\[
\prod_{\lambda \in \mathbb{F}_q} (X - u - \lambda v) = (X - u)^q
\]
we find that \(\overline{f} = f\).

Now chose \(t > 0\) such that \(n^t\) is a principal ideal, say generated by \(a\). Then the \(n^t\)-torsion of \(E\) is defined by the polynomial \(f(f(\cdots (f)))\) (\(t\) iterations), and the \(n^t\)-torsion of \(\overline{E}\) by \(\overline{f}(\overline{f}(\cdots (\overline{f}))\)). Since these are equal, we find
\[
\phi_a = \overline{\phi}_a.
\]
Because \(\overline{\phi}_a\) has positive height, we also derive from
\[
\overline{\phi}_b \overline{\phi}_a = \overline{\phi}_a \overline{\phi}_b
\]
that \(\overline{\phi}_b = \phi_b\) for every \(b \in \mathbb{A}\). Hence, \(\gamma\) and \(\overline{E}\) must be trivial lifts. Now on the trivial lift, we have a two-dimensional space of candidate lifts \(\overline{\alpha}\), namely every candidate is determined by choosing \(v_1\) and \(v_2\). Every such choice yields a \(\Gamma(n)\)-structure on \(\overline{E}\) because of the same formula that was used earlier in this proof.

5.4.8 Proposition. Let \(p\) be the kernel of \(\gamma : \mathbb{A} \rightarrow k\) and assume given an elliptic module over \(\gamma : \mathbb{A} \rightarrow k\) with a \(\Gamma_1(p^\varepsilon)\)-structure \(\alpha\). We want to understand the structure of the space of lifts \((\overline{\gamma}, \overline{E}, \overline{H})\) of \((\gamma, E, H)\). We consider two cases:

i) If the image of \(\alpha\) is étale (\(E\) ordinary), every choice of \(\overline{\gamma}\) allows a unique choice of \(\overline{E}\) which, in turn, has a one-dimensional space of compatible lifts of \(\alpha\).

ii) If \(\alpha\) is local (\(E\) ordinary or supersingular), every choice of \((\overline{\gamma}, \overline{E})\) admits a one-dimensional space of compatible lifts of \(\alpha\).

Proof. Write \(n = p^\varepsilon\). The \(\Gamma_1(n)\)-structure
\[
\alpha : (\mathbb{A} / n) \rightarrow \mathbb{G}_{a,k}(k) = k
\]
is determined by the image of \(1 \in \mathbb{A} / n\). We denote it by \(u \in k\). The lift \(\overline{\alpha}\) of \(\alpha\) is determined by \(\overline{\alpha} = u + \varepsilon v\). Let \(f\) be the monic polynomial corresponding to the \(n\)-torsion divisor, that is, let \(f\) be such that
\[
E[n] = k[X]/(f(X))
\]
If we write \(d\) for \(\deg(n)\), we know that \(f\) has the form
\[
f(X) = a_d X^q^d + a_{d+1} X^q^{d+1} + \ldots + X^q^d
\]
and that \(\alpha\) is a \(\Gamma_1(n)\)-structure is expressed by \(f(u) = 0\). The \(n\)-torsion of a lift \(\overline{E}\) corresponds to a lift of \(f\):
\[
\overline{f} = f + \varepsilon g
\]
Now \( \pi = u + \epsilon v \) corresponds to a \( \Gamma_1(n) \)-structure if \( f'(\pi) = 0 \). Since \( f' = 0 \), this translates to \( g(u) = 0 \).

We conclude that if \( u = 0 \), that is \( \alpha \) is a purely local level structure, every lift of \( (\gamma, E) \) allows a one-dimensional space of lifts of \( \alpha \).

On the other hand, if \( u \neq 0 \), we find that for every lift of \( \gamma \), a unique compatible lift of \( E \) allows for lifts of \( \alpha \). Moreover, this space of lifts of \( \alpha \) is one-dimensional. \( \square \)

5.4.9 PROPOSITION. Let \( p \) be the kernel of \( \gamma : A \to k \) and assume given an elliptic module over \( \gamma : A \to k \) with a \( \Gamma_0(p) \)-structure \( H \). We want to understand the structure of the space of lifts \( (\pi, E, H) \) of \( (\gamma, E, H) \).

As we know, there are two possibilities for \( H \) when \( E \) is ordinary, and one when \( E \) is super-singular.

\[ \begin{align*}
\text{i) } & \text{If } H = E[p]_{\text{red}} \text{ (} E \text{ ordinary), every choice of } \overline{\gamma} \text{ allows a unique choice of } \overline{E} \text{ which, in turn, has a one-dimensional space of compatible lifts of } H. \\
\text{ii) } & \text{If } H = E[p]_{\text{red}} \text{ (} E \text{ ordinary), every choice of } (\overline{\gamma}, \overline{E}) \text{ admits a unique compatible lift of } H \\
\text{iii) } & \text{If } H \text{ is the unique } \Gamma_0(p) \text{-structure on the super-singular elliptic module } E, \text{ only the trivial lift } \overline{\gamma} \text{ of } \gamma \text{ has elliptic modules that allow lifting of the level structure. Moreover, every lift of } E \text{ to an } \overline{E} \text{ over } \overline{\gamma} : A \to k[e] \text{ allows a one-dimensional space of lifts of } H.
\end{align*} \]

Proof. Let \( f(X) \) be the monic \( p \)-torsion polynomial, and \( f(X) + \epsilon g(X) \) be a lift of it. A given \( \Gamma_0(p) \) structure corresponds to a monic polynomial \( h(X) \), dividing \( f(X) \). The proposition is now a purely algebraic exercise: express that a monic lift of \( h \) divides the given lift of \( g \). The reasoning is very similar to that of the previous proposition. \( \square \)

5.4.10 REMARK. It appears to be more difficult to examine higher \( p \)-powers, although the same techniques should work.

5.4.11 THEOREM. The surfaces \( Y(n) \) and \( Y_1(n) \), over the field of constants \( \mathbb{F}_q \) of the curve \( A \) are regular. The surface \( Y_0(n) \) is regular outside super-singular points in fibers above \( p \)'s for which \( p^2 \) is a factor of \( n \).

The curves \( Y(n) \otimes k(p), Y_1(n) \otimes k(p) \) and \( Y_0(n) \otimes k(p) \) are regular if \( n \) and \( p \) are coprime. When \( p \) is a factor of \( n \), \( Y(n) \otimes k(p) \) is not reduced, but \( Y_1(n) \otimes k(p) \) is reduced and regular. When \( n \) and \( p \) are coprime, the curve \( Y_0(np) \otimes k(p) \) is regular outside the supersingular points.

5.5 Compactification

The fine and coarse moduli schemes introduced in the previous section fail to be proper. They can, however, be embedded as open dense subschemes of proper schemes over \( A \). In the case of representable moduli problems, and thus fine moduli schemes, the result is the following, due to Drinfel’d.

5.5.1 THEOREM. Let \( n \) be an admissible ideal. There exists a unique regular scheme \( X(n) \) satisfying:

\[ \begin{align*}
\text{i)} & \ Y(n) \text{ is an open dense subscheme of } X(n), \\
\text{ii)} & \ X(n) \text{ is proper over } A, \\
\text{iii)} & \ X(n) - Y(n) \text{ is finite over } A, \\
\text{iv)} & \ X(n) \text{ is smooth of relative dimension } 1 \text{ over } A(n), \\
\text{v)} & \ X(n) \text{ is smooth of relative dimension } 2 \text{ over } \mathbb{F}_q.
\end{align*} \]

Proof. See [12]. The theorem already appears in Drinfel’d’s paper [3]. \( \square \)
By taking quotients we can extend this theorem to the coarse moduli schemes. We call the resulting $X(n)$, $X_1(n)$ and $X_0(n)$ the compactifications of respectively $Y(n)$, $Y_1(n)$, $Y_0(n)$. We have the following result about regularity at cusps:

5.5.2 Theorem. Let $\mathfrak{n}$ be any ideal, then $X(n)$, $X_1(n)$ and $X_0(n)$ are smooth curves over $\mathbb{A}(n)$ and smooth surfaces over $\mathbb{F}_q$.

Proof. On the affine parts $Y_*(n) \hookrightarrow X_*(n)$ the claimed smoothness has been established already (see 5.4.11). It remains to extend it to the compactification. Let $\mathfrak{m}$ be a principal ideal contained in $\mathfrak{n}$, and assume that $\mathfrak{m}$ has at least two different prime divisors. In [16] we find an explicit description of the formal neighbourhood of cusps of $X(\mathfrak{m})$ above $\mathbb{A}(\mathfrak{m})$. It is the formal spectrum of a ring of the form

$$C = \oplus \mathbb{R}[[X]]$$

where $\mathbb{R}$ is an integrally closed domain, finite over $\mathbb{A}(\mathfrak{m})$, and $\oplus$ a finite direct sum. The action of $\text{GL}(2, \mathbb{A}/\mathfrak{m})$ is simultaneously by permutations of the components, and by normalized $\mathbb{R}$-linear automorphisms of the $\mathbb{R}[[X]]$ (those that send $X$ to a power series of the form $X + \text{higher order terms}$). Now consider a subgroup $G$ of $\text{GL}(2, \mathbb{A}/\mathfrak{m})$, acting stabilly on one of the components, then it follows from the following lemma that the ring of invariants is also of the general form $\oplus \mathbb{R}[[X]]$. Since the curves of the proposition can be obtained as quotients by such $G$ of $X(\mathfrak{m})$, it follows that they are smooth at the cusps.

5.5.3 Lemma. Let $R$ be a Dedekind ring and $G$ a finite group acting faithfully on $R[[X]]$, such that for every $g \in G$ $g(X) = uX$ with $u$ a unit in $R[[X]]$. Then $R[[X]]^G = R[[Y]]$, with $Y = \prod_{g \in G} g(Y)$.

Proof. Clearly $R[[X]]^G \supset R[[Y]]$. Now let $s$ be a place of $R$ and denote by $R_s$ the local ring at $s$ and by $Q_s$ it’s fraction field. Then the fraction field of $R[[X]]$ is $Q_s((X))$. We have that $Q_s((X))^G = Q_s((Y))$. Now since both $R_s[[X]]^G$ and $R_s[[Y]]$ are integrally closed in this field, and since one contains the other we have that $R_s[[X]]^G = R_s[[Y]]$ for every $s$, and hence that $R[[X]]^G = R[[Y]]$.

6. Curves of increasing genus having many points

Fix a prime power $q$. It has been shown by Drinfel’d and Vlăduţ that a curve of genus $g$ over $\mathbb{F}_q$ can have at most

$$(\sqrt{q} - 1 + o(1))g$$

rational points, when $g \to \infty$ (see [13]). An infinite sequence of curves $C_i$ of increasing genus is said to be asymptotically optimal if $\#C_i(\mathbb{F}_q)/g(C_i)$ tends to $\sqrt{q} - 1$.

The above bound is sharp at least when $q$ is a square, since several families of (reductions of) modular curves, shimura curves and drinfeld modular curves are known to be asymptotically optimal over a quadratic extension of their finite field of definition. A paper by Noam Elkies ([4]) gives explicit equations of a few towers of reductions of drinfeld modular curves, and shows that they are asymptotically optimal over the quadratic extension of their field of definition. Moreover, the paper suggests that asymptotic optimality can be proven for more general drinfeld modular curves in the same way Yasutaka Ihara proved a similar result for classical modular curves. Using the necessary tools from the previous chapters, we will formulate and prove such a result here. We note that the result itself already appears in [17], but a correct proof over general rings $\mathbb{A}$ seems to be missing from the literature.

Fix a principal non-zero prime ideal $\mathfrak{p}$ in $\mathbb{A}$ and an ideal $\mathfrak{n} \subset \mathbb{A}$ coprime to $\mathfrak{p}$. Denote the degree of $\mathfrak{p}$ by $m$, so $k(\mathfrak{p}) \cong \mathbb{F}_q^m$. The strategy, due to Ihara, is as follows. Using the modular interpretations, we define a curve
\[ T \text{ on the surface } X_0(n) \times X_0(n) \text{ so that the reduction of } T \text{ modulo } p \text{ is supported at the union of the graphs of the } q^m\text{-th power frobenius morphisms } X_0(n) \otimes k(p) \to X_0(n) \otimes k(p) \text{ and } X_0(n) \otimes k(p) \to X_0(n) \otimes k(p). \]

These two graphs intersect above the \( \mathbb{F}_q^* \)-rational points of \( X_0(n) \otimes k(p) \). Hence the number of singular points of the reduction of \( T \) modulo \( p \) gives a lower bound for the number of rational points of \( X_0(n) \otimes k(p) \). Also, the Hurwitz formula gives an upper bound on the genus of \( X_0(n) \otimes k(p) \) in terms of the genus of \( T \). These two bounds can be linked through a comparison of the Euler-Poincaré characteristics of \( T \) and its reduction, and it turns out that the number of \( q^m \)-rational points on \( X_0(n) \otimes k(p) \) is at least \((q^m - 1)(g - 1)\), \( g \) being the genus of \( X_0(n) \otimes k(p) \). Thus, it follows that the curves \( X_0(n) \otimes k(p) \) are asymptotically optimal when their genus increases.

6.1 Two maps from \( X_0(np) \) to \( X_0(n) \)

Let \( f_1 : Y_0(np) \to Y_0(n) \) denote the forgetful morphism, the map that “forgets” the level \( \Gamma_0(p) \)-structure, that associates with a triple \((E, G, H)\) of an elliptic module \( E \), a level \( \Gamma_0(n) \)-structure \( G \) and a level \( \Gamma_0(p) \)-structure \( H \), a pair \((E, G)\). Denote by \( f_2 : Y_0(np) \to Y_0(n) \) the morphism that associates with a triple \((E, G, H)\) the pair \((E/H, GH/H)\) consisting of the quotient module (see 3.2.4), together with its induced level \( \Gamma_0(n) \)-structure.

These are both maps between coarse moduli spaces, so in order to define them properly, one needs to pass to a fine cover first. For example, one can add a level \( \Gamma(m) \) structure, define the two maps between \( Y_0(np)Y(m) \) and \( Y_0(n)Y(m) \), note that they are \( \text{GL}(2, A/m) \)-equivariant and pass to the quotient. Both maps extend to maps \( X_0(np) \to X_0(n) \) on the compactifications, since the generic fibres are smooth (5.5.2). We will also denote the extensions by \( f_1 \) and \( f_2 \).

Denote by \( T \) the image of \( X_0(np) \) in \( X_0(n) \times X_0(n) \) under \((f_1, f_2)\). Then \( T \) is a two-dimensional integral scheme over \( \mathbb{F}_q \), a closed subscheme of \( X_0(n) \times X_0(n) \) and a flat scheme over \( A \). Denote the normalisation of \( T \) by \( \tilde{T} \). Then the generic fiber of \( \tilde{T} \to A \) is necessarily smooth, since if it had a singularity, it would extend to a singular curve on the surface \( \tilde{T} \).

6.2 The reduction modulo \( p \)

Let \( \Pi \) and \( \Pi' \) denote the graphs of the \( q^m \)-th power frobenius morphisms \( \tau^m : X_0(n) \otimes k(p) \to X_0(n) \otimes k(p) \) and \( \tau^m : X_0(n) \otimes k(p) \to X_0(n) \otimes k(p) \), respectively. Consider \( \Pi, \Pi' \) and \( \Pi \cup \Pi' \) as reduced closed subschemes of the surface \((X_0(n) \otimes k(p)) \times (X_0(n) \otimes k(p)) \) over \( k(p) \).

6.2.1 Proposition. \( T \otimes_A k(p) \) coincides with \( \Pi \cup \Pi' \).

Proof. Let \( E \) be an elliptic module over an algebraic closure \( \overline{k(p)} \) of \( k(p) \), \( G \) a level \( \Gamma_0(n) \)-structure on \( E \) and \( H \) a level \( \Gamma_0(p) \)-structure. Then by 4.3.2 there are two cases to consider.

i) \( H \) is local, then \( H \) is the divisor consisting of \( q^m \) times the zero section. But this is precisely the kernel of the \( q^m \)-th power frobenius isogeny \( E \to E' \), where \( E' \) is the elliptic module obtained by base extension \( \tau^m : \overline{k(p)} \to \overline{k(p)} \). Hence \( E/H \) is isomorphic to \( E' \) and the geometric point corresponding to \((E, G, H)\) maps under \((f_1, f_2)\) to the point corresponding to \((E', G, H')\), where \( G' \) is the level structure obtained by base extension of \( G \).

ii) \( H \) is étale. Denote by \( F \) the elliptic module \( E/H \). Let \( F' \) be the elliptic module obtained by base extension over \( \tau^m \) as above. Then, as before, \( F' \) is isomorphic to the module \( F/q^m[0] \). This, in turn is isomorphic to \( E/E[p] \), and therefore also to \( E \). So the geometric point corresponding to \((E, G, H)\) maps under \((f_1, f_2)\) to the geometric point corresponding to \((E, G), (F, GH/H)\), or
Thus, the image $T$ of $(f_1, f_2)$ coincides with $\Pi \cup \Pi'$ on the open part $(Y_0(n) \otimes \mathbb{k}(p)) \times (Y_0(n) \otimes \mathbb{k}(p))$ of $(X_0(n) \otimes \mathbb{k}(p)) \times (X_0(n) \otimes \mathbb{k}(p))$ and hence also on the entire surface.

6.2.2 REMARK. The level $\Gamma_0(n)$-structure in the above proposition can not be replaced for a $\Gamma_1(n)$-structure. To see this, take $H$ to be étale in the above proof. Then $E$ will still be isomorphic to $F'$, and the induced level $\Gamma_1(n)$-structure on $F'$ will generate the same $\Gamma_0(n)$-structure as before, but there is no way to decide which generator we get.

6.3 Counting the rational points

We are now in position to apply Ihara’s trick in order to prove:

6.3.1 THEOREM. The number of $q^{2m}$-rational points of $X_0(n) \otimes \mathbb{k}(p)$ is at least $(q^m - 1)(g - 1)$, where $g$ equals the genus of $X_0(n) \otimes \mathbb{k}(p)$.

Assuming the theorem, we can simply “increase” $n$ to obtain a tower of curves of increasing genus with asymptotically many rational points over $\mathbb{F}_{q^{2m}}$.

Proof. (Almost identical to [9 §1].) Let $g$ denote the genus of the generic fibre of $X_0(n)$. Then, since $X_0(n) \otimes \mathbb{k}(p)$ is smooth, $g$ is also the genus of the reduction. Denote by $g_0$ the genus of the generic fibre of $\tilde{T}$. Since both projections $T \to X_0(n)$ have degree $q^m + 1$, as can be seen on the fibre above $p$, the Hurwitz formula gives

$$g_0 - 1 \geq (q^m + 1)(g - 1).$$

The set of $\mathbb{F}_{q^{2m}}$-rational points of $X_0(n) \otimes \mathbb{k}(p)$ is in bijection with the set of geometric points in the intersection $\Pi \cap \Pi'$, since it is the set of points invariant under $\tau^m \circ \tau^m$. Such a point in the intersection $\Pi \cap \Pi'$ is called special if it is not a normal point of the two-dimensional scheme $T$.

The fibre above $p$ of $\tilde{T} \to A$ consists of two irreducible components, mapping to $\Pi$ and $\Pi'$. These two components intersect above the special points of $\Pi \cap \Pi'$. Since the Euler-Poincaré characteristic of $\tilde{T}$ is constant along the fibers, we find

$$g_0 - 1 = 2(g - 1) + \#\{\text{special points}\}.$$ 

Combined with the above inequality, this yields

$$\#\{\text{rational points over } \mathbb{F}_{q^{2m}}\} \geq \#\{\text{special points}\} \geq (q^m - 1)(g - 1).$$

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