General polarization modes for the Rosen gravitational wave

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Abstract

Strong-field gravitational plane waves are often represented in either the Rosen or Brinkmann forms. While these two metric ans"atzes are related by a coordinate transformation, so that they should describe essentially the same physics, they rather puzzlingly seem to treat polarization states quite differently. Both ans"atzes deal equally well with + and \( \times \) linear polarizations, but there is a qualitative difference in the way they deal with circular, elliptic and more general polarization states. In this paper we will develop a general formalism for dealing with arbitrary polarization states in the Rosen form of the gravitational wave metric, representing an arbitrary polarization by a trajectory in a suitably defined two-dimensional hyperbolic plane.

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1. Introduction

Strong-field gravitational plane waves (a restriction of the more general \( pp \) waves) are commonly represented in either the Brinkmann form [1]
\[
dx^2 = -2 \, du \, dv + \{ H_{AB}(u)x^A x^B \} \, du^2 + dx^2 + dy^2,
\]
(1)
or the Rosen form [2]
\[
dx^2 = -2 \, du \, dv + g_{AB}(u) \, dx^A \, dx^B,
\]
(2)
where \( x^A = \{ x, y \} \). While these two metric ans"atzes are related by a coordinate transformation [3, 4], and so must describe essentially the same physics, there is a qualitative difference in how they treat circular polarization, elliptic polarization, and more general states of polarization. We shall investigate this puzzle in detail and will ultimately demonstrate a clean way of putting a general polarization state into the Rosen form.
2. Brinkmann form

Consider the general $pp$ spacetime geometry [4–7]

$$ds^2 = -2 du dv + H(u, x, y) du^2 + dx^2 + dy^2.$$  \(3\)

It is then a standard result that the only non-zero component of the Ricci tensor is

$$R_{uu} = -\frac{1}{2} \{ \partial_x^2 H(u, x, y) + \partial_y^2 H(u, x, y) \}.$$  \(4\)

Furthermore (up to the usual index symmetries) the only non-zero components of the Riemann tensor are of the form $R_{uxux}$. Specifically

$$R_{uxux} = -\frac{1}{2} \partial_x^2 H(u, x, y);$$  \(5\)

$$R_{uxuy} = -\frac{1}{2} \partial_x \partial_y H(u, x, y);$$  \(6\)

$$R_{uyuy} = -\frac{1}{2} \partial_y^2 H(u, x, y).$$  \(7\)

Polarization modes are characterized by the relative motion of nearby timelike geodesics under the influence of the imposed $pp$-wave spacetime; as such the polarization modes are sensitive to the individual non-zero components of the Riemann tensor: $R_{uxux}, R_{uxuy}$ and $R_{uyuy}$. This is the basis of the discussion in section 17.2 of [8], which has the net effect that one can safely carry over one’s intuition from weak-field gravitational waves to strong-field gravitational waves in the Brinkmann form.

Gravitational plane waves, as opposed to the more general $pp$-waves [5–7], can be characterized by the fact that $H(u, x, y)$ is a quadratic function of the coordinates $x$ and $y$ [7]. (See also [1, 3].) Then

$$ds^2 = -2 du dv + \{H_{AB}(u)x^A x^B\} du^2 + dx^2 + dy^2.$$  \(8\)

The vacuum gravitational plane wave for arbitrary polarization can now be written down by inspection:

$$ds^2 = -2 du dv + \{[x^2 - y^2]H_x(u) + 2xyH_y(u)\} du^2 + dx^2 + dy^2.$$  \(9\)

Equivalently

$$ds^2 = -2 du dv + r^2 \{\cos(2\phi)H_x(u) + \sin(2\phi)H_y(u)\} du^2 + dr^2 + r^2 d\phi^2.$$  \(10\)

In either of these two forms for the metric the two polarization modes are explicitly seen to decouple and to superimpose linearly—quite similarly to the situation in Maxwell electromagnetism: by choosing $H_x(u)$ and $H_y(u)$ appropriately we can construct not just $+$ and $x$ polarized waves, but also circular polarization, elliptic polarization and even more general polarization states. It is this decoupling that fails in the Rosen form of the metric, and which ultimately is the source of the puzzle.

3. Rosen form

3.1. Most general Rosen form

The 'most general' form of the Rosen metric is [2, 4, 7]

$$ds^2 = -2 du dv + g_{AB}(u) dx^A dx^B.$$  \(11\)
where $x^A = \{x, y\}$. It is a standard result [4], quite easily checked using symbolic manipulation systems such as Maple, that the only non-zero component of the Ricci tensor is

$$ R_{uu} = - \left\{ \frac{1}{2} S^{AB} S''_{AB} - \frac{1}{4} g^{AB} g^{CD} g''_{AB} g^{CD} g''_{CD} \right\}. \tag{12} $$

Less obviously, a brief computation shows that (up to the usual index symmetries) the only non-zero components of the Riemann tensor are

$$ R_{uAuB} = - \left\{ \frac{1}{2} g''_{AB} - \frac{1}{4} g''_{AC} g_{CD} g''_{DB} \right\}. \tag{13} $$

This is the same pattern of non-zero components that occurs in the Brinkmann form, which indicates that some of our intuition regarding polarization modes will also carry over to the Rosen form of the metric. Though these formulae look relatively compact, the matrix inversions implicit in ‘raising the indices’ mean that these quantities are grossly nonlinear functions of the matrix components $g_{AB}(u)$. In particular, in this form of the metric the $+$ and $\times$ linear polarizations do not decouple in any obvious way.

### 3.2. Linear polarization $+$

Consider the strong-field gravity wave metric in the $+$ linear polarization. That is, set $g_{xy} = 0$, so that $g_{AB}$ has only two nontrivial components, $g_{xx}$ and $g_{yy}$, corresponding to oscillations along the $x$ and $y$ axes. The resulting metric can be written in the form

$$ ds^2 = -2 du dv + f^2(u) dx^2 + g^2(u) dy^2. \tag{14} $$

The only non-zero component of the Ricci tensor is

$$ R_{uu} = - \left\{ \frac{f''}{f} + \frac{g''}{g} \right\}. \tag{15} $$

Though the expression for the Ricci tensor is compact, ultimately this form of the metric turns out to be not very useful. If we write the metric in the form

$$ ds^2 = -2 du dv + S^2(u) [e^{+X(u)} dx^2 + e^{-X(u)} dy^2], \tag{16} $$

then

$$ R_{uu} = - \frac{1}{2} \left\{ \frac{S''}{S} + (X')^2 \right\}. \tag{17} $$

Though this may not initially look very promising, it is this version of the metric that permits us to make the most progress. In particular, note that in vacuum we have

$$ X' = 2 \sqrt{-S'}/S, \quad \text{so that} \quad X(u) = 2 \int^u \sqrt{-S'/S} du, \tag{18} $$

and the general vacuum wave for $+$ polarization can be put in the form

$$ ds^2 = -2 du dv + S^2(u) \left\{ \exp \left( 2 \int^u \sqrt{-S'/S} du \right) dx^2 + \exp \left( -2 \int^u \sqrt{-S'/S} du \right) dy^2 \right\}. \tag{19} $$

Note that as expected from the Brinkmann form, we have one free function (per polarization mode).
3.3. Linear polarization ×

Take the strong-field gravity wave metric in the × linear polarization
\[ ds^2 = -2\, du \, dv + \frac{f^2(u) + g^2(u)}{2} \left[ dx^2 + dy^2 \right] + \left[ f^2(u) - g^2(u) \right] dx \, dy. \] (20)

This can be obtained from the + polarization mode (14) simply by performing a 45° rotation
in the x–y plane. The only non-zero component of the Ricci tensor is again
\[ R_{uu} = -\left[ \frac{f''}{f} + \frac{g''}{g} \right]. \] (21)

If we now write this metric in the form
\[ ds^2 = -2\, du \, dv + S^2(u) \left[ \cosh(X(u)) \right] \left[ dx^2 + dy^2 \right] + 2 \sinh(X(u)) \, dx \, dy, \] (22)
then, as for the + mode, we have
\[ R_{uu} = -\frac{1}{2} \left\{ \frac{4}{S'} + (X')^2 \right\}. \] (23)

In vacuum we can again solve for \( X(u) \) and now obtain
\[ ds^2 = -2\, du \, dv + S^2(u) \left[ \cosh \left( 2 \int u \sqrt{\frac{S''}{S}} \, du \right) \left[ dx^2 + dy^2 \right] 
+ 2 \sinh \left( 2 \int u \sqrt{-S''/S} \, du \right) \, dx \, dy \right]. \] (24)

There is again one freely specifiable function for this × linear polarization mode. By rotating
the x–y plane through a fixed but arbitrary angle \( \Theta_0 \), we can easily deal with linear polarization
modes along any desired axis; indeed, we could have obtained this metric directly from
equation (19) by a 45° rotation in the x–y plane. The puzzle arises once we try to deal with
variable amounts of + and × polarization simultaneously.

3.4. Arbitrary polarization

Let us now take an arbitrary, possibly \( u \)-dependent polarization, and consider the following
metric ansatz:
\[ ds^2 = -2\, du \, dv + S^2(u) \left[ \cosh(B(u)) \right] \left[ e^{+X(u)} \, dx^2 + e^{-X(u)} \, dy^2 \right] + 2 \sinh(B(u)) \, dx \, dy. \] (25)

Note that setting \( X(u) = 0 \) corresponds to × polarization, while setting \( B(u) = 0 \) corresponds
to + polarization. Furthermore we have sufficient free functions, namely \{ \( S(u) \), \( B(u) \), \( X(u) \) \},
to completely saturate the arbitrary \( 2 \times 2 \) symmetric matrix \( g_{AB}(u) \). A brief calculation yields
the only non-zero component of the Ricci tensor:
\[ R_{uu} = -\frac{1}{2} \left\{ \frac{S''}{S} + (B')^2 + \cosh^2[B(u)](X')^2 \right\}. \] (26)

Note that \( B(u) \) and \( X(u) \) have not decoupled—however a hint on how to proceed is provided
by noting that the 2-metric
\[ dB^2 + \cosh^2 B \, dX^2 \] (27)
is one of many ways of representing the metric of the hyperbolic plane \( H_2 \).

Now let us try a slightly different representation of the same general metric of
equation (25):
\[ ds^2 = -2\, du \, dv + S^2(u) \left[ \cosh(X(u)) + \cos(\Theta(u)) \sinh(X(u)) \right] \, dx^2 
+ 2 \sin(\Theta(u)) \sinh(X(u)) \, dx \, dy 
+ \left[ \cosh(X(u)) - \cos(\Theta(u)) \sinh(X(u)) \right] \, dy^2. \] (28)
Note that setting $\theta(u) = 0$ corresponds to + polarization, while setting $\theta(u) = \pi/2$ corresponds to $\times$ polarization, and $\theta(u) = \Theta_0/2$ corresponds to linear polarization along axes rotated by an angle $\Theta_0$. Furthermore we again have sufficient free functions, now $\{S(u), X(u), \theta(u)\}$, to completely saturate the arbitrary $2 \times 2$ symmetric matrix $g_{AB}(u)$. Then
\[
R_{uu} = -\frac{1}{2} \left\{ 4 \frac{S''}{S} + (X')^2 + \sinh^2(X(u))(\theta')^2 \right\}. \tag{29}
\]

The vacuum field equations imply
\[
4 \frac{S''}{S} + (X')^2 + \sinh^2(X(u))(\theta')^2 = 0. \tag{30}
\]
Let us introduce a dummy function $L(u)$ and split this into the two equations
\[
4 \frac{S''}{S} + (L')^2 = 0 \tag{31}
\]
and
\[
(L')^2 = (X')^2 + \sinh^2(X(u))(\theta')^2. \tag{32}
\]
The first equation is just the one you would have to solve for a pure + or $\times$ or in fact any linear polarization. The second equation can be rewritten as
\[
dL^2 = dX^2 + \sinh^2(X) d\theta^2, \tag{33}
\]
and is just the statement that $L$ can be interpreted as distance in the two-dimensional hyperbolic plane $H_2$.

**Algorithm.** This gives us now a very straightforward algorithm for arbitrary polarization strong-field gravity waves in the Rosen form.

- Pick an arbitrary $L(u)$ and solve
  \[
  4 \frac{S''}{S} + (L')^2 = 0. \tag{34}
  \]
- Pick an arbitrary curve in the $(X, \theta)$ plane such that $L(u)$ is hyperbolic arc-length along that curve:
  \[
dL^2 = dX^2 + \sinh^2(X) d\theta^2. \tag{35}
  \]
- This construction then solves the vacuum Einstein equations for the metric
  \[
  ds^2 = -2 du dv + S^2(u) \{ [\cosh(X(u)) + \cos(\theta(u)) \sinh(X(u))] dx^2 \\
  + 2 \sin(\theta(u)) \sinh(X(u)) dx dy \\
  + [\cosh(X(u)) - \cos(\theta(u)) \sinh(X(u))] dy^2 \}. \tag{36}
  \]

  In this sense we have completely solved arbitrary polarization strong-field gravity waves in the Rosen form.

Note the similarities (and differences) with regard to Maxwell electromagnetism, (and with respect to the Brinkmann form). In Maxwell electromagnetism the two independent linear polarizations can be specified by
\[
E(u) = E_x(u) \hat{x} + E_y(u) \hat{y}, \tag{37}
\]
with no additional constraints (compare with equation (9)). Thus an electromagnetic wavepacket of arbitrary polarization can be viewed as an arbitrary ‘walk’ in the ($E_x, E_y$) plane. We could also go to a magnitude-phase representation ($E, \theta$) where
\[
\vec{E}(u) = E(u) \cos \theta(u) \hat{x} + E(u) \sin \theta(u) \hat{y}. \tag{38}
\]
(Compare with equation (10).) So an electromagnetic wavepacket of arbitrary polarization can also be viewed as an arbitrary ‘walk’ in the \((E, \theta)\) plane, where the \((E, \theta)\) plane is provided with the natural Euclidean metric

\[
\text{d}L^2 = \text{d}E^2 + E^2 \text{d}\theta^2.
\] (39)

In contrast for the gravitational wave in the Rosen form, we are now dealing with an arbitrary ‘walk’ in the hyperbolic plane \(H_2\), rather than in the Euclidean plane. Furthermore, because of the nonlinearity of general relativity, there is still one remaining differential equation to solve (for the ‘envelope’ \(S(u)\)).

### 3.5. Circular polarization

We can now adopt the above discussion to formulate strong-field circular polarization in the Rosen form. (We emphasize that there is no difficulty whatsoever with weak-field linearized circular polarization; it is only for strong fields that it is difficult to formulate circular polarization for a gravitational wave in the Rosen form.) Circular polarization corresponds to

\[
\theta(u) = \Omega_0 u; \quad X(u) = X_0,
\] (40)

that is, a fixed distortion \(X_0\) with the plane of polarization advancing linearly with the retarded time \(u\). Then

\[
\text{d}s^2 = -2 \text{d}u \text{d}v + S'^2(u) \left\{ [\cosh(X_0) + \cos(\Omega_0 u) \sinh(X_0)] \text{d}x^2 + 2 \sin(\Omega_0 u) \sinh(X_0) \text{d}x \text{d}y \\
+ [\cosh(X_0) - \cos(\Omega_0 u) \sinh(X_0)] \text{d}y^2 \right\}.
\] (41)

The only nontrivial component of the Ricci tensor is then

\[
R_{uu} = -\frac{1}{2} \left\{ \frac{S''}{S} + \sinh^2(X_0) \Omega_0^2 \right\}.
\] (42)

The vacuum field equations imply

\[
S'' = -\frac{\sinh^2(X_0) \Omega_0^2}{4} S,
\] (43)

whence

\[
S(u) = S_0 \cos \left\{ \frac{\sinh(X_0) \Omega_0 (u - u_0)}{2} \right\}.
\] (44)

This now describes a spacetime that has good reason to be called a strong-field circularly polarized gravity wave. Note that the weak-field limit corresponds to \(X_0 \ll 1\) so that for an arbitrarily long interval in the retarded time \(u\), we have \(S \approx S_0\), and without loss of generality we can set \(S \approx 1\). Then

\[
\text{d}s^2 \approx -2 \text{d}u \text{d}v + \text{d}x^2 + \text{d}y^2 + X_0 \{ \cos(\Omega_0 u) \text{d}x^2 - \text{d}y^2 \} + 2 \sin(\Omega_0 u) \text{d}x \text{d}y.
\] (45)

Further generalizations to elliptic polarization are tedious but, given the significantly more general algorithm of the preceding subsection, quite straightforward.

### 3.6. Decoupling the most general Rosen form

Based on what we have seen so far, one might suspect that there is some general decoupling between the overall ‘envelope’ of the gravity wave and the ‘directions of oscillation’. Let us return to considering the metric in the general Rosen form

\[
\text{d}s^2 = -2 \text{d}u \text{d}v + g_{AB}(u) \text{d}x^A \text{d}x^B,
\] (46)
where for generality $x^A, x^B$ represent any arbitrary number of dimensions ($d_\perp \geq 2$) transverse to the $(u, v)$ plane. It is easy to check that the only non-zero component of the Ricci tensor is still

$$R_{uu} = - \left[ \frac{1}{2} g^{AB} \dot{g}_{AB} + \frac{1}{4} g^{AB} g^{CD} \dot{g}_{DA} \right].$$

(47)

Let us now decompose the $d_\perp \times d_\perp$ matrix $g_{AB}$ into an ‘envelope’ $S(u)$ and a unit determinant related to the ‘direction of oscillation’. That is, let us take

$$g_{AB}(u) = S^2(u) \dot{g}_{AB}(u),$$

(48)

where $\det(\dot{g}) = 1$. (A related discussion can be found in section 109 of [9].) Our goal is to see if we can make the overall ‘envelope’ $S(u)$ decouple from $\dot{g}_{AB}(u)$. To start, note that

$$g'_{AB} = 2SS' \dot{g}_{AB} + S'^2 \ddot{g}_{AB},$$

(49)

and

$$g''_{AB} = 2SS'' \dot{g}_{AB} + 2S'S' \ddot{g}_{AB} + 4SS' \dot{g}_{AB} + S'^2 \dddot{g}_{AB}.$$  

(50)

Therefore

$$g^{AB} \dot{g}_{AB} = \frac{1}{S^2} \left[ 2( SS'' + S'S')d_\perp + 4SS'[\dot{g}^{AB} \ddot{g}_{AB}] + S[\dot{g}^{AB} \dddot{g}_{AB}] \right]$$

$$= 2 \left( \frac{S'}{S} S'' + \frac{S'}{S} \right) d_\perp + 4 \frac{S'}{S} [\dot{g}^{AB} \ddot{g}_{AB}] + [\dot{g}^{AB} \dddot{g}_{AB}],$$

(51)

and similarly

$$g^{AB} \dot{g}_{BC} \dot{g}^{CD} \dot{g}_{DA} = g^{AB} [2SS' \dot{g}_{BC} + S'^2 \ddot{g}_{BC}] [\dot{g}^{CD} [2SS' \dot{g}_{DA} + S'^2 \ddot{g}_{DA}]]$$

$$= \frac{1}{S^4} \left[ 4( SS' )^2 d_\perp + 4SS'[\dot{g}^{AB} \ddot{g}_{AB}] + S[\dot{g}^{AB} \dddot{g}_{AB}] \right]$$

$$= 4 \left( \frac{S'}{S} \right)^2 d_\perp + 4 \left( \frac{S'}{S} \right) [\dot{g}^{AB} \ddot{g}_{AB}] + [\dot{g}^{AB} \dddot{g}_{AB}].$$

(52)

Now combine these results:

$$R_{uu} = - \left[ \frac{1}{2} g^{AB} \ddot{g}_{AB} + \frac{1}{4} g^{AB} g^{CD} \dddot{g}_{DA} \right]$$

$$= \left( \frac{S'}{S} \right)^2 d_\perp - 2 \frac{S'}{S} [\dot{g}^{AB} \ddot{g}_{AB}] - \frac{1}{2} [\dot{g}^{AB} \ddot{g}_{AB}]$$

$$+ \left( \frac{S'}{S} \right)^2 d_\perp + \frac{S'}{S} [\dot{g}^{AB} \ddot{g}_{AB}] + \frac{1}{4} [\dot{g}^{AB} \dddot{g}_{BC} \ddot{g}_{DA}]$$

$$= - \frac{S'}{S} d_\perp - \frac{1}{2} [\dot{g}^{AB} \ddot{g}_{AB}] + \frac{1}{4} [\dot{g}^{AB} \ddot{g}_{BC} \ddot{g}_{CD} \dddot{g}_{DA}] + \left( \frac{S'}{S} \right) [\dot{g}^{AB} \dddot{g}_{AB}].$$

(53)

But because we have defined $\det(\ddot{g}) = 1$, we have as a matrix identity

$$[\dot{g}^{AB} \ddot{g}_{AB}] = 0,$$

(54)

and differentiating this one more time

$$[\dot{g}^{AB} \dddot{g}_{AB}] = [\dot{g}^{AB} \dddot{g}_{BC} \ddot{g}_{CD} \dddot{g}_{DA}] = 0.$$  

(55)

Therefore

$$R_{uu} = -d_\perp \frac{S''}{S} - \frac{1}{2} [\dot{g}^{AB} \ddot{g}_{BC} \ddot{g}_{CD} \dddot{g}_{DA}],$$

(56)
or more abstractly
\[ R_{u u} = -d_\perp S'' - \frac{1}{2} \text{tr}[[\hat{\mathbf{g}}^{-1}]\hat{\mathbf{g}}'\hat{\mathbf{g}}^{-1}]}. \] (57)

Note that we have now succeeded in decoupling the determinant \((\det(g) = S^2 S'; \text{effectively the 'envelope' } S(u) \text{ of the gravitational wave})\) from the unit-determinant matrix \(\hat{\mathbf{g}}(u)\). This observation is compatible with all the specific examples considered above. Now consider the set \(S \mathcal{S}(I \mathbb{R}, d_\perp)\) of all unit-determinant real symmetric matrices, and on that set (not a group) consider the Riemannian metric
\[ dL^2 = \text{tr}[[\hat{\mathbf{g}}^{-1}]d[\hat{\mathbf{g}}]^{-1}]d[\hat{\mathbf{g}}]]. \] (58)

Then
\[ R_{u u} = -\frac{1}{2} \left\{ 2d_\perp S''(u) + \left( \frac{dL}{du} \right)^2 \right\}. \] (59)

The vacuum Einstein equations then reduce to
\[ \frac{dL}{du} = \sqrt{-2d_\perp S'' S}; \quad L(u) = \int^u \sqrt{-2d_\perp S'' S} du. \] (60)

That is, an arbitrary polarization vacuum Rosen wave is a random walk in \(S \mathcal{S}(I \mathbb{R}, d_\perp)\), with the distance along the walk \(L(u)\) being related to the envelope function \(S(u)\) as in the discussion above.

To probe the polarization modes in more detail, a completely analogous but slightly more complicated calculation determines the Riemann tensor components to be
\[ R_{u A u B} = -\left\{ S S'' \hat{\mathbf{g}}_{A B} + S^2 \left[ \frac{1}{2} \hat{\mathbf{g}}'_{A B} - \frac{1}{2} \hat{\mathbf{g}}'_{A C} \hat{\mathbf{g}}'_{C D} \hat{\mathbf{g}}'_{D B} \right] + \hat{\mathbf{g}}_{A B} \right\}. \] (61)

Note that this is compatible with equation (53).

4. Discussion

Strong-field gravitational waves, both in Brinkmann [1] form and Rosen [2] form, have been known for some 85 years. Despite this there are still a number of surprises hiding in the Rosen form. In particular, while arbitrary time-dependent polarization mixtures are trivial to deal with in the Brinkmann form, they appear to be much more difficult to implement in the Rosen form. To address this puzzle we have re-analyzed the Rosen gravity wave in terms of an 'envelope' function and two freely specifiable functions corresponding to the individual polarization modes. The vacuum field equations can be reinterpreted in terms of a single differential equation governing the 'envelope' coupled with what is essentially a random walk in polarization space. In particular, we have indicated how to construct a circularly polarized strong-field Rosen form gravity wave, and how to generalize this central idea beyond (3+1) dimensions.

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