Systematics of M-theory spinorial geometry

U Gran, G Papadopoulos and D Roest
Department of Mathematics, King’s College London, Strand, London WC2R 2LS, UK

Received 25 April 2005  
Published 20 June 2005  
Online at stacks.iop.org/CQG/22/2701

Abstract
We reduce the classification of all supersymmetric backgrounds in 11 dimensions to the evaluation of the supercovariant derivative and of an integrability condition, which contains the field equations, on six types of spinors. We determine the expression of the supercovariant derivative on all six types of spinors and give in each case the field equations that do not arise as the integrability conditions of Killing spinor equations. The Killing spinor equations of a background become a linear system for the fluxes, geometry and spacetime derivatives of the functions that determine the spinors. The solution of the linear system expresses the fluxes in terms of the geometry and specifies the restrictions on the geometry of spacetime for all supersymmetric backgrounds. We also show that the minimum number of field equations that is needed for a supersymmetric configuration to be a solution of 11-dimensional supergravity can be found by solving a linear system. The linear systems of the Killing spinor equations and their integrability conditions are given in both a timelike and a null spinor basis. We illustrate the construction with examples.

PACS numbers: 11.25.-w, 11.25.Yb

1. Introduction

In the last ten years, the supersymmetric solutions of ten- and eleven-dimensional supergravities have given new insights into understanding of string theory and M-theory (see, e.g., [1, 2]). Most of the solutions have been found using ansätze adapted to the requirements of physical problems. Recently, the realization that there are new maximally supersymmetric solutions [3] and the rediscovery of some old ones [4, 5] has led to a more systematic exploration of supersymmetric solutions in supergravity theories. The maximally supersymmetric solutions of ten- and eleven-dimensional supergravities have been classified in [6] using the integrability conditions of the Killing spinor equations which leads to the vanishing of the supercovariant curvature. A method\footnote{For a refinement see [7].} based on the Killing spinor bilinear forms has also been used to solve the Killing spinor equations of 11-dimensional supergravity
for backgrounds with one Killing spinor [8, 9]. However this method has not been applied to 11-dimensional backgrounds with more than one supersymmetry.

In [10], a new method to investigate the Killing spinor equations of supergravities has been proposed. It is based on a description of spinors in terms of forms, the gauge symmetry of Killing spinor equations and an oscillator basis in the space of spinors [10]. This has been applied to systematically explore the supersymmetric solutions of 11-dimensional supergravity with one, two, three and four supersymmetries and to solve the Killing spinor equations of IIB supergravity for one Killing spinor [11].

In this paper, we shall show that the method of [10] can be extended further to investigate all supersymmetric 11-dimensional backgrounds\(^2\). For this, we use the linearity of the Killing spinor equations to show that the supercovariant derivative\(^3\) of 11-dimensional supergravity acting on any spinor \(\sigma\) can be decomposed into a linear combination of six ‘irreducible’ components. These six irreducible components are given by the action of the supercovariant derivative, \(D\sigma I\), on six types of spinors

\[
e_1, \ e_{ij}, \ e_{ijk}, \ e_i, e_{ijkl}, \ e_{12345}, \tag{1.1}
\]

which are collectively denoted by \(\sigma I = e_{i_{1i_i}}\), with \(I = 0, \ldots, 5\). These spinors can also be labelled by the irreducible representations of \(U(5)\) on the space \(\Lambda^* (\mathbb{C}^5)\) of forms. We compute \(D\sigma I\). As a result, one can compute \(D\sigma\) for any number of spinors \(\epsilon\) and then use the basis in the space of spinors [10], see also appendix A, to express the Killing spinor equations as a linear system for the geometry, fluxes and spacetime derivatives of the functions that determine the Killing spinors \(\epsilon\). Therefore, we show that the Killing spinor equations for any number of Killing spinors reduce to a linear system and we give all the coefficients and all the unknowns of the system. The solution of this system expresses the fluxes in terms of the geometry and gives the restrictions of the geometry required by supersymmetry.

It has been known for some time that the integrability conditions of the Killing spinor equations imply some of the field equations of supergravity theories, e.g. in maximal supersymmetric backgrounds the integrability conditions of Killing spinor equations imply all the field equations [6]. The first integrability condition of the Killing spinor equations is \(\mathcal{R}_{AB}\epsilon = [D_A, D_B]\epsilon = 0\), where \(\mathcal{R}\) is the curvature of the supercovariant connection. This integrability condition has various components, one of which, \(\mathcal{I}_A\epsilon = \Gamma^{[B}\mathcal{R}_{AB}\epsilon = 0\), contains the field equations of 11-dimensional supergravity [8]. Since the integrability conditions \(\mathcal{R}\epsilon = 0\) and \(\mathcal{I}\epsilon = 0\) of the Killing spinor equations are linear algebraic equations for the Killing spinor \(\epsilon\), they again can be decomposed in terms of the \(\mathcal{R}\sigma I\) and \(\mathcal{I}\sigma I\). We give all the expressions for \(\mathcal{I}\sigma I\). Since the integrability conditions of any number of Killing spinors can be written in terms of \(\mathcal{I}\sigma I\), one can use the basis of [10] to find which components of the field equations are implied as integrability conditions of the Killing spinor equations. In particular, one finds a linear system with the components of the field equations as unknowns and the functions that determine the Killing spinors as coefficients. The components of the field equations that are not determined as solutions of this linear system are those that have to be imposed as additional conditions to the Killing spinor equations for a configuration with any number of supersymmetries to be a solution of the theory. We remark that such an analysis can be done for \(\mathcal{R}\epsilon = 0\). This would be an extension of the method used in [6] to solve the Killing spinor equations for maximally supersymmetric spacetimes\(^3\).

The main aim of this paper is to be used as a manual for systematically constructing all supersymmetric solutions of 11-dimensional supergravity. Because of this, we first present the general formulae for \(D\sigma I\) and \(\mathcal{I}\sigma I\). However, these are rather involved when expressed

---

\(^2\) This includes backgrounds with both \(SU(5)\) and \(\text{Spin}(7) \ltimes \mathbb{R}^8\) \(\times\) \(\mathbb{R}\) invariant spinors.

\(^3\) One may have to consider higher order integrability conditions [12].
in terms of the oscillator basis in the space of spinors; see [10] and appendix A. Because of this, we state the results in tables which have been put in appendices. The construction of the linear systems associated with $D_A \epsilon_h = 0$ and $I \epsilon_h = 0$ for any number of Killing spinors $h = 1, \ldots, N$ can be read off from these tables.

As we have explained, the construction of the linear systems associated with the Killing spinor equations and with the integrability conditions can be done in the basis of [10] for any number of Killing spinors. However, if one or more Killing spinors are null, i.e. they are representatives of the orbit of $Spin(10, 1)$ with stability subgroup $(Spin(7) \times R^5) \times R$, it may be convenient to use another basis in the space of spinors. Such a basis adapted to null spinors has been constructed in [11] in the context of IIB supergravity and it is extended to eleven dimensions in appendix A. We shall refer to the spinor basis presented in [10] as ‘timelike’ and that constructed from IIB supergravity as ‘null’ basis. In the null basis, one can find simple expressions for the representatives of the $(Spin(7) \times R^5) \times R$ orbit. So one expects that the linear systems associated with null Killing spinors will be easier to solve in the null basis than the analogous linear systems derived in the timelike basis. We show that the linear systems associated with the Killing spinor equations and of the integrability conditions in the null basis can be derived from those we have constructed in the timelike basis after a suitable replacement of the time direction with the tenth spatial direction and taking into account the way that $\Gamma^0$ and $\Gamma^5$ act on the spinor basis. The rules of relating the linear systems in the null basis and in timelike basis are given in detail in section 5. It turns out that these rules are very simple. Because of this we shall focus on the timelike case and will only discuss the null case in section 5. Some partial results on the construction of the linear system for the Killing spinor equations in another null basis have been obtained in [13].

To illustrate our construction we solve the Killing spinor equations for backgrounds with two supersymmetries and the most general $SU(4)$ invariant Killing spinors. Special cases have already been investigated in [10]. Then we find for several configurations with one, two, three and four supersymmetries the minimal set of field equations that in addition should be imposed in order to be solutions of 11-dimensional supergravity. In the process, we explain how the tables in the appendices can be used.

Our analysis is in the context of 11-dimensional supergravity. But it can be extended to the effective theory of M-theory which includes higher order corrections, e.g. see [14]. For example, our conclusion about the six types of spinors is not altered by the inclusion of higher order corrections.

This paper is organized as follows. In section 2, we summarize the Killing spinor equations $D \epsilon = 0$ and give the integrability conditions $I \epsilon = 0$ of 11-dimensional supergravity. In section 3, we show how a general spinor is related to the six types of spinors $\sigma_I$, and express the Killing spinor equations $D \epsilon = 0$ and associated integrability conditions $I \epsilon = 0$ in terms of $D \sigma_I$ and $I \sigma_I$, respectively. In section 4, we derive some general formulae that give $D \sigma_I$ and $I \sigma_I$ in terms of the timelike canonical basis (A.8). In section 5, we give $D \sigma_I$ and $I \sigma_I$ in terms of the null canonical basis. In section 6, we summarize the conditions on the geometry and fluxes for the most general background with two supersymmetries and $SU(4)$ invariant spinors, and analyse the geometry of spacetime. In section 7, we analyse the field equations of some backgrounds with one, two and four supersymmetries. In section 8, we solve both the Killing spinor and field equations of a background with four supersymmetries and $SU(4)$ invariant spinors, and in section 9, we give our conclusions. In appendix A, we summarize the properties of $Spin(10, 1)$ spinors. In appendix B, we give the conditions on the geometry and the expressions for the fluxes of backgrounds which admit one $SU(5)$ invariant Killing spinor.

---

4 We denote the tenth direction with $\natural$. 
These results can be found in [10] but are summarized here for convenience. In appendix C, we give the tables with the expressions for $D\sigma_I$ expanded in the basis (A.8). In appendix D, we give the tables with the expressions for $\sigma_I$ expanded in the basis (A.8). In appendix E, we solve the Killing spinor equations for backgrounds which admit two Killing spinors which are invariant under the $SU(4)$ subgroup of $Spin(10, 1)$.

2. Killing spinor equations and integrability conditions

2.1. Killing spinor equations

The Killing spinor equations of 11-dimensional supergravity [15] are the vanishing of the gravitino supersymmetry transformation restricted on the bosonic fields of the theory. The bosonic fields are the metric $g$ and a four-form field strength $F$. The Killing spinors of 11-dimensional supergravity are in the Majorana representation $\Delta_{32}$ of $Spin(10, 1)$. The supercovariant connection of 11-dimensional supergravity is

$$D_A\epsilon = \nabla_A\epsilon + \Sigma_A\epsilon$$

(2.1)

where

$$\nabla_A\epsilon = \partial_A\epsilon + \frac{1}{4}\Omega_{A,BC}^\epsilon\Gamma^{BC}\epsilon,$$

(2.2)

i.e. $\nabla_A$ is the spin covariant derivative induced from the Levi-Civita connection,

$$\Sigma_A = -\frac{1}{288}(\Gamma^A_{\ B_1\cdots B_{10}} - 8\delta^A_{B_1\cdots B_{10}})F_{B_1\cdots B_{10}},$$

(2.3)

and $F$ is the four-form field strength (or flux), $A, B, \ldots = 0, \ldots, 9, 10$ are frame indices. The supercovariant connection is a covariant derivative on the spinor bundle of 11-dimensional spacetime associated with the Majorana representation of $Spin(10, 1)$. However, $D$ is not induced from the tangent bundle because of the term (2.3) which depends on the flux $F$.

As has been explained in [10], the supercovariant connection has gauge symmetry $Spin(10, 1)$ and this can be used to bring the Killing spinors into a canonical or normal form up to an induced Lorentz transformation on the spacetime frame and fluxes $F$. In this way, one can simplify the conditions imposed by supersymmetry of the fluxes and geometry of a background by choosing the Killing spinors to lie at a particular direction. This simplification is possible for backgrounds with one and two supersymmetries. It turns out that the stability subgroup of two generic spinors in $Spin(10, 1)$ is the identity. Therefore, one does not expect that there will be a simplification in the form of a third spinor in a generic background with three supersymmetries. This is unless the conditions on the geometry and on the fluxes imposed by the first two spinors necessitate the vanishing of many components of $\Omega$ and $F$ and so the equations for the third Killing spinor are not involved. In any case there are several special backgrounds with more than two supersymmetries that admit spinors which have a non-trivial stability subgroup in $Spin(10, 1)$.

Since in the basis of gamma matrices we have adopted the frame time direction is distinguished from the rest, it is convenient to decompose the frame indices as $A = (0, i)$, where $i = 1, \ldots, 10$. Then we introduce an orthonormal frame $\{e^A : A = 0, \ldots, 10\}$ and write the spacetime metric as

$$ds^2 = -(e^0)^2 + \sum_{i=1}^{10} (e^i)^2.$$  

(2.4)

In this frame, the four-form field strength $F$ can be expanded in electric and magnetic parts as

$$F = \frac{1}{3!} e^0 \wedge G_{ijk} e^i \wedge e^j \wedge e^k + \frac{1}{4!} F_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l.$$  

(2.5)
The spin (Levi-Civita) connection has non-vanishing components
\[ \Omega_{0,ij}, \quad \Omega_{0,0j}, \quad \Omega_{0,j}, \quad \Omega_{i,jk}. \]  

The Killing spinor equations decompose as
\[ \partial_j \epsilon + \frac{1}{4} \Omega_{0,ij} \Gamma^j \epsilon = \frac{1}{2} \Omega_{i,j} \Gamma^0 \epsilon - \frac{1}{288} (\Gamma_0 \Gamma^{ijkl} F_{ijkl} - 8 G_{ijkl} \Gamma^{ijkl}) \epsilon = 0, \]
\[ \partial_i \epsilon + \frac{1}{4} \Omega_{i,jk} \Gamma^{jk} \epsilon = \frac{1}{2} \Omega_{i,jk} \Gamma^0 \epsilon - \frac{1}{288} (\Gamma_i \Gamma^{jk} G_{jkl} - 8 \Gamma_i \Gamma^{jk} G_{jkl}) \epsilon = 0. \]

This is the form of the Killing spinor equations that we shall use later to derive our results.

2.2. Integrability conditions and field equations

The integrability conditions of the Killing spinor equations
\[ [D_A, D_B] \epsilon = R_{AB} \epsilon = 0, \]  

where \( R \) is the supercovariant curvature which has been computed in [16, 6]. It has been observed in [8] that, using the Bianchi identity of the Riemann curvature of spacetime,
\[ \Gamma^B R_{AB} \epsilon = 0 \]  

can be written as
\[ I_A \epsilon = \left[ E_{AB} \Gamma^B + L_{C_1C_2C_3} (\Gamma_A \Gamma_{C_1C_2C_3} - 6 \delta^C_A \Gamma^{C_1C_2C_3}) + B_{C_1...C_5} (\Gamma_A \Gamma_{C_1...C_5} - 10 \delta^C_A \Gamma^{C_1...C_5}) \right] \epsilon = 0, \]

where
\[ E_{AB} := R_{AB} - \frac{1}{12} \mathcal{F}_{AC_1C_2C_3} \mathcal{F}^{BC_1C_2C_3} + \frac{1}{144} g_{AB} \mathcal{F}^{C_1...C_5} \mathcal{F}_{C_1...C_5}, \]
\[ L_{ABC} := -\frac{1}{36} \ast \left( d \ast \mathcal{F} - \frac{1}{2} \mathcal{F} \wedge \mathcal{F} \right)_{ABC}, \]
\[ B_{A_1...A_5} := \frac{1}{6!} \left( d \mathcal{F} \right)_{A_1...A_5}. \]

The above integrability conditions can be written in terms of the frame \((e^0, e^i)\). This computation is similar to the one for the Killing spinor equations and we shall not repeat it here. It is clear that some of the components of the field equations (and Bianchi identity) are satisfied as integrability conditions of the Killing spinor equations. Sometimes it is customary to impose enough conditions on the field equations and on Bianchi identity \( \mathcal{F} \) such that all Einstein equations are satisfied. This is because the field equations and Bianchi identity of \( \mathcal{F} \) are easier to solve.

3. The six types of spinors

A direct consequence of the construction of the Spin\((10, 1)\) Majorana spinor representation in appendix A is that any Majorana Killing spinor can be written in terms of forms as
\[ \epsilon = f (1 + e_{12345}) + i g (1 - e_{12345}) + \sqrt{2} u^i \left( e_i + \frac{1}{4!} \epsilon_{ijklm} e_{ijklm} \right) + i \sqrt{2} v^i \left( e_i - \frac{1}{4!} \epsilon_{ijklm} e_{ijklm} \right) + \frac{1}{2} u^{ij} \left( e_{ij} - \frac{1}{3!} \epsilon_{ij}^{klm} e_{klm} \right) + \frac{1}{2} v^{ij} \left( e_{ij} + \frac{1}{3!} \epsilon_{ij}^{klm} e_{klm} \right), \]

\[ \text{for an alternative approach see [17].} \]
where \( f, g, u^i, v^i, w^{ij} \) and \( z^{ij} \) are real spacetime functions. The six types of spinors \( e_{i_1 \ldots i_I} \) with \( i = 0, \ldots, 5 \) correspond to the irreducible representation of \( U(5) \) on \( \Lambda^*(\mathbb{C}^5) \) and are denoted by \( \sigma_I \).

The Killing spinor equations for \( \epsilon \) are

\[
D_A \epsilon = \partial_A f (1 + e_{12345}) + i \partial_A g (1 - e_{12345}) + \sqrt{2} \partial_A u^i \left( e_i + \frac{1}{4!} e_{ijklm} e_{jklm} \right) \\
+ i \sqrt{2} \partial_A v^i \left( e_i - \frac{1}{4!} e_{ijklm} e_{jklm} \right) + \frac{1}{2} \partial_A w^{ij} \left( e_{ij} - \frac{1}{3!} e_{ijklm} e_{jklm} \right) \\
+ \frac{1}{2} \partial_A z^{ij} \left( e_{ij} + \frac{1}{3!} e_{ijklm} e_{jklm} \right) + \frac{i}{2} z^{ij} D_A \left( e_{ij} + \frac{1}{3!} e_{ijklm} e_{jklm} \right) = 0. \tag{3.2}
\]

Thus the Killing spinor equations reduce to the evaluation of the supercovariant derivative on the spinors \( \sigma_I \). So it remains to compute the \( I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1,} \), and express the result in the basis (A.8). Note that in some cases it is possible to put some spinors in a canonical or normal form using the \( \text{Spin}(10, 1) \) gauge symmetry of the supercovariant connection \( D; \) see [10]. As a result the spinors depend on fewer functions than those indicated in (3.1). In such cases, it is helpful to consider the orbits of \( \text{Spin}(10, 1) \) in the space of spinors [18, 19].

The same analysis can be done for the integrability condition \( I \epsilon = 0 \) of a Killing spinor \( \epsilon \). Since this condition is linear, we have

\[
I \epsilon = f I (1 + e_{12345}) + i g I (1 - e_{12345}) + \sqrt{2} u^I (e_i + \frac{1}{4!} e_{ijklm} e_{jklm}) \\
+ i \sqrt{2} v^I (e_i - \frac{1}{4!} e_{ijklm} e_{jklm}) + \frac{1}{2} w^{ij} I (e_{ij} - \frac{1}{3!} e_{ijklm} e_{jklm}) \\
+ \frac{i}{2} z^{ij} I (e_{ij} + \frac{1}{3!} e_{ijklm} e_{jklm}). \tag{3.4}
\]

Therefore to find which field equations are determined by the Killing spinor equations, it suffices to compute

\[
I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1}, \, I_A e_{i_1,} \), and then solve the linear system.

4. Linear systems in a timelike basis

4.1. The linear system of killing spinor equations

We would like to explain how one evaluates the supercovariant derivative on an arbitrary basis element

\[
e_{i_1 \ldots i_I} = \frac{1}{2^I/2} \Gamma^{i_1} \ldots \Gamma^{i_I} 1. \tag{4.1}
\]

where the indices \( i_1, \ldots, i_I \) pick out \( I \) holomorphic indices (with \( 0 \leq I \leq 5 \)) from the range \( \alpha = 1, \ldots, 5 \). It will be convenient to distinguish between the indices that do appear in the
basis element (4.1) and those that do not: we split the holomorphic indices $\alpha$ into the indices\(^6\) $a = (i_1, \ldots, i_l)$ and the remaining $5 - l$ indices $\rho$, and similarly for the anti-holomorphic indices $\bar{\alpha}$. Note that $\Gamma^a$ and $\Gamma^\rho$ annihilate the spinor $e_{i_1 \ldots i_l}$, while $\Gamma^\alpha$ and $\Gamma^\bar{\rho}$ act as creation operators. For this reason it is useful to define the new indices $\rho, \sigma, \tau$ consisting of the combination
\[
\rho = (\bar{a}_1, \ldots, \bar{a}_l, p_1, \ldots, p_{5-l}), \quad \bar{\rho} = (a_1, \ldots, a_l, \bar{p}_1, \ldots, \bar{p}_{5-l}),
\]
(4.2)
where $\Gamma^\rho$ and $\Gamma^\bar{\rho}$ are the annihilation and creation operators, respectively, for the spinor $e_{i_1 \ldots i_l}$. Note that the indices $a$ and $\rho$ are identical for $I = 0$, i.e. for the spinor 1. For $I > 0$, i.e. for any other basis element, these indices differ.

In terms of the basis\(^7\)
\[
\{ e_{i_1 \ldots i_l}, \Gamma^\rho e_{i_1 \ldots i_l}, \ldots, \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l} \},
\]
(4.3)
the supercovariant derivative with $A = 0$ can be expanded in the following contributions:
\[
D_0 e_{i_1 \ldots i_l} = \left[ \frac{1}{2} \Omega_{\rho, \bar{\sigma}} \tau + (-1)^{I+1} \frac{i}{24} F_{\bar{\tau}_1 \bar{\tau}_2} \right] e_{i_1 \ldots i_l} + \left[ (-1)^I \frac{i}{2} \frac{1}{6} G_{\rho \sigma} \tau \right] \Gamma^\rho e_{i_1 \ldots i_l}
\]
\[
+ \left[ \frac{1}{4} g_{\rho, \bar{\sigma}} + (-1)^{I+1} \frac{i}{24} F_{\bar{\rho}, \bar{\sigma} \tau} \right] \Gamma^{\rho \sigma} e_{i_1 \ldots i_l} + \left[ \frac{1}{36} G_{\rho, \sigma \tau} \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l} \]
\[
+ \left[ (-1)^{I+1} \frac{i}{288} F_{\bar{\rho}, \sigma} \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l}.
\]
(4.4)
Observe that the component $\Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l}$ vanishes. Similarly, the expression for $A = \rho$ read
\[
D_\rho e_{i_1 \ldots i_l} = \left[ \frac{1}{2} \Omega_{\rho, \sigma} \tau_1 + (-1)^I \frac{i}{4} G_{\rho, \sigma} \tau \right] e_{i_1 \ldots i_l} + \left[ (-1)^I \frac{i}{2} \frac{1}{4} \Omega_{\rho, \bar{\sigma}} \tau \right] \Gamma^{\rho} e_{i_1 \ldots i_l}
\]
\[
- \left[ \frac{1}{24} g_{\rho, \bar{\sigma}} + (-1)^{I+1} \frac{i}{12} G_{\rho, \sigma \tau} \right] \Gamma^{\rho \sigma} e_{i_1 \ldots i_l} + \left[ \frac{1}{4} \frac{1}{8} G_{\rho, \bar{\sigma}} \tau \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l} \]
\[
+ \left[ (-1)^{I+1} \frac{i}{12} g_{\rho, \bar{\sigma}} \sigma \tau \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l} + \left[ \frac{1}{12} \frac{1}{8} G_{\rho, \bar{\sigma}} \sigma \tau \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l} \]
\[
+ \left[ (-1)^{I+1} \frac{i}{288} g_{\rho, \bar{\sigma}} \sigma \tau \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l}.
\]
(4.5)
Finally, for $A = \bar{\rho}$ we find
\[
D_{\bar{\rho}} e_{i_1 \ldots i_l} = \left[ \frac{1}{2} \Omega_{\rho, \sigma} \tau_1 + (-1)^I \frac{i}{12} G_{\rho, \sigma} \tau \right] e_{i_1 \ldots i_l} + \left[ (-1)^I \frac{i}{2} \frac{1}{12} \Omega_{\rho, \bar{\sigma}} \tau \right] \Gamma^{\rho} e_{i_1 \ldots i_l}
\]
\[
+ \left[ \frac{1}{4} \frac{1}{8} G_{\rho, \bar{\sigma}} \tau \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l} + \left[ \frac{1}{12} \frac{1}{8} G_{\rho, \bar{\sigma}} \sigma \tau \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l} \]
\[
+ \left[ (-1)^{I+1} \frac{i}{288} g_{\rho, \bar{\sigma}} \sigma \tau \right] \Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l}.
\]
(4.6)
Observe that the components along $\Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l}$ and $\Gamma^{\rho \sigma \tau} e_{i_1 \ldots i_l}$ vanish.

It is convenient to convert the above expressions from basis (4.3) to the ‘canonical’ basis
\[
\{ 1, \Gamma^a 1, \ldots, \Gamma^{\rho \sigma \tau} 1 \}.
\]
(4.7)
For this, we expand the products of $\Gamma^a$ matrices, which are creation operators on $e_{i_1 \ldots i_l}$, into a sum of products of $\Gamma^a$ and $\Gamma^\rho$ matrices, which are annihilation and creation operators,
\[\text{\footnote{The $i_1, \ldots, i_l$ should not be thought of as indices in this context, but rather as fixed labels for a particular spinor.}}\]
\[\text{\footnote{Note that in this basis $e_{i_1 \ldots i_l}$ is the Clifford algebra vacuum.}}\]
respectively, on 1. Then we act on \( e_{i_1 \cdots i_l} \) with the annihilation operators. In particular, we have

\[
\mathcal{D}_A e_{i_1 \cdots i_l} = \sum_k \left[ \mathcal{D}_A e_{i_1 \cdots i_l} \right]_{p_1 \cdots p_k} \Gamma^{p_1 \cdots p_k} e_{i_1 \cdots i_l}
\]

\[
= \sum_k \sum_{m+n=k} \frac{k!}{m!n!} \left[ \mathcal{D}_A e_{i_1 \cdots i_l} \right]_{a_1 \cdots a_m b_1 \cdots b_n} \Gamma^{a_1 \cdots a_m} \Gamma^{b_1 \cdots b_n} e_{i_1 \cdots i_l}
\]

\[
= \sum_k \sum_{m+n=k} \frac{k!}{m!n!} (-1)^{m/2+nI} \frac{2^{l/2-m}(I-m)!}{(m+n)!} \Gamma^{a_1 \cdots a_m} \Gamma^{b_1 \cdots b_n} e_{i_1 \cdots i_l}
\]

\[
\times \left[ \mathcal{D}_A e_{i_1 \cdots i_l} \right]_{a_1 \cdots a_m b_1 \cdots b_n} \Gamma^{a_1 \cdots a_m} \Gamma^{b_1 \cdots b_n} e_{i_1 \cdots i_l}, \tag{4.8}
\]

with the obvious restrictions \( m \leq I \) and \( n \leq 5 - I \) and the convention that \( e^i_{i_1 \cdots i_l} = 1 \). Using the expressions (4.4), (4.5) and (4.6) for the components of \( \mathcal{D}_A e_{i_1 \cdots i_l} \) in the basis (4.3) which appear in square brackets in (4.8), one can easily compute the components of \( \mathcal{D}_A e_{i_1 \cdots i_l} \) in the canonical basis (4.7). For convenience we give the explicit expressions for the different basis elements in appendix C.

From expression (4.8) one can also derive a relation between the Killing spinors equations from \( e_{i_1 \cdots i_l} \) and \( e_{ij_1 \cdots ij_l} \), whose labels satisfy \( e_{i_1 \cdots i_l ij_1 \cdots ij_l} = 1 \). The key observation is that, in the basis (4.1),

\[
\left( \mathcal{D}_A e_{i_1 \cdots i_l} \right)_{a_1 \cdots a_m b_1 \cdots b_n} = \left( \mathcal{D}_A e_{ij_1 \cdots ij_l} \right)^a_{a_1 \cdots a_m b_1 \cdots b_n}, \tag{4.9}
\]

where the notation, i.e. the division of \( a \) and \( \alpha \) into \( \sigma \) and \( \bar{\sigma} \), is based on \( e_{i_1 \cdots i_l} \) and not on \( e_{ij_1 \cdots ij_l} \) (as it will be in the remainder of this section). Converting both expressions to the canonical basis using (4.8), one finds that the previous relation translates into

\[
\left( \mathcal{D}_A e_{i_1 \cdots i_l} \right)_{a_1 \cdots a_m b_1 \cdots b_n} = \frac{2^{2-m-n}(-1)^{(m+n)/2}(-I/2)(5-m-n)!}{(m+n)!(5-I-n)!} \left( \mathcal{D}_A e_{ij_1 \cdots ij_l} \right)^a_{a_1 \cdots a_m b_1 \cdots b_n}, \tag{4.10}
\]

After the addition of the complex conjugated and dualized version of this expression to its original, one finds that the components of the combination \( e_{i_1 \cdots i_l} + (-1)^{(l+1)/2} e_{ij_1 \cdots ij_l} \) are related to each other:

\[
\left( \mathcal{D}_A e_{i_1 \cdots i_l} + (-1)^{(l+1)/2} \mathcal{D}_A e_{ij_1 \cdots ij_l} \right)_{a_1 \cdots a_m b_1 \cdots b_n} = \frac{2^{2-m-n}(-1)^{(m+n)/2}(-I/2)(5-m-n)!}{(m+n)!(5-I-n)!} \left( \mathcal{D}_A e_{ij_1 \cdots ij_l} + (-1)^{(l+1)/2} \mathcal{D}_A e_{i_1 \cdots i_l} \right)^a_{a_1 \cdots a_m b_1 \cdots b_n}. \tag{4.11}
\]

A similar expression holds for the components of \( i \mathcal{D}_A e_{i_1 \cdots i_l} - \mathcal{D}_A e_{ij_1 \cdots ij_l} \). This relates the Killing spinor equations of any real Majorana spinor (3.1). For this reason one only has to consider half of all equations; in appendix C we give all \( A = \bar{\alpha} \) equations plus the \( A = 0 \) equations coming with less than three \( \Gamma^a \)-matrices.

4.2. The linear system of integrability conditions

As we have explained the integrability condition (2.9) on any Killing spinor, \( \mathcal{I} \epsilon \), can be expressed in terms of \( \mathcal{I} \sigma \). In turn \( \mathcal{I} \sigma \) can be expanded in the basis (4.3). For this, one inserts \( e_{i_1 \cdots i_l} \) (2.9), expands the resulting equation in (4.3) and sets \( A = 0 \) to find that

\[
\mathcal{I} e_{i_1 \cdots i_l} = \left[ (-1)^{l+1} E_{00} - 12L_{00} \sigma - 120B_{00} \sigma^2 \right] e_{i_1 \cdots i_l}
\]

\[
+ \left[ E_{00} + (-1)^{l+1} 6L_{00} \sigma + (-1)^{l+1} 60B_{00} \sigma^2 \right] \Gamma^\sigma e_{i_1 \cdots i_l}
\]

\[
+ \left[ -6L_{00} \sigma_2 - 120B_{00} \sigma_2 \sigma \right] \Gamma^\sigma \sigma_2 e_{i_1 \cdots i_l},
\]
Similarly for $A = \rho$, one finds
\[
\mathcal{I}_\rho e_{\alpha_1...\alpha_l} = \left[\left(-1\right)^{l+1}iE_{0\rho} - 18L_{\rho\sigma}^\alpha - 180B_{\rho\sigma\tau}^\alpha\tau\right]e_{\alpha_1...\alpha_l} + \left[E_{\rho\sigma} + \left(-1\right)^{l+1}6i\bar{g}_{\rho\sigma}L_{0\tau}^\tau + \left(-1\right)^{l}18iL_{0\rho\sigma}\right]e_{\alpha_1...\alpha_l} \\
+ \left[-10B_{0\alpha_1...\alpha_l}\Gamma^{\alpha_1...\alpha_l}_{\alpha_1...\alpha_l}e_{\alpha_1...\alpha_l} + \left(-1\right)^{l+1}iB_{\rho\alpha_1...\alpha_l}\Gamma^{\alpha_1...\alpha_l}_{\alpha_1...\alpha_l}e_{\alpha_1...\alpha_l}\right].
\]
(4.12)

Finally for $A = \bar{\rho}$, we find
\[
\mathcal{I}_{\bar{\rho}} e_{\alpha_1...\alpha_l} = \left[\left(-1\right)^{l+1}iE_{0\rho} - 6L_{\rho\sigma}^\alpha - 60B_{\rho\sigma\tau}^\alpha\tau\right]e_{\alpha_1...\alpha_l} + \left[E_{\rho\sigma} + \left(-1\right)^{l+1}2i6i\bar{g}_{\rho\sigma}L_{0\tau}^\tau + \left(-1\right)^{l}120iL_{0\rho\sigma}\right]e_{\alpha_1...\alpha_l} \\
+ \left[-36i\bar{g}_{\rho\sigma}L_{0\tau}^\tau - 60B_{\rho\sigma\tau}^\alpha\tau\Gamma^{\alpha_1...\alpha_l}_{\alpha_1...\alpha_l}e_{\alpha_1...\alpha_l}\right] + \left(\left(-1\right)^{l+1}6i\bar{g}_{\rho\sigma}L_{0\tau}^\tau + \left(-1\right)^{l}18iL_{0\rho\sigma}\right)\Gamma^{\alpha_1...\alpha_l}_{\alpha_1...\alpha_l}e_{\alpha_1...\alpha_l} \\
+ \left[\left(-1\right)^{l+1}15i\bar{g}_{\rho\sigma}L_{0\tau}^\tau + \left(-1\right)^{l}180iL_{0\rho\sigma}\right]\Gamma^{\alpha_1...\alpha_l}_{\alpha_1...\alpha_l}e_{\alpha_1...\alpha_l}.
\]
(4.13)

Observe that the $\Gamma^{\alpha_1...\alpha_l}_{\alpha_1...\alpha_l}e_{\alpha_1...\alpha_l}$ component of the last integrability condition vanishes. It is straightforward to convert the above expressions to the canonical basis (4.7). This is completely similar to that for the Killing spinor equations in (4.8) and we shall not repeat the expression here. In addition, a relation similar to (4.11) holds for the integrability conditions. In appendix D we give the explicit expressions for $\mathcal{I}_{\alpha l}$ in the canonical basis.

5. Linear systems in a null basis

The construction of the linear systems in the previous section applies to all Killing spinors, i.e. to spinors that represent the orbit of $Spin(10, 1)$ with stability subgroup $SU(5)$ and the spinors that represent the orbit of $Spin(10, 1)$ with stability subgroup $(Spin(7) \times \mathbb{R}^5) \times \mathbb{R}$. However, if it is known that one of the Killing spinors represents the orbit with stability subgroup $(Spin(7) \times \mathbb{R}^5) \times \mathbb{R}$, it may be more convenient to use the null basis of appendix A to construct the linear systems for the Killing spinor equations and the associated integrability conditions. This is because the gauge symmetry of the supercovariant connection can be used to put that spinor along the direction $1 + e_{1234}$; see appendix A.

The timelike and null bases in the space of spinors are oscillator bases. Because of this, the linear system of the Killing spinor equations and of the integrability conditions which we have derived for the timelike basis in the previous section are easily adapted to the null basis. We shall demonstrate this for the linear system of the Killing spinor equations. Since in the null basis the tenth direction, which we denote by $\bar{z}$, is separated from the rest, see appendix A, we decompose the four-form field strength $\mathcal{F}$ as
\[
\mathcal{F} = \frac{1}{3!} e^5 \wedge G_{ijkl} e^i \wedge e^j \wedge e^k + \frac{1}{4!} F_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l,
\]
(5.1)
where $i = 0, 1, 2, \ldots, 9$. We have denoted the tenth component and the remaining components of $\mathcal{F}$ as the electric and magnetic components, respectively, that appear in the decomposition of $\mathcal{F}$ in the timelike basis. The reason for this will become apparent. The spin (Levi-Civita)
connection has non-vanishing components
\[ \Omega_{\alpha\beta}, \quad \Omega_{\alpha\gamma}, \quad \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta}. \] (5.2)

The Killing spinor equations decompose as
\[
\partial_\nu \epsilon + \frac{1}{4} \Omega_{\rho\nu} \Gamma_{i}^{\nu} \epsilon + \frac{1}{2} \Omega_{\rho\sigma} \Gamma^{i}_{\rho} \epsilon = - \frac{1}{288} \left( \Gamma_{i}^{jkl} F_{jkl} - 8 G_{ijkl} \Gamma^{ijl} \right) \epsilon = 0,
\]
\[
\partial_{j} \epsilon + \frac{1}{4} \Omega_{i\nu} \Gamma^{j} \epsilon + \frac{1}{2} \Omega_{i\sigma} \Gamma^{j} \epsilon = - \frac{1}{288} \left( \Gamma_{i}^{jkl} F_{jkl} - 4 \Gamma_{i}^{jkl} G_{jkl} + 24 \Gamma_{jkl} \Gamma^{ijl} - 8 F_{jkl} \Gamma^{ijl} \right) \epsilon = 0. \] (5.3)

Observe that these formulae can be derived from those of the timelike basis in (2.7) after the replacement \( 0 \rightarrow \bar{\rho} \). It is clear from this that the linear system for the null basis associated with the Killing spinor equations can be derived from that of the timelike basis, we have derived, after taking into account the different way that \( \Gamma_{0} \) and \( \Gamma_{1} \) act on the basis spinors.

Every spinor in the null basis can be written as a linear combination of six types of spinors. These spinors are constructed by the creation operators of the null basis acting on the Clifford vacuum 1; see appendix A. In particular, we have that \( \Gamma_{i} = -\Gamma_{0} \Gamma_{1} \cdots \Gamma_{9} \) acts on our null basis of spinors as
\[
\Gamma_{i} \epsilon_{i_{1} \cdots i_{l}} = \Gamma^{i} \epsilon_{i_{1} \cdots i_{l}} = (-1)^{l+1} \epsilon_{i_{1} \cdots i_{l}}. \] (5.4)

This means that the difference between the timelike and the null case consists of replacing \( \Gamma_{0} \) by \( \Gamma_{i} \) in most cases. This amounts to the replacement \( i \rightarrow +1 \). The only exception is the \( F \)-term in the \( D_{a} \) component of the supercovariant derivative, where \( \Gamma_{0} \) is replaced by \( \Gamma_{i} \) and so there is an additional minus sign.

As in the timelike case, it will be convenient to distinguish between the indices that do appear in the basis element \( \epsilon_{i_{1} \cdots i_{l}} \) and those that do not. In particular, we split the indices \( \alpha \) into the indices \( \alpha = (i_{1}, \ldots, i_{l}) \) and the remaining \( 5 - l \) indices \( p \), and similarly for the indices \( \bar{\alpha} \). Subsequently we define the new indices \( \rho, \sigma, \tau \) consisting of the combination\(^8\)
\[
\rho = (\bar{\alpha}_{1}, \ldots, \bar{\alpha}_{l}, p_{1}, \ldots, p_{5-l}), \quad \bar{\rho} = (\bar{\alpha}_{1}, \ldots, \bar{\alpha}_{l}, \bar{p}_{1}, \ldots, \bar{p}_{5-l}),
\] (5.5)

where \( \Gamma_{\rho} \) and \( \Gamma_{\bar{\rho}} \) are the annihilation and creation operators, respectively, for the spinor \( \epsilon_{i_{1} \cdots i_{l}} \). Next consider the basis in the space of spinors associated with the Clifford vacuum \( \epsilon_{i_{1} \cdots i_{l}} \), i.e.
\[
\left\{ \epsilon_{i_{1} \cdots i_{l}}, \Gamma^{\rho} \epsilon_{i_{1} \cdots i_{l}}, \ldots, \Gamma^{\bar{\rho}_{1} \cdots \bar{\rho}_{l}} \epsilon_{i_{1} \cdots i_{l}} \right\}. \] (5.6)

In this basis the supercovariant derivative with \( A = \emptyset \) can be expanded as
\[
D_{\alpha} \epsilon_{i_{1} \cdots i_{l}} = \left[ \frac{1}{2} \Omega_{\alpha\beta} \Gamma_{\beta} + \left( -1 \right)^{l} \frac{1}{24} F_{\beta}^{\alpha_{1} \cdots \alpha_{l}} \right] \epsilon_{i_{1} \cdots i_{l}} + \left. \left( -1 \right)^{l} \frac{1}{2} \Omega_{\bar{\alpha}_{1} \cdots \bar{\alpha}_{l}} \right] \Gamma^{\beta} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{6} G_{\beta\alpha} \right] \Gamma^{\bar{\alpha}} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{288} \right] \Gamma^{\bar{\beta}} \epsilon_{i_{1} \cdots i_{l}}
\]
\[
+ \right. \left( -1 \right)^{l} \frac{1}{24} F_{\beta}^{\alpha_{1} \cdots \alpha_{l}} \right] \Gamma^{\beta} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{36} G_{\beta\alpha} \right] \Gamma^{\bar{\alpha}} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{288} \right] \Gamma^{\bar{\beta}} \epsilon_{i_{1} \cdots i_{l}}.
\] (5.7)

Similarly, the expression for \( A = \rho \) reads
\[
D_{\rho} \epsilon_{i_{1} \cdots i_{l}} = \left[ \frac{1}{2} \Omega_{\rho\sigma} \Gamma_{\sigma} + \left( -1 \right)^{l} \frac{1}{4} G_{\rho\sigma} \right] \epsilon_{i_{1} \cdots i_{l}} + \left. \left( -1 \right)^{l} \frac{1}{2} \Omega_{\rho\bar{\beta}} \right] \Gamma^{\beta} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{4} G_{\rho\sigma} \right] \Gamma^{\beta} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{24} g_{\rho\beta} F_{\beta}^{\alpha_{1} \cdots \alpha_{l}} \right] \Gamma^{\beta} \epsilon_{i_{1} \cdots i_{l}}
\]
\[
+ \left. \left( -1 \right)^{l} \frac{1}{2} \Omega_{\rho\bar{\beta}} \right] \Gamma^{\bar{\alpha}} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{4} G_{\rho\sigma} \right] \Gamma^{\bar{\alpha}} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{24} g_{\rho\beta} F_{\beta}^{\alpha_{1} \cdots \alpha_{l}} \right] \Gamma^{\bar{\alpha}} \epsilon_{i_{1} \cdots i_{l}} + \left. \frac{1}{24} g_{\rho\beta} F_{\beta}^{\alpha_{1} \cdots \alpha_{l}} \right] \Gamma^{\bar{\alpha}} \epsilon_{i_{1} \cdots i_{l}}.
\]
\(^8\) Note that \( \rho \) and \( \bar{\rho} \) are no longer complex conjugate due to the presence of the \((+, -)\) null indices.
Finally, for $A = \tilde{\rho}$ we find

$$D_\rho e_{i_1 \ldots i_4} = \left[ \frac{1}{4} \Omega_{\rho, \sigma}^\alpha + (-1)^2 \frac{1}{12} G_{\rho \sigma} \sigma \right] e_{i_1 \ldots i_4} + \left[ -\frac{1}{4} \Omega_{\rho, \sigma}^\alpha + \frac{1}{12} F_{\rho \sigma} \tau \right] \Gamma^{i_1 \ldots i_4} e_{i_1 \ldots i_4},$$

(5.8)

To go from the basis (5.6) to the ‘canonical’ basis which is associated with the Clifford vacuum 1, one can use the same expressions as for the timelike case. So we shall not repeat the formulae here.

The complex conjugation between the components of the supercovariant derivative found in the timelike case does not extend to the null basis in a straightforward way because $\Gamma^+$ and $\Gamma^-$ are null instead of holomorphic. For this reason, it will be convenient to treat the null indices separately. In what follows all indices only take values in $1, \ldots, 4$. Instead of (4.9), the following relations hold

$$(\mathcal{D}_A e_{i_1 \ldots i_4})_{a_1 \ldots a_4} = (\mathcal{D}_A e_{i_1 \ldots i_4})^*_{\bar{a}_1 \ldots \bar{a}_4}, \quad (\mathcal{D}_A e_{i_1 \ldots i_4})_{a_1 \ldots a_4} = (\mathcal{D}_A e_{i_1 \ldots i_4})^*_{\bar{a}_1 \ldots \bar{a}_4},$$

(5.10)

where $e_{i_1 \ldots i_4} = +1$. Using the relation to the ‘canonical’ basis associated with the Clifford vacuum 1, these imply that

$$(\mathcal{D}_A e_{i_1 \ldots i_4})_{a_1 \ldots a_4} = \frac{2^{m-n}}{(m+n)!} (-1)^{(m+n)/2} (4 - m - n)! \times e_{a_1 \ldots a_4} p_{i_1 \ldots i_4} (\mathcal{D}_A e_{i_1 \ldots i_4})^*_{\bar{a}_1 \ldots \bar{a}_4}.$$  

(5.11)

Adding the complex conjugated and dualized version of this expression to its original, one finds the following expression relating different components of the combination $e_{i_1 \ldots i_4} + (-1)^{1/2} e_{i_1 \ldots i_4}$:

$$(\mathcal{D}_A e_{i_1 \ldots i_4} + (-1)^{1/2} \mathcal{D}_A e_{i_1 \ldots i_4})_{a_1 \ldots a_4} = \frac{2^{m-n}}{(m+n)!} (-1)^{(m+n)/2} (4 - m - n)! \times e_{a_1 \ldots a_4} p_{i_1 \ldots i_4} (\mathcal{D}_A e_{i_1 \ldots i_4} + (-1)^{1/2} \mathcal{D}_A e_{i_1 \ldots i_4})^*_{\bar{a}_1 \ldots \bar{a}_4}.$$  

(5.12)

The same expression holds with an extra $+i$ index on the end of the list of components on both sides. In addition, the same relation holds for the following combinations

$$e_{i_1 \ldots i_4} + (-1)^{1/2} e_{i_1 \ldots i_4}, \quad i e_{i_1 \ldots i_4} = -(-1)^{1/2} e_{i_{1+} \ldots i_4}, \quad i e_{i_1 \ldots i_4} = -(-1)^{1/2} e_{i_{1+} \ldots i_4},$$

(5.13)

which together with $e_{i_1 \ldots i_4} + (-1)^{1/2} e_{i_1 \ldots i_4}$ above span a basis in the space of Majorana spinors for the null case. This concludes the investigation of the complex conjugation relations of the components of the Killing spinor equations in the null basis.

The linear systems associated with the integrability conditions in the null and in the timelike basis are related in a similar way to those of the Killing spinor equations. One
again replaces in the linear system for the integrability conditions $0$ with $\overline{z}$ and $i$ with $+1$. An additional sign appears in the $I_0$ component of the integrability conditions because one replaces $\Gamma_0$ with $\Gamma_1$ as in the Killing spinor equations case. Because of the simplicity of the rules to derive the linear systems associated with the null basis from those of the timelike one, we shall not give further details for the former.

6. $N=2$ backgrounds with $SU(4)$ invariant Killing spinors

6.1. The Killing spinor equations

The most general $SU(4)$ invariant Killing spinors of a $N=2$ background are

$$\eta_1 = f(1 + e_{12345})$$
$$\eta_2 = g_1(1 + e_{12345}) + g_2 i(1 - e_{12345}) + \sqrt{2} g_3 (e_5 + e_{1234} - e_{12345}),$$

where $f$, $g_1$, $g_2$, $g_3$ are real functions of the spacetime which will be determined by the Killing spinor equations. We shall assume that $g_3 \neq 0$ because otherwise the spinors are $SU(5)$ invariant and this case has already been investigated in [10]. The Killing spinor equations of $\eta_1$ are as in the $N = 1$ case. So it remains for us to solve the Killing spinor equations for the second spinor. Using the Killing spinor equations of $\eta_1$, the Killing spinor equations $\mathcal{D}_A \eta_2 = 0$ can be written as

$$(\partial_A g_1 - g_1 \partial_A \log f)(1 + e_{12345}) + i \partial_A g_2 (1 - e_{12345})
+ i g_2 \mathcal{D}_A (1 - e_{12345}) + \sqrt{2} \mathcal{D}_A [g_3 (e_5 + e_{12345})] = 0.$$  

Multiplying the above equation with $g_3^{-1}$, we find that the Killing spinor equations for the second spinor can be rewritten as

$$g_3^{-1} (\partial_A g_1 - g_1 \partial_A \log f + i \partial_A g_2) 1 + g_3^{-1} (\partial_A g_1 - g_1 \partial_A \log f - i \partial_A g_2) e_{12345}
+ \sqrt{2} \partial_A \log g_3 (e_5 + e_{1234}) + i g_3^{-1} g_2 \mathcal{D}_A (1 - e_{12345}) + \sqrt{2} \mathcal{D}_A (e_5 + e_{1234}) = 0.$$  

To proceed one can use the results in appendix C to substitute for $\mathcal{D}_A (e_5 + e_{1234})$ and $i \mathcal{D}_A (1 - e_{12345})$. The resulting expressions have been given in appendix E. It turns out that in solving the resulting linear systems one has to distinguish between $g_2 = 0$ and $g_2 \neq 0$. We will first consider the simplest case with $g_2 = 0$. This splits up in two subcases, depending on whether $g_1$ vanishes or not. If $g_1 = 0$, the results have been given in [10]. Here we shall summarize the $g_1 \neq 0$ case. The conditions on the function $g_3$ and $g_1$ are

$$\partial_0 g_3 = 0, \quad \partial_0 g_3 = \partial_3 \log f, \quad \partial_3 g_3 = \partial_3 \log f,$$  

and

$$\partial_3 g_1 = \partial_0 \log f.$$  

We are left with the two equations (E.27) and (E.28), the first of which gives the time dependence of the function $g_1$:

$$g_3^{-1} \partial_0 g_1 - i \Omega_{0,05} + i \Omega_{0,05} = 0.$$  

The conditions on the $\Omega_{0,0i}$ components are

$$\Omega_{0,05} = -2 \partial_5 \log f, \quad \Omega_{0,0i} = -2 \partial_i \log f.$$  

The conditions on the $\Omega_{0,ij}$ components are

$$\Omega_{0,5i} = \Omega_{0,5i} = \Omega_{0,5i} = \Omega_{0,5i} = 0, \quad \Omega_{0,ij} = \frac{i}{4} (\Omega_{5,\rho \lambda}, -\Omega_{5,\rho \lambda}, \bar{\epsilon}^{\rho \lambda, \rho \lambda}) (6.9)$$

and the traceless part of $\Omega_{0,ij}$ is not determined. The conditions on the $\Omega_{i,j}$ components are

$$\Omega_{i,j} = 0, \quad \Omega_{i,j} = -\Omega_{0,05, \sigma \rho}, \quad \Omega_{i,j} = -\frac{1}{2} \Omega_{0,0i}. \quad (6.10)$$

In addition, we have

$$\Omega_{i,j} = -\Omega_{i,j}, \quad \Omega_{i,j} = -\Omega_{i,j}. \quad (6.11)$$

The conditions on the $\Omega_{5,ij}$ components are

$$\Omega_{5,5j} = \Omega_{5,5j}, \quad \Omega_{5,5j} = \Omega_{5,5j}, \quad \Omega_{5,5} = -\Omega_{0,0}, \quad \Omega_{5,5} = -\Omega_{0,0}. \quad (6.12)$$

We also have the following relations

$$\Omega_{0,ij} + \Omega_{0,ij} - \Omega_{0,55} + \Omega_{5,55} + 2\Omega_{5,\lambda} - 2\Omega_{5,\lambda}, \quad (6.13)$$

and

$$\Omega_{5,5} = \Omega_{5,5}, \quad \Omega_{5,5} = \Omega_{5,5}, \quad \Omega_{5,5} = \Omega_{5,5}. \quad (6.14)$$

All fluxes are expressed in terms of the geometry via the relations summarized in appendix B. In addition, we find that

$$F_{i,j} = -2i \Omega_{0,ij}, \quad F_{i,j} = \frac{1}{2} \Omega_{0,ij} \bar{g}^{\rho \lambda, \rho \lambda}. \quad (6.15)$$

This concludes the analysis of the $N = 2$ SU(4) case with $g_2 = 0$.

The Killing spinor equations for the case with $g_2 \neq 0$ are rather different from those with $g_2 = 0$. The solution of this linear system is described in section E.2. Here, we summarize the conditions on functions that determine the spinors, the geometry and the fluxes.

The conditions on the functions $f, g_1, g_2$ and $g_3$ are

$$\Delta g_3 = 0, \quad g_3^{-1} \Delta g_2 - (\Omega_{0,05} + \Omega_{0,05}) = 0, \quad g_3^{-1} \Delta g_1 - i(\Omega_{0,05} - \Omega_{0,05}) = 0, \quad (6.16)$$

The conditions on the geometry are

$$g_3^{-1} g_2 \left[ \Omega_{0,05} + 2\Omega_{0,\rho}, \rho \right] + 2\Omega_{0,\rho}, \rho = 0, \quad g_3^{-1} g_2 \left[ \Omega_{0,05} + 2\Omega_{0,\rho}, \rho \right] + 2\Omega_{0,\rho}, \rho = 0, \quad (6.17)$$

$$\Omega_{i,j} = \Omega_{i,j}, \quad \Omega_{i,j} = \Omega_{i,j}, \quad \Omega_{i,j} = \Omega_{i,j}, \quad \Omega_{i,j} = \Omega_{i,j}. \quad (6.17)$$
The conditions on the fluxes that arise from the requirement of $N = 1$ supersymmetry have been summarized in appendix B. The additional conditions that arise for two supersymmetries are

$$F_{\rho\sigma35} = -2i\Omega_{\rho,\sigma}\phi$$
$$F_{\beta\kappa\lambda2} = \frac{8i}{3} \Omega_{\beta,\kappa,\lambda} g + \frac{1}{2} (\Omega_{\rho,\sigma}/\Omega_{(\rho,\sigma,\tau)} g) \epsilon e_{\rho/\sigma} g_{\lambda2}.$$  (6.18)

### 6.2. The geometry of spacetime

Using the results of [10], it is straightforward to compute the spacetime form bilinears associated with the Killing spinors (6.1) for both $g_2 = 0$ and $g_2 \neq 0$. These are a zero form

$$\alpha(\eta_1, \eta_2) = -2f g_2.$$  (6.19)

three 1-forms

$$\kappa(\eta_1, \eta_1) = -2f^2 e^0,$$
$$\kappa(\eta_1, \eta_2) = -2f g_1 e^0 + 2\sqrt{2} f g_3 e^{10},$$
$$\kappa(\eta_2, \eta_2) = -2(g_1^2 + g_2^2 + g_3^2) e^0 + 4\sqrt{2} f g_3 e^{10} + 4\sqrt{2} g_2 g_3 e^5.$$  (6.20)

three 2-forms,

$$\omega(\eta_1, \eta_2) = 2f^2 \omega,$$
$$\omega(\eta_2, \eta_2) = 2(g_1^2 + g_2^2) \omega + 4g_3^2 \omega - 4\sqrt{2} g_1 g_3 e^0 \wedge e^5 + 4\sqrt{2} g_2 g_3 e^0 \wedge e^{10},$$
$$\omega(\eta_1, \eta_2) = 2f g_1 \omega - 2\sqrt{2} f g_3 e^0 \wedge e^5.$$  (6.21)

one 3-form

$$\xi(\eta_1, \eta_2) = -2\sqrt{2} f g_3 \omega^{SU(4)} \wedge e^5.$$  (6.22)

one 4-form

$$\zeta(\eta_1, \eta_2) = \frac{f g_2}{\sqrt{2}} \omega \wedge \omega + 2\sqrt{2} f g_3 \{\text{Im} \epsilon - e^0 \wedge \omega^{SU(4)} \wedge e^{10}\},$$  (6.23)

and three 5-forms

$$\tau(\eta_1, \eta_1) = 2f^2 \{\text{Im} \epsilon + \frac{1}{2} e^0 \wedge \omega \wedge \omega\},$$
$$\tau(\eta_1, \eta_2) = 2f g_1 \{\text{Im} \epsilon + \frac{1}{2} e^0 \wedge \omega \wedge \omega\} + 2f g_2 \text{Re} \epsilon - 2\sqrt{2} f g_3 \{e^0 \wedge \omega \wedge \omega^{SU(4)} \wedge e^{10}\},$$
$$\tau(\eta_2, \eta_2) = 2g_1 \{\text{Im} \epsilon + \frac{1}{2} e^0 \wedge \omega \wedge \omega\} + 2g_2 \{-\text{Im} \epsilon + \frac{1}{2} e^0 \wedge \omega \wedge \omega\} + 4g_1 g_2 \text{Re} \epsilon - 4\sqrt{2} g_1 g_3 \{e^0 \wedge \omega \wedge \omega^{SU(4)} \wedge e^{10}\} + 4\sqrt{2} g_2 g_3 \{e^0 \wedge \omega \wedge \omega^{SU(4)} \wedge e^5\}.$$  (6.24)

where

$$\omega = -e^1 \wedge e^6 - e^2 \wedge e^7 - e^3 \wedge e^8 - e^4 \wedge e^9 - e^5 \wedge e^{10},$$
$$\tilde{\omega} = e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9 - e^5 \wedge e^{10},$$

$$\omega^{SU(4)} = e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9,$$
$$\epsilon = (e^1 + ie^6) \wedge \cdots \wedge (e^5 + ie^{10}),$$
$$\epsilon^{SU(4)} = (e^1 + ie^6) \wedge \cdots \wedge (e^4 + ie^9) \wedge (-e^5 + ie^{10}).$$  (6.25)
All the above forms specify the geometry of spacetime. Instead of investigating the properties of all spacetime form bilinears, we shall mostly focus on the properties of the three 1-form bilinears. It is convenient to rescale them with a factor of 1/2 and rewrite them in the Hermitian frame basis as
\[ \kappa(\eta_1, \eta_1) = -f^2 e^0, \]
\[ \kappa(\eta_1, \eta_2) = -f g_1 e^0 - i f g_3 e^5 + i f g_3 e^5, \]
\[ \kappa(\eta_2, \eta_2) = -(g_1^2 + g_2^2 + 2g_3^2)e^0 + 2g_3(g_2 - i g_1)e^5 + 2g_3(g_2 + i g_1)e^5. \]

The fact that the magnetic part of the gauge field equation is implied by the Bianchi identity plus the electric part of the gauge field equation can be easily verified using the conditions summarized in (6.16) and (6.17). In addition it turns out that, if Y and Z mutually commute, i.e. \([X, Y] = 0\) and similarly for the rest of the pairs. In addition, we have that
\[ g(X, X) = -f^4, \quad g(Y, Y) = -f^3 g_1^2 + 2f^2 g_3^3, \]
\[ g(Z, Z) = -[g_1^2 + g_2^2 - 2g_3^2]^2, \quad g(X, Y) = -f g_1, \]
\[ g(X, Z) = -[g_1^2 + g_2^2 + 2g_3^2]f^2, \quad g(Y, Z) = -f g_1^3 + 4f g_1 g_3^2. \]

The vector field X is timelike while as one expects Z is timelike or null.

The Killing vector fields do not commute. So in general one cannot adapt coordinates to all three Killing vectors. The form of the metric can be written by adapting coordinates to one of the Killing vector fields say X.

7. Solutions to the integrability conditions

7.1. \(N = 1\) backgrounds with \(SU(5)\) invariant spinors

The Killing spinor is \(\eta = f(1 + e^{12345})\). The integrability condition on this spinor implies the vanishing of the combination
\[ (\mathcal{L}_A)_a \cdot \alpha - (\mathcal{L}_A e^{12345})_a \cdot \alpha = 0, \]
for \(i = 0, \ldots, 5\). These integrability conditions guarantee the vanishing of the Bianchi components \(B_{\alpha \beta \gamma} \) and \(B_{\alpha \beta \gamma} \). The remaining field equations are subject to the relations

\[ 0 = E_{\alpha \beta} - 12i L_{0a} \epsilon^a_{\alpha \beta}, \quad \epsilon^a_{\alpha \beta} = -20i B_{\alpha \beta} e^a_{\alpha \beta}, \]
\[ 0 = E_{\alpha \beta} - 180i B_{\alpha \beta} e^a_{\alpha \beta}, \]
\[ 0 = E_{\alpha \beta} - 6i g_{\alpha \beta} L_{0a} e^a_{\alpha \beta} + 18i L_{0a} e^a_{\alpha \beta} - 60i g_{\alpha \beta} B_{\alpha \beta} e^a_{\alpha \beta} + 360i B_{\alpha \beta} e^a_{\alpha \beta} - 10 B_{\alpha \beta} e^a_{\alpha \beta}, \]
\[ 0 = E_{\alpha \beta} - 18i L_{0a} e^a_{\alpha \beta} + (80g_{\alpha \beta} B_{\alpha \beta} e^a_{\alpha \beta} - 30 B_{\alpha \beta} e^a_{\alpha \beta}) e^a_{\alpha \beta}, \]
\[ 0 = L_{\alpha \beta} e^a_{\alpha \beta} + 20 B_{\alpha \beta} e^a_{\alpha \beta} \epsilon^a_{\alpha \beta} e^a_{\alpha \beta} \epsilon^a_{\alpha \beta}, \]
\[ 0 = L_{\alpha \beta} + 20 B_{\alpha \beta} \epsilon^a_{\alpha \beta} e^a_{\alpha \beta} + 2i L_{0a} e^a_{\alpha \beta} e^a_{\alpha \beta}. \]

These can be solved by explicitly imposing the components
\[ \{L_{\alpha \beta}, L_{0a}, B_{\alpha \beta} e^a_{\alpha \beta}, B_{\alpha \beta} e^a_{\alpha \beta}, B_{\alpha \beta} e^a_{\alpha \beta}, B_{\alpha \beta} e^a_{\alpha \beta}\}. \]

Therefore, in the \(N = 1\) \(SU(5)\) case, one still needs to impose the above components of the Bianchi identity plus the electric part of the gauge field equation to satisfy all field equations.

\[ \text{9 The fact that the magnetic part of the gauge field equation is implied by } N = 1 \text{ SU}(5) \text{ supersymmetry and the Bianchi identity can also be derived from the bilinear formalism of } [8]. \]
7.2. $N = 2$ backgrounds with $SU(5)$ invariant spinors

The Killing spinors are
\[ \eta_1 = f_1 \eta^{SU(5)}, \quad \eta_2 = f_2 \eta^{SU(5)} + f_3 \theta^{SU(5)}, \quad (7.8) \]
with $f_1$ and $f_3$ non-vanishing. Independent of the functions $f_1$, $f_2$, $f_3$, the integrability conditions arising from these spinors are
\[ (\mathcal{I}_A)_{\alpha_1 \ldots \alpha_5} = (\mathcal{I}_A e^{12345})_{\alpha_1 \ldots \alpha_5} = 0, \quad (7.9) \]
for $i = 0, \ldots, 5$. From these conditions one can derive that the field equations which do not automatically vanish are
\[ \{ E_{00}, E_{0a}, E_{a\beta}, L_{0\alpha\beta}, L_{a\beta\gamma}, B_{0a\beta\gamma\delta}, B_{a\beta\gamma\delta\epsilon} \}, \quad (7.10) \]
where the tilde means traceless part, subject to the relations
\[ 0 = E_{00} - 12i L_{0a}^a, \quad (7.11) \]
\[ 0 = E_{0a} - 180i B_{a\beta}^\beta \gamma^\nu, \quad (7.12) \]
\[ 0 = E_{a\beta} - 6i g_{a\beta} L_{0\nu}^\nu + 18i L_{0a\beta}, \quad (7.13) \]
\[ 0 = L_{a\beta\gamma} - 20 g_{a\beta} B_{\gamma\delta}^\delta \epsilon + 20 B_{a\beta\gamma\delta\epsilon}, \quad (7.14) \]

One can solve these equations by explicitly checking
\[ \{ L_{0a\beta}, B_{0a\beta\gamma\delta}, B_{a\beta\gamma\delta\epsilon} \}, \quad (7.15) \]
after which all other field equations are implied.

7.3. $N = 4$ backgrounds with $SU(4)$ invariant spinors

The Killing spinors are
\[ \eta_1 = f_1 \eta^{SU(5)}, \quad (7.16) \]
\[ \eta_2 = f_2 \eta^{SU(5)} + f_3 \theta^{SU(5)}, \quad (7.17) \]
\[ \eta_3 = f_4 \eta^{SU(5)} + f_5 \theta^{SU(5)} + f_6 \eta^{SU(4)}, \quad (7.18) \]
\[ \eta_4 = f_7 \eta^{SU(5)} + f_8 \theta^{SU(5)} + f_9 \eta^{SU(4)} + f_{10} \theta^{SU(4)}, \quad (7.19) \]
with $f_1$, $f_3$, $f_6$ and $f_{10}$ non-vanishing. In this case, independent of the ten spacetime functions, the integrability conditions (2.9) of the four Killing spinors correspond to the conditions
\[ (\mathcal{I}_A)_{\alpha_1 \ldots \alpha_5} = (\mathcal{I}_A e^{12345})_{\alpha_1 \ldots \alpha_5} = 0, \quad (7.20) \]
\[ (\mathcal{I}_A e^5)_{\alpha_1 \ldots \alpha_5} = (\mathcal{I}_A e^5_{12345})_{\alpha_1 \ldots \alpha_5} = 0, \quad (7.21) \]
for $i = 0, \ldots, 4$. These imply all but the following field equations:
\[ \{ E_{00}, E_{a\beta}, E_{55}, L_{055}, L_{a\rho\nu}, B_{0a\mu\nu\rho}, B_{a\mu\nu\rho\sigma}, B_{a\mu\nu\rho\sigma\delta} \}, \quad (7.22) \]
(where the tilde means traceless part) subject to the relations
\[ 0 = E_{00} - 12i L_{055}, \quad (7.23) \]
\[ 0 = E_{a\beta} - 6i g_{a\beta} L_{055}, \quad (7.24) \]
\[ 0 = E_{55} + 12i L_{055}, \quad (7.25) \]
\[ 0 = L_{a\rho\nu} + 20 B_{a\rho\nu\rho}^\rho. \quad (7.26) \]
These can be solved by requiring the components of the Bianchi identity in (7.22) to vanish and by imposing the field equation $L_{055} = 0$. 

8. Resolved membranes

In this section we will consider the class of solutions which admit Killing spinors as in (7.19) with the restrictions

$$f_2 = f_4 = f_5 = f_7 = f_8 = f_9 = 0$$

as analysed in [10]. We shall show here that the most general solution is a resolved rotating membrane wrapped on a two-torus $T^2$.

We will start by summarizing the conditions for $SU(4)$ backgrounds to admit the four Killing spinors. Firstly, this background has three commuting Killing vectors, one of which is timelike and the other two are spacelike. Because of this, we introduce three coordinates $x^i, i = 0, 1, 2$, adapted to these three Killing vector fields which are thought of as the worldvolume coordinates of a membrane. We define the frames \( e_i \)

$$e^i = f^2(dx^i + \alpha^i),$$

where the $\alpha^i$ are independent of the worldvolume coordinates and only take values in the 8D transverse space. Since the Killing vector fields are orthogonal and of the same length, the metric in this frame reads

$$ds^2 = f^4 g_{ij}(dx^i + \alpha^i)(dx^j + \alpha^j) + 2g_{\lambda\bar{\mu}}e^\lambda e_{\bar{\mu}},$$

where $g_{ij} = \text{diag}(-1, 1, 1, 1)$, $g_{\lambda\bar{\mu}} = \delta_{\lambda\bar{\mu}}$ and $e^\lambda, \lambda = 1, \ldots, 8$, is a Hermitian frame for the metric on the space of orbits $B$ of the three isometries. The eight-dimensional space $B$ is complex and is identified with the transverse space of the membrane. The Killing spinor equations imply [10] that the 4-form field strength can be written as

$$F = -d(e^0 \wedge e^1 \wedge e^2) + \tilde{F}(2,2),$$

where $\tilde{F}(2,2)$ is a traceless (2, 2) form on $B$ and so self-dual. In addition, the components $\Omega_{i,AB}$ of the spin connection satisfy the conditions

$$\Omega_{i,ijk} = 0, \quad \Omega_{i,j\lambda} = 2g_{ij}\partial_\lambda \log f,$$

$$\Omega_{i,\lambda\mu} = 0, \quad \Omega_{i,\lambda\bar{\mu}} = 0, \quad \Omega_{i,\lambda\bar{\mu}} = -\frac{1}{2}f^2(h_{\lambda\bar{\mu}}),$$

while the $\Omega_{\lambda,AB}$ components read

$$\Omega_{\lambda,ij} = 0, \quad \Omega_{\lambda,i\mu} = 0,$$

$$\Omega_{\lambda,\bar{\mu}\bar{\nu}} = -\frac{1}{2}f^2(h_{\lambda\bar{\mu}}), \quad \Omega_{\lambda,\lambda\bar{\mu}\bar{\nu}} = 0,$$

$$\Omega_{\lambda,\lambda\bar{\mu}} = -2g_{\lambda\bar{\mu}}\partial_{\bar{\nu}} \log f, \quad \Omega_{\lambda,\mu\bar{\nu}} = \partial_\lambda \log f,$$

where $da_{ij} = \partial_0 \alpha^i_j - \partial_\mu \alpha^i_j$.

We now turn to the field equations. As explained in section 7.3, the integrability conditions for the $N = 4$ $SU(4)$-invariant Killing spinors imply that one only needs to impose the vanishing of a number of components of the Bianchi equation and one component of the $F$ field equation. Specifically, one has to impose the vanishing of

$$[L_{012}, B_{i\lambda\mu\nu}, B_{j\lambda\mu\nu}],$$

10 The Hermitian frame directions $e^1$ and $e^3$ are related to $e^1$ and $e^2$ as

$$e^3 = (e^1 + ie^2)/\sqrt{2}, \quad e^5 = (e^1 - ie^2)/\sqrt{2}.$$
where the tilde denotes traceless part of the associated quantity. Let us first consider the Bianchi identity. The components with a world-volume index imply independence of $F^{(2,2)}$ of the world-volume Killing directions:

$$\partial_i \tilde{F}^{(2,2)} = 0.$$  

(8.12)

The remaining $(2, 3)$ component of the Bianchi equation implies $F^{(2,2)}$ to be a closed form on $B$, i.e.

$$d_B \tilde{F}^{(2,2)} = 0,$$

(8.13)

where $d_B = e^i \partial_i + e^i \partial_i$. Since $\tilde{F}(2, 2)$ is self-dual, $\tilde{F}(2, 2)$ is also co-closed and so a harmonic $(2, 2)$ form on $B$. The only component of the field equation that one needs to impose is $L_{012}$, which implies

$$D^I \partial_I \log f = \frac{f^4}{6} g_{i j} \omega^i \cdot \omega^j + \frac{1}{12} F^{(2,2)} \cdot \tilde{F}^{(2,2)}, \quad I = \lambda, \bar{\lambda},$$

(8.14)

where the $D_I$ is the Levi-Civita connection of $d\omega^2 = 2\gamma_{\alpha\beta} e^\alpha e^\beta$ of the transverse space $B$ and the inner product of a $k$-form $\phi = \frac{1}{k!} \phi \cdot \phi^{I_1 \ldots I_k}$.

This solution can be written in a more familiar form by rescaling the $e^\lambda$ as $e^\lambda = f^{-1} \tilde{e}^\lambda$ and identifying $H = f^{-6}$. The metric and flux then become

$$d\omega^2 = H^{-2/3} g_{i j} (d\omega^i + \alpha^i)(d\omega^j + \alpha^j) + 2H^{-1/3} g_{\alpha\beta} e^\lambda e^\beta,$$

$$\mathcal{F} = -d(e^0 \wedge e^1 \wedge e^2) + \tilde{F}^{(2,2)},$$

(8.15)

where $\alpha^i = H^{-1/3} (d\omega^i + \alpha^i)$. The components of the spin connection of the rescaled metric $d\tilde{\omega}^2 = g_{i j} \tilde{\omega}^i \cdot \tilde{\omega}^j$ and frame satisfy

$$\tilde{\Omega}^i_{i, \mu} = 0, \quad \tilde{\Omega}^i_{i, \mu} = 0, \quad \tilde{\Omega}^i_{i, \mu} = 0,$$

(8.16)

and hence $d\tilde{\omega}^2$ is a Calabi–Yau metric. The Laplacian equation for $f$ in terms of $H$ becomes

$$-\tilde{D}^i \partial_i H = g_{i j} \omega^i \cdot \omega^j + \frac{1}{12} F^{(2,2)} \cdot \tilde{F}^{(2,2)},$$

(8.17)

where $\tilde{D}_I$ is the Levi-Civita connection of $d\tilde{\omega}^2$ and inner products have been taken with respect to $d\tilde{\omega}^2$. Equation (8.17) for $\alpha^i = 0$ has been explored before in the context of resolved membranes, see e.g., [21–24]. A case $\alpha^0 \neq 0$ has been considered in [8], corresponding to a rotating resolved M2-brane but $d\omega$ was taken to be $(2, 0)$ and $(0, 2)$ instead of $(1, 1)$ and traceless that we have here. The case $\alpha^i \neq 0$, $i = 0, 1, 2$ has been considered in [25] and solutions were found with specific transverse spaces. The interpretation of these solutions is resolved rotating membranes wrapped on a two-torus $T^2$ which fibres over the transverse space $B$. Here we have shown that this is the most general supersymmetric configuration with four supersymmetries for the $SU(4)$-invariant Killing spinors (7.19) subject to (8.1).

It is worth pointing out that on the right-hand side of (8.17) the contribution of the rotation has a different sign from those of the wrapping of the membrane on $T^2$. We can use this to give necessary and sufficient conditions for the existence of non-singular solutions on any compact, connected without boundary, Calabi–Yau manifold $B$. First observe that $\alpha^i$ can be thought of as the connection of a line bundle $L^{(i)}$ over the Calabi–Yau manifold $B$. Since the curvature $\beta^i = d\alpha^i$ is $(1, 1)$, this line bundle is holomorphic. In addition, a necessary and sufficient condition for $L^{(i)}$ to admit a connection $\alpha^i$ such that the curvature $\beta^i$ is traceless, i.e. to satisfy the Donaldson condition, is

$$\int_B \beta^i \wedge \omega \wedge \alpha \wedge \omega = 0,$$

(8.18)

where $\alpha$ is the Kähler form of the Calabi–Yau metric $d\tilde{\omega}^2$ (see, e.g., [20]). Next turn to (8.17). Since the left-hand side of (8.17) is a Laplacian on $H$, one can use Hodge theory to invert the
Laplace operator and solve for $H$. A necessary and sufficient condition for the existence of a (well-defined) solution on $B$ is that the right-hand side expression in (8.17) is orthogonal to the harmonic zero-forms. This translates to the condition
\[ \int_B \text{dvol}\left( g_{ij} \beta^i \cdot \beta^j + \frac{1}{2} \tilde{F}^{(2,2)} \cdot \tilde{F}^{(2,2)} \right) = 0, \tag{8.19} \]
where $\text{dvol}$ is the volume form of the Calabi–Yau metric $d\tilde{\beta}^2$. This relation can be rewritten using the traceless condition of $\beta^i$ and the self-duality condition of $\tilde{F}^{(2,2)}$ as
\[ \int_B \left( -\frac{1}{2} g_{ij} \beta^i \wedge \beta^j \wedge \hat{\omega} \wedge \hat{\omega} + \frac{1}{2} \tilde{F}^{(2,2)} \wedge \tilde{F}^{(2,2)} \right) = 0. \tag{8.20} \]
Observe that the above relation depends on the cohomology class of $\hat{\omega}$, $\beta^i$ and $\tilde{F}^{(2,2)}$ and it may be interpreted as a cancellation of membrane, rotation and wrapping fluxes when integrated over the compact Calabi–Yau manifold $B$. The above condition is the sum of squares and therefore if there is no rotation, i.e. $\alpha^0 = 0$, then $\beta^i = 0$, $i = 1, 2$ and $\tilde{F}^{(2,2)} = 0$ and so there is only a trivial solution, i.e. $H = \text{const}$. However if $\alpha^0 \neq 0$, then there are solutions of (8.20) for non-trivial $\beta^i$ and $\tilde{F}^{(2,2)}$ provided that (8.18) and (8.20) are satisfied. Furthermore observe that $H$ is determined up to a constant in (8.17), and it is bounded because $B$ is compact. Therefore, it is always possible to choose $H$ to be positive, $H > 0$. In such cases, one can find a non-singular solution of 11-dimensional supergravity preserving four supersymmetries with metric and flux given in (8.15). In the context of M-theory, there are corrections to the flux field equation. In particular, one has [26, 27]
\[ d \ast F - \frac{1}{2} F \wedge F = \kappa X_k, \tag{8.21} \]
where $\kappa$ are some units, $X_k = \frac{1}{\pi^2} \left( p_1^2 - 4p_2^2 \right)$, and $p_1$ and $p_2$ are the Pontryagin classes of spacetime. It is clear that in this case the condition (8.20) is modified as
\[ \int_B \left( -\frac{1}{2} g_{ij} \beta^i \wedge \beta^j \wedge \hat{\omega} \wedge \hat{\omega} + \frac{1}{2} \tilde{F}^{(2,2)} \wedge \tilde{F}^{(2,2)} + \kappa X_k \right) = 0, \tag{8.22} \]
and this new condition is required for the existence of non-singular solutions. To summarize, the conditions (8.18) and (8.20) are necessary and sufficient for the existence of a resolved (non-singular), rotating and wrapped membrane solution of 11-dimensional supergravity with transverse space a compact, connected without boundary Calabi–Yau manifold. Incidentally, these types of solutions resemble those found in the context of flux tubes in [28] and it may be worth exploring this further.

9. Concluding remarks

The Killing spinor equations of any background of 11-dimensional supergravity theory have been reduced to the evaluation of the supercovariant derivative $D \sigma_I$ on six types of spinors $\sigma_I$. The expressions for all $D \sigma_I$ have been given in both a timelike and a null spinor basis. In addition the integrability conditions of the Killing spinor equations which encode the field equations of the theory have been investigated. It is shown that these integrability conditions can be expressed as a linear combination of the six types of spinors $\mathcal{I} \sigma_I$. We give the expressions of all $\mathcal{I} \sigma_I$ again in both a timelike and a null spinor basis. As a result, one can determine the field equations of the theory which arise as integrability conditions of the Killing spinor equations. In this way, one can specify the minimal set of additional field equations required for a supersymmetric configuration to be a solution of the supergravity field equations. We have also presented some examples to illustrate our construction. In particular, we have given a class of resolved, rotating, wrapped membranes with transverse
space a Calabi–Yau manifold preserving four supersymmetries. We have also shown that these are the most general supersymmetric solutions for a class of $SU(4)$ invariant Killing spinors.

This paper has given the systematics of how to classify all supersymmetric solutions in 11 dimensions. The above construction can be used to reduce the Killing spinor equations to a linear system for the fluxes, geometry and spacetime derivatives of the functions that determine the Killing spinors. This system is of increasing complexity with the number of Killing spinors that a background admits. Nevertheless, we have determined all the coefficients and unknowns of this linear system for all supersymmetric backgrounds. A similar conclusion applies for the linear system that arises in the integrability conditions which determines the minimal set of field equations which should be satisfied. Therefore, the classification of supersymmetric backgrounds is associated with two linear systems, one is related to the Killing spinor equations and the other to the field equations.

The two linear systems can always be solved. A question arises whether they are tractable for all supersymmetric backgrounds. In the general situation, they will be rather involved. However, some simplifications may occur. The Killing spinors can be simplified by using the gauge symmetry $Spin(10, 1)$ of the supercovariant connection to put them at particular directions in space of spinors, i.e. to put them in a canonical or normal form. This typically reduces the number of functions that the spinors depend on. Further simplifications occur whenever the spinors have some residual symmetry, i.e. some non-trivial stability subgroup in $Spin(10, 1)$. This happens in many supersymmetric backgrounds of interest and in particular in those that appear in compactifications with fluxes. A detailed discussion of this has appeared in [10]. However, it is known that there are backgrounds for which the Killing spinors have the identity in $Spin(10, 1)$ as stability subgroup. This phenomenon occurs even for backgrounds with two supersymmetries. For such backgrounds there is no apparent simplification. Nevertheless, it may be possible in practice to solve these linear systems in general in many cases. For example, since the systems are linear this can be done with the help of computers.

Acknowledgments

The work of UG is funded by The Swedish Research Council and in addition the research of both UG and DR is funded by the PPARC grant PPA/G/O/2002/00475.

Appendix A. Spinors from forms

A.1. A timelike basis

The realization of Majorana spinors of $Spin(10, 1)$ in terms of forms has been described in [10] (see also [29–31]). Here we shall summarize some of the features of the construction. For a detailed account of the construction, see [10].

Let $e_1, \ldots, e_{10}$ be an orthonormal basis in $V = \mathbb{R}^{10}$. Next consider the subspace $U = \mathbb{R}^5$ in $V$ generated by $e_1, \ldots, e_5$. The Euclidean inner product on $V$ can be extended to a Hermitian inner product in $V_{\mathbb{C}} = V \otimes \mathbb{C}$ and then restricted in $U_{\mathbb{C}} = U \otimes \mathbb{C}$ denoted by $\langle \cdot, \cdot \rangle$, i.e. on $U_{\mathbb{C}}$ is

$$\langle z^i e_i, w^j e_j \rangle = \sum_{i=1}^5 (z^i)^* w^i,$$

where $(z^i)^*$ is the standard complex conjugate of $z^i$. The space of $Spin(10)$ Dirac spinors is $\Delta_{\mathbb{C}} = \Lambda^*(U_{\mathbb{C}})$. The above inner product can be easily extended to $\Delta_{\mathbb{C}}$ and it is called the Dirac
inner product on the space of Spin(10) spinors. The gamma matrices act on $\Delta_c$ as

$$\Gamma_7 \eta = e_1 \wedge \eta + e_2 \wedge \eta, \quad i \leq 5$$

$$\Gamma_8 \eta = i e_1 \wedge \eta - i e_2 \wedge \eta, \quad i \leq 5,$$

(A.2)

where $e_{i,j}$ is the adjoint of $e_i \wedge$ with respect to $\langle \cdot , \cdot \rangle$. Moreover we have that the Weyl representations of Spin(10) are $\Delta_{10}^+ = \Lambda^{even} U_C$ and $\Delta_{10}^- = \Lambda^{odd} U_C$. Clearly $\Gamma_i : \Delta_{10}^\pm \to \Delta_{10}^\mp$.

The linear maps $\Gamma_i$ are Hermitian with respect to the inner product $\langle \cdot \rangle$, $\langle \eta, \eta' \rangle = \langle \eta, \Gamma_i \eta' \rangle$, and satisfy the Clifford algebra relations $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij}$. The Majorana $\text{Pin}(10)$ invariant inner product on $\Delta_c$ is

$$B(\eta, \theta) = \langle B(\eta^*), \theta \rangle,$$

(A.3)

where the linear map denoted with the same symbol as the inner product is $B = \Gamma_6 \ldots \Gamma_1$ and $\text{Pin}(10)$. $B$ is skew-symmetric.

The spinor representations of Spin(10, 1) are constructed by first setting $\Gamma_0 = \Gamma_1 \ldots \Gamma_4$. It is easy to see that $\Gamma_0^2 = -1$ as expected and that $\Gamma_0$ anticommutes with $\Gamma_i$. The Dirac representation of Spin(10, 1) is the same as that of Spin(10, 1). The Dirac inner product on Spin(10, 1) representation, $\Delta_c$, is

$$D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle$$

(A.4)

and the $\text{Pin}(10)$ Majorana inner product (A.3) extends to the Majorana inner product of Spin(10, 1). It remains for us to impose the Majorana condition on the Spin(10, 1) representation, $\Delta_c$. This is

$$\eta^* = \Gamma_0 B(\eta), \quad \eta \in \Delta_c.$$

(A.5)

The Spin(10, 1) Majorana spinors $\Delta_{32} = \{ \eta \in \Delta_c \text{ s.t. } \eta^* = \Gamma_0 B(\eta) \}$. For completeness, the spacetime form bilinears associated with the Majorana spinors $\eta, \theta$ are

$$\sigma(\eta, \theta) = \frac{1}{k!} B(\eta, \Gamma_{A_1 \ldots A_k} \theta) e^{A_1} \wedge \cdots \wedge e^{A_k}, \quad k = 0, \ldots, 9; z.$$  

(A.6)

Another ingredient in solving the Killing spinor equations and their integrability conditions is the construction of a basis in the space of Spin(10, 1) Dirac spinors $\Delta_c$. It turns out that

$$\Delta_c = \sum_{k=0}^{5} \Lambda^{0,k} \cdot 1,$$

(A.7)

where denotes Clifford multiplication. Therefore

$$\Gamma^{0,1} \cdot 1, \quad k = 0, \ldots, 5$$

(A.8)

is a basis in the space of spinors $\Delta_c$, where

$$\Gamma_\alpha = \frac{1}{\sqrt{2}} (\Gamma_\alpha + i \Gamma_{\alpha+5}), \quad \Gamma^\alpha = g^{\alpha\beta} \Gamma_\beta, \quad \alpha = 1, \ldots, 5,$$

$$\Gamma_\alpha = \frac{1}{\sqrt{2}} (\Gamma_\alpha - i \Gamma_{\alpha+5}), \quad \Gamma^\alpha = g^{\alpha\beta} \Gamma_\beta, \quad \alpha = 1, \ldots, 5,$$

(A.9)

and $g_{\alpha\beta} = \delta_{\alpha\beta}$. The Clifford algebra relations in this basis are $\Gamma_\alpha \Gamma_\beta + \Gamma_\beta \Gamma_\alpha = 2g_{\alpha\beta}$, $\Gamma_\alpha \Gamma_\beta \Gamma_\gamma \Gamma_\delta = \Gamma_\alpha \Gamma_\delta \Gamma_\gamma \Gamma_\beta + \Gamma_\beta \Gamma_\gamma \Gamma_\delta \Gamma_\alpha$. Observe that $(\Gamma_j + i \Gamma_{j+5})1 = 0$ and similarly $(\Gamma_j - i \Gamma_{j+5})e_1 \wedge \cdots \wedge e_5 = 0$. In particular,

$$e_{12345} = \frac{1}{8 \cdot 5!} e_{a_1 \ldots a_5} \Gamma^{a_1 \ldots a_5} 1,$$

(A.10)

where $e_{12345} = \sqrt{5}$. We shall extensively use this basis for spinors to analyse the Killing spinor equations and their integrability conditions. As in the above equation, throughout the paper we suppress the sign of the Clifford multiplication, e.g. instead of $\Gamma^\alpha \cdot 1$ we write $\Gamma^\alpha$.  

11 From here on, we shall adopt the notation to denote the tenth direction with $z = 10$. 

Systematics of M-theory spinorial geometry
A.2. A null basis

An alternative way to construct the Majorana representation of Spin(10, 1) is to begin from the Spin(9, 1) spinor representations. The realization of the spinor representations of Spin(9, 1) in terms of forms has been presented in [11]. We shall repeat this construction and then shall explain the application to Spin(10, 1). Let $U = \mathbb{R}(e_1, \ldots, e_5)$ be a vector space spanned by $e_1, \ldots, e_5$ orthonormal vectors. The space of Dirac Spin(9, 1) spinors is $\Delta_c = \Lambda^+(U \otimes \mathbb{C})$. The gamma matrices are represented on $\Delta_c$ as

\begin{align}
\Gamma_0 \eta &= -e_5 \wedge \eta + e_5 \wedge \eta, \\
\Gamma_5 \eta &= e_5 \wedge \eta + e_5 \wedge \eta, \\
\Gamma_i \eta &= e_i \wedge \eta + e_i \wedge \eta, \\
\Gamma_{5i} \eta &= i e_i \wedge \eta - i e_i \wedge \eta,
\end{align}

where $\wedge$ is the adjoint of $\wedge$ with respect to the (auxiliary) inner product

$$
\langle z^a e_a, w^b e_b \rangle = \sum_{a=1}^5 (z^a)^* w^a,
$$

on $U \otimes \mathbb{C}$ and then extended to $\Delta_c$. $(z^a)^*$ is the standard complex conjugate of $z^a$. The gamma matrices have been chosen such that $\{\Gamma_i; i = 1, \ldots, 9\}$ are Hermitian and $\Gamma_0$ is anti-Hermitian with respect to the (auxiliary) inner product $(\cdot, \cdot)$.

The above gamma matrices satisfy the Clifford algebra relations $\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}$ with respect to the Lorentzian inner product as expected. The Dirac inner product on the space of spinors $\Delta_c$ is $D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle$ and the $Pin(9, 1)$ invariant (Majorana) inner product is

$$
B(\eta, \theta) = \langle B(\eta)^*, \theta \rangle,
$$

where $B = \Gamma_{06789}$. Observe that $B(\eta, \theta) = -B(\theta, \eta)$.

The Dirac representation of Spin(10, 1) is identified with the Dirac representation of Spin(9, 1). The tenth gamma matrix is chosen as

$$
\Gamma_7 = -\Gamma_{0123456789}.
$$

One can verify that $\Gamma_7^2 = 1$ and that anticommutes with the rest of gamma matrices. The Dirac inner product is $D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle$, i.e. the same as that of the Spin(9, 1). In addition, since $B$ is a $Pin(9, 1)$ invariant inner product, it extends to a $Spin(10, 1)$ invariant Majorana inner product. It remains for us to construct the Majorana representation of Spin(10, 1). For this, we impose the condition that the Dirac conjugate is equal to the Majorana conjugate. It turns out that it is convenient to choose as a reality condition

$$
\eta = -\Gamma_0 B(\eta^*),
$$

or equivalently

$$
\eta^* = \Gamma_{06789} \eta.
$$

The map $C = \Gamma_{06789}$ is also called charge conjugation matrix. In this basis the $(Spin(7) \times \mathbb{R}^8) \times \mathbb{R}$-invariant Majorana spinor is $1 + e_{1234}$. The simplicity of this representative of the Spin(10, 1) orbit with stability subgroup $(Spin(7) \times \mathbb{R}^8) \times \mathbb{R}$ suggests that if one of the Killing spinors is null, then it may be simpler to use this basis to solve the Killing spinor equations.

To solve the Killing spinor equations of 11-dimensional supergravity, it is convenient to use an oscillator basis in the space of spinors $\Delta_c$. For this write

$$
\Gamma_a = \frac{1}{\sqrt{2}}(\Gamma_a + i \Gamma_{a5}), \quad \Gamma_{\pm} = \frac{1}{\sqrt{2}}(\Gamma_5 \pm \Gamma_0), \quad \Gamma_{a} = \frac{1}{\sqrt{2}}(\Gamma_a - i \Gamma_{a5}).
$$
Observe that the Clifford algebra relations in the above basis are $$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2g_{AB}$$, where the non-vanishing components of the metric are $$g_{a\bar{b}} = \delta_{a\bar{b}}; g_{+ -} = 1$$. In addition, we define $$\Gamma^a = g^{a\bar{b}} \Gamma_{\bar{b}}$$. The 1 spinor is a Clifford vacuum, $$\Gamma_\alpha \Gamma_0 = 1$$, and the representation $$\Delta_\epsilon$$ can be constructed by acting on 1 with the creation operators $$\Gamma^\alpha, \Gamma^\ast$$ or equivalently any spinor can be written as

$$\eta = \sum_{\bar{a}} \frac{1}{k!} \phi_i \cdots \phi_k \Gamma^a_1 \cdots \Gamma^a_1 1, \quad \bar{a} = \bar{a}_+, \bar{a}_-,$$

(A.18)
i.e. $$\Gamma^{a_1} \cdots a_1$$, for $$k = 0, \ldots, 5$$, is a basis in the space of (Dirac) spinors.

Observe that both the timelike basis and the null basis of $$\Delta_\epsilon$$ are oscillator bases. Because of this, it is straightforward to adapt the results we have obtained in this paper for the timelike basis for the Killing spinor equations and for their integrability conditions to the null basis.

### Appendix B. N = 1 backgrounds

In this appendix we summarize the solution of the Killing spinor equations for backgrounds that admit one Killing spinor with stability subgroup $$SU(5)$$, i.e. the spacetime 1-form bilinear is timelike. This case has been analysed in [8]. The results, in the form we summarize them below, have appeared in [10].

The conditions on the geometry are

$$\Omega_{0,ij} = \Omega_{i,0j}, \quad 2\partial_0 \log f + \Omega_{0,0\alpha} = 0$$

$$\Omega^\alpha_{\beta\gamma} - \Omega^\gamma_{\beta\alpha} - \Omega^\beta_{\alpha\gamma} = 0.$$  

(B.1)
The electric part of the flux is expressed in terms of the geometry as

$$G_{\alpha\beta\gamma} = -2i\Omega_{\alpha\beta\gamma} + 2ig_{\alpha\beta}\Omega_{0,0\gamma}$$

$$G_{\alpha\beta\gamma\delta} = 6i\Omega_{\alpha\beta\gamma\delta}$$

(B.2)
and the magnetic part of the flux is

$$F_{\alpha\beta\gamma\delta} = \frac{1}{4}(-\Omega_{0,0\alpha} + 2\Omega_{\alpha\beta}\delta)\gamma^\delta\gamma_{\beta\gamma\delta}$$

$$F_{\alpha\beta\gamma\delta} = \frac{1}{2i}(4\Omega_{\alpha\beta\gamma}\delta + 2ig_{\alpha\beta}\Omega_{0,0\delta})\gamma^\delta$$

(B.3)

$$F_{\alpha\beta\gamma\delta} = \frac{1}{4}(-\Omega_{0,0\alpha} + 2\Omega_{\beta\gamma}\delta)\gamma^\delta\gamma_{\beta\gamma\delta} + 3\Omega_{\beta\gamma\delta}\delta + 2\Omega_{\beta\gamma}\delta + 12i\Omega_{\alpha\beta\gamma}\delta + 12i\Omega_{\alpha\beta\gamma}\delta.$$

The traceless (2, 2) part of $$F$$ is not determined by the Killing spinor equations.

The conditions on the geometry imply that the 1-form $$k^\alpha = -f^2 \kappa = f^2 e^0$$ is a timelike Killing vector field and the space of orbits of this vector field has an $$SU(5)$$ structure with $$W_5 + 2df = 0$$, where

$$W_5_{\alpha} = \Omega_{\alpha\beta} - \Omega_{\alpha\beta}, \quad (B.4)$$
is a Gray–Hervella type of class [32]. We use the above results to investigate backgrounds with two supersymmetries.

### Appendix C. Killing spinor equations in the canonical basis

To derive the linear system associated with the Killing spinor equations for the geometry, fluxes and spacetime derivatives of $$f, g, u, v, w, z$$ one has to expand $$D_{\alpha} \sigma_I$$ in the Hermitian basis (A.8) and use (3.2). This computation can be simplified in various ways. First, it is not necessary to compute both $$D_{\alpha} \sigma_I$$ and $$D_{a} \sigma_I$$ because since the spinors $$\epsilon$$ are real the equations
derived from $D_\alpha \epsilon$ are complex conjugate to those of $D_\beta \epsilon$ and so are not independent\(^{12}\). In addition, since $D_0 \epsilon$ is real only half of the relations are independent. These are chosen to be along the basis elements $1, \Gamma^\alpha_1$ and $\Gamma^\alpha \beta_1$. The remaining are related to these by complex conjugation followed by dualization with the $\epsilon$ form $\epsilon$. So again, we shall give only the independent conditions. We remark that one can use these relations between the equations of the linear system to provide a useful check of the result.

It is intended that the results of this appendix to be used as a manual to derive the linear system associated with the Killing spinor equations of any number of spinors. Because of this, we first state the action of the supercovariant derivative $D_A \sigma_I$ on the appropriate irreducible spinor $\sigma_I$ as a title of its subsection. Then we expand $D_A \sigma_I$ in the canonical basis. On the left column, we state the basis element of the oscillator basis ($A, 8$), and in the right column we give the associated component.

\[ C.1. \ D_A 1 \]

Evaluating $D_0 1$ and expanding the result in the basis $(A, 8)$, we find $D_0 1$

\[ 1 \ : \ \frac{1}{2} \Omega_{0, \gamma} \gamma - \frac{i}{24} F_{\gamma \delta} \delta \]

\[ \Gamma^\beta 1 \ : \ \frac{i}{2} \Omega_{0, \gamma} \gamma + \frac{1}{6} G_{\beta \gamma} \gamma \]

\[ \Gamma^\beta_1 \beta_2 1 \ : \ \frac{1}{4} \Omega_{0, \gamma} \gamma - \frac{i}{24} F_{\gamma \delta} \delta. \]

\[ \text{(C.1)} \]

Similarly, computing $D_\alpha 1$ and expanding the result in the basis $(A, 8)$, we get $D_\alpha 1$

\[ 1 \ : \ \frac{1}{2} \Omega_{0, \gamma} \gamma + \frac{i}{12} G_{\beta \gamma} \gamma \]

\[ \Gamma^\beta 1 \ : \ \frac{i}{2} \Omega_{0, \gamma} \gamma + \frac{i}{12} F_{\gamma \delta} \delta \]

\[ \Gamma^\beta_1 \beta_2 1 \ : \ \frac{1}{4} \Omega_{0, \gamma} \gamma + \frac{i}{12} G_{\beta \gamma} \gamma \]

\[ \Gamma^\beta_1 \beta_2 \beta_3 1 \ : \ \frac{1}{72} F_{\gamma \delta} \gamma \]

\[ \Gamma^\beta_1 \beta_2 \beta_3 \beta_4 1 \ : \ 0 \]

\[ \Gamma^\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 1 \ : \ 0 \]

\[ \text{(C.2)} \]

As we have explained the expressions for the remaining basis elements in (C.1) and for $D_\alpha 1$ can be recovered from the above using complex conjugation.

\[ C.2. \ D_A \epsilon_{12345} \]

The time component of the Killing spinor equations yields $D_0 \epsilon_{12345}$

\[ 1 \ : \ 0 \]

\[ \Gamma^\beta 1 \ : \ -\frac{i}{144} F_{\gamma_{\delta_1} \gamma_{\delta_2} \gamma_{\delta_3} \gamma_{\delta_4} \gamma_{\delta_5}} \gamma_{\delta_6} \gamma_{\delta_7} \gamma_{\delta_8} \gamma_{\delta_9} \gamma_{\delta_{10}} \]

\[ \text{(C.3)} \]

\[ \Gamma^\beta_1 \beta_2 1 \ : \ -\frac{1}{72} G_{\gamma_{\delta_1} \gamma_{\delta_2} \gamma_{\delta_3} \gamma_{\delta_4} \gamma_{\delta_5} \gamma_{\delta_6} \gamma_{\delta_7} \gamma_{\delta_8} \gamma_{\delta_9} \gamma_{\delta_{10}}} \]

\(^{12}\) Observe though that $\sigma_I$ are complex spinors and so this complex conjugation relation does not apply for them.
Similarly the $D_\alpha e_{12345}$ yields

$$D_\alpha e_{12345}$$

1 : $-\frac{1}{72} \bar{\epsilon}_\alpha \gamma_\nu \gamma_\rho \gamma_\lambda F_{\nu \rho \gamma \lambda}$

$\Gamma^\beta 1$ : $-\frac{1}{36} \bar{\epsilon}_\alpha \beta \gamma_\nu \gamma_\rho G_{\nu \rho \gamma \lambda}$

$\Gamma^{\hat{\beta}, \beta 2} 1$ : $-\frac{1}{48} \bar{\epsilon}_a \beta_i \beta_j \gamma_\nu \gamma_\rho \gamma_\lambda F_{\nu \rho \gamma \lambda} - \frac{1}{48} a_{\gamma \nu \rho \gamma \lambda} \bar{\epsilon}_a \beta_i \beta_j \gamma_\nu \gamma_\rho \gamma_\lambda \beta_i \beta_j$

$\Gamma^{\hat{\beta}, \hat{\beta}, \beta 3} 1$ : $-\frac{i}{144} \bar{\epsilon}_a \beta_i \beta_j \gamma_\nu \gamma_\rho \gamma_\lambda G_{\gamma \rho \gamma \lambda} + \frac{i}{96} a_{\gamma \nu \rho \gamma \lambda} \bar{\epsilon}_a \beta_i \beta_j \gamma_\nu \gamma_\rho \gamma_\lambda \beta_i \beta_j$

$\Gamma^{\hat{\beta}, \hat{\beta}, \hat{\beta}, \beta 4} 1$ : $-\frac{1}{192} a_{\gamma \nu \rho \gamma \lambda} \bar{\epsilon}_a \beta_i \beta_j \beta_k \beta_\lambda = \frac{1}{24^2 \cdot 4} F_{\gamma \rho \gamma \lambda} \bar{\epsilon}_a \beta_i \beta_j \beta_k \beta_\lambda - \frac{1}{384} a_{\gamma \nu \rho \gamma \lambda} \bar{\epsilon}_a \beta_i \beta_j \beta_k \beta_\lambda$

$\Gamma^{\hat{\beta}, \hat{\beta}, \hat{\beta}, \hat{\beta}, \beta 5} 1$ : $\frac{1}{8 \cdot 5!} \left[ -\frac{1}{2} a_{\gamma \nu \rho \gamma \lambda} + \frac{i}{4} G_{\gamma \rho \gamma} \right] \bar{\epsilon}_a \beta_i \beta_j \beta_k \beta_\lambda$\hspace{1cm} (C.4)

where $\bar{\epsilon}_{\bar{\alpha}_1 \cdots \bar{\alpha}_s} = \sqrt{2} \bar{\epsilon}_{\bar{\alpha}_1 \cdots \bar{\alpha}_s}$ and $\epsilon_{1 \cdots 3} = 1$.

C.3. $\sqrt{2}D_\alpha e_\kappa$

We split up $\alpha$ into $13 \rho$ and $k$, where $\rho$ are the remaining four indices: $\rho = (1, \ldots, \hat{k}, \ldots, 5)$. The time component of the Killing spinor equations yields

$$\sqrt{2}D_\alpha e_\kappa$$

1 : $2 \left( -\frac{i}{2} \Omega_{0,0k} + \frac{i}{12} G_{\kappa \lambda} \right)$

$\Gamma^\tau 1$ : $2 \left( \frac{i}{2} \Omega_{0, tk} + \frac{i}{12} F_{tk \lambda} \right)$

$\Gamma^\tau 1$ : $\frac{1}{2} \Omega_{0, \lambda} = \frac{1}{2} \Omega_{0, \kappa k} + \frac{i}{24} F_{\kappa k \lambda} \sigma^\lambda - \frac{i}{12} F_{\kappa k \lambda}$\hspace{1cm} (C.5)

$\Gamma^{\tau \tau \tau \tau} 1$ : $\frac{1}{6} G_{\tau \tau \tau \kappa}$

$\Gamma^{\rho \kappa 1} : -\frac{i}{2} \Omega_{0, \rho \kappa} + \frac{i}{6} G_{\rho \kappa}$

The different spatial directions, i.e. $\rho$ and $\hat{k}$, yield

$$\sqrt{2}D_\rho e_\kappa$$

1 : $-\frac{i}{4} \Omega_{\rho, 0k} + \frac{i}{6} F_{\rho \kappa \lambda}$

$\Gamma^\tau 1$ : $\Omega_{\rho, tk} = \frac{i}{6} G_{\rho \kappa}$

$\Gamma^\tau 1$ : $\frac{1}{2} \Omega_{\rho, \lambda} = \frac{1}{2} \Omega_{\rho, \kappa k} - \frac{i}{12} G_{\rho \kappa \lambda} + \frac{i}{12} G_{\rho \kappa}$\hspace{1cm} (C.6)

$\Gamma^{\kappa \tau \tau \tau} 1$ : $\frac{1}{12} F_{\rho \kappa \tau \tau \tau k}$

13 Note that $\hat{k}$ is not an index here but rather a fixed label for a particular spinor $e_\kappa$. The same holds for the labels of all other spinors $e_{r, i, j}$ in these tables.
Next we find that \( \sqrt{2} D_\ell e_k \) gives

\[
\begin{align*}
\Gamma^1 & : -\frac{i}{2} \Omega_{\ell,0t} + \frac{1}{12} F_{\rho t,\lambda} + \frac{1}{12} F_{\rho t,kl} \\
\Gamma^{\alpha t} & : 0 \\
\Gamma^{\alpha k} & : \left( \frac{1}{4} \Omega_{\rho, t,\tau} - \frac{i}{24} G_{\rho t,\tau} \right) \\
\Gamma^{\alpha \tau t} & : 0 \\
\Gamma^{\alpha \tau k} & : \frac{1}{72} F_{\rho t,\tau,\tau t} \\
\Gamma^{\alpha \tau \tau \tau} & : 0.
\end{align*}
\]

\[(C.6)\]

Next we find that \( \sqrt{2} D_\ell e_k \) gives

\[
\begin{align*}
1 & : - \left( i \Omega_{\ell,0t} + \frac{1}{12} F_{\kappa,\lambda,\mu} + \frac{1}{3} F_{\kappa,\lambda,\tau} \right) \\
\Gamma^t & : \Omega_{\ell, tk} - \frac{i}{6} G_{t,\tau,\lambda} - \frac{i}{3} G_{t, k,\kappa} \\
\Gamma^k & : \frac{1}{2} \Omega_{\ell, \kappa,\lambda} - \frac{1}{2} \Omega_{\ell, k,\kappa} - \frac{i}{4} G_{kl,\tau} \\
\Gamma^{\tau t} & : -\frac{1}{12} F_{t,\tau,\kappa} - \frac{1}{6} F_{t,\tau, kl} \\
\Gamma^{\tau k} & : -\frac{i}{2} \Omega_{\ell, 0t} - \frac{1}{4} F_{\kappa, kl,\lambda} \\
\Gamma^{\tau \tau} & : 0 \\
\Gamma^{\tau t} & : -\frac{i}{36} G_{t, \tau, kl} \\
\Gamma^{\tau k} & : \frac{1}{4} \Omega_{\ell, k,\kappa} - \frac{i}{8} G_{t, \tau, kl} \\
\Gamma^{\tau \tau t} & : -\frac{1}{144} F_{t, \tau, kl} \\
\Gamma^{\tau \tau k} & : -\frac{1}{24} F_{t, \tau, k} \\
\Gamma^{\tau \tau \tau} & : 0.
\end{align*}
\]

\[(C.7)\]

\[C.4. \quad \sqrt{2} D_\alpha e_{\tau_1 \ldots \tau_4}\]

We split the indices \( \alpha \) into \( \rho \) and \( k \), where \( \rho = (i_1, \ldots, i_4) \) and \( k \) is the missing fifth coordinate. In addition we will use the Levi-Civita symbol \( \epsilon_{\rho_1 \ldots \rho_4} \) which is defined by \( \tilde{\epsilon}_{i_1 \ldots i_4} = \sqrt{2} \). The time component of the Killing spinor equations yields

\[
\begin{align*}
1 & : -\frac{i}{72} F_{\lambda_1, \lambda_2, \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \\
\Gamma^t & : -\frac{1}{18} G_{\lambda_1, \lambda_2, \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \\
\Gamma^k & : 0 \\
\Gamma^{\tau t} & : -\frac{1}{4} \Omega_{\lambda_1, \lambda_2} - \frac{i}{24} F_{\lambda_1, \lambda_2, \sigma} + \frac{i}{24} F_{\lambda_1, \lambda_2, kl} \epsilon^{\lambda_1 \lambda_2} \epsilon_{t_1 t_2} \\
\Gamma^{\tau k} & : \frac{i}{36} F_{\lambda_1, \lambda_2, \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3}.
\end{align*}
\]

\[(C.8)\]
The different spatial directions, i.e. \( \hat{\rho} \) and \( \hat{\kappa} \), yield

\[
\sqrt{2} D_{\hat{\rho}} e_{i_1 \ldots i_4}
\]

\[
1 : - \frac{i}{18} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4}
\]

\[
\Gamma^t 1 : - \left( \frac{1}{12} F_{\hat{\rho} \lambda_1 \lambda_2} + \frac{i}{12} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} - \frac{1}{12} g_{\hat{\rho} \lambda_1} F_{\lambda_2 \lambda_3 \lambda_4} \right) \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

\[
\Gamma^t 1 : - \frac{i}{18} g_{\hat{\rho} \lambda_1} F_{\lambda_2 \lambda_3 \lambda_4} \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4}
\]

\[
\Gamma^{\tau_1 t} 1 : - \frac{i}{2} \left( \frac{1}{4} \Omega_{\hat{\rho} \lambda_1 \lambda_2} + \frac{i}{8} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} - \frac{i}{12} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} \right) \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

\[
\Gamma^{\tau_1 t} 1 : - \frac{i}{24} \left( \frac{1}{4} \Omega_{\hat{\rho} \lambda_1 \lambda_2} + \frac{i}{8} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} - \frac{i}{12} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} \right) \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

\[
\Gamma^{\tau_1 t} 1 : - \frac{i}{2} \left( \frac{1}{4} \Omega_{\hat{\rho} \lambda_1 \lambda_2} + \frac{i}{8} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} - \frac{i}{12} g_{\hat{\rho} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} \right) \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

Next we turn to \( \sqrt{2} D_{\hat{\kappa}} e_{i_1 \ldots i_4} \) to find

\[
\sqrt{2} D_{\hat{\kappa}} e_{i_1 \ldots i_4}
\]

\[
1 : 0
\]

\[
\Gamma^t 1 : - \frac{i}{36} F_{\hat{\kappa} \lambda_1 \lambda_2 \lambda_3} \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

\[
\Gamma^t 1 : 0
\]

\[
\Gamma^{\tau_1 t} 1 : - \frac{i}{12} \left( \frac{1}{4} \Omega_{\hat{\kappa} \lambda_1 \lambda_2} + \frac{i}{8} g_{\hat{\kappa} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} \right) \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

\[
\Gamma^{\tau_1 t} 1 : 0
\]

\[
\Gamma^{\tau_1 t} 1 : \frac{1}{12} \left( \frac{1}{4} \Omega_{\hat{\kappa} \lambda_1 \lambda_2} + \frac{i}{8} g_{\hat{\kappa} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} \right) \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

\[
\Gamma^{\tau_1 t} 1 : 0
\]

\[
\Gamma^{\tau_1 t} 1 : \frac{1}{96} \left( \frac{1}{4} \Omega_{\hat{\kappa} \lambda_1 \lambda_2} + \frac{i}{8} g_{\hat{\kappa} \lambda_1} G_{\lambda_2 \lambda_3 \lambda_4} \right) \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]

\[
\Gamma^{\tau_1 t} 1 : \frac{1}{192} \Omega_{\hat{\kappa} \lambda_1 \lambda_2} \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \tau
\]
C.5. $\mathcal{D}_A e_{ij}$

We split the indices $\alpha$ into $p = (i, j)$ and $a$, which contains the remaining three indices. We also define $\epsilon_{ij} = 1$. Then the time component of the Killing spinor equations yields

$$\mathcal{D}_0 e_{ij}$$

$$1 : -2 \left( \frac{1}{4} \Omega_{0,pq} - \frac{i}{24} F_{pq c} \right) \epsilon^{pq}$$

$$\Gamma^a 1 : -\frac{1}{6} G_{\bar{a}pq} \epsilon^{pq}$$

$$\Gamma^p 1 : \left( \frac{i}{2} \Omega_{0,0q} - \frac{1}{6} G_{qr} + \frac{1}{6} G_{qa} \right) \epsilon^q_p$$

$$\Gamma^{\bar{a}b} 1 : \frac{i}{24} F_{\bar{a}bpq} \epsilon^{pq}$$

$$\Gamma^{\bar{a}p} 1 : \left( \frac{i}{2} \Omega_{\bar{a},0q} - \frac{i}{12} F_{\bar{a}q b} + \frac{i}{12} F_{\bar{a}q r} \right) \epsilon^q_p$$

$$\Gamma^{\bar{a}q} 1 : \frac{1}{4} \left( \frac{i}{2} \Omega_{\bar{a},a} - \frac{1}{2} \Omega_{\bar{a},r} - \frac{i}{24} F_{\bar{a}a b} + \frac{i}{12} F_{\bar{a}a r} \right) \epsilon^a_p$$

The different spatial directions, i.e. $\bar{a}$ and $\bar{p}$, yield

$$\mathcal{D}_a e_{ij}$$

$$1 : -2 \left( \frac{1}{4} \Omega_{a,qr} + \frac{i}{24} G_{aqr} \right) \epsilon^{qr}$$

$$\Gamma^b 1 : -\frac{1}{12} F_{\bar{a}bqr} \epsilon^{qr}$$

$$\Gamma^q 1 : \left( \frac{i}{2} \Omega_{a,0r} + \frac{1}{12} F_{arb} - \frac{1}{12} F_{ars} \right) \epsilon^a_q$$

$$\Gamma^{\bar{b}c} 1 : 0$$

$$\Gamma^{\bar{a}q} 1 : \frac{1}{24} F_{\bar{a}cqs} \epsilon^s_q$$

$$\Gamma^{\bar{a}q} 1 : \frac{1}{4} \left( \frac{i}{2} \Omega_{\bar{a},b} + \frac{i}{12} F_{\bar{a}bc} - \frac{1}{12} F_{\bar{a}bs} \right) \epsilon^{qr}$$

$$\Gamma^{\bar{b}cd} 1 : 0$$

$$\Gamma^{\bar{b}c} 1 : \frac{i}{12} F_{\bar{a}bcs} \epsilon^s_q$$

$$\Gamma^{\bar{a}q} 1 : \frac{1}{24} \left( \frac{i}{2} \Omega_{\bar{a},bc} + \frac{i}{12} G_{abc} \right) \epsilon^{qr}$$

$$\Gamma^{\bar{b}cdq} 1 : 0$$

$$\Gamma^{\bar{b}cd} 1 : 0$$
and

\[ \mathcal{D}_p \epsilon_{ij} \]

\[ 1 : -2 \left( \frac{1}{4} \Omega_{p,qr} + \frac{i}{12} G_{pqr} - \frac{i}{12} \delta_{pq} G_{ra} \right) \epsilon^{qr} \]

\[ \Gamma^b_1 : \left( \frac{1}{12} F_{pqrb} + \frac{1}{12} \delta_{pq} F_{bce} \right) \epsilon^{qr} \]

\[ \Gamma^q_1 : \left( \frac{i}{2} \Omega_{b,0e} + \frac{1}{4} F_{pbe} - \frac{1}{6} F_{pqr} = \frac{1}{24} \delta_{qr} F_{bce} + \frac{1}{12} \delta_{qr} F_{b}^{bc} \right) \epsilon^{br} q \]

\[ \Gamma^{bc}_1 : \frac{i}{12} G_{bce} \epsilon_q^p \]

\[ \Gamma^{bq}_1 : \left( \frac{i}{2} \Omega_{b,0e} + \frac{1}{4} G_{pbe} + \frac{i}{12} \delta_{pq} G_{bce} - \frac{i}{12} \delta_{pq} G_{bc} \right) \epsilon^{pq} \]

\[ \Gamma^{qr}_1 : \frac{i}{4} \left( \frac{i}{2} \Omega_{b,0e} + \frac{1}{4} F_{pbe} - \frac{1}{4} \delta_{qr} G_{bce} - \frac{i}{4} \delta_{qr} G_{b} \right) \epsilon_{qr} \]

\[ \Gamma^{bcd}_1 : -\frac{i}{36} F_{bcdq} \epsilon_q^p \]

\[ \Gamma^{bq}_1 : \left( \frac{1}{4} \Omega_{p,0e} + \frac{1}{8} G_{pbe} \right) \epsilon_{qr} \]

\[ \Gamma^{bq}_{pq} : \frac{1}{96} F_{pbe} \epsilon_{qr} \]

respectively.

C.6. \( \mathcal{D}_A \epsilon_{k\ell m} \)

We split the indices \( \alpha \) into \( a = (k, l, m) \) and \( p \), containing the remaining two indices. The three-dimensional Levi-Civita symbol \( \tilde{\epsilon}_{abc} \) is defined by \( \tilde{\epsilon}_{ijm} = \sqrt{2} \). The time component of the Killing spinor equations yields

\[ \mathcal{D}_n \epsilon_{k\ell m} \]

\[ 1 : -\frac{1}{18} G_{abc} \tilde{\epsilon}_{abc} \]

\[ \Gamma^a_1 : \left( \frac{1}{8} \Omega_{0,bc} - \frac{i}{24} F_{bcd} + \frac{i}{24} F_{bcp} \right) \epsilon^{bc} \tilde{a} \]

\[ \Gamma^b_1 : \frac{i}{36} F_{abc} \epsilon^{abc} \]

\[ \Gamma^{ab}_1 : \frac{1}{4} \left( -\frac{i}{2} \Omega_{0,0c} - \frac{1}{6} G_{cd} + \frac{1}{6} G_{c} \left( F_{bcp} \right) \right) \tilde{\epsilon}_{ab} \]

\[ \Gamma^{ap}_1 : \frac{1}{12} G_{bcp} \tilde{\epsilon}_{bc} \tilde{a} \]

\[ \Gamma^{bp}_1 : 0. \]
The different spatial directions, i.e. $\bar{a}$ and $\bar{p}$, yield

$$D_{\bar{a}}\epsilon_{k\bar{l}m}$$

1 : $-\frac{1}{36}F_{abcd} - \frac{1}{24}g_{ab}F_{cdp}\epsilon^{bcd}$

$\Gamma^{b1}$ : $-\left(\frac{1}{4}\Omega_{b,ad} - i\frac{1}{8}G_{abcd} + i\frac{1}{12}g_{ac}G_{dp}\right)\epsilon^{cd}_{\bar{b}}$

$\Gamma^{q1}$ : $\frac{i}{12}G_{bq\bar{e}}\epsilon^{b\bar{e}}$

$\Gamma^{k1}$ : $\frac{1}{4}\left(-\frac{i}{2}\Omega_{k,0d} - \frac{i}{4}F_{0de} + \frac{i}{4}F_{0dq} - \frac{i}{12}g_{ad}(F_{e}^{e}p^{\bar{f}} - 2F_{e}^{e}p^{\bar{d}} + F_{p}^{p}q^{q})\right)\epsilon^{d}_{\bar{b}}$

$\Gamma^{q1}$ : $\frac{i}{24}F_{bqqr}\epsilon^{b\bar{e}}$

$\Gamma^{c1}$ : $\frac{1}{24}\left(\frac{i}{2}\Omega_{c,0d} + \frac{i}{4}F_{0de} - \frac{i}{4}F_{0dq}\right)\epsilon^{b}_{\bar{c}}$

$\Gamma^{bq1}$ : $\frac{i}{4}\left(-\frac{i}{2}\Omega_{b,ad} + \frac{i}{12}g_{ad}G_{q}\epsilon^{c}_{\bar{d}} + \frac{i}{12}g_{ad}G_{qr}\epsilon^{c}_{\bar{e}}\right)$

$$= \frac{1}{24}\frac{i}{4}G_{q}\epsilon^{c}_{\bar{d}} + \frac{1}{24}\frac{i}{4}G_{qr}\epsilon^{c}_{\bar{e}}$$

$\Gamma^{c1}$ : $\frac{1}{24}\left(\frac{i}{2}\Omega_{c,0d} + \frac{i}{4}F_{0de} - \frac{i}{4}F_{0dq}\right)\epsilon^{b}_{\bar{c}}$

and

$$D_{\bar{p}}\epsilon_{k\bar{l}m}$$

1 : $-\frac{1}{36}F_{bcde}\epsilon^{b\bar{c}d}$

$\Gamma^{b1}$ : $-\left(\frac{1}{4}\Omega_{b,ad} - i\frac{1}{24}G_{bcd}\right)\epsilon^{cd}_{\bar{b}}$

$\Gamma^{q1}$ : $0$

$\Gamma^{k1}$ : $\frac{1}{4}\left(-\frac{i}{2}\Omega_{k,0d} - \frac{i}{12}F_{pde} + \frac{i}{12}F_{pq}\right)\epsilon^{d}_{\bar{b}}$

$\Gamma^{q1}$ : $0$

$\Gamma^{b1}$ : $\frac{1}{24}\left(-\frac{i}{2}\Omega_{b,0d} + \frac{i}{12}G_{p}\epsilon^{e}_{\bar{d}} + \frac{i}{12}G_{pq}\right)\epsilon^{d}_{\bar{b}}$

$\Gamma^{q1}$ : $0$

$\Gamma^{b1}$ : $\frac{1}{24}\left(-\frac{i}{2}\Omega_{b,0d} + \frac{i}{12}G_{pq}\right)\epsilon^{d}_{\bar{b}}$

$\Gamma^{q1}$ : $0$
\[ \Gamma^{bq} 1 = 0 \]
\[ \Gamma^{bc}q 1 = -\frac{1}{24} \left( \frac{i}{2} \Omega^{b}_{a,0q} - \frac{1}{12} F_{\rho q}^{\epsilon} \right) \tilde{e}_{cd} \]
\[ \Gamma^{bq}q 1 = 0 \]
\[ \Gamma^{bc}q 1 = \frac{1}{96} \Omega_{p,q}^{\epsilon} \tilde{e}_{abc} \]

respectively.

**Appendix D. Integrability conditions in canonical basis**

**D.1. \( \mathcal{I}_A 1 \)**

Inserting 1 in (2.9) and expanding in the different \( \Gamma \)-structures, one finds that the integrability conditions with \( A = 0 \) give rise to

\[ \mathcal{I}_0 1 \]
\[ 1 : -iE_{00} - 12L_{0a}^\alpha - 120B_{0a}^\alpha \rho ^6 \]
\[ \Gamma^a 1 : E_{00} - 61L_{0a}^\beta - 60iB_{0a}^\beta \rho ^y \gamma \]
\[ \Gamma^{a\beta} 1 : -6L_{0a}^\beta - 120B_{0a}^\beta \rho ^y \gamma . \]

For \( A = \bar{a} \) we find

\[ \mathcal{I}_{\bar{a}} 1 \]
\[ 1 : -iE_{00} - 6L_{0a}^\rho ^6 - 60B_{0a}^\rho ^\gamma \gamma \]
\[ \Gamma^\rho 1 : E_{00} + 61L_{0a}^\rho ^6 + 120iB_{0a}^\rho ^\gamma \gamma \]
\[ \Gamma^{\rho \beta} 1 : -3L_{0a}^\rho \beta - 60B_{0a}^\rho \beta \delta \]
\[ \Gamma^{\rho \beta} \epsilon 1 : 20iB_{0a}^\rho \beta \delta \]
\[ \Gamma^{\rho \beta \epsilon} 1 : -5B_{0a}^\rho \beta \delta \epsilon \]
\[ \Gamma^{\rho \beta \epsilon} \epsilon 1 : 0. \]

**D.2. \( \mathcal{I}_A e_{12345} \)**

For the basis element \( e_{12345} \) we find the following integrability conditions for \( A = 0 \):

\[ \mathcal{I}_0 e_{12345} \]
\[ 1 : 4iB_{0a\beta \gamma \delta \epsilon} \tilde{e}_{\alpha \beta \gamma \delta \epsilon} \]
\[ \Gamma^a 1 : -20B_{0a\beta \gamma \delta \epsilon} \tilde{e}_{\alpha \beta \gamma \delta \epsilon} \]
\[ \Gamma^{a\beta} 1 : \frac{1}{2} \left( -iL_{\gamma \delta \epsilon} + 20iB_{\gamma \delta \epsilon \phi} \right) \tilde{e}_{\alpha \beta \gamma \delta \epsilon \phi}. \]

For \( A = \bar{a} \) we find

\[ \mathcal{I}_{\bar{a}} e_{12345} \]
\[ 1 : 20i\tilde{g}_{a\beta \gamma \delta \epsilon} B_{0a\beta \gamma \delta \epsilon} \tilde{e}_{\epsilon \beta \gamma \delta \epsilon \phi} \]
\[ \Gamma^\beta 1 : 2 \left( \tilde{g}_{\alpha \gamma} L_{\delta \epsilon \phi} + 20\tilde{g}_{\alpha \gamma} B_{\delta \epsilon \phi \kappa} - 15B_{\delta \epsilon \phi \kappa} \right) \tilde{e}_{\beta \gamma \delta \epsilon \phi} \]
\[ \Gamma^{\beta \gamma} 1 : -\frac{1}{2} \left( 3i\tilde{g}_{\alpha \delta} L_{\epsilon \phi} - 60i\tilde{g}_{\alpha \delta} B_{\epsilon \phi \kappa} - 60iB_{\epsilon \phi \kappa} \right) \tilde{e}_{\rho \gamma \delta \epsilon \phi} \]
\[
\Gamma^{\bar{\rho}\bar{\lambda}} 1 : - \frac{1}{12} \left( -6g_{a\ell} L_{\alpha\phi}^{\lambda} \right. - 9L_{a\phi \phi}^{\lambda} + 60g_{a\phi} B_{\phi \lambda}^{\lambda \phi} + 180B_{a\phi \phi}^{\lambda} \bigg) \tilde{\epsilon}_{\bar{\rho}\bar{\lambda}}^{\phi} \\
\Gamma^{\bar{\rho}\lambda} 1 : \frac{1}{96} \left( E_{\alpha\phi} - 6i g_{a\phi} L_{\alpha\phi}^{\lambda} - 18i L_{a\phi \phi}^{\lambda} + 60i g_{a\phi} B_{\phi \lambda}^{\lambda \phi} + 360i B_{a\phi \phi}^{\lambda} \bigg) \tilde{\epsilon}_{\bar{\rho}\lambda}^{\phi} \\
\Gamma^{\hat{\beta}_{\lambda} \bar{\rho}_{\lambda}} 1 : \left( iE_{\alpha\phi} + 18L_{a\phi}^{\lambda} - 180B_{a\phi \phi}^{\lambda} \bigg) \tilde{\epsilon}_{\hat{\beta}_{\lambda} \bar{\rho}_{\lambda}}^{1} \\
\]  
where \( \tilde{\epsilon}_{1...5} = \sqrt{2} \).

**D.3. \( \sqrt{2}T_{A} e_{k} \)**

Next we consider the contributions from \( \sqrt{2}e_{k} \). We split up \( \alpha \) into \( 14 \) \( \rho \) and \( k \), where \( \rho \) are the remaining four indices: \( \rho = (1, \ldots , \bar{k}, \ldots , 5) \). The \( A = 0 \) integrability conditions amount to

\[
\sqrt{2}T_{A} e_{k} \]

The \( A = \lambda \) integrability conditions on \( \sqrt{2}e_{k} \) yield

\[
\sqrt{2}T_{\hat{\lambda}} e_{k} \]

Finally, the \( A = \bar{k} \) integrability conditions give the following contributions:

\[
\sqrt{2}T_{\bar{k}} e_{k} \]

---

\( ^{14} \) Note that \( k \) is not an index here but rather a fixed label for a particular spinor \( e_{k} \). The same holds for the labels of all other spinors \( e_{i}, \ldots , i \), in these tables.
Next we consider $\sqrt{2} e_{i_1...i_4}$. We split the indices $\alpha$ into $\rho$ and $k$, where $\rho = (i_1, \ldots, i_4)$ and $k$ is the missing fifth coordinate. In addition we will use the Levi-Civita symbol $\varepsilon_{\rho_1...\rho_4}$ which is defined by $\varepsilon_{i_1...i_4} = \sqrt{3}$. The $A = 0$ integrability conditions are

$$
\sqrt{2} \varepsilon_{i_1...i_4}
$$

The $A = \lambda$ integrability conditions on $\sqrt{2} e_{i_1...i_4}$ give rise to

$$
\sqrt{2} \varepsilon_{i_1...i_4}
$$

The $A = k$ integrability conditions on $\sqrt{2} e_{i_1...i_4}$ lead to
\[ \Gamma^a_{\lambda_1 \cdots \lambda_4} : \frac{1}{60} \left( -i E_{0\lambda k} + 6L_{\mu}^{\alpha} \gamma^{ \alpha} \gamma_{\lambda} - 60B_{\mu}^{\alpha} \gamma^{ \alpha} \right) \tilde{\epsilon}_{\lambda_1 \cdots \lambda_4} \]

\( \Gamma^k_1 : 0 \)

\( \Gamma^{\alpha k}_1 : 0 \)

\( \Gamma^{\alpha \beta k}_1 : 0 \)

\( \Gamma^{\lambda_1 \cdots \lambda_4 k}_1 : \frac{1}{60} E_{kl} \tilde{\epsilon}_{\lambda_1 \cdots \lambda_4} \).

(D.10)

**D.5. \( I_A e_{ij} \)**

We now turn to the contributions from \( e_{ij} \). We split the indices \( \alpha \) into \( p = (i, j) \) and \( a \), which contains the remaining three indices. We also define \( e_{ij}^2 = 1 \). The \( A = 0 \) integrability conditions on \( e_{ij} \) give rise to

\[ I_A e_{ij} \]

\( l \) : \(- (12L_{0pq} - 240B_{0ab}^{\alpha} p_q) \epsilon^{pq} \)

\( \Gamma^a_1 : - (6iL_{0a} - 120iB_{ab}^{\alpha} p_q) \epsilon^{pq} \)

\( \Gamma^{ab} : 120B_{0abpq} \epsilon^{pq} \)

\( \Gamma^q_1 : - (12L_{0p} + 120iB_{ab}^{\alpha} p_q - 120iB_{ab}^{\alpha} p_q) \epsilon^{pq} \)

\( \Gamma^{pq} : - (12L_{0ab} + 240B_{0ab}^{\alpha} p_q - 240B_{0ab}^{\alpha} p_q) \epsilon^{pq} \)

The \( A = a \) integrability conditions on \( e_{ij} \) read

\[ I_A e_{ij} \]

\( l \) : \( \frac{1}{2} \epsilon^{pq} \left( 12L_{0pq} + 240B_{0ab}^{\alpha} p_q \right) \)

\( \Gamma^a_1 : - 120i \epsilon^{pq} B_{0abpq} \)

\( \Gamma^{ab} : 60 \epsilon^{pq} B_{0abpq} \)

\( \Gamma^{abc} : 0 \)

\( \Gamma^{abc} : 0 \)

\( \Gamma^{pq} : 60 \epsilon^{pq} B_{0ababc} \)

\( \Gamma^{pq} : 0 \)

\( \Gamma^{pq} : \frac{1}{2} \left( - \frac{1}{4} i L_{0a} - 3L_{0b} - 3L_{0c} - 30B_{ab}^{\alpha} \epsilon^{ \alpha} + 60B_{ab}^{\alpha} \epsilon^{ \alpha} - 30B_{ab}^{\alpha} \epsilon^{ \alpha} \right) \epsilon_{pq} \)

\( \Gamma^{pq} : \frac{1}{2} \left( \frac{1}{2} L_{0abc} + 60iB_{0abc} \epsilon - 60iB_{0abc} \epsilon \right) \epsilon_{pq} \)

\( \Gamma^{pq} : \frac{1}{2} \left( - \frac{1}{2} L_{0abc} + 30B_{0abc} \epsilon \right) \epsilon_{pq} \)

\( \Gamma^{pq} : 0 \)

Finally, the \( A = \bar{p} \) integrability conditions are given by

\[ I_A e_{ij} \]

\( l \) : \( \frac{1}{2} \epsilon^{ir} \left( -24g_{pq} \left( L_{a}^{\alpha} r - L_{s}^{\alpha} s \right) + 36L_{pq} - 240g_{pq} \left( B_{a}^{\alpha} b_{r}^{\alpha} - 2B_{a}^{\alpha} r_{s}^{\alpha} \right) + 720B_{a}^{\alpha} p_{qr} \right) \)

\( \Gamma^a_1 : \frac{1}{2} \epsilon^{ir} \left( -24g_{pq} \left( L_{a}^{\alpha} r - L_{s}^{\alpha} s \right) - 480g_{pq} \left( B_{a}^{\alpha} b_{r}^{\alpha} - B_{a}^{\alpha} r_{s}^{\alpha} \right) + 720iB_{a}^{\alpha} p_{qr} \right) \)

\( \Gamma^{ab} : \frac{1}{2} \epsilon^{ir} \left( -12g_{pq} \left( L_{a}^{\alpha} r - L_{s}^{\alpha} s \right) - 240g_{pq} \left( B_{a}^{\alpha} b_{r}^{\alpha} - B_{a}^{\alpha} r_{s}^{\alpha} \right) + 360B_{ab}^{\alpha} p_{qr} \right) \)
We split the indices \( a \) into \((k,l,m)\) and \(p\), containing the remaining two indices. The three-dimensional Levi-Civita symbol \( \tilde{\varepsilon}_{abc} \) is defined by \( \tilde{\varepsilon}_{kln} = \sqrt{2} \). The \( A = 0 \) integrability conditions for \( e_{klm} \) read

\[
I_{a} e_{klm} = 0
\]

\[
\begin{align*}
1 & : 2 \left( -i L_{abc} - 20i B_{abc p} \right) \tilde{\varepsilon}_{abc} \\
\Gamma^{a} & : (6 L_{abc} - 120 B_{abcd p} + 120 B_{abc p}) \tilde{\varepsilon}_{abc} \\
\Gamma^{b} & : -1/2 \left( - E_{a c} + 6i L_{d p} - 9L_{d p} - 90 B_{a d p} + 180 B_{a d p} \right) \tilde{\varepsilon}_{abc} \\
\Gamma^{p} & : -80 B_{abc p} \tilde{\varepsilon}_{abc} \\
\Gamma^{a p} & : -3i L_{abc} + 60i B_{abc d} - 60i B_{abc p} \tilde{\varepsilon}_{abc} \\
\Gamma^{b p} & : -20i B_{abc d} \tilde{\varepsilon}_{abc}.
\end{align*}
\]

Similarly, for \( A = \bar{a} \) we find

\[
I_{\bar{a}} e_{klm} = 0
\]

\[
\begin{align*}
1 & : -2 \left( 3i g_{a d} L_{cd p} - 60i g_{a d} (B_{o d c d p} + 60i B_{o d c p}) \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{a} & : -1/2 \left( - E_{c d} + 6i L_{d p} - 9L_{d p} + 180 B_{d e f} - 180 B_{d e f} \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{b} & : 1/2 \left( F_{a d} - 6i g_{a d} (L_{o d p} - 18i L_{o d p} + 60i g_{a d} (B_{o d c d p} + 60i B_{o d c p}) \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{d} & : 1/2 \left( F_{a d} - 6i g_{a d} (L_{o d p} - 18i L_{o d p} + 60i g_{a d} (B_{o d c d p} + 60i B_{o d c p}) \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{a b} & : -1/2 \left( -E_{a d} + 6i L_{d p} - 9L_{d p} + 180 B_{d e f} - 180 B_{d e f} \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{a b d} & : -1/2 \left( F_{a d} - 18i L_{o d p} + 360i B_{o d c p} \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{a b p} & : -1/2 \left( F_{a d} - 18i L_{o d p} + 360i B_{o d c p} \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{a p b} & : -1/2 \left( F_{a d} - 18i L_{o d p} + 360i B_{o d c p} \right) \tilde{\varepsilon}_{bcd} \\
\Gamma^{b p a} & : -1/2 \left( F_{a d} - 18i L_{o d p} + 360i B_{o d c p} \right) \tilde{\varepsilon}_{bcd}.
\end{align*}
\]
\[ \Gamma^{\alpha \beta \rho \lambda} = \frac{1}{2} (3i g_{\alpha \beta} L_{\rho \lambda} - 60i g_{\alpha \beta} B_{0 \rho} e_{\lambda} - 180i B_{0 \rho d} p_q) \bar{\epsilon}_{\lambda} d \]
\[ \Gamma^{\alpha b \rho q} = \frac{1}{2} (-9 L_{a \rho q} + 180 B_{ab} e_{q} p_q) \bar{\epsilon}_{hed}. \]  

For \( A = \bar{p} \) the integrability conditions on \( e_{klm} \) lead to

\[ I_p e_{klm} \]

\[ \begin{align*}
1 & : -40i B_{abc} p_{\alpha \beta \rho \lambda} e^{abc} \\
\Gamma^{a 1} & : -\left( -3 L_{bc \rho} + 60 B_{0 \rho d} p_q - 60 B_{bc \rho q} ^a \right) \bar{\epsilon}_{abc} \\
\Gamma^{ab 1} & : \frac{1}{2} \left( E_{c \rho b} + 6i L_{0 c \rho} - 120 B_{0 d} e_{q} p_q + 120i B_{bc \rho q} ^a \right) \bar{\epsilon}_{abc} \\
\Gamma^{abc 1} & : \frac{1}{2} \left( i E_{0 q} + 6 L_{a d} p_q - 60 B_{a d} e_{q} p_q + 120 B_{a d} p_q ^a \right) \bar{\epsilon}_{abc} \\
\Gamma^{q 1} & : 40 B_{abc} \bar{p}_{abc} \bar{\epsilon}_{abc} \\
\Gamma^{a q 1} & : -60 B_{abc} \bar{p}_{q} \bar{\epsilon}_{abc} \\
\Gamma^{ab q 1} & : -\frac{1}{2} \left( 6 L_{a d} p_{q} - 120 B_{a d} p_{q} \right) \bar{\epsilon}_{abc} \\
\Gamma^{abc q 1} & : -\frac{1}{2} \left( E_{pq} - 6i L_{0 pq} + 120i B_{0d} p_q \right) \bar{\epsilon}_{abc} \\
\Gamma^{q r 1} & : 0 \\
\Gamma^{a q r 1} & : 0 \\
\Gamma^{ab q r 1} & : 0. 
\end{align*} \]

Appendix E. The linear system for SU(4) invariant spinors

E.1. The conditions

To solve (6.3), we collect from the appendices above the terms associated with \( D_A (e_5 + e_{1234}) \) and \( i D_A (1 - e_{12345}) \). In addition, we decompose the expressions that arise in \( i D_A (1 - e_{12345}) \) in terms of SU(4) representations. In practice this means of splitting the holomorphic index \( \alpha = \left( \rho, 5 \right) \), where \( \rho = 1, 2, 3, 4 \). Using this, the conditions arising from Killing spinor equations for \( \eta_2 \) involving derivatives along the time direction are

\[ 0 = g_3^{-1} \partial_0 (g_1 + i g_2) - i \Omega_{0 \rho 5} + \frac{1}{3} G_{\rho \tau} \bar{\epsilon}^{\rho \tau \rho 0 5} \]

\[ 0 = i g_3^{-1} g_2 \left[ i \Omega_{0 \rho 5} + \frac{1}{3} (G_{\rho \tau} \bar{\epsilon}^{\rho \tau \rho 0 5}) + \frac{1}{6} F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} - \frac{1}{18} G_{\rho 0 5} \bar{\epsilon}^{\rho 0 5 \rho 0 5} \right] \]

\[ 0 = \partial_0 \log g_3 + i g_3^{-1} g_2 \left[ i \Omega_{0 \rho 5} + \frac{1}{3} G_{\rho \tau} \bar{\epsilon}^{\rho \tau \rho 0 5} + \frac{1}{2} \Omega_{0 \rho 5} + \frac{1}{2} \partial_0 \bar{\epsilon}^{\rho 5 \rho 0 5} + \frac{1}{24} F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} - \frac{1}{12} F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} \right] \]

\[ 0 = i g_3^{-1} g_2 \left[ \frac{1}{2} \Omega_{0 \rho 5} + \frac{i}{12} (F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} + F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5}) \right] + \frac{1}{6} G_{\rho \tau 5} \]

\[ + \left[ - \frac{1}{8} \Omega_{0 \rho 5} + \frac{i}{48} F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} + \frac{i}{48} F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} \right] \]

\[ 0 = i g_3^{-1} g_2 \left[ \Omega_{0 \rho 5} + \frac{i}{6} F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} + \frac{i}{36} F_{\rho 5 \rho} \bar{\epsilon}^{\rho 5 \rho 0 5} \right] + \frac{1}{2} \Omega_{0 \rho 5} + \frac{1}{6} G_{\rho \tau 5} - \frac{1}{6} G_{\rho 5 \rho}. \]  

We have used the conditions on the geometry and fluxes arising from the Killing spinor equations of the first spinor to simplify somewhat the above expression. Similarly the
conditions arising from Killing spinor equations for \( \eta_2 \) involving derivatives along the spatial directions are

\[
0 = g_3^{-1}(\partial_\rho g_1 - g_1 i \partial_\rho \log f + i \partial_\rho g_2) + ig_3^{-1} g_2 \left[ \partial_\rho \log f + (\Omega_{\rho, \sigma} + \Omega_{\rho, \lambda}) \right] + \frac{i}{6} \left( G_{\rho, 0} + G_{\rho, 0} \right) - \frac{i}{6} \left( F_{\rho, \lambda} \right) - \frac{i}{18} g_{\rho, \lambda} G_{\lambda, \lambda} e^{\lambda \mu \nu \lambda} \quad (E.6)
\]

\[
0 = ig_3^{-1} g_2 \left[ \frac{i}{2} \Omega_{\rho, \sigma} + \frac{i}{12} G_{\rho, \sigma} \right] + \frac{i}{12} F_{\rho, \sigma} + \frac{i}{2} \left( \frac{i}{12} \Omega_{\rho, \lambda} + \frac{i}{8} G_{\rho, \lambda} \right) + \frac{i}{12} g_{\rho, \sigma} G_{\lambda, \lambda} \left( e^{\lambda \mu \nu \lambda} \right) \quad (E.7)
\]

\[
0 = \partial_\rho \log g_3 + ig_3^{-1} g_2 \left[ \frac{i}{2} \Omega_{\rho, \sigma} + \frac{i}{12} G_{\rho, \sigma} \right] + \frac{i}{2} \Omega_{\rho, \lambda} - \frac{i}{2} \Omega_{\rho, \mu} - \frac{i}{12} G_{\rho, \lambda} + \frac{i}{12} G_{\rho, \mu} - \frac{1}{18} g_{\rho, \lambda} G_{\lambda, \mu, \lambda} \quad (E.8)
\]

\[
0 = ig_3^{-1} g_2 \left[ \frac{i}{36} F_{\rho, \sigma} + \frac{i}{4} G_{\rho, \sigma} \right] + \frac{i}{2} \left( \frac{i}{12} \Omega_{\rho, \lambda} + \frac{i}{4} F_{\rho, \lambda} \right) - \frac{i}{12} g_{\rho, \sigma} F_{\sigma, \mu} \left( e^{\lambda \mu \nu \lambda} \right) \quad (E.9)
\]

\[
0 = ig_3^{-1} g_2 \left[ \frac{i}{12} F_{\rho, \sigma} + \frac{i}{4} G_{\rho, \sigma} \right] + \frac{i}{4} \left( \frac{i}{12} \Omega_{\rho, \lambda} + \frac{i}{4} F_{\rho, \lambda} \right) - \frac{i}{12} g_{\rho, \sigma} F_{\sigma, \mu} \left( e^{\lambda \mu \nu \lambda} \right) \quad (E.10)
\]

\[
0 = g_3^{-1} \left( \partial_\rho g_1 - g_1 i \partial_\rho \log f + i \partial_\rho g_2 \right) + ig_3^{-1} g_2 \left[ \partial_\rho \log f + (\Omega_{\rho, \sigma} + \Omega_{\rho, \lambda}) \right] + \frac{i}{12} g_{\rho, \sigma} F_{\sigma, \mu} \left( e^{\lambda \mu \nu \lambda} \right) \quad (E.11)
\]

\[
0 = \partial_\rho \log g_3 + \frac{i}{2} \Omega_{\rho, \lambda} + \frac{i}{2} \Omega_{\rho, \mu} - \frac{i}{4} G_{\rho, \lambda} + \frac{i}{4} G_{\rho, \mu} \quad (E.12)
\]

\[
0 = ig_3^{-1} g_2 \left[ \frac{i}{12} F_{\rho, \sigma} + \frac{i}{4} G_{\rho, \sigma} \right] + \frac{i}{4} \left( \frac{i}{12} \Omega_{\rho, \lambda} + \frac{i}{4} F_{\rho, \lambda} \right) - \frac{i}{12} g_{\rho, \sigma} F_{\sigma, \mu} \left( e^{\lambda \mu \nu \lambda} \right) \quad (E.13)
\]

\[
0 = \partial_\rho \log g_3 + \frac{i}{2} \Omega_{\rho, \lambda} - \frac{i}{2} \Omega_{\rho, \mu} - \frac{i}{4} G_{\rho, \lambda} + \frac{i}{4} G_{\rho, \mu} \quad (E.14)
\]

and along the fifth direction we have

\[
0 = g_3^{-1} \left( \partial_\rho g_1 - g_1 i \partial_\rho \log f + i \partial_\rho g_2 \right) + ig_3^{-1} g_2 \left[ \partial_\rho \log f + (\Omega_{\rho, \sigma} + \Omega_{\rho, \lambda}) \right] + \frac{i}{12} g_{\rho, \sigma} F_{\sigma, \mu} \left( e^{\lambda \mu \nu \lambda} \right) \quad (E.15)
\]
\[ 0 = i g_1^{-1} g_2 \left[ i \Omega_{5,0\theta} + \frac{1}{6} F_{5\lambda} \right] \left( \Omega_{5,\sigma 5} - i \frac{3}{6} G_{\sigma \lambda} - \frac{i}{2} G_{\sigma 55} - \frac{1}{36} F_{5\lambda 12 \lambda 12} e^{\lambda = \lambda 12} \right) \] (E.17)

\[ 0 = \partial_5 \log g_3 + \frac{1}{2} \Omega_{5,\lambda} \lambda - \frac{1}{2} \Omega_{5,55} - \frac{i}{4} G_{5\lambda} \] (E.18)

\[ 0 = i g_1^{-1} g_2 \left[ \frac{1}{2} \Omega_{5,\sigma 5} + i \frac{1}{12} G_{5\sigma, \sigma 5} \right] - \left( \frac{1}{12} G_{\lambda, \sigma 5} \right) - \frac{1}{2} \left( \frac{1}{4} \Omega_{5,\lambda 12} + \frac{i}{24} G_{5\lambda 12} \right) e^{\lambda = \lambda 12} \] (E.19)

\[ 0 = \partial_3 \log g_3 + \frac{i}{6} \Omega_{5,0 \sigma} - \frac{1}{4} F_{5\sigma} \] (E.20)

\[ 0 = i g_1^{-1} g_2 \left[ \frac{1}{36} F_{5\sigma, \sigma 5} + i \frac{1}{3} G_{5\sigma, \sigma 5} + \frac{i}{2} \Omega_{5,0 \sigma} - \frac{1}{12} F_{5\lambda} \right] e^{\lambda = \lambda 12} \] (E.21)

\[ 0 = \frac{1}{2} G_{5\sigma, \sigma 5} \] (E.22)

\[ 0 = g_3^{-1} (\partial_3 g_3 - i \partial_3 \log f - i \partial_3 g_2) + ig_3^{-1} g_2 \partial_5 \log f. \] (E.25)

Using the conditions arising from the \( SU(5) \) invariant spinor which have been collected in appendix B, we can substitute for the fluxes \( F \) and rewrite the above equations in terms of the connection. The conditions arising from Killing spinor equations for \( \eta_3 \) involving derivatives along the time direction then become

\[ 0 = g_3^{-1} \partial_0 g_1 - i \Omega_{0,05} + i \Omega_{\sigma,05}, \] (E.26)

\[ 0 = i g_3^{-1} \partial_0 g_2 - \frac{2i}{3} (\Omega_{5,\lambda} \lambda - \Omega_{5,05} + \Omega_{\sigma,05} + \Omega_{0,05}) \] (E.27)

\[ 0 = \frac{1}{3} g_3^{-1} g_2 \left( \Omega_{0,09} - 2 \Omega_{5,\lambda} \lambda - 2 \Omega_{5,05} + \Omega_{0,05} + \Omega_{\sigma,05} \right), \] (E.28)

\[ 0 = 6 g_3^{-1} g_2 \left( \Omega_{0,05} + 2 \left( \Omega_{5,\lambda} \lambda + \Omega_{5,05} + \Omega_{5,05} \right) \right) \] (E.29)

\[ 0 = \frac{1}{3} g_3^{-1} g_2 \left[ \Omega_{0,05} + 2 \left( \Omega_{5,\lambda} \lambda + \Omega_{5,05} + \Omega_{5,05} \right) \right] + \frac{4}{3} \Omega_{0,05} \] (E.30)

\[ 0 = \frac{1}{6} g_3^{-1} \left( 2 \Omega_{5,\lambda 12} + \Omega_{5,\lambda 12} e^{\lambda = \lambda 12} \right) \] (E.31)

\[ 0 = \frac{1}{3} g_3^{-1} g_2 \Omega_{5,\lambda 12} e^{\lambda = \lambda 12} + \frac{1}{6} \left( 2 \Omega_{5,05} + 2 \Omega_{5,05} - 2 \Omega_{5,05} \right). \] (E.32)

where the grouped equations constitute the split into real and imaginary parts.
Similarly, the conditions arising from Killing spinor equations for $\eta_2$ involving derivatives along the spatial $\rho$ directions become
\[
0 = g_3^{-1} \left( \partial_\rho g_1 - g_1 \partial_\rho \log f + i \partial_\rho g_2 \right) + \frac{i}{3} g_3^{-1} g_2 \left( -\frac{1}{2} \Omega_{0,0,\rho} + 4 \Omega_{\rho,\lambda} + 4 \Omega_{\rho,55} \right) \\
- \frac{4i}{3} \Omega_{\rho,05} + \frac{1}{3} \Omega_{\lambda_1 \lambda_2 \lambda_3 \rho} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho,
\]
(E.33)
\[
0 = \frac{i}{3} g_3^{-1} g_2 \left( \Omega_{\lambda_1 \lambda_2 \lambda_3 \rho} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho + 2 \Omega_{\lambda_1 \lambda_2 \rho} \epsilon^{\lambda_1 \lambda_2} \rho \right) + \Omega_{\rho,\sigma 5} - \Omega_{\rho,\sigma 5} \\
+ \frac{1}{3} \Omega_{\rho,\rho 5} + \frac{2}{3} \Omega_{\rho,\sigma 5} - i \frac{1}{3} \Omega_{\lambda_1 \lambda_2 \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho,
\]
(E.34)
\[
0 = \frac{i}{3} g_3^{-1} g_2 \Omega_{\lambda_1 \lambda_2 \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho + \partial_\rho \log g_3 + \frac{1}{2} \Omega_{\rho,\lambda} - \frac{1}{2} \Omega_{5,55} - \frac{1}{6} \Omega_{\sigma,\rho} \\
+ \frac{i}{6} \Omega_{5,55} - \frac{i}{6} \Omega_{5,\rho 5} + \frac{2}{3} \Omega_{5,\rho 5} - \frac{1}{6} \Omega_{0,0,\rho},
\]
(E.35)
\[
0 = \frac{4i}{3} g_3^{-1} g_2 \left( \Omega_{\rho,\sigma 5} - \Omega_{\sigma,\rho 5} \right) + \Omega_{\rho,\lambda_1 \lambda_2 \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho \\
+ \left[ \frac{1}{3} \Omega_{5,55} + \frac{1}{3} \Omega_{5,\rho 5} + \frac{1}{6} \Omega_{0,0,\rho} \right] \epsilon^{\rho,\sigma,\tau},
\]
(E.36)
\[
0 = \frac{i}{3} g_3^{-1} g_2 \left( 2 \Omega_{\rho,55} - \Omega_{5,\rho 5} + \Omega_{\sigma,\rho 5} \right) - \frac{i}{6} \Omega_{\rho,55} + \frac{1}{6} \Omega_{\lambda_1 \lambda_2 \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho \\
+ \frac{1}{6} \Omega_{5,\rho 5} - \frac{1}{6} \Omega_{0,0,\rho},
\]
(E.37)
\[
0 = \frac{i}{6} g_3^{-1} g_2 \left( \Omega_{0,0,5} + 2 \Omega_{5,55} \right) \epsilon^{\rho,\sigma,\tau} + \frac{1}{6} \epsilon^{\rho,\sigma,\tau} F_{\rho,55} \\
- \frac{i}{6} \Omega_{0,0,5} \epsilon^{\rho,\sigma,\tau} - \frac{1}{6} \epsilon^{\rho,\sigma,\tau} F_{\rho,55} - \frac{1}{2} \Omega_{0,0,5},
\]
(E.38)
\[
0 = \frac{i}{6} g_3^{-1} g_2 \left( \Omega_{0,0,5} + 2 \Omega_{5,\rho 5} \right) \epsilon^{\rho,\sigma,\tau} + \frac{1}{6} \epsilon^{\rho,\sigma,\tau} F_{\rho,55} - \frac{1}{8} \Omega_{0,0,5} \epsilon^{\rho,\sigma,\tau} \epsilon^{\rho,\sigma,\tau} \\
+ \frac{1}{8} \Omega_{0,0,5} \epsilon^{\rho,\sigma,\tau} \epsilon^{\rho,\sigma,\tau},
\]
(E.39)
\[
0 = \partial_\rho \log g_3 - \frac{1}{2} \Omega_{\rho,\lambda} - \frac{1}{2} \Omega_{\lambda,\rho} + \frac{1}{2} \Omega_{5,55} + \frac{1}{2} \Omega_{5,\rho 5} - \frac{1}{2} \Omega_{0,0,5},
\]
(E.40)
\[
0 = \frac{1}{3} \left[ -\frac{1}{2} \Omega_{0,0,5} - \frac{1}{2} \Omega_{5,55} - \frac{1}{2} \Omega_{5,\rho 5} + \frac{1}{2} \Omega_{0,0,5} + \frac{1}{2} \Omega_{5,\rho 5} \right] g_{\rho \sigma} + \Omega_{\rho,\sigma 5} + \Omega_{\sigma,\rho 5},
\]
(E.41)
\[
0 = g_3^{-1} \left( \partial_\rho g_1 - g_1 \partial_\rho \log f - i \partial_\rho g_2 \right) + i g_3^{-1} g_2 \partial_\rho \log f + 4 i \Omega_{\rho,55} - \Omega_{\lambda_1 \lambda_2 \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho. 
\]
(E.42)

The conditions arising from Killing spinor equations for $\eta_2$ involving derivatives along the spatial 5 direction become
\[
0 = g_3^{-1} \left( \partial_5 g_1 - g_1 \partial_5 \log f + i \partial_5 g_2 \right) + \frac{i}{3} g_3^{-1} g_2 \left[ -\frac{1}{2} \Omega_{0,0,5} + 4 \Omega_{5,55} \right] \\
+ \frac{8i}{3} \Omega_{0,55} - \frac{4i}{3} \Omega_{0,\lambda},
\]
(E.43)
\[
0 = \frac{i}{3} g_3^{-1} g_2 \Omega_{\lambda_1 \lambda_2 \lambda_3} \epsilon^{\lambda_1 \lambda_2 \lambda_3} \rho + \Omega_{5,\rho 5} - \frac{2}{3} \Omega_{5,55} + \frac{1}{3} \Omega_{5,55} - \frac{1}{3} \Omega_{\rho,\sigma} - \frac{5}{6} \Omega_{0,0,5},
\]
(E.44)
\[
0 = \partial_5 \log g_3 - \Omega_{5,55} - \frac{1}{2} \Omega_{0,0,5}.
\]
(E.45)
\begin{align}
0 &= -\frac{2i}{3} g_3^{-1} g_2(\Omega_{\delta,\sigma,\tau} - \Omega_{\sigma,\tau,\delta}) + \frac{1}{2} \Omega_{\lambda,\mu,\nu,\sigma} \epsilon^{\lambda,\mu,\nu,\sigma} + \frac{5}{6} \Omega_{\lambda,\nu,\rho,\sigma} \epsilon^{\lambda,\nu,\rho,\sigma} \\
&\quad + \frac{1}{3} \Omega_{\lambda,\nu,\rho,\sigma} \epsilon^{\lambda,\nu,\rho,\sigma} - \frac{4}{3} \Omega_{\delta,\sigma,\tau}. 
\end{align} \tag{E.46}

\begin{align}
0 &= ig_3^{-1} g_2 \Omega_{\delta,\sigma,\tau} + 2i \Omega_{\sigma,0,\tau} - \frac{1}{2} \Omega_{\lambda,\lambda,\rho,\lambda} \epsilon^{\lambda,\rho,\lambda}.
\end{align} \tag{E.47}

\begin{align}
0 &= \frac{i}{72} g_3^{-1} g_2(\Omega_{0,0,0} + 2 \Omega_{\lambda,\varepsilon} + 2 \Omega_{\varepsilon,5,5}) \epsilon^{\lambda,\rho,\sigma,\sigma} + \frac{1}{6} \Omega_{\sigma,\sigma,\sigma,1} + \frac{i}{18} \Omega_{5,0,0,5} \epsilon^{\rho,\sigma,\sigma,1}.
\end{align} \tag{E.48}

\begin{align}
0 &= \Omega_{\lambda,\rho,\nu} + \Omega_{\lambda,5,5} + \Omega_{5,5,5} + \frac{1}{2} \Omega_{0,0,0}.
\end{align} \tag{E.50}

\begin{align}
0 &= \Omega_{\lambda,\rho,\nu} + \Omega_{\lambda,5,5} + \Omega_{5,5,5} + \frac{1}{2} \Omega_{0,0,0}.
\end{align} \tag{E.51}

\begin{align}
0 &= g_3^{-1}(\partial_3 g_1 - g_1 \partial_3 \log f - ig_2) + ig_3^{-1} g_2 \partial_3 \log f.
\end{align} \tag{E.52}

**E.2. The solution to the Killing spinor equations with $g_2 \neq 0$**

Here we shall investigate the case $g_2 \neq 0$. Taking the trace of (E.38), we get

\begin{align}
-\frac{2i}{3} g_3^{-1} g_2(2 \Omega_{\lambda,\lambda} + 2 \Omega_{5,5,5} + \Omega_{0,0,0}) - \frac{7i}{3} \Omega_{\lambda,\lambda} - \frac{8i}{3} \Omega_{0,5,5} - \frac{1}{2} F_{55,5} = 0.
\end{align} \tag{E.53}

Substituting

\begin{align}
F_{55,5} = -2i \left( \Omega_{\lambda,\lambda} + 2 \Omega_{5,5,5} \right)
\end{align} \tag{E.54}

from the $N = 1$ results given in appendix B, we find

\begin{align}
-\frac{i}{3} g_3^{-1} g_2[\Omega_{0,5,5} + 2 \Omega_{\lambda,\rho,\nu} + 2 \Omega_{5,5,5}] - \frac{8i}{3} \Omega_{0,5,5} - \frac{2}{3} \Omega_{0,0,0} = 0.
\end{align} \tag{E.55}

Using the above formulae in (E.29), we find

\begin{align}
\partial_3 \log g_3 = 0
\end{align} \tag{E.56}

and in (E.30)

\begin{align}
\Omega_{0,5,5} = 0.
\end{align} \tag{E.57}

From (E.27) and (E.55), we find

\begin{align}
g_3^{-1} \partial_3 g_2 - (\Omega_{0,0,5} + \Omega_{0,0,5}) = 0
\end{align} \tag{E.58}

and (E.26) gives

\begin{align}
g_3^{-1} \partial_3 g_1 - i(\Omega_{0,0,5} - \Omega_{0,0,5}) = 0.
\end{align} \tag{E.59}

Using (E.54) in (E.40) yields

\begin{align}
\Omega_{5,5,5} = 0.
\end{align} \tag{E.60}

Taking the difference between (E.44) and (E.49) gives

\begin{align}
2(\Omega_{5,5,5} - \Omega_{5,5,5}) + 4 \Omega_{5,5,5} + 2 \Omega_{5,5,5} + \Omega_{0,0,5} + \Omega_{0,0,5} = 0,
\end{align} \tag{E.61}

which together with (E.54) yields

\begin{align}
\Omega_{5,5,5} = 0.
\end{align} \tag{E.62}
By instead adding equations (E.44) and (E.49), and using (E.54), we get
\[ \partial \log g_3 - \frac{1}{2} \Omega_{0,05} = \partial \log (g_3 f) = 0. \] (E.63)
Next use (E.54) in (E.42) and (E.51) to find
\[ \partial \log (g_2 f^{-1}) = \partial \log (g_1 f^{-1}) = 0. \] (E.64)
If we combine (E.38) and (E.55) we can solve for one of the components of the $F$ flux in terms of the geometry
\[ i \Omega_{0, \rho \lambda} + \frac{1}{2} F_{\rho \lambda \sigma} = 0. \] (E.65)
The symmetric part of (E.33) implies that
\[ \Omega_{(\rho, \sigma) \bar{\delta}} = \Omega_{(\bar{\rho}, \bar{\sigma}) \delta}. \] (E.66)
Similarly, the symmetric part of (E.36) yields
\[ \Omega_{(\rho, \sigma) \bar{\delta}} = 0 \] (E.67)
and thus
\[ \Omega_{(\rho, \sigma) \bar{\delta}} = \Omega_{(\bar{\rho}, \bar{\sigma}) \delta} = 0. \] (E.68)
Using (E.49) the antisymmetric part of (E.36) yields
\[ -\frac{2i}{3} g_3^{-1} g_2 \Omega_{5, \rho \sigma} - \frac{2i}{3} \Omega_{0, \rho \sigma} - \frac{1}{6} \Omega_{5, \lambda \rho \bar{\delta}} \varepsilon_{\lambda \rho \bar{\delta} \sigma} + \frac{1}{6} \Omega_{5, \lambda \rho \lambda \rho \bar{\delta}} \varepsilon_{\lambda \rho \lambda \rho \bar{\delta} \sigma} = 0, \] (E.69)
which coincides with (E.31) and (E.46). The dual of (E.48) coincides with (E.29), and taking the trace of (E.39) and using the result
\[ F_{55} = -2i \Omega_{0, \lambda \delta} \] (E.70)
from the $N = 1$ solution, we find
\[ i g_3^{-1} g_2 \left( 2 \Omega_{0, \rho, \sigma} + 2 \Omega_{\rho, \bar{\delta}, 55} + \Omega_{5, \lambda \rho \lambda \rho \bar{\delta}} \varepsilon_{\lambda \rho \lambda \rho \bar{\delta} \sigma} \right) \] (E.71)
which combined with (E.29) yields
\[ \Omega_{5, \lambda \rho \lambda \rho \bar{\delta}} \varepsilon_{\lambda \rho \lambda \rho \bar{\delta} \sigma} + 2i \Omega_{0, \rho \delta} = 0. \] (E.72)
Dualizing (E.36) with $\varepsilon_{\rho, \sigma, \delta, \lambda}$ yields
\[ \Omega_{5, \rho \lambda} + \Omega_5_{\rho, \lambda} + \Omega_{5, \lambda \delta} + \frac{1}{2} \Omega_{0, \rho \delta} = 0. \] (E.73)
Taking the sum and difference of (E.33) and (E.44) we find
\[ \Omega_{5, \rho \delta} = \Omega_{5, \rho \delta}. \] (E.74)
and
\[ \frac{2i}{3} g_3^{-1} g_2 \Omega_{5, \lambda \rho \lambda \rho \bar{\delta}} \varepsilon_{\lambda \rho \lambda \rho \bar{\delta} \sigma} + \frac{2}{3} \Omega_{0, \rho \sigma} - \frac{2}{3} \Omega_{5, \rho \delta} - \frac{2}{3} \Omega_{5, \rho \delta} - \frac{5}{3} \Omega_{5, \rho \delta} = 0. \] (E.75)
In the same way the sum and difference between (E.35) and (E.40), and using (E.75), yield
\[ -\Omega_{5, \rho \lambda} + \frac{1}{2} \Omega_{0, \rho \delta} + \Omega_{5, \rho \delta} - \Omega_{5, \rho \delta} = 0 \] (E.76)
and
\[ 2 \partial_\rho \log g_3 - \Omega_{5, \rho \delta} - \frac{3}{2} \Omega_{0, \rho \delta} + \Omega_{5, \rho \delta} + \Omega_{5, \rho \delta} = 0. \] (E.77)
Equation (E.72) can be simplified using the $N = 1$ result
\[ -\Omega_{5, \rho \delta} - \Omega_{5, \rho \delta} - \Omega_{5, \rho \delta} - \Omega_{5, \rho \delta} - \Omega_{5, \rho \delta} = 0 \] (E.78)
and (E.50) yielding
\[ \Omega_{5,5\lambda} = -\Omega_{5,5\lambda}. \]  
(E.79)

Combining (E.75) and (E.50) we find
\[ \Omega_{5,5\sigma} = 0. \]  
(E.80)

The equations (E.70) and (E.46) can be simplified, using (E.50) and (E.71), to
\[ g_3^{-1} g_2 \Omega_{5,5\lambda} = -\Omega_{0,5\lambda} \]  
(E.81)

and
\[ \Omega_{5,5\sigma} = \Omega_{0,5\sigma}. \]  
(E.82)

Using (E.71) and (E.77) the equations (E.74) and (E.76) can be rewritten as
\[ -g_3^{-1} g_2 \Omega_{0,5\rho} = \Omega_{5,5\rho} - \Omega_{5,5\rho} + \Omega_{0,0\rho} = 0 \]  
(E.83)

and
\[ \partial_\rho \log g_3 + \Omega_{5,5\rho} + \Omega_{5,5\rho} = \frac{1}{2} \Omega_{0,0\rho} = 0. \]  
(E.84)

By combining (E.33) and (E.41), using (E.71), we find
\[ \partial_\rho \log (g_1/f) = 0 \]  
(E.85)

and
\[ \partial_\rho \log (g_1/f) - 2 g_3 g_2^{-1} \Omega_{0,5\rho} = 0. \]  
(E.86)

Using the above results (E.39) can be solved for one component of the \( F \) flux
\[ F_{5\lambda_1\lambda_2} - \frac{8i}{3} \Omega_{0,5(\lambda_1,\lambda_2)} = -\frac{1}{2} \left( \Omega_{5,0,\lambda_1}, \Omega_{5,0,\lambda_2} \right) \]  
(E.87)

Taking the sum of (E.31) and (E.34), and using (E.48), yields
\[ \Omega_{5,0,\lambda_1} + \Omega_{5,0,\lambda_2} = 0. \]  
(E.88)

and by substituting the above results back into (E.35) we find
\[ \Omega_{\lambda_1\lambda_2} + \frac{2}{3} \left( \Omega_{0,5\lambda_1} - \Omega_{5,0\lambda_1} + \Omega_{0,0\lambda_1} \right) \]  
(E.89)

\begin{align*}
\Omega_{5,5\lambda_1} & = -\Omega_{5,5\lambda_1} \\
\Omega_{5,5\sigma} & = \Omega_{0,5\sigma} \\
-\Omega_{0,5\rho} & = \Omega_{5,5\rho} - \Omega_{5,5\rho} + \Omega_{0,0\rho} = 0 \\
\partial_\rho \log g_3 & = \Omega_{5,5\rho} - \Omega_{5,5\rho} - \frac{1}{2} \Omega_{0,0\rho} = 0. \\
\partial_\rho \log (g_1/f) & = 0 \\
\partial_\rho \log (g_1/f) - 2 g_3 g_2^{-1} \Omega_{0,5\rho} & = 0. \\
F_{5\lambda_1\lambda_2} & = -\frac{8i}{3} \Omega_{0,5(\lambda_1,\lambda_2)} \\
\Omega_{5,0,\lambda_1} & = \Omega_{5,0,\lambda_2} \\
\Omega_{5,0,\lambda_1} & = \frac{2}{3} \left( \Omega_{0,5\lambda_1} - \Omega_{5,0\lambda_1} + \Omega_{0,0\lambda_1} \right) \\
\end{align*}

\section*{References}

[1] Townsend P K 1995 The eleven-dimensional supermembrane revisited Phys. Lett. B 350 184 (Preprint hep-th/9501068)
[2] Aharony O, Gubser S S, Maldacena J M, Ooguri H and Oz Y 2000 Large \( N \) field theories, string theory and gravity Phys. Rep. 323 183 (Preprint hep-th/9905111)
[3] Blau M, Figueroa-O’Farrill J, Hall C and Papadopoulos G 2002 A new maximally supersymmetric background of IIB superstring theory J. High Energy Phys. JHEP01(2002)047 (Preprint hep-th/0110242)
[4] Chrusciel P T and Kowalski-Glikman J 1984 The isometry group and Killing spinors for the PP wave space-time Phys. Lett. B 149 107
[5] Figueroa-O’Farrill J and Papadopoulos G 2001 Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory J. High Energy Phys. JHEP08(2001)036 (Preprint hep-th/0105308)
[6] Figueroa-O’Farrill J and Papadopoulos G 2003 Maximally supersymmetric solutions of ten-dimensional and eleven-dimensional supergravities J. High Energy Phys. JHEP03(2003)048 (Preprint hep-th/0211089)
[7] Figueroa-O’Farrill J and Papadopoulos G 2002 Pluecker-type relations for orthogonal planes Preprint math.ag/0211170
[7] Mac Conamhna O A P 2004 Refining G-structure classifications Preprint hep-th/0408203
[8] Gauntlett J P and Pakis S 2003 The geometry of $D = 11$ killing spinors J. High Energy Phys. JHEP04(2003)039 (Preprint hep-th/0212008)
[9] Gauntlett J P, Gutowski J B and Pakis S 2003 The geometry of $D = 11$ null killing spinors J. High Energy Phys. JHEP12(2003)049 (Preprint hep-th/0311112)
[10] Gillard J, Gran U and Papadopoulos G 2005 The spinorial geometry of supersymmetric backgrounds Class. Quantum Grav. 22 1033 (Preprint hep-th/0410155)
[11] Gran U, Gutowski J and Papadopoulos G 2005 The spinorial geometry of supersymmetric IIB backgrounds Class. Quantum Grav. 22 2453
[12] Batrachenko A, Liu J T, Varela O and Wen W Y 2004 Higher order integrability in generalized holonomy Preprint hep-th/0412154
[13] MacConamhna O A P 2005 Eight-manifolds with G-structure in eleven dimensional supergravity Preprint hep-th/0504028
[14] Freed D, Harvey J A, Minasian R and Moore G W 1998 Gravitational anomaly cancellation for M-theory fivebranes Adv. Theor. Math. Phys. 2 601 (Preprint hep-th/9803205)
[15] Cremmer E, Julia B and Scherk J 1978 Supergravity in eleven dimensions Phys. Lett. B 76 409
[16] Biran Englert F, de Wit B and Nicolai H 1983 Gauged $N = 8$ supergravity and its breaking from spontaneous compactification Phys. Lett. B 124 45
Biran Englert F, de Wit B and Nicolai H 1983 Phys. Lett. B 128 461 (erratum)
[17] Bellorin J and Ortin T 2005 A note on simple applications of the Killing spinor identities Preprint hep-th/0501246
[18] Bryant R 2000 Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor Preprint math.DG/0004073
[19] Figueroa-O’Farrill J M 2000 Breaking the M-waves Class. Quantum Grav. 17 2925 (Preprint hep-th/9904124)
[20] Green M B, Schwarz J H and Witten E 1987 Superstring Theory vol 2 (Cambridge: Cambridge University Press)
[21] Becker K and Becker M 1996 M-Theory on eight-manifolds Nucl. Phys. B 477 155 (Preprint hep-th/9605053)
[22] Duff M J, Evans J M, Khuri R R, Lu J X and Minasian R 1997 The octonionic membrane Phys. Lett. B 412 281
Duff M J, Evans J M, Khuri R R, Lu J X and Minasian R 1998 Nucl. Phys. Proc. Suppl. 68 295 (Preprint hep-th/9706124)
[23] Cvetic M, Lu H and Pope C N 2001 Brane resolution through transgression Nucl. Phys. B 600 103 (Preprint hep-th/0011023)
[24] Cvetic M, Gibbons G W, Lü H and Pope C N 2003 Ricci-flat metrics, harmonic forms and brane resolutions Commun. Math. Phys. 232 457 (Preprint hep-th/0012011)
[25] Chen C M and Vazquez-Poritz J F 2004 Resolving the M2-brane Preprint hep-th/0403109
[26] Vafa C and Witten E 1995 A one loop test of string duality Nucl. Phys. B 447 261 (Preprint hep-th/9505053)
[27] Duff M J, Liu J T and Minasian R 1995 Eleven-dimensional origin of string/string duality: a one-loop test Nucl. Phys. B 452 261 (Preprint hep-th/9506126)
[28] Emparan R, Mateos D and Townsend P K 2001 Supergravity supertubes J. High Energy Phys. JHEP07(2001)011 (Preprint hep-th/0106012)
[29] Wang McKenzie Y 1989 Parallel spinors and parallel forms Ann. Global Anal. Geom. 7 59
[30] Lawson H B and Michelsohn M-L 1989 Spin Geometry (Princeton, NJ: Princeton University Press)
[31] Harvey F R 1990 Spinors and Calibrations (London: Academic)
[32] Gray A and Hervella L M 1980 The sixteen classes of almost Hermitian manifolds and their linear invariants Ann. Mat. Pura Appl. 282 1