A BAYESIAN APPROACH TO THE OPTIMAL WARRANTY LENGTH FOR PARETO DISTRIBUTED PRODUCT WITH THE GENERAL PROGRESSIVE TYPE-II CENSORING SCHEME

Abstract: The object of the study is to determine the optimal warranty length under free replacement warranty (FRW), pro rata warranty (PRW) and combined warranty policies and the most beneficial warranty scheme to the producer for the product having Pareto life time distribution. A Bayesian approach is used to determine the optimal warranty length based on the general progressive type-II censored data. The optimal warranty is obtained by maximizing the expected utility of the product. A numerical data is presented to exemplify the theory. A simulation study is carried out to check the effect of the hyper parameters on the optimal warranty length and the optimal value of expected utility. From our study we observed that the combined policy gives maximum utility followed by PRW and then by FRW for any choice of the prior parameters. Hence we suggest the producer to adopt the combined policy for such a product.

Keywords: Posterior distribution, warranty policy, economic benefit function, warranty cost function, dissatisfaction cost function, general progressive type-II censoring scheme

1. Introduction

The manufacturers may attract consumers to purchase their products by providing reasonable warranties on the products with the major goal of increasing profits. To increase the profit the important factors are sale volume and the selling price. Sale volume of the product depends not only on the lower price of the product but also depend on the on the quality, reliability and warranty length of the product. A good quality product requires some more cost, which increases the selling price of the product (Scitovszky, 1945). To reduce the selling price producer may produce the product in a very large quantity. To compete with standard product producer should produce the products having good quality and competitive price to fulfill customers expectations. Determination of the appropriate selling price of the product is also an issue for the producer. Jeyakumar and Jevakumar and Robert (2010) considered joint determination of warranty length as well as production quantity under free renewal policy. Quality of the product can be judged by its types warranty and warranty length. Warranty is a contract between the manufacturer and a customer that gives assurance to the customer about the quality of the product. Through warranties, customers are provided guarantees for completely free replacement of the product or partial replacement, even in terms of money for a period of time following the purchase of
product. Thus, a proper warranty plays an important role in increasing sales as well as profit from the products. Such type of work has been done by many authors like Singpurwalla and Wilson (1998). If the manufacturers wish to give compensation to the buyer when failure occurs, the warranty length and the reliability of the product play a significant role on determining the cost of the product. Optimal warranty length in case of the product possessing Rayleigh distributed life time is considered by Wu and Huang (2010). As the Rayleigh distribution has an increasing failure rate over a time, such a study will not be useful for the product having constant or decreasing failure rate. Wu et al. (2006) have considered normal distribution as a product life time model which is suitable only for the product having increasing failure rate. Life time of the product may follow various types of life time distributions like Exponential, Power function, Kumaraswamy distributions. Patel and Patel (2017) have considered a Bayesian approach to optimal warranty length for a Kumaraswamy life time distributed product with general progressive censoring scheme. In this paper we have considered power function life time model for the product having decreasing failure rate.

The knowledge of product reliability is must for a manufacturer to design a cost-effective warranty. Such a knowledge about the reliability of the product can be acquired by conducting life testing experiment. Since the life testing experiments are destructive, which increases the expenses of a producer. To save time and cost censored experiments are conducted. Usually two basic types of censoring schemes are used in life testing experiments. Type-I censoring and Type-II censoring are the most commonly used censoring schemes. Such censoring schemes have been studied by number of authors including Lawless (1982), Gouno et al. (2004), Balakrishnan et al. (2007). There is no facility to withdraw some units, which may be useful for any other purpose, from the survival units during the experiment before the final termination of the test. There are some censoring schemes which allow such type of withdrawal, like progressive type-I, progressive type-II, general progressive type-II, progressive first failure or multiply type-II censoring schemes. Nadi and Gildeh (2016) considered progressive first-failure censoring scheme to estimate the life time performance index for two-parameter exponentially distributed life time product. In this paper we use general progressive type-II censoring in which some failure units are withdrawn from the test.

The aim of this paper is to determine optimal warranty length for the product having Pareto distribution. The information of product reliability is obtained through a general progressive type-II censored life test. The utility function and information are used to determine the warranty length under Bayesian set up. The concept of utility function to determine optimal warranty period as considered by Wu and Huang, (2010) is used. In section 2 the likelihood function for the Pareto distribution is constructed based on the general progressive type-II censored sample. Using gamma conjugate prior distribution for the parameter of the life time distribution, the posterior distribution is obtained. A posterior predictive distribution is derived using the posterior distribution. Section 3 gives the warranty policies. A combined warranty policy based on FRW (Free replacement warranty) and PRW (Pro-rata warranty) are described. Cost functions under the above warranty policies are mentioned. Section 4 provides utility function which is constructed using economic benefit function, warranty cost function and dissatisfaction cost as described by Wu and Huang (2010). Section 5 covers the maximization of expected utility function and optimal warranty. Section 6 provides a numerical example. The sensitivity analysis is also carried out in Section 7 to study the effect of the prior parameters. Some conclusions are drawn in Section 8.
2. Life time model and posterior distribution

The Pareto distribution has its own importance in the life testing experiments. This distribution has been considered by many authors like Aggarwala and Childs (1999), Hassain and Zimmer (2000), Mahmmad et al. (2013), Podder et al. (2004), Shah and Patel (2007) as a life time model.

The probability density function of Pareto distribution is given by

\[ f(x | \theta) = \theta x^{-\theta - 1}, x \geq 1, \theta > 0 \quad (1) \]

Its cumulative distribution function is given by

\[ F(x | \theta) = 1 - x^{-\theta}, x \geq 1, \theta > 0 \quad (2) \]

Hence the failure rate of the distribution becomes

\[ h(x) = \frac{\theta}{x}, x \geq 1, \theta > 0 \quad (3) \]

It is very common that the lifetimes of some test units may not be able to be recorded exactly. For example, in type-II censoring, the test ceases after a predetermined number of failures in order to save time or cost. Moreover, some test units may have to be removed at different stages in the study for various reasons this would lead to a progressive censoring. Progressive Type-II censoring is an important method of obtaining data in lifetime studies. Live units removed early can be readily used in other tests, thereby saving costs to the experimenter. In Statistical inference progressive censoring has received the attention of many authors. Articles by Cohen (1963), Mann (1971), and Viveros and Balakrishnan (1994), Wu et al. (2006) Gajjar and Patel (2008), Patel and Patel (2007), are of some early works on estimation under progressive censoring. Blakrishnan and Sandhu (1996) considered the general progressive censoring scheme to obtain best linear unbiased and maximum likelihood estimator of the parameter of exponential distribution. In this paper we have used such a censoring scheme to determine posterior predictive density function based on Bayesian setup.

Suppose \( n \) units were placed on a life test and first \( r \) failure times \( Y_1, \ldots, Y_r \) are not observed. At failure time \( Y_{r+1}, Y_{r+2} \) units are removed randomly form the survival units on the test, at failure time \( Y_{r+3}, Y_{r+4} \) units are removed randomly form the survival units on the test and so on. Finally, experiment is terminated at the \( m^{th} \) failure at failure time \( Y_m \) with remaining \( R_m \) survivals. Therefore, \( Y_{r+1} \leq \ldots \leq Y_m \) are the lifetimes of the completely observed units to fail and there are \( n_i \) units on test at \((i+1)^{th}\) failure where

\[ n_i = n - i - \sum_{j=r+1}^{i} R_j, \quad i = r+1, \ldots, m-1. \quad (4) \]

Here \( R_{r+1}, R_{r+2}, \ldots, R_m \) are fixed numbers predetermined by the experimenter. The general form of the likelihood function based on the above described general progressive type-II censoring is given by:

\[ L(\theta, x) = \frac{n!}{r!(n-r)!} \left( \prod_{j=r}^{m} n_j \right) \left[ F(t_{r+1}) \right]^{m-1} \prod_{i=r+1}^{m} \left[ f(t_i, \theta) [1 - F(t_i)]^{R_i} \right]^{R_i} \quad (5) \]

Using probability density function and cumulative distribution function from (1) and (2) we have the likelihood function as

\[ L(\theta, x) = c \theta^{m-r} \left( 1 - \frac{\theta}{r+1} \right) \prod_{i=r+1}^{m} \left( \frac{\theta R_i}{x} \right) \prod_{i=r+1}^{m} \left( \frac{\theta}{x} \right)^{R_i} - \theta - 1 \quad (6) \]
where
\[ c = \frac{n!}{r!(n-r)!} \prod_{j=r}^{n-j} \]

To obtain posterior distribution of parameter \( \theta \), here we use the gamma conjugate prior for \( \theta \) as given by
\[ \pi(\theta) = \frac{\delta^\theta \cdot e^{-\delta \theta}}{\Gamma(\nu)} \cdot \theta > 0, \nu > 0, \delta > 0 \]

Here the posterior distribution of the parameter \( \theta \) can be obtained as
\[ \pi(\theta | x) = \frac{L(\theta, x) \pi(\theta)}{\int_{\theta} L(\theta, x) \pi(\theta) \, d\theta} = \frac{\theta^{m+r+v-1} \sum_{j=0}^{r} h_1(j) \cdot \theta^{j} \left( j \log x_{r+1} + A_{rm} + \delta \right)}{\Gamma(m+r+v) \sum_{j=0}^{r} h_1(j) \left( j \log x_{r+1} + A_{rm} + \delta \right)^{m-r+v}} \]

where
\[ A_{rm} = \sum_{i=r+1}^{m} \log x_i \left( R_i + 1 \right) \]
\[ h_1(j) = (-1)^{-j} \binom{r}{j}, j = 0, 1, ..., r \]

From (2.1) and (2.9) the posterior predictive distribution can be obtained using the result
\[ f(t | x) = \int_{0}^{\infty} f(t | \theta) \pi(\theta | x) \, d\theta \]

as
\[ f(t | x) = \sum_{j=0}^{r} \frac{h_1(j) \left( (j \log x_{r+1} + A_{rm} + \delta + \log t)^{m-r+v} \right)}{\sum_{j=0}^{r} h_1(j) \left( (j \log x_{r+1} + A_{rm} + \delta + \log t)^{m-r+v} \right)^{r}} \]

Hence the posterior predictive cumulative distribution function can be obtained as
\[ F(w | x) = \int_{0}^{w} f(t | x) \, dt = \sum_{j=0}^{r} \frac{h_1(j) \left( (j \log x_{r+1} + A_{rm} + \delta + \log t)^{m-r+v} \right)}{\sum_{j=0}^{r} h_1(j) \left( (j \log x_{r+1} + A_{rm} + \delta + \log t)^{m-r+v} \right)^{r}} \]
which can be further simplified by taking

\[ y = \ln t \]

\[
F(w|x) = \frac{\ln w \sum_{j=0}^{r} h_1(j) 1}{(j \log x_{r+1} + A_{rm} + \delta + y)^{m-r+v+1}} dy
\]

\[
= \frac{r \sum_{j=0}^{r} h_1(j)}{(j \log x_{r+1} + A_{rm} + \delta)^{m-r+v}} \left[ - (j \log x_{r+1} + A_{rm} + \delta + \ln w)^{-m+r-v} + (j \log x_{r+1} + A_{rm} + \delta)^{-m+r-v} \right]
\]

\[]

Now consider the integration

\[ t_{11} = \int g(t_0 | \theta) \ dt \]

\[
= \frac{u_1}{l_1} \int_{t_{11}}^{u_1} \frac{(m-r+v)}{t} \left( (j \log x_{r+1} + A_{rm} + \delta + \log t)^{m-r+v+1} \right) dt
\]

\[
= \frac{(m-r+v)}{l_1} \int_{t_{11}}^{u_1} \frac{u_1}{r} \sum_{j=0}^{r} h_1(j) 1}{(j \log x_{r+1} + A_{rm} + \delta + \log t)^{m-r+v+1}} dt
\]

\[
= \frac{(m-r+v) l_0}{l_1} \frac{u_1}{r} \sum_{j=0}^{r} h_1(j) 1}{(j \log x_{r+1} + A_{rm} + \delta)^{m-r+v}}
\]

where:

\[ l_0 = \int_{l_1}^{u_1} \frac{1}{(j \log x_{r+1} + A_{rm} + \delta + \log t)^{m-r+v+1}} dt \]
3. Warranty Policy

Here we have considered a combination of the two commonly used warranty policies namely free replacement warranty and pro rata warranty. Under FRW policy, if a product fails during the warranty period, the product is replaced by another product of the same kind free of charge.

Under PRW policy the manufacturer gives compensation to the buyer on the basis of the failure time during the warranty policy, which may be a linear function of the remaining time of the warranty period.

A combination of these two types of policies is called combined FRW/PRW policy.

Here we assume FRW during the period $[0, w_1)$, and PRW during the period $[w_1, w_2)$ where $w_1 \leq w_2$ are positive values. The reimbursing cost function of an item with time length $t$ for combined FRW/PRW policy is given by

$$C_w(t) = \begin{cases} 
S, & 0 \leq t < w_1 \\
\frac{w_2 - t}{w_2 - w_1}, & w_1 \leq t < w_2 \\
0, & t \geq w_2 
\end{cases}$$

(16)

In case of FRW policy ($w_1 = w_2$) the reimbursing cost function reduces to (17), and under PRW policy ($w_1 = 0$) the reimbursing cost function reduces to (18)

$$\begin{align*}
C_w(t) &= \begin{cases} 
S, & 0 \leq t < w_1 \\
0, & t \geq w_1 
\end{cases} \\
C_w(t) &= \begin{cases} 
\frac{w_2 - t}{w_2}, & 0 \leq t < w_2 \\
0, & t \geq w_2 
\end{cases}
\end{align*}$$

(17)  (18)

where $S$ is the selling price of the product which is cost to the buyer.

This cost function is also called the manufacturer loss associated with setting up a warranty.

4. Utility Function

In the combined FRW/PRW policy, the warranty length, say $w_1$ and $w_2$ are determined for a product. To determine the values of $w_1$ and $w_2$ one has to consider a function of warranty policy that measures the monetary utility when the product fails at time $t$.

Here we consider the utility function, used by Wu and Huang (2010) based on the economic benefit function $B(w_1, w_2)$, the warranty cost function $W(t, w_1, w_2)$ and the dissatisfaction cost function $D(t, w_1, w_2)$ defined as (19). The economic benefit function is proposed as (20).

$$U(t, w_1, w_2) = B(w_1, w_2) - W(t, w_1, w_2) - D(t, w_1, w_2)$$

(19)

$$B(w_1, w_2) = A_1 M (1 - e^{-\frac{w_1 + w_2}{2}})$$

(20)

where $A_1$ is the profit per product obtained by manufacturer and $M$ is the potential number of products to be sold with this warranty policy. The parameter $A_2$ can be derived by solving the equation (21), which is the parameter to control the speed of increment in benefit.

$$\frac{B(0, t_w)}{B(t_w, t_w)} = \frac{1 - e^{-\frac{A_2 t_w}{2}}}{1 - e^{-A_2 t_w}}$$

(21)
The ratio shows the percentage of benefit remains when the manufacturer changes the warranty from FRW to PRW. The warranty cost function \( W(t, w_1, w_2) \) is an item \( C_w(t) \) times the expected number of items that fail under the warranty period. The expected number of failures can be determined using the method given by Wu and Huang (2010) based on the posterior predictive cumulative distribution function under the approach of trinomial distribution. Thus, the warranty cost function can be obtained as (22).

\[
W(t, w_1, w_2) = MF(w_1|x)SI[0,w_1](t) + M[F(w_2|x) - F(w_1|x)S(\frac{w_2-t}{w_2-w_1})I[w_1,w_2](t)
\]

(22)

where \( I_{[a, b]}(t) \) is an indicator function which assumes the value one when \( a \leq t < b \), and zero otherwise.

The dissatisfaction cost is the manufacturer’s indirect cost, when the product fails during the warranty period, or fails during time just after warranty, such cost function is used by Djamaludin et al. (1996).

Under the combined FRW/PRW policy we have used the dissatisfaction cost function considered by Wu and Huang (2010) as (23).

\[
D(t, w_1, w_2) = D_1(t, w_1) + D_2(t, w_1, w_2) + D_3(t, w_2)
\]

(23)

In case of FRW policy, when product fails in the time period \([0, w_1]\), the dissatisfaction cost is a proportion \( q_1(0 < q_1 < 1) \) of the sales price \( S \), multiplied by the expected number of failures, i.e.

\[
D_1(t, w_1) = M\cdot F(w_1|x)\cdot S\cdot q_1\cdot I[0,w_1](t)
\]

(24)

The second component is for the product fails during the time interval \([w_1, w_2]\). Here it is assumed the dissatisfaction cost of an item linearly decreases with time with maximum \( S\cdot q_1 \) and minimum \( S\cdot q_2 \).

Hence,

\[
D_2(t, w_1, w_2) = M\cdot [F(w_2|x) - F(w_1|x)]\cdot S\cdot q_1\cdot (S\cdot q_1 - S\cdot q_2)\cdot \frac{t-w_1}{w_2-w_1}\cdot I[w_1,w_2](t)
\]

(25)

And the third component \( D_3(t, w_2) \) is for the product fails after the expiration of warranty, but the customer may still be unsatisfied with the product unless its lifetime exceeds a specified value \( L, L > w_2 \).

Here \( D_3(t, w_2) \) decreases linearly with time \( t \), reaching to zero when lifetime is \( L \) and given by

\[
D_3(t, w_2) = M\cdot [F(L|x) - F(w_2|x)]\cdot S\cdot q_2\cdot \frac{L-t}{L-w_2}\cdot I[w_2,L](t)
\]

(26)

The value of \( L \) may be considered as the mean or median or percentile of the posterior predictive distribution given in (12).

5. Optimal Warranty

The optimal warranty \((w_1^*, w_2^*)\) is that which maximize the expected value of the utility function \( EU \) with expectation over the posterior predictive distribution.

That is

\[
E(U(T,w_1,w_2)) = \int_{0}^{w} U(t,w_1,w_2) f(t|x)dt
\]

(27)

Using the equation (19) and (21) in the above equation (27), we get the expression for the expected utility function as (28).
After some mathematical manipulation we get the expected utility function as (29).

\[
E(U(T, w_1, w_2)) = \int_0^\infty \left[ B(w_1, w_2) - W[t, w_1, w_2] - D[t, w_1, w_2] \right] f(t | x) dt
\]

(28)

where

\[
E(U(T, w_1, w_2)) = I_1 - I_2 - I_3
\]

(29)

\[
I_1 = \int_0^\infty B(w_1, w_2) f(t | x) dt = MA_1 (1 - e^{-\frac{w_1 + w_2}{2}})
\]

(30)

\[
I_2 = \int_0^\infty W[t, w_1, w_2] f(t | x) dt
\]

Using this formula we have

\[
I_2 = M \left[ S[F(x | w_1)]^2 - S[F(x | w_2)] [I_{2.1} - I_{2.2}] \right]
\]

(31)

\[
= M \left[ \sum_{j=0}^{r} h_1(j) \frac{- (j \log x + 1 + A_{rm} + \delta + \ln w_1)^{-m + r - v}}{+ (j \log x + 1 + A_{rm} + \delta)^m - r + v} \right]^2
\]

\[
- \sum_{j=0}^{r} h_1(j) \frac{1}{(j \log x + 1 + A_{rm} + \delta)^m - r + v}
\]

\[
[ I_{2.1} - I_{2.2} ]
\]

(32)

where,

\[
I_{2.1} = \int \frac{w_2}{w_1} f(\theta | x) dt = \left( \frac{w_2}{w_2 - w_1} \right) [F(w_2 | x) - F(w_1 | x)]
\]
\[
\frac{w_2}{w_2 - w_1} \left[ \sum_{j=0}^{r} h_1(j) \frac{1}{(j \log x_{r+1} + A_{rm} + \delta + \log w_2)^{m+r+v}} \right]
\]
\[
I_{2.2} = \frac{w_2}{w_2 - w_1} \int \frac{f(q|x)}{w_1} dt
\]
\[
= \frac{w_2}{w_2 - w_1} \int \frac{f(q|x)}{w_1} dt
\]
\[
= \frac{1}{w_2 - w_1} (m - r + v) I_{12}
\]
\[
I_{12} = \frac{w_2}{w_1} \sum_{j=0}^{r} h_1(j) \frac{1}{(j \log x_{r+1} + A_{rm} + \delta + \log t)^{m+r+v}} dt
\]
\[
I_3 = M \left[ S q_1 [F(w_1|x)]^2 - S [F(w_2|x) - F(w_1|x)] [I_{3.1} - I_{3.2}] - \left[ S q_2 [F(L|x) - F(w_2|x)] [I_{3.3} - I_{3.4}] \right. \right]
\]
\[
I_{3.1} = \frac{w_2}{w_1} \int_{w_1}^{w_2} q_1 f(q|x) dt = q_1 [F(w_2|x) - F(w_1|x)]
\]
\[
= q_1 \left[ \sum_{j=0}^{r} h_1(j) \frac{1}{(j \log x_{r+1} + A_{rm} + \delta + \log t)^{m+r+v}} \right]
\]
\[
= \frac{w_2}{w_2 - w_1} \int_{w_1}^{w_2} \frac{q_1 - q_2}{w_2 - w_1} \frac{f(q|x)}{dt}
\]
Using we can get

\[
I_{3.2} = \frac{q_1 - q_2}{w_2 - w_1} \left[ \frac{w_2}{w_1} \left[ \int_{\gamma} (\rho (t | x) - \gamma (w_2 | x)) \int_{w_1} \right] \right]
\]

\[
I_{3.3} = \frac{L}{L - w_2} \left[ (F (L | x) - F (w_2 | x)) \right]
\]

\[
I_{3.4} = \frac{1}{L - w_2} \int_{w_2} \left[ \frac{L}{\gamma (t | x) \int_{w_2}} \right]
\]

(38) (39) (40)

and again using (14) we have

\[
I_{3.4} = \frac{1}{L - w_2} \int_{w_2} \frac{L}{\gamma (t | x) \int_{w_2}}
\]

\[
I_{3.4} = \frac{1}{L - w_2} \int_{w_2} \left[ \frac{L}{\gamma (t | x) \int_{w_2}} \right]
\]

(41)

where,

\[
I_{12} = \int_{w_2} \int_{w_2} \frac{1}{\gamma (t | x) \int_{w_2}}
\]

(42)

where \(I_{12}\) and \(I_{2\ell}\) are same as defined the integral in (7).

Using (30) to (36) in (28) we will get an expression for expected utility function.

Thus the optimal warranty \((w_1^*, w_2^*)\) is given by the solution to the optimization problem.
\[(w_1^*, w_2^*) = \arg \max_{w_1 \leq w_2 \in R^+} E(U(T, W_1, W_2))\]

Where \(R^+\) denotes the set of positive real numbers.

This is difficult to solve analytically but computer program can also be prepared to solve it.

**6. Numerical example**

To illustrate the theoretical results we consider the following example:

Let us assume the selling price of the product whose production cost is Rs. 175, fixed by the manufacturer is S=Rs. 250 so that the profit per product becomes \(A_1 = Rs. 75\). We further assume that the manufacturer fixed the proportions of loss from consumer dissatisfaction for timeperiod \([0, w_1]\) as \(q_1 = 0.2\) and for time period \([w_1, w_2]\) as \(q_2 = 0.1\). Suppose that the life time of the product follows Pareto distribution given in (1).

The life times of such 15 products, generated by taking \(\theta = 2\) are given below:

1.019332, 1.140674, 1.165424, 1.183377, 1.212933, 1.325606, 1.423381, 1.426641, 1.52754, 1.552468, 1.69869, 1.831647, 2.121587, 2.387227, 2.640563

From the above data we construct the general progressive type-II censored data with standard notations: \(i = i\)-th failure observed, \(x_i = i\)-th failure observed time, \(R_i = \) number of withdrawals at i-th failure observed, presented in Table 1. Here we have \(n=15, m=9, r=3\).

Manufacturers also assume that the consumer satisfies the product if its life time is at least \(L\) which is the median of the posterior predictive distribution. The standard warranty under the FRW policy is set as \(10^\theta\) percentile of the posterior predictive distribution which is denoted by \(t_w\). Suppose that the manufacturer wishes to set the percentage of benefit remains to be 0.8 (80\%) under combined policy, then putting this value in the equation (21) we get the value of \(A_2\). The values of \(L\) and \(t_w\) are shown in Table 2. The values of \(L\) and \(t_w\) are shown in Table 2 to Table 6 for different values of \(\delta\) and \(\nu\).

**Table 1. General progressive type-II censored data**

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| \(x_i\) | - | - | 1.183377 | 1.423381 | 1.552466 | 1.69869 | 1.831647 | 2.387227 |
| \(R_i\) | - | - | 2 | 2 | 0 | 0 | 1 | 1 |

Based on the above assumptions the optimal warranty length and maximum value of expected utility function (MU) under FRW, PRW and MIX(combined) policies are calculated and the results are shown in the Table 2 to Table 6.

**Table 2. Values of \(L\), \(t_w\) and \(W_1^\ast\), \(W_2^\ast\) and MU under fixed value of \(\delta\) and different values of \(\nu\).**

| \(\delta\) | \(\nu\) | Policy | \(L\) | \(t_w\) | \(W_1^\ast\) | \(W_2^\ast\) | MU |
|---|---|---|---|---|---|---|---|
| 5 | 5 | FRW | 1.724 | 1.085 | 1.048 | - | 51.390025 |
| | | PRW | | | - | 1.2549 | 55.89536 |
| | | MIX | | | 1.032 | 1.0807 | 66.54556 |
Table 2. Values of $L$, $t_w$, $W_1^*$, $W_2^*$ and MU under fixed value of $\delta$ and different values of $v$ (continued)

| $\delta$ | $v$ | Policy | L   | $t_w$ | $W_1^*$ | $W_2^*$ | MU     |
|----------|-----|--------|-----|-------|---------|---------|--------|
| 5        | 10  | FRW    | 1.49| 1.062 | 1.029   | -       | 51.71353 |
|          |     | PRW    | -   | -     | 1.1741  | 55.88600 |
|          |     | MIX     | 1.02| 1.0519| 66.69742 |
|          | 15  | FRW    | 1.369| 1.049| 1.02   | -       | 51.93064 |
|          |     | PRW    | -   | 1.1305  | 55.8753 |
|          |     | MIX     | 1.05 | 1.0386 | 66.79666 |
|          | 20  | FRW    | 1.296| 1.04  | 1.015  | -       | 52.0927 |
|          |     | PRW    | -   | 1.1036  | 55.86541 |
|          |     | MIX     | 1.011| 1.0296 | 66.875817 |
|          | 25  | FRW    | 1.247| 1.034 | 1.012  | -       | 52.2044 |
|          |     | PRW    | -  | 1.0855  | 55.8547 |
|          |     | MIX     | 1.01  | 1.0253 | 66.925033 |

Table 3. Values of $L$, $t_w$, $W_1^*$, $W_2^*$ and MU under fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | L   | $t_w$ | $W_1^*$ | $W_2^*$ | MU     |
|----------|-----|--------|-----|-------|---------|---------|--------|
| 5        | 10  | FRW    | 2.222| 1.127 | 1.094   | -       | 50.96900 |
|          |     | PRW    | -  | 1.4124  | 56.76578 |
|          |     | MIX     | 1.058| 1.1445 | 66.34413 |
|          | 15  | FRW    | 1.794| 1.092 | 1.054  | -       | 51.31276 |
|          |     | PRW    | -   | 1.2795  | 56.989761 |
|          |     | MIX     | 1.036| 1.0902 | 66.51141 |
| 10       | 15  | FRW    | 1.586| 1.072 | 1.037  | -       | 51.57196 |
|          |     | PRW    | -   | 1.2086  | 55.541074 |
|          |     | MIX     | 1.025| 1.064  | 66.63287 |
|          | 20  | FRW    | 1.463| 1.059 | 1.037  | -       | 51.57196 |
|          |     | PRW    | -   | 1.2086  | 54.541074 |
|          |     | MIX     | 1.019| 1.0492 | 66.63287 |
|          | 25  | FRW    | 1.383| 1.05  | 1.021  | -       | 51.91555 |
|          |     | PRW    | -   | 1.1358  | 53.05522 |
|          |     | MIX     | 1.015| 1.0396 | 66.79258 |
Table 4. Values of $L$, $t_w$, $W_1^*$, $W_2^*$ and MU under fixed value of $\delta$ and different values of $\nu$.

| $\delta$ | $\nu$ | Policy | $L$   | $t_w$ | $W_1^*$ | $W_2^*$ | MU       |
|----------|-------|--------|-------|-------|---------|---------|----------|
| 5        | 5     | FRW    | 2.864 | 1.17  | 1.159   | -       | 50.752035|
|          |       | PRW    | 2.16  | 1.123 | 1.089   | -       | 51.00722 |
|          |       | MIX    | 1.836 | 1.096 | 1.055   | 1.1373  | 66.364997|
| 20       | 5     | FRW    | 1.651 | 1.079 | 1.042   | -       | 51.48029 |
|          |       | PRW    | 1.533 | 1.067 | 1.033   | -       | 51.645853|
|          |       | MIX    | 1.386 | 1.057 | 1.022   | 1.0853  | 66.6682  |

Table 5. Values of $L$, $t_w$, $W_1^*$, $W_2^*$ and MU under fixed value of $\delta$ and different values of $\nu$.

| $\delta$ | $\nu$ | Policy | $L$   | $t_w$ | $W_1^*$ | $W_2^*$ | MU       |
|----------|-------|--------|-------|-------|---------|---------|----------|
| 5        | 5     | FRW    | 3.691 | 1.215 | 1.245   | -       | 50.72018 |
|          |       | PRW    | 2.601 | 1.154 | 1.133   | -       | 51.03609 |
|          |       | MIX    | 2.125 | 1.12  | 1.086   | -       | 51.04614 |
| 20       | 5     | FRW    | 1.864 | 1.099 | 1.061   | -       | 51.23527 |
|          |       | PRW    | 1.699 | 1.084 | 1.047   | -       | 51.41336 |
|          |       | MIX    | 1.031 | 1.078 | 1.039   | 1.0989  | 66.47521 |

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Table 6. Values of $L$, $t_w$, $W_1^*$, $W_2^*$ and MU under fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $L$  | $t_w$ | $W_1^*$ | $W_2^*$ | MU      | Std $W_1^*$ | Std $W_2^*$ | Std MU |
|---------|-----|--------|------|-------|---------|---------|---------|------------|------------|--------|
| 5       |     | FRW    | 4.757| 1.262 | 1.355   | -       | 50.86832 | -          | -          | -      |
|         |     | PRW    | -    | -     | -       | 1.9744  | 56.32803 |            |            |        |
|         |     | MIX     | 1.174| 1.4519| 1.174   | 1.4519  | 66.19458 |            |            |        |
| 10      |     | FRW    | 3.132| 1.187 | 1.188   | -       | 50.870394 | -          | -          | -      |
|         |     | PRW    | -    | -     | -       | 1.6543  | 56.998832 |            |            |        |
|         |     | MIX     | 1.105| 1.2633| 1.105   | 1.2633  | 66.204936 |            |            |        |
| 25      | 15  | FRW    | 2.461| 1.145 | 1.119   | -       | 50.87041 |            |            |        |
|         |     | PRW    | -    | -     | -       | 1.4836  | 57.162748 |            |            |        |
|         |     | MIX     | 1.071| 1.1771| 1.071   | 1.1771  | 66.29345 |            |            |        |
| 20      |     | FRW    | 2.103| 1.119 | 1.084   | -       | 51.047785 |            |            |        |
|         |     | PRW    | -    | -     | -       | 1.3796  | 57.58422 |            |            |        |
|         |     | MIX     | 1.052| 1.1303| 1.052   | 1.1303  | 66.38628 |            |            |        |
| 25      |     | FRW    | 1.883| 1.1    | 1.063   | -       | 51.241199 |            |            |        |
|         |     | PRW    | -    | -     | -       | 1.3102  | 57.593349 |            |            |        |
|         |     | MIX     | 1.041| 1.1024| 1.041   | 1.1024  | 66.48072 |            |            |        |

7. Simulation study

In this section we have carried out a simulation study considering the two values of the parameter of the Pareto life time model as $\theta = 2$ and 12 and keep other necessary values same as defined in the numerical example. Also simulation is done 1000 times and the average values of warranty length and maximum value of expected utility function are calculated along with their standard errors in case of all the three policies. All the calculations are done by preparing a computer program in ‘Visual Basic’ language. The results are shown in the Table 7 to Table 16. Table 7 to Table 11 contain optimum warranty length and expected utility function with their standard errors for $\theta = 2$, $n = 20$ and different values of prior parameter $\delta$ and $v$ under FRW, PRW and combined policy and the Table 12 to Table 16 are for $\theta = 12$.

Table 7. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W_1^*$ | $W_2^*$ | MU     | Std $W_1^*$ | Std $W_2^*$ | Std MU |
|---------|-----|--------|---------|---------|--------|------------|------------|--------|
| 5       | 5   | FRW    | 1.2082  | -       | 50.896268 | 0.08067 | -          | 0.08704 |
|         |     | PRW    | -       | 1.7578  | 57.429566 | -          | 0.16433 | 0.69682 |
|         |     | MIX     | 1.0945  | 1.2388  | 66.26433 | 0.03401 | 0.08759 | 0.07806 |
| 10      | 10  | FRW    | 1.1234  | -       | 50.900154 | 0.03196 | -          | 0.13927 |
|         |     | PRW    | -       | 1.50852 | 57.447571 | -          | 0.10935 | 0.55773 |
|         |     | MIX     | 1.0571  | 1.1429  | 66.39094 | 0.0198  | 0.04915 | 0.11284 |
Table 7. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $v$ (continued)

| $\delta$ | $v$ | Policy | $W_1^*$ | $W_2^*$ | MU | Std $W_1^*$ | Std $W_2^*$ | Std MU |
|----------|-----|--------|---------|---------|----|-------------|-------------|--------|
| 15       | 5   | FRW    | 1.1032  | -       | 50.944721 | 0.00733     | -        | 0.50111 |
|          |     | PRW    | -       | 1.37519 | 57.556511 | -           | 0.07988 | 0.45169 |
|          |     | MIX     | 1.0392  | 1.087   | 66.5105   | 0.01329    | 0.03255 | 0.11668 |
| 20       | 1.1 | FRW    | 1.1     | -       | 50.956233 | 0           | -        | 1.03887 |
|          |     | PRW    | -       | 1.29504 | 57.708193 | -           | 0.06198 | 0.3752  |
|          |     | MIX     | 1.0294  | 1.0745  | 66.6075   | 0.00937    | 0.02323 | 0.11094 |
| 25       | 1.1 | FRW    | 1.1     | -       | 50.976124 | 0           | -        | 1.64837 |
|          |     | PRW    | -       | 1.24111 | 57.747196 | -           | 0.05014 | 0.32034 |
|          |     | MIX     | 1.0231  | 1.059   | 66.68464  | 0.00715    | 0.01759 | 0.10328 |

Table 8. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W_1^*$ | $W_2^*$ | MU | Std $W_1^*$ | Std $W_2^*$ | Std MU |
|----------|-----|--------|---------|---------|----|-------------|-------------|--------|
| 5        |     | FRW    | 1.3056  | -       | 50.79160 | 0.10454    | -         | 0.17366 |
|          |     | PRW    | -       | 1.95614 | 58.23363 | -          | 0.17323  | 0.64203 |
|          |     | MIX     | 1.1337  | 1.3425  | 66.21463 | 0.03928   | 0.10621  | 0.02421 |
| 10       | 1.154 | FRW    | 1.1645  | -       | 50.80184 | 0.05013   | -         | 0.08198 |
|          |     | PRW    | -       | 1.64149 | 58.91863 | -         | 0.11588  | 0.53688 |
|          |     | MIX     | 1.0805  | 1.2016  | 66.28942 | 0.0235   | 0.05918  | 0.07448 |
| 15       | 20 | FRW    | 1.1145  | -       | 50.88684 | 0.02175   | -         | 0.10227 |
|          |     | PRW    | -       | 1.47348 | 58.95057 | -         | 0.0847   | 0.44607 |
|          |     | MIX     | 1.0547  | 1.1368  | 66.39713 | 0.01553  | 0.03845  | 0.09167 |
| 20       | 25 | FRW    | 1.1015  | -       | 50.89226 | 0.00418   | -         | 0.32332 |
|          |     | PRW    | -       | 1.37106 | 58.97071 | -         | 0.06568  | 0.37446 |
|          |     | MIX     | 1.0405  | 1.1018  | 66.49518 | 0.01115  | 0.02708  | 0.09434 |
| 25       |     | FRW    | 1.1     | -       | 50.90296 | 0         | -         | 0.6591  |
|          |     | PRW    | -       | 1.30287 | 58.9961  | -         | 0.05307  | 0.32171 |
|          |     | MIX     | 1.0316  | 1.0801  | 66.57804 | 0.00839  | 0.02054  | 0.09127 |
### Table 9. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W_1^*$ | $W_2^*$ | MU   | Std $W_1^*$ | Std $W_2^*$ | Std MU |
|---------|-----|--------|--------|--------|------|------------|------------|-------|
| 5       | 5   | FRW    | 1.4288 | -      | 50.08147 | 0.12795    | -         | 0.31025|
|         |     | PRW    | 1.1779 | 2.16415| 57.96311 | 0.18066    | 0.58591   |
|         |     | MIX    | 1.2347 | 1.43562| 66.22633 | 0.04346    | 0.012397  | 0.04228|
| 10      | 5   | FRW    | 1.2244 | -      | 50.75749 | 0.06338    | -         | 0.05318|
|         |     | PRW    | 1.1779 | 1.78127| 58.54572 | 0.12146    | 0.51092   |
|         |     | MIX    | 1.1071 | 1.2703 | 66.22879 | 0.02652    | 0.06923   | 0.03882|
| 15      | 5   | FRW    | 1.1424 | -      | 50.83473 | 0.03518    | -         | 0.0882 |
|         |     | PRW    | 1.0727 | 1.5762 | 58.61791 | 0.08906    | 0.43241   |
|         |     | MIX    | 1.0938 | 1.1815 | 66.31057 | 0.01758    | 0.04429   | 0.0672 |
| 10      | 5   | FRW    | 1.1092 | -      | 50.92093 | 0.01531    | -         | 0.08312|
|         |     | PRW    | 1.0533 | 1.45103| 58.97843 | 0.069      | 0.37278   |
|         |     | MIX    | 1.0727 | 1.1332 | 66.40187 | 0.01257    | 0.0311    | 0.07729|
| 15      | 5   | FRW    | 1.1008 | -      | 50.93843 | 0.0024     | -         | 0.22862|
|         |     | PRW    | 1.0413 | 1.36773| 58.99363 | 0.05577    | 0.31848   |
|         |     | MIX    | 1.0413 | 1.1039 | 66.48502 | 0.00952    | 0.0233    | 0.07888|

### Table 10. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W_1^*$ | $W_2^*$ | MU   | Std $W_1^*$ | Std $W_2^*$ | Std MU |
|---------|-----|--------|--------|--------|------|------------|------------|-------|
| 5       | 5   | FRW    | 1.5764 | -      | 50.46131 | 0.15039    | -         | 0.43086|
|         |     | PRW    | 2.3801 | 1.6071 | 66.28577 | 0.04629    | 0.14061   | 0.07783|
|         |     | MIX    | 1.2259 | 1.6071 | 66.28577 | 0.04629    | 0.14061   | 0.07783|
| 10      | 5   | FRW    | 1.2987 | -      | 50.81843 | 0.07584    | -         | 0.11528|
|         |     | PRW    | 1.92715| 1.5936 | 59.13768 | 0.1263     | 0.47799   |
|         |     | MIX    | 1.1368 | 1.3495 | 66.20509 | 0.02934    | 0.07916   | 0.01188|
| 15      | 5   | FRW    | 1.8145 | -      | 51.76130 | 0.04449    | -         | 0.0486 |
|         |     | PRW    | 1.6836 | 1.5936 | 59.13768 | 0.09294    | 0.41472   |
|         |     | MIX    | 1.0926 | 1.2324 | 66.25028 | 0.01971    | 0.05047   | 0.04428|
| 10      | 5   | FRW    | 1.1291 | -      | 51.86136 | 0.02661    | -         | 0.08593|
|         |     | PRW    | 1.53474| 1.5936 | 59.41127 | 0.07212    | 0.36209   |
|         |     | MIX    | 1.0677 | 1.169  | 66.3271  | 0.01405    | 0.03528   | 0.0601 |
| 15      | 5   | FRW    | 1.1061 | -      | 51.93533 | 0.01132    | -         | 0.0704 |
|         |     | PRW    | 1.43551| 1.5936 | 59.89447 | 0.05824    | 0.31493   |
|         |     | MIX    | 1.0524 | 1.131  | 66.40592 | 0.0108     | 0.02648   | 0.06668|
Table 11. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $\nu$.

| $\delta$ | $\nu$ | Policy | $W_1^*$ | $W_2^*$ | MU       | Std$W_1^*$ | Std$W_2^*$ | Std MU   |
|----------|-------|--------|---------|---------|----------|-----------|-----------|---------|
| 5        | FRW   | 1.7477 | -       | 50.96146| 0.17091  | -         | 0.52032  |
|          | PRW   | -      | 2.60355 | 57.22238| -        | 0.19356  | 0.47313  |
|          | MIX   | 1.2766 | 1.7662  | 66.38128| 0.04818  | 0.1593   | 0.10332  |
| 25       | FRW   | 1.3859 | -       | 50.97337| 0.08852  | -         | 0.19118  |
| 10       | PRW   | -      | 2.07829 | 57.686805| -         | 0.13051  | 0.44795  |
|          | MIX   | 1.1697 | 1.44    | 66.2144  | 0.03150  | 0.08894  | 0.02539  |
| 15       | FRW   | 1.2359 | -       | 51.74965 | 0.05172  | -         | 0.04074  |
|          | PRW   | -      | 1.79528 | 57.71085 | -        | 0.09634  | 0.40015  |
|          | MIX   | 1.115  | 1.2903  | 66.2146  | 0.02158  | 0.05677  | 0.02251  |
| 20       | FRW   | 1.161  | -       | 51.78152 | 0.03365  | -         | 0.05852  |
|          | PRW   | -      | 1.62166 | 57.84023 | -        | 0.07488  | 0.35456  |
|          | MIX   | 1.0839 | 1.2097  | 66.27042 | 0.01548  | 0.03947  | 0.04396  |
| 25       | FRW   | 1.1207 | -       | 51.8916  | 0.0209   | -         | 0.07557  |
|          | PRW   | -      | 1.50593 | 57.27091 | -        | 0.06053  | 0.31095  |
|          | MIX   | 1.0647 | 1.1612  | 66.3401  | 0.01184  | 0.02914  | 0.05447  |

Table 12. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $\nu$.

| $\delta$ | $\nu$ | Policy | $W_1^*$ | $W_2^*$ | MU       | Std$W_1^*$ | Std$W_2^*$ | Std MU   |
|----------|-------|--------|---------|---------|----------|-----------|-----------|---------|
| 5        | FRW   | 1.1    | 48.292679| 0       | 0.96325  |
|          | PRW   | 1.17999| 54.397966| 0.00284 | 0.04452  |
|          | MIX   | 1.0196 | 1.0506  | 66.71836| 0.00683  |
| 10       | FRW   | 1.1    | 43.935445| 0       | 1.85842  |
|          | PRW   | 1.12884| 54.496211| 0.00166 | 0.03613  |
|          | MIX   | 1.0127 | 1.0335  | 66.84496| 0.00434  |
| 15       | FRW   | 1.1    | 38.269195| 0       | 2.85193  |
|          | PRW   | 1.09691| 54.766834| 0.00794 | 0.07113  |
|          | MIX   | 1.0102 | 1.0258  | 66.9181 | 0.00579  |
| 20       | FRW   | 1.1    | 31.63560 | 0       | 3.91434  |
|          | PRW   | 1.07715| 55.826854| 0       | 0.05596  |
|          | MIX   | 1.01   | 1.0225  | 66.94067| 0.00170  |
| 25       | FRW   | 1.1    | 24.25081| 0       | 4.99327  |
|          | PRW   | 1.06385| 55.914225| 0       | 0.0477   |
|          | MIX   | 1.01   | 1.0206  | 66.9943 | 0.0008   | 0.03468  |
Table 13. Values of $W^*_1$, $W^*_2$, MU, Std $W^*_1$, Std $W^*_2$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W^*_1$ | $W^*_2$ | MU | Std $W^*_1$ | Std $W^*_2$ | Std MU |
|----------|-----|--------|---------|---------|-----|------------|------------|-------|
| 5        |     | FRW 1.1 | 46.67993 | -       | 0   | -          | -          | 0.09053 |
|          |     | PRW -   | 55.016151 | -       | 0.02251 | 0.13607   |
|          |     | MIX 1.041 | 66.47006 | 0.004   | 0.00979 | 0.03249   |
| 10       |     | FRW 1.1 | 46.7765  | -       | 0   | -          | -          | 0.35903 |
|          |     | PRW -   | 55.0231  | -       | 0.01505 | 0.09677   |
|          |     | MIX 1.0256 | 66.6318  | 0.00254 | 0.0058  | 0.0301    |
| 15       |     | FRW 1.1 | 47.791352| -       | 0   | -          | -          | 0.66231 |
|          |     | PRW -   | 55.027895| -       | 0.0111 | 0.07579   |
|          |     | MIX 1.0184 | 66.74239 | 0.00162 | 0.00388 | 0.02642   |
| 20       |     | FRW 1.1 | 49.157105| -       | 0   | -          | -          | 0.99702 |
|          |     | PRW -   | 55.047063| -       | 0.00874 | 0.05969   |
|          |     | MIX 1.0141 | 66.82067 | 0.0013  | 0.00319 | 0.02292   |
| 25       |     | FRW 1.1 | 50.995223| -       | 0   | -          | -          | 1.35378 |
|          |     | PRW -   | 55.868342| -       | 0.00716 | 0.05032   |
|          |     | MIX 1.0114 | 66.87896 | 0.00102 | 0.00239 | 0.0205    |

Table 14. Values of $W^*_1$, $W^*_2$, MU, Std $W^*_1$, Std $W^*_2$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W^*_1$ | $W^*_2$ | MU | Std $W^*_1$ | Std $W^*_2$ | Std MU |
|----------|-----|--------|---------|---------|-----|------------|------------|-------|
| 5        |     | FRW 1.1215 | 50.85341 | -       | 0.00999 | -          | -          | 0.03278 |
|          |     | PRW -   | 55.26216 | -       | 0.02495 | 0.12834   |
|          |     | MIX 1.0697 | 66.30151 | 0.00494 | 0.01247 | 0.02018   |
| 10       |     | FRW 1.1 | 50.92361 | -       | 0   | -          | -          | 0.05776 |
|          |     | PRW -   | 55.36127 | -       | 0.01656 | 0.1012    |
|          |     | MIX 1.0426 | 66.46055 | 0.00284 | 0.00717 | 0.02323   |
| 15       |     | FRW 1.1 | 50.9534  | -       | 0   | -          | -          | 0.18815 |
|          |     | PRW -   | 55.82827 | -       | 0.01219 | 0.07536   |
|          |     | MIX 1.0298 | 66.58526 | 0.00194 | 0.00476 | 0.0226    |
| 20       |     | FRW 1.1 | 50.96380 | -       | 0   | -          | -          | 0.33868 |
|          |     | PRW -   | 56.48382 | -       | 0.00951 | 0.06821   |
|          |     | MIX 1.0225 | 66.6798  | 0.00157 | 0.00363 | 0.02044   |
| 25       |     | FRW 1.1 | 50.9784  | -       | 0   | -          | -          | 0.50654 |
|          |     | PRW -   | 57.24399 | -       | 0.00775 | 0.05431   |
|          |     | MIX 1.018 | 66.75227 | 0.00118 | 0.00284 | 0.0186    |
Table 15. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W_1^*$ | $W_2^*$ | MU     | Std $W_1^*$ | Std $W_2^*$ | Std MU | 
|---------|-----|--------|--------|--------|--------|------------|------------|--------|
| 5       | 5   | FRW    | 1.1956 | -      | 50.71778 | 0.01334    | -          | 0.1007 |
|         |     | PRW    | 1.685  | 55.14570 | -  | 0.02687  | 0.12381 |
|         |     | MIX    | 1.1052 | 1.2644 | 66.21039 | 0.0061  | 0.01571  | 0.00842 |
| 10      | 5   | FRW    | 1.1083 | -      | 50.91495 | 0.00625  | -          | 0.02883 |
|         |     | PRW    | 1.46103 | 56.03845 | - | 0.01791  | 0.09574 |
|         |     | MIX    | 1.0634 | 1.1582 | 66.33262 | 0.00364  | 0.00892  | 0.01655 |
| 20      | 10  | FRW    | 1.1    | -      | 50.94441 | 0        | -          | 0.0377  |
|         |     | PRW    | 1.34199 | 56.37459 | - | 0.01311  | 0.07723 |
|         |     | MIX    | 1.0434 | 1.1091 | 66.45519 | 0.00233  | 0.00573  | 0.01804 |
| 25      | 10  | FRW    | 1.1    | -      | 50.95761 | 0        | -          | 0.12444 |
|         |     | PRW    | 1.26932 | 56.93165 | - | 0.0102  | 0.06599 |
|         |     | MIX    | 1.0326 | 1.0823 | 66.55649 | 0.00169  | 0.0042  | 0.0176 |

Table 16. Values of $W_1^*$, $W_2^*$, MU, Std $W_1^*$, Std $W_2^*$ and Std MU under the fixed value of $\delta$ and different values of $v$.

| $\delta$ | $v$ | Policy | $W_1^*$ | $W_2^*$ | MU     | Std $W_1^*$ | Std $W_2^*$ | Std MU | 
|---------|-----|--------|--------|--------|--------|------------|------------|--------|
| 25      | 5   | FRW    | 1.3001 | -      | 50.76500 | 0.01605  | -          | 0.02121 |
|         |     | PRW    | 1.86147 | 56.79719 | - | 0.02835  | 0.11479 |
|         |     | MIX    | 1.1467 | 1.375 | 66.18847 | 0.00681  | 0.01866  | 0.00163 |
| 10      | 25  | FRW    | 1.1572 | -      | 50.78639 | 0.00852  | -          | 0.01714 |
|         |     | PRW    | 1.59077 | 56.80068 | - | 0.01905  | 0.09378 |
|         |     | MIX    | 1.088  | 1.2198 | 66.24706 | 0.004  | 0.01041  | 0.01015 |
| 15      | 5   | FRW    | 1.1024 | -      | 50.95037 | 0.00332  | -          | 0.0261 |
|         |     | PRW    | 1.43706 | 56.90997 | - | 0.01395  | 0.07835 |
|         |     | MIX    | 1.0598 | 1.1492 | 66.35207 | 0.00668  | 0.00679  | 0.01419 |
| 20      | 25  | FRW    | 1.1    | -      | 50.95766 | 0        | -          | 0.02949 |
|         |     | PRW    | 1.34338 | 56.97975 | - | 0.01082  | 0.06671 |
|         |     | MIX    | 1.0442 | 1.1108 | 66.45164 | 0.00209  | 0.00492  | 0.01479 |
| 25      | 25  | FRW    | 1.1    | -      | 50.98060 | 0        | -          | 0.08617 |
|         |     | PRW    | 1.28092 | 57.00520 | - | 0.00881  | 0.05289 |
|         |     | MIX    | 1.0345 | 1.087 | 66.53715 | 0.00157  | 0.00379  | 0.01472 |
8. Conclusion

In today’s competing market product warranty plays an increasingly significant role in both consumer and commercial transactions. For manufacturer it is important to decide the appropriate warranty length and appropriate warranty policy so that he may increase the demand of his product and hence makes more profit. We have provided an approach to the manufacturers to determine optimal warranty length and warranty policy based on the life time data obtained by conducting life testing experiment using general progressive type-II censoring scheme for the product having Pareto life time distribution. Based on the life data obtained through such a life test a Bayesian predictive distribution is derived to determine the maximum value of utility function and hence an appropriate warranty policy is decided.

Based on the output of the example considered in Section 6 for various combinations of the values of the prior parameters we observed from the Table 2 to Table6 that, for the given data combined policy gives maximum utility followed by PRW and then by FRW policy for any choice of prior parametersδ and ν. We have also examined the effect of change in the value of one prior parameter when the value of other parameter kept fixed. For any fixed value of prior parameter δ as ν increases, maximum utility decreases in all the three types of policies and for keeping ν fixed, as δ increases, maximum utility more or less remains stable. Thus maximum utility has more effect of prior parameter ν compare to the parameter δ.

A simulation study carried out in Section 7 shows a very general effect of the prior parameters on different types of warranty policies. The results are shown in the Table 7 to Table 16. From the Table 7 to Table 16 we observed that combined policy gives maximum utility followed by PRW and then by FRW policy for any choice of prior parameters. For any fixed value of prior parameter δ as ν increases, maximum utility decreases in all the three types of policies and for fixed ν as δ increases, the value of maximum utility fluctuates.

Thus for the product having decreasing failure rate and Pareto life time distribution we suggest to utilize a mixed warranty policy which is a combination of PRW and PRW policies with any values of the prior parameters utilized in the model.

This paper becomes useful to determine optimal warranty of those products which have only Pareto life time distribution which is the limitation of the paper. Many papers are also available to determine the optimal warranty for the product having different life time distributions possessing constant, increasing or decreasing failure rates. The drawback of such papers is to first know the actual life time distribution of the product and then one can apply the appropriate method to decide the optimal warranty. To overcome such a difficulty we are preparing a paper based on the very general life time distribution which may be useful to the product having increasing, constant or decreasing failure rate based on Weibull life time model. But there are many products which possess the failure rate initially decreasing, then after becomes constant for certain period of time and then gradually increases with time. For such types of products these kind of work may not be useful. One has to develop a very general model possessing a bath tub failure rate distribution to determine optimal warranty of such types of product. Very few papers are available in the literature which might be useful to determine the optimum price as well as the optimal warranty of the product; one can do also such kind of work in this direction.

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