Right Unitarity Triangles, Stable CP-violating Phases and Approximate Quark-Lepton Complementarity

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Abstract

Current experimental data indicate that two unitarity triangles of the CKM quark mixing matrix $V$ are almost the right triangles with $\alpha \approx 90^\circ$. We highlight a very suggestive parametrization of $V$ and show that its CP-violating phase $\phi$ is nearly equal to $\alpha$ (i.e., $\phi - \alpha \approx 1.1^\circ$). Both $\phi$ and $\alpha$ are stable against the renormalization-group evolution from the electroweak scale $M_Z$ to a superhigh energy scale $M_X$ or vice versa, and thus it is impossible to obtain $\alpha = 90^\circ$ at $M_Z$ from $\phi = 90^\circ$ at $M_X$. We conjecture that there might also exist a maximal CP-violating phase $\varphi \approx 90^\circ$ in the MNS lepton mixing matrix $U$. The approximate quark-lepton complementarity relations, which hold in the standard parametrizations of $V$ and $U$, can also hold in our particular parametrizations of $V$ and $U$ simply due to the smallness of $|V_{ub}|$ and $|V_{e3}|$.

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I. INTRODUCTION

In the standard model (SM) of electroweak interactions, it is the $3 \times 3$ Cabibbo-Kobayashi-Maskawa (CKM) matrix that provides an elegant and consistent description of the observed phenomena of quark flavor mixing and CP violation [1]. Unitarity is the only but powerful constraint, imposed by the SM itself, on the CKM matrix $V$. This constraint can be expressed as two sets of orthogonality-plus-normalization conditions:

$$\sum_{\alpha} (V_{\alpha i} V_{\alpha j}^*) = \delta_{ij}, \quad \sum_{i} (V_{\alpha i} V_{\beta i}^*) = \delta_{\alpha \beta}, \quad (1)$$

where the Greek subscripts run over the up-type quarks ($u, c, t$) and the Latin subscripts run over the down-type quarks ($d, s, b$). The six orthogonality relations correspond to six triangles in the complex plane, the so-called unitarity triangles. Among them $\Delta_s$, the triangle $\Delta_c$ is most popular because both its three inner angles (defined as $\alpha, \beta$ and $\gamma$ in Fig. 1) and its three sides can well be determined at the $B$-meson factories [3]. The counterpart of $\Delta_s$ is the unitarity triangle $\Delta_c$ (as shown in Fig. 1), which will be measured and reconstructed at the LHC-b [4] and (or) the super-$B$ factory [5]. Note that one of the inner angles of $\Delta_c$ is equal to the inner angle $\alpha$ of $\Delta_s$. Current experimental data [3] tell us that these two triangles are approximately congruent with each other. For example, the inner angles $\xi$ and $\zeta$ of $\Delta_c$ are very close to $\beta$ and $\gamma$ of $\Delta_s$:

$$\xi - \beta \approx \gamma - \zeta \approx \lambda^2 \eta \approx 1^\circ, \quad (2)$$

where $\lambda \approx 0.226$ and $\eta \approx 0.35$ are the well-known Wolfenstein parameters [6] in an $O(\lambda^4)$-expansion of the CKM matrix $V$ [7]. A more striking result is $\alpha \approx 90^\circ$ obtained by the CKMfitter Group [8] and the UTfit Collaboration [9]. If $\alpha = 90^\circ$ holds exactly, then both $\Delta_s$ and $\Delta_c$ will be the right triangles.

The possibility of $\alpha \approx 90^\circ$ was actually conjectured a long time ago in an attempt to explore the realistic texture of quark mass matrices [10], and it has recently been remarked from some different phenomenological points of view [11–13]. Here we are interested in the following questions and possible answers to them:

- What is the immediate consequence of $\alpha = 90^\circ$ on the CKM matrix $V$ and its four independent parameters?

- Could $\alpha \approx 90^\circ$ result from an underlying but more fundamental CP-violating phase $\phi = 90^\circ$ in the quark mass matrices or in the CKM matrix?

- Is the result $\alpha \approx 90^\circ$ or $\phi = 90^\circ$ stable against quantum corrections, for instance, from the electroweak scale $M_Z$ to a superhigh-energy scale $M_X$ (such as the scale of grand unified theories or the scale of neutrino seesaw mechanisms)? In other words, is $\alpha = 90^\circ$ at $M_Z$ possibly a natural low-energy consequence of $\phi = 90^\circ$ at $M_X$ due to the renormalization-group running effect?

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1Here we follow Ref. [2] to name each CKM unitarity triangle by using the flavor index that does not manifest in its three sides.
• Could the $3 \times 3$ Maki-Nakagawa-Sakata (MNS) neutrino mixing matrix $U$ \cite{14} contain a similar maximal CP-violating phase $\varphi = 90^\circ$?

We shall point out that $\alpha = 90^\circ$ simply implies $\text{Re}(V_{tb}V_{ud}V_{td}^*V_{ub}^*) = 0$. Given a very suggestive parametrization of $V$ advocated in Ref. \cite{15}, we show that its CP-violating phase $\phi$ is nearly equal to $\alpha$; i.e., $\phi - \alpha \approx 1.1^\circ$. But we find that both $\phi$ and $\alpha$ are rather stable in the renormalization-group evolution from $M_X$ up to $M_Z$ or vice versa, and thus it is impossible to obtain $\alpha = 90^\circ$ at $M_Z$ from $\phi = 90^\circ$ at $M_X$ by attributing the tiny difference $\phi - \alpha \approx 1.1^\circ$ to radiative corrections. We shall briefly discuss the approximate quark-lepton complementarity relations both in the standard parametrizations of $V$ and $U$ and in our particular parametrizations of $V$ and $U$, and then make a conjecture of the maximal CP-violating phase $\varphi \approx 90^\circ$ for the MNS matrix $U$ at the end of this paper. We hope that some of our points, which might be helpful for building phenomenological models, can soon be tested with more accurate experimental data on quark and lepton flavor mixing parameters.

II. IMPLICATIONS OF $\alpha = 90^\circ$

Let us define the Jarlskog invariant of CP violation $J_q$ for the CKM matrix $V$ \cite{16}:

$$\text{Im} \left( V_{\alpha i}V_{\beta j}^*V_{\alpha j}^*V_{\beta i} \right) = J_q \sum_\gamma \epsilon_{\alpha\beta\gamma} \sum_k \epsilon_{ijk},$$

where the Greek and Latin subscripts run over ($u, c, t$) and ($d, s, b$), respectively. All six unitarity triangles of $V$ have the same area which amounts to $J_q/2$. Triangles $\triangle_s$ and $\triangle_c$ in Fig. 1 correspond to the orthogonality relations

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0,$$
$$V_{tb}V_{ub}^* + V_{ts}V_{us}^* + V_{td}V_{ud}^* = 0.$$  \hspace{1cm} (4)

If $\alpha = 90^\circ$ holds (i.e., both $\triangle_s$ and $\triangle_c$ are right triangles), then we have $\text{Re}(V_{tb}V_{ud}V_{td}^*V_{ub}^*) = 0$ as a straightforward consequence; namely, the rephasing-invariant quartet $V_{tb}V_{ud}V_{td}^*V_{ub}^*$ is purely imaginary. Hence $\alpha = 90^\circ$ implies a certain correlation between the parameters of $V$ in a specific parametrization. Let us illustrate this point by taking two well-known parametrizations of the CKM matrix $V$.

• In the Wolfenstein parametrization of $V$ \cite{6}, we have

$$\text{Re}(V_{tb}V_{ud}V_{td}^*V_{ub}^*) \approx A^2\chi^6 \left[ \rho (1 - \rho) - \eta^2 \right].$$  \hspace{1cm} (5)

So $\alpha = 90^\circ$ coincides with $\eta \approx \sqrt{\rho (1 - \rho)}$ in this parametrization. Taking $\rho \approx 0.135$ as an example, we obtain $\eta \approx 0.34$. Such typical values of $\rho$ and $\eta$ are certainly consistent with current experimental data \cite{3}.

• In the standard parametrization of $V$ recommended by the Particle Data Group \cite{3},

$$\text{Re}(V_{tb}V_{ud}V_{td}^*V_{ub}^*) = \cos^2 \theta_{12} \cos^2 \theta_{13} \sin \theta_{13} \cos^2 \theta_{23} (\tan \theta_{12} \tan \theta_{23} \cos \delta - \sin \theta_{13}).$$  \hspace{1cm} (6)
Then $\alpha = 90^\circ$ leads to $\cos \delta = \sin \theta_{13}/(\tan \theta_{12} \tan \theta_{23})$. Given $\theta_{12} \approx 13^\circ$, $\theta_{13} \approx 0.22^\circ$ and $\theta_{23} \approx 2.4^\circ$ for example [3], the CP-violating phase turns out to be $\delta \approx 66^\circ$. This result is also consistent with the approximate relation $\delta \approx \gamma$ and the present experimental measurement of $\gamma$ [3].

In both cases, however, we see nothing suggestive behind $\alpha = 90^\circ$.

We proceed to consider a different parametrization of $V$ [15], which is more convenient to explore the underlying connection between quark masses and flavor mixing angles:

$$
V = \begin{pmatrix}
  c_u & s_u & 0 \\
  -s_u & c_u & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  e^{-i\phi} & 0 & 0 \\
  0 & c & s \\
  0 & -s & c
\end{pmatrix}
\begin{pmatrix}
  c_d & -s_d & 0 \\
  s_d & c_d & 0 \\
  0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
  c_u s_d c + c_u c_d e^{-i\phi} & s_u c_d c - c_u s_d e^{-i\phi} & s_u s \\
  c_u s_d c - s_u c_d e^{-i\phi} & c_u c_d c + s_u s_d e^{-i\phi} & c_u s \\
  -s_u s & -c_u s & c
\end{pmatrix},
$$

(7)

where $c_{u,d} \equiv \cos \theta_{u,d}$, $s_{u,d} \equiv \sin \theta_{u,d}$, $c \equiv \cos \theta$ and $s \equiv \sin \theta$. The merits of this particular parametrization in understanding quark mass generation and studying heavy flavor physics are striking [15]: (1) it directly follows the chiral expansion of up- and down-type quark mass parametrization in understanding quark mass generation and studying heavy flavor physics closely associated with the light quark sector, in particular with the mass terms of $\mu$ and $d$ quarks; (2) its three mixing angles are simply but exactly related to the precision measurements of $B$-meson physics, $\tan \theta_u = |V_{ub}/V_{cb}|$, $\tan \theta_d = |V_{td}/V_{ts}|$ and $\sin \theta = \sqrt{|V_{ud}|^2 + |V_{cb}|^2}$; (3) the physical meaning of its mixing angles $\theta_u$ and $\theta_d$ can well be interpreted in a variety of quark mass models (see Ref. [2] for a review with extensive references) with the interesting predictions $\tan \theta_u \approx \sqrt{m_u/m_c}$ and $\tan \theta_d \approx \sqrt{m_d/m_s}$; and (4) its CP-violating phase $\phi$ is closely associated with the light quark sector, in particular with the mass terms of $u$ and $d$ quarks. Using Eq. (7) to calculate the inner angle $\alpha$ of $\triangle_s$ and $\triangle_c$, we arrive at

$$
\sin \alpha = \sin \phi \left[ 1 - \left( \tan \theta_u \tan \theta_d \cos \theta \cos \phi + \frac{1}{2} \tan^2 \theta_u \tan^2 \theta_d \cos^2 \theta + \cdots \right) \right]
$$

(8)

with higher-order terms of $\tan \theta_u$ and $\tan \theta_d$ having been omitted. It is clear that $\alpha \approx \phi$ holds to a good degree of accuracy. Taking account of $\theta_u \approx 5.4^\circ$, $\theta_d \approx 11.5^\circ$ and $\theta \approx 2.4^\circ$ for example [17], we obtain either $\alpha \approx 88.9^\circ$ from $\phi = 90^\circ$ or $\phi \approx 91.1^\circ$ from $\alpha = 90^\circ$.

The result $\phi = \alpha \approx 1.1^\circ$ is interesting in the sense that current experimental data might imply $\phi = 90^\circ$ at a superhigh energy scale $M_X$ and $\alpha = 90^\circ$ at the electroweak scale $M_Z$, if radiative corrections happen to compensate for the tiny discrepancy between $\alpha(M_Z)$ and $\alpha(M_X)$.

We shall examine whether this point is true or not in the next section.

Is $\phi$ more fundamental than $\alpha$ in describing the phenomenon of CP violation in the quark sector? The answer to this question should be affirmative if the textures of up- and down-type quark mass matrices ($M_u$ and $M_d$) are parallel and originate from the same underlying

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2It is also worth pointing out that this parametrization is just Euler’s three-dimension rotation matrix if the CP-violating phase $\phi$ is switched off (and a trivial sign rearrangement is made).
dynamics [18]. In this case, \( V = V_u^\dagger V_d \), where \( V_u \) and \( V_d \) are responsible respectively for the diagonalizations of \( M_u^\dagger \) and \( M_d^\dagger \) (i.e., \( V_u^\dagger M_u^\dagger V_u = \text{Diag}\{ m_u^2, m_c^2, m_t^2 \} \) and \( V_d^\dagger M_d^\dagger V_d = \text{Diag}\{ m_d^2, m_s^2, m_b^2 \} \)) and take the following forms:

\[
V_u = \begin{pmatrix}
e^{-i\phi_x} & 0 & 0 \\
0 & c_x & s_x \\
0 & -s_x & c_x \\
\end{pmatrix} \begin{pmatrix}
c_u & -s_u & 0 \\
s_u & c_u & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \\
V_d = \begin{pmatrix}
e^{-i\phi_y} & 0 & 0 \\
0 & c_y & s_y \\
0 & -s_y & c_y \\
\end{pmatrix} \begin{pmatrix}
c_d & -s_d & 0 \\
s_d & c_d & 0 \\
0 & 0 & 1 \\
\end{pmatrix},
\]

where \( c_{x,y} \equiv \cos \theta_{x,y} \) and \( s_{x,y} \equiv \sin \theta_{x,y} \) are defined. It is obvious that \( \theta_y - \theta_x = \theta \) and \( \phi_y - \phi_x = \phi \) hold. Hence \( \phi \) measures the phase difference between up- and down-type quark mass matrices and is the only source of CP violation in the quark sector. Let us make a new phenomenological conjecture of the relationship between \( \theta_{x,y} \) (or \( \phi_{x,y} \)) and \( \theta \) (or \( \phi \)):

\[
\theta_x = -Q_u \theta, \quad \phi_x = -Q_u \phi; \\
\theta_y = -Q_d \theta, \quad \phi_y = -Q_d \phi,
\]

where \( Q_u = +2/3 \) and \( Q_d = -1/3 \) are the electric charges of up- and down-type quarks, respectively. Given the experimental values of \( \theta_u, \theta_d, \theta \) and \( \phi \), it is then possible to determine \( V_u \) and \( V_d \) by using Eqs. (9) and (10). The reconstruction of \( M_u^\dagger M_u^\dagger \) and \( M_d^\dagger M_d^\dagger \) from \( V_u \) and \( V_d \) is straightforward, because the values of six quark masses are all known [19]. If both \( M_u \) and \( M_d \) are taken to be Hermitian or symmetric in a particular flavor basis, then they can directly be reconstructed from quark masses and flavor mixing parameters.

### III. RGE EFFECTS ON \( \phi \) AND \( \alpha \)

The one-loop renormalization-group equations (RGEs) of the CKM matrix elements, together with the RGEs of gauge couplings and the RGEs of Yukawa couplings of quarks and charged leptons, have already been calculated by several authors [20]. Here we focus on the RGE running behaviors of \( |V_{\alpha i}|^2 \) (for \( \alpha = u, c, t \) and \( i = d, s, b \)) by taking account of \( y_u^2 \ll y_s^2 \ll y_t^2 \) and \( y_d^2 \ll y_s^2 \ll y_b^2 \), where \( y_\alpha \) and \( y_i \) stand respectively for the eigenvalues of the Yukawa coupling matrices of up- and down-type quarks. In this excellent approximation, we simplify the results of Ref. [20] and arrive at

\[
16\pi^2 \frac{d}{dt} \begin{pmatrix}
|V_{ud}|^2 & |V_{us}|^2 & |V_{ub}|^2 \\
|V_{cd}|^2 & |V_{cs}|^2 & |V_{cb}|^2 \\
|V_{td}|^2 & |V_{ts}|^2 & |V_{tb}|^2 \\
\end{pmatrix} = 2C_y \begin{pmatrix}
|V_{ud}|^2 & |V_{ub}|^2 \\
|V_{td}|^2 & |V_{tb}|^2 \\
\end{pmatrix} \begin{pmatrix}
|V_{ud}|^2 |V_{ub}|^2 & |V_{us}|^2 |V_{ub}|^2 & -|V_{ub}|^2 (1 - |V_{ub}|^2) \\
|V_{td}|^2 |V_{tb}|^2 & -|V_{td}|^2 |V_{tb}|^2 & -|V_{tb}|^2 (1 - |V_{tb}|^2) \\
\end{pmatrix} \\
+ 2C_y \begin{pmatrix}
|V_{cd}|^2 |V_{td}|^2 & |V_{cb}|^2 |V_{td}|^2 & -|V_{cb}|^2 |V_{tb}|^2 \\
|V_{cd}|^2 |V_{td}|^2 & -|V_{cd}|^2 |V_{tb}|^2 & -|V_{cb}|^2 |V_{tb}|^2 \\
\end{pmatrix} \begin{pmatrix}
|V_{cd}|^2 |V_{td}|^2 & |V_{cb}|^2 |V_{td}|^2 & -|V_{cb}|^2 |V_{tb}|^2 \\
|V_{cd}|^2 |V_{td}|^2 & -|V_{cd}|^2 |V_{tb}|^2 & -|V_{cb}|^2 |V_{tb}|^2 \\
\end{pmatrix},
\]

(11)
where \( t \equiv \ln(\mu/M_Z) \), \( C = -1.5 \) in the SM and \( C = +1 \) in the minimal supersymmetric SM (i.e., MSSM). Therefore,

\[
16\pi^2 \frac{d}{dt} \ln \left| \frac{V_{ub}}{V_{cb}} \right|^2 = -2Cy_b^2 \left( 1 - |V_{tb}|^2 \right) ,
\]

\[
16\pi^2 \frac{d}{dt} \ln \left| \frac{V_{td}}{V_{ts}} \right|^2 = -2Cy_t^2 \left( 1 - |V_{tb}|^2 \right) ,
\]

\[
16\pi^2 \frac{d}{dt} \ln \left( |V_{td}|^2 + |V_{ts}|^2 \right) = -2C \left( y_b^2 + y_t^2 \right) .
\] (12)

Combining Eqs. (7) and (12), we immediately obtain

\[
16\pi^2 \frac{d}{dt} \ln \tan \theta_u = -Cy_b^2 \sin^2 \theta ,
\]

\[
16\pi^2 \frac{d}{dt} \ln \tan \theta_d = -Cy_t^2 \sin^2 \theta ,
\]

\[
16\pi^2 \frac{d}{dt} \ln \theta = -C \left( y_b^2 + y_t^2 \right) ;
\] (13)

or equivalently,

\[
16\pi^2 \frac{d\theta_u}{dt} = -\frac{1}{2} Cy_b^2 \sin 2\theta_u \sin^2 \theta ,
\]

\[
16\pi^2 \frac{d\theta_d}{dt} = -\frac{1}{2} Cy_t^2 \sin 2\theta_d \sin^2 \theta ,
\]

\[
16\pi^2 \frac{d\theta}{dt} = -\frac{1}{2} C \left( y_b^2 + y_t^2 \right) \sin 2\theta .
\] (14)

Let us stress that the simplicity of RGEs of three quark mixing angles is naturally expected for our particular parametrization of \( V \), just because its matrix elements involving \( t \) and \( b \) quarks are very simple and exactly consistent with the \( t \)- and \( b \)-dominance approximations taken for the RGEs of \( |V_{\alpha\beta}|^2 \) [21].

We proceed to derive the RGE of the CP-violating phase \( \phi \) from

\[
16\pi^2 \frac{d}{dt} |V_{ud}|^2 = 2C |V_{ud}|^2 \left( y_b^2 |V_{ub}|^2 + y_t^2 |V_{td}|^2 \right)
\]

\[
= C \sin^2 \theta \left( 2\sin^2 \theta_u \sin^2 \theta_d \cos^2 \theta + 2\cos^2 \theta_u \cos^2 \theta_d \sin 2\theta_u \sin 2\theta_d \cos \theta \cos \phi \right)
\]

\[
\times \left( y_b^2 \sin^2 \theta_u + y_t^2 \sin^2 \theta_d \right) .
\] (15)

Note that the derivative of \( |V_{ud}|^2 \) can be given in terms of the derivatives of \( \theta_u, \theta_d, \theta \) and \( \phi \) as follows:

\[
\frac{d}{dt} |V_{ud}|^2 = \left[ \sin 2\theta_u \left( \sin^2 \theta_d \cos^2 \theta - \cos^2 \theta_d \right) + \cos 2\theta_u \sin 2\theta_d \cos \theta \cos \phi \right] \frac{d\theta_u}{dt}
\]

\[
+ \left[ \sin 2\theta_d \left( \sin^2 \theta_u \cos^2 \theta - \cos^2 \theta_u \right) + \sin 2\theta_u \cos 2\theta_d \cos \theta \cos \phi \right] \frac{d\theta_d}{dt}
\]

\[
- \left[ \sin^2 \theta_u \sin^2 \theta_d \sin 2\theta + \frac{1}{2} \sin 2\theta_u \sin 2\theta_d \sin \theta \cos \phi \right] \frac{d\theta}{dt}
\]

\[
- \left[ \frac{1}{2} \sin 2\theta_u \sin 2\theta_d \cos \theta \sin \phi \right] \frac{d\phi}{dt} .
\] (16)
Substituting Eqs. (14) and (15) into the right- and left-hand sides of Eq. (16), respectively, we simply arrive at
\[ 16\pi^2 \frac{d\phi}{dt} = 0. \] (17)
This result implies that the CP-violating phase \( \phi \) is stable against radiative corrections at the one-loop level and in the approximation of quark mass hierarchies (i.e., \( y_u^2 \ll y_c^2 \ll y_t^2 \) and \( y_d^2 \ll y_s^2 \ll y_b^2 \)). With the help of Eqs. (14) and (17), a straightforward calculation of the derivative of \( \alpha \) given in Eq. (8) leads to
\[ 16\pi^2 \frac{d\alpha}{dt} = 0. \] (18)
Hence the RGE running effect of \( \alpha \) is also negligibly small, implying that the low-energy result \( \phi - \alpha \approx 1.1^\circ \) essentially keeps unchanged even if \( \mu \gg M_Z \) holds. In other words, it is impossible to get \( \alpha = 90^\circ \) at \( M_Z \) from \( \phi = 90^\circ \) at \( M_X \) through the one-loop RGE evolution.

Such a conclusion remains valid at the two-loop level. By using the two-loop RGEs of the CKM matrix elements \[22\], we have carried out a numerical analysis of the running behaviors of \( \phi \) and \( \alpha \) from \( M_X \) to \( M_Z \) (or vice versa) in both the SM and the MSSM \[3\]. Here are our main observations: (1) the RGE running effect of \( \phi \) or \( \alpha \) is too small (less than \( 0.1^\circ \) from \( M_Z \sim 10^2 \) GeV to \( M_X \sim 10^{16} \) GeV) in the SM or in the MSSM with \( \tan \beta > 1.5 \); and (2) it cannot compensate for the small phase difference \( \phi - \alpha \approx 1.1^\circ \) no matter how we adjust the energy scale (and the value of \( \tan \beta \) in the MSSM case).

**IV. QUARK-LEPTON COMPLEMENTARITY**

Compared with the parametrization of the CKM matrix \( V \) given in Eq. (7), a similar parametrization of the MNS matrix \( U \) is also convenient for the description of lepton flavor mixing and CP violation \[4\]:
\[
U = \begin{pmatrix}
c_l & s_l & 0 \\
-s_l & c_l & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
e^{-i\varphi} & 0 & 0 \\
0 & c & s \\
0 & -s & c
\end{pmatrix}
\begin{pmatrix}
c_\nu & -s_\nu & 0 \\
s_\nu & c_\nu & 0 \\
0 & 0 & 1
\end{pmatrix}P
\]
\[
= \begin{pmatrix}
s_l s_\nu c + c_l c_\nu e^{-i\varphi} & s_l s_\nu c - s_l c_\nu e^{-i\varphi} & s_l s_\nu \\
c_l s_\nu c - c_l c_\nu e^{-i\varphi} & c_l s_\nu c + c_l c_\nu e^{-i\varphi} & c_l s_\nu \\
-s_\nu & -c_\nu & 1
\end{pmatrix}P ,
\] (19)

\[3\] H. Zhang and S. Zhou did this numerical exercise for me. Their RGE program has also been used to evaluate the running masses of quarks and leptons at different energy scales \[19\].

\[4\] This parametrization may naturally arise from the parallel (and probably hierarchical) textures of charged-lepton and neutrino mass matrices. It is phenomenologically possible to obtain \( \theta_l \approx \arctan \left( \sqrt{m_e/m_\mu} \right) \approx 4^\circ \) together with a suggestive relationship \( \theta_\nu \approx \arctan \left( \sqrt{m_1/m_2} \right) \) \[23\], where \( m_1 \) and \( m_2 \) are the neutrino masses corresponding to \( \nu_e \) and \( \nu_\mu \) flavors. Furthermore, \( U \) can be decomposed into \( U = U_l^\dagger U_\nu P \) in a way similar to Eqs. (9) and (10) with \( Q_l = -1 \) and \( Q_\nu = 0 \).
where $c_{l\nu} \equiv \cos \theta_{l\nu}$, $s_{l\nu} \equiv \sin \theta_{l\nu}$, $c \equiv \cos \vartheta$ and $s \equiv \sin \vartheta$; and $P$ is a diagonal phase matrix containing two nontrivial CP-violating phases when three neutrinos are Majorana particles. Although the form of $U$ in Eq. (19) is apparently different from that of the standard parametrization of $U$ [3], their corresponding flavor mixing angles ($\theta_{12}$, $\theta_{13}$, $\vartheta$) and ($\vartheta_{12}$, $\vartheta_{13}$, $\vartheta_{23}$) have quite similar meanings in interpreting the experimental data on solar and atmospheric neutrino oscillations. In the limit $\vartheta = \vartheta_{13} = 0$, one easily arrive at $\theta_{12} = \vartheta_{12}$ and $\theta_{23} = \vartheta_{23}$. Note that the tri-bimaximal neutrino mixing pattern [24], which is well consistent with a global fit of current neutrino oscillation data [25], does coincide with this interesting limit (i.e., $\theta_{12} = \vartheta_{13} = 0^\circ$, $\theta_{12} = \arctan(1/\sqrt{2}) \approx 35.3^\circ$ and $\vartheta = \vartheta_{23} = 45^\circ$). Therefore, three mixing angles of $U$ can simply be related to those of solar, atmospheric and reactor neutrino oscillations in the leading-order approximation [21]; i.e., $\vartheta_{\text{sol}} \approx \theta_{\nu}$, $\vartheta_{\text{atm}} \approx \vartheta$ and $\vartheta_{\text{rea}} \approx \vartheta_{l}\sin \vartheta$ as a natural consequence of very small $\vartheta_{l}$.

The above comparison between our parametrization and the standard one indicates that both of them might be suitable for describing the approximate quark-lepton complementarity (QLC) relations [26]. The latter means the following empirical observations in the standard parametrizations of the CKM and MNS matrices:

$$\theta_{12} + \vartheta_{12} \approx 45^\circ, \quad \theta_{23} + \vartheta_{23} \approx 45^\circ,$$

where $\theta_{ij}$ and $\vartheta_{ij}$ (for $1 \leq i < j \leq 3$) represent quark and lepton mixing angles, respectively. Eq. (20) is actually consistent with the present experimental data within $1\sigma$ error bars [3]. Turning to our parametrizations of the CKM and MNS matrices in Eqs. (7) and (19), we find that similar QLC relations can approximately hold within $1\sigma$ error bars:

$$\theta_{d} + \vartheta_{\nu} \approx 45^\circ, \quad \theta + \vartheta \approx 45^\circ.$$

This result seems to be somewhat contrary to the expectation that the QLC relations are convention-dependent and may only hold in a single parametrization for $V$ and $U$ [27]. We believe that the exact QLC relations can only be realized (or assumed) in a unique parametrization for $V$ and $U$, but the approximate ones are possible to show up in different parametrizations. The reason for the latter point is quite simple: the smallest elements of the CKM and MNS matrices are both at their up-right corner (i.e., $|V_{ub}| = \sin \theta_{13} = \sin \theta_{u}\sin \vartheta = \cdots$ and $|V_{e3}| = \sin \vartheta_{13} = \sin \theta_{l}\sin \vartheta = \cdots$), and thus the flavor mixing between the first and second families is approximately decoupled from that between the second and third families. In other words, it is the smallness of $\theta_{13}$ (or $\theta_{u}$) and $\vartheta_{13}$ (or $\vartheta_{l}$) that assures the approximate QLC relations in Eqs. (20) and (21) to hold simultaneously.

Note again that the approximate QLC relations, similar to $\alpha \approx 90^\circ$, are extracted from current experimental data at low energies. One may wonder whether such empirical relations are stable against radiative corrections, or whether they can be exact at a specific energy scale far above $M_Z$. Because quark and lepton flavor mixing angles obey different RGEs in their evolution from $M_Z$ to $M_X$ (or vice versa) [28], we should have $d\theta_{13}/dt + d\vartheta_{12}/dt \neq 0$ and $d\theta_{23}/dt + d\vartheta_{23}/dt \neq 0$ in general [29,30]. This observation is also true for our parametrizations of the CKM and MNS matrices (see Ref. [21] for the explicit RGEs of $\theta_{l}$, $\vartheta_{\nu}$, $\vartheta$ and $\varphi$), no matter whether neutrinos are Dirac particles or Majorana particles.
Finally, let us conjecture that $\varphi = 90^\circ$ holds in the lepton sector. This possibility can actually be realized in some specific neutrino mass models (e.g., $\varphi = 90^\circ$ was first obtained in the so-called “democratic” neutrino mixing scenario [31]). While $\phi$ is rather stable against quantum corrections from one energy scale to another, as already shown in Eq. (18), $\varphi$ is in general sensitive to the RGE effects [21]. Does $\varphi = 90^\circ$ imply that a pair of the leptonic unitarity triangles are right or almost right? The answer to this question depends on the value of $\vartheta_1$ (or equivalently $\vartheta_{13}$ in the standard parametrization of $U$), which has not been fixed by current neutrino oscillation experiments. For illustration, we consider the leptonic unitarity triangle $\triangle_1$ defined by the orthogonality relation $V_{e2}V_{e3}^* + V_{\mu2}V_{\mu3}^* + V_{\tau2}V_{\tau3}^* = 0$ in the complex plane [2]. Denoting the inner angle $\alpha_t \equiv \arg[-(V_{\mu2}V_{\mu3}^*)/(V_{e2}V_{e3}^*)]$ and taking the maximal CP-violating phase $\varphi = 90^\circ$, we find

$$\sin \alpha_t = 1 - \frac{1}{2}s_1^2 c_\varphi^2 s_\varphi^{-2} c^{-2}(s_\varphi^2 - c_\varphi^2 c^2)^2 + \cdots ,$$

where higher-order terms of $s_1$ have been omitted. Then $\alpha_t \approx 89.5^\circ$ can be obtained from Eq. (22) with the typical inputs $\vartheta_1 \approx 5^\circ$, $\vartheta_\nu \approx 34^\circ$ and $\vartheta \approx 45^\circ$. It is easy to see that the value of $\alpha_t$ approaches $\varphi = 90^\circ$ when $\vartheta_1$ approaches zero, but in the limit of $\vartheta_1 = 0^\circ$ there will be no CP violation (i.e., $\varphi$ becomes trivial and can be rotated away from $U$ by rephasing the electron field) and all the leptonic unitarity triangles of $U$ must collapse into lines. This example illustrates that $\varphi = 90^\circ$ implies the existence of two nearly right unitarity triangles ($\triangle_1$ and its counterpart $\triangle_\nu$ defined by the orthogonality relation $V_{e1}V_{\mu1}^* + V_{e2}V_{\mu2}^* + V_{e3}V_{\mu3}^* = 0$) in the lepton sector, similar to the case in the quark sector.

V. SUMMARY AND CONCLUDING REMARKS

In view of the experimental indication that two unitarity triangles of the CKM matrix $V$ are almost the right triangles with $\alpha \approx 90^\circ$, we have explored its possible implications on the phenomenology of quark flavor mixing and quark-lepton complementarity. Taking account of a very suggestive parametrization of $V$, we have shown that its CP-violating phase $\phi$ is nearly equal to $\alpha$ (i.e., $\phi - \alpha \approx 1.1^\circ$). Both $\phi$ and $\alpha$ are stable against the renormalization-group evolution from the electroweak scale $M_Z$ to a superhigh energy scale $M_X$ or vice versa, and thus it is impossible to obtain $\alpha = 90^\circ$ at $M_Z$ from $\phi = 90^\circ$ at $M_X$. We have conjectured that there might also exist a maximal CP-violating phase $\varphi \approx 90^\circ$ in our parametrization of the MNS matrix $U$. The approximate quark-lepton complementarity relations, which hold in the standard parametrizations of $V$ and $U$ (i.e., $\vartheta_{12} + \vartheta_{13} \approx 45^\circ$ and $\vartheta_{23} + \vartheta_{23} \approx 45^\circ$), can also hold in our particular parametrizations of $V$ and $U$ (i.e., $\vartheta_d + \vartheta_\nu \approx 45^\circ$ and $\vartheta + \vartheta \approx 45^\circ$). We have pointed out that the reason for this interesting coincidence simply comes from the smallness of $|V_{ub}|$ and $|V_{e3}|$.

At this point, it is worthwhile to remark that the phenomenological ansatz proposed in Eq. (10) can be elaborated on so as to obtain an explicit texture of quark mass matrices. A similar ansatz can be made for the lepton sector by adopting the parametrization of $U$ advocated in Eq. (19) and decomposing it into $U = U_l^T U_\nu P$ in a way exactly analogous to Eqs. (9) and (10) with $Q_l = -1$ and $Q_\nu = 0$. For simplicity, here we only illustrate how
to reconstruct the Hermitian quark mass matrices $M_u$ and $M_d$ by using Eqs. (9) and (10). After taking account of the smallness of three mixing angles and the hierarchy of six quark masses, we approximately arrive at

$$M_u \approx \begin{pmatrix}
\lambda_u + \lambda_c \theta_u^2 & -\lambda_c \theta_u e^{+i60^\circ} & -\frac{2}{3} \lambda_c \theta_u \theta e^{+i60^\circ} \\
-\lambda_c \theta_u e^{-i60^\circ} & \lambda_c + \frac{4}{9} \lambda_t \theta^2 & -\frac{2}{3} \lambda_t \\
-\frac{2}{3} \lambda_c \theta_u e^{-i60^\circ} & -\frac{2}{3} \lambda_t \theta & \lambda_t
\end{pmatrix},$$

$$M_d \approx \begin{pmatrix}
\lambda_d + \lambda_s \theta_d^2 & -\lambda_s \theta_d e^{-i30^\circ} & +\frac{1}{3} \lambda_s \theta_d e^{-i30^\circ} \\
-\lambda_s \theta_d e^{+i30^\circ} & \lambda_s + \frac{1}{9} \lambda_b \theta^2 & +\frac{1}{3} \lambda_b \theta \\
+\frac{1}{3} \lambda_s \theta_d e^{+i30^\circ} & +\frac{1}{3} \lambda_b \theta & \lambda_b
\end{pmatrix},$$

(23)

where $|\lambda_q| = m_q$ (for $q = u, c, t$ and $d, s, b$), $\theta_u \approx 9.4 \times 10^{-2}$, $\theta_d \approx 2.0 \times 10^{-1}$ and $\theta \approx 4.2 \times 10^{-2}$. Such a parallel texture of up- and down-type quark mass matrices is certainly suggestive and may serve as a phenomenological starting point of model building. For instance, setting $(M_u)_{11} = (M_d)_{11} = 0$ leads to two interesting relations $\theta_u \approx \sqrt{m_u/m_c}$ and $\theta_d \approx \sqrt{m_d/m_s}$.

Although different parametrizations of the CKM matrix $V$ are mathematically equivalent, one of them might be able to make the underlying physics of quark flavor mixing more transparent and to establish simpler connections between the observable quantities and the model parameters. We find that our parametrization of $V$ in Eq. (7) does satisfy the above criterion. We expect that the similar parametrization of the MNS matrix $U$ in Eq. (19) is also useful in describing lepton flavor mixing. Needless to say, much more experimental, phenomenological and theoretical attempts are desirable in order to solve three fundamental flavor puzzles in particle physics — the generation of fermion masses, the dynamics of flavor mixing and the origin of CP violation.

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FIG. 1. The CKM unitarity triangles $\Delta_s$ and $\Delta_c$ defined in the complex plane.