Transport of radiative heat flux by hyperbolic metamaterials

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The transport of heat mediated by thermal photons in hyperbolic multilayer metamaterials is studied using the fluctuational electrodynamics theory. We demonstrate that in comparison to bulk materials the flux inside layered hyperbolic materials can be transported at much longer distances, making these media very promising for thermal management and for near-field energy harvesting.

I. INTRODUCTION

Nanoscale radiative heat transfer has attracted a lot of attention in the last few years because of Polder and van Hove’s prediction of the possibility to observe heat fluxes at subwavelength distances which are several orders of magnitude larger than those obtained by the blackbody theory. Recent experimental results have confirmed these theoretical predictions.

This increased radiative heat transfer in the near-field regime might be used for different applications as for example near-field imaging, nanoscale thermal management by heat flux rectification and amplification, and near-field thermophotovoltaics. In particular, for near-field thermophotovoltaic (nTPV) applications it is desirable to have large heat fluxes which are quasi-monochromatic at the bandgap frequency of the thermophotovoltaic cell. Now, it could be shown theoretically that for phonon-polaritonic materials the heat flux is quasi-monochromatic at the surface phonon-frequency resulting in heat fluxes which can be orders of magnitude larger than the blackbody result due to the large number of contributing surface modes. This is the reason why phonon-polaritonic media are used in most experiments. However, it should be kept in mind that there are also upper limits for this surface mode contribution as shown in Refs. 36,37.

On the other hand, the nanoscale heat flux between two halfspaces separated by a distance d which is due to surface modes is absorbed on a very thin layer of about 0.2d34,35. That means, that when constructing for example a near-field thermophotovoltaic device choosing d = 100 nm most energy is already absorbed in a thin surface layer of about 20 nm. This is very unfavorable for applications in near-field thermophotovoltaic devices, since only the electron-hole pairs in this thin layers can effectively be used for energy conversion.

As could be shown recently, for so called hyperbolic or indefinite metamaterials, which can be constructed by combining layers of a dielectric and a plasmonic/polaritonic material, or by using plasmonic/polaritonic nanowire structures, the nanoscale heat radiation by hyperbolic modes can result in heat fluxes which are on the order of or even larger than the heat flux by surface modes. This is due to a broad band of hyperbolic modes which are, in fact, frustrated total internal reflection modes. Recently, hyperbolic structures were proposed for applications in nTPV.

Having a broad frequency band for nanoscale heat radiation seems to be disadvantageous for nTPV, but this disadvantage is compensated by a striking property of hyperbolic modes: hyperbolic modes are propagating modes inside the hyperbolic metamaterials and therefore can be very large for hyperbolic nano-wire and multilayer structures. Furthermore, our formalism allows for describing near-field thermal radiation, since it tends to overestimate the hyperbolic contribution to the heat flux and it does not correctly describe the surface modes of the composite materials of the hyperbolic structure.

In this paper, we study the penetration depth of the energy flow in multilayer hyperbolic materials using an exact S-matrix method based on the Green’s function formalism combined with fluctuational electrodynamics. We have previously shown that the attenuation length can be very large for hyperbolic nano-wire and multilayer materials using an effective medium description. Here, by using an exact formalism for multilayer structures, only, we proof that the penetration depth inside hyperbolic multilayer materials is indeed much larger than in materials where the heat flux is dominated by surface modes. It can be even larger than predicted by the effective medium theory. Furthermore, our formalism allows...
for calculating the energy flux inside any kind of multilayer structure due to the radiative heat transfer (in far and near field) and therefore it enables one to determine the volume of the absorber in which the thermal radiation is dissipated. Further, it allows for studying deviations from the effective medium theory systematically.

The paper is organized as follows: In Sec. II we briefly outline fluctuational electrodynamics for determining the radiative heat transfer expression. In Sec. III we define the spectral and total attenuation length and introduce the considered geometries together with the predictions of the effective medium theory. We discuss the damping inside a multilayer hyperbolic material and compare numerical results of our exact formalism for a GaN/Ge multilayer structure with the predictions of the effective medium theory. Finally, we give a brief conclusion in Sec. VI.

II. ENERGY FLUX INSIDE A LAYERED MEDIUM

The theoretical description of near-field heat radiation is in most studies based on fluctuational electrodynamics. Within this theory it is assumed that the thermal fluctuating fields of a dielectric body which is assumed to be in local thermal equilibrium at a temperature $T$ are on a macroscopic scale due to fluctuational source current densities. Hence, Maxwell’s equations are augmented by fluctuational Gaussian source currents $J^m$ and $J^s$ yielding

$$\nabla \times E(r, t) = -J^m(r, t) - \frac{\partial B(r, t)}{\partial t}, \quad (1)$$

$$\nabla \times H(r, t) = J^s(r, t) + \frac{\partial D(r, t)}{\partial t}. \quad (2)$$

For nonmagnetic materials the fluctuating magnetic source currents can be neglected $J^m(r, t) = 0$. The source current density $J^s(r, t)$ is assumed to have zero mean value $\langle J^s \rangle = 0$, where the brackets symbolize the ensemble average. Then it is further assumed that the second moment or correlation function of the source currents is given by the fluctuation dissipation theorem of second kind:

$$\langle J^s_\alpha(r, \omega) J^s_\beta(r', \omega') \rangle = 4\pi \omega \Theta(\omega, T) \epsilon_{\text{vac}} \epsilon''_\alpha, \delta(r - r') \delta(\omega + \omega'), \quad (3)$$

where $\Theta(\omega, T) = \hbar \omega / (e^{\hbar \omega/k_B T} - 1)$ and $\epsilon''_\alpha$ is the imaginary part of the permittivity tensor of the considered material; $\epsilon_{\text{vac}}$ is the permittivity of vacuum, $2\pi \hbar$ is Planck’s constant, $k_B$ is Boltzmann’s constant, $\omega$ is the circular frequency, and $\delta$ stands for the delta function. Here, obviously quantum mechanics in form of the fluctuation dissipation theorem enters through the back door into the theoretical description which can therefore be regarded as a semi-classical theory. However, a full quantum mechanical description gives the same results for the observables.

Now, since the fields are linearly related to the sources they can be expressed as

$$E(r, \omega) = i\omega \mu_{\text{vac}} \int_V d^3r' G^E(r, r'; \omega) \cdot J^s(r', \omega), \quad (4)$$

$$H(r, \omega) = i\omega \mu_{\text{vac}} \int_V d^3r' G^H(r, r'; \omega) \cdot J^s(r', \omega) \quad (5)$$

introducing the dyadic Green’s functions $G^E$ and $G^H$. Since we only consider nonmagnetic materials $\mu_{\text{vac}}$ is the permeability of vacuum and of all materials. By means of the fluctuation dissipation theorem we can now derive the mean Poynting vector or Maxwell’s stress tensor, for instance. For some general elaborations on the stress tensor and the Poynting vector within the formalism of fluctuational electrodynamics we refer the interested reader to Ref. Since we are interested in heat radiation we focus on the Poynting vector.

Let us now assume that we have a situation as depicted in Fig. 1. For $z < z_0 = 0$ we have a semi-infinite isotropic material which is at local thermal equilibrium at temperature $T_0$. This halfspace is separated by a vacuum gap of size $d$ from a second halfspace which can be any kind of multilayer structure and which is assumed, for sake of clarity, to be at zero temperature. This assumption means that this medium does not emit thermal photons but it can only scatter and absorb them. However, of course, this medium could be set at any temperature. Straight forwardly we obtain the expression (using the Einstein convention)

$$\langle S_z \rangle = 2\text{Re} \int_0^\infty d\omega \Theta(\omega, T_0) \mu_{\text{vac}}^2 \omega^3 \text{Im}(\epsilon_0) \frac{\pi}{\pi} \times \int_{z' < 0} d^3r' \epsilon_{\alpha\beta}(G^E(r, r') \cdot G^H(r, r'))_{\alpha\beta}, \quad (6)$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric Levi-Civita tensor and $G^E(r, r') (G^H(r, r'))$ are the electric (magnetic) dyadic Green’s functions of the considered geometry with source points $r'$ in the halfspace for $z < 0$ and the observation point $r$ inside the vacuum gap or the second halfspace for $z > d$. Hence, when knowing the dyadic Green’s functions we can determine the Poynting vector which describes the energy transfer by thermal emission at any position within the multilayer structure ($z > d$) and for any separation distance $d$. Note, that $\epsilon_0$ is the permittivity inside the halfspace for $z < 0$.

To determine the mean Poynting vector describing the energy flow, we need to determine the corresponding dyadic Green’s functions for the structure depicted in Fig. 1. These dyadic Green’s function can be determined by a standard procedure. When inserting these expressions into (6) we obtain

$$\langle S_z \rangle = \int_0^\infty d\omega \Theta(\omega, T_0) \sum_{j=p, s} \int_0^\infty d^2k \frac{\pi}{(2\pi)^2} \chi^{(j)}(\omega, \kappa, z), \quad (7)$$

where $\chi^{(j)}$ is the complex dielectric function for the $j$-th medium and $\chi^{(p)}$ and $\chi^{(s)}$ are the permittivity for the inside of the medium and $\omega$ is the angular frequency.
where we have introduced the transmission coefficient in polarization \( j \ (j=s, p) \) of each mode \((\omega, \kappa)\) at a distance \( z \) from the surface as

\[
\mathcal{T}^{(j)}(\omega, \kappa; z) = \frac{\gamma}{|\gamma|} \left[ \text{Re}(c_n^{(j)}(\omega)) \left( e^{-2\gamma_n z} |a_n^{(j)}|^2 - e^{-2\gamma_n' z} |b_n^{(j)}|^2 \right) + i \text{Im}(c_n^{(j)}(\omega)) \left( e^{2\gamma_n' z} a_n^{(j)*} b_n^{(j)} - e^{-2\gamma_n' z} a_n^{(j)*} b_n^{(j)} \right) \right]
\]

(8)

for \( z_{n-1} < z < z_n \). The coefficients in Eq. (8) are

\[
c_n^{(s)} = \gamma_n, \quad c_n^{(p)} = \frac{k_n^2 + |\gamma_0|^2}{|k_0|^2} \frac{\gamma_n k_n^2}{|k_n|^2}
\]

(9)

(10)

with \( k_n^2 = \epsilon_n \frac{\omega^2}{c^2} = \kappa^2 + \gamma_n^2 \) the square of the wave vector and \( \epsilon_n \) the permittivity in the \( n \)-th layer; \( c \) is the vacuum speed of light, \( \kappa = \sqrt{k_x^2 + k_y^2} \) the wave vector component parallel to the surface and \( \gamma_n = k_{x,n} \) the wave vector component in \( z \) direction in the \( n \)-th layer. \( \gamma_n' \) and \( \gamma_n'' \) donate the real and imaginary part and * the complex conjugate of a number. The amplitudes \( a_n^{(j)} \) and \( b_n^{(j)} \) are determined by the S-matrix method described in Appendix A. In the same manner as detailed above, one can determine the heat flux from the medium at \( z > d \) which is assumed to have a temperature \( T_2 \). In this case we obtain the same result as in Eq. (7) but with \( \Theta(\omega, T_0) \) being replaced by \(-\Theta(\omega, T_2)\). Then the total heat flux is the sum of both contributions.

III. ATTENUATION LENGTH OF HEAT FLUX

Let us assume that the temperature of the first half-space \((z < 0)\) is \( T_0 = T + \Delta T \) and that of the multilayer structure \((z > d)\) is \( T_2 = T \) with \( \Delta T \ll T \). One can determine the heat transfer coefficient from \( [7] \)

\[
h(z) = \int_0^{\infty} \frac{d\omega}{2\pi} f(\omega, T) \sum_{j=s, p} \mathcal{T}^{(j)}(\omega, z)
\]

\[
= \int_0^{\infty} \frac{d\omega}{2\pi} H(\omega, z)
\]

(11)

where \( f(\omega, T) = \frac{(\omega^2 / \kappa B T^2)e^{\omega/k_B T}}{(e^{\omega/k_B T} - 1)^2} \) and \( H(\omega, z) \) is the spectral heat transfer coefficient. In this equation \( \mathcal{T}^{(j)}(\omega, \kappa; z) \) is the mean transmission coefficient of all modes at the frequency \( \omega \) over the distance \( z \). The energy flux is then given by \( h \Delta T \). By means of this expression we can define the spectral attenuation length \( l_\alpha \) as the distance \( z \) inside the multilayer structure \((z > d)\) at which the spectral heat transfer coefficient \( H(\omega, z) \) has dropped to \( H(\omega, d/e) \). It is clear from its definition that the asymptotic behavior of heat transfer coefficient at long distance is exponentially decaying. However as we are going to see in the next section, compared with bulk homogeneous materials, the attenuation length of heat flux can be significantly increased in particular layered structures.

A. Heat flux damping inside a layered hyperbolic medium

The formalism introduced above is general and it could be applied to describe heat transport by radiation through any arbitrary layered structures. We focus here our attention on specific media called hyperbolic media. Those media support modes that are governed by an hyperbolic dispersion relation. In this paper we consider hyperbolic media composed of alternated layers of materials (see Fig. 1) whose real parts of dielectric permittivities \( \epsilon_1 \) and \( \epsilon_2 \) are of opposite sign in a given spectral range. According to the effective medium theory, in the longwavelength approximation, the structure is analogous to an uniaxial crystal with a permittivity tensor \( \epsilon = \epsilon_{||}(\mathbf{e}_x \otimes \mathbf{e}_x + \mathbf{e}_y \otimes \mathbf{e}_y) + \epsilon_{\perp} \mathbf{e}_z \otimes \mathbf{e}_z \) of component

\[
\epsilon_{||} = f \epsilon_1 + (1 - f) \epsilon_2,
\]

(12)

in the direction parallel to the surface and

\[
\epsilon_{\perp} = \frac{\epsilon_1 \epsilon_2}{f \epsilon_2 + (1 - f) \epsilon_1},
\]

(13)

along the optical axis \( \mathbf{e}_z \). In this paper these components are plotted in the case of a Gallium Nitride/Germanium (GaIN/Ge) multilayer structure. It can be seen that there are two frequency bands named \( \Delta_1 \) and \( \Delta_2 \).
to the penetration depth if the intensity flux through a homogenized structure is naturally related to the penetration depth if the intensity flux through a homogenized structure is naturally related to the permeability of the material. The vertical lines mark the edges of the two hyperbolic bands. Waves.

The damping of $p$ polarized waves is solutions of the vector wave equation and are given by

$$\gamma_p = \sqrt{\omega^2/c^2 \epsilon_p - k^2}, \quad (14)$$

$$\gamma_p = \sqrt{\omega^2/c^2 \epsilon_p - k^2 \epsilon_p/\epsilon_\perp}. \quad (15)$$

It is worth noting that the attenuation length of heat flux through a homogenized structure is naturally related to the penetration depth if the intensity $\delta_j = \frac{2 \text{Im}(\gamma)}{\text{Re}(\gamma)} (j = s, p)$ of electric and magnetic fields. To get some insight on the flux attenuation mechanism in hyperbolic media we examine below how a plane wave traveling along the $z$ direction is damped. Since these hyperbolic modes are $p$ polarized only, we focus on the damping of $p$ polarized waves.

![Figure 2. Plot of the real parts of the permittivities $\epsilon$, and their phases $\varphi$ and $\varphi_\perp$ from Eqs. (12) and (13) in $\pi$ for an effective GaN/Ge multilayer structure choosing $f = 0.5$. The vertical lines mark the edges of the two hyperbolic bands $\Delta_1$ and $\Delta_2$.](image)

Using the polar representation $\epsilon_p = |\epsilon_p| e^{i\varphi_p}$ and $\epsilon_\perp = |\epsilon_\perp| e^{i\varphi_\perp}$ the $z$ component of the wave vector in polarization $p$ can be recast as

$$\gamma_p = |\epsilon_p|^{1/2} e^{i\varphi_p} k_0 \sqrt{1 - \frac{k^2}{k_0^2} e^{-i\varphi_\perp}}, \quad (16)$$

where $k_0 = \sqrt{|\epsilon_\perp|}$. From this expression we obtain for $\kappa \gg k_0 \sqrt{|\epsilon_\perp|}$

$$\gamma_p \approx i k \sqrt{\frac{2 \text{Im}(\epsilon)}{|\epsilon_\perp|}} e^{i(\varphi - \varphi_\perp)/2}. \quad (17)$$

Taking the imaginary part of this expression yields

$$\text{Im}(\gamma_p) \approx \kappa \sqrt{\frac{|\epsilon|}{|\epsilon_\perp|}} \cos \left(\frac{\varphi_\perp - \varphi}{2}\right). \quad (18)$$

which determines the damping of a plane wave travelling in $z$ direction inside the uni-axial material. Therefore we have small damping inside the anisotropic material (compared to the isotropic case, where $\epsilon_p = \epsilon_\perp$ and hence $\text{Im}(\gamma_p) \approx \kappa$ if

$$\varphi - \varphi_\perp = \pm \pi \quad (19)$$

or if $|\epsilon_p| \ll |\epsilon_\perp|.$ In particular inside a hyperbolic material, where $\text{Re}(\epsilon_p)\text{Re}(\epsilon_\perp) < 0$ the condition on the phases can be fulfilled if the imaginary parts of the permittivities perpendicular and parallel to the optical axis are small, i.e. if $\epsilon_p/\epsilon_\perp \ll 1$ and $\epsilon_p/\epsilon_\perp \ll 1.$ On the other hand, there is also small damping for strong anisotropic materials with $|\epsilon_p| \ll |\epsilon_\perp|.$

In the opposite limit where $\kappa \ll k_0 \sqrt{|\epsilon_\perp|}$ we find

$$\gamma_p \approx |\epsilon_p|^{1/2} e^{i\varphi_p} k_0 \left(1 - \frac{k^2}{2k_0^2} e^{-i\varphi_\perp}\right). \quad (20)$$

and therefore

$$\text{Im}(\gamma_p) \approx |\epsilon_p|^{1/2} k_0 \sin \left(\frac{\varphi_\perp}{2}\right). \quad (21)$$

Hence, if $\varphi \ll 0$ or $|\epsilon_p| \ll 0,$ i.e. losses parallel to the interface are small, we have a large penetration of fields. Note that if $|\epsilon_\perp|$ is small the damping is in both limits small as well.

### B. Results and discussion

Now, we discuss these mechanisms in the infrared range for a simple GaN/Ge periodic structure. For having the broadest possible hyperbolic bands and therefore the largest hyperbolic effect we consider identical widths for the two unit layers, i.e. the filling factor is 0.5. To evaluate the thermal performances of this medium, we assume that its left side is located at a distance $z = d$ from a bulk GaN halfspace maintained at temperature $T = 300 \text{K}$ and we calculate the heat transfer coefficients through the layered structure for different separation gap. The finite multilayer material is assumed to be on a semi-infinite substrate ($z > z_N$) made of the same material as the left halfspace. In the frequency range of interest, the permittivity of Ge layers is set to $\epsilon_{\text{Ge}} = 16$ while for the polar material Gallium Nitride (GaN) it is well described by the Drude-Lorentz model:

$$\epsilon_{\text{GaN}}(\omega) = \epsilon_\infty \frac{\omega_{\text{LO}}^2 - \omega^2 - i\gamma\omega}{\omega_{\text{TO}}^2 - \omega^2 - i\gamma\omega}, \quad (22)$$

where the permittivity at infinite frequency, the damping coefficient, the transverse and longitudinal optical phonon frequencies are given by $\epsilon_\infty = 5.35,$ $\gamma = 1.52 \times 10^{12} \text{rad/s},$ $\omega_{\text{TO}} = 1.06 \times 10^{13} \text{rad/s}$ and $\omega_{\text{LO}} = 1.41 \times 10^{14} \text{rad/s},$ respectively. In Fig. 3 the inverse damping factor $\delta_p = 1/(2\text{Im}(\gamma_p))$ as a function of $\omega$ and $\kappa$ as well as the spectral heat transfer coefficient $H(\omega, z)$ as a
Figure 3. Plot of $\delta_p = 1/(2\text{Im}(\gamma_p))$ (top) as a function of frequency and $\kappa$ and the normalized spectral heat transfer coefficient $H(\omega, z)/H(\omega, z = d)$ (bottom) with respect to the frequency $\omega$ and distance $z$ inside the hyperbolic material. The same parameters as in Fig. 2 are used. The temperature is $T = 300 \text{ K}$ and $d = 100 \text{ nm}$.

Figure 4. Spectral heat transfer coefficient $H(\omega, z)$ for the surface mode resonance frequency of GaN $\omega = 1.36 \cdot 10^{14} \text{ rad/s}$, $d = 100 \text{ nm}$ and $\Lambda = 100 \text{ nm}$ versus the distance $z$. The spectral heat transfer coefficient $H(\omega, d)$ is normalized to the black body value $H_{BB}(\omega) = f(\omega, T)\omega^2/(2\pi c^2)$. The dashed vertical lines mark the distance where $H(\omega, z) = H(\omega, d)/e$ for each case. The thin solid vertical lines are the interfaces of the multilayers. Here and in the following we use $N = 40$, i.e. we have 20 bilayers.

shows the heat transfer coefficient inside a homogenized medium (ii). The green and red dashed lines are the exact results for the layered GaN/Ge medium with Ge (iii) or GaN (iv) as topmost layer. In both cases the heat transfer coefficient is constant inside the Ge layer due to the negligible dissipation inside Ge. The solid vertical lines represent the interfaces of the multilayers and the vertical dashed lines represent the distance at which the spectral heat transfer has dropped to $1/e$ of its value at the interface, i.e. it marks the attenuation length $L_a$ at the given frequency and distance. It can be seen that in the case of bulk GaN (i) the spectral heat transfer coefficient at the surface (at $z = d$) is larger than in the cases of the layered hyperbolic metamaterial structure (ii)-(iv). On the other hand the attenuation length $L_a$ is much smaller for (i) compared to (ii) and (iii). Note that the multilayer structure (iv) with GaN as topmost layer has almost the same properties (regarding the exchanged heat flux as well as the attenuation length) as bulk GaN.

The frequency dependence of heat transfer coefficient and of attenuation lengths is described in Fig. 5 for the four cases (i)-(iv) for a separation gap $d = 100 \text{ nm}$ which corresponds to a distance where heat transfer occurs mainly due to near-field interaction. Note, that the attenuation length for bulk GaN is inside the reststrahlen band ($\omega_{TO} < \omega < \omega_{LO}$) smaller than 200 nm which is due to the strong damping of the surface modes.
Figure 5. (a) and (c) show the spectral heat transfer coefficient $H(\omega, d)$ normalized to the black body value $H_{BB}(\omega) = f(\omega, T)\omega^2/(2\pi c^2)$ for $\Lambda = 10$ nm (top) and $\Lambda = 100$ nm (bottom) keeping the distance fixed at $d = 100$ nm. (b) and (d) show the spectral attenuation length $L_a$.

It can be seen that inside the reststrahlen band where hyperbolic modes and surface modes exist, the hyperbolic structures have an attenuation length which is up to one order of magnitude larger than for bulk GaN. Further, the attenuation length inside the hyperbolic bands scales with the size of the hyperbolic structure which is 200 nm for $\Lambda = 10$ nm and 2 $\mu$m for $\Lambda = 100$ nm. For larger structures one can expect to have an even larger attenuation length as indicated by the effective medium result in Fig. 5. Indeed, in this case an attenuation length on the order of three microns within the hyperbolic region can be found.

On the other hand the spectral heat transfer coefficient is for all structures very similar but for bulk GaN the peak at the surface mode frequency is more pronounced than for the hyperbolic structures. Finally, the deviation between the exact and effective results is small for $\Lambda = 10$ nm as can be expected, since $\Lambda \ll d$ in this case. However, for $\Lambda = 100$ nm, i.e. $\Lambda \approx d$ the deviations between the effective and exact description become important. In particular, the choice of the material of the first layer has a large impact as discussed in detail in Refs. 51, 52, 63.

Now, let us discuss the total heat transfer coefficient $h(d)$ for the different materials. In Fig. 6 we show $h$ for all cases (i)-(iv) for $\Lambda = 100$ nm and $\Lambda = 10$ nm. First it can be seen, that the heat transfer coefficient for the hyperbolic multilayer structure (iv) with GaN as topmost layer gives the same value as bulk GaN (i) for distances smaller than the thickness of the topmost layer as can be
expected\textsuperscript{52}. Furthermore, it can be seen that the heat transfer coefficient for the hyperbolic multilayer structure (iii) with Ge as topmost layer starts to saturate at distances smaller than the thickness of the topmost layer as found in Ref.\textsuperscript{52}. The effective medium result (ii) is between the two different hyperbolic structures (iii) and (iv) and tends to overestimate the heat flux given by the hyperbolic structure (ii) with Ge as topmost layer\textsuperscript{50,51}.

In order to quantify the surface volume in which the most part of the incoming thermal radiation is absorbed, we determine the total attenuation length \( l_a \) for different thicknesses \( d \) of the vacuum gaps. As can be seen in Fig. 6(c) for all shown distances the hyperbolic structures (ii)-(iv) have in general larger penetration depths than the GaN halfspace. Only for distances smaller than the thickness of the topmost layer the result for (iv) with GaN on top coincides with the result of (i), since in this case the heat flux is solely given by the surface modes of the topmost layer. To be more precise, in Fig. 6(b),(d) it can be seen that the result of (iv) coincides with (i) when \( l_a < \Lambda \) means that when the coupling between the layers is negligible. On the other hand the effective hyperbolic structure (ii) has for distances in the near-field regime an attenuation length which is about one order of magnitude larger than that of structure (i). Finally, the attenuation length \( l_a \) of structure (iii) with Ge on top can be even larger than the result predicted by the effective medium theory. This can be easily explained by the fact that the attenuation length is in this case at least \( \Lambda/2 \) since the damping inside the first Ge layer is negligible. However, we find a minimal attenuation length of about 6\( \Lambda \). Hence, the total attenuation length inside the hyperbolic material (ii) can be two orders of magnitude larger than for bulk GaN. Note, that there is a trade-off between large heat transfer coefficients and large attenuation lengths.

IV. CONCLUSION

We have presented an exact formalism to determine the attenuation length of Poynting vector due to near- and far-field thermal radiation inside any kind of multilayer structure. In particular, we have studied the attenuation length inside multilayer hyperbolic structures composed of materials which support surface waves showing that the heat flux can be transported at longer distances than in its bulk constituents. We have shown that the attenuation length inside the investigated hyperbolic structure is about one to two orders of magnitude larger than inside the bulk materials but it highly depends on the choice of the topmost layer material which can strongly screen the heat flux. The long range heat transport could be advantageous for numerous near-field technologies. In particular, it could be used to overcome the tricky problem of the saturation in hole-electron pairs close to the surface in near-field thermophotovoltaic devices. It could also be exploited to develop efficient heat removal systems which are able to extract the huge density of energy confined at the surface of hot bodies.

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Appendix A: T and S matrix

To calculate the Poynting vector in the \( n \)-th layer we need the respective amplitudes \( a^{(n)}_{i} \) and \( b^{(n)}_{i} \). In order to determine these amplitudes we make the following two steps. In a first step we employ the usual continuity conditions for the Green’s tensors at the interfaces \( z = z_{n} \) to get the transfer matrix

\[
\begin{pmatrix}
a_{n} \\
b_{n}
\end{pmatrix} = \begin{pmatrix}
T_1 & T_2 \\
T_3 & T_4
\end{pmatrix} \begin{pmatrix}
a_{n+1} \\
b_{n+1}
\end{pmatrix}
\]

(A1)

for both s- and p- polarized modes connecting the amplitudes of adjacent layers. The matrix elements of the T matrix are given by

\[
T_1 = \frac{1}{l_{n,n+1}} e^{-i(\gamma_{n} - \gamma_{n+1})z_{n}}, \quad T_2 = \frac{r_{n,n+1}}{l_{n,n+1}} e^{-i(\gamma_{n} + \gamma_{n+1})z_{n}},
\]

(A2)

\[
T_3 = \frac{r_{n,n+1}}{l_{n,n+1}} e^{i(\gamma_{n} + \gamma_{n+1})z_{n}}, \quad T_4 = \frac{1}{l_{n,n+1}} e^{i(\gamma_{n} - \gamma_{n+1})z_{n}},
\]

(A3)

where \( r_{i,j} \) and \( t_{i,j} \) are the Fresnel reflection and transmission coefficients for s- and p-polarized modes

\[
\begin{align*}
& r_{i,j}^{(s)} = \frac{\gamma_{i} - \gamma_{j}}{\gamma_{i} + \gamma_{j}}, \quad r_{i,j}^{(p)} = \frac{2\gamma_{i}}{\gamma_{i} + \gamma_{j}}; \\
& t_{i,j}^{(s)} = \frac{e_{j} \gamma_{i} - e_{i} \gamma_{j}}{e_{j} \gamma_{i} + e_{i} \gamma_{j}}, \quad t_{i,j}^{(p)} = \frac{2e_{i} e_{j}}{e_{j} \gamma_{i} + e_{i} \gamma_{j}}.
\end{align*}
\]

(A4)

(A5)

For computational reasons, in a second step we determine the scattering matrix connecting the amplitudes of the incoming and outgoing waves\textsuperscript{64,65}

\[
\begin{pmatrix}
a_{n} \\
b_{0}
\end{pmatrix} = \begin{pmatrix}
S_{1}(n) & S_{2}(n) \\
S_{3}(n) & S_{4}(n)
\end{pmatrix} \begin{pmatrix}
a_{0} \\
b_{n}
\end{pmatrix}
\]

(A6)

with \( S_{1}(0) = 1, \ S_{2}(0) = 0, \ S_{3}(0) = 0, \) and \( S_{4}(0) = 1 \). Using the T matrix in Eq. (A1) and the S matrix for the \( n \)-th layer (A6) we can determine the S matrix for the \((n + 1)\)-th layer

\[
\begin{pmatrix}
a_{n+1} \\
b_{0}
\end{pmatrix} = \begin{pmatrix}
S_{1}(n+1) & S_{2}(n+1) \\
S_{3}(n+1) & S_{4}(n+1)
\end{pmatrix} \begin{pmatrix}
a_{0} \\
b_{n+1}
\end{pmatrix}
\]

(A7)
Figure 6. (a) and (c) show the total heat transfer coefficient $h(d)$ normalized to the black body value $h_{BB} = 6.1 \text{Wm}^{-2}\text{K}^{-1}$ for $\Lambda = 10 \text{nm}$ (top) and $\Lambda = 100 \text{nm}$ (bottom) as a function of distance $d$. (b) and (d) show total attenuation length $l_a$.

With the $(n+1)$-th S-matrix elements given by

\begin{align}
S_1(n+1) &= \frac{S_1(n)}{T_1 - S_2(n)T_3} \quad \text{(A8)} \\
S_2(n+1) &= \frac{S_2(n)T_1 - T_2}{T_1 - S_2(n)T_3} \quad \text{(A9)} \\
S_3(n+1) &= S_3(n) + S_4(n)T_3S_1(n+1) \quad \text{(A10)} \\
S_4(n+1) &= S_4(n)T_3S_2(n+1) + S_4(n)T_4 \quad \text{(A11)}
\end{align}

From the condition $a_0 = 1$ and $b_{N+1} = 0$ we can determine all other amplitudes by means of the S-matrix method.
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