ON CLASSIFICATION OF FINITE DIMENSIONAL COMPLEX
FILIFORM LEIBNIZ ALGEBRAS (PART 2)

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Abstract. The paper is devoted to classification problem of finite dimensional complex none Lie filiform Leibniz algebras. Actually, the observations show there are two resources to get classification of filiform Leibniz algebras. The first of them is naturally graded none Lie filiform Leibniz algebras and the another one is naturally graded filiform Lie algebras. Using the first resource we get two disjoint classes of filiform Leibniz algebras [10]. The present paper deals with the second of the above two classes, the first class has been considered in [2]. The algebraic classification here means to specify the representatives of the orbits, whereas the geometric classification is the problem of finding generic structural constants in the sense of algebraic geometry. Our main effort in this paper is the algebraic classification. We suggest here an algebraic method based on invariants. Utilizing this method for any given low dimensional case all filiform Leibniz algebras can be classified. Moreover, the results can be used for geometric classification of orbits of such algebras.

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1. Introduction

This paper aims to investigate a class of nonassociative algebras which generalizes the class of Lie algebras. These algebras satisfy certain identities that were suggested by J.-L. Loday [4]. When he used the tensor product instead of external product in the definition of the $n$-th cochain, in order to prove the differential property, that is defined on cochains, it sufficed to replace the anticommutativity and Jacobi identity by the Leibniz identity. This is an essential one of the motivation to appear for this class of algebras.

In this paper we suggest an algebraic approach to the classification problem for filiform Leibniz algebras. Utilizing this method for any fixed low dimensional case the corresponding classes of filiform Leibniz algebras can be classified completely. Moreover, the results may be used for geometric classification in the sense of geometric invariant theory [5]. It is assumed that it will be the subject of one of the next papers. For geometric classification of complex nilpotent Leibniz algebras of dimension at most four we refer to [3].

Let $V$ be a vector space of dimension $n$ over an algebraically closed field $K$ ($\text{char} K = 0$). The bilinear maps $V \times V \rightarrow V$ form a vector space $\text{Hom}(V \otimes V, V)$ of dimension $n^3$, which can be considered together with its natural structure of an affine algebraic variety over $K$ and denoted by $\text{Alg}_n(K) \cong K^{n^3}$. An $n$-dimensional algebra $L$ over $K$ may be considered as an element $\lambda(L)$ of $\text{Alg}_n(K)$ via the bilinear mapping $\lambda : L \otimes L \rightarrow L$ defining an binary algebraic operation on $L$; let \{ $e_1, e_2, \ldots, e_n$ \} be a basis of the algebra $L$. Then the table of multiplication of $L$ is
represented by point \((\gamma^k_{ij})\) of this affine space as follow:

\[
\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma^k_{ij} e_k.
\]

\(\gamma^k_{ij}\) are called structural constants of \(L\). The linear reductive group \(GL_n(K)\) acts on \(Alg_n(K)\) by \((g \ast \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))\) (“transport of structure”). Two algebras \(\lambda_1\) and \(\lambda_2\) are isomorphic if and only if they belong to the same orbit under this action. The orbit of \(\lambda\) under this action is denoted by \(O(\lambda)\). It is clear that elements of the given orbit are isomorphic to each other algebras. The classification means to specify the representatives of the orbits. A simple criterion, to decide if the given two algebras are isomorphic, is desired.

2. Preliminaries

**Definition 1.** An algebra \(L\) over a field \(K\) is called a Leibniz algebra if it satisfies the following Leibniz identity:

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y],
\]

where \([\cdot, \cdot]\) denotes the multiplication in \(L\). Let \(\text{Leib}_n(K)\) be a subvariety of \(Alg_n(K)\) consisting of all \(n\)-dimensional Leibniz algebras over \(K\). It is invariant under the above mentioned action of \(GL_n(K)\). As a subset of \(Alg_n(K)\) the set \(\text{Leib}_n(K)\) is specified by system of equations with respect to structural constants \(\gamma^k_{ij}\):

\[
\sum_{l=1}^{n} (\gamma^l_{jk} \gamma^m_{il} - \gamma^l_{ij} \gamma^m_{lk} + \gamma^l_{ik} \gamma^m_{lj}) = 0
\]

It is easy to see that if the bracket in Leibniz algebra happens to be anticommutative then it is a Lie algebra. So Leibniz algebras are “noncommutative” generalization of Lie algebras. As to classifications of low dimensional Lie algebras they are well known. But unless simple Lie algebras the classification problem of all Lie algebras in common remains a big problem. Yu.I.Malcev [6] reduced the classification of solvable Lie algebras to the classification of nilpotent Lie algebras. Apparently the first non-trivial classification of some classes of low-dimensional nilpotent Lie algebras are due to Umlauf. In his thesis [7] he presented the redundant list of nilpotent Lie algebras of dimension at most seven. He gave also the list of nilpotent Lie algebras of dimension less than ten admitting so-called adapted basis (now, the nilpotent Lie algebras with this property are called filiform Lie algebras). It was shown by M.Vergne [8] the importance of filiform Lie algebras in the study of variety of nilpotent Lie algebras laws. Up to now the several classifications of low-dimensional nilpotent Lie algebras have been offered. Unfortunately, many of these papers are based on direct computations and the complexity of those computations leads frequently to errors. We refer the reader to [9] for comments and corrections of the classification errors.

Further if it is not asserted additionally all algebras assumed to be over the field of complex numbers \(C\).

Let \(L\) be a Leibniz algebra. We put:

\[
L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \in N.
\]
Definition 2. A Leibniz algebra $L$ is said to be nilpotent if there exists an integer $s \in N$, such that $L^1 \supset L^2 \supset \ldots \supset L^s = \{0\}$. The smallest integer $s$ for which $L^s = 0$ is called the nilindex of $L$.

Definition 3. An $n$-dimensional Leibniz algebra $L$ is said to be filiform if $\dim L^i = n - i$, for all $2 \leq i \leq n$.

Theorem 1.\cite{[10],[1]}.

Any $(n+1)$-dimensional complex non-Lie filiform Leibniz algebra can be included to one of the following three classes of none Lie filiform Leibniz algebras:

a) $(1^\text{st} \text{class})$:

\[
\begin{align*}
[e_0, e_0] &= e_2, \\
[e_i, e_0] &= e_{i+1}, \\
[e_0, e_1] &= \alpha_3 e_3 + \alpha_4 e_4 + \ldots + \alpha_{n-1} e_{n-1} + \theta e_n,
\end{align*}
\]

$1 \leq i \leq n - 1$

and

\[
\begin{align*}
[e_j, e_1] &= \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + \ldots + \alpha_{n+1-j} e_n,
\end{align*}
\]

$1 \leq j \leq n - 2$

(omitted products are supposed to be zero)

b) $(2^\text{nd} \text{class})$:

\[
\begin{align*}
[e_0, e_1] &= \beta_3 e_3 + \beta_4 e_4 + \ldots + \beta_n e_n, \\
[e_1, e_1] &= \gamma e_n,
\end{align*}
\]

$2 \leq i \leq n - 1$

and

\[
\begin{align*}
[e_j, e_1] &= \beta_3 e_{j+2} + \beta_4 e_{j+3} + \ldots + \beta_{n+1-j} e_n,
\end{align*}
\]

$2 \leq j \leq n - 2$

(omitted products are supposed to be zero)

c) $(3^\text{rd} \text{class})$:

\[
\begin{align*}
[e_0, e_0] &= e_n, \\
[e_1, e_1] &= \alpha e_n, \\
[e_i, e_0] &= e_{i+1}, \\
[e_0, e_1] &= -e_2 + \beta e_n, \\
[e_0, e_1] &= -e_{i+1}, \\
[e_i, e_j] &= -[e_j, e_i] \in \lim < e_{i+1}, e_{i+2}, \ldots, e_n >, \\
[e_n-i, e_i] &= -[e_i, e_{n-i}] = (-1)^i \delta e_n,
\end{align*}
\]

$1 \leq i \leq n - 1$

(omitted products are supposed to be zero)

where \{\{e_0, e_1, e_2, \ldots, e_n\}\} is a basis, $\delta$ is either 1 or 0 for odd $n$ and $\delta = 0$ for even $n$.

In other words, the above proposition means that the set of all $(n+1)$-dimensional complex none Lie filiform Leibniz algebras can be represented as a disjoint union of the above mentioned three classes and the algebras from the difference classes never are isomorphic to each other.

In this paper we will consider the second class of algebras.

Let us denote by $L(\beta)$, the $(n+1)$-dimensional filiform non-Lie Leibniz algebra defined by parameters $\beta = (\beta_3, \beta_4, \ldots, \beta_n, \gamma)$. The set of all $(n+1)$-dimensional complex filiform Leibniz algebras from the second class is denoted by $F\text{Leib}_n$. It is a closed and invariant subset of the variety of nilpotent Leibniz algebras.

Using the method of simplification of the basis transformations in \cite{[1]} the following criterion on isomorphism of two $(n+1)$-dimensional filiform Leibniz algebras was given. We formulate part two of the theorem regarding our case. Namely: let $n \geq 3$. 

Theorem 2. [1] Two algebras $L(\beta_3, \beta_4, \ldots, \beta_n, \gamma)$ and $L'(\beta'_3, \beta'_4, \ldots, \beta'_n, \gamma')$ from $FL_{n+1}$ are isomorphic if and only if there exist $A, B, D \in \mathbb{C}$ such that $AD \neq 0$ and the following conditions hold:

\[
\beta'_1 = \frac{D}{\beta_1} \beta_1, \\
\beta'_k = \frac{1}{4^k} \sum_{i_2=k+3}^{t-1} \left( C_{k-1}^{k-2} A^{k-2} B \beta_{n+2-k} + C_{k-1}^{k-3} B^2 \sum_{i_1=k+2}^{t} \beta_{t+3-i_1} \cdot \beta_{1+i-1-k} + \right. \\
+ C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{t} \beta_{t+3-i_2} \cdot \beta_{2+i-1} \beta_{1+i-k} + \ldots + \\
\left. + C_{k-1}^{k-2} AB^{k-2} \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{t} \beta_{t+3-i_2} \beta_{k-3+3-i_2-i_1} \beta_{1+i-1-k} + \ldots + \\
+ B^{k-1} \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{t} \beta_{t+3-i_2} \beta_{k-3+3-i_2-i_1} \beta_{1+i-1-k} + \ldots + \beta_{3-i_2} \beta_{k-3+3-i_2-i_1} \beta_{1+i-1-k} + \right. \\
\left. \ldots \beta_{3-i_2} \beta_{k-3+3-i_2-i_1} \beta_{1+i-1-k} + \right), \]

where $4 \leq t \leq n - 1$.

\[
\beta'_n = \frac{B}{A} \beta_1 + \frac{1}{4^k} \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{t} \beta_{t+3-i_2} \beta_{2+i-1} \beta_{1+i-k} + \ldots + \\
+ C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{t} \beta_{t+3-i_2} \beta_{2+i-1} \beta_{1+i-k} + \ldots + \\
+ C_{k-1}^{k-2} AB^{k-2} \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{t} \beta_{t+3-i_2} \beta_{k-3+3-i_2-i_1} \beta_{1+i-1-k} + \ldots + \\
+ B^{k-1} \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{t} \beta_{t+3-i_2} \beta_{k-3+3-i_2-i_1} \beta_{1+i-1-k} + \ldots + \beta_{3-i_2} \beta_{k-3+3-i_2-i_1} \beta_{1+i-1-k} + \right), \\
\gamma' = \frac{D}{\beta_1^2} \gamma
\]

Here are the above systems of equalities for some low dimensional cases:

Case of $n = 4$ i.e. $\text{dim} L = 5$:

\[
\begin{cases}
\beta'_3 = \frac{1}{A} \frac{D}{A} \beta_3, \\
\beta'_4 = \frac{1}{A^2} (\beta_4 + \beta_4 - 2 \beta_2^2), \\
\gamma' = \frac{D}{\beta_1^2} (\beta_1^2) \gamma,
\end{cases}
\]

Case of $n = 5$ i.e. $\text{dim} L = 6$:

\[
\begin{cases}
\beta'_3 = \frac{1}{A} \frac{D}{A} \beta_3, \\
\beta'_4 = \frac{1}{A} \frac{D}{A} (\beta_4 - 2 \beta_2^2), \\
\beta'_5 = \frac{1}{A^2} (\beta_5 + \beta_5 - 5 \beta_3 \beta_4 + 5 (\beta_1^2)^2 \beta_3 \beta_4), \\
\gamma' = \frac{D}{\beta_1^2} (\beta_1^2) \gamma,
\end{cases}
\]
Case of $n = 6$ i.e. $\dim \mathbf{L} = 7$:

$$
\begin{align*}
\beta'_3 & = \frac{1}{A^4} D \beta_3, \\
\beta'_4 & = \frac{1}{A^4} D \beta_4 - 2 \frac{B}{A^2} \beta_3^2, \\
\beta'_5 & = \frac{1}{A^4} D \beta_5 - 5 \frac{B}{A^2} \beta_3 \beta_4 + 5 \left( 4\frac{B}{A^2} \right)^2 \beta_3^2. \\
\beta'_6 & = \frac{1}{A^4} D \beta_6 - 6 \frac{B}{A^2} \beta_3 \beta_5 + 21 \left( 4\frac{B}{A^2} \right)^2 \beta_3^2 \beta_4 - 3 \frac{B}{A^2} \beta_3^2 - 14 \left( 4\frac{B}{A^2} \right)^3 \beta_3^4, \\
\gamma' & = \frac{1}{A^2} \left( \frac{D}{A} \right)^2 \gamma,
\end{align*}
$$

(2.0.3)

Case of $n = 7$ i.e. $\dim \mathbf{L} = 8$:

$$
\begin{align*}
\beta'_3 & = \frac{1}{A^4} D \beta_3, \\
\beta'_4 & = \frac{1}{A^4} D \beta_4 - 2 \frac{B}{A^2} \beta_3^2, \\
\beta'_5 & = \frac{1}{A^4} D \beta_5 - 5 \frac{B}{A^2} \beta_3 \beta_4 + 5 \left( 4\frac{B}{A^2} \right)^2 \beta_3^2. \\
\beta'_6 & = \frac{1}{A^4} D \beta_6 - 6 \frac{B}{A^2} \beta_3 \beta_5 + 21 \left( 4\frac{B}{A^2} \right)^2 \beta_3^2 \beta_4 - 3 \frac{B}{A^2} \beta_3^2 - 14 \left( 4\frac{B}{A^2} \right)^3 \beta_3^4, \\
\beta'_7 & = \frac{1}{A^4} D \left( \frac{D}{A} \gamma + \beta_7 - 7 \frac{B}{A^2} \beta_3 \beta_6 + 28 \left( 4\frac{B}{A^2} \right)^2 \beta_3^2 \beta_5 + 28 \left( 4\frac{B}{A^2} \right)^2 \beta_3 \beta_5^2 - 7 \frac{B}{A^2} \beta_3 \beta_5 - 84 \left( 4\frac{B}{A^2} \right)^3 \beta_3^3 \beta_4 + 42 \left( 4\frac{B}{A^2} \right)^4 \beta_5^3 \right) \\
\gamma' & = \frac{1}{A^2} \left( \frac{D}{A} \right)^2 \gamma,
\end{align*}
$$

(2.0.4)

To deal with the classification of $\mathcal{F}Leib_{n+1}$ with respect to the above mentioned action we represent it as a disjoint union of an open and closed (with respect to the Zariski topology) subsets. Moreover each of these subsets are invariant under the corresponding transformations presented in Theorem 2. Then we formulate the solution of the isomorphism problem for the corresponding algebras from the open subset. Similar approach can be used to solve isomorphism problem for the algebras from the corresponding closed subset.

It is not difficult to notice that the expressions for $\beta'_t$, $\gamma'$ in Theorem 2 can be represented in the following form:

$$
\beta'_t = \frac{1}{A^{t-2}} D \psi_t \left( \frac{B}{A}; \beta \right),
$$

where $\beta = (\beta_3, \beta_4, ..., \beta_n, \gamma)$, $3 \leq t \leq n - 1$ and

$$
\begin{align*}
\psi_t(y; z) & = \psi_t(y; z_3, z_4, ..., z_n, z_{n+1}) = z_t - \sum_{k=3}^{n-1} \left( \frac{C_{k-1}^{k-2} y z_{t+2-k} + C_{k-1}^{k-3} y^2 \sum_{i_1=k+2}^{t} z_{t+3-i_1}}{i_1=k+2} \right) \\
& + C_{k-1}^{k-1} y^{k-2} \sum_{i_3=2k-2}^{i_2=2k-2} \sum_{i_1=k-1}^{i_2} z_{t+3-i_1} z_{i_2+3-i_1} z_{i_1-k} + ... + \\
& C_{k-1}^{1} y^{k-2} \sum_{i_2=2k-2}^{t} \sum_{i_1=2k-2}^{i_2-3} z_{t+3-i_1} z_{i_1+3-i_1} + ... + z_{i_1+5-2k}.
\end{align*}
$$

(2.0.5)
The classical problem of the invariant theory \[ 12 \] is the isomorphism problem. If \( K \) is an irreducible then the field of rational invariants can be defined as a quotient field of \( Z \). This is referred to in \[ 11 \] as the "first fundamental problem of invariant theory". If \( \sigma \) is an action of algebraic group \( G \) on a variety \( Z \), then \( Z \) is a \( G \)-variety.

**Definition 4.** An action of algebraic group \( G \) on a variety \( Z \) is a morphism \( \sigma : G \times Z \rightarrow Z \) with

(i) \( \sigma(e, z) = z \), where \( e \) is the unit element of \( G \) and \( z \in Z \).

(ii) \( \sigma(g, \sigma(h, z)) = \sigma(gh, z) \), for any \( g, h \in G \) and \( z \in Z \).

We shortly write \( gz \) for \( \sigma(g, z) \), and call \( Z \) a \( G \)-variety.

**Definition 5.** A morphism \( f : Z \rightarrow K \), (\( K \) is a base field) is said to be invariant if \( f(gz) = f(z) \) for any \( g \in G \) and \( z \in Z \).

The algebra of invariant morphisms on \( Z \) with respect to the action of the group \( G \) is denoted by \( K[Z]^G \). Sometimes this algebra is a finitely generated \( K \)-algebra. This is referred to in \[ 11 \] as the "first fundamental problem of invariant theory". If \( Z \) is an irreducible then the field of rational invariants can be defined as a quotient field of \( K[Z]^G \). It is always finitely generated as a subalgebra of the finitely generated algebra \( K(Z) \). Description the field of rational invariants is another important classical problem of the invariant theory \[ 12 \].

Actually, we use some elements of the algebra of invariant morphisms under the above mentioned adapted action on the variety of filiform Leibniz algebras to solve isomorphism problem.

From here on we assume that \( n \geq 5 \) is a positive integer, since there are complete classifications of complex nilpotent Leibniz algebras of dimension at most four \[ 13, 14 \] (for five-dimensional case see section 4).

Consider the following presentation of \( FLeib_{n+1} \):

\[
(3.0.7) \quad FLeib_{n+1} = U \cup F,
\]
where \( U = \{L(\beta) : \beta_3(4\beta_2^2\beta_5 - 12\beta_3\beta_4\beta_6 + \beta_3^3)(4\beta_3\beta_5 - 5\beta_4^2) \neq 0\} \), \( F = \{L(\beta) : \beta_3(4\beta_2^2\beta_5 - 12\beta_3\beta_4\beta_6 + \beta_3^3)(4\beta_3\beta_5 - 5\beta_4^2) = 0\} \).

Our main interest will be the cases of open sets, "generic algebras", cases.

**Theorem 3.** i) Two algebras \( L(\beta) \) and \( L(\beta') \) from \( U \) are isomorphic if and only if

\[
\vartheta_i \left( \frac{\beta_3(4\beta_2^2\beta_5 - 5\beta_4^2)}{4(\beta_2^2\beta_6 - 3\beta_2\beta_4\beta_5 + 2\beta_4^2)}, \frac{\beta_4}{2\beta_2}, \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} ; \beta \right) = \vartheta_i \left( \frac{\beta_3(4\beta_2^2\beta_5 - 5\beta_4^2)}{4(\beta_2^2\beta_6 - 3\beta_2\beta_4\beta_5 + 2\beta_4^2)}, \frac{\beta_4}{2\beta_2}, \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} ; \beta' \right)
\]

whenever \( i = 3, n - 1 \).

ii) For any \((\lambda_3, \lambda_4, \ldots, \lambda_{n-1}) \in C^{n-3}\) there is an algebra \( L(\beta) \) from \( U \) such that

\[
\vartheta_i \left( \frac{\beta_3(4\beta_2^2\beta_5 - 5\beta_4^2)}{4(\beta_2^2\beta_6 - 3\beta_2\beta_4\beta_5 + 2\beta_4^2)}, \frac{\beta_4}{2\beta_2}, \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} ; \beta \right) = \lambda_i \text{ for all } i = 3, n - 1
\]

**Proof.** i) Let first two algebras \( L(\beta) \) and \( L(\beta') \) be isomorphic that is to say there exist \( A,B,D \in C \) such that \( AD \neq 0 \) and \( \beta' = \vartheta_i \left( \frac{A}{A_0}, \frac{B}{A_0} ; \beta \right) = \vartheta_i \left( \frac{A}{A_0}, \frac{B}{A_0} ; \beta' \right) \).

Consider algebra \( L(\beta') \), where \( \beta' = \vartheta_i \left( \frac{\beta_3(4\beta_2^2\beta_5 - 5\beta_4^2)}{4(\beta_2^2\beta_6 - 3\beta_2\beta_4\beta_5 + 2\beta_4^2)}, \frac{\beta_4}{2\beta_2}, \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} ; \beta \right) \) and \( A_0 = \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} \).

\[
B_0 = \frac{2\beta_3(4\beta_2^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} \quad \text{and} \quad D_0 = \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)}. \quad \text{Since } \beta' = \vartheta_i \left( \frac{A}{A_0}, \frac{B}{B_0} ; \beta' \right)
\]

and \( \beta'' = \vartheta_i \left( \frac{A+A_0B}{A_0D}, \frac{B}{B_0} ; \beta' \right) \). It is easy to check that \( A_0 = \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} \).

Therefore

\[
\vartheta_i \left( \frac{\beta_3(4\beta_2^2\beta_5 - 5\beta_4^2)}{4(\beta_2^2\beta_6 - 3\beta_2\beta_4\beta_5 + 2\beta_4^2)}, \frac{\beta_4}{2\beta_2}, \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} ; \beta \right)
\]

and, in particular,

\[
\vartheta_i \left( \frac{\beta_3(4\beta_2^2\beta_5 - 5\beta_4^2)}{4(\beta_2^2\beta_6 - 3\beta_2\beta_4\beta_5 + 2\beta_4^2)}, \frac{\beta_4}{2\beta_2}, \frac{4(\beta_3^2\beta_6 - 3\beta_3\beta_4\beta_5 + 2\beta_4^2)}{\beta_2^2(4\beta_3\beta_5 - 5\beta_4^2)} ; \beta \right)
\]

for all \( i = 3, n - 1 \).

This procedure can be shown schematically by the following picture:

\[
\beta' \quad \left( \frac{A}{A_0}, \frac{B}{B_0} \right) \quad \vartheta_i \left( \frac{A}{A_0}, \frac{B}{B_0} ; \beta' \right) \quad \left( \frac{A}{A_0}, \frac{B+A_0B}{A_0D} \right)
\]

Conversely, let the equalities
\[
\theta_1\left(\frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2)}{4(\beta_5^2\beta_5^2 - 3\beta_5\beta_5 + 2\beta_5^4)}; \beta\right) = \theta_2\left(\frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2)}{4(\beta_5^2\beta_5^2 - 3\beta_5\beta_5 + 2\beta_5^4)}; \beta'\right)
\]
hold for \( i = 3, n - 1 \). Then it is easy to see that
\[
\theta_2\left(\frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2)}{4(\beta_5^2\beta_5^2 - 3\beta_5\beta_5 + 2\beta_5^4)}; \beta\right) = \theta_2\left(\frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2)}{4(\beta_5^2\beta_5^2 - 3\beta_5\beta_5 + 2\beta_5^4)}; \beta'\right)
\]
for \( i = 1, 2 \) as well and therefore
\[
\theta_1\left(\frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2)}{4(\beta_5^2\beta_5^2 - 3\beta_5\beta_5 + 2\beta_5^4)}; \beta\right) = \theta_2\left(\frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2)}{4(\beta_5^2\beta_5^2 - 3\beta_5\beta_5 + 2\beta_5^4)}; \beta'\right)
\]
which means the algebras \( L(\beta) \) and \( L(\beta') \) are isomorphic to the same algebra and therefore they are isomorphic to each other.

Part ii) can be proved in the same way as the proof of Theorem 3 of [2]. Here are the corresponding invariants for low dimensional cases.

Case of \( \dim L = 6 \):
\[
\theta_1\left(\frac{\beta_5}{\gamma}, \frac{\beta_3}{\beta_4}, \frac{1}{\beta_5}; \beta\right) = \frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2\beta_3 + 2\beta_5^4\gamma)}{4\gamma^2}
\]

Case of \( \dim L = 7 \):
\[
\theta_1\left(\frac{\beta_5}{\gamma}, \frac{\beta_3}{\beta_4}, \frac{1}{\beta_5}; \beta\right) = \frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2\beta_3 + 2\beta_5^4\gamma)}{4\gamma^2}
\]

Case of \( \dim L = 8 \):
\[
\theta_1\left(\frac{\beta_5}{\gamma}, \frac{\beta_3}{\beta_4}, \frac{1}{\beta_5}; \beta\right) = \frac{\beta_5(4\beta_5^3\beta_5 - 5\beta_5^2\beta_3 + 2\beta_5^4\gamma)}{4\gamma^2}
\]
The five-dimensional case.

Proposition 5. ii) The algebras from the set $U$ are isomorphic if and only if

$$\frac{\gamma}{\beta_3^2} = \frac{\gamma'}{\beta_3'^2}.$$ 

Proposition 4. i) Two algebras $L(\beta)$ and $L(\beta')$ from $U_1$ are isomorphic if and only if

$$\frac{\gamma}{\beta_3^2} = \frac{\gamma'}{\beta_3'^2}.$$ 

4. Applications

4.1. The five-dimensional case. This case is specific and can be also completely investigated: $FLeib_5$ can be represented as a disjoint union of several subsets:

$$FLeib_5 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup F,$$

where

$U_1 = \{ L(\beta) \in FLeib_5 : \beta_3 \neq 0 \text{ and } \gamma - 2\beta_3^2 \neq 0 \}$,

$U_2 = \{ L(\beta) \in FLeib_5 : \beta_3 \neq 0, \gamma - 2\beta_3^2 = 0 \text{ and } \beta_4 \neq 0 \}$,

$U_3 = \{ L(\beta) \in FLeib_5 : \beta_3 \neq 0, \gamma - 2\beta_3^2 = 0 \text{ and } \beta_4 = 0 \}$,

$U_4 = \{ L(\beta) \in FLeib_5 : \beta_3 = 0, \gamma \neq 0 \}$,

$U_5 = \{ L(\beta) \in FLeib_5 : \beta_3 = 0, \gamma = 0 \text{ and } \beta_4 \neq 0 \}$,

$F = \{ L(\beta) \in FLeib_5 : \beta_3 = 0, \gamma = 0 \text{ and } \beta_4 = 0 \}$.

As to the isomorphism problem for the algebras from the closed set $F$ a similar procedure as the above can be applied to it as well. We will not consider it here. In the next section we will present final results of classification in five and six-dimensional cases.
4) \( L(0,1,0) : \)
\[
\begin{cases}
[e_1,e_1] = e_3, [e_3,e_1] = e_4, [e_4,e_1] = e_5, [e_1,e_2] = e_5.
\end{cases}
\]
5) \( L(0,0,0) : \)
\[
\begin{cases}
[e_1,e_1] = e_3, [e_3,e_1] = e_4, [e_4,e_1] = e_5
\end{cases}
\]

4.2. The six-dimensional case. The set \( FLeib_6 \) can be represented as a disjoint union of the following subsets:
\[
FLeib_6 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7 \cup U_8 \cup U_9 \cup U_{10} \cup U_{11} \cup F,
\]

where
\[
U_1 = \{ L(\beta) \in FLeib_6 : \beta_3 \neq 0, \beta_4 \neq 0 \text{ and } \gamma \neq 0 \},
\]
\[
U_2 = \{ L(\beta) \in FLeib_6 : \beta_3 \neq 0, \beta_4 \neq 0, \gamma = 0 \text{ and } \beta_3 \beta_4 \neq 4\beta_1^2 \},
\]
\[
U_3 = \{ L(\beta) \in FLeib_6 : \beta_3 \neq 0 \beta_4 \neq 0, \gamma = 0 \text{ and } \beta_3 \beta_4 = 4\beta_1^2 \},
\]
\[
U_4 = \{ L(\beta) \in FLeib_6 : \beta_3 \neq 0, \beta_4 = 0, \gamma \neq 0 \},
\]
\[
U_5 = \{ L(\beta) \in FLeib_6 : \beta_3 \neq 0, \beta_4 = 0, \gamma = 0 \},
\]
\[
U_6 = \{ L(\beta) \in FLeib_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma \neq 0 \},
\]
\[
U_7 = \{ L(\beta) \in FLeib_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma = 0 \text{ and } \beta_5 \neq 0 \},
\]
\[
U_8 = \{ L(\beta) \in FLeib_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma = 0 \text{ and } \beta_5 = 0 \},
\]
\[
U_9 = \{ L(\beta) \in FLeib_6 : \beta_3 = 0, \beta_4 = 0, \beta_5 \neq 0 \text{ and } \gamma \neq 0 \},
\]
\[
U_{10} = \{ L(\beta) \in FLeib_6 : \beta_3 = 0, \beta_4 = 0, \beta_5 \neq 0 \text{ and } \gamma = 0 \},
\]
\[
U_{11} = \{ L(\beta) \in FLeib_6 : \beta_3 = 0, \beta_4 = 0, \beta_5 = 0 \text{ and } \gamma \neq 0 \},
\]
\[
F = \{ L(\beta) \in FLeib_6 : \beta_3 = 0, \beta_4 = 0, \beta_5 = 0 \text{ and } \gamma = 0 \},
\]

**Proposition 7.** i) Two algebras \( L(\beta) \) and \( L(\beta') \) from \( U_1 \) are isomorphic if and only if
\[
\varrho_3 = \frac{2\beta_3\beta_4 \gamma + 4\beta_3^2 \beta_5 - 5\beta_3^2 \beta_4^2}{\gamma^2} = \frac{2\beta_3 \beta_4 \gamma' + 4\beta_3^2 \beta_5' - 5\beta_3^2 \beta_4^2}{\gamma'^2}.
\]

ii) For any \( \lambda \in \mathbb{C} \) there is an algebra \( L(\beta) \) from \( U_1 \) such that \( \varrho_3 = \frac{2\beta_3 \beta_4 \gamma + 4\beta_3^2 \beta_5 - 5\beta_3^2 \beta_4^2}{\gamma^2} = \lambda \).

**Proposition 8.** i) Two algebras \( L(\beta) \) and \( L(\beta') \) from \( U_4 \) are isomorphic if and only if
\[
\varrho_3 = \frac{4\beta_3^2 \beta_5}{\gamma^2} = \frac{4\beta_3^2 \beta_5'}{\gamma'^2}.
\]

ii) For any \( \lambda \in \mathbb{C} \) there is an algebra \( L(\beta) \) from \( U_4 \) such that \( \varrho_3 = \frac{4\beta_3^2 \beta_5}{\gamma^2} = \lambda \).

**Proposition 9.**

a) The algebras from the set \( U_2 \) are isomorphic to the algebra \( L(1,0,1,0) \);
b) The algebras from the set \( U_3 \) are isomorphic to the algebra \( L(1,0,0,0) \);
c) The algebras from the set \( U_9 \) are isomorphic to the algebra \( L(1,1,0,0) \);
d) The algebras from the set \( U_6 \) are isomorphic to the algebra \( L(0,1,0,1) \);
e) The algebras from the set \( U_7 \) are isomorphic to the algebra \( L(0,1,1,0) \);
f) The algebras from the set \( U_8 \) are isomorphic to the algebra \( L(0,1,0,0) \);
g) The algebras from the set \( U_5 \) are isomorphic to the algebra \( L(0,0,0,1) \);
h) The algebras from the set \( U_{10} \) are isomorphic to the algebra \( L(0,0,1,0) \);
k) The algebras from the set \( U_{11} \) are isomorphic to the algebra \( L(0,0,1,1) \);
l) The algebras from the set \( F \) are isomorphic to the algebra \( L(0,0,0,0) \).

**Theorem 10.**
Any 6-dimensional complex filiform Leibniz algebra from $FLeib_6$ is isomorphic to one of the following pairwise nonisomorphic non-Lie filiform complex Leibniz algebras $L = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle$ whose commutation relations are (omitted products are supposed to be zero):

1) $L(1, 0, \lambda, 1) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_4 + \lambda e_6,
   \\
   & [e_2, e_2] = e_6, [e_3, e_2] = e_5, [e_4, e_2] = e_6, \text{where } \lambda \in \mathbb{C}.
   \end{align*}
   \]

2) $L(1, 0, 1, 0) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_4 + e_6,
   \\
   & [e_3, e_2] = e_5, [e_4, e_2] = e_6.
   \end{align*}
   \]

3) $L(1, 0, 0, 0) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_4,
   \\
   & [e_3, e_2] = e_5, [e_4, e_2] = e_6.
   \end{align*}
   \]

4) $L(1, 1, \lambda, 1) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_4 + e_5 + \lambda e_6,
   \\
   & [e_2, e_2] = e_6, [e_3, e_2] = e_5 + e_6, [e_4, e_2] = e_6, \lambda \in \mathbb{C}.
   \end{align*}
   \]

5) $L(1, 1, 0, 0) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_4 + e_5,
   \\
   & [e_3, e_2] = e_5 + e_6, [e_4, e_2] = e_6.
   \end{align*}
   \]

6) $L(0, 1, 0, 0) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_5,
   \\
   & [e_2, e_2] = e_6.
   \end{align*}
   \]

7) $L(0, 1, 1, 0) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_5 + e_6,
   \\
   & [e_3, e_2] = e_6.
   \end{align*}
   \]

8) $L(0, 0, 1, 0) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_1, e_2] = e_5,
   \\
   & [e_3, e_2] = e_6.
   \end{align*}
   \]

9) $L(0, 0, 0, 1) :$
   \[
   \begin{align*}
   & [e_1, e_1] = e_3, [e_3, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6, [e_2, e_2] = e_6.
   \end{align*}
   \]

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