Resonantly driven wobbling kinks

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The amplitude of oscillations of the freely wobbling kink in the $\phi^4$ theory decays due to the emission of second-harmonic radiation. We study the compensation of these radiation losses (as well as additional dissipative losses) by the resonant driving of the kink. We consider both direct and parametric driving at a range of resonance frequencies. In each case, we derive the amplitude equations which describe the evolution of the amplitude of the wobbling and the kink’s velocity. These equations predict multistability and hysteretic transitions in the wobbling amplitude for each driving frequency — the conclusion verified by numerical simulations of the full partial differential equation. We show that the strongest parametric resonance occurs when the driving frequency equals the natural wobbling frequency and not double that value. For direct driving, the strongest resonance is at half the natural frequency, but there is also a weaker resonance when the driving frequency equals the natural wobbling frequency itself. We show that this resonance is accompanied by translational motion of the kink.

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I. INTRODUCTION

The kink of the $\phi^4$ equation has a mode of internal oscillation, commonly referred to as the wobbling mode. To check whether the resonant excitation of the wobbling mode could provide a channel for pumping energy into a kink-bearing system, several authors have studied the dynamics of $\phi^4$ kinks subjected to resonant direct or parametric driving and damping [1, 2, 3, 4]. The damped-driven $\phi^4$ theory arises in a variety of physical contexts, in particular in the description of topological-defect dynamics in media with temporally [5] and spatially [6] modulated parameters or in the presence of fluctuations [7]. Examples include the drift of domain walls in ferromagnets in oscillatory magnetic fields [8]; the Brownian motion of string-like objects on a periodically modulated bistable substrate [10]; ratchet dynamics of kinks in a lattice of point-like inhomogeneities [11] and rectification in Josephson junctions and optical lattices [12]. The damped $\phi^4$ equation driven by noise was used to study the production of topological defects during the symmetry-breaking phase transition [13] and spatiotemporal stochastic resonance in a chain of bistable elements [4].

The mathematical analysis of the damped-driven kinks started with the work of Kivshar, Sánchez and Vázquez [5] who studied the discrete parametrically pumped $\phi^4$ system. Assuming that the driving frequency lies above the phonon band and using the method of averaging, they have discovered the phenomenon of kink death for sufficiently large driving strengths. Next, in an influential paper [8], Sukstanskii and Primak considered the continuous $\phi^4$ equation with a combination of direct and parametric driving. (See also a related discussion in [9].) Using a variant of the Lindstedt-Poincaré technique where the velocity of the kink is adjusted so as to eliminate secular terms at the lowest orders of the perturbation expansion in powers of the wobbling amplitude, they detected a slow unidirectional motion of the kink. In their analysis, Sukstanskii and Primak were not concerned with terms higher than quadratic in the amplitude of the wobbling; in fact, their approach is not suitable to deal with secular terms at the $\epsilon^3$-order of the expansion. Neither can it be utilised in the case of the resonant driving frequency.

The resonant situation was studied by Quintero, Sánchez and Mertens [2, 3, 4] who employed the method of projections. The method assumes a specific functional dependence of the kink on the “collective co-ordinates”, which in this case are the width and position of the kink. Inserting the chosen ansatz in the partial differential equation and projecting the result onto the neutral modes associated with the two degrees of freedom, one obtains a two-dimensional dynamical system, a simplification from the infinite degrees of freedom present in the original partial differential equation. (In the undamped undriven situation, the collective coordinate approach was pioneered by Rice and Mele [15, 16].) The major advantage of the method is that if the collective coordinates have been chosen such that they capture the essentials of the dynamics, one should be able to uncover the very mechanism of the observed nonlinear phenomena — which is otherwise concealed by an infinite number of degrees of freedom (see e.g. [17]). The drawback of the collective coordinate approach, however, is that one cannot know beforehand which degrees of freedom are
essential and which can be omitted without a qualitative impact on the kink’s dynamics. Specifically, the role of radiation (which is neglected in the approach in question) is not obvious and therefore one has no a priori guarantee that a radiationless ansatz is adequate. Another disadvantage is that the resulting two-dimensional dynamical system is amenable to analytical study only in a very special case (when the damping is set to zero [2, 3, 4]).

In this paper, we approach the resonantly driven wobbling kink from a complementary perspective. Instead of trying to guess the most pertinent set of collective variables, we construct the wobbler as a singular perturbation expansion using a sequence of space and time scales. The nonsecularity conditions yield equations for the slow-time evolution of the wobbling amplitude and kink velocity. This asymptotic procedure has already been implemented for the unperturbed kink velocity. This asymptotic procedure has already been implemented for the unperturbed φ^4 equation [21]; here we extend it to include damping and driving terms. We consider both external (direct) and parametric driving, at several resonant frequencies. This includes the cases considered by Quintero, Sánchez and Mertens in [2, 3, 4]. Although the multiscale expansion cannot crystallize the “minimum set” of degrees of freedom accountable for the observed behaviour, it does not suffer from the arbitrariness associated with the choice of collective-coordinate ansatz. The multiscale expansion is asymptotic, i.e. only valid for small amplitudes of the wobbling mode; however, in the small-amplitude limit the expansion provides a faithful description of the wobbler, independent of any assumptions and mode pre-selections. Importantly, it does not neglect the radiation.

In all four cases of the direct and parametric driving considered in this paper, we derive an autonomous system of equations for the amplitude of the wobbling mode and velocity of the kink. In each of the four cases this dynamical system turns out to exhibit stable fixed points corresponding to the nondecaying wobbling of the kink. In some parameter regimes, the amplitude of stable wobbling is nonunique and may undergo hysteretic transitions between two nonzero values. In another case, the wobbling is necessarily accompanied by translational motion of the kink. The conclusions of our asymptotic analysis have been verified in direct numerical simulations of the corresponding partial differential equation. The numerical procedure we shall be using throughout this paper was specified in [21].

An outline of this paper is as follows. In sections II and III we study the parametrically driven wobbling kink. In section II the frequency of the driver is chosen near the natural wobbling frequency of the kink while in III we take the forcing frequency close to double that value. In the next section IV, we compare the mechanisms that are at work in each of the two cases. Subsequently, we consider the kink driven directly — first near half of its natural wobbling frequency (section V) and then close to the wobbling frequency itself (section VI). Our conclusions are summarised in section VII. Here, in particular, we rank the four resonances according to the amplitude of the resulting stationary wobbling and according to the width of the resonant frequency range.

II. 1:1 PARAMETRIC RESONANCE

A. Asymptotic Multiscale Expansion

We start with the parametric driving of the form

$$\frac{1}{2}\phi_{tt} - \frac{1}{2}\phi_{xx} + \gamma\phi_t - [1 + h\cos(\Omega t)]\phi + \phi^3 = 0. \quad (1)$$

This type of a driver was previously considered by Quintero, Sánchez and Mertens [4]. The driving frequency Ω is assumed to be slightly detuned from ω_0, the linear wobbling frequency of the undriven kink:

$$\Omega = \omega_0(1 + \rho).$$

We remind the reader that ω_0 = √3 [21]. Introducing a small parameter ϵ (which will be used to measure the amplitude of the wobbling mode in what follows) we choose the detuning in the form

$$\rho = \epsilon^2 R,$$

where R is of order one. Since the frequency of the free nonlinear wobbling is smaller than ω_0 (see Eq.(44) in [21]), we expect that the strongest resonance will occur not when Ω = ω_0 but for some small negative ρ. [This will indeed be the case; see Eq.(29) below.]

Next, the parameters γ > 0 and h > 0 are the small damping coefficient and driving strength, respectively. We choose the following scaling laws for these parameters:

$$\gamma = \epsilon^2 \Gamma, \quad h = \epsilon^3 H, \quad (2)$$

where Γ and H are quantities of order 1. This choice of scalings will give rise to amplitude equations featuring the driving term of the same order of magnitude as the linear and nonlinear damping terms (so that the stationary wobbling regimes become possible). We assume that the kink moves with a slowly varying, small velocity: v = ϵV where V = V(T_1, T_2, ...) is of order 1.

Before embarking on the perturbation expansions, we transform Eq. (1) to the co-moving reference frame:

$$\frac{1}{2}(1 + \rho)^2\phi_{tt} - v(1 + \rho)\phi_t - \frac{v^2}{2}(1 + \rho)\phi_x - \frac{1 - v^2}{2}\phi_{xx} - \phi + \phi^3 = h\cos(\omega_0 t)\phi + \gamma v\phi_x - \gamma(1 + \rho)\phi_t. \quad (3)$$

Here

$$\xi = x - \int_0^t v(t')dt'.$$

We have also changed t → τ, where

$$\Omega t = \omega_0 \tau.$$
We now expand the field \( \phi(\xi, \tau) \) about the kink \( \phi_0 = \tanh \xi \):

\[
\phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots. \tag{5}
\]

We also define “slow” space and time variables

\[
X_n = \epsilon^n \xi, \quad T_n = \epsilon^n \tau, \quad n = 0, 1, 2, \ldots,
\]

with the standard short-hand notation \( \partial_n = \partial / \partial X_n \), \( D_n = \partial / \partial T_n \). Substituting (5) into the \( \phi^4 \) equation (3), making use of the chain-rule expansions \( \partial / \partial \xi = \partial_0 + \epsilon \partial_1 + \epsilon^2 \partial_2 + \ldots \), \( \partial / \partial \tau = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \ldots \), and equating coefficients of like powers of \( \epsilon \), we obtain a sequence of linear partial differential equations — just as we have done in the case of the free wobbling [21].

As in the case of the undamped undriven \( \phi^4 \) equation, the first-order perturbation is chosen to include only the wobbling mode,

\[
\phi_1 = A(X_1, \ldots; T_1, \ldots) \sech X_0 \tanh X_0 e^{i \omega_0 T_0} + c.c., \tag{6}
\]

while the quadratic correction satisfies the partial differential equation

\[
\frac{1}{2} D_0^2 \phi_2 + L \phi_2 = F_2(X_0, \ldots; T_0, \ldots) \tag{7}
\]

with

\[
L = -\frac{1}{2} \partial_0^2 - 1 + 3 \phi_0^2 = -\frac{1}{2} \partial_0^2 + 2 - 3 \sech^2 X_0
\]

and

\[
F_2 = (\partial_0 \partial_1 - D_0 D_1) \phi_1 - 3 \phi_0 \phi_1^2 + V D_0 \partial_0 \phi_1 + \frac{1}{2} D_1 V \partial_0 \phi_0 - \frac{1}{2} V^2 \partial_0^2 \phi_0. \tag{8}
\]

As in [21], the second-order perturbation is taken to consist just of the harmonics present in the forcing function:

\[
\phi_2 = \phi_2^{(0)} + e^{i \omega_0 T_0} \phi_2^{(1)} + c.c. + e^{2 i \omega_0 T_0} \phi_2^{(2)} + c.c.. \tag{9}
\]

Here the coefficient functions \( \phi_2^{(0)}(X_0) \), \( \phi_2^{(1)}(X_0) \), and \( \phi_2^{(2)}(X_0) \) are found by solving the corresponding ordinary differential equations. The solvability condition for the first of these equations is \( D_1 V = 0 \) and the solution is

\[
\phi_2^{(0)}(0) = 2 |A|^2 \sech^2 X_0 \tanh X_0 \tag{10}
\]

\[
+ \left( \frac{V^2}{2} - 3 |A|^2 \right) X_0 \sech^2 X_0
\]

(see [21]). The solution of the last equation is

\[
\phi_2^{(2)} = A^2 f_1(X_0), \tag{11}
\]

with

\[
f_1(X_0) = \frac{1}{8} \left( 6 \tanh X_0 \sech^2 X_0 + (3 - \tanh^2 X_0 + i k_0 \tanh X_0) \int_0^X e^{i k_0 X_0} \right). \tag{12}
\]

Here the function \( J_2(X_0) \) is defined by the integral

\[
J_2(X_0) = \int_{X_0}^{\infty} e^{i k_0 \xi} \sech^2 \xi \, d\xi \quad (k_0 = \sqrt{8}), \tag{13}
\]

with \( n = 2 \). The constant \( J_2^\infty \) is the asymptotic value of \( J_2(X_0) \) as \( X_0 \to \infty \):

\[
J_2^\infty = \lim_{X_0 \to \infty} J_2(X_0). \tag{14}
\]

Finally, the nonsecularity condition associated with the equation for the coefficient function \( \phi_2^{(1)}(X_0) \) is \( D_1 A = 0 \); with this condition in place, the solution \( \phi_2^{(1)}(X_0) \) is bounded for all \( X_0 \) and decays as \( |X_0| \to \infty \). However, this decay is not fast enough [21]; hence the term \( \phi_2^{(1)}(X_0) e^{i \omega_0 T_0} \) has a quasi-secular behaviour at the infinities and has to be set to zero. This is achieved by imposing the condition [21]

\[
\partial_1 A + i \omega_0 V A = 0. \tag{15}
\]

Proceeding to the order \( \epsilon^3 \), we find the PDE

\[
\frac{1}{2} D_0^2 \phi_3 + L \phi_3 = F_3, \tag{16}
\]

where

\[
F_3 = (\partial_0 \partial_1 - D_0 D_1) \phi_2 + (\partial_0 \partial_2 - D_0 D_2) \phi_1 + \frac{1}{2} (\partial_1^2 - D_1^2) \phi_1 - \phi_1^3 - 6 \phi_0 \phi_1 \phi_2 + V D_0 \partial_0 \phi_2 + V D_0 \partial_1 \phi_1 + \Gamma D_0 \phi_1 + \Gamma V \partial_0 \phi_0 - R D_0^2 \phi_1 + \frac{1}{2} H e^{i \omega_0 T_0} \phi_0 + c.c. \tag{17}
\]

The cubic correction \( \phi_3 \) consists of harmonics present in the function \( F_3 \). The solvability condition for the first harmonic gives the amplitude equation

\[
i D_2 A + \frac{\omega_0 \xi}{2} |A|^2 A - \omega_0 \left( R + \frac{V^2}{2} \right) A = \frac{\pi \omega_0}{8} H - i \Gamma A. \tag{18}
\]
while the solvability condition for the zeroth harmonic produces

\[ D_2V = -2\Gamma V. \]

Letting \( a = \epsilon A \) and keeping in mind that \( D_1A = 0 \) and \( D_1V = 0 \), we can rewrite these two equations in terms of the unscaled variables. This gives a system of two master equations

\[
\dot{a} = -\gamma a - i\omega_0 \alpha + \frac{i}{2} \omega_0 \zeta |a|^2 a - i \frac{\pi}{8} \omega_0 h + O\left(|a|^5\right), \tag{19a}
\]

\[
\dot{v} = -2\gamma v + O\left(|a|^5\right), \tag{19b}
\]

where the overdots indicate differentiation with respect to \( t \) and the complex coefficient \( \zeta \) was evaluated in [21]:

\[
\zeta = \zeta_R + i\zeta_I = -0.8590 + i \times 0.04636. \tag{20}
\]

**B. Reduced Two-Dimensional Dynamics**

Since \( a \) is complex, Eq. (19) defines a dynamical system in three dimensions. However, equation (19) will damp the variable \( v \) until it is of order \( |a|^3 \) and this will make the term \( v^2 a \) negligible in Eq. (19a). Thus, after an initial transient, the dynamics will be determined by the twodimensional system (19a) with \( v = 0 \). Next, the natural wobbling amplitude \( a \) may depend, parametrically, on \( \xi \). However, equation (15) and the fact that \( v \to 0 \) as \( t \) grows, imply that \( a \) may only depend on \( \xi \) via \( X_2, X_3 \), etc. That is, the dependence is weak.

Letting \( a = re^{-i\phi} \), Eq. (19a) yields

\[
\frac{\dot{r}}{r} = -\frac{\omega_0}{2} (\zeta_I r^3 + \gamma (r_0 \sin \theta - r)), \tag{21}
\]

where \( r_0 = \frac{\pi}{2} \omega_0 h/\gamma \). For all \( a \) with \( |a| > r_0 \), the right-hand side of (21) is negative and so no trajectories can escape to infinity. On the other hand, applying Dulac’s criterion (with Dulac’s function equal to a constant), one can easily ascertain that Eq. (19a) with \( v = 0 \) does not have closed orbits. Hence, all trajectories must flow towards fixed points with finite \( |a| \). The fixed points of the system (19a) are given by the equation

\[
(\gamma + i\omega_0 \alpha) a - \frac{i\omega_0 \zeta}{2} |a|^2 a = -\frac{\pi i\omega_0}{8} h. \tag{22}
\]

From Eq. (22), the absolute value of \( a \) satisfies

\[
\mathcal{H}(|a|^2) = h, \tag{23}
\]

where the function \( \mathcal{H}(|a|^2) \) is defined by

\[
\mathcal{H}^2 = \frac{64}{\pi^2} |a|^2 \left[ \left( \frac{\gamma}{\omega_0} + \frac{\zeta_I}{2} |a|^2 \right)^2 + \left( \rho - \frac{\zeta_R}{2} |a|^2 \right)^2 \right]. \tag{24}
\]

Assume, first, that \( \rho > \rho_0 \), where

\[
\rho_0(\gamma) = \frac{1}{\omega_0} \zeta_I - \frac{\sqrt{3}}{\omega_0} \zeta_R \gamma = -1.139\gamma. \tag{25}
\]

In this case \( \mathcal{H}(|a|^2) \) is a monotonically growing function, with the range \((0, \infty)\). Eq. (23) has a single positive root \( |a|^2 \) for any \( h \) and the dynamical system (19) has a single stationary point. This fixed point is always stable.

Now let \( \rho < \rho_0 \). Here, the range of the function \( \mathcal{H}(|a|^2) \) is still \((0, \infty)\); however the function grows for small and large values of \( |a|^2 \) but decreases in the intermediate interval \( |a_-|^2 < |a|^2 < |a_+|^2 \), where

\[
|a_\pm|^2 = \frac{2}{3} \frac{\sqrt{\alpha^2 - 3\beta^2}}{|\zeta|^2}, \tag{26a}
\]

\[
\alpha = \zeta_R \rho - \zeta_I \frac{\gamma}{\omega_0}, \quad \beta = \zeta_I \rho + \frac{\gamma}{\omega_0}. \tag{26b}
\]

Consequently, Eq. (23) has a single root for small and large values of \( h \) and three roots in the intermediate region defined by \( h_+ < h < h_- \), where

\[
h_{\pm} = \mathcal{H}(|a_\pm|^2). \tag{27}
\]

For the dynamical system (19) this implies that there is only one fixed point (which is stable) for small and large \( h \), but as \( h \) approaches the value \( h_+ \) from below or the value \( h_- \) from above, two new fixed points are born in a saddle-node bifurcation. The region \( h_+ < h < h_- \) is characterised by bistability; an adiabatic variation of \( h \) will result in hysteretic transitions between two stable fixed points.

The existence of hysteresis has been verified in the direct numerical simulations of Eq. (1), with \( \gamma = 0.01 \). Starting with \( h = 0 \) we increased \( h \) past \( h_- \), and then reduced it back to zero. At each \( h \)-step, we used the final values of \( \phi(x) \) and \( \phi_t(x) \) from the previous-step simulation as initial conditions for the new run. For each \( h \) we measured the value of \( a \) to which the numerical solution settled after transients died out. The resulting amplitude \( |a| \) is shown in Fig. 1 clearly visible is the hysteresis loop. Fig. 1 corresponds to simulations with \( \rho = -0.03 \); for \( \rho = -0.02 \) the hysteresis loop is smaller and for \( \rho = -0.01 \) it disappears completely. This is consistent with the value of \( \rho_0 \) given by Eq. (25). The value \( h_- = 0.008 \) at which the amplitude was recorded to jump from the bottom to the top branch in Fig. 1 and the value \( h_+ = 0.005 \) at which it dropped back as \( h \) was decreased, are also in agreement with the corresponding predictions of the amplitude equation. Namely, Eqs. (26), (20), and (27) give \( h_- = 8.2 \times 10^{-3} \) and \( h_+ = 4.9 \times 10^{-3} \).

For any given \( h \), equation (23) can be regarded as a quadratic equation for the detuning \( \rho \) where the coefficients are explicit functions of \( |a|^2 \). There are two roots \( \rho_{1,2} \) for \( |a|^2 \) smaller than \( |a_{\text{res}}|^2 \), and none for \( |a|^2 > |a_{\text{res}}|^2 \), where \( |a_{\text{res}}|^2 \) is a unique positive root of the equation

\[
\frac{64}{\pi^2} |a_{\text{res}}|^2 \left( \frac{\gamma}{\omega_0} + \frac{\zeta_I}{2} |a_{\text{res}}|^2 \right)^2 = h^2. \tag{28}
\]
The value $|a_{\text{res}}|$ defined by (28) gives the largest amplitude of the wobbling achievable for the given driving strength $h$. The corresponding value of the detuning,

$$\rho_{\text{res}} = \frac{\xi R}{2} |a_{\text{res}}|^2 < 0,$$

ensures the strongest resonance. As was expected, the strongest resonance is achieved with negative detuning.

### C. 1:1 Parametrically Driven Wobbler

For large times, the asymptotic expansion for the damped-driven wobbler is given by Eq.(48) in [21] where we just need to set $v = 0$ and replace $\omega_0$ with $\Omega$:

$$\phi(x,t) = \tanh \left[ (1 - 3|a|^2)\xi \right] + a \frac{\xi \tanh \xi e^{i\Omega t} + \text{c.c.}}{\sqrt{1 + 2|a|^2 \sech^2 \xi \tanh \xi + a^2 f_1(\xi)e^{i\Omega t} + \text{c.c.}}} + \mathcal{O}(|a|^3).$$

(30)

Here $\xi = x - x_0$, where $x_0$ is a constant determined by initial conditions, and $a$ is a stable fixed point of the dynamical system (19a) with $v = 0$ [a unique fixed point or one of the two stable fixed points depending on whether $h$ is outside or inside the bistability interval $(h_+, h_-)$]. The function $f_1(\xi)$ is given by Eq.(12). The interpretation of different terms in (30) is the same as in the case of the freely wobbling kink [21].

Like the corresponding formula for the free wobbler, the expansion (30) is only valid at distances $|\xi| = \mathcal{O}(1)$. For larger distances one has to use the outer expansions

$$\phi = \pm 1 + e^2 \phi_2 + e^3 \phi_3 + ..., \quad (31)$$

with coefficients $\phi_n$ determined as in section V of [21].

The analysis of the outer equations produces results equivalent to those in [21]: The second-harmonic radiation propagates away from the core of the kink at the group velocity, leaving in its wake a sinusoidal wave with the frequency $2\Omega$, wavenumber $k_0 = \sqrt{8}$ and constant amplitude of the order $|a|^2$.

Unlike the case of the free wobbling of the kink, the frequency of the oscillation is not determined by its amplitude but is locked to the frequency of the driver, $\Omega$. Another difference from the undamped undriven case is that the driven oscillations of the wobbler do not die out as $t \to \infty$. Instead, the amplitude of the oscillations approaches a nonzero constant value which is determined by the parameters of the damping and driving and — in the bistable region — by the initial conditions. On the other hand, the asymptotic velocity of the damped-driven wobbler is zero.

It is interesting to note that unlike in the case of the parametrically driven damped linear oscillator [18] or damped-driven breather of the sine-Gordon or $\phi^4$ equation [19], there is no threshold driving strength in the case of the damped-driven wobbler. No matter how small is $h$, the amplitude $a(t)$ will not decay to zero as $t \to \infty$.

### III. 2:1 PARAMETRIC RESONANCE

#### A. Asymptotic Expansion

It is a textbook fact that the strongest parametric resonance is achieved when the parameter of the oscillator is varied at double its natural frequency. With an eye to the detection of the most efficient driving regime for the wobbling kink, we now consider the driving frequency close to twice its natural wobbling frequency:

$$\frac{1}{2} \phi_{tt} - \frac{1}{2} \phi_{xx} + \gamma \phi_x - \left[ 1 + h \cos(2\Omega t) \right] \phi + \phi^3 = 0. \quad (32)$$

As before, $$\Omega = \omega_0(1 + \rho), \quad \rho = \epsilon^2 R, \quad \gamma = \epsilon^2 \Gamma,$$ but now we use a different scaling for $h$:

$$h = \epsilon^2 H.$$

We transform the equation in exactly the same way as we did in the previous section; this yields

$$\frac{1}{2} (1 + \rho)^2 \phi_{\tau\tau} - \nu \phi_{\xi\xi} - \frac{\nu}{2} (1 + \rho) \phi_x + \frac{\nu^2}{2} (1 + \rho) \phi_{\xi\xi} - \phi + \phi^3 = h \cos(2\omega_0 t) \phi + \gamma \nu \phi_x - \gamma (1 + \rho) \phi_x.$$

The perturbation expansion is unchanged from the undamped undriven case at $\mathcal{O}(\epsilon^1)$. With the addition of the $\epsilon^2$-strong driving, the equation at $\mathcal{O}(\epsilon^2)$ acquires additional terms on the right hand side as compared to
Eq. (7):
\[
\frac{1}{2}D_0^2 \phi_2 + L \phi_2 = F_2(X_0, \ldots; T_0, \ldots) + \frac{H}{2} \phi_0 e^{2i \omega_0 T_0} + \text{c.c.}
\]
Here \( F_2 \) is as in Eq. (8). The zeroth- and first-harmonic components of \( \phi_2 \) are not affected by this extra term. Namely, assuming that the solution is in the form (9) and setting \( D_1 V = 0 \), we get Eq. (10) for \( \varphi_2^{(0)} \) while imposing \( D_1 A = 0 \), Eq. (10) produces \( \varphi_2^{(1)} = 0 \). As for the coefficient function \( \varphi_2^{(2)} \), we obtain
\[
\varphi_2^{(2)} = A^2 f_1(X_0) + H f_2(X_0),
\]
where the function \( f_2(X_0) \) satisfies
\[
(L - 6) f_2(X_0) = \frac{1}{2} \tanh X_0.
\]
We note that the value 6 lies in the continuous spectrum of the operator \( L \), and so in order to determine \( f_2(X_0) \) uniquely, one has to impose two additional conditions fixing the coefficients of two bounded homogeneous solutions that can be added to \( f_2 \). We do this by requiring the absence of incoming radiation. The particular solution of (33) which obeys these radiation boundary conditions is
\[
f_2(X_0) = \frac{1}{12} f_1(X_0) + \frac{1}{24} \tanh X_0 (2 \sech^2 X_0 - 3),
\]
where the function \( f_1(X_0) \) is given by Eq. (12). In what follows, we will use the fact that \( f_2(X_0) \) is an odd function.

The first term in the right-hand side of (33) describes the familiar second-harmonic radiation from the freely wobbling kink. The second term consists of the induced second-harmonic radiation and a standing wave — also excited by the forcing.

At the order \( \epsilon^3 \), we get the equation (14) where \( F_3 \) is given by Eq. (17) with the term \( \frac{1}{2} H e^{2i \omega_0 T_0} \phi_1 \) replaced with \( \frac{1}{2} H e^{2i \omega_0 T_0} \phi_1 \). The amplitude equation for \( A \), which arises as the solvability condition for the first harmonic, is now
\[
iD_2 A + \frac{\omega_0 \zeta}{2} |A|^2 A - \omega_0 \left( \frac{V^2}{2} + R \right) A = \frac{\omega_0 \sigma}{2} H A^* - i \Gamma A,
\]
where
\[
\sigma = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \sech^2 X_0 \tanh^2 X_0 - 6 \sech^2 X_0 \tanh^3 X_0 f_2(X_0) \right] dX_0.
\]
The imaginary part of this integral is
\[
\sigma_I = \frac{1}{12} \zeta_I = 0.003863;
\]
for the real part we find, numerically,
\[
\sigma_R = 0.5958.
\]

The solvability condition for the 0th harmonic gives
\[
D_2 V = -2 \Gamma V.
\]
We finally write the amplitude and velocity equations in terms of the natural variables \( a = e A \) and \( v = \epsilon V \) and the unscaled time \( t \) (as in the previous section):
\[
\dot{a} = -\gamma a - i \omega_0 \rho a + \frac{1}{2} i \omega_0 |a|^2 a
\]
\[
- \frac{1}{2} i \omega_0 \sigma^2 a - \frac{1}{2} i \omega_0 \sigma h a^* + O(|a|^3),
\]
\[
\dot{v} = -2 \gamma v + O(|a|^3).
\]

B. Reduced Dynamics in Two Dimensions

As in the previous case of the 1 : 1 parametrically driven wobbler, the velocity tends to zero as \( t \to \infty \) and the evolution of \( a(t) \) is governed by the dynamical system (37) with \( v = 0 \). This two-dimensional dynamical system does not have periodic orbits, as one can readily check using Dulac’s criterion. Letting \( a = r e^{-i \theta} \) and \( \sigma = |\sigma| e^{i \text{Arg} \sigma} \), Eq. (37) yields
\[
\dot{r} = - \gamma r + \frac{\omega_0}{2} \zeta_I r \left[ \frac{r_0^2}{2} \sin(2 \theta + \text{Arg} \sigma) - r^2 \right],
\]
where \( r_0 \equiv |\sigma| / \zeta_I \). Since the right-hand side of (38) is negative for all \( a \) with \( |a| > r_0 \), no trajectories can escape to infinity. Therefore, all trajectories should flow to one of the fixed points. The fixed points are given by the equation
\[
\gamma a + i \omega_0 \rho a - \frac{i \omega_0 \zeta}{2} |a|^2 a = -i \omega_0 \sigma h a^*. \tag{39}
\]
One fixed point is trivial, \( a = 0 \); this fixed point is stable if \( h < h_+ \), where
\[
\frac{|\sigma|^2}{4} h_+^2 = \left( \frac{\gamma}{\omega_0} \right)^2 + \rho^2, \tag{40}
\]
and unstable otherwise. For the nontrivial points, we get
\[
\mathcal{H}(|a|^2) = h, \tag{41}
\]
where
\[
\mathcal{H} = \frac{4}{|\sigma|^2} \left[ \left( \frac{\gamma}{\omega_0} + \frac{\zeta_I}{2} |a|^2 \right)^2 + \left( \rho - \frac{\zeta_R}{2} |a|^2 \right)^2 \right].
\]
Assume, first, that \( \rho > \rho_0 \), where
\[
\rho_0 = \frac{1}{\omega_0} \frac{\zeta_I}{\zeta_R} \gamma = -0.03146 \gamma. \tag{42}
\]
The function \( \mathcal{H}(|a|^2) \) with \( \rho \) in this parameter range is monotonically growing, from \( h_+ \) to infinity. The equation (41) has one root provided \( h > h_+ \), and no roots
otherwise. Consequently, in the region \( h > h_+ \), the dynamical system \([37]\) has 2 stable fixed points \( a_1 \) and \(-a_1 \), where

\[
|a_1|^2 = \frac{2\alpha + \sqrt{|\sigma|^2 h^2 - 4\beta^2}}{|\zeta|^2}, \tag{43}
\]

with \( \alpha \) and \( \beta \) as in \([26]\). In the region \( h < h_+ \), the (stable) fixed point at the origin is the only fixed point available in the system. (Thus we have a supercritical pitchfork bifurcation as \( h \) is increased through \( h = h_+ \).)

Assume now \( \rho < \rho_0 \). As \(|a|^2\) grows from zero to infinity, the function \( H \) decreases from \( h_+ \) to its lowest value of \( h_- \), where

\[
\frac{|\sigma|^2}{4} h_+^2 = \frac{1}{|\zeta|^2} \left( \frac{\zeta h}{\omega_0} + \zeta \rho \right)^2, \tag{44}
\]

and then increases to infinity. Therefore, for \( \rho < \rho_0 \), the dynamical system \([37]\) has one fixed point at the origin for small driving strengths \( 0 < h < h_- \); 5 fixed points \( a_1, a_2, 0, -a_1, \) and \(-a_2 \) for the intermediate strengths \( h_- < h < h_+ \); and 3 fixed points \( a_1, 0, \) and \(-a_1 \) for \( h > h_+ \). Here \( h_+ \) is given by Eq.\((40)\) and \( h_- \) by \([14]\).

The nontrivial fixed points \( a_1 \) and \( a_2 \) are born in a saddle-node bifurcation at \( h = h_- \). At \( h = h_+ \), a subcritical pitchfork bifurcation occurs; here, the point \( a_2 \) merges with the trivial fixed point. Therefore, out of the two nontrivial fixed points \( a_1 \) and \( a_2 \), the stable one is \( a_1 \), i.e. the fixed point with the larger absolute value — given by Eq.\((38)\).

In summary, for \( h < h_- \) all trajectories flow to the origin; for \( h > h_+ \) they are attracted to the nontrivial fixed points \( \pm a_1 \), and, finally, in the region \( h_- < h < h_+ \) we have a tristability between \( a = 0 \) and \( a = \pm a_1 \).

These predictions of the amplitude equation were compared to results of direct numerical simulations of Eq.\((32)\), with \( \gamma = 0.005 \). As in the previous section, we increased \( h \) past \( h_+ \) and then reduced it to values under \( h_- \). Fig.\(2\) shows the hysteresis loop arising for \( \rho = -0.005 \). For \( \rho = -0.003 \) the hysteresis was less pronounced and for \( \rho = 0 \) it was seen to disappear completely. These observations are consistent with the value of the critical detuning \([22]\) which, for \( \gamma = 0.005 \), equals \( \rho_0 = -1.573 \times 10^{-4} \). The bifurcation values \( h_\pm \) observed in simulations with \( \rho = -0.005 \) (\( h_+ = 0.019 \) and \( h_- = 0.011 \)), are also in agreement with the predictions of the amplitude equation (which gives \( h_+ = 0.01938 \) and \( h_- = 0.01059 \)).

### C. 2:1 Parametrically Driven Wobbler

Restricting ourselves to the \( t \to \infty \) asymptotic behavior of \( \phi(x,t) \), the leading orders of the perturbation expansion in the case of the subharmonic response are

\[
\phi(x,t) = \tanh \left[ (1 - 3|a|^2) \xi \right] + a \text{sech} \xi \tanh \xi e^{i\Omega t} + c.c. + 2|a|^2 \text{sech}^2 \xi \tan \xi + \left[ a^2 f_1(\xi) + b f_2(\xi) \right] e^{2i\Omega t} + c.c. + O \left( |a|^3 \right). \tag{45}
\]

This represents the boundary condition for the amplitudes \( B_{\pm} \); the equations of motion for these variables arise at the order \( a^2 \) and coincide with Eqs.\((33)\) of \([21]\). The solution of these equations with the boundary condition \([17]\) is qualitatively similar to the solution with the
IV. HARMONIC VS SUBHARMONIC PARAMETRIC RESONANCE: QUALITATIVE COMPARISON

With the amount of detail that we had to provide to justify our conclusions and derivations, the resonance mechanisms of the driven wobbling kink may not be easy to crystallise. The purpose of this short section is to discuss the two parametric resonances qualitatively — in particular, to comment on their atypical hierarchy.

We observed that the amplitude $|a|$ is of the order $h^{1/3}$ in the case of the harmonic resonance (i.e. the resonance arising when the driving frequency $\omega_d$ is near $\omega_0$) but only $O(h^{1/2})$ in the case of the subharmonic resonance (the resonance arising for $\omega_d \approx 2\omega_0$). Thus the harmonic resonance is stronger than the subharmonic one, and this is precisely the opposite behaviour to what we might naively expect based on our intuition about the parametric driving.

To explain this surprising behaviour qualitatively, we write the term $h \cos(n \Omega) \phi$ in Eqs. (1) and (32) as $h \cos(n \Omega) \phi_0$ plus terms of order $ha$ and smaller. This representation reveals that what was introduced as a parametric driver is, to the leading order, an external (direct) driving force. This driving force is nonhomogeneous, i.e. its magnitude and direction vary with the distance, and it has odd spatial parity. When $n = 1$, the frequency of this driving force coincides with the natural frequency of the wobbler and its spatial parity coincides with the parity of the wobbling mode (which is also odd). As a result, we have a strong direct resonance.

When $n = 2$, the external force $h \cos(2 \Omega) \phi_0$ is not in resonance with the wobbling frequency. However the function $h \cos(2 \Omega) \phi_0$ acts as a parametric driver on the next term in the expansion of $\phi$ — that is, the product $h \cos(2 \Omega) \phi_1$ has the resonant frequency. Importantly, this term has the “correct”, odd, parity as a function of $\phi$.

In addition, the odd-parity force $h \cos(2 \Omega) \phi_0$ generates odd-parity radiation and the odd-parity standing wave, both with the frequency $2\Omega$. This radiation and standing wave also couple to the wobbling mode, via the term $e^{2i \Omega t} \phi_0 \phi_1$ in Eq. (32). This constitutes a concurrent driving mechanism. Since each of the two mechanisms is indirect (i.e. requires the wobbling mode as a mediator for the frequency halving) and since the resulting effective driving strength is proportional to the amplitude of the wobbling mode (assumed small), the response to the frequency $2\Omega$ is weaker than to $\Omega$.

V. 1:2 DIRECT RESONANCE

A. Perturbation Expansion

We start with the direct driving at half the natural wobbling frequency. This is the following case of the 1:1 direct resonance were previously considered by Quintero, Sánchez and Mertens [2, 3] and so we will be able to compare our results to theirs. The equation is

$$\frac{1}{2} \phi_{tt} - \frac{1}{2} \phi_{xx} + \gamma \phi_t - \phi + \phi^3 = h \cos \left( \frac{\Omega t}{2} \right) ,$$

(48)

where, as in the previous sections, $\Omega = \omega_0(1 + \rho)$. As before, we change the time variable so that $\Omega t = \omega_0 T$ and transform the equation to the moving frame:

$$\frac{1}{2}(1 + \rho)^2 \phi_{\tau \tau} - \nu (1 + \rho) \phi_{\xi \xi} - \frac{\nu^2}{2}(1 + \rho) \phi_{\xi} - \frac{1 - \nu^2}{2} \phi_{\xi \xi}$$

$$- \phi + \phi^3 = h \cos \left( \frac{\omega_0}{2} \tau \right) + v \gamma \phi_{\xi} - \gamma (1 + \rho) \phi_{\tau}. \quad (49)$$

Keeping our standard scalings for the small parameters $\gamma$ and $\rho$,

$$\gamma = e^2 \Gamma, \quad \rho = e^2 R,$$

we choose a fractional-power scaling law for $h$:

$$h = e^{3/2} H.$$ 

This scaling will be shown to produce a balance of damping and driving terms at the leading order in the amplitude equation. Expanding $\phi$ in powers of $e^{1/2}$,

$$\phi = \phi_0 + e \phi_1 + e^{3/2} \phi_{3/2} + e^2 \phi_2 + e^{5/2} \phi_{5/2} + \ldots ,$$

where $\phi_0 = \tanh X_0$, and substituting in (49), we obtain Eq. (1) for $\phi_1$. The partial differential equation arising at $O(e^{3/2})$ is

$$\frac{1}{2} D_{2x}^2 \phi_{3/2} + \mathcal{L} \phi_{3/2} = \frac{H}{2} e^{i(2\omega_0/2)T_0} + c.c.$$

The solution $\phi_{3/2}$ to this equation has the form $\phi_{3/2} = \varphi_{3/2} e^{i(2\omega_0/2)T_0} + c.c.$, where the coefficient function $\varphi_{3/2}$ satisfies the linear nonhomogeneous equation

$$\left( \mathcal{L} - \frac{3}{8} \right) \varphi_{3/2} = \frac{H}{2}.$$ 

Since $\frac{3}{8}$ is not an eigenvalue of the operator $\mathcal{L}$, this equation has a unique bounded solution. To determine it, we note that two homogeneous solutions of this equation, i.e. solutions of $(\mathcal{L} - 3/8)y = 0$, are given by Segur’s formula

$$y_p(X_0) = \frac{1}{(1 + ip)(2 + ip)}\left( e^{ipX_0} \times \frac{1}{(2 - p^2 - 3ip \tanh X_0 - 3sech^2 X_0)} \right)$$

(50)
with \( p = \pm i \sqrt{13/4} \). Using these in the variation of parameters, we obtain

\[
\varphi^{(1/2)}_{3/2} = \frac{1}{6} H (1 - 8 \text{ sech}^2 X_0).
\]

The term \( \varphi^{(1/2)}_{3/2} e^{i(\omega_0 T_0)} \) in the expansion of the wobbling kink represents the background stationary wave induced by the driver.

The equations arising at \( O(\epsilon^2) \) are the same as for the free wobber and the 1 : 1 parametric resonance, Eqs. (7)-8. Hence the coefficients of the harmonic components of \( \varphi_2 \) are the same as in the undamped, undriven case. Namely, imposing the solvability conditions \( D_1 V = 0 \) and \( D_1 A = 0 \), we obtain (10) for \( \varphi^{(0)}_2 \) and (11) for \( \varphi^{(2)}_2 \).

We also impose Eq. (15) to obtain \( \varphi^{(1)}_{5/2} = 0 \).

At the order \( \epsilon^{5/2} \), we have the equation

\[
\frac{1}{2} D^2_0 \phi_{5/2} + \mathcal{L} \phi_{5/2} = -6 \phi_0 \phi_3 \phi_{3/2} + V D_0 \phi_0 \phi_{3/2}.
\]

Its solution consists of the \( \frac{1}{2} \)th and \( \frac{3}{2} \)th harmonics with the coefficient functions

\[
\begin{align*}
\varphi^{(1/2)}_{5/2} &= H A u_a(X_0) + i \omega_0 H V u_b(X_0), \\
\varphi^{(3/2)}_{5/2} &= H A u_c(X_0),
\end{align*}
\]

respectively. Here the functions \( u_a, u_b \) and \( u_c \) satisfy

\[
\begin{align*}
(L - \frac{3}{8}) u_a(X_0) &= -\frac{24}{8} (1 - 8 \text{ sech}^2 X_0) \text{ sech} X_0 \text{ tanh}^2 X_0, \\
(L - \frac{3}{8}) u_b(X_0) &= \frac{24}{16} \text{ sech}^2 X_0 \text{ tanh} X_0,
\end{align*}
\]

and

\[
(L - \frac{27}{8}) u_c(X_0) = -\frac{24}{8} (1 - 8 \text{ sech}^2 X_0) \text{ sech} X_0 \text{ tanh}^2 X_0.
\]

In order to determine \( u_c(X_0) \) uniquely, we impose the radiation boundary conditions. (These are necessary because the value \( \frac{27}{8} \) lies in the continuous spectrum of the operator \( L \).) The functions \( u_a \) and \( u_b \) are even, while \( u_c \) is odd. These three functions can be easily found by solving the above nonhomogeneous boundary-value problems numerically.

Proceeding to the order \( \epsilon^3 \), we find the equation (16), where \( F_3 \) is given by Eq. (17) with the term \( \frac{1}{2} He^{i \omega_0 T_0} \phi_0 + \text{c.c.} \) replaced with \(-3i \phi_0 \phi^{2}_{3/2} \). The solvability conditions for this equation are

\[
D_2 V = -2i TV
\]

for the 0th harmonic, and

\[
D_2 A = -\Gamma A - i \omega_0 R A + \frac{1}{2} \xi \omega_0 |A|^2 A - \frac{1}{2} \omega_0 V^2 A + \frac{60}{169} i \omega_0 \pi H^2
\]

(52)

for the first harmonic. The latter equation includes both the damping and driving terms and so the resulting master equations could be expected to capture the essentials of the nearly-stationary wobbling of the kink (i.e. wobbling in the vicinity of the fixed point of the amplitude equations which arises due to the balance of the damping and driving terms). However the description provided by these amplitude equations — while being qualitatively correct — turns out to be insufficiently accurate when compared to numerical simulations of the full partial-differential equation (18). (The source of this inaccuracy will be clarified below.) In search of greater accuracy, we shall proceed to higher orders.

The solution of Eq. (16) has the form

\[
\varphi_3 = \varphi^{(0)}_3 + \varphi^{(1)}_3 e^{i \omega_0 T_0} + \text{c.c.}
\]

\[
+ \varphi^{(2)}_3 e^{2i \omega_0 T_0} + \text{c.c.} + \varphi^{(3)}_3 e^{3i \omega_0 T_0} + \text{c.c.}
\]

The function \( \varphi^{(2)}_3 \) is calculated to be zero; the coefficient of the 0th harmonic is given by

\[
\varphi^{(0)}_3 = \frac{16}{169} H^2 (45 \text{ sech}^2 X_0 - 3 \text{ tanh} X_0 - 128 \text{ sech}^2 X_0 \text{ tanh} X_0),
\]

and the one for the first harmonic component is

\[
\varphi^{(1)}_3 = -\frac{1}{3} i \omega_0 A X_0 \text{ sech} X_0 \text{ tanh} X_0 + |A|^2 A u_d(X_0) - \frac{3}{8} i \omega_0 (\Gamma + \omega_0 R)(3 - 4 \text{ sech}^2 X_0) + H^2 u_e(X_0),
\]

(53)

where the functions \( u_d(X_0) \) and \( u_e(X_0) \) are the bounded solutions of the following nonhomogeneous equations:

\[
(L - \frac{3}{8}) u_d = \frac{1}{2} \xi \text{ sech} X_0 \text{ tanh} X_0 + 6 \text{ sech} X_0 \text{ tanh}^2 X_0
\]

\[
\times [3 \text{ sech}^2 X_0 - 0.5 \text{ sech}^2 X_0 \text{ tanh} X_0 - f_1(X_0)],
\]

\[
(L - \frac{3}{8}) u_e = -\frac{48}{169} H^2 \text{ tanh} X_0 (1 - 8 \text{ sech}^2 X_0)^2 + \frac{180}{169} \pi \text{ sech} X_0 \text{ tanh} X_0.
\]

Since \( (L - 3/2) \) is a parity-preserving operator while the right-hand sides of the above equations are given by odd functions, and since the homogeneous solution \( y_{ue} = \text{ sech} X_0 \text{ tanh} X_0 \) is also an odd function, it follows that the nonhomogeneous solutions \( u_d \) and \( u_e \) are both odd. This is the only fact about \( u_d \) and \( u_e \) that we will need in this section — we do not need to know any detail of these functions here. Nevertheless, we do evaluate the solution \( u_d \) as it will be required later on, in the study of the 1 : 1 directly driven kink (section VI); we evaluate it using the variation of parameters and numerical integration. The nonhomogeneous solution is defined up to the addition of an arbitrary multiple of \( y_{ue} \); however this extra degree of freedom is fictitious as it can always be eliminated by a suitable rescaling of \( \epsilon \). [Accordingly, the extra term proportional to \( y_{ue} \) cancels in the integral \( \eta \) where it appears in section VI and does not contribute to the amplitude equations (25)].

To eliminate the quasisecond term proportional to \( X_0 \text{ sech} X_0 \text{ tanh} X_0 \) in (54), we set \( \partial_2 A = 0 \).

It will not be necessary to calculate the third harmonic component, \( \varphi^{(3)}_3 \), as this does not contribute to the 0th
or 1st harmonics at $O(\epsilon^4)$, and hence does not affect the $\epsilon^4$-correction to the amplitude equations. Similarly, we shall not calculate $\phi_{7/2}$ as it only contains fractional harmonic components which cannot impinge on the amplitude equations at $O(\epsilon^4)$. Hence we skip the order $\epsilon^{7/2}$.

At $O(\epsilon^4)$, we obtain

$$\frac{1}{2}D_0^2\phi_4 + \mathcal{L}\phi_4 = F_4,$$

where

$$F_4 = (\partial_0\partial_1 - D_0 D_1)\phi_3 + (\partial_0\partial_2 - D_0 D_2)\phi_2 + \frac{1}{2}(\partial_1^2 - D_1^2)\phi_2 + (\partial_0\partial_3 - D_0 D_3)\phi_1 + (\partial_1\partial_2 - D_1 D_2)\phi_1
- 3\phi_2^2\phi_1 - 6\phi_0\phi_3 - 3\phi_0\phi_2^2 + V D_0\partial_0\phi_3 + V D_0\partial_1\phi_2 + V D_0\partial_2\phi_1 + V D_1\partial_0\phi_2 + V D_1\partial_1\phi_1 + V D_2\partial_0\phi_1$$

$$+ \frac{1}{2}D_2 V\partial_0\phi_1 + \frac{1}{2}D_3 V\partial_0\phi_0 - \frac{1}{2}V^2\partial_0^2\phi_2 - V^2\partial_0\partial_1\phi_1 - \Gamma D_0\phi_2 - \Gamma D_1\phi_1 + \Gamma V\partial_0\phi_1 - RD_0^2\phi_2 + VRD_0\partial_0\phi_1.$$

The corresponding solvability conditions are

$$D_3 V = 0$$

and

$$D_3 A = -\frac{i}{2}\omega_0\lambda H^2 A$$

where

$$\lambda = \int_{-\infty}^{\infty} \text{sech} X_0 \tanh X_0 \left[-\frac{90}{169}(45X_0 \text{sech}^2 X_0 - 3 \tanh X_0 - 128 \text{sech}^2 X_0 \tanh X_0) \text{sech} X_0 \tanh^2 X_0 - \frac{24}{13} \text{tanh} X_0 (1 - 8 \text{sech}^2 X_0)u_0(X_0) - \frac{24}{13} \text{tanh} X_0 (1 - 8 \text{sech}^2 X_0)u_c(X_0) - \frac{96}{169} \text{sech} X_0 \tanh X_0 (1 - 8 \text{sech}^2 X_0)^2 \right] dX_0.$$

Numerically,

$$\lambda = \lambda_R + i\lambda_I = -7.4656 - i1.6785.$$

Expanding the derivative $\partial/\partial \tau$ as $D_0 + \epsilon D_1 + \epsilon^2 D_2 + \ldots$ and recalling that $d\tau/dt = 1 + \rho$, we combine equations (52) and (57). We also combine (51) and (56). This yields a system of two master equations:

$$\dot{a} = -\gamma a - i\omega_0 \rho a + i\frac{\omega_0\lambda}{2} |a|^2 a - \frac{\omega_0}{2} a^2 a$$

$$+ i\frac{60}{169} \pi \omega_0 h^2 - \frac{1}{2}i\omega_0 \lambda^2 a + \mathcal{O}(|a|^4), \quad (58a)$$

$$\dot{\nu} = -2\gamma v + \mathcal{O}(|a|^4). \quad (58b)$$

It is essential to combine the slow-scale equations in this way, rather than solving the individual equations with the assumption that the different scales are independent. Solving individual equations separately would be illegitimate because in integrating the equations one is covering more than one timescale. For example, solving Eqs. (56) and (57) we would be integrating over the scale $T_3$ which includes a shorter timescale $T_2$.

All terms in the right-hand side of Eq. (58a) are of the order $|a|^3$, except the last term which is $O(|a|^4)$. This last term is the correction coming from the fourth order of the perturbation expansion. As we have already mentioned, the amplitude equations (58) without this term produce an inaccurate description of the dynamics in the region of interest (i.e. in the vicinity of the fixed points). On the other hand, if the above fourth-order term is included, the predictions of the amplitude equations (58) turn out to be in good agreement with results of the direct numerical simulations of the full partial differential equation (58) — see Fig. 3. The substantial improvement in accuracy is due to the large value of the coefficient $\lambda$.  

FIG. 3: The hysteresis loop in the 1:2 directly driven $\phi^4$ equation, Eq. (15), with $\gamma = 0.001$ and $\rho = -0.002$. The driving strength $h$ is increased from 0.005 to 0.01 in increments of 0.0002 and then reduced back to 0.005 (as indicated by the arrows). Crosses mark results of simulation of the PDE, Eq. (15). The continuous and dashed lines depict the stable and unstable fixed points of the amplitude equation (58a) with $v = 0$. 

| $|a|$ | $h$ |
|---|---|
| 0.02 | 0.005 |
| 0.03 | 0.006 |
| 0.04 | 0.007 |
| 0.05 | 0.008 |
| 0.06 | 0.009 |
| 0.07 | 0.010 |

- $\phi_1 = 0$ 
- $\phi_2 = 0$ 
- $\phi_3 = 0$ 
- $\phi_4 = 0$ 

- $\phi_{7/2} = 0$
B. Reduced Two-Dimensional System

Introducing the notation

\[ \gamma' = \gamma - \frac{\lambda_I}{2} \omega_0 h^2, \quad \rho' = \rho + \frac{\lambda_R}{2} h^2, \quad h' = -\frac{480}{169} h^2, \]

the amplitude equation can be written as

\[ \dot{a} = -\gamma' a - i \omega_0 \rho' a + i \xi \omega_0 |a|^2 a - \frac{i}{2} \pi \omega_0 v^2 a - i \frac{\pi}{8} \omega_0 h' + \mathcal{O}(|a|^5), \quad (59) \]

which has the same form as the amplitude equation for the 1 : 1 parametric resonance, Eq. (19a). Consequently, the dynamics of the 1 : 2 directly driven wobbling kink will have some similarities with the dynamics of the wobbling kink driven by the 1 : 1 parametric force.

According to Eq. (58a), the velocity will be damped until it is so small \( \mathcal{O}(|a|^3) \) that it can be disregarded in Eq. (58a); hence after an initial transient the dynamics will be governed by Eq. (58a) with \( v = 0 \). Similarly to Eq. (19) with \( v = 0 \), equation (58a) does not have closed orbits. All trajectories crossing the circle \(|a| = \frac{60}{\pi} \omega_0 h^2/\gamma \) flow inwards and so no trajectories can escape to infinity. Therefore, all trajectories must flow towards fixed points. If \( a \) is a fixed point, the absolute value \(|a|\) satisfies

\[
\begin{align*}
\frac{1}{4} |\lambda|^2 - \frac{\pi^2}{|a|^2} \left( \frac{60}{169} \right)^2 & = 0, \\
\rho - \frac{1}{2} \lambda_R |a|^2 - \lambda_I \left( \frac{\gamma}{\omega_0} + \frac{1}{2} \xi |a|^2 \right) & = 0, \\
\left( \rho - \frac{1}{2} \xi \lambda_R |a|^2 \right)^2 + \left( \frac{\gamma}{\omega_0} + \frac{1}{2} \xi |a|^2 \right)^2 & = 0. \quad (60)
\end{align*}
\]

The left-hand side of this equation is a bi-quadratic in \( h \). Solving for \( |a| \) in these equations, we find that \( |a| \) is a fixed point of \( \gamma' h^2 \). The lower branch is a fixed point, the absolute value \(|a|\) is a constant determined by the bifurcation values obtained from the numerical simulations of the full PDE. The upper branch is a constant determined by the bifurcation values obtained from the numerical simulations of the full PDE.

The interpretation of terms in (62) is the same as in the previous sections. The frequency of the wobbling, where the wobbling mode is given by the sum of the third and second term in the first line in (62) is locked to double the driving frequency. The term proportional to \( h \) in the second line describes a stationary wave induced by the driver. As in the previous sections, the expansion (62) is

\[ h^2 = \frac{1}{\kappa_0} \left( P_\pm + \sqrt{P_\pm^2 + 2\kappa_0 Q_\pm} \right), \quad (61) \]

where

\[ P_\pm = \frac{1}{18} \left[ \xi (3\beta^2 - 5\alpha^2) + 6\alpha \delta \right] \]

\[ \mp \left( 4\xi - 3\beta \right) \sqrt{\alpha^2 - 3\beta^2} \]

\[ Q_\pm = \frac{1}{27} \left[ \alpha (\alpha^2 + 9\beta^2) \mp (\alpha^2 - 3\beta^2)^{3/2} \right], \]

\[ \kappa_0 = 2 \left( \frac{30\pi}{169} \right)^2 |\xi|^4 = 0.3280, \]

\[ \kappa_1 = \xi R \lambda R + \xi I \lambda I = 6.2747, \]

\[ \delta = |\xi|^2 \left( \lambda_R \rho - \xi I \gamma / \omega_0 \right). \]

and \( \alpha \) and \( \beta \) are defined by Eq. (20b). For \( \gamma = 10^{-3} \) and \( \rho = -2 \times 10^{-3} \), the bifurcation values obtained from Eq. (60) are \( h_+ = 6.74 \times 10^{-3} \) and \( h_- = 7.78 \times 10^{-3} \) while the numerical simulations of the full PDE give \( 6.6 \times 10^{-3} < h_+ < 6.8 \times 10^{-3} \) and \( 7.8 \times 10^{-3} < h_- < 8.0 \times 10^{-3} \). For \( \rho = -3 \times 10^{-3} \), the simulations show a more pronounced hysteresis loop whereas for \( \rho = -1 \times 10^{-3} \), the hysteresis was seen to disappear. (In both cases \( \gamma \) was kept at \( 10^{-3} \).) These observations are consistent with the value of \( \rho_0 \) given by the amplitude equations. [For \( \gamma = 10^{-3} \), Eq. (25) gives \( \rho_0 = -1.1 \times 10^{-3} \).]

C. 1:2 Directly Driven Wobbling

Finally, we produce the first several orders of the perturbation expansion for the 1 : 2 directly driven wobbling kink:

\[
\phi(x, t) = \tanh[(1 - 3|a|^2)\xi] + a \sech \xi \tanh \xi e^{i\Omega t} + c.c. \\
+ \frac{1}{4} h(1 - 8 \sech^2 \xi)e^{i(1/2)\xi} + c.c. \\
+ 2|a|^2 \sech^2 \xi \tanh \xi + a^2 f_1(\xi)e^{2\Omega t} + c.c. + \mathcal{O}(|a|^{5/2}).
\]

(62)

When \( t \) is sufficiently large, the variable \( \xi \) in this expression equals \( x - x_0 \) (where \( x_0 \) is a constant determined by the initial conditions) and \( a \) is a stable fixed point of the system (68a) with \( v = 0 \) at a unique fixed point or one of the two stable fixed points depending on whether \( h \) is outside or inside the bistability interval \((h_+, h_-)\).
only valid at the length scale $|\xi| = \mathcal{O}(1)$. The standard analysis involving outer expansions demonstrates that for larger distances, we have groups of second-harmonic radiation waves moving away from the kink and leaving in their wake a sinusoidal waveform of constant amplitude.

D. Qualitative Analysis

The driving term $h \cos(\frac{\Omega}{2} t)$ is not in resonance with the natural frequency of the wobbler, nor does its parity coincide with the parity of the wobbling mode. Therefore the ability of the $1:2$ direct driving to sustain the wobbling is surprising and requires a qualitative explanation.

The authors of Ref. [2, 3] propose that the mechanism which brings about the unexpected superharmonic resonance is the coupling of the translation mode and the wobbling mode. Our explanation for this phenomenon is rather different and unrelated to the translation mode. The way the driver affects the wobbler is by exciting an even-parity standing wave ($\phi_{3/2}$) at the frequency $\Omega/2$ which then undergoes nonlinear frequency doubling and parity transmutation through the term $\epsilon^2 \phi_0 \phi_{3/2}^3$ in Eq. (13). This latter term serves as an effective driver to the wobbling mode; it has the resonant frequency and “correct” parity.

Since this mechanism involves a two-stage process and the resulting effective driving strength is proportional to $h^2$, this type of driving produces a relatively weak response.

E. Chaotic Wobblers?

The authors of Ref. [2, 3] observed chaotic kink dynamics in numerical simulations of the $1:2$ directly driven wobbling kink. An indirect confirmation of the existence of chaotic motions comes also from the collective-coordinate approach which predicts an unbounded growth of the kink’s width, energy and velocity at resonance [2, 3]. On the other hand, our amplitude equations (15) with $\gamma \neq 0$ reduce to a two-dimensional dynamical system which can obviously not exhibit any chaotic attractors.

To find an explanation for this disagreement, we have carried out a series of numerical simulations of the partial differential equation (13) at a range of driving strengths and damping coefficients. In all our experiments, we confined ourselves to zero detuning, $\rho = 0$. We could not detect any sign of chaotic dynamics for $h$ smaller than a certain minimum value, not even in the undamped case. However for $h$ greater or equal than 0.05 and sufficiently small $\gamma$, our numerical simulations did reveal kinks performing erratic motion, where initially close profiles were seen to diverge exponentially fast. For $h = 0.05, 0.06$ and 0.08, chaos was observed in simulations with $\gamma$ smaller or equal to $10^{-3}, 2 \times 10^{-3}$, and $6 \times 10^{-3}$, respectively, whereas the same sequence of $h$ values paired with $\gamma = 2 \times 10^{-3}, 3 \times 10^{-3}$ and $7 \times 10^{-3}$, respectively, did not feature any chaotic trajectories. Therefore chaotic attractors may only arise when the damping is extremely weak, much weaker than $\mathcal{O}(h^{4/3})$. This is the reason why the chaotic dynamics is not captured by our amplitude equations (15) which have been derived on the assumption that $h = \mathcal{O}(\epsilon^{3/2})$ and $\gamma = \mathcal{O}(\epsilon^2)$.

The description of chaotic motions by means of amplitude equations is a topic of future research. We expect our asymptotic method to remain applicable in this situation, with the appropriate adjustment of the scaling laws of the $a$ and $v$ variables and parameters of the damping and driving.

VI. 1 : 1 DIRECT RESONANCE

A. Multiscale Expansion

Finally, we explore the effect of the direct driving near the natural wobbling frequency of the kink. The equation is

$$\frac{1}{2} \phi_{tt} - \frac{1}{2} \phi_{xx} + \gamma \phi_t - \phi + \phi^3 = h \cos(\Omega t),$$

where $\Omega = \omega_0(1 + \rho)$. We let $a = \epsilon A$ and adopt the following scalings for the three small parameters:

$$h = \epsilon H, \quad \gamma = \epsilon^2 \Gamma, \quad \rho = \epsilon^3 R.$$  (64)

This time, we assume that the velocity is scaled as $v = \epsilon^2 V$ (and not as $v = \epsilon V$). We shall find a non-trivial evolution equation for $v$, and with the above scalings, the leading-order dynamics of $a$ and $v$ will occur on the same timescale. While other scalings could be investigated, the variables $v$ and $a$ would then change on different timescales and so would effectively be decoupled for small $\epsilon$. Therefore, the chosen scalings correspond to the richest, three-dimensional, dynamics. Rescaling the time so that $\Omega t = \omega_0 \tau$ and transforming to the moving frame as in Eq. (31), the equation (63) becomes

$$\frac{1}{2}(1 + \rho)^2 \phi_{\tau\tau} - v(1 + \rho)\phi_{\tau} - \frac{v^2}{2}(1 + \rho)\phi_x - \frac{1}{2} \phi_{xx} - \phi + \phi^3 = h \cos(\omega_0 \tau) + v \gamma \phi_x - \gamma(1 + \rho)\phi_\tau.$$  (65)

We expand $\phi$ as in Eq. (5).

With the driving amplitude of the order $\epsilon$, the linear perturbation consists of the wobbling mode and a standing wave excited by the driver:

$$\phi_1 = [A \text{sech} X_0 \tanh X_0 + H(1 - 2 \text{sech}^2 X_0)] e^{i \omega_0 T_0} + c.c.$$  (71)

Next, at $\mathcal{O}(\epsilon^2)$, we obtain the equation (7) with

$$F_2 = (\partial_0 \partial_1 - D_0 D_1) \phi_1 - 3 \phi_0 \phi_1^2.$$  (73)

Once transients have died out, the solution to Eq. (7) will consist only of the harmonics present in the forcing, i.e. will have the form (9). The solvability condition
for the first-harmonic component is $D_1 A = 0$; assuming that this condition is in place, we obtain $\varphi_{2(1)}(X_0) = -\partial_1 A X_0 \text{sech } X_0 \tanh X_0$. To avoid the quasisecular behaviour at infinity we impose $\partial_1 A = 0$, which results in

$$\varphi_{2(1)}(X_0) = 0.$$  \hfill (66)

The other two harmonic components of the quadratic correction $\varphi_2$ have the coefficients

$$\varphi_{2(0)}(X_0) = |A|^2 \text{sech}^2 X_0 (2 \tanh X_0 - 3 X_0)$$
$$+ H^2 (9 X_0 \text{sech}^2 X_0 - 3 \tanh X_0 - 8 \text{sech}^2 X_0 \tanh X_0)$$
$$- 4 H (A + A^*) \text{sech } X_0 (1 + \text{sech}^2 X_0)$$  \hfill (67)

and

$$\varphi_{2(2)}(X_0) = A^2 f_1(X_0) + AH f_3(X_0) + H^2 f_4(X_0),$$  \hfill (68)

where $f_1$ is as in Eq. (12), and the functions $f_3$ and $f_4$ are defined by

$$f_3(X_0) = \frac{1}{4} \text{sech } X_0 - 4 \text{sech}^3 X_0$$
$$- \frac{15}{32} i k_0 (3 - \tanh^2 X_0 + i k_0 \tanh X_0) [J_1(X_0) - J_{1\infty}] e^{i k_0 X_0}$$
$$+ \frac{15}{32} i k_0 (3 - \tanh^2 X_0 - i k_0 \tanh X_0) J_1(X_0) e^{-i k_0 X_0},$$  \hfill (69)

and

$$f_4(X_0) = -\frac{7}{2} f_1(X_0) + \frac{1}{4} \tanh X_0 (3 - 2 \text{sech}^2 X_0).$$  \hfill (70)

The function $J_1(X_0)$ is given by the integral \hfill (13) with $n = 1$. One can show that $f_3(X_0)$ is an even function and $f_4(X_0)$ is odd.

We note a quasisecular term $(9H^2 - 3|A|^2) X_0 \text{sech}^2 X_0$ in Eq. (67); this term does not lead to the nonuniformity of the expansion as it can be incorporated in the variable width of the kink. [See Eq. (80) below.]

The partial differential equation arising at the order $c^3$, is Eq. (16), with $F_3$ given by

$$F_3 = (\partial_0 \partial_1 - D_0 D_1) \phi_2 + (\partial_0 \partial_2 - D_0 D_2) \phi_1 + \frac{1}{2} (\partial_1^2 - D_1^2) \phi_1 - \phi_1^3$$
$$- 6 \phi_0 \phi_1 \phi_2 + V D_0 \partial_0 \phi_1 + \frac{1}{2} D_1 V \partial_0 \phi_0 - \Gamma D_0 \phi_1 - RD_0^2 \phi_1.$$

The solvability conditions give rise to amplitude equations:

$$D_1 V = 0$$  \hfill (71)

for the zeroth harmonic, and

$$D_2 A + \Gamma A + i \omega_0 RA = \frac{i \omega_0 \zeta}{A^2} A - \frac{i}{2} \pi V H$$
$$+ \frac{i}{2} i \omega_0 \nu H^2 A + \frac{1}{2} i \omega_0 \nu H^2 A^* = 0$$  \hfill (72)



\[ \nu = \int_{-\infty}^{\infty} \text{sech } X_0 \tanh X_0 \left[ -6 \tanh X_0 (1 - 2 \text{sech}^2 X_0) f_3(X_0) - 6 \text{sech } X_0 \tanh^2 X_0 (9 X_0 \text{sech}^2 X_0 - 3 \tanh X_0 $$
$$- 8 \text{sech}^2 X_0 \tanh X_0) + 24 \tanh X_0 (\text{sech } X_0 + \text{sech}^3 X_0) (1 - 2 \text{sech}^2 X_0) - 6 \text{sech } X_0 \tanh X_0 (1 - 2 \text{sech}^2 X_0)^2 \right] dX_0, \]

and

\[ \mu = \int_{-\infty}^{\infty} \text{sech } X_0 \tanh X_0 \left[ 24 \tanh X_0 (1 - 2 \text{sech}^2 X_0) (\text{sech } X_0 + \text{sech}^3 X_0) $$
$$- 3 \text{sech } X_0 \tanh X_0 (1 - 2 \text{sech}^2 X_0)^2 - 6 \text{sech } X_0 \tanh^2 X_0 f_4(X_0) \right] dX_0. \]
Numerically,
\[ \nu = 4.159 - i0.3258, \quad \mu = 1.022 + i0.1623. \]

We note that the velocity enters the amplitude equation (72) as a coefficient in front of one of its two driving terms. On the other hand, Eq. (71) implies that \( V \) does not tend to zero — at least on the timescale \( T_1 \). In order to check whether the velocity decays on a longer timescale and hence whether the translational motion can drive the wobbling, we take the expansion to higher orders.

The cubic correction has the form
\[
\phi_3 = \phi_3^{(0)} + \phi_3^{(1)} e^{i \omega_0 T_o} + c.c.
+ \phi_3^{(2)} e^{2i \omega_0 T_o} + c.c. + \phi_3^{(3)} e^{3i \omega_0 T_o} + c.c.
\]

The 0th harmonic component, \( \phi_3^{(0)} \), is evaluated to be zero, and the coefficient of the first-harmonic component is
\[
\phi_3^{(1)} = |A|^2 A_u(X_0) - (\partial_2^2 A + i \omega_0 V A) X_0 \tanh X_0
+ i \omega_0 VH u_1(X_0) + H |A|^2 u_2(X_0) + HA^2 u_3(X_0)
+ HA^2 u_4(X_0) \left( - \frac{3}{2} \omega_0 \Gamma + i \omega_0 R \right)(3 - 4 \sech^2 X_0)
+ H^2 A u_5(X_0) + H^2 A^* u_6(X_0) + H^3 u_7(X_0). \tag{73}
\]

Here \( u_d(X_0) \) was defined in the previous section as the bounded solution of Eq. (54), and the functions \( u_n(X_0) \) \((n = 1, ..., 7)\) are the bounded solutions of the following nonhomogeneous equations:

\[
\begin{align*}
(L - 3/2)u_1 &= -6 \sech^2 X_0 \tanh X_0 + \frac{3\pi}{4} \sech X_0 \tanh X_0, \\
(L - 3/2)u_2 &= -6 \sech^2 X_0 \tanh^2 X_0(1 - 2 \sech^2 X_0) + 24 \sech^2 X_0 \tanh^2 X_0(1 + \sech^2 X_0)
- 6 \sech^2 X_0 \tanh X_0(2 \tanh X_0 - 3X_0)(1 - 2 \sech^2 X_0) - 6 \sech X_0 \tanh^2 X_0 f_3(X_0), \\
(L - 3/2)u_3 &= 3 \sech^2 X_0 \tanh^2 X_0(7 + 10 \sech^2 X_0) + 12 \sech^2 X_0 \tanh X_0 f_1(X_0), \\
(L - 3/2)u_4 &= -6 \tanh X_0 f_1(X_0), \\
(L - 3/2)u_5 &= -6 \tanh X_0(1 - 2 \sech^2 X_0) f_3(X_0) + 24 \sech X_0 \tanh X_0(1 + \sech^2 X_0)(1 - 2 \sech^2 X_0)
- 6 \sech X_0 \tanh^2 X_0(9 \sech^2 X_0 - 3 \tanh X_0 - 8 \sech^2 X_0 \tanh X_0)
- 6 \sech X_0 \tanh X_0(1 - 2 \sech^2 X_0)^2 - \frac{3}{2} \nu \sech X_0 \tanh X_0, \\
(L - 3/2)u_6 &= -6 \sech X_0 \tanh^2 X_0 f_4(X_0) + 24 \sech X_0 \tanh X_0(1 + \sech^2 X_0)(1 - 2 \sech^2 X_0)
- 3 \sech X_0 \tanh X_0(1 - 2 \sech^2 X_0)^2 - \frac{3}{4} \mu \sech X_0 \tanh X_0,
\end{align*}
\]

and
\[
\begin{align*}
(L - 3/2)u_7 &= -6 \tanh X_0(1 - 2 \sech^2 X_0) f_4(X_0) - 3(1 - 2 \sech^2 X_0)^3
- 6 \sech X_0 \tanh^2 X_0(9 \sech^2 X_0 - 3 \tanh X_0 - 8 \sech^2 X_0 \tanh X_0).
\end{align*}
\]

Like the functions \( u_d \) and \( u_e \) of the previous section, the solutions \( u_n(X_0) \) are defined up to the addition of a multiple of \( y_w \). As in the previous section, this does not provide any extra degrees of freedom and the multiple of \( y_w \) cancels out in the integrals \( \eta \) and \( \chi \) below. The solutions \( u_1, u_5, \) and \( u_6 \) are odd, while \( u_2, u_3, u_4 \) and \( u_7 \) can be chosen to be even functions. The only fact about \( u_1 \) that we need is that it is a real solution; owing to its reality, \( u_1 \) does not contribute to the solvability conditions below. The solutions \( u_n(X_0) \) with \( n = 2, ..., 7 \) are determined using the variation of parameters and numerical integration.

To eliminate the quasisecondary term proportional to \( X_0 \sech X_0 \tanh X_0 \) in \( \phi_3 \), we set \( \partial_2 A = -i \omega_0 V A \).

It is not necessary to calculate the second and third harmonic components, \( \phi_3^{(2)} \) and \( \phi_3^{(3)} \), as these do not contribute to the zeroth harmonic at fourth order in \( \epsilon \), where the leading order behaviour of \( V \) will reveal itself. The equation arising at \( O(\epsilon^4) \) is Eq. (55), with

\[
F_4 = (\partial_0 \partial_1 - D_0 D_1) \phi_3 + (\partial_0 \partial_2 - D_0 D_2) \phi_3 + \frac{1}{2} (\partial_0^2 - D_0^2) \phi_3 + (\partial_0 \partial_3 - D_0 D_3) \phi_1 + (\partial_1 \partial_2 - D_1 D_2) \phi_1
- 3 \phi_1^2 \phi_2 - 6 \phi_0 \phi_1 \phi_3 - 3 \phi_0^2 \phi_3 + V D_0 \partial_0 \phi_2 + V D_0 \partial_1 \phi_1 + \frac{1}{2} D_2 V \partial_0 \phi_0 - \frac{1}{2} V^2 \partial_0^2 \phi_0 - R D_0^2 \phi_2 - \Gamma D_0 \phi_2 + \Gamma V \partial_0 \phi_0.
\]
The solvability condition for the zeroth harmonic yields

\[ D_2 V = -2\Gamma V - \frac{3}{2} \eta H |A|^2 A + \text{c.c.} - \frac{3}{2} \chi H^3 A + \text{c.c.} + \frac{3\pi}{4} \omega a H (\Gamma - i\omega R) A + \text{c.c.} , \]  

(74)

where

\[ \eta = \int_{-\infty}^{\infty} \text{sech}^2 X_0 \left[ 24 \text{sech}^2 X_0 \text{tanh}^2 X_0 (\text{sech} X_0 + \text{sech}^3 X_0) - 3 \text{sech}^2 X_0 \text{tanh}^2 X_0 f_3^* (X_0) \right. \]

\[ - 6 \text{sech}^3 X_0 \text{tanh} X_0 (2 \text{tanh} X_0 - 3 X_0) (1 - 2 \text{sech}^2 X_0) - 6 \text{sech} X_0 \text{tanh} X_0 (1 - 2 \text{sech}^2 X_0) f_1 (X_0) \]

\[ - 6 \text{sech} X_0 \text{tanh} X_0 u_3 (X_0) - 6 \text{sech} X_0 \text{tanh}^2 X_0 u_4 (X_0) \]

\[ - 6 \text{sech} X_0 \text{tanh}^2 X_0 u_3^* (X_0) + 24 \text{sech}^2 X_0 \text{tanh} X_0 (2 \text{tanh} X_0 - 3 X_0) \]

\[ \times (\text{sech} X_0 + \text{sech}^3 X_0) - 6 \text{tanh} X_0 (1 - 2 \text{sech}^2 X_0) u_3 (X_0) - 6 \text{tanh} X_0 f_1 (X_0) f_3^* (X_0) \right] dX_0 , \]

and

\[ \chi = \int_{-\infty}^{\infty} \text{sech}^2 X_0 \left[ - 6 \text{sech} X_0 \text{tanh}^2 X_0 u_7^* (X_0) - 6 \text{tanh} X_0 (1 - 2 \text{sech}^2 X_0) u_5 (X_0) \right. \]

\[ - 6 \text{tanh} X_0 (1 - 2 \text{sech}^2 X_0) u_6^* (X_0) + 24 (1 - 2 \text{sech}^2 X_0)^2 (\text{sech} X_0 + \text{sech}^3 X_0) \]

\[ + 18 \text{sech} X_0 \text{tanh} X_0 (1 + 2 \text{sech}^2 X_0) \times (9 \text{sech} X_0 \text{tanh} X_0 - 3 \text{tanh} X_0 - 8 \text{sech}^2 X_0 \text{tanh} X_0) \]

\[ - 3 (1 - 2 \text{sech}^2 X_0)^2 f_3 (X_0) - 6 \text{tanh} X_0 f_3 (X_0) f_4^* (X_0) - 6 \text{sech} X_0 \text{tanh} X_0 (1 - 2 \text{sech}^2 X_0) f_4^* (X_0) \right] dX_0 . \]

Numerically,

\[ \eta = -2.005 - i 0.3823 , \quad \chi = -12.21 - i 0.5706 . \]

Writing \( a = \epsilon A \) as before, and combining \( D_1 A = 0 \) with Eq. (72) with the help of the chain rule, we obtain the amplitude equation in terms of the unscaled parameters:

\[ \dot{a} = - \gamma a - i \omega_a a + \frac{i}{4} \omega_0 \zeta |a|^2 a + \frac{3\pi}{4} v h \]

\[ - \frac{1}{4} \omega_0 v h^2 a - \frac{1}{2} \omega_0 \psi h^2 a^* + O \left(|a|^5\right) . \]  

(75a)

Similarly, combining Eq. (71) with (74), we arrive at

\[ \dot{v} = - 2 \gamma v - \frac{3}{2} \eta |a|^2 a + \text{c.c.} - \frac{3}{2} \chi H^3 a + \text{c.c.} \]

\[ + \frac{3\pi}{4} i \omega h (\gamma - i \omega_0 \rho) a + \text{c.c.} + O \left(|a|^5\right) . \]  

(75b)

**B. Reduced Dynamics in Three Dimensions**

Thanks to the \( \alpha \)-dependent driving terms in Eq. (75b), the direct driving can sustain the translational motion of the kink — in contrast to the parametrically driven cases we have considered. Fig. 4 shows an example of the kink accelerated by the 1:1 direct driving force which simultaneously excites the wobbling. Note that results from the three-dimensional system (75) are in excellent agreement with predictions of the full partial differential equation — not only after the dynamics have settled to a stationary regime but also during the transient phase.

The detailed analysis of the three-dimensional system (75) will be reported elsewhere; here, we limit ourselves to several basic observations. Firstly, it is straightforward to see that when \( h = 0 \), the trivial fixed point \( a = v = 0 \) is the only attractor available in the system. Secondly, numerical simulations show that as \( h \) is increased for fixed \( \gamma \) and \( \rho \), a nontrivial fixed point bifurcates from the point \( a = v = 0 \) [see Fig. 6(a,b)]. For lower values of \( \rho \), the bifurcation is subcritical (as in the case shown in Fig. 5). For higher \( \rho \), it is supercritical. As \( h \) approaches some critical driving strength \( h_c \), both the \( |a| \) and \( v \) components of the nontrivial fixed point tend to infinity. Finally, in the region \( h > h_c \), there are no stable fixed points. In this region, simulations of the system (75) reveal a blow-up regime, where the functions \( |a(t)|, v(t) \) grow without

**FIG. 4:** An example of the kink being accelerated by the 1:1 direct driving. Here \( h = 0.012, \gamma = 1 \times 10^{-3} \) and \( \rho = 0 \). The crosses are measured values from numerical simulations of the original PDE, while the lines are the predictions of the amplitude equations.
bounds.

To determine the critical value \( h_c \), we assume that the blow-up regime is self-similar, that is, that the growth of \( v \) is pegged to that of \( a \). This assumption can be formalised by expanding \( v \) and \( \text{Arg}(a) \) in powers of large \( |a| \):

\[
v = V_3|a|^3 + V_1|a| + V_{-1}|a|^{-1} + \ldots, \quad (76a)\]

\[
a = |a|e^{-i\theta}, \quad \theta = \theta_0 + \theta_{-2}|a|^{-2} + \theta_{-4}|a|^{-4} + \ldots. \quad (76b)\]

Substituting these expansions in Eq. (75a) and (75b), and equating coefficients of like powers of \( |a| \) to zero, we can evaluate the coefficients \( V_n \) and \( \theta_n \) to any order — this justifies the assumption.

In particular, setting to zero the coefficients of \( |a|^3 \) in Eq. (75a), gives

\[e^{i\theta_0} = -i \frac{\zeta}{|\zeta|}, \quad V_3 = \frac{2\omega_0|\zeta|}{3\pi} \frac{1}{h}. \quad (77)\]

On the other hand, substituting Eqs. (76) in Eq. (75b), we get an equation describing the growth of \(|a(t)|\):

\[
\frac{d}{dt}|a| = \tau|a| + O(|a|^{-1}),
\]

where the growth rate

\[
\tau = -\frac{2}{3}\gamma \frac{\eta e^{-i\theta_0} + \eta^* e^{i\theta_0}}{2V_3} h.
\]

Substituting for \( e^{i\theta_0} \) and \( V_3 \) from (77), this becomes

\[
\tau = \frac{2}{3}\gamma \frac{-3\pi}{2\omega_0|\zeta|} \frac{\eta e^{-i\theta_0} + \eta^* e^{i\theta_0}}{2} h^2. \quad (78)
\]

The growth of \(|a|\) is due to the \( h^2 \) term in Eq. (78) which has a positive coefficient; the growth is damped by the \( \gamma \) term. If we reduce \( h \) keeping \( \gamma \) fixed, then, at \( h = h_c \) where

\[
h_c = \left[ \frac{8\omega_0}{9\pi} \frac{|\zeta|^2}{i(\zeta^* - \zeta \eta)} \right]^{1/2} \gamma^{1/2} = 0.6523\gamma^{1/2}, \quad (79)
\]

the growth rate will become equal to zero. At this point the blow-up regime is replaced by a stable fixed point — which, however, still has large values of \(|a|\) and \( v \). [Reducing \( h \) further, the fixed point will persist but the similarity relations (76) will no longer be valid.] Note that the critical value (79) does not depend on \( \rho \). Numerical simulations of Eqs. (75) carried out for a variety of \( \gamma \) and \( \rho \), reproduce the value of \( h_c \) to high accuracy.

As \( h \) is increased towards \( h_c \) and neither \(|a|\)- nor \( v \)-components of the stationary point are small any longer, the system (75) ceases to provide any reliable description for the dynamics of the wobbler. A natural question
that arises here, is what dynamical regime the kink settles to for $h$ just below $h_c$ and for $h$ above $h_c$. In other words, we want to know what happens when the the wob-bler is driven with the small strength $h$ of order $\gamma^{1/2}$ [so that the conditions (64) are still in place] for which our finite-dimensional approximation is no longer valid. To answer this, we have conducted a series of numerical simu-lations of Eq.(60) with $h$ raised from $h < h_c$ to $h > h_c$. The simulations reveal that in the region inaccessible to our finite-dimensional approximation, the kink settles to wobbling with a constant amplitude which is accompa-nied by its translational motion with a constant velocity. [See Fig(6)] The numerically detected values of $a$ and $v$ are of order $\gamma^{1/3}$ in this region; this accounts for the inadequacy of our approximation which was based on the assumptions $a = O(\gamma^{1/2})$ and $v = O(\gamma)$.

The behaviour of the $1:1$ externally driven wobbling kink above (and just below) the critical value is an issue for which our finite-dimensional approximation is no longer valid. To answer this, we have conducted a series of numerical simulations of Eq.(60) with $h$ raised from $h < h_c$ to $h > h_c$. The simulations reveal that in the region inaccessible to our finite-dimensional approximation, the kink settles to wobbling with a constant amplitude which is accompanied by its translational motion with a constant velocity. [See Fig(6)] The numerically detected values of $a$ and $v$ are of order $\gamma^{1/3}$ in this region; this accounts for the inadequacy of our approximation which was based on the assumptions $a = O(\gamma^{1/2})$ and $v = O(\gamma)$.

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D. Qualitative Analysis

The reason why the external force does not couple directly to the wobbling mode in the case of the 1 : 1 external driving (as it did in the case of the 1 : 1 parametric resonance), is the discrepancy in the parity of the driving profile and the wobbling mode. Instead, there are three indirect amplification mechanisms at work in this case. In each of these, the central role is played by the even-parity standing wave excited by the driver. Firstly, the square of this standing wave couples to the wobbling mode via the term $e^{3\phi}\phi_1$. Secondly, the second and zeroth harmonic of the standing wave as well as the second-harmonic radiation excited by the standing wave couple to the wobbling mode via the term $e^{3\phi}\phi_1\phi_2$. Thirdly, when the kink moves relative to the standing wave, the odd-parity wobbling mode acquires an even parity component which then couples to the standing wave. [This process is accounted for by the term $e^{3\phi}\phi_2$ in Eq. (65).] Since the first two mechanisms rely upon a quadratic superharmonic of the induced standing wave (with the amplitude of the superharmonic being proportional to $h^2$), and since the velocity of the kink (which determines the amplitude of the even component of the wobbling mode in the third mechanism) is small, the 1 : 1 direct resonance is weak.

VII. CONCLUDING REMARKS

In this paper, we have used the asymptotic method to study the wobbling kink driven by four types of resonant driving force, viz., the 1 : 1 and 2 : 1 parametric, and 1 : 2 and 1 : 1 external driving. We have demonstrated the existence of resonance (i.e. the existence of sustained wobbling with nondecaying amplitude despite losing energy to radiation and dissipation) in all four cases. This conclusion (verified in direct numerical simulations of the corresponding partial differential equation) agrees with results of Quintero, Sánchez and Mertens who also demonstrated the existence of the resonance in the 1 : 1 parametrically and 1 : 2 directly driven $\phi^4$ equations [2, 3, 4]. However, we are in disagreement with these authors on the 1 : 1 directly driven, damped equation. Namely, our method does capture the resonance in this case whereas their collective coordinate approach does not. (In fairness to the pioneering work of Quintero, Sánchez and Mertens, their direct numerical simulations did reveal a resonant peak at the frequency of the driver equal to the natural wobbling frequency of the kink — an experimental result which, however, did not reconcile with their collective coordinate predictions [2, 3].)

In each of the four driving regimes that we have considered in this paper, we have derived a system of equations for the complex amplitude of the wobbling coupled to the velocity of the kink. The predictions based on this dynamical system are in agreement with results of the direct numerical simulation of the full partial differential equation. In three out of four cases considered, the velocity of the kink is shown to decay to zero as time advances, as a result of which the dimension of this dynamical system reduces from 3 to 2. Only in one case (the case of the 1 : 1 directly driven wobbler) does the velocity of the kink not necessarily decay to zero. In this latter case the wobbling of the kink is accompanied by its motion with nonzero velocity.

Each of the four dynamical systems derived here give rise to a bifurcation diagram featuring bistability and hysteresis transitions in the wobbling amplitude. In the 1 : 1 parametric and 1 : 2 direct resonances, the bistability is between two nonzero values of the wobbling amplitude, whereas in the 2 : 1 parametrically and 1 : 1 directly driven $\phi^4$ equations, one of the two stable regimes involves a nonzero and the other one a zero amplitude. It is fitting to note here that the collective coordinate approach [2, 3, 4] does not capture the bistability and hysteresis.

In section IV we ranked the two parametric resonances according to the amplitude of the stationary wobbling resulting from the driving with a certain reference strength, $h$. Adding to this hierarchy the two direct resonances produces the following ranking. The 1 : 1 parametric resonance is the strongest of the four cases; in this case the amplitude of the stationary wobbling, $a$, is of the order $h^{1/3}$. The 2 : 1 parametric resonance is second strongest; in this case the kink responds with the wobbling amplitude $a \sim h^{1/2}$. The 1 : 2 direct resonance has $a \sim h^{2/3}$ and the 1 : 1 direct resonance is the weakest: $a \sim h$. (The fact that the harmonic direct resonance is weaker than the superharmonic one, is in agreement with results of [2, 3] where it was established in the undamped situation.)

Our asymptotic approach also allows to rank the resonances according to the widths of the corresponding Arnold tongues on the “driving strength vs driving frequency” plane. The 1 : 1 parametric resonance is the widest one; in this case the resonant region is bounded by the curve $h \sim \rho^{3/2}$. The 2 : 1 parametric resonance is second widest; in this case the Arnold tongue has $h \sim \rho$. The 1 : 2 direct resonance has $h \sim \rho^{5/4}$ and the 1 : 1 direct resonance is the narrowest: $h \sim \rho^{1/2}$. We should also mention that the 1 : 1 parametric and the 1 : 2 direct resonance have no threshold in the strength of the driver whereas the 2 : 1 parametric and 1 : 1 direct resonances occur only if the driving strength exceeds a certain threshold value.

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