On Super Edge-magic Total Labeling of Modified Watermill Graph

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Abstract. An edge-magic total labeling on a graph $G$ is a one-to-one map from $V(G) \cup E(G)$ onto the set of integers $1, 2, \ldots, |V(G)| + |E(G)|$, with the property that, given any edge $uv$, $f(u) + f(\{u, v\}) + f(v) = k$ for every $u, v \in V(G)$, and $k$ is called magic valuation. An edge-magic total labeling $f$ is called super edge-magic total if

$$f(V(G)) = \{1, 2, \ldots, |V(G)|\} \quad \text{and} \quad f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}.$$ 

In this paper we investigate edge-magic total labeling of a new graph called modified Watermill graph. Furthermore, the magic valuation of the modified Watermill graph $WM(n)$ is $k = \frac{1}{2}(21n + 3)$, for $n$ odd, $n \geq 3$.

1. Introduction

All graphs in this paper are finite, simple, and have no loops and multiple edges. A general reference of graph theory can be seen in [1].

Labeling is one of topics in the graph theory. Labeling graph is a map from graph elements to numbers [2], in this paper we discuss about edge total magic labeling which domain is the set of all vertices and edges that map to the natural numbers.

For graph $G$ with vertex-set $V(G)$ and edge-set $E(G)$ an edge-magic total labeling is a bijection $\lambda: V(G) \cup E(G)$ to the set integers $1, 2, \ldots, |V(G)| + |E(G)|$ with the property that, for each edge $\{u, v\}$,

$$f(u) + f(\{u, v\}) + f(v) = k$$

for a fixed integers $k$. Call $f(u) + f(\{u, v\}) + f(v)$ the edge sum of $\{u, v\}$, and $k$ is the magic valuation sum of graph $G$. An edge-magic total labeling $f$ is called super edge-magic total if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$. A graph is called edge-magic total or super edge-magic total if it admits any edge-magic total labeling or super edge-magic total labeling, respectively [3].

Various authors have introduced labeling that generalized the idea of magic square. Labeling was introduced by Sadlák, Sedlack [4] defined a graph to be magic if it had an edge-labeling, with range the real numbers, such that sum of the labels around any vertices equalled constant independent of the choice of vertices.

The notion of edge-magic total graph was introduced and studied by Kotzig and Rosa [5] with a different name as graphs with magic valuation. In 1996 Ringel and Llado [6] redefine this type of labeling called the labeling edge magic. After that Wallis et al (2000) [7], found the concept to
distinguish by another magic labeling. Recently Enomoto et al [8] defines the super edge-magic total labeling.

There are several research studies has been discussed about the super edge-magic total labeling. In [8] proved that every cycle $C_n$, $n \geq 3$ are edge-magic. then in [8] proved that cycle, $n \geq 3$ are Super Edge Labeling if only if $n$ is odd. In [9] show that all paths $P_n$ and all n-suns are edge-magic total. Wijaya and Baskoro [10] studied edge magic total labeling for a product of two graphs. They showed the product $P_n \times C_n$ admits an edge-magic total labeling for $n$ odd, $n \geq 3$. For $n$ even, we only know that $P_2 \times C_n$ is not edge-magic total.

In [11] Ngurah, Baskoro Tomescu gave methods for construction new super edge-magic total graphs from old ones by adding some new pendent edges. They also prove that $K_1 \cup P_n$ is super edge-magic total. Wallis proves that a cycle with one pendent edge is edge-magic total.

Ngurah and Baskoro [3] studies about magic total labeling of generalized Petersen graph. They showed that if $n \geq 3$, then the generalized Petersen Graph $P(n,1)$ has a Super edge-magic total labeling with the magic valuation $k = \frac{1}{2}(11n + 3)$. In this paper, we shall discuss super edge-magic total labelling of modified watermill graph.

2. Result & Discussion
This section explains the research outcome on super edge-magic total labeling of modified watermill graph.

2.1 Sun Graph
Before discussing about the watermill graph, first we have to know about definition of the sun graph. The sun graph $(C_n \circ K_1)$ is a graph that constructed from cycle graph $C_n$ where every vertices on that cycle graph is added a vertex with degree 1 such that every vertices on the sun graph have degree 3, except on the endvertices that have degree 1. The sun graph is the product of corona between two graphs, cycle graph with $n$ vertices $(C_n, n \geq 3)$ and the complement of complete graph with one vertex $(K_1)$. Sun graph is denoted by $C_n \circ K_1$ with $n$ is the number of vertices on cycle graph. Example for $C_6 \circ K_1$ is shown by figure 1.

![Figure 1. Sun Graph $C_6 \circ K_1$.](image)

2.2 Watermill Graph
The Watermill graph is denoted by $WM(n)$ with set of vertices $V(WM(n))$ and set of edges $E(WM(n))$. Watermill graph is defined as follows :

$$WM(n) = \left(V\left(WM(n)\right),E\left(WM(n)\right)\right)$$

with,

$$V(WM(n)) = \{u_1, u_2, ..., u_{2n}, v_1, v_2, ..., v_{2n}\}$$
and

\[ E(WM(n)) = \{ u_{i}u_{i+1} \cup v_{i}v_{i+1} | i = 1, 2, ..., n - 1 \} \cup \{ u_{i}u_{i+n} \cup v_{i}v_{i+n} | i = 1, 2, ..., n \} \]
\[ \cup \{ u_{i}u_{n} \cup v_{1}v_{n} \} \cup \{ u_{i}v_{1} | 1 \leq i \leq n \} \cup \{ u_{i+n}v_{1} | 1 \leq i \leq n \} \]

Watermill graph is a graph formed by two copy sun graphs \((C_n \odot \overline{K}_1)\), where that graphs are made parallel with the parallel vertices is adjacent each other. For example, Watermill graph with \(n = 5\) \((WM(5))\).

![Figure 2. Watermill Graph WM(5).](image)

In this paper we will discuss about super edge-magic total labeling on modified Watermill graph. Modification for this graph is omit some edges on Watermill graph, such that set of vertices and set of edges for this graph is defined as follows,

\[ V(WM(n)) = \{ u_1, u_2, ..., u_{2n}, v_1, v_2, ..., v_{2n} \} \]
\[ E(WM(n)) = \{ u_{i}u_{i+1} \cup v_{i}v_{i+1} | i = 1, 2, ..., n - 1 \} \cup \{ u_{i}u_{i+n} \cup v_{i}v_{i+n} | i = 1, 2, ..., n \} \]
\[ \cup \{ u_{1}u_{n} \cup v_{1}v_{n} \} \cup \{ u_{i}v_{1} | 1 \leq i \leq n \} \cup \{ u_{i+n}v_{1} | 1 \leq i \leq n \} \]
\[ -\{ u_{2(n-i)}v_{2(n-i)} | i = 0, ..., \frac{1}{2}(n - 3) \} \]
For example modified Watermill graph with \( n = 5 \) is given by figure 3.

\[
\text{Figure 3. Modified Watermill Graph } WM(5).
\]

**Theorem**

If \( n \) odd, \( n \geq 3 \), then the modified watermill graph \( WM(n) \) has an edge-magic total labeling with the magic valuation \( k = \frac{1}{2} (21n + 3) \).

**Proof** Label the vertices and edges of \( WM(n) \) in the following way

\[
f(u_i) = \begin{cases} 
  i, & i \leq n, \quad i \equiv 1 \pmod{2} \\
  i + n, & i < n, \quad i \equiv 0 \pmod{2} \\
  i, & i > n, \quad i \equiv 0 \pmod{2} \\
  i - n, & i > n, \quad i \equiv 1 \pmod{2}
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
  \frac{6n - i + 1}{2}, & i \leq n, \quad i \equiv 1 \pmod{2} \\
  \frac{5n - i + 1}{2}, & i < n, \quad i \equiv 0 \pmod{2} \\
  \frac{8n - i}{2}, & i > n, \quad i \equiv 0 \pmod{2} \\
  \frac{9n - i}{2}, & i > n, \quad i \equiv 1 \pmod{2} \\
  \frac{4n}{2}, & i = 2n
\end{cases}
\]

\[
f(v_i v_{i+1}) = 5n + i + 1, \quad i < n
\]

\[
f(v_1 v_n) = 5n + 1
\]
\[ f(v_1v_{i+1}) = \begin{cases} 4n + i + 1, & i < n \\ 4n + 1, & i = n \end{cases} \]

\[
\begin{align*}
\frac{15n - i + 2}{2}, & \quad i \leq n, \quad i \equiv 1 \pmod{2} \\
\frac{14n - i + 2}{2}, & \quad i < n, \quad i \equiv 0 \pmod{2} \\
\frac{14n - i + 3}{2}, & \quad i > n, \quad i \equiv 1 \pmod{2} \\
\frac{13n - i + 3}{2}, & \quad i = n + 1
\end{align*}
\]

\[ f(u_1u_{i+1}) = \frac{19n - 4i + 1}{2}, \quad i < n \]

\[ f(u_1u_n) = \frac{19n + 1}{2} \]

\[ f(u_1u_{i+n}) = \frac{19n - 4i + 3}{2}, \quad i \leq n \]

To prove the function above is bijective, we see that

\[ f(u_i) = \begin{cases} i, & 1, 3, ..., n \\ i + n, & n + 2, n + 4, ..., 2n - 1 \\ i, & n + 1, n + 3, ..., 2n \\ i - n, & 2, 4, ..., n - 1 \end{cases} \]

\[ f_1 = u_i = \{1, 2, 3, ..., 2n\} \]

\[
\begin{align*}
\frac{6n - i + 1}{2}, & \quad 5n + 1, 5n + 3, ..., 3n - 2, 3n - 1, 3n \\
\frac{5n - i + 1}{2}, & \quad 2n + 1, 2n + 2, ..., 5n - 3, 5n - 1 \\
\frac{8n - i}{2}, & \quad 3n + 1, 3n + 2, ..., 7n - 3, 7n - 1 \\
\frac{9n - i}{2}, & \quad 7n + 1, 7n + 2, ..., 4n - 2, 4n - 1 \\
\frac{2}{4n}, & \quad 4n, \quad 4n + 1
\end{align*}
\]

\[ f(v_1v_{i+1}) = 5n + i + 1, \quad 5n + 2, 5n + 3, ..., 6n \]

\[ f(v_1v_n) = 5n + 1, \quad 5n + 1 \]

\[ f(v_1v_{i+n}) = \begin{cases} 4n + i + 1, & 4n + 2, 4n + 3, ..., 5n \\ 4n + 1, & 4n + 1 \end{cases} \]
Let \( f \) be a function that we found based on constructed function that we found before.

Define
\[
\begin{align*}
f(u_i v_i) &= \begin{cases} 
15n - i + 2, & 7n + 1, 7n + 2, \ldots, \frac{15n - 3}{2}, \frac{15n - 1}{2}, \frac{15n + 1}{2} \\
14n - i + 2, & 13n + 3, 13n + 5, \ldots, 7n - 2, 7n - 1, 7n \\
14n - i + 3, & 6n + 2, 6n + 3, \ldots, \frac{13n - 3}{2}, \frac{13n - 1}{2}, \frac{13n + 1}{2} \\
13n - i + 3, & 6n + 1 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
f(u_i u_{i+1}) &= \frac{19n - 4i + 1}{2}, \\
f(u_1 u_n) &= \frac{19n + 1}{2}, \\
f(u_i u_{i+n}) &= \frac{19n - 4i + 3}{2},
\end{align*}
\]

Clearly, every generated labels by the function are different each other.

Define
\[
\begin{align*}
f &\colon f(u_i) \cup f(v_i) \cup f(v_i v_{i+1}) \cup f(v_i v_n) \cup f(v_i v_{i+n}) \cup f(u_i v_i) \cup f(u_i u_{i+1}) \\
&\quad \cup f(u_1 u_n) \cup f(u_i u_{i+n}) = V(WM(n)) \cup E(WM(n)) = \left\{ 1, \ldots, \frac{19n + 1}{2} \right\}
\end{align*}
\]

since,
\[
|V(WM(n)) \cup E(WM(n))| = \frac{19n + 1}{2}
\]

then, \( f \) is bijective.

Clearly, for every \( i \):
\[
f(u_i) + f(u_i u_{i+1}) + f(u_{i+1}) = f(u_i) + f(u_i u_n) + f(u_n)
\]
\[
= f(u_i) + f(u_i u_{i+n}) + f(u_{i+n})
\]
\[
= f(v_i) + f(v_i v_{i+1}) + f(v_{i+1})
\]
\[
= f(v_i) + f(v_i v_n) + f(v_n)
\]
\[
= f(v_i) + f(v_i v_{i+n}) + f(v_{i+n})
\]
\[
= f(u_i) + f(u_i v_i) + f(v_i)
\]
\[
= \frac{1}{2}(21n + 3).
\]

Here is example for super edge-magic total labeling of modified graph \( WM(5) \). For the graph see figure 3.

Let \( G \) is modified graph \( WM(5) \).

Graph \( G \) will be labeled super edge-magic total labeling based on constructed function that we found before.
Labels for $V(G)$:

\[
\begin{align*}
f(u_1) &= 1 & f(v_1) &= 15 \\
f(u_2) &= 7 & f(v_2) &= 12 \\
f(u_3) &= 3 & f(v_3) &= 14 \\
f(u_4) &= 9 & f(v_4) &= 11 \\
f(u_5) &= 5 & f(v_5) &= 13 \\
f(u_6) &= 6 & f(v_6) &= 17 \\
f(u_7) &= 2 & f(v_7) &= 19 \\
f(u_8) &= 8 & f(v_8) &= 16 \\
f(u_9) &= 4 & f(v_9) &= 18 \\
f(u_{10}) &= 10 & f(v_{10}) &= 2
\end{align*}
\]

Label for $E(G)$:

\[
\begin{align*}
f(v_1v_2) &= 27 & f(u_1u_2) &= 46 \\
f(v_2v_3) &= 28 & f(u_1v_1) &= 38 & f(u_2u_3) &= 44 \\
f(v_3v_4) &= 29 & f(u_2v_2) &= 35 & f(u_3u_4) &= 42 \\
f(v_4v_5) &= 30 & f(u_3v_3) &= 37 & f(u_4u_5) &= 40 \\
f(v_1v_5) &= 26 & f(u_4v_4) &= 34 & f(u_1u_5) &= 48 \\
               & & f(u_5v_5) &= 36
\end{align*}
\]

\[
\begin{align*}
f(v_1v_6) &= 22 & f(u_6v_6) &= 31 & f(u_1u_6) &= 47 \\
f(v_2v_7) &= 23 & f(u_6v_7) &= 33 & f(u_2u_7) &= 45 \\
f(v_3v_8) &= 24 & f(u_7v_7) &= 32 & f(u_3u_8) &= 43 \\
f(v_4v_9) &= 25 & f(u_8v_8) &= 32 & f(u_4u_9) &= 41 \\
f(v_5v_{10}) &= 21 & f(u_9v_{10}) &= 32 & f(u_5u_{10}) &= 39
\end{align*}
\]
So we obtained graph $G$ which has been labeled as follows:

**Figure 4.** Labeled Modified Watermill Graph $WM(5)$.

For every edge sum from graph $G$ as follows:

- $f(u_1) + f(u_1u_2) + f(u_2) = 54$
- $f(u_2) + f(u_2u_3) + f(u_3) = 54$
- $f(u_3) + f(u_3u_4) + f(u_4) = 54$
- $f(u_4) + f(u_4u_5) + f(u_5) = 54$
- $f(u_1) + f(u_1u_5) + f(u_5) = 54$
- $f(u_1) + f(u_1u_6) + f(u_6) = 54$
- $f(u_2) + f(u_2u_7) + f(u_7) = 54$
- $f(u_3) + f(u_3u_9) + f(u_9) = 54$
- $f(u_5) + f(u_5u_{10}) + f(u_{10}) = 54$

- $f(v_1) + f(v_1v_2) + f(v_2) = 54$
- $f(v_2) + f(v_2v_3) + f(v_3) = 54$
- $f(v_3) + f(v_3v_4) + f(v_4) = 54$
- $f(v_4) + f(v_4v_5) + f(v_5) = 54$
- $f(v_1) + f(v_1v_5) + f(v_5) = 54$
- $f(v_1) + f(v_1v_6) + f(v_6) = 54$
- $f(v_1) + f(v_1v_6) + f(v_6) = 54$
- $f(v_2) + f(v_2v_7) + f(v_7) = 54$
- $f(v_3) + f(v_3v_9) + f(v_9) = 54$
- $f(v_4) + f(v_4v_9) + f(v_9) = 54$
- $f(v_5) + f(v_5v_{10}) + f(v_{10}) = 5$
As we can see, every sum of labels on an edge and its two endpoints have the same value, which is 54. So the magic value for this graph is \( k = 54 \). This corresponds to the theorem that already proved for magic value,

\[
k = \frac{21n + 3}{2}
\]

\[
k = \frac{21(5) + 3}{2}
\]

\[
k = 54
\]

3. Conclusion

According to result and discussion we found the magic valuation of the modified Watermill graph \( WM(n) \) is \( k = \frac{1}{2}(21n + 3) \), for \( n \) odd, \( n \geq 3 \).

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