Distributed Dynamic Safe Screening Algorithms for Sparse Regularization

Runxue Bao, Xidong Wu, Wenhan Xian and Heng Huang
Electrical and Computer Engineering Department, University of Pittsburgh, PA, USA
{runxue.bao, xidong.wu, wex37, heng.huang}@pitt.edu

Abstract
Distributed optimization has been widely used as one of the most efficient approaches for model training with massive samples. However, large-scale learning problems with both massive samples and high-dimensional features widely exist in the era of big data. Safe screening is a popular technique to speed up high-dimensional models by discarding the inactive features with zero coefficients. Nevertheless, existing safe screening methods are limited to the sequential setting. In this paper, we propose a new distributed dynamic safe screening (DDSS) method for sparsity regularized models and apply it on shared-memory and distributed-memory architecture respectively, which can achieve significant speedup without any loss of accuracy by simultaneously enjoying the sparsity of the model and dataset. To the best of our knowledge, this is the first work of distributed safe dynamic screening method. Theoretically, we prove that the proposed method achieves the linear convergence rate with lower overall complexity and can eliminate almost all the inactive features in a finite number of iterations almost surely. Finally, extensive experimental results on benchmark datasets confirm the superiority of our proposed method.

1 INTRODUCTION
Learning sparse representations plays a significant role in many machine learning and signal processing applications [Lustig et al., 2008; Shevade and Keerthi, 2003; Wright et al., 2009; Wright et al., 2010; Chen et al., 2021; Bian et al., 2021; Zhang et al., 2021]. In the past decades, many models with sparse regularization have achieved great successes in high-dimensional scenarios by encouraging the model sparsity [Tibshirani, 1996; Ng, 2004; Yuan and Lin, 2006; Bao et al., 2019]. Let $A = [a_1, \ldots, a_n]^{\top} \in \mathbb{R}^{n \times p}$, in this paper, we consider the following composite optimization problem:

$$\min_{x \in \mathbb{R}^p} \mathcal{P}(x) := \mathcal{F}(x) + \lambda \Omega(x).$$

(1)

where $x$ is the model coefficient, $\Omega(x)$ is the block-separable sparsity-inducing norm, $\mathcal{F}(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(a_i^\top x)$ is the data-fitting loss, and $\lambda$ is the regularization parameter. We denote $\mathcal{F}_i(x) = f_i(a_i^\top x)$ for simplicity. Given partition $G$ of the coefficients, we denote the sub-matrix of $A$ with the columns of $G_j$ as $A_j \in \mathbb{R}^{n \times |G_j|}$ and have $\Omega(x) = \sum_{j=1}^{q} \Omega_j(x_{G_j})$.

Distributed learning has been actively studied in machine learning community due to its capability for tackling big data computations and numerous large-scale applications. In literature, many distributed learning methods have been proposed to accelerate different optimization algorithms on shared-memory architecture [Langford et al., 2009; Recht et al., 2011; Reddi et al., 2015; Zhao and Li, 2016; Mania et al., 2017; Leblond et al., 2017; Meng et al., 2017; Pedregosa et al., 2017; Zhou et al., 2018] and distributed-memory architecture [Agarwal and Duchi, 2012; Dean et al., 2012; Zheng and Kwok, 2014; Lian et al., 2015; Zhang et al., 2016]. For smooth optimization problems, asynchronous stochastic methods, e.g., Hogwild! [Recht et al., 2011], Lock-Free SVRG [Reddi et al., 2015], AsySVRG [Zhao and Li, 2016], KroMagnon [Mania et al., 2017], and ASAGA [Leblond et al., 2017] were proposed. Furthermore, AsyProxSVRG [Meng et al., 2017], AsyProxSBCDVR [Gu et al., 2018], ProxASAGA [Pedregosa et al., 2017], and AsyMiG [Zhou et al., 2018] were proposed to solve non-smooth composite optimization problems. However, large-scale problems with both massive samples and high-dimensional features widely exist in the era of big data, which still suffer huge burden for computation and memory costs.

By exploiting the sparsity, safe screening is an effective method to accelerate high-dimensional sparse models by pre-identifying inactive features and thereby avoiding the useless computation for training. Safe static screening [Laurent El Ghaoui, 2012] was first proposed for $l_1$ regularized problems, which only performs once prior to the optimization. The strong screening method [Tibshirani et al., 2012] was proposed for Lasso via heuristic strategies and could discard features wrongly. Furthermore, [Wang et al., 2013] proposed the sequential method for Lasso, which needs to estimate the exact dual solution and could be unsafe in practice. Recently, an effective dynamic method [Fercoq et al., 2015] was proposed for Lasso with good performance. Due to its better empirical and theoretical results, many dynamic rules [Shibagaki et al., 2016; Ndiaye et al., 2015; Ndiaye et al., 2016; Ndiaye et al., 2017; Rakotomamonjon et al., 2019] were proposed for a broad class of sparse models with separable
Table 1: Summary of dynamic safe screening methods. “Model” refers to the model it can solve where “MTL & MLR” means multi-task Lasso and multinomial logistic regression. “Safe” represents there are no active features eliminated. “Generalized” means whether it is limited to a specific model. “Distributed” represents whether it can work on distributed-memory architecture. “Scalablity” means whether it is scalable with sample size $n$.

| Reference                  | Model               | Safe | Generalized | Distributed | Scalablity |
|----------------------------|---------------------|------|-------------|-------------|------------|
| [Fercoq et al., 2015]      | Lasso               | Yes  | No          | No          | No         |
| [Ndiaye et al., 2016]      | Sparse-Group Lasso  | Yes  | No          | No          | No         |
| [Shibagaki et al., 2016]   | Sparse SVM          | Yes  | No          | No          | No         |
| [Ndiaye et al., 2017]      | MTL & MLR           | Yes  | No          | No          | No         |
| [Rakotomamonjy et al., 2019]| Proximal Weighted Lasso | Yes | No | No | No |
| [Bao et al., 2020]         | OWL Regression       | Yes  | No          | No          | No         |
| DDSS (Ours)                | Problem (1)         | Yes  | Yes         | Yes         | Yes        |

Table 2: Comparison between PSE and our DDSS algorithm. “Data Sparsity” represents whether it can benefit from the sparsity of data.

| Reference                  | Model               | Dynamic | Safe | Distributed | Scalablity | Data Sparsity |
|----------------------------|---------------------|---------|------|-------------|------------|---------------|
| PSE [Li et al., 2016a]     | Lasso               | No      | No   | No          | No         | No            |
| DDSS (Ours)                | Problem (1)         | Yes     | Yes  | Yes         | Yes        | Yes           |

penalties. Recently, [Bao et al., 2020] introduces a dynamic screening rule to effectively handle the inseparable OWL regression [Bao et al., 2019] via an iterative strategy. Table 1 summarizes existing representative safe dynamic screening methods. However, all of these methods are limited to sequential (single-machine) setting and thus huge requirement of computational complexity hinders its application on large-scale datasets. Therefore, distributed safe dynamic screening algorithms are promising and sorely needed to accelerate existing training methods for large-scale problems.

In this paper, we propose a new distributed dynamic screening method to solve sparse models and apply it on shared-memory (Sha-DDSS) and distributed-memory (Dis-DDSS) architecture, which can accelerate the training process significantly without any loss of accuracy. Specifically, DDSS not only conducts the screening to eliminate inactive features to enjoy the sparsity of the model, but also conducts sparse proximal gradient update with the nonzero partial gradients to enjoy the sparsity of the dataset. To further accelerate DDSS, we also reduce the gradient variance over the active set and thus we can utilize a constant step size to achieve the linear convergence. On distributed-memory system, to reduce the computational burden of the server node, we utilize a decouple strategy to improve the scalability of DDSS w.r.t. the number of workers.

Contributions. The main contributions of our work can be summarized as follows. First, we propose a new distributed dynamic safe screening framework for generalized sparse models, which is easy-to-implement on the both shared-memory and distributed-memory architecture. To the best of knowledge, this is the first work of distributed dynamic safe screening. Second, we rigorously prove the proposed DDSS method can achieve linear convergence rate $O(\log(1/\epsilon))$, reduce the per-iteration cost from $O(p)$ to $O(r)$ where $r \ll p$, and finally achieve a lower overall computational complexity under the strongly convex condition. Third, we prove almost sure finite time identification of the active set to confirm the effectiveness of our DDSS method. Finally, we empirically show that our proposed method can achieve significant acceleration and linear speedup property.

1.1 Related Works

Parallel Static Screening Parallel static screening (PSE) [Li et al., 2016a] is most related work to ours. Table 2 summarizes the advantages of our DDSS method over PSE. First, DDSS is dynamic and conducted during the whole training process. Thus, DDSS can accelerate and meanwhile benefit from the convergence of the optimization algorithm. Second, our DDSS is safe for the training and can guarantee the model accuracy. Third, DDSS is stochastic and can scale well on both samples and features. Fourth, DDSS can enjoy the data sparsity by performing sparse proximal updates to further accelerate the training. Lastly, DDSS can solve Problem (1).

Asynchronous Stochastic Method An asynchronous doubly stochastic method was proposed in [Meng et al., 2017], which performs coordinate updates without considering any sparsity and thus can be very slow. In [Gu et al., 2018], they proposed a asynchronous doubly stochastic method for group regularized learning problems. Moreover, [Leblond et al., 2017] proposed an asynchronous sparse incremental gradient method, which does not enjoy the sparsity of the model and cannot be easily incorporated into the dynamic screening method to accelerate the training. However, our DDSS can simultaneously enjoy the sparsity of the model and the dataset.

2 PROPOSED METHOD

We first propose the distributed dynamic safe screening (DDSS) method and then apply it to shared-memory and distributed-memory architecture respectively.
2.1 Distributed Dynamic Safe Screening

Algorithm 1 Sha-DDSS-Naive

1: Input: $x_{B_0}^0 \in \mathbb{R}^p$, step size $\eta$, inner loops $K$
2: for $s = 0 \rightarrow S - 1$
3: All threads parallelly compute $\nabla F(x_{B_s}^0)$
4: Compute $y_s^*$ by (2) and $B_{s+1}$ from $B_s$ by (3)
5: Update $A_{B_{s+1}} x_{B_{s+1}}^0$
6: For each thread, do:
7: for $t = 0 \rightarrow K - 1$
8: Read $\hat{x}_{B_{s+1}}^t$ from the shared memory
9: Randomly sample $i$ from $\{1, 2, \ldots, n\}$
10: $v_t^i = \nabla f_i (a_i^\top x_{B_{s+1}}^t)$
11: Update $x_{B_{s+1}}^t = \text{prox}_{\eta \lambda s} (\hat{x}_{B_{s+1}}^t - \eta v_t^i)$
end for
13: $x_{B_{s+1}} = x_{B_{s+1}}^K, x_{B_{s+1}}^0 = x_{B_{s+1}}$
end for

To enjoy the sparsity of the model coefficients, we first give a naive implementation of DDSS on shared-memory architecture in Alg. 1 and on distributed-memory architecture in Alg. 2 and 3 respectively. We mainly take the algorithm on shared-memory architecture as the example to illustrate our idea. During the training, DDSS only need solve a subproblem with constantly decreasing size by discarding useless features.

Specifically, Alg. 1 has two loops. We denote the original problem as $P_0$ and the full set as $B_0$. At the $s$-th iteration of the outer loop, we denote the active set as $B_s$ and suppose $B_s$ has $q_s$ active blocks with total $p_s$ active features. Thus, the subproblem $P_s$ is over set $B_s$. Then we can compute the dual variable $y_s^*$

$$y_s^* = -\nabla F(x_{B_s}^0)/\max(1, \Omega^D(A_{B_s}^\top \nabla F(x_{B_s}^0))/\lambda),$$

(2)

where dual norm $\Omega^D(u) = \max_{\Omega(v) \leq 1} \langle v, u \rangle$. With the obtained dual variable, we can eliminate inactive feature blocks (see details in [Ndiaie et al., 2017; Bao et al., 2020]) for $\forall j \in B_s$ as

$$\Omega^D_j(A_j^\top y_s^*) + \Omega^D(A_j) \sqrt{2L(P(x_{B_s}^0) - D(y_s^*))} < n\lambda,$$

(3)

to update $B_{s+1}$ where $L$ is the Lipschitz constant. The dual objective $D(y_s^*)$ of $P_s$ can be computed as:

$$\max D(y_s^*) := -\frac{1}{n} \sum_{i=1}^n f_i^* (-y_s^*),$$

$$\text{s.t. } \Omega^D_j(A_j^\top y_s^*) \leq n\lambda, \; \forall j \in 1, \ldots, q_s.$$

(4)

For the inner loop, all the updates are conducted on $B_{s+1}$. First, each thread inconsistently read $\hat{x}_{B_{s+1}}^t$ from the shared memory and randomly choose sample $i$ to compute the stochastic gradient on $B_{s+1}$. Then, the proximal step is conducted with the stochastic gradient. By (3), we can constantly reduce the model size and the parameter size to accelerate the training by exploiting the sparsity of the model. Since each variable $x_i$ discarded by the screening must be zero for the optimum solution, this method is safe for the training.

Algorithm 2 Dis-DDSS-Naive (Server Node)

1: if flag = True
2: Receive $x_{B_0}^0$ from worker
3: Compute and send gradient $\nabla F_k(x_{B_0}^0) = \sum_{i \in \mathbb{N}_k} \nabla f_i (a_i^\top x_{B_0}^0)$
4: Receive $B_{s+1}$ from server
5: Update $A_i \in \mathbb{N}_k, B_{s+1}, x_{B_{s+1}}^0$
6: else
7: Receive $x_{B_{s+1}}^0$ from server
8: Randomly sample $i$ from $\{1, 2, \ldots, n_k\}$
9: Compute $v_t^i = \nabla f_i (a_i^\top x_{B_{s+1}}^0)$
10: Send $v_t^i$ to server
11: end if

Variance Reduction on the Active Set However, the variance of the gradient estimation in Alg. 1 caused by stochastic sampling does not converge to zero. Thus, we have to use a diminishing step size and can only obtain very little progress for each update. Thus, Alg. 1 can only attain a sublinear convergence rate even when $P$ is strongly convex.

Since the full gradient has been computed by the outer loop for the elimination step, inspired by the variance-reduced technique in [Xiao and Zhang, 2014; Li et al., 2021], we can adjust the gradient estimation over $B_{s+1}$ with the exact gradient from the outer loop without additional computational costs as

$$v_t^i = \nabla f_i (a_{i, B_{s+1}}^\top \hat{x}_{B_{s+1}}^t) - \nabla f_i (a_{i, B_{s+1}}^\top x_{B_{s+1}}^0) + \nabla F(x_{B_{s+1}}^0),$$

(5)

which can guarantee that the variance of stochastic gradients asymptotically converges to zero. Therefore, we can use a constant step size to achieve more progress for each iteration and finally achieve a linear convergence rate for strongly convex function.

Sparse Proximal Gradient Update In practice, sparsity widely exists in large-scale datasets. To utilize the sparsity
that intersect the nonzero coefficients of $\nabla f_i$. Let $n_\mathcal{G}$ be the number of occurrences that $\mathcal{G} \in \Psi_i$, if $n_\mathcal{G} > 0$, we define $d_\mathcal{G} = n/n_\mathcal{G}$. Otherwise, we can ignore that block directly. Thus, we can define diagonal matrix $[D_i]_{\mathcal{G}} = d_\mathcal{G} I[\mathcal{G}]$ for each block $i$ and thus the gradient over set $\mathcal{B}_{s+1}$ can be formulated as

$$v_i^t = \nabla f_i(a_i^t \hat{x}_{B_{s+1}}^t - B_{s+1}^0) + D_i^t \nabla F(x_{B_{s+1}}^t).$$

(6)

Thus, we only need compute a sparse gradient and conduct a sparse update over the active set and the computational cost is further reduced.

On the other hand, the proximal operator of original Problem (1) is computed as

$$\text{prox}_{\eta \lambda \Omega}(x') = \arg \min_{x \in \mathbb{R}^p} \frac{1}{2\eta} \|x - x'\|^2 + \lambda \Omega(x),$$

(7)

which needs to update all the coordinates for each iteration. Considering the sparsity of the dataset again, we only need to update the blocks that contains nonzero partial gradients. Thus, based on the reweighting matrix $D_i$, we use a block-wise weighted norm $\phi_i(x) = \sum_{\mathcal{G} \in \Psi} d_\mathcal{G} \Omega_\mathcal{G}(x)$ to displace $\Omega(x)$. Note it is easy to verify that $\mathbb{E}[\phi_i(x)] = \Omega(x)$. Thus, the new sparse proximal operator can be computed as

$$\text{prox}_{\eta \lambda \phi_i}(x') = \arg \min_{x \in \mathbb{R}^p} \frac{1}{2\eta} \|x - x'\|^2 + \lambda \phi_i(x).$$

(8)

Since we only update the blocks in $\Psi_i$, which could be much less than the one needs a full pass of $p$ coordinates due to the sparsity, we can save much computation and memory cost here. To sum it up, we can conduct the sparse gradient update and sparse proximal operator to accelerate the training by enjoying the sparsity of the dataset.

Decoupled Proximal Update On distributed-memory architecture, multiple workers compute the gradients and send them to the server. The server computes the proximal operator. When the proximal step is time-consuming, doing this in the server would be the computational bottleneck of the whole algorithm. [Li et al., 2016b] proposed a decoupled method to off-load the computational task of the proximal step to workers. Thus, the server only does simple addition computation, which can achieve a sublinear converge rate and perform better than the coupled method.

To relieve the computation cost of the server in our algorithm, the proximal mapping step is computed by workers and the server only needs to do the element-wise computation. The workers conduct the proximal operator as

$$x_{B_{s+1}}^{t+1} = \text{prox}_{\eta \lambda \phi_i}(x_{B_{s+1}}^t - \eta v_i^t),$$

(9)

and send the difference

$$\delta_t^i = \text{prox}_{\eta \lambda \phi_i}(x_{B_{s+1}}^{d(t)} - \eta v_i^t) - x_{B_{s+1}}^{d(t)},$$

(10)

between the parameter $x_{B_{s+1}}^{d(t)}$ and the output of the proximal operator to the server. Therefore, the server only does simple addition computation, which makes the algorithm suitable to parallelize to achieve linear speedup property and can be accelerated via increasing the number of workers.

2.2 DDSS on Shared-Memory Architecture

Algorithm 4 Sha-DDSS

1: for $s = 0$ to $S - 1$ do
2: All threads parallelly compute $\nabla F(x_{B_s}^0)$
3: Compute $y^t$ by (2) and $B_{s+1}$ from $B_s$ by (3)
4: Update $A_{B_{s+1}}, x_{B_{s+1}}^0, \nabla F(x_{B_{s+1}}^0)$
5: For each thread, do:
6: for $t = 0$ to $K - 1$ do
7: Read $x_{B_{s+1}}^t$ from the shared memory
8: Randomly sample $i$ from $\{1, 2, \ldots, n\}$
9: Compute $v_i^t$ by (6)
10: $\delta_t^i = \text{prox}_{\eta \lambda \phi_i}(x_{B_{s+1}}^{t+1} - \eta v_i^t) - x_{B_{s+1}}^{t+1}$
11: $x_{B_{s+1}}^{t+1} = x_{B_{s+1}}^t + \delta_t^i$
12: end for
13: $x_{B_{s+1}} = x_{B_{s+1}}^K x_{B_{s+1}}^0 = x_{B_{s+1}}$
14: end for

On shared-memory architecture, our Sha-DDSS algorithm is in Alg. 4. Suppose we have $l$ cores, in the outer loop, all threads parallelly compute $\nabla F(x_{B_s}^0)$ and $y^t$, and perform the elimination. With new set $B_{s+1}$, we update $A_{B_{s+1}}, x_{B_{s+1}}^0, \nabla F(x_{B_{s+1}}^0)$. In the inner loop, the algorithm optimizes over $B_{s+1}$. Multiple threads update the parameter asynchronously, which means the parameter can be read and written without locks. Specifically, each thread inconsistently read $x_{B_{s+1}}^t$ from the shared memory and then choose sample $i$ from $\{1, 2, \ldots, n\}$. As (6), we compute the gradient $v_i^t$ over $B_{s+1}$. Then we conduct the proximal step, compute the update $\delta_t^i$, and add it to the shared memory.

Notably, first, at the $s$-th iteration, by exploiting the sparsity of the model, our Alg. 4 only solve sub-problem $\mathcal{P}_{s+1}$ over $B_{s+1}$, which is more efficient than training of the full model. Thus, the full gradients at the $s$-th iteration in our algorithm is only computed with $p_i$ coefficients, which is much less than $O(p)$ in practice. Second, by exploiting the sparsity of the dataset, we only conduct sparse gradient update and sparse proximal update, which is very efficient for large-scale real-world datasets. Third, by reducing the gradient variance, we can use a constant step size to improve the convergence and finally achieve a linear convergence rate for strongly convex function. Lastly, all the threads works asynchronously, which is very efficient and easy to parallelize. Hence, the computation and memory costs of large-scale training can be effectively reduced.

2.3 DDSS on Distributed-Memory Architecture

On distributed-memory architecture, our Dis-DDSS algorithm is summarized in Alg. 5 and 6. In Dis-DDSS, suppose we have one server node and $l$ local worker nodes where
each worker stores $n_k$ samples. When the flag is True, in the outer loop of the server node, the server broadcasts the flag and $x^{0}_{B_{s+1}}$ to the workers. At the worker node, worker $k$ receives $x^{0}_{B_{s+1}}$ from the server, computes the gradient over $n_k$ samples, and then sends it to the server node. With the gradients received from all the workers, the server node computes the full gradients and send them to the workers. By (2), the server node computes $y^t$ and then performs the elimination to obtain $B_{s+1}$ and then send it to the workers. The worker node receives $\nabla F(x^0_{B_{s+1}})$ and $B_{s+1}$ from the server. With $B_{s+1}$, the worker updates $A_{i \in n_k, B_{s+1}}, x^{0}_{B_{s+1}}$ and $\nabla F(x^0_{B_{s+1}})$.

**Algorithm 5** Dis-DDSS (Server Node)

1: for $s = 0$ to $S - 1$ do
2: flag = True
3: Broadcast flag and $x^0_{B_{s+1}}$ to all workers
4: Receive gradients from all workers
5: $\nabla F(x^0_{B_{s+1}}) = \frac{1}{n} \sum_{i \in n_k} \nabla F_i(x^0_{B_{s+1}})$
6: Compute $y^t$ by (2)
7: Update $B_{s+1} \subseteq B_s$ by (3)
8: Broadcast $B_{s+1}$ and $\nabla F(x^0_{B_{s+1}})$ to all workers
9: flag = False
10: Broadcast flag to all workers
11: for $t = 0$ to $K - 1$ do
12: Receive $\delta^t_i$ from one worker
13: Update $x^{t+1}_{B_{s+1}} = x^t_{B_{s+1}} + \delta^t_i$
14: end for
15: $x_{B_{s+1}} = x^K_{B_{s+1}}, x^0_{B_{s+1}} = x_{B_{s+1}}$
16: end for

**Algorithm 6** Dis-DDSS (Worker Node $k$)

1: if flag = True then
2: Compute and send gradient $\nabla F_i(x^0_{B_{s+1}}) = \sum_{i \in n_k} \nabla f_i(a_i^0, x^0_{B_{s+1}})$
3: Receive $\nabla F(x^0_{B_{s+1}})$ from server
4: Receive $B_{s+1}$ from server
5: Compute $A_{i \in n_k, B_{s+1}}, x^{0}_{B_{s+1}}, \nabla F(x^0_{B_{s+1}})$
6: end if
7: else
8: Receive $x^{d(t)}$ from server
9: Randomly sample $i$ from $\{1, 2, \ldots, n_k\}$
10: Compute $v_i^t = \nabla f_i(a_i^{d(t)}, x^{d(t)}_{B_{s+1}}) - \nabla f_i(a_i^{0}, x^0_{B_{s+1}}) + D_i x_{B_{s+1}} - \nabla F(x^0_{B_{s+1}})$
11: Send $\delta_i = \text{prox}_{\eta \lambda \phi_i}(v_i^t - x^{d(t)}_{B_{s+1}})$ to server
12: end if

When the flag is False, the server broadcasts the flag to the workers. At the workers, the algorithm optimizes over $B_{s+1}$. Multiple workers update the parameter asynchronously. Specifically, each worker receives stale parameter $x^{d(t)}_{B_{s+1}}$ from the server. The worker first chooses sample $i$ from $\{1, 2, \ldots, n_k\}$ and compute $v_i^t$ over $B_{s+1}$ as (6). Following the decoupling strategy, we compute the proximal step and the update at the worker node. Finally, the worker send it to server. In the inner loop of the server node, with the update information $\delta^t_i$ from workers, $x^{t+1}_{B_{s+1}}$ is updated by $\delta^t_i$ via only simple addition computation.

First, similar to Alg. 4, our Alg. 5 and 6 is very computation-efficient by exploiting the sparsity of the model and the dataset and reducing the gradient variance. Second, the time-consuming proximal step from the server is off-loaded to all the workers, which is very efficient and easy to parallelize. Third, by the eliminating and the sparse update, the communication cost between the server and the workers is also much less than the full updates. Overall, the computation, memory and communication costs could be effectively reduced during the training.

3 THEORETICAL ANALYSIS

In this section, we provide the rigorous theoretical analysis on the convergence and screening ability for DDSS on shared-memory architecture. Note our analysis can be easily extended to distributed-memory architecture. All the proof is provided in the appendix.

3.1 Assumptions, Definitions, and Properties

**Assumption 1** (Strong Convexity). $\Omega(x)$ is convex and block separable. $F(x)$ is $\mu$-strongly convex, i.e., $\forall x, x' \in \mathbb{R}^p$, we have $F(x') \geq F(x) + \nabla F(x)^\top(x' - x) + \frac{\mu}{2}\|x' - x\|^2$.

**Assumption 2** (Lipschitz Smooth). Each $F_i(x)$ is differentiable and Lipschitz gradient continuous with $L$, i.e., $\exists L > 0$, such that $\forall x, x' \in \mathbb{R}^p$, we have $\|\nabla F_i(x) - \nabla F_i(x')\| \leq L \|x - x'\|$.

**Assumption 3** (Bounded Overlapping). There exists a bound $\tau$ on the number of iterations that overlap. The bound $\tau$ means each writing at iteration $t$ is guaranteed to successfully performed into the memory before iteration $t + \tau + 1$.

**Remark 1.** Asm. 1 implies $\mathcal{P}(x)$ is also $\mu$-strongly convex. Asm. 2 implies that $F(x)$ is also Lipschitz gradient continuous. We denote $\kappa := L/\mu$ as the condition number. Asm. 3 means the delay that asynchrony may cause is upper bounded. All the assumptions are commonly seen in asynchronous methods [Pedregosa et al., 2017].

**Definition 1** (Block Sparsity). We denote that the maximum frequency of occurrences $\Delta$ that a feature block belongs to the extended support, which can be formally defined as $\Delta = \max_{a \in B} \left| \{ t : \Psi_t \supseteq \mathcal{G} \} \right| / n$. It is easy to verify that $1/n \leq \Delta \leq 1$.

**Property 1** (Independence). We use the “after read” labeling in [Leblond et al., 2017], which means we update the iterate counter after each thread fully reads the parameters. This means that $x^{d(t)}_{B_{s+1}}$ is the $(t+1)$-th fully completed read. Given the “after read” global time counter, sample $i$, is independent of $x^{d(t)}_{B_{s+1}}$, $\forall t \geq t$.

**Property 2** (Unbiased Gradient Estimation). Gradient $v_i$ is an unbiased estimation of the gradient over set $B_{s+1}$ at $x^{d(t)}_{B_{s+1}}$, which is directly derived from Property 1.
Property 3 (Atomic Operation). The update $x_{B_{n+1}}^{t+1} = x_{B_{n+1}}^t + \delta^t$ to shared-memory in Alg. 4 is coordinate-wise atomic, which can address the overwriting problem caused by other threads.

3.2 Theoretical Results

Theorem 1 (Convergence). Suppose $\tau \leq \frac{1}{10\sqrt{\kappa}}$, let step size $\eta = \min\{\frac{1}{4\kappa L}, \frac{\kappa}{10\tau L}\}$, inner loop size $K = \frac{4\log 3}{\eta \mu}$, we have

$$E\|x_{B_n} - x_{B_n}^*\|^2 \leq (2/3)^S\|x_0 - x^*\|^2.$$  \tag{11}

Remark 2. Theorem 1 shows that DDSS can achieve a linear convergence rate $\log(1/\epsilon)$.

Remark 3. For the case $F(x)$ is nonstrongly convex, we can slightly modify $\Omega(x)$ by adding a small perturbation, e.g., $\mu_f\|x\|^2$ for smoothing where $\mu_f$ is a positive parameter. We can treat $F(x) + \mu_f\|x\|^2$ as the data-fitting loss and then the loss is $\mu_f$-strongly convex. Denote $\kappa := L/\mu_f$, suppose $\tau \leq \frac{1}{10\sqrt{\kappa}}$, let $\eta = \min\{\frac{1}{4\kappa L}, \frac{\kappa}{10\tau L}\}$, $K = \frac{4\log 3}{\eta \mu_f}$, we have $E\|x_{B_n} - x_{B_n}^*\|^2 \leq (2/3)^S\|x_0 - x^*\|^2$. Similarly, the overall computational cost is also very efficient.

Theorem 2 (Screening Ability). Equicorrelation set [Tibshirani, 2013] is defined as $B^s := \{j \in [1, 2, \ldots, q] : \Omega_j^s(A_j^s y^*) = n\lambda\}$. Then, as DDSS converges, there exists an iteration number $S_0 \in \mathbb{N}$, s.t. $\forall s \geq S_0$, any variable block $j \notin B^s$ is eliminated by DDSS almost surely.

Remark 4. Suppose the size of set $B_s$ is $p_s$ and $p^*$ is the size of the active features in $B^s$. Theorem 2 shows we have $p_s$ is decreasing and $\lim_{s \to +\infty} p_s = p^*$.

Remark 5. To sum it up, by the elimination, the cost at each iteration is reduced from $O(p)$ to $O(p_s)$. Moreover, by the sparse update, only the nonzero coefficients of set $B_s$ is updated and thus the cost at each iteration is further reduced from $O(p_s)$ to $O(p_s')$ where $p_s'$ is the size of the nonzero coefficients. In the high-dimensional setting, we have $p^* \ll p$, $p_s \ll p$, and $p_s' \ll p$. Thus, with constantly decreasing $p_s$ and the sparse update, our DDSS can reduced the complexity from $O(p)$ to $O(r)$ where $r$ is the mean of $p_s'$ for $s = 1, 2, \ldots$, which can accelerate the training at a large extent in practice.

4 EXPERIMENTS

4.1 Experimental Setup

We compare our method with other competitive methods on three large-scale datasets. Although DDSS can work more broadly, we focus on Lasso for sparse regression, which is the most popular case for feature screening. Specifically, Lasso solves

$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n \frac{1}{2} (y_i - a_i^T x)^2 + \lambda \|x\|_1.$$  \tag{12}

Figure 1: Convergence results on shared-memory architecture with 8 threads.

Figure 2: Convergence results on distributed-memory architecture with 8 workers.
On shared-memory architecture, we compare six asynchronous methods: 1) PSE-Strong: parallel strong screening in [Li et al., 2016a]; 2) PSE-Safe: parallel static safe screening in [Li et al., 2016a]; 3) ProxASAGA [Pedregosa et al., 2017]; 4) ProxASVRG [Meng et al., 2017]; 5) Sha-DDSS-Naive; 6) Our Sha-DDSS. PSE-Safe and PSE-Strong are parallel static screening. ProxASAGA and ProxASVRG are popular asynchronous method with linear convergence.

On distributed-memory architecture, we compare four asynchronous methods: 1) ProxASAGA [Leblond et al., 2018]; 2) ProxASVRG [Meng et al., 2017]; 3) Dis-DDSS-Naive; 4) Our Dis-DDSS.

Table 3: Real-world datasets in the experiments.

| Dataset       | Sample Size  | Attributes | Sparsity |
|---------------|--------------|------------|----------|
| KDD 2010      | 19,264,097   | 1,163,024  | $7 \times 10^{-6}$ |
| Avazu-app     | 14,596,137   | 1,000,000  | $10^{-5}$ |
| Avazu-site    | 25,832,830   | 1,000,000  | $10^{-5}$ |

We use three large-scale real-world benchmark datasets described in Table 3. All the datasets are from LIBSVM [Chang and Lin, 2011], which can be found at https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/.

We implement all the compared methods in C++. We employ OpenMP and OpenMPI as the parallel framework for shared-memory and distributed-memory architecture respectively. We run all the methods on 2.10 GHz Intel(R) Xeon(R) CPU machines. For the implementation, the inner loop size, ranging from $2 \times 10^5$ to $2 \times 10^9$, and the step size, ranging from $10^{-11}$ to $10^{-13}$, of each method are chosen to obtain the best performance. Parameter $\lambda$ is set as $4 \times 10^{-6} \lambda_{\text{max}}, 2 \times 10^{-3} \lambda_{\text{max}}$, and $1 \times 10^{-3} \lambda_{\text{max}}$ for KDD 2010, Avazu-app, and Avazu-site dataset respectively where $\lambda_{\text{max}}$ is a parameter that, for all $\lambda \geq \lambda_{\text{max}}$, $x^*$ must be 0.

4.2 Experimental Results and Discussions

Convergence Results Figure 1 (a)-(c) provides the convergence results of different methods on shared-memory architecture with 8 threads on three datasets respectively. Our Sha-DDSS-Naive method converges very fast at the initial stage because of the screening ability and slows down later due to its sublinear convergence rate. The results confirm that our Sha-DDSS method always converge much faster than other methods on shared-memory architecture. Figure 2 (a)-(c) provides the convergence results of different methods on distributed-memory architecture with 8 workers on three datasets respectively. The results also confirm that our Dis-DDSS always converge much faster than other methods on distributed-memory architecture.

This is because our method on both shared-memory and distributed-memory architecture can eliminate the features by exploiting the sparsity of the model, perform efficient sparse update by exploiting the sparsity of the dataset, achieve the linear convergence rate by reducing the gradient variance. Our Dis-DDSS also performs the decouple proximal update to reduce the workload of the server and reduces the communication costs.

Linear Speedup Property We evaluate Sha-DDSS with different number of threads on shared-memory architecture and Dis-DDSS with different number of workers on distributed-memory architecture. Figure 3(a)-(c) presents the results of the speedup of Sha-DDSS and Dis-DDSS on three datasets respectively. The results show that our method can successfully achieve a nearly linear speedup when we increase the number of threads or workers, although the performance decreases when the number of processors or workers increases. This is because there are overheads for creating threads and distributing work for OpenMP and communication costs for OpenMPI, which the theoretical analysis does not take into account.

5 CONCLUSION

In this paper, we propose the first distributed dynamic safe screening method for sparse models and apply it on shared-memory and distributed-memory architecture respectively. Theoretically, we prove that our proposed method can achieve a linear convergence rate with lower overall complexity. Moreover, we prove that our method can eliminate almost all the inactive variables in a finite number of iterations almost surely. Finally, extensive experimental results on benchmark datasets confirm the significant acceleration and linear speedup property of our method.
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A Appendix

In the appendix, we provide some basic lemmas and the proof for all the theorems.

A.1 Basic Lemmas

**Lemma 1.** Suppose $F$ is $\mu$-strongly convex, we have:

$$\langle \nabla F(y) - \nabla F(x), y - x \rangle \geq \frac{\mu}{2} \|y - x\|^2 + B_F(x, y)$$  \hspace{1cm} (13)

where $B_F(x, y)$ is the Bregman divergence defined as $B_F(x, y) := F(x) - F(y) - \langle \nabla F(y), x - y \rangle$.

**Proof.** By strong convexity, for any $x, y$, we have:

$$F(y) \geq F(x) + \langle \nabla F(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$$

$$\iff \langle \nabla F(x), x - y \rangle \geq \frac{\mu}{2} \|y - x\|^2 + F(x) - F(y)$$

$$\iff \langle \nabla F(x) - \nabla F(y), x - y \rangle \geq \frac{\mu}{2} \|y - x\|^2 + F(x) - F(y) - \langle \nabla F(y), x - y \rangle$$

$$\iff \langle \nabla F(y) - \nabla F(x), x - y \rangle \geq \frac{\mu}{2} \|x - y\|^2 + B_F(x, y).$$  \hspace{1cm} (14)

This finishes the proof. \hfill $\square$

**Lemma 2.** Suppose $F_i$ is $L$-smooth and convex, we have:

$$\frac{1}{n} \sum_{i=1}^{n} \|\nabla F_i(x) - \nabla F_i(y)\|^2 \leq 2LB_F(x, y).$$  \hspace{1cm} (15)

**Proof.** Based on Eq. 5 of Lemma 4 in [Zhou, 2018], we have

$$\|\nabla F_i(x) - \nabla F_i(y)\|^2 \leq 2L(\nabla F_i(x) - \nabla F_i(y) - \langle \nabla F_i(y), x - y \rangle).$$  \hspace{1cm} (16)

Averaging with $i$, we have

$$\frac{1}{n} \sum_{i=1}^{n} \|\nabla F_i(x) - \nabla F_i(y)\|^2 \leq 2L(\nabla F(x) - \nabla F(y) - \langle \nabla F(y), x - y \rangle)$$  \hspace{1cm} (17)

which is equivalent to

$$\langle \nabla F(x) - \nabla F(y), x - y \rangle \geq \frac{\mu}{2} \|x - y\|^2 + B_F(x, y).$$  \hspace{1cm} (18)

This finishes the proof. \hfill $\square$

**Lemma 3.** Let $x^*$ be the optimal solution of Problem (1), $v^*_i$ is the sparse variance reduced gradient with sample $i$ defined in (6), $g$ is the the sparse gradient mapping for $v^*_i$ and computed as $g = \frac{1}{T}(x - \text{prox}_{\eta\lambda_\Phi_i}(x - \eta v^*_i))$. Then, for any $\beta > 0$ and $x \in \mathbb{R}^p$, we have:

$$\langle g, x - x^* \rangle \geq -\frac{\eta}{2}(\beta - 2)\|g\|^2 - \frac{\eta}{2\beta} \|v^*_i - D_i\nabla F(x^*)\|^2 + \langle v^*_i - D_i\nabla F(x^*), x - x^* \rangle.$$  \hspace{1cm} (19)

**Proof.** Let $x^+ = \text{prox}_{\eta\lambda_\Phi_i}(x - \eta v^*_i)$ and $x^* = \text{prox}_{\eta\lambda_\Phi_i}(x^* - \eta D_i\nabla F(x^*))$, denote $\langle \cdot, \cdot \rangle_{(i)}$ as the inner product restricted to the blocks in $\Psi_i$ and $\|\cdot\|_{(i)}$ as the norm restricted to the blocks in $\Psi_i$, we have:

$$\|x^+ - x^*\|^2_{(i)} - \langle x^+ - x^*, x - \eta v^*_i - x^* + \eta D_i\nabla F(x^*) \rangle_{(i)} \leq 0,$$  \hspace{1cm} (20)

which comes from the firm non-expansiveness of the proximal operator.
Then we have:

\[
\langle \eta g, x - x^* \rangle = (x - x^+, x - x^*)_{(i)}
\]

\[
= \|x - x^*\|^2_{(i)} - (x^+ - x^*, x - x^*)_{(i)}
\]

\[
\geq \|x - x^*\|^2_{(i)} - \langle x^+ - x^*, 2x - \eta v_i^* - 2x^* + \eta D\nabla F(x^*) \rangle_{(i)} + \|x^+ - x^*\|^2_{(i)}
\]

\[
= \|x - x^*\|^2_{(i)} + \langle x^+ - x^*, \eta v_i^* - \eta D\nabla F(x^*) \rangle_{(i)}
\]

\[
= \|x - x^+\|^2_{(i)} + \langle x^+ - x^*, \eta v_i^* - \eta D\nabla F(x^*) \rangle_{(i)} - \langle x - x^+, \eta v_i^* - \eta D\nabla F(x^*) \rangle_{(i)}
\]

\[
\geq \left(1 - \frac{\beta}{2}\right)\|x - x^+\|^2_{(i)} - \eta^2 \left(\frac{2}{\beta}\right) \|v_i^* - D_i\nabla F(x^*)\|^2_{(i)} + \eta \langle v_i^* - D_i\nabla F(x^*), x - x^* \rangle
\]

(21)

where the first inequality is obtained by adding Eq. (20), the second inequality is obtained by Young’s inequality $2\langle a, b \rangle \leq \frac{|a|^2}{\beta} + \frac{|b|^2}{2}$ for arbitrary $\beta > 0$. Finally, the result finishes the proof.

\[
\square
\]

Lemma 4. [Leblond et al., 2017, Proposition 1] For any $u \neq t$, we have

\[
\mathbb{E}|\langle g_u, g_t \rangle| \leq \sqrt{\frac{\Delta}{2}}(\mathbb{E}\|g_u\|^2 + \mathbb{E}\|g_t\|^2).
\]

(23)

Lemma 5. We have following estimations:

\[
\mathbb{E}\|\hat{x}_t^* - x_t^*\|^2 \leq \eta^2 \left(1 + \sqrt{\Delta} \right) \sum_{u=(t-\tau)_+}^{t-1} \mathbb{E}\|g_u^*\|^2
\]

(24)

\[
\mathbb{E}\langle g_t^*, \hat{x}_t^* - x_t^* \rangle \leq \frac{\eta \sqrt{\Delta}}{2} \sum_{u=(t-\tau)_+}^{t-1} \mathbb{E}\|g_u^*\|^2 + \frac{\eta \sqrt{\Delta}}{2} \mathbb{E}\|g_t^*\|^2
\]

(25)

Proof. By Assumption 3, we have the following updates:

\[
\hat{x}_t = x_t - \eta \sum_{u=(t-\tau)_+}^{t-1} G_u^t g(\hat{x}_u, \hat{\alpha}_u, i_u),
\]

(26)

Thus, we have:

\[
\mathbb{E}\|\hat{x}_t - x_t\|^2 \leq \eta^2 \sum_{u,(t-\tau)_+}^{t-1} \mathbb{E}|\langle G_u^tg_u, G_v^tg_v \rangle|
\]

\[
\leq \eta^2 \left(\sum_{u=(t-\tau)_+}^{t-1} \mathbb{E}\|g_u\|^2 + \eta^2 \sum_{u,v=(t-\tau)_+}^{t-1} \mathbb{E}|\langle G_u^tg_u, G_v^tg_v \rangle|\right)
\]

\[
\leq \eta^2 \left(\sum_{u=(t-\tau)_+}^{t-1} \mathbb{E}\|g_u\|^2 + \eta^2 \sum_{u,v=(t-\tau)_+}^{t-1} \mathbb{E}|\langle g_u, g_v \rangle|\right)
\]

\[
\leq \eta^2 \left(\sum_{u=(t-\tau)_+}^{t-1} \mathbb{E}\|g_u\|^2 + \eta^2 \sqrt{\Delta} \sum_{u=(t-\tau)_+}^{t-1} \mathbb{E}\|g_u\|^2\right)
\]

\[
= \eta^2 \left(1 + \sqrt{\Delta} \right) \sum_{u=(t-\tau)_+}^{t-1} \mathbb{E}\|g_u\|^2
\]

(27)
where the fourth inequality is obtained by Lemma 4.
Taking the expectation of \( \langle \hat{x}_t - x_t, g_t \rangle \), we have:

\[
\frac{1}{\eta} \mathbb{E} \langle \hat{x}_t - x_t, g_t \rangle = \sum_{u = (t-\tau)_+}^{t-1} \mathbb{E} \langle G_u g_u, g_t \rangle \\
\leq \sum_{u = (t-\tau)_+}^{t-1} \mathbb{E} \langle g_u, g_t \rangle \\
\leq \frac{\sqrt{\Delta}}{2} \sum_{u = (t-\tau)_+}^{t-1} \mathbb{E} \|g_u\|^2 + \frac{\sqrt{\Delta}}{2} \mathbb{E} \|g_t\|^2.
\]

This finishes the proof. \( \square \)

**Lemma 6.** For any \( x \in \mathbb{R}^n \) and gradient estimator \( v_i^* \) computed by sample \( i \), we have

\[
\mathbb{E} \|v_i^* - D_i \nabla F(x^*)\|^2 \leq 4L \mathbb{E} B_F(\hat{x}_i^*, x^*) + 2L^2 \mathbb{E} \|x_0^* - x^*\|^2
\]

*Proof.* Since \( v_i^* \) is \( \Psi_i \)-sparse, we have

\[
\|v_i^* - D_i \nabla F(x^*)\|^2 = \|v_i^* - D_i \nabla F(x^*)\|^2
\]

\[
= \|\nabla F_i(\hat{x}_i^*) - \nabla F_i(x_0^*) + D_i \nabla F(x_0^*) - D_i \nabla F(x^*)\|^2
\]

\[
\leq 2 \|\nabla F_i(\hat{x}_i^*) - \nabla F_i(x_0^*)\|^2 + 2 \|D_i \nabla F(x_0^*) - D_i \nabla F(x^*) - (\nabla F_i(x_0^*) - \nabla F_i(x^*))\|^2
\]

According to the support set of \( \nabla F_i \) and Lemma 2, we get

\[
\mathbb{E} \|\nabla F_i(\hat{x}_i^*) - \nabla F_i(x^*)\|^2 \leq 2L \mathbb{E} B_F(\hat{x}_i^*, x^*)
\]

On the other hand,

\[
\mathbb{E} \|D \nabla F(x_0^*) - D \nabla F(x^*) - (\nabla F_i(x_0^*) - \nabla F_i(x^*))\|^2
\]

\[
\leq \mathbb{E} \|D \nabla F(x_0^*) - D \nabla F(x^*)\|^2 + \mathbb{E} \|\nabla F_i(x_0^*) - \nabla F_i(x^*)\|^2
\]

\[
\leq 2 \mathbb{E} \|D \nabla F(x_0^*) - D \nabla F(x^*)\|^2 + 2 \mathbb{E} \|\nabla F_i(x_0^*) - D \nabla F(x^*)\|^2
\]

\[
\leq L^2 \mathbb{E} \|x_0^* - x^*\|^2,
\]

where the second equality is derived by the definition of \( D_i \) and the support of \( \nabla F_i \). In the third equality, we take expectation on \( i \) and use \( \mathbb{E} D_i = I_p \). In the first inequality, we use the fact that \( D \) is a diagonal matrix with non-negative entries and hence \( \langle \nabla F(x_0^*) - \nabla F(x^*), D \nabla F(x_0^*) - D \nabla F(x^*) \rangle \geq 0 \). The last inequality comes from Assumption 2. Combining above inequalities, we complete the proof. \( \square \)

### A.2 Proof of Theorem 1

*Proof.* At the \( s - 1 \) iteration, we conduct the elimination step over set \( B_{s-1} \) in the outer loop. Since the eliminated variables are zeroes at the optimal, all the sub-problems have the same optimal solution. We have

\[
\|x_{B_s}^* - x_{B_{s-1}}^*\|^2 = 0,
\]

where the norm is conducted on the corresponding coordinate and the eliminated variables in \( B_s \) are filled with 0.
Meanwhile, denote $\tilde{B}_s = \{ j \in B_{s-1} | j \notin B_s \}$, we have
\[
\| x_{B_{s-1}} - x_{\tilde{B}_{s-1}} \|^2 = \| x_{B_s} - x_{\tilde{B}_s} \|^2 + \| x_{B_s} - x_{\tilde{B}_s} \|^2 .
\]  
(35)

Considering $\| x_{B_s} - x_{\tilde{B}_s} \|^2 \geq 0$, we have
\[
\| x_{B_s} - x_{\tilde{B}_s} \|^2 \leq \| x_{B_{s-1}} - x_{\tilde{B}_{s-1}} \|^2 .
\]  
(36)

Moreover, according to the iteration in the inner loop, we have
\[
\| x^{t+1}_{B_s} - x_{\tilde{B}_s} \|^2 = \| x^{t}_{B_s} - \eta g_t^s - x_{B_{s-1}} \|^2 \\
= \| x^{t}_{B_s} - x_{B_s} \|^2 + \| \eta g_t^s \|^2 - 2\eta \langle g_t^s, x^t_{B_s} - x_{B_s} \rangle \\
= \| x^{t}_{B_s} - x_{\tilde{B}_s} \|^2 + \| \eta g_t^s \|^2 - 2\eta \langle g_t^s, \hat{x}^t_{B_s} - x_{\tilde{B}_s} \rangle + 2\eta \langle g_t^s, \hat{x}^t_{B_s} - x_{B_s} \rangle .
\]  
(37)

Applying Lemma 3 to $B_s$, we obtain
\[
\| x^{t+1}_{B_s} - x_{\tilde{B}_s} \|^2 \leq \| x^{t}_{B_s} - x_{B_s} \|^2 + 2\eta \langle g_t^s, \hat{x}^t_{B_s} - x^t_{B_s} \rangle + \eta^2 (\beta - 1) \| g_t^s \|^2 \\
+ \frac{\eta^2}{\beta} \| v^{t+1} - D_{t,B_s} \nabla F(x_{B_s}) \|^2 - 2\eta \langle v^{t+1} - D_{t,B_s} \nabla F(x_{B_s}), \hat{x}^t_{B_s} - x_{\tilde{B}_s} \rangle .
\]  
(38)

Since $E(D_{t,B_s} = I_{P_s})$, we have
\[
E\langle v^{t+1}_i - D_{t,B_s} \nabla F(x_{B_i}), \hat{x}^t_{B_i} - x_{\tilde{B}_i} \rangle = \langle \nabla F(\hat{x}^t_{B_i}), \nabla F(x_{B_i}), \hat{x}^t_{B_i} - x_{\tilde{B}_i} \rangle \\
\leq \frac{\mu}{2} \| \hat{x}^t_{B_i} - x_{\tilde{B}_i} \|^2 + B_F(\hat{x}^t_{B_i}, x_{\tilde{B}_i}) ,
\]  
(39)

where the inequality is obtained by applying Lemma 1 to the sub-problem $P_s$.

Combining above two inequalities we get
\[
\| x^{t+1}_{B_s} - x_{B_s} \|^2 \leq \| x^t_{B_s} - x_{B_s} \|^2 + 2\eta \langle g_t^s, \hat{x}^t_{B_s} - x^t_{B_s} \rangle + \eta^2 (\beta - 1) \| g_t^s \|^2 \\
+ \frac{\eta^2}{\beta} \| v^{t+1} - D_{t,B_s} \nabla F(x_{B_s}) \|^2 - 2\eta \langle v^{t+1} - D_{t,B_s} \nabla F(x_{B_s}), \hat{x}^t_{B_s} - x_{\tilde{B}_s} \rangle \\
\leq (1 - \frac{\eta \mu}{2}) \| x^t_{B_s} - x_{B_s} \|^2 + 2\eta \langle g_t^s, \hat{x}^t_{B_s} - x^t_{B_s} \rangle + \eta^2 (\beta - 1) \| g_t^s \|^2 \\
+ \frac{\eta^2}{\beta} \| v^{t+1} - D_{t,B_s} \nabla F(x_{B_s}) \|^2 + \eta \mu \| \hat{x}^t_{B_s} - x^t_{B_s} \|^2 - 2\eta B_F(\hat{x}^t_{B_s}, x_{\tilde{B}_s}) ,
\]  
(40)

where in the second inequality we use $\| a + b \|^2 \leq 2 \| a \|^2 + 2 \| b \|^2$.

By considering Lemma 5 and Lemma 6 with $B_s$, we have
\[
E\| x^{t+1}_{B_s} - x_{B_s} \|^2 \leq (1 - \frac{\eta \mu}{2}) E\| x^t_{B_s} - x_{B_s} \|^2 + 2\eta L^2 \eta \| g_t^s \|^2 E\| x^t_{B_s} - x_{B_s} \|^2 + \frac{4L^2 \eta^2}{\beta} E \| g_t^s \|^2 \\
- 2\eta \| g_t^s \|^2 (1 + \sqrt{\Delta}) E \| g_t^s \|^2 \\
+ (\eta^3 \mu (1 + \sqrt{\Delta}) + \eta^2 \Delta) \sum_{u=0}^{t-1} E \| g_u^s \|^2 .
\]  
(41)

Because $4L\eta \leq 1$, $\sqrt{\Delta} \leq \frac{1}{10}$ and $B_F(\cdot, \cdot)$ is non-negative, let $\beta = \frac{1}{2}$, we have
\[
E\| x^{t+1}_{B_s} - x_{B_s} \|^2 \leq (1 - \frac{\eta \mu}{2}) E\| x^t_{B_s} - x_{B_s} \|^2 + 4L^2 \eta \| g_t^s \|^2 E\| x^t_{B_s} - x_{B_s} \|^2 - \frac{2\eta^2}{5} E \| g_t^s \|^2 \\
+ (2\eta^3 \mu + \eta^2 \Delta) \sum_{u=0}^{t-1} E \| g_u^s \|^2 .
\]  
(42)

Applying recursion to above inequality and we can obtain
\[
E\| x^{t}_{B_s} - x_{B_s} \|^2 \leq \left( 1 - \frac{\eta \mu}{2} \right)^K + 8K \eta \| g_0 \| E\| x^0_{B_s} - x_{B_s} \|^2 - \sum_{u=0}^{K-1} \eta^u \| g_u \| E \| g_u \|^2 ,
\]  
(43)
Let \( h(x) = \log(1 + 2ax) - x \log(1 + a) \) for some positive \( a \). We can see \( h(0) = 0 \) and \( h'(x) = \frac{2a}{1 + 2ax} - \log(1 + a) \). If \( ax \leq \frac{1}{2} \), then we have \( h'(x) \geq a - \log(1 + a) \geq 0 \). Therefore, when \( ax \leq \frac{1}{2} \), we always have \( h(x) \geq 0 \) and \( (1 + a)^x \leq 1 + 2ax \). Substitute \( a \) with \( \frac{\eta}{2} \). Since \( \eta \mu \tau \leq \frac{1}{16} \), we have \( \frac{\eta}{2} \mu \tau \leq \frac{1}{2} \). Then we can estimate the lower bound of \( r_u^a \):

\[
 r_u^a \geq (1 - \frac{\eta \mu}{2})^{k-1-u} \left( \frac{2\eta^2}{5} - (2\eta^3\mu + \eta^2\sqrt{\Delta}) \sum_{m=0}^{\tau-1} (1 - \frac{\eta \mu}{2})^{-m} \right).
\]  

(44)

where we also use \( \eta \mu \tau \leq \frac{1}{16} \) and \( \sqrt{\Delta} \tau \leq \frac{1}{16} \). As \( r_u^a \) is positive, we can drop the last term in Eq. (43).

As \( \eta \mu \leq \frac{1}{2} \), it is easy to check that \( \log(1 + \frac{\eta \mu}{2}) \geq \frac{\eta \mu}{2(2-\eta \mu)} \). When \( K = \frac{4\log 3}{\eta \mu} \), we have \( (1 - \frac{\eta \mu}{2})^K \leq \frac{1}{3} \). Since \( \eta \leq \frac{1}{2\kappa L} \), we can prove

\[
\mathbb{E}\|x_{B_s}^* - x_{B_s}^0\|^2 \leq \frac{2}{3}\mathbb{E}\|x_{B_s}^* - x_{B_s}^\star\|^2.
\]  

(46)

Note \( x_{B_s}^K = x_{B_s}^\star \) and \( x_{B_{s-1}}^0 = x_{B_{s-1}}^\star \), combining (36), we have

\[
\mathbb{E}\|x_{B_s}^\star - x_{B_s}^0\|^2 \leq \frac{2}{3}\mathbb{E}\|x_{B_{s-1}}^\star - x_{B_{s-1}}^\star\|^2.
\]  

(47)

Apply recursion to the above inequality, note \( x_{B_0} = x_0 \) and \( x_{B_0}^\star = x^\star \) we can obtain

\[
\mathbb{E}\|x_{B_s}^\star - x_{B_s}^0\|^2 \leq \left(\frac{2}{3}\right)^S\mathbb{E}\|x_0 - x^\star\|^2,
\]  

(48)

which finishes the proof. \( \square \)

### A.3 Proof of Theorem 2

**Proof.** Based on Theorem 1, we know our DDSS method has a converging sequence \( \{x_{B_s}\} \). As DDSS algorithm converges, because of the strong duality, the dual \( y^\star \) and the intermediate duality gap \( P(x_{B_s}) - D(y^\star) \) also converges. For any given \( \epsilon \), \( \exists S_0 \) such that \( \forall S \geq S_0 \), we have

\[
\|y^\star - y^\cdot\|_2 \leq \epsilon,
\]  

(49)

and

\[
\sqrt{2L(P(x_{B_s}) - D(y^\star))} \leq \epsilon.
\]  

(50)

almost surely.

For any \( j \notin B^\star \), we have

\[
\Omega_j^D(A_j^\top y^\star) + \Omega_j^D(A_j)\sqrt{2L(P(x_{B_s}) - D(y^\star))} \\
\leq \Omega_j^D(A_j^\top (y^\star - y^\cdot)) + \Omega_j^D(A_j^\top y^\cdot) + \Omega_j^D(A_j)\sqrt{2L(P(x_{B_s}) - D(y^\star))} \\
\leq 2\Omega_j^D(A_j)\epsilon + \Omega_j^D(A_j^\top y^\star)
\]  

(51)

where the first inequality comes from the triangle inequality and the second inequality comes from (49) and (50).

Hence, if we choose

\[
\epsilon < \frac{n\lambda - \Omega_j^D(A_j^\top y^\star)}{2\Omega_j^D(A_j)},
\]  

(52)

we can ensure the screening test \( \Omega_j^D(A_j^\top y^\star) + \Omega_j^D(A_j)\sqrt{2L(P(x_{B_s}) - D(y^\star))} < n\lambda \) holds for \( j \), which means variable block \( j \) is eliminated at most at this iteration. In (52), since \( j \notin B^\star \), it is easily to verify that \( n\lambda - \Omega_j^D(A_j^\top y^\star) > 0 \). This finishes the proof. \( \square \)