Borehole contraction and depth analysis based on Extended SMP criterion

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Abstract. The excavation of vertical boreholes is involved in several geotechnical engineering constructions. Earth pressure in the soil is released during the drilling process, resulting in soil stress redistribution and borehole contraction along the excavation surface. The stress and displacement responses tend to increase with soil depth, which may degrade the quality of hole formation and cause collapse. Therefore, the concerns of three-dimensional (3-D) stress and displacement states of the soil around the borehole should be addressed during the design and construction of associated engineering. This study proposes exact and simplified analytical solutions based on the Extended Spatially Mobilized Plane criterion to obtain a more accurate borehole contraction law and depth in drained soil. The 3-D stress and displacement values along the borehole depth are obtained by combining the derived plane solution of cylindrical cavity contraction with the classical earth pressure formula. The proposed solution can reasonably consider the influence of the intermediate principal stress and explicitly state its value. The contraction degree of the proposed solution is 7.76% less than the Mohr–Coulomb solution at a depth of 80 m with the specified soil parameters. A parametric analysis shows that the contraction will be reduced by 21.4% and 36.8% if the unloading ratio increases from 0.4 to 0.5 and 0.6, respectively. The borehole can be supported by a high unit weight of slurry in an elastic state. A decrease in soil shear strength impairs borehole stress state and hole formation quality, and internal friction angle affects more than cohesion in deep soil. The effect of dilatancy on borehole contraction is insignificant. Further, the proposed approximate solution, which ignores the elastic strain in the plastic zone, has a slight error in borehole displacement but is convenient for engineering calculation.

Keywords: Borehole drilling; Cylindrical cavity; Extended SMP criterion; Three-dimensional solution; Radius; Intermediate principal stress; Slurry wall protection
1. Introduction

In many geotechnical engineering constructions, such as bored piles\textsuperscript{[1]} and wellbore drillings\textsuperscript{[2]}, the excavation is considered a contraction of vertical boreholes. The theory of cavity expansion has been widely applied to solving geotechnical problems containing drilling holes, since its establishment by pioneering scholars\textsuperscript{[3-5]}. Yu et al.\textsuperscript{[6]} proposed a drained cavity contraction solution based on the Mohr–Coulomb criterion, which provides models for predicting soil behavior around tunnels, but it does not consider the effect of intermediate principal stresses. Chen et al.\textsuperscript{[7]} presented a closed-form elastoplastic solution for wellbore stability problems in Drucker-Prager rocks considering strain hardening or softening. Zhao et al.\textsuperscript{[8]} obtained an elastoplastic solution based on the unified strength theory that considers different degrees of intermediate principal stress, however, the intermediate principal stress magnitude cannot be obtained. Many existing solutions use assumptions to simplify the calculation process and do not account for variations in borehole stress and displacement as well as soil depth.

Matsuoka et al.\textsuperscript{[9]} improved the Spatially Mobilized Plane (SMP) theory to Extended SMP theory in 1995. The Extended SMP criterion can effectively overcome the singularity in the deviatoric stress plane, which is more beneficial in numerical calculation. Based on further research by scholars\textsuperscript{[10–11]}, the Extended SMP criterion under plane strain can be derived as follows:

\[
\begin{align*}
\frac{\sigma_1 + \sigma_0}{\sigma_3 + \sigma_0} &= \frac{1}{4} \left( \frac{8 \tan^2 \varphi + 9}{1 - \frac{8 \tan^2 \varphi + 6 - 2 \sqrt{8 \tan^2 \varphi + 9}}{8 \tan^2 \varphi + 9}} \right)^2 = \alpha \quad \text{MERGEFORMAT (1)} \\
\sigma_3 + \sigma_0 &= \sqrt{(\sigma_1 + \sigma_0)(\sigma_1 + \sigma_0)} \quad \text{MERGEFORMAT (2)} \\
\sigma_0 &= c / \tan \varphi \quad \text{MERGEFORMAT (3)}
\end{align*}
\]

where \(\sigma_1, \sigma_2,\) and \(\sigma_3\) are the major, intermediate, and minor soil principal stresses, respectively, \(\sigma_0\) is the bonding stress point, \(c\) and \(\varphi\) are the soil cohesion and internal friction angle, and \(\alpha\) is the coefficient of the relationship between major and minor principal stresses.

In this study, a cylindrical cavity contraction solution based on the Extended SMP criterion in the fully drained soil is established, and the three-dimensional (3-D) stress and displacement values along the borehole depth are obtained by combining with the classical earth pressure formula. In the elastoplastic derivation stage, the exact and approximate solutions of neglecting elastic strain in the plastic region are both provided to address the effect of our hypothesis on cavity contraction problems. The effects of the borehole-unloading degree, shear strength, and dilatancy of soil on the borehole contraction with depth are addressed in the parametric analysis. The results show that the proposed solution could reasonably consider the influence of the intermediate principal stress, and thus could be used to model the soil responses induced by bored pile excavation, as well as to analyze the stability of drilled wellbores.

2. Problem definition

A single borehole is assumed to exist along the vertical direction in the infinite soil with an initial cylindrical cavity radius, \(a_0\). The horizontal soil stress distribution in the soil mass is homogeneous and transversely isotropic. In addition, the classical formula of earth pressure at rest is adopted in this study and is expressed as follows:

\[
P_0(z) = K_0 \gamma z \quad \text{MERGEFORMAT (4)} \\
K_0 = 1 - \sin \varphi \quad \text{MERGEFORMAT (5)}
\]

where \(P_0\) is the initial horizontal soil stress, \(z\) is the soil depth, \(\gamma\) is the soil’s unit weight, \(K_0\) is the coefficient of earth pressure at rest.

Figure 1 shows that the pressure on the cavity wall in a certain horizontal plane decreases slowly form \(P_0\) to \(P\) (\(0 \leq P < P_0\)), and the cavity radius contracts to \(a\) accordingly. Finally, a plastic region represented as \([a, R]\) is formed around the cavity. \(u(r)\) denotes the radial displacement at a distance, \(r\), from the cavity center. In this cavity contraction model, the soil stress and displacement can be...
solved as a plane strain problem. The 3-D solution of the borehole can be obtained ultimately by substituting equation (4) into the plane solution. Notably, the compression positive convention is used in this study.

Figure 1. Schematic of cavity elastoplastic contraction in a horizontal plane.

3. Cylindrical cavity contraction analysis

3.1. Elastic stage

Considering an element at a radial distance, \( r \), from the center of a cavity, its elastic stresses and strains during unloading must satisfy the equilibrium equation, stress boundary conditions, stress–strain relationship, and geometric equation:

\[
\frac{\partial \sigma_r}{\partial r} + \left( \frac{\sigma_r - \sigma_\theta}{r} \right) = 0
\]

\[
\begin{cases}
\sigma_r(a) = P \\
\sigma_r(\infty) = P_0
\end{cases}
\]

\[
\dot{\epsilon}_r = \left[ (1-\nu^2) / E \right] \left[ \sigma_r - \nu \sigma_\theta / (1-\nu) \right]
\]

\[
\dot{\epsilon}_\theta = \left[ (1-\nu^2) / E \right] \left[ \sigma_\theta - \nu \sigma_r / (1-\nu) \right]
\]

\[
\begin{cases}
\dot{\epsilon}_r = -\dot{\epsilon}_u / \dot{\epsilon}_r \\
\dot{\epsilon}_\theta = -u / r
\end{cases}
\]

where \( \sigma_r \) and \( \sigma_\theta \) are the radial and hoop stresses, respectively, \( \dot{\epsilon}_r \) and \( \dot{\epsilon}_\theta \) are the radial and hoop strain increments, respectively, \( E \) is Young’s modulus, \( \nu \) is Poisson’s ratio, \( \epsilon_r \) and \( \epsilon_\theta \) are the radial and hoop strains.

By combining Equations (6)–(9), the soil elastic stress and displacement fields around the cavity can be described as:

\[
\begin{cases}
\sigma_r = P_0 + (P - P_0)(a / r)^2 \\
\sigma_\theta = P_0 - (P - P_0)(a / r)^2 \\
u = \left[ (P - P_0) / (2G) \right](a / r)^2 r \\
G = E / \left[ 2(1+\nu) \right]
\end{cases}
\]

where \( G \) is the shear modulus.

The hoop stress, \( \sigma_\theta \), vertical stress, \( \sigma_z \), and radial stress, \( \sigma_r \), are usually assumed to be \( \sigma_1 \), \( \sigma_2 \) and \( \sigma_3 \) respectively, for cylindrical cavity contraction as the soil approaches yielding. Thus, the yielding condition of the Extended SMP criterion can be written as follows:
In the outward equation the C analysis the first derive elastic flow strain from the cavity wall when

\[ P = P_0 \left[ 2P_0 + (1 - \alpha) \sigma_0 \right] / (1 + \alpha) \]

\* MERGEFORMAT (14)

3.2. Elastoplastic stage

3.2.1. Stress and displacement analyses in the elastic zone. The plastic zone will expand outward from the cylindrical cavity wall (Figure 2) as \( P \) continues to decrease. A new cavity with a current pressure of \( P_0 \) and a radius of \( R \) can be considered in soil. The soil stress and displacement fields in the elastic zone can be written in the form of Equations (10) and (11):

\[ \begin{align*}
\sigma_r &= P_0 + (1 - \alpha)(P_0 + \sigma_0) / (1 + \alpha)(R / r)^2 \\
\sigma_\theta &= P_0 - (1 - \alpha)(P_0 + \sigma_0) / (1 + \alpha)(R / r)^2 \\
u &= (1 - \alpha)(P_0 + \sigma_0) / \left[ 2(1 + \alpha)G \right](R / r)^2 \end{align*} \]

\* MERGEFORMAT (15)

\* MERGEFORMAT (16)

Particularly,

\[ R_0 / R = 1 - (1 - \alpha)(P_0 + \sigma_0) / \left[ 2(1 + \alpha)G \right] = C_1 \]

\* MERGEFORMAT (17)

3.2.2. Stress analysis in the plastic zone. In the range of \([a, R]\), the soil stress components should satisfy both the stress equilibrium equation and the yield condition. Substituting Equation (13) into (6) and integrating it gives the following:

\[ \begin{align*}
\sigma_r &= C_0 r^{a-1} - \sigma_0 \\
\sigma_\theta &= C_0 a r^{a-1} - \sigma_0 \\
\end{align*} \]

\* MERGEFORMAT (18)

where \( C_0 \) is an integration constant depending on the unloading state, which can be obtained using the two boundary conditions in the cavity wall and yielding surface:

\[ C_0 = (P + \sigma_0) / a^{a-1} = \left( P_0 + \sigma_0 \right) / R^{a-1} \]

\* MERGEFORMAT (19)

Further, the relationship between the plastic zone radius and cavity wall can be expressed as:

\[ R / a = \left[ 2(P_0 + \sigma_0) / \left[ (1 + \alpha)(P + \sigma_0) \right] \right]^{1/a-1} = C_2 \]

\* MERGEFORMAT (20)

3.2.3. Approximate analysis of displacement in the plastic zone. In the case of cylindrical cavity contraction, the nonassociated plastic flow rule can be defined as\(^6\):

\[ \begin{align*}
\dot{\varepsilon}_r^p &\dot{\varepsilon}_\theta^p = -\beta \\
\beta &= \frac{1}{4} \left( \sqrt{8 \tan^2 \psi + 9} - 1 + \sqrt{8 \tan^2 \psi + 6 - 2 \sqrt{8 \tan^2 \psi + 9}} \right)^2 \]

\* MERGEFORMAT (21)

\* MERGEFORMAT (22)

where \( \dot{\varepsilon}_r^p \) and \( \dot{\varepsilon}_\theta^p \) are the radial and hoop plastic strain increments, respectively, \( \psi \) is the soil dilatancy angle. For cohesive soils, \( \psi \) is equal to 0.

In several previous studies, the contribution of elastic deformation for cavity contraction problems in the plastic region is considered minimal and therefore ignored\(^6, 12\). In this section, this assumption will be adopted to derive the approximate displacement solution in the plastic zone. Because the initial strain is 0, it is easy to obtain as follows:

\[ \dot{\varepsilon}_r / \dot{\varepsilon}_\theta = \dot{\varepsilon}_r / \dot{\varepsilon}_\theta = -\beta \]

\* MERGEFORMAT (23)

However, finite strains need to be clearly defined considering the large strain effects in the plastic zone. The logarithmic definition of large strain is used:
Substituting Equation (24) into (23), and integrating both sides in ranges \([a, r] \) and \([a_0, r_0] \), respectively, gives the following:

\[
\begin{align*}
\varepsilon_r &= -\ln\left(\frac{r}{r_0}\right) \quad \varepsilon_\theta &= -\ln\left(\frac{r}{r_0}\right) \\
\end{align*}
\]

\# MERGEFORMAT (24)

Substituting Equation (24) into (23), and integrating both sides in ranges \([a, r] \) and \([a_0, r_0] \), respectively, gives the following:

\[
\begin{align*}
r^{1+\beta} - a^{1+\beta} &= r_0^{1+\beta} - a_0^{1+\beta} \\
\end{align*}
\]

\# MERGEFORMAT (25)

Substituting \( r = R \) and \( r_0 = R_0 \) into the aforementioned equation results in the displacement relationship between the radius of the cavity wall and yielding surface:

\[
a^{1+\beta} - a_0^{1+\beta} = R^{1+\beta} - R_0^{1+\beta} \\
\]

\# MERGEFORMAT (26)

From Equations (17) and (20), the current cavity radius can be expressed as:

\[
a / a_0 = \left[1 - C_1^{1+\beta} \left(1 - C_1^{1+\beta}\right)\right]^{\frac{1}{1+\beta}} \\
\]

\# MERGEFORMAT (27)

3.2.4. Rigorous analysis of displacement in the plastic zone. In this section, the elastic strain of the plastic zone is considered to derive the exact solution of the displacement in the plastic zone. According to equation (21):

\[
\left(\dot{\varepsilon}_r - \dot{\varepsilon}_r^*\right) / \left(\dot{\varepsilon}_\theta - \dot{\varepsilon}_\theta^*\right) = -\beta \\
\]

\# MERGEFORMAT (28)

where \( \dot{\varepsilon}_r^* \) and \( \dot{\varepsilon}_\theta^* \) denote the radial and hoop elastic strain increments.

The elastic stress–strain relation, equation (8), are substituted into (28) as follows:

\[
\dot{\varepsilon}_r + \beta \dot{\varepsilon}_\theta = \frac{1 - \nu^2}{E} \left(1 - \frac{\beta \nu}{1 - \nu}\right) \dot{\sigma}_r + \frac{1 - \nu^2}{E} \left(\beta - \frac{\nu}{1 - \nu}\right) \dot{\sigma}_\theta \\
\]

\# MERGEFORMAT (29)

Integrating both sides of equation (29), considering that total strain is 0 and \( \sigma_r = \sigma_\theta = P_0 \) before unloading, gives the following:

\[
\dot{\varepsilon}_r + \beta \dot{\varepsilon}_\theta = \frac{1 - \nu^2}{E} \left(1 - \frac{\beta \nu}{1 - \nu}\right) \sigma_r + \frac{1 - \nu^2}{E} \left(\beta - \frac{\nu}{1 - \nu}\right) \sigma_\theta - \frac{(1 + \nu)(1 - 2\nu)}{E} P_0 (1 + \beta) \\
\]

\# MERGEFORMAT (30)

By substituting the definition of large strain, equation (24), and stress, equation (18), into the left and right sides of the above equation, respectively, the displacement relationship can be obtained as follows:

\[
-\ln\left(\left(\frac{r}{r_0}\right) / (r / r_0)^\beta\right) = \lambda (r / a)^{a-1} - \zeta \\
\]

\# MERGEFORMAT (31)

\[
\begin{align*}
\lambda &= \frac{1 - \nu^2}{E} (P + \sigma_0) \left[1 - \frac{\beta \nu}{1 - \nu} + \alpha \left(\beta - \frac{\nu}{1 - \nu}\right)\right] \\
\zeta &= \frac{(1 + \nu)(1 - 2\nu)}{E} (P_0 + \sigma_0)(1 + \beta) \\
\end{align*}
\]

\# MERGEFORMAT (32)

Let

\[
y = (r / a)^{a-1} \\
\]

\# MERGEFORMAT (33)

Substituting the new independent variable, equation (33), into equation (31), and integrating it over ranges \([a, r] \) and \([a_0, r_0] \) gives the following:

\[
\left[\exp\left(-\zeta\right) a^{1+\beta} / (\alpha - 1)\right] \int_0^{y^\frac{1}{a - 1}} \exp(\lambda y) y = \left(a^{1+\beta} - a_0^{1+\beta}\right) / (1 + \beta) \\
\]

\# MERGEFORMAT (34)

The above equation can be further simplified using the series:

\[
\exp(\mu r) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} r^n \\
\]

\# MERGEFORMAT (35)

By setting \( r = R \) , \( r_0 = R_0 \) and substituting Equations (17) and (20) into (34), the displacement relation at the cavity wall can be determined:
\[ \frac{a}{a_0} = \left[ (C_1 C_2)^{1+\beta} - \left[ (1 + \beta)/(\alpha - 1) \right] \exp\left( -\zeta \sum_{n=0}^{\infty} A_{n+1} \right) \right]^{1/\alpha} \] \hspace{1cm} \text{MERGEFORMAT (36)}

\[ A_{n+1} = \frac{(-1)^{\alpha - 1} \ln C_2}{(n + 1 + \beta)/(\alpha - 1)} \] \hspace{1cm} \text{MERGEFORMAT (37)}

4. Solution validation and parametric analysis

In this section, an 80-m deep borehole is employed, and the unloading ratio is assumed to be the same as the depth increases, which is taken as \( P/P_0 = 0.3 \). The basic model parameters are chosen as follows: \( a_0 = 1 \text{ m} \), \( p = 19.6 \text{ kN/m}^3 \), \( E = 25 \text{ MPa} \), \( \nu = 0.3 \), \( \varphi = 25^\circ \), \( c = 10 \text{ kPa} \), \( \psi = 0^\circ \).

4.1. Solution validation

To validate and discuss the presented solution’s characteristics, Yu’s Mohr–Coulomb\(^6\) and Zhao’s Unified Strength Theory solutions\(^8\) are compared. In both solutions, the assumption of ignoring elastic strain in the plastic zone was used. Figure 2 shows that the borehole wall displacement in all models increases with depth, with an almost linear variation trend. Figure 3 shows that with a value of 4.04% at the 80-m depth, the contraction degree of the presented exact solution is 7.76% less than Yu’s Mohr–Coulomb solution. Mohr–Coulomb criterion tends to be conservative in cavity contraction problems because it does not consider the intermediate principal stress effect. The approximate Extended SMP solution lies between Zhao’s two solutions, in which the intermediate principal stress coefficient, \( b \), is taken as 0.5 and 1. It indicates that the Extended SMP criterion can reasonably consider the intermediate principal stress effect. The approximate solution leads to a larger borehole radius than the exact solution, however, it is acceptable to some extent and convenient for engineering calculation.

The radial stress is minimized at the borehole wall due to radial unloading and the hoop stress is maximized at the yield surface at each depth (Figure 3). In a particular plane, radial stress outside the borehole and hoop stress within the yield surface of the proposed Extended SMP solution are both larger than those of Yu’s Mohr–Coulomb solution. The yield surface gradually moved further from the borehole wall as soil depth increased, reducing the ratio of the soil stress around the borehole to the initial soil stress, however, the absolute magnitude is much higher than that at a shallower depth.

4.2. Parametric analysis

![Figure 2. Prediction of borehole wall displacement with depth in different models.](image)

![Figure 3. Distribution of stress field at different depths in different models.](image)
4.2.1. Effect of unloading degree on borehole contraction. Figure 4 shows that when the unloading ratio decreases, the slope of the borehole wall displacement curve against depth increases, and the increasing degree becomes more severe. At a depth of 80 m, the contraction will be reduced by 21.4% and 36.8% if the unloading ratio increases from 0.4 to 0.5 and 0.6, respectively.

Figure 5 shows that when the unloading ratio is equal to 0.6 or 0.5, the radial and hoop stresses at the borehole wall remain relatively high because the soil is in a completely elastic or a low degree of elastoplastic state. Further, slurry wall protection technology is usually applied during drilling construction in bored piles. Assuming that the supporting effect of slurry on the borehole wall is mainly provided by its hydrostatic pressure, and this support is constant during the drilling process, then the borehole-unloading ratio can be regarded as being closely related to the slurry weight:

\[
P / P_0 = \frac{\gamma_s}{K_0} \]

where \( \gamma_s \) is the slurry’s unit weight; the unloading ratio is independent of depth. Thus, the selected slurry’s weight should not be too small to maintain the borehole wall elastic or with a low degree of elastoplasticity.

4.2.2. Effect of soil shear strength on borehole contraction. Figure 6 shows that the borehole wall displacement increases with a decrease in soil cohesion and internal friction angle. A moderating segment exists in the shallow soil layer of the displacement curve when \( C = 30 \) kPa, and the moderating segment decreases with cohesion to 20 and 10 kPa. However, cohesion exhibits minimal impact on the slope of the rest of the curve. A decrease in internal friction angle results in a sharp increase in displacement (from 4.04% to 9.02% at depth of 80 m) and has a more direct effect. Figure 7 shows that when the soil strength parameters decrease, the radial and hoop stresses around the borehole exhibit different degrees of decrease. Stress loss around the borehole is more severe in soils with lesser strength under the same unloading ratio, and it is necessary to increase the weight of the slurry to support the borehole wall to ensure engineering safety.
Figure 6. Variation of borehole wall with depth displacement at different internal friction angles and cohesions.

Figure 7. Distribution of stress field at depth of 80 m with different internal friction angles and cohesions.

5. Conclusion
1. The contraction degree in this study is lower than the Mohr–Coulomb solution when the same parameters are used. Extended SMP solution can reasonably consider the influence of the intermediate principal stress. The proposed approximate solution has a slight borehole displacement error but is convenient for engineering calculation.

2. The unloading ratio has a significant influence on borehole radius contraction and stress weakening. It is recommended to increase the unit weight of slurry during drilling.

3. The borehole displacement increases with a decrease in soil shear strength, and the stress state around the borehole weakens. The influence of internal friction angle has a greater impact on deep soil than cohesion.

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