Superstring Threshold Corrections to Yukawa Couplings

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Abstract

A general method of computing string corrections to the Kähler metric and Yukawa couplings is developed at the one-loop level for a general compactification of the heterotic superstring theory. It also provides a direct determination of the so-called Green-Schwarz term. The matter metric has an infrared divergent part which reproduces the field-theoretical anomalous dimensions, and a moduli-dependent part which gives rise to threshold corrections in the physical Yukawa couplings. Explicit expressions are derived for symmetric orbifold compactifications.

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1. Introduction

A fundamental task for the heterotic superstring theory is the determination of the effective action describing the physics of massless string excitations at low energies. This is necessary for the phenomenological applications of string theory, in particular for the unification of gauge interactions. The most general $N = 1$ supergravity action, describing local interactions involving up to two space-time derivatives, is characterized by three functions of chiral superfields: the real Kähler potential $K$ which determines the kinetic terms, the analytic superpotential $W$ related to the Yukawa couplings, and the analytic function $f$ associated with the gauge couplings [1]. From the phenomenological point of view, the most important issue is the dependence of these functions on the moduli-fields. The reason is that the vacuum expectation values (VEVs) of the moduli, which are completely arbitrary in perturbation theory, may drastically affect the values of the observable couplings. This moduli-dependence is restricted by the space-time duality symmetry of the four-dimensional heterotic string theory.

The gauge couplings do not depend on the moduli at the tree-level – they are determined by the dilaton VEV. The moduli-dependence of the radiative corrections to gauge couplings satisfies a non-renormalization theorem [2], namely, it is given entirely by the one loop contributions [3, 2]. The gauge $f$-functions can be determined by using duality symmetry, from the violation of the integrability condition of the corresponding $\Theta$-angles. The gauge group-dependent part of the latter is due to the presence of anomalous diagrams involving massless particles, and can be calculated at the level of the effective field theory [1, 3]. The $f$-functions as well as the related string threshold corrections have been explicitly determined for the symmetric orbifold compactifications, and the general procedure is very clear.

The physical Yukawa couplings depend on the superpotential $W$, as well as on the Kähler potential $K$ through wave functions. At the tree level, both $W$ and $K$ are known
in many cases (including symmetric orbifolds), up to the terms trilinear and quadratic
in matter fields, respectively. In (2,2) models, the $N = 2$ world-sheet supersymmetry of
the bosonic sector implies strong tree-level relations between these two functions \[^\mathbb{E}\]. In
particular, the Kähler manifold of moduli-fields exhibits the so-called special geometry
\[^\mathbb{F}\], which relates the moduli Kähler potential with the matter superpotential.

In this work we study the moduli-dependence of the radiative corrections to physical
Yukawa couplings. As a consequence of non-renormalization theorems, the superpotential
does not receive any loop corrections \[^\mathbb{G}\]. Hence it is sufficient to consider the higher genus
string contributions to the Kähler metric. In section 2, we describe the general method
of the computation of matter and moduli metrics and we obtain the general one-loop
expression. The method is based on the evaluation of the three-point amplitude involving
the two-index antisymmetric tensor and two scalar fields. In section 3, we show that
the above expression coincides with the universal part of the violation of the integrability
condition for $\Theta$-angles. This result is not surprising: in fact our method provides a direct
string evaluation of the so-called Green-Schwarz term \[^\mathbb{H}\]. We also point out a connection
to the $\text{Tr}F(-1)^F$ “index” of the underlying $N = 2$ superconformal theory \[^\mathbb{I}\]. In section
4, we analyze the infrared divergencies present in the matter metric and we identify the
 corresponding contribution to the field-theoretical one-loop beta functions of the Yukawa
couplings. We also define the threshold corrections. In sections 5 and 6, we consider the
example of symmetric orbifolds. In section 5, we derive explicit expressions for the one-loop
metric of the untwisted moduli, and we point out that special geometry is violated beyond
the tree approximation. In section 6, we derive the one-loop metric for the untwisted matter
fields and we determine the threshold corrections to their Yukawa couplings.
2. One-loop Kähler metric and physical Yukawa couplings

In the standard formulation of $N = 1$ supergravity [1], all massless scalars and pseudoscalars form complex fields $z$ which together with their fermionic partners $\chi$ belong to chiral supermultiplets $Z$. They parametrize a Kähler manifold of a non-linear sigma-model with the metric $K_{ij} = \partial_z \partial_{\bar{z}} K(z, \bar{z})$. At the tree-level, the Kähler metric can be extracted from string scattering amplitudes involving four or more complex scalar fields [6]. Such a computation becomes very complicated beyond the tree approximation. On the other hand, direct computation of three-point Yukawa couplings requires fermion vertices which are quite difficult to handle. Another complication is due to the existence of the dilaton which belongs to a very distinct supersymmetry multiplet, together with the two-index antisymmetric tensor which is equivalent to a pseudoscalar axion. Since the dilaton VEV plays the role of the string loop expansion parameter, string loop corrections give rise to kinetic terms that mix the moduli with the dilaton and axion. This complicates the analysis of the moduli-dependence of the Kähler potential. As we explain below, the simplest way to avoid all these problems is to represent the dilaton supermultiplet by a real linear superfield $L$, satisfying the constraint $\mathcal{D}^2 L = \mathcal{D}^2 \bar{L} = 0$, in the rigid supersymmetry notation. The linear representation makes direct use of the antisymmetric tensor, which is natural in view of the form of the corresponding string vertex operator.

In the linear formulation, the kinetic terms originate, up to one-loop order, from the $D$-density of the function [11]:

$$
\mathcal{L} = - \left( \frac{L}{2} \right)^{-1/2} (\Sigma \Sigma)^{3/2} e^{\frac{1}{2} G^{(0)}(Z, \bar{Z})} + \frac{L}{2} G^{(1)}(Z, \bar{Z}),
$$

(2.1)

where $\Sigma$ is the so-called chiral compensator field [1, 12], and $G^{(0)}, G^{(1)}$ are real functions of chiral superfields. As usual, the chiral compensator is fixed by normalizing the gravitational kinetic energy term to $-\frac{1}{2} R$. This reduces to the following condition for the scalar

\footnote{Here, we do not discuss the $z$-independent corrections to the dilaton kinetic term.}
components:

\[ \sigma^3 = \sqrt{l/2} e^{\frac{l}{2}G(0)(z,\bar{z})}, \]  

(2.2)

where \( l \) is the exponential of the dilaton, so that its VEV determines the four-dimensional string coupling constant, \( \langle l \rangle = g^2 \). Note that the chiral compensator \( \sigma \) does not depend on the function \( G^{(1)} \). With this normalization, the bosonic part of the kinetic energy terms is given by:

\[ L_b = -\frac{1}{4l^2} \partial_\mu l \partial^\mu l + \frac{1}{4l^2} h_\mu h^\mu - G_{ij} \partial_\mu z^i \partial^\mu \bar{z}^j - \frac{i}{2} (G_{ij} \partial_\mu z^j - G_{ij} \partial_\mu \bar{z}^j) h^\mu, \]  

(2.3)

where

\[ h^\mu = \frac{1}{2} \epsilon^{\mu \nu \lambda \rho} \partial_\nu b_{\lambda \rho}, \]  

(2.4)

is the dual field strength of the antisymmetric tensor \( b_{\lambda \rho} \). The function \( G \), which determines the Kähler metric, is:

\[ G(l, z, \bar{z}) = G^{(0)}(z, \bar{z}) + l G^{(1)}(z, \bar{z}). \]  

(2.5)

In eq.(2.3) the subscripts of \( G \) denote differentiations with respect to the corresponding fields.

To relate the linear to the standard formulation, in which the dilaton multiplet is represented by a chiral superfield \( S \), one introduces the latter as a Lagrange multiplier and rewrites the Lagrangian as a \( D \)-density of \( L - L(S + \bar{S})/4 \) [11, 14]. The equation of motion for \( S \) imposes the linearity constraint for \( L \). On the other hand, if one uses first the equation of motion for \( L \), one obtains \( L(S, \bar{S}, Z, \bar{Z}, \Sigma, \bar{\Sigma}) \) as a solution of the equation \( \partial_L L = (S + \bar{S})/4 \). The Lagrangian is then given by the \( D \)-density of \( L - L \partial_L L \) which must be identified with the standard Lagrangian \(-\frac{3}{2} \Sigma \bar{\Sigma} e^{-\frac{1}{4} K(S, \bar{S}, Z, \bar{Z})}\). In this way one obtains \( K = -\ln[S + \bar{S} - 2G^{(1)}] + G^{(0)} \). It is now obvious that the presence of moduli-dependent correction \( G^{(1)} \) induces kinetic terms which mix \( S \) with the moduli. The linear formulation provides therefore a very convenient field basis, in which the dilaton does not mix with any other field even in the presence of non-trivial loop corrections, see eq.(2.3).
The Yukawa interactions originate, in the linear formulation, from the field-dependent fermion mass terms contained in the $F$-density of the function $\Sigma^3 W(Z)$. $W$ is the analytic superpotential, which cannot depend on $L$ and therefore it is completely determined at the tree-level, as expected from supersymmetric non-renormalization theorems. With the chiral compensator normalized as in (2.2), the Yukawa interactions between massless matter fields are:

$$\mathcal{L}_Y = -\frac{1}{2} \sqrt{\frac{l}{2}} \frac{e^{G(0)/2}}{W_{ijk}} \chi^i \chi^j z^k + h.c.$$  \hspace{1cm} (2.6)

where the subscripts of $W$ denote differentiations with respect to the corresponding fields. Here, we assumed unbroken supersymmetry with $W = \partial_z W = 0$ in the vacuum. Note that the above expression depends on the tree-level quantities only. The physical Yukawa couplings, defined by the fermion-fermion-scalar S-matrix elements may receive however loop corrections. They arise from the corrections to the Kähler metric [see (2.5)] which change the wave function normalization factors.

For our purposes, the main advantage of using the linear formulation is that it provides a simple way of computing the loop corrections to the Kähler metric, by considering the three-point amplitude involving two complex scalars and the antisymmetric tensor. Inspection of the last term in eq.(2.3) shows that $G^{(1)}_{zz\bar{z}}$ can be determined from the $CP$-odd part of the correlation function:

$$\langle z(p_1) \bar{z}(p_2) b^{\mu\nu}(p_3) \rangle_{odd} = i \epsilon^{\mu\nu\lambda\rho} p_1 \lambda p_2 \rho G^{(1)}_{zz\bar{z}},$$  \hspace{1cm} (2.7)

where $p_1$, $p_2$ and $p_3$ are the corresponding external momenta. Although this amplitude vanishes for on-shell Minkowski momenta, it can be computed for complex Euclidean momenta, as it was done in similar computations for moduli and gauge bosons \[3\].

In the superstring computation, the amplitude (2.7) receives contributions only from the odd spin structures. The one-loop contribution to the Kähler metric is given by:

$$i \epsilon^{\mu\nu\lambda\rho} p_1 \lambda p_2 \rho G^{(1)}_{zz\bar{z}} = \int \frac{d^2 \tau}{\tau_2} \int_{t_1}^{t_2} [d^2 \zeta_i] \langle V_z(p_1, \zeta_1) V_{\bar{z}}(p_2, \zeta_2) V^{(-1)\mu\nu}_b(p_3, \zeta_3) T_F(0) \rangle_{odd} \hspace{1cm} (2.8)$$
where $\tau = \tau_1 + i\tau_2$ is the Teichmüller parameter of the world-sheet torus and $\Gamma$ its fundamental domain. The vertex operators are

\begin{align}
V_z(p, \zeta) &= : (\Phi_z + ip \cdot \psi \Psi_z) e^{ip \cdot X} : , \\
V_{\bar{z}}(p, \zeta) &= : (\Phi_{\bar{z}} + ip \cdot \psi \Psi_{\bar{z}}) e^{ip \cdot X} : , \\
V_b^{-1 \mu \nu}(p, \zeta) &= : i \partial X^{[\mu} \psi^{\nu]} e^{ip \cdot X} : ,
\end{align}

where the square brackets in the last equation denote antisymmetrization in $\mu, \nu$. $X^\mu$ represent the bosonic space-time coordinates and $\psi^\mu$ are their left-moving fermionic superpartners. The complex space-time scalars $z (\bar{z})$ correspond, in the underlying $N = 2$ internal superconformal theory, to chiral (anti-chiral) $N = 2$ supermultiplets. Their lower components are primary fields $\Psi_z (\Psi_{\bar{z}})$ having dimensions $\left( \frac{1}{2}, 1 \right)$ while their upper components $\Phi_z (\Phi_{\bar{z}})$ have dimensions $(1, 1)$ and are given by $\Phi_z = \frac{1}{2\pi i} \oint \Psi_z$. The primary fields $\Psi_z, \Psi_{\bar{z}}$ define the Kähler metric by their two-point function $\Psi_z(\zeta) \Psi_{\bar{z}}(0) \sim G^{(0)}_{z\bar{z}} / \zeta \bar{\zeta}^2$. Finally, the supercurrent $T_F$ insertion and the $(-1)$ ghost picture for the antisymmetric tensor vertex are due to the odd spin-structure of the amplitude [13]. The supercurrent is given by

\begin{equation}
T_F = : \psi^\alpha \partial X^\alpha : + ... ,
\end{equation}

where we omitted the part corresponding to the internal superconformal theory, as well as the ghost part.

The four space-time zero-modes required in the odd spin structure give rise to the kinematic factor $\epsilon^\nu_{\alpha \lambda \rho} p_1 \lambda p_2 \rho$; hence we can set $p_1 = p_2 = p_3 = 0$ everywhere else. The contraction of $\partial X^\mu$ from the antisymmetric tensor vertex (2.11) with $\partial X_\alpha$ from the supercurrent (2.12) gives $\langle \partial X^\mu(\zeta_3) \partial X_\alpha(0) \rangle = -\delta^\mu_\alpha \pi / 4 \tau_2$. These terms combine to yield the kinematical factor $\epsilon^\mu_{\nu \lambda \rho} p_1 \lambda p_2 \rho$ in the amplitude (2.7). After performing the $\zeta_2$ and $\zeta_3$ integrations and taking into account all the normalizations, we find:

\begin{equation}
G_{z\bar{z}}^{(1)} = \frac{1}{8(2\pi)^3} \int_{\Gamma} \frac{d^2 \tau}{\tau_2^2} \int d^2 \zeta \bar{\eta}(\bar{\tau})^{-2} \langle \Psi_z(\zeta) \Psi_{\bar{z}}(0) \rangle_{\text{odd}},
\end{equation}
where \( \eta \) is the Dedekind eta function, and the correlation function is computed in the internal superconformal theory. The general expression (2.13) allows the determination of the one-loop Kähler metric for any four-dimensional heterotic superstring model.

3. Green–Schwarz term

In this section we will relate the one-loop correction to the Kähler metric with the universal part of the violation of the integrability condition for \( \Theta \)-angles. In the linear formulation, there are two sources of gauge kinetic terms. (i) The \( F \)-term of \( \frac{1}{4} f(Z) W^\alpha W_\alpha \), where \( f \) is an analytic function of chiral superfields and \( W^\alpha \) is the gauge field-strength superfield \([1]\). (ii) In the \( D \)-density of the function \( \mathcal{L} \) in (2.1), the linear superfield \( L \) can be replaced by \( L - 2k\Omega \), where \( \Omega \) is the Chern-Simons (real) superfield \( \langle \bar{D}^2\Omega = W^\alpha W_\alpha \rangle \) \([1]\), and \( k \) is an arbitrary constant. This generates, in the component notation, a gauge kinetic term of the form \(-\frac{k}{4} [1 + G^{(1)}] F_{\mu\nu} F^{\mu\nu}\). Moreover, in the bosonic Lagrangian (2.3), \( h^\mu \) is replaced by \( h^\mu - \frac{k}{2} \omega^\mu \), where \( \omega^\mu \) is the gauge topological current \((\partial_\mu \omega^\mu = F_{\mu\nu} \tilde{F}^{\mu\nu})\). Although \( \omega^\mu \) is not gauge invariant, the invariance of the Lagrangian is ensured by the appropriate transformation property of the antisymmetric tensor, which leaves the combination \( h^\mu - \frac{k}{2} \omega^\mu \) inert.

In heterotic superstring theory, the tree-level coupling of the dilaton dictates the constant \( k \) to be equal to the Kač-Moody level \([14]\). The analytic function \( f \) vanishes at the tree-level and receives corrections only at the one-loop \([2]\). The bosonic Lagrangian terms bilinear in gauge fields are contained in:

\[
\mathcal{L}_g = -\frac{1}{4} \Delta F_{\mu\nu} F^{\mu\nu} + \frac{ik}{4} (G_j^{(1)} \partial_\mu z^j - G_j^{\bar{(1)}} \partial_\mu \bar{z}^j) \omega^\mu - \frac{k}{4l^2} h_\mu \omega^\mu + \frac{1}{4} \text{Im} f F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{3.1}
\]

where

\[
\Delta = k \left[ \frac{1}{l} + G^{(1)} \right] + \text{Re} f \tag{3.2}
\]

corresponds to the inverse square of the gauge coupling. The second term in (3.1) is
called Green-Schwarz term because it can be interpreted as the compactification of the
ten-dimensional Chern-Simons term involved in the Green-Schwarz anomaly cancellation
mechanism [3].

The first term in (3.2) shows that if \( G^{(1)}_{z \bar{z}} \neq 0 \), i.e. if the Kähler metric \( G_{z \bar{z}} \) receives
loop corrections, the gauge couplings become non-harmonic functions of complex fields,
\( \Delta_{z \bar{z}} = kG^{(1)}_{z \bar{z}} \). In this case, the pseudoscalar (axionic) couplings to gauge bosons arising
from the Green-Schwarz term violate the integrability condition for the corresponding \( \Theta \)-angles:
\[
\frac{i}{2} (\partial_{z} \Theta_{z} - \partial_{\bar{z}} \Theta_{\bar{z}}) = \Delta_{z \bar{z}} \neq 0,
\]
(3.3)
where the axionic coupling \( \Theta_{z} \) of the pseudoscalar \( \text{Im}z \) to two gauge bosons is defined by
the \( CP \)-odd part of the amplitude:
\[
\langle A^{\mu}(p_1)A^{\nu}(p_2)z(p_3) \rangle_{\text{odd}} = i\epsilon^{\mu\nu\lambda\rho}p_{1\lambda}p_{2\rho} \Theta_{z}.
\]
(3.4)
Note that if \( \Delta_{z \bar{z}} \neq 0 \), then \( \Theta_{z} \) is not a derivative of any function.

The presence of the one-loop Kähler metric in (3.2) is not the only source of violation
of the integrability condition (3.3). This is due to the existence of anomalous couplings,
generated by the loops of massless particles, which give rise to additional non-local terms in
the effective action [13, 14]. Space-time supersymmetry ensures however the validity of the
relation (3.3) between the field-dependent gauge and axionic couplings [3]. In the following,
by analyzing the integrability condition (3.3) of the axionic couplings computed in string
theory, we isolate the local contribution of the Green-Schwarz term from the non-local,
field-theoretical, contribution. In this way, we rederive the formula (2.13) for the one-
loop Kähler metric as the universal (gauge group-independent) part of \( \Delta_{z \bar{z}} \). This method
provides therefore a direct determination of the Green-Schwarz term, whose existence has
been postulated before in the context of modular anomaly cancellation [4].

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2For our purposes, we consider only the case in which \( z \) is neutral with respect to the gauge
bosons \( A^{\mu, \nu} \).
The one-loop string computation of the amplitude \((3.4)\) yields \(16\):

\[
\Theta_z = \frac{i}{(2\pi)^5} \int_G \frac{d^2\tau}{\tau_2} \int d^2\zeta \bar{\eta}(\bar{\tau})^{-2} \langle (Q^2 - \frac{k}{4\pi\tau_2}) T_F(0) \bar{\Psi}_z(\zeta) \rangle_{\text{odd}}, \tag{3.5}
\]

where \(Q\) is the gauge charge operator. Taking the derivative \(\partial_z \Theta_z\) amounts to inserting \(\int d^2\xi \Phi_{\bar{z}}(\xi)\) [see \((2.10)\)] inside the vacuum average \((3.5)\). In order to evaluate the difference \((3.3)\) we use the relation \(\Phi_{\bar{z}} = \frac{1}{2\pi i} \oint_G 2T_F \bar{\Psi}_z\) and deform the contour of integration. The boundary term vanishes due to the periodicity of the supercurrent \(T_F\), and we are left with two contributions. The first arises when the contour encircles \(\Psi_z(\zeta)\) converting it to \(\Phi_z(\zeta)\). This is the same as \(\partial_z \Theta_{\bar{z}}\) and cancels in the difference \((3.3)\). The second contribution comes when the contour encircles \(T_F(0)\) yielding an insertion of the stress-energy tensor of the internal \(N = 2\) superconformal field theory. This reduces to a total derivative with respect to the Teichmüller parameter \(\tau\). We obtain

\[
\Delta_{zz} = -\frac{i}{2(2\pi)^5} \int_G \frac{d^2\tau}{\tau_2} \int d^2\zeta \bar{\eta}^{-2} \left\{ \partial_\tau \langle (Q^2 - \frac{k}{4\pi\tau_2}) \Psi_z(\zeta) \bar{\Psi}_z(0) \rangle_{\text{odd}} ight. \\
+ \left. \partial_\tau \langle \frac{k}{4\pi\tau_2} \Psi_z(\zeta) \bar{\Psi}_z(0) \rangle_{\text{odd}} - \frac{k}{4\pi\tau_2} \partial_\tau \langle \Psi_z(\zeta) \bar{\Psi}_z(0) \rangle_{\text{odd}} \right\}, \tag{3.6}
\]

where the total derivative term proportional to \(k\) has been added and subtracted, so that the first integrand is modular invariant. Since it is a total derivative in \(\tau\), its contribution to the integral comes only from the boundary of the moduli space, namely the degeneration limit \(\tau_2 \to \infty\). In this limit only the \(Q^2\) term contributes due to the presence of massless particles, while the term proportional to \(k\) vanishes due to the extra \(\tau_2\) suppression. The final result is:

\[
\Delta_{zz} = \frac{-i}{2(2\pi)^5} \int_G \frac{d^2\tau}{\tau_2} \partial_\tau \int d^2\zeta \bar{\eta}^{-2} \langle Q^2 \Psi_z(\zeta) \bar{\Psi}_z(0) \rangle_{\text{odd}} \\
+ \frac{k}{8(2\pi)^5} \int_G \frac{d^2\tau}{\tau_2} \int d^2\zeta \bar{\eta}^{-2} \langle \Psi_z(\zeta) \bar{\Psi}_z(0) \rangle_{\text{odd}}. \tag{3.7}
\]

The above derivation of \((3.7)\) is formal because the intermediate expressions contain short-distance singularities. In the Appendix we rederive \((3.7)\) by using the momentum regularization which respects conformal invariance.
The expression (3.7) for the non-harmonicity of gauge couplings contains two parts. The group-dependent part proportional to $Q^2$ was analyzed in Ref. [16] and was shown to reproduce the field theory computation of one-loop anomalous graphs involving massless particles [4, 5]. The universal part (i.e. the term proportional to $k$) should be identified with the Green-Schwarz term. A comparison with (2.13) shows that this term is given by the one loop correction to the Kähler metric, in agreement with the field-theoretical expression (3.2).

We can further relate the one-loop Kähler potential to the quantity $\text{Tr} F(-1)^F$ of the underlying $N = 2$ superconformal theory [10]. In Ref. [16], it was shown that

$$
\Delta = -\frac{i}{32\pi^2} \int \frac{d^2 \tau}{\tau_2} \eta^{-2} \text{Tr} R F(-1)^F(Q^2 - \frac{k}{4\pi \tau_2}) q^{L_0 - \frac{F}{2\pi} q^{L_0 - \frac{F}{2}}},
$$

where $q = e^{2\pi i \tau}$ and the trace is over the Ramond sector of the internal $N = 2$ superconformal theory with left and right central charges $c$ and $\bar{c}$, respectively. Note that the integral (3.8) is infrared divergent. The coefficient of the logarithmic divergence $\frac{d\tau}{\tau_2}$ is:

$$
\frac{1}{32\pi^2}(-3\text{Tr} Q_V^2 + \text{Tr} Q_M^2),
$$

where the two terms are the contribution of gauginos and matter fermions with $U(1)$-charges $F = \pm 3/2$ and $F = \pm 1/2$, respectively. The expression (3.9) coincides with the field-theoretical one-loop $\beta$-function of gauge couplings in $N = 1$ supersymmetric Yang-Mills theory. Comparing the universal term in (3.8), (3.7) and (2.13), we find that the one-loop Kähler potential is given by:

$$
G^{(1)} = \frac{i}{16(2\pi)^3} \int \frac{d^2 \tau}{\tau_2^2} \eta^{-2} \text{Tr} R F(-1)^F q^{L_0 - \frac{F}{2\pi} q^{L_0 - \frac{F}{2}}}.
$$

It is worth noting that the same quantity was studied in Ref. [10], in the massive case, where $F = F_L - F_R$, as a new kind of “index” for $N = 2$ theories. At the conformal point one can take $F = F_L$ or $F = F_R$ since then both $F_L$ and $F_R$ are conserved: in this case one gets an expression which reminds of a loop space generalization of the Ray-Singer torsion...
for Kähler manifolds. Here, we see that in the context of string theory this quantity plays the role of the one-loop contribution to the physical Kähler metric in the effective $N = 1$ supergravity theory.

4. \textit{β}-functions of Yukawa couplings and threshold corrections

In this section, we compute the infrared divergence in the one-loop correction to the matter metric. The coefficients of these divergencies will be identified as the one-loop anomalous dimensions. These also provide the beta-functions of the physical Yukawa couplings, which are entirely given by the wave function renormalization factors, as already explained in Section 2. We also discuss the threshold corrections.

The infrared divergence in the Kähler metric comes from the $\tau_2 \to \infty$ integration limit in the expression (2.13). In this limit the one-loop two-point correlator degenerates to a sum over four-point functions on the sphere, and the coefficient of the logarithmic divergence $\frac{d\tau_2}{\tau_2}$ is:

$$G_{zz}^{(1)}_{\text{div}} = \frac{1}{8(2\pi)^{4}} \lim_{\tau_2 \to \infty} \frac{1}{\tau_2} \int d\tau_1 \bar{\eta}^{-2} \text{Tr}(-1)^F q^{L_0 - \frac{c}{24}} q^{L_0 - \frac{c}{24}} \int d^2 x x^{-\frac{n}{2}} (\bar{v}^R(0) \Psi_z(x) \bar{\Psi}_z(1) v^R(\infty)),$$

(4.1)

where the trace is over all states of the internal conformal theory in the Ramond sector with vertices $v^R$ and $\bar{v}^R$ (conjugate vertices) in the $(-\frac{1}{2})$ ghost picture. The integration of $x$ is in an annulus within radii $|q|^{\frac{1}{2}}$ and $|q|^{-\frac{1}{2}}$ and the factor $x^{-\frac{1}{2}}$ comes from the transformation of the torus coordinates to the annulus coordinates (recall that $\Psi_z$ has dimension $(\frac{1}{2}, 1)$). Inspection of (4.1) shows that only the massless states which have renormalizable interactions, \textit{i.e.} gauginos and matter fermions contribute in the limit [16].

The contribution of gauginos can be explicitly calculated using the fact that the internal part of their vertices is $e^{\pm i\frac{\sqrt{3}}{2}H} \bar{J}/\sqrt{k}$, where $\bar{J}$ is the Kač-Moody current and $H$ bosonizes the
$N = 2 \ U(1)$ current. The result is:

$$- \frac{2}{k} C_2(R) G^{(0)}_{zz},$$

(4.2)

where $C_2(R)$ is the quadratic Casimir of the representation $R$ to which $z$ belongs.

Let us now consider the contribution of matter fields. One way to extract the $\ln |q| \sim \tau_2$ behaviour of the integral over $x$ appearing in (4.1) is to introduce non-zero external momenta and look for poles in $s$ and $u$ in the limit $t \to 0$. We can then follow the approach of Ref.[16] and relate the four-point function appearing in (4.1) to a physical four-point amplitude involving matter scalars:

$$x^{-\frac{1}{2}} \langle V^R_w(p_1,0)V^{(-1)}_z(p_2,x)V^{(-1)}_z(p_3,1)V^R_w(p_4,\infty) \rangle$$

$$= - \frac{1}{u} \langle V^{(-1)}_w(p_1,0)V^{(0)}_z(p_2,x)V^{(0)}_z(p_3,1)V^{(-1)}_w(p_4,\infty) \rangle \ (4.3)$$

where $u = p_2 \cdot p_4$, and $w$ ($\bar{w}$) denote chiral (anti-chiral) matter fields. A similar relation can be obtained when $w$ and $\bar{w}$ are interchanged, with $s = p_1 \cdot p_2$ replacing $-u$ (the minus sign is due to the exchange of space-time fermions). In the trace of (4.1), one should include both the above terms with a relative minus sign to take $(-1)^F$ into account. Using the expressions for physical four scalar amplitudes given in Ref.[6], and taking the trace over $w$'s, we obtain

$$- \frac{2}{k} C_2(R) G^{(0)}_{zz} + \frac{1}{2} W_{zzzz} G^{(0)}_{zzzz}$$

(4.4)

where $W_{zzzz} = \partial_z \partial_{z1} \partial_{z2} W$. Combining the two contributions (4.2) and (4.4), we find the following expression for the wave function renormalization:

$$-2 \gamma_z = \frac{1}{32 \pi^2} \left\{ - \frac{4}{k} C_2(R) + \frac{1}{2} e^{G^{(0)}} W_{zzzz} G^{(0)}_{zzzz} \right\},$$

(4.5)

with no summation over $z$. In the expression (4.5), the tree-level metric has been factorized out and for notational simplicity we write diagonal elements only. The above result reproduces directly the field-theoretical one-loop beta-function of Yukawa couplings [17]. In the comparison one should take into account the normalization factor appearing in the Yukawa
interactions (2.6). Moreover, the comparison with the field-theoretical anomalous dimensions shows that the string computation implicitly uses a gauge in which the superpotential remains unrenormalized.

When the logarithmic divergence in the $\tau_2$ integration is regularized and compared to the field-theoretical DR scheme, it is converted to $\ln \frac{M^2}{\mu^2}$, where $\mu$ is the infrared cutoff corresponding to the low-energy scale and $M$ is the string unification scale [18]. The remaining finite part gives the string threshold corrections to wave function factors:

$$Y_z = \int \frac{d^2\tau_2}{\tau_2} \left\{ \frac{G^{(0)\bar{z}_1}}{8(2\pi)^3} \int d^2\zeta \bar{\eta}^{-2} \langle \Psi_z(\zeta) \bar{\Psi}_{\bar{z}_1}(0) \rangle_{\text{odd}} + 2\gamma_z \right\}.$$  (4.6)

These corrections determine the boundary conditions of the physical Yukawa couplings $\lambda_{ijk}$ at the unification scale:

$$\lambda_{ijk}(M) = \lambda_{ijk}^{\text{tree}} \left[ 1 + g^2(Y_i + Y_j + Y_k) \right]^{-1/2},$$  (4.7)

where $g = \sqrt{\langle I \rangle}$ is the string coupling constant.

We should point out that in contrast to the case of one-loop gauge couplings, the infrared-divergent part of Yukawa couplings may be in principle moduli-dependent, as seen from (4.5). In this case, the moduli-dependence of threshold corrections cannot be consistently defined.

## 5. Moduli metric in orbifold models

In the last two sections we will consider some examples in orbifold models. In particular, we will compute the one-loop correction to Kähler metric of untwisted moduli and matter. Here, we consider the untwisted moduli, which we call generically $T$. The relevant primary fields that appear in (2.13) are:

$$\Psi_T = \psi_L \mathcal{J} \chi$$  (5.1)

---

3Our arguments apply to both $T$-type and $U$-type moduli in the notation of Ref. [3].
where $X$ is a complex coordinate of an internal plane and $\psi_L$ is its left moving fermionic superpartner. $\bar{\Psi}_T$ is the conjugate of $\Psi_T$.

In the orbifold models one has to sum over all sectors of boundary conditions. The untwisted sector, which respects $N = 4$ space-time supersymmetry, gives vanishing contribution due to the internal fermion zero-modes in the odd spin structure. In general there are also sectors that preserve $N = 2$ space-time supersymmetry, which appear when one of the three internal planes is untwisted under the boundary conditions. Such a sector could contribute if the modulus corresponds to a deformation of the untwisted plane. In this case the moduli vertices provide the two fermion zero-modes. Moreover $X$’s in the vertices are replaced by their classical solutions because the quantum correlator being a total derivative does not contribute. After a Poisson-resummation one can show that the result is:

$$G_{TT}^{(1)} = \frac{1}{(T + \bar{T})^2} \mathcal{I}, \quad \mathcal{I} = \int_\Gamma \frac{d^2 \tau}{\tau_2} \partial_{\tau}(\tau_2 Z) F(\bar{\tau}),$$

where $Z$ is the partition function:

$$Z = \sum_{p_L, p_R} q^{\frac{1}{2} p_L^2} \bar{q}^{\frac{1}{2} p_R^2},$$

with $p_L$ and $p_R$ the left and right momenta, respectively, corresponding to the untwisted plane. Their dependence on $T$ and $\bar{T}$ is well-known \[19\]. $F(\bar{\tau})$ is a meromorphic modular form of weight $-2$ in $\tau$ and it does not depend on the moduli $T$.

Using the arguments of Ref.\[4\], we can derive a differential equation for the quantity $\mathcal{I}$ of (5.2). The identity

$$\partial_T \partial_{\bar{T}} Z(\tau, \bar{\tau}) = \frac{4 \tau_2}{(T + \bar{T})^2} \partial_{\tau} \partial_{\bar{\tau}}(\tau_2 Z)$$

implies

$$\partial_T \partial_{\bar{T}} \mathcal{I} = \frac{4}{(T + \bar{T})^2} \int_\Gamma d^2 \tau \{ \frac{i}{\tau_2} \partial_{\tau} \partial_{\bar{\tau}}(\tau_2 Z) + \partial_{\tau} \partial_{\bar{T}}^2(\tau_2 Z) \} F(\bar{\tau}).$$

By doing a partial integration over $\tau$ and noting that the surface terms vanish, we obtain the differential equation:

$$\partial_T \partial_{\bar{T}} \mathcal{I} = \frac{2}{(T + \bar{T})^2} \mathcal{I}.$$
The general solution of the above equation is:

\[ I = \left[ \frac{2}{(T + \bar{T})} - \partial_T \right] f(T) + \text{c.c.}, \quad (5.7) \]

where \( f \) is a complex function of \( T \).

From equation (5.2) it is clear that \( I \) is invariant under the space-time duality transformations \( T \to T + i \) and \( T \to 1/T \) \[20\]. This implies that \( f \) must be a modular function of weight 2 in \( T \). Furthermore, by comparison with (5.2) one can see that \( f \) has no singularity inside the fundamental domain, while at infinity it can at worst have a powerlike singularity in \( T \). This behaviour is inconsistent with the invariance under \( T \to T + i \) and analyticity. Therefore \( f \) must be holomorphic everywhere. Since a modular function of weight 2 which is holomorphic everywhere does not exist, \( f \) must be zero, which in turn implies that \( I \) must vanish. Hence, there are no corrections to the one-loop moduli metric from \( N = 2 \) sectors.

The remaining sectors preserve \( N = 1 \) space-time supersymmetry. In this case, all the internal coordinates are twisted and the integrand in (2.13) is proportional to the product of quantum correlators:

\[ G^{(1)}_{TT} = \frac{1}{8(2\pi)^4} \int_{\Gamma} \frac{d^2 \tau}{\tau_2^2} \int d^2 \zeta \, \bar{\eta}(\bar{\tau})^{-2} \langle \psi_L(\zeta) \bar{\psi}_L(0) \rangle_{\text{odd}} \langle \bar{\partial}X(\zeta) \partial X(0) \rangle. \quad (5.8) \]

The bosonic correlator in (5.8) is a total derivative with respect to \( \bar{\zeta} \). The \( \zeta \) integration can be performed after regularizing the expression by cutting a disk around the origin, and the result, in the \((g, h)\)-twisted sector is \( -\pi \partial_\zeta \ln \tilde{\theta}_{(g, h)}(\tilde{\zeta})|_{\tilde{\zeta}=0} \) times the partition function of the internal superconformal theory in that sector. Here \( \theta_{(g, h)} \) denotes the odd \( \theta \)-function shifted by the corresponding twists. Since the left moving bosonic and fermionic contributions to the partition function cancel in the odd spin structure, the resulting expression depends on \( \bar{\tau} \) only. Summing over all the twisted sectors one obtains:

\[ G^{(1)}_{TT} = \frac{-G^{(0)}_{TT}}{16(2\pi)^4} \int_{\Gamma} \frac{d^2 \tau}{\tau_2^2} E(\bar{\tau}), \quad (5.9) \]
where we have explicitly extracted the tree-level metric $G^{(0)}_{\tau \tau}$ coming from the contractions of $X$’s and $\psi_L$’s. The anti-analytic function $E(\bar{\tau})$ does not depend on the moduli $T$. It is modular invariant in $\bar{\tau}$ and has at most a simple pole singularity at $\bar{q} = 0$. It is therefore a priori proportional to the $j$ invariant up to an additive constant.

In order to compute the $\tau$-integral in (5.9) we can start from the following identity:

$$
\int_{\Gamma} \frac{d^2 \tau}{\tau_2^2} E(\bar{\tau}) = -4 \int_{\Gamma} d^2 \tau E(\bar{\tau}) \partial_\tau \partial_{\bar{\tau}} \ln(\tau_2 \bar{\eta}^2).
$$

The integrand on the r.h.s. is a total derivative in $\tau$ of a $(0,1)$-form, therefore the integral receives contributions only from the boundary $\tau_2 \rightarrow \infty$ of $\Gamma$. The $\tau_1$ integration then picks the constant term in the $\bar{q}$-expansion of

$$
4iE(\bar{\tau})\partial_{\bar{\tau}} \ln[\bar{\eta}(\bar{\tau})].
$$

Actually this argument applies even if $E(\bar{\tau})$ has higher order poles at $\bar{q} = 0$. Hence, the one-loop moduli metric (5.9) is proportional to the tree level metric, with a coefficient determined by (5.11):

$$
G^{(1)}_{\tau \tau} = -G^{(0)}_{\tau \tau} \frac{4 \pi}{4(2\pi)^4} \lim_{\tau_2 \rightarrow \infty} \int d\tau_1 E(\bar{\tau}) \partial_{\bar{\tau}} \ln[\bar{\eta}(\bar{\tau})].
$$

It is very convenient to relate equation (5.12) with the threshold corrections to the gravitational coupling $\Delta^{grav}$, defined as the coefficient of the Gauss-Bonnet term. In Ref.[16] it was shown that:

$$
\Delta^{grav,FT}_{\tau \tau} = \frac{1}{2(2\pi)^5} \lim_{\tau_2 \rightarrow \infty} \int d\tau_1 \partial_{\bar{\tau}} \ln(\bar{\eta}) \int d^2 \zeta \bar{\eta}^{-2} \langle \Psi_T(\zeta) \bar{\Psi}_T(0) \rangle,
$$

where FT denotes the “field-theoretical” part of the threshold correction. In $N = 1$ sectors, the $\zeta$-integral in (5.13) is identical with the one in (5.8), which is equal to $-\pi G^{(0)}_{\tau \tau} E(\bar{\tau})$. Comparing the resulting expression with (5.12), we conclude that the Green-Schwarz term is equal to the contribution of $N = 1$ sectors to the field-theoretical part of the gravitational threshold correction.
For the moduli which have no $N = 2$ subsectors, as in the case of all untwisted moduli of $Z_3$ orbifold, the Green-Schwarz term is equal to the full field-theoretical contribution to the gravitational threshold correction. The latter has been calculated in [16] and shown to be equal to the group-dependent part of the threshold correction to the $E_8$ gauge coupling. This result is not surprising: the universal threshold correction should cancel in this case against the gauge group-dependent contribution of $N = 1$ sectors in (3.7), since $N = 1$ sectors do not contribute to the corresponding axionic coupling $\Theta_T$ [2]. We have verified in several orbifold examples that for such moduli the function $E$ in (5.1) is $2\pi$ times the $j$-function, which yields the coefficient of $G^{(0)}_{TT}$ in (5.9) equal $-30/16\pi^2$; this is in agreement with the previous arguments. The numerical evaluation of the coefficient in (5.9) can also be done for moduli which have $N = 2$ subsectors in various orbifolds and compared with the coefficient of the Green-Schwarz term in the moduli-dependent threshold corrections to gauge couplings [4, 5]. In the case of a modulus corresponding to a $Z_2$-twisted plane, the quantum correlator appearing in (5.9) is proportional to the $\zeta$-derivative of an even $\theta$-function evaluated at $\zeta = 0$. This is zero by the parity properties of $\theta$-functions, implying that there is no one-loop correction to the moduli metric, in agreement with the vanishing of the Green-Schwarz term in this case.

As a result, the Kähler metric of the untwisted moduli in symmetric orbifolds is renormalized at the one-loop order by a finite multiplicative and calculable constant. Symmetric orbifolds are particular examples of $(2, 2)$ compactifications, which possess $N = 2$ worldsheet supersymmetry in both left and right moving sectors. In this case, the gauge group is $E_6 \times E_8$ and the matter fields transform as 27 or $\overline{27}$ under $E_6$ and they are in one-to-one correspondence with the moduli: 27’s are related to (1, 1) moduli and $\overline{27}$’s to (1, 2) moduli. Furthermore, the moduli metric is block-diagonal with respect to these two types of moduli. An interesting consequence of the right-moving $N = 2$ tree-level Ward-identities is the so-called special geometry, which relates the tree-level moduli metric to the Yukawa
couplings \[ R^{(0)}_{acbd} = G^{(0)}_{ac} G^{(0)}_{bd} + G^{(0)}_{ad} G^{(0)}_{bc} - e^{2G^{(0)}_{a}\nabla c d f} G^{(0)}_{e f}, \] (5.14)

where \( R^{(0)}_{acbd} \) is the Riemann tensor of the moduli metric \( G^{(0)} \), and the above equation holds separately for (1,1) and (1,2) moduli. Every term in (5.14) behaves differently under global rescalings of the metric. Consequently, our results imply that special geometry is in general violated beyond the tree approximation. Note that from field-theoretical point of view, special geometry is a consequence of \( N = 2 \) space-time supersymmetry [7]. This is consistent with the fact that orbifold sectors which preserve \( N = 2 \) space-time supersymmetry do not give rise to one-loop corrections to the moduli metric. As a corollary, special geometry is preserved for moduli of planes which are only \( Z_2 \)-twisted.

### 6. Threshold corrections to Yukawa couplings in orbifold models

In this section we consider the radiative corrections to the metric of untwisted matter fields \( z \). The relevant primary fields that appear in (2.13) have a form similar to (5.1) with \( \bar{\partial}X \) replaced by a right-moving fermion bilinear:

\[
\Psi_z = \psi_L \psi_R \lambda, \tag{6.1}
\]

where \( \psi_{L,R} \) are left and right moving internal fermions, and \( \lambda \) is a right moving fermion generating the \( SO(10) \) part of \( E_6 \). As in the case of moduli, \( N = 4 \) sectors give no contribution, while \( N = 1 \) sectors give contribution proportional to the tree-level matter metric with a constant coefficient. Hence, \( N = 1 \) sectors yield moduli-independent corrections to the physical Yukawa couplings (4.6). The moduli-dependent threshold corrections arise therefore from \( N = 2 \) sectors only, similarly to the case of gauge couplings. Moreover, these sectors must leave invariant the plane associated with the matter field \( z \). The reason is that the \( z \) and \( \bar{z} \) vertices should provide the two fermion zero modes of the untwisted plane, otherwise the correlator vanishes. As a result, the threshold correction \( Y_z \) (4.6) may depend
only on the moduli of the plane associated with $z$. The integral (2.13) then becomes:

$$G^{(1)}_{zz} = \frac{G^{(0)}_{zz}}{8(2\pi)^5} \int d^2\tau \frac{Z(\bar{\eta}(\tau))^{-2}}{\tau_2^2} \int d^2\zeta \langle \psi_R(\bar{\zeta})\bar{\psi}_R(0)\lambda(\bar{\zeta})\lambda(0) \rangle,$$

(6.2)

where we extracted the tree-level matter metric from the various contractions. $Z$ is the partition function (5.3) of the untwisted plane.

After summing over even spin structures of the right-moving fermions, the quantum correlator in (6.2) becomes a one-form in $\bar{\zeta}$ with a double pole at $\bar{\zeta} = 0$. Therefore, the normalized correlator can be written as $-\partial^2_{\bar{\zeta}} \ln \bar{\theta}_1$ up to an additive constant. The $\zeta$-integration can then be performed with the result:

$$G^{(1)}_{zz} = -\frac{G^{(0)}_{zz}}{8(2\pi)^3} \int d^2\tau \frac{Z(\bar{\eta}(\tau))^{-2}}{\tau_2^2} \int d^2\zeta \langle \psi_R(\bar{\zeta})\bar{\psi}_R(0)\lambda(\bar{\zeta})\lambda(0) \rangle,$$

(6.3)

where the operator $\Gamma_z$ present in the trace is the analog of the $Q^2$ operator in the corresponding formula for gauge couplings (3.8). Its eigenvalues on massless states contribute to the coefficient of the infrared divergence which, in section 4, was identified with the one loop anomalous dimension $\gamma_z$. Since (6.3) represents the contribution of the $N = 2$ sectors only, it is actually equal to $\frac{\hat{\gamma}_z}{ind}$. Here, $\hat{\gamma}_z$ is the one-loop anomalous dimension coefficient of the $z$-field in the corresponding $N = 2$ space-time supersymmetric theory with the orbifold defined by the little group of the unrotated plane, and $ind$ is the index of this little group in the full orbifold group. After subtracting this infrared divergence, we obtain the threshold correction $Y_z$ (4.6) in a way explained in Section 4.

In order to determine the moduli-dependence of $Y_z$ we follow the method used in the case of gauge couplings [2]. We first differentiate the integral in (5.3) with respect to $T$ and $\bar{T}$, and using the identity (5.4) we obtain:

$$\partial_T \partial_{\bar{T}} Y_z = -\frac{1}{2(2\pi)^3} \frac{1}{(T + \bar{T})^2} \int d^2\tau \partial_\tau (\tau_2 Z(\bar{\eta}(\tau))^{-2}) \int d^2\zeta \langle \psi_R(\bar{\zeta})\bar{\psi}_R(0)\lambda(\bar{\zeta})\lambda(0) \rangle,$$

(6.4)

The second term in the r.h.s. of (6.4), after partial integration with respect to $\tau$, acquires the form of the integral $\mathcal{I}$ appearing in (5.2) and vanishes by the same reasoning. The
first term in the r.h.s. of (6.4) is a total derivative with respect to \( \tau \) and therefore receives contributions from massless states only. The situation becomes similar to the case of gauge couplings when one considers the violation of the integrability condition for \( \Theta \)-angles. Hence, one can replace \( \bar{\eta}^{-2} \text{Tr} \Gamma_z q^{L_0 - \frac{c}{24}} \) by \( \frac{\hat{\gamma}_z}{\text{ind}} \). The boundary integration in (6.4) then gives:

\[
\partial_T \partial_T Y_z = - \frac{2\hat{\gamma}_z}{\text{ind}} \frac{1}{(T + \overline{T})^2}.
\] (6.5)

By integrating the differential equation (6.5) and using duality invariance, one obtains the following formula for the threshold corrections to the wave function factors \( Y_z \) of untwisted matter fields:

\[
Y_z = \frac{2\hat{\gamma}_z}{\text{ind}} \ln|\eta(iT)|^4(T + \overline{T})| + y_z,
\] (6.6)

where \( y_z \) is a moduli-independent constant.

The coefficients \( \hat{\gamma}_z \) are anomalous dimensions of untwisted scalar fields \( z \) in an \( N = 2 \) supersymmetric orbifold. These fields belong to \( N = 2 \) vector supermultiplets. For instance if \( z \) is in the 27 representation of \( E_6 \), in the corresponding \( N = 2 \) theory it belongs to a gauge vector multiplet of \( E_7 \). Consequently, its anomalous dimension \( \hat{\gamma}_z = -\hat{b}_z^i / 2 \), where \( \hat{b}_z^i \) is the corresponding beta function coefficient [3] of any subgroup that transforms \( z \) non-trivially; \( i \) denotes the plane associated with \( z \).

As an example consider the Yukawa coupling between three untwisted 27’s of \( E_6 \). At the tree-level this coupling is:

\[
\lambda_{ijk}^{\text{tree}} = \frac{g}{\sqrt{2}} W_{ijk},
\] (6.7)

where \( W_{ijk} \) are constants which are non zero only if the three 27’s are associated with three different planes. In this case (6.6) combined with (4.7) gives the boundary condition:

\[
\lambda_{ijk}(M) = \frac{g_{E_6}(M)}{\sqrt{2}} W_{ijk} [1 + g_{E_6}^2(M) y_{ijk}]^{-1/2},
\] (6.8)

where \( y_{ijk} \) are moduli-independent constants, and \( g_{E_6}(M) \) is the one-loop \( E_6 \) gauge coupling [3, 2]:

\[
\frac{1}{g_{E_6}^2(M)} = \frac{1}{g^2} - \sum_i \frac{\hat{\gamma}_{E_6}^i}{(\text{ind})^i} \ln[(\eta(iT^i))^4(T^i + \overline{T}^i)] + c_{E_6},
\] (6.9)
with $c_{E_6}$ being another moduli-independent constant. As a result, the boundary relation between the untwisted Yukawa couplings and the $E_6$ gauge coupling at the unification scale does not receive any moduli-dependent corrections at the one loop level.

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**Appendix**

Here, we rederive (3.7) by using the momentum regularization of short-distance singularities. The derivative $\partial_\zbar \Theta_\zbar$ is given by:

$$
i \epsilon^{\mu \nu \lambda \rho} p_1 \lambda p_2 \rho \partial_\zbar \Theta_\zbar =$$

$$\int \frac{d^2 \tau}{\tau_2} \lim_{p_4 \to 0} \int \prod_{i=1}^4 d^2 \zeta_i \left\langle V'^{\mu}(p_1, \zeta_1) V'^{\nu}(p_2, \zeta_2) V^{(-1)}(p_3, \zeta_3) V(p_4, \zeta_4) \mathcal{T}_F(0) \right\rangle_{\text{odd}}^{\text{1PI}},$$

(A.1)

where the vertices are given in (2.9) and by:

$$V^{(-1)}_\zbar(p, \zeta) = : \Psi \zbar e^{ip \cdot X} :,$$

(A.2)

$$V_\zbar^\mu(p, \zeta) = : \bar{J}(\bar{c})(\partial X^\mu + ip \cdot \bar{\psi} \gamma^\mu) e^{ip \cdot X} :,$$

(A.3)

with $\bar{J}$ being the Kač-Moody current. The one-particle irreducible (1PI) amplitude (A.1) is obtained from the full amplitude by subtracting the reducible diagram involving the exchange of the antisymmetric tensor between the tree-level $b$-$A$-$A$ and the one-loop $b$-$z$-$\bar{z}$ vertices. In the $p_4 \to 0$ limit, this subtraction procedure is gauge invariant since the intermediate antisymmetric tensor enters on-shell. More explicitly, in the Feynman gauge for the antisymmetric tensor this amounts to subtracting from the full amplitude the
following expression:

\[ \langle A^\mu(p_1)A^\nu(p_2)z(p_3)\bar{z}(p_4) \rangle_{\text{odd}}^{1\text{PR}} = \frac{k}{2} \epsilon^{\mu\nu\lambda\rho}(p_1 - p_2)\lambda(p_3 - p_4) \rho G_{zz}^{(1)}, \] (A.4)

which is manifestly gauge invariant in the limit of zero \( p_3 \) or \( p_4 \). In the string computation of the four-point function the above expression appears through so-called contact terms. For example one can consider the term in the four point string amplitude involving the space time part of \( T_F \). Contracting \( T_F \) with \( \psi^\mu \) from \( V_A^\mu \) and \( \epsilon^{\mu p_2 X} \) in \( V_A^X \), we find a singularity of the form \( 1/(\zeta - \zeta_1)(\zeta - \zeta_2) \). Furthermore the contraction of the Kač-Moody currents yields a singularity of the form \( k/(\bar{\zeta}_1 - \bar{\zeta}_2)^2 \). To start with, this term is quartic in momenta but the integration of \( \zeta_1 \) and \( \zeta_2 \) gives a pole in \( p_1 \cdot p_2 \) giving rise again to a quadratic term in momenta. Combining the term coming from interchanging the two gauge fields, a straightforward calculation yields the r.h.s. of (A.4) with \( G_{zz}^{(1)} \) given by (2.13). There are also other contact terms in the string amplitude. By contracting \( T_F \) with \( p_1 \cdot \psi \) instead of \( \psi^\mu \) in the above, one gets additional terms that can be interpreted as the ones coming from the second term in (3.1) involving the gauge topological current \( \omega^\mu \). As expected, this term is also not separately gauge invariant. However, as one can check explicitly, after combining all the contact terms, only the kinematic form \( \epsilon^{\mu\nu\lambda\rho}p_1\lambda p_2\rho \) survives which is gauge invariant.

To proceed further, \( \partial_z \Theta_{\bar{z}} \) is defined in a similar way with the vertex of \( \bar{z} \) appearing in the \((-1)\)-ghost picture and the limit \( p_3 \to 0 \).

In order to evaluate the difference (3.3) we can proceed as in [10]. Expressing \( V_{\bar{z}} = \frac{1}{2\pi i} \oint T_F V_{\bar{z}}^{(-1)} \) in (A.4) and deforming the contour integration one finds two contributions. The first arises when the contour encircles \( T_F \) in (A.4) yielding an insertion of the energy-momentum tensor and reduces to a total derivative with respect to the Teichmüller parameter \( \tau \):

\[ \frac{-i}{2(2\pi)^3} \int_T d^2\tau \partial_\tau \int d^2\zeta \bar{\eta}(\bar{\tau})^{-2}\langle (Q^2 - \frac{k}{4\pi\tau_2})\Psi_{\bar{z}}(\zeta)\bar{\Psi}_{\bar{z}}(0) \rangle_{\text{odd}}. \] (A.5)

The second contribution comes when the contour encircles \( V_{\bar{z}}^{(-1)} \) in (A.4) converting it to \( V_z \). This is the same as the expression for \( \partial_z \Theta_{\bar{z}} \) except for the different momentum limit,
namely the limit $p_4 \to 0$. Therefore the computation of $\Delta_{\bar{z}z}$ reduces to evaluating the difference between the two limits: $p_4 \to 0$ and $p_3 \to 0$ of the 1PI part of the four point function $\langle V_A^\mu(p_1) V_A^\nu(p_2) V_z(p_3) V_{\bar{z}}^{(-1)}(p_4) T_F \rangle_{\text{odd}}$. Now the one-particle irreducible part of the amplitude is given by the difference of the full corresponding four-point function and the reducible part (A.4). As stated earlier the full four-point function is gauge invariant and comes with the kinematic structure $\epsilon^{\mu\nu\lambda\rho} p_1_{\lambda} p_2_{\rho}$ which is independent of the two momentum limits and therefore vanishes in the difference. On the other hand the reducible part gives equal contribution with opposite signs in these two limits as can be easily seen from (A.4). Thus in the difference they add up. Combining both these contributions, we reproduce equation (3.7). This derivation makes it clear that the group-dependent contribution to $\Delta_{\bar{z}z}$, which corresponds to the first term of (3.7), comes from the irreducible part of the amplitude (A.4), whereas the universal contribution proportional to the Kač-Moody level arises from the reducible diagrams.
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