Stationary modes and integrals of motion in nonlinear lattices with a $\mathcal{PT}$-symmetric linear part

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Abstract

We consider finite-dimensional nonlinear systems with a linear part described by a parity-time ($\mathcal{PT}$-)symmetric operator. We investigate bifurcations of stationary nonlinear modes from the eigenstates of the linear operator and consider a class of $\mathcal{PT}$-symmetric nonlinearities allowing the existence of families of nonlinear modes. We pay particular attention to situations when the underlying linear $\mathcal{PT}$-symmetric operator is characterized by the presence of degenerate eigenvalues or an exceptional-point singularity. In each of the cases we construct formal expansions for small-amplitude nonlinear modes. We also report a class of nonlinearities allowing the system to admit one or several integrals of motion, which turn out to be determined by the pseudo-hermiticity of the nonlinear operator.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The richness of solutions is a typical feature of nonlinear problems. Since a complete characterization of a vast set of all solutions is not usually possible, the simpler problem of classifying admissible types of solutions of a nonlinear system becomes a relevant and still nontrivial task. In this respect, there is an important distinction between conservative and non-conservative (i.e. dissipative) systems. Conservative nonlinear systems typically possess continuous families of nonlinear stationary solutions (modes) which exist for fixed values of the system parameters. A family of nonlinear modes can be parametrized by an ‘internal’ quantity (or few quantities), like, for instance, the $L^2$-norm of the mode (which can correspond to energy, number of particles or total power, depending on the particular physical statement) or its frequency (i.e. propagation constant or chemical potential). The situation becomes different when one considers a dissipative system characterized by the presence of gain and
loss. In order to admit a stationary solution, such a system requires the absorbed energy to be compensated for exactly by the gain. The requirement of the energy balance imposes an additional constraint on the shape of the solution, which has to have a nontrivial energy flow allowing for the energy transfer from the gain regions to the lossy ones. As a result, for given values of the system parameters, a typical dissipative nonlinear system does not admit continuous families of nonlinear modes. Instead, isolated nonlinear modes appear. From the dynamical point of view, these isolated modes typically behave as attractors (being stable) or repellers (being unstable). In order to obtain a continuous set of physically distinct dissipative nonlinear modes, one has to vary the parameters of the system, i.e. change the system itself. Then, instead of continuous families, characteristic for conservative systems with fixed parameters, the dissipative system admits continuous branches of nonlinear modes, which are obtained through variation of the system parameters. We thus distinguish between parametric families and branches. The described dichotomy is particularly well known in the context of the discussion of dissipative solitons versus conventional solitons in the nonlinear Schrödinger-like and complex Ginzburg–Landau equations [1].

In this context, the physical concept of parity-time (PT) symmetry [2] deserves particular attention. This is because the nonlinear systems with PT symmetry appear to occupy an ‘intermediate position’ between the conventional conservative and dissipative systems. Whereas the systems governed by PT-symmetric operators are of dissipative nature and require a stationary solution to generate a nontrivial energy flow, the exact gain and loss balance inherent to the PT symmetry allows the system to possess in some cases continuous families of nonlinear modes apart from the branches typical to dissipative systems. There are numerous studies where both branches and families of the nonlinear modes in PT-symmetric systems have been reported. In particular, exact solutions were reported for the nonlinear PT-symmetric dimer (a system of two coupled nonlinear oscillators) in [3]. Families of nonlinear discrete modes received particular emphasis in [4–7], while branches of the solutions were obtained for finite lattices such as PT-symmetric trimers and oligomers [5, 8, 9], as well as for infinite PT-symmetric chains [6, 7, 10, 11].

Localized modes (or solitons) are also known to exist in extended nonlinear systems, where they first were found in the presence of a periodic PT-symmetric potential and cubic nonlinearity [12]. Families of nonlinear modes in continuous systems were intensively investigated for gap solitons in Kerr [13] and $\chi^2$ [14] media, in PT-symmetric parabolic potential [15], in defect PT-symmetric lattices [16], in systems with $\chi^{(2)}$ nonlinearity with embedded PT-symmetric defect [17], and in the presence of a PT-symmetric superlattice [18]. Lattices with PT-symmetric nonlinear potentials were proposed in [19, 20]. The combined effect of linear and nonlinear PT-symmetric potentials was explored in discrete [21, 22] and continuous [23] statements. Models of two coupled waveguides (so-called PT-symmetric nonlinear couplers) are also known to support solitons [24] and breathers [25]. Solitons in a system consisting of a necklace of optical PT-symmetric waveguides were recently addressed in [26].

In most of the mentioned studies, the nonlinearity inherent to the system was fixed by the physical statement of the problem. In optical applications one usually considers the cubic (namely, Kerr type) or quadratic ($\chi^2$-type) nonlinearities, while realization of PT symmetry in Bose–Einstein condensates [27–29] also implies Kerr type nonlinearity. In many situations, however, the nonlinearity can be changed, which makes it relevant to study physical systems obeying the same linear properties but having nonlinearities of different types. This leads us to the first goal of the present paper, which is the effect of the type of the nonlinearity on the existence and dynamics of nonlinear modes. In particular, we will study bifurcations of families of nonlinear modes from the eigenstates of the underlying linear operator (i.e. the operator
describing the linear part of the system) and argue that the bifurcations are possible only if the nonlinear operator admits a certain symmetry which can be conveniently referred to as \textit{weak} $\mathcal{PT}$ symmetry. This terminology is to emphasize that the class of weakly $\mathcal{PT}$-symmetric nonlinear operators contains as a subset a class of $\mathcal{PT}$-symmetric nonlinear operators which (by analogy with the definition of $\mathcal{PT}$ symmetry for linear operators) commute with the $\mathcal{PT}$ operator. The second goal of our paper is to study the relevance of $\mathcal{PT}$-symmetric nonlinearities on the dynamical properties of the system. In particular, we will study the possibility for the nonlinear system (1) to admit integrals of motion and show that if the nonlinearity is $\mathcal{PT}$ symmetric (and has a certain additional simple property), then the nonlinear system does admit an integral of motion. On the other hand, we will also argue that integrals of motion can exist if the nonlinear operator is pseudo-Hermitian [30].

Apart from the type of nonlinearity, the properties of the nonlinear modes may strongly depend on the character of the spectrum of the underlying linear operator. Most of the above mentioned studies dealt with the situation when the linear spectrum consists of simple real eigenvalues. Recently, nonlinear modes bifurcating from the doubly degenerate linear eigenstates were reported in [5]. The third goal of the paper is to perform the analysis of bifurcations of stationary nonlinear modes in a situation when the linear operator has a degenerate eigenstate of finite multiplicity or exceptional-point singularities [31]. For both these cases we develop formal asymptotic expansions which describe bifurcations of the nonlinear modes from degenerate linear eigenstates. In order to construct the expansions, we explore the structure of the invariant subspace associated with the multiple eigenvalue. For the case of a semi-simple eigenvalue, we show that its invariant subspace can be spanned by a basis consisting of $\mathcal{PT}$-invariant linearly independent eigenvectors (see proposition 1). For the situation when the geometric multiplicity of the eigenvector is less than its algebraic multiplicity, we show that there exists a $\mathcal{PT}$-invariant generalized eigenvector (proposition 2).

The organization of the paper is as follows. In section 2 we specify the chosen model and make some general remarks related to the subject. Next, we address the effect of nonlinearity on the bifurcation of nonlinear modes from the eigenstates of the linear lattice. We consider the essentially different situations of bifurcations from simple (section 3) and semi-simple (section 4) eigenvalues, as well as nonlinear modes in the presence of the exceptional-point singularity (section 5). In section 6 we address the question of the nonlinearities allowing nonlinear $\mathcal{PT}$-symmetric systems to have integrals of motion. The outcomes are summarized in the conclusion.

2. The model and general remarks

To proceed with our studies, we specify the chosen model. In the present work, we study a nonlinear system of the form

$$i\dot{\mathbf{q}} = -H(\gamma)\mathbf{q} - F(\mathbf{q})\mathbf{q},$$

(1)

where $\mathbf{q} = \mathbf{q}(t)$ is a column-vector of $N$ elements, an overdot stands for the derivative with respect to time $\dot{\mathbf{q}} = d\mathbf{q}/dt$ (or with respect to the propagation distance in optical terminology). The linear part of the finite lattice (1) is described by an $N \times N$ symmetric matrix $H(\gamma)$. We consider $\mathcal{PT}$-symmetric Hamiltonians $H(\gamma)$ [2], meaning that there exist parity, $\mathcal{P}$, and time-reversal, $\mathcal{T}$, operators such that $\mathcal{PT}^2 = I$, $[\mathcal{P}, I] = 0$ and $[\mathcal{PT}, H] = 0$ (hereafter $I$ is the identity operator). The mentioned properties also imply that $(\mathcal{PT})^2 = I$.

As is customary, we define the time-reversal operator $\mathcal{T}$ by complex conjugation, i.e. $\mathcal{T} \mathbf{q} = \mathbf{q}^*$ (hereafter an asterisk stands for complex conjugation), and consider $\mathcal{P}$ to be a matrix
representation of the linear parity reversal operator. The above definition of $T$ ensures that entries of the matrix $\mathcal{P}$ are real [32]. Moreover, $\mathcal{P}T$ symmetry implies $H\mathcal{P} = \mathcal{P}H^*$, and hence

$$H^t = H^* = \mathcal{P}H\mathcal{P},$$

(2)

e i.e. the pseudo-hermiticity of $H(\gamma)$ [30]. We also require the matrix $\mathcal{P}$ to be symmetric (for a discussion of the relevance of this requirement see [32]):

$$\mathcal{P} = \mathcal{P}^t = \mathcal{P}.$$  

(3)

The nonlinear operator $F(q)$ is an $N \times N$ matrix whose elements depend on the field $q$. We focus on the case of cubic nonlinearity when the entries $F_{pq}(q)$ of the matrix $F(q)$ are given as

$$F_{pq}(q) = q^t \mathcal{F}_{pq} q = \sum_{l,m=1}^N f_{pq}^{lm} q_l^* q_m, \quad p, j = 1, 2, \ldots, N.$$  

(4)

i.e. $\mathcal{F}_{pq}$ are $N \times N$ matrices with time-independent entries $f_{pq}^{lm}, l, m = 1, 2, \ldots, N$. In other words, $F_{pq}(q)$ is a linear combination of pair-wise products of the elements of the vectors $q(t)$ and $q^*(t)$ (some of the coefficients $f_{pq}^{lm}$ can be equal to zero). Note that if $f_{pq}^{lm} = (f_{pq}^{lm})^* = f_{lp}^{qm} = f_{pl}^{qm}$ then in the dissipationless limit (i.e. at values $\gamma$ for which $H(\gamma)$ becomes Hermitian) the system (1) is Hamiltonian i.e.

$$(F(q)q)_n = \frac{1}{2} \frac{\partial}{\partial q_n^*} \sum_{j,l,m,p=1}^N q_j^* q_l^* f^{lp}_{jm} q_m q_p, \quad n = 1, 2, \ldots, N.$$  

The particular examples of nonlinearities $F(q)$ considered in this paper are of this type. Nevertheless, the analysis we develop is applicable to a general case, which in particular includes $\mathcal{PT}$-symmetric dissipative nonlinearities [19–22], when the nonlinear operator $F(q)$ results in loss and gain.

Stationary nonlinear modes of system (1) correspond to solutions of the form $q(t) = e^{i\omega t} w$, where $w$ is a time-independent column vector solving the stationary nonlinear problem

$$b w = H(\gamma) w + F(w) w, \quad \text{Im} b = 0.$$  

(5)

The reality of $b$ is required for the existence of stationary nonlinear modes, and the equality $F(w) = F(q)$ readily follows from equation (4). In the linear limit, which formally corresponds to $F(w) = 0$, equation (5) is reduced to a linear eigenvalue problem for the operator $H(\gamma)$:

$$\bar{b} \tilde{w} = H(\gamma) \tilde{w}$$  

(6)

(hereafter tildes denote the eigenvectors and eigenvalues belonging to the linear spectrum). Let us now recall some relevant properties of the linear problem (6) (more details can be found e.g. in [2]). In a certain range of values of the parameter $\gamma$, a non-Hermitian $\mathcal{PT}$-symmetric operator $H = H(\gamma)$ may possess a purely real spectrum. However, for a certain value (or values) of $\gamma$ the system undergoes a spontaneous $\mathcal{PT}$ symmetry breaking, corresponding to the transition from the real spectrum to the complex one (in the latter situation the system is said to be in the phase of broken $\mathcal{PT}$ symmetry). The transition to the broken $\mathcal{PT}$ symmetry phase can be described in terms of an exceptional-point spectral singularity [33, 34].

If an eigenvalue $\bar{b}$ of $H(\gamma)$ is real and possesses exactly one linearly independent eigenvector $\tilde{w}$, then the latter can be chosen to be $\mathcal{PT}$-variant [35], i.e.

$$\mathcal{P}T \tilde{w} = \tilde{w},$$  

(7)

independently of whether the $\mathcal{PT}$ symmetry of $H(\gamma)$ is unbroken or broken. Note that condition (7) fixes the phase of the vector $\tilde{w}$, while the model (1) is phase invariant thanks
to the choice of the cubic nonlinearity of the form (4). Instead of the vector \( \tilde{w} \) fixed by (7), one can consider any eigenvector of the form \( e^{i\varphi} \tilde{w} \), where \( \varphi \) is real. However, due to the phase invariance, this generalization does not lead to physically distinct solutions. Therefore, to simplify the algebra below we hold the definition of the \( \mathcal{PT} \)-invariant mode (7), bearing in mind that modes (either linear or nonlinear) \( w \) and \( e^{i\varphi} \tilde{w} \) are physically equivalent.

Equation (7) trivially leads to the properties \( \tilde{w}^* = \mathcal{T}\tilde{w} = \mathcal{PT}\tilde{w} \). Introducing the inner product as \( \langle a, b \rangle = a^\dagger b = \sum_{j=1}^{n} a_j^* b_j \), we arrive at the conclusion that \( \langle \tilde{w}^*, \tilde{w} \rangle \) is real. Indeed, for any two \( \mathcal{PT} \)-invariant vectors \( a = \mathcal{PT}a \) and \( b = \mathcal{PT}b \) one verifies that

\[
\langle a^*, b \rangle = \langle \mathcal{P} a, b \rangle = \langle \mathcal{P} a, \mathcal{PT} b \rangle = \langle a, T b \rangle = \langle a, b^* \rangle = \langle a^*, b \rangle^*.
\]

(8)

In the situation when the eigenvalue \( \tilde{b} \) is multiple, one should distinguish between two cases. In the first case, the multiple eigenvalue \( b \) (with algebraic multiplicity equal to \( n \)) possesses an invariant subspace spanned by \( n \) linearly independent eigenvectors. In other words, the eigenvalue \( \tilde{b} \) is semi-simple, and the Hamiltonian \( H(\gamma) \) is diagonalizable (provided that all other eigenvalues are also simple or semi-simple). This situation arises, in particular, when two eigenvalues of a parameter-dependent Hamiltonian \( H(\gamma) \) coalesce at some value of the control parameter \( \gamma \) but the two corresponding eigenvalues remain linearly independent. Physically such a situation appears, in particular, when two (or more) identical systems are linearly coupled: the degeneracy occurs when the coupling becomes zero.

The second case corresponds to the situation when the collision of the eigenvalues is accompanied by collisions of eigenvectors. Then the geometric multiplicity of the eigenvalue \( \tilde{b} \) if less than \( \gamma \) the algebraic multiplicity, i.e. the dimension of the invariant subspace of the multiple eigenvalue \( \tilde{b} \) is less than \( n \). This corresponds to so-called exceptional points [31]. As was mentioned above, the relevance of such points is in particular related to the transition to the phase of broken \( \mathcal{PT} \) symmetry (see also the discussion in [36]).

3. Bifurcations of the families of nonlinear modes. A simple eigenvalue

We start by considering families of nonlinear modes bifurcating from a non-degenerate linear eigenstate of the \( \mathcal{PT} \)-symmetric Hamiltonian \( H \). Let \( \tilde{b} \) be a simple real eigenvalue and \( \tilde{w} \) be the corresponding eigenvector solving linear problem (6). According to the discussion in section 2, without loss of generality we can assume that \( \tilde{w} \) is \( \mathcal{PT} \) invariant, i.e. satisfies the condition (7).

In the vicinity of the linear limit the nonlinear modes bifurcating from the eigenstate \( \tilde{w} \) can be described using formal expansions [4]

\[
w = \varepsilon \tilde{w} + \varepsilon^3 w^{(3)} + o(\varepsilon^3) \quad \text{and} \quad b = \tilde{b} + \varepsilon^2 b^{(2)} + o(\varepsilon^2),
\]

(9)

where \( \varepsilon \) is a real small parameter, \( \varepsilon \ll 1 \), and without loss of generality we impose the normalization condition \( \langle \tilde{w}, \tilde{w} \rangle = 1 \). Coefficients \( w^{(3)} \) and \( b^{(2)} \) of the expansions are to be determined.

Substituting expansions (9) into equation (5) and noticing from equation (4) that \( F(w) = \varepsilon^3 F(\tilde{w}) + O(\varepsilon^3) \), one arrives at the following equation for the shift of the eigenvalue \( b^{(2)} \):

\[
b^{(2)} \tilde{w} = (H - \tilde{b})w^{(3)} + F(\tilde{w})\tilde{w}.
\]

(10)

Multiplying this equation by \( \tilde{w}^* \) and using equations (2) and (6), one readily obtains

\[
b^{(2)} = \frac{\langle \tilde{w}^*, F(\tilde{w})\tilde{w} \rangle}{\langle \tilde{w}^*, \tilde{w} \rangle}.
\]

(11)
Taking into account the reality of the product \( \langle \hat{w}^*, \hat{w} \rangle \) (see equations (8)), one notes that the bifurcation of a family of nonlinear modes is possible only if

\[
\text{Im} \langle \hat{w}^*, F(\hat{w}) \hat{w} \rangle = 0, \tag{12}
\]

which is necessary for \( b^{(2)} \) to be real. Thus (12) is a necessary condition for a family of nonlinear modes to bifurcate from the non-degenerate eigenstate of \( H \) corresponding to \( \hat{b} \).

By analogy with the condition (7) which guarantees the reality of the denominator of the right-hand side of equation (11), in order to ensure that the numerator is also real, we introduce the following condition for the nonlinear operator \( F(\hat{w}) \) [4]:

\[
\mathcal{P} \mathcal{T} F(\hat{w}) \hat{w} = F(\hat{w}) \hat{w} \quad \text{for all } \hat{w} \text{ such that } \mathcal{P} \mathcal{T} \hat{w} = \hat{w}. \tag{13}
\]

Obviously, this condition is equivalent to \( [\mathcal{P} \mathcal{T}, F(\hat{w})] = 0 \), where the latter commutator is only considered on the set of vectors satisfying \( \mathcal{P} \mathcal{T} \hat{w} = \hat{w} \). In what follows the nonlinearities obeying (13) will be said to be \textit{weakly} \( \mathcal{P} \mathcal{T} \)-symmetric. The set of weakly \( \mathcal{P} \mathcal{T} \)-symmetric nonlinearities corresponding to the given parity operator \( \mathcal{P} \) will be denoted by \( \text{NL}_{w, \mathcal{P} \mathcal{T}}(\mathcal{P}) \). Thus the weak \( \mathcal{P} \mathcal{T} \) symmetry of the nonlinear operator \( F(\hat{w}) \) appears to be a necessary condition for the system (1) to admit continuous families of nonlinear modes.

The choice of the term \textit{weak} \( \mathcal{P} \mathcal{T} \) symmetry can be understood if one considers a more restrictive condition for the nonlinear operator \( F(\hat{w}) \) to commute with the \( \mathcal{P} \mathcal{T} \) operator for \textit{any} vector \( \hat{w} \):

\[
[\mathcal{P} \mathcal{T}, F(\hat{w})] = 0. \tag{14}
\]

Nonlinear operators \( F(\hat{w}) \) of the latter type will be said to be \( \mathcal{P} \mathcal{T} \) symmetric. The set of \( \mathcal{P} \mathcal{T} \)-symmetric nonlinearities corresponding to the given parity operator \( \mathcal{P} \) will be denoted as \( \text{NL}_{\mathcal{P} \mathcal{T}}(\mathcal{P}) \). Obviously, for a given parity operator \( \mathcal{P} \), the class of weakly \( \mathcal{P} \mathcal{T} \)-symmetric nonlinearities contains the class of \( \mathcal{P} \mathcal{T} \) symmetric nonlinearities as a subset:

\[
\text{NL}_{\mathcal{P} \mathcal{T}}(\mathcal{P}) \subset \text{NL}_{w, \mathcal{P} \mathcal{T}}(\mathcal{P}).
\]

The relevance of the \( \text{NL}_{\mathcal{P} \mathcal{T}}(\mathcal{P}) \) nonlinearities will be discussed in section 6.

As an example, let us consider a finite nonlinear system with \( N = 4 \) (i.e. a quadrimer) and a Kerr nonlinearity, \( F_K(\hat{w}) \), whose elements are defined by

\[
F_{K,pj}(\hat{w}) = \delta_{pj} |w_j|^2
\]

where \( \delta_{pj} \) is the Kronecker delta and \( p, j = 1, \ldots, 4 \). This nonlinearity satisfies the requirement (13) provided that the parity operator is chosen as

\[
\mathcal{P}_{11} = \sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{15}
\]

(hereafter \( \sigma_j \) are the Pauli matrices). However, the nonlinearity \( F_K(\hat{w}) \) does not satisfy the condition (14). Therefore \( F_K \in \text{NL}_{w, \mathcal{P} \mathcal{T}}(\mathcal{P}_{11}) \setminus \text{NL}_{\mathcal{P} \mathcal{T}}(\mathcal{P}_{11}) \).

Note that expansions (9) suggest that the nonlinear modes \( \hat{w} \) bifurcating from the \( \mathcal{P} \mathcal{T} \)-invariant eigenvector \( \hat{w} \) will also be \( \mathcal{P} \mathcal{T} \) invariant: \( \mathcal{P} \mathcal{T} \hat{w} = \hat{w} \). This fact is confirmed in numerous previous studies, see e.g. [4, 6, 7, 9, 11, 13, 27], where a large library of results dedicated to nonlinear modes bifurcating from the non-degenerate (simple) linear eigenstates and supported by the nonlinearities of the \( \text{NL}_{w, \mathcal{P} \mathcal{T}}(\mathcal{P}) \) type can be found. The existence of the families of nonlinear modes was also established through the analytical continuation from the anticontinuum limit [11] (the technique well known in the theory of conservative lattices [37]). Bifurcations of the nonlinear modes from the simple eigenvalues of the underlying linear lattices, as well as from the anticontinuum limit, were recently addressed in a rigorous mathematical framework [6].
4. Bifurcations from the semi-simple eigenvalue

4.1. Structure of the invariant subspace

In the previous section we considered bifurcations of nonlinear modes from a non-degenerate (simple) eigenvalue of the underlying linear problem. Now we turn to the case of a semi-simple eigenvalue, i.e. a multiple eigenvalue whose geometric and algebraic multiplicities are equal. Let \( b \) be a real semi-simple eigenvalue of multiplicity equal to \( n \). This means that there exist \( n \) linearly independent eigenvectors \( \tilde{w}_j, j = 1, \ldots, n \), such that \( \tilde{H} \tilde{w}_j = b \tilde{w}_j \), and there is an \( n \)-dimensional \( H \)-invariant subspace spanned by \( \tilde{w}_j \) such that any vector from this subspace is an eigenvector of the operator \( H \).

Adopting the approach of formal expansions used in section 3 for the case of a simple eigenvalue, we are going to construct a family of nonlinear modes which in the vicinity of the bifurcation from the linear limit behave as \( w = \epsilon \tilde{w} + \cdots \), where \( \tilde{w} \) is some eigenvector of \( H \) corresponding to the eigenvalue \( b \). Looking for \( \mathcal{P}\mathcal{T} \)-invariant nonlinear modes \( \mathcal{P}\mathcal{T} w = w \), one has to find a \( \mathcal{P}\mathcal{T} \)-invariant eigenvector \( \tilde{w} \) for the expansion to be valid. However, while in the case of a simple eigenvalue a \( \mathcal{P}\mathcal{T} \)-invariant eigenvector \( \tilde{w} \) necessarily exists (and is unique up to a multiplier), in the case of a semi-simple eigenvalue \( b \) it is not readily obvious how many \( \mathcal{P}\mathcal{T} \)-invariant eigenvectors exist (if any). Therefore, in the situation at hand it is necessary to explore the structure of the invariant subspace associated with the multiple eigenvalue \( b \). Let us show that in the invariant subspace of \( \tilde{b} \) one can always find a basis of \( n \) linearly independent \( \mathcal{P}\mathcal{T} \)-invariant eigenvectors (independently of whether the \( \mathcal{P}\mathcal{T} \) symmetry of \( H \) is broken or not).

**Proposition 1.** Let \( \tilde{b} \) be a real semi-simple eigenvalue of multiplicity \( n \). Then the invariant subspace of \( \tilde{b} \) has a complete basis \( (u_1, u_2, \ldots, u_n) \) of \( \mathcal{P}\mathcal{T} \)-invariant eigenvectors: \( \mathcal{P}\mathcal{T} u_j = u_j, j = 1, 2, \ldots, n \).

**Proof of proposition 1.** The condition of the proposition implies that there exist \( n \) linearly independent eigenvectors \( \tilde{w}_j, j = 1, 2, \ldots, n \), such that \( \tilde{H} \tilde{w}_j = b \tilde{w}_j \). If the choice of the linearly independent eigenvectors \( \tilde{w}_j \) is arbitrary, then generally speaking \( \mathcal{P}\mathcal{T} \tilde{w}_j \neq \tilde{w}_j \), i.e. eigenvectors \( \tilde{w}_j \) (or some of them) are not \( \mathcal{P}\mathcal{T} \) invariant. Thus, in order to prove the proposition we must find \( n \) linearly independent \( \mathcal{P}\mathcal{T} \)-invariant eigenvectors.

To this end we apply the \( \mathcal{P}\mathcal{T} \) operator to each \( \tilde{w}_j \) and obtain a new set of vectors \( v_j = \mathcal{P}\mathcal{T} \tilde{w}_j, j = 1, \ldots, n \). The \( \mathcal{P}\mathcal{T} \) symmetry of \( H \) and reality of \( \tilde{b} \) imply that \( \tilde{H} v_j = b v_j \), i.e. each \( v_j \) belongs to the invariant subspace of the eigenvalue \( b \). Note also that \( \mathcal{P}\mathcal{T} v_j = w_j \) (thanks to \( (\mathcal{P}\mathcal{T})^2 = I \)). Then the linear independence of vectors \( \tilde{w}_j \) implies that the vectors \( v_j \) are also linearly independent and therefore constitute a basis in the invariant subspace of \( \tilde{b} \). Therefore, there exists an \( n \times n \) nonsingular matrix \( D \) such that \( v_j = \sum_{k=1}^n D_{jk} \tilde{w}_k \).

Let us now introduce a new set of vectors \( u_j, j = 1, 2, \ldots, n \), given as

\[
u_j = e^{i\phi} \tilde{w}_j + e^{-i\phi} v_j = e^{-i\phi} \left( \sum_{k=0}^n D_{jk} \tilde{w}_k + e^{2i\phi} \tilde{w}_j \right) = e^{-i\phi} \sum_{k=0}^n M_{jk} \tilde{w}_k,
\]

where \( \phi \) is an arbitrary (so far) real parameter, and \( M_{jk} \) are the entries of the \( n \times n \) matrix \( M \) defined as \( M = D + e^{2i\phi} I \) (here \( I \) is the \( n \times n \) identity matrix). Then each \( u_j \) belongs to the invariant subspace of \( \tilde{b} \) and is obviously \( \mathcal{P}\mathcal{T} \) invariant. Besides, one can always find \( \phi \) such that the matrix \( M \) is nonsingular (it is necessary and sufficient to choose \( \phi \) such that \( \rho = -e^{2i\phi} \) does not belong to the spectrum of \( D \)). Once \( M \) is nonsingular, the eigenvectors \( u_j \) are linearly independent and therefore constitute a complete \( \mathcal{P}\mathcal{T} \)-invariant basis in the invariant subspace associated with the eigenvalue \( \tilde{b} \). Thus the proposition is proven. \( \square \)
4.2. Expansions for nonlinear modes

Turning now to bifurcations of nonlinear modes, we employ the same expansions as in the case of a simple eigenvalue:

\[ \mathbf{w} = \varepsilon \mathbf{w} + \varepsilon^3 \mathbf{w}^{(3)} + o(\varepsilon^3), \quad \text{and} \quad b = \tilde{b} + \varepsilon^2 \tilde{b}^{(2)} + o(\varepsilon^2). \]  

(16)

However, now the vector \( \mathbf{\tilde{w}} \) is a linear combination of the \( \mathcal{PT} \)-invariant eigenvectors \( \mathbf{u}_j \), constructed in proposition 1: \( \mathbf{\tilde{w}} = \sum_{j=1}^{n} c_j \mathbf{u}_j \). In order to assure that the linear solution \( \mathbf{\tilde{w}} \) is \( \mathcal{PT} \)-invariant, i.e. satisfies (7), we require all the coefficients \( c_j \) to be real. However, an arbitrary set of coefficients \( c_j \) generally speaking does not represent a linear mode \( \mathbf{\tilde{w}} \) allowing for a bifurcation of a family of nonlinear modes \( \mathbf{w} \). The relevant relations among the coefficients can be found by multiplying equation (10) by \( \mathbf{u}^*_j \) leading to conditions as follows:

\[ b^{(2)} = \frac{\langle \mathbf{u}^*_j, F(\mathbf{\tilde{w}}) \mathbf{\tilde{w}} \rangle}{\langle \mathbf{u}^*_j, \mathbf{\tilde{w}} \rangle}, \]  

(17)

where \( b^{(2)} \) is required to be real. One can readily ensure that if the nonlinearity \( F(\mathbf{w}) \) satisfies the requirement (13), i.e. \( F(\mathbf{w}) \in \mathcal{NL}_{\mathcal{PT}}(\mathcal{P}) \), then \( b^{(2)} \) is real. Indeed:

\[ \frac{\langle \mathbf{u}^*_j, F(\mathbf{\tilde{w}}) \mathbf{\tilde{w}} \rangle}{\langle \mathbf{u}^*_j, \mathbf{\tilde{w}} \rangle} = \frac{\langle \mathcal{PT} \mathbf{u}_j, \mathcal{T}(F(\mathbf{\tilde{w}}) \mathbf{\tilde{w}}) \rangle}{\langle \mathcal{PT} \mathbf{u}_j, \mathcal{T} \mathbf{\tilde{w}} \rangle} = \frac{\langle \mathbf{u}^*_j, F(\mathbf{\tilde{w}}) \mathbf{\tilde{w}} \rangle}{\langle \mathbf{u}^*_j, \mathbf{\tilde{w}} \rangle}. \]

Bearing in mind the normalization condition \( \langle \mathbf{\tilde{w}}, \mathbf{\tilde{w}} \rangle = 1 \), the system (17) can be viewed as \( n \) algebraic equations with respect to \( n-1 \) independent real coefficients \( c_j \); the requirement of compatibility of these equation determines \( b^{(2)} \). The existence and a number of the solutions obviously depend on the specific form of the nonlinearity \( F(\mathbf{w}) \), the latter determining the diversity of physically distinct families of nonlinear modes bifurcating from the eigenstates of the linear spectrum.

Finally, we mention a peculiar but physically relevant situation when for a certain set of the coefficients \( c_j \) the eigenstate \( \mathbf{\tilde{w}} \) becomes an eigenvector of the nonlinear eigenvalue problem \( F(\mathbf{\tilde{w}}) \mathbf{\tilde{w}} = \lambda \mathbf{\tilde{w}} \), and the eigenvalue \( \lambda \) is real. Then equations (17) are automatically satisfied provided that \( b^{(2)} = \lambda \), i.e. \( b^{(2)} = \langle \mathbf{\tilde{w}}, F(\mathbf{\tilde{w}}) \mathbf{\tilde{w}} \rangle \). In this case the number of free parameters exceeds the number of the imposed constraints. Such a situation has been encountered for example in the model of a \( \mathcal{PT} \)-symmetric birefringent coupler [5] and occurs in the example considered in the next subsection.

4.3. Example: a \( \mathcal{PT} \)-symmetric coupler of bi-chromatic light

As we already mentioned, the presence of multiply degenerated eigenvalues in the linear spectrum typically occurs when two identical systems are decoupled linearly, and coupling exists only due to the nonlinearity. Let us consider the following example:

\[ iq_1 = i\gamma q_1 + q_2 + (|q_1|^2 + |q_2|^2)q_1, \]

(18a)

\[ iq_2 = -i\gamma q_2 + q_1 + \kappa (|q_2|^2 + |q_4|^2)q_2, \]

(18b)

\[ iq_3 = -i\gamma q_3 + q_4 + (|q_3|^2 + |q_1|^2)q_3, \]

(18c)

\[ iq_4 = i\gamma q_4 + q_3 + \kappa (|q_4|^2 + |q_2|^2)q_4. \]

(18d)

It can be viewed as a model for the propagation of a bi-chromatic light in a \( \mathcal{PT} \)-symmetric coupler. Then \( q_j \) and \( q_{j+2} \) (\( j = 1, 2 \)) are the fields having the propagation constants \( k_1 \) and \( k_2 \) in the \( j \)th waveguide, where self-phase and cross-phase modulation of the two modes propagating in one arm are normalized to 1. The modes \( k_1 \) and \( k_2 \) are subject to gain and absorption in the
first waveguide (and vice versa in the second waveguide). The real parameter \( \kappa_1 \) describes the difference in the Kerr nonlinearities in the waveguides.

The linear Hamiltonian of (18a)–(18d) is described by the matrix

\[
H_{bc}(\gamma_1, \gamma_2) = \begin{pmatrix}
 \gamma_1 & 1 & 0 & 0 \\
 1 & \gamma_2 & 0 & 0 \\
 0 & 0 & -i\gamma_1 & 1 \\
 0 & 0 & 1 & -i\gamma_2 \\
\end{pmatrix}
\]

and the nonlinearity is given by

\[
F_{bc}(q) = \begin{pmatrix}
 |q_1|^2 + |q_3|^2 & 0 & 0 & 0 \\
 0 & \kappa(|q_2|^2 + |q_4|^2) & 0 & 0 \\
 0 & 0 & |q_1|^2 + |q_3|^2 & 0 \\
 0 & 0 & 0 & \kappa(|q_2|^2 + |q_4|^2) \\
\end{pmatrix}.
\]

The subscripts ‘bc’ in equations (19) and (20) stand for ‘bi-chromatic’.

One readily verifies that

\[
H_{bc}(\gamma_1, \gamma_2)
\]

is \( P_{10} T \)-symmetric with respect to

\[
P_{10} = \sigma_1 \otimes \sigma_0 = \begin{pmatrix}
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \\
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

and the nonlinearity is given by

\[
F_{bc}(q) \in \text{NL}_{PT} \subset \text{NL}_{wPT}(P_{10}).
\]

As is clear, the operator \( H_{bc}(\gamma, -\gamma) \) represents two uncoupled linear \( PT \)-symmetric dimers.

\[
H_{bc}(\gamma, -\gamma) = \begin{pmatrix}
 \gamma & 1 & 0 & 0 \\
 1 & \gamma & 0 & 0 \\
 0 & 0 & -\gamma & 1 \\
 0 & 0 & 1 & -\gamma \\
\end{pmatrix}
\]

The physical meaning of the operators \( P_{10} \) and \( P_{01} \) becomes particularly clear in terms of the graph representation [4], as is illustrated in figure 1.

One also verifies that \( F_{bc}(q) \in \text{NL}_{PT}(P_{10}) \subset \text{NL}_{wPT}(P_{10}). \) At \( \kappa = 1 \) one additionally has

\[
F_{bc}(q) \in \text{NL}_{wPT}(P_{01}).
\]
proposition 1) can be written as follows:

\[
\begin{align*}
\mathbf{u}_{+,1} &= \begin{pmatrix} e^{i\psi} \\ e^{-i\psi} \\ e^{-i\psi} \\ e^{i\psi} \end{pmatrix}, & \mathbf{u}_{+,2} &= i \begin{pmatrix} e^{i\psi} \\ e^{-i\psi} \\ -e^{-i\psi} \\ -e^{i\psi} \end{pmatrix}, \\
\mathbf{u}_{-,1} &= \begin{pmatrix} e^{-i\psi} \\ e^{i\psi} \\ e^{i\psi} \\ e^{-i\psi} \end{pmatrix}, & \mathbf{u}_{-,2} &= i \begin{pmatrix} e^{-i\psi} \\ e^{i\psi} \\ -e^{i\psi} \\ -e^{-i\psi} \end{pmatrix},
\end{align*}
\]

where \( \psi \) is defined by the requirements \( \sin(2\psi) = \gamma \) and \( \cos(2\psi) = \sqrt{1 - \gamma^2} \).

Let us search for nonlinear modes bifurcating from the eigenvalue \( \tilde{b}_+ \) (analysis for the eigenvalue \( \tilde{b}_- \) yields similar results). Following the above approach, we search for the linear eigenvector in the form as a linear combination \( \mathbf{w} = c_1 \mathbf{u}_{+,1} + c_2 \mathbf{u}_{+,2} \) where \( c_{1,2} \) are real coefficients. Subject to the normalization \( \langle \mathbf{w}, \mathbf{w} \rangle = 1 \), we obtain \( c_1^2 + c_2^2 = 1/4 \). Straightforward algebra gives

\[
\begin{align*}
\langle \mathbf{u}_{+,1}^*, F_{bc}(\mathbf{w}) \mathbf{w} \rangle &= c_1 \cos(2\psi)(\kappa + 1) + c_2 \sin(2\psi)(\kappa - 1), \\
\langle \mathbf{u}_{+,2}^*, F_{bc}(\mathbf{w}) \mathbf{w} \rangle &= c_1 \sin(2\psi)(\kappa - 1) - c_2 \cos(2\psi)(\kappa + 1), \\
\langle \mathbf{u}_{+,1}^*, \mathbf{w} \rangle &= 4c_1 \cos(2\psi), & \langle \mathbf{u}_{+,2}^*, \mathbf{w} \rangle &= -4c_2 \cos(2\psi).
\end{align*}
\]

Next, following equations (17), we require

\[
\begin{align*}
b^{(2)}(\mathbf{u}_{+,1}^*, \mathbf{w}) &= \langle \mathbf{u}_{+,1}^*, F_{bc}(\mathbf{w}) \mathbf{w} \rangle, \\
b^{(2)}(\mathbf{u}_{+,2}^*, \mathbf{w}) &= \langle \mathbf{u}_{+,2}^*, F_{bc}(\mathbf{w}) \mathbf{w} \rangle.
\end{align*}
\]

Considering first the generic case \( \kappa \neq 1 \), we substitute equations (22a)–(22c) into equations (23a)–(23b) and find that equations (23a) and (23b) are compatible only if \( \sin(4\psi) = 0 \), which means that bifurcations of nonlinear modes are possible only for \( \gamma = 0 \) (i.e. when the dissipation vanishes) or for \( \gamma = \pm 1 \). The latter case corresponds to the point of the phase transition to the broken \( PT \) symmetry, which is described by the exceptional-point singularity and requires a particular analysis (see section 5).

The case \( \kappa = 1 \) corresponds to the peculiar situation when for any choice of \( c_1 \) and \( c_2 \) one has \( F_{bc}(\mathbf{w}) \mathbf{w} = \mathbf{w}/2 \), i.e. equations (23a)–(23b) are automatically satisfied with \( b^{(2)} = 1/2 \). This is however a strongly degenerate case. Looking for solutions in the form \( w_3^* = \chi w_1 \), \( w_4^* = \chi w_2 \) (\( \chi \) is arbitrary real) the system is reduced to the \( PT \)-symmetric nonlinear dimer [3]

\[
\begin{align*}
bw_1 &= i\gamma w_1 + w_2 + (\chi^2 + 1)|w_1|^2 w_1, \\
bw_2 &= -i\gamma w_2 + w_1 + (\chi^2 + 1)|w_2|^2 w_2,
\end{align*}
\]

which supports analytically computable families of nonlinear modes given by the substitution \( w_1 = w_3^* \). Since \( \chi \) is arbitrary, we can obtain a continuous set of solutions even for fixed \( \gamma \) and \( b \).

5. Bifurcations of nonlinear modes in the presence of an exceptional-point singularity

5.1. Expansions for nonlinear modes

Let us turn to a situation when a multiple eigenvalue \( \tilde{b} \) corresponds to the so-called exceptional-point singularity which appears when the coalescence of two (or more) simple eigenvalues
of the Hamiltonian $H(\gamma)$ occurring at specific values of the parameter $\gamma$ is accompanied by collision of the corresponding eigenvectors. Then the geometric multiplicity of the eigenvalue $\tilde{b}$ is less than the algebraic multiplicity, and the Hamiltonian $H(\gamma)$ becomes nondiagonalizable. The presence of exceptional points is a typical feature of $\mathcal{PT}$-symmetric systems, since such points naturally appear at the ‘boundary’ between phases of broken and unbroken $\mathcal{PT}$ symmetries [33, 34]. The nonlinear behavior of $\mathcal{PT}$-symmetric optical lattices near the phase-transition point was recently considered in [38, 39].

We illustrate our ideas by considering an exceptional point where two linear eigenstates coalesce, forming a real multiple eigenvalue $\tilde{b}$ with total multiplicity equal to 2 and having exactly one linearly independent eigenvector $\tilde{w}$. We thus have

$$(H - \tilde{b})\tilde{w} = 0, \quad (H - \tilde{b})v = \tilde{w},$$  \hspace{1cm} (24) $$

where we have introduced a generalized eigenvector $v$. Since $H$ is $\mathcal{PT}$ symmetric and $\tilde{w}$ is the only linearly independent eigenvector corresponding to $\tilde{b}$, we can assume that $\mathcal{PT}\tilde{w} = \tilde{w}$. We can also assume that the generalized eigenvector $v$ is also $\mathcal{PT}$ invariant, i.e. $\mathcal{PT}v = v$. This assumption is valid thanks to the following proposition.

**Proposition 2.** There exists a $\mathcal{PT}$-invariant generalized eigenvector $v$: $\mathcal{PT}v = v$.

**Proof of proposition 2.** Let $u$ be an arbitrarily chosen generalized eigenvector, i.e. $(H - \tilde{b})u = \tilde{w}$. Applying a $\mathcal{PT}$ operator to both sides of the latter equality and using that the operator $H - \tilde{b}$ commutes with $\mathcal{PT}$ and that $\mathcal{PT}\tilde{w} = \tilde{w}$, we find that $\mathcal{PT}u$ is also a generalized eigenvector. Then the $\mathcal{PT}$-invariant generalized eigenvector $v$ can be found as $v = \frac{1}{2}(u + \mathcal{PT}u)$.

Multiplying the second of equations (24) by $\mathcal{PT}= \tilde{w}^*$ and using that $(H - \tilde{b})^\dagger = \mathcal{PT}(H - \tilde{b})\mathcal{PT}$, we find that the eigenvector $\tilde{w}$ is self-orthogonal [40] in the sense of the indefinite inner product [35], i.e. $\langle \mathcal{PT}\tilde{w}, \tilde{w} \rangle = 0$. Therefore, equation (11), as well as the expansions (9) used in section 3 and section 4, no longer work and have to be modified. To this end, we look for modified small-amplitude expansions in the form

$$w = \epsilon \tilde{w} + \epsilon^2 w^{(2)} + \epsilon^3 w^{(3)} + \cdots,$$  \hspace{1cm} (25) $$

$$b = \tilde{b} + \epsilon b^{(1)} + \epsilon^2 b^{(2)} + \cdots.$$  \hspace{1cm} (26) $$

Substituting equations (25)-(26) in (5) and using (24), in the $\epsilon^2$-order we obtain $w^{(2)} = b^{(1)}v$. The third-order equation reads

$$b^{(2)}\tilde{w} + (b^{(1)})^2v = (H - \tilde{b})w^{(3)} + F(\tilde{w})\tilde{w}.$$  \hspace{1cm} (27) $$

Multiplying this equation from the left by $\tilde{w}^*$, we obtain

$$(b^{(1)})^2 = \frac{\langle \tilde{w}^*, F(\tilde{w})\tilde{w} \rangle}{\langle \tilde{w}^*, \tilde{w} \rangle}.$$  \hspace{1cm} (28) $$

Let us now assume that the nonlinear operator $F(w)$ is of the $NL_{\text{sym}}$ type [see (13)]. Then, due to (8), we conclude that the right-hand side of (28) is real. Unlike the case of simple and semi-simple eigenvalues, however, for the condition (28) to have sense one has to require its right-hand side to be non-negative. This additional condition, which does not appear in the case of simple eigenvalues, is indeed restrictive, as we will illustrate in section 5.2.3. Another important feature is that (28) indicates that families bifurcating from the eigenvalue $\tilde{b}$ appear in pairs (corresponding to two opposite signs of $b^{(1)}$).
5.2. Example: a $\mathcal{PT}$-symmetric quadrimer with the nearest-neighbor interactions

5.2.1. ‘Phase diagram’. Let us illustrate the above ideas using the example of a quadrimer with nearest-neighbor interactions. The linear part of (1) is now given by the Hamiltonian

$$H_{nn}(\gamma_1, \gamma_2) = \begin{pmatrix} i\gamma_1 & 1 & 0 & 0 \\ 1 & i\gamma_2 & 1 & 0 \\ 0 & 1 & -i\gamma_2 & 1 \\ 0 & 0 & 1 & -i\gamma_1 \end{pmatrix},$$

(29)

where the subscript ‘nn’ stays for ‘nearest-neighbor’. The linear operator $H_{nn}(\gamma_1, \gamma_2)$ is $\mathcal{PT}$-symmetric with $\mathcal{P}_{11} = \sigma_1 \otimes \sigma_1$ (see equation (15)). The nonlinear part is given by the operator $F_K(w) = \text{diag}(|w_1|^2, |w_2|^2, |w_3|^2, |w_4|^2)$ (where the subscript $K$ stands for Kerr nonlinearity).

Properties of the linear operator $H_{nn}(\gamma_1, \gamma_2)$ can be visualized conveniently by means of the ‘phase diagram’ [4] shown in figure 2, panel (PD). Depending on $\gamma_1$ and $\gamma_2$, the phase diagram features three domains: (i) unbroken or exact $\mathcal{PT}$ symmetry, when all the eigenvalues $\tilde{b}_j$, $j = 1, 2, 3, 4$, of $H_{nn}(\gamma_1, \gamma_2)$ are real; (ii) broken $\mathcal{PT}$ symmetry with two real and two complex conjugated eigenvalues (note that in the particular case $\gamma_1 = \gamma_2$ domain (ii) cannot be encountered); (iii) broken $\mathcal{PT}$ symmetry with all $\tilde{b}_j$ complex. Varying the parameters $\gamma_1$, one can ‘travel’ across the phase diagram visiting domains with different phases. An interesting feature of the phase diagram shown in figure 2 (PD) is that in some cases an increase of the total dissipation brings the system from the phase of broken $\mathcal{PT}$ symmetry to unbroken $\mathcal{PT}$ symmetry. For example, fixing the value of one of the coefficients as $\gamma_1 = 1.2$, one then observes that for small $\gamma_2$ (say $\gamma_2 = 0$) the $\mathcal{PT}$ symmetry is broken, but when $\gamma_2$ becomes sufficiently large, then the system enters the phase of the unbroken $\mathcal{PT}$ symmetry. A further increase of $\gamma_2$ leads to another phase transition and the $\mathcal{PT}$ symmetry becomes broken again. A similar scenario with two $\mathcal{PT}$ transitions was recently reported in [41].
Boundaries separating different domains of the phase diagram correspond to the exceptional points. In particular, the boundaries contain exactly four triple points $T_j$, $j = 1, \ldots, 4$, where the three domains touch. The triple points correspond to the values $\gamma_{1,2}$ for which $\tilde{h}_1, \ldots, 4 = 0$, and the canonical form of $H_m(\gamma_1, \gamma_2)$ consists of the only $4 \times 4$ Jordan block. Depending on how $\gamma_{1,2}$ change in the vicinity of $T_j$, either the unbroken $PT$-symmetric phase or one of the $P^T$ symmetry broken phases arise. All the other points of the boundaries are the double points separating two different phases. They are characterized by the presence of one or two $2 \times 2$ Jordan blocks in the canonical form of $H_m(\gamma_1, \gamma_2)$. The double points can be further sub-classified as belonging to boundaries separating either phases (i) and (ii), or phases (i) and (iii), or phases (ii) and (iii). Here however we do not intend to perform a complete classification and consider only the double points adjacent to phase (i) (i.e. the one with unbroken $P^T$ symmetry) since such points are likely to be more probable ‘candidates’ to give birth to families of stable nonlinear modes (compared to the double points which are not adjacent to phase (i)).

5.2.2. The existence of nonlinear modes. The nonlinear modes that obey $P_{11}T \mathbf{w} = \mathbf{w}$ have the following property: $w_1 = w_1^*, w_2 = w_2^*$. It is a simple exercise to ensure that the nonlinear part of the system $F_k(\mathbf{w})$ is of the $NL_{u,P^T}(P_{11})$ type, i.e. obeys property (13). It was shown [4] that in this case the nonlinear modes can be found as roots of an eight-degree polynomial whose coefficients depend on $b$. The nonlinear modes constitute continuous families, which can be visualized as dependencies $U$ versus $b$, where $U = \frac{1}{2} \sum_{j=1}^{4} |w_j|^2$ can be associated with the norm of the solution (or with the total energy flow in the optical context).

5.2.3. Double points between $T_1$ and $T_2$ (between $T_3$ and $T_4$). Let us now consider bifurcations of nonlinear modes from the double points belonging to the boundary separating phases (i) and (iii). In the panel (PD) of figure 2 such points are situated on the boundary between the points $T_1$ and $T_2$, e.g. point $D_1$ (or on the boundary between $T_3$ and $T_4$ which is considered analogously). The values of $\gamma_{1,2}$ corresponding to such points are given by the equation

$$\left(\gamma_1^2 - \gamma_2^2\right)^2 - 2(\gamma_1^2 + 4\gamma_1\gamma_2 + 3\gamma_2^2) + 5 = 0.$$ (30)

In this case the spectrum of the Hamiltonian $H_m(\gamma_1, \gamma_2)$ consists of two opposite real double eigenvalues $\tilde{b}_\pm, \tilde{b}_- = -\tilde{b}_+$. Both $\tilde{b}_+$ and $\tilde{b}_-$ correspond to a $2 \times 2$ Jordan block. Let $\mathbf{w}_\pm$ and $\mathbf{v}_\pm$ be the eigenvector and the generalized eigenvector corresponding to the eigenvalue $\tilde{b}_+$ (respectively, $\mathbf{w}_-$ and $\mathbf{v}_-$ correspond to $\tilde{b}_-$). Then we can write down two Jordan chains:

$$(H_m - \tilde{b}_\pm)\mathbf{w}_\pm = 0, \quad (H_m - \tilde{b}_\pm)\mathbf{v}_\pm = \mathbf{w}_\pm.$$ (31)

One can easily check that if one chooses $P_{11}T \mathbf{w}_\pm = \mathbf{w}_\pm$, then the eigenvectors $\mathbf{w}_+$ and $\mathbf{w}_-$ can be also chosen to be related by the following relation: $\mathbf{w}_+ = GP_1\mathbf{w}_-$, where $G = \text{diag}(-1, 1, -1, 1)$. Therefore, the generalized eigenvectors are related through the relation $\mathbf{v}_+ = P_{11}GV_-$. This can be checked directly by substitution to equations (31) and using the relations $H_mP_{11}G = G^2P_{11}H_m, G^2 = -I, (P_{11}G)^2 = I$. Then from equation (28) one obtains

$$\left(b^{(1)}_+\right)^2 = \frac{\langle \mathbf{w}_+, F_k(\mathbf{w}_+) \mathbf{w}_+ \rangle}{\langle \mathbf{w}_+, \mathbf{v}_+ \rangle} = \frac{\langle GP_{11}\mathbf{w}_+, F_k(\mathbf{w}_-)GP_{11}\mathbf{w}_- \rangle}{\langle -GP_{11}\mathbf{w}_+, P_{11}GV_- \rangle} = \frac{\langle \mathbf{w}_+, F_k(\mathbf{w}_-) \mathbf{w}_- \rangle}{\langle P_{11}GGP_{11}\mathbf{w}_+, \mathbf{v}_- \rangle} = -\frac{\langle \mathbf{w}_+, F_k(\mathbf{w}_-) \mathbf{v}_- \rangle}{\langle \mathbf{w}_+, \mathbf{v}_- \rangle} = -\left(b^{(1)}_-\right)^2.$$
Here we have additionally used that \( F_K(\tilde{w}_+) = F_K(\tilde{w}_-) \), \((GP_{11})^2 = I, \ PT_{11}^2 = I\), and that the operators \( P_{11}G \) and \( GP_{11} \) are Hermitian. The obtained result indicates that if \( b_+^{(1)} \neq 0 \), then the bifurcation of nonlinear modes is possible only from one eigensate, either from \( b_+ \) or from \( b_- = -b_+ \).

As an example, in figure 2 we report numerically obtained families of nonlinear modes at the double point \( D_1 \) where \( \gamma_1 = \gamma_2 = \sqrt{3}/4 \). Two families of unstable nonlinear modes bifurcate from the positive eigenvalue and no nonlinear modes bifurcate from the negative one (hereafter the stability of nonlinear modes has been investigated by means of numerical analysis of the spectrum of the linearized problem).

5.2.4. Double points between \( T_2 \) and \( T_3 \) (between \( T_1 \) and \( T_3 \)). Such double points lie on the hyperbola \( \gamma_2 = 1 - 1/\gamma_1 \) (for the double points located between the triple points \( T_2 \) and \( T_3 \)), or on \( \gamma_2 = -1 - 1/\gamma_1 \) (for the double points between \( T_1 \) and \( T_3 \)). In this case the spectrum of \( H_{mm}(\gamma_1, \gamma_2) \) contains a double zero eigenvalue \( \tilde{b}_0 = 0 \) corresponding to a \( 2 \times 2 \) Jordan block, and hence

\[
\det H_{mm}(\gamma_1, \gamma_2) = 0. \tag{32}
\]

Two other eigenvalues of \( H_{mm}(\gamma_1, \gamma_2) \) are simple, real, and have opposite values.

The peculiarity of these double points is that due to the particular structure of the linear eigenvector \( \tilde{w} \) and the nonlinearity \( F_K(\tilde{w}) \), equation (28) yields that the coefficient \( b^{(1)} \) is equal to zero. Then for any \( b^{(2)} \) equation (27) has a nontrivial solution \( w^{(3)} \). This also implies that \( w^{(3)} = 0 \). Therefore, for a more detailed description of the families bifurcating from the double eigenvalue \( \tilde{b}_0 \), one has to proceed to next orders of the expansions (25)–(26). The modified expansions read

\[
w = \varepsilon \tilde{w} + \varepsilon^3 w^{(3)} + \varepsilon^4 w^{(4)} + \varepsilon^5 w^{(5)} + \ldots, \tag{33}
\]

\[
b = \varepsilon^2 b^{(2)} + \varepsilon^3 b^{(3)} + \varepsilon^4 b^{(4)} + \ldots, \tag{34}
\]

which in the \( \varepsilon^4 \)-order gives \( w^{(4)} = b^{(3)} v_1 \), while in the \( \varepsilon^5 \)-order we obtain

\[
b^{(4)} \tilde{w} + b^{(2)} w^{(3)} = H_{mm} w^{(5)} + f_5, \tag{35}
\]

where \( f_5 \) is the \( \varepsilon^5 \)-order contribution of the nonlinear term \( F_K(w)w \):

\[
f_5 = \tilde{w} \circ \tilde{w} \circ \tilde{w} \circ \tilde{w} \circ \tilde{w} + \tilde{w} \circ \tilde{w} \circ \tilde{w} \circ w^{(3)}, \tag{36}
\]

where we used ‘\( \circ \)’ to designate element-wise multiplication of vectors. From equation (35) we obtain

\[
b^{(2)}(\tilde{w}^*, w^{(3)}) = (\tilde{w}^*, f_5). \tag{37}
\]

This equation together with equation (27) can be used to compute \( b^{(2)} \). To this end, let us first solve equation (27) with \( b^{(2)} = 0 \), i.e.

\[
H_{mm} w^{(3)} = -F_K(\tilde{w}) \tilde{w}, \tag{38}
\]

with respect to \( w^{(3)} \). In spite of the equality (32), equation (38) does have a \( PT \)-invariant solution. Indeed, if \( u \) is an arbitrary solution of equation (38), then applying a \( PT \) operator to both sides of (38) and using the fact that the nonlinearity \( F_K(w) \) is weakly \( PT \)-symmetric (i.e. obeys (13)), we find that \( PT u \) is also solution of the same equation. Thus a \( PT \)-invariant solution of equation (38) is given by \( u_0 = \frac{1}{2}(u + PT u) \).

On the other hand, to construct a solution of equation (38) we let the first entry of \( w^{(3)} \) be a free parameter and express all other entries of \( w^{(3)} \) in terms of the first one. Introducing \( c = w^{(3)}_1 \) (where \( c \) is the free parameter and \( w^{(3)}_1 \) is the first entry of \( w^{(3)} \), for the next entries
we find $w_2^{(3)} = -i \gamma_1 c - |\tilde{w}_2|^2 \tilde{w}_1$, and $w_3^{(3)} = i \gamma_2 |\tilde{w}_1|^2 \tilde{w}_1 - \gamma_1 c - |\tilde{w}_2|^2 \tilde{w}_2$, where $\tilde{w}_j$ are
the entries of the linear eigenvector $\tilde{w}$. (Here we have also used the relation $\gamma_1 \gamma_2 + 1 = \gamma_1$
which is valid for the double points on the boundary between $T_2$ and $T_3$; for the double points
situated on the boundary between $T_1$ and $T_2$ the resulting expressions will be slightly different
but having the same structure.) Looking for a $\mathcal{PT}$-invariant solution (which is shown to exist)
we require $(w_2^{(3)})^* = w_3^{(3)}$, which results in a system of two linear equations with respect to
$c_1 = \text{Re} c$ and $c_2 = \text{Im} c$:

$$
\gamma_1 (c_1 + c_2) = \text{Re}(|\tilde{w}_1|^2 \tilde{w}_1 - |\tilde{w}_2|^2 \tilde{w}_2^*) - i \gamma_2 |\tilde{w}_1|^2 \tilde{w}_1^*), \\
\gamma_1 (c_1 + c_2) = \text{Im}(|\tilde{w}_1|^2 \tilde{w}_1 - |\tilde{w}_2|^2 \tilde{w}_2^*) - i \gamma_2 |\tilde{w}_1|^2 \tilde{w}_1^*).
$$

(39) (40)

While the determinant of the latter system is zero, the system has to be compatible (otherwise,
the $\mathcal{PT}$-invariant solution would not exist). We set $c_2 = 0$, and then the parameter $c$ is fixed
as follows:

$$
c = \text{Re} c = \frac{1}{\gamma_1} \text{Re}(|\tilde{w}_1|^2 \tilde{w}_1 - |\tilde{w}_2|^2 \tilde{w}_2^*) - i \gamma_2 |\tilde{w}_1|^2 \tilde{w}_1^*).
$$

(41)

Therefore, the $\mathcal{PT}$-invariant solution for equation (38) is written as

$$
u_0 = \begin{pmatrix}
c \\
-i \gamma_1 c - |\tilde{w}_1|^2 \tilde{w}_1 \\
i \gamma_1 c - |\tilde{w}_1|^2 \tilde{w}_1^* \\
c
\end{pmatrix}.
$$

(42)

Note that the choice $c_1 = 0$ was not restrictive since the most general $\mathcal{PT}$-invariant solution
of equation (38) can be found as $w^{(3)} = \nu_0 + d\tilde{w}$, where the particular solution $\nu_0$ is fixed by
equations (41)–(42), and where $d$ is an arbitrary real number, which can always be set equal
to zero by means of rescaling of the small parameter $\varepsilon$.

The solution of equation (27) with arbitrary $b^{(2)}$ can be found as $w^{(3)} = \nu_0 + b^{(2)} \nu$. Substituting
the latter expression into equation (37), one obtains a quadratic equation with respect to $b^{(2)}$
(note that all the coefficients of the latter equation are real and expressed in
terms of $\tilde{w}$, $\nu$ and $\gamma_1, 2$).

Resorting at this stage to numerics, we consider two double points: $D_2 = (\frac{1}{2}, -1)$ and
$D_3 = (\frac{1}{3}, \frac{1}{2})$. For both $D_2$ and $D_3$ we obtained the quadratic equations with respect to $b^{(2)}$
and computed their roots which turned out to be real, nonzero and distinct from each other.
Moreover, for the point $D_2$ both roots are of the same sign, while for $D_3$ the roots have opposite
signs. Therefore, both for $D_2$ and $D_3$ two families of nonlinear modes are expected to bifurcate
from the double eigenvalue $\tilde{b} = 0$. This is confirmed by numerical results in figure 2
where we show families of nonlinear modes both for the points $D_2$ and $D_3$. Bifurcation diagrams for
$D_2$ and $D_3$ are similar locally near the double eigenvalue $\tilde{b} = 0$ in terms of existence: both
for $D_2$ and $D_3$ there are two families bifurcating from $\tilde{b}$. Note however that for the point $D_2$
both families bifurcating from $\tilde{b}$ bifurcate to the right (i.e. to the half-plane $b > 0$), while at
the point $D_3$ one of the families bifurcates to the right and another one bifurcates to the left
(i.e. to the half-plane $b < 0$). This behavior is in agreement with different signs of the roots
of the quadratic equations for $b^{(2)}$. We also note that the stability of the bifurcating families
is different: for the point $D_2$ one of the families is stable (in the vicinity of the bifurcation),
while the other one is unstable; on the other hand, for the point $D_3$ both bifurcating families
are stable in the sufficiently small vicinity of the bifurcation. The existence of stable nonlinear
modes in spite of the presence of the exceptional-point singularity is quite remarkable.
5.2.5. Triple points. There exist exactly four triple points $T_j$, $j = 1, 2, 3, 4$. Each triple point corresponds to the situation when the Hamiltonian $H_{nn}(\gamma_1, \gamma_2)$ has a zero eigenvalue $b_0 = 0$ with multiplicity equal to 4. The canonical Jordan representation of $H_{nn}(\gamma_1, \gamma_2)$ in the triple points is given by a $4 \times 4$ Jordan block. Therefore, our analysis in equations (24)–(28) is not applicable in this case. However, for the sake of completeness of our studies, we report numerical results on the behavior of the nonlinear modes bifurcating from the triple points.

The points $T_2$ and $T_3$ correspond to values of $\gamma_1$ given as real roots of the equation $\gamma_1^4 - 2\gamma_1^2 - 2\gamma_1 + 1 = 0$, and $\gamma_2 = 1 - 1/\gamma_1$. For the triple points $T_1$ and $T_4$ one has the equation $\gamma_1^4 - 2\gamma_1^2 + 2\gamma_1 + 1 = 0$, and $\gamma_2 = -1 - 1/\gamma_1$. Note that for each of the triple points $T_j$ one has $\gamma_j^2 + \gamma_{j+1}^2 = 3$.

In figure 2 we show numerical results for the points $T_1 \approx (-0.37, 1.69)$ and $T_2 \approx (1.68, 0.41)$ which feature different bifurcation diagrams. All the found modes are unstable.

6. Nonlinearities allowing for integrals of motion

6.1. General idea

It is known that the integrals of motion are of fundamental importance for the conservative systems, while they do not necessarily exist for the dissipative ones. Although the dissipative systems typically do not conserve the total energy, they can admit other conserved quantities and even be integrable [1]. For linear $\mathcal{P}\mathcal{T}$-symmetric systems, some integrals of motion can be found in an explicit form [42], but in the nonlinear case the only system with known integrals of motion (to the best of the authors’ knowledge) is the exactly integrable dimer [3] (see also [43] where some of the results stemming from the integrable dynamics of a dimer were generalized to the chain of coupled dimers).

So far, we considered how stationary nonlinear modes depend on the character of the linear eigenstate they bifurcate from and on the type of the nonlinearity $F(q)$. Let us now consider how the choice of nonlinearity $F(q)$ can affect the dynamical properties of the system. More specifically, we address the existence of the conserved quantities of the nonlinear system (1).

In this section, we find a condition which must be satisfied by the nonlinear operator $F(q)$ to support at least one integral of motion. Using this result, we report several integrals of motion for a $\mathcal{P}\mathcal{T}$-symmetric quadrimer.

A motivation for our consideration is the known fact that a linear $\mathcal{P}\mathcal{T}$ system (which formally corresponds to equation (1) with $F(q) = 0$) has an integral which is given as the ‘pseudo-power’ $Q = \langle \mathcal{P}q(t), q(t) \rangle$ [42] (this fact can also be easily verified using equation (2)). However, the equality $Q = 0$ generically does not hold for the nonlinear system (1) with $F(q) \neq 0$. To establish conditions for the nonlinear system to admit an integral of motion, we look for a conserved quantity in the form $Q = \langle Aq(t), q(t) \rangle$ where $A$ is an arbitrary (so far) time-independent linear operator. Then (1) yields

$$i\dot{Q} = -\langle AH + F(q)q, q \rangle + \langle [H^\dagger + F^\dagger(q)]Aq, q \rangle.$$  

(43)

For $Q$ to be a conserved quantity it is sufficient to require

$$AH = H^\dagger A$$  

(44)

and

$$AF(q) = F^\dagger(q)A \quad \text{for all } q.$$  

(45)

A particularly interesting case corresponds to a situation when the properties (44)–(45) hold for $A = \mathcal{P}$, where $\mathcal{P}$ is a parity operator. Then condition (44) is equivalent to the pseudo-hermiticity [30] of $H$, see also equation (2). Equation (45) now gives

$$F^\dagger(q) = \mathcal{P}F(q)\mathcal{P} \quad \text{for all } q.$$  

(46)
i.e., for all \( \mathbf{q} \) the nonlinear operator \( F(\mathbf{q}) \) must be pseudo-Hermitian with respect to the same \( \mathcal{P} \) operator as the linear operator \( H \). The nonlinearities \( F(\mathbf{q}) \) obeying the property (46) will be said to be of pseudo-Hermitian type and the class of such nonlinearities will be denoted \( NL_{ph}(\mathcal{P}) \) (with the respective parity operator \( \mathcal{P} \) indicated in brackets).

The following proposition establishes a simple relation between the pseudo-Hermitian and \( \mathcal{P}\mathcal{T} \)-symmetric nonlinearities introduced in section 3.

**Proposition 3.** If \( F(\mathbf{q}) \in NL_{ph}(\mathcal{P}) \) and \( F(\mathbf{q}) = F^T(\mathbf{q}) \), then \( F \in NL_{ph}(\mathcal{P}). \)

### 6.2. Example: a \( \mathcal{P}\mathcal{T} \)-symmetric quadramer with integrals of motion

Turning now to a particular example, we consider the dynamics of the system (1) described by the linear Hamiltonian \( H_{bc}(\gamma_1, \gamma_2) \) defined by equation (19) and by the nonlinearity \( F_{bc}(\mathbf{q}) \) given by (20). As has already been noted in section 4.3, the linear operator \( H_{bc}(\gamma_1, \gamma_2) \) is \( \mathcal{P}\mathcal{T} \)-symmetric (see equation (21) for the definition of \( \mathcal{P}_{\mathcal{T}} \)), and the nonlinearity \( F_{bc}(\mathbf{q}) \) is \( \mathcal{P}\mathcal{T} \) symmetric, i.e. \( F_{bc}(\mathbf{q}) \in NL_{\mathcal{P}\mathcal{T}}(\mathcal{P}_{\mathcal{T}}) \). Due to proposition 3, the latter fact together with the diagonal structure of \( F_{bc}(\mathbf{q}) \) implies that \( F_{bc}(\mathbf{q}) \in NL_{ph}(\mathcal{P}_{\mathcal{T}}) \) (this can also be checked in a straightforward manner). Hence there exists at least one integral of motion given by

\[
\mathcal{Q}_1 = \langle \mathcal{P}_{\mathcal{T}} \mathbf{q}, \mathbf{q} \rangle = 2 \text{Re} \left( q_1^* q_3 + q_2^* q_4 \right).
\]  

(47)

Furthermore, the model (19)–(20) possesses another integral of motion. To find it, we note that operator \( H_{bc}(\gamma_1, \gamma_2) \) is also \( \mathcal{P}_{\mathcal{T}}\mathcal{T} \)-symmetric with respect to

\[
\mathcal{P}_{\mathcal{T}} = \sigma_2 \otimes \sigma_0 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.
\]  

(48)

Note that while the operator \( \mathcal{P}_{\mathcal{T}} \) is Hermitian, it cannot be considered as a conventional parity operator because it does not commute with the operator \( \mathcal{T} \) which in our case is the complex conjugation. However, one can check that \( F_{bc}^\dagger(\mathbf{w}) = F_{bc}(\mathbf{w}) = \mathcal{P}_{\mathcal{T}} F_{bc}(\mathbf{w}) \mathcal{P}_{\mathcal{T}} \), i.e. \( F_{bc}(\mathbf{q}) \in NL_{ph}(\mathcal{P}_{\mathcal{T}}) \). Thus the second integral of motion is readily found to be

\[
\mathcal{Q}_2 = \langle \mathcal{P}_{\mathcal{T}} \mathbf{q}, \mathbf{q} \rangle = 2 \text{Im} \left( q_1^* q_3 + q_2^* q_4 \right).
\]  

(49)

Obviously, \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) can be combined in a single complex-valued integral

\[
\mathcal{Q} = q_1^* q_3 + q_2^* q_4.
\]  

(50)

The integral (50) can also be obtained from a more general consideration. Indeed, considering the operator

\[
A = \begin{pmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \end{pmatrix}
\]  

(51)

where \( a_{1,2} \) are arbitrary, one can verify that both conditions (44)–(45) hold. Then a conserved quantity is given as

\[
\langle \mathcal{A} \mathbf{q}, \mathbf{q} \rangle = a_1(q_1^* q_3 + q_2^* q_4) + a_2(q_1 q_3^* + q_2 q_4^*).
\]  

(52)

Setting \( a_2 = 0 \) one readily obtains integral (50).

It turns out, however, that in a particular case when the linear part is given by \( H_{bc}(\gamma, -\gamma) \), and \( x = 1 \) in equation (20), the obtained quantity \( \mathcal{Q} \) is not the only integral of motion. Indeed, let us consider the model (18a)–(18d) with \( x = 1 \):

\[
i q_1 = i \gamma q_1 + q_2 + (|q_1|^2 + |q_3|^2) q_1,
\]  

(53a)
dimer in an implicit form have been found in [3].

Next, we introduce the pseudo-electric field

Further, multiplying equations (56) by

Using equations (54) one can express

We also introduce the Stokes vectors $S_j = (S_j^1, S_j^2, S_j^3)$ with $j = 1, 2$, and note that

Next, we introduce the pseudo-electric field $E = (0, 0, 2 \gamma')$, pseudo-magnetic fields $B_0 = (-2, 0, 0)$ and $B = (0, 0, S_2^2 - S_1^2)$, and rewrite the system (53a)–(53d) in the form

Further, multiplying equations (56) by $B$, we obtain ($j = 1, 2$)

and multiplying equations (56) by $B_0$ we obtain

Combining (57) and (58) yields the relations

After subtracting one of equations (59) from the other and using the definitions of the vectors $B$, $B_0$ and $E$, we arrive at

where the integral of motion $J$ is given by

Using equations (54) one can express $J$ only through the components of the three-dimensional Stokes vectors $S_{1,2}$.

Note that in the limit $q_3 = q_4 = 0$, i.e. $S_2^1 = 0$ and $S_2 = 0$, the obtained integral is reduced to one of the known integrals for the $PT$-symmetric dimer [3].

Finally, we note that the system (53a)–(53d) admits a solution in quadratures in the particular case when $Q = 0$. Indeed, this implies that $|q_1|^2|q_3|^2 = |q_2|^2|q_4|^2$, which is equivalent to $S_1^1S_2^2 = S_1^2S_2^1$. Combing the latter relation with equations (55), we have $S_1^0S_2^0 = S_2^0S_1^0$, and hence $S_1^0 = CS_2^0$, where $C$ is a constant. Subsequently $E \cdot S_1 = CE \cdot S_2$, i.e. $S_1^1 = CS_2^1$ and $B = (1 - C)(0, 0, S_2^2)$. The latter formula means that the dynamical equation for $S_2$ is singled out and acquires the form of the equation for the dimer. Solutions for such a dimer in an implicit form have been found in [3].

$\gamma_2 = -i\gamma q_2 + q_1 + (|q_2|^2 + |q_4|^2)q_2,$

$\gamma_3 = -i\gamma q_3 + q_4 + (|q_3|^2 + |q_1|^2)q_3,$

$\gamma_4 = i\gamma q_4 + q_3 + (|q_4|^2 + |q_2|^2)q_4.$
7. Conclusion

In the present paper we have investigated some of the nonlinear properties of finite-dimensional systems respecting $\mathcal{PT}$ symmetry. In contrast to most of the previous studies, our discussion has not been focused on a nonlinearity of any given form. Instead, we have emphasized how nonlinearities of different classes can affect the stationary and dynamical properties of the system. First, we have considered a class of nonlinearities with weak $\mathcal{PT}$ symmetry, the latter appearing to be necessary for the existence of the families of stationary nonlinear modes. We have paid particular attention to the analysis of the bifurcations of nonlinear modes from multiple eigenstates of the underlying linear problem. In a situation when the underlying linear system has a semi-simple eigenstate, we have shown that the invariant subspace associated with the degenerate eigenvalue can always be spanned by a complete basis of $\mathcal{PT}$-invariant eigenvectors. This fact has been used to construct formal expansions for the nonlinear modes bifurcating from the degenerate eigenstate. Next, we considered bifurcations of stationary modes from exceptional-points, which typically occur at the phase transition between unbroken and broken $\mathcal{PT}$ symmetries. We have shown that the generalized eigenvector associated with the multiple eigenvalue can be chosen to be $\mathcal{PT}$ invariant. Then we developed small-amplitude expansions for the bifurcations of nonlinear modes and demonstrated that the bifurcations, as well as stability of the modes, depend on both the nonlinearity and the character of the coalescing eigenstates.

To complete the above picture, we note that the nonlinear modes can also exist in the case when the $\mathcal{PT}$ symmetry of the underlying linear problem is broken, i.e. the spectrum of the underlying linear operator $H(\gamma)$ contains complex eigenvalues. If all the linear eigenvalues are complex, then the nonlinear modes obviously cannot bifurcate from the linear eigenstates, i.e. such nonlinear modes (if they exist) do not have a linear counterpart. Moreover, in the case of a finite-dimensional system (like a $\mathcal{PT}$-symmetric oligomer) nonlinear modes can be stable even if the $\mathcal{PT}$ symmetry of the linear problem is broken [4].

Finally, we turned to the dynamical properties of nonlinear $\mathcal{PT}$-symmetric lattices and indicated another important class of nonlinearities, termed pseudo-Hermitian nonlinearities, which allow the nonlinear system to admit at least one integral of motion. Using this idea, we have found several integrals for a $\mathcal{PT}$-symmetric nonlinear quadrimer and demonstrated that (at least in some cases) it admits a solution in quadratures.

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References

[1] Akhmediev N and Ankiewicz A 2005 Dissipative solitons in the complex Ginzburg–Landau and Swift–Hohenberg equations Dissipative Solitons (Lecture Notes in Physics vol 661) ed N Akhmediev and A Ankiewicz (Berlin: Springer) pp 1–17
[2] Bender C M 2007 Rep. Prog. Phys. 70 947
[3] Ramezani H, Kottos T, El-Ganainy R and Christodoulides D N 2010 Phys. Rev. A 82 043803
[4] Zezyulin D A and Konotop V V 2012 Phys. Rev. Lett. 108 213906
[5] Li K, Zezyulin D A, Konotop V V and Kevrekidis P G 2013 Phys. Rev. A 87 033812
[6] Kevrekidis P G, Pelinovsky D E and Tyugin D Y 2013 SIAM J. Appl. Dyn. Syst. 12 1210
[7] Leykam D, Konotop V V and Desyatnikov A S 2013 Opt. Lett. 38 371
[8] Li K and Kevrekidis P G 2011 Phys. Rev. E 83 066608
[9] Li K, Kevrekidis P G, Malomed B A and Günter U 2012 J. Phys. A: Math. Theor. 45 444021
[10] Suchkov S V, Malomed B A, Dmitriev S V and Kivshar Yu S 2011 Phys. Rev. E 84 046609
[11] Konotop V V, Pelinovsky D E and Zezyulin D A 2012 Europhys. Lett. 100 56006
[12] Musslimani Z H, Makris K G, El-Ganainy R and Christodoulides D N 2008 Phys. Rev. Lett. 100 030402
[13] Nixon S, Ge L and Yang J 2012 Phys. Rev. A 85 023822
[14] Moreira F C, Konotop V V and Malomed B A 2013 Phys. Rev. A 87 013832
[15] Zezyulin D A and Konotop V V 2012 J. Phys. A: Math. Theor. 45 444021
[16] Wang H and Wang J 2011 Opt. Express 19 4030
[17] Moreira F C, Abdullaev F K, Konotop V V and Yulin A V 2012 Phys. Rev. A 86 053815
[18] Zhu X, Wang H, Zheng L X, Li H and He Y J 2011 Opt. Lett. 36 2680
[19] Abdullaev F K, Kartashov Y V, Konotop V V and Zezyulin D A 2011 Phys. Rev. A 83 041805
[20] Zezyulin D A, Kartashov Y V and Konotop V V 2011 Europhys. Lett. 96 64003
[21] Miroshnichenko A E, Malomed B A and Kivshar Yu S 2011 Phys. Rev. A 84 012123
[22] Duannu M, Li K, Horne R L, Kevrekidis P G and Whitaker N 2013 Phil. Trans. R. Soc. A 371 20120171
[23] He Y, Zhu X, Mihalache D, Liu J and Chen Z 2012 Phys. Rev. A 85 013831
[24] Driben R and Malomed B A 2011 Opt. Lett. 36 4223
[25] Barashenkov I V, Suchkov S V, Sukhorukov A A, Dmitriev S V and Kivshar Yu S 2012 Phys. Rev. A 86 053809
[26] Barashenkov I V, Baker L and Alexeeva N V 2013 Phys. Rev. A 87 033819
[27] Graefe E-M 2012 J. Phys. A: Math. Theor. 45 444015
[28] Cartarius H and Wunner G 2012 Phys. Rev. A 86 013612
[29] Kreibich M, Main J, Cartarius H and Wunner G 2013 Phys. Rev. A 87 051601
[30] Mostafazadeh A 2002 J. Math. Phys. 43 205
[31] Kato T 1980 Perturbation Theory for Linear Operators (Berlin: Springer)
[32] Bender C M, Meisinger P N and Wang Q 2003 J. Phys. A: Math. Gen. 36 6791
[33] Graefe E-M and Jones H 2011 Phys. Rev. A 84 013818
[34] Ramezani H, Kottos T, Kovanis V and Christodoulides D 2012 Phys. Rev. A 85 041805
[35] Bender C M, Brody D C and Jones H F 2002 Phys. Rev. Lett. 89 270401
[36] Heiss W D 2012 J. Phys. A: Math. Theor. 45 444016
[37] MacKay R S and Aubry S 1994 Nonlinearity 7 1623
[38] Nixon S, Zhu Y and Yang J 2012 Opt. Lett. 37 4874
[39] Nixon S and Yang J 2013 Opt. Lett. 38 1933
[40] Mostafazadeh A 2002 J. Math. Phys. 43 205
[41] Bender C M and Gianfreda M 2013 Twofold transition in PT-symmetric coupled oscillators arXiv:1305.7107 [hep-th]
[42] Bagchi B, Quesne C and Znojil M 2001 Mod. Phys. Lett. A 16 2047
[43] Kevrekidis P G, Pelinovsky D E and Tyugin D Y 2013 J. Phys. A: Math. Theor. 46 365201