CONFORMALLY EINSTEIN PRODUCT SPACES

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Abstract. We study pseudo-Riemannian Einstein manifolds which are conformally equivalent with a metric product of two pseudo-Riemannian manifolds. Particularly interesting is the case where one of these manifolds is 1-dimensional and the case where the conformal factor depends on both manifolds simultaneously. If both factors are at least 3-dimensional then the latter case reduces to the product of two Einstein spaces, each of the special type admitting a non-trivial conformal gradient field. These are completely classified. If each factor is 2-dimensional, there is a special family of examples of non-constant curvature (called extremal metrics by Calabi), where in each factor the gradient of the Gaussian curvature is a conformal vector field. Then the metric of the 2-manifold is a warped product where the warping function is the first derivative of the Gaussian curvature. Moreover we find explicit examples of Einstein warped products with a 1-dimensional fibre and such with a 2-dimensional base. Therefore in the 4-dimensional case our Main Theorem points towards a local classification of conformally Einstein products. Finally we prove an assertion in the book by A.Besse on complete Einstein warped products with a 2-dimensional base. All solutions can be explicitly written in terms of integrals of elementary functions.

1. Introduction and notations

We consider a pseudo-Riemannian manifold \((M, g)\), which is defined as a smooth \(n\)-manifold \(M\) (here smooth means of class \(C^4\)) together with a pseudo-Riemannian metric of arbitrary signature \((j, n - j), 0 \leq j \leq n\). All manifolds are assumed to be connected. A conformal mapping between two pseudo-Riemannian manifolds \((M, g), (N, h)\) is a smooth mapping \(F : (M, g) \rightarrow (N, h)\) with the property \(F^*h = \varphi^{-2}g\) for a smooth positive function \(\varphi : M \rightarrow \mathbb{R}_+\). In more detail this means that the equation

\[
h_F(x)(dF_x(X), dF_x(Y)) = \varphi^{-2}(x)g_x(X, Y)
\]

holds for all tangent vectors \(X, Y \in T_xM\). Particular cases are homotheties for which \(\varphi\) is constant, and isometries for which \(\varphi = 1\).

A (local) one-parameter group \(\Phi_t\) of conformal mappings of a manifold into itself generates a conformal (Killing) vector field \(V\), sometimes also called an infinitesimal conformal transformation, with \(V = \frac{d}{dt}\Phi_t\). We need to assume that \(V\) itself is of class at least \(C^3\). Conversely, any conformal vector field generates a local one-parameter group of conformal

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mappings. It is well known [45] that a vector field \( V \) is conformal if and only if the Lie derivative \( \mathcal{L}_V g \) of the metric \( g \) in direction of the vector field \( V \) satisfies the equation

\[
\mathcal{L}_V g = 2\sigma g
\]

for a certain smooth function \( \sigma : M \to \mathbb{R} \). Necessarily this conformal factor \( \sigma \) coincides with the divergence of \( V \), up to a constant: \( \sigma = \text{div} V / n \). Particular cases of conformal vector fields are homothetic vector fields for which \( \sigma \) is constant, and isometric vector fields, also called Killing vector fields, for which \( \sigma = 0 \). On a (pseudo-)Euclidean space the divergence of a conformal vector field is always a linear function.

Furthermore it is well known that the image of a lightlike geodesic under any conformal mapping is again a lightlike geodesic and that for any lightlike geodesic \( \gamma \) and any conformal vector field \( V \) the quantity \( g(\gamma', V) \) is constant along \( \gamma \). Conformal vector fields \( V \) with non-vanishing \( g(V, V) \) can be made into Killing fields within the same conformal class of metrics, namely, for the metric \( \tilde{g} = |g(V, V)|^{-1}g \). This is a special case of a so-called inessential conformal vector field.

A vector field \( V \) on a pseudo-Riemannian manifold is called \textit{closed} if it is locally a gradient field, i.e., if locally there exists a function \( f \) such that \( V = \text{grad} f \). Consequently, from Equation 1 and \( \mathcal{L}_V g = 2\nabla^2 f \) we see that a closed vector field \( V \) is conformal if and only

\[
\nabla_X V = \sigma X
\]

for all \( X \) or, equivalently \( \nabla^2 f = \sigma g \). Here \( \nabla^2 f(X, Y) = g(\nabla_X \text{grad} f, Y) \) denotes the Hessian \((0, 2)\)-tensor and \( n\sigma = \Delta f = \text{div} (\text{grad} f) \) is the Laplacian of \( f \). If the symbol \( (\ )^\circ \) denotes the traceless part of a \((0, 2)\)-tensor, then \( \text{grad} f \) is conformal if and only if \( (\nabla^2 f)^\circ \equiv 0 \). This equation

\[
(\nabla^2 f)^\circ = 0
\]

allows explicit solutions in many cases, for Riemannian as well as for pseudo-Riemannian manifolds.

As usual,

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
\]

denotes the \textit{curvature} \((1, 3)\)-tensor. Then the \textit{Ricci tensor} as a symmetric \((0, 2)\)-tensor is defined by the equation

\[
\text{Ric}(X, Y) = \text{trace}(V \mapsto R(V, X)Y).
\]

The associated \((1, 1)\) tensor is denoted by \( \text{ric} \). Thus \( \text{Ric}(X, Y) = g(\text{ric}(X), Y) \). Its trace \( S = \text{trace}(V \mapsto \text{ric}(V)) \) is called the \textit{scalar curvature}. A manifold is \textit{conformally flat}, if every point has a neighborhood which is conformally equivalent to an open subset of a pseudo-Euclidean space.

A pseudo-Riemannian manifold is called an \textit{Einstein space} if the equation

\[
\text{Ric} = \lambda g
\]

holds with a factor \( \lambda = S / n \) which is necessarily constant if \( n \geq 3 \) and which is then called the \textit{Einstein constant}. On a surface, i.e. for \( n = 2 \) a pseudo-Riemannian metric
is called Einstein, if it has constant Gaussian curvature. For convenience the normalized
Einstein constant or normalized scalar curvature will be denoted by \( k = \lambda / (n - 1) \) so that
we have \( k = 1 \) on the unit sphere of any dimension. For \( n = 2 \) we have \( \lambda = k = S/2 = K \)
(Gaussian curvature). For a survey on Einstein spaces in general we refer to [1], [11]. A
short version of this article is [29].

2. Conformally Einstein spaces: Basic equations

We start with the very basic formula for the change of the Ricci tensor under a conformal
change of the metric. This formula provides a way to classify all conformal mappings
between two Einstein spaces. It is – however – more difficult to classify those Riemannian
metrics that are conformally Einstein, see [5], [17], [16], [18], [34]. This is somehow
in contrast with the case of conformally flat metrics. It is clear that a 3-dimensional
metric is conformally Einstein if and only if it is conformally flat. The problem becomes
interesting in dimensions \( n \geq 4 \). Here it is well known that a harmonic Weyl tensor
is a necessary condition for a metric to be conformally Einstein. However, sufficient or
equivalent conditions are difficult to obtain, see [31], [32]. The case of conformally Einstein
Kähler metrics was treated in [12], [33].

Lemma 2.1. The following formula holds for any conformal change \( g \mapsto \bar{g} = \varphi^{-2} g \) of a
metric on an \( n \)-dimensional manifold:

\[
\bar{\text{Ric}} - \text{Ric} = \varphi^{-2} \left( (n - 2) \cdot \varphi \cdot \nabla^2 \varphi + \left[ \varphi \cdot \Delta \varphi - (n - 1) \cdot \| \text{grad} \varphi \|^2 \right] \cdot g \right).
\]

Consequently, the metric \( \bar{g} \) is Einstein if and only if the equation

\[
\varphi \cdot \text{Ric} + (n - 2) \cdot \nabla^2 \varphi = \theta \cdot g
\]

holds for some function \( \theta \) or, equivalently,

\[
\varphi \cdot (\text{Ric})^\circ + (n - 2) \cdot (\nabla^2 \varphi)^\circ = 0
\]

where \(( \ )^\circ\) denotes the trace-free part.

Equation [6] follows from the relationship between the two Levi-Civita connections \( \nabla, \bar{\nabla} \)
associated with \( g \) and \( \bar{g} \):

\[
\bar{\nabla}_X Y - \nabla_X Y = -X(\log \varphi)Y - Y(\log \varphi)X + g(X, Y) \text{grad}(\log \varphi).
\]

Equation [6] is a standard formula, see [11, 1.159] or [23, 8.27].

Corollary 2.2. A metric \( g \) on a manifold \( M \) is (locally or globally) conformally Einstein
if and only if there is a (local or global) positive solution \( \varphi \) of the equation

\[
\varphi \cdot (\text{Ric})^\circ + (n - 2) \cdot (\nabla^2 \varphi)^\circ = 0
\]

Example 2.3. (conformal cylinder, generalized Mercator projection)
Let \( M_* \) be an \( (n - 1) \)-dimensional Einstein space with Einstein constant \( \lambda_* = n - 2 \).
Then the cylinder \( M = \mathbb{R} \times M_* \) with the product metric \( g = dt^2 + g_* \) is conformally
Einstein: The metric \( \bar{g} = \cosh^2 t \cdot g \) is Einstein with \( \bar{\lambda} = n - 1 \). If \( M_* \) is the unit
\( (n - 1) \)-sphere then \((M, g)\) is a cylinder representing the (classical but \( n \)-dimensional)
Mercator projection from the n-sphere without north and south pole. With \( \varphi = \cosh t \) the equation in Corollary 2.2 is satisfied by the block matrix structure

\[
\text{Ric} = \begin{pmatrix} 0 & 0 \\ 0 & \text{Ric}_* \end{pmatrix}, \quad \nabla^2 \varphi = \begin{pmatrix} \varphi'' & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
(\text{Ric})^o = \begin{pmatrix} -(n-1)(n-2)/n & 0 \\ 0 & -n/n g_* \end{pmatrix}, \quad (\nabla^2 \varphi)^o = \begin{pmatrix} (n-1)\varphi/n & 0 \\ 0 & -n/g_* \end{pmatrix}.
\]

In the special case of a compact Einstein space \( M_* \) this generalized Mercator projection is the result of [34, Thm.2.1], see Corollary 2.10. The transition from the conformal cylinder \( \overline{g} = \cosh^{-2} t (dt^2 + g_*) \) to the more familiar version \( \overline{g} = dr^2 + \sin^2 r \, g_* \) in parallel coordinates around the equator with \( r = \pi/2 \) is achieved by the parameter transformation \((-\infty, \infty) \ni t \mapsto r(t) \in (0, \pi) \) with \( \sin r = \cosh^{-1} t \) and \( dr/dt = \cosh^{-1} t \) leading to the Gudermann function \( r(t) = \int_{-\infty}^{t} \cosh^{-1} r \, d\tau = 2 \arctan e^t \). However, \( \overline{g} \) is not complete, as the classical Mercator projection shows.

Similarly, the metric \( \overline{g} = \sin^{-2} t (dt^2 + g_*) \) is an Einstein metric on the cylinder \( \overline{M} = (0, \pi) \times M_\lambda \) with \( \lambda = -(n-1) \) if \( (M_*, g_*) \) is Einstein with \( \lambda_* = -(n-2) \). Moreover, if \( (M_*, g_*) \) is the hyperbolic \((n-1)\)-space then \( (\overline{M}, \overline{g}) \) is isometric to the hyperbolic \( n \)-space with the metric \( \overline{g} = dr^2 + \cosh^2 r \, g_*, \, r \in \mathbb{R} \). The transformation is given by \( \sin t = \cosh^{-1} r \) and \( t(r) = 2 \arctan e^r \). One could call this a hyperbolic Mercator projection. In particular, \( \overline{g} \) is complete if \( g_* \) is Riemannian and complete.

**Corollary 2.4.** (Brinkmann [6])

If \( g \) is an Einstein metric then \( \overline{g} \) is also an Einstein metric if and only if \( (\nabla^2 \varphi)^o = 0 \).

This follows directly from Equation 7. This case of conformal changes between two Einstein metrics was studied by many authors, starting with Brinkmann [6]. The equation \( (\nabla^2 \varphi)^o = 0 \) can be explicitly solved, compare [28]. Roughly the results are the following: As long as \( g(\text{grad} \varphi, \text{grad} \varphi) \neq 0 \), the metric is a warped product \( g = \epsilon dt^2 + (\varphi'(t))^2 g_*, \) with an \((n-1)\)-dimensional Einstein space \( (M_*, g_*) \), \( \epsilon = \pm 1 \), and where \( \varphi \) depends only on \( t \) and satisfies the following differential equations:

\[
\varphi'' + \epsilon k \varphi' = 0, \quad (\varphi'')^2 + \epsilon k (\varphi')^2 = \epsilon k_*
\]

By the second equation the normalized scalar curvature \( k_* \) of \( g_* \) appears as a constant of integration for the first equation. We can integrate the first equation also as \( \varphi'' + \epsilon k \varphi = c \) but this constant \( c \) is not essential: We can add freely a constant to the function \( \varphi \) itself without changing the equation. However, \( c \) becomes essential if \( \varphi \) becomes a conformal factor. If \( g(\text{grad} \varphi, \text{grad} \varphi) = 0 \) on an open subset then we have \( \nabla^2 \varphi = 0 \) and \( \text{Ric} = \text{Ric}^\overline{g} = 0 \), see [28, Thm.3.12].

**Lemma 2.5.** For a real constant \( c \) and a positive function \( \varphi \) we have the following equations for the function \( \varphi^c \):

\[
\text{grad} \varphi^c = c \varphi^{c-1} \text{grad} \varphi
\]

\[
\nabla^2 \varphi^c = c \varphi^{c-1} \nabla^2 \varphi + c(c-1) \varphi^{c-2} d\varphi \otimes d\varphi
\]
Proof. The first equation is obvious from the chain rule $d(\varphi^c) = c\varphi^{c-1} \cdot d\varphi$. For the second equation we calculate
\[
\nabla_X \text{grad}\varphi^c = \nabla_X (c\varphi^{c-1} \cdot \text{grad}\varphi) = c\varphi^{c-1} \cdot \nabla_X \text{grad}\varphi + cX(\varphi^{c-1}) \cdot \text{grad}\varphi
\]
\[
= c\varphi^{c-1} \cdot \nabla_X \text{grad}\varphi + c(c-1)\varphi^{c-2} \cdot d\varphi(X) \cdot \text{grad}\varphi.
\]

Corollary 2.6. If a function $\varphi > 0$ on an $n$-dimensional manifold $(M, g)$ satisfies the equation $\varphi \cdot \text{Ric} + (n-1) \cdot \nabla^2\varphi = 0$ then with $c := \frac{n-1}{n-2}$ the metric $\overline{g} = \varphi^{-2}g$ satisfies the equation
\[
\frac{n-2}{n-1} \cdot \overline{\text{Ric}} = \overline{\varphi} \cdot \overline{g} + \varphi^{-2} \cdot d\varphi \otimes d\varphi
\]
for some scalar function $\overline{\varphi}$. Therefore $\overline{g}$ is a quasi Einstein metric in the sense of [8].

Example 2.7. A quasi Einstein metric conformally equivalent with the hyperbolic space. We start with the hyperbolic space $(\mathbb{H}^n, g_{-1})$ with the metric in polar coordinates $g_{-1} = dr^2 + \sinh^2(r)g_1$ where $g_1$ is the metric on the unit $(n-1)$-sphere. The function $\varphi(r) = \cosh(r)$ satisfies the equation $\nabla^2\varphi = \varphi g_{-1}$, and $\text{Ric} = -(n-1)g_{-1}$. Let $c := \frac{n-1}{n-2}$. Then the metric
\[
\overline{g} = \varphi^{-2}g_{-1} = \cosh^{-2(n-1)/(n-2)}(r)g_{-1}
\]
is quasi Einstein by Corollary 2.6 but not Einstein. More precisely we have
\[
\frac{n-2}{n-1} \cdot \overline{\text{Ric}} = -n \tanh^2(r) \cdot g_{-1} + \cosh^{-2}(r) \cdot d\varphi \otimes d\varphi.
\]

3. Conformally Einstein Products: Main Theorem

In the special case of an Einstein space that is conformally equivalent with a Riemannian product of two manifolds several results have been obtained, see [10], [34], [18], [40], [41]. If the conformal factor depends only on one side this is close to the case of an Einstein warped product which – in general – is not yet solved. If the conformal factor depends on both sides of the product then we can completely classify the solutions. Our Main Theorem 3.2 studies precisely this case, thus generalizing and completing the main result of [9]. Here the 4-dimensional case is special and leads to the class of extremal metrics in the sense of E.Calabi [11, 18.4]. We start with the simplest case of a function that depends only on one real variable.

Proposition 3.1. (The type $\mathbb{R} \times M_\ast$ with a 1-dimensional base)

If $f$ is a non-constant function depending only on the real parameter $t$ then the metric $\overline{g} = f^{-2}(cdt^2 + g_\ast)$ is Einstein if and only if $(M_\ast, g_\ast)$ is an $n$-dimensional Einstein space and $f$ satisfies the ODE $k_\ast f^2 - \epsilon(f')^2 = \overline{k}$.

Proof. We regard $f$ also as a function on the product with $M_\ast$. Then for $g = \epsilon dt^2 + g_\ast$ we have the block matrix structure
\[
\text{Ric} = \begin{pmatrix} 0 & 0 \\ 0 & \text{Ric}_\ast \end{pmatrix}, \quad \nabla^2 f = \epsilon f''(dt^2) = \begin{pmatrix} \epsilon f'' & 0 \\ 0 & 0 \end{pmatrix}
\]
and $\Delta f = \epsilon f''$, $||\text{grad} f||^2 = \epsilon (f')^2$. This implies

$$\text{Ric} = \text{Ric} + (n-1)f^{-1} \nabla^2 f + [\epsilon f^{-1} f'' - \epsilon n f^{-2} (f')^2] \cdot (\epsilon dt^2 + g_s).$$

From the block matrix structure it follows that $\overline{g}$ is Einstein if and only if two equations are satisfied: The first one is the scalar equation

$$\overline{\lambda} = \epsilon n [f f'' - f'^2],$$

the second one is the tensor equation

$$\overline{\lambda} g_s = f^2 \text{Ric}_s + \epsilon [f f'' - n f'^2] g_s$$

where $\overline{\lambda}$ is the Einstein constant of $\overline{g}$. A necessary condition is that $g_s$ is also Einstein with $\text{Ric}_s = \lambda_s g_s$ where $\lambda_s$ depends only on $t$ and $\overline{\lambda}$. Therefore $\lambda_s$ is constant even if $n = 2$. In the Riemannian case $\epsilon = 1$ we obtain $n(f f'' - f'^2) = \overline{\lambda} = f^2 \lambda_s + f f'' - n f'^2$ and therefore

$$f'' = \frac{\lambda_s}{n} f = k_s f.$$

Here the coefficients $k_s = \lambda_s/(n-1)$ and $\overline{k} = \overline{\lambda}/n$ are the normalized scalar curvatures of $g_s$ and $\overline{g}$, respectively.

In the general case we get $\epsilon n [f f'' - f'^2] = \overline{\lambda} = f^2 \lambda_s + f f'' - n f'^2$ and therefore

$$f'' = \epsilon k_s f.$$

We obtain the explicit relationship between $k_s$ and $\overline{k}$ by inserting the last equation into the preceding one

$$k_s f^2 - \epsilon f'^2 = \overline{k}.$$

We remark that an equation of the type $y'' = F(y, y')$ can be integrated in two steps, see [33 §11.VI]. Here $k_s$ appears as one constant of integration. Up to scaling we have to consider only the cases $\overline{k} = 1, 0, -1$. The case $k_s = 0$ leads to linear solutions by $f'' = 0$. Furthermore the cases of $k_s = \pm 1$ are particularly simple to handle. They lead to the standard equation $(f')^2 \pm f^2 = 0$ or $(f')^2 \pm f^2 = \pm 1$ with all possible combinations of signs.

In the Riemannian case $\epsilon = 1$ and with the initial conditions $f(0) = 1, f'(0) = 0$ we have the following solutions:

$f(t) = \cosh t$ if $\overline{k} = 1, k_s = 1$ (this case corresponds to [34 Thm 2.1], compare the Mercator projection in Example 2.3),

$f(t) = \cos t$ if $\overline{k} = -1, k_s = -1$ (compare Example 2.3),

$f(t) = 1$ if $\overline{k} = 0, k_s = 0$.

With the initial conditions $f(0) = 1, f'(0) = 1$ we have the solutions:

$f(t) = e^t$ if $\overline{k} = 0, k_s = 1$ (this case coincides with the flat metric $dr^2 + r^2 g_s$ in polar coordinates with $r = e^{-t}$ if $M_s$ is the unit sphere)

$f(t) = t + 1$ if $\overline{k} = -1, k_s = 0$,

$f(t) = \frac{1}{2\sqrt{2}} [(\sqrt{2} + 1)e^{\sqrt{2}t} + (\sqrt{2} - 1)e^{-\sqrt{2}t}]$ if $\overline{k} = 1, k_s = 2$.

In the last case the initial condition implies $k_s = 2$ leading to the equation $f'' = 2f$. This explains the factor $\sqrt{2}$. 


With the initial conditions \( f(0) = 0, f'(0) = 1 \) we have the solutions:

\[ f(t) = \sinh t \text{ if } \vec{k} = -1, k_\ast = 1, \]
\[ f(t) = \sin t \text{ if } \vec{k} = -1, k_\ast = -1 \text{ (compare Example 2.3),} \]
\[ f(t) = t \text{ if } \vec{k} = -1, k_\ast = 0 \text{ (this is known as the Poincaré halfspace model for the hyperbolic space if } g_\ast \text{ is flat).} \]

All these solutions define admissible metrics only outside the zeros of \( f \). The missing solution with \( f(0) = f'(0) = 0 \) is trivial and occurs only for \( \vec{k} = 0 \). It does not lead to a metric \( \vec{g} \).

**Remark:** In principle this classification is similar to that of Einstein metrics as warped products \( dt^2 + \phi^2(t)g_\ast \) in Corollary 2.4. The differential equations are similar, and so are the solutions. Nevertheless, from the global point of view, there are essential differences.

In more generality we have the following theorem on products of pseudo-Riemannian manifolds of arbitrary dimensions. In the Riemannian case it is stated in [9] but the 4-dimensional case of a product of two surfaces with nonconstant curvature is missing there. Therefore Theorem 1 in [9] is not literally true in dimension four.

**Theorem 3.2.** (Main Theorem on conformally Einstein products)

Let \((M^n, \vec{g})\) and \((M^\ast_n, g_\ast)\) be pseudo-Riemannian manifolds with \( n + n_\ast \geq 3 \). If \( f(y, x) \) is a non-constant function depending on \( y \in M \) and \( x \in M_\ast \) and if the metric \( \vec{g} = f^{-2}(\vec{g} + g_\ast) \) on \( M \times M_\ast \) is Einstein then one of the following cases occurs:

1. \( \vec{g} \) is a warped product, i.e., \( f \) depends only on one of the factors \( M \) or \( M_\ast \). Moreover the fibre is an Einstein space.
2. \( f(y, x) = a(y) + b(x) \) with non-constant \( a \) and non-constant \( b \), and both \((M, g)\) and \((M_\ast, g_\ast)\) are Einstein spaces with normalized scalar curvatures \( \vec{k}, k_\ast \), and \( f \) satisfies the equation \( \vec{\nabla}^2 a = \frac{\Delta_a}{n} \vec{g} \) and, simultaneously, \( b \) satisfies the equation \( \nabla^2 b = \frac{\Delta_b}{n} \vec{g} \).

If \( n \geq 3 \) or \( n_\ast \geq 3 \) then we have necessarily \( \vec{\nabla}^2 a = (-\vec{k}a + c)g \) and, simultaneously, \( \nabla^2 b = (-k_\ast b + c)g_\ast \) with a constant \( c \) and with \( k = -k_\ast \). Such Einstein spaces can be (locally and globally) classified [28].

If \( n = n_\ast = 2 \) then either the Gaussian curvatures are constant and satisfy \( \vec{K} = -K_\ast \), or they satisfy the equations \( \vec{\nabla}^2 K = \frac{\Delta_k}{2} \vec{g} \) and \( \nabla^2 K_\ast = \frac{\Delta_k}{2} g_\ast \). Such metrics are called extremal in [11, 18.4].

Conversely, any Einstein warped product in (1) is conformally equivalent with a product space, and any two Einstein metrics \( \vec{g}, g_\ast \) with constant \( \vec{k} = -k_\ast \) and with solutions \( a(y), b(x) \) of the equations \( \vec{\nabla}^2 a = (-ka + c)g \) and \( \nabla^2 b = (-k_\ast b + c)g_\ast \) lead to an Einstein metric \( \vec{g} = (a + b)^{-2}(\vec{g} + g_\ast) \) on \( M \times M_\ast \) in (2).

If \( n = n_\ast = 2 \) then there are also examples \( M \times M_\ast \) with two surfaces \( M, M_\ast \) that are not of constant curvature, see Example 3.3 for the details. However, by Corollary 3.4 there are no compact examples that are (locally) conformally Einstein.

**Remark:** A complete classification of Einstein warped products is not known, compare [11, 18]. However, Einstein warped products with a 1-dimensional base are easy to classify by the equations in Corollary 2.4 see [22, 25, 28]. For the case of a 2-dimensional base
The Einstein condition for see [11, Thm.9.119]. The similar equations \( \tilde{\nabla}^2 f = -\tilde{k} \tilde{f} \tilde{g} \) and \( \nabla^2 f = \tilde{k} f \ast \) hold for the divergence \( f \) of a non-isometric conformal vector field on a complete non-flat Riemannian product \( M \times M_* \), see [38, Thm.5], [32]. The crucial condition \( \tilde{k} = -k_* \) occurs also in the Fefferman-Graham ambient metric construction on \( M \times M_* \times [0, \varepsilon) \), see [16].

**Proof.** Let \( g = \tilde{g} + g_* \) and \( N = n + n_* - 1 \geq 2 \). Then by Equation 6

\[
f^2(\tilde{\text{Ric}} - \text{Ric}) = (N - 1)f \cdot \nabla^2 f + \left[f \cdot \Delta f - N \cdot \|\text{grad} f\|^2\right] \cdot g.
\]

The Einstein condition for \( \tilde{g} \) implies that \( \nabla^2 f \) admits an orthogonal decomposition into some tensor on \( M \) and another tensor on \( M_* \). In coordinates \( y_1, \ldots, y_n, x_1, \ldots, x_{n_*} \) on \( M \times M_* \) this implies \( \frac{\partial^2 f}{\partial y_i \partial x_j} = 0 \) for any \( i, j \). Therefore \( f \) splits as

\[
f(y, x) = a(y) + b(x)
\]

with functions \( a \) on \( M \) and \( b \) of \( M_* \). This implies

\[
\nabla^2 f = \begin{pmatrix} \tilde{\nabla}^2 a & 0 \\ 0 & \nabla^2 b \end{pmatrix}
\]

and

\[
\|\text{grad} f\|^2 = ||\tilde{\text{grad}} a||^2 + ||\text{grad} b||^2, \quad \Delta f = \tilde{\Delta} a + \Delta_* b.
\]

Therefore with \( \tilde{\text{Ric}} = \tilde{X} \tilde{g} \) Equation 6 can be written in block matrix form as

\[
\tilde{X} \begin{pmatrix} \tilde{g} & 0 \\ 0 & g_* \end{pmatrix} - f^2 \begin{pmatrix} \text{Ric} & 0 \\ 0 & \tilde{\text{Ric}}_* \end{pmatrix} = (N - 1)f \begin{pmatrix} \tilde{\nabla}^2 a & 0 \\ 0 & \nabla^2 b \end{pmatrix} + \left[f \cdot \Delta f - N \cdot \|\text{grad} f\|^2\right] \begin{pmatrix} \tilde{g} & 0 \\ 0 & g_* \end{pmatrix}.
\]

From this equation it is obvious that a constant function \( a \) implies that \( \tilde{g} \) is Einstein and a constant function \( b \) implies that \( g_* \) is Einstein. In each of these cases \( \tilde{g} \) is a warped product metric with an Einstein fibre. This is case (1) in the theorem.

If \( a \) and \( b \) are both non-constant we can pick tangent vectors \( Y \) on \( M \) and \( X \) on \( M_* \) such that \( \nabla_X a \neq 0 \) and \( \nabla_X b \neq 0 \). Then we consider the covariant derivatives \( \nabla_X \) and \( \nabla_Y \) of the last equation with respect to \( g \). By the product decomposition we have \( \nabla_X \tilde{g} = \nabla_Y g_\ast = 0 \) and \( \nabla_X a = \nabla_Y b = 0 \). If \( N \geq 2 \) then \( \tilde{X} \) is constant. Using this and \( f = a + b \) the results are

\[
0 = 2f \nabla_X b \cdot \tilde{\text{Ric}} + (N - 1) \nabla_X b \cdot \tilde{\nabla}^2 a + \left[\nabla_X b \cdot \Delta f + f \nabla_X \Delta a - N \cdot \nabla \|\text{grad} f\|^2\right] \cdot \tilde{g},
\]

\[
0 = 2f \nabla_Y a \cdot \text{Ric}_* + (N - 1) \nabla_Y a \cdot \nabla^2 b + \left[\nabla_Y a \cdot \Delta f + f \nabla_Y \tilde{\Delta} a - \nabla_Y \|\text{grad} f\|^2\right] \cdot g_*,
\]

and, consequently,

\[
0 = 2f \cdot \tilde{\text{Ric}} + (N - 1) \cdot \tilde{\nabla}^2 a + \left[\Delta f + (\nabla_X b)^{-1} \left(f \nabla_X \Delta a - N \cdot \nabla \|\text{grad} f\|^2\right)\right] \cdot \tilde{g},
\]

\[
0 = 2f \cdot \text{Ric}_* + (N - 1) \cdot \nabla^2 b + \left[\Delta f + (\nabla_Y a)^{-1} \left(f \nabla_Y \tilde{\Delta} a - \nabla_Y \|\text{grad} f\|^2\right)\right] \cdot g_*.
\]

Differentiating once more and using \( \nabla_X \tilde{\nabla}^2 a = \nabla_Y \nabla^2 b = 0 \) we obtain

\[
0 = 2 \nabla_X f \cdot \tilde{\text{Ric}} + \nabla_X \left[\Delta f + (\nabla_X b)^{-1} \left(f \nabla_X \Delta a - N \cdot \nabla \|\text{grad} f\|^2\right)\right] \cdot \tilde{g},
\]
\[ 0 = 2\nabla_Y f \cdot \text{Ric}_s + \nabla_Y \left[ \Delta f + (\nabla_Y a)^{-1} \left( f \nabla_Y \Delta a - N \cdot \nabla_Y \|\text{grad} f\|^2 \right) \right] \cdot g_* . \]

This implies that \( \tilde{g} \) and \( g_* \) are Einstein metrics. From the previous equations we get that in addition \( \nabla^2 a \) is a scalar multiple of \( \tilde{g} \) and that \( \nabla^2 b \) is a scalar multiple of \( g_* \). This is precisely the equation in Corollary \[2.4\] On an Einstein space this equation can be completely and explicitly solved, see Corollary \[2.4\] and, in more detail, \[1\], \[38\], \[22\], \[28\]. Moreover it follows that we have Einstein constants \( \tilde{\lambda} \) and \( \lambda_* \) whenever \( n \geq 3 \) and \( n_* \geq 3 \). If \( n = 2 \) or \( n_* = 2 \) no conclusion about the constancy of the curvature is possible. However, if the gradient of \( a \) or \( b \) is isotropic, then it follows \( \Delta a = 0 \) and \( \tilde{k} = 0 \) or \( \Delta_* b = 0 \) and \( k_* = 0 \), respectively. So in the sequel we can assume that the gradients are either spacelike or timelike with \( \epsilon_1 = \text{sign}(\|\text{grad}a\|^2), \epsilon_2 = \text{sign}(\|\text{grad}b\|^2) \).

Now let \( n \geq 3 \) and \( n_* \geq 3 \). Then it is well known \[22\], \[28\] that \( \nabla^2 a = (-\tilde{k} + c)a\tilde{g} \) and simultaneously \( \nabla^2 b = (-k_* + c)b\tilde{g} \) with constants \( c, c_* \). Recall \( \Delta a = \epsilon_1 a\epsilon'' \), \( \Delta_* b = \epsilon_2 b\epsilon'' \) and \( a'' + \epsilon_2 ka' = 0 = b'' + \epsilon_2 k_* b' \) from Corollary \[2.4\] We still have to prove \( \tilde{k} = -k_* \) and \( c = c_* \). Since \( \text{Ric} = (n-1)\tilde{g} \) and \( \text{Ric}_* = (n_* - 1)k_* g_* \), we consider Equation \[11\] and observe that the trace-free part has to vanish. This leads to the equation

\[ (a + b)((n - 1)\tilde{k} - (n_* - 1)k_*) + (n + n_* - 2)(-\tilde{k}a + c + k_* b - c_*) = 0. \]

Since \( a \) depends only on \( y \in M \) and \( b \) depends only on \( x \in M_* \) we conclude that \( k_* = -\tilde{k} \) and, by using this equation, \( c = c_* \). The same is true if \( n = 2 \) and \( n_* \geq 3 \) because a constant scalar curvature on one of the factors implies that the scalar curvature on the other factor is also constant.

If \( n = 1 \) or \( n_* = 1 \) the same holds, see Corollary \[3.9\] below.

If \( n = n_* = 2 \) then we have \( \nabla^2 a = \epsilon_1 a\epsilon'' \tilde{g} \) and \( \nabla^2 b = \epsilon_2 b\epsilon'' g_* \) where \( (\cdot)' \) denotes differentiation with respect to the arc length parameter \( t \) resp. \( s \) on the trajectories of the gradients of \( a, b \), compare Corollary \[2.4\] The metrics are \( \tilde{g} = \epsilon_1 dt^2 \pm a^2 dx^2, g_* = \epsilon_2 ds^2 \pm b^2 dy^2 \). All geometric quantities depend only on \( a', b' \), not on an additive constant of \( a \) or \( b \), e.g. we have the Gaussian curvatures \( K = -\epsilon_1 a''/a', K_* = -\epsilon_2 b''/b' \). Then Equation \[11\] reads as follows:

\[ \tilde{\lambda} = (a + b)^2 K + 2(a + b)\epsilon_1 a'' + (a + b)(2\epsilon_1 a'' + 2\epsilon_2 b'') - 3(\epsilon_1 a'^2 + \epsilon_2 b'^2) \]
\[ \lambda_* = (a + b)^2 K_* + 2(a + b)\epsilon_2 b'' + (a + b)(2\epsilon_1 a'' + 2\epsilon_2 b'') - 3(\epsilon_1 a'^2 + \epsilon_2 b'^2). \]

In particular we have

\[ 0 = (a + b)(K - K_*) + 2(\epsilon_1 a'' - \epsilon_2 b'') = aK + 2\epsilon_1 a'' - (bK_* + 2\epsilon_2 b'') + bK - aK_* \]

where the first summand depends on \( t \) only, the second one on \( s \) only, and the third depends on both simultaneously. By differentiating \( t \) and \( s \) we see that \( b'K' - a'K_*' = 0 \), hence \( K'/a' = K_*'/b' = c \) is constant. Only the case \( c \neq 0 \) is interesting here. By adding some constant to \( a \) and \( b \) we can assume that \( K = ca, K_* = cb \). Therefore from Equation \[12\] we obtain \( ca^2 + 2\epsilon_1 a'' = d \) and \( cb^2 + 2\epsilon_2 b'' = d \) with a constant \( d \). These equations have non-constant solutions, see Example \[3.3\] (4).
Conversely, if \( k_* = \tilde{k} \) is constant and if \( a, b \) satisfy \( \tilde{\nabla}^2 a = (\tilde{\kappa} a + c) g \) and \( \nabla^2 b = (-k_* b + c) g \), then we see that Equation 11 is satisfied because the constant coefficients of \( a \) and \( b \) vanish. This part holds also if \( n = 2, n_* \geq 3 \) because in this case the equations imply that the 2-manifold is of constant curvature. It also holds if \( n = n_* = 2 \) and if the two curvatures are constant. The Einstein constant \( \tilde{\lambda} \) depends on the constant \( \tilde{k} = -k_* \) and the constants of integration \( c, c_1, c_2, d_1, d_2 \) and \( \epsilon_1, \epsilon_2 \) from the various differential equations on \( M, M_* \) according to Equation 8:

\[
a'' + \epsilon_1 \tilde{k} a = \epsilon_1 c, \quad (a'')^2 + \epsilon_1 \tilde{k}(a')^2 = c_1, \quad \epsilon_1 (a')^2 + \tilde{k}a^2 - 2ac = d_1,
\]
\[
b'' + \epsilon_2 k_* b = \epsilon_2 c, \quad (b'')^2 + \epsilon_2 k_*(b')^2 = c_2, \quad \epsilon_2 (b')^2 + k_* b^2 - 2bc = d_2
\]

with the coupling \( c_1 = \tilde{k}d_1 + c^2, \quad c_2 = k_* d_2 + c^2 \). From Equation 11 we obtain

\[
\tilde{\lambda} = (a + b)[(n - 1)\tilde{k}(a + b) + (N - 1)(-\tilde{k}a + c) + n(-\tilde{k}a + c) + n_*(k_* b + c)]
\]
\[
- N(\epsilon_1 a^2 + \epsilon_2 b^2)
\]
\[
= N[-\epsilon_1 a^2 - \tilde{k}a^2 - \epsilon_2 b^2 - k_*b^2 + 2ac + 2bc]
\]
\[
= N(-d_1 - d_2).
\]

Equivalently we get \( \tilde{k} = -(d_1 + d_2) \). Moreover, in the special case of dimensions \( n = n_* = 2 \) there are in fact examples of non-constant curvature and with a non-vanishing Einstein constant \( \tilde{\lambda} \), see Example 3.3(4) below.

**Examples 3.3.**
1. On a Riemannian product the special solution \( a(t) = \cos t, \quad b(s) = \cosh s \) in the situation of Theorem 3.2 realizes the case \( c = 0, \quad \tilde{k} = c_1 = c_2 = d_1 = -d_2 = 1 \) and \( \tilde{\lambda} = 0 \). The same \( \tilde{\lambda} \) is obtained for \( a(t) = \cos t + c, \quad b(s) = \cosh s - c \) with arbitrary \( c \).

2. If \( n \geq 3, n_* \geq 3 \) then special solutions \( a, b \) satisfy

\[
\tilde{\nabla}^2 a = -\tilde{k}a\tilde{g}, \quad \nabla^2 b = -k_* b\tilde{g}.
\]

If in addition the Einstein constants \( \tilde{\lambda} = (n - 1)\tilde{k} \) and \( \lambda_* = (n_* - 1) k_* \) are equal, then \( g \) and \( \tilde{g} \) are both Einstein metrics on \( M \times M_* \). Therefore the conformal transformation \( g \mapsto \tilde{g} = (a + b)^{-2} g \) satisfies the equation in Corollary 2.4. This can actually happen if \( \tilde{k} = k_* = 0 \). A concrete example with a Ricci flat \( M \times M_* \) can be constructed as in Case (3) in the Examples 3.8.

3. (a compact example)
Let \((M, g) = (S^n, g)\) be the unit sphere with normalized scalar curvature \( k = 1 \). Take a second copy \((M_*, g_*) = (S^n, -g)\) with a negative definite metric and with normalized scalar curvature \( k_* = -1 \) and let \( c = 0 \). Then on \( M \) the function \( a(t) = \cos t \) satisfies the
Then we would have a conformally Einstein. It would not help to introduce an additive constant $c$ has zeros. This means that the product metric on $M \times M_s$ is locally but not globally conformally Einstein. It would not help to introduce an additive constant $c$ here because then we would have $a(t) = \cos t + c$ and $b(s) = \cos s - c$.

(4) (Wong [44, Thm.10.1], compare [11, Sect.18], [13], [27, Sect.6])

If $n = n_s = 2$ then the proof of Theorem 3.2 indicates that the Gaussian curvatures $K, K_s$ should satisfy the equation $K^2 + 2K'' = c = K_s^2 + 2K_s''$. Therefore, following Corollary 2.4 we consider the warped product metric $\tilde{g} = dt^2 + K^2(t)dx^2$ admitting the solution

$$\nabla^2 K = \frac{\Delta K}{2} \tilde{g} = K'' \tilde{g}$$

with the Gaussian curvature $-K''/K'$. Since the ODE

$$K'' + KK' = 0$$

holds by differentiating $2K'' + K^2 = c$, $K(t)$ itself becomes the Gaussian curvature of $g$, and $K \tilde{g}$ becomes the Ricci tensor. Conversely, by integration we obtain the constant $c$ satisfying $2K'' + K^2 = c$. Similarly we have the metric $g_s = ds^2 + K_s^2(s)dy^2$ with the ODE

$$K_s'' + K_sK_s' = 0.$$  

There are nonconstant solutions $K(t), K_s(s)$. Particular cases are

$$K(t) = -12t^{-2}, K_s(s) = -12s^{-2}$$

with $c = 0$. Then for any choice of $c$ we can verify the equations in the proof of Theorem 3.2 for the function $f(t, s) = K(t) + K_s(s) = a(t) + b(s)$ (which is nothing else than the scalar curvature of the product metric) and for $X = \partial_s, Y = \partial_t$: We have

$$\nabla_s Xb = K_s', \tilde{\nabla} Y a = K', \Delta a = 2K'' - K_s'' + 2K''_s, ||\text{grad} f||^2 = K'' + K''_s.$$ 

This implies

$$\text{Ric} = K \tilde{g}, \text{Ric}_s = K_s g_s, 2\nabla^2 K = 2K'' \tilde{g} = (c - K^2) \tilde{g}, 2\nabla^2 K_s = 2K_s'' \tilde{g} = (c - K_s^2) g_s.$$ 

By $2K''K' = cK' - K^2 K'$ a second integration step leads to the equation $K'' = cK - \frac{1}{4}K^3 + d$ with a constant $d$, similarly $K_s'' = c_s K_s - c_s K^3 + d_s$. Finally, Equation 11 is satisfied with the Einstein constant

$$\lambda = -(K^3 + K_s^3 + 3K^2 + 3K_s^2) + 3c(K + K_s) = -3(d + d_s):$$

$$(K + K_s)^2 K + 2(K + K_s)K'' + [2(K + K_s)(K'' + K_s'')] - 3(K'' + K_s'') = (K + K_s)^2 K + (K + K_s)(c - K^2) + [(K + K_s)((2c - K^2 - K_s^2) - 3(K'' + K_s'')]

= (K + K_s)[K^2 + K K_s + 3c - 2K^2 - K_s^2] - 3(K'' + K_s'')

= 3c(K + K_s) - (K^3 + K_s^3 + 3K^2 + 3K_s^2)

= \lambda.$$
Similarly we have

$$(K + K_*)^2 K_* + 2(K + K_*)K_*'' + [2(K + K_*)(K'' + K_*''') - 3(K'^2 + K_*^2)] = \overline{\lambda}.$$  

This implies that the metric $\overline{g} = (K + K_*)^{-2}(dt^2 + K^2 dx^2 + ds^2 + K'^2 dy^2)$ is a 4-dimensional Einstein metric. Up to scaling, these examples are unique by the formulas above. Up to scaling, the particular case with $c = d = d_* = 0$ is the Ricci flat metric with $K(t) = -12t^{-2}, K_*(s) = -12s^{-2}$ on an open part of $\mathbb{R}^4$

$$\overline{g} = \frac{t^4 s^4}{(t^2 + s^2)^2} \left( dt^2 + \frac{576}{t^6} dx^2 + ds^2 + \frac{576}{s^6} dy^2 \right).$$  

A picture of a surface of revolution in 3-space with the metric $dt^2 + \frac{576}{t^6} dx^2$ is similar to that of Beltrami’s surface (pseudosphere): If the distance from the axis of revolution is $r(t) = 24t^{-3}$ then the height along the axis is $h(t) = \int_0^t \sqrt{1 - \left(\frac{72}{r^2}\right)^2} d\tau$ with $t \geq t_0$ where $t_0 = \frac{4}{\sqrt{72}}$ is the zero of $h' = \sqrt{1 - r'^2}$. The $t_0$-circle is a singularity in space with $K(t_0) = -12t_0^{-2} = -\sqrt{2}$ and with one vanishing principal curvature, and the curvature $K(t) = -12t^{-2}$ tends to zero from below for $t \to \infty$ with asymptotic principal curvatures $24^{-1} t^3 \approx 1/r$ and $-288t^{-5} \approx Kr$. The surface is complete in the positive $t$-direction. It is not obvious that the product of this surface with itself is conformally Ricci flat.

**Corollary 3.4.** (The compact case)

Let $(M, g) \times (M_*, g_*)$ be compact and assume it is a (locally) conformally Einstein space such that the conformal factor $f$ always depends on both $M$ and $M_*$ simultaneously. Then $M, M_*$ are round spheres, one with a positive definite metric, the other one with a negative definite metric and such that the normalized scalar curvatures satisfy $k_* = -k$. However, there is no conformal factor $f$ without a zero. Consequently, the metric $\overline{g} = f^{-2}(g + g_*)$ does not define (globally) an Einstein metric on a compact manifold.

**Proof.** By Theorem 3.2 the function $f = a + b$ would constitute a nonconstant solution of the differential equations

$$\nabla^2 a = \frac{\Delta a}{n} g \quad \text{or} \quad \nabla^2 b = \frac{\Delta b}{n_*} g_*,$$

respectively. It is well known that topologically the only such compact manifold $M$ or $M_*$ is a sphere, and that the metric must be positive or negative definite. This holds even if the functions $a$ or $b$ are only locally defined. In a neighborhood of every point there is such a solution. However, we note that passing to the negative of a metric does not change the covariant derivative, does not change the Ricci tensor or the Hessian $(0, 2)$-tensor, but it does change the sign of the scalar curvature. In other words: If $\nabla^2 b = -k_* b g_*$ holds then for the metric $g_* = -g_*$ we have $\nabla^2 b = -k_* b g_* = k_b b g_*$. For $n \geq 3, n_* \geq 3$ the Einstein condition then implies that both $(M, \overline{g})$ and $(M_*, g_*)$ are round spheres. The condition $\overline{k} = -k_*$ implies that one is Riemannian, the other one is anti-Riemannian with a negative definite metric. We have the same conclusion if $n = 2, n_* \geq 3$ and for $n = n_* = 2$ if $M, M_*$ are spheres of constant curvature. If $n = 1$ then $a(t)$ would have to be periodic. According to Corollary 3.9 this leads to the same kind of result. In any case the function $f$ will have a zero.
Finally, if \( n = n_* = 2 \) then according to the proof of Theorem 3.2 we have to take the case of two compact surfaces with nonconstant Gaussian curvatures \( K(t) = ca(t), K_*(s) = cb(s) \) into account with a constant \( c > 0 \). The functions \( a, b \) would have to satisfy

\[
\tilde{\nabla}^2 a = \frac{\Delta a}{2} \tilde{g} \quad \text{and} \quad \nabla_*^2 b = \frac{\Delta_* b}{2} g_*.
\]

It is well known that the round 2-sphere is the only compact surface of that kind, and that the metrics would have to satisfy the equations \( \tilde{g} = \pm(dt^2 + a'^2(t)dx^2) \) and \( g_* = \pm(ds^2 + b'^2(s)dy^2) \) with two exceptional points with a removable singularity in these coordinates. The exceptional points correspond precisely to the minima and maxima of the Gaussian curvatures \( K, K_* \). Assume that one of the metrics is Riemannian and that there is such a point \( p \) with \( K' = a = 0 \). Since the singularity in geodesic polar coordinates around \( p \) is removable, we necessarily \( |a''| = A > 0 \) at \( p \). This is infinitesimally the same as for the round sphere or the Euclidean plane or the hyperbolic plane in polar coordinates. Consequently, we have \( a'' = A \) at the minimum and \( a'' = -A \) at the maximum. On the other hand we have \( K = -a''/a' \) and the ODEs

\[ ca^2 + 2a'' = d \quad \text{and} \quad a'^2 = da - \frac{c}{2}a^3 + e \]

with constants \( d \) and \( e \). The first ODE implies \( ca^2 = d - 2A \) at the minimum and \( ca^2 = d + 2A \) at the maximum. Inserting this into the second ODE we obtain

\[ 0 = \sqrt{\frac{d-2A}{c}}(d - \frac{d-2A}{3}) + e \]

at the minimum and

\[ 0 = \sqrt{\frac{d+2A}{c}}(d - \frac{d+2A}{3}) + e \]

at the maximum. Combining these conditions leads to

\[ \sqrt{d-2A(d+A)} = \sqrt{d+2A(d-A)}. \]

However, there is no real solution \( d \) of this equation unless \( A = 0 \) which would imply that \( a \) is constant. Therefore, no Riemannian manifold \((S^2, \tilde{g})\) of this kind exists, and no \((S^2, g_*)\) either. This is not in contradiction with the fact that abstractly there are nontrivial periodic solutions of the ODE \( y^2 + 2y'' = d \) for positive \( d \). However, these necessarily produce surfaces with proper singularities, see Figure 1, compare the correction in [39].

\[ \square \]

**Remark:** The last statement on the product of two surfaces with nonconstant curvature can be found also in [11, Thm.18.4], a statement attributed to E.Calabi. In the Riemannian case no compact example of \( M \times M_* \) is possible at all, except for the product of two curves.

**Corollary 3.5.** (Ruiz [37, Thm.1])

Assume that \( n, n_* \geq 2 \) and \((M^n, g_*)\) is complete and flat, and let \((M^n, g)\) be a compact Riemannian manifold. Then \((M, g) \times (M_*, g_*)\) is not globally conformal to any Einstein metric with a positive Einstein constant.
Figure 1. Extremal surface as a surface of revolution in \(\mathbb{R}^3\) satisfying \(2a'' + a^2 = 2\) and \(a^2 = -a^3/3 + 2a - 4/3\).

**Proof.** This follows from Theorem 3.2 above if we assume that \(f = a + b\) is a globally defined and strictly positive function. Since \(f\) cannot be constant we assume that the product is conformally Einstein with a non-constant conformal factor \(f\). If \(a\) is constant then \(g\) is Einstein. By Equation 11 \(\nabla^2 b\) is a scalar multiple of \(g_*\). Since \(g_*\) is flat we have \(\nabla^2 b = cg_*\) with a constant \(c\). Then Equation 11 implies the equation \(kf = k(a + b) = (N - 1)c\). So either \(g\) is Ricci flat \((k = 0)\) and \(c = 0\) or \(b\) is constant. In the former case \(\nabla^2 b = 0\) implies that \(b\) is either constant or a linear function, necessarily with a zero. This leads to a contradiction on a complete manifold.

If \(b\) is constant then by Ric \(g_* = \nabla^2 b = 0\) Equation 11 tells us that we have \(f^2 \text{Ric} + (N - 1)f\text{grad}f\|\text{grad}f\|^2 = f\Delta a - N||\text{grad}a||^2\) is the Einstein constant \(\lambda\). By the compactness of \(M\) we have \(\int_M \Delta a = 0\), therefore \(-N\int_M f^{-1}||\text{grad}f||^2 = \lambda\int_M f^{-1}\). Since \(f > 0\) we have \(\lambda < 0\).

If both \(a\) and \(b\) are not constant then \(g\) is an Einstein metric and we have \(k = 0\). However, the equation \(\nabla^2 a = cg\) with a constant \(c\) has no solution on a compact manifold except if \(a\) is constant and \(c = 0\). The case of a nonvanishing parallel gradient of \(a\) is impossible. But then \(b\) would be either constant or a linear function. In the former case \(f = a + b\) has a zero, a contradiction in any case. \(\square\)

**Corollary 3.6.** (Cleyton [9, Thm.2])

Assume \(n, n_* \geq 2\), and let \((M^n, g)\) and \((M_*^{n_*}, g_*)\) be complete Riemannian manifolds. If \(M \times M_*\) is globally conformal to an Einstein space with metric \(f^{-2}(g + g_*)\) that is not a warped product, then \(M\) and \(M_*\) are Euclidean spaces, and the conformal factor \(f\) is \(f(y, x) = \frac{1}{2}(||x||^2 + ||y||^2) + d^2\) with a constant \(d \neq 0\).

**Proof.** The proof is somehow contained in the proof of the preceding Corollary. Since the conformal product is assumed not to be a warped product, we have only the third case with a function \(f = a + b\) where both \(a\) and \(b\) are non-constant. By the equation \(\overline{k} = -k_*\) we have two cases: \(k_* = 0\) and \(k_* \neq 0\). In the latter case \(k_* > 0\) one of the equations reads as \(b'' = -k_*b + c\). This implies that \(M_*\) is a sphere of sectional curvature \(k_*\) and that \(b(s) = \sin s + c\) along the trajectories of its gradient. This statement is known as the theorem of Obata [35] and Tashiro [38]. Consequently we have the equation \(a'' = k_*a + c\), so \(a\) is a hyperbolic function minus \(c\) along the trajectories of its gradient.
Example 3.8. Case (2): A parallel vectors, one of them spacelike, the other one timelike.

If we take the cartesian product of two such \( \mathbb{R} \), one can show that the warped product is a warped product metric with a Ricci flat Brinkmann space as base, but the fiber is not Ricci flat. By [28, Thm. 3.12] also.

Furthermore it is well known that the Euclidean space is the only complete Riemannian manifolds admitting a non-constant and everywhere positive function \( a \) with \( \nabla^2 b = cg \) and a constant \( c \neq 0 \). In polar coordinates \( g = ds^2 + b^2(s)g_1 \) around the origin we have \( c^2 = 1 \) and \( b(x) = \frac{1}{2}||x||^2 \) plus a positive constant, similarly for \( a \). The assertion follows.

\( \Box \)

Corollary 3.7. (the “improper case” in the terminology of Brinkmann [6])

Under the same assumptions as for Theorem 3.2 the following holds: If in addition \( \nabla f \) is a null vector field (i.e., if \( ||\nabla f||^2 = 0 \)) on an open subset then either one of the following three cases occurs:

1. \((M \times M_*, \tilde{g})\) is a warped product with a Ricci flat Brinkmann space as the base,
2. \((M, \tilde{g})\) and \((M_*, g_*)\) are Ricci flat Brinkmann spaces carrying a parallel null vector field each. On the other hand the product metric \( g = \tilde{g} + g_* \) itself is Ricci flat, so we have a case of Corollary [2.4] also.
3. Both admit a parallel non-null vector field, one spacelike, the other timelike. The latter case implies that in \( M \) and \( M_* \) both are pseudo-Riemannian products of \( \mathbb{R} \) with a Ricci flat manifold. Hence in the product a 2-dimensional flat factor splits off.

Consequently, if \((M \times M_*, \tilde{g} + g_*)\) is not Ricci flat then we have Case (1), i.e., \( \tilde{g} \) is a warped product metric with a Ricci flat Brinkmann space as base but the fiber is not Ricci flat. By \([11\text{ Cor.9.107]}\) this case cannot occur: We would necessarily have \( f = u \) with the lightlike \( \nabla f = \partial_u \) on the Brinkmann base, furthermore \( \tilde{X} = 0 \) and an Einstein \((M_*, g_*)\) with \( \lambda_* = 0 \). Hence the product metric is always Ricci flat.

Proof: By Theorem 3.2, either case (1) occurs or \( M, M_* \) are Einstein spaces admitting solutions of \( \nabla^2 a = \frac{\Delta}{n} \tilde{g} \) and \( \nabla^2 b = \frac{\Delta}{n} b g_* \) with \( f = a + b \) and \( ||\nabla a||^2 = -||\nabla b||^2 \).

Since one depends only on \( M \), the other one only on \( M_* \), both must be constant. If both are zero then Case (2) occurs by [28, Thm. 3.12]. Otherwise both \( \nabla a \) and \( \nabla b \) are parallel vectors, one of them spacelike, the other one timelike.

Example 3.8. Case (2): A \( pp \)-wave with metric

\[
ds^2 = -2H(u, x, y) du^2 - 2 dudv + dx^2 + dy^2
\]

is Ricci flat if and only if the spatial Laplacian \( \Delta H = H_{xx} + H_{yy} \) vanishes [14, 26]. The vector field \( \nabla u = \partial_x \) is a parallel null vector field, thus the equation \( \nabla^2 u = 0 \) is satisfied.

If we take the cartesian product of two such \( pp \)-waves with parameters \( u_i, v_i, x_i, y_i, i = 1, 2 \), then the function \( f = u_1 + u_2 \) satisfies all the assumptions in Theorem 3.2. This is an example of Case (2) in Corollary 3.7.

Case (1): If \( g_* \) is a Ricci flat \( pp \)-wave as in Case (2) then with \( M = \mathbb{R} \) and \( f(t, x) = t + 1 \) the warped product \( \tilde{g} = f^{-2}(dt^2 + g_*) \) is an Einstein space with Einstein constant \( \lambda = -1 \). This corresponds to one of the cases discussed in Proposition 3.1.
Case (3): The products $\mathbb{R} \times M$ with metric $dt^2 + g$ and $\mathbb{R} \times M_*$ with metric $-d\tau^2 + g_*$ where both $g, g_*$ are Ricci flat provide an example by taking $\mathbb{R} \times M \times \mathbb{R} \times M_*$ with the product metric $ds^2 = dt^2 + g - d\tau^2 + g_*$. The function is $f(t, \tau) = t + \tau$. Obviously a factor $\mathbb{R}^2$ with the Lorentzian metric $dt^2 - d\tau^2$ splits off.

**Corollary 3.9.** (Conformally Einstein products of type $\mathbb{R} \times M_*$) If $f(t, x)$ is a non-constant function of $t \in \mathbb{R}$ and $x \in M_*$ with an $m$-dimensional pseudo-Riemannian manifold $(M_*, g_*)$, $m \geq 2$, and if the metric $\bar{g} = f^{-2}(\epsilon dt^2 + g_*)$ is Einstein then one of the following cases occurs:

1. $f$ depends only on $t$ (the case discussed in Proposition 3.1)
2. $f$ depends only on $x$ (the case discussed in Proposition 4.1)
3. $f(t, x) = a(t) + b(x)$ with non-constant $a$ and non-constant $b$, and $(M_*, g_*)$ is an $m$-dimensional Einstein space with constant normalized scalar curvature $k_*$ (even in dimension 2), and $a$ satisfies the equation $a'' = c_k a + c$ for a constant $c$, and simultaneously $b$ satisfies the equation $\nabla^2 b = \frac{\Delta b}{m} g_*$. Such Einstein spaces can be (locally and globally) classified [22], [28].

**Proof.** This is nothing else than a special case of Theorem 3.2. Independently, the calculation is a bit simpler as follows: Let $g = \epsilon dt^2 + g_*$. Then by Equation 6

$$
\overline{\text{Ric}} - \text{Ric} = (m - 1)f^{-1} \cdot \nabla^2 f + \left[ f \cdot \Delta f - m \cdot \|\text{grad} f\|^2 \right] \cdot \bar{g}
$$

the Einstein condition for $\bar{g}$ implies that $\nabla^2 f$ has the form

$$
\nabla^2 f = \begin{pmatrix}
\frac{\partial^2 f}{\partial t^2} & 0 \\
0 & \ast
\end{pmatrix}
$$

with some $(0, 2)$-tensor on $M_*$ denoted by $\ast$. In coordinates $x_1, \ldots, x_n$ on $M_*$ this implies $\frac{\partial^2 f}{\partial x_i \partial x_i} = 0$ for any $i$. Therefore $f$ splits as

$$
f(t, x) = a(t) + b(x)
$$

with functions $a$ of the real variable $t$ and $b$ of $x \in M_*$. This implies

$$
\nabla^2 f = \epsilon a''(\epsilon dt^2) + \nabla_x^2 b = \begin{pmatrix}
\epsilon a'' & 0 \\
0 & \nabla_x^2 b
\end{pmatrix}
$$

and

$$
\|\text{grad} f\|^2 = \epsilon (a')^2 + \|\text{grad}_x b\|^2, \quad \Delta f = \epsilon a'' + \Delta_x b.
$$

From

$$
\begin{pmatrix}
\epsilon a'' & 0 \\
0 & \nabla_x^2 b
\end{pmatrix} = \epsilon a'' f^2 \bar{g} + \begin{pmatrix}
0 & 0 \\
0 & \nabla_x^2 b - \epsilon a'' g_*
\end{pmatrix}
$$

we see that $\bar{g}$ is Einstein if and only if

$$(a + b)\text{Ric}_* + (m - 1) \left( \nabla_x^2 b - \epsilon a'' g_* \right) = 0.
$$

Splitting the $t$-part from the $x$-part leads to

$$
(13) \quad a\text{Ric}_* - (m - 1)\epsilon a'' g_* = -b\text{Ric}_* - (m - 1)\nabla_x^2 b
$$
with the trace

\[ aS - m(m-1)ea'' = -bS - (m-1)\Delta b. \]

Differentiating by \( t \) leads to

\[ a'S - m(m-1)ea''' = 0, \]

hence \( S = m(m-1)k \) must be constant (unless \( a \) is constant) since it depends only on \( x \). It also follows that \( a \) is a solution of the standard ODE \( a'' = \epsilon k, a' \), in particular \( a'' = \epsilon k, a + c \) with a constant \( c \) and, by the trace of Equation 13 with \( c = k, b + \Delta b/m \).

In particular \( \Delta b \) depends only on \( b \). By inserting this into Equation 13 we obtain

\[ (14) \quad a \left( \text{Ric} - \frac{S}{m} g \right) = (m-1)c g - b \text{Ric} - (m-1)\nabla^2 b \]

Since only the left hand side depends on \( t \) but the right hand side does not, it follows that either \( a \) is constant (then the constant can be incorporated into \( b \), and we can apply Corollary 4.1) or \( g \) is an Einstein metric and the right hand side is zero.

**Case 1:** \( c = 0 \) and \( b = 0 \). This is discussed in Proposition 3.1. We recognize the key equation \( a'' = k, a \) from the proof of Proposition 3.1.

**Case 2:** \( c = 0 \) and \( b \neq 0 \). If \( a \) is constant we have the case of Proposition 4.1. If \( a \) is not constant then \( g \) is Einstein and \( b \text{Ric} + (m-1)\nabla^2 b = 0 \). It follows that \( \nabla^2 b \) is also a scalar multiple of \( g \), hence \( \nabla^2 b = \frac{\Delta b}{m} g \).

**Case 3:** \( c \neq 0 \). If in addition \( a \) is not constant then \( \text{Ric} \) is a multiple of \( g \) and, consequently, \( \nabla^2 b \) is also a scalar multiple of \( g \), hence \( \nabla^2 b = \frac{\Delta b}{m} g \) as in Case 2. On an Einstein space this equation can be completely and explicitly solved, see [22], [28].

**Corollary 3.10.** (Moroianu & Ornea [34, Thm.2.1])

If \( M \) is a compact Riemannian manifold and if \( \tilde{g} = f^{-2}(dt^2 + g) \) is Einstein with \( \tilde{S} > 0 \) and with a non-constant function \( f(t, x) \) that is globally defined and never zero on \( \mathbb{R} \times M \), then one of the following two cases occurs:

1. \( (M, g) \) is a round sphere,
2. \( (M, g) \) is an Einstein space with \( S > 0 \), and the function \( f(t, x) \) is the cosh-function on the real \( t \)-axis, up to constants. In particular \( f \) does not depend on \( x \in M \).

**Remark:** The compactness of \( M \) will be used only for Case 1. In fact that assumption is not essential for Case 2, and our results above show that the question of the authors of [34] at the end of Section 1 can be answered. In general the statement of Case 1 would have to be modified, compare Corollary 3.13 below. See also Example 3.12. It would be even sufficient to assume that \( f \) is defined on an open set containing at least one slice \( \{t_0\} \times M \).

**Proof.** If \( f(t, x) \) depends on \( t \) and on \( x \) then \( (M, g) \) is a round sphere by Proposition 3.9 in combination with the well known theorem that the only compact pseudo-Riemannian Einstein space admitting a non-constant solution of the equation \( \nabla^2 b = \frac{\Delta b}{m} g \) is the round sphere [35], [38], [19], [22], [28], Thm.2.8].
If \( f \) depends only on \( t \) then by Proposition 3.1 \( g^* \) is Einstein. Furthermore the case \( S > 0 \) is only possible for a function of \( \cosh \)-type (up to additive or multiplicative constants). This is a consequence of the ODE \( ff'' - (f')^2 = \kappa > 0 \). Moreover from \( f'' = k_*f \) we get \( k_* > 0 \).

If \( f \) depends only on \( x \) then by Proposition 4.1 \( k^* \) is negative. Then Equation 17 implies that \( \Delta_*f \) is positive at a positive maximum of \( f \), and it is negative at a negative minimum of \( f \). Here we use that \( f \) never vanishes on \( M_* \). This is a contradiction on a compact manifold \( M_* \) because one of these cases must occur. However, compare Example 4.2 with the opposite signs \( k_* > 0 \) and \( k < 0 \). □

We can prove a similar version under a slightly different assumption:

**Corollary 3.11.** If \( M_* \) is compact Riemannian of constant scalar curvature \( S_* \) and if \( \overline{g} = f^{-2}(dt^2 + g_*) \) is Einstein with a non-constant function \( f(t,x) \) that is globally defined and never zero on \( \mathbb{R} \times M_* \), then one of the following two cases occurs:

1. \( (M_*, g_*) \) is a round sphere, and \( \overline{g} \) is of constant sectional curvature,
2. \( (M_*, g_*) \) is an Einstein space with \( S_* > 0 \), and the function \( f(t,x) \) is the \( \cosh \)-function or the exponential function on the real \( t \)-axis, up to constants. We have \( \overline{S} > 0 \) in the first case and \( \overline{S} = 0 \) in the second case.

**Proof.** The first case is the same as in Corollary 3.10. If \( f \) depends on \( t \) and on \( x \) then \( (M_*, g_*) \) is a round sphere. Therefore the product metric on \( \mathbb{R} \times M_* \) is locally conformally flat. It can even be realized as a hypersurface in Euclidean space, namely, as a tube around a straight line. Therefore \( \overline{g} \) is of constant sectional curvature, compare Example 3.17.

In the second case \( (f \) depends only on \( t \)) we have to find all never vanishing global solutions of the ODE \( ff'' - (f')^2 = \kappa \). These are the cases of the exponential function (up to constants) if \( \kappa = 0 \) and hence \( S_* > 0 \) or the \( \cosh \)-function (up to constants) if \( \kappa > 0 \) and hence \( S_* > 0 \). The case \( \kappa < 0 \) cannot occur.

In the third case \( (f \) depends only on \( x \)) we see from Equation 17 that \( \varphi \) is an eigenfunction of the Laplace operator on \( (M_*, g_*) \) for the eigenvalue \(-S_//(n-1)\). This implies \( S_* > 0 \) and

\[
\int_{M_*} \varphi = -\frac{n-1}{S_*} \int_{M_*} \Delta_*\varphi = 0.
\]

This is a contradiction because by assumption \( \varphi \) has no zeros. □

**Example 3.12.** A concrete example for Corollary 3.10 and Corollary 3.11 is the following: Let \( (M_*, g_*) \) be a compact Einstein space with a positive Einstein constant and normalized scalar curvature \( k_* = 1 \), e.g., a complex projective space. Then the metric \( \overline{g} = (\cosh t)^{-2}(dt^2 + g_*) \) is Einstein with normalized scalar curvature \( \kappa = 1 \). This is not of constant curvature if \( g_* \) is not of constant curvature. The resulting metric \( \overline{g} \) coincides with the metric of the classical Mercator projection if \( g_* \) is the metric of the unit sphere, compare Example 2.3. The other cases can be called a non-standard Mercator projection on the product \( \mathbb{R} \times M_* \). This is particularly interesting if \( M_* \) is a sphere and \( g_* \) is a non-standard Einstein metric (i.e., not of constant sectional curvature), see [4], [3]. In these...
cases a compactification by two points (north and south pole) is homeomorphic with the sphere \( S^n \), and it is Einstein except at these two points which are metrical singularities.

Without the assumption of compactness we have the following:

**Corollary 3.13.** If \( \overline{g} = f^{-2}(dt^2 + g_s) \) is an \( n \)-dimensional Riemannian, complete and Einstein metric with a non-constant function \( f(t, x) \) that is globally defined and never zero on \( I \times M_s \) with an open interval \( I \subseteq \mathbb{R} \), then one of the following two cases occurs:

1. \( f \) depends only on \( x \) (the case of Corollary 3.1).
2. \((M_s, g_s)\) is a complete Einstein space with \( S_s \leq 0 \), and the \( f \) depends only on \( t \) (possibly after a change of variables). Moreover, \( f \) is either \( f(t) = \alpha \cos t + \beta \sin t \) on a bounded interval \( I \), or \( f(t) = \alpha t + \beta \) on an unbounded interval \( I \), each time with constants \( \alpha, \beta \). We have \( \overline{S}, S_s < 0 \) in the first case and \( \overline{S} < 0, S_s = 0 \) in the second case.

**Particular cases are the hyperbolic metric \( \overline{g} \) in the form \( \overline{g} = \sin^{-2} t (dt^2 + g_{-1}) \) (compare Example 2.3) or \( \overline{g} = t^{-2}(dt^2 + g_0) \) (the Poincaré half-space).**

**Proof.** If \( f(t, x) = a(t) + b(x) \) depends on both arguments with nonconstant functions \( a \) and \( b \) then \((M_s, g_s)\) is complete Einstein with \( \nabla^2 b = \frac{\Delta_b}{n-1} g_s \) and simultaneously \( a'' = k_s a + c \). If in addition \( k_s > 0 \) then the solution \( a(t) \) is a hyperbolic function. Simultaneously \( b \) is bounded. Therefore \( \overline{g} \) cannot be complete in the \( t \)-direction by the growth of \( a \) at infinity. The case of the Mercator projection shows that \( \overline{g} \) can be extendible to a complete (even compact) manifold. If \( k_s < 0 \) then \( a \) is bounded and \( b \) is a hyperbolic function on \( M_s \) which implies that \( \overline{g} \) cannot be complete. If \( k_s = 0 \) then \( a \) is linear and \( b \) is either linear or quadratic with a zero. In the latter case \( \overline{g} \) is not complete. It remains the case that \( a \) is linear in \( t \) and \( b \) is linear in a variable \( u \) such that \( g_s \) is a Ricci flat warped product \( g_s = ds^2 + b^2(s) g_{ss} \) where \( g_{ss} \) is again Ricci flat. But then \( f = a + b \) is linear in the \((t, s)\)-plane, and we can regard \( g = dt^2 + g_s \) as a metric of type \( g = du^2 + dv^2 + g_{ss}(u) \) such that \( \overline{g} = u^{-2} g \). This is part of case (2) with an unbounded interval \( I \).

If \( f \) depends only on \( t \) then we are in the position of Proposition 3.1. From the list in the proof we see that \( \overline{g} \) cannot be complete unless \( f \) is linear of \( f(t) = \sin t \) or \( f(t) = \cos t \) or a linear combination. Since \( f \) has a zero in any case, the interval \( I \) is unbounded in the linear case but must be bounded in the other case. This is part of Case (2). \( \square \)

**Example 3.14.** (a compact example)

Let \( (M, g) = (S^n, g) \) be the unit sphere with normalized scalar curvature \( k = 1 \). Take a second copy \( (M_s, g_s) = (S^n, -g) \) with a negative definite metric and with normalized scalar curvature \( k_s = -1 \). On \( M \) the function \( a(t) = \cos t \) satisfies the equation \( \nabla^2 a = -kag = -ag \), on \( M_s \) the function \( b(s) = \cos s \) satisfies the equation \( \nabla_s b = -k_s b g_s = b g_s \). By the results of Theorem 3.2 the metric \( \overline{g} = (a + b)^{-2}(g + g_s) \) defines an Einstein space whenever \( a + b \neq 0 \). The Einstein constant is

\[
\overline{\lambda} = (n - 1)(a + b)^2 + (2n - 2)(a + b)(-a) + (a + b)n(b - a) - (2n - 1)(a^2 - b^2) \\
= (a + b)(-2n - 1)(b - a) - (2n - 1)(a^2 - b^2) = (2n - 1)(-a^2 + a^2 + (b^2 + b^2)) = 0.
\]
Unfortunately the function \( f(t, s) = a(t) + b(s) \) has zeros. This means that \( M \times M_* \) is not globally conformally Einstein. It would not help to introduce an additive constant \( c \) here because then we would have \( a(t) = \cos t + c \) and \( b(s) = \cos s - c \).

**Example 3.15.** Here is an example of a complete Riemannian manifold

\[
M = \mathbb{R} \times \mathbb{R} \times \tilde{M} = \mathbb{R} \times M_*
\]

admitting a global solution \( f = a(t) + b(x) \) where both \( a, b \) are non-constant and never zero. Unfortunately \( f \) has zeros. Let \((\tilde{M}, \tilde{g})\) be a complete Ricci flat manifold of dimension \( n - 1 \), and let

\[
a(t) = \cos t - 2, \quad b(s) = e^s + 2.
\]

Then on \( M \) the function

\[
f(t, s, \tilde{x}) = a(t) + b(s)
\]

satisfies all conditions above: The metric \( g_* = ds^2 + e^{2s} \tilde{g} \) on \( M_* = \mathbb{R} \times \tilde{M} \) is a complete Einstein space with \( k_* = -1 \) [22] and \( a, b \) satisfy the equations

\[
a'' = k_* a - 2, \quad b'' = -k_* b - 2, \quad \nabla^2 b = (-k_* b - 2) g_* , \quad c = -2.
\]

By the results above the metric \( \tilde{g} = f^{-2} (dt^2 + ds^2 + e^{2s} \tilde{g}) \) on \( M = \mathbb{R} \times M_* \) is Einstein with \( \tilde{k} = -1 \) whenever \( f(t, s) = \cos t + e^s \neq 0 \). If \( \tilde{g} \) is not flat then \( g_* \) is not of constant curvature and, consequently, \( \tilde{g} \) is not of constant curvature.

**Example 3.16.** A similar example starts with an \((n - 1)\)-dimensional Einstein space \((\tilde{M}, \tilde{g})\) with normalized scalar curvature \(-1\). Then \( M_* = \mathbb{R} \times \tilde{M} \) with the warped product metric \( g_* = ds^2 + \cosh^2(s) \tilde{g} \) is also Einstein with \( k_* = -1 \) [22]. Let

\[
a(t) = \cos t, \quad b(s) = \cosh s.
\]

Then on \( M = \mathbb{R} \times M_* \) the function

\[
f(t, s, \tilde{x}) = a(t) + b(s)
\]

satisfies all conditions above since \( a, b \) satisfy the equations

\[
a'' = k_* a, \quad b'' = -k_* b, \quad \nabla^2 b = (-k_* b) g_* , \quad c = 0.
\]

By the results above the metric \( \tilde{g} = f^{-2} (dt^2 + ds^2 + \cosh^2(s) \tilde{g}) \) on \( M \) is Ricci flat whenever \( f(t, s) = \cos t + \cosh s \neq 0 \). This is the case at least for all \( s \neq 0 \). If \( \tilde{g} \) is not hyperbolic then \( g_* \) is not of constant curvature and, consequently, \( \tilde{g} \) is not of constant curvature.

**Example 3.17.** Here is an example with a compact \( M_* \) and a function depending on two parameters: Let \((M, g_*)\) denote the unit \( n \)-sphere in polar coordinates \( g_* = dr^2 + \sin^2(r) \tilde{g} \) with the unit \((n - 1)\)-sphere \((\tilde{M}, \tilde{g})\). This admits the function

\[
b(r, \tilde{x}) = \cos r
\]

satisfying \( \nabla^2 b = -bg_* \). Then on \( M = \mathbb{R} \times M_* \) the function

\[
f(t, r, \tilde{x}) = a(t) + b(r, \tilde{x}) = \cosh t + \cos r
\]

satisfies all conditions above: The metric \( g_* \) is a complete Einstein space with \( k_* = 1 \), and \( a, b \) satisfy the equations

\[
a'' = k_* a, \quad b'' = -k_* b, \quad \nabla^2 b = (-k_* b) g_* , \quad c = 0.
\]
By the results above the metric \( \overline{g} = f^{-2}(dt^2 + dr^2 + \sin^2(r)\tilde{g}) \) on \( M = \mathbb{R} \times M_* \) is Ricci flat whenever \( f(t, s) \neq 0 \). This is the case for \( t \neq 0 \) or \( r \neq \pi \), i.e. on \( (\mathbb{R} \times M_*) \setminus \{(0,p)\} \) for a point \( p \). \( \overline{g} \) is even flat according to Corollary 3.11. Compare the hyperbolic metric as a warped product \( dt^2 + \cosh^2(t)(dr^2 + \sinh^2(r)\tilde{g}) \).

### 4. Conformally Einstein products of type \( \mathbb{R} \times M_* \)

In this section we come back to some particular cases that were not solved so far. In Proposition 3.1 we discussed the case of a function \( f \) depending on one real parameter. Here we deal with the opposite case: A function depending on the \((n-1)\)-dimensional factor \( M_* \) of the product \( \mathbb{R} \times M_* \). This leads to equations involving the Hessian of the function and the Ricci tensor. If one of them (and, consequently, the other as well) is a multiple of the metric, then we are in the position of Theorem 2.4. The question is whether there are more solutions. In fact they are: We construct iterated warped products of this kind by solving a nonlinear ODE of third order.

**Proposition 4.1.** (The type \( \mathbb{R} \times M_* \) with an \((n-1)\)-dimensional base)

If \( g \) has the form \( g = \pm dt^2 + g_* \) on \( M = \mathbb{R} \times M_* \) and if \( \varphi \) is a never vanishing function defined on the \( n \)-dimensional manifold \((M_*, g_*)\) then \( \overline{g} = \varphi^{-2}g \) is an Einstein metric if and only if the two equations

(15) \[
\varphi \cdot \text{Ric}_* + (n-1) \cdot \nabla^2 \varphi = 0
\]

(16) \[
\|\text{grad} \varphi \|^2 + \varphi^2 k_* + k = 0
\]

are satisfied with a constant \( k \). Here \( k, k_*, \overline{k} \) denote the normalized scalar curvatures of the metrics \( g, g_*, \overline{g} \). The trace of the first equation is the following:

(17) \[
\varphi \cdot S_* + (n-1) \cdot \Delta \varphi = 0
\]

**Proof.** The first equation follows directly from Equation 6 for the \((n+1)\)-dimensional manifold \( M \) by the block matrix structure of Ric and \( \nabla^2 \varphi \)

\[
\text{Ric} = \begin{pmatrix} 0 & 0 \\ 0 & \text{Ric}_* \end{pmatrix}, \quad \nabla^2 \varphi = \begin{pmatrix} 0 & 0 \\ 0 & \nabla^2 \varphi \end{pmatrix}
\]

and \( \overline{\text{Ric}} = n\overline{k}\overline{g} \). Then the trace of Equation 6 leads to the equation

\[
\overline{S} = (n+1)[\varphi \Delta_* \varphi - n\|\text{grad}_* \varphi \|^2].
\]

By inserting Equation 17 and \( \overline{S} = n(n+1)\overline{k}, \ S_* = n(n-1)k_* \) we obtain the equation \( \overline{k} = -\varphi^2 k_* - \|\text{grad}_* \varphi \|^2 \). \( \square \)

**Example 4.2.** A standard example for Corollary 4.1 is the unit sphere \((M_*, g_*)\) with a linear height function \( \varphi(r) = \cos r \) in polar coordinates. Here we have the equations \( \text{Ric}_* = (n-1)g_*, \ \nabla^2 \varphi = -\varphi g_* \) and \( \|\text{grad}_* \varphi \|^2 = \sin^2 r, \ k_* = 1, \ \overline{k} = -1 \). However, the conformally transformed metric \( \overline{g} \) is only defined on the part of \( \mathbb{R} \times S^n \) where \( \cos r \neq 0 \) (as in the case of the Mercator projection).
Corollary 4.3. (Corvino [10] Prop.2.7)
With the same notations as in Corollary 4.1 and with a function $f$ on $M_*$ the metric $\bar{g} = \pm f^2 dt^2 + g_*$ is Einstein if and only if the equation
\begin{equation}
\label{eq:18}
f \cdot \text{Ric}_* - \nabla^2_* f + \Delta_* f \cdot g_* = 0
\end{equation}
holds. Its trace
\begin{equation}
\label{eq:19}
f \cdot S_* + (n-1) \cdot \Delta_* f = 0,
\end{equation}
coincides with Equation 17.

Proof. The metric $\bar{g}$ is a warped product metric with an $n$-dimensional base $(M_*, g_*)$ and a 1-dimensional fibre. It can be regarded as a conformal transformation of the product metric $g = \pm dt^2 + f^{-2} g_*$. In this way Equation 18 follows from Equation 6. For the metric $\hat{g} = f^{-2} g_*$ we have
\[
f \cdot \hat{\nabla}^2 f = f \cdot \nabla^2_* f - ||\text{grad}_* f||^2 g_* + 2 df \otimes df
\]
and
\[
\hat{\text{Ric}} = \text{Ric}_* + \frac{n-2}{f} \nabla^2_* f + \left[ f \Delta_* f - (n-1) ||\text{grad}_* f||^2 \right] \cdot \hat{g}
\]
Each quantity for the metric $g$ is obtained from the same quantity for $\hat{g}$ as a block matrix with a 0 entry in the $t$-component:
\[
\text{Ric} = \begin{pmatrix} 0 & 0 \\ 0 & \text{Ric}_* \end{pmatrix}, \quad \nabla^2 f = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\nabla}^2 f \end{pmatrix}
\]
For the conformally transformed metric $\bar{g} = f^2 g = f^2 (\pm dt^2 + \hat{g})$ we obtain
\[
\text{Ric} = \text{Ric}_* + (n-1) f \nabla^2 f^{-1} + \left[ f^{-1} \Delta f^{-1} - n ||\text{grad} f^{-1}||^2 \right] \cdot \bar{g}
\]
By combining this with the equation
\[
\nabla^2 f^{-1} = 2 f^{-3} df \otimes df - f^{-2} \nabla^2 f
\]
we obtain that $\text{Ric}$ equals some scalar multiple of $\bar{g}$ plus
\[
\begin{pmatrix} 0 & 0 \\ 0 & \text{Ric}_* \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f^{-1} \cdot \Delta_* f \cdot g_* \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & f^{-1} \cdot \nabla^2_* f \end{pmatrix}.
\]
The assertion follows. \qed

Lemma 4.4. (Bourguignon [2], Corvino [10] Prop.2.3)
Assume that the equation $f \cdot \text{Ric} - \nabla^2 f + \Delta f \cdot g = 0$ holds for a never vanishing function $f : (M^n, g) \to \mathbb{R}$. Then the metric $g$ is of constant scalar curvature $S$. Moreover the equation $\Delta f = - \frac{S}{n-1} \cdot f$ holds.

Proof. From the Ricci identity for two commuting vector fields $X, Y$
\[
\nabla_X \nabla_Y \text{grad} f - \nabla_Y \nabla_X \text{grad} f = R(X,Y) \text{grad} f
\]
we obtain the trace
\[
\text{tr}(\nabla_Y \text{grad} f) - \nabla_Y (\text{tr} \text{grad} f) = \text{Ric} \cdot \text{grad} f(Y)
\]
or
\[
div(\nabla^2 f) - d(\Delta f) = \text{Ric} \cdot \text{grad} f.
\]
If we combine this with the divergence of the equation \( f \cdot \text{Ric} - \nabla^2 f + \Delta f \cdot g = 0 \) then we obtain
\[
0 = div(f \cdot \text{Ric}) - div(\nabla^2 f) + div(\Delta f \cdot g)
= f \cdot div(\text{Ric}) + \text{Ric} \cdot \text{grad} f - div(\nabla^2 f) + d(\Delta f)
= f \cdot div(\text{Ric})
= f \cdot d(\frac{1}{2}S)
\]
where the last equality follows from the second Bianchi identity ("the Einstein tensor is divergence free") \cite{23, 6.15}. The condition \( f \neq 0 \) implies the assertion \( dS = 0 \). A discussion of possible zeros of \( f \) is given in \cite{10}. The equation \( \Delta f = -\frac{S}{n-1} \cdot f \) follows from the trace of the other equation, compare Equation 19. □

**Corollary 4.5.** If \( f: M_* \to \mathbb{R} \) is a function on a \( n \)-dimensional manifold \((M_*, g_*)\) such that \( g = \pm f^2 dt^2 + g_* \) is Einstein then \( g_* \) is a metric of constant scalar curvature. In particular \( f \) is an eigenfunction of \( \Delta_* \) for the eigenvalue \(-nk_*\).

In this context we mention without proof the following:

**Proposition 4.6.** (Lafontaine \cite{30, Thm.1.1})
If a compact Riemannian \( 3 \)-manifold \((M_*, g_*)\) admits two linearly independent solutions \( f_1, f_2 \) of Equation 18 (such that \( g = f^2 dt^2 + g_* \) is Einstein for \( i = 1, 2 \) then \((M_*, g_*)\) is isometric to the standard \( 3 \)-sphere or to a standard product \( S^1 \times S^2 \) or \( S^1 \times \mathbb{R}P^2 \).

Notice the opposite sign convention in \cite{30} for the Laplacian.

Now we are going to describe nontrivial solutions of the equations in 4.3. It turns out that there are complete Einstein spaces of this type with a complete \((M_*, g_*)\) that are not warped products with a 1-dimensional base themselves, see Example 4.10.

**Proposition 4.7.** (The Ansatz of iterated warped products)
Let \( g_k \) be an Einstein metric with \( \text{Ric} = k(n-2)g_k \), \( k \in \mathbb{R} \) on an \((n-1)\)-dimensional manifold \( M \) and \( g_* = dx^2 + u^2(x)g_k \) a warped product metric on \( M_* = I \times M \) with an interval \( I \subseteq \mathbb{R} \). Let \( f = f(x) \) be a smooth function on an interval in the real line. Then Equation 18 holds if and only if \( f(x) = au'(x) \) for some constant \( a \neq 0 \) and
\[
2u''u'' + (n-3)uu'u'' - (n-2)u^3 + k(n-2)u' = 0.
\]
Hence for a positive solution \( u \) of Equation 20 the warped product metric
\[
\overline{g} = \pm u^2(x)dt^2 + dx^2 + u^2(x)g_k
\]
on \( I \times I \times M \) is an Einstein metric. Here \( I \) is an interval on which \( u \) and \( u' \) are positive. The metric \( \overline{g} \) can be regarded as a warped product \( g_* + u^2 dt^2 \) with an \( n \)-dimensional base and a 1-dimensional fibre or as a warped product \( dx^2 \pm u^2 dt^2 + u^2 g_k \) with a 2-dimensional base in the \((x,t)\)-plane.
Remark: The classical solutions \( u(x) = x, \sin x, \cos x, e^x, \sinh x, \cosh x \) lead to Einstein warped product metrics \( g_* \) for appropriate choices of \( k \) with a 1-dimensional base. Especially \( u(x) = x \) is a solution with \( k = 1 \) corresponding to cylindrical coordinates in the Ricci flat \( \mathbb{R} \times M_* \) where \( M_* \) is also Ricci flat. This is not interesting by 2.4. We will see below in Proposition 5.7 that there are in fact non-classical solutions leading to complete Einstein metrics on \( \mathbb{R} \times M_* \) with a non-Einstein space \((M_*, g_*)\).

**Proof.** Since \( \nabla_X \partial_x = \frac{u'}{u} X \) we obtain

\[
\nabla^2 f(\partial_x, \partial_x) = f'', \quad \nabla^2 f(X, X) = \frac{f'u'}{u}
\]

where \( X \) denotes a unit tangent vector orthogonal to \( \partial_x \), hence

\[
\Delta f = f'' + (n-1)\frac{f'u'}{u}.
\]

This leads to the equations

\[
-\nabla^2 f(\partial_x, \partial_x) + \Delta f = (n-1)\frac{f'u'}{u}
\]

\[
-\nabla^2 f(X, X) + \Delta f = f'' + (n-2)\frac{f'u'}{u}
\]

and

\[
(21) \quad f \text{Ric}(\partial_x, \partial_x) - \nabla^2 f(\partial_x, \partial_x) + \Delta f = \frac{n-1}{u} (f'u' - f''u)
\]

\[
(22) \quad f \text{Ric}(X, X) - \nabla^2 f(X, X) + \Delta f
\]

\[
= \frac{f}{u^2} \left( k(n-2) - (n-2)u'^2 - uu'' \right) + f'' + (n-2)\frac{f'u'}{u}
\]

Assuming Equation 18 we see that the right hand side must vanish. Hence we conclude from Equation 21 that \( f'/f = u''/u \) and, therefore, \( f = au' \) for some constant \( a \). Without loss of generality let \( a = 1 \). From Equation 22 we conclude Equation 20. Conversely, Equation 20 implies Equation 22 and Equation 18. \( \square \)

**Example 4.8.** (particular non-standard solution)

Equation 20 is satisfied by the function \( u(x) = x^{2/n} \) if \( k = 0 \). In fact we have

\[
u'(x) = \frac{2}{n} x^{(2-n)/n}, \quad u''(x) = \frac{4-2n}{n^2} x^{(2-2n)/n}, \quad u'''(x) = \frac{(4-2n)(2-2n)}{n^3} x^{(2-3n)/n}.
\]

For \( n = 3 \) we obtain the 4-dimensional Einstein metric \( \frac{4}{7} x^{-2/3} dt^2 + dx^2 + x^{4/3} g_0 \) where \((M, g_0)\) is any flat 2-manifold. Obviously the sectional curvature in any \((t, x)\)-plane is not constant, so this is not a space of constant curvature. Moreover, for \( x \to 0 \) we run into a singularity.
Corollary 4.9. (Explicit integration)

Equation 20 is equivalent to any of the following differential equations:

(23) \[ uu'' + \frac{n-2}{2} u'^2 + du^2 = k \frac{n-2}{2} \text{ for some constant } d \]

(24) \[ u^{n-2} \left( u'^2 + \frac{2d}{n} u^2 - k \right) = c \text{ for constants } c \text{ and } d \]

(25) \[ u'^2 + \frac{2d}{n} u^2 - cu^{2-n} = k \text{ for constants } c \text{ and } d \]

(26) \[ \left( u^{n/2} \right)'' + \frac{dn}{2} u^{n/2} = k \cdot \frac{n(n-2)}{4} u^{(n-4)/2} \text{ for some constant } d \]

(27) \[ \left( u^{n/2} \right)'' + \frac{dn}{2} \left( u^{n/2} \right)^2 = k \cdot \frac{n^2}{4} u^{n-2} + e \text{ for constants } d \text{ and } e \]

The two last equations are modified oscillator equations. They can be explicitly solved by integrals of elementary functions. For \( k = 0 \) we obtain a classical oscillator equation for the function \( (u(x)/n)^2/n \). The particular solution in Example 4.8 is the case \( d = 0 \). Equation 25 coincides with the version in [1]. Equation 24 was used in [22] and [25]. Here the case \( c = 0 \) leads to the classical solutions of \( u'^2 + \frac{2d}{n} u^2 = k \). The case \( c = 0 \) in 24 is also the classical case of the oscillator equation for \( u \). In this case the metric \( dx^2 + u'^2 dt^2 \) on the 2-dimensional base is of constant curvature.

Proof. We discuss only the case of a non-constant and non-vanishing solution \( u(x) \).

The equivalence of Equation 20 with Equation 23. We differentiate

\[ \frac{u''}{u} + \frac{(n-2)u'^2}{2u^2} + d = k \frac{n-2}{2u^2} \]

and multiply by \( u^3 \). This leads to

\[ u^2 u'' - uu' u'' + (n-2)uu'' u' - (n-2)u'^3 = -k(n-2)u'. \]

The equivalence of Equation 24 with Equation 23. We multiply Equation 23 by \( 2n^{n-2} u' \) and obtain

\[ (u^{n-2} u'^2)' + \frac{2d}{n} u^n - ku^{n-2} = 0. \]

The equivalence of Equation 23 with Equation 26. Evaluating the second derivative of \( u^{n/2} \) leads to

\[ \frac{n}{2} \left( (u^{n/2})'' + \frac{n-2}{2} u^{(n-4)/2} u'^2 \right). \]

Thus Equation 26 takes the form

\[ \frac{n}{2} u^{(n-4)/2} \left( u'' + \frac{n-2}{2} u'^2 + du^2 \right) = \frac{n}{2} u^{(n-4)/2} \cdot k \frac{n-2}{2}. \]

The equivalence of Equation 26 with Equation 27 follows by differentiating the first order equation and dividing by \( 2(u^{n/2})' \). □
Example 4.10. The simplest examples can be constructed from Equation 26 in the case $k = 0$ and $d = 2/n$. Let $(M, g)$ be a Ricci flat Riemannian manifold, and let $u(x) = (\cosh(x))^{2/n}$. Then the metric $u^2 dt^2 + dx^2 + u^2 g$ is Einstein.

In particular for complete $(M, g)$ this metric $dx^2 + (\cosh(x))^{4/n}g$ provides a complete non-standard solution (i.e., not Einstein) $(\mathbb{R} \times M, g_*)$ in Corollary 4.3. If moreover $n = 4$ and $k = -1/2$ then we have the solution $u^2(x) = e^x + 1$ of Equation 26 with $u \neq 0$ and $u' \neq 0$ everywhere. Therefore this leads to a complete Einstein metric $e^{2x/(e^x + 1)} dt^2 + dx^2 + (e^x + 1)g$

on $\mathbb{R} \times M_* = \mathbb{R}^2 \times M$.

Corollary 4.11. (warped products of constant scalar curvature)

The following conditions are equivalent for a function $u(x)$ and a real constant $k$:

1. $g_k$ is an Einstein metric with $\text{Ric} = k(n - 2)g_k$ on an $(n - 1)$-manifold $M$ and $\bar{g} = u^2(x)dt^2 + dx^2 + u^2(x)g_k$ is Einstein.

2. $g_{(k)}$ is a metric of constant normalized scalar curvature $k$ on an $(n - 1)$-manifold $M$ and the warped product $h_{(k)} = dx^2 + u^2(x)g_{(k)}$ is again a metric of constant scalar curvature.

Proof. The first condition (1) is equivalent to Equation 18 which in turn is equivalent to Equation 20 and to Equation 23 by the calculations above. On the other hand the second condition (2) is equivalent to the same Equation 23, which is a standard formula in this case, see the proof of Theorem 24 in [22] or Lemma 4.6 in [25]. Here $2d/n$ is the constant normalized scalar curvature of $h_{(k)}$. □

By Corollary 4.9 all local solutions can be expressed in terms of elementary functions. Periodic solutions $u(x)$ of Equation 23 lead to a countable number of solutions on the compact manifold $S^1 \times S^{n-1}$, see Ejiri [15]. The simplest one is $u(x) = \sqrt{2 + \cos x}$ for $n = 4, d = \frac{1}{2}, k = 1$. Moreover, $u^2 \frac{\partial}{\partial x}$ is a conformal (locally gradient) vector field [25].

5. 4-dimensional conformally Einstein products: Towards a classification

Our Main Theorem 3.2 provides a rough classification of 4-dimensional conformally Einstein products: These are either warped products or they are of the type $M \times N$ with two 2-manifolds $M, N$ and a particular conformal factor $f(y, x) = a(y) + b(x)$ satisfying $ca'' + 2\epsilon_1 a'' = d = cb'' + \epsilon_2 b''$, compare the proof of the Main Theorem. In the latter case all possible metrics are explicitly classified by the solutions of these two ODEs. It remains to discuss the possible warped products. We recall the following general results on Einstein warped products.

Lemma 5.1. ([10, 9.116]) A warped product metric $g_M = g_B + f^2 g_F$ on $M = B \times F$ with a function $f : B \rightarrow \mathbb{R}$ and with $p = \dim(F), q = \dim(B)$ is Einstein with $\text{Ric}_M = \lambda_M g_M$ if and only if the following three conditions hold:

1. $(F, g_F)$ is Einstein with $\text{Ric}_F = \lambda_F g_F$
\((2)\) \(\text{Ric}_B - \frac{k}{2} \nabla_B^2 f = \lambda_M g_B\)

\((3)\) \(f \Delta f + (p - 1) ||\text{grad} f||^2 + \lambda_M f^2 = \lambda_F\)

If \(q = 2\) then \((3)\) is equivalent to Equation 23 with \(p = n - 1\), \(f = u\), \(\lambda_M = 2d\) and \(\lambda_F = k(n - 2)\), and \((2)\) is equivalent with \((\nabla_B^2 f)^0 = 0\).

**Proof.** The three equations are standard equations for warped products, the so-called O’Neill-equations. In the special case \(q = 2\) we have \(\text{Ric}_B = \lambda_B g_B = K_B g_B\) with the Gaussian curvature \(K_B\), \(\lambda_F = K_F\) and, therefore, \((\nabla_B^2 f)^0 = 0\) and \(\nabla_B^2 f = f^2 g_B\). This implies that \(g_B\) itself is a warped product metric \(g_B = dt^2 + f^2(t) dx^2\) with \(K_B = -f'' / f'\). In \((3)\) we have \(\Delta f = 2 f''\) and \(||\text{grad} f|| = f^2\). Equation 23 follows which is equivalent with Equation 20. Conversely, from Equation 23 we get back to \((3)\) and from the statement of Proposition 4.7 we get back to \((1)\) and \((2)\). \(\square\)

**Corollary 5.2.** (Brinkmann [6])

Let \((M, g)\) be an Einstein space with a warped product metric \(g = \epsilon dt^2 + u^2(t) g_s\). Then \(g_s\) is an Einstein metric and \(u\) satisfies the ODE from Corollary 2.4, i.e.

\[ u'' + \epsilon k u = 0, \quad (u')^2 + \epsilon k u^2 = \epsilon k_s\]

Conversely, these conditions are also sufficient.

**Corollary 5.3.** (1-dimensional base, Brinkmann [6] Thm.III))

A 4-dimensional Einstein warped product with a 1-dimensional base is of constant sectional curvature.

This follows from the proof of Corollary 2.4 together with the fact that a 3-dimensional Einstein fibre must be of constant sectional curvature. This implies that the entire space is of constant sectional curvature. For the Riemannian case compare [11] 9.109 or [22].

**Corollary 5.4.** (2-dimensional base, A.Besse [11] 9.116-9.118])

A 4-dimensional Einstein warped product \(g_M = g_B + u^2 g_F\) with a 2-dimensional base \(B\) has a fibre \(F\) of constant Gaussian curvature \(K_F\) and a base \(B\) of the following type:

The warping function \(u\) on \(B\) is a never vanishing solution of the equation \((\nabla_B^2 u)^0 = 0\), hence the metric on \(B\) is a warped product \(\epsilon dt^2 + u^2(t) dx^2\) itself (an abstract surface of revolution, possibly with exceptional points with \(u' = 0\)). Moreover, \(u\) satisfies the ODE

\[ 2\epsilon uu'' + \epsilon u^2 + \lambda_M u^2 = K_F\]

where \(\lambda_M\) is the Einstein constant of the 4-manifold and \(K_F\) is the constant Gaussian curvature of \(g_F\). One integration step as in 4.9 leads to the equation

\[ u\left(\epsilon u^2 + \frac{\lambda_M}{3} \epsilon u^2 - K_F\right) = c\]

with a constant \(c\). Here the coefficient \(\lambda_M / 3\) is nothing but the normalized scalar curvature \(k_M\) of \(M\).

**Proof.** In our special case \(p = q = 2\) we have \(\text{Ric}_B = \lambda_B g_B = K_B g_B\), \(\lambda_F = K_F\) and, therefore, \((\nabla_B^2 f)^0 = 0\). This implies that \(g_B\) is a warped product metric \(g_B = \epsilon dt^2 + \epsilon u^2 = 0\).
\[ u^2(t)dx^2 \text{ itself with } \epsilon = \pm 1, \quad K_B = -\epsilon u''/u' \text{ and } \nabla^2_B f = \epsilon u''g_B. \] Furthermore (2) implies the equation \[ K_B g_B - 2\epsilon u'' = \lambda_M g_B \text{ or, equivalently,} \]
\[ -\epsilon uu'' - 2\epsilon u'u'' = uu'\lambda_M. \]
From (3) we obtain similarly
\[ 2\epsilon uu'' + \epsilon u'^2 + \lambda_M u^2 = \lambda_F = K_F, \]
and Equation (30) is nothing but the derivative of this equation. Since this is an equation only on \( B \), it follows that \( K_F \) must be constant. Equation (28) follows.

A compact base \( B \) must be a surface of genus zero with two exceptional points with \( f' = 0 \), namely, minimum and maximum of the function \( f \). In fact the function \( f(x) = \cos x \) solves this ODE with \( K_F = 1 \) and \( \lambda_M = 3 \). An example of a compact base is \( dx^2 + \sin^2 x \, dt^2 \) in a warped product \( dx^2 + \sin^2 x \, dt^2 + \cos^2 x \, g_F \). If in this case \( F \) is the unit 2-sphere then \( M \) is the unit 4-sphere in cylindrical coordinates around an equator which is, strictly speaking, a warped product only on a dense subset. Here \( B \) appears as an equatorial 2-sphere of \( M \). In any other case zeros of \( f \) would lead to a contradiction. See also Proposition 5.7. □

Remark: As in the explicit integration above we can transform Equation (28) into the following form:
\[ (u^{3/2})'' + \frac{3\lambda_M}{4}fu^{3/2} = \frac{3K_F}{4}u^{-1/2} \]
which is again a modified oscillator equation for \( f^{3/2} \). We can integrate this equation leading to
\[ ((u^{3/2})')^2 + \frac{\lambda_M}{4}(u^{3/2})^2 = \frac{9K_F}{4}u + c \]
with a constant \( c \).

**Proposition 5.5.** (3-dimensional base)
4-dimensional Einstein warped products with a 3-dimensional base can be constructed along the lines of the iterated warped products in Proposition 4.7. These depend on several real parameters, and in general these metrics are not of constant curvature. Therefore, they cannot be described by warped products with a 1-dimensional base.

Remark: A complete classification of this case would depend on a complete classification of the solutions of Equation (18) on 3-manifolds which does not seem to be known. From Equation (20) we just obtain special families of examples that can also be written as warped products with a 2-dimensional base. More examples of open subsets \( U \) of homogeneous spaces \( M \) can be found in [1, 9.122].

**Corollary 5.6.** 5-dimensional Riemannian Einstein warped products with a 2-dimensional base are products \( B^2 \times F^3(K_F) \) of a surface with a 3-manifold of constant curvature \( K_F \) with the metric \( g_B = dx^2 + u^2(x)dt^2, \; g = g_B + u^2g_F \) where \( u \) satisfies the ODE
\[ 2uu'' + 2u'^2 + \overline{K}u^2 = 2K_F \]
or, equivalently,
\[ (u^2)'' + \overline{K}u^2 = 2K_F. \]
This is a standard oscillator equation for $u^2$. A special case is $u^2(x) = 2 + \cos x$ with $k = 1, K_F = 1$.

Remark: There is no compact base $B$ of this type. The only candidate is a surface of genus zero with the metric $g_B = dx^2 + u'^2(x)dt^2$ where $u$ is periodic and strictly positive with the same value of $|u''|$ at minimum and maximum. Such a solution of the ODE above does not exist by Lemma 5.8. For the example $u(x) = \sqrt{2 + \cos x}$ this can be seen from $u'(0) = u'(\pi) = 0$ and $u''(0) = -1/(2\sqrt{3})$, $u''(\pi) = 1/2$. This is the same phenomenon as for the extremal surfaces above, see Figure 1.

**Proposition 5.7.** (complete Einstein warped products with a 2-dimensional base)
There are essentially three types of complete pseudo-Riemannian Einstein warped products with a 2-dimensional base and an $(n-1)$-dimensional fibre that cannot be written as warped products with a 1-dimensional base and, in particular, that are not of constant curvature. In any case (up to sign) the metric in $n + 1$ dimensions is of the type

$$\bar{g} = \pm dx^2 + u'^2(x)dt^2 + u^2(x)g_k$$

as in Proposition 4.7 with an $(n-1)$-dimensional Einstein metric $g_k$ with normalized scalar curvature $k$ (or metric of constant curvature $k$ if $n = 3$) and a strictly positive function $u(x)$ satisfying the equations in Corollary 4.9.

Type I: $u$ is defined on $\mathbb{R}$ with $u > u_0 > 0, u' > 0$ everywhere, asymptotically we have $u(x) \sim u_0 + e^{ax}$ at $\pm \infty$ with positive constants $a, u_0$. The base is the $(x,t)$-plane or a cylindrical quotient of it.

Type II: $u'$ has a zero at $x_0$ with $u''(x_0) = 1$, and $u > 0$ is defined on $[x_0, \infty)$ with an asymptotic growth at infinity like a linear function. The base is a (rotationally symmetric but non flat) plane in polar coordinates.

Type III: $u'$ has a zero at $x_0$ with $u''(x_0) = 1$, and $u > 0$ is defined on $[x_0, \infty)$ with an asymptotic growth at infinity like an exponential function. The base is a (rotationally symmetric but not flat) plane in polar coordinates.

There is no compact base of this type except for the standard 2-sphere.

Conversely, for any given complete metric $g_k$ of arbitrary signature all three types lead to complete warped products provided the base is Riemannian.

Remark: For Riemannian metrics this classification is mentioned through the types (c), (a), (d) in [1, 9.118]. However, the proof refers to a preprint which apparently never appeared, and the solutions are not given explicitly. Moreover, Type (c) there is not quite convincing, as far as the constant coefficients are concerned. The discussion of this classification in the appendix of [18] is a bit sketchy, in particular on Type (c).

Proof. The special expression for the metric follows from Lemma 5.1 above. In the equations the Einstein constant appears as $2d$. So we recall Equation 24 with $\bar{k} = \frac{2d}{n}$ (normalized scalar curvature of $\bar{g}$):

$$u^{n-2}\left(u'^2 + \bar{k}u^2 - k\right) = c \quad \text{for some constant } c$$
In the particular case $c = 0$ we have the oscillator equation

$$u'^2 + ku^2 - k = 0.$$ 

Up to scaling the standard solutions are

- $u(x) = a \sin x + b \cos x$ if $\overline{k} = k = 1$,
- $u(x) = \sinh x$ if $\overline{k} = -1, k = 1$,
- $u(x) = \cosh x$ if $\overline{k} = k = -1$,
- $u(x) = e^x$ if $\overline{k} = -1, k = 0$,
- $u(x) = x$ if $\overline{k} = k = 0$.

Only the cases $u(x) = \cosh x$ and $u(x) = e^x$ lead to global warped products, in the other cases there can be completions of a dense subset that is a warped product.

The solution $u = e^x$ leads to the warped product $dx^2 + e^{2x}(dt^2 + g_k)$ with a 1-dimensional base. This is complete whenever $g_k$ is complete and Riemannian, compare Case (b) in [1, 9.118]. However, the global warped product $-dx^2 + e^{2x}(dt^2 + g_k)$ is not geodesically complete, see [36, 7.41].

Similarly the Riemannian solution $dx^2 + \sinh^2 x \, dt^2 + \cosh^2 x \, g_k$ is complete if and only if $g_k$ is complete. The base is the hyperbolic plane in geodesic polar coordinates. It is of Type III. If $g_k$ itself is the hyperbolic $(n-1)$-space then the entire space is the hyperbolic $(n+1)$-space in geodesic normal coordinates around this $(n-1)$-subspace.

Similarly, the case $dx^2 + \cosh^2 x \, dt^2 + \sinh^2 x \, g_k$ is a warped product over the hyperbolic plane (with exceptional points for $x = 0$). However, there is a completion only if $g_k$ is the unit sphere. This implies that $\overline{g}$ is the hyperbolic space in geodesic normal coordinates around a geodesic. This space can also be written as a warped product with a 1-dimensional base.

This solution $u = x$ leads to the Ricci flat warped product $dx^2 + dt^2 + x^2 g_k$ where the base is the 2-dimensional Euclidean plane. In polar coordinates $(r, \phi)$ in the same plane this can be transformed into the form $dr^2 + r^2(d\phi^2 + \cos^2 \phi \, g_k)$ with a 1-dimensional base (with exceptional points for $x = 0$). There is a completion only if $g_k$ is the standard unit sphere and, consequently, if $\overline{g}$ is the Euclidean space in cylindrical polar coordinates.

The last case $dx^2 + \sin^2 x \, dt^2 + \cos^2 x \, g_k$ is the only warped product with a compact 2-dimensional base, namely, the unit 2-sphere, compare [20]. There is a completion only if $g_k$ is the unit $(n-1)$-sphere and, consequently, if $\overline{g}$ is the unit $(n+1)$-sphere in geodesic normal coordinates around an equatorial circle.

In the general case $c \neq 0$ we integrate Equation 24 as in [25]: A zero of $u$ implies $c = 0$, so we may require the solution $u(x)$ to be strictly positive everywhere. The equation

$$\left( \frac{du}{dx} \right)^2 = u^{2-n}(c - ku^n + ku^{n-2})$$

for $u$ leads to the expression for the inverse function

$$x(u) = \pm \int_{u_0}^u \sqrt{\frac{v^{n-2}}{c - kv^n + kv^{n-2}}} \, dv.$$
By $u > 0$ there must be a positive zero of the polynomial $P(v) := c - kv^n + kv^{n-2}$ in the denominator. From the derivative $P'(v) = v^{n-3}(k(n-2) - kv^2)$ with at most one positive zero we see that $P$ has either at most two positive zeros of order one or exactly one positive zero of order two (the zero $v = 0$ of $P'$ is not relevant here since $P(0) = c \neq 0$).

**Case 1:** There is one positive zero $u_0$ of order two, i.e., $P(u_0) = P'(u_0) = 0$ and $P(v) > 0$ for all $v > u_0$. Therefore $k = 0$ would imply $k = 0$ and $P' \equiv 0$, a contradiction. Hence we have $k \neq 0$. In this case the integral $\int_{u_0}^u$ is infinite for any $u > u_0$, and the integrand tends to $1/(\sqrt[k]{v})$ for $v \to \infty$, so the integral tends to $1/(\sqrt[k]{k}(\log u - \log u_0)$ for $u \to \infty$ in an asymptotic sense. In particular we have $k < 0$. From $P'(u_0) = 0$ we see that $k = k u_0^2 n/(n - 2) < 0$ as well. From $P(u_0) = 0$ it follows that $c = -2k u_0^2 /(n - 2) > 0$. For the inverse function $u(x)$ this means that $u(x)$ tends to $u_0$ for $x \to -\infty$ and that it is asymptotic to $\exp(\sqrt[k]{k} x)$ for $x \to \infty$. In particular $u(x) > u_0$ and $u'(x) > 0$ everywhere. This is Type I in the proposition.

**Case 2:** There are exactly one positive zero $u_0$ of order one, i.e., $P(u_0) = 0$ and $P(v) > 0$ for all $v > u_0$. Then the integral $\int_{u_0}^u$ converges for any $u > u_0$, and the integrand tends to $1/(\sqrt[k]{k})$ if $k = 0$ and to $1/(\sqrt[k]{k}v)$ otherwise for $v \to \infty$. Consequently the integral tends to $1/(\sqrt[k]{k}u)$ for $u \to \infty$ if $k = 0$, and the integral tends to $1/(\sqrt[k]{k}(\log u - \log u_0)$ for $u \to \infty$ in an asymptotic sense if $k < 0$. For the inverse function $u(x)$ this means that for some finite $x_0$ we have $u'(x_0) = 0$, so that $dx^2 + u^2 dt^2$ describes a metric in the plane in geodesic polar coordinates provided that $u''(x_0) = 1$, a condition that can be satisfied. This is Type II in the proposition for $k = 0$ and Type III for $k < 0$.

**Case 3:** There are precisely two positive zeros $u_1 < u_2$ of order one. For $u > u_2$ and $k < 0$ we obtain the same as in Case 2 for $u > u_0$. For $u_1 < u < u_2$ and $k > 0$ we see that the integral for $x(u)$ converges on either side $u_1$ and $u_2$. This means that $u(x)$ is bounded and attains minimum and maximum (actually it is periodic, Ejiri’s solution [15]). Therefore, if the base is complete and Riemannian, it must be compact. However, a compact base is not possible by the following argument: If $u'' = 1$ at the (positive) minimum then $u'' \neq -1$ at the (positive) maximum, as in the case of the drop surfaces above, compare Corollary 3.4. This follows from Lemma 5.8. For the example $u(x) = 2\sqrt{2} + \cos x$ with $n = 4$ we have $u'' = 1$ at the minimum $x = \pi$ and $u'' = -1/\sqrt{3}$ at the maximum $x = 0$. The same argument shows that there is also no $C$-complete Lorentzian base with indefinitely many critical points of a bounded $u$ in the sense of [24]. If one starts at one critical point with $u'' = 1$ then the next critical point turns out to be a singularity since $u'' \neq -1$: The metric is smooth at every second critical point, the other ones are singularities.

For the converse direction we assume that the 2-dimensional base has a positive definite metric, and $u(x)$ is a strictly positive solution according to the cases I, II, III. Then the base is complete. In particular $u$ is bounded below, that it, $u(x) \geq C > 0$ for a constant $C$. We may assume $C \leq 1$. If $u(x) \leq 1$ then $u/\sqrt{1 + u^2} \geq u/2 \geq C/2$. If $u(x) > 1$ then $u/\sqrt{1 + u^2} \geq 1/2 \geq C/2$, so we obtain $u/\sqrt{1 + u^2} \geq C/2$ in any case. It follows that the integral of $u/\sqrt{1 + u^2}$ is unbounded on either side of the real line. Then the completeness
of the warped product follows from Theorem 3.40 in [7]. For the case of a Lorentz base the situation is more complicated, see [7]. □

Lemma 5.8. (compare [20, Sec.2] with a similar proof)
The differential equation
\[ u'^2 = \alpha - \beta u^2 + \gamma u^{2-n} \]
with arbitrary \( \alpha, \beta, \gamma \in \mathbb{R} \) and an integer \( n \geq 3 \) does not admit any solution \( u : I \to \mathbb{R} \) on an open interval \( I \subseteq \mathbb{R} \) that attains its positive minimum \( a \) and its maximum \( b \) such that \( u'(a) = u'(b) = 0 \) and \( u''(a) = 1, u''(b) = -1 \).

Proof. We assume that \( A = u(a) > 0 \) is the minimum value of the function and \( B = u(b) > A \) is the maximum value such that \( u''(a) = 1, u''(b) = -1 \). First of all \( \gamma = 0 \) would imply that either \( u \) has a zero (if \( \beta \geq 0 \)) or \( u \) is unbounded (if \( \beta < 0 \)) which is impossible by assumption. So we have \( \gamma \neq 0 \). Differentiating
\[ u'^2 = \alpha - \beta u^2 + \gamma u^{2-n} \]
yields
\[ 2u'' = -2\beta u - (n-2)\gamma u^{1-n}. \]
Now \( \beta = 0 \) would imply that \( u'' \) cannot change its sign which is impossible by assumption. Therefore we have \( \beta \neq 0 \). We define the function
\[ g(x) = \alpha - \beta x^2 + \gamma x^{2-n} = x^{2-n}[x^{n-2}(-\beta x^2 + \alpha) + \gamma]. \]
With
\[ g'(x) = -2\beta x - (n-2)\gamma x^{1-n}, \]
and the notations \( u'(x) = g(u(x)) \), \( u''(x) = g'(u(x))/2 \) the equation \( g(A) = 0 \) implies
\[ (31) \quad \gamma = A^{n-2} \left( \beta A^2 - \alpha \right). \]
Then from \( g'(A) = 2u''(A) = 2 \) one obtains
\[ (32) \quad \gamma = -\frac{2A^{n-1}}{n-2} (\beta A + 1). \]
Equation \( 31 \) and Equation \( 32 \) imply:
\[ (33) \quad -\frac{2A^{n-1}}{n-2} (\beta A + 1) = \gamma = A^{n-2} \left( \beta A^2 - \alpha \right). \]
Since \( A \neq 0 \) this is equivalent with the quadratic equation
\[ A^2 + \frac{2}{n\beta} A - \frac{n-2}{n\beta} \alpha = 0. \]
Since \( A > 0 \) by assumption we obtain \( \alpha \beta > 0 \) (in particular \( \alpha \neq 0 \)) and
\[ (34) \quad A = \frac{\sqrt{1 + (n-2)n\alpha \beta} - 1}{n\beta} \]
Analogously
\[ (35) \quad \gamma = B^{n-2} \left( \beta B^2 - \alpha \right) = \frac{2B^{n-1}}{n-2} (1 - \beta B) \]
hence
\[ B^2 - \frac{2}{n\beta} B - \frac{n-2}{n\beta} \alpha = 0 \]
Since \( B > A > 0 \) and \( \alpha \beta > 0 \) we obtain:
\[ B = \sqrt{1 + (n-2)n\alpha\beta + 1} \]
Equation 33 and Equation 35 imply
\[ \frac{A^{n-1}}{B^{n-1}} = \frac{\beta B - 1}{\beta A + 1} \]
With the notation
\[ y := \sqrt{1 + (n-2)n\alpha\beta} > 1 \]
we obtain from Equation 34 and Equation 36:
\[ \left( \frac{y+1}{y-1} \right)^{n-1} = \frac{y+(n-1)}{y-(n-1)} \]
resp.: The positive value \( y \) is a zero of the polynomial
\[ \phi_m(x) := (x-1)^m (x+m) - (x+1)^m (x-m) \]
for \( m = n-1 \). Since for \( m \geq 1 \):
\[ \phi'_m = (m+1)\phi_{m-1} \]
and
\[ \phi_m(0) = \begin{cases} 2m & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases} \]
we obtain that
\[ \phi^{(k)}_m(0) = \begin{cases} 2m(m+1) \cdots (m+2-k) & \text{if } (m-k) \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases} \]
Hence all coefficients \( a_k \) of the polynomial \( \phi_m(x) = \sum_{k=0}^{m-2} a_k x^k \) are non-negative and the coefficients \( a_k \) with \( 0 \leq k \leq n-2 \) and \( m \equiv k \) (mod 2) are positive. Therefore the polynomial \( \phi_m \) has no positive zero. This contradiction finishes the proof. \( \square \)

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