GenEO coarse spaces for heterogeneous indefinite elliptic problems

Niall Bootland, Victorita Dolean, Ivan G. Graham, Chupeng Ma and Robert Scheichl

Abstract Motivated by recent work on coarse spaces for Helmholtz problems, we provide in this paper a comparative study on the use of spectral coarse spaces of GenEO type for heterogeneous indefinite elliptic problems within an additive overlapping Schwarz method. In particular, we focus here on two different but related formulations of local generalised eigenvalue problems and compare their performance numerically. Even though their behaviour seems to be very similar for several well-known heterogeneous test cases that are mildly indefinite, only one of the coarse spaces has so far been analysed theoretically, while the other one leads to a significantly more robust domain decomposition method when the indefiniteness is increased. We present a summary of upcoming results developing such a theory and describe how the numerical experiments illustrate it.

1 Introduction and motivations

For domain decomposition preconditioners, the use of a coarse correction as a second level is usually required to provide scalability (in the weak sense), such that the
iteration count is independent of the number of subdomains, for subdomains of fixed dimension. In addition, it is desirable to guarantee robustness with respect to strong variations in the physical parameters. Achieving scalability and robustness usually relies on sophisticated tools such as spectral coarse spaces [5, 4]. In particular, we can highlight the GenEO coarse space [9], which has been successfully analyzed and applied to highly heterogeneous positive definite elliptic problems. This coarse space relies on the solution of local eigenvalue problems on subdomains and the theory in the SPD case is based on the fact that local eigenfunctions form an orthonormal basis with respect to the energy scalar product induced by the bilinear form.

Our motivation here is to gain a better insight into the good performance of spectral coarse spaces even for highly indefinite high-frequency Helmholtz problems with absorbing boundary conditions, as observed in [3] (for the Dirichlet-to-Neumann coarse space) and more recently in [2] for coarse spaces of GenEO type. While a rigorous analysis for Helmholtz problems still lies beyond reach (see also [6] for the challenges), we present here numerical results, showing the benefits of GenEO-type coarse spaces for the heterogeneous symmetric indefinite elliptic problem

\[-\nabla \cdot (A(x)\nabla u) - \kappa u = f \quad \text{in} \quad \Omega, \quad \text{subject to} \quad u = 0 \quad \text{on} \quad \partial \Omega, \quad (1)\]

in a bounded domain \(\Omega\) with homogeneous Dirichlet boundary conditions on \(\partial \Omega\), thus extending the results of [9] to this case. The coefficient function \(A\) in (1) is a symmetric positive-definite matrix-valued function on \(\Omega \rightarrow \mathbb{R}^{d \times d}\) (where \(d\) is the space dimension) with highly varying but bounded values (\(a_{\min}|\xi|^2 \leq A(x)\xi \cdot \xi \leq a_{\max}|\xi|^2, x \in \Omega, \xi \in \mathbb{R}^d\)) and \(\kappa\) is an \(L^\infty(\Omega)\) function which can have positive or negative values. We assume throughout that problem (1) is well-posed and that there is a unique weak solution \(u \in H_0^1(\Omega)\), for all \(f \in L^2(\Omega)\).

We propose two types of spectral coarse spaces, one built from local spectra of the whole indefinite operator on the left-hand side of (1), and the other built using only the second-order operator in (1). For the latter, the analysis in [1] will apply, while the better performance of the former for large \(\kappa\) provides some insight into the good performance of the \(\mathcal{H}\)-GenEO method introduced in [2] for high-frequency Helmholtz problems, even though it is not amenable to the theory in [1].

The problem (1) involves a Helmholtz-type operator (although this term would normally be associated with the case when \(\kappa\) has a positive sign and (1) would normally be equipped with an absorbing boundary condition rather than the Dirichlet condition here). In the special case \(A = I, \kappa = k^2\) with \(k\) constant, the assumption of well-posedness of the problem is equivalent to the requirement that \(k^2\) does not coincide with any of the Dirichlet eigenvalues of the operator \(-\Delta\) in the domain \(\Omega\). In this case, for large \(k^2\), the solution of (1) will be rich in modes corresponding to eigenvalues near \(k^2\) and thus will have oscillatory behaviour, increasing as \(k\) increases. The Helmholtz problem with \(A = I\) and \(\kappa = \omega^2 n\) (with \(\omega\) real and \(n\) a function), together with an absorbing far-field boundary condition appears regularly in geophysical applications; here \(n\) is the refractive index or ‘squared slowness’ of waves and \(\omega\) is the angular frequency.
To solve discretisations of (1), we consider an additive Schwarz (AS) method with a GenEO-like coarse space and study the performance of this solver methodology for some heterogeneous test cases. GenEO coarse spaces have been shown theoretically and practically to be very effective for heterogeneous positive definite problems. Here, our main focus is to investigate how this approach performs in the indefinite case (1). We now review the underlying numerical methods that are used.

2 Discretisation and domain decomposition solver

We suppose that the domain $\Omega$ is a bounded Lipschitz polygon/polyhedron in 2D/3D. To discretise the problem we use the Lagrange finite element method of degree $p$ on a conforming simplicial mesh $T^h$ of $\Omega$. Denote the finite element space by $V^h \equiv H^1_0(\Omega)$. The finite element solution $u_h \in V^h$ satisfies the weak formulation

$$b(u_h, v_h) = F(v_h), \quad \text{for all } v_h \in V^h,$$

where

$$b(u, v) = \int_{\Omega} (A(x) \nabla u \cdot \nabla v - \kappa uv) \, dx \quad \text{and} \quad F(v) = \int_{\Omega} f v \, dx.$$

Using the standard nodal basis for $V^h$ we can represent the solution $u_h$ through its basis coefficients $\mathbf{u}$ and reduce the problem to solving the symmetric linear system

$$Bu = f$$

where $B$ comes from the bilinear form $b(\cdot, \cdot)$ and $f$ from the linear functional $F(\cdot)$. Note that $B$ is symmetric but generally indefinite. For sufficiently small fine-mesh diameter $h$, problem (3) has a unique solution $u$; see [8]. To solve (3), we utilise a two-level domain decomposition preconditioner within a Krylov method.

Consider an overlapping partition $\{\Omega_j\}_{1 \leq j \leq N}$ of $\Omega$, where each $\Omega_j$ is assumed to have diameter $H_j$ and $H$ denotes the maximal diameter of the subdomains. For each $j$ we define $\bar{\Omega}_j = \{v|_{\Omega_j} : v \in V^h\}$, $V_j = \{v \in \bar{\Omega}_j : \supp(v) \subset \Omega_j\}$, and for $u, v \in \bar{\Omega}_j$

$$b_j(u, v) := \int_{\Omega_j} (A(x) \nabla u \cdot \nabla v - \kappa uv) \, dx \quad \text{and} \quad a_j(u, v) := \int_{\Omega_j} A(x) \nabla u \cdot \nabla v \, dx.$$

Let $R_j^T : V_j \to V^h$, $1 \leq j \leq N$, denote the zero-extension operator, let $R_j^T$ denote its matrix representation with respect to the nodal basis and set $R_j = (R_j^T)^T$. The classical one-level additive Schwarz preconditioner is

$$M_{AS}^{-1} = \sum_{j=1}^N R_j^T B_j^{-1} R_j,$$

where $B_j = R_j B R_j^T$. (4)

It is well-known that one-level additive Schwarz methods are not scalable with respect to the number of subdomains in general, since information is exchanged
only between neighbouring subdomains. Thus, we introduce the two-level additive Schwarz method with GenEO coarse space first proposed in [9]. To this end, for \(1 \leq j \leq N\), let \(\{\phi_j^1, \ldots, \phi_j^\mu_j\}\) be a nodal basis of \(\tilde{V}_j\), where \(\tilde{n}_j = \dim(\tilde{V}_j)\).

**Definition 1 (Partition of unity)** Let \(\text{dof}(\Omega_j)\) denote the internal degrees of freedom (nodes) on subdomain \(\Omega_j\). For any degree of freedom \(i\), let \(\mu_i\) denote the number of subdomains \(\Omega_i\), for which \(i\) is an internal degree of freedom, i.e., \(\mu_i := \#\{j : 1 \leq j \leq N, i \in \text{dof}(\Omega_j)\}\). Then, for \(1 \leq j \leq N\), the local partition of unity operator \(\Xi_j : \tilde{V}_j \to V_j\) is defined by

\[
\Xi_j(v) := \sum_{i \in \text{dof}(\Omega_j)} \frac{1}{\mu_i} \nu_i \phi_i^j, \quad \text{for all} \quad v = \sum_{i=1}^{\tilde{n}_j} \nu_i \phi_i^j \in \tilde{V}_j. \tag{5}
\]

The operators \(\Xi_j\) form a partition of unity, i.e., \(\sum_{j=1}^{N} R_j^T \Xi_j(v|\Omega_j) = v, \forall v \in V^h\) [9].

For each \(j\), we define the following generalised eigenvalue problems:

\[
\begin{align*}
\text{find } p & \in \tilde{V}_j \setminus \{0\}, \lambda \in \mathbb{R} : & a_j(p, v) = \lambda a_j(\Xi_j(p), \Xi_j(v)), & \text{for all } v \in \tilde{V}_j, \quad \tag{6} \\
\text{find } q & \in \tilde{V}_j \setminus \{0\}, \lambda \in \mathbb{R} : & b_j(q, v) = \lambda b_j(\Xi_j(q), \Xi_j(v)), & \text{for all } v \in \tilde{V}_j. \quad \tag{7}
\end{align*}
\]

where \(\Xi_j\) is the local partition of unity operator from Definition 1.

**Definition 2 (\(\Delta\)-GenEO and \(\mathcal{H}\)-GenEO coarse spaces)** For each \(j, 1 \leq j \leq N\), let \(\{p_j^l\}_{l=1}^{m_j}\) and \(\{q_j^l\}_{l=1}^{m_j}\) be the eigenfunctions of the eigenproblems (6) and (7), corresponding to the \(m_j\) smallest eigenvalues, respectively. Then we define the \(\Delta\)-GenEO and \(\mathcal{H}\)-GenEO coarse spaces, respectively, by

\[
\begin{align*}
V^0_{\Delta} & := \text{span}\{R_j^T \Xi_j(p_j^l) : l = 1, \ldots, m_j; j = 1, \ldots, N\} \quad \text{and} \\
V^0_{\mathcal{H}} & := \text{span}\{R_j^T \Xi_j(q_j^l) : l = 1, \ldots, m_j; j = 1, \ldots, N\}. \tag{8} \tag{9}
\end{align*}
\]

Note that here and subsequently, the subscript \(\Delta\) refers to the GenEO coarse space based on [6], the eigenproblem with respect to the ‘Laplace-like’ operator induced by the bilinear form \(a_j\), while the subscript \(\mathcal{H}\) refers to the \(\mathcal{H}\)-GenEO coarse space based on [7], with the ‘Helmholtz-like’ operator appearing in \(b_j\).

Since \(V^0_{\Delta}, V^0_{\mathcal{H}} \subset V^h\), we can introduce the natural embeddings \(R_{0,\Delta}^T : V^0_{\Delta} \to V^h\) and \(R_{0,\mathcal{H}}^T : V^0_{\mathcal{H}} \to V^h\), with matrix representations \(R_{0,\Delta}^T\) and \(R_{0,\mathcal{H}}^T\), respectively, and set \(R_{0,\Delta} = (R_{0,\Delta}^T)^T\) and \(R_{0,\mathcal{H}} = (R_{0,\mathcal{H}}^T)^T\) to obtain the following two-level extensions of the one-level additive Schwarz method [4]:

\[
M_{AS,\Delta}^{-1} = M_{AS}^{-1} + R_{0,\Delta} R_{0,\Delta}^{-1} \quad \text{and} \quad M_{AS,\mathcal{H}}^{-1} = M_{AS}^{-1} + R_{0,\mathcal{H}} R_{0,\mathcal{H}}^{-1} R_{0,\mathcal{H}}, \tag{10}
\]

where \(B_{0,\Delta} := R_{0,\Delta} BR_{0,\Delta}^T\) and \(B_{0,\mathcal{H}} := R_{0,\mathcal{H}} BR_{0,\mathcal{H}}^T\).
3 Theoretical results

The theoretical properties of the preconditioner $M_{AS,\Delta}^{-1}$ are studied in the forthcoming paper [1]. There, the PDE studied is a generalisation of (1), which also allows the inclusion of a non-self-adjoint first order convection term. The important parameters in the preconditioner are the coarse mesh diameter $H$ and the ‘eigenvalue tolerance’

$$\Theta := \max_{1 \leq j \leq N} \left( \frac{1}{m_j+1} \right),$$

where $\{\lambda_m^j : m = 1, 2, \ldots \}$ are the eigenvalues of the generalised eigenproblem (6), given in non-decreasing order. We now highlight a special case of the results in [1].

**Theorem 1** Let the fine-mesh diameter $h$ be sufficiently small. Then there exist thresholds $H_0 > 0$ and $\Theta_0 > 0$ such that, for all $H \leq H_0$ and $\Theta \leq \Theta_0$: the matrices $B_j$ and $B_0,\Delta$ appearing in (4) and (10) are non-singular. Moreover, if problem (3) is solved by GMRES with left preconditioner $M_{AS,\Delta}^{-1}$ and residual minimisation in the energy norm $\|u\|_a := \left( \int_{\Omega} \nabla u \cdot A \nabla u \right)^{1/2}$, then there exists a constant $c \in (0, 1)$, which depends on $H_0$ and $\Theta_0$ but is independent of all other parameters, such that we have the robust GMRES convergence estimate

$$\|r_\ell\|_a^2 \leq \left( 1 - c^2 \right)^\ell \|r_0\|_a^2,$$

for $\ell = 0, 1, \ldots$, where $r_\ell$ denotes the residual after $\ell$ iterations of GMRES.

In fact, the paper [1] will investigate in detail how the thresholds $H_0$ and $\Theta_0$ depend on the heterogeneity and indefiniteness of (1). For example, if the problem is scaled so that $a_{\text{min}} = 1$, then as $\|\kappa\|_\infty$ grows, $H_0$ and $\Theta_0$ have to decrease to maintain the convergence rate of GMRES:

$$H_0 \leq \|\kappa\|_\infty^{-1} \text{ and } \Theta_0 \leq C_{\text{stab}}^{-2} \|\kappa\|_\infty^{-4},$$

where $C_{\text{stab}} = C_{\text{stab}}(A, \kappa)$ denotes the stability constant for problem (1), i.e., the solution $u$ satisfies $\|u\|_{H^1(\Omega)} \leq C_{\text{stab}} \|f\|_{L^2(\Omega)}$ for all $f \in L^2(\Omega)$ and the hidden constants are independent of $h, H, a_{\text{max}}$ and $\kappa$. Thus, as $\|\kappa\|_\infty$ gets smaller, the indefiniteness diminishes and the requirements on $H_0$ and $\Theta_0$ are relaxed.

4 Numerical results

We give results for a more efficient variant of the preconditioner described in §2. Instead of (10), we here use the restricted additive Schwarz (RAS) method, with the GenEO coarse space incorporated using a deflation approach, yielding:
The document contains a mathematical analysis of layered media, focusing on piecewise constant profiles and the use of alternating layers, diagonal layers, and inclusions. The text explains the formulation of the problem, the use of partition of unity operators, and the implementation of the methods using FreeFem software. The mathematical expressions and algorithms are described in detail, including the use of GMRES preconditioning and the role of eigenfunctions in the solution process.
the coefficient $a(x)$ (and hence the problem itself) becomes more complicated as $N$
increases since the geometry of the coefficient remains identical in each subdomain.

In Table 2 we illustrate the effect of increasing $\kappa$, giving iteration counts and (in
brackets) coarse space sizes. Here we see the substantial advantage of $\mathcal{H}$-GenEO
over $\Delta$-GenEO: much better iteration counts are obtained, yet the coarse space size
increases only modestly. As $N$ increases, although the dimension of the coarse space
grows, the number of eigenfunctions per subdomain decreases. For very large $\kappa$,
neither method is fully robust while, for small $\kappa$, both methods perform similarly.
This leads to the interesting question of whether robustness to $\kappa$ can be gained by
taking more eigenfunctions in the coarse space. Table 3 gives results for a sequence
of increasing values of $\kappa$ for the diagonal layers problem, in which we simultaneously
increase $\lambda_{\max}$, indicating (apparent almost) robustness with respect to $\kappa$.

Table 1 GMRES iteration counts with $\lambda_{\max} = \frac{1}{2}$. Left-hand table: Alternating layers problem,
varying $a_{\max}$ and $N$, with fixed $\kappa = 100$ and $n_{\text{glob}} = 400$. Right-hand table: Inclusions
problem, varying $n_{\text{glob}}$ and $N$, with fixed $\kappa = 1000$ and $a_{\max} = 50$.

| $a_{\max}$ \ $N$ | $\Delta$-GenEO | $\mathcal{H}$-GenEO | $a_{\max}$ \ $N$ | $\Delta$-GenEO | $\mathcal{H}$-GenEO |
|------------------|----------------|----------------|------------------|----------------|----------------|
| 10               | 10 9 9 10      | 9 9 9          | 9 9 9            | 9 9 9          | 9 9 9          |
| 50               | 9 9 9 9        | 9 9 9 9        | 200              | 24 16 26 22    | 10 10 10 11   |
| 200              | 9 9 9 9        | 9 9 9 9        | 400              | 23 14 19 18    | 8 9 9 8       |
| 1000             | 9 9 9 9        | 9 9 9 9        | 600              | 21 14 18 19    | 8 9 9 10      |
|                  |                |                | 800              | 22 14 19 20    | 9 9 9 10      |

Table 2 GMRES iteration counts and (in brackets) coarse space dimension for the diagonal layers
problem with $\lambda_{\max} = \frac{1}{2}$, varying $\kappa$ and $N$, with fixed $a_{\max} = 5$ and $n_{\text{glob}} = 600$.

| $\kappa$ \ $N$ | $\Delta$-GenEO | $\mathcal{H}$-GenEO | $\kappa$ \ $N$ | $\Delta$-GenEO | $\mathcal{H}$-GenEO |
|----------------|----------------|----------------|----------------|----------------|----------------|
| 10             | 9 (627)        | 9 (1050)       | 9 (1468)       | 9 (1804)       | 9 (1468)       |
| 100            | 10 (627)       | 9 (1050)       | 9 (1468)       | 9 (1804)       | 9 (1473)       |
| 1000           | 36 (627)       | 43 (1050)      | 35 (1468)      | 28 (1804)      | 13 (674)       |
| 10000          | 215 (627)      | 339 (1050)     | 437 (1468)     | 506 (1804)     | 27 (1256)      |
|                |                |                |                |                |                |

These observations align with the fact that eigenfunctions appear qualitatively
similar for $\Delta$-GenEO and $\mathcal{H}$-GenEO when $\kappa$ is small. As seen in Fig. 2 once $\kappa$
increases the $\mathcal{H}$-GenEO eigenfunctions change: the type of eigenfunctions produced
by $\Delta$-GenEO remain, albeit perturbed, but now we have further eigenfunctions which
include more oscillatory behaviour in the interior of the subdomain; such features are
not found with $\Delta$-GenEO where higher oscillations only appear near the boundary.
Table 3 GMRES iteration counts and (in brackets) coarse space dimension for $H$-GenEO for the diagonal layers problem, varying $N$, with fixed $n_{glob} = 600$ and $a_{\max} = 5$ and increasing eigenvalue threshold $\lambda_{\text{max}}$ as $\kappa$ increases, aiming to control iteration counts as $\kappa$ increases.

| $\lambda_{\text{max}} \kappa \setminus N$ | $16$ | $36$ | $64$ | $100$ |
|------------------------------------------|------|------|------|-------|
| 0.1 10                                  | 23 (108) | 23 (199) | 25 (214) | 23 (324) |
| 0.1 100                                 | 23 (111) | 24 (201) | 28 (223) | 27 (324) |
| 0.2 1000                                | 19 (265) | 27 (418) | 20 (574) | 20 (684) |
| 0.6 10000                               | 24 (1430) | 25 (2129) | 28 (2680) | 15 (3252) |

Fig. 2 Example eigenfunctions on the central subdomain when $N = 25$ and $n_{glob} = 800$: In the first three columns, we plot qualitatively similar eigenfunctions, computed (left-to-right) by $\Delta$-GenEO, $H$-GenEO when $\kappa = 1000$, and $H$-GenEO when $\kappa = 10000$. This illustrates how eigenfunctions of (7) are affected by the indefiniteness in $b_j$, relative to the size of $\kappa$. In addition, as $\kappa$ increases the $H$-GenEO eigenproblem enriches the coarse space with “wave-like” eigenfunctions that are not seen for $\Delta$-GenEO; one of the many examples when $\kappa = 10000$ is plotted in the final column. While the top row explores the homogeneous case, the bottom row demonstrates the effect of heterogeneity in $a(x)$ for the diagonal layers problem: For $N = 25$, $a(x) = a_{\max} = 10$ in the upper-left triangle ($x_2 > x_1$) and $a(x) = a_{\min} = 1$ in the lower-right triangle ($x_2 < x_1$) of the central subdomain. Note that variation in the eigenfunctions is mainly confined to the low coefficient region.

5 Conclusions

In this work we have summarised how the forthcoming analysis in [1] can be applied to a GenEO-type coarse space for heterogeneous indefinite elliptic problems. We provide numerical evidence supporting these results and a comparison with a more effective GenEO-type method for highly indefinite problems but for which no theory is presently available. For mildly indefinite problems these two approaches perform similarly, providing the first theoretical insight towards explaining the good behaviour of the $H$-GenEO method for challenging heterogeneous Helmholtz problems.
References

1. Bootland, N., Dolean, V., Graham, I.G., Ma, C., Scheichl, R.: Overlapping Schwarz methods with GenEO coarse space for non-symmetric and indefinite elliptic equations (2021). In preparation.
2. Bootland, N., Dolean, V., Jolivet, P., Tournier, P.H.: A comparison of coarse spaces for Helmholtz problems in the high frequency regime (2020). ArXiv:2012.02678
3. Conen, L., Dolean, V., Krause, R., Nataf, F.: A coarse space for heterogeneous Helmholtz problems based on the Dirichlet-to-Neumann operator. J Comput Appl Math 271, 83–99 (2014)
4. Dolean, V., Jolivet, P., Nataf, F.: An Introduction to Domain Decomposition Methods: Algorithms, Theory, and Parallel Implementation. SIAM, Philadelphia, PA (2015)
5. Galvis, J., Efendiev, Y.: Domain decomposition preconditioners for multiscale flows in high contrast media. Multiscale Model. Sim. 8(4), 1461–1483 (2010)
6. Gander, M., Ernst, O.: Why it is difficult to solve Helmholtz problems with classical iterative methods. In: I.G. Graham, T.Y. Hou, O. Lakkis, R. Scheichl (eds.) Numerical Analysis of Multiscale Problems, pp. 325–363. Springer, Berlin, Heidelberg (2011)
7. Graham, I.G., Spence, E.A., Vainikko, E.: Domain decomposition preconditioning for high-frequency helmholtz problems with absorption. Math. Comp. 86, 2089–2127 (2017)
8. Schatz, A.H., Wang, J.P.: Some new error estimates for Ritz–Galerkin methods with minimal regularity assumptions. Math. Comput. 213, 19–27 (1996)
9. Spillane, N., Dolean, V., Hauret, P., Nataf, F., Pechstein, C., Scheichl, R.: Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. Numer. Math. 126(4), 741–770 (2014)