On the algebraic solutions of the sixth Painlevé equation related to second order Picard-Fuchs equations

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Abstract

We describe two algebraic solutions of the sixth Painlevé equation which are related to (isomonodromic) deformations of Picard-Fuchs equations of order two.

1 Statement of the result

In this note we describe two algebraic solutions of the following Painlevé VI (PV1) equation

\[ \frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d\lambda}{dt} \]

\[ + \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t^2 - 1)} \left[ \alpha + \frac{\beta t}{\lambda^2} + \frac{t - 1}{(\lambda - 1)^2} + \frac{\delta t(t - 1)}{(\lambda - t)^2} \right]. \]

related to deformations of Picard-Fuchs equations of special type. Recall that the PV1 equation governs the isomonodromic deformations of the second
order Fuchsian equations

\[ x'' + p_1(s)x' + p_2(s)x = 0, \quad ' = \frac{d}{ds} \]

with 5 singular points, one of which is apparent \[4\]. Suppose that the solution of the Fuchsian equation (1) is given by an Abelian integral

\[ x(s) = \int_{\gamma_s} \omega \]

where \( \omega \) is a rational one-form on \( \mathbb{C}^2 \), \( \Gamma_s \subset \mathbb{C}^2 \) is a family of algebraic curves depending rationally on \( s \), and \( \gamma_s \subset \Gamma_s \) is a continuous family of closed loops. Then the equation (1) is said to be of Picard-Fuchs type and its monodromy group is conjugated to a subgroup of \( \text{Gl}_2(\mathbb{Q}) \) (generically \( \text{Gl}_2(\mathbb{Z}) \)). For this reason any continuous deformation

\[ a \rightarrow \Gamma_{s,a} \]

of the family \( \Gamma_s \) induces an isomonodromic deformation of (1). If in addition \( \Gamma_{s,a} \) depends algebraically in \( a \), the coefficients of (1) are also algebraic functions in \( a \), and hence they provide an algebraic solution of \( \mathcal{P}_{VI} \).

Our main result is the following

**Theorem 1** The pencil of \( \mathcal{P}_{VI}(\alpha) \) equations

\[ (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha, -\beta, \gamma, \frac{1}{2} - \delta), s \in \mathbb{C} \]

has a common algebraic solution parameterized as

\[ \lambda = \frac{a^2(2-a)}{a^2 - a + 1}, t = \frac{a^3(2-a)}{2a - 1}, a \in \mathbb{C}. \]

The \( \mathcal{P}_{VI}(\alpha) \) equation with

\[ \alpha = (\frac{1}{8}, \frac{1}{2}, 0, 0) \]

has an algebraic solution parameterized as

\[ \lambda = \frac{a(a-2)(2a^2 + a + 2)}{a^2 - 7a + 1}, t = \frac{a^3(2-a)}{2a - 1}, a \in \mathbb{C} \]
The meaning of these solutions is the following. Consider the family of elliptic curves
\[ \Gamma_s = \{ (\xi, \eta) \in \mathbb{C}^2 : \eta^2 + \frac{3}{2a-1} \xi^4 - \frac{4(a+1)}{2a-1} \xi^3 + \frac{6a}{2a-1} \xi^2 = s \} \tag{5} \]
and let \( \gamma(s) \in H_1(\Gamma_s, \mathbb{Z}) \) be a family of cycles depending continuously on \( s \in \mathbb{C} \). The Abelian integral of first kind
\[ \int_{\gamma(s)} \frac{d\xi}{\eta} \]
satisfies a Picard-Fuchs equation of second order depending on a parameter \( a \), defining an isomonodromy deformation of the equation. This deformation corresponds then to an algebraic solution of \( \mathcal{P}_{VI}(\alpha) \) given by (3). In a similar way, the Abelian integral of second kind
\[ \int_{\gamma(s)} \frac{(3\xi^2 - 2(a+1))\xi d\xi}{\eta} \]
satisfies a Picard-Fuchs equation of second order. The isomonodromy deformation of this equation with respect to \( a \) is described by the solution (4) of \( \mathcal{P}_{VI} \) equation.

Algebraic solutions of \( \mathcal{P}_{VI} \) were found by many authors, e.g. Hitchin[6], Manin[8], Dubrovin-Mazzocco[4], Boalch[3]. Dubrovin and Mazzocco classified all algebraic solutions of the \( \mathcal{P}_{VI} \) equation corresponding to
\[ (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left( \frac{1}{2} (2\mu - 1)^2, 0, 0, 0 \right), \mu \in \mathbb{R}. \]
It turns out that these solutions, up to symmetries, are in a one-to-one correspondence with the regular polyhedra in the three dimensional space. Our solution (3) with
\[ (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1/8, 0, 0, 0) \]
corresponds then to the tetrahedron solution of Dubrovin-Mazzocco (\( \mu = +1/4 \)). It is identified to their solution \( A_3 \) via the Okamoto type transformation (1.24),(1.25), see [4]. It is remarkable that the same solution, but for
\[ (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1/8, 1/8, 1/8, 1/8) \]
was also found by Hitchin[6]. This shows that (3) is a common solution to the family (2) of \( \mathcal{P}_{VI} \) equations. It is clear that for transcendental values of
s in (2) the corresponding isomonodromic family of Fuchs equations (1) can not be of Picard-Fuchs type.

The paper is organized as follows. In the next section we recall briefly, following [7], the relationship between $P_{VI}$ and the isomonodromic deformations of Fuchs equations. In section 3 we deduce the relevant Picard-Fuchs equation and establish the main result.

The present text is an abridged version of [2].

2 The Garnier system and the $P_{VI}$ equation.

Consider a Fuchsian differential equation

$$x'' + p_1(s)x' + p_2(s)x = 0, \quad s = \frac{d}{ds}$$

with five singular points, exactly one of which is apparent. After a bi-rational change of the independent variable $s$ and a linear change of the dependent variable $x$ (involving $s$) we may suppose that the singular points are $0, 1, t, \lambda, \infty$, where the singularity $\lambda$ is apparent and the corresponding Riemann scheme is

$$\begin{pmatrix}
0 & 1 & t & \lambda & \infty \\
0 & 0 & 0 & 0 & \alpha \\
\theta_1 & \theta_2 & \theta_3 & k & \alpha + \theta_\infty
\end{pmatrix}, n \in \mathbb{N}, 2\alpha + \sum_i \theta_i + n = 3.$$

In what follows we shall always suppose that $n = 2$ (which is satisfied generically).

The coefficients $p_1, p_2$ are easily computed to be

$$p_1(s) = \frac{1 - \theta_1}{s} + \frac{1 - \theta_2}{s - 1} + \frac{1 - \theta_3}{s - t} - \frac{1}{t - \lambda}$$

$$p_2(s) = \frac{k}{s(s - 1)} - \frac{t(t - 1)K}{s(s - 1)(s - t)} + \frac{\lambda(\lambda - 1)\mu}{s(s - 1)(s - \lambda)}$$

where $\mu$ is a constant

$$k = \frac{1}{4}\{\left(\sum_{i=1}^{3} \theta_i - 1\right)^2 - \theta_\infty^2\}. $$
The compatibility condition for the singular point $\lambda$ to be apparent reads

$$K = K(\lambda, \mu, t) = \frac{1}{t(t-1)}[\lambda(\lambda - 1)(\lambda - t)\mu^2 - \{\theta_2(\lambda - 1)(\lambda - t) + \theta_3\lambda(\lambda - t) + (\theta_1 - 1)\lambda(\lambda - 1)\}\mu + k\lambda].$$

From the discussion above it is seen that the Fuchs equation (1) depends on the parameters $\theta_0, \theta_1, \theta_t, \theta_\infty, \lambda, \mu, t$. Let us denote this equation by $E_\theta(\lambda, \mu, t)$.

**Theorem 2** $\lambda(t), \mu(t)$ is a solution of the Garnier system

$$\frac{d\lambda}{dt} = \frac{\partial K}{\partial \mu},$$

$$\frac{d\mu}{dt} = -\frac{\partial K}{\partial \lambda},$$

if and only if the induced deformation of $E_\theta(\lambda, \mu, t)$ is isomonodromic.

It is straightforward to check that the sixth Painlevé system $\mathcal{P}_{VI}(\alpha)$ with parameters

$$\alpha = (\frac{1}{2}\theta_\infty^2, \frac{1}{2}\theta_0^2, \frac{1}{2}\theta_1^2, \frac{1}{2}\theta_t^2)$$

is equivalent to the Garnier system. We get therefore the following

**Corollary.** If

$$(t, \lambda, \mu) \rightarrow (t, \lambda(t), \mu(t))$$

is an isomonodromic deformation of $E_\theta(\lambda, \mu, t)$, then $\lambda(t)$ is a solution of $\mathcal{P}_{VI}(\alpha)$ equations with parameters given by (6).

### 3 Picard-Fuchs equations

In this section we restrict our attention to the deformation

$$f_a(\xi, \eta) = \eta^2 + \frac{3}{2a - 1}\xi^4 - \frac{4(a + 1)}{2a - 1}\xi^3 + \frac{6a}{2a - 1}\xi^2, a \in \mathbb{C}$$

of the singularity $\eta^2 + \xi^4$ of type $A_3$, see [1]. The critical values of $f_a(\xi, \eta)$ are

$$0, 1, t = \frac{a^3(2 - a)}{2a - 1}. $$
Consider the locally trivial smooth fibration
\[ f^{-1}(\mathbb{C} \setminus \{0, 1, t\}) \to \mathbb{C} \setminus \{0, 1, t\} \]
whose fibers the affine curves \( \Gamma_s, \) \( s \in \mathbb{C} \setminus \{0, 1, t\} \). Each \( \Gamma_s \) is topologically a torus with two removed points. Hence \( \dim H^1(\Gamma_s, \mathbb{Z}) = \dim H^1_{DR}(\Gamma_s, \mathbb{C}) = 3 \). Therefore if \( \gamma(s) \in H^1(\Gamma_s, \mathbb{Z}) \) is a family of cycles depending continuously on \( s \), then the Abelian integral
\[ I(s) = \int_{\gamma(s)} \omega, \omega = P(\xi, \eta)d\xi + Q(\xi, \eta)d\eta, P, Q \in \mathbb{C}[\xi, \eta] \]
satisfies a Fuchsian differential equation of order three, whose coefficients are polynomials in \( s, a \). In the case when the differential form \( \omega \) has no residues, it satisfies a second order equation. Explicitly, if \( \gamma_1(s), \gamma_2(s) \) is a continuous family of cycles generating the homology group of the compactified elliptic curve \( \Gamma_s \), then the equation reads
\[
\det \begin{pmatrix}
\int_{\gamma_1(s)} \omega & \int_{\gamma_2(s)} \omega \\
(\int_{\gamma_1(s)} \omega)' & (\int_{\gamma_2(s)} \omega)' \\
(\int_{\gamma_1(s)} \omega)'' & (\int_{\gamma_2(s)} \omega)''
\end{pmatrix} = 0.
\]
It follows from the Picard-Lefschetz formula and the moderate growth of the integrals, that the coefficients of the above differential equations are rational in \( s, a \). A local analysis of the singularities shows for instance that
\[
\det \begin{pmatrix}
\int_{\gamma_1(s)} \omega & \int_{\gamma_1(s)} \omega' \\
\int_{\gamma_2(s)} \omega & \int_{\gamma_2(s)} \omega'
\end{pmatrix} = \frac{p(s, a)}{s(s-1)(s-t)}
\]
where \( p(s, a) \) is a polynomial in \( s, a \). If we put \( \omega = dx/y \) then \( \int_{\gamma_1(s)} \omega \) grows no faster than \( s^{1/4-1/2} \) at \( \infty \) (for a fixed \( a \)). Thus
\[
\frac{p(s, a)}{s(s-1)(s-t)}
\]
grows at infinity no faster than \( s^{-1/2-1} \) and hence no faster than \( s^{-2} \). It is expected therefore that \( p(s, a) \) is of degree one in \( s \) and the corresponding root, which we denote by \( \lambda \), is an apparent singularity for the Picard-Fuchs equation in consideration. We are therefore in a position to apply Theorem 2, provided that the deformation of the Fuchs equation with respect to
the parameter $a$ is isomonodromical. Indeed, the monodromy group of our equation is contained in $SL(2, \mathbb{Z})$ which shows that any deformation of this equation is isomonodromical. The Picard-Lefschetz formula shows that the monodromy group in question is generated, up to conjugacy, by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$ (7)

To deduce an explicit formula for the corresponding algebraic solution of $\mathcal{P}_{VI}$ we need explicit formulæ for the Picard-Fuchs equations.

**Lemma 1** Let $\gamma(s) \in H_1(\Gamma_s, \mathbb{Z})$ be a family of cycles depending continuously on $s$. The complete elliptic integrals of first and second kind

$$x(s) = \int_{\gamma(s)} \frac{d\xi}{\eta}, \quad y(s) = \int_{\gamma(s)} \frac{(3\xi^2 - 2(a + 1))\xi d\xi}{\eta}$$

satisfy Picard-Fuchs equations of the form

$$a_0(s)x'' + a_1(s)x' + a_2(s)x = 0$$

$$b_0(s)y'' + b_1(s)y' + b_2(s)y = 0$$

where

$$a_0(s) = s(s-1)((2a-1)s + a^3(a-2))((a^2-a+1)s + a^2(a-2))$$

$$a_1(s) = 2(2a-1)(a^2-a+1)s^3 + (a^6 - 3a^5 + 9a^4 - 19a^3 + 9a^2 - 3a + 1)s^2$$

$$+ 2a^2(a-2)(a^4 - 2a^3 - 2a + 1)s - a^5(a-2)^2$$

$$a_2(s) = (2a-1)[27(a^2-a+1)s^2 - (a-2)(2a^4 - a^3 - 60a^2 - a + 2)s$$

$$+ a^2(a-2)^2(10a^4 + 11a + 10)]/144$$

$$b_0(s) = s(s-1)((2a-1)s + a^3(a-2))((a^2 - 7a + 1)s - a(a-2)(2a^2 + a + 2))$$

$$b_1(s) = (2a-1)s[(a^2 - 7a + 1)s^2 - 2a(a-2)(2a^2 + a + 2)s$$

$$- a(a-2)^2(a^4 + a^3 + a^2 + a + 1)]$$

$$b_2(s) = -(2a-1)[9(a^2 - 7a + 1)s^2 - (a-2)(10a^4 + 31a^3 - 12a^2 + 31a + 10)s$$

$$- a(a-2)^2(2a^2 + a + 2)]/144$$

The proof of the above Lemma is straightforward, see for instance [5]. It is seen that the roots of $a_0(s)$ are 0, 1 and

$$\lambda = \frac{a^2(2-a)}{a^2 - a + 1}, \quad t = \frac{a^3(2-a)}{2a - 1}$$

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which implies the algebraic solution (3). In the same way the roots of $b_0(s)$ provide the solution (4). The Riemann schemes of the Picard-Fuchs equations for $x(s), y(s)$ are given by

$$
\begin{pmatrix}
0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a^2(2-a)}{a^2-a+1} & \infty \\
0 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 2 & \frac{3}{4}
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
0 & 1 & \frac{a^3(2-a)}{2a-1} & \frac{a(a-2)(2a^2+a+2)}{a^2-a+1} & \infty \\
0 & 0 & 0 & 0 & \frac{1}{4} \\
1 & 0 & 0 & 2 & -\frac{1}{4}
\end{pmatrix}.
$$

The Corollary after Theorem 2 implies that the curve (3) is an integral curve of the $P_{VI}(\alpha)$ equation with parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\frac{1}{8}, 0, 0, 0)$, see (6). Similarly, the Fuchsian equation satisfied by the complete elliptic integral of second kind $y(s)$ provides the algebraic solution (4) of $P_{VI}(\alpha)$ with $\alpha = (\frac{1}{8}, \frac{1}{8}, 0, 0)$.

It is remarkable that (3) was found to be a solution of $P_{VI}(\alpha)$ by Hitchin [6][p.177], but for $\alpha = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. After taking the difference between these two equations we obtain the following affine equation of the integral curve (3)

$$
-\frac{t}{\lambda^2} + \frac{t - 1}{(\lambda - 1)^2} - \frac{t(t - 1)}{(\lambda - t)^2} = 0.
$$

This also shows that (3) is a common algebraic solution of the pencil of $P_{VI}(\alpha)$ equations

$$
\alpha = \left(\frac{1}{8}, \frac{s}{8}, \frac{s}{8}, \frac{s}{8}\right), s \in \mathbb{C}.
$$

This completes the proof of Theorem 4.

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