CONSENSUS STABILITY ANALYSIS FOR STOCHASTIC MULTI-AGENT SYSTEMS WITH MULTIPLICATIVE MEASUREMENT NOISES AND MARKOVIAN SWITCHING TOPOLOGIES

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ABSTRACT. We investigate the consensus stability for linear stochastic multi-agent systems with multiplicative noises and Markovian random graphs and investigate the asymptotic consensus in the mean square sense for the systems. To establish the consensus stability for the systems, we analysis the consensus error systems by developing general stochastic differential equation with jumps, matrix theory and algebraic graph theory, and then show that the error consensus in the mean square sense finally tending to zero as time goes on is determined by the strongly connected property of union of topologies. Finally, we provide an example to demonstrate the effectiveness of our theoretical results.

1. Introduction. Over the years, consensus of multi-agent systems has drawn much attention and a mass of research works [27, 29] reported the consensus behaviors from diverse scientific disciplines such as birds, ants, schools, herds etc. Moreover it has abundant of applications such as formation control [5], distributed filtering [24], robotics [11], sensor networks [12] and distributed computation [17].

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Theoretically, consensus is a behavior that all agents always reach a global goal of interest in an appropriate sense as time goes on under some constraints. A resurgent and increased effort has been achieved for this objective on deterministic and stochastic multi-agent systems [1, 4, 7, 8, 9, 16, 18, 21, 23, 25, 26, 35].

In the real world, various uncertainties have different influences on the success of consensus. Matei et al. [22] considered multi-agent systems for both discrete-time and continuous-time cases under Markovian switching topologies respectively and provided a necessary and sufficient condition for average consensus stability. Based on the framework of Markovian switching topologies, You et al. [31] considered consensus for a more general and controllable linear multi-agent systems with a constant gain and gave necessary and sufficient conditions by designing a consensus gain to achieve consensus. Further, Zhang et al. [34] extended the framework in [22] to the systems with additive noises and gave sufficient conditions based on a distributed consensus protocol to attenuate the noises for the asymptotic unbiased average-consensus in both mean square and almost sure senses. Li et al. [13] discussed the same consensus problem in [34], they provided necessary and sufficient conditions under a control protocol based on a new gain which is changing over time. Moreover, they also explored to the cases whose time-varying topological structure is characterized by semi-Markovian switching. Shi et al. [28] considered multi-agent systems with input noises and time-varying information flow without designing a time-vary control gain and achieved robust consensus and integral robust consensus under uniformly joint connectivity and infinitely joint connectivity, respectively. For cases of systems with noises only, Li and Zhang [15] gave necessary and sufficient conditions of asymptotically unbiased mean square average-consensus for the multi-agent systems with additive noises with a time-varying consensus gain. Wang and Elia [30] studied the dynamic behavior of systems with white Gaussian input noises, channel fading ad time-delay. Li et al. [14] considered the systems with multiplicative noises under fixed undirected graph and showed that the unbiased mean square consensus and almost surely average-consensus were achieved with sufficient conditions with respect to connectivity of the topological structure of the systems. More works related to stochastic consensus can be found in [2, 10, 32].

Motivated by the above studies with respect to stochastic consensus, we consider consensus for continuous-time linear systems with noises under Markovian random graphs incorporating a fact which cannot be neglected is that noises often occur in practical systems. Different from Matei et al. [22], here we consider the same continuous-time models but driven by multiplicative measurement noises and investigate the consensus without using consensus control which has not been studied before to the best of our knowledge.

The goal of this paper is to establish the asymptotic consensus for first-order continuous-time multi-agent systems with multiplicative measurement noises under Markovian random graphs by borrowing the ideas from stochastic consensus. Towards the asymptotic consensus, we transform the dynamics of agents into a stochastic error systems. In order to decrease the error systems, we then combine stochastic analysis and tools of symmetrized graph in stochastic Lyapunov analysis to theoretically form a good expression for the ease of analysis. If the graph resulting from the union of graphs which are switched by an ergodic Markov chain is containing a spanning tree to ensure the connectivity of the network to some extent, the error systems in the mean square sense tend to zero as time goes on making sure that all agents may asymptotically agree on their states.
Paper organization. In section 2, we recall some related concepts and basic results on graphs and matrices. In section 3, we present the setup and formulation of the continuous-time linear systems considered in this paper. In section 4, we state our main convergence theorem. In section 5 we provide a simulation example to demonstrate the effectiveness of the stated theorem. In section 6, we give a brief summary of our paper.

Notation. Denote $\mathbb{1}$ be a vector of all ones with $N$ dimension. Let $X$ be a given random variable or vector and $A$ be a matrix, denote the transpose of $X$ and of matrix $A$ by $X^T$ and $A^T$ respectively. Denote $E_X$ be the mathematical expectation of $X$. Denote $\eta_{N,i}$ be the $N$ dimensional vector with the $i$th element being 1 and others being zero. Denote $J_N$ be a given matrix $(1/N)\mathbb{1}\mathbb{1}^T$. Denote $I_N$ be an $N$ dimensional identity matrix.

2. Preliminaries. In this paper, we let $(\Omega, F, \mathbb{P})$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $F_0$ contains all $\mathbb{P}$-null sets). \{w_{ji}(t), i, j = 1, 2, \ldots, N\} are independent standard white noises and adapted in the probability space, $\{\theta(t)\}_{t\geq 0}$ is a right-continuous Markov chain with finite modes collected by a set $\mathcal{S} = \{1, 2, \ldots, s\}$ which is independent of $\{w_{ji}(t), i, j = 1, 2, \ldots, N\}$, and is adapted in the probability space. The generator of $\{\theta(t)\}_{t\geq 0}$ is denoted by $\Lambda = (\lambda_{ij})_{s \times s}$, so that for a sufficiently small $h > 0$,

$$\mathbb{P}\{\theta(t + h) = r|\theta(t) = k\} = \left\{ \begin{array}{ll}
\lambda_{kr}h + o(h), & k \neq r, \\
1 + \lambda_{kk}h + o(h), & k = r,
\end{array} \right. \quad (1)$$

where $o(h)$ is a high-order infinitesimal with respect to infinitesimal $h$ and $\lambda_{kr} \geq 0$ is the transition rate from state $k$ to state $r$ if $k \neq r$ while $\lambda_{kk} = -\sum_{k \neq r} \lambda_{kr}$. From [20], for almost every sample path of $\theta(t)$ in any finite interval of $\mathbb{R}_+ := [0, +\infty)$, we know there are finite simple jumps.

Throughout this paper, we associate to a dynamic system with a communication graph within $N$ agents and present some concepts and preliminary results related.

Let the set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ be the set of agents with node $i$ representing the $i$th agent. The interactive communication topology among the agents is modeled by a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with no self-loops and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A pair $(j, i)$ belongs to the edge set $\mathcal{E}$ if the $j$th agent can receive information from the $j$th agent directly. If edges $(k_1, k_2), (k_2, k_3), \ldots, (k_{m-1}, k_m) \in \mathcal{E}$, we say there is a directed path from node $k_1$ to $k_m$. The weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^N \times N$. For an $i, j \in \mathcal{V}, a_{ij} \geq 0$, and $a_{ij} > 0 \iff (j, i) \in \mathcal{E}$. The graph corresponding to a given adjacency matrix $\mathcal{A}$ is denoted by $\mathcal{G}_\mathcal{A}$. The Laplacian matrix $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}(\sum_{j=1}^N a_{ij}, i = 1, \ldots, N)$. We call $\mathcal{G}$ is a balanced graph if $\sum_{j=1}^N a_{ji} = \sum_{j=1}^N a_{ij}, \ i \in \mathcal{V}$. It is clear that an undirected graph is a balanced digraph. If there exists a directed path from some agent of the systems to the rest of agents, we say that the digraph $\mathcal{G}$ contains a spanning tree.

**Definition 2.1 ([22]).** Let $\mathcal{S} = \{1, 2, \ldots, s\}$ to be the index set of states corresponding to all possible network topologies, and let $\{A(i) \in \mathbb{R}^{N \times N}, i \in \mathcal{S}\}$ be a set of matrices with respect to a set of graphs $\{\mathcal{G}_{A(i)}, i \in \mathcal{S}\}$. We say that the graph $\mathcal{G}_\mathcal{A}$ corresponds to the set $\mathcal{A}$ if it is given by the union of graphs, i.e.,

$$\mathcal{G}_\mathcal{A} \triangleq \bigcup_{i \in \mathcal{S}} \mathcal{G}_{A(i)} = \left( \mathcal{V}, \bigcup_{i \in \mathcal{S}} \mathcal{E}_i, \mathcal{A} \right),$$
where \( A = \sum_{i=1}^{s} A(i) \).

**Lemma 2.2** ([15]). If the graph \( G = \{V, E, A\} \) is undirected, then the corresponding Laplacian matrix \( L \) is symmetric, and has \( N \) real eigenvalues, in an ascending order:

\[
0 = \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_N,
\]

and

\[
\min_{x \neq 0, 1^T x = 0} \frac{x^T L x}{\|x\|^2} = \gamma_2.
\]

We call that \( \gamma_2(L) \) is the algebraic connectivity of \( G \). Moreover, if \( G \) is connected, then \( \gamma_2 > 0 \).

**3. Problem formulation.** Let \( x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{R}^N \), where \( x_i \) represents the state of agent \( i, i \in V \). We consider stochastic systems consisting of \( N \) agents with noise under time-varying communication graphs with respect to different modes which is determined by a continuous-time homogeneous Markov chain with finite stats, and the dynamics of agent \( i \) (\( i \in V \)) at mode \( \theta(t) \) takes the following form:

\[
\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(\theta(t))(x_j(t) - x_i(t)) + \sum_{j=1}^{N} a_{ij}(\theta(t))\sigma_{ji}(x_j(t) - x_i(t))\xi_{ji}(t),
\]

(2)

where \( a_{ij}(\theta(t)) \geq 0 \) represents the information flow forming an adjacency matrix \( A(\theta(t)) \), i.e. \( A(\theta(t)) = (a_{ij}(\theta(t)))_{N \times N} \), in relation to time-dependent network topology corresponding to the switching law determined by the Markov chain \( \theta(t) : [0, \infty) \to \mathbb{S} \) and \( a_{ij}(\theta(t)) = 1 \) or 0, and \( a_{ij}(\theta(t)) = 1 \) if and only if \( (j, i) \in E(\theta(t)) \). \( \xi_{ji}(t) \in \mathbb{R} \) is the measurement noise, parameter \( \sigma_{ji} \geq 0 \) represents the noise intensity. The measurement noise processes \( \{\xi_{ji}(t), i, j \in V\} \) are conditioned as following for the ease of analysis.

**Assumption 1.** The processes \( \{\xi_{ji}(t), i, j \in V\} \) satisfy \( \int_0^t \xi_{ji}(s)ds = w_{ji}(t), t \geq 0 \). Quantized measurements of relative states in consensus problem was first studied in [3], and the stochastic framework regards the quantization uncertainty as white noise [1], under Assumption 1, rewriting the (2) in differential form gives

\[
dx_i(t) = \sum_{j=1}^{N} a_{ij}(\theta(t))(x_j(t) - x_i(t))dt + \sum_{j=1}^{N} a_{ij}(\theta(t))\sigma_{ji}(x_j(t) - x_i(t))dw_{ji}(t).
\]

(3)

For the vector formulation of (3), we introduce the Laplacian matrices \( L(\theta(t)), \theta(t) \in \mathbb{S} \), as follows:

\[
L(\theta(t)) := D(\theta(t)) - A(\theta(t)),
\]

where \( D(\theta(t)) = \text{diag}(\text{deg}_i(\theta(t))), i = 1, \ldots, N \), and \( \text{deg}_i(\theta(t)) = \sum_{j=1}^{N} a_{ij}(\theta(t)), i \in V \). Thus, the vector form of systems (3) is

\[
dx(t) = -L(\theta(t))x(t)dt + \sum_{i=1}^{N} \eta_{Ni} \sum_{j=1}^{N} a_{ij}(\theta(t))\sigma_{ji}(x_j(t) - x_i(t))dw_{ji}(t).
\]

(4)

Next, we introduce the following concept of consensus definition.

**Definition 3.1** ([13]). For any \( x_0 \in \mathbb{R}^n \), the state vector \( x(t) \) evolving according to the systems (3) is said to achieve mean square consensus if \( \lim_{t \to \infty} \mathbb{E}[\|x_i(t) - x_j(t)\|^2] = 0 \), for all \( i, j \in V \).
The Markov process

Assumption 4. The union graph of all available topologies contains a spanning tree.

Assumption 3. The system (4) can achieve mean square consensus if

\[ \text{Assumption 2. All digraphs are assumed to be balanced.} \]

\[ \text{Assumption 4. The Markov process } \{\theta(t)\}_{t \geq 0} \text{ is ergodic.} \]

\[ \text{Theorem 4.1. Suppose that Assumptions 1, 2, 3, 4 hold, then the dynamic system (4) can achieve mean square consensus if } \sigma < \sqrt{N}/(N-1), \text{ where } \sigma = \max\{\sigma_{ij}, i, j \in \mathcal{V}\}. \]

\[ \text{Proof. From Assumption 2, we have } (I_N - J_N) L(k) = L(k)(I_N - J_N), k \in \mathcal{S}. \]

\[ \text{We first consider the error system of the dynamics (4) in the case } k \in \mathcal{S} \text{ as follows:} \]

\[ d\delta(t) = -(I_N - J_N) L(k)x(t)dt + \sum_{i=1}^{N} \eta_{N,i} \sum_{j=1}^{N} a_{ij}(k)\sigma_{ji}(x_j(t) - x_i(t))dw_{ji}(t) \]

\[ = -L(k)\delta(t)dt + \sum_{i=1}^{N} (I_N - J_N)\eta_{N,i} \sum_{j=1}^{N} a_{ij}(k)\sigma_{ji}(\delta_j(t) - \delta_i(t))dw_{ji}(t). \]

\[ \text{Then we investigate the tendency of the error systems in the mean square sense in the following. Define stochastic Lyapunov function candidates } V : \mathbb{R}^N \times \mathcal{S} \rightarrow \mathbb{R} \text{ by } V(\delta) = \delta^T\delta. \]

\[ \text{Applying the Ito formula ([19]), we have} \]

\[ dV(\delta) = d[\delta^T\delta] = V_\delta(\delta)(d\delta) + \frac{1}{2} (d\delta)^T V_{\delta\delta}(\delta)(d\delta) + \sum_{l=1}^{s} \lambda_{kl} V(\delta)dt \]

\[ = 2\delta^T \left[ -L(\theta(t))\delta dt + \sum_{i=1}^{N} (I_N - J_N)\eta_{N,i} \sum_{j=1}^{N} a_{ij}(\theta(t))\sigma_{ji}(\delta_j - \delta_i)dw_{ji}(t) \right] \]

\[ + \left[ -L(\theta(t))\delta dt + \sum_{i=1}^{N} (I_N - J_N)\eta_{N,i} \sum_{j=1}^{N} a_{ij}(\theta(t))\sigma_{ji}(\delta_j - \delta_i)dw_{ji}(t) \right]^T d\delta \]

\[ + \sum_{l=1}^{s} \lambda_{kl}[\delta^T\delta]dt \]
\[
= -[\delta^T (L(\theta(t)) + L^T(\theta(t)))\delta]dt + \frac{N-1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(\theta(t))\sigma_{ij}^2(\delta_j - \delta_i)^2dt
+ 2\sum_{i=1}^{N} \delta^T(I_N - J_N)\eta_{Ni,j} \sum_{j=1}^{N} a_{ij}(\theta(t))\sigma_{ij}(\delta_j - \delta_i)dw_{ij}(t).
\]

Since \(d[V(\delta)\chi_{\{\theta(t)=k\}}] = \chi_{\{\theta(t)=k\}} dV(\delta) + V(\delta) d\chi_{\{\theta(t)=k\}}\), integrating both sides to the equality yielding
\[
\delta^T \delta \chi_{\{\theta(t)=k\}} = \delta^T(0)\delta(0)\chi_{\{\theta(0)=k\}} + \int_0^t \chi_{\{\theta(t)=k\}} dV(\delta) + \int_0^t V(\delta) d\chi_{\{\theta(t)=k\}}.
\]

Do a mathematical expectation to both sides of the equality above, we have
\[
E[\delta^T \delta \chi_{\{\theta(t)=k\}}] = E[\delta^T(0)\delta(0)\chi_{\{\theta(0)=k\}}] + \int_0^t E\left[\chi_{\{\theta(s)=k\}} dV(\delta)\right] + \int_0^t E\left[V(\delta) d\chi_{\{\theta(s)=k\}}\right].
\]

Again, do a differential operation to both sides together with the results of Lemma 4.2 in [6], and we have
\[
dE[\delta^T \delta \chi_{\{\theta(t)=k\}}] = d \int_0^t E\left[\chi_{\{\theta(s)=k\}} dV(\delta)\right] + d \int_0^t E\left[V(\delta) d\chi_{\{\theta(s)=k\}}\right]
= E\left[\chi_{\{\theta(t)=k\}} dV(\delta)\right] + E\left[V(\delta) d\chi_{\{\theta(t)=k\}}\right]
= -E\left[\delta^T (L(k) + L^T(k)) \delta \chi_{\{\theta(t)=k\}}\right] dt + \sum_{i=1}^{s} \lambda_{lk} E[\delta^T \delta \chi_{\{\theta(t)=l\}}] dt + o(dt)
+ \frac{N-1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(k)\sigma_{ij}^2(\delta_j - \delta_i)^2 \chi_{\{\theta(t)=k\}} dt.
\]

Hence, from the formula (7) in [13] under Assumption 4, we have
\[
\frac{dE[\delta^T \delta]}{dt} = \sum_{k=1}^{s} \frac{dE[\delta^T \delta \chi_{\{\theta(t)=k\}}]}{dt}
= -\sum_{k=1}^{s} E[\delta^T (L(k) + L^T(k)) \delta \chi_{\{\theta(t)=k\}}] + \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{lk} E[\delta^T \delta \chi_{\{\theta(t)=l\}}]
+ \frac{N-1}{N} \sum_{k=1}^{s} E\left[\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(k)\sigma_{ij}^2(\delta_j - \delta_i)^2 \chi_{\{\theta(t)=k\}}\right]
= -E\left[\delta^T (L_{un} + L^T_{un}) \delta\right] + \sum_{i=1}^{s} \left(\sum_{k=1}^{N} \lambda_{lk} E[\delta^T \delta \chi_{\{\theta(t)=l\}}]\right)
+ \frac{N-1}{N} E\left[\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(k)\sigma_{ij}^2(\delta_j - \delta_i)^2\right]
= -E\left[\delta^T (L_{un} + L^T_{un}) \delta\right] + \sum_{i=1}^{s} \left(\sum_{k=1}^{N} \lambda_{lk} E[\delta^T \delta]\right)
\]
$$+ \frac{N-1}{N} E \left[ \sum_{k=1}^{s} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(k) \sigma_{ji}^2(\delta_j - \delta_i)^2 \right]$$

$$= - E \left[ \delta^T (L_{un} + L_{un}^T) \delta \right] + \frac{N-1}{N} E \left[ \sum_{k=1}^{s} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(k) \sigma_{ji}^2(\delta_j - \delta_i)^2 \right]$$

$$= - E \left[ \delta^T L \delta \right] + \frac{N-1}{N} E \left[ \sum_{k=1}^{s} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(k) \sigma_{ji}^2(\delta_j - \delta_i)^2 \right].$$

Let $\delta(t) = T_\tau \tilde{\delta}(t)$, $\tilde{\delta}(t) = (\tilde{\delta}_1(t), \tilde{\delta}_2(t), \ldots, \tilde{\delta}_N(t))^T \in \mathbb{R}^N$, then $\tilde{\delta}_1(t) = 0$, $\tilde{\delta}(t) = T_\tau^T \delta(t)$, $\delta(t) = (\delta_1(t), \delta_2(t), \ldots, \delta_N(t))^T$, then from Assumption 3 and Lemma 3.2 in [33], we have $\bar{\delta} = (\phi^T \delta)$, $\delta^T \delta = \tilde{\delta} \delta = \tilde{\delta}^T \tilde{\delta}$ and $\delta^T L \delta = \delta^T \Gamma \delta$. From the preliminaries, we have

$$\frac{dE[\delta^T \delta]}{dt} = \frac{dE[\tilde{\delta}^T \tilde{\delta}]}{dt} = - E \left[ \delta^T \Gamma \tilde{\delta} \right] + \frac{N-1}{N} E \left[ \sum_{k=1}^{s} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2(k) \sigma_{ji}^2(\delta_j - \delta_i)^2 \right]$$

$$\leq - E \left[ \delta^T \Gamma \tilde{\delta} \right] + \frac{N-1}{N} \sigma^2 E \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} (\delta_j - \delta_i)^2 \sum_{k=1}^{s} a_{ij}(k) \right]$$

$$= - E \left[ \delta^T \Gamma \tilde{\delta} \right] + \frac{N-1}{N} \sigma^2 E \left[ \delta^T \Gamma \tilde{\delta} \right]$$

$$= - E \left[ \delta^T \Gamma \tilde{\delta} \right] + \frac{N-1}{N} \sigma^2 E \left[ \tilde{\delta}^T \Gamma \tilde{\delta} \right]$$

(5)

It is noticed that $(\Gamma - \frac{N-1}{N} \sigma^2 \Gamma)$ is positive definite if $\sigma < \sqrt{N/(N-1)}$. So if $\sigma < \sqrt{N/(N-1)}$,

$$\frac{dE[\tilde{\delta}^T \tilde{\delta}]}{dt} \leq - \left( 1 - \frac{N-1}{N} \sigma^2 \right) \gamma_2 E[\tilde{\delta}^T \tilde{\delta}],$$

(6)

which yields

$$E[\tilde{\delta}^T \tilde{\delta}] \leq E[\tilde{\delta}^T (0) \tilde{\delta}(0)] e^{- \left( 1 - \frac{N-1}{N} \sigma^2 \right) \gamma_2 t}.$$ 

(7)

Then, we have $\lim_{t \to \infty} E[\tilde{\delta}^T \tilde{\delta}] = 0$, which leads to $\lim_{t \to \infty} E[\delta^T \delta] = 0$. Finally, we have $\lim_{t \to \infty} E[(x_i(t) - x_j(t))^2] = 0$ for any $i, j \in \mathcal{V}$. By the Definition 3.1, we know that the stochastic multi-agent systems (3) achieve mean square consensus as time goes on. 

\textbf{Remark 1.} This theorem obtains the consensus in the mean square sense asymptotically under an ergodic Markov chain $\{\theta_t\}_{t \geq 0}$ and strongly connected conditions (Assumption 2 and Assumption 3).

(a) If $\sigma_{ji} = 0$, for any $(j, i) \in \mathcal{E}$ and the Markovian switching topologies are considered, the dynamic network (3) degenerates to the noise-free case which has been considered by Matei et al. [22] and the consensus of two different senses (mean square and almost sure) were established by strongly connected condition of the union graphs and irreducible Markov chain. You et al. [31]}
extended the work in [22] to the more general linear systems with a constant gain.

(b) If the topology is fixed, the systems (3) degenerate to a general stochastic differential equations only with multiplicative measurement noises. In such case, Li et al. [14] provided the sufficient conditions for determined single-integrator agent dynamics with a constant gain to be designed and Zong et al. [32] for second-order multi-agent systems with two positive constant gains to be designed.

(c) If the topology is fixed and the communication noises are additive, mean square average-consensus of the single-integrator agent dynamics is considered by Li et al. [15] with a positive time-varying gain to be assumed. Zhang et al. [34] extended [15] to the case of considering Markovian switching topologies. Li et al [13] considered the similar systems in [34] using a different time-varying gain.

5. Simulations. In this section, in order to analyze the influence of noises and switching topologies on to the dynamics of the coupled system (2), we now consider a dynamic of (3) with four agents ($N = 4$) and arbitrary given $x(0) = (4, 1, −5, 3)$. Then $\bar{\sigma} < \sqrt{N/(N - 1)} = 1.1547$. Let $\sigma_{ij} = 1, (i, j) \in \{(i, j) \mid a_{ij}(k) \neq 0, k \in S \}$. Suppose that $S = \{1, 2, 3\}$ and the three different adjacency matrices corresponding to the three different topology graphs are as followings

$$A(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A(3) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (8)$$

Obviously, the corresponding topology graphs are balanced, and

$$L = \begin{pmatrix} 6 & -2 & -2 & -2 \\ -2 & 6 & -2 & -2 \\ -2 & -2 & 6 & -2 \\ -2 & -2 & -2 & 6 \end{pmatrix}. \quad (9)$$

with the eigenvalues $\gamma_1 = 0, \gamma_2 = \gamma_3 = \gamma_4 = 8$. So the union graph of the graphs has a spanning tree. If the switching topologies are compelled by a Markov chain, the generator is chosen as

$$\Lambda = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$ 

Then we know that it will jump between the three topologies reaching every state in the given time in Figure 1 on the left. The dynamic curve of the states of the three agents on the right of Figure 1 shows a sample path of the consensus seeking process, it illustrates the consensus is achieved and consistent with Theorem 4.1.

6. Conclusion. We have investigated considered stochastic consensus of the first-order continuous-time linear multi-agent systems with relative-state dependent measurement noises under time-varying switching topologies. To analysis the time-varying topologies, the time-varying topologies is figured by a continuous-time Markov chain with finite states which corresponds to modes of different directed graphs in the topological structure of the systems. Besides, for each fixed mode,
the noise effecting on an agent is measured by relative states of its neighbor in the fixed topology corresponding the mode with fixed noise intensities. By investigating the connectivity of the directed graphs, the error systems of the systems and the irreducible Markov chain, we give sufficient conditions to achieve the error consensus in the mean square sense for the systems. A simulation example is operated to demonstrate our theoretical results.

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