NOTE ON THE $X_1$-JACOBI ORTHOGONAL POLYNOMIALS.

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Abstract. This note supplements the results in the paper on $X_1$-Jacobi orthogonal polynomials, written by David Gómez-Ullate, Niky Kamran and Robert Milson.

1. Introduction

This note reports on, the $X_1$-Jacobi polynomials, one of the two new sets of orthogonal polynomials considered in the papers [3] and [1], written by David Gómez-Ullate, Niky Kamran and Robert Milson. The other set is named the $X_1$-Laguerre polynomials and is discussed, in similar terms, in the note [2].

These two papers are remarkable and invite comments on the results therein which have yielded new examples of Sturm-Liouville differential equations and their associated differential operators.

The two sets of these orthogonal polynomials are distinguished by:

(i) Each set of polynomials is of the form \( \{P_n(x) : x \in \mathbb{R} \text{ and } n \in \mathbb{N} \equiv \{1, 2, 3, \ldots \}\} \) with \( \deg(P_n) = n \); that is there is no polynomial of degree 0.

(ii) Each set is orthogonal and complete in a weighted Hilbert function space.

(iii) Each set is generated as a set of eigenvectors from a self-adjoint Sturm-Liouville differential operator.

2. $X_1$-Jacobi Polynomials

2.1. Parameters. These polynomials and associated differential equations are detailed in [3 Section 2]

There are two real-valued parameters \( \alpha, \beta \) involved in the differential equation for these orthogonal polynomials. These parameters have to satisfy, see [3 Section 2.1],:

\[
\alpha, \beta \in (-1, \infty) \quad \alpha \neq \beta \quad \text{sgn}(\alpha) = \text{sgn}(\beta)
\]

These conditions give two essentially different cases to consider

\[
\begin{cases}
\text{Case 1} & \beta > \alpha > 0 \\
\text{Case 2} & -1 < \beta < \alpha < 0
\end{cases}
\]

The real parameters \( a, b, c \), see [3 Section 2, (5a) and (5b)], are defined by

\[
a := \frac{1}{2}(\beta - \alpha), \quad b := \frac{\beta + \alpha}{\beta - \alpha}, \quad c := b + a^{-1};
\]
it may be shown, using (2.1), that \( a, b, c \), satisfy
\[
\begin{align*}
\text{Case 1} & : a > 0, \ b > 1, \ c > 1 \\
\text{Case 2} & : a < 0, \ b > 1, \ c < -1.
\end{align*}
\]

2.2. Differential expression. In [3, Section 2, (19) and (20a)] the second-order linear differential equation concerned is given as
\[
(x^2 - 1)y''(x) + 2a \left( \frac{1 - bx}{b - x} \right) [(x - c)y'(x) - y(x)] = \lambda y(x) \text{ for all } x \in (-1, 1),
\]
where the parameter \( \lambda \in \mathbb{C} \) plays the role of a spectral parameter for the differential operators defined below.

This equation (2.5) is not a Sturm-Liouville differential equation; such equations take the form, in this case taking the interval to be \((-1, 1)\),
\[
- (p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (-1, 1),
\]
with \( \lambda \in \mathbb{C} \). Here \( p, q, w : (-1, 1) \to \mathbb{R} \) and satisfy the minimal conditions, [1, Section 3],
\[
\begin{align*}
(i) \quad & p^{-1}, q, w \in L^1_{\text{loc}}(-1, 1) \\
(ii) \quad & w(x) > 0 \text{ for almost all } x \in (-1, 1).
\end{align*}
\]

The equation (2.5) can be transformed into Sturm-Liouville form on multiplication by the weight \( \hat{W}_{\alpha,\beta} \) given in [3, Section 2, (11)], but here denoted by \( w_{\alpha,\beta} \) and defined by
\[
w_{\alpha,\beta}(x) := \frac{(1 - x)^{\alpha}(1 + x)^{\beta}}{(x - b)^2} \text{ for all } x \in (-1, 1).
\]

If now the equation (2.5) is multiplied by \( w_{\alpha,\beta} \) then the Sturm-Liouville form (2.6) is satisfied with
\[
p_{\alpha,\beta}(x) := \frac{(1 - x)^{\alpha+1}(1 + x)^{\beta+1}}{(x - b)^2} \text{ for all } x \in (-1, 1)
\]
and
\[
q_{\alpha,\beta}(x) := 2a \left( \frac{1 - bx}{b - x} \right)(x - c) \frac{(1 - x)^{\alpha}(1 + x)^{\beta}}{(x - b)^2} \text{ for all } x \in (-1, 1).
\]

In passing we note that, for the chosen values of \( \alpha, \beta \) in (2.2), the Sturm-Liouville equation, defining the differential expression \( M_{\alpha,\beta} \),
\[
M_{\alpha,\beta}[y](x) := -(p_{\alpha,\beta}(x)y'(x))' + q_{\alpha,\beta}(x)y(x) = \lambda w_{\alpha,\beta}(x)y(x) \text{ for all } x \in (-1, 1).
\]
is regular at all points of the open interval \((-1, 1)\).

For Case 1 of (2.4) the differential equation (2.11) is singular at both endpoints \( \pm 1 \) since
\[
\int_0^1 \frac{1}{p_{\alpha,\beta}(x)} \, dx = \int_{-1}^0 \frac{1}{p_{\alpha,\beta}(x)} \, dx = +\infty.
\]

For Case 2 of (2.3) the differential equation (2.11) is regular at both endpoints \( \pm 1 \) since
\[
\int_0^1 \frac{1}{p_{\alpha,\beta}(x)} \, dx < +\infty \text{ and } \int_{-1}^0 \frac{1}{p_{\alpha,\beta}(x)} \, dx < +\infty.
\]
The symplectic form for $M_{\alpha,\beta}$ is defined by, for all $k \in (0, \infty)$ and for all $f, g \in D(M_{\alpha,\beta})$,

$$[f, g]_{\alpha,\beta}(x) := f(x)(p_{\alpha,\beta}g')(x) - (p_{\alpha,\beta}f')(x)g(x) \text{ for all } x \in (-1, 1).$$

### 3. Differential operators

The Sturm-Liouville differential expression $M_{\alpha,\beta}$ defines differential operators in the Hilbert function space $L^2((-1, 1); w_{\alpha,\beta})$.

In this space the maximal domain of the differential expression $M_{\alpha,\beta}$ is defined by

$$M_{\alpha,\beta}$$

$$\begin{cases} 
(i) \quad T_{\alpha,\beta} : D(T_{\alpha,\beta}) \subset L^2((-1, 1); w_{\alpha,\beta}) \to L^2((-1, 1); w_{\alpha,\beta}) \\
(ii) \quad D(T_{\alpha,\beta}) := \{f \in D(M_{\alpha,\beta}) : f, w^{-1}M_{\alpha,\beta}[f] \in L^2((-1, 1); w_{\alpha,\beta}) \} \\
(iii) \quad T_{\alpha,\beta}f := w^{-1}M_{\alpha,\beta}[f] \text{ for all } f \in D(T_{\alpha,\beta}). 
\end{cases}$$

All self-adjoint differential operators in $L^2((-1, 1); w_{\alpha,\beta})$ generated by $M_{\alpha,\beta}$ are given by restrictions of the maximal operator $T_{\alpha,\beta}$; these restrictions are determined by placing boundary conditions at the endpoints $-1$ and $+1$, on the elements of $D(T_{\alpha,\beta})$. The number and type of boundary conditions depends upon the endpoint classification of $M_{\alpha,\beta}$ in $L^2((0, \infty); w_k)$; see [1, Section 5].

For the endpoint classification of the differential expression $M_{\alpha,\beta}$ in $L^2((-1, 1); w_{\alpha,\beta})$ we have the results, see again [1, Section 5];

For **Case 1** we have

$$\begin{cases} 
(i) \quad \text{Endpoint } +1 \\
(ii) \quad \text{Endpoint } -1 
\end{cases}$$

The proof of these last results follows using linearly independent solutions $\varphi_1$ and $\varphi_2$ of the equation (2.11) for $\lambda = 0$, see [3, Section 2, (6)],

$$\varphi_1(x) := x - c \text{ for all } x \in (-1, 1)$$

and

$$\varphi_2(x) := \varphi_1(x) \int_0^x \frac{1}{p_{\alpha,\beta}(t)\varphi_1^2(t)} dt = (x - c) \int_0^x \frac{(t - b)^2}{(1 - t)^{\alpha+1}(1 + t)^{\beta+1}(t - c)^2} dt \text{ for all } x \in (-1, 1).$$

Clearly then, recalling the restrictions on the parameters $\alpha, \beta, a, b, c$, asymptotic analysis shows that

$$\varphi_1 \in L^2((-1, 1); w_k) \text{ for all } \alpha, \beta$$
and

\[
\begin{align*}
\varphi_2 &\in L^2((0, 1); w_k) \quad \text{for all } 0 < \alpha < 1 \\
\varphi_2 &\notin L^2((0, 1); w_k) \quad \text{for all } \alpha \geq 1
\end{align*}
\]

(3.7)

\[
\begin{align*}
\varphi_2 &\in L^2((-1, 0); w_k) \quad \text{for all } 0 < \beta < 1 \\
\varphi_2 &\notin L^2((-1, 0); w_k) \quad \text{for all } \beta \geq 1.
\end{align*}
\]

(3.8)

These results imply the endpoint conditions stated (3.2) and (3.3).

For Case 2 we have both \(\varphi_1\) and \(\varphi_2\) belong to \(L^2((-1, 1); w_k)\) for all \(\alpha, \beta\) and this observation gives

\[
\text{limit-circle for } -1 < \beta < \alpha < 0.
\]

(3.9)

To determine the restriction \(A_{\alpha, \beta}\) of the maximal operator \(T_{\alpha, \beta}\) which yields the \(X_1\)-Jacobi orthogonal polynomials as eigenvectors we take the boundary condition function at \(\pm 1\) to be the solution \(\varphi_1\), and use the symplectic form (2.12); here \(f\) is any element of \(D(T_{\alpha, \beta})\);

Case 1

(i) Endpoint +1

\[
\begin{align*}
\text{for } 0 \leq \alpha < 1 & \quad \lim_{x \to +1^-} [f, \varphi_1](x) = 0 \\
\text{or equivalently } & \quad \lim_{x \to +1^-} (1-x)^{\alpha+1}(f(x) - f'(x)(x-c)) = 0
\end{align*}
\]

for \(\alpha \geq 1\) no boundary condition required.

(ii) Endpoint −1

\[
\begin{align*}
\text{for } 0 < \beta < 1 & \quad \lim_{x \to -1^+} [f, \varphi_1](x) = 0 \\
\text{or equivalently } & \quad \lim_{x \to -1^+} (1-x)^{\beta+1}(f(x) - f'(x)(x-c)) = 0.
\end{align*}
\]

for \(\beta \geq 1\) no boundary condition required.

Case 2

\[
\text{for } -1 < \beta < \alpha < 0 \quad \lim_{x \to +1^-} [f, \varphi_1](x) = 0.
\]

(3.12)

The domain \(D(A_{\alpha, \beta})\) of the self-adjoint restriction \(A_{\alpha, \beta}\) is then determined by applying the above boundary conditions in the appropriate cases for the parameters \(\alpha, \beta\) to give

\[
A_{\alpha, \beta}f := w^{-1}_{\alpha, \beta} M_{\alpha, \beta}[f] \quad \text{for all } f \in D(A_{\alpha, \beta}).
\]

(3.13)

The spectrum and eigenvectors of \(A_{\alpha, \beta}\) can be obtained from the results given in [3, Section 2]. The spectrum of \(A_{\alpha, \beta}\) contains the sequence \(\{\lambda_n = n(\alpha+\beta+n) : n \in \mathbb{N}_0\}\); the eigenvectors are given by \(\{\hat{P}_n^{(\alpha, \beta)} : n \in \mathbb{N}_0\}\), the \(X_1\)-Jacobi orthogonal polynomials.

Remark 3.1. (i) The notation \(\lambda_n = n(\alpha+\beta+n)\) for all \(n \in \mathbb{N}_0\) makes good comparison with the eigenvalue notation for the classical Jacobi polynomials; this sequence depends upon the parameters \(\alpha, \beta\).

(ii) We note that \(\hat{P}_n^{(\alpha, \beta)}\) is a polynomial of degree \(n+1\) for all \(n \in \mathbb{N}_0\) and all \(\alpha, \beta\) under consideration.
Note that when the limit-circle condition holds at $\pm 1$, it is essential to check that the polynomials $\{\hat{P}_n^{(\alpha,\beta)}\}$ all satisfy the boundary conditions at $\pm 1$ as required in (3.10), (3.11) and (3.12). Thus it is required that

$$\lim_{x \to 0^+} \left[ \hat{P}_n^{(\alpha,\beta)}, \varphi_1 \right] (x) = 0$$

for all $n \in \mathbb{N}_0$.

This result follows since, using (2.9),

$$\left[ \hat{P}_n^{(\alpha,\beta)}, \varphi_1 \right] (x) = p_{\alpha,\beta}(x) \left[ \hat{P}_n^{(\alpha,\beta)}(x)\varphi'_1(x) - \hat{P}_n^{(\alpha,\beta)'}(x)\varphi_1(x) \right]$$

$$= \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{(x-b)^2} \left[ \hat{P}_n^{(\alpha,\beta)}(x) - \hat{P}_n^{(\alpha,\beta)'}(x)(x+k+1) \right]$$

$$= O((1-x)^{\alpha+1}(1+x)^{\beta+1}) \text{ as } x \to +1^\pm.$$

It is shown in [3, Section 3, Proposition 3.2] that the sequence of polynomials

$$\left\{ \hat{P}_n^{(\alpha,\beta)} : n \in \mathbb{N}_0 \right\}$$

is orthogonal and dense in the space $L^2((-1,1);w_k)$, for all $\alpha, \beta$. This result implies that the spectrum of the operator $A_{\alpha,\beta}$ consists entirely of the sequence of eigenvalues $\{\lambda_n = n(\alpha + \beta + n) : n \in \mathbb{N}_0\}$; from the spectral theorem for self-adjoint operators in Hilbert space it follows that no other point on the real line $\mathbb{R}$ can belong to the spectrum of $A_{\alpha,\beta}$.

**Remark 3.2.** It is to be noted that whilst the Hilbert space theory as given in [1] and [5] provides a precise definition of the self-adjoint operator $A_{\alpha,\beta}$, the information about the particular spectral properties of $A_{\alpha,\beta}$ are to be deduced from the classical analysis results in [3]. Without these results it would be very difficult to deduce the spectral properties of the self-adjoint operator $A_{\alpha,\beta}$, as defined above, in the Hilbert function space $L^2((-1,1);w_k)$.

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