Rotational invariance and the Pauli exclusion principle.

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Abstract

In this article, the rotational invariance of entangled quantum states is investigated as a possible cause of the Pauli exclusion principle. First, it is shown that a certain class of rotationally invariant states can only occur in pairs. This will be referred to as the coupling principle. This in turn suggests a natural classification of quantum systems into those containing coupled states and those that do not. Surprisingly, it would seem that Fermi-Dirac statistics follows as a consequence of this coupling while the Bose-Einstein follows by breaking it. Finally, the experimental evidence to justify the above classification will be discussed. Pacs: 3.65.Bz, 5.30 d, 12.40.Ee

1 Introduction

Rotational invariance in quantum mechanics is usually associated with spin-singlet states. In this article, after having first established a uniqueness theorem relating rotational invariance and spin-singlet states, a statistical classification is carried out. In effect, it will be shown that, within the construct of the proposed mathematical model, rotationally invariant quantum states can only occur in pairs. These pairs will be referred to as isotropically spin-correlated states (ISC) and will be defined more precisely later on. This in turn will suggest a statistical classification procedure into systems containing paired states and those that do not. It will be shown that a system of $n$ coupled and indistinguishable states obey the Fermi-Dirac statistic, while Bose-Einstein statistics will follow when the coupling is broken.

Hopefully, the above results will help deepen our understanding of the spin-statistics theorem first enunciated by Pauli in his 1940 paper: “The Connection Between Spin and Statistics” [1]. At the core of these results is the concept of entanglement and the work of Bell, both of which would appear to suggest that in the case of spin-singlet states, microscopic causality might be violated precisely because of the nature of entanglement and non-locality. This will be discussed in more detail towards the end of the article (section 7).

Throughout the paper the following notation will be used for spin systems: \( \theta \) will represent a polar angle lying within a plane such that \( 0 \leq \theta < 2\pi \). Denote \( |\theta_j - \theta_i| \) by \( \theta_{ij} \) and write a.e. \( \theta \) for “\( \theta \) almost everywhere”.

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$|\psi_{1...n}(\lambda_1,\ldots,\lambda_n)\rangle$ will represent an n-particle state, where $1\ldots n$ represent particles and $\lambda_1\ldots\lambda_n$ represent the corresponding states. However, if there is no ambiguity oftentimes this state will be written in the more compact form $|\psi(\lambda_1,\ldots,\lambda_n)\rangle$ or more simply as $|\psi\rangle$.

$s_n(\theta)$ will represent the spin states of particle $n$ measured in direction $\theta$ where $s_n(\theta) = |\pm\rangle$. In the case of $\theta = 0$, replace $s_n(0)$ with $|+\rangle$ or $|−\rangle$ according to the context, where $+$ and $−$ represent spin up and spin down respectively. Also let $s_n^−(\theta)$ denote the spin state ORTHOGONAL to $s_n(\theta)$.

The wedge product of $n$ 1-forms is given by:

$$a_1 \wedge \ldots \wedge a_n = \frac{1}{n!} \delta _{i_1...i_n} ^{i_1...i_n} a_{i_1} \otimes \ldots \otimes a_{i_n}.$$

Specifically,

$$a_1 \wedge a_2 \wedge a_3 = \frac{1}{3!} (a_1 \otimes a_2 \otimes a_3 + a_2 \otimes a_3 \otimes a_1 + a_3 \otimes a_1 \otimes a_2 - a_2 \otimes a_1 \otimes a_3 - a_3 \otimes a_2 \otimes a_1).$$

2 A Coupling Principle

The concept of isotropically spin-correlated states (to be abbreviated as ISC) is now introduced. This definition is motivated by the probability properties of rotational invariance. Intuitively, $n$ particles are isotropically spin-correlated, if a measurement made in an ARBITRARY direction $\theta$ on ONE of the particles allows us to predict with certainty, the spin value of each of the other $n − 1$ particles for the same direction $\theta$.

**Definition 1** Let $H_1 \otimes H_2$ be a tensor product of two 2-dimensional inner product spaces. Then $|\psi\rangle \in H_1 \otimes H_2$ is said to be rotationally invariant if

$$(R_1(\theta), R_2(\theta))|\psi\rangle = |\psi\rangle,$$

where each

$$R_i(\theta) = \begin{bmatrix} \cos(c\theta) & \sin(c\theta) \\ -\sin(c\theta) & \cos(c\theta) \end{bmatrix}$$

represents a rotation on the space $H_i$ and $c$ is a constant.\[5\]

**Definition 2** Let $H_1,\ldots,H_n$ represent $n$ 2-dimensional inner product spaces. $n$ particles are said to be isotropically spin correlated (ISC) if

(1) for all $\theta$ the two state $|\psi_{ij}\rangle \in H_i \otimes H_j$ is rotationally invariant for all $i, j$ where $i \neq j$ and $1 \leq i, j \leq n$,

(2) for all $\theta$ and each $m \leq n$ the state $|\psi\rangle \in H_1 \otimes \ldots \otimes H_m$ can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}} [s_1(\theta) \otimes s_2(\theta) \ldots \otimes s_m(\theta) \pm s_1^−(\theta)s_2^−(\theta) \ldots \otimes s_m^−(\theta)]$$

(1)
Note that it follows from the definition of ISC states that rotationally invariant states of the form
\[ |\psi > = \frac{1}{2}(|+ > |+ > +| > - > | > + > | > > - > | > + >) \]  
(2)
are excluded. In other words, the existence of ISC states means that if we measure the spin state \( s_1 \) then we have simultaneously measured the spin state for \( s_2 \ldots s_n \) (see Lemma 1). It can also be shown by means of projection operators that the state defined by equation (1) is the only state that can be projected onto the state \( |\psi_{ij} > \in H_i \otimes H_j \) for each \( i, j \). This further highlights its significance.

Two examples of ISC states can be immediately given:
\[ |\psi > = \frac{1}{\sqrt{2}}(|+ > |+ > +| > - > | > >) \]  
(3)
and
\[ |\psi > = \frac{1}{\sqrt{2}}(|+ > | > - > | > >) \]  
(4)
However, what is not apparent is that these are the only ISC states permitted for a system of \( n \)-particles. This is now proven, after having first introduced some probability notation.

Recall that \( s_n(0) = |\pm > \). Associated with these states, define the random variable \( S_n(\theta) : |\pm > \rightarrow \pm 1 \) such that \( P(S_n(\theta) = 1) = P(S_n(\theta) = 1|S_n(0) = 1) = \cos^2(\theta) \) and \( P(S_n(\theta) = -1) = P(S_n(\theta) = -1|S_n(0) = 1) = \sin^2(\theta) \) where \( c \) is constant. In other words, \( P \) is the spectral measure associated with the state \( |\psi(\theta) > = \cos(\theta)|+ > + \sin(\theta)|> > \). Similarly, a joint measure \( P \) can be associated with the state
\[ |\psi(\theta_1, \theta_2) > = \cos(c\theta_1) \cos(c\theta_2)|+ > + \cos(c\theta_1) \sin(c\theta_2)|+ > - > + \sin(c\theta_1) \cos(c\theta_2)|- > + \sin(c\theta_1) \sin(c\theta_2)|- > - > \]  
such that \( P(S_1(\theta_1) = +, S_2(\theta_2) = +) = P(S_1(\theta_1) = -, S_2(\theta_2) = -) = \cos^2(\theta_1) \cos^2(\theta_2) \) etc. In particular, note that \( c\theta_1 = \frac{\pi}{4} \) and \( \theta_2 = \theta \) implies \( P(S_1 = +, S_2 = +) = \frac{1}{2} \cos^2(\theta) \).

**Lemma 1** Let \( |\psi > \) represent an ISC state, \( |\psi_1 > = 1/\sqrt{2}(|+ > +| > - >) \) be the initial state of particle 1 and \( S_n \) be a random variable as defined above. If a measure \( P \) on \( \{S_1, \ldots, S_n\} \) exists then \( P(S_n = \pm|S_1 = \pm) = 1 \) or \( P(S_n = \pm|S_1 = \mp) = 1 \).

**Proof:** From the definition of \( |\psi_1 > \), it follows that \( P(S_1 = +) = P(S_1 = -) = \frac{1}{2} \). Without loss of generality we assume \( (S_1 = +, S_2 = +, \ldots, S_n = +) \). But from the
definition of ISC states \( P(S_1 = +, S_2 = +, \ldots, S_n = +) = \frac{1}{2} \). Also
\[
P(S_1 = +, S_2 = +, \ldots, S_n = +) = P(S_1 = +)P(S_2 = +, \ldots, S_n = +|S_1 = +)
= \frac{1}{2}P(S_2 = +, \ldots, S_n = +|S_1 = +)
= \frac{1}{2}
\]
Hence, \( P(S_n = +|S_1 = +) = P(S_2 = +, \ldots, S_n = +|S_1 = +) = 1 \).

**Remark:** In terms of the physics this means that once the spin-state of particle 1 is measured along an axis then the values of each of the other particle states are also immediately known along the same axis. Secondly, associated with the ISC state as given in (3), random variables \( S_1(\theta) \) and \( S_2(\theta) \) and an associated measure \( P \) can be defined such that \( P(S_1 = +, S_2(\theta) = -) = (1/2)\sin^2(\theta) \) where \( \theta \) can be considered as the angle between \( s_1 \) and \( s_2(\theta) \).

It is now shown that it is impossible to have three or more particles in such ISC states. Specifically, we ask if there can exist \( n \) or more ISC particles which generate a spectral measure \( P \) associated with the random variables \( S_1(\theta_i), S_2(\theta_j), \ldots, S_n(\theta_k) \) and the state
\[
|\psi> = \frac{1}{\sqrt{2}}(s_1 \otimes \ldots \otimes s_n - s_1^- \otimes \ldots \otimes s_n^-)
\]
on the sample space \( H_1 \otimes H_2 \ldots \otimes H_n \)? However, since the existence of \( n \) ISC particles, presupposes the existence of \( n - 1 \) such particles, it is sufficient to prove the theorem for \( n = 3 \). As noted above, such measures clearly exist for \( n = 2 \).

In the interest of clarity, assume without loss of generality, that there are three ISC particles such that \( s_1(\theta_i) = s_2(\theta_i) = s_3(\theta_i) \), for an arbitrary direction \( \theta_i \). This means that the joint spin state for any two of them is given by equation (3). Also, we further refine notation to write, for example, \((+, +, -)\) for \( S_1(\theta_i) = +, S_2(\theta_j) = +, S_3(\theta_k) = - \). These can also be interpreted as measurements in the \( i, j, k \) direction respectively. Also, for the purpose of the argument below, let \( c = 1/2 \). However, the argument can be made to work for any constant value of \( c \), and in a particular can be applied to the spin of a photon, provided \( c = 1 \).

With notation now in place, an argument of Wigner [13] is now adopted to show that isotropically spin-correlated particles must occur in pairs. The proof is by contradiction. Specifically, consider three isotropically spin-correlated particles (see Fig. 1), as explained above. It follows from the probability 1 condition, that three spin

![Figure 1: Three isotropically spin-correlated particles.](image-url)
measurements can be performed, in principle, on the three particle system, in the directions $\theta_i$, $\theta_j$, $\theta_k$. In the above notation, this means that $(S_1(\theta_i), S_2(\theta_j), S_3(\theta_k))$ represent the observed spin values in the three different directions. Recall that $S_n = \pm$ which means that there exists only two possible values for each measurement. Hence, for three measurements there are a total of 8 possibilities in total. In particular,

$$\{(+, +, -), (+, -, +), (-, +, +), (+, +, +), (+, -, -), (+, +, -), (-, +, -), (-, +, +)\} \subset \{(+, +, +), (+, -), (-, +), (-, -, +)\}$$

which is Bell’s inequality. Taking $\theta_{ij} = \theta_{jk} = \frac{\pi}{3}$ and $\theta_{ki} = \frac{2\pi}{3}$ gives $\frac{1}{2} \geq \frac{3}{4}$ which is clearly a contradiction. In other words, three particles cannot all be in the same spin state with probability 1, or to put it another way, isotropically spin-correlated particles with respect to the measure $P$ must occur in pairs.

**Remark:** (1) If the ISC particles are such that $(s_1(\theta_i) = +, s_2(\theta_i) = -, s_3(\theta_i) = +)$ (note same $i$) then regardless of distinguishability or not, the spin measurements in the three different directions $\theta_i$, $\theta_j$, $\theta_k$ can be written as:

$$\{(+, +, -), (+, -, -), (-, +, -), (-, -, -), (+, +, +)\} \subset \{(+, +, +), (+, -), (-, +), (-, -, +)\}$$

The previous argument can now be repeated as above.

(2) Each of the previous arguments apply also to spin 1 particles, like the photons, provided full angle formulae are used instead of the half-angled formulae.

### 3 Pauli exclusion principle

The above results can be cast into the form of a theorem (already proven above) which will be referred to as the “coupling principle”.

**Theorem 1 (The Coupling Principle)** Isotropically spin-correlated particles must occur in PAIRS.

It follows from the coupling principle that multi-particle systems can be divided into two categories, those containing indistinguishable coupled particles and those containing indistinguishable decoupled particles. It now remains to show that a statistical analysis of these two categories generates the Fermi-Dirac and Bose-Einstein statistics respectively.
First note that in the case of ISC particles the two rotationally invariant states can be identified with each other by identifying a spin measurement of \( \pm \) on the second particle in the \( x \) direction with a spin measurement of \( \mp \) in the \(-x\) direction, in such a way as to maintain the rotational invariance. In other words, by replacing the second spinor with its spinor conjugate [4], the state

\[
|\psi > = \frac{1}{\sqrt{2}}(| + > | + > | - > | - >)
\]

can be written as

\[
|\psi(\pi) > = \frac{1}{\sqrt{2}}(| + > | - >_\pi > | - > _\pi > | + > _\pi >)
\]

where the \( \pi \) subscript refers to the fact that the measurements on particles 1 and 2 are made in opposite directions, while maintaining the rotational invariance [4]. The state \(|\psi(\pi) >\) shall be referred to as an improper singlet state. Furthermore, without loss of generality the above identification means it is sufficient to confine oneselfs to singlet states when discussing the properties of ISC particles. It remains to show that the requirement of rotational invariance for ISC particles generates a Fermi-Dirac type statistic. However, before doing so, it is important to emphasize that the Pauli principle has not being assumed but rather is being derived from the usual form of the principle (written in terms of the Slater determinant) by imposing orbital restrictions on the ISC states.

The essential ideas are as follows: The existence of ISC particles means that from the coupling principle (above), the wave function can be written uniquely as a singlet state on the \( H_1 \otimes H_2 \) space. It then follows by imposing the additional requirement that ISC particles occur only in the same orbital, that the usual singlet-state form of the wave function can be extended to the space \( S_1 \otimes S_2 \) where \( S_i = L^2(\mathbb{R}^3) \otimes H_i \). Indeed, it would almost seem to be a tautology stemming from the definition of rotational invariance. Nevertheless, the manner in which the notion of ISC states extend to these spaces needs to be clarified, since it permits an extension of the results to the usual \( L^2 \otimes H \) space associated with quantum mechanics and not just the more restricted spin spaces. Once this extension is made, proof by induction can be used to derive the usual form of the Pauli principle associated with the Slater determinant, for an \( n \)-particle system. Moreover, it is also worth noting that the existence of spin-singlet states in general, and not only in the same orbital, permits more general forms of the exclusion principle (see for example Lemma 3 and Theorem 3).

\(^2\)Whether or not improper singlet states actually exist in nature, remains an open question. If such states were actually discovered then they would throw new light on the “handedness” problem and their existence might possibly be linked to parity violation.
In what remains, let \( s_n = s_n(\theta) \) represent the spin of particle \( n \) in the direction \( \theta \). However, in general the \( \theta \) can be dropped when there is no ambiguity but its presence should always be understood. Also, let \( \lambda_n = (q_n, s_n) \) represent the quantum coordinates of particle \( n \), with \( s_n \) referring to the spin coordinate in the direction \( \theta \) and \( q_n \) representing all other coordinates. In practice, \( \lambda_n = (q_n, s_n) \) will represent the coordinates of the particle in the state \( \psi(\lambda_n) \) defined on the Hilbert space \( S_n = L^2(\mathcal{R}^3) \otimes H_n \), where \( H_n \) represents a two-dimensional spin space of the particle \( n \). We will mainly work with \( \lambda_n \) but sometimes, there will be need to distinguish the \( q_n \) from the \( s_n \). This distinction will also allow the ket to be written as: \( |\psi(\lambda) > = |\psi(q) > s = |\psi(q) > \otimes s \) where \( s \) represents the spinor. With these distinctions made, the notion of orbital is now defined and a sufficient condition for obtaining the usual form of the Fermi-Dirac statistics within the context of our mathematical model, is given.

**Definition 3** Two particles whose states are given by \( |\psi(q_1, s_1) > \) and \( |\psi(q_2, s_2) > \) respectively are said to be in the same \( q \)-orbital when \( q_1 = q_2 \).

The following Lemma allows us to extend the results for ISC particles defined on the space \( H_1 \otimes H_2 \) to the larger space \( S_1 \otimes S_2 \). As mentioned above, the Pauli principle is not being assumed but rather is being deduced by invoking rotational invariance of the ISC particles. Conversely, if the rotational invariance condition is relaxed then the Pauli principle need not apply and as a result many particles can be in the same orbital.

**Lemma 2** Let

\[
|\psi(\lambda_1, \lambda_2) > = c_1|\psi(\lambda_1) > \otimes|\psi_2(\lambda_2) > + c_2|\psi_1(\lambda_2) > \otimes|\psi_2(\lambda_1) > ,
\]

where \( c_1, c_2 \) are independent of \( \lambda_1, \lambda_2 \), represent an indistinguishable two particle system defined on the space \( S_1 \otimes S_2 \). If ISC states for a system of two indistinguishable and non-interacting particles occur only in the same \( q \)-orbital then the system of particles can be represented by the Fermi-Dirac statistics.

**Proof:** The general form of the non-interacting and indistinguishable two particle state is given by

\[
|\psi(\lambda_1, \lambda_2) > = c_1|\psi_1(\lambda_1) > \otimes|\psi_2(\lambda_2) > + c_2|\psi_1(\lambda_2) > \otimes|\psi_2(\lambda_1) > = c_1|\psi_1(q_1) > s_1 \otimes|\psi_2(q_2) > s_2 + c_2|\psi_1(q_2) > s_2 \otimes|\psi_2(q_1) > s_1
\]

where \( c_1, c_2 \) are constants, such that \( c_1^2 = c_2^2 = 1/2 \) for all \( \lambda_1 \) and \( \lambda_2 \). Let \( q_1 = q_2 \) then the particles are in the same \( q \)-orbital and hence rotationally invariant by the ISC condition. Therefore, \( c_1 = -c_2 \) and

\[
|\psi(\lambda_1, \lambda_2) > = \frac{1}{\sqrt{2}}[|\psi(\lambda_1) > \otimes|\psi_2(\lambda_2) > - |\psi_1(\lambda_2) > \otimes|\psi_2(\lambda_1) > ].
\]
The result follows. QED

Remark: (1) The above lemma also applies to improper singlet states, in other words to particles whose spin correlations are parallel to each other in each direction. This can be done by correlating a measurement in direction $\theta$ on one particle, with a measurement in direction $\theta + \pi$ on the other. In this case, the state vector for the parallel and anti-parallel measurements will be found to be:

$$|\psi(\lambda_2, \lambda_2) \rangle = \frac{1}{\sqrt{2}}[|\psi(\lambda_1) > \otimes |\psi(\lambda_2(\pi)) > -|\psi(\lambda_2) > \otimes |\psi(\lambda_1(\pi)) >]$$

where the $\pi$ expression in the above arguments, refer to the fact that the measurement on particle two is made in the opposite sense, to that of particle one.

(2) If the coupling condition (rotationally invariance) is removed then a Bose-Einstein statistic follows and is of the form:

$$|\psi(\lambda_1, \lambda_2) \rangle = \frac{1}{\sqrt{2}}[|\psi(\lambda_1) > \otimes |\psi(\lambda_2) > +|\psi(\lambda_2) > \otimes |\psi(\lambda_1) >].$$

This will be discussed in more detail later. See, for example, Corollary 1 following Lemma 3 and Corollary 2 following Theorem 4. We now prove the following theorem:

**Theorem 2 (The Pauli Exclusion Principle)** A sufficient condition for a system of $n$ indistinguishable and non-interacting particles defined on the space $S_1 \otimes \ldots S_n$ to exhibit Fermi-Dirac statistics is that it contain spin-coupled q-orbitals.

**Proof:** It is sufficient to work with three particles, but it should be clear that the argument can be extended by induction to an $n$-particle system. Consider a system of three indistinguishable particles, containing spin-coupled particles. Using the above notation and applying Lemma 2 to the coupled particles in the second line below, gives:

$$|\psi(\lambda_1, \lambda_2, \lambda_3) \rangle = \frac{1}{\sqrt{3}}\{|\psi(\lambda_1) > \otimes |\psi(\lambda_2, \lambda_3) > +|\psi(\lambda_2) > \otimes |\psi(\lambda_3, \lambda_1) >$$

$$+|\psi(\lambda_3) > \otimes |\psi(\lambda_1, \lambda_2) >\} = \frac{1}{\sqrt{3}}\{|\psi(\lambda_1) > \otimes |\psi(\lambda_2) > \otimes |\psi(\lambda_3) > -|\psi(\lambda_3) > \otimes |\psi(\lambda_2) >\}$$

$$+|\psi(\lambda_2) > \otimes |\psi(\lambda_3) > \otimes |\psi(\lambda_1) > -|\psi(\lambda_1) > \otimes |\psi(\lambda_3) >\}$$

$$+|\psi(\lambda_3) > \otimes |\psi(\lambda_1) > \otimes |\psi(\lambda_2) > -|\psi(\lambda_2) > \otimes |\psi(\lambda_1) >\}$$

$$= \sqrt{3}|\psi_1(\lambda_1) \rangle \wedge |\psi_2(\lambda_2) \rangle \wedge |\psi_3(\lambda_3) \rangle$$

where $\wedge$ represents the wedge product. Thus the wave function for the three indistinguishable particles obeys Fermi-Dirac statistics. The $n$-particle case follows by
As an example, consider the case of an ensemble of \(2n\) identical non-interacting particles with discrete energy levels \(E_1, E_2, \ldots\), satisfying the Fermi-Dirac statistics as above then all occurrences of such a gas would necessarily have a twofold degeneracy in each of the discrete energy levels and the lowest energy would be given by

\[
E = 2E_1 + 2E_2 + 2E_3 + \ldots + 2E_n.
\]

The above theorem applies only under certain conditions. However, both in theory and practice, the spin-singlet state do not have to be confined to the same q-orbital, as for example in the case of the 1s2s-electron configuration in He. This in turn requires a more general formulation of an exclusion principle:

**Lemma 3** Let \(|\psi(\lambda_1, \lambda_2)\rangle \in L^2(\mathbb{R}^3)(q_1, q_2) \otimes H_1 \otimes H_2\) denote the state of two indistinguishable particles where the \(\lambda_1\) and \(\lambda_2\) are as defined above. If the particles are in a spin-singlet state then

\[
|\psi(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}}[|\psi(q_1, q_2)\rangle \otimes s_1 - |\psi(q_2, q_1)\rangle \otimes s_2].
\]

**Proof:** The general form for the indistinguishable two particle state is given by

\[
|\psi(\lambda_1, \lambda_2)\rangle = c_1|\psi(q_1, q_2)\rangle \otimes s_1 + c_2|\psi(q_2, q_1)\rangle \otimes s_2.
\]

Invoking rotational invariance of the spin-singlet state gives \(c_1|\psi_1(q_1, q_2)\rangle = -c_2|\psi(q_2, q_1)\rangle\), from the linear independence of the \(s_1 \otimes s_2\) states. Normalizing the wave function gives \(|c_1| = \frac{1}{\sqrt{2}}\). The result follows. QED

**Corollary 1** Let \(|\psi(\lambda_1, \lambda_2)\rangle \in L^2(\mathbb{R}^3)(q_1, q_2) \otimes H_1 \otimes H_2\) denote the state of two indistinguishable particles where the \(\lambda_1\) and \(\lambda_2\) are as defined above, then

\[
|\psi(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}}[|\psi(q_1, q_2)\rangle \otimes s_1 + |\psi(q_2, q_1)\rangle \otimes s_2].
\]

**Proof:** The general form of the two particle state is given by

\[
|\psi(\lambda_1, \lambda_2)\rangle = c_1|\psi(q_1, q_2)\rangle \otimes s_1 + c_2|\psi(q_2, q_1)\rangle \otimes s_2.
\]

Indistinguishability implies that \(c_1^2 = c_2^2 = 1/2\). Since the particles are not necessarily in a singlet state, then it is possible to consistently choose \(c_1 = c_2 = \frac{1}{\sqrt{2}}\), in this case. The result follows. QED

Lemma 2, Lemma 3 and Corollary 1 taken together can now be used to give a complete classification of the 1s2s state of He. The \(SU(2) \otimes SU(2)\) properties
of spin give the decomposition $2 \otimes 2 = 3 \oplus 1$. Corollary 1 gives the triplet state composition of the $1s2s$ configuration. Lemma 3 gives the singlet state configuration in general. All of these states are experimentally verified [7]. Lemma 2 on the other hand, indicates that the Slater determinant permits ONLY the triplet state of the $1s2s$ configuration and the $1s^2$ singlet. It precludes the $1s2s$ singlet state. Lemma 3 defines the spin singlet state which is antisymmetric under an exchange of spin. In contrast, Lemma 2 defines the spin singlet state which is antisymmetric under complete particle exchange and not just spin alone.

To conclude this section, Lemma 2 and Theorem 2 express a Pauli type exclusion principle which follows naturally from the coupling principle, or equivalently the rotational invariance. Either two particles are ISC and obey a Fermi-Dirac type statistic or they are statistically independent of each other and obey a Bose-Einstein statistic. In the next section, Theorem 2 is generalized to cover interacting n-particle systems.

## 4 Multi-particle interacting systems

Attention is now turned to interacting particles. Once again, the coupling principle is sufficient to derive an exclusion principle in this case. The burden of the proof rests on making full use both of the notions of isotropy and indistinguishability. Isotropy, in fact, will be introduced with the Fock spaces.

Let $F[n] = H(\theta_1) \oplus \ldots \oplus H(\theta_n)$ be a Fock space, where each $H(\theta_i)$ is a two-dimensional Hilbert space. If $s[n] \in F[n]$ then $s[n]$ is said to be a spin n-state. The scalar product of two vectors $s_1[n]$ and $s_2[n]$ in $F[n]$ is defined by $(s_1[n], s_2[n]) = \sum_{k=1}^{n} s_1(\theta_k) s_2(\theta_k)$. We then say that two particles are in the same spin n-state if $s_1[n] = s_2[n]$ and in OPPOSITE spin-states if $s_1(\theta_k) = -s_2(\theta_k)$ for each $k$.

Specifically, consider $n$ interacting particles as defined on the space $L^2(\mathbb{R}^3)(q_1, \ldots, q_n) \otimes F_1[m] \otimes \ldots \otimes F_n[m]$, where $m \geq n$ and $F_k[m]$ represents the Fock spin space of the particle $k$. It should be noted that the use of the Fock space is essential and cannot be replaced by a Hilbert space of the form $H_1 \otimes \ldots \otimes H_n$, otherwise the results below would become meaningless. For example, in the case of spin measurements on three electrons in the same direction $\theta$, two of them must necessarily be in the same spin state. This means that for the above Hilbert space representation, the only possible Fermi-Dirac state for the electrons would be the vacuum state which is impossible. However, by introducing Fock spaces for the spin, not only can three electrons be in different spin states, but only two different statistical structures emerge naturally, namely, those containing coupled particles and those which do not.

For example, in the case of three particles, denoting the spin state $s_k[m]$ by $s_k$,
gives

\[ \sqrt{3!} |\psi(\lambda_1, \lambda_2, \lambda_3) > = |\psi(q_1, q_2, q_3) > s_1 \otimes s_2 \otimes s_3 + |\psi(q_2, q_1, q_3) > s_2 \otimes s_1 \otimes s_3 + |\psi(q_3, q_1, q_2) > s_3 \otimes s_2 \otimes s_1 + |\psi(q_3, q_2, q_1) > s_3 \otimes s_1 \otimes s_2 \]

Now imposing spin-singlet state coupling on \( |\psi(\lambda_1, \lambda_2, \lambda_3) > \) for each pair \( \lambda_i, \lambda_j \) yields \( |\psi(q_{i_1}, q_{i_2}, q_{i_3}) > = |\psi(q_1, q_2, q_3) > \) for every transposition \((i j)\)

\[ \sqrt{3!} |\psi(\lambda_1, \lambda_2, \lambda_3) > = |\psi(q_1, q_2, q_3) > (s_1 \otimes s_2 \otimes s_3 - s_2 \otimes s_1 \otimes s_3 + s_2 \otimes s_3 \otimes s_1 - s_3 \otimes s_2 \otimes s_1 + s_3 \otimes s_1 \otimes s_2 - s_1 \otimes s_3 \otimes s_2). \]

This can be expressed more formally with the following theorem:

**Theorem 3** In a system of \( n \) indistinguishable particles containing a spin-singlet pair, no two particles can be in the same state and the general form of the state function in \( L^2(\mathcal{R}^3)(q_1, \ldots, q_n) \otimes F_1[m] \otimes \ldots \otimes F_n[m] \), where \( m \geq n \) and \( F_k[m] \) is a Fock space, is given by:

\[ |\psi(\lambda_1, \ldots, \lambda_n) > = |\psi(q_1, \ldots, q_n) > \sqrt{n!} s_1 \wedge \ldots \wedge s_n. \]

**Proof:** Let \( \sigma_P \) denote a permutation of the particle states. The general form of an \( n \) particle indistinguishable state for the system under discussion, is given by

\[ |\psi(\lambda_1, \ldots, \lambda_n) > = \frac{1}{\sqrt{n!}} \sum \sigma_P |\psi(q_1, \ldots, q_n) > s_1 \otimes \ldots \otimes s_n. \]

But, indistinguishability and the rotational invariance of the singlet coupling gives for every transposition \((i j)\)

\[ |\psi(q_1, \ldots, q_i \ldots q_j \ldots, q_n) > = - |\psi(q_1, \ldots, q_j \ldots q_i \ldots, q_n) >. \]

This means that \( \sigma_P |\psi(q_1, \ldots, q_n) > = |\psi(q_1, \ldots, q_n) > \) for every even permutation and \( \sigma_P |\psi(q_1, \ldots, q_n) > = - |\psi(q_1, \ldots, q_n) > \) for every odd permutation. It follows immediately that for \( a.e. \) \( \theta \) that

\[ |\psi(\lambda_1, \ldots, \lambda_n) > = |\psi(q_1, \ldots, q_n) > \sqrt{n!} s_1 \wedge \ldots \wedge s_n. \]

The theorem is proven. QED

**Remark:** (1) In the general case, no claim is being made as to the properties of \( |\psi(q_1, \ldots, q_n) > \). Ideally the boundary conditions of each problem will dictate these properties. Also, there is no reason why this state vector should be symmetric or anti-symmetric under a change of coordinates nor are such conditions necessary to
formulate a Pauli-type exclusion principle. In fact, our construct has yielded a Fermi-Dirac type statistic with respect to the spin operator of the system, independently of the properties of $|\psi(q_1, \ldots, q_n)\rangle$. However, it is always possible to impose further restrictions in terms of energy density or locality requirements, to further restrict the structure. (2) The properties of spin-type systems help explain why in chemistry only two electrons share the same orbital, why all chemical bonding involves pairs of electrons and why Cooper pairs occur in the theory of superconductivity.\[9\], \[10\]

5 Bose-Einstein Statistics

In the above discussion rotational invariance has played a key role in formulating a Fermi-Dirac statistic for multi-particle ISC systems as defined by definition 1, 2 and 3. Indeed, from the perspective of this paper, it would seem to be the underlying cause of the Pauli exclusion principle. It now remains to investigate the statistics of multiparticle systems when this condition is relaxed. As noted above the rotational invariance implies that ISC particles can be written in the form of a singlet state, (either proper or improper). Moreover, the definition of indistinguishability means that there is no bias in favor of any of the components of the permutable states. For example, if

$$|\psi(\lambda_1, \lambda_2)\rangle = a|\lambda_1 > \otimes |\lambda_2 > + b|\lambda_2 > \otimes |\lambda_1 >,$$

is permutable then $a^2 = b^2$, otherwise if $|a| > |b|$ (respectively $|b| > |a|$) there would be a bias in favor of the state associated with $a$ (respectively $b$) which, together with the law of large numbers, could then be used to partially distinguish the states.

**Theorem 4** Permutable states for a system of $n$ non-interacting particles, defined on the space $S_1 \otimes \ldots \otimes S_n$, obey either the Fermi-Dirac or the Bose-Einstein statistic.

**Proof:** Let

$$\sigma|\psi(\lambda_1, \ldots, \lambda_n)\rangle = \sum \frac{1}{n!} c_i |\psi(\sigma_i \lambda_1) > \otimes \ldots \otimes |\psi(\sigma_i \lambda_n) >,$$

where $\sigma_i$ represents a permutation of the particles in the states $\lambda_1, \ldots, \lambda_n$. We now claim that if the system of indistinguishable particles are not in the Fermi-Dirac state then

$$\sigma(|\psi(\lambda_1, \ldots, \lambda_n)\rangle = \frac{1}{\sqrt{n!}} \sum \frac{1}{n!} |\psi(\sigma_1 \lambda_1) > \otimes \ldots \otimes |\psi(\sigma_n \lambda_n) >.$$

This follows by noting that if $c_i = \frac{1}{\sqrt{n!}}$ and $c_{i+1} = -\frac{1}{\sqrt{n!}}$ for each $i$ then Fermi-Dirac statistics results. Hence, assume that there is not an exact pairing and that $c_1 = \ldots c_i = \frac{1}{\sqrt{n!}}$ where either $i > \frac{n!}{2}$ or $i < \frac{n!}{2}$ and $c_{i+1} = \ldots = c_n! = -\frac{1}{\sqrt{n!}}$. Then
taking \( \lambda_1 = \lambda_2 \) many of the terms on the right hand side (in fact \( \min\{2i, 2(n! - i)\} \) terms) will cancel, leaving only the excess unpaired positive (negative) terms. If the remaining number of terms in the expansion is less than \( n! \) then \( |\psi(\lambda_1, \ldots, \lambda_n) > \) is NOT invariant under the complete set of permutations, which is a contradiction. It follows that the number of terms must be \( n! \) and nothing vanishes. Hence \( |\psi(\lambda_1, \ldots, \lambda_n) > \) exhibits Bose-Einstein statistics. The result follows.

It might be instructive to apply the above theorem to a three particle wave function that is not of the above type. Consider:

\[
|\psi(\lambda_1, \lambda_2, \lambda_3) > = \frac{1}{\sqrt{3!}} \left\{ |\psi(\lambda_1) > \otimes |\psi(\lambda_2) > \otimes |\psi(\lambda_3) > + |\psi(\lambda_3) > \otimes |\psi(\lambda_2) > \right\} \\
+ |\psi(\lambda_2) > \otimes |\psi(\lambda_3) > \otimes |\psi(\lambda_1) > + |\psi(\lambda_1) > \otimes |\psi(\lambda_3) > \\
+ |\psi(\lambda_3) > \otimes |\psi(\lambda_1) > \otimes |\psi(\lambda_2) > - |\psi(\lambda_2) > \otimes |\psi(\lambda_1) > \right\}.
\]

On putting \( \lambda_1 = \lambda_2 \),

\[
|\psi(\lambda_1, \lambda_2, \lambda_3) > = \frac{1}{\sqrt{3!}} \left\{ |\psi(\lambda_1) > \otimes |\psi(\lambda_2) > \otimes |\psi(\lambda_3) > + |\psi(\lambda_3) > \otimes |\psi(\lambda_2) > \right\} \\
+ |\psi(\lambda_2) > \otimes |\psi(\lambda_3) > \otimes |\psi(\lambda_1) > + |\psi(\lambda_1) > \otimes |\psi(\lambda_3) > \\
\]

which is not invariant under permutations.

**Corollary 2** Let

\[
|\psi(\lambda_1, \ldots, \lambda_n) > = \sum_{1}^{n!} c_i |\psi(\sigma \lambda_1) > \otimes \ldots |\psi(\sigma \lambda_n) >,
\]

where \( c_i \) are independent of \( \lambda_i \), represent an indistinguishable \( n \)-particle system defined on the space \( S_1 \otimes \ldots \otimes S_n \). This system of particles can be represented by the Bose-Einstein statistics.

**Proof:** Normalizing the wave function and using indistinguishability gives \( c_i^2 = c_j^2 \), for each \( i \) and \( j \). If \( c_i = -c_{i+1} \) then each \( q \)-orbital would be a spin-singlet state. But this is not so. Hence \( c_i = c_j \) by the previous lemma and the result follows.

Denote the set of permutations that leave invariant the Bose-Einstein and Fermi-Dirac statistics by \( s_n \) and \( a_n \), respectively. It follows that certain types of mixed statistics can be now described. For example, the 2-electrons of a helium atom, considered together with the 3-electrons in a lithium atom obey \( a_2 \otimes a_3 \) statistics while the 5 electrons in the boron atom obey \( a_5 \) Fermi-Dirac statistics. The electrons in three different helium atoms obey \( a_2 \otimes a_2 \otimes a_2 \) if the helium atoms are considered distinguishable and \( s_3 \circ (a_2 \otimes a_2 \otimes a_2) \) if the atoms are indistinguishable. Finally, if we consider collectively the \( n \) distinct electrons in \( n \) indistinguishable hydrogen atoms then these \( n \) electrons can be described with \( s_n \) Bose-Einstein statistics.
6 Clarification by Contrast

It now remains to discuss the above mathematical results from the perspective of Pauli’s famous paper on spin-statistics [11] and in the overall context of the experimental evidence (discussed next section).

First, in Pauli’s construct the principle of microscopic causality \([\Theta(x), \Theta(y)] = 0\) for \((x - y)^2 < 0\) where \(x\) and \(y\) represent Minkowski 4-vectors) underlies the distinction between bosons and fermions. In contrast, rotationally invariant states (particles) which are key to understanding the model presented, would seem to violate this principle by definition. Specifically, in Pauli’s formulation if \(\sigma_i(x), \sigma_j(y)\) represent the spin operators for particles located at \(x\) and \(y\) respectively then particles which are statistically independent of each other can have their joint spin operators represented by \(I \otimes \sigma_i(x)\) and \(\sigma_j(y) \otimes I\) respectively. This reflects the fact that the joint system represented by the Hilbert spaces \(H_1 \oplus H_2\) can be decomposed into two coherent systems \(H_1\) and \(H_2\), and the joint state for the two particles is a mixture of states from \(H_1\) and \(H_2\). Moreover, since these commute both for time-like and spacelike separations then microscopic causality is obeyed. However, in the case of two particles in a spin-singlet state no such decomposition is possible. The spin-singlet state is a pure state and obeys the principle of superposition, even at spacelike separations. Hence, it cannot be written as a mixture of two states governed by superselection rules. Moreover, a measurement made on one particle of the singlet state means that we have SIMULTANEOUSLY measured the spin of the other particle in the same direction (see Lemma 1), but it does not permit us to make a second independent measurement in a different direction, without destroying the predicted value for the second particle. In other words, for pure singlet states \([\sigma_i(x), \sigma_i(y)] = 0\) for all \(x\) and \(y\) but \([\sigma_i(x), \sigma_j(y)] \neq 0\) for all \(x\) and \(y, i \neq j\), including spacelike intervals. Therefore, it would seem that microscopic causality is violated for entangled states and the existence of such states constitute non-local events. However, it should be emphasized that non-locality does NOT mean a breakdown of cause and effect (at least in this case); it simply means that “alla Pauli” we are dealing with the mathematics of non-commuting operators and the statistics of predetermined correlations which remain correlated even when the distance between the particles is spacelike. Any other usage of the word “non-locality” is NOT intended.

Secondly, if the principle of microscopic causality were to be violated then as a consequence particles in themselves would be neither fermions nor bosons, but rather the relationship between the particles would determine whether the multiparticle system obeyed Fermi-Dirac or Bose-Einstein statistics. In our construct, indistinguishable ISC particles behave as fermions(Theorem 2), but once the same particles are disentangled they behave as bosons (Cor. 2). For this reason, as already noted previously, the consequences of coupling can be applied also to spin 1 particles like photons. In our formulation, singlet-state photons as used in the Aspect et al.
experiment, are an (albeit trivial) instance of Fermi-Dirac statistics.

The above classification is also consistent with Pauli’s own work. He is very aware that if he includes the possibility of non-local events in his schema then his demand that particles with integral spin not obey an exclusion principle would fall apart. To quote him: “For integral spin the quantization according to the exclusion principle is not possible. For this result it is essential the use of the $D_1$ function in place of the $D$ function be, for general reasons, discarded.” In fact, it is precisely the class of $D_1(x,x_0)$ solutions of the second order wave equation ([11] p 720), that underlies non-local events. In his own indirect way, Pauli is indeed affirming that non-locality permits bosons to be second quantized as fermions.

Thirdly, as pointed out above, in Pauli’s original discussion the distinction between entangled states and non-entangled states does not arise. This is understandable, since the significance of entanglement only emerged in the 1960’s with the work of Bell and others. This also has consequences for the theory of angular momentum. Specifically, Pauli assumes that particle wave functions obey the rule $U_1(j_1,k_1)U_2(j_2,k_2) = \sum_{j,k} U(j,k)$ ([11], p717) where $j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$ and $k = k_1 + k_2, k_1 + k_2 - 1, \ldots, |k_1 - k_2|$. In fact, it would appear that this rule only applies to disentangled particles. As a counter example, consider two spin-1/2 electrons in a singlet state. The above decomposition, if it were valid, would mean that $U(1/2)U(1/2) = U(1) \oplus U(0)$ but we know that for a singlet state only the $U(0)$ case arises. This can be seen more clearly, if written in terms of probability theory. Let $S_1, S_2$ be two independent and identically distributed random variables such that for $i = 1$ or $i = 2$, $P(S_i = 1/2) = P(S_i = -1/2) = 1/2$. Let $M(S_1)$ and $M(S_2)$ be the probability moment generating functions of $S_1$ and $S_2$ respectively, then it is easy to show that $M(S_1)M(S_2) = M(S_1 + S_2)$. However, once the independence condition is relaxed the above multiplication rule is no longer valid.

7 Experimental evidence

The experimental justification for accepting the new form of the spin-statistics theorem would appear to come from a wide range of physical phenomena. First, note that the existence of photons in the spin-singlet state seems to support the above formulation. Secondly, we will argue that the new approach offers a more unified and coherent explanation of the phenomenon of paramagnetism. Thirdly, the existence of Cooper pairs in superconductivity can be explained as a specific instance of ISC particles. Fourthly, we discuss baryonic structure from the new perspective. Finally, in keeping with the tradition of theoretical physics, a prediction will be made about the probability distribution for the spin decomposition of a beam of ionized deuterons, a prediction which will distinguish it from the current theory.

(1) Rotational invariance demands the wave function for spin-singlet-state pho-
tons to be of the form

\[ |\psi> = \frac{1}{\sqrt{2}}(|+> - |-> + |>) \].

By definition this is a Fermi-Dirac type statistic. Spin-singlet-state photons were at the heart of the Aspect experiment \cite{Aspect} and hence their existence has already been verified. The exclusion principle is then a tautology in the sense that while photons are in a spin-singlet state then both of them cannot be in the same state. Note, however, that the fermionic state of photons can easily be destroyed by experiment and forced into a Bose-Einstein state. It is a trite (but nevertheless valid) application of the exclusion principle.

(2) The theory of paramagnetism yields two different equations for the magnetic susceptibility, one given by the classical Langevin (Curie) function which makes no reference to the Pauli exclusion principle and the other which is derived as a direct application of the exclusion principle. It would appear that our formulation of the exclusion principle gives an equally apt understanding of the phemenan and would appear to further clarify Pauli’s explanation, by focusing on the unique role of the non-spin-singlet states. Specifically, when the magnetic field is turned on, the spin up component of the spin-singlet state has its energy shifted down by \( \mu B \) while the spin down component has its energy shifted up by \( \mu B \), with the spins being aligned into parallel and anti-parallel states, resulting in a net contribution to the magnetic field of 0. Hence, the paired electrons contribute nothing to the magnetic susceptibility. The remaining unpaired electrons act in such a way that there is an excess of electrons in the spin up state over the spin down state, in order to maintain the common electrochemical potential. Specifically, if we let \( g(\epsilon) \) be the electron density of available states per unit energy range then the total excess energy is given by \( g(\epsilon_F)\mu B \), provided \( \mu B << \epsilon_F \), which is Pauli’s result for paramagnetism. It should also be pointed out that from the perspective of Pauli’s version of the spin-statistics theorem, half-integral-spin particles such as electrons or gaseous-nitric-oxide (NO) molecules, remain as fermions regardless of thermodynamic considerations or of the state they occupy. However, it is generally taken forgranted that as \( kT >> \epsilon_f \) the Pauli principle no longer applies and the magnetic susceptibility is in this case best estimated by using the Boltzmann statistics. This gives rise to the ambivelant situation of referring to particles as fermions, although they are no longer subjected to the Pauli exclusion principle. With our approach, this ambiguity is removed and a more natural and coherent explanation of the transition from Fermi-Dirac to Boltzmann statistics is forthcoming. Essentially, the Boltzmann statistics emerges when the spin coupling which seems to be the underlying causes of the Fermi-Dirac statistics, is first broken and the particles then move apart to become distinguishable and statistically independent. This breaking of the coupling occurs naturally when the temperature is raised, and they become distinguishable and independent

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when the distance separating the particles become large enough to overcome interactions between the particles. As a result, the particles obey Boltzmann statistics and “the Curie law applies to paramagnetic atoms in a low density gas, just as to well separated ions in a solid...”.[6]

(3) The existence of Cooper pairs as spin-singlet states in the theory of superconductors is another instance of the coupling principle at work. Moreover, the fact that 2n superconducting electrons exhibit the statistics of n boson pairs and NOT the usual $a_{2n}$ Fermi-Dirac statistics ([6], p268), normally associated with the exclusion principle, again suggests that the current definition of bosons and fermions in terms of quantum number is inadequate. In contrast, this paper classifies particles into coupled or decoupled particles and then permits various statistics to emerge in accordance with the degree of indistinguishability that is imposed on the system. When complete indistinguishability is imposed on the system then Fermi-Dirac or Bose-einstein statistics will ensue according as to whether the system permits coupled (Theorem 2) or only decoupled particles (Theorem 4) respectively. On the other hand, if complete indistinguishability is relaxed in favor of some type of partial indistinguishability (as with Cooper pairs), we obtain different types of mixed statistics (see for example the last paragraph of section 5).

(4) Spin $\frac{3}{2}$ baryons may be viewed as excited states of spin $\frac{1}{2}$ baryons. In particular, spin $\frac{3}{2}$ quarks may be viewed as composed of three quarks with uncorrelated spin states (statistically independent) while the spin $\frac{1}{2}$ baryon would contain a pair of quarks in a singlet state. Moreover, the need for color to explain the structure of $\Delta^{++}$ and $\Omega^-$ particle, becomes unnecessary in the new approach. Of course, this does not preclude the use of color to give “colorless” baryons. [12]

(5) It is well known that the deuteron ion is in a spin-triplet state. Denote the possible observed spin values $X$ by $+1$, $0$, $-1$ respectively. Conventional quantum mechanics predicts that $P(X = +1) = P(X = 0) = P(X = -1) = \frac{1}{3}$. On the other hand, if we assume that the absence of the spin-singlet state for deuteron ions means that the Bose-Einstein triplet state is composed of two independently distributed spin $\frac{1}{2}$ particles then the model proposed in this paper predicts $P(X = +1) = P(X = -1) = \frac{1}{4}$ and $P(X = 0) = \frac{1}{2}$. This should be testable by passing a beam of neutral deuteron atoms (not molecules) through a Stern-Garlach apparatus.

3Strictly speaking the failure of particles to form a spin-singlet state does not necessarily mean that the subsequent spin values of the triplet state are governed by the laws of independent probability. It may mean that there is some type of dependent but non-deterministic relationship between the particles ($< 1$). This further highlights the importance of performing an experiment like that described above. If decoupled spin states imply statistical independence then classification procedures become very simple. On the other hand, if statistical independence fails to be observed then the Bose-Einstein type statistic would have to be further sub-classified.
8 Conclusion:

In this paper a “spin-coupling principle” is derived which suggests a statistical classification of particles in terms of ISC states (spin-entangled pairs) and non ISC states. These ISC states appear to unify our understanding of atomic orbitals, covalent bonding, paramagnetism, superconductivity, baryonic structure and so on. In summary, subatomic particles seem to form entangled pairs whenever they are free to do so and there appears to be a universal principle at work, although the mechanism behind this coupling would need to be further investigated.

Secondly, in contrast to the current paper, Pauli’s version of the spin-statistics theorem imposes many other conditions on his particle system including Minkowski invariance, locality (“measurements at two space points with a space-like distance can never disturb each other” ([11] p 721)), charge and energy densities. However, the imposition of such extra conditions would seem to be unnecessary in the light of our current understanding of entanglement.

Finally, note that a connection between Bell’s inequality and rotational invariance has been established.

9 Appendix

In this section, an alternate proof of Lemma 2 is presented using a probability argument. As before, let $s_n = s_n(\theta)$ represent the spin of particle $n$ in the direction $\theta$, although the $\theta$ will be dropped when there is no ambiguity. However, its presence should always be understood. Also, let $\lambda_n = (q_n, s_n)$ represent the quantum coordinates of particle $n$, with $s_n$ referring to the spin coordinate in the direction $\theta$ and $q_n$ representing occasionally, in the interest of clarity, all other coordinates. In practice, $\lambda_n = (q_n, s_n)$ will represent the eigenvalues of an operator defined on the Hilbert space $\mathcal{S} = L^2(\mathbb{R}^3) \otimes H_n$, where $H_n$ represents a two-dimensional spin space of the particle $n$. We will mainly work with $\lambda_n$ but sometimes, there will be need to distinguish the $q_n$ from the $s_n$. This distinction will also allow us to write the ket $|\psi(\lambda) > = |\psi(q) > s$ where $s$ represents the spinor.

Lemma 4 Let

$$|\psi(\lambda_1, \lambda_2) > = c_1|\psi_1(\lambda_1) > \otimes |\psi_2(\lambda_2) > + c_2|\psi_1(\lambda_2) > \otimes |\psi_2(\lambda_1) >,$$

where $c_1, c_2$ are independent of $\lambda_1, \lambda_2$, represent an indistinguishable two particle system defined on the space $\mathcal{S}_1 \otimes \mathcal{S}_2$. If ISC states for a system of two indistinguishable and non-interacting particles occur only in the same q-orbital then the system of particles can be represented by the Fermi-Dirac statistics.
Proof: $q$-orbital spin-singlet states implies that $P(\lambda_1 = \lambda_2) \leq P(s_1 = s_2) = 0$. Therefore, $\langle \psi(\lambda_1, \lambda_1) | \psi(\lambda_1, \lambda_1) \rangle = 0$ and hence $|\psi(\lambda_1, \lambda_1)\rangle = 0$, from the inner product properties of a Hilbert space. It follows, that $c_1 = -c_2$ when the particles are coupled and normalizing the wave function gives $|c_1| = \frac{1}{\sqrt{2}}$. The result follows. QED

A slightly less restrictive form of the lemma can also be proven using essentially the same probability argument, as above.

Lemma 5 Let $|\psi(\lambda_1, \lambda_2)\rangle$ denote a two particle state where the $\lambda_1$ and $\lambda_2$ are as above. If the particles are in a spin-singlet state and non-interacting then their joint state can be expressed in the form:

$$|\psi(\lambda_1, \lambda_2)\rangle = \frac{1}{\sqrt{2}}[|\psi(\lambda_1)|\psi(\lambda_2) - |\psi(\lambda_2)|\psi(\lambda_1)].$$

In other words, coupled particles obey Fermi-Dirac statistics.

Proof: The particles are in a pure state of the subspace $H_1 \otimes H_2$ for non interactive and indistinguishable particles. This implies that the general form of the two particle state is given by:

$$|\psi(\lambda_1, \lambda_2)\rangle = c_1|\psi_1(\lambda_1)\rangle \otimes |\psi_2(\lambda_2)\rangle + c_2|\psi_1(\lambda_2)\rangle \otimes |\psi_2(\lambda_1)\rangle.$$ 

Moreover, the particles are in a spin-singlet state and hence $P(q_1, s_1; q_2, s_2) \leq P(s_1 = s_2) = 0$. Therefore, $\langle \psi(q_1, s_1; q_2, s_2) | \psi(q_1, s_1; q_2, s_1) \rangle = 0$ and hence $|\psi(q_1, s_1; q_1, s_2)\rangle = 0$, from the inner product properties of a Hilbert space. It follows, that $c_1 = -c_2$ when the particles are coupled and normalizing the wave function gives $|c_1| = \frac{1}{\sqrt{2}}$. The result follows. QED

References

[1] Alain Aspect, Jean Dalibard, and George Roger, Experimental Test of Bell’s Inequalities Using Time-Varying Analyzers, Phy. Rev. Letts., Vol.49(25).

[2] J.S.Bell, On the Einstein-Podolsky-Rosen paradox, Physics 1, 195-200(1964).

[3] Bjorken and Drell, Relativistic Quantum Fields, 170-172(1965), McGraw-Hill.

[4] Elie Cartan, The Theory Of Spinors, 49(1981), Dover.

[5] Greenberger, Horne, Shimony and Zeilinger, Bells theorem without inequalities, Am. J. Physics 58(12), 1139 (1990).

[6] H.E. Hall, Solid State Physics, 142-146(1974), John Wiley.
[7] H.F. Hameka, *Quantum Theory of the Chemical Bond*, 93(1975), Hafner Press.

[8] Peter Offenhartz, *Atomic and Molecular Orbital Theory*, 118-119(1970), McGraw-Hill.

[9] Paul O’Hara, *The Einstein-Podolsky-Rosen Paradox and The Pauli Exclusion Principle*, in Fundamental Problems in Quantum Theory, 880-881, Annals NYAS, 755 (1995).

[10] Paul O’Hara, *Bell’s Inequality, Statistical Mechanics and Superconductivity*, Conference proceeding of the 5th Wigner Symposium in Vienna, World Scientific, 530-532(1998).

[11] W. Pauli, *The Connection Between Spin and Statistics*, Phys. Rev. 58, 716-722(1940).

[12] Chris Quigg, *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions*, 12(1983), Benjamin Cummins.

[13] Eugene P. Wigner, *On Hidden Variables and Quantum Mechanical Probabilities*, 1970 vol 38(8), American Journal of Physics.