Quantum Frames of Reference and the Noncommutative Values of Observables

Otto C. W. Kong

Department of Physics and Center for High Energy and High Field Physics,
National Central University,
Chung-li, Taiwan 32054

Abstract

Based on a recent relational formulation of quantum reference frame transformations, especially with a case of quantum spatial translations in particular, we analyzed how the ‘value’ of an observable for a fixed state change. That is the exact analog of the classical description, for example, of the value of the $x$-coordinate for a particle decrease by 2 units when we perform a translation of the reference frame putting the new origin at $x = 2$. The essence of the quantum reference frame transformations is to have the quantum fluctuations, and even entanglement, of the physical object which serves as the (new) reference frame, taken into account. We illustrate how the recently introduced notion of the noncommutative values of quantum observables gives such a definite description successfully. Formulations, and an analysis of a case example in qubit systems, of analog transformations for observables with a discrete or finite spectrum is also presented. Issues about the evolving picture of the symmetry system of all quantum reference frame transformations discussed.

Keywords : Quantum Frames of Reference; Noncommutative Values of Observables;

PACS numbers:
I. INTRODUCTION

The basic idea of relativity in physics is that the values of dynamic variables can only be given relative to a frame of reference. Motion is to be seen as motion with respect to a frame of reference, which may be itself in motion when observed from another frame of reference. Physical laws are to be invariant with respect to the symmetry of transformations among the choice of admissible class of frames of references. Naively, a frame of reference is an abstraction of an idealized physical system. In practice, we can only have actual, less than ideal, physical systems to be used. That does not raise any new theoretical concern so long as the latter can be considered classical. Some implications from the quantum nature of the physical system to be used as the frame of reference has been discussed back in 1967 [1]. Before the turn of the century, however, the only more notable papers addressing the topic are, apparently, Refs. [2, 3]. The last one was inspired from consideration of candidate quantized theory of gravity. The general notion of so-called relational formulation of physics has been highlighted. A simple way to put it is that there is no absolute frame of reference, which is really the relativity principle. All physical quantities and states are abstract notions the explicit description of which is reference frame dependent. With respect to quantum physics, described in the usual language, not only the expectation value of an observable is reference frame dependent, its fluctuations around the expectation value, or the whole statistics of results from projective measurements of an observable for a fixed state, would also be. The issue of a quantum frame of references is about the latter. The quantum fluctuations of a measuring equipment, for example, would give results with fluctuations even when measuring what we believe to be a classical object, or what we otherwise considered an eigenstate. We do not want to rush to the conclusion that the notion of a system being quantum or classical is relative. At least all of us humans seem to be classical enough to one another, and most of the macroscopic world looks classical to us too. Classical means, here with negligible quantum fluctuations and entanglement. It suffices to say that quantum frames of reference and transformations are worthy of serious studies.

Advance in experimental quantum physics in a laser environment [4] and otherwise [5] challenges measuring from a physical quantum frame of reference. There are also analyses on plausible applications of the subject matter, as in Refs. [6, 7] for examples. The parallel popularity of theoretical studies on the subject matter and related issues is, for example,
well illustrated by the long lists of references in Refs. [8, 9], which are the key background references for our analysis here. We will refrain from copying over those long lists. Ref. [8], following Ref. [2], focuses much on the conceptually simplest and most fundamentally interesting case of reference frames connected by a spatial translation in which at least one of the frames is not classical. Moreover, it gives a completely relational analysis with interesting and important results. Ref. [9], we think, pushes the particular analysis forwards in an important way. A key inspiration we have taken over from the reference is to have a formulation with the reference frame itself taken into account in the Hilbert space of the relevant states. That allows the presentation, in the example, of the quantum spatial translation as the action of a unitary operator within a single Hilbert space, hence exactly as a symmetry transformation, all in the language of states as in kets and bras. In the classical perspective, reference frame transformations are, mathematically, symmetry transformations as changes of coordinates of the physical space or phase spaces of physical systems. The totality of all admissible reference frame transformations that keeps a dynamical theory invariant is the relativity symmetry. The latter is considered the most fundamental symmetry, especially interpreted as one for the physical space or spacetime. The notion of quantum reference frame transformations asks for a modification or generalization of the perspective. We are not talking about a speculative new kind of quantum symmetry, but what is there in the theory and practical applications of quantum mechanics awaiting a full understanding. We still see many important issues left to be addressed properly, both technically and conceptually. Venturing into the direction is our target task here. Other interesting papers with closely related studies include Refs. [10–13].

The first notable feature of the approach to formulate a quantum spatial translation, as seeing the position observable of a particle $A$ as relative to the position observable of another particle $B$, something like $\hat{x}_A - \hat{x}_B$, is that it involves nontrivial action on the momentum observables. In fact, it is a canonical transformation that preserves the quantum Poisson bracket, effectively the commutator, among observables. After all, symmetries of a quantum system should act as unitary transformations on the Hilbert space. Such a transformation also has a fixed action on all observables, and preserves the Poisson bracket. A careful analysis of the feature show a logical connection with an intuitive perspective of seeing the theory of quantum mechanics as one of particle dynamics on a quantum model the physical space as the phase space [14] with the position and momentum observables as
a kind of noncommutative coordinates. An important related conceptual notion is the noncommutative value of a quantum observable \[15\] which, among other things, admits a rigorous way of seeing such an individual definite quantum translation as a generalization of the classical one of translating by a fixed value of distance. In the classical case, we often consider a simple translation of a coordinate by a fixed amount, like \(x' = x - a\). With or without explicitly thinking about \(a\) as the coordinate value of an object as the new frame of reference, transcribing the description of a physics phenomenon from one using \(x\) to one using the new \(x'\) as a spatial coordinate is effectively a reference frame transformation.

We want to look at the exact analog for the quantum case. When a specific state of \(B\) is known, we want to translate \(\hat{x}_A\) by an ‘amount’ \([\hat{x}_B]_\phi\), not as the variable \(\hat{x}_B\) but, an explicit ‘value’ specific to the state as the analog of the real number \(a\) of the classical case. That ‘value’, however, cannot be a single real number. The latter simply cannot encode the full quantum information about the position of \(B\) at a fixed state \(|\phi\rangle\) including quantum fluctuations and plausible entanglement which are the key interests about quantum reference frame transformations. Recall that a classical reference frame transformation for a quantum system is to be seen as an approximate description, or idealization, in which that state \(|\phi\rangle\) of the new reference frame \(B\) is essentially classical. It is important to note that the latter requires not only that fluctuations in \([\hat{x}_B]_\phi\) be negligible, but the fluctuations in \([\hat{O}]_\phi\) for any observable \(\hat{O}\) be the same. The noncommutative value introduced in Ref.\[14, 15\] as an algebraic representation of the full quantum information a state contains for an observable is here applied to look at the kind of changes in physical quantities, such as \([\hat{x}_B]_\phi\), under the transformations. Explicit illustrations of how that encodes changes in the expectation value, the quantum fluctuations, and entanglement will be presented. In short, the main results presented in this article is the illustrations, through the various examples, of how the change in each noncommutative value answers explicitly the question of how the particular physical quantity for a specific state changes under a quantum reference transformation.

Let us elaborate more on the notion of the noncommutative value as a description of the full quantum information involved. A naive thinking about the full information should, for example, encode the full statistical distribution of the corresponding projective measurements of the observable for a fixed state. That has information about the expectation value as well as the quantum fluctuations around it, including the Heisenberg uncertainty. That may give a good idea on the quantum amount of translation \([\hat{x}_B]_\phi\), only which will
be able to reveal the very interesting quantum features of the transformations studied in Refs. [8, 9]. Though the latter articles do not explicitly discuss such a quantum value translated, the description of the effects of the translations as in how specific kinds of states are transformed are presented, illustrating the ‘quantumness’ of the change in the translated position as involving changes in the quantum fluctuations and even entanglement. Looking at the individual results on the changes of the wavefunctions, the quantum value of change is implicitly there. The notion of a noncommutative value of a quantum observable we introduced recently [14, 15], is exactly a concrete mathematical way to describe the kind of quantum values. In fact, the quantum spatial translation may be one of the best place to reveal the nature of the noncommutative values as formulated from abstract mathematics. We apply a new variant of the formulation of such noncommutative values, which best suits the purpose at hand, below to look at the quantum spatial translations of Refs. [8, 9], aiming at understanding better both the quantum reference frame transformations and the noncommutative values.

In the next section, we put down formulations of the quantum spatial translations and the key relevant results following from Refs. [8, 9], to set the platform for our analysis. Note that our explicit formulation has a part, as the representation of the states in the full Hilbert space including the old and new reference frames, that is different from that of Ref. [9]. We will discuss the merit of the formulation as an improvement on the latter. Sec. III is then devoted to presenting and analyzing the changes in the noncommutative values of the observables under the quantum spatial translations for a couple of illustrative cases. Most readers are probably new to the notion of the noncommutative values. In the beginning of the section, we give an essentially self-contained presentation of a convenient new variant of it. The part together with the explicit functional representation under the use of Schrödinger wavefunction presented in the Appendix should be enough for following and understanding our analyses. The fact that the notion be conceptually new, however, means that one needs to be careful to follow the exact mathematical logic involved to avoid misinterpreting results, and may have to bear with the uncomfortable feeling of dealing with something unfamiliar. We have to beg the readers patience on that.

After that, we take a detour to look at an analog quantum reference frame transformation in a system of qubits in the section to follow. This is particularly interesting both for the theoretical and the practical consideration. We are interested in the general topic
of quantum reference frame transformations. The quantum spatial translation as one of the conceptually most fundamental and well formulated explicitly is really taken as a case example. However, as the translation is basically formulated from a picture of translations of the set of eigenstates, the fact that the position operator has a continuous spectrum distinguishes it from a transformation based on an observable with a discrete, and especially finite set of eigenvalues. For a quantum system with a Hilbert space of finite dimension, that is all we have. The question of the analogous transformations is hence of key interest. In the simple case as given by a system of qubits, the mathematics involves would be easy, the corresponding analysis of the noncommutative values and their changes may have results easier to appreciate and hence helpful especially for more skeptical readers to understand the notion. In a way, our formulation here sketches a basic approach that can gives any quantum reference frame transformation on such systems based on the notion of an observable as the ‘position’ observable of the object taken to be the new frame of reference in the case of the quantum spatial translation. Practically, most of the important experimental studies with good precision on quantum features of systems have been performed on qubit systems. The formulation may then have the theoretical implications checked experimentally.

In section V we discuss various theoretical issues of looking at quantum reference frame transformations as a kind of symmetry transformations, especially coordinate transformations. Comparison with the usual Lie group symmetry picture is discussed. The key feature of plausible entanglement of the system of interest and the old and new frames of reference as physical system is highlighted as what makes the transformations different from the Lie group symmetries as in classical reference frame transformations. However, a noncommutative canonical coordinate picture of the phase space gives the parallel with the classical case and allows a real/complex number coordinate description of the quantum transformations. The noncommutative values of the changes of observables involved illustrate well that we are dealing with a system of symmetries beyond the familiar framework. We are only at the beginning of our effort to fully understand the mathematical structures involved. Some concluding remarks will be presented in the last section. The appendix gives the basics on the notion of noncommutative values with the Schrödinger wavefunction representation of the states, used in our analysis, for the first time.
II. THE SPATIAL TRANSLATION OF QUANTUM FRAME OF REFERENCE

Ref. [8] approaches the issue completely in terms of observables, mostly the position and momentum observables. A spatial translation as a change of relative position coordinates as seen from an inertial (laboratory) frame \(A\) to the relative position coordinates as seen from another frame \(B\), with a third system \(C\) under consideration is presented as a canonical transformation first explicitly given as

\[
\begin{align*}
\hat{x}_b^{(A)} &\rightarrow -\hat{x}_A^{(B)}, \\
\hat{p}_b^{(A)} &\rightarrow -(\hat{p}_A^{(B)} + \hat{p}_C^{(B)}), \\
\hat{x}_c^{(A)} &\rightarrow \hat{x}_C^{(B)} - \hat{x}_A^{(B)}, \\
\hat{p}_c^{(A)} &\rightarrow \hat{p}_C^{(B)}.
\end{align*}
\]

(1)

Note that our notation here is mostly in-line with Ref. [9] instead. While \(A\), particularly, as a frame of reference for position and momentum observables, may have to be a system with some structure under any practical consideration, so long as the transformation considered is concerned, we only have to address its center of mass degrees of freedom, which of course behaves exactly as those for a single particle. Similarly for all, we may simply think about \(A\), \(B\), and \(C\) each as a quantum particle. The expressions for the position and momentum observables each has no components. Generalization to the case that each is a three-vector of independent components would be straightforward. The transformation as a quantum spatial translation is easy to appreciate. The part of the position observables read as classical ones would be exactly what one has in a classical theory. The part of momentum observables is what is required to make the full transformation a canonical one, i.e. to have the Poisson bracket \(\frac{i}{\hbar} \{\cdot, \cdot\}\) or all \(\hat{x}\)-\(\hat{p}\) commutators preserved. Implicitly, the thinking about quantum reference frame transformations has hidden in it an intuitive but formally not so trivial [16] picture of the position and momentum observables as (noncommutative) coordinates of the phase space for the quantum system. Quantum reference frame transformations are symmetry transformations of the latter.

A unitary operator

\[
\hat{S}_x = \hat{P}_{AB} e^{i\hat{x}_b^{(A)}\hat{p}_C^{(A)}},
\]

(2)

where \(\hat{P}_{AB}\) is a parity-swap that sends \(|x\rangle_b \otimes |y\rangle_C\) to \(|-x\rangle_b \otimes |y\rangle_C\), mapping from \(\mathcal{H}_b^{(A)} \otimes \mathcal{H}_C^{(A)}\), the Hilbert space for states of the composite system \(BC\) as described from \(A\), to \(\mathcal{H}_A^{(B)} \otimes \mathcal{H}_C^{(B)}\), the Hilbert space for states of the composite system \(AC\) as described from \(B\), is given to achieve
the above operator transformations as \( \mathcal{O} \rightarrow \hat{S}_x \mathcal{O} \hat{S}_y \). Given the fact that \([\hat{x}_b^{(A)}, \hat{p}_b^{(A)}] = \delta_{ij}, \)
\((\hbar = 1 \text{ units used throughout the paper})\), we have \(e^{i\hat{x}_b^{(A)} \hat{p}_b^{(A)}}\) naively behaves as a translation in \(\hat{x}_c^{(A)}\) by the ‘parameter’ \(\hat{x}_b^{(A)}\) and as a translation in \(\hat{p}_b^{(A)}\) by the ‘parameter’ \(-\hat{p}_c^{(A)}\) and the subsequent action of \(\hat{P}_{AB}\) finishes the job.

Note that on \(\mathcal{H}_b^{(A)} \otimes \mathcal{H}_c^{(A)}\), \(\hat{x}_b^{(A)}\) is really \(\hat{x}_b^{(A)} \otimes \hat{I}_c\) and \(\hat{p}_c^{(A)}\) is really \(\hat{I}_b \otimes \hat{p}_c^{(A)}\), for example. We have, explicitly,

\[
e^{i\hat{x}_b^{(A)} \hat{I}_c} \hat{I}_b \otimes \hat{x}_c^{(A)} e^{-i\hat{x}_b^{(A)} \hat{I}_c} = \hat{I}_b \otimes \hat{x}_c^{(A)} + \hat{x}_b^{(A)} \otimes \hat{I}_c
\]

\[
\rightarrow \hat{I}_b \otimes \hat{x}_c^{(A)} - \hat{x}_b^{(A)} \otimes \hat{I}_c ,
\]

(3)

where the arrow is the action of \(\hat{P}_{AB}\). Similarly, we have

\[
e^{i\hat{p}_b^{(A)} \hat{I}_c} \hat{p}_b^{(A)} \otimes \hat{I}_c e^{-i\hat{p}_b^{(A)} \hat{I}_c} = \hat{p}_b^{(A)} \otimes \hat{I}_c - \hat{I}_b \otimes \hat{p}_c^{(A)}
\]

\[
\rightarrow -\hat{p}_b^{(A)} \otimes \hat{I}_c - \hat{I}_b \otimes \hat{p}_c^{(A)}.
\]

(4)

What we have here in terms of the position observable of \(C\) is a change from its initial position ‘value’ of \(\hat{I}_b \otimes \hat{x}_c^{(A)}\), as an operator on \(\mathcal{H}_b^{(A)} \otimes \mathcal{H}_c^{(A)}\), before the transformation to the final position ‘value’ of \(\hat{I}_b \otimes \hat{x}_c^{(B)}\), as an operator on \(\mathcal{H}_b^{(B)} \otimes \mathcal{H}_c^{(B)}\). We want to think about the change as the difference between the final and the initial ‘values’, for which the formulation here is inadequate even in the abstract as we are dealing with operators on different Hilbert spaces. Moreover, the naive difference between them as seen in Eq.\((1)\) reads \(\hat{x}_b^{(B)} \otimes \hat{I}_c\), or \(-\hat{x}_b^{(A)} \otimes \hat{I}_c\), does not look like having much to do with the position of \(C\). These are puzzles to clarify below.

We give here an alternative formulation of the transformation in terms of position eigenstates, following but improving on the work of Ref.\([9]\). The formulation allows the transformation to be seen more directly as a symmetry transformation within the Hilbert space of \(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C\), with the initial and final frames of reference taken as the ‘states’ \(|0\rangle_A\) and \(|0\rangle_B\), i.e. the zero vectors. We introduce the use of the zero vector under the following considerations. A zero vector of course has no observable physical properties. Any operator acts on it trivially. That corresponds exactly to the idea that a frame of reference does not see itself as a dynamical object, hence cannot have a state with any nontrivial observable properties. We emphasize that it is no enough that the description of a composite state of \(BC\) as observed from \(A\) has no nontrivial content for its own position, \(\hat{x}_A^{(A)}\), when the quantum spatial translation is concerned. For the case one may think the position eigenstate
with eigenvalue zero may do. First of all, the idea of writing that state of $BC$ as observed from $A$ as a vector within $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ should be a notion independent of which particular quantum reference frame transformation one may want to formulate. At least it is more desirable to have such a consistent formulation. In the picture of $\text{Eq.}(1)$ as presented in Ref. [8], there is no $\hat{p}_A^{(A)}$ and $\hat{p}_B^{(B)}$ to consider as there is no $\hat{x}_A^{(A)}$ and $\hat{x}_B^{(B)}$. The zero position eigenstate has no trivial $\hat{x}_A^{(A)}$ with quantum fluctuations, and would lead to similar $\hat{p}_B^{(B)}$ after the quantum translation. One can see that our formulation with the zero vector is free from that and gives consistent results. Of course whatever makes up the physical frame of reference would be observed as a usual object from another frame of reference.

The spatial translation is presented as the action of the unitary operator

$$\hat{U}_x = \mathcal{S}_{AB}^W \hat{1}_A \otimes \int dx' dy' \left| -x' \right\rangle \left\langle x', y' \right| \otimes \left| y' \right\rangle \left\langle y' \right|_C ,$$

which takes a generic state

$$\left| \psi \right\rangle = \left| 0 \right\rangle_A \otimes \int dx dy \left| \psi(x,y) \right\rangle \left\langle x \right|_B \otimes \left| y \right\rangle_C ,$$

to

$$\hat{U}_x \left| \psi \right\rangle = \int dx dy \left| \psi(x,y) \right\rangle \left\langle x, y \right| \otimes \left| y - x \right\rangle_C ,$$

$$= \int \int \int dz dxdy \left| \psi(x+y, x) \right\rangle \left\langle x, y \right| \otimes \left| z \right\rangle \left\langle z \right|_C .$$

$\mathcal{S}_{AB}^W$ is a simple swap sending $\left| z \right\rangle_A \otimes \left| x \right\rangle_B \otimes \left| y \right\rangle_C$ to $\left| x \right\rangle_A \otimes \left| z \right\rangle_B \otimes \left| y \right\rangle_C$. It can further be checked explicitly that

$$\hat{U}_{x} \int \int \int \int \int \left| \psi \right\rangle \left\langle x', y' \right| \left\langle x', y' \right|_C \hat{U} = \int \int \int \int \int \left| \psi \right\rangle \left\langle x, y \right| \left\langle z \right|_C ,$$

$$\hat{U}_{x} \int \int \int \int \int \left| \psi \right\rangle \left\langle x', y' \right| \left\langle x', y' \right|_C \hat{U} = \int \int \int \int \int \left| \psi \right\rangle \left\langle x, y \right| \left\langle z \right|_C ,$$

which are exactly $\hat{U}_{x} \hat{x}_A^{(A)} \hat{U} = -\hat{x}_A^{(A)}$ and $\hat{U}_{x} \hat{x}_C^{(A)} \hat{U} = \hat{x}_C^{(A)} - \hat{x}_A^{(A)}$. Now, we can write explicitly $\hat{x}_C^{(B)} - \hat{x}_C^{(A)}$ as $\hat{x}_C - \hat{U}_{x} \hat{x}_C \hat{U} = \hat{x}_A = -\hat{U}_{x} \hat{x}_B \hat{U}$, or $\hat{x}_C^{(B)} = \hat{U}_{x} \hat{x}_C \hat{U} - \hat{U}_{x} \hat{x}_B \hat{U}$ as the classical analog of $x'_C = x_C - x_B$. Though we focus on the part of the position operators mostly, one can also check explicitly for the momentum part. Some detailed calculations involving the momentum variables in relation to their noncommutative values will actually be presented in the next section. With the noncommutative values, we can also see below how the $\hat{x}_C$
value changes by the value of $-\hat{x}_B$, instead of having only a relation between the operators as the dynamical variables.

Anyway, the formulation in terms of unitary transformation on a Hilbert space, presented with any basis, of course definitely fixed its results on any observable. And the form presented here is certainly unambiguous and easy to apply to states. Actually, $\hat{S}_x$ can be written as a unitary transformation on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ as

$$\hat{S}_x = \int dx' dy' dz' |z'|_A \langle z'_{x'} | -x'_{x}|_B \otimes |y'_{x}|_C \rangle , \quad (9)$$

and $\hat{U}_x$ in the same form differs from it as having a $|z'|_B$ in the first part, with $\hat{S}_x |\psi\rangle = \hat{U}_x |\psi\rangle$.

The forms of $\hat{S}_x$ and $\hat{U}_x$ simply present the transformation without referring to the notion of reference frames. Note however that the unitary operators cannot be applied to any state involving $|0\rangle_B$, though having a nontrivial part for $A$ is admissible and may be of interest.

Ref. [8] presents some very illustrative nice pictures of the effects of the transformation in its figure 3. We gives here explicit analytical expressions for the four cases in that figure. The results are to be used in our analysis below. They are, in terms of simplified notations, as

$$\begin{align*}
(a) : & \quad |x_o\rangle \otimes \int dy \psi(y) |y\rangle \quad \longrightarrow \quad |x_o\rangle \otimes \int dy \psi(y) |y - x_o\rangle ; \\
(a') : & \quad |x_o\rangle \otimes (c |y_1\rangle + s |y_2\rangle) \quad \longrightarrow \quad |x_o\rangle \otimes (c |y_1 - x_o\rangle + s |y_2 - x_o\rangle) ; \\
(b) : & \quad \frac{1}{\sqrt{2}}(|x_1\rangle + |x_2\rangle) \otimes \int dy \psi(y) |y\rangle \\
 & \quad \longrightarrow \quad \frac{1}{\sqrt{2}} \left( |x_1\rangle \otimes \int dy \psi(y) |y - x_1\rangle + |x_2\rangle \otimes \int dy \psi(y) |y - x_2\rangle \right) ; \\
(b') : & \quad \frac{1}{\sqrt{2}}(|x_1\rangle + e^{i\zeta} |x_2\rangle) \otimes (c |y_1\rangle + s |y_2\rangle) \\
 & \quad \longrightarrow \quad \frac{c}{\sqrt{2}} |x_1, y_1 - x_1\rangle + s \frac{c e^{i\zeta}}{\sqrt{2}} |x_1, y_2 - x_1\rangle + \frac{c e^{i\zeta}}{\sqrt{2}} |x_2, y_1 - x_2\rangle + \frac{s e^{i\zeta}}{\sqrt{2}} |x_2, y_2 - x_2\rangle ; \\
(c) : & \quad c |x_1, y_o + x_1\rangle + s |x_2, y_o + x_2\rangle \quad \longrightarrow \quad (c |x_1\rangle + s |x_2\rangle) \otimes |y_o\rangle ; \\
(d) : & \quad \int dx \psi(x) |x, y_o + x\rangle \quad \longrightarrow \quad \int dx \psi(x) |x\rangle \otimes |y_o\rangle ; \quad (10)
\end{align*}$$

where $c \equiv \cos(\theta) e^{-i\zeta}$ and $s \equiv \sin(\theta) e^{i\zeta}$, $0 \leq \theta < \pi$, $0 \leq \zeta < 2\pi$, used to write a generic linear combination of two states. Given the above presentation, the interpretation of the simplified notations should be unambiguous. Case (a) has as the initial state a product of position eigenstate for $B$ and a generic state for $C$ (together with $|0\rangle_A$). The final state maintains
being a product state as shown (involving $|\bar{x}\rangle_A$ and $|0\rangle_B$).\footnote{C, A, and B in the figure (3 of Ref.\cite{8}) correspond to A, B, and C of our notation, respectively.} We present ($a'$) as a restriction of the generic state of $C$ to a linear combination of two position eigenstates, for the purposes of matching to results for a system of qubits. It captures the key features of the generic case. The transformation is much like a classical spatial translation with the ‘classical’ frame of reference for position represented by a position eigenstate when observed from another frame. The quantum nature lying in the fact that the position eigenstate cannot be a momentum eigenstate should be noted. Case ($b'$) is a restriction of ($b$) in exactly the same sense. The cases has the transformation of a product state to one with nontrivial entanglement (between $A$ and $C$), which may be more easily appreciated from ($b'$). Initial state for case ($c$) rather generalized somewhat the one in the figure, as a not necessarily equal combinations of two perfectly correlated parts of products of position eigenstates of $B$, and $C$ with a fixed difference in eigenvalue. The translation to have $B$ as the reference frame gives the final state as a product with the part for $C$ as a simple eigenstate. The perfect correlation makes all the quantum fluctuations of $C$ unobservable from $B$. ($d$) is really just a more general form of ($c$) with the same basic feature. Note that ($b$) is much like the inverse of ($d$). For example, the initial and final state of ($b'$) with $c = 1$ and $s = 0$ can be identified essentially with the final and initial state of ($c$) with $c = 1$ and $s = e^{i\zeta'}$, respectively. We will look at the properties of the initial and final states in the transformations in ($a$) and ($d$) in the next section.

III. CHANGING THE QUANTUM/NONCOMMUTATIVE VALUES OF PHYSICAL QUANTITIES

Let us first give a representation of the noncommutative value of a quantum observable $\hat{\beta}$ on a given physical state. For the $f_\beta(z_n, \bar{z}_n)$ function being the expectation value function of Hermitian operator $\hat{\beta}$, we have

$$V_{\hat{\beta}n} = \partial_n f_\beta = -f_\beta \bar{z}_n + \sum_m \bar{z}_m (\hat{\beta})_m^n ,$$

where $(\hat{\beta})_m^n$ are the matrix element $\langle m|\hat{\beta}|n\rangle$ over an orthonormal basis $\langle m|n\rangle = \delta^n_m$, and $z^n$ the complex coordinates of a normalized state $|\phi\rangle = \sum_n z^n |n\rangle$, $n$ runs over the dimension of $A$. \footnote{C, A, and B in the figure (3 of Ref.\cite{8}) correspond to A, B, and C of our notation, respectively.}
of the Hilbert space for the system under consideration. The set of $z^n$ also serves as the homogeneous coordinates of the projective Hilbert space as a Kähler manifold. One can check that

\begin{align*}
    f_{\hat{\beta} \hat{\gamma}} &= f_{\hat{\beta}} f_{\hat{\gamma}} + \sum_{n} V_{\hat{\beta}_n} V_{\hat{\gamma}_n}, \\
    (\hat{\beta} \hat{\gamma})_n^m &= \sum_{l} (\hat{\beta})_l^m (\hat{\gamma})_n^l, \\
    V_{\hat{\beta}_n} &= -f_{\hat{\beta}} \bar{z}_n + \sum_{m} \bar{z}_m (\hat{\beta} \hat{\gamma})_n^m, \\
    \end{align*}

(12)

where $V_{\hat{\beta}_n} = \partial_n f_{\hat{\beta}}$ is just the complex conjugate of $V_{\hat{\beta}_n}$ for any (Hermitian) operator $\hat{\gamma}$. We can take the noncommutative/quantum value $[\hat{\beta}]_\phi$ as represented by the sequence and complex number values of the quantities \{\(f_{\hat{\beta}}, V_{\hat{\beta}_n}, (\hat{\beta})_n^m\}\}, evaluated on the state. The noncommutative value of an observable as the product $\hat{\beta} \hat{\gamma}$ is then the noncommutative product for two noncommutative values, i.e. $[\hat{\beta} \hat{\gamma}]_\phi = [\hat{\beta}]_\phi [\hat{\gamma}]_\phi$, with elements of the sequence as given by the equations above. The equation gives the explicit definition of the noncommutative product $\star$. For any fixed state, the map from the observable algebra to the noncommutative values, taken as a noncommutative algebra with the product as given, is obviously a homomorphism, maintaining the algebraic relation among the observables in their values. In particular, for $\hat{\beta} = \sum \lambda_m |m\rangle \langle m| \text{ at } |n\rangle$, we have

\begin{align*}
    f_{\hat{\beta}} &= \lambda_n, \\
    V_{\hat{\beta}_m} &= \bar{z}_m (\lambda_m - f), \\
    (\hat{\beta})_l^m &= \delta_l^m \lambda_m .
\end{align*}

So, an eigenstate of an observable always has all corresponding $V_{\hat{\beta}_n}$ being zero, and degenerate eigenstates for an observable have identical noncommutative values. Moreover, $\hat{\beta} = r \hat{I}$ gives $f = r$, have the noncommutative value behaving essentially as a commutative classical real number value. Note that the matrix element $(\hat{\beta})_n^m$ can be expressed in terms of $f_{\hat{\beta}}, V_{\hat{\beta}_n}$ and $\tilde{k}_{\hat{\beta}_m} \equiv \partial_m \partial_n f_{\hat{\beta}}$ [15], hence the full sequence for the noncommutative value can be obtained from a given expectation value function $f_{\hat{\beta}}$ on the projective Hilbert space without knowing $a \text{ priori}$ the explicit operator form of the $\hat{\beta}$. In fact, one can check if a function $f(z_n, \bar{z}_n)$ is indeed an $f_{\hat{\beta}}$ without knowing $\hat{\beta}$ [16, 18]. Moreover, the classical value $r$ as a constant noncommutative value has also $\tilde{k}_{\hat{\beta}_m} = 0$.

The particular representation of the noncommutative value, which is really a single quantity as an element in a noncommutative algebra, is chosen as the optimal one for an easy and more transparent illustration of the theoretical issue address in this article. The sequence
of complex numbers representing a \( [\hat{\beta}]_\phi \) has three parts. The first is the first term which is simply the expectation value. The third part is the matrix elements \( (\hat{\beta})^m_n \), which is of course independent of the state \( |\phi\rangle \). They can be seen here as being there only for the calculation of the \( \ast \kappa \)-product, mostly the part of \( V_{\beta \gamma n} \), from Eq. (12). The \( V_{\beta n} \) part is the key focus in this article. It gives important information about how much the state differs from an eigenstate, hence the quantum nature of the quantity \( [\hat{\beta}]_\phi \). For example, the Heisenberg uncertainty characterizing the spread of the eigenvalue results from projective measurements about the expectation value is given by

\[
(\Delta \beta)^2_\phi = f_{\beta^2} - f_{\beta}^2 = \sum_n |V_{\beta n}|^2.
\]

(13)

For a convenient analysis of the noncommutative value for the position operator on states as described by wavefunction, we give here for the first time the form of the noncommutative value in the formulation with a Hilbert space in uncountable dimension. The first thing is to note that the wavefunction \( \phi(x) \) is really a collection of infinite number of complex number coordinates as \( \langle x | \phi \rangle \), one for each eigenstate \( |x\rangle \) for the value of \( x, -\infty \leq x \leq \infty \). A complex function can be seen as a collection of complex numbers (functional values) one at each point of \( x \). Then, the matrix elements \( (\hat{\beta})^x_{x'} = \langle x' | \hat{\beta} | x \rangle \) are of course are to be expressed together as a two-variable function; for example, \( (\hat{x})^x_{x'} = x \delta(x' - x) \). The coordinate derivatives corresponding to \( V_{\beta n} \) may then be expressed together as a function which is the functional derivative \( \delta \phi f_{\beta} \). That is, we have \( [\hat{\beta}]_\phi = \{ f_{\beta}, \delta \phi f_{\beta}, (\hat{\beta})^x_{x'} \} \) as the noncommutative value. We present in the appendix a full checking of the consistency of the picture for the noncommutative value on the observable algebra.

A word on the practical physical meaning of the noncommutative values is in order. Meaning of the noncommutative value for an observable, or a physical quantity, in physics, like the more familiar commutative real number value, is supposed to be logically fully ingrained in its mathematical properties. It is a definite and state specific algebraic quantity that fully encodes the information about the observable as obtainable from the quantum theory. The commutative real number value does the same for the classical theory, but fails short in the quantum theory. In particular, the set of noncommutative values for all observables of any specific physical state maintains all the algebraic relations among the observables as variable, \textit{i.e.} operators. Beyond that, we have for the classical case the real number value as the value one reads off a measuring apparatus which experimentalists
frequently deal with. That is a practical aspect currently missing for the noncommutative value, but not completely impossible to be achieved. In fact, we believe physical quantities are by nature noncommutative. The notion of the commutative real number values serves only as an approximation useful in the classical setting. The real numbers themselves are abstract symbols not to be found in nature. We read off experimental results from measuring apparatus as real numbers only because we have calibrated the output with a real number scale. And of course there are experimental outputs showing directly as the plot of a function, for example. A (real) function is really a collection of infinite numbers of (real) numbers. The key part of the noncommutative value description we used here, the $V^{\hat{\beta}_n}$, is much like a function. Actually, the corresponding expression under the Schrödinger wavefunction description of the state (see the Appendix), which is applied in this section to look at the quantum spatial translation ($V^{\hat{\beta}_n}(x) = \delta_{\phi f_{\beta}}$), is exactly a complex value function. On the whole some direct experimental determination of the noncommutative value of a physical quantity only awaits the ingenuity of our experimentalists to set up workable schemes for its achievement. However, without something of the kind, and that the notion being conceptually theoretically new, most readers will likely be uncomfortable about its practical physical meaning. We can only rely on the mathematical logic presented to speak for itself.

With the background, we can move on to look at the quantum spatial translations of case (a) and (d) in the language of the noncommutative values. First thing to note is that an expectation value function $f_{\beta}(\phi)$ is of course invariant under any unitary transformation. As the terms in the sequence representing the corresponding noncommutative value are all fixed by the values of the derivatives of $f_{\beta}(\phi)$ for a physical state, the whole noncommutative value should be invariant. For the quantum spatial translation with $\hat{x}_C \rightarrow \hat{x}_C - \hat{x}_A$, the operator $\hat{x}_C$ before and after the transformation are different operators on the same Hilbert space, as position operator formulated on differently defined position eigenstate basis which gives the easily appreciable picture of the translation, as $|y\rangle \rightarrow |y - x\rangle$. The noncommutative values of the observable as position of $C$ for any physical state changes. That is exactly like the translation (of reference frame) in classical physics $x_C \rightarrow x_C - x_A$, the explicit operators describing the same position of $C$ are two different (quantum) position coordinate \textsuperscript{16} which takes different values. However, there is a further subtlety as the $\delta_{\phi f_{\beta}}(\phi)$ and $(\hat{\beta})_{x'}^x$ terms have values which depend on the choice of basis of the Hilbert space. We can only compare
two noncommutative values explicitly in the sequence representations when the latter have the \( \delta \phi f^i_x(\phi) \) and \( \hat{\beta} x' \) terms expressed in the same basis. Say, we have to compare the initial and final value of \( x_c \) through expressing both noncommutative values in either the eigenstate basis before or after the transformation. We illustrate in much details for the case of (a), even explicit showing the invariance of a noncommutative value under the unitary transformation, and some results on the momentum observables. For case (d) then, we are going to present only the key results. The two cases can be seen as typical illustrative examples of cases with or without involving entanglement, respectively. Recall that (c) is not more than a special case of (d), and the key features of case (b) correspond well to that of (d) reading in reverse, as we will discuss.

\[(a) : \; |\phi\rangle = |x_o\rangle \otimes \int dy \; \psi(y) \; |y\rangle \quad \rightarrow \quad |\phi'\rangle = \hat{S}_x |\phi\rangle = |-x_o\rangle \otimes \int dy \; \psi(y) \; |y - x_o\rangle\]

We have an initial state wavefunction \( \phi(x, y) = \delta(x - x_o) \psi(y) \). For the noncommutative value of \( \hat{x}_b \) on the initial state, we have \( [\hat{x}_b]^i_o = \{x_o, \delta \phi f^i_{\hat{x}_b}, x\delta(x'' - x)\delta(y'' - y)\} \) with

\[\delta \phi f^i_{\hat{x}_b} = -\bar{\phi}(x, y)x_o + \int dx''dy'' \; \bar{\phi}(x'', y'')x\delta(x'' - x)\delta(y'' - y) = 0.\]

This is another explicit illustration of \( \delta \phi f^j_x = 0 \) for \( \phi \) being an eigenstate of \( \hat{\beta} \). The noncommutative value of \( \hat{x}_c \) is \([\hat{x}_c]^i_o = \{y_o, \delta \phi f^i_{\hat{x}_c}, y\delta(x'' - x)\delta(y'' - y)\} \), where \( y_o \) denote the value of \( f^i_{\hat{x}_c} \) on \( \phi \) and

\[\delta \phi f^i_{\hat{x}_c} = -\bar{\phi}(x, y)y_o + \int dx''dy'' \; \bar{\phi}(x'', y'')y\delta(x'' - x)\delta(y'' - y) = (y - y_o)\delta(x - x_o)\bar{\psi}(y) .\]

After the transformation, we have the wavefunction \( \phi'(x', y') = \delta(x' + x_o) \psi(y' + x_o) \) for the final state. The new noncommutative values of \( \hat{x}_a \) and \( \hat{x}_c \) are given by \([\hat{x}_a]^f_{\phi'} = \{x'_o, \delta \phi f^f_{\hat{x}_a}, x'\delta(x''' - x')\delta(y''' - y')\} \) and \([\hat{x}_c]^f_{\phi'} = \{y'_o, \delta \phi f^f_{\hat{x}_c}, y'\delta(x''' - x')\delta(y''' - y')\} \), where we have \( x'_o = -x_o, \delta \phi f^f_{\hat{x}_a} = 0, y'_o = y_o - x_o, \) and \( \delta \phi f^f_{\hat{x}_c} = (y' - y'_o)\delta(x' + x_o)\bar{\psi}(y' + x_o) \). Hence, we have a noncommutative value of \( \hat{x}_c - \hat{x}_a \) as

\[ [\hat{x}_c - \hat{x}_a]^f_{\phi'} = [\hat{x}_c]^f_{\phi'} - [\hat{x}_a]^f_{\phi'} = \{y'_o + x_o, (y' - y_o + x_o)\delta(x' + x_o)\bar{\psi}(y' + x_o), (y' - x')\delta(x''' - x')\delta(y''' - y')\} .\]

However, the transformation of course gives \( x' = -x \) and \( y' = y - x_o \), hence, also \( x''' = -x'' \) and \( y''' = y'' - x_o \). We have then the confirmation of \([\hat{x}_a]^f_{\phi'} = \{-x_o, 0, -\delta(x'' - x)\delta(y'' - y)\} = [-\hat{x}_b]^o_\phi \) and \([\hat{x}_c - \hat{x}_a]^f_{\phi'} = \{y_o, (y - y_o)\delta(x - x_o)\bar{\psi}(y), y\delta(x'' - x)\delta(y'' - y)\} = [\hat{x}_c]^f_{\phi} \). The result of \([\hat{x}_c]^f_\phi = [\hat{x}_c]^i_o - [\hat{x}_b]^o_\phi \) is the statement that the transformation
shifts the quantum/noncommutative value of the position observable of C by the quantum/noncommutative value of that of B. In the case, B being an eigenstate, \( \delta \phi f_{\hat{BC}} \) and the uncertainty \( (\Delta \hat{x}_C)_{\phi}^2 \) are unchanged. Actually, without entanglement, the factorization of \( \phi(x, y) \) and \( \phi'(x', y') \) allows the picture of the quantum value of the position of C for any initial state of wavefunction \( \psi(y) \) changes by the fixed noncommutative value of \( [\hat{x}_B]_\delta(y-x_o) = [x_o] \), with the constant real number \( x_o \) taken as a constant 'noncommutative' value.

Let us also check up changes in the noncommutative values of the momentum observables. We have \( [\hat{p}_B]_{\phi}^i = \{0, \delta \phi f_{\hat{p}_B}^i, -i\partial_x \delta(x'' - x)(y'' - y)\} \) and \( [\hat{p}_C]_{\phi}^i = \{p_o, \delta \phi f_{\hat{p}_C}^i, -i\delta(x'' - x)\partial_y \delta(y'' - y)\} \), where \( p_o \) denote the expectation value of \( \hat{p}_C \), \( \partial_x \delta \) is the derivative of the delta function with respect to the variable, \( \delta \phi f_{\hat{p}_B}^i = i\partial_x \delta(x - x_o)\bar{\psi}(y) \), and \( \delta \phi f_{\hat{p}_C}^i = (i\partial_y - p_o)\bar{\psi}(y)\delta(x - x_o) \). We leave details of the results on the noncommutative values of the momentum observable to the appendix. Also given there is an explicit presentation of the transformation on \( \hat{p}_B \), or the \( f_{\hat{p}_B}^i \), from which various results for the noncommutative values verifying \( [\hat{x}_B]_{\phi'}^f = [\hat{x}_B]_{\phi}^i \) and \( [\hat{p}_C]_{\phi'}^f = -[\hat{p}_B]_{\phi}^i - [\hat{p}_C]_{\phi}^i \). From the result, we have explicitly \( [\hat{p}_A]_{\phi'}^f = \{-p_o, \delta \phi f_{\hat{p}_A}^i, -i\partial_x \delta(x'' - x')(y'' - y')\} \) with

\[
\delta \phi f_{\hat{p}_A}^i = -i\partial_x \delta(x - x_o)\bar{\psi}(y) - (i\partial_y - p_o)\bar{\psi}(y)\delta(x - x_o) .
\]

The quantum nature of B as the new reference frame is illustrated in the nontrivial \( \delta \phi f_{\hat{p}_B}^i \), which contributes to the resulted \( \delta \phi f_{\hat{p}_A}^i \) making the latter nontrivial even for the case of trivial \( \delta \phi f_{\hat{p}_B}^i \) as in \( \psi(y) = e^{ip_o y} \), i.e. C in a momentum eigenstate. Moreover, in the case of any nontrivial quantum features for the momenta of B and C, if we have \( \delta \phi f_{\hat{p}_B}^i = -\delta \phi f_{\hat{p}_C}^i \), their contribution to \( \delta \phi f_{\hat{p}_A}^i \) cancel one another. The kind of cancellation, however, requires a perfect correlation of the quantum fluctuations in the two momenta, which happens only with entanglement in B and C as observed in A. The interesting feature is illustrated for the position observables in case (d) which we are moving onto.

\[
(d): \quad |\phi\rangle = \int dx \psi(x) |x, y_o + x\rangle \rightarrow |\phi'\rangle = \hat{S}_x |\phi\rangle = \int dx \psi(x)|-x\rangle \otimes |y_o\rangle
\]

The initial state wavefunction is \( \phi(x, y) = \psi(x)\delta(y - x - y_o) \). We have

\[
\delta \phi f_{\hat{p}_B}^i = (x - x_o)\bar{\psi}(x)\delta(y - x - y_o) ,
\]

\[
\delta \phi f_{\hat{p}_C}^i = (y - y_o - x_o)\bar{\psi}(x)\delta(y - x - y_o) = (x - x_o)\bar{\psi}(x)\delta(y - x - y_o) ,
\]

where \( x_o \), again, denotes the value of \( f_{\hat{p}_B} \), and the value of \( f_{\hat{p}_C} \) is then \( y_o + x_o \). The nature of the results not as products of a function of x and another of y is the signature of the
nontrivial entanglement here seen in the noncommutative values of the observables. In addition, the equality of the two is the signature of their perfect correlation. The final state wavefunction is \( \phi'(x', y') = \psi(-x')\delta(y' - y_o) \), with
\[
\delta_{\phi'} f_{\hat{x}_A}^f = (x' + x_o)\psi(-x')\delta(y' - y_o), \\
\delta_{\phi'} f_{\hat{x}_C}^f = (y' - y_o)\psi(-x')\delta(y' - y_o) = 0,
\]
checking out \( [\hat{x}_i]_{\phi'}^f = [-\hat{x}_i]_{\phi}^f \) and \( [\hat{x}_C - \hat{x}_A]_{\phi'}^f = [\hat{x}_C]_{\phi}^f - [\hat{x}_A]_{\phi}^f \). The result of \( [\hat{x}_C]_{\phi'}^f = [\hat{x}_C]_{\phi}^f - [\hat{x}_A]_{\phi}^f \) has zero \( \delta_{\phi'} f_{\hat{x}_A}^f \) from the cancellation \( \delta_{\phi} f_{\hat{x}_C}^i = \delta_{\phi'} f_{\hat{x}_A}^i \). The perfect correlation between the observables leads to their difference bearing zero uncertainty, as a result of the cancellation of the uncertainties. We can also read the transformation in the reverse, taking the final product state of \( \phi(x', y') \) given in the reference frame of \( B \) as the initial, which would be then expressed as the entangled state of \( \phi(x, y) \) in the reference frame of \( A \) upon the quantum spatial translation. The difference between the \( [\hat{x}_C]_{\phi'}^f \) and \( [\hat{x}_C]_{\phi}^f \) above as \( -[\hat{x}_A]_{\phi}^f = [\hat{x}_A]_{\phi'}^f \), reads as function of \( x' \) and \( y' \), as the Hilbert space coordinates in the position eigenstate basis in the frame of \( B \), shows no entanglement as \( \phi(x', y') \) and \( \delta_{\phi} f_{\hat{x}_A}^i \) or \( \delta_{\phi'} f_{\hat{x}_A}^i \) factorize into a product of functions of \( x' \) and \( y' \). This inverse transformation picture essentially illustrates the key features of case (b), i.e. of turning a product state into one with entanglement. While the difference in \( [\hat{x}_C]_{\phi'}^f \) and \( [\hat{x}_C]_{\phi}^f \) in case (a) above has factorizable expressions in terms of \( x-y \) or \( x'-y' \), here the result has a factorizable expression only in terms of \( x'-y' \). Note that though \( [\hat{x}_C]_{\phi'}^f \) has a nonfactorizable expression in terms of \( x-y \), we cannot say that the latter show entanglement. \( x \) is about the eigenstate or eigenvalue of \( B \) which has no meaning with \( B \) as the reference frame.

**IV. QUANTUM REFERENCE FRAME TRANSFORMATIONS IN QUBIT SYSTEMS**

Here in this section, we want to formulate a transformation on qubit states along the line of the translation by \( \hat{x}_B \) above and study its results. We take as a transformation by \( \hat{\sigma}_x \) (as observed from \( A \)). Considering the analog of Eq. (17), we see a nontrivial feature from the finite dimensional nature of the Hilbert spaces for the individual parts. For each qubit, we have only two base vectors, \(|0\rangle\) and \(|1\rangle\) as eigenstates of \( \hat{\sigma}_x \) with eigenvalues plus and minus 1. The analog of \(|-x')(x')_\phi \rangle \) is clearly \(|0\rangle|1\rangle_\phi \) and \(|1\rangle|0\rangle_\phi \) flipping the sign of the eigenvalues,
hence taking $\hat{\sigma}_y$ to $-\hat{\sigma}_y$. However, one cannot put the eigenvalue shift like $|y' - x'|\langle y'|C\rangle$. If we start with the state $|00\rangle$ (of $BC$), a state for the $C$ part with 0 as the eigenvalue of $\hat{\sigma}_C$ does not exist in $\mathcal{H}_C$. A state with expectation value being 0 then suggests itself. After all, it is about finding some quantum generalization of classical transformations and the classical value actually matches better to the expectation value. For example, neither a position eigenstate nor a momentum eigenstate should be taken as representing the classical state among the quantum ones. It is the canonical coherent state $|x, p\rangle$ characterized as the symmetric minimal uncertainty state labeled by the expectation values. Under that consideration, the unitary transformation given by

$$
|0\rangle_A \otimes |00\rangle \rightarrow |0\rangle_B \otimes \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle),
$$
$$
|0\rangle_A \otimes |01\rangle \rightarrow |0\rangle_B \otimes \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle),
$$
$$
|0\rangle_A \otimes |10\rangle \rightarrow |0\rangle_B \otimes \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle),
$$
$$
|0\rangle_A \otimes |11\rangle \rightarrow |0\rangle_B \otimes \frac{1}{\sqrt{2}}(|10\rangle + |00\rangle),
$$

suggests itself. It gives the following complex coordinate transformation on the phase space (from the ones on the subspace with $|0\rangle_A$ to the ones on the subspace with $|0\rangle_B$):

$$
z'_{00} = \frac{1}{\sqrt{2}}(z_{11} - z_{20}), \quad z'_{01} = \frac{1}{\sqrt{2}}(z_{11} + z_{20}),
$$
$$
z'_{10} = \frac{1}{\sqrt{2}}(z_{00} + z_{20}), \quad z'_{11} = \frac{1}{\sqrt{2}}(z_{00} - z_{20}),
$$

as a canonical transformation. On the basic operators, it gives

$$
\hat{\sigma}_y, \hat{\sigma}_y, \hat{\sigma}_y \rightarrow \hat{\sigma}_A \hat{\sigma}_C, \hat{\sigma}_A \hat{\sigma}_C, -\hat{\sigma}_A ,
$$
$$
\hat{\sigma}_y, \hat{\sigma}_y, \hat{\sigma}_y \rightarrow -\hat{\sigma}_A \hat{\sigma}_C, -\hat{\sigma}_A \hat{\sigma}_C, -\hat{\sigma}_A \hat{\sigma}_C .
$$

The transformation of course preserves the commutation relations among the operators. With the result, the analog results of quantum reference frame transformations based on $\hat{\sigma}_y$ to $-\hat{\sigma}_y$ and $\hat{\sigma}_y$ to $-\hat{\sigma}_y$ are quite obvious.

With the unitary transformation identified, we go next to look at the results for a few illustrative cases of initial states parallel to those analyzed above for the quantum spatial translation. We can check the two lists versus one another and see how well their results
where the explicit coordinates are

\[ (a') : \quad |0\rangle \otimes (c|0\rangle + s|1\rangle) \quad \longrightarrow \quad |1\rangle \otimes \left( \frac{c + s}{\sqrt{2}} |0\rangle + \frac{c - s}{\sqrt{2}} |1\rangle \right), \]

\[ (b') : \quad \frac{1}{\sqrt{2}}([0] + e^{i\zeta'} |1\rangle) \otimes (c|0\rangle + s|1\rangle) \]

\[ \quad \longrightarrow \quad \frac{c - s}{2} (-e^{i\zeta'} |00\rangle + |11\rangle) + \frac{c + s}{2} (e^{i\zeta'} |01\rangle + |10\rangle), \]

\[ (c) : \quad c|00\rangle + s|11\rangle \quad \longrightarrow \quad (s|0\rangle + c|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \]

Let us check explicitly on the noncommutative values for case (c). The initial state has \([\hat{\sigma}_{3y}]^i\) and \([\hat{\sigma}_{3c}]^i\) as given by

\[ \beta = \hat{\sigma}_{3y} : \quad f_\beta = |c|^2 - |s|^2, \quad V_{n\bar{n}} = \{2\bar{c}|s|^2, 0, 0, -2\bar{s}|c|^2\}, \quad (\hat{\beta})_{n\bar{n}} = \{1, 1, -1, -1\}; \]

\[ \beta' = \hat{\sigma}_{3c} : \quad f_\beta' = |c|^2 - |s|^2, \quad V_{n\bar{n}}' = \{2\bar{c}|s|^2, 0, 0, -2\bar{s}|c|^2\}, \quad (\hat{\beta'})_{n\bar{n}} = \{1, -1, 1, -1\}. \]

The identical values of \(f_\beta = f_\beta'\) and \(V_{n\bar{n}} = V_{n\bar{n}}'\) in this case is from the perfect correlation between \(B\) and \(C\). The final state has

\[ \gamma = -\hat{\sigma}_{3y} : \quad f_\gamma' = |c|^2 - |s|^2, \quad V_{n'\bar{n}'} = \{-\sqrt{2}\bar{s}|c|^2, -\sqrt{2}\bar{s}|c|^2, \sqrt{2\bar{c}}|s|^2, \sqrt{2\bar{c}}|s|^2\}, \]

\[ (\hat{\gamma})_{n'\bar{n}'} = \{-1, -1, 1, 1\}; \]

\[ \gamma' = -\hat{\sigma}_{3c} : \quad f_\gamma' = |c|^2 - |s|^2, \quad V_{n'\bar{n}'} = \{-\sqrt{2}\bar{s}|c|^2, -\sqrt{2}\bar{s}|c|^2, \sqrt{2\bar{c}}|s|^2, \sqrt{2\bar{c}}|s|^2\}, \]

\[ (\hat{\gamma'})_{n'\bar{n}'} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]

One can check that

\[ V_{n'\bar{n}'} = \left\{ \frac{1}{\sqrt{2}}(V'_{10} + V'_{11}), \frac{1}{\sqrt{2}}(V'_{10} - V'_{11}), \frac{1}{\sqrt{2}}(V'_{01} - V'_{00}), \frac{1}{\sqrt{2}}(V'_{01} + V'_{00}) \right\}, \]

\[ \quad \text{where the the explicit coordinates are } z'-\text{coordinates as } z'_{n'\bar{n}'} \text{ for } n'\bar{n}' = 00, 01, 10, 11 \text{ while } n\bar{n} \text{ refers to the } z_{n\bar{n}}, n\bar{n} = 00, 01, 10, 11 \text{ as used throughout the section. The matrix elements of observables in the corresponding basis are given directly in the matrix or as a sequence of eigenvalues when the latter is diagonal. The result is exactly } V_{n\bar{n}} = \text{confirming the only not so trivial part of } [\hat{\sigma}_{3y}]^i = [-\hat{\sigma}_{3y}]^f. \]

Similarly, one can easily confirm \(V_{n\bar{n}}' = V_{n\bar{n}}'\), and \([\hat{\sigma}_{3c}]^i = [-\hat{\sigma}_{3c}]^f\). Again, those are nothing more than the confirmation of the noncommutative value as invariant under a unitary transformation. \([\hat{\sigma}_{3c}]^f = \{0, V_{\bar{n}'n'}', \{1, -1, 1, -1\}\}, \text{ with } V_{\bar{n}'n'}' = \left\{ \hat{c}, -\hat{s}, \frac{\hat{c}}{\sqrt{2}}, \frac{\hat{s}}{\sqrt{2}}, -\frac{\hat{c}}{\sqrt{2}} \right\}. \text{ The result, especially when rewritten as } V_{\bar{n}'n'}' = \{0, \hat{c}, -\hat{s}, 0\}, \text{ allows direct comparison versus } [\hat{\sigma}_{3c}]^i. \]
have, for example, the uncertainties $(\Delta \hat{\sigma}_c)^2_j = 2|s|^2|c|^2$ and $(\Delta \hat{\sigma}_{\bar{c}})^2_j = 1$. Product states like our final state for the case have their coordinates factorizable, as for example $z'_{n'\bar{n}'}$ can be written as $z'_{n'\bar{n}'}z'_{n'\bar{n}'}$ which can be traced to the parallel factorizability of the form $V'_{n'\bar{n}'} = V'_{n'\bar{n}'}$. In our case here, we have all $V_{n\bar{n}}$ above being not factorizable illustrating the entanglement of the initial state, while the final state results as are $V'_{n'\bar{n}'}$ all clearly factorizable.

V. TRANSFORMATIONS OF REFERENCE FRAMES, COORDINATES, AND SYMMETRIES

Let us first recall the theoretical features from reference frame transformations in general, and spatial translations in particular, as we understood in classical physics. The group of admissible reference frame transformations is a Lie group that is considered as the relativity symmetry for the theory. The perspective of it as a symmetry of our physical space-time is used to be considered fundamental. In fact, the relativity symmetry and the model of space-time, or spacetime, are really tied together. For example, having Lorentz symmetry requires the Minkowski spacetime as the ‘correct’ model whereas its Galilean limit is to be matched to the Newtonian space-time. Actually, the Newtonian space-time and the Minkowski spacetime can be theoretically constructed each as a coset space of the Galilean and Poincaré groups, respectively (see Ref. [19] for an explicit illustration). Physics is, however, really about dynamics; and dynamics is to be considered on states of physical systems, which does not include space and time themselves in theories not containing gravitational dynamics. In this paper, we focus only on simple quantum mechanics, or the quantum version of Newtonian mechanics. The space of all physical states for a Newtonian particle can also be constructed as a coset space of the Galilean group [19]. This particle phase space splits into the configuration/position space and the momentum space. The single particle configuration space as the space of all possible positions of the particle is what constitutes a model for the physical space. The phase space is naturally a symplectic manifold, which can be seen as a geometric space with symmetry transformations naturally to be formulated as Hamiltonian flows [19]. In fact, the Poisson bracket structure is essentially dictated by the Galilean group as a Lie group [20]. Practical dynamics corresponds to the identification of the physical Hamiltonian, the energy observable, as among all possible generic Hamiltonian functions. Every generator of the Galilean group corresponds to an observable whose
Hamiltonian vector field represents the abstract generator. It generates a one-parameter group of Hamiltonian flows on the phase space preserving the symplectic structure, as well as an automorphism flow on the observable algebra preserving the Poisson bracket \[19\].

The observable algebra itself can be seen as the representation of the universal enveloping algebra, the group algebra, or some generalization of it, fixed by the representations of the generators. Here the observable algebra represented as functions on the phase space with the usual product is commutative.

The (relativity) symmetry theoretical perspective for a theory of particle dynamics works even better in the quantum cases \[21, 22\]. For the simple, ‘nonrelativistic’, quantum mechanics, starting with an irreducible representation of the \(U(1)\) central extension of the Galilean group, formulation of all aspects of the dynamical theory is a natural consequence, which highlights its parallel with the classical theory based on the Hamiltonian dynamical picture \[16, 18\]. We have pointed out that the formulation gives an interesting, intuitive, noncommutative geometrical picture of quantum physics \[14, 16, 22, 23\], with the comprehensive commutative, hence classical, limit retrievable as a contraction of the symmetry and the representation. All that is only within the familiar symmetry, including reference frame, transformation picture based on Lie groups. Not considering a particle with nonzero spin, the irreducible representation of the quantum relativity symmetry is essentially unique.

The representation space is just the familiar Hilbert space, which is usually called the phase space for the quantum particle. The projective Hilbert space as the space of rays in it is the exact symplectic manifold each point of which corresponds to a distinct physical state. While one can have a classical spatial translation, it still has to be formulated as one of the phase space. With the \(U(1)\) central extension effectively incorporating the Heisenberg commutation structure, the extended Galilean group simply does not admit an invariant configuration/position space as a representation. Just like the Minkowski spacetime cannot be split into Lorentz invariant space and time parts, the quantum phase space cannot be split into the configuration and the momentum space parts, though both splitting work under the Newtonian approximation. That is the reason for the nonexistence of the notion of a quantum configuration space. With the perspective that the ‘physical space’ of a theory of particle dynamics, or the proper model of it, can only be a physically meaning notion as the space of all possible position for a free particle, one should conclude that the only proper model of the physical space as behind quantum mechanics is the phase space \[23\].
A quantum reference frame is then always a reference frame for the latter, and hence a
symmetry transformation as coordinate transformation on the phase space. The latter as a
dual structure to the observable algebra admits a noncommutative geometric picture with
the position and momentum observables as a basic set of noncommutative canonical coordi-
nates for the phase space — exactly as one would intuitively expect. That is the
setting to appreciate why a quantum reference frame transformations as one we may want
to think of as simply such a transformation on the physical space like a simple spatial trans-
lation has to be a canonical transformation and there is no way to look at it independent
of the phase space. There is no model of physical space we can think of sensibly with only
the position observables as coordinates. The one with the position and momentum observ-
ables as coordinates is exactly the right model of the physical space as behind the theory
of quantum mechanics, and that intuitive quantum phase space agrees the space of physical
states as usually described as the (projective) Hilbert space of infinite real/complex-number
dimension. Each noncommutative coordinate carries the information of infinite number of
real/complex-number coordinates, an explicit relation between which is offer by a represent-
ation of each noncommutative value of noncommutative coordinate as a sequence of infinite
number of complex numbers, like the one used above.

We summarize the usual Lie group theoretical symmetry picture for quantum and clas-
sical dynamics in Table 1, with the quantum case illustrated in matching correspondence
of the noncommutative coordinate and (commutative) complex number coordinate results.
Each one-parameter group of unitary transformation generated by a Hermitian operator on
the Hilbert space is a Hamiltonian flow and the preserved Poisson bracket among functions
on the latter is the exact correspondence of the operator Poisson bracket given by \( \frac{1}{\hbar} \) times
the commutator. Each observable \( \hat{\beta} \) corresponds to the expectation value function \( f(\hat{\beta}) \) collection of which, with the noncommutative product \( \star \) is an exact copy of the observable algebra
itself. However, a generic Hamiltonian function of \( f(z_n, \bar{z}_n) \) or \( f(\psi) \) can be matched to an
observable as an operator only if the flow generated preserves the full Kähler structure. Natural sets of complex coordinates are obtained as the set of expansion coefficients in terms
of sets of orthonormal basis for the Hilbert space. Such a set of complex coordinates are
canonical in the sense that the real and imaginary parts give exactly pairs of canonical real
coordinates. Some of the notation and results may be easier to appreciate through checking
Appendix. The Lie group symmetries can be seen, as illustrated, in the noncommutative
geometric point of view with the position and momentum observables taken as phase space coordinates. The phase space and the observable algebra as functions on it can be obtained from an irreducible representation of the relativity symmetry in the quantum case. A general symmetry generator, however, can be a generic Hamiltonian function as an observable, outside the Lie algebra of the relativity symmetry. The noncommutative geometric point of view provides also the natural setting to look at the quantum reference frame transformations we are interested in here. An important point to note though is that, unlike the Lie group symmetries, the quantum reference frame transformations for a system cannot be considered on its phase space alone. As we have seen above, the plausibility of nontrivial entanglement between the system and the old or new frame of reference as another physical system demands a more complicated formulation. It is still interesting to compare the new

| generators $G$ | Quantum | Classical |
|---------------|---------|-----------|
| $U_s = e^{\frac{i s}{\hbar} G_s}$ | $[X_i, P_i] = i\hbar$ | $[X_i, P_i] = 0$ |
| coordinates | $\hat{x}_i, \hat{p}_i$ | $x_i, p_i$ |
| observables | $\hat{\beta} = \beta(\hat{x}_i, \hat{p}_i)$ | $f(x_i, p_i)$ |
| $G_{s' = -\hat{p}_i}$, $G_{p_i'} = \hat{x}_i$ | $f(\psi(x_i))$ |
| $U_i^\dagger \hat{x}_i U_i' = \hat{x}_i + x'_i$ | $d(\psi) = \frac{d}{d\psi} G_s |\psi\rangle$ |
| $U_i^\dagger \hat{p}_i U_i' = \hat{p}_i + p'_i$ | $U_{s'}(x_i, p_i) = (x_i + x'_i, p_i)$ |
| $-\frac{1}{\hbar} \tilde{G}_s$ | $U_{p_i'}(x_i, p_i) = (x_i, p_i + p'_i)$ |
| $X_i = \frac{1}{\hbar} [\cdot, \hat{\beta}'] \equiv \frac{1}{\hbar} [\cdot, G_s]$ | $X_f = \{\cdot, f\}$ |
| $X_{\hat{x}_i} = \partial_{\hat{x}_i}$, $X_{\hat{p}_i} = \partial_{\hat{p}_i}$ | $X_{\hat{x}_i} = \partial_{\hat{x}_i}$, $X_{\hat{p}_i} = \partial_{\hat{p}_i}$ |
| $d\hat{\beta}' = -\frac{1}{\hbar} \tilde{G}_s (\beta') ds$ | $df' = -\frac{1}{\hbar} \tilde{G}_s (\beta') ds$ |

TABLE I: Lie group theoretical picture of symmetries in quantum and classical dynamics [16, 19, 22] The symmetry action of each infinitesimal generator and the one-parameter group on the phase space and observable algebra, with observables as functions on the phase space coordinate variables, is given.

We show only, for explicit examples of abstract generators, $X_i$ and $P_i$. Other observables as symmetry generators follow the basic pattern. The physical dynamics in particular is given the Hamiltonian as the energy observable and the corresponding real parameter $s$ is the Newtonian time. Classical results are exact symmetry contraction (naively, $\hbar \to 0$) limits of the quantum results, directly for the observable algebra.
kind of symmetry transformations with the old ones.

From the above, we see that reference frame transformations in quantum mechanics are symmetries of the phase space. For the Lie group theoretical ones as presented in Table 1. We have, including the classical spatial or momentum translations, being given in terms of one-parameter groups of Hamiltonian flows which correspond to automorphism flows on the observable algebra preserving the Poisson bracket. A ‘classical’ spatial translation as given by the unitary operator $e^{ia\hat{p}}$ for a fixed real number $a$ is indeed, like its correspondence in classical mechanics, a canonical transformation. However, it should be emphasized that the real variable $x$ is not a spatial coordinate for quantum mechanics. A spatial coordinate should be an observable and a coordinate to the phase space as a representation of the relativity symmetry. The true spatial coordinate is exact the quantum counterpart of the classical $x$, i.e. $\hat{x}$. The ‘classical’ spatial translation takes $\hat{x}$ to $\hat{x} - a$, with the noncommutative value for each state having only the first term, as the expectation value, in the sequence representation shifted by $a$, hence the quantum fluctuations are unchanged. The action on the wavefunction $\psi(x)$ is the action on the latter as a set of complex number coordinates of the phase space.

The quantum spatial translation considered above shifts the noncommutative value $[\hat{x}_c]_\phi$ for each ‘state of particle $C$’ by a noncommutative value which is taken as the $[\hat{x}_B]_\phi$ with $B$ as the new reference frame. However, the state involved is really a state of at least the composite system of $B$ and $C$. With entanglement, not only that we cannot talk about a state of $C$ in itself, we cannot even talk about a generic state of the system with a fixed $[\hat{x}_B]_\phi$. The transformation hence can only be given as a single unitary operator without any parameter dependence or a notion of generator. Combining such transformations is not a problem. It is straightforward to check that two successive quantum spatial reference frame transformations $A \rightarrow B \rightarrow D$ gives exactly the transformation $A \rightarrow D$ for example, though one has to enlarge the composite system to consider more and more parts. We have seen also that in the special case with no entanglement in both the initial and final states involved, such a description is admissible. But it looks like the only such case has $B$ in a position eigenstate. The case taken with the eigenvalue as a real variable parameter matches exactly to the ‘classical’ spatial translation when effect of the transformation on the new nontrivial state description of the old reference frame as a system is neglected.

Going to the limit when the quantum reference frame considered becomes classical has been well analyzed, for example in Ref. 24. The latter article analyzes the (reference frame
transformation) symmetry picture in quantum physics rigorously though only in terms of applying the symmetry to the physical system, instead of formulating description of the system and the theory about it from the language of representation theory as mentioned above. It tackles in detail how practical considerations of the symmetry transformations as applied between physical frames of reference, with physical quantities defined accordingly as relative to the frames, may realize the mathematical idealization in which no reference frame is explicitly considered. The relevant symmetry is naturally taken essentially as one of a Lie group. Of course the results are qualitatively as one would naively expect. When the reference frame can be well approximated as a classical one, the idealized mathematical symmetry picture of the Lie group, as described above, is well realized. The symmetry transformation picture analyzed there is not one of our changes in the noncommutative values but the more conventional one of changes in the statistical distributions of the projective measurement results. The noncommutative values for the observables obviously reduce to the classical real number values when the physical state descriptions of the reference frames involved can be taken as essentially classical. Moreover, it is interesting to note that the key mathematical objects involved for the physical properties of an observable, as on a state, in the analysis of Ref. [24] is really the expectation value function, from which the corresponding statistical distribution result is to be retrieved. Since the noncommutative value is really like a local representation of the expectation value function, our noncommutative values of changes picture is rather fully compatible with the analysis.

It should be emphasized that the canonical transformations as described in terms of the position and momentum operators [2, 8, 9] are also special cases of real/complex number canonical transformations of the (projective) Hilbert space as a symplectic/Kähler manifold [16]. They are special in the sense that they are taking the noncommutative canonical coordinates to noncommutative canonical coordinates with truly nontrivial changes in the noncommutative values. Of course here we are talking about the phase space of the composite system. Note that a noncommutative value is really an element of the noncommutative algebra of such values, and as such is a fixed abstract quantity as kind of a noncommutative analog of a real number. However, any particular representation of it as a sequence of complex numbers, we can see, is dependent on the choice of basis of the Hilbert space. Especially in relation to the quantum reference frame transformations discussed here, we have issues about choice of coordinates at different levels. The transformations can be seen as transfor-
motions of the set of noncommutative coordinates, hence changes of their noncommutative values. The set of noncommutative coordinates is also a set of basic observables to which any observable in the observable algebra is like a function of. The transformations hence change the representation of all observables, and their noncommutative values. The basis of the Hilbert space defined using a set of eigenstates then changes along with it. To compare two noncommutative values, we have to use sequence representations based on the exact same basis, as done above.

VI. CONCLUDING REMARKS

We have presented above a picture of quantum reference frame transformation, mostly illustrated through the example of a quantum spatial translation, focusing on its effects on the particular properties of individual states transformed, under our improved formulation of the transformation. When the transformation takes a description of the position observable \( \hat{x}_C \) of C from one given under reference frame A, as \( \hat{x}_C^{(A)} \), to one given under reference frame B, as \( \hat{x}_C^{(B)} - \hat{x}_A^{(B)} \), we give an explicit way to describe that relative position and the relative changes in the ‘value’ of \( \hat{x}_C \) as the exact parallel of the familiar solid cases in classical physics.

In the classical case, in the transformation \( x_C^{(A)} \rightarrow x_C^{(B)} - x_A^{(B)} \), the value of \( x_C \) changes by an amount given by \( x_C^{(B)} - x_C^{(A)} = x_A^{(B)} = -x_B^{(A)} \), which is not just the mathematical relation between those observables as variables. When \( x_A^{(B)} \) is known to have a value, say 2, we have \( x_C^{(B)} = x_C^{(A)} + 2 \) which gives explicit results like \( x_C^{(B)} = 5 \) for \( x_C^{(A)} = 3 \). To answer that question of how the particular physical quantity for a specific state changes under the quantum reference transformation, we illustrate that the noncommutative values of (quantum) observables serves the purpose well. In the case of (d) for example, analyzed in section III, the initial value of \( [\hat{x}_C]_{i\phi} \) (for \( \hat{x}_C^{(A)} \)) changes to the final value of \( [\hat{x}_C]^f_{\phi'} \) (for \( \hat{x}_C^{(B)} \)) with the change (as \( [-\hat{x}_B]_{i\phi} \)) in the case can be seen given by the real number expression \( y_o - (y_o + x_o) = -x_o \) and the functional expression \( 0 - (x - x_o)\bar{\psi}(x)\delta(y - x - y_o) = -(x - x_o)\bar{\psi}(x)\delta(y - x - y_o) \). The first one is the relation between the expectation values which changed by an amount \(-x_o\). The functional expression has an initial nontrivial \( V_{\hat{x}_C}(x, y) \), as \( \delta_\phi f_{i\phi}^f \), completely removed in the corresponding final expression, as \( \delta_\phi f_{i\phi}^f \), which is exactly the zero function due to the cancellation of the identical \( V_{\hat{x}_B}(x, y) \) part. One sees no fluctuation in (any classical description of) the position of C, nor any entanglement feature. The initial state of the case
is entangled with an exact correlation between $\hat{x}_C$ and $\hat{x}_B$ but far from any position eigenstate. Under the, not quite correct, classical geometric language, $B$ and $C$ are always separated by the Newtonian distance $y_o$. Observing $\hat{x}_C$ using $B$ as the reference frame hence, intuitively and otherwise, gives $y_o$ as the complete answer. The initial state description (as observed from $A$) is transformed into the final state described as the $\hat{x}_C$ eigenstate. Entanglement between $C$ and either the initial frame of reference $A$ or the final frame of reference $B$ is more the rule than the exception, a noncommutative value of $\hat{x}_B^{(A)}$, for example, may not be fixed without knowing the composite state $BC$ as described from $A$. But with the latter information, we do have a definite noncommutative value for $\hat{x}_B^{(A)}$. In the case there is no entanglement between $B$ and $C$, a definite noncommutative value for $\hat{x}_B^{(A)}$, or any observables of $B$, can be given independent of the state of $C$ as well as in terms of the product state of $BC$.

The information about quantum fluctuations and entanglement concerning any particular physical quantity is fully encoded in the mathematical description of the state. The information is quantum in nature and cannot be fully represented or modeled by a single real number value as in classical physics. The noncommutative value, as an element of a noncommutative algebra, however, encodes that full information. The explicit results on the transformations illustrated that clearly. At least from the theoretical point of view, the noncommutative value of an observable for a fixed state described under a choice of frame of reference and system of coordinates of the phase space, is completely and definitely determined. When a number of observables satisfying a certain algebraic relations as dynamical variables, that exact relation is preserved among their noncommutative values for any particular fixed state, like the (real number) values of the classical observables. Under a quantum reference frame transformation, the state description is changed. The mathematical description of any particular physical quantity of the state has hence to be changed. Unlike a classical reference frame transformation which only changes the expectation value of the observable, the quantum transformation may change every aspect of a state. Such a change as given by the corresponding noncommutative value again encodes the full information involved, from which one can also read off the changes in the quantum fluctuations and entanglement. Without the use of the notion of the noncommutative values, changes in the description of any particular physical properties under a quantum reference frame transformation are only given as state independent observable relations \cite{8, 9}. Tracing how
some states change under the transformation, as for example presented in figure 3 of Ref. [8].

We have discussed how a quantum reference frame transformation is to be seen as a symmetry transformation or coordinate transformation on the quantum phase space. From the perspective of the more familiar Lie group theoretical formulation of relativity symmetry, the Hilbert space picture of the quantum phase space is essentially the only irreducible representation for a spin zero particle. The notion of the configuration space for a single particle system as the space of all possible positions of the particle, hence the physical space, does not exist more than as a part of the phase space. The phase space has to be taken as the geometric model for the physical space. The position and momentum observables $\hat{x}_i$ and $\hat{p}_i$ can be seen, as intuitively expected, to be a kind of coordinates for the phase space. A quantum reference frame transformation is a transformation of such system of noncommutative coordinates hence a symmetry of the phase space. Exactly as in the classical case, they are canonical transformations preserving the corresponding Poisson bracket, though here have to be formulated on the phase space of the composite system with the reference frame(s). The phase space symmetries are unitary transformations of the Hilbert space. However, the quantum reference frame transformation cannot be formulated as individual ones, for example one for each translation of $\hat{x}_A$ by a fixed ‘value’ of $\hat{x}_B$ unless there is no entanglement between $B$ and $C$. Even in the latter case, entanglement of the transformed state of $C$ with the old reference frame $A$ as a quantum system seen from $B$ is often involved. So, we have general to deal with the quantum translation of $x_C^{(A)} \rightarrow x_C^{(B)} - x_A^{(B)}$ for any state of the composite system as a single symmetry transformation.

While composing two quantum reference frame transformations like $x_C^{(A)} \rightarrow x_C^{(B)} - x_A^{(B)}$ followed by $x_C^{(B)} \rightarrow x_C^{(D)} - x_B^{(D)}$ is straightforward, what is more interesting to combine quantum reference frame transformations of different kinds, like a quantum spatial translation and a quantum momentum translation, or even a kind of quantum rotations. One can easily appreciate how a quantum momentum translation can be formulated in a parallel manner, and goes further to the case of three independent pairs of $\hat{x}_i$ and $\hat{p}_i$. The angular momentum
observables can be used to formulate the quantum rotations. Our presentation of quantum reference frame transformations for qubit systems serves as a good illustration on how to proceed in the case of based on observables with a discrete or finite list of eigenvalues. We believe we have essentially given the framework to formulate a quantum reference frame transformation from an initial frame $A$ to a new frame $B$ based on any observable $\mathcal{O}$, i.e., under $\mathcal{O}_A^{(B)} = -\mathcal{O}_B^{(A)}$ and the proper matching transform for $\mathcal{O}_C$. We are hence at the starting point of looking into the system of symmetries of the totality of all quantum reference frame transformations of a given physical system. Further studies in the direction, together with the examinations of the results in terms of the noncommutative values for particular states, is a task on the table.

On the more technical side, besides the key focus of the analyses of the changes in the noncommutative values we have presented an improved formulation of quantum reference frame transformation based on the use of the zero vector to ‘represent’ the reference frame itself. In addition, the transformations given for the qubit systems, as examples of transformations based on the relative notion of an observable with a discrete or finite spectrum, are new. So is the explicit application of the notion of noncommutative values, as well as its presentation, for a system with a finite dimensional Hilbert space. It is our hope that results for such systems may be more accessible experimentally. Together with the example of the quantum, spatial transformation, we believe, they sketch a basic framework to write down any particular quantum reference frame transformations for any system.

Appendix : Noncommutative Values of Position and Momentum Observables with the Schrödinger Wavefunction Representation

With the position eigenstate basis $\hat{x} \ket{x} = x \ket{x}$, we have $\ket{\psi} = \int dx \ket{x} \bra{x} \psi \rangle$ where the wavefunction $\psi(x) \equiv \bra{\psi} x \rangle$ is really an infinite set of coordinates for the physical state $\ket{\psi}$ in the Hilbert space taken as a complex manifold. A (normalized) position eigenstate with eigenvalue $x_o$ is given by the wavefunction $\delta(x - x_o)$, and we have the matrix elements $(\hat{x})_{x'}^x = x \delta(x' - x)$. From

$$f_x(\psi) = \frac{\int dx \bar{\psi}(x) \psi'(x)}{\int dx \bar{\psi}(x) \psi(x)}$$

(18)
taken as a functional of the (normalized) wavefunction, we have the set of infinite coordinate derivatives can be expressed as the functional derivative

\[ V_x(x) = \delta \psi f_x(\psi) \equiv \frac{\delta f_x}{\delta \psi}(\psi) = \bar{\psi}(x - x_o), \]  

(19)

where \( x_o \) here denotes the expectation value of \( f_x \) evaluated for the fixed \( \psi(x) \). Again, we have really one value of \( V_x(x) \) at each \( x \) value matching to the coordinate value of \( \psi(x) \). For the momentum observable, we have

\[ f_p(\psi) = \frac{\int dx \bar{\psi}(x)(-i\partial_x)\psi(x)}{\int dx \bar{\psi}(x)\psi(x)} = \frac{\int dx [i\partial_x \bar{\psi}(x)]\psi(x)}{\int dx \bar{\psi}(x)\psi(x)}, \]  

(20)

which gives

\[ V_p(x) = \delta \psi f_p(\psi) = (i\partial_x - p_o)\bar{\psi}(x), \]  

(21)

where \( p_o \) again denotes the expectation value. In the same spirit, we read the matrix elements \((\hat{p})_{x'}^x \) from

\[ \int dx' dx \bar{\psi}(x')(\hat{p})_{x'}^x \psi(x)\delta(x' - x) = f_p(\psi) = \int dx' dx [i\partial_{x'} \bar{\psi}(x')][\delta(x' - x)\psi(x)] \]

\[ = \int dx' dx \bar{\psi}(x')[-i\partial_{x'}\delta(x' - x)]\psi(x), \]  

(22)

where \( \partial_{x'}\delta(x' - x) = -\partial_x\delta(x' - x) \) are derivatives of the delta function with respect to the variables, and we have the result \((\hat{p})_{x'}^x = -i\partial_{x'}\delta(x' - x) = i\partial_x\delta(x' - x) \). We have then the matrix elements

\[ (\hat{x})_{x'}^x = -i \int dy \left[ \partial_y \delta(y - x)y\delta(x' - y) \right] = -ix'\partial_{x'}\delta(x' - x) \]

\[ (\hat{p}x)_{x'}^x = -i \int dy \delta(y - x)\partial_x\delta(x' - y) = -ix\partial_x\delta(x' - x), \]

\[ (\hat{x}p)_{x'}^x - (\hat{p}x)_{x'}^x = -i(x' - x)\partial_{x'}\delta(x' - x) = i\delta(x' - x), \]  

(23)

and we also have \( f_{xp} = -i \int dx \bar{\psi}(x)x\partial_x\psi(x) \) and \( f_{px} = -i \int dx \bar{\psi}(x)[x\partial_x\psi(x) + \psi(x)] \). The noncommutative product among the noncommutative values corresponding to the formula
of Eq. (12) then gives

\[ V_{\hat{x}\hat{p}}(x) = -i \int dy \tilde{\psi}(y) y \partial_y \delta(y - x) - \tilde{\psi}(x) f_{\hat{x}\hat{p}} \]

\[ = i \int dy \delta(y - x) [\tilde{\psi}(y) + y \partial_y \tilde{\psi}(y)] - \tilde{\psi}(x) f_{\hat{x}\hat{p}} = \tilde{\psi}(x) [i - f_{\hat{x}\hat{p}}] + i x \partial_x \tilde{\psi}(x) , \]

\[ V_{\hat{p}\hat{x}}(x) = -i \int dy \tilde{\psi}(y) x \partial_y \delta(y - x) - \tilde{\psi}(x) f_{\hat{p}\hat{x}} \]

\[ = i \int dy \delta(y - x) x \partial_y \tilde{\psi}(y) - \tilde{\psi}(x) f_{\hat{p}\hat{x}} = -\tilde{\psi}(x) f_{\hat{p}\hat{x}} + i x \partial_x \tilde{\psi}(x) , \]

\[ V_{\hat{x}\hat{p}}(x) - V_{\hat{p}\hat{x}}(x) = \tilde{\psi}(x) [i - f_{\hat{x}\hat{p}} + f_{\hat{p}\hat{x}}] = 0 = \delta\psi(f_{\hat{x}\hat{p}} - f_{\hat{p}\hat{x}}) = V_{\hat{x}\hat{p} - \hat{p}\hat{x}}(x) . \quad (24) \]

The last result is an important consistency check of the fact that the noncommutative value of an observable as a product of two observables is the product of the noncommutative values of the individual observables, and noncommutative value of an observable as the commutator is the commutator of the individual noncommutative values. We check further from

\[ f_{\hat{x}\hat{p}} = f_{\hat{x}} f_{\hat{p}} + \int dx V_{\hat{x}}(x) \tilde{V}_{\hat{p}}(x) , \]

\[ f_{\hat{p}\hat{x}} = f_{\hat{p}} f_{\hat{x}} + \int dx V_{\hat{p}}(x) \tilde{V}_{\hat{x}}(x) , \quad (25) \]

that

\[ f_{\hat{x}\hat{p}} - f_{\hat{p}\hat{x}} = \int dx \left[ V_{\hat{x}}(x) \tilde{V}_{\hat{p}}(x) - V_{\hat{p}}(x) \tilde{V}_{\hat{x}}(x) \right] \]

\[ = \int dy \left[ \tilde{\psi}(y) [y - f_{\hat{x}}] [-i \partial_y - f_{\hat{p}}] \psi(y) - [i \partial_y - f_{\hat{p}}] \tilde{\psi}(y) [y - f_{\hat{x}}] \psi(y) \right] \]

\[ = \int dy \left[ i f_{\hat{x}} [\tilde{\psi}(y) \partial_y \psi(y) + \partial_y \tilde{\psi}(y) \psi(y)] - i [\tilde{\psi}(y) y \partial_y \psi(y) + \partial_y \tilde{\psi}(y) y \psi(y)] \right] \]

\[ = i \int dy \left[ \psi(y) \partial_y [\tilde{\psi}(y) y] - \partial_y \tilde{\psi}(y) y \psi(y) \right] = i . \quad (26) \]

Hence our noncommutative product expressions for the position and momentum observables are fully consistent. A generic observable would be taken as some functions of \( \hat{x} \) and \( \hat{p} \). The noncommutative expression hence gives definite results for at least all observables which can be expressed as a (noncommutative) polynomial in \( \hat{x} \) and \( \hat{p} \). Generalizing to the case of three pairs of \( \hat{x}_i \) and \( \hat{p}_i \) is straightforward.

Here, we give the details of the somewhat subtle calculation of section III for the momentum observable discussed at the end of the analysis for case (a). It shows results of the quantum spatial translation on the momentum observables not otherwise explicitly il-
Ilustrated. But is mostly to be read in reference to the above mentioned analysis.

\[
\hat{S}_x \hat{p}_y \hat{S}^+_x = \int dx'' dy'' dxdy' \left| -x'', y'' - x''\right\rangle_{AC} \langle x'', y''| \hat{p}_y | x, y\rangle_{BC} \langle -x, y - x|_{AC}
\]

\[
= \int dx'' dy'' dxdy' [-i\partial_{x''}\delta(x'' - x)]\delta(y'' - y)\left| -x'', y'' - x''\right\rangle_{AC},
\]

\[
= \int dx'' dy'' dxdy' [i\partial_{x''}\delta(x'' - x)]\delta(y'' - y - x'' + x') \left| x'', y''\right\rangle_{AC},
\]

(27)
giving \( \langle \phi' | \hat{S}_x \hat{p}_y \hat{S}^+_x | \phi' \rangle \) as

\[
\int dx'' dy'' dxdy' [-i\partial_{x''}\delta(x'' - x)]\delta(y'' - y)\delta(x'' + x_o)\bar{\psi}(y'' - x'' + x_o)\delta(x + x_o)\psi(y - x + x_o)
\]

\[
= \int dx'' dy'' dxdy' \delta(x'' - x)\delta(y'' - y)\delta(x + x_o)\bar{\psi}(y - x + x_o)i\partial_{x''}[\delta(x'' + x_o)\bar{\psi}(y'' - x'' + x_o)]
\]

\[
= \int dxdy \delta(x + x_o)[i\partial_x\delta(x + x_o)]|\psi(y - x + x_o)|^2
\]

\[
+ \int dxdy \delta^2(x + x_o)\bar{\psi}(y - x + x_o)i\partial_x\bar{\psi}(y - x + x_o).
\]

(28)
The first term is \(-f^f_{Pa}(\phi')\) and the second term is \(-f^f_{Rc}(\phi')\), which may be more easily seen through substituting \( y' = y - x \) and \( x' = x \) into the direct expressions of

\[
f^f_{Pa}(\phi') = \int dx'dy' [-i\partial_{x'}\delta(x' + x_o)]\delta(x' + x_o)|\psi(y' + x_o)|^2
\]

\[
f^f_{Rc}(\phi') = \int dx'dy' \delta^2(x' + x_o)|\psi(y' + x_o)i\partial_{y'}\bar{\psi}(y' + x_o).
\]

(29)

**Acknowledgments:**

The work is supported by the research grants number 109-2112-M-008-016 and 110-2112-M-008-016 of the MOST of Taiwan.

[1] Aharonov, Y. and Susskind, L. Charge Superselection Rule. Phys. Rev. 155, 1428 (1967).
[2] Aharonov, Y. and Kaufherr, T. Quantum frames of reference. Phys. Rev. D. 30, 368 (1984).
[3] Rovelli, C. Quantum reference systems. Class. Quantum Gravity 8, 317 (1991).
[4] Bartlett, S. D., Rudolph, T. and Spekkens, R. W. Reference frames, superselection rules, and quantum information. Rev. Mod. Phys. 79, 555 (2007).
[5] Katz, B. N., Blencowe, M. P. and Schwab, K. C. Mesoscopic mechanical resonators as quantum non-inertial reference frames. Phys. Rev. A 92, 042104 (2015).
[6] Gour, G. and Spekkens, R. W. The resource theory of quantum reference frames: manipulations and monotones. New J. Phys. 10, 033023 (2008).

[7] Bartlett, S. D., Rudolph, T., Spekkens, R. W. and Turner, P. S. Quantum communication using a bounded-size quantum reference frame. New J. Phys. 11, 063013 (2009).

[8] Giacomini F., Castro-Ruiz, E. and Brukner Č. Quantum mechanics and the covariance of physical laws in quantum reference frames. Nature Commun. 10, 494 (2019)

[9] de la Hamette A. and Galley T.D. Quantum reference frames for general symmetry groups Quantum. 4, 367 (2020).

[10] Angelo, R. M., Brunner, N., Popescu, S., Short, A. and Skrzypczyk, P. Physics within a quantum reference frame. J. Phys. A 44, 145304 (2011).

[11] Pereira, S. T. and Angelo, R. M. Galilei covariance and Einstein’s equivalence principle in quantum reference frames. Phys. Rev. A 91, 022107 (2015).

[12] Smith, A. R. H., Piani, M. and Mann, R. B. Quantum reference frames associated with noncompact groups: The case of translations and boosts and the role of mass. Phys. Rev. A. 94, 012333 (2016).

[13] Ballesteros A., Giacomini F. and Gubitosi G. The group structure of dynamical transformations between quantum reference frames. Quantum 5, 470 (2021).

[14] Kong O.C.W. A Geometric Picture of Quantum Mechanics with Noncommutative Values for Observables. Results in Phys. 19, 103636 (2020).

[15] Kong O.C.W. and Liu W.-Y. The Noncommutative Values of Quantum Observables. Chin. J. Phys. 69, 70-76 (2021).

[16] Kong O.C.W. and Liu W.-Y. Noncommutative Coordinate Picture of the Quantum Phase Space. Chin. J. Phys. 71, 418-434 (2021), and references therein.

[17] Bengtsson I. and Życzkowski K. Geometry of Quantum States, Cambridge University Press, Cambridge (2006).

[18] Cirelli R., Manià A. and Pizzocchero L. Quantum Mechanics as an Infinite-Dimensional Hamiltonian System with Uncertainty Structure. Part I. J. Math. Phys. 31, 2891-2897 (1990).

[19] Kong O.C.W. and Payne J. Special Relativity and its Newtonian Limit from a Group Theoretical Perspective. Symmetry 13, 1925 (2021).

[20] Marsden J.E. and Ratiu T.S. Introduction to Mechanics and Symmetry, Springer (1994).

[21] Bedić S., Kong O.C.W. and Ting H.K. Group Theoretical Approach to Pseudo-Hermitian
Quantum Mechanics with Lorentz Covariance and $c \to \infty$ Limit. Symmetry 13, 22 (2021).

[22] Chew C.S., Kong O.C.W. and Payne J. Observables and Dynamics Quantum to Classical from a Relativity Symmetry and Noncommutative-Geometric Perspective. J. High Energy Phys. Gravit. Cosmol. 5, 553-586 (2019).

[23] Chew C.S., Kong O.C.W. and Payne J. A Quantum Space Behind Simple Quantum Mechanics. Adv. High Energy Phys. 2017, Special Issue on Planck-Scale Deformations of Relativistic Symmetries, 4395918 (2017).

[24] Loveridge, L., Miyadera, T. and Busch, P. Symmetry, reference frames, and relational quantities in quantum mechanics. Found. Phys. 48, 135-198 (2018).

34