Boundedness of fractional integral operators with rough kernels on weighted Morrey spaces

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Abstract
Let $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ be the fractional maximal and integral operators with rough kernels, where $0 < \alpha < n$. In this paper, we shall study the continuity properties of $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ on the weighted Morrey spaces $L^{p,\kappa}(w)$. The boundedness of their commutators with BMO functions is also obtained.

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1 Introduction

Let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, where $S^{n-1}$ denotes the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$ and $s > 1$. For any $0 < \alpha < n$, then the fractional integral operator with rough kernel $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f(x - y) \, dy$$

and a related fractional maximal operator $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha}f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y| \leq r} |\Omega(y') f(x - y)| \, dy,$$

where $y' = y/|y|$ for any $y \neq 0$. In 1971, Muckenhoupt and Wheeden [17] studied the weighted norm inequalities for $T_{\Omega,\alpha}$ with the weight $w(x) = |x|^{\beta}$.

The weak type estimates with power weights for $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ was obtained by Ding in [3]. Later, Ding and Lu [4] considered the weighted norm inequalities for $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ with more general weights. More precisely, they proved

**Theorem A ([3])**. Let $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If $\Omega \in L^s(S^{n-1})$ and $w^{s'} \in A(p/s', q/s')$, then the operators $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ are all bounded from $L^p(w^p)$ to $L^q(w^q)$.

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Let $b$ be a locally integrable function on $\mathbb{R}^n$, then for $0 < \alpha < n$, we shall define the commutators generated by fractional maximal and integral operators with rough kernels and $b$ as follows.

$$[b, M_{\Omega, \alpha}](f)(x) = \sup_{r > 0} \frac{1}{rn^{n-\alpha}} \int_{|x-y| \leq r} |b(x) - b(y)||\Omega(x-y)f(y)| \, dy,$$

$$[b, T_{\Omega, \alpha}](f)(x) = b(x)T_{\Omega, \alpha}f(x) - T_{\Omega, \alpha}(b)(f)(x)$$

$$= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} |b(x) - b(y)||f(y)| \, dy.$$

In 1993, by using the Rubio de Francia extrapolation theorem, Segovia and Torrea [21] obtained the weighted boundedness of commutator $[b, T_{\Omega, \alpha}]$, where $b \in \text{BMO}(\mathbb{R}^n)$ and $\Omega$ satisfies some Dini smoothness condition (see also [20]). In 1999, Ding and Lu [5] improved this result by removing the smoothness condition imposed on $\Omega$. More specifically, they showed (see also [14]).

**Theorem B** ([5]). Let $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Assume that $\Omega \in L^s(S^{n-1})$, $w^{s'} \in A(p/s', q/s')$ and $b \in \text{BMO}(\mathbb{R}^n)$, then the commutator $[b, T_{\Omega, \alpha}]$ is bounded from $L^p(w^q)$ to $L^s(w^q)$.

The classical Morrey spaces $L^{p, \lambda}$ were first introduced by Morrey in [13] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operators on these spaces, we refer the readers to [1, 2, 19]. For the properties and applications of classical Morrey spaces, see [7, 8, 9] and references therein.

In 2009, Komori and Shirai [13] first defined the weighted Morrey spaces $L^{p, \kappa}(w)$ which could be viewed as an extension of weighted Lebesgue spaces, and studied the boundedness of the above classical operators on these weighted spaces. Recently, in [22] and [23], we have established the continuity properties of some other operators on the weighted Morrey spaces $L^{p, \kappa}(w)$.

The purpose of this paper is to discuss the boundedness properties of $M_{\Omega, \alpha}$ and $T_{\Omega, \alpha}$ on the weighted Morrey spaces. Here, and in what follows we shall use the notation $s' = s/(s-1)$ when $1 < s < \infty$ and $s' = 1$ when $s = \infty$. Our main results in the paper are formulated as follows.

**Theorem 1.1.** Suppose that $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$. If $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w^{s'} \in A(p/s', q/s')$, then the fractional maximal operator $M_{\Omega, \alpha}$ is bounded from $L^{p, \kappa}(w^q)$ to $L^{q, \kappa q/p}(w^q)$.

**Theorem 1.2.** Suppose that $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$. If $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w^{s'} \in A(p/s', q/s')$, then the fractional integral operator $T_{\Omega, \alpha}$ is bounded from $L^{p, \kappa}(w^q)$ to $L^{q, \kappa q/p}(w^q)$.

**Theorem 1.3.** Suppose that $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$ and $b \in \text{BMO}(\mathbb{R}^n)$. If $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w^{s'} \in A(p/s', q/s')$, then the commutator $[b, T_{\Omega, \alpha}]$ is bounded from $L^{p, \kappa}(w^q)$ to $L^{q, \kappa q/p}(w^q)$.
2 Notations and definitions

Let us first recall some standard definitions and notations. The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [16]. A weight $w$ is a nonnegative, locally integrable function on $\mathbb{R}^n$, $B = B(x_0, r_B)$ denotes the ball with the center $x_0$ and radius $r_B$. Given a ball $B$ and $\lambda > 0$, $\lambda B$ denotes the ball with the same center as $B$ whose radius is $\lambda$ times that of $B$. For a given weight function $w$, we also denote the Lebesgue measure of $B$ by $|B|$ and the weighted measure of $B$ by $w(B)$, where $w(B) = \int_B w(x) \, dx$. We say that $w \in A_p$, $1 < p < \infty$, if
\[
\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n,
\]
where $C$ is a positive constant which is independent of $B$.

For the case $p = 1$, $w \in A_1$, if
\[
\frac{1}{|B|} \int_B w(x) \, dx \leq C \cdot \operatorname{ess \ inf} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.
\]

For the case $p = \infty$, $w \in A_\infty$ if it satisfies the $A_p$ condition for some $1 < p < \infty$.

We also need another weight class $A(p, q)$ introduced by Muckenhoupt and Wheeden in [18]. A weight function $w$ belongs to $A(p, q)$ for $1 < p < q < \infty$ if there exists a constant $C > 0$ such that
\[
\left( \frac{1}{|B|} \int_B w(x)^q \, dx \right)^{1/q} \left( \frac{1}{|B|} \int_B w(x)^{-p'} \, dx \right)^{1/p'} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n.
\]

A weight function $w$ is said to belong to the reverse Hölder class $RH_r$ if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds
\[
\left( \frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right) \quad \text{for every ball } B \subseteq \mathbb{R}^n.
\]

We state the following results that we will use frequently in the sequel.

Lemma 2.1 ([10]). Let $w \in A_p$ with $p \geq 1$. Then, for any ball $B$, there exists an absolute constant $C > 0$ such that
\[
w(2B) \leq C w(B).
\]

In general, for any $\lambda > 1$, we have
\[
w(\lambda B) \leq C \cdot \lambda^{np} w(B),
\]
where $C$ does not depend on $B$ nor on $\lambda$. 
Lemma 2.2 ([11]). Let $w \in RH_r$ with $r > 1$. Then there exists a constant $C > 0$ such that

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset $E$ of a ball $B$.

Next we shall introduce the Hardy-Littlewood maximal operator, its variant and BMO spaces. The Hardy-Littlewood maximal operator $M$ is defined by

$$M(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls $B$ containing $x$. For $0 < \alpha < n$, $s \geq 1$, we define the fractional maximal operator $M_{\alpha,s}$ by

$$M_{\alpha,s}(f)(x) = \sup_{x \in B} \left(\frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|^s \, dy\right)^{1/s}.$$ 

Moreover, we denote simply by $M_\alpha$ when $s = 1$.

A locally integrable function $b$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx < \infty,$$

where $b_B$ stands for the average of $b$ on $B$, i.e. $b_B = \frac{1}{|B|} \int_B b(y) \, dy$ and the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

Theorem C ([6, 12]). Assume that $b \in BMO(\mathbb{R}^n)$. Then for any $1 \leq p < \infty$, we have

$$\sup_B \left(\frac{1}{|B|} \int_B |b(x) - b_B|^p \, dx\right)^{1/p} \leq C \|b\|_*.$$

We are going to conclude this section by defining the weighted Morrey space and giving the known result relevant to this paper. For further details, we refer the readers to [13].

Definition 2.3 ([13]). Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w$ be a weight function. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) = \{f \in L^p_{\text{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^{\kappa}} \int_B |f(x)|^p w(x) \, dx\right)^{1/p}$$

and the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

In order to deal with the fractional order case, we need to consider the weighted Morrey space with two weights.
Definition 2.4 ([13]). Let $1 \leq p < \infty$ and $0 < \kappa < 1$. Then for two weights $u$ and $v$, the weighted Morrey space is defined by

$$L^{p,\kappa}(u,v) = \{ f \in L^p_{\text{loc}}(u) : \|f\|_{L^{p,\kappa}(u,v)} < \infty \},$$

where $\|f\|_{L^{p,\kappa}(u,v)} = \sup_B \left( \frac{1}{v(B)} \int_B |f(x)|^p u(x) \, dx \right)^{1/p}$.

Theorem D. If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w \in A(p,q)$, then the fractional maximal operator $M_{\alpha}$ is bounded from $L^{p,\kappa}(w^p,w^q)$ to $L^{q,\kappa/q}(w^{q/p}(w^q))$.

Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{A}{B} \leq C$.

3 Proof of Theorem 1.1

Proof of Theorem 1.1. For $\Omega \in L^s(S^{n-1})$, we set

$$\|\Omega\|_{L^s(S^{n-1})} = \left( \int_{S^{n-1}} |\Omega(y')|^s \, d\sigma(y') \right)^{1/s}.$$

From Hölder’s inequality, it follows that

$$M_{\Omega,\alpha} f(x) \leq \sup_{r > 0} \frac{1}{r^{n-\alpha}} \left( \int_{|y| \leq r} |\Omega(y')|^s \, dy \right)^{1/s} \left( \int_{|y| \leq r} |f(x-y)|^{s'} \, dy \right)^{1/s'} \leq C \|\Omega\|_{L^s(S^{n-1})} \cdot \sup_{r > 0} \left( \int_{|y| \leq r} |f(x-y)|^{s'} \, dy \right)^{1/s'} \leq C \|\Omega\|_{L^s(S^{n-1})} M_{\alpha,s'}(f)(x).$$

If we let $p_1 = p/s'$, $q_1 = q/s'$ and $\nu = w^{s'}$, then for $0 < \alpha < n$, $1 \leq s' < n/\alpha$, we have $1/q_1 = 1/p_1 - (\alpha s')/n$ and $0 < \kappa < p_1/q_1$. Also observe that

$$M_{\alpha,s'}(f) = M_{\alpha,s'}(|f|^{s'})^{1/s'}.$$

Hence, by Theorem D, we obtain

$$\|M_{\alpha,s'}(f)\|_{L^{q,\kappa/q}(w^{q/p}(w^q))} = \|M_{\alpha,s'}(|f|^{s'})\|_{L^{q,\kappa/q}(w^{q/p}(w^q))} \leq C \|f|^{s'}\|_{L^{p,\kappa}(w^p,w^q)} \leq C \|f\|_{L^{p,\kappa}(w^p,w^q)},$$

This finishes the proof of Theorem 1.1. \qed
4 Proof of Theorem 1.2

Proof of Theorem 1.2. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ and decompose $f = f_1 + f_2$, where $f_1 = f \chi_{2B}$ and $\chi_{2B}$ denotes the characteristic function of $2B$. Since $T_{\Omega, \alpha}$ is a linear operator, then we can write
\[
\frac{1}{w^{q}(B)^{\kappa/p}} \left( \int_B |T_{\Omega, \alpha}f(x)|^q w(x)^q \, dx \right)^{1/q} \leq \frac{1}{w^{q}(B)^{\kappa/p}} \left( \int_B |T_{\Omega, \alpha}f_1(x)|^q w(x)^q \, dx \right)^{1/q} + \frac{1}{w^{q}(B)^{\kappa/p}} \left( \int_B |T_{\Omega, \alpha}f_2(x)|^q w(x)^q \, dx \right)^{1/q} = I_1 + I_2.
\]

As in the proof of Theorem 1.1, we also set $p_1 = p/s'$, $q_1 = q/s'$ and $\nu = w^{s'}$. Since $\nu \in A(p_1, q_1)$, then we get $\nu^\alpha = w^q \in A_{1+\alpha/p_1'}$ (see [18]). Hence, by Theorem A and Lemma 2.1, we have
\[
I_1 \leq C \cdot \frac{1}{w^{q}(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x)^p \, dx \right)^{1/p} \leq C \|f\|_{L^{p,\alpha}(w^{p}, w^{q})} \cdot w^q(2B)^{\kappa/p} = C \|f\|_{L^{p,\alpha}(w^{p}, w^{q})}.
\]

We now turn to deal with the term $I_2$. An application of Hölder’s inequality gives us that
\[
|T_{\Omega, \alpha}(f_2)(x)| \leq \int_{(2B)^c} \frac{|\Omega(x - y)|}{|x - y|^{n-\alpha}} |f(y)| \, dy \leq \sum_{k=1}^\infty \left( \int_{2^{k+1}B \setminus 2^kB} \frac{|\Omega(x - y)|}{|x - y|^{n-\alpha}} \, dy \right)^{1/s} \left( \int_{2^{k+1}B \setminus 2^kB} \frac{|f(y)|^{s'}}{|x - y|^{(n-\alpha)s'}} \, dy \right)^{1/s'}.
\]

When $x \in B$ and $y \in 2^{k+1}B \setminus 2^{k}B$, then we can easily see that $2^{k-1}r_B \leq |y - x| < 2^{k+2}r_B$. Thus, by a simple computation, we deduce
\[
\left( \int_{2^{k+1}B \setminus 2^kB} |\Omega(x - y)|^{s} \, dy \right)^{1/s} \leq C \|\Omega\|_{L^1(S^{n-1})} |2^{k+1}B|^{1/s}.
\]

We also note that if $x \in B$, $y \in (2B)^c$, then $|y - x| \sim |y - x_0|$. Consequently
\[
\left( \int_{2^{k+1}B \setminus 2^{k}B} \frac{|f(y)|^{s'}}{|x - y|^{(n-\alpha)s'}} \, dy \right)^{1/s'} \leq C \frac{1}{|2^{k+1}B|^{1-\alpha/n}} \left( \int_{2^{k+1}B} |f(y)|^{s'} \, dy \right)^{1/s'}.
\]

Substituting the above two inequalities (2) and (3) into (1), we obtain
\[
|T_{\Omega, \alpha}(f_2)(x)| \leq C \|\Omega\|_{L^1(S^{n-1})} \sum_{k=1}^\infty \frac{1}{2^{k+1}B|^{1-\alpha/n-1/s}} \left( \int_{2^{k+1}B} |f(y)|^{s'} \, dy \right)^{1/s'}.
\]
By using Hölder’s inequality and the definition of \( \nu \in \mathcal{A}(p_1, q_1) \), we can get
\[
\left( \int_{2k+1} |f(y)|^{s'} \, dy \right)^{1/s'} \leq \left( \int_{2k+1} |f(y)|^{p_1 s'} \nu(y)^{p_1} \, dy \right)^{1/(p_1 s')} \left( \int_{2k+1} \nu(y)^{-p_1'} \, dy \right)^{1/(p_1' s')}
\leq C \left( \int_{2k+1} |f(y)|^p w(y)^p \, dy \right)^{1/p} \left( \frac{2^{k+1} B^{1-1/p_1+1/q_1}}{\nu^{q_1}(2^{k+1} B)^{1/q_1}} \right)^{1/s'}
\leq C \|f\|_{L^{p_1, q_1}(w, w^q)} (2^{k+1} B)^{\kappa/p} \cdot \frac{|2^{k+1} B|^{1/s'-1/p+1/q}}{w^q(2^{k+1} B)^{1/q}}
= C \|f\|_{L^{p_1, q_1}(w, w^q)} |2^{k+1} B|^{-1/\alpha/n} \cdot w^q(2^{k+1} B)^{\kappa/p-1/q}.
\]

So we have
\[
|T_{\Omega, \alpha}(f_2)(x)| \leq C \|f\|_{L^{p_1, q_1}(w, w^q)} \sum_{k=1}^{\infty} w^q(2^{k+1} B)^{\kappa/p-1/q},
\]
which implies
\[
I_2 \leq C \|f\|_{L^{p_1, q_1}(w, w^q)} \sum_{k=1}^{\infty} \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1} B)^{1/q-\kappa/p}}.
\]

Observe that \( w^q = \nu^{q_1} \in A_{1+q_1/p_1^*} \), then we know that there exists \( r > 1 \) such that \( w^q \in RH_r \). Thus, it follows directly from Lemma 2.2 that
\[
\frac{w^q(B)\nu}{w^q(2^{k+1} B)^{1/q-\kappa/p}} \leq C \left( \frac{|B|}{2^{k+1} B} \right)^{(r-1)/r}. \tag{5}
\]

Therefore
\[
I_2 \leq C \|f\|_{L^{p_1, q_1}(w, w^q)} \sum_{k=1}^{\infty} \left( \frac{1}{2^{kn}} \right)^{(1-1/r)(1/q-\kappa/p)}
\leq C \|f\|_{L^{p_1, q_1}(w, w^q)},
\]
where the last series is convergent since \( r > 1 \) and \( 0 < \kappa < p/q \). Combining the above estimates for \( I_1 \) and \( I_2 \) and taking the supremum over all balls \( B \subseteq \mathbb{R}^n \), we complete the proof of Theorem 1.2. \( \square \)

### 5 Proof of Theorem 1.3

**Proof of Theorem 1.3.** Fix a ball \( B = B(x_0, r_B) \subseteq \mathbb{R}^n \). Let \( f = f_1 + f_2 \), where \( f_1 = f \chi_{2B} \). Since \([b, T_{\Omega, \alpha}]\) is a linear operator, then we have
\[
\frac{1}{w^q(B)^{\kappa/p}} \left( \int_B \|b, T_{\Omega, \alpha}\| f(x)^q w(x)^q \, dx \right)^{1/q}
\leq \frac{1}{w^q(B)^{\kappa/p}} \left( \int_B \|b, T_{\Omega, \alpha}\| f_1(x)^q w(x)^q \, dx \right)^{1/q} + \frac{1}{w^q(B)^{\kappa/p}} \left( \int_B \|b, T_{\Omega, \alpha}\| f_2(x)^q w(x)^q \, dx \right)^{1/q}
= J_1 + J_2.
\]
As before, we set \( p_1 = p/s', \ q_1 = q/s' \) and \( \nu = w^{s'} \), then \( \nu^\alpha = w^\alpha \in A_{1+q_1/p'_1}. \) Theorem B and Lemma 2.1 imply

\[
J_1 \leq C\|b\| \cdot \frac{1}{w^q(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x)^p \, dx \right)^{1/p} \\
\leq C\|b\| \|f\|_{L^{p,q}(w^\kappa,w^q)} \cdot \frac{w^q(2B)^{\kappa/p}}{w^q(B)^{\kappa/p}} \\
\leq C\|b\| \|f\|_{L^{p,q}(w^\kappa,w^q)}.
\]

In order to estimate the term \( J_2 \), for any \( x \in B \), we first write

\[
|[b,T_{B,a}]f_2(x)| = \left| \int_{(2B)^c} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} [b(x) - b(y)] f(y) \, dy \right| \\
\leq |b(x) - b_B| \int_{(2B)^c} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} |f(y)| \, dy \\
+ \int_{(2B)^c} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} |b(y) - b_B| |f(y)| \, dy \\
= I + II.
\]

For the term I, it follows from the previous estimates (2) and (4) that

\[
I \leq C\|f\|_{L^{p,q}(w^\kappa,w^q)} |b(x) - b_B| \sum_{k=1}^{\infty} \frac{1}{w^q(2^{k+1}B)^{1/q-\kappa/p}}.
\]

Hence

\[
\frac{1}{w^q(B)^{\kappa/p}} \left( \int_B |f(x)|^q w(x)^q \, dx \right)^{1/q} \\
\leq C\|f\|_{L^{p,q}(w^\kappa,w^q)} \frac{1}{w^q(B)^{\kappa/p}} \sum_{k=1}^{\infty} \frac{1}{w^q(2^{k+1}B)^{1/q-\kappa/p}} \cdot \left( \int_B |b(x) - b_B|^q w(x)^q \, dx \right)^{1/q} \\
= C\|f\|_{L^{p,q}(w^\kappa,w^q)} \sum_{k=1}^{\infty} \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1}B)^{1/q-\kappa/p}} \cdot \left( \frac{1}{w^q(B)} \int_B |b(x) - b_B|^q w(x)^q \, dx \right)^{1/q}.
\]

We now claim that for any \( 1 < q < \infty \) and \( \mu \in A_\infty \), the following inequality holds

\[
\left( \frac{1}{\mu(B)} \int_B |b(x) - b_B|^q \mu(x) \, dx \right)^{1/q} \leq C\|b\|_ \ast. \tag{7}
\]

In fact, since \( \mu \in A_\infty \), then there must exist \( r > 1 \) such that \( \mu \in RH_r \). Thus, by Hölder’s inequality and Theorem C, we obtain

\[
\left( \frac{1}{\mu(B)} \int_B |b(x) - b_B|^q \mu(x) \, dx \right)^{1/q} \leq \frac{1}{\mu(B)^{1/q}} \left( \int_B |b(x) - b_B|^{q'r'} \, dx \right)^{1/(q'r')} \left( \int_B \mu(x)^{r'} \, dx \right)^{1/(qr')} \\
\leq C \left( \frac{1}{|B|} \int_B |b(x) - b_B|^{q'r'} \, dx \right)^{1/(qr')} \leq C\|b\|_ \ast,
\]

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which is our desired result. Note that \( w^q \in A_{1+q_1/p'_1} \subset A_\infty \). In addition, we have \( w^{q_1} \in RH_r \) with \( r > 1 \). Hence, by the inequalities (5) and (7), we get

\[
\frac{1}{w^q(B)^{\kappa/p}} \left( \int_B I^q w(x)^q \, dx \right)^{1/q} \leq C \|b\|_* \|f\|_{L^p(\nu \cdot w^q, w^q)} \sum_{k=1}^\infty \left( \frac{1}{2^k} \right)^{(1-1/r)(1/q-\kappa/p)} \\
\leq C \|b\|_* \|f\|_{L^p(\nu \cdot w^q, w^q)}.
\]

(8)

On the other hand

\[
\begin{align*}
&\text{II} \leq \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^kB} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(y) - b_B| |f(y)| \, dy \\
&\leq \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^kB} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(y) - b_{2^{k+1}B}| |f(y)| \, dy \\
&\quad + \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^kB} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b_{2^{k+1}B} - b_B| |f(y)| \, dy \\
&= \text{III} + \text{IV}.
\end{align*}
\]

To estimate III and IV, we observe that when \( x \in B, y \in (2B)^c \), then \(|y - x| \sim |y - x_0|\). Thus, it follows from Hölder’s inequality and (2) that

\[
\text{III} \leq C \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{1-\alpha/n}} \int_{2^{k+1}B \setminus 2^kB} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(y) - b_{2^{k+1}B}| |f(y)| \, dy \\
\leq C \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{1-\alpha/n-1/s}} \cdot \left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{s'} |f(y)|^{s'} \, dy \right)^{1/s'}.
\]

An application of Hölder’s inequality yields

\[
\left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{s'} |f(y)|^{s'} \, dy \right)^{1/s'} \leq \left( \int_{2^{k+1}B} |f(y)|^{p_1 s'} \nu(y)^{p_1} \, dy \right)^{1/(p_1 s')} \left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p_1 s'} \nu(y)^{-p_1} \, dy \right)^{1/(p_1 s')} \\
\leq \left( \int_{2^{k+1}B} |f(y)|^{p_1 s'} \nu(y)^{p_1} \, dy \right)^{1/p_1} \left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p_1 s'} \nu(y)^{-p_1} \, dy \right)^{1/(p_1 s')}.
\]

Since \( \nu \in A(p_1, q_1) \), then we know that \( \nu^{-p_1} \in A_{1+p_1/q_1} \subset A_\infty \) (see [18]). Hence, by using the inequality (7) and the fact that \( \nu \in A(p_1, q_1) \), we obtain

\[
\left( \int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p_1 s'} \nu(y)^{-p_1} \, dy \right)^{1/(p_1 s')} \leq C \|b\|_* \cdot \nu^{-p_1} (2^{k+1}B)^{\frac{1}{1+1/p_1}} \\
\leq C \|b\|_* \cdot \left( \frac{|2^{k+1}B|^{1/q_1 + 1/p_1}}{\nu|2^{k+1}B|^{1/q_1}} \right)^{1/s'}
\]
\[ \|b\|_* \cdot \frac{|2^{k+1}B|^{1/(q'-1/p)+1/q}}{w^q(2^{k+1}B)^{1/q}}. \]

(9)

Consequently, by the above inequality (9), we deduce

\[ \text{III} \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)} \sum_{k=1}^{\infty} \frac{1}{w^q(2^{k+1}B)^{1/q-\kappa/p}}, \]

which implies

\[ \frac{1}{w^q(B)^{\kappa/p}} \left( \int_B \text{III}^q w(x)^q \, dx \right)^{1/q} \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)} \sum_{k=1}^{\infty} \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1}B)^{1/q-\kappa/p}} \]

\[ \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)}. \]

(10)

Since \( b \in BMO(\mathbb{R}^n) \), then a direct calculation shows that

\[ |b_{2^{k+1}B} - b_B| \leq C \cdot k\|b\|_* . \]

Moreover, by Hölder’s inequality, the estimates (2) and (4), we can get

\[ \text{IV} \leq C\|b\|_* \sum_{k=1}^{\infty} k \cdot \frac{1}{|2^{k+1}B|^{1-\alpha/n}} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-y)||f(y)| \, dy \]

\[ \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)} \sum_{k=1}^{\infty} \frac{k}{w^q(2^{k+1}B)^{1/q-\kappa/p}}. \]

Therefore

\[ \frac{1}{w^q(B)^{\kappa/p}} \left( \int_B \text{IV}^q w(x)^q \, dx \right)^{1/q} \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)} \sum_{k=1}^{\infty} k \cdot \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1}B)^{1/q-\kappa/p}} \]

\[ \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)} \sum_{k=1}^{\infty} \frac{k}{2^{kn\delta}} \]

\[ \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)}. \]

(11)

where \( w^q \in RH_r \) and \( \delta = (1 - 1/r)(1/q - \kappa/p) \). Summarizing the estimates (10) and (11) derived above, we thus obtain

\[ \frac{1}{w^q(B)^{\kappa/p}} \left( \int_B \text{III}^q w(x)^q \, dx \right)^{1/q} \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w_p,w_q)}. \]

(12)

Combining the inequalities (6) and (8) with the above inequality (12) and taking the supremum over all balls \( B \subseteq \mathbb{R}^n \), we conclude the proof of Theorem 1.3.

It should be pointed out that \([b,M_{\Omega,\alpha}](f)\) can be controlled pointwise by \([b,T_{[\Omega,\alpha]}(f)]\) for any \( f(x) \). In fact, for any \( 0 < \alpha < n, \ x \in \mathbb{R}^n \) and \( r > 0 \), we
have

\[
[b, T_{[\Omega], \alpha}](|f|)(x) \geq \int_{|y-x| \leq r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(x) - b(y)||f(y)| \, dy
\]

\[
\geq \frac{1}{r^{n-\alpha}} \int_{|y-x| \leq r} |\Omega(x-y)||b(x) - b(y)||f(y)|| \, dy.
\]

Taking the supremum for all \( r > 0 \) on both sides of the above inequality, we get

\[
[b, M_{[\Omega], \alpha}](f)(x) \leq [b, T_{[\Omega], \alpha}](|f|)(x), \quad \text{for all } x \in \mathbb{R}^n.
\]

Hence, as a direct consequence of Theorem 1.3, we finally obtain the following

**Corollary 5.1.** Suppose that \( \Omega \in L^s(S^{n-1}) \) with \( 1 < s \leq \infty \) and \( b \in BMO(\mathbb{R}^n) \). If \( 0 < \alpha < n \), \( 1 \leq s' < p < n/\alpha \), \( 1/q = 1/p - \alpha/n \), \( 0 < \kappa < p/q \) and \( w^\kappa \in A(p/s', q/s') \), then the commutator \( [b, M_{[\Omega], \alpha}] \) is bounded from \( L^p,\kappa(w^p, w^q) \) to \( L^{n,\kappa/p}(w^q) \).

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