On nonlinear Fourier transform: towards the nonlinear superposition

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Abstract. In the paper we consider the nonlinear Fourier transform associated to the AKNS-ZS systems. In particular, we discuss the construction of the nonlinear Fourier modes of this transform by means of a perturbation scheme. The linearization of the AKNS-ZS nonlinear Fourier transform is the usual, linear Fourier transform and the linearization of a nonlinear Fourier mode of frequency $d$ is the linear Fourier mode of the same frequency. We show that the first non-trivial term in the perturbation expression of any nonlinear Fourier mode is given by the dilogarithm function.

1. Introduction

A well-known and important class of integrable partial differential equations are the AKNS-ZS systems. The theory of these systems grew from the pioneering work of Zakharov and Shabat, \cite{19} and Ablowitz, Kaup Newell and Segur, \cite{1}, \cite{2}. The salient members of this class are the nonlinear Schrödinger equation and the sine-Gordon equation in the light-cone coordinates. Being integrable, the AKNS-ZS equations are equivalent to the zero-curvature equations of the form

\[(L_q)_t - (A_q)_x + [L_q, A_q] = 0,\]

where $L_q(x, t; z)$ and $A_q(x, t; z)$ are two suitable $\mathfrak{sl}(2, \mathbb{C})$-valued functions of time $t$, of one spatial variable $x$ and of the spectral parameter $z$. The $L$-matrices of all the zero-curvature conditions, corresponding to the AKNS-ZS equations are of the form

\[L_q(x, t; z) = \begin{pmatrix} \frac{i z}{2} & q(x, t) \\ -\bar{q}(x, t) & -\frac{i z}{2} \end{pmatrix}.\]

A function $q(x, t)$ solves the AKNS-ZS equation at hand if and only if $L_q$, together with the suitably chosen $A_q$, satisfies the zero-curvature condition.

Let $l^2_\mathbb{Z}$ and $L^2[0, 2\pi]$ denote the Hilbert spaces of the square-integrable bi-infinite sequences and of the square integrable complex-valued functions on $[0, 2\pi]$, respectively. Our object of interest in this note will be the nonlinear operator

\[\mathcal{H} : L^2[0, 2\pi] \times l^2_\mathbb{Z} \rightarrow l^2_\mathbb{Z} \times l^2_\mathbb{Z},\]

given as follows. For $q(x) \in L^2[0, 2\pi]$, let $\Phi_q(x; z)$ be the solution of the linear initial problem

\[(\Phi_q)_x = L_q(x; z) \cdot \Phi_q; \quad \Phi_q(0; z) = I,\]
where $I$ denotes the identity $2 \times 2$ matrix. Let us denote

$$
\text{Hol}[q](z) = \Phi_q(x = 2\pi; z) = \begin{pmatrix}
D(z) & F(z) \\
E(z) & G(z)
\end{pmatrix} \in SL(2, \mathbb{C}).
$$

Let now $\sigma = \{\sigma_n\}_{n \in \mathbb{Z}}$ be a sequence in $l^2_{\mathbb{Z}}$. Then $H[q, \sigma]$ is given by

$$
H[q, \sigma](n) = \left( D(z_n), F(z_n) \right) = \left( D(n + \sigma_n), F(n + \sigma_n) \right).
$$

It can be shown that the image of $H$ is indeed in $l^2_{\mathbb{Z}} \times l^2_{\mathbb{Z}}$.

We shall call the operator $H$ the nonlinear Fourier transform. The name is motivated by the fact that the Fourier method for solving the linear partial differential equations can be viewed as the linearization of the inverse scattering method for solving the nonlinear integrable equations. Schematically, the inverse scattering method can be represented by the following diagram:

$$
q(x, 0) \xrightarrow{\text{Hol}} \text{Hol}[q(x, 0)](z) \xrightarrow{\text{time evolution}} \text{Hol}[q(x, t)](z) \xrightarrow{\text{Hol}^{-1}} q(x, t).
$$

The Fourier method for solving linear partial differential equations has the same structure:

$$
q(x, 0) \xrightarrow{F} F[q(x, 0)](z) \xrightarrow{\text{time evolution}} F[q(x, t)](z) \xrightarrow{F^{-1}} q(x, t).
$$

Here $F$ stands for the usual (linear) Fourier transform. It is not difficult to see that the entire holonomy $\text{Hol}[q](z)$ can be reproduced from its first row. Therefore, the operators $H$ and Hol are essentially equivalent, and it makes sense to call $H$ the nonlinear Fourier transform. A similar operator is called nonlinear Fourier transform by Tao and Thiele in their text [17], and also by Fokas (see, e.g. [7], [6], [8]). Various analytical properties of the operators similar to $H$ were studied by many authors. Important results are presented in the papers [9], [10], [4], [3], to name but a few.

The Fourier transform $F$ can be thought of as a decomposition of the initial condition $q(x, 0)$ into the elementary Fourier modes. Calculating the time evolution of an elementary mode is an easy task. To obtain the solution $q(x, t)$, one has to perform the inverse Fourier transform. The inverse scattering method relies upon the same strategy. However, the operators $H$ and $H^{-1}$ are nonlinear, as opposed to $F$ and $F^{-1}$. In particular, finding $H^{-1}$ is a difficult inverse problem. In the case of the differential equations with periodic boundary conditions, the calculation of the time evolution of $H[q(x, 0)](z)$ is also a difficult problem, not explicitly solvable for an arbitrary choice of the initial value $q(x, 0)$. The known solutions of the integrable nonlinear equations with periodic boundary conditions are constructed by means of the algebro-geometric tools, in particular by the use of spectral curves, their Jacobian tori and the associated theta functions. But this approach yields only the so-called finite degrees of freedom solutions. The linear analog of this class are the solutions, expressible with finite Fourier series. A more important limitation of the algebro-geometric approach is the fact that the solutions it provides are not solutions to given initial problems. Of course, one can determine the initial condition, corresponding to a given solution, but only after this solution was found. The initial condition does not figure in any way in the construction of such a solution.

One way to remedy this unfavorable situation might be to follow the steps of the Fourier method for the linear case more closely. For this purpose, one has to construct the nonlinear Fourier modes and then find the nonlinear superposition rules by means of which simpler solutions of a given nonlinear integrable equation yield more complex solutions. In this paper we shall only consider the first of the above two tasks.
The simplest sequence of the complex valued Fourier modes in the linear Fourier analysis \( \{u_d(x)\}_{d \in \mathbb{Z}} \) can be defined by the equation

\[
F[u_d](n) = a_d \, \delta_{n,d},
\]

where \( \delta_{n,d} \) denotes the Kronecker delta and \( a_d \) is a complex constant. The analogous definition of the modes \( \{q_d(x)\}_{d \in \mathbb{Z}} \) in the nonlinear Fourier analysis is then given by

\[
\mathcal{H}[q_d, \sigma](n) = \text{Hol}[q_d](z_n) = I + A_d \, \delta_{n,d},
\]

where \( A_d \) is a matrix in \( SL(2, \mathbb{C}) \). Here \( \sigma = \{\sigma_n\}_{n \in \mathbb{Z}} \) is a suitably chosen square-integrable sequence and \( z_n = n + \sigma_n \), as in (2). The reason for the introduction of \( \sigma \) will be explained in section 2 below. Let us note already now that \( \sigma \) depends on \( A_d \).

We shall see that the Fourier transform \( F \) is the linearization of \( \mathcal{H} \) at the origin of \( L^2[0, 2\pi] \times l^2_\mathbb{Z} \). Therefore it makes sense to try to construct the nonlinear Fourier modes by means of a perturbation scheme. The main result that we describe in this note is the following.

**Theorem 1** Up to the third order in the perturbational parameter \( s \), the nonlinear Fourier mode of frequency \( d \) is given by the formula

\[
q_d(x) = e^{idx} \left[ sA_1 + s^2A_2 + s^3 \left( A_3 + A_4 \left( \text{Li}_2(e^{ix}) + \text{Li}_2(e^{-ix}) \right) \right) \right] + \mathcal{O}(s^4),
\]

where \( A_1, A_2, A_3 \) and \( A_4 \) are suitable complex constants.

The first non-trivial term in the perturbation expansion is of order 3 and it is given by the salient dilogarithm function

\[
\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \log(1-x) \frac{dx}{x}.
\]

In the last decade or two, this classical special function, together with its relatives polylogarithms and multiple polylogarithms, gained much attention in mathematical and physical literature. These functions figure prominently in many important and diverse topics in number theory, geometry, topology, and mathematical physics.Appearances of dilogarithms include K-theory, volumes of hyperbolic three-manifolds and a variety of number theory topics. Probably the most well-known recent occurrences in mathematical physics were provided by W. Nahm and coworkers, [11], [12] who studied certain aspects of conformal field theories, and by L. D. Faddeev and R. M. Kashaev, [5] who introduced the quantum version of the dilogarithm. Their work was followed by a vast number of important contributions by many authors. A beautiful account on some of these topics is the paper [18] by Don Zagier.

Formula (3) above provides another addition to the wealth of interesting occurrences of the dilogarithm. The higher order terms in the perturbation expansion of \( q_d(x) \) are expressible by polylogarithms and multiple polylogarithms. In this paper, we shall confine ourselves to the third-order formula (3), for the sake of brevity.

### 2. Linear and nonlinear Fourier transforms

The essential part of the definition (2) is the linear initial problem

\[
(\Phi_q)_x = L_q(x; z) \cdot \Phi_q; \quad \Phi_q(0; z) = I,
\]

where \( \Phi_q \) is the Fourier transform of the sequence \( \{q_d(x)\}_{d \in \mathbb{Z}} \) and \( L_q(x; z) \) is the linear operator defined by

\[
L_q(x; z) = e^{idx} \left[ sA_1 + s^2A_2 + s^3 \left( A_3 + A_4 \left( \text{Li}_2(e^{ix}) + \text{Li}_2(e^{-ix}) \right) \right) \right] + \mathcal{O}(s^4).
\]
for $L_q(x; z)$ given by (1). The relation to the linear Fourier transform can be exhibited more clearly if we change the gauge of the above equation. Let the gauge transformation be given by the matrix

$$G(x : z) = \begin{pmatrix} e^{-ixz} & 0 \\ 0 & e^{ixz} \end{pmatrix}.$$ 

The matrix $\Phi_q^G(x; z) = G(x; z) \cdot \Phi_q(x; z)$ solves the problem

$$(\Phi_q^G)_x = L_q^G(x; z) \cdot \Phi_q^G; \quad \Phi_q^G(0; z) = I \quad (6)$$

for

$$L_q^G(x; z) = G_q G^{-1}(x; z) + (G \cdot L_q \cdot G^{-1})(x; z) = \begin{pmatrix} 0 & e^{-ixz} q(x) \\ -e^{ixz} q(x) & 0 \end{pmatrix}$$

if and only if $\Phi_q$ solves (5). Let us write the solution $\Phi_q^G$ of (6) in the form of Dyson’s expansion and evaluate it at $x = 2\pi$, and at the integer values $n \in \mathbb{Z}$ of the spectral parameter. We get

$$\text{Hol}^G[q](n) = I + \sum_{k=1}^{\infty} \int_{\triangle_k} L_q^G(x_1) \cdot L_q^G(x_2) \cdots L_q^G(x_k) \, d\vec{x},$$

where the integration domains $\triangle_k$ are the ordered simplices

$$\triangle_k = \{(x_1, x_2, \ldots, x_k); \; 2\pi \geq x_1 \geq x_2 \geq \cdots \geq x_k \geq 0\}.$$ 

For every $n \in \mathbb{Z}$, the matrix $\text{Hol}^G[q](n)$ is an element of $SU(2)$. Let us denote the first row of $\text{Hol}^G[q](n)$ by $(D_Z^G[q](n), F_Z^G[q](n))$. We have

$$D_Z^G[q](n) = 1 + \sum_{k=1}^{\infty} \int_{\triangle_{2k}} e^{-in(x_1-x_2+\cdots-x_{2k})} q(x_1)\overline{q(x_2)} \cdots \overline{q(x_{2k})} \, d\vec{x} \quad (7)$$

$$F_Z^G[q](n) = \sum_{k=1}^{\infty} \int_{\triangle_{2k-1}} e^{-in(x_1-x_2+\cdots+x_{2k-1})} q(x_1)\overline{q(x_2)} \cdots \overline{q(x_{2k-1})} \, d\vec{x} \quad (8)$$

It is proved in [15] that $D_Z$ and $F_Z$ are bounded operators from $L^2[0, 2\pi]$ to $L^2_Z$. Deriving these operators at the origin in the direction $v$ gives

$$D_0D_Z^G[v](n) = 0, \quad D_0F_Z^G[v](n) = \int_0^{2\pi} e^{-inx} q(x) \, dx = F[q](n).$$

So the classical Fourier transform is the linearization at the origin of the bounded operator $F_Z^G$. It is therefore tempting to adopt the operator $F_Z^G$ as the nonlinear Fourier transform. Indeed, this operator has many useful properties. In [15] it is proved that it is not only bounded but also analytic in the vicinity of the origin. In addition to that, it is locally invertible and the inverse $(F_Z^G)^{-1}[[\tau_n]]$ can be iteratively computed to any desired accuracy for any sequence $\{b_n\}$ close enough to the origin of $l^2_Z$.

An important drawback of the operator $F_Z^G$ is the fact that there is no obvious way of constructing the nonlinear Fourier modes by using $F_Z^G$ alone. The equations

$$D_Z[q](n) = 1 + a_d \delta_{n,d}, \quad F_Z[q](n) = b_d \delta_{n,d}, \quad n \in \mathbb{Z} \quad (9)$$

cannot be solved for $q$. This will become clear below. From

$$\Phi_q^G(x; z) = G(x; z) \cdot \Phi_q(x; z)$$

it follows that

$$D_Z[q](n) = (-1)^n D_Z^G[q](n), \quad F_Z[q](n) = (-1)^n F_Z^G[q](n),$$

thus the operators $F_Z^G$ and $F_Z$ suffer from the same problem.
3. Nonlinear Fourier transform and nonlinear Fourier modes

As we mentioned already in the introduction, the defining property of a nonlinear Fourier mode \( q_d \) of frequency \( d \) should be

\[
\text{Hol}[q_d](n) = I + A_d \delta_{n,d}, \quad \text{or equivalently} \quad \text{Hol}^d[q_d](n) = (-1)^n (I + A_d \delta_{n,d}).
\]  

(10)

The reason for this is the obvious stipulation that a Fourier mode must remain a Fourier mode under the time evolution, prescribed by the suitable integrable partial differential equation. The time evolution of the holonomy \( \text{Hol}[q](t; z) \) associated to any initial condition \( q(x) \) and for any value \( z \) of the spectral parameter is always given by the adjoint action of a suitable time-dependent matrix \( M(t; z) \). Explicitly, we have

\[
\text{Hol}[q](t; z) = M(t; z) \cdot \text{Hol}[q](0; z) \cdot M(t; z)^{-1}.
\]

Confining the spectral parameter to the integers, we would expect of a nonlinear Fourier mode \( q_d(x) \) to satisfy the condition

\[
\text{Hol}[q_d](t; n) = M(t; n) \cdot \text{Hol}[q_d](0; n) \cdot M(t; n)^{-1} = I + \tilde{A}_d(t) \delta_{n,d}
\]

for some matrix-valued and time-dependent function \( \tilde{A}_d(t) \). This condition would certainly be satisfied, if (10) held for \( q_d \). However, (10) is not satisfied by any function, because, as we noted above, the system of equations (9) is not solvable. If \( q = F_Z^{-1}(a_d \delta_{n,d}) \), then \( D_Z[q](n) \) is different from 1 for many \( n \), not only for \( n = d \). Thus \( F_Z \) is not a good candidate for the nonlinear Fourier transform if we want to work with the nonlinear Fourier modes modes.

The problem can be remedied by allowing the values of the spectral parameter to vary. So, instead of evaluating at \( n \), we evaluate at \( z_n = n + \sigma_n \) for some complex correction term \( \sigma_n \). It is not difficult to see that

\[
\lim_{n \to \infty} \text{Hol}[q](n) = I,
\]

for every function \( q \in L^2[0, 2\pi] \). Therefore, it makes sense to stipulate that the sequences \( \{\sigma_n\}_{n \in \mathbb{Z}} \) of corrections are square integrable. This brings us to the definition (2)

\[
\mathcal{H}[q, \sigma](n) = \left( D[q](z_n), F[q](z_n) \right) = \left( D[q](n + \sigma_n), F[q](n + \sigma_n) \right)
\]

of the nonlinear Fourier transform.

Without much trouble the following proposition can be proved.

**Proposition 1** For any function \( q \in L^2[0, 2\pi] \) and for any value of the spectral parameter, the holonomy \( \text{Hol}[q](z) \) is determined uniquely by its first row \( (D[q](z), F[q](z)) \), provided the holonomy is close enough to the identity matrix.

The essential result concerning \( \mathcal{H} \) is the following theorem.

**Theorem 2** The nonlinear operator

\[
\mathcal{H} : L^2[0, 2\pi] \times l^2_\mathbb{Z} \longrightarrow l^2_\mathbb{Z} \times l^2_\mathbb{Z}
\]

is locally bounded and invertible in the vicinity of the origin. Moreover, \( \mathcal{H} \) and its local inverse are real analytic operators.

The proof is given in [16]. Important ingredients of this proof are the boundedness and analyticity of the operator \( F_Z \). This is proved in [15]. Strictly speaking, we shall not need this result to prove theorem 1, therefore we omit the proof here. It is important to note, however, that in the above theorem the space \( L^2[0, 2\pi] \) cannot be replaced by \( L^2(\mathbb{R}) \). Muscalu,
Tao and Thiele have shown in [10] that the operator $F$, defined on $L^2(\mathbb{R})$, is not even three times differentiable, let alone analytic. However, it is not too difficult to see that replacing $L^2(\mathbb{R})$ by the Schwartz class resolves this issue.

The above theorem and proposition show that the nonlinear modes can be constructed by means of solving the equations of the form

$$\mathcal{H}[q, \sigma](n) = (1 + a_d \delta_{n,d}, \ b_d \delta_{n,d}),$$

(11)

for $q$ and $\sigma$. We are really only interested in $q$ but we cannot get it without also getting $\sigma$.

We shall need the following specific description of $\mathcal{H}$, also important in the proof of the above theorem.

**Lemma 1** Let $q \in L^2[0, 2\pi]$ and $\{\sigma_n\}_{n \in \mathbb{Z}} \in l^2_{\mathbb{Z}}$. Then

$$D[q, \sigma](n) = (-1)^n e^{i\pi \sigma_n} + \sum_{k=-\infty}^{\infty} P_{n,k}[\sigma] \cdot D_z[q](k) \cdot F[q, \sigma](n) = \sum_{k=-\infty}^{\infty} P_{n,k}[\sigma] \cdot F_z[q](k).$$

Here $P_{n,k}[\sigma]$ is an infinite matrix, given by

$$P_{n,k}[\sigma] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n-k)x} e^{-i\sigma_n x} \, dx.$$

**Proof:** We shall sketch the proof only for $D$. The proof for $F$ is the same. From the gauge relation between $D^G$ and $D$, and from Dyson’s expansion we get

$$D[q](z) = e^{i\pi z} \left(1 + \sum_{k=1}^{\infty} \int_{\triangle_{2k}} e^{-iz(x_1-x_2-\ldots-x_{2k})} q(x_1)q(x_2)\ldots q(x_{2k}) \, d\vec{x}\right).$$

In every term of the above sum we can introduce the new variable $u = x_1 - x_2 + \ldots - x_{2k}$ and express the integrals in the form

$$\int_{\triangle_{2k}} e^{-iz(x_1-x_2-\ldots-x_{2k})} q(x_1)q(x_2)\ldots q(x_{2k}) \, d\vec{x} = \int_0^{2\pi} e^{-izu} T_{2k}(u) \, du,$$

where

$$T_{2k}(u) = \int_{D(u)} q(x_1)q(x_2)\ldots q(x_{2k}) \, d\vec{x}$$

and

$$D(u) = \{(x_1, x_2, \ldots, x_{2k}) \in \triangle_{2k}; x_1 - x_2 + \ldots - x_{2k} = u\}.$$

We can therefore write

$$D[q](z) = e^{i\pi z} \left(1 + \int_0^{2\pi} e^{-izu} \tilde{T}(u) \, du\right) = e^{i\pi z} + \int_{-\pi}^{\pi} e^{-izv} \tilde{T}(v + \pi) \, dv.$$

(12)

Above we denoted $\tilde{T}(u) = \sum_{k=1}^{\infty} T_{2k}(u)$ and introduced the new integration variable $v = u - \pi$.

For every $n$, we can expand the function $e^{-iz_{n}x}$, $z_{n} = n + \sigma_{n}$ into the Fourier series:

$$e^{-iz_{n}x} = \sum_{k=-\infty}^{\infty} P_{n,k}[\sigma] e^{-ik}, \quad P_{n,k}[\sigma] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iz_{n}x} e^{ikx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n-k)x} e^{-i\sigma_{n}x} \, dx.$$
Inserting this expansion into (12) and a short calculation give
\[
D[q, \sigma](n) = D[q](z_n) = (-1)^n e^{i\pi \sigma_n} + \sum_{k=-\infty}^{\infty} P_{n,k}[\sigma] \cdot D_\mathbb{Z}[q](k)
\]
as claimed. Above, we used
\[
\int_{-\pi}^{\pi} e^{-ikv} \tilde{T}(v + \pi) \, dv = (-1)^k \int_{0}^{2\pi} e^{-iku} \tilde{T}(u) = D_\mathbb{Z}[q](k).
\]

4. One-parameter families of Fourier modes

Fourier modes appear in one-parameter families. In the case of the linear Fourier transform, this parameter is related to the energy (or the amplitude) of the mode. Concretely, if we set the values of the spectral parameter to be the integers, the complex valued linear Fourier modes are
\[
u_d(x) = a_d e^{idx}; \quad d \in \mathbb{Z}, \quad a_d \in \mathbb{C}.
\]

In cases of linear integrable partial differential equations, the decomposition of solutions into the linear Fourier modes actually provides the action-angle variables of the system (see [13] and [14] for the case of the Klein-Gordon equation). Roughly speaking, the modulus of \(a_d\) is an action and the argument of \(a_d\) is the associated angle variable. In the nonlinear systems the situation is more involved. Clearly, the dependence of nonlinear Fourier modes of the action and angle variables will not be linear. Still, the linear case suggests that it should make sense to construct the one-parameter families of the nonlinear Fourier modes and that the parameter can be thought of as the energy (or some function of the energy) of the mode. Our strategy will therefore be the following. Let \(s \mapsto (a_d(s), b_d(s))\) be a real analytic curve in \(\mathbb{C}^2\), such that \((a_d(0), b_d(0)) = (0, 0)\). We are searching for the curve \(s \mapsto e\ell_d(s) = (q(x; s), \{\sigma_n(s)\}) = \mathcal{H}^{-1}(a_d(s), b_d(s))\) in \(L^2[0, 2\pi] \times \mathbb{Z}_2\). By theorem 2, a unique such curve exists for every \((a_d(s), b_d(s))\) lying close enough to the identity. In addition, the curve \(s \mapsto e\ell_d(s)\) is analytic. It can therefore be expressed in the form of a convergent power series
\[
\mathcal{H}^{-1}(a_d(s), b_d(s)) = (q(x; s), \{\sigma_n(s)\}_{n \in \mathbb{Z}}) = \left(\sum_{m=1}^{\infty} \frac{s^m}{m!} q^{(m)}(x), \sum_{m=1}^{\infty} \frac{s^m}{m!} \{\sigma_n^{(m)}\}_{n \in \mathbb{Z}}\right).
\]

Indeed, formula (7) shows that also the curves \(s \mapsto q(x; s)\) and \(s \mapsto \{\sigma_n(s)\}\) begin at the origin, so \(q^{(0)}(x) = 0\) and \(\{\sigma_n^{(0)}\} = 0\).

**Theorem 3** Let \(s \mapsto (a_d(s), b_d(s))\) in \(L^2 \times \mathbb{Z}_2\) be an analytic curve as above. Then the terms \(q^{(m)}(x)\) and \(\sigma_n^{(m)}\) given by (13) can be calculated iteratively from the equations
\[
\frac{d^{m}}{ds^{m}}|_{s=0} \mathcal{H}(q(x; s), \{\sigma_n(s)\}_{n \in \mathbb{Z}}) = (a_d^{(m)}, b_d^{(m)}), \quad m = 1, 2, \ldots.
\]

Above \(a_d^{(m)}, b_d^{(m)}\) denote the terms in the Taylor expansion
\[
(a_d(s), b_d(s)) = \left(\sum_{m=1}^{\infty} \frac{s^m}{m!} a_d^{(m)}, \sum_{m=1}^{\infty} \frac{s^m}{m!} b_d^{(m)}\right).
\]
The above theorem is proved in [16]. The general formulae for \( q^{(m)} \) and \( \sigma_n^{(m)} \) are quite involved and we shall not present them here.

We shall now prove the main result of this paper.

**Proof of theorem 1:** In order to prove our theorem 1, it is enough to solve explicitly equations \( (14) \) for \( m = 1, 2, 3 \). Solving these equations illustrates nicely the plausibility of theorem 3, and also shows that the formulae for the general \( q^{(m)} \), \( \sigma_n^{(m)} \) demand much more space and work.

Lemma 1 tells us that for every \( m \in \mathbb{N} \) the equation \( (14) \) is equivalent to the system

\[
\begin{align*}
(-1)^n \frac{d^n}{ds^n} |_{s=0} e^{i\pi \sigma_n(s)} &+ \sum_{k=-\infty}^{\infty} \sum_{l=0}^{m} \binom{m}{l} P_{n,k}^{(l)}[\sigma] \cdot D_Z^{(m-l)}[q](k) = a_d^{(m)} \delta_{n,d} \quad (15) \\
\sum_{k=-\infty}^{\infty} \sum_{l=0}^{m} \binom{m}{l} P_{n,k}^{(l)}[\sigma] \cdot F_Z^{(m-l)}[q](k) & = b_d^{(m)} \delta_{n,d}, \quad (16)
\end{align*}
\]

where

\[
\begin{align*}
P_{n,k}^{(l)}[\sigma] & = \frac{d^l}{ds^l} |_{s=0} P_{n,k}[\sigma(s)], & D_Z^{(l)}[q] & = \frac{d^l}{ds^l} |_{s=0} D_Z[q(x; s)], & F_Z^{(l)}[q] & = \frac{d^l}{ds^l} |_{s=0} F_Z[q(x; s)].
\end{align*}
\]

From our assumption that the analytic curve \( s \mapsto (a_q(s), b_d(s)) \) starts at the origin, it follows that

\[
D_Z^{(0)}[\sigma] = D_Z^{(1)}[\sigma] = 0, \quad F_Z^{(0)}[q] = 0.
\]

Let us denote \( \tilde{\sigma}_n^{(l)} = i\pi \sigma_n^{(l)} \). We shall also use the following expressions which are easily computable from \( (7) \) and \( (8) \):

\[
F_Z^{(1)}[q](n) = (-1)^n F[q^{(1)}](n), \quad F_Z^{(2)}[q](n) = (-1)^n F[q^{(2)}](n)
\]

and

\[
F_Z^{(3)}[q](n) = (-1)^n F[q^{(3)}](n) + (-1)^{(n+1)} \int_{\Delta_3} e^{-i(n_1 x_1 - x_2 + x_3)} q^{(1)}(x_1) \overline{q^{(1)}(x_2)} q^{(1)}(x_3) \, d\vec{x}.
\]

As before, \( F \) denotes the linear Fourier transform. On the diagonal part of the holonomy we shall need only the expression

\[
D_Z^{(2)}[n] = (-1)^n \int_{\Delta_2} e^{-i(n_1 x_1 - x_2)} q^{(1)}(x_1) \overline{q^{(1)}(x_2)} \, d\vec{x}.
\]

Below, we shall solve the first five of the equations \( (15) \) and \( (16) \), two at a time.

\( s^1 \):

\[
\begin{align*}
\tilde{\sigma}_n^{(1)} & = a_d^{(1)} (-1)^n \delta_{n,d} \\
F[q^{(1)}](n) & = b_d^{(1)} (-1)^n \delta_{n,d}.
\end{align*}
\]

The solution of the above is

\[
\begin{align*}
\sigma_n^{(1)} & = (-1)^{d+1} \pi a_d^{(1)} \delta_{n,d} = \tilde{\sigma}_n^{(1)} \delta_{n,d} \\
q^{(1)}(x) & = (-1)^d b_d^{(1)} e^{idx} = \tilde{\sigma}_n^{(1)} e^{idx}.
\end{align*}
\]
where calculation shows that the sequence \( P_\alpha \) where \( \alpha \) is the matrix

\[
F[q^{(2)}](n) + 2(P^{(1)}[\sigma] \cdot F[q^{(1)}])(n) = \tilde{b}^{(2)}_d \delta_{n,d}.
\]

The solution is

\[
\sigma^{(2)} = \frac{\alpha_2}{n-d}(1 - \delta_{n,d}) + \beta_2 \delta_{n,d}
\]

\[
q^{(2)}(x) = \gamma_2 e^{idx},
\]

where \( \alpha_2, \beta_2 \) and \( \gamma_2 \) are suitably chosen complex constants.

\[\text{s}^3:\]

\[
F_2[q^{(3)}, q^{(1)}] + 3P^{(1)}[\sigma^{(1)}] \cdot F[q^{(2)}] + 3P^{(2)}[\sigma^{(1)}, \sigma^{(2)}] \cdot F[q^{(1)}] = b^{(3)}_d. \tag{17}
\]

Calculation shows that the sequence \( P^{(1)}[\sigma^{(1)}]F[q^{(2)}] \) is equal to zero. For \( P^{(2)} \) we get

\[
P^{(2)}[\sigma] = -i\sigma^{(1)}m^{(1)}_{n,k} + (-i\sigma^{(1)}_n)^2 m^{(2)}_{n,k},
\]

where

\[
m^{(1)}_{n,k} = \frac{1}{2\pi} \int_{-\pi}^\pi x e^{i(n-k)x} \, dx = \frac{i(-1)^{n+k}}{k-n}, \quad m^{(2)}_{n,k} = \frac{1}{2\pi} \int_{-\pi}^\pi x^2 e^{i(n-k)x} \, dx = \frac{2(-1)^{n+k}}{(n-k)^2}.
\]

Because \( F[q^{(1)}](n) \) is different from zero only for \( n = d \), we are interested only in \( d \)-th row of the matrix \( P^{(2)} \). Collecting everything and inserting into (17) gives

\[
F[q^{(3)}](n) = \frac{\alpha_3}{(n-d)^2}(1 - \delta_{n,d}) + \beta_3 \delta_{n,d},
\]

for a suitable pair of constants \( \alpha_3 \) and \( \beta_3 \). By applying the inverse (linear) Fourier transform \( F^{-1} \), we finally get

\[
q^{(3)}(x) = e^{idx} \left( \beta_3 + \alpha_3 \left( \text{Li}_2(e^{ix}) + \text{Li}_2(e^{-ix}) \right) \right).
\]

Collecting the \( q^{(1)}, q^{(2)} \) and \( q^{(3)} \) together yields the one parameter family of the elementary modes \( q_d(x; s) \) of frequency \( d \) up to the third power of the perturbation parameter

\[
q_d(x; s) = e^{idx} \left[ sA_1 + s^2 A_2 + s^3 \left( A_3 + A_4 \left( \text{Li}_2(e^{ix}) + \text{Li}_2(e^{-ix}) \right) \right) \right] + O(s^4).
\]

The above proves our theorem 1.

We conclude with a brief remark on the choice of the parameter \( s \) and on the role of the constants \( A_i \).

**Remark 1** The form of the expression (3) does not change if we replace the parameter \( s \) by another \( t = f(s) \), where \( f \) is an analytic function, and \( f(0) = 0 \). Reparametrization affects only the choice of the constants \( A_1, \ldots, A_4 \). The freedom that we have in choosing the constants \( A_i \) shows that there exists a wealth of different nonlinear Fourier modes. This makes sense, since the \( L \)-matrix (1) appears in the zero-curvature conditions for many different integrable equations. However, the first nontrivial power in the perturbation expansion is the third power and the first nontrivial term is given by the dilogarithm in all the cases.
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