Holographic Superconductors

En-Jui Chang,¹, Chia-Jui Chou,¹ and Yi Yang¹

¹Department of Electrophysics, National Chiao Tung University, Hsinchu, ROC

Abstract

We study a non-minimal holographic superconductor model in both non-backreaction and full-backreaction cases using an analytic method. We calculate the condensate of the dilaton and the critical temperature of the phase transition. We also study the properties of the electric conductivity in various parameters.

*Electronic address: phyenjui@gmail.com
†Electronic address: agoodmanjerry.ep02g@nctu.edu.tw
‡Electronic address: yiyang@mail.nctu.edu.tw
I. INTRODUCTION

The perturbation technique is a standard approach in the conventional quantum field theories. However, it only works for the weakly coupled systems. It is well known that the traditional BCS theory [1] which describes the weakly coupled electron-phonon interaction is very successfully. However, the high critical temperature $T_c$ superconductivity involves the strong couplings beyond the BCS theory.

The AdS/CFT duality [2-6] is a powerful tool to study a $d$-dimensional strongly coupled conformal field theories (CFT) by studying its $d + 1$-dimensional dual gravitational theory in an Anti de Sitter (AdS) space, and vice versa. It is interesting to study the strongly coupled superconductivity via the AdS/CFT duality, and this kind of the superconductors...
models are called the holographic superconductors (HSC) [7–13].

In the last decade, the HSC has been studied in both the non-backreaction case [8, 11, 14–45] and the backreaction case [9, 12, 46–60]. Most of these studies are based on the numerical methods [45, 47–51, 54, 61–75], while some others used the analytic methods [52, 53, 55, 57, 58, 76–87]. In [77–87], an approximate analytic method, matching method, has been developed to study the holographic correspondence in the non-backreaction case.

In this work, we will study a non-minimal holographic superconductor model with two parameters using the matching method in both the non-backreaction and the full-backreaction cases. By solving the system analytically, we study the condensate of the dilaton which acts as a order parameter of a second order phase transition. The critical temperatures of phase transition are calculated. We find that the condensate of dilaton in the non-backreaction case is not consistent with numeric results, but the one in the full-backreaction case produces the correct results. In addition, we study the electric conductivity using an analytic truncating method.

This paper is organized as follow: In section II, we introduce the EMD system and describe the matching method to solve the system. In section III, we study the condensate of dilaton in the non-backreaction case. In section IV, we study both the condensate and the electric conductivity in the full backreaction case. We summarize our results in section V.

II. EINSTEIN-MAXWELL-SCALAR SYSTEM

In this paper, we consider a 4-dimensional Einstein-Maxwell-Scalar (EMS) system, which includes a gravity field $g_{\mu\nu}$, a Maxwell field $A_\mu$ and a charged complex scalar field $\Phi = \phi e^{i\theta}$. After fixing the St"uckelberg field $\theta = 0$, the action can be expressed as,

$$S = \frac{1}{16\pi G_4} \int d\tau d^3x \sqrt{-g} \left[ R - \frac{f(\phi)}{4} F^{\mu\nu}F_{\mu\nu} + \frac{6}{\ell^2} U(\phi) - \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi) - \frac{J(\phi)}{2} A^\mu A_\mu \right],$$

(2.1)

where $f(\phi)$ is the gauge kinetic function which describes the interaction between the gauge field and the scalar field, $U(\phi)$ is the scalar potential and $J(\phi)$ is the extended St"uckelberg
function that preserve the gauge invariant. In this paper, we choose

\[ f(\phi) = 1 + \frac{\alpha^2}{2} \phi^2, \]  

\[ U(\phi) = 1 - \frac{\ell^2}{12} m^2 \phi^2, \]  

\[ J(\phi) = q^2 \phi^2. \]

The similar system has been studied numerically in [48]. There are four free parameters \((\alpha, q, m, \ell)\) in the action. We will analytically study the system by using the matching method and compare our results with the ones in the numerical method.

The equations of motion are obtained by varying the action with the different fields,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left[ R - \frac{f(\phi)}{4} F^{\rho\sigma} F_{\rho\sigma} + \frac{6}{\ell^2} U(\phi) - \frac{1}{2} (\partial^\rho \phi \partial_\rho \phi) - \frac{J(\phi)}{2} A^\rho A_\rho \right] \]

\[ - \frac{f(\phi)}{2} F_{\mu\rho} F^\rho_{\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{J(\phi)}{2} A_\mu A_\nu = 0, \]  

\[ \nabla_\mu [f(\phi) F^{\mu\nu}] - J(\phi) A^\nu = 0, \]  

\[ \nabla^2 \phi - \frac{1}{4} \frac{\partial f(\phi)}{\partial \phi} F^{\mu\nu} F_{\mu\nu} + \frac{6}{\ell^2} \frac{\partial U(\phi)}{\partial \phi} - \frac{1}{2} \frac{\partial J(\phi)}{\partial \phi} A^\mu A_\mu = 0. \]

Since we are going to study the thermodynamic properties in the HSC system at the finite temperature, we consider a black hole background that asymptotic to the AdS space. Without loss of generality, we consider the following ansatz of an isotropic black hole,

\[ ds^2 = -g(r) e^{-\chi(r)} dt^2 + \frac{dr^2}{g(r)} + r^2 (dx^2 + dy^2), \]  

\[ A = A_t(r) dt, \]  

\[ \phi = \phi(r), \]

where \(r\) is the holographic radius which corresponds to the energy scale in the dual field theory. The Hawking temperature of the black hole can be calculated as,

\[ T = \left. \frac{1}{4\pi} g(r)^{'} e^{-\chi(r)} \right|_{r=r_H}, \]

where the black hole horizon \(r_H\) is defined by \(g(r_H) = 0\).

Plugging the ansatz (2.4) into the equations of motion (2.3) leads to the following equa-
tions of motion for the background fields:

\[
\chi' + \frac{r}{2} \phi'^2 + \frac{r J}{2 g^2} e^\chi A_t^2 = 0, \tag{2.6a}
\]

\[
\frac{g'}{rg} + \frac{1}{4 r^2} + \frac{1}{4} \phi'^2 - \frac{3}{\ell^2 g} U(\phi) + \frac{f}{4 g} e^\chi A_t^2 + \frac{J}{4 g^2} e^\chi A_t^2 = 0, \tag{2.6b}
\]

\[
A_t'' + \left( \frac{f'}{f} + \frac{2}{r} + \frac{\chi'}{2} \right) A_t' - \frac{J}{g f} A_t = 0, \tag{2.6c}
\]

\[
\phi'' + \left( \frac{g'}{g} + \frac{2}{r} - \frac{\chi'}{2} \right) \phi' + \left( \frac{6 U'}{\ell^2 g} + \frac{e^\chi A_t^2 J}{2 g} + \frac{e^\chi A_t^2 J'}{2 g^2} \right) \frac{1}{\phi'} = 0, \tag{2.6d}
\]

where the prime represents the derivative with respect to \( r \).

It is easy to verify that the RN-AdS black hole is a simple solution of the above equations of motion (2.6),

\[
g(r) = \frac{r^2}{\ell^2} - \frac{1}{r} \left( \frac{r_H^3}{\ell^2} + \frac{\rho^2}{4 r_H^2} \right) + \frac{\rho^2}{4 r^2}, \tag{2.7a}
\]

\[A_t = \rho \left( \frac{1}{r_H} - \frac{1}{r} \right), \tag{2.7b}\]

\[\chi(r) = \phi(r) = 0. \tag{2.7c}\]

In the following, we will find an approximate analytic solution of the equations of motion (2.6) by the matching method. (If we have the exact solution, we can get the series expansion to all orders at any point in principle. These series expansion are identical at any points. Obviously, there is no closed form to the solutions of the equations of motion (2.6).) Instead of solving the equations of motion (2.6) exactly, we solve for the background fields near the horizon at \( r = r_H \) and near the boundary at \( r = \infty \), respectively. Given the boundary conditions at either the horizon or the boundary, the background fields can be determined order by order. We then match the two solutions near the horizon and the boundary at some intermediate points \( r_m \in (r_H, \infty) \) by requiring the continuous or even the smooth conditions for the background fields. Obviously, the solutions by matching are only the good approximation at the horizon and the boundary, but not the exact solutions of the equations of motion for every \( r \). Nevertheless, since the core of the holographic correspondence is to relate the filed theory on the boundary and the near horizon geometry of the bulk, we believe that the detail behavior of the solution at the intermediate part does not affect the physics too much. We thus can trust the physics from the results of this approximate matching solution at least qualitatively.
It is convenient to make a coordinate transformation from $r$ to $z = r_H/r \in (0, 1)$ with the boundary at $z = 0$ and the black hole horizon at $z = 1$. In this work, to be concrete, we choose the matching point as $z_m = 3/4$. With the new coordinate $z$, the equations of motion for the background fields become,

$$\chi' - \frac{z}{2} \phi' \phi - \frac{r_H^2 B^2}{2z^3 g^2} = 0,$$  \hspace{1cm} (2.8a)

$$g' - \left(\frac{1}{z} + \frac{\chi'}{2}\right) g - \frac{zf}{4} \left( \frac{B'}{\chi/2} - \frac{\chi}{2} B \right)^2 + \frac{3r_H^2}{z^3} U = 0,$$  \hspace{1cm} (2.8b)

$$B'' + \left(\frac{f'}{f} - \frac{\chi'}{2}\right) B' - \left(\frac{\chi''}{2} + \frac{\chi' f'}{2 f} + \frac{1}{z^4} \frac{r_H^2 J}{g f}\right) B = 0,$$  \hspace{1cm} (2.8c)

$$\phi'' + \left(\frac{g'}{g} - \frac{\chi'}{2}\right) \phi' + \left[ \frac{(B' - \frac{\chi}{2} B)^2 f'}{2g} + \frac{r_H^2 B^2 J'}{2z^4 g^2} + \frac{6r_H^2 U'}{z^4 \ell^2 g} \right] \frac{1}{\phi'} = 0.$$  \hspace{1cm} (2.8d)

where the prime represents the derivative with respect to $z$ now, and we have defined a new field $B^2 = e^{\chi} A_t^2$ for convenience.

The series expansions of the background fields near the horizon can be written as:

$$\chi (z) = \sum_{k=0}^{\infty} \chi_k (1 - z)^k,$$  \hspace{1cm} (2.9a)

$$g (z) = \frac{r_H^2}{\ell^2 z^2} \left[ 1 - z^3 + \sum_{k=0}^{\infty} g_k (1 - z)^k \right],$$  \hspace{1cm} (2.9b)

$$B (z) = \sum_{k=0}^{\infty} B_k (1 - z)^k,$$  \hspace{1cm} (2.9c)

$$\phi (z) = \sum_{k=0}^{\infty} \phi_k (1 - z)^k,$$  \hspace{1cm} (2.9d)

In the above expansions, because the differential equations for $\chi$ and $g$ are first order and the differential equations for $B$ and $\phi$ are second order, there are six coefficients ($\chi_0, g_0, B_0, B_1, \phi_0, \phi_1$) can be chosen as the boundary conditions at the horizon $z = 1$, while the other higher ordered coefficients can be obtained from these six coefficients by solving the equations of motion order by order.

Specially, the boundary condition to define the horizon $g(1) = 0$ leads $g_0 = 0$. Furthermore, the regularity boundary conditions at the horizon require,

$$B_0 = 0,$$  \hspace{1cm} (2.10a)

$$\phi_1 = -\frac{1}{g'} \left( \frac{\partial U}{\partial \phi} + \frac{B_1}{2} \frac{\partial f}{\partial \phi} \right) \bigg|_{z=1},$$  \hspace{1cm} (2.10b)
In this work, for the least approximation, we only take the expansions at the horizon up to the order \( k = 1 \) for the fields \( \chi \) and \( g \), and to the order \( k = 2 \) for the fields \( B \) and \( \phi \). The series expansion of the background fields near the horizon become,

\[
\begin{align*}
\chi(z) & = \chi_0 + \chi_1 (1 - z), \\
g(z) & = \frac{r_H^2}{\ell^2 z^2} [1 - z^3 + g_1 (1 - z)], \\
B(z) & = B_1 (1 - z) + B_2 (1 - z)^2, \\
\phi(z) & = \phi_0 + \phi_1 (1 - z) + \phi_2 (1 - z)^2,
\end{align*}
\]

(2.11)

where \( \phi_1 \) is given in Eq. (2.10b) and only three coefficients \((\chi_0, B_1, \phi_0, \ldots)\) left to be given.

At the boundary \( z = 0 \), the asymptotic behavior of the fields are,

\[
\begin{align*}
\chi(z) & \sim \chi^{(0)} + \frac{\Delta^+}{4\ell^2 r_H^2} \Delta^+ z^{2\Delta^+}, \\
g(z) & \sim \begin{cases} \\
\frac{r_H^2}{\ell^2 z} - \frac{2M}{r_H^2} z + \frac{\Delta^+}{4\ell^2 r_H^2} \frac{D^+}{2\Delta^+ - 2} z^{2\Delta^+ - 2} & \frac{1}{2} < \Delta^+ \leq \frac{3}{2}, \\
\frac{r_H^2}{\ell^2 z} - \frac{2M}{r_H^2} z & \frac{3}{2} < \Delta^+,
\end{cases} \\
B(z) & \sim \mu - \frac{\rho}{r_H^2} z, \\
\phi(z) & \sim \frac{D^-}{r_H^2} z^{\Delta^-} + \frac{D^+}{r_H^2} z^{\Delta^+}
\end{align*}
\]

(2.12)

where

\[
\Delta^\pm = \frac{3 \pm \sqrt{9 + 4m^2\ell^2}}{2}.
\]

(2.13)

are the conformal dimensions for the scalar fields with mass \( m \).

Similarly, in the above asymptotic forms, there are six coefficients \((\chi^{(0)}, M, \mu, \rho, D^-_\Delta, D^+_\Delta)\) can be chosen as the boundary conditions at the boundary \( z = 0 \). The asymptotic coefficients \( \mu \) and \( \rho \) in \( B(z) \) are chemical potential and charge density. The asymptotic coefficients \( D^-_\Delta \) and \( D^+_\Delta \) in \( \phi(z) \) represent the condensates.

To satisfy the conformal symmetry, we should fix either \( D^-_\Delta = 0 \) or \( D^+_\Delta = 0 \) to obtain a stable solution. Both boundary conditions are allowed and we choose \( D^-_\Delta = 0 \) for convention in this work. Setting \( m^2\ell^2 = -2 \), we have \( \Delta^- = 1 \) and \( \Delta^+ = 2 \). With our choices, the
asymptotic behavior of the background fields at the boundary \( z = 0 \) becomes,

\[
\chi(z) \sim \chi(0) + \frac{1}{2} \frac{D^2}{r^4_H} z^4, \tag{2.14a}
\]

\[
g(z) \sim \frac{r_H^2}{\ell^2 z^2} - \frac{2M}{r_H z}, \tag{2.14b}
\]

\[
B(z) \sim \mu - \frac{\rho}{r_H} z, \tag{2.14c}
\]

\[
\phi(z) \sim \frac{D_2}{r_H^2} z^2, \tag{2.14d}
\]

where only five coefficients \( (\chi(0), M, \mu, \rho, D_2) \) left to be given.

III. NON-BACKREACTION

We first consider the case of non-backreaction, i.e. we treat the field \( \phi \) and \( B \) as the probe fields which do not affect the spacetime background. By setting \( \phi = B = 0 \), the equations of motion (2.8) for the background fields reduces to

\[
\chi' = 0, \tag{3.1a}
\]

\[
g' - \frac{g}{z} + \frac{3r_H^2}{z^3 \ell^2} = 0, \tag{3.1b}
\]

which admits the simple AdS-Schwarzschild black hole solution,

\[
\chi(z) = 0, \tag{3.2a}
\]

\[
g(z) = \frac{r_H^2}{\ell^2 z^2} (1 - z^3), \tag{3.2b}
\]

with the black hole temperature \( T = 3r_H^2 / \ell^2 \).

Next, we are going to solve the equations of motion for the probe fields \( B \) and \( \phi \),

\[
B'' + \frac{f'(z)}{f} B' - \frac{1}{z^4} \frac{r_H^2 J}{g^3} B = 0, \tag{3.3a}
\]

\[
\phi'' + \frac{g'}{g} \phi' + \left[ \frac{B'^2}{2g} + \frac{r_H^2 B^2 J'}{2z^4 g^2} + \frac{6r_H^2 U''}{z^4 \ell^2 g} \right] \frac{1}{\phi'} = 0. \tag{3.3b}
\]

There are two scaling symmetries in the equations of motion (3.3) with the form \( C \rightarrow s^{-n_C} C \), where \( C \) is one of the variables \( (r, t, \vec{x}, g, B, \ell, q) \), \( s \) is a scaling factor and \( n_C \) is the scaling dimension for \( C \). The scaling dimensions for the two scaling symmetries are listed in Table 1.
TABLE I: The two scaling symmetries of the equations of motion (3.3) in the case of non-backreaction.

|   | Sym. | $n_r$ | $n_t$ | $n_\hat{r}$ | $n_q$ | $n_B$ | $n_\ell$ | $n_q$ |
|---|------|-------|-------|-------------|-------|-------|---------|-------|
| I |      | 1     | -1    | -1          | 2     | 1     | 0       | 0     |
| II|      | 1     | 0     | 1           | 0     | 0     | 1       | -1    |

We can use the above two scaling symmetries to set $\ell = 1$ and $\rho = \text{constant}$. In the case of non-backreaction, the series expansions of the fields $B$ and $\phi$ near the horizon in Eqs.(2.11) becomes,

$$B(z) = a (1 - z) + B_2 (1 - z)^2,$$

$$\phi(z) = b + \phi_1 (1 - z) + \phi_2 (1 - z)^2,$$

where we have renamed $B_1 = a$ and $\phi_0 = b$ in (2.11). $a$ and $b$ will be imposed as the boundary values at the horizon $z = 1$. Plug the series expansions (3.4) into the equations of motion (3.3), $B_2, \phi_1$ and $\phi_2$ can be solved in terms of the boundary values $a$ and $b$ order by order,

$$B_2 = \frac{1}{6} \left( \frac{(4\alpha^2 + 2q^2) r_H^2}{r_H^2 (\alpha^2 b^2 + 2)} \right) b^2 a,$$

$$\phi_1 = \frac{1}{6} \frac{b (\alpha^2 a^2 + 4r_H^2)}{r_H^2},$$

$$\phi_2 = \frac{1}{48} \frac{b}{r_H^2 (\alpha^2 b^2 + 2)} \left\{ \left( \frac{32\alpha^2 b^2}{3} + \frac{64}{3} \right) r_H^4 + 4a^2 \left[ \frac{1}{3} \alpha^4 b^2 + \left( b^2 q^2 - \frac{10}{3} \right) \right. \alpha^2 + \frac{2}{3} q^2 \right] r_H^2 + a^4 \alpha^4 \left( \alpha^2 b^2 - \frac{2}{3} \right) \right\}.$$

At the boundary $z = 0$, the asymptotic behavior of the fields $B$ and $\phi$ are the same as in Eqs. (2.14),

$$B(z) \sim \mu - \frac{\rho}{r_H} z,$$

$$\phi(z) \sim D_2 \frac{z^2}{r_H^2},$$

where $\mu$ and $D_2$ will be imposed as the boundary values at the boundary $z = 0$.

The two boundary values $a$ and $b$ at the horizon $z = 1$ and the two boundary values $\mu$ and $D_2$ at the boundary $z = 0$ are related by the matching conditions of the fields $B$ and $\phi$ at a matching point $z_m$. 

9
To match the series expansions of the fields $B(z)$ and $\phi(z)$ at the horizon in Eq. (3.4) and that at the boundary in Eq. (3.6) smoothly at a matching point $z_m$, we require the following four constraint equations,

$$\mu - \frac{\rho}{r_H} z_m = a (1 - z_B) + B_2 (1 - z_m)^2,$$

(3.7a)

$$- \frac{\rho}{r_H} = -a - 2B_2 (1 - z_B),$$

(3.7b)

$$\frac{D_2}{r_H^2} z_m^2 = b + \phi_1 (1 - z_m) + \phi_2 (1 - z_m)^2,$$

(3.7c)

$$\frac{2D_2}{r_H^2} z_m = -\phi_1 - 2\phi_2 (1 - z_m).$$

(3.7d)

The above constraint equations can be solved as,

$$\mu = a + B_2 (1 - z_m) (1 + z_m),$$

(3.8a)

$$\rho = [a + 2B_2 (1 - z_m)] r_H,$$

(3.8b)

$$D_2 = \frac{b}{12z_m} [12r_H^2 - (\alpha^2 a^2 + 4r_H^2)(1 - z_m)],$$

(3.8c)

$$0 = 2b + \phi_1 (2 - z_m) + 2\phi_2 (1 - z_m).$$

(3.8d)

To be concrete, we take $z_m = 3/4$ in the following calculation. The two parameters $a$ and $b$ can be analytically solved from Eqs. (3.8b) and (3.8d) once the charge density $\rho$ is given,

$$b^2 = \frac{-24r_H(ar_H - \rho)}{2a(8\alpha^2 + q^2)r_H^2 - 12\alpha^2 r_H \rho + \alpha^4 \alpha^3},$$

(3.9)

with $a$ satisfies a quartic equation,

$$\alpha^4 a^4 - 4r_H^2 (-2\alpha^2 + q^2)a^2 - 48r_H \rho \alpha^2 a + 304r_H^4 = 0.$$

(3.10)

The condensate $D_2$ is the order parameter of the superconducting phase transition. Above a critical temperature $T_c$, the order parameter $D_2 \propto b$ is zero, which from Eq. (3.9) implies $ar_H - \rho = 0$ at the critical temperature $T_c$. From Eq. (3.10), we obtain the critical temperature by taking $a = \rho/r_H$,

$$T_c = \frac{3}{4\pi} \left( \frac{10\alpha^2 + q^2 + \sqrt{24\alpha^4 + 20\alpha^2 q^2 + q^4}}{152} \right)^{1/4} \rho^{1/2}.$$

(3.11)

For $\alpha = q = 0$, $T_c$ vanishes, there is no phase transition at this special point. For $\alpha \gg q$, the critical temperature $T_c \propto \alpha^{1/2}$; While for $\alpha \ll q$, the critical temperature $T_c \propto q^{1/2}$. 


The critical temperature $T_c$ is proportional to $\sqrt{\rho}$ and increases with $\alpha$ and $q$ monotonously. $T_c/\sqrt{\rho}$ vs. the parameters $(\alpha, q)$ is plotted in Fig. 1. The behavior of the analytic expression of the critical temperature in Eq. (3.11) is consistent with the numeric result in [48].

Near the critical temperature $T_c$, the condensate behaves as $D_2 = AT_c^2(1 - T/T_c)^{1/2}$, which indicates that the phase transition at $T_c$ is a second order phase transition. The coefficient $A$ depends on the parameters $\alpha$, $q$ and $\rho$, see Eq. (A.1) in the Appendix.

The condensate $D_2$ vs. $T/T_c$ for $\alpha = q = 1$ can be calculated from Eq. (3.8c) and is plotted as the solid line in Fig. 2. We see that the low temperature behavior of condensate in the non-backreaction calculation (red line) is not consistent with the numeric calculation in [48]. The blue line in Fig. 2 is the condensate near the critical temperature and is exploited to the low temperature region.

The inconsistency of the condensate around the low temperature is due to the non-backreaction approximation we took in this section. In the next section, we will study the full backreaction case.
FIG. 2: The behavior of the condensate $D_2$ with the couplings $\alpha = 1$, $q = 1$. The red line is the condensate directly from Eq. (3.8c), while the blue line is the approximate condensate near the critical temperature $T_c$.

IV. FULL-BACKREACTION

In this section, we consider the case of full-backreaction by including the backreacted effects of the fields $B$ and $\phi$. It is necessary to solve the fields $\chi$ and $g$ in the metric (2.4a) as well as the fields $B$ and $\phi$ together from the full equations of motion (2.8).

Similarly to the non-backreaction case, there are three scaling symmetries in the equations of motion (2.8) in the full-backreaction case. The scaling dimensions of these variables for the scaling symmetries are listed in Table II.

| Sym. | $n_T$ | $n_t$ | $n_g$ | $n_B$ | $n_{\chi}$ | $n_q$ |
|------|-------|-------|-------|-------|-------------|-------|
| I    | 1     | 1     | 0     | 0     | 2           | 0     |
| II   | 1     | -1    | -1    | 0     | 2           | 1     |
| III  | 1     | 1     | 0     | 0     | 0           | 1     |
|      |       |       |       |       | -1          |       |

TABLE II: Scaling symmetries in the equations of motion (2.8) with full-backreaction.

As in the non-backreaction case, we can use the three scaling symmetries to set the parameters $\ell = 1$, $\rho = \text{constant}$ and $\chi^{(0)} = 0$ in Eq. (2.12a), which are the necessary boundary conditions to ensure the asymptotic AdS at the boundary.
A. Matching Solutions

Similarly as what we have done in the non-backreaction case, we will solve the fields \((\chi, g, B, \phi)\) in the full-backreaction case by using the matching method.

Up to the order next to the initial values, the series expansions of the fields \((\chi, g, B, \phi)\) near horizon in (2.11) becomes,

\[
\begin{align*}
\chi(z) &= \chi_0 + \chi_1 (1 - z), \\
g(z) &= \frac{r_H^2}{z^2} \left[1 - z^3 + g_1 (1 - z)\right], \\
B(z) &= a (1 - z) + B_2 (1 - z)^2, \\
\phi(z) &= b + \phi_1 (1 - z) + \phi_2 (1 - z)^2,
\end{align*}
\]

where we have renamed \(B_1 = a\) and \(\phi_0 = b\) in (2.11). In the above expansion, \(\chi_0, a\) and \(b\) need to be imposed as the boundary values at the horizon \(z = 1\). Plug the series expansions (4.1) into the equations of motion (2.8), we can solve for the coefficients \((\chi_1, g_1, B_2, \phi_2)\) in Eqs. (4.1) in terms of the boundary values \((\chi_0, a, b)\).

\[
\begin{align*}
g_1 &= \frac{1}{8r_H^2} \left(4b^2r_H^2 - a^2(\alpha^2b^2 + 2)\right), \\
\phi_1 &= \frac{-(a^2\alpha^2 + 4r_H^2)b}{2(g_1 + 3)r_H^2}, \\
\chi_1 &= \frac{-\phi_1^2(g_1 + 3)^2r_H^2 - a^2}{2(g_1 + 3)^2r_H^2}, \\
B_2 &= \frac{a((1/4\chi_1(g_1 + 3)\alpha^2 + q^2)b^2 - \phi_1\alpha^2(g_1 + 3)b + 1/2(g_1 + 3)\chi_1)}{(\alpha^2b^2 + 2)(g_1 + 3)}, \\
\phi_2 &= \frac{1}{8(g_1 + 3)^2r_H^2} \left[(((\chi_1 + 4)b - \phi_1)g_1 + (3\chi_1 + 15)b - 3\phi_1)\alpha^2 - 2bq^2)a^2 - 4B_2\alpha^2(g_1 + 3)a + 12r_H^2(1/12g_1^2\chi_1\phi_1 + 1/2(\chi_1 + 7/3)\phi_1g_1 + b + \phi_1(3/4\chi_1 + 7/2))\right].
\end{align*}
\]

The detailed forms are listed in (B.2).

At the boundary \(z = 0\), the asymptotic behavior of the fields \((\chi, g, B, \phi)\) was listed in Eq. (2.14), where \(M, \mu\) and \(D_2\) will be imposed as the boundary values at the boundary \(z = 0\).

The three boundary values \((\chi_0, a, b)\) at the horizon \(z = 1\) and the three boundary values \((M, \mu, D_2)\) at the boundary \(z = 0\) are related by six matching conditions of the fields \((\chi, g, B, \phi)\) at a matching point \(z_m\).
To match the asymptotic behaviors of the fields at the horizon in Eq. (2.14) and that at the boundary in Eq. (4.1) smoothly at a matching point $z_m$, we require the following six constraint equations,

\[ \frac{1}{2} \left( \frac{D_2}{r_H^2} z_m^2 \right)^2 = \chi_0 + \chi_1 (1 - z_m), \]  

\[ \frac{1 - 2 M}{z_m^2} r_H^2 = \frac{1 - z_m^3}{z_m^2} r_H^2 + g_1 (1 - z_m) z_m^2 r_H^2, \]  

\[ \mu - \frac{\rho}{r_H} z_m = a (1 - z_m) + B_2 (1 - z_m)^2, \]  

\[ \mu - \frac{\rho}{r_H} = -a - 2 B_2 (1 - z_m), \]  

\[ \frac{D_2}{r_H^2} z_m^2 = b + \phi_1 (1 - z_m) + \phi_2 (1 - z_m)^2, \]  

\[ 2 \frac{D_2}{r_H^2} z_m = -\phi_1 - 2 \phi_2 (1 - z_m), \]

which can be solved as

\[ \chi_0 = -\chi_1 (1 - z_m) + \frac{1}{2} \left( \frac{D_2}{r_H^2} \right)^2 z_m^4, \]  

\[ M = \frac{1}{2} - g_1 (1 - z_m) \frac{z_m^3}{2}, \]  

\[ \mu = a + B_2 (1 - z_m) (1 + z_m), \]  

\[ \rho = \left[ a + 2 B_2 (1 - z_m) \right] r_H, \]  

\[ D_2 = \frac{2b + \phi_1 (1 - z_m)}{2z_m} r_H^2, \]  

\[ 0 = 2b + \phi_1 (2 - z_m) + 2 \phi_2 (1 - z_m). \]

The same as in the non-backreaction case, we take $z_m = 3/4$ in the following. Eq. (4.4f) can be expressed as a cubic equation of $a^2$, see Eq. (B.6), which can be solved analytically in term of $b^2$.

We check the stability of the analytic hairy black hole solution by comparing its free energies with the free energies of RN black holes are obtained in Eqs. (2.7),

\[ F_{\text{Hairy}} = V (-2M + \mu \rho), \]  

\[ F_{\text{RN}} = \frac{V}{r_H} \left( -r_H^4 + \frac{3 \rho^2}{4} \right). \]
For the given couplings $\alpha$ and $q$, as well as a charge density $\rho$, the free energies $F_{\text{Hairy}}$ and $F_{\text{RN}}$ vs. $T/T_c$ are plotted in Fig. 3.

![Graph](image)

**FIG. 3:** The free energies $F_{\text{Hairy}}$ (blue) and $F_{\text{RN}}$ (red) with $\alpha = 5$, $q = 1$, and $\rho = \text{constant}$ for $T \leq T_c$.

When $T < T_c$, the free energy of the hairy black hole is always lower than that of the RN black hole and vice versa. Thus for $T > T_c$, the system is in the RN black hole phase with the scalar field $\phi = 0$; while for $T < T_c$, the system transits to the hairy black hole phase with $\phi \neq 0$ that indicates the condensation.

**B. Condensate**

In this section, we investigate the condensate $D_2$, the order parameter of the superconducting phase transition, in more details. The behavior of the order parameter $D_2 \propto b$ around the critical temperature is the same as that in the non-backreaction case, $D_2 \propto (1 - T/T_c)^{1/2}$. However, the fixed-$\rho$ condition has been modified at the critical temperature ($B_2|_{T=T_c} \neq 0$). The condensate $D_2$ vs the ratio of the temperature $T/T_c$ for different parameters $\alpha$ and $q$ are plotted in Fig. 4 and 5.
FIG. 4: The condensate to temperature diagram with the coupling $q = 1, 3, 5, 7$ from bottom to top and (a) $\alpha = 1$, (b) $\alpha = 5$.

FIG. 5: The condensate to temperature diagram with the coupling $\alpha = 10^{-4}, 1, 3, 5, 7$ from top to bottom and (a) $q = 1$, (b) $q = 5$.

Form Fig. 4 and Fig. 5, we see that, for a fixed $\alpha$, the condensate $D_2$ increases as the parameter $q$ increases, while for a fixed $q$, the condensate $D_2$ decreases as the parameter $\alpha$ increases. The behavior of the condensate $D_2$ depending on the parameters $\alpha$ and $q$ is consistent with the numeric result in [48].

As we did in the non-backreaction case, the critical temperature in the full-backreaction case can be obtained analytically, but the expression is much more complicated. The behavior of the critical temperature $T_c$ in the full-backreaction case is also similar to that in the non-backreaction case and is plotted in Fig. 6.
FIG. 6: The critical temperature $T_c$ vary with the couplings $\alpha, q$.

C. Conductivity

In this section, we compute the conductivity of this system by adding an external source $A_x$ perturbatively. The metric is thus modified by a perturbative non-diagonal component $g_{tx}$. Defining $A_x = \tilde{A}_x (r) e^{-i\omega t}$ and $g_{tx} = \tilde{g}_{tx} (r) e^{-i\omega t}$, the equations of motion for the perturbative field $\tilde{A}_x (r)$ and $\tilde{g}_{tx} (r)$ are

$$
\tilde{A}_x'' + \left( g' g - \frac{\chi'}{2} + \frac{f'}{f} \right) \tilde{A}_x' + \left( \frac{\omega^2}{g^2} e^{\chi} - \frac{J}{g f} \right) \tilde{A}_x + \frac{\tilde{A}_t'}{g} e^{\chi} \left( \tilde{g}_{tx}' - \frac{2}{r} \tilde{g}_{tx} \right) = 0,
$$

(4.6a)

$$
\tilde{g}_{tx}' - \frac{2}{r} \tilde{g}_{tx} + f \tilde{A}_t' \tilde{A}_x = 0,
$$

(4.6b)

which lead to a homogeneous linear differential equation for $\tilde{A}_x$,

$$
\tilde{A}_x'' + \left( g' g - \frac{\chi'}{2} + \frac{f'}{f} \right) \tilde{A}_x' + \left[ \left( \frac{\omega^2}{g^2} - \frac{f \tilde{A}_t'^2}{g} \right) e^{\chi} - \frac{J}{g f} \right] \tilde{A}_x = 0.
$$

(4.7)
Making a coordinate transformation $z = r_H/r$ and defining $e^\chi \tilde{A}_x^2 = C^2$, the equation for $\tilde{A}_x$ becomes

$$C'' + \left( g' - \frac{3}{2} \chi' + \frac{f'}{f} + \frac{2}{z} \right) C' + \left[ -\frac{1}{2} \chi'' + \frac{1}{2} \chi^2 - \frac{1}{2} \chi' \left( \frac{g'}{g} + \frac{f'}{f} + \frac{2}{z} \right) \right] C = 0.$$  

(4.8)

To realize the structure of the Eq. (4.8), we make a further coordinate transformation,

$$du = -\frac{r_H e^\chi}{g z^2} dz,$$  

(4.9)

which transforms the horizon at $z = 1$ to $u = -\infty$ and the boundary at $z = 0$ to $u = 0$.

We integrate the coordinate transformation (4.9) term by term to get

$$u = -\int \left( \frac{p_0}{1-z} + p_1 + \cdots \right) dz = p_0 \ln(1-z) + p_1 (1-z) + \cdots,$$  

(4.10)

where $p_i$’s are the expansion coefficients near the horizon at $z = 1$.

By defining a new field $\Psi = \sqrt{f} e^{\chi/2} C$, the Eq. (4.8) can be brought to the form of the Schrodinger equation,

$$\frac{d^2 \Psi}{du^2} + [\omega^2 - V(u)] \Psi = 0,$$  

(4.11)

where the potential is

$$V(u) = g \left[ f \left( \frac{d(e^{\chi/2} B)}{du} \right)^2 + \frac{J}{f} e^{-\chi} \right] + \frac{1}{\sqrt{f}} \frac{d^2 \sqrt{f}}{du^2},$$  

(4.12)

with $V(0) = V(-\infty) = 0$.

The Schrodinger equation (4.11) with the potential (4.12) is a standard one-dimensional scattering problem, and the wave function $\Psi(u)$ near the boundary at $u \sim 0$ behaves as

$$\Psi(u) = e^{-i\omega u} + Re^{i\omega u},$$  

(4.13)

where $R$ is the reflection coefficient. The conductivity can thus be written as

$$\sigma(\omega) = \frac{1 - R}{1 + R}.$$  

(4.14)

At the horizon at $u = -\infty$, the in-falling wave boundary condition admits a non-reflection wave function,

$$\Psi(u) = Te^{-i\omega u}.$$  

(4.15)
where $T$ is the transmission coefficient.

In the $z$ coordinate, the in-falling wave function becomes

$$
\Psi = Te^{-i\omega[p_0 \ln(1-z) + p_1 (1-z)]}.
$$

(4.16)

Therefore, near the horizon, the field $C(z)$ can be expanded as,

$$
C(z) = c_0 e^{-i\omega[p_0 \ln(1-z) + p_1 (1-z)]} \cdot [1 + c_1 (1 - z) + \ldots]
\approx c_0 e^{-i\omega[p_0 \ln(1-z) + P_1 (1-z)]} \cdot [1 + C_1 (1 - z)],
$$

(4.17)

where in the second line we assumed a truncated form of the field $C(z)$ with $P_1$ and $C_1$ being the effective coefficients by truncating all the higher-order terms at the horizon. Therefore, $P_1$ and $C_1$ contain the effects far from the horizon and will be determined by Eq. (4.17) and the boundary condition at $z = 0$. The approximated truncating solution is of course different from the true solution, but we will see that some important properties are preserved in this approximation.

At the boundary, the asymptotic form of $C(z)$ is

$$
C(z) = C^{(0)} + \frac{C^{(1)}}{r_H} z + \cdots.
$$

(4.18)

Expanding the field $C(z)$ in Eq. (4.17) at the boundary $z = 0$, the conductivity can be calculated as follows [9],

$$
\sigma(\omega) = \frac{1}{i\omega} \frac{C^{(1)}}{C^{(0)}} = r_H (p_0 + P_1) - \frac{r_H}{i\omega} \left( \frac{C_1}{1 + C_1} \right).
$$

(4.19)

In the high frequency limit $\omega \to \infty$, the conductivity should approach to one due to the asymptotic AdS geometry, the above expression gives

$$
P_1 = -p_0 + \frac{1}{r_H}.
$$

(4.20)

The coefficients $p_0$ and $C_1$ can be now calculated by plugging the the field $C(z)$ in Eq. (4.17) into the equation of motion (4.8). The exact expressions of $p_0$ and $C_1$ are listed in the Appendix as Eqs. (B.3a) and (B.3b).

The DC conductivity can be obtained from the low frequency expansion of the conductivity (4.19),

$$
\sigma(\omega) = \Re(\sigma) + i\Im(\sigma),
$$

(4.21)

$$
\Re(\sigma) = \sigma_0 + O(\omega^2),
$$

(4.22)

$$
\Im(\sigma) = \sigma_{-1} \omega^{-1} + O(\omega).
$$

(4.23)
where both $\sigma_0$ and $\sigma_{-1}$ are real and they are given in Eq. (B.7) in the Appendix. At $\omega = 0$, the imaginary part of the conductivity $\Im(\sigma)$ is proportional to $\omega^{-1}$. By Kramers-Kronig relations, this implies that the real part of the conductivity $\Re(\sigma)$ behaves as a delta function at $\omega = 0$, i.e. DC superconductivity.

In the high frequency, the conductivity can be expanded as,

$$\sigma(\omega) = 1 + \frac{\ii \tau H}{\omega} + O(\omega^{-2}),$$

which gives $\sigma(\omega) \to 1$ as $\omega \to \infty$ that has been fixed by the choice of $P_1$ in Eq. (4.20).

The real and imaginary parts of the conductivity vs. frequency are plotted in Fig. 7 - 9.

FIG. 7: The real and imaginary part of the conductivity with $\alpha = 1$ and $q = 1, 3, 5$ from left to right. The different curves in each figure are respect to different temperature ratio.
FIG. 8: The real and imaginary part of the conductivity with $\alpha = 3$ and $q = 1, 3, 5$ from left to right. The different curves in each figure are respect to different temperature ratio.

FIG. 9: The real and imaginary part of the conductivity with $\alpha = 5$ and $q = 1, 3, 5$ from left to right. The different curves in each figure are respect to different temperature ratio.
V. CONCLUSION

In this paper, we studied a non-minimal holographic superconductor model by considering the Einstein-Maxwell-Dilaton system. The model has two adjustable parameters $\alpha$ and $q$. $\alpha$ describes the coupling between the Maxwell field and the dilaton, and $q$ represents the charge of the dilaton.

For various $\alpha$ and $q$, we solved the system analytically by an approximate method, the matching method. In this method, we only solved the equations of motion near the boundary and horizon, then matched the two asymptotic solutions smoothly at an intermediate matching point $z_m$ between the boundary and the horizon. The matching solution ignores much details in the bulk, but surprisingly possesses many important properties which are consistent with the numeric results.

We studied both non-backreaction and the full-backreaction cases. We found phase transitions with the similar critical temperature $T_c$ in both cases. The critical temperature increases with $\alpha$ and $q$ monotonously and behaves as $T_c \propto \alpha^{1/2}$ for small $q$ and $T_c \propto q^{1/2}$ for small $\alpha$, which is consistent with the numeric results. The condensate near the critical temperature behaves as $(1 - T/T_c)^{1/2}$ which indicates that the phase transition is second order.

In the non-backreaction case, the behavior of the condensate at low temperature blows up quickly and is not make sense. Nevertheless, we showed that in the full-backreaction case, the low temperature behavior of the condensate is rectified. For a fixed $\alpha$, the condensate increases as the parameter $q$ increases, while for a fixed $q$, the condensate decreases as the parameter $\alpha$ increases. The behavior of the condensate depending on the parameters $\alpha$ and $q$ is consistent with the numeric result.

In addition, we developed an approximate analytic method to calculate the electric conductivity. We expand the perturbative field $A_x$ near the horizon obliged on the in-falling boundary condition of the Schrodinger equation at the horizon. We then truncated the expansion to the linear order and determined the truncating coefficients by the equation of motion and the boundary condition at the boundary. We showed that the imaginary part of the conductivity suffers a $1/\omega$ divergence at small frequency that implies the Dirac $\delta$ function behavior in the real part of the conductivity, i.e. DC superconductor. Furthermore, we showed that the asymptotic values of the real conductivity in the small frequency
limit increases as the temperature raised. However, we did not observe the "Drude Peak" behavior at the low frequency as in the numeric calculation [48].

In summary, we studied the condensation and electric conductivity a holographic superconductor model by using approximate analytic methods. The approximate solutions ignore some details of the system, but preserve many important physical properties. Since the analytic solutions are much more useful to investigate the critical phenomena than the numeric ones, it is worth to study various holographic models by using these analytic methods in the future.

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Appendices

A. CONDENSATE NEAR CRITICAL TEMPERATURE

Expanding the condensate around the critical temperature $T_c$, we obtain the approximate condensate for $T \leq T_c$,

$$D_2 = AT_c^2 \left( -\frac{T - T_c}{T_c} \right)^{1/2},$$  \hspace{1cm} (A.1a)

$$A = \frac{11264\pi^4 T_c^4 - 81\alpha^2 \rho^2}{5184\pi^2 T_c^4} \left( -\frac{T_c}{T_1} \right)^{1/2},$$  \hspace{1cm} (A.1b)

$$T = T_c + T_1 b^2 + O(b^4),$$  \hspace{1cm} (A.1c)

$$T_1 = \frac{\pi^4 \left( -2125824\pi^4 T_c^4 \alpha^2 - 77824\pi^4 T_c^4 q^2 + 22356\alpha^4 \rho^2 + 5670\alpha^2 q^2 \rho^2 + 243q^4 \rho^2 \right) T_c^5}{1944\rho^2 \left( \pi^4 T_c^4 q^2 + 10\pi^4 T_c^4 \alpha^2 - \frac{81\alpha^4 \rho^2}{512} \right)},$$  \hspace{1cm} (A.1d)
**B. FORMULAS WITH FULL-BACKREACTION**

The near horizon coefficients with full-backreaction are listed here. The coefficients for the condensate background, \(\text{[4.1]}\), are written as follows:

\[
\chi_1 = \frac{-8 \left[ 16b^2r_H^4 + 4a^2(2\alpha^2b^2 + 1)r_H^2 + a^4\alpha^4b^2 \right]}{-4(b^2 + 6)r_H^2 + a^2(\alpha^2b^2 + 2)}^2, \quad \text{(B.2a)}
\]

\[
g_1 = \frac{4br_H^2 - a^2(\alpha^2b^2 + 2)}{8r_H^2}, \quad \text{(B.2b)}
\]

\[
B_2 = \frac{-6a}{(\alpha^2b^2 + 2)\left[ -4(b^2 + 6)r_H^2 + a^2(\alpha^2b^2 + 2) \right]^2} \cdot \left\{ -16b^2 \left[ \frac{1}{3}(\alpha^2 + q^2)b^2 + 2 \left( 2\alpha^2 + q^2 - \frac{1}{3} \right) \right] r_H^4 \right.
\]
\[
+ \frac{4}{3}a^2 \left[ \alpha^2(2\alpha^2 + q^2)b^4 - (12\alpha^4 - 9\alpha^2 - 2q^2)b^2 + 2 \right] r_H^2
\]
\[
+ a^4\alpha^4(\alpha^2b^2 + 2)b^2 \right\}, \quad \text{(B.2c)}
\]

\[
\phi_1 = \frac{4b(4r_H^2 + \alpha^2\alpha^2)}{-4(b^2 + 6)r_H^2 + a^2(\alpha^2b^2 + 2)}, \quad \text{(B.2d)}
\]

\[
\phi_2 = \frac{-4b}{(\alpha^2b^2 + 2)\left[ -4(b^2 + 6)r_H^2 + a^2(\alpha^2b^2 + 2) \right]^3} \left\{ -64(b^2 + 12)(\alpha^2b^2 + 2)r_H^6 \right.
\]
\[
+ 16a^2 \left[ \alpha^4(b^4 + 12b^2 + 6)b^2 - \alpha^2 \left( (3q^2 - 2)b^4 + (18q^2 - 37)b^2 - 60 \right) - 2q^2b^2 - 12q^2 \right]
\]
\[
+ 10r_H^4 - 8a^4 \left[ \alpha^6(b^6 + \frac{7b^4}{2} + 9b^2) - \alpha^4 \left( \frac{3q^2}{2} - 4 \right) b^4 - \frac{27b^2}{2} + 6 \right]
\]
\[
- \left( 4(q^2 - 1)b^2 - 21 \right) \alpha^2 - 2q^2 \right] r_H^2 + a^6\alpha^2(\alpha^2b^2 + 2) \cdot \left( (\alpha^2b^2 + 2)^2 + 2\alpha^2(2\alpha^2b^2 - 1) \right) \right\}. \quad \text{(B.2e)}
\]

The coefficients of the near horizon expansion of the perturbed field \(C(z)\) are:

\[
p_0 = \frac{e^\frac{\chi_0}{2}}{(g_1 + 3)r_H}, \quad \text{(B.3a)}
\]

\[
C_1 = \frac{\Gamma}{\Delta}, \quad \text{(B.3b)}
\]
where,
\[
\begin{align*}
\Gamma &= -2 \left( \alpha^2 (\chi_1 - 2)b^2 + 2 \alpha^2 b^2 \phi_0 + 2 \chi_1 - 4 \right) \omega^2 e^{\chi_0} \\
&+ 4(g_1 + 3) \left\{ \left[ 2 \alpha^2 \left( -\omega^2 p_0^2 - i \left( \frac{3}{8} \chi_1 + \frac{1}{4} \right) \omega p_0 + \frac{\chi_1}{8} \right) b^2 + i \alpha^2 \omega p_0 \phi_0 (1 - i \omega p_0) b \\
- 4 \omega^2 p_0^2 - i \left( \frac{3}{2} \chi_1 + 1 \right) \omega p_0 + \frac{\chi_1}{2} \right] g_1 + \left[ -9 \alpha^2 \omega^2 p_0^2 - i \left( \frac{9}{4} \chi_1 + 3 \right) \alpha^2 \omega p_0 \\
+ q^2 + \frac{3}{4} \alpha^2 \chi_1 \right] b^2 + 3 i \alpha^2 \omega p_0 \phi_0 (1 - i \omega p_0) b + 18 \left( \frac{1}{3} - i \omega p_0 \right) \left( \frac{\chi_1}{4} - i \omega p_0 \right) \right] r_H^2 \\
+ i \omega (\alpha^2 b^2 + 2) \left( \frac{1}{2} - i \omega p_0 \right) (g_1 + 3) r_H + \frac{1}{4} \alpha^2 (\alpha^2 b^2 + 2)^2 \right\}, \quad (B.4)
\end{align*}
\]
\[
\Delta = 2(\alpha^2 b^2 + 2) \left( e^{\chi_0} \omega^2 + r_H^2 (g_1 + 3)^2 (1 - i \omega p_0)^2 \right). \quad (B.5)
\]

The cubic equation of \( a^2 \) can be written as:
\[
0 = -128(\alpha^2 b^2 + 2) \cdot \left( b^6 + \frac{31 b^4}{2} + 77 b^2 + 114 \right) r_H^6 + 96 a^2 \left[ \alpha^4 b^6 + \left( \frac{65}{6} \alpha^4 + 4 \alpha^2 \right) b^6 \\
+ \left( 32 \alpha^4 + \left( q^2 + \frac{127}{3} \right) \alpha^2 + 4 \right) b^4 + \left( 28 \alpha^4 + \left( 6 q^2 + \frac{335}{3} \right) \alpha^2 + \frac{2 q^2}{3} + \frac{124}{3} \right) b^2 \\
+ 40 \alpha^2 + 4 q^2 + \frac{302}{3} \right] r_H^4 - 24 \alpha^4 \left[ \alpha^6 b^8 + \left( \frac{37 \alpha^6}{6} + 6 \alpha^4 \right) b^6 + \left( \frac{23 \alpha^6}{3} + (q^2 + 35) \alpha^4 \\
+ 12 \alpha^2 \right) b^4 - \left( 6 \alpha^6 - 31 \alpha^4 - \left( \frac{8 q^2}{3} + 66 \right) \alpha^2 - 8 \right) b^2 + 4 \alpha^4 + 26 \alpha^2 + \frac{4 q^2}{3} + \frac{124}{3} \right] r_H^2 \\
\cdot 2 a^6 (\alpha^2 b^2 + 2) \cdot \left[ \alpha^6 b^6 + 3 \alpha^4 \left( \frac{a^2}{2} + 2 \right) b^4 - 2 \alpha^2 (2 \alpha^4 - 3 \alpha^2 - 6) b^2 + 2 \alpha^2 (\alpha^2 + 3) + 8 \right]. \quad (B.6)
\]

The DC expansion of the conductivity is:
\[
\sigma_{DC}(\omega) = i \sigma_1 \omega^{-1} + \sigma_0 + O(\omega), \quad (B.7)
\]
\[
= i \left( 1 - \frac{2(\alpha^2 b^2 + 2)(g_1 + 3)}{\Xi} r_H^2 \right) r_H \omega^{-1} + \left( 1 - \frac{4 p_0 (\alpha^2 b^2 + 2) (g_1 + 3)}{\Xi} r_H^3 + \frac{\Phi}{\Xi^2} \right) \right) \\
+ O(\omega), \quad (B.8)
\]
where,

$$\Xi = \left[ \left( \alpha^2 + \frac{4q^2}{(\chi_1 + 2)(g_1 + 3)} \right) b^2 + 2 \right] (\chi_1 + 2)(g_1 + 3) r_H^2 + a^2 \left( \alpha^2 b^2 + 2 \right)^2,$$

(B.9)

$$\Phi = 6 \left\{ p_0 \left[ \left( (\chi_1 + 2) g_1 + 3 \chi_1 + 8 \right) b - \frac{4}{3} \phi_1 (g_1 + 3) \right] b \alpha^2 + \left( 2 \chi_1 + 4 \right) g_1 + 6 \chi_1 + 16 \right\} r_H$$

$$- \frac{2}{3} \left( \alpha^2 b^2 + 2 \right) (g_1 + 3) \left( \alpha^2 b^2 + 2 \right) (g_1 + 3) r_H^4.$$

(B.10)

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