Hilbert Space Representation of an Algebra of Observables for q-Deformed Relativistic Quantum Mechanics

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Abstract

Using a representation of the q-deformed Lorentz algebra as differential operators on quantum Minkowski space, we define an algebra of observables for a q-deformed relativistic quantum mechanics with spin zero. We construct a Hilbert space representation of this algebra in which the square of the mass $p^2$ is diagonal.

1 Introduction

The concept of Lie groups and Lie algebras has found a natural generalization in the framework of non-commutative Hopf algebras and quantum groups. This has posed the question, whether these can appear as symmetries of physical theories. In this paper we want to address the problem of quantum symmetric relativistic quantum mechanics with spin zero.

In ordinary relativistic quantum mechanics, we have a Hilbert space of states which is a representation of the Poincaré algebra. Acting on this space, we have the algebra of observables generated by position and momentum coordinates, which are essentially self-adjoint operators defined on a common domain, which is dense in the Hilbert space. They satisfy the Heisenberg algebra $[x^i, p^j] = ig^{ij}$. Position and momentum space are both isomorphic to Minkowski space and carry a representation of the Lorentz group.
In this paper, we define a deformation of the algebra of observables. The position and momentum coordinates generate algebras, which are isomorphic as ∗-comodule algebras of the q-deformed Lorentz group. Introducing an algebra of angular momentum operators, which is a representation $U_q(SL(2,\mathbb{C}))$, we construct a Hilbert space representation of this algebra of observables. Coordinates and momenta are represented on this Hilbert space by unbounded operators.

## 2 Preliminaries

### 2.1 $SL_q(2,\mathbb{C})$ and Quantum Lorentz Group

The complex quantum group $SL_q(2,\mathbb{C})$ ([4]) is a ∗-Hopf algebra constructed by taking two copies of the quantum group $SL_q(2)$ (cf. [3]) with generators $(t_{\alpha \beta})_{\alpha, \beta=1,2}$ and $(t_{\alpha \beta})_{\alpha, \beta=1,2}$ which are connected by the mixed relations $\hat{R}^{\alpha \beta \gamma} t^{\gamma \delta \sigma} = t^{\alpha \mu \beta \nu} \hat{R}^{\mu \nu \rho \sigma}$, where $\hat{R}$ denotes the $\hat{R}$-matrix of $SL_q(2)$. The involution is $(t_{\alpha \beta})^* := S(t^{\alpha \beta})$, $S$ being the antipode of $SL_q(2)$.

The quantum Lorentz group $SO_q(3, 1)$ (cf. [4]) is the ∗-sub-Hopf algebra $SL_q(2,\mathbb{C})$ generated by the elements $M_{(\alpha \beta) (\gamma \delta)} := t^{\alpha \gamma} t^{\beta \delta}$ with relations and Hopf structure induced by $SL_q(2,\mathbb{C})$. Its $\hat{R}$-matrix is a product of $SL_q(2)$-$\hat{R}$-matrices. Using a single index $i = 1, 2, 3, 4$ instead of the double index $(\alpha, \beta) = (1, 1), (1, 2), (2, 1), (2, 2)$, the reality condition for the generators is given by $(M^i_j)^* = \eta^i_b S(M^b_a) \eta^a_i$, where the matrix $\eta^i_j$ is essentially given by the $\hat{R}$-matrix of $SL_q(2)$. The $\hat{R}$-matrix has the projector decomposition characteristic of orthogonal quantum groups:

$$\hat{R}^{ij} = q P_S^{ij} - q^{-1} P_A^{ij} + q^{-3} P_1^{ij},$$

(1)

$P_S$, $P_A$ and $P_1$ being the symmetric, antisymmetric and trace projector, respectively. $P_1^{ij}$ can be expressed in terms of the q-Minkowski metric $C_{ij}$ as $P_1^{ij} = Q^{-1} C^{ij} C_{kl}$ with $Q := C^{ij} C_{ij} = [2]_q^2$. Here we have introduced the q-numbers $[x]_q := (q^x - q^{-x})/(q - q^{-1})$.

### 2.2 Quantum Minkowski Space and Differential Calculus

A comodule algebra for $SO_q(3, 1)$ can be defined as the unital $\mathbb{C}$-algebra generated by the coordinates $x^i$ with relations $P_A^{ij} k_{ij} x^k x^j = 0$. This is the algebra of functions on quantum Minkowski space and is denoted $A_x(M_q)$. The coaction $\delta$ of $SO_q(3, 1)$ is given by $\delta(x^i) := M^i_j \otimes x^j$. For $\delta$ to become a homomorphism of ∗-algebras, we have to define the involution on $A_x(M_q)$ by $(x^i)^* := x^k C_{kl} q^l_i$. The element $r^2 := C_{ij} x^i x^j$ is central in $A_x(M_q)$. Another convenient set of generators $t, x, y, z$ for $A_x(M_q)$ is given by

$$t := q^{-1}/[2]_q (q^{-1} x^2 - x^3), \quad z := q^{-1}/[2]_q (-q x^2 - x^3),$$

$$y := x^1, \quad x := -x^4.$$
The generator $t$ becomes central in $A_x(M_q)$ and factorization with respect to the relation $t = 0$ yields an $SO_q(3)$-comodule algebra (cf. [3]).

We can now introduce partial derivatives acting on $A_x(M_q)$ as linear operators. We define the partial derivative $\partial^i \in \text{End}_Q(A_x(M_q))$, $i = 1, \ldots, 4$, by its action $\partial^i(1) := 0$ on the unit element $1 \in A_x(M_q)$ and by the Leibniz rule

$$\partial^i(x^j f) := C_{ij} f + q\hat{R}^{-1ij}_{kl} x^k \partial^l(f) \quad \forall f \in A_x(M_q),$$

which determines $\partial^i$ on the whole algebra $A_x(M_q)$. The partial derivatives satisfy $P_{\mathcal{A}}^{ij}_{kl} \partial^k \partial^l = 0$ and $\delta(\partial^i(f)) = (M^i_j \otimes \partial^j) \circ \delta(f)$, $\forall f \in A_x(M_q)$.

Therefore the algebra $A_\partial$ generated by $(\partial^i)$ is isomorphic to $A_x(M_q)$ as an $SO_q(3,1)$-comodule algebra. The coordinates $x^i$ act on $A_x(M_q)$ as linear operators by left multiplication. So we can consider the algebra $A_{x,\partial} := \langle (x^i)_{i=1,\ldots,4}, (\partial^i)_{i=1,\ldots,4} \rangle > \text{End}_Q(A_x(M_q))$ with relations

$$P_{\mathcal{A}}^{ij}_{kl} x^k x^l = P_{\mathcal{A}}^{ij}_{kl} \partial^k \partial^l = 0,$$

$$\partial^i x^j = C_{ij} + q\hat{R}^{-1ij}_{kl} x^k \partial^l.$$ (4)

Given this algebra, we can recover the Leibniz rule and therefore the action of $\partial^i$ as a differential operator on $A_x(M_q)$. Next, we want to define a $*$-structure on $A_{x,\partial}$. To that end, we first note, that there exists an operator $\Lambda \in A_{x,\partial}$ (cf. [3]) satisfying $\Lambda(1) = 1$, $\Lambda x^i = q^2 x^i \Lambda$ and $\Lambda \partial^i = q^{-2} \partial^i \Lambda$. Consequently $\Lambda \in \text{End}_Q(A_x(M_q))$ is a positive and invertible operator, so we can define an invertible operator $\mu := q^2 \Lambda^{1/2}$. We can now introduce the algebra $D(M_q)$ generated by the coordinates $(x^i)$, the derivatives $(\partial^i)$ and the operators $\mu$ and $\mu^{-1}$, and we call it algebra of differential operators on quantum Minkowski space. In this extended algebra we introduce the elements $\bar{\partial}^i$ by

$$\bar{\partial}^i := \Lambda^{-1} (\partial^i + q^3 x^i \Delta),$$

and it was shown in [3] that these differential operators satisfy the relations of the second possible covariant differential calculus on $A_x(M_q)$, i.e. we have

$$P_{\mathcal{A}}^{ij}_{kl} \bar{\partial}^k \bar{\partial}^l = 0$$

and $\bar{\partial}^i x^j = C_{ij} + q^{-1} \hat{R}^{ij}_{kl} x^k \bar{\partial}^l$. The involution on the partial derivatives can now be defined by $(\partial^i)^* := -q^{-4} \bar{\partial}^k C_{kl} \eta^i_l$, such that $D(M_q)$ finally becomes a $*$-algebra in which $\mu^* = \mu^{-1}$.

### 2.3 Regular Functionals and Vector Fields

In [3] it was shown that the dual algebra $SL_q(2,\mathcal{C})^*$ of $SL_q(2,\mathcal{C})$ contains a $*$-sub-Hopf algebra $U_{\hat{R}}$, called algebra of regular functionals. $U_{\hat{R}}$ is generated by functionals $(L^{\pm \alpha}_\beta)_{\alpha,\beta=1,2}$ and $(\hat{L}^{\pm \alpha}_\beta)_{\alpha,\beta=1,2}$. The action of these functionals is defined using the $\hat{R}$-matrix of $SL_q(2)$:

$$L^{\pm \alpha}_\beta(1) := \delta^\alpha_\beta \quad L^{\pm \alpha}_\beta(\gamma^\gamma_\delta) := \hat{R}^{\pm 1\alpha}_\beta \delta^\gamma_\delta \quad L^{\pm \alpha}_\beta(\tilde{\gamma}^\gamma_\delta) := \hat{R}^{\pm 1\alpha}_\beta \delta^\gamma_\delta$$ (6)
and the same relations for $\hat{L}^{\pm \alpha \beta}$ with $t$ and $\hat{t}$ exchanged. These definition is extended to products by $L^{\pm \alpha \beta}(ab):= L^{\pm \alpha \mu}(a)L^{\pm \mu \beta}(b), \forall a, b \in SL_q(2, \mathbb{C})$, and the same for $\hat{L}^{\pm \alpha \beta}$. The commutation relations in $U_R$ and the Hopf algebra structure are a direct consequence of these definitions (cf. [4]). The involution on the generators of $U_R$ is $(L^{\pm \alpha \beta})^\dagger = S(\hat{L}^{\pm \alpha \beta})$. Using the properties of the $SL_q(2)-R$-matrix, some of these generators can be eliminated, and it turns out that $U_R$ is generated by $L^{+1}_1, L^{+2}_2, \hat{L}^{+1}_1, L^{-1}_1, L^{-2}_1, L^{-2}_2$ with $L^{+2}_2 = (L^{+1}_1)^{-1}$. Moreover, in a certain minimal extension also $L^{-1}_1$ is invertible, so we are essentially left with six generators (cf. [4]). We can now also define the algebra of vector fields as the unital $\mathbf{C}$-algebra generated by the elements $Y^{\alpha \beta} := L^{+\alpha \gamma}S(L^{-\gamma \beta}) \in U_R$ and $\hat{Y}^{\alpha \beta} := \hat{L}^{+\alpha \gamma}S(\hat{L}^{-\gamma \beta}) \in U_R$. This algebra was introduced in [5, 6], and it was shown that for $SL_q(2, \mathbb{C})$ it is a sub-Hopf algebra of $U_R$ and that a certain natural extension of the algebra of vector fields is isomorphic to $U_R$. The left invariant vector fields are then given by $X := 1/\lambda(1 - Y)$. Let us finally mention that there exist two Casimir operators $C_1$ and $C_2$ in $U_R$.

3 Angular Momentum Representation of $U_R$

3.1 Angular Momentum Algebra

In the sequel we will always assume $q \geq 1$.

In the undeformed case we have a representation of the Lie algebra of $SL(2, \mathbb{C})$ in terms of antisymmetric generalized angular momentum operators acting on functions over Minkowski space. In this section, we will define the analogue of this in the q-deformed framework, i.e. a representation of the quantum universal enveloping algebra of $SL_q(2, \mathbb{C})$ (which is essentially given by $U_R$) by differential operators on quantum Minkowski space $A_x(M_q)$. In [6] such a representation was found for the closely related case of $SO_q(N)$ and, apart from the star structure, these results can be applied to the case of $SL_q(2, \mathbb{C})$.

Therefore we first introduce the elements $u^{ij} \in D(M_q)$ as

$$u^{ij} := (1 + q^{-4})C^{ij} + q^{-4}(q^{-1}\hat{R}^{ij}_{kl}x^k\partial^l - q\hat{R}^{-1ij}_{kl}x^k\partial^l)$$  \hspace{1cm} (7)$$

and define

$$V^{ij} := \mu \mathcal{P}^{ij}_{kl}u^{kl} = q^{-4}\lambda\mathcal{P}^{ij}_{kl}\mu x^k\partial^l, \hspace{1cm} (8)$$

$$U := \mu \mathcal{C}^{ij}u^{ij} = (1 + q^{-4})\mu((q + q^{-1})^2\mathbf{1} + (q^{-4} - 1)C^{ij}x^i\partial^j). \hspace{1cm} (9)$$

These differential operators commute with $r^2$ and we are led to the following

**Definition 3.1** The $*$-subalgebra $\hat{U}_q := \langle(V^{ij})_{i,j=1,...,4}, U \rangle \subset D(M_q)$ with involution $*$ given by

$$V^{ij} = V^{kl}C^{km}C^{ln}n^p_i\eta^{p}_{lj}, \hspace{1cm} (10)$$

is called angular momentum algebra on quantum Minkowski space.
As the rank of the antisymmetric projector in four dimensions is six, only six of the generators $V^{ij}$ are linearly independent. To determine the action of the elements of $\hat{U}_q$ as differential operators on $A_x(\hat{U}_q, \mathcal{M}_q)$, we consider the algebra

$$A_x(\hat{U}_q, \mathcal{M}_q) := \langle 1, (V^{ij}), U, (x^j) \rangle_{i,j=1,\ldots,4} \subset D(\mathcal{M}_q).$$

Using the relations in $D(\mathcal{M}_q)$, we can derive the relations in $A_x(\hat{U}_q, \mathcal{M}_q)$, which determines the action of the generators of $\hat{U}_q$ as differential operators. The result is (cf. [6]):

$$V^{ij} x^k = -[2]_q P^{ij}_{mn} x^m V^{nk} + \frac{q\lambda}{(1 + q^4)[2]_q} P^{ij}_{mn} C^{nk} x^m U,$$

$$U x^k = (q^4 - q^{-4}) \left( \frac{1}{\lambda[2]_q} x^k U - \frac{q}{2} C_{mn} x^m V^{nk} \right).$$

Furthermore the relations in $A_x(\mathcal{M}_q)$ imply the following identities:

$$0 = x^3 V^{12} + x^2 V^{31} + \lambda x^2 V^{14} - \frac{\lambda}{[2]_q} x^1 V^{14} - \frac{2q-1}{[2]_q} x^1 V^{23},$$

$$0 = x^4 V^{12} + q^{-2} x^1 V^{24} - \frac{q-1}{[2]_q} (q^2 + q^{-2}) x^1 V^{14} - \frac{q^{-2} \lambda}{[2]_q} x^2 V^{23},$$

$$0 = x^4 V^{31} + x^1 V^{43} + \lambda x^1 V^{24} - \frac{q^2 \lambda^2}{[2]_q} x^2 V^{14},$$

$$0 = x^2 V^{14} x^2 V^{23} + \frac{2q}{[2]_q} x^3 V^{14} - \frac{\lambda}{[2]_q} x^3 V^{23},$$

$$0 = x^3 V^{24} + q^{-2} x^2 V^{43} - \frac{q^2 \lambda}{[2]_q} x^4 V^{14} - \frac{2q}{[2]_q} x^4 V^{23}.$$
and
\[ 1 = \frac{1}{(1 + q^{-4})^2[2]_q^2} U^2 - \frac{q^6}{2[2]_q^2} C_{ij} C_{kl} V^{ik} V^{lj}. \] (17)

The second equation yields two identities which read with \( i = 1,2 \)
\[ 1 + M_i^2 - \frac{2}{[2]_q^2} M_i C + \frac{q\lambda^2}{[2]_q^2} (V_i^- V_i^+ + q^{-2} V_i^+ V_i^-) = 0, \] (18)
and \([16]\) gives another six relations \((i=1,2)\):
\[ V_i^\pm M_i = q^{-2} M_i V_i^\pm, \]
\[ q^{-1} V_i^- V_i^- - q V_i^- V_i^+ = \frac{1}{\lambda^2}(1 - M_i^2). \] (19)

Note that the algebras generated by \(\{V_1^+, V_1^-, M_1\}\) or \(\{V_2^+, V_2^-, M_2\}\) alone are isomorphic to the algebra of left invariant vector fields on the quantum group \(SU_q(2)\) as defined in \([9]\). This is exactly analogous to the undeformed case, but unlike the classical case, these two subalgebras do not commute:
\[ V_1^+ V_2^- = q^2 V_2^- V_1^+, \]
\[ V_1^+ V_2^+ = q^{-2} V_2^+ V_1^+, \]
\[ V_1^- V_2^- = q^{-2} V_2^- V_1^+, \]
\[ V_2^- M_1 = M_1 V_2^-, \]
\[ V_1^+ M_2 = M_2 V_1^+, \] (20)
\[ M_2 M_1 - M_1 M_2 = \lambda^3 V_1^+ V_2^- , \]
\[ M_1 V_2^+ - V_2^+ M_1 = q\lambda V_1^+ ([2]_q M_2 - C) , \]
\[ M_2 V_1^- - V_1^- M_2 = q\lambda V_2^- (C - [2]_q M_1), \]
\[ V_2^+ V_1^- - q^{-2} V_1^- V_2^+ = \frac{[2]_q}{\lambda}(q M_2 M_1 + q^{-1} M_1 M_2 \]
\[ - (M_1 + M_2) C + \frac{1}{[2]_q} C^2) . \]

\([16]\) means that \( C \) is central \(\hat{U}_q \). Finally, the commutation relations of the scaling operator \( \mu \) with the coordinates imply that it commutes with all the generators of \(\hat{U}_q \). The extension of \( A_x(\hat{U}_q, M_q) \) by \( \mu \) and \( \mu^{-1} \) will be denoted by \( A_{x,\mu}(\hat{U}_q, M_q) \). From \([16]\) we obtain the involution on the generators as \((V_i^+)^* = -q V_i^-, (V_i^-)^* = -q^{-1} V_i^+, (M_1)^* = M_2 \) and \( C^* = C \). Actually there is a natural extension of the Algebra \(\hat{U}_q \). We define the element \( H \in \hat{U}_q \) by
\[ H := M_1 M_2 - q^{-1} \lambda^2 V_1^+ V_2^- . \] (21)

\( H \) satisfies
\[ H(x^1, x^2, x^3, x^4) = (q^{-2} x^1, x^2, x^3, q^2 x^4) H, \] (22)
which together with $H(1) = 1$ implies that $H$ is a positive and invertible differential operator on $A_x(M_q)$. Therefore the operator $H^{1/2}$ and its inverse are well defined by their commutation relations with the coordinates following from (22). The extension of $U_q$ by $H^{\pm 1/2}$ will be denoted by $U_q$.

### 3.2 Differential Representation of $U_R$

In the undeformed case the algebra of angular momentum operators on Minkowski space is a representation of the Lie algebra of $SL(2, \mathbb{C})$, i.e. a representation of the left invariant vector fields on the Lie group. It turns out that a similar statement is true in the deformed case. To see this, we consider an algebra representation $\pi : U_\mathbb{R} \to \text{End}_x(A_x(M_q)), \ a \mapsto \pi(a) =: a_\pi$, of $U_\mathbb{R}$ by linear operators on $A_x(M_q)$. $a_\pi$ is defined for an arbitrary $f \in A_x(M_q)$ by

$$a_\pi(f) = (S^{-1}(a) \otimes \text{id}) \circ \delta(f), \quad (23)$$

where $\delta$ is the $SO_q(3,1)$-coaction defined in section 1. $U_\pi := \pi(U_\mathbb{R})$ is a subalgebra of $A_x(M_q)$. It turns out that $U_\pi = U_q$ and we obtain the

**Proposition 3.2** The algebra $U_q \subset \text{End}_x(A_x(M_q))$ is a $*$-representation of the algebra $U_\mathbb{R}$. In this representation the two Casimir operators of $U_\mathbb{R}$ coincide, i.e. $\pi(C_1) = \pi(C_2) = C$.

In particular, it turns out that $\pi$ maps the algebra of vector fields on $SL_q(2, \mathbb{C})$ to $U_q$. However, $A_x(M_q)$ being a comodule algebra, $U_\pi$ and therefore $U_q$ is also a bialgebra and even a Hopf algebra. Defining the mappings $\Delta : U_\pi \to U_\pi \otimes U_\pi, \varepsilon : U_\pi \to \mathbb{C}$ and $S_\pi : U_\pi \to U_\pi$ by

$$(\Delta_\pi(a_\pi))(f \otimes g) := a_\pi(f g) \quad \forall f, g \in A_x(M_q),$$

$$(\varepsilon_\pi(a_\pi)) := i(a(1)), \quad \pi(1),$$

$$(S_\pi(a_\pi)) := \pi(S^{-1}(a)),$$

where $i : A_x(M_q) \to \mathbb{C}$ projects the subalgebra of $A_x(M_q)$ generated by $1 \in A_x(M_q)$ onto the complex numbers, we deduce $\forall a \in U_\mathbb{R} : \Delta_\pi(a_\pi) = (\pi \otimes \pi)(\tau \circ \Delta(a)), \varepsilon_\pi(a_\pi) = a(1)$, where $\tau$ denotes the transposition of tensor components and $\Delta$ the coproduct in $U_\mathbb{R}$, and get the following

**Proposition 3.3** $U_\pi$ together with the coproduct $\Delta_\pi$, the counit $\varepsilon_\pi$, the antipode $S_\pi$ and the involution $a_\pi^* := \pi(a^*)$ is a $*$-Hopf algebra.

$U_q$ contains a $*$-subalgebra corresponding to vector fields on $SU_{q^{-1}}(2)$. It is generated by $H^{\pm 1/2}, X^\pm$ with

$$X^+ := \left( \frac{q}{[2]_q} \right)^{1/2} (qV_1^+ C - q^2V_1^+M_2 - q^{-1}V_2^+ M_1),$$

$$X^- := \left( \frac{q^{-1}}{[2]_q} \right)^{1/2} (-V_2^- C + q^{-1}V_2^- M_1 - q^{-1}V_1^- M_1),$$

$$X^- := \left( \frac{q^{-1}}{[2]_q} \right)^{1/2} (-V_2^- C + q^{-1}V_2^- M_1 - q^{-1}V_1^- M_1)$$

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and involution $H^* = H$ and $(X^+)^* = qX^-$. It will be denoted $\mathcal{U}_q^3$, its extension by the generators of $A_x(\mathcal{M}_q)$ by $A_x(\mathcal{U}_q^3, \mathcal{M}_q)$. The relations in $\mathcal{U}_q^3$ are those already encountered in (19). The element $W \in \mathcal{U}_q^3$, related to the central element $S^2 \in U_{q^{-1}}(SU(2))$ (cf. [9]) by $W := q\pi(\lambda^2 S^2 + [2]_q 1)$ is the central Casimir operator of $\mathcal{U}_q^3$. It satisfies

$$HW = q^2 1 + H^2 + q^2 \lambda^2 [2]_q X^- X^+, \quad W^* = W. \quad (27)$$

Finally, we give the relations determining $H$, $X^\pm$ and $W$ as operators on $A_x(\mathcal{M}_q)$. They read

$$H(t, y, z, x) = (t, q^{-2} y, z, q^2 x) H,$$
$$X^+(t, y, z, x) = (t, y, z, x) X^+ + (0, 0, q^{-1} y, -z) H$$
$$X^-(t, y, z, x) = (t, y, z, x) X^- + (0, z, -q x, 0) H \quad (28)$$
$$W(t, y, x, z) = (t, q^2 y, z, q^{-2} x) W + \lambda [2]_q (0, -y, q \lambda z, q \lambda x) H + \lambda^2 [2]_q [(0, q^2 z, -q x, 0) X^+ + (0, 0, q y, -z) X^-]. \quad (29)$$

As $H$ and $X^\pm$ are a representation of vector fields of the q-deformed rotation group, they commute with the generator $t \in A_x(\mathcal{M}_q)$ corresponding to the time coordinate in the limit $q \to 1$.

4 Algebra of Observables

We want to define momentum coordinates $p^i$, such that the algebra generated by the momenta is isomorphic to $A_x(\mathcal{M}_q)$ as a $\ast$-comodule algebra of $SO_q(3,1)$. Unlike the undeformed case, the partial derivatives $\partial^i$ are not closed under involution.

Lemma 4.1 The momentum operators $(p^j)_{j=1,\ldots,A} \in D(\mathcal{M}_q)$ defined by

$$p^j := -i(\partial^j + q^{-4} \bar{\partial}^j) \quad (30)$$

generate the momentum quantum space $A_p(\mathcal{M}_q)$ of functions over q-Minkowski space, i.e. they satisfy

$$P_A^{ij} p^k p^l = 0, \quad (p^j)^* = p^k C^{ij}_{kl} p^l. \quad (31)$$

The complete symmetry between coordinates and momenta as $\ast$-comodule algebras of $SO_q(3,1)$ is established by the following

Lemma 4.2 The algebra $A_p(\mathcal{U}_q, \mathcal{M}_q)$ generated by $(p^j)$ and $\mathcal{U}_q$ is isomorphic to $A_x(\mathcal{U}_q, \mathcal{M}_q)$ with the isomorphism $i$ given on the generators by $i(p^j) = x^i, i(V^{ij}) = V^{ij}$ and $i(U) = U$. 
Now we are in the position to introduce the algebra of observables (cf. [6]) by the following

**Definition 4.3** The algebra of observables $\mathcal{O} := \langle 1, (x^i)_{i=1,...,4}, (p^j)_{j=1,...,4} \rangle \subset D(\mathcal{M}_q)$ is the algebra generated by coordinates and momenta.

**Proposition 4.4** In $\mathcal{O}$ we have relations

$$p^i x^j - q \hat{R}^{-1} x^k p^l = -iu^{ij}$$

where $u^{ij} \in D(\mathcal{M}_q)$ are the elements defined in [7].

The next aim is to construct a Hilbert space representation of this algebra of observables, which can be interpreted as the state space for $q$-deformed relativistic quantum mechanics. To make the connection with the angular momentum algebra, we rewrite (32) as

$$p^i x^j - q \hat{R}^{-1} x^k p^l = -i\mu^{-1} (V^{ij} + U).$$

(33)

It was shown in [3] that we can reconstruct the partial derivatives given the coordinates and the angular momentum algebra with the help of the following identities in $A_{x,\mu}(\mathcal{U}_q, \mathcal{M}_q)$:

$$r^2 \partial^i = \frac{1 + q^2}{1 - q^2} x^i + \frac{1}{q^8 - 1} x^i U^\mu + \frac{1 + q^{-2}}{2(q^2 - 1)} C_{kl} x^k V^{li} \mu,$$

$$r^2 \bar{\partial}^i = -\frac{1 + q^2}{q^2 - 1} x^i + \frac{1}{q^8 - 1} x^i U^\mu^{-1} + q^2 \frac{1 + q^2}{2(1 - q^{-2})} C_{kl} x^k V^{li} \mu^{-1},$$

(34)

(35)

Due to the complete symmetry between coordinates and momenta, the same relations hold with coordinates and momenta exchanged. This means that we can find Hilbert space representations $\mathcal{O}$ by constructing Hilbert space representations of $A_{x,\mu}(\mathcal{U}_q, \mathcal{M}_q)$ with invertible $r^2$. This will be done in the next section.

## 5 Hilbert Space Representation of $\mathcal{O}$

### 5.1 Representation of $A_p(\mathcal{U}_q^3, \mathcal{M}_q)$

In analogy to the undeformed case we expect a relativistic one particle state to be an element of an irreducible representation of the Poincaré algebra. As a maximal set of commuting observables we take the square of the momentum $p^2 := C_{ij} p^i p^j$ which is the square of the particle mass, the energy $p^0$ (corresponding to the generator $t$ in the coordinate algebra), the angular momentum $W$ and the 3-component of angular momentum $H$. In the limit $q \to 1$ this is a complete set of commuting observables for spin 0. We will construct an
irreducible $*$-representation of $A_p(U_q, M_q)$ as a direct sum of irreducible representations of $A_p(U_q^3, M_q)$.

The hermitean elements $p^2, p^0$ are central in $A_p(U_q^3, M_q)$. In an irreducible representation we can therefore choose them as multiples of the identity operator. We define for arbitrary $M^2, E \in \mathbb{R}$ a linear space

$$H^{M,E} := \text{Lin}\{a|M, E, 0, 0) : a \in A_p(U_q^3, M_q)\},$$

where the cyclic vector $|M, E, 0, 0)$ has the properties

$$X^\pm|M, E, 0, 0) := 0, \quad H|M, E, 0, 0) := |M, E, 0, 0).$$

(37)

$$p^2|M, E, 0, 0) := M^2|M, E, 0, 0), \quad p^0|M, E, 0, 0) := E|M, E, 0, 0)$$

(38)

i.e. it carries a scalar representation of $U_q^3$.

The commutators (28) imply $H^{M,E} = \text{Lin}\{a|M, E, 0, 0) : a \in A_p(M_q)\}$. Considering the vector $p^2|M, E, 0, 0) \in H^{M,E}$ with $l \in \mathbb{N}_0$, we find it to be an eigenvector of $H$ and $W$ with eigenvalue $l$ for both operators which is annihilated by $X^+$. Therefore it is a highest weight for the irreducible representation of $U_q(SU(2))$ characterized by the eigenvalue $l$ of the Casimir $W$.

If we define $\gamma_{E,l}|M, E, l, l) := p^2|M, E, 0, 0)$, then a basis of this representation is given by the states $|M, E, l, m)$ with $m = -l, \ldots, l$ defined by

$$|M, E, l, m) := q^{m+1}\left(\frac{[2]_q}{[l + m + 1]_q[l - m]_q}\right)^{-1/2} X^-|M, E, l, m + 1).$$

(39)

Being eigenvectors to different eigenvalues of $H$, these states are linearly independent.

**Lemma 5.1** The set $B := \{|M, E, l, m) : m \in \mathbb{Z} \} \text{mit } l \in \mathbb{N}_0, \text{ } m \in \mathbb{Z} \} \text{ is a linear basis of } H^{M,E}$.

For $\langle M, E, l', m'|M, E, l, m)$ := $\delta_{l,l'}\delta_{m,m'}$ to be a consistent definition of a scalar product on $H^{M,E}$, the constant $\gamma_{E,l} = \langle M, E, 0, 0|y^+y^l|M, E, 0, 0)$ must be positive. Using the definition of the central element $p^2$ and the relations (28), we get a recursion formula for $\gamma_{E,l}$:

$$\gamma_{E,l+1} = \frac{q^{2}[l + 1]_q}{[2l + 3]_q}\left(\frac{1}{\{l + 1\}_q}\right)^{2 - \{l + 1\}_q M^2} \gamma_{E,l}$$

(40)

with $\{x\}_q := \frac{x^q - x^{-q}}{q + q^{-1}}$. As $\gamma_{E,0} = 1$ this fixes $\gamma_{E,l}$ for all $l \in \mathbb{N}_0$. We have to distinguish two cases:

i) $M^2 \leq 0 : \quad \gamma_{E,l} > 0 \quad \forall l \in \mathbb{N}_0.$

(41)

ii) $M^2 > 0 : \quad \gamma_{E,l} \xrightarrow{l \to \infty} -\infty.$

(42)
In the first case there is no restriction on the values of $E$, but for the physically interesting case of real mass the requirement of positivity forces the recursion to terminate, i.e. for each fixed value of $M^2 > 0$ there must be an $N \in \mathbb{N}_0$, such that $\gamma_{E,l} = 0 \; \forall \; l > N$. This means that in this case we get only the discrete energy eigenvalues

$$E = \pm \{N + 1\}_q M.$$  

(43)

In the sequel we take $M^2 > 0$. We can then use the positive integer $N$ to label the states by $|M, N, l, m\rangle$. The sign of the energy eigenvalues is an additional invariant of the representation, because all the generators of $A_p(U_q^3, \mathcal{M}_q)$ commute with $p^0$. So we have two orthonormal bases which span the linear spaces

$$\mathcal{H}^{M,N}_{\pm} := \text{Lin}\{\pm, M, N, l, m\} : N, l \in \mathbb{N}_0, \; l \leq N, \; m \in \mathbb{Z}, \; |m| \leq l\}. \tag{44}$$

Completing $\mathcal{H}^{M,N}_{\pm}$ to Hilbert spaces $\mathcal{H}^{M,N}_{\pm}$ with respect to the scalar product defined above, we get

Lemma 5.2 For every $M^2 > 0$ and $N \in \mathbb{N}_0$ the Hilbert spaces $\mathcal{H}^{M,N}_{\pm}$ are irreducible $\star$-representations of the algebra $A_p(U_q^3, \mathcal{M}_q)$.

5.2 Representation of $A_p(U_q, \mathcal{M}_q)$

We consider now the linear space

$$\mathcal{H}^M_{\pm} := \bigoplus_{N=0}^{\infty} \mathcal{H}^{M,N}_{\pm} \tag{45}$$

with scalar product $\langle \pm, M, N, l', m' | \pm, M, N, l, m \rangle := \delta_{NN'} \delta_{ll'} \delta_{mm'}$. To show that these spaces are representations of $A_p(U_q, \mathcal{M}_q)$, it is sufficient to determine the action of the generators of $U_q$ on the $U_q^3$-invariant vectors $|\pm, M, N, 0, 0\rangle$, because the action on an arbitrary vector $|\pm, M, N, l, m\rangle$ can then be deduced using the relations in $A_p(U_q, \mathcal{M}_q)$. Making use of the commutators of the generators of $U_q$ and $U_q^3$ and of the discreteness of the spectrum of $p^0$, we make the following ansatz:

$$V_k^{|\pm, M, N, 0, 0\rangle} = \sum_{i=0}^{\infty} c^{|\pm, N, k, i} |\pm, M, i, 1, 1\rangle,$$

$$V_k^{|\pm, M, N, 0, 0\rangle} = \sum_{i=0}^{\infty} c^-|\pm, N, k, i |\pm, M, i, 1, -1\rangle,$$

$$M_k|\pm, M, N, 0, 0\rangle = \sum_{i=0}^{\infty} c_0^{|\pm, N, k, i} |\pm, M, N, 1, 0\rangle + c'|_{\pm, N, k, i} |\pm, M, N, 0, 0\rangle,$$

with $k = 1, 2$. This also determines the action of the Casimir $C$, because one can prove $C|\pm, M, N, 0, 0\rangle = (qM_1 + q^{-1}M_2)|\pm, M, N, 0, 0\rangle$. The commutation
relations of the generators of $U_q$ with $X^+$ and $X^-$ can be used to eliminate all but two coefficients:

$$
\begin{align*}
    c^\prime_\pm, N, 1, i &= c^\prime_\pm, N, 2, i, \\
    c^+, N, 1, i &= (q[2]_q)^{1/2} \frac{1}{\chi} c_\pm, N, 1, i, \\
    c^-, N, 1, i &= -(q^{-1}[2]_q)^{1/2} \frac{1}{\chi} c_\pm, N, 1, i, \\
    c^+, N, 2, i &= -q^2 c_\pm, N, 1, i, \\
    c^+, N, 2, i &= q^3(q[2]_q)^{1/2} \frac{1}{\chi} c_\pm, N, 1, i, \\
    c^-, N, 2, i &= -(q[2]_q)^{1/2} \frac{1}{\chi} c_\pm, N, 1, i.
\end{align*}
$$

(47)

Dropping the index 1 we denote the remaining coefficients by $c_\pm, N, i$ and $c^\prime_\pm, N, i$.

The commutation relations of the generators of $U_q$ with $p^0$ can be used to determine the possible values of $i$. For $i \geq 1$ we find

$$
i = N \pm 1.
$$

(48)

Furthermore we get the result that $i = 0$ can only appear for $N = 1$ which is consistent with (48). Moreover we get for $i \geq 1$ relations between $c^\prime_\pm, N, i$ and $c_\pm, N, i$. Finally, using these relations, the identity $H(\pm, M, N, 0, 0) = (M_1 M_2 - q^{-1} \lambda^2 V_1^+ V_2^-)|\pm, M, N, 0, 0\rangle = |\pm, M, N, 0, 0\rangle$ and the fact that we want a $\ast$-representation of $A_p(U_q, M_q)$ on $H^M$, we can determine all the coefficients. Choosing a common phase to make the coefficients real, we obtain as the final result:

$$
\begin{align*}
    c^\prime_\pm, N, N + 1 &= c^\prime_\pm, N + 1, N = \frac{1}{[2]_q} \text{ für } N \geq 0, \\
    c^\prime_\pm, N, N - 1 &= c^\prime_\pm, N - 1, N = \frac{1}{[2]_q} \text{ für } N \geq 1, \\
    c_\pm, N, N + 1 &= \pm q^{-1}[2]_q \left( \frac{[N + 3]_q}{[3]_q[N + 1]_q} \right)^{1/2} \text{ für } N \geq 0, \\
    c_\pm, N, N - 1 &= \pm q^{-1}[2]_q \left( \frac{[N - 1]_q}{[3]_q[N + 1]_q} \right)^{1/2} \text{ für } N \geq 1.
\end{align*}
$$

(49) (50) (51) (52)

Checking now the consistency of this solution with all the relations in $A_p(U_q, M_q)$, and completing $H_\pm$ to the Hilbert space $H^M_\pm$ we get

**Proposition 5.3** The Hilbert space $H^M_\pm$ is an irreducible $\ast$-representation of the algebra $A_p(U_q, M_q)$.

This is in particular a representation of the q-deformed Poincaré algebra (cf. [8]).
5.3 Representation of $O$

For fixed $M^2 > 0$ the operator $p^2$ is constant and positive on $H^M_\pm$ and the operators $p := (p^2)^{1/2}, p^{-1} := (p^2)^{-1/2}$ and $p^0/p|\pm, M, N, l, m\rangle = \pm \{N + 1\}_q$ are well-defined. $\mu$ does not commute with $p^2$, so it connects different irreducible representations.

Let $M_0 > 0$ be fixed, $M_k := M_0 q^k$ with $k \in \mathbb{Z}$ and

$$H_\pm := \bigoplus_k H^M_k$$

with scalar product $\langle \pm, M_{k'}, N', l', m'|\pm, M_k, N, l, m\rangle := \delta_{k'k} \delta_{N'N} \delta_{l'l} \delta_{m'm}$. Denoting the Hilbert space completion of $H_\pm$ by $H_\pm$, we have

**Proposition 5.4** $H_\pm$ is an irreducible $\ast$-representation of $A_{p,\mu}(U_q, M_q)$ with $\mu$ acting as $\mu|\pm, M_k, N, l, m\rangle = |\pm, M_{k-1}, N, l, m\rangle$. $p^2$ is invertible on $H_\pm$, so this is in particular a representation of the algebra of observables $O$.

In this paper, we have found a Hilbert space representation for a deformation of the Heisenberg algebra. In quantum mechanics the operators $(x^i)$ and $(p^i)$ must be observables, i.e. their restriction to a common domain must be essentially self-adjoint. The operators $p^2$ and $p^0$, being diagonal in this representation, are already essentially self-adjoint on the Hilbert space. This implies also that all the momenta have this property. The question however is, whether $r^2$ and $t$ are essentially self-adjoint. To solve this problem, we have to diagonalize $r^{2\dagger}$. The transformation between the eigenvectors of $p^2$ and $r^2$ is then a generalized Fourier transformation. With the help of the operators $(x^i)$ one could then try to discuss the problem of locality.

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