The competition numbers of Johnson graphs with diameter four

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Abstract

In 2010, Kim, Park and Sano studied the competition numbers of Johnson graphs. They gave the competition numbers of $J(n, 2)$ and $J(n, 3)$. In this note, we consider the competition number of $J(n, 4)$.

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1. Introduction

The notion of a competition graph was introduced by Cohen [1] as a means of determining the smallest dimension of ecological phase space. The competition graph $C(D)$ of a digraph $D$ is a simple undirected graph which has the same vertex set as $D$ and an edge between vertices $x$ and $y$ if and only if there exists a vertex $u \in D$ such that $(x, u)$ and $(y, u)$ are arcs of $D$. For any graph $G$, $G$ together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Roberts [7] defined the competition number $k(G)$ of a graph $G$ to be the smallest number $k$ such that $G$ together with $k$ isolated vertices is the competition graph of an acyclic digraph. Opsut [4] showed that the computation of the competition number of a graph is an NP-hard problem. In the study of competition graphs, it has been one of important problems to determine the competition numbers for various graph classes. In [3], Kim, Park and Sano studied the competition numbers of Johnson graphs. In particular, they gave the following results.

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**Theorem 1.1** (See [3]). For \( n \geq 4 \), we have \( k(J(n, 2)) = 2 \).

**Theorem 1.2** (See [3]). For \( n \geq 6 \), we have \( k(J(n, 3)) = 4 \).

They also asked about the exact value of the competition number of \( J(n, 4) \). In this note, we give a partial answer to the question. Our result is the following.

**Theorem 1.3.** For \( n \geq 8 \), we have \( k(J(n, 4)) \in \{7, 8, 9\} \).

2. Preliminaries

Throughout this note, we use the notations given in [3]. We denote an \( n \)-set \( \{1, \ldots, n\} \) by \([n]\) and the set of all \( d \)-subsets of \( n \)-set by \( \binom{[n]}{d} \). The Johnson graph \( J(n, d) \) is an undirected graph whose vertex set is \( \{v_X \mid X \in \binom{[n]}{d}\} \), and two vertices \( v_X \) and \( v_Y \) are adjacent if and only if \( |X \cap Y| = d - 1 \). Since \( J(n, d) \) is isomorphic to \( J(n, n - d) \), we always assume \( n \geq 2d \).

For a digraph \( D \), a sequence \( v_1, \ldots, v_n \) of the vertex set \( V(D) \) is called an acyclic ordering of \( D \) if \( (v_i, v_j) \in A(D) \) implies \( i < j \). It is well known that a digraph \( D \) is acyclic if and only if there exists an acyclic ordering of \( D \).

For a digraph \( D \) and a vertex \( v \) of \( D \), we define the out-neighborhood \( P_D(v) \) of \( v \) in \( D \) to be the set \( \{w \in V(D) \mid (v, w) \in A(D)\} \). A vertex in the out-neighborhood of a vertex \( v \) in a digraph \( D \) is called a prey of \( v \) in \( D \).

For a graph \( G \) and a vertex \( v \) of \( G \), we define the neighborhood \( N_G(v) \) of \( v \) in \( G \) to be the set \( \{u \in V(G) \mid uv \in E(G)\} \). We also use \( N_G(v) \) to stand for the subgraph induced by its vertices.

For a clique \( S \) of a graph \( G \) and an edge \( e \) of \( G \), we say \( e \) is covered by \( S \) if both of the endpoints of \( e \) are contained in \( S \). An edge clique cover of a graph \( G \) is a family of cliques such that each edge of \( G \) is covered by some clique in the family. The edge clique cover number \( \theta_E(G) \) of a graph \( G \) is the minimum size of an edge clique cover of \( G \). An edge clique cover of \( G \) is called a minimum edge clique cover of \( G \) if its size is equal to \( \theta_E(G) \). A vertex clique cover of a graph \( G \) is a family of cliques such that each vertex of \( G \) is contained in some clique in the family. The vertex clique cover number \( \theta_V(G) \) of a graph \( G \) is the minimum size of a vertex clique cover of \( G \).

A minimum edge clique cover of \( J(n, d) \) is given in [3] as follows. For each \( Y \in \binom{[n]}{d-1} \), we define

\[
S_Y = \{v_X \mid X = Y \cup \{j\} \text{ for } j \in [n] \setminus Y\}.
\]

Then \( \{S_Y \mid Y \in \binom{[n]}{d-1}\} \) is the collection of cliques of maximum size. We denote it by \( \mathcal{F}_d^n \). Note that \( \mathcal{F}_d^n \) is an edge clique cover of \( J(n, d) \).

**Lemma 2.1** (See Section 3 of [3]). We have \( \theta_E(J(n, d)) = \binom{n}{d-1} \), and \( \mathcal{F}_d^n \) is a minimum edge clique cover of \( J(n, d) \).

3. Main results

In this section, we give a lower bound for the competition number of \( J(n, d) \) and an upper bound for the competition number of \( J(n, 4) \).

**Lemma 3.1** (See Lemma 3 of [3]). We have \( \theta_V(N_{J(n,d)}(x)) = d \).
Lemma 3.2 (See Theorem 4 of [3]). For any two adjacent vertices \(v_{x_1}\) and \(v_{x_2}\) of \(J(n, d)\), we have \(|P_D(v_{x_1}) \setminus P_D(v_{x_2})| \geq d - 1\).

Theorem 3.1. For \(n \geq 2d \geq 8\), we have \(k(J(n, d)) \geq 2d - 1\).

Proof. We denote \(k(J(n, d))\) by \(k\). Then there exists an acyclic digraph \(D\) such that \(C(D) = J(n, d) \cup I_k\), where \(I_k = \{z_1, z_2, \ldots, z_k\}\) is a set of isolated vertices.

Let \(x_1, x_2, \ldots, x_{\binom{n}{2}}, z_1, z_2, \ldots, z_k\) be an acyclic ordering of \(D\). Put \(v_1 = x_{\binom{n}{2}}, v_2 = x_{\binom{n}{2} - 1}\) and \(v_3 = x_{\binom{n}{2} - 2}\). It follows from Lemma 3.1 that \(\theta_V(N_{J(n, d)}(x_i)) = d\) for \(1 \leq i \leq \binom{n}{2}\). So, \(v_i\) has at least \(d\) distinct prey in \(D\), that is,

\[
|P_D(v_i)| \geq d. \tag{1}
\]

Since \(x_1, x_2, \ldots, x_{\binom{n}{2}}, z_1, z_2, \ldots, z_k\) is an acyclic ordering of \(D\), we have

\[
P_D(v_1) \cup P_D(v_2) \cup P_D(v_3) \subseteq I_k \cup \{v_1, v_2\}. \tag{2}
\]

First of all, we assume that \(v_1\) and \(v_2\) are not adjacent in \(J(n, d)\). Then \(v_1\) and \(v_2\) do not have a common prey in \(D\), that is,

\[
P_D(v_1) \cap P_D(v_2) = \emptyset. \tag{3}
\]

It follows from (1), (2) and (3) that

\[
k + 1 \geq |P_D(v_1) \cup P_D(v_2)| = |P_D(v_1)| + |P_D(v_2)| \geq 2d.
\]

So, we have \(k \geq 2d - 1\).

Next, we assume that \(v_1\) and \(v_2\) are adjacent in \(J(n, d)\). Then \(v_1\) and \(v_2\) have at least one common prey in \(D\), that is,

\[
|P_D(v_1) \cap P_D(v_2)| \geq 1. \tag{4}
\]

Now we divide our consideration into four cases:

1. \(v_1\) and \(v_3\) are not adjacent, and \(v_2\) and \(v_3\) are not adjacent;
2. \(v_1\) and \(v_2\) are adjacent, and \(v_2\) and \(v_3\) are not adjacent;
3. \(v_1\) and \(v_3\) are not adjacent, and \(v_2\) and \(v_3\) are adjacent;
4. \(v_1\) and \(v_3\) are adjacent, and \(v_2\) and \(v_3\) are adjacent.

In the first case, we have

\[
k + 2 \geq |P_D(v_1) \cup P_D(v_2) \cup P_D(v_3)| \quad \text{(by (2))}
= |P_D(v_3)| + |P_D(v_1) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \cap P_D(v_2)|
\geq d + d - 1 + d - 1 + 1 \quad \text{(by (1), Lemma 3.2 and (4))}
= 3d - 1.
\]

So, we have \(k \geq 3d - 3\).
In the second case, we have

\[ k + 2 \geq |P_D(v_1) \cup P_D(v_2) \cup P_D(v_3)| \quad \text{(by (2))} \]
\[ = |P_D(v_3) \setminus P_D(v_1)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \setminus P_D(v_2)| + |P_D(v_1) \cap P_D(v_2)| \]
\[ \geq d - 1 + d - 1 + d - 1 + 1 \quad \text{(by Lemma 3.2 and (4))} \]
\[ = 3d - 2. \]

So, we have \( k \geq 3d - 4. \)

In the third case, we have

\[ k + 2 \geq |P_D(v_1) \cup P_D(v_2) \cup P_D(v_3)| \quad \text{(by (2))} \]
\[ = |P_D(v_3) \setminus P_D(v_2)| + |P_D(v_1) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \cap P_D(v_2)| \]
\[ \geq d - 1 + d - 1 + d - 1 + 1 \quad \text{(by Lemma 3.2 and (4))} \]
\[ = 3d - 2. \]

So, we have \( k \geq 3d - 4. \)

In the fourth case, we have

\[ k + 2 \geq |P_D(v_1) \cup P_D(v_2) \cup P_D(v_3)| \quad \text{(by (2))} \]
\[ \geq |P_D(v_3) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \setminus P_D(v_3)| \]
\[ = d - 1 + d - 1 + d - 1 \quad \text{(by Lemma 3.2)} \]
\[ = 3d - 3. \]

So, we have \( k \geq 3d - 5. \)

Since \( d \geq 4, \) it holds \( 3d - 5 \geq 2d - 1. \) Therefore, we have \( k(J(n, d)) \geq 2d - 1. \) \( \square \)

Now we give an order \( \prec \) on the vertex set of \( J(n, d) \) as follows. Take two distinct elements \( v_{X_1} \) and \( v_{X_2} \) in \( \{v_X \mid X \in \binom{[n]}{d}\} \). Let \( X_1 = \{i_1, \ldots, i_d\} \) and \( X_2 = \{j_1, \ldots, j_d\} \), where \( i_1 < \cdots < i_d \) and \( j_1 < \cdots < j_d \). Then we define \( v_{X_1} \prec v_{X_2} \) if there exists \( t \in \{1, \ldots, d\} \) such that \( i_s = j_s \) for \( 1 \leq s \leq t - 1 \) and \( i_t < j_t. \)

**Theorem 3.2.** For \( n \geq 8, \) we have \( k(J(n, 4)) \leq 9. \)

**Proof.** We define a digraph \( D \) as follows:

\[
V(D) = V(J(n, 4)) \cup I_9
\]
where \( I_9 = \{ z_1, \ldots, z_9 \} \), and

\[
A(D) = \bigcup_{i=1}^{n-4} \bigcup_{j=i+1}^{n-3} \bigcup_{k=j+1}^{n-2} \{ (x, v_{i,j,k+1,k+2}) \mid x \in S_{i,j,k} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{n-4} \bigcup_{j=i+1}^{n-3} \{ (x, v_{i,j+1,j+1,j+3}) \mid x \in S_{i,j,n-1} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{n-5} \bigcup_{j=i+1}^{n-4} \{ (x, v_{i,j+1,j+2,j+4}) \mid x \in S_{i,j,n} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{n-6} \bigcup_{j=i+1}^{n-5} \{ (x, v_{i+1,i+2,i+3,i+4}) \mid x \in S_{i,n-3} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{n-6} \{ (x, z_8) \mid x \in S_{n-5,n-1,n} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{n-6} \{ (x, v_{i+1,i+2,i+3,i+6}) \mid x \in S_{i,n-2} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{n-5} \{ (x, v_{i+1,i+2,i+3,i+5}) \mid x \in S_{i,n-2,n-1} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{3} \{ (x, z_i) \mid x \in S_{n-5+i,n-1,n} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{2} \{ (x, z_{i+3}) \mid x \in S_{n-5+i,n-2,n} \in \mathcal{F}_4^n \}
\]

\[
\bigcup_{i=1}^{2} \{ (x, z_{i+5}) \mid x \in S_{n-5+i,n-2,n-1} \in \mathcal{F}_4^n \}.
\]

It is easy to see that

\[
\mathcal{F}_4^n = \{ S_{i,j,k} \mid i = 1, \ldots, n-4; j = i+1, \ldots, n-3; k = j+1, \ldots, n-2 \}
\]

\[
\bigcup \{ S_{i,j,n-1}, S_{i,j,n} \mid i = 1, \ldots, n-4; j = i+1, \ldots, n-3 \}
\]

\[
\bigcup \{ S_{i,n-1,n}, S_{i,n-2,n}, S_{i,n-2,n-1} \mid i = 1, \ldots, n-5 \}
\]

\[
\bigcup \{ S_{n-4,n-1,n}, S_{n-3,n-1,n}, S_{n-2,n-1,n} \}
\]

\[
\bigcup \{ S_{n-4,n-2,n}, S_{n-3,n-2,n} \} \cup \{ S_{n-4,n-2,n-1}, S_{n-3,n-2,n-1} \}.
\]

By the definition of \( \prec \), for \( x \) in the cliques in \( \mathcal{F}_4^n \) one can check that \( (x, y) \in A(D) \) if and only if either \( x = v_X \) and \( y = V_Y \) with \( X \prec Y \), or \( x = v_X \) and \( y = z_i \) with \( X \in S_{n-4,n-1,n} \) \cup
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\[ S_{\{n-3,n-1,n\}} \cup S_{\{n-2,n-1,n\}} \cup S_{\{n-4,n-2,n\}} \cup S_{\{n-3,n-2,n\}} \cup S_{\{n-4,n-2,n-1\}} \cup S_{\{n-3,n-2,n-1\}} \cup S_{\{n-5,n-1,n\}} \cup S_{\{n-5,n-2,n\}} \] and \(1 \leq i \leq 9\). Thus, we have \( C(D) = J(n, 4) \cup I_9\). This completes the proof. \( \square \)

By Theorems 3.1 and 3.2, we have Theorem 1.3.

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