The Yamabe operator and invariants on octonionic contact manifolds and convex cocompact subgroups of $F_4(-20)$

Yun Shi $^1$ · Wei Wang $^2$

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Abstract
An octonionic contact (OC) manifold is always spherical. We construct the OC Yamabe operator on an OC manifold and prove its transformation formula under conformal OC transformations. An OC manifold is scalar positive, negative or vanishing if and only if its OC Yamabe invariant is positive, negative or zero, respectively. On a scalar positive OC manifold, we can construct the Green function of the OC Yamabe operator and apply it to construct a conformally invariant tensor. It becomes an OC metric if the OC positive mass conjecture is true. We also show the connected sum of two scalar positive OC manifolds to be scalar positive if the neck is sufficiently long. On the OC manifold constructed from a convex cocompact subgroup of $F_4(-20)$, we construct a Nayatani-type Carnot–Carathéodory metric. As a corollary, such an OC manifold is scalar positive, negative or vanishing if and only if the Poincaré critical exponent of the subgroup is less than, greater than or equal to 10, respectively.

Keywords Octonionic contact (OC) manifolds · The Biquard connection · The OC Yamabe operator · Transformation formula under conformal OC transformations · Convex cocompact subgroups of $F_4(-20)$ · The octonionic Heisenberg group

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Wei Wang
wwang@zju.edu.cn
Yun Shi
shiyun@zust.edu.cn

1 Department of Mathematics, Zhejiang University of Science and Technology, Hangzhou 310023, China
2 Department of Mathematics, Zhejiang University, Hangzhou 310027, China
1 Introduction

Biquard [5] introduced notions of quaternionic and octonionic contact manifolds. Recently, it is an active direction to study quaternionic contact manifolds (see [3, 18–22, 30, 40, 44] and reference therein). An octonionic contact manifold \((M, g, \mathfrak{l})\) is a 15-dimensional manifold \(M\) with a codimension 7 distribution \(H\) locally given as the kernel of a \(\mathbb{R}^7\)-valued 1-form \(\Theta = (\theta_1, \ldots, \theta_7)\), on which \(g\) is a Carnot–Carathéodory metric, where \(\mathfrak{l} := (I_1, \ldots, I_7)\) with \(I_\beta \in \text{End}(H)\) satisfying the octonionic commutating relation (2.6). Note that \(\mathfrak{l}\) is a rank-7 bundle which consists of endomorphisms of \(H\) locally generated by 7 almost complex structures \(I_1, \ldots, I_7\) on \(H\). They are Hermitian compatible with the metric:

\[
g(I_\beta^*, I_\beta^*) = g(\cdot, \cdot),
\]

and satisfy the compatibility condition

\[
g(I_\beta X, Y) = d\theta_\beta(X, Y),
\]

for any \(X, Y \in H, \beta = 1, \ldots, 7\). Biquard [5] introduced a canonical connection on an OC manifold, which is called Biquard connection now.

As pointed by Biquard [5], any octonionic contact manifold is spherical by a theorem of Yamaguchi [46]. \(f : (M, g, \mathfrak{l}) \to (M', g', \mathfrak{l}')\) is called OC conformal if locally we have \(f^* g' = \phi g\) for some positive function \(\phi > 0\) and \(f^* \mathfrak{l}' = \Psi \mathfrak{l}\) for some \(\text{SO}(7)\) valued smooth function \(\Psi\) [5]. The conformal class of OC manifolds is denoted by \([M, g, \mathfrak{l}]\). The purpose of this paper is to investigate the conformal geometry of OC manifolds as we have done for spherical CR manifolds [42] and spherical qc manifolds [40]. The main difficulty to investigate OC manifolds comes from algebra. The associative algebras \(\mathbb{R}, \mathbb{C}\) and \(\mathbb{H}\) are replaced by the non-associative octonion algebra \(\mathbb{O}\), and the classical Lie groups \(\text{SO}, \text{SU}\) and \(\text{Sp}\) are replaced by the exceptional Lie group \(\text{F}_4(-20)\). For example, it is more complicated to describe explicitly actions of \(\text{F}_4(-20)\), on the octonionic hyperbolic space as isometries and on the octonionic Heisenberg group as conformal transformations, which are given in Sect. 2.

We give the transformation formula of scalar curvatures under a conformal transformation in Sect. 3.

**Theorem 1.1** The scalar curvature \(s_g\) of the Biquard connection of \((M, g, \mathfrak{l})\) with \(\bar{g} = \phi^{\frac{4}{Q-2}} g\) satisfies the OC Yamabe equation:

\[
L_g \phi = s_g \phi^{\frac{Q+2}{Q-2}}, \quad b = \frac{4(Q-1)}{Q-2} = \frac{21}{5},
\]

where \(L_g := b\Delta_g + s_g\) is the OC Yamabe operator and \(\Delta_g\) is the sub-Laplacian associated to the Carnot–Carathéodory metric \(g\) and \(Q = 22\) is the homogeneous dimension of \(M\).

**Corollary 1.1** The OC Yamabe operator \(L_g\) satisfies the transformation formula

\[
L_g \phi = \phi^{\frac{Q+2}{Q-2}} L_g (\phi f),
\]
if \( \tilde{g} = \phi^{-\frac{4}{Q-2}} g \) and \( f \in C^\infty(M) \).

As in the locally conformally flat, CR and qc cases, for a connected compact octonionic manifold \((M, g, I)\), we have the following trichotomy: there exists an OC metric \( \tilde{g} \) conformal to \( g \) which has either positive, negative or vanishing scalar curvature everywhere. Denote by \( G_{\tilde{g}}(\xi, \cdot) \) the Green function of the OC Yamabe operator with the pole at \( \xi \), i.e., \( L_{\tilde{g}} G_{\tilde{g}}(\xi, \cdot) = \delta_\xi \), where \( \delta_\xi \) is the Dirac function at the point \( \xi \). On a scalar positive OC manifold, the Green function of the OC Yamabe operator \( L_g \) always exists, and is its singular part, if we identify a neighborhood of \( \xi \) with an open set of the octonionic Heisenberg group \( H \) with the OC metric \( g = \phi^{-\frac{4}{Q-2}} g_0 \). Here, \( g_0 \) is the standard OC metric on the octonionic Heisenberg group and \( C_Q \) is a positive constant (4.1).

**Theorem 1.2** Let \((M, g, I)\) be connected, compact, scalar positive OC manifold, which is not OC equivalent to the standard sphere. Define \( \text{can}(g) := A^2_g \), where

\[
A_g(\xi) = \lim_{\eta \to \xi} \left| G_g(\xi, \eta) - \rho_g(\xi, \eta) \right|^{\frac{1}{Q-2}},
\]

if \( g = \phi^{-\frac{4}{Q-2}} g_0 \) on a neighborhood \( U \) of \( \xi \). Then, \( \text{can}(g) \) is well defined and depends only on the conformal class \([M, g, I]\).

As in the locally flat [39], CR [6, 29] and qc [40] cases, we propose the following OC positive mass conjecture: Let \((M, g, I)\) be a compact scalar positive OC manifold with \( \dim M = 15 \). Then,

1. For each \( \xi \in M \), there exists a local OC diffeomorphism \( C_\xi \) from a neighborhood of \( \xi \) to the octonionic Heisenberg group \( H \) such that \( C_\xi(\xi) = \infty \) and

\[
\left( C_\xi^{-1} \right)^\# \left( G_g(\xi, \cdot) \phi^{-\frac{4}{Q-2}} g \right) = h(\phi^{-\frac{4}{Q-2}} g_0),
\]

where

\[
h(\eta) = 1 + A_g(\xi)\eta^{Q+2} + O(\eta^{Q+1}),
\]

near \( \infty \), and \( g_0 \) is the standard OC metric on \( H \); \( A_g(\xi) \) is called the OC mass at the point \( \xi \).

2. \( A_g(\xi) \) is nonnegative. It is zero if and only if \((M, g, I)\) is OC equivalent to the standard sphere.

We also introduce the connected sum of two OC manifolds and prove that the connected sum of two scalar positive OC manifolds is also scalar positive if the neck is sufficiently long.

In the last section, we recall definitions of a convex cocompact subgroup of \( F_4(-20) \) and the Patterson–Sullivan measure. For a discrete subgroup \( \Gamma \) of \( F_4(-20) \), the limit set of \( \Gamma \) is
\[ \Lambda(\Gamma) = \overline{\Gamma q} \cap \partial \mathcal{U}, \tag{1.7} \]

for \( q \) in the Siegel domain \( \mathcal{U} \), where \( \overline{\Gamma q} \) is the closure of the orbit of \( q \) under \( \Gamma \), and

\[ \Omega(\Gamma) = \partial \mathcal{U} \setminus \Lambda(\Gamma) \tag{1.8} \]

is the maximal open set where \( \Gamma \) acts discontinuously. It is known that \( \Omega(\Gamma)/\Gamma \) is a compact OC manifold when \( \Gamma \) is a convex cocompact subgroup of \( F_{4(-20)} \). The Poincaré critical exponent \( \delta(\Gamma) \) of a discrete subgroup \( \Gamma \) is defined as

\[ \delta(\Gamma) = \inf \left\{ s > 0; \sum_{\gamma \in \Gamma} e^{-s d(z, \gamma(w))} < \infty \right\}, \]

where \( z \) and \( w \) are two points in \( \mathcal{U} \) and \( d(\cdot, \cdot) \) is the octonionic hyperbolic distance on \( \mathcal{U} \). \( \delta(\Gamma) \) is independent of the particular choice of points \( z \) and \( w \). For any convex cocompact subgroup \( \Gamma \) of \( F_{4(-20)} \), there exists a probability measure \( \tilde{\mu}_\Gamma \) supported on its limit set \( \Lambda(\Gamma) \), called the Patterson–Sullivan measure, such that

\[ \gamma^* \tilde{\mu}_\Gamma = |\gamma'|^{\delta(\Gamma)} \tilde{\mu}_\Gamma \]

for any \( \gamma \in \Gamma \) (cf. [10]), where \( |\gamma'| \) is the conformal factor. Define \( \mu_\Gamma = \chi \tilde{\mu}_\Gamma \) by choosing a suitable factor \( \chi \) in (5.3). The conformal factor of \( \mu_\Gamma \) is the same as that \( \Gamma \) acts on the octonionic Heisenberg group. We define a \( C^\infty \) function on \( \Omega(\Gamma) \) by

\[ \phi_\Gamma(\xi) := \left( \int_{\Lambda(\Gamma)} G_0^\kappa(\xi, \eta)d\mu_\Gamma(\eta) \right)^{\frac{1}{\kappa}}, \quad \kappa = \frac{2\delta(\Gamma)}{Q-2}, \tag{1.9} \]

where \( G_0(\xi, \eta) \) is the Green function of OC Yamabe operator on the octonionic Heisenberg group with the pole at \( \xi \). Then

\[ g_\Gamma := \phi_\Gamma^{\frac{Q}{Q-2}} g_0, \tag{1.10} \]

is invariant under \( \Gamma \), which is the OC generalization of Nayatani’s canonical metric in conformal geometry [32]. See [33, 40, 42] for CR case and qc case, respectively.

**Theorem 1.3** Let \( \Gamma \) be a convex cocompact subgroup of \( F_{4(-20)} \) such that \( \Lambda(\Gamma) \neq \{ \text{point} \} \). Then, the scalar curvature of \( (\Omega(\Gamma)/\Gamma, g_\Gamma, \mathbf{1}) \) is positive (or negative, or zero) everywhere if and only if \( \delta(\Gamma) < 10 \) (or \( \delta(\Gamma) > 10 \), or \( \delta(\Gamma) = 10 \)).

## 2 Preliminaries of octonions, Jordan algebra, \( F_{4(-20)} \), the octonionic hyperbolic space and the octonionic Heisenberg group

### 2.1 Octonions and Jordan algebra

In this paper, we denote by \( a, b, c, \ldots \) numbers in \( Z_8 := \{0, \ldots, 7\} \) and by \( a, b, c, \ldots \) numbers in \( Z_8^+ := \{1, \ldots, 7\} \). The octonion algebra has a basis \( \{e_a\} \) satisfying the relation \( e_a e_0 = e_0 e_a, a \in Z_8 \), and

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The constants $\epsilon_{\alpha \beta \gamma}$ in (2.1) are completely antisymmetric in $\alpha, \beta, \gamma$, and equal the value $+1$ for $(\alpha, \beta, \gamma) \in \Omega$ (cf., e.g., [41]), where

$$\Omega := \{(1, 2, 3), (2, 4, 6), (4, 3, 5), (3, 6, 7), (6, 5, 1), (5, 7, 2), (7, 1, 4)\}. \quad (2.2)$$

$\epsilon_{\alpha \beta \gamma}$ is nonzero only for $(\alpha, \beta, \gamma)$ to be a permutation of a triple in $\Omega$. We may choose $\Omega$ differently (cf., e.g., [2]). By (2.2), it is direct to see that, for fixed $(\alpha, \beta)$, $\epsilon_{\alpha \beta \gamma}$ is non-vanishing only when $\gamma$. So, $\epsilon_{\alpha \beta \gamma}$ in (2.1) is the same as $\sum_{\gamma=1}^8 \epsilon_{\alpha \beta \gamma} \epsilon_\gamma$.

The octonion algebra $O$ is neither commutative nor associative. For $x, y, z \in O$, define an associator: $\{x, y, z\} := (xy)z - x(yz)$. Besides, the octonions obey some weak associative laws, such as the so-called Moufang identities (cf., e.g., (2.5) in [41]):

$$(uvu)x = u(v(u)x), \quad x(uy) = ((ux)v)u, \quad u(xy)u = (ux)(yu),$$

for any $u, v, x, y \in O$. In particular, we have

$$uvu = (uv)u = u(vu).$$

Namely, the octonion algebra is alternative, i.e., $\{u, v, u\} = \{u, u, v\} = \{v, u, u\} = 0$ for any $u, v \in O$.

**Proposition 2.1** (cf. [41, Proposition 2.1] and references therein) For any $a, b, c, d \in \mathbb{Z}_8$, we have

$$\{e_a, e_b, e_c\} := (e_a e_b) e_c - e_a (e_b e_c) = 2 e_{abcd} e_d. \quad (2.3)$$

where $e_{abcd}$ is totally antisymmetric, and equals to $1$ for $(a, b, c, d) \in \Lambda$, where

$$\Lambda = \{(5, 4, 6, 7), (7, 3, 5, 1), (1, 6, 7, 2), (2, 5, 1, 4), (4, 7, 2, 3), (3, 1, 4, 6), (6, 2, 5, 3)\}.$$

$e_{abcd}$ is nonzero only when $(a, b, c, d)$ is the permutation of a quadruple in $\Lambda$.

Denote by $M(n, O)$ all $(n \times n)$ matrices with entries in $O$ and $I_n = \text{diag}(1, \ldots, 1) \in M(n, O)$ the unit matrix. For $A \in M(n, O)$, $A'$ denotes the transposed matrix of $A$. The Jordan algebra

$$\mathcal{J} := \{X \in M(3, O)|D_1 X = \bar{X}' D_1\},$$

has $F_{4(-20)}$ as its automorphism group (cf. [1, 31, 36]), where

$$D_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (2.4)$$

**Proposition 2.2** ([17, Lemma 14.66]) $\text{Spin}(7)$ is generated by $\{L_\mu : \mu \in S^6 \subset \text{Im} O\}$, where $L_\mu x = \mu x$, for $x \in O$.

It is characterized as the subgroup of $SO(8)$ which conjugates $\mathbb{R}^7 = \{L_\mu : \mu \in \text{Im} O\}$ to itself (cf. [3]). We have a decomposition of $\mathfrak{so}(8)$ into irreducible $\mathfrak{so}(7)$ modules in the form

$$e_\alpha e_\beta = -\delta_{\alpha \beta} + \epsilon_{\alpha \beta \gamma} e_\gamma, \quad (2.1)$$
\[ \wedge^2 \mathcal{O} = \mathfrak{so}(8) = \mathfrak{so}(7) \oplus \mathbb{R}^7. \]  
(2.5)

In the sequel, we will use the Einstein convention of repeated indices. If we identify \( \mathcal{O} \) with \( \mathbb{R}^8 \), the left multiplication by \( e_\beta \) is a linear transformation on \( \mathbb{R}^8 \), given by a \((8 \times 8)\)-matrix \( I_\beta = L_{e_\beta} \). Namely, by identifying \( x = x_a e_a \) with \( x = (x_0, \ldots, x_7) \), we have

\[ e_\beta x = (I_\beta x)_a e_a. \]

\((I_\beta x)_a \) is the \( a \)-th entry of the vector in \( \mathbb{R}^8 \) corresponding to \( e_\beta x \). \( I_1, \ldots, I_7 \) do not satisfy the commutating relation \((2.1)\) of octonions because of the non-associativity of \( \mathcal{O} \).

**Proposition 2.3** (cf. [41, Proposition 3.1, 3.2]) Suppose \( e_a e_b = e_\beta, \ a, b \in \mathbb{Z}_{\geq 1} \). Then, we have

\[ I_a I_\beta = I_\beta - N_{ab}, \]
(2.6)

where \( N_{ab} \) are \((8 \times 8)\)-matrices with \((N_{ab})_{dc} = 2e_{abcd} \).

**Proof** Note that \((e_a e_b)x = e_\beta x = (I_\beta x)_c e_c \), and

\[ e_a (e_b x) = e_a ((I_\beta x)_c e_c) = (I_\beta x)_c \cdot (I_a)_{dc} e_d = (I_a I_\beta x)_d e_d. \]

We find that

\[ (I_a I_\beta x)_d e_d = e_a (e_b e_c) x_c = (e_a e_b)(e_c x_c) - 2e_{abcd} e_d x_c = e_\beta (x_c e_c) - 2e_{abcd} e_d x_c = (I_\beta x)_d e_d - (N_{ab} x)_d e_d. \]

Then, \((2.6)\) follows. \( \square \)

Set

\[ E^\beta = \begin{pmatrix} 0 & -\nu^\beta \\ \nu^\beta & e^\beta \end{pmatrix}, \quad \beta = 1, \ldots, 7. \]
(2.7)

Here, \( e^\beta \) are \((7 \times 7)\)-matrices with

\[ e^\beta_{\alpha \gamma} = e_{\gamma \alpha \beta}, \quad \nu^\beta = (\delta^\beta_1, \ldots, \delta^\beta_7) \in \mathbb{R}^7. \]

Note that \( e_\beta x_\gamma e_\gamma = e_{\gamma \alpha \beta} x_\gamma e_\gamma - x_\beta = e_{\gamma \alpha \beta} x_\gamma e_\gamma - x_\beta = (e^\beta_{\alpha \gamma} x_\gamma) e_\gamma - x_\beta \). So, \((I_\beta)_{0a} \) is the \((b, a)\)-th entry of \( E^\beta \) given by \((2.7)\).

**Proposition 2.4** \( E^\beta \)'s are antisymmetric matrices satisfying \( 1 \) \( (E^\beta)^2 = -I_{8 \times 8} \); \( 2 \) \( E^a E^\beta = -E^\beta E^a \), for \( a \neq \beta \in \mathbb{Z}_{\geq 1} \).

**Proof** The proposition is proved in [41, Proposition 3.1, 3.3].

1. \(-x = (e_\beta e_\beta)x = e_\beta (I_\beta x)_a e_a = (I^\beta x)_a e_a \), i.e., \((E^\beta)^2 = -I_{8 \times 8} \).
2. It follows from \((2.6)\) that \( E^a E^\beta \) are also antisymmetric by the antisymmetry of \( E^\beta \) and \( N \).

\( \square \)
2.2 The octonionic hyperbolic space

Let us recall basic facts of octonionic hyperbolic space (cf. [1, 36, 38]). Define

$$\mathbb{O}^3_0 = \left\{ \mathbf{v} = \begin{pmatrix} y \\ x \\ z \end{pmatrix} : x, y, z \text { all lie in some associative subalgebra of } \mathbb{O} \right\}.$$  

$v \sim w$ if $v = w \lambda$ for some $\lambda$ in an associative subalgebra of $\mathbb{O}$ containing the entries $x, y, z$ of $v$. The map from $\mathbb{O}^3_0$ to the set of equivalent classes is the analogue of right projection and so we denote by $PO^3_0$ the set of right equivalent classes. Define a map $\pi_{D_1} : \mathbb{O}^3_0 \rightarrow J$ by

$$\pi_{D_1}(\mathbf{v}) = \mathbf{v} v^* D_1 = \begin{pmatrix} y \bar{x} & y \bar{x} & |y|^2 \\ x \bar{y} & x \bar{y} & |x|^2 \\ |z|^2 & z \bar{y} \end{pmatrix},$$

where $D_1$ is given by (2.4) and $v^* = (\bar{y}, \bar{x}, \bar{z})$. If $v = (x, y) \in \mathbb{O}^2$, let $\bar{v}$ denote the column vector

$$\bar{v} = \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \in \mathbb{O}^3.$$

Set $j : \mathbb{O}^2 \rightarrow M(3, \mathbb{O})$ with

$$j(v) = \pi_{D_1}(\bar{v}) = \bar{v} v^* D_1 = \begin{pmatrix} y \bar{y} & |y|^2 \\ x \bar{x} & x \bar{x} & |x|^2 \\ 1 & \bar{x} & \bar{y} \end{pmatrix}, \quad j(\infty) = \pi_{D_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for $v = (x, y) \in \mathbb{O}^2$. Then, we can define

$$D_\pm := \{ \bar{v} \in \mathbb{O}^3_0; \pm \text{tr}(j(v)) > 0 \}, \quad D_0 := \{ \bar{v} \in \mathbb{O}^3_0; \text{tr}(j(v)) = 0 \}.$$  

$H^2_\mathbb{O} := D_-/\sim \subset PO^3_0$ is the octonionic hyperbolic space. Recall that $\text{tr}(j(v)) = \text{tr}(\bar{v} v^* D_1) = v^* D_1 \bar{v}$, which is the analog of the Hermitian form in quaternionic case. We define a bilinear form on $\mathbb{O}^2$ (cf. [36, p. 87]) by

$$\langle j(v), j(w) \rangle = \frac{1}{2} \text{Re} \text{tr}(j(v)j(w) + j(w)j(v)).$$

Then, we have $\langle j(v), j(w) \rangle = |v^* D_1 \bar{w}|^2$. Define

$$(v, w) := \frac{|v^* D_1 \bar{w}|}{|v^* D_1 \bar{v}|^\frac{1}{2} |\bar{w} D_1 \bar{v}|^\frac{1}{2}}.$$  

The metric on $H^2_\mathbb{O}$ is given by (cf. [36, p. 88])

$$ds^2 = -4 \frac{|v^* D_1 \bar{v}||d\bar{v} D_1 d\bar{v}| - |v^* D_1 d\bar{v}|^2}{|v^* D_1 \bar{v}|^2},$$

(2.10)
at point \( v \in D_\ast \), and the distance \( d(\cdot, \cdot) \) is given by

\[
cosh \left( \frac{d(v, w)}{2} \right) = (v, w).
\]

(2.11)

\( D_\ast \) is exactly the \textit{octonionic Siegel domain}:

\[
\mathcal{U} = \{ (x, y) \in \mathbb{O}^2 : 2 \text{Re} \ y + |x|^2 < 0 \}.
\]

We introduce the positive definite form \( \langle v, w \rangle = v_1 \bar{w}_1 + v_2 \bar{w}_2 \) on \( \mathbb{O}^2 \) and the ball model for octonionic hyperbolic space \( B^{16} = \{ v \in \mathbb{O}^2; \langle v, v \rangle < 1 \} \). The \textit{Cayley transform} is the map from the sphere \( S^{15} \) minus the southern point to the boundary of the Siegel domain

\[
\partial \mathcal{U} = \{ (x, y) \in \mathbb{O}^2 : 2 \text{Re} \ y + |x|^2 = 0 \}
\]

defined by

\[
C : S^{15} \to \mathcal{U}, \quad (v_1, v_2) \mapsto \left( \sqrt{2}(1 + v_2)^{-1}v_1, -(1 - v_2)(1 + v_2)^{-1} \right).
\]

\section{2.3 The octonionic Heisenberg group}

The octonionic Heisenberg group \( \mathcal{H} \) is \( \mathbb{O} \oplus \text{Im} \mathbb{O} \) equipped with the multiplication given by

\[
(x, t) \cdot (y, s) = (x + y, t + s + 2 \text{Im}(x\bar{y})),
\]

(2.12)

where \( (x, t), (y, s) \in \mathbb{O} \oplus \text{Im} \mathbb{O} \). Note that

\[
x\bar{y} = x_0y_0 - x_0y_\gamma e_\gamma + y_0x_\gamma e_\gamma - x_0y_\gamma e_\alpha e_\beta e_\gamma,
\]

(2.13)

by (2.1), i.e., \( \text{Im}(x\bar{y}) = \left( -x_0y_\gamma + y_0x_\gamma - x_0y_\gamma e_\alpha e_\beta e_\gamma \right)e_\gamma \). Therefore, the multiplication of the octonionic Heisenberg group in terms of real variables can be written as

\[
(x, t) \cdot (y, s) = \left( x + y, t + s + 2E^\beta_{ab}x_ay_b \right),
\]

where \( x = (x_0, \ldots, x_7) \in \mathbb{R}^8, t = (t_1, \ldots, t_7) \in \mathbb{R}^7 \), and \( E^\beta \) are given by (2.7).

Since the definition of our octonionic Heisenberg group is a bit different from that in [41] (with \( \text{Im}(\bar{x}y) \) replaced by \( \text{Im}(x\bar{y}) \)), so are \( E^\beta \)'s. The norm of the octonionic Heisenberg group \( \mathcal{H} \) is defined by

\[
\| (x, t) \| := (|x|^4 + |t|^2)^{\frac{1}{4}}.
\]

(2.14)

By definition,

\[
X_a = \frac{\partial}{\partial x_a} + 2E^\beta_{ba}x_b \frac{\partial}{\partial t_\beta}, \quad a = 0, 1, \ldots, 7,
\]

(2.15)

are the left invariant vector fields on \( \mathcal{H} \). The standard \( \mathbb{R}^7 \)-valued contact form of the group is

\[
\Theta_0 := dt - x \cdot d\bar{x} + dx \cdot \bar{x}.
\]

(2.16)

If we write \( \Theta_0 = (\theta_{0,1}, \ldots, \theta_{0,7}) \), then we have
θ₀; β = dt β − 2E^β ba x^b dx_a, \quad (2.17)

by using (2.13). The standard Carnot–Carathéodory metric on the group is \( g_0(X_a, X_b) = δ_{ab} \).

The transformations \( I_β \) on \( H_0 \) are given by \( I_β X_a = E^β_{ba} X_b \). Let \( \mathbb{V} \) be the Biquard connection associated to this standard OC structure on \( \mathcal{H} \), which is the flat model of OC manifolds, i.e.,

\[ \nabla_{X_a} X_b = 0, \quad (2.18) \]

for any \( X_a, X_b \) in (2.15) and its scalar curvature and torsion are identically zero. The sub-Laplacian on \( \mathcal{H} \) is \( \Delta_0 = − \sum'_{a=0} X^2_a \). We can identify \( \mathcal{H} \) with the boundary of the Siegel domain, by using the projection

\[ \pi : \partial \mathcal{U} \rightarrow \mathcal{H}, \quad (y, z) \mapsto \left( \frac{y}{\sqrt{2}}, \frac{|y|^2}{2} + z \right) = (x, t). \quad (2.19) \]

### 2.4 The exceptional group \( F_4(-20) \)

It is the group of isometries of octonionic hyperbolic space \( H^2_O \). The group is generated by

\[ T = \begin{pmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \quad D_δ = \begin{pmatrix} δ & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{δ} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.20) \]

where \( δ \neq 0 \in \mathbb{R} \), and

\[ S_μ : \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} μȳμ \\ μx \\ 1 \end{pmatrix}, \quad (2.21) \]

for unit imaginary octonion \( μ \) (cf. [36, p. 88]). The group generated by transformations \( \{ S_μ : μ \in \text{Im } O \} \) is the compact group \( \text{Spin}(7) \). We remark that in general \( S_μ \circ S_ν \neq S_{μν} \) for unit imaginary octonions \( μ, ν, F_4(-20) \), acts on \( O^3 \) as left action of matrices (2.20)–(2.21), and the induced action of \( F_4(-20) \) on \( O^2 \subset PO^3_0 \) is

\[ γ(ν) = (γ(ν_1)γ(ν_2)γ(ν_3)^{-1}, 1). \]

Then, we have the following transformations of \( \mathcal{H} \):

1. **Dilations**: for given positive number \( δ \), define

\[ D_δ : (x, t) \rightarrow (δx, δ^2 t), \quad δ > 0; \quad (2.22) \]

2. **Left translations**: for given \( (y, s) \in \mathcal{H} \), define

\[ τ_{(y,s)}(x, t) = (y, s) \cdot (x, t); \quad (2.23) \]

3. **Rotations**: for given unit imaginary octonion \( μ \), define
\[ S_\mu(x, t) = (\mu x, \mu t \tilde{\mu}); \quad \text{(2.24)} \]

\[ R(x, t) = \left(-\frac{1}{|x|^2 - t} x, -\frac{t}{|x|^4 + |t|^2}\right). \quad \text{(2.25)} \]

Recall that for a nilpotent Lie group \( G \) with Lie algebra \( \mathfrak{g} \), a continuous map \( f : \Omega \to G \) is \( P \)-differentiable at \( p \) if the limit

\[ Df(p)(q) = \lim_{\delta \to 0^+} D\delta^{-1} \circ L_{f(p)}^{-1} \circ D\delta(q) \]

exists, uniformly for \( q \) in compact subsets of \( G \), where \( L_p \) is a left translation by \( p \) in \( G \); if \( Df(p) \) exists, then it is a strata-preserving homomorphism of \( G \). Then, there is a Lie algebra homomorphism \( df(p) : \mathfrak{g} \to \mathfrak{g} \) such that \( Df(p) \circ \exp = \exp df(p) \). We call \( Df(p) \) the \( P \)-derivative and \( df(p) \) the \( P \)-differential of \( f \) at \( p \).

We have the following version of Liouville-type theorem in the case of the octonionic Heisenberg group. Denote the Carnot–Carathéodory distance \( d_{cc}(p, q) := \inf \int_0^1 |\gamma'(t)| dt \) for any \( p, q \in \mathcal{H} \), where \( \gamma : [0, 1] \to \mathcal{H} \) is taken over all Lipschitzian horizontal curves, i.e., \( \gamma'(t) \in H_{\gamma(t)} \) almost everywhere. Define balls \( B_{cc}(x, r) := \{ y \in \mathcal{H} : d_{cc}(x, y) < r \} \).

**Theorem 2.1** (OC Liouville-type theorem) Every conformal contact transformation between open subsets of \( \mathcal{H} \) is the restriction of the action of an element of \( F_{4(-20)} \). Here, conformal mapping is in the sense of sub-Riemannian manifold, i.e., \( f^*g_0 = \phi^2 g_0 \) for some bounded positive smooth function \( \phi \).

**Proof** Since \( f^*g_0 = \phi^2 g_0 \), it is direct to see that \( f \) is locally quasiconformal, i.e., we can write \( f : \Omega \to \mathcal{H} \) for some domain \( \Omega \subset \mathcal{H} \), and for any \( y \in \Omega \) there exists a constant \( k, r_0 > 0 \) such that

\[ B_{cc}(f(x), r/k) \subset f(B_{cc}(x, r)) \subset B_{cc}(f(x), kr), \]

for some \( r < r_0, x \in B_{cc}(y, r_0) \subset \Omega \). It follows from Pansu’s well-known rigidity theorem [35, Corollary 11.2], the \( P \)-differential of \( f \) must be a similarity. Cowling and Ottazzi proved that ([9, Theorem 4.1]): let \( G \) be a Carnot group, \( \Omega \) be a connected open subset of \( G \), and let \( f : \Omega \to G \) be a conformal mapping in the sense that \( P \)-differential is a similarity. Then, if \( G \) is the Iwasawa \( N \) group of a real-rank-one simple Lie group, \( f \) is the restriction to \( \Omega \) of the action of an element of this associate Lie group on \( N \cup \{ \infty \} \). Thus, \( f \) extends analytically to a conformal map on \( G \) or \( G \backslash \{ p \} \) for some point \( p \). Since the octonionic Heisenberg group is an Iwasawa \( N \) group of \( F_{4(-20)} \), the theorem follows directly.

**Proposition 2.5** For \( \gamma \in F_{4(-20)} \), we have

\[ \gamma^* g_0 = \phi^2 g_0, \quad \text{(2.26)} \]

for some positive smooth function \( \phi \). In particular, \( \phi(x, t) = \delta, 1, 1 \) and \( (|x|^4 + |t|^2)^{-\frac{1}{2}} \) for the dilation \( D_\delta \), left translation \( \tau_{(x,s)} \), rotation \( S_\mu \) and the inversion \( R \), respectively.
**Proof** The definition of similarity implies that \( g_0(\omega X_a, f_s X_a) = \phi^2(x, t)g(X_a, X_a) \) for any \( X_a \) given in (2.15). So, (2.26) follows. It is obvious that the proposition holds for dilations \( D_\delta \), left translation \( \tau_{(y, t)} \) and rotation \( S_\mu \). Here, we only need to prove (2.26) for the inversion \( R \).

Since given a point \( (x, t) \in \mathcal{H} \), we can choose suitable \( D_\delta, \tau_{(y, t)} \) and \( S_\mu \) such that \( S_\mu \circ D_\delta \circ \tau_{(y, t)}(x, t) = (1, 0, \ldots, 0, t') \) for some \( t' \in \text{Im} \mathcal{O} \), it is sufficient to prove the result at point \( (1, 0, \ldots, 0, t) \). For \( (x, t) = (1, 0, \ldots, 0, t) \), we have

\[
(R, X_1) F(R(x, t)) = \frac{d}{dk} F(R((1, 0, \ldots, 0, t) \cdot (\kappa, 0, \ldots, 0))) \bigg|_{k=0}
\]

\[
= \frac{d}{dk} F(R(1 + \kappa, 0, \ldots, 0, t)) \bigg|_{k=0} = \frac{d}{dk} F(\frac{\mathcal{E}_0}{\rho}, \ldots, \frac{\mathcal{E}_7}{\rho}, \frac{-t}{\rho}) \bigg|_{k=0},
\]

for any smooth function \( F \) with

\[
\mathcal{E}_0 = (1 + \kappa)^4, \quad \mathcal{E}_\beta := t_\beta(1 + \kappa), \quad \rho = (1 + \kappa)^4 + |t|^2, \quad \beta = 1, \ldots, 7.
\]

Then, the right-hand side of (2.27) equals to

\[
\frac{1 - 3|t|^2}{(1 + |t|^2)^2} F_0 + \frac{t_\beta(3 - |t|^2)}{(1 + |t|^2)^2} F_\beta + 4 \frac{t_\beta}{(1 + |t|^2)^2} F_{7+\beta}
\]

\[
= A_0^\beta \left( F_\beta + 2E_{kl}^\beta \left( \frac{\mathcal{E}_k}{1 + |t|^2} \right) F_{7+\beta} \right) (R(x, t)),
\]

for any smooth function \( F \) with

\[
A_0^\beta = \frac{1 - 3|t|^2}{(1 + |t|^2)^2}, \quad A_0^\beta = \frac{t_\beta(3 - |t|^2)}{(1 + |t|^2)^2}.
\]

Since

\[
-A_0^\beta E_{kl}^\beta \mathcal{E}_k \bigg|_{\kappa=0} = -A_0^\beta E_{kl}^\beta \frac{1}{1 + |t|^2} - A_0^\beta E_{kl}^\beta \frac{t_a}{1 + |t|^2}
\]

\[
= -A_0^\beta E_{0l}^\beta \frac{1}{1 + |t|^2} - A_0^\beta E_{al}^\beta \frac{1}{1 + |t|^2}
\]

\[
= -A_0^\beta E_{00}^\beta \frac{1}{1 + |t|^2} - A_0^\beta E_{0a}^\beta \frac{1}{1 + |t|^2} - A_0^\beta E_{al}^\beta \frac{t_a}{1 + |t|^2}
\]

\[
= \frac{(|t|^2 - 3)t_a}{(1 + |t|^2)^2} E_{0a}^\beta + \frac{(1 - 3|t|^2)t_a}{(1 + |t|^2)^2} E_{al}^\beta + \frac{(3|t|^2 - 3}{(1 + |t|^2)^2} E_{ay}^\beta t_a t_y
\]

\[
= \frac{-2}{(1 + |t|^2)^2} E_{0a}^\beta t_a = \frac{2t_\beta}{(1 + |t|^2)^2}.
\]

The third identity holds by \( E_{00}^\beta = 0 \), and the fourth identity holds by \( E_{ay}^\beta t_a t_y = 0 \). Thus, \( R, X_1 \big|_{R(1, 0, \ldots, 0, t)} = A_0^\beta X_1 \big|_{R(1, 0, \ldots, 0, t)} \). We can easily check \( \sum_{\beta=0}^7 (A_0^\beta)^2 = \frac{1}{1 + |t|^2} \). The proposition is proved.

As mentioned before, an OC manifold \((M, g, \mathfrak{L})\) is always spherical, i.e., it is locally conformally OC equivalent to an open set of the octonionic Heisenberg group with standard OC structure. A conformal class can be described topologically as a manifold whose
coordinate charts are given by open subsets of the octonionic Heisenberg group and elements of $\text{F}_4(-20)$ as transition maps. So, we can omit $l$ in the notion $(M, g, l)$ of an OC manifold.

3 The OC Yamabe operator and its transformation formula under conformal transformations

3.1 The Biquard connection

**Theorem** ([5, Theorem B]) For an OC manifold with Carnot–Carathéodory metric $g$ on $H$, there exists a unique connection $\nabla$ on $H$ and a unique supplementary subspace $V$ of $H$ in $TM$, such that

(i) $\nabla$ preserves the decomposition $H \oplus V$ and the metric;
(ii) for $X, Y \in H$, one has $T_{X,Y} = -[X, Y]_V$;
(iii) $\nabla$ preserves the $\text{Spin}(7)$-structure on $H$;
(iv) for $R \in V$, the endomorphism $- \cdot (T_{R_\alpha})_H$ of $H$ lies in $\mathfrak{so}_7$;
(v) the connection on $V$ is induced by the natural identification of $V$ with the subspace $\mathbb{R}^{l'}$ of the endomorphisms of $H$.

Since the Biquard connection preserving Carnot–Carathéodory metric on it, we have

$$\nabla(\mathbb{d}\theta_\alpha) \in \Lambda^1H \otimes \mathfrak{so}(8) = \Lambda^1H \otimes (\mathfrak{so}(7) \oplus \mathbb{R}^7),$$

and that the connection preserves the $\text{Spin}(7)$ structure if its component in $\Lambda^1H \otimes \mathbb{R}^7$ vanish (cf. [5, p. 84]). It satisfies the following properties proved by Biquard [5, Proposition II.1.7, II.1.9]. Recall that Reeb vector fields $R_\alpha, \alpha \in \mathbb{Z}_{8+}$, satisfy $i_{R_\alpha} d\theta_\alpha|_H = 0, i_{R_\alpha} d\theta_\beta|_H = -i_{R_\beta} d\theta_\alpha|_H$ for $\alpha \neq \beta$ and $\theta_\beta(R_\alpha) = \delta_{\alpha\beta}$.

**Proposition 3.1** (1) The Biquard connection satisfies

$$\nabla(\mathbb{d}\theta_\alpha) = -(i_{R_\beta} \mathbb{d}\theta_\alpha)|_H \otimes \mathbb{d}\theta_\beta,$$

and in particular, $(i_{R_\beta} \mathbb{d}\theta_\alpha)|_H = -(i_{R_\alpha} \mathbb{d}\theta_\beta)|_H$ and $(i_{R_\alpha} \mathbb{d}\theta_\alpha)|_H = 0$. By isomorphism $\mathbb{d}\theta_\alpha \rightarrow R_\alpha$, we have

$$\nabla R_\alpha = -(i_{R_\beta} \mathbb{d}\theta_\alpha)|_H \otimes R_\beta. \quad (3.1)$$

(2) It is a metric connection on $V$ and

$$\nabla_X R = [X, R]_V, \quad X \in H, R \in V. \quad (3.2)$$

**Proposition 3.2** (1) $\mathfrak{so}(7) \cong \text{span}\{I_\alpha I_\beta; 1 \leq \alpha < \beta \leq 7\}$;

(2) $\mathbb{R}^7 \cong \text{span}\{I_\alpha\}$ is an $\mathfrak{so}(7)$ module.
Proof

(1) Recall that \((N^a_{\alpha\beta})_{\gamma\delta} = 2\epsilon_a{\delta}\gamma\delta\) is given by (2.3), \(\{N^a_{\alpha\beta};1 \leq \alpha < \beta \leq 7\}\) are linearly independent, because \(N^a_{\alpha\beta}\) is \((8 \times 8)\)-antisymmetric matrix with only 4 entries not zero, and for any fixed \(\alpha < \beta\), the nonzero elements are at different entries (cf. Proposition 2.1). As \(I_a\) is an \((8 \times 8)\)-antisymmetric matrix with \((0, a)\) entry to be \(-1\), while \((0, a)\) entry of \(N^a_{\alpha\beta}\) is always 0 for any \(a\), so \(\{I_a;\alpha = 1, \ldots, 7\}\) or \(\{N^a_{\alpha\beta};1 \leq \alpha < \beta \leq 7\}\) are linearly independent. \(\{I_a;1 \leq \alpha < \beta \leq 7\}\) is closed under Lie brackets. Since

\[
[I_a I_\beta, I_\gamma] = I_a I_\beta I_\gamma - I_\gamma I_\beta I_\alpha = 0,
\]

for \(\alpha, \beta, \gamma, \delta\) different, and

\[
[I_a I_\beta, I_\gamma] = I_a I_\beta I_\gamma - I_\gamma I_a I_\beta = 2I_\beta I_\gamma,
\]

for \(\alpha, \beta, \gamma\) different. Moreover, \(\{I_a I_\beta;1 \leq \alpha < \beta \leq 7\}\) are linearly independent by \(I_a I_\beta = I_\gamma - N^a_{\alpha\beta}\) if \(\epsilon_a{\delta}\gamma\delta\). Therefore, \(\{I_a;\alpha = 1, \ldots, 7\}\) or \(\{I_a I_\beta;1 \leq \alpha < \beta \leq 7\}\) are linearly independent. Note that \(\mathbb{R}^7 = \text{span}\{I_a;\alpha = 1, \ldots, 7\}\), \(\mathfrak{so}(8) \cong \mathbb{R}^7 \oplus \mathfrak{so}(7)\), and \(\dim \mathfrak{so}(7) = 21 = \dim \{I_a I_\beta;1 \leq \alpha < \beta \leq 7\}\). Then, we have \(\mathfrak{so}(7) \cong \text{span}\{I_a I_\beta;1 \leq \alpha < \beta \leq 7\}\).

(2) We have \([I_a I_\beta, I_\gamma] = I_a I_\beta I_\gamma - I_\gamma I_a I_\beta = 0\), for \(\alpha, \beta, \gamma\) different and \([I_a I_\beta, I_\delta] = I_a I_\beta I_\delta - I_\delta I_a I_\beta = 2I_\beta I_\delta \in \mathbb{R}^7\). The proposition is proved.

Proposition 3.3 The connection coefficient of the Biquard connection is \(\mathfrak{so}(7)\)-valued.

Proof We write the connection coefficients as \(\Gamma_{ab}^c\), i.e., Proposition 3.1 (1) implies that \(\nabla V_a V_b = \Gamma_{ab}^c V_c\) for a local frame \(\{V_a\}\). Since \(d\theta_a(\cdot, \cdot) = g(I_a \cdot, \cdot)\), we have

\[
\nabla I_a |_H = -(i_{R^H_a} d\theta_a I_a) |_H \otimes I_\beta,
\]

which is equivalent to

\[
\Gamma_{ab}^c I_a I_b - \Gamma_{ab}^d I_a I_b^c = -d\theta_a(R^H_{ab}, V_a) I_{\beta;\cdot},
\]

i.e., \([\Gamma_{ab}, I_a] \in \mathbb{R}^7\). On the other hand, \([\Gamma_{ab}, I_a] = ([\Gamma_{ab}]_{\mathfrak{so}(7)}, I_a) + ([\Gamma_{ab}]_{\mathfrak{so}(7)}, I_a)\), and \([([\Gamma_{ab}]_{\mathfrak{so}(7)}, I_a) \in \mathbb{R}^7\) by Proposition 3.2. But \([([\Gamma_{ab}]_{\mathfrak{so}(7)}, I_a) \notin \mathbb{R}^7\) if \(\Gamma_{ab} \notin \mathbb{R}^7\). So, we must have \(\Gamma_{ab} \in \mathbb{R}^7\). The curvature of Biquard connection is defined by \(R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}\) and Ricci curvature is defined by \(\text{Ric}(X, Y) = g(R(V_a, X) Y, V_a)\) for any \(X, Y \in \mathbb{H}\), where \(V_a\) is local orthonormal basis of horizontal subspace \(H\). The scalar curvature is \(s_g = \text{tr}^H \text{Ric}\).

3.2 The Biquard connection under conformal transformations

When the octonionic structure \(\mathfrak{L}\) is rotated by \(\text{SO}(7)\), the Carnot–Carathéodory metric also satisfies (1.1)–(1.2) for 1-form \(\Theta\) rotated. Hence, the Reeb vectors are also rotated by definition, and the Biquard connection is the same. So, when consider conformal transformations, we can fix the octonionic structure \(\mathfrak{L}\).
**Proposition 3.4** Under the conformal change \( \tilde{g} = f^2g_0 \) on the octonionic Heisenberg group, the scalar curvature becomes

\[
sl_{\tilde{g}} = -f^{-2}(2Q - 2)tr^{H}(\nabla K) + (Q - 1)(Q - 2)|K|^2
\]

\[
= -f^{-2}(42tr^{H}(\nabla K) + 420|K|^2),
\]

where \( K := f^{-1}df \).

To prove this proposition, we need to know the transformation formulae of Reeb vector fields and the Biquard connection under the OC conformal transformation. Recall that the wedge product of 1-forms \( \phi \) and \( \psi \) is given by

\[
(\phi \wedge \psi)(X, Y) := \phi(X)\psi(Y) - \phi(Y)\psi(X),
\]

for any vector field \( X \) and \( Y \). Then, we have \( d\phi(X, Y) = X\phi(Y) - Y\phi(X) - \phi([X, Y]) \). For \( X, Y \in H \), define \( X \wedge Y \) as the endomorphism of \( H \) by

\[
X \wedge Y(Z) := g(X, Z)Y - g(Y, Z)X
\]

for any \( Z \in H \).

**Proposition 3.5** The Reeb fields of \( \tilde{g} = f^2g \) associate with \( \tilde{\theta}_a = f^2\theta_a, a = 1, \ldots, 7 \) are the vectors

\[
\tilde{R}_a = f^{-2}(R_a + r_a), \quad r_a = -2I_aK^a,
\]

where the vector \( K^a \in H \) is defined by \( K(X) := \langle K^a, X \rangle \) for any \( X \in H \). The Biquard connection of \( \tilde{g} \) satisfies

\[
\tilde{\nabla}_X = \nabla_X + A_X, \quad \text{for} \ X \in H,
\]

with

\[
A_X = K(X) + U_X := K(X) + (I_aK^a,X)I_a + K^a \wedge X + I_aK^a \wedge I_aX.
\]

**Proof** By Proposition 3.1, the vector field \( \tilde{R}_a \) is characterized by \( i_{\tilde{R}_a}d\tilde{\theta}_a|_H = 0 \), \( i_{\tilde{R}_a}d\tilde{\theta}_a|_H = -i_{\tilde{R}_a}d\tilde{\theta}_a|_H \) and \( \tilde{\theta}_a(R_a) = \delta_{a\beta} \). Since \( f^{-2}d\tilde{\theta}_a = d\theta_a + 2K \wedge \theta_a \), we have (3.6) immediately. As \( \tilde{T}_{X,Y} = -[X, Y]_{\tilde{g}} \) \( = d\tilde{\theta}_a(X, Y)\tilde{R}_a \), for \( X, Y \in H \), we have

\[
g(A_X Y, Z) - g(A_Y X, Z) = g\left(\tilde{T}_{X,Y} - T_{X,Y}, Z\right) = g\left(d\theta_a(X, Y)r_a, Z\right)
\]

\[
= 2d\theta_a(X, Y)K(I_aZ),
\]

while \( \tilde{T}_{X,Y} = 0 \) yields

\[
0 = \tilde{\nabla}_X \tilde{g}(Y, Z) = X(\tilde{g}(Y, Z)) - f^2g(\nabla_X Y + A_X Y, Z) - f^2g(Y, \nabla_X Z + A_X Z)
\]

\[
= 2f^2K(X)g(Y, Z) - f^2g(A_X Y, Z) - f^2g(Y, A_X Z),
\]

i.e.,

\[
g(A_X Y, Z) + g(Y, A_X Z) = 2K(X)g(Y, Z).
\]

Alternating \( X, Y, Z \), we have
Take the sum of the first two equations in (3.10)–(3.11) and then minus the last one to get
\begin{align}
g(A_X Y, Z) + g(Z, A_Y X) &= -2d\theta_a(X, Z)K(I_a Y) - 2d\theta_a(Y, Z)K(I_a X) \\
&\quad + 2K(X)g(Y, Z) + 2K(Y)g(Z, X) - 2K(Z)g(X, Y)
\end{align}
(3.12)
by using (3.9). Then, (3.8) is the sum of (3.9) and (3.12).

Denote by $D_{\mathfrak{so}(7)}$ and $D_{\mathbb{R}^7}$ the projection to $\mathfrak{so}(7)$ and $\mathbb{R}^7$ for $D \in \mathfrak{so}(8)$ in the decomposition (2.5) with respect to the Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{so}(8)$:
\begin{equation}
\langle D, F \rangle = tr(F^*D) = \frac{1}{8} \sum_{a=0}^7 \langle D(V_a), F(V_a) \rangle \quad \text{for} \quad D, F \in \mathfrak{so}(8), \quad \text{where} \quad \{V_a\}_{a=0}^7 \quad \text{is a local orthonormal basis of} \ H.
\end{equation}
By (2.5), we have
\begin{equation}

D_{\mathfrak{so}(7)} = D - D_{\mathbb{R}^7},
\end{equation}
for any 1-form $D$ with value in $\mathfrak{so}(8)$, where $D_{\mathbb{R}^7} = \sum_{a=1}^7 D_a I_a$ with
\begin{equation}
D_a = \frac{1}{8} \sum_{\alpha} \langle DV_a, I_a V_\alpha \rangle.
\end{equation}

**Proposition 3.6** The Biquard connection $\tilde{\nabla}$ satisfies
\begin{align}
\tilde{\nabla}_{R_a} &= \nabla_{R_a} + A_{R_a} := \nabla_{R_a} + K(R_a) + (U_{R_a})_{\mathfrak{so}(7)}, \\
\tilde{T}_{R_a + r_a X} &= T_{R_a X} + K(R_a)X - 2|K|^2 I_a X - (U_{R_a})_{\mathbb{R}^7} X,
\end{align}
where
\begin{equation}
U_{R_a} X = 2I_a \nabla X K^\sharp + 4 \sum_{\beta=1}^7 \langle I_\beta K^\sharp, X \rangle I_{a\beta} K^\sharp - 4 \sum_{\beta \neq a} \langle I_{a\beta} K^\sharp, X \rangle I_{\beta\sharp} K^\sharp.
\end{equation}

**Proof** Let us write
\begin{equation}
\tilde{\nabla}_B = \nabla_B + K(B) + \mathcal{A}(B)
\end{equation}
for the connection acting on the horizontal subbundle for $B \in V$ and some $\mathcal{A} \in \Lambda^1 \otimes \mathfrak{gl}(8)$, where $\nabla$ is the Biquard connection associated to $\tilde{g}$ and $\tilde{V} = \text{span}(\tilde{R}_a)_{a=1}^7$ given by (3.6). This connection preserves $f^*g$ with torsion given by
\begin{align}
\tilde{T}_{R_a + r_a X} &= \tilde{\nabla}_{R_a + r_a X} - \tilde{\nabla}_X (R_a + r_a) - [R_a + r_a, X] \\
&= \tilde{\nabla}_{R_a + r_a X} - [R_a + r_a, X]_{H/\tilde{\nabla}} \\
&= \nabla_{R_a} X + K(R_a)X + \tilde{\nabla}_{r_a X} - [R_a, X]_{H/\tilde{\nabla}} - [r_a, X]_{H/\tilde{\nabla}} + \mathcal{A}(R_a) X,
\end{align}
by (3.2) for $\tilde{\nabla}$, where $H/\tilde{\nabla}$ denotes the projection onto $H$ along the direction of $\tilde{\nabla}$. On the one hand,
\begin{equation}
\tilde{\nabla}_{r_a X} - [r_a, X]_{H/\tilde{\nabla}} = \tilde{\nabla}_{X r_a},
\end{equation}
by the definition of $\tilde{\nabla}$, and on the other hand
\[
\nabla_{R_a} X - [R_a, X]_{H/\tilde{\nabla}} = T_{R_a, X} + [R_a, X]_{H/\tilde{\nabla}} - [R_a, X]_{H/\tilde{\nabla}} = T_{R_a, X} - d\theta_{\tilde{\mu}(R_a, X)r_{\beta}}.
\]
(3.19)
Inserting these two identities into (3.17), we get
\[
\tilde{T}_{R_a+r_a, X} = T_{R_a, X} + K(R_a)X + \nabla_x r_a - d\theta_{\tilde{\mu}(R_a, X)r_{\beta}} + \omega(R_a)X.
\]
We can choose an orthonormal frame such that $\nabla I_\alpha(p) = 0$ for any $\alpha$ at a fixed point $p$, and thus, we have $d\theta_{\tilde{\mu}}(R_{\beta}, X) = 0$ for any $X \in H$. To calculate $\nabla_x r_a$, note that
\[
\nabla_x I_\alpha = i_{R_a + r_a} (\nabla_{\theta_\beta} + 2K \wedge \theta_\beta) X_\beta = (\nabla_{\theta_\beta}(r_a, X) - 2K(X)\delta_\alpha \beta) X_\beta
\]
\[
= -2 \sum_{\beta \neq \alpha} X_\beta^\alpha \mathcal{I}_\beta^\alpha K^\beta X_\beta
\]
by (3.3).

By Proposition 3.5, we have
\[
\nabla_x K^\beta = \nabla_x K^\alpha + |K|^2 X + 2X_\beta^\alpha X_\beta K^\beta.
\]
Thus,
\[
\nabla_x (I_a^\alpha K^\beta) = I_a^\alpha \nabla_x K^\beta + |K|^2 I_a^\alpha X + 2X_\beta^\alpha X_\beta I_a^\alpha \mathcal{I}_\beta^\alpha K^\beta - 2 \sum_{\beta \neq \alpha} X_\beta^\alpha \mathcal{I}_\beta^\alpha K^\beta X_\beta
\]
Therefore,
\[
\tilde{T}_{R_a+r_a, X} = T_{R_a, X} + K(R_a)X - 2|K|^2 I_a^\alpha X - U_{R_a} X + \omega(R_a)X,
\]
by $r_a = -2I_a^\alpha K^\beta$, where $U_{R_a}$ is defined in (3.15). Since $\tilde{T}_{R_a} \cdot$ is an endomorphism of $H$ lying in $\mathfrak{s}o_7$, (3.14) follows. \hfill \Box

### 3.3 Proof of Proposition 3.4.

Recall that $\{X_a\}_{a=1}^7$ is the standard orthonormal basis (2.15) of $H_0$ on $\mathcal{H}$ and $V$ is flat, i.e., $\nabla_{X_a} X_b = 0$. Denote by $s_\mathcal{H}$ the scalar curvature of $g_0$ and by $X_a = f^{-1}X_a$ the local orthonormal basis of $\mathcal{H}$. By formulae in Proposition 3.5, Proposition 3.6 and direct calculation, we have
\[
f^2 s_\mathcal{H} = f^2 \left\langle \tilde{R}(X_a, X_b)X_b, \tilde{X}_b \right\rangle_{f^2 g_0} = \left\langle \tilde{R}(X_a, X_b)X_b, X_a \right\rangle_{g_0}
\]
\[
= \left\langle \nabla X_a, \nabla X_b \right\rangle - \left\langle \nabla X_a, \nabla X_b \right\rangle_{\nabla X_a, X_b} + \left\langle \nabla X_b, \nabla X_a \right\rangle_{\nabla X_a, X_b}
\]
\[
= \left\langle \nabla X_a (K(X_b)X_b + U_{X_a} X_b) - \nabla X_b (K(X_a)X_a + U_{X_b} X_a) - A_{X_b} X_a, X_a \right\rangle
\]
\[
= \left\langle X_a K(X_b)X_b + \nabla X_a (U_{X_b} X_a) + U_{X_b} (U_{X_a} X_b) - X_b K(X_a)X_b
\]
\[
- \nabla X_b (U_{X_b} X_a) - U_{X_b} (U_{X_a} X_b) - A_{X_a} X_b, X_a \right\rangle
\]
(3.20)
where we used the flatness of $\mathcal{H}$ in the third and fourth identities. We write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{g_0}$ for simplicity. Note that $\langle X_a K(X_b)X_b - X_b K(X_a)X_a, X_a \rangle = 0$. As

\[\square\] Springer
\[ \langle \nabla_{X_a}(U_{X_b} X_b), X_a \rangle = 2 \langle I_a \nabla_{X_a} K^\sharp, X_b \rangle \langle I_a X_b, X_a \rangle + \langle \nabla_{X_a} K^\sharp, X_b \rangle \langle X_b, X_a \rangle - 8 \langle \nabla_{X_a} K^\sharp, X_a \rangle = -14 \langle \nabla_{X_a} K^\sharp, X_a \rangle + \langle \nabla_{X_a} K^\sharp, X_a \rangle - 8 \langle \nabla_{X_a} K^\sharp, X_a \rangle = -21 X_a K(X_a). \]

Here and in the following, we use \( \nabla I_a = 0, \nabla X_a = 0 \) and \( \langle X_a, X_b \rangle = \delta_{ab} \) repeatedly. We also have used the property that for any fixed \( a \) and \( a \), there exists a unique \( b \) such that \( \langle I_a X_b, X_a \rangle \) non-vanishing and equal to \( \pm 1 \), i.e., \( I_a X_b = \pm X_a \). For example,

\[ \sum_{a,b} \langle I_a X_a, X_b \rangle \langle I_b X_b, X_a \rangle = - \sum_{a,b} \langle X_a, I_a X_b \rangle \langle I_b X_b, X_a \rangle = - \sum_a \langle X_a, I_a X_b \rangle^2 = -56. \]

Similarly, by \( U_{X_a} \) being antisymmetric, we have

\[ \langle U_{X_a}(U_{X_b} X_b), X_a \rangle = - \langle U_{X_b} X_b, U_{X_a} X_a \rangle = -2 \langle I_a K^\sharp, X_b \rangle \langle I_a X_b, X_a \rangle + K(X_b) X_b - 8 K^\sharp, 2 \langle I_b K^\sharp, X_a \rangle \langle I_b X_b, X_a \rangle + K(X_a) X_a - 8 K^\sharp \]

\[ = -4 \langle I_a K^\sharp, X_b \rangle \langle I_b K^\sharp, X_a \rangle \langle I_a X_b, I_b X_a \rangle - 2 \langle I_b K^\sharp, X_b \rangle \langle I_a X_b, X_a \rangle K(X_b) + 16 \langle I_a K^\sharp, X_b \rangle \langle I_a X_b, K^\sharp \rangle - 2 \langle I_b K^\sharp, X_b \rangle \langle X_b, I_b X_a \rangle K(X_b) - K(X_b) K(X_a) + 8 K(X_b) K(X_a) + 16 \langle K^\sharp, I_b X_a \rangle \langle I_b K^\sharp, X_a \rangle + 8 K(X_b) K(X_a) - 64 \langle K^\sharp, K^\sharp \rangle \]

\[ = (-196 + 14 - 112 + 14 - 1 + 8 - 112 + 8 - 64) |K|^2 = -441 |K|^2. \]

Here, \( |K|^2 = \sum_{a=0}^{7} \left| \langle K^\sharp, X_a \rangle \right|^2 = \langle K^\sharp, K^\sharp \rangle \). We also have

\[ - \langle \nabla_{X_a}(U_{X_b} X_b), X_a \rangle = - \langle I_a \nabla_{X_a} K^\sharp, X_b \rangle \langle I_a X_b, X_a \rangle - X_b (K(X_b)) \langle X_a, X_a \rangle + \langle X_a, X_b \rangle \langle \nabla_{X_a} K^\sharp, X_a \rangle \]

\[ - \langle I_a \nabla_{X_a} K^\sharp, X_b \rangle \langle I_a X_a, X_a \rangle + \langle I_a \nabla_{X_a} K^\sharp, X_a \rangle = -7 \langle \nabla_{X_a} K^\sharp, X_a \rangle - 8 X_b (K(X_b)) + \langle \nabla_{X_a} K^\sharp, X_a \rangle - 7 \langle \nabla_{X_a} K^\sharp, X_a \rangle \]

\[ = -21 X_a K(X_a), \]

and

\[ - \langle U_{X_a}(U_{X_b} X_b), X_a \rangle = \langle U_{X_b} X_b, U_{X_a} X_a \rangle \]

\[ = \langle I_a K^\sharp, X_a \rangle I_a X_b + \langle K^\sharp, X_a \rangle X_a - \langle X_a, X_b \rangle K^\sharp + \langle I_a K^\sharp, X_b \rangle I_a X_a - \langle I_a X_a, X_b \rangle I_a K^\sharp, \]

\[ \langle I_b K^\sharp, X_b \rangle I_b X_a + \langle K^\sharp, X_a \rangle X_b - \langle X_b, X_a \rangle K^\sharp + \langle I_b K^\sharp, X_a \rangle I_b X_b - \langle I_b X_b, X_a \rangle I_b K^\sharp \]

\[ = (-35 + 7 + 56 + 35 + 1 - 1 + 7 - 7 + 7 - 1 + 8 + 7 + 56 + 7 + 7 - 35 - 35 - 7 - 35 - 56) |K|^2 = 21 |K|^2. \]
Here, we have use for example
\[
\left\langle I_a K^\sharp, X_a \right\rangle I_a X_b, \left\langle I_b K^\sharp, X_b \right\rangle I_b X_a \right\rangle = -\left\langle K^\sharp, I_a X_a \right\rangle \left\langle I_b K^\sharp, X_b \right\rangle \left\langle X_b, I_a I_b X_a \right\rangle \\
= \left\langle K^\sharp, I_a X_a \right\rangle \left\langle K^\sharp, I_a X_a \right\rangle \left\langle X_b, I_a X_b \right\rangle - \sum_{\alpha \neq \beta} \left\langle K^\sharp, I_a X_a \right\rangle \left\langle K^\sharp, I_\beta X_\beta \right\rangle \left\langle X_b, I_a I_\beta X_a \right\rangle \\
= 7|K|^2 - 42|K|^2 = -35|K|^2.
\]

By Propositions 3.5, 3.6, and \(-[X_a, X_b] = T_{X_a X_b} = \sum_a d\theta_a(X_a, X_b)R_a\), we have
\[
-\left\langle A_{[X_a, X_b]}X_b, X_a \right\rangle = d\theta_a(X_a, X_b)\left\langle A_{R_a}X_b, X_a \right\rangle \\
= d\theta_a(X_a, X_b)\left\langle (K(R_a) + (U_{R_a})_{\theta(7)})X_b, X_a \right\rangle \\
= \left\langle I_a X_a, X_b \right\rangle \left\langle (K(R_a)X_b, X_a) + 2\left\langle I_a \nabla K^\sharp \right\rangle_{\theta(7)}(X_b), X_a \right\rangle \\
+ 4 \sum_\beta \left\langle \left[ (I_\beta K)I_a I_\beta K^\sharp \right]_{\theta(7)}(X_b), X_a \right\rangle - 4 \sum_{\beta \neq \alpha} \left\langle \left[ (I_\beta I_a K)I_\beta K^\sharp \right]_{\theta(7)}(X_b), X_a \right\rangle \right\rangle.
\]

(3.21)

We claim (3.21) vanishes. Set \((I_\beta K)(X_b) := \left\langle I_\beta K^\sharp, X_b \right\rangle\). Then substitute the above identities into (3.20) to get (3.4). Note that the projection of \(I_a \nabla K^\sharp\) to \(\mathbb{R}^7\) is \((I_a \nabla K^\sharp)_\beta I_\beta\), then we have
\[
\left\langle I_a X_a, X_b \right\rangle \left\langle \left( I_a \nabla K^\sharp \right)_\mathbb{R}^7, (X_b), X_a \right\rangle = \frac{1}{8} \left\langle I_a X_a, X_b \right\rangle \left\langle I_a \nabla X_a K^\sharp, I_a X_a \right\rangle \left\langle I_a X_a, X_a \right\rangle \\
= -7\text{tr}H \nabla K,
\]

and so
\[
\left\langle I_a X_a, X_b \right\rangle \left\langle \left( I_a \nabla K^\sharp \right)_{\theta(7)}, (X_b), X_a \right\rangle \\
= \left\langle I_a X_a, X_b \right\rangle \left\langle \left( I_a \nabla K^\sharp \right)_{\theta(7)}, (X_b), X_a \right\rangle - \left\langle \left( I_a \nabla K^\sharp \right)_{\mathbb{R}^7}, (X_b), X_a \right\rangle = 0.
\]

We have
\[
\left\langle I_a X_a, X_b \right\rangle \left\langle \sum_\beta \left\langle I_\beta K^\sharp, X_b \right\rangle \left\langle I_a I_\beta K^\sharp, X_a \right\rangle - \sum_{\beta \neq \alpha} \left\langle I_\beta I_a K^\sharp, X_b \right\rangle \left\langle I_\beta K^\sharp, X_a \right\rangle \right\rangle \\
= -\left\langle I_a X_a, X_b \right\rangle \left\langle I_a K^\sharp, X_b \right\rangle \left\langle K^\sharp, X_a \right\rangle = -7|K|^2,
\]

and
\[
\left\langle I_a X_a, X_b \right\rangle \left\langle \sum_\beta \left\langle \left[ (I_\beta K)I_a I_\beta K^\sharp \right]_{\mathbb{R}^7}, (X_b), X_a \right\rangle - \sum_{\beta \neq \alpha} \left\langle \left[ (I_\beta I_a K)I_\beta K^\sharp \right]_{\mathbb{R}^7}, (X_b), X_a \right\rangle \right\rangle \\
= \frac{1}{8} \left\langle I_a X_a, X_b \right\rangle \left\langle (I_\beta K)^2, X_n \right\rangle \left\langle I_a I_\beta K^\sharp, I_\beta X_n \right\rangle \left\langle I_\beta X_n, X_a \right\rangle \\
- \frac{1}{8} \sum_{\beta \neq \alpha} \left\langle I_a X_a, X_b \right\rangle \left\langle (I_\beta K)^2, X_n \right\rangle \left\langle I_a I_\beta K^\sharp, I_\beta X_n \right\rangle \left\langle I_a X_b, X_a \right\rangle = -7|K|^2.
\]
The right-hand side does not vanish only when $\alpha = \gamma$. So
\[
\langle I_a X_a, X_b \rangle \left\{ \sum_\beta \left( \left[ (I_\beta K) I_\beta K^2 \right]_{\mathfrak{so}(7)} (X_b), X_a \right) - \sum_{\beta \neq \alpha} \left( \left[ (I_\beta I_\alpha K) I_\beta K^2 \right]_{\mathfrak{so}(7)} (X_b), X_a \right) \right\} = 0.
\]
Thus, (3.21) vanishes. The Proposition is proved. \hfill \Box

### 3.4 The transformation formula for the OC Yamabe operator

Since $\nabla$ preserves $H$, there exist 1-forms $\omega_\alpha^b$, such that $\nabla_Y V_a = \omega_\alpha^b(Y) V_b$, for $Y \in H$, where $\{V_a\}$ is a local basis. Write $\omega_\alpha^b(V_b) = \Gamma_{b\alpha}^\gamma V_\gamma$. Then $\nabla_Y V_b = \Gamma_{ab}^c V_c$. For 1-form $\omega \in \Omega(M)$, $(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$. The Carnot–Carathéodory metric $g$ induces a dual metric on $H^*$, denoted by $\langle \cdot, \cdot \rangle_g$. Then, we define an $L^2$ inner product $\langle \cdot, \cdot \rangle_{g}$ on $\Gamma(H^*)$ by
\[
\langle \omega, \omega' \rangle_{g} := \int_M \langle \omega, \omega' \rangle_{g} dV_g,
\]
where the volume form $dV_g$ is
\[
dV_g := \theta_1 \land \ldots \land \theta_7 \land (d\theta_\beta)^4, \tag{3.22}
\]
$\beta = 1, \ldots, 7$, if we write $\Theta = (\theta_1, \ldots, \theta_7)$ locally.

**Proposition 3.7** The volume element $dV_g$ only depends on $g$, not on $\beta$ or the choice of the $\mathbb{R}^7$-valued contact form $\Theta = (\theta_1, \ldots, \theta_7)$.

**Proof** Let 1-forms $\{\theta^a\}$ be the basis dual to $\{V_a\}$. Since
\[
d\theta_\beta(V_a, V_b) = g(I_\beta V_a, V_b) = g(E_\alpha^b V_c, V_b) = E_{ba}^\beta,
\]
we have the structure equation
\[
d\theta_\beta = \frac{1}{2} E_{ba}^\beta \theta^a \land \theta^b, \mod \theta_1, \ldots, \theta_7, \beta = 1, \ldots, 7, \text{where } E^\beta \text{ is given in (2.7). It is direct to check that}
\]
\[
(d\theta_\beta)^4 = \left( \frac{1}{2} E_{ba}^\beta \theta^a \land \theta^b \right)^4 = 4! \theta^0 \land \ldots \land \theta^7, \mod \theta_1, \ldots, \theta_7. \tag{3.23}
\]
Let $\tilde{\Theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_7)$ be another contact form satisfying $d\tilde{\theta}_\beta(X, Y) = g(I_\beta X, Y)$. We can write $\tilde{\theta}_\beta = c_{\beta a} \theta_a$, for some $\text{SO}(7)$-valued function $(c_{a\beta})$ and simultaneously $\tilde{I_\beta} = c_{\beta a} I_a$, for $\beta = 1, \ldots, 7$. Dually, we have $\tilde{V}_a = k_{ab} V_b$, for some induced $\text{SO}(8)$-valued function $(k_{ab})$, and the dual basis $\{\tilde{\theta}^a\}$ such that $\tilde{\theta}^a = k_{ab} \theta^b$ for $a = 0, \ldots, 7$. In fact, we have $\tilde{\theta}^0 \land \ldots \land \tilde{\theta}^7 = \det(k_{ab}) \theta^0 \land \ldots \land \theta^7$ and $\det(k_{ab}) = 1$ by $(k_{ab}) \in \text{SO}(8)$. By (3.23), we have
\[
dV_{\tilde{g}} = \tilde{\theta}_1 \land \ldots \land \tilde{\theta}_7 \land (d\tilde{\theta}_\beta)^4 = 4! \det(c_{a\beta}) \theta_1 \land \ldots \land \theta_7 \land \tilde{\theta}^0 \land \ldots \land \tilde{\theta}^7 = \theta_1 \land \ldots \land \theta_7 \land (d\theta_\beta)^4. \tag{3.24}
\]
The proposition is proved. \hfill \Box
Denote $d_b := \text{pr} \cdot d$, where $\text{pr}$ is the projection from $T^*M$ to $H^*$. We define the sub-Laplacian $\Delta_g$ associated to Carnot–Carathéodory metric $g$ by

$$\int_M \Delta_g u \cdot \mathrm{d}V_g = \int_M \langle d_b u, d_b v \rangle \mathrm{d}V_g$$

(3.25)

for $u, v \in C_c^\infty(M)$, where $\mathrm{d}V_g$ is the volume form. The sub-Laplacian $\Delta_g$ has the following expression (see Proposition 2.1 in [44] for the qc case.)

**Proposition 3.8** Let $\{V_a\}_{a=0}^7$ be a local orthonormal basis of $H$. Then, locally for $u \in C_c^\infty(M)$, we have

$$\Delta_g u = (-V_a V_a u + \Gamma_a^{bb} V_a u).$$

(3.26)

**Proof** Let $\{\theta^a\}_{a=0}^7$ be the dual basis of $\{V_a\}_{a=0}^7$ for $H^*$ and $\theta^a|_V = 0$ for each $a$. Let $V^*_a$ be the formal adjoint operator of $V_a$. By Stokes formula

$$\int_M V_a \cdot \mathrm{d}V_g = \int_M i_{V_a} \mathrm{d}u \cdot \mathrm{d}V_g = \int_M v \mathrm{d}u \wedge i_{V_a} \mathrm{d}V_g = -\int_M u \mathrm{d}v \wedge i_{V_a} \mathrm{d}V_g - \int_M u \mathrm{d}v \mathrm{d}V_g - \int_M u \mathrm{d}v \wedge i_{V_a} \mathrm{d}V_g.$$  

(3.27)

Here, we have used identities $\int_M i_{V_a} (\mathrm{d}u \wedge \mathrm{d}V_g) = 0$. Note that

$$d\theta^a = \frac{1}{2}(-\Gamma^{ab}_{bb} + \Gamma^{ba}_{bb}) \theta^b \wedge \theta^a \text{ mod } \theta_1, \ldots, \theta_7.$$

Note that

$$d(i_{V_a} \mathrm{d}V_g) = (-1)^a 4! (-1)^{a+1} \theta_1 \wedge \ldots \wedge \mathrm{d}\theta_a \wedge \ldots \wedge \theta_7 \wedge \hat{\theta}_2 \wedge \ldots \wedge \theta_7$$

$$+ \sum_{b \neq a} \mathrm{d}\theta^b \wedge i_{V_b} i_{V_a} \mathrm{d}V_g.$$  

(3.28)

The first term on the right side of (3.28) is zero since it is annihilated by the Reeb vectors $R_a$ by $i_{R_a} \mathrm{d}\theta_a = 0$. Since

$$d\theta^b = \frac{1}{2}(-\Gamma^{ba}_{bb} + \Gamma^{ab}_{ba}) \theta^b \wedge \theta^a + \frac{1}{2}(-\Gamma^{ab}_{bb} + \Gamma^{ba}_{ab}) \theta^b \wedge \theta^a + \ldots,$$

we get

$$d(i_{V_a} \mathrm{d}V_g) = \sum_{b \neq a} (-\Gamma^{ab}_{bb} + \Gamma^{ba}_{ab}) \mathrm{d}V_g = -\sum_{b \neq a} \Gamma^a_{bb} \mathrm{d}V_g.$$  

(3.29)

So, by (3.27) and (3.29), we have $V^*_a = -V_a + \Gamma_a^{bb}$. Therefore, (3.26) follows.  

The scalar curvature $s_{\tilde{g}}$ for the metric $\tilde{g} = e^{2h} g_0$ satisfies

$$s_{\tilde{g}} = e^{-2h} \left( 42 \Delta_{g_0} h - 420 \sum_{a=0}^7 (X_a h)^2 \right).$$

(3.30)
We can also write the transformation law in the following form.

**Corollary 3.1** The scalar curvature $s_{\tilde{g}}$ of the Biquard connection for $\tilde{\tilde{g}} = \phi^{\frac{4}{Q-2}}g_0$ on the octonionic Heisenberg group satisfies the OC Yamabe equation:

$$b\Delta_{s_{\tilde{g}}}\phi = s_{\tilde{g}}\phi^{\frac{Q+2}{Q-2}}, \quad b = \frac{4(Q - 1)}{Q - 2} = \frac{21}{5}. $$

Now let us derive the transformation formula of the OC Yamabe operator on general spherical OC manifolds. See [34] for such derivation for the pseudo-Riemannian case, [43] for the CR case and [40] for the qc case.

**Proposition 3.9** Let $(M, \tilde{g}, \tilde{\mathcal{L}})$ and $(M, g, \mathcal{L})$ be two OC manifolds. Let $\tilde{g} = \phi^{\frac{4}{Q-2}}g$ for some positive smooth function $\phi$ on $M$. Then

$$\Delta_{\tilde{g}}(\phi \cdot f) = \Delta_{\tilde{g}}\phi \cdot f + \phi^{\frac{Q+2}{Q-2}}\Delta_{\tilde{g}}f \quad (3.31)$$

for any smooth function $f$ on $M$.

**Proof** Note that $\Delta_{\tilde{g}}$ and the OC Yamabe operator is independent of the choice of compatible $\tilde{\mathcal{L}}$. Let $\Theta = (\theta_1, \ldots, \theta_7)$ be a $\mathbb{R}^7$-valued 1-form associated to $(M, g, \mathcal{L})$ and let $\tilde{\Theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_7)$ be associated to $(M, \tilde{g}, \tilde{\mathcal{L}})$. For any real function $h$ on $M$, we have

$$\langle \Delta_{\tilde{g}}(\phi \cdot f), h \rangle_{\tilde{\tilde{g}}} = \langle d_b(\phi \cdot f + \phi d_b f, d_b h)_{\tilde{g}} + \langle d_b f, \phi \cdot d_b h \rangle_{\tilde{g}} = \langle d_b(\phi \cdot f \cdot d_b h),\phi \cdot d_b h \rangle_{\tilde{g}} \quad (3.32)$$

Let us calculate the second term in the right side of (3.32). By our assumption and Proposition 3.7, we just need to consider $\tilde{\Theta} = \phi^{\frac{4}{Q-2}}\Theta$. Then, for a fixed $\beta$, we have

$$d\tilde{\theta}_\beta = d(\phi^{\frac{4}{Q-2}}\theta_\beta) = \frac{4}{Q-2} \phi^{\frac{4}{Q-2}} d\phi \wedge \theta_\beta + \phi^{\frac{4}{Q-2}} d\theta_\beta. \quad (3.33)$$

So, we get $dV_{\tilde{g}} = \phi^{\frac{20}{Q-2}}dV_{\tilde{g}}$. Consequently, for 1-forms $\omega_1, \omega_2 \in H^*$, we have

$$\langle \omega_1, \omega_2 \rangle_{\tilde{g}} = \int_M \langle \omega_1, \omega_2 \rangle dV_{\tilde{g}} = \langle \phi^{-2}\omega_1, \omega_2 \rangle_{\tilde{g}}. \quad (3.34)$$

Now we find that

$$\langle d_b f, \phi \cdot d_b h - h \cdot d_b \phi \rangle_{\tilde{g}} = \langle \phi^{-2}d_b f, \phi \cdot d_b h - h \cdot d_b \phi \rangle_{\tilde{g}} = \langle d_b f, d_b(\phi^{-1}h) \rangle_{\tilde{g}} = \int_M \Delta_{\tilde{g}} f \cdot \phi^{-1}dV_{\tilde{g}} = \int_M \phi^{\frac{Q+2}{Q-2}}\Delta_{\tilde{g}} f \cdot dV_g.$$

The proposition is proved. $\square$

**Proof of Theorem 1.1** By choosing an open set of the octonionic Heisenberg group with standard OC metric as a local coordinate, we can write $\tilde{g} = \phi^{\frac{4}{Q-2}}g_0, g = \phi^{\frac{4}{Q-2}}g_0$ locally. Then, $\tilde{g} = \phi^{\frac{4}{Q-2}}g$ with $\phi = \phi_1\phi_2^{-1}$. Applying (3.31), we have
\[ \Delta_{\nu_0} \phi_1 = \Delta_{\nu_0} (\phi_2 \cdot \phi_1 \phi^{-2}_1) = \Delta_{\nu_0} \phi_2 \cdot \phi_1 \phi^{-2}_1 + \phi_2 \frac{\nu_2}{\nu_2} \Delta_{\nu} (\phi_1 \phi^{-2}_2). \]

Thus, we have

\[ s_\nu \phi_1^{\nu_2} = s_\nu \phi_2^{\nu_2} \cdot \phi + b \phi^{\nu_2} \Delta_{\nu} \phi \]

by using Corollary 3.1, i.e., \( b \Delta_{\nu} \phi + s_\nu \phi = s_\nu \phi^{\nu_2}. \) The theorem is proved. \( \square \)

**Proof of Corollary 1.1** By using (1.3) and (3.31), we have

\[ L_\nu (\phi f) = b \Delta_{\nu} (\phi f) + s_\nu \phi f = b \left( \Delta_{\nu} \phi \cdot f + \phi^{\nu_2} \Delta_{\nu} f \right) + s_\nu \phi f = \phi^{\nu_2} \left( b \Delta_{\nu} f + s_\nu f \right). \]

The result follows. \( \square \)

Define the OC Yamabe invariant

\[ \lambda(M, g) := \inf_{u > 0} \int_M \left( b |\nabla_u u|^2 + s_g u^2 \right) dV_g, \quad (3.34) \]

where \( |\nabla_u f|^2 = \sum_{\alpha = 0}^7 |V_\alpha f|^2 \) if \( \{ V_\alpha \} \) is a local orthogonal basis of \( H \) under the Carnot–Carathéodory metric \( g. \) It is an invariant for the conformal class of OC manifolds. There is a natural OC Yamabe problem as in the (contact) Riemannian, CR and qc cases (cf., e.g., [21, 25, 45] and references therein). The Yamabe-type equation on groups of Heisenberg type, including the octonionic Heisenberg group, has been studied in [14].

### 4 The Green function of the OC Yamabe operator and conformal invariants

#### 4.1 The Green function

It is a continuous function \( G_g : M \times M \setminus \text{diag}M \to \mathbb{R} \) such that

\[ \int_M G_g(\xi, \eta) L_\nu u(\eta) dV_g(\eta) = u(\xi) \]

for all \( u \in C^0_0(M). \) Namely, \( L_\nu G_g(\xi, \cdot) = \delta_\nu. \)

The explicit form of the fundamental solution of the sub-Laplacian on H-type groups, including the octonionic Heisenberg groups, is known (cf., e.g., [4, 26]). It can be checked directly as in the Appendix in [40].

**Proposition 4.1** The Green function of the OC Yamabe operator \( L_0 = b \Delta_0 \) on the octonionic Heisenberg group \( \mathcal{H} \) with the pole at \( \xi \) is

\[ G_0(\xi, \eta) := \frac{C_0}{\| \xi^{-1} \eta \|^{\nu_2-2}}, \]

for \( \xi \neq \eta, \xi, \eta \in \mathcal{H}, \) where \( \| \cdot \| \) is the norm on \( \mathcal{H} \) defined by (2.14) and

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\[ C_Q^{-1} = (Q + 2)(Q - 2)b \int_{\mathbb{R}^{15}} \frac{|x|^2}{(|x|^4 + |t|^2 + 1)^{\gamma}} \, dV_0, \quad (4.1) \]

where \(dV_0\) is Lebesgue measure.

**Proposition 4.2** For a connected compact OC manifold \((M, g, \mathfrak{l})\), we have trichotomy: there exists an OC metric \(\tilde{g}\) conformal to \(g\) which has either positive, negative or vanishing scalar curvature everywhere.

**Proof** The OC Yamabe operator \(L_g\) is a formally self-adjoint and subelliptic differential operator. So, its spectrum is real and bounded from below. Let \(\lambda_1\) be the first eigenvalue of \(L_g\) and let \(\phi\) be an eigenfunction of \(L_g\) with eigenvalue \(\lambda_1\). Then, \(\phi > 0\) and is \(C^\infty\) as the qc case [40]. The scalar curvature of \((M, \tilde{g}, \mathfrak{l})\) with \(\tilde{g} = \phi^{-2/3}g\) is \(s_{\tilde{g}} = \lambda_1 \phi^{-2/3}\) by the OC Yamabe equation (1.3). In particular, \(s_{\tilde{g}} > 0\) (resp. \(s_{\tilde{g}} < 0\), resp. \(s_{\tilde{g}} \equiv 0\)) if \(\lambda_1 > 0\) (resp. \(\lambda_1 < 0\), resp. \(\lambda_1 = 0\)). On the other hand, if \(\tilde{g}\) has scalar curvature \(s_{\tilde{g}} > 0\) (resp. \(s_{\tilde{g}} < 0\), resp. \(s_{\tilde{g}} \equiv 0\)), the first eigenvalue \(\hat{\lambda}_1\) of \(L_{\tilde{g}}\) obviously satisfies \(\hat{\lambda}_1 > 0\) (resp. \(\hat{\lambda}_1 < 0\), resp. \(\hat{\lambda}_1 = 0\)). \(\square\)

For \(\xi \in \mathcal{H}\) and \(\epsilon > 0\), define a ball \(B(\xi, \epsilon) := \{\eta \in \mathcal{H} : \|\xi^{-1} \cdot \eta\| < \epsilon\}\) in the octonionic Heisenberg group.

**Proposition 4.3** Let \((M, g, \mathfrak{l})\) be a connected compact OC manifold with positive scalar curvature and let \(U\) be a sufficiently small open set. Then, the function \(\tilde{G}_g(\xi, \eta) - \rho_g(\xi, \eta)\) can be extended to a \(C^\infty\) function on \(U \times U\), where \(\rho_g(\cdot, \cdot)\) is given by (1.5).

**Proof** Suppose that \(\tilde{U} \subset \tilde{U} \subset \mathcal{H}\) and \(g = \phi^{1/3}g_0\) on \(\tilde{U}\). We choose a sufficiently small \(\rho\) such that \(B(\xi, \rho) \subset \tilde{U}\) for any \(\xi \in U\). We can construct the Green function as follows. For \(\xi, \eta \in U\), define

\[ \tilde{G}(\xi, \eta) = \tilde{G}(\xi^{-1} \eta), \]

where \(\tilde{G}\) is the cut-off fundamental solution of \(L_0 = -b \sum_{a=0}^7 X_a^2\), i.e., \(\tilde{G}(\tilde{\eta}) = \frac{C_Q}{\|\tilde{\eta}\|^2} f(\tilde{\eta})\) for \(\tilde{\eta} \in \mathcal{H}\) with \(f \in C^\infty(\mathcal{H})\) satisfying \(f \equiv 1\) on \(B(0, \frac{\xi}{2})\) and \(f \equiv 0\) on \(B(0, \rho)\). Then,

\[ L_0 \tilde{G}(\tilde{\eta}) = \delta_0 - b X_a \left( \frac{C_Q}{\|\tilde{\eta}\|^2} \right) X_a f(\tilde{\eta}) + \frac{C_Q}{\|\tilde{\eta}\|^2} L_0 f(\tilde{\eta}) =: \delta_0 + \tilde{G}_1(\tilde{\eta}) \quad (4.2) \]

by \(X_a f \equiv 0\) on \(B(0, \frac{\xi}{2})\). Here, \(\delta_0\) is the Dirac function at the origin with respect to the measure \(dV_0\) and \(\tilde{G}_1\) is defined by the last equality in (4.2). Set \(G_1(\xi, \eta) := G_1(\xi^{-1} \eta)\) for \(\xi, \eta \in U\). Then, \(G_1(\xi, \eta) \in C^\infty(\tilde{U} \times \tilde{U})\) and for each \(\xi \in U\), \(G_1(\xi, \cdot)\) can be naturally extended to a smooth function on \(M\). By transformation law (1.4) and left invariance of \(X_a\), we find that

\[ L_g \left( \phi(\xi)^{-1} \phi(\cdot)^{-1} \tilde{G}(\xi, \cdot) \right) = \phi(\xi)^{-1} \phi(\cdot)^{-1} \frac{\alpha_{\xi^2}}{\alpha_2} L_0 \tilde{G}(\xi, \cdot) \]

\[ = \phi(\xi)^{-1} \phi(\cdot)^{-1} \frac{\alpha_{\xi^2}}{\alpha_2} \left( \delta_0(\xi^{-1} \cdot) + G_1(\xi, \cdot) \right) \]

\[ = \delta_\xi + \phi(\xi)^{-1} \phi(\cdot)^{-1} \frac{\alpha_{\xi^2}}{\alpha_2} G_1(\xi, \cdot), \]
on $U$ for $\xi \in U$, where $\delta_\xi$ is the Dirac function at point $\xi$ with respect to the measure $dV_g = \phi^{\frac{n-2}{2}} dV_0$. Now set
\[
G(\xi, \eta) := \phi(\xi)^{-1} \phi(\eta)^{-1} \tilde{G}(\xi, \eta) + G_2(\xi, \eta)
\]  
(4.3)
for $\eta \in M$, where $G_2(\xi, \eta)$ satisfies
\[
L_g G_2(\xi, \cdot) = -\phi(\xi)^{-1} \phi(\cdot) \frac{\partial^{2-2} G_1(\xi, \cdot)}{\partial \xi^2}.
\]  
(4.4)
$G_2(\xi, \cdot)$ exists since $L_g$ is invertible in $L^2(M)$. $G_2(\xi, \cdot) \in C^\infty(M)$ for fixed $\xi \in U$ by the sub-elliptic regularity of $L_g$. $G_2(\cdot, \eta)$ is also in $C^\infty(U)$ by differentiating (4.4) with respect to the variable $\xi$ repeatedly. Now we have $L_g \tilde{G}(\xi, \cdot) = \delta_\xi$, i.e., $G(\xi, \eta)$ is the Green function of $L_g$.

By (4.3), $G_g(\xi, \eta) = G(\phi(\xi), \phi(\eta))$, i.e., $G(\xi, \eta)$ is the Green function of $L_g$.

We have the following transformation formula of the Green functions under conformal OC transformations.

**Proposition 4.4** Let $(M, g, \mathcal{U})$ be a connected, compact, scalar positive OC manifold and $G_g$ be the Green function of the OC Yamabe operator $L_g$. Then
\[
G_g(\xi, \eta) = \frac{1}{\phi(\xi)\phi(\eta)} G_g(\xi, \eta)
\]  
(4.5)
is the Green function of the OC Yamabe operator $L_g$, for $\tilde{g} = \phi^{\frac{4}{n-2}} g$.

**Proof** By the transformation law (4.4), we find that
\[
\int_M \frac{G_g(\xi, \eta)L_g u(\eta)}{\phi(\xi)\phi(\eta)} dV_g = \frac{1}{\phi(\xi)} \int_M \frac{1}{\phi(\eta)} G_g(\xi, \eta)\phi(\eta)^{-\frac{4}{n-2}} L_g(\phi u)(\eta)\phi(\eta)^{\frac{20}{n-2}} dV_g
\]
\[
= \frac{1}{\phi(\xi)} \int_M G_g(\xi, \eta)L_g(\phi u) dV_g = u(\xi)
\]
for any $u \in C^\infty_0(M)$. The proposition follows from the uniqueness of the Green function.

\[\square\]

### 4.2 An invariant tensor on a scalar positive OC manifold

**Proof of Theorem 1.2** We will verify that $\mathcal{A}^2_g$ is independent of the choice of local coordinates and $\mathcal{A}^2_{g\tilde{g}}$ is independent of the choice of $g$ in the conformal class $[g]$. Suppose $\tilde{g} = \Phi^{\frac{4}{n-2}} g$. Let $U \subset M$ be an open set and let $\rho : U \to V \subset \mathcal{H}$ and $\tilde{\rho} : U \to \tilde{V} \subset \mathcal{H}$ be two coordinate charts such that
\[
g = \rho^*(\phi_1^{\frac{4}{n-2}} g_0), \quad \tilde{g} = \tilde{\rho}^*(\phi_2^{\frac{4}{n-2}} g_0),
\]
for two positive function $\phi_1$ and $\phi_2$. Then, $f = \tilde{\rho} \circ \rho^{-1} : V \to \tilde{V}$ and
\[
f^* g_0|_{\xi'} = \phi_1^{\frac{4}{n-2}}(\xi') g_0|_{\xi'} \quad \text{with} \quad \phi(\xi') = \phi_1(\xi') \phi_2^{-1}(f(\xi')) \Phi(\rho^{-1}(\xi'))
\]  
(4.6)
for $\xi' \in V$. We claim the following the transformation law of the Green function on the octonionic Heisenberg group under a conformal OC transformation:

$$\frac{1}{\|f(\xi')^{-1}f(\eta')\|^{Q-2}} = \frac{1}{\phi(\xi')} \cdot \frac{1}{\|\xi'^{-1}\eta'\|^{Q-2}},$$

(4.7)

for any $\xi', \eta' \in V$. Apply this to $\xi' = \rho(\xi), \eta' = \rho(\eta)$ and $f = \tilde{\rho} \circ \rho^{-1}$ to get

$$\frac{1}{\|\tilde{\rho}(\xi)^{-1}\tilde{\rho}(\eta)\|^{Q-2}} = \frac{1}{\phi(\rho(\xi))} \cdot \frac{1}{\|\rho(\xi)^{-1}\rho(\eta)\|^{Q-2}}$$

and so

$$A_{k_{\xi}}(\xi) = \lim_{\eta \to \xi} \left| G_{k_{\xi}}(\xi, \eta) - \frac{1}{\phi(\rho(\xi))} \cdot \frac{C_{Q}}{\|\rho(\xi)^{-1}\rho(\eta)\|^{Q-2}} \right|^{\frac{1}{Q-2}}$$

$$= \lim_{\eta \to \xi} \left| \frac{G_{k_{\xi}}(\xi, \eta)}{\Phi(\xi)\Phi(\eta)} - \frac{1}{\Phi(\rho(\xi))\Phi(\rho(\eta))} \cdot \frac{C_{Q}}{\|\rho(\xi)^{-1}\rho(\eta)\|^{Q-2}} \right|^{\frac{1}{Q-2}}$$

$$= \Phi^{-\frac{\sigma}{Q-2}}(\xi) \lim_{\eta \to \xi} \left| G_{k_{\xi}}(\xi, \eta) - \frac{1}{\phi(\rho(\xi))} \cdot \frac{C_{Q}}{\|\rho(\xi)^{-1}\rho(\eta)\|^{Q-2}} \right|^{\frac{1}{Q-2}}$$

$$= \Phi^{-\frac{\sigma}{Q-2}}(\xi) A_{k_{\xi}}(\xi).$$

Consequently, we have $A_{k_{\xi}} = A_{k_{\zeta}}$. It remains to check (4.7). By OC Liouville-type Theorem 2.1, $f$ is a restriction to $V$ of an OC automorphism of $\mathcal{H}$, denoted also by $f$. By the transformation law (1.4), for functions $\tilde{\phi} := \phi \circ f^{-1}, \tilde{u} := u \circ f^{-1}$ on $\tilde{V}$, we have

$$L_{0}(\tilde{\phi}^{-1}\tilde{u})\bigg|_{f(\eta')} = L_{g}(\phi^{-1}u)\bigg|_{\eta'} = \phi^{-\frac{\sigma}{Q-2}}(\eta') L_{0}(u)\bigg|_{\eta'},$$

(4.8)

$$f^{*}dV_{0}\bigg|_{f(\eta')} = \phi^{\frac{\sigma}{Q-2}}(\eta') dV_{0}\bigg|_{\eta'}.$$
See [28] for the identity (4.7) on the Euclidean space and see [42] on the Heisenberg group and [40] on the quaternionic Heisenberg group. The OC positive mass conjecture implies that \( A_t \) is non-vanishing. Then \( A^2_t g \) is a conformally invariant OC metric. This invariant metric was given by Habermann-Jost [15, 16] for the locally conformally flat case, by the second author for the CR case [42] and by us [40] for the qc case, respectively.

### 4.3 The connected sum of two scalar positive OC manifolds

Let \((M, g, \mathfrak{l})\) be a OC manifold of dimension 15 with two punctures \( \eta_1, \eta_2 \), or disjoint union of two connected OC manifolds \((M_i, g_i, \mathfrak{l}_i)\) with one puncture \( \eta_i \in M_i \) each, \( i = 1, 2 \). Let \( U_1 \) and \( U_2 \) be two disjoint neighborhoods of \( \eta_1 \) and \( \eta_2 \), respectively. Let

\[
\psi_i : U_i \to B(0, 2), \quad i = 1, 2,
\]

be local coordinate charts such that \( \psi_i(\eta_i) = 0 \). For \( t < 1 \), define

\[
U_j(t, 1) := \{ \eta \in U_j; t < \| \psi_j(\eta) \| < 1 \}, \quad U_j(t) := \{ \eta \in U_j; \| \psi_j(\eta) \| < t \},
\]

\( i = 1, 2 \). For any \( t \in (0, 1), A \in \text{Spin}(7) \), we can form a new OC manifold \( M_{t, A} \) by removing the closed balls \( \overline{U_i(t)} \), \( i = 1, 2 \), and gluing \( U_1(t, 1) \) with \( U_2(t, 1) \) by the conformal OC mapping \( \Psi_{t, A} : U_1(t, 1) \to U_2(t, 1) \) defined by

\[
\Psi_{t, A}(\eta) = \psi_2^{-1} \circ D_\eta \circ R \circ A \circ \psi_1(\eta), \quad \text{for} \ \eta \in U_1(t, 1),
\]

where \( R : \{ \zeta \in \mathbb{H}; t < \| \zeta \| < 1 \} \to \{ \zeta \in \mathbb{H}; 1 < \| \zeta \| < \frac{1}{t} \} \) is the inversion in (2.25). Note that \( \Psi_{t, A} \) is a conformal OC mapping \( U_1(t, 1) \) to \( U_2(t, 1) \), which identifies the inner boundary of \( U_1(t, 1) \) with the outer boundary \( U_2(t, 1) \) and vice versa. Let \( \pi_{t, A} : (M_1 \setminus \overline{U_1(t)}) \cup (M_2 \setminus \overline{U_2(t)}) \to M_{t, A} \) be a canonical projection. We call \( M_{t, A} \) the connected sum of \( M_1 \) and \( M_2 \). We denote this OC manifold by \((M_{t, A}, g, \mathfrak{l}_{t, A})\), where \( g \) is a metric in the conformal class. As in the locally conformally case, the connected sums are expected to be not isomorphic for some different choices of \( t, A \) [24]. Scalar positive OC manifolds are abundant by the following proposition.

**Proposition 4.5** If \( t \) is sufficiently small, the connected sum \((M_{t, A}, g, \mathfrak{l}_{t, A})\) is scalar positive.

Proposition 4.5 follows from Proposition 4.6. As preparation, we prove the following lemma firstly.

**Lemma 4.1** For \( g|_\xi = \frac{g_{0|_\xi}}{\| \xi \|^2}, \xi = (x, t) \), we have

\[
x_g(\xi) = (Q - 2)(Q - 1)\frac{\| x \|^2}{\| \xi \|^2},
\]

(4.11)

**Proof** Note that

\[
X_a \| \xi \|^4 = 4| x |^2 x_a + 4\mathbf{p}_b x_b t_\phi,
\]

(4.12)

by using the expression of the vector field \( X_a \) in (2.15). Then
\[
\left| \nabla_0 \| \xi \|^4 \right|^2 = (X_\alpha \| \xi \|^4 \cdot (X_\alpha \| \xi \|^4) = 16 \left( |x|^4 x_\alpha \cdot x_\alpha + E_{\beta\alpha}^\beta E_{\beta\alpha}^\beta x_\alpha x_\beta t_{\beta'} \right) = 16 \| \xi \|^4 |x|^2,
\]

(4.13)

by using (4.12) and \( E_{\beta\alpha}^\beta E_{\beta\alpha}^\beta = (E\beta E\beta')_{bb'} \), antisymmetry of \( E\beta E\beta' \) of \( \beta \neq \beta' \) and \( (E\beta)^2 = -\text{id} \) in Proposition 2.4. Similarly, by (4.12), we get

\[
\Delta_0 \| \xi \|^4 = - \sum \left( 8x_\alpha^2 + 4 |x|^2 + 8E_{\beta\alpha}^\beta E_{\beta\alpha}^\beta x_\alpha x_\beta \right) = -4(Q + 2) |x|^2.
\]

(4.14)

We can write \( g = \phi^4 g_0 \) with \( \phi = \| \xi \|^{-\frac{Q-2}{2}} \). It follows from the transformation formula (1.3) of the scalar curvatures that

\[
s_g = \phi^{-\frac{Q-2}{2}} b \Delta_0 \phi = \| \xi \|^{-\frac{Q-2}{2}} b \Delta_0 \| \xi \|^{-\frac{Q-2}{2}} \\
= -\frac{Q-2}{8} b \| \xi \|^{-\frac{Q-2}{2}} \left( \frac{Q+6}{8} \| \xi \|^{-\frac{Q+14}{2}} \left| \nabla \| \xi \|^4 \right| + \| \xi \|^{-\frac{Q+6}{2}} \Delta_0 \| \xi \|^4 \right) \\
= (Q-2)(Q-1) \frac{|x|^2}{\| \xi \|^2}
\]

by (4.13) and (4.14) for \( b = 4 \frac{Q-1}{Q-2} \).

Proposition 4.6 If \( t \) is sufficiently small, we have \( \lambda(M_{tA}, g, \mathcal{I}_{tA}) > 0 \).

Proof See [27] for the Riemannian case, [42] for the spherical CR case (see also [7] for a different proof) and [40] for the spherical qc case. In [8, 11], it is generalized to the non-spherical CR case. The proof of this proposition is similar to the spherical CR and qc case. Let \( M_0 = M_1 \setminus \{ \xi_1 \} \cup M_2 \setminus \{ \xi_2 \} \), and let \( \tilde{g} \) be an OC metric on \( M_0 \). Then, by multiplying a positive function \( \mu \in C^\infty(M) \setminus \{ \xi_1, \xi_2 \} \), we can assume \( g = \mu \tilde{g} \) satisfying

\[
\left( \psi_i^{-1} \right)^* g|_x = \frac{g_0|_x}{\| \xi \|^2} \text{ on } B(0,2) \setminus \{0\},
\]

where \( \psi_i : U_i \rightarrow B(0,2), \ i = 1, 2, \) are coordinate charts in (4.9). It is easy to see that gluing mapping \( \Psi_{tA} \) in (4.10) preserves the metric \( \frac{g_0|_x}{\| \xi \|^2} \) on \( t < \| \xi \| < \frac{1}{\gamma}, \ 0 < \frac{2}{3} t < 1 \), by the transformation formula and Proposition 2.5. \( \frac{g_0|_x}{\| \xi \|^2} \) is invariant under the rotation \( A \) and the inversion \( R \). Hence, we can glue \( g \) by \( \Psi_{tA} \) to obtain a OC metric that coincides with \( g \) on \( M_1 \setminus \overline{U}_1(t) \cup M_2 \setminus \overline{U}_2(t) \). We denote the resulting OC metric also by \( g \) by abuse of notations, and the connected sum by \( (M_1, g, \mathcal{I}) \). Here, we omit the subscripts \( A \) for simplicity.

\( (M_0, g, \mathcal{I}) \) has two cylindrical ends. We can identify the ball with cylindrical end by the mapping

\[
\Psi : B(0,1) \rightarrow [0, \infty) \times \Sigma, \quad \xi = D_{e^{-u}}(\eta) \mapsto \left( \ln \frac{1}{\| \xi \|}, \frac{\xi}{\| \xi \|} \right) = (u, \eta),
\]

(4.15)

where \( \Sigma = \{ \eta \in \mathcal{H} \| \eta \| = 1 \} \) is diffeomorphic to the sphere \( S^4 \). Define a Carnot–Carathéodory metric

\[
\tilde{g}|_{\Psi(\xi)} = \left( \Psi^{-1} \right)^* \left( \frac{g_0|_x}{\| \xi \|^2} \right)
\]

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on $[0, \infty) \times \Sigma$ and $\tilde{\Theta} = (\Psi^{-1})^* \Theta_0 = (\tilde{\theta}_1, \ldots)$ is a compatible contact form. 

$\left( B(0,1) \setminus \{0\}, \frac{g_0}{\|x\|^4}, \|x\| \right)$ is OC equivalent to $([0, \infty) \times \Sigma, \tilde{g}, \|x\|)$. Since $(\Psi^{-1})^* dV_g = \frac{dV_0}{\|x\|^4}$ is invariant under rescaling, it is easy to see that the measure $\tilde{\theta}_1 \wedge \ldots \wedge \tilde{\theta}_7 \wedge (d\tilde{\theta})^4$ is invariant under translation $(u', \xi) \to (u' + u_0, \xi)$ on $[0, \infty) \times \Sigma$. As a measure, we have

$$\tilde{\theta}_1 \wedge \ldots \wedge \tilde{\theta}_7 \wedge (d\tilde{\theta})^4 = dudS_\Sigma,$$  

where $dS_\Sigma$ is a measure on $\Sigma$. Set $l = \ln \frac{1}{l}$, and write

$$(M_0, g, \|x\|) = ([0, \infty) \times \Sigma, \tilde{g}, \|x\|) \cup (\hat{M}, g, \|x\| \cup ([0, \infty) \times \Sigma, \tilde{g}, \|x\|),$$

where $\hat{M} = M \setminus (U_1(1) \cup U_2(1)).$ We identify two pieces of $([0, l) \times \Sigma, \tilde{g}, \|x\|$ to get $(M_1, g, \|x\|).$

For $\eta \in \Sigma$, we can write $\eta = (x_\eta, t_\eta) \in \mathcal{X}$ for some $x_\eta \in \mathbb{R}^8, t_\eta \in \mathbb{R}^7$. Then, the OC Yamabe invariant holds. But it is still independent of the variable $\eta$. Now define a Lipschitz function $f_i \in C^\infty(M)$ such that

$$\int_{M} (b|\nabla_{g_i} f|^2 + s_{g_i} f^2) dV_g < \lambda(M_1, g, \|x\|) + \frac{1}{l}, \quad \text{with} \quad \int_{M} f_i^{\frac{20}{19}} dV_g = 1. \quad (4.19)$$

Put $A_1 = -\min \{0, \min_{x \in \Sigma} s_{g_i} \}$ Vol$(\hat{M})^{\frac{1}{\gamma-2}}$, which is uniformly bounded by Vol$(M, g)$. Thus, by using H"older’s inequality we get from (4.19) that

$$\int_{[0, \infty) \times \Sigma} (b|\nabla_{g_i} f|^2 + s_{g_i} f^2) dS_\Sigma < \lambda(M_1, g, \|x\|) + \frac{1}{l} + A_1,$$

(cf. Lemma 6.2 in [27]). Note that $s_{g_i}$ is nonnegative on $[0, \infty) \times \Sigma$ by (4.18). Therefore, there exists $l_1 \in [1, l - 1]$ such that

$$\frac{1}{2} \int_{l_1 - 1}^{l_1 + 1} du \int_{\Sigma} (b|\nabla_g f|^2 + s_g f^2) dS_\Sigma < \frac{\lambda(M_1, g, \|x\|) + \frac{1}{l} + A_1}{l - 2},$$

i.e., we have the estimate

$$\int_{l_1 - 1}^{l_1 + 1} du \int_{\Sigma} (|\nabla_g f(u, \eta)|^2 + |x_\eta|^2 f^2(u, \eta)) dS_\Sigma(\eta) < \frac{C}{l}, \quad (4.20)$$

by the scalar curvature of $\tilde{g}$ in (4.18), where $C$ is a constant independent of $l$ (because the OC Yamabe invariants $\lambda(M_1, g, \|x\|)$ for $l > 1$ have a uniform upper bound by choosing a test function). It is different from the Riemannian case that the scalar curvature of $\frac{20\lambda}{\|x\|^4}$ is not constant. But it is still independent of the variable $u$. Now define a Lipschitz function $F_i$ on $M_0$ by $F_i = f_i$ on $[0, l_1] \times \Sigma \cup \hat{M} \cup [0, l - l_1] \times \Sigma$ and
\[ F_t(u, x) = \begin{cases} 
(l_\ast + 1 - u)f_t(u, x) & \text{for } (u, x) \in [l_\ast, l_\ast + 1] \times \Sigma, \\
0 & \text{for } (u, x) \in [l_\ast + 1, \infty) \times \Sigma,
\end{cases} \] (4.21)

and similarly on \([l - l_\ast, \infty) \times \Sigma\).

By definition, \(|\nabla_\theta F_t| = |\nabla_\phi f_t|\) and \(F_t^2 = f_t^2\) hold on \([0, l_\ast) \times \Sigma \cup \tilde{M} \cup [0, l - l_\ast) \times \Sigma\). On the other hand, note that \(|\nabla_\theta F_t| \leq |\nabla_\phi u|[f_t] + |\nabla_\phi f_t|\) pointwisely on \((l_\ast, l_\ast + 1) \times \Sigma\) by definition. By (4.17), (4.18) and estimate (4.20), we find that

\[
\int_{(l_\ast, l_\ast + 1) \times \Sigma} (b|\nabla_\theta F_t|^2 + s_\theta F_t^2) \, du \, d\Sigma \\
\leq C' \int_{(l_\ast, l_\ast + 1) \times \Sigma} (|\nabla_\phi f_t|^2 + |x_{\phi}^t|^2 f_t^2) \, du \, d\Sigma(\eta) \leq \frac{B'}{l}.
\]

Therefore, we get

\[
\int_{M_0} (b|\nabla_\theta F_t|^2 + s_\theta F_t^2) \, dv_g < \lambda(M_t, g, \mathbb{I}_t) + \frac{B}{l},
\]

for some constant \(B\) independent of \(l\).

Obviously from (4.19) and the definition of \(F_t\), we get \(\int_{M_0} F_t^2 \geq \int_{M_0} F_t^2 \geq \int_{M_0} \geq 1\). Therefore,

\[
\inf_{F > 0} \frac{\int_{M_0} (b|\nabla_\theta F|^2 + s_\theta F^2) \, dv_g}{(\int_{M_0} F_t^2 \, dv_g)^{\frac{2}{n-2}}} < \lambda(M_t, g, \mathbb{I}_t) + \frac{B}{l}, \quad (4.22)
\]

where the infimum is taken over all nonnegative Lipschitz functions with compact support. It follows from the definition of the Yamabe invariant that the left side is greater than or equal to \(\lambda(M, g, \mathbb{I})\). Then, \(\lambda(M_t, g, \mathbb{I}_t)\) is positive if \(l\) is sufficiently large, i.e., \(t\) is sufficiently small. We complete the proof. \(\square\)

5 The convex cocompact discrete subgroups of \(F_4(-20)\)

5.1 Convex cocompact subgroups of \(F_4(-20)\)

A group \(G\) is called discrete if the topology on \(G\) is discrete. We say that \(G\) acts discontinuously on a space \(X\) at point \(w\) if there is a neighborhood \(U\) of \(w\), such that \(g(U) \cap U = \emptyset\) for all but finitely many \(g \in G\).

Let \(\Gamma\) be a discrete subgroup of \(F_4(-20)\). It is known that the limit set \(\Lambda(\Gamma)\) in (1.7) does not depend on the choice of \(q \in U\) (cf. [12, Proposition 1.4, 2.9]). The limit set \(\Lambda(\Gamma)\) of all limit points is closed and invariant under \(\Gamma\). The radial limit set of \(\Gamma\) is

\[
\Lambda^r(\Gamma) := \left\{ \xi \in \Lambda(\Gamma) \left| \liminf_{T \to \infty} d(\xi_T, \gamma(0)) < \infty, \gamma \in \Gamma \right. \right\},
\]

where \(\xi_T\) refers to the point on the ray from 0 to \(\xi\) for which \(d(0, \xi_T) = T\) and \(d(\cdot, \cdot)\) is the octonionic hyperbolic distance. \(\Gamma\) is called a Kleinian group if \(\Omega(\Gamma)\) defined in (1.8) is non-empty. A Kleinian group is called elementary if \(\Lambda(\Gamma)\) contains at most two points. \(\Gamma\) is
called convex cocompact if $\hat{M}_\Gamma = (\mathcal{U} \cup \Omega(\Gamma))/\Gamma$ is a compact manifold with boundary. In this case, $\Omega(\Gamma)/\Gamma$ is a compact smooth OC manifold.

**Proposition 5.1** (cf. [10, p. 528]) Suppose that $\Gamma$ is a convex cocompact group of $F_{4(-20)}$. Then,

1. The radial limit set coincides with the limit set.
2. Any small deformation of the inclusion $i : \Gamma ' \to F_{4(-20)}$ maps $\Gamma '$ isomorphically to a convex cocompact group.

### 5.2 The Patterson–Sullivan measure

**Theorem 5.1** (cf. [10, p. 532]) For any convex cocompact subgroup $\Gamma$ of $F_{4(-20)}$, there exists a probability measure $\tilde{\mu}_\Gamma$ supported on $\Lambda(\Gamma)$ such that

$$\gamma^* \tilde{\mu}_\Gamma = |\gamma'|^{\delta(\Gamma)} \tilde{\mu}_\Gamma$$

for any $\gamma \in \Gamma$, where $|\gamma'|$ is a conformal factor.

See [13, 42] for Patterson–Sullivan measure for the complex case and [40] in the quaternionic case. We need to know the explicit conformal factor $|\gamma'|$ for our purpose later. Fix a reference point $v = (0, -1) \in \mathcal{U}$, where $\tilde{v} = (-1, 0, 1)^t$. Let us recall the definition of Patterson–Sullivan measure in [37]. Define a family of measures as

$$\tilde{\mu}_{s, z} := \frac{\sum_{\gamma \in \Gamma} e^{-\frac{1}{2} s \cdot d(z, \gamma(w))} \delta_{\gamma(w)}}{\sum_{\gamma \in \Gamma} e^{-\frac{1}{2} s \cdot d(v, \gamma(w))}}$$

for some fixed $w \in \mathcal{U}$, where $d(\cdot, \cdot)$ is the hyperbolic distance defined in (2.11), $\delta_{\gamma(w)}$ is the Dirac measure supported at point $\gamma(w)$ and $v = (0, -1)$. For each $s > \delta(\Gamma)$, this is a finite positive measure concentrated on $\Gamma w \subset \Gamma \mathcal{U}$. Since the set of all probability measures on $\Gamma \mathcal{U}$ is compact (cf. [10, p. 532]), there is a sequence $s_j$ approaching $\delta(\Gamma)$ from above such that $\tilde{\mu}_{s_j, z}$ approaches a limit $\tilde{\mu}_{s, z}$. After rewriting the coefficients, we may assume that the denominator in the definition of $\tilde{\mu}_{s, z}$ diverges at $s = \delta(\Gamma)$. Thus, we replace the above expression by

$$\tilde{\mu}_{s, z} = \frac{\sum_{\gamma \in \Gamma} a_{\gamma} e^{-\frac{1}{2} s \cdot d(z, \gamma(w))} \delta_{\gamma(w)}}{L(s, v)}$$

where $L(s, v) = \sum_{\gamma \in \Gamma} a_{\gamma} e^{-\frac{1}{2} s \cdot d(v, \gamma(w))}$, with the $a_{\gamma}$ chosen so that the denominator converges for $s > \delta(\Gamma)$ and diverges for $s \leq \delta(\Gamma)$. The definition of the measure $\tilde{\mu}_s$ does not depend on $w \in \mathcal{U}$ and the choice of $a_{\gamma}$ (cf. [10, p. 532]). The Patterson–Sullivan measure is the weak limit of these measures:

$$\tilde{\mu}_{\Gamma, z} = \lim_{s_j \to \delta(\Gamma)^+} \tilde{\mu}_{s_j, z}.$$ 

For any $\gamma \in F_{4(-20)}$ and any $f \in C(\mathcal{U})$, we have
\[(\gamma^* \bar{\mu}_{s, z})(f) = \frac{\sum_{\gamma \in \Gamma} a_{\gamma} e^{-\frac{\gamma^*}{2} d(z, \gamma(w))} \gamma^* \delta_{\gamma(w)}(f)}{L(s_i, v)} = \frac{\sum_{\gamma \in \Gamma} a_{\gamma} e^{-\frac{\gamma^*}{2} d(r^{-1}(z), r^{-1}(\gamma(w)))} f(r^{-1}(\gamma(w)))}{L(s_i, v)} = \frac{\sum_{\gamma \in \Gamma} a_{\gamma} e^{-\frac{\gamma^*}{2} d(r^{-1}(z), \gamma(w))} f(\gamma(w))}{L(s_i, v)} = \tilde{\mu}_{\gamma^{-1}(z)}(f)\]

by the invariance of the octonionic hyperbolic distance \(d(\cdot, \cdot)\) under the action of \(F_4(-20)\).

It is easy to see that \(\{a_{\gamma}\}\) is also such sequence satisfying the definition for fixed \(\gamma\). Let \(s_i \to \delta^+\). We get

\[\gamma^* \bar{\mu}_{s, z} = \bar{\mu}_{\gamma^{-1}(z)}\]

The **Buseman function** is defined by

\[b_\xi(x) = \lim_{t \to \infty} (d(x, \sigma(t)) - t), \quad (5.2)\]

where \(\sigma : [0, \infty) \to H^2_0\) is a geodesic ray asymptotic to \(\xi\). Note that

\[(\gamma(z)_1, \gamma(z)_2, \gamma(z)_3)' \sim (\gamma(\overline{z}_1), \gamma(\overline{z}_2), \gamma(\overline{z}_3)', 1)'\]

Recall that we have the following the Radon–Nikodym relation (cf. [47, p. 77, 81]):

\[\frac{d\tilde{\mu}_{\gamma^{-1}(z)}}{d\bar{\mu}_{\gamma, z}} |\xi| = e^{\frac{\delta}{2} (b_\xi(\gamma(z)) - b_\xi(\gamma^{-1}(z)))}\]

and

\[b_\xi(z) - b_\xi(w) = \lim_{t \to \infty} (d(z, \sigma(t)) - d(w, \sigma(t))) = 2 \lim_{t \to \infty} (\cosh^{-1}(z, \sigma(t)) - \cosh^{-1}(w, \sigma(t))) = 2 \lim_{t \to \infty} (\cosh(z, \sigma(t)) + \sqrt{(z, \sigma(t))^2 - 1}) = 2 \lim_{t \to \infty} (\cosh(w, \sigma(t)) + \sqrt{(w, \sigma(t))^2 - 1}) = 2 \lim_{t \to \infty} \left| \frac{\overline{D_1 \xi}}{|\overline{D_1 \xi}|} \right|^2\]

for \(\overline{D_1 \xi} \in \partial U\). The second last identity holds by \(\cosh^{-1} s = \ln (s + \sqrt{s^2 - 1})\) for \(s > 0\) and \((z, \sigma(t)), (w, \sigma(t)) \to \infty\).

For \(z = (0, -1), \eta = (\eta_1, \eta_2) \in U\), denote

\[\varphi(\eta) := |\overline{z^* D_1 \eta}|^2, \quad \chi(\eta) := \varphi(\eta) \overline{\delta(\eta)}\]

where \(\overline{z} = (-1, 0, 1)\), \(\overline{\eta} = (\eta_2, \eta_1, 1)\) and \(D_1\) is given by (2.4). Then, we have

\[\varphi(\gamma(\eta)) = \overline{\zeta^* D_1 \gamma(\eta)} |^2\]

Then, we have
\[
\frac{d\mu_{\Gamma,\gamma^{-1}(\xi)}}{d\mu_{\Gamma,z}} = \lim_{\eta \to \xi} \left( \frac{|\bar{z}^* D_1 \bar{\eta}|}{|\gamma^{-1}(\xi)|^{\frac{1}{2}}} \right) \left( \frac{|\gamma^{-1}(\xi)|^{\frac{1}{2}} D_1 \bar{\eta}|}{|\bar{z}^* D_1 \bar{\eta}|} \right) = \lim_{\eta \to \xi} \left( \frac{(z, \eta)}{|(\gamma^{-1}(\xi), \eta)|} \right) \left( \frac{\delta(\Gamma)}{\delta(\Gamma)} \right) = \lim_{\eta \to \xi} \left( \frac{|\bar{z}^* D_1 \eta|}{|\bar{z}^* D_1 \eta|} \right) \left( \frac{|\gamma^{-1}(\xi)|^{\frac{1}{2}} D_1 \bar{\eta}|}{|\gamma^{-1}(\xi)|^{\frac{1}{2}} D_1 \bar{\eta}|} \right) = 1, \tag{5.5}
\]

where \( \eta \in \mathcal{U} \) and \(|(\cdot, \cdot)| = \cosh \left( \frac{1}{2} d(\cdot, \cdot) \right) \) is invariant under \( F_{4(-20)} \). The last identity holds since

\[
\frac{|\gamma(\eta)|^* D_1 \gamma(\eta)}{|\bar{\eta}^* D_1 \bar{\eta}|} = \frac{2 \text{Re} \left[ \gamma(\bar{\eta}_1) \gamma(\bar{\eta}_3) \right] + |\gamma(\bar{\eta}_2)|^2}{|\gamma(\bar{\eta}_3)|^2 \left( 2 \text{Re} \eta_2 + |\eta_1|^2 \right)} = \frac{1}{|\gamma(\bar{\eta}_3)|^2}, \tag{5.6}
\]

by

\[
2 \text{Re} \left[ \gamma(\bar{\eta}_1) \gamma(\bar{\eta}_3) \right] + |\gamma(\bar{\eta}_2)|^2 = 2 \text{Re} \eta_2 + |\eta_1|^2,
\]

which can be checked directly for all dilation \( D_\delta \), left translations \( \tau_{(\cdot, \cdot)} \), rotation \( S_\mu \) and inversion \( R \) in (2.20)–(2.21). So, if we define \( \mu_{\Gamma} := \mu_{\Gamma,z} = \chi \mu_{\Gamma,z} \) with \( \chi \) given by (5.3), we have

\[
\gamma^* d\mu_{\Gamma}(\xi) = \frac{1}{|\gamma(\bar{\xi}_3)|^{\delta(\Gamma)}} d\mu_{\Gamma}(\xi). \tag{5.7}
\]

Note that \( \frac{1}{|\gamma(\bar{\xi}_3)|} \) coincides with \( \phi \) given in (2.5), i.e., \( \gamma^* g_0 = \phi^2 g_0 \) with \( \phi = \frac{1}{|\gamma(\bar{\xi}_3)|} \). For example, let \( \gamma \) be the inversion given in (2.20), then we have

\[
\gamma(\bar{\xi}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \xi_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\xi_1 \\ \xi_2 \end{pmatrix},
\]

i.e., \(|\gamma(\bar{\xi}_3)|^2 = |\xi_2|^2 = |x|^2 - t^2 = |x|^4 + |t|^2 \) by (2.19).

**Remark 5.1** In the CR and qc cases [40, 42], we use the ball model and choose \( z \) to be the origin. Then, \( \mu_{\Gamma,z} \) automatically has conformal factor as the action of the group on the sphere. But in the OC case, we use the flat model on which we have an extra conformal factor \( \chi \) given by (5.3). But the flat model has many advantages.
5.3 An invariant OC metric of Nayatani type

When the OC manifold is $\Omega(\Gamma)/\Gamma$ for some convex cocompact subgroup $\Gamma$ of $F_4(-20)$, we can construct an invariant OC metric $g_\Gamma$, which is the OC generalization of Nayatani’s canonical metric in conformal geometry [32]. See [42] and [40] for CR case and qc case, respectively.

Recall $g_\Gamma = \phi_\Gamma^{G_0} g_0$ defined in (1.10). Since

$$G_0(\gamma(\xi), \gamma(\zeta)) = |\gamma(\tilde{\xi})_3|^{\frac{G_0-2}{2}} |\gamma(\tilde{\zeta})_3|^{\frac{G_0-2}{2}} G_0(\xi, \zeta)$$

by the conformal factor (5.7), Proposition 2.5 and the transformation formula (4.5) of the Green functions, we have

$$\phi_\Gamma(\gamma(\tilde{\xi})) = \left( \int_{\Lambda(\Gamma)} G_0^{G_0-2}(\gamma(\tilde{\xi}), \zeta) d\mu_\Gamma(\zeta) \right)^{\frac{G_0-2}{2}} = \left( \int_{\Lambda(\Gamma)} G_0^{G_0-2}(\gamma(\tilde{\xi}), \gamma(\xi)) d\gamma^* \mu_\Gamma(\xi) \right)^{\frac{G_0-2}{2}}$$

$$= \left( \int_{\Lambda(\Gamma)} |\gamma(\tilde{\xi})_3|^{\frac{G_0-2}{2}} (G_0^{G_0-2}(\xi, \zeta) d\mu_\Gamma(\zeta)) \right)^{\frac{G_0-2}{2}} = |\gamma(\tilde{\xi})_3|^{\frac{G_0-2}{2}} \phi_\Gamma(\xi).$$

(5.8)

Therefore, (5.8) together with Proposition 2.5 and (5.7) implies that $\gamma^* g_\Gamma = g_\Gamma$. So, it induces an OC metric on the compact OC manifold $\Omega(\Gamma)/\Gamma$.

The proof of Theorem 1.3 is similar to the CR case [42, p. 265] and qc case [40, p. 302]; we omit details.

References

1. Allcock, D.: Reflection groups on the octave hyperbolic plane. J. Algebra. 213, 467–98 (1999)
2. Baez, J.: The octonions. Bull. Am. Math. Soc. 39(2), 145–205 (2002)
3. Barilari, D., Ivanov, S.: A Bonnet-Myers type theorem for quaternionic contact structures. Calc. Var. Partial Differ. Equ. 58(1), 161–215 (2019)
4. Bonfiglioli, A., Uguzzoni, F.: Nonlinear Liouville theorems for some critical problems on H-type groups. J. Funct. Anal. 207(1), 161–215 (2004)
5. Biquard, O.: Métriques d’Einstein asymptotiquement symétriques. Astérisque 265 (2000)
6. Cheng, J.-H., Chiu, H.-L., Yang, P.: Uniformization of spherical CR manifolds. Adv. Math. 255, 182–216 (2014)
7. Cheng, J.-H., Chiu, H.-L.: Connected sum of spherical CR manifolds with positive CR Yamabe constant. J. Geom Anal. 29, 3113–3123 (2019)
8. Cheng, J.-H., Chiu, H.-L., Ho, P.-T.: Connected sum of CR manifolds with positive CR Yamabe constant. J. Geom Anal. 31, 298–311 (2021)
9. Cowling, M., Ottazzi, A.: Conformal maps of Carnot groups. Ann. Acad. Sci. Fenn. Math. 40(1), 203–213 (2015)
10. Corlette, K.: Hausdorff dimension of limit sets I. Invent. Math. 102, 521–541 (1990)
11. Dietrich, G.: Contact structures, CR Yamabe invariant, and connected sum. Trans. Am. Math. Soc. 374(2), 881–897 (2021)
12. Eberlein, P.O.: Visibility manifold. Pac. J. Math. 46, 45–109 (1973)
13. Epstein, C., Melrose, R., Mendoza, G.: Resolvent of the Laplacian on strictly pseudoconvex domains. Acta. Math. 167, 1–106 (1991)
14. Garofalo, N., Vassilev, D.: Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type. Duke Math. J. 106, 411–448 (2001)
15. Habermann, L., Jost, J.: Green functions and conformal geometry. J. Differ. Geom. 53, 405–442 (1999)
16. Habermann, L.: Riemannian Metrics of Constant Mass and Moduli Spaces of Conformal Structures, Lecture Notes in Mathematics, vol. 1743. Springer, Berlin (2000)
17. Harvey, F.: Spinors and calibrations. Perspect. Math. 9, 173 (1990)
18. Ivanov, S., Minchev, I., Vassilev, D.: Extremals for the Sobolev inequality on the seven-dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem. J. Eur. Math. Soc. 12(4), 1041–1067 (2010)
19. Ivanov, S., Vassilev, D.: Extremals for the Sobolev Inequality and the quaternionic contact Yamabe Problem. London, World Scientific (2011)
20. Ivanov, S., Minchev, I., Vassilev, D.: Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem. Mem. Am. Math. Soc. 231, 1086 (2014)
21. Ivanov, S., Minchev, I., Vassilev, D.: Quaternionic contact hypersurfaces in hyper-Kähler manifolds. Ann. Mater. Pura Appl. 196, 245–267 (2017)
22. Ivanov, S., Petkov, A.: The qc Yamabe problem on non-spherical quaternionic contact manifolds. J. Math. Pures Appl. 118(9), 44–81 (2018)
23. Izeki, H.: Limits sets of Kleinian groups and conformally flat Riemannian manifolds. Invent. Math. 122, 603–625 (1995)
24. Izeki, H.: The Teichmüller distance on the space of flat conformal structures. Conform Geom. Dyn. 2, 1–24 (1998)
25. Jerison, D., Lee, J.M.: The Yamabe problem on CR manifolds. J. Differ. Geom. 25, 167–197 (1987)
26. Kaplan, A.: Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. Trans. Am. Math. Soc. 258, 147–153 (1980)
27. Kobayashi, O.: A Riemannian metric invariant under Möbius transformations in \( \mathbb{R}^n \), Lecture Notes in Mathematics, vol. 1351, pp. 223–235. Springer, Berlin (1988)
28. Li, Z.: Uniformization of spherical CR manifolds and the CR Yamabe problem. Proc. Symp. Pure Math. 54, 299–305 (1993)
29. Mostow, G.D.: Strong rigidity of locally symmetric spaces, (No. 78). Princeton University Press, Princeton (1973)
30. Nayatani, S.: Patterson-Sullivan measure and conformally flat metrics. Math. Z. 225, 115–131 (1997)
31. Nayatani, S.: Discrete groups of complex hyperbolic isometries and pseudo-Hermitian structures. Anal. Geom. Several Complex Var. 3, 209–237 (1997)
32. Orsted, B.: Conformally invariant differential equations and projective geometry. J. Funct. Anal. 44, 1–23 (1981)
33. Pansu, P.: Métriques de Carnot-Carathéodory et quasisométries des espaces symétriques de rang un. Ann. Math. 129, 1–60 (1989)
34. Parker, John R.: Hyperbolic spaces, Jyväskylä lectures in Mathematics, (2008)
35. Patterson, S.J.: The limit set of a Fuchsian group. Acta. Math. 136, 241–273 (1976)
36. Platis, I.D.: Cross-ratios and the Ptolemaean inequality in boundaries of symmetric spaces of rank 1. Geom. Dedicata. 169(1), 187–208 (2014)
37. Shio, Y., Wang, W.: On conformal qc geometry, spherical qc manifolds and convex cocompact subgroups of mathrm Sp(n+1,1). Ann. Global Anal. Geom. 49(3), 271–307 (2016)
38. Wang, W.-F.: On octonionic regular functions and the Szegö projection on the octonionic Heisenberg group. Complex Anal. Oper. Theory 8(6), 1285–1324 (2014)
39. Wang, W.: Canonical contact forms on spherical CR manifolds. J. Eur. Math. Soc. 5, 245–273 (2003)
40. Wang, W.: Representations of SU(p, q) and CR geometry I. J. Math. Kyoto Univ. 45, 759–780 (2005)
41. Wang, W.: The Yamabe problem on quaternionic contact manifolds. Ann. Mat. Pura Appl. 186, 359–380 (2007)
42. Wu, F.-F.: On the Yamabe problem on contact Riemannian manifolds. Ann. Global Anal. Geom. 46, 456–506 (2019)
43. Yamaguchi, K.: Differential systems associated with simple graded Lie algebras. Adv. Stud. Pure Math. 22, 413–494 (1993)
44. Yue, C.-B.: Mostow rigidity of rank 1 discrete groups with ergodic Bowen-Margulis measure. Invent. Math. 125, 75–102 (1996)