A NOTE ON THE HODGE CONJECTURE

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Abstract. This paper presents an approach that may lead to finding a counterexample to the Hodge conjecture. As an example of what kind of a class the counterexample could be the paper presents a (2,2)-form on a product of two K3 surfaces. It represents a nonzero cohomology class and does not arise as a linear combination with rational coefficients of cohomology classes of algebraic subvarieties. However, it is not a Hodge class, and does not give a counterexample. The approach in a modified form may work on some other space.

1. Introduction

The Hodge conjecture is one of the better known open problems in mathematics and was chosen as one of the Millennium Prize problems by the Clay Mathematics Institute [1]. It states that every Hodge class on a projective complex manifold \( X \) is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of \( X \). There are some good introductions to the topic available through the Internet, like [2] and [3]. Counterexamples and proofs of the Hodge conjecture in special cases are known, like [4]. There have been some arxiv papers trying to prove the Hodge conjecture false [5], or true [6] but these attempts have turned out to have errors.

This manuscript proposes an approach for finding a counterexample to the Hodge conjecture. As an example we show that there exists a well-known (2,2)-form on the product of two K3 surfaces that has many of the properties that a counterexample to the Hodge conjecture should have, but it turns out that the class is not rational and thus not Hodge. It may be possible to modify the argument to a better result.

The idea is simple. It is known that if \( Z \) is a complex submanifold of \( X \) of codimension \( k \) then the inclusion map \( i : Z \to X \) induces a cohomology class \( [Z] \) in \( H^{k,k}(X) \). In local coordinates the cohomology class \( [Z] \) is then a form of the type

\[
\psi = f(z, \bar{z}) dz^I \wedge d\bar{z}^J \tag{1.1}
\]

where \((z, \bar{z}) = (z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)\) are the local coordinates. As \( \psi \in H^{k,k}(X) \) there are the same number \( k \) of indices in the multi-indices \( I \) and \( J \). A natural question is how the cohomology class knows which are \( z^j \) and \( \bar{z}^j \) coordinates e.g. if \( Z \) is embedded into a product space? The simple reason why there are equally many coordinates of the type \( dz^j \) and \( d\bar{z}^j \) in (1.1) is that these differentials always appear in pairs: if there is \( dz^j \) there is also \( d\bar{z}^j \). It is a consequence of the way the tangent space of the submanifold \( Z \) (in this case a subvariety is also a submanifold) is mapped in the tangent space of \( X \) at a given point. Thus, we cannot get a form

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of the type where the index sets $I$ and $J$ are different, e.g. of the type

$$\psi = f(z, \bar{z})dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4$$

(1.2)

from a cohomology class of a subvariety. It is possible to show that we also cannot get this kind of a form from a $\mathbb{Q}$-linear combination. Then we only need to find this kind of a harmonic form. Such a form can be created by taking the nowhere vanishing $(2,0)$-form of a K3 surface. The complex conjugate of this form is a $(0,2)$-form and the wedge product of these forms can be shown to be a harmonic $(2,2)$-form in the product space $K3 \times K3$. The paper realizes this idea. Section 2 introduces the concepts and has the main goal of calculating $\ast \psi$ in local coordinates. Section 3 has the construction of the $(2,2)$-form. Lemmas 3.1 and 3.2 show that for a particular type of a form $\psi$ we get $\ast \psi$ which is quite similar to $\psi$ in form. Lemma 3.3 shows that the form is harmonic. Actually, we created the form so that both $d\psi = 0$ and $d(\ast \psi) = 0$ for trivial reasons. In Lemma 3.4 we prove that a form of the type (1.2) cannot come as a $\mathbb{Q}$-linear combination from the cohomology classes of submanifolds. In Lemma 3.5 we show that such a form exists on $X = K3 \times K3$, which is a projective complex manifold of complex dimension 4. We also show that the form is not zero in $H^{2,2}(X)$. This is where we need the surface to be K3 and e.g. an Abelian surface would not work. Complex conjugation sends a harmonic form to a harmonic form, and thus we have a wedge of two harmonic forms, but the wedge of two harmonic forms is not necessarily harmonic - it can be exact. We need the nowhere vanishing $(2,0)$-form in order to conclude that the wedge product is harmonic. However, this form is not rational and thus does not represent a Hodge class.

2. Notations and concepts

We will use the notations in Kodaira [7] page 147. Local coordinates of a complex manifold $M$ of complex dimension $n$ at the base point $z_0$ are denoted by $z^1, \ldots, z^n$, $\bar{f}(z^1, \ldots, z^n)$ is the complex conjugate of $f(z^1, \ldots, z^n)$, and the Hodge star operation is denoted by $\ast$.

Let $\varphi$ and $\psi$ be $C^\infty(p,q)$-forms in a complex manifold $M$.

$$\varphi = \frac{1}{plq!} \sum \varphi_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q}(z)dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}$$

$$\psi = \frac{1}{plq!} \sum \psi_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q}(z)dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}$$

(2.1)

The inner product is defined as

$$(\varphi, \psi)(z) = \frac{1}{plq!} \sum \varphi_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q}(z)\bar{\psi}_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q}(z)$$

(2.2)

and

$$(\varphi, \psi) = \int_M (\varphi, \psi)(z)\frac{\omega^n}{n!}$$

(2.3)

where the volume element $\frac{\omega^n}{n!}$ is

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2} g(z)dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$$

(2.4)

Here

$$g(z) = \det(g_{\alpha\beta}(z))_{\alpha, \beta=1,\ldots,n}$$

(2.5)
Lemma 3.1. Let $p = q = 2$, $n = 4$, and
\[
\psi_{13\bar{3}4}(z) = 4f(z^1, z^2, \bar{z}^3, \bar{z}^4)
\]
\[
\psi_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2}(z) = 0 \quad \text{if} \quad (\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2) \neq (12\bar{3}\bar{4})
\]

and let $f(z^1, z^2, \bar{z}^3, \bar{z}^4)$ be a holomorphic function. Then in local coordinates
\[
\psi = f(z^1, z^2, \bar{z}^3, \bar{z}^4)dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4
\]
\[
\psi^\ast = g(z)f(\bar{z}^1, \bar{z}^2, z^3, z^4)dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2
\]
Proof. The first claim is obvious since
\[ \psi = \frac{1}{4} \psi_{1234}(z) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \] (3.3)

For the second claim we calculate
\[ *\psi = (i)^4(-1)\frac{4}{4!}3+2 \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} g(z)\psi_{1234}(z) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \] (3.4)

Since
\[ \psi^{\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2}(z) = \sum_{\lambda_1, \lambda_2, \mu_1, \mu_2} g^{\lambda_1 \lambda_2} g^{\alpha_1 \alpha_2} g^{\bar{\beta}_1 \mu_1} g^{\bar{\beta}_2 \mu_2} \bar{\psi}_{\lambda_1 \lambda_2 \mu_1 \mu_2}(z) \]
\[ = \sum_{\lambda_1, \lambda_2, \mu_1, \mu_2} \delta_{\lambda_1 \lambda_1} \delta_{\lambda_2 \alpha_2} \delta_{\bar{\beta}_1 \mu_1} \delta_{\bar{\beta}_2 \mu_2} \bar{\psi}_{\lambda_1 \lambda_2 \mu_1 \mu_2}(z) \] (3.5)
\[ = \psi^{\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2}(z) = \begin{cases} \bar{\psi}_{1234}(z) & \text{if } (\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2) = (1234) \\ 0 & \text{otherwise} \end{cases} \]

Thus
\[ *\psi = g(z)\bar{\psi}_{1234}(z) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \]
\[ = g(z)\bar{f}(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \] (3.6)
and since \( f(z^1, z^2, \bar{z}^3, \bar{z}^4) \) is holomorphic \( \bar{f}(z^1, z^2, \bar{z}^3, \bar{z}^4) = f(z^1, \bar{z}^2, z^3, \bar{z}^4). \)

Lemma 3.2. Let \( p = q = 2, n = 4, \) and
\[ \psi_{1234}(z) = 4f(z^1, z^2, \bar{z}^3, \bar{z}^4) \]
\[ \psi_{\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2}(z) = 0 \quad \text{if} \quad (\alpha_1 \alpha_2 \bar{\beta}_1 \bar{\beta}_2) \neq (1234) \] (3.7)
and let \( f(z^1, z^2, \bar{z}^3, \bar{z}^4) \) be a holomorphic function. We can select the metric such that in local coordinates
\[ \psi = f(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \]
\[ *\psi = f(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \] (3.8)

Proof. At the base point \( z_0 \) we can select the metric \( g(z_0) = \det(g_{\alpha \beta})_{\alpha, \beta = 1, \ldots, n} \) such that \( g_{\alpha \beta} = \delta_{\alpha \beta}. \) Then \( g(z_0) = \det(g_{\alpha \beta})_{\alpha, \beta = 1, \ldots, n} = 1. \) We can make the same selection at all points \( z \) and thus \( g(z) = 1. \)

Lemma 3.3. Let \( \psi \) be a \((2,2)\)-form in a complex manifold \( M. \) Let \( \psi \) be expressed in local coordinates as
\[ \psi = f(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \] (3.9)
where \( f(z^1, z^2, \bar{z}^3, \bar{z}^4) \) is holomorphic. Let us assume that the complex dimension of \( M \) is four, \( M \) is compact and without boundary. Then \( \Delta \psi = 0. \)

Proof. For compact manifolds without boundary
\[ \Delta \psi = 0 \quad \iff \quad d\psi = 0 \quad \text{and} \quad \delta \psi = 0 \] (3.10)

Let us calculate \( d\psi \)
\[ d\psi = \sum_{j=1}^{4} \frac{\partial f}{\partial z^j}(z^1, z^2, \bar{z}^3, \bar{z}^4) dz^j \wedge dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 = 0 \] (3.11)
The coordinate systems at \( p \) and the complex structure of \( M \) can be completed to a coordinate system of \( \mathbb{T}_N \) submanifold of \( \mathbb{A} \) there is an orthonormal coordinate transform \( \text{Proof.} \)

\[
\frac{\partial f}{\partial z^j}(z^1, z^2, z^3, \bar{z}^4) = 0
\]

(3.12) since in the coordinate system \((z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)\) the coordinates \( z^j \) and \( \bar{z}^j \) are considered independent. The second assertion follows from the definition of the codifferential: if \( *: \Omega^k \to \Omega^{n-k} \) is the Hodge star operator then \( \delta: \Omega^k \to \Omega^{k-1} \) is defined by

\[
\delta \psi = (-1)^{n(k+1)+1}(\ast d\ast)\psi
\]

(3.13) Inserting \( n = 4, k = 2 \), yields

\[
\delta \psi = -(\ast d\ast)\psi = -d(\ast \psi)
\]

(3.14) We may assume that the metric is chosen such that \( g(z) = 1 \). By Lemma 3.2

\[
\ast \psi = f(z^1, z^2, z^3, z^4)dz^3 \wedge dz^4 \wedge dz^1 \wedge d\bar{z}^2
\]

(3.15) As in the previous case, we conclude that

\[
d(\ast \psi) = \sum_{j=1}^{4} \frac{\partial f}{\partial z^j}(z^1, z^2, z^3, \bar{z}^4)dz^j \wedge dz^3 \wedge dz^4 \wedge dz^1 \wedge d\bar{z}^2 = 0
\]

(3.16) \( \square \)

**Lemma 3.4.** A \((2,2)\)-form of the type \( f(z, \bar{z})dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 \) is not a linear combination of cohomology classes deriving from complex submanifolds of complex codimension 2 in a complex submanifold of complex dimension 4.

**Proof.** Let \( N \) be a complex manifold of dimension \( n \) and let \( M \) be a complex submanifold of \( N \) of complex dimension \( k \). Let \( i : M \to N \) be the inclusion map and the complex structure of \( M \) be induced by the complex structure of \( N \). Let \( p \in M \) be a point in \( M \) and \((x^1, y^n), \ldots, (x^k, y^n)\) be a local coordinate system in \( TM_p \) where \( M \) is considered as a real manifold of dimension \( 2k \). The coordinate system can be completed to a coordinate system of \( TN_{i(p)} = TN_p \) by adding \( 2(n-k) \) coordinate vectors. This yields a coordinate system

\[
(x', y') = (x^1, y^n, \ldots, x^k, y^1, x^k+1, y^k+1, \ldots, x^n, y^n)
\]

to \( TN_p \). We can assume that the coordinates are orthonormal. The manifold \( N \) can be considered as a real manifold of dimension \( 2n \) and \( TN_p \) be given a local coordinate system

\[
(x, y) = (x^1, y^1, \ldots, x^n, y^n)
\]

\( p = i(p) = (0,0) \)

There is an orthonormal coordinate transform \( A : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \), \( A \in SO(2n) \), such that

\[
[x' y']^T = A[x y]^T
\]

(3.17) The coordinate systems at \( TN_p \) can be chosen such that

\[
z^j = x^j + iy^j, \quad \bar{z}^j = x^j - iy^j \quad j = 1, \ldots, n
\]

\[
z'^j = x'^j + iy'^j, \quad \bar{z}'^j = x'^j - iy'^j \quad j = 1, \ldots, n
\]

(3.18)
and there is an orthonormal coordinate transform $B: \mathbb{C}^n \to \mathbb{C}^n$ such that for
\[
(z, \bar{z}) = (z^1, \bar{z}^1, \ldots, z^n, \bar{z}^n) \\
(z, \bar{z}') = (z'^1, \bar{z}'^1, \ldots, z'^m, \bar{z}'^m)
\]
the transform takes $(z, \bar{z})$ to $(z', \bar{z}')$, $[z' \bar{z}']^T = B[z \bar{z}]^T$. As the complex structure of $M$ is induced from the complex structure of $N$, the following statement holds
\[
\text{if } (z', \bar{z}') = (x'^1, x'^n, \ldots, x'^m, x'^n) \text{ i.e. } y'^j = 0 \text{ for } j = 1, \ldots, n \text{ then } (z, \bar{z}) = (x^1, x^1, \ldots, x^n, x^n) \text{ i.e. } y^j = 0 \text{ for } j = 1, \ldots, n
\]
It follows that if
\[
z'^j = \sum_m a_{m,j} z^m
\]
then
\[
\bar{z}'^j = \sum_m a_{m,j} \bar{z}^m
\]
that is, $a_{m,j} \in \mathbb{R}$ for all $j$ and $m$. The Poincaré dual $[M]$ of $M$ considered as a $2k$-dimensional real manifold in the $2n$-dimensional real manifold $N$ satisfies the condition that $[M]$ capped with the fundamental class of $N$ is the homology class of $M$. Thus, $[M]$ is a form of the type
\[
\varphi(x', y') dx'^{k+1} \wedge dy'^{k+1} \wedge \cdots \wedge dx^n \wedge dy^n
\]
Since $x'^j$ and $y'^j$ are independent, the class $[M]$ is represented at $p$ by a form of the type
\[
\varphi(z', \bar{z}') dz'^{k+1} \wedge d\bar{z}'^{k+1} \wedge \cdots \wedge dz^n \wedge d\bar{z}^n
\]
In coordinates $\{z^1, \bar{z}^1, \ldots, z^n, \bar{z}^n\}$ the form can be expressed by inserting $dz'^j$ and $d\bar{z}'^j$ as linear combinations (with real coefficients) of $dz^k$ and $d\bar{z}^k$, $k = 1, \ldots, n$. Let us consider a (finite) linear combination (with rational coefficients) at the point $p \in N$ of forms that represent cohomology classes that are dual to complex sub-manifolds $M_m \subset N$ for some index set $m \in A \subset \mathbb{N}$, where each $M_m$ has complex codimension $2k$. Each form is expressed in local coordinates
\[
(z'^1, \ldots, z'^m, \bar{z}'^1, \ldots, \bar{z}'^m)
\]
of $TN_p$ as
\[
\varphi_m = \varphi_m(z'^1, \ldots, z'^m, \bar{z}'^1, \ldots, \bar{z}'^m) dz'^{k+1, m} \wedge d\bar{z}'^{k+1, m} \wedge \cdots \wedge dz'^{m, m} \wedge d\bar{z}'^{m, m}
\]
The forms $\varphi_m$ can be expressed in the coordinates $(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$ of $TN_p$ by inserting $dz'^j,m$ and $d\bar{z}'^j,m$ as linear combinations (with real coefficients) of $dz^k$ and $d\bar{z}^k$, $k = 1, \ldots, n$. Let $v = \sum_j c_j x^j$ be a real vector, $c_j \in \mathbb{R}$. It can be expressed as $v = \sum_j c_{j,m} x^{j,m}$ where $c_{j,m} \in \mathbb{R}$. For each $M_m$ the projections
\[
Pr_{+v}([M_m]) = \sum_{j=k+1}^n \varphi_m(z'^1, \ldots, z'^m, \bar{z}'^1, \ldots, \bar{z}'^m) Pr_v(dz'^j,m)
\]
\[
Pr_{-v}([M_m]) = \sum_{j=k+1}^n \varphi_m(z'^1, \ldots, z'^m, \bar{z}'^1, \ldots, \bar{z}'^m) Pr_v(d\bar{z}'^j,m)
\]
satisfy $Pr_{+,v}([M_m]) - Pr_{-,v}([M_m]) = 0$ where $Pr_v(u)$ is the projection of $u$ on the direction of the vector $v$. We can generalize the functions $Pr_{+,v}$ and $Pr_{-,v}$ to act on any form

$$\psi_m = \sum_j \psi_{j,m}(z^{1,m}, \ldots, z^{n,m}, \bar{z}^{1,m}, \ldots, \bar{z}^{m,m}) dz^{j,1,m} \wedge d\bar{z}^{j,1,m}$$

(3.27)

where $I_{j,m}, J_{j,m} \subset \{1, m, \ldots, n, m\}$ are multi-indices, as

$$Pr_{+,v}(\psi_m) = \sum_j \sum_{s,m \in I_{j,m}} \psi_{j,m} Pr_v(dz^{s,m})$$

$$Pr_{-,v}(\psi_m) = \sum_j \sum_{s,m \in J_{j,m}} \psi_{j,m} Pr_v(d\bar{z}^{s,m})$$

(3.28)

These functions are linear, and if $\psi$ is a linear combination with rational coefficients of terms of the type (3.29) then

$$Pr_{+,v}(\psi) - Pr_{-,v}(\psi) = 0$$

(3.29)

This equation does not depend on the choice of the local coordinates as long as they are related by an orthogonal transform with real entries. Thus, the equation holds also when the forms are expressed in the coordinates $\{z^1, \ldots, z^n, \bar{z}^n\}$. If a form of the type

$$\alpha = f(z, \bar{z}) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4$$

(3.30)

is a linear combination with rational coefficients of classes $[M_m]$ then it also must satisfy the equation

$$Pr_{+,v}(\alpha) - Pr_{-,v}(\alpha) = 0$$

(3.31)

Clearly, it does not satisfy the equation directly. We must check if there is a form in the same cohomology class as $\alpha$ which satisfies the equation. Thus, let $\alpha'$ be of the form

$$\alpha' = f(z, \bar{z}) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4 + \beta$$

(3.32)

where $\beta = 0$ in $H^{2,2}(N)$. This means that $\beta$ is exact but the missing term that must be added to $\alpha$ so that $\alpha$ satisfies the equation (3.36) is

$$f(z, \bar{z}) dz^3 \wedge dz^4 \wedge d\bar{z}^1 \wedge d\bar{z}^1$$

(3.33)

which is not zero in $H^{2,2}(N)$ by Lemma 3.3. Thus, as $\beta$ is exact, $\alpha'$ does not satisfy the equation (3.36) and Lemma 3.5 holds.

\[\square\]

**Lemma 3.5.** Let $X$ be a product of two K3 surfaces. Then $X$ allows a $(2,2)$-form of the type

$$f(z^1, z^2) f(\bar{z}^3, \bar{z}^4) dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4$$

(3.34)

where $f(z^1, z^2)$ is holomorphic and nowhere vanishing.

**Proof.** The existence of a nowhere vanishing 2-form is often taken as the definition of a K3 surface, the additional condition guaranteeing that a 2-dimensional complex manifold $X$ is a K3 surface is that the manifold $X$ is connected. This nowhere vanishing 2-form $\lambda$ is the generator of $H^{2,0}(X)$, i.e., every other element $\alpha \in H^{2,0}(X)$ can be expressed as $\alpha = c\lambda$ for some $c \in \mathbb{C}$. The complex conjugate $\bar{\lambda}$ of $\lambda$ is the generator of $H^{0,2}(X)$ as is shown by the Hodge duality pairing. Let the $(2,0)$-form $\lambda$ be expressed in local coordinates as

$$\lambda = f(z^1, z^2) dz^1 \wedge dz^2$$

(3.35)
Then \( f(z^1, z^2) \) is holomorphic and nowhere vanishing. If \( M = K3 \times K3 \) then there are two nowhere vanishing 2-forms \( \lambda_1 \) and \( \lambda_2 \). We can make a \((2,2)\)-form as a wedge product of

\[
\lambda_1 = f(z^1, z^2)dz^1 \wedge dz^2
\]

and

\[
\lambda_2 = \tilde{f}(z^3, z^4)d\bar{z}^3 \wedge d\bar{z}^4 = f(\bar{z}^3, \bar{z}^4)d\bar{z}^3 \wedge d\bar{z}^4
\]

as \( f(z^3, z^4) \) is holomorphic. Thus, we define

\[
\psi = f(z^1, z^2)dz^1 \wedge dz^2 \wedge f(\bar{z}^3, \bar{z}^4)d\bar{z}^3 \wedge d\bar{z}^4 = f(z^1, z^2, \bar{z}^3, \bar{z}^4)dz^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge d\bar{z}^4
\]

The form \( \psi \) satisfies \( \Delta \psi = 0 \) by Lemma 3.3. We still have to show that it is not zero in \( H^{2,2}(X) \). Typical ways to show this are calculating periods or intersection products, but we will show it differently. For any complex manifold \( M \) the Hodge star gives an isomorphism from \( H^k(M) \) to \( H^{n-k}(M) \) and the Poincaré duality gives an isomorphism from \( H_k(M) \) to \( H^{n-k}(M) \). Especially, when \( n = 4 \), \( H^2(X) \) and \( H_2(X) \) are isomorphic and if \( X = S_1 \times S_2 \) \( H^2(S_1) \) and \( H_2(S_2) \), \( i = 1,2 \) are isomorphic (for any coefficients, so coefficients are supressed in the notations). Let this isomorphism be \( \Psi : H_2(X) \to H^2(X) \). Let \( C \) be a 2-chain in \( S_1 \) and let \( (C, p_t) \in X \) be a family of pairs where \( p_t \) is a path in \( S_2 \). This family \( C \times [0,1] \) defines a homotopy from \( (C, t_0) \) to \( (C, t_1) \). By the homomorphism we have also a family \( \lambda_1 \wedge \lambda_2(P_t) \) parametrized by \( t \in [0,1] \). Then \( P_t \) defines a path in \( S_2 \). This is where we need the nowhere vanishing \((2,0)\)-form. If there existed a point \( P_1 \) such that \( \lambda_2(P_1) = 0 \) then \( \psi = 0 \) at any point \( (Q, P_1) \in S_1 \times S_2 \). As the preimage of zero is a point, this would yield a homotopy from \( C \) to a point, i.e., \( (C, p_0) = 0 \) in \( \pi_2(X) \). Consequently \( (C, p_0) = 0 \) in \( H_2(X) \) and \( \Psi(C, p_0) = (\lambda_1, P_0) = 0 \) in \( H^2(X) \). As there is no such point \( P_1 \), \( (C, p_0) \) is not contractible in \( X \) and is therefore nonzero in \( H_2(X) \). Its image under \( \Psi \) is therefore nonzero in \( H^2(X) \). Thus, \( \psi \) is a harmonic form in \( H^{2,2}(X) \). 

We have found a nonzero form from \( H^{2,2}(M) \) that is not a linear combination of with rational coefficients of the cohomology classes of complex subvarieties of \( M \). However, we should still show that the cohomology class of this form is a Hodge class in order to present a counterexample to the Hodge conjecture. Unfortunately the form does not represent a Hodge class since we cannot show that it is rational. We remember that singular cohomology of a complex manifold of complex dimension \( n \) is defined as the singular cohomology of the underlying real manifold of dimension \( 2n \). This yields \( H^*(X; \mathbb{Z}) \). In order to get \( H^*(X; k) \), where the field \( k \) is \( \mathbb{Q} \) or \( \mathbb{R} \), or \( \mathbb{C} \), we form the tensor product of \( H^*(X; \mathbb{Z}) \) with \( k \). When the base classes of \( \mathbb{Z} \) are selected, the classes of the tensor products are linear combinations of the base classes with coefficients in \( k \). We can multiply the created \((2,2)\)-class with any number in \( k \) and get a harmonic class. If the class is a multiple of a base vector, we can always multiply it with a suitable number to get a class in \( H^4(X; \mathbb{Q}) \). However, if it is a linear combination with generic coefficients, multiplication by one number does not give a rational class. This seems to be the case with this \((2,2)\)-form. 

Theorem 2 in [8] on page 54, also published in [9], shows that the Hodge conjecture holds for certain K3xK3 spaces, showing that in the general case the constructed \((2,2)\)-form cannot be a Hodge class. The space of Hodge classes \( B(H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q})) \) is identified up to a Tate twist with \( \text{End}_{Hdg}(H^2(S, \mathbb{Q})) \) on
A NOTE ON THE HODGE CONJECTURE

Zarhin’s theorem is used to characterize $\text{End}_{Hdg}(T)$ on page 16, and Mukai’s theorem is used in the proof of Theorem 2, as in the proofs of other theorems in [8]. Mukai’s theorem requires that the endomorphism $\varphi : T(S_1) \to T(S_1)$ is a Hodge isometry. A Hodge isometry maps $H^{2,0}(X)$ to $H^{2,0}(X)$, as is clear e.g. from [10] page 211. This method in [8] seems to derive from a paper of D. Morrison [11]. The endomorphism in the presented paper is complex conjugation, which sends $H^{2,0}(X)$ to $H^{0,2}(X)$ and is thus not a Hodge isometry. Mukai’s theorem thus cannot be used, but the identification of the space of Hodge classes with endomorphisms preserving the Hodge structure still holds, and consequently the $(2,2)$-form created in Lemma 3.5 does not represent a Hodge class.

I thank Bert van Geemen for clarifying the theorem is used in [8] and pointing out that I have not shown that the created $(2,2)$-class is rational.

As a conclusion, the presented paper does not give a counterexample to the Hodge conjecture. This is not surprising since the method always seemed far too easy for such a fundamental result. Maybe the approach has some interest and can lead to a better attack.

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