Deformation quantization and quantum coadjoint orbits of SL(2,\mathbb{R}).

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Abstract

In this article we describe the coadjoint orbits of SL(2,\mathbb{R}). After choosing polarizations for each orbits, we pointed out the corresponding quantum coadjoint orbits and therefore unitary representations of SL(2,\mathbb{R}) via deformation quantization.

1 Introduction

Let us recall that quantization is a process associating to each Poisson manifold $M$ a Hilbert space $H$ of so-called quantum states, to each classical quantity $f \in C^\infty(M)$ a quantum quantity $Q(f) \in \mathcal{L}(H)$, i.e., a continuous, perhaps unbounded, normal operator which is auto-adjoint if $f$ is a real-valued function such that

$$Q(\{f, g\}) = \frac{i}{\hbar}[Q(f), Q(g)],$$

$$Q(1) = Id_H.$$

There are some approaches to this problem, such as Feynman path integral quantization, pseudo differential operator quantization, geometric quantization, etc... In Fedosov deformation quantization, the quantization is considered as the deformation of the structure of the Poisson algebra of classical
observables via a family of associated algebras indexed by the so-called deformation parameter rather than a radical change in the nature of the observables.

It is interesting to construct quantum objects corresponding to the classical ones. It is well-known that the coadjoint orbits are almost all the classified flat G-symplectic manifolds. A natural question is to associate to coadjoint orbits some quantum systems called quantum coadjoint orbits. Following Kontsevich’s result, every Poisson structure can be quantized. However, this quantization is only formal and it is difficult to calculate exactly the corresponding quantum objects and representations in concrete cases. Recently, Do Ngoc Diep and Nguyen Viet Hai, in [5], [6], described the quantum coadjoint orbits and representations of MD and MD$_4$ groups. However, the problem for SL(2,$\mathbb{R}$) is still open. Although all the irreducible unitary representations of SL(2,$\mathbb{R}$) are well-known, the correspondence of them with coadjoint orbits are not yet clarified. In this paper, we shall use Fedosov deformation quantization to find out $\star$-product formulae and representation of SL(2,$\mathbb{R}$). The algebras of smooth functions on coadjoint orbits of SL(2,$\mathbb{R}$), deformed by exactly computed $\star$-products give us series of quantum coadjoint orbits: quantum elliptic hyperboloids, quantum upper (lower) half-hyperboloids, quantum upper (lower) cones, etc...These quantum objects, as we know, appear here for the first time.

The paper is organized as follows. We describe coadjoint orbits in §2. In §3 we compute for each coadjoint orbit a polarization. The deformation $\star$-products are computed in §4 and in the last section §5, we show the relation with the unitary dual of SL(2,$\mathbb{R}$).

For notation, we refer readers to [10] or [1], [3], [6].

2 Coadjoint orbits of SL(2,$\mathbb{R}$)

Recall that SL(2,$\mathbb{R}$) is a Lie group with Lie algebra consisting of 2 by 2 matrices with trivial trace. It admits a natural basis of three generators:

\[ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

subject to relations: $[H,X]=2Y$, $[H,Y]=-2X$, $[X,Y]=-2H$. Denote by $X^*$, $H^*$, $Y^*$ the dual basis of $\mathfrak{g}^*$. Because the Killing form is non-degenerate, we can
identity \( g \) with \( g^\ast \) in such a way that \( \hat{X}(Y) = \frac{1}{4}B(X,Y) = \frac{\text{Tr(ad}X\text{.ad}Y)}{4} \). This isomorphism maps \( X \) into \( 2X^\ast \), \( H \) into \( 2H^\ast \), \( Y \) into \( -2Y^\ast \).

Naturally, the coadjoint action of \( \text{SL}(2,\mathbb{R}) \) on \( g^\ast \) is given by:

\[
\langle K(g)F, Z \rangle = \langle F, \text{Ad}(g^{-1})Z \rangle \quad \forall F \in g^\ast, g \in G, \text{ and } Z \in g.
\]

where \( g \) is a G-space vi Ad-action. However, there is a natural isomorphism of G-spaces.

**Proposition 2.1**: Operator \( X \mapsto \hat{X} \) is an smooth \( G \)-equivariant isomorphism between G-spaces. In another words, \( \hat{\text{Ad}}(g)X = K(g)\hat{X} \).

It is well-known that \( \text{GL}(2,\mathbb{R}) \) is a direct product of \( \text{SL}(2,\mathbb{R}) \) and \( R^\ast = R \setminus \{0\} \), and therefore each \( B \in \text{GL}(2,\mathbb{R}) \) can be decomposed as the product of an element from \( \text{SL}(2,\mathbb{R}) \) and \( \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) or \( \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) with \( \lambda \in R^\ast_+ \).

Due to the equivariant isomorphism of \( g \) with Ad-action and \( g^\ast \) with K-action, we study the adjoint orbits in place of coadjoint orbits of \( g^\ast \). It is well-known that every matrix \( B \in \text{sl}(2,\mathbb{R}) \) can be reduced to one of the following normal forms:

\[
\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

We obtain the following description of the geometry of coadjoint orbits which is folklore but we could not locate a precise computation from research literature.

**Theorem 2.2** Each coadjoint orbit of \( \text{SL}(2,\mathbb{R}) \) is one of the forms:

(a) **Elliptic hyperboloid**: \( \Omega^1 = \{ 2xX^\ast + 2hH^\ast - 2yY^\ast \mid x^2 + h^2 = y^2 + \lambda^2, \lambda \neq 0 \} \),

(b) **Upper half-cones**: \( \Omega^2_+ = \{ 2xX^\ast + 2hH^\ast - 2yY^\ast \mid x^2 + h^2 = y^2, y > 0 \} \),

**Lower half-cones**: \( \Omega^2_- = \{ 2xX^\ast + 2hH^\ast - 2yY^\ast \mid x^2 + h^2 = y^2, y < 0 \} \),

**One point**: \( \Omega^2_0 = \{ 0 \} \),

(c) **Upper half-hyperboloid**: \( \Omega^3_+ = \{ 2xX^\ast + 2hH^\ast - 2yY^\ast \mid x^2 + h^2 = y^2 - \lambda^2, y > 0 \} \),

\[ \text{3} \]
Lower half-hyperboloid: $\Omega^3 = \{ 2xX^* + 2hH^* - 2yY^* \mid x^2 + h^2 = y^2 + \lambda^2, y < 0 \}$.

**Proof.** We describe the geometry of adjoint orbits corresponding to $\Omega^1_\lambda$, $\Omega^2_-$ and $\Omega^3_\lambda, +$. The case of other orbits can be analogously treated. The adjoint orbit corresponding to $\Omega^1_\lambda$ contains $\left( \begin{array}{cc} \lambda & 0 \\ 0 & -\lambda \end{array} \right)$. By a direct computation, for $S = \left( \begin{array}{cc} u & v \\ s & t \end{array} \right) \in SL(2, \mathbb{R})$, we have

$$
\left( \begin{array}{cc} h & x + y \\ x - y & -h \end{array} \right) = S \left( \begin{array}{cc} 0 & \lambda \\ \lambda & 0 \end{array} \right) S^{-1} = \left( \begin{array}{cc} \lambda(ut + sv) & -2\lambda uv \\ 2\lambda st & -\lambda(ut + sv) \end{array} \right).
$$

Hence, $h = \frac{u}{\lambda} = ut + sv$, $\frac{x+y}{\lambda} = -2uv$, $\frac{x-y}{\lambda} = 2st$ and therefore, $\frac{x^2-y^2}{\lambda^2} + \frac{h^2}{\lambda^2} = -4uvst + (ut+sv)^2 = (ut-sv)^2 = 1$. Moreover, the coadjoint orbit containing $2\lambda H^*$ is

$$\{ 2xX^* + 2hH^* - 2yY^* \mid x^2 + h^2 - y^2 = \lambda^2 \}.$$

It is exactly the elliptic hyperboloid. The adjoint orbit corresponding to $\Omega^2_-$ containing $\left( \begin{array}{cc} 0 & -\lambda \\ \lambda & 0 \end{array} \right)$. By a direct computation, for $S = \left( \begin{array}{cc} u & v \\ s & t \end{array} \right) \in SL(2, \mathbb{R})$, we have

$$
\left( \begin{array}{cc} h & x + y \\ x - y & -h \end{array} \right) = S \left( \begin{array}{cc} 0 & -\lambda \\ \lambda & 0 \end{array} \right) S^{-1} = \left( \begin{array}{cc} \lambda(ut - sv) & -\lambda(u^2 + v^2) \\ \lambda(s^2 + t^2) & -\lambda(ut - sv) \end{array} \right).
$$

Hence, $h = \frac{u}{\lambda} = vt + us$, $\frac{x+y}{\lambda} = -(u^2 + v^2)$, $\frac{x-y}{\lambda} = s^2 + t^2$. And therefore, $\frac{x^2-y^2}{\lambda^2} + \frac{h^2}{\lambda^2} = 1$ for $0 \geq x + y, x - y \geq 0$. Moreover, the coadjoint orbit containing $2\lambda Y$ is $\{ 2xX^* + 2hH^* - 2yY^* \mid x^2 + h^2 = y^2 - \lambda^2, y < 0 \}$. It is exactly one of the two connected components of the elliptic hyperboloid.

Let us consider the adjoint orbit corresponding to $\Omega^2_+$ containing $\left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$.

By direct computation, for $S \in SL(2, \mathbb{R})$, we have:

$$
\left( \begin{array}{cc} h & x + y \\ x - y & -h \end{array} \right) = S \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) S^{-1} \left( \begin{array}{cc} vt & -v^2 \\ t^2 & -vt \end{array} \right).
$$

Hence, $h=vt$, $x + y = -v^2$, $x - y = t^2$. And therefore $x^2 + h^2 - y^2 = 0, 0 \geq x+y, x-y \geq 0$. Note that $(x, h, y) \neq (0, 0, 0)$. The coadjoint orbit containing
$X^* + Y^*$ is
\[ \{2xX^* + 2hH^* - 2yY^* \mid x^2 + h^2 = y^2, y > 0 \} \] It is really the upper half-cones.
The theorem is proved.

3 Complex Polarizations of K-orbits of $\text{SL}(2, \mathbb{R})$

Before quantizing coadjoint orbits we do first describe some polarizations on orbits. Let us recall some basis concepts concerning polarization, see [4].

Let $G$ be a Lie group. A complex polarization of orbit $\Omega_F$ at $F \in \Omega_F$ is a quadruple of $(\eta, \mathfrak{h}, U, \rho)$ such that:

1. $\eta$ is a subalgebra of the complex Lie algebra $\mathfrak{g}_C = \mathfrak{g} \otimes \mathbb{C}$ containing $\mathfrak{g}_F$.
2. The subalgebra $\eta$ is invariant under the action of all the operators of type $Ad_{\mathfrak{g}_C} x, x \in G_F$.
3. The vector space $\eta + \bar{\eta}$ is complexification of real subalgebra Lie $\mathfrak{m} = (\eta + \bar{\eta}) \cap \mathfrak{g}$.
4. All subgroup $M_0, H_0, M, H$ are closed, where, by definition $M_0$ (resp., $H_0$) is the connected subgroup of $G$ with Lie algebra $\mathfrak{m}$ (resp., $\mathfrak{h} := \eta \cap \mathfrak{g}$) and $M := G_F.M_0, H := G_F.H_0$.
5. $U$ is an irreducible representation of $H_0$ in some Hilbert space $H$ such that: 1. The restriction $U \mid_{G_F \cap H_0}$ is some multiple of $\chi_F$ where by definition $\chi_F(expX) \mid_{G_F \cap H_0} := exp(2\pi \sqrt{-1}\langle F, X \rangle)$; 2. The Nelson condition is satisfied. See [4], 10.5, theorem 3.
6. The Pukanszky condition is satisfied: $F + \eta^\perp \subset \Omega_F$, see [10], §15.3

Denote by $\rho$ the one dimension representation $2\pi \sqrt{-1}\langle F, X \rangle$ of Lie algebra $\eta$.

Let $C^\infty(G, \eta, H, \rho, U)$ be the set of common solutions of
\[ f(hg) = U(h).f(g) \]
\[ (L_X - \rho(X))f = 0 \quad X \in \eta \]

**Remark 1** The condition 5 and 6 are often included in order to obtain irreducible representations.

In this section, we establish complex polarization for K-orbits.
3.1 Polarization of $\Omega^1_\lambda$

Let us consider a point $\hat{F} = 2\lambda H^* \in \Omega^1_\lambda$, the complex subalgebra $\eta = \langle H, X + Y \rangle^C$. The representation $U = e^{2\pi i\langle F, . \rangle}$ of $\mathfrak{h} = \eta \cap \mathfrak{g}$ can be extended to $H = H^0 \cup \varepsilon H^0$ as $U(\varepsilon) = \pm 1$. Let $\rho$ be the natural extension of $dU$ to $\eta$.

**Proposition 3.1** $(\eta, \rho, U)$ is a polarization of $\Omega^1_\lambda$.

**Proof.** It is easy to see that the stabilizer $G_F = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$ consists of two connected components corresponding to $a > 0$ and $a < 0$. Obviously, its Lie algebra is $\mathfrak{g}_F = \langle H \rangle$. The Ad-orbit passing through $F = \lambda H$ contains two lines $\{ F + t(X \mp Y) \}$. Clearly, these lines are the images of ones $\{ \hat{F} + t(X^* \pm Y^*) \}$ passing through $\hat{F}$ on $\Omega^1_\lambda$ under the isomorphism generated by Killing form. Chose $\eta = \langle H, X + Y \rangle^C$. We can see Pukansky satisfied. Note that $[H, X + Y] = 2(X + Y)$ so $\eta$ is a invariant Lie algebra under Ad-action of $G_F$. We also deduce $\mathfrak{h} = \eta \cap \mathfrak{g} = \mathfrak{m} = \langle H, X + Y \rangle, \bar{\eta} = \eta, \bar{\mathfrak{m}}_C = \eta + \bar{\eta} = \eta$. Chose $\rho(A) = 2\pi i \langle \hat{F}, A \rangle$ with $A \in \eta$ is holomorphic representation of $\eta$. We have, $\rho(aH + b(X + Y)) = 4\pi i \lambda a$. Because $G_F$ has two connected components, $H = G_F.H^0 = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \neq 0 \right\}$.

By an exact computation, we have
\[
\exp \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} = \exp \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^a & b \frac{e^a - e^{-a}}{2} \\ 0 & e^{-a} \end{pmatrix}.
\]

Thus, $U \left( \exp \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right) = e^{4\pi i \lambda a}$ or $U \left( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right) = e^{4\pi i \lambda a}$ for all $\lambda > 0$.

On the other hand, $H = H^0 \cup \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).H^0$, and so we can extend $U$ onto $H$ following $U \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \pm I$. Corresponding to characters of $H/H^0 = \mathbb{Z}_2$, we obtain thus two unitary representations of $H$: $U \left( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right) = |\alpha|^{4\pi i \lambda}$ and $U \left( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \right) = |\alpha|^{2\pi i \lambda} \cdot \text{sgn}(\alpha)$.

3.2 Polarization of orbit $\Omega^2_+$

Let us consider a point $\hat{F} = X^* - Y^* \in \Omega^2_+$, the complex subalgebra $\eta = \langle H, X + Y \rangle^C$. The representation $U = e^{2\pi i\langle F, . \rangle}$ can be extended to $H =
H^0 \cup \varepsilon H^0 \text{ as } U(\varepsilon) = \pm 1. \text{ Let } \rho \text{ be the natural extension of } dU \text{ to } \eta.

**Proposition 3.2** \((\eta, \rho, U, \rho)\) is a polarization of \(\Omega^2_+\).

**Proof.** It is easy to see that the stabilizer \(G_F = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\} ; a \in \{-1, 1\}\) consists of two connected components corresponding to \(a > 0\) and \(a < 0\) with Lie subalgebra \(\mathfrak{g}_F = \langle X + Y \rangle\). Chose \(\eta = \langle H, X + Y \rangle_C\). Due to \([H, X + Y] = 2(X + Y), \eta\) is a invariant Lie algebra under the Ad action of \(G_F\).

As known \(\eta^\perp = \langle X^* - Y^* \rangle\): functional on \(\mathfrak{g}\) such that vanishes on \(\eta\) when extented to complexification of \(\mathfrak{g}\). We also imply \(\mathfrak{h} = \eta \cap \mathfrak{g} = \langle H, X + Y \rangle\), \(\bar{\eta} = \eta = \mathfrak{m}_C\) and \(0\) is the one-dimension representation of \(\eta\). Naturally, \(H = H^0 \cup \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. H^0 \) and \(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = I.\) It follows \(U \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \pm I.\)

Following to characters of \(H/H^0\) we obtain two unitary representations of \(H:\)

\[ U \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = 1 \text{ and } U \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \text{sgn}(\alpha). \]

By analogy, we obtain the same result for \(\Omega^2_-\).

### 3.3 Polarization for \(\Omega^3_{\lambda^+,+}\)

Let us consider a point \(\hat{F} = 2H^* \in \Omega^3_{\lambda^+,+}\), the complex subalgebra \(\eta = \langle Y, X + iH \rangle_C\). Because of the fact that the stabilizer \(SO(2, \mathbb{R})\) of \(\hat{F}\) is not simply connected, \(U = e^{2\pi i \langle F, \cdot \rangle}\) can be extented to \(H\) only if the orbit is integral.

**Proposition 3.3** \((\eta, \rho, U, \rho)\) is a polarization of \(\Omega^3_{\lambda^+,+}\) and this orbit is integral if and only if \(\lambda\) is of the form \(\lambda = \frac{k}{8}\).

**Proof.** It is trivial that the stabilizer \(G_F = SO(2, \mathbb{R})\) with Lie algera \(\mathfrak{g}_F = \langle Y \rangle\) is connected but not simply connected. By choosing \(\eta = \langle Y, X + iH \rangle_C\), \(\mathfrak{m}_C = \mathfrak{g}, \mathfrak{h} = \eta \cap \mathfrak{g}\), \(\eta\) admits an one-dimension representation \(\rho \begin{pmatrix} -ia & a + b \\ -a + b & ia \end{pmatrix} = -4\pi i \lambda a\), which has the restriction on \(\mathfrak{h}\), \(\rho \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = -4\pi i \lambda a\). On the other hand,

\[
\exp \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}.
\]
Thus
\[
U \left( \begin{array}{cc}
\cos a & \sin a \\
-\sin a & \cos a
\end{array} \right) = e^{-4\pi i \lambda a}.
\]
Because SO(2,\mathbb{R}) is not simply connected, \( U \) may not exist. The necessary and sufficient condition is \( \lambda = \frac{k}{8} \). The orbit \( \Omega^3_{\lambda-} \) can be treated analogously and we gain the same result. A corollary of polarization for all co-adjoint orbits is the representation of SL(2,\mathbb{R}) on the Hilbert space of partial holomorphic, square-integrable sections of induced vector bundle. See e.g [1], [4]. We follow another approach by deformation quantization.

4 Quantum coadjoint orbits of SL(2,\mathbb{R})

We shall work from now on for the fixed coadjoint orbit \( \Omega^1_{\lambda} \). Following the scheme from [5], [6], first we study the geometry of this orbit and introduce some canonical coordinates in it. It’s well known that coadjoint orbits are isomorphism to the homogeneous spaces \( G/G_F \) which are symplectic manifolds. We’ll introduce a coordinate system on this orbit and it turns out to be a Darboux one. Each \( A \in \mathfrak{g} \) can be considered as a linear functional \( \tilde{A} \) on coadjoint orbits, as a subset of \( \mathfrak{g}^* \), \( \tilde{A}(F) = \langle F, A \rangle \). It is also well known that this function is just the Hamilton function associated with the Hamiltonian vector field \( \xi_A \) generated by \( A \) following the formula:

\[
\xi_A(f)(x) = \frac{d}{dt}f(x \exp(tA))|_{t=0}
\]

The Kirillov form \( \omega_F \) is defined by the formula

\[
\omega_F(\xi_A, \xi_B) = \langle F, [A,B] \rangle
\]

It is known as the flatness of the coadjoint orbits that the correspondence \( A \mapsto \tilde{A} \) is a Lie homomorphism. Motivated by the constructed polarizations, \( \Omega^1_{\lambda} \) can be parameterized as

\[
\begin{align*}
x = M(p,q) &= p \cos(q) - \lambda \sin(q); \\
h = N(p,q) &= p \sin(q) + \lambda \cos(q); \\
y = P(p,q) &= p;
\end{align*}
\]

\( M, N, P \) satisfy

\[
M_q = -N; N_q = M; M_p = \cos(q); N_p = \sin(q); M \cdot \cos(q) + N \cdot \sin(q) = p; \quad (1)
\]
Let us consider the mapping \( \psi : (p, q) \mapsto 2M(p, q)X^* + 2N(p, q)H^* - 2P(p, q)Y^* \). Clearly, \((R^2, \Omega^1_\lambda, \psi)\) is an universal covering space.

**Proposition 4.1** \( \psi \) is a symplectomorphism and Hamiltonian \( \tilde{A} \) in coordinates \((p, q)\) is of the form:

\[
\tilde{A}(F) = \langle F, A \rangle = (2a_1 \cos q + 2b_1 \sin q - 2c_1)p + (-2a_1 \sin q + 2b_1 \cos q)\lambda
\]

**Proof:** Each \( F \in \Omega^1_\lambda \) is of the form \( 2MX^* + 2NH^* - 2PY^* \). From this it follows that the Hamiltonian function generated by invariant vector field \( \xi_A \) is

\[
\tilde{A}(F) = \langle F, A \rangle = 2a_1 M + 2b_1 N - 2c_1 P.
\]

It follows therefore

\[
\tilde{A}(F) = 2a_1 (p \cos q - \lambda \sin q) + 2b_1 (p \sin q + \lambda \cos q) - 2c_1 p.
\]

On \( R^2 \) there are two symplectic structures: the first one is the Kirillov form induced by mapping \( \psi \) and the second is the canonical symplectic form \( dp \wedge dq \). We prove their coincidence by observing their values at invariant vector fields are equal.

Note that \( \omega_F(\xi_A, \xi_B) = \langle F, [A, B] \rangle \)

\[
= \langle 2M X^* + 2NH^* - 2PY^*, 2(b_1 c_2 - b_2 c_1)X + 2(c_1 a_2 - c_2 a_1)H - 2(a_1 b_2 - a_2 b_1)Y \rangle
\]

\[
= 4M(b_1 c_2 - b_2 c_1) + 4N(c_1 a_2 - c_2 a_1) + 4P(a_1 b_2 - a_2 b_1).
\]

On the other hand,

\[
(dp \wedge dq)(\xi_A, \xi_B) = \{\tilde{A}, \tilde{B}\} = \frac{\partial b}{\partial p} \frac{\partial b}{\partial q} - \frac{\partial a}{\partial q} \frac{\partial a}{\partial p}
\]

\[
= 4(b_1 c_2 - b_2 c_1)N_q + 4(c_1 a_2 - c_2 a_1)(-M_q) + 4(a_1 b_2 - a_2 b_1)(M_p N_q - N_p M_q).
\]

Then \( \omega_F(\xi_A, \xi_B) = (dp \wedge dq)(\xi_A, \xi_B) \).

The theorem is therefore proven.

**Remark 2** The case of different orbits can be treated similarly with a small correction. With the orbits \( \Omega^3_{\lambda^+} \) and \( \Omega^3_{\lambda^+} \), clearly we can’t find out a affine subspace of a half dimensions, thus there can’t exist a coordinate as above. However, a good approach is considering the complexification of orbits and we obtain \((C \times C, \Omega^1_{\lambda}, \psi)\) as universal complex symplectic covering space, only by replacing \( \lambda \) by \( i\lambda \). The orbits \( \Omega^2_{\lambda^+}, \Omega^2_{\lambda^+} \) can be viewed as a part of the case \( \Omega^1_{\lambda^+} \) and \( \Omega^3_{\lambda^+} \) when \( \lambda = 0 \).

From now, because of the similarity, we’ll deal mainly with the orbits \( \Omega^1_{\lambda} \). The other orbit can be treated with a simple modification.
Theorem 4.2 With $A, B \in \mathfrak{g}$, the Moyal $\star$-product satisfies

$$i\tilde{A}\star i\tilde{B} - i\tilde{B}\star i\tilde{A} = i\{A, B\}$$

Proof: Consider two arbitrary elements $A = a_1X + b_1H + c_1Y, B = a_2X + b_2H + c_2Y \in \mathfrak{g}$, By the Moyal-Weyl formular,

$$i\tilde{A}\star i\tilde{B} = \sum_{k=0}^{\infty} P^k(i\tilde{A}, i\tilde{B}). \frac{1}{k!} \left(\frac{1}{2i}\right)^k,$$

with $P^k(i\tilde{A}, i\tilde{B}) = -\bigwedge^{i_1j_1} \bigwedge^{i_2j_2} \ldots \bigwedge^{i_kj_k} \partial_{i_1i_2\ldots i_k} \tilde{A} \partial_{j_1j_2\ldots j_k} \tilde{B}$

It’s easy, then, to see that:

$$P^0(i\tilde{A}, i\tilde{B}) = -\tilde{A} \cdot \tilde{B},$$

$$P^1(i\tilde{A}, i\tilde{B}) = -\left(\bigwedge^{12} \frac{\partial \tilde{A}}{\partial p} \frac{\partial \tilde{B}}{\partial q} + \bigwedge^{21} \frac{\partial \tilde{A}}{\partial q} \frac{\partial \tilde{B}}{\partial p}\right) = -\{\tilde{A}, \tilde{B}\},$$

By proposition 4.1, $\tilde{A}, \tilde{B}$ are linear functions of $p$. Thus for $k \geq 2$, we have

$$P^2(i\tilde{A}, i\tilde{B}) = -\left(\bigwedge^{12} \frac{\partial \tilde{A}}{\partial p} \frac{\partial \tilde{B}}{\partial q} + \bigwedge^{21} \frac{\partial \tilde{A}}{\partial q} \frac{\partial \tilde{B}}{\partial p}\right) = -\{\tilde{A}, \tilde{B}\},$$

We get

$$i\tilde{A}\star i\tilde{B} - i\tilde{B}\star i\tilde{A} = (P^1(i\tilde{A}, i\tilde{B}) - P^1(i\tilde{B}, i\tilde{A})) \frac{1}{2i} + (P^2(i\tilde{A}, i\tilde{B}) - P^2(i\tilde{B}, i\tilde{A})) \frac{1}{2i} = i\{\tilde{A}, \tilde{B}\} = i[\tilde{A}, \tilde{B}].$$

The theorem can be proved analogously on $\Omega^2_\times, \Omega^2_\times$ and $\Omega^3_{\lambda,C}$.

Remark 3 Consider the canonical representation of quantum algebra $(C^\infty(\Omega), \star)$ on itself which is a Frchet Poisson algebra by left $\star$-multiplication defined by:

$$l_f : C^\infty(\Omega) \to C^\infty(\Omega),$$

$$g \mapsto f \star g.$$

Then, $C^\infty(\Omega)$ can be viewed as a algebra of pseudo-diiffential operators on $C^\infty(\Omega)$. On the other hand, the corespondence $A \mapsto \tilde{A}$ is a Lie algebra homomorphism. Thus, we can consider the repersentation of Lie algebra
Following the Moyal-Weyl formula, we have \( L_{\text{sl}(2,\mathbb{R})} \) on dense subspace \( L^2(\mathbb{R} \times [0, 2\pi), \frac{d\theta}{2\pi})^\infty \) of smooth functions by left \( \star \)-multiplication by \( i\hat{A} \). This representation is then extended to the whole space \( L^2(\mathbb{R} \times SO(2,\mathbb{R}), \frac{dpdq}{2\pi}) \) by \( \mathfrak{g} \). We study now the convergence of the formal power series. In order to do this, we look at the \( \star \)-product of \( i\hat{A} \) as the \( \star \)-product of symbols and define the differential operators corresponding to \( i\hat{A} \). It is easy to see that the resulting correspondence is a representation of \( \mathfrak{g} \) by pseudo-differential operators.

On \( \Omega_\lambda = \{ 2xX^* + 2hH^* - 2yY^* \mid x^2 + h^2 = y^2 + \lambda^2 \} \) the following results hold:

**Lemma 4.3**

1. \( \mathcal{F}_p(\partial_p \mathcal{F}_p^{-1}(f)) = i^{-1}(x)f \),
2. \( \mathcal{F}_p(p \mathcal{F}_p^{-1}(f)) = i\partial_x(f) \),
3. \( P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \sum k(-1)^{k-1} \tilde{A}_{q-p} \partial_{p-q} \mathcal{F}_p^{-1}(f) + (-1)^{k} \tilde{A}_{q-p} \partial_{p} \mathcal{F}_p^{-1}(f) \).

**Proof.** The two first formulas are well-known from the theory of Fourier transforms. If \( k \geq 2 \) then by theorem 4.1 it implies \( \tilde{A} \) is a linear function of \( p \). Because one of the coordinates is linear, if two of index \( i_1, i_2, \ldots, i_k \) equals to 1 then \( \partial_{i_1, i_2, \ldots, i_k} \tilde{A} = 0 \). Therefore, for all \( k \geq 2 \):

\[
\begin{align*}
P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) &= \sum \wedge^{i_1} \wedge^{i_2} \ldots \wedge^{i_k} \tilde{A}_{i_1 \ldots i_n} \partial_{j_1 \ldots j_n} \mathcal{F}_p^{-1}(f) \\
&= \sum \wedge^{21} \wedge^{12} \ldots \wedge^{21} \tilde{A}_{q-p} \partial_{p-q} \mathcal{F}_p^{-1}(f) + \wedge^{21} \wedge^{21} \tilde{A}_{q-p} \partial_{p} \mathcal{F}_p^{-1}(f).
\end{align*}
\]

It is clear that \( \wedge^{-1} = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \), so we get \( \wedge^{12} = 1, \wedge^{21} = -1 \). It deduces

\[
P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \sum (-1)^{k-1} \tilde{A}_{q-p} \partial_{p-q} \mathcal{F}_p^{-1}(f) + (-1)^{k} \tilde{A}_{q-p} \partial_{p} \mathcal{F}_p^{-1}(f).
\]

With \( k=0 \) hay \( k=1 \), clearly, the lemma is also satisfied. Apply this lemma, we have the following theorem:

**Theorem 4.4** If we set \( s=q-\frac{x}{2} \), for each compactly supported smooth function \( f \in C_0^\infty(\mathbb{R}^2) \) we have

\[
\hat{l}_A = \mathcal{F}_p \circ l_A \circ \mathcal{F}_p^{-1} = (a_1 \cos s + b_1 \sin s - c_1) \partial_s + (-a_1 \sin s + b_1 \cos s)(2\lambda i + 1)
\]

**Proof.**

Following the Moyal-Weyl formula, we have

\[
\hat{l}_A(f) = \mathcal{F}_p \circ l_A \circ \mathcal{F}_p^{-1}(f) = i\mathcal{F}_p(\sum_{k=0}^{\infty} \left( \frac{1}{2l} \right)^k \frac{1}{k!} P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)),
\]

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By the lemma,

\[
\hat{A}(f) = i \mathcal{F}_p \left( \sum_{k=0}^{\infty} \left( \frac{1}{2i} \right)^k \frac{1}{k!} (-1)^{k-1, k} \bar{A}_{q \ldots p \ldots q} \partial_{p \ldots p} \mathcal{F}_p^{-1}(f) + \sum_{k=0}^{\infty} \left( \frac{1}{2i} \right)^k \frac{1}{k!} (-1)^{k} \bar{A}_{q \ldots q} \partial_{p \ldots p} \mathcal{F}_p^{-1}(f) \right) = I + J,
\]

Note the fact that \( \bar{A} \) is a linear function of \( p \). Therefore \( \bar{A}_{q \ldots p \ldots q} \) is a function of only variable \( p \).

\[
I = i \mathcal{F}_p \sum_{k=0}^{\infty} \left( \frac{1}{2i} \right)^k \frac{1}{k!} (-1)^{k-1, k} \bar{A}_{q \ldots p \ldots q} \partial_{p \ldots p} \mathcal{F}_p^{-1}(f))
\]

\[
= i \sum_{k=0}^{\infty} \left( \frac{1}{2i} \right)^k \frac{1}{k!} (-1)^{k-1, k} \mathcal{F}_p(\bar{A}_{q \ldots p \ldots q} \partial_{p \ldots p} \mathcal{F}_p^{-1}(f))
\]

\[
= i \sum_{k=1}^{\infty} \left( \frac{1}{2i} \right)^k \frac{1}{(k-1)!} (-1)^{k-1, (k-1)} \bar{A}_{q \ldots p \ldots q} \partial_{p} f
\]

\[
= \frac{1}{2} \partial_{p} \bar{A}(q - \frac{x}{2}).\partial_{q}(f).
\]

Set \( \bar{A} = p \). \( M+N \), where \( M, N \) depend only \( q \), by exact computations, we have

\[
J = i \sum_{k=0}^{\infty} \left( \frac{1}{2i} \right)^k \frac{1}{k!} (-1)^{k \cdot M(q - \ldots q) \cdot \mathcal{F}_p^{-1}(f))}
\]

\[
= i \sum_{k=0}^{\infty} \left( \frac{i}{2} \right)^k \frac{1}{k!} ((i \cdot \partial_{x} M^{(k)}) + N^{(k)}).f
\]

\[
= i \sum_{k=0}^{\infty} \left( \frac{i}{2} \right)^k \frac{1}{k!} i \partial_{x} M^{(k)}(q).f + i \sum_{k=0}^{\infty} \left( \frac{i}{2} \right)^k \frac{1}{k!} N^{(k)}(q).f
\]

\[
= - \sum_{k=0}^{\infty} \left( \frac{-x}{2} \right)^k \frac{M^{(k)}(q)}{k!} \partial_{x} f - \sum_{k=0}^{\infty} \left( \frac{-x}{2} \right)^k \frac{M^{(k)}(q)}{k!} \partial_{x} f + i \sum_{k=0}^{\infty} \left( \frac{-x}{2} \right)^k \frac{N^{(k)}(q)}{k!} \partial_{x} f
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{-x}{2} \right)^k \frac{M^{(k+1)}(q)}{k!} f - M(q - \frac{x}{2}) \partial_{x} f + iN(q - \frac{x}{2}) f
\]

\[
= \frac{1}{2} . M'(q - \frac{x}{2}) f - M(q - \frac{x}{2}) \partial_{x} f + iN(q - \frac{x}{2}) f.
\]
Finally, we have the exact formula of the corresponding quantized operator:

\[
\hat{A}(f) = \frac{1}{2} \partial_q \tilde{A}(q - \frac{x}{2}) \partial_q (f) + \frac{1}{2} M'(q - \frac{x}{2}) f - M(q - \frac{x}{2}) \partial_x f + i N(q - \frac{x}{2}) f
\]

\[
= M(q - \frac{x}{2}) (\frac{1}{2} \partial_q - \partial_x) f + \frac{1}{2} M'(q - \frac{x}{2}) f + i N(q - \frac{x}{2}) f.
\]

Put \( q - \frac{x}{2} = s; q + \frac{x}{2} = t \), it follows \( \partial_s = \partial_q - 2 \partial_x \). Recall that

\[
\tilde{A}(F) = 2a_1 (p \cos q - \lambda \sin q) + 2b_1 (p \sin q + \lambda \cos q) - 2c_1 p. \quad M(q) = 2a_1 (p \cos q - \lambda \sin q) + 2b_1 (p \sin q + \lambda \cos q) - 2c_1,
\]

\[
N(q) = -2\lambda a_1 \sin q + 2\lambda b_1 \cos q, \quad M'(q) = \frac{N(q)}{2}.
\]

Therefore,

\[
\hat{A}(f) = \frac{1}{2} M(s) \partial_s f + \left( \frac{N(s)}{2} + i Ns \right) f = (a_1 \cos s + b_1 \sin s - c_1) \partial_s + (-a_1 \sin s + b_1 \cos s)(2\lambda i + 1).
\]

The theorem is proved.

By analogy, we get the same results for all two-dimension coadjoint orbits. Note that, following the virtual of the polarizations chosen for orbits, we obtain the representation of \( \text{sl}(2, \mathbb{R}) \) on \( L^2 \)-space on \( \text{SO}(2, \mathbb{R}) \).

## 5 Relation with unitary dual of SL(2, \mathbb{R})

We recall some basic results of constructing unitary dual of \( \text{SL}(2, \mathbb{R}) \) by the classical methods, see e.g. [1].

Consider the subgroup \( H = \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) \) associated with one-dimension representation \( \rho_s \left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) = a^{s+1} \). Let \( \phi_s \) be the induced representation of \( \rho_s \) on to \( \text{SL}(2, \mathbb{R}) \). Clearly, the space of induced vector bundle is isomorphic to the space \( H_s \) of function on \( G \) satisfies \( f(hg) = \rho_s(h) f(g) \) with restriction on \( K \) lying on \( L^2(K) \), also isomorphic to \( L^2(K) \) where \( K = \text{SO}(2, \mathbb{R}) \simeq G/H \).

Let \( T \) be the representation of \( G \) on \( C^\infty(G) \) defined by \( T(g_1) f(g) = f(g g_1) \). The infinitesimal representation of \( T \) determined by \( L(A) f(g_0) = \frac{\partial}{\partial t} T(e^{tA}) f(g_0) \big|_{t=0} \). By the Iwasawa decomposition, each \( g \) of \( \text{SL}(2, \mathbb{R}) \) can be viewed as the product \( g = \left( \begin{array}{cc} \frac{1}{\sqrt{y}} & 0 \\ 0 & \sqrt{y} \end{array} \right) \cdot \left( \begin{array}{cc} y & x \\ 0 & 1 \end{array} \right) \cdot \left( \begin{array}{cc} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array} \right) \).
So a function on $G$ can be viewed as function of $x$, $y$, $\theta$. We obtain the explicit formulas of $L$ as:

\[ L_X = (s + 1) \sin 2\theta - \cos 2\theta \partial_\theta, \]

\[ L_H = (s + 1) \cos 2\theta + \sin 2\theta \partial_\theta, \]

\[ L_Y = \partial_\theta. \]

From this, by considering the algebraic vector subspaces of $L^2(K)$, it can imply all the irreducible unitary representations of $SL(2, \mathbb{R})$ of discrete series, principal series, the complementary series as in [11]. In order to prove the equivalence of two approaches, it is enough to show that the corresponding infinitesimal representations of Lie algebra $sl(2, \mathbb{R})$ are the same.

**Theorem 5.1** The representations $\hat{l}$ obtained from deformation quantization are coincided with the infinitesimal representation $L$ of Lie algebra corresponding to discrete series, principal series, the complementary series of $SL(2, \mathbb{R})$.

**Proof:** We know that $f(x, y, \theta) = y^{\frac{s+1}{2}} f(\theta)$.

So, $\frac{\partial f}{\partial y} = \frac{s+1}{2y} f$, $\frac{\partial f}{\partial \theta} = 0$.

Thus $2y \partial_y = s + 1$, $\partial_x = 0$. We obtain the explicit form of representation: for $A = a_1 X + b_1 H + c_1 Y$

\[ L_A = (-a_1 \cos(2\theta) + b_1 \sin 2\theta + c_1) \partial_\theta + (s + 1)(a_1 \sin 2\theta + b_1 \cos 2\theta) \]

setting $s = 2\lambda v - 2\theta = s$. Then

\[ L_A = (a_1 \cos s + b_1 \sin s - c_1) \partial_s + (-a_1 \sin s + b_1 \cos s)(2\lambda + 1) = \hat{l}_A \]

The proof is therefore achieved.

**Remark 4** We demonstrated how irreducible unitary representations of $SL(2, \mathbb{R})$ could be obtained from deformation quantization. It is reasonable to refer to the algebras of functions on coadjoint orbits with corresponding $\star$-product as a quantum ones, namely quantum elliptic hyperboloids ($C^\infty(\Omega^1_\lambda), \star_h$), quantum elliptic cones ($C^\infty(\Omega^2_{\pm}), \star_h$), two folds quantum hyperboloids ($C^\infty(\Omega^3_\lambda), \star_h$) etc.

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