Deformations of representations of fundamental groups of open Kähler manifolds

Philip A. Foth

March 1, 2022

Abstract

Given a compact Kähler manifold, we consider the complement $U$ of a divisor with normal crossings. We study the variety of unitary representations of $\pi_1(U)$ with certain restrictions related to the divisor. We show that the possible singularities of this variety as well as of the corresponding moduli space of irreducible representations are quadratic. In the course of our proof we exhibit a differential graded Lie algebra which reflects our deformation problem.

Contents

1 Introduction 2
2 Forms with logarithmic poles 4
3 Construction of the DGLA and formality 8
4 The deformation theory of flat bundles 12
5 Moduli of representations of $\pi_1(U)$ 16

1Sloan Doctoral Dissertation Fellow
1 Introduction

In the present paper we study the spaces of unitary representations of fundamental groups of open Kähler manifolds. More precisely, let $X$ be a compact Kähler manifold and let $U$ be the complement of a divisor with normal crossings $D \subset X$, such that there exists a unique decomposition $D = \bigcup_{i=1}^r D_i$ into the union of $r$ smooth irreducible complex-analytic subvarieties of $X$. In fact the Hironaka resolution of singularities theorem [8] states that every smooth quasi-projective variety is birational to a manifold of this type.

Let us fix an $r$-tuple of conjugacy classes $\mathcal{N} = (C_1, \ldots, C_r)$ in $U(N)$, a base point $b \in U$, simple loops $\gamma_i$ based at $b$ and encircling $D_i$ for $1 \leq i \leq r$ defining classes $[\gamma_i] \in \pi_1(U, b)$. We consider the space $\tilde{\mathcal{R}}(\pi_1(U, b), U(N))_N$ of representations $\rho$ of $\pi_1(U, b)$ into $U(N)$ such that for $1 \leq i \leq r$ the image $\rho([\gamma_i])$ belongs to the conjugacy class $C_i$. This space has the natural structure of a real algebraic variety.

For an algebraic variety $Y$ and a point $y \in Y$ we say that $Y$ is quadratic at $y$ if there exists an analytic embedding $\phi$ of a neighborhood $U$ of $y$ into a vector space such that $\phi(y) = 0$ and $\phi(U)$ is given as the zero locus of a finite number of homogeneous quadratic equations. We prove

**Theorem 4.5** Let $\rho : \pi_1(U, b) \to U(N)$ be a representation such that $\rho([\gamma_i]) \in C_i$ for $1 \leq i \leq r$, then the space $\tilde{\mathcal{R}}(\pi_1(U, b), U(N))_N$ is quadratic at $\rho$.

This theorem was proved by Goldman and Millson [7] in the compact case, i.e. when $D$ is empty. Working in the context of integrable connections, Biquard [1] showed that when $r = 1$ (i.e. smooth divisor case) the singularities of the space of isomonodromic deformations of an integrable logarithmic connection are quadratic.

If we restrict our attention to irreducible representations, then we can consider the moduli space

$$\mathcal{M}_N := \tilde{\mathcal{R}}^{\text{irr}}(\pi_1(U, b), U(N))_N/U(N),$$

where the group $U(N)$ acts on the space of representations by conjugation. This space was the subject of a work of Jean-Luc Brylinski and the author [2]. In fact, it was shown there that when a certain obstruction vanishes, the tangent space to $\mathcal{M}_N$ at a point $[\rho] \in \mathcal{M}_N$ is identified with the first
intersection cohomology group $IH^1(X, \tilde{g})$, where $\tilde{g}$ is the local system on $U$ corresponding to the representation $Ad \circ \rho : \pi_1(U, b) \to u(N)$. This was used to exhibit a symplectic form on the smooth locus of $\mathcal{M}_N$, which turns out to be a Kähler form when $X$ is projective. Another possible description of the tangent space $T_{[\rho]}\mathcal{M}_N$ is the first relative group cohomology group $H^1(\pi_1(U), (\Gamma_i), u(N))$, where $\Gamma_i$ be a cyclic subgroup of $\pi_1(U)$ generated by $\gamma_i$ and $Ad \circ \rho$ give $u(N)$ the structure of a $\pi_1(U, b)$-module.

A straightforward consequence of the above Theorem is that if $\rho$ is an irreducible representation satisfying the above hypotheses, then $\mathcal{M}_N$ is quadratic at $[\rho]$.

Let us take a representation $\rho$ defining a point $[\rho] \in \mathcal{M}_N$. Consider the pairing

$$H^1(\pi_1(U), (\Gamma_i), u(N)) \times H^1(\pi_1(U), (\Gamma_i), u(N)) \to H^2(\pi_1(U), (\Gamma_i), u(N))$$

given by the cup product in relative group cohomology together with the Lie bracket as a coefficient pairing.

**THEOREM 5.2** (cf. [7]) A point $[\rho] \in \mathcal{M}_N$ is a smooth point of $\mathcal{M}_N$ if the above pairing is identically zero.

Following the ideology developed by Schlessinger-Stasheff [10], Deligne, Goldman-Millson [7] we construct a differential graded Lie algebra (DGLA) $B^\bullet(\tilde{g})$ which is the controlling DGLA for our deformation problem. This algebra consists of smooth differential forms on $U$ with coefficients in $\tilde{g}$ with certain conditions on asymptotics of the forms along $D$. We say that a DGLA is formal if it is quasi-isomorphic to its cohomology algebra.

**THEOREM 3.5** The DGLA $B^\bullet(\tilde{g})$ is formal.

This is similar to the fundamental result of Deligne-Griffiths-Morgan-Sullivan [3] about the formality of the de Rham algebra of a compact Kähler manifold. One of the important ingredients in the course of our proof is an analogue of the $\bar{\partial}$-Poincaré lemma. The DGLA $B^\bullet(\tilde{g})$ is naturally bigraded by holomorphic and anti-holomorphic degrees. Our Lemma 3.4 tells us that locally the complex

$$\ldots \nabla'' \to B^{p,q-1}(\tilde{g}) \nabla'' \to B^{p,q}(\tilde{g}) \nabla'' \to B^{p,q+1}(\tilde{g}) \nabla'' \to \ldots$$
is exact.

The section of our paper which relates this DGLA and our deformation problem heavily relies on the theory which Goldman and Millson developed in the compact case. The deformation theory that we study is equivalent to the deformation theory of flat unitary connections such that the monodromy around \( i \)-th irreducible component of the divisor lies in a specified conjugacy class \( C_i \). As in [7] we formulate the equivalence theorem in terms of functors from the category of Artin local algebras.

We must mention that Kapovich and Millson in [9] studied the relative deformation theory of representation with relation to the local deformations of linkages.

**Acknowledgments.** I am very grateful to Jean-Luc Brylinski for the excellent guidance which I and this paper indubitably benefited from. I would like to thank John Millson for important conversations, James Stasheff for useful comments, and Indranil Biswas for drawing my attention to the results of Timmerscheidt.

## 2 Forms with logarithmic poles

Let \( X \) be a compact Kähler manifold of complex dimension \( d \) endowed with a Kähler form \( \lambda \) and let \( D \) be a normal crossing divisor on \( X \) which can be represented as a union of \( r \) smooth irreducible complex-analytic subvarieties of codimension 1:

\[
D = \bigcup_{i=1}^{r} D_i.
\]

We let \( G = U(N) \) and \( \mathfrak{g} = u(N) \) and we pick a non-degenerate symmetric invariant bilinear form \( B(\cdot, \cdot) \) on \( \mathfrak{g} \). We also fix a set of \( r \) conjugacy classes in \( G \):

\[
\mathcal{N} = (C_1, ..., C_r),
\]

one for each component of the divisor. We let \( U := X \setminus D \), we let the map \( j : U \hookrightarrow X \) be the inclusion, and we pick a base point \( b \in U \) and we denote \( \pi_1(U) := \pi_1(U, b) \). Let \( \gamma_i \) be a simple loop encircling \( D_i \) for \( 1 \leq i \leq r \) and let \( \Gamma_i \simeq \mathbb{Z} \) be a subgroup of \( \pi_1(U) \) generated by \([\gamma_i]\) - the class of \( \gamma_i \) in \( \pi_1(U) \).
We consider the space \( \tilde{\mathcal{R}}(\pi_1(U), G)_N \) of representations 

\[ \rho : \pi_1(U) \to G \]

such that \( \rho([\gamma_i]) \in \mathcal{C}_i \). This space has the structure of a real algebraic variety.

We also consider the moduli space \( \mathcal{M}_N \) of irreducible representations of this type, i.e.

\[ \mathcal{M}_N = \tilde{\mathcal{R}}^{irr}(\pi_1(U), G)_N / G, \]

where \( G \) acts by conjugation. In general these spaces are not smooth, and we shall show that in fact all possible singularities are quadratic: i.e. a neighbourhood of each point \([\rho] \in \mathcal{M}_N\) is analytically isomorphic to a neighbourhood of the origin of \( \mathcal{L} \) in \( E \). Here \( E \) is a vector space and \( \mathcal{L} \) is the zero locus of a quadratic form \( Q \) in \( E \) which in general is given by several quadratic equations. A point is smooth then if \( Q = 0 \) identically on \( E \).

Let \( D^{(m)} \) be the subvariety consisting of points which belong to at least \( m \) irreducible components of the divisor \( D \), e.g. \( D^{(0)} = X \) and \( D^{(1)} = D \). Let \( \tilde{D}^{(m)} \to D^{(m)} \) be normalization maps and let

\[ v^m : \tilde{D}^{(m)} \to D^{(m)} \hookrightarrow X \]

be the composition of normalizations with inclusions. We also let \( \tilde{C}^{(m)} := (v^m)^{-1}(D^{(m+1)}) \), and it is a divisor with normal crossings on \( \tilde{D}^{(m)} \) (possibly empty).

Given a representation \( \rho \) such that \([\rho] \in \mathcal{M}_N\) we can consider \( g \) as a \( \pi_1(U) \)-module via the adjoint representation followed by \( \rho \). We also let \( \tilde{g} \) stand for the local system corresponding to \( g \), let \( V \) be the flat bundle over \( U \) corresponding to the local system \( \tilde{g} \), and finally let \( (\tilde{V}, \nabla) \) be the Deligne extension of \( V \). Here \( \tilde{V} \) is a holomorphic vector bundle over \( X \) and \( \nabla \) is a holomorphic connection with logarithmic singularities along \( D \). We refer the reader to [3] for the details. The connections are extended as usual to holomorphic forms with logarithmic poles.

There exist higher residues \( Res_m(\tilde{V}) \) which give morphisms of complexes:

\[ Res_m(\tilde{V}) : \Omega_X^\bullet(D) \otimes \tilde{V} \to v^m_*(\Omega_{\tilde{D}^{(m)}}^\bullet(\tilde{C}^{(m)}) \otimes \tilde{V}_m)[-m]. \]

Here \( \tilde{V}_m \) is the unique vector subbundle of \( (v^m)^*\tilde{V} \) equipped with a unique holomorphic integrable connection \( \nabla_m \) with logarithmic poles along \( \tilde{C}^{(m)} \).
such that
\[ \text{Ker} \nabla_{m|\tilde{D}(m)} \subset \tilde{C}(m) = (v_m)^{-1}((j_\ast \tilde{g})|_{\tilde{D}(m)} \setminus \tilde{D}(m+1)). \]
This means that
\[ (\bar{V}_m, \nabla_m) \]
is the canonical extension of
\[ (v_m)^{-1}((j_\ast \tilde{g})|_{\tilde{D}(m)} \setminus \tilde{D}(m+1)). \]
As usual, we denote by \( \Omega^\bullet_X(D) \) the complex of sheaves of holomorphic forms on \( U \) with logarithmic poles along \( D \). The fact that \( \text{Res}_m(\bar{V}) \) is a morphism of complexes means that
\[ \text{Res}_m(\bar{V}) \circ \nabla = \nabla_m \circ \text{Res}_m(\bar{V}). \]
Let us define
\[ \tilde{\Omega}^\bullet_X(\bar{V}) := \text{Ker} \text{Res}_1(\bar{V}); \]
it is a complex of sheaves of holomorphic forms on \( X \) with coefficients in \( \bar{V} \) which have logarithmic poles along \( D \) and have no residues. Residue maps take values in the monodromy invariant part of the local system, so when all the eigenvalues of monodromy transformation are different from 1, the complexes \( \tilde{\Omega}^\bullet_X(\bar{V}) \) and \( \Omega^\bullet_X(D) \otimes \bar{V} \) coincide.

The complex \( \tilde{\Omega}^\bullet_X(\bar{V}) \) admits a different description \[\text{[11]}\]. First, on \( U \) we let \( \tilde{\Omega}^p_X(\bar{V})|_U := \Omega^p_U \otimes \bar{V}|_U. \) For each \( x \in D \) we pick a small polycylinder \( \Delta \) containing \( x \) such that \( D \cap \Delta \) is given by \( z_1 \cdots z_l = 0 \) in local holomorphic coordinates \( z_1, ..., z_d \). Let \( T_i, 1 \leq i \leq l, \) be the monodromy transformation of \( \tilde{g}_{\Delta \setminus D} \) around \( D_i. \) Let \( \Omega^1_X(\bar{V})|_\Delta \subset \Omega^\bullet_X(D) \otimes \bar{V}|_\Delta \) be generated over \( \mathcal{O}_\Delta \) by \( \Omega^1_\Delta \otimes \mathcal{O}_\Delta \bar{V}|_\Delta \) and
\[ \Omega^1_\Delta(D_i) \otimes \mathbb{C} (\text{Ker}(T_i - \text{Id}))^\perp, \]
where \( (\text{Ker}(T_i - \text{Id}))^\perp \) is the orthogonal complement of the local subsystem \( \text{Ker}(T_i - \text{Id}) \) of \( \tilde{g}_{\Delta \setminus D_i} \), and \( 1 \leq i \leq l. \) Similarly we define the groups \( \tilde{\Omega}^p_X(\bar{V})|_\Delta. \) Those sheaves glue together nicely to a subcomplex \( (\tilde{\Omega}^\bullet_X(\bar{V}), \nabla) \) of the complex of sheaves of holomorphic differential forms on \( X \) with logarithmic poles along \( D. \)

We will use the notation \( d' \) and \( d'' \) respectively for the holomorphic and anti-holomorphic covariant derivatives corresponding to \( \nabla. \)
We define $L^{p,q}(\tilde{\mathfrak{g}}(2))$ to be the sheaf of measurable $(p, q)$-forms $\alpha$ on $U$ with values in $\tilde{\mathfrak{g}}$ such that both $\alpha$ and $(d' + d'')\alpha$ are locally square-integrable. Here as usual we have the Poincaré metric near $D$ and the Kähler metric on $U$ which are compatible. It means that the Kähler metric near $x \in D$ has the same asymptotic form as the Poincaré metric on $\Delta \cap D$, where $\Delta$ is a small polycylinder centered at $x$ (cf. [12]). Set

$$L^{p,q}(U, \tilde{\mathfrak{g}}(2)) := \Gamma(U, L^{p,q}(\tilde{\mathfrak{g}}(2))).$$

We let

$$H^{p,q}(U, \tilde{\mathfrak{g}})(2) := H^q(L^{p,\bullet}(U, \tilde{\mathfrak{g}}(2)), d'').$$

Analogously one defines the space $H^k(U, \tilde{\mathfrak{g}})(2)$. Let us recall the following result due to Timmerscheidt [11]:

**PROPOSITION 2.1**

(a) $H^{p,q}(U, \tilde{\mathfrak{g}})(2)$ is conjugate-isomorphic to $H^{q,p}(U, \tilde{\mathfrak{g}}')(2)$, where $\tilde{\mathfrak{g}}'$ is the local system dual to $\tilde{\mathfrak{g}}$ and

$$H^k(U, \tilde{\mathfrak{g}})(2) \simeq \bigoplus_{p+q=k} H^{p,q}(U, \tilde{\mathfrak{g}})(2).$$

(b) The complexes of sheaves $\tilde{\Omega}^p_X(\tilde{V})$ and $L^{p,\bullet}(\tilde{\mathfrak{g}})(2)$ are quasi-isomorphic; thus

$$H^q(X, \tilde{\Omega}^p_X(\tilde{V})) \simeq H^{p,q}(U, \tilde{\mathfrak{g}})(2).$$

(c) The spectral sequence

$$E_1^{p,q} = H^q(X, \tilde{\Omega}^p_X(\tilde{V}))$$

degenerates at $E_1$, abuts to $H^{p+q}(X, j_*\tilde{\mathfrak{g}})$ with

$$H^k(X, j_*\tilde{\mathfrak{g}}) \simeq \bigoplus H^q(X, \tilde{\Omega}^p_X(\tilde{V})).$$

(d) We have a conjugate linear isomorphism

$$H^q(X, \tilde{\Omega}^p_X(\tilde{V})) \simeq H^p(X, \tilde{\mathfrak{g}}'^q_X(\tilde{V}')),$$

where $\tilde{V}'$ is the canonical extension of $V'$ - the flat bundle corresponding to $\tilde{\mathfrak{g}}'$. 

7
3 Construction of the DGLA and formality

In this section everything is defined over the field of complex numbers. A differential graded Lie algebra (DGLA) \((L, d)\) consists of a graded Lie algebra

\[ L = \bigoplus_{i \geq 0} L^i, \quad [\cdot, \cdot] : L^i \times L^j \to L^{i+j}, \]

satisfying for \(\alpha \in L^i, \beta \in L^j,\) and \(\gamma \in L^k:\)

\[ [\alpha, \beta] + (-1)^{ij}[\beta, \alpha] = 0, \]

\[ (-1)^{ki}[\alpha, [\beta, \gamma]] + (-1)^{ij}[\beta, [\gamma, \alpha]] + (-1)^{jk}[\gamma, [\alpha, \beta]] = 0 \]

together with a derivation \(d\) of degree 1 (also called a differential):

\[ d : L^i \to L^{i+1}, \quad d \cdot d = 0, \quad d[\alpha, \beta] = [d\alpha, \beta] + (-1)^i[\alpha, d\beta]. \]

The cohomology of any DGLA is a DGLA too, considered with zero differential. We say that two DGLAs \((L_1, d_1)\) and \((L_2, d_2)\) are quasi-isomorphic if there exists a third DGLA \((L_3, d_3)\) and DGLA homomorphisms \(i\) and \(p:\)

\[ (L_1, d_1) \xleftarrow{i} (L_3, d_3) \xrightarrow{p} (L_2, d_2) \]

such that both \(i\) and \(p\) induce isomorphisms in cohomology. We also say that a DGLA is formal if it is quasi-isomorphic to its cohomology.

Next we construct a double complex version of a DGLA \((B^{\bullet, \bullet}(\tilde{g}), d', d'')\) such that the corresponding ordinary DGLA \((B^{\bullet}(\tilde{g}), d' + d'')\) controls deformations of \([\rho_0] \in \mathcal{M}_N\). The space \(B^{p,q}(\tilde{g})\) is defined as the space of global sections of a sheaf \(B^{p,q}(\tilde{g})\). If \(x \in U\), and \(\Delta\) is a small polydisc around \(x\), then the sections of \(B^{p,q}(\tilde{g})(\Delta)\) are the smooth differential forms of type \((p, q)\) on \(\Delta\) with coefficients in \(\tilde{g}\). Let us now consider a point \(x \in D\) and a polydisc \(\Delta\) containing this point such that \(\Delta \cap D\) is given in local coordinates \(z_1, ..., z_d\) by \(z_1 \cdots z_k = 0\). Locally, since we have a semi-simple local system and a normal crossing divisor, the local fundamental group is commutative (and isomorphic to \(\mathbb{Z}^k\)) and the local system splits into the direct sum of local systems of rank one. Therefore, we may assume that \(\tilde{g}\) has rank 1. We also may assume that the monodromy around \(D_i := \{z_i = 0\}\) is non-trivial with the eigenvalue \(e^{-\sqrt{-1}\theta_i}\) with \(0 < \theta_i < 2\pi\). Otherwise, we can ignore the components of the divisor with trivial monodromy. Any ordinary homogeneous differential form \(\omega\) of type \((p, q)\) we represent as

\[ \omega = f(z_1, ..., z_d)dz_1^{\delta_1} \wedge \cdots \wedge dz_d^{\delta_d} \wedge dz_1^{\sigma_1} \wedge \cdots \wedge dz_d^{\sigma_d}, \]
\[ \delta_i, \sigma_j \in \{0, 1\}, \quad \sum \delta_i = p, \quad \sum \sigma_i = q. \]

**Definition 3.1** We say that a differential form \( \omega \) of type \((p, q)\) given by
\[
\omega = f(z_1, \ldots, z_d) \, dz_1^{\delta_1} \wedge \cdots \wedge dz_d^{\delta_d} \wedge dz_1^{\sigma_1} \wedge \cdots \wedge dz_d^{\sigma_d}
\]
satisfies condition C if \( f(z_1, \ldots, z_d) \) is smooth over \( \Delta \setminus D \), is bounded for \( p = q = 0 \), and for \( p + q > 0 \) has the following asymptotics:
\[
f(z_1, \ldots, z_n) = O(r_1^{\varepsilon - \delta_1 - \sigma_1} \cdots r_k^{\varepsilon - \delta_k - \sigma_k})
\]
for some \( \varepsilon > 0 \).

If we have locally on \( \Delta \setminus D \) a multi-valued horizontal section \( s: \nabla s = 0 \) then the monodromy operator \( T_i \) corresponding to \( D_i \) acts on it as \( T_i s = e^{-\sqrt{-1} \theta_i} s \). Now we will say that \( \alpha \otimes s \in B^{p,q}(\tilde{\mathfrak{g}})(\Delta) \) for a homogeneous differential form \( \alpha \) of type \((p, q)\) if \( \alpha, d'\alpha, d''\alpha \), and \( d'd''\alpha \) all satisfy condition C.

It is clear that defined in such a way spaces of sections glue nicely together to give a double complex of sheaves \( B^{\bullet, \bullet}(\tilde{\mathfrak{g}}) \) and we get a double complex \((B^{\bullet, \bullet}(\tilde{\mathfrak{g}}), d', d'')\) by taking the spaces of global sections of the above double complex of sheaves. One easily checks that the cup-product of differential forms combined with Lie bracket as the coefficient pairing endowes \( B^{\bullet, \bullet}(\tilde{\mathfrak{g}}) \) with the structure of DGLA.

Our next goal is to prove the following

**Proposition 3.2** The spectral sequence defined by the filtration associated to either degree of the double complex \((B^{p,q}(\tilde{\mathfrak{g}}), d', d'')\) degenerates at \( E_1 \). The two induced filtrations on \( H^k \) are \( k \)-opposite.

We need to establish certain results analogous to the classical \( d'' \)-Poincaré Lemma. First, we prove the result in complex dimension one. Let \( \Delta \subset \mathbb{C} \) be a a disc centered at zero, let \( \overline{\Delta} \) be its closure, and let \( \Delta^\ast \) be the disc \( \Delta \) punctured at zero. The number \( \varepsilon \) is always assumed to be a small positive real number.

**Lemma 3.3** Given a complex \( C^\infty \) function \( f \) in an open punctured neighbourhood \( W_1^\ast \) of \( \overline{\Delta} \) such that the asymptotics of \( f \) at zero is as \( f = O(r^{\varepsilon - k}) \),
there exists a $C^\infty$ function $g = O(r^{\varepsilon-k+1})$ in an open punctured neighbourhood $W_2^* \subset W_1^*$ of $\Delta$ such that
\[
\frac{\partial g}{\partial \bar{z}} = f
\]
in $W_2^*$. Moreover, if $f$ is $C^\infty$ or has asymptotics like above in some additional parameters, then $g$ can be chosen to have the same properties.

**Proof.** The main idea of the proof is that the controlled asymptotics at the origin allows us to use Fourier analysis, which turns out to be an appropriate tool in this context. We notice that the function $h(z) = r^{4-\varepsilon} f$ is of class $C^2$, and $h, \frac{\partial h}{\partial r}, \frac{\partial^2 h}{\partial r^2}$ vanish at the origin.

We decompose
\[
h(z) = \sum_{n \in \mathbb{Z}} e^{\sqrt{-1}n\phi} h_n(r),
\]
where
\[
h_n(z) = \frac{e^{-\sqrt{-1}n\phi}}{2\pi} \int_0^{2\pi} h(e^{\sqrt{-1}\phi} z) e^{-\sqrt{-1}n\phi} d\phi
\]
(clearly, it is a function of $r$). Each function $h_n(r)$ is also of class $C^2$ with vanishing $h_n, \frac{\partial h_n}{\partial r}, \frac{\partial^2 h_n}{\partial r^2}$ at the origin. Moreover, integration by parts shows that
\[
\sup ||h_n(r)|| \leq \frac{C}{n^2} \sup ||\frac{\partial^2 h_n}{\partial r^2}||
\]
for a universal constant $C$, which shows that the above series is absolutely convergent.

Now we let
\[
g_n(r) = -r^{4-\varepsilon-n} \int_r^1 2\rho^{n-4+\varepsilon} h_n(\rho) d\rho
\]
and we immediately see that $g_n(r)$ is of class $C^3$ and vanishes at zero together with its first, second, and third derivatives. Furthermore, one sees easily that the series
\[
j(z) = \sum_{n \in \mathbb{Z}} e^{\sqrt{-1}(n-1)\phi} g_n(r)
\]
is absolutely convergent together with its first derivative on compact subsets of $W_1$. Finally, we let $g(z) = j(z)r^{\varepsilon-4}$ and using the identity
\[
\frac{\partial}{\partial \bar{z}} = \frac{e^{\sqrt{-1}\phi}}{2} \frac{\partial}{\partial r} + \frac{\sqrt{-1}e^{\sqrt{-1}\phi}}{2r} \frac{\partial}{\partial \phi}
\]
we conclude that the function $g(z)$ is a well-defined smooth $C^\infty$ function in $W_2^*$ and satisfies $\frac{\partial g(z)}{\partial \bar{z}} = f(z)$. The fact that the function $g(z)$ has the desired asymptotics is a consequence of our explicit formulae.

The following statement is local in nature and is a direct consequence of the above Lemma 3.3 and the standard separation of variables arguments which can be found e.g. in Vol.I E, Theorem 3.

**LEMMA 3.4** For $q > 0$, $x \in X$ let $\omega$ be an element of $B^{p,q}(\tilde{g})$ defined in an open neighbourhood of a polydisc $\Delta$ containing $x$ such that $d''\omega = 0$. Then there exists $\alpha \in B^{p,q-1}(\tilde{g})$ defined in an open neighbourhood of $\Delta$ such that $\omega = d''\alpha$.

This Lemma allows us to identify $E_1$ of the spectral sequence of the double complex $B^{p,q}(\tilde{g})$ corresponding to the holomorphic filtration. The local exactness of the columns of the $E_{p,0} := B^{p,q}(\tilde{g})$ is equivalent to Lemma 3.4 and the statement that the kernel of the sheaf homomorphism $d'' : B^{p,0}(\tilde{g}) \to B^{p,1}(\tilde{g})$

is the sheaf $\tilde{\Omega}_X^p(V)$. For simplicity, we will discuss the case $d = 1$ in a polydisc $\Delta \ni x \in D$ when the monodromy around $z := z_1 = 0$ is equal to $e^{-\sqrt{-1}\theta}$, where $0 \leq \theta < 2\pi$. Thus if we have a multi-valued horizontal section $s$ such that $\nabla s = 0$ then the monodromy operator $T$ acts on it as $Ts = e^\theta s$.

Let us consider the uni-valued section $\mu = z^{ \theta/2\pi} s$ which spans the canonical (Deligne [3]) holomorphic extension $\tilde{V}$ of $\tilde{g}$. We have:

$$\nabla \mu = \frac{\theta}{2\pi} \frac{dz}{z} \otimes \mu$$

and the Leibnitz rule also imply that for any local section $f(z) \in B^{1,0}(\tilde{g})(\Delta)$, we have

$$d''(f \otimes \mu) = d'' f \otimes \mu$$

and the asymptotics of $f \otimes \mu$ is clearly $O(e^{-1-\frac{\theta}{2\pi}})$. Thus $d'' f = 0$ implies that $f$ is necessarily a holomorphic 1-form with at most logarithmic singularities at $z = 0$ when $\theta > 0$, and is a holomorphic one-form in $\Delta$ when $\theta = 0$.

Similar arguments apply straightforwardly in higher dimension situations to prove

$$E_{1}^{p,q} = H^{q}(X, \tilde{\Omega}_X^p(V)).$$
The construction of our bigraded DGLA $B^{\bullet\bullet}(\tilde{g})$ was performed in order to ensure a certain symmetry with respect to complex conjugation. In particular, if we consider our double complex with the anti-holomorphic filtration, then the spectral sequence $\bar{E}$ associated to it will have $\bar{E}_1^{p,q} = H^p(\bar{X}, \tilde{\Omega}_X^q(\bar{V}))$. Here $\bar{V}$ is the anti-holomorphic extension of $\tilde{g}$ and $\bar{X}$ is the manifold $X$ considered with the complex conjugate complex structure. For example, when $d = 1$ and we have rank one local system $\tilde{g}$ with horizontal multi-valued section $s$ as above then $\bar{V}$ is spanned by $z^{1-\frac{q}{2\pi}}s$.

It follows that each of our spectral sequences identifies with the spectral sequence for the $L_2$ double complex, considered in Proposition 2.1. More precisely, $E_1^{p,q}$ identifies with $H^{p,q}(U, \tilde{g})(2)$, and $E_1^{p,q}$ with $H^{p,q}(U, \tilde{g}')(2)$. Hence, by Proposition 2.1 the spectral sequences degenerate at $E_1$ and the induced filtrations on $E_1 = \bar{E}_1 = E_{\infty}$ produce $k$-opposite filtrations on $H^k(U, \tilde{g})(2)$. This completes the proof of Proposition 3.2.

Next, we apply Proposition 5.17 from [5] which tells us that our Proposition 3.2 is in fact equivalent to the $d'd''$-Lemma for the DGLA $B^{p,q}(\tilde{g})$. Repeating verbatim the proof of The Main Theorem from the same source [5] then assures us that for the diagram

$$(B^\bullet(\tilde{g}), d' + d'') \xleftarrow{i_*} \text{Ker}[d' : B^\bullet(\tilde{g}) \to B^\bullet(\tilde{g})], d'' \xrightarrow{p} (H^\bullet(X, j_*\tilde{g}), 0)$$

the maps $i^*$ and $p^*$ induce isomorphisms on cohomology. We conclude with

**THEOREM 3.5** The DGLA $(B^\bullet(\tilde{g}), d' + d'')$ is formal.

In fact, the complex of $L_2$ differential forms $L^\bullet(U, \tilde{g})(2)$ is not necessarily itself a DGLA, and this is the main reason that we had to construct the DGLA $B^\bullet(\tilde{g})$ which has the right cohomology and therefore adequately reflects our deformation problem. We further notice that we have a natural inclusion of double complexes: $B^{p,q}(\tilde{g}) \hookrightarrow L^{p,q}(U, \tilde{g})(2)$. We immediately have

**PROPOSITION 3.6** This inclusion induces an isomorphism on cohomology.

### 4 The deformation theory of flat bundles

In this section we follow ideas in the paper of Goldman and Millson [7]. Basically we shall adapt key results of their works to our situation, which is
different, because our Kähler manifold is not compact and we have restrictions on monodromy transformations around $D_i$'s. We shall adopt categorical language and our major claims will be stated as equivalences of certain groupoids. For complete and exhaustive treatment applied in many situations we refer the reader to [7].

First, we introduce an Artin local algebra $A$ over $\mathbb{C}$ (which has residue field $\mathbb{C}$) with maximal ideal $m$. We assume $A$ to be unital so there are well-defined maps $\mathbb{C} \hookrightarrow A \twoheadrightarrow \mathbb{C}$ of inclusion and projection onto the residue field.

Let us have a unitary irreducible representation $\rho_0: \pi_1(U) \rightarrow G$ defining a point $[\rho] \in \mathcal{M}_A$, i.e. $\rho_0$ sends the class $[\gamma_0]$ of a designated simple loop $\gamma_i$ around $D_i$ to the fixed conjugacy class $C_i \subset G$ for $1 \leq i \leq r$. This representation defines a flat unitary bundle $(W, \nabla_0)$ on $U$ with the property that the monodromy transformation around $D_i$ lies in $C_i$, $1 \leq i \leq r$. In this situation the bundle $V$ corresponds to the bundle of skew-adjoint endomorphisms of $W$.

Let us consider the groupoid $\tilde{\mathcal{F}}_A(\nabla_0)$ such that

- its objects are $A$-linear integrable smooth connections $\nabla$ over $U$ with values in $V \otimes A$
- these connections deform $\nabla_0$, i.e. $p(\nabla) = \nabla_0$
- any element $\nabla \in \tilde{\mathcal{F}}_A(\nabla_0)$ has the property that $\alpha := i(\nabla_0) - \nabla$, $d'\alpha$, $d''\alpha$, and $d'd''\alpha$ satisfy condition C from Section 3, where the map $i$ is induced by the inclusion $i: \mathbb{C} \rightarrow A$.

The morphisms of this groupoid are given by the group of gauge equivalences $\mathcal{G}_A$ with values in $A$, i.e. smooth bundle automorphisms of $V \otimes A$ over $X$ which preserve $\nabla$.

Next we recall the DGLA $(B^\bullet(\tilde{g}), d' + d'')$ and the quadratic map

$$Q_A: B^1(\tilde{g}) \otimes m \rightarrow B^2(\tilde{g}) \otimes m, \quad Q_A(\alpha) = (d' + d'')\alpha + \frac{1}{2}[\alpha, \alpha].$$

The subspace $Q_A^{-1}(0)$ is preserved by the natural action of $\exp(B^0(\tilde{g}) \otimes m)$. This action makes the set of objects $Q_A^{-1}(0)$ into a groupoid $\tilde{C}_A$. We also notice that $d' + d''$ is the decomposition of the covariant derivative corresponding to $\nabla_0$. 

13
Given any $\nabla_0$, Goldman and Millson in [7] define uniquely an element $\tilde{\nabla}_0 \in \tilde{\mathcal{F}}_A(\nabla_0)$ called the extension of $\nabla_0$. Take $\nabla \in \tilde{\mathcal{F}}_A(\nabla_0)$; the difference $\nabla - \tilde{\nabla}_0$ is an element of $B^1(\tilde{g}) \otimes \mathfrak{m}$, because it clearly has the right asymptotics modulo $\mathfrak{m}$ by definition of $\tilde{\mathcal{F}}_A(\nabla_0)$. Proposition 6.6 from [7] now translates as

**Proposition 4.1** The above correspondence defines an isomorphism of groupoids

$$\tilde{\mathcal{F}}_A(\nabla_0) \to \tilde{\mathcal{C}}_A$$

depending naturally upon $A$.

To throw a bridge between deformations of a flat irreducible connection $\nabla_0$ and the corresponding representation $\rho_0$ we shall need another groupoid $\tilde{\mathcal{R}}_A(\rho_0)$ whose objects are representations of $\pi_1(U, b)$ into $G_A$ - the group of $A$-points of $G$ which deform $\rho_0$ (i.e. $p(\rho) = \rho_0$) subject to the condition that for $\rho \in \tilde{\mathcal{R}}_A(\rho_0)$ the element $\rho(\gamma_i)$ is conjugate to $\rho_0(\gamma_i)$ for $1 \leq i \leq r$. We also require that if the intersection of several components $D_{i_1}, ..., D_{i_s}$ of the divisor is non-empty, then $\rho(\gamma_{i_1}), ..., \rho(\gamma_{i_s})$ are simultaneously conjugate in the group $G_A$ to $\rho_0(\gamma_{i_1}), ..., \rho_0(\gamma_{i_s})$ respectively. The set of morphisms in this groupoid is provided by the action of the group $\exp(\mathfrak{g} \otimes \mathfrak{m})$.

We shall need the following well-known direct consequence of Nakayama’s lemma:

**Lemma 4.2** Let $A$ be a noetherian local algebra over $\mathbb{C}$ and let $f_A : M_A \to N_A$ be an $A$-linear map between two free finitely generated $A$-modules. Then if

$$f_A \otimes Id : M_A \otimes_A \mathbb{C} \to N_A \otimes_A \mathbb{C}$$

is an isomorphism, then $f_A$ is an isomorphism as well.

Let us define the map $\varepsilon_b : G_A \to G_A$ given as the evaluation at the base point $b$. We recall the monodromy functor [7] (where it is called holonomy, though) $(\text{mon}_b, \varepsilon_b)$ between $\tilde{\mathcal{F}}_A(\nabla_0)$ and $\mathcal{R}_A(\rho_0)$ defined naturally. Now we will adapt Proposition 6.3 from [7] to our purposes:

**Proposition 4.3** The functor

$$(\text{mon}_b, \varepsilon_b) : \tilde{\mathcal{F}}_A(\nabla_0) \to \tilde{\mathcal{R}}_A(\rho_0)$$

is well-defined and is an equivalence of groupoids.
Proof. To prove that this functor is well-defined as well as surjective on isomorphism classes, we shall construct an inverse image $\text{mon}^{-1}_b(\rho)$ for any $\rho \in \tilde{\mathcal{R}}_A(\rho_0)$ and show that it does belong to $\tilde{\mathcal{F}}_A(\nabla_0)$. (There does not exists a natural quasi-inverse functor to $(\text{mon}_b, \varepsilon_b)$, though.) Our procedure is as follows: first we have the standard construction of a flat bundle $W_A$ out of a representation $\pi_1(U, b) \to G_A$. Then we consider the canonical extension $\bar{W}_A$ which is a holomorphic bundle over $X$ with an $A$-action such that all fibers are free $A$-modules. Let us consider for any $l \in \pi_1(U, b)$ the following diagram:

$$
\begin{array}{ccc}
A^N & \xrightarrow{\varphi(l)} & A^N \\
\downarrow p & & \downarrow p \\
C^N & \xrightarrow{\rho_0(l)} & C^N
\end{array}
$$

This diagram is clearly commutative since $p(\rho) = \rho_0$. This means that $\bar{W}_A \otimes_A C$ is isomorphic to $\bar{W}_C$. Then Lemma 4.2 implies that $\bar{W}_A$ is (non-uniquely) isomorphic to $\bar{W}_C \otimes C$. Besides, since $\rho(\gamma_i)$ is conjugate in $G_A$ to $\rho_0(\gamma_i)$ for $1 \leq i \leq r$ the eigenvalues (which a priori belong to $A$) do not change and therefore the local system $\tilde{g} \otimes A$ as well as $\tilde{g}$ splits locally into a direct sum of local system of rank 1 (over $A$). Now one sees directly from the arguments used in 3.4 that the residues of $\text{mon}_b^{-1}(\nabla)$ are the same as of $\nabla_0$. For the rest of the proof one copies arguments from the proof of Proposition 6.3 in [7].

Now we are ready to continue this section with a result similar in all ways to Theorem 6.8 of [7]. Let $L'$ denote the augmentation ideal

$$L' := \text{Ker}[B^*(\tilde{g}) \to \tilde{g}],$$

where $k$ is the differential of the map $\varepsilon_b : \exp(B^0(\tilde{g})) \to G$ at the identity. Analogously to the groupoid $\tilde{C}_A$ one defines the groupoid $\tilde{C}_A(L').$

**THEOREM 4.4** The analytic germ of $\tilde{\mathcal{R}}(\pi_1(U), G)_N$ at $\rho_0$ pro-represents the functor

$$A \to \text{Iso}(\tilde{C}_A(L')).$$

Now the proof of Theorem 1 in [7] immediately implies that the functor

$$A \to \text{Iso}(\tilde{C}_A(L'))$$

15
from the above Theorem is pro-represented by the analytic germ of the quadratic cone consisting of all \( u \in H^1(X, j_* \tilde{g}) \) such that \([u, u] = 0\). Therefore we conclude

**THEOREM 4.5** Let \( \rho \) be a representation \( \pi_1(U) \to G \) such that \( \rho(\gamma_i) \in C_i \). Then \( \tilde{R}(\pi_1(U), G)_N \) is quadratic at \( \rho \).

5 Moduli of representations of \( \pi_1(U) \)

In this section we deduce the consequences of results obtained earlier in this paper to find out the geometric structure of the moduli space \( \mathcal{M}_N \). This space was studied by J.-L. Brylinski and the author in [2]. In that paper the condition of vanishing of the second relative group cohomology group \( H^2(\pi_1(U), (\Gamma_i)_i, g) \) was imposed to conclude that \( \mathcal{M}_N \) is smooth at a point \([\rho_0] \in \mathcal{M}_N\); here again \( g \) is a \( \pi_1(U) \)-module via the adjoint representation followed by \( \rho_0 \). Under the same assumption, the tangent space \( T_{[\rho]} \mathcal{M}_N \) was identified as the first intersection cohomology group (middle perversity) \( IH^1(X, \tilde{g}) \). Other possible descriptions of \( T_{[\rho]} \mathcal{M}_N \) include relative group cohomology group \( H^1(\pi_1(U), (\Gamma_i)_i, g) \) the first cohomology group \( H^1(X, j_* \tilde{g}) \) of our controlling DGLA, and the first \( L_2 \) cohomology group \( H^1(U, \tilde{g})_2 \). In the same paper [2] it was proven that if the smooth locus is reduced then it is symplectic and in the case when \( X \) is quasi-projective, then it actually is a Kähler manifold.

When we pass to the quotient

\[ \mathcal{M}_N = \tilde{R}^{irr}(\pi_1(U), G)_N / G, \]

due to the fact that we consider only irreducible representations and therefore all \( G \)-orbits on \( \tilde{R}^{irr}(\pi_1(U), G)_N \) are compact and equidimensional, we do not acquire additional singularities. Therefore Theorem [2.13] combined with Section 3 of [2] imply

**THEOREM 5.1** The singularities of the moduli space \( \mathcal{M}_N \) are quadratic.

Let us take a point \([\rho] \in \mathcal{M}_N\); as was found in [2] for this point to be smooth, vanishing of an infinite number of obstructions in \( H^2(\pi_1(U), (\Gamma_i)_i, g) \)
is sufficient. The first obstruction in this series is given by the pairing

$$H^1(\pi_1(U), (\Gamma_i), g) \times H^1(\pi_1(U), (\Gamma_i), g) \to H^2(\pi_1(U), (\Gamma_i), g), \quad (5.1)$$

defined as cup product in group cohomology together with Lie bracket as coefficient pairing. Since we know now that singularities of the space $\mathcal{M}_N$ are quadratic, this obstruction is the only one.

**THEOREM 5.2** A point $[\rho] \in \mathcal{M}_N$ is a smooth point of $\mathcal{M}_N$ if the pairing $[5.1]$ is identically zero.

The same conclusion about smoothness can be drawn if the pairing

$$[,] : H^1(X, j_*\tilde{g}) \times H^1(X, j_*\tilde{g}) \to H^2(X, j_*\tilde{g})$$

is identically zero.

**References**

[1] O. Biquard, Fibrés de Higgs at connexions intégrables: la cas logarithmique (diviseur lisse), *Ann. scient. Éc. Norm. Sup.*, 30, 1997, 41-96

[2] J.-L. Brylinski and P. A. Foth, Moduli of Flat Bundles on Open Kähler Manifolds, preprint [alg-geom/9703011](https://arxiv.org/abs/alg-geom/9703011)

[3] P. Deligne, Equations Différentielles a Points Singuliers Réguliers, *Lect. Notes in Math.* 163, Springer-Verlag, 1970

[4] P. Deligne, Théorie de Hodge II, *Publ. Math. IHES*, 40, 1971, 5-58

[5] P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan, Real homotopy theory of Kähler manifolds, *Invent. Math.*, 29, 1975, 245-274

[6] R. Gunning, Introduction to holomorphic functions of several variables, Vol. I, II, III, *Wadsworth & Brooks/Cole*, 1990

[7] W. Goldman and J. Millson, The deformation theory of representations of fundamental groups of compact Kähler manifolds, *Publ. Math. IHES*, 67, 1988, 43-96

17
[8] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, *Ann. of Math.*, 79, 1964, no. 1 & 2

[9] M. Kapovich and J. Millson, The relative deformation theory of representations and flat connections and deformations of linkages in constant curvature spaces, *Composito Math.*, 103, 1996, 287-317

[10] M. Schlessinger and J. Stasheff, Deformation theory and rational homotopy type, *U. of North Carolina preprint*, 1979; short version: The Lie algebra structure of tangent cohomology and deformation theory, *J. of Pure and Applied Alg.*, 38, 1985, 313-322

[11] K. Timmerscheidt, Hodge decomposition for unitary local sysems, Appendix to: H. Esnault, E. Viehweg, Logarithmic de Rham complexes and vanishing theorems, *Invent. Math.*, 86, 1986, 161-194

[12] S. Zucker, Hodge theory with degenerating coefficients: $L_2$-cohomology in the Poincaré metric, *Ann. Math.*, 109, 1979, 415-476

Department of Mathematics
Penn State University
University Park, PA 16802
foth@math.psu.edu

*AMS subj. class.*: primary 32G08; secondary 20L05, 17B70.