CONVEX-CONCAVE BODY IN $\mathbb{RP}^3$ CONTAINS A LINE

A. KHOVANSKII, D. NOVIKOV

Abstract. We define a class of $L$-convex-concave subsets of $\mathbb{RP}^3$, where $L$ is a projective line in $\mathbb{RP}^3$. These are sets whose sections by any plane containing $L$ are convex and concavely depend on this plane. We prove a version of Arnold hypothesis for these sets, namely we prove that each such set contains a line.

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Date: March 29, 2022.
Khovanskii’s work is partially supported by Canadian Grant N 0GF0156833. Novikov’s work was supported by the Killam grant of P. Milman and by James S. McDonnell Foundation.
1. Introduction

Consider a connected a closed hypersurface $M$ without a boundary embedded to $\mathbb{R}P^n$. Suppose that the second fundamental form of $M$ is everywhere negatively defined. It means that in some affine coordinates in $\mathbb{R}P^n$ the hypersurface is locally defined as $x_n = -x_1^2 - ... - x_{n-1}^2 + \text{higher order terms}$. A well-known theorem claims then that $M$ bounds a convex body in $\mathbb{R}P^n$, i.e. doesn’t intersects some hyperplane $H \subset \mathbb{R}P^n$ and bounds a convex body in the affine space $\mathbb{R}P^n \setminus H$.

Arnold (in [1]) conjectured that an analogue of this fact holds for any hypersurface with an everywhere non-degenerate second fundamental form. We will say that a quadratic form in $\mathbb{R}^{n-1}$ has signature $(n-k-1,k)$ if its restriction to some $k$-dimensional linear subspace is negatively defined and its restriction to some $n-k-1$-dimensional linear subspace is positively defined.

**Conjecture 1** (Arnold Conjecture). Consider a domain $U \subset \mathbb{R}P^n$, bounded by a connected smooth hypersurface $B$. Suppose that the second fundamental form of $B$ is non-degenerate at any point of $B$ and has signature $(n-k-1,k)$ (necessarily the same for all points) with respect to the outward normal. Then there exist a projective subspace $L^k$ of dimension $k$ contained in $U$ and a projective subspace $L^{n-k-1}$ of dimension $n-k-1$ not intersecting $U$.

**Example 1.** Domain $U = \{(x_0, ..., x_n) \in \mathbb{R}P^n | x_0^2 + x_1^2 + x_2^2 - x_3^2 - ... - x_n^2 \leq 0\}$, bounded by a quadric, satisfies to the conditions and conclusions of this conjecture.

**Example 2.** For $k = n-1$ the conditions imposed on $B$ in the conjecture coincide with the conditions of the theorem, and the claim of the conjecture means existence of a hypersurface not intersecting $U$ and of an interior point of $U$. So for $k = n-1$ the conjecture follows from the theorem above.

1.1. **Affine version of the Arnold conjecture.** There is an affine version of the Arnold conjecture: in the statement of the conjecture $\mathbb{R}P^n$ is changed to $\mathbb{R}^n$ and projective subspaces to the affine one. We prove it (in [2]) for surfaces asymptotically approaching to the quadratic cone $K = \{x_1^2 + ... + x_k^2 - x_{k+1}^2 - ... - x_n^2 = 0\}$ as $|x_n| \to +\infty$. This condition in particular guarantees the smoothness of the closure of these surfaces after embedding in $\mathbb{R}P^n$.

However, in the case of slightly different asymptotical behavior the claim is wrong already for $k = 1, n = 3$. Consider a union $K' \subset \mathbb{R}^3$ of moved apart halves of $K$ (e.g. $K' = \{(x,y,z) \mid x^2 + y^2 = (|z| - 1)^2, |z| \geq 1\}$). We construct (in [2]) an example of a domain in $\mathbb{R}^3$ not containing lines, satisfying conditions of the affine version of Arnold conjecture and which boundary asymptotically, as $|z| \to \infty$, approaches $K'$.

However, the closure of such domains in $\mathbb{R}P^3$ will be non-smooth. Moreover, it cannot be made smooth by small perturbation without creating points of degeneracy of the second fundamental form.

1.2. **L-convex-concave subsets of $\mathbb{R}P^3$.** In this paper we prove the first non-trivial case $(k = 1, n = 3)$ of the Arnold conjecture in some additional assumptions. Namely, for any projective subspace $L \subset \mathbb{R}P^n$ we define a class of $L$-convex-concave subsets of $\mathbb{R}P^n$.

**Definition 1.** A closed set $A \subset \mathbb{R}P^n$ is $L$-convex-concave if
1. $A \cap L = \emptyset$.
2. for any projective subspace $N \subset \mathbb{R}P^n$ of dimension $\dim L + 1$ and containing $L$ the intersection $A \cap N$ is convex,
3. for any projective subspace $T \subset L$ of dimension $\dim L - 1$ the complement to the image of $\pi(A)$ under projection $\pi : \mathbb{R}P^n \setminus T \to \mathbb{R}P^n/T$ is an open convex set.

In general the boundary of a $L$-convex-concave subset of $\mathbb{R}P^n$ need not be smooth, so the class of $L$-convex-concave domains is not included into the class of domains described in the Arnold conjecture. However, any $L$-convex-concave set after a suitable arbitrarily small perturbation will have a smooth and non-degenerate boundary and will satisfy conditions of the Arnold conjecture.

The inverse inclusion is also wrong: not all domains satisfying the conditions of Arnold conjecture are $L$-convex-concave for some $L$. The difference is twofold. First, in the very definition of the $L$-convex-concave domain we postulate the existence of one of the subspaces whose existence is claimed in the Arnold conjecture. Second, in the definition of $L$-convex-concave domains we suppose that all its sections by subspaces containing $L$ as a hyperplane are convex, which is a very strong assumption.

An analogue of the Arnold conjecture for $L$-convex-concave domains is the following

**Conjecture 2.** Any $L$-convex-concave domain $A \subset \mathbb{R}P^n$ contains a projective subspace of dimension equal to $n - \dim L - 1$.

In this paper we prove the first nontrivial case of this conjecture:

**Theorem 1.** Any $L^1$-convex-concave set $A \subset \mathbb{R}P^3$, $\dim L^1 = 1$, contains a projective line.

### 1.3. Structure of the paper

The proof of this theorem belongs in fact to the realm of the convex geometry. It heavily exploits the two fundamental theorems of the convex geometry: Helly theorem and the Browder theorem. Proof is partly guided by the general ideology of the Chebyshev best approximation. In particular, one of the key ingredients of the proof is an analogue of the Chebyshev alternance, see Lemma and Theorem.

Further we will consider only bodies $L$-convex-concave with respect to some fixed once and forever real projective line $L$. So we will use the term convex-concave for the $L$-convex-concave bodies.

Also, we will use an equivalent definition of a convex-concave set. Namely, in it is shown that the convex-concave subsets of $\mathbb{R}P^3$ can be characterized in the following way.

**Definition 2.** A body $B \in \mathbb{R}P^3$ is called projective convex-concave with respect to a line $L$ (further called infinite line) not intersecting $B$ if

- sections of $B$ by planes passing through this line (further called horizontal planes) are all convex and
- for any three such horizontal sections through any point of any of them passes a line intersecting two another.

**Remark 1.** One can define an affine analogue of projective convex-concave sets. Namely, a body $B \in \mathbb{R}^3$ is called affine convex-concave if, first, its horizontal sections
are all convex and, second, for any three horizontal sections through any point of the middle one passes a line intersecting two another.

In [6] we build a counterexample to an affine version of Arnold conjecture by smoothening a suitable affine convex-concave body.

The proof is organized as follows. In §2 we show that it is enough to prove existence of a line intersecting any five sections of the body, see Theorem 2. This is a standard application of the Helly theorem. From the other hand, using Browder theorem, we prove that for any four sections we can find a line intersecting all of them, see Theorem 4.

Starting from §3 we are dealing with five fixed sections of a convex-concave body. The general idea is simple. Fix an Euclidean metric on some affine cart in $\mathbb{R}P^3$ containing all five sections and take a line closest to these five sections (the Chebyshev line). Our goal is to prove that one can always find a line which lies closer to these five sections, unless the Chebyshev line intersects all five sections.

More exact, in §3 we introduce the Euclidean metric, define the Chebyshev line and prove its basic properties. On planes containing sections arise five half-planes with the property that any line lying closer to five sections than the Chebyshev line should intersect all these half-planes. The opposite is almost true. Namely, any line intersecting these half-planes (further called good deformation) produce a line closer to the sections than the Chebyshev line, see Lemma 5. So all we need to prove is the existence of a line intersecting these five half-planes, which depends on the projective properties of their mutual position only. These properties are the main object of further investigations.

At this stage a split occurs. We impose a condition of genericity on the collection of these half-planes (namely, their boundaries should be pairwise non-parallel) and deal further with non-degenerate cases only. In degenerate cases existence of the good deformation follows from Theorem 4 due to a remarkable self-duality of the condition of $L$-convex-concavity, see §3.4 and [5].

In §4 and §5 we investigate combinatorial properties of a collection of five half-planes corresponding to a Chebyshev line, forgetting for a moment the convex-concavity condition. In other words, we consider a more general problem of properties of a line closest to five convex figures on five parallel planes. This reduces to a purely combinatorial problem about possible arrangements of rooks on a chess board. We find an equivalent of the classical condition of Chebyshev alternance for our situation. Namely, only six possible combinatorial types of collections of half-planes are possible, see Theorem 8.

In §6 for each of these six types we prove existence of a good deformation using the convex-concavity condition. More exact, each of these combinatorial types have some continuous parameters (e.g. distances between sections). If a configuration of half-planes arose from a Chebyshev line, then these parameters should satisfy some inequalities. In other words, only part of the space of parameters corresponds to Chebyshev alternances. It turns out that configurations of half-planes arising from sections of a convex-concave body belong to the complement to this part.

Namely, using the combinatorial properties of each case, we are able to prove existence of a line intersecting four of the half-planes in a some particular sectors. These sectors are chosen in such a way that the line intersecting them should necessarily intersect the fifth half-plane and the existence of a good deformation follows.
2. Applications of the Helly theorem and of the Browder theorem

In this section we first introduce a linear structure on the set of all lines not intersecting the line $L$. We prove that the Theorem 1 follows from the fact that for any five sections of a convex-concave body there is a line intersecting all of them. Another result claims that for any four sections there is a line intersecting all of them.

2.1. Linear structure on the set of all non-horizontal lines. We will call a line non-horizontal if it doesn’t intersect the infinite line. We choose coordinates in a complement to some horizontal plane in such a way that the infinite line lies in the projective plane $\{z = 0\}$. In these coordinates non-horizontal lines have a parametrization of the type $x = az + b, y = cz + d$. This correspondence $\{\text{non-horizontal line}\} \to (a, b, c, d)$ defines coordinates on the set $U$ of all non-horizontal lines.

Remark 2. These coordinates are correlated with the affine structure in horizontal planes: intersection of a convex combination of two lines with a horizontal plane is a convex combination (with the same coefficients) of intersections of these two lines with this plane. Therefore the affine structure defined by these coordinates is independent of the choice of coordinates and depends on the choice of the infinite line only (however, the linear structure, i.e. the line with coordinates $(0,0,0,0)$ (=z-axis), can be chosen arbitrarily).

Denote by $U_t$ the set of all non-horizontal lines intersecting a horizontal section $S_t = B \cap \{z = t\}$ of a projective convex-concave body $B$. From the last remark we immediately see that

Lemma 1. $U_t$ is closed and is convex in the coordinates introduced above.

The inverse is also true. Namely, for any horizontal plane $\{z = t\}$ there is a map $\phi_t : U \to \{z = t\}$ mapping a non-horizontal line to its point of intersection with this plane.

Lemma 2. This map preserves convexity, i.e the image of a convex set is again a convex set.

2.2. Non-horizontal lines and sections of a convex-concave body.
2.2.1. Five sections: Helly theorem.

Theorem 2. The Theorem 1 follows from the following claim:

\[ \forall t_1, t_2, t_3, t_4, t_5 \in \mathbb{R} \quad \bigcap_{i=1}^{5} U_{t_i} \neq \emptyset \]

In other words, it is enough to prove that for any five horizontal sections \( S_i \) of \( \mathcal{B} \) there exists a line intersecting all of them.

Proof. Indeed, the Theorem 1 is equivalent to \( \bigcap_{t_i} U_{t_i} \neq \emptyset \). Since \( U_{t_i} \) are convex subsets of \( \mathbb{R}^4 \), the claim is almost a particular case \( (n = 4) \) of the classical Helly theorem:

Theorem 3 (Helly theorem, see [3, 4]). Intersection of a finite family of closed convex sets in \( \mathbb{R}^n \) is nonempty if and only if intersection of any \( n + 1 \) of them is nonempty.

The only problem is that the family \( U_{t_i} \) is not finite. However, one can circumvent this technicality using the fact that

Lemma 3. Intersection of any two different \( U_{t_i} \) is compact.

Indeed, any line belonging to \( U_{t_i} \cap U_{t_j} \) is uniquely defined by its points of intersection with these two sections, so \( U_{t_i} \cap U_{t_j} \) is homeomorphic to \( S_{t_1} \times S_{t_2} \), which is compact.

So, take a compact \( K = U_1 \cap U_0 \) and consider a family of sets \( \bar{U}_{t_i} = K \setminus U_{t_i} \). These sets are relatively open in \( K \). We want to prove that \( \bigcap_{t_i} U_{t_i} \neq \emptyset \). If not, then \( \bar{U}_{t_i} \) is a covering of \( K \), so we can take a finite family of \( \bar{U}_{t_i} \) covering \( K \). It means that the intersection of a finite family consisting of the corresponding \( U_{t_i} \) and \( U_1 \) and \( U_0 \) will be empty. This is impossible by Helly theorem if intersection of any five of \( U_{t_i} \) is nonempty.

2.3. Four sections: Browder theorem. It turns out that the convex-concavity condition (even the affine one) guarantees existence of a line passing through any four sections. We will prove this in slightly more general assumptions.

Theorem 4. Let \( A, B, C, D \) be four compact convex non-empty sets in \( \mathbb{R}^n \) satisfying the following condition:

1. \( A \subset \{ x_n = t_1 \} \), \( B \subset \{ x_n = t_2 \} \), \( C \subset \{ x_n = t_3 \} \), \( D \subset \{ x_n = t_4 \} \), where \( t_i \) are pairwise different;
2. through any point of \( B \) passes a line intersecting both \( A \) and \( C \), and
3. through any point of \( C \) passes a line intersecting both \( B \) and \( D \).

Then there exists a line intersecting all four bodies.

Remark 3. Here we use only part of conditions provided by convex-concavity.

We will use a Browder theorem — a fixed-point theorem for upper semi-continuous set-valued mappings, see [2].

Let \( f : X \rightarrow \text{Set}(X) \) be a mapping from \( X \) to the set of all subsets of \( X \).

Definition 3. \( f \) is called upper semi-continuous on \( X \) if for any \( x_0 \in X \) and any open set \( G \) containing \( f(x_0) \) there exists a neighborhood \( U \) of \( x_0 \) such that \( f(x) \subset G \) for all \( x \in U \).
Lemma 5. Mapping Lemma 4. \( f \)

\[
\{ \text{set of all points of } U \} \subset V
\]

Proof. The proof is the same for both \( x \) it means that the line passing through \( D \) both \( B \) one-point subsets of \( B \) and any \( \epsilon > 0 \) there exist a \( \delta > 0 \) such that if \( U' \subset N_\delta(U) \) then \( h_1(U') \subset G = N_{2\epsilon}(h_1(U)) \). The mapping \( h_2 \) is also upper semi-continuous.

Remark 4. For single-valued maps this property means continuity.

Our theorem follows from the following result of Browder:

Theorem 5 (see [2]). Let \( X \) be a non-empty compact convex set in a real, locally convex, Hausdorff topological vector space \( E \). Let \( f \) be an upper-semicontinuous set-valued mapping defined on \( X \) such that for each \( x \in X \), \( f(x) \) is a non-empty closed convex subset in \( X \). Then there exists a point \( \hat{x} \in X \) with \( \hat{x} \in f(\hat{x}) \).

We will apply this theorem to the composition \( f : B \to CSet(B) \) of the tautological map \( B \to CSet(B) \) and two maps \( h_1 : CSet(B) \to CSet(C) \) and \( h_2 : CSet(C) \to CSet(B) \), where \( CSet(B) \) and \( CSet(C) \) are sets of all compact convex subsets of \( B \) and \( C \) correspondingly. Namely, for \( U \subset B \) we define \( h_1(U) \subset C \) as set of all points of \( C \) which lie on a line intersecting both \( A \) and \( U \). Similarly, for \( V \subset C \) we define \( h_2(V) \subset B \) as set of all points of \( B \) which lie on a line intersecting both \( D \) and \( V \). These maps are completely defined by their restrictions to the one-point subsets of \( B \) and \( C \) correspondingly, namely \( h_i(U) = \bigcup_{x \in U} h_i(\{x\}) \).

Check first that our result indeed follows from the Theorem 5. Suppose that \( x \in f(x) \). It means that \( x \in h_2(y) \) for some point \( y \in h_1(\{x\}) \). By definition of \( h_i \) it means that the line passing through \( x \) and \( y \) intersects both \( A \) and \( D \), q.e.d.

We have to check that \( f(x) \) satisfies conditions of Theorem 5.

By convex-concavity \( f(x) \) is non-empty for all \( x \in B \).

Lemma 4. \( f(x) = h_2(h_1(\{x\})) \) is upper semi-continuous.

We will prove that both \( h_1 \) and \( h_2 \) are upper semi-continuous in the sense defined below, and the claim will follow from the fact that the composition of upper semi-continuous maps is again upper semi-continuous. Denote by \( N_\delta(U) = \{ x | \text{dist}(x, U) < \delta \} \) the \( \delta \)-neighborhood of \( U \).

Lemma 5. Mapping \( h_1 \) is upper semi-continuous in the following sense: for any \( U \subset CSet(B) \) and any \( \epsilon > 0 \) there exist a \( \delta > 0 \) such that if \( U' \subset N_\delta(U) \) then \( h_1(U') \subset G = N_{2\epsilon}(h_1(U)) \). The mapping \( h_2 \) is also upper semi-continuous.

Proof. The proof is the same for both \( h_1 \) and \( h_2 \), so we prove it for \( h_1 \) only. By definition \( h_1(U) = \bigcup_{x \in U} h_1(\{x\}) \). Therefore by compactness of \( U \) it is enough to prove that for any \( b \in B \) and any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( \text{dist}(b', b) < \delta \) then \( h_1(b') \subset N_\epsilon(h_1(b)) \).

Note that \( h_1(b) = C \cap A_b \), where \( A_b \) is a compact continuously depending on \( b \) in Hausdorff metric (\( A_b \) and \( A_{b'} \) differ by a shift).

The claim follows form the fact that an intersection of a compact with another compact continuously depending on parameters depends upper semi-continuously.
on parameters. Let’s prove this fact. Let $V = h_1(b)$, and $A'_b = A_b \setminus N_\epsilon(V)$. Let $0 < \alpha < \min(\epsilon, \text{dist}(A'_b, C))$. For $b'$ close enough to $b$ we have $A'_{b'} \subset N_\alpha(A_b)$ and $h_1(b') = C \cap A_{b'} \subseteq C \cap N_\alpha(A_b) \subseteq (C \cap N_\alpha(A'_b)) \cup (C \cap N_\alpha(N_\epsilon(V))) \subseteq C \cap N_{\alpha + \epsilon}(V) \subset G$.

The second inclusion is true by continuous dependence of $A_b$ on $b$, the third is true since $A_b \subset A'_b \cup N_\epsilon(V)$, the fourth is true since $C \cap N_\alpha(A'_b) = \emptyset$ by choice of $\alpha$ and the last one is true since $\alpha + \epsilon < 2\epsilon$.

To satisfy the last condition of the Theorem 5 we have to check that $f(x)$ is a closed convex subset of $B$.

**Lemma 6.** $h_1(U)$ is compact convex set as soon as $U$ is compact convex set.

*Proof.* Indeed, the set of lines intersecting both $U$ and $A$ is convex (as intersection of two convex closed sets) and compact (since a line is defined by its two points of intersection with $U$ and $A$, which are both compact), so the set of points of intersections of these lines with $\{x_n = t_3\}$ is also convex and compact. But $h_1(U)$ is exactly the intersection of this set with $C$, so it is also convex and compact. □

**Remark 5.** From a Leray theorem and the previous result we get that the set of non-horizontal lines intersecting at least one of the chosen five sections is homotopically equivalent to a ball or to a sphere according to the existence or nonexistence of a line passing through all five sections. We know that there exist affine convex-concave bodies (see introduction and 6) without a line inside, so the case of a sphere is possible. This sphere divides the set of all non-horizontal lines into two connected parts. As a corollary we see that for some five sections of these affine convex-concave body (in our example in 6 these are just line segments) there is a line not intersecting them which cannot be moved to infinity without intersecting the sections.

3. Chebyshev line

By the previous section all we need to prove is that through any five horizontal sections of the convex-concave body passes a line. We fix them from now on. We choose a sixth horizontal plane $L$ (not containing sections), choose affine coordinates in $\mathbb{R}^3 \cong \mathbb{R} P^3$ and, using a standard scalar product, introduce a metric on horizontal planes. Using this metric we define a Chebyshev line — a line minimizing the maximal distance from its point of intersection with a plane of the section to the section. On each plane containing a section we choose a half-plane containing the section with boundary passing through the point of intersection of the Chebyshev line with the plane and perpendicular to the shortest segment joining this point to the section.

In this and the next section we investigate combinatorial conditions imposed on the configuration of these half-planes by the fact that the Chebyshev lines minimizes the maximal distance to the sections.

3.1. The Chebyshev line. Denote by $S_1, S_2, S_3, S_4$ and $S_5$ the five sections of a convex-concave body $B \in \mathbb{R}^3$ cut by five horizontal planes $L_i$, i.e. $S_i = B \cap L_i$. Choose coordinates $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w})$ in $\mathbb{R}^3$ in such a way that the infinite line has equation $\tilde{z} = \tilde{w} = 0$ and $S_i \subset \{\tilde{w} \neq 0\} \cong \mathbb{R}^3$. We take standard coordinates $(x = \frac{\tilde{w}}{\tilde{w}}, y = \frac{\tilde{y}}{\tilde{w}}, z = \frac{\tilde{z}}{\tilde{w}})$ in $\{\tilde{w} \neq 0\} \simeq \mathbb{R}^3$. In these coordinates the planes $L_i$
are given by equations $L_i = \{ z = t_i \}$. We take metric on $L_i$ induced by a scalar product

$$(x_1, y_1, z_1), (x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2.$$ 

Suppose that there is no line intersecting all five sections $S_i$ (otherwise there is nothing to prove).

**Definition 4.** The (non-horizontal) line $\ell$ minimizing the $\max_{i=1,...,5} \dist(\ell \cap L_i, S_i)$ (where $L_i$ are the horizontal planes containing $S_i$) will be called a Chebyshev line.

The existence of this line follows from compactness of sections. Further we will denote $a_i = \ell \cap L_i$ and by $s_i \in S_i$ the point of $S_i$ closest to $a_i$.

**Lemma 7** (Chebyshev property). The $\dist(a_i, S_i) = \dist(a_i, s_i)$ are all equal.

**Proof.** Indeed, let one of them, say $\dist(a_1, S_1)$ is strictly smaller than all others. By the Browder theorem Theorem 4 there exists a line $\ell_1$ which intersects all four remaining sections. Therefore for small values of $\epsilon$ the points of intersections of the line $\ell_\epsilon = (1 - \epsilon)\ell + \epsilon \ell_1$ lies closer to $S_1$ than $a_i$ for $i = 2, 3, 4, 5$. From the other hand, $\dist(\ell_\epsilon \cap L_1, S_1)$ changes continuously with $\epsilon$. So for small $\epsilon > 0$ we get $\max_{i=1,...,5} \dist(\ell_\epsilon \cap L_i, S_i) < \max_{i=1,...,5} \dist(a_i, S_i)$, which contradicts to the Chebyshev property of $\ell$. \hfill \Box

**Corollary 1.** The Chebyshev line $\ell$ doesn’t intersect $S_i$ if there is no line intersecting all $S_i$.

Further, in order to simplify the notations, we will suppose that the coordinates are chosen in such a way that the Chebyshev line coincides with the $z$ axis. Indeed, a linear transformation of the type $(x, y, z) \rightarrow (x - (az + b), y - (cz + d), z)$ doesn’t change metric in horizontal planes, so the Chebyshev line for the shifted sections will be the shifted Chebyshev line. From the other side, using a transformation of this type we can move any non-horizontal line to the $z$-axis.

### 3.2. Five half-planes.

The Chebyshev condition on the line $\ell$ says that one cannot find five points $a'_i \in L_i$ lying on a line and such that $\dist(a'_i, S_i) < \dist(\ell \cap L_i, S_i)$. Here we describe explicitly what the second requirement means.

For each $a_i = \ell \cap L_i$ we can indicate an angle of desirable directions in $L_i$: if $a_i$ moves in this direction then the $\dist(a_i, S_i)$ decreases. These are directions forming an acute angle with the direction $\overrightarrow{a_iS_i}$. So arises the half-plane $H_i = \{ x \in L_i \mid (\overrightarrow{a_i}\cdot\overrightarrow{a_iS_i}) \geq 0 \}$. The vector $\overrightarrow{a_iS_i}$ is orthogonal to its boundary and is directed inward.

Another description of $H_i$ is as follows: the function $f(x) = \dist(x, S_i)$ is a smooth function everywhere on $L_i \setminus S_i$, so in particular for $x = a_i$. After identification of $T_{a_i} L_i$ and $L_i$ the half-plane $H_i$ is described as $\{ df_{a_i}(\cdot) \geq 0 \}$.

We will need further the following evident statement, see Figure 3:

**Lemma 8.** Let $H$ be a half-plane in $L_i$ bounded by a line passing through $a_1$ and normal to $\overrightarrow{a_1H}$. Suppose that $S_1 \subset H$. Then $n \in H_1$. 
3.3. **Good deformations.** Here we describe lines (further called good deformations) whose existence contradicts to the fact that the Chebyshev line \( \ell \) doesn’t intersect the sections \( S_i \). Our goal from now is to prove their existence.

**Lemma 9.** If \( \ell \) is the Chebychev line for \( S_i \) and \( H_i \) are as above, then there exist no line intersecting interiors of all \( H_i \).

**Proof.** Suppose there exists a line \( \tilde{\ell} \) intersecting interiors of all \( H_i \). Then \( z \)-axis cannot be a Chebyshev line since

\[
\max_i \text{dist}(\{(\tilde{\ell} + (1 - \epsilon)\ell) \cap L_i, S_i\}) < \max_i \text{dist}(a_i, S_i)
\]

for \( \epsilon > 0 \) - small enough. In other words, moving the Chebyshev line in the direction of \( \tilde{\ell} \) in the space of all non-horizontal lines decreases its distance to \( S_i \).

Indeed, all we have to check is that \( \frac{d}{d\epsilon} \left|_{\epsilon=0} \text{dist}(b_{\lambda, \mu}^{\lambda, \mu}, S_i) \right. \) are non-negative for \( i = 2, 3, 4, 5 \). Therefore for some positive \( c_1, c_2 \) we have \( \frac{d}{d\epsilon} \big|_{\epsilon=0} \text{dist}(b_{\lambda, \mu}^{\lambda, \mu}, S_i) > 0 \)

for \( i = 1, 2, 3, 4, 5 \), i.e. for small \( \epsilon > 0 \) the line \( \epsilon c_1 \ell_1 + \epsilon c_2 \ell_{2345} + (1 - \epsilon c_1 - \epsilon c_2)\ell \) is closer to \( S_i \) than the Chebyshev line - contradiction.

**Remark 6.** The use of convex-concave property of the sections is almost unnecessary: any four parallel half-planes with pairwise non-parallel (see below) sides can be intersected by a line, which is as good as \( \ell_{2345} \) for the proof.
3.4. **Degenerate cases.** In what follows we will always impose the following genericity assumption on $H_i$: we assume that $\partial H_i$ are pairwise non-parallel (i.e. do not intersect in $\mathbb{R}P^3$).

For the degenerate cases (with some of the boundaries $\partial H_i$ being parallel) the proof of existence of a good deformation is reduced via duality considerations to the Theorem 4, see [5]. This is done in the following way:

1. First, we circumscribe convex polygons $P_i$ with $\leqslant 8$ sides around $S_i$. The sides are tangent to $S_i$ and parallel to the boundaries of $H_i$.
2. Second, we build the maximal (by inclusion) convex-concave body $P$ with sections $P_i$. It exists since $S_i$ were sections of a convex-concave body. $P$ is the union of all points $a \in \mathbb{R}P^3$ with the property that through any two $P_i$ and the point $a$ passes a line.
3. Third, we consider a dual $\tilde{P}$ of $P$ with respect to a special duality constructed in [5]. $\tilde{P}$ is also a convex-concave body. Sections of $\tilde{P}$ correspond to projections of $P$. We prove that $\tilde{P}$ is constructed from four convex figures in the way described in (2). By Theorem 4 there exists a line intersecting all four of them and therefore this line lies inside $\tilde{P}$.
4. The dual of this line lies inside $P$ and therefore intersects all $P_i$. Since $P_i \subset H_i$, this line is a good deformation.

4. **Combinatorial properties of half-planes arising from a Chebyshev line**

In this and the next chapters we investigate combinatorial properties of mutual position of the five half-planes constructed above. We do not use in this chapter the convex-concavity of the sections $S_i$ (so the results are valid for any five convex compact figures lying on five horizontal planes), and use only part of conditions implied by the fact that $\ell$ is the Chebyshev line for $S_i$. Namely, we use, first, the absence of lines interior of all $H_i$ and intersecting $\ell$, and, second, the genericity assumption of §3.4. We single out six combinatorial types of configurations of half-planes satisfying these two assumptions.

The settings we deal with can be described in projective terms. Namely, in $\mathbb{R}P^3$ we are given a configurations consisting of

1. five different projective planes $L_i$, all containing the same line (further called infinite line),
2. five half-planes $H_i \subset L_i$ - parts of these planes - containing convex (with respect to the infinite line) figures $S_i$ together satisfying convex-concavity condition. Boundary of each half-plane consists of the infinite line and some other line. The other lines are pairwise nonintersecting by genericity assumption of §3.4.
3. a line $\ell$ intersecting all these other lines and not intersecting the infinite line.

In this chapter we encode combinatorial properties of configurations by a purely combinatorial code, leaving temporarily aside continuous parameters of the problem (like distances between $L_i$). This encoding can be done in several ways, so to each configuration correspond several codes. The configurations we need have the property that none of the corresponding codes is trivial. In the next chapter we will see that there are at most six such configurations.
4.0.1. Coding. We will code combinatorial properties of configurations using projections from points \( x \in \ell \) to horizontal planes. As a result we will get a code — a permutation of numbers 1, 2, 3, 4, 5 with signs.

The line \( \ell \) is an affine part of a projective line \( \tilde{\ell} \cong \mathbb{RP}^1 \cong \mathbb{S}^1 \). This projective line is divided into 5 intervals by its points of intersection with half-planes \( H_i \). We choose a point \( M \in \ell \) from one of these intervals and orientation on \( \ell \). We enumerate the points of intersections of the half-planes with \( \ell \) starting from \( M \) according to the chosen orientation, thus enumerating the half-planes by numbers 1, 2, 3, 4, 5.

Consider a projection \( \pi : \mathbb{R}^3 \setminus L_M \to L_1 \) (where \( L_M \) is the horizontal plane passing through \( M \)). Take an orientation on the circle \( \mathbb{S}^1 \subset L_1 \) centered at \( a_1 = \ell \cap L_1 \) and a point \( N \in \mathbb{S}^1 \setminus \cup \pi(\partial H_i) \). Thus we get an enumeration of the set of 10 points \( \mathbb{S}^1 \cap (\cup \pi(\partial H_i)) \) (note that by non-degeneracy assumption none of \( \partial H_i \) are parallel).

We can now write down a sequence of five numbers with signs (further called a code) which will encode the combinatorial properties of the configuration: on the \( i \)-th place of this sequence stands the number of the half-plane which boundary projects onto the \( i \)-th point on \( \mathbb{S}^1 \) taken with + if the projection contains the point \( N \) and with − otherwise.

![Figure 4. This projection and choices of \( N \) and of orientation of \( \mathbb{S}^1 \) correspond to the code \( 3 + 2 − 1 + 4 + 5+ \)](image-url)

Remark 7. On the figures we denote the boundaries of \( \pi(H_i) \) by their numbers. The arrows point inward the projections of the corresponding half-planes.

4.0.2. Equivalent codes. In the coding procedure described above we made several choices. As a result we get several codes for the same situation. The resulting classes are in fact orbits of a group acting on the set of all possible codes.

This group is generated by two pairs of generators. The first pair corresponds to the choices made on \( \mathbb{S}^1 \).

The first generator, denoted by \( \beta_1 \), corresponds to the moving the point \( N \) to the previous interval. It acts on the code by cyclic permutation of the numbers and changing the sign of the last element: the \( i \)-th number goes to the \((i + 1)\)-th place except the first one which moves to the fifth place and changes sign, e.g. \( \beta_1(1 + 2 + 3 + 4 − 5) = 5 + 1 + 2 + 3 + 4− \).
The second generator, denoted by $\beta_2$, corresponds to the change of orientations on the circle. It acts on codes by symmetry: we should put the $i$-th number on the $5-i$-th place preserving the sign (e.g. $\beta_2(1+2+3+4-5) = 5-4-3+2+1+)$.

The second pair corresponds to the choices made on the Chebyshev line $\ell$. In general, changing the position of the center of the projection or the orientation results not only in change of enumeration of half-planes but also in the different choices of the plane to which we project. So in order to describe the effect of moving the point $M$ to the next interval or changing the orientation of $\ell$ we have to identify somehow the planes of projections.

The third generator of the group, denoted by $\alpha_1$, corresponds to the moving the point $M$ to the point $M'$ in the previous interval. If we identify planes $L_1$ and $L_2$ using the projections from $M'$ and make the same (upon this identification) choice of $N$ and of the orientation of $S_1$, then $\alpha_1$ acts on codes by changing $1+$ to $2+$, $2+$ to $3+$, $3+$ to $4+$, $4+$ to $5+$, $5+$ to $1-$, $\ldots$, $5-$ to $1+$ (e.g. $\alpha_1(1+2+3+4-5) = 2+3+4+5-1+)$). In other words, the numeration shifts by 1 and the image of the fifth (from the $M$) half-plane flips.

The fourth generator, denoted by $\alpha_2$, corresponds to the change of orientation of $\ell$. After identifying $L_1$ and $L_5$ by projection from $M$ action of $\alpha_2$ reduces to the renaming of the planes. So $\alpha_2$ acts on codes by interchanging 5 with 1 and 4 with 2 with signs preserved (e.g. $\alpha_2(1+2+3+4-5) = 5+4+3+2-1-)$.

It is easy to see from this geometrical description that $\alpha_1\beta_1\alpha^{-1}_1\beta^{-1}_2 = \alpha_1^{10} = \alpha_2^{2} = \beta_1^{10} = \beta_2^{2} = \text{Id}$, $\alpha_2\alpha_1\alpha_2 = \alpha_1^{-1}$ and $\beta_2\beta_1\beta_2 = \beta_1^{-1}$, i.e. the group generated by $\alpha_i$ and $\beta_i$ is $D_5 \oplus D_5$.

4.1. Cases of evident good deformation: trivial codes and Chebyshev property.
4.1.1. *Trivial codes.* There are cases (i.e. a combinatorial types of intersections of projections of $H_i$) which are forbidden for Chebyshev lines. These are in particular the cases when for some choice of $M$, projections of all $H_i$ have nontrivial intersection (i.e. more than one point). Indeed, in this case a good deformation which will intersect the Chebyshev line can be easily found.

**Theorem 6.** Configuration corresponding to a Chebyshev line cannot be coded by a code containing $1^+, 2^+, 3^+$ and $4^+$.

**Proof.** Suppose first that by choosing a point $M \in \ell$ and a point $N \in S_i$ we get a code consisting of positive numbers only, i.e a permutation of $1^+, 2^+, 3^+$, and $5^+$. By definition it means that the line connecting $M$ and $N$ intersects all $H_i$ at their interior, i.e. is a good deformation.

If the code contains $5^-$, then, after applying $\alpha_1^-$, we get an equivalent code with positive only entries, thus reducing to the previous case.

4.1.2. *Another easy case: the Chebyshev property.* The following lemma uses for the first (and the last) time the Euclidean metric. More exact, it uses the definition of $H_i$ as the set of all points $x \in L_i$ such that the scalar product $(a_i \cdot s_i, a_i \cdot s_i)$ is positive (where $s_i \in S_i$ is the point of $S_i$ closest to $a_i$). We will need this lemma only in the last chapter, when we consider the six nontrivial codes.

**Theorem 7.** No half-plane $H \subset L_1$ such that $a_1 \in \partial H$ can contain $S_1$, $\pi(S_2)$, $\pi(S_3)$ and $\pi(S_4)$ simultaneously.

**Proof.** Denote by $N$ the endpoint of inward normal $a_1 N$ to $\partial H$.

We are given that $\pi(S_i) \subset H$ for $i = 1, 2, 3, 4$. Therefore $\pi(s_i) \in H$, so, by Lemma 8, $N \in \pi(H_i)$.

If $N \notin \pi(\partial H_i)$ then the code corresponding to $N$ contains $1^+$, $2^+$, $3^+$ and $4^+$ and we are done by the previous lemma.

If not, we can slightly move the point $N$ and get the same result. Namely, suppose that $N \in \pi(\partial H_i)$ for some $i$. Since (by genericity assumption) none of $\pi(\partial H_i)$ coincide, $N$ cannot lie on more than one $\pi(\partial H_i)$. Therefore slightly moving $N$ inward this $\pi(H_i)$ we get a point $N'$ corresponding to a code containing $1^+$, $2^+$, $3^+$ and $4^+$, which is forbidden by the Theorem 6.

**Remark 8.** This lemma generalizes the following simple geometrical fact:

**Lemma 11.** There is no half-space $H \subset \mathbb{R}^3$ with the Chebyshev line on its boundary containing all five sections $S_i$.

Indeed, in this case in each plane $L_i$ we will get a figure like in Lemma 8, so a line obtained from a Chebyshev line by a small parallel translation in the direction of the inward normal to $\partial H$ will lie closer to all sections.
In this section we single out all non-trivial codes, i.e. not equivalent to the named in Theorem 6. Though the number of codes is huge (namely $3840 = 2^55!$), there are only six equivalency classes not containing trivial codes. They are listed in the Theorem 8 below.

5.1. From a code to a corresponding chessboard. It is easier to visualize codes as a position of five rooks on a $5 \times 5$ chess board. This is done as follows: in the first column we put the rook in the row which number is equal to the first number in the code. The color of the rook is white if this first number has sign $+$ and black otherwise. We continue like this for the second, third, fourth and fifth column (so if we forget the colors, the rooks position is exactly the graph of the permutation given by the code).

5.2. How the symmetry group acts on rooks positions. We described above an action of some symmetry group on codes. In the chess board realization the action of this group is remarkably simple:
- $\beta_1$ acts by moving the fifth column to the first place and changes the color of the rook standing in this column;
- $\alpha_1$ acts in a similar way but with rows: $\alpha_1$ moves the fifth row to the first place and changes the color of the rook standing in this row;
• $\alpha_2$ acts by symmetry with respect to the vertical line;
• $\beta_2$ acts by symmetry with respect to the horizontal line.

Figure 10. Action of generators of the group

5.3. Six equivalence classes consisting of nontrivial arrangements only.
The trivial codes correspond to the arrangements of white rooks only, which will be called trivial arrangements. Our goal is to exclude rooks arrangements equivalent to trivial ones. This is done in this subsection by a

Lemma 12. Any arrangement non-equivalent to a trivial case is equivalent to an arrangement with only one black rook. Moreover, this rook can be supposed to stand not on the border of the board.

Proof. Pick any arrangement which is not equivalent to a trivial one. The $\beta_1^5$ simply changes all colors to the opposite ones, so we can assume that the number of black rooks is equal to one or two. The first case is what we need, so suppose that there are two black rooks. If one of them stands on the first or the last row, then using $\alpha_1^{\pm1}$ we can change its color without changing the color of others, so leaving only one black rook. Similar statement holds for columns and $\beta_1$.

So we can suppose that both black rooks are in the inner $3 \times 3$ square. Then we get at least two white rooks on the border. Take the fifth row. It contains one rook. Therefore a first or a fifth column should contain another white rook and moving this column and the fifth row (i.e. acting by $\beta_1\alpha_1$ or by $\beta_1^{-1}\alpha_1$) we arrive to a situation with four black rooks, which is equivalent (by $\beta_1^5$) to a situation with one rook only.

This black rook cannot stand on the border since otherwise by one move ($\alpha_1^{\pm1}$ or $\beta_1^{\pm1}$) we arrive to a trivial situation.

Using the symmetries $\alpha_2$ and $\beta_2$, we can assume that the black rook occupies one of the four squares $(2, 2), (2, 3), (3, 2), (3, 3)$.
5.3.1. *The case* $(2, 2)$. Consider first the case of the black rook on the square $(2, 2)$.

**Lemma 13.** If one of the squares $(1, 1), (1, 5), (5, 1)$ is occupied, then the position is trivial.

*Proof.* Indeed, in these cases $\beta_1^{-1} \alpha_1^{-2}$ or $\beta_1^{-2} \alpha_1^{-1}$ or $\beta_1 \alpha_1^{-2}$ correspondingly transforms the position to a trivial one. \(\square\)

Therefore in a position not equivalent to a trivial one the white rook of the first column can occupy one of the squares $(1, 3)$ or $(1, 4)$ only and the square $(5, 1)$ is empty.

5.3.2. *White on* $(1, 3)$ *and Black on* $(2, 2)$. This leaves four configurations:

- C1 $3 + 2 - 1 + 4 + 5 +$
- C2 $3 + 2 - 1 + 5 + 4 +$
- C3 $3 + 2 - 4 + 1 + 5 +$
- C4 $3 + 2 - 5 + 1 + 4 +$

5.3.3. *White on* $(1, 4)$ *and Black on* $(2, 2)$. These are another four possibilities (remind that $(5, 1)$ is empty):

- D2 $4 + 2 - 1 + 5 + 3 +$
- D3 $4 + 2 - 3 + 1 + 5 +$
- D4 $4 + 2 - 5 + 1 + 3 +$
- D5 $4 + 2 - 1 + 3 + 5 +$

But D4 becomes trivial after $\alpha_1^{-3} \beta_1^2$, and D2 becomes D3 after $\beta_1^2 \alpha_2 \beta_2$. Moreover, after $\alpha_1^3 \beta_1^{-3}$ C4 becomes D2. So the only new configuration is D5.

![Figure 11](image-url) **Figure 11.** D4 becomes trivial after $\alpha_1^{-3} \beta_1^2$. 
5.3.4. **Black on** (2, 3). Similarly to Lemma 13, the white rook in the first column cannot stand on the first or the last row. In other words, in a position with black rook on (2, 3) and not equivalent to a trivial one the squares (1, 5) and (1, 1) are empty. Indeed, $\alpha_1^{-1}\beta_1^{-2}$ correspondingly trivialize these arrangements.

So the only places the white rook can stand on are (1, 2) or (1, 4). These positions are in fact equivalent by $\alpha_2$, so we can consider the positions with a white rook on (1, 4) and the black rook on (2, 3).

But these positions are equivalent by $\alpha_2\beta_2\beta_1^{-2}$ to the positions with the black rook on (2, 2), so are in fact considered above.

5.3.5. **Black on** (3, 2). These arrangements are also equivalent to arrangements with the (only) black rook on (2, 2). The proof repeats word-by-word the proof above with change of $\beta$ to $\alpha$ and of $\alpha$ to $\beta$ everywhere. This is because the actions of the group is symmetric with respect to diagonal (though this symmetry isn’t itself in the group).

5.3.6. **Black rook on** (3, 3). The complement of the square to the third row and the third column consists of four two-by-two squares.

**Lemma 14.** If the arrangement is not equivalent to a trivial one, then each square contains exactly one rook.

**Proof.** Indeed, if not, then one of them contains two rooks and the opposite should necessarily contain the other two (since in each row and in each column stands exactly one rook). Applying $\beta_2$ if necessary, one can suppose that these are the lower left and the upper right squares. Then $\alpha_1^2\beta_1^3$ transforms arrangement to a trivial one.

**Lemma 15.** If one of rooks stands in the corner (i.e. on (1, 1), (1, 5), (5, 1) or (5, 5)), then the situation is equivalent to a situation with the only black rook standing on (2, 2) (i.e. is in fact considered above).
Figure 14. Triviality of the case of the black rook on (3,3) and one of the squares containing two rooks.

Proof. Using $\alpha_2$ and $\beta_2$, if necessary, we can suppose that the white rook stands on (1,1). Then we get a situation with the only black rook on (2,2) after $\alpha_1^{-1}\beta_1^{-1}$.

Corollary 2. All configurations with one of white rooks in the inner $3 \times 3$ square are trivial or have a rook in a corner.

Proof. Suppose that (2,2) is occupied and the position is neither trivial nor with a rook in a corner. Then the $2 \times 2$ square contain one rook each. Then the squares (1,4) and (4,1) are occupied, since the corners are empty and the second row and second column already contain a rook. Therefore the only remaining square for the fourth rook is in the corner (5,5), which is forbidden.

Figure 15. The case of the black rook on (3,3) and one of the rooks in the corner is equivalent to the series C.

The only remaining positions are $4 + 1 + 3 - 5 + 2 +$ and $2 + 5 + 3 - 1 + 4 +$, which are equivalent by $\alpha_2$ or $\beta_2$.

5.3.7. The final list. It consists of six variants.

Theorem 8. A configuration corresponding to a Chebyshev line should be equivalent to a configuration described by one of the following codes

$C1 \ 3 + 2 - 1 + 4 + 5 + \quad C2 \ 3 + 2 - 1 + 5 + 4 + \quad C3 \ 3 + 2 - 4 + 1 + 5 + \quad C4 \ 3 + 2 - 5 + 1 + 4 + \quad D5 \ 4 + 2 - 1 + 3 + 5 + \quad E6 \ 4 + 1 + 3 - 5 + 2 +$

6. Non-triviality of a code and convex-concavity imply existence of good deformation

In this chapter we consider the six nontrivial cases of Theorem 8. Each case has several continuous parameters (e.g., angles between $\partial H_i$, distances between $L_i$), and only for some choice of parameters the configuration of half-planes arises from a Chebyshev line. In other words, for only part of the parameter space parameterizing this combinatorial type the corresponding configuration of half-planes do not admit a good deformation. Indeed, the Theorem 8 excludes only codes admitting
a good deformation intersecting the Chebyshev line $\ell$, and do not deal with good deformations not intersecting $\ell$.

In what follows we show that the configurations of half-planes arising from sections $S_i$ of a convex-concave body all admit a good deformation. Therefore they cannot correspond to a Chebyshev line, so the assumption that the Chebyshev line doesn’t intersect the sections leads to a contradiction.

More exact, we extract from the convex-concavity condition some inequality between double ratio of angles between $\partial H_i$ and double ratios of distances between $L_i$ in some particular combinatorial assumptions. This inequality implies existence of a line intersecting four from half-planes $H_i$ in some particular sectors. For five from the six cases of Theorem 8 these assumptions are satisfied, and moreover the resulting line automatically intersects the fifth half-plane. The sixth case E6 simply cannot occur for convex-concave sections.

The main tool in the proofs is the Theorem 4, only applied now to some parts of the sections $S_i$. The only Euclidean property we will need is the Theorem 7, which statement is projective. So we can move the center of projection to infinity, and the projection becomes a parallel projection $\pi : \mathbb{R}^3 \to L_1$ along the $z$-axis, with $S_i$ are ordered by their $z$-coordinate.

We will also use a linear structure defined on $L_1$ defined by the coordinates $x$ and $y$ (i.e. we take the point $a_1$ as the origin).

6.1. **Sectorial Browder Theorem.** We will denote by $\pi(H_i)^c$ for the closure of $L_1 \setminus \pi(H_i)$. We define half-spaces $B_i = \pi^{-1}(\pi(H_i))$ and denote by $B_i^c$ the closure of their complements.

**Theorem 9.** Suppose that

1. $H_1 \cap \pi(H_4)^c \subset \pi(H_2)^c$ and
2. $\pi(H_3) \cap \pi(H_2) \subset \pi(H_4)^c$.

Suppose moreover that $S_1 \cap \pi(H_4)^c \neq \emptyset$. Then $\pi(S_2) \cap \pi(H_3)$, $\pi(S_3) \cap \pi(H_2)$ and $\pi(S_4) \cap H_1$ are also non-empty and there exists a straight line $L$ intersecting $S_1 \cap B_4^c$, $S_2 \cap B_3$, $S_3 \cap B_2$ and $S_4 \cap B_1^c$.

In our notations the conditions (1) and (2) mean existence of the subsequence $1 + 2 - 3 + 4-$ in a sequence coding the configuration. In applications below the condition $S_1 \cap \pi(H_4)^c \neq \emptyset$ will follow from the Lemma 21 below.

**Proof.** First we prove two combinatorial lemmas:

**Lemma 16.** $H_1 \cap \pi(H_4)^c \subset \pi(H_3)$
other words, we have to check that

\( A \) as

\[
\text{Proof. Suppose that } H_1 \cap \pi(H_4)^c \not\subset \pi(H_3). \text{ Since boundaries of the half-planes are pairwise different, there is a point } x \text{ lying in the interior of } (H_1 \cap \pi(H_4)^c) \setminus \pi(H_3).
\]

Then \(-x \in H_1^i \cap \pi(H_4) \cap \pi(H_3) \subset \pi(H_2) \cap \pi(H_3) \) by assumption and also \(-x \in \pi(H_4) - \) contradiction.

\[
\text{Lemma 17. } \pi(H_2) \cap \pi(H_3) \subset \pi(H_1)^c
\]

\[
\text{Proof. As before, take } x \text{ in the interior of } \pi(H_2) \cap \pi(H_3) \cap H_1. \text{ Then } x \in \pi(H_4)^c \cap \pi(H_1) \text{ by the assumption (2) and therefore } x \in \pi(H_2)^c \text{ by the assumption (1) - contradiction.}
\]

Our claim will be proved by applying the Theorem to \( S_1 \cap \pi(H_4)^c \) as \( B \), \( S_2 \cap B_3 \) as \( A \), \( S_3 \cap B_2 \) as \( C \) and \( S_4 \cap B_1^c \) as \( D \). Let’s check conditions of Theorem. In other words, we have to check that

1. a line passing through \( S_1 \cap B_1^c \) and intersecting \( S_2 \) and \( S_3 \) (existing by convex-concavity) intersects \( S_2 \cap B_3 \) and \( S_3 \cap B_2 \) and
2. a line passing through \( S_3 \cap B_2 \) and intersecting \( S_1 \) and \( S_4 \) (existing by convex-concavity) intersects \( S_4 \cap B_1^c \) and \( S_1 \cap B_2^c \).

(Clearly \( S_1 \cap B_1^c, S_2 \cap B_3, S_3 \cap B_2 \) and \( S_4 \cap B_1^c \) are compact and convex).

Let a line intersects \( S_1 \cap B_1^c \) and \( S_2 \) and \( S_3 \) at points \( c_1, c_2 \) and \( c_3 \) accordingly. Necessarily \( c_2 \) lies between \( c_1 \) and \( c_3 \). We know that \( c_1 \in S_1 \cap \pi(H_4)^c \subset H_1 \cap \pi(H_4)^c \subset \pi(H_2)^c \cap \pi(H_3) \). Since \( c_1, c_3 \in B_3 \), so \( c_2 \in B_3 \) (so \( S_2 \cap B_3 \) is non-empty). Similarly, \( c_1 \in B_2^c \) and \( c_2 \in B_2 \), so \( c_3 \in B_2 \) (and \( S_3 \cap B_2 \) is non-empty). So the first claim is proved.

Similarly, let a line intersects \( S_3 \cap B_2 \) and \( S_4 \) and \( S_1 \) at points \( c_3, c_4 \) and \( c_1 \) accordingly. As before, \( s_3 \subset B_4 \cap B_2^c \). Since \( c_4 \in B_4 \) and \( c_3 \in B_2^c \), so \( c_1 \in S_1 \cap \pi(H_4)^c \). Since \( c_3 \in B_2^c \), so \( c_4 \in S_4 \cap B_3^c \) (so in particular \( S_4 \cap B_3^c \) is not empty). The second claim follows.

\[
\text{6.2. Double ratios. After projecting a configuration satisfying conditions of the Theorem to the plane } L_1 \text{ we obtain a figure below.}
\]

Here \( L \) is the projection of the line existing by Theorem. By \( A', B', C', D' \) we denote intersections of \( L \) with \( S_i \) and by \( A, B, C, D \) intersections of \( L \) with \( \partial(\pi(H_i)) \).
The existence of the line $L$ implies an inequality between the double ratio of distances between $L_i$ and the double ratio of directions of boundaries of $H_i$. Namely, denote the double ratio $\frac{AB}{AC} : \frac{BD}{BC}$ of points $A, B, C, D$ by $(A, B, C, D)$. Then $(A'B'C'D')$ is exactly the double ratio of distances between $L_i$:

$$(A, B, C, D) = \frac{h_1}{h_1 + h_2} : \frac{h_1 + h_2}{h_3},$$

where $h_i$ is the distance between $L_i$ and $L_{i+1}$. $(A, B, C, D)$ is the double ratio of directions of boundaries of $\partial H_i$ and the following inequality holds:

**Corollary 3.** In assumptions of Theorem 9 the double ratio of distances between $L_i$ is strictly smaller than the double ratio of directions of $\partial H_i$:

$$(A', B', C', D') > (A, B, C, D)$$

**Proof.** Indeed, the configuration of the points $A', B', C', D'$ is obtained from the points $ABCD$ by the movements which only increase the above double ratio:

1. $(A, B, C, D) < (A', B, C, D)$ since $\frac{A'B}{A'C} > \frac{AB}{AC}$,
2. $(A', B, C, D) < (A', B', C, D)$ since $\frac{A'B'}{A'C'} > \frac{AB}{AC}$,
3. $(A', B', C, D) < (A', B', C', D)$ since $\frac{C'D}{C'D'} > \frac{CD}{CD'}$,
4. $(A', B', C', D') < (A', B', C', D)$ since $\frac{C'D'}{C'D'} > \frac{CD}{CD'}$.

The equality is possible only if all points $A', B', C', D'$ lies on the corresponding lines, which is impossible since, for example, the point $B'$ lies in $\pi(S_2)$ which is included in the interior of the half-plane $\pi(H_2)$, so $B' \neq B$ and the inequality in (2) is strict. \hfill \Box

**Lemma 18.** With conditions as above suppose that four points $A'', B'', C'', D''$ lies on a line $L'$ and

- $A'' \in \partial H_1 \setminus \pi(H_4)$
- $C'' \in \partial(\pi(H_3)) \setminus \pi(H_4)$
- $D'' \in \partial(\pi(H_4)) \setminus H_1$

Suppose moreover, that $A''B'' : B''C'' : C''D'' = A'B' : B'C' : C'D'$. Then $B''$ lies in the interior of $\pi(H_2) \cap \pi(H_3)$.

**Proof.** This follows directly from the inequality Lemma 3. Indeed, let $\overline{B} = L' \cap \partial(\pi(H_2))$. Then $(A'', B, C', D') = (A, B, C, D) < (A', B', C', D') = (A'', B'', C'', D'')$. This is equivalent to $\frac{A''B}{B'D''} < \frac{A'B}{B'D'}$, which is possible only if $B''$ is between $\overline{B}$ and $D''$, i.e. $B'' \in \pi(H_2)$. Since $B'' \in [A''C'']$, also $B'' \in \pi(H_3)$.

\hfill \Box
The Lemma means that the line, which existence is claimed in Theorem 9, can be moved in such a way that it will still intersect the interior of \( H_2 \) and will also intersect boundaries of \( H_1, H_3 \) and \( H_4 \).

6.3. The six non-trivial configurations: contradiction with convex-concavity.

We will call by stencil any five points \( c_1, c_2, c_3, c_4, c_5 \in L_1 \) which are projections points of intersections of some line \( l \subset \mathbb{R}P^3 \) with \( L_i, c_i = l \cap L_i \). Note that \( |c_1c_2| : |c_2c_3| : |c_3c_4| : |c_4c_5| \) is the same for all stencils and is equal to \( h_1 : h_2 : h_3 : h_4 \) where \( h_i \) are the distances between \( L_i \) and \( L_{i+1} \). Evidently this is a necessary and sufficient condition for five points in \( L_1 \) lying on a line in this order to be a projection of points of intersection of some line in \( \mathbb{R}P^3 \) with the planes \( L_i \).

A projection of a good deformation is a stencil with an additional property \( c_i \in \pi(H_i), \) with at least one of \( c_i \) lying in the interior of \( \pi(H_i) \). Vice versa, any such stencil is a projection of a good deformation.

We can reformulate the Lemma 18 using these notations.

Lemma 19. Suppose that
1. \( H_1 \cap \pi(H_2)^c \subseteq \pi(H_2)^c \),
2. \( \pi(H_3) \cap \pi(H_2) \subseteq \pi(H_4)^c \) and
3. \( S_1 \cap \pi(H_4)^c \neq \emptyset \).

Then there exists a stencil such that
1. \( c_1 \in \partial H_1 \cap \pi(H_4)^c \),
2. \( c_2 \) lies in the interior of \( \pi(H_2) \cap \pi(H_3) \),
3. \( c_3 \in \partial \pi(H_4) \cap \pi(H_4)^c \) and
4. \( c_4 \in \partial \pi(H_4) \cap \pi(H_1)^c \).

Similar statements hold for all strictly increasing subsequence of 12345 consisting of four numbers (i.e. 1245 or 1345 etc. instead 1234).

6.3.1. Chebyshev property. Here we prove that one of consequences of the Chebyshev property formulated in Lemma 7 is that the set \( S_1 \cap \pi(H_4)^c \) in Lemma 18 is never empty.

Lemma 20. If \( S_1 \subseteq \pi(H_4) \) or \( S_1 \subseteq \pi(H_5) \) then the configuration is trivial.

Proof. Indeed, in the first case \( \pi(S_2) \) and \( \pi(S_3) \) also lie in \( \pi(H_4) \) by convex-concavity. Indeed, any point of \( S_2 \) lies on a segment with endpoints on \( S_1 \) and \( S_4 \), and projection of such a segment lies entirely in \( \pi(H_4) \). The same is true for
S_3, so by Lemma 7 the configuration is trivial. In the second case \( S_i \subset \pi(H_5) \) for \( i = 1, 2, 3, 4, 5 \) and again by Lemma 7 the configuration is trivial.

In cases C1, C3, C4 and D5 the Lemma 19 and the Lemma 20 give immediately existence of a stencil which is a projection of a good deformation.

6.3.2. The C1 case. This is the case \( 3 + 2 - 1 + 4 + 5+ \). We will consider an equivalent (after \( \beta_1^{-2} \beta_2 \)) variant \( 1 + 2 - 3 + 5 - 4- \).

If \( S_1 \cap \pi(H_4) = \emptyset \) then the configuration is trivial by Lemma 20. So \( S_1 \cap \pi(H_4) \neq \emptyset \) and the Lemma 19 is applicable to the subsequence \( 1 + 2 - 3 + 4- \) of the code.

In the resulting stencil \( c_4 \in \pi(H_3) \) and \( c_1 \notin \pi(H_5) \). Indeed, the sector \( H_1 \cap \pi(H_4)^c \) is the smallest sector bounded by boundaries of half-planes and containing the point \( N \). Since \( N \notin \pi(H_5) \), so \( H_1 \cap \pi(H_4)^c \cap \pi(H_5) = \emptyset \). This means that \(-c_4, c_1 \notin \pi(H_5)\).

Therefore the point \( c_5 \) of the stencil lies in \( \pi(H_5) \). Therefore the line projecting to this stencil is a good deformation.

6.3.3. The C3 case. This is the case \( 3 + 2 - 4 + 1 + 5+ \). We will consider the equivalent (after \( \beta_2 \alpha_1^2 \)) case of \( 1 + 2 - 5 - 3 + 4- \).

As above, \( S_1 \notin \pi(H_4) \) by Lemma 20, so we can apply the Lemma 19 to the the subsequence \( 1 + 2 - 3 + 4- \) of the code, exactly as in the case C1. As before, \( c_1 \) lies on \( \partial H_1 \cap \pi(H_4)^c \) and therefore in \( \pi(H_5)^c \). Also, \( c_4 \in \partial \pi(H_4) \cap H_1^c \) and therefore \( c_4 \in \pi(H_5) \). So \( c_5 \) also lies in \( \pi(H_5) \) since \( c_5 \) and \( c_1 \) lie from different sides of \( c_4 \).

Therefore the stencil given by Lemma 19 is a projection of a good deformation.
6.3.4. The $C_4$ case. This is the case of $3 + 2 - 5 + 1 + 4 +$. We will consider the equivalent (after $b_1^2 b_2 a_1^{-1}$) case of $1 + 2 - 3 + 5 - 4 +$.

As before, by Lemma 20, $S_1 \not\subset B_5$. We apply Lemma 19 to the subsequence $1 + 2 - 3 + 5 -$ and get a stencil with $c_1 \in \partial H_1 \cap \pi(H_5)^c$, $c_2 \in \pi(H_2)$, $c_3 \in \partial \pi(H_3) \cap \pi(H_5)^c$ and $c_5 \in \partial \pi(H_5) \cap H_1^c$. Then $c_4 \in \pi(H_4)$. Indeed, $c_1, c_5 \in \pi(H_4)$ and $c_4$ lies between $c_1$ and $c_5$. So this stencil is a projection of a good deformation.

6.3.5. The $D_5$ case. This is the case of $4 + 2 - 1 + 3 + 5 +$. It is equivalent (after $a_1^2 b_2$) to the case $1 + 4 - 2 - 3 + 5 -$.

Again, $S_1 \not\subset B_5$ by Lemma 20. We apply Lemma 19 to the subsequence $1 + 2 - 3 + 5 -$ and get a stencil with $c_1 \in \partial H_1 \cap \pi(H_5)^c$, $c_2 \in \pi(H_2)$, $c_3 \in \partial \pi(H_3) \cap \pi(H_5)^c$ and $c_5 \in \partial \pi(H_5) \cap H_1^c$. Now $c_4 \in \pi(H_4)$ follows from the fact that $c_3, c_5 \in \pi(H_4)$ and $c_4$ lies between $c_3$ and $c_5$. So this stencil is a projection of a good deformation.
In two last cases we should exhibit a little more inventiveness.

The case C2 requires double application of the Lemma 19, whereas in E6 the combinatorial properties of the intersections contradict to the Theorem 9.

6.3.6. The C2 case. This is the case of $3 + 2 - 1 + 5 + 4^+$. After applying $\alpha_1^3\beta_1\beta_2$ it will transform to an equivalent variant $1 + 2^-3^-4^-5$. By Lemma 20 $S_1 \cap \pi(H_3)^c \neq \emptyset$. Applying Lemma 19 to the sequences $1 + 2 - 4 + 5$ and $1 + 3 - 4 + 5$ we see that there are two stencils, one with points $c_1, c_2, c_3, c_4, c_5$ and another with points $c_1', c_2', c_3', c_4', c_5'$. The following conditions hold:

1. $c_1, c_1' \in \partial H_1 \cap \pi(H_3)^c$;
2. $c_2 \in \pi(H_2) \cap \pi(H_4)$;
3. $c_3' \in \pi(H_3) \cap \pi(H_4)$;
4. $c_4, c_4' \in \partial \pi(H_4) \cap \pi(H_5)^c$ and
5. $c_5, c_5' \in \partial \pi(H_5) \cap \pi(H_1)^c$.

But any two stencils satisfying (1), (4) and (5) differ only by a dilatation centered at the origin and these dilatations preserve $\pi(H_i)$. So $c_3 \in \pi(H_3) \cap \pi(H_4)$ and we get the stencil which is a projection of a good deformation.

6.3.7. The E6 case. This is the case of $4 + 1 + 3 - 5 + 2^+$. It is equivalent (by $\beta_1^4$) to $1 - 3 + 5 - 2 - 4^+$. Recall that $B_i = \pi^{-1}(H_i)$.

Suppose first that $S_1 \cap \pi(H_3) \neq \emptyset$. Similar to the proof of the Theorem 9, we will apply the Theorem 9 to $S_1 \cap \pi(H_3)$ as $B$, $S_4 \cap B_2$ as $C$ and $S_2$ and $S_3$ as $A$ and $D$ correspondingly and will arrive to contradiction.

Construct two mappings, $h_1 : CSet(S_1 \cap B_3) \to CSet(S_4 \cap B_2)$ and $h_2 : CSet(S_4 \cap B_2) \to CSet(S_1 \cap B_3)$, as in Namely, take a point $b \in S_1 \cap B_3$. There is a line passing through this point and section $S_2$ and intersecting the section $S_4$ at point...
Since \( \pi(b) \in \pi(S_1) \cap \pi(H_3) \subset \pi(H_2)^c \) and evidently \( \pi(S_2) \subset \pi(H_2) \), we conclude that \( \pi(c) \in \pi(S_4) \cap \pi(H_2) \), i.e. \( c \in S_4 \cap \pi(H_2) \). The mapping \( h_1 \) is the extension to the closed subsets of \( S_1 \cap B_3 \) of the mapping sending the points \( a \) to the set of all such \( c \). Similarly, to define \( h_2 \) take any point \( c \in S_4 \cap B_2 \). There is a line passing through this point and intersecting the section \( S_3 \) and the section \( S_1 \) at a point \( a \). Since \( \pi(c) \subset \pi(H_4) \cap \pi(H_2) \subset \pi(H_3)^c \) and \( \pi(S_3) \subset \pi(H_3) \), we get that \( a \in S_1 \cap B_3 \).

In virtue of the Theorem 4 this proves existence of a line intersecting \( S_1 \cap \pi(H_3) \), \( S_2 \), \( S_3 \) and \( S_4 \cap B_2 \).

But this line cannot exist. Indeed, denoting the projections of the intersection points by \( c_1, c_2, c_3, c_4 \) we see that \( c_2, c_4 \in \pi(H_4) \cap \pi(H_2) \subset \pi(H_3)^c \) and therefore the point \( c_3 \) – lying between \( c_2 \) and \( c_4 \) – should also belong to \( B_3 \), which contradicts to \( c_3 \in \pi(H_3) \).

Figure 20. The case of \( S_1 \cap \pi(H_3) \neq \emptyset \) is impossible.

Therefore \( S_1 \subset \pi(H_3)^c \). By convex-concavity we get that \( \pi(S_4), \pi(H_5) \subset \pi(H_3) \) (any point of these sections is an endpoint of a segment intersecting \( S_3 \) with another endpoint in \( S_1 \)). Therefore \( \pi(S_3) \subset \pi(H_5) \cap \pi(H_3) \subset \pi(H_2) \) and \( \pi(S_4) \subset \pi(H_4) \cap \pi(H_3) \subset \pi(H_2)^c \).

This is incompatible with the existence of lines joining \( S_5, S_4 \) and \( S_2 \) given by convex-concavity condition. Indeed, take any segment intersecting \( S_2, S_4 \) and \( S_5 \) at points \( s_2, s_4 \) and \( s_5 \) correspondingly. Its projection \([\pi(s_2), \pi(s_5)]\) has both ends in \( \pi(H_2) \), so \( \pi(s_4) \in \pi(H_2) \) as well, which contradicts to \( \pi(s_4) \in \pi(S_4) \subset \pi(H_2)^c \).

Figure 21. The case of \( S_1 \subset B_2^c \) is impossible (the previous picture is rotated by \( 180^\circ \)).
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Department of Mathematics, Purdue University, West Lafayette IN
E-mail address: dmitry@math.purdue.edu

Department of Mathematics, Toronto University, Toronto, Canada
E-mail address: askold@math.toronto.edu