Backstepping boundary control: an application to the suppression of flexible beam vibration

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Abstract. This paper presents a backstepping boundary control for vibration suppression of flexible beam. The applications are such as industrial robotic arms, space structures, etc. Most slender beams can be modelled using a shear beam. The shear beam is more complex than the conventional Euler-Bernoulli beam in that a shear deformation is additionally taken into account. At present, the application of this method in industry is rather limited, because the application of controllers to the beam is difficult. In this research, we use the shear beam with moving base as a model. The beam is cantilever type. This design method allows us to deal directly with the beam’s partial differential equations (PDEs) without resorting to approximations. An observer is used to estimate the deflections along the beam. Gain kernel of the system is calculated and then used in the control law design. The control setup is anti-collocation, i.e. a sensor is placed at the beam tip and an actuator is placed at the beam moving base. Finite difference equations are used to solve the PDEs and the partial integro-differential equations (PIDEs). Control parameters are varied to see their influences that affect the control performance. The results of the control are presented via computer simulation to verify that the control scheme is effective.

1. Introduction
Flexible beams constitute an important problem in many applications such as space structures, industrial robotic arms, cantilever cranes, helicopter rotor, astronomical telescopes, etc. In this paper, we consider a model of the undamped shear beam. Most of the slender beams can be represented by this model. The beam system is distributed parameter in nature, so it is governed by partial differential equations (PDEs). The model consists of a wave equation coupled with a second-order-in-space ODE or can be alternatively represented as a fourth-order-in-space/second-order-in-time PDE. The shear beam is more complex than the Euler-Bernoulli model and slightly simpler than the Timoshenko model. There are many models for flexible beams such as Euler-Bernoulli, Rayleigh, shear and Timoshenko beam equations. Derivations and comparisons of these beam models can be found in [1].

Boundary control is preferable for controlling PDE systems since actuation and sensing are only through the boundary conditions [2]. For its historical development, the reader is referred to [3]. Morgul presented results for boundary control of infinite dimensional systems of Euler-Bernoulli [4]. The
passivity property of nonlinear Euler-Bernoulli beams has been studied in Fard [5]. The method guarantees finite gain $L_2$ stability and passivity of closed-loop systems. Sasaki [6] used a Lyapunov functional of the system to derive the control law by minimizing the time rate of change of the functional at every point in time and using neural networks in tuning a control gain.

Krstic et al. presented the backstepping boundary control for undamped shear beams with an anti-collocated setup in which the sensing and the actuation are at different locations [7, 8, 9, 10]. Gain kernels for both controller and observer are found and used for control law design. The applications include atomic force microscopy (AFM) where the piezo actuation is applied at the beam base. The industrial application is rather limited. Ali and Padhi presented active control of Euler-Bernoulli beams [11]. They proposed two state feedback controllers based on optimal dynamic inversion techniques. Boonkumkrong N. and Kuntanapreeda S. applied the method of backstepping boundary control to the thermal system experiment [12].

This work is motivated by the interests in stabilizing vibrating slender bodies. The structure is modeled by a completely undamped shear beam model. The design is a combination of the classical damping boundary feedback idea with backstepping boundary control. The works of Fard, Sasaki and Krstic are the inspiration of this paper [5, 6, 7].

At present, the application of the backstepping boundary control is rather limited, because the application of controllers to the beam is difficult. An objective of this paper is to apply the method to the problem of the free vibration suppression of the undamped shear beam with controllers applied to the beam by using a moving base. This is new method to apply the backstepping boundary controllers to the system.

The rest of this article is organized as follows. The next section presents a mathematical model of the system. Brief explanations of the beam models are also provided. Backstepping boundary controller design is given in Section 3. In Section 4, an observer design is presented. Numerical studies are presented in Section 5. The results of simulation are given in Section 6. This article is concluded in the last section.

2. Mathematical models

The undamped shear beam model can be expressed by a second-order-in-time, fourth-order-in-space PDE, as follows [1]

$$aw_{tt}(x,t) - \varepsilon w_{xxxx}(x,t) + w_{xxxx}(x,t) = 0$$

where $w(x,t)$ is the beam deflection, $\varepsilon$ is a constant inversely proportional to the shear modulus and $\alpha$ is a positive constant. The subscripts $t$ and $x$ denote the partial differentiation with respect to time and space, respectively. The deflection due to the beam mass is neglected.

The beam model can also be represented by a wave equation PDE coupled with a second-order-in-space ODE as follows [1, 14]

$$\varepsilon w_{tt}(x,t) = w_{xx}(x,t) - \varphi_a(x,t), \quad x \in (0,1)$$

$$\varphi_{xx}(x,t) - b^2 \varphi(x,t) + b^2 w_t(x,t) = 0$$

where $\varphi$ is the rotating angle as a result of the bending moment of the beam and $b = \sqrt{\alpha/\varepsilon}$. The derivations of beam models equation (1) and equations (2) - (3), can be found in [14]. In [14], one can verify that both models are equivalent.

In figure 1, the beam is free at one end $(x = 0)$ and has the following boundary conditions [1]:

$$w_t(0,t) - \varphi(0,t) = 0$$

$$\varepsilon w_{tt}(x,t) - \varepsilon w_{xxxx}(x,t) + w_{xxxx}(x,t) = 0$$
\[ \varphi_x(0,t) = 0. \]  

Equations (4) and (5) mean that the shear force and the moment are both zero at the free end, respectively. From the Newton’s 2\textsuperscript{nd} law of motion, the boundary condition of the other end \((x = 1)\) with a controller \(U(t)\) applied at the moving base is as follows \[6\]

\[ m_b \frac{d^2 w_b(t)}{dt^2} = -(w_x(1,t) - \varphi(1,t)) + U(t), \]  

where \(m_b\) is the mass of the moving base, \(w_b(t)\) is the displacement of the base, the terms in the parentheses on the right-hand side of equation (6) represent a shear force exerted by the beam on the base \[4\]. The beam model, equations (2) - (3), is used to design the control law. Note that \(w_b(t) = w(1,t)\), i.e. the displacements of the beam right end and the moving base are the same.

![Shear beam with a controller](image)

**Figure 1.** Shear beam with a controller.

### 3. Backstepping Boundary Control

In this section, backstepping boundary control method is presented and the control law is then formulated. The system, equations (2) - (3), will be first re-written in a hyperbolic partial integro-differential equation (PIDE), as the following,

\[
\begin{align*}
\varepsilon \, w_t(x,t) & = w_x(x,t) + b^2 w(x,t) - b^2 \cosh(bx) w(0,t) + b^2 \int_0^x \sinh \left[ b(x-y) \right] w(y,t) \, dy \\
& - b \frac{\cosh(bx)}{\cosh(b)} \left( \varphi(1,t) - b \sinh(b) \, w(0,t) + b^2 \int_0^1 \cosh \left[ b(1-y) \right] w(y,t) \, dy \right) \\
\end{align*}
\]

**Equation (7)**

\[
w_x(0,t) = \frac{1}{\cosh(b)} \{ \varphi(1,t) - b \sinh(b) \, w(0,t) + b^2 \int_0^1 \cosh \left[ b(1-y) \right] w(y,t) \, dy \}. \tag{8}
\]

Equations (7) and (8) are obtained by solving the ODE equation (3) as a two-point boundary value problem using Laplace transform in the spatial variable \(x\) as follows \[7\]

\[ \varphi(x,t) = \cosh(bx) \varphi(0,t) - b \int_0^x \sinh \left( b(x-y) \right) w_y(y,t) \, dy. \tag{9} \]
The term \( \varphi(0,t) \) in equation (9) can be expressed in term of \( \varphi(1,t) \) by evaluating equation (9) at \( x = 1 \) to get

\[
\varphi(1,t) = \cosh(b) \varphi(0,t) - b \int_0^t \sinh(b(1 - y)) w_y(y,t) \, dy. \tag{10}
\]

Solve equation (10) for \( \varphi(0,t) \),

\[
\varphi(0,t) = \frac{1}{\cosh(b)} \{ \varphi(1,t) + b \int_0^t \sinh(b(1 - y)) w_y(y,t) \, dy \}. \tag{11}
\]

Integrate by parts the integral term on the right side of equation (11) to get

\[
\varphi(0,t) = \frac{1}{\cosh(b)} \{ \varphi(1,t) - b \sinh(b) w(0,t) + b^2 \int_0^t \cosh(b(1 - y)) w(y,t) \, dy \}. \tag{12}
\]

The integral term on the right-hand side of equation (12) is not spatially casual because the upper limit of integration is 1. To put the system into a strictly feedback form, we eliminate this integral by choosing the first controller as follows

\[
\varphi(1,t) = \sinh(b) w(0,t) - b^2 \int_0^t \cosh(b(1 - y)) w(y,t) \, dy. \tag{13}
\]

So that \( \varphi(0,t) = 0 \) in equation (12). Then, equation (9) becomes

\[
\varphi(x,t) = \sinh(bx) w(0,t) - b^2 \int_0^t \cosh(b(x - y)) w(y,t) \, dy. \tag{14}
\]

Note that the upper limit of integration is now \( x \).

Differentiating \( \varphi(x,t) \) with respect to \( x \) and substituting the results into the wave equation (2), we get the system in the strictly feedback form for control design,

\[
\varepsilon \, w_x(x,t) = w_{xx}(x,t) - b^2 \cosh(bx) w(0,t) + b^3 \int_0^x \sinh(b(x - y)) w(y,t) \, dy. \tag{15}
\]

\[
w_x(0,t) = 0. \tag{16}
\]

Next, use the following transformation [7],

\[
v(x,y) = w(x,y) - \int_0^x k(x,y) w(y,t) \, dy, \tag{17}
\]

where is \( k(x,y) \) is the gain kernel of the system. Equation (17) is used to map the system equation (2) into the following exponentially stable target system

\[
\varepsilon \, v_{x}(x,t) = v_{xx}(x,t), \tag{18}
\]
v_x(0,t) = c_0 v(0,t), \quad (19)  
\text{and}  
v_x(1,t) = -c_1 v_y(1,t) \quad (20)

where \( c_0 \) and \( c_1 \) are design parameters. Stability proof of equations (18)-(20) can be found in [9].

Substituting the transformation equation (17) into the target system, equations (18)-(20), we can derive the following PDE for gain kernel \( k(x,y) \)[13]:

\[
k_{xx}(x,y) = k_{yy}(x,y) + b^2 \sinh (b(x-y)) + b^2 \int_0^x k(x,\xi) w(\xi,t) \, d\xi \quad (21)
\]

\[
k(x,x) = \frac{\lambda}{2} x - c_0 \quad (22)
\]

\[
k_x(x,0) = b^2 \int_0^x k(x,\xi) \cosh (b\xi) \, d\xi - b^2 \cosh(bx). \quad (23)
\]

The second boundary controller is obtained by differentiating equation (17) with respect to \( x \) and setting \( x = 1 \)[13]:

\[
w_x(1,t) = k(1,1) w(1,t) + \int_0^1 k_x(1,y) w(y,t) \, dy - c_1 w_y(1,t) + c_1 \int_0^1 k(1,y) w_y(y,t) \, dy \quad (24)
\]

Stability proof of the feedback control can be found in [10]. Gain kernels \( k(1,y) \) and \( k_x(1,y) \) of control law equation (24) are shown in figures 2 and 3, respectively.

![Figure 2 The gain kernel \( k(1,y) \).](image1)

![Figure 3 The gain kernel \( k_x(1,y) \).](image2)

In backstepping boundary control method, we use the controller, which consists of two equations, i.e. equation (13) and equation (24), to control the beam. Figure 4 shows the transformation from cantilever beam to target system. The first part of controller, \( \varphi(1,t) \) converts the free end into a moving end, i.e. \( \varphi(0,t) = 0 \), and then the second part of the controller, \( w_y(1,t) \), converts the beam into taut string with a stiff spring at the tip, equation (19) and a tuned damper at the base, equation (20).

The first controller, equation (13) is combined to the beam model, equations (7)-(8), so the controller, equation (24) is applied at the moving base as follows
\[ U(t) = k(1,1) w(1,t) + \int_{0}^{t} k_y(x,y) w(y,t) \, dy - c_i w_i(1,t) \]

\[ + c_i \int_{0}^{t} k(1,y) w_i(y,t) \, dy + m_b \frac{d^2 w(1,t)}{dt^2} \]  

(25)

4. Observer Design

For the backstepping boundary control method, the states along the beam are needed for control law calculation, but the only possible measurement of the system is at the boundary \( x = 0 \). The Luenberger-like observer is used to estimate beam deflections along the beam [10].

Consider the following beam model:

\[ \varepsilon w''(x,t) = w_{xx}(x,t) + b^3 \int_{0}^{x} \sinh(b(x-y)) \, w(y,t) \, dx - b^2 \cosh(bx) \, w(0) \]

\[-b \sinh(bx) \alpha(0) \]

\[ w_x(0,t) = \varphi(0,t) \]  

(26)

(27)

The Luenberger-like observer is given by [10]

\[ \varepsilon \hat{w}_{xx}(x,t) = \hat{w}_{xx}(x,t) + b^2 \hat{w}(x,t) + b^3 \int_{0}^{x} \sinh(b(x-y)) \, w(y,t) \, dx \]

\[-b^2 \cosh(bx) \, w(0) - b \sinh(bx) \varphi(0) + p_y(x,0) [w(0,t) - \hat{w}(0,t)] \]

\[ \hat{w}_x(0,t) = \varphi(0) + p(0,0) [w(0,t) - \hat{w}(0,t)] - \tilde{c}_0 [w_i(0,t) - w_i(0,t)] \]

\[ \hat{w}(1,t) = w(1,t). \]  

(28)

(29)

The observer gains \( p_y(x,0) \) in equation (18) and \( p(0,0) \) in equation (29) are determined by solving the PDE [10],

\[ p_{yy}(x,y) = p_{xx}(x,y) + b^2 p(x,y) + b^3 \sinh(b(x-y)) + b^3 \int_{y}^{x} p(\xi, y) \sinh(b(x-\xi)) \, d\xi \]

(30)

(31)
\[ p(x, x) = \frac{b^2}{2} (x - 1) \]  
\[ p(1, y) = 0. \]  

Defining the observer error as \( \tilde{w} = w - \hat{w} \) and subtracting equations (28)-(29) from equations (26)-(27), we obtain the observer error:

\[ \varepsilon \tilde{w}_x(x, t) = \tilde{w}_{xx}(x, t) + b^2 \tilde{w}(x, t) + b^3 \int_0^x \sinh(b(x - y)) \tilde{w}(y, t) \, dx + p_y(x, 0) \tilde{w}(0, t) \]  
\[ \tilde{w}_y(0, t) = -p(0, 0) \tilde{w}(0, t) + \tilde{c}_0 \tilde{w}_t(0, t) \]  
\[ \tilde{w}(1, t) = 0. \]  

Using the following transformation,

\[ \tilde{w}(x, t) = \tilde{v}(x, t) - \int_0^x p(x, y) \tilde{v}(y, t) \, dy \]  

to convert the error system equations (34)-(36) into

\[ \varepsilon \tilde{v}_x(x, t) = \tilde{v}_{xx}(x, t) \]  
\[ \tilde{v}_y(0, t) = \tilde{c}_0 \tilde{v}_t(0, t) \]  
\[ \tilde{v}(1, t) = 0, \]  

which is known to be exponentially stable (see [10]).

The gain kernel PDEs can be solved numerically, but in this paper, we solve them numerically. The plot of observer gain kernel \( p(x, 1) \) and \( p_y(x, 1) \) are shown in figures 5 and 6, respectively.

![Figure 5](image-url). Figure 5. The observer gain kernel \( p(x, 1) \).

![Figure 6](image-url). Figure 6. The observer gain kernel \( p_y(x, 1) \)
5. Numerical Calculations
The beam model with strictly feedback form, equation (15) is a second-order-in-time, second-order-in-space partial integro-differential equation (PIDE) and is used in a numerical calculation. It is a simple second-order partial differential equation with the integration terms, and can be easily solved with finite difference.

The highest order of the PIDE that we study in this paper is second order, so we have the following finite-difference approximations [15]:

\[
\frac{\partial^2 w}{\partial x^2} = \frac{w^m_{n+1} - 2w^m_n + w^n_{n-1}}{(\Delta x)^2} + O[(\Delta x)^2] \\
\frac{\partial^2 w}{\partial t^2} = \frac{w^m_{n+1} - 2w^m_n + w^n_{n-1}}{(\Delta t)^2} + O[(\Delta t)^2]
\]

(41) (42)

For the integration approximation, we use the trapezoidal integration rule as follows,

\[
\int_a^b f(x) \, dx = \frac{\Delta x}{2} f(a) + \sum_{i=a+1}^{b-1} f(i) + \frac{\Delta x}{2} f(b)
\]

(43)

Substituting equations (41)-(43) into equation (15), we get the following finite difference equation,

\[
w^m_{n+1} = \frac{1}{\varepsilon} \left( w^m_{n+1} - 2w^m_n + w^n_{n-1} + \frac{(\Delta t)^2}{\varepsilon} b^2 w^m_n - \frac{(\Delta t)^2}{\varepsilon} b^2 \cosh(b x(m)) w^n_1 \right) + \frac{(\Delta t)^2}{\varepsilon} b \left\{ \frac{\Delta x}{2} \sinh[b(x(m) - y(j))] w(1) + \Delta x \sum_{j=2}^{m-1} \sinh[b(x(m) - y(j))] w(j) \right\}
\]

(44)

where \(\Delta x\) and \(\Delta t\) are the spatial and the temporal increments indexed by \(m\) and \(n\), which start from 1 to \(M\) and \(N\), respectively, and \(r = \Delta t/\Delta x\). Figure 7 shows the calculation grid with spacing \(\Delta x\) in the row and \(\Delta t\) in the column. The dark circles (the first and second rows from the bottom) are the values obtained by the initial conditions and the grey circle \((w^m_n)\) is the point to be calculated.

With the boundary condition, equation (16), one can obtain the first element \((w^m_{n+1})\) of each row and from the controller, equation (25) and the boundary-condition equation (6), the last element of the row \((w^M_n)\) is obtained.

![Figure 7. Calculation grid](image)
6. Simulations
In this section, the simulation results are presented. The shear beam is simulated using the finite difference equations mentioned in the last section. The beam started to vibrate under the following initial conditions (see figure 8) [8]:

\[ w(x,0) = 0.1 (1-x)^2 \sin (1.6 \pi (1-x)) \]  \hspace{1cm} (45)
\[ w_t(x,0) = -0.1(1-x)^2 \sin (1.6 \pi (1-x)) \]  \hspace{1cm} (46)

where equation (45) is the initial displacement, and equation (46) is the initial velocity at \( t = 0 \).

The beam was simulated with the parameter of beam material \( \varepsilon = 1 \) and \( b = 0.6 \). The mass of the moving base was fixed at \( m_p = 0.01 \), and the spring and damping constants were set at \( c_0 = 1 \) and \( c_1 = 1 \) , respectively. The parameter for observer equation (29) is set at \( \tilde{c}_0 = 15 \) [8]. The following values are used: the grid size \( M = 20 \) (in space), the time step \( \Delta t = 0.01 \) and the final time \( T = 30 \). Since there is no damping term in the beam model and no external force exerted, the beam with zero Dirichlet boundary condition at \( x = 1 \), i.e. with a fixed moving base, will vibrate perpetually as shown in figure 9. The tip displacement at \( x = 0 \) is shown in figure 10. This is the uncontrolled case.

The simulation of the shear beam with the control law, equation (25) is shown. The simulations of the shear beam with full-state control and with observer are shown in figures 11 and 12, respectively.
We see that the responses are the same. Note that the initial conditions for observer are 50% more than the values of equations (45) and (46).

The control action and tip displacement are shown in figures 13 and 14, respectively. The settling time is about 20.

6.1 Parameter Tunings

In this subsection, we study parameter changes that affect the control performance. First, we fix damping constant at $c_j = 1.0$ and vary spring constants at various values, $c_0$ at 0.5, 1.0 and 2.0. Figure 15 shows tip displacements.

In the case of a weaker value of damping, i.e. $c_0 = 0.5$, the response takes a longer time to settle at $t = 30$ and for the strong values of damping constant, it takes about $t = 20$. 

![Figure 11. Shear beam simulation with control](image1)

![Figure 12. Shear beam simulation with observer](image2)

![Figure 13. Control action, $w(1,t)$](image3)

![Figure 14. Tip displacement, $w(0,t)$](image4)

![Figure 15. Tip displacement with various spring constants ($c_0$)](image5)

![Figure 16. Tip displacements with various damping constants ($c_1$)](image6)
In the second case, we fix spring constant at \( c_0 = 1.0 \) and vary damping constants at various values, \( c_1 \) at 0.5, 1.0 and 2.0. Figure 16 shows tip displacements of the beam. In the case of a weak value of damping, i.e. \( c_1 = 0.5 \), the response takes a longer time to settle at \( t = 30 \) and for the strong values of damping constant, it takes about \( t = 20 \).

7. Conclusion

Backstepping boundary control is applied to the problem of vibration suppression of the undamped shear beam. The beam model consists of a wave equation, coupled with a second-order-in-space ODE. An observer is used to estimate the deflections along the beam. Gain kernel of the system is calculated and then used in the control law design. Finite difference equations are used to solve the PDEs and the partial integro-differential equations (PIDEs). Numerical results for the control of a shear beam are presented to verify that the control scheme is effective in suppressing the vibration of the beam, and the parameters, i.e. \( c_0 \) and \( c_1 \), can be tuned to obtain the control with the desired behaviours.

For real implementation or experiment, an electromagnetic linear actuator/motor can be used to move or slide the base. The speed of the actuator depends on the frequency of the vibration.

For further research, the other boundary control methods such as passivity-based control method will be applied to the shear beam and their performance will be compared.

References

[1] Han S M, Benaroya H and Wei T 1999 *J. of Sound and Vibration*, **225** pp 935-988
[2] Smyshlyaev A and Miroslav M. 2004 *IEEE Trans. on Automatic Control* **49** pp 2185-2202
[3] Padhi R and Ali Sk F 2009 *Annual reviews in Control* **33** pp 59-68
[4] Morgul O 1992 *Automatica* **28** pp 1255-1260
[5] Fard M P 2002 *Modelling, Identification and Control* **23** pp 239-258
[6] Sasaki M, Asai H, Kawafuku M and Hori Y 2000 *Proc. of the IEEE Int. Conf. on Systems, Man and Cybernetics* pp 3259-3264
[7] Krstic M, Balogh A and Smyshlyaev A 2006a *Proc. of the 17th Int. Symp. Math. Theory of Networks and Systems*
[8] Krstic M, Balogh A and Smyshlyaev A 2006b *Proc. of the 45th IEEE Conf. on Decision & Control.*
[9] Krstic M, Siranosian A, Balogh A and Guo BZ 2007 *Proc. of the 2007 American Control Conf.*
[10] Krstic M, Guo BZ 2008 *J. of Control Optimization* **47** pp 553–574
[11] Ali SF and Padhi R 2009 *J. of Systems and Control Engineering* **223** pp 657-672
[12] Boonkumkrong N and Kuntanapreeda S 2014 *J. of Systems and Control Engineering* **228** pp 295-302
[13] Krstic M, Smyshlyaev A 2008 *Boundary Control of PDEs: A Course on Backstepping Designs* (Philadelphia: SIAM)
[14] Rao S S 2011 *Mechanical Vibrations* (New Jersey: Prentice Hall)
[15] Mathews J H and Fink K D 1999 *Numerical Methods Using MATLAB* (New Jersey: Prentice Hall)