CONFORMAL SCALAR PROPAGATION AND HAWKING RADIATION

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Abstract. The construction of the conformal scalar propagator which has been obtained in the preceding two projects as an analytic function of the Schwarzschild black-hole space-time is completed with a boundary condition imposed by the physical context through contour integration in the exterior vicinity of the event horizon. It is shown that, as a consequence of the semi-classical character which the emitted quanta have in that exterior vicinity, the particle production by the Schwarzschild black hole which was formally established in the preceding project is identical to thermal Hawking radiation. By extension, it is established that such a particle production corresponds to a spectrum which detracts from thermality by the amount predicted by Parikh and Wilczek if energy conservation is properly imposed as a constraint on scalar propagation. The results obtained herein support the case made by S. Hawking on the relation between quantum propagation and observation of particles produced by a black hole.

“Du siehst, mein Sohn, zum Raum wird hier die Zeit!”
(“You see my son, in here time becomes space!”)
From Richard Wagner’s opera “Parsifal”

I. Introduction

The transition amplitude between two events is essential to the investigation of a quantum field’s dynamical behaviour in any curved space-time. The primary reason is the intimate relation which exists on general theoretical grounds between the Feynman propagator and the expectation value \( < T_{\mu\nu} > \) in any specific vacuum state. In a black-hole space-time the Feynman propagator receives additional significance if only because it provides a concrete mathematical description of the consolidated intuitive approach to the original rigorous frequency-mixing derivation of black-hole radiation [1]. That approach advances on the heuristic description of vacuum activity in terms of the transient and unobservable existence of positive-energy pairs of a particle and its anti-particle, very much in the spirit of Dirac’s old hole theory. Since, with respect to spatial infinity, the interior region of the black hole necessarily contains negative-energy states a positive-energy particle changes to a negative-energy one upon crossing the event horizon. In effect, the “uncertainty principle” inherent in this heuristic approach does not necessitate re-annihilation in the event that one of the two particles produced by vacuum fluctuations in the exterior region crosses the event horizon. The “uncertainty principle” is still identically upheld if the remaining positive-energy particle transits

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to spatial infinity or else registers its existence on a detector situated at any distance above the event horizon. For a Schwarzschild black hole of mass $M$ in Plank units ($G = c = h = k = 1$) this heuristic approach to Hawking radiation yields the same rigorously established thermal spectrum which corresponds to the Hawking temperature

$$T_H = \frac{1}{8\pi M}$$

Inherent in this result is the assumption that the background geometry for $r \geq 2M$ remains fixed. Such a semi-classical approximation is, in turn, justified by the fact that the rate of change of the metric $\sim M^{-3}$ is very slow compared to the typical frequency of the radiation $\sim M^{-1}$ as long as $M$ is much bigger than the Planck mass. It would, for that matter, be a valid approximation to neglect the time dependence of the metric and to calculate the emission from a sequence of stationary metrics [2].

The thermal spectrum which emerges as a result of the stated semi-classical approximation lies at the core of the centrally important issue of unitary evolution in the presence of event horizons. An alternative approach to black-hole radiation which is based entirely on the above-stated heuristic pair production as an expression of vacuum fluctuations in a Schwarzschild black-hole space-time yields a correction to the thermal spectrum by dismissing the comparison between the metric’s rate of change and the frequency of the emitted radiation in favour of energy conservation for each emitted quantum particle [3]. That approach has the distinctive feature of treating each emitted quantum particle as the result of such tunneling across the event horizon as is characterised by a barrier which crucially depends on the tunneling particle itself [4]. The detachment from thermality emerges as a consequence of the fact that energy conservation causes the event horizon to contract to a smaller Schwarzschild radius in the event of emission. In effect, in the exterior vicinity of the event horizon the geometry is invariably dynamic. An immediate consequence of such an approach is the fact that the emitted particles are also the result of tunneling from the interior region across the event horizon forward in each particle’s proper time, a feature which is absent in the heuristic treatment of black-hole radiation based on the stated semi-classical approximation.

Certain features of the heuristic pair-production approach to Hawking radiation - both, in the context of the assumption of a semi-classical geometry and in that of a dynamic geometry - deserve attention in their own merit. First - contrary to a common claim - the radiation emitted by a black hole is not a consequence of the gravitational field. It is, instead, the exclusive consequence of space-time’s causal structure determined by the presence of the event horizon. This general property of black-hole radiation is inherent in the preceding analysis and is more rigorously analyzed in [5]. Second, such an approach is relevant only to the Hartle-Hawking and to the Unruh vacuum states respectively. Its character as heuristic and intuitive stems from the fact that it does not define those vacuum states but is, instead, the result of those states’ instability. Third, a crucial aspect of the pair-production approach is the fact that - as a consequence of infinite time dilation - observers situated close to the event horizon in decreasing order of the Schwarzschild radial coordinate necessarily register each particle emitted by the black hole in an increasingly semi-classical state of very high energy. In the vicinity of the event horizon itself the corresponding uncertainties in the probability distributions of
each emitted particle tend to zero. Consequently, in the vicinity of the event horizon the semi-classical limit $\hbar \to 0$ naturally characterises the emitted particles. This is an equivalent semi-classical description to that which underlies the derivation in [3].

It is precisely this third feature which shall constitute the basis of the following analysis. In [6] and [7] the propagator for a massless conformal scalar field in a Schwarzschild black-hole space-time has been obtained in the Hartle-Hawking state for a finite range of values of the Schwarzschild radial coordinate above and below the Schwarzschild radius $r_S = 2M$ respectively. That propagator has the unique feature of being an explicitly analytic function of the background space-time geometry. As that propagator coincides with the exact propagator on the entire Schwarzschild black-hole geometry only in that finite range of values it is ideally suited for the description of observations made in a local frame established close to the event horizon. In fact, by properly matching the massless conformal scalar propagator in the interior region of the Schwarzschild black hole with that in the exterior region across the event horizon it was formally established in the heuristic context of pair production that, within a finite advance of his proper time, an observer situated at any $r > 2M$ registers particles emitted by the Schwarzschild black hole [7]. In what follows such a physical effect will be rigorously established. It will be shown that - always with respect to such an observer - the propagator established in [6] and [7] results in both, the thermal spectrum of Hawking radiation if for $r \geq 2M$ the geometry is treated as static in the context of the stated semi-classical approximation and the corrected spectrum obtained in [3] if energy conservation is accordingly enforced. For that matter, these results will be identically valid also for the "inertial" observer at spatial infinity $r \to \infty$.

A classic calculation which derives black-hole radiation and its thermal spectrum in terms of a transition amplitude in a Schwarzschild black-hole space-time has been accomplished in [8]. However, the inherent semi-classicality which characterises the emitted particles close to the event horizon is not manifest. As the analysis herein will advance on the necessary match of the expressions in [7] and [6] across the horizon that inherent semi-classicality will be explicit. The semi-classical character of the emitted particles will, in turn, constitute the basis for the derivation of both, the thermal spectrum of black-hole radiation in the absence of horizon-related evolution and in the detraction from the thermal spectrum which energy conservation enforces.

An aspect of central importance in the ensuing analysis is the boundary condition which stems from the physical demand that the emitted particles correspond to positive-energy states propagating forward in time. It will be shown that such a boundary condition constitutes an essential aspect of the massless conformal scalar propagator and that it completes that propagator’s construction which was achieved in [6] and [7] on the Schwarzschild black-hole geometry.

The analytic expressions obtained for the propagator in [6] and, especially, in [7] are particularly involved and apparently intractable. However, in addition to the calculation which established black-hole radiation in [7] the present derivation will also provide yet another example as to how, despite appearances, the causal structure of the Schwarzschild black-hole space-time can substantially simplify the relevant calculations.

II. The Thermal Spectrum
The transition amplitude for a field $\Phi$ between two events in any space-time is equivalent to the functional integral over all paths which intercept the associated two space-time points. In Euclidean signatures obtained by analytically extending the temporal coordinate to imaginary values this is

$$W_E = \int D[\Phi] e^{-S[\Phi, \nabla\alpha\Phi]}$$

where in the positive definite action functional $S[\Phi, \nabla\alpha\Phi]$ the gradient $\nabla\alpha$ is defined with respect to the space-time metric $g_{\mu\nu}$. At the semi-classical limit $\hbar \to 0$ (1) reduces to the expansion

$$W_E = e^{-S_0[\Phi_0, \nabla\alpha\Phi_0] + O(\hbar)}$$

with $S_0[\Phi_0, \nabla\alpha\Phi_0]$ being the classical action and with $O(\hbar)$ describing higher-loop corrections to it. The physical content inherent in (1) and (2) will constitute the basis of the following analysis.

The Schwarzschild metric is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The analytical extension $\tau = +it$ of the real-time coordinate $t$ in imaginary values results in a Euclidean, positive definite metric for $r > 2M$. The apparent singularity which persists at $r = 2M$ can be removed by introducing the new radial coordinate

$$\rho = 4M \sqrt{1 - \frac{2M}{r}}$$

Upon replacing

$$\beta = 4M$$

the metric in the new coordinates is

$$ds^2 = \rho^2(\frac{1}{\beta^2})d\tau^2 + \left(\frac{4r^2}{\beta^2}\right)^2d\rho^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Through the identification of $\frac{\tau}{4M}$ with an angular coordinate of period $2\pi$ the coordinate singularity at $r = 2M$ corresponds to the origin $\rho = 0$ of polar coordinates and is thus trivially removed. This procedure results in a complete singularity-free positive definite Euclidean metric which is periodic in the imaginary-time coordinate $\tau$. The period $8\pi M$ of that coordinate is the underlying cause of the thermal radiation at temperature $T = (8\pi M)^{-1}$ emitted by the Schwarzschild black hole.

By imposing the requirement $\rho^2 << \beta^2$ the propagator for a massless conformal scalar field $\Phi$ in the Hartle-Hawking state expressed as an explicitly analytic function of the exterior region of the Schwarzschild black-hole space-time for a certain range of values of the radial variable $r$ above $r = 2M$ has been established in [6] to be
$$D(x_2 - x_1) =$$

$$\frac{2}{\beta} \frac{1}{\sqrt{p_1 p_2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1) \sum_{p=0}^{\infty} e^{i \frac{\pi}{4} (2u + 3)(\rho_2 - \rho_1)} \int_{u_0[p]}^{\infty} \frac{\cos \frac{\pi}{4} (4u + 2p + 3)(\rho_2 - \rho_1)}{\pi^2 u^2 + 4(l^2 + l + 1)}$$

$$- \frac{2}{\beta^2} \frac{1}{\sqrt{\rho_1}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{p=0}^{\infty} \int_{u_0[p]}^{\infty} \frac{\cos \frac{\pi}{4} (4u + 2p + 3)(\rho_2 - \rho_1)}{\pi^2 u^2 + 4(l^2 + l + 1)} \frac{J_p(2i \sqrt{l^2 + l + 1})}{J_p(2i \sqrt{l^2 + l + 1})} e^{i \frac{\pi}{4} (2u + 3)(\rho_2 - \rho_1)} Y_{lm}(\theta_2, \phi_2) Y_{lm}^*(\theta_1, \phi_1) ;$$

$$(7) \quad u_0 \gg p ; \ u'_0 \gg p ; \ \frac{\pi u}{\beta \rho_2} \gg p$$

with a range of validity specified by

$$(8) \quad 0 \leq \rho_i^2 \leq \frac{\beta^2}{100} ; \ i = 1, 2$$

or, equivalently, by

$$(9) \quad 2M \leq r \leq 2.050M$$

The expression which amounts to the first term on the right side in $(7)$ is the singular part $D_{as}(x_2 - x_1)$ of $D(x_2 - x_1)$. In terms of the eigenvalues and eigenfunctions which the associated elliptic operator has in the Euclidean sector of the Schwarzschild metric $D_{as}(x_2 - x_1)$ constitutes the asymptotic expression of $D(x_2 - x_1)$ and as such contains the singularity at the coincidence space-time limit $x_2 \rightarrow x_1$ [6]. The expression which amounts to the second term constitutes the boundary part $D_b(x_2 - x_1)$ of $D(x_2 - x_1)$ which ensures the boundary condition of vanishing propagation on the boundary hypersurface $\rho = \beta$ of the Euclidean Schwarzschild black-hole geometry.

In the interior region $r < 2M$ of $(3)$ the analytical extension to imaginary time also involves the extension $\theta \rightarrow i \tilde{\theta}$ as a result of which $(3)$ reduces to the negative-definite [7]

$$(10) \quad ds^2 = -\left[\frac{2M}{r} - 1\right]d\tau^2 + \frac{1}{\frac{2M}{r} - 1} dr^2 + r^2 (d\tilde{\theta}^2 + sinh^2 \tilde{\theta} d\phi^2)$$

Through the analytical extension of $(7)$ into the interior region of the Schwarzschild black hole the massless conformal scalar propagator $D^{(int)}(x_2 - x_1)$ has been established in [7] for a certain range of values of $r$ below $r = 2M$. In view of the extensive character of that Green function and of the fact that only its singular part is relevant to the ensuing analysis the associated boundary part $D_b^{(int)}(x_2 - x_1)$ shall not be cited. With $\tau = \pm i \xi$ and $\rho = \pm i \zeta$ that singular part itself is
\[ D_{\alpha \beta}^{(int)}(x_2 - x_1) = \frac{i e^{-\frac{3\pi}{\beta}(\zeta_2 - \zeta_1)}}{\sqrt{\zeta_1 \zeta_2}} \int_{\epsilon - \infty}^{\infty} d[\bar{p}] \frac{1}{1 - e^{-2\pi |\bar{p}|}} e^{i \frac{\pi}{4}(\zeta_2 - \zeta_1)} e^{i \frac{\pi}{\beta} \left[ |\bar{p}| + u_0[|\bar{p}|] \right] (\zeta_2 - \zeta_1)} \times \]

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(i\bar{\theta}_2, \phi_2) Y_{lm}^*(i\bar{\theta}_1, \phi_1) \int_{\epsilon - \infty}^{\infty} dw \frac{e^{-i \frac{\pi}{2} w(\zeta_2 - \zeta_1)}}{\pi^2 (u_0[|\bar{p}|] - w)^2 - 4(l^2 + l + 1)} \]

\[ + \frac{i}{\beta} \frac{e^{\frac{3\pi}{\beta}(\zeta_2 - \zeta_1)}}{\sqrt{\zeta_1 \zeta_2}} \int_{\epsilon - \infty}^{\infty} d[\bar{p}] \frac{1}{1 - e^{-2\pi |\bar{p}|}} e^{i \frac{\pi}{4}(\zeta_2 - \zeta_1)} e^{i \frac{\pi}{\beta} \left[ |\bar{p}| + u_0[|\bar{p}|] \right] (\zeta_2 - \zeta_1)} \times \]

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(i\bar{\theta}_2, \phi_2) Y_{lm}^*(i\bar{\theta}_1, \phi_1) \int_{\epsilon - \infty}^{\infty} dw \frac{e^{-i \frac{\pi}{2} w(\zeta_2 - \zeta_1)}}{\pi^2 (u_0[|\bar{p}|] + w)^2 - 4(l^2 + l + 1)} \]

\[ - \frac{i}{\beta} \frac{e^{-\frac{3\pi}{\beta}(\zeta_2 - \zeta_1)}}{\sqrt{\zeta_1 \zeta_2}} \int_{\epsilon - \infty}^{\infty} d[\bar{p}] \frac{1}{1 - e^{-2\pi |\bar{p}|}} e^{i \frac{\pi}{4}(\zeta_2 - \zeta_1)} e^{-i \frac{\pi}{\beta} \left[ |\bar{p}| + u_0[|\bar{p}|] \right] (\zeta_2 - \zeta_1)} \times \]

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(i\bar{\theta}_2, \phi_2) Y_{lm}^*(i\bar{\theta}_1, \phi_1) \int_{\epsilon - \infty}^{\infty} dw \frac{e^{i \frac{\pi}{2} w(\zeta_2 - \zeta_1)}}{\pi^2 (u_0[|\bar{p}|] - w)^2 - 4(l^2 + l + 1)} \]

\[ - \frac{i}{\beta} \frac{e^{-\frac{3\pi}{\beta}(\zeta_2 - \zeta_1)}}{\sqrt{\zeta_1 \zeta_2}} \int_{\epsilon - \infty}^{\infty} d[\bar{p}] e^{i \theta} e^{-i \frac{3\pi}{2} w(\zeta_2 - \zeta_1)} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(i\bar{\theta}_2, \phi_2) Y_{lm}^*(i\bar{\theta}_1, \phi_1) \times \]

\[ \left( \int_{0}^{\infty} dw \frac{e^{-i \frac{\pi}{2} w(\zeta_2 - \zeta_1)}}{\pi^2 (u_0[|\bar{p}|] e^{i\theta} - i w)^2 + 4(l^2 + l + 1)} \right) \]

with a range of validity for \( D^{(int)}(x_2 - x_1) \) specified by

\[ \rho_{int}^2 \leq \frac{\beta^2}{100} \]

and accordingly, in terms of \( r \), by

\[ 1.980 M \leq r \leq 2M \]
The derivation of Hawking radiation has been established as a consequence of (7) and (11) in \[7\]. The essence of the situation is depicted in Fig.1 and will be briefly outlined. In the context of the pair-related approach described in the introduction the negative-energy (anti)-particle advances toward the future singularity forward in its proper time. Equivalently, a positive-energy particle emerges from the future singularity and transits backward in its proper time across the event horizon. A quantum of the scalar field $\Phi$ emerges at some point $B$ of the future singularity, transits as a positive-energy state backward in its proper time across the future event horizon $H^+$ where it is scattered by the background curvature at some point $C$ in the vicinity of the latter and continues its transition forward in time always as a positive-energy state before it registers its existence at space-time point $A \equiv (t_1', r_1', \theta_1', \phi_1')$ on a detector whose world line is characterised by the fixed radial coordinate $r_1'$ \[3\]. “Forward in time” now has the significance of forward in both, the particle’s proper time and the Schwarzschild coordinate time $t$. As $t$ is, at once, the distant observer’s proper time such a significance is of essence to the boundary condition of forward-time propagation for positive energy states which shall be analyzed in due course. The value $r_1'$ falls within the range stated in (9). The syncopated segment $BZ$ represents quantum propagation at radial values below the lower bound $r_1 = 1.980M$ in (13). The segment which is depicted by the continuous line from $Z \equiv (t_1, r_1, \theta_1, \phi_1)$ to some point on the future event horizon represents the contribution to the interior propagator which stems from the range of validity stated in (13). Consequently, the segment $ZCA$ represents the massless conformal scalar propagator whose singular part is given by (11) in the interior region of the Schwarzschild black hole and by (7) altogether in the corresponding exterior region respectively.

The probability for the emergence and transition of the quantum particle from point $Z$ in the interior region to the observation point $A$ is the square of the magnitude of the transition amplitude from $Z$ to $A$. That transition amplitude is itself equivalent to the functional integral over all paths which intercept $Z$ and $A$ on condition that they do not pass through the interior of the collapsing matter or extend to past infinity $J^- \[8\]$. As the Schwarzschild metric is regular on the event horizon when expressed in Kruskal coordinates the associated action functional is rigorously defined. The functional integral can thus be given the concrete meaning which it has in (1) through the analytic continuation of the Kruskal coordinates to a domain in which the Schwarzschild metric has signature $+4$.

The transition amplitude between $Z$ and $A$ expressed in terms of the scalar propagators in the exterior region and in the interior region respectively is a superposition over a complete set of states defined on the future event horizon $H^+$ of the extended Kruskal geometry \[7\]. Consequently, in the Euclidean sector of the Schwarzschild metric which is respectively expressed by (6) and by (10) in the context of (4) the transition amplitude $D(x'_1 - x_1)$ between the interior space-time point $(i\xi_1, i\zeta_1, i\theta_1, \phi_1)$ associated with $Z$ and the exterior space-time point $(\tau'_1, \rho'_1, \theta'_1, \phi'_1)$ associated with $A$ is necessarily expressed in terms of (7) and (11) through superposition over a complete set of states defined on the

\[1\]Backward, in terms of the Schwarzschild time-like coordinate $r < 2M$.

\[2\]Actually scalar fields distinguish neither between the two arrows of time nor, for that matter, between positive and negative energy states. This fact, however, is irrelevant to this heuristic picture.
Figure 1. Particle production by the Schwarzschild black hole in the Hartle-Hawking vacuum state. The segment \(ZCA\) corresponds to radial distances from the future event horizon \(H^+\) within which the analytic Green functions in [7] and [6] coincide with the exact massless scalar propagator. The fixed radial distance of the observer whose world line is also depicted in the diagram lies within the range of that coincidence.

Two-dimensional sphere at \(\rho = 0 \Leftrightarrow r = 2M\). That superposition is itself applied on the understanding that - with the angular dependence on \(\theta\) and \(\phi\) suppressed - the coordinate singularity has caused through (4) the two sections which respectively correspond to \(r \geq 2M\) and to \(r \leq 2M\) to be connected only at the point \(\rho = 0\).

In this context \(D(x_1' - x_1)\) reduces to\(^3\)

\[
< \rho_1', \theta_1', \phi_1', \tau_1'|i\zeta_1, i\theta_1, \phi_1, i\xi_1 > =
\]

\[
(2M)^2 \int_{-1}^{1} d\cos \theta_2 \int_{0}^{2\pi} d\phi_2 \left[ D_{as}(x_1' - x_2)D_{as}^{(int)}(x_2 - x_1) \right] +
\]

\[
(2M)^2 \int_{-1}^{1} d\cos \theta_2 \int_{0}^{2\pi} d\phi_2 \left[ D_{b}(x_1' - x_2)D_{b}^{(int)}(x_2 - x_1) \right]
\]

\(^3\)This corrects an obviously inadvertent and irrelevant typesetting error in the corresponding expression in [7].
and formally establishes through its asymptotic part $D_{as}(x'_1 - x_1)$ that within a finite advance of his proper time the observer registers at point $A$ radiation emitted by the Schwarzschild black hole.

Crucial to the derivation of black-hole radiation is the behaviour which the singular parts $D^{(ext)}_{as}(x_2 - x_1)$ and $D_{as}(x'_1 - x_2)$ respectively have as a consequence of the background geometry’s causal structure. Specifically, at $r_2 \to 2M^{-} \Leftrightarrow \zeta_2 \to 0$ the two time-independent terms in (11), being the only two non-vanishing terms, combine in the singular part of the propagator in (14) with $\lim_{r_2 \to 0} D_{as}(x'_1 - x_2)$ in such a manner as to result in \[7\]

$$D_{as}(x'_1 - x_1) = (2M)^2 \int_{-1}^{1} d\cos\theta_2 \int_{0}^{2\pi} d\phi_2 \left[ \frac{i}{8\pi M \sqrt{\zeta_1}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(i\hat{\theta}_2, \phi_2) Y_{lm}^*(i\hat{\theta}_1, \phi_1) \times \right.$$

$$\left. [e^{-i\frac{\pi}{4}u_0[0] \zeta_1}] \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta e^{i\theta} e^{-i\frac{\pi}{4}M \zeta_1} \int_{0}^{\infty} dw \frac{e^{\frac{iM}{16}w e^{-i\theta} \zeta_1}}{\pi M \rho_1} \right] \times$$

$$\left. e^{i\frac{\pi}{4}u_0[0] \zeta_1} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta e^{i\theta} e^{i\frac{\pi}{4}M \zeta_1} \int_{0}^{\infty} dw \frac{e^{\frac{iM}{16}w e^{-i\theta} \zeta_1}}{\pi M \rho_1} \right] \times$$

$$\frac{1}{M^2} \frac{1}{32\pi^2} \frac{1}{\sqrt{\rho_1}} \sum_{k=0}^{\infty} \sum_{p>0}^{\infty} e^{i\frac{\pi}{16}M n(8\pi M)} e^{-i\frac{\pi}{16}M \tau_1} \frac{2k+1}{p} P_k(\cos \gamma)$$

(15)

where $\cos \gamma = \cos \theta_2 \cos \theta_1 + \sin \theta_2 \sin \theta_1 \cos (\phi_2 - \phi_1)$ and $\tau_2 \to \infty \Rightarrow n \to \infty$, a fact which will be thoroughly analyzed in what follows. In \[7\] the singular part of the exterior propagator as appears in the last line of (15) also involves a multiplicative factor $C << 1$. The origin of that factor has been analyzed in \[6\] and its presence in \[7\] characterises the exterior propagator’s behaviour when one end-point of propagation is specified arbitrarily close to the event horizon. In $D_{as}(x'_1 - x_1)$ that factor cancels essentially against the corresponding factor inherent in the boundary condition that (11) diverge as $\zeta_2^{-\frac{1}{2}}$ at $\zeta_2 \to 0$.

The transition amplitude $D(x'_1 - x_1)$ in (14) is - with $R = 0$ - equivalent to (1) with

$$S[\Phi, \nabla \Phi] = \frac{1}{2} \int d^4 x [-g(x)]^{\frac{1}{2}} g^{\mu\nu}(x) \partial_\mu \Phi(x) \partial_\nu \Phi(x)$$

on condition that the Hartle-Hawking vacuum state of the field $\Phi(x)$ has been excited only to the stated quantum particle. The semi-classical limit is, in fact, inherent in the transition amplitude stated in (15) since in the exterior vicinity of the event horizon it is $\tau_2 \to \infty$ with $\tau_2 = n(8\pi M) ; n \to \infty$. In the present physical context this is, indeed, the essence of the semi-classical situation $\hbar \to 0$ which, as stated in the introduction, characterises the pair-production approach in the vicinity of the event horizon with significance which decreases as $\rho'_1$ increases. In effect, in the exterior vicinity of the event horizon the entire transition amplitude $D(x'_1 - x_1)$ in (14) reduces to (2). In addition, the fact that in the exterior vicinity of the event horizon the imaginary-time
coordinate $\tau_2$ is an integer multiple of $8\pi M$ is a necessary and sufficient condition for the thermal character of Hawking radiation at $T = (8\pi M)^{-1}$. This shall also be amply demonstrated in what follows.

No less important an aspect of black-hole radiation than the stated semi-classicality is the state of very high energy in which - as stated in the introduction - each emitted particle is in the exterior vicinity of the event horizon. In turn, this attribute has an immediate implication for the elliptic operator from whose eigenvalues and eigenstates the massless conformal scalar propagator was constructed in the Euclidean sector of the Schwarzschild metric [6]. That is, the dominant contribution to the semi-classical limit of $D(x'_1 - x_1)$ which describes the amplitude for the emitted particle to be detected in a local frame close to the event horizon comes from eigenvalues and eigenstates of very high order. Consequently, in addition to reducing to (2) in that frame, the entire transition amplitude $D(x'_1 - x_1)$ for a quantum of a scalar field also reduces to $D_{as}(x'_1 - x_1)$.

In order to explore the behaviour of (15) and, consequently, of the transition amplitude in (14) at the stated semi-classical limit it is necessary to convert the infinite series over $p$ to a contour integral. In anticipation of that it is

$$\sum_{k=0}^{\infty} \sum_{p>p_0} e^{i\frac{\pi}{4M}n(8\pi M)} e^{-i\frac{\pi}{4M}r'_1} \frac{2k+1}{p} P_k(\cos\gamma) =$$

(17) $$e^{i\frac{\rho_{n+1}}{4M}n(8\pi M)} \sum_{k=0}^{\infty} (2k+1) P_k(\cos\gamma) \sum_{p>p_0} \frac{1}{p} e^{i\frac{\rho-(p_0+1)}{4M}n(8\pi M)} e^{-i\frac{\rho}{4M}r'_1}$$

Upon extending $p \in R$ to $\tilde{p} \in C$ the residue theorem implies that (17) can be expressed as

(18) $$\frac{1}{2i} e^{i\frac{\rho_{n+1}}{4M}n(8\pi M)} \sum_{k=0}^{\infty} (2k+1) P_k(\cos\gamma) \oint_{c_{\tilde{p}}} d\tilde{p} \frac{(-1)^{\tilde{p}}}{\sin(\pi \tilde{p})} \frac{1}{\tilde{p}} e^{i\frac{\rho-(p_0+1)}{4M}n(8\pi M)} e^{-i\frac{\rho}{4M}r'_1}$$

where the infinite contour $c_{\tilde{p}}$ encompasses all integers bigger than some $p_0 \geq 1$ as in Fig.2.

Inspection of the expression in (18) reveals that at $n \to \infty$ the integrand oscillates with infinite frequency unless $\tilde{p} - (p_0 + 1) = |\tilde{p} - (p_0 + 1)|e^{i\theta}$ is, for all intents and purposes, a constant. Consequently, the only contribution to (18) stems from the relevant segment of $c_{\tilde{p}}$’s semi-circle along which $\cos\theta \sim -1$.

Since

(19) $$e^{i\frac{\rho-(p_0+1)}{4M}n(8\pi M)} = e^{i\frac{|\rho-(p_0+1)|}{4M}n(8\pi M)\cos\theta} e^{-i\frac{|\rho-(p_0+1)|}{4M}n(8\pi M)\sin\theta}$$

it follows that at $n \to \infty$ (17) reduces to

$$\frac{1}{2i} e^{i\frac{\rho_{n+1}}{4M}n(8\pi M)} \sum_{k=0}^{\infty} (2k+1) P_k(\cos\gamma) \times$$
\begin{align*}
    i\left|\tilde{p} - (p_0 + 1)\right| \int_{\pi-\epsilon}^{\pi+\epsilon} d\theta e^{i\theta} \frac{(-1)^{\tilde{p}}}{\sin(\pi \tilde{p})} \frac{1}{\tilde{p}} e^{i\tilde{p} - (\tilde{p} + 1)} n(8\pi M) \cos \theta e^{-\omega'(8\pi M)} e^{-i\tau'_{0+1}}
\end{align*}

with the value of the positive constant $\epsilon$ being close to zero and with

\begin{align*}
    \omega' = \frac{|\tilde{p} - (p_0 + 1)|}{4M} (n \sin \theta)
\end{align*}

For $\pi - \epsilon < \theta < \pi + \epsilon$ it is $\sin \theta \sim \pi - \theta = \chi \in [-\epsilon, \epsilon]$. As $\chi \to 0^+$ the limit $n \to \infty$ coincides with $\lim_{\chi \to 0^+} \frac{\xi}{\chi} = c > 0$ and imposes the restriction $0 < \omega' < \infty$. This restriction remains identically valid as integration over $\theta$ extends to negative values of $\chi$ which lie in a neighborhood of zero. This is the immediate consequence of the positive (counterclockwise) orientation of $c_{\tilde{p}}$ and will be further analyzed below.

The replacement of (20) in (15) reduces the latter to

\begin{align*}
    D_{as}(x'_1 - x_1) \sim e^{-8\pi M \omega'}
\end{align*}

Since, as stated, in the local frame defined close to the event horizon $D(x'_1 - x_1)$ reduces to $D_{as}(x'_1 - x_1)$ (22) also implies that in that frame it is

\begin{align*}
    D(x'_1 - x_1) \sim e^{-8\pi M \omega'}
\end{align*}

which is manifestly consistent with (2).
This allows for the evaluation of the probability $\Gamma(x'_1 - x_1)$ that an excitation of the massless conformal scalar field transit between $x_1$ and $x'_1$. Since the latter is the product between $D(x'_1 - x_1)$ and its conjugate expression it follows that

$$\Gamma(x'_1 - x_1) \sim e^{-8\pi M \omega}; \quad \omega = 2\omega'$$

The positive quantity $\omega$ in (24) can now be identified with the energy of the emitted particle. Such an identification is the necessary consequence of the positive orientation which was imposed on the contour $c_p$ in (18). A negative orientation of $c_p$ would have resulted in a physically unacceptable negative $\omega$. It can be seen, for that matter, that the positive orientation of $c_p$ signifies a boundary condition which stems from the physical demand that the emitted particles correspond to positive-energy states which propagate forward in time. The formal statement made in connection to (15) to the effect that the Schwarzschild black hole radiates must be supplemented with the stated boundary condition. It is that boundary condition which reflects a non-vanishing contribution to contour integration at $\tau_2 \to \infty$ while enforcing the stated demand of forward-time propagation for positive-energy states. In turn, that boundary condition which is essentially imposed on the future event horizon $H^+$ is additional to the boundary condition for vanishing propagation at spatial infinity imposed on the solution to the Green equation in [6] and completes the construction of the massless conformal scalar propagator which was achieved in [6] and [7] in the exterior and interior region of the Schwarzschild black hole respectively.

It should be remarked at this point that - in line with expectations - the identification of $\omega$ with the energy of the emitted particle advances (23) into coincidence with (2) since $8\pi M \omega'$ now has dimensions of action and - up to the logarithm of the entire factor which multiplies the exponential - it is, indeed, the Euclidean semi-classical action for the particle emitted by the Schwarzschild black hole in the Hartle-Hawking state.

An issue which merits attention in its own right is that raised in [9]. S. Hawking has effectively argued that it is because the particles which constitute the black-hole radiation emanate from the interior of the black hole about which an external observer has no knowledge that such an observer can not predict the amplitudes for them to be emitted. He can only predict the probabilities of emission without the phases. At first sight it would appear that the result expressed by (15) and (23) contradicts that statement. Such an appearance is deceptive for two reasons. The first is inherent in the factor $u_0[0]$, explicitly featured in (15). The origin of that factor can be traced to the separation of the exterior propagator in a singular part and a boundary part and has been thoroughly analyzed in [9]. As that separation is arbitrary so is $u_0[0]$. Any change in the latter necessarily effects a change in the boundary part of the exterior propagator in such a manner as to leave the exterior propagator intact. Not so, however, in the case of particle emission from the Schwarzschild black hole. As stated, the high-energy state of the strongly semi-classical particles emitted in the exterior vicinity of the event horizon brings the entire transition amplitude for each quantum of a scalar field into coincidence with that amplitude’s singular part. Consequently, $u_0[0]$ can no longer be arbitrary. The determination of $u_0[0]$ can now only be the result of a boundary condition imposed on the singular part of the exterior propagator. In turn, that boundary condition necessarily
relates to the physical context in the interior region about which the external observer has no knowledge and is, for that matter, unattainable.

The second ambiguity inherent in the transition amplitude expressed by (15) and (23) relates to the fact that in the immediate exterior vicinity of the event horizon the imaginary-time coordinate $\tau_2$ is an integer multiple of $8\pi M$. The precise significance of that statement is $\tau_2 = (n + c)(8\pi M) \; ; \; c \in R$ with $\lim_{\rho_2 \to 0} n = \infty$. As the preceding analysis reveals the presence of the constant $c$ will introduce yet another multiplicative phase factor in the entire factor which multiplies the exponential in (23). That factor is indeterminate since knowledge of the value of $c$ can, again, only be elicited by the physical context in the interior region and is, for that matter, unattainable by the exterior observer.

In conclusion, the transition probability in (24) signifies a Boltzmann factor for a particle of energy $\omega$ at Hawking temperature $T = \frac{1}{8\pi M}$. That, in turn, expresses the thermal character of Hawking radiation. This result has been established in a local frame close to the Schwarzschild event horizon. The Schwarzschild radial coordinate $r$ associated with that frame falls within the range of validity stated in (9). Through further superposition over a complete set of states defined on a hypersurface which includes the event of that particle’s registration in that frame the same result can trivially be extended to the “inertial” observer at $r \to \infty$.

III. The Detraction

The preceding analysis can now be extended to the detraction from the thermal character in which the tunneling approach in [3] and [4] results. As stated in the introduction that approach has the distinctive feature of treating each emitted quantum particle as the result of such tunneling across the event horizon as is characterised by a barrier which crucially depends on the tunneling particle itself. Specifically, whereas in the original approach analyzed in the previous section the emission of a particle of energy $\omega$ necessarily corresponds to only one entangled quantum state (that is, to only one pair of quantum particles) which originates just above the event horizon in a fixed exterior geometry, as in Fig.1, the approach in [3] and [4] involves two possible entangled quantum states one of which corresponds to the description in Fig.1 with emitted energy equal to $\omega$ and the other one of which originates just below the event horizon and corresponds to the tunneling of a particle of energy $\omega$ forward in the particle’s proper time. In the context of either entangled state energy conservation causes the event horizon to contract to a smaller Schwarzschild radius.

The event horizon in its initial location and in the final location to which it contracts yields respectively two null hypersurfaces each of which qualifies for a classical turning point. The separation of these two hypersurfaces corresponds to the classically inaccessible region through which the semi-classical particle must tunnel as a condition for its emission. For that matter, this approach to black-hole radiation actively involves the event horizon and, indeed, causes its exterior vicinity to be stationary but not static. This renders the expression of the Schwarzschild metric in Schwarzschild coordinates in (3) inadequate for the description of such a situation. A particularly suitable coordinate system which renders that stationary character of the exterior region manifest and which, at once, is regular at $r = 2M$ is that of Gullstrand-Painlevé coordinates.
with the temporal coordinate $t$ being distinct from the corresponding Schwarzschild radial coordinate in (3).

The self-gravitating character of the emitted particle of energy $\omega$ eventually modifies the background geometry to

$$(26) \quad ds^2 = -(1 - \frac{2(M - \omega)}{r})dt^2 + 2\sqrt{\frac{2(M - \omega)}{r}}dtdr + dr^2 + r^2d\Omega^2$$

An essential digression is in order at this point. In Planck units $M$ has dimensions of length. For that matter, so does $\omega$ in (26). Consequently, $\omega$ in (26) does not coincide with $\omega$ in (24) since - as (21) explicitly shows - the latter has dimensions of mass. In spite of that, the $\omega$ which appears in (26) unduly multiplies the expression $M - \frac{\omega}{2}$ in the final result for the emission probability $\Gamma \sim e^{-8\pi\omega(M - \frac{\omega}{2})}$ and in all associated expressions which appear in [3] resulting in length-dimensionality of order two in the exponent. Taken at “face value” such a result is incorrect as it identically contradicts the fact that the exponent which it features must have dimensions of action. A careful consideration of the mass dimensionality in the derivation cited in [3] reveals, indeed, that the $\omega$ which multiplies $M - \frac{\omega}{2}$ in that exponent is of mass-dimensionality of order one and coincides, for that matter, with the $\omega$ in (24) whereas the $\omega$ which appears in $M - \frac{\omega}{2}$ and in (26) is of length dimensionality of order one. In order to bring my result into line with that in [3] I shall, in what follows, adhere to the stated convention which the authors of [3] have made.

The contraction of the event horizon is now manifest in (26). A particle of positive energy $\omega$ emerges at the future singularity and transits backward in its proper time to a point at $r_{in} = 2M - \epsilon; \epsilon \to 0^+$ whence it tunnels through the contracting future event horizon forward in its proper time to $r_{out} = 2(M - \omega) + \epsilon; \epsilon \to 0^+$. The separation between the event horizon at Schwarzschild radius equal to $2M$ and that at $2(M - \omega)$ defines the classically inaccessible region of this tunneling process. The other entangled state associated with the emission of a particle of energy $\omega$ emerges at space-time point $C$ just above the future event horizon and involves two semi-classical particles each of which is of positive energy $\omega$. Upon crossing the event horizon inward one of the two particles reverses the sign of its energy to $-\omega$ with respect to spatial infinity. Equivalently, that particle emerges at the future singularity in a positive-energy state and propagates backward in its proper time until space-time point $C$ where it is scattered by the background curvature in forward-time propagation to the observer at space-time point $A$. Again, energy conservation causes the event horizon to contract from its initial location at Schwarzschild radius equal to $2M$ to a final location at Schwarzschild radius equal to $2(M - \omega)$. The distinction between these two entangled states has no operational significance for the observer whose world line is depicted in Fig.1. Since both entangled quantum states contribute to the total probability of emission his observation
at space-time point $A$ consists in the particle of energy $\omega$ itself and in the above-stated contraction of the event horizon.

The evaluation of the probability that a scalar particle of energy $\omega$ be emitted in a physical context which is characterised by an evolving event horizon - equivalently, the evaluation of that particle’s transition probability - in terms of the propagators in [6] and [7] can be accomplished through a careful analysis of the relation between that evolution and those propagators. To that end, all aspects of the analytical procedure which was, in the previous section, applied to (15) will be re-assessed in the present physical context. In what follows the emitted scalar quantum will be treated as a spherical shell. Such an approach is justified by the fact that the emission of a scalar-field configuration by the Schwarzschild black hole occurs primarily as an s-wave [10].

As observed from spatial infinity the sum-total of the hole’s mass and that corresponding to the energy of the emitted spherical shell is constant while the mass of the hole varies according to the energy of the emitted shell. By extension, such will also be the case in the frame of the observer at $A$ in Fig.1. Energy conservation is the underlying cause of that situation as a consequence of which in the course of emission the background space-time geometry becomes dynamic in the exterior vicinity of the event horizon and the massless semi-classical particle transits on a null geodesic. That geodesic corresponds to the metric in (26) and describes the motion of a particle of energy $\omega$ in the stationary region of space-time bounded by the original location of the contracting event horizon at $r_s = 2M$ and by its final location at $r'_s = 2(M - \omega)$ [3]. This is the essence of the situation in the context of either entangled state.

In the transition amplitude which corresponds to (15) the quantity $M$ determines the background curvature of the Schwarzschild black-hole geometry on which the quantum scalar excitation propagates. In the present physical context the background curvature is determined by the constant total mass $(M - \omega) + \omega$ as observed at $A$. Since this coincides with the mass of the Schwarzschild black hole prior to the emission of the spherical shell so does the quantity $M$ in the transition amplitude which corresponds to (15). In fact, all space-time points in the exterior region retain their $r$-values in the course of the shell’s propagation. That concerns, all the same, the set of points at $r = 2M$ which defined the event horizon prior to the emission. As a consequence, in the event of emission $\rho'_1$ also retains the value specified by (4).

In the present context, for that matter, the mathematical expression in (15) remains - at this stage - identically the same. If the analytical evaluation in the previous section were to be applied in this context the same arbitrary positive constant $\omega'$ would emerge. This, directly calls the relation of $\omega'$ to the emitted energy $\omega$ into question. The present physical context, which is centrally determined by energy conservation, involves two entangled quantum states as opposed to the one represented in Fig.1. Both of these states contribute to the transition amplitude. This fact, however, concerns exclusively the difference in the probability for emission of energy equal to $\omega$ in the two distinct contexts. Since in either context that probability concerns a particle of energy equal to

\[\text{[10].} \]
ω it follows that, pursuant to (24), the transition amplitude which is determined by the present physical context is also characterised by ω' = \( \frac{\pi}{2} \).

The expression in (24) signifies the characteristic Boltzmann factor for the thermal emission of a particle of energy ω at temperature \( T = [8\pi M]^{-1} \) and is, in the present context, physically inconsistent in view of the emitted particle's self-gravitating character. Such an inconsistency is a direct consequence of the fact that the interpretation of the physical content of (15) is incomplete. In order to properly take the reduction of the hole's mass by ω into consideration use must be made of the fact that, prior to the emission, in the vicinity of the event horizon it is \( \tau_2 = n(8\pi M) \); \( n \to \infty \). The replacement of the temporal period which appears in this mathematical statement by any positive constant leaves the latter invariant. In the present physical context which is centrally characterised by a contracting event horizon that positive constant will necessarily be determined by the constraint which energy conservation imposes on the system. The latter demands that the initial mass \( M \) of the black hole be reduced by ω in the event of emission of energy equal to \( \omega \). Since - as shown above - the emitted energy \( \omega \) corresponds to \( \frac{\pi}{2} \) in the transition amplitude it follows that the hole’s mass \( M \) must, in the temporal period \( 8\pi M \) which appears in (15), be reduced by the same amount. The replacement of \( 8\pi M \) by \( 8\pi (M - \frac{\omega}{2}) \) does not contradict the stated constancy of the total mass \( M \) because the temporal period in the Euclidean sector of the Schwarzschild metric is determined exclusively by the mass of the black hole, which - as stated - is variable.

Equivalently, the demand for energy conservation must naturally be imposed on the Euclidean semi-classical action which - pursuant to the analysis in section II - is equal to \( \frac{\pi}{2}(8\pi M) \) and inherent in (15). Since, in that action, the emitted energy \( \omega \) corresponds to \( \frac{\pi}{2} \) the expression \( n(8\pi M) \) which appears in the temporal sector of (15) must, as a result of the demand for conservation of energy, be replaced by \( n8\pi (M - \frac{\omega}{2}) \).

In effect, the constraint imposed on the transition amplitude between points \( Z \) and \( A \) by the demand for conservation of energy yields

\[
D(x_1' - x_1) = \int_{-1}^{1} d\cos \theta_2 \int_0^{2\pi} d\phi_2 \left[ \frac{i}{8\pi M} \sqrt{\zeta_1} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(i\theta_2, \phi_2) Y_{lm}^*(i\theta_1, \phi_1) \times \right.
\]

\[
\left[ e^{-i\frac{\pi}{4\pi}u_0[0]_1} \right] \frac{3\pi}{2} \int \frac{d\theta e^{i\theta} e^{-i\frac{3\pi}{16\pi}e^{-i\theta} \zeta_1}}{2\pi^2(u_0[0])e^{i\theta} + i(w)^2 + 4(l^2 + l + 1)}
\]

\[
e^{i\frac{\pi}{4\pi}u_0[0]_1} \zeta_1 \frac{3\pi}{2} \int \frac{d\theta e^{i\theta} e^{-i\frac{3\pi}{16\pi}e^{-i\theta} \zeta_1}}{2\pi^2(u_0[0])e^{i\theta} - i(w)^2 + 4(l^2 + l + 1)} \times
\]

\[
(27) \quad \frac{1}{8\pi^2} \sqrt{\rho_1} \sum_{k=0}^{\infty} \sum_{p > p_0 > 0} e^{i\frac{\pi}{4\pi}n8\pi(M - \frac{\omega}{2})} e^{-i\frac{\pi}{4\pi}r_1} \frac{2k + 1}{p} P_k(\cos \gamma)
\]

It can be seen, for that matter, that the reduction of the hole's mass effectively defines a new temporal period which reflects the associated contraction of the event horizon and, thereby, the dynamic character of the background geometry as a consequence of vacuum activity. Clearly, the semi-classical propagation along the null geodesics between
\( r_{\text{in}} = 2M - \epsilon \) and \( r_{\text{out}} = 2(M - \omega) + \epsilon \) and between \( r_{\text{out}} = 2M + \epsilon \) and \( r_{\text{in}} = 2(M - \omega) - \epsilon \) respectively for either entangled quantum state is inherent in (27) and translates to the reduction of the temporal period from \( 8\pi M \) to \( 8\pi (M - \frac{\omega}{2}) \).

The transition amplitude in (27) is now consistent with energy conservation, with the concomitant consequence of a contracting event horizon and with a Schwarzschild geometry which corresponds to a constant total mass in the course of emission. Consequently, in the context of the demand for energy conservation (27) is the correct expression for the massless conformal scalar propagator between points \( Z \) and \( A \).

The formal procedure which was applied to (15) may now be, consistently, applied to (27). That procedure reduces the transition amplitude in (27) to

\[
D(x'_1 - x_1) \sim e^{-8\pi(M - \frac{\omega}{2})}
\]

The expression in (27) - and accordingly that in (28) - is the desired transition amplitude for the emission of a self-gravitating particle of energy \( \omega \) by the Schwarzschild black hole of mass \( M \). In addition to being consistent with energy conservation the expression in (28) also naturally reduces to that in (23) in the event that \( \omega \ll M \). Crucial to the derivation of the transition amplitude in (27) has been the demand for transition between states of the same total energy.

The transition probability \( \Gamma(x'_1 - x_1) \) for an excitation of the massless conformal scalar field between \( x_1 \) and \( x'_1 \) is the product between \( D(x'_1 - x_1) \) and its conjugate expression. From (28) it follows that

\[
\Gamma(x'_1 - x_1) \sim e^{-8\pi \omega(M - \frac{\omega}{2})}
\]

which coincides with the result in [3].

The detraction from the thermal character of Hawking radiation enforced by energy conservation is, in (29), manifest in the \( \omega^2 \)-term. As was also the case in the thermal context of the previous section crucial to the derivation of this result has been the boundary condition of positive orientation for the contour along which the corresponding integration occurs in the complex plane. It is invariably that boundary condition which enforces the physical demand that the emitted particles correspond to positive-energy states which propagate forward in time.

**IV. Conclusions**

The propagator for a conformal scalar field in the Hartle-Hawking state was developed in [6] as an explicitly analytic function of the background geometry for a finite range of values of the Schwarzschild radial coordinate above the Schwarzschild radius. That propagator was analytically extended into the interior region of the Schwarzschild black hole in [7] in which it was also established that the boundary condition which the causal structure of the Schwarzschild black-hole space-time imposes on the propagator in the interior region results in particle production by the Schwarzschild black hole. The entire analysis in this project has established that the particle production which the massless conformal scalar propagator in [6] and [7] describes corresponds to Hawking radiation of thermal character if the exterior background geometry is treated as a fixed background to quantum propagation and to Hawking radiation which detracts from thermality if
energy conservation is enforced. Central to the derivation of Hawking radiation and its character in either case is the semi-classical limit $\hbar \to 0$ which characterises each emitted scalar quantum in the exterior vicinity of the event horizon. That, in turn, imposes the condition that in the exterior vicinity of the event horizon the imaginary temporal coordinate be, in the thermal case, an integer multiple of its period $8\pi M$. Although this statement remains formally valid if the radiation detracts from thermality the demand for energy conservation enforces the condition that in the vicinity of the event horizon the imaginary temporal coordinate be, instead, an integer multiple of the effective period $8\pi(M - \frac{\omega^2}{2})$ for each emitted quantum of energy $\omega$.

Crucial to the derivation of Hawking radiation from the scalar propagators in [6] and [7] is the boundary condition that the contour of integration in, what is essentially, the energy complex plane be of positive orientation. That boundary condition is imposed on the propagator in the exterior vicinity of the event horizon and enforces the physical demand that particles emitted by the black hole correspond to positive-energy states which propagate forward in time. That boundary condition completes, for that matter, the construction of the massless conformal scalar propagator on the Schwarzschild black-hole space-time which was initiated in [6]. The above-stated boundary condition imposed by space-time’s causal structure in the interior region, that imposed in the exterior vicinity of the event horizon by the demand for forward-time propagation of positive-energy states and the boundary condition of vanishing propagation at spatial infinity are the three boundary conditions which characterise the scalar propagator on the Schwarzschild black-hole geometry.

The derivation of Hawking radiation from the scalar propagators in [6] and [7] supports the case made by S. Hawking to the effect that an external observer can only predict the probability for particle emission but not the amplitude for that emission. The scalar propagators in [6] and, especially, in [7] are particularly involved. However, the derivation of Hawking radiation - both, of thermal character and of a character which detracts from thermality - is indicative of the potential for enormous simplification which the causal structure of space-time has in any calculational context which describes conformal scalar propagation on the Schwarzschild black-hole geometry.

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