REGULAR ATTRACTORS OF ASYMPOTOTICALLY AUTONOMOUS STOCHASTIC 3D BRINKMAN-FORCHHEIMER EQUATIONS WITH DELAYS

QIANGHENG ZHANG AND YANGRONG LI*

School of Mathematics and Statistics
Southwest University
Chongqing 400715, China

(Communicated by Alain Miranville)

Abstract. We study asymptotically autonomous dynamics for non-autonomous stochastic 3D Brinkman-Forchheimer equations with general delays (containing variable delay and distributed delay). We first prove the existence of a pullback random attractor not only in the initial space but also in the regular space. We then prove that, under the topology of the regular space, the time-fibre of the pullback random attractor semi-converges to the random attractor of the autonomous stochastic equation as the time-parameter goes to minus infinity. The general delay force is assumed to be pointwise Lipschitz continuous only, which relaxes the uniform Lipschitz condition in the literature and includes more examples.

1. Introduction. Let \( \Omega \) be a bounded domain in the cube Euclid space with smooth boundary \( \partial \Omega \). We consider the following stochastic 3D Brinkman-Forchheimer equations with general delays and time-dependent forces:

\[
\begin{aligned}
&du - \nu \Delta u dt + \nabla pdt = (f(t, u_t) + g(t, x) - h(u)) dt + \varphi(x) dW, \\
&\nabla \cdot u = 0 \text{ on } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad t > \tau, \\
&u(\tau + \xi, x) = \psi(\xi, x), \xi \in [-\varrho, 0], x \in \Omega, \tau \in \mathbb{R},
\end{aligned}
\]

where \( h(u) = (\alpha + \beta |u| + \gamma |u|^2)u \), \( u = (u_1, u_2, u_3) \) is the velocity vector, \( u_t \) denotes the delay shift, \( p \) is the pressure and \( f \) is the general delay force (which includes variable delay and distributed delay).

The deterministic BF equations describe the motion of fluid flow in a saturated porous medium, see [9, 19]. All positive constants \( \nu, \alpha, \beta, \gamma \) have their physical meanings: \( \nu \) is the Brinkman effective viscosity, \( \alpha \) is the Darcy coefficient (viscosity divided by permeability), \( \beta \) and \( \gamma \) are Forchheimer coefficients. The BF equation is the limit of Navier-Stokes equations when the coefficient (density) of the fluid term \( c(u \cdot \nabla)u \) goes to zero.

The dynamics of deterministic BF equations has been studied in [10, 11, 14, 18, 21, 24, 26]. In this article, we consider the long term behavior of the BF delay model...
when the fluid flow is affected by random factors such as the Brownian motion \( W \) on a probability space \((\Omega, \mathcal{F}, P)\).

As the usual discussions [4, 23] for other delay equations, we take the phase spaces by two spaces of vector-valued continuous functions:

\[
H_\varrho = C([-\varrho, 0], H), \quad V_\varrho = C([-\varrho, 0], V),
\]

where \( H \) and \( V \) are subspaces of \((L^2(\mathcal{O}))^3\) and \((H^1_0(\mathcal{O}))^3\), respectively.

The random attractor is an important tool that describes the dynamics of stochastic PDEs, see [2, 7, 15] for the autonomous case and [3, 20, 22] for the non-autonomous case.

The preliminary objective in this article is to prove the existence of pullback random attractors in \( H_\varrho \). While, a further objective is to establish the regularity of random attractors. Two objectives will be achieved by proving the existence of a bi-spatial random attractor in \((H_\varrho, V_\varrho)\).

More precisely, we will prove the existence of a \((H_\varrho, V_\varrho)\)-random attractor \( A = \{ A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) such that, under the topology of \( V_\varrho \), each \( A(\tau, \omega) \) is compact and attracts all initial data in \( H_\varrho \). In addition, we will prove the measurability of \( \omega \to A(\tau, \omega) \) not only in the initial space \( H_\varrho \) but also in the end space \( V_\varrho \), which expands the discussions [15, 17] for the measurability of a bi-spatial attractor in the initial space only.

Due to the time-dependence of the pullback random attractor \( A(\tau, \omega) \), a natural problem is to consider the following asymptotic convergence of random attractors:

\[
\lim_{\tau \to -\infty} \text{dist}_X(A(\tau, \omega), A_\infty(\omega)) = 0, \quad \forall \omega \in \Omega, \tag{1.2}
\]

where \( A_\infty(\omega) \) is the random attractor for the autonomous stochastic BF equation (two time-independent forces \( f_\infty, g_\infty \) instead of time-dependent forces \( f, g \) in (1.1)).

We will prove the convergence (1.2) under the topology of \( X = V_\varrho \) as well as \( X = H_\varrho \).

The subject of asymptotic autonomy of (deterministic) pullback attractors as in (1.2) was discussed by Kloeden et al. [12, 13] and Li et al. [16] in the initial space \( X = H \) (for deterministic non-delay equations). The method in these references can be expanded to prove the asymptotic convergence (1.2) of random attractors in the initial space \( X = H_\varrho \). However, the subject about asymptotic autonomy of attractors in the regular space seems to be new even in the deterministic non-delayed case. Some difficulties arise from proving the asymptotic autonomy (1.2) in the end space \( V_\varrho \).

On one hand, we need to prove the bi-spatial random attractor \( A \) is backward compact in the end space \( X = V_\varrho \) (other than in \( H_\varrho \) only), more precisely, the backward union \( \cup_{s \leq \tau} A(s, \omega) \) is relatively compact in \( V_\varrho \), which needs us to overcome the difficulty of non-compactness of \( V_\varrho \to H_\varrho \) (although the embedding is continuous).

On the other hand, as we know, in any abstract results about the upper semi-continuity of random attractors, the convergence of random dynamical systems is a key criterion. However, we can verify the convergence of solutions in the initial space \( H_\varrho \) only:

\[
\lim_{\tau \to -\infty} \| \Phi(t, \tau, \omega)v_\tau - \Phi_\infty(t, \omega)v \|_{H_\varrho} = 0, \quad \text{as} \quad \| v_\tau - v \|_{H_\varrho} \to 0. \tag{1.3}
\]

The convergence (1.3) can be applied to show the asymptotic autonomy of random attractors in \( H_\varrho \). It is slightly surprise that the convergence (1.3) in \( H_\varrho \) (together with the backward asymptotic compactness in \( V_\varrho \)) can be applied to prove the upper
Uhlenbeck equation:

$$dz_t = \omega_t \cdot dW_t$$

We use some weaker assumptions than [11, 14] for the delay force $f$. In fact, we assume that $f$ is pointwise Lipschitz continuous (instead of the uniform Lipschitz continuity) with a positive function $L_f(\cdot)$ of Lipschitz bounds. We need not to restrict the upper bound of the Lipschitz bound $L_f(\cdot)$, which relaxes the conditions in the literature [4, 23].

Another difficulty arises from proving the measurability of pullback attractors even in $H_\epsilon$ (let alone in $V_\epsilon$). In our problem, the absorbing set is an uncountable union of some random sets (pointwise absorbing sets) and thus its measurability is unknown. But we can prove that the attractors do not change under two special union of some random sets (pointwise absorbing sets) and thus its measurability is also unknown. But we can prove that the attractors do not change under two special.

The paper is organized as follows. In section 2, we define a non-autonomous random dynamical system via the well-posedness of equations. In section 3, we derive the backward uniform estimates of solutions. In section 4, we prove the backward compactness as well as existence of a bi-spatial random attractor in $(H_\epsilon, V_\epsilon)$. In section 5, we establish the asymptotic autonomy (1.2) of random attractors in the regular space $V_\epsilon$ and provide two important examples for the general delay force, which include variable delay and distributed delay.

2. Stochastic Brinkman-Forchheimer equations with delays. We need to reset the stochastic equation (1.1). As usual [1], we identify the scalar Wiener process $W(\cdot, \omega)$ with the path $\omega(\cdot)$ on the measurable flow $(\Omega, \mathcal{F}, P, \theta)$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with the compact-open topology, $\mathcal{F}$ is the Borel $\sigma$-algebra, $P$ is the two-sided Wiener measure and the basic flow is defined by $\theta_t(\omega(\cdot)) = \omega(\cdot + t) - \omega(t)$ for $t \in \mathbb{R}$. Then the unique stationary solution of the Ornstein-Uhlenbeck equation: $dz + zdW$ can be denoted by $z(t, \omega) = z(\theta_t(\omega))$ such that the path $t \rightarrow z(\theta_t(\omega))$ is continuous and for $P$-a.s. $\omega \in \Omega$,

$$\lim_{t \to \pm \infty} \frac{z(\theta_t(\omega))}{t} = \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t z(\theta_r(\omega)) dr = 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t |z(\theta_r(\omega))| dr = \mathbb{E}|z|. \quad (2.1)$$

By the change of variables

$$v(t, \tau, \omega, \phi) = u(t, \tau, \omega, \psi) - \varphi(x)z(\theta_t(\omega)), \quad t \geq \tau, \quad \omega \in \Omega, \quad (2.2)$$

Eq. (1.1) can be rewritten as the following random equation:

$$\begin{aligned}
\frac{\partial v}{\partial t} - v \Delta v + \alpha v + \nabla p + \beta |v + \varphi z|(v + \varphi z) + \gamma |v + \varphi z|^2(v + \varphi z) \\
= f(t, v_t + \varphi z(\theta_{t+}(\omega)) + g(t, x) + z(\theta_{t}(\omega))(\nu \Delta \varphi - \alpha \varphi + \varphi),
\end{aligned} \quad \nabla \cdot v = 0 \text{ on } \mathcal{O}, \quad v = 0 \text{ on } \partial \mathcal{O}, \quad t > \tau,

v(\tau + \xi, x) := \phi(\xi, x), \quad \xi \in [-\rho, 0], \quad x \in \mathcal{O}, \quad \tau \in \mathbb{R},

\text{where the delay shift is defined by } u_t(\xi) = u(t + \xi) \text{ for all } \xi \in [-\rho, 0] \text{ and } t \geq \tau. \text{ Let } L^q(\mathcal{O}) = (L^q(\mathcal{O}))^3, \mathbb{H}^q(\mathcal{O}) = (H^q(\mathcal{O}))^3 \text{ and denote by}

$$H = \{u \in L^2(\mathcal{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathcal{O}} = 0\}, \quad V = \{u \in H^1(\mathcal{O}) : \nabla \cdot u = 0, u|_{\partial \mathcal{O}} = 0\},$$

where $n$ is the unit outward normal vector at $\partial \mathcal{O}$. Then $H$ is a closed subspace of $L^2(\mathcal{O})$ whose norm is denoted by $|| \cdot ||$. $V$ is a separable Hilbert space with the inner
product and norm: For all \( u = (u_1, u_2, u_3), \, v = (v_1, v_2, v_3) \in V, \)
\[
((u, v)) = \sum_{i=1}^{3} \int_{\Omega} \nabla u_i \cdot \nabla v_i \, dx, \quad \|u\|_{V}^2 = ((u, u)).
\]

For each memory time \( \varrho > 0, \) we denote by \( H_{\varrho} := C([-\varrho, 0], H) \) and \( V_{\varrho} := C([-\varrho, 0], V) \) equipped with the norms:
\[
\|\varrho\|_{H_{\varrho}} = \sup_{\xi \in [-\varrho, 0]} \|\varrho(\xi)\|, \quad \|\varrho\|_{V_{\varrho}} = \sup_{\xi \in [-\varrho, 0]} \|\varrho(\xi)\|_{V}.
\]

The delay force \( f: \mathbb{R} \times H_{\varrho} \to L^{2}(\mathcal{O}) \) is assumed to fulfill the following conditions.

\(\text{(F1)}\) For each \( \varrho \in H_{\varrho}, \) the mapping \( t \to f(t, \varrho) \) is measurable from \( \mathbb{R} \) to \( L^{2}(\mathcal{O}). \)

\(\text{(F2)}\) \( f(t, 0) = 0 \) for all \( t \in \mathbb{R}. \)

\(\text{(F3)}\) There is a positive continuous function \( L_{f}(\cdot) \) such that
\[
\|f(t, \varrho_1) - f(t, \varrho_2)\| \leq L_{f}(t) \|\varrho_1 - \varrho_2\|_{H_{\varrho}}, \quad \forall t \in \mathbb{R}, \quad \varrho_1, \varrho_2 \in H_{\varrho},
\]
where \( L_{f}(\cdot) \) is \textit{backward limitable}:
\[
\lim_{\lambda \to +\infty} \sup_{s \leq \tau} \int_{-\infty}^{s} e^{\lambda(r-s)} L_{f}^{2}(r) \, dr = 0, \quad \forall \tau \in \mathbb{R}. \quad (2.5)
\]

\(\text{(F4)}\) There are two positive constants \( m_{f} \) and \( c_{f} > 0 \) such that for all \( t \geq \tau \) and \( u, v \in C([\tau - \varrho, t]; H), \)
\[
\int_{\tau}^{t} e^{m_{f}r} \|f(r, u_{r}) - f(r, v_{r})\|^{2} \, dr \leq c_{f}^{2} \int_{\tau - \varrho}^{t} e^{m_{f}r} \|u(r) - v(r)\|^{2} \, dr. \quad (2.6)
\]

Two examples of \( f \) with \( \text{(F1)}-\text{(F4)} \) will be given later. We also need make the assumptions about the time-dependent force \( g. \)

\(\text{(G1)}\) \( g \in L^{2}_{\text{loc}}(\mathbb{R}, H) \) is \textit{backward limitable}:
\[
\lim_{b \to +\infty} \sup_{s \leq \tau} \int_{-\infty}^{s} e^{b(r-s)} \|g(r)\|^{2} \, dr = 0, \quad \forall \tau \in \mathbb{R}. \quad (2.7)
\]

Lemma 2.1. Assume \( g \in L^{2}_{\text{loc}}(\mathbb{R}, H) \) is backward limitable, then \( g \) is backward tempered:
\[
G(b) := \sup_{s \leq \tau} \int_{-\infty}^{s} e^{b(r-s)} \|g(r)\|^{2} \, dr < +\infty, \quad \forall b > 0, \quad \tau \in \mathbb{R}. \quad (2.8)
\]

Proof. By \( (2.7), \) there is a \( b_{0} > 0 \) such that
\[
G(b_{0}) = \sup_{s \leq \tau} \int_{-\infty}^{s} e^{b_{0}(r-s)} \|g(r)\|^{2} \, dr = \sup_{s \leq \tau} \int_{-\infty}^{0} e^{b_{0}r} \|g(r + s)\|^{2} \, dr < +\infty.
\]
If \( b \geq b_{0}, \) we have \( G(b) \leq G(b_{0}) < +\infty. \) If \( 0 < b < b_{0}, \)
\[
G(b) = \sup_{s \leq \tau} \int_{-\infty}^{s} e^{b(r-s)} \|g(r)\|^{2} \, dr = \sup_{s \leq \tau} \sum_{n=0}^{+\infty} \int_{s-n}^{s-n-1} e^{b(r-s)} \|g(r)\|^{2} \, dr
\]
\[
\leq \sum_{n=0}^{+\infty} e^{-bn} \sup_{s \leq \tau} \int_{s-n}^{s-n-1} \|g(r)\|^{2} \, dr \leq \sum_{n=0}^{+\infty} e^{-bn} e^{b_{0}} \sup_{s \leq \tau} \int_{s-n}^{s} e^{b_{0}(r-s)} \|g(r)\|^{2} \, dr
\]
\[
= \frac{e^{b_{0}}}{1 - e^{-b}} \sup_{s \leq \tau} \int_{-\infty}^{0} e^{b_{0}r} \|g(r + s)\|^{2} \, dr < +\infty.
\]
Then \( (2.8) \) holds true for any rate \( b > 0. \) \qed
The above lemma indicates the backward tempered condition in [25] need not to be repeatedly assumed. Similarly, by (2.5) we have

\[
\sup_{s \leq r} \int_{-\infty}^{s} e^{b(r-s)} L^2_f(r) dr < +\infty, \ \forall b > 0, \ \tau \in \mathbb{R}. \quad (2.9)
\]

Let \( \hat{P} \) be the Helmholtz-Leray orthogonal projection from \( L^2(\Omega) \) onto \( H \). Applying \( \hat{P} \) to Eq. (2.3), we obtain that the following abstract equation:

\[
\begin{align*}
\begin{cases}
\frac{\partial v}{\partial t} + \nu Av + \hat{P}(\alpha v + \beta v + \gamma v + \varphi z + \gamma |v + \varphi z|^2(v + \varphi z)) \\
v(\tau + \xi, x) = \phi(\xi, x), \ \xi \in [-\rho, 0],
\end{cases}
\end{align*}
\]

(2.10)

where \( A = -\hat{P} \Delta \) is defined by \( \langle Au, v \rangle = \langle (u, v) \rangle \) and we always assume \( \varphi \in V \cap H^2(\mathcal{O}) \).

As the equation (2.10) is a deterministic equation parameterized by \( \omega \in \Omega \), by the standard Faedo-Galerkin method, one can prove the well-posedness (see [11, 14]).

**Lemma 2.2.** Suppose (F1)-(F4) hold and \( g \in L^2_{loc}(\mathbb{R}, H) \). Then for all \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( \phi \in H_\varphi \), Eq. (2.10) has a unique weak solution

\[
v \in C([\tau - \rho, +\infty); H) \cap L^2_{loc}(\tau, +\infty; V) \cap L^4_{loc}(\tau, +\infty; \mathbb{L}^4(\mathcal{O})),
\]

such that \( v(\tau + \xi, \tau, \omega, \phi) = \phi(\xi) \) for all \( \xi \in [-\rho, 0] \). Furthermore, for any \( \varepsilon > 0 \) and \( T > \tau + \varepsilon \)

\[
v \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A)),
\]

where \( D(A) = H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}) \).

We define a mapping \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H_\varphi \rightarrow H_\varphi \) by

\[
\Phi(t, \tau, \omega) \phi = v_{t+s}(-, \tau, \theta_{-\tau} \omega, \phi), \ t \geq 0, \ \tau \in \mathbb{R}, \ \omega \in \Omega. \quad (2.11)
\]

By the same method as given in [8], we have \( \omega \rightarrow \Phi(t, \tau, \omega) \phi \) is \( \mathcal{F} \)-measurable. Furthermore, the uniqueness of solutions implies the cocycle property: \( \Phi(0, \tau, \omega) = id_{H_\varphi} \) and

\[
\Phi(t + s, \tau, \omega) = \Phi(t, \tau + s, \theta_s \omega) \Phi(s, \tau, \omega), \ t, s \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega.
\]

Hence, \( \Phi \) is a continuous cocycle in the sense of [22]. By (2.2), we can also define a continuous cocycle \( \Psi \) associated with (1.1) is defined by

\[
\Psi(t, \tau, \omega) \psi = v_{t+s}(-, \tau, \theta_{-\tau} \omega, \psi), \ t \geq 0, \ \tau \in \mathbb{R}, \ \omega \in \Omega.
\]

However, we only consider \( \Phi \) in this paper since it is equivalent to \( \Psi \) (see [5, Theorem 3.4]). Let \( \mathcal{D} = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) be a family of bounded nonempty subsets of \( H_\varphi \), we claim that \( \mathcal{D} \) is tempered if

\[
\lim_{t \rightarrow +\infty} e^{-bt} \|D(\tau - t, \theta_{-\tau} \omega)\|^2_{H_\varphi} = 0, \ \forall \tau \in \mathbb{R}, \omega \in \Omega, b > 0, \quad (2.12)
\]

and \( \mathcal{D} \) is backward tempered if

\[
\lim_{t \rightarrow +\infty} e^{-bt} \sup_{s \leq \tau} \|D(s - t, \theta_{-t} \omega)\|^2_{H_\varphi} = 0, \ \forall \tau \in \mathbb{R}, \omega \in \Omega, b > 0. \quad (2.13)
\]

We now take two attraction universes. One is the universe \( \mathcal{D} \), which is consist of all tempered families in \( H_\varphi \):

\[
\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : \mathcal{D} \text{ satisfies (2.12)} \}.
\]
Another is the universe \( \mathcal{B} \), which be the class of all backward tempered families in \( H_\theta \):
\[
\mathcal{B} = \{ \mathcal{B}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : \mathcal{B} \text{ satisfies (2.13)} \}.
\]
Then both universes are inclusion-closed such that \( \mathcal{B} \subset \mathcal{D} \). In particular, \( \mathcal{B} \) is also backward-closed: if \( \mathcal{B} \in \mathcal{B} \), then \( \mathcal{B} \in \mathcal{B} \) with \( \mathcal{B}(\tau, \omega) = \bigcup_{s \leq \tau} \mathcal{B}(s, \omega) \).

We will use the following Gagliardo-Nirenberg inequality:
\[
\|D^j u(t)\|_p \leq c\|u(t)\|^{1-\gamma}_q \|D^j u(t)\|_r^\gamma, \forall u \in W^{l,r}(\mathcal{O}) \cap L^p(\mathcal{O}),
\]
where \( \mathcal{O} \subset \mathbb{R}^n, 1 \leq p, q, r \leq +\infty, 0 \leq j < l, \frac{j}{r} \leq \gamma \leq 1 \) and
\[
\frac{1}{p} - \frac{j}{n} = \gamma(\frac{1}{r} - \frac{l}{q}) + (1 - \gamma)\frac{1}{q}.
\]

3. Backward uniform estimates of solutions. From now on, we denote by \( c > 0 \) an intrinsic constant and \( C(\omega) > 0 \) an intrinsic random variable.

**Lemma 3.1.** Suppose (F1)-(F4) and (G1) hold. Let \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), then we have the following conclusions:

(i) For each \( \mathcal{D} \in \mathcal{D} \) there is a \( \bar{T} = \bar{T}(\mathcal{D}, \tau, \omega) \geq 2q + 4 \) such that
\[
\sup_{\xi \in [-2q-4,0]} \|v(\tau + \xi, \tau - t, \theta_{-\omega}, \phi)\|^2 \leq c\bar{R}(\tau, \omega),
\]
for all \( t \geq \bar{T} \) and \( \phi \in \mathcal{D}(\tau - t, \theta_{-\omega}) \), where
\[
\bar{R}(\tau, \omega) \equiv \int_0^\infty e^{m(1 + |z(\theta_{r}\omega)|)^6 + \|g(\tau + r)\|^2}dr < +\infty,
\]
where \( m_f \) is given in (2.6).

(ii) For each \( \mathcal{B} \in \mathcal{B} \) there exists a \( T = T(\mathcal{B}, \tau, \omega) \geq 2q + 4 \) such that
\[
\sup_{s \leq \tau} \|v(s + \xi, s - t, \theta_{-\omega}, \phi)\|^2 \leq cR(\tau, \omega),
\]
for all \( t \geq T \) and \( \phi \in \mathcal{B}(s - t, \theta_{-\omega}) \) with \( s \leq \tau \), where
\[
R(\tau, \omega) = \sup_{s \leq \tau} \bar{R}(s, \omega) < +\infty.
\]

**Proof.** Taking the inner product of (2.10) with \( v(r, s - t, \theta_{-\omega}, \phi) \) in \( H \), we have
\[
\frac{1}{2}\frac{d}{dr}\|v(r)\|^2 + v\|v\|^2_\theta + \alpha\|v\|^2 + \beta(|v + \varphi z(\theta_{r-s}\omega)(v + \varphi z(\theta_{r-s}\omega))|v + \varphi z(\theta_{r-s}\omega)), v)
\]
\[
+ \gamma(|v + \varphi z(\theta_{r-s}\omega)|^2(v + \varphi z(\theta_{r-s}\omega)), v)
\]
\[
= (f(r, v_r + \varphi z(\theta_{r-s}+\omega)) + g(r, v_r) + z(\theta_{r-s}\omega)(-A\varphi - \alpha\varphi + \varphi, v)).
\]
Note that
\[
\gamma \int_\mathcal{O} |v + \varphi z(\theta_{r-s}\omega)|^2(v + \varphi z(\theta_{r-s}\omega))v(r)dx
\]
\[
= \gamma \|v(r)\|^4_\mathcal{O} + \gamma \int_\mathcal{O} 3\varphi z(\theta_{r-s}\omega)v^3 + 3\varphi^2 z^2(\theta_{r-s}\omega)v^2 + \varphi^3 z^3(\theta_{r-s}\omega)vdx.
\]
It follows from the Young inequality, \( \varphi \in V \cap H^2(\mathcal{O}) \) and \( V \hookrightarrow L^p(2 \leq p \leq 6) \) that
\[
- \gamma \int_\mathcal{O} 3\varphi z(\theta_{r-s}\omega)v^3 + 3\varphi^2 z^2(\theta_{r-s}\omega)v^2 + \varphi^3 z^3(\theta_{r-s}\omega)vdx
\]
\[
\leq \frac{\gamma}{8}\|v(r)\|^4 + \frac{1}{2}\|v(r)\|^2 + c(1 + |z(\theta_{r-s}\omega)|^6)
\]
and similarly

\[-\beta((v + \varphi z(\theta_{r-s}\omega))(v + \varphi z(\theta_{r-s}\omega)), v) \leq \beta \int_{\mathcal{O}} (v + \varphi z(\theta_{r-s}\omega))^2 |v| dx\]

\[\leq \frac{\gamma}{8} \|v(r)\|^4 + \frac{1}{2} \|v(r)\|^2 + c(1 + |z(\theta_{r-s}\omega)|^4).\]

We also have

\[(f(r, v_r + \varphi z(\theta_{r-s+}\omega)) + g(r, \cdot), v)\]

\[\leq \left( \frac{c_f}{2} + \frac{1}{2} \|v(r)\|^2 + \frac{1}{2c_f} \|f(r, v_r + \varphi z(\theta_{r-s+}\omega))\|^2 + \frac{1}{2} \|g(r)\|^2\]

and

\[z(\theta_{r-s}\omega)(-\lambda \varphi - \alpha \varphi + \varphi, v) \leq \frac{1}{2} \|v(r)\|^2 + c|z(\theta_{r-s}\omega)|^2.\]

Then we obtain

\[\frac{d}{dr} \|v(r)\|^2 + 2\nu \|v(r)\|_V^2 + 2\alpha \|v(r)\|^2 + \frac{3\gamma}{2} \|v(r)\|^4\]

\[\leq c(1 + |z(\theta_{r-s}\omega)|^6) + (4 + c_f) \|v(r)\|^2 + \|g(r)\|^2 + \frac{1}{c_f} \|f(r, v_r + \varphi z(\theta_{r-s+}\omega))\|^2.\]

Note that

\[(4 + c_f - 2\alpha) \|v(r)\|^2 = -(m_f + 2c_f) \|v(r)\|^2 + (m_f + 4 + 3c_f - 2\alpha) \|v(r)\|^2\]

\[\leq -(m_f + 2c_f) \|v(r)\|^2 + \frac{\gamma}{2} \|v(r)\|^4 + c.\]

Hence we obtain

\[\frac{d}{dr} \|v(r)\|^2 + 2\nu \|v(r)\|_V^2 + m_f \|v(r)\|^2 + \gamma \|v(r)\|^4\]

\[\leq -(m_f + 2c_f) \|v(r)\|^2 + c(1 + |z(\theta_{r-s}\omega)|^6 + \|g(r)\|^2) + \frac{1}{c_f} \|f(r, v_r + \varphi z(\theta_{r-s+}\omega))\|^2.\]

Multiplying (3.6) by $e^{m_f r}$, and then integrating the result over $[s - t, s + \xi]$ with $\xi \in [-2\vartheta - 4, 0]$ and $t > 2\vartheta + 4$ yields

\[e^{m_f(s+\xi)} \|v(s + \xi)\|^2 + 2\nu \int_{s-t}^{s+\xi} e^{m_f r} \|v(r)\|_V^2 dr + \gamma \int_{s-t}^{s+\xi} e^{m_f r} \|v(r)\|^4 dr\]

\[\leq e^{m_f(s-t)} \|\phi\|^2_{H_\nu} - 2c_f \int_{s-t}^{s+\xi} e^{m_f r} \|v(r)\|^2 dr + \frac{1}{c_f} \int_{s-t}^{s+\xi} e^{m_f r} \|f(r, v_r + \varphi z(\theta_{r-s+}\omega))\|^2 dr\]

\[+ c \int_{s-t}^{s+\xi} e^{m_f r} (1 + |z(\theta_{r-s}\omega)|^6 + \|g(r)\|^2) dr.\]

We now main treat the delay term. By (2.6) and $f(\cdot, 0) = 0$ we get

\[\frac{1}{c_f} \int_{s-t}^{s+\xi} \|f(r, v_r + \varphi z(\theta_{r-s+}\omega))\|^2 dr\]

\[\leq c_f \int_{s-t}^{s+\xi} e^{m_f r} \|v(r) + \varphi z(\theta_{r-s}\omega)\|^2 dr\]

\[\leq 2c_f \int_{s-t}^{s+\xi} e^{m_f r} \|v(r)\|^2 dr + ce^{m_f(s-t)} \|\phi\|_{H_\nu}^2 + c \int_{s-t}^{s+\xi} e^{m_f r} |z(\theta_{r-s}\omega)|^2 dr.\]
Hence, we have, for all $\xi \in [-2\rho - 4, 0]$,
\[
\|v(s + \xi, s - t, \theta_{-s}, \omega, \phi)\|^2 + \int_{s-t}^{s+\xi} e^{m_f(r-s)}(\|\nu(r)\|_V^2 + \|v(r)\|_V^4)dr
\leq ce^{-m_f t}\|\phi\|^2_{H_\delta} + c\int_{-\infty}^{0} e^{m_f r}(1 + |z(\theta_{-r}, \omega)|^6 + \|g(r + s)\|^2)dr,
\tag{3.7}
\]

where $R(T, \omega)$ is given by (3.2). Hence we obtain (3.1) holds true.

(ii) If $\phi \in B(s - t, \theta_{-\tau})$, then, by (2.13), there exists $T = T(B, \tau, \omega) \geq 2\rho + 4$ such that for all $t \geq T$,
\[
e^{-m_f t}\sup_{s \leq \tau} \|\phi\|^2_{H_\delta} \leq e^{-m_f t} \sup_{s \leq \tau} \|\phi\|^2_{H_\delta} \leq R(\tau, \omega),
\]
where $R(\tau, \omega)$ is defined by (3.4). Then we obtain (3.3) as desired. By (2.1) and (2.8), it is easy to see that $R(\tau, \omega)$ and $R(\tau, \omega)$ are finite. The proof is complete. \qed

In addition, we also have the following auxiliary estimate:
\[
\sup_{s \leq \tau} \int_{s - \rho - 3}^{s} (\|\nu(r)\|_V^2 + \|v(r)\|_V^4)dr \leq c\rho(T, \omega).
\tag{3.8}
\]

Indeed, by (3.7) we obtain for all $t \geq T$ ($T$ is given in (3.3)),
\[
\int_{s-t}^{s} e^{m_f r}(\|\nu(r)\|_V^2 + \|v(r)\|_V^4)dr \geq \int_{s-\rho-3}^{s} e^{m_f r}(\|\nu(r)\|_V^2 + \|v(r)\|_V^4)dr
\geq e^{-m_f (e+3)} \int_{s-\rho-3}^{s} (\|\nu(r)\|_V^2 + \|v(r)\|_V^4)dr,
\]
which implies (3.8) holds true.

Lemma 3.2. Suppose (F1)-(F4) and (G1) hold. For each $B \in \mathfrak{S}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have for all $\phi \in B(s - t, \theta_{-\tau})$ with $s \leq \tau$ and $t \geq T$,
\[
\sup_{s \leq \tau} \sup_{\xi \in [-\rho - 2, 0]} \|v(s + \xi, s - t, \theta_{-s}, \omega, \phi)\|^2_{V^2} + \sup_{s \leq \tau} \sup_{\xi \in [-\rho - 2, 0]} \|v(r)\|_V^4 dr
\leq C(\omega)(1 + R(\tau, \omega))(1 + L(\tau) + G(\tau)),
\tag{3.9}
\]
and
\[
\sup_{s \leq \tau} \int_{s-\rho}^{s} \|\frac{\partial}{\partial r} v(r, s - t, \theta_{-s}, \omega, \phi)\|^2 dr \leq C(\omega)(1 + R(\tau, \omega))(1 + L(\tau) + G(\tau)),
\tag{3.10}
\]
where
\[
L(\tau) := \sup_{s \leq \tau} \int_{-\infty}^{s} e^{m_f (r-s)} L_f^2(r)dr, \quad G(\tau) := \sup_{s \leq \tau} \int_{-\infty}^{s} e^{m_f (r-s)} \|g(r)\|_V^2 dr.
\tag{3.11}
\]

Proof. Multiplying (2.10) by $\frac{\partial}{\partial r} v(r, s - t, \theta_{-s}, \omega, \phi)$ yields
\[
\|\frac{\partial}{\partial r} v(r)\|^2 + \frac{\nu}{2} \frac{d}{dr} \|\nu(r)\|_V^2 + \gamma(\nu + \varphi_3(\theta_{-t-s})^2(\nu + \varphi_3(\theta_{-t-s})^2, \frac{\partial}{\partial r} v(r))
= -\alpha(\nu, \frac{\partial}{\partial r} v(r)) - \beta((\nu + \varphi_3(\theta_{-t-s})^2(\nu + \varphi_3(\theta_{-t-s})^2, \frac{\partial}{\partial r} v(r))
\]
\[
+ (f(r, v_r + \varphi z(\theta_{r-s} \omega)), \frac{\partial}{\partial r} v(r)) + (g(r, \cdot), \frac{\partial}{\partial r} v(r))
+ z(\theta_{r-s} \omega)(-\nu A \varphi - \alpha \varphi + \varphi, \frac{\partial}{\partial r} v(r)). \\
\tag{3.12}
\]

We can rewrite the third term as
\[
\gamma((v + \varphi z(\theta_{r-s} \omega))^2(v + \varphi z(\theta_{r-s} \omega)), \frac{\partial}{\partial r} v(r))
= \gamma(v^3 + 3\varphi z(\theta_{r-s} \omega)v^2 + 3\varphi^2 z^2(\theta_{r-s} \omega)v + \varphi^3 z^3(\theta_{r-s} \omega), \frac{\partial}{\partial r} v(r))
= \frac{\gamma}{4} \frac{d}{dr} ||v(r)||_4^4 + \gamma(3\varphi z(\theta_{r-s} \omega)v^2 + 3\varphi^2 z^2(\theta_{r-s} \omega)v + \varphi^3 z^3(\theta_{r-s} \omega), \frac{\partial}{\partial r} v(r)).
\]

By (2.4), the Young inequality, \( \varphi \in V \cap H^2(\mathcal{O}) \) and \( V \hookrightarrow L^p(\mathcal{O})(2 \leq p \leq 6) \) we obtain
\[
\gamma((v + \varphi z(\theta_{r-s} \omega))^2(v + \varphi z(\theta_{r-s} \omega)), \frac{\partial}{\partial r} v(r)) \\
\leq \frac{1}{10} \frac{d}{dr} ||v(r)||_4^4 + c|z(\theta_{r-s} \omega)|^2 ||v(r)||_4^4 + c|z(\theta_{r-s} \omega)|^4 ||v(r)||_4^2 + c|z(\theta_{r-s} \omega)|^6,
\]
which further implies that
\[
\gamma((v + \varphi z(\theta_{r-s} \omega))^2(v + \varphi z(\theta_{r-s} \omega)), \frac{\partial}{\partial r} v(r)) \\
\geq \frac{\gamma}{4} \frac{d}{dr} ||v(r)||_4^4 - \frac{1}{10} \frac{d}{dr} ||v(r)||_4^2 - c|z(\theta_{r-s} \omega)|^2 ||v(r)||_4^4 \\
- c|z(\theta_{r-s} \omega)|^4 ||v(r)||_4^2 - c|z(\theta_{r-s} \omega)|^6.
\]

Similarly, we have
\[
- \alpha(v, \frac{\partial}{\partial r} v(r)) - \beta((v + \varphi z(\theta_{r-s} \omega))(v + \varphi z(\theta_{r-s} \omega)), \frac{\partial}{\partial r} v(r))
\leq \frac{1}{5} \frac{d}{dr} ||v(r)||_4^4 + c|1 + |z(\theta_{r-s} \omega)||^2 ||v(r)||_4^2 + c|z(\theta_{r-s} \omega)|^4, \\
(f(r, v_r + \varphi z(\theta_{r-s} \omega)) + g(r, \cdot), \frac{\partial}{\partial r} v(r))
\leq \frac{1}{10} \frac{d}{dr} ||v(r)||_4^4 + cL^2(r) ||v_r + \varphi z(\theta_{r-s} + \omega)||_{H^2}^2 + c||g(r)||_2^2,
\]
\[
z(\theta_{r-s} \omega)(-\nu A \varphi - \alpha \varphi + \varphi, \frac{\partial}{\partial r} v(r)) \\
\leq \frac{1}{10} \frac{d}{dr} ||v(r)||_4^4 + c|z(\theta_{r-s} \omega)|^2 ||v(r)||_4^2.
\]

Substituting (3.13)-(3.16) into (3.12) we have
\[
\frac{d}{dr} ||v(r)||_4^4 + \frac{d}{dr} ||v(r)||_4^2 + \frac{\gamma}{2} ||v(r)||_4^4 \\
\leq c(1 + |z(\theta_{r-s} \omega)|^2) \cdot (1 + ||v(r)||_2^2 + ||v(r)||_4^4 + L^2(r) ||v_r + \varphi z(\theta_{r-s} + \omega)||_{H^2}^2 + ||g(r)||_2^2).
\]

Integrating (3.17) on \([\zeta, s + \xi] \) with \( \zeta \in [s + \xi - 1, s + \xi] \) and \( \xi \in [-\rho - 2, 0] \), and then integrating this result on \([s + \xi - 1, s + \xi] \) w.r.t. \( \zeta \) we get
\[
||v(s + \xi)||_{L^4}^4 + ||v(s + \xi)||_{L^2}^2 \\
\leq C(\omega) \int_{s-\rho-3}^s (1 + ||v||_2^2 + ||v||_4^4 + ||v||_{L^4}^4)dr \\
+ C(\omega) \int_{s-\rho-3}^s (L^2(r) ||v_r + \varphi z(\theta_{r-s} + \omega)||_{H^2}^2 + ||g(r)||_2^2)dr.
\]
\[
\tag{3.18}
\]
By (3.3) we obtain for all $s \leq \tau$ and $\xi \in [-\rho, 2, 0]$,
\[
\int_{s-\rho}^{s} \left( 1 + \|v(r)\|^2 + L_f^2(r)\|v_r + \varphi z(\theta_{r-s} + \omega)\|^2_{H_e} \right) dr
\leq (1 + \|v(s + \xi)\|^2 + (e^{m_f(e+3)}\|v(s + \xi)\|^2 + C(\omega)) \int_{s-\rho}^{s} e^{m_f(r) L_f^2(r)} dr
\leq C(\omega)(1 + R(\tau, \omega))(1 + L(\tau))
\] (3.19)
where $L(\tau)$ is defined by (3.11). Inserting (3.8) and (3.19) into (3.18) yields
\[
\sup_{s \leq \tau} \sup_{\xi \in [-\rho, 2, 0]} \|v(s + \xi)\|^2_{V} + \sup_{s \leq \tau} \|v(s + \xi)\|^4_{4}
\leq C(\omega)(1 + R(\tau, \omega))(1 + L(\tau) + G(\tau)).
\] (3.20)
where $G(\tau)$ is defined by (3.11). Hence (3.9) holds. Integrating (3.17) over $[s - \rho, s]$ yields
\[
\int_{s-\rho}^{s} \left( 1 + \|v(r)\|^2 + \|v(r)\|^4 + L_f^2(r)\|v_r + \varphi z(\theta_{r-s} + \omega)\|^2_{H_e} + \|g(r)\|^2 \right) dr.
\] (3.21)
Substituting (3.8), (3.19) and (3.20) into (3.21) yields (3.10) holds. \qed

**Lemma 3.3.** Suppose (F1)-(F4) and (G1) hold. For each $B \in \mathcal{B}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have for all $\phi \in \mathcal{B}(s-t, \theta_{-\rho} \omega)$ with $s \leq \tau$ and $t \geq T$,
\[
\sup_{s \leq \tau} \int_{s-\rho}^{s} \|Av(r)\|^2 dr \leq C(\omega)(1 + R(\tau, \omega))^3 \left(1 + L(\tau) + G(\tau)\right)^3.
\] (3.22)

**Proof.** Multiplying (2.10) by $Av(r, s-t, \theta_{-s} \omega, \phi)$ yields
\[
\frac{1}{2} \frac{d}{dr} \|v(r)\|^2_{V} + \nu \|Av\|^2 + \alpha \|v(r)\|^2_{V} + \beta(\|v + \varphi z(\theta_{r-s} \omega)(v + \varphi z(\theta_{r-s} \omega))\), Av)
+ \gamma(\|v + \varphi z(\theta_{r-s} \omega)\|^2 + \varphi z(\theta_{r-s} \omega)), Av)
= (f(r, v_r + \varphi z(\theta_{r-s} \omega)), Av) + (g(r, \cdot), Av) + z(\theta_{r-s} \omega)(-\nu A\varphi - \alpha \varphi + \varphi, Av).
\]
Notice that By the Hölder inequality,
\[
- \beta(\|v + \varphi z(\theta_{r-s} \omega)(v + \varphi z(\theta_{r-s} \omega))\), Av)
\leq \frac{\nu}{8} \|Av\|^2 + c \|v + \varphi z(\theta_{r-s} \omega)\|^4_{4} \leq \frac{\nu}{8} \|Av\|^2 + c(\|v(r)\|^4_{4} + |z(\theta_{r-s} \omega)|^4).
\]
Similarly,
\[
\gamma(\|v + \varphi z(\theta_{r-s} \omega)\|^2 + \varphi z(\theta_{r-s} \omega)), Av) \leq \frac{\nu}{8} \|Av\|^2 + c \|v(r)\|^6_{6} + c |z(\theta_{r-s} \omega)|^6.
\]
By $\varphi \in D(A)$ we have
\[
z(\theta_{r-s} \omega)(-\nu A\varphi - \alpha \varphi + \varphi, Av) \leq \frac{\nu}{8} \|Av\|^2 + c |z(\theta_{r-s} \omega)|^2.
\]
By the Young inequality, $f(\cdot, 0) = 0$ and (2.4) we obtain
\[
(f(r, v_r + \varphi z(\theta_{r-s} + \omega)), Av) + (g(r, \cdot), Av)
\leq \frac{\nu}{8} \|Av\|^2 + c(\|f(r, v_r + \varphi z(\theta_{r-s} + \omega))\|^2 + \|g(r)\|^2
\].
\[ \leq \frac{\nu}{8} \| Av \|^2 + c L_f^2(r) \| v_r + \varphi z(\theta_{r-s} \omega) \|^2_{H_{\omega}} + c \| g(r) \|^2. \]

Hence, we obtain
\[ \frac{d}{dr} \| v(r) \|^2 + \nu \| Av \|^2 \leq c \left( 1 + \| v(r) \|_2^4 + \| v(r) \|_6^6 + \| z(\theta_{r-s} \omega) \|_6^6 \right) \]
\[ + c \left( L_f^2(r) \| v_r + \varphi z(\theta_{r-s} \omega) \|^2_{H_{\omega}} + \| g(r) \|^2 \right) . \]

Integrating (3.23) over \([s - \varrho - 1, s]\) we get
\[ \int_{s - \varrho - 1}^s \| Av(r) \|^2 dr \]
\[ \leq c \int_{s - \varrho - 1}^s \left( 1 + \| v(r) \|_2^4 + \| v(r) \|_6^6 + \| z(\theta_{r-s} \omega) \|_6^6 \right) dr \]
\[ + c \int_{s - \varrho - 1}^s \left( L_f^2(r) \| v_r + \varphi z(\theta_{r-s} \omega) \|^2_{H_{\omega}} + \| g(r) \|^2 \right). \]

By (3.9) we have
\[ \sup_{s \leq T} \int_{s - \varrho - 1}^s (\| v(r) \|_2^4 + \| v(r) \|_6^6) dr \]
\[ \leq (\varrho + 1) \sup_{s \leq T} \sup_{\xi \in [-\varrho - 1, 0]} (\| v(s + \xi) \|_2^4 + \| v(s + \xi) \|_6^6) \]
\[ \leq C(\omega)(1 + R(\tau, \omega))^3 (1 + L(\tau) + G(\tau))^3. \]

It follows from (3.19) and (3.25) that
\[ \sup_{s \leq T} \int_{s - \varrho - 1}^s \| Av(r) \|^2 dr \leq C(\omega)(1 + R(\tau, \omega))^3 (1 + L(\tau) + G(\tau))^3. \]

The proof is complete. \[\square\]

It is easy to see that each \( v \in V \) has the following orthogonal decomposition:
\[ v = P_k v \oplus (I - P_k) v = v_{k,1} + v_{k,2}, \]
for each \( k \in \mathbb{N}, \)
where \( P_k : H \rightarrow H_k = \text{span}\{e_1, e_2, \ldots, e_k\} \) is the orthogonal projection and \( \{e_k\}_{k=1}^{\infty} \subset V \) is the family of eigenfunctions for \( A \) in \( H \) with the corresponding eigenvalues: \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \rightarrow +\infty \) as \( k \rightarrow +\infty. \)

**Lemma 3.4.** Suppose (F1)-(F4) and (G1) hold. For each \( B \in \mathcal{B}, \tau \in \mathbb{R}, \omega \in \Omega \) and any \( \varrho > 0 \), there exists a \( \delta > 0 \) with \( |\xi_1 - \xi_2| < \delta \) and \( \xi_1, \xi_2 \in [-\varrho, 0] \) such that for all \( t \geq T \) and \( \phi \in B(s-t, \theta_{-\omega} \omega) \) with \( s \leq \tau, \)
\[ \sup_{s \leq T} \| P_k v(s + \xi_1, s - t, \theta_{-\omega} \omega, \phi) - P_k v(s + \xi_2, s - t, \theta_{-\omega} \omega, \phi) \|_V < \varrho. \]

**Proof.** Notice that \( \| v_{k,1} \|_V^2 \leq \lambda_k \| v_{k,1} \|^2 \) and suppose that \( \xi_1 \leq \xi_2. \) Hence, we have
\[ \sup_{s \leq T} \| v_{k,1}(s + \xi_1, s - t, \theta_{-\omega} \omega, \phi) - v_{k,1}(s + \xi_2, s - t, \theta_{-\omega} \omega, \phi) \|_V \]
\[ \leq \lambda_k^2 \sup_{s \leq T} \| v_{k,1}(s + \xi_1) - v_{k,1}(s + \xi_2) \| \leq \lambda_k^2 \sup_{s \leq T} \left( \int_{s - \varrho}^s \| \frac{\partial}{\partial r} v(r, s - t, \theta_{-\omega} \omega, \phi) \|^2 dr \right)^{\frac{1}{2}} \]
\[ \leq \lambda_k^2 \sup_{s \leq T} \left( \int_{s - \varrho}^s \| \frac{\partial}{\partial r} v(r, s - t, \theta_{-\omega} \omega, \phi) \|^2 dr \right)^{\frac{1}{2}} |\xi_1 - \xi_2|^{\frac{1}{2}}. \]
Then, by (3.10), for any $\varepsilon > 0$ there exists a $\delta > 0$ with $|\xi_1 - \xi_2| < \delta$ such that

$$\sup_{s \leq \tau} \|v_{k,1}(s + \xi_1, s-t, \theta_{-s}\omega, \phi) - v_{k,1}(s + \xi_2, s-t, \theta_{-s}\omega, \phi)\|_V < \varepsilon,$$

which implies (3.26) holds true. \hfill \square

**Lemma 3.5.** Suppose (F1)-(F4) and (G1) hold. For each $B \in \mathcal{B}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any $\varepsilon > 0$, there exists a $K = K(\varepsilon, \tau, \omega) \in \mathbb{N}$ such that for all $k \geq K$, $t \geq T$ and $\phi \in \mathcal{B}(s-t, \theta_{-t}\omega)$ with $s \leq \tau$,

$$\sup_{s \leq \tau} \sup_{\xi \in [-\varepsilon, 0]} \|v_{k,2}(s + \xi, s-t, \theta_{-s}\omega, \phi)\|_V^2 < \varepsilon. \quad (3.27)$$

**Proof.** Taking the inner product of (2.10) with $Av_{k,2}(r, s-t, \theta_{-s}\omega, \phi)$ in $H$, we have

$$\frac{d}{dr} \|v_{k,2}(r)\|_V^2 + \nu \|Av_{k,2}\|^2 + 2\alpha \|v_{k,2}\|^2 \leq c(1 + \|z(\theta_{-s}\omega)\|^6 + \|v\|^4 + \|v\|^8 + L_2^2(r)\|v_r + \varphi z(\theta_{r-s+\omega})\|_{H_\omega}^2 + \|g(r)\|^2).$$

Therefore, by $\|Av_{k,2}\|^2 \geq \lambda_k \|A^1 v_{k,2}\|^2$, we get

$$\frac{d}{dr} \|v_{k,2}(r)\|_V^2 \leq c\|v\|^4 + \|v\|^8 + L_2^2(r)\|v_r + \varphi z(\theta_{r-s+\omega})\|_{H_\omega}^2 + \|g(r)\|^2).$$

By (2.14) for $p = 4, j = 0, q = 3, r = 2, l = 2$, we have

$$\|v\|^4 \leq c\|v\|_A^4 \|Av\|_A^2.$$

By (2.14) for $p = 6, j = 0, q = 4, r = 2, l = 2$, we also have

$$\|v\|^6 \leq c\|v\|_A^{24} \|Av\|_A^6.$$

Hence, we have

$$\frac{d}{dr} \|v_{k,2}(r)\|_V^2 \leq c\|v\|^4 + \|v\|^8 + L_2^2(r)\|v_r + \varphi z(\theta_{r-s+\omega})\|_{H_\omega}^2 + \|g(r)\|^2).$$

Integrating this inequality on $[\zeta, s + \xi]$ with $\zeta \in [s + \xi - 1, s + \xi]$ and $\xi \in [-\varphi, 0]$, and then integrating this result on $[s + \xi - 1, s + \xi]$ w.r.t. $\zeta$ yields

$$e^{\nu \lambda_k(s+\xi)} \|v_{k,2}(s + \xi, s-t, \theta_{-s}\omega, \phi)\|_V^2 \leq C \int_{s+\xi-1}^{s+\xi} e^{\nu \lambda_k(r)}(1 + \|v\|_A^4 \|Av\|_A^2 + \|v\|_A^{24} \|Av\|_A^6 + L_2^2(r)\|v_r + \varphi z(\theta_{r-s+\omega})\|_{H_\omega}^2 + \|g(r)\|^2)dr.$$

which implies

$$\|v_{k,2}(s + \xi, s-t, \theta_{-s}\omega, \phi)\|_V^2 \leq C \int_{s-1}^{s} e^{\nu \lambda_k(r-s)}(1 + \|v\|_A^4 \|Av\|_A^2 + \|v\|_A^{24} \|Av\|_A^6 + L_2^2(r)\|v_r + \varphi z(\theta_{r-s+\xi+\omega})\|_{H_\omega}^2 + \|g(r)\|^2)dr. \quad (3.28)$$
4. Existence and backward compactness of bi-spatial random attractors.

4.1. Preliminaries on bi-spatial random attractors. We review some basic concepts related to bi-spatial pullback random attractors. Let $\Phi$ be a cocycle over $\Omega$ and $(X, \| \cdot \|_X)$ be a Banach space. A family of nonempty subsets $D(\tau, \omega)$ of $X$ parameterized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$ is called a bi-parametric set in $X$ over $\mathbb{R} \times \Omega$. Such a bi-parametric set $D(\cdot, \cdot)$ is called

- **random** if for each $\tau \in \mathbb{R}$, the subset $D(\tau, \cdot)$ of $X \times \mathcal{B}(X)$-measurable;
- **compact, closed, bounded** if every component $D(\tau, \cdot)$ is compact, closed and bounded, respectively;

Note that
\[\sup_{s \leq \tau, \xi \in [-\rho, 0]} \int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} dr \leq \frac{1}{\nu_\lambda_k} \to 0 \text{ as } k \to +\infty, \quad (3.29)\]

It follows from (3.9) and (3.22) that for all $s \leq \tau$ and $\xi \in [-\rho, 0],
\[\int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} \| \psi(r+\xi) \|_{\mathcal{L}(1)}^{10} \| Av(r+\xi) \|_2^{10} \| v(r+\xi) \|_2^{10} \| Av(r+\xi) \|_2^{10} \| g(r+\xi) \|_2^{10} dr \]
\[\leq \sup_{r \in [s-\rho-1, s]} \| v_{k, 2}(r) \|_V^{10} \sup_{\xi \in [-\rho, 0]} \int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} \| Av(r+\xi) \|_1^{10} dr + \sup_{r \in [s-\rho-1, s]} \| v_{k, 2}(r) \|_V^{10} \sup_{\xi \in [-\rho, 0]} \int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} \| Av(r+\xi) \|_1^{10} dr \]
\[\leq \sup_{r \in [s-\rho-1, s]} \| v_{k, 2}(r) \|_V^{10} \left( \int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} dr \right)^{10} \left( \int_{s-1}^{s} \| Av(r) \|_2^{10} dr \right)^{10} + \sup_{r \in [s-\rho-1, s]} \| v_{k, 2}(r) \|_V^{10} \left( \int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} dr \right)^{10} \left( \int_{s-1}^{s} \| Av(r) \|_2^{10} dr \right)^{10} \]
\[\leq \left( \frac{2}{3\nu_\lambda_k} \right)^{10} C(\omega)(1 + R(\tau, \omega))^{10}(1 + L(\tau) + G(\tau))^{10} \to 0 \text{ as } k \to +\infty. \quad (3.30)\]

For the forcing term, by (2.7) we get
\[\sup_{s \leq \tau, \xi \in [-\rho, 0]} \int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} \| g(r+\xi) \|_2 dr \leq \sup_{s \leq \tau} \int_{s-\infty}^{s} e^{\nu_\lambda_k(r-s)} \| g(r) \|_2^2 dr \to 0 \text{ as } k \to +\infty. \quad (3.31)\]

For the delay term, from (2.5) and (3.3) we have
\[\sup_{s \leq \tau, \xi \in [-\rho, 0]} \int_{s-1}^{s} e^{\nu_\lambda_k(r-s)} L^2_f(r+\xi) \| v_{r+\xi} + \varphi(z(\theta_{r-s+\xi+\omega})) \|_H dr \leq C(\omega)(1 + R(\tau, \omega)) \sup_{s \leq \tau} \int_{s-\infty}^{s+\xi} e^{\nu_\lambda_k(r-s+\xi)} L^2_f(r) dr \]
\[\leq C(\omega)(1 + R(\tau, \omega)) \sup_{s \leq \tau} \int_{s-\infty}^{s} e^{\nu_\lambda_k(r-s)} L^2_f(r) dr \to 0 \text{ as } k \to +\infty. \quad (3.32)\]

Therefore, by (3.28)-(3.32) we obtain (3.27) as desired. \qed
Proof. We first prove (2.11) if we take $D$ be another Banach space. This inclusion property holds for the cocycle as defined in (2.11) if we take $X = H_q$ and $Y = V_q$.

**Definition 4.1.** $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback $(X, Y)$-random attractor for $\Phi$ if for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$

1. $A(\tau, \omega) \subset Y$ such that it is compact in $X \cap Y$;
2. $A(\tau, \cdot)$ is random only not in $X$ but also in $Y$;
3. $A$ is invariant under the cocycle $\Phi$;
4. $A$ attracts every element $D \in \mathcal{D}$ under the topology of $X \cap Y$, i.e. for each $D \in \mathcal{D}$,

$$\lim_{t \to +\infty} \text{dist}_{X \cap Y}(\Phi(t, -t, \theta_{-t}\omega)D(\tau - t, \theta_{-t}\omega), A(\tau, \omega)) = 0.$$ 

We remark that a $(X, Y)$-attractor must be an $(X, X)$-attractor and a $(Y, Y)$-attractor, the latter needs the embedding $Y \hookrightarrow X$.

**Definition 4.2.** A set-map $K : \mathbb{R} \times \Omega \to 2^X \setminus \{\emptyset\}$ is a $\mathcal{D}$-pullback absorbing set for $\Phi$ if for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$ there is a $T := T(\tau, \omega, D) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega)D(\tau - t, \theta_{-t}\omega) \subset K(\tau, \omega), \quad \forall t \geq T.$$ 

Furthermore, $K$ is increasing if $K(\tau_1, \omega) \subset K(\tau_2, \omega)$ for $\tau_1 \leq \tau_2$ and $K$ is random $\mathcal{D}$-pullback absorbing set if $K$ is random.

**Definition 4.3.** The cocycle $\Phi$ is said to be backward $(X, Y)$-$\mathcal{D}$-pullback asymptotically compact if for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, $\{\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega)x_n\}_{n \in \mathbb{N}}$ has a convergence subsequence in $Y$ whenever $s_n \leq \tau$, $t_n \to +\infty$ and $\{x_n\} \subseteq D(s_n - t_n, \theta_{-t_n}\omega)$.

### 4.2. Backward compactness of bi-spatial random attractors.

**Lemma 4.4.** Suppose (F1)-(F4) and (G1) hold. We have the following conclusions:

(i) The cocycle $\Phi$ in (2.11) has a $\mathcal{D}$-pullback random absorbing set $\tilde{K} \in \mathcal{D}$, defined by

$$\tilde{K}(\tau, \omega) = \{\vartheta \in H_q : \|\vartheta\|_{H_q}^2 \leq c\tilde{R}(\tau, \omega)\}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega. \quad (4.1)$$

(ii) The cocycle $\Phi$ in (2.11) has a backward $\mathcal{B}$-pullback absorbing set $K \in \mathcal{B}$, defined by

$$K(\tau, \omega) = \{\vartheta \in H_q : \|\vartheta\|_{H_q}^2 \leq cR(\tau, \omega)\} = \bigcup_{s \leq \tau} \tilde{K}(s, \omega), \quad \forall \tau \in \mathbb{R}, \omega \in \Omega, \quad (4.2)$$

where $\tilde{R}(\tau, \omega)$ and $R(\tau, \omega)$ are given in (3.2) and (3.4), respectively.

**Proof.** We first prove $\tilde{K} \in \mathcal{D}$ and $K \in \mathcal{B}$. By (2.1), there exists a positive constant $T_1 \geq T$ ($T$ is given by (3.3)) such that for all $t \geq T_1$

$$|2z(\theta_t\omega)| \leq t. \quad (4.3)$$
Notice that $R(\tau, \omega)$ is increasing as $\tau$ increases. By (4.3), we obtain for all $t \geq T_1$
\[
e^{bt} \sup_{s \leq \tau} \|K(s - t, \theta_{-s} \omega)\|_{H_0}^2 \leq ce^{-bt} \sup_{s \leq \tau} R(s - t, \theta_{-s} \omega)
\leq ce^{-bt} R(\tau - t, \theta_{-s} \omega) = ce^{-bt} \sup_{s \leq \tau} \tilde{R}(s, \theta_{-s} \omega) \leq ce^{-bt} \sup_{s \leq \tau} \tilde{R}(s, \theta_{-s} \omega)
\leq ce^{-bt} \sup_{s \leq \tau} \int_0^0 e^{mtr} (1 + |z(\theta_{t-r} \omega)|^2 + \|g(r + s)^2\|^2 dr
\leq ce^{-bt} \sup_{s \leq \tau} \int_0^0 e^{mtr} (1 + |t - r|^2 + \|g(r + s)^2\|^2 dr \to 0 \text{ as } t \to +\infty. \tag{4.4}
\]
where we use (2.8), which implies $K \in \mathcal{B}$. Notice that $e^{-bt} \sup_{s \leq \tau} \tilde{R}(s, \theta_{-s} \omega) \geq e^{bt} \tilde{R}(\tau - t, \theta_{-s} \omega)$. So we obtain $\tilde{K} \in \mathcal{D}$.

We then show that $\tilde{K}$ is a random set. It is easy to know that $\omega \to \tilde{K}(\tau, \omega)$ is measurable since it is the integral of some random variables. Hence, $\tilde{K}$ is a random set. However, the measurability of $K$ is unknown.

Finally, by (3.1) and (3.3) we have $\tilde{K}$ is a $\mathcal{D}$-pullback absorbing set and $K$ is a backward $\mathcal{B}$-pullback absorbing set.

\begin{lemma}
Suppose (F1)-(F4) and (G1) hold. Then the cocycle $\Phi$ in (2.11) is backward $(H_0, V_\epsilon)\mathcal{B}$-pullback asymptotically compact.
\end{lemma}

\begin{proof}
Based on Definition 4.3 and the Ascoli-Arzelà theorem, we divide the proof into two steps.

\textbf{Step 1.} We prove $\{\Phi(t_n, s_n - t_n, \theta_{-t_n} \omega, \phi_n)\}_{n \in \mathbb{N}}$ in $V_\epsilon$ is equi-continuous from $[-\varrho, 0]$ to $V$.

Notice that there exists a $N \in \mathbb{N}$ such that $t_n \leq T$ (T is given in Lemma (3.1)) for all $n \geq N$ due to $t_n \to +\infty$. By (2.26) and (2.27), we have for all $n \geq N$ and $k_0 \geq K$
\[
\|\Phi(t_n, s_n - t_n, \theta_{-t_n} \omega)\phi_n\|_V
\leq \|v(s_n + \xi_n, s_n - t_n, \theta_{-s_n} \omega, \phi_n) - v(s_n + \xi_n, s_n - t_n, \theta_{-s_n} \omega, \phi_n)\|_V
\leq \|P_{k_0}v(s_n + \xi_n) - P_{k_0}v(s_n + \xi_n)\|_V + \|(I - P_{k_0})v(s_n + \xi_n)\|_V
\leq \|v(s_n + \xi_n)\|_V + \|(I - P_{k_0})v(s_n + \xi_n)\|_V < 3\varepsilon.
\]
Hence, we obtain $\{\Phi(t_n, s_n - t_n, \theta_{-t_n} \omega, \phi_n) : n \geq N\}$ is equi-continuous from $[-\varrho, 0]$ to $V$. On the other hand, the finite set $\{\Phi(t_n, s_n - t_n, \theta_{-t_n} \omega, \phi_n) : n < N\}$ of continuous functions is obviously equi-continuous from $[-\varrho, 0]$ to $V$. Then, we have $\Phi(t_n, s_n - t_n, \theta_{-t_n} \omega, \phi_n)$ is equi-continuous.

\textbf{Step 2.} For each fixed $\xi \in [-\varrho, 0]$, we prove $\Phi(t_n, s_n - t_n, \theta_{-t_n} \omega, \phi_n)\xi = v(s_n + \xi, s_n - t_n, \theta_{-s_n} \omega, \phi_n)$ is pre-compact in $V$.

By (3.22), we obtain that $\{P_{k_0}v(s_n + \xi_n, s_n - t_n, \theta_{-s_n} \omega, \phi_n)\}_{n \in \mathbb{N}}$ is bounded in $V$ and thus pre-compact in the $k_0$-dimensional subspace $V_{k_0}$. Then, there is an index subsequence $n^*$ of $n$ such that $\{P_{k_0}v(s_n^* + \xi_n, s_n^* - t_n^*, \theta_{-s_n^*} \omega, \phi_n^*)\}_{n^* \in \mathbb{N}}$ is a Cauchy sequence in $V_{k_0}$. On the other hand, let $n^*, m^*$ large enough, we have
\[
\|v(s_n^* + \xi_n, s_n^* - t_n^*, \theta_{-s_n^*} \omega, \phi_n^*) - v(s_m^* + \xi_n, s_m^* - t_m^*, \theta_{-s_m^*} \omega, \phi_m^*)\|_V
\leq \|P_{k_0}v(s_n^* + \xi_n, s_n^* - t_n^*, \theta_{-s_n^*} \omega, \phi_n^*) - P_{k_0}v(s_m^* + \xi_n, s_m^* - t_m^*, \theta_{-s_m^*} \omega, \phi_m^*)\|_V
\quad + \|(I - P_{k_0})v(s_n^* + \xi_n)\|_V + \|(I - P_{k_0})v(s_m^* + \xi_n)\|_V < 3\varepsilon.
\]
Then \( \{v(s_n^*, \xi, s_n^*-t_n^*, \theta_{-s_n^*}\omega, \phi_n^*)\}_{n^* \in \mathbb{N}} \) is a Cauchy sequence in \( V \), and so the proof of Step 2 is complete. \( \square \)

**Theorem 4.6.** Suppose (F1)-(F4) and (G1) hold. Then the following results are true:

(i) The cocycle \( \Phi \) in (2.11) has a unique \( \mathcal{D} \)-pullback random attractor \( \tilde{A} \in \mathcal{D} \) in \( H_\theta \), defined by

\[
\tilde{A}(\tau, \omega) = \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, \tau-t, \theta_{-t}\omega)\K(\tau-t, \theta_{-t}\omega)^{H_\theta}, \quad \tau \in \mathbb{R}, \quad \omega \in \Omega. \tag{4.5}
\]

Moreover, \( \tilde{A} \) is also a unique \( \mathcal{D} \)-pullback \((H_\theta, V_\theta)\)-random attractor.

(ii) The cocycle \( \Phi \) in (2.11) has a unique \( \mathcal{B} \)-pullback \((H_\theta, V_\theta)\)-attractor \( A \in \mathcal{B} \), defined by

\[
A(\tau, \omega) = \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, \tau-t, \theta_{-t}\omega)\K(\tau-t, \theta_{-t}\omega)^{H_\theta}, \quad \tau \in \mathbb{R}, \quad \omega \in \Omega. \tag{4.6}
\]

Furthermore, \( \mathcal{A}(\cdot, \cdot) \) is backward compact in \( V_\theta \) (and thus in \( H_\theta \)).

(iii) \( \tilde{A} = A \) and so \( \tilde{A}(\tau, \cdot) \) is a random set in \( V_\theta \) as well as in \( H_\theta \).

**Proof.** (i) Using the similar method as in Lemma 4.5, we can obtain that \( \Phi \) is \((H_\theta, V_\theta)\)-\( \mathcal{D} \)-pullback asymptotically compact. Since \( V_\theta \hookrightarrow H_\theta \), we know that the combination \((H_\theta, V_\theta)\) is limit-identical and \( \Phi \) is \( \mathcal{D} \)-pullback asymptotically compact in \( H_\theta \). By (i) of Lemma 4.4, \( \Phi \) has a \( \mathcal{D} \)-pullback random absorbing set \( \K \in \mathcal{D} \) in \( H_\theta \). Hence all conditions of [22, Theorem 22.3] are fulfilled. Then \( \Phi \) has a unique \( \mathcal{D} \)-pullback random attractor \( \tilde{A} \), defined by (4.5).

Moreover, we also need to prove \( \tilde{A}(\tau, \cdot) \) is a random set in \( V_\theta \). By Lemma 3.2, \( \Phi \) has a \( \mathcal{D} \)-pullback absorbing set \( \K_1 \) in \( V_\theta \) defined by

\[
\K_1(\tau, \omega) = \{ \varphi \in V_\theta : \|\varphi\|^2_{V_\theta} \leq C(\omega)(1 + \tilde{R}(\tau, \omega))(1 + L(\tau) + G(\tau)) \}.
\]

Let \( \K_2 = \K \cap \K_1 \). Since \( C(\omega)(1 + \tilde{R}(\tau, \omega)) \) is measurable in \( \omega \), \( \K_2 \) is a \( \mathcal{D} \)-pullback random absorbing set in \( V_\theta \). This along with \( \Phi \) is \((H_\theta, V_\theta)\)-\( \mathcal{D} \)-pullback asymptotically compact implies that all conditions of [8, Theorem 19] are satisfied and so \( \tilde{A}(\tau, \cdot) \) is a random set in \( V_\theta \). Then \( \tilde{A} \) is also a unique \( \mathcal{D} \)-pullback \((H_\theta, V_\theta)\)-random attractor.

(ii) Similarly to (i), we can obtain that \( \Phi \) has a unique \( \mathcal{B} \)-pullback attractor \( A \in \mathcal{B} \) given by (4.6) and \( A \) is also a unique \( \mathcal{B} \)-pullback \((H_\theta, V_\theta)\)-attractor. Furthermore, the measurability of \( A \) is unknown due to the measurability of \( \K \) is unknown.

Next, we only need prove \( A \) is backward compact in \( V_\theta \). Let \( u_n \in \cup_{\tau \leq \tau} \A(s, \omega) \) for each \( \tau \in \mathbb{R} \). Then there exists \( s_n \leq \tau \) such that \( u_n \in \A(s_n, \omega) \). For each \( n \), by the invariance of \( A \), we can find a \( v_n \in \A(s_n - t_n, \theta_{-t_n}\omega) \) such that

\[
u_n = \Phi(t_n, s_n - t_n, \theta_{-t_n}\omega)v_n,
\]

where \( t_n \to +\infty \) as \( n \to +\infty \). Since \( A \in \mathcal{B} \) and \( \Phi \) is backward \((H_\theta, V_\theta)\)-\( \mathcal{B} \)-pullback asymptotically compact, \( \{u_n\} \) has a convergence subsequence. Then we obtain that \( A \) is backward compact in \( V_\theta \).

(iii) It is easy to prove \( \tilde{A} \subset A \). Notice that \( A \in \mathcal{B} \subset \mathcal{D} \). By the invariance of \( A \) and the \( \mathcal{D} \)-pullback attraction of \( \tilde{A} \) we have

\[
dist(A(\tau, \omega), \tilde{A}(\tau, \omega)) = dist(\Phi(\tau-t, \theta_{-t}\omega)A(\tau-t, \theta_{-t}\omega), \tilde{A}(\tau, \omega)) \to 0\]
as \( t \to +\infty \), which implies \( \tilde{A} \supset A \). Then we obtain that \( \tilde{A} = A \) and so \( A(\tau, \cdot) \) is a random set in \( V_\varepsilon \).

5. Asymptotic autonomy in the regular space. In this section, we assume that the delay force \( f \) with the property \((F1)-(F4))\) is independent of the parameter and thus the equation \((1.1)\) can be read as

\[
\begin{aligned}
&\frac{du}{dt} - (\nu \Delta u - \alpha u - \nabla p)dt = (-\beta |u|u - \gamma |u|^2u + f(u_t) + g(t, x))dt + \varphi(x)dW, \\
&\nabla \cdot u = 0 \text{ on } \mathcal{O}, \quad u = 0 \text{ on } \partial \mathcal{O}, \quad t > \tau, \\
&u(\tau + \xi, x) = u_\tau(\xi, x), \quad \xi \in [-\varrho, 0], \quad x \in \mathcal{O}, \tau \in \mathbb{R},
\end{aligned}
\]

(5.1)

Similarly to (2.10), Eq. (5.1) can be converted into the following random equation:

\[
\begin{aligned}
&\frac{\partial u}{\partial t} + \nu Au + \tilde{P}(\alpha_0 \tilde{u} + \tilde{v} + \varphi \tilde{z}) = P(f(v_t + \varphi(z(\theta_{t+\omega}) + g(t, x))) + z(\theta_{t+\omega})(-\nu A\varphi + \tilde{P}(\alpha_0 \varphi + \varphi)), \\
&v(\tau + \xi, x) = v_\tau(\xi, x), \quad \xi \in [-\varrho, 0], \quad x \in \mathcal{O}.
\end{aligned}
\]

Both equations (5.1) and (5.2) are still non-autonomous. By Theorem 4.6, the cocycle \( \Phi \) for the equation (5.2) has a \( \mathcal{D} \)-pullback \((H_\varepsilon, V_\varepsilon)\)-random attractor \( A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \), which is backward compact in \( V_\varepsilon \).

We use \( g_\infty \in H \) to replace the time-dependent force \( g \in L^2_{loc}(\mathbb{R}, H) \) in (5.1) and obtain the corresponding autonomous equation:

\[
\begin{aligned}
&\frac{d\tilde{u}}{dt} - (\nu \Delta \tilde{u} - \alpha \tilde{u} - \nabla p)dt = (-\beta |\tilde{u}|\tilde{u} - \gamma |\tilde{u}|^2\tilde{u} + f(\tilde{u}_t) + g_\infty(x))dt + \varphi(x)dW, \\
&\nabla \cdot \tilde{u} = 0 \text{ on } \mathcal{O}, \quad \tilde{u} = 0 \text{ on } \partial \mathcal{O}, \quad t > 0, \\
&\tilde{u}(\xi, x) = u_\tau(\xi, x), \quad \xi \in [-\varrho, 0], \quad x \in \mathcal{O}.
\end{aligned}
\]

(5.3)

Let \( \tilde{u}(t, \omega, \tilde{u}_0) = \tilde{v}(t, \omega, \tilde{v}_0) - \varphi(z(\theta_{t+\omega}), \omega) \), we obtain

\[
\begin{aligned}
&\frac{d\tilde{v}}{dt} + \nu A\tilde{v} + \tilde{P}(\alpha_0 \tilde{v} + \tilde{v} + \varphi \tilde{z}) = \tilde{P}(f(\tilde{v}_t + \varphi(z(\theta_{t+\omega}), \omega) + g_\infty(x)) + z(\theta_{t+\omega})(-\nu A\varphi + \tilde{P}(\alpha_0 \varphi + \varphi)), \\
&\tilde{v}(\xi, x) = \tilde{v}_\tau(\xi, x), \quad \xi \in [-\varrho, 0].
\end{aligned}
\]

(5.4)

Similarly to (2.11), the equation (5.4) can generate a continuous random dynamical system \( \Phi_\infty \). By the same method as in Theorem 4.6, one can prove that \( \Phi_\infty \) has a \((H_\varepsilon, V_\varepsilon)\)-random attractor \( A_\infty = \{A_\infty(\omega) : \omega \in \Omega\} \in \mathcal{D}_\infty \), where \( \mathcal{D}_\infty \) is the collection of all tempered families in \( H_\varepsilon \), more precisely, \( \mathcal{D}_\infty \in \mathcal{D}_\infty \) if and only if

\[
\lim_{t \to +\infty} e^{-bt}\|\mathcal{D}_\infty(\theta_{t+\omega})\|_{H_\varepsilon} = 0, \quad \forall b > 0, \quad \omega \in \Omega.
\]

In order to prove the convergence from \( A(\tau, \omega) \) to \( A_\infty(\omega) \) as \( \tau \to -\infty \), we need to assume the convergence of the forces.

\textbf{(G2)} \( g_\infty \in L^2(\mathcal{O}) \) satisfies

\[
\lim_{\tau \to -\infty} \int_{-\infty}^{\tau} \|g(r) - g_\infty\|^2 dr = 0.
\]

(5.5)

Moreover, we will use the following inequality:

\[
|x|^{p-2}x - |y|^{p-2}y, x - y\geq 2^{2-p}|x - y|^p, \forall x, y \in \mathbb{R}^n, \quad p \geq 2.
\]

(5.6)

\textbf{Lemma 5.1.} Assume \((F1)-(F4)\) and \textbf{(G2)} hold. If \( \|v_\tau - \tilde{v}_0\|_{H_\varepsilon} \to 0 \) as \( \tau \to -\infty \), then

\[
\begin{aligned}
&\lim_{\tau \to -\infty} \Phi(t, \tau, \omega) v_\tau - \Phi_\infty(t, \omega) \tilde{v}_0 \|_{H_\varepsilon}^2 \\
= &\lim_{\tau \to -\infty} \|v_{t+\tau}(\cdot, \tau, \theta_{-\tau} \omega, v_\tau) - \tilde{v}_\tau(\cdot, \omega, v_\tau)\|_{H_\varepsilon}^2 = 0
\end{aligned}
\]

(5.7)
for all \( t > 0 \) and \( \omega \in \Omega \).

**Proof.** For each \( \tau \in \mathbb{R} \), we consider the difference of two solutions:

\[
V^\tau(r) = v(r + \tau, \theta_{-\tau} \omega, v_r) - \tilde{v}(r, \omega, \tilde{v}_0), \quad \forall r \geq -\varrho.
\]

Subtracting (5.4) from (5.2) yields for all \( r \geq 0 \),

\[
\begin{align*}
\frac{\partial V^\tau(r)}{\partial r} + \nu AV^\tau(r) + \alpha \tilde{P} V^\tau(r) &+ \beta \tilde{P} ((v(r + \tau) + \varphi z(\theta_r \omega))(v(r + \tau) + \varphi z(\theta_r \omega))) \\
&- \beta \tilde{P} ((\tilde{v}(r) + \varphi z(\theta_r \omega))(\tilde{v}(r) + \varphi z(\theta_r \omega))) \\
&+ \gamma \tilde{P} ((v(r + \tau) + \varphi z(\theta_r \omega))^2(v(r + \tau) + \varphi z(\theta_r \omega))) \\
&- \gamma P ((\tilde{v}(r) + \varphi z(\theta_r \omega))^2(\tilde{v}(r) + \varphi z(\theta_r \omega))) \\
&= \tilde{P} (f(v_{r+\tau} + \varphi z(\theta_{r+\tau} \omega)) - f(\tilde{v}_r + \varphi z(\theta_{r+\tau} \omega)) + (g(r + \tau) - g_\infty)).
\end{align*}
\]

Taking the inner product of (5.8) with \( V \) in \( H \), we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dr} \| V^\tau(r) \|^2 &+ \nu \| V^\tau(r) \|_V^2 + \alpha \| V^\tau(r) \|^2 \\geq \beta ((v(r + \tau) + \varphi z(\theta_r \omega))(v(r + \tau) + \varphi z(\theta_r \omega))) \\
&- \beta ((\tilde{v}(r) + \varphi z(\theta_r \omega))(\tilde{v}(r) + \varphi z(\theta_r \omega)), V^\tau(r)) \\
&+ \gamma ((v(r + \tau) + \varphi z(\theta_r \omega))^2(v(r + \tau) + \varphi z(\theta_r \omega))) \\
&- \gamma ((\tilde{v}(r) + \varphi z(\theta_r \omega))^2(\tilde{v}(r) + \varphi z(\theta_r \omega)), V^\tau(r)) \\
&= \beta ((u(r + \tau)u(r + \tau) - |\tilde{u}(r)|^2, v(r + \tau) + \varphi z(\theta_r \omega) - (\tilde{v}(r) + \varphi z(\theta_r \omega))) \\
&+ \gamma ((u(r + \tau))^2u(r + \tau) - |\tilde{u}(r)|^2, v(r + \tau) + \varphi z(\theta_r \omega) - (\tilde{v}(r) + \varphi z(\theta_r \omega))) \\
&\geq \beta \int_\Omega |u(r + \tau) - \tilde{u}(r)|^2 dx + \frac{\gamma}{2} \int_\Omega |u(r + \tau) - \tilde{u}(r)|^4dx \geq 0.
\end{align*}
\]

Notice that \( L_f(\cdot) \equiv L_f \) (in (2.4)) is a positive constant in this section. Hence, by (2.4) and the Young inequality, we have

\[
\frac{d}{dr} \| V^\tau(r) \|^2 \leq \| V^\tau(r) \|^2 + c \| v_{r+\tau} - \tilde{v}_r \|^2 + c \| g(r + \tau) - g_\infty \|^2.
\]

Integrating (5.9) over \([0, t]\) yields

\[
\| V^\tau(t) \|^2 \leq \| V^\tau(0) \|^2 + \int_0^t \| V^\tau(r) \|^2 dr + c \int_0^t \| v_{r+\tau} - \tilde{v}_r \|^2 H_u dr \\
+ c \int_0^t \| g(r + \tau) - g_\infty \|^2 dr, \quad \forall t \in [0, T], \; T > 0.
\]
Notice that
\[ \int_0^t \| v_{t+r} - \tilde{v}_t \|^2_{H_v} \, dr = \int_0^t \| V^\tau_r \|^2_{H_v} \, dr, \quad \forall t \in [0, T], \quad T > 0. \]  
(5.11)

It follows from (5.10) and (5.11) that for all \( t \in [0, T] \),
\[ \| V^\tau(t) \|^2 \leq c \int_0^t \| V^\tau_r \|^2_{H_v} \, dr + \| v_t - \tilde{v}_0 \|^2_{H_v} + c \int_0^T \| g(r + \tau) - g_\infty \|^2 \, dr. \]

Note that \( \| V^\tau(t) \|^2 \leq \| v_t - \tilde{v}_0 \|^2_{H_v} \) for \( t \in [-\varrho, 0] \). Hence we have, for all \( t \in [0, T] \),
\[ \| V^\tau_r \|^2_{H_v} \leq c \int_0^t \| V^\tau_r \|^2_{H_v} \, dr + \| v_t - \tilde{v}_0 \|^2_{H_v} + c \int_0^T \| g(r + \tau) - g_\infty \|^2 \, dr. \]

Using the Gronwall inequality (see e.g. [6, page 167]), we obtain, for all \( t \in [0, T] \),
\[ \| V^\tau_r \|^2_{H_v} \leq c e^{\psi T} (\| v_t - \tilde{v}_0 \|^2_{H_v} + \int_{-\infty}^{\tau + T} \| g(r) - g_\infty \|^2 \, dr). \]

By (5.5) in (G2) and the initial assumption, we have
\[ \| V^\tau_r \|^2_{H_v} = \| v_t(\cdot, \tau, \theta, \omega, v_\tau) - \tilde{v}_t(\cdot, \omega, \tilde{v}_0) \|^2_{H_v} \to 0, \quad \text{as } \tau \to -\infty. \]
The proof is complete. \( \square \)

By Lemma 5.1, the convergence of solutions holds true under the topology of the initial space \( H_v \) only. However, we can prove the asymptotic convergence of random attractors under the topology of the end space \( V_0 \) as follows.

**Theorem 5.2.** Suppose (F1)-(F4) and (G1)-(G2) hold. Then we have
\[ \lim_{\tau \to -\infty} \text{dist}_{V_0}(A(\tau, \omega), A_{\infty}(\omega)) = 0, \quad \omega \in \Omega. \]
(5.12)

**Proof.** We first prove the \( \mathcal{D} \)-pullback absorbing set \( K \) (as defined by (4.2)) is recurrent. More precisely, we claim \( K_{bu} \in \mathcal{D}_\infty \), where
\[ K_{bu}(\omega) := \bigcup_{s \leq 0} K(s, \omega), \quad \forall \omega \in \Omega. \]

Let \( b > 0 \). Since \( \tau \to K(\tau, \omega) \) is increasing, by the similar method as in (4.4), we have
\[ e^{-bt} \| K_{bu}(\theta - \tau \omega) \|^2_{H_v} = e^{-bt} \| \bigcup_{s \leq 0} K(s, \theta - \tau \omega) \|^2_{H_v} = e^{-bt} \| K(0, \theta - \tau \omega) \|^2_{H_v} \]
\[ = e^{-bt} \| K(0, \theta - \tau \omega) \|^2_{H_v} \leq c e^{-bt} \| R(0, \theta - \tau \omega) \| = c e^{-bt} \sup_{s \leq 0} \| R(s, \theta - \tau \omega) \| \]
\[ \leq c e^{-bt} \sup_{s \leq 0} \int_{-\infty}^0 e^{m/r} (1 + |z(\theta - \tau \omega)|^6 + \| g(r + s) \|^2) \, dr \]
\[ \leq c e^{-bt} \sup_{s \leq 0} \int_{-\infty}^0 e^{m/r} (1 + |t - r|^6 + \| g(r + s) \|^2) \, dr \to 0, \quad \text{as } t \to +\infty, \]

in view of the assumption (G1). Hence \( K_{bu} \in \mathcal{D}_\infty \) as desired.

We then prove (5.12) by contradiction. Suppose (5.12) is not true. Then there exist \( \delta > 0 \), \( \tau_n \to -\infty \) and \( \omega_0 \in \Omega \) such that for all \( n \in \mathbb{N} \)
\[ \text{dist}_{V_0}(A(\tau_n, \omega_0), A_{\infty}(\omega_0)) \geq 4\delta. \]

For each \( n \in \mathbb{N} \), we can find a \( u_n \in A(\tau_n, \omega_0) \) such that
\[ \text{dist}_{V_0}(u_n, A_{\infty}(\omega_0)) \geq \text{dist}_{V_0}(A(\tau_n, \omega_0), A_{\infty}(\omega_0)) - \delta \geq 3\delta. \]
(5.13)
Without loss of the generality we assume \( \tau_n \leq 0 \) for all \( n \in \mathbb{N} \). Then we have
\[
\{u_n\}_{n \in \mathbb{N}} \subset \bigcup_{s \leq 0} A(s, \omega_0).
\]
By Theorem 4.6, the attractor \( A \) is backward compact in \( V_\circ \). Hence \( \bigcup_{s \leq 0} A(s, \omega_0) \) is pre-compact in \( V_\circ \) and so is \( \{u_n\}_{n \in \mathbb{N}} \). Therefore, passing to a subsequence,
\[
\lim_{n \to +\infty} \|u_n - u_0\|_{V_\circ} = 0, \quad \text{for some } u_0 \in \bigcup_{s \leq 0} A(s, \omega_0) V_\circ.
\] (5.14)

Since the attractor is included into the absorbing set, it follows that
\[
\lim_{n \to +\infty} \|u_n - u_0\|_{V_\circ} = 0.
\]

On the other hand, since the embedding \( V \hookrightarrow H \) is continuous, the embedding \( V_\circ \hookrightarrow H_\circ \) is continuous too and thus \( \|v_n - v_0\|_{H_\circ} \leq c\|v_n - v_0\|_{V_\circ} \to 0 \). By Lemma 5.1 we obtain
\[
\lim_{n \to +\infty} \|\Phi(T, \tau_n - T, \theta_T \omega_0) v_n - \Phi(T, \theta_T \omega_0) v_0\|_{H_\circ} = 0.
\]

This means \( \|u_n - \Phi(T, \theta_T \omega_0) v_0\|_{H_\circ} \to 0 \). By the uniqueness of limits, we see from (5.14) that
\[
u_0 = \Phi(T, \theta_T \omega_0) v_0.
\]

Hence, by combining (5.14) and (5.16), we have, if \( n \) is large enough then
\[
\text{dist}_{V_\circ}(u_n, A_\infty(\omega_0)) \leq \|u_n - u_0\|_{V_\circ} + \text{dist}_{V_\circ}(\Phi(T, \theta_T \omega_0) v_0, A_\infty(\omega_0)) < 2\delta
\]
which contradicts (5.13). The proof is complete.

To close this paper, we provide two important examples for the general non-autonomous delay force \( f(t, u_t) \) with (F1)-(F4).

**Example 1 (Variable delay).** Let \( F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a measurable function such that
\[
F(t, 0) = 0, \quad |F(t, y) - F(t, z)| \leq L_F(t)|y - z|, \quad \forall t, y, z \in \mathbb{R},
\] (5.17)
where $L_F(\cdot)$ is positive, continuous and increasing such that $L := \lim_{t \to +\infty} L_F(t) < +\infty$. Let
\[
    f(t, u_t)(x) := F(t, u(t - \rho(t), x)), \ \forall t \in \mathbb{R}, \ x \in \mathcal{O},
\]
where $\rho(\cdot)$ is a positive function such that $\rho(\cdot) \in C^1(\mathbb{R})$ and
\[
    \varrho := \sup_{t \in \mathbb{R}} \rho(t) < +\infty, \quad \rho_* := \sup_{t \in \mathbb{R}} \rho'(t) < 1.
\]
It is easy to show that $f$ is pointwise Lipschitz continuous with the Lipschitz coefficient $L_F(\cdot)$ (i.e. (2.4) holds). Since $L_F(\cdot)$ is increasing, there is a $\lambda_0 > 0$ such that
\[
    \int_{-\infty}^{0} e^{\lambda_0} \sup_{s \leq \tau} L_F^2(r + s) dr \leq L^2 \int_{-\infty}^{0} e^{\lambda_0} dr = \frac{L^2}{\lambda_0} < +\infty, \quad \forall \tau \in \mathbb{R}.
\]
By the Lebesgue dominated convergence theorem we have
\[
    \lim_{\lambda \to +\infty} \sup_{s \leq \tau} \int_{-\infty}^{s} e^{\lambda(r-s)} L_F^2(r) dr \leq \lim_{\lambda \to +\infty} \int_{-\infty}^{0} e^{\lambda} \sup_{s \leq \tau} L_F^2(r + s) dr = \int_{-\infty}^{0} \lim_{\lambda \to +\infty} e^{\lambda} \sup_{s \leq \tau} L_F^2(r + s) dr = 0, \quad \forall \tau \in \mathbb{R}.
\]
Then $L_F(\cdot)$ is backward limitable and thus (F3) holds true. We now verify (F4),
\[
    \int_{\tau}^{t} e^{\lambda r} \| f(r, u_r) - f(r, v_r) \|^2 dr
    = \int_{\tau}^{t} e^{\lambda r} \| F(r, u(r - \rho(r))) - F(r, v(r - \rho(r))) \|^2 dr
    \leq L^2 \int_{\tau}^{t} e^{\lambda r} \| u(r - \rho(r)) - v(r - \rho(r)) \|^2 dr
    \leq \frac{L^2 e^{\lambda \tau} \rho_*}{1 - \rho_*} \int_{\tau}^{t} e^{\lambda r} \| u(r) - v(r) \|^2 dr,
\]
Hence (F4) holds with $c_f = L e^{\lambda \tau} \rho_* / (1 - \rho_*)^{-1/2}$.

**Example 2** (Distributed delay). Suppose $F$ is given as in (5.17) and $\varrho > 0$ is arbitrary. Consider
\[
    f(t, u_t)(x) := \int_{-\varrho}^{0} F(t + r, u(t + r, x)) dr, \quad \forall t \in \mathbb{R}, \ x \in \mathcal{O}.
\]
Since $L_F(\cdot)$ is increasing,
\[
    \| f(t, u_t) - f(t, v_t) \|^2 = \int_{\mathcal{O}} \left| \int_{-\varrho}^{0} (F(t + r, u(t + r, x)) - F(t + r, v(t + r, x))) dr \right|^2 dx
    \leq \int_{\mathcal{O}} \left( \int_{-\varrho}^{0} L_F(t + r) |u(t + r, x) - v(t + r, x)| dr \right)^2 dx
    \leq \varrho L_F^2(t + 0) \int_{\mathcal{O}} \int_{-\varrho}^{0} |u(t + r, x) - v(t + r, x)|^2 dr dx
    \leq \varrho^2 L_F^2(t) \sup_{r \in [-\varrho, 0]} \int_{\mathcal{O}} |u(t + r, x) - v(t + r, x)|^2 dx = \varrho^2 L_F^2(t) \| u_t - v_t \|_H^2.
\]
Since $L_F(\cdot)$ is backward limitable (see (5.18)), the Lipschitz bound $L_f(\cdot) := \varrho L_F(\cdot)$ is backward limitable and thus (F3) holds true. Note that

$$\int_t^\tau e^{s\tau} \| f(u_r) - f(v_r) \|^2 dr$$

$$= \int_t^\tau e^{s\tau} \int_\Omega \left[ \int_0^s \left| F(r + s, u(r + s, x)) - F(r + s, v(r + s, x)) \right| ds \right]^2 dx dr$$

$$\leq \int_t^\tau e^{s\tau} \int_\Omega \left( \int_0^s L_F(r + s) \left| u(r + s, x) - v(r + s, x) \right| ds \right)^2 dx dr$$

$$\leq \varrho L^2 \int_t^\tau e^{s\tau} \int_\Omega \int_0^s \left| u(r + s) - v(r + s) \right|^2 ds dx dr$$

$$= \varrho L^2 \int_0^\tau e^{s\tau} \int_\Omega \int_{t-s}^{t+s} \left| u(s) - v(s) \right|^2 ds dx dr$$

$$\leq \varrho e^{2L^2} \int_{t-\tau}^{t+\tau} e^{s\tau} \| u^0 - v^0 \|^2 dr.$$

Hence (F4) holds with $c_f = \varrho Le^{2\tau/2}$.

**Remark 1.** The above two examples are suitable for the existence of pullback random attractors. To deduce the asymptotic autonomy, we need to assume that the function $F$ in the two examples is independent of the time $t$ and that $\rho(\cdot) \equiv \varrho$ (the constant delay) in Example 1. In this case, the delay term $f(u_t)$ is independent of the time as assumed in this last section.

On the other hand, it remains open whether the asymptotic autonomy of attractors holds provided $f = f(t, u_t)$.

**Acknowledgements.** The authors thank the anonymous referees for their pertinent comments and suggestions, which greatly improved the earlier manuscript.

**REFERENCES**

[1] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differ. Equ.*, 246 (2009), 845–869.

[2] Z. Brzózniak, M. Capiński and F. Flandoli, Pathwise global attractors for stationary random dynamical systems, *Probab. Theory Relat. Fields*, 95 (1993), 87–102.

[3] T. Caraballo and J. A. Langa, On the upper semicontinuity of cocycle attractors for nonautonomous and random dynamical systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 10 (2003), 491–513.

[4] T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differ. Equ.*, 205 (2004), 271–297.

[5] T. Caraballo and K. Lu, Attractors for stochastic lattice dynamical systems with a multiplicative noise, *Front. Math. China*, 3 (2008), 317–335.

[6] A. N. Carvalho, J. A. Langa and J.C. Robinson, Attractors for Infinite-Dimensional Nonautonomous Dynamical Systems, Applied Mathematical Sciences, vol.182, Springer, 2013.

[7] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dyn. Differ. Equ.*, 9 (1997), 307–341.

[8] H. Cui, J. A. Langa and Y. Li, Measurability of random attractors for quasi strong-to-weak continuous random dynamical systems, *J. Dyn. Differ. Equ.*, 30 (2018), 1873–1898.

[9] R. C. Gilver and S. A. Altabelli, A determination of effective viscosity for the Brinkman-Forchheimer flow model, *J. Fluid Mech.*, 370 (1994), 258–355.

[10] V. Kalantarov and S. Zelik, Smooth attractors for the Brinkman-Forchheimer equations with fast growing nonlinearities, *Commun. Pure Appl. Anal.*, 11 (2012) 2037–2054.
[11] J. R. Kang and J. Y. Park, Uniform attractors for non-autonomous Brinkman-Forchheimer equations with delay, Acta Math. Sin., 29 (2013) 993–1006.
[12] P. E. Kloeden and J. Simsen, Attractors of asymptotically autonomous quasi-linear parabolic equation with spatially variable exponents, J. Math. Anal. Appl., 425 (2015), 911–918.
[13] P. E. Kloeden, J. Simsen and M. S. Simsen, Asymptotically autonomous multivalued Cauchy problems with spatially variable exponents, J. Math. Anal. Appl., 445 (2017), 513–531.
[14] L. Li, X. Yang, X. Li, X. Yan and Y. Lu, Dynamics and stability of the 3D Brinkman-Forchheimer equation with variable delay (1), Asymptot. Anal., 113 (2019), 167–194.
[15] Y. Li, A. Gu and J. Li, Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations, J. Differ. Equ., 258 (2015), 504–534.
[16] Y. Li, L. She and R. Wang, Asymptotically autonomous dynamics for parabolic equation, J. Math. Anal. Appl., 459 (2018), 1106–1123.
[17] Y. Li and J. Yin, A modified proof of pullback attractors in a Sobolev space for stochastic Fitzhugh-Nagumo equations, Discrete Contin. Dyn. Syst. Ser. B, 21 (2016), 1203–1223.
[18] P. A. Markowich, E. S. Titi and S. Trabelsi, Continuous data assimilation for the three-dimensional Brinkman-Forchheimer-extended Darcy Model, Nonlinearity, 29 (2016), 1292–1328.
[19] D. A. Nield, The limitations of the Brinkman-Forchheimer equation in modeling flow in a saturated porous medium and at an interface, Int. J. Heat Fluid Flow, 12 (1991), 269–272.
[20] L. Shi, X. Wang and D. Li, Limiting behavior of non-autonomous stochastic reaction-diffusion equations with colored noise on unbounded thin domains, Commun. Pure Appl. Anal., 19 (2020), 5367–5386.
[21] B. Wang and S. Lin, Existence of global attractors for the three-dimensional Brinkman-Forchheimer equation, Math. Meth. Appl. Sci., 31 (2008), 1479–1495.
[22] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, J. Differ. Equ., 253 (2012), 1544–1583.
[23] X. Wang, K. Lu and B. Wang, Random attractors for delay parabolic equations with additive noise and deterministin nonautonomous forcing, SIAM J. Appl. Dyn. Syst., 14 (2015), 1018–1047.
[24] X. Yang, L. Li, X. Yan and L. Ding, The structure and stability of pullback attractors for 3D Brinkman-Forchheimer equation with delay, Electron. Res. Arch., 28 (2021), 1395–1418.
[25] J. Yin, A. Gu and Y. Li, Backwards compact attractors for non-autonomous damped 3D Navier-Stokes equations, Dyn. Partial Differ. Equ., 14 (2017), 201–218.
[26] Y. You, C. Zhao, S. Zhou, The existence of uniform attractors for 3D Brinkman-Forchheimer equations, Disc. Cont. Dyn. Syst., 32 (2012), 3787–3800.

Received March 2021; revised June 2021; early access July 2021.
E-mail address: zqh157@email.swu.edu.cn
E-mail address: liyr@swu.edu.cn