Two-dimensional electron transport
in the presence of magnetic flux vortices

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Abstract

We have considered the conductivity properties of a two dimensional electron gas (2DEG) in two different kinds of inhomogeneous magnetic fields, i.e. a disordered distribution of magnetic flux vortices, and a periodic array of magnetic flux vortices. The work falls in two parts. In the first part we show how the phase shifts for an electron scattering on an isolated vortex, can be calculated analytically, and related to the transport properties through a force balance equation. In the second part we present numerical results for the Hall conductivity of the 2DEG in a periodic array of flux vortices. We find characteristic peaks in the Hall conductance, when plotted against the filling fraction. It is argued that the peaks can be interpreted in terms of "topological charge" piling up across local and global gaps in the energy spectrum.
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I. INTRODUCTION

Over the last decade the two dimensional electron gas (2DEG) have been exposed to a wide range of physical experiments, in which the electrons have been perturbed by different configurations of electrostatic potentials, with or without a homogeneous perpendicular magnetic field. These experiments have shown new kinds of oscillations in the magnetoconductivity, with a periodicity not given by the geometry of the Fermi surface, as is the case with the Shubnikov-de Haas oscillations, but given by the interaction of the two length scales given respectively by the magnetic length, and by the spatial structure of the potential, e.g. the Weiss oscillations. More recently, there have been increasing interest in systems where the 2DEG is exposed to an inhomogeneous perpendicular magnetic field. In such systems the inhomogeneities in the magnetic field acts as perturbations of the 2DEG, relative to the homogeneous magnetic field, where the band structure consists of the dispersionless Landau bands. The inhomogeneous magnetic field appears in the Hamiltonian in the form of a non trivial vector potential. In the case of a periodic variation in the magnetic field, it is possible to construct a periodic vector potential, if and only if the flux through the unit cell of the field is equal to a rational number, when measured in units of the flux quantum $\phi_0 = h/e$. Under these special circumstances Bloch states can be used as a basis for the calculation of response properties of the electron gas.

In this paper we have considered a special class of spatially varying magnetic fields which consists of flux vortices, that are either distributed at random or placed in a regular lattice structure. A system consisting of a 2DEG penetrated by a random distribution of magnetic flux vortices, have been experimentally realized by Geim et al. They made a sandwich construction of a GaAs/GaAlAs sample with a 2DEG at the interface, and a type II superconducting lead film (electrically disconnected from the 2DEG). When the system was placed in an external magnetic field, and cooled below the transition temperature of the film, the magnetic field penetrated the film, and thereby the 2DEG, in the form of Abrikosov vortices. When the external magnetic field is weak, below 100G, the vortices will
be well separated, and the 2DEG therefore sees a very inhomogeneous magnetic field. In the experiments conducted by Geim et al. the flux pinning in the film was strong, resulting in a disordered distribution of flux vortices. This is the physical situation which we investigate in Sec. II below. In very clean films of type II superconducting material, the flux vortices will order in a periodic array, i.e. an Abrikosov lattice, and thereby create a periodic magnetic field at the 2DEG. This is the situation which we analyse in Sec. III.

Several authors have investigated the transport properties of 2DEG’s in different kinds of inhomogeneous magnetic fields. Peeters and Vasilopoulos have made a theoretical study of the magnetoconductivity in a 2DEG in the presence of a magnetic field, which was modulated weakly and periodically along one direction. They found large oscillations in the longitudinal resistivity as a function of the applied magnetic field strength. These oscillations are due to the interference between the two length scales given respectively by the period of the lateral variation of the magnetic field, and by the magnetic length corresponding to the average background field. The oscillations are reminiscent of the Weiss oscillations, but have a higher amplitude and a shifted phase, relative to the magnetoresistance oscillations induced by the periodic electrostatic potential.

The problem of how the transport properties of the 2DEG is modified by the presence of a random distribution of flux vortices, have been treated earlier by A. V. Khaetskii, and also by Brey and Fertig. The approach used by these authors are closely related to the one we have presented in Sec. II, i.e. based on the scattering theory, and the Boltzmann transport equation. Khaetskii considered the scattering in certain limiting cases, including the semiclassical, and Brey and Fertig calculated the transport properties numerically.

In Sec. III we will address the “paradox” of how the Hall effect can disappear in the following situation: We imagine a 2DEG in a regular 2D-lattice of flux vortices, with the magnetic field from a single vortex exponentially damped with an exponential length \( \xi \), in units of the lattice spacing. We take the total flux from a single vortex to be \( \phi_0/2 \), as is the case when the vortices come from a superconductor. When \( \xi \gg 1 \), the field is homogeneous and the Hall conductivity is \( \sigma_H = pe^2/h \), where \( p \) is the number of filled
bands. In the other limit i.e. when $\xi \to 0$, the time reversal symmetry is restored, and the Hall conductivity vanish. The fact that the system has time reversal symmetry when the vortices are infinitely thin, can be seen by subtracting a Dirac string carrying one quantum of magnetic flux $\phi_0$, from each flux vortex. The introduction of the Dirac strings can not change any physical quantities, and the procedure therefore establishes that the system, with infinitely thin vortices with flux $+\phi_0/2$, is equivalent with the system with reversed flux $-\phi_0/2$, through each vortex. The paradoxical situation arises because it is known from general arguments\textsuperscript{7}, that the contribution to the total Hall conductivity from a single filled nondegenerate band is a topological invariant, and therefore cannot change gradually. The situation is even more clear cut if we imagine a periodic array of exponential flux vortices carrying one flux quantum $\phi_0$ each. Then the limit $\xi = \infty$ corresponds to a homogeneous magnetic field, while the opposite limit $\xi = 0$ corresponds to free particles. Incidentally this scheme can be used to establish an interpolating path between the multi-fractal structure known as Hofstadters butterfly\textsuperscript{8} which is a plot of the allowed energy levels for electrons on a lattice in a homogeneous magnetic field, and the corresponding plot for lattice-electrons in no field, which is completely smooth.

The plan of the paper is as follows. In Sec. II we calculate the longitudinal and transverse resistivities of the 2DEG, in the limit where the mean distance between the vortices is so long that correlation effects are negligible. First we review the classical scattering theory in Sec. II A, before we discuss the quantum dynamics in Sec. II B. The longitudinal and transverse resistivities are discussed in Sec. II B 3, and resonance scattering is demonstrated in Sec. II C. The case of a 2DEG in a periodic array of flux vortices, is the subject of Sec. III. In the first part, Sec. III A–III D, of this section the general theory of electron motion in a periodic magnetic field is reviewed, and in the second part, Sec. III E–III F, we present the results of the numerical calculations.
II. SINGLE VORTEX SCATTERING

In this section we will consider the introduction of magnetic flux vortices into a 2 dimensional electron gas, in the limit where each vortex can be treated as an individual scattering center. The vortices are assumed to be distributed at random, homogeneously over the sample. The average separation between the vortices is assumed so large, that we can neglect interference from multiple scattering events.

A. The Classical Cross Section

We will start by calculating the differential cross section for a charged particle scattering on a flux vortex within the framework of classical mechanics. This will provide a reference frame, and allow us to speak unambiguously about the classical limit. In the calculations we will use an ideal vortex, which has a circular cross section with constant magnetic field inside, and zero magnetic field outside.

\[ B(r) = \begin{cases} \frac{\phi}{\pi r_0^2} & \text{for } r < r_0 \\ 0 & \text{for } r > r_0. \end{cases} \]  

Here \( r_0 \) is the radius, and \( \phi \) is the total flux carried by the vortex. The classical orbit is found as the solution to Newton’s equation of motion with the force given by the Lorentz expression \( F = e v \times B \), for a particle of charge \( e \). It consists, as is well known, of straight line segments outside the vortex, and an arc of a circle inside, with radius of curvature given by the cyclotron radius \( l_c = v / \omega_c \), with \( v \) being the particle velocity, and \( \omega_c = \frac{eB}{m} \) the cyclotron frequency. Because the orbit inside the vortex is an arc of a circle, and it is impossible to draw a circle that only cut the circumference of the vortex once, it follows that a particle which initially is outside the vortex, can never become trapped in the vortex. The geometry of the classical scattering process shown in Fig. 1. is determined by the dimensionless parameter \( \gamma = l_c / r_0 \), which is the ratio between the radius of the cyclotron orbit, and the radius of the flux vortex. The definitions of the reduced impact parameter
\[ \beta = b/r_0, \text{ and the angles } \phi, \psi, \theta, \text{ follows from Fig. 1.} \] By inspecting the figure, it is observed that the following relations hold

\[ \beta = \sin \phi \tag{2} \]
\[ \tan \psi = \frac{\gamma + \beta}{\sqrt{1 - \beta^2}} \tag{3} \]
\[ \gamma \sin \frac{\theta}{2} = \sin(\psi - \phi) \text{sign}(\gamma + \beta), \tag{4} \]

where the sign of \( \gamma \) is dictated by the direction of the magnetic field inside the vortex. Hereafter we take \( \gamma \) to be positive. After a small amount of arithmetic \( \phi \) and \( \psi \) are eliminated, and we have

\[ \sin \frac{\theta}{2} = \text{sign}(\gamma + \beta)\sqrt{\frac{1 - \beta^2}{\gamma^2 + 2\gamma\beta + 1}}. \tag{5} \]

This relation gives the scattering angle \( \theta = \theta(\beta) \) as a function of the impact parameter \( \beta \). It is observed that for \( 0 < \gamma < 1 \) the scattering angle \( \theta \) sweeps through the interval \([-\pi, \pi]\), for \( \beta \in [-1, 1] \), while for \( \gamma > 1 \) we have \( \theta \in [0, \theta_0] \), with \( \theta_0 = 2 \arcsin(1/\gamma) = \theta(-1/\gamma) \). Furthermore \( d\theta/d\beta = 0 \) for \( \beta = -1/\gamma \) producing a singularity in the differential scattering cross section at \( \theta_0 \).

The differential cross section \( d\sigma/d\theta \) gives the total weight of impact parameters, which give scattering into the direction \( \theta \). Equation 3 has at most two solutions, which are easily found to be

\[ \beta_{\pm}(\theta) = -\gamma \sin^2 \theta/2 \pm \cos \theta/2 \sqrt{1 - \gamma^2 \sin^2 \theta/2}. \tag{6} \]

Furthermore we have

\[ \frac{d\beta_{\pm}}{d\theta} = -\frac{\gamma}{2} \sin \theta \mp \sin \theta/2 \frac{1 + \gamma^2 \cos \theta}{2\sqrt{1 - \gamma^2 \sin^2 \theta/2}}. \tag{7} \]

The differential cross section is for \( 0 < \gamma < 1 \) given by

\[ \frac{d\sigma}{d\theta} = \begin{cases} r_0 \left| \frac{d\beta_-}{d\theta} \right| & \text{for } \theta < 0 \\ r_0 \left| \frac{d\beta_+}{d\theta} \right| & \text{for } \theta > 0, \end{cases} \tag{8} \]
and for $\gamma > 1$, it is given by the expression

$$
\frac{d\sigma}{d\theta} = \begin{cases} 
    r_0 \left| \frac{d\beta_-}{d\theta} \right| + r_0 \left| \frac{d\beta_+}{d\theta} \right| & \text{for } 0 < \theta < \theta_0 \\
    0 & \text{otherwise}
\end{cases}
$$

Examples of cross-sections and trajectories are shown in Fig. 2. The integrated cross section $\sigma_{\text{tot}}$ is equal to the total weight of impact parameters corresponding to particles which hit the vortex. It is equal to the diameter of the vortex $\sigma_{\text{tot}} = 2r_0$, as is always the case in classical scattering. If the classical cross section is substituted into the collision integral of the Boltzmann transport equation, it is straightforward as shown in App. A, to obtain expressions for the transverse and longitudinal resistivities induced by a random distribution of vortices, in the limit where the density of vortices goes to zero, thereby eliminating the effects of multiple scattering events. Let $B$ denote the average flux density applied to the sample, and $\rho_H^0 = B/n_e e$ the Hall resistivity of a 2DEG in a homogeneous magnetic field $B$. We then have

$$
\rho_{xy}/\rho_H^0 = \zeta_{xy}^{\text{clas}}, \quad \rho_{xx} - \rho_{\text{imp}}/\rho_H^0 = \zeta_{xx}^{\text{clas}},
$$

with

$$
\zeta_{xy}^{\text{clas}} = \frac{2\gamma}{r_0} \int_{-\pi}^{\pi} d\theta \frac{d\sigma}{d\theta},
$$

$$
\zeta_{xx}^{\text{clas}} = \frac{2\gamma}{r_0} \int_{-\pi}^{\pi} d\theta \frac{d\sigma}{d\theta} (1 - \cos \theta).
$$

The $\zeta^{\text{clas}}$ parameters are a function of the sole parameter $\gamma = l_c/r_0$. This is indeed what one would expect, since the transport coefficients calculated within a classical framework can not depend on the flux quantum $\phi_0$. The functional behavior of the $\zeta$-quantities, are shown in Fig. 3. We will postpone the discussion of the experimental consequences of these curves, until we have calculated the same quantities quantum mechanically.

**B. Quantum Dynamics**

In this section we shall consider the electron scattering on a magnetic flux vortex within the framework of quantum theory. We will calculate the electron wave function, from which
the longitudinal and transverse conductivities can be found from the theory of Sec. II B 3. The quantum nature of the electron radically alters the picture of the scattering process, when the wavelength of the electron is comparable to, or longer than the diameter of the vortex. In the limit of very small electron wavelength, the scattering can essentially by described by the laws of geometrical optics, and thereby classical mechanics.

1. The Scattering Wave Function

We will again take an idealized cylindrical vortex, with constant magnetic field inside, and zero field outside, Eq. [11]. This vortex is completely symmetric under any rotation about the center axis. This symmetry can also be made a symmetry of the Hamiltonian, by choosing the right gauge when writing down the vector potential. In cylindrical coordinates $A = e_r A_r + e_\theta A_\theta$, we have

$$B(r) = \partial_r A_\theta - \frac{1}{r} \partial_\theta A_r + \frac{A_\theta}{r}. \quad (13)$$

When $B$ is invariant under rotation, this equation has the simple solution

$$A_r = 0, \quad A_\theta(r) = \frac{1}{r} \int_0^r dr' r' B(r'), \quad (14)$$

which in our case give the vector potential $A = e_\theta A_\theta$ with

$$A_\theta(r) = \begin{cases} \frac{\phi r}{2\pi r_0^2} & \text{for } r < r_0 \\ \frac{\phi}{2\pi r} & \text{for } r > r_0. \end{cases} \quad (15)$$

The Hamiltonian is given by the expression

$$H = \frac{1}{2m} (p - eA)^2 = -\frac{\hbar^2}{2m} \left\{ \partial_r^2 + \frac{1}{r} \partial_r + \left( \frac{1}{r} \partial_\theta - \frac{ie}{\hbar} A_\theta \right)^2 \right\}. \quad (16)$$

Here we have taken the charge of the particle to be $e$. The Hamiltonian is rotationally invariant, and therefore commutes with the angular momentum about the symmetry axis,
$L_z$. Consequently $L_z$ and $H$ have common eigenstates. The canonical momentum operator of a particle in a magnetic field is $p = mv + qA = \frac{\hbar}{i} \nabla$, and the operator for the angular momentum about the $z$-axis is $L_z = [r \times p]_z = \frac{\hbar}{i} \partial_\theta$. The eigenstates of $L_z$ are $e^{il\theta}$, where $l$ must be an integer in order not to have cuts in the wave function. We can now separate the variables of the common eigenstates of $L_z$ and $H$, and write

$$\phi_{kl}(r, \theta) = R_{kl}(r) e^{il\theta}, \quad (17)$$

where $k$ is an energy label $E = \hbar^2 k^2 / 2m$. Let us introduce the flux quantum $\phi_0 = \hbar/e$ and the dimensionless fraction $\alpha = \phi / \phi_0$. The differential equations for the radial part of the wave function, takes a particularly simple form if we write them down in dimensionless variables $\xi = r/r_0$ and $\kappa = kr_0$. The variable $\kappa$ measures the size of the vortex compared to the electron wavelength. With these definitions, the equation for the radial part of the wave function for $\xi < 1$ is

$$R'' + \frac{1}{\xi} R' + \left( \kappa^2 - \left( \frac{l}{\xi} - \alpha \xi \right)^2 \right) R = 0. \quad (18)$$

And for $\xi > 1$ we have

$$R'' + \frac{1}{\xi} R' + \left( \kappa^2 - \frac{(l - \alpha)^2}{\xi^2} \right) R = 0. \quad (19)$$

Inside the vortex an analytical solution to the radial equation can be found by following the procedure of L. Page, see also the review of Olariu and Popescu. The solution to the radial equation inside the vortex, which is regular as $\xi \to 0$, is

$$\phi_{kl}(\xi, \theta) = C \xi^{|l|} e^{-\frac{1}{2} \alpha \xi^2} M \left[ \frac{1}{2}(|l| + 1 - l - \frac{(kr_0)^2}{2\alpha}), |l| + 1, \alpha \xi^2 \right] e^{il\theta}. \quad (20)$$

Here $M$ is a confluent hypergeometric function (solution to Kummer’s equation), in the notation of Abramowitz and Stegun, and $C$ is a normalization constant which we will not need to evaluate. At the Landau quantization energies the confluent hypergeometric function $M$ reduces to an associated Laguerre polynomial.

Outside the vortex the radial equation is just the differential equation for Bessel functions of the first kind. We have for $\xi > 1$
\[ \phi_{kl}(\xi, \theta) = (a_l J_{l-\alpha}(kr) + b_l Y_{l-\alpha}(kr)) e^{i\theta}. \] (21)

The two constants \( a_l, b_l \) are found from the requirement, that the wave function has to be continuously differentiable at the boundary of the vortex. Let us write

\[ a_l = c_l \cos \delta_l \quad b_l = -c_l \sin \delta_l, \] (22)

then we have the following asymptotic expression for the total wave function \( \psi(r, \theta) \) in the region \( kr \gg 1 \)

\[ \psi(r, \theta) = \sqrt{\frac{2}{\pi kr}} \sum_l c_l \cos(x_l + \delta_l) e^{i\theta}. \] (23)

Here we have introduced the abbreviation \( x_l = kr - (\pi/2)(l - \alpha) - \pi/4 \). The scattering phase shift \( \delta_l \), is found from the condition that the wave function has to be continuously differentially at the boundary of the vortex. The constant \( c_l \) can be expressed in terms of the phase shift, via the scattering condition, i.e. that all the incoming particle current must be described by the wave function \( \psi_i \) which far away from the vortex describes a uniform current directed towards the vortex. In the geometry with which we are concerned, the particles come in along the positive \( x \)-axis, towards the vortex placed at the origin, as shown in Fig. 4. The state describing a uniform current in the direction of the negative \( x \)-axis is

\[ \psi_i = e^{-ikr \cos \theta + i\alpha \theta}. \] (24)

This expression is observed to have a cut whenever \( \alpha \) is not integer, however this cut will not be present in the total wave function \( \psi \). There will be a counter cut in the outgoing wave function \( \psi_s = \psi - \psi_i \), to render the total wave function completely smooth. This cut only arises because we insists in splitting the total wave function into incoming and outgoing terms. We take the angle \( \theta \) to have values in the interval \( -\pi < \theta < \pi \), which places the cuts in \( \psi_{i,s} \) along the negative real axis. With this choice the asymptotic expansion of \( \psi_i \), for \( kr \gg 1 \), in terms of partial waves, takes the form
\[ \psi_i = \frac{1}{\sqrt{2\pi kr}} \sum_l \left[ e^{ikr-i\pi/4} \cos \pi(l-\alpha) + e^{-ikr+i\pi/4} \right]. \] (25)

The scattering condition is now implemented by demanding that the coefficient of \( e^{-ikr} \), in respectively the expansion of \( \psi \) and \( \psi_i \) must be identical for each \( l \), resulting in the identity

\[ c_l = e^{i\delta_l-i\pi(l-\alpha)/2}. \] (26)

2. Phase Shifts for Scattering on Vortex with Finite Radius

Let us now calculate the phase shifts for a charged particle scattering on a cylindrical flux vortex with finite radius. The phase shifts are derived from the condition that the logarithmic derivative of the radial wave function must be continuous at the boundary of the vortex

\[ \frac{1}{R_i} \frac{dR_i^<}{d\xi} \bigg|_{\xi=1} = \frac{1}{R_i} \frac{dR_i^>}{d\xi} \bigg|_{\xi=1}. \] (27)

Inside the vortex we have with the abbreviations \( h_l \equiv (|l|+1-l-(kr_0)^2/2\alpha)/2 \) and \( g_l \equiv |l|+1 \)

\[ E_l \equiv \frac{1}{R_i} \frac{dR_i^<}{d\xi} \bigg|_{\xi=1} = |l| - \alpha + 2e^{h_l} M[h_l+1,g_l+1,\alpha] g_l M[h_l,g_l,\alpha]. \] (28)

Outside the vortex the logarithmic derivative reads

\[ \frac{1}{R_i} \frac{dR_i^>}{d\xi} \bigg|_{\xi=1} = \frac{\dot{j}_l - \delta_l \tan \delta_l}{J_{l-\alpha}(\kappa) - Y_{l-\alpha}(\kappa) \tan \delta_l}, \] (29)

where we have introduced the abbreviations \( z_l = (kr_0/2)(Z_{l-\alpha-1}(kr_0) - Z_{l-\alpha+1}(kr_0)) \) with \( (z,Z) = (j,J) \) or \( (y,Y) \). It is now simple to solve for \( \delta_l \)

\[ \tan \delta_l = \frac{j_l - E_l J_{l-\alpha}(kr_0)}{y_l - E_l Y_{l-\alpha}(kr_0)}. \] (30)

The tan \( \delta_l \)'s are the bricks from which cross sections and transport coefficients can be build. The presented curves have all been calculated with phase shifts found by this expression with the help of Mathematica, which have implementations of all the involved special functions.
3. Momentum Flow and Force Balance

In order to express the transverse and longitudinal resistivities of the 2DEG in terms of matrix elements of the force, we make a simple force balance argument. For a more comprehensive discussion of the force balance, we refer to E. B. Hansen.

We consider a 2DEG carrying a current of density \( j_x \), along the \( x \)-direction, i.e. \( j_y = 0 \). The distribution function of the electrons is then

\[
n(k) = n^0(k - \delta k),
\]

with \( n^0(k) = \Theta(\epsilon_F - \epsilon_k) \), and \( \delta k \) is related to the current density \( j_x = (ev_F^2/2\hbar)g(\epsilon_F)\delta k \), where \( g \) is the density of states. The forces acting on an electron in the state \( k \) can be separated into two classes. First, there are the forces due to the electric fields \( E_x, E_y \) present in the sample. Secondly, there are the forces due to scattering processes, which due to the rotational symmetry of the scatterers – vortices and impurities – can be decomposed into a force \( F_L \) parallel to \( k \), and a force \( F_T \) at right angles to \( k \). If we let \( \theta_k \) denote the angle between \( k \) and the \( x \)-axis, we can write the scattering force on an electron in the state \( k \), projected along the coordinate axes as

\[
F_x(k) = F_L(k) \cos \theta_k - F_T(k) \sin \theta_k,
\]

\[
F_y(k) = F_L(k) \sin \theta_k + F_T(k) \cos \theta_k.
\]

When we have a steady state transport situation, the total force on the 2DEG must vanish. With \( \mu = x, y \), and \( n_e \) denoting the density of the 2DEG, we therefore have the equation

\[
en_e E_\mu = \sum_k F_\mu(k)n(k).
\]

And thereby

\[
en_e E_\mu = \frac{v_F}{\hbar} g(\epsilon_F)\delta k \int \frac{d\theta_k}{2\pi} F_\mu(k_F, \theta_k) \cos \theta_k.
\]

Along the \( x \)- and \( y \)-directions respectively, this leads to

\[
en_e E_x = \frac{v_F}{\hbar} g(\epsilon_F) \delta k \frac{\sin \theta_k}{2\pi} F_L(k_F, \theta_k),
\]

\[
en_e E_y = \frac{v_F}{\hbar} g(\epsilon_F) \delta k \frac{\cos \theta_k}{2\pi} F_L(k_F, \theta_k).
\]
\[
\frac{E_x}{j_x} = \frac{F_L(k_F)}{ne^2v_F} \tag{36}
\]
\[
\frac{E_y}{j_x} = \frac{F_T(k_F)}{ne^2v_F} \tag{37}
\]

The ordinary impurities etc. do not give rise to a net transverse force on the electrons. Consequently, only the vortices contribute to the transverse force \(F_T(k_F)\) on an electron, on the Fermi surface. This force will, in the limit of low vortex density, be proportional to the total number of vortices. In this limit we can therefore calculate the force as the total number of vortices \(N_\alpha\), times the force from a single vortex

\[
F_T(k_F) = N_\alpha \left\langle \Psi | \hat{F}_y | \Psi \right\rangle \tag{38}
\]

Here \(N_\alpha = An_\alpha = AB/\alpha \phi_0\), with \(B\) equal to the average flux density applied to the sample.

Further more, \(\hat{F} = (e/2)[v \times B + B \times v]\) is the operator corresponding to the Lorentz force inside the vortex, and \(\Psi\) is the normalized wave function of the electron, i.e. \(\left\langle \Psi | \Psi \right\rangle = 1\). The electron wave function \(\Psi\) is an eigenstate of the one electron – one vortex system. It describes an electron which initially travels along the \(x\)-axis, or in other words, the expectation value of the velocity is \(\left\langle \Psi | \hat{v}_\mu | \Psi \right\rangle = \delta_{\mu x} v_F\), and it is labeled by the k-vector \(k = (k_F, 0)\). Note, that \(\Psi\) is identical to the wave function \(\psi\) of Sec. II B 1 apart from the normalization \(\left\langle \psi | \psi \right\rangle = A\).

The transverse resistivity of the 2DEG can now be expressed as

\[
\rho_{xy} = \frac{E_y}{j_x} = \rho_H^0 \zeta_{xy}, \tag{39}
\]

where \(\rho_H^0 = B/n_e e\) is the Hall resistivity of electrons in a homogeneous magnetic field of magnetic flux density \(B\), and

\[
\zeta_{xy} = \frac{\left\langle \psi | \hat{F}_y | \psi \right\rangle}{2\pi \alpha \hbar v_F}. \tag{40}
\]

Similarly, we have for the longitudinal resistivity

\[
\rho_{xx} - \rho_{\text{imp}} = \rho_H^0 \zeta_{xx} = \frac{\hbar}{e^2 n_e^2} 2\pi \alpha \zeta_{xx}. \tag{41}
\]

where we have singled out the contribution from the vortices, and
\[ \zeta_{xx} = \frac{\langle \psi| \hat{F}_x |\psi \rangle}{2\pi \alpha \hbar v_F}. \] (42)

The matrix elements \( \langle \psi|\hat{F}_\mu|\psi \rangle \) can be calculated by numerically performing the integral, which is limited to the range of the magnetic field, i.e. to the core of the vortex. We have used this procedure, which is rather laborious for the computer, to check the results obtained by the following much faster procedure.

In a continuum the flow of kinematical momentum is described by a continuity equation, which at the classical level takes the form

\[ \partial_t k_\mu + \partial_\nu \pi_{\nu \mu} = F_\mu. \] (43)

Here \( k_\mu = mv_\mu \) is the kinematical momentum (which in the presence of a magnetic field differ from the generalized momentum \( p_\mu = mv_\mu + eA_\mu \)) and \( \pi_{\nu \mu} = mv_\nu v_\mu \) is the tensor describing the flow of kinematical momentum. The quantum mechanical equivalent of Eq. 43 is

\[ \partial_t k_\mu + \partial_\nu \pi_{\nu \mu} = \Psi^* \hat{F}_\mu \Psi, \] (44)

where now

\[ k_\mu = m\Psi^* v_\mu \Psi, \] (45)

\[ \pi_{\nu \mu} = \frac{m}{2} \left[ \Psi^* (v_\nu v_\mu \Psi) + (v_\mu \Psi)(v_\nu \Psi)^* \right], \] (46)

and \( \hat{F} \) is the operator corresponding to the Lorentz force. The interpretation of Eq. 44 as a generalization of Eq. 43 applies strictly speaking only to the real part. For a derivation of Eq. 44 we refer to the review of Olariu and Popescu. The scattering situation we are considering is in a steady state and therefore \( \partial_t k_\mu = 0 \). We have by Gauss law

\[ \langle \psi|\hat{F}_\mu|\psi \rangle = \int d^2 r \partial_\nu \pi_{\nu \mu}(r) = \int_{-\pi}^{\pi} r d\theta \pi_{r \mu}(r), \] (47)

where the index \( \nu \) refers to the radial component, and the last integral is along any circle of radius \( r > r_0 \), (the Lorentz force is only non-vanishing inside the vortex). Let us first consider the \( y \)-component of the force. As \( \pi_{\nu \mu} \) is a tensor, we can write \( \pi_{r \theta} = \pi_{rr} \sin \theta + \pi_{r \theta} \cos \theta \). Furthermore the velocity operators are (outside the vortex core)
\[ v_r = \frac{\hbar}{m} \frac{\partial_r}{i} \tag{48} \]
\[ v_\theta = \frac{\hbar}{mr} \left( \frac{\partial_\theta}{i} - \alpha \right) \tag{49} \]

and due to the extra factor of \( r \) in \( v_\theta \), only \( \pi_{rr} \) will contribute to the integral, Eq. 47, in the limit \( r \to \infty \). In the asymptotic region we have

\[ \pi_{rr} = \frac{\hbar^2 k}{\pi mr} \sum_m \sum_l e^{i(\delta_{l+m} - \delta_l)} i^m \cos[\delta_{l+m} - \delta_l - \frac{\pi}{2} m] e^{in\theta}, \tag{50} \]

and

\[ \langle \psi | \hat{F}_y | \psi \rangle = \int_{-\pi}^{\pi} r d\theta \pi_{rr} \sin \theta, \tag{51} \]

resulting after some algebra in

\[ \zeta_{xy} = \frac{1}{2\pi \alpha} \sum_l \sin[2(\delta_{l+1} - \delta_l)]. \tag{52} \]

Similarly, we have in the \( x \)-direction \( \pi_{rx} = \pi_{rr} \cos \theta - \pi_{r\theta} \sin \theta \), and

\[ \langle \psi | \hat{F}_x | \psi \rangle = \int_{-\pi}^{\pi} r d\theta \pi_{rr} \cos \theta, \tag{53} \]

resulting in the expression

\[ \zeta_{xx} = \frac{1}{\pi \alpha} \sum_l \sin^2[\delta_{l+1} - \delta_l]. \tag{54} \]

In order to calculate the \( \zeta \)-quantities it is convenient to express them in terms of \( t_l \equiv \tan \delta_l \)

\[ \zeta_{xx} = \frac{1}{\pi \alpha} \sum_l \frac{(t_{l+1} - t_l)^2}{(1 + t_l^2)(1 + t_{l+1}^2)}, \tag{55} \]
\[ \zeta_{xy} = \frac{1}{\pi \alpha} \sum_l \frac{(t_{l+1} - t_l)(1 + t_l t_{l+1})}{(1 + t_l^2)(1 + t_{l+1}^2)}. \tag{56} \]

The \( \zeta_{xy} \) expression can be divided into two terms of different origin. Define \( \Delta \zeta_{xy} \) and \( \Delta_{xy}(\alpha) \) to be respectively

\[ \Delta \zeta_{xy} \equiv \frac{2}{\pi \alpha} \sum_l \frac{(t_{l+1} - t_l)t_l t_{l+1}}{(1 + t_l^2)(1 + t_{l+1}^2)}, \tag{57} \]
\[ \Delta_{xy}(\alpha) \equiv \zeta_{xy} - \Delta \zeta_{xy} = \frac{(t_{l+1} - t_l)(1 - t_l t_{l+1})}{(1 + t_l^2)(1 + t_{l+1}^2)}. \tag{58} \]
Then we have
\[
\Delta_{xy}(\alpha) = \frac{1}{\pi \alpha} \sum_l \left[ \frac{t_{l+1}}{1 + t_{l+1}^2} - \frac{t_l}{1 + t_l^2} \right]
\]
\[
= \frac{1}{\pi \alpha} \left[ \sin \delta_\infty \cos \delta_\infty - \sin \delta_{-\infty} \cos \delta_{-\infty} \right]
= \frac{\sin 2\pi \alpha}{2\pi \alpha}.
\] (59)

Here we have used the fact that the sum is telescopic, and the following properties of the phase shifts. For any given finite energy of the electron, the \(l\)’th partial wave, which is an eigenstate of \(L_z\) with eigenvalue \(\hbar l\), will have an insignificantly small probability density inside a radius \(b_l\), determined by \(\hbar |l| \sim b_l \hbar k\), i.e. \(b_l \sim |l|/k\). As \(|l|\) grows to infinity the \(l\)’th partial wave will not be able to detect any difference between a vortex of radius \(r_0 \ll b_l\), and an Aharonov-Bohm string, and consequently \(\delta_l \to \delta_l^{AB}\) for \(l \to \pm\infty\). The Aharonov-Bohm phase shifts are
\[
\delta_l^{AB} = \begin{cases} 
0 & \text{for } l > \alpha \\
\pi(l - \alpha) & \text{for } l < \alpha,
\end{cases}
\] (60)

which can be seen by comparing the wave functions given respectively in Eq. 21 and Eq. B1.

By inserting the Aharonov-Bohm phase shift into \(\Delta \zeta_{xy}\), it is seen that \(\Delta \zeta_{xy}\) vanishes in this limit. In this form \(\zeta_{xy}\) is therefore separated into a term due to the Aharonov-Bohm effect, and a correction term due to the effects of the finite radius of the vortex. Moreover the sum over \(l\) in the correction term, converges rapidly for increasing \(|l|\). The \(\Delta_{xy}(\alpha)\) term show, that even though the magnetic field is only non-vanishing at a single point in the Aharonov-Bohm limit, the electron still experiences a transverse force. This force was first described by S. V. Iordanskii\(^{15}\). Let us remark that the sum appearing in \(\zeta_{xx}\) is already convergent. The residual value of \(\zeta_{xx}\) in the Aharonov-Bohm limit is
\[
\zeta_{xx}^{AB} = \frac{1}{\pi \alpha} \sin^2 \pi \alpha.
\] (62)
Curves showing $\zeta_{xy}$ as a function of $kr_0$ for different values of the flux $\alpha$, have been plotted in Fig. 5. For $kr_0 \gg 1$ all the curves go to $\zeta_{xy} = 1$, which is the classical value as is seen in Fig. 3. In the Aharonov-Bohm limit $kr_0 \ll 1$, the curves go to a residual value depending on the flux $\zeta_{xy} \rightarrow \zeta_{xy}^{AB} = \sin 2\pi\alpha/2\pi\alpha$. The $\zeta_{xy}$-curve for $\alpha = 1/2$ we can compare with the experimental Hall factor measured by Geim et al. The overall qualitative behavior is in good agreement. To make a quantitative test we have fitted our curve to the experimental curve by tuning the radius of the vortex $r_0$. The best fit is obtained for a vortex radius $r_0 = 50\text{nm}$, and this is significantly smaller than the exponential length estimated by Geim to be 100nm.

Curves showing $2\pi\alpha\zeta_{xx}$ as a function of $kr_0$ for different values of the flux $\alpha$, is shown in Fig. 6. The magneto-resistivity, given by

$$\frac{d\rho_{xx}}{dB} = \frac{1}{ne}\zeta_{xx},$$

is constant, corresponding to the fact that the resistivity is proportional to the density of vortices, in the low $B$-field limit. The magnitude of the magneto-resistivity do not show perfect agreement with the values measured by Geim et al. For the 2DEG with the lowest density that has been measured by Geim, $n_e = 3.65 \cdot 10^{10}\text{cm}^{-2}$, Geim reports $d\rho_{xx}^{\text{meas}}/dB = 0.06\Omega/G$. In this case Eq. (63) predicts $d\rho_{xx}/dB = 0.55\Omega/G$, for $r_0 = 50\text{nm}$.

In order to investigate the sensitivity of the $\zeta$’s to the radial distribution of magnetic field in the vortex, we have calculated them with an exponential distribution

$$B_{\exp}(r) = \frac{\alpha\phi_0}{2\pi r_0^2} e^{-r/r_0}.$$  

The calculation is performed by solving the radial Shrödinger equation numerically, and imposing an artificial boundary on the vortex, outside which the amount of flux is negligible. The phase shifts are then found from the condition that the wave function much be smooth at the boundary. From the phase shifts the $\zeta$-quantities can be calculated as for the step function vortex. The calculation shows almost no difference between the $\zeta$-values, corresponding to the two different choices, indicating that the $\zeta$-quantities are not very sensitive to the shape of the vortices.
C. Multi Flux Quantum Vortex and Resonance Scattering

When the total amount of magnetic flux inside the vortex is increased, the \( \zeta \)-quantities acquire more structure. To illustrate this we have plotted \( \zeta_{xy} \) and \( 2\pi \alpha \zeta_{xx} \) in Fig. 7 for a flux vortex carrying a total of \( \alpha = 10 \) flux quanta. The structure seen in the plots is an effect of the resonant scattering which takes place when the energy of the incoming particle is close to one of the Landau quantization energies corresponding to the magnetic field strength inside the vortex. The magnetic field in the vortex vanishes outside a finite range – the radius of the vortex – and there are therefore no real Landau levels in the sense of stationary eigenstates, but only metastable states. The resonance condition \( (p + 1/2)\hbar \omega_c = \hbar^2 k^2 / 2m \) translates for fixed \( \alpha \) into

\[
k_p r_0 = 2 \sqrt{|\alpha| (p + \frac{1}{2})}, \quad p = 0, 1, 2, \ldots
\]

These values are in excellent agreement with the resonances seen in Fig. 7, where the first eight resonances corresponding to \( p = 0, \ldots, 7 \), are clearly distinguished. At the resonance energies the typical time the particle spends in the scattering region, i.e. inside the vortex, is much longer than it is away from resonance. The time the particle spends inside the vortex at a resonance, is the lifetime of the corresponding metastable state. The inverse lifetime is proportional to the width of the resonance, that is strictly speaking the width of the peak in the partial wave cross section \( \sigma_l \), corresponding to the \( l \) quantum number of the metastable Landau state.

It is easy to interpret the peaks in the \( \zeta_{xx} \)-curve in Fig. 7, appearing at the resonance energies. Because when the electron spends longer time in the scattering region, it loses knowledge of where it came from, resulting in an enhanced probability of being scattered in the backwards direction. The \( \zeta_{xy} \)-curve is a measure of asymmetric scattering, and we can therefore interpret the dips seen in Fig. 7 along the same line of reasoning. The electron spends longer time in the scattering region, thereby losing knowledge of what is left and what is right. Every time a new “channel” is opened the asymmetry of the scattering is suppressed, thus resulting in a sawtooth like curve.
III. HALL EFFECT IN A REGULAR ARRAY OF FLUX VORTICES

A. Introduction

Recently measurements were made by Geim et al.\cite{2,3} of the Hall resistivity of low density 2DEG’s in a random distribution of flux vortices, at very low magnetic field strengths. A profound suppression of the Hall resistivity was found, for 2DEG’s with Fermi wavelengths of the same order of magnitude as the diameter of the flux vortices. This indicates that we are dealing with a phenomenon of quantum nature. These measurements were made by placing a thin lead film on top of a GaAs/GaAlAs heterostructure. When a perpendicular magnetic field is applied, the magnetic field penetrates the superconducting lead film and also the heterostructure in the form of flux vortices each carrying half a flux quantum \( \phi_0/2 \) of magnetic flux. Due to the strong flux vortex pinning in the films Geim have used, the vortices were positioned in a random configuration.

In this section we will consider the hypothetical experiment where one instead of a “dirty” film, places a perfectly homogeneous type II superconducting film, on top of the 2DEG. The film do not have to be made of a material which is type II superconducting in bulk form. A film of a type I superconducting material will also display a mixed state if the thickness of the film is below the critical thickness \( d < d_c \). Experimentally perfect Abrikosov flux vortex lattices have been observed in thin films of lead with thickness \( d < d_c \approx 0.1 \mu m \).\cite{16}

If one succeeds to make such a sandwich construction, one has an ideal system for investigating how a 2 dimensional electron gas behaves in a periodic magnetic field. When the magnetic field exceeds \( H_{c1} \), which can be extremely low, the superconducting film will enter the mixed phase, and form an Abrikosov lattice of flux vortices. The Abrikosov lattice in the superconductor will give rise to a periodic magnetic field at the 2DEG, and moreover as the strength of the applied magnetic field is varied the only difference at the 2DEG, is that the lattice constant of the periodic magnetic field varies. The Abrikosov lattice is most often a triangular lattice with hexagonal symmetry, although other lattices have been observed.
(e.g. square) in special cases where the atomic lattice structure impose a symmetry on the flux lattice. For simplicity, the model calculations we have made are for a system with a square lattice of flux vortices, but we do not expect this to influence the overall features of the results. From the point of view of the 2DEG it is important that the flux vortices carry half a flux quantum \( \frac{h}{2e} = \frac{\phi_0}{2} \) due to the 2\( e \)-charge of the Cooper pairs in the superconductor. The magnetic field from a single flux vortex falls off exponentially with the distance from the center of the vortex. This exponential decay is characterized by a length \( \lambda_s \), which essentially is the London length of the superconductor, proportional to one over the square root of the density of Cooper pairs. The length \( \lambda_s \) can be varied by changing the temperature, or the material of the superconductor.

The other characteristic lengths of the system are the Fermi wavelength \( \lambda_F = \sqrt{\frac{2\pi}{n_e}} \), where \( n_e \) is the density of the 2DEG, the lattice constant \( a \) of the periodic magnetic field, and the mean free path \( l_f = v_F \tau \). The mean free path we assume to be very large compared to \( a \) and \( \lambda_s \). The magnetic field is only varying appreciably when \( a \) is larger than \( \lambda_s \). This means that the magnetic flux density of the applied field should be appreciably less than \( \phi_0/(\pi \lambda_s^2) \). In the limit where \( \lambda_s \ll a, \lambda_F \), the vortices can be considered magnetic strings, and the electrons experiences a periodic array of Aharonov-Bohm scatterers. In this case the value of the flux through each vortex is crucial. If for instance the flux had been one flux quantum \( \phi_0 \), the electrons would not have been able to feel the vortices at all. But in the real world the vortices from the superconductor carry \( \frac{\phi_0}{2} \) of flux, and therefore this limit is nontrivial. The electrons has for instance a band structure quite different from that of free electrons. In the mathematical limit of infinitely thin vortices each carrying half a flux quantum, there cannot be any Hall effect. This is most easily seen by subtracting one flux quantum from each vortex to obtain a flux equal to minus a half flux quantum through each vortex. As we have discussed earlier the introduction of the Dirac strings can not change any physics, and the procedure therefore shows that the system is equivalent to it’s time reversed counterpart, thereby eliminating the possibility of a Hall effect.

In this study we have ignored electron-electron interaction and the electron spin through-
out, in order to keep the model simple. From the point of view of the phenomena we are
going to describe, the effect of the electron spin will be to add various small corrections.

B. Electrons in a periodic magnetic field

1. Magnetic translations

It is a general result for a charged particle in a spatially periodic magnetic field $B(x, y)$, that the eigenstates of the system can be labeled by Bloch vectors taken from a Brillouin zone, if and only if the flux through the unit cell of the magnetic field is a rational number $p/q$ times the flux quantum. The standard argument for this fact is made by introducing magnetic translation operators. To introduce magnetic translation operators in an inhomogeneous magnetic field we first make the following observation. The periodicity of the field can be stated $B(r + R) = B(r)$, for $R$ belonging to a Bravais lattice. But this implies that the difference between the vector potentials $A(r + R)$ and $A(r)$ must be a gauge transformation

$$\nabla \times \{A(r + R) - A(r)\} = B(r + R) - B(r) = 0. \quad (66)$$

We introduce the gauge potential $\chi_R$ and write

$$A(r + R) = A(r) + \nabla \chi_R(r). \quad (67)$$

The function $\chi_R$ is only defined modulo an arbitrary additive constant which have no physical effect. The Hamiltonian of the electrons is

$$H = \frac{1}{2m} (p - eA(r))^2. \quad (68)$$

The ordinary translation operators $T_R = \exp[\frac{i}{\hbar} R \cdot p]$ do not commute with the Hamiltonian, because they shift the argument of the vector potential from $r$ to $r + R$, but as we just have seen this can be undone with a gauge transformation. We therefore introduce the magnetic
translation operators as the combined symmetry operation of an ordinary translation and a gauge transformation

\[ M_R = \exp[-i\frac{e}{\hbar}\chi_R(r)] \exp[i\frac{\hbar}{R} \cdot p]. \] (69)

The operator \( M_R \) is unitary, as it is the product of two unitary operators, and therefore has eigenvalues of the form \( e^{i\lambda} \). Let us denote the primitive vectors of the Bravais lattice \( a \) and \( b \). We can find common eigenstates of \( M_a, M_b \) and \( H \), if and only if they all commute with each other. The magnetic translations each commute with the Hamiltonian by construction, and furthermore we have

\[ M_a M_b = \exp[2\pi i \frac{\phi}{\phi_0}] M_b M_a. \] (70)

If the flux \( \phi \) through the unit cell is a rational number \( p/q \) (\( p \) and \( q \) relatively prime) times the flux quantum \( \phi_0 \), \( M_{qa} \) and \( M_b \) commute. In this case the cell spanned by \( qa \) and \( b \) is called the magnetic unit cell. Let us define \( c = qa \). The possible eigenvalues of \( M_c \) are phases \( e^{2\pi i k_1} \), where we can restrict \( |k_1| < \frac{1}{2} \), and equivalently for \( M_b \). We can therefore label the common eigenstates \( |k, n\rangle \), where \( k = k_1 c^* + k_2 b^* \), and \( c^*, b^* \) are the primitive vectors of the reciprocal lattice. The vector \( k \) is restricted to the magnetic Brillouin zone. An arbitrary magnetic translation of an eigenstate with a Bravais lattice vector \( R = nc + mb \) can now be written \( M_R |k, n\rangle = (M_c)^n (M_b)^m |k, n\rangle = \exp[i k \cdot R] |k, n\rangle \), showing that the eigenstate is a Bloch state. In this case we can speak of energy bands forming a band structure in the usual sense. When the flux through the elementary unit cell \((a, b)\) is an irrational number of flux quanta, the situation is different. The irrational number can be reached as the limit where \( p \) and \( q \) get very large, and consequently the Brillouin zone get very small and collapses in the limit.

2. The Dirac vortex viewpoint

In this section we will show, how it is possible to argue in a slightly different way from the previous section, and hereby in a simpler way obtain the vector potential of a periodic
magnetic field. Let us again assume a rectangular unit cell \((a, b)\), \(B(x+a, y) = B(x, y + b) = B(x, y)\) etc., to keep the notation simple. The magnetic field enter the Hamiltonian only through the vector potential. The question is therefore if one can choose a gauge such that the vector potential will be translationally invariant relative to a unit cell \((c, d)\) \(A(x+c, y) = A(x, y+d) = A(x, y)\) etc. It is clear that if such a periodic \(A\)-field exists, then the total flux \(\Phi_{cd}\) through the unit cell \((c, d)\) will be zero, as it is given by the line integral of \(A\) around the boundary of the unit cell \((c, d)\)

\[
\Phi_{cd} = \oint_{\partial(c,d)} A \cdot dl, \tag{71}
\]

which is zero by the periodicity. We remark that due to the relation \(B = \nabla \times A\), the cell \((c, d)\) will be bigger than or equal to \((a, b)\). If the flux through the unit cell of the magnetic field is not zero, but equal to a rational number times the flux quantum, \(\Phi_{ab} = p/q \phi_0\) \((p\) and \(q\) relatively prime), a trick can be applied to make the flux \(\Phi_{cd}\) become zero. It is a basic fact, apparently first observed by Dirac\(^17\), that a particle with charge \(e\) cannot feel an infinitely thin solenoid carrying a flux equal to an integer multiple of the flux quantum \(\phi_0 = h/e\). Such a stringlike object carrying one flux quantum is sometimes called a Dirac vortex. To find the periodic vector potential take an enlarged unit cell \((c, d) = (qa, b)\), so that \(\Phi_{cd} = p\phi_0\) and put by hand \(p\) counter Dirac vortices through the cell, to obtain zero net flux. Then a divergence free vector potential can be build for instance by Fourier transform

\[
B(Q) = \frac{1}{cd} \int_{(c,d)} d^2r \exp[-iQ \cdot r]B(r), \tag{72}
\]

\[
A(r) = \sum_{Q \neq 0} \begin{pmatrix}
  iQ_y \\
  -iQ_x
\end{pmatrix} \frac{\exp[iQ \cdot r]}{Q^2} B(Q), \tag{73}
\]

where the sum is over \(Q\) in the reciprocal lattice. Here we have used continuum notation, but it is straightforward to write down the lattice equivalents of the expressions.
C. Lattice calculation of Hall conductivity

We have calculated the Hall conductance of the 2DEG in the vortex field by a numerical lattice method. This we do because the calculations then reduces to linear algebra operations on finite size matrices, which can be implemented in a program on a computer. The idea is to consider an electron moving on a discrete lattice, rather than in continuum space. We know, although we are not going to prove it here, that in the limit where the discrete lattice becomes fine-grained compared to all other characteristic length of the system, the continuum theory is recovered. Here we assume that the original Bravais lattice has square lattice symmetry, with a lattice parameter which we call $a$. The discrete micro lattice is then introduced as a fine-grained square lattice inside the unit cell of the Bravais lattice. The lattice parameter of the micro lattice we take as $a/d$, where $d$ is some large number, in order to keep the two lattices commensurable. The condition that we have to impose on the micro lattice, in order that it is a good approximation to the continuum, can then be stated

\[ a/d \ll a, \lambda_F, \lambda_s, \cdots. \]  

(74)

In the numerical calculations we have made, we have taken $d = 10$.

The tight-binding calculations are made with the Hamiltonian

\[ H = - \sum_{ij\tau\tau'} t_{i+\tau,j+\tau'} c_j^{\dagger} c_{j+\tau'}. \]  

(75)

Here $i, j$ are Bravais lattice vectors, and $\tau, \tau'$ are vectors indicating the sites in the basis. The matrix elements $t_{i+\tau,j+\tau'}$ are taken non-zero only between nearest neighbour sites. The matrix element between two nearest neighbour sites $\tau$ and $\tau'$ are complex variables $t_{\tau,\tau'+e_\mu} = t e^{i A_\mu(\tau)}$ with a phase given by the vector potential $A_\mu(\tau)$ residing on the link joining the sites. The translation invariance of the Hamiltonian can then be stated $t_{i+\tau+l,j+\tau'+l} = t_{i+\tau,j+\tau'}$, for all vectors $l$ belonging to the Bravais lattice. Let us introduce the system on which our calculations were made as an example. Fig. 8 shows the unit cell with its internal structure i.e. the basis. There are $N = d \cdot d$ sites in the basis. The length of the links we write
as \(a/d\), where \(a\) is the side of the unit cell, with area \(\Omega = a^2\). The vectors \(\tau = (\tau_1, \tau_2)\frac{a}{d}\), \(\tau_1, \tau_2 = 0, 1, \ldots d - 1\) are offsets into the basis, while the vectors \(i = (i_1, i_2)a, i_1, i_2 \in \mathbb{Z}\) indicate the cells in the Bravais lattice. The operator \(c^\dagger_{i+\tau}\), for a given \(\tau\), is defined on the Bravais lattice, and accordingly it can be resolved as a Fourier integral over the Brillouin zone as

\[
c^\dagger_{j+\tau} = \frac{\Omega}{(2\pi)^2} \int_{BZ} d^2 q e^{-i q \cdot (j+\tau)} c^\dagger_q. \tag{76}
\]

Inserting this and using the translation invariance, the Hamiltonian can be rewritten as

\[
H = \frac{\Omega}{(2\pi)^2} \int_{BZ} d^2 k H_k, \tag{77}
\]

where we have introduced

\[
H_k = -\sum_{j, \tau, \tau'} t_{\tau, j+\tau'} e^{ik \cdot (j+\tau'-\tau)} c^\dagger_{k, \tau} c_{k, \tau'}. \tag{78}
\]

It is seen that \(H_k\) only mixes the \(N\) states \(|k\tau\rangle\), i.e. it is a \(N \times N\) matrix. The \(N\) eigenvalues of \(H_k\) are the energies of the \(N\) tight binding Bloch states with wave vector \(k\). Let us denote the eigenstates of \(H_k\) by \(u^\alpha_k\)

\[
H_k u^\alpha_k = E^\alpha_k u^\alpha_k \tag{79}
\]

where \(\alpha = 1, 2, \ldots N\) and \(E^\alpha_k \leq E^{\alpha+1}_k\). From the \(N\) dimensional vector \(u^\alpha_k\) we can construct the eigenstate \(\Psi^\alpha_k\) of the Hamiltonian \(H\)

\[
\langle j + \tau | \Psi^\alpha_k \rangle = e^{ik \cdot (j+\tau)} u^\alpha_k(\tau). \tag{80}
\]

It is straightforward to verify that this is the correct Bloch eigenstate of \(H\). The band structure can be calculated directly by diagonalizing the \(N \times N\) matrices, \(H_k\), for representative choices of \(k\) in the Brillouin zone. Before one can compare the spectrum obtained from this calculation with that of a continuum system, a scaling of the energies is required. To scale the energy to the spectrum of a particle with an effective mass \(m\), we have to take \(t = \hbar^2 d^2/ma^2\), and \(\epsilon^\alpha(k) = E^\alpha_k + 4t\).
The Hall conductivity can be calculated by the same method as in the homogeneous magnetic field\(^7\). We have used a single particle Kubo formula to calculate the Hall conductance

\[
\sigma_{xy} = \frac{i\hbar}{A_0} \sum_{E^\alpha < E_F < E^\beta} \frac{(J_x)^{\alpha\beta}(J_y)^{\beta\alpha} - (J_y)^{\alpha\beta}(J_x)^{\beta\alpha}}{(E^\alpha - E^\beta)^2},
\]

(81)

where \(J_x, J_y\) are the currents in the \(x, y\) directions, and the sum is over single particle states \(|\alpha, k\rangle\) with energies below and above the Fermi level \(E_F\). The area of the system is denoted \(A_0\). All quantities are diagonal in \(k\), and therefore this index is suppressed. The summation is composed of a discrete sum over bands, and an integral over the Brillouin zone for each band. The Brillouin zone shown in Fig. 9 is doubly connected because the states on the edges is to be identified according to the translation invariance. This gives the Brillouin zone the topology of a torus \(T^2\), with two basic non-contractible loops. The current operator can be written

\[
\mathbf{J} = \frac{\Omega}{(2\pi)^2} \int_{BZ} d^2 k \mathbf{J}_k,
\]

(82)

where \(\mathbf{J}_k = (e/\hbar)(\partial H_k/\partial k)\). By use of some simple manipulations and completeness, it is straightforward to rephrase Eq. 81

\[
\sigma_{xy} = \frac{ie^2}{\hbar A_0} \sum_{E^\alpha < E_F} \left( \left\langle \frac{\partial \alpha}{\partial k_x} \left| \frac{\partial \alpha}{\partial k_y} \right\rangle - \left\langle \frac{\partial \alpha}{\partial k_y} \frac{\partial \alpha}{\partial k_x} \right\rangle \right) \right),
\]

(83)

where \(|\beta, k\rangle\) is shorthand for \(\frac{\partial}{\partial k_\mu} |\beta, k\rangle\). This formula was first derived by Thouless, Kohmoto, Nightingale and den Nijs\(^7\), for a noninteracting two-dimensional electron gas in a periodic scalar potential, and a commensurate perpendicular magnetic field. It requires some comments to be meaningful. In order to calculate \(\frac{\partial}{\partial k_\mu} \) it is necessary to consider the difference \((|\alpha, k + \delta k_\mu\rangle - |\alpha, k\rangle)/\delta k_\mu\). But this difference is not well defined as it stands, as the phase of the states is arbitrary. Rather than representing the state \(u_k\) by a single vector in \(\mathbb{C}^N\), it should be represented by a class of vectors which differ one from another only by a phase. These equivalence classes are sometimes called rays. To compare states locally, we
need to project this $U(1)$ degree of freedom out. This is done by demanding the wave function to be real and positive, when evaluated in a fixed point, i.e. $u_k^\alpha(\tau_i) = \langle \tau_i|\alpha, k \rangle \in \mathbb{R}_+$. If the wave function happens to be zero in $\tau_i$, some other point $\tau_j$ must be used. When a band has a non-zero Hall conductivity, it is not possible to find a single $\tau$ which work for all the states in the Brillouin zone. The change from $\tau_i$ to $\tau_j$ which shifts the phase of the states, is analogous to a “gauge transformation” on the Brillouin zone, of the set of states. The special combination of terms which appear in Eq. 81 is gauge invariant with respect to these special “gauge transformations”. If we let $|\chi^\alpha\rangle$ denote a state which is obtained from $|\alpha, k\rangle$ by fixing the phase according to the above scheme, the following formula for the contribution to the Hall conductivity from a single band $\alpha$, is well defined

$$\sigma_{xy}^\alpha = \frac{e^2}{h} \frac{1}{2\pi i} \int_{BZ} d^2k \left\{ \left\langle \frac{\partial \chi^\alpha}{\partial k_y} \right| \frac{\partial \chi^\alpha}{\partial k_x} \right\} - \left\langle \frac{\partial \chi^\alpha}{\partial k_x} \right| \frac{\partial \chi^\alpha}{\partial k_y} \right\}. \quad (84)$$

It has been shown in detail by Kohmoto, for the homogeneous magnetic field case, that this expression is equal to minus $e^2/h$ times the first Chern number of a principal fiber bundle over the torus. As the first Chern number is always an integer, this has the physical consequence that whenever the Fermi energy lies in an energy gap, the Hall conductance is quantized. We will use this result to interpret certain peaks in the $\sigma_{xy}$-spectra we have calculated. When the Fermi level is not in an energy gap of the system, we will have to use Eq. 83 to calculate $\sigma_{xy}$. As we shall see, in this case there is no topological quantization of the Hall conductivity.

### D. Energy band crossing

In this section we study the effect on the Hall conductivity of an energy band crossing. This has previously been discussed in different contexts by several authors.

When the shape of the magnetic field is varied, controlled by some outer parameter $\xi$, it will happen for certain parameter values $\xi_0$, that two bands cross, see Fig. 10. This is the consequence of the Wigner-von Neumann theorem, which states that three parameters
are required in the Hamiltonian in order to produce a degeneracy not related to symmetry. Here the parameters are $k_x, k_y$ and the outer parameter $\xi$, which in our calculation is the exponential length of the flux vortices from the superconductor. When the energy difference $E^+ - E^-$ between the two bands considered is much smaller than the energy distance to the other bands, the Hamiltonian can be restricted to the subspace spanned by the two states $|+, k_0^0\rangle$ and $|-, k_0^0\rangle$. The point in the Brillouin zone where the degeneracy occur we denote $k^0$. The Hamiltonian $H(k)$ is diagonal for $k = k^0$, and we denote the diagonal elements respectively $E_0 + \epsilon$ and $E_0 - \epsilon$. For small deviations of $k$ from $k^0$ the lowest order corrections to the Hamiltonian is off-diagonal elements $\Delta(k)$ linear in $k - k^0$. Without essential loss of generality we can assume that $\epsilon$ is independent of $k$. Then $\epsilon$ plays the role of the outer parameter controlling the band crossing. The Hamiltonian is then approximated by

$$H(k) = \begin{pmatrix} \epsilon & \Delta^* \\ \Delta & -\epsilon \end{pmatrix} + E_0.$$  \hspace{1cm} (85)

The off-diagonal element is expanded as

$$\Delta(k) = \alpha(k_x - k_x^0) + \beta(k_y - k_y^0)$$  \hspace{1cm} (86)

where we have introduced the matrix elements $\alpha = \frac{\partial}{\partial k_x} \langle -, k_0^0 | H_k | +, k_0^0 \rangle$, and $\beta = \frac{\partial}{\partial k_y} \langle -, k_0^0 | H_k | +, k_0^0 \rangle$.

We want to find the consequences of the energy band degeneracy, on the topological Hall quantum numbers of the bands. Let us define

$$B_{\pm}(k) = \left\{ \left\langle \frac{\partial \pm}{\partial k_y} \left| \frac{\partial \pm}{\partial k_x} \right\rangle - \left\langle \frac{\partial \pm}{\partial k_x} \left| \frac{\partial \pm}{\partial k_y} \right\rangle \right\} \right\}.$$  \hspace{1cm} (87)

Then the interesting quantities are the integrals of $B_+(k)$ and $B_-(k)$, around a small neighbourhood of the degeneracy point $k^0$. It turns out, that it is the two numbers $\alpha$ and $\beta$ that control what happens.

In general $\alpha$ and $\beta$ will be nonzero complex numbers — nonzero because we have assumed the degeneracy to be of first order. Let us first consider the degenerate case where $\alpha$ and $\beta$
are linearly dependent, i.e. $\alpha/\beta$ is real, or otherwise stated $\text{Im}(\alpha^*\beta) = 0$. Then by a linear transformation we can write the Hamiltonian

$$H(\kappa) = (\kappa_1 + \kappa_2)\sigma^1 + \epsilon\sigma^3 = \begin{pmatrix} \epsilon & \kappa_1 + \kappa_2 \\ \kappa_1 + \kappa_2 & -\epsilon \end{pmatrix},$$

(88)

where $\kappa_1, \kappa_2$ are rescaled momentum variables, with dimension of energy. The $\sigma^\mu$'s refers to the Pauli matrices. The eigenvectors of this Hamiltonian can be chosen real, and this will clearly make $B_\pm(\kappa)$ vanish. In this case we therefore conclude that there is no exchange of topological charge. Here the word topological charge is used to denote Hall quanta.

Let us now treat the general case where $\alpha$ and $\beta$ are linearly independent, i.e. $\text{Im}(\alpha^*\beta) \neq 0$. In this case we can, by a linear transformation, write $\Delta(\kappa) = \kappa_1 + i\kappa_2 = \kappa e^{i\theta}$, which defines the scaled variables $\kappa, \theta$. This reduces the Hamiltonian to the form

$$H(\kappa, \theta) = \kappa_1\sigma^1 + \kappa_2\sigma^2 + \epsilon\sigma^3 = \begin{pmatrix} \epsilon & \kappa e^{-i\theta} \\ \kappa e^{i\theta} & -\epsilon \end{pmatrix}. $$

(89)

Let us define $\lambda = \sqrt{\epsilon^2 + \kappa^2}$. Then the eigenvalues of $H(\kappa, \theta)$ are $\pm \lambda$ and the two corresponding eigenstates are

$$|\pm, \kappa\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm \sqrt{1 \mp \epsilon/\lambda} \\ e^{i\theta} \sqrt{1 \mp \epsilon/\lambda} \end{pmatrix}. $$

(90)

In order to calculate the integrals of the $B_\pm$-functions we need to express them in terms of the $\kappa, \theta$-variables. The Jacobian of the transformation is given by the expression

$$dk_x dk_y = \frac{\kappa dkd\theta}{|\alpha_r \beta_i - \alpha_i \beta_r|},$$

(91)

and

$$B_\pm(\kappa, \theta) = \frac{\alpha_r \beta_i - \alpha_i \beta_r}{\kappa} \left\{ \left\langle \frac{\partial \pm}{\partial \theta} \left| \frac{\partial \pm}{\partial \kappa} \right. \right\rangle - \left\langle \frac{\partial \pm}{\partial \kappa} \left| \frac{\partial \pm}{\partial \theta} \right. \right\rangle \right\}$$

$$= \pm (\alpha_r \beta_i - \alpha_i \beta_r) \frac{i\epsilon}{2\lambda^3},$$

(92)

where the indices $r, i$ refer to the real and imaginary parts respectively. We can now calculate the contribution to the Hall conductivity from each of the bands, from the area around $k^0$. 
given by $|\kappa| < \kappa_c$, where $\kappa_c$ is some local cutoff parameter which limit the integration to the area where the approximation leading to the Hamiltonian Eq. 85 is valid

$$\Delta \sigma_{xy}^{\pm} = \frac{e^2}{h} \frac{1}{2\pi i} \int dk_x \int dk_y B_{\pm}(k)$$

$$= \pm \frac{e^2}{h} \frac{1}{2\pi i} \text{sign}[\text{Im}(\alpha^* \beta)] \int_0^{\kappa_c} d\kappa \int_0^{2\pi} d\theta \frac{i\kappa \epsilon}{2\lambda^3}$$

$$= \pm \frac{e^2}{2h} \text{sign}[\alpha^* \beta] \frac{\epsilon}{|\epsilon|} \int_0^{\kappa_c/|\epsilon|} \frac{u \, du}{(1 + u^2)^{3/2}}$$

$$= \pm \frac{e^2}{2h} \text{sign}[\text{Im}(\alpha^* \beta)] \text{sign}[\epsilon], \quad (93)$$

where the last equality sign is valid when $\kappa_c/|\epsilon| \gg 1$, i.e. close to the crossing where $\epsilon = 0$. Here the factor $\text{sign}[\epsilon]$ signals that the two bands exchange exactly one topological conductivity quantum $e^2/h$, at the crossing. This is not surprising, because we know that the total contribution from the states in a single band is always an integer times the quantum $e^2/h$. Moreover it is readily seen that in the hypothetical situation of a $n'$th order degeneracy, i.e. one for which $\Delta = \kappa^n e^{i n \theta}$, $n$ quanta are exchanged. The total topological charge in the band structure is conserved. The topological charge can flow around and rearrange itself inside a band, but only be exchanged between bands in lumps equal to an integer multiple of the conductivity quantum $e^2/h$. When we gradually shrink the radius of the flux vortices to zero, the Hall effect has to disappear. There are two mechanisms with which the Hall effect can be eliminated. The first is by moving the topological charge up through the band structure by exchanging quanta, resulting in a net upward current of topological charge, eventually moving the charge up above the Fermi surface, where it has no effect. The second mechanism is by rearranging the topological charge inside the bands, so that each band has a large negative charge in the bottom, and a large positive charge in the top, but arranged in such a clever way that charge neutrality is more or less retained for all energies. This second mechanism will also give a net displacement of topological charge up above the Fermi energy, because in general the Fermi surface cuts a great many bands, and for all these bands the large negative charge, which they have in their bottom part, will be uncompensated. To use the language of electricity theory, we can say that every band...
gets extremely polarized, resulting in a net upward displacement current, in analogy with the situation in a strongly polarized dielectric. Our numerical calculations indicate, that it is the second mechanism which is responsible for the elimination of the Hall effect, as the radius of the vortices shrinks to zero.

When two bands are nearly degenerate for some \( k^0 \), each of the bands have concentrated half a quantum in a small area in k-space around \( k^0 \), and in general the topological charge piles up across local and global gaps in the energy spectrum. This is the reason for the oscillatory and spiky behavior of the Hall conductance as a function of electron density, that is seen on the calculated spectra below. It is also the reason why the numerical integration involved in the actual evaluation of the Hall conductivity, is more tricky than one could wish.

E. Numerical results

1. Transverse Conductivity

We have calculated the transverse conductivity \( \sigma_{xy} \) as a function of the integrated density of states, for electrons in a square lattice of flux vortices, for a series of varying cross sectional shapes of the flux vortices. Each of the field configurations consists of a square lattice of flux vortices with a given exponential length \( \lambda_s \). The parameter which vary from calculation to calculation, is the dimensionless ratio \( \xi = \lambda_s/a \), where \( a \) is the length of the edges of the quadratic unit cell. The unit cell is shown on Fig. 8 and contain, as we have already discussed, two vortices and a counter Dirac vortex. In order to do the tight binding calculation a micro lattice is introduced in the unit cell. In all the numerical calculations we present, the micro lattice is \( 10 \times 10 \). This gives 100 energy bands distributed symmetrically about the center on an energy scale. Out of these only the lower part, say band 1 to 20, approximate the real energy bands well, while the rest is significantly affected by the finite size of the micro lattice. A careful examination of the vector potential reveal that the symmetry of the Hamiltonian
is very high for the particular choice of unit cell shown in Fig. 8. The field from a single vortex we have taken as \( B_0 e^{-((|\tau_x|+|\tau_y|)/\lambda s)} \) instead of the more realistic \( B_0 e^{-|\tau|/\lambda s} \). With this choice the vector potential can be written down analytically in closed form. This makes the calculations simpler, and does not break any symmetry that is not already broken by the introduction of the micro lattice. (We have made calculations of the band structure, with both kinds of flux vortices, and the differences are indeed very small). The energy spectrum is invariant under the changes \((k_x, k_y) \mapsto (\pm k_x, \pm k_y), (\pm k_y, \pm k_x)\). This fact is exploited to present the band structures in an economic way. The labels \( \Gamma, X \) and \( M \) correspond to the indicated points in the Brillouin zone, Fig. 9.

In Fig. 11 we have plotted a selection of typical band structures which illustrates the crossover from the completely flat Landau bands in the homogeneous magnetic field \( \xi = \infty \), to the band structure of electrons in a square lattice of Aharonov-Bohm scatterers with \( \alpha = 1/2 \) at \( \xi = 0 \). The band structures have been found by direct numerical diagonalization of the Hamiltonian. (See also Fig. 14).

In Fig. 12 some typical results of the numerical calculations of the Hall conductivity are shown. In general we have no reason to expect, that the Hall conductivity should be isotropic as a function of the angle between the current and the flux vortex lattice. The results we present is for a current running along the diagonal of the square lattice, i.e. along the \( x \)-axis in Fig. 8. It is seen, that whenever there is a gap in the spectrum, the Hall conductivity gives the quantized value in agreement with the discussion in the last section. At Fermi energies not lying in a gap \( \sigma_{xy} \) always tend to be lower than the value it has in the homogeneous field. And in the limit \( \xi \mapsto 0 \), \( \sigma_{xy} \) vanishes altogether. In this limit the electron sees a periodic array of Aharonov-Bohm scatterers, each carrying half a flux quantum, and there is no preferential scattering to either side. We observe that when the flat Landau bands starts to get dispersion, the contribution to the Hall effect is no longer distributed equally in the Brillouin zone. Instead it piles up across local and global gaps in the spectrum resulting in the spiky \( \sigma_{xy} \) spectra. The density of topological charge is plotted as a function of the filling fraction in Fig. 13. It is seen that for \( \xi \to 0 \) the distribution gets
strongly polarized, with a negative contribution to the Hall effect at the bottom part of the bands, and a positive contribution at the uppermost part of the bands.

In Fig. 12, the filling fraction is limited to values below 10, but the same behavior continues for larger values of the filling fraction. We have observed the spiky behavior up to $\nu = 30$, and have no reason to believe that it should not continue to larger values. The limitations on our calculations, comes from the fact, that we are only able to handle matrices of limited size in the numerical calculations. Filling fractions of order 30 correspond in this context, to very low density electron gases, and consequently our calculations belong to the “quantum” regime, i.e. to the regime where $\lambda_F \gg \lambda_s$. According to the discussion in Sec. I, we expect a cross-over to a semiclassical regime, for electron gases with higher density, where $\lambda_F \ll \lambda_s$, with a qualitatively different behavior.

It is an important question whether it is possible to observe these features of the Hall conductivity in experiments. The conditions, which are necessary, are that the mean free path $l$ is long compared to all other lengths, and that $a, \lambda_s < \lambda_F$. The last condition indicates that we are concerned with quantum magneto transport, in the sense that the vector potential is included in the proper quantum treatment of the electrons. This is in contrast to the case $\lambda_F \ll a, \lambda_s$ where the electron transport can be treated as the semiclassical motion of localized wave packets in a slowly varying magnetic field. Let us assume, in order to make some estimates, that the superconductor has a London length somewhat less than 100nm, resulting in an exponential length of the vortices $\lambda_s$ of about 100nm at the 2DEG, after the broadening due to the distance between the superconductor and the 2DEG has been taken into account. In order to have a variation in the magnetic field we should have $a > \lambda_s$, and to be in the quantum regime $\lambda_F > a, \lambda_s$. This gives the estimate for the electron density $n_e \sim 10^{10}$ cm$^{-2}$, which is not unrealistic. The effect of the impurities is (to first order), to give the electrons a finite lifetime. This gives a finite longitudinal conductivity $\sigma_{xx}$ and broadens the density of states. It also introduces localized states at the band edges (Lifshitz tails). If the field is nearly homogeneous, we have the standard quantum Hall picture with mobility edges above and below every Landau band,
resulting in the formation of plateaus in $\sigma_{xy}$, which only can be observed at much higher magnetic fields, of order $10^5$ G, where the filling fraction is of order one. On the other hand when the amount of impurities is low, that is $k_F l \gg 1$, all these effects will be small, and we expect that the features of $\sigma_{xy}$, shown in Fig. [2] will have observable consequences in

$$\rho_{xy} = \frac{\sigma_{xy}}{(\sigma_{xx}^2 + \sigma_{xy}^2)}.$$ 

In Fig. [4] we have plotted the band structure of electrons in a square lattice of vortices carrying one flux quantum each. The calculation has been made with the same basis as the band structures shown in Fig. [1], the only difference is that all fluxes have been multiplied by a factor of 2, as can be seen from the spacing between the Landau bands. In the limit where $\xi \to 0$, and the vortex lattice becomes a regular array of Aharonov-Bohm scatterers, we recover the well known band structure of free electrons. This is in agreement with our discussion of the AB-vortex in Sec. [1].

2. Exchange of topological quanta

An example of exchange of topological quanta between neighbouring bands is shown in Fig. [13]. The figure is an enlargement of the band structure around the $X$ point, showing an accidental degeneracy between the 3’ed and 4’th band, which occur about $\xi = 0.035$. Also indicated is the Hall conductance of each band in units of $e^2/h$, found by numerical integration. At the degeneracy it is not possible to define the Hall conductance for the individual bands. On the Brillouin zone torus there are two $X$ points $X_1$, $X_2$, and the bands have a first order degeneracy in each. The numerical integration shows that two topological quanta are transfered from the lower to the upper band, and this is in full agreement with the discussion of Sec. [III]. Exchange of topological quanta between bands is a common phenomena as $\xi$ is varyed, and this particular example has only been chosen as an illustration of the general phenomena.
When the vortices come from a thin film of superconducting material, which have many impurities and crystal lattice defects acting as pinning centers for the vortices, the distribution of vortices will be disordered rather than forming a regular Abrikosov lattice. We expect the effect of the disorder to be, to wash out the distinctive features of the band structure, i.e. to average out the characteristic fluctuations in \(\sigma_{xy}\), leaving a smooth curve in Fig. 12 with a characteristic dimensionless proportionality constant \(s(\xi)\), in the form \(\sigma_{xy} = (e^2/h)s(\xi)\nu\), where \(\nu\) is the number of electrons in the magnetic unit cell \(\nu = n_e a^2 = n_e \phi_0 / B\) (the filling fraction). With this conjecture we can estimate the normalized Hall conductivity \(s(\xi)\) by making a linear fit to the calculated \(\sigma_{xy}(\nu)\) distribution. In the experimental situation \(\lambda_s\) is constant, and this makes \(s(\xi)\) a function of the applied magnetic field through the relationship \(\xi = \lambda_s / a = \lambda_s \sqrt{B/\phi_0}\). In Fig. 16 we have plotted \(s(B)\) for a vortex exponential length \(\lambda_s = 80\text{nm}\). The \(s(B)\)-curve shows essentially that the Hall effect of a dilute distribution of vortices is strongly suppressed compared to the Hall effect of a homogeneous magnetic field with the same average strength. This is in good qualitative agreement with what is seen in the experiments of Geim et al.\(^2\). When doing experiments, one is not directly measuring the conductivities, but rather the resistivities \(\rho_{xx}, \rho_{xy}\). The experiments of Geim et al. cover the parameter range from \(\lambda_F \ll \lambda_s\) at high 2DEG densities, down to the value \(\lambda_F / \lambda_s = 0.7\) for the 2DEG with the lowest density experimentally obtainable, where the new phenomena begin to occur. Our numerical calculations belongs to the other side of this cross-over where \(\lambda_F \gg \lambda_s\). The physical picture of this cross-over can be stated as follows. On the high density side the magnetic field varies slowly over the size of an electron wave packet for electrons at the Fermi energy, with the result that the wave packet more or less behaves as a classical particle. On the other side of the crossover \(\lambda_F \gg \lambda_s\) the magnetic field varies rapidly over the length-scale of a wave packet, for an electron at the Fermi energy, and this introduces new phenomena of an essential quantum character.
F. Superlattice potential

The general picture we have outlined so far of energy bands having dispersion, with the dispersion giving rise to a non trivial behavior of the Hall conductivity, is not limited to the inhomogeneous magnetic field. The dispersion could have another origin for instance a superlattice potential\(^2\). To illustrate this a series of calculations have been made on a 2DEG in a homogeneous magnetic field, and a scalar potential which we have taken as a square lattice cosine potential. This system have commensurability problems because of the two “interfering” length scales, given respectively by the magnetic length \(l_B = \sqrt{\hbar/eB}\), and the period of the superlattice potential \(a\). To make things as simple as possible we have fixed the period of the cosine potential \(a\), and the magnetic field strength \(B\), and only varied the amplitude of the cosine potential. Furthermore the flux density of the magnetic field is tuned so that the flux through one unit cell of the cosine potential is exactly one flux quantum. The cosine potential is

\[
U(x, y) = V_0(\cos 2\pi x/a + \cos 2\pi y/a),
\]

and the magnetic field \(B = \phi_0/a^2\). The dimensionless parameter controlling the shape of the energy band structure is in this case

\[
v = \frac{V_0}{\hbar \omega_c}.
\]

Examples of the \(\sigma_{H}\)-spectra are shown in Fig. [17]. It is observed that although the spectra look different from the vortex lattice spectra, they have the same spiky nature. The spikes have the same interpretation as in the vortex lattice system. Local spikes are due to local gaps in the spectra. That is when to bands are close to each other for some \(k\) vector in the Brillouin zone, the result is a pile up of topological charge across the gap, and this gives a spike in the \(\sigma_{H}\)-spectra when the Fermi energy is swept across the gap. Global spikes, that is, spikes which go all the way up to the diagonal line indicating the Landau limit, are due to global gaps in the energy spectrum, combined with the topological quantization.
IV. CONCLUSION

In Sec. II of this paper we have considered the longitudinal and the transverse resistivities of a 2DEG in a disordered distribution of flux vortices, within the theoretical framework where each scattering event is treated independently, and the electrons are non-interacting. The general features observed in experiments are in agreement with the results we have outlined, but we do not have perfect quantitative agreement. The radius of the vortices, is estimated by Geim to be $r_0 \simeq 100\text{nm}$, while we find the best fit between the calculated and the measured Hall resistivity for $r_0 \simeq 50\text{nm}$.

In Sec. III we have considered a new kind of experiment where a 2DEG is placed in a periodic magnetic field varying on a length scale $\lambda_s$, comparable to (or less than) the Fermi wavelength $\lambda_F$ of the electrons. In this limit, where it is necessary to include explicitly the vector potential in a quantum treatment of the electron motion, we expect the 2DEG to exhibit new phenomena. We have presented numerical results for a non-interacting 2DEG without impurities showing characteristic spikes of the Hall conductivity versus filling fraction, which can be understood in terms of local and global energy gaps in the spectrum.

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APPENDIX A: SOLUTION OF THE BOLTZMANN EQUATION

The Boltzmann equation, linearized in the external electric field, reads

$$-e\mathbf{v} \cdot \mathbf{E} \frac{\partial f^0}{\partial \epsilon} = n_\alpha \int \frac{d^2q}{(2\pi)^2} \left\{ f_{p+q} w_{p+q \rightarrow p} - f_p w_{p \rightarrow p+q} \right\} - \frac{f_p - f^0}{\tau_{\text{imp}}}.$$  (A1)
Here $n_\alpha$ is the density of vortices, and $w_{k \rightarrow k'}$ is the scattering probability for scattering on a single vortex. The transition probabilities $w_{p \rightarrow p+q}$ are potentially asymmetrical quantities, due to the time reversal breaking magnetic field in the vortices. As argued by B. I. Sturman, the correct form of the collision integral, even in the absence of detailed balance, is the one given in Eq. [11]. The electron-vortex scattering will be elastic. In order to solve the Boltzmann equation we Fourier transform, and write

$$f(k, \theta) = \sum_{n=-\infty}^{\infty} e^{-i n \theta} f_n(k)$$ \hspace{1cm} (A2)

$$w(k, \theta) = \sum_{n=-\infty}^{\infty} e^{-i n \theta} w_n(k),$$ \hspace{1cm} (A3)

where $\theta$ is the angle between $k$ and $E$. The electron-vortex collision integral is diagonal when Fourier transformed, and we get the following equation for the $n$'th component of the distribution function

$$-evE \frac{\partial f_0}{\partial \epsilon} \frac{1}{2} (\delta_{n,1} + \delta_{n,-1}) = -n_\alpha \{w_0 - w_n\} f_n - \frac{f_n - f_0 \delta_{n,0}}{\tau_{imp}}.$$ \hspace{1cm} (A4)

The current is given by

$$j = -2e \int \frac{d^2k}{(2\pi)^2} v f(k, \theta) = -e^2 \varepsilon_F E \left\{ \frac{\text{Re}\left[\frac{1}{w_1 - w_0}\right]}{\frac{\pi \hbar^2 n_\alpha}{\alpha}} \right\},$$ \hspace{1cm} (A5)

from which the conductivities can be read off. The resistivities are found by inverting the conductivity tensor

$$\rho_{xy} = \rho_H \frac{k_F}{\alpha} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \sin \theta w(\theta),$$ \hspace{1cm} (A6)

$$\rho_{xx} = \rho_H \frac{k_F}{\alpha} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (1 + \cos \theta) w(\theta).$$ \hspace{1cm} (A7)

For a vortex with flux $\alpha \phi_0$ we have $k_F/\alpha = 2\gamma/r_0$, with $\gamma = l_c/r_0$ and the cyclotron radius $l_c = v/\omega_c$. 

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APPENDIX B: INTEGRAL REPRESENTATION OF THE AHARONOV-BOHM WAVE FUNCTION, AND SCATTERING CROSS SECTION

In the standard geometry (Fig. 4) the Aharonov-Bohm wave function\(^{13}\) is

\[\psi = \sum_m (-i)^{m-\alpha} J_{m-\alpha}(kr)e^{im\theta}. \quad (B1)\]

By introducing the integral representation of the Bessel functions, the sum can be effectuated with the result

\[\psi = \int_{\gamma} d\tau \frac{2}{2\pi} e^{-ikr \cos \tau} \left[ \frac{e^{i\alpha \tau}}{1 - e^{i(\tau - \theta)}} - \frac{e^{-i\alpha \tau}}{1 - e^{-i(\tau + \theta)}} \right], \quad (B2)\]

where the contour \(\gamma\) can be deformed into the contour shown in Fig. 18. The integral can now be separated naturally into two terms. The integral along the real axis consists of a principal value integral which vanish because the integrand is odd, and a contribution from the two poles

\[e^{-ikr \cos \theta + i\alpha \theta} \equiv \psi^i, \quad (B3)\]

which is nothing but the incoming part of the wave function. We therefore immediately have the following expression for the scattered part of the wave function

\[\psi^s \equiv -2 \sin \pi \alpha \int_0^{\infty} dt \frac{e^{ikr \cosh t} e^{i\theta}}{2\pi} \frac{\cosh \alpha t + \cosh (1 - \alpha) t}{\cos \theta + \cosh t}. \quad (B4)\]

At this level the splitting \(\psi = \psi^i + \psi^s\) is exact. Although it may not be apparent, the cut in \(\psi^i\) for \(\theta = \pm \pi\) is accompanied by a counter cut it \(\psi^s\), as is necessary in order for \(\psi\) to be smooth. In the “wave zone” \(kr \gg 1\) it is possible to simplify \(\psi^s\) by expanding the integral asymptotically

\[\psi^s = -\sin \pi \alpha e^{-ikr \cos \theta + i\theta/2} \frac{G(\sqrt{2kr \cos \theta/2})}{G(0)}, \quad (B5)\]

where \(G\), which is related to the error function, is given by

\[G(x) = \int_x^{\infty} ds e^{is^2}. \quad (B6)\]
This expression can be further simplified when \( \sqrt{2kr \cos \theta/2} \gg 1 \), i.e., asymptotically in all but the forward direction

\[
\psi^s = -\sin \pi \alpha \frac{e^{ikr+i\theta/2+i\pi/4}}{\sqrt{2\pi kr \cos \theta/2}} \equiv -F(\theta)\frac{e^{ikr+i\pi/4}}{\sqrt{r}},
\]

leading to the scattering cross section

\[
|F(\theta)|^2 = \frac{1}{2\pi k \cos^2 \theta/2} \sin^2 \pi \alpha \cos^2 \theta/2.
\]

Although this differential cross section is singular in the forward direction, it is completely symmetric and can therefore not give rise to a Hall effect if substituted into the Boltzmann transport equation, Eqns. \([A6]-[A7]\).

**APPENDIX C: INTERFERENCE BETWEEN THE INCOMING AND SCATTERED PARTS OF THE WAVE FUNCTION**

In this appendix we want to argue that it is not the ordinary scattering cross section which should be used in the Boltzmann equation, but rather a redefined “cross section” which take into account the interference between the scattered and incoming parts of the electron wave function.

The particle density current, of the total wave function \( \psi \)

\[
j = \text{Re}[\psi^* \mathbf{v} \psi]
\]

can naturally be separated into the terms

\[
j = \underbrace{\text{Re}[\psi^*_i \mathbf{v} \psi_i]}_{j^i} + \underbrace{\text{Re}[\psi^*_i \mathbf{v} \psi_s + \psi^*_s \mathbf{v} \psi_i]}_{j^u} + \underbrace{\text{Re}[\psi^*_s \mathbf{v} \psi_s]}_{j^s}.
\]

In terms of the density current the ordinary scattering cross section is

\[
\frac{d\sigma}{d\theta} = \lim_{r \to \infty} \frac{r j^s(r, \theta)}{j_0},
\]

where the index \( r \) refers to the radial component, and \( j_0 \) is the incoming current. As is well known the scattering cross section satisfies the optical theorem.
\[ \sigma = 2\sqrt{\frac{2\pi}{k}} \text{Im}[F(\pi)], \quad (C4) \]

where \( F(\theta) \) is defined as in Eq. \[B7\]. The optical theorem follows directly from particle number conservation \( \nabla \cdot \mathbf{j} + \partial \rho / \partial t = 0 \). Inside a conductor there are no collimators, or any other devises to separate \( \psi^s \) from \( \psi^i \) at large distances from the scatterer, and consequently the physically correct form of the scattering cross section in this case is

\[ \frac{d\tilde{\sigma}}{d\theta} = \lim_{r \to \infty} \frac{r(j_r - j_r^i)}{j_0} = \lim_{r \to \infty} \frac{r(j_r^s + j_r^i)}{j_0}. \quad (C5) \]

The equivalent of the optical theorem for this modified cross section is

\[ \tilde{\sigma} = 0. \quad (C6) \]

The interference contribution \( r j_r^i / j_0 \) is only different from zero in the forward direction \( \theta = \pm \pi \), because this is the only direction with constructive interference. For all Hamiltonians with the mirror symmetry \( \theta \to -\theta \), the difference between \( \sigma \) and \( \tilde{\sigma} \) will be proportional to \( \delta(\theta - \pi) \), and will never show up in the transport coefficients \( \rho_{xx} \) and \( \rho_{xy} \), because of the geometrical factors:

\[ \rho_{xx} \propto \int d\theta (1 + \cos \theta) \frac{d\tilde{\sigma}}{d\theta}, \quad (C7) \]

\[ \rho_{xy} \propto \int d\theta \sin \theta \frac{d\tilde{\sigma}}{d\theta} = 0, \quad (C8) \]

where \( 1 + \cos \theta \sim (\pi - \theta)^2 / 2 \) for \( \theta \sim \pi \), thus regularizing the integral. But if the Hamiltonian is asymmetric in \( \theta \), the interference term can have a contribution which is proportional to the derivative of a delta function \( \delta'(\theta - \pi) \), as we will now show. We do this by directly calculating the quantity

\[ \lim_{r \to \infty} \frac{k}{2\pi \alpha} \frac{r}{j_0} \int_{-\pi}^{\pi} d\theta \sin \theta j_r^i(r, \theta) \equiv \Delta^{AB}(\alpha). \quad (C9) \]

After some algebra, and dropping terms which integrate to zero, we have

\[ \Delta^{AB}(\alpha) = \frac{kr \sin \pi \alpha}{\pi \alpha} \int_{-\pi}^{\pi} d\theta \sin \theta \cos \theta \sin(1/2 - \alpha) \theta \left[ S(\sqrt{2kr} \cos \theta/2) - C(\sqrt{2kr} \cos \theta/2) \right], \quad (C10) \]
Where $S, C$ are the Fresnel sine and cosine integrals\cite{footnote}. For $ kr \to \infty$ we can evaluate the integral asymptotically
\[
\Delta^{AB}(\alpha) = 4 \frac{\sin \pi \alpha \cos \pi \alpha}{\pi \alpha} \int_0^\infty dx \ x [C(x) - S(x)] = \sin \frac{2\pi \alpha}{2\pi \alpha}.
\] (C11)

This is exactly the residual value of $\zeta_{xy}$ in the Aharonov-Bohm limit — due to the Iordanskii force. We conclude that, in this particular example, using the Boltzmann transport equation with the ordinary (quantum mechanical) scattering cross section, leads to the erroneous conclusion of a vanishing Hall resistance. In order to obtain the correct result it is necessary to redefine the cross section to take into account the interference between the scattered and incoming parts of the electronic wave function.
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FIGURES

FIG. 1. The scattering geometry for classical scattering on an idealized cylindrical vortex with constant magnetic field inside and no field outside. Here the radius of the vortex has been taken as the unit of length.

FIG. 2. Differential cross section (divided by $r_0$), and classical trajectories for four different values of the parameter $\gamma = l_c/r_0 = 0.025, 0.4, 1.0, 2.5$.

FIG. 3. The Hall efficiency factor $\zeta_{xy}(\gamma)$, and the resistance efficiency factor $\zeta_{xx}(\gamma)$ calculated from the classical cross section for scattering on a magnetic flux tube. The parameter $\gamma$ is given by the cyclotron radius divided by the radius of the flux tube $\gamma = l_c/r_0$.

FIG. 4. The geometry of the scattering situation.

FIG. 5. The efficiency of a dilute distribution of vortices in producing Hall effect, compared to a homogeneous magnetic field with the same average flux density. Curves show $\zeta_{xy}$ for $\alpha = 1/4, 1/2, 3/4, 1$.

FIG. 6. Resistance efficiency of single vortex. Curves show $2\pi \alpha \zeta_{xx}$, for $\alpha = 1/4, 1/2, 3/4, 1$.

FIG. 7. These plots of $\zeta_{xy}$ and $2\pi \alpha \zeta_{xx}$ for a vortex with $\alpha = 10$, shows a striking structure of resonances at the values of $kr_0$ corresponding to the Landau quantization energies.

FIG. 8. (A) The unit cell with basis. The large circles indicates the position of the Abrikosov vortices, and the small circle indicate the position of the Dirac vortex with the counter flux. The micro lattice shown here is $6 \times 6$, whereas all the numerical results we have presented are obtained with a micro lattice of $10 \times 10$ sites. (B) Four concatenated unit cells, showing the square lattice of Abrikosov vortices.

FIG. 9. The Brillouin zone with symmetry labels.
FIG. 10. Schematic energy band crossing, controlled by an outer parameter $\gamma \propto \xi - \xi_0$.

FIG. 11. Band structures for 2D electrons in a square lattice of Abrikosov vortices with $\alpha = 1/2$. The different plots show band structures corresponding to various values of the parameter $\xi$ equal to the ratio between the exponential length of the magnetic field from a single vortex, and the lattice parameter.

FIG. 12. Calculated Hall conductivity versus filling fraction, for various values of the ratio $\xi = \lambda_s/a$. These calculations are made on the same system as the band structures shown in Fig. 11, that is, a square lattice of Abrikosov vortices with $\alpha = 1/2$. Each of the spectra are made by calculating the total Hall conductivity $\sigma_H(\epsilon_F)$ and the integrated density of states $\nu(\epsilon_F)$, for 2000 equidistant values of the Fermi energy $\epsilon_F$. The $x$-axis indicates the integrated density of states in units of filled bands. The $y$-axis indicates the total Hall conductivity in units of the conductivity quantum $e^2/h$. The diagonal line in the plots indicate the Hall conductivity in a homogeneous magnetic field.

FIG. 13. The density of Hall effect, or “topological charge”, plotted as function of the filling fraction, for $\alpha = 1/2$.

FIG. 14. Band structures for 2D electrons in a square lattice of vortices carrying one flux quantum each, $\alpha = 1$. The different plots shows band structures corresponding to various values of the parameter $\xi$ equal to the ratio between the exponential length of the magnetic field from a single vortex, and the lattice parameter. The band structures have been calculated using the basis shown in Fig. 8, with the only difference that here the flux through each of the vortices are $\phi_0$, and the counter Dirac flux is $-2\phi_0$.

FIG. 15. Exchange of topological quanta. The figure shows an enlargement of the $\alpha = 1/2$ band structure around the $X$ point, where a degeneracy between the 3’rd and 4’th band occur. (See Fig. 11). The parameter $\gamma$ appearing in the figure is defined as $\gamma = \xi - \xi_0$, with $\xi_0 = 0.035$. The numbers give the Hall conductance of the bands in units of $e^2/h$, found by numerical integration.
FIG. 16. The normalized Hall conductivity $s(B)$ for vortices of exponential length $\lambda_s = 80$ nm, and $\alpha = 1/2$.

FIG. 17. Calculated Hall conductivity versus filling fraction, for a 2DEG in a homogeneous magnetic field, and a square lattice cosine potential, in the special case where the magnetic flux density is exactly equal to one flux quantum per unit cell area. The dimensionless parameter $v$ indicated in the plots, is equal to the amplitude of the cosine potential divided by the Landau energy $\hbar \omega_c$. The $x$-axis indicates the integrated density of states in units of filled bands. The $y$-axis indicates the total Hall conductivity in units of the conductivity quantum $e^2/h$. The diagonal lines in the plots indicate the Hall conductivity in a homogeneous magnetic field, without any potential.

FIG. 18. The contour of integration.