The minimal degree of permutation representations of finite groups
Amirim Honors Program final project

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1 Introduction

In this thesis we study the following property, $\mu(G)$, of a finite group $G$:

Definition 1. $\mu(G) = \min\{n \mid G \text{ embeds in } S_n\}$.

By Cayley’s theorem, $\mu(G) \leq |G|$. We start, after the introduction, with an explicit formula for $\mu(G)$ when $G$ is abelian. This formula and its proof first appeared in [1]. We give a different proof. The formula shows that for abelian groups $G$ and $H$,

$$\mu(G \times H) = \mu(G) + \mu(H)$$

(1)

The equality (1) was established in [2] for nilpotent groups (and even more: for groups $G$ which contain a nilpotent subgroup $G_0$ such that $\mu(G) = \mu(G_0)$). We extend it in Section 5 to the class $CS$ of groups for which the socle is central (and even more: for groups $G$ which contain a subgroup $G_0$ which belongs to $CS$ such that $\mu(G) = \mu(G_0)$). We also study when $\mu(G) = |G|$ and begin to explore the compression ratio $cr(G) = \frac{|G|}{\mu(G)}$. In [1], it was determined when $cr(G) = 1$. We refine it by showing that if $cr(G) > 1$ then $cr(G) \geq 1.2$ (this bound is tight).

2 Background: Permutation Representations

Given a finite group $G$. A homomorphism $\rho : G \to S_n$ is called a permutation representation of $G$. In case $\rho$ is a monomorphism, we say $\rho$ is a faithful representation. The number $n$ is called the degree of the representation $\rho$. Any subgroup $H \leq G$ induces a transitive permutation representation of $G$ by the action of $G$ on the left cosets of $H$. That is, it induces a representation $\rho : G \to S_{\text{Sym}(G/H)}$, defined by $\rho(g) = (xH \mapsto gxH)$ for any $g \in G$. The degree of $\rho$ is $|G/H| = |G : H|$. The representation $\rho$ is faithful if and only if $\text{Core}_G(H) = 1$. More generally, any multiset $\{H_1, ..., H_m\}$ of subgroups of $G$ induces a representation $\rho : G \to S_{\text{Sym}(G/H_1)} \times \cdots \times S_{\text{Sym}(G/H_m)} \to S_{[G:H_1] + \cdots + [G:H_m]}$ defined by $\rho(g) = ((x_1H_1, ..., x_mH_m) \mapsto (gx_1H_1, ..., gx_mH_m))$ for any $g \in G$. The representation $\rho$ is faithful if and only if $\text{core}_G(\cap_{i=1}^m H_i) = \cap_{i=1}^m \text{core}_G(H_i) = 1$. The degree of $\rho$ is $\sum_{i=1}^m [G : H_i]$ and $\rho$ has $m$ transitive constituents. Moreover, any permutation representation of $G$ is equivalent to a permutation representation induced by some multiset of subgroups in the way described above: Given a permutation representation $\rho$, an equivalent representation is induced by $\{H_1, ..., H_m\}$, where $H_i$ is the point stabilizer of $\alpha_i$ and $\{\alpha_1, ..., \alpha_m\}$ are representatives of the transitive constituents of $\rho$. This correspondence between permutation representations and multisets of subgroups allows us to refer to such multisets as a permutation representation and vice
versa. We will use both viewpoints interchangeably. A more detailed description of these basic results can be found in [5] (Chapter 2, p. 13).

3 The Basics

Given a representation \( R = \{H_1, \ldots, H_m\} \) of a finite group \( G \), we denote by \( \mu_G(R) \) the degree of \( R \) as a representation of \( G \). By the discussion in Section 2, we have

\[
\mu_G(R) = \sum_{i=1}^{m} |G : H_i|.
\]

Thus we have a formula for the function \( \mu \) given by \( \mu(G) = \min\{\sum_{H \in R}|G : H| \mid R \) is a collection of subgroups of \( G \) with \( \cap_{H \in R} \text{core}_G(H) = 1 \} \).

For any two nontrivial finite groups \( G \) and \( H \), we have \( \mu(G \times H) \leq \mu(G) + \mu(H) \) because for any pair of faithful representations, \( R_1 = \{G_1, \ldots, G_n\} \) and \( R_2 = \{H_1, \ldots, H_m\} \), of \( G \) and \( H \) respectively, we can construct the faithful representation \( R = \{G_1 \times H_1, \ldots, G_n \times H_n, G \times H_1, \ldots, G \times H_m\} \) of \( G \times H \), and \( \mu_{G \times H}(R) = \mu_G(R_1) + \mu_H(R_2) \).

We proceed to explore some interaction between a representation of a group and representations of its subgroups and then more specifically - between a representation of a direct product and representations of each of its factors. One natural way to get a representation of \( H \) is to restrict the representation of \( G \) to the elements of \( H \). We now define a different way to induce a representation on a subgroup that will be useful for our purposes.

**Definition 2** (induced representation). Let \( G \) be a finite group, let \( R = \{G_1, \ldots, G_n\} \) be a representation of \( G \) and let \( H \leq G \) be a subgroup of \( G \). Then the representation \( R_H = \{G_1 \cap H, \ldots, G_n \cap H\} \) of \( H \) is called the induced representation by \( R \) on \( H \).

Warning: Even if \( R \) is a faithful representation of \( G \), \( R_H \) is not necessarily a faithful representation of \( H \). For example, consider \( G = S_3 \), \( R = \{(1 2)\} \) and \( H = \langle (1 2) \rangle \).

**Definition 3** (faithful decomposition). Let \( G_1, \ldots, G_n \) be finite groups and let \( R \) be a faithful representation of \( \prod_{i=1}^{n} G_i \). We say that \( (G_1, \ldots, G_n)_R \) admits a faithful decomposition as \( R = \{\pi_i^{-1}(G_1^{(i)}), \ldots, \pi_i^{-1}(G_n^{(i)})\} \) for some faithful representation \( \{G_1^{(i)} : \ldots : G_n^{(i)}\} \) of \( G_i \).

**Definition 4** (weak faithful decomposition). Let \( G_1, \ldots, G_n \) be finite groups and let \( R \) be a faithful representation of \( \prod_{i=1}^{n} G_i \). We say that \( (G_1, \ldots, G_n)_R \) admits a weak faithful decomposition as \( R = \{\pi_i^{-1}(G_i^{(i)})\} \) for each \( 1 \leq i \leq n \), the induced representation \( R_G^{(i)} \) is a faithful representation of \( G_i \).

It is easy to see that if \( (G_1, \ldots, G_n)_R \) admits a faithful decomposition then it also admits a weak faithful decomposition as the names imply. If \( G \) and \( H \) are nontrivial finite groups and \( (G, H)_R \) admits a faithful decomposition as \( R = R' \uplus R'' \), we immediately conclude that

\[
\mu_{G \times H}(R) = \mu_G(R'_G) + \mu_H(R''_H) \tag{2}
\]

We now show that even if we only require \( (G, H)_R \) to admit a weak faithful decomposition, we still get one inequality between the two sides of equality (2).

**Lemma 1** (weak decomposition inequality). Let \( G \) and \( H \) be nontrivial finite groups such that \( (G, H)_R \) admits a weak faithful decomposition as \( R = R' \uplus R'' \). Then \( \mu_{G \times H}(R) \geq \mu_G(R'_G) + \mu_H(R''_H) \).

**Proof.** For each \( K \in R' \), we have \( [G : K \cap G] = \frac{|G|}{|K \cap G|} \leq \frac{|G|}{|K \cap G|} \frac{|G||H|}{|K|} = \frac{|G||H||K \cap G|}{|K||K \cap G|} = \frac{|G||H|}{|K|} = [G \times H : K] \). Similarly, for each \( K \in R'' \), we have \( [H : K \cap H] \leq [G \times H : K] \). Finally,

\[
\mu_{G \times H}(R) = \sum_{K \in R'} [G \times H : K] \geq \sum_{K \in R'} [G : K \cap G] + \sum_{K \in R''} [H : K \cap H] = \mu_G(R'_G) + \mu_H(R''_H) \text{ as desired.} \]
Lemma 2. Let $G$ and $H$ be finite groups. Then if there is a minimal-degree faithful representation $R$ of $G \times H$ such that $(G, H)_R$ admits a weak faithful decomposition then $\mu(G \times H) = \mu(G) + \mu(H)$. The other direction is easy and is discussed in the beginning of this section.

Proof. Let $(G, H)_R$ admit a weak faithful decomposition as $R = R' \cup R''$. We have $\mu(G \times H) = \mu_{G \times H}(R) \geq \mu_G(R_G) + \mu_H(R_H') \geq \mu(G) + \mu(H)$ where inequality (a) is due to Lemma 1.

Lemma 3. Let $G$ and $H$ be nontrivial finite groups such that gcd$(|G|, |H|) = 1$ and let $R = \{K_1, \ldots, K_n\}$ be a minimal-degree faithful representation of $G \times H$. Then $(G, H)_R$ admits a faithful decomposition.

Proof. Since gcd$(|G|, |H|) = 1$, by Lemma 17 in the appendix, for any $1 \leq i \leq n$ we have $K_i = G_i \times H_i$ for some $G_i \leq G$, $H_i \leq H$. Let $1 \leq i_0 \leq n$. We have $K_{i_0} = G_{i_0} \times H_{i_0} = (G \times H) \cap (G_{i_0} \times H)$. Thus $R' = (R \setminus \{K_{i_0}\}) \cup \{(G \times H) \cap (G_{i_0} \times H)\}$ is a faithful representation of $G$ (because $R$ is). Since $R$ is of minimal degree, we have $0 \leq \mu_G(R') - \mu_G(R) = -|G \times H : G_{i_0} \times H_{i_0}| + |G \times H : G_{i_0} \times H| + |G \times H : G \times H_{i_0}| = |G : G_{i_0}|[H : H_{i_0}] - |G : G_{i_0}|[H : H_{i_0}]$. That is, $|G : G_{i_0}| + [H : H_{i_0}] \geq |G : G_{i_0}|[H : H_{i_0}]$. Thus either $|G : G_{i_0}| = 1$ or $[H : H_{i_0}] = |G : G_{i_0}|$. The latter is impossible since gcd$(|G|, |H|) = 1$, thus either $K_{i_0} = G \times H_{i_0}$ or $K_{i_0} = G_{i_0} \times H$. So $R = R' \cup R''$ where $R' = \{G_1 \times H, \ldots, G \times H\}$ and $R'' = \{G \times H_1, \ldots, G \times H\}$. We have $1 = \cap_{i=1}^n \text{core}_{G \times H}(K_i) = (\cap_{i=1}^n \text{core}_{G \times H}(G_i \times H)) \cap (\cap_{i=1}^n \text{core}_{G \times H}(G \times H_i)) = (\cap_{i=1}^n \text{core}_{G \times H}(G_i \times H)) \cap (\cap_{i=1}^n \text{core}_{G \times H}(G \times H_i)) = (\cap_{i=1}^n \text{core}_{G \times H}(G_i \times H)) \cap (\cap_{i=1}^n \text{core}_{G \times H}(G \times H_i))$. Therefore, $\text{core}_{G \times H}(\cap_{i=1}^n G_i) = \text{core}_{G \times H}(\cap_{i=1}^n H_i) = 1$, so $\{G_1, \ldots, G_n\}$ and $\{H_1, \ldots, H\}$ are faithful representations of $G$ and $H$ respectively and so $(G, H)_R$ admits a faithful decomposition as claimed.

Theorem 4 (coprime additivity of $\mu$). Let $G$ and $H$ be finite groups such that gcd$(|G|, |H|) = 1$, then $\mu(G \times H) = \mu(G) + \mu(H)$.

Proof. By Lemma 3 we have that any minimal-degree faithful representation of $G \times H$ admits a faithful decomposition. In particular, there is a faithful representation which admits a faithful decompositions and therefore admits a weak faithful decomposition and thus by Lemma 2 we have $\mu(G \times H) = \mu(G) + \mu(H)$.

To conclude this section we prove another basic result that shows that any finite group has a minimal-degree representation with a certain useful property. Recall that in any lattice $L$, an element $x \in L$ is called meet-irreducible if for any two elements $y, z \in L$, $x = y \wedge z$ implies $x = y$ or $x = z$.

The following result first appeared as Lemma 1 in [1].

Proposition 5 (existence of a minimal-degree representation by meet-irreducible subgroups). Let $G$ be a finite group. Then there is a minimal-degree faithful permutation representation of $G$, given by $\{G_1, \ldots, G_n\}$, such that for any $1 \leq i \leq n$, $G_i$ is meet-irreducible in the subgroup lattice of $G$.

Proof. Let $R = \{K_1, \ldots, K_m\}$ be any minimal-degree faithful permutation representation of $G$. That is - $\mu_G(R) = \mu(G)$. First we note that

$$\text{for any } 1 \leq i < j \leq m, \text{ we have } K_i \not\subseteq K_j \text{ and } K_j \not\subseteq K_i \quad (3)$$

In particular $R$ is a set (not a multiset).

We will iteratively alter $R$ until all of the subgroups in it are meet-irreducible. On one hand we will prove that each iteration keeps $R$ faithful of minimal degree. On the other hand we will show that this iterative process terminates after some finite number of steps. Together these 2 claims prove the existence of a minimal-degree faithful representation with the desired property.
We now describe the iterative process. As long as there is a meet-reducible subgroup of $G$ in $R$ we do the following: Let $K \in R$ be such a meet-reducible group. So there are subgroups $M$ and $L$ of $G$ such that $K$ is a proper subgroup of both $M$ and $L$, but $K = M \cap L$. Therefore $R' = (R \setminus \{K\}) \cup \{M, L\}$ is a faithful representation of $G$ with $\mu_G(R') = \mu(G) - [G : K] + [G : M] + [G : L] = \mu(G) + [G : K](-1 + \frac{1}{|M:K|} + \frac{1}{|L:K|}) \leq \mu(G) + [G : K](-1 + \frac{1}{2} + \frac{1}{2}) = \mu(G)$. Thus $R'$ is still a minimum-degree faithful representation.

It remains to show that this process eventually terminates. By property (3) we know that it is not possible to get the same representation in 2 different iterations. But $G$ is finite, and thus so is its subgroup lattice and therefore so is the number of subsets of its subgroup lattice and therefore the process does eventually terminate.

It should be noted that the above proof shows that for a group $G$ of odd order, any minimal-degree faithful representation is given by a collection of meet-irreducible subgroups. We will not use that fact.

4 The value of $\mu(G)$ for an abelian group $G$

In this section we show how to compute the value of $\mu(G)$ for a finite abelian group $G$. To describe the formula, we first need to recall that any finite abelian group is isomorphic to the direct product of cyclic groups, each of prime-power order. That is, if $G$ is a nontrivial finite abelian group then $G \cong \prod_{i=1}^{n} \mathbb{Z}_{p_i^{e_i}}$ for some integer $n \geq 1$, primes $p_1, \ldots, p_n$ and integers $e_1, \ldots, e_n \geq 1$. This decomposition of $G$ is called the primary decomposition of $G$. Further, the primary decomposition of $G$ is unique up to the order of the factors. This allows us to give a formula for $\mu(G)$ in terms of the numbers $n, p_1, \ldots, p_n$ and $e_1, \ldots, e_n$. We can now state the result of this section: For an arbitrary finite abelian group $G$, isomorphic to $\prod_{i=1}^{n} \mathbb{Z}_{p_i^{e_i}}$, as above, we have $\mu(G) = \sum_{i=1}^{n} p_i^{e_i}$. This is the content of Theorem 8 This result was first proved Theorem 2 of [11] by induction on the number of factors in the primary decomposition of $G$. We give a new, different, proof.

We begin with some notation:

Definition 5 (the function $m$). Let $G$ be a finite abelian group. Let the unique primary decomposition of $G$ be $G \cong \prod_{i=1}^{n} \mathbb{Z}_{p_i^{e_i}}$ for some $n \geq 1$, primes $p_1, \ldots, p_n$ and integers $e_1, \ldots, e_n \geq 1$. Then we define $m(G) := \sum_{i=1}^{n} p_i^{e_i}$

Lemma 6 (properties of $m$). Let $K$ and $L$ be finite abelian groups and let $K = \prod_{i=1}^{n} \mathbb{Z}_{p_i^{d_i}}$, $L = \prod_{i=1}^{m} \mathbb{Z}_{q_i^{e_i}}$ be their primary decompositions, then:

- (cardinality bound) $m(K) \leq |K|$.
- (additivity) $m(K \times L) = m(K) + m(L)$.
- (monotonicity) If $H \leq K$ then $m(H) \leq m(K)$.

Proof. 

$m(K) = \sum_{i=1}^{n} p_i^{d_i} \leq \prod_{i=1}^{n} p_i^{d_i} \leq |K|$.

$m(K \times L) = \sum_{i=1}^{n} p_i^{d_i} + \sum_{i=1}^{m} q_i^{e_i} = m(K) + m(L)$

Lemma 21 in the appendix states that if $H$ is a subgroup of the finite abelian group $K = \prod_{i=1}^{n} \mathbb{Z}_{p_i^{d_i}}$, then $H = \prod_{i=1}^{n} \mathbb{Z}_{a_i^{d_i}}$ for some integers $a_1, \ldots, a_n$ such that $0 \leq a_i \leq d_i$ for each $1 \leq i \leq n$. Thus, $m(H) = m(\prod_{i=1}^{n} \mathbb{Z}_{a_i^{d_i}}) = \sum_{\substack{1 \leq i \leq n \quad |a_i| \neq 0}} a_i^{d_i} \leq \sum_{1 \leq i \leq n} a_i^{d_i} \leq \sum_{1 \leq i \leq n} p_i^{d_i} = m(K)$.

□
**Lemma 7** (minimal degree of an abelian $p$-group). Let $G$ be a finite abelian $p$-group. Then $\mu(G) = m(G)$.

**Proof.** Let the primary decomposition of $G$ be $G \cong \prod_{i=1}^{n} \mathbb{Z}_{p_i^{a_i}}$. We prove both $\mu(G) \leq m(G)$ and $\mu(G) \geq m(G)$ to conclude the desired equality:

- $\mu(G) \leq m(G)$: We need to construct a faithful permutation representation of $G$ of degree $m(G)$. For any $1 \leq j \leq n$ define $H_i = \prod_{i=1}^{j-1} \mathbb{Z}_{p_i^{a_i}} \times \prod_{i=j+1}^{n} \mathbb{Z}_{p_i^{a_i}}$. Then the representation $R = \{H_1, \ldots, H_n\}$ of $G$ is faithful because $\cap_{i=1}^{n} K_i = 1$ and its degree is $d_G(R) = \sum_{i=1}^{n} |G : H_i| = \sum_{i=1}^{n} p_i^{a_i} = m(G)$, as required.

- $\mu(G) \geq m(G)$: We need to take an arbitrary faithful representation of $G$ and prove that its degree is no less than $m(G)$. Let $\{H_1, \ldots, H_m\}$ be a faithful representation of $G$. It is sufficient to justify the following chain of equalities and inequalities:

  \[
  \mu_G(\{H_1, \ldots, H_m\}) = \sum_{i=1}^{m} [G : H_i] = \sum_{i=1}^{m} |G/H_i| \geq \sum_{i=1}^{m} m(G/H_i) = m(\prod_{i=1}^{m} (G/H_i)) \geq m(G).
  \]

  Steps (a), (b) and (c) are due to the properties of the function $m$ stated in Lemma 6. Inequality (a) follows from the cardinality bound of $m$. Equality (b) follows from the additivity of $m$. In order to show that inequality (c) follows from the monotonicity of $m$ we need to show that $G$ embeds in $\prod_{i=1}^{m} (G/H_i)$ which we do as follows:

  The function $\phi : G \to \prod_{i=1}^{m} (G/H_i)$ defined by $\phi(g) = (gh_1, \ldots, gh_m)$ is a homomorphism. We have $ker(\phi) = \cap_{i=1}^{m} H_i = \cap_{i=1}^{m} core_G(H_i) = 1$. Equality (d) follows because $G$ is abelian and equality (e) follows because the representation $\{H_1, \ldots, H_m\}$ is faithful. Thus $\phi$ is an embedding. Thus $G$ embeds in $\prod_{i=1}^{m} (G/H_i)$ as desired. This completes the proof.

\[ \square \]

**Theorem 8** (minimal degree of an abelian group). Let $G$ be a finite abelian group. Then $\mu(G) = m(G)$.

**Proof.** The group $G$ is a direct product of abelian $p$-groups $G = \prod_{i=1}^{n} G_i$, where $G_i$ is a $p_i$-group for some distinct primes $p_1, \ldots, p_n$. We now have $\mu(G) = \mu(\prod_{i=1}^{n} G_i) \overset{(1)}{=} \prod_{i=1}^{n} \mu(G_i) \overset{(2)}{=} \prod_{i=1}^{n} m(G_i) \overset{(3)}{=} m(\prod_{i=1}^{n} G_i) = m(G)$, where equality (1) follows from the coprime additivity of $\mu$ proved in Lemma 6 equality (2) follows from the additivity of $m$ for abelian $p$-groups proved in Lemma 7, equality (3) follows from the additivity of the function $m$ stated in the second part of Lemma 6.

Note that if $G$ and $H$ are finite abelian groups, then by Theorem 8 we have $\mu(G \times H) = \mu(G) + \mu(H)$. A larger collection of groups for which this formula holds is the subject of the next two sections.

## 5 Additivity of $\mu$ for central socle groups

This section generalizes a result first proved in [2]. Some of the ideas presented here are based on ideas which first appeared in [2].

Recall that the socle of a group $G$, denoted $Soc(G)$, is the subgroup generated by all minimal normal subgroups of $G$. The socle of a finite group is always a direct product of simple groups and thus, if $G$ is a finite group and $Soc(G)$ is abelian, then $Soc(G)$ is the direct product of elementary abelian groups.

**Definition 6** (Central socle groups). The collection $CS$ is defined as the collection of all nontrivial finite groups for which the socle is central. That is, $CS := \{G \mid G$ is a nontrivial finite group and $Soc(G) \leq Z(G)\}$. 


Further, if primes dividing and let prove that admits a weak faithful decomposition. We will show that if Lemma 2, it is enough to construct a minimal-degree faithful representation, as we have already seen in Lemma 5, such a representation exists. We thus let be a minimal-degree faithful representation of given by meet-irreducible subgroups and proceed to show that admits a weak faithful representation. To do so, we consider the representation induced by on given by meet-irreducible subgroups and . Finally, we show that if is the set of subgroups in which induce and is the set of subgroups in which induce then is a weak faithful decomposition of , as desired.

Before executing the plan described above, we compare it to the proof given in [2]. In order to compare the two proofs we must describe the method of [2] using the terminology presented in Section 3. Both proofs start with a minimal-degree representation of given by meet-irreducible subgroups. In [2], it is assumed that is nilpotent and thus decomposes as faithful representations of -groups whose direct product is . The proof in [2] then proceeds in a method similar to the one used in our paper to show that each of these representations decomposes to faithful representations of a factor coming from and a factor coming from , using the fact that the socle of a -group is a vector space. Our paper refines this ideas by only requiring the socle to be central and immediately considering the representation induced on (which is a direct product of vector spaces when it is central). We then show that the representation induced on decomposes to faithful representations of and and show that when we go back up to and we get faithful representations of and . To summarize the comparison, decomposing the representation entirely down at the socle, instead of first decomposing to -groups and then decomposing at the socle of each of them, is what allows us to generalize the result proved in [2].

Lemma 9 (properties of the induced representation on the socle). Let be a group belonging to and let be a minimal-degree faithful representation of . Let be the set of primes dividing . Then the induced representation of on decomposes faithfully as (denote where is a minimal-degree representation of ).

1. is faithful and decomposes faithfully as (denote ).

2. For each , has no redundant transitive constituents. That is, for any we have

Further, if is meet-irreducible in the subgroup lattice of for each , then:
3. For each \(1 \leq j \leq m\) and \(1 \leq i \leq n_j\), we have \(\dim(G_i^{(j)} \cap Z(G)[p_j]) = \dim(Z(G)[p_j]) - 1\).

Proof. \(1\). We first show that \(R_{\text{Gen}(G)}\) is a faithful representation. Assume, for the sake of contradiction, that \(\cap_{i=1}^{n_0} G_i \cap \text{Gen}(G) \neq 1\). We have \(\cap_{i=1}^{n_0} G_i \cap \text{Gen}(G) \leq G\) because \(\cap_{i=1}^{n_0} (G_i \cap \text{Gen}(G)) \leq Z(G)\). But \(\cap_{i=1}^{n_0} G_i \cap \text{Gen}(G) \leq \cap_{i=1}^{n_0} G_i\). Thus \(1 \neq \cap_{i=1}^{n_0} G_i \cap \text{Gen}(G) \leq \text{core}_G \cap_{i=1}^{n_0} G_i\) in contradiction with the faithfulness of \(R = \{G_1, \ldots, G_m\}\). So \(R_{\text{Gen}(G)}\) is faithful, and thus by Lemma 3 it decomposes into faithful representations of \(Z(G)[p_1], \ldots, Z(G)[p_m]\) as desired.

2. Fix some \(1 \leq j \leq m\) and \(1 \leq i_0 \leq n_j\) and assume, for the sake of contradiction, that \(\cap_{K \in R_j \setminus \{G_i^{(j)}\}} (K \cap Z(G)[p_j]) = 1\). That is, \(\cap_{K \in R_j \setminus \{G_i^{(j)}\}} G_i^{(j)} \cap Z(G)[p_j] = 1\). Thus, by the faithful decomposition of \((Z(G)[p_1], \ldots, Z(G)[p_m])_{R_{\text{Gen}(G)}}\) proved in conclusion (1), we get \(\cap_{K \in R_j \setminus \{G_i^{(j)}\}} G_i \cap \text{Gen}(G) = 1\). Then \(\text{core}_G \cap_{K \in R_j \setminus \{G_i^{(j)}\}} G_i \cap \text{Gen}(G) = 1\). Therefore \(\text{core}_G \cap_{K \in R_j \setminus \{G_i^{(j)}\}} G_i = 1\). Thus \(R \setminus \{G_i^{(j)}\}\) is a faithful representation of \(G\), contradicting the minimality of \(R\).

3. Fix some \(1 \leq j \leq m\) and \(1 \leq i_0 \leq n_j\) and assume, for the sake of contradiction, that \(\dim(G_i^{(j)} \cap Z(G)[p_j]) = \dim(Z(G)[p_j])\). Then \(G_i^{(j)} \cap Z(G)[p_j] = Z(G)[p_j]\). Thus, \(\cap_{i \neq i_0} (G_i^{(j)} \cap Z(G)[p_j]) = \cap_{i=1}^{n_0} (G_i^{(j)} \cap Z(G)[p_j]) = 1\), by conclusion (1), contradicting conclusion (2).

Assume, again - for the sake of contradiction, that \(\dim(G_i^{(j)} \cap Z(G)[p_j]) < \dim(Z(G)[p_j])\). Then \(\dim(Z(G)[p_j])/G_i^{(j)} \cap Z(G)[p_j]) \geq 2\). But \(Z(G)[p_j]/G_i^{(j)} \cap Z(G)[p_j]) \cong Z(G)[p_j]G_i^{(j)}/G_i^{(j)}\) by the second isomorphism theorem. So \(Z(G)[p_j]G_i^{(j)}/G_i^{(j)}\) is a vector space spanned by a basis \(z_1 G_i^{(j)}, \ldots, z_r G_i^{(j)}\) for some \(r \geq 2\) and \(z_1, \ldots, z_r \in Z(G)[p_j]\). In particular, \(z_1 G_i^{(j)}\) and \(z_2 G_i^{(j)}\) are linearly independent and therefore \(\text{span}\{z_1 G_i^{(j)}\} \cap \text{span}\{z_2 G_i^{(j)}\} = \{G_i^{(j)}\}\) and \(\text{span}\{z_1 G_i^{(j)}\} \neq \{G_i^{(j)}\}\). So \(G_i^{(j)} < \langle z_1 G_i^{(j)}, z_2 G_i^{(j)}\rangle\) and \(z_1 G_i^{(j)} \cap \langle z_2 G_i^{(j)}\rangle = G_i^{(j)}\) contradicting the fact that \(G_i^{(j)}\) is meet-irreducible.

\(\square\)

Lemma 10 (lifting a representation back up from \(\text{Gen}(G)\) to \(G\)). Let \(G\) be a group belonging to \(\text{CS}\). Let \(R = \{G_1, \ldots, G_m\}\) be a representation of \(G\). Then if the induced representation \(R_{\text{Gen}(G)}\) is a faithful representation of \(\text{Gen}(G)\), then \(R\) is a faithful representation of \(G\).

Proof. Assume, for the sake of contradiction, that \(\text{core}_G \cap_{i=1}^{n_0} G_i \neq 1\). Then \(\text{core}_G \cap_{i=1}^{n_0} G_i \cap \text{Gen}(G) \neq 1\). But then \(1 \neq \text{core}_G \cap_{i=1}^{n_0} G_i \cap \text{Gen}(G) = \text{core}_G \cap_{i=1}^{n_0} (G_i \cap \text{Gen}(G)) \leq \text{core}_{\text{Gen}(G)} \cap_{i=1}^{n_0} (G_i \cap \text{Gen}(G))\) in contradiction to the faithfulness of \(R_{\text{Gen}(G)}\).

\(\square\)

Lemma 11. Let \(r \geq 1\) be an integer and let \(p_1, \ldots, p_r\) be distinct primes. Suppose that for each \(1 \leq i \leq r\) we have:

- \(G_i\) and \(H_i\) are elementary abelian \(p_i\)-groups of dimensions \(m_i\) and \(n_i\) respectively.
- \(\{v_1^{(i)}, \ldots, v_{m_i+n_i}^{(i)}\}\) is a basis for the vector space \(G_i \times H_i\).
- \(K_{j,i}^{(i)} = \text{span}\{v_k^{(i)} | 1 \leq k \leq m_i + n_i \text{ and } k \neq j\}\) for each \(1 \leq j \leq m_i + n_i\).

Let \(G = \prod_{i=1}^{r} G_i\) and \(H = \prod_{i=1}^{r} H_i\) and let \(R = \{K_{j,i}^{(i)} | 1 \leq j \leq m_i + n_i\}\) be a (faithful) representation of \(G \times H\). Then \((G,H)_R\) admits a weak faithful decomposition.
Proof. By the formula for $\mu$ for abelian groups given in Lemma 8, we know that $R$ is a minimal-degree faithful representation of $G \times H$. Therefore, since the orders of $G_i$ and $H_j$ are coprime whenever $i \neq j$ we conclude, by Lemma 9, that $(G_1 \times H_1, \ldots, G_r \times H_r)_R$ admits a faithful decomposition. It is thus sufficient to prove, for each $1 \leq i \leq r$, that $(G_i, H_i)_{R_{G_i \times H_i}}$ admits a weak faithful decomposition. Fix some $1 \leq j_0 \leq r$ and denote $m = m_{i_0}$, $n = n_{i_0}$ and $v_j = v_j^{(i_0)}$ for each $1 \leq j \leq m_{i_0} + n_{i_0}$. Form a matrix $M$ that has $\{v_1, \ldots, v_{m+n}\}$ as its rows. By applying the matrix decomposition whose definition and existence are given in Lemma 24 in the appendix to the invertible matrix $M$ with parameter $m$, we can conclude that we can assume that the matrix $M$ is of the form: $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A$ is an $m \times m$ matrix, $D$ is an $(n-m) \times (n-m)$ matrix and both $A$ and $D$ are invertible. Partition $R$ as $R = R' \cup R''$ where $R' = \{K_1, \ldots, K_m\}$ and $R'' = \{K_{m+1}, \ldots, K_{m+n}\}$. Now $R'_G$ is a faithful representation of $G$ because $\cap_{K \in R'_G} K' = \cap_{K \in R} (K \cap G) = (\cap_{K \in R} K) \cap G = \text{span}\{v_{m+1}, \ldots, v_{m+n}\} \cap G \subseteq 1$ where equality (a) is due to the fact that the submatrix $D$ of $M$ is invertible. Similarly, $R''_H$ is a faithful representation of $H$ and thus $(G, H)_R$ admits a weak faithful decomposition as claimed.

Lemma 12. Let $G$ and $H$ be groups belonging to CS, then $\mu(G \times H) = \mu(G) + \mu(H)$.

Proof. Let $R = \{K_1, \ldots, K_n\}$ be a minimal-degree faithful representation of $G \times H$. By Lemma 5 we can assume that $K_1, \ldots, K_n$ are all meet-irreducible. The properties of the the induced representation on $\text{Soc}(G \times H)$ proved in Lemma 9 together with the linear algebra result proved in Lemma 24 in the appendix show that $\text{Soc}(G)$, $\text{Soc}(H)$ and the representation $R_{\text{Soc}(G \times H)}$ fit the hypothesis of Lemma 11 and therefore $(R_{\text{Soc}(G)}, R_{\text{Soc}(H)})_{R_{\text{Soc}(G \times H)}}$ admits a weak faithful decomposition. Thus, by Lemma 11, $(G, H)_R$ admits a weak faithful decomposition too. Therefore, by Lemma 11 we conclude that $\mu(G \times H) = \mu(G) + \mu(H)$ as claimed.

6 A larger collection for which $\mu$ is additive

We now extend the collection $CS$ to a larger collection for which the function $\mu$ is additive. The extended collection, denoted $CSE$, is defined as the collection of groups $G$ for which there is a subgroup $H \leq G$ such that $H \in CS$ and $\mu(H) = \mu(G)$. This extension idea first appeared in [2], where a collection $\mathfrak{C}$ was similarly defined as the collection of groups $G$ for which there is a nilpotent subgroup such that $\mu(H) = \mu(G)$.

The collection $\mathfrak{C}$ is obviously a subcollection of CSE since the collection of nilpotent groups is a (proper) subcollection of CS. We show $\mathfrak{C}$ is a proper subcollection of CSE by giving an example of a group in $CSE$ (actually, in CS, which is subcollection of CSE) that does not belong to $\mathfrak{C}$.

We begin by proving that $CSE$ is closed under taking direct products and that $\mu$ is additive for groups belonging to $CSE$.

Lemma 13. Let the groups $G$ and $H$ belong to the collection $CSE$. Then:

1. $G \times H$ belongs to $CSE$.
2. $\mu(G \times H) = \mu(G) + \mu(H)$.

Proof. On one hand $\mu(G \times H) \leq \mu(G) + \mu(H)$. On the other hand, since $G$ and $H$ belong to $CSE$, there are subgroups $G_1$ and $H_1$ of $G$ and $H$ respectively such that $G_1$ and $H_1$ both belong to $CS$ and $\mu(G_1) = \mu(G)$ and $\mu(H_1) = \mu(H)$. Therefore $\mu(G \times H) \geq \mu(G_1 \times H_1) = \mu(G_1) + \mu(H_1) = \mu(G) + \mu(H)$. Thus $\mu(G \times H) = \mu(G) + \mu(H)$, proving conclusion (2). Therefore inequality (a) is in fact an equality and
thus \( \mu(G_1 \times H_1) = \mu(G \times H) \) which proves conclusion (1) because the subgroup \( G_1 \times H_1 \) of \( G \times H \) belongs to \( CS \) since \( CS \) is closed under taking direct products.

We proceed to show that the binary icosahedral group \( SL(2,5) \) belongs to \( CSE \), but not to \( \emptyset \). First, \( SL(2,5) \) belongs to \( CS \) (and thus to \( CSE \)) because its only proper normal subgroup is its center, \{+1, −1\}. To show that \( SL(2,5) \) does not belong to \( \emptyset \) we first note that its nilpotent subgroups are isomorphic to cyclic groups of orders 1, 2, 3, 4, 5, 6 and 10, or to \( Q_8 \). Of these groups, the one with the largest minimal-degree is \( Q_8 \), for which \( \mu(Q_8) = 8 \) (see Lemma 10). Thus, it is sufficient to show that \( \mu(SL(2,5)) > 8 \). In fact, we will show that \( \mu(SL(2,5)) = 24 \). Recall that \( Z(SL(2,5)) = \{-1, +1\} \) is normal in \( SL(2,5) \). Thus any faithful representation of \( SL(2,5) \) must be given by a collection of subgroups of \( SL(2,5) \) of which at least one does not contain the element −1. But, the element −1 is the only element of order 2 in \( SL(2,5) \). Therefore, any subgroup of \( SL(2,5) \) of even order contains the element −1. Thus, any faithful representation of \( SL(2,5) \) must be given by a collection of subgroups of which at least one is of odd order. But the largest subgroup of \( SL(2,5) \) of odd order is a cyclic group of order 5. So we can already conclude that the degree of any faithful representation of \( SL(2,5) \) is at least \( 120/5 = 24 \). Conversely, any subgroup of \( SL(2,5) \) of odd order does not contain \( Z(SL(2,5)) \), which is the unique minimal normal subgroup of \( SL(2,5) \). Therefore the representation \( \{Z_5\} \) is a minimal-degree representation of \( SL(2,5) \) and its degree is 24.

7 Semidirect Products

Lemma 14. Let \( G \) and \( H \) be nontrivial finite groups. Then \( \mu(G \rtimes H) \leq |G| + \mu(H) \).

Proof. It is sufficient to embed \( G \times H \) in \( Sym(G) \times H \). Let the multiplication in \( G \times H \) be defined by \((g_1, h_1)(g_2, h_2) = (g_1\varphi_{g_1}, g_2, h_1h_2)\) for any \( g_1, g_2 \in G \) and \( h_1, h_2 \in H \) where \( \varphi : H \to Aut(G) \) is a homomorphism. We show that \( \rho : G \rtimes H \to Sym(G) \times H \) defined by \( \rho(g_0, h_0) = ((g \mapsto g_0\varphi_{g_0}(g)), h_0) \) is a monomorphism. The function \( \rho \) is a homomorphism because \( \rho((g_1, h_1)(g_2, h_2)) = \rho((g_1\varphi_{g_1}, g_2, h_1h_2)) = (g \mapsto (g_1\varphi_{g_1}, g_2, h_1h_2)(g), h_1h_2) = (g \mapsto g_1\varphi_{g_1}(g_2, h_1h_2)(g), h_1h_2) = (g \mapsto g_1\varphi_{g_1}(g_2, h_2)(g), h_1h_2) = \rho(g_1, h_1)\rho(g_2, h_2) \). The homomorphism \( \rho \) is injective because \( \rho(g_0, h_0) = (id, 1) \) implies \( (g \mapsto g_0\varphi_{h_0}(g), h_0) = (id, 1) \), that is \( h_0 = 1 \) and \( (g \mapsto g_0\varphi_{g_2}(g)) = (g \mapsto g_0\varphi_{h_0}(g)) = id. \) Thus we must have \( g_0 = 1 \) and so \( ker(\rho) = (id, 1) \).

8 Compression Ratio

Definition 7 (compression ratio). Let \( G \) be a finite group. Then the compression ratio of \( G \) is defined as
\[
cr(G) = \frac{|G|}{\mu(G)}
\]
For any finite group \( G \) we have \( \cr(G) \geq 1 \) since \( \mu(G) \leq |G| \) by Cayley’s theorem.

Lemma 15 (monotonicity of compression ratio). Let \( G \) be a finite group and let \( H \leq G \) be a subgroup of \( G \). Then \( \cr(H) \leq \cr(G) \)

Proof. The inequality \( \cr(H) \leq \cr(G) \) is equivalent to the inequality \( \mu(G) \leq |G : H|\mu(H) \). Therefore, it is sufficient to construct a faithful permutation representation of \( G \) of degree \( |G : H|\mu(H) \). Let \( \{H_1, \ldots, H_n\} \) be a minimal-degree permutation representation of \( H \). That is \( core_H(\cap_{i=1}^n H_i) = 1 \) and \( \sum_{i=1}^n [H : H_i] = \mu_H(\{H_1, \ldots, H_n\}) = \mu(H) \). The representation \( \{H_1, \ldots, H_n\} \) can also be viewed as a representation of \( G \). We show that it is faithful and of the desired degree: The faithfulness of \( \{H_1, \ldots, H_n\} \) as a representation of \( G \) follows because \( core_G(\cap_{i=1}^n H_i) \leq core_H(\cap_{i=1}^n H_i) = 1 \) and thus \( core_G(\cap_{i=1}^n H_i) = 1 \). The degree of
\{H_1, \ldots, H_n\} as a representation of G is \(\mu_G(\{H_1, \ldots, H_n\}) = \sum_{i=1}^n [G : H_i] = \sum_{i=1}^n [G : H][H : H_i] = [G : H] \sum_{i=1}^n [H : H_i] = [G : H] \mu_H(\{H_1, \ldots, H_n\}) = [G : H] \mu(H)\). 

A finite group G is called incompressible if \(cr(G) = 1\). The following characterization of incompressible groups is due to [1]. We strengthen the conclusion described in [1] by stating that if a group has a compression ratio larger than 1, then its compression ratio is at least 1.2 (this is tight because \(cr(\mathbb{Z}_6) = 1.2\).

**Theorem 16** (incompressible groups). Let G be a nontrivial finite group. The following conditions are equivalent:

- The group G is incompressible (that is, \(cr(G) = 1\)).
- The group G is of one of the following types:
  1. Cyclic group of prime power order
  2. Generalized quaternion group of order \(2^n\) (for \(n \geq 3\))
  3. The Klein four-group \(V_4\)

Further, \(cr(G) < 1.2\) if and only if \(cr(G) = 1\).

**Proof.** We begin by showing that groups of types (a), (b) or (c) are incompressible. If G is of type (c) then by the formula for the function \(\mu\) for abelian groups given in Theorem 8 we have \(\mu(G) = 2^2 = 4 = |G|\) and thus G is incompressible. Assume now that G is either of type (a) or of type (b). Then G has a unique minimal subgroup \(H\). The subgroup \(H\) must be normal in G. Let \(R = \{G_1, \ldots, G_m\}\) be a faithful representation of G. Assume, for the sake of contradiction, that for any \(1 \leq i \leq m\) we have \(G_i \neq 1\). Then, for any \(1 \leq i \leq m\), we have \(H \leq G_i\). Thus \(H \leq \cap_{i=1}^m G_i\). But H is normal in G and therefore \(1 \neq H \leq \text{core}_G(\cap_{i=1}^m G_i)\) in contradiction to the faithfulness of the representation R. Therefore, there is some \(1 \leq i_0 \leq m\) such that \(G_{i_0} = 1\). Thus \(\mu_G(R) = \sum_{i=1}^m [G : G_{i_0}] \geq [G : G_{i_0}] = [G : 1] = |G|\). Finally, since R is an arbitrary faithful representation of G we get \(\mu(G) = |G|\) and thus G is incompressible.

To complete the proof we need to show that if \(cr(G) < 1.2\) then G is of one of the types (a), (b) or (c). Assume that \(cr(G) < 1.2\). We first show that if \(H\) and \(K\) are nontrivial subgroups of G satisfying \(H \cap K = 1\), then both \(H\) and \(K\) are of order 2. Assume for the sake of contradiction that \(|H| \geq 3\). The representation \(R = \{H, K\}\) of G is faithful because \(H \cap K = 1\). Thus we have \(\mu(G) \leq \mu(R) = [G : H] + [G : K] = [G](1/|H| + 1/|K|) \leq [G](1/3 + 1/2) = (5/6)|G|\). Therefore, \(cr(G) \geq 1.2\), contradicting the assumption that \(cr(G) < 1.2\). Thus any two nontrivial subgroups of G intersecting trivially must be both of order 2. In particular, there cannot be two elements in G of distinct prime orders and therefore G is a p-group for some prime p. If p is an odd prime, then G is a group of odd-order which has a unique subgroup of order \(p\) and thus G is of type (a) (see [2], p. 118, Theorem 15). If \(p = 2\), that is, G is a 2-group, we consider two cases: If there is an element \(g\) in G of order 4 then \(g^2\) must be the unique element of order 2 in G and thus we conclude that G is either of type (a) or of type (b) (again, by [2], p. 118, Theorem 15). If, on the other hand, no element of G is of order 4 then G in an elementary abelian 2-group. That is \(G = \mathbb{Z}_2^n\) for some \(n \geq 1\) and thus, by the formula for the function \(\mu\) for abelian groups given in Theorem 8 we have \(\mu(G) = 2n\). But \(|G| = 2^n\). Thus \(1.2 > cr(G) = 2^n/(2n)\) and thus either \(n = 1\) or \(n = 2\). That is, G is either of type (a) or of type (c).

Note that if we further assume that G is of odd order, then, by similar reasoning, we get \(cr(G) < 1.5\) if and only if \(cr(G) = 1\).

We believe it would be interesting to continue the study of the compression ratio by answering questions similar to the following:

Is there a function \(f : \mathbb{R} \to \mathbb{R}\) such that whenever \(cr(G) \leq r\) there must be a solvable subgroup of G of index \(\leq f(r)\)?
9 Appendix

9.1 Group Theory

Lemma 17. Let $G$ and $H$ be finite groups such that $\gcd(|G|, |H|) = 1$ and let $K \leq G \times H$. Then for some $G' \leq G$, $H' \leq H$ we have $K = G' \times H'$.

Proof. Let $G' = \pi_1(K)$, $H' = \pi_2(K)$ where $\pi_i$ is the projection of the $i$th coordinate. Obviously $K \subset G' \times H'$. For the reverse inclusion, let $(g, h) \in G' \times H'$. Then there are $g' \in G$, $h' \in H$ such that $(g', h)(g, h) \in K$. Since $\gcd(|G|, |H|) = 1$ and by the chinese remainder theorem, there exist integers $e_1$, $e_2$ such that $e_1 \equiv 1 \pmod{|G|}$, $e_2 \equiv 0 \pmod{|H|}$, $e_1 \equiv 0 \pmod{|G|}$, $e_2 \equiv 1 \pmod{|H|}$. Thus, $(g, 1) = (g, h)^{e_1} \in K$ and $(1, h) = (g', h)^{e_2} \in K$ and so $(g, h) = (g, 1)(1, h) \in K$. □

Definition 8. If $G$ is abelian group and $m > 0$ is an integer, then $G[m] := \{x \in G \mid mx = 0\}$

Note that if $p$ is prime then $G[p]$ is a vector space (over $\mathbb{Z}_p$).

Definition 9. Let $G$ be a finite group. The the socle of $G$, denoted $\text{Soc}(G)$, is the subgroup of $G$ generated by the nontrivial minimal normal subgroups of $G$.

Lemma 18. Let $G$ be a finite group for which $\text{Soc}(G) \leq Z(G)$. Then $\text{Soc}(G) = \prod_{p|\text{Z}(G)} Z(G)[p]$.

Proof. On one hand, any cyclic central subgroup of prime order of $G$ is a minimal normal subgroup. On the other hand, since $\text{Soc}(G) \leq Z(G)$, any minimal normal subgroup of $G$ must be central and thus must be cyclic of prime order. So $\text{Soc}(G)$ is the subgroup generated by all central elements of $G$ of prime order which is $\prod_{p|\text{Z}(G)} Z(G)[p]$ as claimed. □

Lemma 19. Let $G$ and $H$ be finite groups. Then $\text{Soc}(G \times H) = \text{Soc}(G) \times \text{Soc}(H)$.

Proof. On one hand, any minimal normal subgroup of $G$ or $H$ is a minimal normal subgroup of $G \times H$ and thus $\text{Soc}(G) \times \text{Soc}(H) \leq \text{Soc}(G \times H)$. We proceed to show the reverse inclusion: Let $1 \neq N \leq G \times H$ be a minimal normal subgroup of $G \times H$. We need to show that $N \leq \text{Soc}(G) \times \text{Soc}(H)$. Since $N \subseteq \pi_1(N) \times \pi_2(N)$, it is sufficient to prove that $\pi_1(N) \leq \text{Soc}(G)$ and $\pi_2(N) \leq \text{Soc}(H)$. To do so, we will show that $\pi_1(N)$ and $\pi_2(N)$ are minimal normal subgroups of $\text{Soc}(G)$ and $\text{Soc}(H)$ respectively. First, $\pi_1(N)$ is a normal subgroup of $G$ because for any $g \in \pi_1(N)$ there exists some $h \in N$ such that $(g, h) \in N$ and therefore, for any $g_0 \in G$ it holds that $(g_0gg_0^{-1}, h) = (g_0, 1)(g, h)(g_0, 1)^{-1} \in N$ and therefore $g_0gg_0^{-1} \in \pi_1(N)$. Similarly, $\pi_2(N)$ is a normal subgroup of $H$. For minimality, assume, for the sake of contradiction, that there exists $1 \neq N_1 < \pi_1(N)$ such that $N_1 \nsubseteq G$. Then surely $(N_1 \times \pi_2(N)) \cap N \neq N$ and $(N_1 \times \pi_2(N)) \cap N \leq G \times H$ as an intersection of normal subgroups. But since $1 \neq N_1 \leq \pi_1(N)$, there exists some $1 \neq g \in N_1$ and $h \in \pi_2(N)$ such that $(g, h) \in N$. Therefore $(N_1 \times \pi_2(N)) \cap N \neq 1$ in contradiction with the minimality of $N$. So $\pi_1(N)$ is a minimal normal subgroup of $G$. Similarly, $\pi_2(N)$ is a minimal normal subgroup of $H$. This completes the proof. □

Definition 10. If $G$ is abelian $p$-group and $t \geq 0$ is an integer then $g(G, t) := \text{the number of factors of order } \geq p^t \text{ in the primary decomposition of } G$.

Lemma 20. Let $G$ be a finite abelian $p$-group and $t \geq 0$ an integer. Then $g(G, t) = \log_p(|G|p^t : G[p^{t-1}]|)$. 

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Proof. Denote \( h(G, t) := \log_p([G[p^t] : G[p^{t-1}][G^t]]) \). Fix some integer \( t \geq 0 \). On one hand, for a finite cyclic p-group \( K \), we easily have \( g(K, t) = h(K, t) \). On the other hand, if \( K \) and \( H \) are finite abelian p-groups we have \( g(K \times H, t) = g(K, t) + g(H, t) \) and \( h(K \times H, t) = h(K, t) + h(H, t) \). Thus the equality between \( g \) and \( h \) is proved by induction.

Lemma 21. If \( H \leq G = \prod_{i=1}^{n} \mathbb{Z}_{p^{\alpha_i}} \) for some prime \( p \) and integers \( d_1, \ldots, d_n \) then there exist integers \( c_1 \leq d_1, \ldots, c_n \leq d_n \) such that \( H \cong \prod_{i=1}^{n} \mathbb{Z}_{p^{c_i}} \).

Proof. An equivalent formulation of the proposition is: for any \( t \geq 0 \), \( g(H, t) \leq g(G, t) \). By Lemma 20 this is equivalent to \( [H[p^t] : H[p^{t-1}]] \leq [G[p^t] : G[p^{t-1}]] \). To prove this we note that: \( [H[p^t] : H[p^{t-1}]] = [H[p^t]/(H[p^t] \cap G[p^{t-1}])]/G[p^{t-1}] \leq [G[p^t]/G[p^{t-1}]]/G[p^{t-1}] \). Thus the isomorphism follows from the second isomorphism theorem. This completes the proof.

9.2 Linear Algebra

Consider an \( n \times n \) matrix \( A \), a list of row indices \( r = (r_1, \ldots, r_k) \) and a list of column indices \( c = (c_1, \ldots, c_k) \). We define two submatrices of \( A \): a \( k \times k \) submatrix \( S(A; r, c) \) and an \( (n-k) \times (n-k) \) submatrix \( S'(A; r, c) \). The submatrix \( S(A; r, c) \) is obtained by keeping the entries of the intersection of any row belonging to the list \( r \) and any column belonging to the list \( c \). The submatrix \( S'(A; r, c) \) is obtained by keeping the entries of the intersection of any row \textit{not} belonging to the list \( r \) and any column \textit{not} belonging to the list \( c \). To simplify the formula given in the next Lemma, we let the index of the first row and the first column be 0.

Lemma 22 (Laplace’s Determinant Expansion Theorem). Let \( A \) be an \( n \times n \) matrix. Let \( c = (c_1, \ldots, c_k) \) be a list of \( k \) column indices, where \( 1 \leq k < n \) and \( 0 \leq c_1 < c_2 < \cdots < c_k < n \). Then, the determinant of \( A \) is given by \( \det(A) = (-1)^{|c|} \sum_r (1)^{|r|} \det S(A; r, c) \det S'(A; r, c) \) where \( |c| = c_1 + \ldots + c_k \), \( |r| = r_1 + \ldots + r_k \) and the summation is over all \( k \)-tuples \( r = (r_1, \ldots, r_k) \) for which \( 0 \leq r_1 < \cdots < r_k < n \).

Proof. See [7].

The proof of the following lemma is due to Robert Israel [6].

Lemma 23. Let \( M \) be an \( n \times n \) invertible matrix and let \( 1 \leq m \leq n-1 \) be an integer. Then it is possible to permute the rows of \( M \) to obtain a matrix \( M' \) of the form \( M' = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) where \( A \) is an \( m \times m \) matrix and \( D \) is an \( (n-m) \times (n-m) \) matrix and both \( A \) and \( D \) are invertible.

Proof. Assume, for the sake of contradiction, that no permutation of the rows of \( M \) brings it to the desired form. Then, by plugging \( c = (0, 1, 2, \ldots, m-1) \) into Laplace’s Expansion Theorem (Lemma 22), we conclude that \( \det(M) \) is a sum of terms of the form \( \pm \det(A) \det(D) \) such that in each term either \( \det(A) \) is zero or \( \det(D) \) is zero. Therefore \( \det(M) = 0 \) in contradiction the fact that \( M \) is invertible.

Lemma 24. Let \( V \) be a vector space of finite dimension and let \( V_1, \ldots, V_n \) be subspaces of \( V \) such that:

1. \( \cap_{i=1}^{n} V_i = \{0\} \).
2. For any \( 1 \leq i \leq n \) we have \( \cap_{j \neq i} V_j \neq \{0\} \).
3. For any \( 1 \leq i \leq n \) we have \( \dim(V_i) = \dim(V) - 1 \).
Then, \( n = \dim(V) \) and there exists a basis \( \{v_1, \ldots, v_n\} \) of \( V \) such that for any \( 1 \leq i \leq n \), \( V_i = \text{span}\{v_j \mid 1 \leq j \leq n \wedge j \neq i\} \)

Proof. First, for any \( 1 \leq i \leq n \), we have \( 1 \leq \dim(\cap_{j \neq i} V_j) = \dim(V_i + \cap_{j \neq i} V_j) - \dim(V_i) + \dim(\cap_{j=1}^n V_j) \leq \dim(V) \leq \dim(V) - 1 \). Thus \( \dim(\cap_{j \neq i} V_j) = 1 \). Second, for any \( 1 \leq i \leq n \) we have \( (\cap_{j \neq i} V_j) \cap \text{span}(\cup_{j \neq i} \cap k \neq j V_k) \subset (\cap_{j \neq i} V_j) \cap V_i = \cap_{j=1}^n V_j = \{0\} \). Combining these 2 facts we conclude that there exists a linearly independent set \( \{v_1, \ldots, v_n\} \subset V \) such that for any \( 1 \leq i \leq n \) we have \( \cap_{j \neq i} V_j = \text{span}\{v_i\} \). Now, for any \( 1 \leq i \leq n \) we have \( \text{span}\{v_j \mid j \neq i\} = \sum_{j \neq i} \cap_{k \neq j} V_k \subset V_i \). Thus, for any \( 1 \leq i \leq n \) we have \( v_i \notin V_i \), because otherwise we would have \( v_i \in \cap_{j=1}^n V_j = \{0\} \), a contradiction. To summarize, we have found a linearly independent set \( \{v_1, \ldots, v_n\} \subset V \) such that for any \( 1 \leq i, j \leq n \) it holds that \( v_j \in V_i \iff i \neq j \). Assume, for the sake of contradiction, that \( \{v_1, \ldots, v_n\} \) does not span \( V \) and let \( w \in V \) be such that \( w \notin \text{span}\{v_1, \ldots, v_n\} \). Thus, for any \( 1 \leq i \leq n \) we have \( w \notin V_i \), because otherwise we would have \( \dim(V) - 1 = \dim(V_i) = \dim(V_i + \text{span}\{v_i, w\}) - \dim(\text{span}\{v_i, w\}) \leq \text{div}(V) - 2 \), a contradiction. So \( \{v_1, \ldots, v_n\} \) is a basis for \( V \) and therefore \( n = \dim(V) \) and for each \( 1 \leq i \leq n \) we have \( V_i = \text{span}\{v_j \mid j \neq i\} \) as desired. \( \square \)

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