Connection between maximum-work and maximum-power thermal cycles

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Abstract

We propose a new connection between maximum-power Curzon-Ahlborn thermal cycles and maximum-work reversible cycles. This linkage is built through a mapping between the exponents of a class of heat transfer laws and the exponents of a family of heat capacities depending on temperature. This connection leads to the recovery of known results and to a wide and interesting set of new results for a class of thermal cycles. Among other results we find that it is possible to use analytically closed expressions for maximum-work efficiencies to calculate good approaches to maximum-power efficiencies.

PACS numbers: 05.70.Ln Non-equilibrium and irreversible thermodynamics; 05.70.-a Thermodynamics; 84.60.Bk Performance characteristics of energy conversion systems, figure of merit.

As it is well known, in 1975 [1] Curzon-Ahlborn (CA) found that an endoreversible Carnot-like thermal engine in which the isothermal branches of the cycle are not in thermal equilibrium with their corresponding heat reservoirs at absolute temperatures $T_1$ and $T_2 < T_1$ has an efficiency at the maximum-power (MP) regime given by

$$\eta_{CA} = 1 - \sqrt{\frac{T_2}{T_1}}. \quad (1)$$

This equation was obtained by using the so-called Newton law of cooling to model the heat exchanges between the heat reservoirs and the working fluid along the isothermal branches of the cycle. In fact, Eq. (1) was obtained previously by Novikov [2] in a context very close to finite-time thermodynamics (FTT). Later, within the context of FTT some authors [3–7] demonstrated that Eq. (1) is only valid for the Newtonian heat exchanges. As a matter of fact the CA-efficiency stems from taking into account constant conductances and a linear dependence between the heat fluxes and the working substance temperatures along the isothermal branches of the cycle [4–7]. Once the linearity in the heat transfer law is dropped, the square root term in the MP-efficiency ($\eta_{MP}$) is lost. Very recently, this fact has been widely confirmed by many authors working on the first order irreversible thermodynamics [8–10], microscopic [11] and mesoscopic [12] heat engines. On the other hand, Eq. (1) also was obtained for some reversible thermal cycles performing at maximum-work (MW) regime, such as the Otto and Joule-Brayton cycles [13]. These coincident results for the CA, Otto and Joule-Brayton cycles motivated Landsberg and Leff (LL) to propose that the CA-efficiency possesses a nearly universal behavior for a certain class of thermodynamic cycles operating at MW. This result was achieved by means of a generalized cycle which reduces to the Otto, Joule-Brayton, Diesel and some other known cycles [14]. Clearly, the kind of universality of the CA-efficiency claimed by LL is not of the class of the true universality of Carnot efficiency, $\eta_C$ [15]. The square root term (SRT) observed in the CA-efficiency can be found in other processes of energy conversion, such as the so-called water powered machine, which mixes two steady streams of hot and cold water to produce an output stream of warm water at maximum kinetic energy [16]. In fact, the role of the SRT of temperatures is more general and it appears also in some irreversible processes such as the irreversible cooling or heating of an ideal gas initially at temperature $T_i$ in contact with a series of auxiliary reservoirs to reach the final temperature $T_f$ of a main heat reservoir, the SRT appears when the generation of entropy of this process is minimized [17]. As it can be seen, the SRT is found in several thermal processes (reversible or irreversible) subject to some extremal conditions. A result less known is that corresponding to the way the square root is lost in the case of reversible cycles operating at MW regime. In 1989 [14] LL first studied a cycle formed by two adiabatic processes and two paths with constant heat capacities $C > 0$ of the working fluid (see Fig. 1). This reversible cycle operating under MW conditions has an efficiency given by $\eta_{CA} = 1 - \sqrt{\tau}$, where $\tau = T_+ / T_-$ is the ratio between the minimum and maximum temperatures of the cycle (see Fig. 1). Actually, the first author in finding this expression for a MW engine was Chambadal [18]. LL [14] generalized the model of Fig. 1, to encompass a family of symmetric and asymmetric reversible cycles which have a MW efficiency that do not deviate from $\eta_{CA}$ more than 14%. This behavior was referred to as a near universality property of $\eta_{CA}$. However, for the case of reversible cycles performing at MW, we will demonstrate that the CA-efficiency is lost when constant heat capacities are not used, in a similar way as it occurs in FTT, where the SRT in the CA-efficiency is related only to constant conductances and to a linear law of heat transfer.

We start with a heat capacity of the form $C = aT^n$ where $a$ is a constant and $n$ a real number. Following the cycle
depicted in Fig. 1, we have after integrating the heat capacity over the temperature,

\[ Q_{in} = \begin{cases} 
    a \ln \left( \frac{T_{+}}{T_{n+1}} \right) 
    & n = -1 \\
    \frac{aT_{n+1}}{n+1} \left[ 1 - \left( \frac{T_{+}}{T_{n+1}} \right)^{n+1} \right] 
    & n \neq -1 
\end{cases} 
\]  

and

\[ Q_{out} = \begin{cases} 
    b \ln \left( \frac{T_{-}}{T_{n+1}} \right) 
    & n = -1 \\
    \frac{bT_{n+1}}{n+1} \left[ 1 - \left( \frac{T_{-}}{T_{n+1}} \right)^{n+1} \right] 
    & n \neq -1 
\end{cases} 
\]  

where \( b \) is also a constant and might be different from \( a \). This case represents a more general scenario for cyclic processes following Fig. 1. The adjustable temperatures \( T \) and \( T' \) are coupled because the fluid’s entropy change per reversible cycle is zero, i.e.

\[ \Delta S = \int_{T}^{T_{+}} \frac{C_{d}dT}{T} + \int_{T}^{T_{-}} \frac{C_{d}dT}{T} = 0, \]  

which leads to

\[ T' = \begin{cases} 
    T_{+} \left( \frac{T_{-}}{T_{+}} \right)^{\gamma} 
    & n = 0 \\
    \left[ T_{+}^{n} + \gamma (T_{-}^{n} - T_{+}^{n}) \right]^{\frac{1}{n}} 
    & n \neq 0 
\end{cases} \]  

where \( \gamma = \frac{b}{a} \). Because the change in the total internal energy is zero, the work done per cycle \( W = |Q_{in}| - |Q_{out}| \) satisfies

\[ W = \begin{cases} 
    aT_{+} \left[ 1 - \left( \frac{T_{+}}{T_{-}} \right)^{\gamma} \right] + b(T_{-} - T) 
    & n = 0 \\
    a \ln \left[ 1 + \gamma T_{+} \left( T_{-}^{-1} - T_{+}^{-1} \right) \right] 
    & n = -1 \\
    b \ln \left( \frac{T_{+}}{T_{-}} \right) 
    & n \neq -1 \\
    \frac{aT_{n+1}}{n+1} \left\{ 1 - \left[ 1 + \gamma \left( \frac{T_{n}^{n} - T_{+}^{n}}{T_{n+1}^{n+1}} \right) \right]^{\frac{n+1}{n}} \right\} 
    & n \neq 0, -1 \\
    + \frac{bT_{n+1}^{n+1}}{n+1} \left( 1 - \frac{T_{n}^{n+1}}{T_{n+1}^{n+1}} \right) 
\end{cases} \]  

By maximizing \( W \) with respect to \( T \) we find that \( T^{*} \) and \( T'^{*} \) are given by

\[ T^{*} = T'^{*} = \begin{cases} 
    \left( \frac{T_{+}^{n} + T_{-}^{n}}{1 + \gamma T_{+} \left( T_{-}^{n} - T_{+}^{n} \right)} \right)^{\frac{1}{n}} 
    & n = 0 \\
    \left( \frac{T_{+}^{n} + T_{-}^{n}}{1 + \gamma T_{+} \left( T_{-}^{n} - T_{+}^{n} \right)} \right)^{\frac{1}{n}} 
    & n \neq 0, -1 
\end{cases} \]  

Then, as a result, the named symmetric cycles (\( \gamma = 1 \)) and asymmetric cycles (\( \gamma \neq 1 \)) fulfill that \( T^{*} = T'^{*} \). Therefore, from these expressions we immediately find the efficiency \( \eta = W/Q_{in} \) under MW conditions; that is,

**Figure 1:** \( T - S \) diagram of a cycle formed with two adiabats and two other processes with \( C > 0 \).

**Figure 2:** \( \eta_{MW} \) vs \( n \) with \( \gamma = 1 \) and \( \tau = 2/5 \). The cases \( n = 0, -1/2 \) reproduce the well-known CA efficiency. Notice that there is a maximum at \( n = -1/4 \).

Clearly, Eq. 5 only reproduces the \( \eta_{CA} \)-efficiency for a few combinations of \( \gamma \) and \( n \). In Fig. 2 we depict the plot of \( \eta_{MW} \) versus the exponent \( n \) considering \( \gamma = 1 \) (symmetric scenario). In this figure we observe that \( \eta_{CA} \) is obtained only for two values of \( n \), the known case \( n = 0 \) (constant heat capacity) and also at \( n = -1/2 \), which is a novel case with MW-efficiency given by \( \eta_{CA} \). Another relevant case is that with \( n = -1/4 \), where the MW-efficiency takes its maximum value. By fixing the value of \( \tau \) it is possible to find out (Fig. 3) that the CA-efficiency does not have a special behavior with respect to other efficiencies, in the sense that it does not represent the maximum value for an efficiency at MW. In Fig. 3, we can observe how the case \( n = 0 \) only reaches the value \( \eta_{CA} \) for \( \gamma = 1 \); that is, the case of symmetric cycles with constant heat capacities performing at MW [13, 14]. Besides, in this figure we can see how the \( \eta_{CA} \) value for MW-efficiency is also obtained for asymmetric cases (\( \gamma \neq 1 \)) and \( n \neq 0 \). It is quite remarkable that the unique case independent of \( \gamma \) is that with \( n = -1/2 \); that is \( \eta_{MW} (n = -1/2) = \eta_{CA} \) for any value of \( \gamma \). In any case, for a cycle as shown in Fig. 1 with \( C = aT^{-1/2} \) in the
corresponding MP (dotted-line) efficiencies. The depicted efficiencies are for symmetric cases $\gamma = 1$: a) $n = -2$ in Eqs. (2) and (3) and $k = -1$ in Eqs. (11) and (12); b) $n = -1/2$ and $k = 1/2$; c) $n = 0$ and $k = 1$.

process 1 $\rightarrow$ 2 and $C = kT^{-1/2}$ in the process 3 $\rightarrow$ 4, the true cycle is characterized by the $\eta_{CA}$ at MW for any value of $\gamma$. However, within the context of FTT, the cycle with MP-efficiency given by $\eta_{CA}$ and independent of $\gamma' = \beta/\alpha$ (being $\alpha$ and $\beta$ heat conductances, see Eqs. (11) and (12)) is indeed the Curzon and Ahlborn cycle. Through their papers on reversible cycles performing at MW, LL [13, 14] established a bridge with FTT-MP-cycles, basically by means of the CA-efficiency. In recent years several authors [8, 10, 11, 12, 23] have renewed the interest on the CA-efficiency. The discussion on this famous formula has been mainly turned on to its possible universal nature within the context of finite-time cycles. Practically since the beginning of FTT as an active discipline, the limited validity of the CA-efficiency was established [8, 15]. Nonetheless, at the end of the 1980’s the CA-efficiency was found linked to MW reversible cycles [13, 12] and recently with its role in microscopic [11], mesoscopic [12] and macroscopic thermal cycles [8, 9], and therefore the CA-efficiency has gained new insights, that have to do with a possible universality at least at the first few terms in the Taylor expansion of the efficiency as a function of $\eta_{C} = 1 - \tau$ [12, 19, 21, 23]. Clearly, Eq. (8) admits this treatment, leading to a series in terms of $\eta_{C}$ as functions of $\gamma$ and $n$ at any level of approximation. The series for the efficiencies in Eq (8) are given by the following expressions.

$$\eta^* = \left\{\begin{array}{ll}
\frac{\eta_C^2}{2} + \frac{1}{16} + \frac{\eta_C^2}{6} + O[\eta_C]^4 & n = 0 \\
\frac{\eta_C^2}{2} + \frac{1}{8} + \frac{\eta_C^2}{6} + O[\eta_C]^4 & n = -1 \\
\frac{\eta_C^2}{2} + \frac{1}{2} + \frac{\eta_C^2}{6} + O[\eta_C]^4 & n \neq 0, -1
\end{array}\right.$$

(9)

The linear term is really the same in every case, that strengthen the idea that this is a characteristic for cycles operating in the maximum work regime. When we restrict $\gamma$ to the symmetric case, $\gamma = 1$, then the efficiencies are

$$\eta^* = \left\{\begin{array}{ll}
\frac{\eta_C^2}{2} + \frac{1}{16} + \frac{\eta_C^2}{6} + O[\eta_C]^4 & n = 0 \\
\frac{\eta_C^2}{2} + \frac{1}{8} + \frac{\eta_C^2}{6} + O[\eta_C]^4 & n = -1 \\
\frac{\eta_C^2}{2} + \frac{1}{2} + \frac{\eta_C^2}{6} + O[\eta_C]^4 & n \neq 0, -1
\end{array}\right.$$

(10)

In this case the coincidence extends up to quadratic order in $\eta_{C}$, such as it occurs for MP strong coupling models that possess a left-right symmetry [21, 28]. Following the spirit of the articles of LL [13, 14], we may wonder: Could it be possible to link the results obtained with heat capacities depending on temperature given by Eq. (8) with finite-time cycles of the CA-type? We suggest that this connection is indeed possible. As it is well known, in reversible cycles of the Otto and Joule-Brayton type, $Q_{in}$ and $Q_{out}$ correspond to quasi-static processes accomplished by means of an infinite set of auxiliary heat reservoirs that lead the working substance temperature from $T$ to $T_{+}$ and from $T'$ to $T_{-}$, respectively. These heat quantities are calculated by means of integrals that lead to Eqs. (2) and (3). In the case of FTT-cycles as the CA-engine, $Q_{in/out}$ per cycle are given by irreversible heat transfer laws depending on the conductances ($\alpha$, $\beta$) and the temperatures of the corresponding heat reservoirs and the working substance. As asserted by Wang and Tu [22], for the CA-cycle, along both “isothermal” branches, the effective temperature of the working substance can vary. Thus, we can propose that in a $T-S$ plane, the CA-cycle follows a topologically equivalent diagram as that of Fig. 1. For FTT-cycles, heat transfer laws of the form

$$\dot{Q}_{in} = \alpha T_{1}^{k} \left[ 1 - \left( \frac{T_{1W}}{T_{1}} \right)^{k} \right],$$

(11)

$$\dot{Q}_{out} = \beta T_{2}^{k} \left[ \left( \frac{T_{2W}}{T_{2}} \right)^{k} - 1 \right].$$

(12)
are typically used, where $T_{1W/2W}$ are the working substance temperatures, $T_{1/2}$ are the heat reservoir temperatures and $k$ is a real number. Although evidently the conceptual meaning of the heat quantities ($Q_{in/out}$) is different within the framework of reversible cycles and FTT cycles respectively, it is quite remarkable how their corresponding efficiencies have a good agreement for not arbitrary couples of $n$ and $k$ values. As is well known, $Q_{in/out}$, power output and MP-efficiency for CA-engines with heat transfer laws given by Eqs. (11) and (12), are numerically calculated in an easy and direct manner by maximizing the power output $P$ given by the following equation:

$$
P(\eta) = \frac{\alpha T_1 \gamma \eta \left[(1 - \eta)^k - \tau^k\right]}{1 - \eta + \gamma (1 - \eta)^k}.
$$

(13)

with respect to the efficiency $\eta$. In Fig. 4, we can observe the excellent fitting between three MW-efficiencies and their corresponding FTT-MP-cases. Clearly, the mapping between $n$ and $k$ is given by $k \to n + 1$, which stems from associating the exponents of Eqs. (2) and (3) with those of Eqs. (11) and (12).

The fitting between $\eta_{MW}$ and $\eta_{MP}$ goes from excellent (symmetric cases) to very good (asymmetric cases) (see Figs. 4 and 5), in such a way that the analytical closed expressions of $\eta_{MW}$ given by Eq. (8) can be used as reliable first approximations for the FTT corresponding cycles, which commonly have to be calculated by means of numerical methods.

Another remarkable fact is that the MW-efficiency when $n = -2$ and the corresponding MP-efficiency at $k = -1$ are exactly the same for any value of $\gamma$ (from now on $\gamma = \gamma'$), that is,

$$\eta_{MW} (n = -2) = \eta_{MP} (k = -1) = \frac{\eta_C}{2 - \left(1 - \sqrt{\frac{2 + \gamma}{1 + \gamma}}\right)}.
$$

(14)

Which has two limits: $\gamma \to 0$ and $\gamma \to \infty$ bounding the possible values of the efficiency for a given $\tau$, at $n = -2$ ($k = -1$)

$$\lim_{\gamma \to \infty} (\eta_{MW}) = \frac{\eta_C}{2} < \eta_{MW/MP} < \lim_{\gamma \to 0} (\eta_{MW}) = \frac{\eta_C}{2 - \eta_C}.
$$

(15)

Recently, some authors have underscored the importance of these limits (first found in [5]), which have been reported within different contexts, such as, a stochastic heat engine [11], a low-dissipation Carnot engine [20] and for a linear irreversible Carnot-like heat engine [22]. However, we shall see below that these limits are only of a particular validity among a numerous set of limits for different values of $k$ (or $n$). On the basis of Eq. (8) we obtain the limits of $\eta_{MW}$ for $\gamma \to 0$ and $\gamma \to \infty$ which bound the values of $\eta_{MW}$. These $\chi$-shaped curves (continuous curves) are depicted in Fig. 5 where the corresponding $\eta_{MP}$ curves (large dashed) are also showed along with the symmetric cases for both efficiencies. We have used well-known FTT numerical methods to plot the $\eta_{MP}$ curves [24–26].

Some interesting facts can be remarked from this figure: as mentioned before, for $n = -2$ ($k = -1$) both efficiencies $\eta_{MW}$ and $\eta_{MP}$ have the same limits when $\gamma \to 0$ and $\gamma \to \infty$. Interestingly, the $\chi$-shaped curve corresponding to the $\eta_{MW}$ efficiencies have an exact specular symmetry with respect to the value $n = -1/2$ ($k = 1/2$). At this point, the upper and lower limits of the efficiency are the same because $\eta_{MW} (n = -1/2)$ does not depend on $\gamma$. This specular symmetry has as a consequence that both limits given by Eq. (15) are also found in the MW-efficiency for $n = 1$ ($k = 2$, see Fig. 5). On the other hand, the $\chi$-shaped curve for the $\eta_{MP}$ efficiencies does not have a specular symmetry with respect to the crossing point at $k = 1$, where both limits ($\gamma \to 0$ and $\gamma \to \infty$) are the same because for $k = 1$ $\eta_{MP} = \eta_{CA}$ is independent of $\gamma$. This lack of symmetry precludes that the limits given by Eq. (15) appear for any other $k \neq -1$. It should be noted that at the left side and at the right side of the crossing points over the MW and MP $\chi$-shaped curves, the lower and upper limits are interchanged. For the specular symmetric MW case, when exchanging $n \to -(n + 1)$ ($k \to -(k - 1)$), both limits have the same value, but they are inverted ($\gamma \to 0$ is replaced by $\gamma \to \infty$ and vice versa). There is another fact of great interest about the $\chi$-shaped curves. For heat transfer laws with approximately $k \in (-10, 10)$, we can observe that in some regions $\eta_{MW} < \eta_{MP}$. This inequality is not an artifact of numerical solutions for $\eta_{MP}$, because in that region exist some cases where the inequality is an exact analytical result, for example, for $k = 1/2$ ($n = -1/2$). However, clearly, if in addition to the heat fluxes many other irreversibilities are considered, the above mentioned inequality should be inverted. For the Stefan-Boltzmann case, that is, $k = 4$ ($n = 3$); the upper and lower limits, both for $\eta_{MW}$ and $\eta_{MP}$ are not the limits given by Eq. (15) and additionally they are inverted, in such a way that for $\gamma' \to \infty$ a Müser-type engine ($\beta \approx \alpha$) is obtained [27] and $\eta_{MW} > \eta_{MP}$. In Fig. 5 a large range of values of $k$ (or $n$) is considered, showing that at the same limits $\gamma \to 0$, $\gamma \to \infty$ and $\gamma \to 1$ the values of the MP and MW–efficiencies are very close to each other, strengthening the idea that the analytical forms of the MW–efficiencies are a good approximation to the corresponding MP efficiencies. In addition, for all symmetric cases $\eta_{MW} \geq \eta_{MP}$ and both convex curves ($\gamma = 1$) tend to zero when $|n|, |k| \to \infty$. It is noteworthy that the superior branches of the $\chi$–shaped curves for both $\eta_{MW}$ and $\eta_{MP}$ tend asymptotically to $\eta_{C}$ and both inferior branches tend to zero when $|n|, |k| \to \infty$. These results are consequence of Eq. (8) and the numerical solutions of FTT–MP–efficiencies. The physical consistency
of all these asymptotic limits stems from the restrictions imposed by Eqs. and the first and second laws of thermodynamics.

In summary, in this article we have demonstrated that there exists a strong and rich relationship between a class of reversible MW–cycles and finite–time MP–cycles of the CA–type. This connection opens a wide spectrum of interesting results containing known facts and some new findings. All of our results suggest that behind the remarkable agreement of the mentioned connection there is a kind of extended endoreversibility contained in Figs. (1) and (5) for very low-dissipation cycles, as those of very large compression ratios.

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