ON POWER BOUNDED OPERATORS THAT ARE QUASIAFFINE TRANSFORMS OF SINGULAR UNITARIES

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Abstract. In [9] a question is raised: if a power bounded operator is quasisimilar to a singular unitary operator, is it similar to this unitary operator? For polynomially bounded operators, a positive answer to this question is known [1], [13]. In this paper a positive answer is given in some particular cases, but in general an answer remains unknown.

1. Introduction

Let $\mathcal{H}$ be a (complex, separable) Hilbert space, and let $T$ be a (linear, bounded) operator acting on $\mathcal{H}$. An operator $T$ is called power bounded if

$$\sup_{n \geq 0} \|T^n\| < \infty.$$ 

Let $T$ and $R$ be operators on spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and let $X: H \to K$ be a linear bounded transformation such that $X$ intertwines $T$ and $R$, that is, $XT = RX$. If $X$ is unitary, then $T$ and $R$ are called unitarily equivalent, in notation: $T \cong R$. If $X$ is invertible, that is, the inverse $X^{-1}$ is bounded, then $T$ and $R$ are called similar, in notation: $T \approx R$. If $X$ is a quasiaffinity, that is, $\ker X = \{0\}$ and $\operatorname{clos} X \mathcal{H} = \mathcal{K}$, then $T$ is called a quasiaffine transform of $R$, in notation: $T \prec R$. If $T \prec R$ and $R \prec T$, then $T$ and $R$ are called quasisimilar, in notation: $T \sim R$. Recall that if $T$ and $R$ are unitary operators and $T \prec U$, then $T \sim U$ ([19, II.3.4]).

It is well known that if $T$ is a contraction, that is, $\|T\| \leq 1$, and $U$ is a singular unitary operator, then the relation $T \prec U$ implies that $T \cong U$.

It is known that if $T$ is a power bounded operator and $U$ is a unitary operator whose spectral measure is pure atomic, then the relation $T \prec U$ implies that $T \approx U$ ([1], [17], see also Example 2.6).

Also, if $T$ is polynomially bounded, that is, there exists a constant $M$ such that $\|p(T)\| \leq M \sup \{|p(z)|, |z| \leq 1\}$ for every polynomial $p$, and $U$ is singular unitary, then the relation $T \prec U$ implies that $T \approx U$ [1], [13]. We sketch the proof for the case, where the (closed) spectrum of $U$ has Lebesgue measure zero. Denote by $\mathbb{D}$ and by $\mathbb{T}$ the unit disk and the unit circle, respectively. For a polynomially bounded operator $T$ the functional calculus on the disk algebra $\mathcal{A}(\mathbb{D})$ is well defined. Let $E \subset \mathbb{T}$ be a closed set of zero Lebesgue measure. We denote by $C(E)$ the space of continuous functions on $E$. We set $\mathcal{I}(E) = \{f \in \mathcal{A}(\mathbb{D}) : f = 0$ on $E\}$ and $\mathcal{A}(E) = \mathcal{A}(\mathbb{D})/\mathcal{I}(E)$. It is known that the natural imbedding $\mathcal{A}(E) \to C(E)$ is an isometrical isomorphism [3, ch.6]. Applying this fact to a polynomially bounded operator $T$ such that $T \prec U$, where $U$ is unitary with spectrum

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E, we obtain that \(\|T^{-n}\| \leq M\) for \(n \in \mathbb{N}\) (where \(M\) is a constant from the condition on polynomially boundedness of \(T\)). Thus, we can apply [18] and conclude that \(T \approx U\).

If we suppose only that \(T\) is power bounded, then it seems natural to consider the functional calculus on the subalgebra

\[ A^+(\mathbb{T}) = \{f(z) = \sum_{n \geq 0} \hat{f}(n) z^n, \quad \|f\| = \sum_{n \geq 0} |\hat{f}(n)| < \infty\} \]

of the Wiener algebra

\[ A(\mathbb{T}) = \{f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n, \quad \|f\| = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty\}. \]

Let \(E \subset \mathbb{T}\) be a closed set of zero Lebesgue measure. As before, we set

\[ I^+(E) = \{f \in A^+(\mathbb{T}) : f = 0 \text{ on } E\}, \quad A^+(E) = A^+(\mathbb{T})/I^+(E), \]

and we consider the natural imbedding \(A^+(E) \to C(E)\). But there exist many closed sets \(E\) of zero Lebesgue measure for which this imbedding is not an isomorphism [2], [6], [8], [11]. Let us consider \(AA^+\)-sets [6], [7]. Namely, for a closed set \(E \subset \mathbb{T}\) of zero Lebesgue measure, we set

\[ I(E) = \{f \in A(\mathbb{T}) : f = 0 \text{ on } E\}, \quad A(E) = A(\mathbb{T})/I(E), \]

and consider the natural imbedding

\[ (1.1) \quad A^+(E) \to A(E). \]

If the imbedding (1.1) is onto, then the set \(E\) is called an \(AA^+\)-set. For an \(AA^+\)-set \(E\) the imbedding (1.1) is invertible; we will denote by \(K(E)\) the norm of its inverse.

**Lemma 1.1.** Suppose that \(T\) is a power bounded operator, \(U\) is a unitary operator, \(T \prec U\) and the spectrum \(E\) of \(U\) is an \(AA^+\)-set. Then \(T \approx U\) and

\[ (1.2) \quad \sup_{n \geq 0} \|T^n\| \leq K(E) \sup_{n \geq 0} \|T^n\|, \]

where \(K(E)\) is the norm of the inverse to the imbedding (1.1).

**Proof.** We set \(M = \sup_{n \geq 0} \|T^n\|\). If \(f \in A^+(\mathbb{T})\), then \(f(T)\) is well defined, and \(\|f(T)\| \leq M\|f\|_{A^+(\mathbb{T})}\). Since \(T \prec U\), we have \(f(T) \prec f(U)\). If \(f \in I^+(E)\), then \(f(U) = 0\), and from the relation \(f(T) \prec f(U)\) we conclude that \(f(T) = 0\). Thus, the functional calculus for \(T\) on the algebra \(A^+(E)\) is well defined. Let \(k \in \mathbb{N}\) be fixed. Since \(E\) is an \(AA^+\)-set, there exists a function \(f_k \in A^+(\mathbb{T})\) such that \(\hat{f}_k(\zeta) = \zeta^{-k}\) for every \(\zeta \in E\). Clearly, \(f_k(U) = U^{-k}\). We have \(f_k(T)T^k = T^k f_k(T) \prec U^k f_k(U) = I\), and we conclude that \(f_k(T)T^k = T^k f_k(T) = I\). Thus, \(T^k\) is invertible, and \(T^{-k} = f_k(T)\). Furthermore,

\[ \|T^{-k}\| = \|f_k(T)\| \leq M\|f_k\|_{A^+(E)} \leq MK(E)\|\zeta^{-k}\|_{A(E)} \leq MK(E). \]

\(\square\)

Again, there exist many closed sets of zero Lebesgue measure that are not \(AA^+\)-sets. Moreover, for an \(AA^+\)-set \(E\), the norm \(K(E)\) of the inverse to imbedding (1.1) can be arbitrary large [5], [7]. A question arises if estimate (1.2) exact. It will be shown in this paper that, for sets satisfying some
metric condition (see Definition 3.1), this estimate is not exact. Namely, for every $K > 0$ there exists an $AA^+$-set $E$ such that $K(E) \geq K$ [5], [7], but the estimate of the left part of (1.2) depends only on $\sup_{n \geq 0} \| T^n \|$ (Theorem 3.5).

The paper is organized as follows. In Section 2, some general propositions on power bounded operators that are quasisimilar to singular unitaries are proved. In Section 3, these propositions are applied to unitary operators whose spectral measure is supported on the sets satisfying some metric condition.

In the rest of Introduction, the notation and definitions are given.

By $I$ the identity operator is denoted; if it is needed, the space on which it acts will be mentioned in the lower index. Let $T$ be a power bounded operator on a Hilbert space $\mathcal{H}$. $T$ is of class $C_1$, if $\inf_{n \geq 0} \| T^n x \| > 0$ for every $x \in \mathcal{H}$, $x \neq 0$. $T$ is of class $C_0$, if $\lim_{n \to \infty} \| T^n x \| = 0$ for every $x \in \mathcal{H}$. By $C_1$ and $C_0$ the classes of power bounded operators $T$ such that $T^*_n \in C_1$ and $T^*_n \in C_0$, respectively, are denoted. As usually, $C_{11} = C_1 \cap C_1$, $C_{10} = C_1 \cap C_0$, and so on. By $\text{Lat} T$, $\text{Hyplat} T$, and $\text{Hyplat}_1 T$ the invariant subspace lattice, the hyperinvariant subspace lattice, and the lattice of hyperinvariant subspaces for power bounded operator $T$ such that the restrictions of $T$ on these subspaces belong to the class $C_{11}$, respectively, are denoted (see [9]).

Let $U$ be a unitary operator on a Hilbert space $\mathcal{H}$. There exists a finite positive Borel measure $\mu$ on the unit circle $\mathbb{T}$ having the following property: for every $x, y \in \mathcal{H}$ there exists a function $f_{x,y} \in L^1(\mu)$ such that $(U^n x, y) = \int_{\mathbb{T}} \zeta^n f_{x,y}(\zeta) d\mu(\zeta)$ for every $n \in \mathbb{Z}$ (of course, $\int_{\mathbb{T}} f_{x,y} d\mu = (E(\sigma x), y)$, where $E$ is the (operator valued) spectral measure of $U$, and $\sigma \subset \mathbb{T}$ is a Borel set). The measure $\mu$ is called the scalar spectral measure of $U$. In other words, the (operator valued) spectral measure and the scalar spectral measure of a unitary operator are mutually absolutely continuous. A unitary operator is called singular, if its spectral measure is singular with respect to Lebesgue measure (arc length measure) on $\mathbb{T}$. Recall that every invariant subspace of a singular unitary operator is reducing, that is, its orthogonal complement is also invariant, therefore, the restriction of a singular unitary operator on its invariant subspace is a unitary operator. Let a sequence $\{n_k\}_k$ and a function $\varphi \in L^\infty(\mu)$ be such that $\zeta^{n_k} \to_k \varphi$ in the weak-star topology on $L^\infty(\mu)$. Then $U^{n_k} \to_k \varphi(U)$ in the weak operator topology.

Let $T$ be an operator on a Hilbert space. Then $T$ is similar to a unitary operator if and only if $\sup_{n \in \mathbb{Z}} \| T^n \| < \infty$ ([18], see also [19]). We will use this fact in what follows without additional references.

To conclude Introduction, we note that in [14] an example of a power bounded operator which is quasisimilar to a unitary operator and is not similar to a contraction is constructed, but the unitary operator from this example is the bilateral shift of infinite multiplicity.

2. Some general results

In this section, we prove the main result of the paper (Theorem 2.11), which claims that countable orthogonal sum of “good” (in the sense of the present paper) unitary operators rests “good”.

The following simple lemmas are given for convenience of references.
Lemma 2.1. Let $T : \mathcal{H} \to \mathcal{H}$ be a power bounded operator. We set $M = \sup_{n \geq 0} \|T^n\|$. Then

$$\limsup_n \|T^n x\| \leq M \liminf_n \|T^n x\| \quad \text{for every } x \in \mathcal{H}. $$

Proof. Let $x \in \mathcal{H}$ be fixed, and let the sequences $\{n_k\}_k$, $\{\ell_j\}_j$ be such that

$$\limsup_n \|T^n x\| = \lim_j \|T^{\ell_j} x\|, \quad \liminf_n \|T^n x\| = \lim_k \|T^{n_k} x\|. $$

There exists a sequence $\{j_k\}_k$ such that $\ell_{j_k} > n_k$, $k = 1, 2, \ldots$. We have

$$\limsup_k \|T^{n_k} x\| = \lim_k \|T^{\ell_{j_k}} x\| = \lim_k \|T^{\ell_{j_k} - n_k} T^{n_k} x\|$$

$$\leq M \lim_k \|T^{n_k} x\| = M \liminf_n \|T^n x\|. $$

□

Lemma 2.2. 1) Let $U$ be a unitary operator having the following property:

(*) if $T$ is a power bounded operator such that $T \prec U$, then $T \approx U$.

Let $\mathcal{E} \subseteq \text{lat} U$. Then $U|_{\mathcal{E}}$ has property (*).

2) Let $U$ be a unitary operator having the following property:

(**) if $T$ is a power bounded operator such that $T \sim U$, then $T \approx U$.

Let $\mathcal{E} \subseteq \text{lat} U$. Then $U|_{\mathcal{E}}$ has property (**).

Proof. Let $U$ be a unitary operator having property (*). Then $U$ is singular. Indeed, if $U$ has an absolutely continuous part, then one can find a contraction $T$ such that $T \sim U$, but $T$ is not similar to $U$ [19, II.3.5, VI.3.5, IX.1.2]. Since $U$ is singular, every invariant subspace $\mathcal{E}$ of $U$ is reducing, that is, $\mathcal{E}^\perp$ is also invariant.

Let $U$ has property (*), let $\mathcal{E} \subseteq \text{lat} U$, and let $T_1$ be a power bounded operator such that $T_1 \prec U|_{\mathcal{E}}$. We set $T = T_1 \oplus U|_{\mathcal{E}^\perp}$. Clearly, $T \prec U|_{\mathcal{E}^\perp} \oplus U|_{\mathcal{E}} \approx U$, therefore, $T \approx U$. Since $T_1$ is the restriction of $T$ on its invariant subspace, $T_1$ is similar to the restriction of $U$ on some of its invariant subspace, thus, $T_1$ is similar to some unitary operator. Since $T_1 \prec U|_{\mathcal{E}}$, we have $T_1 \approx U|_{\mathcal{E}} [19, II.3.4]$, see Introduction of the paper. Part 1 of the lemma is proved. The proof of part 2 is analogous. □

Lemma 2.3. Suppose that $R, A : \mathcal{K} \to \mathcal{K}$ are operators and $\{n_k\}_k$ is a sequence such that $R^{n_k} \to_k A$ in the weak operator topology. Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a power bounded operator, and $X : \mathcal{H} \to \mathcal{K}$ is a quasiaffinity such that $XT = RX$. Then there exists an operator $B : \mathcal{H} \to \mathcal{H}$ such that $T^{n_k} \to_k B$ in the weak operator topology, $XB = AX$, and $\|B\| \leq \sup_{n \geq 0} \|T^n\|$.

Proof. We set $M = \sup_{n \geq 0} \|T^n\|$. Let $x \in \mathcal{H}$ be fixed. If $g \in \mathcal{K}$, then

$$(T^{n_k} x, X^* g) = (XT^{n_k} x, g) = (R^{n_k} X x, g) \to_k (AX x, g).$$

Since

$$(2.2) \quad \sup_{k \geq 0} \|T^{n_k} x\| \leq M \|x\| < \infty$$

and $X^* \mathcal{K}$ is dense in $\mathcal{H}$, we have that $\lim_k (T^{n_k} x, y)$ exists for every $y \in \mathcal{H}$. By the Banach–Steinhaus theorem, there exists $h \in \mathcal{H}$ such that

$$\lim_k (T^{n_k} x, y) = (h, y) \quad \text{for every } y \in \mathcal{H}. $$
We set $Bx = h$. From (2.2) we conclude that $\|Bx\| \leq M\|x\|$. It is clear from the definition of $B$ and from (2.1) that $T^n_k \rightarrow_k B$ in the weak operator topology and $XB = AX$. 

The following lemma shows that, for some unitary operators, the assumption on quasisimilarity in the question regarded in the paper can be replaced by the assumption that a power bounded operator is a quasisi form of this unitary. We note that unitary operators satisfying the conditions of Theorems 2.5, 2.12, and 3.5 satisfy the conditions of Lemma 2.4, too.

**Lemma 2.4.** Let $U : \mathcal{K} \rightarrow \mathcal{K}$ be a unitary operator having the following properties:

1) if $x \in \mathcal{K}$ is such that $U^n x \rightarrow_n 0$ in the weak topology, then $x = 0$;
2) if $T$ is a power bounded operator such that $T \sim U$, then $T \approx U$.

Then if $T$ is a power bounded operator such that $T \prec U$, then $T \approx U$.

**Proof.** Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a power bounded operator. By [9, Lemma 1], there exists $H_1 \in \text{Lat } T$ having the following properties: $T |_{H_1} \in C_1$, $P_{H_0 H_1} T |_{H_0 H_1} \in C_0$, and if $E \in \text{Lat } T$ is such that $|E| \in C_1$, then $E \subset H_1$ (by $P_{H_0 H_1}$, the orthogonal projection on the space $H \cap H_1$ is denoted; the definitions of classes $C_1$ and $C_0$ are recalled in Introduction). The space $H_1$ is called the $C_1$-subspace of $H$ with respect to $T$.

Now let $X : \mathcal{H} \rightarrow \mathcal{K}$ be a quasiaffinity such that $XT = UX$. We set $T_1 = T |_{H_1}$ and $K_1 = \text{clos } X H_1$. Clearly, $U |_{K_1}$ is unitary and $T_1 \prec U |_{K_1}$. By the definition of the space $H_1$, we have $T_1 \in C_1$, therefore, $T_1$ is quasisimilar to a unitary operator [9]. From the relation $T_1 \prec U |_{K_1}$ and [19, II.3.4] (see Introduction) we conclude that $T_1 \sim U |_{K_1}$. By assumption 2) and Lemma 2.2, $T_1 \approx U |_{K_1}$.

Since $T_1 = T |_{H_1}$ is similar to a unitary operator, there exists $H_0 \in \text{Lat } T$ such that $H = H_1 \cap H_0$ (see [10]; although the proposition in [10] is formulated for contractions, application of results from [9] allows to repeat the proof for power bounded operators). We set $T_0 = T |_{H_0}$. We will show that $T_0 \in C_0$. Let $H_0$ be the $C_1$-subspace of $H_0$ with respect to $T_0$. We set $H_1 = H_1 \cap H_0$. We have $H_1 \in \text{Lat } T$ and $T |_{H_1} \in C_1$. Since $H_1$ is the $C_1$-subspace of $H$ with respect to $T$, we have $H_1 \subset H_0$. Thus, we conclude that $H_0 = \{0\}$, that is, $T_0 \in C_0$.

We set $K_0 = \text{clos } X H_0$, and we apply [14, Lemma 3.2] to the operators $T_0$ and $U |_{K_0}$. Namely, from the relations $T_0 \prec U |_{K_0}$ and $T_0 \in C_0$ we obtain that $(U |_{K_0})^n \rightarrow_n 0$ in the weak operator topology. But it means that $U^n x \rightarrow_n 0$ in the weak topology for every $x \in K_0$. By assumption 1) of the lemma, $K_0 = \{0\}$, therefore, $H_0 = \{0\}$.

Thus, $H_1 = H$, $K_1 = K$, and $T_1 = T \approx U$. 

The following theorem is a modification of [1, Theorem 2].

**Theorem 2.5.** Suppose that $U$ is a unitary operator, $A$ is an invertible operator, and $\{n_k\}_k$ is a sequence such that $U^{n_k} \rightarrow_k A$ in the weak operator topology. Let $T$ be a power bounded operator such that $T \sim U$. We set $M = \sup_{n \geq 0} \|T^n\|$. Then $T \approx U$ and $\sup_{n \leq 0} \|T^n\| \leq \|A^{-1}\| M^2$.

**Proof.** Since $T \sim U$, we have that $U$ is unitarily equivalent to the isometric asymptote of $T$. Denote by $\mathcal{H}$ and $\mathcal{K}$ the spaces on which $T$ and $U$ act, and
by \(X : \mathcal{H} \to \mathcal{K}\) the canonical intertwining mapping. It is easy to see from the definition of \(X\) and properties of a Banach limit that
\[
\liminf_n \|T^n x\| \leq \|X x\| \leq \limsup_n \|T^n x\| \quad \text{for every } x \in \mathcal{H}
\]
(see [9]). Clearly, \(\|X\| \leq M\). Of course, \(U, A, T\) and \(X\) satisfy the conditions of Lemma 2.3, therefore, there exists an operator \(B : \mathcal{H} \to \mathcal{H}\) such that \(T^n_k \to_k B\) in the weak operator topology, \(X B = A X\), and \(\|B\| \leq M\). Since \(A\) is invertible, we have \(X = A^{-1} X B\), therefore,
\[
\|X x\| \leq \|A^{-1}\| \|M\| \|B x\| \quad \text{for every } x \in \mathcal{H}.
\]
Clearly, \(T B = B T\).

Let \(\ell \in \mathbb{N}\) be fixed. Let \(x \in \mathcal{H}\). Since \(T^n_k x \to_k B x\) weakly,
\[
\|B x\| \leq \liminf_k \|T^n_k x\| = \liminf_k \|T^n_k - \ell T^\ell x\| \leq \limsup_n \|T^n T^\ell x\|
\]
\[
\leq M \liminf_n \|T^n T^\ell x\| \leq M \|X T^\ell x\| \leq \|A^{-1}\| M^2 \|B T^\ell x\| = \|A^{-1}\| M^2 \|T^\ell B x\|
\]
(we apply Lemma 2.1, (2.3), and (2.4)). We obtain that
\[
\|B x\| \leq \|A^{-1}\| M^2 \|T^\ell B x\| \quad \text{for every } x \in \mathcal{H} \text{ and for every } \ell \in \mathbb{N}.
\]

Since \(T \sim U\), there exists a quasi-affinity \(Y : \mathcal{K} \to \mathcal{H}\) such that \(Y U = T Y\).
It is easy to see from this intertwining relation and the conditions \(U^n_k \to_k A\) and \(T^n_k \to_k B\) in the weak operator topology that \(Y A = B Y\). We have \(\mathcal{H} = \text{clos} Y A \mathcal{K} = \text{clos} B Y \mathcal{K}\), thus, \(\text{clos} B \mathcal{H} = \mathcal{H}\); from this equality and (2.5) we conclude that
\[
\|x\| \leq \|A^{-1}\| M^2 \|T^\ell x\| \quad \text{for every } x \in \mathcal{H} \text{ and for every } \ell \in \mathbb{N}.
\]
From (2.6) we conclude that \(T\) is left invertible. The left invertibility of \(T\) and the relation \(T \sim U\) imply that \(T\) is invertible. From (2.6) we have that
\[
\|T^{-\ell}\| \leq \|A^{-1}\| M^2 \quad \text{for every } \ell \in \mathbb{N}.
\]
By [18], \(T\) is similar to a unitary. Since \(T \sim U\), this unitary is unitarily equivalent to \(U\).

Now we mention some known examples of unitary operators that satisfy the conditions of Theorem 2.5.

Example 2.6. [1], [17]. Let the spectral measure of a unitary operator \(U\) be pure atomic, that is, there exist countable families of Hilbert spaces \(\{\mathcal{H}_j\}_j\) and of points \(\{\lambda_j\}_j \subset \mathbb{T}\) such that \(U \cong \oplus_j \lambda_j I_{\mathcal{H}_j}\). Using the well-known diagonal process, one can find a sequence \(\{n_k\}_k\) and a family of points \(\{\xi_j\}_j \subset \mathbb{T}\) such that \(\lambda_j^{n_k} \to_k \xi_j\) for every \(j\). We set \(A = \oplus_j \xi_j I_{\mathcal{H}_j}\), then \(U^{n_k} \to_k A\) for every \(x \in \oplus_j \mathcal{H}_j\). Consequently, \(U^{n_k} \to_k A\) in the weak operator topology. Clearly, \(A\) is unitary, therefore, \(\|A^{-1}\| = 1\).

Example 2.7. Let \(E \subset \mathbb{T}\) be a set of absolute convergence, or, in other terminology, of type \(N\) (see [8], [11]), that is, there exists a sequence \(\{a_n\}_{n=1}^\infty\) such that \(a_n \geq 0\) for every \(n \geq 1\), \(\sum_{n=1}^\infty a_n = \infty\), and \(\sum_{n=1}^\infty a_n |\Im \zeta^n| < \infty\) for every \(\zeta \in E\). Note that we do not suppose that \(E\) is closed. Moreover, if a sequence \(\{a_n\}_{n=1}^\infty\) as above is fixed and the set \(E = \{\zeta \in \mathbb{T} : \sum_{n=1}^\infty a_n |\Im \zeta^n| < \infty\}\) is infinite, then \(\text{clos} E = \mathbb{T}\) [8, VII.XI].

Let \(\mu\) be a finite positive Borel measure on \(\mathbb{T}\) such that \(\mu(\mathbb{T}) = \mu(E)\).
Then there exists a sequence \(\{n_k\}_k\) such that \(\zeta^{n_k} \to_k 1\) a.e. with respect to \(\mu\) (see [8, §VII.7], [11, XIII.2.3]). We sketch the proof briefly. Let \(\{a_n\}_{n=1}^\infty\)
be a sequence such that $a_n \geq 0$ for every $n \geq 1$, $\sum_{n=1}^{\infty} a_n = \infty$, and $\sum_{n=1}^{\infty} |a_n| \text{Im} \zeta^n | < \infty$ for a.e. $\zeta$ with respect to $\mu$. By the dominated convergence theorem,

$$\int T \left( \sum_{n=1}^{N} a_n |\text{Im} \zeta^n| \right) / \left( \sum_{n=1}^{N} a_n \right) d\mu(\zeta) \to_{N \to \infty} 0,$$

therefore, $\liminf_n \int_{T} |\text{Im} \zeta^n| d\mu(\zeta) = 0$. Thus, there exists a sequence $\{n_k\}_k$ such that $\sum_k \int_{T} |\text{Im} \zeta^{n_k}| d\mu(\zeta) < \infty$. Since

$$\sum_k \int_{T} |\text{Im} \zeta^{n_k}| d\mu(\zeta) = \int_{T} \left( \sum_k |\text{Im} \zeta^{n_k}| \right) d\mu(\zeta),$$

we conclude that $\sum_k |\text{Im} \zeta^{n_k}| < \infty$ for a.e. $\zeta$ with respect to $\mu$, therefore, $\zeta^{n_k} \to_k 1$ a.e. with respect to $\mu$.

Now, let $U$ be a unitary operator, $\mu$ be its scalar spectral measure, and let there exist a set $E$ of absolute convergence such that $\mu(T) = \mu(E)$. Then, as it was proved just above, there exists a sequence $\{n_k\}_k$ such that $\zeta^{n_k} \to_k 1$ a.e. with respect to $\mu$, therefore, $U^{n_k} \to_k I$ in the weak operator topology.

**Lemma 2.8.** Let $\mu$ be a finite positive Borel measure on $\mathbb{T}$ such that

$$\limsup_{n} |\hat{\mu}(n)| = \mu(T),$$

where $\hat{\mu}(n)$ are the Fourier coefficients of $\mu$. Then there exist a sequence $\{n_k\}_k$ and a point $\xi \in \mathbb{T}$ such that $\zeta^{n_k} \to_k \xi$ in the weak-star topology on $L^\infty(\mu)$.

**Proof.** Without loss of generality, we may assume that $\mu(T) = 1$. Let $\{\ell_k\}_k$ be a sequence such that $\limsup_n |\hat{\mu}(\ell_k)| = \lim_k |\hat{\mu}(\ell_k)|$. There exist a subsequence $\{n_k\}_k$ of $\{\ell_k\}_k$ and a function $\varphi \in L^\infty(\mu)$ such that $\zeta^{n_k} \to_k \varphi$ in the weak-star topology on $L^\infty(\mu)$. Since $||\zeta^{n_k}||_\infty = 1$, we have $||\varphi||_\infty \leq 1$. Furthermore,

$$1 = \lim_k |\hat{\mu}(n_k)| = \lim_k \left| \int T \zeta^{n_k} d\mu(\zeta) \right| = \left| \int T \varphi d\mu \right| \leq \int T |\varphi| d\mu \leq 1.$$

Thus, $\int T |\varphi| d\mu = 1$, therefore, $|\varphi(\zeta)| = 1$ for a.e. $\zeta$ with respect to $\mu$. We set $\xi = \int T \varphi d\mu$, then $1 = \int T \varphi d\mu = \int T \text{Re}(\bar{\zeta} \varphi) d\mu$. Since $\text{Re}(\bar{\zeta} \varphi) \leq 1$, we conclude that $\varphi(\zeta) = \xi$ for a.e. $\zeta$ with respect to $\mu$. \hfill $\square$

**Corollary 2.9.** If the scalar spectral measure of a unitary operator $U$ satisfies the conditions of Lemma 2.8, then there exist a sequence $\{n_k\}_k$ and a point $\xi \in \mathbb{T}$ such that $U^{n_k} \to_k \xi I$ in the weak operator topology.

**Example 2.10.** [4, Theorem 2]. There exist unitary operators $U$ such that $U^{\kappa I} \to_k \kappa I$ in the weak operator topology for some sequences $\{n_k\}_k$ and some $\kappa \in \mathbb{C}$, $0 < |\kappa| < 1$.

The following theorem is the main result of the paper.

**Theorem 2.11.** Suppose that $M, C$ are positive constants, $\{K_j\}_j$ is no more than countable family of Hilbert spaces, and $U_j: K_j \to K_j$ are unitary operators. We set $K = \oplus_j K_j$ and $U = \oplus_j U_j$. Suppose that

(i) every operator $U_j$ has the following property: if $R$ is an operator such that $\sup_{n \geq 0} \|R^n\| \leq M$ and $R \sim U_j$, then $\sup_{n \leq 0} \|R^n\| \leq C$.  


(ii) $K_j \in \text{Hyplat}_U$ for every $j$.

Let $T$ be an operator such that $\sup_{n \geq 0} \|T^n\| \leq M$ and $T \sim U$. Then $T \approx U$ and $\sup_{n \leq 0} \|T^n\| \leq M^2C^3$.

Proof. Since $T \sim U$, we have that $U$ is unitarily equivalent to the isometric asymptote of $T$. Denote by $H$ the space on which $T$ acts, and by $X : H \rightarrow K$ the canonical intertwining mapping for $T$ and $U$ (see [9]). Denote by

$$q : \text{Hyplat}_1 T \rightarrow \text{Hyplat}_U, \quad qM = \text{clo} X^*M, \quad M \in \text{Hyplat}_1 T,$$

the lattice isomorphism between $\text{Hyplat}_1 T$ and $\text{Hyplat}_U$ (see [9]). We set $M_j = q^{-1}K_j$ and $T_j = T|_{M_j}$ for every $j$. By [9], $T_j \sim U_j$, therefore, by assumption (i), $T_j \approx U_j$ and

$$\sup_{n \leq 0} \|T^n_j\| \leq C \quad \text{for every } j.$$

It is obvious from the above relations that $\oplus_j T_j \approx U$. We will show that $T \approx \oplus_j T_j$.

From the estimate on $\|T^n\|$ for $n \geq 0$, we have that $\|X\| \leq M$. From assumption (i) and the properties of $X$ (see (2.3) and (2.8)) we have that $\|Xx\| \geq \frac{1}{C} \|x\|$ for every $x \in M_j$ and every $j$. Let $\{x_j\}_j$ be a finite family such that $x_j \in M_j$. Then

$$\frac{1}{C^2} \sum_j \|x_j\|^2 \leq \sum_j \|Xx_j\|^2 = \left\|X\left(\sum_j x_j\right)\right\|^2 \leq M^2\|\sum_j x_j\|^2.$$

We obtain that

$$\frac{1}{M^2C^2} \sum_j \|x_j\|^2 \leq \left\|\sum_j x_j\right\|^2$$

for any finite family $\{x_j\}_j$ such that $x_j \in M_j$.

Clearly, $U^*$ and $T^*$ satisfy the conditions of the theorem. Denote by $X_s : H \rightarrow K$ the canonical intertwining mapping for $T^*$ and $U^*$ and by $q_s$ a lattice isomorphism between $\text{Hyplat}_1 T^*$ and $\text{Hyplat}_U^*$ $\text{Hyplat}_U$ defined analogously to (2.7). We set $M'_j = q_s^{-1}K_j$ for every $j$. Analogously to (2.9) we obtain that

$$\frac{1}{M^2C^2} \sum_j \|y_j\|^2 \leq \left\|\sum_j y_j\right\|^2$$

for any finite family $\{y_j\}_j$ such that $y_j \in M'_j$.

Now, we will show that the families $\{M_j\}_j$ and $\{M'_j\}_j$ are biorthogonal, that is,

$$M_j \perp M'_k, \quad \text{if } j \neq k.$$

Let $j \neq k$, and let $x \in M_j$ and $y \in M'_k$. By [9], $q^{-1}\mathcal{E} = \text{clo} X^*_s\mathcal{E}$ for every $\mathcal{E} \in \text{Hyplat}_U$. Thus, $x = \lim_n X^*_s g_n$, where $g_n \in K_j$, and $(x, y) = \lim_n (X^*_s g_n, y) = \lim_n (g_n, X_s y) = 0$, because $X_s y \in K_k$.

From (2.10), (2.11) and [15, VI.4], [16, C.3.1] we conclude that

$$\left\|\sum_j x_j\right\|^2 \leq M^2C^2 \sum_j \|x_j\|^2$$

for any finite family $\{x_j\}_j$ such that $x_j \in M_j$. 


For convenience, we give a proof of (2.12) below. By (2.10), the mapping
\[ Z : \mathcal{H} \to \oplus_j \mathcal{M}_j, \quad Z\left(\sum_j y_j\right) = \oplus_j y_j, \quad \text{where} \quad y_j \in \mathcal{M}_j, \]
is a linear bounded transformation and \( \|Z\| \leq MC \). Denote by \( P_j : \mathcal{M}_j' \to \mathcal{M}_j \) the restriction on \( \mathcal{M}_j' \) of the orthogonal projection on \( \mathcal{M}_j \). We set \( P = \oplus_j P_j \), then, evidently, \( \|P\| \leq 1 \). Let us regard the linear bounded transformation
\[ (PZ)^* : \oplus_j \mathcal{M}_j \to \mathcal{H}. \]
Clearly,
\[ \|(PZ)^*\| = \|PZ\| \leq MC. \]
Using the biorthogonality of the families \( \{\mathcal{M}_j\}_j \) and \( \{\mathcal{M}_j'\}_j \), it is easy to see that \( (PZ)^* \) acts by the formula
\[ (PZ)^*(\oplus_j x_j) = \sum_j x_j, \quad \text{where} \quad x_j \in \mathcal{M}_j. \]
Now, (2.12) follows from (2.13).

From (2.9) and (2.12) we conclude that the mapping
\[ Y : \mathcal{H} \to \oplus_j \mathcal{M}_j, \quad Y\left(\sum_j x_j\right) = \oplus_j x_j, \quad \text{where} \quad x_j \in \mathcal{M}_j, \]
is a linear bounded invertible transformation, \( \|Y\| \leq MC, \|Y^{-1}\| \leq MC \), and, evidently, \( YT = (\oplus_j T_j)Y \). Therefore, \( T \approx \oplus_j T_j \) and
\[ \|T^n\| \leq \|Y^{-1}\| \|(\oplus_j T_j)^n\| \|Y\| \leq M^2 C^3 \text{ for all } n \leq 0. \]

The following theorem can be regarded as a generalization of Lemma 1.1. It is evident from the definition of the Wiener algebra \( A(\mathbb{T}) \) of functions on \( \mathbb{T} \) that \( A(\mathbb{T}) \) is isometrically isomorphic to the space \( \ell^1 \); therefore, the dual space \( A(\mathbb{T})^* \) of \( A(\mathbb{T}) \) is isometrically isomorphic to \( \ell^\infty \). The sequences from \( \ell^\infty \) regarded as elements of \( A(\mathbb{T})^* \) are called pseudomeasures \([2],[6],[7],[8],[11]\). The inclusion \( A(\mathbb{T}) \subset C(\mathbb{T}) \) implies that \( C(\mathbb{T})^* \subset A(\mathbb{T})^* \), that is, if \( \mu \) is a complex Borel measure on \( \mathbb{T} \), then \( \mu \in A(\mathbb{T})^* \), and for \( f = \sum_{n \in \mathbb{Z}} \hat{f}(n)\zeta^n \in A(\mathbb{T}) \) we have
\[ \langle f, \mu \rangle = \int_{\mathbb{T}} f(\zeta) d\mu(\zeta) = \int_{\mathbb{T}} \left( \sum_{n \in \mathbb{Z}} \hat{f}(n)\zeta^n \right) d\mu(\zeta) = \sum_{n \in \mathbb{Z}} \hat{f}(n)\hat{\mu}(-n), \]
where \( \hat{\mu}(n) = \int_{\mathbb{T}} \zeta^{-n} d\mu(\zeta) \). In other words, a complex Borel measure on \( \mathbb{T} \) is a pseudomeasure. But there exist pseudomeasures that are not measures, that is, there exist sequences from \( \ell^\infty \) that are not sequences of the Fourier coefficients of any complex Borel measure on \( \mathbb{T} \)[2],[6],[8],[11].

The duality between functions \( f = \sum_{n \in \mathbb{Z}} \hat{f}(n)\zeta^n \in A(\mathbb{T}) \) and pseudomeasures \( \mu = \{\hat{\mu}(n)\}_{n \in \mathbb{Z}} \in \ell^\infty \) is given by the formula \( \langle f, \mu \rangle = \sum_{n \in \mathbb{Z}} \hat{f}(n)\hat{\mu}(-n) \). The following description of \( AA^+ \)-sets is a consequence of this duality. A
closed set $E \subset T$ is an $\text{AA}^+$-set if and only if there exists a constant $K(E)$ such that
\begin{equation}
\sup_{n \in \mathbb{Z}} |\hat{\mu}(n)| \leq K(E) \lim_{n \to \infty} |\hat{\mu}(n)|
\end{equation}
for every pseudomeasure $\mu$ such that $\langle f, \mu \rangle = 0$ for every $f \in I(E)$ (see [6], [7]). Theorem 2.12 shows that in Lemma 1.1 condition (2.14) on pseudomeasures (which is contained in Lemma 1.1 in a nonobvious form) can be replaced by the condition on positive measures. Note that for positive measures on some set $E \sup_{n \in \mathbb{Z}} |\hat{\mu}(n)| = \mu(E)$.

**Theorem 2.12.** Let $K > 0$ be a constant, and let $E \subset T$ be a Borel set such that $\mu(E) \leq K \limsup_{n \to \infty} |\hat{\mu}(n)|$ for every positive finite Borel measure $\mu$ on $T$ such that $\mu(E) = \mu(T)$. Let $U$ be a unitary operator, let $\nu$ be the scalar spectral measure of $U$, and let $\nu(E) = \nu(T)$. Suppose that $T$ is a power bounded operator such that $T \sim U$. Then $T \approx U$ and
\[ \sup_{n \leq 0} \|T^n\| \leq K^3 \left( \sup_{n \geq 0} \|T^n\| \right)^8. \]

**Proof.** Let $E$ be a set having the property described in the theorem, let $\nu$ be a positive finite Borel measure on $T$ such that $\nu(E) = \nu(T)$, and let $0 < c < \frac{1}{K}$. Applying the Zorn lemma, it is easy to see that there exists no more than countable family $\{E_j\}_j$ of Borel sets such that $E = \bigcup_j E_j$ and for every $j$ there exist a sequence $\{n_{jk}\}_k$ and a function $\varphi_j \in L^\infty(E_j, \nu)$ such that $\zeta^{n_{jk}} \to \varphi_j$ in the weak-star topology on $L^\infty(E_j, \nu)$ and $|\varphi_j| \geq c$ a.e. on $E_j$ with respect to $\nu$.

Now let $\nu$ be the scalar spectral measure of a unitary operator $U$, and let $0 < c < \frac{1}{K}$ be fixed. Let $\{E_j\}_j$ be the family of Borel sets constructed just above. We set $E'_1 = E_1$ and $E'_j = E_j \setminus E_{j-1}$ for $j > 1$; of course, the sets $\{E'_j\}_j$ are mutually disjoint. For every $j$, denote by $U_j$ the restriction of $U$ on its spectral subspace corresponding to the set $E'_j$. Clearly, $U_j$ and $\varphi_j(U_j)$ satisfy the conditions of Theorem 2.5, and $\|((\varphi_j(U_j))^{-1})\| \leq \frac{1}{c}$ for every $j$. Therefore, $U = \oplus_j U_j$ satisfies the conditions of Theorem 2.11. We set $M = \sup_{n \geq 0} \|T^n\|$; by Theorems 2.5 and 2.11, $T \approx U$ and
\[ \sup_{n \leq 0} \|T^n\| \leq M^2 \left( \frac{1}{c} M^2 \right)^3 = \frac{1}{c^3} M^8. \]
To conclude the proof of the theorem, let $\frac{1}{c}$ tend to $K$. \qed

### 3. On the sets satisfying some metric condition

In this section we apply the main result of the paper to unitary operators whose spectral measures are supported on sets satisfying some metric condition.

**Definition 3.1.** ([5], [7]). Let $E \subset T$ be a closed set. For $\varepsilon > 0$, denote by $N_\varepsilon$ the smallest number of closed arcs of length $\varepsilon$ whose union contains $E$. We set
\[ \alpha(E) = \liminf_{\varepsilon \to 0} \frac{N_\varepsilon}{\log \frac{1}{\varepsilon}}. \]
It is proved in [5, 7] that if \( \alpha(E) < \infty \), then \( E \) is an \( AA^+ \)-set; however, the norm \( K(E) \) of the inverse to the imbedding (1.1) can be arbitrarily large. In this section we show that for \( AA^+ \)-sets \( E \) such that \( \alpha(E) < \infty \) the estimate of the left part of (1.2) does not depend on \( K(E) \).

We will need the following technical lemma.

**Lemma 3.2.** Let \( E \subset \mathbb{T} \) be a closed set such that \( \alpha(E) < \infty \), and let \( \delta > 0 \). Then

\[
E = \bigcup_{j=1}^{\ell} \{\xi_j\} \cup \bigcup_k E_k,
\]

where \( \ell < \infty \), \( \xi_j \in E \), \( j = 1, \ldots, \ell \), and \( \{E_k\}_k \) is no more than countable family of closed mutually disjoint subsets of \( E \) such that \( \alpha(E_k) \leq \delta \).

**Proof.** First, we prove the lemma for \( \delta = \alpha(E)/2 \). If \( E' \subset \mathbb{T} \) is a closed set which does not contain nonempty open arcs and \( d > 0 \), then \( E' \) can be represented in the form \( E' = \bigcup_{k=1}^{K} E'_k \), where \( K < \infty \) and \( \{E'_k\}_{k=1}^{K} \) is a family of closed mutually disjoint subsets of \( E' \) such that \( \operatorname{diam} E'_k \leq d \), \( k = 1, \ldots, K \) (recall that \( \operatorname{diam} E = \sup \{||\zeta - \xi|| : \zeta, \xi \in E\} \)).

Let a sequence \( \{d_n\}_n \) be such that \( d_n > 0 \), \( n = 1, 2, \ldots, \), and \( d_n \to 0 \). We represent \( E \) in the form \( E = \bigcup_{k=1}^{K_1} E_{1k} \), where \( \{E_{1k}\}_{k=1}^{K_1} \) is a family of closed mutually disjoint nonempty subsets of \( E \) such that \( \operatorname{diam} E_{1k} \leq d_1 \), \( k = 1, \ldots, K_1 \). If \( \varepsilon > 0 \) is less than the minimum of distances between the sets \( E_{1k} \), then \( N_{\varepsilon}(E) = \sum_{k=1}^{K_1} N_{\varepsilon}(E_{1k}) \). Therefore, \( \alpha(E) \geq \sum_{k=1}^{K_1} \alpha(E_{1k}) \).

From this estimate we conclude that there is at most one index \( k \) such that \( \alpha(E_{1k}) > \alpha(E)/2 \). If \( \alpha(E_{1k}) \leq \alpha(E)/2 \) for every index \( k \), then the lemma is proved for \( \delta = \alpha(E)/2 \). If there exists an index \( k \), \( 1 \leq k \leq K_1 \), such that \( \alpha(E_{1k}) > \alpha(E)/2 \), say \( k = 1 \), then we apply the above procedure to the number \( d_2 > 0 \) and the set \( E_{11} \), and so on. The following two cases are possible.

The first case: in some step, say \( n \), we have that \( \alpha(E_{nk}) \leq \alpha(E)/2 \) for every index \( k \), \( k = 1, \ldots, K_n \), thus, the lemma is proved for \( \delta = \alpha(E)/2 \).

The second case: there exists a sequence of closed sets \( \{E_n\}_n \) such that \( \operatorname{diam} E_{1n} \leq d_n \), \( E_{n+1,1} \subset E_{1n} \), the set \( E'_n = E_{1n} \setminus E_{n+1,1} \) is a closed nonempty set, and \( \alpha(E_{1n}) > \alpha(E)/2 \) for every \( n \). Since \( d_n \to 0 \), the intersection \( \cap_n E_{1n} \) is a singleton. Finally, \( E_{n+1,1} \cup E'_n = E_{1n} \subset E \), and, since \( E_{n+1,1} \) and \( E'_n \) are closed and disjoint, \( \alpha(E) \geq \alpha(E_{1n}) \geq \alpha(E_{n+1,1}) + \alpha(E'_n) > \alpha(E)/2 + \alpha(E'_n) \), and we conclude that \( \alpha(E'_n) \leq \alpha(E)/2 \).

Thus, the lemma is proved for \( \delta = \alpha(E)/2 \), that is, the representation of \( E \) in the form

\[
E = \{\xi\} \cup \bigcup_k E_k,
\]

where \( \{E_k\}_k \) is no more than countable family of closed mutually disjoint subsets of \( E \) such that \( \alpha(E_k) \leq \alpha(E)/2 \) and \( \{\xi\} = \cap_n E_{1n} \), if the second case takes place, is obtained. Now we apply the already proved part of the lemma to every set \( E_k \) and obtain the needed representation of \( E \) in which (new) sets \( E_k \) satisfy the condition \( \alpha(E_k) \leq \alpha(E)/4 \), and so on. On some step, say \( j \), we get that \( \alpha(E)/2^j \leq \delta \). \( \square \)
 Remark 3.3. In the proof of Lemma 3.2, the second case actually can take place. Let $0 < \alpha \leq 1/2$ and $E_j = \{e^{i\frac{2\pi}{n}}\} \cup \bigcup_{n=j}^{\infty} \{e^{i\sum_{k=1}^{n} a_k}\}$. It is easy to see that $\alpha(E_j) = \frac{1}{\log n}$ for every $j = 1, 2, \ldots$ and $\bigcap_{j=1}^{\infty} E_j = \{e^{i\frac{2\pi}{n}}\}$.

The following lemma is actually from [7] and [8, §VII.8]. For convenience, we sketch the proof.

Lemma 3.4. Suppose that $K \in \mathbb{N}$, $K \geq 3$, and $E \subset \mathbb{T}$ is a closed set such that $\alpha(E) < 1/\log K$. Then $\liminf_n \|e^n - 1\|_{C(E)} \leq 2\sin \frac{\pi}{K-1}$.

Proof. There exists a sequence $\{\varepsilon_n\}_n$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ and

\[ N_n = N_{\varepsilon_n} \leq \frac{1}{\log K} \log \frac{1}{\varepsilon_n}. \]

We take a sequence $\{Q_n\}_n \subset \mathbb{N}$ such that $Q_n \to \infty$ and

\[ Q_n \left( K \left( \frac{K-1}{K} \right) N_n \right) \to_n 0. \]

Let $n$ be fixed. Denote by $t_{n1}, \ldots, t_{nN_n}$ real points such that $e^{2\pi i t_{nj}}$, $j = 1, \ldots, N_n$, are the centerpoints of closed arcs of length $\varepsilon_n$ whose union contains $E$. We apply the Dirichlet theorem (see [8, appendix §V.2]) to real points $t_{n1}, \ldots, t_{nN_n}$ and natural numbers $K - 1$ and $Q_n$. We obtain $q_n \in \mathbb{N}$ and $p_{n1}, \ldots, p_{nN_n} \in \mathbb{Z}$ such that

\[ Q_n \leq q_n \leq Q_n (K-1)^{N_n} \quad \text{and} \quad |q_n t_{nj} - p_{nj}| \leq \frac{1}{K-1}, \quad j = 1, \ldots, N_n. \]

Let $\zeta \in E$. There exist a real point $t$ and an index $j$, $1 \leq j \leq N_n$, such that $\zeta = e^{2\pi i t}$ and $|t - t_{nj}| \leq \varepsilon_n/2$. From this estimate, (3.1) and (3.3), we have

\[ |q_n t - p_{nj}| \leq \frac{Q_n}{2} \left( \frac{K-1}{K} \right)^{N_n} + \frac{1}{K-1}. \]

Since $\zeta = e^{2\pi i t}$ is an arbitrary point of $E$, from (3.2) and (3.4) we conclude that $\liminf_n \|e^{q_n n} - 1\|_{C(E)} \leq 2\sin \frac{\pi}{K-1}$. \hfill \qed

Theorem 3.5. Suppose that $\{E_k\}_k$ is no more than countable family of closed subsets of $\mathbb{T}$ such that $\alpha(E_k) < \infty$ (where the quantity $\alpha$ is defined in Definition 3.1). Suppose that $U$ is a unitary operator, $\mu$ is its scalar spectral measure, and $\mu(T) = \mu(\cup_k E_k)$. Let $T$ be a power bounded operator such that $T \sim U$. Then $T \approx U$ and $\sup_{n \geq 0} \|T^n x\| \leq (\sup_{n \geq 0} \|T^n x\|)^3$.

Proof. We set $M = \sup_{n \geq 0} \|T^n x\|$ and $E = \cup_k E_k$. We denote by $\mu_a$ and $\mu_c$ the pure atomic and continuous parts of $\mu$, respectively, and we denote by $U_a$ and $U_c$ the correspondent parts of $U$. It is easy to see that $U_a$ satisfies condition (i) of Theorem 2.11 with $C = M^2$ (see Example 2.6).

We fix a natural number $K \geq 3$. Applying Lemma 3.2 with $\delta < 1/\log K$ to every set $E_k$, we obtain the representation of $E$ in the form

\[ E = \bigcup_j \{\xi_j\} \cup \bigcup_{\ell} E_{K\ell}, \]

where $\alpha(E_{K\ell}) < 1/\log K$ and the sets of indeces $j$ and $\ell$ are no more than countable. Let $U_{K\ell}$ be the restriction of $U_c$ on its spectral subspace corresponding to the set $E_{K\ell}$. Applying Theorem 2.5 and Lemma 3.4, we will
show that $U_{K\ell}$ satisfies condition (i) of Theorem 2.11 with

$$C = C_K = \frac{1}{1 - 2 \sin \frac{\pi}{K-1}} M^2.$$  

By Lemma 3.4, there exists a sequence $\{\eta_n\}_n$ (which depends on $K$ and $\ell$) such that $\lim_n \|\xi^{\eta_n} - 1\|_{C(E_{K\ell})} \leq 2 \sin \frac{\pi}{K-1}$. There exist a subsequence $\{p_n\}_n$ of $\{\eta_n\}_n$ and a function $\varphi \in L^{\infty}(E_{K\ell}, \mu_c)$ such that $\xi^{p_n} \to \varphi$ in the weak-star topology on $L^{\infty}(E_{K\ell}, \mu_c)$. Thus, $U_{K\ell}^{p_n} \to \varphi(U_{K\ell})$ in the weak operator topology. Since $\lim_n \|\xi^{p_n} - 1\|_{\infty} \leq 2 \sin \frac{\pi}{K-1}$, we have that $|\varphi| \geq 1 - 2 \sin \frac{\pi}{K-1}$ a.e. on $E_{K\ell}$ with respect to $\mu_c$. By Theorem 2.5, $U_{K\ell}$ satisfies condition (i) of Theorem 2.11 with the same constant $C = C_K$.

Since we do not suppose that the sets $E_k$ are mutually disjoint, we need to change the sets $E_{K\ell}$ in representation (3.5). Namely, we set $E'_{K1} = E_{K1}$ and $E'_{K\ell} = E_{K\ell} \setminus E_{K,\ell-1}$ for $\ell > 1$. Now the sets $E'_{K\ell}$ are mutually disjoint. Let $U'_{K\ell}$ be the restriction of $U_c$ on its spectral subspace corresponding to the set $E'_{K\ell}$. Applying Lemma 2.2 to $U_{K\ell}$ we conclude that $U'_{K\ell}$ satisfies the condition (i) of Theorem 2.11 with the same constant $C = C_K$. Thus, $U$ has the representation

$$U = U_a \oplus \bigoplus_{\ell} U'_{K\ell},$$

and we conclude that $U$ satisfies the conditions of Theorem 2.11. Therefore, $T \approx U$ and

$$\sup_{n \leq 0} \|T^n x\| \leq M^2 C_K^3,$$

where $C_K$ is defined in (3.6) and $K$ is an arbitrary natural number, $K \geq 3$.

To conclude the proof of the theorem, we note that $M^2 C_K^3 \to M^3$ when $K \to \infty$. □

**Corollary 3.6.** In Theorem 3.5 the condition $T \sim U$ can be replaced by $T \prec U$.

**Proof.** We will show that $U$ satisfies condition 1) of Lemma 2.4. Then the corollary will be proved. Denote by $\mathcal{K}$ the space on which $U$ acts and by $\mu$ the scalar spectral measure of $U$. Following [12] we set

$$\mathcal{Z}(U) = \{x \in \mathcal{K} : U^n x \to 0 \text{ n.a.e.} \} \text{ in the weak topology.}$$

It is easy to see that $\mathcal{Z}(U)$ is a hyperinvariant subspace for $U$, therefore, there exists a set $\tau \subset \mathbb{T}$ such that $\mathcal{Z}(U)$ is the spectral subspace of $U$ corresponding to $\tau$. Let us suppose that $\mu(\tau) > 0$. Then there exists an index $k$ such that $\mu(\tau \cap E_k) > 0$. The spectral subspace of $U$ corresponding to $\tau \cap E_k$ is contained in $\mathcal{Z}(U)$, therefore, the Fourier coefficients of $\mu|_{\tau \cap E_k}$ tend to zero. But, by assumption, the set $E_k$ is an $AA^+$-set, and if the Fourier coefficients of a measure supported on $E_k$ tend to zero, then this measure must be zero itself [6], [7] (see (2.14)), a contradiction. □

**References**

[1] T. Ando and K. Takahashi, On operators with unitary $\rho$-dilations, *Ann. Polon. Math.*, 66 (1997), 11–14.
[2] C. C. Graham and O. C. McGehee, *Essays in commutative harmonic analysis*, Springer, Berlin, Heidelberg, New York, 1979.

[3] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[4] A. Iwanik, M. Lemańczyk and C. Mauduit, Piecewise absolutely continuous cocycles over irrational rotations, *J. London Math. Soc. (2)*, 59 (1999), 171–187.

[5] J. P. Kahane, A metric condition for a closed circular set to be a set of uniqueness, *J. Approx. Theory*, 2 (1969), 233–236.

[6] J. P. Kahane, *Séries de Fourier absolument convergentes*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[7] J. P. Kahane and Y. Katznelson, Sur les algèbres de restrictions des séries de Taylor absolument convergentes à un fermé du cercle, *J. Analyse Math.*, 23 (1970), 185–197.

[8] J. P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, Paris, 1994.

[9] L. Kérchy, Isometric asymptotes of power bounded operators, *Indiana Univ. Math. J.*, 38 (1989), 173–188.

[10] L. Kérchy, On the inclination of hyperinvariant subspaces of $C_{11}$-contractions, *The Gohberg Anniversary Collection, Volume II: Topics in Analysis and Operator Theory*, Oper. Theory Adv. Appl., Birkhäuser, Basel, 41 (1989), 345–351.

[11] L. A. Lindahl and F. Poulsen (Eds.), *Thin sets in harmonic analysis*, Marcel Dekker, New York, 1971.

[12] W. Mlak, Decompositions of polynomially bounded operators, *Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys.*, 21 (1973), 317–322.

[13] W. Mlak, Algebraic polynomially bounded operators, *Ann. Polon. Math.*, 29 (1974), 133–139.

[14] V. Müller and Y. Tomilov, Quasisimilarity of power bounded operators and Blum–Hanson property, *J. Funct. Anal.*, 246 (2007), 385–399.

[15] N. K. Nikolskii, *Treatise on the shift operator*, Springer, Berlin, 1986.

[16] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Volume II: Model operators and systems*, Math. Surveys and Monographs 93, AMS, 2002

[17] M. Radjabalipour, Some results on power bounded operators, *Indiana Univ. Math. J.*, 22 (1973), 673–677.

[18] B. Sz.-Nagy, On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math. (Szeged)*, 11 (1947), 152–157.
[19] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert spaces*, North Holland, Amsterdam, 1970.

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