NO TRIANGLE CAN BE CUT INTO SEVEN CONGRUENT TRIANGLES

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Abstract. We give a short and direct proof of the theorem in the title, and prove the theorem for 11 as well as 7. By previous work of others, the problem reduces to a number of cases, and each case can be resolved by a computation.

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1. Introduction

An $N$-tiling of triangle $ABC$ by triangle $T$ is a way of writing $ABC$ as a union of $N$ triangles congruent to $T$, overlapping only at their boundaries. The triangle $T$ is the “tile”. We consider the problem of cutting a triangle into $N$ congruent triangles. Shortly we shall give a number of examples of $N$-tilings, for various small values of $N$. These examples will be tilings that have, for the most part, been known a very long time. But it will be obvious that $N = 7$ is not in this list of examples, nor is $N = 11$.

These two values of $N$ are the focus of this self-contained paper. We will prove here, without using the machinery developed in three longer papers or any very deep theories at all, that there are no 7-tilings or 11-tilings. Originally it was the question of 7-tilings that attracted us to this subject. This question could easily have been understood by the Greek geometers working in Alexandria with Euclid three centuries BCE, and possibly could have been solved by them too. We were able to give a purely Euclidean proof, but it was very long and complicated. Once sufficient machinery is developed, non-existence of tilings for many $N$, including all primes congruent to 3 mod 4, is a consequence, but also that development is long and complicated. Therefore we were happy, in October 2018, to discover a short and simple proof of the non-existence of any 7-tiling, which we present here. It was also possible to treat $N = 11$ with very little extra work–something we could not do with a purely Euclidean proof. One might say that here Descartes is victorious over Euclid, as algebra and computation is shorter and more efficient than geometry. Following Euclid we could do $N = 7$, but not 11.

We checked most of the algebra both by hand and by computer, using SageMath [12], and we provide the short snippets of code we used. In only one place is it too laborious to do by hand.

These results fit into a larger research program, begun by Lazkovich [4]. He studied the possible shapes of tiles and triangles that can possibly be used in tilings,

\[\text{SageMath code, being written in Python, needs to contain tabs for indentation. When you cut and paste from a pdf file, you will get spaces, not tabs. Therefore you must paste into a file and supply tabs using the Unix utility unexpand. Also single quote marks are a different character in pdf. For very short snippets you may find it easier just to retype.}\]
and obtained results that will be described below. It is our contribution to focus attention on \( N \) as well. One may say that Laczkovich studied the pair \((ABC, T)\), and we want to refine his work to study the triple \((ABC, T, N)\).

2. SOME EXAMPLES OF TILINGS

Figures 1 through 4 show the simplest examples of \( N \)-tilings.

**Figure 1.** Two 3-tilings

**Figure 2.** A 4-tiling, a 9-tiling, and a 16-tiling

The method illustrated for \( N = 4, 9, \) and 16 generalizes to any perfect square \( N \). While the two exhibited 3-tilings clearly depend on the exact angles of the triangle, *any* triangle can be decomposed into \( n^2 \) congruent triangles by drawing \( n - 1 \) equally spaced lines parallel to each of the three sides of the triangle, as illustrated in Fig. 3. Moreover, the large (tiled) triangle is similar to the small triangle (the “tile”). We call such a tiling a *quadratic tiling*.

**Figure 3.** A quadratic tiling of an arbitrary triangle

It follows that if we have a tiling of a triangle \( ABC \) into \( N \) congruent triangles, and \( m \) is any integer, we can tile \( ABC \) into \( Nm^2 \) triangles by subdividing the first tiling, replacing each of the \( N \) triangles by \( m^2 \) smaller ones. Hence the set of \( N \) for which an \( N \)-tiling of some triangle exists is closed under multiplication by squares.
Sometimes it is possible to combine two quadratic tilings (using the same tile) into a single tiling, as shown in Fig. 4. We will explain how these tilings are constructed. We start with a big right triangle resting on its hypotenuse, and divide it into two right triangles by an altitude. Then we quadratically tile each of those triangles. The trick is to choose the dimensions in such a way that the same tile can be used throughout. If that can be done then evidently \( N \), the total number of tiles, will be the sum of two squares, \( N = n^2 + m^2 \), one square for each of the two quadratic tilings. On the other hand, if we start with an \( N \) of that form, and we choose the tile to be an \( n \) by \( m \) right triangle, then we can construct such a tiling. We call these tilings “biquadratic.” More generally, a biquadratic tiling of triangle \( ABC \) is one in which \( ABC \) has a right angle at \( C \), and can be divided by an altitude from \( C \) to \( AB \) into two triangles, each similar to \( ABC \), which can be tiled respectively by \( n^2 \) and \( m^2 \) copies of a triangle similar to \( ABC \). A larger biquadratic tiling, with \( n = 5 \) and \( m = 7 \) and hence \( N = 74 \), is shown in at the right of Fig. 4.

Since \( 5 = 2^2 + 1^2 \), the simplest case of a biquadratic tiling is \( N = 5 \). The second 5-tiling in Fig. 5 shows that a biquadratic tiling can sometimes be more complicated than a combination of two quadratic tilings. Symmetry can permit rearranging some of the tiles. The symmetrical tile used in Fig. 6 also allows for variety.

If the original triangle \( ABC \) is chosen to be isosceles, and is then quadratically tiled, then each of the \( n^2 \) triangles can be divided in half by an altitude; hence any isosceles triangle can be decomposed into \( 2n^2 \) congruent triangles. If the original triangle is equilateral, then it can be first decomposed into \( n^2 \) equilateral triangles, and then these triangles can be decomposed into 3 or 6 triangles each, showing that any equilateral triangle can be decomposed into \( 3n^2 \) or \( 6n^2 \) congruent triangles. For example we can 12-tile an equilateral triangle in two different ways, starting with...
a 3-tiling and then subdividing each triangle into 4 triangles (“subdividing by 4”),
or starting with a 4-tiling and then subdividing by 3.

The examples above do not exhaust all possible tilings, even when $N$ is a square. For example, Fig. 8 shows a 9-tiling that is not produced by those methods; again this seems attributable to symmetry permitting a rearrangement of tiles in a quadratic tiling.

There is another family of $N$-tilings, in which $N$ is of the form $3m^2$, and both the tile and the tiled triangle are 30-60-90 triangles. We call these the “triple-square” tilings. The case $m = 1$ is given in Fig. 11, the case $m = 2$ makes $N = 12$. There are two ways to 12-tile a 30-60-90 triangle with 30-60-90 triangle. One is to first quadratically 4-tile it, and then subtile the four triangles with the 3-tiling of Figure 1. This produces the first 12-tiling in Fig. 9. Somewhat surprisingly, there is another way to tile the same triangle with the same 12 tiles, also shown in Fig. 9. The next member of this family is $m = 3$, which makes $N = 27$. Two 27-tilings are shown in Fig. 10. Similarly, there are two 48-tilings (not shown).

Until October 12, 2008, we did not know any more complicated tilings than those illustrated above (and there also none in [11]). Then we found the beautiful
Figure 9. Two 12-tilings

Figure 10. Two 27-tilings

27-tiling shown in Fig. 11. This tiling is one of a family of $3k^2$ tilings (the case $k = 3$). The next case is a 48-tiling, made from six hexagons (each containing 6 tiles) bordered by 4 tiles on each of 3 sides. In general one can arrange $1+2+\ldots+k$ hexagons in bowling-pin fashion, and add $k+1$ tiles on each of three sides, for a total number of tiles of $6(1+2+\ldots+k)+3(k+1) = 3k(k+1)+3(k+1) = 3(k+1)^2$. Fig. 12 shows more members of this family, which we call the “hexagonal tilings.”

Figure 11. A 27-tiling due to Major MacMahon 1921, rediscovered 2011

Whenever there is an $N$-tiling of the right triangle $ABM$, there is a $2N$-tiling of the isosceles triangle $ABC$. Using the biquadratic tilings (see Fig. 5 and Fig. 4).

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2In January, 2012, I bought a puzzle at the exhibition at the AMS meeting, which contained the tiling in Fig. 11 as part of a tiling of a larger hexagon. The tiling is attributed to Major Percy Alexander MacMahon (1854-1929) [9].
and triple-square tilings (see Fig. 9 and Fig. 10), we can produce $2N$-tilings when $N$ is a sum of squares or three times a sum of squares. We call these tilings “double biquadratic” and “hexquadratic”. For example, one has two 10-tilings and two 26-tilings, obtained by reflecting Figs. 4 and 5 about either of the sides of the triangles shown in those figures; and one has 24-tilings and 54-tilings obtained from Figs. 8 and 9. Note that in the latter two cases, $ABC$ is equilateral.

In the case when the sides of the tile $T$ form a Pythagorean triple $n^2 + m^2 + k^2 = N/2$, then we can tile one half of $ABC$ with a quadratic tiling and the other half with a biquadratic tiling. The smallest example is when the tile has sides 3, 4, and 5, and $N = 50$. See Fig. 17. One half is 25-tiled quadratically, and the other half is divided into two smaller right triangles which are 9-tiled and 16-tiled quadratically. This shows that the tiling of $ABC$ does not have to be symmetric about the altitude.

3. Definitions and notation

We give a mathematically precise definition of “tiling” and fix some terminology and notation. Given a triangle $T$ and a larger triangle $ABC$, a “tiling” of triangle $ABC$ by triangle $T$ is a set of triangles $T_1, \ldots, T_n$ congruent to $T$, whose interiors are disjoint, and the closure of whose union is triangle $ABC$.

Let $a$, $b$, and $c$ be the sides of the tile $T$, and angles $\alpha$, $\beta$, and $\gamma$ be the angles opposite sides $a$, $b$, and $c$. The letter “$N$” will always be used for the number of triangles used in the tiling. An $N$-tiling of $ABC$ is a tiling that uses $N$ copies of some triangle $T$. The meanings of $N$, $\alpha$, $\beta$, $\gamma$, $a$, $b$, $c$, $A$, $B$, and $C$ will be fixed throughout this paper, and we assume $\alpha \leq \beta \leq \gamma$, when there is no other assumption about $\alpha$ and $\beta$, such as $3\alpha + 2\beta = \pi$.

4. History

In our gallery of examples, we saw quadratic and biquadratic tilings in which the tile is similar to $ABC$, and also hexagonal tilings. These involve $N$ being square, a sum of two squares, or three times a square. Snover et. al. [10] took up the challenge of showing that these are the only possible values of $N$. The following
Theorem 1 (Snover et. al. [10]). Suppose $ABC$ is $N$-tiled by tile $T$ similar to $ABC$. If $N$ is not a square, then $T$ and $ABC$ are right triangles. Then either

(i) $N$ is three times a square and $T$ is a 30-60-90 triangle, or

(ii) $N$ is a sum of squares $e^2 + f^2$, the right angle of $ABC$ is split by the tiling, and the acute angles of $ABC$ have rational tangents $e/f$ and $f/e$, and these two alternatives are mutually exclusive.

Soifer’s book [11] appeared in 1990, with a second edition in 2009. He considered two “Grand Problems”: for which $N$ can every triangle be $N$-tiled, and for which $N$ can every triangle be dissected into similar, but not necessarily congruent triangles. (The latter eventually became a Mathematics Olympiad problem.) The 2009 edition has an added chapter in which the biquadratic tilings and a theorem of Laczkovich occur.

Mikhail Laczkovich published six papers [3, 4, 5, 6, 7] on triangle and polygon tilings. According to Soifer, the 1995 paper was submitted in 1992. Laczkovich, like Soifer, studied dissecting a triangle into smaller similar triangles, not congruent triangles as we require here. If those similar triangles are rational (i.e., the ratios of their sides are rational) then if we divide each of them into small enough quadratic subtilings, we can achieve an $N$-tiling into congruent triangles, but of course $N$ may be large. Laczkovich proved little about $N$, focusing instead on the shapes of $ABC$ (or more generally, convex polygons) and of the tile. His theorems, for example, do not address the possibility of an $N$-tiling (of some $ABC$ by some tile) for any particular $N$, but they do give us an exhaustive list of the possible shapes of $ABC$ and the tile, which we will need in our proof that there is no 7-tiling. This list can be found in §5 (of this paper). However, his theorem published in the last chapter of [11] does mention $N$. It states that given an integer $k$, there exists an $N$-tiling for some $N$ whose square-free part is $k$.

5. LACZKOVICH

A basic fact is that, apart from a small number of cases that can be explicitly enumerated, if there is an $N$-tiling of $ABC$ by a tile with angles $(\alpha, \beta, \gamma)$, then the angles $\alpha$ and $\beta$ are not rational multiples of $\pi$. This theorem is Theorem 5.1 of [4]. Laczkovich calls the angles of the tile commensurable if each of them is a rational multiple of $\pi$. He states his theorem conversely to the way we just described it: if there is a tiling of $ABC$ by a tile $T$ with commensurable angles, then the pair $(ABC, T)$ belongs to a specific, fairly short list. It is important to note that Laczkovich’s list in Theorem 5.1 is about dissections of $ABC$ into similar, not necessarily congruent, triangles. His subsequent Theorem 5.3 shows that three possibilities for dissecting the right isosceles triangle $ABC$ into similar triangles are impossible with congruent tiles. That is stated in the proof, but not in the statement, of Theorem 5.3.

Laczkovich’s list of possibilities from the cited 1995 paper is given in Table 1. In the table, the triples giving the angles of the tile are $(\alpha, \beta, \gamma)$ after a suitable permutation, i.e., they are unordered triples. The reader who checks with [4] will need to remember that we have deleted the entries for the right isosceles $ABC$ mentioned above.
Table 1. Laczkovich’s 1995 list of tilings by tiles with commensurable angles

| $ABC$                  | the tile                          |
|------------------------|-----------------------------------|
| $(\alpha, \beta, \gamma)$ similar to $ABC$ | $\gamma = \pi/2$                  |
| $(\alpha, \alpha, 2\beta)$                   | $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$ |
| equilateral            | $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{12})$ |
| equilateral            | $(\frac{\pi}{3}, \frac{19\pi}{30}, \frac{13\pi}{30})$ |
| equilateral            | $(\frac{\pi}{3}, \frac{7\pi}{30}, \frac{13\pi}{30})$ |

In subsequent work, specifically Theorem 3.3 of [7], Laczkovich proved that the table can be considerably shortened: the tilings of the equilateral triangle mentioned in the last three rows cannot occur (when the tiles are required, as in this paper, to be congruent rather than just similar). Thus the final version is as shown in

Table 2. Laczkovich’s 2012 list of tilings by tiles with commensurable angles

| $ABC$                  | the tile                          |
|------------------------|-----------------------------------|
| $(\alpha, \beta, \gamma)$ similar to $ABC$ | $\gamma = \pi/2$                  |
| $(\alpha, \alpha, 2\beta)$                   | $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$ |
| equilateral            | $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{12})$ |

It is possible to prove by direct computation that the last two rows of Table 1 do not correspond to actual tilings. Namely, the area equation for the equilateral triangle with side $X$ tells us $X^2 = Nbc$, if angle $\alpha = \pi/3$. Then writing $X = pa + qb + rc$ and calculating $(a,b,c) = (\sin \alpha, \sin \beta, \sin \gamma)$ for the specific angles involved, we get equations in certain algebraic number fields, that one then has to show impossible. For example,

\[(1) \quad \left( p \left( \xi - \frac{1+\sqrt{5}}{8} \right) + q \frac{\sqrt{3}}{2} + r \left( \xi + \frac{1+\sqrt{5}}{8} \right) \right)^2 = N \left( \frac{3}{8} - \frac{\sqrt{5}}{8} \right) \]

One interesting thing about this approach is that SageMath is fully capable of performing all the required calculations, including determining whether certain expressions lie in certain algebraic number fields or not. We did not succeed, however, in entirely eliminating the row mentioning $\pi/12$ by computation; in that case, using the area equation as described only tells us that $N$ is six times a square. That would be enough for this paper, where we only need that $N$ cannot be 7 or 11; but Laczkovich entirely eliminated that possible tiling as well as the other two.
6. The coloring equation

In this section we introduce a tool that is useful for some, but not all, tiling problems. Suppose that triangle \(ABC\) is tiled by a tile with angles \((\alpha, \beta, \gamma)\) and sides \((a, b, c)\), and suppose there is just one tile at vertex \(A\). We color that tile black, and then we color each tile black or white, changing colors as we cross tile boundaries. Under certain conditions this coloring can be defined unambiguously, and then, we define the “coloring number” to be the number of black tiles minus the number of white tiles. An example of such a coloring is given in Fig. 13.

**Figure 13.** A tiling colored so that touching tiles have different colors.

The following theorem spells out the conditions under which this can be done. In the theorem, “boundary vertex” refers to a vertex that lies on the boundary of \(ABC\) or on an edge of another tile, so that the sum of the angles of tiles at that vertex is \(\pi\). At an “interior vertex” the sum of the angles is \(2\pi\).

**Theorem 2.** Suppose that triangle \(ABC\) is tiled by the tile \((a, b, c)\) in such a way that

(i) There is just one tile at \(A\).

(ii) At every boundary vertex an odd number of tiles meet.

(iii) At every interior vertex an even number of tiles meet.

(iv) The numbers of tiles at \(B\) and \(C\) are both even, or both odd.

Then every tile can be assigned a color (black or white) in such a way that colors change across tile boundaries, and the tile at \(A\) is black. Let \(M\) be the number of black tiles minus the number of white tiles. Then the coloring equation

\[ X \pm Y + Z = M(a + b + c) \]

holds, where \(Y\) is the side of \(ABC\) opposite \(A\), and \(X\) and \(Z\) are the other two sides. The sign is + or − according as the number of tiles at \(B\) and \(C\) is odd or even.
Proof. Each tile is colored black or white according as the number of tile boundaries crossed in reaching it from A without passing through a vertex is even or odd. The hypotheses of the theorem guarantee that color so defined is independent of the path chosen to reach the tile from A. The total length of black edges, minus the total length of white edges, is \( M(a + b + c) \), since \( a + b + c \) is the perimeter of each tile. Each interior edge makes a contribution of zero to this sum, since it is black on one side and white on the other. Therefore only the edges on the boundary of \( ABC \) contribute. Now sides \( X \) and \( Y \) contain only edges of black tiles, by hypotheses (i) and (ii). Side \( Y \) is also black if the number of tiles at \( B \) and \( C \) is odd, and white if it is even. Hence the difference in the total length of black and white tiles is \( X \pm Y + Z \), with the sign determined as described. That completes the proof.

7. Possible values of \( N \) in tilings with commensurable angles

We wish to add a third column to Laczkovich’s Table 2 giving the possible forms of \( N \) if there is an \( N \)-tiling of \( ABC \) by the tile in that row. For example, when \( ABC \) is similar to the tile, then \( N \) must be a square, so we put \( n^2 \) in the third column. While we are at it, we add a fourth column with a citation to the result, and delete the rows corresponding to the tilings of the equilateral triangle that we have proved impossible. The revised and extended table is Table 3. All the entries in this table except the last one give necessary and sufficient conditions on \( N \) for the tilings to exist. The last one gives necessary conditions for certain tilings that probably do not actually exist.

Table 3. \( N \)-tilings by tiles with commensurable angles, with form of \( N \)

| \( ABC \)          | the tile                  | form of \( N \) | citation |
|-------------------|---------------------------|----------------|----------|
| \((\alpha, \beta, \gamma)\) similar to \( ABC \) | \( n^2 \)             | 10             |          |
| \((\alpha, \beta, \gamma)\) similar to \( ABC \), \( \gamma = \pi/2 \) | \( e^2 + f^2 \) | 10             |          |
| \((\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2})\) similar to \( ABC \) | \( 3n^2 \)             | 10             |          |
| \((\alpha, \alpha, 2\beta)\) \( \gamma = \pi/2 \) | \( 2n^2 \)             | Theorem 4      |          |
| \((\alpha, \alpha, 2\beta)\) | \((\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2})\) | \( n^2 \) | Theorem 4 |
| \((\frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3})\) | \((\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2})\) | \( 6n^2 \) | Theorem 4 |
| equilateral \((\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})\) | \( 6n^2 \)             | Theorem 4      |          |
| equilateral \((\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3})\) | \( 3n^2 \)             | Theorem 4      |          |

Theorem 3. Suppose \((\alpha, \beta, \gamma)\) are all rational multiples of \( 2\pi \), and triangle \( ABC \) is \( N \)-tilled by a tile with angles \((\alpha, \beta, \gamma)\). Then \( ABC \), \((\alpha, \beta, \gamma)\), and \( N \) correspond to one of the lines in Table 3.

Proof. As discussed above, Laczkovich characterized the pairs of tiled triangle and tile, as given in Table 2. It remains to characterize the possible \( N \) for each line.
In several cases lines in Table 2 split into two or more lines in Table 3, which supplies the required possible forms of values of $N$. That table lists in its last column citations to the literature or theorems in this paper for each line. Finally, we have deleted the rows of Table 2 corresponding to the tilings that are impossible by Theorem 3.3 of [7]. That completes the proof.

8. Laczkovich’s second table

Laczkovich also studied the case when not all the angles of the tile are rational multiples of $\pi$. Again a finite number of cases can arise. This is Theorem 4.1 of [4], and the list of cases is given in Table 4.

Table 4. Tilings when not all angles are rational multiples of $\pi$.

| $ABC$ | the tile |
|--------|----------|
| $(\alpha, \beta, \gamma)$ similar to $ABC$ | $\alpha = \pi/3$ |
| $\text{equilateral}$ | $\gamma = \pi/2$ |
| $(\alpha, \alpha, 2\beta)$ | $\gamma = 2\alpha$ |
| $(2\alpha, \beta, \alpha + \beta)$ | $3\alpha + 2\beta = \pi$ |
| $(2\alpha, \alpha, 2\beta)$ | $3\alpha + 2\beta = \pi$ |
| $\text{isosceles}$ | $3\alpha + 2\beta = \pi$ |
| $(\alpha, \alpha, \pi - 2\alpha)$ | $\gamma = 2\pi/3$ |
| $(\alpha, 2\alpha, \pi - 3\alpha)$ | $\gamma = 2\pi/3$ |
| $(\alpha, 2\beta, 2\alpha + \beta)$ | $\gamma = 2\pi/3$ |
| $(\alpha, \alpha + \beta, \alpha + 2\beta)$ | $\gamma = 2\pi/3$ |
| $(2\alpha, 2\beta, \alpha + \beta)$ | $\gamma = 2\pi/3$ |
| $\text{equilateral}$ | $\gamma = 2\pi/3$ |

Table 2 and 4 together constitute an exhaustive list of tilings. If we have some conditions on the tile, such as for example $3\alpha + 2\beta = \pi$, then we look to see what entries in Table 2 satisfy those conditions. That gives some tilings with commensurable angles. Then we look in the other table for tilings in which not all the angles are rational multiples of $2\pi$. To fix the ideas we spell out the details for the case $3\alpha + 2\beta = \pi$.

Lemma 1. Let $3\alpha + 2\beta = \pi$. Suppose there is an $N$-tiling of triangle $ABC$ by tile $T$ with angles $(\alpha, \beta, \gamma)$. Suppose also that $ABC$ is not similar to $T$. Then $\alpha$ and $\beta$ are not rational multiples of $\pi$, and every linear relation between $\pi, \alpha, \beta$ is a multiple of $3\alpha + 2\beta = \pi$.

Proof. Suppose there is an $N$-tiling as in the statement of the lemma. Then if angles of the tile are all rational multiples of $\pi$, the pair $ABC$ and the tile must occur in Table 2. So we have to check if any of the triples in that table satisfy $3\alpha + 2\beta = \pi$. And they do not, so that completes the proof.
Remark. The reader of Laczkovich’s paper [4] should beware: Theorem 5.1 includes the triple \((\pi/4, \pi/8, 5\pi/8)\), which does satisfy \(3\alpha + 2\beta = \pi\). But as discussed above, it is included since the theorem is about dissections into similar triangles, and Theorem 5.3 of [4] rules it out for tilings into congruent triangles. Hence we have deleted it from Table 2 and do not need to consider it here.

9. Adding a column for \(N\) to Laczkovich’s second table

The research program that we have been pursuing in this subject is to study triples \((ABC, T, N)\) instead of just pairs \((ABC, T)\), where there is an \(N\)-tiling of \(ABC\) by tile \(T\). Another way to say that is that we wish to add a third column to Laczkovich’s second table, entering the possible forms of \(N\) in that column, as we did to the first table. This has proved to be a longer business than I had originally imagined, although also more interesting, since several new tilings have been discovered in the process, and this research program is not complete. The point of the present paper is that we have pursued it far enough to reach the goal of showing that 7-tilings are impossible. Presently, we can supply entries in the third column down to the cases with \(\gamma = 2\pi/3\), but some of them are only necessary conditions, not necessary and sufficient, leaving open many questions about particular values of \(N\) that are not ruled out by those necessary conditions.

10. Some number-theoretic facts

The facts in this section may not be well-known to all our readers, and their proofs are short.

Lemma 2. A quotient of sums of two rational squares is a sum of two rational squares.

Proof. A sum of two rational squares is the square of the absolute value of some complex number. The quotient of the absolute values is the absolute value of the quotient. Explicitly:

\[ \frac{a^2 + b^2}{c^2 + d^2} = \frac{|a + bi|^2}{|c + di|^2} = \frac{(a + bi)^2}{(c + di)^2} = \frac{(a + bi)(c - di)^2}{c^2 + d^2} = \left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{bc - ad}{c^2 + d^2}\right)^2 \]

That completes the proof of the lemma.

The following lemma identifies those relatively few rational multiples of \(\pi\) that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over \(\mathbb{Q}\).

Lemma 3. Let \(\zeta = e^{i\theta}\) be algebraic of degree \(d\) over \(\mathbb{Q}\), where \(\theta\) is a rational multiple of \(\pi\), say \(\theta = 2m\pi/n\), where \(m\) and \(n\) have no common factor.
Then \( d = \varphi(n) \), where \( \varphi \) is the Euler totient function. In particular if \( d = 4 \), which is the case when \( \tan \theta \) is rational and \( \sin \theta \) is not, then \( n \) is 5, 8, 10, or 12; and if \( d = 8 \) then \( n \) is 15, 16, 20, 24, or 30.

**Remark.** For example, if \( \theta = \pi/6 \), we have \( \sin \theta = 1/2 \), which is of degree 1 over \( \mathbb{Q} \). Since \( \cos \theta = \sqrt{3}/2 \), the number \( \zeta = e^{i\theta} \) is in \( \mathbb{Q}(i, \sqrt{3}) \), which is of degree 4 over \( \mathbb{Q} \). The number \( \zeta \) is a 12-th root of unity, i.e. \( n \) in the theorem is 12 in this case; so the minimal polynomial of \( \zeta \) is of degree \( \varphi(12) = 4 \). This example shows that the theorem is best possible.

**Remark.** The hypothesis that \( \theta \) is a rational multiple of \( \pi \) cannot be dropped. For example, \( x^4 - 2x^3 + x^2 - 2x + 1 \) has two roots on the unit circle and two off the unit circle.

**Proof.** Let \( f \) be a polynomial with rational coefficients of degree \( d \) satisfied by \( \zeta \). Since \( \zeta = e^{i2\pi/n} \), \( \zeta \) is an \( n \)-th root of unity, so its minimal polynomial has degree \( d = \varphi(n) \), where \( \varphi \) is the Euler totient function. Therefore \( \varphi(n) \leq d \). If \( \tan \theta \) is rational and \( \sin \theta \) is not, then \( \sin \theta \) has degree 2 over \( \mathbb{Q} \), so \( \zeta \) has degree 2 over \( \mathbb{Q}(i) \), so \( \zeta \) has degree 4 over \( \mathbb{Q} \). The stated values of \( n \) for the cases \( d = 4 \) and \( d = 8 \) follow from the well-known formula for \( \varphi(n) \). That completes the proof of (ii) assuming (i).

**Corollary 1.** If \( \sin \theta \) or \( \cos \theta \) is rational, and \( \theta < \pi \) is a rational multiple of \( \pi \), then \( \theta \) is a multiple of \( 2\pi/n \) where \( n \) is 4, 5, 8, 10, or 12.

**Proof.** Let \( \zeta = \cos \theta + i \sin \theta = e^{i\theta} \). Under the stated hypotheses, the degree of \( \mathbb{Q}(\zeta) \) over \( \mathbb{Q} \) is 2 or 4, since \( \mathbb{Q}(\zeta) = \mathbb{Q}(\cos \theta, \sin \theta, i) \). Hence, by Lemma 3, \( \theta \) is a multiple of \( 2\pi/n \), where \( n \) is 5, 8, 10, or 12 (if the degree is 4) or \( n = 4 \) (if the degree is 2).

### 11. Isosceles \( ABC \), Formulas for \( a \) and \( b \)

The tilings of an isosceles \( ABC \) will be analyzed in two cases, according to whether the angles of the tile are all rational multiples of \( \pi \) or not. In this section, we derive some formulas and facts about tilings of an isosceles \( ABC \) that will be used in both cases.

**Lemma 4.** Suppose isosceles triangle \( ABC \) with base angles \( \beta \) is \( N \)-tiled by tile \((\alpha, \beta, \pi/2)\) with sides \((a, b, 1)\). Then

(i) The following formulas give \( a \) and \( b \) in terms of \( \lambda = \sqrt{N/2} \).

\[
\begin{align*}
a &= p(\lambda - q) + \frac{r\sqrt{p^2 + r^2 - (\lambda - q)^2}}{p^2 + r^2} \\
b &= \frac{r(\lambda - q)}{p^2 + r^2} + \frac{p\sqrt{p^2 + r^2 - (\lambda - q)^2}}{p^2 + r^2}
\end{align*}
\]

Here \((p, q, r)\) are determined by the decomposition of \( AB \) into tile edges,

\[X = pa + rb + q.\]

(ii) \( a \) and \( b \) are both in \( \mathbb{Q}(\sqrt{N/2}) \), and

(iii) If \( q = 0 \) then \( a/b \) is rational and \( N/2 \) is a sum of two squares. Specifically, \( N/2 = p^2 + r^2 \), and \( \tan \beta = r/p \).
Proof. The length of $AB$ is $X = pa + rb + q$. From the area equation $2X^2 = N$ we have

$$2(pa + rb + q)^2 = N$$

Then

$$\lambda = pa + rb + q$$
$$rb = \lambda - pa - q$$
$$r^2b^2 = (\lambda - pa - q)^2$$
$$r^2(1 - a^2) = (\lambda - pa - q)^2 \quad \text{since} \, b^2 = 1 - a^2 \quad \text{since} \, a = \sin \alpha \, \text{and} \, b = \sin \beta$$

Writing it as a polynomial in $a$ we have

(2) \quad 0 = a^2(p^2 + r^2) - 2ap(\lambda - q) + (\lambda - q)^2 - r^2

We have $p^2 + r^2 \neq 0$, since otherwise $2X^2 = 2q^2 = N$, contradiction. Solving (2) by the quadratic formula,

$$a = \frac{p(\lambda - q)}{p^2 + r^2} \pm \sqrt{\frac{(\lambda - q)^2p^2 + (p^2 + r^2)(r^2 - (\lambda - q)^2)}{p^2 + r^2}}$$

That is the formula for $a$ given in part (i) of the lemma. The formula for $b$ can be derived similarly, interchanging $p$ and $r$. The formula for $b$ contains $\mp$ instead of $\pm$, because the equation $a^2 + b^2 = 1$ implies that the signs in the equations for $a$ and $b$ must be opposite.

Define

$$\mu := \sqrt{p^2 + r^2 - (\lambda - q)^2}$$

First we observe that $\mu^2$ is irrational if and only if $q \neq 0$, since $\mu^2 = p^2 + r^2 + N/2 + q^2 + 2q\lambda$, whose irrational part is $2q\lambda$. Writing the formulas for $a$ and $b$ in terms of $\mu$, we have

(3) \quad a = \frac{p(\lambda - q) \mp r\mu}{p^2 + r^2}

(4) \quad b = \frac{r(\lambda - q) \mp p\mu}{p^2 + r^2}

We now argue by cases according as $q = 0$ or $q \neq 0$. First assume $q \neq 0$. Suppose, for proof by contradiction, that $\mu$ does not belong to $\mathbb{Q}(\lambda)$. The definition of $\mu$ shows that $\mu$ is quadratic over $\mathbb{Q}(\lambda)$, so if it does not belong to $\mathbb{Q}(\lambda)$, then the degree of $F = \mathbb{Q}(\mu, \lambda)$ over $\mathbb{Q}(\lambda)$ is exactly 2. Since $\lambda$ is quadratic irrational, the degree of $F$ over $\mathbb{Q}$ is 4. Then I say $\{1, \lambda, \mu, \mu\lambda\}$ are linearly independent over $\mathbb{Q}$. For suppose

$$\mu\lambda = A + B\lambda + C\mu \quad \text{with} \, A, B, C \in \mathbb{Q}.$$ 

Then $C \neq 1$ because $\lambda$ is irrational, so $\mu = (A + B\lambda)/(1 - C)$ belongs to $\mathbb{Q}(\lambda)$, contrary to our assumption.
\( \lambda \) and \( \mu \) are rational linear combinations of 1, \( a \) and \( b \), as can be seen by using (3) and (4) to verify the following equations:

(5) \[ \lambda = pa + rb + q \]

(6) \[ \mu = ra - pb \]

The verification is simple by hand, and can also be done immediately by the SageMath code in Fig. 14.

**Figure 14. Code to check (5) and (6)**

```python
var('p,q,r,lambda,mu')
mu = sqrt(p^2+r^2-(lambda-q)^2)
a = (p*(lambda-q)+ r*mu)/(p^2 + r^2)
b = (r*(lambda-q)- p*mu)/(p^2 + r^2)
eq1 = p*a + r*b + q -lambda
eq2 = r*a - p*b -mu
eq1 = eq1.full_simplify()
eq2 = eq2.full_simplify()
print(eq1) # 0 is printed
print(eq2) # 0 is printed
```

Equation (6) implies

(7) \[ r \neq 0 \]

as we now show. Assume, for proof by contradiction, that \( r = 0 \). Then by (6), \( \mu = -pb \leq 0 \); but \( \mu \) is a square root, so that implies \( \mu = 0 \). But under the assumption \( q \neq 0 \), which is currently in force, \( \mu \) is irrational. That is a contradiction. Therefore \( r \neq 0 \) as claimed.

Multiplying (5) and (6) shows that \( \lambda \mu \) is a rational linear combination of \( \{1, a, b, ab, a^2, b^2\} \). Since \( b^2 = 1 - a^2 \) we can delete \( b^2 \) from that set. We will next show that \( b \) can be expressed as a rational linear combination of the others, so that we can delete \( b \) too. Squaring (5) we have

\[
\frac{N}{2} = \lambda^2 = (pa + rb + q)^2
\]

\[
= p^2a^2 + r^2b^2 + q^2 + 2prab + 2pqa + 2rqb
\]

\[= (p^2 - r^2)a^2 + (r^2 + q^2) + 2prab + 2pqa + 2rqb \text{ since } b^2 = 1 - a^2 \]

Isolating the \( b \) term,

\[2rqb = (N/2 - r^2 - q^2) - (p^2 - r^2)a^2 - 2prab - 2pqa\]

We have assumed \( q \neq 0 \), and by (7), we also have \( r \neq 0 \). Therefore we can solve for \( b \):

(8) \[ b = \frac{N/2 - r^2 - q^2}{2rq} - a^2 \left( \frac{p^2 - r^2}{2rq} \right) - ab \left( \frac{\mu}{q} \right) - a \left( \frac{p}{r} \right) \]

Thus every element of the basis \( \{1, \lambda, \mu, \lambda \mu\} \) is a rational linear combination of \( \{1, a, a^2, ab\} \). Therefore \( \{1, a, a^2, ab\} \) is a basis of \( \mathbb{F} \).
Recall that $X$ is the length of $AB$ and $Y$ the length of $AC$. Then since $ABC$ is isosceles with base angles $\beta$, 

$$Y = 2X \cos \beta = 2X \sin \alpha = 2aX.$$ 

$X$ and $Y$ are integer linear combinations of $\{a, b, c\}$ because they are composed of tile edges. As above, we take $c = 1$, $a = \sin \alpha$, $b = \sin \beta = \cos \alpha$. Then we have 

$$X = pa + rb + q$$

(9) 

$$2aX = 2pa^2 + 2rab + 2qa$$

The edge decomposition of $AC$ gives us, for three non-negative integers $(t, s, u)$, 

$$Y = ta + sb + u$$

We put in the value of $b$ from (8): 

$$Y = u + s \left( \frac{N/2 - r^2 - q^2}{2rq} \right) + a \left( t - \frac{sp}{r} \right) - ab \left( \frac{sp}{q} \right) - a^2 \left( \frac{s(p^2 - r^2)}{2rq} \right)$$

(10) 

Since $Y = 2aX$, we may equate the coefficients of the basis elements $\{1, a, a^2, ab\}$ in (9) and (10). From the coefficient of $ab$ we have 

$$2r = -sp/q$$

Since $(r, p, q)$ are all non-negative, that implies $r = 0$. But that contradicts (7). That completes the proof in case $q \neq 0$.

We now take up the remaining case, $q = 0$. Then we have 

$$\mu = \sqrt{p^2 + r^2 - \lambda^2} = \sqrt{p^2 + r^2 - N/2}.$$ 

Then $\mu$ is the square root of a rational number, and belongs to the quadratic field $\mathbb{Q}(\lambda)$. Therefore $\mu$ is a rational multiple of $\lambda$. That is, $\mu = t\lambda$ for some rational $t$.

The formulas for $a$ and $b$ become 

$$a = \frac{p\lambda}{p^2 + r^2} \pm \frac{rt\lambda}{p^2 + r^2}$$

(11) 

$$b = \frac{r\lambda}{p^2 + r^2} \pm \frac{pt\lambda}{p^2 + r^2}$$

(12) 

making it evident that $a$ and $b$ are in $\mathbb{Q}(\lambda)$ and $a/b$ is rational. Then 

$$\mu^2 = t^2\lambda^2$$

$$p^2 + r^2 - N/2 = t^2N/2$$

(13) 

$$N/2 = \frac{p^2 + r^2}{t^2 + 1}$$

By Lemma[2] $N/2$ is a sum of two squares.

However, the lemma makes an additional claim: $N/2$ is not just the sum of some two squares, but specifically of $p^2 + r^2$. Since $q = 0$ we have $X = pa + rb$; starting
with the area equation \( \frac{N}{2} = X^2 \), we have
\[
\frac{N}{2} = (pa + rb)^2 = \left( p\left( \frac{p\lambda}{p^2 + r^2} \pm \frac{rt\lambda}{p^2 + r^2} \right) + r\left( \frac{r\lambda}{p^2 + r^2} + \frac{pt\lambda}{p^2 + r^2} \right) \right)^2
\]
\[
= \left( \frac{p^2 + r^2t^2 + p^2 + r^2t^2}{p^2 + r^2} \right)^2 \lambda^2
\]
\[
= (1 + t^2)^2 \frac{N}{2}
\]
Canceling \( \frac{N}{2} \) from both sides we have
\[
1 = (1 + t^2)^2
\]
Therefore \( t = 0 \). Then (13) becomes
\[
\frac{N}{2} = p^2 + r^2
\]
as claimed in the lemma. With \( t = 0 \), (11) and (12) become
\[
a = \frac{p\lambda}{p^2 + r^2}
\]
\[
b = \frac{r\lambda}{p^2 + r^2}
\]
Dividing we see \( b/a = \tan \beta = p/r \). That completes the proof of the lemma.

12. ISOSCELES \( \triangle ABC \) TILED BY A RIGHT TRIANGLE WITH \( \alpha / \pi \) RATIONAL

**Theorem 4.** Suppose \( \triangle ABC \) is isosceles with base angles \( \alpha \), and \( \triangle ABC \) is tiled by a right triangle similar to half of \( \triangle ABC \). If \( \alpha \) is a rational multiple of \( \pi \), then \( N \) is even and either
- (i) \( N/2 \) is a square, or
- (ii) \( N/2 \) is a twice a square (that is, \( N \) is a square) and \( \alpha = \pi/4 \), or
- (iii) \( N/2 \) is three times a square and \( \alpha = \pi/6 \).

**Remark.** One possible tiling under case (iii) of the theorem is illustrated in Fig. 15.

**Proof.** The proof reduces quickly to well-known facts about cyclotomic fields. Suppose that \( \alpha \) is a rational multiple of \( \pi \). By Lemma 4, \( \cos \alpha \) and \( \sin \alpha \) belong to
Therefore $e^{i\alpha}$ has degree 2 or 4 over $\mathbb{Q}$. We can therefore apply Corollary 1 to conclude that $\alpha = 2\pi/n$, where $n = 5, 8, 10, \text{ or } 12$.

From the area equation $2X^2 = N$ we have

\begin{equation}
2(pa + rb + q)^2 = N
\end{equation}

In case $n = 8$ we have $\alpha = \pi/4$; hence the left hand side of (14) belongs to $\mathbb{Q}(\sqrt{2})$; hence $\sqrt{N/2}$ belongs to $\mathbb{Q}(\sqrt{2})$. Then $\sqrt{N/2}$ has the form $u + v\sqrt{2}$ with $u$ and $v$ rational. Squaring both sides we have $N/2 = u^2 + 2u^2 + 2uv\sqrt{2}$. Hence $uv = 0$. In case $v = 0$ then $N/2$ is a square. In case $u = 0$ then $N/2$ is twice a square.

In case $n = 12$, $\alpha = \pi/6$, so $\cos \alpha = \sqrt{3}/2$ and $\sin \alpha = 1/2$; hence the left hand side belongs to $\mathbb{Q}(\sqrt{3})$; hence $\sqrt{N/2}$ belongs to $\mathbb{Q}(\sqrt{3})$. Then $\sqrt{N/2}$ has the form $u + v\sqrt{3}$ with $u$ and $v$ rational. Squaring both sides we have $N/2 = u^2 + 3v^2 + 2uv\sqrt{3}$. Hence $uv = 0$. Hence either $u = 0$ or $v = 0$. In case $u = 0$ then $N/2$ is three times a square (which is possible, for example by doubling one of the tilings in Fig. 10, producing a 54-tiling). In case $v = 0$ then $N/2$ is a square.

In case $n = 10$ we have $\alpha = \pi/5$. Then $\cos \alpha = (1/4)(1 + \sqrt{5})$, and

$$\sin \alpha = \frac{1}{2} \sqrt{\frac{1}{2}(5 - \sqrt{5})}$$

But by Lemma 1 $\sin \alpha$ and $\cos \alpha$ are both of degree 2 over $\mathbb{Q}$, so $\sin \alpha$ must belong to $\mathbb{Q}(\cos \alpha)$. Hence $\sqrt{(5 - \sqrt{5})/2}$ belongs to $\mathbb{Q}(\cos \alpha) = \mathbb{Q}(\sqrt{5})$. This is impossible, as SageMath can tell us using this code:

```python
x = sqrt((5 - sqrt(5))/2)
K.<a> = QuadraticField(5)
x = sqrt((5 - a)/2)
print("Is x in Q(sqrt(5))?")
print(x in K)
```

It is not difficult to verify that result by hand if desired, by showing that there are no rational numbers $u$ and $v$ such that $\sqrt{(5 - \sqrt{5})/2} = u + v\sqrt{5}$. Thus the case $n = 10$ cannot actually arise.

In case $n = 5$ we have $\alpha = 2\pi/5$ and

$$\sin \alpha = \frac{1}{2} \sqrt{\frac{1}{2}(5 + \sqrt{5})}$$

$$\cos \alpha = \frac{1}{4}(-1 + \sqrt{5})$$

and in this case also $\sin \alpha$ does not belong to $\mathbb{Q}(\cos \alpha)$, so by Lemma 4 this case cannot actually arise. That completes the proof of the theorem.

13. **Isosceles $ABC$ tiled by a right triangle with $\alpha/\pi$ irrational**

We have two ways to tile an isosceles triangle by a right triangle: either tile each of its two halves by a quadratic tiling, in which case $N$ is twice a square, or tile each of its halves with a biquadratic tiling, in which case $N$ is twice a sum of squares. See Figs. 10 and 17. These are the only possible values of $N$ when $\alpha$ is not a rational multiple of $\pi$, although we shall not prove that in this paper, but stick to our goal of proving $N$ cannot be 7 or 11.
Figure 16. $N$ is a twice a square or a twice a sum of squares. 50 is both.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure16}
\caption{Figure 16. 50 is both twice a square and twice a sum of squares.}
\end{figure}

Is it possible to have more complicated tilings without essential edges? Yes, because when two tiles share their hypotenuses, they form a rectangle, and we can just draw the diagonal of that rectangle the other way. In this way we can produce (exponentially) many different tilings, but they differ only in this trivial way. And sometimes, as shown in Fig. 18, even those rectangles can be rotated. That figure also shows that a tiling need not necessarily include the altitude of $ABC$.

Figure 18. There are many ways to rearrange the tiles.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure18}
\caption{Figure 18. There are many ways to rearrange the tiles}
\end{figure}

In the tilings based on two biquadratic tilings, there are no $c$ edges on $AB$ and $BC$, while in the tilings based on two quadratic tilings, there are only $c$ edges. There are of course some hybrid tilings when a square is also a sum of squares, in which $AB$ falls under one case and $BC$ under the other. If $N/2$ is not a square (as is the case for the biquadratic tilings) then there are no $c$ edges on $AB$ and $BC$, as we see in the biquadratic tilings (and prove in the next section). All these tilings,
in which $N/2$ is a sum of squares, involve essential edges (where tiles of different lengths occur on the two sides of an internal line). One sees such linear relations in two of the tilings illustrated in Fig. 16.

Laczkovich studied the possible shapes of tiles that can tile an isosceles triangle, but did not characterize the possible $N$. We have developed a theory of tilings of isosceles (but not equilateral) $ABC$ by a tile with $\gamma = \pi/2$. That theory provides a complete characterization of the possible $N$, a proof that the tile is necessarily rational, and since $N$ has to be even, it is certainly not prime. That work is long and has yet to appear in a peer-reviewed journal. In the spirit of this paper we eschew theoretical complications, instead giving a short computational proof that $N$ cannot be 7 or 11, or for that matter 19.

13.1. **The area equation for $ABC$ isosceles tiled by a right triangle.** Let $ABC$ be isosceles with base angles $\beta$ at $A$ and $C$. Let the sides of the tile be $(a, b, c)$. Without loss of generality we may assume

$$(a, b, c) = (\cos \alpha, \sin \alpha, 1).$$

The vertex angle at $B$ is $2\alpha$. Let $X$ be the length of the equal sides $AB$ and $BC$. Then twice the area of $ABC$ is equal to $N$ times twice the area of the tile:

$$X^2 \sin 2\alpha = Nab$$
$$X^2 2\sin \alpha \cos \alpha = Nab$$
$$2nX^2 ab = Nab$$
$$2x^2 = N \text{ area equation}$$

This will be our main tool.

13.2. **The $N/3$ lemma for tilings of isosceles $ABC$.** How many tiles can be supported by the equal sides $AB$ and $AC$ of isosceles $ABC$? In this section we show that one of the two supports fewer than $N/3$ tiles.

**Lemma 5.** Let isosceles $ABC$ with base angles $\alpha$ (at $A$ and $C$) be $N$-tiled by a tile with angles $(\alpha, \beta, \pi/2)$. Assume that $\alpha$ is not a rational multiple of $\pi$, and none of $(a, b, c)$ is an integer multiple of $a$ or an integer multiple of $b$. Then no tile has one vertex on $AB$ and another on $BC$.

**Remark.** Without the last hypotheses, the lemma would not be true. See Fig. 18.

**Proof.** Suppose, for proof by contradiction, that $E$ lies on $AB$ and $F$ lies on $BC$ and $EF$ is an edge of one tile in the tiling. Then triangle $BEF$ is tiled, so angles $BEF$ and $BFE$ are composed of tile angles. Since the vertex angle at $B$ is $2\alpha$, the sum of those two angles is $2\beta$. Since $\alpha$ is not a rational multiple of $\pi/2$, angles $BEF$ and $BFE$ are each equal to $\beta$. That means that more than one tile is supported on the north side of $EF$, since one tile cannot have two $\beta$ angles. Then the south side of $EF$ supports just one tile, say Tile 1.

Suppose Tile 1 has its $a$ edge on $EF$. Since $a < c$, there are no $c$ edges on the north side of $EF$. Hence north of $EF$ are only $b$ edges, so $a$ is an integer multiple of $b$ (which is contrary to hypothesis). Similarly, if Tile 1 has its $b$ edge on $EF$, then $b$ is an integer multiple of $a$, which again is contrary to hypothesis. Finally, if Tile 1 has its $c$ edge on $EF$, then since $a + b > c$, $c$ is an integer multiple of $a$ or an integer multiple of $b$, also contrary to hypothesis. These contradictions complete the proof of the lemma.
Lemma 6. Let isosceles $ABC$ with base angles $\alpha$ (at $A$ and $C$) be $N$-tiled by a tile with angles $\alpha, \beta, \pi/2$. Suppose $N/2$ is not a square, and $\alpha$ is not a rational multiple of $\pi$. Then fewer than $N/3$ tiles are supported on side $AB$, or fewer than $N/3$ tiles are supported on side $BC$.

Proof. According to the area equation, $X^2 = N/2$. If $N/2$ is not a square, then $X$ is not rational. Since $X = pa + qb + r$ for some integers $p$ and $q$, at least one of $a$ and $b$ is irrational. We argue by cases.

Case 1: $p = r = 0$ and $X = qb$. This is impossible, since the tile at $A$ has its $\beta$ angle at $A$ and hence has its $a$ or $c$ edge on $AB$.

Case 2: $q = r = 0$ and $X = pa$. This is impossible, since the tile at $B$ has its $\alpha$ angle at $B$.

Case 3: $p = q = 0$ and $X = rc$. But $c = 1$, so then $X$ is rational, and $N/2 = X^2$ is a rational square, contrary to hypothesis.

Case 4: $b = ma$ for some integer $m$. Then $X = \ell a + r$, and we may assume $\ell > 0$ since otherwise Case 3 applies. Then $a$ is irrational, since otherwise $X$ is rational and $N/2 = X^2$ is a square, contrary to hypothesis. The equation $1 = a^2 + b^2 = (1 + m^2)a^2$ implies $a = \sqrt{1/(1 + m^2)}$, so $\mathbb{Q}(a)$ is a quadratic field. We have

$$\frac{N}{2} = X^2 = \ell^2a^2 + 2\ell ra + r^2 = \frac{\ell^2}{1 + m^2} + r^2 + 2\ell ra$$

Regarding this as an equation in $\mathbb{Q}(a)$, we have a contradiction, since the irrational part is zero on the left and $2\ell > 0$ on the right. That disposes of Case 4.

Case 5: $a = mb$ for some integer $m$. Interchange $a$ and $b$ in the argument for Case 4. That disposes of Case 5.

Case 6: $c = 1 = ma$ for some integer $m$. Then $X = \ell a + b$ for some integer $\ell$, and we may assume $\ell > 0$ since otherwise Case 3 applies. Then $b$ is irrational and

$$\frac{N}{2} = X^2 = (\ell a + b)^2 = \ell^2a^2 + 2\ell ab + b^2 = \frac{\ell^2}{m^2} + (1 - 1/m^2) + 2\ell b$$

Regarding this as an equation in $\mathbb{Q}(b)$, the irrational part is zero on the left and $\ell m > 0$ on the right, contradiction. That disposes of Case 6.

Case 7: The only remaining possibility is that none of $(a, b, c)$ is an integer multiple of another one. Then Lemma 5 can be applied below.

Let $P$ be a vertex of the tiling lying on the interior of a side of $ABC$. If only two tiles meet at $P$ then they cannot have different angles, since any two of $(\alpha, \beta, \gamma)$ make together less than $\pi$. But if they have the same angle at $P$, that angle would be a right angle, contrary to hypothesis. Therefore at least three tiles meet at each such vertex $P$.

Let $n$ and $m$ be, respectively, the total number of tiles with an edge or vertex on $AB$, and the total number of tiles with an edge or vertex on $BC$. Then $n + m \leq N$,
since $N$ is the total number of tiles, and by Lemma 6 no tile contributes to both $n$ and $m$. Therefore either $n \leq N/2$ or $m \leq N/2$. Relabeling $A$ and $C$ if necessary, we can assume without loss of generality that $n \leq N/2$. Now let $p$ and $q$, respectively, be the number of tiles supported by $AB$, and the number of tiles with one and only one vertex on $AB$. Then $q \geq p - 1$ (it might be strictly greater if some vertices have more than three tiles sharing that vertex). Therefore

\[
\begin{align*}
p - 1 & \leq q \\
p + q & = n \leq N/2 \\
p & \leq N/2 - q \\
p & \leq N/2 - (p - 1) \\
2p & \leq N/2 + 1 \\
p & \leq N/4 + 2 \\
p & < N/3 \quad \text{since } N/4 + 2 < N/3 \text{ because } 2 < N
\end{align*}
\]

In the last line, $N > 2$ since if $N = 2$ then one tile would share vertices $A$ and $B$, and one tile would share vertices $A$ and $C$, and one of those tiles would have two $\alpha$ angles, so $\alpha = \pi/4$, so $\gamma = 2\alpha = \pi/2$, contradicting the hypothesis that the tile has no right angle. That completes the proof of the lemma.

13.3. $N$ cannot be 7 or 11 for an isosceles $ABC$ tiled by a right angle.

**Theorem 5.** Let the isosceles triangle $ABC$ with angles $(\alpha, \alpha, \alpha + \beta)$ be $N$-tiled by a tile with angles $(\alpha, \alpha, 2\gamma)$. Suppose $\alpha$ is not a rational multiple of $\pi$. Then $N$ is not equal to 7, 11, 14, or 19.

**Proof.** Let the sides of the tile be $(a, b, c)$. We do not assume they are rational, but without loss of generality we can assume $c = 1$. Because there is a tiling, there are non-negative integers $(p, q, r)$ such that $X = pa + qb + r$. Putting this expression for $X$ into the area equation, we have

\[
2(pa + qb + r)^2 = N
\]

Since $a^2 + b^2 = 1$, we set $b = \sqrt{1 - a^2}$:

\[
2(pa + q\sqrt{1 - a^2} + r)^2 = N
\]

SageMath cannot solve the equation in this form, so we bring it to an explicit polynomial form first by hand.

\[
\begin{align*}
2(pa + r)^2 + 2q^2(1 - a^2) + 4(pa + r)q\sqrt{1 - a^2} & = N \\
4(pa + r)q\sqrt{1 - a^2} & = N - 2(pa + r)^2 - 2q^2(1 - a^2) \\
16(pa + r)^2q^2(1 - a^2) & = (N - 2(pa + r)^2 - 2q^2(1 - a^2))^2
\end{align*}
\]

Having gotten rid of the square root, the equation is now in a form that SageMath is happy with, so we abstain from further simplification. By Lemma 6 if there is an $N$-tiling of this form, we will have (possibly after changing the labels on $A$ and $C$)

\[p + q + r < N/3.\]

Therefore, given a particular $N$, we can ask SageMath to check all such triples $(p, q, r)$ and try to solve the above equation for $a$ with $0 < a < 1$. Should it find no solution, then there can be no such $N$-tiling. Fig. 19 shows the code, which prints
Figure 19. SageMath code to check for tilings of isosceles $ABC$ by a right triangle

```python
def check_isosceles(N):
    var('x')
    Nover3 = int(N/3)
    p = 3
    for p in range(0,Nover3):
        for q in range(0,Nover3):
            for r in range(0,Nover3):
                if p + q + r >= N/3:
                    continue
                print(p,q,r)
                eq = (N - 2*(p*x+r)^2 - 2*q^2*(1-x^2))^2 \
                    - 4*q^2*(1-x^2)*(p*x+r)^2
                answers = solve(eq,x)
                for A in answers:
                    a = A.rhs()
                    if not a in RR:
                        continue # only interested in real a
                    if a <= 0 or a >= 0:
                        continue
                    b = sqrt(1-a^2).simplify()
                    print(a,b)
```

out any solution it finds. We ran this code for $N = 7, 11, 14, 19$, and it found no solutions. That completes the proof of the theorem.

14. USEFUL LEMMAS

In this section, we collect some facts that will be applied when we start eliminating the possibilities for 7-tilings and 11-tilings case by case, according to the cases of Laczkovich's second table.

14.1. Angles.

**Lemma 7.** Suppose triangle $ABC$ is $N$-tiled by a tile in which $3\alpha + 2\beta = \pi$. Then $\gamma > \pi/2$.

**Proof.**

\[
\begin{align*}
\pi &= 3\alpha + 2\beta \\
    &= \alpha + 2(\alpha + \beta) \\
    &= \alpha + 2(\pi - \gamma) \\
\gamma &= \frac{\pi}{2} + \frac{\alpha}{2} > \frac{\pi}{2}
\end{align*}
\]

That completes the proof.

**Lemma 8.** Let triangle $ABC$ be $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$. Suppose that either $3\alpha + 2\beta = \pi$ and $ABC$ is not isosceles with base angles $\alpha$, or $\gamma = 2\pi/3$. Then no tile has its $\gamma$ angle at a vertex of $ABC$. 
Proof. By Lemma 1, $\alpha$ and $\beta$ are not rational multiples of $\pi$. Hence the angles of $ABC$ are linear integral combinations of $\alpha$, $\beta$, and $\gamma$. First assume $3\alpha + 2\beta = \pi$. Then the angles of $ABC$ are each equal to $\alpha$, $2\alpha$, $\alpha + \beta$, $\beta$, or $2\beta$. Of these angles, all but $2\beta$ are less than $\gamma$, as we now show. Then $\gamma = \beta + 2\alpha$, and

$$\begin{align*}
\alpha &< \beta + 2\alpha = \gamma \\
\beta &< \beta + 2\alpha < \gamma \\
\alpha + \beta &< \beta + 2\alpha = \gamma \\
2\alpha &< \beta + 2\alpha = \gamma.
\end{align*}$$

Since $ABC$ is not similar to the tile, there cannot be a $\gamma$ angle alone at any vertex, since that would leave $\alpha + \beta$ for the other two vertices, making $ABC$ similar to the tile, since $\alpha$ is not a rational multiple of $\beta$.

Since all the possible angles but $2\beta$ are less than $\gamma$, it only remains to deal with the case where angle $C$ is equal to $2\beta$ and $\gamma < 2\beta$, and there is a tile with its $\gamma$ angle at $C$. We do not have $2\beta = \gamma$, by Lemma 1. Then there must be another tile at $C$ as well. If the angle of that tile at $C$ is $\alpha$, then the total angle at $C$ is at least $\gamma + \alpha = 2\alpha + \beta + \alpha = 3\alpha + \beta$, leaving only $\beta$ for the other two angles of $ABC$. But that is impossible, since $\alpha$ is not a rational multiple of $\beta$. If the second angle at $C$ is $\beta$, then the total angle at $C$ is at least $\gamma + \beta = 2\alpha + 2\beta$, leaving just $\alpha$ for the other two angles, which is again impossible. Hence the second angle at $C$ cannot be $\beta$. That completes the proof under the assumption $3\alpha + 2\beta = \pi$.

We now take up the case $\gamma = 2\pi/3$. Then the possible angles of $ABC$ are $\alpha$, $\beta$, $\alpha + \beta$, $\alpha + 2\beta$, $2\alpha + \beta$, $3\alpha$, and $3\beta$. All but $3\alpha$ and $3\beta$ are less than $2\alpha + 2\beta = \gamma$, so a $\gamma$ angle tile can occur, if at all, only at a vertex angle of $3\alpha$ or $3\beta$. Suppose vertex $C$ has angle $3\alpha$ and there is a $\gamma$ angle of a tile at $C$. Then $\gamma < 3\alpha$ and angles $A$ and $B$ together are $\pi - 3\alpha < \pi - \gamma$, which is impossible since the three angles of $ABC$ add up to $\pi$. Similarly if vertex $C$ has angle $3\beta$ and $\gamma < 3\beta$. That completes the proof of the lemma.

14.2. Two $c$ edges on each side of $ABC$.

Lemma 9. Suppose triangle $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ and $\gamma > \pi/2$. Suppose all the tiles along one side of $ABC$ do not have their $c$ sides along that side of $ABC$. Then there is a tile with a $\gamma$ angle at one of the endpoints of that side of $ABC$.

Proof. Let $PQ$ be the side of $ABC$ with no $c$ sides of tiles along it. Then the $\gamma$ angle of each of those tiles occurs at a vertex on $PQ$, since the angle opposite the side of the tile on $PQ$ must be $\alpha$ or $\beta$. Let $n$ be the number of tiles along $PQ$; then there are $n - 1$ vertices of these tiles on the interior of $PQ$. Since $\gamma > \pi/2$, no vertex on the boundary has more than one $\gamma$ angle. By the pigeonhole principle, there is at least one tile whose $\gamma$ angle is not at one of those $n - 1$ interior vertices; that angle must be at $P$ or $Q$. That completes the proof of the lemma.

Lemma 10. Suppose triangle $ABC$ is $N$-tiled by a tile $T$ with angles $(\alpha, \beta, \gamma)$.

Suppose

(i) $\gamma > \pi/2$, and

(ii) $\alpha$ is not a rational multiple of $\pi$, and

(iii) Every angle of triangle $ABC$ is less than $\gamma$. 


Then there are at least two $c$ edges of tiles on side $AC$.

Remarks. One can prove by the same method that the $c$ edges must occur in adjacent blocks of at least two edges, but we found no use for that result.

Proof. By hypothesis (ii), every boundary vertex $P$ (except $A$, $B$, and $C$) that has a $\gamma$ angle (i.e., some tile with a vertex at $P$ has its $\gamma$ angle at $P$) touches exactly three tiles, which contribute angles of $\alpha$, $\beta$, and $\gamma$. By Lemma 9, each side of $ABC$ has at least one $c$ edge. The present lemma, however, claims more: there must be at least two $c$ edges. Suppose, to the contrary, that there is just one $c$ tile, Tile 1, with an edge on one side $EF$ of triangle $ABC$. Then all the other tiles with an edge on $EF$ have a $\gamma$ angle on $EF$. We visualize $EF$ as horizontal with triangle $ABC$ above, and use the word “north” and “northwest” accordingly. See Fig. 20.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure20}
\caption{Proof of Lemma 10: another tile won’t fit next to Tile 2}
\end{figure}

Since there cannot be a $\gamma$ angle at the vertices of $ABC$, it follows that both the tiles on $AC$ adjacent to Tile 1 (if there are two, or otherwise, only the one) have their $\gamma$ angles adjacent to Tile 1. Let $PQ$ be the $c$ edge of Tile 1 lying on $AC$. Let $R$ be the northern vertex of Tile 1. Suppose (without loss of generality) that Tile 1 has its $\beta$ angle at $Q$. Then the side $PR$ of Tile 1, opposite $Q$, has length $b$. Let Tile 2 be the tile adjacent to $PR$.

Since the hypotheses of the theorem remain true if (the names of) $\alpha$ and $\beta$ are interchanged, we may assume without loss of generality that $\alpha < \beta$. Then by the law of sines, $a < b$. Since $\gamma > \pi/2$ we also have $a < c$ (by the law of cosines).

Assume, for proof by contradiction, that neither $P$ nor $Q$ is a vertex of $ABC$. Then there exist Tile 0 and Tile 3 on $AC$ sharing vertices $P$ and $Q$ with Tile 1. Tile 2, between Tile 0 and Tile 1, must have its $\beta$ angle at $P$, since Tile 1 has its $\alpha$ angle there and Tile 0 has its $\gamma$ angle at $P$. There is an open $\alpha$ angle between Tile 1 and Tile 3; let Tile 4 be the tile that fills that notch. Then Tile 4 has its $b$ or $c$ edge along $QR$. Since Tile 1 has its $a$ edge along $QR$ and $a < b$ and $a < c$, the edge of Tile 4 on $QR$ extends past $R$. Then the segment $PR$ is of length $b$ and its northwest side is composed of a number of tile edges, starting with Tile 2 at $P$. These must all be $a$ edges, since $a$ is the only edge less than $b$. Since the tiles northwest of $PR$ all have their $a$ edges on $PR$, they all have a $\gamma$ angle on $PR$. But Tile 2 does not have its $\gamma$ angle at $P$, since Tile 0 has its $\gamma$ angle at $P$. And the last tile cannot have its $\gamma$ angle at $R$, since Tile 4 extends along $QR$ past $R$,
and Tile 1 has its $\gamma$ angle at $R$. So if there are $n$ tiles northwest of $PR$, there are only $n-1$ possible places for their $\gamma$ angles, contradicting the pigeon-hole principle. This contradiction proves that one of $P$ or $Q$ is a vertex of $ABC$.

Now we argue by cases.

Case 1: $Q$ is a vertex of $ABC$, i.e., $Q = C$. If the angle of $ABC$ at $Q$ is strictly between $\beta$ and $2\beta$, then Tile 4 must have its $\alpha$ angle at $Q$, and we argue exactly as before. If the angle of $ABC$ at $Q$ is exactly $\beta$, then we argue as above, except that $RQ$ is now extended past $R$ by one side of $ABC$ rather than an edge of Tile 4. The argument about the $\gamma$ angles of the tiles northwest of $PR$ is unchanged, if $P$ is not a vertex of $ABC$. If $P$ is a vertex of $ABC$, then we still can argue that Tile 2 must have its $a$ side on $PR$, because it cannot fit next to Tile 1 with its $b$ or $c$ side on $PR$.

Therefore we may assume that the angle of $ABC$ at $Q$ is at least $2\beta$, and that Tile 4 has its $\beta$ angle at $Q$ and its $a$ edge against Tile 1. Hence there is a double angle at $Q$. Then by hypothesis (iv), $b$ is not a multiple of $a$. Tile 4 cannot have its $\gamma$ angle at $Q$, by hypothesis (iii). Therefore Tile 4 has its $\gamma$ angle at $R$, and since $\gamma > \pi/2$ by hypothesis (i), $PR$ does not extend past $R$ as part of the tiling. The tiles northwest of $PR$ must all have their $a$ edges on $PR$, since $a$ is the only edge less than $b$. Similarly, the tiles supported by the west edge of Tile 4 must all have their $a$ edges against that west edge, which has length $b$. All those tiles northwest of $PR$ have their $\gamma$ angles on $PR$ (since they have their $a$ edges on $PR$), and by the pigeonhole principle those $\gamma$ angles are all at the northwest. Therefore the tile supported by $PR$ at $R$, call it Tile 5, has its $\gamma$ angle there. Since Tiles 1 and 4 already have their $\gamma$ angles at $R$, Tile 5 shares an edge with Tile 4, and as just shown that edge has length $a$. But now Tile 5 has two $a$ edges, contradiction. That completes Case 1.

Case 2: $P$ is a vertex of $ABC$, and $Q$ is not. Then Tile 4 is placed as shown in the figure. Therefore the angle of $ABC$ at vertex $P$ must be greater than $\alpha$, since if it were equal to $\alpha$, Tile 4 would not lie inside $ABC$. Then Tile 2 exists, and Tile 2 must have its $a$ side on $PR$, because it cannot fit next to Tile 1 with its $b$ or $c$ side on $PR$. From there the argument proceeds as before. That completes Case 2.

That completes the proof of the lemma.

If there are enough tiles on the boundary of $ABC$ then $N$ must be at least 12. How many is “enough”? As it turns out we do not need a precise answer; the following lemma is helpful enough and easy to prove. No doubt the number 10 can be improved, but this is good enough.

**Lemma 11.** Let $ABC$ be $N$-tiled, and suppose the total number of tiles with an edge on the boundary of $ABC$ is at least $k$, with at least two tiles on each side of $ABC$, and only one tile at $B$, and a total of five tiles at the vertices of $ABC$. Suppose $\gamma \neq \pi/2$. Then $N \geq k + 2$.

**Proof.** We must produce at least two non-boundary tiles. Case 1, two vertices, say $A$ and $B$, of $ABC$ have only one tile each. Since $\gamma \neq \pi/2$, at three tiles (at least) meet at each boundary vertex. Therefore, the tile that shares an edge with the tile at $A$ is not a boundary tile, and the same for the tile next to the tile at $B$. That makes at least $k + 2$ tiles.

Case 2, only $B$ has a single tile, while vertices $A$ and $C$ have two tiles each. Then the tile adjacent to the tile at $B$ is a non-boundary tile. Consider the two
tiles at vertex $A$, say Tile 1 and Tile 2. If they do not share a common edge then one of them, say Tile 1, has a shorter edge along their common boundary. Then the tile adjacent to that edge is not a boundary tile, and hence it is a second non-boundary tile. If they do share a common edge, then let Tile 3 and Tile 4 be the tiles adjacent to Tile 1 and Tile 2, respectively. At most one of Tile 3 and Tile 4 can have a boundary extending past the common interior vertex $E$ of Tile 1 and Tile 2, and the one that does not cannot be a boundary tile. Hence it is a second non-boundary tile. That completes the proof of the lemma.

15. The case $3\alpha + 2\beta = \pi$

Three of the rows of Table 4 fall under the case $3\alpha + 2\beta = \pi$, with $\alpha$ not a rational multiple of $\pi$. For some of those cases we have proved necessary and sufficient conditions for the existence of an $N$-tiling; and for all of them we have strong necessary conditions. In other words, we have added a fourth column to Table 4 at least for the rows corresponding to $3\alpha + 2\beta = \pi$. From those entries we can simply read off that $N = 7$ and $N = 11$ are impossible. In fact $N = 28$ is the smallest possible $N$. But the proofs, which are unpublished, occupy approximately a hundred pages, and we wish to make the present paper self-contained (except for its dependence on [4] and [10]). Therefore we give a short, self-contained, algebraic and computational proof that $N$-tilings do not exist when $N < 12$ and $3\alpha + 2\beta = \pi$ and $\alpha$ is not a rational multiple of $\pi$. 

An important tool in the analysis of these tilings is the “coloring equation” given in Theorem 2. That theorem applies here, as we now show. If $3\alpha + 2\beta = \pi$ and $\alpha$ is not a rational multiple of $\pi$, then every boundary vertex is composed of three tiles ($\alpha + \beta + \gamma$) or five tiles ($3\alpha + 2\beta$), and every interior vertex is either a “center” with four tiles ($3\gamma + \beta$) or has six tiles ($2\alpha + 2\beta + 2\gamma$) or eight tiles ($4\alpha + 3\beta + \gamma$) or ten tiles ($6\alpha + 4\beta$).

Since there are five tiles at the angles of $ABC$, by renaming the vertices we may assume that only one tile is at $B$. Let $(X, Y, Z)$ be the lengths of sides $AB$, $BC$, and $AC$. Then we have the “coloring equation”

$$M(a + b + c) = X + Z \pm Y$$

where the + sign is taken if the angles at $A$ and $C$ have an odd number of tiles, and the − sign is taken if they have an even number.

Besides the coloring equation, we have the “area equation”, which says that the area of $ABC$ is equal to $N$ times the area of the tile. We use the formula for the area of a triangle that says twice the area is the product of two adjacent sides and the sine of the included angle. By the law of sines, $a/c = \sin \alpha / \sin \gamma$. Then the area equation can be written

$$XZ \sin \alpha = Nbc \sin \alpha$$

$$XZ = Nbc \quad \text{if \ angle \ } B = \alpha$$

$$XZ = Nac \quad \text{if \ angle \ } B = \beta$$

**Definition 1.** Let a triangle have angles $(\alpha, \beta, \gamma)$. We define

$$s = 2 \sin (\alpha/2)$$

This definition is useful because the ratios $a/c$ and $b/c$ can be expressed simply in terms of $s$, as shown in the following lemma.
Lemma 12. Suppose $3\alpha + 2\beta = \pi$. Let $s = 2 \sin \alpha / 2$. Then we have
\[
\begin{align*}
\sin \gamma &= \cos \frac{\alpha}{2} \\
\frac{a}{c} &= s \\
\frac{b}{c} &= 1 - s^2
\end{align*}
\]

Proof. Since $\gamma = \pi - (\alpha + \beta)$, we have
\[
\begin{align*}
\sin \gamma &= \sin(\pi - (\alpha + \beta)) \\
&= \sin(\alpha + \beta) \\
&= \cos(\pi / 2 - (\alpha + \beta)) \\
&= \cos \frac{\alpha}{2} \quad \text{since } \pi/2 - \beta = 3\alpha/2
\end{align*}
\]
Then $c = \sin \gamma = \cos \alpha / 2$, and $a = \sin \alpha = 2 \sin (\alpha / 2) \cos (\alpha / 2)$. Hence
\[
\frac{a}{c} = 2 \sin \alpha / 2.
\]
Since $3\alpha + 2\beta = \pi$, we have
\[
\begin{align*}
\sin \beta &= \sin(\pi / 2 - 3\alpha / 2) \\
&= \cos(3\alpha / 2) \\
&= 4 \cos^3 \frac{\alpha}{2} - 3 \cos \frac{\alpha}{2}
\end{align*}
\]
Hence
\[
\frac{b}{c} = 4 \cos^2 (\alpha / 2) - 3
\]
\[
= 4(1 - \sin^2 \alpha / 2) - 3
\]
\[
= 1 - 4 \sin^2 \alpha / 2
\]
Then we have
\[
\begin{align*}
\frac{a}{c} &= s \\
\frac{b}{c} &= 1 - s^2
\end{align*}
\]
establishing the second equation of the lemma. That completes the proof of the lemma.

Theorem 6. Suppose $3\alpha + 2\beta = \pi$, and triangle $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ not similar to $ABC$, and $\alpha$ is not a rational multiple of $\pi$. Then $N \geq 12$.

Proof. We first discuss the possibility of applying of Lemma[10] Do the hypotheses hold? By Lemma[8] no tile has a $\gamma$ angle at a vertex of $ABC$; and by Lemma[7] $\gamma > \pi / 2$. Since the tile is not similar to $ABC$, and $\alpha$ is not a rational multiple of $\pi$, each angle of $ABC$ is less than $\gamma$. Therefore Lemma[7] is applicable.

We now explain the idea of the proof. The tiling provides an expression for each side of $ABC$ as a linear combination of $abc$. Thus
\[
\begin{align*}
X &= pa + qb + rc \\
Z &= ua + vb + wc \\
Y &= ka + \ell b + mc
\end{align*}
\]
Substitute these expressions for \((X, Y, Z)\) in the coloring equation. With \(P = p + u \pm k, Q = q + v \pm \ell, R = r + w \pm m\) we have \(M(a + b + c) = Pa + Qb + Rc\). Dividing by \(c\) and use \(a/c = s\) and \(b/c = 1 - s^2\) we have
\[
M(2 + s - s^2) = Ps + Q(1 - s^2) + R
\]
For given \((M, P, Q, R)\) that quadratic can be solved for \(s\) (provided its discriminant is nonnegative). The area equation too can be expressed in terms of \(s\), and we can check if it is satisfied for the \(s\) from the coloring equation. For a given \(N\), we need to consider only values of the integer parameters between 0 and \(N\), so this search will terminate. Moreover, as discussed in the first paragraph of this proof, Lemma 10 tells us that we can restrict the search by only examining values of \(r, w,\) and \(m\) that are at least 2, provided \(ABC\) is isosceles or \(s\) is rational. Finally, Lemma 11 allows us to not consider cases in which there would be ten or more boundary tiles, i.e., when \(p + q + r + u + v + w + k + \ell + m \geq 10\). SageMath code to carry out this plan for isosceles \(ABC\) with base angles \(\alpha\) or \(\beta\) is exhibited in Fig. 21. Run that code passing 7 as the function parameter, and then again passing 11. It runs in about 12 seconds, and produces no output except the reassuring progress reports as \(M\) changes. That shows that there is no 7 or 11 tiling in the case of isosceles \(ABC\) with angles \(\alpha\) at \(A\) and \(B\), or \(\beta\) at \(A\) and \(B\), i.e., when the coloring equation is \(M(a + b + c) = X + Y + Z\).

The other possible shapes of \(ABC\) satisfy the coloring equation \(M(a + b + c) = X - Y + Z\). That code differs from the code in Fig. 21 in two respects. First, because of the minus sign in the coloring equation, negative values of \((P, Q, R)\) are allowed, and the upper limits of \((P, Q, R)\) go up to \(N, N - |P|,\) and \(N - |P| - |R|\), respectively, and the values of \((k, \ell, m)\) are preceded by a minus sign, with a continue statement inserted to reject negative values. Second, we are only allowed to assume each side contains at least two \(c\) edges in case \(s\) is rational, so the variable \texttt{looplimit} has to be recalculated each time \(s\) is recalculated, and set to 2 if \(s\) is rational, and otherwise to 1. Although it adds a page to the length of the paper, we also include enough of this code so that any reader can reproduce our results. See Fig. 22.

To prove the theorem as stated, we ran both programs for all \(N\) between 3 and 11, inclusive. The second program is slower, requiring 27 seconds for \(N = 7\), over three minutes for \(N = 11\), and about 8 minutes for all values 3 to 11. But it gets the answer: no solutions are found. That completes the proof.

Remark. This method cannot be used for \(N = 14\) or 19, as some solutions are found. As discussed in [20] below, we do have a proof that there is no 14-tiling or 19-tiling with \(3\alpha + 2\beta = \pi\); but the short direct computational proof given here will not work.
Figure 21. SageMath code used in the proof of Theorem 6. $ABC$ isosceles

```python
def oct22(N):  # ABC isosceles
    var('P,Q,R,M,s,p,q,r,u,v,w,k,ell,m')
    epsilon = 0.0000001
    lowerlimit = 2  # each side has at least 2 c edges
    for M in range(1,N):
        print("M=%d" %M)
        for P in range(0,N):
            for Q in range(0,N-P):
                for R in range(6,N-P-Q):
                    eq1 = M*(2+s-s^2) - P*s - Q*(1-s^2) - R
                    discriminant = (M-P)^2 - 4*(Q-M)*(2*M-Q-R)
                    if discriminant < 0:
                        continue
                    answers = solve(eq1,s)
                    for x in answers:
                        if x.rhs() <= 0 or x.rhs() >= 1:
                            continue
                        for r in range(lowerlimit,R+1):
                            for w in range(lowerlimit,R-r):
                                m = R -r-w
                                if m < lowerlimit:
                                    continue
                                for p in range(0,P+1):
                                    for u in range(0,P-p):
                                        k = P-p-u;
                                        for q in range(0,Q+1):
                                            for v in range(0,Q-q):
                                                ell = Q-q-v
                                                boundarytiles = p+q+r+u+v+w+k+ell+m
                                                if boundarytiles >= N-2:
                                                    continue
                                                X = r + p*S + q*(1-S^2)
                                                Y = w + u*S + v*(1-S^2)
                                                area1 = abs( X*Y - N*(1-S^2))
                                                area2 = abs(X*Y - N*S)
                                                if n(area1) < epsilon:  # B = alpha
                                                    print("alpha",N,M,p,q,r,u,v,w,k,ell,m)
                                                    print(area1)
                                                if n(area2) < epsilon:  # B = beta
                                                    print("beta",N,M,p,q,r,u,v,w,k,ell,m)
```
Figure 22. SageMath code used in the proof of Theorem 6

def oct22b(N):  # case when ABC is not isosceles
    var(’P,Q,R,M,s,p,q,r,u,v,w,k,ell,m’)
    epsilon = 0.0000001
    for M in range(1,N):
        print("M=%d" %M)
        for P in range(-N,N+1):
            for Q in range(-(N-abs(P)),N-abs(P)+1):
                for R in range(-(N-abs(P)-abs(Q)),N-abs(P)-abs(Q)+1):
                    eq1 = M*(2+s-s^2) - P*s - Q*(1-s^2) - R
                    discriminant = (M-P)^2 - 4*(Q-M)*(2*M-Q-R)
                    if discriminant < 0:
                        continue
                    answers = solve(eq1,s)
                    for x in answers:
                        S = x.rhs()
                        if S <= 0 or S >= 1:
                            continue
                        lowerlimit=1;  #will be set to 2 when S is rational
                        if S < 1-S^2 and (S in QQ or not (1-S^2)/S in ZZ):
                            lowerlimit = 2
                        else:
                            if 1-S^2 < S and (not (S/(1-S^2) in ZZ)):
                                lowerlimit = 2
                            else:
                                lowerlimit = 1
                        for r in range(lowerlimit,R+1):
                            for w in range(lowerlimit,R-r):
                                m = -(R-r-w)
                                if m < lower3limit:
                                    continue
                                for p in range(0,P+1):
                                    for u in range(0,P-p):
                                        k = -(P-p-u);
                                        if k < 0:
                                            continue
                                        for q in range(0,Q+1):
                                            for v in range(0,Q-q):
                                                ell = -(Q-q-v)
                                                if ell < 0:
                                                    continue
                                                # ... the rest as in the previous figure
16. The case $\gamma = 2\pi/3$ and $\alpha$ not a rational multiple of $\pi$

In this case, $\alpha + \beta = \pi/3$, so a boundary vertex can be composed of angles contributed by 3 or 6 tiles. Hence it is not in general possible to color the tiles black and white in a way that leads to a “coloring equation.”

There are several shapes possible for $ABC$, listed in Table 4, but for our purposes here there are just two cases to consider: either one of the vertices of $ABC$ has just one tile (in which case we rename $\alpha$ and $\beta$ so that the standalone angle is $\alpha$, and we rename the vertices so it occurs at $A$), or there are two tiles at each of the three vertices, in which case we may assume that the angle at $A$ is $\alpha + \beta$. We do not need to consider the case $ABC$ similar to the tile, so no $\gamma$ angles occur at the vertices of $ABC$. The shape of the tile can be expressed using the law of cosines, since $\cos(2\pi/3) = -\frac{1}{2}$, by the equation

$$c^2 = a^2 + b^2 + ab.$$  (19)

For example, $(3, 5, 7)$ and $(8, 7, 13)$ are rational tiles satisfying this equation.

Although there is no coloring equation, we do still have the “area equation” that says the area of $ABC$ is $N$ times the area of the tile. That equation takes different forms depending on the shape of $ABC$. In case the angle at $A$ is $\alpha$, and the sides $AB$ and $AC$ have length $X$ and $Y$, the area equation is $XY \sin \alpha = Nab \sin \alpha$. After canceling $\sin \alpha$ we have

$$XY = Nab$$  (20)

We do have some general results about this kind of tiling, but the theory is incomplete and we do not go into it here. For our present purposes it suffices to show that any such tiling requires at least 12 tiles; that is Theorem 7 below.

**Theorem 7.** Let triangle $ABC$ be $N$- tiled by a tile with angles $(\alpha, \beta, 2\pi/3)$, not similar to $ABC$, and suppose $\alpha$ is not a rational multiple of $\pi$. Then $N \geq 12$. In particular, $N$ is not equal to 7 or 11.

**Remark.** The idea of the proof of this case is that, because of Lemma 10, each side of $ABC$ is at least $2c$ in length, and that makes the area more than the area of 12 tiles. See Fig. 23 which illustrates an equilateral $ABC$ with six tiles placed, and more area remaining than six tiles can cover. (This figure is only illustrative.) We first proved this theorem by a geometrical argument about placing tiles, but algebra is shorter and simpler. Both ideas can be seen in Fig. 23: It is geometrically impossible to complete the tiling, and also the untiled area is more than the area of six tiles.
Figure 23. The case of Theorem 7 when there are exactly six boundary tiles

Proof. Since the tile is not similar to $ABC$, and $\alpha$ is not a rational multiple of $\pi$, there can be no $\gamma$ angle at a vertex of $ABC$. Then there must occur a total of six tiles at the vertices of $ABC$, contributing three $\alpha$ angles and three $\beta$ angles to make up the angles of $ABC$. Lemma 10 is applicable, since no vertex of $ABC$ can have a $\gamma = 2\pi/3$ angle, so each side of $ABC$ has at least two $c$ edges. That is the key idea of this proof.

We divide the proof into two cases. Case 1: One vertex of $ABC$ has an angle $\delta$ with $\pi/3 \leq \delta \leq 2\pi/3$. The point of that inequality is that it implies $\sin \pi/3 < \sin \delta$.

Let $X$ and $Y$ be the lengths of the sides adjacent to that angle. Then we have the area equation

$$XY \sin \delta = Nab \sin \frac{2\pi}{3}$$

Since $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \leq \sin \delta$, we have

$$XY \leq Nab$$

According to Lemma 10 there are at least two $c$ edges on each side of $ABC$. Hence $X \geq 2c$ and $Y \geq 2c$. Therefore $Nab \geq XY \geq 4c^2$. Therefore

$$(21) \quad 3ab \geq \frac{12c^2}{N}$$

Recall (19):

$$c^2 = a^2 + b^2 + ab$$

$$(22) \quad c^2 - 3ab = a^2 + b^2 - 2ab = (a-b)^2 > 0$$

Substituting on the left from (21) we have

$$c^2 \left(1 - \frac{12}{N}\right) > 0$$

We have strict inequality since the hypothesis that $\alpha$ is not a rational multiple of $\pi$ implies $a \neq b$. Since $c^2 > 0$ we have

$$1 - \frac{12}{N} > 0 \quad \frac{N}{N} > 12$$

That completes the proof in Case 1.

Case 2: Every vertex angle of $ABC$ is either more than $2\pi/3$ or less than $\pi/3$. They cannot all be less than $\pi/3$ since they add up to $\pi$. Therefore one angle is
more than $2\pi/3$. Renaming the vertices if necessary, we can assume the angle at $B$
more than $2\pi/3$. Renaming $\alpha$ and $\beta$ if necessary, we can assume $\alpha < \beta$. Then
the angles at $A$ and $C$ are either $(\alpha, \alpha)$ or $(\alpha, 2\alpha)$ or $(2\alpha, \alpha)$, since otherwise the
angle at $B$ is $\leq \pi - (\alpha + \beta) = 2\pi/3$.

I say that no tile has one vertex on $AB$ and another vertex on $BC$. Suppose, to
the contrary, that Tile 1 has vertex $P$ on $AB$ and vertex $Q$ on $BC$. Then Tile 1
does not have a vertex at $B$, since the tiles at $B$ have at least one vertex interior
to $ABC$. Consider triangle $PBQ$. Angle $B$ is either $3\beta$ or $3\beta + \alpha$, either of which
is more than $2\pi/3$, so $|\cos B| > 1/2$. Hence angles $P$ and $Q$ of triangle $PBQ$ are
acute. Consider the rays emanating from $B$ along the dividing lines between tiles
with vertices at $B$. All three or four of the tiles at $B$ have one $c$ edge emanating
along such a ray, and the whole tile is contained in $PBQ$. If one of the tiles at $B$
supported by $PB$ or $BQ$ has its $c$ edge there, then $PB \geq c$ or $BQ \geq c$. Otherwise
let $BR$ be the ray containing the other edge of the tile supported by $PB$ at $B$, with
$R$ on $PQ$. In triangle $BRP$, $BR$ is opposite an acute angle $P$, and $PB$ is opposite
an angle greater than $\pi/2$, since that angle is $\pi$ minus angle $P$ minus angle $PBR$,
and both the subtracted angles are acute. Since the greater side is opposite the
greater angle, $BP > BR \geq c$. Thus either $BP \geq c$ or $BQ \geq c$. Relabeling $P$ and
$Q$ if necessary, we can assume $BP \geq c$. Since $BQ$ is also composed of tile edges, at
least $BQ \geq a$. Then by the law of cosines we have

$$PQ^2 = BP^2 + BQ^2 - 2PB \cdot QB \cos B$$
$$= BP^2 + BQ^2 + 2PB \cdot QB |\cos B|$$
$$\geq c^2 + a^2 + 2ac |\cos B|$$
$$= c^2 + a^2$$
$$> c^2$$

Hence the length of $PQ$ is greater than $c$, and hence cannot be just one tile edge.
Hence, as I said, no tile has one vertex on $AB$ and another vertex on $BC$.

I say that $c$ is not a linear integral combination of $(a, b)$ unless it is a multiple of
$a$. For suppose $c = ua + vb$. Then

$$c^2 = a^2 + b^2 + ab$$
$$(ua + vb)^2 = a^2 + b^2 + ab$$

$$(u^2 - 1)a^2 + (v^2 - 1)b^2 + (2uv - 1)ab = 0$$

which is a contradiction if both $u$ and $v$ are positive. Therefore $u = 0$ or $v = 0$. If
$v = 0$, then $c$ is a multiple of $a$ as claimed. Therefore we may assume $c = vb$ with
$v > 1$. Then by the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$
$$> b^2 + c^2 - 2bc$$
$$= b^2(v^2 + 1 - 2v) \quad \text{since } c = vb$$
$$= b^2(v - 1)^2$$
$$> b^2 \quad \text{since } v > 1$$

Hence $a > b$, contradiction. Hence, as I said, $c$ is not a linear integral combination
of $(a, b)$ unless it is a multiple of $a$.

We now divide into further cases. Case 2A: each of the two edges $AB$ and $BC$
supports at least three tiles. Then, there are two “notches” between the three tiles,
so there are five tiles touching $AB$ in at least one point, and five tiles touching $BC$. None of these have been double-counted, since no tile has a vertex on $AB$ and a vertex on $BC$, so that is ten tiles. If $ABC$ is isosceles then there are four tiles with vertices at $B$, two of which we have not yet counted, so that makes 12. If $ABC$ is not isosceles then there are three tiles with vertices at $B$, and one more with a vertex at $A$ or $C$, making again two uncounted tiles for 12 total. That completes the proof in Case 2A.

Case 2B: $AB$ supports exactly two tiles, and $ABC$ has angle $\alpha$ at $A$. As already noted, by Lemma 10, there are at least two $c$ edges on each side of $ABC$, so both the tiles supported by $AB$ have their $c$ edges on $AB$. See Fig. 24 for an illustration of the following argument. Let Tile 1 be the tile at $A$; then Tile 1 has its $b$ edge on $AC$. Let Tile 2 be the tile east of Tile 1. Tile 2 shares the $a$ edge of Tile 1, since that edge terminates at both ends on the boundary of $ABC$. Tile 2 cannot have its $\gamma$ angle to the south, since that would make two $\gamma$ angles at a vertex on the boundary. Therefore it has its $\gamma$ angle on $AB$ and its $\beta$ angle on $AC$. Let Tile 3 be the tile north of Tile 2. Then Tile 3 is supported by $AB$ and hence has its $c$ edge on $AB$, and shares the $b$ edge of Tile 2 on its southern border. Let Tile 4 be the tile east of Tile 3. Then Tile 4 has a vertex at $B$. Either Tile 4 has its $b$ or $c$ edge along Tile 3 (either of which extends beyond Tile 3, since $a < b < c$), or it has its $a$ edge shared with Tile 3 and its $\gamma$ angle to the southwest. In all three of those cases, it terminates the line forming the southeast border of Tile 2. Let Tile 5 be the tile south of Tile 2, with its southwest vertex $P$ on $AC$ shared with Tile 1.

Suppose, for proof by contradiction, that Tile 5 does not share the $c$ edge of Tile 2. Since its $\alpha$ angle is towards the west, it has its $b$ edge along Tile 2. Since the tiles south of Tile 2 must terminate at the eastern vertex of Tile 2, the remaining $c - b$ of the southeast border of Tile 2 must be filled by some tile edges. But then $c$ would be an integral linear combination of $a$ and $b$, including at least one $b$, which is impossible, as proved above. That contradiction completes the proof that Tile 5 shares the $c$ edge of Tile 2. Hence Tile 5 has its $b$ edge on $AC$, as shown in Fig. 24.

Then Tiles 1, 2, 3, 5 are definitely as shown in Fig. 24.

Now consider $BQ$, the eastern border of Tiles 3 and 5. On the west side of $BQ$ are two $a$ edges. These cannot be matched on the east by two $a$ edges, since then the $\gamma$ angles of those two tiles would occur on $BQ$, either both to the north, or both to the south, but both are impossible. If Tile 4 has its $b$ edge on $BQ$ then $2a - b$ is an integer linear combination of $(a, b, c)$, which is impossible. Hence Tile 4 has its $c$ edge on $BQ$. Hence $c = 2a$ and Tile 4 has a vertex on $AC$ as shown in

Figure 24. Case 2B of Theorem 7
Let Tile 6 be south of Tile 4. Then Tile 6 has its $\beta$ angle at $Q$ and hence does not have its $b$ edge along Tile 4. Since $2a = c > b$, the tile or tiles south of Tile 4 do not terminate at the eastern boundary of Tile 4, but continue to the east.

Let Tile 7 be the tile east of Tile 4. Then Tile 7 must share its $a$ edge with Tile 4, since it cannot extend to the south. But that is impossible, as then it would have a $\gamma$ angle on the shared $a$ edge, but that cannot occur at $B$ on the north or at the southeast vertex of Tile 4 either. We have reached a contradiction. That completes the proof in Case 2B.

Case 2C: The angle of $ABC$ at $A$ is $2\alpha$ and $AB$ supports exactly two tiles. Then again the two tiles on $AB$ have their $c$ edges on $AB$. As before let Tile 1 have a vertex at $A$. Let Tile 8 be the other tile with a vertex at $A$, south of Tile 1. Then Tile 8 either extends east of Tile 1, or has its $\gamma$ angle at the shared eastern vertex $P$.

Tile 2 cannot be placed with its $\gamma$ angle also at $P$, making $P$ a “center” with three $\gamma$ angles, since in that case the $b$ side of Tile 2 would extend past Tile 8, which is impossible as that would go outside $ABC$. Therefore, whatever the position of Tile 8, Tile 2 must be placed as before, with its $a$ edge shared with Tile 1. Then Tile 3 must be placed as before also, and as before Tile 4 must terminate the southern boundary of Tile 2 from extending eastwards. Let Tile 5 be the tile south of Tile 2. Since the southern boundary of Tile 2 is terminated at both ends. Assume, for proof by contradiction, that Tile 5 does not share its $c$ edge with Tile 2. Then $c$ is a combination of tile edges $a$ and $b$, which implies that $c$ is a multiple of $a$. Then Tile 5 has its $a$ edge against Tile 2. Then its $\beta$ angle or $\gamma$ angle is at $P$, which implies that Tile 8 shares its $b$ edge with Tile 1 and has its $\gamma$ angle at $P$.

But then, there is not room for Tile 5 to also have its $\gamma$ angle at $P$. Hence Tile 5 has its $\beta$ angle at $P$. Then there exists Tile 9 between Tile 8 and Tile 5, with the $\alpha$ angle of Tile 9 at $P$. Tile 9 lies next to the $a$ edge of Tile 8, but since Tile 9 has its $\alpha$ angle at $P$, it must have its $b$ or $c$ side next to Tile 9, which is impossible as that side would extend outside $ABC$. That contradiction completes the proof that Tile 5 does share its $c$ edge with Tile 2. Hence Tile 5 must occur in the position shown in Fig. 24. That is, Fig. 24 correctly shows Tiles 1, 2, 3, 5 also in Case 2C, regardless of the position of Tile 8 (which is not shown in the figure).

Now, in Case 2B, the southern vertex of Tile 4 had to be the southeastern vertex of Tile 5. If that is so, then as before $c = 2a$, and the proof is completed as in Case 2B. Let $P$ and $Q$ be the southeastern vertices of Tiles 1 and 5, respectively. If some tile south of $APQ$ blocks line $BQ$ from continuing south of $PQ$, then the southern vertex of Tile 4 is $Q$, and as in Case 2B, $c = 2a$ and the proof can be completed. Hence, we can assume $BQ$ does extend through $APQ$. Then the tiles south of Tiles 1 and 5 must share the $b$ edges of those tiles. But that is impossible, as those two tiles would have their $\gamma$ angles to the east, crossing $BQ$.

That contradiction completes the proof in Case 2C.

If Case 2 holds then either $AB$ or $BC$ supports exactly two tiles. Renaming $A$ and $C$ if necessary, we can assume it is $AB$ that supports exactly two tiles. Then either Case 2B or Case 2C applies. That completes the proof of the theorem.

Laczkovich proved that $N$-tilings of the kind discussed in this section exist, but did not actually exhibit any. Although in [4], he did not explicitly consider $N$ at all, he did consider $N$ in a theorem that he allowed Soifer to publish in [11]. In that theorem, he proved that for tilings of an equilateral triangle by a tile $(\alpha, \beta, 2\pi/3)$, the square-free part of $N$ could be anything desired. Following Laczkovich’s ideas,
we found the tiling of Fig. 25 with $N = 10935$. Other tilings that we found require more than 32,000 tiles (and so are too big to draw nicely on a normal page). What the smallest possible $N$ is, we have no idea. For all we know, the construction method used for this tiling might yield a smaller $N$ for some tile with very large sides; or there might be a much more efficient tiling construction yet to be discovered. In 2012 it was not known if there is an $N$-tiling of the equilateral triangle for every sufficiently large $N$, or if instead there are arbitrarily large $N$ for which, like 7 and 11, there is no $N$-tiling at all. In unpublished work, we have proved $N$ cannot be prime.

**Figure 25.** $N = 10935$. The tile is $(3, 5, 7)$. $ABC$ is equilateral.

17. Tilings of an isosceles triangle with $\gamma = 2\alpha$

In this section we take up the row of Laczkovich’s second table in which $ABC$ is isosceles with base angles $\alpha$ and is tiled by a tile with $\gamma = 2\alpha$, and $\alpha$ is not a rational multiple of $\pi$. The condition $\gamma = 2\alpha$ can also be written as $3\alpha + \beta = \pi$. Unlike the similar-looking condition $3\alpha + 2\beta = \pi$, this condition does not imply $\gamma > \pi/2$. The vertex angle of $ABC$ is then $\pi - 2\alpha = \alpha + \beta$.

Laczkovich [4] proves that, given any tile with $\gamma = 2\alpha$ and $\alpha$ not a rational multiple of $\pi$, an isosceles triangle can be dissected into triangles similar to the tile. Following the steps of his proof with the tile $(4, 5, 6)$, one finds the dissection
Figure 26. Laczkovich’s dissection of isosceles $ABC$ into triangles similar to $(4, 5, 6)$ can be used to produce a $5861172$-tiling.  

shown in Fig. 26 To make an $N$-tiling, we have to tile each of these triangles and the parallelogram with many copies of the same tile. Along each edge in the figure there is an arithmetical condition to satisfy. Working out those conditions, we find that more than five million tiles are required: $5861172$ to be precise. It is not possible to print such a large tiling (unless one could use the side of a large building), and we do not know a smaller one. But at least, some such tilings do exist. Indeed, many such tilings exist.

The theory of such tilings has progressed far enough (in unpublished work) to prove that that tile is necessary rational, and from that plus the characterization of the tile given above, one can show that $N$ cannot be prime. But again, the proof that the tile is rational is complicated, so here we will show by a simple computation that $N$ cannot be 7 or 11, without appealing to the rationality of the tile. Even though we can prove that $N$ cannot in general be prime, we cannot prove (yet) that $N$ has to be even, and in fact we have no good reason even to conjecture that. One difficulty in advancing the theory is that there is no “coloring equation” in this case, since there might be three or there might be four tiles meeting at a given boundary vertex.

17.1. Characterization of the tile. By the law of cosines,

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$= a^2 + b^2 - 2ab \cos 2\alpha \quad \text{since} \quad \gamma = 2\alpha$$

$$= a^2 + b^2 - 2ab(2\cos^2 \alpha - 1)$$

$$= a^2 + b^2 + 2ab - 4ab \cos^2 \alpha$$

By the law of sines, $\sin \alpha/a = \sin \gamma/c = \sin 2\alpha/c = 2 \sin \alpha \cos \alpha/c$, so $\cos \alpha = c/(2a)$. Hence

$$c^2 = a^2 + b^2 + 2ab - bc^2/a$$

$$= (a + b)^2 - bc^2/a$$

$$c^2(1 + b/a) = (a + b)^2$$

$$c^2 = a(a + b)$$

(23)
Triangles with $\gamma = 2\alpha$ correspond to solutions of this equation with $c < a + b$ and $b < a + c$ and $a < b + c$. For example, $(4,5,6)$, and $(9,7,12)$.

**Lemma 13.** Suppose the triangle with sides $(a,b,c)$ has angles $(\alpha, \beta, 2\alpha)$. Then for some real number $t$ with $0 < t < 1$ we have

$$(a, b, c) = c(t, 1 - \frac{t^2}{t}, 1)$$

Consequently $b = \frac{c^2}{a} - a$.

**Remark.** With $t = 2/3$ we have $(a,b,c) = (4,5,6)$.

**Proof.** By (23) we have

$$c^2 = a^2 + ab$$

This can be written as

$$b^2 + (2c)^2 = (2a + b)^2$$

as is apparent upon expanding the right side. (Although that is a trivial algebraic identity, the idea to make this particular rewriting is not completely obvious: I found it in [8].) Dividing by the right side we have

$$\left(\frac{b}{2a + b}\right)^2 + \left(\frac{2c}{2a + b}\right)^2 = 1$$

Choosing $\theta$ so that

$$\cos \theta = \frac{b}{2a + b}$$

it follows that

$$\sin \theta = \frac{2c}{2a + b}$$

Choosing $t = \tan(\theta/2)$ we then have

$$\cos \theta = \frac{2t}{1 + t^2}$$

$$\sin \theta = \frac{1 - t^2}{1 + t^2}$$

and hence

$$\frac{1 - t^2}{1 + t^2} = \frac{b}{2a + b} \quad (24)$$

$$\frac{2t}{1 + t^2} = \frac{2c}{2a + b}$$

Then

$$\frac{b}{c} = \frac{1 - t^2}{t} \quad (25)$$
Inverting (24) we have
\[
\frac{1 + t^2}{2t} = \frac{2a + b}{2c} = \frac{a + b}{c} + \frac{1 - t^2}{2t} \quad \text{by (25)}
\]
Subtracting the last term from both sides,
\[
\frac{a}{c} = \frac{1 + t^2}{2t} - \frac{1 - t^2}{2t} = t
\]
This equation, together with (25), implies \((a, b, c) = c(t, (1 - t^2)/t, 1)\), as claimed in the statement of the lemma. Then \(t = a/c\), so
\[
b = \frac{1 - t^2}{c} = \frac{1 - (a/c)^2}{a/c} = \frac{c^2/a - a}{a/c}
\]
That completes the proof of the lemma.

17.2. The number of tiles on a side of \(\triangle ABC\).

**Lemma 14.** Let isosceles \(\triangle ABC\) with base angles \(\alpha\) (at \(A\) and \(C\)) be \(N\)-tiled by a tile with angles \((\alpha, \beta, 2\alpha)\), and the tile not a right triangle. Then no tile has one vertex on \(AB\) and another on \(BC\).

**Proof.** By Lemma 1, \(\alpha\) is not a rational multiple of \(\pi\), and \(2\alpha\) cannot be written in any other way as a linear combination of \(\alpha\), \(\beta\), and \(\gamma\). Suppose some tile has an edge \(EF\) with \(E\) on \(AC\) and \(F\) on \(BC\). Consider the triangle \(BEF\). Since it has angle \(\alpha + \beta\) at \(B\) and is tiled, its angles at \(E\) and \(F\) must both be \(\alpha\), since otherwise \(2\alpha\) could be written as a linear combination of \(\alpha\) and \(\beta\) other than \(2\alpha\). Then the north side of \(EF\) cannot be covered by a single tile, since if it were, that tile would have two \(\alpha\) angles, one at \(E\) and another at \(F\). There the north side of \(EF\) supports at least two tiles. North of \(EF\) there cannot be only \(a\) edges on \(EF\), since there are \(\alpha\) angles at \(E\) and \(F\).

I say that north of \(EF\), there cannot be only \(b\) edges. Suppose the contrary. Then each of the tiles supported by \(EF\) on its north side has a \(\gamma\) angle on \(EF\). But at \(E\) and \(F\) there are \(\alpha\) angles, not \(\gamma\) angles. By the pigeonhole principle there is a vertex on \(EF\) with two \(\gamma\) angles. But that is impossible, as the only ways to write \(\pi\) as a sum of tile angles are \(\alpha + \beta + \gamma\) and \(3\alpha + \beta\). Hence, as I said, north of \(EF\) there cannot be only \(b\) edges.

I say also that north of \(EF\), there cannot be only \(c\) edges. Suppose the contrary. Then each of the tiles supported by \(EF\) on its north side has a \(\beta\) angle with vertex on \(EF\). The tile at \(E\) has its \(\alpha\) angle at \(E\), so its \(\beta\) angle is to the east. The tile at \(F\) has its \(\alpha\) angle at \(F\), so its \(\beta\) angle is to the west. By the pigeonhole principle, there is a vertex on \(EF\) where two tiles north of \(EF\) have their \(\beta\) angles. That is impossible, since the only ways to write \(\pi\) as a sum of tile angles are \(\alpha + \beta + \gamma\)
and $3\alpha + \beta$. Two $\beta$ angles are not permitted. Hence, as I said, north of $EF$ there cannot be only $c$ edges.

Now we argue by cases.

Case 1: The length of $EF$ is $a$. Then for some non-negative integers $p$ and $q$ we have $a = pb + qc$. If $p = 0$, then all the tiles north of $EF$ have their $c$ edges on $EF$, which has been disproved above. Hence $p \neq 0$. Similarly, if $q = 0$, all the tiles north of $EF$ have their $b$ edges on $EF$, which has been disproved above. Hence $q \neq 0$.

Then $a = pb + qc$ with $p$ and $q$ both positive. Then $a > b + c$, which is impossible since $(a, b, c)$ are the sides of a triangle. That completes Case 1.

Case 2: The length of $EF$ is $c$. Then for some integers $p$ and $q$ we have $c = pa + qb$. We have $p = 0$ since $c < a$, by Lemma 13. Then $c = qb$. Then all the tiles supported by $EF$ on its north side have their $b$ edges on $EF$, which has been disproved. That completes Case 2.

Case 3: The length of $EF$ is $b$. Then for some integers $p$ and $q$ we have $b = pa + qc$. We have $p = 0$ since $b < a$, by Lemma 13. Then $b = qc$. Then all the tiles supported by $EF$ on its north side have their $c$ edges on $EF$, which has been disproved. That completes Case 3, and that completes the proof of the lemma.

**Lemma 15.** Let isosceles $ABC$ with base angles $\alpha$ (at $A$ and $C$) be $N$-tiled by a tile with angles $\alpha, \beta, 2\alpha$. Suppose $N/2$ is not a square, and $\alpha$ is not a rational multiple of $\pi$. Then fewer than $N/3$ tiles are supported on side $AB$, or fewer than $N/3$ tiles are supported on side $BC$.

*Proof.* Same as the proof of Lemma 6 except referencing Lemma 14 instead of Lemma 5.

**Lemma 16.** Let $ABC$ be an isosceles triangle with base angles $\alpha$. Let $\gamma = 2\alpha$. Suppose $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ and $\alpha$ is not a rational multiple of $\pi$. Let $(a, b, c)$ be the sides of the tile. Then among the tiles supported by $AB$ there is at least one with its $b$ edge on $AB$, and at least one with its $c$ edge on $AB$, and the same for $BC$.

*Proof.* Suppose there is no $c$ edge on $AB$. Then every tile supported by $AB$ has a $\gamma$ angle on $AB$. But there are no $\gamma$ angles at $A$ or $B$, since $\alpha < \gamma$, and the vertex angle $\alpha + \beta$ cannot be decomposed into any other integral linear combination of $\alpha$ and $\beta$. By the pigeonhole principle, there is a vertex $P$ on $AB$ with two $\gamma$ angles at $P$, contradiction. Hence there is at least one $c$ edge on $AB$.

Now suppose there is no $b$ edge on $AB$. Then every tile supported by $AB$ has a $\beta$ angle on $AB$. But there are no $\beta$ angles at $A$ or $C$, since $\alpha < \gamma$ and $\alpha$ is not a rational multiple of $\beta$. By the pigeonhole principle, there is a vertex $P$ on $AB$ with two $\beta$ angles at $P$, contradiction. Hence there is at least one $c$ edge on $AB$. That completes the proof of the lemma.

**Lemma 17.** Let $ABC$ be an isosceles triangle with base angles $\alpha$. Let $\gamma = 2\alpha$. Suppose $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ and $\alpha$ is not a rational multiple of $\pi$. Let $(a, b, c)$ be the sides of the tile. Let $Y$ be the length of the base $AC$. Suppose $Y = pa + qb + rc$. Then

$$p + q + r < (N - 3)/2$$
Proof. The angle at vertex $B$ is $\alpha + \beta$, so there are two tiles with vertices at $B$. By relabeling $A$ and $C$ if necessary, we may assume without loss of generality that the tile with a vertex at $B$ and supported by $AB$ has its $\alpha$ angle at $B$. Call that Tile 1.

Tile 1 does not have a vertex at $A$, since the tile at $A$ must have its $\alpha$ angle at $A$. Call that tile Tile 2. If every tile supported by $AB$ has an $\alpha$ angle on $AB$, then by the pigeonhole principle, there is a vertex $P$ on $AB$ with at least two $\alpha$ angles. Therefore Tile 1 and Tile 2 do not share a common vertex, since then there would be no such vertex $P$. Therefore there are at least three tiles supported by $AB$. Let Tile 3 be the tile supported by $AB$ that shares a vertex $P$ with Tile 2. I say that Tile 3 does not have a vertex on $AC$. Suppose, to the contrary, that it does have such a vertex. Call that vertex $Q$. Then $PQ$ is greater than the eastern edge of Tile 2, which is $a$. So $PQ$ is equal to $b$ or $c$. But $PQ$ is less than the northeast edge of Tile 3. Hence $PQ$ is equal to $b$ and Tile 3 has its $\gamma$ angle at $P$. Hence the angle between Tile 2 and Tile 3 at $P$ is $\alpha$. That angle must be filled by a single tile, say Tile 4, but Tile 4 cannot have its $c$ edge ending at $P$, since $PQ = b < c$, and Tile 4 cannot have its $c$ edge adjacent to Tile 2, since that edge has length $a$ and terminates at $AC$. We have reached a contradiction from the assumption that Tile 3 has a vertex on $AC$; hence, as I said above, the two tiles with vertices at $B$ do not have a vertex on $AC$.

I say that the two tiles with vertices at $B$ do not have a vertex on $AC$. Without loss of generality, we can assume Tile 1 supported by $AB$ has angle $\alpha$ at $B$ and Tile 2 supported by $BC$ has its $\beta$ angle at $B$. Assume, for proof by contradiction, that Tile 1 has a vertex $P$ on $AC$. Triangle $ABP$ has $\alpha$ angles at $B$ and $A$ so angle $APB$ is $\alpha + \beta$, triangle $ABP$ is isosceles, and $AP = BP = c$. Since angle $APB$ is $\alpha + \beta$, there are exactly two tiles filling that angle, one of which is Tile 1. Then Tile 1 has its $\beta$ angle at $P$, since its $\alpha$ angle is at $B$. Then the tile at $A$ has two $\alpha$ angles, one at $P$ and one at $A$, contradiction. Hence Tile 1 does not have a vertex on $AC$.

Assume, for proof by contradiction, that Tile 2 has a vertex $Q$ on $AC$. Then angle $BQC$ is equal to $\gamma$. Assume, for proof by contradiction, that angle $BQC$ is filled by a single tile. Then Tile 2 is $BQC$, $BQ$ has length $a$, and $BC$ has length $c$, which contradicts Lemma 16. Hence angle $BQC$ is not filled by a single tile. Then Tile 2 does not have its $\gamma$ angle at $Q$. Hence it has its $\alpha$ angle at $Q$. Let $R$ be the eastern vertex of Tile 2, so $R$ lies on $BC$. Since $\gamma = 2\alpha$, angle $RQC$ is $\alpha$. Then triangle $RQC$ is isosceles, with base angles $\alpha$ at $Q$ and $C$. $RQ$ is the southeast edge of Tile 2, opposite the $\beta$ angle at $B$, so $RQ$ has length $b$. Since $QRC$ is isosceles, $RC = QR = b$. Since $BR$ is the edge of Tile 2 opposite its $\alpha$ angle, we have $BR = a$. Then $BC = BR + RC = a + b$. Since $a < c$, we have $BR < c + b$. But that contradicts Lemma 16. That contradiction shows that Tile 2 does not have a vertex on $AC$. Hence, as I said above, the two tiles with vertices at $B$ do not have a vertex on $AC$.

Now let us count tiles. Suppose there are $k$ tiles supported by $AC$. Between those tiles are $k - 1$ more tiles, each with a vertex on $AC$. In addition we have two tiles with vertices at $B$, Tile 3 supported by $AB$ without a vertex on $AC$, and another tile between Tile 3 and Tile 1. That makes $k + (k - 1) + 4 = 2k + 3$ tiles. Hence $2k + 3 \leq N$, the total number of tiles. Then $k \leq (N - 3)/2$. That completes the proof of the lemma.

Lemma 18. Let $ABC$ be an isosceles triangle with base angles $\alpha$. Let $\gamma = 2\alpha$. Suppose $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ and $\alpha$ is not a rational
multiple of \( \pi \). Let \((a, b, c)\) be the sides of the tile. Let \(Y\) be the length of the base \(AC\). Suppose \(Y = pa + qb + rc\). Then

(i) It is not the case that \(p = q = r = 1\), and
(ii) \(r \neq 0\) and \(q \neq 0\) (at least one \(c\) edge and one \(b\) edge on \(AC\)), and
(iii) at least three tiles are supported by \(AC\), and
(iv) It is not the case that \(p = q = 0\) (not only \(c\) edges on \(AC\)).

**Proof.** Ad (i). Suppose \(p = q = r = 1\). Then there are exactly three tiles supported by \(AC\). Number them Tile 1, Tile 2, and Tile 3, in order west to east. Tiles 1 and 3 have their \(a\) angles at \(A\) and \(C\). Hence Tile 2 is the one with its \(a\) edge on \(AC\). Relabeling \(A\) and \(C\) if necessary, we can assume that Tile 3 has its \(c\) edge on \(AC\) and Tile 1 has its \(b\) edge on \(AC\). Then Tile 1 has its \(\gamma\) angle at its eastern vertex \(P\), which is shared with Tile 2. Then Tile 2 has its \(\beta\) angle at \(P\) and its \(\gamma\) angle at its eastern vertex \(Q\). Then Tile 3 has its \(\beta\) angle at \(Q\) (since two \(\gamma\) angles at \(Q\) is impossible). The remaining angle at \(Q\) is exactly \(\alpha\), so there is just one more tile at \(Q\), say Tile 4, which adjoins both Tile 2 and Tile 3. It does not have its \(a\) edge against Tile 3, since it has its \(\alpha\) angle at \(Q\). It cannot have its \(c\) edge against Tile 3, since that edge terminates at \(BC\) and \(c > a\). Hence Tile 4 has its \(b\) edge against Tile 3. Hence \(b < a\) and the remaining part of the western edge of Tile 3 has length \(a - b\), which must be tiled only by \(b\) edges. Let \(R\) be the northern vertex of Tile 3, so that \(QR\) is the western edge of Tile 3. Then each of the tiles supported on the west by \(QR\) has its \(b\) edge on \(QR\). Each of those tiles has its \(\gamma\) angle on \(QR\) and each must have its \(\gamma\) angle to the northeast, since there cannot be two \(\gamma\) angles at the same vertex. But at \(R\), there is already a \(\gamma\) angle south of \(QR\) belonging to Tile 3. Then there are two \(\gamma\) angles at the boundary vertex \(R\), contradiction. That completes the proof of part (i) of the lemma.

Ad (ii). Suppose, for proof by contradiction, that there are no \(c\) edges on \(AC\). Then every tile supported by \(AC\) has a \(\gamma\) angle on \(AC\). Since there are \(a\) angles, not \(\gamma\) angles, at \(A\) and \(C\), by the pigeonhole principle there is a vertex on \(AC\) with at least two \(\gamma\) angles, contradiction.

Now suppose, for proof by contradiction, that there are no \(b\) edges on \(AC\). Then every tile supported by \(AC\) has a \(\beta\) angle on \(AC\). Since there are \(\alpha\) angles, not \(\beta\) angles, at \(A\) and \(C\), by the pigeonhole principle there is a vertex on \(AC\) with at least two \(\beta\) angles, contradiction. That completes the proof of (ii).

Ad (iii). Suppose, for proof by contradiction, that there are fewer than three tiles supported by \(AC\). By part (ii), there are at least two. Therefore there are exactly two, Tile 1 with a vertex at \(A\) and Tile 2 with a vertex at \(C\). By part (ii), we may assume without loss of generality that Tile 1 has its \(b\) edge on \(AC\) and Tile 2 has its \(c\) edge on \(AC\). Let \(P\) be the shared vertex; then Tile 1 has its \(\gamma\) angle at \(P\) and Tile 2 has its \(\beta\) angle at \(P\). Then there is just one tile, Tile 3, between Tile 1 and Tile 2, and it has its \(\alpha\) angle at \(P\). Therefore its \(c\) side lies either against Tile 1 or Tile 2, both of which have their \(a\) sides against Tile 3 and terminated at the boundary of \(ABC\). But since \(c > a\), that is impossible. That completes the proof of (iii).

Ad (iv). Suppose, for proof by contradiction, that all the tiles supported by \(AC\) have their \(c\) edges on \(AC\). Then they each have their \(\beta\) angle on \(AC\). Since there are \(\alpha\) angles at \(A\) and \(C\), by the pigeonhole principle there is a vertex on \(AC\) with
two $\beta$ angles, contradiction. That completes the proof of (v) and the proof of the lemma.

**Lemma 19.** Let $ABC$ be an isosceles triangle with base angles $\alpha$. Let $\gamma = 2\alpha$. Suppose $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ and $\alpha$ is not a rational multiple of $\pi$. Let $(a, b, c)$ be the sides of the tile. Then each side $AB$ and $BC$ supports at least three tiles.

**Proof.** Suppose, for proof by contradiction, that there are fewer than three tiles supported by $AB$. We relabel $A$ and $C$ if necessary so that the tile supported by $AB$ at $B$ has its $\alpha$ angle at $B$. By Lemma 16, there are at least two. Therefore there are exactly two, Tile 1 with a vertex at $A$ and Tile 2 with a vertex at $B$. By Lemma 16, one of Tile 1 has its $b$ edge on $AB$ and the other its $c$ edge. Let $P$ be the shared vertex; then one of Tile 1 and Tile 2 has its $\gamma$ angle at $P$ and the other has its $\beta$ angle at $P$. Then there is just one tile, Tile 3, between Tile 1 and Tile 2, and it has its $\alpha$ angle at $P$. Therefore its $c$ side lies either against Tile 1 or Tile 2, both of which have their $a$ sides against Tile 3. The $c$ side of Tile 3 cannot lie against Tile 1, since that side terminates on $AB$ and $AC$ and $a < c$. Hence the $c$ side of Tile 3 lies against Tile 2. Since $c > a$, that side extends past Tile 2. The $b$ side of Tile 3 lies against Tile 1. The remaining part of the $a$ edge of Tile 1 can then be tiled only with $b$ edges, so $a$ is an integer multiple of $b$. In particular $a > b$. Let Tile 4 be the other tile with a vertex at $B$; then Tile 4 has its $\beta$ angle at $B$. Its $c$ side cannot lie against Tile 2, since $c > a$ and the way is blocked by Tile 3. Let $R$ be the southeast vertex of Tile 2. Since $PB$, as the eastern edge of Tile 2, is either equal to $b$ or $c$, and $PR = a$, $BR$ is equal to $b$ or $c$. If $BR$ is equal to $b$, it cannot be tiled, since $a > b$. Hence $BR$ is not equal to $b$. Hence $BR = c$. Now the western edge of Tile 4 is either $b$ or $c$.

Assume, for proof by contradiction, that the western edge of Tile 4 is $c$. Then Tile 4 has a vertex at $R$ and its $\alpha$ angle at $R$. Then angle $PRB$ is $\beta + \alpha$, since Tile 2 has its $\beta$ angle at $R$ opposite its $b$ edge $PB$, and Tile 4 has its $\beta$ angle at $R$ opposite its $b$ edge on $BC$. But then there are two $\beta$ angles at $R$, and the remaining portion of $\pi$ above $PR$ extended cannot be expressed as an integral combination of tile angles. Hence the western edge of Tile 4 is not $c$. If it is $b$ then because the rest of $BR$ has to be tiled, $c - b$ is an integral linear combination of $a$ and $b$; but $a$ is an integer multiple of $b$, so $c$ is an integer multiple of $b$. Now we have $c = \lambda b$ and $a = \mu b$ for some integers $\lambda$ and $\mu$. We will show this is impossible:

$$b = \frac{c^2}{a} - a \quad \text{by Lemma 13}$$

$$= \frac{(\lambda b)^2}{\mu b} - \mu b$$

$$= \frac{\lambda^2 b^2}{\mu} - \mu b$$

$$b \mu = \frac{\lambda^2 b - \mu^2 b}{\mu}$$

$$\mu = \frac{\lambda^2 - \mu^2}{\mu^2 + \mu - \lambda^2} = 0$$

This is a quadratic equation in $\mu$ whose discriminant is $1 + 4\lambda^2 = 1 + (2\lambda)^2$. This is not a square, since the next square after $(2\lambda)^2$ is $(2\lambda + 1)^2 > 1 + (2\lambda)^2$. Hence we have reached a contradiction. That contradiction proves that $AB$ supports at least three tiles.
It remains to prove that $BC$ also supports at least three tiles. That is not yet proved, because we chose $AB$ to be the side on which the top tile has an $\alpha$ angle. Suppose, for proof by contradiction, that $BC$ nevertheless supports only two tiles. Then $X = b + c$, by Lemma 16. But since $AB$ also has length $X$, and supports three tiles, two of which have their $b$ and $c$ edges on $AB$, we have $X > b + c$, contradiction. That completes the proof of the lemma.

17.3. $N \neq 7$ when $\gamma = 2\alpha$ and $ABC$ is isosceles, proved without a computer.

**Theorem 8.** Let $ABC$ be an isosceles triangle with base angles $\alpha$. Let $\gamma = 2\alpha$. Suppose $ABC$ is $N$-tiled by a tile with angles $(\alpha, \beta, \gamma)$ and $\alpha$ is not a rational multiple of $\pi$. Then $N \geq 12$.

**Remark.** We first give a proof that does not involve a computer program; in the next section we give a computer program that does the same job. The program, however, presumes the lemmas above, just as this proof does. The program shows that 12 is the best possible number that can be obtained based on the area equation and the boundary-tiling lemmas above.

**Proof.** We count tiles. Let $k$ be the number of tiles supported by $AC$. Then by Lemma 15 $k \geq 3$. By Lemma 19 at least three tiles are supported by each of $AB$ and $BC$. Between the tiles on $AB$ there are at least two more tile with a vertex on $AB$, and similarly on $BC$, and there is no danger of double-counting these tiles, by Lemma 14. Counting the tiles we have at least 5 with edges or vertices on $AB$, at least 5 with edges or vertices on $BC$, and at least 3 on $AC$, but we have double-counted the tiles at $A$ and $C$, so that makes a total of at least $5 + 5 + 3 - 2 = 11$. For all we know the tiles with only a vertex on $AC$ have already been counted—they may also have a vertex on $AB$ or $BC$. Hence $N$ cannot be 7.

It seems that we have barely missed proving the theorem for $N = 11$. But we can fix that. Above we subtracted two because we did not dare to count the two tiles (or more?) tiles that have a vertex but not an edge on $AC$, for fear that they might also have a vertex on $AB$ or $BC$ and thus have already been counted. We now show that we are entitled to count at least one of them, which will push the count of tiles to 12. Suppose, for proof by contradiction, that there are fewer than 12 tiles. Then there are exactly three tiles supported by $AC$, say Tile 1, Tile 2, and Tile 3, in order west to east. If there are fewer than 12 tiles then there must be just one tile, Tile 4, adjacent to Tile 1, whose $a$ edge is shared with Tile 1 and touches both $AB$ and $AC$. There must also be just one tile, Tile 5, adjacent to Tile 3, whose $a$ edge is shared with Tile 3 and touches both $AC$ and $BC$. Let $P$ be the vertex shared by Tiles 1 and 4, and $Q$ the vertex shared by Tiles 3 and 5. None of those tiles has an $a$ angle at $P$ or $Q$, since their $a$ edges end at $P$ and $Q$. Since there cannot be two $\beta$ angles, or two $\gamma$ angles, at any boundary vertex, Tiles 1 and 4 together contribute a $\beta$ and a $\gamma$ angle at $P$, and Tiles 3 and 5 contribute a $\beta$ and a $\gamma$ angle at $Q$. Therefore the remaining angle to be filled at $P$ and $Q$ is $\alpha$ in both cases. But Tile 2 has only one $a$ angle, so it cannot contribute an $a$ at both $P$ and $Q$. We have reached a contradiction. That completes the proof of the theorem.

17.4. A computer program to deal with larger $N$. In this section we exhibit a computer program that implements the following plan, when given input $N$:

For $(p, q, r)$ each ranging from 0 to $N/3$, set $X = pa + qb + r$. Then $X$ would be the length of $AB$ in a hypothetical tiling, scaled so that $c = 1$. Solve the quartic
equation
\[ X^2 = Nab = Na(1/a - a) \]
for \( a \). Here \( b = 1/a - a \) by Lemma \ref{lem:13} when \( c = 1 \). Reject any complex solutions and any solutions in which \( a \) does not lie between 0 and 1. For each of the remaining solutions, check whether \( Y = X/a \) (the length of \( AC \)) can be written as an integer linear combination of \( (a, b, 1) \). If not then go on to the next solution. If none of the solutions of the quartic permit \( Y \) to be so expressed, then there can be no tiling.

This plan needs some refinements, because as just sketched it does find some possible solutions. Those refinements consist in incorporating all the restrictions proved in Lemmas \ref{lem:16} \ref{lem:19} and \ref{lem:18}. After that, the plan works beautifully—for \( N = 7 \) and 19. But unexpectedly, we found solutions for some other low values of \( N \):

**Theorem 9.** Let \( ABC \) be an isosceles triangle with base angles \( \alpha \). Let \( \gamma = 2\alpha \). Suppose \( ABC \) is \( N \)-tiled by a tile with angles \((\alpha, \beta, \gamma)\) and \( \alpha \) is not a rational multiple of \( \pi \). Then \( N \) is 12, 18, 27, 32, or more than 32. In particular, \( N \) cannot be 7, 11, 14, or 19.

**Remarks.** This theorem is proved by sheer computation (plus of course the preliminary lemmas above); it does not use the fact (proved in unpublished work) that the tile must be rational. Nevertheless, the solutions found all have \((a, b, c)\) rational.

**Proof.** Let \( X \) be the length of the equal sides \( AB \) and \( BC \). Let the sides of the tile be \((a, b, c)\). We do not assume they are rational. The plan is to search, not for a tiling, but only for an \((a, b, c)\) that satisfies the area equation \( X^2 = Nab \) and permits the sides of \( ABC \) to be written as integral linear combinations of \((a, b, c)\). If none exist, then there is no \( N \)-tiling. We do not know \( X \) in advance, but since \( X \) has the form \( pa + qb + rc \), and we have the bound \( p + q + r < N/3 \), there are only finitely many possibilities for \((p, q, r)\) and we just check them all. The details follow.

Twice the area of \( ABC \) is \( X^2 \sin(\alpha + \beta) = X^2 \sin \gamma \). Twice the area of the tile is \( Nab \sin \gamma \). Since \( \sin(\alpha + \beta) = \sin(\pi - \gamma) = \sin \gamma \), the area equation can be written as
\[ X^2 = Nab \quad \text{area equation} \]
Without loss of generality, we may scale the tile and the triangle \( ABC \) so that \( c = 1 \). Then, according to Lemma \ref{lem:13} we have \( b = c^2/a - a = 1/a - a \). Hence the area equation becomes
\[ X^2 = Na(1/a - a) \]
\[ = Na(1 - a^2) \]
Because there is a tiling, there are non-negative integers \((p, q, r)\) such that \( X = pa + qb + r \). Putting this expression for \( X \) into the area equation, we have
\[ (pa + qb + r)^2 = Na(1 - a^2) \]
Setting \((a, b, c) = (a, 1/a - a, 1)\) as discussed above, we have
\[ (pa + q(1/a - a) + r)^2 = Na(1 - a^2) \]
Multiplying both sides by \( a^2 \) to bring it to polynomial form,
\[ (pa^2 + q(1 - a^2) + ra)^2 = Na^2(1 - a^2) \]
Table 5. Candidates that satisfy area and boundary-tiling constraints

| $N$  | $(a, b, c)$ | $X$         | $Y$         |
|------|-------------|-------------|-------------|
| 20   | (4, 5, 6)   | $a + 2b + c$| $a + 5b + c$|
| 27   | (1, 3, 2)   | $b + 3c$    | $4b + 6c$   |
| 28   | (9, 7, 12)  | $a + 3b + c$| $a + 6b + 5c$|
| 32   | (1, 8, 3)   | $2a + b + 2c$| $5b + 8c$   |
| 36   | (9, 16, 15)| $a + 3b + c$| $7b + 8c$   |
| 44   | (25, 11, 30)| $a + 5b + c$| $a + 9b + 8c$|

This is a fourth-degree equation in $a$, so SageMath can solve it explicitly, for each particular $(p, q, r)$. By Lemma 15 we have

$$p + q + r < N/3.$$ 

Therefore, given a particular $N$, we can ask SageMath to check all such triples $(p, q, r)$ and try to solve the above equation for $a$ with $0 < a < 1$. Should it find no solution, then there can be no such $N$-tiling, since if there is a tiling then $a < c = 1$ since $\alpha < \gamma$. We ran this code for $N = 7, 11, 19$, and it did find a few solutions.

Therefore, as mentioned above, we refined the code, building in the restrictions in the lemmas proved above. Figs. 27 and 28 show the code. The lines that correspond to the lemmas are explained in the comments. The function `check()` exhibited in Fig. 28 does the checking of whether $Y$ can be tiled. Some remarks about the details of that function follow.

The length of $AC$ is $Y = X/a$ by the law of sines. To write `check()`, we must know a bound on the number of tiles that can be supported by $AC$. We have $N/3$ as a bound for the number of tiles supported by $AB$, but there is no obvious way to improve on the crude bound $N$, so we just use that bound.

We can run the code for $N = 7, 11, 19$, and only `True` is printed. Hence there is no tiling for those values of $N$. Running this code for more values of $N$, we found solutions as shown in Table 5 (and of course many more). That completes the proof of the theorem.

Remarks about the code. Note that decimal values of $Y$, $a$, and $b$ are checked, rather than the complicated symbolic expressions produced by `solve()`. That speeds up the computation considerably.
Figure 27. SageMath code for isosceles \( ABC \) and tile with \( \gamma = 2\alpha \)

```python
def GammaEqualsTwoAlphaCase(N):
    var('x')
    Nover3 = int(N/3)
    for p in range(0,Nover3):
        for q in range(1,Nover3):
            for r in range(1,Nover3):
                if p + q + r >= N/3:
                    continue
                eq = (p*x^2 + q*(1-x^2) + r*x)^2 - N*x^2*(1-x^2)
                answers = solve(eq,x)
                for A in answers:
                    a = A.rhs()
                    if a.has(x):
                        print("oops") # x should not occur in the solution
                    if not a in RR:
                        continue # only interested in real a
                    if n(a) <= 0 or n(a) >= 1:
                        continue
                    b = (1/a-a).simplify()
                    X = p*a + q*b + r
                    Y = n(X/a)
                    if check(Y,N,n(a),n(b))==false:
                        return false
    return true
```
def check(Y,N,a,b):
    bound = N
    print("checking")
    for p in range(0,bound):
        for q in range(1,bound):  # at least one b edge
            for r in range(1,bound):  # at least one c edge
                if p+q+r > bound:    # we proved this bound  
                    continue
                if p+q+r <= 2:       # at least three tiles on AC
                    continue
                if p==0 and q==0:    # only c edges on AC
                    continue
                if p==1 and q==1 and r==1:  # proved impossible
                    continue
                test = Y-p*a-q*b-r
                if abs(test) < 0.0000001:
                    print("yes")
                    print(Y,p,q,r)
                    return false
    return true
18. Tilings of an equilateral triangle with \( \alpha/\pi \) irrational

Laczkovich’s Theorem 3.1 \([4]\) says that, given a rational tile with an angle \( \pi/3 \), and the other angles not rational multiples of \( \pi \), that tile will tile some an equilateral \( ABC \). There are infinitely many such rational tiles, as Laczkovich proves. The two simplest ones are \((7, 5, 8)\) and \((7, 3, 8)\). Similarly when the tile has a \( 2\pi/3 \) angle.

Hence there are plenty of tilings of the kind considered here.

Laczkovich’s second table has an entries for the cases when \( ABC \) is equilateral and the tile has either a \( \pi/3 \) or a \( 2\pi/3 \) angle. The second table assumes not all the angles are rational multiples of \( \pi \), so this entry also assume that \( \beta \) is not a rational multiple of \( \pi/3 \). In Theorem 7, we proved that if the tile has a \( 2\pi/3 \) angle then \( N > 12 \). Thus, for the purpose of proving \( N \) cannot be 7 or 11, we are already done with that case. Nevertheless, for very little extra work, we can push the lower limit on \( N \) higher for both cases of tilings of an equilateral triangle.

The main difference between the two cases is that when the tile has a \( \pi/3 \) angle, we can color the tiles black and white in such a way that the coloring theorem applies. That is not possible when the tile has a \( 2\pi/3 \) angle, for example because it is possible (and necessary) for three tiles to meet at some vertex where all three have a \( 2\pi/3 \) angle, and three tiles at one vertex cannot be colored.

In unpublished work, we have interesting results about the existence or non-existence of such tilings, including a proof that the tile ratios \( b/a \) and \( c/a \) can be computed from \( N \) and \( M \), and that \( N \) cannot be prime, which certainly covers the cases 7 and 11. These proofs will not be presented here; instead we treat the problem computationally. Of course that covers only sufficiently small \( N \). But it at least deals with \( N = 7 \) and 11.

There are two computational approaches to the problem of tiling an equilateral triangle; one uses the coloring equation and hence applies only to the case of a tile with a \( \pi/3 \) angle, but the other uses only the area equation, and applies just as well to both cases. The latter method, however, needs to use the fact that the tile is rational, which Laczkovich proved in 2012 \([7]\), Theorem 3.3. The approach via the coloring equation does not use that result, which we may count in its favor, but the method based on the area equation works better, so we present the code for that method.

We will label the angles of the tile so that \( \gamma \) is the \( \pi/3 \) angle or the \( 2\pi/3 \) angle. Then we have the law of cosines:

\[
\begin{align*}
a^2 &= b^2 + c^2 - 2bc \cos \gamma \\
a^2 &= b^2 + c^2 \pm bc
\end{align*}
\]

since \( \cos \gamma = \pm \pi/3 \). The plus sign corresponds to the case \( \gamma = 2\pi/3 \) and the minus sign corresponds to \( \gamma = \pi/3 \). The area equation is the same in both cases, since \( \sin \gamma = \sin \pi/3 \) in either case. If \( X \) is the length of each side of \( ABC \),

\[
\begin{align*}
X^2 \sin \pi/3 &= Nab \sin \gamma \\
X^2 &= Nab
\end{align*}
\]

In case \( \gamma = \pi/3 \), we also have the coloring equation

\[
M(a + b + c) = 3X
\]

where \( M \) is the coloring number of the tiling.
Both computational approaches depend on writing
\[ X = pa + qb + rc \]
for non-negative integers \((p, q, r)\); this expression describes how the sides of \(ABC\) are composed of tile edges.

The following lemma gives some useful restrictions on the possibilities for \((p, q, r)\).

**Lemma 20.** Let the equilateral triangle \(ABC\) be tiled by a tile with \((\alpha, \beta, \gamma)\) and sides \((a, b, c)\), where \(\gamma\) is either \((\pi/3)\) or \((2\pi/3)\). Possibly after a relabeling of the vertices of \(ABC\), let \(X = pa + qb + rc\) where \(X\) is the length of \(AB\). Then

(i) If \(\gamma = \pi/3\), then \(p \geq 1\) and \(q \geq 1\)

(ii) \(r \geq 2\)

**Proof.** Ad (i). If \(\gamma = \pi/3\), then at the vertices of \(ABC\) there are tiles with altogether three \(a\) edges and three \(b\) edges. Therefore at least one side of \(ABC\) has one \(a\) edge and one \(b\) edge at its endpoints. Choosing that side for the decomposition of \(X\), we have \(p \geq 1\) and \(q \geq 1\). We relabel the vertices so that side is \(AB\).

If \(\gamma = 2\pi/3\), we do not assert \(p \geq 1\) and \(q \geq 1\). But we do have, by Lemma 10, that there are at least two \(c\) edges on each side of \(ABC\), so \(r \geq 2\).

Now suppose \(\gamma = \pi/3\). We have \(r \geq 1\), since if there are no \(c\) edges on \(AB\), then every tile supported by \(AB\) has a \(\gamma\) angle on \(AB\), which contradicts the pigeonhole principle since the edges at the endpoints are \(a\) or \(b\) edges. I say that \(r \geq 2\) holds also in case \(\gamma = \pi/3\). Indeed if there is only one \(c\) edge on \(AB\), and a total of \(n\) tiles supported on \(AB\), then there are \(n-1\) tiles with a \(\gamma\) angle on \(AB\), and \(n-1\) possible vertices for them, so the one tile with a \(c\) edge is bordered by two tiles with their \(\gamma\) angles adjacent. Let \(PQ\) be the \(c\) edge of Tile 3 supported by \(AB\), and let Tiles 1,2,3,4,5 be adjacent tiles in order, so that Tiles 1,3,5 are supported by \(AB\). Tiles 1 and 5 have their \(\gamma\) angles at \(P\) and \(Q\) respectively. Since \(\gamma = \pi/3\) we have \(\alpha + \beta = 2\pi/3\), so \(\alpha < \gamma < \beta\). Hence \(a < c < b\). Renaming \(P\) and \(Q\) if necessary, we may assume that Tile 3 has its \(\alpha\) angle at \(P\) and its \(\beta\) angle at \(Q\). Then Tile 2 has its \(\beta\) angle at \(P\) and Tile 4 has its \(\alpha\) angle at \(Q\). Let \(R\) be the third vertex of Tile 3. Then \(RQ\) has length \(a\) since it is opposite the \(\alpha\) angle of Tile 3. The adjacent edge of Tile 4 is not the \(a\) edge, since Tile 4 has its \(\alpha\) angle at \(Q\). Hence Tile 2 has its \(b\) edge matching the \(b\) edge of Tile 3 along \(PR\), since the \(c\) edge is too long to fit, and if the \(a\) edge were there, the remaining part \(b-a\) cannot be tiled unless \(b\) is an integer multiple of \(a\). But \(b\) cannot be an integer multiple of \(a\), for then by the law of cosines,

\[
c^2 = a^2 + b^2 - ab = a^2 + (ma)^2 - a(ma) = a^2
\]

Hence \(c = a\). Hence by the law of sines \(\alpha = \gamma\), contradicting the assumption that \(\alpha\) is not a rational multiple of \(\pi\). Hence Tile 2 has its \(b\) edge along \(PR\). But that is impossible, since it has its \(\beta\) angle at \(P\). That completes the proof of the lemma.

Our algorithm is going to check all possible values of \((p, q, r)\) and try to solve the area equation. The time that takes will clearly depend on how large \((p, q, r)\) can be in terms of \(N\). Of course each of \((p, q, r)\) is at most \(N\) since there are at most \(N\) tiles altogether. But for efficiency of computation, we want a better bound. We
improve the bound in the next two lemmas by a factor of 6; still crude, but enough for our purposes.

**Lemma 21.** Let the equilateral triangle $ABC$ be $N$-tiled by a tile with sides $(a, b, c)$ and $\alpha/\pi$ irrational. If $\gamma = 2\pi/3$ then no tile touches two different sides of $ABC$. If $\gamma = \pi/3$ then exactly three tiles touch two different sides of $ABC$.

*Proof.* First assume $\gamma = 2\pi/3$. Since we can relabel the vertices of $ABC$, it suffices to show that no tile touches both $AB$ and $BC$. We have $a < c$ and $b < c$ since $\alpha + \beta = \pi/3 < \gamma$. Renaming $\alpha$ and $\beta$ if necessary, we may suppose $\alpha < \beta$. Suppose $PQ$ is a tile edge with $P$ on $AB$ and $Q$ on $AC$. Then angle $APQ$ plus angle $AQP$ is $2\pi/3$, since the sum of the angles in triangle $APQ$ is $\pi$. Then one of those two angles is $\leq \pi/3$. Without loss of generality we can assume it is angle $APQ$. Then angle $APQ$ is either equal to $\pi/3$ or to $\beta$. If $AQ$ does not support a tile with its $c$ edge on $AQ$, then each tile supported by $AQ$ has its $\gamma$ angle on $AQ$, and by the pigeonhole principle, the tile at $Q$ has its $\gamma$ angle at $Q$, contradiction. Therefore $AQ \geq c$. In triangle $APQ$, the angle opposite $PQ$ (namely $\pi/3$) is greater than or equal to the angle opposite $AQ$ (namely angle $APQ$). Therefore the length $x$ of $PQ$ is greater than or equal to the length of $AQ$. Hence $x \geq c$. But $x$ is the length of a tile edge, and $c$ is the longest tile edge, so $x = c$ and we have equality throughout, i.e., $AQ = c$ and triangle $APQ$ is equilateral. Now we have an equilateral triangle $APQ$ tiled by some number $n$ of tiles, with the side of $APQ$ equal to $c$. The area equation tells us $c^2 = nab$. Let $g = \gcd(a, b)$. Then $g^2$ divides $c$; but $(a, b, c)$ have no common factor, so $g = 1$. By the law of cosines,

$$
c^2 = a^2 + b^2 + ab
$$

$$(n - 1)ab = a^2 + b^2 \quad \text{since } c^2 = nab$$

Taking this equation mod $a$ we find $b^2 \equiv 0 \bmod a$. Since $b$ and $a$ are relatively prime, that is a contradiction. That completes the proof in case $\gamma = 2\pi/3$.

Now we assume $\gamma = \pi/3$. Then $\alpha < \gamma < \beta$, so $a < c < b$. At each vertex of $ABC$, there is a single tile with its $\gamma$ angle at the vertex. Its $c$ side therefore does touch two sides of $ABC$. We have to prove that no other tile touches both $AB$ and $AC$. Suppose to the contrary that $P$ lies on $AB$ and $Q$ on $AC$ and $PQ$ is an edge of a tile in the tiling. Let $x$ be the length of $PQ$, so $x$ is one of $(a, b, c)$. Assume, for proof by contradiction, that triangle $APQ$ is equilateral. What is the length $x$ of the sides of $APQ$? Since the tiles at $A$ have their $\gamma$ angles at $A$, they have their $a$ or $b$ edges on $AP$ and $AQ$. If either has its $b$ edge there, then $x \geq b$, so $x = b$. If neither has its $b$ edge there, then they both have their $b$ edges along a line from $A$ to an interior point on $PQ$. Such a line is shorter than the side of an equilateral triangle, so again $x \geq b$. In this case $x > b$, which is impossible since $x$ is one of $(a, b, c)$ and $b$ is the largest of these. Therefore $x = b$. Now triangle $APQ$ is $n$-tiled for some $n$, and the area equation tells us

$$
b^2 = nab
$$

$$
b = na$$

By the law of cosines

$$
c^2 = a^2 + b^2 - ab
$$

$$
= a^2 + (na)^2 - a(na)
$$

$$
= a^2$$
Hence $a = c$, contradiction. Therefore triangle $APQ$ is not equilateral.

Now consider angles $APQ$ and $AQP$; one of these angle is less than or equal to $\pi/3$, since the sum of the angles of triangle $APQ$ is $\pi$ and angle $A$ is $\pi/3$. Since triangle $APQ$ is not equilateral, one of those two angles is strictly less than $\pi/3$. Without loss of generality we may assume it is angle $APQ$. Since $PQ$ is part of the tiling, angle $APQ$ must be $\alpha$. Then angle $AQP$ is $\beta$, and triangle $APQ$ is similar to the tile. Then for some $\lambda > 0$ we have $x = \lambda c$ and $AQ = \lambda b$ and $AP = \lambda a$. Since $x$ is one of $(a, b, c)$ we consider the possibilities one by one. If $x = c$ then $\lambda = 1$ and $AQ = b$ and $AP = a$. Then the two tiles at $A$ form a parallelogram with sides $a$ and $b$, and $PQ$ is the diagonal, hence not a part of the tiling. That rules out the case $x = c$. If $x = b$ then $\lambda = b/c$ and $AQ = b^2/c$ and $AP = ab/c$. Since $a < b < c$ that would make $AP < a$, which is impossible since $AP$ must support at least one tile. That rules out $x = b$. Therefore $x = a$. Then $\lambda = a/c$ and $AQ = ab/c < a$, which is also impossible. That completes the proof of the lemma.

**Lemma 22.** Let the equilateral triangle $ABC$ be $N$-tiled by a tile with sides $(a, b, c)$, and let $X$ be the length of $AB$. Suppose $X = pa + qb + rc$. Then $p + q + r \leq N/6 + 1$.

**Proof.** There are three tiles at each boundary vertex of $ABC$. Suppose there are $k$ tiles supported by $AB$. Then there are $2k - 1$ tiles with an edge or vertex on $AB$.

First we assume $\gamma = 2\pi/3$. Then there are two tiles at each vertex, and by Lemma 21 no tile has a vertex on two different sides of $ABC$. Then there will be no double-counting of tiles when we triple that number: there are at least $3(2k - 1)$ different tiles with an edge or vertex on the boundary. Hence $6k - 3 \leq N$, so $k = p + q + r(e(N + 3)/6 \leq N/6 + 1$.

Now assume $\gamma = \pi/3$. Then when we triple the number $3(2k - 1)$, we have double-counted the single tiles at the vertices of $ABC$, and also there are three tiles with a vertex on two sides of $ABC$ that will be double-counted. But by Lemma 21 only those three tiles touch two different sides of $ABC$. Hence

\[
3(2k - 1) - 6 \leq N
6k \leq N + 6
p + q + r \leq N/6 + 1
\]

That completes the proof of the lemma.

**Lemma 23.** Let the equilateral triangle $ABC$ be $N$-tiled by a tile with $\gamma = \pi/3$ or $2\pi/3$, and $\alpha$ not a rational multiple of $\pi$. Then $N \geq 40$. If $N \leq 75$ then the possible values of $N$ and the associated tiles are given in Table 6.

**Proof.** After a suitable relabeling of the vertices of $ABC$ so that Lemma 20 will apply, let $X$ be the length of $AB$ and let the tiling determine the integers $(p, q, r)$ such that $X = pa + qb + rc$. According to the area equation (27) we have

\[
X^2 = Nab
(pa + qb + rc)^2 = Nab
\]

By the law of cosines (20), we have

\[
c = \sqrt{a^2 + b^2 \pm ab}
\]

Putting that into the area equation, we have

\[
(pa + qb + r\sqrt{a^2 + b^2 \pm ab})^2 = Nab
\]
Table 6. Possible $N$ and $(a, b, c)$ for equilateral tilings

| $N$  | $\gamma$   | the tile       |
|------|------------|----------------|
| 40   | $\frac{\pi}{3}$ | (5, 8, 7)          |
| 54   | $\frac{2\pi}{3}$ | (3, 8, 7)          |
| 56   | $\frac{2\pi}{3}$ | (7, 8, 13)         |
| 60   | $\frac{2\pi}{3}$ | (3, 5, 7)          |
| 65   | $\frac{\pi}{3}$  | (9, 65, 61)        |
| 66   | $\frac{2\pi}{3}$ | (11, 24, 31)       |
| 70   | $\frac{\pi}{3}$  | (7, 40, 37)        |
| 80   | $\frac{2\pi}{3}$ | (5, 16, 19)        |
| 84   | $\frac{\pi}{3}$  | (16, 20, 19)       |
| 85   | $\frac{\pi}{3}$  | (17, 80, 73)       |

Define $s := a/b$ and divide the equation by $b^2$:

$$(ps + q + r\sqrt{s^2 + 1} \pm s)^2 = Ns$$

Expanding the left side we have

$$(ps + q)^2 + r^2(s^2 + 1 \pm s) + 2(ps + q)r\sqrt{s^2 + 1 \pm s} = Ns$$

$$2(ps + q)r\sqrt{s^2 + 1 \pm s} = Ns - (ps + q)^2 - r^2(s^2 + 1 \pm s)$$

Squaring both sides we have

$$(29)\quad 4(ps + q)^2r^2(s^2 + 1 \pm s) = (Ns - (ps + q)^2 - r^2(s^2 + 1 \pm s))^2$$

That is a fourth-degree polynomial equation in $s$. Since the tile is known to be rational, and since we can assume $a < b$, we are looking for rational solutions $s$ with $0 < s < 1$. If there is a tiling, there will be such a solution. Since SageMath can solve quartic equations, and test whether the solutions are rational, we are almost finished: we just need to bound the possible values of $(p, q, r)$.

We make use of Lemma 22 to use the (still crude) bound $N/6 + 1$. Now the algorithm is simple: given $N$, check all the possibilities $(p, q, r)$ that satisfy the conditions of Lemma 20 and satisfy $p + q + r \leq N/6 + 1$. Solve (29). Reject solutions that are not real and solutions that are not rational and solutions that are not between 0 and 1. If there are no solutions remaining, then there is no $N$-tiling. If there is a solution, the method is inconclusive. The code is shown in Fig. 29. We put the function defined there in an outer loop and ran it over $N$ from 3 to 47. It finds solutions only for the values of $N$ mentioned in the statement of the lemma. That completes the proof, at least, if you believe the code is correctly written and correctly executed.

We took the following steps to check this code for correctness. We wrote this code independently in SageMath and in C, and, commenting out the line to reject rational solutions, printed out the solutions that are found. The C code only works with floating-point numbers, so to compare the results, we had SageMath numerically evaluate its exact solutions. The C and SageMath code printed out the same decimal values, which we found reassuring. Since the C code works with
def feb24(N, an):  # an = -1 for pi/3, 1 for 2pi/3
    # search for equilateral gamma = pi/3 or 2pi/3 solutions
    var('s')
    if an == -1:  # gamma at A,B,C
        Abound = 1  # at least one a edge
        Bbound = 1  # at least one b edge
    else:  # alpha + beta at A,B,C
        Abound = 0
        Bbound = 0
    upperbound = N/6 + 2
    for p in range(Abound, upperbound):
        for q in range(Bbound, upperbound):
            for r in range(2, upperbound):  # at least 2 c edges in either case
                if p+q+r >= floor(upperbound):
                    continue
                eq = 4*(p*s + q)^2*r^2*(s^2+1+an*s)
                - (N*s-(p*s+q)^2 - r^2*(s^2+1+an*s))^2
                answers = solve(eq, s)
                for t in answers:
                    S = t.rhs();
                    if not S in RR:
                        continue
                    if not S in QQ:
                        continue  # tile is known to be rational
                    if S <= 0 or 1 <= S:  # we can assume a < b so s<1
                        continue
                    A = S.numerator()
                    B = S.denominator()
                    C = B * sqrt(S^2+1+an*s)
                    if not C in QQ:
                        continue;
                    if not C in ZZ:
                        print("oops")  # it better be in ZZ!
                        print(A,B,C);
                    g = gcd(A,gcd(B,C));
                    a = A/g
                    b = B/g
                    C = C/g
                    print("found (%d, %d, %d)" %(a,b,c))
    return true

floating-point numbers, it is not so easy to have it reject irrational solutions, so we
could not replace the SageMath code with more efficient C code.
19. No 7-tilings

We break the proof that there are no 7-tilings or 11-tilings into two cases, according as the angles are commensurable or not. All the required cases have already been dealt with: it only remains to put the pieces together.

**Theorem 10.** Suppose \((\alpha, \beta, \gamma)\) are all rational multiples of \(2\pi\). Then there is no 7-tiling of any triangle \(ABC\) by a tile with angles \((\alpha, \beta, \gamma)\). Moreover, there is no \(N\)-tiling by such a tile for \(N = 11, 14, 19, 31\) or any number which is neither a square, sum of squares, or 2, 3, or 6 times a square.

**Remark.** Any odd \(N\) which is not divisible by 3 but is divisible by some prime congruent to 3 mod 4 meets the conditions of the theorem.

**Proof.** Assume, for proof by contradiction, that there is such a tiling. By Theorem 3, the pair \(ABC\) and \((\alpha, \beta, \gamma)\) (after a suitable renaming of the angles) occurs in Table 3. But 7 does not match any of the forms of \(N\) listed in that table, which are the forms listed in the final sentence of the theorem. That completes the proof.

Finally we have arrived at the main theorem.

**Theorem 11.** There are no 7-tilings or 11-tilings.

**Proof.** Suppose, for proof by contradiction, that triangle \(ABC\) is \(N\)-tiled by a tile with angles \((\alpha, \beta, \gamma)\), with \(N = 7\) or 11. We also note the cases where the proof works for \(N = 14\) and 19. Since \(N\) is not a square or a sum of two squares, then by [10], \(ABC\) is not similar to the tile. Then according to Theorem 10, not all the angles \((\alpha, \beta, \gamma)\) are rational multiples of \(\pi\). Then according to [4], the tiling must correspond to one of the rows in Table 4 in this paper.

For our purpose these rows will be combined into five cases: Either \(3\alpha + 2\beta = \pi\), or \(\gamma = 2\pi/3\), or \(ABC\) is isosceles with base angles \(\alpha\) and \(\gamma = \pi/2\), or \(ABC\) is isosceles with base angles \(\alpha\) and \(\gamma = 2\alpha\), or \(ABC\) is equilateral and \(\alpha = \pi/3\).

In case \(ABC\) is equilateral and \(\alpha = \pi/3\), Lemma 23 tells us there is no 7-tiling or 11-tiling (but the proof does not work for \(N = 14\) or \(N = 19\)).

In case \(ABC\) is isosceles with base angles \(\alpha\) and \(\gamma = \pi/2\), and \(\alpha\) is not a rational multiple of \(\pi\), then Theorem 5 tells us that \(N\) cannot be 7, 11, 14, or 19. If \(\alpha\) is a rational multiple of \(\pi\), then Theorem 4 tells us that \(N/2\) is a square or a sum of squares; in particular \(N\) is even. But 7, 11, and 19 are odd, and 14/2 is not square or a sum of squares.

In case \(ABC\) is isosceles with base angles \(\alpha\) and \(\gamma = 2\alpha\), Theorem 8 tells us that \(N\) cannot be 7 or 11. Theorem 9 confirms that result and also checks \(N = 14\) and 19.

In case \(3\alpha + 2\beta = \pi\), Theorem 6 tells us there is no 7-tiling or 11-tiling. In case \(\gamma = 2\pi/3\), by Theorem 7 there is no 7-tiling or 11-tiling. That completes the proof of the theorem.

20. Concluding Remarks

This paper has successfully avoided the need to appeal to the hundred pages of theoretical work on the case \(3\alpha + 2\beta = \pi\), as well as more than thirty pages each on the isosceles and the equilateral cases. Instead we have used algebraic and computation shortcuts that work only for small values of \(N\).
In this section we nevertheless mention some results of those lengthier investigations. First, in each of the three cases ($3\alpha + 2\beta = \pi$, isosceles, equilateral), we used techniques pioneered by Laczkovich to prove that the tile has to be rational. Then we used the area equation and (for the $3\alpha + 2\beta = \pi$ case and one equilateral case) the “coloring equation” to derive necessary conditions. We used these equations to prove that $N$ cannot be prime in almost all the cases. The exceptions are as follow: There are the biquadratic tilings of a right triangle, in which case if $N$ is prime it must be congruent to 1 mod 4. And in case $ABC$ is isosceles with base angles $\alpha$, and $\gamma = 2\pi/3$, and $\alpha/\pi$ is not rational, we were unable to settle the question, although if $N$ is prime in that case, $N = 2b + a$.

Generally we want to we expand the lines with in Table 4 by adding a third column with restrictions on the possible form of $N$. In some cases this is a necessary and sufficient condition; in others it is only a necessary condition. Where the necessary and sufficient conditions do not match, there are open questions. Whether there are yet-undiscovered tilings, or our necessary conditions are too weak, we do not know. Table 7 gives a summary of what we know about $N \leq 100$.

Figures 30 and 31 show some examples of tilings unknown before 2011 and 2018, respectively. We also have many examples of larger tilings.

Very little is known about the possible values of $N$ for tilings of isosceles and equilateral triangles; proving that $N$ cannot be prime is an advance, since for example in 2012 it was not known whether there are arbitrarily large $N$ such that no equilateral triangle can be $N$-tiled by a tile whose angles are not all rational multiples of $\pi$. Similar for the case of tiling an isosceles $ABC$ by a tile with $\gamma = 2\alpha$. Now we know $N$ can’t be prime, but we still don’t know if $N$ can be even or not, and we don’t know if $N$ can be less than five million, although Laczkovich proved that any such tile must tile some isosceles $ABC$, so there do exist a lot of such tilings—we just don’t know how big (or small) $N$ can be.
Table 7. Knowledge about tilings with \( N \leq 100 \) as of March, 2019

| \( ABC \) shape     | the tile          | known \( N \) values \( \leq 100 \) | least unknown \( N \) |
|---------------------|-------------------|-------------------------------------|----------------------|
| equilateral         | \( \left( \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3} \right) \) | \( 6n^2 \) 24, 72, 96           |                      |
|                     | \( \left( \frac{\pi}{7}, \frac{\pi}{7}, \frac{2\pi}{3} \right) \) | \( 3n^2 \) 12, 27, 48, 75       |                      |
|                     | \( (\alpha, \beta, \frac{\pi}{7}) \) | 5861172 ? 40?                    | 40?                  |
|                     | \( (\alpha, \beta, \frac{2\pi}{3}) \) | 10395 ? 40?                      | 40?                  |
| right(\( \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{4} \)) | \( \left( \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{4} \right) \) | \( n^2, 3n^2 \) 4, 9 \ldots 81, 100 |                      |
| right(\( \alpha, \beta, \frac{\pi}{4} \)) | \( (\alpha, \beta, \frac{\pi}{4}) \) | \( N = e^2 + f^2 \)             |                      |
| isosceles-\( \alpha \) | \( \left( \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2} \right) \) | \( 6n^2 \) 24, 72, 96           |                      |
|                     | \( (\alpha, \beta, \frac{\pi}{2}) \) | \( 2n^2 \) 2, 8, 18\ldots       |                      |
|                     | \( \left( \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2} \right) \) | \( 6n^2 \) 24, 72, 96           |                      |
|                     | \( (\alpha, \beta, 2\alpha) \) | not p or 2p 20?                  |                      |
|                     | \( (\alpha, \beta, 2\pi) \) | 1878500 37? 71?                  |                      |
| isosceles-\( \beta \) | \( 3\alpha + 2\beta = \pi \) | 84 70?                           |                      |
| isosceles-\( \alpha + \beta \) | \( 3\alpha + 2\beta = \pi \) | 48 45? 72? 75? 99?               |                      |
| \( (\alpha, 2\alpha, 2\beta) \) | \( 3\alpha + 2\beta = \pi \) | 77                               |                      |
| \( 2\alpha, \beta, \alpha + \beta \) | \( 3\alpha + 2\beta = \pi \) | 28                               |                      |
| \( (\alpha, \alpha + \beta, \alpha + 2\beta) \) | \( (\alpha, \beta, 2\pi) \) | 13?                              |                      |
| \( \alpha, 2\alpha, 3\beta \) | \( (\alpha, \beta, \frac{2\pi}{3}) \) | 13?                              |                      |
| \( (2\alpha, 2\beta, \alpha + \beta) \) | \( (\alpha, \beta, \frac{2\pi}{3}) \) | 13?                              |                      |
| any \( ABC \) similar to \( ABC \) | \( n^2 \) 4, 9, 16, \ldots 100 |                      |                      |

Figure 31. Tilings with \( N = 44 \) and 48, with \( 3\alpha + \beta = \pi \) and tile \( (2, 3, 4) \)
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