Dimension Elevation in Müntz Spaces: A New Emergence of the Müntz Condition

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Abstract

We show that the limiting polygon generated by the dimension elevation algorithm with respect to the Müntz space \( \text{span}(1, t^{r_1}, t^{r_2}, \ldots, t^{r_m}, \ldots) \), with \( 0 < r_1 < r_2 < \ldots < r_m < \ldots \) and \( \lim_{n \to \infty} r_n = \infty \), over an interval \([a, b]\) converges to the underlying Chebyshev-Bézier curve if and only if the Müntz condition \( \sum_{i=1}^{\infty} \frac{1}{r_i} = \infty \) is satisfied. The surprising emergence of the Müntz condition in the problem raises the question of a possible connection between the density questions of nested Chebyshev spaces and the convergence of the corresponding dimension elevation algorithms. The question of convergence with no condition of monotonicity or positivity on the pairwise distinct real numbers \( r_i \) remains an open problem.

Keywords: Müntz spaces, Chebyshev blossoming, Dimension elevation, Chebyshev-Bernstein bases, Gelfond-Bernstein bases, Schur functions, Chebyshev-Bézier curves

1. Introduction

Degree elevation of Bézier curves is a standard technique in computer aided curve design. It consists of iteratively expressing a Bézier curve of a fixed degree in the Bernstein bases of the linear spaces of polynomials of higher degrees. The process generates a sequence of control polygons which converges uniformly to the underlying Bézier curve [16]. Degree elevation, or more appropriately, dimension elevation, can be generalized to any infinite nested sequence of linear spaces in which an analogue notion of Bernstein basis can be defined. More precisely, let \( E_\infty = (u_1, u_2, \ldots, u_n, \ldots) \) be an infinite sequence of sufficiently differentiable functions \( u_i \) over an interval \([a, b]\) and such that for every \( n \geq 1 \), the linear space \( DE_n = \text{span}(u'_1, u'_2, \ldots, u'_n) \) is an extended Chebyshev space of dimension \( n \) over \([a, b]\) [12]. Then, for any \( n \geq 1 \), the linear space \( E_n = \text{span}(1, u_1, \ldots, u_n) \) possesses a so-called Chebyshev-Bernstein basis \( B_k^n \), \( k = 0, \ldots, n \) and characterized as the unique normalized basis of \( E_n \) such that for every \( k \in \{1, \ldots, n\} \), the function \( B_k^n \) has \( k \) zeros at \( a \) and \((n - k)\) zeros at \( b \) [13]. A Chebyshev-Bézier curve, \( \Gamma \), in the space \( E_n \) can be expressed in the Chebyshev-Bernstein bases of the spaces \( E_n \) and \( E_{n+1} \) over an interval \([a, b]\) as

\[
P(t) = \sum_{i=0}^{n} B_i^n(t) P_i, \quad P_{i+1}^n = \sum_{i=0}^{n+1} B_i^{n+1}(t) P_i^1, \quad P_i, P_i^1 \in \mathbb{R}^s; \quad s \geq 1.
\]
The defining endpoint conditions of Chebyshev-Bernstein bases show that the points \( P^1_i \) are related to the points \( P_i \) as follows:

\[
P^1_0 = P_0, \quad P^1_{n+1} = P_n
\]

and for \( i = 1, ..., n \), there exist real numbers \( \xi_i \in [0, 1] \) such that

\[
P^1_i = (1 - \xi_i)P_{i-1} + \xi_iP_i. \tag{1}
\]

Iterating the process of expressing the curve \( \Gamma \) in the Chebyshev-Bernstein bases of the nested sequence of spaces \( E_{n+1} \subset E_{n+2} \subset ... \subset E_m \subset ... \) generates a sequence of control polygons. Since dimension elevation is a corner cutting scheme, the generated sequence converges to a Lipschitz-continuous curve \([6]\).

However, a difficult question is to characterize the sequences \( E_\infty \) in which the limiting polygon converges uniformly to the underlying Chebyshev-Bézier curve. Here, we report our complete solution to the problem when \( E_\infty \) is the Müntz sequence \( E_\infty = (t^{r_1}, t^{r_2}, ..., t^{r_m}, ...) \) with \( 0 < r_1 < r_2 < ... < r_m < ... \) and \( \lim_{n \to \infty} r_n = \infty \). More precisely, we have

**Theorem 1.** The limiting polygon generated by the dimension elevation algorithm with respect to the Müntz space \( \text{span}(1, t^{r_1}, t^{r_2}, ..., t^{r_m}, ...) \) with \( 0 < r_1 < r_2 < ... < r_m < ... \) and \( \lim_{n \to \infty} r_n = \infty \) over an interval \([a, b] \subset [0, \infty)\) converges uniformly to the underlying non-constant Chebyshev-Bézier curve if and only if

\[
\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty. \tag{2}
\]

Let us compare our theorem with the celebrated original Müntz Theorem on the density of Müntz spaces \([1, 8, 15]\).

**Theorem 2. (Müntz Theorem)** Let \( (r_1, r_2, ..., r_n, ...) \) be an infinite strictly increasing sequence of positive real numbers such that \( \lim_{n \to \infty} r_n = \infty \). The Müntz space \( \text{span}(1, t^{r_1}, ..., t^{r_m}, ...) \) is a dense subset of \( C([0, 1]) \) (the linear space of continuous functions on \([0, 1]\) endowed with the uniform norm) if and only if

\[
\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty.
\]

The emergence of the Müntz condition \((2)\) in both Theorem 1 and Theorem 2 is rather surprising and may suggest a deep connection between the problem of density of nested Chebyshev spaces and the convergence of the associated dimension elevation algorithms. For nested Müntz spaces over the interval \([0, 1]\), a hypothesis of equivalence is ruled out by the fact that the condition \( \lim_{n \to \infty} r_n = \infty \) can be dropped in Müntz Theorem \([2, 5]\), while such condition is necessary for the convergence of the dimension elevation over \([0, 1]\) to the underlying curve (See Theorem 5). However, the necessity of the condition \( \lim_{n \to \infty} r_n = \infty \) in Theorem 1 remains an open problem.

It is interesting to note that as much as the classical Weierstrass approximation Theorem is a special case of Müntz Theorem for the exponents \( r_n = n \), the classical convergence theorem of the degree elevation of Bézier curves is a consequence of Theorem 1 for the same set of exponents.

The Müntz condition \((2)\) also appears in different contexts other than the density questions of Müntz spaces, such as in Biernack Theorem on entire functions \([1]\), in Ramm-Horváth Theorem on the inverse scattering problem \([9]\) or in the famous Erdős conjecture on arithmetic progressions \([7]\).
The strategy for the proof of Theorem 1. For nested M"untz spaces, the analytical form of $\xi_i$ in (1) can be expressed in terms of a quotient of generalized Schur functions that depend on the interval parameters $a$ and $b$ [2]. Iterating the dimension elevation process using these quotients leads to complicated expressions that hinder a direct proof of Theorem 1. The limiting curve generated by the dimension elevation in M"untz spaces over an interval $[a, b]$ depends only on the shape parameter $b/a [12]$. Therefore, we only need to prove Theorem 1 over an interval $[a, 1]$ with $0 < a < 1$. Taking into account the special role of the origin in the density questions of M"untz spaces [5], we could first look at the dimension elevation algorithm in M"untz spaces over the interval $[0, 1]$. An apparent obstruction to such strategy is the fact that M"untz spaces does not possess Chebyshev-Bernstein bases over the interval $[0, 1]$. Fortunately, as we will show in this work, the pointwise limits of the Chebyshev-Bernstein bases over an interval $[a, 1]$ as $a$ goes to zero do exist and in fact coincide with the Gelfond-Bernstein bases of M"untz spaces [3]. The dimension elevation algorithm for the Gelfond-Bernstein bases over the interval $[0, 1]$ can be easily expressed. This allows us to prove Theorem 1 over the interval $[0, 1]$. To prove Theorem 1 over an interval $[a, 1]$ such that $0 < a < 1$, we use the notion of Chebyshev blossoming in M"untz spaces in order to show that the control polygon of the dimension elevation over $[a, 1]$ can be obtained by a generalized de Casteljau algorithm from the control polygon of the dimension elevation over the interval $[0, 1]$. The necessity of the M"untz condition is proved with the aid of the bounded Chebyshev inequality in M"untz spaces. Although the initial steps of the proof of Theorem 1, namely the proof of the theorem over the interval $[0, 1]$, appeared in our work in [3, 4], we review here of all the steps of the proof in order to give a consistent presentation and show how we extended our methods to deal with the away from the origin case.

2. Chebyshev-Bernstein Bases in M"untz Spaces

Throughout this work, we will denote by $\Lambda_\infty = (r_0 = 0, r_1, ..., r_m, ...)$ an infinite sequence of strictly increasing real numbers. For any integer $n$, we denote by $\Lambda_n$ the finite subsequence $\Lambda_n = (r_0 = 0, r_1, ..., r_n)$ and denote by $E(\Lambda_n) = \text{span}(t^{r_0} = 1, t^{r_1}, ..., t^{r_n})$ the associated M"untz space. To give an explicit expression for the Chebyshev-Bernstein basis of the linear space $E(\Lambda_n)$ over an interval $[a, b]$, we introduce the following definitions and terminology

**Definition 1.** A finite sequence of real numbers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is termed a real partition if it satisfies

$$\lambda_1 > \lambda_2 - 1 > \lambda_3 - 2 > ... > \lambda_n - (n - 1) > -n.$$ 

The generalized Schur function indexed by a real partition $\lambda$ is defined as the continuous extension of the function defined for pairwise distinct real values $u_1, u_2, ..., u_n$ by [14]

$$S_{\lambda}(u_1, ..., u_n) = \frac{\det(u_i^{\lambda_j + n - j})_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n}(u_i - u_j)}.$$ 

We use the notation $S_{\lambda}(u_1^{m_1}, u_2^{m_2}, ..., u_k^{m_k})$ to mean the evaluation of the generalized Schur function in which the argument $u_1$ is repeated $m_1$ times,
the argument \( u_2 \) is repeated \( m_2 \) times and so on. When the elements of the finite sequence \( \lambda \) are positive integers, we recover the classical notion of integer partitions whose associated Schur functions \( S_\lambda(u_1, \ldots, u_n) \) are elements of the ring \( \mathbf{Z}[u_1, \ldots, u_n] \). The value \( S_\lambda(1^n) \) can be computed for integer partitions using the hook-length formula \([17]\) or in general by the formula

\[
S_\lambda(1^n) = \frac{\prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k - j + k)}{\prod_{j=1}^{n} (j-1)!}.
\]

**Definition 2.** For a finite sequence \( \Lambda_n = (r_0, r_1, \ldots, r_n) \) of \((n+1)\) real numbers such that \( 0 = r_0 < r_1 < \ldots < r_n \), we define the real partition \( \lambda = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) \) associated with the finite sequence \( \Lambda_n \) by

\[
\lambda_k = r_n - r_{k-1} - (n-k+1) \quad \text{for} \quad k = 1, \ldots, n+1.
\]

We also denote by \( \lambda^{(0)} \) the real partition \( \lambda^{(0)} = (\lambda_2, \lambda_3, \ldots, \lambda_{n+1}) \) termed the bottom partition of \( \lambda \).

With the above definitions, we can prove the following theorem \([2]\).

**Theorem 3.** The Chebyshev-Bernstein basis \( (B_{0, \Lambda_n}^n, \ldots, B_{n, \Lambda_n}^n) \) over an interval \([a, b]\) of the Müntz space \( E(\Lambda_n) \) is given by

\[
B_{k, \Lambda_n}^n(t) = \frac{S_{\lambda}(1^{n+1})}{S_{\lambda}(1^n)} B_{k}^0(t) \frac{S_{\lambda}(a^{n-k}, b^k) t^1 \cdot S_{\lambda}(a^{n-k}, b^k, \frac{t}{b-a})}{S_{\lambda}(a^{n+1-k}, b^k) S_{\lambda}(a^{n-k}, b^k+1)},
\]

where \( B_{k}^0 \) is the classical Bernstein basis of the polynomial space over the interval \([a, b]\), \( \lambda \) the real partition associated with \( \Lambda_n \) and \( \lambda^{(0)} \) the bottom partition of \( \lambda \).

**2.1. Gelfond-Bernstein Bases as Pointwise Limits of Chebyshev-Bernstein Bases**

Let \( f \) be a smooth real function defined on an interval \( I \). For any real numbers \( x_0 < x_1 < \ldots < x_n \) in the interval \( I \), the divided difference \([x_0, \ldots, x_n] f\) of the function \( f \) supported at the point \( x_i, i = 0, \ldots, n \) is recursively defined by \([x_0] f = f(x_0)\) and

\[
[x_0,x_1,\ldots,x_n] f = \frac{[x_1,\ldots,x_n] f - [x_0,x_1,\ldots,x_{n-1}] f}{x_n - x_0} \quad \text{if} \quad n > 0.
\]

Consider, now, the function \( f_t \) defined on \([0, +\infty[\) as

\[
\begin{cases}
  f_t(x) = t^x & \text{for} \quad t > 0, \quad x \geq 0 \\
  f_0(0) := 1, \quad f_0(x) = 0 & \text{for} \quad x > 0.
\end{cases}
\]

**Definition 3.** For a finite sequence \( \Lambda_n = (0 = r_0, r_1, \ldots, r_n) \) of strictly increasing positive real numbers, the Gelfond-Bernstein basis of the Müntz space \( E(\Lambda_n) \) with respect to the interval \([0, 1]\) is defined by

\[
H_{k, \Lambda_n}^n(t) = (-1)^{n-k} r_{k+1} \cdots r_n [r_k, \ldots, r_n] f_t \quad \text{for} \quad k = 0, \ldots, n-1
\]

and

\[
H_{n, \Lambda_n}^n(t) = t^r_n,
\]

where \( f_t \) is the function defined in \([3]\).
The relation between the Chebyshev-Bernstein basis and the Gelfond-Bernstein basis of a given Müntz space is given by the following theorem [3].

**Theorem 4.** Let \( \Lambda_n = (0 = r_0, r_1, ..., r_n) \) be a finite sequence of strictly increasing real numbers. We denote by \( B^n_k,\Lambda_n \), \( k = 0, ..., n \) the Chebyshev-Bernstein basis of the Müntz space \( E(\Lambda_n) \) over an interval \([a, 1]\). Then, for \( k = 0, ..., n \) and \( t \in [0, 1] \) we have

\[
\lim_{a \to 0} B^n_k,\Lambda_n (t) = H^n_k,\Lambda_n (t).
\]

The proof of Theorem 4 is based on applying to the explicit expression of the Chebyshev-Bernstein basis in Theorem 3 the following splitting formula for generalized Schur functions: If \( \eta = (\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_h) \) is a real partition then we have

\[
\lim_{\epsilon \to 0} S^\eta(z_1, ..., z_k, \epsilon y_1, ..., \epsilon y_h) = S^{\lambda}(z_1, ..., z_k) S^{\mu}(y_1, ..., y_h),
\]

where \( \lambda \) and \( \mu \) are the real partitions \( \lambda = (\lambda_1, ..., \lambda_k) \) and \( \mu = (\mu_1, ..., \mu_h) \) and where \( |\mu| \) denotes \( \mu_1 + \mu_2 + ... + \mu_h \). The proof also relies on the following interesting connection between generalized Schur functions and divided differences of the function \( f_t \) in [3]. Namely, for any finite sequence \( \Lambda_n = (0 = r_0, r_1, ..., r_n) \) of strictly increasing real numbers we have

\[
[r_0, r_1, ..., r_n] f_t = \frac{(-1)^n}{r_1 r_2 ... r_n} (1 - t)^n \frac{S^\lambda(1, t^n)}{S^{\lambda(0)}},
\]

where \( \lambda \) is the real partition associated with the finite sequence \( \Lambda_n \) and \( \lambda(0) \) the bottom partition of \( \lambda \).

### 2.2. Dimension Elevation in Müntz Spaces over the Interval \([0, 1]\)

To express the corner cutting scheme associated with the dimension elevation of Gelfond-Bézier curves, we should express the Gelfond-Bernstein basis of \( E(\Lambda_n) \) in terms of the Gelfond-Bernstein basis of \( E(\Lambda_{n+1}) \). Such expression is given by the following proposition [3].

**Proposition 1.** For \( k = 0, ..., n \), and for any \( t \in [0, 1] \) we have

\[
H^n_k,\Lambda_n (t) = \frac{r_{n+1} - r_k}{r_{n+1} r_{n+1}} H^n_{k+1},\Lambda_{n+1} (t) + \frac{r_k + 1}{r_{n+1} r_{n+1}} H^n_{k+1},\Lambda_{n+1} (t).
\]

Let \( P \) be an element of the Müntz space \( E(\Lambda_n) \) written as

\[
P(t) = \sum_{k=0}^{n} H^n_k,\Lambda_n (t) P_k = \sum_{k=0}^{m} H^m_k,\Lambda_m (t) b^m_k, \quad m > n.
\]

By Proposition 4, the control points \( b^m_k = P^m_{k-n} \) in (4) can be computed using the following corner cutting scheme: For \( i = 0, 1, ..., n \), we set \( P^i_0 = P_i \) and for \( j = 1, 2, ..., m-n, \) we construct iteratively new polygons \( P^j_0, P^j_1, ..., P^j_{n+j} \) using the inductive rule

\[
P^j_0 = P^j_0, \quad P^j_{n+j} = P^j_{n+j-1} \quad (5)
\]

and for \( i = 1, ..., n + j - 1 \)

\[
P^j_i = \frac{r_i}{r_{n+j}} P^j_{i-1} + \left( 1 - \frac{r_i}{r_{n+j}} \right) P^j_{i-1} \quad (6)
\]
When the real numbers \( r_i \) are given by \( r_i = i \) for every index \( i \), then the corner cutting scheme (5) and (6) leads to the classical degree elevation algorithm, in which it is well known that the limiting control polygon converges to the underlying Bézier curve as \( m \) goes to infinity [16]. Consider the case where \( r_i = i \) for \( i = 1, \ldots, n \) and \( r_i = 2i \) for \( i > n \). Figure 2 shows the obtained polygons after 100 iterations with \( n = 3 \). It is clear from the figure that the limiting polygon does not converge to the Bézier curve with control points \((P_0, P_1, P_2, P_3)\). Now, consider, for example, the limiting polygon of the corner cutting scheme for the case \( n = 3 \) and in which \( r_1 = 2, r_2 = 4, r_3 = 10 \) and \( r_i = 2i + 5 \) for \( i > 3 \). Figure 3 shows the generated polygons from 100 iterations and the Gelfond-Bézier curve associated with the Müntz space \( E(F, \Lambda_n) = \text{span}(1, t^{r_1}, t^{r_2}, t^{r_3}) = \text{span}(1, t^2, t^4, t^{10}) \) and control polygon \((P_0, P_1, P_2, P_3)\). The figure suggests that the limiting polygon converges to the Gelfond-Bézier curve.

To exhibit the importance of the condition \( \lim_{s \to \infty} r_s = \infty \) in the dimension elevation of Gelfond-Bézier curves, Figure 4 shows the limiting polygon for the case \( n = 3, r_1 = 1, r_2 = 2, r_3 = 3 \) and \( r_i = 4 - \frac{i}{2} \) for \( i > 3 \). The limiting polygon does not converge to the Bézier curve with control polygon \((P_0, P_1, P_2, P_3)\). In fact we can prove the following theorem [4].

**Theorem 5.** The limiting polygon generated from a non-constant polygon \((P_0, P_1, \ldots, P_n)\) in \( \mathbb{R}^s, s \geq 1 \) using the corner cutting scheme (5) and (6) with respect to the sequence \( \Lambda_n \) converges uniformly to the Gelfond-Bézier curve associated with the Müntz space \( E(\Lambda_n) \) and control polygon \((P_0, P_1, \ldots, P_n)\) if and only if the real numbers \( r_i \) satisfy the conditions

\[
\lim_{s \to \infty} r_s = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{r_i} = \infty.
\]

For the rest of the paper, we adopt the following notation: for a given element \( P \) of the Müntz space \( E(\Lambda_n) \), we denote by \( \eta_i(P; \Lambda_n, [0, 1]) \) the Gelfond-Bézier control points of \( P \) with respect to the interval \([0, 1]\) and we denote by
Figure 2: The sequence of polygons generated from 100 iterations of the corner cutting scheme (5) and (6) and parameters \( n = 3, r_1 = 1, r_2 = 2, r_3 = 3 \) and \( r_i = i^2 \) for \( i \geq 4 \). The red curve is the Bézier curve associated with the control polygon \((P_0, P_1, P_2, P_3)\).

Figure 3: The sequence of polygons generated from 100 iterations of the corner cutting scheme (5) and (6) and parameters \( n = 3, r_1 = 2, r_2 = 4, r_3 = 10 \) and \( r_i = 2i + 5 \) for \( i \geq 4 \). The red curve is the Gelfond-Bézier curve associated with the Müntz space \( \text{span}(1, t^2, t^4, t^{10}) \) and control polygon \((P_0, P_1, P_2, P_3)\).

Figure 4: The sequence of control polygons generated from 100 iterations of the corner cutting scheme (5) and (6) and parameters \( n = 3, r_1 = 1, r_2 = 2, r_3 = 3 \) and \( r_i = 4 - \frac{1}{i} \) for \( i \geq 4 \). The red curve is the Bézier curve associated with the control polygon \((P_0, P_1, P_2, P_3)\).
\(\eta_i(P, \Lambda_n, [a, 1])\) the Chebyshev-Bézier control points of \(P\) with respect to the interval \([a, 1]\). One of the main tools in proving Theorem 5, which will be needed later, is the fact that for any element \(P\) of \(E(\Lambda_n)\) we have
\[
\lim_{m \to \infty} \|P(\eta_i(t^{r_1}, \Lambda_m, [0, 1])^{1/r_1}) - \eta_i(P, \Lambda_m, [0, 1])\|_\infty = 0
\]
(7) uniformly in \(i\).

3. Dimension elevation over an interval \([a, 1]\)

Given two intervals \([a, 1] \subset [b, 1]\), \(b > 0\), we can infer from the de Casteljau algorithm of Chebyshev-Bézier curves that for any element \(P\) of \(E(\Lambda_n)\) there exist real numbers \(s_j^{(i)} \in [0, 1]; j = 0, ..., n\) (independent of \(P\)) such that
\[
\sum_{j=0}^n s_j^{(i)} = 1
\]
and
\[
\eta_i(P, \Lambda_n, [a, 1]) = \sum_{j=0}^n s_j^{(i)} \eta_j(P, \Lambda_n, [b, 1]).
\]
(8)

Although the latter statement remains true for \(b = 0\), its proof is not obvious and requires the generalization of the notion of blossoming in Müntz spaces over the interval \([0, 1]\) and the use of the relation between the Gelfond-Bernstein bases and the Chebyshev-Bernstein bases as stated in Theorem 4. We have [4]

**Theorem 6.** Let \(P\) be an element of the Müntz space \(E(\Lambda_n)\). Then for any \(i = 0, 1, ..., n\) there exist real numbers \(s_j^{(i)} \in [0, 1]; j = 0, ..., n\) (independent of \(P\)) such that
\[
\eta_i(P, \Lambda_n, [a, 1]) = \sum_{j=0}^n s_j^{(i)} \eta_j(P, \Lambda_n, [0, 1])\]
with \(\sum_{j=0}^n s_j^{(i)} = 1\).

Using Theorem 6, we can prove the following theorem.

**Theorem 7.** Under the Müntz condition [2] on the sequence \(\Lambda_\infty\), for any positive integer \(k \leq m\)
\[
\lim_{m \to \infty} |\eta_i(t^{r_1}, \Lambda_m, [a, 1])^{r_k} - \eta_i(t^{r_x}, \Lambda_m, [a, 1])| = 0
\]
uniformly in \(i\).

**Proof.** Consider the parametric curve \(\Gamma : (t^{r_1}, t^{r_x})\) over the interval \([a, 1]\). Since \(r_k > r_1\), \(\Gamma\) is the graph of a convex function. As a parametric curve in the Müntz space \(E(\Omega)\) with \(\Omega = (0, r_1, r_k)\), it is thus located above its control polygon. This remain true when considering \(\Gamma\) as a parametric curve in \(E(\Lambda_m)\) (as dimension elevation is a corner cutting scheme). Therefore, we have for \(i = 0, ..., m\)
\[
\eta_i(t^{r_1}, \Lambda_m, [a, 1])^{r_k} \geq \eta_i(t^{r_x}, \Lambda_m, [a, 1]).
\]
(9)

From Theorem 6 we have
\[
\eta_i(t^{r_1}, \Lambda_m, [a, 1])^{r_k} - \eta_i(t^{r_x}, \Lambda_m, [a, 1]) = \left(\sum_{j=0}^m s_j^{(i)} \eta_j(t^{r_1}, \Lambda_m, [0, 1])\right)^{r_k} - \sum_{j=0}^m s_j^{(i)} \eta_j(t^{r_x}, \Lambda_m, [0, 1]).
\]

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Therefore, by Jensen Inequality and Inequality \(0 \leq \eta(t^r, \Lambda_m, [a, 1]) \leq \eta(t^r, \Lambda, [a, 1])\), we have
\[
0 \leq \eta(t^r, \Lambda_m, [a, 1]) \leq \eta(t^r, \Lambda, [a, 1]) \leq \eta(t^r, \Lambda, [0, 1]) - \eta(t^r, \Lambda, [0, 1]).
\]

The proof is concluded upon using Equation \(7\). \(\square\)

**Corollary 1.** Under the M"untz condition \((3)\) on the sequence \(\Lambda_\infty\), for any \(P \in E(\Lambda_n)\) and for any positive integer \(k \leq m\), we have
\[
\lim_{m \to \infty} ||P(\eta(t^r, \Lambda_m, [a, 1])^{1/r}) - \eta(P, \Lambda_m, [a, 1])||_\infty = 0
\]
uniformly in \(i\).

**Proof.** Denote by \(P\) an element of \(E(\Lambda_n)\) given by \(P(t) = \sum_{k=0}^{n} t^r A_k\). We have
\[
||P(\eta(t^r, \Lambda_m, [a, 1])^{1/r}) - \eta(P, \Lambda_m, [a, 1])||_\infty \leq \sum_{k=0}^{n} ||A_k||_\infty ||(\eta(t^r, \Lambda_m, [a, 1])^{r/r}) - (\eta(t^r, \Lambda, [a, 1]))||_\infty.
\]
We conclude the proof using Theorem \(7\). \(\square\)

Using the Widder-Hirschman-Gelfond Theorem \([11, 10]\), we proved in \([4]\) that the point set \(D_m = \{\eta_i(t^r, \Lambda_m, [a, 1])^{1/r}, i = 0, \ldots, m\}\) form a dense subset of the interval \([0, 1]\) as \(m\) goes to infinity. Applying the generalized de Casteljau algorithm to the function \(t^r\) to compute its control points over the interval \([a, 1]\) for its control points over the interval \([0, 1]\) and using the density property of the point set \(D_m\), we also have the following

**Theorem 8.** Under the M"untz condition \((2)\), the point set \(D_m = \{\eta_i(t^r, \Lambda_m, [a, 1])^{1/r}, i = 0, \ldots, m\}\) form a dense subset of the interval \([a, 1]\) as \(m\) goes to infinity.

We are now in a position to prove the main Theorem \(7\) when the sequence \(\Lambda_\infty\) satisfies the M"untz condition \((2)\). In this case, if we denote by \(P\) an element of \(E(\Lambda_n)\), we have to show that given a point \(t \in [a, 1]\) and a sequence of real numbers \(\eta_{\infty}(t^{1/r}, \Lambda_m, [a, 1])^{1/r}\) that converges to \(t\) as \(m\) goes to infinity (this is possible thanks to the density result in Theorem \(8\)), the point \(b_{\infty}(t) = \eta_{\infty}(P, \Lambda_m, [a, 1])\) converges to \(P(t)\) as \(m\) goes to infinity uniformly on \(t\). We have
\[
\max_t ||P(t) - b_{\infty}(t)||_\infty \leq \max_t ||P(t) - P(\eta_{\infty}(t^{1/r}, \Lambda_m, [a, 1])^{1/r})||_\infty + \max_{i=\infty} ||P(\eta_{\infty}(t^{1/r}, \Lambda_m, [a, 1])^{1/r}) - b_{\infty}(t)||_\infty.
\]

The function \(P\) is continuous in the compact interval \([a, 1]\), thus
\[
\max_t ||P(t) - P(\eta_{\infty}(t^{1/r}, \Lambda_m, [a, 1])^{1/r})||_\infty \to 0 \text{ as } m \to \infty,
\]
and Corollary [1] shows that
\[ \max_{t_m(t)} ||P(\eta_{m}(t)(t^{r_1}, \Lambda_m, [a, 1])^{1/r_1} - b^{m}_{t_m(t)}||_{\infty} \to 0 \text{ as } m \to \infty. \]

This concludes the proof of the “if” part of Theorem [1].

**Remark 1.** Using Equation [8] it can be proven that if the limiting polygon generated by the dimension elevation with respect to a sequence \( \Lambda_{\infty} \) over an interval \([a, b]\) converges to the underlying Chebyshev-Bézier curve, then the limiting polygon over any interval \([c, d]\) \(\subset [a, b]\) also converges to the underlying curve.

To prove that the Müntz condition (2) is necessary in Theorem [1] we need the following bounded Chebyshev inequality [1, 8, 5]

**Theorem 9. (Bounded Chebyshev’s Inequality)** Let us assume that the sequence \( \Lambda_{\infty} = (r_0 = 0, r_1, r_2, \ldots) \) satisfies \( \sum_{i=1}^{\infty} 1/r_i < \infty \). Then for any real-valued element \( P \in E(\Lambda_{\infty}) \) and for each \( \epsilon > 0 \) there is a constant \( c(\Lambda_{\infty}, \epsilon) > 0 \) depending only on \( \Lambda_{\infty} \) and \( \epsilon \) (and not on the number of terms in \( P \)) and such that
\[ ||P'||_{[0,1-\epsilon]} \leq c(\Lambda_{\infty}, \epsilon)||P||_{[1-\epsilon,1]} \cdot (10) \]

To prove that the Müntz condition (2) is necessary in Theorem [1] we proceed as follows: Let \( P \) be a non-constant element of \( E(\Lambda_n) \) expressed in the Chebyshev-Bernstein basis over an interval \([a, 1]\) as
\[ P(t) = \sum_{i=0}^{m} B_{i,\Lambda_m}^{m}(t)b_{i}^{m} \quad m \geq n. \]

Without loss of generality, we can assume that \( P'(a) \neq 0 \) (otherwise, we work on an interval \([b, 1]\) such that \([a, 1] \subset [b, 1]\) and \( P'(b) \neq 0 \) and invoke Remark 1). We have [2]
\[ P'(a) = (B_{0,\Lambda_m}^{m})'(a)(b_{0}^{m} - b_{1}^{m}). \]

Using the bounded Chebyshev inequality (10) with \( \epsilon = 1 - a \), we have
\[ |(B_{0,\Lambda_m}^{m})'(a)| \leq c(\Lambda_{\infty}, \epsilon)||B_{0,\Lambda_m}^{m}||_{[a,1]} = c(\Lambda_{\infty}, \epsilon) \]

for any \( m \geq n \). Therefore,
\[ \lim_{m \to \infty} ||b_{0}^{m} - b_{1}^{m}||_{\infty} \geq \frac{||P'(a)||_{\infty}}{c(\Lambda_{\infty}, \epsilon)} > 0. \]

This shows that the dimension elevation leads to a limiting curve with a segment of non-zero length as part of the curve. Thereby, the limiting polygon cannot converge to the underlying Chebyshev-Bézier curve.

**4. Concluding Remarks**

For a sequence of distinct real positive numbers \( \Lambda_{\infty} = (r_0 = 0, r_1, \ldots, r_n, \ldots) \) (with no monotonicity condition on the \( r_i \)) the space \( E(\Lambda_{\infty}) \) is dense in \( C([0, 1]) \) if and only if [5]
\[ \sum_{k=1}^{\infty} \frac{r_k}{r_k^2 + 1} = \infty. \]
Gelfond-Bézier curves are too "degenerate" at the origin to study the dimension elevation algorithm in case we have no condition of monotonicity on the real numbers $r_i$. For instance, if we consider the case $n = 3$, $r_1 = 1$, $r_2 = 2$, $r_3 = 3$ and $r_j = 1/j$, for $j > 3$ and we start with a control polygon $(P_0, P_1, P_2, P_3)$ then the control polygon obtained by a dimension elevation to the order $m$ is not obtained by a corner cutting scheme similar to (5) and (6), but instead the algorithm collapses the first $m - 3$ control points to $P_0$ while the remaining control points are given by $(P_1, P_2, P_3)$ [3]. Therefore, an analogue formulation as in (11) for the dimension elevation in Mûntz spaces over $[0,1]$ is unlikely. However, if we consider the dimension elevation algorithm of Gelfond-Bézier curves away from the origin, i.e., over an interval $[a, 1]$ with $a > 0$, then the Gelfond-Bernstein basis coincides with the Chebyshev-Bernstein basis, the degeneracy at the origin disappears and the algorithm leads to a family of corner cutting schemes without imposing any condition of monotonicity on the real numbers $r_i$. Unfortunately, such family of corner cutting schemes involves rather complicated coefficients expressed in term of generalized Schur functions [2]. It will be interesting to find, for the away from the origin case, conditions on the real number $r_i$ for the convergence of the dimension elevation algorithm to the underlying curve. In the theory of the density of Mûntz spaces over an interval $[a, b]$ with $a > 0$, with no condition of positivity nor monotonicity on the pairwise distinct real numbers $r_i$, the corresponding Mûntz space is a dense subset of $C([a, b])$ if and only if the real numbers $r_i$ satisfy the so-called full Mûntz condition (12):

$$\sum_{r_k \neq 0} \frac{1}{|r_k|} = \infty.$$  

(12)

An interesting question is then: Let $[a, b]$ be an interval with $a > 0$ and let $r_i$ be a sequence of pairwise distinct real numbers without any condition of positivity or monotonicity. Can we claim that the corresponding dimension elevation algorithm over $[a, b]$ converges to the underlying curve if and only if the sequence $r_i$ satisfies the full Mûntz condition (12)?

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