Abstract. We prove Thom isomorphism theorem in twisted groupoid \( K \)-theory for real or complex equivariant vector bundles over the unit space of a locally compact groupoid. To this intent, we first discuss equivariant bivariant \( K \)-theory for groupoid actions by Morita equivalences on real or complex \( C^* \)-algebras, and establish its functorial properties in the category of locally compact second-countable groupoids and Hilsum-Skandalis morphisms. We treat the real and the complex cases in the unified framework of \textit{Reality}, that is, groupoids equipped with involutions acting by Morita equivalences on “real” \( C^* \)-algebras.

Contents

1. Introduction 1
2. Preliminaries and conventions 3
3. Equivariant \( C^* \)-correspondences 3
4. The \( \text{KKR}_G \)-bifunctor 5
5. Functoriality in the Hilsum-Skandalis category 8
6. \( \text{KKR}_G \)-equivalence 12
7. Bott periodicity 12
8. Twisting by Real Clifford bundles and Stiefel-Whitney classes 14
9. Thom isomorphism in twisted groupoid \( K \)-theory 19
References 20

1. Introduction

Equivariant \( K \)-theory for groupoid actions by automorphisms was introduced by Le Gall in his thesis [16] generalising Kasparov’s work in equivariant \( K \)-theory for groups [13]. A groupoid \( \mathcal{G} \rightarrow X \) is said to \textit{act by automorphisms} on a \( C^* \)-algebra \( A \) if there is a non-degenerate \( * \)-homomorphism from \( C_0(X) \) to the centre \( \mathcal{ZM}(A) \) of the multiplier algebra of \( A \) (\( A \) is then said a \( C(X) \)-algebra) and an isomorphism of \( C(X) \)-algebras \( \alpha : s^*A \rightarrow r^*A \) such that the induced \( * \)-isomorphisms \( \alpha_g : A_{s(g)} \rightarrow A_{r(g)} \) satisfy \( \alpha_g \circ \alpha_h = \alpha_{gh} \) whenever \( g \) and \( h \) are composable. In that case \( A \) is called a \( \mathcal{G} \)-algebra [16]. Given two \( \mathcal{G} \)-algebras \( A \) and \( B \), Le Gall has defined a group \( \text{KKG}(A,B) \) which is functorial in \( B \), co-functorial in \( A \) and functorial in \( \mathcal{G} \) with respect to generalised morphisms. These groups are the main setting for the study of Baum-Connes conjecture for groupoids [23] and play a prominent role in twisted \( K \)-theory of Lie groupoids, or more generally, for differentiable stacks [24, 25].

The idea of this paper grew out of a nice construction arisen in [25] where, in order to investigate additive structure and geometric picture of twisted \( K \)-theory for stacks, the authors were led to \textit{weaken} the definition of groupoid action on a \( C^* \)-algebra by considering \textit{generalised actions} instead of actions by automorphisms; \textit{i.e.} the \( * \)-isomorphisms \( \alpha_g \) are replaced by Morita equivalences. More precisely, a \textit{generalised action} of the groupoid \( \mathcal{G} \rightarrow X \) on \( A \) is a Fell bundle [14] \( A \rightarrow \mathcal{G} \) such that \( A \) is isomorphic to \( C_0(X;A|\mathcal{G}) \). If \( A \) and \( B \) are \( C^* \)-algebras endowed with generalised
$\mathfrak{g}$-actions, a group $KK_{\mathfrak{g}}(A, B)$ is then defined by means of correspondences and connections \([5, 13]\).

The first step of this paper is to study the above theory from scratch in a very general framework combining real and complex $\mathbb{Z}_2$-graded $C^*$-algebras. This is done by considering the simple idea of Real $C^*$-algebra. A Real graded $C^*$-algebra is a $C^*$-algebra $A$ endowed with a $\mathbb{Z}_2$-grading $A = A_0 \oplus A_1$ and an involution $\tau : A \rightarrow A$ of $C^*$-algebras compatible with the grading. In other words, $A$ is the complexification of a graded real (real in the usual sense) $C^*$-algebra. More generally, Real graded Banach spaces (or algebras) are defined in a similar way. It is a trivial result that the category of Real graded $C^*$-algebras is isomorphic to the category of graded real $C^*$-algebras under the operation of taking the complexification of a graded real $C^*$-algebra in one way, and the "realification" of a Real graded $C^*$-algebra (its invariant part under the involution), in the other. It follows that any theory built upon the category of Real graded $C^*$-algebras apply naturally to both the category of graded complex $C^*$-algebras and the category of graded real $C^*$-algebras. In view of this observation, G. Kasparov focused attention on $C^*$-algebras with involutions when proving fundamental results in KK-theory in his pioneering paper \([12]\).

We precisely deal with a Real groupoid $\mathfrak{g}$ and Real graded $C^*$-algebras $A, B$ acted upon by $\mathfrak{g}$ through graded Fell bundles $A, B$ with involutions over $\mathfrak{g}$. Adjusting the constructions of \([25, \text{Section 6.2}]\) to the Real case, we define the groups $KKR_{\mathfrak{g}}(A, B)$ and give the usual first properties in a straightforward way. Then, we prove that $KK_{\mathfrak{g}}$ is co-functorial with respect to $\mathfrak{g}$ in the category $\mathfrak{g}$ whose objects are locally compact second-countable Real groupoids and whose morphisms are Hilsum-Skandalis (or generalised) morphisms with involutions. We use this property to establish Bott periodicity in this generalised equivariant KK-theory. Note that these results were not contained in \([25]\).

In the second step of the paper, we use the constructions and results of generalised equivariant KKR-theory to establish some results in twisted K-theory of topological Real groupoids, especially the Thom isomorphism theorem. Recall that in \([17]\), the author has associated to any locally compact Hausdorff second-countable Real groupoid $\mathfrak{g} \xrightarrow{\pi} X$ the abelian group $\widehat{\text{Br}}\mathbb{R}(\mathfrak{g})$, called the Real graded Brauer group, of Morita equivalence classes of Real graded Dixmier-Douady bundles (i.e. complex graded Dixmier-Douady bundles \([24]\) with involutions) over $\mathfrak{g}$. Associated to any representative $A$ of a class $[A] \in \text{Br}\mathbb{R}(\mathfrak{g})$, there is a Real graded $C^*$-algebra $\mathfrak{A} \subset \mathfrak{g}$. The $[A]$-twisted K-groups of the Real groupoid $\mathfrak{g}$ is defined as $\text{KR}^A_{\mathfrak{g}}(\mathfrak{g}^*) := \text{KR}_{\pi - q}(\mathfrak{A} \rtimes \mathfrak{g})$ \([12, 19]\). These groups depend, up to isomorphisms, only on the class of $\mathfrak{A}$. Let $V \rightarrow X$ be a $\mathfrak{g}$-equivariant Euclidean vector bundle with involution locally induced from that of the Real space $\mathbb{R}^{|\mathfrak{g}|}$ (this is the space $\mathbb{R}^p \times \mathbb{R}^q$ equipped with the involution $(x, y) \mapsto (x, -y)$); $V$ is called a Real Euclidean vector bundle of type $p - q$ over $\mathfrak{g} \xrightarrow{\pi} X$. Associated to such a bundle, there are two different Real graded Dixmier-Douady bundles $\mathfrak{C}(V)$ (the Clifford bundle associated to $V$) and $A\mathfrak{C}$ over $\mathfrak{g}$. However, we show that for all $A \in \text{Br}\mathbb{R}(\mathfrak{g})$ and all Real Euclidean vector bundle of type $p - q$ over $\mathfrak{g} \xrightarrow{\pi} X$,

$$\text{KR}^A_{\mathfrak{c} + \mathfrak{C}}(\mathfrak{g}^*) \cong \text{KR}^{A \rtimes \mathfrak{g}}(\mathfrak{g}^*).$$

Although this result is a generalisation of \([24, \text{Proposition 2.5}]\), we propose here a very different approach to prove it. Indeed, the idea consists mainly of comparing the classes of $A\mathfrak{C}$ and $\mathfrak{C}(V)$ in the Brauer group. This is done by working on the cohomological picture of $\widehat{\text{Br}}\mathbb{R}(\mathfrak{g})$. Specifically, it is shown in \([17]\) that for a groupoid with involution $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$, there is an isomorphism of abelian groups

$$\text{DD} : \widehat{\text{Br}}\mathbb{R}(\mathfrak{g}) \xrightarrow{\otimes} \mathcal{H}R^0(S^*, Z_2) \oplus \mathcal{H}R^1(S^*, Z_2) \rtimes \mathcal{H}R^2(S^*, S^1),$$

where $\mathcal{H}R^*$ is equivariant groupoid cohomology with respect to involutions, and where $S^1$ is endowed with the involution given by complex conjugation, $Z_2$ and $Z_8$ are given the trivial involutions (see \([18]\)). If $\text{DD}(\mathcal{A}) = (n, \alpha, \beta)$, we say that $\mathcal{A}$ is a Real graded D-D bundle of type
n mod 8. We then compare the values of $\text{Cl}(V)$ and $A_V$ in the right hand side of the above isomorphism by introducing Stiefel-Whitney classes $w(V)$ for Real Euclidean vector bundles over a Real groupoid.

In the last section of the paper, we prove Thom isomorphism theorem in twisted K-theory of Real groupoids. It says that for a Real Euclidean vector bundle $V$ over $\mathcal{G} \to X$ with projection $\pi : V \to X$, and a Real graded Dixmier-Douady bundle $A$, there is an isomorphism of abelian groups

$$KR^*_{\pi,A}(\mathcal{G}) = KR^*_{\pi,A+\text{Cl}(V)}(\mathcal{G}).$$

Moreover, $V$ is KR–oriented, that is $w(V) = 0$, then $KR^*_{\pi,A}(\mathcal{G}) = KR^*_{\pi,A}(\mathcal{G})$. As far as we know, this result was known only in the case of twistings by Azumaya bundles over topological spaces [6,11] and in the case of twisted K-theory of bundle gerbes [4].

2. Preliminaries and conventions

2.1. By a Real graded $C^*$-algebra ( [12]) we mean a graded complex $C^*$-algebra $A = A_0 \oplus A_1$ equipped with an involution $A \ni a \mapsto \bar{a} \in A$ of $C^*$-algebra compatible with the grading. Real Banach spaces (algebras) are defined in a similar fashion. If $A$ is a Real graded $C^*$-algebra, a graded Hilbert $A$-module (say left) $E$ is Real graded if it is a Real graded Banach spaces with the property that its involution satisfies $\bar{a} \cdot e = \bar{a} \cdot e$ and $\langle e, f \rangle = \langle \bar{e}, f \rangle$ for all $a \in A$ and $e, f \in E$.

2.2. Recall [18] that a groupoid $\mathcal{G} \to X$ is Real if it is endowed with a groupoid isomorphism $\tau : \mathcal{G} \to \mathcal{G}$ such that $\tau^2 = 1$. A Real graded Fell bundle (resp. u.s.c. Fell bundle) over $\mathcal{G}$ is a Fell bundle (resp. u.s.c. Fell bundle) [14,25] $\pi : \mathcal{E} \to \mathcal{G}$ such that every fibre $\mathcal{E}_g$ is a graded complex Banach space, and there is an involution $\mathcal{E} \ni e \mapsto \bar{e} \in \mathcal{E}$ compatible with the grading on the fibres and satisfying $\tau(\pi(e)) = \pi(\bar{e})$. Note that if $\mathcal{E} \to \mathcal{G}$ is a Real graded (u.s.c.) Fell bundle, then for all $g \in \mathcal{G}$, the graded complex Banach space $\mathcal{E}_g$ is a graded Morita $\mathcal{E}_{\tau(g)},\mathcal{E}_{\tau(g)}$–equivalence. The reduced $C^*$-algebra $C_r^*(\mathcal{G};\mathcal{E})$ ([14]) associated to a Real graded Fell bundle is naturally equipped with the structure of Real graded $C^*$-algebra. We refer to [19] for more details on Real graded Fell bundles and their $C^*$-algebras.

2.3. Throughout the paper, all our $C^*$-algebras, Hilbert modules, (u.s.c.) Fell bundles, continuous field of $C^*$-algebras or Banach spaces, Dixmier-Douady bundles, are assumed Real graded, unless otherwise stated. All our groupoids are Real, locally compact, Hausdorff and second-countable, unless otherwise stated. Thus, to avoid annoying repetitions, we will often omit the terms "Real" and "Real graded". We will often simply write $\mathcal{G}, \mathcal{F}$, etc., for the Real groupoids $\mathcal{G} \to X$, $\mathcal{F} \to Y$, etc. Moreover, we assume the reader is familiar with the language of groupoids, groupoid actions on spaces, (graded) extensions, and groupoid cohomology, which is contained in many articles [15,19,24,25].

3. Equivariant $C^*$-correspondences

Let $A$ and $B$ be $C^*$-algebras. A $C^*$-correspondence from $A$ to $B$ is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a Hilbert (right) $B$-module, and $\varphi : A \to \mathcal{L}(\mathcal{E})$ is a non-degenerate homomorphism of $C^*$-algebras. We then view $\mathcal{E}$ as a Real graded left $A$-module by $a \cdot e := \varphi(a)e$. When there is no risk of confusion we will write $A\mathcal{E}B$ for the $C^*$-correspondence $(\mathcal{E}, \varphi)$. We also say that $\mathcal{E}$ is a (Real graded) $A,B$-correspondence.

If $(\mathcal{E}, \varphi)$ and $(\mathcal{F}, \psi)$ are $C^*$-correspondences from $A$ to $B$ and from $B$ to $C$, respectively, we define the $C^*$-correspondence $(\mathcal{F}, \psi) \circ (\mathcal{E}, \varphi)$ from $A$ to $C$ by $(\mathcal{E} \otimes_{\varphi} \mathcal{F}, \mathcal{F} \otimes \psi)$; this $C^*$-correspondence is called the composition of $(\mathcal{F}, \psi)$ by $(\mathcal{E}, \varphi)$.

An isomorphism of $C^*$-correspondences from $A\mathcal{E}B$ to $A\mathcal{F}B$ is a Real degree 0 unitary $u \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ such that $u \circ \varphi(a) = \psi(a) \circ u$ for all $a \in A$. 

Definition 3.1. Let $\mathcal{G}$ be a groupoid, and $A$ a $C^*$-algebra. A generalized $\mathcal{G}$-action on $A$ consists of a u.s.c. Fell bundle $\mathcal{A} \to \mathcal{G}$ such that $A \cong C_0(X; \mathcal{A})$, where, as usual, we have denoted $C_0(X; \mathcal{A})$ for $C_0(X; \mathcal{A}_{X})$.

Example 3.2. If $A \to X$ is a u.s.c. field of $C^*$-algebras, then the u.s.c. Fell bundle $s^*A \to \mathcal{G}$ is a generalized $\mathcal{G}$-action on $A = C_0(X; A)$.

Remark 3.3. As mentioned in [25, §6.2], a generalized action is in fact an action by Morita equivalences, which justifies the terminology. Indeed, if $\mathcal{A}$ is a generalized $\mathcal{G}$-action on $A$, then from the properties of Fell bundles we see that for $g \in \mathcal{G}$, $\mathcal{A}_g^{-1}$ is a graded $\mathcal{A}_{s(g)} \mathcal{A}_{t(g)}$-Morita equivalence.

Denote by $i : \mathcal{G} \to \mathcal{G}$ the inversion map. If $A$ is a Real graded $C^*$-algebra endowed with a generalized $\mathcal{G}$-action $\mathcal{A}$, we define $\mathcal{T}_b(i^*\mathcal{A})$ as the Real graded Banach algebra of norm-bounded continuous functions vanishing at infinity $a' : \mathcal{G} \ni g \mapsto a'_g \in \mathcal{A}_{g^{-1}}$; the Real structure is given by $(a'_g) := (\overline{a'_g})$, and the grading is inherited from that of $\mathcal{A}$. Observe that $\mathcal{T}_b(i^*\mathcal{A})$ is naturally a Real graded (right) Hilbert $r^*A$-module under the module structure

$$(a' \cdot a)_g := a'_g \cdot a_g, \quad \text{for } a' \in \mathcal{T}_b(i^*\mathcal{A}), a \in r^*A = C_0(\mathcal{G}; r^*(\mathcal{A}_{\mathcal{G}})),$$

and the graded scalar product

$$(a', a'')_g := (a'_g)^* \cdot a''_g, \quad \text{for } a', a'' \in \mathcal{T}_b(i^*\mathcal{A}).$$

Also, $\mathcal{T}_b(i^*\mathcal{A})$ has the structure of Real graded $s^*A$-module by setting

$$(\xi \cdot a'_g)_g := (\xi(g)) \cdot a'_g, \quad \text{for } \xi \in s^*A, a' \in \mathcal{T}_b(i^*\mathcal{A}), \text{ and } g \in \mathcal{G}.$$

Suppose now that $(\mathcal{E}, \varphi)$ is a $C^*$- correspondence from $A$ to $B$, and $\mathcal{A}$ and $\mathcal{B}$ are generalized $\mathcal{G}$-actions on the Real graded $C^*$-algebras $A$ and $B$, respectively. Then, it is easy to check that we have two $C^*$-correspondences $s^*A(\mathcal{T}_b(i^*\mathcal{A}) \mathcal{B}_rA r^*E))_r \mathcal{B}$ and $s^*A(s^*E \mathcal{T}_b(i^*\mathcal{B}))_r \mathcal{B}$ with respect to the maps

$$\text{Id} \otimes r^* \varphi : s^*A \to \mathcal{L}_{r^*E}(\mathcal{T}_b(i^*\mathcal{A}) \mathcal{B}_rA r^*E),$$

and

$$s^* \varphi \otimes \text{Id} : s^*A \to \mathcal{L}_{r^*E}(s^*E \mathcal{T}_b(i^*\mathcal{B}))_r \mathcal{B},$$

respectively. The link between these two correspondences "measures" the $\mathcal{G}$-equivariance of $A \mathcal{B}$. In particular, we give the following definition.

Definition 3.4. Let $A, B, \mathcal{A}, \mathcal{B}, E$ be as above. A $C^*$-correspondence $A \mathcal{B}$ is said $\mathcal{G}$-equivariant if there is an isomorphism of $C^*$-correspondences

$$W \in \mathcal{L}(s^*E \otimes \mathcal{T}_b(i^*\mathcal{B}), \mathcal{T}_b(i^*\mathcal{A}) \mathcal{B}_rA r^*E),$$

such that for every $(g, h) \in \mathcal{G}^2$, the following diagram commutes

$$
\begin{align*}
\begin{array}{c}
E_{s(h)} \otimes \mathcal{B}_{B_{\theta(h)}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{s^{-1}} \\
\downarrow W_{h} \otimes \text{Id}_{\mathcal{B}_{B_{\theta(h)}}} \\
E_{s(g)} \otimes \mathcal{B}_{B_{\theta(g)}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}} \mathcal{B}_{s^{-1}}
\end{array}
\end{align*}
\begin{array}{c}
\mathcal{A}_{r^{-1}} \otimes \text{Id}_{\mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{s^{-1}} \\
\downarrow \text{Id}_{\mathcal{A}_{r^{-1}} \mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{s^{-1}} \\
\mathcal{A}_{r^{-1}} \otimes \text{Id}_{\mathcal{B}_{B_{\theta(g)}}}
\end{array}
\begin{array}{c}
\mathcal{A}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}} \mathcal{B}_{s^{-1}} \\
\downarrow W_{g} \\
\mathcal{A}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}}
\end{array}
\end{align*}

\begin{align*}
\mathcal{A}_{r^{-1}} \otimes \text{Id}_{\mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{s^{-1}} \\
\downarrow \text{Id}_{\mathcal{A}_{r^{-1}} \mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{s^{-1}}
\end{align*}

\begin{align*}
\mathcal{A}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}} \mathcal{B}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}} \mathcal{B}_{s^{-1}} \\
\downarrow W_{g}
\end{align*}
\begin{align*}
\mathcal{A}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}}
\end{align*}

where the isomorphisms $\mathcal{A}_{r^{-1}} \otimes \text{Id}_{\mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{s^{-1}} \cong \mathcal{A}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}} \mathcal{B}_{s^{-1}}$ and $\mathcal{A}_{r^{-1}} \otimes \text{Id}_{\mathcal{B}_{B_{\theta(h)}}} \mathcal{B}_{s^{-1}} \cong \mathcal{A}_{r^{-1} \mathcal{B}_{B_{\theta(g)}}} \mathcal{B}_{s^{-1}}$ come from the properties of Fell bundles.
Lemma 3.5. Let $A$, $B$, and $C$ be $\mathcal{C}$-algebras endowed with generalized $\mathcal{G}$-actions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively. If $A \mathcal{E}_B$, and $B \mathcal{F}_C$ are $\mathcal{G}$-equivariant $\mathcal{C}$-correspondences, their composition $\mathcal{F} \mathcal{E} \mathcal{G}$ is a $\mathcal{G}$-equivariant Real graded $A, C$-correspondence. Therefore, there is a category $\mathcal{C}_{\mathcal{G}}$, whose objects are Real graded $\mathcal{C}$-algebras endowed with generalized $\mathcal{G}$-actions, and whose morphisms are isomorphism classes of equivariant correspondences.

Proof. Suppose $W' \in \mathcal{L}_B(s' A \mathcal{E}_B \mathcal{F}_B(i' \mathcal{C}), \mathcal{F}_B(i' \mathcal{C}) \mathcal{E}_B \mathcal{F}_B(r' \mathcal{C}))$ is an isomorphism of Real graded $s' A, r' B$-correspondences and $W'' \in \mathcal{L}_C(s' \mathcal{E}_B \mathcal{F}_C(i' \mathcal{C}), \mathcal{F}_C(i' \mathcal{C}) \mathcal{E}_B \mathcal{F}_C(r' \mathcal{C}))$ is an isomorphism of $s' B, r' C$-correspondences implementing $\mathcal{G}$-equivariance. We define the isomorphism of Real graded $s' A, r' C$-correspondences

$$W : s' (E \otimes_C \mathcal{F}) \mathcal{E}_B \mathcal{F}_B(i' \mathcal{C}) \longrightarrow \mathcal{F}_B(i' \mathcal{C}) \mathcal{E}_B \mathcal{F}_B(r' \mathcal{C})$$

by setting $W = (W' \mathcal{E}_B \mathcal{F}_B \mathcal{F}_C(i' \mathcal{C}) \circ (\text{Id}_{s' \mathcal{E}_B} \mathcal{E}_B \mathcal{F}_C(i' \mathcal{C}))$, via the identification

$$s' (E \otimes_C \mathcal{F}) \mathcal{E}_B \mathcal{F}_B(i' \mathcal{C}) \cong s' (E \otimes_B \mathcal{F}) \mathcal{E}_B \mathcal{F}_B(i' \mathcal{C})$$

and

$$r' (E \otimes_C \mathcal{F}) \mathcal{E}_B \mathcal{F}_B(r' \mathcal{C}) \cong r' (E \otimes_B \mathcal{F}) \mathcal{E}_B \mathcal{F}_B(r' \mathcal{C}).$$

Now it is straightforward that commutativity of the diagram (1) holds for $W$.

\[\square\]

4. The $\text{KKR}_\mathcal{G}$-bifunctor

To define the equivariant KKR-groups, we need some more notions (cf. [5, Appendix A], [25, Definition 6.5]). Let $A$, $B$ be Real graded $\mathcal{C}$-algebras. Let $\mathcal{E}_1$ be Real graded Hilbert $A$-module, and $\mathcal{E}_2$ a Real graded $B, C$-correspondence. Put $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$. For $\xi \in \mathcal{E}_1$, let $T_\xi \in \mathcal{L}_B(\mathcal{E}_2, \mathcal{E})$ be given by $T_\xi(\eta) := \xi \mathcal{A}_1 \mathcal{A}_2$ (with adjoint given by $T_\xi(\xi_1) \mathcal{A}_1 \mathcal{A}_2 \eta := \langle \xi, \xi_1 \rangle \eta$). Observe that $T_\xi = T_{\xi^*}$ so that $T_\xi$ is Real if and only if $\xi$ is.

Now let $A, B, \mathcal{E}_1$, and $\mathcal{E}_2$ as above. Let $F_2 \in \mathcal{L}(\mathcal{E}_2)$, and $F \in \mathcal{L}(\mathcal{E})$. We say that $F$ is an $F_2$-connection for $\mathcal{E}_1$ if for every $\xi \in \mathcal{E}_1$:

$$T_\xi F_2 - (-1)^{|\xi||F_2|} F T_\xi \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}),$$

$$F_2 T_\xi - (-1)^{|\xi||F_2|} F T_\xi \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}).$$

Remark 4.1. It is easy to check that $F$ is an $F_2$-connection for $\mathcal{E}_1$ if and only if for every $\xi \in \mathcal{E}_1$, $\ [	heta_\xi, F_2 \oplus F] \in \mathcal{K}(\mathcal{E}_2 \oplus \mathcal{E})$, where $\theta_\xi := \begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix} \in \mathcal{L}_B(\mathcal{E}_2 \oplus \mathcal{E})$ ([22, Definition 8]).

Definition 4.2. Let $A$ and $B$ be Real graded $\mathcal{C}$-algebras endowed with generalized $\mathcal{G}$-actions. A $\mathcal{G}$-equivariant (or just equivariant if $\mathcal{G}$ is understood) Kasparov $A, B$-correspondence is a pair $(\mathcal{E}, F)$ where $\mathcal{E}$ is a $\mathcal{G}$-equivariant $\text{Rg} A, B$-correspondence, $F$ is a Real operator of degree 1 in $\mathcal{L}(\mathcal{E})$ such that for all $a \in A$,

(i) $a(F - \mathcal{F}) \in \mathcal{K}(\mathcal{E});$

(ii) $a(F^2 - 1) \in \mathcal{K}(\mathcal{E});$

(iii) $[a, F] \in \mathcal{K}(\mathcal{E});$

(iv) $W(s' F \mathcal{E}_B \mathcal{F}_B(i' \mathcal{C})) W' \in \mathcal{L}(\mathcal{E}_B(i' \mathcal{C}) \mathcal{F}_B(r' \mathcal{C}))$ is an $r' F$-connection for $\mathcal{F}_B(i' \mathcal{C})$.

We say that $(\mathcal{E}, F)$ is degenerate if the elements

$$a(F - \mathcal{F}), a(F^2 - 1), [a, F], \text{ and } [\theta_\xi, r' F \oplus W(s' F \mathcal{E}_B \mathcal{F}_B(i' \mathcal{C})) W']$$

are 0 for all $a \in A, \xi \in \mathcal{F}_B(i' \mathcal{C})$.

Remark 4.3. Notice that conditions (i)-(iii) are the usual ones presented in any text about KKR-theory. To digest condition (iv), suppose $\mathcal{G}$ acts on $A$ and $B$ by automorphisms and $A \mathcal{E}_B$ is a $\mathcal{G}$-equivariant $A, B$-correspondence. Then the isomorphism $W$ induces a continuous family of graded isomorphisms $\mathcal{W}_a : \mathcal{E}_{a(\mathcal{G})} \longrightarrow \mathcal{E}_{a(\mathcal{G})}$ via the identifications

$$\mathcal{E}_{a(\mathcal{G})} \mathcal{B}_{a(\mathcal{G})} \mathcal{B}_{a(\mathcal{G})} \cong \mathcal{E}_{a(\mathcal{G})} \mathcal{B}_{a(\mathcal{G})} \mathcal{B}_{a(\mathcal{G})} \cong \mathcal{E}_{a(\mathcal{G})}, \text{ and } \mathcal{A}_{a(\mathcal{G})} \mathcal{B}_{a(\mathcal{G})} \mathcal{B}_{a(\mathcal{G})} \cong \mathcal{E}_{a(\mathcal{G})}.$$
Note that from the commutativity of (1), $\overline{W}_{gh} = \overline{W}_g \circ \overline{W}_h$ for all $(g, h) \in G^2$; so that $\overline{W}$ is a $\mathcal{G}$-action by automorphisms on $\mathcal{E}$. Moreover, it is straightforward to see that the map $\varphi : A \to \mathcal{L}(\mathcal{E})$ is $\mathcal{G}$-equivariant; i.e. $\overline{W}_g \varphi(a) \overline{W}_h = \varphi(a_ga)$ for all $g \in \mathcal{G}$ and $a \in A_{\mathcal{G}}$. Now, if $(\mathcal{E}, F)$ is a $\mathcal{G}$-equivariant Kasparov $A, B$-correspondence, then condition (iv) of Definition 4.2 implies that $\overline{W}_g F(\mathcal{G}) \overline{W}_h - F(\mathcal{G}) \in \mathcal{K}(\mathcal{E}(\mathcal{G}))$ for all $g \in \mathcal{G}$ (take $\xi = 0$), so that we recover Le Gall’s definition of an equivariant Kasparov bimodule ([16, Definition 4.1.2]). We will then refer to condition (iv) as the condition of invariance modulo compacts.

**Definition 4.4.** Two equivariant Kasparov $A, B$-correspondences $(\mathcal{E}_i, F_i), i = 1, 2$ are unitarily equivalent if there exists an isomorphism of $A, B$-correspondences $u \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ such that $F_2 = uF_1u^*$; in this case we write $(\mathcal{E}_1, F_1) \sim_u (\mathcal{E}_2, F_2)$. The set of unitarily equivalence classes of equivariant Kasparov $A, B$-correspondences is denoted by $\text{ER}_G(A, B)$.

Let the $C^*$-algebra $A$ be endowed with a generalised $\mathcal{G}$-action $\mathcal{A}$. Then the $C^*$-algebra $A[0, 1] := C([0, 1], A)$ (with the grading $(A[0, 1])^t = A'[0, 1], i = 0, 1$, and Real structure $\hat{f}(t) := f(t)$) is equipped with the generalised $\mathcal{G}$-action given by the Real graded u.s.c. Fell bundle $\mathcal{A} [0, 1] \to \mathcal{G}$ with $((\mathcal{A} [0, 1])_g = \mathcal{A} [0, 1]_g$.

**Definition 4.5.** Let $A$ and $B$ be $C^*$-algebras endowed with generalised $\mathcal{G}$-actions. A homotopy in $\text{ER}_G(A, B)$ is an element $(\mathcal{E}, F) \in \text{ER}_G(A, B[0, 1])$. Two elements $(\mathcal{E}, F), i = 0, 1$ of $\text{ER}_G(A, B)$ are said to be homotopically equivalent if there is a homotopy $(\mathcal{E}, F)$ such that $(\mathcal{E} \hat{\otimes}_{\mathcal{G}t} \mathcal{B}, F \hat{\otimes}_{\mathcal{G}t} \text{Id}) \sim \mathcal{E}(\mathcal{G}, \mathcal{F})$, and $(\mathcal{E} \hat{\otimes}_{\mathcal{G}t} \mathcal{B}, F \hat{\otimes}_{\mathcal{G}t} \text{Id}) \sim \mathcal{E}(\mathcal{G}, \mathcal{F})$, where for all $t \in [0, 1]$, the evolution map $\mathcal{E}(\mathcal{G}, \mathcal{F}) : [0, 1] \to B$ is the surjective $^*\text{-homomorphism} \mathcal{E}(\mathcal{G}, \mathcal{F}) : t(f) := f(t)$. The set of homotopy classes of elements of $\text{ER}_G(A, B)$ is denoted by $\text{KKR}_G(A, B)$.

**Example 4.6.** Let $A$ be a $C^*$-algebra equipped with a generalised Real $\mathcal{G}$-action. Then there is a canonical element $1_A \in \text{KKR}_G(A, A)$ given by the class of the equivariant Kasparov $A, A$-correspondence $(A, 0)$, where $A$ is naturally viewed as a $A, A$-correspondence via the homomorphism $A \to \mathcal{L}_A(A) = A$ defined by left multiplication by elements of $A$.

**Definition 4.7.** Given two elements $x_1, x_2 \in \text{KKR}_G(A, B)$, their sum is given by $x_1 \oplus x_2 = (\mathcal{E}_i \oplus \mathcal{E}_2, F_1 \oplus F_2)$, where $(\mathcal{E}_i, F_i), i = 1, 2$ is any representative of $x_i$.

Let $A, B$, and $D$ be $C^*$-algebras endowed with the generalized $\mathcal{G}$-actions $\mathcal{A}, \mathcal{B}$, and $\mathcal{D}$, respectively. Then the $C^*$-algebras $A \hat{\otimes}_{\mathcal{G}} C_{\mathcal{D}}(X) \cong C_\mathcal{D}(X; \mathcal{A} \hat{\otimes} \mathcal{D} \mathcal{B})$ and $B \hat{\otimes}_{\mathcal{G}} C_{\mathcal{D}}(X) \cong C_\mathcal{D}(X; \mathcal{B} \hat{\otimes}_{\mathcal{G}} \mathcal{D})$ are provided with the generalized Real $\mathcal{A} \hat{\otimes}_{\mathcal{G}} \mathcal{D}$-correspondence given by the Real graded u.s.c. Fell bundles $\mathcal{A} \hat{\otimes}_{\mathcal{G}} \mathcal{D} \to \mathcal{G}$ and $\mathcal{B} \hat{\otimes}_{\mathcal{G}} \mathcal{D} \to \mathcal{G}$. Now if $A \mathcal{E} \mathcal{B}$ is a $\mathcal{G}$-equivariant $C^*$-correspondence via the non-degenerate homomorphism $\varphi : A \to \mathcal{L}(\mathcal{E})$, then $\mathcal{E} \hat{\otimes}_{\mathcal{G}} A \hat{\otimes}_{\mathcal{D}} C_{\mathcal{D}}(X) \cong D$ is a $\mathcal{G}$-equivariant $A \hat{\otimes}_{\mathcal{D}} C_{\mathcal{D}}(X) \cong D$-correspondence via the map

$$\varphi \hat{\otimes}_{\mathcal{D}} \text{Id}_A \hat{\otimes}_{\mathcal{D}} \text{Id}_D : A \hat{\otimes}_{\mathcal{D}} C_{\mathcal{D}}(X) \to \mathcal{L}(\mathcal{E}) \hat{\otimes}_{\mathcal{G}} (C_{\mathcal{D}}(X) \cong D).$$

It follows that Le Gall’s construction in [16, Définition 4.2.1] can easily be extended to our generalised actions settings.

**Definition 4.8.** Let $A, B$, and $D$ be as above. We define a group homomorphism $\tau_D : \text{KKR}_G(A, B) \to \text{KKR}_G(A \hat{\otimes}_{\mathcal{G}} D, B \hat{\otimes}_{\mathcal{G}} D)$ by setting $\tau_D([\mathcal{E}, F]) := [(\mathcal{E} \hat{\otimes}_{\mathcal{G}} A \hat{\otimes}_{\mathcal{D}} C_{\mathcal{D}}(X) \cong D, F \hat{\otimes}_{\mathcal{G}} 1_A \hat{\otimes}_{\mathcal{D}} 1_D)]$, for $[\mathcal{E}, F] \in \text{KKR}_G(A, B)$.

The following can be proven as in the standard case where no generalised actions are involved (see [12, §4]).

**Proposition 4.9.** Under the operations of direct sum, $\text{KKR}_G(A, B)$ is an abelian group. Moreover, the assignment $(A, B) \mapsto \text{KKR}_G(A, B)$ is a bifunctor, covariant in $B$ and contravariant in $A$, from the category $\text{Corr}_G$ to the category $\mathfrak{Ab}$ of abelian groups.
Note that the inverse of an element $x \in \text{KKR}_G(A, B)$ is given by the class of $(-\mathcal{E}, -F)$, where $(-\mathcal{E}, F)$ is a representative of $x$. $-\mathcal{E}$ is the Real graded $A, B$-correspondence given by $\mathcal{E}$ with the opposite grading (i.e. $(-\mathcal{E})^i = \mathcal{E}^{1-i}, i = 0, 1$) and the same Real structure, and the non-degenerate homomorphism of Real graded $C^*$-algebras $-\varphi : A \to \mathcal{L}(-\mathcal{E})$ defined by

$$-\varphi(a) := \begin{pmatrix} 0 & 1 \varepsilon_1 \\ 1 & 0 \end{pmatrix} a, \quad \forall a \in A.$$ 

Also, as in the usual case, degenerate elements are homotopically equivalent to $(0, 0)$, so that they represent the zero element of $\text{KKR}_G(A, B)$.

**Remark 4.10.** One recovers Kasparov’s $\text{KKR}(A, B)$ of [12] by taking the Real groupoid $\mathcal{G}$ to be the point. Indeed, in this case we may omit condition (iv) of Definition 4.2 since, thanks to (1), for the automorphism $W : \mathcal{E} \to \mathcal{E}$ is indeed equal to the identity.

Higher $\text{KKR}_G$-groups are defined in an obvious way. Given a $C^*$-algebra $A$ endowed with a generalised $\mathcal{G}$-action $\mathcal{A} \to \mathcal{G}$, the u.s.c. Fell bundle

$$\mathcal{A} \hat{\otimes} \text{Cl}_{p,q} := \coprod_{g \in \mathcal{G}} \mathcal{A}_g \hat{\otimes} \text{Cl}_{p,q}$$

over $\mathcal{G}$ is a generalised $\mathcal{G}$-action on the Real graded $C^*$-algebra

$$A_{p,q} := A \otimes \text{Cl}_{p,q} \cong C_0(X; \mathcal{A} \hat{\otimes} \text{Cl}_{p,q}).$$

**Definition 4.11.** Let $A, B$ be $C^*$-algebras endowed with generalised $\mathcal{G}$-actions. Then, the higher $\text{KKR}_G$-groups $\text{KKR}_{\mathcal{G}, j}(A, B)$ are defined by

$$\text{KKR}_{\mathcal{G}, j}(A, B) = \text{KKR}_{\mathcal{G}, j}^j(A, B) := \begin{cases} \text{KKR}_G(A_{j,0}, B) \cong \text{KKR}_G(A, B_{0,0}), & \text{if } j \geq 0 \\ \text{KKR}_G(A_{-j,0}, B) \cong \text{KKR}_G(A, B_{0,0}), & \text{if } j \leq 0 \end{cases}$$

Let us outline the construction of the Kasparov product in groupoid-equivariant $\text{KKR}$-theory for generalised actions. To do this, we need the following

**Theorem 4.12.** (cf. [25, Theorem 6.9]). Let $A, D$ be separable $C^*$-algebras endowed with generalised $\mathcal{G}$-actions. Let $(\mathcal{E}_1, F_1) \in \text{ER}_{\mathcal{G}}(A, D)$, $(\mathcal{E}_2, F_2) \in \text{ER}_{\mathcal{G}}(D, B)$. Denote by $\mathcal{E}$ the $\mathcal{G}$-equivariant $A, B$-correspondence $\mathcal{E} = \mathcal{E}_1 \hat{\otimes} \mathcal{E}_2$. Then the set $F_1 \hat{\otimes} F_2$ of Real operators $F \in \mathcal{L}(\mathcal{E})$ such that

- $(\mathcal{E}, F) \in \text{ER}_{\mathcal{G}}(A, B)$;
- $F$ is a $F_2$-connection for $\mathcal{E}_1$;
- $\forall a \in A, a[F_1 \hat{\otimes} 1, F]a^* \geq 0 \mod \mathcal{K}(\mathcal{E})$

is non-empty.

From this theorem, the Kasparov product

$$\hat{\otimes}_{\mathcal{G}, D} : \text{KKR}_G(A, D) \hat{\otimes}_D \text{KKR}_G(D, B) \to \text{KKR}_G(A, B)$$

of $[(\mathcal{E}_1, F_1)] \in \text{KKR}_G(A, D)$ and $[(\mathcal{E}_2, F_2)] \in \text{KKR}_G(D, B)$ is defined by

$$[(\mathcal{E}_1, F_1)] \hat{\otimes}_D [(\mathcal{E}_2, F_2)] := [(\mathcal{E}, F)],$$

where $\mathcal{E} := \mathcal{E}_1 \hat{\otimes} \mathcal{E}_2$ and $F \in F_1 \hat{\otimes} F_2$. It is not hard to see that as in the complex case where the $C^*$-algebras are equipped with $\mathcal{G}$-actions by automorphisms (see [16]), this product is well-defined, bilinear, associative, homotopy-invariant, covariant with respect to $B$ and contravariant with respect to $A$.

More generally, we have

**Theorem 4.13.** Let $A_1, A_2, B_1, B_2$ and $D$ be separable $C^*$-algebras endowed with generalised $\mathcal{G}$-actions. Then, the product (2) induces an associative product

$$\text{KKR}_{\mathcal{G}, i}(A_1, B_1 \hat{\otimes}_{C_0(X)} D) \hat{\otimes}_D \text{KKR}_{\mathcal{G}, i}(D \hat{\otimes}_{C_0(X)} A_2, B_2) \to \text{KKR}_{\mathcal{G}, i+j}(A_1 \hat{\otimes}_{C_0(X)} A_2, B_1 \hat{\otimes}_{C_0(X)} B_2).$$
Proof. The proof is almost the same as that of [12, Theorem 5.6].

Moreover, as in the usual case (cf. [13]), there are descent morphisms

\[ j_3 : \text{KKR}_3(A, B) \to \text{KKR}(C(\mathcal{S}; \mathcal{A}), C(\mathcal{S}; \mathcal{B})); \]

\[ j_3, \text{red} : \text{KKR}_3(A, B) \to \text{KKR}(C(\mathcal{S}; \mathcal{A}), C(\mathcal{S}; \mathcal{B})), \]

compatible with the Kasparov product.

5. Functoriality in the Hilsum-Skandalis category

Recall [19] that a (Real) generalised morphism (or a Hilsum-Skandalis morphism) from a Real groupoid \( \Gamma \) to the Real groupoid \( \mathcal{S} \) consists of a Real space \( Z \ni z \mapsto z \in Z \), two continuous maps \( Y \longrightarrow Z \longrightarrow X \) which are equivariant with respect to the Real structures, a continuous left (resp. right) action of \( \Gamma \) (resp. of \( \mathcal{S} \)) on \( Z \) respecting the involutions \( i.e. \cdot g = \tau(g) \cdot \tau(g) \) where the product makes sense), making \( Z \) a generalised morphism in the usual sense (see for instance [7, 25]).

There is a category \( \mathcal{M}_\mathcal{S} \) whose objects are Real groupoids and whose morphisms are (Morita equivalence classes of) Real generalised morphisms. This category was shown in [18, Proposition 1.37] to be isomorphic to a category \( \mathcal{M}_\mathcal{S} \), which is much more exploitable. The category \( \mathcal{M}_\mathcal{S} \) has same objects as \( \mathcal{M}_\mathcal{S} \). If \( \Gamma, \mathcal{S} \) are Real groupoids, \( \text{Hom}_{\mathcal{M}_\mathcal{S}}(\Gamma, \mathcal{S}) \) consists of Morita equivalence classes of compositions of the form

\[ \Gamma \longrightarrow T[U] \xrightarrow{f} \mathcal{S} \]

where \( U = (U_i)_{i \in I} \) is a Real open cover of \( Y \) (that is, \( I \) has involution \( i \mapsto \tilde{i} \) satisfying \( U_i = \tau(U_i), \forall i \in I \)), and \( f \) is a strict Real morphism; i.e. equivariant with respect to the involution \( (i, \gamma, j) \mapsto (\tilde{i}, \tau_\gamma(\tilde{j})) \) on the cover groupoid \( \Gamma[U] \) (induced from the those of \( \Gamma \) and \( \mathcal{S} \)). Such a morphism will be represented by the couple \( (U, f) \).

In this section we show that \( \text{KKR}(\Lambda) \) is functorial in the category of locally compact second-countable Real groupoids and generalised Real morphisms. We first need to show that it is functorial with respect to strict Real morphisms.

Let \( f : \Gamma \longrightarrow \mathcal{S} \) be a strict morphism of Real groupoids and \( A \) a \( C^* \)-algebra endowed with the generalised Real \( \mathcal{S} \)-action \( \mathcal{A} \longrightarrow \mathcal{S} \). Then the pull-back \( f^* \mathcal{A} \longrightarrow \Gamma \) defines a generalised Real \( \Gamma \)-action on the \( C^* \)-algebra \( f^* A = C_0(Y; f^* \mathcal{A}) \). Let \( B \) be another \( C^* \)-algebra together with a generalised Real \( \mathcal{S} \)-action \( \mathcal{B} \). Suppose \( \Lambda E_B \) is a \( C^* \)-correspondence. Then under the identifications

\[ f^* A = A \hat{\otimes}_{C_0(X)} C_0(Y), \]

\[ f^* B = B \hat{\otimes}_{C_0(X)} C_0(Y), \]

\[ f^* E = E \hat{\otimes}_{C_0(X)} C_0(Y), \]

we see that \( f^* E \) is a Real graded \( f^* A, f^* B \)-correspondence. Further, assume that \( \Lambda E_B \) is \( \mathcal{S} \)-equivariant with respect to the isomorphism \( W : s_r^* E \hat{\otimes}_{C_0(\mathcal{S})} T(j_\mathcal{S} \mathcal{B}) \longrightarrow T(j_\mathcal{S} \mathcal{A}) \hat{\otimes}_{C_0(\mathcal{S})} \Lambda^* E \). Then, by using the following identifications (compare with [16, p.65])

\[ \tag{5} \]

\[ \begin{align*}
  i_t^* (f^* \mathcal{A}) &= f^* (i_t^* \mathcal{A}), & i_t^* (f^* \mathcal{E}) &= f^* (i_t^* \mathcal{E}), \\
  i_t^* (f^* A) &= s_r^* A \hat{\otimes}_{C_0(\mathcal{S})} C_0(\Gamma), & r_t^* (f^* A) &= s_r^* A \hat{\otimes}_{C_0(\mathcal{S})} C_0(\Gamma), \\
  s_t^* (f^* E) &= r_t^* E \hat{\otimes}_{C_0(\mathcal{S})} C_0(\Gamma), & r_t^* (f^* E) &= r_t^* E \hat{\otimes}_{C_0(\mathcal{S})} C_0(\Gamma),
\end{align*} \]

where the Real action of \( C_0(\mathcal{S}) \) on \( C_0(\Gamma) \) is induced by \( f \) in an obvious way, we get an isomorphism

\[ f^* W : s_r^* (f^* E) \hat{\otimes}_{\Lambda^* F, F, \Lambda^* F} T(j_\mathcal{S} f^* \mathcal{B}) \longrightarrow T(j_\mathcal{S} f^* A) \hat{\otimes}_{\Lambda^* F, F, \Lambda^* F} f^* E, \]

making \( f^* E \) into a \( \mathcal{S} \)-equivariant \( f^* A, f^* B \)-correspondence. Hence equivariant KKR-theory has a functorial property in the category \( \mathcal{M}_\mathcal{S} \).
**Definition and Lemma 5.1.** Let $f : \Gamma \rightarrow \mathcal{S}$ be a strict morphism of Real groupoids. Let $A$ and $B$ be $C^*$-algebras endowed with generalised Real $\mathcal{S}$-actions. Then we define a group homomorphism

$$f^* : \text{KKR}_{\mathcal{S}}(A, B) \rightarrow \text{KKR}_{\Gamma}(f^* A, f^* B)$$

by assigning to a $\mathcal{S}$-equivariant Kasparov $A, B$-correspondence $(E, F) \in \text{ER}_{\mathcal{S}}(A, B)$ the pair

$$f^*(E, F) := (f^* E, f^* F) \in \text{ER}_{\Gamma}(f^* A, f^* B),$$

where under the identifications (4), we put $f^* F = \tilde{F} \circ C_{\mathcal{S}}(\Gamma) \text{Id}_{C_0(\Gamma)}$. Moreover, the map $f^*$ is natural with respect to the Kasparov product (2) in the sense that if $D$ is another $C^*$-algebra equipped with a generalised Real $\mathcal{S}$-action, then

$$f^*(x \otimes f^* D f^*(y)) = f^*(x \otimes f^* D f^*(y)), \quad x \in \text{KKR}_{\mathcal{S}}(A, D), \ y \in \text{KKR}_{\mathcal{S}}(D, B).$$

**Proof.** The proof is the same as those of Lemma 6.1.1 and Proposition 6.1.3 in [16]. □

More generally, in order to establish functoriality in the category $\mathcal{R}\mathcal{S}$ we need the following proposition which is a generalisation of [16, Proposition 3.1.3].

**Proposition 5.2.** Let $\mathcal{S} \rightarrow X$ be a locally compact second-countable Real groupoid, and let $\mathcal{U} = \{U_j\}_{j \in J}$ a Real open cover of $X$. Denote by $i : \mathcal{S}[\mathcal{U}] \rightarrow \mathcal{S}$ the canonical inclusion. For all Fell bundle (resp. u.s.c. Fell bundle) $\mathcal{A} \rightarrow \mathcal{S}[\mathcal{U}]$, there exist a Fell bundle (resp. u.s.c. Fell bundle) $i^*_1 \mathcal{A} \rightarrow \mathcal{S}$ and an isomorphism of Fell bundles (resp. u.s.c. Fell bundles)

$$i^*_1 \mathcal{A} \overset{\simeq}{\rightarrow} \mathcal{A}$$

over $\mathcal{S}[\mathcal{U}] \rightarrow \coprod_{j \in J} U_j$.

**Proof.** Without loss of generality, we may suppose $\mathcal{U}$ is locally finite since $X$ is paracompact. We use the following notations: as usual, we write $\mathcal{S} \times_{U_j} G$ for $(j_0, g, j_1) \in \mathcal{S}[\mathcal{U}]$, and $x_j$ for $(j, x) \in \coprod_{j \in J} U_j$; let $\pi : \mathcal{A} \rightarrow \mathcal{S}[\mathcal{U}]$ be the projection of the given Real graded Fell bundle (resp. u.s.c. Fell bundle); an element $a \in \pi^{-1}(s_{U_{j_0}j_0})$ will be written $a_{j_0j_1}$.

For $x \in X$ we denote by $I_x$ the finite subset of $j \in J$ such that $x \in U_j$, and for $g \in \mathcal{S}$ let $I_g$ the finite subset of pairs $(j_0, j_1) \in J \times J$ such that $g \in \mathcal{S} \times_{U_{j_0}j_0} G$. Put

$$i^*_1 \mathcal{A}_g := \bigoplus_{(j_0, j_1) \in I_g} \mathcal{A} \times_{U_{j_0}j_1} G \quad \text{ and } \quad i^*_1 \mathcal{A} := \coprod_{g \in \mathcal{S}} i^*_1 \mathcal{A}_g.$$

Then $i^*_1 \mathcal{A} \rightarrow \mathcal{S}$ is a Real graded Banach bundle (resp. u.s.c. Banach bundle), with the projection $i^*_1 \pi$ defined by

$$i^*_1 \pi((g, a_{j_0j_1}))_{(j_0, j_1) \in I_g} := g.$$

Moreover, for $(g_1, g_2) \in \mathcal{S}[\mathcal{U}]$, the pairing

$$I_{g_1} \times_{I_{g_1}} I_{g_2} \rightarrow I_{g_1 \times g_2}$$

$$((j_0, j_1), (j_1, j_2)) \mapsto (j_0, j_2)$$

where $I_{g_1} \times_{I_{g_1}} I_{g_2} := \{(j_0, j_1), (j_1, j_2) \in I_{g_1} \times I_{g_2}\}$, enables us to define a multiplication on $i^*_1 \mathcal{A}$

$$i^*_1 \mathcal{A}_g \otimes_{i^*_1 \mathcal{A}_{g_1}} i^*_1 \mathcal{A}_{g_2} \rightarrow i^*_1 \mathcal{A}_{g_1 \times g_2}$$

generated by $a_{j_0j_1} \otimes b_{j_1j_2} \mapsto (ab)_{j_0j_2}$. One easily verifies that together with this multiplication, $i^*_1 \mathcal{A} \rightarrow \mathcal{S}$ is a Real graded Fell bundle (resp. u.s.c. Fell bundle) that satisfies the desired isomorphism. □

**Definition 5.3.** Let $\mathcal{S} \rightarrow X$ and $\mathcal{U}$ be as above. Given a $C^*$-algebra $A$ with a generalised Real $\mathcal{S}[\mathcal{U}]$-action $\mathcal{A}$, we denote by $i^*_1 A$ the $C^*$-algebra $C_0(X; i^*_1 \mathcal{A})$ endowed with the generalised Real $\mathcal{S}$-action $i^*_1 \mathcal{A} \rightarrow \mathcal{S}$. 

Proposition 5.4. Let $A, B$ be $C^*$-algebras endowed with generalised Real $\mathcal{G}[\mathcal{J}]$-actions $\mathcal{A}$ and $\mathcal{B}$, respectively. Assume $\mathcal{E}$ is a $\mathcal{G}[[\mathcal{J}]]$-equivariant $A,B$-correspondence. Then there exists a $\iota A, \iota B$-correspondence $\iota \mathcal{E}$ and an isomorphism of $B$-correspondences $\iota' \iota \mathcal{E} \cong \mathcal{E}$.

Proof. The map $\iota : B \rightarrow \iota B$ sending $\phi \in C_0([\mathcal{J}], U)\mathcal{B}$ to the function $\iota \phi : C_0(X; \iota \mathcal{B})$ given by

$$\iota \phi(x) := (\phi((j, x)))_{j \in \mathcal{J}},$$

is a surjective Real graded $^*$-homomorphism. We then can define the push-out $\iota \mathcal{E}$ of the Hilbert $B$-module $\mathcal{E}$ via $\iota$. Let us recall [9, §1.2.2.] the definition of the Real graded Hilbert $\iota B$-module $\iota \mathcal{E}$. Let $N_{\text{r}} = \{ \xi \in \mathcal{E} \mid \iota((\xi, \xi)_{\mathcal{B}}) = 0 \}$; denote by $\hat{\xi}$ the image of $\xi \in \mathcal{E}$ in the quotient space $\mathcal{E}/N_{\text{r}}$, the latter being a Real graded $\iota B$-module by setting

$$\hat{\xi} \cdot \iota(b) := \hat{\xi} b, \quad \hat{\xi} \in \mathcal{E}/N_{\text{r}}, \quad \iota(b) \in \iota B.$$

Then define the Hilbert $\iota B$-module $\iota \mathcal{E}$ as the completion of $\mathcal{E}/N_{\text{r}}$ with respect to the Real graded $\iota B$-valued pre-inner product

$$\langle \xi, \eta \rangle_{\iota \mathcal{E}} := \iota((\xi, \eta)), \quad \xi, \eta \in \mathcal{E}.$$

For $T \in \mathcal{L}(\mathcal{E})$, let $\iota T$ be the unique operator in $\mathcal{L}(\iota \mathcal{E})$ making the following diagram commute

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\iota} & \iota \mathcal{E} \\
\downarrow T & & \downarrow \iota T \\
\mathcal{E} & \xrightarrow{\iota} & \iota \mathcal{E}
\end{array}$$

where the horizontal arrows are the quotient map; i.e.,

$$\iota T(\hat{\xi}) = \hat{T(\xi)}, \quad \hat{\xi} \in \iota \mathcal{E}. \quad (6)$$

Hence, the map $\varphi : A \rightarrow \mathcal{L}(\mathcal{E})$ implementing the $A,B$-correspondence gives rise to a non-degenerate $^*$-homomorphism $\iota \varphi : \iota A \rightarrow \mathcal{L}(\iota \mathcal{E})$ such that

$$(\iota \varphi)(\iota(a)) := \iota(\varphi(a)), \quad \forall a \in A. \quad (7)$$

Therefore, $\iota \mathcal{E}$ is a $\iota A$-$\iota B$-correspondence. It is not hard to check that $\iota \mathcal{E}$ is isomorphic to $C_0(X; \tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}} \rightarrow X$ is the unique u.s.c. field of Banach algebras with fibre $(\iota \mathcal{E})_x = \bigoplus_{j \in \mathcal{J}} \mathcal{E} \otimes \mathcal{B}_{(j)}$; indeed, since $\mathcal{E}$ is a Real graded Hilbert $B$-module, thanks to [25, Appendix A] there is a unique topology on the u.s.c. field $\mathcal{F} = \bigoplus_{j \in \mathcal{J}} \mathcal{E} \otimes \mathcal{B}_{(j)}$ such that $\mathcal{E} \cong C_0([\mathcal{J}], U)\mathcal{B}$ the $\mathbb{Z}_2$-grading and the Real structure on $\mathcal{F}$ is the obvious one; so, by Proposition 5.2, we get the Real graded u.s.c. field $\tilde{\mathcal{E}}$.

Let $W : s^*\mathcal{E} \otimes_{\mathcal{F}^B} \mathcal{F}_B(\iota' \mathcal{A}) \rightarrow \mathcal{F}_B(\iota' \mathcal{A}) \otimes_{\mathcal{F}^A} \iota^* \mathcal{E}$ be the isomorphism of $C^*$-correspondences implementing the $\mathcal{G}[\mathcal{J}]$-equivariance. Then from the identifications

$$s'((\iota A)) = \iota(s' A), \quad r'((\iota A)) = \iota(r' A),$$

$$s'((\iota B)) = \iota(s' B), \quad r'((\iota B)) = \iota(r' B),$$

$$\iota'(\iota' \mathcal{A}) = \iota(\iota' \mathcal{A}), \quad \iota'(\iota' \mathcal{B}) = \iota(\iota' \mathcal{B}),$$

$$s'((\iota \mathcal{E})) = \iota(s' \mathcal{E}), \quad r'((\iota \mathcal{E})) = \iota(r' \mathcal{E}),$$

we get

$$s'((\iota \mathcal{E}) \otimes_{\mathcal{F}^B} \mathcal{F}_B(\iota' \mathcal{B})) \cong \iota((s^* \mathcal{E} \otimes_{\mathcal{F}^B} \mathcal{F}_B(\iota' \mathcal{B}))), \quad \text{and}$$

$$\mathcal{F}_B(\iota' (\iota' \mathcal{A})) \otimes_{\mathcal{F}^A} \iota(\iota' \mathcal{E}) \cong \iota((\mathcal{F}_B(\iota' \mathcal{A}) \otimes_{\mathcal{F}^A} \iota(\iota' \mathcal{E}))).$$

Thus, $W$ induces an isomorphism of $s'((\iota A))$, $r'(\iota B)$-correspondences

$$\iota W : s'((\iota \mathcal{E}) \otimes_{\mathcal{F}^B} \mathcal{F}_B(\iota' (\mathcal{B})) \rightarrow \mathcal{F}_B(\iota' (\iota' \mathcal{A}) \otimes_{\mathcal{F}^A} \iota(\iota' \mathcal{E})), $$
defined in a similar fashion as (7); so that \( \iota W \) is compatible with the partial product of \( \mathcal{J} \) in the sense of the commutative diagram (1). That the \( \mathcal{J}[\mathcal{U}] \)-equivariant \( A, B \)-correspondences \( \mathcal{E} \) and \( \iota^* \mathcal{E} \) are isomorphic is an immediate consequence of the construction of \( \iota \mathcal{E} \) and \( \iota W \).

Lemma 5.5. Suppose \( x = (\mathcal{E}, F) \in \text{ER}_{\mathcal{GU}}(A, B) \). Then \( \iota x := (\iota \mathcal{E}, \iota F) \in \text{ER}_{\mathcal{GU}}(\iota A, \iota B) \), where the operator \( \iota F \in \mathcal{L}(\iota \mathcal{E}) \) is given by (6).

Proof. This is a matter of algebraic verifications. For instance, the map \( \iota : \mathcal{L}(\mathcal{E}) \longrightarrow \mathcal{L}(\iota \mathcal{E}) \) respects the degree and the Real structures. Moreover \( \iota \) sends \( \mathcal{K}(\mathcal{E}) \) onto \( \mathcal{K}(\iota \mathcal{E}) \) because \( \iota(\theta_{\mathcal{E}, \mathcal{E}}) = \theta_{\iota \mathcal{E}, \iota \mathcal{E}} \) for all \( \xi, \eta \in \mathcal{E} \), \( (\iota T_1)(\iota T_2) = \iota(T_1 T_2) \), and \( [\iota T_1, \iota T_2] = \iota[T_1, T_2] \), \( \forall T_1, T_2 \in \mathcal{L}(\mathcal{E}) \); thus conditions (i)-(iii) in Definition 4.2 are satisfied by the pair \((\iota \mathcal{E}, \iota F)\). To verify condition (iv), let us put \( \mathcal{E}_1 = T_p(\iota^* \mathcal{E}), \mathcal{E}_2 = r^* \mathcal{E}, \) and \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \). Then \( \text{K}(\mathcal{E}_2 \otimes \iota \mathcal{E}) = \mathcal{K}(\iota \mathcal{E}_1 \otimes \mathcal{E}) \), and we have seen in the proof of Proposition 5.2 that \( \iota \mathcal{E}_1 \otimes \mathcal{E} = \iota(\iota \mathcal{E}_1 \otimes \mathcal{E}) \). It follows that for \( \xi \in \mathcal{E}_1 \) and \( \eta \in \mathcal{E}_2 \), one has

\[
\iota(T_\xi) = \hat{T}_\xi(\eta) = \hat{\xi} \otimes \hat{\eta},
\]

which means that \( \iota(T_\xi) = T_{\hat{\xi}} \in \mathcal{L}(\iota \mathcal{E}_1, \iota \mathcal{E}) \), and hence \( \iota(\theta_\mathcal{E}) = \theta_{\iota \mathcal{E}} \) (recall notations used in Remark 4.1). Then,

\[
[\theta_{\mathcal{E}, \iota^*}(r^* F) \otimes \iota (W(s^* F \otimes_{\mathcal{E}} \text{Id}))W^*] = \iota([\theta_\mathcal{E}, r^* F \otimes W(s^* F \otimes_{\mathcal{E}} \text{Id})]W^*) \in \mathcal{K}(\iota \mathcal{E}_2 \otimes \iota \mathcal{E});
\]

therefore, \( \iota W(s^* (\iota F) \otimes_{\mathcal{E}} \text{Id}) \iota W^* = \iota(W(s^* F \otimes_{\mathcal{E}} \text{Id})W^*) \) is an \( r^* \iota F \)-connection for \( \iota \mathcal{E}_1 = T_p(\iota^* \mathcal{E}) \).

The following result can be proved with similar arguments as in [16, Théoréme 6.2.1], so we omit the proof.

Theorem 5.6. Let \( \mathcal{J} \longrightarrow \mathcal{X}, \mathcal{U}, A \) and \( B \) be as previously. Then the canonical Real inclusion \( \mathcal{J}[\mathcal{U}] \hookrightarrow \mathcal{J} \) induces a group isomorphism

\[
\iota^* : \text{KKR}_{\mathcal{J}}(\iota A, \iota B) \longrightarrow \text{KKR}_{\mathcal{GU}}(A, B),
\]

whose inverse is

\[
\iota : \text{KKR}_{\mathcal{GU}}(A, B) \ni [(\mathcal{E}, F)] \mapsto [(\iota \mathcal{E}, \iota F)] \in \text{KKR}_{\mathcal{J}}(\iota A, \iota B).
\]

Corollary 5.7. ([16, Corollaire 6.2.1]) The isomorphism \( \iota : \text{KKR}_{\mathcal{GU}}(A, B) \longrightarrow \text{KKR}_{\mathcal{J}}(\iota A, \iota B) \) is natural with respect to Kasparov product; i.e.,

\[
\iota(x \otimes_{\mathcal{D}} y) = \iota(x \otimes_{\mathcal{D}} y), \forall x \in \text{KKR}_{\mathcal{GU}}(A, D), y \in \text{KKR}_{\mathcal{J}}(\mathcal{D}, B).
\]

Now Theorem 5.6 enables us to define the pull-back of a \( C^* \)-algebra endowed with a generalised action along a morphism in the category \( \mathcal{Y} \).

Definition 5.8. Let \( \mathcal{Z} : \Gamma \longrightarrow \mathcal{J} \) be a generalised Real homomorphism, \( A \) and \( B \) be \( C^* \)-algebras endowed with generalised Real \( \mathcal{J} \)-actions. Let \( (\mathcal{U}, f) \) be a representative of a morphism in \( \mathcal{Y} \) realising \( \mathcal{Z} \). Let \( \iota : \Gamma[\mathcal{U}] \longrightarrow \Gamma \) be the canonical inclusion. Define the pull-back of \( A \) and \( B \) along \( \mathcal{Z} \) by

\[
\mathcal{Z}' \mathcal{B} := \iota f^* \mathcal{A}, \quad \text{and} \quad \mathcal{Z}' \mathcal{B} := \iota f^* \mathcal{B}.
\]

Then we define the pull-back homomorphism

\[
\mathcal{Z}' : \text{KKR}_{\mathcal{J}}(A, B) \longrightarrow \text{KKR}_{\Gamma}(\mathcal{Z}' \mathcal{A}, \mathcal{Z}' \mathcal{B}),
\]

as the composite

\[
\text{KKR}_{\mathcal{J}}(A, B) \xrightarrow{f^*} \text{KKR}_{\mathcal{GU}}(f^* A, f^* B) \xrightarrow{\iota} \text{KKR}_{\Gamma}(\iota f^* A, \iota f^* B).
\]

Notice that by similar arguments as in [16] (Corollaire 3.2.1), \( \mathcal{Z}' \) is well defined, that is, its construction does not depend on the choice of the pair \([\mathcal{U}, f]\), and that it is functorial in \( \mathcal{Y} \) and natural with respect to Kasparov product [16, Théoréme 6.2.2]. In particular, if \( \mathcal{Z} \) is a Morita equivalence, then \( \mathcal{Z}' \) is an isomorphism.
6. KKR${}_G$-equivalence

The notion of KK-equivalence, which has already been treated in many textbooks and papers, provides important features in the study of K-amenability and Baum-Connes conjecture (see Blackadar’s book [3, Definition 19.1.1]). It also gives a powerful way to establish Bott periodicity in K-theory of C*-algebras.

This notion as well as the results around it extend in a very natural way to the more general setting of equivariant KK-theory for generalised Real groupoid actions.

**Definition 6.1.** Let $\mathcal{G} \rightarrow X$ be a Real groupoid, $A$ and $B$ be C*-algebras endowed with generalised $\mathcal{G}$-actions. We say that an element $x \in KKR_G(A, B)$ is a $KKR_G$-equivalence if there is $y \in KKR_G(B, A)$ such that

$$x \hat{\otimes}_{\mathcal{G}} y = 1_A,$$

and

$$y \hat{\otimes}_{\mathcal{G}} x = 1_B.$$

A and $B$ are said KKR$_G$-equivalent if there exists a KKR$_G$-equivalence in $KKR_G(A, B)$.

**Lemma 6.2.** Assume $x \in KKR_G(A, B)$ is a KKR$_G$-equivalence. Then for any Real graded C*-algebra $D$ endowed with a generalised Real $\mathcal{G}$-action, the maps

$$x \hat{\otimes}_{\mathcal{G}} D(B, D) \rightarrow KKR_G(A, D),$$

and

$$(\cdot) \hat{\otimes}_{\mathcal{G}} x : KKR_G(D, A) \rightarrow KKR_G(D, B),$$

are isomorphisms which are natural in $D$ by associativity.

The proof is almost the same as that of [12, Theorem 4.6]. For instance, the map

$$KKR_G(A, D) \ni z \mapsto y \hat{\otimes}_{\mathcal{G}} D \in KKR_G(B, D)$$

is an inverse of the first homomorphism. See also [3, §19.1]).

**Proposition 6.3.** KKR$_G$-equivalence is functorial in $\mathcal{G}$ in the following sense: if $Z : \Gamma \rightarrow \mathcal{G}$ is a generalised Real homomorphism, and if $A$ and $B$ are KKR$_G$-equivalent, then $Z^*A$ and $Z^*B$ are KKR$_\Gamma$-equivalent.

**Proof.** By naturality of $Z^*$ with respect to Kasparov product, we have

$$1_{Z^*A} = Z^*(x \hat{\otimes}_{\mathcal{G}} A y) = Z^*(x) \hat{\otimes}_{\Gamma, Z^*} Z^*(y),$$

and

$$1_{Z^*B} = Z^*(y \hat{\otimes}_{\mathcal{G}} B x) = Z^*(y) \hat{\otimes}_{\Gamma, Z^*} Z^*(x);$$

therefore $Z^*(x) \in KKR_\Gamma(Z^*A, Z^*B)$ is a KKR$_\Gamma$-equivalence. \qed

7. Bott periodicity

In this section we establish Bott periodicity in KKR$_G$-theory. We first need some definitions and constructions.

**Definition 7.1.** Let $\mathcal{G} \rightarrow X$ be a Real groupoid. A Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \rightarrow X$ is a Euclidean vector bundle $\pi : E \rightarrow X$ of rank $p+q$ equipped with a Real $\mathcal{G}$-action (with respect to $\pi$) such that the Euclidean metric is $\mathcal{G}$-invariant and the Real space $E$ is locally homeomorphic to $\mathbb{R}^{p+q}$; that is to say, for every $x \in X$, there is a Real open neighborhood $U$ of $x$ and a Real homeomorphism $h_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{p+q}$, where $U \times \mathbb{R}^{p+q}$ is provided with the Real structure $(x, t) \mapsto (x, t)$. This is equivalent to the existence of a Real open cover $\mathcal{U} = (U_j)_{j \in J}$ and a family of homeomorphisms $h_j : \pi^{-1}(U_j) \rightarrow U_j \times \mathbb{R}^{p+q}$ such that the following diagram commute

\[
\begin{array}{ccc}
\pi^{-1}(U_j) & \xrightarrow{h_j} & U_j \times \mathbb{R}^{p+q} \\
\tau \downarrow & & \downarrow \tau \times (1_p - 1_q) \\
\pi^{-1}(U_j) & \xrightarrow{h_j} & U_j \times \mathbb{R}^{p+q}
\end{array}
\]  

(8)
For $p, q \in \mathbb{N}$ with $n = p + q \neq 0$, we define the Real group $O(p + q)$ to be the orthogonal group $O(n)$ equipped with the involution induced from $\mathbb{R}^{p,q}$ (we identify $M_{p,q}(\mathbb{R})$ with $\mathcal{L}(\mathbb{R}^{p,q})$, the latter is then a Real space). Similarly one defines the Real group $SO(p + q)$

**Definition 7.2.** Associated to any Real Euclidean vector bundle $E$ of type $p - q$ over the Real groupoid $\mathcal{G} \longrightarrow X$, there is a generalised Real homomorphism

$$\mathcal{F}(E) : \mathcal{G} \longrightarrow O(p + q),$$

where $\mathcal{F}(E)$ is the frame bundle of $E \longrightarrow X$.

**Remark 7.3.** The above definition does make sense, for the fibre of the $O(p + q)$-principal bundle $\mathcal{F}(E) \longrightarrow X$ at a point $x \in X$ identifies to the $\mathbb{R}$-linear space $\text{Isom}(\mathbb{R}^{p,q}, E_x)$ of $\mathbb{R}$-linear isomorphisms; so that $\mathcal{G}$ acts on $\mathcal{F}(E)$ by $g \cdot (s(g), q) \mapsto (r(g), g \cdot q)$, for $q \in \text{Isom}(\mathbb{R}^{p,q}, E_{\bar{s}(q)})$, where $(g \cdot q)(t) := g \cdot q(t), t \in \mathbb{R}^{p,q}$. $\mathcal{F}(E)$ is equipped with the Real structure $(x, q) \mapsto (\bar{x}, \bar{q})$, where $\bar{q}(t) := q(t), t \in \mathbb{R}^{p,q}$. It is clear that the actions of $\mathcal{G}$ and $O(p + q)$ are compatible with this involution.

**Example 7.4.** The trivial bundle $\mathcal{E} = X \times \mathbb{R}^{p,q} \longrightarrow X$ is a Real Euclidean vector bundle of type $p - q$ over $\mathcal{G} \longrightarrow X$ with respect to the Real $\mathcal{G}$-action $g \cdot (s(g), t) \mapsto (r(g), t)$. The associated generalised Real homomorphism $\mathcal{F}(\mathcal{E} \times \mathbb{R}^{p,q})$ from $\mathcal{G} \longrightarrow O(p + q)$ is isomorphic to the trivial Real $O(p + q)$-principal $\mathcal{G}$-bundle $X \times O(p + q) \longrightarrow X$; we denote it by $\mathcal{F}_{\mathcal{G}}$.

Recall that $\mathcal{C}_{p,q} := \mathcal{C}(\mathbb{R}^{p,q}) := \mathcal{C}_{p,q} \otimes \mathbb{R} \subset \mathcal{C}(\mathbb{R}^{p,q}) \otimes \mathbb{R}$ is the Real graded Clifford $C^*$-algebra, where the Real structure is $x \otimes \mathcal{R} \lambda \mapsto x \otimes \mathcal{R} \bar{\lambda}$, and where the involution "bar" in $\mathcal{C}(\mathbb{R}^{p,q})$ is induced from that of $\mathbb{R}^{p,q}$. The Real action of $O(p + q)$ on $\mathbb{R}^{p,q}$ induces a Real $O(p + q)$-action on $\mathcal{C}_{p,q}$.

Recall that Kasparov has defined in [12, §5] a $\text{KKR}_{O(p+q)}$-equivalence

$$\alpha_{p,q} \in \text{KKR}_{O(p+q)} \left( C_0(\mathbb{R}^{p,q}), \mathcal{C}_{p,q} \right), \quad \text{and} \quad \beta_{p,q} \in \text{KKR}_{O(p+q)} \left( \mathcal{C}_{p,q}, C_0(\mathbb{R}^{p,q}) \right),$$

providing a $\text{KKR}_{O(p+q)}$-equivalence between $C_0(\mathbb{R}^{p,q})$ and $\mathcal{C}_{p,q}$.

We will use these elements to prove Bott periodicity in generalised Real groupoid equivariant KK-theory as well as the Thom isomorphism in twisted K-theory which will be discussed in the next section.

**Theorem 7.5** (Bott periodicity). Let $\mathcal{G} \longrightarrow X$ be a locally compact second-countable Real groupoid, and let $A$ and $B$ be $C^*$-algebras endowed with generalised $\mathcal{G}$-actions. Then the Kasparov product with $\mathcal{F}_{p,q} \alpha_{p,q} \in \text{KKR}_{\mathcal{G}} \left( C_0(X) \otimes C_0(\mathbb{R}^{p,q}), C_0(X) \otimes C_0(\mathbb{R}^{p,q}) \right)$ defines an isomorphism

$$\text{KKR}_{\mathcal{G},i+p+q}(A, B) \cong \text{KKR}_{\mathcal{G},i}(A(\mathbb{R}^{p,q}), B),$$

where $A(\mathbb{R}^{p,q}) = C_0(\mathbb{R}^{p,q}; A) = C_0(\mathbb{R}^{p,q}) \otimes A$.

**Proof.** First of all notice that the pullbacks $\mathcal{F}_{p,q}(C_0(\mathbb{R}^{p,q}))$ and $\mathcal{F}_{p,q}(\mathcal{C}_{p,q})$ via the generalised homomorphism $\mathcal{F}_{p,q} : \mathcal{G} \longrightarrow O(p + q)$ are isomorphic to $C_0(X; C_0(\mathbb{R}^{p,q})) = C_0(X) \otimes C_0(\mathbb{R}^{p,q})$ and $C_0(X) \otimes \mathcal{C}_{p,q} = C_0(X) \otimes \mathcal{C}_{p,q}$, respectively. These are then (Real graded) $C^*$-algebras endowed with generalised Real $\mathcal{G}$-actions. Since $\alpha_{p,q} \in \text{KKR}_{O(p+q)}(C_0(\mathbb{R}^{p,q}), \mathcal{C}_{p,q})$ is a $\text{KKR}_{O(p+q)}$-equivalence, its pullback $\mathcal{F}_{p,q}^* \alpha_{p,q} \in \text{KKR}_{\mathcal{G}}(C_0(X) \otimes C_0(\mathbb{R}^{p,q}), C_0(X) \otimes C_0(\mathbb{R}^{p,q}))$ is a $\text{KKR}_{\mathcal{G}}$-equivalence, thanks to Proposition 6.3. Hence, from Lemma 6.2, the Kasparov product

$$\text{KKR}_{\mathcal{G}} \left( C_0(X) \otimes C_0(\mathbb{R}^{p,q}), C_0(X) \otimes \mathcal{C}_{p,q} \right) \otimes C_0(X) \otimes C_0(\mathbb{R}^{p,q}) \cong \text{KKR}_{\mathcal{G},i}(C_0(X) \otimes C_0(\mathbb{R}^{p,q}) \otimes A, B)$$

$$\mathcal{F}_{p,q}^* \alpha_{p,q} \otimes C_0(X) \otimes C_0(\mathbb{R}^{p,q}) \cong \text{KKR}_{\mathcal{G},i}(A(\mathbb{R}^{p,q}), B \otimes C_0(X) C_0(X))$$
is an isomorphism. We therefore have the desired isomorphism since $C_0(X) \otimes_{C_0(X)} Cl_{p,q} \otimes A \cong A \otimes Cl_{p,q}$. \hfill \Box

8. Twisting by Real Clifford bundles and Stiefel-Whitney classes

In this section we use the previous constructions to prove some new results in twisted groupoid $K$-theory. First recall [17] that a Real graded $S^1$-extension over the Real groupoid $\mathcal{G}$ is a graded extension (see [24]) $(\hat{\mathcal{G}}, \Gamma, Z)$, such that the groupoids $\hat{\mathcal{G}}, \Gamma$ are Real, $Z$ is a Real generalised morphism $Z : \Gamma \to \mathcal{G}$, and if the groups $S^1, Z_2$ are given the involution by complex conjugation and the trivial involution, respectively, all of the maps

$$
\begin{array}{ccc}
S^1 & \longrightarrow & \hat{\Gamma} \\
\pi \downarrow & \quad & \downarrow \\
& Z_2 & 
\end{array}
$$

are equivariant. The set $\widetilde{\text{Ext}}(\mathcal{G}, S^1)$ of Morita equivalence classes of Real graded $S^1$-extension over $\mathcal{G}$ has the structure of abelian group. Moreover, by [18, Theorem 2.60], there is an isomorphism $d d : \widetilde{\text{Ext}}(\mathcal{G}, S^1) \xrightarrow{\cong} \text{HR}^1(\mathcal{G}, Z_2) \times \text{HR}^2(\mathcal{G}, S^1)$, which is natural in the category $\mathcal{G}$. It follows that there is a natural isomorphism $\text{BrR}(\mathcal{G}) \xrightarrow{\cong} \widetilde{\text{Ext}}(\mathcal{G}, S^1)$ (note that the construction of this isomorphism is explicitly given in [17]), where the left hand side is the subgroup of $\widetilde{\text{Br}}R(\mathcal{G})$ consisting of Real graded D-D bundles of type $0$.

Let $n = p + q \in \mathbb{N}^*$. The group $\text{Pin}(p + q)$ is defined as

$$
\text{Pin}(p + q) := \left\{ \gamma \in Cl_{p,q} \mid \varepsilon(\gamma)v\gamma^* \in \mathbb{R}^{p,q}, \forall v \in \mathbb{R}^{p,q}, \text{and } \gamma\gamma^* = 1 \right\},
$$

where $\varepsilon$ is the canonical $Z_2$-grading of $Cl_{p,q}$. It is known (§IV.4) that

$$
\text{Pin}(p + q) \cong \{ \gamma = x_1 \cdots x_k \in Cl_{p,q} \mid x_i \in S^{p,q}, 1 \leq k \leq 2n \},
$$

It follows that $\text{Pin}(p + q)$ is a Real group with respect to the involution induced from $S^{p,q}$; i.e., $\bar{\gamma} = \bar{x}_1 \cdots \bar{x}_k$, for $\gamma \in \text{Pin}(p + q)$. Of course, this involution is equivalent to that induced from $Cl_{p,q}$. Moreover, the surjective homomorphism $\pi : \text{Pin}(p + q) \longrightarrow O(p + q), \gamma \longmapsto \pi_{\gamma},$ where $\pi_\gamma(v) := \varepsilon(\gamma)v\gamma^*$ for $v \in \mathbb{R}^{p,q}$, is clearly Real. Notice that $\ker \pi = \{ \pm 1 \} = Z_2$. Hence, there is a canonical Real graded $Z_2$-central extension of the Real groupoid $O(p + q)$.

$$
\begin{array}{ccc}
Z_2 & \longrightarrow & \text{Pin}(p + q) \\
\pi \downarrow & \quad & \downarrow \\
& O(p + q) & 
\end{array}
$$

where the homomorphism $\delta : O(p + q) \longrightarrow Z_2$ is the unique map such that $\det A = (-1)^{\delta(A)}$ (compare with [24, §2.5]). Let

$$
\text{Pin}'(p + q) := \text{Pin}(p + q) \times_{\{ \pm 1 \}} S^1,
$$

be endowed with the Real structure $[(\gamma, \lambda)] \longmapsto [(\bar{\gamma}, \bar{\lambda})]$, where as usual, the "bar" operation in $S^1$ is the complex conjugation. Then the above Real graded $Z_2$-central extension induces a Real graded $S^1$-central extension $\mathcal{T}_{p,q}$

$$
\begin{array}{ccc}
S^1 & \longrightarrow & \text{Pin}'(p + q) \\
\pi \downarrow & \quad & \downarrow \\
& O(p + q) & 
\end{array}
$$

(9)
Let $V$ be a Real Euclidean vector bundle of type $p - q$ over $\mathbb{S}^1 \times X$. Then there is a $\text{Br}_q \mathbb{S}^1$-central extension $\mathbb{E}^q(V)$ obtained by pulling back $\mathbb{T}_{p,q}$ via the generalized Real homomorphism $\mathbb{F}(V) : \mathbb{S}^1 \to O(p + q)$.

**Definition 8.1.** Let $V$ be a Real Euclidean vector bundle of type $p - q$ over $\mathbb{S}^1 \times X$. We define its associated Real graded D-D bundle as the Real graded D-D bundle $A_V$ of type 0 over $\mathbb{S}^1 \times X$ whose image in $\text{Ext}(\mathbb{S}^1, \mathbb{S}^1)$ is $[\mathbb{E}^q(V)]$ via the isomorphism $\hat{\text{Br}}_q(\mathbb{S}^1) \to \text{Ext}(\mathbb{S}^1, \mathbb{S}^1)$.

**Lemma 8.2.** Let $V$ and $V'$ be Real Euclidean vector bundles of type $p - q$ and $p' - q'$, respectively, over $\mathbb{S}^1 \times X$. Then the Real graded D-D bundles $A_V \otimes A_V'$ and $A_V \otimes A_V'$ are Morita equivalent.

**Proof.** Considering the Real homomorphisms $\text{Pin}^*(p + q) \times \text{Pin}^*(p' + q') \to \text{Pin}^*(p + p') + (q + q')$ and $O(p + q) \times O(p' + q') \to O(p + p') + (q + q')$ (cf. [12]) and a Real open cover of $X$ trivialising both $V$ and $V'$ (and hence the direct sum $V \oplus V'$), one easily checks that $(\mathbb{F}(V) \otimes \mathbb{T}_{p,q}) \otimes (\mathbb{F}(V') \otimes \mathbb{T}_{p',q'}) \sim (\mathbb{F}(V \oplus V')) \otimes \mathbb{T}_{p+p'q+q'}$. □

**Theorem 8.4.** Let $V$ be a Real Euclidean vector bundle of type $p - q$ over $\mathbb{S}^1 \times X$. Then for all $A \in \text{Br}_q(\mathbb{S}^1)$, we have

$$\text{KR}_A^* (\mathbb{C}(V)) \cong \text{KR}_A^{*+q-p} (\mathbb{S}^1).$$

We shall mention that this result is proven by J.-L. Tu in the complex case (see [24, Proposition 2.5]). However, the approach we will be using here to prove it is very different from that used by Tu.

Indeed, our proof requires the construction of *generalised Stiefel-Whitney classes* of a Real vector bundles over a Real groupoid. Recall that associated to any Real vector bundle $V$ over a locally compact paracompact space $X$, there are cohomology classes $w_i(V) \in H^i(X, \mathbb{Z}_2)$ called the $i^{\text{th}}$ *Stiefel-Whitney classes* of $V$ (see for instance [8, Chap.17 §2]). For instance $w_1(V)$ is the constraint for $V$ being *oriented*, and $w_2(V)$ is the constraint for $V$ being Spin– (we will say more about that later).

We have already seen that a Real Euclidean vector bundle $V$ of type $p - q$ gives rise to a generalised Real homomorphism $\mathbb{F}(V) : \mathbb{S}^1 \to O(p + q)$. In fact, Real Euclidean vector bundles arise this way: given $P : \mathbb{S}^1 \to O(p + q)$, $V := P \times_{O(p + q)} \mathbb{R}^p \to X$ is a Real Euclidean vector bundle of type $p - q$. There then is a bijection between the set $\text{Vect}_{p+q}(\mathbb{S})$ of isomorphism classes of Real Euclidean vector bundles of type $p - q$ and the set $\text{Hom}_{\mathbb{R}^q}(\mathbb{S}, O(p + q))$, hence with $\mathbb{H}^1(\mathbb{S}, O(p + q))$, thanks to [18, Proposition 2.49].

Let $\epsilon$ be a Real $O(p + q)$-valued 1-cocycle over $\mathbb{S}$ realizing $\mathbb{F}(V)$. This can be considered as a Real family of continuous maps $\xi_{(j, k)} : U^1_{(j, k, j, k)} \to O(p + q)$ such that

$$\xi_{(j, k)}(\gamma_1) \xi_{(j, k, j, k)}(\gamma_2) = \xi_{(j, k, k, k)}(\gamma_1 \gamma_2), \quad (\gamma_1, \gamma_2) \in U^2_{(j, j, k, k)},$$

(10)
where $\mathcal{U} = \{ U_j \}_{j \in J}$ is a Real open cover of $X$ (indeed, if $f : \mathcal{G} [\mathcal{U}] \to O(p + q)$ is a Real homomorphism realising $\mathcal{F}(V)$, then one can take $\xi_{(j, j_1)}(\mathcal{G} (j, j_1)) := f(\mathcal{G} (j, j_1))$). We may suppose that the simplicial Real cover $\mathcal{U}_*$ of $\mathcal{G}_*$ is "small" enough so that we can pick a Real family of continuous maps $\xi_{(j, j_1)} : U_1^{(j, j_1)} \to \text{Pin}^r(p + q)$ which are a Pin$^r(p + q)$-lifting of $(\xi_{(j, j_1)})$ through the Real projection $\pi : \text{Pin}^r(p + q) \to O(p + q)$; i.e., $\pi(\xi_{(j, j_1)}(\gamma)) = \xi_{(j, j_1)}(\gamma), \forall \gamma \in U_1^{(j, j_1)}$. In view of equation (10), we have

$$\xi_{(j, j_1)}(\gamma_1)\xi_{(j, j_2)}(\gamma_2) = \alpha_{(j, j_1, j_2)}(\gamma_1, \gamma_2)\xi_{(j, j_2)}(\gamma_1)\xi_{(j, j_2)}(\gamma_2), \forall (\gamma_1, \gamma_2) \in U_2^{(j, j_1, j_2)}, \tag{11}$$

for some $\omega_{(j, j_1, j_2)}(\gamma_1, \gamma_2) \in S^1$. The elements $\omega_{(j, j_1, j_2)}(\gamma_1, \gamma_2)$ clearly define a Real family of continuous functions $\omega_{(j, j_1, j_2)} : U_2^{(j, j_1, j_2)} \to S^1$ which are easily checked to be an element of $ZR^2_{\mathbb{S}}(\mathcal{U}_*, S^1)$.

**Definition 8.5.** Let $V$ be a Real Euclidean vector bundle of type $p - q$ over $\mathcal{G} \to X$. Let $\epsilon$ be the class of $\mathcal{F}(V)$ in $\check{H}R^1(\mathbb{S}_*, O(p + q))$.

(a) The first generalized Stiefel-Whitney class $w_1(V) \in \check{H}R^1(\mathbb{S}_*, \mathbb{Z}_2)$ as $w_1(V) := \epsilon \circ \alpha$, where $\alpha : O(p + q) \to \mathbb{Z}_2$ is the homomorphism defined in (9).

(b) The second generalized Stiefel-Whitney class $w_2(V)$ is the class in $\check{H}R^2(\mathbb{S}_*, S^1)$ of the Real 2-cocycle $\omega$ uniquely determined by equation (11).

We define $w(V) := (w_1(V), w_2(V)) \in \check{H}R^1(\mathbb{S}_*, \mathbb{Z}_2) \times \check{H}R^2(\mathbb{S}_*, S^1)$.

**Remark 8.6.** Note that $w_1(V) = 0$ implies that $\mathcal{F}(V)$ is actually a Real $SO(p + q)$-principal bundle over $\mathcal{G} \to X$, which means that $V$ is oriented. Moreover, the Real family $(\omega_{(j, j_1, j_2)})$ is nothing but the obstruction for the Real $O(p + q)$-valued 1-cocycle $\epsilon$ to lift to a Real Pin$^r(p + q)$-valued 1-cocycle $\xi$; or in other words, it is the obstruction for $\mathcal{F}(V)$ to lift to a Real Pin$^r(p + q)$-principal bundle over $\mathcal{G} \to X$.

**Example 8.7.** Denote by $\theta^{p, q}$ the trivial Euclidean vector bundle $\mathbb{R}^{p, q}$ over $O(p + q)$ and $\mathcal{F}(V)$ has $\mathbb{R}^{p, q}$-valued 1-cocycle corresponding to the Real graded $S^1$-central extension $T_{p, q} \in \text{ExtR}(O(p + q), S^1)$ of (9). Moreover, $w(\theta^{p, q}) = dd(T_{p, q})$.

**Definition 8.8.** A Real Euclidean vector bundle $V$ of type $p - q$ over $\mathcal{G} \to X$ admits a Spin$^c$-structure if $w(V) = 0$; in this case we say that $V$ is Spin$^c$. We also say that $V$ is KR-oriented (following Hilsum-Skandalis terminology [7], see also H. Schröder’s book [21]).

Taking the involution of $\mathcal{G}$ to be the trivial one, the next result is actually the groupoid equivariant analogue of Plymen’s [20, Theorem 2.8].

**Proposition 8.9.** Let $V$ be a Real Euclidean vector bundle of type $p - q$ over $\mathcal{G} \to X$.

1. We have $\text{DD}(\text{Cl}(V)) = w(V)$. Hence, $V$ is KR-oriented if and only if $\text{Cl}(V)$ is trivial.
2. If $p = q \in \mathbb{N}^*$, then $\text{DD}(\text{Cl}(V)) = \text{DD}(\text{A}_V) = w(V)$. Therefore, if $p = q$, the following statements are equivalent:
   (i) $V$ is KR-oriented.
   (ii) The Real $D$-$D$ bundle $\text{Cl}(V) \to X$ is trivial.
   (iii) The Real $D$-$D$ bundle $\text{A}_V \to X$ is trivial.

**Proof.** 1. Since the map sending a Real graded $S^1$-central extension onto its Dixmier-Douady class is a natural isomorphism (see [17]), we have a commutative diagram

$$\begin{array}{ccc}
\text{ExtR}(O(p + q), S^1) & \xrightarrow{dd} & \text{ExtR}(\mathcal{G}, S^1) \\
\check{H}R^1(O(p + q), \mathbb{Z}_2) \times \check{H}R^2(O(p + q), S^1) & \xrightarrow{F(V) \times F(V')} & \check{H}R^1(\mathbb{S}_*, \mathbb{Z}_2) \times \check{H}R^2(\mathbb{S}_*, S^1)
\end{array}$$
Hence, if \( f : \mathcal{S}[U] \to O(p + q) \) is a Real homomorphism realising \( \mathbb{F}(V) \), then
\[
DD(A_V) = dd(f^*T_{p,q}) = (f^*w_1(\theta^{p,q}), f^*w_2(\theta^{p,q})) = w(V),
\]
where the first equality comes from the very definition of the Real graded D-D bundle \( A \), the second one follows from Example 8.7, and the last one is a simple interpretation on the construction of \( w_1 \) and \( w_2 \).

2. If \( p = q \), then \( Cl_{p,p} \) is the type 0 Real graded elementary C*-algebra \( \mathcal{K}(\hat{H}) \), where
\[
\hat{H} = \mathbb{C}^{2p-1} \oplus \mathbb{C}^{2p-1}.
\]
Identifying the Real space \( \mathbb{R}^{p,p} \) with \( \mathbb{C}^p \) endowed with the coordinatewise complex conjugation, there is a degree conserving Real representation
\[
\lambda : \text{Pin}^c(p + p) \to \hat{U}(\hat{H}),
\]
induced by the Real map \( \mathbb{C}^p \to \mathcal{L}(\Lambda^\infty \mathbb{C}^p) \cong \mathcal{L}(\hat{H}) \) given by exterior multiplication (cf. [12]). This gives us a projective Real representation
\[
\text{Ad}_\lambda : O(p + p) \to \hat{\text{PU}}(\hat{H})
\]
given by \( \text{Ad}_\lambda(\gamma)(T) := \lambda(\tilde{\gamma})T\lambda(\tilde{\gamma})^{-1} \), for \( T \in \mathcal{K}(\hat{H}) \), where \( \tilde{\gamma} \in \text{Pin}^c(2p) \) is an arbitrary lift of \( \gamma \in O(p + p) \), and where we have used the well-known identification \( \hat{\text{PU}}(\hat{H}) \cong \text{Aut}^0(\mathcal{K}(\hat{H})) \). We thus have commutative diagrams

\[
\begin{array}{ccc}
S^1 & \xrightarrow{\text{Pin}^c(p + p)} & O(p + p) \\
\lambda & & \text{Ad}_\lambda \\
\downarrow & & \downarrow \\
S^1 & \xrightarrow{\hat{U}(\hat{H})} & \hat{\text{PU}}(\hat{H})
\end{array}
\]

which yields to an equivalence of Real graded \( S^1 \)-central extensions
\[
(\text{Ad}_\lambda)^* \mathbb{E}_{\mathcal{K}(\hat{H})} \sim \mathcal{T}_{p,p} \in \text{Ext}(O(p + q), S^1) \tag{12}
\]
where \( \mathbb{E}_{\mathcal{K}(\hat{H})} \) is the Real graded \( S^1 \)-central extension \( (\hat{U}(\hat{H}), \text{deg}) \) of the Real groupoid \( \hat{\text{PU}}(\hat{H}) \).

Now let \( (U_i, h_i) \) be a Real trivialisation of \( V \), with transition functions \( \alpha_{ij} : U_{ij} \to O(p + q) \). Then \( V \) is isomorphic to the Euclidean Real vector bundle over \( \mathcal{S} \)
\[
\prod_i U_i \times \mathbb{C}^p / \sim \to X,
\]
where \( (x, t)_i \sim (x, \alpha_{ij}(x)t)_j \), for \( x \in U_{ij} \), endowed with the Real \( \mathcal{S} \)-action
\[
g \cdot [(s(g), t)_i] := [(r(g), \alpha_{ij,ji}(g)t)]_j, g \in U_{ij}^1,\]
Here \( [(x, t)]_i \) denotes the class of \( (x, t)_i \in U_i \times \mathbb{C}^p \) in \( \prod_i U_i \times \mathbb{C}^p / \sim \).

Moreover, by universality of Clifford algebras (cf. [10, Chap.IV, §4]), the Real family of homeomorphisms \( h_i : V|_{U_i} \to U_i \times \mathbb{C}^p \) induces a Real family of homeomorphisms
\[
\text{Cl}(V)|_{U_i} \to U_i \times \text{Cl}_{p,p} = U_i \times \mathcal{K}(\hat{H})
\]
with transition functions \( \text{Ad}_\lambda \alpha_{ij}(\cdot) : U_{ij} \to \hat{\text{PU}}(\hat{H}) \). Since the Real action of \( \mathcal{S} \) on the Rg D-D bundle \( \text{Cl}(V) \to X \) is induced from the Real action of \( \mathcal{S} \) on \( V \to X \), it follows that \( \text{Cl}(V) \) is isomorphic to
the Real graded D-D bundle

\[ \prod_i U_i \times K(\hat{\mathcal{H}})/_{\sim} \]

where the equivalence relation is \((x, a)_i \sim (x, Ad_{A_i}(a(x))a)_i\) for \(x \in U_{ij}\), with the Real \(S\)-action by graded automorphisms

\[ g \cdot [(s(g), a)]_{ji} := [(\tau(g), Ad_{A_i}(\epsilon_{(ji,j)})a)]_{ji} \quad g \in U^{1}_{(j_0,j_1)}. \]

Therefore, if \(P : S \to \hat{PU}(\hat{\mathcal{H}})\) is the generalised classifying morphism ([17, Section 66]) for \(\text{Cl}(V)\), it corresponds to the class of \(Ad_{A_i} \in H^R(\mathbb{S}, \hat{PU}(\hat{\mathcal{H}}))\), thanks to [17, Proposition 7.4]. Putting this in terms of generalised Real homomorphisms, there is a commutative diagram in the category \(\mathcal{H}\)

\[ \begin{array}{ccc}
\mathcal{H} & \xrightarrow{p} & \hat{PU}(\hat{\mathcal{H}}) \\
\downarrow & & \downarrow \\
O(p + p) & \xrightarrow{Ad_1} & \hat{PU}(\hat{\mathcal{H}})
\end{array} \]

This combined with (12) implies

\[ P^*E_{\mathcal{H}(\hat{\mathcal{H}})} \sim \mathcal{F}(V)^*T_{p,p}. \]

Hence, \(DD(\text{Cl}(V)) = dd([P^*E_{\mathcal{H}(\hat{\mathcal{H}})}]) = dd([\mathcal{F}(V)^*T_{p,p}]) = dd(A_1) = w(V)\), where the third and fourth equalities come from the first statement of the proposition. \(\square\)

To see how things work in the general case, observe first that if \(V_1, V_2\) are Real Euclidean vector bundles of types \(p_1 - q_1\) and \(p_2 - q_2\), respectively, then \(\text{Cl}(V_1 \oplus V_2) \to X\) is a Real graded D-D bundle of type \((q_1 + q_2) - (p_1 + p_2)\) because \(\text{Cl}(V_1 \oplus V_2) \equiv \text{Cl}(V_1) \otimes_X \text{Cl}(V_2)\) (cf. [12, §2.15] or [2]).

Now let \(V\) be a Real Euclidean vector bundle of type \(p - q\) over \(S \to X\). Then we define \(\hat{V} := V \oplus 1^D\), and \(\text{Cl}(V) \in \text{BrR}_0(\mathbb{S})\) as the Rg D-D bundle of type 0 defined by

\[ \text{Cl}(V) := \text{Cl}(\hat{V}) \to X, \]

with the obvious Real \(S\)-action. Notice that this definition is a direct adaptation of [1, §3.5]. Moreover, we have the following

**Theorem 8.10.** Let \(V\) be a Real Euclidean vector bundle of type \(p - q\) over \(S \to X\). Then

\[ DD(\text{Cl}(V)) = (q - p, w_1(V), w_2(V)). \]  \hspace{1cm} (13)

**Proof.** We have \(DD(\text{Cl}(V)) = DD(A_V)\), thanks to Proposition 8.9 2). Applying Lemma 8.2 and Lemma 8.3, we get \(DD(\text{Cl}(V)) = DD(A_V)\). Furthermore, \(\text{Cl}(V)\) is clearly Morita equivalent to \(\text{Cl}(V) \otimes_X \text{Cl}(1^{p,q})\). Therefore,

\[ DD(\text{Cl}(V)) = DD(\text{Cl}(1^{p,q})) + DD(\text{Cl}(1^{p,q})) = DD(A_V) + (q - p, 0, 0). \]

We conclude by applying Proposition 8.9 1). \(\square\)

By using the fact \(DD\) is a group homomorphism, we immediately deduce from the above theorem that

**Corollary 8.11.** If \(V\) and \(V'\) are Real Euclidean vector bundles over \(S \to X\) then

\[ w_1(V \oplus V') = w_1(V) + w_1(V'), \quad \text{and} \quad w_2(V \oplus V') = (-1)^{w_1(V)+w_1(V')}w_2(V) \cdot w_2(V'). \]
Proof of Theorem 8.4. As a consequence of Theorem 8.10, one has

$$\text{KR}^*_{\text{Cl}(V)}(\mathcal{G}^*) \cong \text{KR}^*_{\mathcal{A}_V + \mathcal{C}_{\text{fr}}}(\mathcal{G}^*),$$

so we conclude by using Kasparov product in KKR-theory (recall that by definition we have \(\text{KR}^*_{\mathcal{A} + \mathcal{B}}(\mathcal{G}^*) = \text{KKR}^*_{\mathcal{A},(\mathcal{A} \otimes \mathcal{B}) \ltimes \pi}(\mathcal{G})\)).

\[\square\]

9. Thom isomorphism in twisted groupoid K-theory

We start this section by some observations about Spin\(^c\) Real Euclidean vector bundles. Let

$$\text{Spin}(p + q) := \text{Pin}(p + q) \cap \mathcal{C}^0_{p,q}$$

(cf. [2, 10, 12]). The restriction of the projection \(\text{Pin}(p + q) \rightarrow O(p + q)\) induces a surjective Real homomorphism

$$\text{Spin}(p + q) \rightarrow SO(p + q)$$

with kernel \(\mathbb{Z}_2\), where \(SO(p + q)\) is equipped with the Real structure induced from \(O(p + q)\). Moreover, there is a Real (trivially) graded \(S^1\)-central extension \(\mathcal{T}_{p,q}^r\)

$$S^1 \rightarrow \text{Spin}^c(p + q) \rightarrow SO(p + q)$$

over the Real groupoid \(SO(p + q) \rightarrow \mathbb{Z}_2\), where \(SO(p + q) \rightarrow \mathbb{Z}_2\) is the zero map, and where

$$\text{Spin}^c(p + q) := \text{Spin}(p + q) \times_{\mathbb{Z}_2} S^1.$$

Now suppose \(V\) is a Real Euclidean vector bundle of type \(p - q\) over \(\mathcal{G} \rightarrow X\). If \(w_1(V) = 0\), then \(F(V)\) reduces to a generalised Real homomorphism from \(\mathcal{G}\) to \(SO(p + q)\). So, \(\mathcal{A}_V\) comes from the Real graded \(S^1\)-central extension \(F(V) \cdot \mathcal{T}_{p,q}^r\). Moreover, \(V\) being KR-oriented means that \(F(V)\) actually is a Real Spin\(^c\)-principal bundle over \(\mathcal{G} \rightarrow X\), hence a generalised Real homeomorphism from \(\mathcal{G} \times X\) to \(\text{Spin}^c(p + q)\).

The following result generalises the Thom isomorphism theorem in twisted orthogonal K-theory known in the case of topological spaces (see Karoubi and Donovan [6] and Karoubi [11]) and in twisted K-theory of bundle gerbes proved by A. Carey and B.-L. Wang in [4].

**Theorem 9.1.** Let \(\mathcal{G} \rightarrow X\) be a locally compact Hausdorff second-countable Real groupoid with Real Haar system. Let \(\pi : V \rightarrow X\) be a Real Euclidean vector bundle of type \(p - q\) over \(\mathcal{G} \rightarrow X\), and let \(\mathcal{A} \in \text{BrR}(\mathcal{G})\). Then there is a canonical group isomorphism

$$\text{KR}^*_{\pi^*\mathcal{A}}((\pi^*\mathcal{G})^*) \cong \text{KR}^*_{\mathcal{A} + \mathcal{C}_{\text{fr}}(V)}(\mathcal{G}^*).$$

Furthermore, if \(V\) is KR-oriented, then there is a canonical isomorphism

$$\text{KR}^*_{\pi^*\mathcal{A}}((\pi^*\mathcal{G})^*) \cong \text{KR}^{\pi\mathcal{A},p+q}(\mathcal{G}^*),$$

where as usual, the Real groupoid \(\pi^*\mathcal{G} \rightarrow V\) is the pullback of \(\mathcal{G} \rightarrow X\) via the projection \(\pi\).

**Proof.** From Theorem 8.4 we have \(\text{KR}^*_{\mathcal{A} + \mathcal{C}_{\text{fr}}(V)}(\mathcal{G}^*) \cong \text{KR}^{\pi\mathcal{A},p+q}(\mathcal{G}^*)\); in particular if \(V\) is Spin\(^c\), \(\mathcal{A}_V = 0\) and \(\text{KR}^*_{\mathcal{A} + \mathcal{C}_{\text{fr}}(V)} \cong \text{KR}^{\pi\mathcal{A},p+q}(\mathcal{G}^*)\), which implies that the isomorphism (15) deduces from isomorphism (14). Let us show the latter. The KKR\(_{(p+q)}\)-equivalence

$$\alpha_{p,q} \in \text{KKR}(\mathcal{C}_0(\mathbb{R}^{p,q}), \mathcal{C}(\mathbb{R}^{p,q}))$$

then we conclude by using Kasparov product in KKR-theory (recall that by definition we have \(\text{KR}^*_{\mathcal{A} + \mathcal{B}}(\mathcal{G}^*) = \text{KKR}^*_{\mathcal{A},(\mathcal{A} \otimes \mathcal{B}) \ltimes \pi}(\mathcal{G})\)).

\[\square\]
induces by functoriality in the category \( \mathfrak{H} \) a \( \text{KKR}_v \)-equivalence
\[ F(V)^* \alpha_{p,q} \in \text{KKR}_v(C_0(V) \hat{\otimes} \mathcal{C}_0(X), C_0(X; \mathcal{C}(V))). \]
Thus, by the identifications of \( C^\ast \)-algebras with generalized Real \( \mathcal{G} \)-actions
\[ C_0(X; A) \cong C_0(X) \hat{\otimes} \mathcal{C}_0(X) \mathcal{C}_0(X; A), \quad \text{and} \quad C_0(V; \pi^\ast A) \cong C_0(V) \hat{\otimes} \mathcal{C}_0(X) \mathcal{C}_0(X; A), \]
we get a \( \text{KKR}_v \)-equivalence
\[ \bar{\alpha}_V \in \text{KKR}_v(C_0(V; \pi^\ast A), C_0(X; A) \otimes X \mathcal{C}(V)), \]
by taking \( \alpha_V \) to be the Kasparov product of \( F(V)^* \alpha_{p,q} \) with the canonical \( \text{KKR}_v \)-equivalence
\[ 1_{C_0(X; A)} \in \text{KKR}_v(C_0(X; A), C_0(X; A)). \]
Therefore, we obtain a \( \text{KKR} \)-equivalence
\[ \alpha_V := j_{\mathcal{G}, \text{red}}(\bar{\alpha}_V) \in \text{KKR}_v(C_0(V; \pi^\ast A) \rtimes \mathcal{G}, C_0(X; A) \rtimes X \mathcal{C}(V) \rtimes \mathcal{G}), \]
where \( j_{\mathcal{G}, \text{red}} \) is the descent morphism and we are done. \( \square \)

References

[1] A. Alekseev and E. Meinrenken, Dirac structures and Dixmier-Douady bundles, Int. Math. Res. Not. IMRN 4 (2012), 904–956. MR2889163
[2] F. M. Atiyah, R. Bott, and A. Shapiro, Clifford modules, Topology 3 (1964), no. suppl. 1, 3–38. MR0167985 (29 #5250)
[3] B. Blackadar, K-Theory for operator algebras, Mathematical Sciences Research Institute Publications, vol. 5, Springer-Verlag, New York, 1986. MR859867
[4] A. L. Carey and B.-L. Wang, Thom isomorphism and push-forward map in twisted K-theory, J. K-Theory 1 (2008), no. 2, 357–393, doi: 10.1017/is007011015kt011. MR2434190 (2009g:55005)
[5] J. Cuntz and G. Skandalis, Mapping cones and exact sequences in K-theory, J. Operator Theory 15 (1986), no. 1, 163–180. MR816237
[6] P. Donovan and M. Karoubi, Graded Brauer groups and K-theory with local coefficients, Inst. Hautes Études Sci. Publ. Math. 38 (1970), 5–25. MR0282363 (43 #8075)
[7] M. Hilsum and G. Skandalis, Morphismes K-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes), Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 3, 325–390 (French, with English summary). MR925720
[8] D. Husemoller, Fibre bundles, 3rd ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994. MR1249482 (94k:55001)
[9] K. K. Jensen and K. Thomsen, Elements of KK-theory, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1993. MR1249482 (94k:19008)
[10] M. Karoubi, K-theory, Classics in Mathematics, Springer-Verlag, Berlin, 2008. An introduction; Reprint of the 1978 edition; With a new postface by the author and a list of errata. MR2458205 (2009i:19001)
[11] ———, Twisted K-theory—old and new, K-theory and noncommutative geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 117–149, doi: 10.4171/061-1/5. MR2513335 (2010h:19010)
[12] G. G. Kasparov, The operator K-functor and extensions of \( C^\ast \)-algebras, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 3, 571–636, 719 (Russian); English transl., Math. USSR-Izv. 16 (1981), no. 3, 513–572 (1981). MR582160
[13] ———, Equivariant KK-theory and the Novikov conjecture, Invent. Math. 91 (1988), no. 1, 147–201, doi: 10.1007/BF01404917. MR918241
[14] A. Kumjian, Fell bundles over groupoids, Proc. Amer. Math. Soc. 126 (1998), no. 4, 1115–1125, doi: 10.1090/S0002-9939-98-04240-3. MR1443836 (98j:46055)
[15] A. Kumjian, P. S. Muhly, J. N. Renault, and D. P. Williams, The Brauer group of a locally compact groupoid, Amer. J. Math. 120 (1998), no. 5, 901–954. MR1646047 (2000b:46122)
[16] P.-Y. Le Gall, Théorie de Kasparov équivariante et groupoïdes, Thèse de Doctorat, Université Paris VII, 1994.
[17] E. M. Moutuou, The graded Brauer group of a groupoid with involution (2012), eprint. arXiv: 1202.0257.
[18] ———, On groupoids with involutions and their cohomology (2012), eprint. arXiv: 1202.0155.
[19] E.M. Moutuou, Twisted groupoid KK–Theory, Ph.D. thesis, Université de Lorraine - Metz, and Universität Paderborn, 2012, http://www.theses.fr/2012LORR0942.
[20] R. J. Plymen, Strong Morita equivalence, spinors and symplectic spinors, J. Operator Theory 16 (1986), no. 2, 305–324. MR860349 (88d:58112)
[21] H. Schröder, K-theory for real \( C^\ast \)-algebras and applications, Pitman Research Notes in Mathematics Series, vol. 290, Longman Scientific & Technical, Harlow, 1993. MR1267059 (95f:19006)
[22] G. Skandalis, *Some remarks on Kasparov theory*, J. Funct. Anal. 56 (1984), no. 3, 337–347. MR 743845

[23] J.-L. Tu, *The Baum–Connes conjecture for groupoids*, C*-algebras (Münster, 1999), Springer, Berlin, 2000, pp. 227–242. MR 1798599

[24] ———, *Twisted K-theory and Poincaré duality*, Trans. Amer. Math. Soc. 361 (2009), no. 3, 1269–1278. MR 2457398

[25] J.-L. Tu, Ping Xu, and C. Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 6, 841–910 (English, with English and French summaries). MR 2119241

Institut Élie Cartan de Lorraine - Metz, Université de Lorraine et CNRS, Ile de Saulcy, 57000 Metz

Current address: School of Mathematics, University of Southampton, Highfield, Southampton SO17 1BJ

E-mail address: E.MohamedMoutuou@soton.ac.uk