Manipulating the critical temperature for the superfluid phase transition in trapped atomic Fermi gases

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Since the observation of Bose-Einstein condensation in trapped atomic gases [1], there has been increasing interest in the possibility of observing the Bardeen-Cooper-Schrieffer (BCS) phase transition to the superfluid state [2] in dilute fermionic alkali gases [3, 4, 5, 6, 7, 8, 9, 10]. Currently, experimental efforts in cooling fermionic atoms of $^{6}$Li [12, 13] and $^{40}$K [14, 15] to the quantum degenerate regime have made significant progress, reaching temperatures as low as $0.2T_F$ where $T_F$ is the Fermi temperature. However, these temperatures are still far beyond the critical temperature required for the BCS phase transition, which is at least an order of magnitude lower [3, 4, 5, 6, 7, 8].

On the other hand, the current techniques used to cool fermionic atoms lose their utility when the temperature becomes much less than $T_F$. For example, evaporative cooling, which has been used for $^{40}$K, becomes ineffective for temperatures much less than $T_F$ due to Pauli blocking [16] while sympathetic cooling of the fermions using bosons, as done with $^{6}$Li [17], loses its effectiveness when the heat capacity of the bosons falls below that of the fermions. Due to these technical obstacles in cooling fermionic atoms, it seems that one must seek other avenues to reach the BCS superfluid state.

Recently there have been a number of proposals to study the possibility of achieving a higher $T_C$ by increasing the strength of the attractive interactions, i.e., the scattering length between the fermions using either a Feshbach resonance [1, 18] or photoassociation [19]. Since $T_C$ is a function of the attractive interaction between fermions and the density of states at the Fermi energy, $\rho_F$, higher transition temperatures can also be achieved by increasing the density of states for the fermions at the Fermi surface. In bulk superconductors, the only way to modify the density of states is by changing the spatial dimensionality of the system. However, in atomic systems the density of states can be controlled through the shape and strength of the external trapping potential.

The ability of an external trapping potential to change the critical temperature for a phase transition by modifying the density of states, $\rho(c)$ (see Eq. (3) below), was first considered by Bagvato et al. for Bose-Einstein condensation [20]. They showed that the critical temperature for Bose-Einstein condensation, $T_BEC$, as a function of the number of atoms confined in a power law trap is calculated in the Thomas-Fermi approximation. For anharmonic traps, $T_C$ can be increased by several orders of magnitude in comparison to a harmonic trap. Our theoretical results indicate that, in practice, one could manipulate the critical temperature for the BCS phase transition by shaping the traps confining the atomic Fermi gases.
two-body interactions and \( \alpha = 2^{7/3}eC^{-7/3}/\pi \approx 0.28 \) with \( C \) being the Euler's constant \[8, 14, 13\]. The density of states at the Fermi surface, in terms of the Fermi momentum \( \hbar k_F \), is

\[
\rho_F = \frac{mk_F}{2\pi^2\hbar^2}.
\]

From Eq. (1) we can see that by raising the density of states at the Fermi surface (or equivalently, by raising \( k_F \)), there will be an exponential increase in the transition temperature. Note that even though we restrict ourselves to the case of \( s \)-wave pairing, our results can easily be extended to pairing via higher partial waves since in those cases the critical temperature has a similar exponential dependence on \( \rho_F \) \[19\].

The goal of this paper is to determine the critical temperature for the inhomogeneous Fermi gas where the atoms in both hyperfine states are subject to the trapping potential,

\[
U(r) = \varepsilon_1 \left| \frac{r}{a} \right|^p + \varepsilon_2 \left| \frac{y}{b} \right|^l + \varepsilon_3 \left| \frac{z}{c} \right|^q,
\]

and \( p, l, \) and \( q \) are positive integers. We refer to the magnitude of the exponents \( p, l, \) and \( q \) as the confining power of the trap. For simplicity, we assume that the trapping potential is independent of the hyperfine state, hence the density of fermions is the same for both states, \( n(r) \). If \( n(r) \) varies sufficiently slowly, then we can make a local density (or Thomas-Fermi) approximation by assuming that at each point in space the Fermi gas can be treated as being homogenous \[20\]. In that case, a local value of the critical temperature, \( T_C(r) \), can be calculated using Eq. (3). For temperatures \( T \ll T_F \), the local density approximation is valid provided the average distance between atoms, \( \sim k_F^{-1} \), is much less than the distances over which \( n(r) \) changes significantly. Since the changes in \( n(r) \) are determined by \( U(r) \) for which the characteristic length scales are \( a, b, \) and \( c \) and we have the condition \( k_F^{-1} \ll (abc)^{1/3} \). In addition, for \( T \ll T_C \), the coherence length for the Cooper pairs, \( \xi_0 = \hbar^2 k_F/\pi n \Delta_0 \) where \( \Delta_0 = (\pi e^{-C})k_B T_C \) is the BCS gap at \( T = 0 \), should also be much less than \( (abc)^{1/3} \) \[3, 4\].

In the Thomas-Fermi approximation, the chemical potential, \( \mu \), is replaced with a local value, \( \mu(r) = \mu - U(r) + gn(r) \) so that the fermions can be treated at each \( r \) as an ideal homogenous gas with chemical potential \( \mu(r) \). The term \( gn(r) \) represents the Hartree-Fock potential experienced by each atom due to the atoms in the opposite hyperfine state. Since we are interested in temperatures \( T \ll T_F \), the chemical potential may be approximated by the Fermi energy, \( \mu \approx E_F = k_B T_F = \hbar^2 k_F^2/2m \), since for low temperatures \( \mu \approx E_F + O(T/T_F)^2 \) \[21\]. Correspondingly, we can express \( \mu(r) \) in terms of a local Fermi wave number, \( k_F(r) \), defined as

\[
\frac{\hbar^2 k_F^2(r)}{2m} = E_F - U(r) + gn(r).
\]

Using the result \( n(r) = k_F^2(r)/(6\pi^2) \) for an ideal Fermi gas, then gives an expression for the density

\[
n(r) \left( 1 - \frac{q}{A^{2/3}} n^{1/3}(r) \right)^{3/2} = \frac{1}{A} \left( E_F - U(r) \right)^{3/2},
\]

where \( A \equiv 6\pi^2 (\hbar^2/2m)^{3/2} \). For a dilute gas where \( |\alpha n(0)|^{1/2} \ll 1 \), the Hartree-Fock term in Eqs. (3) and (4) can be neglected in a first approximation. The Fermi energy is then determined by the requirement that the total number of atoms in each spin component be conserved,

\[
N_F = \frac{1}{A} \int_{V(E_F)} \left[ E_F - U(r) \right]^{3/2} d^3r,
\]

where the integration volume \( V(E_F) \) is the volume available to a classical particle with total energy \( E_F \), i.e. \( E_F \geq U(r) \). Equation (5) can easily be solved for \( E_F \) for the case of the power law trap \( 2 \) and gives,

\[
E_F = \left[ \frac{Ae^{1/p}e^{1/3}l^{1/3}q}{(1 + 1/p)(1 + 1/q)\Gamma(5/2)} \left( \frac{N_F}{8abc} \right)^{1/\delta} \right]
\]

where \( \Gamma(x) \) is the Gamma function and

\[
\delta = 3/2 + 1/p + 1/l + 1/q.
\]

Note that the limit \( p, l, q \to \infty \) corresponds to a box with volume \( 8abc \) \[3\]. For a harmonic oscillator, \( p = l = q = 2 \), \( 8abc \) is equal to \( 2^{9/2}e \xi_0 \delta \xi_0 \), where \( \xi_0 = \sqrt{\hbar/m \omega_i} \) is the harmonic oscillator length along the axis of the trap with frequency \( \omega_i \). Therefore, \( \bar{n} = N_F/(8abc) \) can be used to define a characteristic atomic density. Note that, with the exception of the rigid box, \( \bar{n} \) does not correspond to the average density of atoms in the trap. In what follows, we will assume that \( \bar{n} \) is fixed and independent of \( \delta \).

Equation (6) along with Eq. (2) can be used to calculate \( T_C(r) \). It is clear from Eq. (3) that \( k_F(r) \) is a maximum at the center of the trap and hence, the density of states at the local Fermi surface, \( \rho_F(r) = mk_F(r)/(2\pi^2\hbar^2) \), is a maximum there. Consequently, \( T_C(r) \) is largest at \( r = 0 \). Physically, this means that as the temperature is lowered Cooper pairing first occurs at the center of the trap and then spreads to the edges of the trap as the temperature is lowered still further. Therefore at \( r = 0 \) and neglecting the Hartree-Fock term in Eq. (3), the transition temperature is simply,

\[
T_C(0) = a \exp \left( -\frac{\hbar \pi}{\alpha |a| \sqrt{8m E_F}} \right).
\]

Evaluating the critical temperature at the center of the trap has the added benefit that the Thomas-Fermi approximation is expected to be most accurate here.

From Eqs. (3) and (4) we see that for a given value of \( \bar{n}, T_C(0)/T_F \) is an increasing function of \( 1/\delta \). Furthermore, \( T_F = E_F/k_B \) is also an increasing function
of $1/\delta$. Therefore, increasing the confining power of the trap increases not only the ratio of the critical temperature to the Fermi temperature but also the absolute value of the critical temperature. More importantly, from the perspective of current experiments, anharmonic traps for which $p, l, q \geq 3$ will result in higher values of $T_C$ (assuming that all the other terms in Eq. (3) are the same). This is the central result of this paper.

To illustrate the effect of varying the confining power of the trap, we consider the case of $p = l = q = 3$ for the values of 1 through 5 and $\infty$. We choose values of $\epsilon_i$ and $a, b, c$ that correspond to parameters used in current experiments with harmonic traps. In Tables I and II we calculate the Fermi temperature, $T_F$, as well as $T_C(0)$ for $^{6}\text{Li}$ and $^{40}\text{K}$, respectively. For $^{6}\text{Li}$ we consider an isotropic trap with $N_F = 10^5$ and $\epsilon_i = h\omega = h(2\pi \times 100 s^{-1})$, which gives $a = b = c = \sqrt{2\hbar m/\omega} = 5.8 \mu m$ and $\bar{n} = 6.4 \times 10^{13} cm^{-3}$. For $^{40}\text{K}$, we use values from the experiment by DeMarco and Jin [14]. This gives $N_F = 3.5 \times 10^5$ and $\epsilon_1 = \epsilon_2 = h\omega_r = h(2\pi \times 127 s^{-1})$ and $\epsilon_3 = h\omega_z = h(2\pi \times 19.5 s^{-1})$. This gives values for the characteristic lengths of $a \approx b \approx 2 \mu m$ and $c = 5.09 \mu m$ and a characteristic density of $\bar{n} = 2.15 \times 10^{15} cm^{-3}$. Bohn has recently showed that there exists an experimentally accessible Feshbach resonance for two of the hyperfine states of $^{40}\text{K}$ that could be accessed to create a scattering length of $a = -1000a_0 [8]$. We therefore adopt this value for $a$ since in the absence of a Feshbach resonance, the scattering lengths for $^{40}\text{K}$ would result in unreasonably small values of $T_C$.

Tables I and II illustrate the dramatic effect that $p$ has on $T_F$ and $T_C(0)$. For both $^{6}\text{Li}$ and $^{40}\text{K}$, there is a two order of magnitude increase in the Fermi temperature as $p$ is increased from 1 to $\infty$. Furthermore, in going from a harmonic trap ($p = 2$) to a rigid box ($p = \infty$) the Fermi temperatures increase by factors of 24 and 36 for $^{6}\text{Li}$ and $^{40}\text{K}$, respectively. Even more striking is the change in $T_C(0)/T_F$, which increases by three orders of magnitude over the full range of $p$. Altogether, this implies that by increasing the confining power of the trap ($p$) one could, in principle, increase $T_C(0)$ by as much as three orders of magnitude in comparison to a harmonic potential. Note that in our calculation, we take $E_F$ as the Fermi energy of an ideal Fermi gas. Including the attractive atom-atom interaction will decrease $E_F$. However, one must now include the Hartree-Fock term, $g_0(n(0))$, in Eq. (6), which raises the critical temperature. We checked this effect numerically and find that for the parameters used in the paper, including the atom-atom interactions increases $T_C(0)$ by a factor of 2 to 3.

Physically, the increase in $T_F$ and $T_C(0)$ with increasing confining power is a result of the trap being able to confine the atoms to a smaller total volume, which thereby increases the local density. For a rigid box, the atoms are confined to a volume of $8abc$ regardless of the value of $E_F$. For a harmonic trap, the total volume occupied by the gas corresponds to the extent of the wave function for an atom in the highest occupied state with energy $E_F > \hbar \omega$, which has a volume much larger than the volume of the ground state wave function given by $\sim \hbar \omega$. In general, the total volume is given by $V(E_F)$. For the isotropic form of the power law potential, $U(r) = \varepsilon(r/a)^p$, the volume scales as $V(E_F) \sim N_F^{2/p+2}$. This explains the increase in $T_F$ and $T_C$ since, in the limit of a homogenous gas, they depend only on the density of the fermions.

Alternatively, the increase in $T_C$ can also be explained by examining the density of states for an atom with total energy between $\epsilon$ and $\epsilon + de$ [7],

$$\rho(\epsilon) = \frac{1}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_{V(\epsilon)} \sqrt{\epsilon - U(r)} d^3r. \quad (8)$$

Increasing the confining power of the trap reduces the volume of phase space available to an atom with energy $\epsilon$, $V(\epsilon)$, and as a result, the number of states is reduced. In fact, it is easy to show that $\rho(\epsilon) \sim \epsilon^{p-1}$. The reduction in the density of states causes $E_F$ to be increased since the fermions are forced to occupy higher energy states in order to accommodate all $N_F$ atoms. In the Thomas-Fermi approximation, the density of states on the local Fermi surface at the center of the trap is $\rho_F(0) = mk_F(0)/2\pi^2\hbar^2 \sim \sqrt{E_F}$. Consequently, an increase in the Fermi energy, $E_F$, increases the local density of states and therefore $T_C(0)$, as can be seen from Eq. (6).

Both Eq. (1) and Eq. (6) are valid in the limit of a dilute gas, $|a|n(0) \ll 1$. This approximation begins to break down for $p > 5$ when $|a|n(0) \gtrsim 0.2$ for both cases considered here. There are, however, two reasons why this need not be of any great concern. First, the neglect of the Hartree-Fock mean-field in Eq. (6) underestimates the actual density of the gas at the center of the trap because the interactions are attractive. The fact that attractive interactions increase the local density for trapped gases is well known for Bose-Einstein condensates [22] and has been previously noted for fermions [3]. Second, Heiselberg has shown that in the regime of intermediate densities, $|a|n > 1$, $T_C$ becomes a finite fraction of $T_F$ with values of $T_C/T_F \gtrsim 0.1$ [23]. Therefore, for large confining powers, our results underestimate the actual value of $T_C(0)$.

Finally we remark that from an experimental point of view, changing the confining power of the trapping potential appears to be realistic. The generation of power law traps would most easily be accomplished using far-detuned optical dipole traps, which have the added benefit of producing a confining potential that is independent of the hyperfine state of the atoms. In particular, the generation of higher-order Bessel beams with radial intensity profiles proportional to $J_l^2(k_r r)$, where $J_l$ is a Bessel function of integer order and $k_r$ is the radial component of the wavevector, have recently been produced using an axicon [24]. Bessel beams with $l = 1$ and 4 radii for the hollow core of the beam on the order of tens of micrometers were created. For blue-detuned beams
and small $k_rr$, the radial optical dipole potential experienced by the atoms is proportional to $(k_r)^2$. Two perpendicular intersecting Bessel beams with $l > 1$ could be used to create an anharmonic potential. It is worth noting that evaporative cooling of a two-component gas of $^6$Li to temperatures below $T_F$ in an optical trap has recently been demonstrated experimentally \cite{13}.

In conclusion we have examined the effect that a power-law trapping potential has on the BCS transition temperature. We have shown that by increasing the confining power of the trap, one can obtain values of $T_C$ that are several orders of magnitude larger than the corresponding harmonic trap. The origin of the increase is the ability of tighter traps to confine the atoms to a smaller total volume.

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\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$p = l = q$ & $E_F(J)$ & $T_F(K)$ & $T_C(0)/T_F$ & $T_C(0)(K)$ \\
\hline
1 & $1.90 \times 10^{-30}$ & $1.38 \times 10^{-7}$ & $1.62 \times 10^{-4}$ & $2.24 \times 10^{-11}$ \\
2 & $5.56 \times 10^{-30}$ & $4.03 \times 10^{-7}$ & $3.58 \times 10^{-3}$ & $1.44 \times 10^{-9}$ \\
3 & $1.06 \times 10^{-29}$ & $7.69 \times 10^{-7}$ & 0.0115 & $8.84 \times 10^{-9}$ \\
4 & $1.63 \times 10^{-29}$ & $1.18 \times 10^{-6}$ & 0.022 & $2.60 \times 10^{-8}$ \\
5 & $2.20 \times 10^{-29}$ & $1.59 \times 10^{-6}$ & 0.031 & $4.93 \times 10^{-8}$ \\
$\infty$ & $1.36 \times 10^{-28}$ & $9.82 \times 10^{-6}$ & 0.116 & $1.14 \times 10^{-6}$ \\
\hline
\end{tabular}
\caption{Same as Table I for $^{40}$K.}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$p = l = q$ & $E_F(J)$ & $T_F(K)$ & $T_C(0)/T_F$ & $T_C(0)(K)$ \\
\hline
1 & $1.71 \times 10^{-30}$ & $1.24 \times 10^{-7}$ & $3.93 \times 10^{-4}$ & $4.87 \times 10^{-11}$ \\
2 & $5.76 \times 10^{-30}$ & $4.17 \times 10^{-7}$ & $7.8 \times 10^{-3}$ & $3.26 \times 10^{-9}$ \\
3 & $1.19 \times 10^{-29}$ & $8.62 \times 10^{-7}$ & 0.023 & $1.98 \times 10^{-8}$ \\
4 & $1.93 \times 10^{-29}$ & $1.40 \times 10^{-6}$ & 0.04 & $5.54 \times 10^{-8}$ \\
5 & $2.72 \times 10^{-29}$ & $1.97 \times 10^{-6}$ & 0.054 & $1.06 \times 10^{-7}$ \\
$\infty$ & $2.12 \times 10^{-28}$ & $1.54 \times 10^{-5}$ & 0.155 & $2.37 \times 10^{-6}$ \\
\hline
\end{tabular}
\caption{$T_F$ and $T_C(0)$ as functions of the confining power of the isotropic power-law trap, for $^6$Li.}
\end{table}

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