The Quasi-Sure Limit of Convex Combinations of Nonnegative Measurable Functions

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This study shows that, for a sequence of nonnegative valued measurable functions, a sequence of convex combinations converges to a nonnegative function in the quasi-sure sense. This can be used to prove some existence results in multiprobabilities models, and an example application in finance is discussed herein.

1. Introduction

Stochastic models are widely used in practical applications and most are built on a probability space. However, in practice, situations with many possible probabilities occur. A few examples include the uncertain drift or uncertain volatility cases in economic models. These multiprobabilities are called ambiguity, model uncertainty, or Knight uncertainty. Reference [1] emphasized the significant distinction between risk and ambiguity and [2] showed that the distinction between risk and ambiguity is behaviorally significant. To study economic problems while considering ambiguity, mathematical models must be built on a multiprobability space.

Reference [3] first developed the theory of nonlinear $g$-expectation which nontrivially generalizes the classical linear expectation from a probability space to a space with a set of uniformly absolute continuous probabilities. Considering this theory, [4] studied the stochastic differential recursive utility with drift ambiguity. However, many economic and financial problems involve significant volatility uncertainty, which is characterized by a family of nondominated probability measures. Motivated by volatility uncertainty in statistics, risk measures, and super-hedging in finance, [5] introduced a nonlinear expectation, called the G-expectation, which can be regarded as the upper expectation of a specific family of nondominated probability measures. Subsequently, [6, 7] introduced a new type of “G-Brownian motion” and presented the related calculus of Itô’s type. Reference [8] developed a representation of the G-expectation and G-Brownian motion. Reference [9] studied the martingale representation theorem for the G-expectation. Reference [10] determined the properties of hitting times for the G-martingales. References [11, 12] studied the G-BSDEs in G-expectation space.

This paper provides a convergence result in the multiprobability space. In such spaces, under the upper expectation of $E[\cdot]$ defined in (4), the corresponding Fatou’s lemma and dominated convergence theorem no longer hold. Therefore, the convergence result reported in this paper will be useful for future studies.

This paper is organized as follows. In Section 2, we prove that, for a sequence of nonnegative measurable functions, there is a sequence of convex combinations which converges to a nonnegative function in the quasi-sure sense. In Section 3, we use the results of Section 2 to prove an existence result in a multiprobabilities model.

2. Main Results

Let $\Omega$ be a complete separable metric space, $\mathcal{B}(\Omega)$ the Borel $\sigma$-algebra of $\Omega$, and $\mathcal{M}$ the collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$. Let $L^0(\Omega)$ be the space of all $\mathcal{B}(\Omega)$-measurable real functions.

Consider a given subset $\mathcal{P} \subseteq \mathcal{M}$.

Denote

$$c(A) = \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$
Then, $c(\cdot)$ is a Choquet capacity; see [13–15]. A set $A$ is called polar if $c(A) = 0$, and we say a property holds “quasi-surely” (q.s.) if it holds outside a polar set. Let $X_n, X \in L^0(\Omega)$, $n \in \mathbb{N}$. The sequence $X_n$ is said to converge in capacity $c$ to $X$, denoted by $X_n \xrightarrow{c} X$, if
\[
\lim_{n \to \infty} c \left( \|X_n - X\| \geq \varepsilon \right) = 0, \quad \forall \varepsilon > 0.
\]

The sequence $X_n$ is said to mutually converge in capacity $c$, if
\[
\lim_{n,m \to \infty} c \left( \|X_m - X_n\| \geq \varepsilon \right) = 0, \quad \forall \varepsilon > 0.
\]

The upper expectation $E[\cdot]$ of $\mathcal{P}$ is defined as follows (see [16]): for each $X \in L^0(\Omega)$ such that $E^P[X]$ exists for each $P \in \mathcal{P}$,
\[
E[X] = \sup_{P \in \mathcal{P}} E^P[X].
\]

**Lemma 1.** Let $X_n, X \in L^0(\Omega)$, $n \in \mathbb{N}$. Then,

(i) $X_n \to X, n \to \infty$, q.s., if
\[
c \left( \bigcap_{n=1}^{\infty} \left\{ \|X_n - X\| \geq \varepsilon \right\} \right) = 0, \quad \forall \varepsilon > 0.
\]

(ii) $X_n$ is Cauchy q.s., if
\[
c \left( \bigcap_{n=1}^{\infty} \left\{ \|X_n - X_n\| \geq \varepsilon \right\} \right) = 0, \quad \forall \varepsilon > 0.
\]

**Proof.** (i) We choose $\{\varepsilon_k\}$, such that $\varepsilon_k \to 0$ and $\varepsilon_k \to 0$, $k \to \infty$.

\[
c \left( \|X_n - X\| \to 0 \right)^c = c \left( \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ \|X_n - X\| \geq \varepsilon_k \right\} \right) \leq \sum_{k=1}^{\infty} c \left( \bigcap_{n=1}^{\infty} \left\{ \|X_n - X\| \geq \varepsilon_k \right\} \right) = 0.
\]

Thus, we obtain $X_n \to X$ q.s.

(ii) Similar to proof (i), we choose $\{\varepsilon_k\}$, such that $\varepsilon_k \to 0$ and $\varepsilon_k \to 0$, $k \to \infty$.

\[
c \left( \|X_n - X_n\| \to 0 \right)^c = c \left( \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ \|X_n - X_n\| \geq \varepsilon_k \right\} \right) \leq \sum_{k=1}^{\infty} c \left( \bigcap_{n=1}^{\infty} \left\{ \|X_n - X_n\| \geq \varepsilon_k \right\} \right) = 0.
\]

Therefore, $X_n$ is Cauchy q.s. \hfill \Box

**Lemma 2.** Let $X_n, X \in L^0(\Omega)$, $n \in \mathbb{N}$, $\{e_n\}$ be a positive number sequence, and $e_n \to 0$.

(i) If
\[
\sum_{n=1}^{\infty} c \left( \|X_n - X\| \geq e_n \right) < \infty,
\]
then $X_n \to X$ q.s.

(ii) If
\[
\sum_{n=1}^{\infty} c \left( \|X_n - X_n\| \geq e_n \right) < \infty,
\]
then $X_n$ is Cauchy q.s.

**Proof.** (i) For any $\varepsilon > 0$, there exists $n_0$ such that, for all $n \geq n_0$, $e_n < \varepsilon$. Then, we have
\[
c \left( \bigcap_{n=1}^{\infty} \left\{ \|X_n - X\| \geq \varepsilon \right\} \right) \leq \sum_{n=1}^{\infty} c \left( \|X_n - X\| \geq e_n \right)
\]
\[
\leq \sum_{n=1}^{\infty} c \left( \|X_n - X\| \geq e_n \right).
\]

Letting $k \to \infty$ on the right side of the above inequality yields
\[
c \left( \bigcap_{n=1}^{\infty} \left\{ \|X_n - X\| \geq \varepsilon \right\} \right) = 0,
\]
and by Lemma 1, we have $X_n \to X$ q.s.

(ii) Similar to proof (i). \hfill \Box

**Lemma 3.** Let $X_n \in L^0(\Omega)$, $n \in \mathbb{N}$.

(i) If $X_n \xrightarrow{c} X$, then there exists a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k} \to X$ q.s.

(ii) If $X_n, n \in \mathbb{N}$, mutually converges in capacity $c$, then there exist a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ and $X \in L^0(\Omega)$ such that $X_{n_k} \to X$ q.s.

**Proof.** (i) Since $X_n \xrightarrow{c} X$, for each $\varepsilon_k = 1/2^k, k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$, such that
\[
c \left( \left\{ \|X_n - X\| \geq \frac{1}{2^k} \right\} \right) < \frac{1}{2^k}. \quad n \geq n_k.
\]

We can choose $n_k \uparrow \infty, k \to \infty$. Then, we have
\[
\sum_{k=1}^{\infty} c \left( \left\{ \|X_n - X\| \geq \frac{1}{2^k} \right\} \right) < \infty.
\]

By Lemma 2, $X_{n_k} \to X, k \to \infty$ q.s.

(ii) Similar to proof (i), there exists a subsequence $\{X_{n_k}\}$, which is Cauchy q.s. Then, there exists $X \in L^0(\Omega)$ such that $X_{n_k} \to X$ q.s.

If $L^0(\Omega)$ is the space of all $\mathcal{B}(\Omega)$-measurable functions in $R \cup \{+\infty\}$, the sequence $X_n$ mutually converges in capacity $c$, if
\[
\lim_{m,n \to \infty} c \left( \left\{ \|X_m - X_n\| \geq \varepsilon \text{ and } \min (X_m, X_n) \leq e^{-1} \right\} \right)
\]
\[
= 0, \quad \forall \varepsilon > 0.
\]

The results of the above lemmas still hold. \hfill \Box
Let \( \text{conv}(X_n, X_{n+1}, \ldots) \) denote the convex combination of \( X_n, X_{n+1}, \ldots \). Using a similar argument as in Lemma A1.1 of [17], we obtain the following.

**Theorem 4.** Let \((X_n)_{n \geq 1}\) be a sequence of \([0, \infty]\) valued measurable functions. There exists a sequence \( g_n \in \text{conv}(X_n, X_{n+1}, \ldots) \) such that \((g_n)_{n \geq 1}\) converges to a \([0, \infty]\) valued function \( g \) a.s.

**Proof.** Let \( u : R_+ \cup \{0\} \cup \{+\infty\} \to [0, 1] \) be defined as \( u(x) = e^{-x} \). Define \( s_n \) as
\[
s_n = \inf \{ u(g) \mid g \in \text{conv}(X_n, X_{n+1}, \ldots) \} \quad (16)
\]
and choose \( g_n \in \text{conv}(X_n, X_{n+1}, \ldots) \) so that
\[
u(g_n) \leq s_n + \epsilon_n, \quad (17)
\]
where \( 0 \leq \epsilon_n \to 0 \).

It is clear that \( s_n \) is a bounded increasing sequence, so there exists \( s_0 \) such that \( \lim_{n \to \infty} u(g_n) = s_0 \).

On the compact (metrisable) space \([0, \infty)\), \((X_n)_{n \geq 1}\) is Cauchy if and only if for each \( \alpha > 0 \) there is \( n_0 \) so that for all \( n, m \geq n_0 \) we have \( |X_n - X_m| \leq \alpha \) or \( \min(X_n, X_m) \geq \alpha^{-1} \).

From the properties of \( u(x) \), we have that for \( \alpha > 0 \) \( \beta > 0 \) so that \( |x - y| > \alpha \) and \( \min(x, y) \leq \alpha^{-1} \) imply \(((1/2)u(x) + (1/2)u(y) > u(x + y)/2) + \beta \).

For a given \( \alpha > 0 \), we can take \( \beta \) as above and, with the convexity of \( u \), we obtain
\[
\beta 1_{\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\}} + u \left( \frac{g_n + g_m}{2} \right) \leq \frac{1}{2} u(g_n) + \frac{1}{2} u(g_m). \quad (18)
\]
Then,
\[
\beta 1_{\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\}} \\
\geq u \left( \frac{g_n + g_m}{2} \right) - \frac{1}{2} u(g_n) - \frac{1}{2} u(g_m). \quad (19)
\]
Without loss of generality, we can set \( m \geq n \). By (16), \( u(g_n + g_m)/2) \geq s_n \). Taking the expectation about each \( P \in \mathcal{P} \) and letting \( n \to \infty \), we obtain
\[
\frac{1}{2} \lim_{n \to \infty} P \left( \{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\} \right) \geq \frac{1}{2} \lim_{n \to \infty} E^p \left[ u \left( \frac{g_n + g_m}{2} \right) - \frac{1}{2} u(g_n) - \frac{1}{2} u(g_m) \right]. \quad (20)
\]
Thus, we have
\[
\lim_{n \to \infty} P \left( \{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\} \right) = 0. \quad (21)
\]
Therefore,
\[
\lim_{n \to \infty} E \left( \left| g_n - g_m \right| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1} \right) = 0.
\]
That is, \( g_n \) mutually converges in capacity \( c \), and, by Lemma 3, there exist a subsequence \( (g_{n_k})_{k \geq 1} \) and a \([0, \infty]\) valued function \( g \) such that \( g_{n_k} \to g \), a.s.

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**3. Application of Theorem 4**

We set, for \( p > 0 \),
\[
\mathcal{L}^p = \{ X \in L^p(\Omega) : E[|X|^p] < \infty \},
\]
\[
\mathcal{N} = \{ X \in L^0(\Omega) : X = 0, \text{q.s.} \},
\]
and denote \( 0^p = \mathcal{L}^p / \mathcal{N} \).

Consider the following optimization problem in finance:
\[
\inf_{X \in \mathcal{L}^p} E[\|\xi - X\|],
\]
under the constraint \( E[X_T] \leq x \), where \( X_T \) is the terminal wealth of the hedging portfolio at terminal time \( T \), \( L^p_T = \{ X : 0 \leq X \leq \xi \} \); the nonnegative contingent claim which the investor attempts to hedge; \( E[|\xi|^2] < \infty \); \( l \) is the loss function which is an increasing convex function defined on \([0, \infty)\), and \( x \) is the constraint regarding the initial wealth. The corresponding hedging problem in the single probability model was introduced in [18] and is referred to as efficient hedging.

We use the result of the Theorem 4 to prove the existence of the solution of problem (25).

**Theorem 5. There is a solution \( \bar{X}_T \in L^1_T \) to problem (25).**

**Proof.** Let \( L^1_T \) consist of elements of \( L^1_T \) that satisfy \( E[X_T] \leq x \) and let \( (X^n_T) \) be a minimizing sequence for (25) in \( L^1_T \).

By Theorem 4, there exists a sequence \( X^n_T \) belonging to \( \text{conv}(X^n_T, X^{n+1}_T, \ldots) \) such that \( X^n_T \to \bar{X}_T \), a.s. Since \( X^n_T \in L^1_T \), i.e., \( 0 \leq X^n_T \leq \xi \) and \( E[X^n_T] \leq x \), we obtain \( 0 \leq X^n_T, \bar{X}_T \leq \xi \), and \( E[X^n_T] \leq x \), and
\[
E \left[ \bar{X}_T \right] = \sup_{P \in \mathcal{P}} E^p \left[ \lim_{n \to \infty} X^n_T \right] = \sup_{P \in \mathcal{P}} \lim_{n \to \infty} E^p \left[ X^n_T \right]
\]
\[
\leq \sup_{P \in \mathcal{P}} \lim_{n \to \infty} E \left[ X^n_T \right] = \lim_{n \to \infty} E \left[ X^n_T \right] \leq x,
\]
where the second equality sign is a result of the dominated convergence theorem under probability \( P \). Therefore, \( \bar{X}_T \in L^1_T \).
Similarly, since $\overline{X}^n_T \to \overline{X}_T$, q.s., we have

$$\mathbb{E}[l(\xi - \overline{X}_T)] = \sup_{P \in \mathcal{P}} \mathbb{E}^P[l(\xi - \overline{X}_T)]$$

$$= \sup_{P \in \mathcal{P}} \mathbb{E}^P\left[\lim_{n \to \infty} l(\xi - \overline{X}^n_T)\right]$$

$$\leq \sup_{P \in \mathcal{P}} \lim_{n \to \infty} \mathbb{E}^P[l(\xi - \overline{X}^n_T)]$$

$$= \lim_{n \to \infty} \mathbb{E}^P[l(\xi - \overline{X}^n_T)]$$

By the convexity of $\mathbb{E}[\cdot]$ and function $l$ and $\overline{X}^n_T$ belonging to $\text{conv}\{X^m_T, X_T, \ldots\}$, we can conclude that $\mathbb{E}[l(\xi - \overline{X}^n_T)]$ is not larger than the corresponding convex combination of $\mathbb{E}[l(\xi - X^m_T)]$, $m \geq n$. And because $(X^n_T)$ is a minimizing sequence for (25) in $L^2_\xi$,

$$\lim_{n \to \infty} \mathbb{E}[l(\xi - X^n_T)] = \inf_{X_t \in \mathcal{L}^2_\xi} \mathbb{E}[l(\xi - X_T)].$$

Then we have

$$\mathbb{E}[l(\xi - \overline{X}_T)] \leq \inf_{X_t \in \mathcal{L}^2_\xi} \mathbb{E}[l(\xi - X_T)].$$

So

$$\mathbb{E}[l(\xi - \overline{X}_T)] = \inf_{X_t \in \mathcal{L}^2_\xi} \mathbb{E}[l(\xi - X_T)].$$

\[\blacksquare\]

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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