MULTIPLIER IDEALS OF ANALYTICALLY IRREDUCIBLE PLANE CURVES

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Abstract. Let $S$ be a Puiseux series of the germ of an analytically irreducible plane curve $Z$. We construct a set of polynomials $F = \{F_1, \ldots, F_{g-1}\}$ associated to $S$. Using these polynomials as building blocks, we describe a set of generators of multiplier ideals of the form $\mathcal{I}(\alpha Z)$ with $0 < \alpha < 1$ a rational number.

1. Introduction

Let $X$ be a smooth complex variety and $D$ be an effective $\mathbb{Q}$-divisor on $X$. The multiplier ideal associated to $D$ is defined as

$$\mathcal{I}(D) = \varphi_* \mathcal{O}_Y(K_{Y/X} - [\varphi^* D])$$

where $\varphi : Y \to X$ is a log resolution of $(X, D)$ (see [Laz04, Definition 9.2.1]). Multiplier ideals and their vanishing theorems are very useful in many areas, but an explicit formula for multiplier ideals is hard to give except in several special cases, see for example [Bli04], [How11], [Mus06], [Tei07], [Tei08], [Tho14], [Tho16]. On the other hand, singularities of plane curves have been studied for a long time and there is a main way to describe them by using infinitely near points, Puiseux series, and invariants such as characteristic exponents, multiplicity sequences, intersection multiplicities, etc. In this case, there are many results that describe partial information associated to multiplier ideals, like the log-canonical threshold or more generally jumping numbers, by analyzing characteristic exponents or the contribution of exceptional divisors of a resolution of singularity, see [AAD16], [GHM16], [HJ18], [Igu77], [Jär07], [Jär11], [Kuw99], [Nai09], [ST06], [Tuc10]. By analyzing properties and invariants of infinitely near points, such as proximity, free (or satellite) points and multiplicity sequences, and using unique factorization theorems for complete primary valuation ideals in a regular local ring of dimension 2, Järvillehto [Jär11] provided a thorough description of the jumping numbers of multiplier ideals of a simple complete ideal, in particular, of an analytically irreducible plane curve. In [HJ18], the authors generalize the formula to any complete ideal in a regular local ring of dimension 2. Taking a different path, Naie [Nai09] developed independently a complete description of the jumping numbers for analytically irreducible plane curves in terms of the Zariski exponents. From a different perspective, Tucker [Tuc10] presented an algorithm to find jumping numbers for any plane curve. In [AAD16], the authors improved Tucker’s algorithm to compute more efficiently the jumping numbers of any ideal $\mathfrak{a}$ in a two-dimensional local ring $\mathcal{O}_{X, \mathfrak{a}}$ with a rational singularity. Moreover, given a fixed log resolution $\varphi : X' \to X$ of $\mathfrak{a}$ and any jumping number $\lambda$ of $\mathfrak{a}$, this algorithm managed to compute an antinef divisor $D_{\lambda, \varphi}$ such that

$$\mathcal{I}(\mathfrak{a}^\lambda) = \varphi_* \mathcal{O}_{X'}(-D_{\lambda, \varphi})$$

where $\mathcal{I}(\mathfrak{a}^\lambda)$ is the multiplier ideal associated to the ideal $\mathfrak{a}$ and the coefficient $\lambda$ (see [Laz04, Definition 9.2.3]).

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In this paper, we give an explicit formula for multiplier ideals of $\mathbb{Q}$-divisors associated to $Z$, where $X = \mathbb{C}^2$ and $Z$ is an analytically irreducible plane curve singularity at the origin $O$, in terms of a Puiseux series of $Z$. This extends Järvilehto’s results mentioned above. However, some of his calculations are used as an ingredient in our proof.

Fix an irreducible plane curve singularity $O \in Z$ and fix local coordinates $x, y$ around $O$ such that the curve $Z$ is not tangent to the $y$-axis. Denote by $(n; m_1, \ldots, m_g)$ the characteristic sequence of $Z$ (see Definition 2.6). Let $\pi$ be the the standard resolution of $Z$, which is the unique birational morphism that consists of the smallest number of blow-ups with $\pi^*Z$ having normal crossing support. We define a function $\rho : \mathbb{C}\{x, y\} \to \mathbb{Q}_+$ in terms of the standard resolution $\pi$ (see Definition 2.18) so that, for $0 < \alpha < 1$,

$$G \in \mathcal{J}(\alpha Z) \iff \rho(G) > \alpha, \quad \forall G \in \mathbb{C}\{x, y\}.$$  

Our aim is to find a set of generators of these multiplier ideals. An application of classical results (see Proposition 2.20) describes how a partial sum of one of the Puiseux series $S$ of $Z$ determines the standard resolution $\pi$. Another generalization of classical results, which is probably well known to experts, indicates that there is a standard form of the power series $f$ that defines $Z$ (see Proposition 2.26). Based on these two ingredients, we show that there exists a set of standard factors $F = \{F_1, F_2, \ldots, F_g-1\}$ (see Definition 2.23) such that the set

$$\{x^{p_1} y^{p_0} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}} \mid \rho(x^{p_1} y^{p_0} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}) > \alpha\}$$

is a set of generators of $\mathcal{J}(\alpha Z)$. Consequently, we give a full description of multiplier ideals associated to the irreducible plane curve singularity $Z$ as follows.

**Main Theorem.** There exists a formula for the multiplier ideals $\mathcal{J}(\alpha Z)$ with $0 < \alpha < 1$ in terms of a set of standard factors $F_1, \ldots, F_{g-1}$. For the precise statement, see Theorem 3.1.

By describing multiplier ideals completely, we also derive a formula for all the jumping numbers of $Z$ in Corollary 3.2, which recovers [Jär11, Theorem 9.4]. We need neither Zariski’s unique factorization theorem (see [ZS60, Appendix 5, Theorem 3]) nor Lipman’s unique factorization theorem (see [Lip94, Corollary 3.1]), which are used as key ingredients in [Jär11]. A complete calculation using the method in this paper is given in Example 3.4.

After finishing this paper, the author received a manuscript [Dur18] [VD19], which provides a formula for generators of multiplier ideals of plane branches by a different method.

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2. **Preliminary**

2.1. **Puiseux series.** Denote by $\mathbb{C}\langle x \rangle$ the field of formal Laurent series

$$\sum_{i \geq r} c_i x^{\frac{i}{n}}$$
with \( r, n \in \mathbb{Z} \), \( n \geq 1 \) (see [Cas00, §1.2] for the construction of this field).

**Definition 2.1.** [Cas00, §1.2, §2.2]

- For any
  \[ S = \sum_{i \geq r} c_i x^{\frac{i}{n}} \in \mathbb{C}[[x]], \]
we define the order in \( x \) of \( S \) to be
  \[ o_x(S) = \begin{cases} \infty, & \text{if } S = 0; \\ \min \{ i | c_i \neq 0 \} / n, & \text{otherwise} \end{cases} \]

- *Puiseux series* are all such series \( S \) with \( o_x(S) > 0 \).

- For any Puiseux series \( S \), we can write \( S = \sum_{i > 0} c_i x^{\frac{i}{n}} \) such that \( n \) is coprime to \( \gcd \{ i | c_i \neq 0 \} \). Then \( n \) is called the *polydromy order* of \( S \) and denoted as \( n = \nu(S) \).

- For any Puiseux series \( S = \sum_{i > 0} c_i x^{\frac{i}{n}} \), let
  \[ [S]_{< l} := \sum_{0 < i \frac{i}{n} < l} c_i x^{i \frac{i}{n}} \]
  and
  \[ [S]_{\leq l} := \sum_{0 < i \frac{i}{n} \leq l} c_i x^{i \frac{i}{n}}. \]

- For \( n = \nu(S) \) and each \( n \)-th root of unity \( \epsilon \), we call the series
  \[ \sigma_\epsilon(S) = \sum_{i \geq r} \epsilon^i c_i x^{\frac{i}{n}} \]
a *conjugate* of \( S \). Let \( f \in \mathbb{C}[[x, y]] \). We say a Puiseux series \( S \) is a *\( y \)-root* of \( f \) if \( f(x, S) = 0 \). Its conjugates are also Puiseux series. If \( S \) is a \( y \)-root of \( f \), then all conjugates of \( S \) are \( y \)-roots of \( f \), too. The set of all conjugates of \( s \) will be called the *conjugacy class* of \( S \). We set
  \[ f_S = \prod_{i=1}^{\nu(S)} (y - S^i) \in \mathbb{C}[[x]][y], \]
where \( S_1, \ldots, S^{\nu(S)} \) are conjugates in the conjugacy class of \( S \). We know \( f_S \in \mathbb{C}[[x]][y] \) since all its coefficients are invariant by conjugation.

- Let \( O \) be the origin of \( \mathbb{C}^2 \) and fix local coordinates \( x, y \) at \( O \). Let \( f \) be the equation of the germ of a plane curve \( Z \) at \( O \), that is to say, \( Z = (f = 0) \), \( f \in \mathbb{C}\{x, y\} \) and \( f(0, 0) = 0 \). We call the \( y \)-roots of \( f \) the *Puiseux series of the germ* \( Z \).

- We say a Puiseux series \( S \) is *modified* if \( o_x(S) \) is not an integer.

The following lemmas illustrates the relation between irreducible germs of plane curves and conjugacy classes of Puiseux series.

**Lemma 2.2.** [Cas00, Corollary 2.2.4] Let \( Z \) be the germ of a plane curve at \( O \) and fix local coordinates \( x, y \) such that \( Z \) does not contain the germ of the \( y \)-axis. Then \( Z \) is irreducible if and only if all its Puiseux series are in a single conjugacy class.
**Lemma 2.3.** [Cas00, Corollary 1.8.5] Let \( f \in \mathbb{C}\{x,y\} \) with no factor \( x \). Then \( f \) is irreducible if and only if \( f = uFS \), with \( u \in \mathbb{C}\{x,y\} \) a unit and \( S \) a convergent Puiseux series.

2.2. The Newton-Puiseux algorithm.

**Definition 2.4.** Fix a system of orthogonal coordinates \( \alpha, \beta \) of the plane \( \mathbb{R}^2 \). For any element

\[
 f = \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} x^\alpha y^\beta \in \mathbb{C}[[x,y]],
\]

we denote by

\[
 \Delta(f) = \{(\alpha, \beta) \mid c_{\alpha, \beta} \neq 0\}
\]
a discrete set of points and call it the **Newton diagram of** \( f \). We consider the convex hull \( \bar{\Delta}(f) \) of \( \Delta(f) + (\mathbb{R}^+)^2 \) and call the union of compact faces of \( \bar{\Delta}(f) \) the **Newton polygon of** \( f \), denoted by \( N(f) \). Notice that \( N(f) \) may be a single vertex. Suppose the vertices of \( N(f) \) are \( P_i = (\alpha_i, \beta_i) \), \( i = 0, \ldots, k \) with \( \alpha_{i-1} < \alpha_i \) and \( \beta_{i-1} > \beta_i \), \( i = 1, \ldots, k \). Then we define the height of \( N(f) \) to be \( h(N(f)) := \beta_0 \).

We will review the Newton-Puiseux algorithm which provides all \( y \)-roots of a given formal power series \( f \in \mathbb{C}[[x,y]] \). Details of algorithm can be consulted in textbooks about singularities of plane curve (see for example [Cas00, §1.4]).

**The Newton-Puiseux Algorithm.** Fix a series

\[
 f(x,y) = \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta}^{(0)} x^\alpha y^\beta \in \mathbb{C}[[x,y]].
\]

Assume further \( h(N(f)) > 0 \), then the Newton polygon \( N(f) \) ends in the \( \alpha \)-axis.

**Step 1:** If \( N(f) \) ends above the \( \alpha \)-axis. Then we get a \( y \)-root \( S = 0 \) and the algorithm stops here. Otherwise, if \( N(f) \) ends in the \( \alpha \)-axis, then we choose a side \( \Gamma_0 \) of \( N(f) \). Set \( \beta' = \min\{\beta | (\alpha, \beta) \in \Gamma_0\} \) and denote by \((\alpha', \beta')\) the corresponding point on \( \Gamma_0 \). Then we associate a polynomial

\[
 F_{\Gamma_0} = \sum_{(\alpha, \beta) \in \Gamma_0} c_{\alpha, \beta}^{(0)} Z^{\beta - \beta'} \in \mathbb{C}[Z]
\]
to the side \( \Gamma_0 \). Choose a root \( a \) of \( F_{\Gamma_0} \). Write down the equation of \( \Gamma_0 \) as \( na + mb = k \), where \( \gcd(n,m) = 1 \). The coefficients \( n, m \) and the root \( a \) determine a coordinate change

\[
\begin{align*}
x &= x_1^n \\
y &= x_1^m (a + y_1).
\end{align*}
\]

Then we set \( f(x,y) = x_1^k f_1(x_1, y_1) \). Denote

\[
 f_1(x_1, y_1) = \sum_{\alpha_1, \beta_1 \geq 0} c_{\alpha_1, \beta_1}^{(1)} x_1^{\alpha_1} y_1^{\beta_1}.
\]

**Step 2:** Inductively, for \( i \geq 1 \), if the Newton polygon \( N(f_i) \) ends in the \( \alpha_1 \)-axis, then we choose a side \( \Gamma_i \) of \( N(f_i) \). Similar as above we associate a polynomial \( F_{\Gamma_i} \) to the side \( \Gamma_i \) and choose a root \( a_i \) of \( F_{\Gamma_i} \). The equation of \( \Gamma_i \) will be \( n_i \alpha + m_i \beta = k_i \) where \( \gcd(n_i,m_i) = 1 \). The new variables \( x_{i+1}, y_{i+1} \) are given by the rules

\[
\begin{align*}
x_i &= x_{i+1}^{n_i} \\
y_i &= x_{i+1}^{m_i} (a_i + y_{i+1}).
\end{align*}
\]
We denote by \( f_i = x_i^{k_i} f_{i+1} \).

The algorithm will stop whenever \( N(f_i) \) ends above the \( \alpha_i \)-axis. In this case, we will get a \( y \)-root

\[
S = x^m (a + x_1^{m_1} (a_1 + \cdots + x_{i-1}^{m_{i-1}} (a_{i-1} + 0) \cdots)).
\]

Otherwise, the algorithm may keep going on and we can only write down the initial part of the \( y \)-root as

\[
S = ax^m + a_1 x_1^{m_1} + \cdots + a_i x_i^{m_i} + \cdots.
\]

The following lemma will be useful for finding the standard form of the power series of an irreducible plane curve.

**Lemma 2.5.** [Cas00, Proposition 1.5.7] A Puiseux series \( S \) is a \( y \)-root of \( f \) if and only if it is obtained from \( f \) by the Newton-Puiseux algorithm.

### 2.3. The characteristic sequence and the multiplicity sequence.

We recall further invariants associated to Puiseux series.

**Definition 2.6.**
- For any Puiseux series \( S = \sum_{j \geq n} c_j x_j^{d_j} \) that is not an integral power series and with polydromy order \( \nu(S) = n \), it can be written as

\[
S = \sum_{t \leq \frac{d_i}{n}} c_{tn} x^t + a_1 x_1^{\frac{m_1}{n}} + \sum_{\eta \in (d_1)} b_\eta x_\eta + a_2 x_2^{\frac{m_2}{n}} + \cdots + a_g x_g^{\frac{m_g}{n}} + \sum_{j > m_g} c_j x_j^{d_j},
\]

where \( d_0 = n, \ d_i = \gcd(m_i, \ldots, m_1, n), \ m_i \neq (d_i-1) \) and \( d_{g-1} > d_g = 1 \). Then the sequence \((n; m_1, \ldots, m_g)\) is called the characteristic sequence of \( S \). Given any germ of irreducible plane curve singularity \( Z \), we define the characteristic sequence of \( Z \) to be the characteristic sequence of any \( y \)-root of \( Z \). It is well defined since all \( y \)-roots of the irreducible curve \( Z \) are conjugate by Lemma 2.2 and hence have the same characteristic sequence. The characteristic sequence of \( Z \) is independent of the choice of coordinates \( x, y \) around \( O \) as long as \( Z \) is not tangent to the \( y \)-axis.

- For the germ of a plane curve \( Z \), there exists a unique “smallest” resolution of the singularity which is a birational morphism \( \pi \) consisted by the smallest number of blow-ups

\[
\pi : Y = Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_1} Y_0 = \mathbb{C}^2
\]

with \( \pi^* Z \) having normal crossing support. We call this resolution the standard resolution of \( Z \) (see [BK86, Definition, page 498]). The multiplicity of the strict transform of \( Z \) at each center \( q_i \) of the blow-up \( \pi_i \) is denoted as \( M_i \). Then the sequence \((M_1, \ldots, M_k)\) is called the multiplicity sequence of \( Z \).

**Remark 2.7.** Fix local coordinates \( x, y \) at \( O \). Then the irreducible curve singularity \( O \in Z \) is not tangent to \( y \)-axis if and only if the characteristic sequence of \( Z \) \((n; m_1, \ldots, m_g)\) satisfies that \( n < m_1 \).

In this paper, we will always assume \( Z \) is not tangent to the \( y \)-axis, then the characteristic sequence is independent of the choice of coordinates \( x, y \) around \( O \).
The following theorem of Enriques and Chisini [EC24] tells us that the multiplicity sequence of $Z$ and the characteristic sequence of $Z$ determine each other if $Z$ is an irreducible plane curve singularity.

**Theorem 2.8** (Enriques-Chisini Theorem). Given $(n; m_1, \ldots, m_g)$ the characteristic sequence of an irreducible plane curve singularity $Z$, we get the following chain of $g$ Euclidean algorithms:

\[
\begin{cases}
  m_1 = h_{1,0}n + r_{1,1} \\
  n = h_{1,1}r_{1,1} + r_{1,2} \\
  \vdots \\
  r_{1,k_1-1} = h_{1,k_1}r_{1,k_1} \\
  m_2 - m_1 = h_{2,0}r_{1,k_1} + r_{2,1} \\
  r_{1,k_1} = h_{2,1}r_{2,1} + r_{2,2} \\
  \vdots \\
  r_{2,k_2-1} = h_{2,k_2}r_{2,k_2+2} \\
  \vdots \\
  m_g - m_{g-1} = h_{g,0}r_{g-1,k_2} + r_{g,1} \\
  \vdots \\
  r_{g,k_g-1} = h_{g,k_g}r_{g,k_g},
\end{cases}
\]

where we set $n = r_{1,0}$, $r_{i,k_i} = r_{i+1,0}$, and $1 \leq r_{i,j} < r_{i,j-1}$ for any $1 \leq i \leq g$ and $1 \leq j \leq k_i$. Then the multiplicity sequence $(M_1, \ldots, M_k)$ of $Z$ is equal to

\[
\left(\frac{h_{1,0}}{n}, \ldots, \frac{h_{1,1}}{n}, \frac{h_{1,k_1}+h_{2,0}}{r_{1,k_1}}, \ldots, \frac{h_{g,k_g}}{r_{g,k_g}}, \ldots, \frac{r_{g,k_g}}{r_{g,k_g}}\right).
\]

Conversely, given the multiplicity sequence of an irreducible plane curve singularity $Z$, one can recover the characteristic sequence of $Z$ by the chain of Euclidean algorithms.

### 2.4. Proximity matrix and multiplier ideals.

We recall the proximity relation of infinitely near points and its connection to multiplier ideals.

**Definition 2.9.** [Cas00, §3.3] Let $O$ be a point on a smooth surface $Y$. We call points on the exceptional divisor $E$ of blowing up $O$ on $Y$ **points in the first infinitesimal neighborhood of $O$ on $Y$**. Inductively, for $i > 0$, we define the points in the $i$-th infinitesimal neighborhood of $O$ on $Y$ to be the points in the first infinitesimal neighborhood of some point in the $(i - 1)$-th infinitesimal neighborhood of $O$. We call a point a **infinitely near point of $O$ on $Y$** if it is in the $i$-th infinitesimal neighborhood of $O$ on $Y$ for some $i > 0$.

**Definition 2.10.** Let $p, q$ be points equal or infinitely near to $O$. The point $q$ is said to be **proximate to $p$** if and only if it belongs, as an ordinary or infinitely near point, to the exceptional divisor $E^p$ of blowing up the point $p$ and we denote it by $q \succ p$. The **proximity matrix** of a sequence of infinitely near points $q_1, \ldots, q_k$ of $O$ on $\mathbb{C}^2$ reads

\[
P := (p_{i,j})_{k \times k}, \quad \text{where } p_{i,j} = \begin{cases} 
1, & \text{if } i = j; \\
-1, & \text{if } q_i \succ q_j; \\
0, & \text{otherwise.}
\end{cases}
\]
Notation 2.11. Set $\gamma_0 = 0$ and for $1 \leq i \leq g$, set integers $\gamma_i, \tau_{i-1}$ such that
\[
\gamma_i = \gamma_{i-1} + \sum_{j=0}^{k_i} h_{i,j}, \quad \text{and} \quad \tau_{i-1} = \gamma_{i-1} + h_{i,0} + 1
\]
where $h_{i,j}$ are given in (2).

Definition 2.12. [Cas00, §3.6] An infinitely near point $p$ of $O$ is called a free point of $O$ if it is proximate to just one point equal or infinitely near to $O$. Otherwise, $p$ is called a satellite point of $O$.

Remark 2.13. Let
\[
\pi : Y = Y_k \longrightarrow Y_{k-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = \mathbb{C}^2
\]
be the standard resolution of $Z$. Let $q_1, \ldots, q_k$ be the centers of the blow-ups $\pi_1, \ldots, \pi_k$ in the standard resolution of $Z$. Then $q_r$ is a free point of $O$ when $\gamma_j < r \leq \tau_j$ for some $0 \leq j \leq g - 1$, and $q_r$ is a satellite point of $O$ when $\tau_j < r \leq \gamma_{j+1}$ for some $0 \leq j \leq g - 1$. Centers $q_{\gamma_1} < q_{\gamma_2} < \cdots < q_{\gamma_g}$ (or $q_{\gamma_1} < q_{\gamma_1} < \cdots < q_{\gamma_g}$) are exactly all the terminal satellite (or free) points of the point basis $(M_1, \ldots, M_k)$, respectively, defined in [Jär11, Definition 3.1].

Definition 2.14. Let $Z$ be the germ of an irreducible plane curve and let $q_1, \ldots, q_k$ be the centers of the blow-ups $\pi_1, \ldots, \pi_k$ in the standard resolution of $Z$. Denote by $P$ the proximity matrix of infinitely near points $q_1, \ldots, q_j$ of $O$. We call the matrix $P^{-1} = (x_{i,j})_{k \times k}$ the inverse proximity matrix of $Z$ and denote by
\[
X_i = [x_{i,1}, \ldots, x_{i,k}]
\]
the $i$th row of $P^{-1}$.

By elementary computations we get the following formulas:

Lemma 2.15. We have, for any $1 \leq i \leq g$,
\[
X_{\gamma_i} = \frac{h_{i,0}}{d_i}, \frac{n_i}{d_i}, \frac{r_{i,1}}{d_i}, \frac{r_{i,1}}{d_i}, \frac{r_{i,j-1,k_{i-1}}}{d_i}, \frac{r_{i,1,k_{i-1}}}{d_i}, \ldots, \frac{r_{i,1,k_{i-1}}}{d_i}, \frac{r_{i,1,k_{i-1}}}{d_i}, \ldots, \frac{r_{i,1,k_{i-1}}}{d_i}, \frac{r_{i,1,k_{i-1}}}{d_i}, 0, \ldots, 0,
\]
and for any $0 \leq j \leq g - 1$,
\[
X_{\tau_j} = \frac{h_{j,0}}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, \frac{n_j}{d_j}, 0, \ldots, 0.
\]

Notation 2.16. Let $G \in \mathbb{C}[x,y]$. Denote by $C_G$ the divisor defined by $G$ and let
\[
\text{ord}(G) = [\text{mult}_{q_1} C_G, \ldots, \text{mult}_{q_k} C_G],
\]
where we denote by $C_G$ the strict transform of $C$ in $Y_1, \ldots, Y_k$.

The following lemma is probably well known to experts, but we shall give a proof for the convenience of readers.
Lemma 2.17. Let
\[ \pi : Y = Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_1} Y_0 = \mathbb{C}^2 \]
be the standard resolution of \( Z \) and \( P^{-1} \) be the inverse proximity matrix of \( Z \). Then
\[ \pi^*C_G = \tilde{C}_G + (\text{ord}(G) \cdot X_1)E_1 + \cdots + (\text{ord}(G) \cdot X_k)E_k. \]

Proof. By Definition 2.14, the \( i \)th column of \( P^{-1} \) is denoted by \( [x_{1,i}, \ldots, x_{k,i}]^T \). We claim, for any \( 1 \leq i \leq k-1 \),
\begin{equation}
(\pi_k \circ \pi_{k-1} \cdots \pi_{i+1})^*E_i = x_{1,i}E_1 + \cdots + x_{k,i}E_k.
\end{equation}
For \( i = k-1 \), we need to show \( \pi_k^*E_{k-1} = x_{1,k-1}E_1 + \cdots + x_{k,k-1}E_k \). We know the center \( q_k \) is always proximate to \( q_{k-1} \), so \( \pi_k^*E_{k-1} = E_{k-1} + E_k \). On the other hand, by the definition of \( P \) and \( P^{-1} \) and by elementary calculations, we get \( x_{1,k-1} = \cdots = x_{k-2,k-1} = 0 \) and \( x_{k-1,k-1} = x_{k,k-1} = 1 \). Hence we proved the base case \( i = k-1 \). Inductively, suppose for \( j \leq i \leq k-1 \), we have
\begin{equation}
(\pi_k \circ \pi_{k-1} \cdots \pi_{i+1})^*E_i = x_{1,i}E_1 + \cdots + x_{k,i}E_k.
\end{equation}
Let \( [p_{1,j-1}, \ldots, p_{k,j-1}]^T \) be the \((j-1)\)th column of \( P \). Since for \( l > j-1 \),
\[ q_{l,j-1} = \begin{cases} -1, & \text{if } q_l > q_{j-1}; \\ 0, & \text{otherwise}, \end{cases} \]
we know
\[ \pi_l^*E_{j-1} = E_{j-1} - p_{l,j-1}E_j. \]
By the assumptions (4), we have
\[ (\pi_k \circ \cdots \circ \pi_j)^*E_{j-1} = E_{j-1} + (-x_{j,j}p_{j,j-1})E_j + (-x_{j+1,j}p_{j,j-1} - x_{j+1,j+1}p_{j+1,j-1})E_{j+1} + \cdots + (-x_{k,j}p_{k,j-1} - \cdots - x_{k,k}p_{k,j-1})E_k \]
\[ = x_{1,j-1}E_1 + \cdots + x_{k,j-1}E_k, \]
where the last equality is give by \( PP^{-1} = I \). By induction we proved the claim. Then by the definition of \( \text{ord}(G) \), we proved that
\[ \pi^*C_G = \tilde{C}_G + \sum_{i=1}^k \text{mult}_{q_i} \tilde{C}_G(\pi_k \circ \pi_{k-1} \cdots \pi_{i+1})^*E_i \]
\[ = \tilde{C}_G + (\text{ord}(G) \cdot X_1)E_1 + \cdots + (\text{ord}(G) \cdot X_k)E_k. \]
\[ \square \]

Let the standard resolution \( \pi \) be the log resolution \( \varphi \) in (1). Then the multiplier ideal of the pair \((\mathbb{C}^2, cZ)\), for any \( c \in \mathbb{Q}_+ \), is equal to
\[ \mathcal{J}(cZ) = \pi_* \mathcal{O}_Y(K_Y/\mathbb{C}^2 - [\pi^*(cZ)]). \]

Since the singularity of \( Z \) is isolated at \( O \), the ideal sheaf \( \mathcal{J}(\alpha Z) \) is trivial away from \( O \), hence we are only interested in the stalk at \( O \). By Lemma 2.15, for \( 0 < \alpha < 1 \), the ideal can be written as
\begin{equation}
\mathcal{J}(\alpha Z) = \left\{ G \in \mathbb{C}\{x, y\} \mid G^{(i)} + 1 + a^{(i)} > \alpha b^{(i)}, \forall 1 \leq i \leq k \right\}.
\end{equation}
Then we define a function \( \rho \) and by
where \( E \) denote by (sequence is 2.14 curve singularity \( O \)).

Lemma 2.19. We can simplify the formula for Definition 2.18. where

\[
\rho(G) = \min_{1 \leq i \leq k} \left\{ \frac{G^{(i)} + a^{(i)} + 1}{b^{(i)}} \right\}.
\]

The following lemma is a direct consequence of [Jär11, Proposition 7.14] and Lemma 2.17.

Lemma 2.19. We can simplify the formula for \( \rho \) as

\[
(6) \quad \rho(G) = \min_{1 \leq i \leq g} \left\{ \frac{\text{ord}(G) \cdot X_{\gamma_i} + a^{(\gamma_i)} + 1}{b^{(\gamma_i)}} \right\}.
\]

2.5. The position of points and standard factors. Fix a germ of an irreducible plane curve \( O \in Z \) and fix local coordinates \( x, y \). The following result indicates necessary and sufficient conditions on the Puiseux series of an irreducible germ \( Z' \) for it to go through some of the centers of blow-ups in the standard resolution of \( Z \).

Proposition 2.20. [Cas00, Propositions 5.7.1, 5.7.3, 5.7.5] Fix a germ of an irreducible plane curve singularity \( O \in Z \) and local coordinates \( x, y \) at \( O \) such that the Puiseux series of \( Z \) are modified and \( Z \) is not tangent to the \( y \)-axis. Fix one of its Puiseux series \( S \) and the characteristic sequence is \( (n; m_1, \ldots, m_g) \). Let \( d_i = \gcd(n,m_1,\ldots,m_i) \). Denote by \( \pi \) the standard resolution of \( Z \) and by \( q_1, \ldots, q_k \) centers of blow-ups consisting of \( \pi \). Use the same notations as in Notation 2.14. Consider another irreducible germ \( Z' \) and denote by \( q'_1, q'_2, \ldots, q'_k' \) its centers of blow-ups in its standard resolution.

(a) For a free point \( q_j \) with \( \gamma_j < j \leq \tau_i \) for some \( 0 \leq i \leq g - 1 \), \( q'_j = q_j \) if and only if \( Z' \) has a Puiseux series \( S' \) such that the partial sums

\[
[S']_{\leq m_{i+1} + (j-\gamma_i - 1)d_{i+1}} = [S]_{\leq m_{i+1} + (j-\gamma_i - 1)d_{i+1}}.
\]

Furthermore, there is a projective absolute coordinate in the first neighborhood of \( q_j \) such that the satellite point (the point on the \( y \)-axis if \( q_j = O \)) has coordinate \( \infty \) and for any \( a \in \mathbb{C} \), \( Z' \) goes through the point of coordinate \( a \) if and only if

\[
[S']_{\leq m_{i+1} + (j-\gamma_i - 1)d_{i+1}} + ax^{m_{i+1} + (j-\gamma_i - 1)d_{i+1}}.
\]

(b) The center \( q''_i = q_{\gamma_i} \) for some \( i \) and has \( q'_{\gamma_i + 1} \) being a free point of \( Z' \) if and only if \( Z' \) has a Puiseux series \( S' \) such that

\[
[S']_{\leq m_i} + ax^{m_i} + \cdots
\]

for some \( a \in \mathbb{C} - \{0\} \). Moreover, one may choose an absolute projective coordinate \( z \) in the first neighborhood of \( q_{\gamma_i} \) so that, for any \( a \neq 0 \), \( Z' \) goes through the point of coordinate \( z \) in the first neighborhood of \( q_{\gamma_i} \) if and only if \( a^{\frac{m_i}{q_{\gamma_i}}} = z \).
(c) Fix a satellite point $q_j$ with $\tau_i < j \leq \gamma_i + 1$ for some $0 \leq i \leq g - 1$. More precisely, either there exists $1 \leq t < k_{i+1}$ such that
$$j - \tau_i = h_{i+1,1} + \cdots + h_{i+1,t-1} + r$$
where $2 \leq r \leq h_{i+1,t} + 1$ or $t = k_{i+1}$ and $2 \leq r = j + h_{i+1,t} - \gamma_{i+1} \leq h_{i+1,t}$.
Then $q_j' = q_j$ if and only if $Z'$ has a Puiseux series $S'$ of the form
$$S' = [S]_{\frac{m_i}{n_i}} + bx^{\frac{m_i}{n_i}}' + \cdots$$
where $n' = \nu(S') = \frac{\Delta m_i}{d_i}$ for some $d \in \mathbb{N}$ is the polydromy order of $S'$ and $m'$ satisfies the following condition:

If we write $\frac{m_i}{d_i}$ as a continued fraction in the form
$$\frac{m_i}{d_i} = h_i' + \frac{1}{h_{i+1}' + \cdots + \frac{1}{h_{i+t}'}}$$
then either $h_{i+1}' = h_i', \ldots, h_{i+t-1}' = h_{i+1}, h_{i+t}' \geq r - 1$ if $t' > t$, or $h_{i+1}' = h_{i+1}, \ldots, h_{i+t-1}' = h_{i+1}, h_{i+t}' \geq r$ if $t' = t$.

**Corollary 2.21.** Let $Z_1, Z_2$ be two irreducible plane curve singularities and fix coordinate $x, y$ such that $Z_1$ and $Z_2$ are not tangent to $y$-axis in coordinates $x, y$. Assume $Z_1$ and $Z_2$ have the same characteristic sequence $(n; m_1, \ldots, m_g)$. Then $Z_1$ and $Z_2$ have the same standard resolution if and only if there are a $y$-root $S_1$ of $Z_1$ and a $y$-root $S_2$ of $Z_2$ with $[S_1]_{\frac{m_i}{n_i}} = [S_2]_{\frac{m_i}{n_i}}$.

**Lemma 2.22.** [Jär11, Lemma 8.5] If $a \leq b$ are positive integers and $\gcd(a, b) = 1$, then for any positive integer $u$ there exists positive integers $s$ and $t$ such that $sa + tb = ab + u$.

**Proposition-Definition 2.23.** Let $S$ be a modified Puiseux series of an irreducible germ $Z$ with the characteristic sequence $(n; m_1, \ldots, m_g)$. Denote by
$$d_i = \gcd(m_i, \ldots, m_1, n), \quad \forall \ i = 1, \ldots, g$$
and $h_{i,0}$ is given in (2). Then there exist a set of polynomials
$$H = \{H_{i,j} \mid 1 \leq i \leq g - 1, 0 \leq j \leq h_{i+1,0}\}$$
with
$$H_i := \sum_{j=0}^{h_{i+1,0}} H_{i,j}$$
such that, for any $1 \leq i \leq g - 1$, $0 \leq j \leq h_{i+1,0}$, or $i = g, j = 0$, the polynomial
$$F_{i,j} := \left((y^{\frac{d_i}{n_i}} + H_{1,0})^{\frac{d_i}{d_1}} + H_{2,0}^{\frac{d_i}{d_2}} + \cdots + H_{i-1,0}^{\frac{d_i}{d_{i-1}}} + H_{i,0} + H_{i,1} + \cdots + H_{i,j}\right)^{\frac{d_i-1}{d_i}} + H_{i,0}$$
has a $y$-root $[S_i]_{\frac{m_i+jd_i}{n_i}} = [S]_{\frac{m_i+jd_i}{n_i}}$ and the polydromy order $\nu(S_i) = \frac{n_i}{d_i}$, and $H_{i,j}$ is a linear combination of $x^\alpha y^\beta F_{1,0}^{\delta_1} \cdots F_{i-1,0}^{\delta_{i-1}}$ satisfying, for any $1 \leq l \leq g$,
$$\text{ord}(x^\alpha y^\beta F_{1,0}^{\delta_1} \cdots F_{i-1,0}^{\delta_{i-1}}) \cdot X_{\gamma_l} \geq X_{\gamma_{l+j}} \cdot X_{\gamma_l}.$$
For $1 \leq i \leq g - 1$, denote by $F_i = F_{i,h_{i+1,0}}$. Then we call the set $F = \{F_1, \ldots, F_{g-1}\}$ a set of standard factors of $S$.

Proof. By Lemma 2.22, there exist positive integers $s, t$ such that $s \frac{m_1}{d_1} + t \frac{m_1}{d_1} = \frac{nm_1}{d_1} + 1$. By running the Newton-Puiseux algorithm, we get

$$y^{\frac{m_1}{d_1}} - a_1^{\frac{m_1}{d_1}} x^{\frac{m_1}{d_1}} = b_{m_1+d_1} x^s (y/a_1)^t$$

has a $y$-root with a partial sum

$$a_1 x^{\frac{m_1}{d_1}} + b_{m_1+d_1} x^{\frac{m_1+d_1}{d_1}}.$$

Therefore

$$H_{1,0} = -a_1^{\frac{m_1}{d_1}} x^{\frac{m_1}{d_1}}$$

and $H_{1,1} = -b_{m_1+d_1} x^s (y/a_1)^t$ satisfied the desired conditions. Inductively, for $1 \leq q \leq h_{2,0}$, suppose we have constructed

$$F_{1,q} = y^{\frac{m_1}{d_1}} - a_1^{\frac{m_1}{d_1}} x^{\frac{m_1}{d_1}} + H_{1,1} + \cdots + H_{1,q}$$

which has a $y$-root with a partial sum $[S] \leq \frac{m_1+q+1}{n}$, and suppose that after taking coordinate changes

$$\begin{cases} 
  x = x_1^{n/d_1} = x_2^{n/d_1} = \cdots = x_q^{n/d_1}; \\
  y = x_1^{m_1/d_1} (y_1 + a_1) = \cdots = x_q^{m_1/d_1} (y_q + b_{m_1+(q+1)d_1} x^{q+1} + \cdots + b_{m_1+d_1} x^{q+1} + a_1),
\end{cases}$$

we can write

$$y^{\frac{m_1}{d_1}} - a_1^{\frac{m_1}{d_1}} x^{\frac{m_1}{d_1}} + H_{1,1} + \cdots + H_{1,q} = x_q^{\frac{m_1}{d_1}+q} \left( \lambda_1 y_q + \sum_{\alpha+\beta>1} c^{(q)}_{\alpha,\beta} x^{\alpha} y^{\beta+1}_{q+1} \right).$$

By Lemma 2.22 we can construct $H_{1,q+1}$, a linear combination of $x^s y^t$ with suitable coefficients such that , after taking coordinate changes,

$$\begin{cases} 
  x_{q+1} = x_{q+2}, \\
  y_{q+1} = x_{q+1} (y_{q+2} + b_{m_1+(q+1)d_1}),
\end{cases}$$

we have $F_{1,q+1} = F_{1,q} + H_{1,q+1}$ is of the form

$$F_{1,q+1} = x_{q+2}^{\frac{m_1}{d_1}+q} \left( \lambda_1 y_{q+2} + \sum_{\alpha+\beta>1} c_{(q+1)}^{(q)} x^{\alpha} y^{\beta+2}_{q+2} \right).$$

By the Newton-Puiseux Algorithm and by Lemma 2.2, we know that $F_{1,q+1}$ has a $y$-root with a partial sum

$$a_1 x^{\frac{m_1}{n}} + b_{m_1+d_1} x^{\frac{m_1+d_1}{n}} + \cdots + b_{m_1+(q+1)d_1} x^{\frac{m_1+(q+1)d_1}{n}}.$$

By induction, we proved that there exists polynomials $H_{1,0}, \ldots, H_{1,h_{2,0}}$ satisfying the desired conditions. Notice that $\text{ord}(x^s y^t) \cdot X_{\gamma_l} = \frac{d_1}{d_1} s + \frac{m_1}{d_1} t$, so we obtain that $H_{1,j}$ is a linear combination of $x^s y^t$ satisfying, for any $1 \leq l \leq g$,

$$\text{ord}(x^s y^t) \cdot X_{\gamma_l} \geq \frac{d_1}{d_1} \frac{nm_1}{d_1} + j \geq X_{\gamma_l+j} \cdot X_{\gamma_l}.$$

Now we want to show there exists $H_{2,0}$ such that $F_1^{d_2} + H_{2,0}$ has a $y$-root with a partial sum

$$a_1 x^{\frac{m_1}{n}} + \sum_{\eta \in (d_1)} b_\eta x^{\frac{m_1}{n}} + a_2 x^{\frac{m_2}{n}},$$
and $H_{2,0}$ is a linear combination of $x^\alpha y^\beta F_{1}^{s_1}$ satisfying,

$$\text{ord}(x^\alpha y^\beta F_{1}^{s_1}) \cdot X_{\gamma_1} \geq X_{\gamma_2} \cdot X_{\gamma_1}.$$ 

By the previous arguments, we have

$$F_1 = y^{\frac{n}{d_1}} + H_1 = x_{h_{2,0}+1}^{n m_1 + h_{2,0}} \left( \lambda_1 y_{h_{2,0}+1} + \sum_{\alpha + \beta > 1} c^{(h_{2,0})}_{\alpha, \beta} x_{h_{2,0}+1}^{\alpha} y_{h_{2,0}+1}^{\beta} \right).$$

By Lemma 2.22, for any positive integer $w$, there exists a positive integer $s, t, u$ such that

$$\text{ord}((s, t, u) F^{s_1}_{1, h_{2,0}} + w) = \sum_{\alpha + \beta > 1} c^{(h_{2,0})}_{\alpha, \beta} x_{h_{2,0}+1}^{\alpha} y_{h_{2,0}+1}^{\beta} + w.$$

So, by setting $r = m_2 - m_1 - h_{2,0}d_1$, we can construct a linear combination $H^{(1)}_{2,0}$ of $s^t y^u$ such that

$$F_1^{d_1} + H^{(1)}_{2,0} = x_{h_{2,0}+1}^{n m_1 + h_{2,0}} \left( \lambda_1 y_{h_{2,0}+1} + a_2 \lambda_1 x_{h_{2,0}+1}^{r} + \sum_{\alpha + \beta > 1} c^{(h_{2,0}+1)}_{\alpha, \beta} x_{h_{2,0}+1}^{\alpha} \right).$$

For any positive integer $w$, by Lemma 2.22, there exist positive integers $u, v$ such that

$$u\frac{m_2 - m_1}{d_2} + v\frac{d_1}{d_2} = \frac{m_2 - m_1}{d_2} \cdot \frac{d_1}{d_2} + t.$$ 

Choose some $0 < u < \frac{d_1}{d_2}$. Then

(7) $$u\left(\frac{n m_1 + m_2 - m_1}{d_1 d_2} \right) + \left(\frac{d_1}{d_2} - u - 1\right) \frac{n m_1}{d_1 d_2} + \frac{m_1}{d_1 d_2} + v \frac{d_1}{d_2} = \frac{nm_1 + (m_2 - m_1)d_1}{d_2} + w.$$ 

By Lemma 2.22 again, there exist positive integers $s, t$ such that

(8) $$s \frac{n}{d_2} + t \frac{m_1}{d_2} = \frac{nm_1 + (m_2 - m_1) d_1}{d_2} + v \frac{d_1}{d_2}.$$ 

Combining (7) and (8) we obtain

$$u\left(\frac{n m_1 + m_2 - m_1}{d_1 d_2} \right) + \left(\frac{d_1}{d_2} - u - 1\right) \frac{m_1}{d_1 d_2} + s \frac{n}{d_2} + t \frac{m_1}{d_2} = \frac{nm_1 + (m_2 - m_1)d_1}{d_2} + w.$$ 

Set $s' = \left(\frac{d_1}{d_2} - u - 1\right) \frac{m_1}{d_1 d_2} + s$, then we obtain positive integers $s', t, u$ such that

$$\frac{n}{d_2} \frac{s'}{d_2} + \frac{m_1}{d_2} \frac{t}{d_2} + \frac{nm_1 + d_1(m_2 - m_1)}{d_1 d_2} u = \frac{nm_1 + d_1(m_2 - m_1)}{d_2} + w.$$ 

Thus we can construct $H^{(2)}_{2,0}$, a linear combination of $x^\alpha y^\beta F_{1}^{s_1}$ such that, after coordinate changes,

$$x_{h_{2,0}+1} = x_{h_{2,0}+1}^{1 + \frac{d_1}{d_2}},$$

$$y_{h_{2,0}+1} = y_{h_{2,0}+1}^{1 + \frac{d_1}{d_2}}.$$ 

we have

$$F_1^{d_1} + H^{(1)}_{2,0} + H^{(2)}_{2,0} = x_{2,0}^{d_1} \left(\lambda_2 y_{2,0} + \sum_{\alpha + \beta > 1} c^{(2,0)}_{\alpha, \beta} x_{2,0}^{\alpha} y_{2,0}^{\beta} \right).$$

By elementary calculations, we get, for $1 \leq l \leq g$,

$$\text{ord}(x^\alpha y^\beta F_{1, h_{2,0}}^{s_1}) \cdot X_{\gamma_l} \geq X_{\gamma_2} \cdot X_{\gamma_1}.$$
Following very similar arguments as above, we showed that there exists $F_1, \ldots, F_{g-1}$ that satisfied the desired conditions. \hfill \square

**Example 2.24.** Let $Z = (y^6 - 6x^2y^5 + 9x^4y^4 - 2x^5y^3 + 6x^7y^2 + x^{10} - 9x^{11} = 0)$ be the germ of a plane curve. After running the Newton-Puiseux Algorithm, we know that $Z$ is irreducible. The characteristic sequence of $Z$ is $(n; m_1, m_2) = (6, 10, 13)$ and it has a $y$-root $S$ with

$$[S]_{x>4} = x^3 + x^2 + x^{14}. $$

We may construct a set of standard factors of $S$ to be

$$F = \{F_1 = y^3 - x^5 - 3x^2y^2\}. $$

The choice of standard factors is not unique. For example, we can choose $F_1 = y^3 - x^5 - 3x^2y^2 + ax^6y^t$ with $3s + 5t > 16$ and $a \in \mathbb{C}$.

To compute $\text{ord}(x^{ps}y^{p_0}F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}) \cdot X_{\gamma_1}$ and $\rho(x^{ps}y^{p_0}F_1^{p_1} \cdots F_{g-1}^{p_{g-1}})$ efficiently, we shall use the following formulas:

**Lemma 2.25.** Denote by $(n; m_1, \ldots, m_g)$ the characteristic sequence of $Z$ and by $(M_1, \ldots, M_k)$ the multiplicity sequence of $Z$. Let $\pi : Y \to Y_0 = \mathbb{C}^2$ be the standard resolution of $Z$ and set

$$K_{Y/Y_0} = a^{(1)}E_1 + \cdots + a^{(k)}E_k, \quad \gamma \cdot Z = \tilde{Z} + b^{(1)}E_1 + \cdots + b^{(k)}E_k. $$

Then, for any $1 \leq i \leq g$ and $1 \leq j \leq g - 1$, we have

$$a^{(\gamma_i)} + 1 = \frac{m_i + n}{d_i}, \quad b^{(\gamma_i)} = \frac{M_1^2 + \cdots + M_i^2}{d_i}, $$

and for any set of standard factors $F = \{F_1, \ldots, F_{g-1}\}$ we have

$$\text{ord}(F_j) \cdot X_{\gamma_i} = \begin{cases} \frac{b^{(\gamma_i)}}{d_j} = \frac{M_1^2 + \cdots + M_i^2}{a_id_j} & \text{when } i \leq j; \\ \frac{b^{(\gamma_i)}}{d_i} + \frac{m_{i+1} - m_j}{d_j} = \frac{M_1^2 + \cdots + M_{i+1}^2}{a_id_j} & \text{when } i > j. \end{cases} $$

**Proof.** Since $K_{Y_i} - \pi_i^*K_{Y_{i-1}} = E_i$, by (3), we get

$$a^{(\gamma_i)} = [1, \ldots, 1] \cdot X_{\gamma_i}. $$

By Lemma 2.17, we have

$$b^{(\gamma_i)} = [M_1, \ldots, M_k] \cdot X_{\gamma_i}. $$

Since $F_1$ has a $y$-root $S_i$ with $[S_i]_{x>\frac{n_{i+1} + h_{i+1,0}d_i}{n}} = [S]_{x>\frac{n_{i+1} + h_{i+1,0}d_i}{n}}$ and the polydromy order is $\frac{n_i + h_{i+1,0}d_i}{n}$, by Proposition 2.20, we get $\text{ord}(F_1) = X_{\gamma_i}$. Then, using formulas in Lemma 2.15 and (10), we obtain the desired results. \hfill \square

### 2.6. Standard form of the power series of an irreducible plane curve.

When the initial term is fixed, i.e., for any Puiseux series starting with the term $a_1x^{d}$, with the polydromy order equal to $n$, the polynomial solved by this Puiseux series is, up to multiplicative constant, of the form

$$(y^{n'} - a_1^{n'}x^{m'})^d + \sum_{\alpha + \beta > nm} c_{\alpha, \beta}x^\alpha y^\beta$$

where $d = \gcd(n, m)$ and $n' = \frac{n}{d}$, $m'_1 = \frac{m}{d}$ (see for example [Cas00, Proposition 2.2.5]). The following proposition is a generalization of this result, giving a standard form when we fix a partial sum of a Puiseux series of $Z$. The idea of the proof is similar.
Proposition 2.26. Let $S$ be a modified Puiseux series of an irreducible germ $Z$ with the characteristic sequence $(n;m_1,\ldots,m_g)$. Let $F = \{F_1,\ldots,F_{g-1}\}$ be a set of standard factors of $S$ and let $F_{i,j}$ be the polynomials as defined in Proposition 2.23. Denote by
\[ d_i = \gcd(m_i,\ldots,m_1,n), \quad \forall \ i = 1,\ldots,g. \]
Then, up to a multiplicative unit factor in $C\{x,y\}$, for any $1 \leq i \leq g - 1$, $0 \leq j \leq h_{i+1,0}$, or $i = g$, $j = 0$, the power series $f$ of $Z$ is of the form
\[ F_{i,j}^{d_i} + F_{\text{ext}}^{(i,j)}, \]
where the power series $F_{\text{ext}}^{(i,j)}$ is a linear combination of $x^\alpha y^\beta F_1^{\delta_1} F_2^{\delta_2} \cdots F_{g-1}^{\delta_{g-1}}$ satisfying, for $1 \leq l \leq g$,
\[ \text{ord}(x^\alpha y^\beta F_1^{\delta_1} F_2^{\delta_2} \cdots F_{g-1}^{\delta_{g-1}}) \cdot X_\alpha > \text{ord}(F_{i,j}^{d_i}) \cdot X_\alpha. \]

Proof. Let $f$ be a power series of $Z$. By Lemma 2.5 the $y$-root $S$ is obtained from $f$ by the Newton-Puiseux algorithm. Since the term with the smallest fractional power is $a_1 x^{\frac{m_1}{n}}$, we know from Step 0 of the Newton-Puiseux algorithm (see (2.2)) that $N(f)$ has a single side $\Gamma_0$ (because $Z$ is irreducible) and the slope of $\Gamma_0$ is $-\frac{m_1}{n}$. By Lemma 2.2, any root of $F_{\Gamma_0}$ is equal to $e^{m_1}a_1$ for some $e$ such that $e^n = 1$, the first coefficient of one of the $y$-root $\sigma_e(s)$. Since $d_1 = \gcd(n,m_1)$, we know all roots of $F_{\Gamma_0}$ are of the form $e^{\alpha_1} a_1$ where $e^{\frac{\alpha_1}{n}} = 1$. We know the degree of $F_{\Gamma_1}$ is $h(N(f)) = \nu(s) = n$. This implies that the equation of $\Gamma_0$ is
\[ \frac{n}{d_1} \alpha + \frac{m_1}{d_1} \beta = \frac{nm_1}{d_1}, \]
and
\[ F_{\Gamma_0} = (Z^{\frac{m_1}{d_1}} - a_1^{\frac{\alpha_1}{n}})^{d_1} \]
up to a multiplicative constant. From this equation we know $f$ is of the form
\[ (y^{\frac{n}{d_1}} - \frac{n}{d_1} x^{\frac{m_1}{d_1}})^{d_1} + \sum_{\nu + \mu_1 \beta > \mu_1 n} c_{\alpha,\beta} x^{x\alpha} y^{\beta}. \tag{9} \]
Now we want use the similar argument on the next nonzero term in the $y$-root $S$ to get more details of the standard from of $f$. Following the algorithm, the first coordinate change in the process of producing the $y$-root $s$ in the algorithm is given as
\[ \begin{cases} \begin{align*} x &= x_1^{\frac{m_1}{d_1}} \\ y &= x_1^{\frac{m_1}{d_1}} (y_1 + a_1). \end{align*} \end{cases} \tag{10} \]
We get the form (9) is equal to
\[ x_1^{-\frac{nm_1}{d_1}} \left[ ((y_1 + a_1)^{\frac{n}{d_1}} - \frac{n}{d_1} a_1)^{d_1} + \sum_{\nu + \mu_1 \beta > \mu_1 n} c_{\alpha,\beta} x_1^{\frac{nm_1 + \beta - \mu_1 n}{d_1}} (y_1 + a_1)^{\beta} \right]. \]
Following the Newton-Puiseux algorithm, we set $f_1 = x_1^{-\frac{nm_1}{d_1}} f$. Then the degree of $f_1$ is $d_1$ and $h(N(f_1)) = d_1$. Then there are two cases.

Case 1: The next nonzero fractional power term is $b_{m_1 + rd_1} x^{\frac{m_1 + rd_1}{n}}$, with the integer $1 \leq r \leq h_{2,0}$. Then from Step 1 of the algorithm we know that $N(f_1)$ has a single side $\Gamma_1$ and the slope of $\Gamma_1$ is $-\frac{d_1}{m_1 + rd_1} = -\frac{1}{r}$. By Lemma 2.5, any root of $F_{\Gamma_1}$ is equal to $e^{\frac{m_1 + rd_1}{n}} b_{m_1 + rd_1}$ where $e^n = 1$ and $e^{m_1} = 1$.  

(since the first coefficient is a fixed number \(a_1\)). Hence we get \(b_{m_1 + rd_1}\) is the only root of \(F_{\Gamma_1}\) and the degree of \(F_{\Gamma_1}\) is \(d_1 = h(N(f_1))\). Therefore the equation of \(\Gamma_1\) is

\[ \alpha_1 + r\beta_1 = r, \]

and

\[ F_{\Gamma_1} = (Z - b_{m_1 + rd_1})^{d_1}. \]

Therefore the series \(f_1\) of variables \(x_1, y_1\), is of the form

\[ (y_1 - b_{m_1 + rd_1}^{-1})^{d_1} + \sum_{\alpha_1 + r\beta_1 > rd_1} c^{(1)}_{\alpha_1, \beta_1} x_1^{\alpha_1} y_1^{\beta_1}. \]

We claim that \(f\) is of the form

\[ (y_n^{\alpha_1} - a_n^{\alpha_1} x_1^{\alpha_1} + H_{1,r})^{d_1} + \sum_{\alpha_1 + r\beta_1 > rd_1} c^{(1)}_{\alpha_1, \beta_1} x_1^{\alpha_1} y_1^{\beta_1}. \]

where \(y_n^{\alpha_1} - a_n^{\alpha_1} x_1^{\alpha_1} + H_{1,r}\) has a \(y\)-root with initial terms \(a_1 x^{\frac{m_1}{n}} + b_{m_1 + rd_1} x^{\frac{m_1 + rd_1}{n}}\) and with polydromy order \(\frac{\beta_1}{d_1}\). Indeed, up to a multiplicative constant,

\[ D := f - (y_n^{\alpha_1} - a_n^{\alpha_1} x_1^{\alpha_1} + H_{1,r})^{d_1} = x_1^{\frac{m_1}{n}} \cdot \sum_{\alpha_1 + r\beta_1 > rd_1} c^{(1)}_{\alpha_1, \beta_1} x_1^{\alpha_1} y_1^{\beta_1}. \]

Denote by \(\tilde{N}\) the Newton polygon of the difference \(D\), and pick a vertex \(x_1^{\alpha_1} y_1^{\beta_1}\) of \(\tilde{N}\). By (10) we observe that, the term \(c^{(1)}_{\alpha_1, \beta_1} x_1^{\alpha_1} y_1^{\beta_1}\) must be provided by polynomials, up to multiplicative constants, of the form \(x^{\alpha_1} y^{\beta_1} (y_n^{\alpha_1} - a_n^{\alpha_1} x_1^{\alpha_1})^t\) such that

\[ \frac{n}{d_1} \alpha + \frac{m_1}{d_1} \beta + \frac{m_1 n}{d_1} t = s. \]

Combining with the condition \(s - \frac{m_1 n}{d_1} t > rd_1\), we obtain

\[ n\alpha + m_1 \beta + \frac{m_1 n + rd_1}{d_1} t > m_1 n + rd_1. \]

By substracting a suitable linear combination of polynomials of such form from the difference \(D\), all terms supported on \(\tilde{N}\) are eliminated and we can do the similar argument for the new difference. This process will terminate in finite steps since there are finitely many \((\alpha, \beta)\) such that

\[ n\alpha + m_1 \beta \leq m_1 n + rd_1^2. \]

Thus the difference \(D\) is of the form

\[ \sum_{n\alpha + m_1 \beta + \frac{m_1 n + rd_1}{d_1} t > m_1 n + rd_1^2} c^{(1)}_{\alpha, \beta, \delta_1} x^{\alpha_1} y^{\beta_1} (y_n^{\alpha_1} - a_n^{\alpha_1} x_1^{\alpha_1})^{\delta_1}. \]

**Case 2:** Suppose that the next nonzero term of \(S\) is \(a_2 x^{\frac{m_2}{n}}\). Then \(N(f_1)\) has a single side \(\Gamma_1\) with the slope \(-\frac{d_1}{m_2 - m_1}\), and \((d_1, m_2 - m_1) = d_2\). This implies that

\[ F_{\Gamma_1} = (Z - \frac{d_1}{a_2^{m_2}})^{d_2}. \]

We get the series \(f_1\) is of the form

\[ (y_1^{\frac{d_1}{n}} - a_2^{\frac{d_1}{n}} x_1^{\frac{m_2 - m_1}{d_2}})^{d_2} + \sum_{d_1 \alpha_1 + (m_2 - m_1) \beta_1} c^{(1)}_{\alpha_1, \beta_1} x_1^{\alpha_1} y_1^{\beta_1}. \]
By a very similar argument as in Case 1, the series $f$ is of the form
\[
\left((y^{\frac{n}{d_2}} - a_{\frac{n}{d_1}, \frac{m_1}{d_1}} \frac{d_1}{d_2} + H_{2,0})^{d_2} + \sum_{\alpha + m_1 \beta + F_i^{(\gamma_2)} \delta_1 > d_2 F_1^{(\gamma_2)}} c_{\alpha, \beta, \delta_1, \delta_2}^{(1)} x^\alpha y^\beta (y^{\frac{n}{d_2}} - a_{\frac{n}{d_1}, \frac{m_1}{d_1}} \frac{d_1}{d_2})^{\delta_1}
\]
where $(y^{\frac{n}{d_1}} - a_{\frac{n}{d_1}, \frac{m_1}{d_1}} \frac{d_1}{d_2})^{d_2} + H_{2,0}$ has a $y$-root with initial terms $a_1 x^{\frac{m_1}{d_1}} + a_2 x^{\frac{m_2}{d_1}}$ and with polydromy order $\frac{n}{d_2}$, and $H_{2,0}$ is a linear combination of $x^\alpha y^\beta$ satisfying $ns + m_1 t = \frac{m_1 n + d_1 (m_2 - m_1)}{d_2} = \frac{d_1}{d_2} F_1^{(\gamma_2)}$.

By repeating arguments very similar to the ones in Case 1 and Case 2, we conclude that $f$ is of the form
\[
F_1^{d_1} + F_{ext}^{(1)}
\]
where $F_{ext}^{(1)}$ is a linear combination of $x^\alpha y^\beta F_1^{d_1}$ satisfying
\[
\alpha + m_1 \beta + (\text{ord}(F_1) \cdot X_{g_1}) \delta_1 > d_1 (\text{ord}(F_1) \cdot X_{g_1}).
\]
Inductively, by very similar arguments, for any $1 \leq i \leq g - 1$, we obtain that $f$ is of the form
\[
F_{i,j}^{d_i} + F_{ext}^{(i,j)}
\]
where $F_{ext}^{(i,j)}$ is a linear combination of $x^\alpha y^\beta F_1^{d_1} \cdots F_{i-1}^{d_{i-1}}$ satisfying, for $1 \leq l \leq g$,
\[
\text{ord}(x^\alpha y^\beta F_1^{d_1} \cdots F_{i-1}^{d_{i-1}} \cdot X_{g_1}) > \text{ord}(F_{i,j}^{d_i}) \cdot X_{g_1}.
\]

**Definition 2.27.** Let $S$ be a modified Puiseux series of an irreducible germ $Z$. Fixing a set of standard factors $F = \{F_1, \ldots, F_{g-1}\}$ of $S$ (see Definition 2.23), for any $c \in \mathbb{Q}$, we define an ideal $I_S^{\leq c}$ generated by all terms $x^{p_x, y^{p_y}} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}$ satisfying the condition
\[
\rho(x^{p_x, y^{p_y}} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}) \geq c,
\]
and an ideal $I_S^{\prec c}$ generated by all terms $x^{p_x, y^{p_y}} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}$ satisfying the condition
\[
\rho(x^{p_x, y^{p_y}} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}) > c.
\]

**Remark 2.28.** For any $c \in \mathbb{Q}$, the ideal $I_S^{\leq c}$ (or $I_S^{\prec c}$) is independent of the choice of the set of standard factors of $S$. Indeed, let $F = \{F_1, \ldots, F_{g-1}\}$ and $F' = \{F'_1, \ldots, F'_{g-1}\}$ be two sets of standard factors of $S$. We have
\[
\rho(x^{p_x, y^{p_y}} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}) = \rho(x^{p_x, y^{p_y}} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}})
\]
for any powers $p_x, p_y, p_1, \ldots, p_{g-1}$. Since $F_i$ has a $y$-root $S_i$ with $[S_i]_{m_i} = [S]_{m_i}$ and with the characteristic sequence $(\frac{m_1}{d_1}, \ldots, \frac{m_{g-1}}{d_1})$, by Proposition 2.23, we can choose $\{F_1, \ldots, F_{g-1}\}$ to be a set of standard factors of $S_i$. By Proposition 2.26, the difference $F_i - F'_i$ can be written as a linear combination of $x^\alpha y^\beta F_1^{d_1} \cdots F_{i-1}^{d_{i-1}}$ satisfying, for any $1 \leq l \leq g$,
\[
\text{ord}(x^\alpha y^\beta F_1^{d_1} \cdots F_{i-1}^{d_{i-1}} \cdot X_{g_1}) > \text{ord}(F_i) \cdot X_{g_1} = \text{ord}(F_i) \cdot X_{g_1}
\]
and hence $\rho(F_i - F'_i) > \rho(F_i)$. Therefore the ideal $I_S^{\leq c}$ is well defined for any $c \in \mathbb{Q}$.

---

1For the convenience of calculations in Remark 3.3, we denote by $p_0$ the power of $y$. 
3. The main theorem

3.1. Main result. We can now state the main result of the paper. Recall that the standard factors $F_1, \ldots, F_{g-1}$ are defined in Definition 2.23 and, for any $c \in \mathbb{Q}$, the ideal $\mathcal{O}_S^\geq c$ is defined in Definition 2.27.

**Theorem 3.1.** Let $S$ be a modified Puiseux series of an irreducible germ $Z$ which is not tangent to the $y$-axis. Let $F = \{F_1, \ldots, F_{g-1}\}$ be a set of standard factors of $S$. Then for $0 < \alpha < 1$, we have $\mathfrak{J}(\alpha Z) = \mathcal{O}_S^\geq \alpha$.

**Proof.** We start by proving that

$$G \in \mathcal{O}_S^\geq \rho(G), \forall G \in \mathbb{C}\{x, y\}.$$

First, let $G \in \mathbb{C}\{x, y\}$ be an irreducible series. By Lemma 2.2 and Lemma 2.3, it defines a germ of an irreducible plane curve $C_G$. Let

$$\pi : Y = Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_1} Y_0 = \mathbb{C}^2$$

be the standard resolution of $Z$ and $q_1, \ldots, q_k$ be the centers of the blow-ups $\pi_1, \ldots, \pi_k$, and similarly, let

$$\pi' : Y' = Y_{k'} \xrightarrow{\pi_{k'}} Y'_{k'-1} \xrightarrow{\pi_{k'-1}} \cdots \xrightarrow{\pi'_1} Y'_1 \xrightarrow{\pi'_1} Y_0 = \mathbb{C}^2$$

be the standard resolution of $C_G$ and $q'_1, \ldots, q'_{k'}$ be the centers of the blow-ups $\pi'_1, \ldots, \pi'_{k'}$. If $C_G$ is tangent to the $y$-axis, then we reverse the order of coordinates and find a $x$-root $S'$ of $G$, which is a fractional power series of $y$ with polydromy order $m'$ and of the form

$$c_{m'y} + \cdots + c_{hm'y^h} + a'y^{\frac{n'}{m'}} + \cdots$$

where $m' < n'$ and $h = \lceil \frac{m'}{n'} \rceil$. Applying Proposition 2.26 to $S'$ with coordinates $y, x$, and set $\gcd(n', m') = d'$, we obtain that $G$ is of the form

$$\left((x - c_{m'y} + \cdots + c_{hm'y^h})^{\frac{m'}{d'}} - (a')^{\frac{d}{d'}y^{\frac{n'}{m'}}}ight)^{d'} + \sum_{n'\alpha + m'\beta > n'm'} c_{\alpha, \beta}y^\alpha(x - c_{m'y} + \cdots + c_{hm'y^h})^\beta.$$

So $G \in (x, y)^{n'm'}$. We know for any $x^s y^t \in (x, y)^{n'm'}$, $\text{ord}(x^s y^t) \cdot X_{\gamma_i} = \frac{sn + mt}{m'} \geq \frac{m'n'}{d'} = \text{ord}(x^{n'm'}) \cdot X_{\gamma_i}$. Hence $G \in \mathcal{O}_F^{\rho(x^{n'm'})}$. On the other hand, we see that $\text{ord}(G)$ is of the form $(m', 0, \ldots, 0)$ (since after the first blow-up the strict transform of $G$ will never pass the center $q_2$). By Lemma 2.15, we get $\text{ord}(G) \cdot X_{\gamma_i} = \frac{m'n'}{d'}$. Then by Lemma 2.15 and (6), we have $\rho(G) = \rho(x^{n'm'})$. Hence we obtain $G \in \mathcal{O}_S^{\rho(G)}$. Now we suppose that $G$ is not tangent to $y$-axis. Denote by $(n'; m'_1, \ldots, m'_{g'})$ the characteristic sequence of $S'$ and denote by

$$\left(M'_1, \ldots, M'_{k'}\right) = (n', \ldots, n', r'_{1,1}, \ldots, r'_{g'_{k'}})$$

the multiplicity sequence of $S'$, where $n'$ appears $h'_{1,0}$ times, $r'_{1,1}$ appears $h'_{1,1}$ times, and so on, where $h'_{i,j}$ and $r'_{i,j}$ are invariants provided by the chain of $g'$ Euclidean algorithm similar to (2).

Similar to Notation 2.11, let $\gamma'_0 = 0$ and for $1 \leq i \leq g'$, let

$$\gamma'_i = \gamma'_{i-1} + \sum_{j=0}^{k_i} h'_{i,j}, \quad \text{and} \quad \tau'_{i-1} = \gamma'_{i-1} + h'_{i,0} + 1.$$
We claim that, $G$ can be written as a linear combination of $x^{\alpha_{i}}y^{\beta_{i}}F_{1}^{\delta_{1}} \cdots F_{g-1}^{\delta_{g-1}}$ satisfying
\[ \text{ord}(x^{\alpha_{i}}y^{\beta_{i}}F_{1}^{\delta_{1}} \cdots F_{g-1}^{\delta_{g-1}}) \cdot X_{\gamma_{i}} \geq \text{ord}(G) \cdot X_{\gamma_{i}}. \]

Suppose first that $q_{1} = q'_{1}$, $\ldots$, $q_{j} = q'_{j}$ and $q_{j+1} \neq q'_{j+1}$ with $j < \min\{k, k'\}$. Suppose further that $q_{j} = q'_{j}$ is a free point of $Z$ and $Z'$, and suppose that $q_{j+1} \neq q'_{j+1}$ are free points of $Z$ and $Z'$, respectively. Then there exists $0 \leq i \leq g - 1$ such that $\gamma_{i} = \gamma'_{j} < j < \min\{\tau_{i}, \tau'_{i}\}$. Let $r = j - \gamma_{i}$.

By Proposition 2.20, we know that $G$ has a $y$-root $S'$ which is of the form
\[ S' = [S]_{\leq m + rd} \cdot x^{m_{i} + rd} + \cdots \]
where $b \neq 0$. By Proposition 2.26, we find $G$ is of the form
\[ (F_{i-1}^{d_{i-1}} + H_{i,0} + \cdots + H_{i,r} + cH_{i,r})^{d_{i}} + F_{\text{ext}}^{(i)} \]
where $F_{\text{ext}}^{(i)}$ is a linear combination of $x^{\alpha_{i}}y^{\beta_{i}}F_{1}^{\delta_{1}} \cdots F_{g-1}^{\delta_{g-1}}$ satisfying
\[ \text{ord}(x^{\alpha_{i}}y^{\beta_{i}}F_{1}^{\delta_{1}} \cdots F_{g-1}^{\delta_{g-1}}) \cdot X_{\gamma_{i}} \geq \text{ord}(G) \cdot X_{\gamma_{i}}. \]

On the other hand, we have
\[ \text{ord}(G) = [n', \ldots, n', d'_{i}, \ldots, d'_{i}, d'_{i}, \ldots, d'_{i}, 0, \ldots, 0]. \]
So by Lemma 2.15, we get
\[ \text{ord}(G) \cdot X_{\gamma_{i}} = \begin{cases} d_{i} \cdot \frac{M_{1}^{2} + \cdots + M_{l}^{2}}{d_{i}d_{i}} & l \leq i \\ d_{i} \cdot \frac{M_{1}^{2} + \cdots + M_{l}^{2}}{d_{i}d_{i}} & l > i \end{cases} \]

By Proposition D:standardfactors, we know $F_{i,r}$ has a $y$-root has a partial sum equal to $[S]_{\leq m + rd}$. and with the polydromy order $\frac{1}{l}$. By Proposition 2.20, we then know $F_{i,r}$ goes through at least $q_{1}, \ldots, q_{j}$. So we get $\text{ord}(F_{i,r}) \cdot X_{\gamma_{i}} \geq \frac{1}{l} \text{ord}(G) \cdot X_{\gamma_{i}}$. Therefore the claim is true in this case. For the other cases, following a very similar argument, we proved the claim is true. This implies that $G \in \mathcal{O}_{S}^{\geq \rho(G)}$.

When $G$ is reducible, we can write $G = G_{1}^{r_{1}} \cdots G_{m}^{r_{m}}$ as a decomposition of irreducible factors. Without loss of generality, assume $m = 2$. By the argument above we have two inequalities: for $1 \leq l \leq g$, and $i = 1, 2$, $G_{i}$ is a linear combination of $x^{\alpha_{i}}y^{\beta_{i}}F_{1}^{\delta_{i,1}} \cdots F_{g-1}^{\delta_{i,g-1}}$ satisfying
\[ \alpha_{i} \frac{n}{d_{i}} + \beta_{i} \frac{m_{1}}{d_{i}} + \sum_{w=1}^{g-1} \text{ord}(F_{w}) \cdot X_{\gamma_{i}} \delta_{i,w} \geq \text{ord}(G_{i}) \cdot X_{\gamma_{i}}. \]

So $G_{1} \cdot G_{2} \in \mathcal{O}_{S}^{\geq \rho(G_{1}G_{2})}$ and hence $G \in \mathcal{O}_{S}^{\geq \rho(G)}$.

Following Definition 2.18 and by (5), we know for $0 \leq \alpha < 1$,
\[ G \in \mathcal{J}(\alpha Z) \iff \rho(G) > \alpha. \]

Therefore $\mathcal{J}(\alpha Z) \subset \mathcal{O}_{S}^{\geq \alpha}$. Conversely, by Definition 2.18 and (5), we know that generators of $\mathcal{O}_{S}^{\geq \alpha}$ have $\rho$-value greater than $\alpha$ and hence are in the ideal $\mathcal{J}(\alpha Z)$. So $\mathcal{O}_{S}^{\geq \alpha} \subset \mathcal{J}(\alpha Z)$. Therefore we obtain $\mathcal{J}(\alpha Z) = \mathcal{O}_{S}^{\geq \alpha}$. \(\square\)
Corollary 3.2. Let $S$ be a modified Puiseux series of an irreducible germ $Z$ which is not tangent to the $y$-axis. Let $F = \{F_1, \ldots, F_{g-1}\}$ be a set of standard factors of $S$. The set of jumping numbers of $Z$ between 0 and 1 is 

$$\{\rho(x^{p_x} y^{p_0} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}) \mid \forall p_x, p_0, p_1, \ldots, p_{g-1} \in \mathbb{N}\} \cap (0, 1).$$

Remark 3.3. Let $(M_1, \ldots, M_k)$ be the multiplicity sequence of $Z$. Set $d_0 = n$ and 

$$B_\nu = \frac{M_1^2 + \cdots + M_{\nu-1}^2}{d_{\nu-1}}$$

so that $B_\nu$ is the same as the notation $b_\nu$ in [Jär11]. By Lemma 2.25, for any $p_x, p_0, p_1, \ldots, p_{g-1} \in \mathbb{N}$, we denote by 

$$\rho(x^{p_x} y^{p_0} F_1^{p_1} \cdots F_{g-1}^{p_{g-1}}) = \min_{1 \leq i \leq g} \Omega_i$$

where, for $1 \leq l \leq g$,

$$\Omega_l = \frac{m_l + n + np_x + \sum_{j=0}^{l-1} B_j p_j}{d_{l-1} B_l} + \sum_{j=l}^{g-1} \frac{p_j}{d_j}.$$ 

Notice that 

$$M_1^2 + \cdots + M_{\nu-1}^2 = m_1 n + d_1 (m_2 - m_1) + d_2 (m_3 - m_2) + \cdots + d_{\nu-1} (m_\nu - m_{\nu-1}).$$

By elementary calculations, we can write 

$$\Omega_l = \frac{p_{l-1} + 1}{d_{l-1}} + \frac{t_l + 1}{B_l} + \sum_{j=l}^{g-1} \frac{p_j}{d_j}$$

with $t_l \in \mathbb{N}$, and we obtain

$$\Omega_l \leq \Omega_{l+1} \iff \frac{p_{l-1} + 1}{d_{l-1}} + \frac{t_l + 1}{B_l} \leq \frac{1}{d_l}.$$ 

So the above corollary recovers formulas given in [Jär11, Theorem 9.4].

Example 3.4. Consider the germ 

$$Z = (y^4 - 4x^2 y^3 + 4x^4 y^2 - 2x^3 y^2 + 4x^5 y - 4x^6 y + x^6 = 0).$$

After running the Newton-Puiseux Algorithm for $Z$, we know it is irreducible since it has a single conjugacy class of Puiseux series. The characteristic sequence of $Z$ is $(4;6,9)$. Observe that $Z$ has a Puiseux series $S$ with a partial sum 

$$x^\frac{3}{2} + x^2 + x^\frac{9}{4}$$

and choose a set of standard factors of $S$

$$F = \{F_1 = y^2 - x^3 - 2x^2 y\}.$$ 

By Theorem 3.1, a set of generators of $\mathfrak{I}(\alpha Z)$ are all the polynomials of the form $x^{p_x} y^{p_0} F_1^{p_1}$ satisfying 

$$\min \left\{\frac{5 + 2p_x + 3p_0 + 6p_1}{12}, \frac{13 + 4p_x + 6p_0 + 15p_1}{30}\right\} > \alpha.$$ 

Then we can describe the multiplier ideals $\mathfrak{I}(\alpha Z)$ with $0 < \alpha < 1$ explicitly as in Table 1 below.
3.2. A question. It is well known that two irreducible plane curves are topologically equivalent if and only if they have the same characteristic sequence (see for example [BK86, Theorem 21]). Järvilehto [Jär11] proved that the data of jumping numbers of multiplier ideals between 0 and 1 of the irreducible plane curve is the same as the data of the characteristic sequence, and hence determines the topological equivalence class.

We notice that $I_0(\alpha Z) = I_0(\alpha Z')$ for all $0 < \alpha \leq 1$ after a possible holomorphic change of coordinates is not a sufficient condition for the analytic equivalence of $Z$ and $Z'$. For example, set $Z = (y^5 - x^6 = 0)$ and $Z' = (y^5 - x^6 - 5x^4y^2 = 0)$. Then by Theorem 3.1, $I_0(\alpha Z) = I_0(\alpha Z')$ for any $0 < \alpha \leq 1$. (Note that $I_0(\alpha Z) = \mathcal{O}((\alpha - \epsilon)Z) = \mathcal{O}_{\frac{\alpha}{2}}$, where $0 < \epsilon \ll 1$.) However, $Z$ and $Z'$ are not analytically isomorphic (see [Zar86, Chapter V §34]).

In [MP16], [MP18a] and [MP18b], the authors defined Hodge ideals, which contain richer information about the singularity than multiplier ideals. It is natural to ask if they can determine a more subtle equisingularity equivalence class. Popa asked the following:

**Question 3.5.** [Pop19] Assume that $Z$ and $Z'$ are two germs of irreducible plane curve singularities with the same characteristic sequence. Are $Z$ and $Z'$ analytically equivalent if and only if $I_p(\alpha Z) = I_p(\alpha Z')$ for all $p \geq 0$ and all $0 < \alpha \leq 1$, after a possible holomorphic change of coordinates? Is it in fact enough to consider only $p = 0, 1$?

### Table 1. Multiplier ideals for Example 3.4

| jumping number $\xi_i$ | multiplier ideal $I(\alpha Z), \xi_{i-1} \leq \alpha < \xi_i$ |
|------------------------|---------------------------------------------------------------|
| $\frac{5}{12}$         | $\mathbb{C}\{x, y\}$                                          |
| $\frac{17}{30}$        | $(x, y)$                                                     |
| $\frac{19}{30}$        | $(x^2, y)$                                                  |
| $\frac{21}{30}$        | $(x, y)^2$                                                  |
| $\frac{23}{30}$        | $(x^3, xy, y^2)$                                             |
| $\frac{25}{30}$        | $(x^3, x^2y, y^2)$                                           |
| $\frac{27}{30}$        | $(y^2 - x^3, x^2y, x^4, xy^2, y^3)$                           |
| $\frac{11}{12}$        | $(y^2 - x^3 - 2x^2y, x^4, x^3y, xy^2, y^3)$                  |
| $\frac{29}{30}$        | $(x^3y, x^4, xy^2, y^3)$                                    |
| $\frac{1}{12}$         | $(xy^2 - x^4, x^5, x^3y, xy^2, y^3)$                         |

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