HYPER CONTRACTIVITY ON THE UNIT CIRCLE FOR ULTRASPHERICAL MEASURES: LINEAR CASE

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Abstract. In this paper we extend complex uniform convexity estimates for \( \mathbb{C} \) to \( \mathbb{R}^n \) and determine best constants. Furthermore we provide the link to log-Sobolev inequalities and hypercontractivity estimates for ultraspherical measures.

1. Introduction

The starting point of this paper is the Bonami’s sharp complex convexity estimate (A. Bonami [Bon70, Chapter III, Theorem 7])

\[
\int_{S^1} |x + a\zeta| \, dm(\zeta) \geq \left( |x|^2 + \frac{1}{2}a^2 \right)^{1/2} \quad \text{for } x \in \mathbb{R}^2, a \in [0, \infty),
\]

where \( S^1 \) denotes the unit circle in \( \mathbb{R}^2 \) and \( m \) the usual Haar measure on \( S^1 \) with \( m(S^1) = 1 \). W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann [DGTJ84, Proposition 3.1] presented a proof of (1) based on the power series representation of elliptic integrals. Separately S. Janson, applying F. Weissler’s logarithmic Sobolev inequalities on the circle, obtained (in particular) that for any \( \alpha \in (0, 2] \),

\[
\left( \int_{S^1} |x + a\zeta|^{\alpha} \, dm(\zeta) \right)^{1/\alpha} \geq \max\{a, |x|\}^{\alpha/2} \quad \text{for } x \in \mathbb{R}^2, a \in [0, \infty),
\]

and \( \frac{\alpha}{2} \) is the best (i.e., largest) real constant satisfying (2). In 2007, Aleksandrov [Ale07] presented an elegant short proof of (2). The proofs in [Wei80], respectively [Ale07] of (2) are complex analytic in nature; they don’t seem to work in higher dimensions, where \( S^1 \) is replaced by the unit sphere in \( \mathbb{R}^n \) and where \( m \) is replaced by \( \sigma \), the normalized Haar measure on the unit sphere in \( \mathbb{R}^n \).

Recently, [LMS19] we recorded a proof of (2), based on Green’s identities and sub-harmonicity estimates such as,

\[
\int_{S^1} |x + a\zeta|^{\beta} \, dm(\zeta) \geq \max\{|a, |x||, \beta \in \mathbb{R}(!), \ x \in \mathbb{R}^2, a \in [0, \infty),
\]

and in the present paper we prove that the extension of (2) to dimensions \( n \geq 3 \) holds true. The cases \( n = 3 \) and \( n \geq 4 \) are treated separately. For \( n = 3 \) we were able to adjust the argument in [LMS19]. In dimensions four and higher, our proof is guided by the link between the integral-term appearing in (5) and Riesz potential operators on \( \mathbb{R}^n \), acting on the surface measure \( \sigma \).

In Section 3, we investigate the hypercontractivity for ultraspherical measures on the unit circle

\[
d\nu_m(z) = c_m |\sin(\theta)|^m d\theta, \quad z = e^{i\theta} \in S^1, \quad \nu_m(S^1) = 1, \quad m > -1,
\]

considering the “linear polynomials” on \( S^1 \) given by \( f(z) = a + bz \).

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For $m = -1$ by definition we set $d\nu_{-1}(z) = \frac{1}{2}(\delta_1(z) + \delta_{-1}(z))$. We are interested in real numbers $m, p, q, r$ with $0 < p \leq q < \infty$ and $r \in \mathbb{R}$ such that
\begin{equation}
||1 + rbz||_{L^q(S^1, d\nu)} \leq ||1 + bz||_{L^p(S^1, d\nu)} \quad \text{for all} \ b \in \mathbb{R}.
\end{equation}
Considering $b \to 0$ in (3) one easily obtains a necessary condition on 4-tuple $(m, p, q, r)$, namely,
\begin{equation}
|r| \leq \sqrt{\frac{p + m}{q + m}}.
\end{equation}

If $m = -1$ then we are in the setting of a celebrated theorem of Bonami [Bon70], also known as Bonami–Beckner–Gross "two-point inequality", which says that (4) implies (3) when $(m, p, q, r) = (-1, p, q, r)$ and $q \geq p > 1$. A theorem of Weissler [Wei80] shows that (4) implies (3) when $(m, p, q, r) = (0, p, q, r)$ and $q \geq p > 0$. Extension of (2) to higher dimensions, the main theorem of our paper, in an equivalent way can be restated as (4) implies (3) when $(m, p, q, r) = (n - 2, p, 2, r)$ with $n \geq 2$, $n \in \mathbb{N}$, and $2 \geq p > 0$. In Section 3 using log-Sobolev inequalities for ultraspherical measures, we show that (4) implies (3) for the 4-tuples $(m, p, q, r)$ with $q \geq p \geq 6$ and all $m \geq -1$. Despite of partial progresses the description of all 4-tuples $(m, p, q, r)$ for which the hypercontractivity (3) holds true remains an open question.

2. Main Theorem

$S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, $\sigma$ the normalized Haar measure on $S^{n-1}$, $B^n_r(x)$ the open ball in $\mathbb{R}^n$ with radius $r > 0$, centered at $x \in \mathbb{R}^n$, and we set for convenience, $B^n_{1} = B^n(0)$.

**Theorem 2.1.** Let $n \in \mathbb{N}$ with $n \geq 2$. Let $p \in (0, 2]$ and $\lambda \leq \frac{np-2}{n}$. Then,
\begin{equation}
\int_{S^{n-1}} |x - az|^p \, d\sigma(z) \geq \left( |x|^2 + \lambda a^2 \right)^{\frac{p}{2}} \quad \text{for} \ x \in \mathbb{R}^n, a \in [0, \infty),
\end{equation}
and $\frac{np-2}{n}$ is the best (i.e. largest) constant satisfying (5).

We move to the front the elementary observation that $\frac{np-2}{n}$ is the best (i.e. largest) constant satisfying (5). For $x \in \mathbb{R}^2$ with $|x| = 1$, and $a, \lambda \in \mathbb{R}^+$, define
\begin{equation}
I(a) = \int_{S^{n-1}} |x - az|^p \, d\sigma(z), \quad \text{and} \quad g(a) = \left( 1 + \lambda a^2 \right)^{\frac{p}{2}},
\end{equation}
Assuming that (5) holds true, for $\lambda > 0$ we have
\begin{equation}
I(a) \geq g(a) \quad \text{for} \ a \geq 0.
\end{equation}

We now show that (7) implies that $\lambda \leq \frac{np-2}{n}$. Clearly we have $I(0) = 1$, $g(0) = 1$, $g'(0) = 0$ and $g''(0) = p/\lambda$. Next, since
\begin{align*}
\partial_x |x - az|^p &= p|x - az|^{p-2} z \cdot (az - x)
\end{align*}
we have $I'(0) = 0$. Hence (7) implies that $I''(0) \geq g''(0)$. Calculating further
\begin{align*}
\partial_x^2 |x - az|^p &= p(p - 2)|x - az|^{p-4}(z \cdot (az - x))^2 + p|x - az|^{p-2}|z|^2,
\end{align*}
and invoking the integral identity
\begin{align*}
\int_{S^{n-1}} |(x \cdot z)|^2 \, d\sigma(z) &= \frac{1}{n}
\end{align*}
gives $I''(0) = \frac{p(p-2)}{n} + p$. Thus $I''(0) \geq g''(0)$, implies that $\lambda \leq \frac{np-2}{n}$.

Before turning to the proof of Theorem 2.1 we determine the parameters $n$ and $q$ for which $x \mapsto |x|^q$ is a subharmonic mapping on $\mathbb{R}^n$, and draw consequences (analogous to Jensen’s formula in complex analysis).
Lemma 2.1. Let \( n \in \mathbb{N} \) and \( q \in \mathbb{R} \). The function \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, x \mapsto |x|^q \) is subharmonic, if and only if \( q \geq \max\{0, 2 - n\} \) or \( q \leq \min\{0, 2 - n\} \), and then

\[
\int_{S^{n-1}} |x - az|^q \, d\sigma(z) \geq \max\{|a, |x||^q \quad \text{for } a \in \mathbb{R}, x \in \mathbb{R}^n.
\]

Proof. For \( i \in \{1, \ldots, n\} \) we have

\[
d_i f(x) = q x_i |x|^{q-1}, \quad d_2^2 f(x) = q|x|^{q-2} + q(q - 2)x_i^2 |x|^{q-4}
\]

and therefore

\[
\Delta f(x) = q(n + q - 2)|x|^{q-2}.
\]

Clearly the sign of the factor \( q(n + q - 2) \) determines if \( f \) is subharmonic or not.

We next turn to verifying that \( q(n + q - 2) \geq 0 \) implies \( \mathbf{8} \). If \( a < |x| \), the mean value property of subharmonic functions directly yields

\[
\int_{S^{n-1}} |x - az|^q \, d\sigma(z) \geq |x|^q.
\]

To treat the case \( a > |x| \), we define \( H_a : B_n^a(0) \to \mathbb{R} \) by

\[
H_a(x) := \int_{S^{n-1}} |x - az|^q \, d\sigma(z).
\]

and notice, that \( H_a \) is subharmonic and rotational invariant, i. e. there exists a function \( h_a : [0, a) \to \mathbb{R} \), such that

\[
H_a(x) = h_a(|x|) \quad \text{for } x \in [0, a).
\]

Using subharmonicity and rotational invariance together with the representation of the Laplace operator in \( n \)-dimensional spherical coordinates, we obtain

\[
0 \leq \Delta H_a(x) = |x|^{1-n} \partial_r \left( r^{n-1} \partial_r H_a(r)(|x|) \right) \quad \text{for } |x| \in (0, a).
\]

This yields

\[
r^{n-1} \partial_r H_a(r) \geq 0 \quad \text{for } r \in [0, a)
\]

and consequently

\[
h_a(r) \geq h_a(0) = a^q \quad \text{for } r \in [0, a).
\]

Hence for \( a > |x| \) we have \( H_a(x) = h_a(|x|) \geq a^q \), and hence

\[
\int_{S^{n-1}} |x - az|^q \, d\sigma(z) \geq a^q.
\]

\[\square\]

Proof. We now prove that \( \mathbf{5} \) holds true for \( \lambda := \frac{n^2 + p^2 - 2}{n} \). Since the case \( n = 2 \) is already known, we consider \( n \geq 3 \). An application of the divergence theorem yields that

\[
\int_{S^{n-1}} |x - az|^p \, d\sigma(z) = 1 + a^2 p(p + n - 2) \int_0^1 \int_0^1 \left( \frac{r}{t} \right)^{n-1} \int_{S^{n-1}} |x - az|^{p-2} \, d\sigma(z) \, dr \, dt.
\]
Indeed, put \( f : \mathbb{R}^n \to \mathbb{R}, f(y) := |x - ay|^p \) and define a vectorfield \( X \) by \( X(y) := \nabla f(y) \). Then \( \text{div} X = t \Delta f(ty) \) and by the divergence theorem
\[
\int_{\mathbb{R}^n} f(tz) \, \text{d} \sigma(z) = \int_{\mathbb{R}^n} X(z) \cdot z \, \text{d} \sigma(z)
\]
\[
\frac{1}{n \text{Vol}_n(B_2^n)} \int_{B_2^n} \Delta f(ty) \, \text{d} y
\]
(11)
\[
= \frac{1}{n \text{Vol}_n(B_2^n)} \int B_2^n \Delta f(y) \, \text{d} y
\]
\[
= \frac{1}{n \text{Vol}_n(B_2^n)} \int_{\mathbb{R}^n} t^{n-1} \Delta f(tz) \, \text{d} \sigma(z).
\]
Integrating the identity (11) from \( t = 0 \) to \( t = 1 \) and invoking (9) gives (10). Define
\[
H(a, x) := \int_{S^{n-1}} |x - az|^{p-2} \, \text{d} \sigma(z).
\]
Then \( H(a, \cdot) \) is rotational invariant, i.e. there exists a function \( h : [0, \infty)^2 \to \mathbb{R} \), such that \( H(a, x) = h(a, |x|) \). By (10) and re-scaling we have
\[
\int_{S^{n-1}} |x - az|^p \, \text{d} \sigma(z) = 1 + p(p + n - 2) \int_0^1 \int_0^1 t^{1-n} u^{n-1} h(u, 1) \, \text{d} u \, \text{d} t.
\]
(12)
The proof of Theorem 2.1 will be obtained by proving suitable lower estimates for the volume integral appearing on the right hand side of (12). We will distinguish the case where \( x \mapsto |x|^p \) is sub-harmonic (corresponding to \( n = 3 \) and \( p \leq 1 \)), and the case where sub-harmonicity fails (corresponding to \( n \geq 4 \) or \( p > 1 \)).

2.1. Case \( n = 3 \) and \( p \leq 1 \). First note that
\[
1 + p(p + 1) \int_0^a \int_0^1 t^{-2} u^2 \max[1, u]^{p-2} \, \text{d} u \, \text{d} t = \begin{cases} \int_0^a t^{-2} \max[1, u]^{p-2} \, \text{d} u, & a \in [0, 1], \\ \int_0^a t^{-2} \max[1, u]^{p-2} \, \text{d} u, & a > 1. \end{cases}
\]
(13)
Indeed, (13) follows from a direct calculation separating the cases \( a \leq 1 \) and \( a > 1 \).

Case \( a \leq 1 \): We calculate
\[
\int_0^a \int_0^1 t^{-2} u^2 \max[1, u]^{p-2} \, \text{d} u \, \text{d} t = \int_0^a t^{-2} \int_0^1 u^2 \, \text{d} u \, \text{d} t = \frac{a^2}{6},
\]
which yields (13) for \( a \leq 1 \).

Case \( a > 1 \): We calculate
\[
\int_0^a \int_0^1 t^{-2} u^2 \max[1, u]^{p-2} \, \text{d} u \, \text{d} t
\]
\[
= \int_0^1 \int_0^a t^{-2} u^2 \max[1, u]^{p-2} \, \text{d} u \, \text{d} t + \int_0^1 \int_0^a t^{-2} \int_0^a u^2 \max[1, u]^{p-2} \, \text{d} u \, \text{d} t + \int_0^1 \int_0^a t^{-2} \int_0^a u^p \max[1, u]^{p-2} \, \text{d} u \, \text{d} t
\]
\[
= \frac{1}{6} - \frac{a^{-1} - 1}{3} + \frac{a^p - 1}{(p + 1)p} + \frac{a^{-1} - 1}{(1 + p)}
\]
which yields (13) for \( a \geq 1 \), by arithmetic.

Since \( x \mapsto |x|^{p-2} \) is subharmonic, for \( n = 3 \) and \( p \in (0, 1] \), Lemma 2.1 yields \( h(a, x) \geq \max[1, a]^{p-2} \). Applying this estimate to (12) and invoking (13) we obtain
\[
\int_{S^2} |x - az|^p \, \text{d} \sigma(z) \geq \begin{cases} \int_0^a t^{-2} \max[1, u]^{p-2} \, \text{d} u, & a \leq 1 \\ \int_0^a t^{-2} \max[1, u]^{p-2} \, \text{d} u, & a > 1 \end{cases}
\]
(14)
Defining
\[ g(a) := \begin{cases} 1 + \frac{p(p+1)}{6}a^2, & a \in [0, 1] \\ a^p + \frac{p(2-p)}{3a} - \frac{p(1-p)}{2}, & a^2 \in (1, \frac{3}{2-p}) \end{cases}, \]

it suffices to show
\[ \int_{S^2} |x - az|^p \, d\sigma(z) \geq g(a) \geq \left( 1 + \frac{p+1}{3}a^2 \right)^{\frac{p}{2}}. \]

We first consider \( a^2 \geq \frac{3}{2-p} \). In that case we have
\[ \int_{S^2} |x - az|^p \, d\sigma(z) \geq a^p \geq \left( 1 + \frac{p+1}{3}a^2 \right)^{\frac{p}{2}}. \]

Indeed, by Lemma 2.1, we get
\[ \int_{S^{n-1}} |x| \, d\sigma(x) = 2 \] for \( p \geq 1 \), \( x \mapsto |x|^p \) is subharmonic. Taking into account that \(|x| = 1\) and \( a^2 > \frac{3}{2-p} \), Lemma 2.1 yields
\[ \int_{S^{n-1}} |x - az|^p \, d\sigma(z) \geq \max[a, 1]^p \geq a^p. \]

To obtain the second estimate in (16) note that \( a^2 \geq \frac{3}{2-p} \) holds if and only if \( a^2 \geq 1 + \frac{p+1}{3}a^2 \).

We now turn to the case \( a^2 < \frac{3}{2-p} \). By (14) in this case it remains to show the second inequality of (16). If moreover \( a \in [0, 1] \) this is just Bernoulli’s inequality. If finally \( a^2 \in \left( 1, \frac{3}{2-p} \right) \) we proceed as follows: For \( p \in (0, 1] \), we define
\[ \phi(t) := t^\frac{p}{2} + \frac{p(2-p)}{3} \frac{p(1-p)}{2} - \left( 1 + \frac{p+1}{3}t \right)^{\frac{p}{2}}. \]

We show that \( \phi(t) \geq 0 \) for \( t \in \left( 1, \frac{3}{2-p} \right) \). Indeed, since \( t < \frac{3}{2-p} \) holds if and only if \( t < 1 + \frac{p+1}{3}t \), we get
\[ \phi'(t) = \frac{p}{2} t^{\frac{p}{2}-1} - \frac{p(2-p)}{6} t^{-\frac{1}{2}} - \frac{p(1-p)}{2} \left( 1 + \frac{p+1}{3}t \right)^{\frac{p}{2}} \geq 0. \]

Due to \( \phi(1) \geq 0 \), this implies \( \phi(t) \geq 0 \) for \( t \in \left( 1, \frac{3}{2-p} \right) \). Summing up for \( p \in (0, 1] \) and \( t = a^2 \in \left( 1, \frac{3}{2-p} \right) \) we have
\[ a^p + \frac{p(2-p)}{3a} - \frac{p(1-p)}{2} \geq \left( 1 + \frac{p+1}{3}a^2 \right)^{\frac{p}{2}}. \]
2.2. **Case** \( n > 3 \) or \( p > 1 \). Since we cannot apply Lemma 2.1, we need another lower bound for \( h(a,1) \). In order to accomplish that we use the formula

\[
(17) \quad r^{-\zeta} = \frac{1}{\Gamma\left(\frac{\zeta}{2}\right)} \int_0^\infty t^{-\frac{\zeta}{2}} \exp\left(-\frac{r^2}{t}\right) dt = \frac{1}{\Gamma\left(\frac{\zeta}{2}\right)} \int_0^\infty t^{-\frac{\zeta}{2}-1} \exp\left(-r^2 t\right) dt,
\]

which holds for all \( r > 0 \) and \( \Re \zeta > 0 \). Putting \( \zeta := 2-p \), i. e. \( p = 2-\zeta \), we get

\[
H(a,x) = \int_{S^{n-1}} |x-az|^{p-2} d\sigma(z)
= \frac{1}{\Gamma\left(\frac{\zeta}{2}\right)} \int_0^\infty \int_{S^{n-1}} t^{-\frac{\zeta}{2}-1} \exp\left(-\frac{|x-az|^2}{t}\right) d\sigma(z) dt
= \frac{1}{\Gamma\left(\frac{\zeta}{2}\right)} \int_0^\infty \left(\int_{S^{n-1}} \exp\left(\frac{2ax \cdot z}{t}\right) d\sigma(z)\right) t^{-\frac{\zeta}{2}-1} \exp\left(-\frac{1+a^2}{t}\right) dt.
\]

We are thus left with finding a good lower bound for

\[
\int_{S^{n-1}} \exp(\lambda x \cdot z) d\sigma(z) = \int_{S^{n-1}} \cosh(\lambda |x \cdot z|) d\sigma(z),
\]

where \( \lambda > 0 \) and \( |x| = 1 \). The obvious bound is 1, which eventually turns out not to be sufficient for \( p < \frac{4}{n+2} \), so we take the second Taylor approximation: \( \cosh s \geq 1 + \frac{s^2}{2} \). By (17) and the functional equation of the gamma function we conclude

\[
h(a,1) \geq \left(1 + a^2\right)^{-\frac{\zeta}{2}} + \frac{2a^2}{n\Gamma\left(\frac{\zeta}{2}\right)} \int_0^\infty t^{-\frac{\zeta}{2}-3} \exp\left(-\frac{1+a^2}{t}\right) dt
= \left(1 + a^2\right)^{-\frac{\zeta}{2}} + \frac{2a^2 \Gamma\left(\frac{\zeta}{2} + 2\right)}{n\Gamma\left(\frac{\zeta}{2}\right)} \left(1 + a^2\right)^{-\frac{\zeta}{2}-2}
= \left(1 + a^2\right)^{-\frac{\zeta}{2}-1} \left(1 + \frac{(4-p)(2-p) a^2}{2n} \left(1 + a^2\right)^2\right) =: \psi(a).
\]

According to (12) it remains to prove that

\[
1 + p(p + n - 2) \int_0^a \int_0^t t^{1-n} u^{n-1} \psi(u) du dt \geq \left(1 + \frac{p+n-2}{n} a^2\right)^\frac{\zeta}{2}.
\]

We set \( c := \frac{u^p + 2}{n} \) and show

\[
F(a) := 1 + p(p + n - 2) \int_0^a \int_0^t t^{1-n} u^{n-1} \psi(u) du dt - \left(1 + ca^2\right)^\frac{\zeta}{2} \geq 0.
\]

Since \( F(0) = 0 \), this follows from \( F' \geq 0 \), i. e.

\[
u \int_0^a u^{n-1} \psi(u) du - a^n \left(1 + ca^2\right)^{\frac{\zeta}{2}-1} \geq 0,
\]

which in turn follows from

\[
n a^{n-1} \psi(a) - \vartheta^a \left(1 + ca^2\right)^{\frac{\zeta}{2}-1} \geq 0.
\]

Rearranging terms this amounts to

\[
1 + \frac{(4-p)(2-p) a^2}{2n(1+a^2)^2} - \left(1 + a^2\right)^{\frac{\zeta}{2}} + \frac{a^2 c(2-p)}{n(1+a^2)} \left(1 + a^2\right)^{\frac{\zeta}{2}-2} \geq 0.
\]
Put $x := (1 + ca^2)/(1 + a^2)$, then $x \in (c, 1)$ and

$$a^2 = \frac{1 - x}{x - c}, \quad 1 + a^2 = \frac{1 - c}{x - c}, \quad \text{and} \quad \frac{a^2}{1 + a^2} = \frac{1 - x}{1 - c}.$$ 

Thus we have to show that

$$1 + \frac{(4 - p)(2 - p)(1 - x)(x - c)}{2n(1 - c)^2} = x^q - 1 + \frac{c(2 - p)(1 - x)}{n(1 - c)} x^{q - 2} \geq 0$$

i.e.

$$x^q - 1 + \frac{n(1 - c) + (2 - p) x (x - c)}{2n(1 - c)^2} = \frac{1 - c - (1 - c^2)(1 - x)}{1 - c + \left(2 - \frac{p}{x}\right)(1 - x)(x - c)}.$$

Considering $n \geq 4$ or $p > 1$, we have $c = 1 - \frac{2 - p}{n} \geq \frac{1}{2}$. So eventually it suffices to prove that given $q := 2 - \frac{p}{x} \in [0, 1]$, then for all $(x, y) \in [0, 1]^2$ satisfying $x \geq y \geq \frac{1}{2}$, we have

$$x^q(1 - y + q(1 - x)(x - y)) \geq 1 - y - (1 - y^2)(1 - x).$$

The function $q \mapsto x^q(1 - y + q(1 - x)(x - y))$ is decreasing. Indeed, the derivative of the logarithm with respect to $q$ is

$$\frac{(1 - x)(x - y)}{1 - y + q(1 - x)(x - y)} - \log \frac{1}{x} \leq \frac{(1 - x)(x - y)}{1 - y} - \log \frac{1}{x} \leq 1 - x - \log \frac{1}{x} \leq 0,$$

where we simply used the fact $y \leq x \leq 1$. Thus we only have to prove, that, assuming $\frac{1}{2} \leq y \leq x \leq 1$, he have

$$x^2(1 - y + 2(1 - x)(x - y)) - 1 + y + (1 - y^2)(1 - x) \geq 0.$$

The left-hand side is a polynomial in $x$ of order 4, which factorizes to

$$(1 - x)(x - y)(2x^2 + y - 1).$$

Due to the conditions on $x$ and $y$, this is obviously non-negative.

However the polynomial is negative for $y \leq x < 1/2$ and thus the inequality (18) doesn’t hold for small values of $p$ and $n \in [2, 3]$. Hence the above argument does not apply to dimension two and three!

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**Figure 2.** Plots of $x \mapsto x^q - \frac{1 - y - (1 - y^2)(1 - x)}{1 - y + q(1 - x)(x - y)}$ for $y \in [0.5, 0.3]$ and $q \in [1, 1.1, \ldots, 2]$. 
3. Hypercontractivity for ultrasperical measures on the unit circle

The main theorem (Theorem 2.1) in an equivalent form we can restate as for $0 < p < 2$, $n ≥ 2, |r| ≤ \sqrt{\frac{p+n-2}{n}}$, $r ∈ R$, we have

$$\|x + ay\|_{L^p(S^{n-1}, d\sigma(y))} ≤ \|x + ay\|_{L^p(S^{n-1}, d\sigma(y))} \quad \text{for all} \quad x ∈ R^n, a ∈ R.$$  \hspace{1cm} (19)

In this section we consider an extension of (19), namely, let $n ≥ 2$, and $0 < p < q < ∞$. We are interested to find the largest possible constant $C = C(n, p, q) > 0$ such that for all $r ∈ R, |r| ≤ C(p, q, r)$ we have

$$\|x + ary\|_{L^q(S^{n-1}, d\sigma(y))} ≤ \|x + ary\|_{L^q(S^{n-1}, d\sigma(y))} \quad \text{for all} \quad x ∈ R^n, a ∈ R.$$ \hspace{1cm} (20)

First we prove a theorem on the unit circle for ultrasperical measures

$$dv_m(z) = c_m|\sin(\theta)|^m d\theta \quad \text{for all} \quad m > -1,$$

where $z = e^{i\theta} ∈ S^1$, and the scalar $c_m = \frac{Γ(m/2 + 1)}{Γ(1/2)Γ(1 + m/2)}$ is chosen in such a way that $v_m(S^1) = 1$. For $m = -1$ we set $dv_{-1}(z) = \frac{1}{2}(δ_{-1}(z) + δ_1(z))$.

**Theorem 3.1.** Let $m ≥ -1$ and $6 ≤ p ≤ q$. We have

$$\|1 + r az\|_{L^p(S^1, dv_m)} ≤ \|1 + b z\|_{L^p(S^1, dv_m)} \quad \text{for all} \quad b ∈ R.$$ \hspace{1cm} (21)

if and only if $|r| ≤ \sqrt{\frac{p+m}{q+m}}$.

Let us show that the theorem implies

**Corollary 3.1.** For any $6 ≤ p ≤ q$, all integers $n ≥ 2$, and any real $|r| ≤ \sqrt{\frac{p+n-2}{q+n-2}}$ the inequality (20) holds true.

Indeed, without loss of generality we can assume $|x| = 1$ in (20). Next, for $y = (y_1, ..., y_n) ∈ S^{n-1}$ and $λ = \frac{n-2}{2}$, we have

$$\|x + ay\|_{L^p(S^{n-1}, d\sigma(y))} = \int_{S^{n-1}} (1 + 2a(x, y) + a^2)^{p/2} d\sigma(y) =$$

$$\frac{Γ(1 + λ)}{Γ(1/2)Γ(λ + 1/2)} \int_{-1}^{1} (1 + 2at + a^2)^{p/2} (1 − t^2)^{(λ−(1/2))} dt \quad \text{(t = \cos(θ))} =$$

$$\frac{Γ(1 + λ)}{Γ(1/2)Γ(λ + 1/2)} \int_{0}^{π} (1 + 2a\cos(θ) + a^2)^{p/2} \sin^{2λ}(θ)dθ =$$

$$\int_{S^1} [1 + az]^p dv_{2λ}(z) = \|1 + az\|_{L^p(S^1, dv_{n+2})}^p.$$ Similarity we have $\|x + ary\|_{L^q(S^{n-1}, d\sigma(y))} = \|1 + r az\|_{L^q(S^1, dv_{n+2})}$. Thus the inequalities (20) and (21) are the same with $m = n − 2$.

Next we prove Theorem 3.1.

**Proof.** As the measure $dv_{-1}(z) = \frac{1}{2}(δ_{-1}(z) + δ_1(z))$ is the weak* limit of of the measures $dv_m(z)$ when $m → -1$, $m > -1$, without loss of generality we can assume that $m > -1$ in the theorem.

First we show that the assumption $|r| ≤ \sqrt{\frac{p+m}{q+m}}$ is necessary for the hypercontractivity (21). Indeed, notice that

$$\int_{S^1} (Re(z))^2 dv_m(z) = c_m \int_{0}^{2π} \cos^2(θ)|\sin(θ)|^m dθ = 1 - c_m |\sin(θ)|^m dθ = 1 - \frac{m+1}{m+2} = 1 - \frac{m+1}{m+2}.$$
Therefore

\[
\|1 + b z\|_{L^p(S^1, d\nu_m)} = \left( \int_{S^1} \left(1 + b^2 \right) |d\nu_m(z)| \right)^{1/p} \]

\[
\left( \int_{S^1} 1 + \frac{p}{2} (2b |\mathcal{R}(z)| + b^2) + \frac{p}{4} - 1 \right) 4b^2 |\mathcal{R}(z)|^2 + o(b^2) |d\nu_m(z)| \right)^{1/p} =
\]

\[
\left( 1 + \frac{p}{2} b^2 + \frac{p(p-2)}{2} b^2 \right) (|\mathcal{R}(z)|^2 |d\nu_m|)^{1/p} = 1 + \frac{b^2}{2} + \frac{p-2}{m+2} b^2 + o(b^2) = 1 + \frac{b^2}{2} \frac{m + p}{m + 2} + o(b^2).
\]

So the inequality \( \|1 + rbz\|_{L^p(S^1, d\nu_m)} \leq \|1 + b z\|_{L^p(S^1, d\nu_m)} \) implies \( r^2 \frac{m + q}{m + 2} \leq \frac{m + p}{m + 2} \). Since \( p, q > -m \) we obtain i.e., \( |r| \leq \sqrt{\frac{m + p}{m + q}} \).

Next we show that the necessary condition \( |r| \leq \sqrt{\frac{p + m}{q + m}} \) is sufficient for (21). Since \( q \geq 1 \) and \( d\nu_m(z) \) is even measure, i.e., \( d\nu_m(z) = d\nu_m(-z) \), we see that the map \( r \mapsto \|1 + rbz\|_{L^p(S^1, d\nu_m)} \) is even convex function on \( \mathbb{R} \), hence it is nondecreasing on \([0, \infty)\). Thus it suffices to prove (21) in the case when \( r = \sqrt{\frac{p + m}{q + m}} \). Let \( m = 2\lambda \).

After rescaling \( b \) as \( b \mapsto \frac{b}{\sqrt{2}} \), we can rewrite (21) as follows

\[
\left( \int_{-1}^{1} \left(1 + \frac{2bt}{\sqrt{q + 2\lambda} + \frac{b^2}{q + 2\lambda}} \right)^{2/q} d\mu_\lambda(t) \right)^{1/2} \leq \left( \int_{-1}^{1} \left(1 + \frac{2bt}{\sqrt{p + 2\lambda} + \frac{b^2}{p + 2\lambda}} \right)^{2/p} d\mu_\lambda(t) \right)^{1/2},
\]

where \( d\mu_\lambda(t) = 2\mathcal{C}_2, (1 - t^2)^{2 - 1/2} d\) is a probability measure on \([-1, 1]\). Rescaling \( b \) as \( b \mapsto \frac{b}{\sqrt{2}} \) we see that the inequality (22) simply means that the map

\[
s \mapsto \left( \int_{-1}^{1} \left(1 + \frac{2bt}{\sqrt{s + \lambda} + \frac{b^2}{s + \lambda}} \right)^{2/s} d\mu_\lambda(t) \right)^{1/2}
\]

is nonincreasing on \((3, \infty)\) (here \( s = \mu/2 \)). If we differentiate in \( s \), then after a certain calculation we see that it suffices to show the following log-Sobolev inequality: put \( f(t) = 1 + \frac{2b}{\sqrt{s + \lambda} + \frac{b^2}{s + \lambda}} \), then

\[
\int f^s \ln f^s d\mu_\lambda - \int f^s d\mu_\lambda \ln \int f^s d\mu_\lambda \leq -s^2 \int f^{s-1} \frac{d}{ds} f^{-1} d\mu_\lambda.
\]

Therefore, if we let \( b(s + \lambda)^{-1/2} = \tilde{b} \) and \( g(t) = 1 + 2bt + \tilde{b}^2 \), then our log-Sobolev inequality rewrites as follows

\[
\int g^s \ln g^s d\mu_\lambda - \int g^s d\mu_\lambda \ln \int g^s d\mu_\lambda \leq \frac{s^2}{s + \lambda} \int g^{s-1} (bt + \tilde{b}^2) d\mu_\lambda.
\]

The log-Sobolev inequality of Mueller-Weissler [MW82, p 277] for \( d\mu_\lambda \) states that

\[
\int g^s \ln g^s d\mu_\lambda - \int g^s d\mu_\lambda \ln \int g^s d\mu_\lambda \leq \frac{s^2}{2(s + \lambda)} \frac{2\lambda + 1}{2(\lambda + 1)} \int (g')^2 g^{s-2} d\mu_{\lambda+1}
\]

\[
= \frac{s^2}{4(\lambda + 1)} \int (g')^2 g^{s-2} d\mu_{\lambda+1}.
\]
Thus we need to show that
\[ \int (g')^2 g^{s-2} d\mu_{\lambda+1} \leq \frac{4(\lambda + 1)}{s + \lambda} \int g^{s-1} (\hat{b} t + \hat{b}^2) d\mu_{\lambda}. \]

After an integration by parts we can rewrite the left hand side of the last inequality as \( \frac{4(\lambda + 1)}{s + \lambda} \int g^{s-1} \hat{b} d\mu_{\lambda} \) (here we used the fact that \( \frac{\lambda^2 (\lambda + 1)}{\lambda + 1} = \frac{\lambda^2}{s + \lambda} \)). Hence, to prove (21) it suffices to show that
\[ \frac{1}{s - 1} \int g^{s-1} t d\mu_{\lambda} \leq \frac{1}{s + \lambda} \int g^{s-1} (t + \hat{b}) d\mu_{\lambda} \]
holds true. We can rewrite (25) as
\[ \int g^{s-1} d\mu_{\lambda} \geq \int \frac{t(\lambda + 1)}{b(s - 1)} g^{s-1} d\mu_{\lambda}. \]

Integrating the right hand side by parts we see that it is enough to show
\[ \int (1 + 2 at + a^2)^{s-1} d\mu_{\lambda}(t) \geq \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda+1}(t), \]
for all \( a = \hat{b} > 0 \). We claim that it suffices to consider the case when \( a \in (0, 1) \). Indeed, otherwise we can write
\[ \int (1 + 2 at + a^2)^{s-1} d\mu_{\lambda}(t) = a^2(s-1) \int (a^2 + 2a^2 t + 1)^{s-1} d\mu_{\lambda}(t) \]
\[ \geq a^2(s-1) \int (a^2 + 2a^2 t + 1)^{s-2} d\mu_{\lambda+1}(t) \]
\[ = a^2 \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda+1}(t) \]
\[ \geq \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda+1}(t). \]

The inequality \( \frac{s-1}{s-2} \geq 1 \) implies
\[ \int (1 + 2 at + a^2)^{s-1} d\mu_{\lambda}(t) \geq \left( \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda}(t) \right)^{\frac{s-1}{s-2}}. \]

Next, by Jensen’s inequality we have \( \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda} \geq (1 + a^2)^{s-2} \geq 1. \) Therefore
\[ \left( \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda}(t) \right)^{\frac{s-1}{s-2}} \geq \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda}(t). \]

Therefore, we need to show that \( \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda}(t) \geq \int (1 + 2 at + a^2)^{s-2} d\mu_{\lambda+1}(t). \)
The inequality trivially holds true if \( s = 3 \). Considering the linear function \( F(t) = 1 + 2 at + a^2 \), it suffices to show that
\[ (26) \quad \int_0^\infty r^{s-3} \lambda_{\lambda}(t \in [-1, 1]: F(t) > r) dr \geq \int_0^\infty r^{s-3} \lambda_{\lambda+1}(t \in [-1, 1]: F(t) > r) dr. \]

Consider \( h(u) = \lambda_{\lambda}(t \in [-1, 1]: t > u) - \mu_{\lambda+1}(t \in [-1, 1]: t > u) \). Clearly \( h(-1) = h(0) = h(1) = 0 \). Also
\[ h'(u) = -2c_2 \lambda (1 - u^2)^{\lambda - 1/2} + 2c_2 \lambda^2 (1 - u^2)^{\lambda + 1/2} = 2c_2 \lambda (1 - u^2)^{\lambda - 1/2}. \]

It follows that \( h(u) \leq 0 \) on \([0, 1]\) and \( h(u) \geq 0 \) on \([-1, 0]\). Therefore \( \phi(r) = \mu_{\lambda}(t \in [-1, 1]: F(t) > r) - \mu_{\lambda+1}(t \in [-1, 1]: F(t) > r) \) changes sign only once i.e., there exists
$r_0 \in [0, \infty)$ such that $\varphi(r) \leq 0$ on $[0, r_0]$ and $\varphi(r) \geq 0$ on $[r_0, \infty)$. If $r_0 = 0$ then (26) trivially holds true. If $r_0 > 0$, then we have
\begin{equation}
\int_0^\infty \left( \left( \frac{r}{r_0} \right)^{s-3} - 1 \right) \varphi(r) \, dr \geq 0
\end{equation}
because the integrand has nonnegative sign. Therefore, inequality (27) together with $\int_0^\infty \varphi(r) \, dr = 0$ implies $\int_0^\infty r^{s-3} \varphi(r) \, dr \geq 0$.
\[\square\]

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