Density matrix of strongly coupled quantum dot - microcavity system

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Abstract. Any two-level quantum system can be used as a quantum bit (qubit)-the basic element of all devices and systems for quantum information and quantum computation. Recently it was proposed to study the strongly coupled system consisting of a two-level quantum dot and a monoenergetic photon gas in a microcavity-the strongly coupled quantum dot-microcavity (QD-MC) system for short, with the Jaynes-Cumming total Hamiltonian, for the application in the quantum information processing. Different approximations were applied in the theoretical study of this system. In this work, on the basis of the exact solution of the Schrodinger equation for this system without dissipation we derive the exact formulae for its density matrix. The realization of a qubit in this system is discussed. The solution of the system of rate equation for the strongly coupled QD-MC system in the presence of the interaction with the environment was also established in the first order approximation with respect to this interaction.

Keywords: Quantum bit, quantum information, quantum dot, rate equations.

1. Introduction
Cavity quantum electrodynamics (CQED) studies the properties of atoms coupled to discrete photon modes in high-Q cavities. Such systems are of great interest in the study of the fundamental quantum mechanics of open systems, the engineering of quantum states and the study of measurement-induced decoherence, and have also been proposed as possible candidates for use in quantum information processing and transmission [1-10]. This work is devoted to the study of the density matrix of the QD-MC system with the Jaynes-Cumming Hamiltonian. For the system without the dissipative interaction of the environment we derive exact expressions of the density matrix. In the presence of the weak dissipative interaction of the system with the environment the system of rate equations can be solved by means of the perturbation theory. We derive the expressions of their solution in the first order approximation.

2. System without decoherence
Consider the simplest model of strongly coupled system of a two-level quantum dot (QD) and monoenergetic photons in a microcavity (MC) with the Jaynes-Cumming total Hamiltonian

\[ H_s = E_e c_e^* c_e + E_g c_g^* c_g + \omega \gamma + f \left( \gamma c_e^* c_g + \gamma c_g^* c_e \right), \]  

where \( c_e \) and \( c_e^* \) are the destruction and creation operators for the electrons at the energy level
$E_g (E_e - E_g = E > 0)$, and similarly for $c_g$ and $c_g^\ast$. $\gamma$ and $\gamma^\prime$ are the photon destruction and creation operators, $\omega$ is the photon energy and $f > 0$ is the effective constant of the electron-photon interaction. In the Hilbert space of state vectors there are following vector subspaces invariant under the action of the Hamiltonian:

1-d subspace $V_1$ with unit vector

$$|\phi\rangle = |0\gamma, g\rangle,$$  \hspace{1cm} (2)

2-d subspaces $V_{2}^{(n)}, n \geq 1$, with two basis vectors in each subspace:

$$|\phi_1^{(n)}\rangle = |(n-1)\gamma, e\rangle = \frac{(\gamma^{\prime})^{n-1}}{\sqrt{n!}} c_e^{\ast} |\text{vacuum}\rangle,$$  \hspace{1cm} (3)

$$|\phi_2^{(n)}\rangle = |n\gamma, g\rangle = \frac{(\gamma^{\prime})^{n}}{\sqrt{n!}} c_g^{\ast} |\text{vacuum}\rangle.$$

(4)

Each pair of two states in any 2-d invariant subspace $V_{2}^{(n)}$ can be used as a physical realization of a qubit. In the absence of the damping, the eigenstates and eigenvalues of the Jaynes-Cumming Hamiltonian are

$$\psi^{(n)} = A^{(n)} |\phi_1^{(n)}\rangle + B^{(n)} |\phi_2^{(n)}\rangle, \alpha = +, -, \psi^{(0)} = |\phi\rangle,$$

(5)

$$E^{(n)} = \frac{\Delta}{2} + E_g + n\omega \pm \frac{1}{2} \frac{\omega}{\Omega_n}, E^{(0)} = E_g,$$

(6)

where

$$A^{(n)} = \left[ \frac{1}{2} \left( 1 \mp \frac{\Delta}{\Omega_n} \right) \right]^{1/2},$$

(7)

$$B^{(n)} = \pm \left[ \frac{1}{2} \left( 1 \mp \frac{\Delta}{\Omega_n} \right) \right]^{1/2},$$

(8)

$$\Omega_n = \left( \Delta^2 + 4f_n^2 \right)^{1/2}, \quad f_n = \sqrt{n!}, \quad \Delta = E - \omega.$$

(9)

In the 2-d subspaces $V_{2}^{(n)}, n \geq 1$, we have following 4 the elements of the density matrix

$$\rho_{(g,n)(g,n)} = \langle g,n | \rho | g,n \rangle \equiv \rho^{(n)}_{gg},$$

$$\rho_{(g,n)(e,(n-1))} = \langle g,n | \rho | e,(n-1) \rangle \equiv \rho^{(n)}_{ge},$$

$$\rho_{(e,(n-1))(g,n)} = \langle e,(n-1) | \rho | g,n \rangle \equiv \rho^{(n)}_{eg},$$

$$\rho_{(e,(n-1))(e,(n-1))} = \langle e,(n-1) | \rho | e,(n-1) \rangle \equiv \rho^{(n)}_{ee}.$$

(10)

Using exact solution (5) of the Schrodinger equation for this system without dissipation, we derive exact formulae for its density matrix

$$\rho^{(n)}_{11}(t) = \left[ 1 - \frac{2f_n^2}{\Omega_n^2} \right] \rho^{(n)}_{11}(0) + \frac{f_n}{\Omega_n} \left[ \frac{\Delta}{\Omega_n} \right] \rho^{(n)}_{12}(0) + \frac{2f_n^2}{\Omega_n^2} \rho^{(n)}_{22}(0),$$

$$\rho^{(n)}_{12}(t) = \left[ 1 - \frac{2f_n^2}{\Omega_n^2} \right] \rho^{(n)}_{12}(0) + \frac{f_n}{\Omega_n} \left[ \frac{\Delta}{\Omega_n} \right] \rho^{(n)}_{21}(0) + \frac{2f_n^2}{\Omega_n^2} \rho^{(n)}_{22}(0).$$

(11)
\[ \rho_{zz}^{(n)}(t) = \frac{2f_z^2}{\Omega_z^2} \left[ 1 - \cos(\Omega_z t) \right] \rho_{zz}^{(n)}(0) - \frac{f_z}{\Omega_z} \left( \frac{\Delta}{\Omega_z} \left[ 1 - \cos(\Omega_z t) \right] - i \sin(\Omega_z t) \right) \rho_{12}^{(n)}(0) \]
\[ - \frac{f_z}{\Omega_z} \left( \frac{\Delta}{\Omega_z} \left[ 1 - \cos(\Omega_z t) \right] + i \sin(\Omega_z t) \right) \rho_{21}^{(n)}(0) + \left( 1 - \frac{2f_z^2}{\Omega_z^2} \left[ 1 - \cos(\Omega_z t) \right] \right) \rho_{22}^{(n)}(0), \] (12)

\[ \rho_{12}^{(n)}(t) = \frac{f_z}{\Omega_z} \left( \frac{\Delta}{\Omega_z} \left[ 1 - \cos(\Omega_z t) \right] + i \sin(\Omega_z t) \right) \rho_{11}^{(n)}(0) - \frac{2f_z^2}{\Omega_z^2} \left[ 1 - \cos(\Omega_z t) \right] \rho_{11}^{(n)}(0) \]
\[ + \frac{2f_z^2}{\Omega_z^2} + \left( 1 - \frac{2f_z^2}{\Omega_z^2} \right) \cos(\Omega_z t) - i \frac{\Delta}{\Omega_z} \sin(\Omega_z t) \right) \rho_{12}^{(n)}(0), \] (13)

\[ \rho_{21}^{(n)}(t) = \frac{f_z}{\Omega_z} \left( \frac{\Delta}{\Omega_z} \left[ 1 - \cos(\Omega_z t) \right] - i \sin(\Omega_z t) \right) \rho_{11}^{(n)}(0) - \frac{2f_z^2}{\Omega_z^2} \left[ 1 - \cos(\Omega_z t) \right] \rho_{11}^{(n)}(0) \]
\[ + \frac{2f_z^2}{\Omega_z^2} + \left( 1 - \frac{2f_z^2}{\Omega_z^2} \right) \cos(\Omega_z t) + i \frac{\Delta}{\Omega_z} \sin(\Omega_z t) \right) \rho_{21}^{(n)}(0). \] (14)

Let us represent the density matrix in the form
\[ \rho = \begin{pmatrix} \rho_{(x,x)(x,x)} & \rho_{(x,x)(x,x-1)} \\ \rho_{(x,x-1)(x,x)} & \rho_{(x,x-1)(x,x-1)} \end{pmatrix} = \sigma_1 \rho_x^{(n)} + \sigma_2 \rho_y^{(n)} + \sigma_3 \rho_z^{(n)} + I \rho_0^{(n)}, \] (15)

where \( \sigma_1, \sigma_2, \sigma_3, \) and \( I \) are the Pauli matrices and the identity matrix. From equations (11)-(15) it follows that
\[ \rho_x^{(n)} = \frac{4f_z^2}{\Omega_z^2} + \left( 1 - \frac{4f_z^2}{\Omega_z^2} \right) \cos(\Omega_z t) \rho_x^{(n)}(0) - \frac{\Delta}{\Omega_z} \sin(\Omega_z t) \rho_y^{(n)}(0) - \frac{2f_z^2}{\Omega_z^2} \left[ 1 - \cos(\Omega_z t) \right] \rho_z^{(n)}(0), \] (16)
\[ \rho_y^{(n)} = -\frac{\Delta}{\Omega_z} \sin(\Omega_z t) \rho_x^{(n)}(0) + \cos(\Omega_z t) \rho_y^{(n)}(0) - \frac{2f_z^2}{\Omega_z^2} \sin(\Omega_z t) \rho_z^{(n)}(0), \] (17)
\[ \rho_z^{(n)} = -\frac{2f_z^2}{\Omega_z^2} \left[ 1 - \cos(\Omega_z t) \right] \rho_z^{(n)}(0) + \frac{2f_z^2}{\Omega_z^2} \sin(\Omega_z t) \rho_y^{(n)}(0) + \left( 1 - \frac{4f_z^2}{\Omega_z^2} \left[ 1 - \cos(\Omega_z t) \right] \right) \rho_z^{(n)}(0). \] (18)

Formulae (16)-(18) show that the density matrix at any \( t > 0 \) is completely determined by the initial values \( \rho_{\alpha}^{(n)}(0), \) \( \alpha = x, y, z. \) At the resonance \( \omega = E, \) we have
\[ \rho_x^{(n)}(t) = \rho_x^{(n)}(0), \] (19)
\[ \rho_y^{(n)} = \cos(\Omega_z t) \rho_y^{(n)}(0) - \sin(\Omega_z t) \rho_y^{(n)}(0), \] (20)
\[ \rho_z^{(n)} = \cos(\Omega_z t) \rho_z^{(n)}(0) + \sin(\Omega_z t) \rho_y^{(n)}(0). \] (21)

3. System with decoherence

Coupling to additional uncontrollable degrees of freedom leads to energy relaxation and dephasing in the system. In the Born-Markov approximation, the density matrix is determined by the quantum Liouville equation [3,4,5]
\[ \frac{d \rho}{dt} = -i [H, \rho] + L \rho, \] (22)

where \( L \rho \) describes the effect of the environment on the system.
\begin{equation}
L \rho = \lambda_\gamma \left(L_\gamma^\dagger \rho \right) + \lambda_\sigma \left(L_\sigma^\dagger \rho \right) + \lambda_\rho \left(L_\rho^\dagger \rho \right), \tag{23}
\end{equation}

with
\begin{align}
L_\gamma^\dagger \rho &= \gamma \rho \gamma - \frac{1}{2} \gamma^\dagger \gamma \rho - \frac{1}{2} \rho \gamma \gamma^\dagger, \tag{24} \\
L_\sigma^\dagger \rho &= \sigma \rho \sigma - \frac{1}{2} \sigma^\dagger \sigma \rho - \frac{1}{2} \rho \sigma \sigma^\dagger, \tag{25} \\
L_\rho^\dagger \rho &= \sigma \rho \sigma - \frac{1}{2} \sigma^\dagger \sigma \rho - \frac{1}{2} \rho \sigma \sigma^\dagger. \tag{26}
\end{align}

In equation (23), \( \lambda_\gamma \) is the resonator rate of photon loss, \( \lambda_\sigma \) is the qubit energy dissipation rate and \( \lambda_\rho \) is the qubit pure dephasing rate. Equation (22) is obtained in the Markov approximation. For high-quality factor systems, like high-Q transmission line resonators or (most) superconducting qubit, this approximation is accurate as the system is probing the environment in a very small frequency bandwidth [3-5].

Suppose that at the initial time moment the system is excited only up to the states in the subspace \( \mathcal{V}_N^{(N)} \). In this case the matrix elements of \( \rho \) in subspaces with higher energies vanish. Substituting the expressions (23)-(26) into equation (22), we derive the following system of rate equations
\begin{align}
\frac{d}{dt} \rho_{11}^{(N)} &= -\gamma_{1N} \rho_{11}^{(N)} - f_N \left[ \rho_{12}^{(N)} - \rho_{21}^{(N)} \right], \tag{27} \\
\frac{d}{dt} \rho_{12}^{(N)} &= -\gamma_{2N} \rho_{12}^{(N)} + f_N \left[ \rho_{12}^{(N)} - \rho_{21}^{(N)} \right], \tag{28} \\
\frac{d}{dt} \rho_{21}^{(N)} &= (\Delta - i \gamma_{1N}) \rho_{21}^{(N)} - f_N \left[ \rho_{11}^{(N)} - \rho_{22}^{(N)} \right], \tag{29} \\
\frac{d}{dt} \rho_{22}^{(N)} &= -(\Delta + i \gamma_{1N}) \rho_{22}^{(N)} + f_N \left[ \rho_{11}^{(N)} - \rho_{22}^{(N)} \right], \tag{30}
\end{align}

where
\begin{align}
\gamma_{1N} &= \frac{2N-1}{2} \lambda_\gamma + \lambda_\sigma + 2 \lambda_\rho, \quad \gamma_{1N} = (N-1) \lambda_\gamma + \lambda_\sigma, \quad \gamma_{2N} = N \lambda_\gamma. \tag{31}
\end{align}

The decoherence constants \( \lambda_\gamma, \lambda_\sigma \) and \( \lambda_\rho \) are very small in comparison with the energy differences between the energy levels. In the first order approximation we obtain the following solution of the rate equations
\begin{equation}
\rho_{ij}^{(N)} (t) = \sum_{kl} \frac{G_{ij,kl}^{(N)} (t) \rho_{kl}^{(N)} (0)}{i}, \text{with } ij, kl = 11, 12, 21, 22, \tag{32}
\end{equation}

with
\begin{equation}
G_{ij,kl}^{(N)} (t) = A_{ij,kl}^{(N)} \exp(-\Gamma_{1N} t) + B_{ij,kl}^{(N)} \exp(-\Gamma_{2N} t) + C_{ij,kl}^{(N)} \exp(-\Gamma_{1N} t) \cos(\Omega_N t) + i D_{ij,kl}^{(N)} \exp(-\Gamma_{1N} t) \sin(\Omega_N t), \tag{33}
\end{equation}

where
\begin{align}
A_{11,11}^{(N)} &= \frac{(\Gamma_{1N} - \gamma_{2N}) \Delta^2 + 2(\Gamma_{1N} - \gamma_{1N}) \Gamma_{1N}^2}{(\Gamma_{1N} - \Gamma_{2N}) \Omega_N^2}, \\
A_{11,12,11} &= A_{11,21}^{(N)} = A_{12,11}^{(N)} = A_{21,11}^{(N)} = \frac{(\Gamma_{1N} - \gamma_{2N}) f_N \Delta}{(\Gamma_{1N} - \Gamma_{2N}) \Omega_N^2}, \\
A_{11,22} &= A_{22,11}^{(N)} = \frac{2(\Gamma_{1N} - \gamma_{1N}) f_N^2}{(\Gamma_{1N} - \Gamma_{2N}) \Omega_N^2}, \\
A_{12,21} &= A_{21,12}^{(N)} = A_{21,21}^{(N)} = A_{21,21}^{(N)} = \frac{2 \Gamma_{1N} - (\gamma_{1N} + \gamma_{2N})}{(\Gamma_{1N} - \Gamma_{2N}) \Omega_N^2} f_N^2, \tag{34}
\end{align}
\[ A_{12,22}^{(N)} = A_{21,22}^{(N)} = A_{22,21}^{(N)} = A_{22,22}^{(N)} = \frac{(\Gamma_{1N} - \gamma_{1N})f_N \Delta}{(\Gamma_{1N} - \Gamma_{2N}) \Omega_N^2}, \quad A_{22,22}^{(N)} = \frac{(\Gamma_{1N} - \gamma_{1N})(\gamma_{1N}^2 + 2(\Gamma_{1N} - \gamma_{1N})f_N^2)}{(\Gamma_{1N} - \Gamma_{2N}) \Omega_N^2}. \]

\[ B_{11,11}^{(N)} = \frac{(\Gamma_{2N} - \gamma_{2N})\Delta^2 + 2(\Gamma_{2N} - \gamma_{2N})f_N^2}{(\Gamma_{2N} - \Gamma_{1N}) \Omega_N^2}, \quad B_{11,22}^{(N)} = B_{12,12}^{(N)} = B_{12,22}^{(N)} = B_{22,22}^{(N)} = \frac{2(\Gamma_{2N} - \gamma_{2N})f_N^2}{(\Gamma_{2N} - \Gamma_{1N}) \Omega_N^2}. \]

\[ B_{11,22}^{(N)} = \frac{(\Gamma_{2N} - \gamma_{2N})f_N^2}{(\Gamma_{2N} - \Gamma_{1N}) \Omega_N^2}, \quad B_{12,12}^{(N)} = B_{22,22}^{(N)} = \frac{2(\Gamma_{2N} - \gamma_{1N})(\gamma_{1N}^2 + 2(\Gamma_{2N} - \gamma_{1N})f_N^2)}{(\Gamma_{2N} - \Gamma_{1N}) \Omega_N^2}. \]

\[ B_{11,21}^{(N)} = B_{12,21}^{(N)} = B_{22,21}^{(N)} = B_{21,21}^{(N)} = \frac{2(\Gamma_{2N} - \gamma_{1N})(\gamma_{1N}^2 + 2(\Gamma_{2N} - \gamma_{1N})f_N^2)}{(\Gamma_{2N} - \Gamma_{1N}) \Omega_N^2}. \]

\[ C_{11,11} = -C_{11,22} = -C_{12,21} = C_{12,22} = C_{22,21} = -C_{22,22} = \frac{2f_N^2}{\Omega_N^2}, \quad C_{12,12} = C_{21,21} = \frac{\Omega_N^2 - 2f_N^2}{\Omega_N^2}. \]

\[ D_{11,11}^{(N)} = D_{11,22}^{(N)} = D_{12,21}^{(N)} = D_{12,22}^{(N)} = D_{22,22}^{(N)} = 0, \quad D_{11,12}^{(N)} = D_{21,22}^{(N)} = D_{22,11}^{(N)} = D_{22,12}^{(N)} = \frac{f_N}{\Omega_N}. \]

\[ \Gamma_{1N} = \frac{1}{2}[1 + \sqrt{1 - k_N}] \Gamma_N, \quad \Gamma_{2N} = \frac{1}{2}[1 - \sqrt{1 - k_N}] \Gamma_N. \]

\[ \Delta^2 = \frac{\gamma_{1N} + \gamma_{2N} + 2f_N^2 \gamma_{1N} + \gamma_{2N} + 2 \gamma_N}{\Omega_N^2}, \quad \Gamma_N^* = \frac{f_N^2}{\Omega_N^2} - (\gamma_{1N} + \gamma_{2N}) + \left(1 - 2 \frac{f_N^2}{\Omega_N^2}\right) \gamma_N. \]

\[ k_N = 4 \left(\Delta^2 + 4f_N^2\right)^2 \gamma_{1N}^2 \gamma_{2N}^2 + 2f_N^2 \gamma_{1N} \gamma_{2N} + 2 \gamma_N^2 \right] \left[\Delta^2 \left(\gamma_{1N} + \gamma_{2N}^2 + 2 f_N^2 \gamma_N + 2 \gamma_N\right) \right]. \]

Due to the term \( L\rho \) in equation (22) in the system of the rate equations for the matrix elements in the subspace \( V_2^{(N-1)} \) there are the contributions from the known matrix elements in the subspace \( V_2^{(N)} \).

In general, for any set of matrix elements (10) with \( n < N \) we have following rate equations

\[ \frac{d\rho_{11}^{(n)}}{dt} = -i\gamma_{11} \rho_{11}^{(n)} - f_{\gamma} \left[ \rho_{12}^{(n)} - \rho_{21}^{(n)} \right] + i\lambda_N \rho_{11}^{(n+1)}, \]

\[ \frac{d\rho_{12}^{(n)}}{dt} = -i\gamma_{12} \rho_{22}^{(n)} + f_{\gamma} \left[ \rho_{12}^{(n)} - \rho_{21}^{(n)} \right] + i\lambda_N \left[ \rho_{11}^{(n+1)} + \lambda_N \rho_{12}^{(n+1)} \right], \]

\[ \frac{d\rho_{12}^{(n)}}{dt} = (\Delta - i\gamma_{12}) \rho_{12}^{(n)} - f_{\gamma} \left[ \rho_{12}^{(n)} - \rho_{21}^{(n)} \right] + i\lambda_N \left[ \rho_{11}^{(n+1)} + \lambda_N \rho_{12}^{(n+1)} \right], \]

\[ \frac{d\rho_{12}^{(n)}}{dt} = (\Delta + i\gamma_{12}) \rho_{12}^{(n)} + f_{\gamma} \left[ \rho_{12}^{(n)} - \rho_{21}^{(n)} \right] + i\lambda_N \left[ \rho_{11}^{(n+1)} + \lambda_N \rho_{12}^{(n+1)} \right]. \]

Beside of the matrix elements of the form (10) there are also matrix elements between two states belonging to different subspaces \( V_2^{(n)} \) and \( V_2^{(m)} \), \( n \neq m \) as well as between the states in \( V_2^{(n)} \) and \( V_1 \). The solutions of rate equations for all above-mentioned matrix elements were derived and will be published elsewhere.
4. Conclusion
The rate equations for the density matrix of the strongly coupled QD-MC system were studied. It was shown that if the interaction of this system with the environment is neglected, then in each 2-dimensional invariant subspace $V_2^{(n)}, n > 1$, 4 matrix elements of density matrix $\rho$ satisfy a closed exactly solvable system of 4 rate equations. In this case the pair of two states in each subspace $V_2^{(n)}$ can be considered as a qubit. In the presence of the interaction of the system with the environment the rate equations for the matrix elements in $V_2^{(n-1)}$ always contain the contribution of those in $V_2^{(n)}$. The solution of the rate equation in the first order approximation with respect to the system-environment interaction was also established.

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