Nearly-Linear Time Approximate Scheduling Algorithms

Shi Li *

Abstract

We study nearly-linear time approximation algorithms for non-preemptive scheduling problems in two settings: the unrelated machine setting, and the identical machine with job precedence constraints setting. The objectives we study include makespan, weighted completion time, and $L_q$ norm of machine loads. We develop nearly-linear time approximation algorithms for the studied problems with $O(1)$-approximation ratios, many of which match the correspondent best known ratios achievable in polynomial time.

Our main technique is linear programming relaxation. For problems in the unrelated machine setting, we formulate mixed packing and covering LP relaxations of nearly-linear size, and solve them approximately using the nearly-linear time solver of Young [47]. We show the LP solutions can be rounded within $O(1)$-factor loss. For problems in the identical machine with precedence constraints setting, the precedence constraints cannot be formulated as packing or covering constraints. To achieve the claimed running time, we define a polytope for the constraints, and leverage the multiplicative weight update (MWU) method with an oracle which always returns solutions in the polytope.

Along the way of designing the oracle, we encounter the single-commodity maximum flow problem over a directed acyclic graph $G = (V, E)$, where sources and sinks have limited supplies and demands, but edges have infinite capacities. We develop a $1 - \frac{1}{1 + \epsilon}$-approximation for the problem in time $O\left(\frac{|E|}{\epsilon} \log |V|\right)$, which may be of independent interest.

1 Introduction

Scheduling theory is an important sub-area of combinatorial optimization, operations research and approximation algorithms. Over the past few decades, advanced techniques have been developed to design approximation algorithms for numerous scheduling problems, among which mathematical relaxation (e.g., linear/convex/semi-definite programming relaxations) is a prominent one. The algorithms based on the technique follow a two-step framework: solve some linear/convex/semi-definite program relaxation for the problem to obtain a fractional solution, and round it into an integral schedule. The main focus of the algorithm design in the literature has been the best approximation ratios that can be achieved in polynomial time. Many known LPs used have size much larger than that of the input, and a general convex/semi-definite program requires a large polynomial time to solve, making many of these algorithms impractical.

To address the issue, we initiate a systematic study of nearly-linear time approximation algorithms for classic non-preemptive scheduling problems. We focus on two of the most well-studied settings:

- **Unrelated machine setting.** We are given a set $J$ of $n$ jobs, a set $M$ of $m$ machines, a bipartite graph $G = (J, M, E)$ between $J$ and $M$, and a processing time $p_{i,j} \in \mathbb{Z}_{>0}$ for every $(j, i) \in E$, indicating the time it takes to process job $j$ on machine $i$. The output of a problem in this setting is an assignment $\sigma : J \to M$ of jobs to machines so that $(j, \sigma(j)) \in E$ for every $j \in J$. In the solution $\sigma$, we process each job $j$ on machine $\sigma(j)$. The goal of a problem is to minimize some objective function. We consider three objectives that have been extensively studied in the literature:

  - **Makespan ($C_{\text{max}}$).** This is the maximum completion time over all jobs, which is defined as $C_{\text{max}} := \max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{i,j}$. As this is also the maximum load over all machines, the problem is also called the load balancing problem.

*Department of Computer Science and Engineering, University at Buffalo. The work is in part supported by NSF-grant CCF-1844890.
- **Weighted Completion Time** ($\sum_j w_j C_j$). In this objective, we are additionally given a weight $w_j \in \mathbb{Z}_{>0}$ for every job $j \in J$, and the goal of the problem is to minimize $\sum_{j \in J} w_j C_j$, where $C_j$ is the completion time of $j$ on its assigned machine. When the assignment $\sigma: J \to M$ of jobs to machines is decided, it is well-known that the Smith’s rule\(^1\) gives the optimum schedule on each machine $i$. This gives the weighted completion time of $\sum_{i \in M} \sum_{j : \sigma(j) = i} \min\{w_j p_{i,j'}, w_j p_{i,j}\}$ for the assignment $\sigma$, where $j = j'$ is allowed in the second summation.

- **$L_q$-norm of machine loads for some $q \geq 1$ ($L_q(\text{loads})$)**. This is the $L_q$ norm of the load vector over all machines, that is, $L_q(\text{loads}) := \left( \sum_{i \in M} \left( \sum_{j \in \sigma^{-1}(i)} p_{i,j} \right)^q \right)^{1/q}$. When $q = \infty$, the objective becomes the makespan.

- **Identical machine with job precedence constraints setting**. In this setting, we are given a set $J$ of $n$ jobs, each job $j \in J$ with a processing time $p_j \in \mathbb{Z}_{>0}$, and the number $m \geq 1$ of identical machines. There are precedence constraints of the form $j \prec j'$, indicating that the job $j'$ can only start after job $j$ completes. The output of a problem in the setting is a completion time vector $(C_j)_{j \in J} \in \mathbb{Z}^J_{\geq 0}$, meaning that a job $j \in J$ is processed during the time interval $[C_j - p_j, C_j]$. We need $C_j \geq p_j$ for every $j \in J$, $C_j \leq C_j' - p_j$ for every $j \prec j'$, and every integer $t \geq 1$ is contained in $[C_j - p_j, C_j]$ for at most $m$ jobs $j \in J$.\(^2\) We consider the weighted completion time objective in this setting. That is, we are additionally given a weight $w_j \in \mathbb{Z}_{\geq 0}$ for every $j \in J$ and the goal is to minimize $\sum_{j \in J} w_j C_j$.

It is convenient for us to use the classic three-field notation $\alpha|\beta|\gamma$ in [16] to denote scheduling problems studied in this paper, where $\alpha$ indicates the machine model, $\beta$ gives the set of additional constraints, and $\gamma$ is the objective. In the first setting we have $\alpha = R$, meaning that the machines are unrelated. The problems with the three different objectives are denoted as $R || C_{\text{max}}$, $R || \sum_j w_j C_j$, and $R || L_q(\text{loads})$ respectively. The second setting is described by letting $\alpha = P$ and $\beta \in \{\text{prec}\}$, which means all the machines are identical and jobs have precedence constraints. The problem with the weighted completion time objective is then denoted as $P|\text{prec}|\sum_j w_j C_j$. We will also consider special cases of these problems, whose notations will be given when they are introduced.

There is a rich literature on designing approximation algorithms for these problems. For the unrelated makespan minimization problem, i.e., $R || C_{\text{max}}$, the classic result of Lenstra, Shmoys and Tardos [29] gives a 2-approximation, which remains the state-of-the-art result. The problem is NP-hard to approximate within a factor of better than 1.5. Plotkin, Shmoys and Tardos [35] studied fast approximation algorithms for the problem, as an application of their packing and covering LP solver. They developed a randomized $(2 + \epsilon)$-approximation algorithm in time $O((mn)^{1/3})$.\(^3\) So their algorithm is nearly-linear if $|E| = \Theta(mn)$. Much work on the problem has focused on a special case of the unrelated machine setting, i.e., the restricted assignment setting [45, 21, 22], where there is an intrinsic size $p_j \in \mathbb{Z}_{>0}$ for every $j \in J$, and for every $(j, i) \in E$ we have $p_{i,j} = p_j$. We use $\alpha = P_{\text{restricted}}$ in the three-field notation to denote this special case.

For the unrelated machine weighted completion time problem, i.e., $R || \sum_j w_j C_j$, many independent rounding algorithms achieve an approximation ratio of 1.5 [38, 42, 39, 31, 30]. Bansal, Svensson and Srinivasan [6] showed that the barrier of 1.5 is inherent for this type of algorithms. To overcome the barrier, they developed a novel dependent rounding scheme and a lifted SDP relaxation for the problem, leading to a $(1.5 - 1/2160000)$-approximation algorithm. The ratio has been improved to $1.5 - 1/6000$ by Li [31] and then to the current best ratio of 1.488 by Im and Shadloo [20]. Both [31] and [20] are based on some time-indexed LP relaxation for the problem.

Awerbuch et al. [3] introduced the unrelated machine problem to minimize the $L_q$ norm of machine loads, i.e., $R || L_q(\text{loads})$, in the online setting. They developed an algorithm with a competitive ratio of $O(q)$, which is the best possible for online algorithms. In the offline setting, Azar and Epstein [4] developed a convex-programming based algorithm, that achieves an approximation ratio of 2 for any $q > 1$ (the case

\(^1\)By this rule, we schedule jobs $j$ assigned to a machine $i$ using non-decreasing order of $p_{i,j}/w_j$.

\(^2\)It is a folklore that if the last property is satisfied, we can assign $\{(C_j - p_j) : j \in J\}$ to $m$ machines so that the intervals assigned to each machine are disjoint.

\(^3\)In this paper, we use $\tilde{O}(\cdot)$ to hide a factor that is poly-logarithmic in the input size of the instance being considered, which will be clear from the context, and polynomial in $1/\epsilon$. An algorithm is nearly linear if its running time is $O_\epsilon(\text{input size})$. 

2
Queyranne and Schulz [34] gave approximation ratios of 3 and 4 for the special case and the general problem bounded by a polynomial. Also recall that for these problems in the third row of Table 1. We remark that for the two problems 1

The goal of this paper is to design fast approximation algorithms for scheduling problems, and as is typical, we often expensive.

There is a vast literature on the problem of minimizing weighted completion time in the identical machine with job precedence constraints setting, i.e., the problem \( P[\text{prec}] \sum_j w_j C_j \). A special case of the problem where there is only \( m = 1 \) machine, denoted as \( 1[\text{prec}] \sum_j w_j C_j \), is already non-trivial. Hall et al. [17] developed a 2-approximation for the problem, which is the best possible under some stronger version of the unique game conjecture introduced by Bansal and Khot [5]. Another special case that is considered moderately in the literature is when all jobs have unit-size, denoted as \( P[\text{prec}, p_j = 1] \sum_j w_j C_j \). Munier, Queyranne and Schulz [34] gave approximation ratios of 3 and 4 for the special case and the general problem \( P[\text{prec}] \sum_j w_j C_j \) respectively. The ratios were improved to \( 1 + \sqrt{2} \) and \( 2 + 2 \ln 2 \) by Li [31]. Most algorithms [17, 34, 37, 31] for \( P[\text{prec}] \sum_j w_j C_j \) and the two special cases use the following framework: Solve some linear/convex program to obtain an order of the jobs respecting the precedence constraints. For every job in this order, schedule it as early as possible, without violating the precedence and \( m \)-machine constraints.

Most of the results we discussed focus on the best approximation ratio that can be achieved in polynomial time. Often the algorithms require a large polynomial running time to run. The running time for LP-based algorithms might be large for two reasons. First the size of the LP might already be large compared to size of the input. Second, the running time of a general LP solver is large. To solve a linear program with \( \bar{n} \) variables, \( \bar{m} \) constraints and \( \bar{N} \) non-zero coefficients up to a precision of \( \epsilon \), recent candidate algorithms include those by Lee and Sidford [26] with running time \( \tilde{O}\left((\bar{N} + \bar{m}^2)\sqrt{\bar{m}} \log \frac{1}{\epsilon}\right) \), and Lee, Song and Zhang [26] with running time \( \tilde{O}(\bar{n}^2 \log \frac{1}{\epsilon}) \) [27]4, where \( \omega \approx 2.373 \) is the current best exponent for matrix multiplication.

In a typical case where the number \( \bar{n} \) of variables is linear in the input size, the running time is superquadratic. To solve a general convex program, one can use the interior point or ellipsoid methods, which are often expensive.

### 1.1 Our Results

The goal of this paper is to design fast approximation algorithms for scheduling problems, and as is typical, we aim at nearly-linear time algorithms. In the unrelated machine setting, \( G = (J, M, E) \) denotes the bipartite graph between \( J \) and \( M \), and a nearly-linear time is one of order \( \tilde{O}(|E|) \). For the identical machine with precedence constraints setting, we use \( \kappa \) to denote the number of precedence constraints. A nearly-linear time algorithm runs in time \( \tilde{O}(n + \kappa) \). Unlike the polynomial running time scenario, we can not assume \( \kappa \) is transitive, as it will increase the number of precedence constraints to quadratic. Moreover, the best known algorithm computing the transitive closure of the precedence constraints takes \( O(n\kappa) \) time [36].

Our main results are given in Theorem 1.1 and Table 1. Many of the approximation ratios we obtain match the best known approximation ratios achieved in polynomial time:

**Theorem 1.1.** There are approximation algorithms for the problems listed in the first column of Table 1, with respective approximation ratios given in the second column of the table. The algorithms run in nearly-linear time except for those for \( 1[\text{prec}] \sum_j w_j C_j \) and \( P[\text{prec}] \sum_j w_j C_j \), which run in time \( \tilde{O}(\alpha(n + \kappa) \log p_{\max}) \), where \( p_{\max} := \max_{j \in J} p_j \) is the maximum job size.

For comparison, we also list the current best known approximation ratios achieved in polynomial time for these problems in the third row of Table 1. We remark that for the two problems \( 1[\text{prec}] \sum_j w_j C_j \) and \( P[\text{prec}] \sum_j w_j C_j \), the running time is \( \tilde{O}(n + \kappa) \log p_{\max} \). So the time is nearly-linear only when \( p_{\max} \) is bounded by a polynomial. Also recall that \( P_{\text{restricted}} \) is the restricted assignment setting, a special case of the unrelated machine setting.

---

4The result requires that the LP does not have redundant constraints.
Table 1: The problems we studied, the nearly-linear time approximation ratios we give, and their best known polynomial time approximation ratios. The running time for the algorithms for the two problems marked * is $O_{\epsilon}(\log n + \log p_{\max})$. The algorithms for the other problems run in nearly-linear time.

Along the way of designing some oracle in the identical machine with precedence constraints setting, we developed a nearly-linear time $(1 + \epsilon)$-approximation algorithm for the single commodity network flow problem in directed acyclic graphs, with bounded supplies and demands on sources and sinks, but infinite capacities on edges. As the result might be of independent interest, we summarize it in a separate theorem.

**Theorem 1.2.** Let $\epsilon > 0$ and the network flow instance $(G, S, T, a, b)$ be given as above. There is an $O(E \frac{\log |V|}{\epsilon})$-time algorithm that outputs a valid flow $f$ whose value is at least $\frac{1}{1 + \epsilon}$ times that of the maximum flow for the instance.

We remark that the condition that $G$ is acyclic is with out loss of generality. For a general directed graph $G$, we can find strongly connected components of $G$, and contract each component into a single node.

### 1.2 Our Techniques

All of our algorithms are based on linear programming: We design an LP relaxation of nearly-linear size, solve it in nearly linear time to obtain a $(1 + \epsilon)$-approximate solution, and round it into an integral schedule in nearly linear time. For different problems, the novelty comes from different steps.

For problems in the unrelated machine setting, the LPs we formulate are of the mixed packing and covering form, which can be solved by the algorithm of Young [47] in time $O\left(\frac{N \log m}{\epsilon^2}\right) = \tilde{O}_{\epsilon}(N)$, where $m$ and $N$ are the number of constraints and non-zero coefficients in the given LP respectively. Therefore, if additionally the LP has nearly-linear size, it can be solved in nearly-linear time.

For $R||C_{\text{max}}$, the natural LP relaxation has $O(|E|)$ size and thus can be solved in $\tilde{O}_{\epsilon}(|E|)$ time. We remark that the $O_{\epsilon}(mn)$-running time of [35] comes from both solving the LP, and rounding the LP solution. So even with the nearly-linear time mixed covering and packing LP solver, the algorithm of [35] still requires $O_{\epsilon}(mn)$ time. To round the fractional solution, we apply the grouping technique of [41] for the so called generalized assignment problem, but with a $(1 + \epsilon)$-slack. This gives us a bipartite graph $H = (J, V, E_H)$ satisfying
\(|N_H(J')| \geq (1+\epsilon)|J'|\) for every \(J' \subseteq J\), where \(N_H(J')\) is the set of neighbors of \(J'\) in \(H\). The property allows us to find a matching in \(H\) that covers \(J\) in nearly-linear time, which leads to a \((2+\epsilon)\)-approximate solution, matching the current best approximation of \(2\) in [29].

For the problem \(R||\sum_j w_j C_j\), our starting point is the time-indexed LP of Li [31], in which there is a variable \(x_{j,i,t}\) indicating if a job \(j\) is scheduled on the machine \(i\) and has completion time \(t\), and constraints that at most one job is processed at any time on any machine. To reduce the size of the LP to \(\hat{O}_\epsilon(|E|)\), we round all completion times to integer powers of \(1+\epsilon\). As the completion times are not precise, we could not impose the same packing constraints as in [31]. Instead, we require that on any machine \(i\) and for any time \(t\) that is an integer power of \(1+\epsilon\), the total volume of jobs processed by the time \(t\) on \(i\) is at most \(t\). We show this is sufficient to achieve the \(1.5+\epsilon\) approximation based on independent rounding.

For \(R||L_q(\text{loads})\), our LP has the same constraints as \(R||\sum_j w_j C_j\), but a different objective. We show the LP is at least as strong as the convex program of Azar and Epstein [4] and thus it can estimate the value of the instance up to a factor of \(\alpha(q)\), for the approximation ratio \(\alpha(q)\) obtained in Kumar et al. [25]. However, we do not know how to achieve the approximation ratio of \(\alpha(q)\) in [25] or even the ratio of \(2\) in [4] in nearly-linear time. Their algorithms use iterative rounding, which seems hard to implement in nearly-linear time. By losing another factor of \(2\), we can reduce the problem to the bipartite matching problem with \((1+\epsilon)\)-slack. This leads to the \((4+\epsilon)\)-approximation ratio. The rounding algorithm is similar to that of Chakraborty and Swamy [8] for the problem with an arbitrary symmetric norm. For the restricted assignment setting, the extra factor of \(2\) loss is not needed. For the case \(q=2\), the independent rounding procedure of [4], which still gives an approximation ratio of \(\sqrt{2}\), can be implemented in nearly-linear time.

Then we proceed to our techniques for the weighted completion time problems in the identical machine with precedence constraints setting, i.e., the problem \(P|\text{prec}|\sum_j w_j C_j\) and its special cases. Due to the precedence constraints, the LP relaxations do not have the mixed packing and covering form anymore. Nevertheless, the multiplicative weight update (MWU) framework can still be applied. We enclose the precedence constraints in a polytope \(Q\). In each iteration of the MWU framework, we guarantee that all these constraints are satisfied, i.e., the vector we obtain is in \(Q\). Other than the precedence constraints, we have \(\hat{O}_\epsilon(\log p_{\max})\) packing inequalities correspond the \(m\)-machine constraint. This is due to that we can round completion times to integer powers of \(1+\epsilon\).

The number of iterations the MWU framework takes is \(\hat{O}_\epsilon(m)\), where \(m\) is the number of packing constraints in the LP, without counting the constraints for \(Q\). Fortunately we have \(m = \hat{O}_\epsilon(\log p_{\max})\). To obtain the claimed \(\hat{O}_\epsilon((n+\kappa)\log p_{\max})\) time, we need to run each iteration of MWU in nearly linear time. The bottleneck comes from finding a vector in \(Q\) satisfying one aggregated packing constraint, that maximizes a linear objective with non-negative coefficients.

A key technical contribution of our paper is an oracle for the problem. For an appropriately defined directed acyclic graph \(G = (V,E)\), the polytope \(Q\) can be formulated as \(\{y \in [0,1]^V : y_v \leq y_u, \forall (v,u) \in E\}\). For two given row vectors \(a, b \in \mathbb{R}_{\geq 0}^V\), the aggregated LP in each iteration of MWU is: \(\max ay\) subject to \(y \in Q\) and \(by \leq 1\). Using LP duality, the problem is reduced to the single commodity maximum flow problem in Theorem 1.2. We have bounded supplies and demands on sources and sinks, but infinite capacities on edges. When allowing a \((1+\epsilon)\)-approximation for the scheduling problem, we need to find a flow whose value is at least the maximum value for the instance with sink capacities scaled by \(\frac{1}{1+\epsilon}\). We remark that the source capacities are not scaled by \(\frac{1}{1+\epsilon}\) in the instance we compare against. The weighted completion time of the LP solution is a fixed term minus a linear function over the solution with non-negative coefficients. Scaling source capacities imposes a multiplicative factor loss on the linear function, which does not transform to the same factor loss on the weighted completion time. So, the goal stated in Theorem 1.2 is slightly easier, in which we compare against the optimum value of the instance with both \(a\) and \(b\) scaled by \(\frac{1}{1+\epsilon}\).

An overview for the approximation algorithm for the network flow problem will be given in Section 3.3.
1.3 Other Related Work

The makespan minimization problem in the identical machine setting with precedence constraints, i.e., the problem $P|\text{prec}|C_{\text{max}}$, is another classic problem in scheduling theory. The seminal work of Graham [15] gives a simple greedy algorithm that achieves a $2$-approximation. On the negative side, Lenstra and Rinnooy Kan [28] proved a $(4/3-\varepsilon)$-hardness for the problem. Under the stronger version of the Unique Game Conjecture (UGC) introduced by Bansal and Khot [5], Svensson [44] showed that the problem is hard to approximate within a factor of $2-\varepsilon$ for any $\varepsilon > 0$. Much work has focused on the special case where $m = O(1)$ and all jobs have size $1$ [30, 14, 32], for which obtaining a PTAS is a long-standing open problem.

The multiplicative weight update (MWU) method for solving linear programs has played an important role in a wide range of applications. Some of its foundational work can be found in a beautiful survey by Arora, Hazan and Kale [2]. There has been a vast literature on solving packing, covering, and mixed packing LPs approximately to a factor of $1+\varepsilon$ using iterative methods [40, 35, 33, 46, 13, 24, 23, 47, 1, 11]. In particular, to solve a mixed packing and covering LP with $\bar{n}$ variables, $\bar{m}$ constraints and $\bar{N}$ non-zero coefficients, the algorithm of Young [47] returns $(1+\varepsilon)$-approximation deterministically in $O\left(\frac{\bar{N}\ln\bar{m}}{\varepsilon^2}\right)$ time. The dependence on $\varepsilon$ has been improved slightly by Chekuri and Quanrud [11], who gave a randomized algorithm with running time $\tilde{O}\left(\frac{\bar{N}}{\varepsilon^2} + \frac{\bar{n}}{\varepsilon^2} + \frac{\bar{N}}{\varepsilon}\right)$, where $\tilde{O}(\cdot)$ hides a poly-logarithmic factor.

**Organization** The rest of the paper is organized as follows. In Section 2, we define some elementary notations used across the paper, and describe the result of Young [47] on solving mixed packing and covering LPs, and a template solver for packing LPs over an “easy” polytope. In Section 3, we give our algorithms for the problems to minimize weighted completion time in the identical machine with precedence constraints setting, i.e., $P|\text{prec}||\text{w}\sum w_jC_j$ and its special cases. In Section 4, we describe the algorithms for problems in the unrelated machine setting. The missing proofs in the three sections can be found in Appendices A, B and C respectively. An important component of the algorithms in Section 3 is the nearly-linear time $(1+\varepsilon)$-approximation for the network flow problem with capacities on sources and sinks and infinite edge capacities. We give in Section 3.3 an overview of the algorithm, which also proves Theorem 1.2, and defer the details to Appendix D. In Section E, we show how to handle the case where input integers are not polynomially bounded for different problems. The other missing proofs, including how to derandomize rounding algorithms, how to use the self-balancing binary search tree data structure to run the list scheduling algorithm in nearly linear time, can be found in Appendix F.

2 Preliminaries

We use bold lowercase letters to denote vectors, and their correspondent italic letters to denote their coordinates. We use bold uppercase letters to denote matrices. $\mathbf{0}$ and $\mathbf{1}$ are used to denote the all-0 and all-1 vectors whose domain can be inferred from the context. Given a template vector $v$ over some finite domain, and a subset $S$ of the domain, let $v(S) := \sum_{e \in S} v_e$ be the sum of $v$-values over elements in $S$.

Given an (undirected) graph $H = (V_H, E_H)$, we use $\delta_H(v), N_H(v), \delta_H(U), N_H(U)$ to respectively denote the sets of incident edges of $v \in V_H$, neighbors of $v$, edges between the set $U \subseteq V_H$ and $V_H \setminus U$, and vertices in $V_H \setminus U$ with at least one neighbor in $U$, in the graph $H$. Given a directed graph $H = (V_H, E_H)$, for every $v \in V_H$, we use $\delta_H^+(v)$ and $\delta_H^-(v)$ to denote the sets of outgoing and incoming edges of $v$ respectively. For every $U \subseteq V_H$, let $\delta_H^+(U) := \{(u, v) \in E_H : u \in U, v \notin U\}$ and $\delta_H^-(U) := \{(u, v) \in E_H : u \notin U, v \in U\}$ be the sets of edges from $U$ to $V_H \setminus U$ and from $V_H \setminus U$ to $U$ respectively. When $H = G$ for the graph $G$ in the context (which can be undirected or directed), we omit the subscript $H$ in the notations.

For cleanness of exposition, we use $\tilde{O}(\cdot)$ to hide factors that are polynomial in $\frac{1}{\varepsilon}$ and poly-logarithmic in the size of the input. As we gave the first nearly-linear time algorithms for the studied problems, the hidden factors are small compared to the improvements we make. For all (undirected or directed) graphs $H = (V_H, E_H)$ we deal with, we assume every vertex is incident to at least one edge so $|E_H| = \Omega(V_H)$; one can verify that isolated vertices can be handled easily. For any $a \in \mathbb{R}$, we define $(a)_+$ as $\max\{a, 0\}$.
2.1 Nearly-Linear Time Mixed Packing and Covering LP Solver

A mixed packing and covering LP is an LP of the following form:

\[
\text{find } x \text{ such that } x \geq 0, \quad Px \leq 1 \quad \text{and} \quad Cx \geq 1, \quad \text{(MPC)}
\]

where \( P \in \mathbb{R}^{\tilde{m} \times \tilde{n}} \) and \( C \in \mathbb{R}^{\tilde{m} \times \tilde{n}} \) for some positive integers \( \tilde{n}, \tilde{m}_P, \tilde{m}_C \). Let \( \tilde{m} = \tilde{m}_P + \tilde{m}_C \) and \( \tilde{N} \) be the total number of non-zeros in \( P \) and \( C \). Young [47] developed a nearly-linear time algorithm that solves (MPC) approximately:

**Theorem 2.1** ([47]). Given an instance of (MPC) and \( \epsilon > 0 \), there is an \( O\left(\frac{\tilde{N}\ln \tilde{m}}{\epsilon^2}\right) \)-time algorithm that either claims (MPC) is infeasible, or outputs a \( x \in \mathbb{R}^{\tilde{N}}_{\geq 0} \) such that \( Px \leq (1 + \epsilon)1 \) and \( Cx \geq \frac{1}{1 + \epsilon} \).

2.2 Template Packing LP Solver over a Simple Polytope

In this section, we describe a template MWU-based LP solver for packing linear program with an additional requirement that the solution is inside an “easy” polytope \( \mathbb{Q} \). The framework we describe here is introduced in [9] and later reformulated in [10].

Let \( P \in \mathbb{R}^{\tilde{m} \times \tilde{n}} \) be a non-negative matrix, with \( \tilde{N} \) non-zero entries. Let \( a \in \mathbb{R}^\tilde{n} \) be a row vector, and \( Q \subseteq \mathbb{R}^\tilde{n}_{\geq 0} \) be a polytope which is defined by “easy” constraints. We focus on the following linear program:

\[
\max ax \quad \text{subject to} \quad x \in \mathbb{Q} \quad \text{and} \quad Px \leq 1. \quad \text{(P}_Q\text{)}
\]

Throughout the paper, we guarantee (P\_Q) is feasible.

**Definition 2.2.** Let \( \epsilon \in (0, 1), \phi > 0 \) be two parameters. An \((\epsilon, \phi)\)-approximate solution to (P\_Q) is a vector \( x \in \mathbb{Q} \) satisfying \( Px \leq (1 + \epsilon)1 \) and \( ax \geq ax^* - \phi \), where \( x^* \in \mathbb{Q} \) is the optimum solution to (P\_Q).

As a hindsight, we only allow a loss of an additive factor \( \phi \) in the objective function of the LP for \( P|\text{prec}||\sum_j w_jC_j| \), which will be set to be a polynomially small term. As is typical in a MWU framework, we need to solve the following LP where the constraints \( Px \leq 1 \) are aggregated into one constraint \( by \leq 1 \), where \( b \in \mathbb{R}^{\tilde{n}}_{\geq 0} \) is a row vector:

\[
\max ay \quad \text{subject to} \quad y \in \mathbb{Q} \quad \text{and} \quad by \leq 1. \quad \text{(1)}
\]

Again we guarantee (1) is always feasible.

**Definition 2.3.** Let \( \epsilon \in (0, 1), \phi > 0 \) be two parameters. An \((\epsilon, \phi)\)-approximate solution to (1) is a vector \( y \in \mathbb{Q} \) satisfying \( by \leq 1 + \epsilon \) and \( ay \geq ay^* - \phi \), where \( y^* \) is the optimum solution to the LP. An \((\epsilon, \phi)\)-oracle for (1) is an algorithm that, given an instance of (1), and \( \epsilon \in (0, 1), \phi > 0 \), outputs an \((\epsilon, \phi)\)-approximate solution \( y \) to (1).

The template LP solver is described in Algorithm 1, where we use \( P_i \) to denote the \( i \)-th row vector of \( P \). By our assumption that (P\_Q) is feasible, the instance of (1) defined in every execution of Step 3 is also feasible. The performance of the algorithm is summarized in the following theorem.

**Theorem 2.4.** Algorithm 1 will return an \( (O(\epsilon), \phi)\)-approximate solution \( x \) to (P\_Q), within \( O\left(\frac{\tilde{m}\log \tilde{m}}{\epsilon^2}\right) \) iterations of Loop 2.

For each iteration of loop 2, the steps other than Step 3 takes \( O(\tilde{N}) \) time. Therefore, the running time of Algorithm 1 is \( O\left(\frac{\tilde{m}\log \tilde{m}\tilde{N}}{\epsilon^2}\right) \), plus the time for running the oracle \( O\left(\frac{\tilde{m}\log \tilde{m}}{\epsilon^2}\right) \) times.
Algorithm 1 LP Solver for \((P_\mathcal{O})\)

Input: an instance of \((P_\mathcal{O})\), \(\epsilon \in (0, 1)\), \(\phi > 0\), and \((\epsilon, \phi)\)-oracle \(\mathcal{O}\) for (1)

Output: an \((O(\epsilon), \phi)\)-approximate solution \(x\) for \((P_\mathcal{O})\)

\[\begin{align*}
1: & \quad t \leftarrow 0, \rho \leftarrow \frac{\ln m}{\ln \phi}, \quad \mathbf{x}^{(0)} \leftarrow 0 \in \mathbb{R}^m_{\geq 0}, \quad \mathbf{u}^{(0)} \leftarrow 1 \in \mathbb{R}^m_{> 0} \quad \Rightarrow \quad \mathbf{x}^{(i)}'s \text{ are column vectors and } \mathbf{u}^{(i)}'s \text{ are row vectors} \\
2: & \quad \text{while } t < 1 \text{ do} \\
3: & \quad \delta \leftarrow \min \left\{ \min_{i \in [m]} \frac{1}{\rho} \mathbf{P}^i \mathbf{y}, 1 - t \right\} \\
4: & \quad \text{for every } i \in [m] \text{ do } u_i^{(t+\delta)} \leftarrow u_i^{(t)} \cdot \exp \left( \delta \rho \cdot \mathbf{P} \mathbf{y} \right) \\
5: & \quad \mathbf{x}^{(t+\delta)} \leftarrow \mathbf{x}^{(t)} + \delta \mathbf{y}, t \leftarrow t + \delta \\
6: & \quad \text{return } x := \mathbf{x}^{(1)}
\end{align*}\]

3 Identical Machine Precedence Constrained Scheduling to Minimize Weighted Completion Time

In this section, we give our nearly-linear time algorithms for \(P|\text{prec}| \sum_j w_j C_j\) and its two special cases \(1|\text{prec}| \sum_j w_j C_j\) and \(P|\text{prec}, p_j = 1| \sum_j w_j C_j\). We describe the LP in Section 3.1 and the rounding algorithms in Section 3.2. The oracle for solving (1) in the MWU framework is described in Section 3.3, with the key component on solving the network flow problem and the proof of Theorem 1.2 deferred to Section D.

3.1 Linear Programming Relaxation

In this section, we describe the LP relaxation for \(P|\text{prec}| \sum_j w_j C_j\). To concentrate on the main ideas, we assume \(p_{\text{max}} := \max_{j \in J} p_j\) is bounded by \(\text{poly}(n)\), and defer the general case to Appendix E.1. We remark that a direct implementation of the algorithm would give a \(\tilde{O}(n + \kappa)^3 p_{\text{max}}\) running time; additional ideas are needed to reduce the \(\log^3 p_{\text{max}}\) term to \(\log p_{\text{max}}\).

For every \(j \in J\), let \(q_j\) be the maximum total length of jobs in a precedence chain ending at \(j\). This can be computed in \(O(n + \kappa)\) time using dynamic programming. We define a list of completion times as follows: let \(\tau_0 = 0, \tau_d = (1 + \epsilon)^{d-1}\) for every integer \(d \geq 1\). Let \(D\) be the smallest integer such that \(\tau_D \geq p(J)\). Then \(D = \tilde{O}\left(\frac{\log n}{\epsilon}\right)\) since we assumed \(p(J) = \text{poly}(n)\). For every integer \(d \in [0, D - 1]\) we define \(\eta_d := \tau_{d+1} - \tau_d\).

Let \(d_{\text{min}}^j = 0\) and \(d_{\text{max}}^j = D\) for every \(j \in J\). Later in the super-polynomial \(p_{\text{max}}\) case, we define \(d_{\text{min}}^j\)'s and \(d_{\text{max}}^j\)'s differently. The linear program is defined by the objective (2) and constraints (3-7).

\[
\begin{align*}
\min & \quad w(J)\tau_D - \sum_{j \in J} \sum_{d=1}^{D-1} \eta_d x_{j,d} \\
\text{s.t.} & \quad x_{j,d} \leq x_{j,d+1} \quad \forall j \in J, d \in [0, D] \quad \text{(3)} \quad x_{j,d} = 0 \quad \forall j \in J, d \in [0, d_{\text{min}}^j] \text{ or } \tau_d < q_j \quad \text{(6)} \\
\text{s.t.} & \quad x_{j,d} \geq x_{j',d} \quad \forall j < j', d \in [0, D] \quad \text{(4)} \quad x_{j,d} = 1 \quad \forall j \in J, d \in [d_{\text{max}}^j, D] \quad \text{(7)} \\
\sum_{j \in J} p_j x_{j,d} & \leq m\tau_d \quad \forall d \in [D] \quad \text{(5)}
\end{align*}
\]

In the correspondent 0/1-integer program, \(x_{j,d}\) is intended to indicate whether \(j\) has completion time at most \(\tau_d\). (3) says if \(j\) has completion time at most \(\tau_d\), then it has completion time at most \(\tau_{d+1}\). (4) requires that for two jobs \(j < j'\), if \(j'\) has completion time at most \(\tau_d\), then so does \(j\). (5) requires the total size of jobs with completion time at most \(\tau_d\) to be at most \(m\tau_d\) for every \(d \in [0, D]\). (6) says a job can not complete before \(q_j\). For the \(p_{\text{max}} = \text{poly}(n)\) setting, the condition \(d \in [0, d_{\text{min}}^j]\) is redundant.\(^5\) (7) says a job must

\(^5\)We may assume there are no jobs \(j\) with \(q_j = 0\) since they can be removed.
Therefore, in each iteration the oracle takes $\tilde{O}(1)$ iterations. So, the running time of the algorithm is $\tilde{O}(1)$ and $\bar{O}(1)$ if we have the constraint $x_{j,d} \leq x_{j',d'}$ in (3) or (4). Define

$$Q := \{x \in [0,1]^V : x_v \leq x_u, \forall (v,u) \in E\}.$$ 

Let $P \in \mathbb{R}_{\geq 0}^{[0,\bar{D}] \times V}$ so that (5) can be written as $Px \leq 1$. Notice that each $v \in V$ participates in exactly one row of $P$ and thus $P$ has $\tilde{N} := |V|$ non-zeros. Let $a_{j,d} = w_j \eta_d$ for every $(j,d) \in V$. Then minimizing (2) is equivalent to maximizing $ax$. Our LP becomes $\max ax$ subject to $x \in Q$ and $Px \leq 1$, which is exactly $(P_Q)$. Let $x^*$ be the optimum solution to the LP.

To apply Algorithm 1, we need an $(\epsilon, \phi)$-oracle for (1) with some appropriate value of $\phi$. This is summarized in the following theorem, which we prove in Section 3.3 and Appendix D.

**Theorem 3.2.** Let $G = (V,E)$ be a directed acyclic graph and $Q := \{y \in [0,1]^V : y_v \leq y_u, \forall (v,u) \in E\}$. Let $b, a \in \mathbb{R}_{\geq 0}^V$ be two row vectors. Let $y^*$ be the $y \in Q$ satisfying $by \leq 1$ with the maximum $ay$. Let $\epsilon \in (0,1)$ be a time point and let $\eta \in (0,|a|_1/2)$. Then, in $\tilde{O}_\epsilon \left( |E| \cdot \log^2 \frac{|Q|}{\phi} \right)$ time, we can find a $y \in Q$ satisfying $by \leq 1 + \epsilon$ and $ay \geq ay^* - \phi$.

We run Algorithm 1 on our instance of $(P_Q)$ defined by $Q, P$ and $a$, with the $(\epsilon, \phi)$-oracle given in Theorem 3.2 to output an $O(\epsilon, \phi)$-approximate solution to $(P_Q)$, where $\phi = \epsilon \cdot w(J) \leq \epsilon \cdot \opt$. Then the $x$ returned by the template LP solver has $x \in Q, Px \leq (1 + O(\epsilon))1$ and $ax \geq ax^* - \phi$. Then, we have $w(J)\tau_D - ax \leq w(J)\tau_D - ax^* + \phi = lp + \phi \leq (1 + \epsilon)\opt + \epsilon \cdot \opt = (1 + 2\epsilon)\opt$.

The running time of Algorithm 1, excluding Step 3, is $O \left( \frac{m \ln \tilde{m} \cdot \tilde{N}}{\epsilon^2} \right) = \tilde{O}_\epsilon(n)$, as $m = D = O \left( \frac{\log n}{\epsilon} \right) = \tilde{O}_\epsilon(1)$ and $\tilde{N} = O \left( \frac{n \log n}{\epsilon^2} \right) = \tilde{O}_\epsilon(n)$. Also $|a|_1 \leq \poly(n) \cdot \phi$ as all job sizes are polynomially bounded.

Therefore, in each iteration the oracle takes $\tilde{O}_\epsilon(|E|) = \tilde{O}_\epsilon(n + \kappa)$ time, and there are at most $O \left( \frac{\tilde{m} \log \tilde{m}}{\epsilon^2} \right) = \tilde{O}_\epsilon(1)$ iterations. So, the running time of the algorithm is $\tilde{O}_\epsilon(n + \kappa)$, assuming $p_{\max} = \poly(n)$.

Before proceeding to the next section, we summarize the properties of our $x \in [0,1]^J \times [0,\bar{D}]$. Its value to (2) is at most $(1 + O(\epsilon))\opt$. $x$ satisfies all constraints in LP(2), except (5), which is satisfied with a factor of $1 + O(\epsilon)$ on the right side.

### 3.2 Rounding Algorithms

After we obtain the solution $x$, we round it to an integral one using problem-dependent algorithms. For every $j \in J$, we define

$$C_j := \sum_{d=1}^{D} \tau_d (x_{j,d} - x_{j,d-1}) = \sum_{d=1}^{D-1} (\tau_d - \tau_{d+1}) x_{j,d} + \tau_D = \tau_D - \sum_{d=1}^{D-1} \eta_d x_{j,d}$$

to be the fractional completion time of $j$. Then $x$ has value $\sum_{j \in J} w_j C_j$ to the LP(2).

**Claim 3.3.** For a job $j \in J$, we have $C_j \geq q_j$. For two jobs $j < j'$, we have $C_j \leq C_{j'}$.

**Lemma 3.4.** Let $C^* \geq 0$ be a time point and let $J' := \{j \in J : C_j \leq C^*\}$. Then, we have

$$p(J') \leq (2 + O(\epsilon))mC^*.$$

The lemma immediately gives us a $\tilde{O}_s(n + \kappa)$-time $(2 + O(\varepsilon))$-approximation for $P|\text{prec}|\sum_j w_j C_j$. We schedule the jobs on the single machine in non-decreasing order of $C_j$ values, guaranteeing that if $j < j'$ then $j$ is scheduled before $j'$. Then the completion time $\tilde{C}_j$, of a job $j^*$ is at most $p(\{j \in J: C_j \leq C_j^*\}) \leq (2 + O(\varepsilon))C_j^*$. The weighted completion time of the schedule then is at most $(2+O(\varepsilon))\sum_j w_j \tilde{C}_j \leq (2+O(\varepsilon))\text{opt}$ as the value of $x$ to LP(2) is at most $(1 + O(\varepsilon))\text{opt}$.

When $m > 1$, we use a simple job-driven list scheduling algorithm as in [31]. In addition to the set $J$ of jobs with job sizes and precedence constraints, we are given a vector $(F_j)_{j \in J} \in \mathbb{R}_{\geq 0}^J$ that respects the precedence constraints: For every $j < j'$ we have $F_j \leq F_{j'}$. Notice it is possible that $F_j = F_{j'}$ for $j < j'$.

In the algorithm, for every job $j$ in non-decreasing order of $F_j$ values, breaking ties so that if $j' < j''$ then $j'$ is handled before $j''$, we schedule $j$ as early as possible without violating the $m$-machine constraint and the precedence constraints. The pseudo-code is given in Algorithm 2. In the algorithm, the congestion of a set of scheduling intervals is the maximum number of intervals in the set covering a same unit-time slot.

### Algorithm 2 list-scheduling($(F_j)_{j \in J}$)

**Input:** a vector $(F_j)_{j \in J} \in \mathbb{R}_{\geq 0}^J$ respecting the precedence constraints  
**Output:** a schedule of jobs, given by starting times $(\tilde{S}_j)_{j \in J}$ and completion times $(\tilde{C}_j = \tilde{S}_j + p_j)_{j \in J}$

1. for every $j \in J$ in non-decreasing order of $F_j$, breaking ties first using $<$ and then arbitrarily do  
   2. $t \leftarrow \max_{j < j'} \tilde{C}_j$, assuming the maximum of an empty set is 0  
   3. find the minimum $t' \geq t$ such that we can schedule $j$ in interval $(t', t' + p_j)$, without increasing the congestion of the scheduling intervals to $m + 1$  
   4. $\tilde{S}_j \leftarrow t'$, $\tilde{C}_j \leftarrow t' + p_j$, and schedule $j$ in $(\tilde{S}_j, \tilde{C}_j)$  
5. return $((\tilde{C}_j)_{j \in J})$

To guarantee that the algorithm runs in $\tilde{O}_s(n + \kappa)$ time, we need to show how to find the $t'$ in Step 3 in amortized $O(\log n)$ time. This is done by maintaining two self-balancing binary search trees. We defer the details to Section F.2.

Throughout this section, we fix a job $j^* \in J$ and analyze the completion time $\tilde{C}_{j^*}$ of the job in the constructed schedule. We focus on the moment where $\tilde{S}_{j^*}$ and $\tilde{C}_{j^*}$ are decided; that is, the end of the iteration in which we handle $j^*$. We call the scheduled constructed so far the schedule of interest (jobs handled after $j^*$ are not scheduled yet). In the schedule, a unit time slot $(t - 1, t]$ is said to be busy if exactly $m$ jobs have scheduling intervals covering $(t - 1, t]$; otherwise we say $(t - 1, t]$ is idle. Let $T_{\text{busy}}$ and $T_{\text{idle}}$ be the number of busy and idle unit-time slots before $\tilde{C}_{j^*}$, w.r.t the schedule of interest. Then $\tilde{C}_{j^*} = T_{\text{busy}} + T_{\text{idle}}$.

**Claim 3.5.** $T_{\text{busy}} \leq \frac{1}{m}p(\{j \in J: F_j \leq F_{j^*}\})$.

**Proof.** The total size of jobs in the schedule of interest is at most $p(\{j \in J: F_j \leq F_{j^*}\})$. So, $mT_{\text{busy}} \leq p(\{j \in J: F_j \leq F_{j^*}\})$. Dividing both sides by $m$ gives the claim. \qed

**Lemma 3.6.** [34, 31] When all jobs have unit sizes, we have $T_{\text{idle}} \leq q_{j^*}$.

So, if we let $F_j = C_j$ for every $j$ (notice that $(C_j)_{j \in J}$ respects the precedence constraints), and apply Claim 3.5 and Lemma 3.6, we have

$$\tilde{C}_{j^*} = T_{\text{busy}} + T_{\text{idle}} \leq \frac{1}{m}(\{j \in J: C_j \leq C_{j^*}\}) + q_{j^*} \leq (2 + O(\varepsilon))C_{j^*} + C_{j^*} = (3 + O(\varepsilon))C_{j^*}.$$ 

The second inequality used Lemma 3.4. This gives us a $\tilde{O}_s(n + \kappa)$-time $(3 + O(\varepsilon))$-approximation for $P|\text{prec}, p_j = 1|\sum_j w_j C_j$. In Section F.1, we show the approximation ratio of $1 + \sqrt{2}$ due to [31] can be recovered using our LP relaxation.

Finally, we focus on the general problem $P|\text{prec}|\sum_j w_j C_j$. As our LP is weaker, we could not recover the approximation ratios of 4 in [34] or $2 + 2\ln 2$ in [31]. Instead, we obtain a worse ratio of $6 + O(\varepsilon)$. 

10
Lemma 3.7. Let $\theta \in (0, 1)$ be a number such that for every $j < j'$, we have $F_{j'} - F_j \geq \theta p_j$. Then $T_{idle} \leq F_j^* + p_j^*$. 

[34] used $F_j = C_j - \frac{p_j}{2}$ to obtain their 4-approximation for the problem. However we are not guaranteed that the vector $(C_j - \frac{p_j}{2})_{j \in J}$ respects the precedence constraints. Instead, we define $F_j = C_j + q_j - p_j$ for every $j \in J$ and call this scheduling $(F_j)_{j \in J}$. $q_j - p_j$ is the maximum size of jobs in a precedence chain ending at some predecessor of $j$. If $j < j'$ then $F_{j'} - F_j = (C_{j'} + q_{j'} - p_{j'}) - (C_j + q_j - p_j) \geq q_{j'} - p_{j'} - q_j + p_j \geq p_j$. So, $(F_j)_{j \in J}$ respects the precedence constraints, and it satisfies the condition in Lemma 3.7 with $\theta = 1$. By the lemma, we have $T_{idle} \leq F_{j'} + p_{j'} = C_{j'} + q_{j'} \leq 2C_{j'}$.

The first and third inequalities used Claim 3.5 and Lemma 3.4. Therefore, we have $C_j^* \leq (6 + O(\epsilon))C_{j^*}$, which gives a $O(n + \epsilon)$-time $(6 + O(\epsilon))$-approximation for $P|\text{prec}|\sum_j w_j C_j$, assuming $p_{\text{max}} = \text{poly}(n)$.

3.3 Approximate Oracle for (1): Proof of Theorem 3.2

In this section, we prove Theorem 3.2. Throughout this section, we fix $G = (V,E)$, $Q, b, a, y^*, \epsilon$ and $\phi$ as in Theorem 3.2, among which $y^*$ is not given to our algorithm. For any directed graph $H = (V_H, E_H)$, and two subsets $U, U' \subseteq V_H$, $U \rightarrow_H U'$ holds if there is a path from some vertex in $U$ to some vertex in $U'$ in $H$. If there is no such a path, then $U \not\rightarrow_H U'$ holds. If $U$ or $U'$ is a singleton set, we can replace it with the vertex it contains. When $H = G$, the subscripts $H$ in the notations defined above can be omitted.

Let $S = \{s \in V : a_s > 0\}$ and $T = \{t \in V : b_t > 0\}$. We prove that the following properties can be assumed w.l.o.g.

**Lemma 3.8.** To prove Theorem 3.2, we can w.l.o.g assume

- $S \cap T = \emptyset$ and there are no edges from $S$ to $T$,
- $\delta^-(s) = \emptyset$ for every $s \in S$,
- $\delta^+(t) = \emptyset$ for every $t \in T$, and
- for every $v \in V$, we have $S \rightarrow v$ and $v \rightarrow T$.

Finally, we can w.l.o.g replace the constraint $y \in [0, 1]^V$ by $y_s \leq 1$ for every $s \in S$ and $y_t \geq 0$ for every $t \in T$.

With Lemma 3.8, the LP in Theorem 3.2 becomes LP(8), which has constraints (9-12).
Definition 3.9. For any \( \gamma \geq 0 \), we use NFP\(_{\gamma} \) to denote the following single-commodity flow problem. We are given the network \( G = (V, E, ) \) with sources \( S \) and sinks \( T \). Each source \( s \in S \) has a supply of \( a_s > 0 \), each sink \( t \in T \) has a demand \( \gamma t \), and the capacities of all edges in \( G \) are infinite.

So, a valid flow for NFP\(_{\gamma} \) is a vector \( f \in \mathbb{R}^E \) satisfying \( f(\delta^+(s)) \leq a_s \) for every \( s \in S \), \( f(\delta^+(t)) \leq \gamma t \) for every \( t \in T \), and \( f(\delta^+(v)) = f(\delta^-(v)) \) for every \( v \in V \setminus (S \cup T) \). Let \( F_{\gamma} \subseteq \mathbb{R}^E \) denote the set of valid flows \( f \) for NFP\(_{\gamma} \). The value of a flow \( f \in F_{\gamma} \), denoted as \( \text{val}(f) \), is defined as \( f(\delta^+(S)) = f(\delta^-(T)) \).

The value of the dual LP(13) is the minimum, over all \( \gamma \geq 0 \), of \( \gamma + |a|_1 - \text{opt}_\gamma \), where \( \text{opt}_\gamma \) is the value of the maximum flow for NFP\(_{\gamma} \). To understand better why \( \gamma + |a|_1 - \text{opt}_\gamma \) is an upper bound on the value of (8), assume in the optimum flow for NFP\(_{\gamma} \), each \( s \in S \) sends \( a_s^* \) units flow. Then, the flow, (10) and (12) imply \( a'y \leq \gamma by \), which is at most \( \gamma \) by (9). Then \( (a-a')y \leq |a-a'_1| = |a|_1 - \text{opt}_\gamma \) by (11). If we are not concerned with the running time, then NFP\(_{\gamma} \) is equivalent to a fractional maximum bipartite matching problem, on the bipartite graph between \( S \) and \( T \) where \((s, t)\) exists if and only if \( s \sim t \).

We state the main theorem that solves NFP\(_{\gamma} \), approximately. As we can only lose an additive factor in the main value of Theorem 3.2, the values in \( a \) need to be respected. Values in \( b \) can be approximated within a factor of \( 1 + O(\epsilon) \). So the theorem is slightly stronger than Theorem 1.2. In the statement, for a subset \( S' \subseteq S \) of sources, \( T(S') := \{ t \in T : S' \sim t \} \) denotes the set of sinks that can be reached from \( S' \).

Theorem 3.10. Let \( \gamma \geq 0, \epsilon > 0 \). There is an \( O \left( \frac{1}{\epsilon} \cdot |E| \cdot |V| \cdot \log \frac{|a|_1}{\epsilon} \right) \)-time algorithm that outputs a flow \( f \in F_{\gamma} \) and a set \( S' \subseteq S \) such that \( a(S \setminus S') + \frac{\gamma b(T(S'))}{1+\epsilon} \leq \text{val}(f) + \frac{\phi}{2} \).

The value of the maximum flow for NFP\(_{\gamma} \) is \( \min_{S \subseteq S} (a(S \setminus S') + \gamma b(T(S'))) \), by the maximum flow minimum cut theorem. The theorem finds a flow whose value is at least that of the maximum flow for the instance with \( b \) replaced by \( \frac{b}{1+\epsilon} \). We show that Theorem 3.10 implies Theorem 3.2 in Appendix B. We leave the proof of Theorems 3.10 and 1.2, which involves many technical definitions, to Appendix D. In the remainder of this section, we give some high-level ideas behind the proofs.

We first consider the case where the graph \( G \) is a directed bipartite graph from \( S \) to \( T \). Later in Lemma 4.1 we show that if we have a \((1+\epsilon)\)-factor slack in the condition of Hall’s bipartite matching theorem, then we can find a perfect bipartite matching in nearly-linear time. Indeed, the proof extends to the problem of fractional \( b \)-matching where the maximum matching is not necessarily perfect. Consider the problem NFP\(_{\gamma/(1+\epsilon)} \) with the maximum flow \( f^* \). If we are allowed to scale up the capacities of the sinks \( T \) by a factor of \( 1+\epsilon \), then in nearly-linear time, we can find a flow \( f \in F_{\gamma} \) with \( \text{val}(f) \geq \text{val}(f^*) \). Moreover, we can find a “witness” for the inequality: A set \( S' \subseteq S \) such that \( a(S \setminus S') + \frac{2b(T(S'))}{1+\epsilon} \leq \text{val}(f) \). This is the inequality guaranteed by Theorem 4.3 without the term \( \phi/3 \), which is introduced for a small technicality issue.

To find the flow \( f \) and the witness set \( S' \subseteq S \), we show that if there is no augmenting path of length \( 2L + 1 \), where \( L = O(\log |L|/\epsilon^2) \), for the flow \( f \) w.r.t the problem NFP\(_{\gamma} \), then \( \text{val}(f) \geq \text{val}(f^*) \). The witness \( S' \) can be identified easily from the proof of the inequality. So, we can use the shortest augmenting path algorithm of Hopcroft and Karp [18] or Dinic [12] to find such a flow \( f^* \). More specifically, in each iteration one can in nearly-linear time find a “blocking flow” in the residual graph for \( f \) restricted to shortest augmenting paths. Augmenting \( f \) by the flow increases the length of the shortest augmenting path by at least \( 2 \).

Now, we move to the case where \( G = (V \supseteq S \cup T, E) \) is general graph as in Theorem 3.10. Let \( H = (S \cup T, E_H) \) be the transitive closure of \( G \) where \((s, t) \in E_H \) if and only if \( s \sim t \). Then the maximum flow problem NFP\(_{\gamma} \) over \( G \) is equivalent to that over \( H \). However, we can not construct and maintain \( H \) explicitly as it may have quadratic number of edges. Instead, we try to mimic the shortest augmenting path algorithm. A shortest augmenting path in \( H \) corresponds to an augmenting path in \( G \) with the minimum number of switches between forward and backward edges. Then it is tempting to run the algorithm where in each iteration we do the following: Construct the residual graph for the flow \( f \), restrict it to the augmenting paths with the minimum number of switches, find a blocking flow in the graph and use it to augment \( f \).

However, unlike the bipartite graph case, the above operations do not necessarily increase the minimum number of switches in an augmenting path for \( f \). This is due to the interference between the forward and backward “segments” of an augmenting path. To address the issue, we separate two forward and backward
segments, using the structure of “handled graphs”: A handle is a copy of a sub-graph of \( G \), with sources and sinks identified with those in \( G \). A handled graph \( G' \) is the graph \( G \) with many handles. When augmenting the flow \( f \) over the graph \( G' \), we only consider the forward edges in \( G \), and backward edges in the handles of \( G' \). This way, the forward and backward edges have disjoint supports. By carefully constructing the handles, we show that augmenting \( f \) by a blocking flow in the handled graph can increase the minimum number of switches in an augmenting path increases by at least 2. Repeating the procedure \( L \) times leads to a flow for which any augmenting path has more than \( 2L \) switches.

The proof of Theorem 1.2 is the same as that of Theorem 3.2 except we use a smaller \( L \) and a simpler and well-known argument: If a fractional bipartite matching does not have an augmenting path of length at most \( 2L + 1 \) with \( L = \lfloor \frac{1}{\epsilon} \rfloor \), then the matching is a \( \frac{1}{1+\epsilon} \)-approximation.

4 Nearly-Linear Time Approximation Algorithm for Unrelated Machine Scheduling Problems

In this section, we consider the unrelated machine setting, and describe the nearly-linear time approximation algorithms for the problems with makespan, weighted completion and \( L_q \)-norm of machine loads respectively in Sections 4.1, 4.2 and 4.3. Recall the common setting: we are given a bipartite graph \( G = (J, M, E) \) and a \( p_{j,i} \in \mathbb{Z}_{\geq 0} \) for every \((j,i) \in E\). Recall that \( N(j), N(i), \delta(j) \) and \( \delta(i) \) denote the set of neighbors or incident edges of a job \( j \in J \) or a machine \( i \in M \), in the graph \( G \).

4.1 Makespan Minimization

We give the nearly-linear time \( (2+\epsilon) \)-approximation algorithm for the makespan minimization problem. Via a standard technique described in Appendix F.3, we can focus on the following promise version:

- We are given a number \( P \geq \text{opt} \), where \( \text{opt} \) is the optimal makespan of the instance, and our goal is to construct an assignment of makespan at most \((2 + O(\epsilon))P\).

For some \((j,i) \in E\) with \( p_{j,i} > P \), we can remove \((j,i)\) from \( E \), as we are guaranteed that the optimum solution does not use the edge. The following is the natural LP relaxation for the problem:

\[
\sum_{j \in N(i)} p_{j,i} x_{j,i} \leq P \quad \forall i \in M \quad (20) \quad \sum_{i \in N(j)} x_{j,i} \geq 1 \quad \forall j \in J \quad (21) \quad x_{j,i} \geq 0 \quad \forall (j,i) \in E \quad (22)
\]

In the correspondent integer program, \( x_{j,i} \in \{0, 1\} \) for every \((j,i) \in E\) indicates whether the job \( j \) is assigned to machine \( i \). (20) requires that the makespan of the schedule be at most \( P \), (21) requires every job to be scheduled. In the linear program, we replace the requirement that \( x_{j,i} \in \{0, 1\} \) with the non-negativity constraint (22).

By the promise that \( P \geq \text{opt} \), the LP is feasible. Therefore, applying Theorem 2.1, we can solve the LP in \( O((|E|)) \) time to obtain an approximate solution \( x \in [0, 1]^E \). By scaling, we can assume (21) holds with equalities, and (20) holds with right side replaced by \((1 + O(\epsilon))P\).

To round the solution to an integral assignment in \( O((|E|))\)-time, we use the grouping idea from [41]: For each machine \( i \in M \), we break the fractional jobs assigned to \( i \) into groups, each containing \( \frac{1}{1+\epsilon} \) fractional jobs. This gives us a bipartite graph \( H \) between jobs and groups. Any perfect matching (i.e., a matching covering all jobs \( J \)) will give a \( (2 + O(\epsilon)) \)-approximation for the makespan problem. In \( H \), every subset \( J' \subseteq J \) of jobs has at least \((1 + \epsilon)|J'|\) neighbors. The \( (1 + \epsilon) \)-factor allows us to design a \( O((|E|)) \)-time algorithm to find a matching covering all jobs \( J \), as stated in the following lemma:

**Lemma 4.1.** Assume we are given a bipartite graph \( H = (S,T,E_H) \) and \( \epsilon > 0 \) such that \( |N_H(S')| \geq (1 + \epsilon)|S'| \) for every \( S' \subseteq S \). In \( O\left(\frac{|E_H|}{\epsilon} \log |S|\right) \)-time, we can find a matching in \( H \) covering all vertices in \( S \).

With the lemma, we can prove the following theorem using the grouping technique from [35]:

13
Theorem 4.2. Given \( x \in [0, 1]^E \) satisfying \( x(\delta(j)) = 1 \) for every \( j \in J \), and \( \epsilon \in (0, 1) \), there is an \( O\left( \frac{|E| \log n}{\epsilon^2} \right) \)-time algorithm that outputs an assignment \( \sigma : J \to M \) of jobs to machines such that \( (j, \sigma(j)) \in E \) and \( x_{j, \sigma(j)} > 0 \) for every \( j \in J \), and for every \( i \in M \), we have

\[
\sum_{j \in \sigma^{-1}(i)} p_{i,j} \leq (1 + \epsilon) \sum_{j \in N(i)} p_{i,j} \cdot x_{j,i} + \max_{j \in \sigma^{-1}(i)} p_{i,j}. \quad \text{(Assume the maximum over } \emptyset \text{ is 0.)}
\]

We can then apply Theorem 4.2 with the solution \( x \) we obtained from solving LP(20-22). Clearly we have \( \max_{j \in \sigma^{-1}(i)} p_{i,j} \leq P \) for every \( i \in M \). So, the total load on any machine \( i \) is at most \( P + (1 + \epsilon) \cdot \sum_{j \in N(i)} p_{i,j} x_{j,i} \leq P + (1 + \epsilon) \cdot (1 + O(\epsilon))P = (2 + O(\epsilon))P \), as (20) is satisfied with right side replaced by \( (1 + O(\epsilon))P \). This finishes the analysis of the algorithm for \( R||C_\max \).

4.2 Weighted Completion Time

In this section we give the \( \tilde{O}_\epsilon(|E|) \)-time \((1 + \epsilon)\)-approximation algorithm for the weighted completion time problem. In Section 4.2.1 we formulate the LP for the problem. Then in Section 4.2.2, we demonstrate that the LP solution can be rounded to obtain a \((1 + \epsilon)\)-approximation.

To deliver the key techniques more clearly, we first assume all processing times and weights are polynomially bounded. The general case is handled in Appendix E.2. As in Section 4.1, we focus on the promise version of the problem: We are given a number \( \Phi \geq \text{opt} \) and our goal is to construct a schedule of weighted completion time at most \((1 + O(\epsilon))\Phi\), where \text{opt} is the optimum weighted completion time for the instance.

4.2.1 Linear Programming Relaxation

We round completion times to integer powers of \( 1 + \epsilon \). As before let \( \tau_0 = 0, \tau_d = (1 + \epsilon)^{d-1} \) for every \( d \geq 1 \), and let \( D \) be the smallest integer such that \( \tau_d \geq n \max_{(j,i) \in E} p_{i,j} \). So \( D = O\left( \frac{\log n}{\epsilon^2} \right) = \tilde{O}_\epsilon(1) \). Let \( \mathbb{D} = \{(j, i, d) : (j, i) \in E, d \in [D], \tau_d \geq p_{i,j}\} \). The variables in our linear program are \( x \in [0, 1]^\mathbb{D} \), where each \( x_{j,i,d} \) indicates if \( j \) is scheduled on \( i \) and completes in \( (\tau_{d-1}, \tau_d] \), in the corresponding integer program. Notice that if this happens, then \( \tau_d \geq p_{i,j} \).

To capture the capacity constraints, it is convenient to make the following definition: For any real numbers \( 0 \leq p \leq C \) and \( \theta > 0 \), we define

\[
\rho(p, C, \theta) := \min \{(p + \theta - C)_{+}, p\}.
\]

The meaning of \( \rho(p, C, \theta) \) is as follows. Suppose a job is scheduled on some machine with completion time \( C \), and it has length \( p \) on the machine. Then \( \rho(p, C, \theta) \) is the volume of the job processed before time \( \theta \). The function is non-decreasing on \( \theta \) and non-increasing on \( C \).

With the definitions, we can describe the LP relaxation, as follows:

1. \[
\frac{1}{1 + \epsilon} \sum_{(j,i,d) \in \mathbb{D}} w_j \tau_d x_{j,i,d} \leq \Phi \quad \text{(23)}
\]
2. \[
\sum_{i,d:(j,i,d) \in \mathbb{D}} x_{j,i,d} \geq 1 \quad \forall j \in J \quad \text{(25)}
\]
3. \[
\sum_{j,i,d:(j,i,d) \in \mathbb{D}} \rho(p_{i,j}, \tau_d, \tau_r)x_{j,i,d} \leq \tau_r \quad \forall i \in M, r \in [D] \quad \text{(24)}
\]
4. \[
x_{j,i,d} \geq 0 \quad \forall (j,i,d) \in \mathbb{D} \quad \text{(26)}
\]

(23) bounds the weighted completion time of the schedule: If \( j \) has completion time in \( (\tau_{d-1}, \tau_d] \), then its completion time is at least \( \frac{\tau_d}{1 + \epsilon} \). To see (24), suppose a job \( j \) is scheduled on \( i \) and has completion time in \( (\tau_{d-1}, \tau_d] \). Then the volume of \( j \) scheduled on \( i \) before time \( \tau_r \) is at least \( \rho(p_{i,j}, \tau_d, \tau_r) \). So the inequality is valid as the total volume of jobs scheduled on \( i \) before \( \tau_r \) is at most \( \tau_r \). (25) is valid since every \( j \) is scheduled. In the linear program, we only require the variables to be non-negative (Constraint (26)).

\[
\text{Notice that here we require the completion time to be in } (\tau_{d-1}, \tau_d], \text{ while in Section 3, } x_{j,i,d} \text{ indicates if } j \text{ has completion time at most } \tau_d.
\]
We count the number of non-zero coefficients on the left-side of the LP, which is dominated by (24). Since there are at most \( D \) choices for \( r \) and \( d \), the number is at most \( O(D^2|E|) = \tilde{O}_c(|E|) \). Using Theorem 2.1, we can solve the LP approximately to obtain a solution \( x \in [0,1]^m \) in running time \( \tilde{O}_c(|E|) \). By scaling, we assume (25) is satisfied with equalities, and (23) and (24) are satisfied with right side multiplied by \( 1 + O(\epsilon) \).

### 4.2.2 Rounding Algorithm for Weighted Completion Time

In this section, we round the LP solution \( x \in [0,1]^m \) to a schedule for the weighted completion time problem. The rounding algorithm and analysis are very similar to those in [31], except that we use our weaker LP. For every \( j \), we randomly choose a pair \( (i_j, d_j) \) so that \( \Pr[(i_j, d_j) = (i, d)] = x_{j,i,d} \), and assign job \( j \) to machine \( i_j \). This process is well-defined because (25) holds with equalities. Then we choose a real number \( \theta_j \) uniformly at random in \((\tau_{d_j} - p_{i_j,j}, \tau_{d_j})\). The decisions for all jobs \( j \) are independent. On each machine \( i \), we process all jobs \( j \) assigned to it in increasing order of \( \theta_j \) values. This finishes the description of the rounding algorithm. The running time of the algorithm is the number of variables in LP(23-26), plus \( O(n \log n) \) for sorting jobs in increasing order of \( \theta_j \) values. Once the assignment of jobs to machines is decided, it is the optimum to use the Smith’s rule to process the jobs. So our analysis works even if we are using the possibly sub-optimum schedule according to \( \theta_j \) values.

**Lemma 4.3.** The expected total weighted completion time of the schedule produced by the rounding algorithm is at most \((1.5 + O(\epsilon))\Phi\).

So, this finishes the analysis of the \((1.5 + O(\epsilon))\)-approximation ratio. Indeed, the algorithm can be derandomized to run in \( \tilde{O}_c(|E|) \) time. We describe it in Appendix F.4.

### 4.3 \( L_q \)-Norm of Machine Loads

In this section, we consider the \( L_q \)-norm of loads problem. By a simple transformation which we describe in Appendix E.3, we can assume all job sizes are between 1 and \( \text{poly}(n) \). Let \( \text{opt} \) be the \( L_q \)-norm of loads of the optimum schedule. Following a similar processing step as in Section 4.1, we assume we are given a parameter \( P \geq \text{opt} \) and our goal is to find a schedule with \( L_q \)-norm of loads being \( O(1) \cdot P \). We remove edges \((j,i) \in E\) with \( p_{i,j} > P \).

Again, define \( \tau_0 = 0, \tau_d = (1 + \epsilon)^{d-1} \) for every \( d \geq 1 \) as before. Let \( D \) be the smallest integer such that \( \tau_D \geq P \). Let \( \mathbb{D} = \{(j,i,d) : (j,i) \in E, d \in [D], \tau_d \geq p_{i,j}\} \). As \( D = \tilde{O}_c(1) \) we have \(|\mathbb{D}| = \tilde{O}_c(|E|)\).

The value of our linear program corresponds to the \( L_q \)-cost of an assignment, i.e., the \( L_q \)-norm of loads, raised to the power of \( q \). So it is more convenient for us to introduce a parameter \( \Phi := P^q \). The LP is almost the same as that for weighted completion, except that we have a different objective function. For every \((j,i,d) \in \mathbb{D}\), we define

\[
c_{j,i,d} := \tau_d^q - (\tau_d - p_{i,j})^q. \tag{27}
\]

To understand the definition, consider a job \( j \) processed on machine \( i \) with completion time \( t_d \). Then \( j \) increases the \( L_q \)-cost of the assignment by \( \tau_d^q - (\tau_d - p_{i,j})^q \). The LP is LP(27-30), with the only difference from LP(23-26) being that between (27) and (23).

\[
\frac{1}{(1 + \epsilon)q} \sum_{(j,i,d) \in \mathbb{D}} c_{j,i,d} x_{j,i,d} \leq \Phi \tag{27}
\]

\[
\sum_{i,d : (j,i,d) \in \mathbb{D}} x_{j,i,d} \geq 1 \quad \forall j \in J \tag{29}
\]

\[
\sum_{j,d : (j,i,d) \in \mathbb{D}} \rho(p_{i,j}, \tau_d, \tau_r) x_{j,i,d} \leq \tau_r \quad \forall i \in M, r \in [D] \tag{28}
\]

\[
x_{j,i,d} \geq 0 \quad \forall (j,i,d) \in \mathbb{D} \tag{30}
\]

To see that the LP is valid, it suffices to notice that if a job \( j \) is scheduled on \( i \) with completion time \( C \in (\tau_{d-1}, \tau_d) \), we have \( C^q - (C - p_{i,j})^q \geq \tau_{d-1}^q - (\tau_{d-1} - p_{i,j})^q = \frac{1}{(1+\epsilon)q} (\tau_d^q - (\tau_d - (1 + \epsilon)p_{i,j})^q) \geq \frac{1}{(1+\epsilon)^2q} c_{j,i,d} \).

\footnote{The proof requires that \( d \geq 2 \). But if \( d = 1 \) then \( p_{i,j} = C = 1 \) and \( C^q - (C - p_{i,j})^q = \tau_1^q - (\tau_1 - p_{i,j})^q = c_{j,i,1} \).}
Before describing this algorithm, we show some special cases for which we can obtain a better approximation. However, we could not design a nearly-linear time rounding algorithm achieving the approximation ratio.

The following two inequalities hold for the vector $y$:

\[ \text{Lemma 4.4}. \]

\[ \text{Notice that the third term in the max operator in the definition of } T \text{ is at most } P. \]

\[ [\text{4}] \text{ showed there is a polynomial time that rounds } y \text{ into an integral schedule with } L_q \text{-norm of loads being at most } 2T. \]

\[ [25] \text{ improved the ratio of } 2 \text{ to } \alpha(q) \text{ for some function } \alpha : [1, \infty) \rightarrow [1, 2). \]

We prove the following lemma, which proves that $T \leq (1 + O(\epsilon))P$:

\[ \text{Lemma 4.4}. \] The following two inequalities hold for the vector $y \in [0, 1]^E$ we constructed:

\[ \sum_{(j,i) \in E} p_{i,j}y_{j,i} \leq (1 + \epsilon)O(q)\Phi \] \hspace{1cm} (33)

\[ \sum_{i \in M} g_i^q \leq (1 + \epsilon)O(q)\Phi \] \hspace{1cm} (34)

So the result of [25] shows that our LP gives an $\alpha(q)$-estimation of the optimum $L_q$ norm for an instance. However, we could not design a nearly-linear time rounding algorithm achieving the approximation ratio.

Instead, we design a nearly-linear time rounding algorithm with a weaker approximation ratio of $4 + \epsilon$. Before describing this algorithm, we show some special cases for which we can obtain a better approximation ratio. For the case $q = 2$, Azar and Epstein [4] showed that independent rounding leads to a $\sqrt{2}$-approximation:

\[ \text{Theorem 4.5} \ (4). \] Let $q = 2$, $y \in [0, 1]^E$ with $y(\delta(j)) = 1$ for every $j \in J$. Let $q_i$'s and $T$ be defined as in (31) and (32). Let $\sigma : J \rightarrow M$ be the assignment obtained by randomly and independently assigning jobs, so that $j$ is assigned to $i$ with probability $y_{j,i}$. Then, $\mathbb{E} \left[ \sum_{i \in M} \left( \sum_{j \in \sigma^{-1}(i)} p_{i,j} \right)^2 \right] \leq 2T^2$.

Due to the concavity of the $\sqrt{\cdot}$ function, we have $\mathbb{E} \left[ \sqrt{\sum_{i \in M} \left( \sum_{j \in \sigma^{-1}(i)} p_{i,j} \right)^2} \right] \leq \sqrt{2T}$. As $T \leq (1 + O(\epsilon))P$ for the $y$ we obtained, we obtain a $O((|E|)$-time algorithm $(\sqrt{2} + \epsilon)$-approximation for $R||L_2(\text{loads})$. In Section F.5, we show how to derandomize the algorithm.

The second special case for which we can get a better approximation ratio is the restricted assignment model, in which all the $p_{i,j}$'s for the same $j$ have the same value $p_j$. In this case we can use the Theorem 4.2 to obtain an assignment $\sigma : J \rightarrow M$ such that $y_{j,\sigma(j)} > 0$ for every $j \in J$, and for every $i \in M$ we have

\[ \sum_{j \in \sigma^{-1}(i)} p_j \leq (1 + \epsilon) \sum_{j \in N(i)} p_jy_{j,i} + \max_{j \in \sigma^{-1}(i)} p_j. \] \hspace{1cm} (35)

\[ \text{Lemma 4.6}. \] The $\sigma : J \rightarrow M$ obtained has $L_q$ norm of loads being at most $(2 + O(\epsilon))P$.

Finally, we consider the general case. The algorithm is very similar to that of [8] which gives a 4-approximation for the unrelated machine load balancing problem with an arbitrary symmetric norm. First, we modify the vector $y \in [0, 1]^E$ to obtain a vector $y' \in [0, 1]^E$ as follows. For every $j \in J$, we sort all the machines $i$ with $(j, i) \in E$ in non-decreasing order of $p_{i,j}$ values. Let the order be $i_1, i_2, \cdots, i_K$. Then for every $k = 1, 2, 3, \cdots, K$, let $y'_{j,i_k} = \min \left\{ (1 - \sum_{k'=1}^{k-1} y_{j,i_{k'}})_, 2y_{j,i_k} \right\}$. Roughly speaking, we partition
the 1 fractional of machines $j$ is assigned to according to $y$ into $1/2$ fraction of the fast machines and $1/2$
fraction of slow machines. To obtain the fractional assignment of $j$ in $y'$, we only keep the $1/2$ fraction of fast
machines, and scale the fractions by 2.

We apply Theorem 4.2 over $y'$ to obtain an assignment $\sigma : J \rightarrow M$ of jobs to machines satisfying the
condition of the lemma, in time $O_\epsilon(|E|)$. That is, for every $i \in M$ we have

$$\sum_{j \in \sigma^{-1}(i)} p_{i,j} \leq (1 + \epsilon) \sum_{j \in N(i)} p_{i,j} y'_{j,i} + \max_{j \in \sigma^{-1}(i)} p_{i,j}.$$ (36)

**Lemma 4.7.** The $\sigma : J \rightarrow M$ obtained has $L_q$ norm of loads being at most $\left( (2^q - 1 + 2q)^{1/q} + O(\epsilon) \right) P$.

**Acknowledgment**

The author would like to thank Richard Peng for pointing out the references to blocking flows and the link
cut tree data structure.

**References**

[1] Zeyuan Allen-Zhu and Lorenzo Orecchia. Nearly-linear time positive lp solver with faster convergence
rate. In *Proceedings of the Forty-Seventh Annual ACM Symposium on Theory of Computing*, STOC
'15, page 229–236, New York, NY, USA, 2015. Association for Computing Machinery.

[2] Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-
algorithm and applications. *Theory of Computing*, 8(6):121–164, 2012.

[3] Baruch Awerbuch, Yossi Azar, Edward F. Grove, Ming Yang Kao, P. Krishnan, and Jeffrey Scott Vitter.
Load balancing in the lp norm. *Annual Symposium on Foundations of Computer Science - Proceedings*,
pages 383–391, December 1995. Proceedings of the 1995 IEEE 36th Annual Symposium on Foundations
of Computer Science ; Conference date: 23-10-1995 Through 25-10-1995.

[4] Yossi Azar and Amir Epstein. Convex programming for scheduling unrelated parallel machines. In *Proceedings of the Thirty-Seventh Annual ACM Symposium on Theory of Computing*, STOC '05, page
331–337, New York, NY, USA, 2005. Association for Computing Machinery.

[5] Nikhil Bansal and Subhash Khot. Optimal long code test with one free bit. In *Proceedings of the
2009 50th Annual IEEE Symposium on Foundations of Computer Science*, FOCS '09, pages 453–462,
Washington, DC, USA, 2009. IEEE Computer Society.

[6] Nikhil Bansal, Aravind Srinivasan, and Ola Svensson. Lift-and-round to improve weighted completion
time on unrelated machines. In *Proceedings of the Forty-eighth Annual ACM Symposium on Theory of
Computing*, STOC ’16, pages 156–167. ACM, 2016.

[7] Deeparnab Chakrabarty and Chaitanya Swamy. Approximation algorithms for minimum norm and
ordered optimization problems. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory
of Computing*, STOC 2019, page 126–137, New York, NY, USA, 2019. Association for Computing
Machinery.

[8] Deeparnab Chakrabarty and Chaitanya Swamy. Simpler and better algorithms for minimum-norm load
balancing. In ESA, 2019.

[9] Chandra Chekuri, T.S. Jayram, and Jan Vondrak. On multiplicative weight updates for concave and
submodular function maximization. In *Proceedings of the 2015 Conference on Innovations in Theoretical
Computer Science*, ITCS ’15, page 201–210, New York, NY, USA, 2015. Association for Computing
Machinery.
[10] Chandra Chekuri and Kent Quanrud. Near-linear time approximation schemes for some implicit fractional packing problems. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’17, page 801–820, USA, 2017. Society for Industrial and Applied Mathematics.

[11] Chandra Chekuri and Kent Quanrud. Randomized mwu for positive lps. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’18, page 358–377, USA, 2018. Society for Industrial and Applied Mathematics.

[12] E. A. Dinic. Algorithm for solution of a problem of maximal flow in a network with power estimation. Doklady Akademii Nauk SSSR, 194(4):1277–1280, 1970.

[13] Naveen Garg and Jochen Könemann. Faster and simpler algorithms for multicommodity flow and other fractional packing problems. SIAM Journal on Computing, 37(2):630–652, 2007.

[14] R. L. Graham. Bounds on multiprocessing timing anomalies. SIAM Journal on Applied Mathematics, 17(2):416–429, 1969.

[15] R. L. Graham, E. L. Lawler, J. K. Lenstra, and A. H. G. Rinnooy Kan. Optimization and approximation in deterministic sequencing and scheduling: a survey. Ann. Discrete Math., 4:287–326, 1979.

[16] Leslie A. Hall, Andreas S. Schulz, David B. Shmoys, and Joel Wein. Scheduling to minimize average completion time: Off-line and on-line approximation algorithms. Math. Oper. Res., 22(3):513–544, August 1997.

[17] John E. Hopcroft and Richard M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. SIAM Journal on Computing, 2(4):225–231, 1973.

[18] Oscar H. Ibarra and Shlomo Moran. Deterministic and probabilistic algorithms for maximum bipartite matching via fast matrix multiplication. Inf. Process. Lett., 13(1):12–15, 1981.

[19] Sungjin Im and Maryam Shadloo. Weighted completion time minimization for unrelated machines via iterative fair contention resolution [extended abstract]. In Proceedings of the Thirty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’20, page 2790–2809, USA, 2020. Society for Industrial and Applied Mathematics.

[20] Klaus Jansen and Lars Rohwedder. On the configuration-lp of the restricted assignment problem. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’17, page 2670–2678, USA, 2017. Society for Industrial and Applied Mathematics.

[21] Klaus Jansen and Lars Rohwedder. A quasi-polynomial approximation for the restricted assignment problem. SIAM Journal on Computing, 49(6):1083–1108, 2020.

[22] Christos Koufogiannakis and N. Young. A nearly linear-time ptas for explicit fractional packing and covering linear programs. Algorithmica, 70:648–674, 2013.

[23] V. S. Anil Kumar, Madhav V. Marathe, Srinivasan Parthasarathy, and Aravind Srinivasan. A unified approach to scheduling on unrelated parallel machines. J. ACM, 56(5), August 2009.

[24] Yi-Tat Lee and Aaron Sidford. Efficient inverse maintenance and faster algorithms for linear programming. In 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, pages 230–249, 2015.
[27] Yin Tat Lee, Zhao Song, and Qiuyi Zhang. Solving empirical risk minimization in the current matrix multiplication time. In Alina Beygelzimer and Daniel Hsu, editors, Proceedings of the Thirty-Second Conference on Learning Theory, volume 99 of Proceedings of Machine Learning Research, pages 2140–2157, Phoenix, USA, 25–28 Jun 2019. PMLR.

[28] J. K. Lenstra and A. H. G. Rinnooy Kan. Complexity of scheduling under precedence constraints. Oper. Res., 26(1):22–35, February 1978.

[29] Jan Karel Lenstra, David B. Shmoys, and Éva Tardos. Approximation algorithms for scheduling unrelated parallel machines. Mathematical Programming, 46:259–271, 1990.

[30] Elaine Levey and Thomas Rothvo. A (1+ε)-approximation for makespan scheduling with precedence constraints using lp hierarchies. SIAM Journal on Computing, 50(3):STOC16–201–STOC16–217, 2021.

[31] Shi Li. Scheduling to minimize total weighted completion time via time-indexed linear programming relaxations. SIAM Journal on Computing, 49(4):FOCS17–409–FOCS17–440, 2020.

[32] Shi Li. Towards PTAS for Precedence Constrained Scheduling via Combinatorial Algorithms, pages 2991–3010. 2021.

[33] Michael Luby and Noam Nisan. A parallel approximation algorithm for positive linear programming. In Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing, STOC ’93, page 448–457, New York, NY, USA, 1993. Association for Computing Machinery.

[34] Alix Munier, Maurice Queyranne, and Andreas S. Schulz. Approximation Bounds for a General Class of Precedence Constrained Parallel Machine Scheduling Problems, pages 367–382. Springer Berlin Heidelberg, 1998.

[35] Serge A. Plotkin, David B. Shmoys, and Éva Tardos. Fast approximation algorithms for fractional packing and covering problems. Mathematics of Operations Research, 20(2):257–301, 1995.

[36] Paul Purdom. A transitive closure algorithm. BIT Numerical Mathematics, 10:76–94, 1970.

[37] Maurice Queyranne and Andreas S. Schulz. Approximation bounds for a general class of precedence constrained parallel machine scheduling problems. SIAM J. Comput., 35(5):1241–1253, May 2006.

[38] Andreas S. Schulz and Martin Skutella. Scheduling unrelated machines by randomized rounding. SIAM J. Discret. Math., 15(4):450–469, April 2002.

[39] Jay Sethuraman and Mark S. Squillante. Optimal scheduling of multiclass parallel machines. In Proceedings of the Tenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’99, pages 963–964. Society for Industrial and Applied Mathematics, 1999.

[40] Farhad Shahrokhii and D. W. Matula. The maximum concurrent flow problem. J. ACM, 37(2):318–334, April 1990.

[41] David B. Shmoys and Éva Tardos. An approximation algorithm for the generalized assignment problem. 62(1–3):461–474, February 1993.

[42] Martin Skutella. Convex quadratic and semidefinite programming relaxations in scheduling. J. ACM, 48(2):206–242, March 2001.

[43] Daniel D. Sleator and Robert Endre Tarjan. A data structure for dynamic trees. Journal of Computer and System Sciences, 26(3):362–391, 1983.

[44] Ola Svensson. Conditional hardness of precedence constrained scheduling on identical machines. In Proceedings of the Forty-second ACM Symposium on Theory of Computing, STOC ’10, pages 745–754, New York, NY, USA, 2010. ACM.
A Missing Proofs in Section 2

Theorem 2.4. Algorithm 1 will return an \((O(\epsilon), \phi)\)-approximate solution \(x\) to \((P_Q)\), within \(O\left(\frac{m \log m}{\epsilon^2}\right)\) iterations of Loop 2.

Proof. Focus on one iteration of Loop 2. Let \(t\) be the value of \(t\) at the beginning of the iteration, \(y\) and \(\delta\) be the \(y\) and \(\delta\) obtained in Step 3 and 4 in the iteration respectively. Then we have

\[
|u^{(t+\delta)}| = \sum_{i \in [m]} u^{(t+\delta)}_i = \sum_{i \in [m]} u^{(t)}_i \exp(\delta \rho \cdot P_i y) \leq \sum_{i \in [m]} u^{(t)}_i (1 + (1 + \epsilon) \cdot \delta \rho \cdot P_i y) = |u^{(t)}| + (1 + \epsilon) \cdot \delta \rho \cdot u^{(t)} \cdot P_j y \leq |u^{(t)}| + (1 + \epsilon)^2 \delta \rho \cdot |u^{(t)}| \leq |u^{(t)}| \exp((1 + \epsilon)^2 \delta \rho).
\]

The inequality in the first line is by that \(\delta \rho \cdot P_i y \in [0, 1]\) for every \(i \in [m]\) and \(e^\theta \leq 1 + \epsilon \theta + (\epsilon \theta)^2 \leq 1 + \epsilon \theta + \epsilon^2 \theta\) for every \(\epsilon \in [0, 1]\) and \(\theta \in [0, 1]\). The first inequality in the second line is by that \(\frac{u^{(t)}(i)}{|u^{(t)}(i)|} P_j y = \beta y \leq 1 + \epsilon\).

Combining the inequality over all iterations, we have

\[
|u^{(t)}| \leq |u^{(0)}| \exp((1 + \epsilon)^2 \delta \rho) = m \cdot \exp((1 + \epsilon)^2 \delta \rho). \tag{37}
\]

For every \(i \in [m]\), we have \(u^{(1)}_i = \exp(\epsilon \rho \cdot P_i x)\), where \(x := x^{(1)}\) is the returned solution. So, by (37), we have \(\exp(\epsilon \rho \cdot P_i x) \leq m \cdot \exp((1 + \epsilon)^2 \epsilon \rho)\), which implies \(P_i x \leq \frac{\ln m}{\epsilon^2} + (1 + \epsilon)^2 \leq (1 + \epsilon)^2 + \epsilon = 1 + O(\epsilon)\).

In the end \(x = x^{(1)}\) is a convex combination of vectors \(y\) obtained in all iterations. As each \(y\) is in \(Q\), we have \(x \in Q\). Moreover, for the instance of (1) in any iteration, \(x^*\) is a valid solution. So, the optimum solution \(y^*\) to the instance of (1) has \(\alpha y^* \geq \alpha x^*\), and the \(y\) returned by the oracle has \(\alpha y \geq \alpha y^* - \phi \geq \alpha x^* - \phi\). This implies our final \(x\) has \(\alpha x \geq \alpha x^* - \phi\). Therefore, \(x\) is a \((O(\epsilon), \phi)\)-approximate solution to \((P_Q)\).

It remains to bound the number of iterations that Loop 2 can take. In every iteration of loop 2 except for the last one, some \(i\) has \(\frac{1}{\alpha^2 \cdot \rho \cdot y_i} = \delta\), i.e., \(\delta \rho \cdot P_j y = \epsilon\). We say \(u_i\) is increased fully in the iteration. Notice by (37), each \(u_i\) can be increased fully in at most \(\frac{\ln \left(\frac{m}{\epsilon} \exp((1 + \epsilon)^2 \rho)\right)}{\epsilon^2} = \ln \left(\frac{m}{\epsilon} \cdot (1 + \epsilon)^2 \rho\right) = O\left(\frac{\ln m}{\epsilon^2}\right)\) iterations. This bounds the number of iterations by \(O\left(\frac{m \log m}{\epsilon^2}\right)\) as there are \(m\) different values of \(i\).

B Missing Proofs in Section 3

Lemma 3.1. \(lp \leq (1 + \epsilon) opt\).

Proof. Let \(\tilde{x}^* \in \{0, 1\}^{J \times [0,D]}\) be the solution correspondent to the optimum schedule: \(\tilde{x}_{j,d}^* \in \{0, 1\}\) indicates if \(j\) has completion time at most \(\tau_d\) in the schedule. Then, we have

\[
\text{opt} \geq \frac{1}{1 + \epsilon} \sum_{j \in J} \sum_{d=1}^D \tilde{x}_{j,d}^* \cdot \tau_d = \frac{1}{1 + \epsilon} \sum_{j \in J} \sum_{d=1}^{D-1} \tilde{x}_{j,d}^* \cdot (\tau_d - \tau_{d+1}) + \tau_D
\]

\[
= \frac{1}{1 + \epsilon} \sum_{j \in J} w_j \left(\tau_D - \sum_{d=1}^{D-1} \eta_d \tilde{x}_{j,d}^*\right) \geq \frac{lp}{1 + \epsilon}.
\]
To see the inequality in the first line, note that a job $j \in J$ with $\hat{x}_{j,d-1}^* = 0$ and $\hat{x}_{j,d}^* = 1$ has completion time at least $\frac{D-j}{D}$. The equality in the line is by rearranging of terms, and that $\hat{x}_{j,0}^* = 0$ and $\hat{x}_{j,D}^* = 1$ for every $j \in J$. The equality in the second line is by the definition of $\eta_d$'s. The inequality in the line is by that $\bar{x}^*$ is a valid solution to the LP. Therefore, we have $lp \leq (1 + \epsilon)opt$, finishing the proof of the lemma.

Claim 3.3. For a job $j \in J$, we have $C_j \geq q_j$. For two jobs $j < j'$, we have $C_j \leq C_{j'}$.

Proof. To see the first statement, notice that $x_{j,d} = 0$ if $\tau_d < q_j$. Thus $C_j = \sum_{d=1}^{D} \tau_d(x_{j,d} - x_{j,d-1}) \geq \sum_{d=1}^{D} q_j(x_{j,d} - x_{j,d-1}) = q_j$. The second statement follows from that $C_j = \tau_D - \sum_{d=1}^{D-1} \eta_d x_{j,d}, C_{j'} = \tau_D - \sum_{d=1}^{D-1} \eta_d x_{j',d}$ and that $x_{j,d} \geq x_{j',d}$ for every $d \in [0, D]$. □

Lemma 3.4. Let $C^*$ be a time point and let $J' := \{ j \in J : C_j \leq C^* \}$. Then, we have

$$p(J') \leq (2 + O(\epsilon))mC^*.$$ 

Proof. Let $\xi$ be the $1 + O(\epsilon)$ term so that $x$ satisfies (5) with the right-side replaced by $\xi m \tau_d$. Let $D'$ be the minimum number such that $\xi m \tau_{D'} \geq p(J')$. Then $1 \leq D' \leq D$. If $D' = 1$, then we have $p(J') \leq \xi m \leq (1 + O(\epsilon))mC^*$ if $C^* \geq 1$; if $C^* < 1$ then $J' = \emptyset$ by Claim 3.3. So, we can assume $2 \leq D' \leq D$.

$$C^* p(J') \geq \sum_{j \in J'} C_j p_j = \sum_{j \in J'} \left( \tau_D - \sum_{d=0}^{D-1} \eta_d x_{j,d} \right)$$

$$= p(J') \tau_D - \sum_{j \in J'} \eta_d \sum_{d=0}^{D-1} p_j x_{j,d}$$

$$\geq p(J') \tau_D - \sum_{d=0}^{D-1} \eta_d \min \left\{ \xi m \tau_d, p(J') \right\} = p(J') \tau_D - \left( \xi m \sum_{d=0}^{D-1} \eta_d + \sum_{d=D'}^{D-1} \eta_d p(J') \right)$$

$$\geq p(J') \tau_D - \xi m \int_{t=0}^{T_{D'}} t dt - (\tau_D - \tau_{D'})p(J') = p(J') \tau_{D'} - \frac{\xi m \tau_{D'}^2}{2} \geq \frac{(1 - \epsilon)p(J') \tau_{D'}}{2} \geq \frac{(1 - \epsilon)p^2(J')/(\xi m)}{2}.$$

The inequality in the third line holds as $\sum_{j \in J'} \eta_d x_{j,d} \leq \sum_{j \in J} p_j x_{j,d} \leq \xi m \tau_d$ and $\sum_{j \in J'} p_j x_{j,d} \leq p(J')$. To see the inequality in the fourth line, notice that $\tau_d \eta_d \leq \int_{t=\tau_d}^{\tau_{d+1}} dt$ for every $d \in [0, D'-1]$. The first inequality in the last line used that $\xi m \tau_{D'} = \xi m (1 + \epsilon) \tau_{D'-1} < (1 + \epsilon)p(J')$ by the choice of $D'$. The second inequality in the line used that $\tau_{D'} \geq p(J')/(\xi m)$.

Therefore, we have $p(J') \leq \frac{2k m C^*}{1 + \epsilon} = (2 + O(\epsilon))mC^*$, as $\xi = 1 + O(\epsilon)$. □

Lemma 3.7. Let $\theta \in (0, 1)$ be a number such that for every $j < j'$, we have $F_{j'} - F_j \geq \theta p_j$. Then $T_{\text{idle}} \leq F_{j^*} + p_{j^*}$.

Proof. We first revisit the tools built in [34] and [31] that bound $T_{\text{idle}}$ when job sizes are arbitrary. The following lemma was proved in the two papers. (See, e.g., Lemma 2.2 in [31].)

Lemma B.1 ([34], [31]). Let $j \in J$ be a job in the schedule of interest with $\bar{S}_j > 0$. Then we can find a job $j'$ such that either

- (B.1a) $j' < j$ and $(\bar{C}_j, \bar{S}_j)$ is busy, or
- (B.1b) $F_{j'} \leq F_j, \bar{S}_{j'} < \bar{S}_j$ and $(\bar{S}_{j'}, \bar{S}_j)$ is busy.

We start from $j = j^*$ and repeat the following process. While $\bar{S}_j > 0$, we find a job $j'$ satisfying either (B.1a) or (B.1b), and update $j \leftarrow j'$. Notice that $F_j$ and $\bar{S}_j$ only decrease from iteration to iteration. $\bar{S}_j$
decreases from the initial value of $\tilde{S}_j$ to the final value of 0, and $F_j$ decreases from the initial value of $F_j$ to some non-negative number.

In each iteration, we show that the number of idle slots in $(\tilde{S}_j, \tilde{S}_j)$ is at most $\frac{F_j - F_{j'}}{\phi}$. In case (B.1a), we get at most $C_j - \tilde{S}_j = p_j$ units of idle time in $(\tilde{S}_j, \tilde{S}_j)$, and $p_j \leq \frac{F_j - F_{j'}}{\phi}$. In case (B.1b), there are no idle slots in $(\tilde{S}_j, \tilde{S}_j)$, and $0 \leq \frac{F_j - F_{j'}}{\phi}$. So, the total number of idle time slots before $\tilde{S}_j$ is at most $\frac{F_j}{\phi}$, implying that the total amount of idle time before $\tilde{C}_j$ is at most $\frac{F_j}{\phi} + p_{j'}$. This finishes the proof of Lemma 3.7. □

The following two proofs involve Theorem 3.2, so we repeat the theorem below for convenience.

**Theorem 3.2.** Let $G = (V, E)$ be a directed acyclic graph and $Q := \{y \in [0, 1]^V : y_v \leq y_u, \forall (v, u) \in E\}$. Let $b, a \in \mathbb{R}_{\geq 0}^V$ be two row vectors. Let $y^*$ be the $y \in Q$ satisfying $by \leq 1$ with the maximum $ay$. Let $\epsilon \in (0, 1), \phi \in (0, |a|_1/2)$. Then, in $\tilde{O}_\epsilon \left(|E| \cdot \log^2 \frac{|a|_1}{\phi}\right)$ time, we can find a $y \in Q$ satisfying $by \leq 1 + \epsilon$ and $ay \geq ay^* - \phi$.

**Lemma 3.8.** To prove Theorem 3.2, we can w.l.o.g assume

- $S \cap T = \emptyset$ and there are no edges from $S$ to $T$,
- $\delta^-(s) = \emptyset$ for every $s \in S$,
- $\delta^+(t) = \emptyset$ for every $t \in T$, and
- for every $v \in V$, we have $S \leadsto v$ and $v \leadsto T$.

Finally, we can w.l.o.g replace the constraint $y \in [0, 1]^V$ by $y_v \leq 1$ for every $s \in S$ and $y_v \geq 0$ for every $t \in T$.

**Proof.** W.l.o.g, we assume every vertex $v \in V$ has $S \leadsto v$. It is the best to set $y_v = 0$ for the vertices $v$ with $S \not\leadsto v$ and remove them. Similarly, we assume every vertex $v \in V$ has $v \leadsto T$: It is the best to set $y_v = 1$ for vertices $v$ with $v \not\leadsto T$ and remove them.

Now for every $s \in S$, we can add a new vertex $s'$ and a new edge $(s', s)$ to $G$. We set $a_{s'} = a_s, b_{s'} = 0$, change $a_s$ to 0, and update $S$ to $S \cup \{s'\} \setminus \{s\}$. This does not change the instance since we have $y_{s'} \leq y_s$ and it is the best to set $y_{s'} = y_s$. Similarly, for every $t \in T$, we add a new vertex $t'$ and a new edge $(t, t')$ to $G$. We set $a_{t'} = 0, b_{t'} = b_t$, change $b_t$ to 0, and update $T$ to $T \cup \{t'\} \setminus \{t\}$. This does not change the instance since we have $y_{t'} \geq y_t$ and it is the best to $y_{t'} = y_t$. After this modification, the four properties in the list of the lemma are satisfied.

We show w.l.o.g the constraint $y \in [0, 1]^V$ can be replaced by $y_v \leq 1, \forall s \in S$ and $y_v \geq 0, \forall t \in T$. Fixing $(y_v)_{v \in S}$, we can assume $y_v = \max_{s \in S, s \leadsto v} y_s, \forall v \in V \setminus S$, as this will minimize $by$. As we have $S \leadsto v$, the condition $y_v \leq 1, \forall v \in V$ is implied by that $y_v \leq 1, \forall v \in S$. Similarly, fixing $(y_v)_{v \in T}$, we can assume $y_v = \min_{t \in T, v \leadsto t} y_t, \forall v \in V \setminus T$, as it maximizes $ay$. So, the condition $y_v \geq 0, \forall v \in V$ is also implied.

**Proof of Theorem 3.2 using Theorem 3.10.** To gain insights into the proof, we first ignore the running time requirement. Let $\Psi$ be the value of LP(8). Then by LP duality, we have $\Psi = \min_{\gamma \geq 0} (\gamma + |a|_1 - \text{opt}_\gamma)$, where $\text{opt}_\gamma$ is the value of the optimum flow for NFP$_\gamma$. By MFMC theorem, for every $\gamma \geq 0$, there is a set $S' \subseteq S$ such that $\text{opt}_\gamma = a(S \setminus S')\gamma b(T(S'))$. So, for every $\gamma \geq 0$, there is a set $S' \subseteq S$ such that $\gamma + a(S') - \gamma b(T(S')) \geq \Psi$. Then one can show that there are two subsets $S', S'' \subseteq S$ and a real number $z \in [0, 1]$ such that $\gamma + z(a(S') - \gamma b(T(S'))) + (1 - z)(a(S'') - \gamma b(T(S''))) \geq \Psi$ for every $\gamma \geq 0$. That means the coefficient $1 - zb(T(S')) + (1 - z)b(T(S''))$ is non-negative, and $za(S') + (1 - z)a(S'') \geq \Psi$. Then setting $y_v = z1_{S' \subseteq v} + (1 - z)1_{S'' \subseteq v}, \forall v \in V$ gives a solution to LP(8) of value $za(S') + (1 - z)a(S'') \geq \Psi$.

Now we take the running time and the approximation parameters $\epsilon$ and $\phi$ into account. We define an interesting set $\Gamma$ of $\gamma$’s as follows. Start from $\gamma = \frac{\phi}{3}$ and $\Gamma = \{\gamma\}$. While $\gamma < |a|_1$ we do the following:

\[\gamma \leftarrow (1 + \epsilon)\gamma, \Gamma \leftarrow \Gamma \cup \{\gamma\}.\]

Notice that we have $|\Gamma| = O\left(\frac{\log(|a|_1/\phi)}{\epsilon}\right) = \tilde{O}_\epsilon \left(\log \frac{|a|_1}{\phi}\right)$. For every $\gamma \in \Gamma$, let $\text{opt}_\gamma$ be the value of the optimum flow for NFP$_\gamma$. We use Theorem 3.10 to find a flow $\Gamma' \subseteq \Gamma$, and a set $S'_\gamma \subseteq S$ with $a(S \setminus S'_\gamma) + \frac{\gamma b(T(S'_\gamma))}{1 + \epsilon} \leq \text{val}(\Gamma') + \frac{\phi}{3} \leq \text{opt}_\gamma + \frac{\phi}{3}$. Then:

\[ay^* \leq \min_{\gamma \in \Gamma} (|a|_1 - \text{opt}_\gamma + \gamma) \leq \min_{\gamma \in \Gamma} \left(|a|_1 - a(S \setminus S'_\gamma) - \frac{\gamma b(T(S'_\gamma))}{1 + \epsilon} + \gamma\right) + \frac{\phi}{3} = 22\]
The first inequality is by that for a fixed $\gamma \in \Gamma$, LP(13) can attain value $|a_0| - \text{opt}_1 + \gamma$.

Define $\Psi := \min_{\gamma \in \Gamma} \left( a(S_\gamma') - \frac{\gamma b(T(S_\gamma'))}{1 + \epsilon} + \gamma \right) + \frac{\phi}{3}$.

Lemma B.2. The value of the following LP is at least $\Psi - \frac{2\phi}{3}$: maximize $\bar{a}z$ subject to $z \in \mathbb{R}^F_{\geq 0}, |z|_1 \leq 1$ and $\frac{b_\gamma z}{(1+\epsilon)^2} \leq 1$.

Proof. Assume otherwise. By duality, there is an $\alpha \in [0, \Psi - \frac{2\phi}{3}]$ such that $\frac{a_\gamma}{(1+\epsilon)^2} + (\Psi - \frac{2\phi}{3} - \alpha)1 \geq \bar{a}$. Consider the largest $\gamma \in \Gamma$ that is at most $\alpha$, or consider $\gamma = \frac{\phi}{3}$ if $\alpha < \frac{\phi}{3}$. So, $\alpha < (1+\epsilon)\gamma$ and $\gamma \geq \gamma - \frac{2\phi}{3} > \gamma - \frac{2\phi}{3}$. Then $\frac{a_\gamma}{(1+\epsilon)^2} + \Psi - \frac{2\phi}{3} - \alpha \geq \bar{a}_\gamma$ implies $\frac{\gamma b_\gamma}{1+\epsilon} + \Psi - \gamma > \bar{a}_\gamma$, which is $\Psi > \bar{a}_\gamma - \frac{\gamma b_\gamma}{1+\epsilon} + \gamma$. This contradicts the definition of $\Psi$.

There are only two non-trivial linear constraints in LP in Lemma B.2. So its value can be attained by a solution $z$ to the LP, with value at least $\Psi - \frac{2\phi}{3}$.

For every $v \in V$, let $y_v := \sum_{\gamma : S_\gamma' \ni v} z_\gamma$. Then $y_v \leq |z|_1 \leq 1$. Clearly $y_v \leq y_u$ for every $(v,u) \in E$ as $S_\gamma' \ni v$ implies $S_\gamma' \ni u$. $ay = \sum_{\gamma \in \Gamma} z_\gamma a(S_\gamma') = \bar{a}z \geq \Psi - \frac{2\phi}{3}$. Similarly, $by = b_\gamma z \leq (1+\epsilon)^2$. Notice that $ay^* \leq \Psi + \frac{\phi}{3}$ by (38), so $ay \geq ay^* - \phi$. The running time of the algorithm is $O\left( (|E|) (|V|) \log \frac{|a_0|}{\phi} \right)$ as we need to run the algorithm in Theorem 3.10 for $|\Gamma| = O\left( (|S|) \log \frac{|a_0|}{\phi} \right)$ times.

\section{Missing Proofs in Section 4}

Lemma 4.1. Assume we are given a bipartite graph $H = (S,T,E_H)$ and $\epsilon > 0$ such that $|N_H(S')| \geq (1+\epsilon)|S'|$ for every $S' \subseteq S$. In $O\left( \frac{|E_H|}{\epsilon} \log |S| \right)$-time, we can find a matching in $H$ covering all vertices in $S$.

Proof. Let $L = \left[ \log_{1+\epsilon} |S| \right] + 1 > \log_{1+\epsilon} |S|$. Then we use the shortest-augmenting path algorithm of Hopcroft and Karp [18] to find a matching for which there is a augmenting path of length at most $2L + 1$. The running time of the algorithm can be made to $O(|E_H| |L| = O\left( \frac{|E_H|}{\epsilon^2} \log |S| \right))$. It remains to show the following lemma:

Lemma C.1. Let $F$ be a matching in $H$ for which there is no augmenting path of length at most $2L + 1$. Then all vertices in $S$ are matched in the matching $F$.

Proof. Let $\bar{H}$ be the residual graph of $H$ w.r.t. the $F$: $\bar{H}$ is a directed graph over $S \cup T$, for every edge $(s,t) \in E_H$, we have $(s,t) \in \bar{H}$, and for every $(s,t) \in \bar{F}$, we have $(t,s) \in \bar{H}$. We say a vertex in $S$ is free if it is unmatched in $F$. For every integer $\ell \in [0, L]$, define $S^{\ell}$ ($T^{\ell}$ resp.) to be the set of vertices in $S$ ($T$, resp.) to which there exists a path in $\bar{H}$ of length at most $2\ell$ ($2\ell + 1$, resp.) from a free vertex. So, we have $S^0 \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq S^L$ and $T^0 \subseteq T^1 \subseteq T^2 \subseteq \cdots \subseteq T^L$.

Notice that $T^\ell = N_H(S^{\ell})$ for every $\ell \in [0, L]$. So for every $\ell \in [0, L]$, we have $(1+\epsilon)|S^{\ell}| \leq |T^\ell|$ by the condition of the lemma. All vertices in $T^L$ are matched by our assumption that there are no augmenting paths of length at most $2L + 1$. So for every $\ell \in [0, L - 1]$, we have $|T^\ell| \leq |S^{\ell+1}|$ as all vertices in $T^\ell$ are matched to $S^{\ell+1}$.

Combining the two statements gives us $(1+\epsilon)|S^{\ell}| \leq |S^{\ell+1}|$ for every $\ell \in [0, L - 1]$. Thus $|S^L| \geq (1+\epsilon)L |S^0|$, which contradicts the definition of $L$ and that $|S^0| \geq 1, |S^L| \leq |S|$.

This finishes the proof of Lemma 4.1.
Theorem 4.2. Given \( x \in [0, 1]^E \) satisfying \( x(\delta(j)) = 1 \) for every \( j \in J \), and \( \epsilon \in (0, 1) \), there is an \( O \left( \frac{|E|}{\epsilon} \log n \right) \)-time algorithm that outputs an assignment \( \sigma : J \rightarrow M \) of jobs to machines such that \((j, \sigma(j)) \in E \) and \( x_{j, \sigma(j)} > 0 \) for every \( j \in J \), and for every \( i \in M \), we have

\[
\sum_{j \in \sigma^{-1}(i)} p_{i,j} \leq (1 + \epsilon) \sum_{j \in N(i)} p_{i,j} x_{j,i} + \max_{j \in \sigma^{-1}(i)} p_{i,j}. \quad \text{(Assume the maximum over } \emptyset \text{ is 0.)}
\]

Proof. We construct a bipartite graph \( H = (J, V, E_H) \), starting with \( V = \emptyset \) and \( E_H = \emptyset \). For every machine \( i \in M \), we run the following procedure. See Figure 1 for an illustration. (The notations defined in the paragraph depend on \( i \); if a notation does not contain \( i \) in the subscript, it will only be used locally, in this paragraph.) Let \( D_i \) be the number of jobs \( j \) with positive \( x_{j,i} \) values. Let \( j_1, j_2, \ldots, j_{D_i} \) be these jobs \( j \), sorted in non-increasing order of \( p_{i,j} \); that is, we have \( p_{i,j_1} \geq p_{i,j_2} \geq \cdots \geq p_{i,j_{D_i}} \). For every integer \( d \in [0, D_i] \), we define \( Z_d = \sum_{i=d}^{d+1} x_{j_d,i} \). Let \( R_i = \lceil (1 + \epsilon)Z_d \rceil = \lceil (1 + \epsilon)x(\delta(i)) \rceil \). For every \( r = 1, 2, 3, \ldots, R_i \), we create a vertex \((i, r)\) and add it to \( V \). We add to \( E_H \) an edge between \( j_d, d \in [D_i] \) and \((i, r), r \in [R_i] \) if \( (Z_{d-1}, Z_d) \cap \left( \frac{r}{1 + \epsilon}, \frac{r+1}{1 + \epsilon} \right) \neq \emptyset \), and we define \( y_{j_d, (i, r)} \) to be the length of the interval. This finishes the construction of \( H = (J, V, E_H) \), along with a vector \( y \in \left( 0, \frac{1}{1 + \epsilon} \right)^{|E_H|} \).

![Figure 1: Construction of the \( H \) for the machine \( i \in M \). In the bipartite graph between \( \{j_1, j_2, \ldots, j_{D_i}\} \) and \( \{(i, 1), (i, 2), \ldots, (i, D_i)\} \), there is an edge between \( j_d \) and \((i, r)\) iff the interval correspondent to \( j_d \) intersects the interval \( (\frac{r}{1 + \epsilon}, \frac{r+1}{1 + \epsilon}) \).](image)

The number of edges in \( H \) for each \( i \) is at most \( D_i + R_i - 1 \leq |\delta(i)| + (1 + \epsilon)x(\delta(i)) \). Therefore the total number of edges we created in \( H \) is at most \(|E| + (1 + \epsilon)|J| = O(|E|) \). For every \((j, i) \in E\), we have \( \sum_{r=1}^{R_i} y_{j, (i, r)} = x_{j,i} \). This implies that for every \( j \in J \), we have \( y(\delta_H(j)) = 1 \). For every \((i, r) \in V\), we have \( y(\delta_H((i, r))) \leq \frac{1}{1 + \epsilon} \), and the inequality holds with equality except when \( r = R_i \).

For every set \( J' \subseteq J \), we have \(|N_H(J')| \geq (1 + \epsilon)|J'| \), as we can view \( y \) as a fractional matching in \( H \) where every \( j \in J \) is matched to an extent of 1 and every \((i, r) \in V \) is matched to an extent of at most \( \frac{1}{1 + \epsilon} \). Then we can use Lemma 4.1 to find a matching in \( H \) that covers all jobs \( J \). The running time of the algorithm is \( O \left( \frac{|E_H|}{\epsilon} \log n \right) = O \left( \frac{|E|}{\epsilon} \log n \right) \). The matching gives an assignment \( \sigma : J \rightarrow M \): If \( j \) is matched to \((i, r)\), then define \( \sigma(j) = i \). Fix some \( i \in M \) with \( \sigma^{-1}(i) \neq \emptyset \); we upper bound \( \sum_{j \in \sigma^{-1}(i)} p_{i,j} \):

\[
\sum_{j \in \sigma^{-1}(i)} p_{i,j} \leq \max_{j \in \sigma^{-1}(i)} p_{i,j} + \sum_{r=2}^{R_i} \sum_{j \in N_H((i,r))} p_{i,j} x_{j,i} \leq \max_{j \in \sigma^{-1}(i)} p_{i,j} + (1 + \epsilon) \sum_{r=2}^{R_i} \sum_{j \in N_H((i,r-1))} p_{i,j} y_{j, (i, r-1)} \leq \max_{j \in \sigma^{-1}(i)} p_{i,j} + (1 + \epsilon) \sum_{r=1}^{R_i} \sum_{j \in N_H((i,r))} p_{i,j} y_{j, (i, r)} = \max_{j \in \sigma^{-1}(i)} p_{i,j} + (1 + \epsilon) \sum_{j \in \sigma^{-1}(i)} p_{i,j} x_{j,i}.
\]

To see the first inequality in the first line, notice that the job \( j' \) matched to \((i, 1)\) (if it exists) has \( p_{i,j'} \leq \max_{j \in \sigma^{-1}(i)} p_{i,j} \), and the job \( j' \) matched to each \((i, r), r \in [2, R_i] \), has \( p_{i,j'} \leq \max_{j \in \delta_H((i,r))} p_{i,j} \). Consider the
second inequality. For every \( r \in [2, R_i] \), any \( j \in \delta_H((i, r)) \) and any \( j' \in \delta_H((i, r - 1)) \), we have \( p_{i,j} \leq p_{i,j'} \). Moreover, for every \( r \in [2, R_i] \), we have \( y_\delta((i, r - 1)) = \frac{1}{1+\epsilon} \). The inequality in the second line follows from replacing \( r \) with \( r + 1 \). The equality holds since for every \((j, i) \in E\) we have \( \sum_{r:(j, (i, r)) \in E_H} y_{j, (i, r)} = x_{j, i} \). □

**Lemma 4.3.** The expected total weighted completion time of the schedule produced by the rounding algorithm is at most \((1.5 + O(\epsilon))\Phi\).

**Proof.** Let \( \tilde{C}_j \) denote the completion time of \( j \) in the solution produced by the algorithm. We bound \( \mathbb{E}[\tilde{C}_j] \).

First, we condition on \( i, d_j \) and \( \theta_j \). The starting time of \( j \) is the total length of jobs \( j' \) assigned to \( i \) with \( \theta_{j'} < \theta_j \) (notice that we can assume no two jobs have the same \( \theta \) value since the probability it happens is 0). Therefore, for any \( i, d, \theta \), and the smallest \( r \in [D] \) such that \( \theta \leq \tau_r \), we have

\[
\mathbb{E}[\tilde{C}_j|i = i, d_j = d, \theta_j = \theta] = p_{i,j} + \sum_{j' \in J \setminus \{j\}, d'} y_{j', i, d'} \cdot \frac{\rho(p_{i,j'}, \tau_{d'}, \theta)}{p_{i,j'}} \cdot p_{i,j'} = p_{i,j} + \sum_{j' \in J \setminus \{j\}, d'} y_{j', i, d'} \cdot \rho(p_{i,j'}, \tau_{d'}, \theta) \leq p_{i,j} + (1 + O(\epsilon)) \tau_r \leq p_{i,j} + (1 + O(\epsilon)) \theta. \tag{39}
\]

The equality in the first line holds as the second term on its right side is the expected total size of jobs that are scheduled on \( i \) before \( j \). The first equation in the second line is by the definition of \( \rho(\cdot, \cdot, \cdot) \) and the way we choose \( \theta_{j'} \). The second inequality in the third line follows from that (24) holds with a \( 1 + O(\epsilon) \) factor. A subtle issue is when \( \theta \leq 1 \) and \( r = 1 \). As no jobs terminate in \((0, 1)\), we have \( \rho(p_{i,j'}, \tau_{d'}, \theta) \leq \theta \cdot \rho(p_{i,j'}, \tau_{d'}, 1) \).

So, \( \sum_{j' \in J \setminus \{j\}, d'} y_{j', i, d'} \cdot \rho(p_{i,j'}, \tau_{d'}, \theta) \leq \theta \cdot \sum_{j' \in J \setminus \{j\}, d'} y_{j', i, d'} \cdot \rho(p_{i,j'}, \tau_{d'}, 1) \leq (1 + O(\epsilon)) \theta_j \); (39) still holds.

Then, de-conditioning on \( \theta_j = \theta \), we get

\[
\mathbb{E}[\tilde{C}_j|i = j, d_j = d] \leq p_{i,j} + (1 + O(\epsilon)) \left( \tau_d - \frac{p_{i,j}}{2} \right) \leq \tau_d + (1 + O(\epsilon)) \left( \tau_d - \frac{\tau_d}{2} \right) = (1.5 + O(\epsilon)) \tau_d.
\]

The first inequality used (39) and that \( \mathbb{E}[\theta_j|i = j, d_j = d] = \tau_d - \frac{p_{i,j}}{2} \). The second inequality used that \( p_{i,j} \leq \tau_d \), if the event that \( i_j = i \) and \( d_j = d \) happens with positive probability.

Finally, de-conditioning on \( i_j \) and \( d_j \), we have

\[
\mathbb{E}[\tilde{C}_j] \leq (1.5 + O(\epsilon)) \sum_{i,d} x_{i,j,d} \tau_d.
\]

Therefore, we have

\[
\mathbb{E} \left[ \sum_{j \in J} w_j \tilde{C}_j \right] \leq (1.5 + O(\epsilon)) \sum_{j,i,d} w_j x_{j,i,d} \tau_d \leq (1.5 + O(\epsilon)) \cdot (1 + O(\epsilon)) \Phi = (1.5 + O(\epsilon)) \Phi.
\]

The second inequality is by that (23) holds a \((1 + O(\epsilon)) \) term on the right-side. □

**Lemma 4.4.** The following two inequalities hold for the vector \( y \in [0, 1]^E \) we constructed:

\[
\sum_{(j, i) \in E} p_{i,j}^2 y_{j, i} \leq (1 + \epsilon)^{O(\epsilon)} \Phi \tag{33}
\]

\[
\sum_{i \in M} g_i^2 \leq (1 + \epsilon)^{O(\epsilon)} \Phi \tag{34}
\]
Figure 2: Definitions of \( f_d \)'s and \( \psi(p, d, d') \)'s. (a) illustrates the definition of \( f_d \), which is the slope of the line segment connecting the two dots. (b) shows the definition of \( \psi(p, d, d') \). \( \psi(p, d, d'') \) and \( \psi(p, d, d') \) are the lengths of the two rectangles.

**Proof.** To see (33), notice that the derivative \( q \theta^q - 1 \) of the function \( \theta^q \) w.r.t \( \theta \) is monotone increasing. Therefore, \( c_{j,i,d} = (\tau_d - p_{i,j})^q \geq p_{i,j}^q \) for every \( (j, i, d) \in D \).

\[
\sum_{(j,i) \in E} p_{i,j}^q y_{j,i} = \sum_{(j,i,d) \in D} p_{i,j}^q x_{j,i,d} \leq \sum_{(j,i,d) \in D} c_{j,i,d} x_{j,i,d} \leq (1 + \epsilon)^{O(q)} \Phi.
\]

This finishes the proof of (33).

The proof of (34) is more involved. For every \( d, d' \in [D] \) and \( p \geq 0 \) with \( \tau_d \geq p \), we let \( \psi(p, d, d') := \rho(p, \tau_d, \tau_{d'}) - \rho(p, \tau_d, \tau_{d'-1}) \): If a job has completion time \( d \) and length \( p \) on its assigned machine, then \( \psi(p, d, d') \) is the volume of the job processed in \( (\tau_d, \tau_{d'-1}, \tau_{d'}) \), i.e., the length of intersection of \( (\tau_d - p, \tau_d) \) and \( (\tau_{d'-1}, \tau_{d'}) \). For every \( d \in [D] \), \( f_d = \frac{\tau_d - \tau_d}{\tau_d - \tau_{d-1}} \). Notice that we have \( 1 = f_1 \leq f_2 \leq f_3 \leq \cdots \leq f_D \). Define \( f'_1 = f_1 = 1 \) and for every integer \( d \in [2, D] \), define \( f'_d = f_d - f_{d-1} \geq 0 \). See Figure 2 for the definitions of \( f_d \)'s and \( \psi(p, d, d') \)'s.

With the notations defined, we can start to prove (34). Let \( \xi \) be the \( 1 + O(\epsilon) \) term such that (28) holds with right side being \( \xi \tau_d \).

\[
(1 + \epsilon)^{O(q)} \Phi \geq \sum_{(j,i,d) \in D} x_{j,i,d} c_{j,i,d} = \sum_{(j,i,d) \in D} x_{j,i,d} \left[ \frac{\tau_d^q}{\tau_d^q} - (\tau_d - p_{i,j})^q \right]
\]

\[
\geq \sum_{(j,i,d) \in D} x_{j,i,d} \sum_{d'=1}^D \psi(p_{i,j,d}, d', d') f_{d'} = \sum_{(j,i,d) \in D} x_{j,i,d} \sum_{d'=1}^D \psi(p_{i,j,d}, d') \sum_{d'=1}^D f_{d'}
\]

\[
= \sum_{i \in M} \sum_{d''=1}^D f_{d''} \sum_{j, d : (j,i,d) \in D} x_{j,i,d} \sum_{d'=1}^D \psi(p_{i,j,d}, d', d'')
\]

\[
= \sum_{i \in M} \sum_{d''=1}^D f_{d''} \sum_{j, d : (j,i,d) \in D} x_{j,i,d} (\rho(p_{i,j}, \tau_d, \tau_D) - \rho(p_{i,j}, \tau_{d'-1}, \tau_{d'-1})) \geq \sum_{i \in M} \sum_{d''=1}^D f_{d''} (g_i - \xi \tau_{d'-1})_+.
\]

The inequality in the first line is by that \( x \) satisfies (27) with right-side replaced by \( (1 + O(\epsilon)) \Phi \). The equality is by the definition of \( c_{j,i,d'} \)'s. To see the inequality in the second line, we break each \( \tau_d^q - (\tau_d - p_{i,j})^q \) into a sum of quantities, one for each \( d' \) with \( \psi(p_{i,j,d}, d') > 0 \): The quantity for \( d' \) is \( \tau_d^q - \max\{\tau_d - p_{i,j}, \tau_d - p_{i,j}\} \). When \( \psi(p_{i,j,d}, d') > 0 \), we have \( \psi(p_{i,j,d}, d') = \tau_d - \max\{\tau_d - p_{i,j}, \tau_d - p_{i,j}\} \). By the definition of \( f_{d'} \) and that \( q \geq 1 \), the quantity for \( d' \) is at least \( \psi(p_{i,j,d}, d') f_{d'} \). The equality in the second line is by the definition
of $f'_{d'}$. The equality in the third line is by reordering of variables. The equality in the last line is by the relation between $\psi(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$. The inequality in the line used $g_i = \sum_{j,i,d} x_{j,i,d} \rho(p_{i,j}, \tau_d, \tau_D)$ is the fractional load on $i$, and that (28) holds with a factor of $\xi$ on the right side.

In this paragraph we fix $i$ and lower bound the quantity $\sum_{d''=1}^D f'_{d''}(g_i - \xi \tau_{d''-1})_+.$

$$\sum_{d''=1}^D f'_{d''}(g_i - \xi \tau_{d''-1})_+ = \sum_{d''=1}^D (f_{d''} - f_{d''-1})(g_i - \xi \tau_{d''-1})_+$$

$$= \sum_{d''=1}^D f_{d''}((g_i - \xi \tau_{d''-1})_+ - (g_i - \xi \tau_{d''}))_+ = \sum_{d''=1}^D f_{d''} \min \{\xi \tau_{d''} - \xi \tau_{d''-1}, (g_i - \xi \tau_{d''-1})_+\}. \quad (40)$$

Above, we assumed $f_0 = 0$ and used that $(g_i - \xi \tau_D)_+ = 0$. Let $r$ be the index such that $\xi \tau_r \leq g_i < \xi \tau_{r+1}$. Then, the right-side of (40) is at least

$$\xi \sum_{d''=1}^r f_{d''}(\tau_{d''} - \tau_{d''-1}) = \xi \sum_{d''=1}^r (\tau_{d''}^q - \tau_{d''-1}^q) = \xi \tau_r^q > \xi \left(\frac{g_i}{\xi} \right)^q.$$ 

One subtle case is when $g_i < \xi$ and $r = 0$. In this case, the right-side of (40) is $g_i = \xi (g_i/\xi) \geq \xi (g_i/\xi)^q$.

Therefore, we have proved that $(1 + \epsilon)^{O(q)} \Phi \geq \xi \sum_{i \in M} \left(\frac{g_i}{\xi} \right)^q$. As $\xi = 1 + O(\epsilon)$, we have $\sum_{i \in M} g_i^q \leq (1 + \epsilon)^{O(q)} \Phi$, which is exactly (34).

**Lemma 4.6.** The $\sigma : J \to M$ obtained has $L_q$ norm of loads being at most $(2 + O(\epsilon))P$.

**Proof.** We have

$$\sum_{i \in M} \left(\sum_{j \sigma^{-1}(i)} p_j\right)^q \leq \sum_{i \in M} \left((1 + \epsilon) \sum_{j \in N(i)} p_j y_j,i + \max_{j \sigma^{-1}(i)} p_j\right)^q$$

$$\leq 2^{q-1} \left((1 + \epsilon)^q \sum_{i \in M} \left(\sum_{j \in N(i)} p_j y_j,i\right)^q + \sum_{i \in M} \left(\max_{j \sigma^{-1}(i)} p_j\right)^q\right)$$

$$\leq 2^{q-1} \left((1 + \epsilon)^q \sum_{i \in M} g_i^q + \sum_{j \in J} p_j^q\right) \leq 2^{q-1} \left((1 + \epsilon)^{O(q)} \Phi + \Phi\right) = 2^q (1 + \epsilon)^{O(q)} \Phi.$$ 

The inequality in the first line is by (35). The one in the second line used that $(a + b)^q \leq 2^{q-1}(a^q + b^q)$ for any $a, b \geq 0$. The first inequality in the third line used the definition of $g_i$’s and that each $j$ is assigned to only one machine. The second inequality used (34) and that $\sum_{j \in J} p_j^q \leq T^q = \Phi$. Therefore, we have

$$\left(\sum_{i \in M} \left(\sum_{j \in \sigma^{-1}(i)} p_j\right)^q\right)^{1/q} \leq 2(1 + O(\epsilon))P = (2 + O(\epsilon))P.$$ 

**Lemma 4.7.** The $\sigma : J \to M$ obtained has $L_q$ norm of loads being at most $\left(2^{2q-1} + 2q+O(\epsilon)\right)P$.

**Proof.** We have

$$\sum_{i \in M} \left(\sum_{j \sigma^{-1}(i)} p_{i,j}\right)^q \leq \sum_{i \in M} \left((1 + \epsilon) \sum_{j \in N(i)} p_{i,j} y_{j,i} + \max_{j \sigma^{-1}(i)} p_{i,j}\right)^q$$

$$\leq 2^{q-1} \left((1 + \epsilon)^q \sum_{i \in M} \left(\sum_{j \in N(i)} p_{i,j} y_{j,i}\right)^q + \sum_{i \in M} \left(\max_{j \sigma^{-1}(i)} p_{i,j}\right)^q\right)$$

$$\leq 2^{q-1} \left((2 + 2\epsilon)^q \sum_{i \in M} \left(\sum_{j \in N(i)} p_{i,j} y_{j,i}\right)^q + \sum_{j \in J} p_j^q\right)$$

$$\leq 2^{q-1} \left((2 + 2\epsilon)^q \sum_{i \in M} \left(\sum_{j \in N(i)} p_{i,j} y_{j,i}\right)^q + 2 \sum_{j \in J} \sum_{i \in N(j)} y_{j,i} p_{i,j}^q\right)$$

27
\[ 2^{q-1}\left(2^q(1 + \epsilon)^{O(q)}\Phi + 2(1 + \epsilon)^{O(q)}\Phi\right) = (2^{2q-1} + 2^q)(1 + \epsilon)^{O(q)}\Phi. \]

The inequality in the first line used (36). The one in the second follows from \((a + b)^q \leq 2^{q-1}(a^q + b^q)\) for every \(a, b \geq 0\). The inequality in the third line used \(y' \leq 2y\). The one in the fourth line follows from that every \((j, j')\) in the support of \(y'\) has \(\sum_{i \in N(j): p_{i,j} \geq 0} y_{j,i} \geq 1/2\), and \((\sigma(j), j)\) is in the support. The last line follows from (33) and (34).

So we have

\[
\left( \sum_{i \in M} \left( \sum_{j \in \sigma^{-1}(i)} p_{i,j} \right)^q \right)^{1/q} \leq (2^{2q-1} + 2^q)^{1/q}(1 + O(\epsilon))P. \]

\[\square\]

D Approximate Network Flow Algorithm: Proofs of Theorems 3.10 and 1.2

In this section, we first prove Theorem 3.10, which is repeated below. Theorem 1.2 can be proved with small changes and we defer it to the end of the section.

**Theorem 3.10.** Let \(\gamma \geq 0, \epsilon > 0\). There is an \(O\left(\frac{1}{\epsilon} |E| \cdot |V| \cdot \log |\mathcal{N}| \cdot \log \left|\frac{n}{\epsilon}\right|\right)\)-time algorithm that outputs a flow \(f \in \mathcal{F}_\gamma\) and a set \(S' \subseteq S\) such that \(\text{val}(f) + \frac{\text{val}(S')}{1+\epsilon} \leq \text{val}(f) + \phi\).

In Theorem 3.10, \(\gamma\) is fixed. So we omit the subscript \(\gamma\) in \(\mathcal{F}_\gamma\) and simply use \(\mathcal{F}\) for \(\mathcal{F}_\gamma\).

D.1 Handled Graphs and Shortcut Graphs

In this section, we introduce two important structures: handled graphs and shortcut graphs.

**Definition D.1.** Let \(\hat{G} = (\hat{V}, \hat{E})\) be a sub-graph of \(G\). A directed graph \(\tilde{G} = (\tilde{V}, \tilde{E})\) is a copy of \(\hat{G}\) if

- \(\tilde{V} \cap (S \cup T) = \hat{V} \cap (S \cup T)\),
- \(\tilde{V} \setminus (S \cup T)\) and \(\hat{V} \setminus (S \cup T)\) are disjoint, and
- there is a bijection \(\pi : \tilde{V} \to \hat{V}\) such that \(\pi(v) = v\) for every \(v \in \tilde{V} \cap (S \cup T) = \hat{V} \cap (S \cup T)\), and \((u, v) \in \tilde{E}\) if and only if \((\pi(u), \pi(v)) \in \hat{E}\) for every \(u, v \in \hat{V}\).

For every \(v \in \hat{V}\), we say \(\pi(v)\) is the pre-image of \(v\).

So \(\tilde{G}\) satisfies the definition if it is obtained from \(\hat{G}\) by copying everything except the sources and sinks.\(^8\) Throughout the paper, we shall use \(\pi\) to denote the function that maps all vertices in all copies of sub-graphs we ever defined to their pre-images. For convenience, we also let \(\pi(v) = v\) for every \(v \in V\). We guarantee the \(\pi(v)\) of every \(v\) is known to our algorithm.

**Definition D.2.** Let \(G^1 = (V^1, E^1), G^2 = (V^2, E^2), \ldots, G^k = (V^k, E^k)\) be \(k\) copies of sub-graphs of \(G\) for some integer \(k \geq 0\), such that for every \(1 \leq i < j \leq k\) we have \(V^i \cap V^j \subseteq S \cup T\). We say \(G' = (V', E')\), where \(V' = V^1 \cup V^2 \cup V^3 \cup \cdots \cup V^k\) and \(E' = E^1 \cup E^2 \cup E^3 \cup \cdots \cup E^k\), is a handled graph, and \(G^1, G^2, \ldots, G^k\) are called the handles of \(G'\).

See Figure 3 for an illustration of handled graphs. Notice that the handles of \(G'\) do not improve the connectivity from sources to sinks: For every \(s \in S\) and \(t \in T\), we have \(s \not\sim_{G'} t\) if and only if \(s \not\sim t\). The usefulness of handled graphs will be discussed later.

We extend the definition of a valid flow to sub-graphs, copies of sub-graphs and handled graphs:

\(^8\)There are no edges from sources to sinks in \(G\) so the first condition in the definition implies \(E \cap \hat{E} = \emptyset\).
Definition D.3. Let $G' = (V', E')$ be a sub-graph of $G$, or a copy of a sub-graph of $G$, or a handled graph. A valid flow $f'$ for $G'$ is a vector in $\mathbb{R}^{|E'_0|}$ satisfying $f'(\delta^+_{G'}(v)) \leq a_v$ for every $v \in V' \cap S$. $f'(\delta^-_{G'}(t)) \leq \gamma b_t$ for every $t \in V' \cap T$, and $f'(\delta^+_{G'}(v)) = f'(\delta^-_{G'}(v))$ for every $v \in V' \setminus (S \cup T)$. Let $\mathcal{F}_{G'}$ be the set of all valid flows for $G'$. The value of a $f' \in \mathcal{F}_{G'}$, denoted as $\text{val}(f')$, is defined as $f'(\delta^+_{G'}(V' \cap S)) = f'(\delta^-_{G'}(V' \cap T))$.

Given $f' \in \mathcal{F}_{G'}$ for some $G'$, we define the support of $f'$, denoted as $\text{supp}(f')$, as the following sub-graph $G'' = (V'', E'')$ of $G'$: $E''$ is the set of edges with positive $f'$ values, and $V''$ is the set of vertices incident to at least one edge in $E''$. We use $V_{\text{supp}(f')}$ and $E_{\text{supp}(f')}$ to denote the vertices and edges in $\text{supp}(f')$ respectively.

Definition D.4 (Shortcut Edges and Graphs). Given $s \in S, t \in T$ with $s \rightarrow t$, we say $(s, t)$ is a shortcut edge. Let $G' = (V', E')$ be a handled graph and $f' \in \mathcal{F}_{G'}$. For any $s \in S, t \in T$, we say $(t, s)$ is a backward shortcut edge w.r.t $f'$ if $s \leftarrow f'(t)$. A shortcut edge w.r.t $f'$ is defined as either a forward shortcut edge or a backward shortcut edge w.r.t $f'$.

Let $G' = (V', E')$ be a handled graph and $f' \in \mathcal{F}_{G'}$, the shortcut graph $H$ for $f'$ is the graph $(S \cup T, E_H)$, where $E_H$ is the set of shortcut edges w.r.t $f'$.

Notice that we can not maintain a shortcut graph explicitly since its size might be quadratic in $|V|$.

Definition D.5 (Augmenting Shortcut Path and Alternating Shortcut Path). Let $G' = (V', E')$ be a handled graph and $f' \in \mathcal{F}_{G'}$, and let $H$ be the shortcut graph for $f'$. We say a source $s \in S$ is satisfied w.r.t $f'$ if $f(\delta^+(s)) = a_s$ and unsatisfied otherwise. We say a sink $t \in T$ is saturated w.r.t $f'$ if $f(\delta^-(t)) = b_t$ and unsatisfied otherwise.

An alternating shortcut path in $H$ is a simple path starting from an unsatisfied vertex $s \in S$. An alternating shortcut path in $H$ is said to be an augmenting shortcut path in $H$ if it ends at an unsatisfied vertex $t \in T$. 

Figure 3: A handled graph. $G$ is the graph contained in the gray rectangle. $S$ and $T$ are respectively the vertices in the left and right small white rectangles. The thick edges are edges in handles. $v_1, v_2, v_3, v_4, v_5$ and $v_6$ are copies of $v_1, v_2, v_3, v_4, v_5$ and $v_6$ respectively.

Figure 4: $S^i$'s, $T^i$'s, $G^i,+$'s and $G^i,−$'s defined in Algorithm 3. The light gray components are inside $G$, and the dark gray components are copies of $G$. 

29
D.2 Long Augmenting Shortcut Paths Imply \((1 + \epsilon)\)-Approximate Flow

In this section, we prove that if we have a flow \(f'\) in some handled graph whose shortcut graph does not contain a short augmenting path, then we can find a set \(S' \subseteq S\) satisfying the property of Theorem 3.10. In the proof and throughout the rest of Section D, we use \(S^{\leq i}\) as a shorthand for \(\bigcup_{i \leq i} S^{i}\). Let \(\gamma_b\) be the size of the algorithm.

Proof. Let \(H = (S \cup T, E_H)\) be the shortcut graph for \(f'\). For every integer \(\ell \in [0, L]\), let \(S^{\ell}\) be the set of vertices in \(S\) to which the shortest alternating path in \(H\) has length exactly \(2\ell\), and let \(T^{\ell}\) be the set of vertices in \(T\) to which the shortest alternating path in \(H\) has length exactly \(2\ell + 1\). Let \(S^{L+1} = S \setminus S^{\leq L}\) and \(T^{L+1} = T \setminus T^{\leq L}\), so that \((S^{\ell})_{\ell \in [0, L+1]}\) and \((T^{\ell})_{\ell \in [0, L+1]}\) form partitions of \(S\) and \(T\) respectively.

We find the \(\ell^* \in [0, L]\) with minimum \(a(S^{> \ell^*}) + \frac{2\gamma_b(T^{\leq \ell^*})}{1 + \epsilon}\) and output \(S' := S^{\leq \ell^*}\). We prove that \(a(S \setminus S^{\ell^*}) + \frac{2\gamma_b(T^{S^{\ell*}})}{1 + \epsilon} \leq \val(f') + \frac{\phi}{3}\). Assume towards the contradiction that \(a(S^{> \ell^*}) + \frac{2\gamma_b(T^{\leq \ell^*})}{1 + \epsilon} > \val(f') + \frac{\phi}{3}\). Then, for every \(\ell \in [0, L]\), we have \(a(S^{> \ell^*}) + \frac{2\gamma_b(T^{\leq \ell})}{1 + \epsilon} > \val(f') + \frac{\phi}{3}\) by the way we choose \(\ell^*\), which is \(\val(f') - a(S^{> \ell^*}) + \frac{\phi}{3} < \frac{2\gamma_b(T^{\leq \ell})}{1 + \epsilon}\).

We prove \(\gamma_b(T^{\leq \ell}) \leq \val(f') - a(S^{\leq \ell+1})\) for every \(\ell \in [0, L]\). In the flow \(f'\), all sinks in \(T^{\leq \ell}\) are saturated as there are no augmenting path of length at most \(2\ell + 1\) in \(H\). So \(\gamma_b(T^{\leq \ell})\) units flow are sent to \(T^{\leq \ell}\) in \(f'\). The senders of this flow are in \(S^{\leq \ell+1}\). Moreover, as the sources in \(S^{> \ell+1}\) are saturated, the sources in \(S^{\leq \ell+1}\) sent \(\val(f') - a(S^{> \ell+1})\) units of flow. Hence \(\gamma_b(T^{\leq \ell}) \leq \val(f') - a(S^{> \ell+1})\).

Therefore, we have proved that \(\val(f') - a(S^{> \ell+1}) > (1 + \epsilon) \left(\val(f') - a(S^{> \ell}) + \frac{\phi}{3}\right)\) for every \(\ell \in [0, L]\). This gives us \(\val(f') = \val(f') - a(S^{L+1}) > (1 + \epsilon) L+1 \left(\val(f') - a(S^{L+1}) + \frac{\phi}{3}\right) \geq (1 + \epsilon) L+1 \frac{\phi}{3}\). However, as \(\val(f') \leq |\mathcal{A}|\), we have a contradiction by our definition of \(L\).

To construct the partitions \((S^{\ell})_{\ell \in [0, L+1]}\) and \((T^{\ell})_{\ell \in [0, L+1]}\), we define the following directed graph. We have vertices \(\{(v, o) : v \in V', o \in \{0, 1\}\}\). For every \(t \in T\), we have an edge \((t, (0), (1))\) of length 1. For every \(s \in T\), we have an edge \(((s, 1), (s, 0))\) of length 1. For every \((v, u) \in E'\), we have an edge \(((v, 0), (u, 0))\) of length 0. For every \((v, u) \in \supp(f')\), we have an edge \(((u, 1), (v, 1))\) of length 0. Then \((S^{\ell})_{\ell \in [0, L+1]}\) and \((T^{\ell})_{\ell \in [0, L+1]}\) can be constructed using a variant of BFS that takes care of length-0 edges. The running time of the algorithm is \(O(|E'|)\).

With Lemma D.6, it remains to construct a handled graph \(G'\) and a flow \(f' \in \mathcal{F}^{G'}\) satisfying the condition of the lemma. To achieve the goal, we need more definitions and tools.

D.3 Sub-Flows, Projections and Blocking Flows

Definition D.7. Let \(G' = (V', E')\) be a handled graph, and \(f' \in \mathcal{F}^{G'}\). Let \(S' \subseteq S\) be a subset of sources. Then, a sub-flow of \(f'\) sent from \(S'\) is a flow \(f'' \in \mathcal{F}^{G'}\) satisfying

\[
\begin{align*}
(D.7a) \quad & f''_e \leq f'_e \text{ for every } e \in E', \\
(D.7b) \quad & f''_e = f'_e \text{ for every } e \in \delta^+_G(S'), \text{ and} \\
(D.7c) \quad & f''_e = 0 \text{ for every } e \in \delta^-_G(S' \setminus S'). 
\end{align*}
\]

Let \(T' \subseteq T\) be a subset of sinks. Similarly, a sub-flow of \(f'\) received by \(T'\) is a flow \(f'' \in \mathcal{F}^{G'}\) satisfying

\[
\begin{align*}
(D.7d) \quad & f''_e \leq f'_e \text{ for every } e \in E', \\
(D.7e) \quad & f''_e = f'_e \text{ for every } e \in \delta^-_G(T'), \text{ and} \\
(D.7f) \quad & f''_e = 0 \text{ for every } e \in \delta^-_G(T \setminus T').
\end{align*}
\]
Lemma D.8. Let $G' = (V', E')$ be a handled graph, and $f' \in \mathcal{F}^G$, and $S' \subseteq S$. Then we can find a sub-flow $f''$ of $f'$ sent from $S'$ in time $O(|S'| + |\supp(f')| \cdot \log |V'| + |E_{\supp(f')}|)$.

Proof. Initially, we have $f'(\delta_{G'}^+(s))$ units of commodity at any $s \in S'$, and 0 units of commodity elsewhere. We then process the vertices $V' \setminus T$ in topological order one by one. When processing a vertex $v \in V' \setminus T$, we push the commodity at $v$ to its out-neighbors along edges in $\delta_{G'}^-(v)$, with the only constraint being $f'' \leq f'$.

We assume the topological ordering of $G$ is computed at the beginning of the whole algorithm and for every $v \in V$ we know the rank of $v$ in the ordering. Notice that the rank of vertices in $G$ can be extended to rank of vertices in $G'$. Then we use a priority-queue data structure to store the vertices which hold the commodity, with rank being the priority function. The overall running time of the algorithm can be bounded by $O(|S'| + |\supp(f')| \cdot \log |V'| + |E_{\supp(f')}|)$.

Similarly, the following lemma can be proved:

Lemma D.9. Let $G' = (V', E')$ be a handled graph, and $f' \in \mathcal{F}^G$, and $T' \subseteq T$. Then we can find a sub-flow $f''$ of $f'$ received by $T'$ in time $O(|T'| + |\supp(f')| \cdot \log |V'| + |E_{\supp(f')}|)$.

Definition D.10. Let $G' = (V', E')$ be a handled graph, and $f' \in \mathcal{F}^G$. Let $\tilde{G} = (V, \tilde{E})$ be a sub-graph of $G$, or a copy of a sub-graph of $G$. If for every $(u, v) \in \supp(f')$, we have some $(\hat{u}, \hat{v}) \in \tilde{E}$ with $\pi(u) = \pi(\hat{u})$ and $\pi(v) = \pi(\hat{v})$, then we say $f'$ can be projected to $\tilde{G}$. (Notice that the $(\hat{u}, \hat{v})$ is unique, if it exists.) Otherwise, we say $f'$ can not be projected to $\tilde{G}$. In the former case, we define the projection of $f'$ to $\tilde{G}$ to be the vector $f \in \mathbb{R}_{\geq 0}^{\tilde{E}}$ satisfying:

$$\hat{f}_{u,v} = f'(\{(u,v) \in E' : \pi(u) = \pi(\hat{u}), \pi(v) = \pi(\hat{v})\}), \quad \forall (\hat{u}, \hat{v}) \in \tilde{E}.$$

Clearly, in the above definition, the projection $\hat{f}$ of $f'$ to $\tilde{G}$ has $\hat{f} \in \mathcal{F}^{\tilde{G}}$ and $\val(\hat{f}) = \val(f')$.

Dinitic [12] introduced the notion of blocking flows, which is a $s$-$t$ flow such that every $s$-$t$ path in the graph $G$ has an edge that is full.

Definition D.11 ([12]). Let $R = (V_R, E_R)$ be a directed graph with two special vertices $s^*, t^* \in V_R$ such that $\delta^-(s^*) = \emptyset$ and $\delta^+(t^*) = \emptyset$. Let $c \in [0, \infty]^{E_R}$ be a capacity vector on $E_R$. A blocking flow in $(R, c)$ is a vector $g \in \mathbb{R}_{\geq 0}^{E_R}$ satisfying $g \leq c$, $g(\delta^+(v)) = g(\delta^-(v))$ for every $v \in V_R \setminus \{s^*, t^*\}$, and every path $P$ from $s^*$ to $t^*$ in $R$ contains an edge $e$ with $g_e = c_e$.

So, a flow $g$ is a block flow if we can not increase its value by only increasing $g_e$ values. In an influential paper of Sleator and Tarjan [43], they developed the dynamic tree (also known as link cut tree) data structure and showed how it can be used to find a blocking flow in $O(|E_R| \log |V_R|)$-time.

Theorem D.12 ([43]). Let $R = (V_R, E_R)$, $s^*, t^*$ and $c$ as defined in Definition D.11. There is $O(|E_R| \log |V_R|)$-time algorithm that finds a blocking flow in $(R, c)$.

### D.4 Augmenting using Shortest Shortcut Paths

Now we show how to find $G'$ and $f'$ satisfying the property of Lemma D.6. We maintain $G'$ and $f' \in \mathcal{F}^G$ and repeatedly augment $f'$ along shortest augmenting paths in the shortcut graph $H'$ for $f'$. During the procedure, we need to change the handled graph $G'$ from iteration to iteration. The following core theorem states that we can increase the length of the shortest augmenting path in $H'$ by 2 in nearly-linear time.

Theorem D.13. Let $G^0 = (V^0, E^0)$ be a handled graph, $f^0 \in \mathcal{F}^{G^0}$, and $H^0$ be the shortcut graph for $f^0$. Let $\ell \geq 0$ be an integer such that the length of the shortest augmenting path in $H^0$ is at least $2\ell + 1$. Given $\ell, G^0, f^0$ and $\epsilon > 0$, there is an $O(|E^0| \log |V^0|)$-time algorithm that outputs a handled graph $G' = (V', E')$ with $|V'| \leq 3|V|$ and $|E'| \leq 3|E|^9$ and a flow $f' \in \mathcal{F}^{G'}$, such that in the shortcut graph $H'$ for $f'$, the shortest augmenting path has length at least $2\ell + 3$. 

31
The algorithm is described in Algorithm 3. In the algorithm, the sub-graph of $G$ between a set $S' \subseteq S$ and $T' \subseteq T$ is defined as the sub-graph of $G$ induced by $\{v \in V : S' \rightarrow v, v \rightarrow T'\}$.

In Steps 1 to 5, we define $S^0$, $T^0$, $G^{t,+}$'s, $G^{t,-}$'s and $G'$. See Figure 4 for an illustration. By our assumption that every augmenting path in $H'$ has length at least $2\ell + 1$, all sinks in $T^{<\ell}$ are saturated by the initial $f^0$. Notice that $G^{t,+}$'s are sub-graphs of $G$, but $G^{t,-}$'s are copies of sub-graphs and are included in $G'$ as handles. In Loop 6 of Algorithm 3, we construct the flows $f^{(i,+)}$, $f^{(i,-)}$ and $f^{(i,-)}$'s. The following claimed can be proved via mathematical induction:

**Claim D.14.** Focus on the iteration $i$ of Loop 6.

- At the beginning of the iteration, $f^0$ is a flow from $S^{\geq i}$ to $T^{\geq i}$.
- $f^{(i,+)}$ can be projected to $G^{t,+}$.
- $f^{(i-)}$ can be projected to $G^{t-}$.

**Proof.** Assume that at the beginning of iteration $i$ of Loop 6, $f^0$ is a flow from $S^{\geq i}$ to $T^{\geq i}$; this holds for $i = 0$. Then $f^{(i,+)}$ is a flow from $S^i$ to $T^i$ since $S^i \not\subset G'$, $T^{\geq i}$. So it can be projected to $G^{t,+}$. Then before Step 10 in the iteration, $f^0$ is a flow from $S^{\geq i+1}$ to $T^{\geq i}$. $f^{(i,-)}$ is a flow from $S^{t+1}$ to $T^i$ since $S^{t+1} \not\subset \text{supp}(f^0)$. So, it can be projected to $G^{t,-}$. In the end of the iteration $i$ and thus at the beginning of iteration $i + 1$, $f^0$ is a flow from $S^{\geq i+1}$ to $T^{\geq i+1}$.

9We use concrete constants here to avoid abuse of $O(\cdot)$ notation caused by applying the theorem repeatedly.
In Step 13, we construct $f'$ by summing up $f^{(i,+)}$s and $f^{(i,-)}$s. Then, we define a residual graph $R$, find a blocking flow $g$ in the graph, augment $f'$ using $g$ in Steps 14, 15 and 16. We return $(G', f')$ in the end.

Now we can prove the key lemma that establishes the correctness of the algorithm:

**Lemma D.15.** Let $H'$ be the shortcut graph w.r.t $f'$ returned by Algorithm 3. Then any augmenting path in $H'$ has length at least $2\ell + 3$.

**Proof.** To avoid confusion, we use $f'$ be the flow $f'$ obtained after Step 13, that is, before it is augmented. We can use $f'$ to be the final $f'$ returned by the algorithm. Let $H'$ and $H'$ be the shortcut graphs for $f'$ and $f'$ respectively. Let $S^{\ell+1} := S \setminus S^{\ell}$ and $T^{\ell+1} := T \setminus T^{\ell}$, so that $(S_i)_{i \in [0,\ell+1]}$ and $(T_i)_{i \in [0,\ell+1]}$ are partitions of $S$ and $T$ respectively.

The following two properties hold:

(P1) If a forward edge in $H'$ connects $S^{i}$ to $T^{i'}$, then we have $i' \leq i$.

(P2) If a backward edge in $H'$ connects $T^{i}$ to $S^{i'}$, then $i' \in \{i, i + 1\}$.

(P1) follows from the definition of $S^i$'s and $T^{i'}$'s. (P2) follows from that $G^{i,+}$'s and $G^{i,-}$'s are internally disjoint, and that $f'$ has support in these graphs.

If we focus on the sequence $S^{0}, T^{0}, S^{1}, T^{1}, \ldots, S^{\ell}, T^{\ell}$ of vertex sets, every edge in $H'$ can only increase the position of the vertex in the sequence by 1. All unsatisfied sources are in $S^0$ and all unsaturated sinks are in $T^{\ell} \cup T^{\ell+1}$. Therefore, an augmenting path in $H'$ has length at least $2\ell + 1$. For it to have length exactly $2\ell + 1$, it can only use *useful* shortcut edges in $H'$: A forward shortcut edge is useful if it connects $S^i$ to $T^{i'}$ for some $i \in [0,\ell]$, and a backward shortcut edge is useful if it connects $T^{i'}$ to $S^{i+1}$ for some $i \in [0,\ell - 1]$. Moreover, if there is a backward edge from $T^{i'}$ to $S^{i+1}$ in $H'$, the correspondent path from $S^{i+1}$ to $T^{i'}$ in $\text{supp}(f')$ must be completely in $G^{\ell+1}$.

We then consider how augmenting $f'$ using $g$ to $f'$ changes the set of backward shortcut edges. The set of backward shortcut edges created by paths in the handles can only shrink, since the augmenting operation can only decrease $f_{z}$ values of edges in the handles. The backward shortcut edges created by paths in $G^{i,+}$ are not useful, as they connect $T^{i}$ to $S^{i}$. Therefore the set of useful shortcut edges can only shrink from $H'$ to $H'$. Suppose there is an augmenting path of length $2\ell + 1$ in $H'$. It must be an augmenting path in $H'$. However, as $g$ is a blocking flow in $R$, one of the (backward) shortcut edge in the path must be broken in $H'$, a contradiction. Therefore, there are no augmenting paths of length $2\ell + 1$ in $H'$. The lemma follows since an augmenting shortcut path has length being an odd number. \hfill \Box

We remark that it is crucial for us to make $G^{i,-}$'s and $G$ internally disjoint using handles. If we let $G^{i,-}$'s be sub-graphs of $G$, then increasing flow values in $G^{i,+}$ may create new backward shortcut edges from $T^{i}$ to $S^{i+1}$. Though augmenting $f'$ by $g$ destroys the old augmenting paths of length $2\ell + 1$ in $H'$, it may create new ones. At the other extreme, we could decompose $f^{(i,-)}$'s completely into paths, and maintain a set of source-sink pairs, each with the amount of flow sent between them. But this way we could not bound the number of such pairs as the algorithm proceeds. Moreover, in the end, we have to realize the flows sent between the pairs in $G$. Even assuming in the end we have, say, $O(|E|)$ $(s,t)$-pairs with positive amount of flow sent in between, we do not know how to realize the flows in $G$ in nearly-linear time. So, the handled graph gives an approach between the two extremes, which can guarantee that the length of the shortest augmenting shortcut paths increases from iteration to iteration, and that the graphs we maintain have nearly-linear size.

We now show the algorithm has nearly-linear running time. The following are two simple but useful observations we can make:

**Observation D.16.** For each vertex $v \in V$, let $i$ be the smallest integer such that $S^{i} \rightharpoonup v$. Then $v$ is not in $G^{i,+}$ for any $i' \neq i$. $v$ does not appear in $G^{i,-}$ as a copy for any $i' \notin \{i - 1, i\}$.

**Proof.** For any $i' < i$, we have $S^{i'} \not\rightarrow v$. All the vertices $t \in T$ with $v \rightharpoonup t$ are included in $T^{\leq i}$. So, for every $i' > i$ we have $v \not\rightharpoonup T^{i'}$. Therefore the two statements follow. \hfill \Box

**Observation D.17.** Any $v \in V^{\circ}$ is in the support of at most 3 flows in $\{f^{(i,+)} : i \in [0,\ell]\} \cup \{f^{(i,-)} : i \in [0,\ell - 1]\}$. 

33
Proof. Let \( i \) be the smallest integer such that \( S^i \sim_{G^o} v \). Then, \( S'^i \not\sim_{G^o} v \) for every \( i' < i \). Also \( v \not\sim_{G^o} T' \) for every \( i' > i \) since otherwise \( S^i \sim_{G^o} T' \), which implies \( S^i \sim_{G} T' \), a contradiction. So, if \( v \in V_{\text{supp}(f^{(i'+1)})} \), then \( i' = i \). If \( v \in V_{\text{supp}(f^{(i'-1)})} \), then \( i' \in \{i-1, i\} \).

Steps 1 and 2 can be implemented in \( O(|E'|) \) time using BFS. By Observation D.16, we have \(|V'| \leq 3|V|\) and \(|E'| \leq 3|E|\), and Step 3, 4 and 5 can be implemented in time \( O(|E'|) \). Constructing the sub-flows \( f^{(i,+)} \) and \( f^{(i,-)} \) takes time \( O(|S^i| + |V_{\text{supp}(f^{(i,+)})}| \cdot \log |V^o| + |E_{\text{supp}(f^{(i,+)})}|) \) and \( O(|T^i| + |V_{\text{supp}(f^{(i,-)})}| \cdot \log |V^o| + |E_{\text{supp}(f^{(i,-)})}|) \) respectively. By Observation D.17, Loop 6 takes time \( O(|V^o| \cdot \log |V^o| + |E^o|) \). Steps 13 and 14 take time \( O(|V^o|) \). The bottleneck of the algorithm is Step 16, which runs in time \( O(|E'| \cdot \log |V^o|) \) using Theorem D.12. This finishes the proof of Theorem D.13.

D.5 Finishing Proof of Theorem 3.10

In this section, we can wrap up the proof of Theorem 3.10. Let \( L := \left\lceil \log_{1+\epsilon} \frac{3|a|_1}{\varphi} \right\rceil \) as in Lemma D.6. We run Algorithm 4 defined below:

**Algorithm 4** Construction of \( f \in \mathcal{F} \) and \( S' \subseteq S \) satisfying properties of Theorem 3.10

1. \( G^{(0)} \leftarrow G, f^{(0)} \leftarrow \) all-0 vector over domain \( E \)
2. for every \( \ell = 0 \) to \( L := \left\lceil \log_{1+\epsilon} \frac{3|a|_1}{\varphi} \right\rceil \) do
   3. \( (G^{(\ell+1)}, f^{(\ell+1)}) \leftarrow \text{inc-len}(\ell, G^{(\ell)}, f^{(\ell)}) \)
4. return the projection \( f \) of \( f^{(L+1)} \) to \( G \), and \( S' \) obtained using Lemma D.6 over \( G^{(L+1)} \) and \( f^{(L+1)} \)

By Theorem D.13, for every \( \ell \in [0, L+1] \), we have that \( G^{\ell} \) is a handled graph, \( f^{\ell} \in \mathcal{F}^{G^{\ell}} \) and the shortest augmenting path in the shortcut graph for \( f^{\ell} \) has length at least \( 2\ell + 1 \). Then by Lemma D.6, we can find a set \( S' \subseteq S \) with \( a(S \setminus S') + \frac{\gamma \delta(T(S'))}{1+\epsilon} \leq \text{val}(f^{\ell+1}) + \frac{\varphi}{3} = \text{val}(f) + \frac{\varphi}{3} \). By Theorem D.13, all handled graphs \( G^{\ell} \) constructed have at most \( 3|V| \) vertices and \( 3|E| \) edges. So, the overall running time is \( O(L \cdot |E| \cdot \log |V|) = O \left( \frac{1}{\epsilon} \cdot |E| \cdot \log |V| \cdot \log \frac{|a|_1}{\varphi} \right) \). This finishes the proof of Theorem 3.10.

D.6 Proof of Theorem 1.2

Finally, we prove Theorem 1.2, which is repeated below.

**Theorem 1.2.** Let \( \epsilon > 0 \) and the network flow instance \((G, S, T, a, b)\) be given as above. There is an \( O\left(\frac{|E|}{\epsilon} \cdot \log |V|\right)\)-time algorithm that outputs a valid flow \( f \) whose value is at least \( \frac{1}{1+\epsilon} \) times that of the maximum flow for the instance.

The instance in Theorem 1.2 is indeed NFP1 and thus we fix \( \gamma = 1 \). We modify the proof of Lemma D.6 to prove the following lemma:

**Lemma D.18.** Let \( G' = (V', E') \) be a handled graph and \( f' \in \mathcal{F}^{G'} \). Let \( L = \left\lceil \frac{1}{\epsilon} \right\rceil \). Assume the shortcut graph \( H \) for \( f' \) does not contain an augmenting path of length at most \( 2L + 1 \). Then there is a set \( S' \subseteq S \) such that \( a(S \setminus S') + b(T(S')) \leq \text{val}(f') + \epsilon \cdot \text{val}(f') \).

**Proof.** We can define \((S^\ell)_{\ell \in [0, L+1]} \) and \((T^\ell)_{\ell \in [0, L+1]} \) as in the proof of Lemma D.6. If there is no such \( S' \), then using the same argument as in the proof, with \( 1 + \epsilon \) replaced by \( 1 \) and \( \frac{\varphi}{3} \) replaced by \( \epsilon \cdot \text{val}(f') \), we can prove \( \text{val}(f') - a(S^{\ell+1}) \geq \text{val}(f') - a(S^{\ell}) + \epsilon \cdot \text{val}(f') \) for every \( \ell \in [0, L] \). This implies \( \text{val}(f') = \text{val}(f') - a(S^{L+1}) \geq \text{val}(f') - a(S^{0}) + (L+1) \cdot \epsilon \cdot \text{val}(f') \geq (L+1) \cdot \epsilon \cdot \text{val}(f') \), which leads to a contradiction by our definition of \( L \).

Therefore, the optimum value of NFP1 is at most \( (1 + \epsilon)\text{val}(f') \), implying that the projection \( f \) of \( f' \) to \( G \) is a \( \frac{1}{1+\epsilon} \)-approximation to NFP1. Running Algorithm 4 with \( L = \left\lceil \frac{1}{\epsilon} \right\rceil \), we obtain an \( O(L \cdot |E| \cdot \log |V|) = O \left( \frac{1}{\epsilon} \cdot |E| \cdot \log |V| \right) \) time algorithm that outputs such a flow \( f \). This finishes the proof of Theorem 1.2.
E Handling Super-Polynomial Integers in Input

In this section, we show how to handle the cases when the sizes and/or weights are super-polynomial in n, for the two problems \( P|\text{prec}||\sum_j w_j C_j \) and \( R||\sum_j w_j C_j \).

E.1 Handling Super-Polynomial \( p_{\text{max}} \) for \( P|\text{prec}||\sum_j w_j C_j \)

The main modification in this case is that we define \( d_j^{\text{min}} \) and \( d_j^{\text{max}} \) differently. First, we need a poly\((n)\)-approximation for the scheduling instance. Recall that \( q_j \) is the maximum total size of jobs in a precedence chain ending at \( j \). The optimum schedule has weighted completion time at least \( \Phi := \sum_j w_j q_j \). On the other hand, if we schedule all the jobs in non-decreasing order of \( q_j \) values on one machine (even in case we have \( m \) machines) so that jobs respect the precedence constraints, the completion time of a job \( j \) is at most \( nq_j \) and thus the weighted completion time of the schedule is at most \( n\Phi \). Therefore, we have \( \Phi \leq \text{opt} \leq n\Phi \), where \( \text{opt} \) is the optimum weighted completion time for the given instance.

For every \( j \in J \), and let \( \tilde{w}_j := \max_{j' \succ_j^*} w_{j'} \) be the maximum weight of a job that directly or indirectly succeeds \( j \); \( j' \succ_j^* j \) means there is a precedence chain from \( j \) to \( j' \); we assume \( j \succ_j^* j \). \( \tilde{w}_j \)'s can be computed in \( O(|E|) \) time using dynamic programming. We still define \( \tau_0 = 0 \) and \( \tau_d = (1 + \epsilon)^{d-1} \) for every integer \( d \geq 1 \). For every \( j \in J \), define

\[
\tilde{d}_j^{\text{min}} := \max \left\{ d : \tau_d \leq \frac{\epsilon \Phi}{n\tilde{w}_j} \right\} \quad \text{and} \quad \tilde{d}_j^{\text{max}} := \min \left\{ d : \tau_d \geq \frac{n\Phi}{w_j} \right\}.
\]

Let \( D = \max_{j \in J} \tilde{d}_j^{\text{max}} \). We use \( \tilde{w}_j \) instead of \( w_j \) in the definitions to guarantee that \( (\tilde{d}_j^{\text{min}})_{j \in J} \) and \( (\tilde{d}_j^{\text{max}})_{j \in J} \) respect the precedence constraints.

We still use LP(2), but with the new definitions of \( d_j^{\text{min}} \) and \( d_j^{\text{max}} \) values. In the linear program that we actually solve, we only have a variable \( x_{j,d} \) for every \( j \in J \) and integer \( d \in (d_j^{\text{min}}, d_j^{\text{max}}) \), since the other variables are fixed to 0 or 1. In the analysis, it is convenient for us to keep a variable \( x_{j,d} \) for every \( j \in J \) and \( d \in [0, D] \).

We need to argue the validity of LP(2) again since now we forced \( x_{j,d} = 0 \) for \( d \leq \tilde{d}_j^{\text{min}} \). In the correspondent 0/1-integer program with the requirement, \( x_{j,d} \) is intended to indicate whether \( d > \tilde{d}_j^{\text{min}} \) and \( j \) has completion time at most \( \tau_d \). (3) says if \( j \) has completion time at most \( \tau_d \) and \( d > \tilde{d}_j^{\text{min}} \), then it has completion time at most \( \tau_{d+1} \) and \( d + 1 > \tilde{d}_j^{\text{min}} \). (4) requires that for two jobs \( j < j' \) and \( d > \tilde{d}_j^{\text{min}} \), if \( j' \) has completion time at most \( \tau_d \), then so does \( j \) and \( d > \tilde{d}_j^{\text{min}} \). This is valid since \( \tilde{w}_j \geq \tilde{w}_{j'} \), which implies \( \tilde{d}_j^{\text{min}} \leq \tilde{d}_{j'}^{\text{min}} \). (5) is valid since the total size of jobs with completion time at most \( \tau_d \) is at most \( m\tau_d \) for every \( d \in [0, D] \) in any valid solution. (6) is from the intended meaning of \( x_{j,d} \)'s and that a job \( j \) can not complete before time \( q_j \). (7) is valid by the definition of \( \tilde{d}_j^{\text{max}} \): If a job \( j \) has completion time more than \( \tau_{d_j^{\text{max}}} \), then it incurs a weighted completion time of more than \( n\Phi \geq \text{opt} \).

Then we show that forcing \( x_{j,d} = 0 \) if \( d \leq \tilde{d}_j^{\text{min}} \) only incurs a multiplicative factor of \( 1 + \epsilon \). Recall that \( \text{lp} \) and \( \text{opt} \) are respectively the values of LP(2) and the scheduling instance.

**Lemma E.1.** \( \text{lp} \leq (1 + \epsilon) \text{opt} \).

**Proof.** Let \( \tilde{x}^* \in \{0, 1\}^{J \times [0, D]} \) be the solution correspondent to the optimum schedule: \( \tilde{x}^*_{j,d} \in \{0, 1\} \) indicates if \( d > \tilde{d}_j^{\text{min}} \) and \( j \) has completion time at most \( \tau_d \) in the schedule. Then, we have

\[
\text{opt} \geq \sum_{j \in J} w_j \left( \sum_{d=1}^{D} \tilde{x}^*_{j,d} \frac{\tau_d}{1 + \epsilon} - \tau_{d_j^{\text{min}}} \right) \geq \frac{1}{1 + \epsilon} \sum_{j \in J} w_j \left( \sum_{d=1}^{D} \tilde{x}^*_{j,d} (\tau_d - \tau_{d+1}) \right) - \epsilon \cdot \text{opt} = \frac{1}{1 + \epsilon} \sum_{j \in J} w_j \left( \tau_D - \sum_{d=1}^{D-1} \eta_d \tilde{x}^*_{j,d} \right) - \epsilon \cdot \text{opt} \geq \frac{\text{lp}}{1 + \epsilon} - \epsilon \cdot \text{opt}.
\]
To see the first inequality in the first line, focus on a job $j \in J$ and the $d$ such that $\hat{x}_{j,d} = 1$ and $\hat{x}_{j,d-1} = 0$. If $d-1 > d_{j}^{\text{min}}$, then the completion time of $j$ is in $(\tau_{d-1}, \tau_{d}]$ and thus is at least $\frac{\tau_{d-1} + \tau_{d}}{2}$. Otherwise $d-1 = d_{j}^{\text{min}}$ and the completion time of $j$ is at least 1. In either case, the term inside the parentheses lower bounds the completion time. The second inequality in the line holds since $\sum_{j \in J} w_{j} \tau_{j}^{\text{min}} \leq \epsilon \cdot \text{opt}$, as $w_{j} \tau_{j}^{\text{min}} \leq \hat{w}_{j} \tau_{j}^{\text{min}} \leq \frac{\epsilon}{n} \leq \frac{\epsilon \cdot \text{opt}}{n}$ for every $j \in J$. The other arguments are the same as those in the proof of Lemma 3.1. In the end, we have $l_{p} \leq (1 + \epsilon)^{2} \text{opt}$, finishing the proof of the lemma. 

Again, we define the directed graph $G = (V, E)$ in Section 3.1: $(j, d) \in V$ if and only if $x_{j,d}$ is not fixed to 0 or 1 in LP(2), and there is an edge from $(j, d)$ to $(j', d')$ if we have a constraint $x_{j,d} \leq x_{j',d'}$ in (3) or (4). For every $j \in J$, we have $d_{j}^{\text{max}} - d_{j}^{\text{min}} = \tilde{O} \left( \frac{\log n}{\epsilon} \right) = \tilde{O}_{\epsilon}(1)$. Therefore we have $|V| \leq \tilde{O}_{\epsilon}(n)$. The numbers of constraints in (3) and (4) are respectively $\tilde{O}_{\epsilon}(n)$ and $\tilde{O}_{\epsilon}(\kappa)$. So $|E| \leq \tilde{O}_{\epsilon}(n + \kappa)$. Each variable appears in exactly one constraint in (5), the matrix $P$ defining (5) has the number of non-zeros being $\tilde{N} = \tilde{O}_{\epsilon}(n)$. Let $a_{j,d} = w_{j} \eta_{d}$ for every variable $(j, d) \in V$. Then the LP is equivalent to max $\text{ax}$ subject to $x \in \mathbb{Q} := \{x \in [0, 1]^{V} : x_{v} \leq x_{u} \forall (v, u) \in E\}, Px \leq 1$.

We set $\phi = \epsilon \cdot \text{opt}$. Then for every $(j, d) \in V$, we have $a_{j,d} = w_{j} \eta_{d} \leq \hat{w}_{j} \tau_{j}^{\text{max}} - \tau_{d-1} \leq n \Phi \leq \frac{n}{\epsilon} \cdot \phi$, as $d < d_{j}^{\text{max}}$. Therefore, we have $|a|_{1} \leq \text{poly}(n) \cdot \phi$. Then each time the oracle given in Theorem 3.2 still takes time $\tilde{O}_{\epsilon}(n + \kappa)$. However, Loop 2 in the template algorithm need to run for $\tilde{O} \left( \frac{\log n}{\epsilon} \right) = \tilde{O}_{\epsilon}(D) = \tilde{O}_{\epsilon}(\log p_{\text{max}})$ iterations, where $\tilde{m}$ is the number of rows of $P$, i.e., the number of constraints in (5). Overall, the running time of the algorithm is $\tilde{O}_{\epsilon}(n + \kappa) \log p_{\text{max}}$.

### E.2 Handling Arbitrary Processing Times and Weights for $R|| \sum_{j} w_{j} C_{j}$

In this section, we consider the unrelated machine weighted completion time problem, and remove the assumption that all weights and processing times are bounded by a polynomial function of $n$.

**Preprocessing** First we need a simple poly($n$)-approximation for the problem, and this can be done easily:

**Lemma E.2.** Assigning each job $j$ to the machine $i$ with the smallest $p_{i,j}$ leads to a $\frac{n+1}{2}$-approximation for the weighted completion time problem.

**Proof.** Let $\sigma : J \rightarrow M$ be the assignment that assigns each job $j$ to the machine $i$ with the smallest $p_{i,j}$. Notice that $Q := \sum_{j \in J} w_{j} p_{\sigma(j),j}$ is a lower bound for the weighted completion time of any schedule.

Using Smith’s rule, it is well known that the weighted completion time of $\sigma$ is

$$
\sum_{i,j,j'} \min \left\{ w_{j} p_{i,j'}, w_{j'} p_{i,j} \right\} \leq \sum_{(j,j')} \min \left\{ w_{j} p_{\sigma(j'),j}, w_{j'} p_{\sigma(j),j} \right\} 
$$

$$
\leq \frac{1}{2} \sum_{(j,j')} \left( w_{j} p_{\sigma(j),j} + w_{j'} p_{\sigma(j'),j'} \right) = \frac{n+1}{2} \sum_{j \in J} w_{j} p_{\sigma(j),j} = \frac{n+1}{2} Q.
$$

Above, $(j,j')$ is over all subsets of $J$ of size 1 (in case $j = j'$) or 2. The summations are well-defined since all the terms inside are symmetric w.r.t $j$ and $j'$. The second inequality holds as $w_{j} p_{\sigma(j),j} + w_{j'} p_{\sigma(j'),j'} \geq 2 \sqrt{w_{j} p_{\sigma(j),j} w_{j'} p_{\sigma(j'),j'}} \geq 2 \min \left\{ w_{j} p_{\sigma(j'),j'}, w_{j'} p_{\sigma(j),j} \right\}$. 

Then, we can assume we are given an upper bound $\Phi$ on the optimum weighted completion time and our goal is to find a schedule with weighted completion time $(1.5 + O(\epsilon))\Phi$. If some $(j, i) \in E$ has $w_{j} p_{i,j} > \Phi$, then we can remove $(j, i)$ from $E$ since it can not be used. For any job $j \in J$ for which there exists a machine $i \in M$ such that $w_{j} p_{i,j} > \frac{\epsilon}{n} \Phi$, we can then remove $j$ from $J$ (but keeping $n$ unchanged), and in the end we
insert $j$ to this machine $i$ using the Smith’s rule. Let $J'$ be the set of remaining jobs and $J''$ be the set of jobs removed and inserted back in the end. Assume we have a schedule for $J'$ with total weighted completion time at most $(1.5 + O(\epsilon)) \cdot \Phi = O(1) \cdot \Phi$, and it obeys the Smith’s rule. We prove

**Claim E.3.** Inserting $J''$ to the schedule for $J'$ increases the weighted completion time by at most $O(\epsilon) \cdot \Phi$.

*Proof.* For two jobs $j, j' \in J$, we use $j \sim j'$ to denote that $j$ and $j'$ are assigned to the same machine. Let $p_{j}^{i'}$ be the processing time of $j$ on its assigned machine in the final schedule. The completion time of $j$ is in $\sqrt{\frac{\sum_{j \in J'} \sum_{j' \in J} p_{j}^{i'} p_{j'}^{i'}}{\sum_{j \in J} \sum_{j' \in J} p_{j}^{i'} p_{j'}^{i'}}} \leq \sqrt{n} \cdot \sqrt{\frac{\sum_{j \in J'} \sum_{j' \in J} w_{j} p_{j}^{i'} p_{j'}^{i'}}{\sum_{j \in J} \sum_{j' \in J} w_{j} p_{j}^{i'} p_{j'}^{i'}}} \leq \sqrt{n} \cdot \frac{\sqrt{\Phi} \cdot c \Phi}{\sqrt{n}} \leq O(\epsilon) \cdot \Phi.

Both inequalities in the second line used that every $j' \in J''$ has $w_{j} p_{j}^{i'} \leq c \Phi \cdot \frac{\Phi}{\sqrt{n}}$. The second inequality also used that $\sqrt{\sum_{i} \frac{a_{i}^{2}}{|\sum_{i} a_{i}|}} \leq \sqrt{\frac{\sum_{i} a_{i}^{2}}{|\sum_{i} a_{i}|}}$ for any $a_{1}, a_{2}, \ldots, a_{n} \geq 0$. The first inequality in the third line used that the schedule for $J'$ has weighted completion time at most $O(1) \cdot \Phi$. $\blacksquare$

Therefore, after removing jobs $J''$ from $J$, we can assume for every machine $(j, i) \in E$, we have $w_{j} p_{i,j} \in (\frac{\Phi}{\sqrt{n}}, \Phi)$ for $B = \frac{n^{3}}{\sqrt{\Phi}} = \text{poly}(n, \frac{1}{\epsilon})$.

**Lemma E.4.** In any schedule that respects the Smith’s rule, if $j$ is scheduled on $i$, then its completion time is in $[p_{i,j}, n \sqrt{B} p_{i,j}]$.

*Proof.* The completion time of $j$ is at least $p_{i,j}$. On the other hand, if $j'$ is scheduled before $j$ on the same machine $i$, then we have $\frac{p_{i,j}^{2}}{w_{j}} \leq \frac{p_{i,j}^{2}}{w_{j}}$. This implies that $p_{i,j} = \frac{p_{i,j}^{2}}{w_{j}^{2}} \leq B p_{i,j} w_{j} \cdot \frac{p_{i,j}^{2}}{w_{j}} = B p_{i,j}^{2}$. So, $p_{i,j} \leq \sqrt{B} p_{i,j}$. This implies that the completion time of $j$ is at most $n \sqrt{B} p_{i,j}$. $\blacksquare$

**Modifications to LP (23-26)** With the lemma, we can then show how to modify LP (23-26) to make our running time nearly-linear.

1. **Restricting the set of variables.** With Lemma E.4, we include $(j, i, d) \in \mathbb{D}$ only if $[p_{i,j}, n \sqrt{B} p_{i,j}] \cap (\tau_{d-1}, \tau_{d}) \neq \emptyset$. There are at most $O\left(\frac{\log(n \sqrt{B})}{\epsilon}\right) = \tilde{O}(1)$ different elements $(j, i, d) \in \mathbb{D}$ for a fixed $(j, i) \in E$. Thus, $|\mathbb{D}| \leq O_{\epsilon}(|E|)$.

2. **Using $(1 - \epsilon)p_{i,j}$ (instead of $p_{i,j}$) in (24) and the rounding algorithm.** We then scale $p_{i,j}$ values by $1 - \epsilon$ in (24) and in the rounding algorithm: We replace $\rho(p_{i,j}, \tau_{d}, \tau_{r})$ by $\rho((1 - \epsilon)p_{i,j}, \tau_{d}, \tau_{r})$ on the left-side of (24), and we choose $\theta_{j}$ randomly from $(\tau_{d_{j}} - (1 - \epsilon)p_{i,j}, \tau_{d_{j}})$ in the rounding algorithm. Notice that we still require $\tau_{d} \geq p_{i,j}$ for $(j, i, d) \in \mathbb{D}$; so we are not just handling an isomorphic instance.

$$
\sum_{j,d,(j,i,d) \in \mathbb{D}} \rho((1 - \epsilon)p_{i,j}, \tau_{d}, \tau_{r}) x_{j,i,d} \leq \tau_{r}, \quad \forall i \in M, r \in [D] \tag{24'}
$$

(24') remains valid. One can easily show that the $(1 - \epsilon)$ factor only leads to a $1 + O(\epsilon)$ factor loss in the end. So, the expected weighted completion time of the constructed schedule is still bounded by $(1.5 + O(\epsilon)) \Phi$. 

37
3. Restricting $(j, i, d)$ tuples on the left-side of (24'). Finally, we change the left side of (24'): We include a term $(j, i, d) \in D$ on the left side for in the constraint (24') for $(i, r)$ only if $\tau_d \geq \frac{x}{n} \tau_r$. The final version of constraint (24) becomes the following:

$$\sum_{j,d:(j,i,d)\in\mathbb{D},\tau_d \geq \frac{x}{n} \tau_r} \rho((1-\epsilon)p_{i,j}, \tau_d, \tau_r)x_{j,i,d} \leq \tau_r, \quad \forall i \in M, r \in [D] \quad (24')$$

Fix a $(i, r)$ pair considered in the (24'). On one hand, the inequality remains valid. On the other hand, if (24') holds, then (24) holds with the right side replaced by $(1 + \epsilon)\tau_r$, which can be ignored. This is true since if $x$ satisfies (25) and (26), then

$$\sum_{j,d:(j,i,d)\in\mathbb{D},\tau_d < \frac{x}{n} \tau_r} \rho((1-\epsilon)p_{i,j}, \tau_d, \tau_r)x_{j,i,d} \leq \sum_{as \text{ before } \tau_d} x_{j,i,d} \leq \sum_{as \text{ before } \frac{x}{n} \tau_r} x_{j,i,d} \leq \epsilon \tau_r.$$ 

Claim E.5. The number of non-zero coefficients on the left-side of (24') is $\tilde{O}(x(|E|)).$

Proof. We bound the number of $(j, i, d, r)$ tuples for which $i \in M, r \in D, (j, i, d) \in \mathbb{D}, \tau_d \geq \frac{x}{n} \tau_r$ and $\rho((1-\epsilon)p_{i,j}, \tau_d, \tau_r) > 0$. Fix $(j, i, d) \in \mathbb{D}$. For $\rho((1-\epsilon)p_{i,j}, \tau_d, \tau_r) > 0$ to hold, we must have $\tau_r \geq \tau_d - (1-\epsilon)p_{i,j} \geq \tau_d - (1-\epsilon)\tau_d = \epsilon \tau_d$. Also, we need $\tau_r \leq \frac{\epsilon}{\tau} \tau_d$ for the tuple to be considered. Therefore, there are $O\left(\log_2 \frac{D}{\tilde{x}}\right) = \tilde{O}(1)$ different values of $r$ satisfying the condition for a fixed $(j, i, d) \in \mathbb{D}$. As $|D| = \tilde{O}(x(|E|))$, the number of non-zero coefficients on the left-side of (24') is at most $\tilde{O}(x(|E|))$. \hfill \square

Therefore, we can solve the modified LP, which contains (23), (24'), (25) and (26), in $\tilde{O}(x(|E|))$ time.

E.3 Making $p_{\text{max}}$ polynomially bounded for $R||L_q(\text{loads})$

We can assume $P$ is bounded by a polynomial by losing an approximation factor of $1 + \epsilon$: let $P' = \max_{j \in J} \min_{i \in \delta(j)} p_{i,j}$. Then clearly the optimum schedule to the input instance has $L_q$ norm of loads between $P'$ and $nP'$. So, we can remove $(j, i) \in E$ from $E$ if $p_{i,j} > nP'$ and $\sum_{d=1}^{D} \tau_d (x_{j,i,d} - x_{j,d-1}) = \sum_{d=1}^{D} \tau_d (x_j - x_{j,d-1}) = \int_{\theta=0}^{1} D_j^\theta \theta$. Our algorithm for the problem chooses $\theta$ uniformly at random from $(0, 1)$, and then call list-scheduling($(D_j^\theta)_{j\in J}$) and output the returned schedule. Let $\hat{C}_j$ be the completion time of the job $j$ in the constructed schedule. Focus on a job $j^*$ in $J$ from now on and we bound $E_{\theta}\left[\frac{\hat{C}_j}{C_j}\right]$.

We shall use $g(\theta) = D_j^\theta$, for every $\theta \in (0, 1)$ and so $C_j^* = \int_{\theta=0}^{1} g(\theta) \theta d\theta$. Let $g(0) = \lim_{\theta \rightarrow 0^+} g(\theta)$; that is, $g(0)$ is $\tau_d$ for the smallest $d$ such that $x_{j^*,d} > 0$. By (6), we have $g(0) \geq q_{j^*}$. For every $j \in J$ and $\theta \in [0, 1]$, define $h_j(\theta) := x_{j,d}$ for the $d$ satisfying $\tau_d = g(\theta)$. This is the fraction of job $j$ that is completed when $\theta$ fraction of job $j^*$ is completed. Thus, we have

$$\sum_{j^* \in J} h_j(\theta) \leq (1 + O(\epsilon)) mg(\theta)$$

for every $\theta \in [0, 1]$. Notice that $D_j^\theta \leq D_j^\theta$, if and only if $h_j(\theta) \geq \theta$. So, by Claim 3.5, Lemma 3.6 and that $q_{j^*} \leq g(0)$, we have $\hat{C}_j \leq g(0) + \frac{1}{m} \sum_{j \in J} h_j(\theta) \geq \theta$. Thus, we can bound $\frac{\hat{C}_j}{C_j}$ by the supreme of

$$\int_{\theta=0}^{1} g(\theta) \theta d\theta$$

by

$$\frac{g(0) + \frac{1}{m} \sum_{j \in J} \int_{\theta=0}^{1} 1_{h_j(\theta) \geq \theta} g(\theta) d\theta}{\int_{\theta=0}^{1} g(\theta) \theta d\theta} \quad (41)$$

38
Lemma F.1 ([31]). The supreme of (41) satisfying the three properties is at most \((1 + \sqrt{2})(1 + O(\epsilon))\).

Indeed, [31] proved that the supreme is exactly \(1 + \sqrt{2}\) if the \((1 + O(\epsilon))\)-term in (41.3) is 1. So, we can define \(g' = (1 + O(\epsilon))g\) and \(g'\) satisfies the property holds with \((1 + O(\epsilon))\) replaced to 1; moreover (41.1) and (41.2) remain satisfied for \(g'\). Then the supreme of

\[
g'(0) + \frac{1}{2} \sum_{j \in J} \int_{\theta=0}^{\theta} \mathbf{1}_{b_j(\theta) > 0} d\theta \]

is \(1 + \sqrt{2}\). So, the supreme of (41) is at most \((1 + \sqrt{2})(1 + O(\epsilon)) = 1 + \sqrt{2} + O(\epsilon)\).

F.2 Implementation of list-scheduling for Identical Machine Precedence Constrained Scheduling in \(O((n + \kappa) \log n)\) Time

We define two data structures. The first data structure, which we call Idle-Intervals, maintains the set of idle unit-time slots. (Recall that a slot \((t - 1, t]\) is idle if the number of jobs processed during the slot is at most \(m - 1\), and busy otherwise.) Initially, all unit-time slots \((t - 1, t]\), \(t \geq 1\) are idle. The data structure supports the following two operations:

- **find** \((t, p)\): given two integer \(t \geq 0\) and \(p > 0\), return the smallest \(t' \geq t\) such that \((t', t' + p]\) is idle, i.e., all unit-time slots in \((t', t' + p]\) are idle.
- **remove** \((t, t')\): given an idle interval \((t, t']\), mark all unit-time slots in \((t, t']\) as busy.

To implement the data structure, we maintain the set \(I\) of inclusion-wise maximal idle intervals. We store \(I\) in a self-balancing binary-search tree (BST) with the left-to-right order. For each node in the BST, we maintain the maximum length of intervals in \(I\) stored in the sub-tree rooted at the node.

With this data structure, both operations can be done in \(O(\log s)\) time, where \(s\) is the maximum possible size of \(I\). For **find** \((t, p)\), we first try to find the interval \(I \in I\) containing \(t\). If it exists and containing \((t, t + p]\), we return \(t\). If the algorithm does not return, we find the left-most interval \(I \in I\) to the right of \(t\) with length at least \(D\). Both steps can be implemented in \(O(\log s)\)-time. For **remove** \((t, t')\), we need to find the interval in \(I = (t, t'] \in I\) containing \((t, t']\), which is guaranteed to exist, remove \(I\) from \(I\), adding \((t, t]\) and/or \((t', t']\) to \(I\), if they are not empty. So **remove** \((t, t')\) can be done in \(O(\log s)\) time.

The second data structure, which we call Critical-Counters, maintains a set \(\mathcal{T}\) of critical time points, and a counter \(m_t\) for every \(t \in \mathcal{T}\), which is an integer in \([0, m]\). In the list-scheduling algorithm, a time point \(t\) is critical if \(t = 0\) or some job completes at \(t\). Notice that the starting time \(\bar{S}_j\) of a job \(j\) is either 0 or the completion time \(\bar{C}_j\) of some other job \(j'\), \(m_t\) for a \(t \in \mathcal{T}\) is supposed to be the number of jobs \(j\) such that \((t, t + 1] \subseteq (\bar{S}_j, \bar{C}_j]\). The data structure supports the following operations:

- **insert** \((t)\): if \(t \notin \mathcal{T}\), then we add \(t\) to \(\mathcal{T}\), with \(m_t = m_{t'}\) where \(t'\) is the last critical time point before \(t\).
- **increase** \((\tau, \tau')\): increase the \(m_t\) values of all time points \(t \in \mathcal{T} \cap [\tau, \tau']\) by 1, and return the list of those points \(t\) whose new \(m_t\) values become \(m\), as well as their respective next critical time point. It is guaranteed that before the updates, every \(t \in \mathcal{T} \cap [\tau, \tau']\) has \(m_t < m\).

We again use a self-balancing BST to store \(\mathcal{T}\) and their \(m_t\) values, with the natural integer order for the time points. We setup some notations first before describing the values maintained at the nodes of the tree. For each node \(v\) in the BST, let \(A_v\) be the set of ancestor nodes of \(v\) in the BST, including \(v\) itself, let \(\Lambda_v\) be the descendant nodes of \(v\), including \(v\). For every \(u \in \Lambda_v\), let \(P(v, u)\) be the set of nodes in the path from \(v\) to \(u\) in the BST. For every node \(v\) in the BST, let \(t_v\) be the time point stored at \(v\), and for every \(t \in \mathcal{T}\), let \(v_t\) be the node in the BST storing \(t\). So \(v_t = v\) if and only if \(t_v = t\). We maintain three values for each node \(v\): \(m_{\text{inc}}(v), m_{\text{self}}(v)\) and \(m_{\text{max}}(v)\). We guarantee the following two properties:
For every $t \in \mathcal{T}$, we have $m_t = \sum_{v \in A_v} m_{inc}(v) + m_{self}(v_t)$.

For every $v$ in the BST, $m_{max}(v) = \max_{u \in \Lambda_v} \left( \sum_{u' \in P(v, u)} m_{inc}(u') + m_{self}(u) \right)$, which is equal to $\max_{u \in \Lambda_v} m_u - \sum_{v' \in A_v \setminus \{v\}} m_{inc}(v')$.

Given $m_{inc}(v)$’s and $m_{self}(v)$’s for all $v$, the $m_{max}(v)$’s can be defined as follows: for every node $v$ in the BST with left child $v'$ and right child $v''$, let $m_{max}(v) = \max(m_{max}(v'), m_{max}(v''), m_{self}(v)) + m_{inc}(v)$, where we assume if $v'$ or $v''$ does not exist, then its $m_{max}$ value is $-\infty$. This guarantees that if we rotate the tree, $m_{max}(v)$’s can be updated efficiently.

Again let $s$ be the maximum possible size of $\mathcal{T}$. Then, insert($t$) takes $O(\log s)$ time. It takes $O(\log s)$-time for increase($\tau$, $\tau'$) to update the data structure: Let $\mathcal{T}_t = \{ t_u : u \in \Lambda_u \}$. We need to update the information for a node $v$ only if $\mathcal{T}_u \cap [\tau, \tau') \neq \emptyset$ and $\mathcal{T}_u \setminus [\tau, \tau') \neq \emptyset$, or $v$ is the topmost node with $\mathcal{T}_u \subseteq [\tau, \tau')$. In the former case, we may need to update $m_{self}(v)$ and $m_{max}(v)$. In the latter case, we shall increase $m_{inc}(v)$ and $m_{max}(v)$ by 1. There are only $O(\log s)$ nodes $v$ whose information will be updated. It takes $O((r + 1) \log s)$-time for increase($\tau$, $\tau'$) to return the list of critical time points $t \in [\tau, \tau')$ with $m_t = m$, where $r$ is the number of such points.

Now we show how to implement the list-scheduling algorithm using the two data structures. Consider the iteration for scheduling $j$. To find the minimum $t' \geq t$ such that $(t', t' + p_j)$ is idle, we call $t' \leftarrow \text{Idle-Intervals}.\text{find}(t, p_j)$, and $\text{Critical-Counters}.\text{insert}(t' + p_j)$. To schedule $j$ in $(t', t' + p_j)$, we call $\text{Critical-Counters}.\text{increase}(t', t', p_j)$. For every returned time point $\tau$ and its next critical point $\tau'$, we call $\text{Idle-Intervals}.\text{remove}((\tau, \tau'))$. $\text{Idle-Intervals}.\text{find}$ and $\text{Critical-Counters}.\text{insert}$ are called once for every $j$, and each of the two operations has running time $O(\log n)$, as there are at most $n + 1$ critical time points. Every critical time point is returned at most once by $\text{Critical-Counters}.\text{increase}$, since once a $t \in \mathcal{T}$ has $m_t = m$, $m_t$ will never be increased in the future. This also implies that $\text{Idle-Intervals}.\text{remove}$ will only be called $O(n)$ times. Therefore, the running time of the list-scheduling algorithm is $O(\kappa + n \log n)$ using the two data structures.

### F.3 Reducing $R\| C_{\text{max}}$ to Promise Version

We first show how to reduce the general problem to the promise version. Let $\mathcal{A}$ be the algorithm for the promise version of the problem. If we are given a $P \geq \text{opt}$ to $\mathcal{A}$, the it will successfully output a schedule of makespan at most $(2 + O(\epsilon))P$. However, when $P < \text{opt}$, the algorithm may or may not succeed.

Assigning each job $j$ to the machine $i$ with the smallest $p_{j,i}$ value gives us an $m$-approximation. Then we can create a geometric sequence of $\lceil \log_{1 + \epsilon} m \rceil + 1$ numbers $P$, such that one of them has $\text{opt} \leq P < (1 + \epsilon)\text{opt}$. Via binary search among these numbers\footnote{Simply enumerating all values of $P$ is sufficient, but binary search gives a better dependence.}, we can run $\mathcal{A}$ for $O(\log(\lceil \log_{1 + \epsilon} n \rceil + 1)) = O(\log \log n)$ times, to find a $P < (1 + \epsilon)\text{opt}$ for which $\mathcal{A}$ succeeds. So, the schedule for this $P$ has makespan at most $(2 + O(\epsilon))P \leq (2 + O(\epsilon))\text{opt}$.

### F.4 Derandomization of Rounding Algorithm for $R\| \sum_j w_j C_j$

We briefly talk about how to derandomize the rounding algorithm for $R\| \sum_j w_j C_j$, while keeping its running time $O(\sqrt{|E|})$. Now it is more convenient for us to use the Smith’s rule to process the jobs. For every $(j, i) \in E$, we define $y_{j,i} := \sum_{d} x_{j,i,d}$ to be the fraction of job $j$ assigned to $i$. Thus the following randomized algorithm leads to a $(1.5 + O(\epsilon))$-approximation in expectation, as it is as good as the algorithm we described: For every job $j \in J$, we assign it to a machine randomly and independently, such that the probability $j$ is assigned to $i$ is $y_{j,i}$. Then for each machine, we schedule the jobs assigned to it using Smith’s rule. The expected weighted completion time given by the algorithm is $\text{cost} := \sum_{j \in J, i \in I, j \prec_i j'} y_{j,i} y_{j',i} p_{j,i} w_{j'} + \sum_{(j,i) \in E} y_{j,i} p_{j,i} w_{j,i}$, where $j \prec_i j'$ denotes that job $j$ will be scheduled before $j'$ on machine $i$ according to Smith’s rule, if they are both assigned to $i$. 

40
We derandomize the above rounding algorithm using conditional expectation. We start with the initial vector \( y \in [0, 1]^E \), and for every job \( j \) (in arbitrary order), we make the assignment of \( j \) integral. To do so, we need to choose some \( i \in N(j) \), change \( y_{j,i} \) to 1, and change \( y_{j,i'} \) to 0 any other machine \( i' \neq i \). We choose the \( i \) to minimize cost. To obtain a \( \tilde{O}_\epsilon(|E|) \)-time algorithm, we need to perform the operation in \( \tilde{O}_\epsilon(|N(j)|) \) time. To achieve the goal, we maintain two range query trees of size \( O(|N(i)|) \) for each \( i \in M \). With the range trees, the following operations can be done in \( \tilde{O}_\epsilon(1) \) time:

- Given an edge \((j, i) \in E\), return \( \sum_{j', j' \prec i, j} y_{j', i} p_{i,j'} \), or \( \sum_{j', j \prec i, j} y_{j', i} w_{j'} \).
- Given an edge \((j, i) \in E\), update \( y_{j,i} \) to 0 or 1.

With the range trees, the running time for each \( j \) is \( \tilde{O}_\epsilon(|N(j)|) \), and the total time to initialize the range trees is \( \sum_{i \in M} \tilde{O}_\epsilon(|N(i)|) = \tilde{O}_\epsilon(|E|) \). So the derandomized algorithm has running time \( \tilde{O}_\epsilon(|E|) \).

### F.5 Derandomization of Rounding Algorithm for \( R||L_2(\text{loads}) \)

The rounding algorithm can be easily derandomized in \( O(|E|) \)-time, via conditional expectation. We know that the expectation of the \( L_2^2 \) of loads of \( \sigma \) is \( Q := \sum_{i,j \neq j'} y_{j,i} y_{j',i} p_{i,j} p_{i,j'} + \sum_{i,j} y_{j,i} p_{i,j}^2 \). For every \( j \in J \) in arbitrary order, we make \( (y_{j,i})_{i \in M} \) integral, so as to minimize the quantity \( Q \). To do so, we maintain the fractional load of every machine \( i \), which is \( \sum_{i' \in N(i)} y_{j,i'} p_{i,j'} \). Then, for each \( j \), finding the best way to assign \( j \), and updating the fractional loads take \( O(|N(j)|) \) time. So, the whole rounding algorithm takes \( O(|E|) \)-time.