Scaling Laws of Cognitive Networks

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Abstract

We consider a cognitive network consisting of $n$ random pairs of cognitive transmitters and receivers communicating simultaneously in the presence of multiple primary users. Of interest is how the maximum throughput achieved by the cognitive users scales with $n$. Furthermore, how far these users must be from a primary user to guarantee a given primary outage. Two scenarios are considered for the network scaling law: (i) when each cognitive transmitter uses constant power to communicate with a cognitive receiver at a bounded distance away, and (ii) when each cognitive transmitter scales its power according to the distance to a considered primary user, allowing the cognitive transmitter-receiver distances to grow. Using single-hop transmission, suitable for cognitive devices of opportunistic nature, we show that, in both scenarios, with path loss larger than 2, the cognitive network throughput scales linearly with the number of cognitive users. We then explore the radius of a primary exclusive region void of cognitive transmitters. We obtain bounds on this radius for a given primary outage constraint. These bounds can help in the design of a primary network with exclusive regions, outside of which cognitive users may transmit freely. Our results show that opportunistic secondary spectrum access using single-hop transmission is promising.

I. INTRODUCTION

The scaling laws of the capacity of ad-hoc wireless networks has been an active area of research. Initiated by the work of Gupta and Kumar [1], this area of research has been pursued under a variety of wireless channel models and communication protocol assumptions [2]–[13]. These papers usually assume $n$ pairs of homogeneous devices, thrown at random in a plane, wishing to communicate. Each transmitter has a single receiver, which may be located anywhere in the network. The underlying question is how the total network throughput (also called the sum rate), or equivalently the per-user throughput, scales as the number of communication pairs
$n \to \infty$. This is accomplished by either letting the density of nodes stay fixed and the area increase with $n$ (extended network), or by fixing the network area and letting the density increase with $n$ (dense network). As transmitter-receiver pairs are selected at random in the network, a packet may require transmission over multiple hops to reach its destination.

The throughput scaling in ad hoc networks depends greatly on the node distribution and the physical-layer processing capability, more specifically the ability to cooperate among nodes. In the interference-limited regime, in which no cooperation is allowed (except simple decode-and-forward), and all nodes treat other signals as interference, the per-node throughput scales at most as $1/\sqrt{n}$ [1]. If the nodes are uniformly distributed, a simple nearest-neighbor forwarding scheme achieves a $1/(n \log(n))$ per-node throughput [1]. When the nodes are distributed according to a Poisson point process, a backbone-based routing scheme achieves the per-node scaling of $1/\sqrt{n}$ [11], meeting the upper bound. On the other hand, when nodes are able to cooperate, a much different scaling law emerges. Specifically, a hierarchical scheme can achieve a linear growth in the sum rate, corresponding to a constant per-node throughput [13]. A key step in this scheme is MIMO cooperation among nodes, which requires joint encoding and decoding. The development of these scaling laws show that the assumptions about the network and the nodes’ signal processing capability are crucial to the scaling law.

In this paper, instead of considering a homogeneous ad hoc wireless network, we study a cognitive network consisting of two types of users: primary and cognitive. Recent introduction of secondary spectrum licensing necessitates the study of such cognitive networks. The cognitive users opportunistically access the now-exclusive but under-utilized spectrum of the primary users, while ensuring that any performance degradation to the primary users is within an acceptable level. Other scenarios in which two networks operate concurrently are also applicable.

Consider for example a TV station broadcasting in a now-exclusive, licensed band. This band is wasted in geographic locations barely covered by the TV signal. This prompts questions such as: can we allow other devices (cognitive users) to transmit in the same band as the TV (primary users), provided their interference to any TV receiver is at “an acceptable level”? If so, what is the minimum distance from the TV station at which these devices can start transmitting? What are the maximum rates that these devices can achieve by transmitting in the TV band?

We formulate this problem from an information theoretic viewpoint for a network with multiple primary and multiple cognitive users. We define the “acceptable interference level” to be a
threshold on the probability that the received signal (or rate) of a primary user is below a certain level, provided that the primary receiver (Rx) is within a radius of interest from its transmitter (Tx). This is analogous to the concept of outage capacity. The radius of interest specifies the primary exclusive region (PER), which are non-overlapping for different primary transmitters. These PERs are void of cognitive transmitters, but they can contain cognitive receivers. We consider an extended network in which the cognitive users are uniformly distributed such that their density is a constant.

Because of the opportunistic nature of the cognitive users, we consider a network and communication model different from the previously mentioned ad hoc networks. We assume that each cognitive transmitter communicates with a receiver within a bounded distance $D_{\text{max}}$, using single-hop transmission. Being different from multi-hop communication in ad hoc networks, single hop communication appears suitable for cognitive devices which are mostly short-range. Our results, however, are not limited to short-range communication. There can be other cognitive devices (transmitters and receivers) in between a Tx-Rx pair. (This is different from the local scenarios of ad hoc networks, in which every node is talking to its neighbor.) If the transmit power model of the cognitive users is constant, then the maximum Tx-Rx distance $D_{\text{max}}$ stays constant. (In practice, we may preset a $D_{\text{max}}$ based on a large network and use the same value for all networks of smaller sizes.) If we allow the cognitive devices to scale its power according to the distance to the primary user, then $D_{\text{max}}$ may scale with the network size by a feasible exponent. Other assumptions include a protected band around each receiver (primary or cognitive) to ensure that any interfering transmitter is not at the same point as the interfered receiver. Assuming no cooperation, the cognitive receivers simply treats other users’ signals as interference.

Within such a network, for both cases of constant and varying cognitive transmit power, we find that the cognitive users’ throughput scales linearly in the number of users $n$. Equivalently, as $n \to \infty$, the per user capacity remains constant. Our results thus indicate that an initial approach to building a scalable cognitive network should involve limiting cognitive transmissions to a single hop. This scheme appears reasonable for secondary spectrum usage, which is opportunistic in nature.

The impact of cognitive users on the primary user is captured in the expected amount of interference from the cognitive users. We derive upper and lower bounds on this interference and show that the average interference remains bounded irrespective of the number of cognitive
users. Based on this interference bounds, we provide an upper bound on the radius of the PER that satisfies the outage constraint on the primary user’s rate. The bound also allows us to study the interdependence and trade-offs between the PER radius, the protected band around each primary receiver and the primary transmit power.

The paper structure is as follows. In Section II we introduce our network model and formulate the problem. In Section III we study the throughput scaling of the cognitive users with constant transmit power in the presence of multiple primary users. In Section IV we examine the outage constraint on a single primary user and derive an upper bound on the radius of the primary exclusive region. In Section V we investigate the option of allowing the cognitive users to scale the transmit power according to the distance to the primary user. In Section VI we make our conclusions.

II. Problem Formulation

Consider a cognitive network with two types of users: primary and cognitive users. We address two main questions. First, how the total throughput of the cognitive users scales with network size, given the presence of the primary users. Second, at what distance from the primary users can these cognitive users operate to ensure a maximum outage probability for the primary user. The users are not allowed to cooperate, hence the network is interference-limited. We will first discuss the network model and the channel and signal models. We then formulate specifically each of the two criteria: cognitive users throughput, and the allowable distance from the primary user.

A. Network model

We consider an extended network with all transmitters and receivers located on a plane. With fixed nodes densities, the network size grows with the number of nodes. As a specific instance, we consider a circular network with radius $R$. To scale the number of cognitive and primary users, we let $R$ increase. Other shapes also produce a similar scaling law.

We introduce our network model in Figure 1. Within the network, there are $m$ primary users and $n$ cognitive users. Let $Tx_p^i$ and $Rx_p^i$ denote a primary transmitter and its intended receiver ($i = 1, 2, \cdots, m$), and $Tx_c^j$ and $Rx_c^j$ for the cognitive transmitter and receiver ($j = 1, 2, \cdots, n$). Each primary transmitter is located at the center of a primary exclusive region (PER) with radius
Fig. 1. A cognitive network consists of multiple primary users and multiple cognitive users. Each primary transmitter $Tx^i_p$ is at the center of a primary exclusive region (PER) with radius $R_0$, which contains its intended receiver. These PERs are non-overlapping. Surrounding each PER is a protected band of width $\epsilon_p > 0$. Outside the PERs and the protected bands, $n$ cognitive transmitters are distributed randomly and uniformly with density $\lambda$. The placement model for the cognitive users is illustrated in Figure 2.

$R_0$; the corresponding primary receiver can be anywhere within this region. This model is based on the premises that the primary receiver location may not be known to the cognitive users. Such a setup is typical in broadcast scenarios, such as often found in cellular or TV networks. Hence we choose to center the PER circle on the primary transmitter $Tx^i_p$ (for example, a base station) rather than the receiver $Rx^i_p$. These PERs of radius $R_0$ are non-overlapping. WOLG, we also assume that there is a PER at the center of the network, with the transmitter $Tx^1_p$ and receiver $Rx^1_p$. Other than that, we make no specific assumptions about the placement of the PERs, or the primary transmitters. This means their locations are arbitrary.

Around each primary receiver we assume there is a circle of radius $\epsilon_p > 0$ in which no interfering cognitive transmitter may lie. (The cognitive receivers, however, can lie within this $\epsilon_p$ circle.) Because the location of the primary receiver is unknown to the cognitive transmitters, this assumption results in a protected band of width $\epsilon_p$ around each PER, inside which no cognitive
Fig. 2. Cognitive user model: Each cognitive transmitter Tx$^c_i$ wishes to transmit to a single cognitive receiver Rx$^c_i$, which lies within a distance $\leq D_{\text{max}}$ away. Each cognitive receiver has a protected circle of radius $\epsilon_c > 0$, in which no interfering transmitter may operate. We will later design this radius $\epsilon_p$ in conjunction with $R_0$ and other system parameters to meet certain primary outage constraints.

Similarly, we assume that all cognitive receivers have a protected circle of radius $\epsilon_c > 0$ around them, in which no interfering, either primary or cognitive, transmitters may lie ($\epsilon_c$ may be different from $\epsilon_p$). These practical constraints simply ensure that the interfering transmitters and receivers are not located at exactly the same point.

All cognitive transmitters are distributed outside the PERs encircled by an $\epsilon_p$-band. We assume that the cognitive transmitters are randomly and uniformly distributed with constant density $\lambda$. The cognitive receivers, however, can be anywhere in the network (subject to the $\epsilon_c$ protected distance), including inside the PERs. We assume that each cognitive receiver is within a $D_{\text{max}}$ distance from its transmitter. Depending on the transmit power of the cognitive users, $D_{\text{max}}$ may scale with the network size (as analyzed later). Figure 2 provides an example of a such cognitive Tx-Rx layout.

Table I summarizes the network notation.

The introduced network is a general model with multiple primary and multiple cognitive users, which we will use to study the scaling law of the cognitive users. We then use a special case of this model – with only a single primary user at the center – to study the radius of the
Primary user $i$th transmitter and receiver $\text{Tx}_p^i$, $\text{Rx}_p^i$
Cognitive user $i$th transmitter and receiver $\text{Tx}_c^i$, $\text{Rx}_c^i$
Outer radius for cognitive transmission $R$
Number of primary users $m$
Number of cognitive users $n$
Primary exclusive region radius $R_0$
Maximum cognitive $\text{Tx}_c^i$-$\text{Rx}_c^i$ distance $D_{\text{max}}$
Minimum $\text{Tx}_c^i$-$\text{Rx}_c^k$ distance $\epsilon_p$
Minimum $\text{Tx}_c^i$-$\text{Rx}_c^k$ distance ($i \neq k$) $\epsilon_c$
Cognitive user density $\lambda$

TABLE I
NETWORK NOTATION.

primary exclusive region (the minimum distance from a primary user at which a cognitive user can operate).

B. Channel and signal models

We consider a path-loss only model for the wireless channel. Given a distance $d$ between the transmitter and the receiver, the channel $h$ is therefore given as

$$h = \frac{A}{d^{\alpha/2}}$$

where $A$ is a frequency-dependent constant and $\alpha$ is the power path loss. In subsequent analysis, we normalize $A$ to be 1 for simplicity. We consider $\alpha > 2$ which is typical in practical scenarios.

We assume that the channels between different transmitters and receivers are independent. Furthermore, they all undergo independent zero-mean additive white Gaussian noise of power $\sigma^2$. We define the notation for selected channels in Table II.

For the signal model, we assume no multiuser detection. Thus each user, either primary or cognitive, has no knowledge of other users’ signals and treats their interference as noise. We assume that each primary user signal is constrained by a constant power $P_0$, and each cognitive user by $P$. (Later on we will consider two cases: $P$ is constant, and $P$ is variable with distance). Furthermore, the signals of different users are statistically independent. With a large number of
users, independent and power-constrained, their interference to a receiver will be (approximately) Gaussian. Thus the optimal transmit signals for both types of users are zero-mean Gaussian.

C. The cognitive network throughput

Consider the transmission rate of a cognitive user in the presence of other cognitive users and multiple primary users. Denote $I_{ci}$ and $I_{pi}$ ($i = 0, \ldots, n$) as the total interference power to cognitive user $i$ from other cognitive transmitters and from the primary transmitters, respectively. Based on Table II these interference powers can be written as

$$I_{ci} = \sum_{j \neq i} P|h_{ji}^c|^2, \quad I_{pi} = \sum_{j=1}^{n} P_0|g_{ji}^c|^2.$$  \hfill (2)

With Gaussian signaling and transmit power $P$, the rate of cognitive user $i$ can then be written as

$$C_i = \log \left( 1 + \frac{P|h_{ii}^c|^2}{I_{ci} + I_{pi} + \sigma^2} \right), \quad i = 1 \ldots n.$$ \hfill (3)

Because of the random placement of cognitive users, $I_{ci}$ is a random variable. The rate $C_i$ therefore is also random.

Define the average total throughput of the cognitive users as

$$S_n = \sum_{i=1}^{n} E[C_i].$$ \hfill (4)

An equivalent measure is the per-user throughput defined as

$$T_n = \frac{1}{n} S_n.$$ \hfill (5)

We are interested in how the average sum rate (4), equivalently the per-user throughput (5), scales as $n \to \infty$. 

| Channel from cognitive $Tx_i^c$ to cognitive $Rx_j^c$ | $h_{ij}$ |
| Channel from primary $Tx_i^p$ to cognitive $Rx_j^c$ | $g_{ij}$ |
| Channel from the center primary $Tx_1^p$ to its primary $Rx_1^p$ | $h_0$ |
| Channel from cognitive $Tx_i^c$ to primary $Rx_1^p$ | $g_i$ |

TABLE II

Channel notation.
D. The primary exclusive region

To study the radius of the primary exclusive region, we consider a special case of the network with only a single PER at the center. In other words, we consider only Tx$_p^1$ at the center of the network and its receiver Rx$_p^1$ within a radius $R_0$ from the primary transmitter. The main reason is that we focus on the impact on a primary user of the addition of cognitive users. Without these cognitive users, the primary network would operate with noise and the usual interference from the other primary users. Hence this special case can also be thought of as approximating the noise power to include the interference from other primary users to the considered user.

The radius $R_0$ of the primary exclusive region is determined by the outage constraint on the primary user given as

$$\Pr[\text{primary user’s rate } \leq C_0] \leq \beta$$

where $C_0$ and $\beta$ are pre-chosen constants. This constraint guarantees the primary user a rate of at least $C_0$ for all but $\beta$ fraction of the time.

Denote $h_0$ as the channel of the considered primary user, and $g_i$ as the channel from cognitive transmitter $i$ to this user’s receiver (as in Table II). The interference power from the cognitive users to the considered primary user is

$$I_0 = \sum_{i=1}^{n} P |g_i|^2$$

Again this interference power is random because of the random placement of the cognitive users. With Gaussian signaling, the rate of this primary user can be written as

$$C_p = \log \left( 1 + \frac{P_0 |h_0|^2}{I_0 + \sigma^2} \right).$$

This rate is random because of random interference $I_0$. The outage constraint can now be written as

$$\Pr \left[ \log \left( 1 + \frac{P_0 |h_0|^2}{I_0 + \sigma^2} \right) \leq C_0 \right] \leq \beta.$$  

Since we consider channels with only path loss, outages occur here are not because of fading as in traditional schemes, but because of the random placement of cognitive users.
E. Single-primary network with cognitive power scaling

When considering a network with a single primary user at the center, a feasible option is to allow the cognitive transmitters to scale their power according to the distance from the primary user. Specifically, the transmit power $P$ of a cognitive user is now a function of the radius $r$, at which this cognitive user is located, as

$$P(r) = P_c r^\gamma$$

for some constant power $P_c$ and a feasible power exponent $\gamma$ (which will be analyzed later). Similar to the case of constant cognitive transmit power, we will also examine the throughput scaling of the cognitive users and the primary exclusive radius in this case (albeit both objectives with a single-primary network model).

III. THE SCALING LAW OF A COGNITIVE NETWORK WITH CONSTANT POWER

In this section, we study the throughput of the cognitive users with constant transmit power, assuming multiple primary users. In particular, we examine the throughput scaling law as the number of cognitive users $n$ increases to infinity. We first establish upper and lower bounds to the per-user throughput and then show that both bounds scale with the same order, which then becomes the scaling order of the throughput itself.

A. Lower bound on the cognitive per-user capacity

To derive a lower bound on the capacity of a cognitive user, we study an upper bound on the interference to a cognitive receiver. This includes the interference from the primary users and from the cognitive users.

1) Interference from the primary users: We assume that the primary users must be spaced such that the primary exclusive regions (with radius $R_0$) are non-overlapping. Two PERs closest to each other may have the boundaries (circles) touching at one point, as an example shown in Figure 1. We shall examine the densest placement of the primary users to upper bound the interference from them to a cognitive receiver.

Consider boundary circles of the PER with radius $R_0$. The tightest circle packing is according to the hexagon lattice [14], as shown in Figure 3, in which the three bold circles represents the PERs with radius $R_0$. Since each cognitive receiver has a protected radius of $\epsilon_c$, the worst
case cognitive receiver will be on a circle of radius $\epsilon_c$ around a primary transmitter. We are interested in the interference from all the primary transmitters, located on the hexagon lattice, to this cognitive receiver.

To calculate the interference, we superimpose x-y axes on the hexagon topology with the origin at a primary transmitter and with normalized length (1 unit length = $R_0$), as shown in Figure 3. Consider the circle of radius $\epsilon_c$ around the origin. (For practical reasons, we consider $\epsilon_c < R_0$.) Let A be a point on this circle at an angle $\theta$ to the x-axis. The interference from all primary transmitters to A is

$$I_A = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{\left[ (2\sqrt{3}k - \epsilon_c \cos \theta)^2 + (2m - \epsilon_c \sin \theta)^2 \right]^{\alpha/2}}$$

$$+ \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{\left[ (\sqrt{3}(2k+1) - \epsilon_c \cos \theta)^2 + (2m+1 - \epsilon_c \sin \theta)^2 \right]^{\alpha/2}}. \quad (9)$$

For any $\theta \in [0, 2\pi]$, each of the above summations is bounded for $\alpha > 2$, as shown in the Appendix. Let $I_P$ be the worst-case interference from the primary users,

$$I_P = \max_{\theta} I_A \quad (10)$$
then $I_p$ is also bounded. Thus the total interference from all primary users to any cognitive receiver is bounded.

2) **Worst-case interference from cognitive users:** An upper bound is obtained by filling all primary exclusive regions with cognitive transmitters. Since the cognitive user density is constant, this filling increases the number of cognitive users at most by a scaling factor (the ratio between the area of the PERs and the area occupied by cognitive users).

Now consider a uniform network of $n$ cognitive users. The worst case interference would then be to a cognitive receiver at the center of the network (without loss of generality assumed to be $Rx^1_c$). From the considered receiver, draw a circle of radius $R$ that covers all other cognitive transmitters. With constant user density of $\lambda$ users per unit area, then $R^2$ grows linearly with $n$ (in other words, $R^2$ is of order $n$).

To see that this case is indeed the worst interference from cognitive users, consider another cognitive receiver ($Rx^2_c$) that is not at the center of the network. Again draw a circle of radius $R$ centered at $Rx^2_c$. Since this receiver is not at the center of the network, the circle will not cover all cognitive transmitters. The interference to $Rx^2_c$ is then increased by moving all the transmitters from outside this new circle (area A in Figure 4) to inside the circle (area B in Figure 4), resulting in the same interference as to $Rx^1_c$.

Consider an interfering cognitive transmitter located randomly within the circle of radius $R$ from the considered receiver. With uniform distribution, the distance $r$ between this interfering
transmitter and this receiver has the density

\[ f_r(r) = \frac{2r}{R^2 - \epsilon_c^2}, \quad \epsilon_c \leq r \leq R. \]  \hspace{1cm} (11)

The average interference from this transmitter to the considered receiver therefore is

\[ I_{\text{avg,1}} = \int_{\epsilon_c}^{R} \frac{2rP}{(R^2 - \epsilon_c^2)r^\alpha} dr = \frac{2P}{(R^2 - \epsilon_c^2)(\alpha - 2)} \left( \frac{1}{\epsilon_c^{\alpha - 2}} - \frac{1}{R^{\alpha - 2}} \right). \]  \hspace{1cm} (12)

The average total interference from all other cognitive transmitters to the considered receiver then becomes

\[ I_{\text{avg,n}} = nI_{\text{avg,1}}. \]

But \( \lambda \pi (R^2 - \epsilon_c^2) = n \), thus

\[ I_{\text{avg,n}} = \frac{2\pi \lambda P}{(\alpha - 2)} \left( \frac{1}{\epsilon_c^{\alpha - 2}} - \frac{1}{R^{\alpha - 2}} \right). \]  \hspace{1cm} (13)

For any cognitive receiver, its average interference is upper-bounded by \( I_{\text{avg,n}} \), that is

\[ E[I_i] \leq I_{\text{avg,n}}. \]  \hspace{1cm} (14)

As \( n \to \infty \), provided that \( \alpha > 2 \), this average interference to the cognitive receiver at the center approaches a constant as

\[ I_{\text{avg,n}} \xrightarrow{n \to \infty} \frac{2\pi \lambda P}{(\alpha - 2)\epsilon_c^{\alpha - 2}} \triangleq I_\infty. \]  \hspace{1cm} (15)

3) Lower bound on the cognitive per-node throughput: Now consider the rate of the \( i \)-th cognitive user given in (3). Since the distance between a cognitive transmitter and its intended receiver is bounded by \( D_{\text{max}} \), we have \(|h_{ii}|^2 \geq 1/D_{\text{max}}^\alpha\). Denote the minimum cognitive received power as \( P_{r,\text{min}} = P/D_{\text{max}}^\alpha \). Given that the interference from the primary users is bounded by \( I_P \) in (10), then

\[ C_i \geq \log \left( 1 + \frac{P_{r,\text{min}}}{\sigma^2 + I_P + I_i} \right). \]  \hspace{1cm} (16)

Noting that \( \log(1 + a/x) \) is convex in \( x \) for \( a > 0 \), by Jensen’s inequality, we have

\[ E \log \left( 1 + \frac{a}{X} \right) \geq \log \left( 1 + \frac{a}{EX} \right). \]
Thus the average rate of each cognitive user satisfies

\[
E[C_i] \geq \log \left( 1 + \frac{P_{r,\min}}{\sigma^2 + IP + E[I_i]} \right) = \log \left( 1 + \frac{P_{r,\min}}{\sigma^2 + IP + I_{\text{avg},n}} \right).
\]

As \( n \to \infty \), the lower bound approaches a constant as

\[
E[C_i] \geq \log \left( 1 + \frac{P_{r,\min}}{\sigma^2 + IP + I_\infty} \right) \triangleq \bar{C}_1
\]

where \( I_\infty \) is defined in (15). Thus the average per-user rate of a cognitive network remains at least a constant as the number of users increases.

### B. Upper bound on the network sum capacity

A trivial upper-bound can be obtained by removing the interference from all other cognitive users. Assuming that the capacity of a single cognitive user under noise alone is bounded by a constant, then the total network capacity grows at most linearly with the number of users.

### C. Linear scaling law of the cognitive network average throughput

From the above lower and upper bounds, we conclude that the average sum throughput of the cognitive network grows linearly in the number of users

\[
E[S_n] = nK\bar{C}_1
\]

for some constant \( K \), where \( \bar{C}_1 \) defined in (18) is the achievable average rate of a single cognitive user under constant noise and interference power. In other words, the average per-user rate stays constant as the number of users increases.

### D. The concentration of the network throughput around its mean

Given that the average cognitive users’ throughput scales linearly with the number of users, the concentration of the throughput around its mean is also of interest. This concentration provides the probability that the throughput of a specific network (with a realization of the cognitive user locations) scales at the same rate as the mean throughput. Suppose this specific throughput can be written as \( S_n = E[S_n] + \Delta = nK\bar{C}_1 + \Delta \) for some real \( \Delta \). Then we need to show that with high probability, \( \frac{|\Delta|}{n} \) approach 0 as \( n \to \infty \).
Specifically, for a $\delta > 0$, we examine
\[
P_\delta \triangleq \Pr \left[ \frac{1}{n} |S_n - E[S_n]| \geq \delta \right],
\]
where $S_n = \sum C_i$ as given in (4). Since we consider only channels with path loss, the rate $C_i$ of each cognitive user in (3) is bounded. Thus $C_i$ is a random variable with finite mean and finite variance. Furthermore, the $C_i$ are i.i.d. Thus by the central limit theorem, as $n \to \infty$, the sum throughput (4) can be approximated as a Gaussian random variable. Then the following inequalities hold:
\[
P_\delta \leq \frac{1}{\sqrt{2\pi}} \int_{n\delta \sqrt{\text{var}(S_n)}}^{\infty} e^{-z^2/2} dz
\]
\[
\leq \frac{\sqrt{\text{var}(S_n)}}{n\delta \sqrt{2\pi}} \int_{n\delta \sqrt{\text{var}(S_n)}}^{\infty} e^{-w^2} dw
\]
\[
= \frac{\sqrt{\text{var}(S_n)}}{n\delta \sqrt{2\pi}} \exp \left( -\frac{n^2 \delta^2}{2\text{var}(S_n)} \right),
\]
where the notation $\leq$ indicates that the inequality holds in the limit as $n \to \infty$. Now suppose $K_2 > 0$ is an upper bound on the variance of $C_i$, that is $\text{var}(C_i) \leq K_2$, then $\text{var}(S_n) = n\text{var}(C_i) \leq nK_2$. Therefore
\[
\frac{\sqrt{\text{var}(S_n)}}{n\delta \sqrt{2\pi}} \exp \left( -\frac{n^2 \delta^2}{2\text{var}(S_n)} \right) \leq \frac{\sqrt{nK_2}}{n\delta \sqrt{2\pi}} \exp \left( -\frac{n^2 \delta^2}{2nK_2} \right) \xrightarrow{n \to \infty} 0
\]
This means that any deviation of the throughput of a specific network from its mean scales sub-linearly. Thus with high probability, the total throughput of the cognitive users in a specific network scales linearly with the number of users.

IV. THE PRIMARY EXCLUSIVE REGION

We next study the relation between the primary exclusive region radius $R_0$ and the primary receiver guard band width $\epsilon_p$. For this part, we focus on a single primary transmitter $\text{Tx}_p^1$ at the center of the network of radius $R$. This primary transmitter is surrounded by a single PER and an $\epsilon_p$-width transmission-free band, as shown in Figure 5. Outside this PER are the cognitive users. The cognitive transmitters must lie outside the circle of radius $R_0 + \epsilon_p$. In other words, these transmitter cannot be placed in the transmission-free $\epsilon_p$-band, which is a valid assumption in all scenarios in which the cognitive transmitter is forbidden to be placed in exactly the same location as the primary receiver. (Ideally, there needs to be only a protected circle of radius
Fig. 5. Worst-case interference to a primary receiver: the receiver is on the boundary of the primary exclusive region of radius $R_0$. We seek to find $R_0$ to satisfy the outage constraint on the primary user.

$\epsilon_p$ around the primary receiver. But since we assume that the cognitive users may not know the location of this primary receiver, we impose a whole transmission-free band.) A cognitive receiver, however, can lie in the $\epsilon_p$ band. To study the primary exclusive region, we consider the worst case scenario in which the primary receiver $\text{Rx}_p^1$ is at the edge of this region, on the circle of radius $R_0$, as shown in Figure 5. The outage constraint must also hold in this (worst) case, and we find a bound on $R_0$ to ensure this.

Consider interference at the primary receiver on the boundary of the PER from a cognitive transmitter at radius $r$ and angle $\theta$. The distance $d(r, \theta)$ (the distance depends on $r$ and $\theta$) between this interfering transmitter and the primary receiver satisfies

$$d(r, \theta)^2 = r^2 + R_0^2 - 2R_0r \cos \theta.$$ 

For uniformly distributed cognitive users, $\theta$ is uniform in $[0, 2\pi]$, and $r$ has the density

$$f_r(r) = \frac{2r}{R^2 - (R_0 + \epsilon_p)^2}.$$

The expected interference power experienced by the primary receiver from all $n = \lambda \pi (R^2 - (R_0 + \epsilon_p)^2)$ cognitive users is then given as

$$E[I_0] = n \int_{R_0 + \epsilon_p}^{R} \int_{0}^{2\pi} \frac{P}{d(r, \theta)\alpha} f_r(r) f_\theta(\theta) \, dr \, d\theta$$

$$= \int_{R_0 + \epsilon_p}^{R} \int_{0}^{2\pi} \frac{\lambda r P \, dr \, d\theta}{(r^2 + R_0^2 - 2R_0r \cos \theta)\alpha/2}. \quad (20)$$
For \( \alpha = 2k \) with integer \( k \), we can calculate \( E[I_0] \) analytically. As an example, for \( \alpha = 4 \), we obtain the values of \( E[I_0] \) as

\[
E[I_0]_{\alpha=4} = \lambda \pi P \left[ \frac{R^2}{(R^2 - R_0^2)^2} + \frac{(R_0 + \epsilon_p)^2}{\epsilon_p^2(2R_0 + \epsilon_p)^2} \right].
\]

(21)

The derivation is in the Appendix. Letting \( R \to \infty \), this average interference becomes

\[
E[I_0]_{\alpha=4} = \lambda \pi P \left[ \frac{(R_0 + \epsilon_p)^2}{\epsilon_p^2(2R_0 + \epsilon_p)^2} \right]
\]

(22)

Next, we derive bounds on this expected interference power \( E[I_0] \) at the primary receiver for a general \( \alpha \). We use these bounds to analyze the interference versus the radius \( R_0 \) and the path loss \( \alpha \). We then relate the outage probability to the average interference through the Markov inequality and establish an explicit dependence of \( R_0 \) on \( \epsilon_p \) and other design parameters.

A. Upper and lower bounds on the average interference

In this subsection we obtain two lower bounds and an upper bound on \( E[I_0] \).

1) A first lower bound on \( E[I_0] \): A first lower bound on \( E[I_0] \) can be established by re-centering the network at the primary receiver \( Rx_p^1 \). We then make a new exclusive region of radius \( 2R_0 \), and a new outer radius of \( R - R_0 \), both centered at \( Rx_p^1 \), as shown in Figure 6. The set of cognitive users included in the new ring will be a subset of the original, making the interference a lower bound as

\[
E[I_0]_{LB1} = \int_{2R_0 + \epsilon_p}^{R - R_0} \frac{2\pi \lambda P r}{r^\alpha} \, dr.
\]

\[
= \frac{2\pi \lambda P}{\alpha - 2} \left( \frac{1}{(2R_0 + \epsilon_p)^{\alpha-2}} - \frac{1}{(R - R_0)^{\alpha-2}} \right).
\]

(23)

As \( R \to \infty \), this bounds approach the limit:

\[
E[I_0]_{LB1}^\infty = \frac{2\pi P \lambda}{\alpha - 2} \frac{1}{(2R_0 + \epsilon_p)^{\alpha-2}}
\]

(24)

2) A second lower bound on \( E[I_0] \): Another lower bound on the interference can be derived by approximating the interference region by two half-planes, similar to [15]. As illustrated in Figure 7, consider only interference from the cognitive users in the two half-planes \( P_A \) and \( P_B \) which touch the circle of radius \( R_0 + \epsilon_p \). Consider a line in \( P_A \) that makes an angle \( \phi \) at \( Rx_p^1 \), the distance \( d \) from any point on this line to \( Rx_p^1 \) satisfies \( \frac{\epsilon_p}{\cos(\phi)} \leq d < \infty \). Since the cognitive users are distributed uniformly, as \( R \to \infty \), the distribution of \( d \) becomes similar to the distribution
Fig. 6. A lower bound on the expected interference at the primary Rx is obtained by forming a cognitive-free circle of radius $2R_0$ around the primary receiver and reducing the network radius, now centered at the primary receiver, to $R - R_0$. All cognitive transmitters now lie within these two new boundaries.

Fig. 7. Another lower bound on the expected interference at the primary Rx is obtained by approximating the interference region by two half-planes $P_A$ and $P_B$. The region between these planes is free from cognitive transmitters.

of $r$ given in (11), and $\phi$ will be uniform in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Similar analyses hold for $P_B$. Hence the average total interference from the cognitive users in $P_A$ and $P_B$ to $\text{Rx}_1$ is

$$E[I_0]_{\text{LB2}} = P\lambda \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{r_p \cos(\phi)}{2}}^{R - \epsilon_0} \frac{r \, dr \, d\phi}{r^\alpha} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2R_0 + \epsilon_p}^{R - \epsilon_0} \frac{r \, dr \, d\phi}{r^\alpha} \right)$$

$$= \frac{P\lambda}{\alpha - 2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\cos^{\alpha-2}(\phi)}{\epsilon_p^{\alpha-2}} + \frac{\cos^{\alpha-2}(\phi)}{(2R_0 + \epsilon_p)^{\alpha-2}} - \frac{1}{R^{\alpha-2}} \right) d\phi$$

(25)
Denote
\[ A(\alpha) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\alpha-2}(\phi) \, d\phi. \quad (26) \]

For an integer \( \alpha \), we can compute \( A(\alpha) \) in closed form. We demonstrate a table for some values of \( A(\alpha) \) in the Appendix, which we use in simulations. For other \( \alpha \), numerical evaluation of \( A(\alpha) \) is possible. We now can write the second lower bound on the average interference as
\[ E[I_0]_{LB2} = \frac{P \lambda}{\alpha - 2} \left( \frac{A(\alpha)}{\epsilon_p^{\alpha-2}} + \frac{A(\alpha)}{(2R_0 + \epsilon_p)^{\alpha-2}} - \frac{\pi}{R^{\alpha-2}} \right). \quad (27) \]

When \( R \to \infty \), this lower bound approaches
\[ E[I_0]_{LB2}^\infty = \frac{P \lambda A(\alpha)}{\alpha - 2} \left( \frac{1}{\epsilon_p^{\alpha-2}} + \frac{1}{(2R_0 + \epsilon_p)^{\alpha-2}} \right). \quad (28) \]

Since this bound takes into account the interfering transmitters close to the primary receiver, for a small \( \epsilon_p \) or large \( R_0 \), this lower bound is tighter than the previous one in (24).

3) An upper bound on \( E[I_0] \): For the upper bound, similar to the first lower bound, we recenter the network at the primary receiver. We now reduce the exclusive region radius, centered at \( \text{Rx}_p^1 \), to \( \epsilon_p \) and extend the outer network radius, also centered at \( \text{Rx}_p^1 \), to \( R_0 + R \), as in Figure 8. The set of cognitive transmitters contained within these two new circles is a superset of the original, creating an upper bound on the interference as
\[ E[I_0]_{UB} = \int_{\epsilon_p}^{R_0 + R} \frac{2\pi P \lambda r}{r^\alpha} \, dr = \frac{2\pi P \lambda}{\alpha - 2} \left( \frac{1}{\epsilon_p^{\alpha-2}} - \frac{1}{(R + R_0)^{\alpha-2}} \right). \]

As \( R \to \infty \), this upper bound becomes
\[ E[I_0]_{UB}^\infty = \frac{2\pi P \lambda}{\alpha - 2} \frac{1}{\epsilon_p^{\alpha-2}}. \quad (29) \]

B. Comparisons of the bounds on the expected interference power

We now compare the upper bound in (29) and the lower bounds in (24) and (28) for various values of \( R_0 \) and \( \alpha \), while fixing \( \lambda = 1, P = 1, \) and \( \epsilon_p = 2 \) and assuming an infinite network \((R \to \infty)\). For \( \alpha = 3 \), Figure 9 shows that for small \( R_0 \), lower bound 1 is better than lower bound 2, as expected. The exact expression for the expected interference for \( \alpha = 4 \) in (22) provides a lower bound on the interference for \( \alpha = 3 \). For \( \alpha = 4 \), Figure 10 shows the upper and lower bounds compared to the exact expression of (22). We see that lower bound 2 is asymptotically tight as \( R_0 \to \infty \). For \( \alpha = 5 \), Figure 11 shows the upper and lower bounds. In this case, the exact expected interference expression for \( \alpha = 4 \) yields an upper bound on the expected interference, which is tighter than the upper bound in (29) for large \( R_0 \).
Fig. 8. An upper bound on the expected interference at the primary Rx is obtained by forming a cognitive-free circle of radius $\epsilon_p$ around the primary receiver and enlarging the network radius, centered at the primary receiver, to $R + R_0$. All cognitive transmitters now lie within these new boundaries.

Fig. 9. Upper (29), lower bound 1 (24), lower bound 2 (28) for $\alpha = 3$, $\lambda = 1$, $P = 1$, $\epsilon_p = 2$. In this case the exact expression for $\alpha = 4$ is a lower bound on the expected interference.
Fig. 10. Upper (29), lower bound 1 (24), lower bound 2 (28) for $\alpha = 4$, $\lambda = 1$, $P = 1$, $\epsilon_p = 2$. In this case we have the exact expression for $\alpha = 4$, which we compare to the other bounds to give an indication of their tightness.

Fig. 11. Upper (29), lower bound 1 (24), lower bound 2 (28) for $\alpha = 5$, $\lambda = 1$, $P = 1$, $\epsilon_p = 2$. In this case the exact expression for $\alpha = 4$ is an upper bound on the expected interference.

C. The primary exclusive radius

The above bounds on the expected interference can be used to bound the radius $R_0$ of the primary exclusive region. In particular, for a given outage capacity $C_0$, the primary outage
constraint (7) can be written as

\[ P_e = \Pr \left[ \log_2 \left( 1 + \frac{P_0}{I_0 + \sigma^2} \right) \leq C_0 \right] \]

\[ = \Pr \left[ I_0 \geq \frac{P_0/R_0^\alpha}{(2C_0 - 1) - \sigma^2} \right]. \]

Assuming that the primary network operates in the region that there is no outage due to noise, then

\[ \frac{P_0/R_0^\alpha}{(2C_0 - 1) - \sigma^2} \geq 0 \]

\[ \iff R_0 \leq \left( \frac{P_0}{\sigma^2(2C_0 - 1)} \right)^{1/\alpha} \triangleq R_0^u. \]  \hspace{1cm} (30)

If \( R_0 \) is larger than \( R_0^u \), the receivers at the edge of the PER will be in outage because of noise alone. Thus \( R_0^u \) is the maximum radius to ensure that the outage constraint holds even without any cognitive users.

Assuming that \( R_0 \) satisfies (30), we can apply Markov’s inequality to bound the outage probability as

\[ P_e \leq \frac{E[I_0]}{R_0/R_0^u} - \sigma^2. \]

Assuming an infinite network \( (R \to \infty) \), using the upper bound on \( E[I_0] \) in \( (29) \), we can further bound \( P_e \) as

\[ P_e \leq \frac{2\pi P \lambda}{\alpha - 2} \frac{1}{\sigma^2(2C_0 - 1)} \left( \frac{P_0/R_0^u}{2C_0 - 1} - \sigma^2 \right)^{-1}. \]

Bounding this probability by the outage constraint \( \beta \), we get

\[ R_0^\alpha \leq \frac{P_0}{(2C_0 - 1)} \left( \frac{2\pi P \lambda}{\beta(\alpha - 2)} \frac{1}{\sigma^2} + \frac{1}{\sigma^2} \right)^{-1}. \]  \hspace{1cm} (31)

This bound is always smaller than the bound in (30). Thus, as expected, the maximum distance that we can guarantee an outage probability for a primary receiver will be reduced in the presence of cognitive users.

When \( \alpha \) is an even integer, we can use the exact value of \( E[I_0] \) in the Markov inequality to obtain a tighter bound on \( R_0 \). Using the example for \( \alpha = 4 \) in (22), we obtain an implicit equation for all exclusive region radii \( R_0 \) such that (7) holds as

\[ \frac{(R_0 + \epsilon_p)^2}{\epsilon_p^2(2R_0 + \epsilon_p)^2} \leq \frac{\beta}{\lambda \pi P} \left( \frac{P_0/R_0^4}{2C_0 - 1} - \sigma^2 \right). \]  \hspace{1cm} (32)
Equations (31) and (32) provide a relation among the system parameters: $P_0$ (the primary transmit power), $P$ (the cognitive users’ power), $C_0$ (the outage capacity), $\beta$ (the outage probability), $\lambda$ (the cognitive user density), $\sigma^2$ (the noise power), and $R_0$ (the exclusive region radius). These equations may be of particular interest when designing the primary system to guarantee the primary outage constraint $\Pr[\text{primary user’s rate} \leq C_0] \leq \beta$. By fixing several of the parameters, we can obtain relations among the others. Specifically, we relate the primary outage target rate $C_0$ to the capacity without interference $C = \log_2(1 + P_0/\sigma^2)$ as $C_0 = \eta C$, where $0 \leq \eta \leq 1$ represents the fraction of the interference-free capacity that we wish to guarantee with probability $\beta$ in a sea of cognitive users.

As an example, we plot in Figure 12 the relation between the exclusive region radius $R_0$ and the guard-band width $\epsilon_p$ for various values of the outage capacity $C_0$, while fixing all other parameters according to (32) for $\alpha = 4$. The plots show that $R_0$ increases with $\epsilon_p$, and the two are of approximately the same order. This is intuitive since at the primary receiver there is a trade-off between the interference seen from the secondary users, which is of a minimum distance $\epsilon_p$ away, and the desired signal strength from the primary BS, which is of the distance $R_0$ away. The larger the $\epsilon_p$, the less interference, and thus the further away the primary receiver may lie from the base station. We also notice that as $C_0$ increases, $R_0$ decreases for the same $\epsilon_p$. This is again intuitive: as we require a higher capacity, the relative interference (to the desired signal) must be reduced, which is achieved by reducing $R_0$ for a fixed $\epsilon_p$. Finally, as $\epsilon_p \to \infty$, $R_0$ approaches the limit of the interference-free bound in (30) for $\alpha = 4$.

Alternatively, we can fix the guard band $\epsilon_p$ and the secondary user power $P$ and seek the relation between the primary power $P_0$ and the exclusive radius $R_0$ that can support the outage capacity $C_0$. In Figure 13 we plot this relation according to (32) for $\alpha = 4$. The fourth-order increase in power here is inline with the path loss $\alpha = 4$. Interestingly, a small increase in the gap band $\epsilon_p$ can lead to a large reduction in the required primary transmit power $P_0$ to reach a receiver at a given radius $R_0$ while satisfying the given outage constraint.

V. SINGLE PRIMARY USER NETWORK WITH COGNITIVE DISTANCE-DEPENDENT POWER SCALING

In this section, we consider the case in which cognitive transmitters can scale their power according to the distance to a single primary user located at the center of the network. Such a
Fig. 12. The relation between the exclusive region radius $R_0$ and the guard band $\epsilon_p$ according to (31) for $\lambda = 1$, $P = 1$, $P_0 = 100$, $\sigma^2 = 1$, $\beta = 0.1$ and $\alpha = 3$.

Fig. 13. The relation between the BS power $P_0$ and the exclusive region radius $R_0$ according to (31) for $\lambda = 1$, $P = 1$, $\sigma^2 = 1$, $\beta = 0.1$, $C_0 = 3$ and $\alpha = 3$. 
model is relevant when primary transmitters are spread out, or whenever the interference from other primary users is negligible. Suppose that the network consists of a single primary user with the transmitter at the center and the receiver within a radius $R_0$, where we assume $R_0 > 1$ in this section. Then, intuitively the cognitive transmitters further away from the center may transmit at a higher power without significantly increasing the interference to the primary receiver. We confirm that this cognitive power scaling does not affect the cognitive user scaling law. Furthermore, it allows $D_{max}$, the maximum distance between a cognitive transmitter and receiver, to grow with the network size. We also explore the impact of distance-dependent cognitive power scaling on the expected interference at the primary receiver and on the PER radius $R_0$.

Assume the transmit power of a cognitive user at radius $r$ is

$$P = P_c r^\gamma$$

(33)

where $P_c$ is a constant, and $\gamma$ is the cognitive power exponent. As shown later, for the interference from the cognitive users to stay bounded, we require that $\gamma < \alpha - 2$.

A. The effect of cognitive power scaling on $D_{max}$

Let the maximum distance between a cognitive Tx and Rx be $D_{max}$. Then the cognitive channel gain is given by $|h_{ii}|^2 \geq 1/D_{max}^\alpha$. The received power at the cognitive receiver will be

$$P_r = \frac{P}{|h_{ii}|^2} \leq \frac{P_c r^\gamma}{D_{max}^\alpha}.$$

(34)

Consider the ratio $\frac{r^\gamma}{D_{max}^\alpha}$. If we constrain this ratio to be lower-bounded by a constant, then $D_{max}$ can grow with distance such that

$$D_{max} \leq K_d r^{\gamma/\alpha}.$$

(35)

for some constant value $K_d$. Therefore, when we let the transmit power of cognitive users scale with distance, then the maximum distance between a cognitive Tx and Rx can also scale with distance. Noting that $\gamma < \alpha - 2$, and since $r \geq R_0 + \epsilon_p > 1$, we have

$$D_{max} \leq K_d r^{\gamma/\alpha} < K_d r^{1-2/\alpha}.$$

(36)

Thus depending on the path loss $\alpha$, the cognitive Tx-Rx distance can grow with an exponent of upto $1 - 2/\alpha$. For a large $\alpha$, this growth is almost at the same rate as the network.
B. Cognitive throughput scaling

We first examine the throughput scaling of the cognitive users as $n \to \infty$. Consider a cognitive receiver, the interference from the single primary user will be bounded due to the finite transmit power and the exclusive radius $R_0$. The interference from the other cognitive users, however, is different.

1) Interference from other cognitive users: The scaled cognitive transmit power affects the interference to a cognitive receiver from other cognitive users. Similar to the development in (12) and (13), this average interference in the worst-case is now upper bounded by

$$I_{av, n} = 2\pi \lambda \int_{r_c}^{R} r^\gamma r^{-\alpha} dr = \frac{2\pi \lambda}{\alpha - 2 - \gamma} \left( \frac{1}{e^{\alpha - 2 - \gamma}} - \frac{1}{R^{\alpha - 2 - \gamma}} \right).$$

If $\gamma < \alpha - 2$, then this average interference is bounded as $R \to \infty$. Thus we can let the cognitive users transmit with higher power the further they are from the primary user, as long as their power scaling satisfies $\gamma < \alpha - 2$. This power scaling therefore is dependent on the propagation environment.

With $\gamma < \alpha - 2$, as $n \to \infty$, the average interference from other cognitive users approaches

$$I^{(\gamma)}_\infty = \frac{2\pi \lambda}{\alpha - 2 - \gamma} \frac{1}{e^{\alpha - 2 - \gamma}}.\tag{38}$$

This can be seen as the interference with an effective path loss decreased to $\alpha - \gamma$. Therefore, the scaling in transmit power of cognitive users takes the advantage of the margin in a high path loss when signal power decays faster than 2.

2) Lower bound on the cognitive network throughput: The rate of a cognitive user (3) can now be written as

$$C_i = \log \left( 1 + \frac{P_c r^\gamma |h_{ii}|^2}{I_{pi} + \sigma^2 + I_{ci}} \right).\tag{39}$$

Recall that $I_{pi} \leq I_P$ (10). Together with the bounds on $D_{\text{max}}$ in (35) and on the receive power in (34), we now have

$$C_i \geq \log \left( 1 + \frac{P_c}{(I_P + \sigma^2 + I_{ci}) K_d} \right)^{\gamma} \tag{40}$$

Again applying the Jensen inequality, the average rate of each cognitive user satisfies

$$E[C_i] \geq \log \left( 1 + \frac{P_c}{(I_P + \sigma^2 + E[I_i]) K_d} \right)$$
As \( n \to \infty \), the lower bound approaches a constant as

\[
E[C_i] \geq \log \left( 1 + \frac{P_{r,\text{min}}}{(I_P + \sigma^2 + I_\infty^{(\gamma)})K_\alpha^2} \right) \triangleq \bar{C}_1^{(\gamma)}
\]

where \( I_\infty^{(\gamma)} \) is given in (38). Again the average per-user throughput of this cognitive network remains at least a constant as \( n \to \infty \). Applying a concentration analysis similar to Section III-D we conclude that with cognitive transmit power scaling (33), the sum throughput of the cognitive users in any network also scales linearly with the number of users.

C. Effect of cognitive power scaling on the interference at the primary receiver

Similar to Section IV, we examine the effect on the expected interference \( E[I_0] \) at the primary receiver of having cognitive transmitters scale their power according to the distance from the primary user as in (33). This expected interference may now be expressed as

\[
E[I_0] = n \int_{R_0+\epsilon_p}^{R} \int_0^{2\pi} \frac{P_{c}\gamma^\alpha f_r(r)f_\theta(\theta)}{d(r,\theta)^\alpha} \, dr \, d\theta \\
= \int_{R_0+\epsilon_p}^{R} \int_0^{2\pi} \frac{\lambda r^{\gamma+1}P_c}{(r^2 + R_0^2 - 2R_0r^2 \cos \theta)^{\alpha/2}} \, dr \, d\theta.
\]

(42)

We notice the additional factor \( r^\gamma \) in the numerator. Similarly, the two lower bounds and single upper bound derived in Section IV may also be changed to reflect the power scaling. The bounds, in terms of \( \gamma \), may be expressed as (43), (44) and (45), where we require that \( \gamma < \alpha - 2 \), and \( R \to \infty \) for simplicity.

\[
E[I_0]_{LB1}^\infty(\gamma) = \frac{2\pi P_c \lambda}{\alpha - 2 - \gamma} \left( \frac{1}{(2R_0 + \epsilon_p)^{\alpha-2-\gamma}} \right)
\]

(43)

\[
E[I_0]_{LB2}^\infty(\gamma) = \frac{P_c \lambda A(\alpha - \gamma)}{\alpha - 2 - \gamma} \left( \frac{1}{\epsilon_p^{\alpha-2-\gamma}} + \frac{1}{(2R_0 + \epsilon_p)^{\alpha-2-\gamma}} \right)
\]

(44)

\[
E[I_0]_{UB}^\infty(\gamma) = \frac{2\pi P_c \lambda}{\alpha - 2 - \gamma} \left( \frac{1}{\epsilon_p^{\alpha-2-\gamma}} \right)
\]

(45)

These bounds may be interpreted as follows. For a given path loss \( \alpha \) and acceptable power scaling of \( \gamma \) (such that the cognitive users may achieve the same linear scaling law as when power scaling was not employed), these bounds correspond to those of a channel with no power scaling and path loss \( \alpha^* = \alpha - \gamma \). Again, a network with power scaling may be thought of as an equivalent network without power scaling but with a slower decay of the power with distance (a
smaller path loss parameter). As an example, for $\alpha = 5$, the plots of bounds in Figure 11 apply when $\gamma = 0$, in Figure 10 when $\gamma = 1$ and in Figure 9 when $\gamma = 2$.

VI. CONCLUSION

As secondary spectrum usage is rapidly approaching, it is important to study the potential of cognitive radios and cognitive transmission from a network perspective. In this paper, we have determined the sum-rate scaling of a network of one-hop cognitive transmitter-receiver pairs which simultaneously communicate, while probabilistically guaranteeing the primary user link a minimum rate. With simultaneous one-hop cognitive transmissions, we show that the sum-rate of cognitive users scales linearly in the number of cognitive links $n$ as $n \to \infty$. This result holds in presence of multiple primary users, when the cognitive transmitters use constant power. The same result also holds in the presence of a single primary user, when the cognitive transmitters scale their power according to the distance from the primary user. Then using the outage constraint on the primary user, we derive bounds on the radius of a primary exclusive region (PER) around each primary transmitter. These bounds help in the design of a primary network with PERs such that, outside these regions, uniformly distributed cognitive transmitters may freely transmit while not harming the primary user.

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**APPENDIX**

**BOUNDS ON THE INTERFERENCE FROM THE PRIMARY USERS**

To show that $I_A$ in (9) is bounded, we use the following inequalities:

$$\frac{1}{(x^2 + y^2)^{\alpha/2}} \leq \frac{2^{\alpha/2}}{(x + y)^{\alpha}}, \quad x + y > 0$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(ak + b)^\alpha} \leq \frac{1}{b^\alpha} + \int_0^{\infty} \frac{1}{(ax + b)^\alpha} dx, \quad b > 0$$

$$= \frac{1}{b^\alpha} + \frac{1}{(\alpha - 1)a} \frac{1}{b^{\alpha - 1}}, \quad b > 0.$$

Applying these inequalities to the following generic sum with $a > 0$, $c > 0$ and $b + d > 0$ as:

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{[(ak + b)^2 + (cm + d)^2]^\alpha/2} \leq \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{\alpha/2}}{(ak + b + cm + d)^\alpha}$$

$$\leq 2^{\alpha/2} \sum_{m=0}^{\infty} \left( \frac{1}{(cm + b + d)^\alpha} + \frac{1}{(\alpha - 1)a} \frac{1}{(cm + b + d)^{\alpha - 1}} \right)$$

$$\leq 2^{\alpha/2} \left[ \frac{1}{(b + d)^\alpha} + \frac{1}{(\alpha - 1)c} (b + d)^{\alpha - 1} + \frac{1}{(\alpha - 1)a} \left( \frac{1}{(b + d)^{\alpha - 1}} + \frac{1}{(\alpha - 2)c} \frac{1}{(b + d)^{\alpha - 2}} \right) \right].$$

Thus for $\alpha > 2$, this summation is bounded.
Now consider $I_A$ in (9). Denote $I_{A1}$ as the first double-summation, then it can be rewritten as

$$I_{A1} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(2\sqrt{3}k - \epsilon_c \cos \theta)^2 + (2m - \epsilon_c \sin \theta)^2}^{\alpha/2}$$

$$+ \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(2\sqrt{3}k + \epsilon_c \cos \theta)^2 + (2m + \epsilon_c \sin \theta)^2}^{\alpha/2} - \frac{1}{\epsilon_c^d}$$

Since $\epsilon_c < 1$ (normalized to $R_0$), then $|\epsilon_c \cos \theta| \leq 1$ and $|\epsilon_c \sin \theta| \leq 1$ for all $\theta \in [0, 2\pi]$. For each of the double-summations in $I_{A1}$, after separating out the first three finite terms corresponding to $(k = 0, m = 0)$, $(k = 0, m = 1)$, and $(k = 1, m = 0)$, then the rest can be rewritten in the form (46) with $b + d > 0$, which is bounded. Similarly for the second summation in $I_A$ (9). Therefore for any $\theta$, $I_A$ (9) is bounded.

**Calculation of the exact $E[I_0]$ when $\alpha = 4$**

For $a > |b|$, from pg. 383 [16], we obtain

$$\int_0^{2\pi} \frac{dx}{(a + b \cos(x))^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

In the integral of interest (20) we have $a = R_0^2 + r^2$ and $b = -2R_0r$, and so $R_0^2 + r^2 > 2R_0r$ as needed. Thus, the expected interference from all cognitive users is given by (47).

$$E[I_0] = \lambda \pi P \int_{R_0 + \epsilon_p}^{R} \int_0^{2\pi} \frac{2r \, dr \, d\theta}{2\pi (R_0^2 + r^2 - 2R_0r \cos \theta)^2} = \lambda \pi P \int_{R_0 + \epsilon_p}^{R} \frac{2r(r^2 + R_0^2)}{(r^2 - R_0^2)^3} \, dr$$

$$= \lambda \pi P \left[ \frac{r^2 + R_0^2}{2(r^2 - R_0^2)^2} - \frac{1}{2(r^2 - R_0^2)} \right]_{R_0 + \epsilon_p}^R$$

$$= \lambda \pi P \left[ \frac{R^2}{2(R^2 - R_0^2)^2} + \frac{(R_0 + \epsilon_p)^2}{\epsilon_p^2(2R_0 + \epsilon_p)^2} \right]$$

(47)

Thus, if we let the number of users $n \to \infty$, or equivalently, as $R \to \infty$, the total interference experienced by the primary receiver when on the edge of the primary exclusive region approaches the constant

$$E[I_0]_\infty = \frac{\lambda \pi P(R_0 + \epsilon_p)^2}{\epsilon_p^2(2R_0 + \epsilon_p)^2}.$$

**Evaluation of $A(\alpha)$**

The lower bounds on the expected value and variance of the interference depend on the function

$$A(\alpha) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\alpha - 2}(\phi) \, d\phi$$

This function may be easily calculated (see for example pg. 161 of [16]) for integral values of $\alpha$. For completeness, and reference for our simulations, here is a table of $A(\alpha)$.

| $\alpha$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $A(\alpha)$ | $\pi$ | $\pi/2$ | $4/3$ | $3\pi/8$ | $16/15$ | $5\pi/16$ | $32/35$ | $35\pi/128$ |