From Frame-like Wavelets to Wavelet Frames keeping approximation properties and symmetry

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Abstract
For a given symmetric refinable mask obeying the sum rule of order \( n \), an explicit method is suggested for the construction of mutually symmetric almost frame-like wavelet system providing approximation order \( n \). A transformation based on the lifting scheme is described that allows to improve almost frame-like wavelets to dual wavelet frames and preserve other properties. A direct method for the construction of dual wavelet frames providing approximation order \( n \) and mutual symmetry properties is also discussed. For an abelian symmetry group \( \mathcal{H} \), a technique providing the \( \mathcal{H} \)-symmetry property for each wavelet function is given for the above three methods.

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Introduction
Multivariate nonseparable wavelet systems have its applications in 2-D tomography, 3-D rotational angiography and other fields (see, e.g.[6],[17], [18], [25] and the references therein). Such wavelets are more suitable for applications than separable wavelets. But in the multivariate case, the problem of the construction of wavelets with desired properties is more complicated comparing to the univariate case. One of the main difficulties is that there is no simple and effective algorithm for the matrix extension problem. A number of papers are devoted to the construction of nonseparable wavelets for some concrete situations.

A symmetry property for wavelets is highly desired in applications since symmetric wavelets efficiently work with the edges of images and reduce the amount of computations. Symmetric wavelets are connected with linear-phase filter banks, which are more applicable for images since they produce no phase distortion. But due to the different kinds of symmetry, it is more complicated to construct symmetric wavelets in general case, in contrast to the univariate case where the general techniques are known (see [14], [24]). In the multivariate case methods for the construction of point symmetric wavelets can be found in [7] and [17]. Some schemes in different setups for the construction of highly symmetric wavelet systems were presented in [1], [15], [16], [19] and the references therein. The construction of wavelet masks for the interpolatory case was considered in [21].

The paper is devoted to the construction of multivariate wavelet frames with important for applications properties, i.e. good approximation order and symmetry properties. Let \( \mathcal{H} \) be a symmetry group, \( M \) be a dilation matrix. For a given \( \mathcal{H} \)-symmetric refinable mask obeying the sum rule of order \( n \), in Theorem 5 an algorithm is suggested for the construction of almost frame-like wavelet system providing approximation order \( n \) and having mutual symmetry properties. The number of wavelet generators is equal to \( m \) or \( m - 1 \), \( m = | \det M | \). Frame-like wavelets were introduced in [20]. These systems are not dual wavelet frames but

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preserve important properties of frames. Among the benefits, there is a simplification of the construction, since we do not need to provide vanishing moments for all wavelet functions (this condition is required for dual wavelet frames). Using the lifting scheme it is possible to improve almost frame-like wavelet system to dual wavelet frame, preserving the symmetry and approximation properties. Also, we discuss a direct approach for the construction of dual wavelet frames. This approach is based on Algorithm 1 from [29]. In this case, for a given \( \mathcal{H} \)-symmetric refinable mask obeying the sum rule of order \( n \), in Theorem 7 the method is given for the construction of dual wavelet frames such that wavelet masks are mutually symmetric and have vanishing moments of order \( n \). The number of wavelet generators, in this case, is equal to \( m \) or \( m + 1 \). But the support of the constructed wavelet functions can be large and this is not good for applications. For an abelian symmetry group \( \mathcal{H} \) and under some additional assumptions, a symmetrization step can be done for the three setups described above. As a result, all wavelet masks have the \( \mathcal{H} \)-symmetry properties keeping other properties unchanged.

The paper is organized as follows. In Section 1 we give some basic notations and definitions including the notion of a symmetry group and connected notions. Section 2 is devoted to the construction of wavelet masks with mutual symmetry properties for three cases: frame-like wavelets, frames based on the lifting scheme, frames based on Algorithm 1 from [29]. Section 3 describes a symmetrization step which allows to construct fully \( \mathcal{H} \)-symmetric wavelet systems in the same three cases. In Section 4 several examples are presented.

1. Basic notations and definitions

We use the standard multi-index notations. For \( x, y \in \mathbb{R}^d \), \( (x, y) = \sum_{i=1}^{d} x_i y_i \), \( \mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d \) and \( x \geq y \) if \( x_j \geq y_j, j = 1, \ldots, d \). \( \mathbb{Z}^d_+ := \{ x \in \mathbb{Z}^d : x \geq \mathbf{0} \} \). If \( \alpha, \beta \in \mathbb{Z}^d_+ \), \( b \in \mathbb{R}^d \), we set \( |\alpha| = \sum_{j=1}^{d} \alpha_j \), \( \alpha! = \prod_{j=1}^{d} \alpha_j! \), \( \langle \alpha \rangle = \prod_{j=1}^{d} \frac{\alpha_j!}{\alpha!} \). We assume that \( \mathcal{H} \) is an abelian symmetry group and connected notions. Section 2 is devoted to the construction of wavelet masks with mutual symmetry properties for three cases: frame-like wavelets, frames based on the lifting scheme, frames based on Algorithm 1 from [29]. Section 3 describes a symmetrization step which allows to construct fully \( \mathcal{H} \)-symmetric wavelet systems in the same three cases. In Section 4 several examples are presented.

A finite set \( \mathcal{H} \) of \( d \times d \) unimodular matrices (i.e. integer matrices with determinant equal to \( \pm 1 \)) is a symmetry group, if \( \mathcal{H} \) forms a group under the matrix multiplication. We say that a dilation matrix \( M \) is appropriate for \( \mathcal{H} \), if \( M^{-1}EM \in \mathcal{H} \), \( \forall E \in \mathcal{H} \). Or, equivalently, for each \( E \in \mathcal{H} \) there exists \( E' \in \mathcal{H} \) such that

\[
EM = M'E' \quad \text{or} \quad M^{-1}E = E'M^{-1}.
\]

A compactly supported distribution \( f \) is called \( \mathcal{H} \)-symmetric with respect to a center \( C \in \mathbb{R}^d \), if

\[
\hat{f}(C) = \hat{f}(E^*C) e^{2\pi i (EC-C)}, \quad \forall E \in \mathcal{H}, \quad \xi \in \mathbb{R}^d.
\]

For trigonometric polynomials we use a bit different definition. We say that \( c \in \mathbb{R}^d \) is an appropriate symmetry center for \( \mathcal{H} \), if \( c - Ec \in \mathbb{Z}^d \), \( \forall E \in \mathcal{H} \). A trigonometric polynomial \( t(\xi) = \sum_{k \in \mathbb{Z}^d} h_k e^{2\pi i k \cdot \xi} \), \( h_k \in \mathbb{C} \), is \( \mathcal{H} \)-symmetric with respect to an appropriate center \( c \), if

\[
t(\xi) = e^{2\pi i (c - Ec) \cdot \xi})^* t(E^*c), \quad \forall E \in \mathcal{H}.
\]
A function/distribution $\varphi$ is called \textit{refinable} if there exists a 1-periodic function $m_0 \in L_2([0,1]^d)$ (mask, also refinable mask, low-pass filter) such that
\[ \hat{\varphi}(\xi) = m_0(M^{-1}\xi)\hat{\varphi}(M^{-1}\xi). \] (3)
This condition is called the refinement equation. It is well known (see, e.g., [23, § 2.4]) that for any trigonometric polynomial $m_0$ satisfying $m_0(0) = 1$ there exists a unique (up to a factor) compactly supported solution of the refinement equation (3) in the space of tempered distributions $S'$. Throughout the paper we assume that any refinable mask $m_0$ is a trigonometric polynomial and $m_0(0) = 1$. For an appropriate $M$ for $H$, it is known that refinable mask $m_0$ is $H$-symmetric with respect to an appropriate center $c$ if and only if the corresponding refinable function is $H$-symmetric with respect to $(M - I_d)^{-1}c$ (see [3]).

Let us fix the set of digits $D(M)$. For any trigonometric polynomial $t$ there exists a unique set of trigonometric polynomials $\tau_k$, $k = 0, \ldots, m - 1$, such that
\[ t(\xi) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (\xi_k\xi)} \tau_k(M^{*}\xi), \] (4)
where $s_k \in D(M)$. Equality (4) is the \textit{polyphase representation} of $t$. Trigonometric polynomial $\tau_k$ is called the \textit{polyphase component} of $t$ corresponding to the digit $s_k$.

Let $t$ be a trigonometric polynomial, $n \in \mathbb{N}$. We say that $t$ obey the \textit{sum rule of order} $n$ with respect to dilation matrix $M$, if $D^\beta t(M^{-1}\xi) |_{\xi = s} = 0$, $\forall s \in D(M^*) \setminus \{0\}$, $\forall \beta \in \Delta_n$. We say that $t$ has \textit{vanishing moments of order} $n$ if $D^\beta t(0) = 0$, $\forall \beta \in \Delta_n$.

1.1. \textit{Symmetry groups and digits}

A general method for the construction of multivariate $H$-symmetric masks that obey the sum rule of an arbitrary order is given in [22]. The key feature used in the construction is the description of how a symmetry group acts on the set of digits. Let a center $c$ and a dilation matrix $M$ be appropriate for $H$. It is known that any $\alpha \in \mathbb{Z}^d$ can be uniquely represented as $\alpha = M\beta + s$, where $\beta \in \mathbb{Z}^d$, $s \in D(M)$. This fact yields that for each digit $s \in D(M)$ and matrix $E \in \mathcal{H}$ there exist a unique digit $q \in D(M)$ and a unique vector $r_k^E \in \mathbb{Z}^d$ such that
\[ Es = Mr_k^E + q + Ec. \] (5)
The indices of $r_k^E$ mean that vector $r_k^E$ depends on digit $s$ and matrix $E$.

The coset corresponding to digit $s \in D(M)$ we denote by $\langle s \rangle$, i.e. $\langle s \rangle = M\mathbb{Z}^d + s$. Denote by $\mathcal{D} := \{ \langle s \rangle, s \in D(M) \}$ the set of cosets. Define a group action of $\mathcal{H}$ on $\mathcal{D}$ as follows
\[ E\langle s \rangle := \{ EM\beta + Es + c - Ec, \beta \in \mathbb{Z}^d \}, \quad E \in \mathcal{H}, \quad s \in D(M). \]
Note that $E\langle s \rangle$ is also a coset, i.e. there exists $q \in D(M)$ such that $E\langle s \rangle = \langle q \rangle$. Next, we introduce suitable notations that will be used throughout the paper. These notations are illustrated in [22] and [21].

- $\mathcal{H}\langle s \rangle = \{ E\langle s \rangle, E \in \mathcal{H} \}$ is the orbit of $\langle s \rangle \in \mathcal{D}$. The orbits are disjoint, $\mathcal{H}\langle s \rangle \subseteq \mathcal{D}$.
- The set $\Lambda \subset \mathcal{D}$ contains representatives from each orbit. $\mathcal{D} = \bigcup_{\langle s \rangle \in \Lambda} \mathcal{H}\langle s \rangle$.
- For convenience, redenote the elements of the set $\Lambda$ by $\langle s_{p,0} \rangle$, where $p = 0, \ldots, \#\Lambda - 1$.
- The set $\mathcal{H}_{p,0} = \{ F \in \mathcal{H} : F\langle s_{p,0} \rangle = \langle s_{p,0} \rangle \}$ is the stabilizer of $\langle s_{p,0} \rangle$; $\mathcal{H}_{p,0} \subset \mathcal{H}$.
- The set $\mathcal{E}_p$ contains a complete set of representatives of $\mathcal{H}/\mathcal{H}_{p,0}$; $\mathcal{E}_p \subset \mathcal{H}$.
- The elements of the orbit $\mathcal{H}\langle s_{p,0} \rangle$ we denote by $\langle s_{p,i} \rangle$, $i = 0, \ldots, \#\mathcal{E}_p - 1$.
- For a fixed index $p$, the matrices of the set $\mathcal{E}_p$ we denote by $E^{(i)}$ such that $E^{(i)}\langle s_{p,0} \rangle = \langle s_{p,i} \rangle$, $i = 0, \ldots, \#\mathcal{E}_p - 1$. Note that $E^{(0)} = I_d$. 

3
• The digit corresponding to the coset \( s_{p,i} \) we choose such that

\[
E^{(i)} s_{p,0} + c - E^{(i)} c =: s_{p,i}, \quad i = 1, \ldots, \#\mathcal{E}_p - 1.
\]  

Note that for a fixed \( p, p = 0, \ldots, \#\mathcal{E} - 1 \), symmetry group \( \mathcal{H} \) can be uniquely represented as follows \( \mathcal{H} = \mathcal{E}_p \times \mathcal{H}_{p,0} \), i.e., for each matrix \( K \) in \( \mathcal{H} \) there exist matrices \( E \in \mathcal{E}_p \) and \( F \in \mathcal{H}_{p,0} \) such that \( K = EF \). The sets \( \mathcal{E}_p, \mathcal{H}_{p,0} \) can be considered as the “coordinate axes” of symmetry group \( \mathcal{H} \). For each \( p \) these “coordinate axes” of \( \mathcal{H} \) can be different. Also note how \( \mathcal{H} \) acts on a digit \( s_{p,0} \in D(M) \). If \( E^{(i)} \in \mathcal{E}_p \), then \( E^{(i)} s_{p,0} \) is defined in (6). If \( F \in \mathcal{H}_{p,0} \), then \( F(s_{p,0}) = (s_{p,0}) \) and

\[
F s_{p,0} = M r^F_{p,0} + s_{p,0} + F c - c,
\]

where \( r^F_{p,0} \in \mathbb{Z}^d \). Notice that \( r^F_{p,0} = M^{-1} (c - s_{p,0}) - M^{-1} F (c - s_{p,0}) \).

Now we formulate the \( \mathcal{H} \)-symmetry condition (2) for a trigonometric polynomial in terms of its polyphase components. Alongside with the standard enumeration of the polyphase components of trigonometric polynomial \( t \), we also use enumeration corresponding to the new enumeration of digits: \( \tau_{p,i}(\xi), i = 0, \ldots, \#\mathcal{E}_p - 1 \), \( p = 0, \ldots, \#\mathcal{E} - 1 \).

**Lemma 1.** [21, Lemma 8] A trigonometric polynomial \( t \) is \( \mathcal{H} \)-symmetric with respect to an appropriate center \( c \) if and only if its polyphase components \( \tau_{p,i} \) satisfy

\[
\tau_{p,0}(\xi) = e^{2\pi i (r^F_{p,0}, \xi)} \tau_{p,0}(M^{-1} F M)^* \xi, \quad \text{for all } F \in \mathcal{H}_{p,0};
\]

\[
\tau_{p,i}(\xi) = \tau_{p,0}(M^{-1} E^{(i)} M)^* \xi, \quad E^{(i)} \in \mathcal{E}_p, \quad \text{for all } i \in \{0, \ldots, \#\mathcal{E}_p - 1\}.
\]

for each \( p \in \{0, \ldots, \#\mathcal{E} - 1\} \).

### 2. Construction of symmetric wavelets

A family of functions \( \{f_\alpha\}_{\alpha \in \mathbb{N}} \) (\( \mathbb{N} \) is a countable index set) in a Hilbert space \( H \) is called a frame in \( H \) if there exist constants \( A, B > 0 \) such that \( A\|f\|_2^2 \leq \sum\limits_{\alpha} |\langle f, f_\alpha \rangle|^2 \leq B\|f\|_2^2 \), \( \forall f \in H \). If \( \{f_\alpha\}_\alpha \) is a frame in \( H \), then every \( f \in H \) can be decomposed as \( f = \sum \langle f, f_\alpha \rangle f_\alpha \), where \( \{f_\alpha\}_\alpha \) is a dual frame in \( H \). Comprehensive characterization of frames can be found in [23]. Wavelet frames are of great interest in many applications, especially in signal processing. For more information about multivariate wavelet frames see [30].

For \( \psi^{(\nu)} \in S', \nu = 1, \ldots, r \), a system \( \{\psi^{(\nu)}\}_j \) is called a wavelet system. We say that wavelet system \( \{\psi^{(\nu)}_{j,k}\} \) has vanishing moments of order \( n \in \mathbb{N} \) (or has the \( VM^n \) property) if \( D^\beta \psi^{(\nu)}(0) = 0 \), \( \forall \beta \in \Delta_n \), \( \nu = 1, \ldots, r \).

A general scheme for the construction of compactly supported MRA-based wavelet systems (in particular, wavelet frames in \( L_2(\mathbb{R}^d) \)) was developed in [26, 27] (the Unitary Extension Principle). To construct a pair of such wavelet systems one starts with two compactly supported refinable functions \( \varphi, \tilde{\varphi}, \tilde{\varphi}(0) = 1, \tilde{\varphi}(0) = 1 \), (or its masks \( m_0, \tilde{m}_0 \), respectively, which are trigonometric polynomials). Then one finds trigonometric polynomials \( m_{\nu}, \tilde{m}_{\nu}, \nu = 1, \ldots, r, r \geq m - 1 \), called wavelet masks, such that the following polyphase matrices

\[
\mathcal{M} := \{\mu_{\nu k}\}_{\nu = 0}^{r, \nu = 0, m - 1}, \quad \tilde{\mathcal{M}} := \{	ilde{\mu}_{\nu k}\}_{\nu = 0}^{r, \nu = 0, m - 1} \quad \text{satisfy} \quad \mathcal{M}^* \tilde{\mathcal{M}} = I_m,
\]

Here \( \mu_{\nu k}, \tilde{\mu}_{\nu k} \), \( k = 0, \ldots, m - 1 \), are the polyphase components of the wavelet masks \( m_{\nu}, \tilde{m}_{\nu} \) for all \( \nu = 0, \ldots, r, r \geq m - 1 \). The wavelet functions \( \psi^{(\nu)}(\xi) = m_{\nu}(\mathcal{M}^{\ast -1}\xi)\tilde{\varphi}(\mathcal{M}^{\ast -1}\xi), \psi^{(\nu)}(\xi) = \tilde{m}_{\nu}(\mathcal{M}^{\ast -1}\xi)\tilde{\varphi}(\mathcal{M}^{\ast -1}\xi) \). If the wavelet functions \( \psi^{(\nu)}, \tilde{\psi}^{(\nu)}, \nu = 1, \ldots, r, r \geq m - 1 \), are constructed as above, then the set of the functions \( \{\psi^{(\nu)}_{j,k}\}, \{\tilde{\psi}^{(\nu)}_{j,k}\} \)
is said to be a compactly supported MRA-based dual wavelet system generated by the refinable functions \( \varphi, \tilde{\varphi} \) (or their masks \( m_0, \tilde{m}_0 \)).

Suppose that some compactly supported refinable functions \( \varphi, \tilde{\varphi} \in L_2(\mathbb{R}^d) \) generate a MRA-based dual wavelet system \( \{ \psi_{jk}^{(\nu)} \}, \{ \tilde{\psi}_{jk}^{(\nu)} \}, \nu = 1, \ldots, r, r \geq m - 1 \). A necessary (see \cite{29}, Theorem 1) and sufficient (see \cite{29}, Theorems 2.2, 2.3) condition for the wavelet system \( \{ \psi_{jk}^{(\nu)} \}, \{ \tilde{\psi}_{jk}^{(\nu)} \} \) to be a pair of dual wavelet frames in \( L_2(\mathbb{R}^d) \) is that each wavelet system \( \{ \psi_{jk}^{(\nu)} \} \) and \( \{ \tilde{\psi}_{jk}^{(\nu)} \} \) has vanishing moments at least of order 1. Or, equivalently, the corresponding wavelet masks \( m_\nu, \tilde{m}_\nu \) have vanishing moments at least of order 1. Also, it is known that good approximation properties for the corresponding wavelet systems are provided by the VM property for the dual wavelet system \( \{ \tilde{\psi}_{jk}^{(\nu)} \} \) (see, e.g., \cite{29}, Theorem 4). Although algorithms for the construction of dual wavelet frames with vanishing moments were developed (see, \cite{29}), but it is not easy to provide various types of symmetry for different dilation matrices. If we reject the frame requirements and aim to provide the VM property only for dual wavelet system \( \{ \tilde{\psi}_{jk}^{(\nu)} \}, \nu = 1, \ldots, r \), then the method can be simplified. The necessary conditions with the constructive proof are given in Lemma 2.\cite{29}:

**Lemma 14** Let \( \varphi, \tilde{\varphi} \) be compactly supported refinable distributions, \( m_0 \) obeys the sum rule of order \( n \) and

\[
D^\beta \left( 1 - m_0(\xi)\overline{m_0}(\xi) \right) \bigg|_{\xi=0} = 0 \quad \forall \beta \in \Delta_n.
\]

(10)

then there exist a MRA-based dual wavelet system \( \{ \psi_{jk}^{(\nu)} \}, \{ \tilde{\psi}_{jk}^{(\nu)} \}, \nu = 1, \ldots, m \), such that wavelet system \( \{ \psi_{jk}^{(\nu)} \} \) has the VM property.

A technique for the extension of the polyphase matrices realizes as follows. Let us denote the rows of the polyphase components of masks \( m_0 \) and \( \tilde{m}_0 \) by \( P = (\mu_{00}, \ldots, \mu_{0,m-1}) \), \( \tilde{P} = (\tilde{\mu}_{00}, \ldots, \tilde{\mu}_{0,m-1}) \) and extend them with the elements \( 1 - PP^* \) and 1 accordingly. Then the explicit formulas for the matrix extension are given by

\[
N = \begin{pmatrix} P & 1 - PP^* \\ U & -U \tilde{P}^* \end{pmatrix}, \quad \tilde{N} = \begin{pmatrix} \tilde{P} & 1 \\ \tilde{U} - \tilde{U}P^* & \tilde{U} \tilde{P}^* \end{pmatrix}.
\]

(11)

Here \( U, \tilde{U} \) are \( m \times m \) matrices consisting of trigonometric polynomials such that \( U\tilde{U}^* \equiv I_m \). For instance, we can take \( U = \tilde{U} = I_m \). It follows easily that \( N\tilde{N}^* \equiv I_{m+1} \). It is important to note that dual wavelet frames cannot be constructed using such extension technique due to the fact that in the matrix \( \tilde{N} \) the upper right element is equal to one. Therefore, wavelet masks \( m_\nu, \nu = 1, \ldots, m \) do not have any order of vanishing moments according with \cite{29} Theorem 8. Nevertheless, for appropriate functions \( f \) we can consider frame-type decompositions with respect to this system.

Let \( \{ \psi_{jk}^{(\nu)} \}, \{ \tilde{\psi}_{jk}^{(\nu)} \} \) be a MRA-based dual wavelet system and \( A \) be a class of functions \( f \) such that \( \langle f, \tilde{\varphi}_{0k} \rangle, \langle f, \tilde{\psi}_{jk}^{(\nu)} \rangle \) have meaning (e.g., \( f \) in the Schwartz space \( S \)) if \( \varphi, \tilde{\varphi} \in S' \); \( f \in L_p \) if \( \varphi, \tilde{\varphi} \in L_q \), \( \frac{1}{p} + \frac{1}{q} = 1 \). We say that dual wavelet system \( \{ \psi_{jk}^{(\nu)} \}, \{ \tilde{\psi}_{jk}^{(\nu)} \} \) is almost frame-like if

\[
f = \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{0k} \rangle \varphi_{0k} + \sum_{j=0}^{\infty} \sum_{\nu=1}^{m} \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\psi}_{jk}^{(\nu)} \rangle \psi_{jk}^{(\nu)} \quad \forall f \in A,
\]

(12)

where the series in \( 12 \) converge in some natural sense. Here we give the result from \cite{29}.

**Theorem 3.**\cite{29} \( \text{Theorem 12} \) Let \( f \in S, \varphi, \tilde{\varphi} \in S' \), \( \varphi, \tilde{\varphi} \) be compactly supported and refinable, \( \tilde{\varphi}(0) = \tilde{\varphi}(0) = 1 \). Then the MRA-based dual wavelet system \( \{ \psi_{jk}^{(\nu)} \}, \{ \tilde{\psi}_{jk}^{(\nu)} \}, \nu = 1, \ldots, r \), generated by \( \varphi, \tilde{\varphi} \) is almost frame-like, i.e., \( \text{12} \) holds with the series converging in \( S' \).

If, moreover, \( \varphi, \tilde{\varphi} \) are as in Lemma 2 \( \varphi \in L_2(\mathbb{R}^d) \), then \( \text{12} \) holds with the series converging in \( L_2 \)-norm and the corresponding almost frame-like system has approximation order \( n \) (see \cite{29} Theorem 16). In the next sections we show that starting from symmetric refinable masks, the symmetry property for the wavelet masks can be provided.
2.1. Dual masks

Given an $\mathcal{H}$-symmetric refinable mask $m_0$, the first step for the construction of symmetric wavelets is the construction of an appropriate $\mathcal{H}$-symmetric dual refinable mask $\tilde{m}_0$. The method is given by the following theorem.

**Theorem 4.** Let a dilation matrix $M$ and a center $c$ be appropriate for a symmetry group $\mathcal{H}$, $n \in \mathbb{N}$. Suppose $m_0$ is an $\mathcal{H}$-symmetric with respect to the center $c$ mask that obeys the sum rule of order $n$. Then there exists a dual mask $\tilde{m}_0$ that is $\mathcal{H}$-symmetric with respect to the center $c$ and satisfies condition $\text{(10)}$.

**Proof.** Set $(2\pi i)^{[\beta]} \lambda_\beta := D^\beta m_0(0)$, $\beta \in \Delta_n$. And let numbers $\tilde{\lambda}_\beta$ satisfy

$$
\sum_{\alpha \in \mathcal{D}, \delta} (-1)^{[\beta] - \alpha} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \lambda_\alpha \overline{\lambda}_{\beta - \alpha} = 0, \quad \forall \beta \in \Delta_n.
$$

(13)

Numbers $\tilde{\lambda}_\beta$ can be found recursively from $\text{(13)}$. Define mask $\tilde{m}_0$ as follows

$$
\tilde{m}_0(\xi) = \frac{1}{#\mathcal{H}} \sum_{E \in \mathcal{H}} G(E^* \xi) e^{2\pi i (c - Ec, \xi)},
$$

(14)

where $G(\xi)$ is a trigonometric polynomial such that $D^\beta G(0) = (2\pi i)^{[\beta]} \tilde{\lambda}_\beta$ for all $\beta \in \Delta_n$. It is not hard to check that $\tilde{m}_0(\xi)$ is $\mathcal{H}$-symmetric with respect to the point $c$. Let us show that condition $\text{(10)}$ is also valid. When $\beta = 0$, then $\lambda_0 = \tilde{\lambda}_0 = 1$, and $m_0(\xi) = \tilde{m}_0(\xi) = 1$. For $\beta \neq 0$ condition $\text{(10)}$ will be valid if

$$
D^\beta \left( m_0(\xi) G(E^* \xi) e^{2\pi i (c - Ec, \xi)} \right) |_{\xi = 0} = 0, \quad \forall \beta \in \Delta_n, \beta \neq 0, \quad \forall E \in \mathcal{H}.
$$

(15)

Since $m_0$ is $\mathcal{H}$-symmetric, then $m_0(\xi) G(E^* \xi) e^{2\pi i (c - Ec, \xi)} = m_0(E^* \xi) G(E^* \xi)$. Due to the higher chain rule with the linear change of variables, equalities $\text{(15)}$ are equivalent to

$$
D^\beta \left( m_0(\xi) G(E^* \xi) \right) |_{\xi = 0} = 0, \quad \forall \beta \in \Delta_n, \beta \neq 0.
$$

These equalities are valid, since $D^\beta G(0) = (2\pi i)^{[\beta]} \tilde{\lambda}_\beta$, $\forall \beta \in \Delta_n$, and numbers $\tilde{\lambda}_\beta$ satisfy $\text{(13)}$. \hfill \Box

For applications it is important to get refinable masks with the minimal coefficient support (the coefficient support of trigonometric polynomial $t = \sum_k h_k e^{2\pi i (k, \xi)}$ is defined by $\text{coefsupp}(t) = \{ k \in \mathbb{Z}^d : h_k \neq 0 \}$). The general recipe for the construction of an $\mathcal{H}$-symmetric dual refinable mask with the minimal coefficient support is based on a fact that for any vector $k \in \text{coefsupp}(G)$, the set of integer points $\{ E(k + c - Ec : E \in \mathcal{H}) \}$ should be a subset of $\text{coefsupp}(\tilde{m}_0)$ (see $\text{(14)}$). Thus, for some fixed $k_1 \in \mathbb{Z}^d$, one should check whether the system of linear equations $D^\beta G(0) = (2\pi i)^{[\beta]} \tilde{\lambda}_\beta$, $\forall \beta \in \Delta_n$, is solvable with $\text{coefsupp}(G) = K_1 := \{ E k_1 + c - Ec : E \in \mathcal{H} \}$. If yes, then one solves the system. If not, one should add another point $k_2 \in \mathbb{Z}^d$ and check the solvability of the system with $\text{coefsupp}(G) = K_1 \cup \{ E k_2 + c - Ec : F \in \mathcal{H} \}$ and so on.

2.2. Symmetric frame-like wavelets

Assume we have two $\mathcal{H}$-symmetric masks $m_0$ and $\tilde{m}_0$ such that $\text{(10)}$ is valid. Let us consider the construction of wavelet systems using the basic matrix extension $\text{(11)}$ with $U = \tilde{U} = I_m$. Namely,

$$
\mathcal{N} = \begin{pmatrix} P & 1 - P \tilde{P}^* \\ I_m & -\tilde{P}^* \end{pmatrix}, \quad \tilde{\mathcal{N}} = \begin{pmatrix} \tilde{P} & 1 \\ I_m - P^* \tilde{P} & -P^* \end{pmatrix},
$$

(16)

where $P$ and $\tilde{P}$ are the rows of the polyphase components of masks $m_0$ and $\tilde{m}_0$. For convenience, we use the following enumeration of wavelet masks. Let $\mu_{0,p,i}$ and $\tilde{\mu}_{0,p,i}$, $i = 0, \ldots, \#\mathcal{E}_p - 1$, $p = 0, \ldots, \#\Lambda - 1$, be the polyphase components of $m_0$ and $\tilde{m}_0$,

$$
T_p = (\mu_{0,p,0}, \ldots, \mu_{0,p,\#\mathcal{E}_p - 1}), \quad \tilde{T}_p = (\tilde{\mu}_{0,p,0}, \ldots, \tilde{\mu}_{0,p,\#\mathcal{E}_p - 1}).
$$
Set $P = (T_0, \ldots, T_{\#A-1})$, $\tilde{P} = (\tilde{T}_0, \ldots, \tilde{T}_{\#A-1})$. The polyphase components for wavelet masks $m_\nu$ and $\tilde{m}_\nu$ are contained in the submatrices $I_m$ and $I_m - P^* \tilde{P}$ in (16). Note that $I_m - P^* \tilde{P}$ is a block matrix

$$I_m - P^* \tilde{P} = \begin{pmatrix}
I_{N_0-1} - T_0^* \tilde{T}_0 & -T_{01}^* \tilde{T}_1 & \cdots & -T_{0(\#A-1)}^* \tilde{T}_{\#A-1} \\
-T_{10}^* \tilde{T}_0 & I_{N_1-1} - T_1^* \tilde{T}_1 & \cdots & -T_{1(\#A-1)}^* \tilde{T}_{\#A-1} \\
\vdots & \ddots & \ddots & \vdots \\
-T_{(\#A-1)0}^* \tilde{T}_0 & \cdots & I_{N(\#A-1)-1} - T_{(\#A-1)^2}^* \tilde{T}_{(\#A-1)^2} 
\end{pmatrix}$$

(17)

where $N_p = \#E_p$. The rows of the submatrix we enumerate by double index $(p, i)$. The $(p, i)$th row is the $i$th row in the $p$th block. A wavelet mask corresponding to the $(p, i)$th row is denoted by $\tilde{m}_{(p, i)}$. Analogously, wavelet masks $m_{(p, i)}$ are enumerated.

**Theorem 5.** Let a dilation matrix $M$ and a center $c$ be appropriate for a symmetry group $\mathcal{H}_s$, $s \in \mathbb{N}$. Suppose $m_0$ and $\tilde{m}_0$ are $\mathcal{H}$-symmetric with respect to the center $c$ masks such that mask $m_0$ obeys the sum rule of order $n$, mask $\tilde{m}_0$ satisfies condition (14). Then wavelet masks $m_{(p, i)}$ and $\tilde{m}_{(p, i)}$ constructed using matrix extension (16) have the following symmetry properties:

**W1** $m_{(p, 0)}$, $\tilde{m}_{(p, 0)}$ are $\mathcal{H}_{p,0}$-symmetric with respect to the center $s_{p,0}$,

**W2** $m_{(p, i)}(\xi) = m_{(p, 0)}(E^{(i)*}\xi)e^{2\pi i (c-E^{(i)}\xi)}$, $\tilde{m}_{(p, i)}(\xi) = \tilde{m}_{(p, 0)}(E^{(i)*}\xi)e^{2\pi i (c-E^{(i)}\xi)}$, $E^{(i)} \in E_p$,

where $i = 0, \ldots, \#E_p - 1$, $p = 0, \ldots, \#A - 1$. Wavelet masks $\tilde{m}_{(p, i)}$ have vanishing moments of order $n$. The corresponding MRA-based dual wavelet system is almost frame-like in $S'$.

**Proof.** Wavelet masks from submatrix $I_m - P^* \tilde{P}$ in (16) are given by

$$\tilde{m}_{(p, i)}(\xi) = \frac{1}{\sqrt{m}}e^{2\pi i (s_{p, i}\xi)} - \tilde{\mu}_{0,p,i}(M^*\xi)\tilde{m}_0(\xi), \quad i = 0, \ldots, \#E_p - 1, p = 0, \ldots, \#A - 1.$$

Then mask $m_{(p, 0)}$ is $\mathcal{H}_{p,0}$-symmetric with respect to the center $s_{p,0}$. Indeed, due to (7) and (8) for all $F \in \mathcal{H}_{p,0}$ we have

$$\tilde{m}_{(p, 0)}(F^*\xi) = \frac{1}{\sqrt{m}}e^{2\pi i (F s_{p,0}\xi)} - \tilde{\mu}_{0,p,0}(M^*\xi)\tilde{m}_0(\xi^*F^*\xi),$$

$$= \tilde{m}_{(p, 0)}(\xi)e^{2\pi i (F s_{p,0} - s_{p,0}\xi)}.$$

Also, by (6) and (11) we have

$$\tilde{m}_{(p, 0)}(E^{(i)*}\xi) = \frac{1}{\sqrt{m}}e^{2\pi i (E^{(i)}s_{p,0}\xi)} - \tilde{\mu}_{0,p,0}(M^*E^{(i)*}\xi)\tilde{m}_0(E^{(i)*}\xi),$$

$$= \tilde{m}_{(p, 0)}(\xi)e^{2\pi i (E^{(i)}c - c\xi)}.$$

Similarly, it can be checked that the same symmetric properties are valid for wavelet masks $m_{(p, i)}$, since

$$m_{(p, i)}(\xi) = \frac{1}{\sqrt{m}}e^{2\pi i (s_{p, i}\xi)}, \quad i = 0, \ldots, \#E_p - 1, p = 0, \ldots, \#A - 1.$$

Vanishing moments of order $n$ for the dual wavelet masks $\tilde{m}_{(p, i)}$ are provided by Lemma 2.

Thus, the matrix extension (16) leads to the wavelet masks that are mutually symmetric, i.e. some wavelet masks are reflected or rotated copies of the others. Note that [W1] and [W2] by direct computations imply that $m_{(p, i)}$ and $\tilde{m}_{(p, i)}$ are $\mathcal{H}_{p,i}$-symmetric with respect to the center $s_{p,i}$. Proposition 2.1 in [12] allows to compute the symmetry centers of the corresponding wavelet functions. Notice that if for some $\mathcal{H}$
and $M$ we get that $\mathcal{E}_p = \{I_d\}$ for all $p = 0, \ldots, \#\Lambda - 1$, then all wavelet masks constructed by Theorem 5 are $H$-symmetric.

The number of wavelet generators for almost frame-like wavelet systems in Theorem 6 can be reduced to $m - 1$ in the following case. Suppose for some $k = 0, \ldots, m - 1$, $\mu_0 \equiv \sqrt{m}$ Then we can take $\tilde{\mu}_0 \equiv \sqrt{m}$ and $\tilde{\mu}_l \equiv 0$, for $l \neq k$. Therefore, $\tilde{m}_0 (\xi) = e^{2\pi i (s_k, \xi)}$ and by Remark 15 in [20], since $\sum_{l=0}^{m-1} \mu_0(\xi)\tilde{\mu}_l(\xi) \equiv 1$, condition (10) is valid. The matrix extension (10) leads to a wavelet system with $m - 1$ wavelet generators.

From the point of view of Theorem 1.1 in [13], the wavelet system constructed by Theorem 5 can be a dual wavelet frame in a pair of dual Sobolev spaces depending on the Sobolev smoothness exponents of $\varphi$ and $\tilde{\varphi}$.

2.3. Lifting scheme: frame-like wavelets to frames

The lifting scheme was introduced by W. Sweldens [31]. It is a tool for designing wavelets and performing the discrete wavelet transform. The lifting scheme has lots of different applications and useful properties. One important for us feature is that the lifting scheme allows to improve properties of a given wavelet system. In particular, it allows to provide additional vanishing moments for wavelet masks (see, e.g., [2]). Since vanishing moments are necessary and sufficient conditions for a wavelet system to be a dual wavelet system, it allows to improve properties of a given wavelet system.

Let refinable masks $m_0, \tilde{m}_0$ and wavelet masks $m_\nu, \tilde{m}_\nu, \nu = 1, \ldots, r$ be such that the corresponding polyphase matrices $\mathcal{M}, \mathcal{M}$ satisfy $\mathcal{M}^* \mathcal{M} = I_m$. Let $L_1, \ldots, L_r$ be trigonometric polynomials. Define $r \times r$ matrices $\mathcal{L}, \tilde{\mathcal{L}}$ as follows:

$$\mathcal{L} := \begin{pmatrix} L & 0 \\ I_{r-1} & \end{pmatrix}, \quad \tilde{\mathcal{L}} := \begin{pmatrix} 1 & \tilde{\mathcal{L}} \\ 0 & I_{r-1} \end{pmatrix},$$

where $L = (L_1, \ldots, L_r)$. It is easy to check that $\mathcal{L}^* \tilde{\mathcal{L}} = I_r$. Define new polyphase matrices $\mathcal{M}_{\text{new}} := \mathcal{L} \mathcal{M}$ and $\tilde{\mathcal{M}}_{\text{new}} := \tilde{\mathcal{L}} \mathcal{M}$. Note that the equality $\mathcal{M}_{\text{new}}^* \mathcal{M}_{\text{new}} = I_m$ is preserved. Denote the elements of the new matrices as follows: $\mathcal{M}_{\text{new}} = \{\mu_{ij}^n\}_{i=0}^{m-1}, \nu=0, r, \mathcal{M}_{\text{new}} = \{\tilde{\mu}_{ij}^n\}_{i=0}^{m-1}, \nu=0, r$. The transformation formulas for the polyphase components are the following:

$$\mu_{0,j}^n (\xi) = \mu_{0,j} (\xi), \quad \mu_{ij}^n (\xi) = \mu_{ij} (\xi) + L_\nu (\xi) \mu_{0,j} (\xi),$$

$$\tilde{\mu}_{0,j}^n (\xi) = \tilde{\mu}_{0,j} (\xi) - \sum_{i=1}^{r} L_i (\xi) \tilde{\mu}_{i,j} (\xi), \quad \tilde{\mu}_{ij}^n (\xi) = \tilde{\mu}_{ij} (\xi).$$

Let new masks $m_\nu^n, \tilde{m}_\nu^n$ be constructed from the new polyphase components by (41). Thus, the transformation formulas for the masks are

$$m_\nu^n (\xi) = m_\nu (\xi), \quad m_\nu^n (\xi) = m_\nu (\xi) + L_\nu (\xi^* m_\nu (\xi),$$

$$\tilde{m}_\nu^n (\xi) = \tilde{m}_\nu (\xi) - \sum_{i=1}^{r} L_i (\xi^* \tilde{m}_\nu (\xi), \quad \tilde{m}_\nu^n (\xi) = \tilde{m}_\nu (\xi).$$

The above transformation of the masks is called the lifting scheme transformation. Note that the choice of trigonometric polynomials $L_\nu, \nu = 1, \ldots, r$ is not restricted.

The aim is to find the lifting scheme transformation such that all new masks preserve their symmetry properties and all new wavelet masks have vanishing moments at least of order 1.

**Theorem 6.** Let a dilation matrix $M$ and a center $c$ be appropriate for a symmetry group $H, n \in \mathbb{N}$. Suppose $m_0$ and $\tilde{m}_0$ are as in Theorem 5. Suppose $m_{(p,i)}, \tilde{m}_{(p,i)}$ are wavelet masks constructed using Theorem 5. Assume that trigonometric polynomials $L_{p,i}$ satisfy $L_{p,i}(0) = -m_{(p,i)}(0)$ and

$$L_{p,0}(M^* F^* M^{*-1} \xi) = L_{p,0}(\xi) e^{2\pi i (r_{p,0} F \xi)}, \quad \forall F \in H_{p,0},$$

(19)
Define a mapping $\tilde{\imath}$ uniquely defined such that
\begin{equation}
L_{p,0}(M^*E^{(i)}*M^{*-1}\xi) = L_{p,i}(\xi), \quad E^{(i)}\in \mathcal{E}_p,
\end{equation}
for $i = 0, \ldots, \#\mathcal{E}_p - 1$, $p = 0, \ldots, \#\Lambda - 1$. New masks $m_0^n, \tilde{m}_0^n, m_{(p,i)}^n, \tilde{m}_{(p,i)}^n$, defined by the lifting scheme transformation preserve symmetry properties of masks $m_0, \tilde{m}_0, m_{(p,i)}, \tilde{m}_{(p,i)}$, respectively. New wavelet masks $m_{(p,i)}^n$ have vanishing moments at least of order 1. If new refinable functions $\varphi, \tilde{\varphi}$ corresponding to new refinable masks $m_0^n, \tilde{m}_0^n$ are in $L_2(\mathbb{R}^d)$, then the resulting wavelet system is a dual wavelet frame.

**Proof.** Vanishing moments at least of order 1 for wavelet masks $m_{(p,i)}^n$, are provided by conditions $L_{p,i}(0) = -m_{(p,i)}(0)$. This follows from (18).

Next, we show that the symmetry properties are preserved. For $m_{(p,0)}^n$ condition [W1] in Theorem 5 is preserved, since conditions (18) and (20) yield
\begin{equation}
m_{(p,0)}(E^{(i)}\xi) = m_{(p,0)}(E^{(i)}\xi) + L_{(p,0)}(M^*E^{(i)}*E^{(i)})m_0(\xi) = m_{(p,0)}(\xi)e^{2\pi i(F_{p,0}-s_{p,0},\xi)} + L_{(p,0)}(M^*E^{(i)}*E^{(i)})m_0(\xi) \quad \text{for } i = 0, \ldots, \#\mathcal{E}_p - 1,
\end{equation}
Next, condition [W2] in Theorem 5 for $m_{(p,i)}^n$ is preserved, since conditions (18) and (20) yield
\begin{equation}
m_{(p,i)}(E^{(i)}\xi) = m_{(p,i)}(E^{(i)}\xi) + L_{(p,0)}(M^*E^{(i)}*E^{(i)})m_0(\xi) = m_{(p,i)}(\xi)e^{2\pi i(E^{(i)}c-c,\xi)} + L_{(p,i)}(M^*E^{(i)}*E^{(i)})m_0(\xi) \quad \text{for } i = 0, \ldots, \#\mathcal{E}_p - 1.
\end{equation}

Now, we show that new refinable mask $\tilde{m}_0^n$ remains $\mathcal{H}$-symmetric with respect to the point $c$. Let us fix $p$. Recall that $\mathcal{H}$ can be uniquely represented as follows $\mathcal{H} = \mathcal{E}_p \times \mathcal{H}_{p,0}$. Suppose $K \in \mathcal{H}$ and $E^{(i)} \in \mathcal{E}_p$. Define a mapping $j(\cdot, p, K)$ from the set of indices $\{0, \ldots, \#\mathcal{E}_p - 1\}$ to itself, where index $j = j(i, p, K)$ is uniquely defined such that $KE^{(i)} = E^{(j)}F$.

For $K \in \mathcal{H}$, consider
\begin{equation}
\tilde{m}_0^n(K^*\xi) = \tilde{m}_0(\xi)e^{2\pi i(Kc-c,\xi)} - \sum_{p=0}^{\#\Lambda-1} \sum_{i=0}^{\#\mathcal{E}_p-1} L_{p,i}(M^*K^*\xi)\tilde{m}_{p,i}(K^*\xi).
\end{equation}

Let us fix indices $p$ and $i$. Then $KE^{(i)} = E^{(j)}F$, where $F \in \mathcal{H}_{p,0}$, $E^{(j)} \in \mathcal{E}_p$, $j = j(i, p, K)$. With conditions on $L_{p,i}$, we obtain
\begin{align*}
L_{p,i}(M^*K^*\xi) &= L_{p,0}(M^*E^{(i)}*M^{*-1}\xi) = L_{p,0}(M^*E^{(i)}*E^{(i)}*\xi) \\
&= L_{p,0}(M^*F^{(i)}*\xi)e^{-2\pi i(M_{p,0}^*E^{(i)}*\xi)} = L_{p,0}(M^*\xi)e^{-2\pi i(E^{(i)}M_{p,0}^*\xi)},
\end{align*}
where $-E^{(j)}M_{p,0}^*F = E^{(j)}s_{p,0} - E^{(j)}c - KE^{(i)}(s_{p,0} - c).$ Using [W1] and [W2] we get
\begin{align*}
\tilde{m}_{(p,i)}(K^*\xi) &= \tilde{m}_{(p,i)}(E^{(i)}K^*\xi)e^{2\pi i(c-E^{(i)}c,K^*\xi)} \\
&= \tilde{m}_{(p,i)}(F^{(i)}K^*\xi)e^{2\pi i(c-E^{(i)}c,K^*\xi)} \\
&= \tilde{m}_{(p,i)}(E^{(i)}K^*\xi)e^{2\pi i(F_{p,0}-s_{p,0},E^{(i)}c)E^{(i)}c+2\pi i(c-E^{(i)}c,K^*\xi)} \\
&= \tilde{m}_{(p,j)}(\xi)e^{2\pi i(E^{(i)}c-c,\xi)2\pi i(E^{(j)}(F_{p,0}-s_{p,0},E^{(i)}c)E^{(i)}c+2\pi i(Kc-KE^{(i)}c,\xi)}.
\end{align*}
So, finally we have
\begin{equation}
\tilde{m}_{(p,i)}(K^*\xi) = \tilde{m}_{(p,j)}(\xi)L_{p,j}(M^*\xi)e^{2\pi i(R,\xi)}.
\end{equation}
where
\[
R = -E^{(j)} M_{p,0} + E^{(j)} c - E^{(j)}(F_{p,0} - s_{p,0}) + Kc - KE^{(i)}c
\]
\[
= E^{(j)} s_{p,0} - E^{(j)} c - KE^{(i)}(s_{p,0} - c) + E^{(j)} c - c + KE^{(i)} s_{p,0} - E^{(j)} s_{p,0} + Kc - KE^{(i)} c
\]
\[
= Kc - c.
\]

Thus, by (23) we obtain that \( \tilde{\mu}_0^c \) is \( \mathcal{H} \)-symmetric with respect to the center \( c \).

Condition (19) means that \( L_{p,0} = M^{-1}H_{p,0}M \)-symmetric with respect to the center \( M^{-1}(s_{p,0} - c) \), \( p = 0, \ldots, \#\Lambda - 1 \). Such trigonometric polynomials satisfying condition \( L_{p,0}(0) = -m_{(p,0)}(0) \) can be easily constructed.

### 2.4. Symmetric wavelet frames

In [29, Algorithm 1] an algorithm for the construction of dual wavelet frames was suggested. Here we modify this algorithm, such that the constructed wavelets have the symmetry properties.

Firstly, we give a constructive description of the method. Let a dilation matrix \( M \) and a center \( c \) be appropriate for \( \mathcal{H} \), \( n \in \mathbb{N} \).

**Algorithm 1**

**Step 1.** Find a refinable mask \( m_0 \) that is \( \mathcal{H} \)-symmetric with respect to the center \( c \) and obeys the sum rule of order \( n \). Let \( (2\pi i)^{\beta} \tilde{c}_\beta := D^\beta m_0(0), \beta \in \Delta_n \).

**Step 2.** Next, define numbers \( \tilde{c}_\beta, \beta \in \Delta_n \), such that (13) is valid and find a trigonometric polynomial \( \tilde{m}_0 \) that is \( \mathcal{H} \)-symmetric with respect to the center \( c \), obeys the sum rule of order \( n \) and \((2\pi i)^{\beta} \tilde{c}_\beta := D^\beta \tilde{m}_0(0)\).

**Step 3.** Let \( \tilde{p}_0^0, \ldots, \tilde{p}_{m-1}^0 \) be the polyphase components of \( \tilde{m}_0 \). Set
\[
\sigma := \sum_{l=0}^{m-1} \tilde{p}_l \tilde{p}_l^*, \quad \tilde{\mu}_k := (2 - \sigma)\tilde{p}_k^*, k = 0, \ldots, m-1,
\]
\[
\mu_{0m} := (1 - \sigma), \quad \tilde{\mu}_{0m} := 1 - \sigma.
\]

**Step 4.** Set \( r = m \) if \( \sigma \equiv 1 \), otherwise we set \( r = m + 1 \), \( \mu_{0,m+1} \equiv 0, \tilde{\mu}_{0,m+1} \equiv 0 \). Let \( N := \{\mu_{vk}\}_{v,k=0}^r \), \( \tilde{N} := \{\tilde{\mu}_{vk}\}_{v,k=0}^r \). Then matrix extension can be realized as follows
\[
N := \begin{pmatrix} P & 0 & \mu_{0m} \\ I_m - \tilde{P}^*P & 0 & \tilde{\mu}_{0m} \\ -\mu_{0m}\mu_{0m} & 0 & \mu_{0m} \end{pmatrix}, \quad \tilde{N} := \begin{pmatrix} \tilde{P} & 0 & P^* \\ I_m - P^*\tilde{P} & \tilde{\mu}_{0m} & P^* \\ -\mu_{0m}\mu_{0m} & \mu_{0m}P^* & P^* \end{pmatrix},
\]
if \( r = m \), or
\[
N := \begin{pmatrix} P & \mu_{0m} & 0 \\ I_m - \tilde{P}^*P & 0 & \tilde{\mu}_{0m}P^* \\ -\mu_{0m}\mu_{0m} & 1 - \mu_{0m}\mu_{0m} & \tilde{\mu}_{0m} \end{pmatrix}, \quad \tilde{N} := \begin{pmatrix} \tilde{P} & 0 & \tilde{\mu}_{0m}P^* \\ I_m - P^*\tilde{P} & \mu_{0m} & \tilde{\mu}_{0m} \tilde{P}^* \\ -\mu_{0m}\mu_{0m} & 1 - \mu_{0m}\mu_{0m} & \mu_{0m} \end{pmatrix},
\]
if \( r = m + 1 \)
if \( r = m + 1 \), where \( P = (\mu_{00}, \ldots, \mu_{0,m-1}) \), \( \tilde{P} = (\tilde{\mu}_{00}, \ldots, \tilde{\mu}_{0,m-1}) \). It is not difficult to see that the matrices satisfy \( N^*N = I_r \). This yields that the columns of the polyphase matrices \( M, \tilde{M} \) are biorthonormal. Wavelet masks \( m_\nu, \tilde{m}_\nu, \nu = 1, \ldots, r \), are constructed from the polyphase components. \( \diamond \)

The next Theorem states that wavelet masks constructed by Algorithm 1 have the mutual symmetry properties. Again, for convenience, we enumerate wavelet masks using double index as in Theorem 5. For the submatrices \( I_m - \tilde{P}^*P \) and \( I_m - P^* \tilde{P} \) we use notations as in (17).

**Theorem 7.** Let a dilation matrix \( M \) and a center \( c \) be appropriate for a symmetry group \( H \), \( n \in \mathbb{N} \). Suppose refinable and wavelet masks are constructed by Algorithm 1. Then wavelet masks \( m_{(p,i)}, \tilde{m}_{(p,i)} \) have the following symmetry properties:

- \( m_{(p,0)}, \tilde{m}_{(p,0)} \) are \( H_{p,0} \)-symmetric with respect to the center \( s_{p,0} \);
- \( m_{(p,i)} = m_{(p,0)}(E^{(i)}*\xi)e^{2\pi i(c-E^{(i)}\cdot \xi)}, \tilde{m}_{(p,i)} = \tilde{m}_{(p,0)}(E^{(i)}*\xi)e^{2\pi i(c-E^{(i)}\cdot \xi)}, E^{(i)} \in \mathcal{E}_p, \)

\[ i = 0, \ldots, \#\mathcal{E}_p - 1, p = 0, \ldots, \#\Lambda - 1. \]

If \( r = m + 1 \), then wavelet masks \( m_r \) and \( \tilde{m}_r \) defined by the last rows of matrices \( N \) and \( \tilde{N} \) are \( H \)-symmetric with respect to the center \( c \). All wavelet masks have vanishing moments of order \( n \). If refinable functions \( \varphi, \tilde{\varphi} \) corresponding to refinable masks \( m_0, \tilde{m}_0 \) are in \( L_2(\mathbb{R}^d) \), then the resulting wavelet system is a dual wavelet frame.

**Proof.** First, we prove that \( \sigma \) defined in (24) is \( H \)-symmetric with respect to the origin. Let us rewrite \( \sigma \) using another enumeration of the polyphase components and show that \( \sigma(M^*K^*M^{*-1}\xi) = \sigma(\xi) \) for all \( K \in H \).

Thus,

\[
\sigma(M^*K^*M^{*-1}\xi) = \sum_{i=0}^{\#\Lambda -1} \sum_{p=0}^{\#\mathcal{E}_p -1} \mu_{0,p,i}(M^*K^*M^{*-1})_{0,p,i} \tilde{\mu}_{0,p,i}(M^*K^*M^{*-1})_{0,p,i} \nonumber
\]

\[
\sigma(M^*K^*M^{*-1}\xi) = \sum_{i=0}^{\#\Lambda -1} \sum_{p=0}^{\#\mathcal{E}_p -1} \mu_{0,p,i}(M^*E^{(i)}*K^*M^{*-1})_{0,p,i} \tilde{\mu}_{0,p,i}(M^*E^{(i)}*K^*M^{*-1})_{0,p,i} \nonumber
\]

Let us fix \( p \). Recall that \( H \) can be uniquely represented as follows \( H = \mathcal{E}_p \times H_{p,0} \). Recall that for a matrix \( K \) and fixed \( p \) there exist unique matrices \( E^{(i)} \in \mathcal{E}_p \) and \( F \in H_{p,0} \) such that \( KE^{(i)} = E^{(j)}F \), where \( j = j(p,i,K) \). Using (25), (26) and continuing the above equalities, we get

\[
\sigma(M^*K^*M^{*-1}\xi) = \sum_{i=0}^{\#\Lambda -1} \sum_{p=0}^{\#\mathcal{E}_p -1} \mu_{0,p,i}(M^*F^*E^{(i)}(p,i,K)*M^{*-1}\xi) \tilde{\mu}_{0,p,i}(M^*F^*E^{(i)}(p,i,K)*M^{*-1}\xi) \nonumber
\]

\[
\sigma(M^*K^*M^{*-1}\xi) = \sum_{i=0}^{\#\Lambda -1} \sum_{p=0}^{\#\mathcal{E}_p -1} \mu_{0,p,j(p,i,K)}(\xi) \tilde{\mu}_{0,p,j(p,i,K)}(\xi) = \sigma(\xi) \nonumber\]

Since \( \sigma \) is \( H \)-symmetric with respect to the origin, then polyphase components \( \tilde{\mu}_{0\mu} \) and \( \tilde{\mu}_{0\kappa} \) defined in (25) have the same symmetry properties. Therefore, dual refinable mask \( \tilde{m}_0 \) is \( H \)-symmetric with respect to the center \( c \). Symmetry properties of wavelet masks \( m_{(p,i)}, \tilde{m}_{(p,i)} \) defined by submatrices \( I_m - \tilde{P}^*P \) and \( I_m - P^* \tilde{P} \) in (26) or (27) can be proved analogously to the proof of Theorem 5. If \( r = m + 1 \), then the last wavelet masks \( m_r \) and \( \tilde{m}_r \) defined by the last rows of matrices \( N \) and \( \tilde{N} \) are \( H \)-symmetric with respect to the center \( c \). Indeed, since

$$
m_r(\xi) = -\mu_{0m}(M^*\xi)m_0(\xi), \quad \tilde{m}_r(\xi) = -\tilde{\mu}_{0m}(M^*\xi)\tilde{m}_0(\xi),$$

then for all \( K \in H \)

$$m_r(K^*\xi) = -\mu_{0m}(M^*K^*M^{*-1}M^*\xi)m_0(K^*\xi)$$

$$= -\mu_{0m}(M^*\xi)m_0(\xi)e^{2\pi i(Kc-\xi)}$$

$$= m_r(\xi)e^{2\pi i(Kc-\xi)}$$

11
and analogously with $\tilde{m}_n$.

Vanishing moments for all wavelet masks are provided by [29, Theorem 8].

Note that wavelet masks $\tilde{m}_{(p,i)}$, $\tilde{m}_{(p,j)}$ are $H_{p,i}$-symmetric with respect to the center $s_{p,i}$.

The resulting wavelet masks constructed by Theorem 7 in [29] have quite huge coefficient supports. It can be reduced if we will not require vanishing moments of order $n$ for all wavelet functions. According to Theorem 7 in [29], the vanishing moments of order $n$ for dual wavelet masks are provided by sum rule of order $n$ for $m_0$ and $D^3(1 - \sigma)(0) = 0$ for all $|\beta| < n$. These vanishing moments should be kept, since they provide approximation order $n$ for the resulting wavelet system. The vanishing moments of order $n$ for wavelet masks are provided by sum rule of order $n$ for $\tilde{m}_0$ and $D^3(1 - \sigma)(0) = 0$ for all $|\beta| < n$. Here, the order of vanishing moments can be reduced up to 1 (since we need only to keep the frame condition). So, we can require the sum rule of order 1 for $\tilde{m}_0$, preserving $D^3(1 - \sigma)(0) = 0$ for all $|\beta| < n$. In some cases, this can be done (see Example 2 in Section 4).

3. Symmetrization

The aim of this section is to construct wavelet masks such that all of them are $H$-symmetric in a sense. To do that we extend the definition of $H$-symmetric trigonometric polynomials. Let $t$ be a trigonometric polynomial. Then $t$ has the $H$-symmetry property if for each matrix $E \in H$

$$t(E^*\xi) = \varepsilon_E e^{2\pi i (r_E E) t(\xi)},$$

where $\varepsilon_E \in C$, $|\varepsilon_E| = 1$, $r_E \in Z^d$. This definition is a generalization of [2]. For example, the new definition includes antisymmetric trigonometric polynomials and trigonometric polynomials, which do not have a symmetry center common to all matrices $E \in H$.

Let $H$ be an abelian symmetry group. Assume we have a row of the polyphase components of an $H$-symmetric mask. Next, we find a unitary transformation of the row such that each element of the new row has the $H$-symmetry property. For a fixed $p$, $H_{p,0}$ and $E_p$ are an abelian subgroups of $H$ and $E_p$ can be expressed as the direct product of cyclic subgroups. Let $\gamma_p$ be the number of cyclic subgroups of $E_p$. Then there exist unique prime numbers $N_{p,j}$, $j = 1, \ldots, \gamma_p$ and matrices $K_1, \ldots, K_{\gamma_p} \in E_p$ such that

$$E_p = \{I_d, K_1, \ldots, K_1^{N_{p,1}-1}\} \times \cdots \times \{I_d, K_{\gamma_p}, \ldots, K_{\gamma_p}^{N_{p,\gamma_p}-1}\}.$$

Thus, fixed element $E \in E_p$ can be uniquely represented as follows $E = \prod_{j=1}^{\gamma_p} K_{\gamma_p}^{k_j}$, where $k_j$ are some integers from the sets $\{0, \ldots, N_{p,j} - 1\}$ respectively. So, this matrix $E$ we denote by $E^{(k)}$ where $k = (k_1, \ldots, k_{\gamma_p})$ is a number in a mixed radix number system for set of numbers $\{0, \ldots, \#E_p - 1\}$ with the base $(N_{p,1}, \ldots, N_{p,\gamma_p})$. For two matrices $E^{(k)}, E^{(l)} \in E_p$ their product can be written as follows $E^{(k)} E^{(l)} = E^{(k\oplus l)}$, where $k \oplus l$ is an addition on the set $\{0, \ldots, \#E_p - 1\}$ defined as follows:

$$k \oplus l = ((k_1 + l_1) \mod N_{p,1}, \ldots, (k_{\gamma_p} + l_{\gamma_p}) \mod N_{p,\gamma_p}).$$

For the integers $N_{p,j}$ we denote $\varepsilon_{N_{p,j}} := e^{2\pi i / N_{p,j}}$. For any $p \in \{1, \ldots, \#H - 1\}$, define the matrix of the discrete Fourier transform $W_{N_{p,j}} = \frac{1}{\sqrt{N_{p,j}}} e^{2\pi i N_{p,j}^{-1} kl}$. It is known that $W_{N_{p,j}}$ is a unitary and symmetric matrix, i.e. $W_{N_{p,j}}^*, W_{N_{p,j}} = I_m, W_{N_{p,j}}^2 = W_{N_{p,j}}$. Define $W_p = W_{N_{p,1}} \otimes \cdots \otimes W_{N_{p,\gamma_p}}$, where operation $\otimes$ is the Kronecker product. $W_p$ is a $(\#E_p)^{\times} \times (\#E_p)$ unitary matrix. Some properties of the matrix $W_p$ are

$$[W_p]_{k,l}[W_p]_{n,l} = [W_p]_{k\oplus n,l}, \quad [W_p]_{k,l}[W_p]_{k,l} = 1, \quad k, l, n = 0, \ldots, \#E_p - 1.$$

The next Lemma states that $W_p$ symmetrizes the part of the row of polyphase components.
Lemma 8. Suppose \( m_0 \) is an \( \mathcal{H} \)-symmetric with respect to the an appropriate center \( c \) mask and \( \mu_{0,p,i} \) are its polyphase components. For a fixed \( p \), a row \( T_p \) is defined by \( T_p = (\mu_{0,p,0}, \ldots, \mu_{0,p,\#E_p-1}) \).

Suppose that \( r_{p,0}^F = M^{-1}Emr_{p,0}^F \) for all \( F \in \mathcal{H}_{p,0} \) and \( E \in \mathcal{E}_p \). Then each element of the row \( T'_p := T_pW_p \) has the \( \mathcal{H} \)-symmetry property, i.e. for any \( K \in \mathcal{H} \)

\[
\mu_{0,p,r}(M^{-1}K M)^* \xi = \overline{W_{p,r}}^{0} \mu_{0,p,r}^{0}(\xi)e^{-2\pi i (r'_{p,0}, \xi)}, \quad r = 0, \ldots, \#E_p - 1, \tag{28}
\]

where \( K = E^{(i)}F, F \in \mathcal{H}_{p,0} \) and \( E^{(i)} \in \mathcal{E}_p \).

According with this Lemma we need a special assumption

\[
r_{p,0}^F = M^{-1}Emr_{p,0}^F, \quad \forall F \in \mathcal{H}_{p,0}, \forall E \in \mathcal{E}_p \tag{29}
\]

to ensure the \( \mathcal{H} \)-symmetry property for all components of the row \( T'_p \). Notice that this assumption is used only to provide \( M^{-1}H_{p,0}M \)-symmetry for \( \mu_{0,p,r}^{0}(\xi) \).

3.1. Symmetrization for frame-like wavelets

Define a block diagonal unitary matrix \( W \) as follows: \( W = \text{diag}(W_0, \ldots, W_{\#A-1}) \). Let \( m_0 \) be an \( \mathcal{H} \)-symmetric mask. Suppose (29) is valid for all \( p = 0, \ldots, \#A - 1 \). Then by Lemma 8 matrix \( W \) symmetrizes the row \( P = (T_0, \ldots, T_{\#A-1}) \) of the polyphase components of \( m_0 \), namely all elements of the row

\[
P^* := \bar{W}P = (T_0W_0, \ldots, T_{\#A-1}W_{\#A-1})
\]

have the \( \mathcal{H} \)-symmetry property. Wavelets with the \( \mathcal{H} \)-symmetry property can be constructed using matrix \( W \).

Theorem 9. Let a dilation matrix \( M \) and a center \( c \) be appropriate for an abelian symmetry group \( \mathcal{H} \).

Suppose \( m_0 \) and \( \tilde{m}_0 \) are \( \mathcal{H} \)-symmetric with respect to the center \( c \) masks such that mask \( m_0 \) obeys the sum rule of order \( n \), mask \( \tilde{m}_0 \) satisfies condition (10). Suppose that condition (29) is valid for all \( p = 0, \ldots, \#A - 1 \). Then there exist wavelet masks \( m_{(p,i)} \) and \( \tilde{m}_{(p,i)} \) which have the \( \mathcal{H} \)-symmetry property, wavelet masks \( \tilde{m}_{(p,i)} \) having vanishing moments of order \( n \), \( i = 0, \ldots, \#E_p - 1, p = 0, \ldots, \#A - 1 \). The corresponding MRA-based dual wavelet system is almost frame-like in \( S' \).

Proof. Let \( \mu_{0,p,i} \) and \( \mu_{0,p,i} \) be the polyphase components of \( m_0 \) and \( \tilde{m}_0 \), \( i = 0, \ldots, \#E_p - 1, p = 1, \ldots, \#A - 1 \), and let \( T_p = (\mu_{0,p,0}, \ldots, \mu_{0,p,\#E_p-1}) \), and \( \tilde{T}_p = (\tilde{\mu}_{0,p,0}, \ldots, \tilde{\mu}_{0,p,\#E_p-1}) \). Set \( P = (T_0, \ldots, T_{\#A-1}) \), \( \tilde{P} = (\tilde{T}_0, \ldots, \tilde{T}_{\#A-1}) \). Let us consider matrix extension (11) with \( U = \bar{U} = W^* \)

\[
N = \begin{pmatrix} P & 1 - P\bar{P}^* \\ W^* & -W^*\bar{P}^* \end{pmatrix}, \quad \overline{N} = \begin{pmatrix} \bar{P} & 1 \\ W^* - W^*P^*\bar{P} & -W^*P^* \end{pmatrix}. \tag{30}
\]

Let us consider submatrix \( W^* - W^*P^*\bar{P} \). It is a block matrix

\[
W^* - W^*P^*\bar{P} = \begin{pmatrix}
W_{00}^* - (T_0W_0)^*\tilde{T}_0 & -(T_0W_0)^*\tilde{T}_1 & \cdots & -(T_0W_0)^*\tilde{T}_{\#A-1} \\
-(T_1W_1)^*\tilde{T}_0 & W_{10}^* - (T_1W_1)^*\tilde{T}_1 & \cdots & -(T_1W_1)^*\tilde{T}_{\#A-1} \\
\vdots & \vdots & \ddots & \vdots \\
-(T_{\#A-1}W_{\#A-1})^*\tilde{T}_0 & \cdots & \cdots & W_{\#A-1,0}^* - (T_{\#A-1}W_{\#A-1})^*\tilde{T}_{\#A-1}
\end{pmatrix}
\]

Denote the elements of the submatrix as \( W^* - W^*P^*\bar{P} = \{\tilde{\mu}_{(p,i),(t,j)}\}_{t=0,\ldots,\#A-1,j=0,\ldots,\#E_j-1} \). With this enumeration the element \( \tilde{\mu}_{(p,i),(t,j)} \) in the submatrix is the element in the block \((p,t)\) with position \((i,j)\).

Namely, if \( p \neq t \) then

\[
\tilde{\mu}_{(p,i),(t,j)}(\xi) = -(T_pW_p)^*\tilde{T}_{t,j}|_{\xi} = -\mu_{0,p,i}(\xi) \tilde{\mu}_{0,t,j}(\xi);
\]

13
if \( p = t \) then
\[
\tilde{\mu}_{(p,t),(p,t)}(\xi) = [W_p - (T_pW_p)^*T_p]_{i,j} = [W_p]_{i,j} - \tilde{\mu}_{0,p,i}(\xi) \tilde{\mu}_{0,p,j}(\xi),
\]
where \( \mu_{0,p,i} \), \( i = 0, \ldots, \#\mathcal{E}_p - 1 \), are the elements of the row \( T_pW_p \), \( p = 0, \ldots, \#\mathcal{L} - 1 \). All \( \mu_{0,p,i} \) have the \( \mathcal{H} \)-symmetry property by Lemma \( \ref{lem:HSymmetry} \).

Next, we collect wavelet masks by \( \ref{eq:waveletmasks} \). For fixed \( p = 0, \ldots, \#\mathcal{L} - 1, i = 0, \ldots, \#\mathcal{E}_p - 1 \), we get
\[
\tilde{m}_{(p,i)}(\xi) = \frac{1}{\sqrt{m}} \sum_{j=0}^{\#\mathcal{E}_p-1} \frac{[W_p]_{i,j}}{[W_p]_{i,j}} e^{2\pi i (s_{p,j} \cdot \xi)} - \mu_{0,p,i}(M^*\xi) \tilde{m}_0(\xi).
\]

Check that \( \tilde{m}_{(p,i)} \) has the \( \mathcal{H} \)-symmetry property. For a fixed \( F \in \mathcal{H}_{p,0} \), by \( \ref{eq:waveletmasks} \), and by the properties of \( W_p \) we obtain
\[
\tilde{m}_{(p,i)}(F^*\xi) = \frac{1}{\sqrt{m}} \sum_{j=0}^{\#\mathcal{E}_p-1} \frac{[W_p]_{i,j}}{[W_p]_{i,j}} e^{2\pi i (F_{s_{p,j}} \cdot \xi)} - \mu_{0,p,i}(M^*F^*\xi) \tilde{m}_0(F^*\xi)
\]
\[
= e^{2\pi i (F_{s_{p,a}} - s_{p,a} \cdot \xi)} \tilde{m}_{(p,i)}(\xi),
\]

since \( F_{s_{p,j}} = F(E^{(j)} s_{p,0} + c - E^{(j)} c) = s_{p,j} + F s_{p,0} - s_{p,0} \) by \( \ref{eq:trig_polynomial} \), \( \ref{eq:waveletmasks} \). For a fixed \( E^{(k)} \in \mathcal{E}_p \), by \( \ref{eq:waveletmasks} \), and by the properties of \( W_p \) we obtain
\[
\tilde{m}_{(p,i)}(E^{(k)} \xi) = \frac{1}{\sqrt{m}} \sum_{j=0}^{\#\mathcal{E}_p-1} \frac{[W_p]_{i,j}}{[W_p]_{i,j}} e^{2\pi i (E^{(k)} s_{p,j} \cdot \xi)} - \mu_{0,p,i}(M^*E^{(k)}\xi) \tilde{m}_0(E^{(k)}\xi)
\]
\[
= [W_p]_{i,k} e^{2\pi i (E^{(k)} c - c \cdot \xi)} \tilde{m}_{(p,i)}(\xi),
\]

since \( E^{(k)} s_{p,j} = s_{p,j} \otimes k + E^{(k)} c - c \) by \( \ref{eq:trig_polynomial} \). Therefore, for a fixed \( K \in \mathcal{H} \), with \( K = E^{(k)} F \), \( E^{(k)} \in \mathcal{E}_p \), \( F \in \mathcal{H}_{p,0} \),
\[
\tilde{m}_{(p,i)}(K^*\xi) = [W_p]_{i,k} e^{2\pi i (K_{c-c} + Mr_{p,a} \xi)} \tilde{m}_{(p,i)}(\xi).
\]

Thus, we get the \( \mathcal{H} \)-symmetry property for the wavelet mask \( \tilde{m}_{(p,i)} \). It is not hard to see that the wavelet masks \( m_{(p,i)} \) have the same \( \mathcal{H} \)-symmetry property as \( \tilde{m}_{(p,i)} \).

The vanishing moments of order \( n \) for the wavelet masks \( \tilde{m}_{(p,i)} \) are provided by Theorem \( \ref{thm:vanishing_moments} \).

Note that if \( \ref{eq:waveletmasks} \) is not valid for some \( p \), then we can take the corresponding matrix \( W_p \) equal to the identity matrix. Thus, for this index \( p \) the corresponding set of wavelet masks \( m_{(p,i)} \), \( \tilde{m}_{(p,i)} \) will remain mutually symmetric.

3.2. Symmetrization for lifting scheme

In this subsection we suggest a lifting scheme transformation preserving the \( \mathcal{H} \)-symmetry property of the constructed by Theorem \( \ref{thm:lifting} \) wavelet masks. The corresponding trigonometric polynomials for the transformation we defined as follows: let \( \mathcal{L}_p = (L_{p,0}, \ldots, L_{p,#\mathcal{E}_p-1}) \). Define \( \mathcal{L}'_p = \mathcal{L}_p^*W_p^* \) and denote the elements of the row \( \mathcal{L}'_p = (L'_{p,0}, \ldots, L'_{p,#\mathcal{E}_p-1}) \).

Theorem 10. Let a dilation matrix \( M \) and a center \( c \) be appropriate for an abelian symmetry group \( \mathcal{H} \), \( n \in \mathbb{N} \). Suppose \( m_0 \) and \( \tilde{m}_0 \) are as in Theorem \( \ref{thm:lifting} \) and condition \( \ref{eq:waveletmasks} \) is valid for all \( p = 0, \ldots, \#\mathcal{L} - 1 \). Suppose \( m_{(p,i)} \), \( \tilde{m}_{(p,i)} \) are wavelet masks constructed using Theorem \( \ref{thm:lifting} \). Assume that trigonometric polynomials \( L_{p,i} \) are as in Theorem \( \ref{thm:lifting} \). New masks \( m_{(p,i)}^n \), \( \tilde{m}_{(p,i)}^n \) defined by the lifting scheme transformation with trigonometric polynomials \( L'_{p,i} \) have the same symmetry properties as masks \( m_{(p,i)} \), \( \tilde{m}_{(p,i)} \). New wavelet masks \( m_{(p,i)}^n \) have vanishing moments at least of order 1. If new refinable functions \( \varphi, \tilde{\varphi} \) corresponding to new refinable masks \( m_0^n, \tilde{m}_0^n \) are in \( L^2(\mathbb{R}^d) \), then the resulting wavelet system is a dual wavelet frame.
Proof. By direct computations we can state that
\[ L_{p,i}((M^{-1}E^{(k)}M)^*\xi) = [W_p]_{k,i}L'_{p,i}(\xi), \quad \forall E^{(k)} \in \mathcal{E}_p \]
and
\[ L_{p,i}((M^{-1}FM)^*\xi) = L'_{p,i}(\xi)e^{2\pi i(r_p^p,\xi)} \quad \forall F \in \mathcal{H}_{p,0}, \]
i = 0, \ldots, \#E_p - 1, p = 0, \ldots, \#\Lambda - 1. Therefore, we obtain
\[ m_{n(p,i)}^{(k)}(F^*\xi) = e^{2\pi i(F^*\nu_0 - s_0,\xi)}m_{n(p,i)}^{(k)}(\xi) \]
and
\[ m_{n(p,i)}^{(k)}(E^{(k)}^*\xi) = [W_p]_{k,i}e^{2\pi i(E^{(k)e-c,\xi})}m_{n(p,i)}^{(k)}(\xi), \]
i = 0, \ldots, \#E_p - 1, p = 0, \ldots, \#\Lambda - 1. Thus, wavelet mask \( m_{n(p,i)}^{(k)} \) has the same symmetry properties as \( m_{n(p,i)}^{(k)} \). Also, by direct computations it can be checked that \( \tilde{m}_0^* \) remains \( \mathcal{H} \)-symmetric with respect to the center \( c \).

3.3. Symmetrization for frames

Symmetrization step also can be done in Theorem 7. Again, we assume that \( \mathcal{H} \) is an abelian symmetry group and (29) is valid for all \( p = 0, \ldots, \#\Lambda - 1 \). For \( r = m \), instead of matrix extension (26) we consider
\[ Q := \begin{pmatrix} P & 0 \\ W^* - W^* \tilde{P}^* P & 0 \end{pmatrix}, \quad \tilde{Q} := \begin{pmatrix} \tilde{P} & 0 \\ W^* - W^* \tilde{P}^* P & W^* P^* \end{pmatrix}, \quad (31) \]
where matrix \( W \) is a symmetrization matrix. Note that \( Q^* \tilde{Q} = I_r \), since \( Q = \text{diag}(1, W^*) \mathcal{N} \) and \( \tilde{Q} = \text{diag}(1, W^*) \tilde{\mathcal{N}}. \)

For \( r = m + 1 \), instead of matrix extension (24) we consider
\[ \mathcal{N} := \begin{pmatrix} P & \mu_{0m} \\ W^* - W^* \tilde{P}^* P & \frac{\mu_{0m}}{1 - \mu_{0m}} \end{pmatrix}, \quad \tilde{\mathcal{N}} := \begin{pmatrix} \tilde{P} & \tilde{\mu}_{0m} \\ W^* - W^* \tilde{P}^* P & \frac{\tilde{\mu}_{0m}}{1 - \tilde{\mu}_{0m}} \end{pmatrix}, \quad (32) \]
Note that \( Q^* \tilde{Q} = I_r \), since \( Q = \text{diag}(1, W^*, 1) \mathcal{N} \), \( \tilde{Q} = \text{diag}(1, W^*, 1) \tilde{\mathcal{N}} \).

Theorem 11. Let a dilation matrix \( M \) and a center \( c \) be appropriate for an abelian symmetry group \( \mathcal{H} \), \( n \in \mathbb{N} \). Suppose masks \( m_\nu, \tilde{m}_\nu, \nu = 0, \ldots, r \) are constructed by Algorithm 1 where the matrix extension is realized by (27), if \( r = m \), or by (28), if \( r = m + 1 \). Then wavelet masks \( m_{(p,i)} \) and \( \tilde{m}_{(p,i)} \) have the \( \mathcal{H} \)-symmetry properties, i.e. for \( K \in \mathcal{H} \), with \( K = E^{(k)}F, E^{(k)} \in \mathcal{E}_p, F \in \mathcal{H}_{p,0}, \)
\[ m_{(p,i)}(K^*\xi) = [W_p]_{k,i}e^{2\pi i(Kc-c+Mr_{p,0}^p,\xi)}m_{(p,i)}(\xi) \]
and analogously for \( \tilde{m}_{(p,i)} \), \( i = 0, \ldots, \#E_p - 1, p = 0, \ldots, \#\Lambda - 1 \). If \( r = m + 1 \), then the last wavelet masks \( m_\nu \) and \( \tilde{m}_\nu \), defined by the last rows of matrices \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \) are \( \mathcal{H} \)-symmetric with respect to the center \( c \). All wavelet masks have vanishing moments of order \( n \). If refinable functions \( \varphi, \tilde{\varphi} \) corresponding to refinable masks \( m_0, \tilde{m}_0 \) are in \( L_2(\mathbb{R}^d) \), then the resulting wavelet system is a dual wavelet frame.

The proof can be done analogously to the proof of Theorem.
wavelet masks are $m_{L}$ with coefficient support in $[0, 2]^3$. Let $\nu_{2}(f) = \sup \left\{ \nu \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} : \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2(1 + \|\xi\|)^{\nu} d\xi < \infty \right\}$. Below, the Sobolev smoothness exponent is calculated by Theorem 7.1 in [11].

1. Let $\mathcal{H}$ be a hexagonal symmetry group on $\mathbb{Z}^2$, namely,

$$\mathcal{H} = \left\{ \pm I_2, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

c = 0, $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The set of digits is $D(M) = \{ s_0 = (0,0), s_1 = (0,1), s_2 = (-1,0), s_3 = (-1,-1) \}$, $m = 4$. Let us construct an interpolatory refinable mask that is $\mathcal{H}$-symmetric with respect to the origin and obeys the sum rule of order $n = 3$. The digits are renumbered as $s_{0,0} = (0,0), s_{1,0} = (0,1), s_{1,1} = (-1,0), s_{1,2} = (-1,-1)$. And $\mathcal{H}_{0,0} = \mathcal{H}$, $\mathcal{E}_0 = \{I_2\}$.

$$\mathcal{H}_{1,0} = \left\{ \pm I_2, \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\},$$

$$\mathcal{E}_1 = \{E^{(0)}, E^{(1)}, E^{(2)}\} = \left\{ I_2, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\}.$$ According to Theorem 10 in [21], mask $\tilde{m}_0$ can be constructed as follows

$$m_0 :$$

$$\begin{pmatrix}
0 & 0 & 0 & -\frac{1}{64} & 0 & 0 & -\frac{1}{64} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{9}{64} & \frac{3}{64} & \frac{3}{64} & 0 \\
-\frac{1}{64} & 0 & \frac{9}{64} & \frac{3}{64} & \frac{3}{64} & 0 & -\frac{1}{64} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{64} & 0 & 0 & -\frac{1}{64} & 0 & 0 & 0
\end{pmatrix}$$

with coefficient support in $[-3,3]^2 \cap \mathbb{Z}^2$. Note that $\mu_{00} = \frac{1}{2}$. The corresponding refinable function $\varphi$ is in $L_2(\mathbb{R}^2)$ since $\nu_2(\varphi) \geq 1.76585$. Dual mask can be taken as $\tilde{m}_0 \equiv 1$. Then by Theorem 5 we get wavelet masks

$$m_{1,0} :$$

$$\begin{pmatrix}
0 & 0 & 0 & -\frac{1}{64} & 0 & 0 & -\frac{1}{64} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{9}{64} & \frac{3}{64} & \frac{3}{64} & 0 \\
\frac{1}{512} & 0 & -\frac{9}{512} & -\frac{3}{512} & -\frac{3}{512} & 0 & \frac{1}{512} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{512} & 0 & -\frac{9}{512} & -\frac{3}{512} & -\frac{3}{512} & 0 & \frac{1}{512}
\end{pmatrix},$$

with coefficient support in $[-3,3] \times [-4,4] \cap \mathbb{Z}^2$. $m_{1,1}(\xi) = m_{1,0}(E^{(1)} \xi), m_{1,2}(\xi) = m_{1,0}(E^{(2)} \xi).$ The dual wavelet masks are $\tilde{m}_{1,0}(\xi) = \frac{1}{4} e^{2\pi i \xi_2}, \tilde{m}_{1,1}(\xi) = \frac{1}{4} e^{-2\pi i \xi_1}, \tilde{m}_{1,2}(\xi) = \frac{1}{4} e^{-2\pi i (\xi_1 + \xi_2)}.$ Since the symmetry group is not abelian, the symmetrization step cannot be done here. Thus, we get mutually symmetric frame-like wavelet system providing approximation order 3. Now, using the lifting scheme, we improve this system to dual wavelet frames keeping the symmetry and approximation properties. Let $L_1(\xi) = -\frac{1}{8} - \frac{1}{8} e^{2\pi i \xi_1}, L_2(\xi) = -\frac{1}{8} - \frac{1}{8} e^{-2\pi i \xi_1}, L_3(\xi) = -\frac{1}{8} - \frac{1}{8} e^{-2\pi i (\xi_1 + \xi_2)}.$ Then the new dual wavelet masks are

$$\tilde{m}_{1,0}(\xi) = e^{2\pi i \xi_2} \left( 1 - \frac{9}{16} e^{2\pi i \xi_2} - \frac{9}{16} e^{-2\pi i \xi_2} + \frac{1}{16} e^{2\pi i \xi_2} + \frac{1}{16} e^{-2\pi i \xi_2} \right),$$

4. Examples

In this section we give several examples which illustrate the results of the paper. For a compactly supported distribution $f \in \mathcal{S}$, its Sobolev smoothness exponent is defined to be $\nu_{2}(f) = \sup \left\{ \nu \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} : \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2(1 + \|\xi\|)^{\nu} d\xi < \infty \right\}$.
\[ m_{1,1}(\xi) = m_{1,0}(E(1)^*\xi), m_{1,2}(\xi) = m_{1,0}(E(2)^*\xi). \] 

The new dual refinable mask is

\[
\tilde{m}_0 : \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{128} & 0 & 0 & 0 & \frac{1}{128} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{128} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{128} & 0 & -\frac{1}{128} & \frac{1}{128} & \frac{1}{128} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{128} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{128} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

with coefficient support in \([-4,4]^2 \cap \mathbb{Z}^2\). Since \(\tilde{\varphi}\) is in \(L_2(\mathbb{R}^d)\) (\(\nu_2(\varphi) \geq 0.1566\)), we get mutually symmetric dual wavelet frame in \(L_2(\mathbb{R}^d)\) providing approximation order 3.

2. 1. Let \(\mathcal{H}\) be a point symmetry group on \(\mathbb{Z}^2\), namely, \(\mathcal{H} = \{ \pm I_2 \}, c = (1/2,0), M = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \).

The set of digits is \(D(M) = \{ s_0 = (0,0), s_1 = (-1,0), s_2 = (1,0) \}, m = 3\). Let us construct a refinable mask that is \(\mathcal{H}\)-symmetric with respect to \(c\) and obey the sum rule of order \(n = 2\). The digits are renumbered as \(s_{0,0} = (-1,0), s_{1,0} = (0,0), s_{1,1} = (1,0)\). And \(\mathcal{H}_{0,0} = \mathcal{H}, \mathcal{E}_0 = \{ I_2 \}, \mathcal{H}_{1,0} = \{ I_2 \}, \mathcal{E}_1 = \mathcal{H}\). According to Theorem 10 in [21], mask \(m_0\) can be constructed as follows

\[
m_0 : \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{128} \\
0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{128} & 0 & 0 & 0
\end{pmatrix}
\]

with coefficient support in \([-1,2] \times [-2,2] \cap \mathbb{Z}^2\). The corresponding refinable function \(\varphi\) is in \(L_2(\mathbb{R}^2)\) since \(\nu_2(\varphi) \geq 0.776\). The utility dual mask can be taken as \(\tilde{m}_0\) satisfying \(D^3(1 - \sigma)(0) = 0\) for \(|\beta| < n\). Then by Algorithm 1 we get dual mask

\[
\tilde{m}_0 : \begin{pmatrix}
0 & 0 & 0 & \frac{1}{128} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{128} & \frac{7}{128} & \frac{1}{128} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{128} & -\frac{1}{128} & \frac{1}{128} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{128} & \frac{1}{128} & \frac{1}{128} \\
\frac{11}{128} & -\frac{11}{128} & \frac{1}{128} & 0 & 0 & 0 & 0 \\
\frac{11}{128} & -\frac{11}{128} & \frac{1}{128} & 0 & 0 & 0 & 0 \\
\frac{11}{128} & -\frac{11}{128} & \frac{1}{128} & 0 & 0 & 0 & 0 \\
\frac{11}{128} & -\frac{11}{128} & \frac{1}{128} & 0 & 0 & 0 & 0
\end{pmatrix}
\]

with coefficient support in \([-2,3] \times [-3,3] \cap \mathbb{Z}^2\). Note that \(\tilde{\varphi} \in L_2(\mathbb{R}^d)\) since \(\nu_2(\tilde{\varphi}) \geq 0.503\). Also, \(\tilde{m}_0\) obeys the sum rule of order 1. Wavelet function and dual wavelet functions can be constructed by the technique in Algorithm 1. Then wavelet functions have vanishing moments of order 1, dual wavelet functions have vanishing moments of order 2.

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