Rigidity of flag supermanifolds

Elizaveta Vishnyakova

Abstract

We prove that under certain assumptions a supermanifold of flags is rigid, this is its complex structure does not admit any non-trivial small deformation. Moreover under the same assumptions we show that a supermanifold of flags is unique non-split supermanifold with given retract.

1 Introduction

It is a classical result that any flag manifold is rigid, see [Bott]. In other words its complex structure does not possess any non-trivial small deformation. In general this statement is false for a flag supermanifold, see [BO, Va]. For instance the projective superspace $\mathbb{CP}^1|\mathbb{C}$ where $m \geq 4$, see [Va], and the super-grassmannian $\text{Gr}_{2[2,1]}$, see [BO], are not rigid. In [O4] it was proved that the super-grassmannian $\text{Gr}_{m|n,k|l}$ is rigid if $m, n, k, l$ satisfy the following conditions

$$0 < k < m, \quad 0 < l < n,$$

$$(k,l) \neq (1, n-1), (m-1, 1), (1, n-2), (m-2, 1), (2, n-1), (m-1, 2). \quad (1)$$

The idea of the paper [O4] is to compute 1-cohomology with values in the tangent sheaf showing their triviality. This implies the rigidity of the super-grassmannian in this case.

In this paper we compute 1-cohomology with values in the tangent sheaf $\mathcal{T}$ of a flag supermanifold showing their triviality. Therefore we prove that under certain conditions any flag supermanifold is rigid. We use the results of [O4] and the fact that a supermanifold of flags of length $n$ is a superbundle with base space a supergrassmannian and fiber a flag supermanifold of length $n-1$.

Acknowledgements: E. V. was partially supported by FAPEMIG, grant APQ-01999-18, and by Tomsk State University, Competitiveness Improvement Program.
2 Main definitions

2.1 Supermanifolds

We use the word “supermanifold” in the sense of Berezin and Leites, see [BL, L, M1] for detail. Throughout the paper, we will be interested in the complex-analytic version of the theory. More precisely, a complex-analytic superdomain of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space $\mathcal{U} = (\mathcal{U}_0, \mathcal{F}_{\mathcal{U}_0} \otimes \wedge (m))$, where $\mathcal{F}_{\mathcal{U}_0}$ is the sheaf of holomorphic functions on an open set $\mathcal{U}_0 \subset \mathbb{C}^n$ and $\wedge (m)$ is the exterior (or Grassmann) algebra over $\mathbb{C}$ with $m$ generators. A complex-analytic supermanifold of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ that is locally isomorphic to a complex superdomain of dimension $n|m$. An example of a complex-analytic supermanifold is the ringed space $(\mathcal{M}_0, \wedge \mathcal{E})$, where $\mathcal{M}_0$ is a complex-analytic manifold and $\mathcal{E}$ is a holomorphic locally free sheaf on $\mathcal{M}_0$. In this case $\dim \mathcal{M} = n|m$, where $n = \dim \mathcal{M}_0$ and $m$ is the rank of $\mathcal{E}$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ be a complex-analytic supermanifold and $\mathcal{J}_\mathcal{M} = (\mathcal{O}_\mathcal{M})_1 + (\mathcal{O}_\mathcal{M})_2$ be the subsheaf of ideals generated by odd elements in $\mathcal{O}_\mathcal{M}$. We set $\mathcal{F}_\mathcal{M} := \mathcal{O}_\mathcal{M}/\mathcal{J}_\mathcal{M}$. Then $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$ is a usual complex-analytic manifold. It is called the reduction or underlying space of $\mathcal{M}$. Usually we will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$ and $\mathcal{F}_{\mathcal{M}_0}$ instead of $\mathcal{F}_\mathcal{M}$. A morphism of supermanifolds is a morphism of the corresponding $\mathbb{Z}_2$-graded ringed spaces. If $f : \mathcal{M} \to \mathcal{N}$ is a morphism of supermanifolds, then we denote by $f_0$ the morphism of the underlying spaces $\mathcal{M}_0 \to \mathcal{N}_0$ and by $f^*$ the morphism of the structure sheaves $\mathcal{O}_\mathcal{N} \to (f_0)_*(\mathcal{O}_\mathcal{M})$.

Denote by $\mathcal{T}_\mathcal{M}$ the tangent sheaf or the sheaf of vector fields of $\mathcal{M}$. Since the sheaf $\mathcal{O}_\mathcal{M}$ is $\mathbb{Z}_2$-graded, the tangent sheaf $\mathcal{T}_\mathcal{M}$ is also $\mathbb{Z}_2$-graded. Furthermore we have the following filtration in the structure sheaf $\mathcal{O}_\mathcal{M}$:

$$\mathcal{O}_\mathcal{M} = \mathcal{J}_\mathcal{M}^0 \supset \mathcal{J}_\mathcal{M}^1 \supset \mathcal{J}_\mathcal{M}^2 \supset \ldots .$$

This filtration induces the following filtration in $\mathcal{T}_\mathcal{M}$

$$\mathcal{T}_\mathcal{M} = (\mathcal{T}_\mathcal{M})_{(-1)} \supset (\mathcal{T}_\mathcal{M})_{(0)} \supset (\mathcal{T}_\mathcal{M})_{(1)} \supset \cdots ,$$

where

$$(\mathcal{T}_\mathcal{M})_{(p)} = \{ v \in \mathcal{T}_\mathcal{M} \mid v(\mathcal{O}_\mathcal{M}) \subset \mathcal{J}_\mathcal{M}^p, \mathcal{v}(\mathcal{J}_\mathcal{M}) \subset \mathcal{J}_\mathcal{M}^{p+1} \} \text{ for } p \geq 0$$

Here and everywhere in the paper for simplicity we use the notation $\mathcal{U}_i := (U_i, \mathcal{O}_\mathcal{N}|_{U_i})$ for a chart on a supermanifold $\mathcal{N}$. We say that $\{ \mathcal{U}_i \}$ is an open covering of $\mathcal{N}$ if $\{ U_i \}$ is an open covering of $\mathcal{N}_0$.

Let $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ be complex-analytic supermanifolds.

**Definition 1** The quadruple $(\mathcal{M}, \mathcal{B}, \pi, \mathcal{S})$ is called a superbundle with total space $\mathcal{M}$, base space $\mathcal{B}$, projection $\pi : \mathcal{M} \to \mathcal{B}$ and with fiber $\mathcal{S}$, if $\pi : \mathcal{M} \to \mathcal{B}$ is locally
trivial. In other words there exists an open covering \( \{ U_i \} \) of the supermanifold \( B \) and isomorphisms \( \psi_i : \pi^{-1}(U_i) \to U_i \times S \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\pi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times S \\
\downarrow \pi & & \downarrow \text{pr}_{U_i} \\
U_i & \xrightarrow{} & U_i
\end{array}
\]

where \( \text{pr}_{U_i} \) is the projection onto the first factor.

Here we denote by \( \pi^{-1}(U_i) \) the following supermanifold \((\pi^{-1}(U_i), \mathcal{O}_M|_{\pi^{-1}(U_i)})\). Sometimes for simplicity we will denote the superbundle \((M, B, \pi, S)\) just by \( M \).

Let \( M \) be a superbundle.

**Definition 2** A global vector field \( v \in H^0(M_0, T_M) \) is called projectible with respect to \( \pi \) if there exists a vector field \( v' \in H^0(B_0, T_B) \) such that

\[
v(\pi^*(f)) = v'(f) \quad \text{for all} \quad f \in \mathcal{O}_B.
\]

The vector field \( v' \) is called the projection of \( v \). If \( v' = 0 \), the vector field \( v \) is called vertical. All vertical vector fields form a Lie subsuperalgebra in \( H^0(M_0, T_M) \).

The following theorem was proven in [B]:

**Theorem 3** Let \( \pi : M \to B \) be the projection of a superbundle with fiber \( S \). If \( \mathcal{H}^0(S_0, \mathcal{O}_S) \simeq \mathbb{C} \), then any vector field on \( M \) is projectible with respect to \( \pi \).

### 2.2 Split supermanifolds

A supermanifold \( M \) is called split if \( \mathcal{O}_M \simeq \bigwedge \mathcal{E} \) for a locally free sheaf \( \mathcal{E} \) on \( M_0 \). In this case \( \mathcal{O}_M \) is a \( \mathbb{Z} \)-graded sheaf. More precisely, \( \mathcal{O}_M = \bigoplus_p (\mathcal{O}_M)_p \), where \( (\mathcal{O}_M)_p \) is the image of \( \bigwedge^p \mathcal{E} \) by the isomorphism \( \bigwedge \mathcal{E} \to \mathcal{O}_M \). Moreover, \( \mathcal{O}_M \) is a sheaf of \( \mathcal{F}_M \)-modules, where as above \( \mathcal{F}_M \) is the structure sheaf of the underlying space \( M_0 \). Note that \( \mathcal{O}_M \simeq \bigwedge(\mathcal{O}_M)_1 \). If \( M \) is split, then the tangent sheaf \( T_M \) is also a locally free sheaf of \( \mathcal{F}_M \)-modules. Furthermore, the sheaf \( T_M \) is \( \mathbb{Z} \)-graded

\[
(T_M)_p = \{ v \in T_M \mid v((\mathcal{O}_M)_q) \subset (\mathcal{O}_M)_{p+q} \text{ for all } q \geq 0 \}, \quad p \geq -1. \tag{2}
\]

To any split supermanifold \( M \) we can assign the following exact sequence of sheaves, see [O2]

\[
0 \to (A_M)_p \xrightarrow{\alpha} (T_M)_p \xrightarrow{\beta} (C_M)_p \to 0, \tag{3}
\]

where

\[
(A_M)_p := (\mathcal{O}_M)_1^* \otimes \bigwedge^p (\mathcal{O}_M)_1 \quad \text{and} \quad (C_M)_p := \Theta_{M_0} \otimes \bigwedge^p (\mathcal{O}_M)_1.
\]
Here $\Theta_{M_0}$ is the tangent sheaf of the manifold $M_0$, the map $\alpha$ assigns the corresponding vector field from $(T_M)_p$ to any morphism $(\mathcal{O}_M)_1 \to \bigwedge^{p+1}(\mathcal{O}_M)_1$, and $\beta$ is the restriction of a vector field $v \in (T_M)_p$ to the subsheaf $\mathcal{F}_M$.

Split supermanifolds form a category. More precisely, objects of this category are all split supermanifolds $M$ with a fixed isomorphism $\mathcal{O}_M \cong \bigwedge E$ for a certain locally free sheaf $E$ on $M_0$, and morphisms are all morphisms of ringed spaces preserving the $\mathbb{Z}$-gradings. There is a functor gr from the category of supermanifolds to the category of split supermanifolds. Let us briefly describe this construction. Let $M$ be any supermanifold. As above, denote by $J_M \subset \mathcal{O}_M$ the subsheaf of ideals generated by odd elements of $\mathcal{O}_M$. Then by definition $\text{gr}_M$ is the split supermanifold with the structure sheaf

$$\mathcal{O}_{\text{gr}_M} = \bigoplus_{p \geq 0} (\mathcal{O}_{\text{gr}_M})_p,$$

$$(\mathcal{O}_{\text{gr}_M})_p := \mathcal{J}_M^p / \mathcal{J}_M^{p+1}, \quad \mathcal{J}_M^0 := \mathcal{O}_M.$$

In this case $(\mathcal{O}_{\text{gr}_M})_1$ is a locally free sheaf and there is a natural isomorphism of $\mathcal{O}_{\text{gr}_M}$ onto $\bigwedge (\mathcal{O}_{\text{gr}_M})_1$. If $\psi = (\psi_0, \psi^*): M \to N$ is a morphism of supermanifolds, then $\text{gr}(\psi) = (\psi_0, \text{gr}(\psi^*))$, where $\text{gr}(\psi^*): \mathcal{O}_N \to \mathcal{O}_M$ is defined by

$$\text{gr}(\psi^*)(f + \mathcal{J}_N^p) := \psi^*(f) + \mathcal{J}_M^p$$

for $f \in (\mathcal{J}_N)^{p-1}$.

Recall that by definition every morphism $\psi$ of supermanifolds is even and as a consequence $\psi^*$ sends $\mathcal{J}_N^p$ into $\mathcal{J}_M^p$.

If $T_M$ is the tangent sheaf of a supermanifold $M$ and $T_{\text{gr}_M}$ is the tangent sheaf of $\text{gr}_M$, then as above the tangent sheaf $T_{\text{gr}_M}$ is $\mathbb{Z}$-graded

$$T_{\text{gr}_M} = \bigoplus_{p \geq -1} (T_{\text{gr}_M})_p,$$

and

$$(T_{\text{gr}_M})_p \cong (T_M)_p / (T_M)_{p+1}$$

for $p \geq -1$, see [O1].

A split superbundle is a superbundle in the category of split supermanifolds. More precisely, it is a superbundle $(\mathcal{M}, \mathcal{B}, \pi, \mathcal{S})$, where $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ are split and $\pi$ preserves the $\mathbb{Z}$-gradings. If $\pi: \mathcal{M} \to \mathcal{B}$ is a superbundle with fiber $\mathcal{S}$ then $\text{gr} \pi: \text{gr} \mathcal{M} \to \text{gr} \mathcal{B}$ is a split superbundle with fiber $\text{gr} \mathcal{S}$.

### 2.3 Retract of a split superbundle and its tangent sheaf

In this section $(\mathcal{M}, \mathcal{B}, \pi, \mathcal{S})$ is a split superbundle, this is $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ are split supermanifolds with fixed $\mathbb{Z}$-gradings and $\pi^*: \mathcal{O}_B \to (\pi_0)_*(\mathcal{O}_M)$ preserves these $\mathbb{Z}$-gradings. To simplify notations we write in this section $\mathcal{O}$ instead of $\mathcal{O}_M$ and $T$ instead of $T_M$. 

---

[O1]: [O1] Reference

---

4
Denote by $J_B$ the sheaf of ideals in $\mathcal{O}$ generated by $\pi^*((\mathcal{O}_B)_1)$. By definition of a split superbundle $J_B$ is a sheaf of $\mathbb{Z}$-graded ideals. We have the following filtration in $\mathcal{O}$

$$\mathcal{O} = J_B^0 \supset J_B^1 \supset \ldots.$$  
(4)

Since $\mathcal{M}$ is split, we have $\mathcal{O} = \bigoplus_p \mathcal{O}_p$ and Filtration (4) gives rise to the following filtration in any $\mathcal{O}_p$

$$\mathcal{O}_p = \mathcal{O}_{p(0)} \supset \mathcal{O}_{p(1)} \supset \ldots \supset \mathcal{O}_{p(p)} \supset \mathcal{O}_{p(p+1)} = 0,$$  
(5)

where $\mathcal{O}_{p(q)} = J_B^q \cap \mathcal{O}_p$. We set

$$\hat{\mathcal{O}}_{pq} := \mathcal{O}_{p(q)} / \mathcal{O}_{p(q+1)}$$  
and

$$\hat{\mathcal{O}} := \bigoplus_{p,q \geq 0} \hat{\mathcal{O}}_{pq}.$$  

The sheaf $\hat{\mathcal{O}}$ is a sheaf of superalgebras with respect to the following multiplication:

$$(f + \mathcal{O}_{p(q+1)})(g + \mathcal{O}_{p'(q'+1)}) = fg + \mathcal{O}_{p+p',q+q'+1},$$  

where $f \in \mathcal{O}_{p(q)}$ and $g \in \mathcal{O}_{p'(q')}$.

The sheaves $\hat{\mathcal{O}}_{11}$ and $\hat{\mathcal{O}}_{10}$ are locally free. Moreover,

$$\hat{\mathcal{O}}_{pq} = \bigwedge^p \hat{\mathcal{O}}_{10} \otimes \bigwedge^q \hat{\mathcal{O}}_{11}.$$  

In particular the ringed space $\hat{\mathcal{M}} := (\mathcal{M}_B, \hat{\mathcal{O}})$ is a split supermanifold. Since, $\pi^*((\mathcal{O}_B)_p) \subset \mathcal{O}_{p(p)} = \hat{\mathcal{O}}_{pp}$, the morphism $\hat{\pi} : \hat{\mathcal{M}} \to B$ is defined, where $\hat{\pi}^*(f) = \pi^*(f)$ for $f \in \mathcal{O}_B$. The morphism $\hat{\pi} : \hat{\mathcal{M}} \to B$ preserves the fixed $\mathbb{Z}$-grading in $\mathcal{O}_B$ and the $\mathbb{Z}$-grading $\hat{\mathcal{O}}_p := \bigoplus \mathcal{O}_{pq}$ in $\hat{\mathcal{O}}$. Hence, $\hat{\mathcal{M}}$ is a split superbundle. We will call the superbundle $\hat{\pi} : \hat{\mathcal{M}} \to B$ the retract of the split superbundle $\pi : \mathcal{M} \to B$.

Further consider the tangent sheaf $\mathcal{T}$ of the split supermanifold $\mathcal{M}$. The filtration (4) induces the following filtration in $\mathcal{T}_p$

$$\mathcal{T}_p = \mathcal{T}_{p(-1)} \supset \mathcal{T}_{p(0)} \supset \ldots \supset \mathcal{T}_{p(q)} \supset \ldots \supset \mathcal{T}_{p(p+1)} \supset \mathcal{T}_{p(p+2)} = 0.$$  
(6)

where

$$\mathcal{T}_{p(q)} = \{ v \in \mathcal{T}_p \mid v(J_B) \subset J_B^{q+1}, \ v(\mathcal{O}) \subset J_B^q \}, \ q \geq -1.$$  

We put

$$\hat{\mathcal{T}} = \bigoplus_{p,q \geq -1} \hat{\mathcal{T}}_{pq}, \text{ where } \hat{\mathcal{T}}_{pq} = \mathcal{T}_{p(q)} / \mathcal{T}_{p(q+1)}.$$  

The sheaf $\hat{\mathcal{T}}$ possesses a Lie superalgebra structure that is induced by the Lie superalgebra structure on $\mathcal{T}$. Moreover, $\hat{\mathcal{T}}$ has a natural geometric interpretation. Denote by $Der\hat{\mathcal{O}}$ the sheaf of derivations of $\hat{\mathcal{O}}$. It is a sheaf of Lie superalgebras. Clearly,

$$Der\hat{\mathcal{O}} = \bigoplus_{p,q \geq -1} Der_{pq} \hat{\mathcal{O}},$$

5
where

\[ \text{Der}_{pq} \hat{\mathcal{O}} = \{ u \in \text{Der} \hat{\mathcal{O}} \mid u(\hat{\mathcal{O}}_{st}) \subset \hat{\mathcal{O}}_{s+t+q}, \forall s, t \in \mathbb{Z} \}, \quad p, q \geq -1, \]

**Lemma 4** We have \( \hat{T} \simeq \text{Der} \hat{\mathcal{O}} \) as sheaves of Lie superalgebras and \( \hat{T}_{pq} \simeq \text{Der}_{pq} \hat{\mathcal{O}} \).

Proof. Let us take \( u \in (\mathcal{T}_{pq})_x \), where \( x \in M_0 \). Since

\[ u((\mathcal{O}_{r(s)})_x) \subset (\mathcal{O}_{r+p, (s+q)})_x \quad \text{and} \quad u((\mathcal{O}_{r(s+1)})_x) \subset (\mathcal{O}_{r+p, (s+q+1)})_x, \]

we see that \( u \) determines the \( (p, q) \)-derivation \( \hat{u} : (\hat{\mathcal{O}}_{rs})_x \rightarrow (\hat{\mathcal{O}}_{r+p, s+q})_x \). Further, it is easy to see that \( \hat{u} = 0 \) if and only if \( u \in (\mathcal{T}_{pq+1})_x \). Therefore, the map

\[ \alpha_{pq} : \hat{T}_{pq} \rightarrow \text{Der}_{pq} \hat{\mathcal{O}}, \quad u + (\mathcal{T}_{pq+1})_x \mapsto \hat{u} \]

is injective. Clearly, locally we can find pre-images of basic elements. Hence, \( \alpha_{pq} \) is surjective. Moreover, by definitions of all structures, we see that

\[ \bigoplus_{pq} \alpha_{pq} : \bigoplus_{pq} \hat{T}_{pq} \rightarrow \bigoplus_{pq} \text{Der}_{pq} \hat{\mathcal{O}} \]

is a homomorphism of sheaves of Lie superalgebras. □

We will identify \( \hat{T} \) and \( \text{Der} \hat{\mathcal{O}} \) using the isomorphism \( \bigoplus \alpha_{pq} \) from Lemma 4.

Let us rewrite exact Sequence (3) for \( \hat{T}_{p} \). Recall that \( \hat{\mathcal{O}}_{pq} = \bigoplus_{q} \hat{\mathcal{O}}_{pq} \). We get

\[ 0 \rightarrow (\hat{\mathcal{O}}_{10} \oplus \hat{\mathcal{O}}_{11})^* \otimes (\hat{\mathcal{O}}_{10} \oplus \hat{\mathcal{O}}_{11})^* \rightarrow \hat{T}_{p} \rightarrow \Theta \otimes (\hat{\mathcal{O}}_{10} \oplus \hat{\mathcal{O}}_{11}) \rightarrow 0. \]

Here \( \Theta \) is the tangent sheaf of the manifold \( M_0 \). This exact sequence is a direct sum of the following exact sequences.

\[ 0 \rightarrow \mathcal{A}_{pq} \rightarrow \hat{T}_{pq} \rightarrow \mathcal{C}_{pq} \rightarrow 0, \quad (7) \]

where

\[ \mathcal{A}_{pq} := \hat{\mathcal{O}}_{10}^* \otimes \bigwedge^p \hat{\mathcal{O}}_{10} \otimes \bigwedge^q \hat{\mathcal{O}}_{11} + \hat{\mathcal{O}}_{11}^* \otimes \bigwedge^p \hat{\mathcal{O}}_{10} \otimes \bigwedge^q \hat{\mathcal{O}}_{11}, \]

\[ \mathcal{C}_{pq} := \Theta \otimes \bigwedge^p \hat{\mathcal{O}}_{10} \otimes \bigwedge^q \hat{\mathcal{O}}_{11}. \]

The sheaves \( \mathcal{A}_{pq} \) and \( \mathcal{C}_{pq} \) possess another description in terms of factor sheaves. More precisely, by definition we have

\[ \mathcal{A}_p = \mathcal{O}_1^* \otimes \bigwedge^p \mathcal{O}_1 \quad \text{and} \quad \mathcal{C}_p = \Theta \otimes \bigwedge^p \mathcal{O}_1, \]
These sheaves possess the following filtrations

\[ A_p = A_{p(-1)} \supset A_{p(0)} \supset \ldots, \quad C_p = C_{p(0)} \supset C_{p(1)} \supset \ldots, \]

where

\[ A_{p(q)} = \mathcal{O}_{10}^* \otimes \bigwedge^{p-q+1} \mathcal{O}_{1(0)} \otimes \bigwedge^{q} \mathcal{O}_{1(1)} + \mathcal{O}_{1}^* \otimes \bigwedge^{p-q} \mathcal{O}_{1(0)} \otimes \bigwedge^{q+1} \mathcal{O}_{1(1)} \]

and

\[ C_{p(q)} = \mathcal{O}_{1(0)} \otimes \bigwedge^{q} \mathcal{O}_{1(1)}. \]

Note that \( \mathcal{O}_{10}^* \subset \mathcal{O}_1^* \).

**Lemma 5** We have \( A_{pq} = A_{p(q)}/A_{p(q+1)} \) and \( C_{pq} = C_{p(q)}/C_{p(q+1)} \).

**Proof.** The idea of the proof is similar to the idea of the proof of Lemma 4. \( \square \)

### 3 Exact sequences associated to the tangent sheaf of a split superbundle

Let \( \pi : \mathcal{M} \to B \) be an arbitrary superbundle with fiber \( S \) and \( \text{gr} \pi : \text{gr} \mathcal{M} \to \text{gr} B \) be the corresponding split superbundle with fiber \( \text{gr} S \). Denote also by \( \hat{\pi} : \hat{\mathcal{M}} \to \hat{B} \) the retract of \( \text{gr} \pi : \text{gr} \mathcal{M} \to \text{gr} B \). From now on we fix the following notations

\[
\begin{align*}
\mathcal{O}, \quad \tilde{\mathcal{O}} &= \bigoplus_{p \geq 0} \mathcal{O}_p, \quad \hat{\tilde{\mathcal{O}}} = \bigoplus_{p, q \geq 0} \hat{\mathcal{O}}_{pq}, \quad \mathcal{O}_B, \quad \tilde{\mathcal{O}}_B = \mathcal{O}_B = \bigoplus_{p \geq 0} \mathcal{O}(\mathcal{O}_B)_p, \\
\mathcal{O}_S, \quad \hat{\tilde{\mathcal{O}}}_S = \hat{\mathcal{O}}_S = \bigoplus_{p \geq 0} (\hat{\mathcal{O}}_S)_p
\end{align*}
\]

for the structure sheaves of \( \mathcal{M}, \text{gr} \mathcal{M}, \hat{\mathcal{M}}, B, \text{gr} B = \hat{B}, S \) and \( \text{gr} S = \hat{S} \), respectively. To simplify notations sometimes we will denote the bundle projections \( \pi, \text{gr} \pi \) and \( \hat{\pi} \) simply by \( \pi \). From now on we also denote by

\[
\begin{align*}
\mathcal{T}, \quad \hat{\mathcal{T}} &= \bigoplus_{p \geq -1} \mathcal{T}_p, \quad \text{and} \quad \hat{\mathcal{T}} = \bigoplus_{p, q \geq -1} \mathcal{T}_{pq}, \\
\mathcal{T}_B, \quad \hat{\mathcal{T}}_B = \mathcal{T}_B = \bigoplus_{p \geq -1} (\mathcal{T}_B)_p, \\
\mathcal{T}_S, \quad \hat{\mathcal{T}}_S = \mathcal{T}_S = \bigoplus_{p \geq -1} (\mathcal{T}_S)_p
\end{align*}
\]

the tangent sheaves of \( \mathcal{M}, \text{gr} \mathcal{M}, \hat{\mathcal{M}}, B, \text{gr} B = \hat{B}, S \) and \( \text{gr} S = \hat{S} \), respectively. A local vector field \( v \in \mathcal{T} \) is called *vertical* if \( v(\pi^*(\mathcal{O}_B)) = 0 \). Clearly, the commutator of two vertical vector fields is again a vertical vector field. Denote by \( \mathcal{T}^\circ \subset \mathcal{T} \),
\[ \hat{T}_p^v \subset \hat{T}_p \text{ and } \hat{T}_{pq}^v \subset \hat{T}_{pq} \text{ the subsheaves of vertical vector fields, respectively.} \]

The sheaves \( T^v \) and \( \hat{T}_p^v \) possess the following filtrations:

\[ T^v = T^v_{(1)} \supset T^v_{(2)} \supset \cdots, \quad \hat{T}_p^v = \hat{T}_p^v_{(1)} \supset \hat{T}_p^v_{(2)} \supset \cdots, \]

where \( T^v_{(p)} := T^v \cap T^v_{(p)} \) and \( \hat{T}_p^v_{(q)} := \hat{T}_p^v \cap T^v_{(q+1)} \).

**Lemma 6** We have \( \hat{T}_p^v \simeq T^v_{(p)} / T^v_{(p+1)} \) and \( \hat{T}_{pq}^v \simeq \hat{T}_p^v_{(q)} / \hat{T}_p^v_{(q+1)} \).

Proof. Consider the natural map \( T^v_{(p)} \to \hat{T}_p^v \). It is surjective and its kernel is \( T^v_{(p+1)} \).

For \( \hat{T}_{pq}^v \) the argument is similar. \( \square \)

We put \( T^h = T / T^v \), \( \hat{T}_p^h = \hat{T}_p^v / \hat{T}_p^v_{(p)} \) and \( \hat{T}_{pq}^h = \hat{T}_{pq}^v / \hat{T}_{pq}^v_{(q)} \).

Denote also \( T^h_{(p)} = T_{(p)} / T^v_{(p)} \) and \( \hat{T}^h_{(p)} = \hat{T}_{(p)}^v / \hat{T}_{(p)}^v_{(q)} \).

Since we have the natural inclusions \( T_{(p+1)} \to T_{(p)} \) and \( T^v_{(p+1)} \to T^v_{(p)} \), we can define the sheaf morphism \( \Phi : T^h_{(p+1)} \to T^h_{(p)} \). Furthermore, the sheaf morphisms \( T^v_{(p)} \to \hat{T}_p^v \) and \( T^v_{(p+1)} \to \hat{T}_{pq}^v \) give rise to the morphism \( \Psi : T^h_{(p)} \to \hat{T}^h_{p} \). We will need the following lemmas:

**Lemma 7** The following diagram is commutative, lines and columns are exact:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \to & \hat{T}_p^v & \xrightarrow{\chi} & \hat{T}_p^h & \xrightarrow{\delta} & \hat{T}_h^v & \to & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \to & T^v_{(p)} & \xrightarrow{\theta} & T^v_{(p+1)} & \xrightarrow{\gamma} & T^v_{(p+1)} & \to & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \to & T^h_{(p+1)} & \xrightarrow{\lambda} & T^h_{(p+1)} & \to & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Proof. The exactness of all lines and first two columns follows from definitions and Lemma 6. Let us show the exactness of \( 0 \to T^h_{(p+1)} \xrightarrow{\Phi} T^h_{(p)} \xrightarrow{\Phi} \hat{T}^h_{(p)} \to 0 \) for example in the term \( T^h_{(p)} \). Let us take \( v \in T^h_{(p)} \) such that \( \Psi(v) = 0 \). We have to show that there exists \( w \in T^h_{(p+1)} \) such that \( \Phi(w) = v \). Using commutativity and exactness we get the following. There exists \( x \in T^h_{(p)} \) such that \( \gamma(x) = v \). Since \( \delta \circ \epsilon(x) = 0 \), there exists \( z \in \hat{T}_p^v \) such that \( \chi(z) = \epsilon(x) \). Further there exists \( t \in T^v_{(p)} \)
such that $\sigma(t) = z$. Consider $\tilde{x} = x - \theta(t)$. We have $\gamma(\tilde{x}) = \gamma(x) - \gamma(\theta(t)) = \gamma(x) = v$. Hence, $\tilde{x}$ is also a representant of $v$ in $\mathcal{T}_{(p)}$. Moreover, $\epsilon(\tilde{x}) = \epsilon(x) - \epsilon(\theta(t)) = 0$, therefore, $\exists t \in \mathcal{T}_{(p+1)}$ such that we can set $w := \lambda(r)$. This observation completes the proof. \hfill \square

Lemma 8 The following diagram is commutative, lines and columns are exact:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & \mathcal{T}_{pq}^v & \mathcal{T}_{pq}^h \\
\uparrow & \uparrow & \uparrow \\
0 & \mathcal{T}_{p(q)}^v & \mathcal{T}_{p(q)}^h \\
\uparrow & \uparrow & \uparrow \\
0 & \mathcal{T}_{p(q+1)}^v & \mathcal{T}_{p(q+1)}^h \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0 \\
\end{array}
$$

Proof. The proof is similar to the proof of Lemma 7. \hfill \square

We put

$$
A_{pq}^v = \hat{O}_{10}^* \otimes \bigwedge^q \hat{O}_{11} \otimes \bigwedge^{p-q+1} \hat{O}_{10}, \quad C_{pq}^v = \Theta^v \otimes \bigwedge^q \hat{O}_{11} \otimes \bigwedge^{p-q} \hat{O}_{10},
$$

where $\Theta^v$ is the sheaf of vertical vector fields on $M$ with respect to $\pi_0$.

Lemma 9 The following diagram is commutative, the lines and the columns are exact:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & \mathcal{C}_{pq}^v & \mathcal{C}_{pq}^h \\
\uparrow & \uparrow & \uparrow \\
0 & \mathcal{T}_{pq}^v & \mathcal{T}_{pq}^h \\
\uparrow & \uparrow & \uparrow \\
0 & \mathcal{A}_{pq}^v & \mathcal{A}_{pq}^h \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0 \\
\end{array}
$$

where

$$
A_{pq}^h = \hat{O}_{11}^* \otimes \bigwedge^{q+1} \hat{O}_{11} \otimes \bigwedge^{p-q} \hat{O}_{10}, \quad C_{pq}^h = \Theta/\Theta^v \otimes \bigwedge^q \hat{O}_{11} \otimes \bigwedge^{p-q} \hat{O}_{10}.
$$
Proof. By definition, $A_{pq} = A^h_{pq} + A^v_{pq}$. Further, from exactness of the sequence

$$0 \rightarrow \Theta^v \rightarrow \Theta \rightarrow \Theta^h \rightarrow 0$$

it follows the exactness of

$$0 \rightarrow C^v_{pq} \rightarrow C_{pq} \rightarrow C^h_{pq} \rightarrow 0.$$ 

The proof of exactness of

$$0 \rightarrow A^v_{pq} \rightarrow \widehat{T}^v_{pq} \rightarrow C^v_{pq} \rightarrow 0$$

is similar to the proof of exactness of (3) in \[O2]. The rest of the proof is similar to the proof of Lemma [7]. □

We set

$$\Theta = \lim_{\rightarrow} (\Theta_{pq})_{pq}.$$ 

for the bundle $\mathcal{M}$ we have the corresponding trivialization domain $U := (U_0, \mathcal{O}_{\mathcal{M}|U_0})$ of the bundle $gr\mathcal{M}$ and denote by $\psi_U$ a trivialization isomorphism

$$\tilde{\psi}_U : gr\pi^{-1}(U) \rightarrow U \times gr\mathcal{S}.$$ 

For the bundle $\widehat{\mathcal{M}}$ we have the corresponding trivialization domain $U$ of the bundle $\widehat{\mathcal{M}}$ and the following trivialization isomorphism:

$$\widehat{\psi}_U : \widehat{\pi}^{-1}(U) \rightarrow U \times \widehat{\mathcal{S}}.$$ 

We will identify $\widehat{\pi}^{-1}(U)$ with $U \times \widehat{\mathcal{S}}$ using this isomorphism. Denote by $\pi_S$ the projection $U \times \widehat{\mathcal{S}} \rightarrow \widehat{\mathcal{S}}$.

A (local) description of the sheaves introduced above is given in the following proposition.

**Proposition 10** Let $\pi = (\pi_0, \pi^*) : \mathcal{M} \rightarrow \mathcal{B}$ be a superbundle and $\widehat{\pi} = (\widehat{\pi}_0, \widehat{\pi}^*) : \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{B}}$ be the retract of $gr\mathcal{M}$. Then we have

1. $\mathcal{O}_{11} \simeq \pi^*((\mathcal{O}_B)_1)$;
2. $A^h_{pq} \simeq \bigwedge^{p-q} \mathcal{O}_{10} \otimes \pi^*((\mathcal{A}_B)_q)$ and $A^v_{pq}|_{\widehat{\pi}^{-1}(U)} \simeq \bigwedge^{q} \mathcal{O}_{11}|_U \otimes \pi_S^*(\mathcal{A}_S)_{p-q}$;
3. $C^h_{pq} \simeq \bigwedge^{p-q} \mathcal{O}_{10} \otimes \pi^*((\mathcal{C}_B)_q)$ and $C^v_{pq}|_{\widehat{\pi}^{-1}(U)} \simeq \bigwedge^{q} \mathcal{O}_{11}|_U \otimes \pi_S^*(\mathcal{C}_S)_{p-q}$;
4. $\widehat{T}^h_{pq} \simeq \bigwedge^{p-q} \mathcal{O}_{10} \otimes \pi^*((\mathcal{T}_B)_q)$ and $\widehat{T}^v_{pq}|_{\widehat{\pi}^{-1}(U)} \simeq \bigwedge^{q} \mathcal{O}_{11}|_U \otimes \pi_S^*(\mathcal{T}_S)_{p-q}$.

The tensor product $\otimes$ is taken over the corresponding sheaf of holomorphic functions.

**Proof.** The proof follows from the definitions. □
4 Supermanifold of flags

Denote by $F_k^m$ the manifold of flags of type $k = (k_0, k_1, \ldots, k_r)$ in $\mathbb{C}^m$, where $0 \leq k_r \leq \cdots \leq k_1 \leq k_0 = m$. Let us describe an atlas on $F_k^m$ that we will adapt for supercase. Let $\mathbb{C}^m \supset W_1 \supset \cdots \supset W_r$ be a flag of type $k$. We choose a basis $B_s$ in each $W_s$ and assume that $B_0 = (e_1, \ldots, e_m)$ is the standard basis in $\mathbb{C}^m$. Further, for any $s = 1, \ldots, r$, we define the matrix $X_s \in \text{Mat}_{k_{s-1}, k_s}(\mathbb{C})$ in the following way: the columns of $X_s$ are the coordinates of the vectors from $B_s$ with respect to the basis $B_{s-1}$. Since $\text{rk} \, X_s = k_s$, the matrix $X_s$ contains a non-degenerate minor of size $k_s$.

For each $s = 1, \ldots, r$ let us fix a $k_s$-tuple $I_s \subset \{1, \ldots, k_{s-1}\}$. We put $I = (I_1, \ldots, I_r)$. Denote by $U_I$ the set of flags from $F_k^m$ satisfying the following conditions: there exist bases $B_s$ such that $X_s$ contains the identity matrix of size $k_s$ in the lines with numbers from $I_s$. Clearly any flag from $U_I$ is uniquely determined by those elements of $X_s$ that are not contained in the identity matrix. Furthermore, any flag is contained in a certain $U_I$. The elements of $X_s$ that are not contained in the identity matrix are the coordinates of a flag from $U_I$ in the chart determined by $I$. Hence the local coordinates in $U_I$ are determined by $r$-tuple $(X_1, \ldots, X_r)$.

Rename $X_{I_s} := X_s$. If $J = (J_1, \ldots, J_s)$, where $J_s \subset \{1, \ldots, k_{s-1}\}$ and $|J_s| = k_s$, then the transition functions between the charts $U_I$ and $U_J$ are given by:

$$X_{J_s} = X_{I_s} C_{I_s, J_s}^{-1}, \quad X_{J_s} = C_{I_{s-1}, J_{s-1}} X_{I_s} C_{I_{s-1}, J_{s-1}}^{-1}, \quad s \geq 2,$$

where $C_{I_s, J_s}$ is the submatrix of $X_{I_s}$ formed by the lines with numbers from $J_1$ and $C_{I_{s-1}, J_{s-1}, J_s}$, $s \geq 2$, is the submatrix of $C_{I_{s-1}, J_{s-1}} X_{I_s}$ formed by lines with numbers from $J_s$.

Let us give a similar description of a classical flag supermanifold in terms of atlases and local coordinates, see also [V1][V2]. Let us take $m, n \in \mathbb{N}$ and let

$$k = (k_0, k_1, \ldots, k_r) \quad \text{and} \quad l = (l_0, l_1, \ldots, l_r)$$

be two $(r+1)$-tuples such that

$$0 \leq k_r \leq \ldots \leq k_1 \leq k_0 = m, \quad 0 \leq l_r \leq \ldots \leq l_1 \leq l_0 = n, \quad 0 < k_r + l_r < \ldots < k_1 + l_1 < k_0 + l_0 = m + n.$$

Let us define the supermanifold $F_k^{m|n}$ of flags of type $(k|l)$ in the superspace $V = \mathbb{C}^{m|n}$. The underlying manifold of $F_k^{m|n}$ is the product $F_k^m \times F_l^n$ of two manifolds of flags of type $k$ and $l$ in $\mathbb{C}^m = V_0$ and $\mathbb{C}^n = V_1$.

For each $s = 1, \ldots, r$ let us fix $k_s$- and $l_s$-tuples of numbers $I_{s0} \subset \{1, \ldots, k_{s-1}\}$ and $I_{s1} \subset \{1, \ldots, k_{s-1}\}$. We put $I_s = (I_{s0}, I_{s1})$ and $I = (I_1, \ldots, I_r)$. Our goal
now is to construct a superdomain $\mathcal{W}_I$. To each $I_s$ let us assign a matrix of size $(k_{s-1} + l_{s-1}) \times (k_s + l_s)$

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ H_s & Y_s \end{pmatrix}, \quad s = 1, \ldots, r. \quad (11)$$

Assume that the identity matrix $E_{k_s+l_s}$ is contained in the lines of $Z_{I_s}$ with numbers $i \in I_{s0}$ and $k_{s-1} + j, \ j \in I_{s1}$. Here $X_s \in \text{Mat}_{k_{s-1}+k_s}(\mathbb{C})$, $Y_s \in \text{Mat}_{l_{s-1}+l_s}(\mathbb{C})$, where $\text{Mat}_{a,b}(\mathbb{C})$ is the space of matrices of size $a \times b$ over $\mathbb{C}$. By definition, the entries of $X_s$ and $Y_s$, $s = 1, \ldots, r$, that are not contained in the identity matrix form the even coordinate system of $\mathcal{W}_I$. The non-zero entries of $\Xi_s$ and $H_s$ form the odd coordinate system of $\mathcal{W}_I$.

Thus we have defined a set of superdomains on $F^m_k \times F^n_l$ indexed by $I$. Note that the reductions of these superdomains cover $F^m_k \times F^n_l$. The local coordinates of each superdomain are determined by the $r$-tuple of matrices $(Z_{I_1}, \ldots, Z_{I_r})$. Let us define the transition functions between two superdomains corresponding to $I = (I_s)$ and $J = (J_s)$ by the following formulas:

$$Z_{J_s} = Z_{I_s}C_{I_s,J_s}^{-1}, \quad Z_{J_s} = C_{I_{s-1}J_{s-1}}Z_{I_s}C_{I_s,J_s}^{-1}, \quad s \geq 2, \quad (12)$$

where $C_{I_s,J_s}$ is the submatrix of $Z_{I_s}$ that consists of the lines with numbers from $J_1$, and $C_{I_s,J_s}$, $s \geq 2$, is the submatrix of $C_{I_{s-1}J_{s-1}}Z_{I_s}$ that consists of the lines with numbers from $J_s$. Gluing the superdomains $\mathcal{W}_I$, we define the supermanifold of flags $F^{m|n}_{k|l}$. In the case $r = 1$ this supermanifold is called a super-grassmannian. In the literature the notation $\text{Gr}_{m|n,k|l}$ is sometimes used.

The supermanifold $F^{m|n}_{k|l}$ is $GL_{m|n}(\mathbb{C})$-homogeneous. The action is given by

$$(L, (Z_{I_1}, \ldots, Z_{I_r})) \mapsto (\tilde{Z}_{I_1}, \ldots, \tilde{Z}_{I_r}),$$

$$\tilde{Z}_{I_s} = LZ_{I_s}C_{I_s}^{-1}, \quad \tilde{Z}_{I_s} = C_{I_{s-1}}Z_{I_s}C_{I_s}^{-1} \quad (13).$$

Here $L$ is a coordinate matrix of $GL_{m|n}(\mathbb{C})$, $C_1$ is the invertible submatrix of $LZ_{I_1}$ that consists of the lines with numbers from $J_1$, $C_s$, $s \geq 2$, is the invertible submatrix of $C_{I_{s-1}}Z_{I_s}$ that consists of the lines with numbers from $J_s$. If $0 < k_r < \ldots < k_1 < m, \ 0 < l_r < \ldots < l_1 < n$ we will say that the flag supermanifold $F^{m|n}_{k|l}$ has generic type.

5 Retract of the superbundle $F^{m|n}_{k|l}$

Let $\mathcal{G}$ be a Lie supergroup, $\mathcal{H}^1$ and $\mathcal{H}^2$ be two Lie subsupergroups of $\mathcal{G}$ such that $\mathcal{H}^2$ is a Lie subsupergroup of $\mathcal{H}^1$. (More information about Lie supergroups can be found for instance in [V3, V4].) Then the homogeneous superspace $\mathcal{M} = \mathcal{G}/\mathcal{H}^2$ is
a superbundle with base $\mathcal{B} = \mathcal{G}/\mathcal{H}^1$ and fiber $\mathcal{F} = \mathcal{H}^1/\mathcal{H}^2$. In \cite{V3} it was proved that

$$\text{gr} \mathcal{M} = \text{gr} \mathcal{G}/\text{gr} \mathcal{H}^2, \quad \text{gr} \mathcal{B} = \text{gr} \mathcal{G}/\text{gr} \mathcal{H}^1.$$ 

Moreover, $\text{gr} \mathcal{H}^2$ is a Lie subsupergroup of $\text{gr} \mathcal{H}^1$. The superbundle $\tilde{\pi}: \text{gr} \mathcal{G}/\text{gr} \mathcal{H}^2 \rightarrow \text{gr} \mathcal{G}/\text{gr} \mathcal{H}^1$, which we will also denote by $\tilde{\pi}: \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{B}}$, is the split superbundle corresponding to $\mathcal{M} \rightarrow \mathcal{B}$, see Subsection 2.3. Denote by $\pi: \hat{\mathcal{M}} \rightarrow \hat{\mathcal{B}}$ the retract of $\tilde{\pi}: \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{B}}$. As above we denote by $\mathcal{H}^1_0$ and $\mathcal{H}^2_0$ the underlying spaces of $\mathcal{H}^1$ and $\mathcal{H}^2$, respectively.

Let $G$ be a Lie group and $H$ be its Lie subgroup. Consider a locally free sheaf $\mathcal{E}$ on the homogeneous manifold $G/H$ and denote by $\mathcal{E}$ the corresponding vector bundle. (Recall that there is a one-to-one correspondence between locally free sheaves on $G/H$ and vector bundles over $G/H$.) Assume that $\mathcal{E}$ is a homogeneous vector bundle, see \cite{A}. In this case sometimes we will call $\mathcal{E}$ a homogeneous locally free sheaf. Recall that there is a bijection between homogeneous bundles $\mathcal{E}$ over $G/H$ and representations of $H$ in the fiber $\mathcal{E}_{eH}$, see \cite{A}. For simplicity sometimes we will write: the representation of $H$ corresponding to the homogeneous locally free sheaf $\mathcal{E}$ meaning this correspondence.

Denote by $\alpha$ the representation of the Lie group $\mathcal{H}^1_0$ corresponding to the homogeneous locally free sheaf $(\mathcal{O}_B)_1$ and by $\beta$ the representation of $\mathcal{H}^2_0$ corresponding to the homogeneous locally free sheaf $\hat{\mathcal{O}}_1$. (As above we denote by $\hat{\mathcal{O}}$ and by $\hat{\mathcal{O}}$ the structure sheaf of $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}$, respectively.) Recall that

$$\hat{\mathcal{O}}_1 = \hat{\mathcal{O}}_{11} + \hat{\mathcal{O}}_{10} = \pi^*((\mathcal{O}_B)_1) + (\hat{\mathcal{O}})_1/\pi^*((\mathcal{O}_B)_1).$$

Hence the locally free sheaf $\hat{\mathcal{O}}_1$ corresponds to the representation

$$\alpha|_{\mathcal{H}^2_0} + \beta/(\alpha|_{\mathcal{H}^2_0})$$

of Lie group $\mathcal{H}^2_0$. In particular, $\hat{\mathcal{M}}$ is a homogeneous supermanifold. Note that a split supermanifold is homogeneous if and only if the corresponding bundle is homogeneous, see \cite{V3}.

Let $\mathcal{M} := \mathbb{F}_{k|l}^{m|n}$ and $r \geq 2$. The flag supermanifold $\mathcal{M}$ is a superbundle with base $\mathcal{B} := \mathbb{F}_{k|l}^{m|n}$ and fiber $\mathcal{S} := \mathbb{F}_{k'|l'}^{k|l}$, where

$$k' = (k_1, \ldots, k_r) \quad \text{and} \quad l' = (l_1, \ldots, l_r).$$

In coordinates (11) the bundle projection $\pi: \mathcal{M} \rightarrow \mathcal{B}$ is given by

$$(Z_{l_1}, \ldots, Z_{l_r}) \mapsto (Z_{l_1}).$$

Denote by $\tilde{\pi}: \hat{\mathcal{M}} \rightarrow \hat{\mathcal{B}}$ the corresponding to $\mathbb{F}_{k|l}^{m|n}$ split superbundle with fiber $\hat{\mathcal{S}}$, see Subsection 2.3. Denote also by $\pi: \hat{\mathcal{M}} \rightarrow \hat{\mathcal{B}}$ the retract of $\tilde{\pi}: \hat{\mathcal{M}} \rightarrow \hat{\mathcal{B}}$. Consider the superdomain $Z_1$ in $\mathbb{F}_{k|l}^{m|n}$ corresponding to

$$I_{s\bar{0}} = (k_{a-1} - k_a + 1, \ldots, k_{a-1}), \quad I_{s\bar{1}} = (l_{a-1} - l_a + 1, \ldots, l_{a-1}).$$
where $s = 1, \ldots, r$, see Section 4. Denote by $x$ the origin of $Z_f$. We also denote by $P$ and by $p$ the underlying Lie group and Lie superalgebra of the super-stabilizer of $x$ for the action (13) of $GL_{m,n}(\mathbb{C})$. Let $R$ be the reductive part of $P$. A direct calculation shows that $R$ has the following form

$$R = \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & \vdots \\ 0 & \cdots & \ddots \end{pmatrix} \times \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & \vdots \\ 0 & \cdots & \ddots \end{pmatrix} \right\},$$

(17)

where $A_i \in GL_{k_i-1-k_i}(\mathbb{C})$ and $B_i \in GL_{l_i-1-l_i}(\mathbb{C})$, where $k_{r+1} = l_{r+1} := 0$ and $i = 1, \ldots, r+1$. Denote by $\rho_i$ the standard representations of the group $GL_{k_i-1-k_i}(\mathbb{C})$ and by $\sigma_i$ the standard representations of the Lie group $GL_{l_i-1-l_i}(\mathbb{C})$.

The sheaf $\mathcal{O}_1$ is a homogeneous locally free sheaf on $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})/P$. Let us compute the corresponding representation $\vartheta$ of $P$. The fiber over $x$ of the corresponding to $\mathcal{O}_1$ vector bundle is isomorphic to $(\mathfrak{gl}_{m,n}(\mathbb{C})_1/p_1)^*$ as $P$-modules, see [V3] for details. A direct computation shows that

$$p_1 = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C \in \text{Mat}_{m,n}(\mathbb{C}), \; D \in \text{Mat}_{n,m}(\mathbb{C}) \right\} \subset \mathfrak{gl}_{m,n}(\mathbb{C})_1,$$

where

$$C = \begin{pmatrix} C_1 & 0 & 0 \\ * & C_2 & \vdots \\ * & \cdots & \ddots \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 & 0 \\ * & D_2 & \vdots \\ * & \cdots & \ddots \end{pmatrix},$$

where $C_i \in \text{Mat}_{k_i-1-k_i,l_i-1-k_i}(\mathbb{C})$ and $D_i \in \text{Mat}_{l_i-1-k_i,k_i-1-k_i}(\mathbb{C})$.

Denote by $\varphi$ the representation of $P$ corresponding to the locally free sheaf $\mathcal{O}_{11}$ and by $\psi$ the representation of $P$ corresponding to the locally free sheaf $\mathcal{O}_{10}$.

**Lemma 11** We have

$$\varphi|_R = \sum_{i>1} \rho_i^* \otimes \sigma_i + \sum_{i>1} \sigma_i^* \otimes \rho_i, \quad \psi|_R = \sum_{1<i<j} \rho_i^* \otimes \sigma_j + \sum_{1<i<j} \sigma_i^* \otimes \rho_j,$$

$$\vartheta|_R = \varphi|_R + \psi|_R = \sum_{i<j} \rho_i^* \otimes \sigma_j + \sum_{i<j} \sigma_i^* \otimes \rho_j.$$

**Proof.** Let us compute the representation $\vartheta^*|_R$ of $R$ in $\mathfrak{gl}_{m,n}(\mathbb{C})_1/p_1$ for example for $r = 1$. We identify the vector space $\mathfrak{gl}_{m,n}(\mathbb{C})_1/p_1$ with

$$p_1 = \left\{ \begin{pmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{pmatrix} \mid \tilde{C} \in \text{Mat}_{m,n}(\mathbb{C}), \; \tilde{D} \in \text{Mat}_{n,m}(\mathbb{C}) \right\} \subset \mathfrak{gl}_{m,n}(\mathbb{C})_1,$$

where

$$\tilde{C} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{D} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix},$$

14
\[ M \in \text{Mat}_{k_0-k_1,l_1-l_2}(\mathbb{C}) \text{ and } N \in \text{Mat}_{l_0-l_1,k_1-k_2}(\mathbb{C}). \] We have

\[
\begin{pmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & B_1 & 0 \\
0 & 0 & 0 & B_2
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & M \\
0 & 0 & 0 & 0 \\
0 & 0 & N & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 & 0 & A_1M - MB_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & N - NA_2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Hence, \( \theta^*|R = \rho_1 \otimes \sigma^*_1 + \sigma_1 \otimes \rho^*_2 \). For \( r \geq 2 \) the proof is similar. To obtain \( \varphi|R \) and \( \psi_R \) we use Formula (14). \( \square \)

Recall that we denoted by \( \Theta \) and by \( \Theta^v \) the sheaf of vector fields and the sheaf of vertical vector fields on the manifold \( M_0 \), respectively. Denote by \( \tau, \tau^v \) and \( \tau^h \) the representation of \( P \) corresponding to the sheaves \( \Theta, \Theta^v \) and \( \Theta^h = \Theta/\Theta^v \), respectively. (Recall that we have a natural representation of \( P \) in the fiber over the origin \( x \) of \( Z_l). \)

**Lemma 12** We have

\[
\tau|R = \sum_{i<j} (\rho_i \otimes \rho_j^* + \sigma_i \otimes \sigma_j^*), \quad \tau^v|R = \sum_{1<i<j} (\rho_i \otimes \rho_j^* + \sigma_i \otimes \sigma_j^*), \\
\tau^h|R = \sum_{1<i<j} (\rho_i \otimes \rho_j^* + \sigma_i \otimes \sigma_j^*).
\]

**Proof.** The representation \( \tau|R \) is the isotropy representation. Recall that the isotropy representation is isomorphic to the natural representation of \( P \) in the vector space \( \mathfrak{gl}_{m|n}(\mathbb{C})/p_0 \). Let us compute this representation in the case \( r = 2 \). We identify \( \mathfrak{gl}_{m|n}(\mathbb{C})/p_0 \) with

\[
\left\{ \begin{pmatrix} \ast & U_{12} & U_{13} \\ \ast & \ast & U_{23} \\ \ast & \ast & \ast \end{pmatrix} \times \begin{pmatrix} \ast & V_{12} & V_{13} \\ \ast & \ast & V_{23} \\ \ast & \ast & \ast \end{pmatrix} \right\},
\]

where \( U_{ij} \in \text{Mat}_{k_i-k_{i-1},k_{j-1}-k_j}(\mathbb{C}) \) and \( V_{ij} \in \text{Mat}_{l_i-l_{i-1},l_{j-1}-l_j}(\mathbb{C}) \). We have

\[
\begin{pmatrix}
A_1 & 0 & 0 \\
0 & A_2 & 0 \\
0 & 0 & A_3
\end{pmatrix} \begin{pmatrix}
\ast & U_{12} & U_{13} \\ \ast & \ast & U_{23} \\ \ast & \ast & \ast
\end{pmatrix} = 
\begin{pmatrix}
\ast & A_1U_{12} - U_{12}A_2 & A_1U_{13} - U_{13}A_3 \\
\ast & \ast & A_2U_{23} - U_{23}A_3 \\
\ast & \ast & \ast
\end{pmatrix},
\]

\[
\begin{pmatrix}
B_1 & 0 & 0 \\
0 & B_2 & 0 \\
0 & 0 & B_3
\end{pmatrix} \begin{pmatrix}
\ast & V_{12} & V_{13} \\ \ast & \ast & V_{23} \\ \ast & \ast & \ast
\end{pmatrix} = 
\begin{pmatrix}
\ast & A_1V_{12} - V_{12}A_2 & A_1V_{13} - V_{13}A_3 \\
\ast & \ast & A_2V_{23} - V_{23}A_3 \\
\ast & \ast & \ast
\end{pmatrix}.
\]

15
Therefore
\[\tau|_R = \sum_{i<j}(\rho_i \otimes \rho^*_j + \sigma_i \otimes \sigma^*_j),\]

For \(\tau^v|_R\) and \(\tau^h|_R\) we respectively have
\[\tau^v|_R = \rho_2 \otimes \rho^*_1 + \sigma_2 \otimes \sigma^*_3, \quad \tau^h|_R = \rho_1 \otimes \rho^*_3 + \rho_3 \otimes \rho^*_1 + \sigma_1 \otimes \sigma^*_2 + \sigma_2 \otimes \sigma^*_3.\]

For \(r \geq 2\) the proof is similar. □

6 Vector fields on the retract of flag supermanifold

Recall that \(\mathcal{M} = F_{k|l}^{m|n}\), where \(r \geq 2\), is a superbundle with base space \(\mathcal{B} = F_{k|l}^{m|n}\) and fiber \(\mathcal{S} = F_{k'|l'}^{k|l}\), where \(k', l'\) are defined in (15). As above we denote by \(\tilde{\pi} : \mathcal{M} \to \tilde{\mathcal{B}}\) the corresponding to \(\mathcal{M}\) split superbundle with fiber \(gr\, \mathcal{S}\), see Section 5. The aim of this section is to compute the Lie superalgebra of holomorphic vector fields on the retract \(\mathcal{M}\) of the flag supermanifold \(\mathcal{M}\) for \(r \geq 2\).

In Section 2.2 we defined the functor \(gr\). Recall that \(Gr_{m|n,k|l}\) is the super-grassmannian of type \((k|l)\). We denote the Lie superalgebra of holomorphic vector fields on \(gr\, \mathcal{G}_{m|n,k|l}\) by \(v(gr\, \mathcal{G}_{m|n,k|l})\). Consider the following Cartan subalgebra in the Lie algebra \(gl_{m|n}(\mathbb{C})_0\)
\[\mathfrak{h} := \{\text{diag}(\mu_1, \ldots, \mu_m)\} \oplus \{\text{diag}(\lambda_1, \ldots, \lambda_n)\}.\]

The Lie superalgebra \(v(gr\, \mathcal{G}_{m|n,k|l})\) was computed in [OS1].

Theorem 13 [OS1, Theorem 4, Lemma 3] Let \(0 < k < m\) and \(0 < l < n\). Then
\[v(gr\, \mathcal{G}_{m|n,k|l}) = v(gr\, \mathcal{G}_{m|n,k|l})_{-1} \oplus v(gr\, \mathcal{G}_{m|n,k|l})_0,\]

where
\[v(gr\, \mathcal{G}_{m|n,k|l})_{-1} \simeq gl_{m|n}(\mathbb{C})_1\]
as \(gl_{m|n}(\mathbb{C})_0\)-modules. For \(v(gr\, \mathcal{G}_{m|n,k|l})_0\) we have the following possibilities.

1. if \(v(gr\, \mathcal{G}_{m|n,k|l})_0 \simeq gl_{m|n}(\mathbb{C})_0\) (as Lie algebras) for (1) \(1 < k < m - 1\) and \(1 < l < n - 1\); (2) \(k = 1\) and \(l < n - 1\); (3) \(k = m - 1\) and \(l > 1\); (4) \(k < m - 1\) and \(l = 1\); (5) \(k > 1\) and \(l = n - 1\).

2. if \((1)\ k = n - l = 1\ and \(m - k > 1\ or\ (2)\ k = n - l = 1\ and \(l > 1\), then
\[v(gr\, \mathcal{G}_{m|n,k|l})_0 \simeq gl_{m|n}(\mathbb{C})_0 \oplus \mathfrak{k}\]
(as \(gl_{m|n}(\mathbb{C})_0\)-modules), where \(\mathfrak{k}\) is the irreducible \(gl_{m|n}(\mathbb{C})_0\)-module with the highest weight \(-\mu_{m-1} - \mu_m + \lambda_1 + \lambda_2;\)
3. if (1) \( l = m - k = 1 \) and \( n - l > 1 \), or (2) \( l = m - k = 1 \) and \( k > 1 \), then
\[
\mathfrak{v}(\text{gr} \text{Gr}_{m|n,k|l})_0 \simeq \mathfrak{gl}_{m|n}(\mathbb{C})_0 \oplus \mathfrak{k}
\]
(as \( \mathfrak{gl}_{m|n}(\mathbb{C})_0 \)-modules), where \( \mathfrak{k} \) is the irreducible \( \mathfrak{gl}_{m|n}(\mathbb{C})_0 \)-module with the highest weight \( \mu_1 + \mu_2 - \lambda_{n-1} - \lambda_n \);

4. if \( m = n = 2, \ k = l > 0 \), then
\[
\mathfrak{v}(\text{gr} \text{Gr}_{m|n,k|l})_0 \simeq \mathfrak{gl}_{m|n}(\mathbb{C})_0 \oplus \mathfrak{k} \oplus \mathfrak{k}'
\]
(as \( \mathfrak{gl}_{m|n}(\mathbb{C})_0 \)-modules), where \( \mathfrak{k}, \mathfrak{k}' \) are the irreducible \( \mathfrak{gl}_{m|n}(\mathbb{C})_0 \)-modules with the highest weights \(-\mu_1 - \mu_2 + \lambda_1 + \lambda_2 \) and \( \mu_1 + \mu_2 - \lambda_1 - \lambda_2 \) respectively;

Note that in [OS1] Theorem 4 there is a mistake in the statement. To obtain the result of Theorem 13 we need to use [OS1] Theorem 4 and [OS1] Lemma 3. Further, the case \( k = 0 \), when the super-grassmannian is split, was considered by A. Onishchik and A. Serov. A complete description of the Lie superalgebras of holomorphic vector fields on super-grassmannians can be found in [V1]. The Lie superalgebra of holomorphic vector fields on a flag supermanifold \( \mathcal{M} = \text{F}^{|m|n}_{k|l} \) of length \( r > 1 \) was computed in generic case in [V1]. Using the method developed in [V1] we can calculate the Lie superalgebra of holomorphic vector fields on \( \mathcal{M} = \text{gr} \text{F}^{|m|n}_{k|l} \). Since this computation is similar we will omit some details.

In what follows we will consider flag supermanifolds under the following conditions on the flag type:
\[
0 < k_r < \cdots < k_1 < k_0 = m \quad \text{and} \quad 0 < l_r < \cdots < l_1 < l_0 = n \quad (18)
\]
and
\[
(k_r, l_r) \neq (1, l_{r-1} - 1), (k_r - 1, 1), (1, l_{r-1} - 2),
(k_{r-1} - 2, 1), (2, l_{r-1} - 1), (k_r - 1, 2). \quad (19)
\]

Assumption (18) is related to the fact that under this conditions \( \mathfrak{v}(\mathcal{M}) \simeq \mathfrak{gl}_{m|n}(\mathbb{C}) \), see [V1]. We assume (19) due to our use of induction and results of [OS1]. Indeed, results [OS1] were obtained under conditions (19) for a super-grassmannian.

Denote by \( \mathfrak{v}(\tilde{\mathcal{M}}) \) the Lie superalgebra of holomorphic vector fields on \( \tilde{\mathcal{M}} \).

**Theorem 14** Assume that \( r > 1 \) and \( \text{F}^{|m|n}_{k|l} \) satisfies the conditions (18) and (19). Then \( \mathfrak{v}(\tilde{\mathcal{M}}) = \mathfrak{v}(\tilde{\mathcal{M}})_{-1} \oplus \mathfrak{v}(\tilde{\mathcal{M}})_0 \), where
1. \( \mathfrak{v}(\tilde{\mathcal{M}})_{-1} \simeq \mathfrak{gl}_{m|n}(\mathbb{C})_1 \) (as \( \mathfrak{gl}_{m|n}(\mathbb{C})_0 \)-modules);
2. \( \mathfrak{v}(\tilde{\mathcal{M}})_0 \simeq \mathfrak{gl}_{m|n}(\mathbb{C})_0 \) (as Lie algebras).

**Proof.** In [V3] it was shown that \( H^0(\mathcal{M}_0, \tilde{\mathcal{O}}) \simeq \mathbb{C} \) if (18) hold true. Hence by Theorem 4 any vector field on \( \mathcal{M} \) is projectible with respect to \( \tilde{\pi} \). It follows that we have a homomorphism \( \Pi : \mathfrak{v}(\mathcal{M}) \to \mathfrak{v}(\tilde{\mathcal{O}}) \) of Lie superalgebras. The idea of [V1] was to compute the kernel and the image of \( \Pi \).
Denote by $W = (\tilde{\pi}_0)_* T_v$ the direct image of the sheaf of vertical vector fields on $\tilde{M}$. It is clear that $\text{Ker } \Pi = H^0(M_0, W)$. The sheaf $W$ possesses the following filtration:

$$W = W(0) \supset W(1) \supset \ldots,$$

where $W(p) := (J^B)^p W$ and $J^B$ is the ideal in $\hat{O}$ generated by odd elements in $\tilde{\pi}^*(\tilde{O}_B)$. We put $\tilde{W}_p := W(p)/W(p+1)$. The sheaves $\tilde{W}_p$ are homogeneous locally free sheaves of $\mathcal{F}_{B_0}$-modules, see [V1] for details. The 0-cohomology group of $\tilde{W}_p$ can be computed using the Borel - Weil - Bott Theorem. Denote by $P_B$ the stabilizer of the origin of the chart $Z_{I_1}$, see (16) and below. A direct computation shows that the dominant highest weight of the representation of $P_B$ corresponding to $W_0$ is $0$ ($\times 2$).

Therefore by the Borel - Weil - Bott Theorem $H^0(B_0, \tilde{W}_0) = \mathbb{C}$. Similarly to [V1] we can prove that $H^0(B_0, W(0)) = 0$. The idea here is to use [V1, Lemma 2] and the fact that $H^0(B_0, W(0))$ is an ideal in the Lie superalgebra of vector fields on $\tilde{M}$. Therefore the homomorphism $\Pi : v(\tilde{M}) \to v(B)$ is injective.

Let us show that $\Pi$ is surjective. By Theorem 13 on the base space we have $v(B) \simeq \mathfrak{gl}_{m|n}(\mathbb{C})$ (as $\mathfrak{gl}_{m|n}(\mathbb{C})_0$-modules). Further we have the following embedding $\mathfrak{pgl}_{m|n}(\mathbb{C}) = v(M) \hookrightarrow v(\tilde{M})$, which image is equal to the Lie superalgebra of all fundamental vector fields on $\tilde{M}$. Moreover on any split supermanifold there is the grading operator which we denote by $\tilde{z}$. This operator acts in the sheaf $\hat{O}$ in the following way: $\tilde{z}(f) = pf$, where $f \in \hat{O}_p$. The operator $\tilde{z}$ is not fundamental since there is no such operators on $M$. Further by definition of $\tilde{z}$ we have $[\tilde{z}, \tilde{\mathcal{T}}_o] = 0$.

Hence,

$$v(\tilde{M})_0 \simeq \mathfrak{pgl}_{m|n}(\mathbb{C})_0 \oplus \langle \tilde{z} \rangle \simeq \mathfrak{gl}_{m|n}(\mathbb{C})_0.$$

By Theorem 13 under the conditions of our theorem all vector fields on $\tilde{B}$ are fundamental with respect to the action of $\text{gr} (\mathfrak{gl}_{m|n})$ except of the grading operator. Therefore all holomorphic vector fields on $\tilde{B}$ can be lifted to $\tilde{M}$. In other words the homomorphism $\Pi$ is surjective. □

As a corollary we get.

**Corollary 15** Assume that $r > 0$ and (18) and (19) hold true. Then

$$H^0(M_0, A_p) = \begin{cases} \{0\}, & p \neq 0, -1; \\ \mathbb{C}^2, & p = 0; \\ \mathfrak{gl}_{m|n}(\mathbb{C})_1, & p = -1, \end{cases}$$

where $A_p = \hat{O}^*_1 \otimes \bigwedge^p \hat{O}_1$.

**Proof.** Consider exact Sequence (13). It determines the exact sequence of cohomology groups

$$0 \to H^0(M_0, A_p) \to H^0(M_0, \tilde{T}_p) \to H^0(M_0, C_p), \ p \geq -1.$$
Using Theorem 14 we conclude that

\[ H^0(\mathcal{M}_0, \mathcal{A}_p) = \{0\}, \quad p \neq 0, -1. \]

Further for \( p = -1 \) we have \( \mathcal{T}_{-1} \simeq \mathcal{A}_{-1} \), hence

\[ H^0(\mathcal{M}_0, \mathcal{A}_{-1}) \simeq H^0(\mathcal{M}_0, \mathcal{T}_{-1}) \simeq \mathfrak{sl}_m(\mathbb{C})_1. \]

Consider the case \( p = 0 \). It is a classical result that the Lie algebra of holomorphic vector fields \( H^0(\mathcal{M}_0, \Theta) \) on \( \mathcal{M}_0 \) is isomorphic to \( \mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C}) \), see for example [A] for details. Hence,

\[ H^0(\mathcal{M}_0, \mathcal{C}_0) = H^0(\mathcal{M}_0, \Theta) \simeq \mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C}). \]

The following diagram is commutative

\[
\begin{array}{ccc}
H^0(\mathcal{M}_0, \mathcal{T}_0) & \rightarrow & H^0(\mathcal{M}_0, \Theta) \\
\downarrow & & \downarrow \\
H^0(\mathcal{B}_0, (\mathcal{T}_{\mathcal{B}_0})_0) & \rightarrow & H^0(\mathcal{B}_0, \Theta_{\mathcal{B}_0})
\end{array}
\]

The vertical right arrow is an isomorphism of Lie algebras since these Lie algebras are both isomorphic to \( \mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C}) \) and they coincide with Lie algebras of fundamental vector fields. The vertical left arrow is an isomorphism of Lie algebras by Theorem 13 and Theorem 14. Further, the map \( H^0(\mathcal{B}_0, (\mathcal{T}_{\mathcal{B}_0})_0) \rightarrow H^0(\mathcal{B}_0, \Theta_{\mathcal{B}_0}) \) is surjective, see [OST]. It follows that the map \( H^0(\mathcal{M}_0, \mathcal{T}_0) \rightarrow H^0(\mathcal{M}_0, \mathcal{C}_0) \) is surjective too. Our assertion follows from Theorem 14. \( \square \)

### 7 Some known results about vector bundles

Let us recall some results about vector bundles over usual complex-analytic manifolds. Let \( X \) and \( Y \) be complex-analytic manifolds, \( \mathcal{G} \) and \( \mathcal{H} \) be coherent analytic sheaves on \( X \) and \( Y \), respectively. Assume in addition that \( X \) is a Stein manifold and \( Y \) is compact. By the Cartan Theorems A and B we have \( H^p(X, \mathcal{G}) = 0 \) for \( p > 0 \) and by the Cartan-Serre Theorem \( H^p(Y, \mathcal{H}) \) is finite dimensional. Now consider the direct product \( X \times Y \) and the natural projections \( \pi_X : X \times Y \rightarrow X \) and \( \pi_Y : X \times Y \rightarrow Y \). As usual we denote by \( \pi_X^*(\mathcal{G}) \) and by \( \pi_Y^*(\mathcal{H}) \) the pullback sheaves. These sheaves are sheaves of \( \mathcal{F}_{X \times Y} \)-modules, where \( \mathcal{F}_{X \times Y} \) is the sheaf of holomorphic functions on \( X \times Y \). The following theorem was proved in [K].

**Theorem 16 (Küneth formula)** Assume that \( X \) is Stein, \( Y \) is compact, \( \mathcal{G} \) and \( \mathcal{H} \) are locally free sheaves on \( X \) and \( Y \), respectively. Then we have the following isomorphism

\[ H^k(X \times Y, \pi_X^*(\mathcal{G}) \otimes_{\mathcal{F}_{X \times Y}} \pi_Y^*(\mathcal{H})) \simeq H^0(X, \mathcal{G}) \otimes H^k(Y, \mathcal{H}). \quareq \]
Let $\pi : M \to B$ be a complex-analytic bundle with compact fiber $F$ and $E \to B$ be a holomorphic vector bundle. We denote by $\Gamma(V, E)$ the vector space of all sections of $E$ over an open set $V \subset B$ and by $\pi^*E \to M$ the pullback bundle.

**Lemma 17** For any open set $V \subset B$ we have

$$\Gamma(\pi^{-1}(V), \pi^*E|_{\pi^{-1}(V)}) \simeq \Gamma(V, E|_V).$$

**Proof.** The result is a consequence of the fact that

$$H^0(\pi^{-1}(V), \pi^*(F_B)) = H^0(V, F_B),$$

where $F_B$ is the sheaf of holomorphic functions over $B$. Formula (21) follows from Theorem 16. □

We will need also the following lemma that we prove here for completeness.

**Lemma 18** Let $G$ be a sheaf of $\mathcal{F}_X$-modules on a complex-analytic manifold $X$ and $U = \{U_i\}$ be an open covering of $X$. Assume that $H^1(U_i, G) = \{0\}$ for any $U_i \in U$. Then

$$H^1(X, G) = H^1(U, G).$$

In other words in this case we can compute 1st cohomology of $G$ using the open covering $U$.

**Proof.** We put

$$\mathcal{H}^p := \prod_{i_0, \ldots, i_p} j_*(G|_{U_{i_0, \ldots, i_p}}), \quad p \geq 0,$$

where $U_{i_0, \ldots, i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$ and $j : U_{i_0, \ldots, i_p} \to X$ is the natural embedding. Any $\mathcal{H}^p$ is a sheaf over $X$ in a natural way. Consider the following complex

$$0 \to G \overset{i}{\to} \mathcal{H}^0 \overset{d^0}{\to} \mathcal{H}^1 \overset{d^1}{\to} \cdots,$$

where $d^p$ is induced by the Čech coboundary operator. We have

$$H^q(X, \mathcal{H}^p) \simeq \prod_{i_0, \ldots, i_p} H^q(U_{i_0, \ldots, i_p}, G).$$

Therefore, $H^1(X, \mathcal{H}^0) = \{0\}$. Let us construct an isomorphism $\tau : H^1(U, G) \to H^1(X, G)$. Consider the following exact sequence of sheaves over $X$

$$0 \to G \overset{i}{\to} \mathcal{H}^0 \overset{d^0}{\to} \text{Ker} d^1 \to 0,$$

and the corresponding long exact sequence

$$\to H^0(X, \mathcal{H}^0) \overset{d^0}{\to} H^0(X, \text{Ker} d^1) \overset{\delta}{\to} H^1(X, G) \to H^1(X, \mathcal{H}^0) = \{0\}.$$

20
It follows that $\delta$ induces an isomorphism

$$\tilde{\delta}: H^1(\mathcal{U}, \mathcal{G}) = H^0(X, \text{Ker} d^1)/ \text{Im} d^0 \to H^1(X, \mathcal{G}).$$

The proof is complete. $\square$

If we have a map of complex-analytic manifolds $p: M \to N$ and $\mathcal{R}$ is a sheaf on $M$, then we denote by $p_*(\mathcal{R})$ the direct image of the sheaf $\mathcal{R}$. We will need the following lemma.

**Lemma 19** Let $\pi: \mathcal{M} \to \mathcal{B}$ be a superbundle with compact fiber $\mathcal{S}$ and $\dim(\mathcal{B}) = n|m$. Then the sheaves

$$(\pi_0)_*(A_{pq}^v), \quad (\pi_0)_*(C_{pq}^v), \quad (\pi_0)_*(\hat{T}_{pq}^v)$$

are locally free sheaves of $\mathcal{F}_{\mathcal{B}_0}$-modules of rank

$$N_q \dim H^0(S_0, (\mathcal{A}_\mathcal{S})_{p-q}), \quad N_q \dim H^0(S_0, (\mathcal{C}_\mathcal{S})_{p-q}), \quad N_q \dim \mathfrak{v}(\mathcal{S})_{p-q},$$

respectively. Here $N_q := \dim \wedge^q(m)$, where $\wedge(m)$ is the Grassmann algebra with $m$ generators.

**Proof.** Assume that the bundle $\hat{\mathcal{M}}$ is trivial over $\mathcal{U} \subset \hat{\mathcal{B}}$. Let us prove for example that $(\pi_0)_*(\hat{T}_{pq}^v)$ is a locally free sheaf of $\mathcal{F}_{\mathcal{B}}$-modules. Let $V \subset \mathcal{U}_0$ be open. By Theorem 16 and Proposition 10 we have

$$H^0(V, (\pi_0)_*(\hat{T}_{pq}^v)) = H^0(\pi_0^{-1}(V), \hat{T}_{pq}^v) \cong H^0(\pi_0^{-1}(V), \pi^\mathfrak{v}_B(O_B^q) \otimes \pi^\mathfrak{s}_S(\hat{T}_{p-q})) \cong H^0(V, (\hat{O}_B)_q) \otimes H^0(S_0, (\hat{T}_S)_{p-q}).$$

We see that $(\pi_0)_*(\hat{T}_{pq}^v)|_{\mathcal{U}_0}$ is a free sheaf of $\mathcal{F}_{\mathcal{B}}$-modules. The cases $(\pi_0)_*(A_{pq}^v)$ and $(\pi_0)_*(C_{pq}^v)$ are similar. $\square$

### 8 1st cohomology group with values in $\hat{T}_p$

In this section we will compute cohomology group with values in the sheaves $\mathcal{A}_p$ for all $p$ and $\mathcal{C}_p$ for $p \neq 2$. More precisely, we will show that

$$H^1(\mathcal{M}_0, \mathcal{A}_p) = \{0\}, \quad p \geq -1, \quad \text{and} \quad H^1(\mathcal{M}_0, \mathcal{C}_p) = \{0\}, \quad p \neq 2.$$

Using these results and Sequence 13 we will obtain that $H^1(\mathcal{M}_0, \hat{T}_p) = \{0\}$, where $p \neq 2$. Further, we will compute $H^1(\mathcal{M}_0, \hat{T}_{2q}), H^1(\mathcal{M}_0, \hat{T}_{2q}^h)$ and $H^1(\mathcal{M}_0, \hat{T}_{2q})$ and at the end we will show that $H^1(\mathcal{M}_0, \hat{T}_2) = \mathbb{C}$.
8.1 1st cohomology group with values in $A^v_{pq}$, $A^h_{pq}$ and $A_p$

As above consider the following Cartan subalgebra in $\mathfrak{gl}_{m|n}(\mathbb{C})_0 \simeq \mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C})$

$$\mathfrak{h} := \{ \text{diag}(\mu_1, \ldots, \mu_m) \} \oplus \{ \text{diag}(\lambda_1, \ldots, \lambda_n) \}.$$  

Recall that the reductive part $R$ of $P$ has the following form see (17)

$$R \simeq \bigoplus_{i=1}^{r+1} \mathfrak{gl}_{k_i-1-k_i}(\mathbb{C}) \oplus \bigoplus_{i=1}^{r+1} \mathfrak{gl}_{l_i-1-l_i}(\mathbb{C}).$$

We fix the following system of positive roots

$$\Delta^+ = \Delta^+_1 \cup \Delta^+_2,$$

where

$$\Delta^+_1 = \{ \mu_i - \mu_j, i < j \} \quad \text{and} \quad \Delta^+_2 = \{ \lambda_p - \lambda_q, p < q \}.$$  

We denote by $\Phi = \Phi_1 \cup \Phi_2$, where

$$\Phi_1 = \{ \alpha_1, \ldots, \alpha_{m-1} \}, \quad \alpha_i = \mu_i - \mu_{i+1}, \quad \Phi_2 = \{ \beta_1, \ldots, \beta_{n-1} \}, \quad \beta_j = \lambda_j - \lambda_{j+1},$$

the system of simple roots. Further, denote by $\mathfrak{h}^*(\mathbb{R})$ the real subspace in $\mathfrak{h}^*$ spanned by $\mu_j$ and $\lambda_i$. Consider the scalar product $(\ ,\ )$ in $\mathfrak{h}^*(\mathbb{R})$ such that the vectors $\mu_j, \lambda_i$ form an orthonormal basis. An element $\delta \in \mathfrak{h}^*(\mathbb{R})$ is called dominant if $(\delta, \alpha) \geq 0$ for all $\alpha \in \Delta^+$. Denote by $\zeta$ the half-sum of positive roots. This is

$$\zeta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \frac{1}{2} \sum_{i=1}^{m} (m - 2i + 1) \mu_i + \frac{1}{2} \sum_{j=1}^{n} (n - 2j + 1) \lambda_j. \quad (22)$$

Following Bott, we say that $\delta \in \mathfrak{h}^*(\mathbb{R})$ has index 1 if $(\delta + \zeta, \alpha) > 0$ for all $\alpha \in \Delta^+$ except for one root $\beta \in \Delta^+$ for which $(\delta + \zeta, \beta) < 0$. We call $\delta \in \mathfrak{h}^*(\mathbb{R})$ singular if $(\delta + \zeta, \alpha) = 0$ for some $\alpha \in \Delta^+$.

Further we will identify $\mathfrak{gl}_{k_i-1-k_i}(\mathbb{C})$ or $\mathfrak{gl}_{l_i-1-l_i}(\mathbb{C})$ with the corresponding subalgebra in $R$. And we will mark with the superscript $i$ an element in $\{ \mu_j \}$ or $\{ \lambda_j \}$ if this element is in

$$(\mathfrak{h} \cap \mathfrak{gl}_{k_i-1-k_i}(\mathbb{C}))^* \quad \text{or} \quad (\mathfrak{h} \cap \mathfrak{gl}_{l_i-1-l_i}(\mathbb{C}))^*,$$

respectively. For example if $\mu_j \in (\mathfrak{h} \cap \mathfrak{gl}_{k_i-1-k_i}(\mathbb{C}))^*$ we will write $\mu_j^i$ instead of $\mu_j$.

From Proposition 10, Theorem 14 and Corollary 15 we obtain the following result.

**Proposition 20** Assuming (18) and (19), we have

$$(\pi_0)_*(\widehat{T}^v_{pq}) = 0 \quad \text{and} \quad (\pi_0)_*(A^v_{pq}) = 0,$$

for $p - q \neq -1, 0$.  

22
Proof. Under assumptions of the proposition by Theorem 14 we have the equality $H^0(S_0, (\overline{T_S})_{p-q}) = 0$ and by Corollary 15 we get $H^0(S_0, (\mathcal{A}_S)_{p-q}) = 0$. The result follows from Proposition 10. □

As above we denote by $P_B$ the underlying Lie group of the stabilizer of the origin of $Z_{I_1}$, where $Z_I = (Z_{I_1})$ are as in [16]. Denote also by $R_B$ the reductive part of $P_B$ and by $\phi_B$ the representation of $P_B$ corresponding to the homogeneous locally free sheaf $(\mathcal{O}_B)_1$. In [OS1] it was proved that the representation $\phi_B$ is completely reducible and

$$\phi_B|_{R_B} = \rho_1 \otimes \varsigma_2 + \sigma_1 \otimes \varsigma_2,$$

where $\rho_1$ and $\sigma_1$ are standard representations of $GL_{m-k_1}(\mathbb{C})$ and $GL_{n-l_1}(\mathbb{C})$, respectively; $\varsigma_2$ and $\varsigma_2$ are standard representations of $GL_{k_1}(\mathbb{C})$ and $GL_{l_1}(\mathbb{C})$, respectively. Note that $\phi_B$ is equal to $\theta$ from Lemma 11 for $r = 1$.

Lemma 21 We assume (18) and (19). The locally free sheaves of $F_B$-modules

$$(\pi)_*(\hat{T}_{-10}), \quad (\pi)_*(\hat{\mathcal{T}}_{00}), \quad (\pi)_*(\hat{A}_{-10}), \quad (\pi)_*(\hat{A}_{00})$$

on $B_0$ are homogeneous. The corresponding representations of $P_B$ are completely reducible and they respectively are:

$$\chi_{\hat{T}_{-10}}|_{R_B} = \phi_2 \otimes \varsigma_2 + \phi_2 \otimes \varsigma_2, \quad \chi_{\hat{T}_{00}}|_{R_B} = \rho_2 + \varsigma_2 + 1 + 1,$$

$$\chi_{\hat{A}_{-10}}|_{R_B} = \phi_2 \otimes \varsigma_2 + \phi_2 \otimes \varsigma_2, \quad \chi_{\hat{A}_{00}}|_{R_B} = 1 + 1,$$

where 1 is the trivial one dimensional representation, $\rho_2$ and $\varsigma_2$ are adjoint representations of $GL_{k_1}(\mathbb{C})$ and $GL_{l_1}(\mathbb{C})$, respectively.

Proof. The proposition follows from Proposition 10 Theorem 14, the isomorphism $\hat{T}_{-10} \simeq A_{-10}$ and Corollary 15. Further a direct computation shows that the nilpotent part of $P_B$ acts trivially. Hence we obtain completely reducibility □

Corollary 22 The homogeneous locally free $F_B$-sheaves $(\pi)_*(\hat{T}_{pq})$ and $(\pi)_*(\hat{A}_{pq})$ correspond to the following representations of $P_B$

$$\bigwedge^q \phi_B \otimes \chi_\hat{T}_{p-q,0}, \quad \bigwedge^q \phi_B \otimes \chi_\hat{A}_{p-q,0},$$

respectively.

Proof. The result is a consequence of the following observation

$$\hat{T}_{pq} \simeq \bigwedge^q \hat{\pi}^*(\hat{\mathcal{O}}_B)_1 \otimes \hat{T}_{p-q,0}, \quad \hat{A}_{pq} \simeq \bigwedge^q \hat{\pi}^*(\hat{\mathcal{O}}_B)_1 \otimes \hat{A}_{p-q,0}. \quad □$$

In [V3] the following Lemma was proved.
Lemma 23 Assuming (18), we have

\[ H^0(\mathcal{M}_0, \mathcal{O}_p) = \begin{cases} \{0\}, & p \neq 0, \\ \mathbb{C}, & p = 0. \end{cases} \]

Now we need the following theorem.

Theorem 24 We assume (18) and (19). Then

\[ H^1(\mathcal{M}_0, \mathcal{O}) = \{0\} \quad \text{and} \quad H^1(\mathcal{M}_0, \mathcal{O}) = \{0\}. \]

Proof. We use induction on \( r \). For the case \( \mathcal{M} = \text{Gr}_{m|n,kl} \) by [OS1] we have \( \mathcal{O} \simeq \bigwedge \mathcal{E}_{\phi_B} \), where \( \mathcal{E}_{\phi_B} \) is the homogeneous locally free sheaf corresponding to the following representation of \( P_B \)

\[ \phi_B|_{R_B} = \rho_1^* \otimes \sigma_2 + \sigma_1^* \otimes \phi_2. \]

Here \( \rho_1, \phi_2, \sigma_1, \sigma_2 \) are standard representations of \( GL_{m-k}(\mathbb{C}), GL_k(\mathbb{C}), GL_{n-l}(\mathbb{C}), GL_l(\mathbb{C}) \), respectively. Let us show that \( H^1(\mathcal{M}_0, \bigwedge^q \mathcal{E}_{\phi_B}) = \{0\} \) for \( q \geq 0 \).

Any weight of the representation \( \bigwedge^q \phi_B \) has the form \( \gamma = \gamma_0 + \gamma_1 \), where

\[ \gamma_0 = -\mu_1^1 - \cdots - \mu_{i_a}^1 + \mu_2^1 + \cdots + \mu_{j_b}^2, \quad \gamma_1 = -\lambda_1^1 - \cdots - \lambda_{i_1}^1 + \lambda_2^1 + \cdots + \lambda_{j_2}^2, \quad a + b = q. \]

Assume that \( \gamma_0 \) is dominant. Then under assumption \( 0 < k < m \), we get that \( a = b = 0 \). In this case \( \gamma_1 \) cannot have index 1. Similarly we get that if \( \gamma_1 \) is dominant \( \gamma_0 \) cannot have index 1. Summing up under our assumptions the weight \( \gamma \) cannot have index 1 and therefore by Borel-Weil-Bott theorem we have

\[ H^1(\mathcal{M}_0, \mathcal{O}_q) = H^1(\mathcal{M}_0, \bigwedge^q \mathcal{E}_{\phi_B}) = \{0\}, \quad q \geq 0. \]

Further we have the following exact sequence

\[ 0 \rightarrow \mathcal{O}_{(p+1)} \rightarrow \mathcal{O}_{(p)} \rightarrow \mathcal{O}_p \rightarrow 0. \]

For enough big \( p \) we have the natural isomorphism \( \mathcal{O}_p \simeq \mathcal{O}_{(p)} \) and therefore for enough big \( p \) we have \( H^1(\mathcal{M}_0, \mathcal{O}_{(p)}) \simeq H^1(\mathcal{M}_0, \mathcal{O}_p) = \{0\} \). By induction we obtain \( H^1(\mathcal{M}_0, \mathcal{O}_{(p)}) = \{0\} \) for any \( p \).

Now assume that \( \mathcal{M} \) is a flag supermanifold of length \( r > 1 \). Then \( \mathcal{M} \) is a superbundle with base space \( \mathcal{B} = \text{Gr}_{m|n,kl} \) and with fiber \( \mathcal{S} \), where \( \mathcal{S} \) is a flag supermanifold of length \( r-1 \). Denote by \( \pi : \tilde{\mathcal{M}} \rightarrow \mathcal{B} \) the projection of the split superbundle. Let \( \{\mathcal{U}\} \) be a covering of \( \tilde{\mathcal{B}} \) and assume that the bundle \( \mathcal{M} \) is trivial over any \( \mathcal{U} \). Assume by induction that \( H^1(\mathcal{S}_0, \mathcal{O}_S) = \{0\} \). Then by Theorem [16] and by induction we have

\[ H^1(\pi^{-1}(\mathcal{U}), \mathcal{O}_{\mathcal{S}}|_{\pi^{-1}(\mathcal{U})}) = H^0(\mathcal{U}_0, \mathcal{O}_B|_{\mathcal{U}_0}) \otimes H^1(\mathcal{S}_0, \mathcal{O}_S) = \{0\}. \]
Hence by Lemma 18
\[ H^1(\mathcal{M}_0, \tilde{\mathcal{O}}) \simeq H^1(\pi^{-1}(\mathcal{U}), \tilde{\mathcal{O}}) \simeq H^1(\{U\}, \pi_*(\tilde{\mathcal{O}})). \]

By Lemma 23 we have \( H^0(S_0, \tilde{\mathcal{O}}_S) \simeq \mathbb{C}. \) Therefore by Theorem 16 \( \tilde{\pi}_*(\tilde{\mathcal{O}}) = \tilde{\mathcal{O}}_B. \) Hence
\[ H^1(\mathcal{M}_0, \tilde{\mathcal{O}}) = H^1(B_0, \tilde{\mathcal{O}}_B) = \{0\}. \]

The argument in the case of \( \mathcal{O} \) is similar to the case \( r = 1. \) \( \square \)

In [O4, Propositions 1, 2, 3] the following lemma was proved.

**Lemma 25** Assume that \( r = 1 \), and (18) and (19) hold true. Then for \( \mathcal{M} = \text{Gr}_{m[n, k]l} \) we have
\[ H^1(\mathcal{M}_0, \mathcal{A}_p) = \{0\}, \quad p \geq -1. \]

Now we can prove the following theorem.

**Theorem 26** Assume that \( r \geq 2 \) and that (18) and (19) hold true. Then
\[ H^1(\mathcal{M}_0, \mathcal{A}_{pq}^v) = \begin{cases} \mathbb{C}^2, & (p, q) = (0, 1), \\ \{0\}, & (p, q) \neq (0, 1), \end{cases} \]
\[ H^1(\mathcal{M}_0, \mathcal{A}_{pq}^h) = \{0\} \text{ for all } p, q. \]
\[ H^1(\mathcal{M}_0, \mathcal{A}_p) = \{0\} \text{ for all } p. \]

**Proof.** We use induction on \( r \). For \( r = 1 \) the sheaves \( \mathcal{A}_{pq}^v \) and \( \mathcal{A}_{pq}^h \) are not defined and \( H^1(\mathcal{M}_0, \mathcal{A}_p) = \{0\} \) by Lemma 25. Assume by induction that for a flag supermanifold of the length \( r - 1 \) we have \( H^1(\mathcal{M}_0, \mathcal{A}_p) = \{0\} \) for all \( p \). Let us prove the statement for the length \( r \). We have the following exact sequence
\[ 0 \to \mathcal{A}_{p(q+1)} \to \mathcal{A}_{p(q)} \to \mathcal{A}_{pq} \to 0, \tag{23} \]
where \( \mathcal{A}_{pq} = \mathcal{A}_{pq}^v + \mathcal{A}_{pq}^h \). Let us compute 1-cohomology group for the sheaves \( \mathcal{A}_{pq}^v \) and \( \mathcal{A}_{pq}^h \).

**Step 1: 1st cohomology group with values in \( \mathcal{A}_{pq}^h \).** Let \( \{\mathcal{U}\} \) be a covering of \( \mathcal{B} \) and assume that the bundle \( \mathcal{M} \) is trivial over any \( \mathcal{U} \). By Theorem 24 we have \( H^1(S_0, (\mathcal{O}_S)_{p-q}) = \{0\} \) for \( p-q \geq 0 \). Therefore, by Theorem 16 and by Proposition 10 we have \( H^1(\pi^{-1}(\mathcal{U}_0), \mathcal{A}_{pq}^h) = 0 \) and hence by Lemma 18 we get
\[ H^1(\mathcal{M}_0, \mathcal{A}_{pq}^h) = H^1(\pi^{-1}(\mathcal{U}_0), \mathcal{A}_{pq}^h). \]

Further by Theorem 16 by Proposition 10 and by Lemma 23 we get
\[ H^0(\pi^{-1}(\mathcal{U}_0), \mathcal{A}_{pq}^h) = H^0(S_0, \bigwedge^{p-q} (\mathcal{O}_B)_1) \otimes H^0(\mathcal{U}_0, (\mathcal{A}_B)_q) = \{0\} \text{ for } p \neq q, \]
and

\[ H^0(\pi^{-1}(U_0), A^h_{pp}) = H^0(U_0, (A_S)_p) \text{ for } p = q. \]

Therefore, \( H^1(\{\pi^{-1}(U_0)\}, A^h_{pp}) = H^1(\{U\}, (A_S)_p) \) and hence \( H^1(M_0, A^h_{pp}) = \{0\} \) by Lemma \( \text{25} \). Summing up we proved that

\[ H^1(M_0, A^h_{pq}) = \{0\}, \ p, q \geq -1. \]

**Step 2: 1st cohomology group with values in \( A^v_{pq} \).** Further by the induction assumption, \( H^1(S_0, (A_S)_{p-q}) = \{0\} \). Hence by Proposition \( \text{10} \) and by Theorem \( \text{16} \) we have \( H^1(\pi^{-1}(U_0), A^v_{pq}) = \{0\} \). Therefore by Lemma \( \text{18} \) to compute 1st cohomology we can use the covering \( \{\pi^{-1}(U_0)\} \). Now consider the homogeneous locally free sheaf \( \pi_a(A^v_{pq}) \) of \( \mathcal{F}_S \)-modules. The corresponding representation of \( P_S \) is \( \wedge^q \phi_B \otimes \chi_{A^v_{pq-0}} \), see Lemma \( \text{21} \) and its Corollary. Let us apply the Borel-Weil-Bott Theorem.

Consider first of all the case \( p - q = -1 \). Any weight of \( \wedge^q \phi_B \otimes \chi_{A^v_{pq-0}} \) has the form \( \Lambda = \Lambda_0 + \Lambda_1 \), where

\[
\Lambda_0 = -\mu_1^1 - \cdots - \mu_a^1 + \mu_1^s + \cdots + \mu_b^s + \begin{cases} (1) & -\mu_i^s, \\ (2) & \mu_j^s, \end{cases} \ a + b = q, \ s > 1.
\]

\[
\Lambda_1 = -\lambda_1^1 - \cdots - \lambda_b^1 + \lambda_1^s + \cdots + \lambda_a^s + \begin{cases} (1) & \lambda_i^s, \\ (2) & -\lambda_i^s, \end{cases}
\]

Assume that \( \Lambda \) has index 1. Consider the case when \( \Lambda_0 \) is dominant and \( \Lambda_1 \) has index 1. In case (1) it follows that either (1.A) \( a = 0, \ b = 1, \ \Lambda_0 = 0; \) or (1.B) \( a = b = 0. \) In case (1.B) the weight \( \Lambda_1 \) is singular. In case (1.A) the weight \( \Lambda_1 \) is singular except of \( \Lambda_1 = -\lambda_{n-1}^1 + \lambda_{n-1}^2, \) which has index 1. In case (2), \( \Lambda_0 \) is not dominant. Summing up, \( \Lambda = -\lambda_{n-1}^1 + \lambda_{n-1}^2 \) is the unique weight of index 1. We see that \( \Lambda \) is a highest weight of the representation \( \wedge^q \phi_B \otimes \chi_{A^v_{pq-0}} \) for \( q = 1. \)

In case if \( \Lambda_1 \) is dominant and \( \Lambda_0 \) has index 1 similarly as above we get another highest weight \( \Lambda = -\mu_{m-k}^1 + \mu_{m-k}^2 \) of \( \wedge^q \phi_B \otimes \chi_{A^v_{pq-0}} \) of index 1 for \( q = 1. \)

By the Borel-Weil-Bott Theorem we get

\[ H^1(M_0, A^v_{q-1,0}) = H^1(B_0, \pi_a(A^v_{q-1,0})) = \begin{cases} \mathbb{C}^2, & q = 1; \\ \{0\}, & q \neq 1. \end{cases} \]

Consider now the case \( p - q = 0 \). Any weight of the representation \( \wedge^q \phi_B \otimes \chi_{A^v_{pq-0}} \) corresponding to the locally free sheaf \( \pi_a(A^v_{pq}) \) has the form \( \Lambda = \Lambda_0 + \Lambda_1 \), where

\[
\Lambda_0 = -\mu_1^1 - \cdots - \mu_a^1 + \mu_1^s + \cdots + \mu_b^s, \ a + b = q, \ s > 1.
\]

\[
\Lambda_1 = -\lambda_1^1 - \cdots - \lambda_b^1 + \lambda_1^s + \cdots + \lambda_a^s,
\]

Assume that \( \Lambda \) has index 1. Consider the case when \( \Lambda_0 \) is dominant and \( \Lambda_1 \) has index 1. Hence \( a = b = 0 \) and \( \Lambda_1 = 0. \) We get a contradiction with the assumption that \( \Lambda \) has index 1. The case when \( \Lambda_1 \) is dominant and \( \Lambda_0 \) has index 1 is similar.
Therefore the representation $\bigwedge^q \phi_B \otimes \chi_{A_{0,0}^v}$ does not have weights of index 1. We get that

$$H^1(M_0, A_{qq}^v) = H^1(B_0, \pi_*(A_{qq}^v)) = \{0\}.$$ 

For $(p, q)$ such that $p - q \neq -1, 0$ we get using Corollary 15

$$H^0(\pi^{-1}(U), A_{qq}^v) = \{0\}.$$ 

Hence,

$$H^1(M_0, A_{pq}^v) = \{0\}.$$ 

**Step 3: 1st cohomology group with values in $A_p$.** Now from the exact sequence 23 we get that $H^1(M_0, A_p) = \{0\}$, where $p \neq 0$. Further,

$$H^1(M_0, A_{0(1)}) = H^1(M_0, A_{01}) = H^1(M_0, A_{01}^v \oplus A_{01}^h) = \mathbb{C}^2.$$ 

We have the following long exact sequence

$$0 \to H^0(M_0, A_{0(1)}) \to H^0(M_0, A_{00}) \to H^0(M_0, A_{00}) \to H^1(M_0, A_{01}) \to H^1(M_0, A_{00}) \to H^1(M_0, A_{00}) = 0.$$ 

From Corollary of Theorem 14 we know that $H^1(M_0, A_0) = \mathbb{C}^2$. A direct calculation shows that

$$H^0(M_0, A_{0(1)}) = \{0\}, \quad H^0(M_0, A_{00}) = \mathbb{C}^2.$$ 

Let us compute $H^0(M_0, A_{00})$. Clearly, $A_{00} = \hat{\mathcal{O}}_{11} \otimes \hat{\mathcal{O}}_{11} + \hat{\mathcal{O}}_{10} \otimes \hat{\mathcal{O}}_{10}$. As above using the Borel-Weil-Bott Theorem, we obtain

$$H^0(M_0, \hat{\mathcal{O}}_{11} \otimes \hat{\mathcal{O}}_{11}) = H^0(B_0, \hat{\mathcal{O}}_B \otimes \hat{\mathcal{O}}_B) = \mathbb{C}^2, \quad H^0(M_0, \hat{\mathcal{O}}_{10} \otimes \hat{\mathcal{O}}_{10}) = H^0(B_0, \hat{\mathcal{W}}_0) = \mathbb{C}^2.$$ 

Hence, $H^1(M_0, A_{00}) = \{0\}$. From the exact sequence

$$\{0\} = H^1(M_0, A_{00}) \to H^1(M_0, A_{0(-1)}) \to H^1(M_0, A_{0(-1)}) = \{0\}$$

it follows that $H^1(M_0, A_0) = H^1(M_0, A_{0(-1)}) = \{0\}$. The proof is complete. □

### 8.2 1st cohomology group with values in $C_{pq}^v, C_{pq}^h$ and $C_p$

By Theorem 14, Corollary 15 and Theorem 26 we get.

**Lemma 27** Assume that $r \geq 1$ and that (18) and (19) hold true. Then

$$H^0(M_0, C_p) = \begin{cases} \mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C}), & p = 0, \\ \{0\}, & p \neq 0. \end{cases}$$
In case \( r = 1 \) this result was obtained in [OS1].

**Proof.** The result is a consequence of Theorem 14, Corollary 15, Theorem 26 and the following exact sequence

\[
0 \to H^0(\mathcal{M}_0, \mathcal{A}_p) \to H^0(\mathcal{M}_0, \overline{T}_p) \to H^0(\mathcal{M}_0, \mathcal{C}_p) \to H^1(\mathcal{M}_0, \mathcal{A}_p) = 0. \square
\]

By Proposition 10, Theorem 16 and by Lemma 27, we get.

**Lemma 28** We have \( \pi_*(C^v_{pq}) = \{0\} \) for \( p \neq q \).

Further we need the following lemma, which is a consequence of Lemma 27 and Proposition 10.

**Lemma 29** The representations of \( P_\mathcal{B} \) corresponding to the sheaves \( \pi_*(C^v_{0p}) \) and \( \pi_*(C^v_{pq}) \) have the following form

\[
\chi_{C^v_{pq}}|_R = \begin{cases} 0, & p \neq 0, \\ Ad_{e_2} + Ad_{e_3}, & p = 0. \end{cases}, \quad \chi_{C^v_{pq}}|_R = \bigwedge^q \phi_{\mathcal{B}} \otimes \chi_{C^v_{p-q,0}},
\]

respectively.

In [O4] for \( \text{Gr}_{m|n,k|l} \) the following lemma was proved.

**Lemma 30** Assume that \( r = 1, 0 < k < m, 0 < l < n \) and \( (k, l) \neq (1, n - 1), (m - 1, 1) \). Then

\[
H^1(\mathcal{M}_0, \mathcal{C}_p) = \begin{cases} \mathbb{C}^2, & p = 2, \\ \{0\}, & p \neq 2. \end{cases}
\]

We are ready to prove the following theorem.

**Theorem 31** Assume that \( r \geq 2 \) and that (18) and (19) hold true. Then

\[
H^1(\mathcal{M}_0, \mathcal{C}_p) = \{0\} \quad \text{for} \quad p \neq 2
\]

and

\[
H^1(\mathcal{M}_0, \mathcal{C}^h_{pq}) = \begin{cases} \{0\}, & (p, q) \neq (2, 2), \\ \mathbb{C}^2, & (p, q) = (2, 2). \end{cases}, \quad H^1(\mathcal{M}_0, \mathcal{C}^v_{pq}) = \{0\}, \quad (p, q) \neq (2, 0).
\]

**Proof.** Consider the following exact sequence

\[
0 \to \mathcal{C}^v_p \to \mathcal{C}_p \to \mathcal{C}^h_p \to 0. \quad (24)
\]

Let us compute the 1-st cohomology of the sheaves \( \mathcal{C}^v_p \) and \( \mathcal{C}^h_p \), where \( p \neq 2 \).

Recall that by Proposition 10 we have

\[
\mathcal{C}^h_{pq} \simeq \bigwedge^{p-q} \hat{\mathcal{O}}_{10} \otimes \pi^*((\mathcal{C}_{\mathcal{B}})_q), \quad \mathcal{C}^v_{pq}|_{\overline{\pi}^{-1}(\mathcal{U})} \simeq \bigwedge^q \hat{\mathcal{O}}_{11}|_{\mathcal{U}} \otimes \pi^*_5(\mathcal{C}_{\mathcal{S}})_{p-q}.
\]
Step 1, 1st cohomology group with values in $C^h_{pq}$. By Theorem 24 and by Lemma 18 to compute $H^1(\mathcal{M}_0, C^h_{pq})$ we can use the covering $\{\pi^{-1}(U)\}$. Further, by Lemma 23 and Theorem 16 we have $H^0(\{\pi^{-1}(U)\}, C^h_{pp}) = \{0\}$ for $p \neq q$ and $H^0(\{\pi^{-1}(U)\}, C^h_{pp}) = H^0(\mathcal{B}_0, (\mathcal{E}_p)_p)$. Therefore by Lemma 30 we have

$$H^1(\mathcal{M}_0, C^h_{pq}) = \begin{cases} \{0\} & (p, q) \neq (2, 2), \\ \mathbb{C}^2 & (p, q) = (2, 2). \end{cases}$$

Step 2, 1st cohomology group with values in $C^v_{pq}$ for $p - q \neq 2$. For $p - q \neq 2$ by induction we assume that $H^1(\mathcal{S}_0, (\mathcal{S}_{p-q}) = \{0\}$. Hence in this case we can use Theorem 16 and Lemma 18. Consider the locally free sheaf $\pi_*(C^v_{pq})$ of $F_{\mathcal{B}}$-modules. Note that by Lemma 27 we have $H^0(\mathcal{U}_0, \pi_*(C^v_{pq})) = \{0\}$ for $p - q \neq 0$. Hence,

$$H^1(\mathcal{M}_0, C^v_{pq}) = \{0\} \text{ for } p - q \neq 2, 0.$$

Further the representation $\chi_{C^v_{pq}}$ of $P_{\mathcal{B}}$ in a fiber of $\pi_*(C^v_{pq})$ is completely reducible, since the nilradical of $P_{\mathcal{B}}$ acts trivially. We use the Borel–Weil–Bott Theorem. By Lemma 29 any weight of $\chi_{C^v_{pq}}$ for $p - q = 0$ has the form $\Lambda = \Lambda_0 + \Lambda_1$, where

$$\Lambda_0 = -\mu_1^1 - \cdots - \mu_a^1 + \mu_j^2 + \cdots + \mu_a^2 + \begin{cases} 1. \mu_j^2 - \mu_i^2, \\ 2. 0, \end{cases} a + b = q,$$

$$\Lambda_1 = -\lambda_j^1 - \cdots - \lambda_b^1 + \lambda_i^2 + \cdots + \lambda_a^2 + \begin{cases} 1. 0, \\ 2. \lambda_j^2 - \lambda_i^2, \end{cases}$$

The weights $\Lambda_0$ and $\Lambda_1$ are not dominant for any $a, b$. Therefore $\Lambda$ cannot have index 1 and

$$H^1(\mathcal{M}, C^v_{pq}) = \{0\}, \ p - q \neq 2.$$

Step 3, 1st cohomology group with values in $C^v_{pq}$ for $p - q = 2$ and with values in $C^r_p$. Note that in this case $q \geq 0$. Assume that $q > 0$. Then by Lemma 32 below the representation of $P$ corresponding to $C^v_{pq}$ does not have weights of index 1. Therefore we have only one possibility $(p, q) = (2, 0)$. Summing up, we get the following result

$$H^1(\mathcal{M}_0, C^v_{pq}) = \{0\} \text{ for } (p, q) \neq (2, 0).$$

This implies that $H^1(\mathcal{M}_0, C^r_p) = \{0\}$ for $p \neq 2$. Hence we get

$$H^1(\mathcal{M}_0, C^r_p) = \{0\} \text{ for } p \neq 2, \square$$

Lemma 32 Assume that $r \geq 2$, that (18) and (19) hold true, $q > 0$ and $p - q = 2$. Then the representation $\tau^v \otimes \Lambda^q \varphi \otimes \Lambda^{p-q} \phi$ corresponding to the sheaf $C^v_{pq}$ does not have weights of index 1.

Proof. We use notations of Lemma 11 and Lemma 12. Recall that we denoted by $\varphi$ the representation of $P$ corresponding to the homogeneous locally free sheaf $\mathcal{O}_{11}$.
and by $\psi$ the representation of $P$ corresponding to the homogeneous locally free sheaf $\mathcal{O}_{10}$. A weight $\Lambda$ of the representation

$$\tau^r \otimes q \otimes \bigwedge^2 \psi$$

of $P$ has the form $\Lambda = \Lambda_0 + \Lambda_1$, where

$$\Lambda_0 = -\mu_{i_1}^1 - \cdots - \mu_{i_a}^1 + \mu_{j_1}^{2^2} + \cdots + \mu_{j_b}^{2^2} + \left\{ \begin{array}{l} -\mu_{i_1}^{\nu_1} - \mu_{j_1}^{\nu_2} \quad (A), \\ -\mu_{i_1}^{\nu_1} + \mu_{j_1}^{\nu_2} \quad (B), \\ + \mu_{i_1}^{\geq 3} + \mu_{j_1}^{\geq 3} \quad (C) \end{array} \right\} + \left\{ \begin{array}{l} \mu_1^{\nu_3} - \mu_1^{\geq 3} \quad (1), \\ 0 \quad (2), \end{array} \right\} ,$$

and

$$\Lambda_1 = -\lambda_{j_1}^1 - \cdots - \lambda_{i_b}^1 + \lambda_{i_a}^{2^2} + \cdots + \lambda_{i_a}^{2^2} + \left\{ \begin{array}{l} \lambda_{j_1}^{\geq 3} + \lambda_{j_1}^{\geq 3} \quad (A), \\ -\lambda_{i_1}^{\kappa_1} + \lambda_{j_1}^{\geq 2} \quad (B), \\ -\lambda_{i_1}^{\geq 2} - \lambda_{j_1}^{\kappa_2} \quad (C) \end{array} \right\} + \left\{ \begin{array}{l} 0 \quad (1), \\ \lambda_{i_a}^{\kappa_3} - \lambda_{j_1}^{\geq 3} \quad (2). \end{array} \right\}$$

Here $a + b = q$ and $1 < \nu_i, \kappa_i < r + 1$.

Assume that $\Lambda_0$ is dominant and $\Lambda_1$ has index 1. Since $q > 0$ we have $a > 0$ or $b > 0$. Let us write $\Lambda_0$ in the following form

$$\Lambda_0 = \sum \alpha_i \mu_i.$$ 

Then $\Lambda_0$ is dominant if and only if $\alpha_i \geq \alpha_{i+1}$ for any $i$. If $a > 0$ and $\Lambda_0$ is dominant we have only one possibility: (A.1) with $a = 1$, $b = 0$, $r = 2$. However in this case $\Lambda_1$ is singular.

Now consider the case $a = 0$ and $b > 0$. In case (A.1) if $\Lambda_0$ is dominant we have the following possibilities:

(I) $b = 1$ and $\Lambda_0 = -\mu_1^{r+1}$;

(II) $b = 2$ and $\Lambda_0 = 0$.

In case (I) the weight $\Lambda_1$ is singular. In case (II) the weight $\Lambda_1$ is singular or has index greater then 1.

In case (A.2) if $\Lambda_0$ is dominant we have the following possibility:

(I) $b = 2$ and $\Lambda_0 = 0$.

In this case the weight $\Lambda_1$ is singular or has index greater then 1.

In other cases, this is the cases (B.1), (B.2), (C.1) and (C.2), the weight $\Lambda_0$ is not dominant if $b > 0$. The proof is complete.□

From Theorem 26 and Theorem 31 the following result follows.
Theorem 33 Assume that \( r \geq 2 \) and that (18) and (19) hold true. Then
\[
H^1(M_0, T_p) = \{0\}, \quad \text{where} \quad p \neq 2,
\]
and
\[
H^1(M_0, \hat{T}_{pq}^h) = \begin{cases} 
\{0\} & (p,q) \neq (2,2), \\
\mathbb{C}^2 & (p,q) = (2,2), \\
\{0\} & (p,q) \neq (2,0), (0,1), \\
\mathbb{C}^2 & (p,q) = (0,1), \\
? & (p,q) = (2,0).
\end{cases}
\]

Proof. The result follows from the exact sequence (3), Theorem 26, Theorem 31, Lemma 28 and Lemma 9.

8.3 1st cohomology group with values in the sheaf \( \hat{T}_{22} \)

Let us prove first the following theorem.

Theorem 34 Assuming (18) and (19), we have \( H^1(M_0, \hat{T}_{22}^h) = \mathbb{C} \).

Proof. By Proposition 10 we have
\[
\hat{T}_{pq}^h \simeq \bigwedge^p \mathcal{O}_{10} \otimes \pi^*((\hat{T}_B)_q).
\]
We use Lemma 18 and covering \( \{U\} \) of \( B \) as above. By Theorem 24 and Theorem 10 we obtain that
\[
H^1(\pi^{-1}(U_0), \hat{T}_{pq}^h) = 0.
\]
Further by Lemma 23 and Theorem 10 we have
\[
H^1(\{\pi^{-1}(U)\}, \hat{T}_{pq}^h) = \{0\} \quad \text{for} \quad p \neq q
\]
and
\[
H^1(\{\pi^{-1}(U)\}, \hat{T}_{pq}^h) = H^1(\{U\}, (\hat{T}_B)_p).
\]
Now our result follows from the corresponding result for super-grassmannians, in [O4, Theorem 1]. □

Let us recall some results of Bott [Bott], see also [O4, Section 3], that we will essentially use to prove the next theorem. Let \( E \) be a homogeneous bundle over \( B_0 \) and \( x_0 \in B_0 \) be the origin of \( Z_B \subset B_0 \), see (16). Denote by \( E_{x_0} \) the \( P_B \)-module corresponding to \( E \). In other words \( E_{x_0} \) is the fiber of \( E \) over \( x_0 \). Denote also by \( \mathfrak{P} \) and by \( \mathfrak{R} \) the Lie algebras of \( P_B \) and \( R_B \), respectively. We have the following isomorphisms [Bott, Theorem I, Corollary 2 of Theorem W2]
\[
H^1(B_0, E)_{\mathfrak{g}_{\text{id}}(\mathcal{C})_0} \simeq H^1(\mathfrak{P}, \mathfrak{R}, E_{x_0}) \simeq H^1(\mathfrak{n}_B, E_{x_0}) R_B,
\]
(25)
where $n_B$ is the nilradical of the Lie algebra of $P_B$. Now we are ready to prove the following theorem.

**Theorem 35** Assuming (18) and (19), we have $H^1(\mathcal{M}_0, \widehat{T}_{22}) = \{0\}$.

**Proof.** *Step 1.* $H^1(\mathcal{M}_0, \widehat{T}_{22})$ as invariant Lie algebra cohomology. The following sequence of sheaves

$$0 \to \pi_* (\widehat{T}_{22}) \to \pi_* (\widehat{T}_{22}^v) \to \pi_* (\widehat{T}_{22}^h) \to 0. \quad (26)$$

is exact. Indeed, from Proposition 10 and Theorem 33 it follows that $H^1(U_0, \pi_* (\widehat{T}_{22}^v)) = \{0\}.$

Further we use the long exact sequence over any $V \subset U_0$. We also can choose $\mathcal{U}$ such that Sequence (26) is split over $\mathcal{U}$. In other words we have

$$\pi_* (\widehat{T}_{22})|_{\mathcal{U}} = \pi_* (\widehat{T}_{22}^v)|_{\mathcal{U}} \oplus \pi_* (\widehat{T}_{22}^h)|_{\mathcal{U}}$$

for any $\mathcal{U}$. As above we get that

$$H^1(\mathcal{U}, \pi_* (\widehat{T}_{22}^v)) = 0, \quad H^1(\mathcal{U}, \pi_* (\widehat{T}_{22}^h)) = 0.$$

Therefore we can use Lemma 18 for the sheaf $\widehat{T}_{22}$. By Theorem 33, Theorem 34, the observations above and Sequence (26) we obtain

$$0 \to H^1(B_0, \pi_* (\widehat{T}_{22}^v)) \to H^1(B_0, \pi_* (\widehat{T}_{22}^h)) = \mathbb{C}. \quad (27)$$

Denote by $T$, $T^v$ and by $T^h$ the vector bundles corresponding to the locally free sheaves $\pi_* (\widehat{T}_{22})$, $\pi_* (\widehat{T}_{22}^v)$ and $\pi_* (\widehat{T}_{22}^h)$ on $B_0$, respectively. Denote also by $T_B$ the vector bundle corresponding to the locally free sheaf $(\widehat{T}_B)_2$ on $B_0$. Further using (27) and (25) we get

$$H^1(B_0, \pi_* (\widehat{T}_{22})) = H^1(B_0, \pi_* (\widehat{T}_{22}^v)) \oplus_{m|n(C)|_0} H^1(n_B, T_{x_0})^{R_B}.$$

**Step 2. Computation of $H^1(n_B, T_{x_0})^{R_B}$.** A direct computation shows that there is the following $R_B$-invariant decomposition

$$T_{x_0} = T_{x_0}^v \oplus T_{x_0}^h = T_{x_0}^v \oplus (T_B)_{x_0}.$$

Consider the chart corresponding to the coordinate matrices $Z_{I_s}$ given by (16). Let us write $Z_{I_s}$ explicitly

$$Z_{I_s} = \begin{pmatrix} X_s & \Xi_s \\ E & 0 \\ H_s & Y_s \\ 0 & E \end{pmatrix}, \quad s = 1, \ldots, r., \quad (28)$$
where \(X_s = (x^s_{ij}), Y_s = (y^s_{ij}), \Xi_s = (\xi^s_{ij})\) and \(H_s = (\eta^s_{ij})\). In [15] Proof of Theorem 1] a basis of vector space \(C^1(\mathfrak{n}_B, (\mathbb{T}_B)_{x_0})^{R_\mathbb{S}}\) of cochains was calculated. This basis contains only the following two elements

\[
c(\epsilon_{\alpha\beta}) = a \sum_{ij} \eta^1_{\alpha i} \xi^1_{\beta j} \frac{\partial}{\partial y^1_{ij}}; \quad c(f_{\alpha\beta}) = b \sum_{ij} \xi^1_{\alpha i} \eta^1_{\beta j} \frac{\partial}{\partial x^1_{ij}}, \quad a, b \in \mathbb{C}.
\]

Here \(\epsilon_{\alpha\beta} \in \mathfrak{n}_B^1 \cong \text{Mat}_{k_1, m-k_1}(\mathbb{C})\), \(\epsilon_{\alpha\beta} \in \mathfrak{n}_B^2 \cong \text{Mat}_{l_1, n-l_1}(\mathbb{C})\) and \(\mathfrak{n}_B = \mathfrak{n}_B^1 \oplus \mathfrak{n}_B^2\), where \(\mathfrak{n}_B^1\) and \(\mathfrak{n}_B^2\) are irreducible \(R_\mathbb{S}\)-modules with highest weights \(-\mu_{m-k_1} + \mu_{m-k_1+1}\) and \(-\lambda_{n-l_1} + \lambda_{n-l_1+1}\), respectively. From the proof of Theorem [26] Step 2, and from the proof of Theorem [31] Step 2, it follows that the representation of \(R_\mathbb{S}\) in \(\mathbb{T}_x\) does not have weights of index 1. In particular it follows that this representation does not have weights \(-\mu_{m-k_1} + \mu_{m-k_1+1}\) and \(-\lambda_{n-l_1} + \lambda_{n-l_1+1}\). Therefore any cochain \(d \in C^1(\mathfrak{n}_B, (\mathbb{T}_B)_{x_0})^{R_\mathbb{S}}\) has values in \((\mathbb{T}_B)_{x_0}\). Therefore we have \(C^1(\mathfrak{n}_B, (\mathbb{T}_B)_{x_0})^{R_\mathbb{S}} \cong C^1(\mathfrak{n}_B, (\mathbb{T}_B)_{x_0})^{R_\mathbb{S}}\).

Let us show that from \(\delta(c) = 0\) for \(c \in C^1(\mathfrak{n}_B, (\mathbb{T}_B)_{x_0})^{R_\mathbb{S}}\) it follows that \(c = 0\). In other words let us show that the vector space of cocycles \(Z^1(\mathfrak{n}_B, (\mathbb{T}_B)_{x_0})^{R_\mathbb{S}}\) is trivial. By definition \(c \in Z^1(\mathfrak{n}_B, (\mathbb{T}_B)_{x_0})^{R_\mathbb{S}}\) if and only if

\[
(\delta c)(x, y) = xc(y) - yc(x) = 0 \quad \text{for any } x, y \in \mathfrak{n}_B.
\]

Let us calulate the fundamental vector fields on \(\hat{\mathcal{M}}\) corresponding to the matrices \(U_1 \in \mathfrak{n}_B^1\) and \(U_2 \in \mathfrak{n}_B^2\) defined by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
E & 0 & 0 & 0 & 0 & 0 \\
0 & E & 0 & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & 0 & E & 0 & 0 \\
0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & 0 & 0 & 0 & E
\end{pmatrix} \quad \begin{pmatrix}
X^{11} & X^{12} & \Xi^{11} & \Xi^{12} \\
X^{11} & X^{12} & \Xi^{11} & \Xi^{12} \\
X^{21} & X^{22} & \Xi^{21} & \Xi^{22} \\
X^{21} & X^{22} & \Xi^{21} & \Xi^{22}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
E & 0 & 0 & 0 \\
0 & E & 0 & 0 \\
0 & 0 & E & 0 \\
0 & 0 & 0 & E \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \begin{pmatrix}
\eta^1 & \eta^2 \\
\eta^3 & \eta^4 \\
\eta^5 & \eta^6 \\
\eta^7 & \eta^8
\end{pmatrix}.
\]

Here \(U_1 \in \text{Mat}_{k_1-k_2, m-k_1}(\mathbb{C})\) and \(U_2 \in \text{Mat}_{l_1-l_2, n-l_1}(\mathbb{C})\). The first matrix is divided in blocks of size \((m-k_1, k_1-k_2, k_2, n-l_1, l_1-l_2, l_2)\), other matrices are divided in blocks accordingly. We will denote by \(U_i^*\) the fundamental vector fields on \(\hat{\mathcal{M}}\) corresponding to \(U_i\). Consider \([29]\). As we have seen above \(c(x) \in (\mathbb{T}_B)_{x_0}\). Further any fundamental vector field is projectible. Therefore we can decompose \([29]\) into the horizontal and the vertical parts. Our idea is to show that the vertical part of \([29]\) does not hold true.
After a direct computation we get $\hat{U}_i^* = (\hat{U}_i^*)^h + (\hat{U}_i^*)^v$, where

$$
(\hat{U}_i^*)^v = \sum_{\alpha \beta} (U_1 X^{11} X^2 + U_1 X^{12} )_{\alpha \beta} \frac{\partial}{\partial x^2_{\alpha \beta}} + \sum_{\alpha \beta} (U_1 X^{11} \Xi^2 + U_1 \Xi^{11} Y^2 + U_1 \Xi^{12} )_{\alpha \beta} \frac{\partial}{\partial \xi_{\alpha \beta}};
$$

$$
(\hat{U}_i^*)^v = \sum_{\alpha \beta} (U_2 Y^{11} Y^2 + U_2 Y^{12} )_{\alpha \beta} \frac{\partial}{\partial y_{\alpha \beta}} + \sum_{\alpha \beta} (U_2 Y^{11} H^2 + U_2 H^{11} X^2 + U_2 H^{12} )_{\alpha \beta} \frac{\partial}{\partial \eta_{\alpha \beta}}.
$$

Denote by $\hat{U}_i^*$ the fundamental vector field on $\hat{M}$ corresponding to $U_i$. We have $\hat{U}_i^* = (\hat{U}_i^*)^h + (\hat{U}_i^*)^v$, where

$$
(\hat{U}_i^*)^v = \sum_{\alpha \beta} (U_1 X^{11} X^2 + U_1 X^{12} )_{\alpha \beta} \frac{\partial}{\partial x^2_{\alpha \beta}} + \sum_{\alpha \beta} (U_1 X^{11} \Xi^2 )_{\alpha \beta} \frac{\partial}{\partial \xi_{\alpha \beta}};
$$

$$
(\hat{U}_i^*)^v = \sum_{\alpha \beta} (U_2 Y^{11} Y^2 )_{\alpha \beta} \frac{\partial}{\partial y_{\alpha \beta}} + \sum_{\alpha \beta} (U_2 Y^{11} H^2 )_{\alpha \beta} \frac{\partial}{\partial \eta_{\alpha \beta}}.
$$

Therefore

$$
(\hat{c}_{ij}^*)^v = \sum_{\alpha \beta} x^{11}_{ja\alpha \beta} \frac{\partial}{\partial x^2_{ij}} + \sum_{\alpha \beta} x^{12}_{ja\alpha \beta} \frac{\partial}{\partial y^2_{ij}} + \sum_{\alpha \beta} x^{11}_{ja\alpha \beta} \frac{\partial}{\partial \eta^2_{ij}};
$$

$$
(\hat{f}_{kl}^*)^v = \sum_{\alpha \beta} y^{11}_{la\alpha \beta} \frac{\partial}{\partial x^2_{kl}} + \sum_{\alpha \beta} y^{12}_{la\alpha \beta} \frac{\partial}{\partial y^2_{kl}} + \sum_{\alpha \beta} y^{11}_{la\alpha \beta} \frac{\partial}{\partial \eta^2_{kl}}.
$$

Let us compute the vertical part of $\delta c$. We have

$$
(\hat{c}_{ij}^*)^v c(f_{kl}) = -b \left( \sum_{\alpha \beta} \xi^{11}_{jk\alpha \beta} \eta^{11}_{la\alpha \beta} \frac{\partial}{\partial x^2_{ij}} + \sum_{\alpha \beta} \xi^{11}_{jk\alpha \beta} \eta_{la\alpha \beta} \frac{\partial}{\partial y^2_{ij}} + \sum_{\alpha \beta} \xi^{11}_{jk\alpha \beta} \eta^{11}_{la\alpha \beta} \frac{\partial}{\partial \eta^2_{ij}} \right).
$$

$$
(\hat{f}_{kl}^*)^v c(e_{ij}) = -a \left( \sum_{\alpha \beta} \eta_{li\alpha \beta} \xi^{11}_{ja\alpha \beta} \frac{\partial}{\partial x^2_{ij}} + \sum_{\alpha \beta} \eta_{li\alpha \beta} \xi^{12}_{ja\alpha \beta} \frac{\partial}{\partial y^2_{ij}} + \sum_{\alpha \beta} \eta_{li\alpha \beta} \xi^{11}_{ja\alpha \beta} \frac{\partial}{\partial \eta^2_{ij}} \right).
$$

From (29) we get $a = b = 0$, therefore $Z^1(n_\mathcal{B}, T_{x_0})^{R_\mathcal{B}} = \{ 0 \}$, hence

$$
H^1(\mathcal{M}_0, \hat{T}_{22}) = H^1(n_\mathcal{B}, T_{x_0})^{R_\mathcal{B}} = \{ 0 \}.
$$

The proof is complete □
8.4 1st cohomology group with values in the sheaf \( \hat{\mathcal{T}}_{20} \)

We start with the following lemma.

**Lemma 36** Assuming (18) and (19), we have

\[
H^1(\mathcal{M}_0, \mathcal{C}_{20}^v) = H^1(\mathcal{M}_0, \mathcal{C}_{20}^v)^{\psi_{m|\nu}(\mathbb{C})_0} \quad \text{and} \quad H^1(\mathcal{M}_0, \hat{\mathcal{T}}_{20}^v) = H^1(\mathcal{M}_0, \hat{\mathcal{T}}_{20}^v)^{\psi_{m|\nu}(\mathbb{C})_0}.
\]

**Proof.** The second statement follows from the first one, from Theorem 26, Theorem 31 and from the following exact sequence

\[
0 \to H^1(\mathcal{M}_0, \mathcal{A}_{20}^v) = \{0\} \to H^1(\mathcal{M}_0, \hat{\mathcal{T}}_{20}^v) \to H^1(\mathcal{M}_0, \mathcal{C}_{20}^v).
\]

Let us prove the first statement. Consider the representation \( \tau^v \otimes \bigwedge^2 \psi \) of \( P \), see Lemma 11 and Lemma 12 corresponding to the homogeneous locally free sheaf \( \mathcal{C}_{20}^v = \Theta^v \otimes \bigwedge^2 \mathcal{O}_{20} \). A weight of this representation has the form \( \Lambda = \Lambda_0 + \Lambda_1 \), where

\[
\Lambda_0 = \begin{cases}
1. \mu_i^p - \mu_j^q, & 1 < p < q; \\
2. 0;
\end{cases}
\]

\[
\Lambda_1 = \begin{cases}
1. 0; \\
2. \lambda_i^p - \lambda_j^q, & 1 < p < q;
\end{cases}
\]

For \( \Lambda_0 \) is dominant and \( \Lambda_1 \) has index 1. In case (1.A) the weight \( \Lambda_1 \) is singular. In case (1.B) the weight \( \Lambda_0 \) cannot be dominant. In case (1.C) our assumptions we have only one possibility \( \Lambda_0 = 0 \) and \( \Lambda_1 = -\lambda_i^s + \lambda_j^t + 1 \). It is easy to see that the reflection \( s \) corresponding to the root \( \lambda_i^s - \lambda_j^{s+1} \) maps \( \Lambda + \zeta \), where \( \zeta \) is as in (23), to the weight \( \zeta \). Since \( s(\Lambda + \zeta) - \zeta = 0 \). In other words the weight \( \Lambda + \zeta \) corresponds to 1-dimensional trivial \( \mathfrak{gl}_{m|\nu}(\mathbb{C})_0 \)-module.

In cases (2.A) and (2.B) the weight \( \Lambda_0 \) is not dominant. In case (2.C) if \( \Lambda_0 \) is dominant we have only one possibility \( \Lambda_0 = 0 \). For \( \Lambda_1 \) we also have only one possibility \( \Lambda_1 = -\lambda_i + \lambda_{i+1} \) for some \( i \). This weight also corresponds to 1-dimensional trivial \( \mathfrak{gl}_{m|\nu}(\mathbb{C})_0 \)-module. □

The following theorem was proved in [O4].

**Theorem 37** [O4, Theorem 1] Assume that (18) and (19) hold true and \( r = 1 \). Then

\[
H^1(\mathcal{M}_0, \hat{\mathcal{T}}_2) = \mathbb{C}.
\]

Now we are ready to prove the following theorem.
Theorem 38 Assume that (18) and (19) hold true. If \( H^1(S_0, \hat{T}_S) = C \), then

\[ H^1(M_0, \hat{T}_{20}) = C. \]

Proof. Assume that \( H^1(S_0, \hat{T}_S) = C \). By Lemma 30 we need to calculate \( H^1(M_0, T_{20})^{\hat{\theta}_{min}(C)} \). Denote by \( P_S \) a copy of the group \( P \) for the supermanifold \( S \) and by \( R_S \) the reductive part of \( P_S \). Denote also by \( T_{20}^{\nu = 0} \) and \( (T_S) \) the vector bundles corresponding to the locally free sheaves \( \hat{T}_{20} \) and \( (\hat{T}_S) \). Let us prove the equality

\[ H^1(n, (T_{20})_{x_0})^R = H^1(n_S, ((T_S)_{x_0})^{R_S}, \]

where \( x_0 = P, \bar{x}_0 = P_S \) and \( n_S \) is the nilradical of the Lie algebra of \( P_S \). As usual we denote by \( E_x \) a fiber of a vector bundle \( E \). Note that by Proposition 10 we have

\[ (T_{20})_{x_0} = (T_{x_0})_{x_0}. \]

The representation of \( P_S \) in \( ((T_S)_{x_0})_{x_0} \) is equal to the restriction of the representation of \( P \) in \( (T_{20})_{x_0} \) to the subgroup \( P_S \subset P \). The \( R \)-module \( n \) is a completely reducible \( R \)-module with highest weights

\[-\mu_{\max}^p + \mu_{\min}^q, \quad -\lambda_{\max}^p + \lambda_{\min}^q, \quad p < q, \quad p, q \in \{1, \ldots, r + 1\}.\]

The \( R_S \)-module \( n_S \) is a completely reducible \( R_S \)-module with highest weights

\[-\mu_{\max}^p + \mu_{\min}^q, \quad -\lambda_{\max}^p + \lambda_{\min}^q, \quad p < q, \quad p, q \in \{2, \ldots, r + 1\}.\]

We denoted by \( \mu_{\max}^p \) the element \( \mu_i^p \) with maximal possible index \( i \) from the block \( p \) and by \( \mu_{\min}^q \) the element \( \mu_j^p \) with minimal possible index \( j \) from the block \( q \). The same agreement we use for \( \lambda \)'s.

Further we note that the weights of \( P \) in a fiber of \( (T_{20})_{x_0} \) is equal to the weights of \( P \) in the fiber of the sheaf \( A_{20}^v \) plus the weights of \( P \) in the fiber of the sheaf \( C_{20}^v \).

Above we calculated the representations corresponding to the homogeneous sheaves \( A_{20}^v \) and \( C_{20}^v \). Using Lemma 11 and Lemma 12 we get that they have the following form

\[ \phi^3 \otimes \bigwedge^3 \phi \quad \text{and} \quad \tau^2 \otimes \bigwedge^2 \phi, \]

respectively. We see that the corresponding weights do not contain elements \( -\mu_{\max}^1 \) and \( -\lambda_{\max}^1 \). Moreover we have the following decomposition of \( R \)-modules \( n = n_B \oplus n_S \), where \( n_S \) is a completely reducible \( R \)-module with highest weights \( -\mu_{\max}^1 + \mu_{\min}^q \) and \( -\lambda_{\max}^1 + \lambda_{\min}^q \), where \( q > 1 \). (Compare also with the proof of Theorem 35.)

Now we see that we have the equality of vector spaces of cochains

\[ C^1(n, (T_{20})_{x_0})^R = C^1(n_S, ((T_S)_{x_0})^{R_S}. \]

Let us prove that \( n_B \) acts trivially in \( (T_{20})_{x_0} \). Any fundamental vector field on \( \tilde{\mathcal{N}} \) has the form \( v = v^h + v^c \), where \( v^h \in \tilde{T}^h_{00} \) and \( v^c \in \tilde{T}^c_{00} \). The part \( v^h \) acts on \( (T_{20})_{x_0} \).
trivially since sections of the sheaf $\hat{T}_{20}^v$ do not depend on coordinates $Z_{I_1}$. Further the part $v^u$ also acts trivially on $(T_{20}^u)_{x_0}$. Indeed, $n_S$ contains all matrices of the following form
\[
\begin{pmatrix}
0 & 0 \\
C_1 & 0 \\
C_2 & 0
\end{pmatrix}
\times
\begin{pmatrix}
0 & 0 \\
C_1 & 0 \\
C_2 & 0
\end{pmatrix},
\]
where $C_1 \in \text{Mat}_{k_1 \times m-k_1}(\mathbb{C})$ and $C_2 \in \text{Mat}_{l_1 \times n-l_1}(\mathbb{C})$. In coordinates $(Z_I)$, see (16) for definition of $I$, after a short calculation we get
\[
\begin{pmatrix}
E + C_1X^1 & C_1\Xi^1 \\
C_2H^1 & E + C_2Y^1
\end{pmatrix}
Z_{I_2}, Z_{I_3}, \ldots.
\]
To get the action in $(T_{20}^v)_{x_0}$ we need to put $\Xi^1 = H^1 = 0$ and $X^1 = Y^1 = 0$ in (30). Now we see that this action is trivial. Therefore the action of $n$ and of $n_S$ in the fibers $(T_{20}^v)_{x_0}$ and $((T_S)_{x_0})$ coincide. Therefore,
\[
H^1(n, (T_{20}^v)_{x_0})^R = H^1(n_S, ((T_S)_{x_0})^R_S,
\]
The result follows. □

We will use the notation $V \leq \mathbb{C}$ for a vector space $V$ over $\mathbb{C}$. This meant that $V$ is either equal to $\mathbb{C}$ or $\{0\}$. From Theorem 38 we get.

**Corollary 39** Assume (18) and (19). If $H^1(S_0, (\hat{T}_S)_{2}) = \mathbb{C}$, then $H^1(M_0, \hat{T}_{20}) \leq \mathbb{C}$.

**Proof.** Consider the following exact sequence
\[
0 \rightarrow T_{20}^v \rightarrow T_{20} \rightarrow T_{20}^h \rightarrow 0.
\]
Therefore we get
\[
\mathbb{C} \rightarrow H^1(M_0, \hat{T}_{20}) \rightarrow 0. \quad \square
\]
We are ready to prove one of our main results.

**Theorem 40** Assuming (18) and (19), we have
\[
H^1(M_0, \hat{T}_2) = \mathbb{C}
\]

**Proof.** Under Theorem’s assumption the flag supermanifold $\mathcal{M}$ is not split. This fact can be deduced for instance from results [V1], where the Lie algebras of holomorphic vector fields were computed. The idea is the following: a supermanifold is split if and only if it possesses a so called grading vector field. (Details about grading vector fields can be found in [V5].)

Further in [Gr] it was shown that the classes of non-split supermanifolds are in one-to-one correspondence with orbits of a certain group acting on $H^1(M_0, \hat{T}_2)$, and the point 0 corresponds to the split supermanifold. Therefore from the fact that a flag
supermanifold is not split it follows that \( H^1(\mathcal{M}_0, \tilde{T}_2) \) is not zero. Now consider the following exact sequence of sheaves

\[
0 \to \tilde{T}_{2(q+1)} \to \tilde{T}_{2(q)} \to \tilde{T}_{2q} \to 0.
\]

For \( q = 3 \) we have an isomorphism \( \tilde{T}_{2(3)} \cong \tilde{T}_{23} \). Therefore,

\[
H^1(\mathcal{M}_0, \tilde{T}_{2(3)}) = H^1(\mathcal{M}_0, \tilde{T}_{23}) = \{0\}.
\]

Further

\[
0 \to H^1(\mathcal{M}_0, \tilde{T}_{2(2)}) \to H^1(\mathcal{M}_0, \tilde{T}_{22}) = \{0\},
\]

hence, \( H^1(\mathcal{M}_0, \tilde{T}_{2(2)}) = \{0\} \). Similarly, \( H^1(\mathcal{M}_0, \tilde{T}_{2(1)}) = \{0\} \). From the exact sequence

\[
0 \to H^1(\mathcal{M}_0, \tilde{T}_{2(0)}) \to H^1(\mathcal{M}_0, \tilde{T}_{20}) \leq \mathbb{C}
\]

it follows that \( H^1(\mathcal{M}_0, \tilde{T}_{2(0)}) \leq \mathbb{C} \). Finally we obtain

\[
\leq \mathbb{C} \to H^1(\mathcal{M}_0, \tilde{T}_{2(-1)}) \to H^1(\mathcal{M}_0, \tilde{T}_{20}) = \{0\}.
\]

Therefore, \( H^1(\mathcal{M}_0, \tilde{T}_2) = H^1(\mathcal{M}_0, \tilde{T}_{2(-1)}) \leq \mathbb{C} \)

**Corollary 41** Under the conditions of Theorem \( \text{40} \), the flag supermanifold \( \mathcal{M} \) is the unique non-split supermanifold up to isomorphism which corresponds to the split supermanifold \( \hat{\mathcal{M}} \).

Now we are ready to prove another main result.

**Theorem 42** Assuming \( \text{(18)} \) and \( \text{(19)} \), we have \( H^1(\mathcal{M}_0, \mathcal{T}) = \{0\} \).

**Proof.** The following sequence

\[
0 \to \mathcal{T}_{(p+1)} \to \mathcal{T}_{(p)} \to \mathcal{T}_p \to 0
\]

is exact. Hence, \( H^1(\mathcal{M}_0, \mathcal{T}_{(p)}) = \{0\} \) for \( p \leq 3 \) and \( H^1(\mathcal{M}_0, \mathcal{T}_{(2)}) \leq \mathbb{C} \). From \( H^0(\mathcal{M}_0, \tilde{T}_1) = \{0\} \) and from \( H^1(\mathcal{M}_0, \tilde{T}_1) = \{0\} \) we get \( H^1(\mathcal{M}_0, \mathcal{T}_{(1)}) \simeq H^1(\mathcal{M}_0, \mathcal{T}_{(2)}) \).

Further, for \( p = 0 \) we have

\[
0 \to H^0(\mathcal{M}_0, \mathcal{T}_{(1)}) \to H^0(\mathcal{M}_0, \mathcal{T}_{(0)}) \to H^0(\mathcal{M}_0, \tilde{T}_0) \to \mathbb{C}
\]

\[
H^1(\mathcal{M}_0, \mathcal{T}_{(1)}) \to H^1(\mathcal{M}_0, \mathcal{T}_{(0)}) \to H^1(\mathcal{M}_0, \tilde{T}_0)
\]

Under our assumption \( H^0(\mathcal{M}_0, \mathcal{T}_{(0)}) \simeq \mathfrak{gl}_{m|n}(\mathbb{C}) \). (This result was obtained in \[\text{VI}\]). Further by Theorem \( \text{14} \) \( H^0(\mathcal{M}_0, \tilde{T}_0) \simeq \mathfrak{gl}_{m|n}(\mathbb{C}) \). The image \( H^0(\mathcal{M}_0, \mathcal{T}_{(0)}) \to H^0(\mathcal{M}_0, \tilde{T}_0) \) has codimension 1 and contains the grading operator. We also proved that \( H^1(\mathcal{M}_0, \tilde{T}_0) = \{0\} \). From these facts we conclude that \( H^1(\mathcal{M}_0, \mathcal{T}_{(0)}) = \{0\} \).

Therefore,

\[
0 \to H^1(\mathcal{M}_0, \mathcal{T}_{(-1)}) \to H^1(\mathcal{M}_0, \tilde{T}_{-1}) = 0.
\]

The result follows. □
References

[A] Akhiezer D. N. Homogeneous complex manifolds. (Russian) Current problems in mathematics. Fundamental directions, Vol. 10, 223-275, 283.

[B] Bashkin M.A. Vector fields on a direct product of complex supermanifolds. (In Russian.) Sovremennye problemy matematiki i informatiki, V. 3. Yaroslavl’, YarGU, 2000. Pages 11-16.

[BL] Berezin F.A., Leites D.A. Supermanifolds. Soviet Math. Dokl. 16, 1975, 1218-1222.

[Bott] Bott R. Homogeneous vector bundles. Ann. Math. 66 (1957), 203-248.

[BO] Bunegina V.A., Onishchik A.L. Two families of flag supermanifolds. Differential Geometry and its Applications Volume 4, Issue 4, December 1994, Pages 329-360.

[Gr] Green P. On holomorphic graded manifolds. Proc. Amer. Math. Soc. 85, no. 4, 1982, Pages 587-590.

[K] Kaup L. Eine Künethformel für Fréchetgarben, Math. Zeitschr. 97, 1967, pp 158-168.

[L] Leites D.A. Introduction to the theory of supermanifolds. Russian Math. Surveys 35 (1980), 1-64.

[M1] Manin Yu.I. Gauge field theory and complex geometry, Grundlehren der Mathematischen Wissenschaften, V. 289, Springer-Verlag, Berlin, second edition, 1997.

[M2] Manin Yu.I. Topics in Noncommutative geometry. Princeton University Press, 1991.

[O1] Onishchik A.L. A construction of non-split supermanifolds. Ann. Global Anal. Geom. 16 (1998), no. 4, 309-333.

[O2] Onishchik A.L. Transitive Lie superalgebras of vector fields. Reports Dep. Math. Univ. Stockholm 26, 1987, 1-21.

[O3] Onishchik A.L. Non-split supermanifolds associated with the cotangent bundle. Universite de Poitiers, Departement de Math., N 109. Poitiers, 1997.

[O4] Onishchik A.L. On the rigidity of super-Grassmannians. Ann. Global Anal. Geom. 11 (1993), no. 4, 361-372.

[OS1] Onishchik A.L., Serov A.A. Holomorphic vector fields on Super-Grassmannians. Adv. in Soviet Mathematics. V. 5. Providence, AMS, 1995. P. 113-129.
[OS2] Onishchik A.L., Serov A.A. Vector fields and deformations of isotropic super-Grassmannians of maximal type, Lie Groups and Lie Algebras: E.B. Dynkin’s Seminar, Amer. Math. Soc. Transl. Ser. 2, vol. 169, Amer. Math. Soc., Providence, RI, 1995, pp. 75-90.

[OS3] Onishchik A.L., Serov A.A. On isotropic super-Grassmannians of maximal type associated with an odd bilinear form, E. Schrödinger Inst. for Math. Physics, Preprint No. 340, Vienna 1996.

[OS] Penkov I.B., Skornyakov I.A. Projectivity and D-affineness of flag supermanifolds, Russ. Math. Surv. 40, (1987) P. 233-234.

[P] Penkov I.B. Borel-Weil-Bott theory for classical Lie supergroups. (Russian) Translated in J. Soviet Math. 51 (1990), no. 1, 2108–2140. Itogi Nauki i Tekhniki, Current problems in mathematics. Newest results, Vol. 32, 71–124, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988.

[Va] Vaintrob A.Yu. Deformations of complex superspaces and coherent sheaves on them. J. Soviet Math. 51 (1990) 2069-2083.

[V1] Vishnyakova E.G. Vector fields on $\mathfrak{gl}_{m|n}(\mathbb{C})$-flag supermanifolds. Journal of Algebra, Volume 459, 2016, Pages 1-28.

[V2] Vishnyakova E.G. Vector fields on II-symmetric flag supermanifolds. São Paulo Journal of Mathematical Sciences, 2016, 1-16.

[V3] Vishnyakova E. G. On holomorphic functions on a compact complex homogeneous supermanifold. Volume 350, Issue 1, 15 January 2012, Pages 174-196.

[V4] Vishnyakova E.G. On complex Lie supergroups and split homogeneous supermanifolds. Transformation Groups 16 (2011), no. 1, 265-285.

[V5] Vishnyakova E.G. The Splitting Problem for Complex Homogeneous Supermanifolds. Journal of Lie theory 25(2), 2015, 459-476.

E. V.: Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, Av. Antônio Carlos, 6627, CEP: 31270-901, Belo Horizonte, Minas Gerais, BRAZIL, and Laboratory of Theoretical and Mathematical Physics, Tomsk State University, Tomsk 634050, RUSSIA,

e-mail: VishnyakovaE@googlemail.com