QUOTIENTS BY COMPLEX CONJUGATION FOR REAL COMPLETE INTERSECTION SURFACES

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Abstract. Quotients $Y = X / \text{conj}$ by the complex conjugation $\text{conj}: X \to X$ for complex surfaces $X$ defined over $\mathbb{R}$ tend to be completely decomposable when they are simply connected, i.e., split into connected sums $\#_n \mathbb{CP}^2 \#_m \overline{\mathbb{CP}}^2$ if $w_2(Y) \neq 0$, or into $\#_n (S^2 \times S^2)$ if $w_2(Y) = 0$. The author proves this property for complete intersections which are constructed by method of a small perturbation.

§1. Introduction

We mean by a Real variety (Real curve, Real surface etc.) a pair $(X, \text{conj})$, where $X$ is a complex variety and $\text{conj}: X \to X$ an anti-holomorphic involution called the real structure or the complex conjugation. Given an algebraic variety over $\mathbb{R}$ we consider the set of its complex points with the natural complex conjugation (the Galois transformation) as the corresponding Real variety. The fixed point set of $\text{conj}$ will be denoted by $X_{\mathbb{R}}$ and called the real part of $X$. We put $Y = X / \text{conj}$ and confuse in what follows $X_{\mathbb{R}}$ with its image $q(X_{\mathbb{R}})$ under the quotient map $q: X \to Y$.

If $(X, \text{conj})$ is a nonsingular Real curve then $Y$ is a compact surface with the boundary $X_{\mathbb{R}}$ and its topological type depends, obviously, only on the genus of $X$, the number of components in $X_{\mathbb{R}}$ and orientability of $Y$; the latter depends on vanishing of the fundamental class $[X_{\mathbb{R}}] \in H_1(X; \mathbb{Z}/2)$. It is also trivial enough that any compact surface with nonempty boundary can appear as $Y$ for some Real curve.

The subject of the author’s interest is topology of $Y$ in the case of nonsingular Real surfaces. It is not difficult to see that $Y$ in this case is a closed 4-manifold with the map $q$ being a 2-fold covering branched along $X_{\mathbb{R}}$. Moreover, $Y$ inherits from $X$ an orientation and a smooth structure making $q$...
smooth and orientation preserving. The natural question is to describe the
diffeomorphism types of 4-manifolds which can arise as the quotients $Y$ for
Real surfaces $X$. Another interesting question is if the topological types
of $X$ and $X_{\mathbb{R}}$ together with some information about the fundamental class
$[X_{\mathbb{R}}] \in H_2(X; \mathbb{Z}/2)$ determine the topology of $Y$ like in the case of curves.

It turns out that in all the known examples if $Y$ is simply connected then
it splits into a connected sum of copies of $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ or $S^2 \times S^2$. Let call this
property CDQ-property (complete decomposability for quotients) and call the
Real surfaces satisfying it CDQ-surfaces. Note that $Y$ is simply connected,
for example, if $X$ is simply connected and $X_{\mathbb{R}} \neq \emptyset$.

CDQ-property is known for all Real rational [3] and all Real $K3$ surfaces
[2] except the cases when $X_{\mathbb{R}} = \emptyset$. Further, CDQ-property is proved for a
plenty of double planes, for a family of elliptic surfaces and in fairly general
setting for doubles of CDQ-surfaces branched along certain “double” curves
(for details and more history see [2]).

In the present paper we show that CDQ-surfaces can be found among
complete intersections in $\mathbb{CP}^{n+2}$ of arbitrary multi-degrees $(d_1, \ldots, d_n)$. More
precisely, it appears that the straightforward construction of Real complete
intersections by a small perturbation method produces CDQ-surfaces.

This gives some arguments for the following relatively moderate conjecture:
any deformation type of simply connected complex algebraic surfaces contains
a CDQ-surface.

§2. Main results

Let $(V, \text{conj})$ be a Real variety. A holomorphic linear bundle $p: L \to V$
will be called a Real bundle if it is supplied with an anti-linear involution
$\text{conj}_L: L \to L$, which covers $\text{conj}$:

$$
\begin{array}{ccc}
L & \xrightarrow{\text{conj}_L} & L \\
p \downarrow & & \downarrow p \\
V & \xrightarrow{\text{conj}} & V
\end{array}
$$

A section $f: V \to L$ is called real if $\text{conj}_L \circ f = f \circ \text{conj}$. The zero divisor
of $f$ is, clearly, a Real subvariety of $V$.

Assume now that $(V, \text{conj})$ is a Real nonsingular and connected 3–fold,
$L_i \to V$, $i = 1, 2$, are real linear bundles and $f_i: V \to L_i$ are real sections
which zero divisors $X_0^{(i)}$ are nonsingular and intersecting transversally. Assume
further that $f: V \to L$ is a real section of $L = L_1 \otimes L_2$ which zero
divisor $X$ intersects transversally the surfaces $X_0^{(i)}$, $i = 1, 2$, and the curve
Consider the section \( f_\varepsilon : V \to L_0 \), \( f_\varepsilon = f_1 \otimes f_2 + \varepsilon f \), \( \varepsilon \in \mathbb{R} \), and denote by \( X_\varepsilon \) its zero divisor, which is nonsingular for a sufficiently small \( \varepsilon > 0 \), as it can be easily seen.

**Theorem 1.** If \( X_0^{(1)} \) and \( X_0^{(2)} \) are CDQ-surfaces, \( A \) is connected, \( A_\mathbb{R} \neq \emptyset \) and \( \varepsilon > 0 \) is small enough then \( X_\varepsilon \) is CDQ-surface as well.

We prove this theorem in §3 and discuss in the rest of this section some of its implications.

Let \((V, \text{conj})\) be as above. We call a real linear bundle \( L \) **CDQ-bundle** if it is very ample and admits a real section with a nonsingular CDQ zero divisor.

**Lemma 2.** If \( L \) is a CDQ-bundle then its multiples \( L \otimes d \), \( d \geq 1 \), are CDQ-bundles as well.

**Proof.** Let \( X_0^{(1)} \) be a CDQ-divisor of \( L \). We prove by induction on \( d \) that there exists a CDQ-divisor, \( X_0^{(2)} \), of \( L \otimes d \) which intersects \( X_0^{(1)} \) transversally along a curve having nonempty real part. This claim is trivial for \( d = 1 \), since we can perturb \( X_0^{(1)} \) so that the result will intersect \( X_0^{(1)} \) transversally and contain a given real point of it.

Suppose that \( X_0^{(2)} \) satisfies the induction assumption. By Lefschetz Theorem \( A \) is connected. A generic real section of \( L \otimes (d+1) \) has zero divisor \( X \) transversal to \( X_0^{(i)} \) and to \( X_0^{(1)} \cap X_0^{(2)} \), hence, we can apply Theorem 1 and get a CDQ-divisor \( X_\varepsilon \) by a perturbation of \( X_0 \cup X_0^{(2)} \) via \( X \). We can also choose \( X \) containing a real point of \( X_0^{(1)} \), since \( L \otimes (d+1) \) is very ample. Then, for a sufficiently small \( \varepsilon > 0 \), \( X_\varepsilon \) intersects \( X_0^{(1)} \) transversally and \( X_\varepsilon \cap X_0^{(1)} = X \cap X_0^{(1)} \) has nonempty real part. \( \square \)

**Theorem 3.** For arbitrary integers \( n, d_1, \ldots, d_n \geq 1 \) there exists a CDQ-surface \( X \subset \mathbb{C}P^{n+2} \) which is a complete intersection of multi-degree \((d_1, \ldots, d_n)\).

**Proof.** Induction on \( n \). \( \mathbb{C}P^2 / \text{conj} \) is diffeomorphic to \( S^4 \) (see, e.g., [1]), therefore, \( O(1)_{\mathbb{C}P^2} \) is a CDQ-bundle. By Lemma 2, \( O(d)_{\mathbb{C}P^3} \), \( \forall d \geq 1 \), is a CDQ-bundle as well. Assume now that we have given a complete intersection of Real hypersurfaces, \( X = H_1 \cap \cdots \cap H_n \subset \mathbb{C}P^{n+2} \), of multi-degree \((d_1, \ldots, d_n)\) and that \( X \) is CDQ-surface. Choose hypersurfaces \( H'_i \subset \mathbb{C}P^{n+3} \), \( i = 1, \ldots, n \), so that \( H'_i \cap \mathbb{C}P^{n+2} \) and the intersection \( V = H'_1 \cap \cdots \cap H'_n \) is transversal. Then the bundle \( L \to V \) induced from \( O(1)_{\mathbb{C}P^{n+3}} \) is CDQ-bundle, hence, \( X \) is its zero divisor. By Lemma 2, \( L \otimes d \) is also CDQ-bundle, hence, there exists a CDQ complete intersection of multi-degree \((d_1, \ldots, d_n, d)\). \( \square \)

**Remark.** The method used for Theorem 3 can be applied similarly to complete intersections in weighted projective spaces or in products of projective spaces and yields also CDQ-surfaces of arbitrary multi-degree.
§3 Proof of Theorem 1

Denote by \( N(i) \) a conj-invariant tubular neighborhood of \( A \) in \( X_0(i) \) and put \( \overline{N(i)} = N(i)/\text{conj} \). \( B = A/\text{conj} \), \( Y(i) = X_0(i)/\text{conj} \) and \( Y_\varepsilon = X_\varepsilon/\text{conj} \). It can be easily seen that \( \overline{N(i)} \) is a regular neighborhood of \( B \) in \( Y(i) \). Let \( 2k \) denote the number of imaginary points in \( A \cap X \).

**Proposition 4.** There exists a diffeomorphism \( \overline{\varphi} : \partial \overline{N(1)} \to \overline{N(2)} \), such that \( Y_\varepsilon \cong ((Y(1) - \overline{N(1)}) \cup \overline{\varphi}(Y(2) - \overline{N(2)})) \# k \mathbb{C}P^2 \).

Let see first how Theorem 1 follows from the above proposition.

Since \( A \) is connected and has nonempty real part, \( B \) is a compact connected surface with a nonempty boundary, hence, \( \overline{N(i)} \) are handlebodies with one 0-handle and several 1-handles embedded into \( Y(i) \). It is well known that if we glue a pair of simply connected 4-manifolds, \( Y(i), i = 1, 2 \), along the boundary of such handlebodies the result is diffeomorphic to \( Y(1) \# Y(2) \# g \mathbb{Z} \), where \( g = \text{rank}(H_1(B)) \) and \( Z = S^2 \times S^2 \) or \( \mathbb{C}P^2 \# \mathbb{C}P^2 \) (see, e.g., [4]). This implies complete decomposability of \( Y_\varepsilon \) if \( Y(i) \) are completely decomposable.

The proof of Proposition 4 follows closely the idea of [4]. By blowing up \( \hat{V} \to V \) we lift the pencil \( X_t = (1 - t)X_0 + tX \) to a real fibering \( \hat{V} \to \mathbb{C}P^1 \). Then we apply the deformation theorem of [4] in the equivariant version. Specifically, assume that \( \varepsilon \in \mathbb{R} \) and \( \varepsilon > 0 \) is sufficiently small. Then \( X_\varepsilon \) intersects \( X_0(i), i = 1, 2 \) transversally along the curve \( C_i = X_0(i) \cap X \). Consider first the blow-up, \( \hat{V} \to V \), along \( C_1 \) and denote by \( \hat{C}_i, \hat{X}_0(i) \), and \( \hat{X}_t \) the proper images of \( C_i, X_0(i) \) and \( X_t \). The pencil \( \hat{X}_t \) has the base-curve \( \hat{C}_2 \), therefore, the next blow-up \( \hat{V} \to \hat{V} \) along \( \hat{C}_2 \) gives a fibering over \( \mathbb{C}P^1 \) with fibers \( \hat{X}_t \) (here and below we mark by a hat the proper image in \( \hat{V} \)).

The projections \( \hat{X}_\varepsilon \to X_\varepsilon, \hat{X}_0(1) \to X_0(1) \) are biregular, as well as \( \hat{X}_0(2) \to \hat{X}_0(2) \), whereas \( \hat{X}_0(2) \to X_0(2) \) is the blow-up at \( C_1 \cap X_0(2) = A \cap X \).

The real structure on \( V \) can be obviously lifted to the real structure, \( \text{conj}_\hat{V} : \hat{V} \to \hat{V} \), and we have \( \hat{V}(1) \cong Y(1), \hat{Y}_\varepsilon \cong Y_\varepsilon \) and \( \hat{V}(2) \cong Y(2) \# k(\mathbb{C}P^2) \), where \( \hat{V}(1), \hat{Y}_\varepsilon, \hat{V}(2) \) denote the quotients by \( \text{conj}_\hat{V} \) of \( \hat{X}_0(1), \hat{X}_\varepsilon \) and \( \hat{X}_0(2) \). The latter diffeomorphism follows from that blow-ups at real points do not change the diffeomorphism type of the quotient, since \( \mathbb{C}P^2 / \text{conj} \cong S^4 \), whereas a pair of blow-ups at conjugated imaginary points descends to a blow-up in the quotient. Restrictions give diffeomorphisms between \( N(i) \) and regular neighborhoods of \( \hat{A} / \text{conj}_\hat{V} \) in \( \hat{X}_0(i) / \text{conj}_\hat{V} \) for \( \hat{A} = \hat{X}_0(1) \cap \hat{X}_0(2) \).

To complete the proof we use the deformation theorem [4]. Recall the statement of one of its corollaries.
Let \( f: W \to \Delta \) be a nonconstant proper holomorphic mapping of a 3-fold \( W \) into a disc, \( \Delta \subset \mathbb{C} \), around zero. Assume that \( f \) has a critical value only at zero and the zero divisor \( X_0 \) of \( f \) consists of two nonsingular irreducible components \( X_0^{(i)}, i = 1, 2, \) of multiplicity 1 crossing transversally along a nonsingular irreducible curve \( A \). Suppose that \( U \subset W \) is a sufficiently small tubular neighborhood of \( A \), so that \( N^{(i)} = U \cap X_0^{(i)} \) is a tubular neighborhood of \( A \) in \( X_0^{(i)}, i = 1, 2 \). Then there exists a bundle isomorphism \( \varphi: \partial N^{(1)} \to \partial N^{(2)} \), reversing orientations of fibers, such that \( X_t = f^{-1}(t) \) is diffeomorphic to \( (X_0^{(1)} - N^{(1)}) \cup \varphi(X_0^{(2)} - N^{(2)}) \) for a non-critical value \( t \in \Delta \).

In the equivariant version of this theorem we assume in addition that \( W \) is supplied with a real structure \( \text{conj}_W: W \to W \) and \( f \circ \text{conj}_W = \text{conj} \circ f \), where \( \text{conj}: \Delta \to \Delta \) is the complex conjugation on \( \mathbb{C} \). Then one can choose the neighborhood \( U \) to be \( \text{conj}_W \)-invariant, make the isomorphism \( \varphi \) \( \text{conj}_W \)-equivariant and the diffeomorphism \( X_t \cong (X_0^{(1)} - N^{(1)}) \cup \varphi(X_0^{(2)} - N^{(2)}) \) commute with the involutions of complex conjugation in \( X_t \) and \( X_0^{(i)} - N^{(i)} \).

To prove the latter one should repeat the arguments of [4] with some, not essential modifications: we need to choose a \( \text{conj}_W \)-invariant metric on \( W \) and instead of the fibering \( U \cap X_t \to A \) considered in [4] deal with its quotient, \( U \cap X_t / \text{conj}_W \to B \), and then apply similarly the arguments on the reduction of the structure group.

**References**

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