Counting perfect matchings of cubic graphs in the geometric dual

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Abstract
Lovász and Plummer conjectured, in the mid 1970’s, that every cubic graph G with no cutedge has an exponential in |V(G)| number of perfect matchings. In this work we show that every cubic planar graph G whose geometric dual graph is a stack triangulation has at least \(3\varphi^{\left|V(G)\right|/72}\) distinct perfect matchings, where \(\varphi\) is the golden ratio. Our work builds on a novel approach relating Lovász and Plummer’s conjecture and the number of so called groundstates of the widely studied Ising model from statistical physics.

1 Introduction

A graph is said to be cubic if each vertex has degree 3 and bridgeless if it contains no cutedges. As early as in 1891 Petersen proved that every cubic bridgeless graph has a perfect matching. Nowadays, this famous theorem is obtained indirectly using major results such as Hall’s Theorem from 1935 and Tutte’s 1-factor Theorem from 1947.

In the mid-1970’s, Lovász and Plummer asserted that for every cubic bridgeless graph with \(n\) vertices, the number of perfect matchings is exponential in \(n\). The best result known is a superlinear lower bound by Esperet, Kardos and Král’ [3].

The conjecture remains open despite considerable attempts to prove. So far, there are three classes of cubic graphs for which the conjecture has been proved. For bipartite graphs, the assertion was shown by Voorhoeve [9] who proved: Every cubic bipartite graph with \(n\) vertices has at least \(6(4/3)^{\frac{n}{2}}\) perfect matchings. This result was later extended to \(k\)—regular bipartite graphs by Schrijver [8]. The conjecture was positively solved for the class of planar graphs by Chudnovsky and Seymour [2] who showed: Every cubic bridgeless planar graph with \(n\) vertices has at least \(2^c n\) perfect matchings, where \(c = 1/65597852\). Oum [7] recently established the conjecture for the class of claw-free cubic graphs: Every claw-free cubic bridgeless graph with \(n\) vertices has at least \(2^{n/12}\) perfect matchings.

In what follows, we restrict to the class of planar graphs. We suggest to study the conjecture of Lovász and Plummer in the dual setting. This relates the conjecture to a phenomenon well known in statistical physics, namely to the degeneracy of the Ising model on totally frustrated triangulations of the plane.

A planar graph is a triangulation if each face is bounded by a cycle of length 3. Note that the dual graph \(G^*\) of a cubic bridgeless planar graph \(G\) is a triangulation. A set \(M\) of edges of a triangulation \(\Delta\) is intersecting if \(M\) contains exactly one edge of each face of \(\Delta\). Clearly, \(M\) is an intersecting set of \(G^*\) if and only if \(M\) is a perfect matching of \(G\). Now, with the previous definitions, we can reformulate the conjecture of Lovász and Plummer for

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the class of planar graphs as follows: Each planar triangulation has an exponential number of intersecting sets of edges.

Next, let us consider the Ising model. Given a triangulation $\Delta = (V, E)$ we associate the coupling constant $c(e) = -1$ with each edge $e \in E$. For any $W \subseteq V$, a spin assignment of $W$ is any function $s : W \rightarrow \{1, -1\}$ and $1, -1$ are called spins. A state of $\Delta$ is any spin assignment of $V$. The energy of a state $s$ is defined as $-\sum_{e=(u,v) \in E} c(e) s(u) s(v)$. The states of minimum energy are called groundstates. The number of groundstates is usually called the degeneracy of $\Delta$, denoted $g(\Delta)$, and it is an extensively studied quantity (for regular lattices) in statistical physics. Given a state $s$ of $\Delta$ we say that edge $\{u, v\}$ is frustrated by $s$ or that $s$ frustrates edge $\{u, v\}$ if $s(u) = s(v)$. Clearly, each state frustrates at least one edge of each face of $\Delta$. A state is a groundstate if it frustrates the smallest possible number of edges.

We say that a state $s$ is satisfying for a face $f$ of a planar triangulation $\Delta$, if there is exactly one edge $e = \{u_1, u_2\}$ in the boundary of $f$ that is frustrated by $s$. Moreover, we say that $s$ is a satisfying state of $\Delta$ if $s$ is satisfying for every inner face $f$ of $\Delta$. Clearly, the set of edges frustrated by a satisfying state which is also satisfying for the outer face is an intersecting set. Hence, the number of satisfying states which are also satisfying for the outer face, is at most twice the number of intersecting sets of edges. The converse also holds: if we delete an intersecting set of edges from a planar triangulation, then we get a bipartite graph and its bipartition determines a satisfying spin assignment which is also satisfying for the outer face. Given that any planar triangulation $\Delta$ has an intersecting set, (induced by a perfect matching in its dual), it follows that $s$ is a satisfying state of $\Delta$ which is also satisfying for its outer face if and only if $s$ is a groundstate of $\Delta$. Summarizing, the degeneracy is twice the number of intersecting sets. Hence, Chudnovsky and Seymour’s result can be reformulated as follows: Each planar triangulation has an exponential (in the number of vertices) degeneracy. This motivated Jiménez, Kiwi and Loebl [5] to consider the problem of lower bounding the degeneracy of triangulations of an $n$-gon, as well as the use of the (transfer matrix) method for achieving their goal. Since the dual of triangulations of $n$-gons are seldom cubic graphs, the results of [5] do not directly relate to Lovász and Plummer’s conjecture, not even for a subfamily of cubic graphs. In this article, we further develop the approach proposed in [5] and establish the feasibility of using it to attack Lovász and Plummer’s conjecture for a non-trivial subclass of cubic graphs. More precisely, the subclass of cubic bridgeless planar graphs whose geometric dual are stack triangulations (also called 3-trees [11, page 167]). Specifically, provided $\varphi = (1 + \sqrt{5})/2 \approx 1.6180$ denotes the golden ratio, we establish the following:

**Theorem 1** The degeneracy of any stack triangulation $\Delta$ with $|\Delta|$ vertices is at least $6\varphi^{(|\Delta|+3)/36}$.

As a rather direct consequence of the preceding theorem we obtain the following result.

**Corollary 2** The number of perfect matchings of a cubic graph $G$ whose dual graph is a stack triangulation is at least $3\varphi^{|V(G)|}/72$.

Note that the preceding result applies to a subclass of graphs for which Chudnovsky and Seymour’s [2] work already establishes the validity of Lovász and Plummer’s conjecture, albeit for a smaller rate of exponential growth and arguably by more complicated and involved arguments. We believe that the main relevance of this work is that it validates the feasibility of the alternative approach proposed in [5] for approaching Lovász and Plummer’s conjecture.

### 1.1 Organization

The paper is organized as follows. We provide some mathematical background in Section 2. Then, in Section 3 we describe a bijection between rooted stack triangulations and colored rooted ternary trees — this bijection allows us to work with ternary trees instead of triangulations. In Section 4 we first introduce the concept of degeneracy
vector in stack triangulations. This vector satisfies that the sum of its coordinates is the number of satisfying states of the stack triangulation. We also introduce the concept of root vector of a ternary trees and show that via the aforementioned bijection, the degeneracy vector of a stack triangulation $\Delta$ is the same as the root vector of the associated colored rooted ternary tree. In Section 5 we adapt to our setting the transfer matrix method as used in statistical physics in the study of the Ising Model. Some essential results are also established. In Section 6 we prove the main results of this work. In Section 7 we conclude with a brief discussion and comments about possible future research directions.

2 Preliminaries

We now introduce the main concepts and notation used throughout this work.

2.1 Stack triangulations

Let $\Delta_0$ be a triangle. For $i \geq 1$, let $\Delta_i$ be the plane triangulation obtained by applying the following growing rule to $\Delta_{i-1}$.

**growing rule:** Given a plane triangulation $\Delta$,

1. Choose an inner face $f$ from $\Delta$,
2. Insert a new vertex $u$ at the interior of $f$.
3. Connect the new vertex $u$ to each vertex of the boundary of $f$.

Clearly, the number of vertices of $\Delta_n$ is $n + 3$. The collection of $\Delta_n$’s thus obtained are called stack triangulation. Among others, the set of stack triangulations coincides with the set of plane triangulations having a unique Schnyder Wood (see [4]) and is the same as the collection of planar 3-trees (see [1, page 167]).

Consider now a stack triangulation $\Delta_1$ and for $i \geq 2$, let $\Delta_i$ be the plane triangulation obtained by applying the growing rule to $\Delta_{i-1}$ restricting Step 1 so the face chosen is one of the three new faces obtained by the application of the growing rule to $\Delta_{i-2}$. For $n \geq 1$, we say that $\Delta_n$ is a stack-strip triangulation (for an example see Figure 1). Clearly, stack-strip triangulations are a subclass of stack triangulations.

![Figure 1: Example of stack-strip triangulation (numbers correspond to the order in which nodes are added by the growing rule).](image)

Let $\Delta_n$ be a stack triangulation with $n \geq 0$ and $\Delta_0$ be the starting plane triangle in its construction. If we prescribe the counterclockwise orientation to any edge of $\Delta_0$, we say that $\Delta_n$ is a rooted stack triangulation (see Figure 2).
2.2 Ternary trees

A rooted tree is a tree $T$ with a special vertex $v \in V(T)$ designated to be the root. If $v$ is the root of $T$, we denote $T$ by $T_v$. A rooted ternary tree is a rooted tree $T_v$ such that all its vertices have at most three children. From now on, let $X$ be an arbitrary set with three elements. We say that a rooted ternary tree $T_v$ is colored by $X$ (or simply colored) if: (1) each non-root vertex is labeled by an element of $X$, and (2) for every vertex of $V(T)$ all its children have different labels.

3 From stack triangulations to ternary trees

It is well known that stack triangulations are in bijection with ternary trees (see [6]). For our purposes, the usual bijection is not enough (we need a more precise handle on the way in which triangular faces touch each other). The main goal of this section is to precisely describe a one-to-one correspondence better suited for our purposes.

3.1 Bijection

Let $\Delta_n$ be a rooted stack triangulation with $n \geq 1$ and $\Delta_0$ be the starting plane triangle in its construction. We will show how to construct a colored rooted ternary tree $T(\Delta_n)$ which will be in bijective correspondence with $\Delta_n$. Throughout this section, the following concept will be useful.

Definition 1 Let $\Delta$ be a rooted stack triangulation. Let $\tilde{\Delta}$ be the rooted stack triangulation obtained by prescribing the counterclockwise orientation to exactly one edge of each inner face of $\Delta$. We refer to $\tilde{\Delta}$ as an auxiliary stack triangulation of $\Delta$.

Note that in an auxiliary stack triangulation of $\Delta$, we allow inner faces of $\Delta$ to have edges oriented clockwise as long as exactly one of its edges is oriented counterclockwise. It is also allowed to have edges with both orientations. We now, describe the key procedure in the construction of $T(\Delta_n)$. For $i \in \{1, \ldots, n\}$, let $f_i$, $u_i$ and $\Delta_i$, denote the chosen face, the new vertex and the output corresponding to the $i$-th application of the growing rule in the construction of $\Delta_n$. Initially, $i = 1$ and $\tilde{\Delta}_0$ is $\Delta_0$ with one of its edges oriented counterclockwise.

Labeling procedure:

Step 1: Let $\tilde{e}_f$ be the counterclockwise oriented edge of $f_i$. The orientation of $\tilde{e}_f$ induces a counterclockwise ordering of the three new faces around $u_i$, starting by the face that contains $\tilde{e}_f$, say $f_i(1)$. Let $f_i(2)$ and $f_i(3)$ denote the second and third new faces according to the induced order. For each $j \in \{1, 2, 3\}$, we say that $f(j)$ is in position $j$ or that $j$ is the position of $f_i(j)$. (See Figure 2)
Step 2: For each \( j \in \{2, 3\} \), take the unique edge \( e_{f_i}(j) \) in \( E(f_i) \cap E(f_i(j)) \) and prescribe the counterclockwise orientation to this edge (see Figure 3). For all other faces of \( \Delta_i \) not contained in \( f_i \), keep the same counterclockwise oriented edge. (Observe that for each \( j \in \{1, 2, 3\} \), the triangle \( f_i(j) \) has a prescribed counterclockwise orientation in one of its three edges. Moreover, note that \( \vec{e}_{f_i} = \vec{e}_{f_i(1)} \).)

\[
\begin{align*}
\vec{e}_{f_i} & = \vec{e}_{f_i(1)} \\
\end{align*}
\]

Figure 3: Labeling procedure. Left to center, step 1. Center to right, step 2.

The set \( \Theta_{\Delta_n} = \{(f_i, u_i, f_i(1), f_i(2), f_i(3))\}_{i \in \{1, \ldots, n\}} \) will be henceforth referred to as the growth history of \( \Delta_n \). Note that, for \( j \in \{1, 2, 3\} \), each face \( f_1(j) \) together with its oriented edge induce a rooted stack triangulation, henceforth denoted \( \Delta_n^j \), on the vertices of \( \Delta_n \) that lie on the boundary and interior of \( f_1(j) \).

We are ready to describe \( T(\Delta_n) \) in terms of the growth history of \( \Delta_n \):

**Combinatorial description of** \( T(\Delta_n) \): Let \( X = \{1, 2, 3\} \). Let \( V(T(\Delta_n)) = \{u_1, \ldots, u_n\} \). Let \( u_1 \) be the root of \( T(\Delta) \). For \( i \in \{2, \ldots, n\} \), \( u_i \) is a child of vertex \( u_j \) if there is a \( k \in \{1, 2, 3\} \) such that \( f_i = f_j(k) \). The label of \( u_i \) is \( k \). For an example see Figure 4.

Figure 4: Example of the bijection between rooted stack triangulations and colored rooted ternary trees.

In particular, we have proved the following result.

**Proposition 3** Let \( \Delta_n \) be a rooted stack triangulation. The colored ternary tree \( T(\Delta_n) \) rooted on \( v \), satisfies the following statements:

1. If \( \Delta_n^i \) has 3 vertices for all \( i \in \{1, 2, 3\} \), then \( T(\Delta_n) \) has exactly one vertex \( v \) (its root).

2. If there are \( i, j \in \{1, 2, 3\} \) with \( i \neq j \) such that \( \Delta_n^i \) and \( \Delta_n^j \) have 3 vertices and \( \Delta_n^k \) with \( k \in \{1, 2, 3\} \setminus \{i, j\} \) has at least 4 vertices, then the root \( v \) has exactly one child \( w \) labeled by \( k \). Moreover, the root of \( T(\Delta_n^k) \) is \( w \), where \( T(\Delta_n^k) \) is the colored sub-ternary tree of \( T(\Delta_n) \) induced by \( w \) and its descendants.
3. If there is an $i \in \{1, 2, 3\}$ such that $\Delta_n^i$ has 3 vertices and $j, k \in \{1, 2, 3\} \setminus \{i\}$ with $j \neq k$ such that $\Delta_n^j$ and $\Delta_n^k$ have at least 4 vertices, then the root $v$ has exactly two children $w_j$ and $w_k$ labeled by $j$ and $k$, respectively. Moreover, for every $t \in \{j, k\}$, the root of $T(\Delta_n^t)$ is $w_t$, where $T(\Delta_n^t)$ is the colored sub-ternary tree of $T(\Delta_n)$ induced by $w_t$ and its descendants.

4. If $\Delta_n^i$, $i \in \{1, 2, 3\}$, has at least 4 vertices, then the root $v$ has three children $w_1, w_2$ and $w_3$ labeled by 1, 2 and 3, respectively. Moreover, for every $i \in \{1, 2, 3\}$, the root of $T(\Delta_n^i)$ is $w_i$, where $T(\Delta_n^i)$ is the colored sub-ternary tree of $T(\Delta_n)$ induced by $w_i$ and its descendants.

4 Transfer Method

The main tool we use to carry out our work, is an adaptation of a method (well known among physicist) called the transfer matrix method. In [5], we directly apply the transfer matrix method to obtain the number of satisfying states of triangulations of a convex $n$-gon. In this work we develop the technique further by considering transfer vectors instead of transfer matrices.

4.1 Methodology

In general terms, our aim is to obtain for each stack triangulation $\Delta$ a vector $\nu_\Delta$ in $\mathbb{R}^4$ such that the sum of its coordinates equals twice the number of satisfying states of $\Delta$. We now elaborate on this. Let $n \geq 1$ and $\Delta_n$ be a rooted stack triangulation. Let $\Delta_0 = (v_1, v_2, v_3)$ denote the starting triangle in the construction of $\Delta_n$ such that $\{v_1, v_2\}$ is the oriented edge with $v_1$ the tail and $v_2$ the head. We wish to construct a vector $\nu_{\Delta_n} \in \mathbb{R}^4$ such that its coordinates are indexed by the ordered set $I = \{+++, +++, ++-, +-, -, --, --+\}$. For every $\phi \in I$, the $\phi$-th coordinate of $\nu_{\Delta_n}$, denoted $\nu_{\Delta_n}[\phi]$, is defined as the number of satisfying states of $\Delta_n$ when the spin assignment of $(v_1, v_2, v_3)$ is equal to $\phi$. The vector $\nu_{\Delta_n}$ will be called the degeneracy vector of $\Delta_n$. In particular, $\nu_{\Delta_0} = (0, 1, 1, 1)^t$ is the degeneracy vector of a triangle. Clearly, for every $\phi \in I$ we have the relation

$$\nu_{\Delta_n}[\phi] = \nu_{\Delta_n}[-\phi] \quad (1)$$

Let $\Theta_{\Delta_n} = \{(f_i, u_i, f_i(1), f_i(2), f_i(3))\} \in \{1, \ldots, n\}$ be the growth history of $\Delta_n$. Let $v$ denote $u_1$. Recall that $f_i(j)$ induces a rooted stack triangulation $\Delta_n^j$, according to the growth history of $\Delta_n$, (see Subsection 4.1): the oriented edge of $\Delta_n^1$ is $\{v_1, v_2\}$ with $v_1$ its tail and $v_2$ its head; the oriented edge of $\Delta_n^2$ is $\{v_2, v_3\}$ with $v_2$ its tail and $v_3$ its head; and the oriented edge of $\Delta_n^3$ is $\{v_3, v_1\}$ with $v_3$ its tail and $v_1$ its head. The following result shows how to express the degeneracy vector of $\Delta_n$ in terms of the degeneracy vectors $\nu_{\Delta_n^1}$, $\nu_{\Delta_n^2}$, and $\nu_{\Delta_n^3}$.

**Proposition 4** For each $j \in \{1, 2, 3\}$, let $\nu_{\Delta_n^j} = (\nu_{\Delta_n^j})_{k \in \{0, 1, 2, 3\}}$. Then,

$$\nu_{\Delta_n} = \begin{pmatrix}
\nu_{\Delta_n}^0 & \nu_{\Delta_n}^1 & \nu_{\Delta_n}^2 & \nu_{\Delta_n}^3 \\
\nu_{\Delta_n}^0 & \nu_{\Delta_n}^1 & \nu_{\Delta_n}^2 & \nu_{\Delta_n}^3 \\
\nu_{\Delta_n}^0 & \nu_{\Delta_n}^1 & \nu_{\Delta_n}^2 & \nu_{\Delta_n}^3 \\
\nu_{\Delta_n}^0 & \nu_{\Delta_n}^1 & \nu_{\Delta_n}^2 & \nu_{\Delta_n}^3
\end{pmatrix}.$$  

**Proof:** Let $\phi \in I$. Note that $\nu_{\Delta_n}[\phi]$ equals the sum of the number of satisfying states of $\Delta_n$ when $(v_1, v_2, v_3, v)$ are assigned spins $(\phi, +)$ and $(\phi, -)$. For a given spin assignment to $(v_1, v_2, v_3, v)$, the number of satisfying states of $\Delta_n$, is obtained by multiplying the number of satisfying states of each $\Delta_n^j$ when the spin assignment of its outer faces agree with the fixed spins assigned to $(v_1, v_2, v_3, v)$. 

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First, consider the case where \( \phi = +++ \). If \( v \)'s spin is +, then
\[
\Delta_n^4[+++] \cdot \Delta_n^3[+++] \cdot \Delta_n^3[+++] = v_1^0 v_2^0 v_3^0.
\]

If \( v \)'s spin is -, then
\[
\Delta_n^4[+--] \cdot \Delta_n^3[+--] \cdot \Delta_n^3[+--] = v_1^1 v_2^1 v_3^1.
\]

Hence, \( \Delta_n[\phi] = v_1^0 v_2^0 v_3^0 + v_1^1 v_2^1 v_3^1 \).

Now, consider the case where \( \phi = +++ \). If \( v \)'s spin is+, then
\[
\Delta_n^4[+++] \cdot \Delta_n^3[++] \cdot \Delta_n^3[++] = v_1^2 v_2^2 v_3^2.
\]

Recalling that by identity 1 we have that \( \Delta_n^2[++] = \Delta_n^3[-++] \) and \( \Delta_n^3[-++] = \Delta_n^3[++] \), if \( v \)'s spin is-, then
\[
\Delta_n^4[+--] \cdot \Delta_n^3[+--] \cdot \Delta_n^3[+--] = v_1^1 v_2^1 v_3^1.
\]

Hence, \( \Delta_n[\phi] = v_1^0 v_2^0 v_3^0 + v_1^1 v_2^1 v_3^1 \).

The other two remaining cases, where \( \phi \) equals \( -++ \) and \( -++ \), can be similarly dealt with and left to the interested reader.

\[ \blacksquare \]

4.2 Root vectors of ternary trees

We will now introduce the concept of root vector of a colored rooted ternary tree. Then, we will see that \( \mathbf{v}_{\Delta} \) is the degeneracy vector of the rooted stack triangulation \( \Delta \) if and only if \( \mathbf{v}_{\Delta} \) is the root vector of the colored rooted ternary tree \( T(\Delta) \).

Let \( T \) be a colored rooted ternary tree. For any node \( u \) of \( T \setminus \{v\} \), we denote by \( l_u \in \{1, 2, 3\} \) its label.

**Definition 2** Let \( T \) be a colored ternary tree rooted at \( v \). We recursively define the root vector \( \mathbf{v} \in \mathbb{R}^4 \) of \( T \) associated to \( v \) according to the following rules:

**Rule 0:** \( \mathbf{v} = (1, 1, 1, 1)^t \) when \( v \) does not have any children.

**Rule 1:** If \( v \) has exactly one child \( u \) with \( \mathbf{u} = (u_s)_{s=0,\ldots,3} \), then \( \mathbf{v} \in [\mathbf{u}] \) where
\[
[\mathbf{u}] = \{ (u_1, u_0 + u_1, u_3, u_2)^t, (u_1, u_3, u_2, u_0 + u_1)^t, (u_1, u_2, u_0 + u_1, u_3)^t \}.
\]

The choice of \( \mathbf{v} \) depends on the label of \( u \); if \( l_u = i \), \( \mathbf{v} \) is the \( i \)-th vector in \( [\mathbf{u}] \).

**Rule 2:** If \( v \) has two children \( u \) and \( w \) with \( \mathbf{u} = (u_s)_{s=0,\ldots,3} \), \( \mathbf{w} = (w_s)_{s=0,\ldots,3} \), and \( (l_u, l_w) \in \{(1, 2), (2, 3), (3, 1)\} \), then
\[
\mathbf{v} \in \left\{ \begin{pmatrix} u_1 w_1 \\ u_0 w_2 + u_1 w_3 \\ u_3 w_2 \\ u_2 w_1 + u_3 w_0 \end{pmatrix}, \begin{pmatrix} u_1 w_1 \\ u_3 w_2 \\ u_3 w_0 + u_2 w_1 \\ u_1 w_3 + u_0 w_2 \end{pmatrix}, \begin{pmatrix} u_1 w_1 \\ u_3 w_2 \\ u_0 w_2 + u_1 w_3 \\ u_3 w_2 \end{pmatrix} \right\}.
\]

The choice of \( \mathbf{v} \) depends on \( (l_u, l_w) \); if \( l_u = i \), \( \mathbf{v} \) is the \( i \)-th vector in the last set.

**Rule 3:** If \( v \) has three children \( u \), \( w \) and \( z \) with \( \mathbf{u} = (u_s)_{s=0,\ldots,3} \), \( \mathbf{w} = (w_s)_{s=0,\ldots,3} \), \( \mathbf{z} = (z_s)_{s=0,\ldots,3} \), and \( (l_u, l_w, l_z) = (1, 2, 3) \), then
\[
\mathbf{v} \in \left\{ \begin{pmatrix} u_0 w_0 z_0 + u_1 w_1 z_1 \\ u_0 w_2 z_3 + u_1 w_3 z_2 \\ u_2 w_3 z_0 + u_3 w_2 z_1 \\ u_2 w_1 z_3 + u_3 w_0 z_2 \end{pmatrix} \right\}.
\]
The following result establishes that determining the degeneracy vector of rooted stack triangulations is equivalent to determining the root vector of colored rooted ternary trees.

**Lemma 5** Let $n \geq 1$ and $\Delta_n$ be the rooted stack triangulation $\Delta_n$. Then, the root vector of the colored ternary tree $T(\Delta_n)$ in bijection with $\Delta_n$ equals the degeneracy vector of $\Delta_n$.

**Proof:** By induction on $n$. For the base case $n = 1$; the stack triangulation $\Delta_1$ is isomorphic to $K_1$ and $T(\Delta_1)$ is a vertex. It is clear that $\Delta_1[\phi] = 1$ for all $\phi \in I$, and the root vector of $T(\Delta_1)$ is obtained by Rule 0 in Definition 2.

Now, let $\Delta_n$ be a rooted stack triangulation with $n > 1$. We denote by $v$ the root of $T(\Delta_n)$. We separate the proof in cases according to the number of vertices of the rooted stack triangulations $\Delta_{n_i} = \Delta_n^i$ with $i \in \{1, 2, 3\}$. We note that if $n_i = 0$ for every $i \in \{1, 2, 3\}$, then $n = 1$. Thus, we can assume that $n_i \geq 1$ for at least one index $i \in \{1, 2, 3\}$. We now consider three possible situations.

First, assume there are $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$ such that $n_i = n_j = 0$ and $n_k \geq 1$. By definition of the degeneracy vector, we have that $v_{\Delta_{n_i}} = v_{\Delta_{n_j}} = (0, 1, 1, 1)^t$. Let $v_{\Delta_{n_k}} = (v_k^i)_{i \in \{0, 1, 2, 3\}}$. According to Proposition 4 we have that

$$v_{\Delta_{n_k}} = \begin{cases} \left( \begin{array}{c} v_1^1 \\ v_1^0 + v_1^1 \\ v_1^2 \\ v_1^3 \\
\end{array} \right), & \left( \begin{array}{c} v_2^1 \\ v_2^0 + v_2^1 \\ v_2^2 \\ v_2^3 \\
\end{array} \right), & \left( \begin{array}{c} v_3^1 \\ v_3^0 + v_3^1 \\ v_3^2 \\ v_3^3 \\
\end{array} \right) \end{cases}$$

where $v_{\Delta_{n_k}}$ is the $k$-th vector in the set above. Statement 2 of Proposition 3 says that $T(\Delta_{n_k})$ is labeled by $k$ and rooted on $w$, where $w$ is the unique child of $v$. Given that $1 \leq n_k < n$, by induction we get that $w = v_{\Delta_{n_k}}$. By Definition 2 we know that $v$ is obtained from $w$ by application of Rule 1. Hence, $v = v_{\Delta_{n_k}}$.

Assume now that there is an $i \in \{1, 2, 3\}$ such that $n_i = 0$ and $j, k \in \{1, 2, 3\} \setminus \{i\}$ with $j \neq k$ such that $n_j, n_k \geq 1$. We have that $v_{\Delta_{n_i}} = (0, 1, 1, 1)^t$. Consider $v_{\Delta_{n_j}} = (v_j^i)_{i \in \{0, 1, 2, 3\}}$ and $v_{\Delta_{n_k}} = (v_k^j)_{i \in \{0, 1, 2, 3\}}$. Proposition 4 implies that

$$v_{\Delta_{n_i}} = \begin{cases} \left( \begin{array}{c} v_1^1 v_3^1 \\ v_2^0 v_3^0 + v_2^1 v_3^1 \\ v_2^2 v_3^2 + v_2^3 v_3^3 \\
\end{array} \right), & \left( \begin{array}{c} v_1^0 v_3^1 \\ v_2^0 v_3^1 + v_1^1 v_3^1 \\ v_2^2 v_3^2 + v_1^1 v_3^3 \\
\end{array} \right), & \left( \begin{array}{c} v_1^0 v_2^1 \\ v_1^2 v_2^2 + v_1^3 v_2^3 \\
\end{array} \right) \end{cases}$$

where $v_{\Delta_{n_i}}$ is the $i$-th vector in the set above. Statement 3 of Proposition 3 guarantees that the root $v$ of $T(\Delta)$ has exactly two children $w$ and $u$, labeled $j$ and $k$, respectively. Moreover, $T(\Delta_n^j)$ and $T(\Delta_n^k)$ are rooted on $w$ and $u$, respectively. We know that $1 \leq n_j < n$ and $1 \leq n_k < n$, then by induction, $w = v_{\Delta_{n_j}}$ and $u = v_{\Delta_{n_k}}$. If we now apply Rule 2 of Definition 2 we get $v = v_{\Delta_{n_i}}$.

Finally, assume that $n > n_j \geq 1$ for every $j \in \{1, 2, 3\}$. Suppose that $v_{\Delta_{n_j}} = (v_j^i)_{i \in \{0, 1, 2, 3\}}$ for each $j \in \{1, 2, 3\}$. Statement 4 of Proposition 3 and the induction hypothesis imply that the root $v$ of $T(\Delta)$ has three children $w_1, w_2$ and $w_3$ such that $w_j = v_{\Delta_{n_j}}$ for each $j \in \{1, 2, 3\}$. By Proposition 4 and since $v$ is derived by applying Rule 3 of Definition 2 the desired conclusion follows.

5 **Colored Rooted Ternary Trees**

The goal of this section is to prove a result that we should refer to as the Main Lemma which shows that the degeneracy of stack triangulations is exponential in the number of its nodes.
We now introduce notation that will be useful when dealing with rooted ternary trees. We denote by $|T|$ the number of vertices of the ternary tree $T$. For any node $u$ of $T$, we denote by $T_u$ the colored rooted sub-ternary tree of $T$ rooted at $u$ and induced by $u$ and its descendants. Also, we denote by $P_{w,w}$ any path with end nodes $w$ and $w$. Moreover, $|P_{w,w}| = |P_{w,w}| - 1$ denotes the length of $P_{w,w}$.

5.1 Remainders

In this subsection we introduce the concept of remainder of a rooted ternary tree and prove some useful and fundamental claims related to this concept. We will show that after removing remainders from a rooted ternary tree we are still left with a tree of size at least a third of the original one. The root vertex of the derived remainder free tree will provide a component wise lower bound on the components of the root vertex of the original rooted ternary tree. The underlying motivation for this section is that lower bounding the components of a root vertex is significantly easier for remainder free rooted ternary trees.

**Definition 3** Let $v$ be a leaf of $T$ and $w$ be its father. Consider the following cases:

I.- If $u \neq v$ is a child of $w$, then $|T_u| \geq 3$.

II.- If $T_w$ is just the edge $wv$, then the father of $w$, say $y$, has two children $w$ and $u$, where $|T_u| \geq 3$.

*If Case I holds, we say that $\{v\}$ is a remainder of $T$ and that $w$ is the generator of $\{v\}$. If Case II holds we say that $\{v, w\}$ is a remainder of $T$ and that $y$ is its generator. We say that $T$ is reminder free if it does not contain any remainder. We denote the set of remainders of $T$ by $R(T)$ and by $G(R(T))$ the set of its generators.*

See Figures 5 and 6 for an illustration of the distinct situations encompassed by each of the preceding definition’s cases.

![Figure 5](attachment:image5.png)  
**Figure 5:** Structure of $T_w \subseteq T$ having a remainder $v$ of $T$ with generator $w$. Case where $w$ has three children (left) and two children (right).

![Figure 6](attachment:image6.png)  
**Figure 6:** Structure of $T_y \subseteq T$ having a remainder $\{v, w\}$ with generator $y$. 
Proposition 6 Let $T$ be a rooted ternary tree. Then, $|R(T)| = |G(R(T))|$. 

Proof: It is enough to show that any vertex $w \in G(R(T))$ is the generator of exactly one remainder of $T$. For the sake of contradiction, suppose that $w$ is the generator of at least two remainders of $T$, say $S_1$ and $S_2$. We consider three possible cases which cover all possible scenarios: (i) $S_1 = \{v\}$ and $S_2 = \{u\}$, (ii) $S_1 = \{v, \tilde{v}\}$ and $S_2 = \{u, \tilde{u}\}$, and (iii) $S_1 = \{v\}, S_2 = \{u, \tilde{u}\}$. 

If $S_1 = \{v\}$ and $S_2 = \{u\}$, then by Case I of Definition 3 we get that $|T_v| \geq 3$. If $S_1 = \{v, \tilde{v}\}$ and $S_2 = \{u, \tilde{u}\}$, then by Case II of Definition 3 we get that $|T_v| \geq 3$. If $S_1 = \{v\}$ and $S_2 = \{u, \tilde{u}\}$, then by Case III of Definition 3 we have that $|T_v| \geq 3$. Hence, all feasible cases lead to contradictions. 

Let $V_{R(T)}$ denote the subset of vertices of $T$ which belong to the elements of $R(T)$, i.e. $V_{R(T)} = \cup_{S \in R(T)}\{v : v \in S\}$. 

Lemma 7 Let $T$ be a rooted ternary tree. Then, $\tilde{T} = T \setminus V_{R(T)}$ is remainder free. 

Proof: For the sake of contradiction, assume $S$ is a remainder of $\tilde{T}$. We consider three scenarios depending on which case of Definition 3 holds for $S$. 

First, assume $S = \{v\}$ satisfies Case I of Definition 3 and the father of $v$ in $\tilde{T}$ has tree children $v, u, z$ with $|\tilde{T}_u|, |\tilde{T}_z| \geq 3$. Clearly, $|T_u| \geq |\tilde{T}_u|$ and $|T_z| \geq |\tilde{T}_z|$. Since $v$ has no children in $\tilde{T}$, it follows that $v \notin G(R(T))$. Thus, $v$ is a leaf of $T$ and $\{v\} \in R(T)$. 

Assume now that $S = \{v\}$ satisfies Case II of Definition 3 and $v$'s father in $\tilde{T}$, say $w$, has two children $v, u$ with $|\tilde{T}_u| \geq 3$. We have $|T_u| \geq |\tilde{T}_u| \geq 3$. Moreover, since $v$ is a leaf of $\tilde{T}$, it must also hold that $v$ is a leaf of $T$ (otherwise, all of $v$'s children in $T$ must belong to some reminder, a situation that is not possible). If $w$ has three children in $T$, say $v, u, z$, then $\{z\} \in R(T)$. This implies that $|T_v| \geq 3$, contradicting the fact that $v$ is a leaf of $T$. Hence, $w$ has two children in $T$. It follows that $\{v\} \in R(T)$. 

Finally, assume $S = \{v, \tilde{v}\}$ satisfies Case III of Definition 3. Let $w$ be the generator of $S$ and the father of $\tilde{v}$ in $\tilde{T}$. Then, $w$ has two children $\tilde{v}, u$ in $\tilde{T}$ with $|\tilde{T}_u| \geq 3$. We again have that $|T_u| \geq |\tilde{T}_u| \geq 3$ and that $v$ is a leaf of $T$. Assume $w$ has three children in $T$, say $\tilde{v}, u, z$. Then, $\{z\} \in R(T)$, implying that $|T_v| \geq 3$, and hence $\tilde{v} \in G(R(T))$. Therefore, $|T_v| \geq 3$, but this cannot happen because $v$ is a leaf of $T$. Thus, $w$ must have only two children in $T$. If $\tilde{v}$ has exactly two children in $T$, then $\tilde{v} \in G(R(T))$ and $|T_v| \geq 3$, contradicting again the fact that $v$ is a leaf. If $\tilde{v}$ has only one child, then $\{v, \tilde{v}\} \in R(T)$, which contradicts the fact that $\tilde{v}$ is a node of $\tilde{T}$. 

Since all possible scenarios lead to a contradiction, the desired conclusion follows. 

Lemma 8 Let $\tilde{T} = T \setminus V_{R(T)}$. Then, $|\tilde{T}| \geq |T|/3$. 

Proof: Follows from the fact that $G(R(T))$ and $R(T)$ are disjoint, that each element $S \in R(T)$ is of cardinality at most 2, and Proposition 6. 

5.2 Counting satisfying states 

In this section, we establish properties of the root vectors of colored rooted ternary trees and relate them to characteristics of colored tree. Informally, for some special classes of colored rooted ternary trees, we obtain lower bounds for the sum of the coordinates of its associated rooted vectors.
Recall that \( \psi = (1 + \sqrt{5})/2 \approx 1.6180 \) denotes the golden ratio. For \( s \in \{0, \ldots, 3\} \), let \( e_s \in \mathbb{N} \) and \( e = (\psi^s)_{s=0,\ldots,3} \). Define

\[
\Psi(e) = 2 \sum_{j=1}^{3} e_j, \quad \text{and} \quad \Phi(e) = \Psi(e) - \{|s| e_s > e_0\}.
\]

Henceforth, for a vector \( \nu \) we let \([\nu]\) denote the collection of all vectors obtained by fixing the first coordinate of \( \nu \) and permuting the remaining coordinates in an arbitrary way. Note that if \( e = (\psi^e_s)_{s=0,\ldots,3} \) with \( e_0, e_1, e_2, e_3 \in \mathbb{N} \), then for all \( \tilde{e} \in [e] \) we have that \( \Psi(\tilde{e}) = \Psi(e) \) and \( \Phi(\tilde{e}) = \Phi(e) \). For a set \( S \) of vectors, we let \([S]\) denote the union of the sets \([\nu]\) where \( \nu \) varies over \( S \).

Given vectors \( x = (x_s)_{s=0,\ldots,3} \) and \( y = (y_s)_{s=0,\ldots,3} \), we write \( x \geq y \) if \( x_s \geq y_s \) for all \( s \in \{0, \ldots, 3\} \).

**Proposition 9** Let \( T_v \) be a colored rooted ternary tree with \(|T_v| = 2\). Then, there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( v \geq e = (\psi^e_s)_{s=0,\ldots,3} \) and \( \Psi(e) = 2 \).

**Proof:** Clearly \( T_v \) is a rooted tree on \( v \) with exactly one child \( w \) which is a leaf of \( T_v \). In other words, \( T_v = P_{w,v} \) with \(|P_{w,v}| = 1\). We observe that by applying Rules 0 and 1, we get that \( w = (1, 1, 1, 1)^t \) and \( v \in [(1, 2, 1, 1)^t] \). Given that \( 1 = \varphi^0 \) and \( 2 \geq \varphi^1 \), it is easy to see that the desired vector \( e \) belongs to \([\varphi^0, \varphi^1, \varphi^0, \varphi^0]^t\).

**Proposition 10** Let \( T_v \) be a colored rooted ternary tree with \(|T_v| = 3\). Then, there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( v \geq e = (\psi^e_s)_{s=0,\ldots,3} \) and \( \Psi(e) = 4 \).

**Proof:** Since \(|T_v| = 3\), either \( T_v = P_{w,v} \) with \(|P_{w,v}| = 2\), or \( v \) has exactly two children \( w \) and \( u \), which are leaves of \( T_v \).

In the first scenario, applying Rule 0 once and Rule 1 twice, we get that \( v \in [(2, 3, 1, 1)^t, (1, 2, 2, 1)^t] \). Given that \( 1 = \varphi^0, 2 \geq \varphi^1 \) and \( 3 \geq \varphi^2 \), we can take \( e \in [(\varphi^1, \varphi^2, \varphi^0)^t, (\varphi^0, \varphi^1, \varphi^0)^t] \) satisfying the statement.

In the second scenario, applying Rule 0, we get that \( w \) and \( u \) are vectors all of whose coordinates are 1. Applying Rule 2, we see that \( v \in [(1, 2, 1, 2)^t] \). Given that \( 1 = \varphi^0 \) and \( 2 \geq \varphi^1 \), the desired vector \( e \) may be chosen from the set \([(\varphi^0, \varphi^1, \varphi^0, \varphi^1)^t] \).

**Proposition 11** Let \( T_v \) be a colored rooted ternary tree with \(|T_v| = 4\). Then, there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( v \geq e = (\psi^e_s)_{s=0,\ldots,3} \) and \( \Psi(e) \geq 6 \).

**Proof:** The tree \( T_v \) may be one of the four trees depicted in Figure 7 Each case is analyzed separately below (in the order in which they appear in Figure 7).

![Figure 7: All rooted ternary trees with 4 vertices.](image)
For the first case, note that by Rule 0 we have that \( \mathbf{w}, \mathbf{u} \) and \( \mathbf{z} \) are vectors all of whose coordinates are 1. Thus, by Rule 3, we get that \( \mathbf{v} = (2, 2, 2, 2)^t \). Hence, \( \mathbf{v} \geq \mathbf{e} \) where \( \mathbf{e} = (\varphi^1, \varphi^1, \varphi^1)^t \).

For the second case, by Rule 0 we have that all coordinates of \( \mathbf{u} \) and \( \tilde{\mathbf{w}} \) are 1. Thus, by Rule 1, \( \mathbf{w} \in [(1, 2, 1, 1)^t] \). Then, by Rule 2, we get that \( \mathbf{v} \in [(2, 3, 1, 2)^t, (1, 2, 1, 3)^t, (1, 3, 2, 2)^t] \). Given that \( 1 = \varphi^0, 2 \geq \varphi^1 \) and \( 3 \geq \varphi^2 \) the result follows.

For the third case, note that \( |T_{\mathbf{w}}| = 3 \) and that the structure of \( T_{\mathbf{w}} \) is the same as the second one considered in the proof of Proposition \[10\]. Hence, we know that \( \mathbf{w} \in [(1, 2, 1, 1)^t] \). By Rule 1, we get that \( \mathbf{v} \in [(2, 3, 1, 2)^t, (1, 2, 1, 3)^t, (1, 3, 2, 2)^t] \). Given that \( 1 = \varphi^0, 2 \geq \varphi^1 \) and \( 3 \geq \varphi^2 \) the claimed result follows.

We leave the last case to the interested reader.

![Figure 8: The tree \( T_v \) of Proposition 12](image)

**Proposition 12** Let \( T_v \) be a colored rooted ternary tree with \( |T_v| = 5 \) and where \( v \) has two children which are not leaves. Then, there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( \mathbf{v} \geq \mathbf{e} = (\varphi^e_s)_{s=0,\ldots,3} \) and \( \Psi(\mathbf{e}) \geq 8 \).

**Proof:** Assume \( T_v \) is as depicted in Figure 8. Clearly, \( \mathbf{w}, \mathbf{u} \in [(1, 2, 1, 1)^t] \). By Rule 2 we get that \( \mathbf{v} \in [(4, 3, 1, 3)^t, (1, 3, 3, 2)^t, (2, 2, 1, 3)^t, (1, 3, 3, 3)^t, (1, 3, 4, 3)^t] \). The desired conclusion follows since \( 1 = \varphi^0, 2 \geq \varphi^1, 3 \geq \varphi^2, 4 \geq \varphi^2 \) and \( 5 \geq \varphi^3 \).}

**Proposition 13** Let \( T_v \) be a colored rooted ternary tree such that \( v \) has three children \( u, w \) and \( z \). Suppose that \( 1 \leq |T_u| \leq 3 \) and \( 1 \leq |T_w| \leq 3 \). Then,

- If \( |T_z| = 2 \), there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( \mathbf{v} \geq \mathbf{e} = (\varphi^e_s)_{s=0,\ldots,3} \) and \( \Psi(\mathbf{e}) \geq 8 \).
- If \( |T_z| = 3 \), there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( \mathbf{v} \geq \mathbf{e} = (\varphi^e_s)_{s=0,\ldots,3} \) and \( \Psi(\mathbf{e}) \geq 10 \).

**Proof:** We first note if \( |T_x| \geq 2 \), then \( x \geq (1, 1, 1, 1)^t \). This implies that it is enough to prove both statements for the case \( |T_u| = 1 \) and \( |T_w| = 1 \). Observe that by Rule 0, we have that \( \mathbf{u} = \mathbf{w} = (1, 1, 1, 1)^t \).

For the first statement, assume \( |T_z| = 2 \). By Rule 1, we have that \( \mathbf{z} \in [(1, 2, 1, 1)^t] \). Then, by Rule 3 we have that \( \mathbf{v} \in [(3, 3, 2, 2)^t, (2, 3, 3, 2)^t] \). The result follows, since \( 2 \geq \varphi^1 \) and \( 3 \geq \varphi^2 \).

Assume now that \( |T_z| = 3 \). From the proof of Proposition 10 we know that \( \mathbf{z} \in [(2, 3, 1, 1)^t, (1, 2, 2, 1)^t] \). Then, by Rule 3 we have that \( \mathbf{v} \in [(5, 2, 5, 2)^t, (3, 4, 3, 4)^t, (3, 3, 3, 3)^t, (2, 4, 2, 4)^t] \). The desired conclusion follows since \( 2 \geq \varphi^1, 3 \geq \varphi^2, 4 \geq \varphi^2 \) and \( 5 \geq \varphi^3 \).}

**Lemma 14** Let \( T = T_v \) be a colored rooted ternary tree, such that \( T_v = T_{\tilde{\mathbf{v}}} \cup P_{\tilde{\mathbf{v}}, \mathbf{v}} \) where \( P_{\tilde{\mathbf{v}}, \mathbf{v}} \) is non-trivial. If \( \mathbf{\tilde{v}} \geq \mathbf{\tilde{e}} = (\varphi^{\tilde{e}}_s)_{s=0,\ldots,3} \) with \( \tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \in \mathbb{N} \), then there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( \mathbf{v} \geq \mathbf{e} = (\varphi^e_s)_{s=0,\ldots,3} \) and \( \Phi(\mathbf{e}) \geq \Phi(\mathbf{\tilde{e}}) + ||P_{\tilde{\mathbf{v}}, \mathbf{v}}|| \).
Proof: It is enough to prove the result for \( P_{\tilde{v},v} \) of length 1. By Rule 1, we get that \( v \in \tilde{v} \) where \( \tilde{v} \geq \hat{e} \) for some \( \hat{e} \in [(\varphi^{e_1}, \varphi^{e_2}, \varphi^{e_3}, \varphi^{e_4})] \). Assume \( \hat{e} \) is the vector within the double brackets (the other cases are similar).

We now consider several scenarios:

- **Case \( \hat{e}_1 > \hat{e}_0 + 1 \):** Clearly, \( v \geq e = (\varphi^{e_1}, \varphi^{e_2}, \varphi^{e_3}, \varphi^{e_4}) \). Moreover, \( \Psi(e) = \Psi(\hat{e}) \) and \( \{(s | e_s > e_0)\} \leq \{(s | \tilde{e}_s > \tilde{e}_0)\} - 1 \). Hence, \( \Phi(e) \geq \Phi(\hat{e}) + 1 \).

- **Case \( \hat{e}_1 \in (\hat{e}_0, \hat{e}_0 + 1) \):** If \( \hat{e}_1 = \hat{e}_0 \), then \( \varphi^{e_1} + \varphi^{e_0} = 2 \varphi^{e_1} \geq \varphi^{e_1+1} \). Since \( 1 + \varphi = \varphi^2 \), if \( \hat{e}_1 = \hat{e}_0 + 1 \), then \( \varphi^{e_1} + \varphi^{e_0} = \varphi^{e_1+1} \). Hence, \( v \geq e = (\varphi^{e_1}, \varphi^{e_1+1}, \varphi^{e_2}, \varphi^{e_3}) \). Moreover, \( \Psi(e) = \Psi(\hat{e}) + 2 \) and \( \{(s | e_s > e_0)\} \leq \{(s | \tilde{e}_s > \tilde{e}_0)\} + 1 \). Hence, \( \Phi(e) \geq \Phi(\hat{e}) + 1 \).

- **Case \( \hat{e}_1 \leq \hat{e}_0 - 1 \):** Since \( 1 + \varphi = \varphi^2 \), if \( \hat{e}_1 = \hat{e}_0 - 1 \), then \( \varphi^{e_1} + \varphi^{e_0} = \varphi^{e_1+2} \). If \( \hat{e}_1 \leq \hat{e}_0 - 2 \), then \( \varphi^{e_1} + \varphi^{e_0} \geq \varphi^{e_1+2} \). Hence, \( v \geq e = (\varphi^{e_1}, \varphi^{e_1+2}, \varphi^{e_2}, \varphi^{e_3}) \). Moreover, \( \Psi(e) = \Psi(\hat{e}) + 4 \) and \( \{(s | e_s > e_0)\} \leq \{(s | \tilde{e}_s > \tilde{e}_0)\} + 3 \). Hence, \( \Phi(e) \geq \Phi(\hat{e}) + 1 \).

The following result is an immediate consequence of Lemma 14.

**Corollary 15** Let \( T = T_v \) be a colored rooted ternary tree, such that \( T_v = T_0 \cup P_{\tilde{v},v} \) where \( P_{\tilde{v},v} \) is non-trivial. If \( \tilde{v} \geq \hat{e} \), then an \( e \) exists such that \( v \geq e \) and

\[
\Psi(\hat{e}) \geq \Psi(e) + \max\{||P_{\tilde{v},v}|| - 3, 0\}.
\]

**Lemma 16** Let \( T_v \) be a colored rooted ternary tree, such that \( v \) has two children \( w \) and \( u \). If \( w \geq e^w = (\varphi^{e_1^w})_{s=0,...,3} \) with \( e_0^w, e_1^w, e_2^w, e_3^w \in \mathbb{N} \) and \( u \geq e^u = (\varphi^{e_1^u})_{s=0,...,3} \) with \( e_0^u, e_1^u, e_2^u, e_3^u \in \mathbb{N} \), then there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( v \geq e = (\varphi^{e_s})_{s=0,...,3} \) and \( \Psi(e) = \Psi(e^w) + \Psi(e^u) \).

**Proof:** Since \( w \geq e^w \) and \( u \geq e^u \), by Rule 2 we have that \( v \geq \tilde{v} \) where

\[
\tilde{v} \in \left\{ \begin{pmatrix} \varphi^{e_1} + e_1 \\ \varphi^{e_0} + e_2 \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \end{pmatrix}, \begin{pmatrix} \varphi^{e_1} + e_1 \\ \varphi^{e_0} + e_2 \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \end{pmatrix}, \begin{pmatrix} \varphi^{e_1} + e_1 \\ \varphi^{e_0} + e_2 \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \end{pmatrix} \right\}.
\]

Moreover, \( \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \geq \varphi^{e_0} + e_1 + e_3 \) and \( \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \geq \varphi^{e_0} + e_1 + e_3 \), so depending on the value of \( \tilde{v} \) we can take

\[
e \in \left\{ \begin{pmatrix} \varphi^{e_1} + e_1 \\ \varphi^{e_0} + e_2 \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \end{pmatrix}, \begin{pmatrix} \varphi^{e_1} + e_1 \\ \varphi^{e_0} + e_2 \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \end{pmatrix}, \begin{pmatrix} \varphi^{e_1} + e_1 \\ \varphi^{e_0} + e_2 \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \\ \varphi^{e_0} + e_2 + \varphi^{e_1} + e_3 \end{pmatrix} \right\},
\]

and obtain that \( v \geq e \) and \( \Psi(e) = \Psi(e^w) + \Psi(e^u) \).

**Lemma 17** Let \( T_v \) be a colored rooted ternary tree, such that \( v \) has three children \( w \), \( u \) and \( z \). If \( w \geq e^w = (\varphi^{e_1^w})_{s=0,...,3} \) with \( e_0^w, e_1^w, e_2^w, e_3^w \in \mathbb{N} \), \( u \geq e^u = (\varphi^{e_1^u})_{s=0,...,3} \) with \( e_0^u, e_1^u, e_2^u, e_3^u \in \mathbb{N} \) and \( z \geq e^z = (\varphi^{e_1^z})_{s=0,...,3} \) with \( e_0^z, e_1^z, e_2^z, e_3^z \in \mathbb{N} \), then there are \( e_0, e_1, e_2, e_3 \in \mathbb{N} \) such that \( v \geq e = (\varphi^{e_s})_{s=0,...,3} \) and \( \Psi(e) = \Psi(e^w) + \Psi(e^u) + \Psi(e^z) \).
Proof: By Rule 3 we have that
\[
\nu = \left( \varphi v^u + e_0^v + e_0^w + \varphi e_0^v + e_1^v + e_1^w \right) \geq \left( \varphi e_0^v + e_1^v \right).
\]

Let \( e \) be the last vector in the preceding expression and note that \( \Psi(e) = \Psi(e^w) + \Psi(e^u) + \Psi(e^\varepsilon) \). \( \blacksquare \)

5.3 Main Lemma

The main result of this work, i.e. Theorem 11 will follow almost directly from the next key claim which roughly says that the root vector \( \nu \) of a colored rooted ternary remainder free tree \( T = T_v \) either has large components relative to the size of \( T_v \), or \( T_v \) corresponds to a short path \( P_{v,u} \) and a tree \( T_{\bar{u}} \) whose root vertex \( \bar{v} \) has large components relative to the size of \( T_{\bar{v}} \).

Lemma 18 Let \( T = T_v \) be a colored rooted ternary remainder free tree such that \(|T| \geq 4\). Then, there is a path \( P_{v,u} \) such that \( T_v = T_{\bar{v}} \cup P_{v,u} \) with \( 0 \leq |P_{v,u}| \leq 5 \) (if \( |P_{v,u}| = 0 \), then \( \bar{v} = v \) and \( T_{\bar{v}} = T_v \)) and there are \( e_0^v, e_1^v, e_2^v, e_3^v \in \mathbb{N} \) such that
\[
|\bar{v}| \geq e^\varepsilon = (\varphi e^\varepsilon)_{s=0,\ldots,3}, \quad \text{and} \quad \Psi(e^\varepsilon) \geq \frac{|T_v| + 7}{2}.
\] (2)

Proof: We proceed by induction on \(|T|\). For the base case \(|T_v| = 4\), by Proposition 11 there exists an \( e \leq v \) such that \( \Psi(e) \geq 6 > (|T_v| + 7)/2 \). Let \( T_v \) be a colored rooted ternary remainder free with \(|T_v| \geq 5\). We separate the proof in three cases depending on the number of children of the root \( v \). It is clear that for any node \( u \) of \( T_v \), the tree \( T_u \) is a colored rooted ternary remainder free tree.

Case 1 (\( v \) has one child \( w \)): We have \(|T_w| = |T_v| - 1 \geq 4\). By induction \( T_w = T_{\bar{w}} \cup P_{\bar{w},w} \) with \( 0 \leq |P_{\bar{w},w}| \leq 5 \) and \( \bar{w} \) satisfying (2). If \( |P_{\bar{w},\bar{w}}| < 5 \), then \( T_{\bar{v}} = T_{\bar{w}} \cup P_{\bar{w},w} \) and thus it satisfies the desired property. By Corollary 15 we know that there is an \( e \leq \nu \) such that \( \Psi(e) \geq \Psi(e^{\bar{w}}) + |P_{\bar{w},\bar{w}}| - 3 \). Given that \(|T_w| = |T_{\bar{w}}| + |P_{\bar{w},\bar{w}}| + 1\), if \(|P_{\bar{w},\bar{w}}| = 5\), then there is an \( e \leq \nu \) such that
\[
\Psi(e) \geq \Psi(e^{\bar{w}}) + |P_{\bar{w},\bar{w}}| - 3 \geq \frac{|T_{\bar{w}}| + 7}{2} + |P_{\bar{w},\bar{w}}| - 3 = \frac{|T_{\bar{v}}| + 7}{2}.
\]

Therefore, \( T_{\bar{v}} \) satisfies the desired property.

Case 2 (\( v \) has two children \( w \) and \( u \)): First, note that \( T_w \) and \( T_u \) have size at least 2 (otherwise we would have, say \(|T_w| = 1\) and \(|T_u| = |T_v| - |T_w| - 1 \geq 3\), implying that \( w \) is a remainder of \( T \), a contradiction). If \(|T_w| = 2\), then \(|T_u| = 2\) (otherwise, \(|T_u| \geq 3\), implying that there is a remainder \( S \) of \( T \) such that \( w \in S \), a contradiction). Since \(|T_w| = |T_u| = 2\), by Proposition 12 we have that there is an \( e \leq \nu \) such that \( \Psi(e) = 8 > (|T_v| + 7)/2 \).

Hence, we assume that \(|T_w|, |T_u| \geq 3\). If \(|T_w| = |T_u| = 3\), by Proposition 10 and Lemma 16 we get that there is an \( e \leq \nu \) such that \( \Psi(e) = 8 > (|T_v| + 7)/2 \).

We now assume that \(|T_w| = 3\) and \(|T_u| \geq 4\). By induction, \( T_{\bar{u}} = T_{\bar{w}} \cup P_{\bar{w},\bar{u}} \) with \( 0 \leq |P_{\bar{w},\bar{u}}| \leq 5 \) and \( \bar{u} \) satisfying (2). By Lemma 16 there is an \( e \leq \nu \) such that \( \Psi(e) = \Psi(e^w) + \Psi(e^u) \). By Proposition 10

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Corollary [15] and the fact that $|T_v| = |T_u| + ||P_{u,u}|| + 4,$

$$\Psi(e) \geq \Psi(e^w) + \Psi(e^\bar{u}) + \max\{|P_{\bar{u},u}|| - 3, 0\}$$

$$\geq 4 + \frac{|T_{\bar{u}}| + 7}{2} + \max\{|P_{\bar{u},u}|| - 3, 0\}$$

$$= \frac{|T_{\bar{u}}| + 7}{2} + \frac{4 - |P_{\bar{u},u}|| - 3, 0}{2} + \max\{|P_{\bar{u},u}|| - 3, 0\}$$

$$\geq \frac{|T_{\bar{u}}| + 7}{2} + \frac{1}{2} \max\{3 - |P_{\bar{u},u}||, |P_{\bar{u},u}|| - 3, 0\}$$

$$\geq \frac{|T_{\bar{u}}| + 7}{2}.$$ 

Hence, $T_v$ satisfies the desired property.

Finally, we assume that $|T_u|, |T_u| \geq 4$. By induction, $T_w = T_{\bar{u}} \cup P_{\bar{w},w}$ and $T_u = T_{\bar{u}} \cup P_{\bar{u},u}$ where $0 \leq |P_{\bar{w},w}||, |P_{\bar{u},u}|| \leq 5$ and $\bar{u}, \bar{w}$ satisfying (2). By Lemma [16], there is an $e \leq v$ such that $\Psi(e) = \Psi(e^w) + \Psi(e^\bar{u})$. By Corollary [15] and given that $|T_v| = |T_{\bar{u}}| + |T_{\bar{u}}| + |P_{\bar{w},w}|| + |P_{\bar{u},u}|| + 1,$

$$\Psi(e) \geq \Psi(e^\bar{w}) + \Psi(e^\bar{v}) + \max\{|P_{\bar{w},w}|| - 3, 0\} + \max\{|P_{\bar{u},u}|| - 3, 0\}$$

$$\geq \frac{|T_{\bar{u}}| + 7}{2} + \frac{|T_{\bar{u}}| + 7}{2} + \max\{|P_{\bar{w},w}|| - 3, 0\} + \max\{|P_{\bar{u},u}|| - 3, 0\}$$

$$= \frac{|T_{\bar{u}}| + 7}{2} + \frac{1}{2} \max\{|P_{\bar{w},w}|| - 3, 3 - |P_{\bar{w},w}||\} + \frac{1}{2} \max\{|P_{\bar{u},u}|| - 3, 3 - |P_{\bar{u},u}||\}$$

$$\geq \frac{|T_{\bar{u}}| + 7}{2}.$$ 

Hence, $T_v$ satisfies the desired property.

Case 3 ($v$ has three children $w$, $u$ and $z$): Since $|T_v| \geq 5$, it can not happen that $|T_u| = |T_u| = |T_z| = 1$. If $1 \leq |T_w| \leq 3$, then $1 \leq |T_u| \leq 3$ and $2 \leq |T_z| \leq 3$, we have that: if $|T_z| = 2$, then $|T_v| \leq 9$ and by the first statement of Proposition [13] there is a vector $e \leq v$ such that $\Psi(e) \geq 8 = 16/2 \geq (|T_w| + 7)/2$; if $|T_z| = 3$, then $|T_v| \leq 10$ and by the second statement of Proposition [13] there is a vector $e \leq v$ such that $\Psi(e) \geq 10 > 17/2 \geq (|T_v| + 7)/2$. Therefore, $T_v$ satisfies the desired property.

We now assume that at least one of the children of $v$ induces a subtree with at least 4 vertices.

- If $1 \leq |T_w|, |T_u| \leq 2$ and $|T_z| \geq 4$, then by Rules 0, 1 and 3, we have
  $$v \geq \begin{pmatrix} \varphi^{e_0} + \varphi^{e_1} \\ \varphi^{e_5} + \varphi^{e_2} \\ \varphi^{e_3} + \varphi^{e_1} \\ \varphi^{e_4} + \varphi^{e_3} \end{pmatrix}, \text{ or } v \geq \begin{pmatrix} \varphi^{e_0} + \varphi^{e_1} \\ \varphi^{e_2} + \varphi^{e_3} \\ \varphi^{e_5} + \varphi^{e_1} \\ \varphi^{e_4} + \varphi^{e_3} \end{pmatrix}, \text{ or } v \geq \begin{pmatrix} \varphi^{e_0} + \varphi^{e_1} \\ \varphi^{e_2} + \varphi^{e_3} \\ \varphi^{e_5} + \varphi^{e_1} \\ \varphi^{e_4} + \varphi^{e_3} \end{pmatrix}.$$

If $e_3 = e_2$, given that $2 > \varphi$, we may choose the vector $e$ from the set

$$\begin{pmatrix} \varphi^{e_0} \\ \varphi^{e_5} + 1 \\ \varphi^{e_1} \\ \varphi^{e_2} + 1 \end{pmatrix}, \begin{pmatrix} \varphi^{e_0} \\ \varphi^{e_5} + 1 \\ \varphi^{e_1} \\ \varphi^{e_2} + 1 \end{pmatrix}, \begin{pmatrix} \varphi^{e_0} \\ \varphi^{e_5} + 1 \\ \varphi^{e_1} \\ \varphi^{e_2} + 1 \end{pmatrix}.$$ 

If not, we have $e_3 \geq e_2 + 1$ (analogously $e_2 \geq e_3 + 1$) and given that $\varphi + 1 = \varphi^2$, we may choose the vector $e$ from the set

$$\begin{pmatrix} \varphi^{e_0} \\ \varphi^{e_5} + 2 \\ \varphi^{e_1} \\ \varphi^{e_2} + 2 \end{pmatrix}, \begin{pmatrix} \varphi^{e_0} \\ \varphi^{e_5} + 2 \\ \varphi^{e_1} \\ \varphi^{e_2} + 2 \end{pmatrix}, \begin{pmatrix} \varphi^{e_0} \\ \varphi^{e_5} + 2 \\ \varphi^{e_1} \\ \varphi^{e_2} + 2 \end{pmatrix}.$$ 

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Therefore, for any choice of $e$ we get that $\Psi(e) = \Psi(e^z) + 4$. By induction, $T_z = T_z \cup P_{z,z}$ with $0 \leq ||P_{z,z}|| \leq 5$ and $\tilde{z}$ satisfying (2). Since $|T_v| \leq |T_z| + ||P_{z,z}|| + 5$, by Corollary 15

$$\Psi(e) \geq 4 + \Psi(e^z) + \max\{|P_{z,z}|, 0\}$$

$$\geq \frac{|T_z| + 7}{2} + 4 + \max\{|P_{z,z}|, 0\}$$

$$\geq \frac{|T_z| + 7}{2} + 3 - \frac{|P_{z,z}|}{2} + \max\{|P_{z,z}|, 0\}$$

$$= \frac{|T_z| + 7}{2} + \frac{1}{2} \max\{3 - |P_{z,z}|, |P_{z,z}| - 3\}$$

Hence, $T_v$ satisfies the desired property.

- If $2 \leq |T_w|, |T_u| \leq 3$ and $|T_z| \geq 4$. By Lemma 17 there is an $e \leq v$ such that $\Psi(e) = \Psi(e^w) + \Psi(e^z) + \Psi(e^z)$. By Proposition 9 and Proposition 10, we have $\Psi(e^w) = 2(|T_w| - 1)$ and $\Psi(e^z) = 2(|T_u| - 1)$. By induction, $T_z = T_z \cup P_{z,z}$ and $T_z = T_z \cup P_{z,z}$ with $0 \leq ||P_{z,z}|| \leq 5$ and $\tilde{u}, \tilde{z}$ satisfying (2). Since $|T_v| = |T_w| + |T_u| + |T_z| + ||P_{z,z}|| + 1$, by Corollary 15

$$\Psi(e) \geq 2(|T_w| - 1) + 2(|T_u| - 1) + \Psi(e^z) + \max\{|P_{z,z}|, 0\}$$

$$\geq \frac{|T_z| + 7}{2} + 2(|T_w| + |T_u|) - 4 + \max\{|P_{z,z}|, 0\}$$

$$\geq \frac{|T_z| + 7}{2} + \frac{3}{2} (|T_w| + |T_u|) - \frac{|P_{z,z}|}{2} - \frac{9}{2} + \max\{|P_{z,z}|, 0\}$$

$$\geq \frac{|T_z| + 7}{2} + \frac{3}{2} \max\{3 - |P_{z,z}|, |P_{z,z}| - 3\}$$

Hence, $T_v$ satisfies the desired property.

- The case $|T_w| = 1, |T_u| = 3$, and $|T_z| \geq 4$ can not happen, since it would imply that $\{w\}$ is a remainder of $T$.

- If $2 \leq |T_w| \leq 3$ and $|T_u|, |T_z| \geq 4$. By Lemma 17 there is an $e \leq v$ such that $\Psi(e) = \Psi(e^w) + \Psi(e^z) + \Psi(e^z)$. By Proposition 9 and Proposition 10, we have $\Psi(e^w) = 2(|T_w| - 1)$. By induction, $T_z = T_z \cup P_{z,z}$ and $T_z = T_z \cup P_{z,z}$ with $0 \leq ||P_{z,z}|| \leq 5$ and $\tilde{u}, \tilde{z}$ satisfying (2). Since $|T_v| = |T_w| + |T_u| + |T_z| + ||P_{z,z}|| + 1$, by Corollary 15

$$\Psi(e) \geq 2(|T_w| - 1) + \Psi(e^z) + \Psi(e^z) + \max\{|P_{z,z}|, 0\} + \max\{|P_{z,z}|, 0\}$$

$$\geq \frac{|T_z| + 7}{2} + \frac{|T_z| + 7}{2} + 2(|T_w| - 1)$$

$$\geq \frac{|T_z| + 7}{2} + \frac{3}{2} (|T_w| - 2) + \frac{6}{2} - \frac{|P_{z,z}|}{2} - \frac{|P_{z,z}|}{2}$$

$$\geq \frac{|T_z| + 7}{2} + \frac{3}{2} \max\{3 - |P_{z,z}|, |P_{z,z}| - 3\}$$

$$\geq \frac{|T_z| + 7}{2} + \frac{1}{2} \max\{3 - |P_{z,z}|, |P_{z,z}| - 3\}$$
Hence, $T_v$ satisfies the desired property.

- If $|T_w|, |T_u|, |T_z| \geq 4$. Similar to the preceding case.

6 Proof of Main Results

Proof of Theorem 1: Recall that $T(\Delta_n)$ is a colored rooted ternary tree on $|\Delta_n| - 3$ nodes such that its root vector $v$ is equal to the degeneracy vector of $\Delta_n$. By Lemmas 7 and 8, the rooted colored ternary tree $\tilde{T}(\Delta_n) = T(\Delta_n) \setminus V_{R(T(\Delta_n))}$ is remainder free and $|\tilde{T}(\Delta_n)| \geq |T(\Delta_n)|/3$. Clearly, the root vector $\tilde{v}$ of $\tilde{T}(\Delta_n)$ is such that $v \geq \tilde{v}$. The Main Lemma guarantees that there are $e_0, e_1, e_2, e_3 \in \mathbb{N}$ such that $v \geq e = (\phi^{e_{\phi}})_{s=0,\ldots,3}$ and
\[
\Psi(e) \geq \frac{(|\tilde{T}(\Delta_n)| - 5) + 7}{2} = \frac{|\tilde{T}(\Delta_n)| + 2}{2} \geq \frac{|T(\Delta_n)| + 6}{12} = \frac{|\Delta_n| + 3}{12}.
\]
Moreover, we know that $\Delta_n[\phi] = \Delta_n[-\phi]$ for all $\phi \in \{+,-\}^3$. Hence, the degeneracy of $\Delta_n$ is at least $2 \sum_{s=1}^3 \phi^{e_{\phi}} \geq 6\phi^{\frac{1}{3}}\Psi(e) \geq 6\phi^{(|\Delta_n|+3)/36}$.

Proof of Corollary 2: Let $G$ be a cubic planar graph such that its geometric dual graph is the stack triangulation $\Delta$. We know that the number of perfect matchings of $G$ is equal to half of the degeneracy of $\Delta$. From Euler’s formula we get that $2|\Delta| = |G| - 4$. Therefore, by Theorem 1 we have that the number of perfect matchings of $G$ is at least $3\phi^{(G)}/72$.

7 Final Comments

The approach followed throughout this work seems to be specially well suited for calculating the degeneracy of triangulations that have some sort of recursive tree like construction, e.g. 3-trees. It would be interesting to identify other such families of triangulations where similar methods allowed to lower bound their degeneracy. Of particular relevance would be to show that the approach we follow in this work can actually be successfully applied to obtain exponential lower bounds for non-trivial families of non-planar bridgeless cubic graphs.

As already mentioned, our arguments are motivated by the transfer matrix method as used by statistical physicists. We believe that most of the arguments we developed throughout this work can be stated in more combinatorial terms, except maybe for our Main Lemma. It might eventually be worthwhile to clarify the implicit combinatorial structure of our proof arguments.

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