Long-time behavior of stochastically perturbed neuronal networks

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Our investigation is specially motivated by the stochastic version of a common model of potential spread in a dendritic tree. We do not assume the noise in the junction points to be Markovian. In fact, we allow for long-range dependence in time of the stochastic perturbation. This leads to an abstract formulation in terms of a stochastic diffusion with dynamic boundary conditions, featuring fractional Brownian motion. We prove results on existence, uniqueness and asymptotics of weak and strong solutions to such a stochastic differential equation.

Keywords: diffusion on network, fractional Brownian motion, strong solutions of infinite dimensional stochastic differential equations, invariant measures.
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1. Introduction

This paper is concerned with a model of stochastically perturbed parabolic network equations. We consider a network, i.e. a metric graph, on whose nodes we impose continuity conditions, complemented by dynamic or Kirchhoff-type laws that also incorporate stochastic noise. We employ known results concerning well-posedness and asymptotics of the deterministic model associated with the equations in order to discuss solvability of the stochastic differential equation and convergence properties of the stochastic convolution process.

There is a well established theory concerning stochastic equations in infinite dimensional spaces with additive Wiener noise, that can be found for instance in [11]. Our aim is to provide similar results for perturbation driven by a fractional Brownian motion. Since this noise is neither a semimartingale nor a Markov process, a different approach to a stochastic calculus with respect to it should be chosen. Our first result, Theorem 3.8 provides the existence of a weak solution for our problem. In this case we extend the established results for Wiener noise. More interesting, and less classical, is the existence of strong solutions provided in Theorem 3.10 Even for the Wiener noise such a result, in the infinite dimensional case, is known only in some special cases; as a matter of fact, our assumption $H > 3/4$, together with the ambientation $W(t) \in D((-A)^{\alpha})$ for any $\alpha < 1/4$,
which leads to $H + \alpha > 1$, is the analog of the assumption $W(t) \in D((-A)^{\alpha})$ for some $\alpha > 1/2$ required in [1] (since the Wiener case corresponds to $H = 1/2$). We shall also mention the paper [18], which inspired our proof, but where the stronger requirement $W(t) \in D(A)$ is made.

In Subsection 2.1 we are going to introduce the deterministic model of the time evolution of potential in a dendritical tree according to Rall’s linear cable theory. Successively, we are going to perturb it by stochastic terms that are represented by a fractional Brownian motion, as delineated in Subsection 2.2. Section 3 is devoted to the stochastic analysis for a network which contains only active nodes, while the general setting is sketched in Section 4.

2. Biological motivations

The motivation of this paper is provided by a stochastic model of diffusion for neuronal activity in passive dendritic fibers. The electrical behaviour of a neuron has been long studied, beginning with the pioneering experiments of H. von Helmholtz, more than 150 years ago. In the 1950s, the ground-breaking papers of, among other, A.L. Hodgkin and A.F. Huxley [14] and W. Rall [30] have represented a major breakthrough in the modern neurophysiology. Ever since, experimental results have dramatically increased. Successively, more and more interest has been revived by stochastic models – these may help in gaining a better understanding of neuronal behaviour and in predicting neuronal activity.

A typical neuron cell consists of four parts: the dendrites, the synapses, the soma, and the axon. Dendrites are the input stage of a neuron and receive input from other neurons through axo-dendritic synaptical junctions. Synapses also connect dendrites to each other (leading to so-called dendro-dendritic interactions): in graph theoretical language, the dendritic network is actually a tree converging into the soma, the cell’s body. In [30, 31] Rall introduced the so-called “lumped soma model”, representing (a part of) a dendritic tree as a lumped R-C circuit, thus allowing for an internal electrical activity of the soma. In Rall’s model as well as in most of its generalizations (and in particular in leaky integrate-and-fire neuronal models, see [15]) it is the soma’s duty to elaborate these inputs and transmits them to other neurons through the axon. In this way Rall was able, under strong (geometric) symmetry and (electrotonic) isotropy assumptions, to describe the behaviour of a whole dendritic tree by a cable equation on a single “equivalent cylinder” (i.e., an individual interval of finite length) equipped with a dynamic boundary condition. Later on, multicylinder models have been proposed already in the 1970s in [32–33] in order to weaken the theoretical assumptions on the topology of the dendritic tree, thus describing more general classes of individual neurons. Such models have been extensively discussed in the 1990s, see e.g. [20] and subsequent papers. Aim of the present paper is to further generalize a model of Rall-type to a non-deterministic one, in order to allow for noise in the boundary, i.e., for non-deterministic inputs in the somata.
The starting point for the present analysis is represented by the recent article [26]. There, a thorough investigation of the deterministic version of such model has been performed: beside well-posedness results, several qualitative properties of solutions have been shown. Our mathematical approach also applies some of the methods of that paper.

Rall’s linear cable theory applies to the behaviour of the passive dendritic tree: the propagation of action potentials along an axon is modelled as a semilinear diffusion with a zero-order term that accounts for describing how the cell expends energy in order to propagate the signal, see below. On the other hand, it is known that in axons action potentials are initiated and propagated with fixed asymptotical speed and profile in the outward direction: an effect of regenerative self-excitation is thus needed while modelling axons. Such a phenomenon is commonly described by a coupled system of a semilinear diffusion equation and a nonlinear ordinary differential equation. Such a system has been introduced by R. FitzHugh and J. Nagumo as a simplification of the original Hodgkin–Huxley model: for a survey of this theory we refer, e.g., to [37]. Thus, the behaviour of the neuron in the dendritic tree and the axon is significantly different: as in [20] and [26] we will not attempt to introduce axons into the model considered in this note. A (naive) hybrid axo-dendritic model has been studied in a deterministic setting in [7].

Most of the aforementioned papers deal with a compartmental model of the neuron, subsequently investigated by numerical methods. Aim of this note is to provide a more theoretical approach based on an interplay of operator theory and stochastic analysis.

Stochastic fluctuations of synaptic activity and post-synaptic elaboration of electronic potential is directly related to chaotic interferences in the streaming of charge carrying molecules. This chaotic streaming is classically modeled as a sequence of independent random variables; averaging on time, in dependence of the time scale, leads to choose a Brownian motion (or more generally a Levy process) as a model for the stochastic input process. Possibly combined with active properties of axons, this may lead to chaotic triggering of action potentials. Several articles have actually dwelled on the stochastic dynamics of axons within the framework of the Hodgkin–Huxley or FitzHugh–Nagumo axonal models, see e.g. [38] or [36], mainly concerned with the issue of voltage fluctuations near threshold or the more theoretical approach in [4].

Less attention has seemingly been paid to the non-deterministic aspects of sub-threshold stochastic behaviour, either in passive or active fibers. However, there seem to be good reasons to perform such an analysis: in particular, “many computations putatively performed in the dendritic tree (coincidence detection, multiplication, synaptic integration and so on) occur in the sub-threshold regime” (quoted from [22]). A model for passive dendrites of infinite length with distributed as well as synaptic current noise has been proposed in [17] [10], whereas more experimental studies have been presented in [12]. A simplified model of a network with active nodes is proposed in [5]. Perhaps the clearest outcome of such investigations is that voltage noise increases with depolarization, cf. [16].
Although our biological motivation leads us to consider a parabolic network equation, our results (with possibly the exception of the existence of strong solutions) are not confined to the 1-dimensional case. In fact, by means of the same techniques we may consider with minor changes also a stochastic initial-boundary value problem whose deterministic part is a diffusion equation with dynamic (a.k.a. Wentzell-Robin) boundary conditions on a bounded domain. The same remark applies to the stochastic analysis provided in Section 3 which could have been stated in abstract form and applied in different contests.

2.1. Derivation of the equation. The basic object of our investigations, i.e., the dendritic network is modelled by a finite, connected graph $G$ with $m$ edges $e_1, \ldots, e_m$ and $n$ vertices $v_1, \ldots, v_n$.

Let $V$ denote the electric potential on the whole dendritic network, including the junctions (synapses and further ramification points) and the soma. Then $V = V(t, x)$ is a function of the position $x$ along the network and of time $t$ only. Of course, $V$ denotes the deviation from resting potential, which we rescale to 0 (in concrete neurophysical measurements it is of approx. $-70mV$). With a somewhat coarse but common approximation, we will assume the cell’s body to be isopotential.

By $V_j$ we denote the electric potential on the edge $e_j$, i.e., $V_j(t, x) := V(t, e_j(x))$. Up to considering suitable rescaling diffusion parameters in the equations, we may and do parametrize the edges as intervals of unitary length. Actually, we avoid to introduce more general network elliptic operators: in fact, under uniform ellipticity assumptions this case is known to present no serious mathematical challenges over the basic case of a plain network Laplacian.

Following Rall’s passive cable theory, the transmission of post-synaptic potential in dendritic trees can be mathematically described by a linear cable equation: on each edge $e_j$ (i.e., each minimal set of dendritic elements that can be reduced to an equivalent cylinder, cf. [20]), we consider the partial differential equation

$$
\frac{\partial V_j}{\partial t}(t, x) = \frac{\partial^2}{\partial x^2} V_j(t, x) - p_j(x) V_j(t, x), \quad t \geq 0, \ x \in (0, 1),
$$

We borrow from graph theory some basic notions, in order to describe the behaviour of the system at ramification points and to describe the architecture of the dendritic network in a compact form. We consider a graph $G$ with vertex set $V$ and edge set $E$ throughout. We impose dynamical and time-independent conditions in the nodes $v_1, \ldots, v_{n_0}$ and $v_{n_0+1}, \ldots, v_n$, respectively. All nodes of the network are connected by the above introduced edges $e_j$. We denote by $\mathcal{J} := \mathcal{J}^+ - \mathcal{J}^-$ the $n \times m$ incidence matrix of the graph, where

\footnote{In biological applications a dendritic network is in fact a tree, in the graph theoretical sense, and therefore only one soma needs to be considered, i.e., $n_0 = 1.$}
\( J^+ := (\iota^+_{ij}) \) and \( J^- := (\iota^-_{ij}) \) are the *incoming* and *outgoing* incidence matrices defined by

\[
\iota^+_{ij} := \begin{cases} 
1 & \text{if } e_j(0) = v_i, \\
0 & \text{otherwise},
\end{cases} \quad \text{and} \quad \iota^-_{ij} := \begin{cases} 
1 & \text{if } e_j(1) = v_i, \\
0 & \text{otherwise},
\end{cases}
\]

respectively. We also denote by \( \Gamma(v_j) \) the set of all indices of those edges having an endpoint in \( v_j \), i.e.,

\[
\Gamma(v_i) = \{ j \in \{1, \ldots, m\} \mid e_j(0) = v_i \text{ or } e_j(1) = v_i \}.
\]

Motivated by Rall's lumped soma model (see e.g. \[30\] and \[20\]), we specify the behaviour of the system at the edges' boundaries by imposing several kinds of conditions in the nodes.

- Both somata and synapses can be considered as isopotential. In other words, in all nodes \( v_i, i = 1, \ldots, n \), the potential \( V \) must satisfy the continuity assumption

\[
V_j(t, v_i) = V_k(t, v_i) =: q^V_i \quad \text{for all } j, k \in \Gamma(v_i), \quad t \geq 0.
\]

- In the somata \( v_1, \ldots, v_{n_0} \), the potential \( q^V_i(t) = V(t, v_i) \) undergoes internal dynamics, subject to internal electrical activity and a stochastic feedback from the dendritic network. In the spirit of Rall's lumped soma model (see e.g. \[20\]) we impose general, possibly absorbing (and also possibly non-local) nodal conditions. They can be formulated as a Langevin-type equation

\[
\frac{d}{dt} q^V_i(t) = - \sum_{j=1}^{m} \iota_{ij} \frac{\partial V_j}{\partial x}(t, v_i) - \sum_{h=1}^{n} b_{ih} q^V_h(t) + \sum_{h=1}^{n} c_{ih} \dot{Z}_h(t), \quad t \geq 0,
\]

where the last addend on the right hand side is a sum, over the edges incident in \( v_i \), of (formal) derivatives of the stochastic inputs acting there. We shall give a precise definition and complete assumptions on the noisy terms \( Z^h \) in Section 2.2.

- Motivated by the intrinsic randomness of external stimuli, we impose in the synapses and other ramification nodes \( v_i, i = n_0 + 1, \ldots, n \), a version of a Kirchhoff's law which is also perturbed by a stochastic term \( Z_i \). Such a condition can be equivalently formulated as

\[
\sum_{j=1}^{m} \iota_{ij} \frac{\partial V_j}{\partial x}(t, v_i) + \sum_{h=1}^{n} b_{ih} q^V_h = \sum_{h=1}^{n} c_{ih} Z_h(t), \quad t \geq 0.
\]
Summing up, the biological model we want to discuss is governed by the system of
stochastic initial-boundary value problems
\begin{equation}
\begin{cases}
\dot{V}_j(t,x) = V_j''(t,x) - p_j(x)V_j(t,x), & t \geq 0, \ x \in (0,1), \\
V_j(t,v_i) = V_{\ell}(t,v_i) =: q_i^V, & t \geq 0, \ j, \ell \in \Gamma(v_i), \\
 q_i^V(t) = - m \sum_{j=1}^{m} \iota_{ij} V_j''(t,v_i) - \sum_{h=1}^{n} b_{ih} q_h^V + \sum_{h=1}^{n} c_{ih} \dot{Z}_h(t), & t \geq 0, \ i = 1, \ldots, n_0, \\
\sum_{h=1}^{n} b_{ih} q_h^V = - m \sum_{j=1}^{m} \iota_{ij} V_j''(t,v_i) + \sum_{h=1}^{n} c_{ih} \dot{Z}_h(t), & t \geq 0, \ i = n_0 + 1, \ldots, n, \\
V_j(0,x) = V_{0j}(x), & x \in (0,1), \ j = 1, \ldots, m, \\
V(0,v_i) = q_{0i}, & i = 1, \ldots, n_0.
\end{cases}
\end{equation}

For the sake of notational simplicity, let us introduce the Kirchhoff operators $K_a, K_p$
mapping $H^2(0,1;\mathbb{C}^m)$ into $\mathbb{C}^{n_0}$ and $\mathbb{C}^{n-n_0}$, respectively, defined by

$$K_a V := \left( - \sum_{j=1}^{m} \iota_{ij} V_j''(t,v_1) \right) \quad \text{and} \quad K_p V := \left( - \sum_{j=1}^{m} \iota_{n_0+j} V_j''(t,v_{n_0+1}) \right).$$

Thus, the vector $K_a V$ (resp., $K_p V$) represents the differences between incoming and out-
going flows in each of the active (resp., passive) nodes of the network.

We introduce the matrices

$$B_a = \left( b_{ih} \right)_{i=1,\ldots,n_0, \ h=1,\ldots,n}, \quad B_p = \left( b_{(n_0+i)h} \right)_{i=1,\ldots,n-n_0, \ h=1,\ldots,n} \quad \text{and} \quad B = \begin{pmatrix} B_a \\ B_p \end{pmatrix} = \left( b_{ih} \right)_{i=1,\ldots,n}.$$ 

Roughly speaking, $B_a$ and $B_p$ encode the inhibitory and excitatory properties of active
and passive nodes under the influence of the whole system. Also, in order to model the
stochastic terms we introduce the matrix

$$C_{aa} = \left( c_{ih} \right)_{i=1,\ldots,n, \ h=1,\ldots,n_0},$$

as well as

$$C_a = \left( c_{ih} \right)_{i=1,\ldots,n_0, \ h=1,\ldots,n}, \quad C_p = \left( c_{(n_0+i)h} \right)_{i=1,\ldots,n-n_0, \ h=1,\ldots,n} \quad \text{and} \quad C = \begin{pmatrix} C_a \\ C_p \end{pmatrix} = \left( c_{ih} \right)_{i=1,\ldots,n}.$$ 

With no loss of generality, we may and do assume that the entries of the matrix $C$ are non-
negative real numbers and that there exist some entries in $C_a$ which are strictly positive.
In particular, the stochastic terms in (2.3)–(2.4) account for both the intrinsic cellular noise
and the external input signals. Observe that although the possibility of non-local conditions does not seem to be realistic in biological system, it may be interpreted as some form of external boundary feedback control of the system – e.g., with the aim of stabilization.

2.2. Fractional Brownian motion. As customary in network applications, we use stochastic processes in order to model the input of our system. Although the stochastic processes used in this field were often assumed to be Markovian, other considerations show that real inputs may exhibit long-range dependence, i.e., the behaviour of the process at time \( t \) does depend on the whole history up to time \( t \). Another property that stochastic inputs show, also in telecommunication networks, is \textit{self-similarity}: their behaviour is stochastically the same, up to a space scaling, on changing the time scale. These features are common in physiological systems and have been already applied to the neuroscience, see e.g. [27, 9], and the contributions in [34, Part II]. Among the processes which exhibit these properties, we propose to model our stochastic input with fractional Brownian motion (fBm for short). This class of processes also verify \textit{stationarity} of the increments and continuity of trajectories.

A 1-dimensional fractional Brownian motion is a centered Gaussian process \( \{B^H(t), t \geq 0\} \), such that

\[
\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2H}(t^{2H} + s^{2H} - |t - s|^{2H}).
\]

The constant \( H \) is called \textit{Hurst parameter} and takes value in \((0,1)\). The construction of fBm was proposed by Mandelbrot and van Ness [21] using the representation

\[
B^H(t) = \int_0^t (t - s)^{H-1/2} dB_s + \int_{-\infty}^0 (t - s)^{H-1/2} - |s|^{H-1/2}) dB_s
\]

where \( B = \{B_t, t \geq 0\} \) is a standard Brownian motion. We shall be interested in a different characterization of fBm, based on the observation that, given the fBm \( B^H(t) \), there exists a unique Brownian motion \( W(t) \), adapted to the same filtration, and a kernel \( K_H(t,s) \) such that the identity

\[
B^H(t) = \int_0^t K_H(t,s) dW(s).
\]

holds. In case \( H = \frac{1}{2} \), \( B^{1/2}(t) = W(t) \) is a standard Brownian motion. In this paper we restrict our interest to a particular class of fBm and impose the following.

**Assumption 2.1.** We assume that there exists a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) endowed with a filtration \( \{\mathcal{F}_t, t \geq 0\} \) that satisfies the standard assumptions, i.e., it contains all negligible subsets of \( \Omega \) and it is right-continuous.

On this space we are given a family of independent standard Brownian motions \( \{W_h(t), t \geq 0\}_{h=1, \ldots, n} \); then we consider, for each \( h = 1, \ldots, n \), the fBm

\[
Z_h(t) = \int_0^t K_H(t,s) dW_h(s), \quad t \geq 0.
\]
We let \( Z^a(t) = (Z_1(t), \ldots, Z_{n_0}(t)) \) and \( Z^p(t) = (Z_{n_0+1}(t), \ldots, Z_n(t)) \) denote the stochastic inputs in active and passive nodes, respectively, and \( Z(t) = (Z^a(t), Z^p(t)) \) be the n-dimensional stochastic input process; we refer to \( Z(t) \) as a n-dimensional fBm.

The Hurst parameter \( H \) belongs to \((\frac{1}{2}, 1)\).

2.3. Wiener integrals for fBm. For any given Hilbert space \( \Xi \), we are interested in the introduction of Wiener-type integrals for \( \Xi \)-valued, square integrable functions, defined with respect to a fractional Brownian motion \( Z(t) \). This is a known topic in literature and we follow the stochastic calculus of variations approach, see the monograph [2] and the references therein; we shall sketch below the notation and the main results.

In the Brownian case \( H = \frac{1}{2} \), the space of Gaussian random variables defined as \( \xi = \int_R f(s) dW_s \) is isomorphic to the space \( L^2(\mathbb{R}; \Xi) \). This isometry is classically proved first for step functions and then is extended to the whole space. A similar construction was first investigated in [29] for the fractional Brownian motion. Let \( \mathcal{E}(\Xi) \) be the set of step functions \( \Phi : \mathbb{R} \to \Xi \)

\[
\Phi(t) = \sum_{i=0}^{m} \Phi_i 1_{[t_i, t_{i+1}]};
\]

the stochastic integral \( I(\Phi) \) is defined by setting

\[
I(\Phi) = \int_R \Phi(s) dB^H(s) = \sum_{i=0}^{m} \Phi_i (B^H(t_{i+1}) - B^H(t_i))
\]

and it is a Gaussian random variable with zero mean and covariance that can be computed using the covariance matrix of \( B^H \). The set of Gaussian random variables defined by the elements of \( \mathcal{E}(\Xi) \) is a subset of the first Wiener chaos

\[
\mathcal{H}_1 = \{ X \in L^2(\Omega; \Xi) | \exists (f_n)_{n \geq 0} \subset \mathcal{E}(\Xi) : \lim_{n \to \infty} I(f_n) = X \text{ in } L^2(\Omega; \Xi) \}
\]

generated by \( B^H \). The reproducing kernel Hilbert space (RKHS for short) \( \Lambda \) is defined as the closure of \( \mathcal{E}(\Xi) \) with respect to the scalar product

\[
(1_{[0,t]} | 1_{[0,s]}) = R(t, s);
\]

the mapping \( I : 1_{[0,t]} \to B^H(t) \) defines an isometry between \( \Lambda \) and \( \mathcal{H}_1 \).

As opposite to the Brownian motion case, for \( H > \frac{1}{2} \) the RKHS \( \Lambda \) for the fBm \( B^H(t) \) cannot be identified with a space of functions; however, \( \Lambda \) contains (linear subspaces which are isometric to) inner product spaces of functions. Proceeding as in [29] we can prove that this space can be identified with the space \( |\mathcal{H}| \) of functions in \( L^1(\mathbb{R}; \Xi) \cap L^2(\mathbb{R}; \Xi) \) endowed with the norm

\[
\|\phi\|_{|\mathcal{H}|}^2 = \alpha_H \int_R \int_R |t - r|^{2H-2} |\phi(r)| \Xi |\phi(t)| \Xi \, dr \, dt.
\]
This space contains the class of step functions $\mathcal{E}(\Xi)$ and the isometry
\[ E \left| \int_{\mathbb{R}} \phi(t) \, dB^H(t) \right|_\Xi^2 = \| \phi \|^2_{L^2[\Xi]} \]
holds.

3. The abstract formulation

Before discussing the complete stochastic differential equation in Section 4, we begin by considering the system with the following simplified form of (2.4):
\begin{equation}
\sum_{j=1}^{m} \frac{\partial V_j(t, v_i)}{\partial x} + \sum_{h=1}^{n} b_{ih} q_h V = 0, \quad t \geq 0.
\end{equation}

To begin with, we introduce the Hilbert spaces $X := (L^2(0,1))^m$ and $\partial X_a := \mathbb{C}^{n_0}$. On the domain
\[ D(A) := \left\{ v := \begin{pmatrix} V \\ q_a \end{pmatrix} \in (H^2(0,1))^m \times \mathbb{C}^{n_0} \quad \text{s. th.} \quad \exists q^V \in \mathbb{C}^n \quad \text{with} \quad (J^+)^\top q^V = V(0), \right. \]
\[ \left. (J^-)^\top q^V = V(1), \quad (q_1^V, \ldots, q_{n_0}^V) = (q_{a1}, \ldots, q_{an_0}), \quad \text{and} \quad K_p V = B_p q^V \right\} \]
we define the operator $A$ by
\begin{equation}
A v := A \begin{pmatrix} V \\ q_a \end{pmatrix} := \begin{pmatrix} V'' - pV \\ K_a V - B_a q^V \end{pmatrix},
\end{equation}
where $V''$ denotes the vector $(V_1'', \ldots, V_m'') \in X$ consisting of (one-dimensional) second derivatives of the real-valued entries of the vector-valued function $V = (V_1, \ldots, V_m) \in (H^2(0,1))^m$.

One can directly check that the initial value problem associated with (2.1)–(2.2)–(2.3)–(3.1) can be equivalently formulated as an abstract stochastic Cauchy problem
\begin{equation}
\begin{cases}
dv(t) = Av(t) \, dt + C_a \, dZ^a(t), & t \geq 0, \\
v(0) = v_0,
\end{cases}
\end{equation}
where the initial value is given by $v_0 := (V_0, q_0)^T \in X := X \times \partial X_a$ and $C_a$ maps every vector $q \in \partial X_a$ in $(0, C_{aa} q)^T \in X$.

Our first aim is to collect some results on the underlying deterministic model, which will show well-posedness and further qualitative properties of our system. Although formally new, the results of this section can essentially be proved combining the ideas of [24] and [26], where parabolic network equations with passive only, non-local interactions and local, active node conditions have been considered, respectively.
In order to prove the generation property of the operator $A$, we introduce a Hilbert space
\[
V := \left\{ v := \begin{pmatrix} V \\ q_a \end{pmatrix} \in (H^1(0, 1))^m \times \mathbb{C}^{n_0} \text{ s. th. } \exists q^V \in \mathbb{C}^n \text{ with } (J^+)^\top q^V = V(0), \right. \\
(J^-)^\top q^V = V(1), \ (q^V_1, \ldots, q^V_{n_0}) = (q^V_{a1}, \ldots, q^V_{am}) \right\}
\]
and the sesquilinear form $a : V \times V \rightarrow \mathbb{C}$ defined by
\[
a(u, v) := a_0(u, v) + a_1(u, v) := (U'|V')_X + ((pU'|V')_X + (Bq^U'|q^V)_0X_a).
\]

We emphasize that this form is in general not symmetric, but it is always densely defined. Mimicking the proofs of [24, Lemma 3.4] and [26, Lemma 3.3] one can prove that the operator associated with $a$ is $A$. Observe that $a$ and hence $A$ are self-adjoint if and only if the scalar matrix $B$ is hermitian, i.e., if and only if mutual interaction of the nodes are symmetric.

Combining the techniques presented in [26] and [24] for discussing dynamic and non-local boundary conditions, respectively, we can prove the following.

**Proposition 3.1.** Let $B \in M_n(\mathbb{C})$ and $p \in (L^1(0, 1))^m$. Then the operator $A$ generates a strongly continuous, analytic and compact semigroup $(S(t))_{t \geq 0}$ on the Hilbert space $X := X \times \partial X_a$.

**Proof.** We first observe that the leading term in the form $a$, i.e., $a_0$, is clearly $X$-elliptic and continuous. Furthermore, it follows from the Gagliardo–Nirenberg inequality that the space
\[
C(G) := \left\{ v := \begin{pmatrix} V \\ q_a \end{pmatrix} \in (C([0, 1]))^m \times \mathbb{C}^{n_0} \text{ s. th. } \exists q^V \in \mathbb{C}^n \text{ with } (J^+)^\top q^V = V(0), \right. \\
(J^-)^\top q^V = V(1), \ (q^V_1, \ldots, q^V_{n_0}) = (q^V_{a1}, \ldots, q^V_{am}) \right\}
\]
\[
\simeq \left\{ V \in (C([0, 1]))^m \text{ s. th. } \exists q^V \in \mathbb{C}^n \text{ with } (J^+)^\top q^V = V(0), \ (J^-)^\top q^V = V(1) \right\}
\]
of continuous functions over the network is embedded in an interpolation space of order $\frac{1}{2}$ between $V$ and $X$, and obviously $a_1 : C(G) \times C(G) \rightarrow \mathbb{C}$ is bounded. Thus, by a suitable perturbation argument, cf. [25] Lemma 2.1, one concludes that also their sum $a$ is $X$-elliptic and continuous. Thus, by [28] Prop. 1.51, Thm. 1.52] the associated operator $A$ generates a strongly continuous, analytic semigroup. Moreover, by Rellich–Khondrakov’s theorem the imbedding $V \hookrightarrow X$ is compact and accordingly the semigroup is compact. \(\Box\)

**Remark 3.2.** If coefficients $b_{ij}$ and $p_j(x)$ are real constant, resp. real valued functions, then $S(t)$ maps real valued functions in real valued functions. thus, despite of the choice of complex valued function spaces, we may appeal to this remark in order to justify the application of our results to biological models.
All further properties of the semigroup essentially depend on the matrix $B$: e.g., it is possible to characterize asymptotics of the semigroup generated by $A$ by means of positivity of the $L^1$-coefficient $p$ and/or of the definiteness of the (in general non-hermitian) matrix $B$, cf. [8, Thm. 2.3].

**Proposition 3.3.** If $p \geq 0$ and $B$ is positive semidefinite, then the following assertions hold.

1. $(S(t))_{t \geq 0}$ is contractive, i.e., $\|S(t)\| \leq 1$ for all $t \geq 0$, and hence mean ergodic.
2. If additionally $p = 0$ and $B^*1 \neq 0$, or else if $B = 0$ and $p \neq 0$, then $(S(t))_{t \geq 0}$ is strongly stable, i.e., $\lim_{t \to \infty} S(t)v = 0$ for all $v \in X$.
3. If additionally $p \geq p_0 > 0$ and/or $B$ is positive definite, then $(S(t))_{t \geq 0}$ is uniformly exponentially stable, i.e. $\|S(t)\| \leq e^{-\omega t}$ for some $\omega > 0$ and all $t \geq 0$.

By compactness of $(S(t))_{t \geq 0}$ and an abstract criterion due to Ouhabaz, see [28, Thm. 2.6] we obtain the following. We do not sketch here the proof and refer the reader to [26] and [24] for technical details.

**Corollary 3.4.** If $p = 0$ and $B = 0$, then $(S(t))_{t \geq 0}$ converges towards the (strictly positive) projection onto the 1-dimensional eigenspace spanned by the constant vector $1 \in X \times \partial X_a$.

Assume for a moment that $A$ is negative definite. Then, since $A$ is a sectorial operator, we can introduce the scale of interpolation spaces defined with respect to the domain of the operator $A$. The leading term of the form associated with $A$ is symmetric, i.e., $A$ is self-adjoint up to dropping the boundary conditions given by $B$, which are only relevant for higher powers of $A$.

More precisely, it follows by the spectral theorem that complex interpolation spaces $X_\alpha := [D(-A), X]_\alpha$ are equivalent to the domains of fractionary powers $(-A)^\alpha$ of $-A$.

Factoring $A$ as in [23, Lemma 2.7] one can (up to similarity) decouple its domain into the product $H_0(G) \times \partial X_a$. Here $H_0(G)$ denotes the space of $(H^2(0,1))^m$-functions with Dirichlet boundary conditions in the active nodes $v_1, \ldots, v_m$. For $\alpha < \frac{1}{4}$, it follows that $D((-A)^\alpha)$ is isomorphic to $H^{2\alpha}(G) \times \partial X_a$.

In general, it is always possible to perform the same steps by defining the complex interpolation spaces $X_\alpha := [D(-\lambda I - A), X]_\alpha$ for a suitable constant $\lambda$.

### 3.1. Stochastic differential equations with additive fBm.

In this section we solve the stochastic differential equation (3.3)

\[
\begin{aligned}
\text{d}v(t) &= Av(t) \text{d}t + C_a \text{d}Z^a(t), \quad t \geq 0, \\
v(0) &= v_0,
\end{aligned}
\]

where $v_0 := (v_0, q_0)^T \in X := X \times \partial X_a$. 
Definition 3.5. Following the Wiener case treated for instance in [DPZ92] we call an $X$-valued adapted process $v = \{v(t), t \in [0,T]\}$ a strong solution to (3.3) if $v$ has a version such that

(S1) for almost all $t \in [0,T]$, $\mathbb{P}(v(t) \in D(A)) = 1$;

(S2) for any $t \in [0,T]$, $\mathbb{P} \left( \int_0^t |Av(s)|_X^2 \, ds < +\infty \right) = 1$, and

(S3) for any $t \in [0,T]$ there holds

\[ v(t) = v_0 + \int_0^t Av(s) \, ds + C_a Z^a(t). \]

Definition 3.6. Similarly, we define a weak solution to (3.3) if

(W1) for any $t \in [0,T]$, $\mathbb{P} \left( \int_0^t |v(s)|_X^2 \, ds < +\infty \right) = 1$, and

(W2) for any $t \in [0,T]$ and any $y \in D(A^*)$ it holds

\[ \langle v(t), y \rangle = \langle v_0, y \rangle + \int_0^t \langle v(s), A^* y \rangle \, ds + \langle C_a Z^a(t), y \rangle. \]

3.2. Existence of weak solutions. In order to provide the existence of a weak solution, our main interest lies in the stochastic convolution process

\[ W_A(t) := \int_0^t S(t-\sigma) C_a \, dZ^a(\sigma). \]

The key point is to find estimates for the variance, like

\[ \mathbb{E}|W_A(t)|^2 \leq \sum_{k=1}^{\infty} \left\| |S(t-\sigma) C_a e_k|_{X^2} 1_{(0,t)}(\sigma) \right\|_\Lambda^2 \]

where $\Lambda$ is the RKHS associated with the fBm $Z^a$. This means that in case $H = 1/2$ (the Wiener case) $\Lambda$ is the space $L^2(\mathbb{R})$ and in case $H > 1/2$ we can take instead of $\Lambda$ the space $[g]$. Since $S(t)$ is a semigroup, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S(t)\|_{L(\mathbb{R})} \leq Me^{\omega t}$; however, Proposition 3.1 implies that we can choose $M = 1$ and $\omega < 0$. It follows that

\[ \mathbb{E}|W_A(t)|^2 \leq M \|C_a\|_{H_S} \|e^{\omega t} 1_{(0,t)}(\sigma)\|_\Lambda^2; \]

in the Wiener case, the norm is given by $\int_0^t e^{2\omega s} \, ds = \frac{1}{2\omega}(e^{\omega t} - 1) \leq \frac{1}{2|\omega|}$; on the other hand, in the fBm case, the estimate becomes

\[ \int_0^t \int_0^t e^{\omega s} e^{\omega r} |s-r|^{2H-2} \, ds \, ds \leq \text{Const.} \frac{1}{|\omega|^{2H}} \left| 1 - e^{2\omega t} \right| \]
and we get
\begin{equation}
\sup_{t \in [0, T]} \mathbb{E}|W_A(t)|^2 \leq \text{Const.} \|C_a\|_{HS}^2.
\end{equation}

With a little effort, it is possible to be more precise on the regularity (in space) of the stochastic convolution process. We remark that the case $H = 1/2$ is treated in [11, Proposition 6.17].

**Lemma 3.7.** Assume that the operator $A$ is injective and dissipative. Hence, we can consider the fractional powers of the operator $A$ and the family of spaces $X_\alpha$ with equivalent norm $\|v\|_{X_\alpha} := \|(-A^\alpha v)\|_X$.

Let $H > 1/2$ and assume that $H - \alpha > 0$. Then the stochastic convolution process verifies
\begin{equation}
\mathbb{E} \int_0^T \|W_A(t)\|^2_{X_\alpha} dt \leq \text{Const.}
\end{equation}

**Proof.** We get
\begin{align*}
\mathbb{E} \int_0^T \|W_A(t)\|^2_{X_\alpha} dt &\leq \sum_{k=1}^{n_0} \int_0^T \|S(t - \sigma)C_a e_k\|_{X_\alpha} \|1_{(0,t)}(\sigma)\|^2_{\Lambda} d\sigma dt \\
&\leq |C_a e_k|^2 \sum_{k=1}^{n_0} \int_0^t \int_0^t \frac{M_\alpha^2}{\sigma^\alpha \theta^\alpha} |\sigma - \theta|^{2H-2} d\sigma d\theta \\
&\leq M_\alpha^2 C_{\alpha,H} |C_a e_k|^2 t^{H-\alpha}
\end{align*}

and for any finite $T > 0$ the quantity in (3.9) is finite. □

**Theorem 3.8.** If $A$ generates a $C_0$-semigroup $(S(t))_{t \geq 0}$, then there exists a unique weak solution to (3.3) and it has the representation
\begin{equation}
v(t) = S(t)v_0 + \int_0^t S(t - s)C_a dZ^a(s), \quad t \in [0, T].
\end{equation}

**Proof.** Fix $t \in [0, T]$. Let us first notice the following identity which holds for every $\Phi \in C^1([0, t]; D(A^\alpha))$:
\begin{equation}
\langle v(t), \Phi(t) \rangle - \langle v_0, \Phi(0) \rangle = \int_0^t \langle v(s), \Phi(s) \rangle ds + \int_0^t \langle v(s), A^\alpha \Phi(s) \rangle ds + \int_0^t \langle \Phi(s), C_a dZ^a(s) \rangle.
\end{equation}
Assume now that \( u \) is a weak solution to (3.3). Choosing \( \Phi(s) = S^*(t-s)y_{1_{(0,t)}(s)} \) in (3.11) we get
\[
\langle v(t), y \rangle - \langle v_0, S^*(t)y \rangle = \int_0^t \langle v(s), dS^*(t-s)y \rangle ds + \int_0^t \langle v(s), A^*S^*(t-s)y \rangle ds + \int_0^t \langle S^*(t-s)y, \mathcal{C}_a dZ^a(s) \rangle
\]
that is
\[
\langle v(t), y \rangle - \langle \mathcal{S}(t)v_0, y \rangle = \int_0^t \langle y, S(t-s)\mathcal{C}_a dZ^a(s) \rangle
\]
which implies uniqueness of the weak solution.

We now proceed to show that the process \( u \) defined in (3.10) is a weak solution to (3.3). Taking in mind formula (3.6) we compute
\[
\int_0^t \langle S(\sigma)v_0 + \int_0^\sigma S(\sigma-s)\mathcal{C}_a dZ^a(s), A^*y \rangle d\sigma
= \int_0^t \langle v_0, S(\sigma)A^*y \rangle d\sigma + \int_0^t \int_0^\sigma \langle S(\sigma-s)\mathcal{C}_a dZ^a(s), A^*y \rangle d\sigma
= \int_0^t \int_0^\sigma \langle v_0, S^*(\sigma)y \rangle d\sigma + \int_0^t \int_0^{t-s} \frac{d}{d\sigma}S^*(\sigma)y d\sigma, \mathcal{C}_a dZ^a(s) \rangle
= \langle S(t)v_0, y \rangle - \langle v_0, y \rangle + \int_0^t \langle S^*(t-s)y - y, \mathcal{C}_a dZ^a(s) \rangle
= \langle S(t)v_0, y \rangle - \langle v_0, y \rangle + \int_0^t \langle y, S(t-s)\mathcal{C}_a dZ^a(s) \rangle - \mathcal{C}_a Z^a(t)
\]
and the representation (3.10) yields to the result. \( \square \)

**Corollary 3.9.** Assume that \( A \) is a bounded linear operator. Then the stochastic convolution process

(3.12) \[ W_A(t) := \int_0^t S(t-s)\mathcal{C}_a dZ^a(s) \]

verifies

(3.13) \[ W_A(t) = \int_0^t A W_A(s) ds + \mathcal{C}_a Z^a(t). \]

### 3.3. Existence of a strong solution.

**Theorem 3.10.** Let \( A \) be a sectorial operator and assume that \( \mathcal{C}_a \) maps \( X \) into \( X_\alpha \) for any \( \alpha < 1/4 \). Assume that the Hurst parameter \( H \) of the fBm \( Z^a(t) \) verifies \( H > 3/4 \) and take the initial condition \( v_0 \in D(A) \). Then there exists a unique strong solution to equation (3.3).
Up to rescaling, since we deal with a finite time interval \([0, T]\), we can assume that \(A\) is dissipative and injective.

**Yosida approximations.** Before to proceed to prove the existence of a strong solution for (3.3), we need to introduce the family of operators \(A_n, n \in \mathbb{N}\), defined by

\[
A_n = nAR(n, A) = n^2R(n, A) - nI.
\]

This family has been introduced in Yosida’s own proof of the celebrated Hille–Yosida generation theorem. It enjoys remarkable properties: the operators \(A_n\) are bounded; they commute with one another as well as with \(A\) and with the resolvent operators of \(A\); and finally

\[
\lim_{n \to \infty} S_n(t)x = S(t)x \quad \text{for all } x \in X \quad \text{and} \quad \lim_{n \to \infty} A_n x = Ax \quad \text{for all } x \in D(A),
\]

by [13, Lemma II.3.4].

For future references we need a lemma concerning the speed of convergence of the semi-groups generated from Yosida approximations to \(S(t)\).

**Lemma 3.11.** Let \(A\) be a sectorial operator. Let also \(C \in L(X, X_\alpha)\) for some \(\alpha \in (0, 1)\). Define, for any \(n \in \mathbb{N}\), the sequence of functions \(f_n(\sigma)\) setting

\[
f_n(\sigma) = \|A_n(S_n(\sigma) - S(\sigma))C\|_{L(X)}.
\]

Then for given \(T > 0\) it holds

\[
\lim_{n \to \infty} \sup_{\sigma \in (0, T)} |f_n(\sigma)| = 0.
\]

**Proof.** Write

\[
\|A_n(S_n(\sigma) - S(\sigma))C\|_{L(X)} = n\|A^{1-\alpha}R(n, A)(S_n(\sigma) - S(\sigma))A^\alpha C\|_{L(X)}
\]

\[
\leq n\|A^{1-\alpha}R(n, A)\|_{L(X)}\|(S_n(\sigma) - S(\sigma))\|_{L(X)}\|A^\alpha C\|_{L(X)}
\]

We estimate, for \(\omega = s(A)\) the spectral bound of \(A\) and some \(\epsilon > 0\), for any \(n > \omega + \epsilon\),

\[
\|A^{1-\alpha}R(n, A)\|_{L(X)} \leq \int_0^\infty e^{-nt}\|S(t)A^{1-\alpha}\|_{L(X)} \, dt
\]

\[
\leq \int_0^\infty e^{-nt}e^{(-\omega+\epsilon)t}t^{\alpha-1} \, dt
\]

\[
\leq \Gamma(\alpha) \frac{1}{(n - \omega + \epsilon)^\alpha}.
\]

We now recall that in [3] it is proved that \(\|R(\lambda, A_n) - R(\lambda, A)\| \leq \frac{(1+M_1)^2}{n}\). Define \(S(t)\) by the Dunford-Taylor integral

\[
S(\sigma) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda\sigma}R(\lambda, A) \, d\lambda.
\]
Then we get
\[ \| (S_n(\sigma) - S(\sigma)) \|_{L(\mathcal{X})} \leq \left\| \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda \sigma} [R(\lambda, A) - R(\lambda, A_n)] d\lambda \right\|_{L(\mathcal{X})} \]

where \( \Gamma \) is a piecewise smooth curve consisting of three pieces: a segment \( \Gamma_1 = \{ re^{-i(\theta - \epsilon)} : 1 \leq r \leq \infty \} \), a segment \( \Gamma_2 = \{ re^{i(\theta - \epsilon)} : 1 \leq r \leq \infty \} \) and the arc \( \Gamma_3 = \{ e^{i\beta} : - (\theta - \epsilon) \leq \beta \leq (\theta - \epsilon) \} \). Take the norm inside the integral; it follows
\[ \| (S_n(\sigma) - S(\sigma)) \|_{L(\mathcal{X})} \leq \frac{1}{2\pi} \int_{\Gamma} |e^{\lambda \sigma}| \| [R(\lambda, A) - R(\lambda, A_n)] \|_{L(\mathcal{X})} d\lambda \]
\[ \leq \frac{(1 + M_1)^2}{2n\pi} \int_{\Gamma} |e^{\lambda \sigma}| d\lambda = \frac{(1 + M_1)^2}{2n\pi} \int_{\Gamma_2} |e^{\lambda \sigma}| d\lambda \leq \frac{(1 + M_1)^2}{n}. \]

\( \square \)

**Proof of Theorem 3.10** It is possible to take the initial condition \( \varphi_0 = 0 \), possibly considering the process \( \tilde{\varphi}(t) = \varphi(t) - S(t)\varphi_0 \). Then we shall prove that \( \varphi(t) = W_{\mathcal{X}}(t) \) is a strong solution of (3.3) with zero initial condition.

We notice that, by Corollary 3.9, the identity
\[ W_{\mathcal{X}}(t) = \int_0^t S_n(t-s)C \sigma dZ(s) = \int_0^t A_n W_{\mathcal{X}}(s) ds + C \sigma Z^n(t). \]
holds. We shall now send \( n \) to infinity in order to get (3.4).

As a first step we prove that
\[ \lim_{n \to \infty} \sup_{t \in [0,T]} E|W_{\mathcal{X}}(t) - W_{\mathcal{X}}(t)|^2 = 0. \]

Notice that
\[ W_{\mathcal{X}}(t) - W_{\mathcal{X}}(t) = \int_0^t [S_n(t-s) - S(t-s)C \sigma dZ(s) \]
and setting \( q_k := C \sigma_k \) (where \( \{e_k\} \) denotes the canonical basis of \( \partial X_a \)) one concludes that
\[ E|W_{\mathcal{X}}(t) - W_{\mathcal{X}}(t)|^2 = E \left| \sum_{k=1}^{n_0} \int_0^t [S_n(t-s) - S(t-s)]q_k dZ_k(s) \right|^2 X \]
\[ = \sum_{k=1}^{n_0} \| [S_n(\cdot) - S(\cdot)]q_k \|_{X}^2 X. \]

Hence, since it holds that \( \psi(s) := [S_n(s) - S(s)]e_k \) converges to 0 as \( n \to \infty \), uniformly in \( s \in [0,T] \) and
\[ \| \psi(\cdot) I_{(0,t)}(\cdot) \|_{\mathcal{Q}}^2 = \int_0^t \int_0^t |\psi(r)||\psi(s)||s - r|^{2H-2} dr ds \leq \frac{T^{2H}}{2H(2H-1)} \sup_{s \in [0,T]} |\psi(s)|^2, \]
the claim (3.14) holds.

Next we show that $W_A(t) \in D(A)$, $\mathbb{P}$-a.s., for all $t \in [0,T]$. In the case of a $Q$-Wiener process $W(t)$, the thesis follows, for instance, from an application of [DPZ92: Proposition 4.15]. In our case, the assumptions can be modified: in fact, we require

$$\sum_{k=1}^{n_0} \|A(\sigma)c_\alpha e_k|_{\Re (\sigma)}\|^2_{[\gamma]} < +\infty.$$  (3.15)

We can then mimic the proof of the aforementioned result by Da Prato–Zabczyk and obtain the claimed result. Just observe that $c_\alpha e_k$ belongs to $X_\alpha$ for any $\alpha < \frac{1}{4}$, hence $|A(\sigma)c_\alpha e_k|_{X} \leq \text{Const.} t^{\alpha-1}, \quad t \in [0,T], \alpha < 1/4.$

A direct computation shows that

$$\int_0^T \int_0^T r^{\alpha-1}s^{\alpha-1}|r-s|^{2H-2}dr\,ds = \text{Const.} T^{2\alpha+2H-2}$$  (3.16)

provided $\alpha + H - 1 > 0$; notice that the existence of an $\alpha$ which satisfies the bounds $1-H < \alpha < 1/4$ only holds if $H > 3/4$. 

Now, our last claim is to prove the following convergence

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E} |A_nW_A(t) - AW_A(t)|^2_{X} = 0.$$  (3.17)

Together with (3.14), this will imply that we can pass to the limit in (3.13) and prove (3.5), hence the claim follows.

We split the estimate in two parts, since

$$\mathbb{E} |A_nW_A(t) - AW_A(t)|^2_{X} \leq 2(\Phi_{n,1}(t) + \Phi_{n,2}(t))$$  (3.18)

with

$$\Phi_{n,1}(t) = \mathbb{E} |A_nW_A(t) - A_nW_A(t)|^2_{X}$$

$$\Phi_{n,2}(t) = \mathbb{E} |A_nW_A(t) - AW_A(t)|^2_{X}.$$

Notice that

$$\mathbb{E} |A_nW_A(t) - A_nW_A(t)|^2_{X} \leq C, \|A_n (S_n(t-s) - S(t-s)) c_\alpha|_{L(X)}\|^2_{[\gamma]}$$

and this term was treated in Lemma (3.11). We can follow the same steps to get

$$\Phi_{n,2}(t) = \mathbb{E} |A_nW_A(t) - AW_A(t)|^2_{X} = \sum_{k=1}^{n_0} \|A - A_n|S(t-s)c_\alpha e_k|_{X}\|^2_{[\gamma]}$$
Now since \([A - A_n]x \to 0\) as \(n \to \infty\) for any \(x \in D(A)\), it is sufficient to show that each term is uniformly bounded by an integrable function and apply dominated convergence theorem in order to obtain the desired convergence. Further, recall that \(A_n\) is a bounded operator; hence, proceeding as above, we get the bound (compare with (3.16))

\[
\left\| [A - A_n]S(s)C_a e_k |x1_{(a,t)}(s) \right\|_{\mathcal{H}}^2 < +\infty.
\]

In conclusion we obtain that both \(\Phi n,1(s)\) and \(\Phi n,2(s)\) converge to 0 as \(n \to \infty\) uniformly in \(s \in [0,T]\) and the last claim is proved.

3.4. Long time behaviour. In this subsection we will be concerned with the asymptotic behavior of \(W_A(t)\) for \(t \to \infty\). Let us recall from [11, Theorem 11.7] the equivalence between the existence of an invariant measure and the boundedness of the covariance operator \(Q_t\) for \(W_A(t)\) at all times \(t \geq 0\). In the truly dissipative case we obtain the following.

**Proposition 3.12.** Assume that \(p \geq p_0 > 0\) and/or \(B\) is positive definite. Then \(W_A(t)\) converges as \(t \to \infty\) to a centered Gaussian random variable \(\mu\). Furthermore, \(\mu\) is an invariant measure for equation (3.3).

**Proof.** The assertion follows if we prove that

\[
\sup_{t>0} Tr Q_t < \infty,
\]

where \(Q_t\) is the covariance operator of \(W_A(t)\). We have discussed this quantity in previous section and we know that

\[
\mathbb{E}|W_A(t)|_X^2 = \sum_{k=1}^{n_0} \left\| |S(\sigma)C_a e_k |X1_{(a,t)}(\sigma) \right\|_\Lambda^2.
\]

Now since in our assumptions, by Proposition 3.3(3) the semigroup is uniformly exponentially stable, there exist \(M \geq 1\) and \(\omega > 0\) such that

\[
Tr Q_t \leq M \|C_a\|_{HS} \int_0^\infty e^{-2\omega \sigma} d\sigma \leq \text{Const.}
\]

In case \(Z_t\) is a fBm, we use in the space \(\Lambda\) the norm \(|\mathcal{H}|\) and we get (compare (3.7)), for \(t\) large,

\[
\mathbb{E}|W_A(t)|_X^2 \leq M \sum_{k=1}^{n_0} \left\| |S(\sigma)C_a e_k |X1_{(a,t)}(\sigma) \right\|_\Lambda^2 \leq \text{Const.} \|C_a\|_{HS} \left[ \frac{1}{\omega^{2H}} + \frac{t^{2H-2}}{|\omega|^2} \right].
\]

This concludes the proof. □
We study now the uniqueness of the invariant measure and we make precise the weak convergence of the associated transition probabilities. Notice that we can appeal again to the theoretical results in [11, Ch. 11]; hence, our results are based on the properties of the semigroup $S(t)$ obtained before.

**Proposition 3.13.** Assume that $p \geq 0$ and $B$ is positive semidefinite; then the following assertions hold.

1. If additionally $p = 0$ and $B^* \mathbf{1} \neq 0$, or else if $B = 0$ and $p \neq 0$, then there exists at most one invariant measure $\mu$ for (3.3).
2. If additionally $p \geq p_0 > 0$ and/or $B$ is positive definite, then there exists one invariant measure $\mu$ for (3.3). Further, for all initial condition $\mathbf{v}_0$ the solution $\mathbf{v}(t)$ converges in law and in $L^2(\Omega; \mathbb{X}^2)$ to $\mu$.

In different terms, the situation that we obtain is that there exists an equilibrium state (which can be thought of as resulting in the long term behavior) for the neuronal network. It may be interesting, in this connection, to study large deviations to estimate the probability of onsets of chaotic impulses, compare [35, 5].

The situation is different in case there is no dissipation acting in the network. Then by Corollary 3.4 the semigroup converges toward a projection and for some $t_0$ there holds

$$\|S(t)C_a e_k\| \geq \frac{1}{2} |1| \sum_k \langle 1, C_a e_k \rangle \quad t > t_0$$

which is bounded below by the sum of the entries of $k$-th column of the matrix $C_a$. Since at least one entry of $C_a$ is strictly positive, as $t$ goes to infinity it follows

$$\sum_{k=1}^{n_0} \|S(\sigma)C_a e_k |1_{(0,t)}(\sigma)\| \geq \text{Const.} \|1_{(t_0,t)}(\sigma)\| \geq \text{Const.} (t - t_0)^{2H}.$$  

Therefore, the trace of the covariance operator $Q_t$ becomes unbounded as $t \to \infty$ independently of the kind of noise that we consider; this (negative) result is interesting enough to be recorded in a proposition.

**Corollary 3.14.** In case no dissipation occurs in the network ($p \equiv 0$ and $B \equiv 0$) then there does not exist an invariant probability measure for the system.

4. **General stochastic perturbation**

In this section we briefly discuss a possible approach to inhomogeneous boundary value problem. This method has been proposed in [19] and has been extended to stochastic differential equations in [6].
Let $\partial X_p^2 := \mathbb{C}^{n-n_0}$ and define a boundary operator $R : D(A) \to \partial X_p^2$ by

$$Rv = R(V, q_V^T) := K_p V - B_p q_V.$$

Consider the space

$$D(\mathcal{A}) := \{ v := (v_{\phi}) \in D(A) \times \partial X_p^2 \ s.t. \ \exists q^V \in \mathbb{C}^n \ with \ (J^+)^\top q^V = V(0), \ (J^-)^\top q^V = V(1), \ (q_{n_0}^V, \ldots, q_{n_0}^V) = (q_{a_1}, \ldots, q_{a_n}), \ and \ Rv = \phi^V \}.$$

and the operator on $X^2 \times \partial X_a^2 \times \partial X_p^2$ defined by

$$\mathcal{A}v := \begin{pmatrix} Av \\
\phi^V \end{pmatrix}.$$

Then, it is known (cf. [19, § 3]) that $\mathcal{A}$ generates an analytic semigroup because $A$ does so. Such a semigroup is never stable, since it acts as the identity on the second component of the vectors in $(X^2 \times \partial X_a^2) \times \partial X_p^2$. In fact, even if $(S(t))_{t \geq 0}$ is uniformly exponentially stable, the semigroup generated by $\mathcal{A}$ is given by

$$S(t) := \begin{pmatrix} S(t) \\
I \end{pmatrix},$$

where $D_0^{A,R}$ is a so-called abstract Dirichlet operator, cf. [19, Lemma 3.3].

We can now examine the case of a system featuring the presence of (stochastic) inputs in the passive nodes. The system is described by the operator matrix $\mathcal{A}$ on the space $(X^2 \times \partial X_a^2) \times \partial X_p^2$. Let $Z(t)$ be the $n$-dimensional stochastic process which models the input in the (active and passive) nodes, and $\mathbb{C} = (0 \ C_a \ C_p)^T$ be the covariance operator of $Z(t)$, where $C_a$ and $C_p$ are the matrices defined in Section 2. Then the stochastic model can be written in the form

$$du(t) = \mathcal{A}u(t) \ dt + \mathbb{C} dZ(t)$$

$$u(0) = u_0 \tag{4.1}$$

which is solved in mild form by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\mathbb{C} dZ(s). \tag{4.2}$$

Proceeding as in Section 3.1 we obtain the following result.

**Theorem 4.1.** Assume that $u_0 \in (X^2 \times \partial X_a^2) \times \partial X_p^2$. Then the process $u = \{u(t), t \in [0, T]\}$ is a weak solution of (4.1).

Assume further that $u_0 \in D(\mathcal{A})$ and $H > 3/4$. Then the weak solution is a strong solution of (4.1).
We can be more precise in formula (4.2) since we know the representation of the semigroup \( S(t) \):

\[
S(t) := \begin{pmatrix} S(t) & (I - S(t))D_0^{A,R} \\ 0 & I \end{pmatrix},
\]

hence

\[
u(t) = \begin{pmatrix} u_a(t) \\ C_p Z(t) \end{pmatrix},
\]

where

\[
u_a(t) = \int_0^t S(t - s) C_a \, dZ(s) + \int_0^t (I - S(t - s)) D_0^{A,R} C_p \, dZ(s).
\]

In this case, whenever the matrix \( C_p \) is not identically zero, we are again in presence of a gaussian process with finite trace class covariance operator at any time but not bounded as \( t \to \infty \) and, similarly to Corollary 3.14 we obtain the following result.

**Corollary 4.2.** If the behaviour of passive nodes is affected by some stochastic inputs (i.e., the matrix \( C_p \) is not identically zero), then there does not exist an invariant probability measure for the system.

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