Stochastic resonance in Gaussian quantum channels

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We determine conditions for the presence of stochastic resonance in a lossy bosonic channel with a nonlinear, threshold decoding. The stochastic resonance noise benefit occurs if and only if the detection threshold is outside of a “forbidden interval.” We show how noise benefits can occur in different settings: when transmitting classical messages through a lossy bosonic channel, when transmitting over an entanglement-assisted lossy bosonic channel, and when discriminating channels with different loss parameters. Moreover, we consider a setting in which noise can benefit the faithful transmission of a qubit over a lossy bosonic channel with a particular encoding and decoding. In the latter case, we measure noise benefits in terms of the average channel fidelity and the entanglement preserved between a reference system and the channel output. In all cases, we assume Gaussian noise, allowing us to improve upon the forbidden-interval conditions found in earlier work.

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I. INTRODUCTION

It is common wisdom to consider noise as a nuisance. This is especially the case in quantum information science [23, 31], where noise is often an obstacle to the robust implementation of quantum information processing protocols. However, the idea that noise can sometimes play a constructive role has recently started to penetrate this field. The so-called stochastic resonance (SR) effect is the most common example of this kind [9, 12, 14, 18, 21], and it can be present in biological systems [7, 19, 26, 29]. SR occurs in a nonlinear system when noise benefits the system. It can take place in both classical and quantum systems that use noise to help detect faint signals. In a quantum communication setting, this possibility has been put forward in Refs. [3, 4] and more recently in Refs. [10, 30, 32]. It is important to determine general criteria for the occurrence of SR in quantum information protocols.

Continuous variable systems are usually confined to Gaussian states and processes, and SR effects are not expected in any linear processing of such systems. However, information is often available in digital (discrete) form, and therefore it must be subject to “dequantization” at the input of a continuous Gaussian channel and “quantization” at the output [15, 24]. These processes are usually involved in the conversion of digital to analog signals and vice versa. Since these mappings are few-to-many and many-to-few, they are inherently non-linear, and we can thus expect the occurrence of the SR effect in this case. The simplest model representing such a situation is one in which two bit values are encoded into the positive and negative parts of a real continuous alphabet and subsequently decoded by a threshold detection [19]. In the quantum framework, the continuous alphabet is realized by coherent-state amplitudes [6, 13].

Hence, we shall consider a bit encoded into squeezed-coherent states with different amplitudes that are subsequently sent through a Gaussian quantum channel (specifically, a lossy bosonic channel [11]). At the output, the states are subjected to threshold measurement of their amplitude. In addition to such a setting, we shall consider one involving entanglement shared by a sender and receiver as well as one involving quantum channel discrimination. Finally, we also consider a quantum communication setting. For all of these settings, we determine conditions for the occurrence of the SR effect. These appear as forbidden intervals for the threshold detection values, in analogy with the results of Refs. [19, 32] that are here extended to other schemes.

II. NOISE BENEFITS IN CLASSICAL COMMUNICATION

Let us consider a lossy bosonic quantum channel with transmissivity \( \eta \in (0, 1) \) [11, 16]. Our aim is to evaluate the probability of successful decoding, considered as a performance quantifier, when sending classical information through such a channel.

Let us suppose that the sender uses as input a displaced and squeezed vacuum. Working in the Heisenberg picture, the input variable of the communication setup is expressed by the operator:

\[
\hat{q} e^{-r} - \alpha_{q}(-1)^{X},
\]

encoding a bit value \( X \in \{0, 1\} \), where \( \hat{q} \) is the position quadrature operator, \( \alpha_{q} \in \mathbb{R}_{+} \) is the displacement amplitude, and \( r \geq 0 \) is the squeezing parameter [13].

The output observable resulting from sending the above state through a lossy bosonic channel with transmissivity \( \eta \) is as follows [10]:

\[
\sqrt{\eta} (\alpha_{q}(-1)^{X} + \hat{q} e^{-r}) + \sqrt{1-\eta} \hat{q},
\]

where \( \hat{q} \) is the position quadrature operator, and \( \alpha_{q} \) is the displacement amplitude.
where \( \hat{q}_E \) is the position quadrature operator of an environment mode (assumed to be in the vacuum state for the sake of simplicity).

At the receiver’s end, let us consider the possibility of adding a random, Gaussian-distributed displacement \( \nu_q \in \mathbb{R} \) to the arriving state. Then, the output observable becomes as follows:

\[
\sqrt{\eta} (-\alpha_q (-1)^X + \hat{q}e^{-r}) + \sqrt{1-\eta} \hat{q}_E + \nu_q.
\]

Upon measurement of the position quadrature operator, the following signal value \( S_x \) results

\[
S_x = \sqrt{\eta} (-\alpha_q (-1)^X + q e^{-r}) + \sqrt{1-\eta} \hat{q}_E + \nu_q. \tag{2}
\]

Following Ref. [22], we define a random variable summing up all noise terms:

\[
N \equiv \sqrt{\eta} q e^{-r} + \sqrt{1-\eta} \hat{q}_E + \nu_q. \tag{3}
\]

Let \( \mathcal{N}(\mu, \sigma^2) \) denote the density for a Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \):

\[
\mathcal{N}(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}.
\]

The density \( P_N(n) \) of random variable \( N \) is then as follows:

\[
P_N(n) = \mathcal{N}(0, \eta e^{-2r}/2) \ast \mathcal{N}(0, \sigma^2/2) \ast \mathcal{N}(0, (1-\eta)/2), \tag{4}
\]

where \( \ast \) denotes convolution. The density \( P_N(n) \) is a convolution because random variables \( q e^{-r}, \nu_q \), and \( q_E \) are independent.

Notice that the noise term \( [3] \) does not depend on the encoded value \( X \) and neither does its probability density. From \([4]\) we explicitly get

\[
P_N(n) = \mathcal{N}(0, (1-\eta + \sigma^2 + \eta e^{-2r})/2). \tag{5}
\]

The output signal \( [2] \) can now be written as

\[
S_x = -\sqrt{\eta} \alpha_q (-1)^X + N.
\]

The receiver then thresholds the measurement result with a threshold \( \theta \in \mathbb{R} \) to retrieve a random bit \( Y \) where

\[
Y \equiv H(-\sqrt{\eta} \alpha_q (-1)^X + N - \theta), \tag{6}
\]

and \( H \) is the Heaviside step function defined as \( H(x) = 1 \) if \( x \geq 0 \) and \( H(x) = 0 \) if \( x < 0 \).

To evaluate the probability of successful decoding, we compute the conditional probabilities

\[
P_{Y|X}(0|0) = \int_{-\infty}^{+\infty} [1 - H(-\sqrt{\eta} \alpha_q + n - \theta)] P_N(n) \ dn = 1 - P_{Y|X}(1|0), \tag{7}
\]

\[
P_{Y|X}(1|1) = \int_{-\infty}^{+\infty} H(\sqrt{\eta} \alpha_q + n - \theta) P_N(n) \ dn = 1 - P_{Y|X}(0|1). \tag{8}
\]

Using \([5]\), we find

\[
P_Y|X(0|0) = \frac{1}{2} + \frac{1}{2} \operatorname{erf} \left( \frac{\theta + \sqrt{\eta} \alpha_q}{\sqrt{1-\eta + \sigma^2 + \eta e^{-2r}}} \right),
\]

\[
P_Y|X(1|1) = \frac{1}{2} - \frac{1}{2} \operatorname{erf} \left( \frac{\theta - \sqrt{\eta} \alpha_q}{\sqrt{1-\eta + \sigma^2 + \eta e^{-2r}}} \right),
\]

where \( \operatorname{erf}(z) \) denotes the error function:

\[
\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z \exp \left\{ -x^2 \right\} \ dx.
\]

This situation is identical to the one treated in Ref. [22], and the forbidden interval can be determined in a simple way by looking at the probability of successful decoding (note that others have also considered the probability-of-success (or error) criterion [24, 26]). The probability of success is defined as

\[
P_s \equiv P_X(0)P_{Y|X}(0|0) + P_X(1)P_{Y|X}(1|1). \tag{9}
\]

Setting \( P_X(0) = \phi \) and \( P_X(1) = (1 - \phi) \), the probability of success is as follows:

\[
P_s = \frac{1}{2} + \frac{1}{2} \phi \left( \frac{\theta + \sqrt{\eta} \alpha_q}{\sqrt{1-\eta + \sigma^2 + \eta e^{-2r}}} \right)
\]

\[
- \frac{1}{2} (1 - \phi) \left( \frac{\theta - \sqrt{\eta} \alpha_q}{\sqrt{1-\eta + \sigma^2 + \eta e^{-2r}}} \right). \tag{10}
\]

Our goal is to find the maximum of \([10]\) over \( \alpha_q, r, \theta \) for fixed \( \eta \) and see how it changes as a function of \( \sigma \). This leads us to the following proposition:

**Proposition 1 (The forbidden interval)** The probability of success \( P_s \) shows a non-monotonic behavior versus \( \sigma \) if \( \theta \not\in [\theta_-, \theta_+] \), where \( \theta_\pm \) are the two roots of the following equation:

\[
\frac{4 \sqrt{\eta} \alpha_q \theta}{(1-\phi)(-\sqrt{\eta} \alpha_q + \theta) + \phi (\sqrt{\eta} \alpha_q + \theta)} = \exp \left[ \frac{4 \sqrt{\eta} \alpha_q \theta}{1-\eta + e^{-2r\eta}} \right], \tag{11}
\]

such that \( \theta_- \leq -\alpha_q \sqrt{\eta} < +\alpha_q \sqrt{\eta} \leq \theta_+ \).

**Proof.** We consider \( P_s \) as a function of \( \sigma^2 \). In order to have a non-monotonic behavior for \( P_s(\sigma^2) \), we must check for the presence of a maximum. By imposing

\[
\frac{dP_s(\sigma^2)}{d\sigma^2} = 0 \tag{12}
\]

we obtain Eq. \([11]\). Two necessary conditions for this equation to admit a solution are

\[
\theta > \sqrt{\eta} \alpha_q, \tag{13}
\]

and

\[
\left( \frac{1-\phi}{\phi} \right) \left( \frac{\theta - \sqrt{\eta} \alpha_q}{\theta + \sqrt{\eta} \alpha_q} \right) \leq 1. \tag{14}
\]
Explicitly the critical value of $\sigma^2$, for which (12) is satisfied, reads

$$\sigma^2 = -1 + \eta - e^{-2r\eta} + \frac{4\sqrt{\eta}\alpha_1}{\ln\left[\eta(\sqrt{\eta}\alpha_1 + \theta)/(1 - \eta)(-\sqrt{\eta}\alpha_1 + \theta)\right]}.$$  \hspace{1cm} (15)

An inspection of Eq. (11), shows that $\sigma^2$ is positive iff either $\theta < \theta_-$ or $\theta > \theta_+$, where $\theta_\pm$ are the two roots of the equation $\sigma^2 = 0$. Because of the condition (13), they are such that $\theta_- \leq -\alpha_1\sqrt{\eta} < +\alpha_1\sqrt{\eta} \leq \theta_+$. Conversely, if $\theta < \theta_-$ or $\theta > \theta_+$, it follows that $\sigma^2 > 0$, and hence $P_s(\sigma^2)$ is non-monotonic.

The above proposition improves upon the theorem from Ref. [32] in several important ways, due to the assumption that the noise is Gaussian, allowing us to analyze it more carefully. First, (15) gives the optimal value of the noise that leads to the maximum success probability if the threshold is outside of the forbidden interval (though, note that other works have algorithms to learn the optimal noise parameter [22]). Second, there is no need to consider an infinite-squeezing limit, as was the case in Ref. [32], in order to guarantee the non-monotonic signature of SR. Finally, the theorem demonstrates that SR can occur for both subthreshold and suprathreshold signals, whereas the theorems from Ref. [32] apply only to subthreshold signals.

As an example, Figure 1 shows the probability of success $P_s$ as a function of $\sigma$ for various values of the threshold $\theta$ around the high signal level $\alpha_q\sqrt{\eta}$. Identical behavior can be found for values of $\theta$ around the low signal level $-\alpha_q\sqrt{\eta}$.

Figure 2 plots the forbidden interval in the $\theta, r$ plane. We can see that increasing the squeezing level reduces the width of the forbidden interval up to $2\sqrt{\eta}\alpha_q$. Similarly, Figure 3 shows the forbidden interval in the $\theta, \alpha_q$ plane. Increasing the amplitude enlarges the width of the forbidden interval up to $2\sqrt{\eta}\alpha_q$.

### III. NOISE BENEFITS IN ENTANGLEMENT-ASSISTED CLASSICAL COMMUNICATION

Let us consider the same channel as in the previous section, but we now assume that the sender and receiver share an entangled state, namely, a two-mode squeezed vacuum [13], before communication begins. This situation is somewhat similar to the communication scenario in super-dense coding [2, 6], with the exception that we have continuous variable systems and thresholding at the receiver. Let mode 1 (resp. 2) belong to the sender (resp. receiver). The sender displaces her share of the entanglement by the complex number $-\alpha_q(-1)^{X_q} - i\alpha_p(-1)^{X_p}$ in order to transmit the two bits $X_q$ and $X_p$. The resulting displaced squeezed vacuum operators are as follows:

$$\begin{align*}
(\hat{q}_1 - \hat{q}_2)e^{-r} - \alpha_q(-1)^{X_q}, \quad (16) \\
(\hat{p}_1 + \hat{p}_2)e^{-r} - \alpha_p(-1)^{X_p}, \quad (17)
\end{align*}$$

where $\hat{q}, \hat{p}$ are the position, momentum quadrature operators, $r \geq 0$ is the squeezing strength, $\alpha_q, \alpha_p \in \mathbb{R}$ are
The receiver then thresholds the measurement result with a threshold \( \theta_q \in \mathbb{R}_+ \), and he retrieves a random bit \( Y_q \) where

\[
Y_q \equiv H \left( -\sqrt{\eta} \alpha_q (1 - X_q) + N - \theta_q \right),
\]

and \( H \) is the Heaviside step function.

Proceeding as in Section II we obtain the following input/output conditional probabilities

\[
P_{Y_q|X_q}(0|0) = \frac{1}{2} + \frac{1}{2} \text{erf} \left[ \frac{\theta_q + \sqrt{\eta} \alpha_q}{\sqrt{1 - \eta + \sigma_q^2 + (1 + \eta)e^{-2r}}} \right],
\]

\[
P_{Y_q|X_q}(1|1) = \frac{1}{2} - \frac{1}{2} \text{erf} \left[ \frac{\theta_q - \sqrt{\eta} \alpha_q}{\sqrt{1 - \eta + \sigma_q^2 + (1 + \eta)e^{-2r}}} \right].
\]

Then, writing \( P_{X_q}(0) = \varphi_q \) and \( P_{X_q}(1) = 1 - \varphi_q \), the probability of success reads

\[
P_{s,q} = \frac{1}{2} + \frac{1}{2} \varphi_q \text{erf} \left[ \frac{\theta_q + \sqrt{\eta} \alpha_q}{\sqrt{1 - \eta + \sigma_q^2 + (1 + \eta)e^{-2r}}} \right] - \frac{1}{2} (1 - \varphi_q) \text{erf} \left[ \frac{\theta_q - \sqrt{\eta} \alpha_q}{\sqrt{1 - \eta + \sigma_q^2 + (1 + \eta)e^{-2r}}} \right].
\]

Analogously, for the quadrature \( \hat{p}_1 + \hat{p}_2 \), we have

\[
P_{s,p} = \frac{1}{2} + \frac{1}{2} \varphi_p \text{erf} \left[ \frac{\theta_p + \sqrt{\eta} \alpha_p}{\sqrt{1 - \eta + \sigma_p^2 + (1 + \eta)e^{-2r}}} \right] - \frac{1}{2} (1 - \varphi_p) \text{erf} \left[ \frac{\theta_p - \sqrt{\eta} \alpha_p}{\sqrt{1 - \eta + \sigma_p^2 + (1 + \eta)e^{-2r}}} \right].
\]

We have assumed that the noise terms added to the quadratures \( \hat{q}_1 - \hat{q}_2 \) and \( \hat{p}_1 + \hat{p}_2 \) have variances \( \sigma_q^2 \) and \( \sigma_p^2 \) respectively.

We finally arrive at the following proposition:

**Proposition 2 (The forbidden rectangle)** The probability of success \( P_s = P_{s,q} P_{s,p} \) shows a non-monotonic behavior vs \( \sigma_q, \sigma_p \) if \( \theta_q \notin [-\theta_q, \theta_q] \) or \( \theta_p \notin [-\theta_p, \theta_p] \) where \( \theta_{\pm} \) are the roots of the following equation:

\[
\varphi_q \left( \sqrt{\eta} \alpha_q + \theta_q \right) \left( 1 - \varphi_p \right) \left( \sqrt{\eta} \alpha_p + \theta_p \right) = \exp \left[ -\frac{4 \sqrt{\eta} \alpha_q \theta_q}{1 - \eta + (1 + \eta)e^{-2r}} \right],
\]

such that \( \theta_{\pm} \leq -\alpha \sqrt{\eta} \leq +\alpha \sqrt{\eta} \leq \theta_{\pm} \) (here \( \bullet \) stands for either \( q \) or \( p \)).

**Proof.** If we consider the probability of success as function of \( \sigma_q^2 \) and \( \sigma_p^2 \), in order to have a non-monotonic behavior for \( P_s(\sigma_q^2, \sigma_p^2) \) we must check for the presence of a maximum in either the \( \sigma_q^2 \) or \( \sigma_p^2 \) variable. By solving

\[
\frac{\partial P_s(\sigma_q^2, \sigma_p^2)}{\partial \sigma_q^2} = P_{s,q} P_{s,p} = 0,
\]

the displacement amplitudes, and \( X_q, X_p \in \{0, 1\} \) are bit value random variables.

Since \( \hat{q}_1 - \hat{q}_2 \) commutes with \( \hat{p}_1 + \hat{p}_2 \), it suffices to analyze the output for [10]. After the sender transmits her share of the entanglement through a lossy bosonic channel with transmissivity \( \eta \in (0, 1) \), the operator describing their state is as follows:

\[
\left( \sqrt{\eta} \hat{q}_1 - \hat{q}_2 \right) e^{-r} - \sqrt{\eta} \alpha_q (-1)^{X_q} + \sqrt{1 - \eta} \hat{q}_E + \nu_q,
\]

where \( \hat{q}_E \) is the position quadrature operator of the environment mode (assumed to be in the vacuum state for the sake of simplicity).

At the receiver’s end, let us again consider the possibility of adding a random, Gaussian-distributed displacement \( \nu_q \in \mathbb{R} \) to the arriving state. Then the output observable becomes as follows:

\[
\left( \sqrt{\eta} \hat{q}_1 - \hat{q}_2 \right) e^{-r} - \sqrt{\eta} \alpha_q (-1)^{X_q} + \sqrt{1 - \eta} \hat{q}_E + \nu_q.
\]

Repeating the steps of Section [II] we have

\[
S_{X_q} = -\sqrt{\eta} \alpha_q (-1)^{X_q} + N,
\]

where

\[
N \equiv \left( \sqrt{\eta} \hat{q}_1 - \hat{q}_2 \right) e^{-r} + \sqrt{1 - \eta} \hat{q}_E + \nu_q,
\]

and

\[
P_N (n) = \mathcal{N} \left( 0, \eta e^{-2r}/2 \right) \circ \mathcal{N} \left( 0, e^{-2r}/2 \right) \circ \mathcal{N} \left( 0, \sigma_q^2/2 \right) \circ \mathcal{N} \left( (1 - \eta)/2 \right),
\]

so that

\[
P_N (n) = \mathcal{N} \left( 0, (1 - \eta + \sigma_q^2 + (1 + \eta)e^{-2r})/2 \right).
\]
we get (provided $P_{s,p} > 0$)

$$\sigma_{q*}^2 = -1 + \eta - (1 + \eta)e^{-2r} + \frac{4\sqrt{\eta}q\theta_\eta}{\ln\left(1-\nu_\eta(\theta_\eta-\sqrt{\eta}\theta_\eta)\right)}.$$ 

It must be that $\sigma_{q*}^2 > 0$ because this parameter is the variance of the noise. This is realized when either $\theta < \theta_-$ or $\theta > \theta_+$, where $\theta_{\pm}$ are the two roots of the equation $\sigma_{q*}^2 = 0$ such that $\theta_- < -\alpha\sqrt{\eta} < +\alpha\sqrt{\eta} < \theta_+$. 

Conversely, if $\theta < \theta_-$ or $\theta > \theta_+$ it follows that $\sigma_{q*}^2 > 0$, hence $P_s$ is non-monotonic.

Analogous reasonings hold for the maximum in $\sigma_{p*}^2$. 

It is worth remarking in the above proposition that either of the conditions $\theta_\eta \notin [-\theta_{q-}, \theta_{q+}]$ or $\theta_\eta \notin [-\theta_{p-}, \theta_{p+}]$ (or both) have to be satisfied in order to have a non-monotonic behavior for $P_s$. This follows because $P_s$ is a function of two variables $\sigma_{q*}^2$ and $\sigma_{p*}^2$, hence it suffices that the partial derivative with respect to one of them has a maximum to have a non-monotonic behavior. As consequence we have a “forbidden rectangle” rather than “forbidden stripes” in the $\theta_\eta, \theta_\eta$, plane.

It is also worth noticing the difference of the equation in the above proposition with that in the proposition of the previous section. Here the squeezing factor $e^{-2r}$ is multiplied by $(1 + \eta)$ rather than simply $\eta$. This is because we now have two squeezed modes, one of which is attenuated by the lossy channel.

**IV. NOISE BENEFITS IN CHANNEL DISCRIMINATION**

Let us now consider two lossy quantum channels with transmissivities $\eta_0, \eta_1 \in (0,1)$ (suppose wlog $\eta_0 > \eta_1$). Our aim is to distinguish them. Then, suppose to use as probe (input) a squeezed and displaced vacuum operator

$$\hat{q} e^{-r} + \alpha_q,$$ 

where $\hat{q}$ is the position quadrature operator, $r$ is the squeezing parameter, and $\alpha_q \in \mathbb{R}$ the displacement amplitude.

The transmission through the lossy channel with transmissivity $\eta_X$, $X = 0, 1$, can be considered as the encoding of a bit value random variable $X = 0, 1$ occurring with probability $P_X$. The output observable after transmission is then

$$\sqrt{\eta_X} (\alpha_q + \hat{q} e^{-r}) + \sqrt{1 - \eta_X} \hat{q}_E,$$

where $\hat{q}_E$ is the position quadrature operator of the environment mode (assumed to be in the vacuum state for the sake of simplicity).

At the receiver’s end, let us consider the possibility of adding a random, Gaussian-distributed displacement $\nu_q \in \mathbb{R}$ to the arriving state. Then the output observable becomes

$$\sqrt{\eta_X} (\alpha_q + \hat{q} e^{-r}) + \sqrt{1 - \eta_X} \hat{q}_E + \nu_q.$$ 

Upon measurement of the position quadrature operator, the signal value is

$$S_X = \sqrt{\eta_X} (\alpha_q + \hat{q} e^{-r}) + \sqrt{1 - \eta_X} q_E + \nu_q.$$ 

We define a conditional random variable $N|X$ summing all noise terms:

$$N|X = \sqrt{\eta_X} q e^{-r} + \sqrt{1 - \eta_X} q_E + \nu_q.$$ 

The density $P_{N|X}(n|x)$ of random variable $N|X$ is

$$P_{N|X}(n|x) = \mathcal{N}(0, \eta_x e^{-2r}/2) \circ \mathcal{N}(0, \sigma^2/2) \circ \mathcal{N}(0, (1 - \eta_x)/2).$$ 

Notice that the noise term explicitly depends on the encoded value $X$ and so does its probability density. From (28), we explicitly obtain

$$P_{N|X}(n|x) = \mathcal{N}(0, (1 - \eta_x + \sigma^2 + \eta_x e^{-2r})/2).$$ 

The output signal can now be written as

$$S_X = \sqrt{\eta_X} \alpha_q + N|X.$$ 

The receiver then thresholds the measurement result with a threshold $\theta \in \mathbb{R}_+$ to retrieve a random bit $Y$ where

$$Y \equiv H(\theta - \sqrt{\eta_X} \alpha_q - N|X),$$ 

and $H$ is the unit Heaviside step function. In this case, the receiver assigns $Y = 1$ if the threshold $\theta$ is larger than the output signal $S_X$, and he assigns $Y = 1$ if it is less than the signal value.

The final detected bit $Y$ should be the same as the encoded bit $X$. Hence, the probability of success reads like (9) where now

$$P_{Y|X}(0|0) = \int_{-\infty}^{+\infty} [1 - H(\theta - \sqrt{\eta_0} \alpha_q - n)] P_{N|X}(n|0) \ dn,$$

$$P_{Y|X}(1|1) = \int_{-\infty}^{+\infty} H(\theta - \sqrt{\eta_1} \alpha_q - n) P_{N|X}(n|1) \ dn.$$ 

Using (29) we obtain

$$P_{Y|X}(0|0) = \frac{1}{2} \left[ 1 - \text{erf}\left(\frac{\theta - \sqrt{\eta_0} \alpha_q}{\sqrt{1 - \eta_0 + \sigma^2 + \eta_0 e^{-2r}}}\right)\right],$$

$$P_{Y|X}(1|1) = \frac{1}{2} \left[ 1 + \text{erf}\left(\frac{\theta - \sqrt{\eta_1} \alpha_q}{\sqrt{1 - \eta_1 + \sigma^2 + \eta_1 e^{-2r}}}\right)\right].$$ 

Then, writing $P_X(0) = \varphi$ and $P_X(1) = 1 - \varphi$, we get

$$P_s = \frac{1}{2} - \frac{1}{2} \varphi \text{erf}\left(\frac{\theta - \sqrt{\eta_0} \alpha_q}{\sqrt{1 - \eta_0 + \sigma^2 + \eta_0 e^{-2r}}}\right) + \frac{1}{2} (1 - \varphi) \text{erf}\left(\frac{\theta - \sqrt{\eta_1} \alpha_q}{\sqrt{1 - \eta_1 + \sigma^2 + \eta_1 e^{-2r}}}\right).$$ 

(30)

Our goal is to find the maximum of (30) over $\alpha_q, r, \theta$ for fixed $\eta_0, \eta_1$ and see how it changes as a function of $\sigma^2$.

In the simplest case of $r = 0$, we have the following proposition:
Proposition 3 (The forbidden interval) The probability of success $P_s$ shows a non-monotonic behavior as a function of $\sigma^2$ iff $\theta \notin [\theta_-, \theta_+]$, where $\theta_{\pm}$ are the two roots of the following equation:

$$\varphi(\alpha_q \sqrt{\eta_0} - \theta) = \frac{(1 - \varphi)(\alpha_q \sqrt{\eta_1} - \theta)}{\ln \left[ \varphi(\sqrt{\eta_0} \alpha_q - \theta)/\varphi(\sqrt{\eta_1} \alpha_q - \theta) \right]},$$

such that $\theta_- \leq \sqrt{\eta_1} \alpha_q < \sqrt{\eta_0} \alpha_q \leq \theta_+.$

**Proof.** We consider the probability of success as a function of $\sigma^2$. In order to have a non-monotonic behavior for $P_s(\sigma^2)$, we must check for the presence of a maximum. By solving

$$\frac{dP_s(\sigma^2)}{d\sigma^2} = 0,$$

we obtain

$$\sigma^2_{\ast} = -1 + \frac{\alpha_q^2 (\eta_0 - \eta_1) - 2 \alpha_q \theta (\sqrt{\eta_0} - \sqrt{\eta_1})}{\ln \left[ \varphi(\sqrt{\eta_0} \alpha_q - \theta)/\varphi(\sqrt{\eta_1} \alpha_q - \theta) \right]},$$

It must be the case that $\sigma^2_{\ast} > 0$. By analyzing (32), we find that $\sigma^2_{\ast} = 0$ admits two roots $\theta_-$ and $\theta_+$, such that $\theta_- \leq \sqrt{\eta_1} \alpha_q < \sqrt{\eta_0} \alpha_q \leq \theta_+.$$ Conversely, if $\theta_+ < \theta_-$ or $\theta > \theta_+$ it follows that $\sigma^2_{\ast} > 0$. Hence $P_s(\sigma)$ is non-monotonic.

If $r \neq 0$, (31) is not algebraic—hence we did not succeed in providing an analytical expression for $\sigma^2_{\ast}$. However, numerical investigations show a qualitative behavior of the forbidden interval’s boundaries identical to that shown in Figures 2 and 3 (notice that here $-\sqrt{\eta_0} \alpha_q$ and $\sqrt{\eta_0} \alpha_q$ are replaced by $\sqrt{\eta_1} \alpha_q$ and $\sqrt{\eta_0} \alpha_q$, respectively).

V. NOISE BENEFITS IN QUANTUM COMMUNICATION

Let us now consider a setting in which a noise benefit can occur in the transmission of a qubit ($Q$). The aim is to first encode a qubit state into a bosonic mode state, and then send it through the lossy channel, and finally coherently decode, with a threshold mechanism, the output bosonic mode state into a qubit system at the receiving end. We should qualify that it is unclear to us whether one would actually exploit the encodings and decodings given in this section, but regardless, the setting given here provides a novel scenario in which a noise benefit can occur for a quantum system. We might consider the development in this section to be a coherent version of the settings in the previous sections. Also, it is in the spirit of a true “quantum stochastic resonance” effect hinted at in Ref. 8.

We work in the Schrödinger picture, and consider an initial state $|\varphi\rangle_Q \otimes |0\rangle_B$, where $|\varphi\rangle_Q = a|0\rangle_Q + b|1\rangle_Q$ is an arbitrary qubit state and $|0\rangle_B$ is the zero-eigenstate of the position-quadrature operator of the bosonic field. Here for the sake of simplicity we are going to work with infinite-energy position eigenstates rather than with squeezed-coherent states.

Suppose that the encoding take place through the following unitary controlled-operations:

$$U_{1B}^{QB} = |0\rangle_Q \langle 0| \otimes e^{-i\hat{p}_B x_0} + |1\rangle_Q \langle 1| \otimes e^{i\hat{p}_B x_0},$$

$$U_{2B}^{QB} = I_Q \otimes \int_{x \geq 0} |x\rangle_B \langle x| dx + X_Q \otimes \int_{x < 0} |x\rangle_B \langle x| dx,$$

with $\hat{p}_B$ the canonical momentum operator of the bosonic system, $| \pm x_0 \rangle$ the generalized eigenstate of the canonical position operator $\hat{p}_B$, and WLOG we assume $x_0 \in \mathbb{R}_+$. This encoding is the coherent version of encoding a binary number into an analog signal.

The effect of such operations on the initial states is

$$U_2^{QB} U_1^{QB} |\varphi\rangle_Q \otimes |0\rangle_B = |0\rangle_Q \otimes (a|x_0\rangle_B + b|-x_0\rangle_B).$$

Now, with the bosonic mode state factored out from the qubit state, it can be sent through the lossy channel. For the sake of analytical investigation, we consider a channel with unit transmittivity (an identity channel). Then, the output state simply reads

$$\rho_B = (a|x_0\rangle_B + b|-x_0\rangle) \rho_{\pi} (a|x_0\rangle_B + b|-x_0\rangle),$$

At this point we consider the possibility of adding Gaussian noise before the (threshold) decoding stage. This is modeled as a Gaussian-modulated displacement of the quadrature $\hat{q}_B$. The resulting state will be

$$\rho_{B'} = \int dq \ G_{\sigma^2}(q) \ D(q, 0) \rho_B D^\dagger(q, 0),$$

where $D(q, 0)$ is the displacement operator (displacing only in the $\hat{q}_B$ direction), and $G_{\sigma^2}(q)$ is a zero-mean, normalized Gaussian random variable with variance $\sigma^2$.

Now, the state $\rho_{B'}$ is decoded into a qubit system $Q'$ initially prepared in the state $|0\rangle_{Q'}$ through the following controlled-unitary operations involving a coherent threshold mechanism

$$V_{1B'}^{Q'} = I_Q \otimes \int_{x \geq \theta} |x\rangle_B \langle x| dx + X_Q \otimes \int_{x < \theta} |x\rangle_B \langle x| dx,$$

$$V_{2B'}^{Q'} = |0\rangle_Q \langle 0| \otimes e^{i\hat{p}_B x_0} + |1\rangle_Q \langle 1| \otimes e^{-i\hat{p}_B x_0}.$$

Clearly, if $\theta = 0$ the decoding unitaries (36), (37) are the inverse of the encoding ones (33), (34). They hence allow unit fidelity encoding/decoding if $\sigma^2 = 0$. However, if $\theta \neq 0$, there could be a nonzero optimal value of $\sigma^2$. 
The final qubit state is

\[ \sigma' = E(\varphi)_Q(\varphi) \]

\[ = \text{Tr}_{B'} \left\{ V_2^{Q'B'} V_1^{Q'B'} (|0\rangle \langle 0|_Q \otimes \rho_{B'}) \right\} \]

\[ = \left[ |a|^2 (1 - \Pi_<) + |b|^2 \Pi_> \right] |0\rangle_Q \langle 0| + \left[ |a|^2 \Pi_< + |b|^2 (1 - \Pi_> \right] |1\rangle_Q \langle 1| + (1 - \Pi_< - \Pi_>) (a\bar{b}|0\rangle_Q (1| + \bar{a}b|1\rangle_Q (0|) \]

where

\[ \Pi_< = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\theta - x_0}{\sqrt{2\sigma^2}} \right), \]

\[ \Pi_> = \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{\theta + x_0}{\sqrt{2\sigma^2}} \right). \]

Then we can calculate the average channel fidelity

\[ \langle F \rangle = \int d\varphi \langle \varphi | \sigma' | \varphi \rangle_Q, \]

where \( d\varphi \) is the uniform measure induced by the Haar measure on \( SU(2) \).

Using (35), (36), (37), (38) and (43) we finally obtain

\[ \langle F \rangle = \frac{1}{2} + \frac{1}{4} \left[ \text{erf} \left( \frac{\theta + x_0}{\sqrt{2\sigma^2}} \right) - \text{erf} \left( \frac{\theta - x_0}{\sqrt{2\sigma^2}} \right) \right]. \]

Then, we have the following proposition:

**Proposition 4 (The forbidden interval)** The average channel fidelity \( \langle F \rangle \) shows a non-monotonic behavior as a function of \( \sigma \) iff \( \theta \notin [-x_0, +x_0] \).

**Proof.** We consider \( \langle F \rangle \) as a function of \( \sigma \). We assume without loss of generality that \( \theta > 0 \). In order to have a non-monotonic behavior, we must check for the presence of a local maximum. The derivative \( \frac{d\langle F \rangle}{d\sigma^2} \) is as follows:

\[ \frac{d\langle F \rangle}{d\sigma^2} = e^{-\frac{\theta^2 - x_0^2}{2\sigma^2}} \left[ \left( \theta + x_0 \right) e^{-\frac{\theta^2}{\sigma^2}} - \left( \theta - x_0 \right) e^{\frac{x_0^2}{\sigma^2}} \right]. \]

Then, the condition \( \frac{d\langle F \rangle}{d\sigma^2} = 0 \) implies that

\[ \frac{\theta - x_0}{\theta + x_0} = e^{-\frac{2\theta x_0}{\sigma^2}}, \]

which has the following unique solution:

\[ \sigma_* = \sqrt{ \frac{2\theta x_0}{\ln \left( \frac{\theta + x_0}{\theta - x_0} \right)} } \]

iff \( \theta - x_0 > 0 \). Finally, taking into account also the possibility that \( \theta, x_0 < 0 \), we conclude that a critical value of \( \sigma \) exists iff \( \theta \notin [-x_0, +x_0] \).

**VI. CONCLUSION**

In conclusion, we have determined necessary and sufficient conditions for noise benefits when transmitting classical and quantum information over a lossy bosonic channel or discriminating lossy channels (all of which have nonlinear coding or decoding processes). Specifically, we have considered a bit encoded into coherent states with different amplitudes that are subsequently sent through a lossy bosonic channel and decoded at the output by threshold measurement of their amplitudes (without and with the assistance of entanglement shared by sender and receiver). We have considered the discrimination of lossy bosonic channels having different loss parameters. In all
SR effects appear whenever the threshold lies outside of the different forbidden intervals that we have established in our four propositions. If it lies inside of a forbidden interval, then the SR effect does not occur. Thus the SR effect should be useful in situations where the signals are so faint that they are indistinguishable within a finite measurement resolution. For instance, this is the case in homodyne detection, if the square of the average signal times the overall detection efficiency (which accounts for the detector’s efficiency, the fraction of the field being measured, etc.) is below the vacuum noise strength [33].

Finally, we have also considered the transmission of quantum information, represented by a qubit which is encoded into the state of a bosonic mode and then decoded according to a threshold mechanism. The found SR effects, in terms of the average channel fidelity and of the output entanglement (quantified by the logarithmic negativity), represent a clear signature of noise benefits in quantum communication.

**REFERENCES**

[1] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. Physical Review A, 54(5):3824–3851, November 1996.

[2] Charles H. Bennett and Stephen J. Wiesner. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. Physical Review Letters, 69(20):2881–2884, November 1992.

[3] Garry Bowen and Stefano Mancini. Noise enhancing the classical information capacity of a quantum channel. Physics Letters A, 321(1):1–5, 2004. arXiv:quant-ph/0311126.

[4] Garry Bowen and Stefano Mancini. Stochastic resonance effects in quantum channels. Physics Letters A, 352(4-5):272–275, April 2006. arXiv:quant-ph/0512099.

[5] Samuel L. Braunstein and H. Jeff Kimble. Dense coding for continuous variables. Physical Review A, 61(4):042302, March 2000.

[6] Samuel L. Braunstein and Peter van Loock. Quantum information with continuous variables. Reviews of Modern Physics, 77:513–577, 2005.

[7] A. R. Bulsara, E. W. Jacobs, T. Zhou, F. Moss, and L. Kiss. Stochastic resonance in a single neuron model: Theory and analog simulation. Journal of Theoretical Biology, 152(4):531–555, October 1991.

[8] Adi R. Bulsara. No-nuisance noise. Nature, 437(13):962–963, October 2005.

[9] Adi R. Bulsara and Luca Gammaitoni. Tuning in to noise. Physics Today, 49:39–45, 1996.

[10] Filippo Caruso, Susana F. Huelga, and Martin B. Plenio. Noise-enhanced classical and quantum capacities in communication networks. Physical Review Letters, 105(19):190501, November 2010. arXiv:1003.5877.

[11] Jens Eisert and Michael M. Wolf. Quantum Information with Continuous Variables of Atoms and Light, chapter Gaussian quantum channels, pages 23–42. Imperial College Press, London, 2007. arXiv:quant-ph/0505151.

[12] Luca Gammaitoni, Peter Hänggi, Peter Jung, and Fabio Marchesoni. Stochastic resonance. Reviews of Modern Physics, 70(1):223–287, 1998.

[13] Christopher C. Gerry and Peter L. Knight. Introductory Quantum Optics. Cambridge University Press, 2005.

[14] Igor Goychuk and Peter Hänggi. Quantum stochastic resonance in parallel. New Journal of Physics, 1(14), 1999.

[15] Robert M. Gray and D. L. Neuhoff. Quantization. IEEE Transactions on Information Theory, 44(6):2325–2383, October 1998.

[16] Alexander S. Holevo and Reinhard F. Werner. Evaluating capacities of bosonic Gaussian channels. Physical Review A, 63(3):032312, February 2001. arXiv:quant-ph/0012067.

[17] A. Jamiolkowski. Linear transformations which preserve trace and positive semidefiniteness of operators. Reports on Mathematical Physics, 3(4):275–278, December 1972.

[18] Bart Kosko. Noise. Viking/Penguin, 2006.

[19] Bart Kosko and Sanya Mitaim. Stochastic resonance in noisy threshold neurons. Neural Networks, 14:755–761, 2003.

[20] Mark D. McDonnell and Derek Abbott. What is stochastic resonance? Definitions, misconceptions, debates, and its relevance to biology. PloS Computational Biology, 5(5):1–9, May 2009.

[21] Mark D. McDonnell, Nigel G. Stocks, Charles E. M. Pearce, and Derek Abbott. Stochastic Resonance: From Suprathreshold Stochastic Resonance to Stochastic Signal Quantization. Cambridge University Press, 2008.

[22] Sanya Mitaim and Bart Kosko. Adaptive stochastic

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**FIG. 5:** The logarithmic negativity vs $\sigma$ for $x_0 = 0.3$ and several values of $\theta$ inside and outside the forbidden interval. From top to bottom, $\theta = 0.20$, $\theta = 0.25$, $\theta = 0.29$ (inside the forbidden interval), $\theta = 0.31$, $\theta = 0.35$, $\theta = 0.40$ (outside the forbidden interval).

In these cases, the performance is evaluated in terms of success probability (the complement of error probability to one). It is worth remarking that the mutual information is a monotonic function of this probability, and so the same conclusions can be drawn in terms of mutual information.
Proceedings of the IEEE, 86(11):2152–2183, November 1998.

[23] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.

[24] Ashok Patel and Bart Kosko. Error-probability noise benefits in threshold neural signal detection. Neural Networks, 22(5-6):697–706, July-August 2009.

[25] Martin B. Plenio. Logarithmic negativity: A full entanglement monotone that is not convex. Physical Review Letters, 95(9):090503, August 2005.

[26] David Rousseau, G. V. Anand, and Francois Chapeau-Blondeau. Noise-enhanced nonlinear detector to improve signal detection in non-Gaussian noise. Signal Processing, 86(11):3456–3465, November 2006.

[27] Seymour Stein and J. Jay Jones. Modern communication principles. McGraw Hill, New York, 1967.

[28] Nigel G. Stocks. Suprathreshold stochastic resonance in multilevel threshold systems. Physical Review Letters, 84(11):2310–2313, March 2000.

[29] Kurt Wiesenfeld and Frank Moss. Stochastic resonance and the benefits of noise: From ice ages to crayfish and squids. Nature, 373:33–36, 1995.

[30] Mark M. Wilde. Can classical noise enhance quantum transmission? Journal of Physics A: Mathematical and Theoretical, 42(32):325301, July 2009. arXiv:0801.0096.

[31] Mark M. Wilde. From Classical to Quantum Shannon Theory. June 2011. arXiv:1106.1445.

[32] Mark M. Wilde and Bart Kosko. Quantum forbidden-interval theorems for stochastic resonance. Journal of Physics A: Mathematical and Theoretical, 42(46):465309, October 2009. arXiv:0801.3141.

[33] Howard M. Wiseman and Gerard J. Milburn. Quantum theory of field-quadrature measurements. Physical Review A, 47(1):642–662, January 1993.