Heat kernel and number theory on NC-torus

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Abstract

The heat trace asymptotics on the noncommutative torus, where generalized Laplacians are made out of left and right regular representations, is fully determined. It turns out that this question is very sensitive to the number-theoretical aspect of the deformation parameters. The central condition we use is of a Diophantine type. More generally, the importance of number theory is made explicit on a few examples. We apply the results to the spectral action computation and revisit the UV/IR mixing phenomenon for a scalar theory. Although we find non-local counterterms in the NC $\phi^4$ theory on $T^4$, we show that this theory can be made renormalizable at least at one loop, and may be even beyond.

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1 Introduction

The importance of heat kernel techniques in spectral analysis (see [26, 32]) or in quantum field theory is known for a long time (see for instance the references in [40]). This type of expansion is particularly very useful for the control of anomalies and loop divergences. Naturally, its extension to noncommutative theories using for instance the Moyal product instead of the pointwise one, has also begun a long time ago (see reviews [18, 37, 39] and also [10]). The idea, originally due to Heisenberg, behind this generalization is that it could help to suppress some divergences. Unfortunately, a consequence of this idea is that the situation is as difficult as in the classical setting, or even worse since some UV/IR mixing can occur, except in some peculiar cases where the renormalisability of the model is proved [29]. Meanwhile, the noncommutative geometry (NCG) pioneered by Alain Connes [12] has shown its capacity to cover isospectral deformations like the deformation of a classical torus into the celebrated noncommutative torus (nc-torus). While many physical ideas coming from string theory have justified a systematic study of noncommutative quantum field theory, the interest of NCG stems also from its mathematical roots. In particular, the spectral action introduced by Chamseddine–Connes refers to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of an algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ and a given Dirac operator $\mathcal{D}$ which generates the inner fluctuations corresponding to gauge potentials. This spectral action is simply $\text{Tr}(\Phi(\mathcal{D}_A/\Lambda))$ where $\Phi$ is a positive even function, $\Lambda$ is a mass scale parameter, $\mathcal{D}_A = \mathcal{D} + A$ and $A$ is a one-form.

This torus depends on parameters through a deformation matrix $\Theta$ and it appears that the heat asymptotics are very sensitive to it. In particular, some Diophantine conditions, see \[13\], \[16\], are necessary to control the number-theoretic deviation from rational numbers. We also investigate a situation beyond this condition which yields precise aspects of number theory.

To control the heat trace asymptotic we apply a two-step procedure. First, we define a trace (cf. \[11\]), which is indeed proportional to the Dixmier trace, and calculate it through the Fourier coefficients (cf. \[18\]). Then we prove that being expressed in terms of this trace the heat trace asymptotics for generalized Laplacians look precisely the same as in the commutative case.

We first apply the control of the heat trace asymptotic to the spectral action computation. This action has been partially computed in [24], but here we pay attention to the natural existing real structure $J$ of the triple (\[13\]): now $\mathcal{D}_A = \mathcal{D} + A + \epsilon JA J^{-1}$, so we are in the most difficult situation where exist simultaneously left and right regular representations (see also [27] for further physical motivations). We show here that this full spectral action is the expected one for a 4-dimensional nc-torus. The amazing fact is that unlike for arbitrary generalized Laplacians, where non-standard terms appear in the heat kernel expansion (typically of the form ‘product of traces’), for the square of the covariant Dirac operator, such weird terms are absent. Thus the formula \[57\] we obtain is the expected one, up to some numerical coefficients.

Then, we apply our results to the study of a scalar field in an nc-4-torus. We show that the divergent part of the effective action does not reproduce the structure of the classical one, that is, the divergences cannot be cancelled by a proper couplings re-definition. Nevertheless, the theory can be made renormalizable at one loop by adding to the classical action a non-local term, which perfectly fits in with the philosophy of [29]. We conjecture that the modified theory is renormalizable to all orders in perturbation theory.

The paper is organized as follows: we recall in Section 2 some useful facts on nc-tori and study a trace, in fact a Dixmier trace, applied to operators like $L(a)R(b)$, $a, b \in \mathcal{A}$, where $L$ (resp. $R$) is the left (resp. right) multiplication, giving a full asymptotic of $\text{Tr}(L(a)R(b)e^{-tP})$.
as $t \to 0$ for a generalized Laplacian $P$. Section 3 touches on toric noncommutative manifolds, not necessarily compact. The spectral action is computed in Section 4 and the last section is devoted to the study of divergences of a scalar field theory.

Since the proof of the asymptotics of the heat trace is technical, it is postponed to Appendix A while some consequences of number theory in this setting are developed in Appendix B.

## 2 Heat trace asymptotic on NC-torus

### 2.1 Traces and number theory

Let $C^\infty(\mathbb{T}_\Theta^n)$ be the smooth noncommutative $n$-torus associated to a skewsymmetric deformation matrix $\Theta \in M_n(\mathbb{R})$ (see [11], [35]). This means that $C^\infty(\mathbb{T}_\Theta^n)$ is the algebra generated by $n$ unitaries $u_i$, $i = 1, \ldots, n$ subject to the relations

$$u_i u_j = e^{i\Theta_{ij}} u_j u_i,$$

and with Schwartz coefficients: using the Weyl elements $U_k := e^{-\frac{i}{2}k \cdot \chi k} u_1^{k_1} \cdots u_n^{k_n}$, $k \in \mathbb{Z}^n$, relation (1) reads

$$U_k U_q = e^{-\frac{i}{2}k \cdot \Theta q} U_{k+q},$$

where $\chi$ is the matrix restriction of $\Theta$ to its upper triangular part. Thus unitary operators $U_k$ satisfy $U_k^* = U_{-k}$ and a typical element $a \in C^\infty(\mathbb{T}_\Theta^n)$ can be written as $a = (2\pi)^{-n/2} \sum_{k \in \mathbb{Z}^n} a_k U_k$, where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^n)$. We use this non-standard normalization in order to simplify upcoming formulas.

Let $\tau$ be the (unique) normalized faithful trace on $C^\infty(\mathbb{T}_\Theta^n)$ defined by

$$\tau(a) := (2\pi)^{-n/2} a_0$$

and $\mathcal{H}_\tau$ be the GNS Hilbert space obtained by completion of $C^\infty(\mathbb{T}_\Theta^n)$ with respect of the norm induced by the scalar product $\langle a, b \rangle := \tau(a^* b)$.

On $\mathcal{H}_\tau$, we consider the left and right regular representations of $C^\infty(\mathbb{T}_\Theta^n)$ by bounded operators, that we denote respectively by $L(\cdot)$ and $R(\cdot)$.

An easy consequence of the associativity of the algebra is the commutativity of these two representations, namely $L(a) R(b) = R(b) L(a)$, for all $a, b \in C^\infty(\mathbb{T}_\Theta^n)$.

Let also $\delta_\mu$, $\mu = 1, \ldots, n$ be the $n$ (pairwise commuting) canonical derivations, defined by

$$\delta_\mu(U_k) := ik_\mu U_k.$$ (3)

They extend to unbounded operators on $\mathcal{H}_\tau$ (with suitable domain), and let

$$\Delta := -g^{\mu\nu} \delta_\mu \delta_\nu \geq 0$$

be the associated Laplacian where we assume that the metric $g^{\mu\nu}$ on $\mathbb{T}^n$ is constant.

In the following, for $k, q \in \mathbb{Z}^n$, we denote $k \cdot q := g_{\mu\nu} k^\mu q^\nu$ and $|k|^2 := k \cdot k$. Naturally, the matrix $\Theta$ satisfies $k \cdot \Theta q = -\Theta k \cdot q$.

There is (at least) one analogous description of $C^\infty(\mathbb{T}_\Theta^n)$ given in terms of Rieffel star-product [36]. If $\alpha$ denotes the (periodic) action of $\mathbb{R}^n$ on $\mathbb{T}^n$, then $C^\infty(\mathbb{T}_\Theta^n) \simeq (C^\infty(\mathbb{T}^n), \star)$, where the star-product $\star$ is defined by the following oscillatory integrals:

$$f \star h := (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} d^n y d^n z e^{-iyz} \frac{1}{2\Theta y} \alpha_\star(\overline{f}) \alpha_\star(h), \quad f, g \in C^\infty(\mathbb{T}^n),$$ (4)
yielding relation \(2\) on the Fourier modes,

\[
e^{ikx} * e^{iqx} = e^{-\frac{1}{2}k, \Theta q} e^{i(k+q)x}.
\]

A counterpart of eq. \(3\) reads \(\partial_\mu e^{ikx} = ik_\mu e^{ikx}\). In this description, the above defined Laplacian is nothing else but the ordinary one (associated to the constant metric \(g\)) and the trace \(\tau\) is the normalized integral,

\[
\tau(f) := (2\pi)^{-n} \int_{\mathbb{T}^n} d^n x f(x) = (\text{vol } \mathbb{T}^n)^{-1} \int_{\mathbb{T}^n} d^n x \sqrt{g} f(x), \quad f \in C^\infty(\mathbb{T}^n).
\]

We can consider using a non-flat metric, but we need (for later use) a severe restriction on it: the \(\mathbb{R}^n\)-action must be isometric. Thus only constant metrics are allowed.\(^1\)

The purpose of this section is to establish the small-\(t\) asymptotics of the function

\[
t \to \text{Tr}(L(l)R(r)e^{-tP}),
\]

where \(P\) is a generalized Laplacian, i.e. \(P\) has the form

\[
P := -(g^{\mu\nu} \nabla_\mu \nabla_\nu + E),
\]

where

\[
\nabla_\mu := \delta_\mu + \omega_\mu := \delta_\mu + L(\lambda_\mu) - R(\rho_\mu) ; \quad E := L(l_1) - R(r_1) + L(l_2)R(r_2),
\]

and \(l, r, \lambda_\mu, \rho_\mu, l_1, r_1 \in C^\infty(\mathbb{T}_\Theta^n)\). The arbitrary choice of the sign \(-R\) will be justified in (49). One can also take more general forms of \(E\) and \(\omega\). For example, \(\omega_\mu\) can contain a term like \(R(\rho'_\mu)L(l')\) with some smooth \(\rho'_\mu\) and \(l'\). Such modifications change very little in our considerations below.

For that, the central asymptotic to compute is the one of the function

\[
t \to \text{Tr}(L(l) R(r) e^{-t\Delta}), \quad l, r \in C^\infty(\mathbb{T}_\Theta^n). \tag{8}
\]

Indeed, after an expansion of the semi-group \(e^{-tP}\), viewed as an unbounded perturbation of the heat operator \(e^{-t\Delta}\), the only other asymptotics we need are

\[
t \to \text{Tr}(L(l) e^{-t\Delta}), \quad t \to \text{Tr}(R(r) e^{-t\Delta}), \tag{9}
\]

but we have shown in [24, 41] that they have same asymptotics as their commutative (\(\Theta = 0\)) counterparts.

Note that the heat semi-group \(e^{-t\Delta}\) is trace-class, since it is diagonal in the orthonormal basis \(\{U_k\}_{k \in \mathbb{Z}^n}\), with eigenvalues \(e^{-t|k|^2}\). Of course, the same property holds for \(e^{-tP}\), as shown in the next lemma, based on a simple application of the Duhamel expansion.

**Lemma 2.1.** For any \(r, l, \lambda_\mu, \rho_\mu, l_1, r_1 \in C^\infty(\mathbb{T}_\Theta^n)\), the operator \(e^{-tP}\) is trace-class for \(t > 0\).

\(^1\)There are very few attempts to deal with metrics which are not constant in the non-commutative directions. So far, one was able to obtain expressions for the heat trace asymptotics as formal power series in deviations of the metric from the flat one only [42]. Similar difficulties appear if the metric is matrix-valued [2].
Proof. We are going to use Duhamel’s expansion for the semi-group generated by \( P \), viewed as an unbounded perturbation of \( \Delta \). We write

\[
P = \Delta - B,
\]

where \( B = 2g^\mu{}^\nu\omega_\mu\delta_\nu + C \) with \( C = g^\mu{}^\nu(\omega_\mu\omega_\nu + \omega_\nu\omega_\mu) + E \) and \( \omega_\nu\omega_\mu = L(\delta_\mu\lambda_\nu) - R(\delta_\mu\rho_\nu). \)

From the Duhamel principle

\[
e^{-t(A+B)} = e^{-tA} - t \int_0^1 e^{-st(A+B)} B e^{-(1-s)tA} ds,
\]

we first formally write

\[
e^{-tP} = \sum_{j=0}^{\infty} (-t)^j E_j(t), \tag{10}
\]

where

\[
E_0(t) := e^{-t\Delta} \text{ and } E_j(t) := \int_{\Delta_j} e^{-st\Delta} B e^{-(s_2-s_1)t\Delta} \cdots B e^{-(1-s_j)t\Delta} d^j s.
\]

Here \( \Delta_j \) denotes the ordinary \( j \)-simplex:

\[
\Delta_j := \{ s \in \mathbb{R}^j; 0 \leq s_1 \leq \cdots \leq s_j \leq 1 \} \simeq \{ s \in \mathbb{R}^{j+1}; s_1 \geq 0, \sum_{i=0}^j s_i = 1 \}.
\]

We prove convergence of the expansion (10) in the trace-norm and for reasonably small \( t \): from the Hölder inequality for Schatten classes, we have

\[
\|E_j(t)\|_1 \leq \int_{\Delta_j} \|e^{-s_0t\Delta}\|_{s_0^{-1}} \|Be^{-s_1t\Delta}\|_{s_1^{-1}} \cdots \|Be^{-s_jt\Delta}\|_{s_j^{-1}} \cdot d^j s.
\]

where \( B = 2\omega^\mu\delta_\mu + C \) and \( \omega^\mu, C \) are bounded. By functional calculus,

\[
\|\delta_\mu e^{-s_i t\Delta}\|_{s_i^{-1}} \leq \|\delta_\mu e^{-s_i t\Delta/2}\|\|e^{-s_i t\Delta/2}\|_{s_i^{-1}}
\]

\[
\leq c(g)(es_it)^{-1/2}(\text{Tr } e^{-t\Delta/2})^{s_i},
\]

using the inequality \( \|f(\delta_1, \cdots, \delta_n)||_{op} \leq ||f||_{\infty} \) where \( f(x) = x^\mu e^{-x.x} \) which follows from \( f(\delta_1, \cdots, \delta_n)U_k = f(ik)U_k \). So

\[
\|E_j(t)\|_1 \leq \int_{\Delta_j} (\text{Tr } e^{-t\Delta})^{s_0} \left( \|C\| (\text{Tr } e^{-t\Delta})^{s_1} + 2c(g) \sum_\mu \|\omega^\mu\| (es_1t)^{-1/2}(\text{Tr } e^{-t\Delta/2})^{s_1} \right) \cdots
\]

\[
\cdots \left( \|C\| (\text{Tr } e^{-t\Delta})^{s_j} + 2c(g) \sum_\mu \|\omega^\mu\| (es_jt)^{-1/2}(\text{Tr } e^{-t\Delta/2})^{s_j} \right) d^j s
\]

\[
\leq \text{Tr}(e^{-t\Delta/2}) \int_{\Delta_j} \left( \|C\| + 2c(g) \sum_\mu \|\omega^\mu\| (es_1t)^{-1/2} \right) \cdots
\]

\[
\cdots \left( \|C\| + 2c(g) \sum_\mu \|\omega^\mu\| (es_jt)^{-1/2} \right) d^j s.
\]

5
Using
\[ \int_{\Delta_j} \prod_{i=1}^{j} s_i^{-1/2} \, ds \leq 2^{j-1}, \]
the last expression can be estimated for \( t \leq e^{-1} \) (since \( s_j \leq 1 \)) by
\[ t^{-(j-1)/2} e^{-(j-1)/2} 2^{j-1} \left( \|C\| + 2c(g) \sum_{\mu} \|\omega_{\mu}\| \right)^{j-1} \text{Tr}(e^{-t\Delta/2}). \]

Thus
\[ \left\| \sum_{j=0}^{\infty} (-t)^j E_j(t) \right\|_1 \leq t \text{Tr}(e^{-t\Delta/2}) \sum_{j=0}^{\infty} \left( \frac{2}{\sqrt{e}} (\|C\| + 2c(g) \sum_{\mu} \|\omega_{\mu}\|) \sqrt{t} \right)^{j-1}, \]
which is finite for
\[ 0 < t < \frac{e}{4(\|C\| + 2c(g) \sum_{\mu} \|\omega_{\mu}\|)^2} := t_0. \]

Finally, for \( t_0 \leq t \leq 2t_0 \), note that
\[ \|e^{-tP}\|_1 \leq \|e^{-(t-t_0)P}\|_1 \|e^{-t_0P}\|, \]
and the result follows inductively.

One can probably prove this Lemma also by using some estimates involving Sobolev spaces, cf. [41]. However, the Duhamel principle is quite important in its own right for physical applications. We shall use [10] below in Sec. 5. The convergence of the Duhamel expansion is necessary to construct the covariant perturbation series in the approach of Barvinsky and Vilkovisky [4].

Note also that the Duhamel expansion has been used to compute one-loop divergencies in a more general framework of NCQFT, namely on Moyal plane with degenerate but non-constant \( \Theta \) [23].

Let us define the functional \( \text{Sp} \) on \( \mathcal{L}(\mathcal{H}_\tau) \), given for a bounded operator \( A \) by
\[ \text{Sp}(A) := \lim_{t \to 0^+} (4\pi t)^{n/2} \text{Tr} \left( Ae^{-t\Delta} \right), \quad (11) \]

Note that this definition is not sensitive to the kernel of \( \Delta \) (which is \( \mathbb{C}U_0 \)), so we may change \( \Delta \) in \( \Delta + 1 \) or assume that \( \Delta \) is invertible.

We will show that \( \text{Sp}(L(\cdot)R(\cdot)) \), as a functional on \( C^\infty(\mathbb{T}_\Theta) \times C^\infty(\mathbb{T}_\Theta) \), is indeed a finite and faithful trace in each argument, namely it vanishes whenever one of its arguments is a commutator, see Lemma 2.4. This should not be surprising once one knows that

\( \text{Sp}(A) \) is a multiple of the Dixmier trace of \( A(1 + \Delta)^{-n/2} \).

Indeed, from the knowledge of the eigenvalues of \( \Delta \) and the boundedness of \( A \), one has \( A(1 + \Delta)^{-n/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_\tau) \), i.e., \( \sum_{k=1}^{N} \mu_k(A(1 + \Delta)^{-n/2}) = O(\ln N) \) where \( \mu_k(X) \) are the ordered singular values of \( X \). It follows then (see [14, p. 236] with immediate modifications) that
\[ \text{Tr}_\omega(A(1 + \Delta)^{-n/2}) = \frac{1}{\Gamma(n/2 + 1)} \lim_{t \to 0} \rho^{n/2} \text{Tr}(A e^{-t\Delta}) = \frac{1}{(4\pi)^{n/2}\Gamma(n/2 + 1)} \text{Sp}(A). \]

However, we leave this feature now since our goal is to find an algorithm to obtain analytic expressions for the heat coefficients. For that, Dixmier-trace technology is not so helpful.
From relations (1) and using the orthonormal basis \( \{ U_k \}_{k \in \mathbb{Z}^n} \) to compute the trace, we find for \( l = (2\pi)^{-n/2} \sum q_l Q_l, r = (2\pi)^{-n/2} \sum q_r Q_r \) in \( C^\infty(\mathbb{T}^n_{\Theta}) \),

\[
\text{Tr}(L(l) R(r) e^{-t\Delta}) = \sum_{k \in \mathbb{Z}^n} \tau(U_k^* L(l) R(r) e^{-t\Delta} U_k) = \sum_{k \in \mathbb{Z}^n} e^{-t|k|^2} \tau(U_k^* l U_k r) \\
= (2\pi)^{-n} \sum_{k, q_1, q_2 \in \mathbb{Z}^n} e^{-t|k|^2} l_{q_1} r_{q_2} \tau(U_{-k} U_{q_1} U_k U_{q_2}) \\
= (2\pi)^{-n} \sum_{q, k \in \mathbb{Z}^n} l_q r_{-q} e^{-ik.\Theta q} e^{-t|k|^2},
\]

which after Poisson resummation reads

\[
\text{Tr}(L(l) R(r) e^{-t\Delta}) = \sqrt{g} (4\pi t)^{-n/2} \sum_{q, k \in \mathbb{Z}^n} l_q r_{-q} e^{-|\Theta q - 2\pi k|^2/4t} \\
= \sqrt{g} \sum_{q, k \in \mathbb{Z}^n} l_q r_{-q} K(t, \Theta q - 2\pi k)
\]

(12)

where \( K(t, x) := (4\pi t)^{-n/2} e^{-|x|^2/4t} \) is the heat kernel of \( \mathbb{R}^n \) with metric \( g^{\mu\nu} \).

To proceed further, we need to impose some restrictions on the matrix \( \Theta \). Whereas it does not for the asymptotics of (12), the number-theoretical aspect of \( \Theta \) has huge consequences for the asymptotics of (11).

To explain what this is about, let us review what happens in the (nondegenerate) two-dimensional case, where \( \Theta = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), with \( \theta \) an arbitrary real number. (Actually, up to an isomorphism, the range of \( \theta \) can be reduced to the interval \([0, \frac{1}{2}]\).) In this case, there are two distinct situations to consider [21]. When \( \theta \) is rational (relatively to \( 2\pi \), then in the sum (12), only terms with \( q \) multiple of \( \theta \)-denominator, will contribute to the small-\( t \) asymptotics. When \( \theta \) is irrational (again relatively to \( 2\pi \)), one can guess that only the zero-mode will contribute to the small-\( t \) asymptotics of \( \text{Sp}(L(l)R(r)) \).

This is indeed true, provided one has a control on the sum

\[
\sum_{k \in \mathbb{Z}^2} \sum_{q, k \in \mathbb{Z}^2} l_q r_{-q} e^{-|\theta q - 2\pi k|^2/4t} / (4\pi t)^{n/2},
\]

(13)
i.e., provided one can measure how far from rationals \( \theta \) is, since \( |\theta q - 2\pi k| \) could be in principle arbitrarily close to 0.

This control is precisely given by a Diophantine condition.

**Definition 2.2.** A number \( \theta \) is said to satisfy a Diophantine condition (relatively to \( 2\pi \)) if there exists two constants \( C > 0, \beta \geq 0 \) such that for all \( q \in \mathbb{Z}^* \)

\[
\|\theta q\|_T := \inf_{k \in \mathbb{Z}} |\theta q - 2\pi k| \geq \frac{C}{|q|^{1+\beta}} \iff |1 - \cos(\theta q)| \geq \frac{2C^2}{|q|^{1+2\beta+\beta^2}}
\]

(14)

In fact, \( \inf_{p \in \mathbb{Z}} |\theta q - p| \leq |\sin(\pi \theta q)| \) : actually, for a given \( q \), their exists an integer \( p_0 \) such that \( |\theta q - p_0| \leq \frac{1}{2} \) and since we work under modulus, we may assume that \( 0 \leq \theta q - p_0 \leq \frac{1}{2} \). Since \( (\sin x)/x \) is decreasing on \([0, \frac{\pi}{2}]\), we get \( \sin(\pi (\theta q - p_0)) \geq 2(\theta q - p_0) \) and the above equivalence by taking the square of the inequality with \( \sin^2(\frac{\theta q}{2}) = \frac{1}{2}(1 - \cos(\theta q)) \).
In other words, this condition states that the inverse torus norm of $\theta q$ is a temperate distribution over $\mathbb{Z}^\ast$. This is exactly what we need since in equation [13], the complex coefficients $l_q$, $r_q$ are of Schwartz-class by the smoothness assumption. Note that this condition is not too restrictive since the set of irrational numbers satisfying a Diophantine condition is of full Lebesgue measure.

In the general Poisson case, one can always change the coordinates on $\mathbb{T}^n$, $y = Bx$ with a constant matrix $B$, so that the new coordinates $y$ are $2\pi$-periodic again, and the Poisson matrix $\Theta = B^T \Theta B$ becomes

$$\Theta = 0_l \bigoplus_{\{i: \theta_i \in 2\pi \mathbb{Q} \}} \theta_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bigoplus_{\{j: \theta_j \in \mathbb{R} \setminus 2\pi \mathbb{Q} \}} \theta_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{15}$$

where $0_l$ is the zero matrix of size $l$ and $n = l + m_1 + m_2$. Here $m_1$ is the size of the rational part (with $\theta_i = p_i/q_i$, $i = 1, \ldots, m_1$) and $m_2$ of the irrational one. We define then $\mathcal{Z} = \mathbb{Z}^l \times q_1\mathbb{Z} \times \cdots \times q_{m_1}\mathbb{Z} \times \{(0, \ldots, 0) \in \mathbb{Z}^{m_2}\}$. The rest of $\mathbb{Z}^n$ is denoted by $\mathcal{K}$ and is split into two parts, $\mathcal{K}_{\text{per}} \simeq \{(k_1, \ldots, k_{m_1}) \in \mathbb{N}^{m_1}, 1 \leq k_i \leq q_i - 1, i = 1, \ldots, m_1\}$ and $\mathcal{K}_{\text{inf}} \simeq \mathbb{Z}^{m_2} \setminus \{0\}$. $\mathcal{K}_{\text{per}}$ is a finite set which lays in an $\mathbb{R}$-linear space generated by $\mathcal{Z}$. We shall omit the bar over $\Theta$ in what follows.

One can avoid the use of a particular coordinate system. Then $\mathcal{Z}$ is defined as the set of $q \in \mathbb{Z}^n$ such that $(2\pi)^{-1}\Theta q \in \mathbb{Z}^n$. This definition reveals the actual meaning of this set. One can also figure out how to give coordinate independent definitions of $\mathcal{K}_{\text{per}}$ and $\mathcal{K}_{\text{inf}}$.

Now we are ready to formulate our restriction on $\Theta$. We assume the following:

The matrix $\Theta$ satisfies a Diophantine condition with respect to $\mathcal{K}_{\text{inf}}$, i.e., there are two positive constants $C$ and $\beta$ such that

$$\inf_{k \in \mathbb{Z}^n} |\Theta q - 2\pi k| \geq \frac{C}{|q|^{1+\beta}} \text{ for all } q \in \mathcal{K}_{\text{inf}}. \tag{16}$$

In the standard definition of a Diophantine condition for an $l$-tuple of real numbers, one assumes that $1 + \beta \geq l$. Our consideration is valid in a more general case for any positive $1 + \beta$, so we do not need this restriction as far as such $\Theta$ exist; note, for instance, that there exists a set of full Lebesgue measure of $l$-tuples of Roth type, that is of $\alpha \in \mathbb{R}^l$ such that for all $\epsilon > 0$, there exists $C_{\epsilon}$ with $\inf_{k \in \mathbb{Z}^l} |\alpha q - 2\pi k| \geq \frac{C_{\epsilon}}{|q|^{1+\beta}}$, see [31].

We remark that it does not matter whether one uses the metric $g^{\mu\nu}$ or the normalized diagonal metric in norm-values of [16] since it can be absorbed in the constant $C$.

With this restriction on $\Theta$ we can prove the following formula which governs the asymptotic behavior of the trace at $t \to 0^+$.

$$\text{Tr} \left( L(l) R(r) e^{-i\Delta t} \right) = \frac{\sqrt{g}}{(4\pi l)^{n/2}} \sum_{q \in \mathcal{Z}} l_q r_{-q} + \text{e.s.t.} \tag{17}$$

where e.s.t. denotes some exponentially small terms in $t$, i.e., the terms which vanish faster than any power of $t$ as $t \to 0^+$. The proof, which is somewhat technical, is postponed to Appendix [A].

Equation (17) immediately yields the following Theorem.

**Theorem 2.3.** Assume $\Theta$ satisfies condition [16]. Then for any $l, r \in C^\infty(\mathbb{T}^n_0)$,

$$\text{Sp} (L(l) R(r)) = \sqrt{g} \sum_{q \in \mathcal{Z}} l_q r_{-q}. \tag{18}$$
The explicit expression \((18)\) makes it possible to show rather directly that \((11)\) indeed defines a trace in each variable: it vanishes whenever \(l\) or \(r\) is a commutator:

**Corollary 2.4.** Let \(l, r, s \in C^\infty(T^n_\Theta)\). Then,

\[
\text{Sp}(L(l)R([r, s])) = 0.
\]

**Proof.** From commutation relations \((2)\), we find

\[
[r, s] = -2i(2\pi)^{-n} \sum_{k, q \in \mathbb{Z}^n} r_k s_q \sin(\frac{1}{2}k, \Theta q) U_{k+q},
\]

and thus

\[
[r, s]_k = 2i(2\pi)^{-n/2} \sum_{q \in \mathbb{Z}^n} r_{k-q} s_q \sin(\frac{1}{2}q, \Theta k),
\]

which is zero whenever \(k \in \mathbb{Z}\) since it is equivalent to \((2\pi)^{-1}\Theta k \in \mathbb{Z}^n\).

**Remark 2.5.** We have the following relations between the functional \(\text{Sp}\) and the trace \(\tau\):

\[
\text{Sp}(L(a)) = \text{Sp}(R(a)) = (\text{vol } T^n) \tau(a),
\]

for all \(a \in C^\infty(T^n_\Theta)\) and any \(\Theta\), and when \(\mathbb{Z} = \{0\}\) (i.e. pure Diophantine case), then

\[
\text{Sp}(L(l) R(r)) = (\text{vol } T^n) \tau(l) \tau(r),
\]

which makes transparent the statement of the corollary.

This completes our study of the trace \((11)\). Below we present several relations similar to \((17)\) which will be used in the next section. One can show (see Appendix A) that

\[
\text{Tr} \left( (L(l)R(r))^{\mu_1 \ldots \mu_m} e^{-t\Delta} \right) = 0 + \text{e.s.t.},
\]

(19)

where the notation \([(L(l)R(r))^{\mu_1 \ldots \mu_m}] means that the vector indices are distributed between \(l\) and \(r\). For higher derivatives we have

\[
\text{Tr} \left( (L(l)R(r))^{\mu_1 \ldots \mu_m} \delta_{\mu_1} \ldots \delta_{\mu_m} e^{-t\Delta} \right) = \sum_{k \in \mathbb{Z}^n} \tau(U_k^* [L(l)R(r)]^{\mu_1 \ldots \mu_m} \delta_{\mu_1} \ldots \delta_{\mu_m} e^{-t\Delta} U_k)
\]

\[
= i^m \sum_{k \in \mathbb{Z}^n} k_{\mu_1} \ldots k_{\mu_m} \tau(U_k^* [L(l)R(r)]^{\mu_1 \ldots \mu_m} e^{-t\Delta} U_k)
\]

\[
= i^m C_{\mu_1 \ldots \mu_m}^{(m)} \text{Tr} \left( [L(l)R(r)]^{\mu_1 \ldots \mu_m} e^{-t\Delta} \right),
\]

(20)

One can calculate the tensors \(C_{\mu_1 \ldots \mu_m}^{(m)}\) by varying \((17)\) or \((18)\) with respect to the metric \(g^{\mu\nu}\) (this is a standard way to include derivatives in the heat trace expansion, cf. [6].) All \(G^{(2j+1)}\) are exponentially small and can be neglected in our analysis. For even \(m\), corresponding tensors \(G^{(m)}\) are obtained from the following recursion relation:

\[
G_{\mu_1 \ldots \mu_{2p}}^{(2p)} = -\frac{1}{p} \delta_{\mu_1} \ldots \mu_{2p} G_{\mu_1 \ldots \mu_{2p}}^{(2p)},
\]

(21)
with $G^{(0)} = \sqrt{g}$. One has to take into account that $g^{\mu \nu}$ is symmetric, so that not all of the components are indeed independent, and

$$
\frac{\delta}{\delta g^{\rho \sigma}} g^{\rho \sigma} = -\frac{1}{2} (g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}).
$$

(22)

For example,

$$
G^{(2)}_{\mu \nu} = \sqrt{g} g^{\mu \nu},
$$

$$
G^{(4)}_{\mu \nu \rho \sigma} = \sqrt{g} (g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho} + g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}).
$$

(23)

### 2.2 Heat trace asymptotics for generalized Laplacians

From expression (5) of generalized Laplacian $P = -g^{\mu \nu} \nabla_\mu \nabla_\nu - E$, we need also to define the associated curvature:

$$
\hat{\Omega}_{\mu \nu} := \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu = L(\Omega^L_{\mu \nu}) - R(\Omega^R_{\mu \nu}),
$$

and higher “covariant derivatives” of $E$ and $\hat{\Omega}$ as repeated commutators with $\nabla$. For example, $E_{\mu \nu} := [\nabla_\mu, E]$, $E_{\mu \nu \rho \sigma} := [\nabla_\nu, E_{\mu \rho \sigma}]$.

Actually, there are two covariant derivatives, $\nabla^L_\mu := \delta_\mu + L(\lambda_\mu)$ and $\nabla^R_\mu := \delta_\mu - R(\rho_\mu)$ and two gauge symmetries in the problem. One of these symmetries acts on “left” fields $\lambda_\mu$, $l_1$, $l_2$, the other acts on “right” fields $\rho_\mu$, $r_1$, $r_2$. The gauge group has a direct product structure. Explicit expressions for the symmetry transformations can be found in [43]. The functional $\text{Sp}$ of any polynomial of $E$, $\hat{\Omega}$ and derivatives is gauge invariant. Full invariance will mean that also all vector indices are contracted in pairs. This is precisely the class of invariants which will appear in the heat trace asymptotics. Due to the product structure of the gauge group there are gauge invariants which do not belong to the class we have just described. For example, $\text{Sp}(L(l_1))$ is such an invariant.

We will see that in spectral action picture, there is only one gauge group, namely the automorphism group of the algebra but lifted to the spinor bundle via the charge conjugation operator. In such application, the representation really looks like the adjoint one (see Sec. 4).

We shall need the notion of **canonical mass dimension**. We assign canonical mass dimension 2 to $E$ and $\hat{\Omega}$, and canonical mass dimension 1 to each derivative. Canonical mass dimension of any monomial is the sum of canonical mass dimensions of all factors.

The heat trace asymptotic is then given by the following

**Theorem 2.6.** Let $P$ be as defined above. Then

i) There is a full asymptotic expansion of the heat trace

$$
\text{Tr} \left( L(l) R(r) e^{-tP} \right) \sim \sum_{k=0}^{\infty} a_k(l, r; P) t^{(k-n)/2}.
$$

(24)

ii) The coefficients $a_k$ can be expressed as

$$
a_k(l, r; P) = \sum_{\alpha} b_\alpha \text{Sp}(L(l) R(r) A_\alpha),
$$

(25)

where $A_\alpha$ are independent invariant free polynomials of canonical mass dimension $k$ of $E$, $\hat{\Omega}$ and their covariant derivatives. The numbers $b_\alpha$ are constants. The odd-numbered coefficients $a_{2j+1}$ vanish.
Consequently, odd numbered heat kernel coefficients $a_k$ the canonical mass dimensions of the monomials as defined above. Besides, monomials contribute to the coefficient $a_k$ sum over $L$. Being commuted through an operator of the type $F$ in this operator by their derivatives, e.g., $(f, h)$ from 2 power of 2 becomes an operator of (left and right) multiplication by the connection, i.e., $(\nabla^2)$ and all momenta $k$ from the expression inside the NC-torus trace in (30). Therefore, one can apply (20) to obtain instead of (30) the following expression

$$G^{(m)}_{\mu_1 \cdots \mu_m} \sum_k e^{-t|k|^2} \tau \left( U^*_k L(l) R(r) F(E, (\nabla - ik))^{\mu_1 \cdots \mu_m} U_k \right).$$

Let us now evaluate the power of $t$ corresponding to each monomial. If $F$ contains an $N_E$-th power of $E$, an $N_P$-th power of $(\nabla - ik)^2$ and an $N_K$-th power of $2ik^\mu (\nabla^\mu - ik^\mu)$, then $F$ itself contains $t^{N_E+N_P+N_K}$. Explicit multipliers $k^\mu$ which we put in front of $F$ in (30) come from $2ik^\mu (\nabla^\mu - ik^\mu)$ only. Consequently, $m = N_K$. For odd $m$, the tensors $G^{(m)}$ vanish up to exponentially small terms, while for an even $m$ the tensor $G^{(m)}$ is proportional to $t^{-m/2}$. The sum over $k$ brings another $t^{-n/2}$. Altogether, we have $t^{N_E+N_P+N_K/2-n/2}$. This means that such monomials contribute to the coefficient $a_p(l, r, P)$ with $p = 2N_E+2N_P+N_K$, which are precisely the canonical mass dimensions of the monomials as defined above. Besides, $N_K$ should be even. Consequently, odd numbered heat kernel coefficients $a_{2j+1}$ vanish.

**Proof.** Existence of the trace follows from Lemma 2.1. The proof of the second statement (below) can be considered as a constructive proof for the existence of the asymptotic expansion.

To evaluate the asymptotic behavior of the trace on the left hand side of (24) we use the canonical basis $\{U_k\}$ of $\mathcal{H}_r$. This is a standard procedure used in quantum field theory for a long time (cf. [33]), which was recently applied to noncommutative theories [41, 43]: one first factors out a global $e^{-t|k|^2}$ term,

$$\text{Tr} \left( L(l) R(r) e^{-tF} \right) = \sum_k \tau \left( U^*_k L(l) R(r) e^{-tF} U_k \right) = \sum_k e^{-t|k|^2} \tau \left( U^*_k L(l) R(r) e^{t((\nabla^\mu - ik^\mu)(\nabla^\mu - ik^\mu) + 2ik^\mu (\nabla^\mu - ik^\mu) + E)} U_k \right).$$

Then, one expands the exponential in (29) as a power series in $E$ and $(\nabla - ik)$. As a result, one gets a sum of monomials of the form

$$\sum_k e^{-t|k|^2} k_{\mu_1} \cdots k_{\mu_m} \tau \left( U^*_k L(l) R(r) F(E, (\nabla - ik))^{\mu_1 \cdots \mu_m} U_k \right).$$

We stress that it is important that each $\nabla$ appear in the combination $(\nabla - ik)$. We take in $F$ all $(\nabla - ik)$ one by one starting with the rightmost $(\nabla - ik)$ and push them to the right. Being commuted through an operator of the type $L(f)R(h)$, $(\nabla - ik)$ replaces the functions $f, h$ in this operator by their derivatives, e.g., $(\nabla - ik)L(f) = L(f)(\nabla - ik) + L(\nabla^L f)$, where only the Leibniz rule satisfied by the derivations $\delta$ has been used. When $(\nabla - ik)$ hits $U_k$, it becomes an operator of (left and right) multiplication by the connection, i.e., $(\nabla^\mu - ik^\mu) U_k = (L(\lambda^\mu) - R(\rho^\mu)) U_k$. In this way, one can remove all derivative operators (which are replaced by left and right multiplication operators) and all momenta $k$ from the expression inside the NC-torus trace in (30). Therefore, one can apply (20) to obtain instead of (30) the following expression

$$G^{(m)}_{\mu_1 \cdots \mu_m} \sum_k e^{-t|k|^2} \tau \left( U^*_k L(l) R(r) F(E, (\nabla - ik))^{\mu_1 \cdots \mu_m} U_k \right).$$
Now we have to prove that the heat kernel coefficients are of the form declared in the Theorem, i.e., that they are invariant polynomials constructed from $E, \Omega_{\mu\nu}$ and their derivatives. We have already proved that the expression inside the trace $\tau$ in (31) is in fact a multiplication operator (i.e., a combination of left and right regular representation operators) which does not contain $k$. Together with gauge invariance of the heat trace this could have been enough to get the statement. However, since the gauge group has a product structure, there are more gauge invariants than we expect to find in the heat trace asymptotics. Let us collect all monomials $F_a(E, (\nabla - ik))^{\mu_1 \ldots \mu_m}$ of a given (even) canonical mass dimension $p$ which appear in the expansion of the exponential of (29), and consider the sum

$$\tau \left( U^*_k L(l) R(r) \sum_a G^{(m)}_{\mu_1 \ldots \mu_m} F_a(E, (\nabla - ik))^{\mu_1 \ldots \mu_m} U_k \right).$$

As we have demonstrated above, $F_a$ are free polynomials of $E, \omega$ and their derivatives $\delta_\mu E, \delta_\mu \omega, \delta_\mu \omega, \delta_\mu \omega$ etc. The same procedure as above can be carried out for an arbitrary Laplace type operator $\hat{P}$ acting on smooth sections of an arbitrary (non-abelian) vector bundle over the commutative torus $\mathbb{T}^n$. The free polynomials of the endomorphism $E$, the connection $\omega$, which characterize $\hat{P}$, and their derivatives are in one-to-one correspondence with the polynomials in (32). In the case of $\hat{P}$ we know that all terms can be recombined into covariant derivatives and field strengths thus giving standard heat kernel coefficients. This is a purely combinatorial statement, which does not depend on the nature of $\nabla$ (or $\bar{\nabla}$) and $E$ (or $\bar{E}$). Therefore, the same recombination can be done also in the noncommutative case considered here. This completes the proof of the second assertion.

The third point is easy, it simply means that independent invariants remain independent when reduced to “pure left” or “pure right” cases. The coefficients in front of these invariants can therefore be read off from “pure left” heat kernel coefficients [41] on the torus. This includes (26), (27), (28) and even $a_6$ which is not given explicitly in the present work.

The interested reader can calculate also higher terms in the heat trace asymptotics by using the expressions for $a_8$ [1] and $a_{10}$ [38] obtained in the commutative case.

3 Toward the asymptotics for toric noncommutative manifolds

This section is devoted to the study of the asymptotic (17) in a more general setting of noncommutative spaces. We will concentrate here on toric noncommutative manifolds, $C^\infty(M_\Theta)$, also called periodic isospectral deformations (the aperiodic case [25], akin to Moyal plane, will be studied elsewhere). This class of quantum spaces can be thought as a curved space generalization of NC-tori. They were originally defined by Connes and Landi [15] (from cohomological considerations) within a twisted product approach (that we will follow here) and later by Connes and Dubois-Violette [16] in a more intrinsic way via fixed-point algebra techniques.

We first recall the definition of $C^\infty_c(M_\Theta)$: let $(M, g)$ be a Riemannian (compact or not) $n$-dimensional manifold without boundary. Consider $\alpha : \mathbb{T}^l \to \text{Isom}(M, g)$, a smooth isometric action of a $l$-torus on $M$, typically given by the maximal abelian subgroup of the isometry group of the manifold (the interesting class is $l \geq 2$). This action induces a spectral (Peter–Weyl) decomposition of any smooth function with compact support $f \in C^\infty_c(M)$

$$f = \sum_{r \in \mathbb{Z}^l} f_r, \quad \text{such that} \quad \alpha_z(f_r) = e^{-ir.z} f_r, \quad \forall z \in \mathbb{T}^l,$$
where the action by automorphism of the $l$-torus on $C_c^\infty(M)$ (also denoted $\alpha$) is given by 

$$(\alpha_zf)(p) := f(\alpha_{-z}(p)).$$

It is important to notice that this expansion is convergent in the sup-norm $\|\cdot\|_\infty$ (in fact $\|f\|_\infty$ is a Schwartz sequence for $f \in C_c^\infty(M)$.)

By analogy with the noncommutative torus, given a skewsymmetric $l \times l$ matrix $\Theta$, one can deform the algebra $C_c^\infty(M)$ to a noncommutative one $C_c^\infty(M_\Theta)$, defining the following twisted product on pairs of homogeneous elements

$$f_r \star g_s = e^{-\frac{i}{2}r.\Theta s} f_r g_s.$$ 

(33)

It should be clear that this product can also be realized via the Rieffel star-product [11] associated to the action $\alpha$ [36].

In this setting, the natural trace $\tau$ of this algebra is the integral with Riemannian volume form

$$\tau(.) = \int_M (.) \mu_g,$$

(34)

and all first-order differential operators which commute with the action $\alpha$ form a Lie algebra of derivations.

On the Hilbert space $\mathcal{H}$ of square integrable functions on $M$ with Riemannian volume form $\mu_g$, one defines left and right twisted multiplication operators $L(l), R(r), l, r \in C^\infty_c(M)$, by

$$L(l)\psi = l \star \psi, \quad R(r)\psi = \psi \star r,$$

for all $\psi \in \mathcal{H}$.

Because the Peter–Weyl expansion is sup-norm convergent, those operators are bounded for (at least) smooth compactly supported functions.

For any isometric $\mathbb{R}^l$-action on a Riemannian manifold, from the expression of the kernel of the operators $L(l), R(r)$ (see for instance [21]) and using the heat kernel $K_t(p,p')$ of the scalar Laplacian associated with the metric $g$ and its volume form $\mu_g$, one can compute

$$\text{Tr}(L(l) R(r) e^{-t\Delta}) = (2\pi)^{-l} \int_M \mu_g(p) \int d^l y d^l z l(z) r(\alpha_z(p)) K_t(\alpha_{-z}(p), p).$$

After a Peter–Weyl expansion of the functions $l, r$, this reads

$$\text{Tr}(L(l) R(r) e^{-t\Delta}) = \sum_{q \in \mathbb{Z}^l} \int_M \mu_g(p) l_q(p) r_{-q}(p) K_t(\alpha_{-q}(p), p).$$

(35)

For a large class of manifolds (see [17, 25] for a review of sufficient conditions), the behavior of the off-diagonal heat kernel is controlled by the geodesic distance function:

$$\frac{1}{(4\pi t)^{n/2}} e^{-d_g(p,p')/4t} \leq K_t(p,p') \leq C \frac{1}{(4\pi t)^{n/2}} e^{-d_g(p,p')/4(1+c)t},$$

(36)

where $d_g$ is the geodesic distance and $C, c$ are positive constants.

This estimate and the fact that the metric on the orbits of the torus action is constant (since $T^l$-action is isometric) show that the previous discussion on the role of the arithmetic nature of the deformation parameters applies also in this setting (since the geodesic distance on the orbits is the torus one). But in this framework this is not the end of the story.

Indeed, such an action is not necessarily free (for example it is for NC-torus but not for Connes–Landi spheres), that is, there may exist fixed or rather singular points for the action.
For instance, on a neighborhood of a fixed point, we see that in the integral \( (35) \), we are left with the heat kernel on the diagonal (i.e., the dumping factor of the exponential of geodesic distance disappears). This has certainly some consequences for the power-\( t \) expansion. In view of \( (36) \), note that the lack of freedom for the action can be rephrased in term of non-local integrability of the function \( p \mapsto d_g^{-2}(\alpha_g(p), p) \) for certain \( 0 \neq y \in \mathbb{T}^l \), in the neighborhood of singular points.

At this level of generality, we are only able to treat the free torus-action case, where one can easily derive the asymptotic of \( (35) \). From previous techniques, the estimate \( (36) \) and under the Diophantine assumption \( (16) \), one gets

\[
\text{Tr} \left( L(l) R(r) e^{-t\Delta} \right) = \frac{1}{(4\pi t)^{n/2}} \sum_{q \in \mathbb{Z}} \int_M \mu_g(p) \, l_q(p) \, r_{-q}(p) \, k_t(p, p) + \text{e.s.t.}
\]

\[
= \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} t^k \sum_{q \in \mathbb{Z}} \int_M \mu_g(p) \, l_q(p) \, r_{-q}(p) \, a'_{2k}(p) + \text{e.s.t.},
\]

where \( a'_{2k}(p) \) are the local heat kernel coefficients for the scalar Riemannian Laplacian.

For a non-free torus action, it seems to be difficult to outstrip the qualitative level in general, i.e., to express the asymptotic of \( (35) \) in terms of geometric invariants. We will instead treat the Diophantine assumption \( (16) \), one gets

\[
\text{Tr} \left( L(l) R(r) e^{-t\Delta} \right) = \frac{1}{(4\pi t)^{n/2}} \sum_{q \in \mathbb{Z}} \int_M \mu_g(p) \, l_q(p) \, r_{-q}(p) \, k_t(p, p) + \text{e.s.t.}
\]

\[
= \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} t^k \sum_{q \in \mathbb{Z}} \int_M \mu_g(p) \, l_q(p) \, r_{-q}(p) \, a'_{2k}(p) + \text{e.s.t.},
\]

where \( a'_{2k}(p) \) are the local heat kernel coefficients for the scalar Riemannian Laplacian.

For a non-free torus action, it seems to be difficult to outstrip the qualitative level in general, i.e., to express the asymptotic of \( (35) \) in terms of geometric invariants. We will instead treat the (quite simple but non-trivial) example of the ambient space of Connes–Landi 3-sphere \( S^3_{\theta} \) [15].

One standard way to construct this ambient space goes as follow. One parameterizes \( \mathbb{R}^4 \) (with standard metric) in spherical \( (\phi_1, \phi_2, \psi) \), \( \phi_i \in \mathbb{T}, \psi \in [0, \pi/2] \) (with non-trivial boundary conditions) and radial \( R \in [0, +\infty) \) coordinates. That is to say, in terms of Cartesian coordinates:

\[
x_1 = R \cos \psi \cos \phi_1, \quad x_2 = R \cos \psi \sin \phi_1, \quad x_3 = R \sin \psi \cos \phi_2, \quad x_4 = R \sin \psi \sin \phi_2.
\]

Then, one twists the product via the \( \mathbb{T}^2 \)-action

\[
y.(R, \psi, \phi_1, \phi_2) = (R, \psi, \phi_1 + y_1 \text{ mod } 2\pi, \phi_2 + y_2 \text{ mod } 2\pi), \quad y \in \mathbb{R}^2.
\]

In other words, we are mapping the commutative generators \( u_i = e^{2\pi i \phi_i}, i = 1, 2, \) to those of the noncommutative 2-torus (of course \( S^3_{\theta} \) is obtained by imposing the sphere relation on the generators).

For the question of the asymptotic of \( (35) \), it is more convenient to move to another coordinate system. It allows to identify the ambient space of \( S^3_{\theta} \) with the ambient space of \( T^2_{\theta} \).

This is achieved by setting

\[
R_1 = R \cos \psi, \quad R_2 = R \sin \psi,
\]

which leads to a parameterization of \( \mathbb{R}^4 \) in double polar coordinates \( (R_1, \phi_1; R_2, \phi_2) \). Thus, it corresponds to twist the product of the commutative algebra \( \mathcal{S}(\mathbb{R}^4) \) via the action of \( \mathbb{T}^2 \) given by the two \( SO(2) \)-rotations (which generate the maximal compact Abelian subgroup of the isometry group of \( \mathbb{R}^4 \)). In such a case, the only interesting situation is when \( \Theta = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), with \( \theta \) irrational.

We make the Fourier transform in the two angular directions and leave two radial coordinates \( R_1, R_2 \) as they are. From the expression of the heat kernel of \( \mathbb{R}^4 \) parameterized by \( (R_1 \cos \phi_1, R_1 \sin \phi_1, R_2 \cos \phi_2, R_2 \sin \phi_2) \) and action \( (\phi'_1, \phi'_2).(R, \phi_1, \phi_2) := (R, \phi_1 + \phi'_1 \text{ mod } 2\pi, \phi_2 + \phi'_2 \text{ mod } 2\pi) \), we have

\[
K_t(\alpha_{-\theta q}(p), p) = \frac{1}{(4\pi t)^2} e^{-\left( r_1^2(1-\cos \theta q_2) + r_2^2(1-\cos \theta q_1) \right)^2/2t},
\]

14
The sum and the integral are convergent if $\psi$ control of $\psi$, and we obtain for (35)

$$
\text{Tr}(L(l)R(r)e^{-t\Delta}) = \frac{c}{t^2} \int_{\mathbb{R}^+ \times \mathbb{R}^+} d^2r_1r_2 \sum_{q_1 \neq 0} l_q(r) r^{-q}(r) e^{-\left(r_1^2(1-\cos \theta q_2)+r_2^2(1-\cos \theta q_1)\right)/2t}. 
$$

(38)

The term with $q = 0$ in (38) gives our “standard” result. Other terms give (in the asymptotics) contributions from the “singularities” in $r_1 = 0$, $r_2 = 0$.

There is a relation between oscillations of a smooth function in the angular directions and its behavior near the origin of the coordinate system $r = 0$. Consider a smooth complex function $\psi$ on $\mathbb{R}^2$. Let us restrict it to the unit disc in $\mathbb{R}^2$. We are going to expand $\psi$ is a series of eigenfunctions of the Laplace operator on the disc. Consider a polar coordinate system $(r, \phi)$ centered at the origin. Let $\psi^0$ be a restriction of $\psi$ to the boundary of the disc, and $\psi^0(\phi) = \sum_l \psi^0_l e^{il\phi}$. Then $\psi^0_l e^{il\phi}$ is a (smooth) zero mode of the Laplacian, and $\psi(r, \phi) := \psi(r, \phi) - \psi^0_l(\phi)$ is a smooth function satisfying Dirichlet boundary conditions on the boundary of the disc. $\psi$ can be expanded in a sum of non-zero eigenfunctions of the Laplacian, which are $e^{il\phi}J_l(|r|\lambda)$, where $J_l$ is the Bessel function, and the eigenvalues $\lambda$ are defined by the boundary condition $J_l(\lambda) = 0$. The Taylor expansion for $J_l(\lambda)$ around $\lambda = 0$ starts with the powers $\lambda^{l+k}$, $k \in \mathbb{N}_0$. Since $\psi$ is smooth, its harmonic expansion is rapidly convergent, and we can conclude that each Fourier mode $\psi_l$ behaves near $r = 0$ as

$$
\psi_l(r) = r^l(\psi_l^{(0)} + r^2\psi_l^{(1)} + \ldots).
$$

(39)

Let us see in detail what happens near $r_1 = 0$. For that we have to look at the sum over $q_1 = 0$, $q_2 \neq 0$. The corresponding terms in (38) read

$$
\frac{c}{t^2} \int d^2r_1r_2 \sum_{q_2 \neq 0} f_{q_2}(r) e^{-r_1^2(1-\cos \theta q_2)/2t},
$$

(40)

where $f_{q_2}(r) := l_{0,q_2}(r)r_{0,-q_2}(r)$. Fix $\epsilon_1 > 0$. It is easy to see that the integral $\int_{\epsilon_1}^{\infty} dr_1$ gives an exponentially small term. For $r_1 \leq \epsilon_1$ we use the Taylor expansion of

$$
f_{0,q_2}(r) = f_{0,q_2}^{(0)}(r) + r^2f_{0,q_2}^{(1)}(r) + \ldots
$$

Then (40) takes the form (up to higher order terms):

$$
\frac{c}{t^2} \int dr_2 \sum_{q_2 \neq 0} \left( f_{0,q_2}^{(0)}(r_2) \frac{2t}{1-\cos \theta q_2} + f_{0,q_2}^{(1)}(r_2) \frac{4t^2}{(1-\cos \theta q_2)^2} + \ldots \right)
$$

(41)

The sum and the integral are convergent if $\theta$ satisfies a Diophantine condition. Recall the control of $(1-\cos \theta q_i)^{-1}$ by (14). It is interesting to note that already the $1/t$ term receives a contribution from the singularity.

Next, let $q_1 \neq 0$, $q_2 \neq 0$. The contributions from the integrals $\int_{\epsilon_2}^{\infty} dr_1$ and $\int_{\epsilon_2}^{\infty} dr_2$ are exponentially small. Therefore, we restrict ourselves to the integral $\int_{\epsilon_1}^{\infty} \int_{\epsilon_2}^{\infty} dr_1 dr_2$ where we use again the Taylor expansion in the radii. The Taylor expansion of any smooth function starts with $r_1^{q_1}r_2^{q_2}$ and the Taylor expansion for $l_q \cdot r^{-q}$ starts with $r_1^{2q_1}r_2^{2q_2}$. The corresponding terms contribute to the heat kernel coefficients with

$$
\frac{1}{t^2} \left( \frac{2t}{1-\cos \theta q_2} \right)^{q_1+1} \left( \frac{2t}{1-\cos \theta q_1} \right)^{q_2+1}
$$

(42)
so that the modifications start with $t^2$. One can easily evaluate corresponding terms (which
describe the effect of the singularity at $r_1 = r_2 = 0$).

The asymptotic we obtain strongly depends on the functions $l$ and $r$ through their Taylor
coefficients in a neighborhood of singular points. This is typical for the heat trace asymptotics
if boundaries or singularities are present, cf. [26,40].

### 4 Spectral action for NC-tori

Within noncommutative geometry, the spectral action introduced by Chamseddine–Connes plays
an important role [8]. More precisely, given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an algebra
acting on the Hilbert space $\mathcal{H}$ and $D$ is a Dirac-like operator (see [12,28]), they proposed a
physical action depending only on the spectrum of the covariant Dirac operator

$$D_A := D + A + \epsilon JAJ^{-1}$$

where $A$ is a one-form represented on $\mathcal{H}$, i.e. it is of the form

$$A = \sum_i a_i [D, b_i],$$

where $a_i, b_i \in \mathcal{A}$, $J$ is a real structure on the triple corresponding to charge conjugation and
$\epsilon \in \{1, -1\}$ depending on the dimension of this triple and comes from the commutation relation

$$JD = \epsilon DJ.$$ (45)

This action is

$$S(D_A, \Phi) := \text{Tr}(\Phi(D_A^2/\Lambda^2))$$

where $\Phi$ is any positive function viewed as a cut-off which could be replaced by a step function
up to some mathematical difficulties surmounted in [20]. This means that $\Phi$ counts the spectral
values of $|D_A|$ less than the mass scale $\Lambda$ (note that the resolvent of $D_A$ is compact since, by
assumption, the same is true for $D$).

In [24], the spectral action on NC-tori has been computed only for operators of the form
$D + A$. Thanks to our previous result, we can fill the gap and compute (46) in full generality.

We need to fix notations: Let $A_{\Theta} := C^\infty(T^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m + 1$ (i.e., $m = \lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$), the square integrable sections of the trivial
spin bundle over $T^n$.

Each element of $A_{\Theta}$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$. The Tomita conjugation

$$J_0(a) := a^*$$

satisfies $[J_0, \partial_\mu] = 0$ since $J_0 \partial_\mu U_k = J_0(ik_\mu)U_k = -ik_\mu U_{-k} = \partial_\mu U_{-k} = \partial_\mu J_0 U_k$. Besides, it induces the analogous operator on $\mathcal{H}$,

$$J := J_0 \otimes C_0$$

where $C_0$ is an operator on $\mathbb{C}^{2^m}$. The Dirac operator is defined by

$$D := -i e^\mu_a \delta_\mu \otimes \gamma^a = -i \delta_\mu \otimes \gamma^\mu, \quad \gamma^\mu,$$
explaining the construction of one-forms $A$.

\[ JD = (J_0 \otimes C_0)(-i \partial_\mu \otimes \gamma^\mu) = J_0(-i) \partial_\mu \otimes C_0 \gamma^\mu = i \partial_\mu J_0 \otimes \gamma^\mu C_0, \]

which by (15) is equal to $\epsilon(-i \partial_\mu)J_0 \otimes \gamma^\mu C_0$. Moreover, $C_0^2 = \pm 1_{2m}$ depending on the parity of $m$. Finally, one introduces the chirality (which in the even case is $\chi := id \otimes (-i)^m \gamma^1 \cdots \gamma^n$) and this yields that $(A_\Theta, H, D, J, \chi)$ satisfies all axioms of a spectral triple, see [12, 28].

The unitary elements $u$ of $A_\Theta$ (or of its generated C*-algebra) play an important role since they reflect the inner automorphisms of $A_\Theta$. For instance $U_u := (u \otimes 1_{2m})J(u \otimes 1_{2m})J^{-1}$ is a unitary on $H$ (with $U_u = (u^* \otimes 1_{2m})J(u^* \otimes 1_{2m})J^{-1}$) such that

\[ U_u DU_u^* = D + u \otimes 1_{2m}[D, u^* \otimes 1_{2m}] + \epsilon Ju \otimes 1_{2m}[D, u^* \otimes 1_{2m}]J^{-1} \]

explaining the construction of one-forms $A$ in (14) thus satisfying $U_u A U_u^* = u \otimes 1_{2m} Au^* \otimes 1_{2m}$. These properties follow from the axioms: for all $a, b \in A_\Theta$, $[a \otimes 1_{2m}, Jb \otimes 1_{2m}J^{-1}] = 0$ and $[[D, a \otimes 1_{2m}], Jb \otimes 1_{2m}J^{-1}] = 0$.

In conclusion, the fact that the perturbed Dirac operator must satisfy condition (45) (which is equivalent to $H$ being endowed with a structure of $A_\Theta$-bimodule: for $a, b \in A_\Theta$ and $\psi \in H$, $a, \psi, b := a \otimes 1_{2m} Jb^* \otimes 1_{2m}J^{-1} \psi$), yields the necessity of a symmetrized covariant Dirac operator: $D_A = D + A + \epsilon J A J^{-1}$.

Note that for $a \in A_\Theta$, using $J_0 L(a)J_0^{-1} = R(a^*)$,

\[ \epsilon J(L(a) \otimes \gamma^\mu)J^{-1} = \epsilon R(a^*) \otimes C_0 \gamma^\mu C_0^{-1} = -R(a^*) \otimes \gamma^\mu, \]

and that the representation $L$ and the antirepresentation $R$ are $\mathbb{C}$-linear, commute and satisfy

\[ [\delta_\mu, L(a)] = L(\delta_\mu a), \quad [\delta_\mu, R(a)] = R(\delta_\mu a). \]

Choosing an arbitrary selfadjoint one-form $A$, it can be written as

\[ A = L(-i A_\mu) \otimes \gamma^\mu, \quad A_\mu = -A^*_\mu \in A_\Theta \]

and using (47)

\[ D_A = -i (\delta_\mu + L(A_\mu) - R(A_\mu)) \otimes \gamma^\mu. \]

Defining

\[ A_\mu := L(A_\mu) - R(A_\mu), \]

we get

\[ D_A^2 = -g^{\mu\nu}(\delta_\mu + A_\mu) + \tilde{A}_\mu + \tilde{A}_\nu 1_{2m} - \frac{1}{2} \Omega_{\mu\nu} \otimes \gamma^{\mu\nu} \]

where $\gamma^{\mu\nu} := \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ and

\[ \Omega_{\mu\nu} := [\delta_\mu + A_\mu, \delta_\nu + A_\nu] = L(F_{\mu\nu}) - R(F_{\mu\nu}) \]

where

\[ F_{\mu\nu} := \delta_\mu(A_\nu) - \delta_\nu(A_\mu) + [A_\mu, A_\nu]. \]

Gathering all results,

\[ D_A^2 = -g^{\mu\nu}(\delta_\mu + L(A_\mu) - R(A_\mu))(\delta_\nu + L(A_\nu) - R(A_\nu)) \otimes 1_{2m} - \frac{1}{2} (L(F_{\mu\nu}) - R(F_{\mu\nu})) \otimes \gamma^{\mu\nu} \]

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Now, comparing (50) and (5), we can apply the previous result with the following replacement in (6) and (7)

\[
\begin{align*}
L(\lambda_i) &\to L(A_\mu) \otimes 1_{2^m}, \\
R(\rho_i) &\to R(A_\mu) \otimes 1_{2^m}, \\
L(l_1) &\to -\frac{1}{2} L(F_{\mu\nu}) \otimes \gamma^{\mu\nu}, \\
R(r_1) &\to -\frac{1}{2} R(F_{\mu\nu}) \otimes \gamma^{\mu\nu}, \\
L(l_2) &\to 0, \\
R(r_2) &\to 0.
\end{align*}
\]  

(51)

or in Theorem 2.6

\[
\begin{align*}
\nabla_\mu &\to (\delta_\mu + L(A_\mu) - R(A_\mu)) \otimes 1_{2^m}, \\
E &\to -\frac{1}{2} (L(F_{\mu\nu}) - R(F_{\mu\nu})) \otimes \gamma^{\mu\nu}, \\
\hat{\Omega}_{\mu\nu} &\to (L(F_{\mu\nu}) - R(F_{\mu\nu})) \otimes 1_{2^m}, \\
L(l) &\to 1, \\
R(r) &\to 1.
\end{align*}
\]  

(52)

It is interesting to note that, in the commutative case when \(\Theta = 0\), \(L = R\) thus \(D_A = D\) for any selfadjoint one-form \(A\): the Dirac operator does not fluctuate.

We will derive the spectral action by Laplace transform techniques such as in [34], see [44] for details on Laplace transform (alternatively one can follow [20]). We assume that the function \(\Phi\) has the following property:

\[
\Phi \in C^\infty(\mathbb{R}^+) \text{ is the Laplace transform of } \hat{\psi} \in \mathcal{S}(\mathbb{R}^+) := \{ g \in \mathcal{S}(\mathbb{R}) : g(x) = 0, x \leq 0 \}
\]  

(53)

Thus, any function with this property has necessarily an analytic extension on the right complex plane and is a Laplace transform. Consequently, any \(m\)-differentiable function \(\psi\) such that \(\psi^{(m)} = \Phi\) is the Laplace transform of a function \(\hat{\psi}\) and by differentiation, it satisfies

\[
\Phi(z) = \psi^{(m)}(z) = (-1)^m \int_0^\infty e^{-t z} t^m \hat{\psi}(t) \, dt, \quad \Re z > 0.
\]

One can invoke dominated convergence (see [24]), to obtain:

\[
\begin{align*}
\text{Tr} \left( \Phi(D_A^2/\Lambda^2) \right) &= (-1)^m \int_0^\infty \text{Tr} \left( e^{-t D_A^2/\Lambda^2} \right) t^m \hat{\psi}(t) \, dt \\
&= (-1)^m \int_0^\infty \sum_{k=0}^m \Lambda^{n-2k} \tilde{a}_{2k} t^{m+k-n/2} \hat{\psi}(t) \, dt + O(\Lambda^{n-2(m+1)}) \\
&= \sum_{k=0}^m \Lambda^{n-2k} \Phi_{2k} \tilde{a}_{2k} + O(\Lambda^{n-2(m+1)}),
\end{align*}
\]

where \(\Phi_{2k}\) is defined by

\[
\Phi_{2k} := (-1)^m \int_0^\infty t^{m+k-n/2} \hat{\psi}(t) \, dt, \quad (54)
\]

and

\[
\tilde{a}_{2k} := a_{2k}(1,1; D_A^2) \quad (55)
\]

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Then the following expansion of the spectral action holds:

\[
\Phi_{2k} = \begin{cases} 
\frac{1}{(m-k)!} \int_0^\infty \Phi(t) t^{m-1-k} \, dt, & \text{for } k = 0, \ldots, m-1, \\
(-1)^k \Phi(k-m)(0), & \text{for } k = m, \ldots, n.
\end{cases}
\]  

(56)

For \( n \) odd, the coefficients \( \Phi_{2k} \) have less explicit forms because they involve fractional derivatives of \( \Phi \), so in this case, it is better to stick with definition (54).

Let us summarize:

**Theorem 4.1.** Let \( \mathcal{D}_A = \mathcal{D} + \mathcal{A} + \epsilon \mathcal{J} \mathcal{A} \mathcal{J}^{-1} \) and \( A = L(-iA_\mu) \otimes \gamma^\mu \) a hermitian one-form where \( A_\mu^* = -A_\mu \in \mathcal{A}_\Theta = C^\infty(\mathbb{T}^n_\Theta) \). Let \( \Phi \in C^\infty(\mathbb{R}^+) \) be a positive function satisfying condition (53). Then the following expansion of the spectral action holds:

\[
S(\mathcal{D}_A, \Phi) = \sum_{k=0}^{\lfloor n/2 \rfloor} \Lambda^{n-2k} \Phi_{2k} \tilde{a}_{2k} + O(\Lambda^{n-2(\lfloor n/2 \rfloor + 1)}),
\]

where the \( \Phi_{2k} \) are defined in (54) or (56) depending on the dimension and the \( \tilde{a}_{2k} \) are defined in (57) with replacement (51).

More precisely,

\[
\tilde{a}_0 = 2^{\lfloor n/2 \rfloor} \pi^{n/2}, \\
\tilde{a}_2 = 0, \\
\tilde{a}_4 = (4\pi)^{-n/2} \left[ \frac{1}{2} \mathrm{Sp}(E^2) + \frac{1}{12} \mathrm{Sp}(\hat{\Omega}^{\mu\nu} \hat{\Omega}_{\mu\nu}) \right].
\]

Moreover, all terms in \( \tilde{a}_{2k} \) linear in \( l_1, r_1 \) are zero.

**Proof.** The last assertions follow from \( \tilde{a}_0 = (4\pi)^{-n/2} \mathrm{Sp}(1 \otimes 1_2^m) \), \( \tilde{a}_2 = (4\pi)^{-n/2} \mathrm{Sp}(E) \), \( \tilde{a}_4 = (4\pi)^{-n/2} \left[ \frac{1}{2} \mathrm{Sp}(E^2) + \frac{1}{6} \mathrm{Sp}(g^{\mu\nu} E_{\mu\nu}) + \frac{1}{12} \mathrm{Sp}(\hat{\Omega}^{\mu\nu} \hat{\Omega}_{\mu\nu}) \right] \) and \( \mathrm{Tr}(\gamma^{\mu}) = 0 \), so all linear terms in \( E \) are of zero trace.

Now comes an amazing fact: in four dimensions, the non-standard terms (those with products of traces) simply disappear. Indeed, when \( n = 4 \), with \( g = \text{diag}(1,1,1,1) \) and \( \Theta \) a direct sum of two “Diophantine” matrices, we find, up to negative powers of \( \Lambda \),

\[
S(\mathcal{D}_A, \Phi) = (4\pi)^{-n/2} \left( \Phi_0 \Lambda^4 \mathrm{Sp}(1) - \frac{\Phi(0)}{6} \mathrm{Sp}(L(F^{\mu\nu} F_{\mu\nu}) + R(F^{\mu\nu} F_{\mu\nu}) - 2L(F^{\mu\nu}) R(F_{\mu\nu})) \right)
\]

\[
= 4\pi^{n/2} \left( \Phi_0 \Lambda^4 - \frac{\Phi(0)}{3} \left( \tau(F^{\mu\nu} F_{\mu\nu}) - \tau(F^{\mu\nu}) \tau(F_{\mu\nu}) \right) \right).
\]

But \( F^{\mu\nu} \) is a sum of derivatives plus a commutator, so is of zero trace. Thus, the spectral action has the standard form:

\[
S(\mathcal{D}_A, \Phi) = 4\pi^{n/2} \left( \Phi_0 \Lambda^4 - \frac{\Phi(0)}{3} \tau(F^{\mu\nu} F_{\mu\nu}) \right) + O(\Lambda^{-2}).
\]  

(57)

The only difference appears in the numerical value of the coefficients.

What happens for generic compact toric NC manifolds (we add the assumption that \( M \) carries a spin structure as well)?

Even lacking a trace asymptotic expansion of the semi-group generated by a generalized Laplacian (i.e., an analogue of theorem 2.6), we are able to finish in the 4-d pure Diophantine
case, using the symmetries we have at disposal. First, it should be clear from examples treated in previous section that the supplementary terms coming from the ‘singular points’ actually do not appear. Indeed, they should appear only in the sub-leading order of a given term, but here the only one we have is the ‘Yang–Mills’ one, which is already the last with non-negative power of $\Lambda$ (i.e., such correction terms appears in 4-d with negative powers of $\Lambda$).

In summary, the torus action being free or not has no serious implication for the structure of the spectral action in dimension four or less. One should emphasize that this is no longer true in higher dimensions.

What previous examples also do show is that, up to sub-leading order terms,

$$\text{Tr}(L(l) R(r) e^{-lD^2}) = \int_M \mu_g(p) l_0(p) r_0(p) \tilde{K}_t(p,p),$$

where $\tilde{K}_t$ is the on-diagonal kernel of $e^{-lD^2}$. However, $a_0 = 0$ whenever $a$ is either a commutator or a derivative (with respect to the action $\alpha$) in $C^\infty(M_\Theta)$. Thus the same conclusion holds, namely, that the asymptotics of $\text{Tr}[\Phi((D + A + \epsilon JAJ^{-1})^2/\Lambda^2)]$ can be easily derived from those of $\text{Tr}[\Phi((D + A)^2/\Lambda^2)]$, in 4-d with Diophantine deformation matrix. Note that the latter is easily computable from the classical asymptotics of the kernel $\tilde{K}_t(p,p)$.

To compute the spectral action, it can also be convenient to use the full force of [9].

## 5 NC-QFT: Structure of divergences for a scalar field theory

Let us consider a real scalar field $\phi$ in a four-dimensional NC torus with the classical action

$$S[\phi] = \int d^4x \sqrt{g} \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi \star \phi \star \phi \star \phi \right),$$

(58)

where $\lambda$ is a coupling constant. Here we change to the notations which are more common in quantum field theory and write the star products ([4] or [33]) and partial derivatives. To calculate the effective action in this theory, we split $\phi = \varphi - \delta \phi$ into a background part $\varphi$ and quantum fluctuations $\delta \phi$. Then one expands $S[\phi]$ about the background. The first term, $S[\varphi]$, simply gives the classical approximation to the effective action. The second term, which is proportional to the first derivative of $S[\varphi]$ is canceled by external sources. The quadratic term can be rewritten as

$$S^{(2)}[\varphi, \delta \phi] = \frac{1}{2} \int d^4x \sqrt{g} (\delta \phi) P(\delta \phi),$$

where (cf. [22])

$$P = -g^{\mu \nu} \partial_\mu \partial_\nu + \frac{\lambda}{3} [R(\varphi \star \varphi) + L(\varphi \star \varphi) + L(\varphi) R(\varphi)] + m^2.$$  

(59)

Note that $P > 0$ for $\lambda > 0$, since $-g^{\mu \nu} \partial_\mu \partial_\nu \geq 0$ and $\varphi^* = \varphi$ so that

$$R(\varphi \star \varphi) + L(\varphi \star \varphi) + L(\varphi) R(\varphi) = \frac{1}{2} (L(\varphi) + R(\varphi))^* (L(\varphi) + R(\varphi)) + \frac{1}{2} L(\varphi)^* L(\varphi) + \frac{1}{2} R(\varphi)^* R(\varphi).$$

This operator corresponds to the following choice in [57]

$$\lambda_\mu = \rho_\mu = 0, \quad l_1 = -r_1 = -\frac{\lambda}{6} \varphi \star \varphi - \frac{m^2}{2}, \quad l_2 = -r_2 = \sqrt{\frac{\lambda}{6}} \varphi.$$

Formally, the one-loop effective action reads

$$W = \frac{1}{2} \ln \det P.$$
This expression has to be regularized. We shall use the zeta-function regularization \([19,30]\). The zeta function can then be defined as an \(L^2\)-trace

\[
\zeta(s, P) = \text{Tr}(P^{-s}).
\]

The pole structure of the zeta function is determined by the asymptotic properties of heat trace at \(t \to 0\) (see, e.g. \([26]\)). Due to Theorem 2.6, the zeta function of \(P\) has only simple poles and is regular at \(s = 0\). There is a useful relation

\[
a_k(P) = \text{Res}_{s=n-k} \left( \Gamma(s) \zeta(s, P) \right).
\]

In particular, \(a_n = \zeta(0, P)\).

The regularized effective action is defined as

\[
W(s) = -\frac{1}{2} \mu^2 \Gamma(s) \zeta(s, P).
\]

The regularization is removed in the limit \(s \to 0\). \(\mu\) is a dimensional constant introduced to keep the regularized effective action dimensionless. Because of a pole of the \(\Gamma\)-function, \(W_s\) is divergent at \(s \to 0\). The divergent part of the effective action reads

\[
W_{\text{div}}(s) = -\frac{1}{2} \zeta(0, P) = -\frac{1}{2\pi} a_4(P).
\]

Let us assume that \(\Theta\) satisfies a Diophantine condition. Then, by eq. (28),

\[
a_4(P) = \frac{1}{32\pi^2} \left[ v m^4 - \frac{\lambda m^2}{3} \left( 2 \int d^4 x \sqrt{g} \varphi^2 + v^{-1} \left( \int d^4 x \sqrt{g} \varphi^2 \right)^2 \right) + \frac{\lambda^2}{36} \left( 2 \int d^4 x \sqrt{g} \varphi_4^4 + 3 v^{-1} \left( \int d^4 x \sqrt{g} \varphi^2 \right)^2 + 4 v^{-1} \int d^4 x \sqrt{g} \varphi^4 \int d^4 x \sqrt{g} \varphi^4 \right) \right].
\]  

(60)

Here \(v = \text{vol} \mathbb{T}^4 = (2\pi)^4 \sqrt{g}\), \(\varphi^k\) denotes the \(k\)-th star-power of \(\varphi\). For example, \(\varphi^3 := \varphi \ast \varphi \ast \varphi\).

The theory is called form-renormalizable if all divergences in the effective action can be compensated by redefinitions of couplings in the classical action, i.e., if the divergent part of the effective action repeats the structure of the classical action. The term \(m^4\) in (60) causes no problem as it does not depend on the field (it is said that such terms are removed by a renormalization of the cosmological constant). The terms with \(m^2\varphi^2\) and \(\varphi^4\) can be removed by suitable renormalization of \(m^2\) and \(\lambda\) in (68). The remaining non-local terms cannot be renormalized away. Therefore, the model (58) is not form-renormalizable.

It is instructive to consider an infinite-volume limit of (60). Let us introduce normalized coordinates \(y^a = e^a_\mu x^\mu\), where \(e^a_\mu\) is a constant vierbein, \(e^a_\mu e^\alpha_\nu = g_{\mu\nu}\), and let us assume that \(\varphi(y^a)\) is kept constant as \(v \to \infty\). Then all nonlocal terms vanish in the limit \(v \to \infty\). This is consistent with the conclusion of \([43]\) that the counterterms for \(\phi^4\) on \(\mathbb{R}^4\) are local in the zeta function regularization if \(\Theta\) is nondegenerate. Note that, for a degenerate \(\Theta\), this is no longer true \([22]\). This is because the IR divergence developed in the non-planar sector of the 2-point function (proportional to \(|\Theta\xi|^2\) in momentum space), turns out to be non-locally integrable and thus the associated Green function does not define a temperate distribution.

Of course, one can add several terms to the classical action which repeat the structure of the one-loop divergences. However, such terms change, in general, also the divergent part of the effective action and bring up new structures. Can this process be closed after several steps? Below we show that, at least in one-loop approximation, the answer is affirmative.
First of all, we impose antiperiodic conditions in one of the coordinates, say $x^1$ on the field $\phi$ (and also on the background field $\varphi$, and on quantum fluctuations $\delta \phi$):

$$\phi(x^1, x^2, x^3, x^4) = -\phi(x^1 + 2\pi, x^2, x^3, x^4).$$  \hfill (61)

In the language of NC-QFT, this corresponds to considering a field theory on a finite projective module over the noncommutative torus. This Hermitian module is well defined. One first lifts the torus action on the commutative vector bundle and then defines the module structure on the smooth sections via a star product made out of the lifted torus action (see [16]).

This anti-periodic condition cancels all terms with linear integral of the field, $\int d^4x \phi = 0$. By condition (61) also the Laplacian spectrum is changed, $\Delta \sim p_\mu p^\mu, p \in \mathbb{Z}^n_{(1/2)} := \mathbb{Z}^n + (1/2, 0, 0, 0)$.

However, as $t \to 0$, $\sum_{k \in \mathbb{Z}^{n}_{(1/2)}} e^{-\frac{(k+1/2)^2}{2}t} = \sqrt{\pi t} + \text{e.s.t.}$ and all asymptotic relations derived above remain true after obvious modifications. Since quantum fluctuations are anti-periodic, one has to take in (12) $k \in \mathbb{Z}^{n}_{(1/2)}$. The fields $l$ and $r$ are powers of the background field $\varphi$. Therefore, $r$ and $l$ may be both periodic and anti-periodic.

Thus, $q \in \mathbb{Z}^{n}_{(1/2)} \cup \mathbb{Z}^{n}_{(1/2)}$ in that equation (but only half of the Fourier coefficients may be non-zero in each particular case). The Diophantine condition also should be understood with respect to $k \in \mathbb{Z}^{n}_{(1/2)}$ and $q \in \mathbb{Z}^{n}_{(1/2)}$.

To deal with the remaining non-local terms in (60) we add to the classical action a non-local part

$$\delta S[\phi] = \frac{\tilde{\lambda}}{2\sqrt{g}} \left[ \int d^4x \sqrt{g} \phi^2 \right]^2,$$  \hfill (62)

where $\tilde{\lambda}$ is a new coupling constant. Renormalization of $\tilde{\lambda}$ allows to remove all existing one-loop divergences. It is important to make sure that no new types of divergences appear. Because of (62), the quadratic form of the action receives the contribution

$$\delta S^{(2)}[\varphi, \delta \phi] = \frac{2\tilde{\lambda}}{\sqrt{g}} \left[ \int d^4x \sqrt{g} \varphi \cdot (\delta \phi) \right]^2 + \frac{\tilde{\lambda}}{\sqrt{g}} \int d^4x \sqrt{g} \varphi^2 \int d^4x \sqrt{g} (\delta \phi)^2.$$  \hfill (63)

The second term in (63) is harmless. Due to that, the term $m^2$ in (59) is replaced by a background-dependent but still coordinate-independent mass term:

$$m^2 \to \tilde{m}^2 = m^2 + 2\tilde{\lambda} \int d^4x \varphi^2.$$  \hfill (64)

Let us assume $\tilde{\lambda} > 0$ so that after this replacement the spectrum of $P$ remains positive. Next we substitute $\tilde{m}^2$ for $m^2$ in (63) and take into account that $\int d^4x \varphi = 0$ to see that the divergent part of the effective action does not receive any new structure beyond those which are already present in $S + \delta S$.

The first term on the right hand side of (63) does not contribute to the divergences at all. Let us denote by $\tilde{P}$ the operator (59) with the replacement (64). Then, the operator acting on quantum fluctuations reads

$$P = \tilde{P} + B, \quad B = 4\tilde{\lambda}g^{-1/2} |\varphi\rangle \langle \varphi|$$

where $|\varphi\rangle \langle \varphi|$ is a rank one operator (with suggestive notations). $B$ being proportional to a projector to a one-dimensional subspace of the Hilbert space, it is clear that the insertions of $B$
improve ultraviolet behavior of quantum amplitudes. To make this argument more precise we use the Duhamel principle ([10]):

\[ e^{-t(P+B)} = \sum_{j=0}^{\infty} (-t)^j \beta_j. \]

For \( j \geq 1 \),

\[
\text{Tr} \beta_j = \int_{\Delta_j} d^2s \text{Tr} \left( B e^{-(s_2-s_1)tP} B \ldots e^{-(1-s_j+s_1)tP} \right) = \frac{(4\lambda)^j}{g^{j/2}} \int_{\Delta_j} d^2s \langle \varphi | e^{-(s_2-s_1)tP} | \varphi \rangle \ldots \langle \varphi | e^{-(1-s_j+s_1)tP} | \varphi \rangle.
\]

Since under our assumptions the operator \( \bar{P} \) is positive, \( |\langle \varphi | e^{-sP} | \varphi \rangle| \leq \| \varphi \|_{L^2}^2 \) for \( s \geq 0 \). Thus

\[
|\text{Tr} \beta_j| \leq \frac{|(4\lambda)^j\| \varphi \|_{L^2}^2|^j}{j!g^{j/2}},
\]

and this means that the series expansion

\[
\text{Tr}(e^{-t(P+B)}) - \text{Tr}(e^{-t\bar{P}}) = \sum_{j=1}^{\infty} (-t)^j \text{Tr} \beta_j
\]

is absolutely convergent in the UV regime, and

\[
\text{Tr}(e^{-t(P+B)}) - \text{Tr}(e^{-t\bar{P}}) = O(t),
\]

i.e., the operator \( B \) does not contribute to the coefficient \( a_4(\bar{P}+B) \) or to the one-loop divergences.

The action \( S + \delta S \) with anti-periodic boundary conditions is indeed renormalizable at one loop. The canonical mass dimension of the coupling constant \( \tilde{e} \) is +4. Standard power-counting arguments show that the insertions of the interactions with \( \tilde{e} \) can only improve the ultraviolet behavior of corresponding Feynman diagrams (since in the momentum cut-off regularization each positive power of \( \tilde{e} \) should be accompanied by negative powers of the cut-off momentum). Although this power-counting may break down for noncommutative theories, it is nevertheless natural to conjecture that the theory with \( S + \delta S \) will remain renormalizable also at higher orders of the loop expansion.

The need to add non-standard terms to the NC action in order to achieve renormalizability is not really surprising (see [29]). Note that in the approach of [29], the Diophantine condition does not play any role. The difference probably comes from the fact that we are working with compact noncommutative directions right from the beginning, while in [29] the ”compactification” appears dynamically due to the presence of an oscillator potential. Physical consequences of adding the non-local term (62) to the action are still unclear to us. Since we consider the case of a compact Euclidean manifold (a torus) there are no immediate problems with causality (note that on \( \mathbb{R}^n \), no terms like (62) are required for the one-loop renormalization [43]). Alternatively, noncommutative theories may be viewed as effective low-energy theories, so that renormalizability is not required. In this case one needs a self-consistent subtraction scheme only. An example of such a scheme is given by the large-mass subtraction [5], which also uses the asymptotic expansion of the heat trace.
The technical tools developed in the previous sections are sufficient to analyze other fields (spinors, gauge fields, etc). In this paper, we restricted ourselves to abelian gauge fields. An extension to non-abelian gauge fields can be done rather straightforwardly. Probably, even an extension to superfields can be achieved since the technique used in superspace [3] is similar to the one presented here.

Appendix

A Proof of asymptotic formulae

Here we prove equation (17). Starting from (12)

\[ \text{Tr} \left( L(l) R(r) e^{-t\Delta} \right) = \sqrt{g} (4\pi t)^{-\frac{n}{2}} \sum_{q \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} l_q r_{-q} e^{\frac{|\Theta_q - 2\pi k|^2}{4t}}, \]

we split the sum over \( q \) as

\[ \sum_{q \in \mathbb{Z}^n} = \sum_{q \notin \mathcal{K}} + \sum_{q \in \mathcal{K}}. \quad (65) \]

Consider the sum over \( \mathcal{Z} \) first. Since for \( q \in \mathcal{Z} \) the vector \( \Theta_q/2\pi \) belongs to \( \mathbb{Z}^n \), one can shift \( k \) in the subsequent sum by \( \Theta_q/2\pi \). This yields

\[ \sqrt{g} (4\pi t)^{-\frac{n}{2}} \sum_{q \in \mathcal{Z}} \left( l_q r_{-q} + \sum_{0 \neq k \in \mathbb{Z}^n} l_q r_{-q} e^{-\frac{\pi^2|k|^2}{t}} \right) = (4\pi t)^{-\frac{n}{2}} \sum_{q \in \mathcal{Z}} l_q r_{-q} + \text{e.s.t.} \]

since

\[ \left| \sum_{q \in \mathcal{Z}} l_q r_{-q} \sum_{0 \neq k \in \mathbb{Z}^n} e^{-\frac{\pi^2|k|^2}{t}} \right| \leq 2 e^{-\frac{n^2}{4t}} \sum_{q \in \mathcal{Z}} |l_q r_{-q}| \sum_{k \in \mathbb{N}^n} e^{-\frac{\pi^2(|k|^2 + 2k)}{t}} \leq C e^{-\frac{n^2}{4t}}, \]

because \( \sum_{k \in \mathbb{N}^n} e^{-\pi^2(|k|^2 + 2k)/t} \) is uniformly bounded in \( t \), and \( \{l_q r_{-q}\} \) is a Schwartz sequence.

Next we have to consider the second sum in (65). For each \( q \in \mathcal{K} \) we choose \( k_0(q) \in \mathbb{Z}^n \) which minimizes the distance to \( \Theta_q/(2\pi) \) (if there are several such \( k \), we can take any one of them.) For \( k \neq k_0 \), one can estimate \( |\Theta_q - 2\pi k|^2 \geq 4\pi^2(c_1 + c_2|k - k_0|^2) \). From now on, \( c_i \)'s denote some positive constants. Therefore, the terms with \( k \neq k_0 \) give only exponentially small terms in the second sum of (65). It remains to evaluate the sum

\[ S = \sqrt{g} (4\pi t)^{-n/2} \sum_{q \in \mathcal{K}} l_q r_{-q} e^{-|\Theta_q - 2\pi k_0(q)|^2/4t}. \quad (66) \]

Note that \( \Theta_q - 2\pi k_0(q) \neq 0 \) in (66). We split the sum in (66) into the sums over \( \mathcal{K}^{\text{per}} \) and over \( \mathcal{K}^{\text{inf}} \). The first sum consists of a finite number of exponentially small terms. Consequently, it is exponentially small itself. For \( q \in \mathcal{K}^{\text{inf}} \) we use Diophantine condition (16) to obtain

\[ |S| \leq \sqrt{g} (4\pi t)^{-n/2} \sum_{q \in \mathcal{K}^{\text{inf}}} |l_q r_{-q}| e^{-C^2/4|q|^{2(1+\beta)}} + \text{e.s.t.} \]

Let us again divide the sum into two parts. The first one (which we denote \( S_{\leq} \)) is taken over a cube \( |q_\mu| \leq Q \) except for \( q = 0 \). The rest is denoted \( S_{>}. \)
We estimate $S_>$ first. We would like to show that as $t \to 0$ this sum vanishes faster than $t^{p-\frac{n}{2}}$ for arbitrary positive $p$ (for convenience $\frac{n}{2}$ is extracted explicitly). Since $l$ and $r$ are smooth, their Fourier coefficients are of rapid decay. This means that the partial sums of $|l_q r_q|$ outside of the cube vanish faster than any power of $Q$, i.e., if $Q$ is sufficiently large,

$$\sum_{|q| > Q} |l_q r_q| \leq c_3 Q^{-m}$$

(67)

for any chosen $m$. For reasons which will become clear later, we choose $Q = c_4 t^{-1/(4(1+\beta))}$, $m = 4(1 + \beta)p + 1$. The inequality (67) holds for a sufficiently large $c_4$ and a sufficiently small $t$. Now we estimate

$$|S_>| t^{-p+\frac{n}{2}} \leq c_4 (4\pi)^{-\frac{n}{2}} t^{-p} Q^{-m} = c_5 t^{\frac{1}{1+\beta}}$$

so that the left hand side vanishes for any $p$ as $t \to 0$.

We now turn to $S_\leq$. This is a finite sum with at most $Q^n$ terms. The Fourier coefficients entering this sum are bounded by a constant, $|l_q r_q| \leq c_6$. We also have

$$-\frac{C^2}{4|q|^{2(1+\beta)}t} \leq -\frac{C^2}{4|Q|^{2(1+\beta)}t} = -\frac{c_7}{t^{1/2}}$$

Therefore,

$$|S_\leq| t^{-p+\frac{n}{2}} \leq c_6 (4\pi)^{-\frac{n}{2}} t^{-p} (2Q)^n e^{-c_7/t^{1/2}}.$$

This expression vanishes at $t \to 0$ because of the exponential damping. This completes the proof of (17).

The proof of (19) goes in the same way. The main observation is that the sum over $K \setminus \{0\}$ remains exponentially small, while the “main” terms (produced by $q \in Z$) are zero since the first derivative of the geodesic distance vanishes in the coincidence limit.

**B Beyond the Diophantine condition**

This section is an attempt to understand what happens if $\Theta$ is ‘in between’ rational numbers and “Diophantine numbers”. Consider the simplest case: $\mathbb{T}^2$ with

$$\Theta^{\mu\nu} = \theta \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right),$$

and $g = \text{diag}(1,1)$.

To proceed, we need some results from number theory [7]. Let $f : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ be a continuous function such that $x \to x^2 f(x)$ is non-increasing. Consider the set

$$\mathcal{F}(f) := \{ \theta \in \mathbb{R} : |\theta q - p| < qf(q) \text{ for infinitely many rational numbers } \frac{q}{p} \} \}$$

The elements of $\mathcal{F}(f)$ are termed $f$-approximable. Note that we cannot expect the above estimate to be valid for all rational numbers $\frac{q}{p}$ since for all irrational numbers $\theta$, the set of fractional values of $(\theta q)_{q \geq 1}$ is dense in $[0, 1]$.

Then, there exists an uncountable set of real numbers $\theta/(2\pi)$ which are $f$-approximable but not $cf$-approximable for any $0 < c < 1$, see [7, Exercise 1.5].

Let us choose

$$f(x) = (2\pi x)^{-1} e^{-cx},$$

(68)

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c_8 > 1, and fix a constant c_9 < 1. Let us pick a \( \theta \) which is \( f \)-approximable, but not \( c_9 f \)-approximable. We restrict our attention to the functions \( l \) and \( r \) which do not depend on the second coordinate \( x^2 \) and \( l_q r_q = l_{-q} r_q \). Then, according to (12),

\[
\text{Tr} \left( L(l) R(r)e^{-t\Delta} \right) = \frac{1}{4\pi t} \left( l_0 r_0 + 2 \sum_{q_1 > 1} l_{q_1,0} r_{-q_1,0} \sum_{k_2 \in \mathbb{Z}} e^{-\left(\theta q_1 - 2\pi k_2\right)^2 / 4t} \right)
\]

up to exponentially small terms. The first term in parentheses is our standard result. Let us consider the “correction” term only. Let \( k_2^{(0)}(q_1) \) be an integer which minimizes the distance to \( \theta q_1 / 2\pi \). The sum over \( k_2 \neq k_2^{(0)}(q_1) \) is exponentially small. Consequently, the correction term becomes

\[
T(t) := \frac{1}{2\pi t} \sum_{q_1 > 1} l_{q_1,0} r_{-q_1,0} e^{-\left(\theta q_1 + 2\pi k_2^{(0)}(q_1)\right)^2 / 4t} + \text{e.s.t.} \quad (69)
\]

For a Diophantine \( \theta \), the whole correction term \((69)\) is exponentially small. For \( \theta / (2\pi) \in \mathbb{Q} \) this term in \( \mathcal{O}(1/t) \). Below we work out two explicit examples and show that for the values of \( \theta \) which we consider in this section the correction term is, in general, neither one nor the other.

**Example 1.** Let us take

\[
l_{q_1,0} r_{-q_1,0} = e^{-\alpha |q_1|}.
\]

According to our assumption, \( \theta / (2\pi) \) is not \( c_9 f \)-approximable. Consequently, for all but a finite number of \( q_1 \in \mathbb{N} \), \( |\theta q_1 - 2\pi k_2^{(0)}(q_1)| > c_9 e^{-c_8 q_1} \). Then we can estimate (69) as

\[
T(t) \leq \frac{1}{2\pi t} \sum_{q_1 = 0}^{\infty} e^{-\alpha |q_1|} \exp \left( -\frac{c_9^2 e^{-2c_8 q_1}}{4t} \right) + \text{e.s.t.}
\]

(adding or removing any finite number of terms in this sum does not change it up to e.s.t.). Now we use the Euler–Maclaurin formula to transform this sum to an integral (with exponentially small correction terms):

\[
\frac{1}{2\pi t} \int_0^\infty dq e^{-\alpha q} e^{-\frac{c_9^2 e^{-2c_8 q}}{4t}}
\]

This integral can be easily evaluated:

\[
T(t) \leq (2\pi c_8)^{-1} \Gamma \left( \frac{\alpha}{2c_8} \right) c_9^{-\frac{\alpha}{2c_8}} \cdot t^{-\frac{\alpha}{2c_8}} \cdot t^{-1 + \frac{\alpha}{2c_8}} + \text{e.s.t.} \quad (70)
\]

**Example 2.** In our second example we shall obtain a lower bound on \( T(t) \), though for a different choice of \( l \) and \( r \). According to our assumption, \( \theta / (2\pi) \) is \( f \)-approximable with \( f \) given by (68). Therefore, for an infinite set of \( q_j, j = 1, 2, \ldots \)

\[
0 < |\theta q_j - 2\pi k^{(0)}(q_j)| < e^{-c_8 q_j} \quad (71)
\]

We suppose that the \( q_j \) are ordered, \( q_j < q_{j+1} \). Consequently \( j \leq q_j \) and (71) yields

\[
|\theta q_j - 2\pi k^{(0)}(q_j)| < e^{-c_8 q_j}.
\]

Let us now take

\[
l_{q_j,0} r_{-q_j,0} = \delta_{q_j,0} e^{-\alpha j}.
\]
By repeating the arguments from previous example, one obtains

\[ T(t) \geq (2\pi c_8)^{-1}\Gamma\left(\frac{\alpha_2}{4c_8}\right) \cdot t^{-1+\frac{\alpha_2}{4c_8}} + \text{e.s.t.} \quad (72) \]

The two estimates (70), (72) suggest that for non-Diophantine irrational \( \theta/(2\pi) \) one has to expect power-law corrections to the asymptotics (17). These power-law corrections are unstable in the sense that they crucially depend on the asymptotic behavior of \( l_q r_{-q} \) for large \( q \).

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