Reduction of a symplectic-like Lie algebroid with momentum map and its application to fiberwise linear Poisson structures

Juan Carlos Marrero\textsuperscript{1}, Edith Padrón\textsuperscript{1} and Miguel Rodríguez-Olmos\textsuperscript{2}

\textsuperscript{1} Unidad Asociada ULL-CSIC, Geometría Diferencial y Mecánica Geométrica, Departamento de Matemática Fundamental, Facultad de Matemáticas, Universidad de la Laguna, La Laguna, Tenerife, Canary Islands, Spain
\textsuperscript{2} Departamento de Matemática Aplicada IV, Universidad Politécnica de Cataluña, Barcelona, Spain

E-mail: jcmarrer@ull.es, mepadron@ull.es and miguel.rodriguez.olmos@upc.edu

Received 22 December 2011, in final form 3 March 2012
Published 4 April 2012
Online at stacks.iop.org/JPhysA/45/165201

Abstract
This paper addresses the problem of developing an extension of the Marsden–Weinstein reduction process to symplectic-like Lie algebroids, and in particular to the case of the canonical cover of a fiberwise linear Poisson structure, whose reduction process is the analog to cotangent bundle reduction in the context of Lie algebroids.

Dedicated to the memory of Jerrold E Marsden

PACS numbers: 02.40.Yy, 02.40.Hw, 02.40.Ma
Mathematics Subject Classification: 53D17, 53D20, 37J15, 53D05

1. Introduction

1.1. Preliminaries

A smooth and proper action of a Lie group $G$ on a symplectic manifold $(M, \Omega)$ is called a Hamiltonian if $G$ acts by symplectomorphisms and it admits a coadjoint equivariant momentum map $J : M \to g^*$ satisfying the compatibility condition

$$\Omega(\xi_M, \cdot) = d(J(\cdot), \xi) \quad \forall \xi \in g,$$

where $\xi_M \in \mathfrak{X}(M)$ is the fundamental vector field corresponding to the Lie algebra element $\xi$. The Marsden–Weinstein symplectic reduction process introduced in [20] states that if $\mu$ is a regular value of $J$, and $G_{\mu}$, the stabilizer of $\mu$ for the coadjoint representation, acts freely and properly on $J^{-1}(\mu)$, then the quotient $J^{-1}(\mu)/G_{\mu}$ is a smooth manifold with a naturally induced ‘reduced’ symplectic form $\Omega_{\mu}$. The identity characterizing $\Omega_{\mu}$ is

$$\iota^*_\mu \Omega = \pi^*_\mu \Omega_{\mu}.$$
where $J_\mu : J^{-1}(\mu) \hookrightarrow M$ and $\pi_\mu : J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_\mu$ are the natural inclusion and projection, respectively.

One can look at the particular and important case when our symplectic structure is not on $M$, but on its cotangent bundle $T^*M$, and the action of $G$ is the cotangent lift of an action on $M$. In this case, the lifted action is automatically Hamiltonian with respect to the canonical symplectic form on $T^*M$ given in trivializing local coordinates by $\Omega_c = dx^i \wedge dy^i$. It can be shown that an equivariant momentum map for this action is given by

$$J(\alpha_s)(\xi) = \alpha_s(\xi_M(x)) \quad \text{for all} \quad \alpha_s \in T^*_s M, \xi \in \mathfrak{g}. \quad (1.1)$$

The reduction theory for lifted actions on cotangent bundles was first studied in [26], where only the case of Abelian actions was addressed. The general case was treated in [1] and [12]. The set of results emerging from those and other references is usually known as cotangent bundle reduction. We will expose here the basic lines of this subject and refer to [18, 23] for a more detailed survey. We will assume from now on that the action of $G$ on $M$ is free and proper.

For the case of lifted actions, due to the particularities of the fibered geometry, we can distinguish different situations for the choice of momentum value. These cases are

1. $\mu = 0$, there is a symplectomorphism $\phi_0 : (J^{-1}(0)/G, \Omega_0) \rightarrow (T^*(M/G), \Omega_c)$.
2. $G_\mu = G$, there is a symplectomorphism $\phi_\mu : (J^{-1}(\mu)/G_\mu, \Omega_\mu) \rightarrow (T^*(M/G_\mu), \Omega_c - B_\mu)$.
3. General $\mu$. There is a symplectic embedding $\phi_\mu : (J^{-1}(\mu)/G_\mu, \Omega_\mu) \rightarrow (T^*(M/G_\mu), \Omega_c - B_\mu)$.

In the last two cases, the magnetic term $B_\mu$ is the pullback by the cotangent bundle projection of a closed two-form on $M/G_\mu$. This two-form is obtained, for example, via the choice of a principal connection for the fibration $M \rightarrow M/G_\mu$.

Note also that in the case $G_\mu = G$ (which corresponds to values of $J$ for which their coadjoint orbits are trivial), we have $M/G_\mu = M/G$. Therefore, topologically, all the reduced spaces for momentum values $\mu$ with trivial coadjoint orbits are equivalent to the same space $T^*(M/G)$, and their symplectic forms differ only possibly in the terms $B_\mu$.

This paper develops a generalization of the reduction theory reviewed above for general symplectic manifolds and the particular case of cotangent bundles, to the setup of Lie algebroids. For general Marsden–Weinstein reduction, this happens when one substitutes the tangent bundle $TM$ of a symplectic manifold $M$ by a more general symplectic vector bundle $\mathcal{A}$ over $M$ (a symplectic-like Lie algebroid). For cotangent bundle reduction, the generalization consists of substituting $TM$ by a general Lie algebroid $\mathcal{A}$, and $T(T^*M)$ by a special construction called the canonical cover (or the prolongation of $\mathcal{A}$ over $A^*$ following the terminology of [13]) of $A^*$, which happens to be a symplectic-like Lie algebroid. The latter generalization is a particular case of the former, and both cases coincide with Marsden–Weinstein reduction and cotangent bundle reduction, respectively, when $A$ is just the tangent bundle of $M$. In the remainder of this section, we will give an overview of the new results of this paper.
1.2. Reduction for Lie algebroids

A Lie algebroid is a natural generalization of the tangent bundle to a manifold. It consists of a vector bundle $A \to M$ equipped with a certain geometric structure that allows us to generalize on the one hand, the Lie algebra of vector fields on $M$ to a Lie algebra structure $[\cdot, \cdot]$ on the space of sections of $A$, and on the other, the exterior derivative on differential forms to the derivation $d^A$ of the exterior algebra of multi-sections of $A^*$. The general theory of Lie algebroids is reviewed in section 2. We remark that giving a Lie algebroid structure on the vector bundle $A$ is equivalent to giving a linear Poisson bivector on the dual vector bundle $A^*$ of $A$.

In order to study the reduction process for a Lie algebroid $A \to M$, we introduce in subsection 3.1 the notion of an action by complete lifts on $A$ as an action $\Phi : G \times A \to A$ of a Lie group $G$ by vector bundle automorphisms of a Lie group $G$ on $A$ together with a Lie algebra anti-morphism $\psi : \mathfrak{g} \to \Gamma(A)$ such that the infinitesimal generator of $\xi \in \mathfrak{g}$ with respect to $\Phi$ is just the complete lift of $\psi(\xi)$; or equivalently, an action $\Phi : G \times A^* \to A^*$ on the dual vector bundle $A^*$ by Poisson automorphisms such that the infinitesimal generator of $\xi$ is just the Hamiltonian vector field (with respect to the linear Poisson structure on $A^*$) of the linear function associated with the section $\psi(\xi) \in \Gamma(A)$. The standard example of an action by complete lifts on the Lie algebroid $TM$ is the tangent lift of an action on $M$.

If $\Phi : G \times A \to A$ is a free and proper action of a connected Lie group $G$ on the Lie algebroid $A$ by complete lifts, then in section 3 we construct an affine action $\Phi^T : TG \times A \to A$ of the tangent Lie group $TG$ such that the orbit space $A/TG$ is a Lie algebroid over the reduced manifold $M/G$ corresponding to the induced action $\phi : G \times M \to M$ of $G$ on the base manifold $M$ of the Lie algebroid $A$. Moreover, we prove that the projection $\tilde{\pi} : A \to A/TG$ is a Lie algebroid morphism (see theorem 3.6).

1.3. Reduction for symplectic-like Lie algebroids

The main idea behind the generalization of symplectic reduction to Lie algebroids consists of realizing that a symplectic manifold can be seen as a Lie algebroid endowed with a symplectic vector space structure on each fiber varying smoothly. Under this point of view, the Lie algebroid is nothing but the tangent bundle of the symplectic manifold. Therefore, the setup for this paper will be a symplectic-like Lie algebroid, i.e. a Lie algebroid $A \to M$ equipped with a non-degenerate smooth 2-section $\Omega \in \Gamma(\wedge^2 A^*)$ satisfying $d^A \Omega = 0$ and an action $\Phi : G \times A \to A$ of a Lie group $G$ by complete lifts on $A$.

The main result of subsection 3.2 is to obtain a Lie algebroid version of the Marsden–Weinstein reduction for symplectic manifolds. Firstly, we will consider a momentum map $\mathcal{J} : M \to \mathfrak{g}$ for the action $\phi : G \times M \to M$ which allows us to define an equivariant map $\mathcal{J}^T : A \to \mathfrak{g}^* \times \mathfrak{g}^*$ for the affine action $\Phi^T : TG \times A \to A$. Then, in theorem 3.11, we describe the Lie algebroid analog of the Marsden–Weinstein reduction scheme. It states that under a regularity condition involving a value $\mu \in \mathfrak{g}^*$ of $\mathcal{J}$, the quotient $A^\mu := (\mathcal{J}^T)^{-1}(0, \mu)/TG^{\mu}$ is a symplectic-like Lie algebroid over $\mathcal{J}^{-1}(\mu)/G^{\mu}$. If $\Omega$ is the symplectic-like section on $A$ and $\pi^\mu : (\mathcal{J}^T)^{-1}(0, \mu) \to A^\mu$ and $\tau^\mu : (\mathcal{J}^T)^{-1}(0, \mu) \to A$ are the canonical projection and inclusion, respectively, then the reduced symplectic-like section $\Omega^\mu$ on $A^\mu$ is characterized by the condition

$$\pi^\mu_\ast \Omega^\mu = \tau^\mu_\ast \Omega.$$
It is well known that the base manifold of a symplectic-like Lie algebroid has an induced Poisson structure (see [13, 11, 16]). Then, as a consequence of the reduction theorem for symplectic-like Lie algebroids, it is shown in theorem 3.13 that the Poisson structures on the base manifolds of the original and reduced symplectic-like Lie algebroids are related in a similar way. Namely, if $\{\cdot, \cdot\}$ denotes the Poisson structure on $M$ induced by $\Omega$ and $\{\cdot, \cdot\}_\mu$ is the corresponding structure on $J^{-1}(\mu)/G_\mu$ induced by the reduced symplectic-like section $\Omega_\mu$, then

$$\{\tilde{f}, \tilde{g}\}_\mu \circ \pi_\mu = \{f, g\} \circ \iota_\mu,$$

where $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ and $\iota_\mu : J^{-1}(\mu) \to M$ are the canonical projection and the inclusion, respectively, $\tilde{f}, \tilde{g}$ are functions on $J^{-1}(\mu)/G_\mu$ and $f, g$ are $G$-invariant extensions to $M$ of $\tilde{f} \circ \pi_\mu$ and $\tilde{g} \circ \pi_\mu$, respectively. That is, the reduction obtained on the base manifold of the Lie algebroid is just the Marsden–Ratiu reduction for Poisson manifolds [19].

In [2], a theory of reduction for Courant algebroids is presented. A symplectic-like Lie algebroid $A$ induces a Lie bialgebroid and therefore a Courant algebroid on $A \oplus A^*$ (see [14]). Then, one may apply this Courant reduction process to $A \oplus A^*$ and could recover, after a long computation, some results described in section 3.2. However, we focus our study in the reduction of the particular case of symplectic-like Lie algebroids which allows us to obtain more explicit results on this type of reduction.

### 1.4. Reduction for canonical covers of fiberwise linear Poisson structures

Section 4 studies, within the framework of symplectic-like Lie algebroids, the situation equivalent to cotangent bundle reduction. In this case, the generalization goes as follows. First, the cotangent bundle over a manifold $M$ is replaced by $A^*$, the dual of a Lie algebroid $A \to M$, and then we consider the canonical cover of $A^*$ (also known as the prolongation of $A$ over $A^*$), denoted by $T^A A^*$. This is a natural construction on the dual of a Lie algebroid, which happens to be in a canonical way, a symplectic-like Lie algebroid with the base manifold $A^*$. If $A$ is the tangent bundle of $M$, then $T^A A^*$ is just $T^* M$. If there is a suitable action of a Lie group $G$ by complete lifts on $A$, this action can be further lifted to the canonical cover of $A^*$, in a natural way, and this lifted action happens to be a morphism of symplectic-like Lie algebroids. Furthermore, one may define an equivariant momentum map on $A^*$ (the base space of $T^A A^*$) in a similar way as how the classical momentum map (1.1) on $T^* M$ is introduced. In general, it is not possible to find an equivariant momentum map for a Poisson action (see, for instance, [8]). However, for the case of the Poisson action $\Phi^* : G \times A^* \to A^*$ associated with an action $\Phi : G \times A \to A$ by complete lifts, an equivariant momentum map is described.

Applying the reduction theory of symplectic-like Lie algebroids just developed, we know that the reduction of $T^A A^*$ at any momentum value is again a symplectic-like Lie algebroid. However, as in the situation of cotangent bundle reduction, it is expected that the extra properties of the symplectic-like Lie algebroid, in this case the prolonged fibered structure, will be recovered in the quotient in some way. This is the content of the results of section 5, for which the obtained new results reduce to the standard cotangent bundle reduction theory in the case that the starting Lie algebroid $A \to M$ is the standard Lie algebroid $TM$. In subsection 5.1, it is shown (theorem 5.1) that if $\mu = 0$, there is a symplectic-like Lie algebroid isomorphism between the reduced symplectic-like Lie algebroid $(J^T)^{-1}(0, 0)/TG$ and $T^A A^*_0$. Here $A_0 \to M/G$ is a Lie algebroid with total space $A/TG$. The case $G_\mu = G$ is studied in subsection 5.2. There it is shown that $(J^T)^{-1}(0, \mu)/TG_\mu$ is also isomorphic to $T^A A^*_0$, but in this case this isomorphism is canonical between the symplectic-like Lie algebroids if the canonical symplectic-like section on $T^A A^*_0$ is modified by the addition of a twisting term which consists
of the lift to $T^hA^*_\mu$ of a closed 2-section of $A^*_\mu$. This is the content of theorem 5.2. Finally, subsection 5.3, in its main result, theorem 5.3 shows that for the most general momentum values, $(J^\tau)^{-1}(0,\mu)/TG_\mu$ is canonically embedded as a Lie subalgebroid of $T^h_{A,\mu}A^*_\mu$, where $A_{0,\mu}$ is the Lie algebroid $A/TG_\mu$ over $M/G_\mu$, and the prolongation $T^h_{A,\mu}A^*_\mu$ is equipped with its canonical symplectic-like section minus a magnetic term, just as in the $G_\mu = G$ case.

As far as we know, there is similar research being done independently by Martínez [22]. In addition, in the same direction, some similar results in the more general setting of Lie bialgebroids has been discussed in [25].

2. Lie algebroids

Let $A$ be a vector bundle of rank $n$ over the manifold $M$ of dimension $m$ and let $\tau: A \rightarrow M$ be its bundle projection. Denote by $\Gamma(A)$ the $\mathcal{C}^\infty(M)$-module of sections of $\tau: A \rightarrow M$. A Lie algebroid structure $([-,-],\rho)$ on $A$ is a Lie bracket $[-,-]$ on the space $\Gamma(A)$ and a bundle map $\rho: A \rightarrow TM$, called the anchor map, such that we also denote by $\rho: \Gamma(A) \rightarrow \mathcal{X}(M)$ the homomorphism of $\mathcal{C}^\infty(M)$-modules induced by the anchor map satisfying

$$\langle X, fY \rangle = f\langle X, Y \rangle + \rho(X)(f)Y \quad \text{for } X, Y \in \Gamma(A) \quad \text{and } f \in \mathcal{C}^\infty(M).$$

(2.1)

The triple $(A, [-,-], \rho)$ is called a Lie algebroid over $M$ (see [15]). In such a case, the anchor map $\rho: \Gamma(A) \rightarrow \mathcal{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(A), [-,-])$ and $(\mathcal{X}(M), [-,-])$. If $(A, [-,-], \rho)$ is a Lie algebroid, one can define a cohomology operator, which is called the differential of $A$, $d^A: \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^{k+1} A^*)$ as follows:

$$(d^A\mu)(X_0, \ldots, X_k) = \sum_{i=0}^k (-1)^i \rho(X_i)(\mu(X_0, \ldots, \hat{X}_i, \ldots, X_k))$$

$$+ \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)$$

(2.2)

for $\mu \in \Gamma(\wedge^k A^*)$ and $X_0, \ldots, X_k \in \Gamma(A)$. Moreover, if $X \in \Gamma(A)$ one may introduce, in a natural way, the Lie derivate for multisections of $A^*$ with respect to $X$, as the operator $L^A_X: \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^k A^*)$ given by $L^A_X = i_X \circ d^A + d^A \circ i_X$ (see [15]). The Lie derivative of a multisection $P \in \Gamma(\wedge^k A)$ of $A$ with respect to $X$ is the $k$-section $L^A_X P$ on $A$ characterized by

$$L^A_X P(\alpha_1, \ldots, \alpha_k) = \rho(X)(P(\alpha_1, \ldots, \alpha_k)) - \sum \limits_i P(\alpha_1, \ldots, L^A_X \alpha_i, \ldots, \alpha_k)$$

with $\alpha_i \in \Gamma(A^*)$.

If $(A, [-,-], \rho)$ is a Lie algebroid, we have a natural linear Poisson structure $\Pi_A$ on the dual vector bundle $A^*$ characterized as follows:

$$\{X, Y\}_{\Pi_A} = -\{\hat{X}, \hat{Y}\},$$

$$\{\hat{X}, fH \circ \tau_H\}_{\Pi_A} = -\rho(X)(fH) \circ \tau_H,$$

$$\{fH \circ \tau_H, hH \circ \tau_H\}_{\Pi_A} = 0,$$

(2.3)

for $X, Y \in \Gamma(A)$ and $fH, hH \in \mathcal{C}^\infty(M)$. $\tau_H: A^* \rightarrow M$ being the canonical projection. Here, $\hat{X}$ and $\hat{Y}$ denote the linear functions on $A^*$ induced by the sections $X$ and $Y$, respectively. Conversely, if $A^*$ is endowed with a linear Poisson structure $\Pi_{A^*}$, then it induces a Lie algebroid structure on $A$ characterized by (2.3) (see [4]).

Now, suppose that $(A, [-,-], \rho)$ and $(A', [-,-], \rho')$ are Lie algebroids over $M$ and $M'$, respectively, and that $F: A \rightarrow A'$ is a vector bundle morphism over the map $f: M \rightarrow M'$. Then, $F$ is said to be a Lie algebroid morphism if

$$d^A(F^*\alpha') = F^*(d^A\alpha') \quad \text{for } \alpha' \in \Gamma(\wedge^k (A')^*) \quad \text{and for all } k.$$  

(2.4)
Here, $F^*a'$ denotes the section of the vector bundle $\wedge^k A^* \to M$ defined by
\[
(F^*(a'))_x(a_1, \ldots, a_k) = a'_f(a_1, \ldots, F(a_k))
\] (2.5)
for $x \in M$ and $a_1, \ldots, a_k \in A_x$.

If $F : A \to A'$ is a vector bundle isomorphism over a diffeomorphism $f : M \to M'$, then the dual isomorphism $F^* : (A')^* \to A^*$ over $f^{-1} : M' \to M$ is defined as follows:
\[
[F^*(\alpha'_x)](a_{f^{-1}(x)}) = \alpha'_f(F(a_{f^{-1}(x)}))
\]
for $x' \in M'$, $\alpha'_x \in (A')^*_x$ and $a_{f^{-1}(x)} \in A_{f^{-1}(x)}$.

Moreover, we have that $F$ is a Lie algebroid isomorphism if and only if $F^*$ is a Poisson isomorphism, that is,
\[
\{f' \circ F^*, g' \circ F^*\} \pi_x = \{f', g'\} \pi_{f^{-1}(x)} \circ F^* \text{ for } f', g' \in C^\infty((A')^*)
\]
If $F$ is a Lie algebroid morphism, $f$ is an injective immersion and $F|_{A_x} : A_x \to A'_f(x)$ is injective for all $x \in M$, then $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ is said to be a Lie subalgebroid of $(A', \llbracket \cdot, \cdot \rrbracket, \rho')$.

Let $\tilde{\pi} : A \to A'$ be an epimorphism of vector bundles over $\pi : M \to M'$, i.e. $\pi$ is a submersion and for each $x \in M$, $\tilde{\pi}_x : A_x \to A'_{\pi(x)}$ is an epimorphism of vector spaces. If $X : M \to A$ is a section of $A$, we said that $X$ is $\tilde{\pi}$-projectable if there is $X' \in \Gamma(A')$ such that the following diagram is commutative:

\[
\begin{array}{c}
A \\
\downarrow \pi \\
M
\end{array}
\xymatrix{
A \\
\tilde{\pi} \\
A'}
\begin{array}{c}
X \\
\downarrow \pi \\
X'
\end{array}
\xymatrix{
M \\
\tilde{\pi} \\
M'}
\]

In the next proposition, we will describe the necessary and sufficient conditions to obtain a Lie algebroid structure on $A'$ such that $\tilde{\pi}$ is a Lie algebroid morphism.

**Proposition 2.1** ([10]). Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid and $\tilde{\pi} : A \to A'$ an epimorphism of vector bundles. Then, there is a Lie algebroid structure on $A'$ such that $\tilde{\pi}$ is a Lie algebroid epimorphism if and only if the following conditions hold.

(i) $[X, Y]$ is a $\tilde{\pi}$-projectable section of $A$, for all $X, Y \in \Gamma(A)$ $\tilde{\pi}$-projectable sections of $A$.

(ii) $[X, Y] \in \Gamma(\ker \tilde{\pi})$, for all $X, Y \in \Gamma(A)$ with $X \in \Gamma(A)$ a $\tilde{\pi}$-projectable section of $A$ and $Y \in \Gamma(\ker \tilde{\pi})$.

An equivalent dual version of this result was proved in [3].

Let $X$ be a section of the Lie algebroid $A$. The *vertical lift* of $X$ is the vector field on $A$ given by $X'(a) = X(\tau(a))\beta_a$ for $a \in A$, where $\beta_a : A_{\tau(a)} \to T_aA_{\tau(a)}$ is the canonical isomorphism of vector spaces.

On the other hand, there is a unique vector field $X^c$ on $A$, the *complete lift* of $X$ to $A$, such that $X^c$ is $\tau$-projectable on $\rho(X)$ and $X^c(\tilde{\alpha}) = \mathcal{L}_{X^c}^\alpha$ for all $\alpha \in \Gamma(A^*)$ (see [5, 6]). Here $\tilde{\beta}$, with $\beta \in \Gamma(A^*)$, is the linear function on $A$ induced by $\beta$.

We have that, for all $X, Y \in \Gamma(A)$,
\[
[X^c, Y^c] = [\llbracket X, Y \rrbracket]^c, \quad [X^c, Y^c] = [\llbracket X, Y \rrbracket]^c, \quad [X^c, Y^c] = 0.
\] (2.6)

The flow of $X^c \in \chi(A)$ is related with the Lie algebroid structure of $A$ as follows.

**Proposition 2.2** ([7, 21]). Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid over $M$ and $X$ a section of $A$. Then, for all $P \in \Gamma(\wedge^kA)$ (respectively, $\alpha \in \Gamma(\wedge^kA^*)$)
(i) there exists a local flow \( \varphi_s : A \to A \) which covers smooth maps \( \tilde{\varphi}_s : M \to M \) such that
\[
L^A_\psi P = \frac{d}{ds}((\varphi_s)_*P)_{|s=0} \quad \text{respectively,} \quad L^A_\alpha = \frac{d}{ds}((\varphi_s)^*\alpha)_{|s=0},
\]
(2.7)

(ii) \( L^A_\psi P = 0 \) if and only if \( (\varphi_0)_*P = P \),

(iii) \( L^A_\alpha = 0 \) if and only if \( \tilde{\psi}_s^*\alpha = \alpha \),

(iv) the vector field \( X^c \) on \( A \) is complete if and only if the vector field \( \rho(X) \) on \( M \) is complete.

Here, \( (\varphi_0)_*P \) is the section of \( \wedge^k A \to M \) defined by
\[
((\varphi_0)_*P)(x)(\alpha_1, \ldots, \alpha_k) = P(\tilde{\varphi}_s^{-1}(x))(\varphi_s^*(\alpha_1), \ldots, \varphi_s^*(\alpha_k))
\]
for all \( x \in M \) and \( \alpha_1, \ldots, \alpha_k \in A^*_x \).

If \( X \) is a section of \( A \), we define the complete lift of \( X \) to \( A^* \), as the vector field \( X^{\alpha} \) on \( A^* \) which is \( \tau^* \)-projectable on \( \rho(X) \) and \( X^{\alpha}(\tilde{Y}) = \|X, Y\| \), for all \( Y \in \Gamma(A) \) (see [13]). If \( \{\varphi_i\} \) is the local flow of \( X^\alpha \), then the local flow of \( X^{\alpha} \) is \( \{\varphi^*_i\} \).

If \( (x^i) \) are local coordinates on \( M \) and \( \{e_I\} \) is a local basis of sections of \( A \), then we have the local functions \( \rho_I^I, C_{IJ} \) (the structure functions of \( A \)) on \( M \) which are characterized by
\[
\rho(e_I) = \rho_I^I \frac{\partial}{\partial x^I}, \quad \|e_I, e_J\| = C_{IJ} e_K.
\]

If \( (x^i, y^j) \) (respectively, \( (x^\prime, y^\prime) \)) denote the local coordinates on \( A \) (respectively, \( A^* \)) induced by the local basis \( \{e_I\} \) (respectively, the dual basis \( \{e_I\} \)) then, for a section \( X = X^\prime e_I \) of \( A \), the vector fields \( X^\alpha \), \( X^\alpha \) and \( X^{\alpha} \) are given by
\[
X^v = X^I \frac{\partial}{\partial x^I}, \quad X^c = X^I \rho_I^I \frac{\partial}{\partial x^I} + \left( \rho_I^I \frac{\partial X^I}{\partial x^I} - X^K C_{IK} \right) y^J \frac{\partial}{\partial y^J},
\]
\[
X^{\alpha} = X^I \rho_I^I \frac{\partial}{\partial x^I} - \left( \rho_I^I \frac{\partial X^K}{\partial x^I} + C_{IK} X^J \right) y^J \frac{\partial}{\partial y^J}. \quad (2.8)
\]

3. Reduction of symplectic-like Lie algebroids in the presence of a momentum map

3.1. Actions by complete lifts for Lie algebroids

Let \( (A, \|\cdot\|, \rho) \) be a Lie algebroid over the manifold \( M \) and let \( \tau : A \to M \) be the corresponding vector bundle projection. We consider a left action \( \Phi : G \times A \to A \) by vector bundle automorphisms of a connected Lie group \( G \) on \( A \). Then, \( \Phi \) induces a linear left action \( \Phi^* : G \times A^* \to A^* \) given by
\[
(\Phi^* g)_A x = ((\Phi g(\cdot, x))_{|_{\text{even}}})^* A^* \to A^*_\phi(\xi) \quad \text{for} \ g \in G \ \text{and} \ x \in M,
\]
where \( \phi : G \times M \to M \) is the corresponding action of \( G \) on \( M \).

We note that \( \Phi : G \times A \to A \) is an action by complete lifts if there is a Lie algebra anti-

morphism \( \psi : g \to \Gamma(A) \) such that the infinitesimal generator of \( \xi \in g \) with respect to \( \Phi \) is just the complete lift of \( \psi(\xi) \) to \( A \). Note that this condition implies that \( \xi_M = \rho(\psi(\xi)) \), where \( \xi_M \) is the infinitesimal generator of the action \( \phi : G \times M \to M \) with respect to \( \xi \). Moreover, \( \psi(\xi)^* \) is a morphic vector field in the sense of [17] and therefore, for all \( g \in G \), \( \Phi^*_g : A \to A \) is a Lie algebroid automorphism. Thus, the induced action \( \Phi^*: G \times A \to A \) of \( G \) on \( A^* \) is Poisson with respect to the corresponding linear Poisson structure on \( A^* \). Furthermore, we have the following result.

7
Let (A, [·, ·], ρ) be a Lie algebroid on the manifold M, and \( \Phi : G \times A \to A \) an action by vector bundle automorphisms of a connected Lie group G on A and \( \psi : g \to \Gamma(A) \) a Lie algebra anti-morphism. Then, \( \Phi : G \times A \to A \) is an action of the Lie group G on A by complete lifts with respect to \( \psi \) if and only if \( \Phi^*: G \times A^* \to A^* \) is an action on \( A^* \) by Poisson morphisms such that the infinitesimal generator \( \xi_A^* \) associated with \( \xi \in g \) is just the Hamiltonian vector field corresponding to the linear function \( \hat{\psi}(\xi) \) associated with the section \( \psi(\xi) \in \Gamma(A) \).

**Proof.** Denote by \( \Pi_{\xi^*} \), the linear Poisson structure on \( A^* \). We will prove that the Hamiltonian vector field

\[
H_{(\hat{\psi}(\xi))}^{\Pi_{\xi^*}} \in \mathfrak{X}(A^*)
\]

is just the infinitesimal generator \( \xi_A^* \) of \( \xi \) with respect to the action \( \Phi^* \). In fact, if \( f \in C^\infty(M) \) and \( X \in \Gamma(A) \), using (2.3), we have that

\[
H_{(\hat{\psi}(\xi))}^{\Pi_{\xi^*}} (f \circ \tau_x) = (f \circ \tau_x)(\hat{\psi}(\xi))\Pi_{\xi^*} = \rho(\psi(\xi))(f) \circ \tau_x = (\psi(\xi))^* (f \circ \tau_x)
\]

\[
H_{(\hat{\psi}(\xi))}^{\Pi_{\xi^*}} (\hat{X}) = \{\hat{X}, \hat{\psi}(\xi)\}\Pi_{\xi^*} = \{\hat{\psi}(\xi), X\} = (\psi(\xi))^*(\hat{X})
\]

Here, \( \{\cdot, \cdot\}\Pi_{\xi^*} \) is the Poisson bracket associated with \( A^* \). Thus, \( H_{(\hat{\psi}(\xi))}^{\Pi_{\xi^*}} = (\psi(\xi))^* \).

On the other hand, the flow of \( (\psi(\xi))^* \in \mathfrak{X}(A) \) is \( \{\Phi_{\exp(\{\cdot, \cdot\})} : A \to A\}_{t \in \mathbb{R}} \) if and only if the flow of \( (\psi(\xi))^* \in \mathfrak{X}(A^*) \) is \( \{\Phi_{\exp(-\{\cdot, \cdot\})} : A^* \to A^*\}_{t \in \mathbb{R}} \), and, in consequence, we have that the proposition holds.

**Examples 3.2.**

(i) If \( A = TM \) and \( \Phi = T\phi \) is the tangent lift of the action \( \phi : G \times M \to M \), then it is clear that \( \Phi \) is an action by complete lifts with Lie anti-morphism

\[
\psi : g \to \mathfrak{X}(M), \quad \psi(\xi) = \xi_M^*.
\]

(ii) Let G be Lie group. If \( (g, [\cdot, \cdot]_g) \) is the Lie algebra associated with \( G \), then we have, in a natural way, a Lie algebroid structure on \( g \times TM \to M \), where the Lie bracket and the anchor map are characterized by

\[
[[\xi_1, X_1], [\xi_2, X_2]] = ([\xi_1, \xi_2]_g, [X_1, X_2]), \quad \rho(\xi, X) = X
\]

for all \( \xi_1, \xi_2, \xi, \in g \) and \( X_1, X_2, X \in \mathfrak{X}(M) \).

Now, consider a free and proper action \( \phi : G \times M \to M \) of \( G \) on the manifold \( M \). We denote by \( \Phi : G \times (g \times TM) \to g \times TM \) and \( \psi : g \to \Gamma(g \times TM) \cong C^\infty(M, g) \times \mathfrak{X}(M) \) the action of \( G \) on \( g \times TM \) and the Lie algebra anti-morphism, respectively, given by

\[
\Phi_{t\xi}(v_x) = (Ad^G_t\xi, T_t\phi_x(v_x)) \quad \xi \in g \text{ and } v_x \in T_xM
\]

\[
\psi(\xi) = (-\xi, \xi_M^*)
\]

where \( Ad^G : G \times g \to g \) is the adjoint action of \( G \) on \( g \). Note that if \( ad^G : g \to g \) denotes the infinitesimal generator of the adjoint action for \( \xi \in g \), then the infinitesimal generator of \( \xi \in g \) with respect \( \Phi \) is just \( (ad^G_t, \xi_M^*) \). Thus, \( \Phi \) is a free and proper action by complete lifts with respect to \( \psi \).

**Remark 3.3.** Suppose that we have an action \( \Phi : G \times A \to A \) of a Lie group \( G \) on a Lie algebroid \( A \) by complete lifts with respect to \( \psi : g \to \Gamma(A) \) such that the corresponding action \( \phi : G \times M \to M \) on \( M \) is free. In such a case, for all \( x \in M \), \( \psi_x : g \to A_x \) is injective. Indeed, if \( \xi, \xi' \in g \) satisfy \( \psi_x(\xi) = \psi_x(\xi') \), we have that \( \xi_M(x) = \rho(\psi_x(\xi)) = \rho(\psi_x(\xi')) = \xi_M'(x) \) which implies, using the fact that \( \phi \) is free, that \( \xi = \xi' \).
Next, we will prove that each action of a connected Lie group $G$ over a Lie algebroid $A$ by complete lifts induces an affine action of the Lie group $TG$ over $A$. Previously, we recalled some facts which were related to the Lie group structure of $TG$.

If $G$ is a Lie group then $TG$ is also a Lie group. In fact, if $\cdot : G \times G \to G$ denotes the multiplication of $G$, then the tangent map $T\cdot : TG \times TG \to TG$ of $\cdot$ is such that $(TG, T\cdot)$ is a Lie group. Moreover, $TG$ may be identified with the Cartesian product $G \times g$, where $(g, [\cdot, \cdot]_g)$ is the corresponding Lie algebra associated with $G$. This identification is given by

$$TG \to G \times g, \quad X_g \mapsto (g, (T\cdot |_{g-1})(X_g)) \in G \times g.$$

$l_{g^{-1}} : G \to G$ being the left translation by $g^{-1}$ on $G$. The corresponding Lie group structure on $G \times g$ is defined as follows:

$$(g, \xi) \cdot (g', \xi') = ((g \cdot g'), \xi' + Ad_{g'}^{G}(\xi))$$ (3.1)

and its associated Lie algebra is $g \times g$ with the Lie bracket

$$\{([\xi, \eta]), ([\xi', \eta'])\}_g = ([\xi, \xi'], [\xi, \eta'])_g - ([\xi', \eta'], [\xi', \eta])_g.$$

Here $Ad^G : G \times g \to g$ denotes the adjoint action of $G$.

Moreover, if $\text{CoAd}^G : (G \times g) \times (g^* \times g^*) \to g^* \times g^*$ is the left coadjoint action of $TG \cong G \times g$ on the dual space of the Lie algebra $(g \times g, [\cdot, \cdot]_g)$, then

$$\text{CoAd}^G_{(g, \xi)}(\mu, \mu^\prime) = (\text{CoAd}^G_{g}(\mu' + \text{coad}^G_{g}(\mu)), \text{Coad}^G_{g}(\mu'', x_{g})$$ (3.3)

for $(g, \xi) \in G \times g$ and $(\mu', \mu'') \in g^* \times g^*$, where $\text{CoAd}^G : G \times g^* \to g^*$ is the left coadjoint action associated with $G$ and $\text{coad}^G : g \times g^* \to g^*$ is the corresponding infinitesimal left coadjoint action.

The following proposition describes how $\psi$ works with respect to the action $\Phi$.

**Proposition 3.4.** Let $\Phi : G \times A \to A$ be an action of a connected Lie group $G$ on the Lie algebroid $A$ by complete lifts with respect to $\psi : g \to \Gamma(A)$. Then,

$$\Phi_g(\psi(Ad^G_{g'}(x))(x)) = \psi(\xi)(\phi_g(x))$$ (3.4)

for all $\xi \in g$, $g \in G$ and $x \in M$.

**Proof.** We organize the proof in two steps.

**First step.** Suppose that $G$ is a connected and simply connected Lie group. Consider the map

$$\psi^c : g \times g \to \mathfrak{X}(A), \quad \psi^c(\xi, \eta) = (\psi(\xi))^c + (\psi(\eta))^c.$$

Using (2.6), we may prove easily that $\psi^c$ is an infinitesimal action of $TG$ over $A$, that is, $\psi^c$ is $\mathbb{R}$-linear and

$$\psi^c([([\xi, \eta]), ([\xi', \eta'])_g)]_g = [\psi^c(\xi, \eta), \psi^c(\xi', \eta')].$$

Then, since the vector field $\psi^c(\xi, \eta) \in \mathfrak{X}(A)$ is complete, from Palais theorem (see [24]), there is a unique action $\Phi^T : TG \times A \to A$ from $TG \cong G \times g$ such that for all $(\xi, \eta) \in g \times g$

$$(\xi, \eta)_A = \psi^c(\xi, \eta) = (\psi(\xi))^c + (\psi(\eta))^c.$$

Here $(\xi, \eta)_A \in \mathfrak{X}(A)$ is the infinitesimal generator of $(\xi, \eta)$ with respect to the action $\Phi^T$.

Now, suppose that $g = \exp_{G}(\eta)$. Then, we have that

$$\Phi_g(\psi(Ad^G_{g'}(\xi))(x)) = \Phi^T((g, 0_b), \psi(Ad^G_{g'}(\xi))(x))$$ (3.5)

with $0_b$ being the zero of $g$. In fact, one can prove that

$$\pi : \mathbb{R} \to G \times g, \quad s \mapsto \pi(s) = (\exp_{G}(s\eta), 0_b)$$ (3.6)
is a one-parameter subgroup and \( \frac{d\psi}{dt}|_{t=0} = (\eta, 0_g) \). So, \( \Phi^T((\exp_G(s\eta), 0_g), \psi(Ad_{g^{-1}}^G, \xi)(x)) \) is just the integral curve of \( \Phi^T(\eta, 0_g) \) at the point \( \psi(Ad_{g^{-1}}^G, \xi)(x) \in A_\xi \). In consequence,  
\[
\Phi^T((\exp_G(s\eta), 0_g), \psi(Ad_{g^{-1}}^G, \xi)(x)) = \Phi(\exp_G(s\eta), \psi(Ad_{g^{-1}}^G, \xi)(x)).
\]
In particular, when \( s = 1 \), we obtain (3.5).

Furthermore,  
\[
\psi(Ad_{g^{-1}}^G, \xi)(x) = \Phi^T((e, Ad_{g^{-1}}^G, \xi), 0_g),
\]
where \( 0_g \) is the zero of \( A_\xi \) and \( e \) is the identity element of \( G \).

In fact, in order to prove (3.7), we consider the one-parameter subgroup  
\[
\pi': \mathbb{R} \to G \times g, \quad s \mapsto \pi'(s) = (e, sAd_{g^{-1}}^G, \xi).
\]
Then, \( \frac{d\pi'}{dt}|_{t=0} = (0_g, Ad_{g^{-1}}^G, \xi) \in g \times g \) and, therefore, \( \Phi^T((e, sAd_{g^{-1}}^G, \xi), 0_g) \) is just the integral curve of \( \psi^e((0_g, Ad_{g^{-1}}^G, \xi)) = (\psi(Ad_{g^{-1}}^G, \xi))^e \in \mathcal{X}(A) \) at the point \( 0_g \in A_\xi \), i.e.  
\[
\Phi^T((e, sAd_{g^{-1}}^G, \xi), 0_g) = s\psi(Ad_{g^{-1}}^G, \xi)(x).
\]
In particular, if \( s = 1 \) we obtain (3.7).

Now, from (3.1), (3.5) and (3.7), we deduce that  
\[
\Phi_\xi(\psi(Ad_{g^{-1}}^G, \xi)(x)) = \Phi^T((g, 0_g), \Phi^T((e, Ad_{g^{-1}}^G, \xi), 0_g))
\]
\[
= \Phi^T((g, 0_g) \cdot (e, Ad_{g^{-1}}^G, \xi), 0_g)
\]
\[
= \Phi^T((e, \xi) \cdot (g, 0_g), 0_g) = \Phi^T((e, \xi, \psi(Ad_{g^{-1}}^G, \xi)(x))).
\]
On the other hand, using (3.5) (with \( \xi = 0_g \)), it follows that \( \Phi^T((g, 0_g), 0_g) = 0_{\Phi_\xi(x)} \). In addition, from (3.7) (with \( g = e \)), we obtain that \( \Phi^T((e, \xi, 0_{\Phi_\xi(x)}) = \psi(\xi)(\Phi_\xi(x)). \) This proves (3.4) for \( g = \exp_G(\eta) \). Finally, using that \( G \) is connected, we conclude that (3.4) holds for all \( g \in G \).

Second step. Now, we suppose that \( G \) is a connected Lie group with Lie algebra \( g \). Denote by \( \tilde{G} \) the universal covering of \( G \) and by \( \tilde{\eta} \) its corresponding Lie algebra. Then, the covering projection \( p: \tilde{G} \to G \) is a local isomorphism of Lie groups and the map  
\[
\tilde{\Phi}: \tilde{G} \times A \to A, \quad \tilde{\Phi}(\tilde{g}, a_\xi) = \Phi(p(\tilde{g}), a_\xi)
\]
is an action of \( \tilde{G} \) over \( A \) by complete lifts with respect to the Lie algebra anti-morphism \( \tilde{\psi} = \psi \circ T \rho: \tilde{g} \to \Gamma(A) \). So, for all \( g \in G, x \in M \) and \( \xi \in g \), there are \( \tilde{g} \in \tilde{G} \) and \( \tilde{\xi} \in \tilde{g} \) such that  
\[
p(\tilde{g}) = g \text{ and } (T_2 p)(\tilde{\xi}) = \xi.
\]
Here, \( \tilde{\xi} \) is the identity element of \( \tilde{G} \). Therefore, using the first step  
\[
\tilde{\Phi}_\xi(\psi(Ad_{\tilde{g}^{-1}}^G, \tilde{\xi})(x)) = \psi(\xi)(\Phi_\xi(x)).
\]
Finally, since \( (T_2 p)(Ad_{\tilde{g}^{-1}}^G, \tilde{\xi}) = Ad_{g^{-1}}^G, \xi \), we obtain (3.4). \( \square \)

As we previously claimed, from an action of \( G \) on \( A \) by complete lifts, we can define an affine action of \( TG \) on \( A \) as it is described in the following theorem.

**Theorem 3.5.** Let \( \Phi: G \times A \to A \) be an action of the connected Lie group \( G \) by complete lifts on the Lie algebroid \( A \) with respect to \( \psi: g \to \Gamma(A) \). Then,
\[
\Phi^T: (G \times g) \times A \to A, \quad \Phi^T((g, \xi), a_\xi) = \Phi_\xi(a_\xi + \psi(\xi)(x))
\]
defines an affine action of \( TG \cong G \times g \) over \( A \). Moreover, if \( (\xi, \eta) \in g \times g \), its infinitesimal generator \( (\xi, \eta)_A \) with respect to the action \( \Phi^T \) is  
\[
(\xi, \eta)_A = (\psi(\xi))(\eta) + (\psi(\eta))x.
\]
Proof. Equation (3.4) allows us to prove that $\Phi^T : (G \times \mathfrak{g}) \times A \to A$ is an affine action of $TG \cong G \times \mathfrak{g}$ over $A$. In fact,

$$\Phi^T((g, \xi) \cdot (h, \eta), a) = \Phi^T\left((g \cdot h, \eta + Ad^G_{h^{-1}}(\xi), a)\right) = \Phi_{\Phi(h)}(a) + \Phi_h\left(\psi(\eta + Ad^G_{h^{-1}}(\xi))(x)\right)$$

$$= \Phi_{\Phi(h)}(\Phi_h(a)) + \Phi_h\left(\psi(\eta)(x) + \psi\left(Ad^G_{h^{-1}}(\xi)(x)\right)\right)$$

$$= \Phi_{\Phi(h)}(\Phi_h(a)) + \Phi_h\left(\psi(\eta)(\Phi_h(x))\right)$$

$$= \Phi^T((g, \xi), \Phi^T((h, \eta), a))$$

and

$$\Phi^T((e, 0_g), a) = \Phi_e(a) + \Phi_e(\psi(0_g)(x)) = a.$$ 

Moreover, using the one-parameter subgroups defined in (3.6) and (3.8), one may conclude easily that the infinitesimal generator $(\xi, \eta)_\lambda$ of $\mathfrak{g} \times \mathfrak{g}$ with respect to $\Phi^T$ is

$$(\xi, \eta)_\lambda = (\xi, 0_g)_\lambda + (0_g, \eta)_\lambda = (\psi(\xi))^\lambda + (\psi(\eta))^\lambda.$$

Note that, under the same hypotheses as in theorem 3.5, the action $\Phi : G \times A \to A$ is free and proper if and only if the corresponding action $\phi : G \times M \to M$ on $M$ is free and proper.

Moreover, if $\Phi : G \times A \to A$ is free and proper, then so is $\Phi^T : TG \times A \to A$.

On the other hand, we recall that the space of orbits $N/H$ of a free and proper action of a Lie group $H$ on a manifold $N$ is a quotient differentiable manifold and the canonical projection $\pi : N \to N/H$ is a surjective submersion (see [1]). With this, we prove a preliminary reduction result.

Theorem 3.6. Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid on $M$ and $\Phi : G \times A \to A$ a free and proper action of a connected Lie group $G$ on $A$ by complete lifts with respect to the Lie algebra anti-morphism $\psi : \mathfrak{g} \to \Gamma(A)$. Then, $A/TG$ is a Lie algebroid over $M/G$ and the projection $\tilde{\pi} : A \to A/TG$ is a Lie algebroid epimorphism.

Proof. Since $\Phi : G \times A \to A$ is an action by Lie algebroid automorphisms $A/G$ is a Lie algebroid over $M/G$ with vector bundle projection $\tau : G/A \to M/G$ (see [10, 15]). The space of sections of this vector bundle may be identified with that of $G$-invariant sections $\Gamma(A)^G$ of $A$. Under this identification the bracket and the anchor map of the Lie algebroid structure on $A/G$ is just

$$\llbracket X, Y \rrbracket_{A/G} = \llbracket X, Y \rrbracket, \quad \rho_{A/G}(X(\pi(x))) = T_x\pi(\rho(X(x))$$

for all $X, Y \in \Gamma(A)^G$ and $x \in M$, where $\pi : M \to M/G$ is the quotient projection corresponding to the induced action $\phi : G \times M \to M$.

On the other hand, using that $\phi : G \times M \to M$ is free, we have that $\psi_\xi : \mathfrak{g} \to A_x$ is injective (see remark 3.3). Thus, for all $x \in M$, we have that

$$\dim(\psi_\xi(\xi)/\xi \in \mathfrak{g}) = \dim \mathfrak{g}$$

for all $x \in M$.

Therefore, since $\psi : \mathfrak{g} \to \Gamma(A)$ is a Lie algebra anti-morphism, we deduce that

$$\psi(\mathfrak{g}) = \bigcup_{x \in M} \psi_\xi(\xi)/\xi \in \mathfrak{g}$$

is a Lie subalgebroid of $A$ over $M$. Moreover, from proposition 3.4, we have that the Lie group $G$ acts by Lie algebroid automorphisms on $\psi(\mathfrak{g})$. So, one may induce a Lie algebroid structure on the quotient vector bundle $\psi(\mathfrak{g})/G$ such that $\psi(\mathfrak{g})/G$ is a Lie subalgebroid of $A/G$. Now, we will show that it is also an ideal.
If \( X \in \Gamma(A) \) is \( G \)-invariant then \( X \circ \phi_g = \Phi_g \circ X \) for all \( g \in G \). Thus, the flow \( \Upsilon_t(\xi) : A \to A \) of the vector field \((\psi(\xi))^\ast\) and the flow \( \phi_t(\xi) : M \to M \) of \((\rho \circ \psi)(\xi)\) satisfy the following property:

\[
\Upsilon_t(\xi) \circ X = X \circ \phi_t(\xi) \quad \text{for all } t \in \mathbb{R}.
\]

Equivalently,

\[
\hat{X} \circ \Upsilon_t(\xi)^\ast = \hat{X},
\]

where \( \Upsilon_t(\xi)^\ast \) is the dual morphism of \( \Upsilon_t(\xi) : A \to A \) and \( \hat{X} \) is the linear function associated with the section \( X \).

Therefore, we have that

\[
\frac{d}{dt}_{t=0} (\hat{X} \circ \Upsilon_t(\xi)^\ast) = 0.
\]

Since \( \Upsilon_t(\xi)^\ast \) is the flow of the vector field \( \psi(\xi)^\ast \), the previous equation is equivalent to the relation

\[
\|\psi(\xi), X\| = 0 \quad \text{for all } \xi \in g.
\]

(3.11)

On the other hand, let \( Y \) be a \( G \)-invariant section of \( \psi(g) \) and \( \{\xi_i\} \) a basis of \( g \). Then,

\[
Y = \sum Y^i \psi(\xi_i),
\]

with \( Y^i \) real functions on \( A \). Moreover, using proposition 3.4 and the fact that \( \psi_2 \) is injective, we have that

\[
Y^i \circ \phi_g = (AdG)^g_{\xi_i}(g)Y^i,
\]

(3.12)

where \( AdG_{\xi_i} = (AdG)^g_{\xi_i}(g)\). Hence, from (3.11),

\[
\|X, Y\| = \sum \rho(X)(Y^i)\psi(\xi_i).
\]

Now, using that \( X \) and \( Y \) are \( G \)-invariant sections, we obtain that \( \|X, Y\| \) is a \( G \)-invariant section and, as a consequence, \( \|X, Y\| \) is a \( G \)-invariant section of the vector bundle \( \psi(g) \to M \). Thus, \( \psi(g)/G \) is indeed an ideal of \( A/G \) of constant rank (since the action \( \Phi \) is free) and therefore, the quotient vector bundle \( (A/G)/\psi(g)/G \) admits a Lie algebroid structure over \( M/G \).

Finally, we have that this vector bundle is isomorphic to \( A/TG \) and, thus, a Lie algebroid structure on \( A/TG \) is induced in such a way that this isomorphism is a Lie algebroid isomorphism. In fact, using (3.4), one may prove that \( \Phi \) induces a free and proper action \( \Phi : G \times A/\psi(g) \to A/\psi(g) \) on \( A/\psi(g) \) such that the projection \( \Psi_1 : A \to A/\psi(g) \) is equivariant, with respect to the action \( \Phi^T \) and \( \Phi \). So, \( \Psi_1 \) induces a smooth map \( \Psi_1 : A/TG \to (A/\psi(g))/G \). Moreover, one easily proves that \( \Psi_1 \) is a one-to-one correspondence. On the other hand, the map

\[
\Psi_2 : (A/\psi(g))/G \to (A/G)/(\psi(g)/G), \quad [\tilde{\Psi}_1(a)] \mapsto \tilde{\Psi}_2([a])
\]

is bijective, where \( \tilde{\Psi}_2 : A/G \to (A/G)/(\psi(g)/G) \) is the corresponding smooth map. Consequently, the reduced vector bundle \( A/TG \to M/G \) is isomorphic to the vector bundles

\[
(A/\psi(g))/G \to M/G \quad \text{and} \quad (A/G)/(\psi(g)/G) \to M/G
\]

(3.13)

Note that, using the above isomorphisms, the space of sections of the vector bundle \( A/TG \to M/G \) may be identified with the quotient space

\[
\Gamma(A)/\Gamma((\psi(g))/G).
\]
where $\Gamma(A)^G$ (respectively, $\Gamma(\psi(g))^G$) is the space of $G$-invariant sections on $A$ (respectively, $\psi(g)$). Under this identification the Lie algebroid structure $(\| \cdot \|_{A/TG}, \rho_{A/TG})$ on $A/TG$ is characterized by

$$
\| [X,Y]_{A/TG} \|_{A/TG} = \| X,Y \|_{A/TG},
\rho_{A/TG}(\|X\|_{A/TG}) \circ \pi = T\pi \circ \rho(X) \quad \text{for } X,Y \in \Gamma(A)^G.
$$

(3.14)

This implies that the canonical projection $\tilde{\pi} : A \rightarrow A/TG$ is a Lie algebroid epimorphism. □

**Examples 3.7.**

(i) In the case when $A = TM$ and $\Phi = T\phi$ is the tangent lift of the action $\phi : G \times M \rightarrow M$, the reduced Lie algebroid $TM/TG$ from the previous theorem is isomorphic to $T(M/G)$ with its standard Lie algebroid structure.

(ii) For the case $A = g \times TM$ from (ii) in examples 3.2, we obtain that the reduced Lie algebroid $(g \times TM)/TG$ with respect to the action $\Phi^T : (G \times g) \times (g \times TM) \rightarrow g \times TM$ given by

$$
\Phi^T((g, \tilde{\xi}), (\xi, v_{\xi})) = (Ad^g_{\tilde{\xi}}(\xi - \tilde{\xi}), T_{\xi}g(v_{\xi} + \tilde{\xi}_M(x)))
$$

can be identified with the Atiyah Lie algebroid $TM/G$ induced by the principal bundle $\pi : M \rightarrow M/G$.

We recall the construction of this last Lie algebroid. Firstly, we denote by $\tau : TM \rightarrow M$ the projection of $TM$ on $M$ which is equivariant with respect to the tangent lift action $T\phi : G \times TM \rightarrow TM$ and $\phi : G \times M \rightarrow M$. The sections of the induced vector bundle $\tau/G : TM/G \rightarrow M/G$ can be identified with the $G$-invariant vector fields on $M$. Moreover, the set of $G$-invariant vector fields is closed with respect to the Lie bracket of vector fields. Using this fact, one can define the Lie algebroid structure $(\| \cdot \|_{TM/G}, \rho_{TM/G})$ on $TM/G$ by

$$
\| X,Y \|_{TM/G} = [X,Y], \quad \rho_{TM/G}(X(x)) = T_{\xi}(X(x))
$$

for $X,Y$ $G$-invariant vector fields of $M$ and $x \in M$. The corresponding Lie algebroid is known as *Atiyah Lie algebroid* (see [13]).

Now, we have the following vector bundle epimorphism:

$$
F : (g \times TM)/TG \rightarrow TM/G, \quad F([(\xi, v_{\xi})]_{TG}) = [v_{\xi} + \xi_M]_{G}. \quad (3.15)
$$

Note that $F$ is well defined. Indeed, for all $\xi, \tilde{\xi} \in g, g \in G$ and $v_{\xi} \in T_{\xi}M$, one has that

$$
F\left(Ad^G_{\tilde{\xi}}(\xi - \tilde{\xi}, T_{\xi}g(v_{\xi} + \tilde{\xi}_M(x)))\right)_{TG} = [T_{\xi}g(v_{\xi} + \tilde{\xi}_M(x)) + (Ad^G_{\tilde{\xi}}(\xi - \tilde{\xi}))_{Ad^G_{\tilde{\xi}}}(v_{\xi} + \tilde{\xi}_M(x))]_{G} = [v_{\xi} + \xi_M(x)]_{G}
$$

Moreover, since

$$
[(\xi, v_{\xi})]_{TG} = [(0, v_{\xi} + \xi_M(x))]_{TG}, \quad \text{for all } v_{\xi} \in T_{\xi}M \text{ and } \xi \in g,
$$

we deduce that $F$ is a vector bundle isomorphism.

On the other hand, using (3.16), we obtain that if $\{X_i\}$ is a local basis of $G$-invariant vector fields on $M$, then $\{[(0, X_i)]_{TG}\}$ is a local base of $\Gamma((g \times TM)/TG)$. This fact allows us to prove that $F$ is a Lie algebroid isomorphism.
3.2. Reduction of symplectic Lie algebroids

A Lie algebroid \((A, \llbracket \cdot, \cdot \rrbracket, \rho)\) on the manifold \(M\) is symplectic-like if there is a nondegenerate 2-section \(\Omega \in \Gamma(\wedge^2 A^*)\) on \(A^*\) which is closed, i.e. \(d^A\Omega = 0\). In such a case, for each function \(f : M \to \mathbb{R}\) on \(M\) we have the Hamiltonian section on \(A\) which is characterized by
\[
i_{\rho(\xi)}\Omega = d^A f.
\]

The base space \(M\) of a symplectic-like Lie algebroid \(A\) is a Poisson manifold, where the Poisson bracket on \(M\) is given by
\[
\{f, g\} = \rho(\partial_g^A(f)) \quad f, g \in C^\infty(M)
\]
(see [13, 11, 16]).

Note that if \(f \in C^\infty(M)\), then the Hamiltonian vector field of \(f\) with respect to the Poisson structure on \(M\) is \(\rho(\partial_g^A)\). Thus, the solutions of Hamilton’s equations for \(f\) are the integral curves of the vector field \(\rho(\partial_g^A)\).

Now, in the rest of this section, we suppose that \((A, \llbracket \cdot, \cdot \rrbracket, \rho, \Omega)\) is a symplectic-like Lie algebroid over \(M\), that \(\Phi : G \times A \to A\) is an action of a connected Lie group \(G\) on \(A\) by complete lifts with respect to the Lie algebra anti-morphism \(\psi : g \to \Gamma(A)\) and that \(J : M \to g^*\) is an equivariant smooth map, i.e.
\[
Coad^A_\psi(J(x)) = J(\phi_\xi(x)), \quad \forall x \in M, \quad \forall g \in G.
\]

The action \(\Phi\) is said to be a Hamiltonian action with the momentum map \(J : M \to g^*\) if
\[
\Phi^*_g(\Omega) = \Omega \quad \text{and} \quad i_{\psi(\xi)\Omega} = d^A J_\xi, \quad \text{for all } g \in G \text{ and } \xi \in g.
\]
where \(J_\xi\) is the real function on \(M\) given by
\[
J_\xi(x) = J(x)\xi \quad \text{for } x \in M.
\]

Now, let \(J^T : A \to (g \times g)^* \cong g^* \times g^*\) be the map given by
\[
J^T(a) = ((TJ \circ \rho)(a), J(\tau(a))).
\]

Then, we have

**Lemma 3.8.** The map \(J^T : A \to g^* \times g^*\) is equivariant for the action \(\Phi^T : TG \times A \to A\).

**Proof.** Let \((g, \xi) \in G \times g \cong TG\) and \(a \in A\). Since \(J\) is equivariant, we have that
\[
TJ(\rho(\psi(\xi)(x))) = coad^A_\psi(J(x)) \quad \text{for } \xi \in g \text{ and } x \in M.
\]

Moreover, using that \(\Phi_\xi\) is a Lie algebroid morphism over \(\phi_\xi\) and that \(J\) is equivariant, we obtain
\[
TJ \circ \rho \circ \Phi_\xi = TJ \circ T\phi_\xi \circ \rho = coad^A_\psi \circ TJ \circ \rho.
\]

As a consequence, from (3.3), (3.21), (3.22) and (3.23), we have that
\[
Coad^A_{\Phi_\xi}(J^T(a)) = Coad^A_{\Phi_\xi}(TJ(\rho(a)), J(\tau(a)))
\]
\[
= (coad^A_\psi(TJ \circ \rho(a)) + coad^A_\psi(coad^A_\psi(J(\tau(a))))), coad^A_\psi(J(\tau(a))))
\]
\[
= (J^T(\rho(a)), (J^T(\rho(a)), J(\tau(a))))
\]
\[
= (TJ(\rho(a)) \Phi_\xi(a), (J(\tau(\Phi_\xi(a))))).
\]

Hence, \(J^T\) is equivariant with respect to \(\Phi^T\). \(\square\)
Proposition 3.9. Let \((A, [\cdot, \cdot], \rho, \Omega)\) be a symplectic-like Lie algebroid over \(M\), \(\Phi : G \times A \to A\) a Hamiltonian action of a connected Lie group \(G\) on \(A\) with Lie algebra anti-morphism \(\psi : \mathfrak{g} \to \Gamma(A)\) and equivariant momentum map \(J : M \to \mathfrak{g}^*\). Let \(\mu \in \mathfrak{g}^*\) be a regular value of \(J\) such that \(T_x J \circ \rho : A_x \to T_x \mathfrak{g}^*\) has a constant rank for all \(x \in J^{-1}(\mu)\). Then,

(i) \((JT)^{-1}(0, \mu)\) is a Lie subalgebroid of \(A\) over \(J^{-1}(\mu)\);

(ii) the restriction \(\psi_\mu : \mathfrak{g}^*_\mu \to \Gamma(A)\) of \(\psi\) to the isotropy algebra \(\mathfrak{g}^*_\mu\) of \(\mu\) with respect to the coadjoint action acts on \((JT)^{-1}(0, \mu)\) by constant rank for all \(x \in J^{-1}(\mu)\);

(iii) the isotropy Lie group \(G_\mu\) of \(\mu\) with respect to the coadjoint action acts on \((JT)^{-1}(0, \mu)\) by constant rank for all \(x \in J^{-1}(\mu)\);

(iv) the action of \(G_\mu\) on the Lie subalgebroid \((JT)^{-1}(0, \mu)\) induces an affine action \(\Phi_\mu : TG_\mu \times (JT)^{-1}(0, 0) \to (JT)^{-1}(0, 0)\).

\[\psi\] is a vector subbundle of \(A\) on \(J^{-1}(\mu)\) of rank \(\text{dim} G\), where \(n = \text{rank} A\). On the other hand, \((JT)^{-1}(0, \mu)\) is a Lie algebra morphism; then, it is straightforward to see that the restriction \(\tau_{J(JT)^{-1}(0, \mu)} : (JT)^{-1}(0, \mu) \to J^{-1}(\mu)\) of \(\tau : A \to M\) to \((JT)^{-1}(0, \mu)\) is a Lie subalgebroid of \((A, [\cdot, \cdot], \rho)\).

(ii) If \(x \in J^{-1}(\mu)\) and

\[\xi \in \mathfrak{g}^*_\mu = \{\eta \in \mathfrak{g}/\text{coad}_G^G \mu = 0\},\]

then, since \(J\) is an equivariant map, we have that

\[T_x J(\rho(\psi(\xi)(x))) = T_x J(\xi_M(x)) = \text{coad}_G^G (\mu) = 0.\]

Thus, the restriction of \(\psi(\xi)\) to \(J^{-1}(\mu)\) is a section of the vector bundle \((JT)^{-1}(0, \mu) \to J^{-1}(\mu)\).

(iii) Using the equivariance of \(J\) and the fact that \(\Phi_\mu\) is a Lie algebroid automorphism, for any \(g \in G\) we have that the action \(\Phi : G \times A \to A\) induces an action \(\Phi_\mu\) of \(G_\mu\) on \((JT)^{-1}(0, \mu)\). In fact, if \(g \in G_\mu, a \in (JT)^{-1}(0, \mu)\) and \(x \in J^{-1}(\mu)\), then

\[J(\Phi_g(x)) = \text{coad}_G^G (J(x)) = \text{coad}_G^G (\mu) = \mu\]

and

\[(T \circ \rho) (\Phi(a)) = T (J \circ \psi_\mu)(\rho(a)) = T (\text{coad}_G^G \circ J)(\rho(a)) = 0.\]

Moreover, from (ii), we have that \(\Phi_\mu\) is an action by complete lifts with respect to the Lie algebra anti-morphism \(\psi_\mu\).

(iv) It is a direct consequence of (iii) and theorem 3.5.

Let \(\mu \in \mathfrak{g}^*\) be a regular value of \(J : M \to \mathfrak{g}^*\) such that \(T_x J \circ \rho : A_x \to T_x \mathfrak{g}^*\) has a constant rank for all \(x \in J^{-1}(\mu)\). Suppose that the corresponding action \(\Phi_\mu : G_\mu \times J^{-1}(\mu) \to J^{-1}(\mu)\) is free and proper. Then, using theorem 3.6 and proposition 3.9, we obtain that \(A_\mu = (JT)^{-1}(0, \mu)/G_\mu\) is a Lie algebroid over \(J^{-1}(\mu)/G_\mu\). In the following result, we will prove that \(A_\mu\) is a symplectic-like Lie algebroid. For this purpose, we will need the following properties.
Lemma 3.10. Let \((A, \ll, \cdot, \rho, \Omega)\) be a symplectic-like Lie algebroid over the manifold \(M\) and \(\Phi : G \times A \to A\) a Hamiltonian action of a connected Lie group \(G\) on \(A\) with an equivariant momentum map \(J : M \to \mathfrak{g}^*\) and associated Lie algebra anti-morphism \(\psi : \mathfrak{g} \to \Gamma(A)\). If \(\mu \in \mathfrak{g}^*\), then for any \(x \in M\),

\[(i) (\psi \rho)_x(g_x) = \psi_x(g) \cap \ker(T_xJ \circ \rho_x);
(ii) \ker(T_xJ \circ \rho_x) = (\psi_x(g)) = [\{a_x \in A_x^\Omega, b_x \} = 0, \forall b_x \in \psi_x(g)].\]

Proof.

(i) It is an immediate consequence of the fact that \(J\) is equivariant.

(ii) If \(a_x \in A_x\), using (2.2) and (3.19), we deduce that

\[\Omega(a_x, \psi(\xi)(x)) = -(i_{\psi(\xi)}\Omega)(a_x) = -(d^4 J_x)(\xi)(a_x) = -(T x J(\rho_x(a_x)))(\xi)\]

for all \(\xi \in \mathfrak{g}\). Thus, one concludes immediately (ii) from this relation. \(\square\)

The following result may be seen as the analog of Marsden–Weinstein reduction theorem for symplectic-like Lie algebroids.

Theorem 3.11. Reduction theorem of symplectic-like Lie algebroids. Let \((A, \ll, \cdot, \rho, \Omega)\) be a symplectic-like Lie algebroid and \(\Phi : G \times A \to A\) a Hamiltonian action of a connected Lie group \(G\) on \(A\) with an equivariant momentum map \(J : M \to \mathfrak{g}^*\) and associated Lie algebra anti-morphism \(\psi : \mathfrak{g} \to \Gamma(A)\). Suppose that \(\mu \in \mathfrak{g}^*\) is a regular value of \(J : M \to \mathfrak{g}^*\) such that \(T x J \circ \rho_x : A_x \to T_x \mathfrak{g}^*\) has a constant rank for all \(x \in J^{-1}(\mu)\) and the restricted action \(\varphi / \mu : G \mu \times J^{-1}(\mu) \to J^{-1}(\mu)\) is free and proper. Then, \(A_\mu = (J^T)^{-1}(0, \mu)/TG_\mu\) is a symplectic-like Lie algebroid over \(J^{-1}(\mu)/G_\mu\) with a symplectic-like section \(\Omega_\mu\) characterized by the condition

\[\Xi_\mu = T_\mu \Omega,
\]

where \(\Xi_\mu : (J^T)^{-1}(0, \mu) \to A_\mu\) is the canonical projection and \(\varphi / \mu : (J^T)^{-1}(0, \mu) \to A\) is the canonical inclusion.

Proof. Since \(A_\mu\) is a Lie algebroid over \(J^{-1}(\mu)/G_\mu\), one needs to prove that this algebroid is symplectic-like.

Let \(\Omega_\mu = \Xi_\mu \Omega\) be the 2-cocycle on the Lie subalgebroid \((J^T)^{-1}(0, \mu) \to J^{-1}(\mu)\) induced by \(\Omega\). We will prove that \(\Omega_\mu\) induces a symplectic-like 2-section \(\Xi_\mu\) over \(A_\mu\).

Suppose that \(X_\mu, Y_\mu \in \Gamma(A_\mu)\). Then, we may choose two sections \(\tilde{X}_\mu, \tilde{Y}_\mu \in \Gamma((J^T)^{-1}(0, \mu))\) such that the following diagram is commutative:

\[\begin{array}{ccc}
J^{-1}(\mu) & \xrightarrow{\tilde{X}_\mu, \tilde{Y}_\mu} & (J^T)^{-1}(0, \mu) \\
\pi_\mu \downarrow & & \downarrow \pi_\mu \\
J^{-1}(\mu)/G_\mu & \xrightarrow{X_\mu, Y_\mu} & A_\mu
\end{array}\]

We will see that \(\Omega_\mu(X_\mu, Y_\mu)\) is a \(G_\mu\)-invariant function (or, equivalently, a \(\pi_\mu\)-basic function).

Denote by \((\ll, \cdot, (J^T)^{-1}(0, \mu), \rho_{(J^T)^{-1}(0, \mu)})\) the Lie algebroid structure on \((J^T)^{-1}(0, \mu) \to J^{-1}(\mu)\).

As we know, the vertical bundle of \(\pi_\mu\) is generated by the vector fields on \(J^{-1}(\mu)\) of the form \(\rho_{(J^T)^{-1}(0, \mu)}(\psi_x(\xi))\), with \(\xi \in \mathfrak{g}_\mu\).
Now, we have that
\[
(\rho(\mathcal{J}^{-1}(0,\mu))(\psi_\mu(\xi)))(\bar{\Omega}_\mu(\tilde{X}_\mu, \tilde{Y}_\mu)) = \left(\mathcal{L}^{(\mathcal{J}^{-1}(0,\mu))}_{\psi_\mu(\xi)}\right)(\bar{\Omega}_\mu(\tilde{X}_\mu, \tilde{Y}_\mu)) \\
+ \bar{\Omega}_\mu(\mathcal{L}^{(\mathcal{J}^{-1}(0,\mu))}_{\psi_\mu(\xi)}(\tilde{X}_\mu, \tilde{Y}_\mu)) + \bar{\Omega}_\mu(\mathcal{L}^{(\mathcal{J}^{-1}(0,\mu))}_{\psi_\mu(\xi)}(\tilde{X}_\mu, \tilde{Y}_\mu)).
\]

Thus, for all \(X_\mu, Y_\mu \in \Gamma(A_\mu)\) there is a function \(\Omega_\mu(X_\mu, Y_\mu)\) on \(J^{-1}(\mu)/G_\mu\) such that
\[
\Omega_\mu(X_\mu, Y_\mu) \circ \pi_\mu = \pi_\mu(\tilde{X}_\mu, \tilde{Y}_\mu).
\]

Note that the function \(\Omega_\mu(X_\mu, Y_\mu)\) does not depend on the chosen sections \(\tilde{X}_\mu, \tilde{Y}_\mu \in \Gamma((\mathcal{J}^{-1}(0,\mu))\) which project on \(\tilde{X}_\mu\) and \(\tilde{Y}_\mu\), respectively. In fact, from lemma 3.10, we have that
\[
\ker(\bar{\Omega}_\mu(x)) = (\ker \bar{\pi}_\mu)_{(\mathcal{J}^{-1}(0,\mu))} = (\psi_\mu)_x(\mathfrak{g}_\mu) \text{ for all } x \in J^{-1}(\mu).
\]

Therefore, the map
\[
\Gamma(A_\mu) \times \Gamma(A_\mu) \to C^\infty(J^{-1}(\mu)/G_\mu), \quad (X_\mu, Y_\mu) \mapsto \Omega_\mu(X_\mu, Y_\mu)
\]
defines a section \(\Omega_\mu\) of the vector bundle \(\Lambda^2 A^*_\mu \to J^{-1}(\mu)/G_\mu\) and it is clear that
\[
\bar{\pi}^* \Omega_\mu = \bar{\pi}^* \Omega = \bar{\Omega}_\mu.
\]
This implies that
\[
\bar{\pi}^*(d^A_{\mathfrak{h}_\mu} \Omega_\mu) = d^{(\mathcal{J}^{-1}(0,\mu))}(\bar{\pi}^* \Omega_\mu) = d^{(\mathcal{J}^{-1}(0,\mu))} \bar{\pi}^* \Omega = \bar{\pi}^*(d^A \Omega) = 0
\]
and since \(\bar{\pi}_\mu : (\mathcal{J}^{-1}(0, \mu) \to A_\mu = (\mathcal{J}^{-1}(0, \mu))/TG_\mu\) is an epimorphism of vector bundles, we conclude that \(d^A \Omega_\mu = 0\).

Finally, we will prove that \(\Omega_\mu\) is non-degenerate. In fact, if \(x \in J^{-1}(\mu)\) and \(\vec{u}_x \in \ker(T_xJ \circ \rho_\mu)\) is such that
\[
\Omega_\mu(\pi_\mu(x))(\bar{\pi}_\mu(\vec{u}_x), u_{\pi_\mu(x)}) = 0, \quad \forall u_{\pi_\mu(x)} \in (A_\mu)_{\pi_\mu(x)},
\]
then
\[
\bar{\Omega}(x)(\vec{u}_x, \vec{u}_x) = 0 \quad \text{for all } \vec{u}_x \in \ker(T_xJ \circ \rho_\mu).
\]
Consequently (see lemma 3.10)
\[
\vec{u}_x \in (\ker(T_xJ \circ \rho_\mu))^\perp = \psi_\mu(\mathfrak{g})
\]
and thus,
\[
\vec{u}_x \in (\psi_\mu)_x(\mathfrak{g}_\mu)
\]
which implies that \(\bar{\pi}_\mu(\vec{u}_x) = 0\). □

**Remark 3.12.** In the particular case when \(M\) is a symplectic manifold and \(A\) is the standard symplectic-like Lie algebroid \(TM \to M\), then theorem 3.11 reproduces the classical Marsden–Weinstein reduction result for the symplectic manifold \(M\).
Theorem 3.13. Under the hypotheses of theorem 3.11, if \( \{ \cdot, \cdot \}_\mu \) is the Poisson bracket on \( J^{-1}(\mu)/G_\mu \), we have that

\[
[f, g]_{\mu} \circ \pi_\mu = [f, g] \circ i_\mu
\]  
(3.24)

for \( f, g \in C^\infty(J^{-1}(\mu)/G_\mu) \), where \( i_\mu : J^{-1}(\mu) \to M \) is the canonical inclusion and \( f, g \in C^\infty(M) \) are arbitrary \( G \)-invariant extensions of \( f \circ \pi_\mu \) and \( g \circ \pi_\mu \), respectively.

Proof. From theorem 3.11, we deduce that \( (\bar{A}_\mu, [], , \rho_\mu, \Omega_\mu) \) is a symplectic-like Lie algebroid on \( J^{-1}(\mu)/G_\mu \). Then, one can define a Poisson structure on \( J^{-1}(\mu)/G_\mu \) as in (3.17).

We will prove that the associated Poisson bracket \( \{ \cdot, \cdot \}_\mu \) satisfies (3.24).

If \( f, g : J^{-1}(\mu)/G_\mu \to \mathbb{R} \) are two real functions on \( J^{-1}(\mu)/G_\mu \) and \( f : M \to \mathbb{R}, g : M \to \mathbb{R} \) are arbitrary \( G \)-invariant extensions of \( f \circ \pi_\mu \) and \( g \circ \pi_\mu \), respectively, for any \( \xi \in g \) satisfying \( \rho(\psi(\xi))(f) = \rho(\psi(\xi))(g) = 0 \), we have that

\[
\Omega(f(\psi(\xi))) = \Omega(g(\psi(\xi))) = 0.
\]

or, equivalently,

\[
\Omega(H_\mu^f(\psi(\xi))) = \Omega(H_\mu^g(\psi(\xi))) = 0.
\]

Therefore, \( H_\mu^f(x), H_\mu^g(x) \in \psi_\mu(\xi)^{-1} = (J^T)^{-1}(0, \mu) \) for all \( x \in J^{-1}(\mu) \).

On the other hand, if \( x \in J^{-1}(\mu) \) and \( a_x \in (J^T)^{-1}(0, \mu) \), then, using theorem 3.11, the fact that \( (\bar{\pi}_\mu, \pi_\mu) \) is a Lie algebroid epimorphism and that \( (J^T)^{-1}(0, \mu) \to J^{-1}(\mu) \) is a Lie subalgebroid of \( A \), one deduces that

\[
\Omega_\mu(\bar{\pi}_\mu(H^f_\mu(x)), \bar{\pi}_\mu(a_x)) = \Omega(H^f_\mu(x), a_x) = (d^A f)(a_x)
\]

\[
= (d^{J^T} (f \circ \pi_\mu))(a_x) = (d^A f)(\bar{\pi}_\mu(a_x))
\]

\[
= \Omega_\mu(H^f_\mu(\pi_\mu(x)), \pi_\mu(a_x)).
\]

So since \( \Omega_\mu \) is non-degenerate,

\[
\bar{\pi}_\mu(H^f_\mu(x)) = H^f_\mu(\pi_\mu(x)).
\]

Thus, using again that \( (\bar{\pi}_\mu, \pi_\mu) \) is an epimorphism of Lie algebroids, we conclude that

\[
T_x \bar{\pi}_\mu(\rho_{J^T}^{-1}(0, \mu)(H^f_\mu(x))) = \rho_{A_x}(H^f_\mu(\pi_\mu(x))), \quad \forall x \in J^{-1}(\mu).
\]

Therefore, if \( x \in J^{-1}(\mu) \)

\[
[\bar{f}, \bar{g}]_\mu(\pi_\mu(x)) = -(\rho_{A_x}(H^f_\mu(\pi_\mu(x))))\bar{g}
\]

\[
= -(T_x \pi_\mu(\rho_{J^T}^{-1}(0, \mu)(H^f_\mu(x))))\bar{g}
\]

\[
= -(\rho_{J^T}^{-1}(0, \mu)(H^f_\mu(x)))(\bar{g} \circ \pi_\mu)
\]

\[
= -(\rho(H^f_\mu(x)))\bar{g} = \{f, g\}(x).
\]

\[\square\]
The symplectic-like Lie algebroid structure on canonical cover of the fiberwise linear Poisson structure of $A$.

In fact, it is a particular case of a type of symplectic-like Lie algebroids, the prolongation of a Lie algebroid $A$ on its dual $A^*$ in the terminology of [13], which may be considered as the canonical cover of the fiberwise linear Poisson structure of $A^*$.

In this section, we will describe this last canonical cover and will prove that if a Lie group $G$ acts freely and properly on $A$ by complete lifts, then one may introduce an equivariant momentum map with respect to a certain canonical action by complete lifts on its canonical cover.

The vector bundle $T^*A^* \to A^*$. Let $(A, \llbracket \cdot, \cdot \rrbracket, \rho)$ be a Lie algebroid of rank $n$ over a manifold $M$ of dimension $m$ with $\tau: A \to M$ the associated vector bundle projection and let $F : M' \to M$ be a smooth map from a manifold $M'$ to $M$. If $x' \in M'$, we consider the vector subspace

$$(T^*A^*)_x' = \{ (a, v) \in A_{F(x')}(\mathbb{T}) \times T_{x'}M' \mid \rho(a) = T_{x'}F(v) \}$$

of $A_{F(x')}(\mathbb{T}) \times T_{x'}M'$ of dimension $n + m' - \text{dim}(\rho(A_{F(x')})) + \text{dim}(T_{x'}F(T_{x'}M'))$, where $m'$ is the dimension of $M'$. If we suppose that $\text{dim}(\rho(A_{F(x')})) + \text{dim}(T_{x'}F(T_{x'}M'))$ is constant over $F(M')$ (for instance, if $F$ is a submersion), then $T^*A'M'$ is a vector bundle over $M'$ which is called the prolongation of $A$ over $F$ (see [9, 13]). In this case, a section $\mathcal{X}$ of $T^*A'M' \to M'$ is said to be projectable if there exist a section $X$ of $A$ and a vector field $V$ on $M'$, $F$-projectable over $\rho(X)$, satisfying

$$(\mathcal{X})(m') = (X(F(m')) + V(m'))$$

for all $m' \in M'$.

The section $\mathcal{Z}$ will be denoted by $\mathcal{Z} = (X, V)$. Note that one may choose a local basis $\{Z_I\}$ of $\Gamma(T^*A^*)$ such that, for all $I$, $Z_I$ is a projectable section.

On the other hand, a section $\mathcal{Y}$ of the dual vector bundle $(T^*A^*)^* \to M'$ is said to be projectable if there exist a section $\alpha$ of $A^*$ and a 1-form $\beta$ of $M'$ such that

$$(\mathcal{Y})(X, V) = \alpha(X) \circ F + \beta(V),$$

for $(X, V)$ a projectable section of $T^*A'M'$.

In such a case, we will use $\mathcal{Y} = (\alpha, \beta)$. Note that one may choose a local basis $\{Z^I\}$ of $\Gamma((T^*A^*)^*)$ such that, for all $I$, $Z^I$ is a projectable section.

A particular case is when the function $F$ is the dual bundle projection $\tau_x : A^* \to M$ of the Lie algebroid $A$. Then, the prolongation $\tau_{T^*A^*} : T^*A^* \to A^*$ of $A$ over $\tau_x : A^* \to M$ is called the $A$-tangent bundle of $A^*$. In such a case, $\text{dim}(\rho(A_x) + T_{\omega_x}(T_{0_x}A^*))$ is just the dimension of $M$ for all $x_0 \in A^*_0$, and the rank of $T^*A^*$ is $2n$.

A basis of local sections of the vector bundle $\tau_{T^*A^*} : T^*A^* \to A^*$ is defined as follows. If $(x')$ are local coordinates on an open subset $U$ of $M$, $\{e_I\}$ is a basis of sections of the vector bundle $\tau^{-1}(U) \to U$ and $(x', y_I)$ are the corresponding local coordinates on $A^*$, then $\{X_I, Y^I\}$ is a local basis of $\Gamma(T^*A^*)$, where $X_I$ and $Y^I$ are the projectable sections defined by

$$X_I = \left( e_I, \frac{\partial}{\partial x'} \right), \quad Y^I = \left( 0, \frac{\partial}{\partial y_I} \right).$$

The symplectic-like Lie algebroid structure on $T^*A^* \to A^*$. The vector bundle $T^*A^*$ admits a Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket, \rho_{T^*A^*})$ which is characterized by the following conditions:

$$\llbracket (X, V), (X', V') \rrbracket_{T^*A^*} = \llbracket [X, Y], [V, V'] \rrbracket, \quad \rho_{T^*A^*}(X, V) = V$$

for $(X, V), (X', V')$ projectable sections of $T^*A^*$. 


If $dT^A\Lambda^+$ is the differential associated with this Lie algebroid structure, then

$$
\begin{align*}
\frac{df}{dT^A\Lambda^+}(X_1, V_1) &= df(V_1) \\
\frac{df}{dT^A\Lambda^+}(\alpha, \beta)((X_1, V_1), (X_2, V_2)) &= d^A\alpha(X_1, X_2) \circ \tau_\alpha + d^B(V_1, V_2),
\end{align*}
$$

(4.2)

where $f: A^* \to \mathbb{R}$ is a smooth function, $(\alpha, \beta) \in \Gamma((T^A\Lambda^+)^*)$ and $(X_i, V_i) \in \Gamma(T^A\Lambda^+)$ are projectable sections of $(T^A\Lambda^+)^*$ and $T^A\Lambda^+$, respectively.

The canonical section $\lambda_A$ of the dual bundle to $T^A\Lambda^+$ (which is called the Liouville section associated with the Lie algebroid $\Lambda$) may be defined as follows:

$$
\lambda_A(a, v_a) = \alpha(a) \quad \text{for} \quad \alpha \in A^* \quad \text{and} \quad (a, v_a) \in T^A \Lambda^*.
$$

(4.3)

The section $\Omega_A$ of $\wedge^2(T^A\Lambda^*)^* \to A^*$ given by

$$
\Omega_A = -\frac{df}{dT^A\Lambda^+}\lambda_A
$$

is nondegenerate and $\frac{df}{dT^A\Lambda^+}\Omega_A = 0$. Thus, $\Omega_A$ is a symplectic-like section of the Lie algebroid $T^A\Lambda^* \to A^*$ which is called the canonical symplectic-like section associated with the Lie algebroid $\Lambda$. The Poisson structure on the base space $A^*$ induced by this symplectic-like section is just the linear Poisson structure on $A^*$ associated with the Lie algebroid $\Lambda$ (see [13]). For this reason, $T^A\Lambda^*$ may be considered as the canonical cover of the fiberwise linear Poisson structure on $A^*$. If $\{I, J\}$ is the local basis of sections of $T^A\Lambda^*$ described in (4.1), the local expressions of $\lambda_A$ and $\Omega_A$ are

$$
\lambda_A = y_I\lambda^I, \quad \Omega_A = \lambda^I \wedge \lambda^J + \frac{1}{2}C^k_{ij}\lambda^i \wedge \lambda^j
$$

where $\{\lambda^I, \lambda^J\}$ is the dual basis of $\{\lambda_i, \lambda_j\}$ and $C^k_{ij}$ are the local structure functions of the bracket $[\cdot, \cdot]$ (for more details, see [13]).

**Examples 4.1.**

(i) Note that if $A$ is the standard Lie algebroid $TM$, then the symplectic-like Lie algebroid $T^A\Lambda^* \to A^*$ may be identified with the standard Lie algebroid $T(TM) \to T^*M$ and, under this identification, $\Omega_A$ is the canonical symplectic $\Omega_M$ structure of $T^*M$.

(ii) For the case $A = g \times TM$ from (ii) in examples 3.2, we have that $T^A\Lambda^* \to A^*$ can be identified with $T^g g^* \oplus T(TM) \to g^* \times T^*M$, i.e.

$$(g \times Tg^*) \oplus T(TM) \to g^* \times T^*M.$$  

Under this identification, the symplectic-like structure $\Omega_{g\times TM}$ on $T^g g^* \times T^*M$ is just $\Omega_g \oplus \Omega_M$, where $\Omega_M$ is the standard symplectic 2-form on $T^*M$ and $\Omega_g$ is the symplectic-like structure on $g \times Tg^* \to g^*$ characterized by

$$
(\Omega_g)_{\eta_0}((\xi, \eta), (\xi', \eta')) = \eta'(\xi) - \eta(\xi') + \eta_0[\xi, \xi']_g
$$

for all $\eta_0, \eta, \eta' \in g^*$ and $\xi, \xi' \in g$.  

The action of a Lie group $G$ on $T^A\Lambda^* \to A^*$ by complete lifts. Now, suppose that $\Phi: G \times A \to A$ is a free and proper action of a connected Lie group $G$ by complete lifts with respect to the Lie algebra anti-morphism $\psi: g \to \Gamma(A)$. Denote by $\phi: G \times M \to M$ the corresponding action on $M$ and by $\Phi^*: G \times A^* \to A^*$ the left dual action on $A^*$. In what follows, we will describe a free and proper canonical action by complete lifts on $T^A\Lambda^*$ induced by $\Phi$.  

20
Proposition 4.2. Let $\Phi : G \times A \to A$ be a free and proper action of a connected Lie group $G$ on the Lie algebroid $A$ by complete lifts with respect to $\psi : \mathfrak{g} \to \Gamma(A)$. Then the map $(\Phi, T\Phi^\ast) : G \times T^A A^\ast \to T^A A^\ast$ given by

$$(\Phi, T\Phi^\ast)(g, (\alpha_\ast, v_{\alpha_\ast})) = (\Phi_g(\alpha_\ast), (T_{\alpha_\ast} \Phi_g^\ast)(v_{\alpha_\ast})), \quad \alpha_\ast \in A^\ast \quad \text{and} \quad (\alpha_\ast, v_{\alpha_\ast}) \in T_{\alpha_\ast} A^\ast \quad (4.4)$$

defines a free and proper left canonical action of $G$ on the symplectic-like Lie algebroid $T^A A^\ast$ by complete lifts with respect to the Lie algebra anti-morphism $\psi^T : \mathfrak{g} \to \Gamma(T^A A^\ast)$ defined by

$$\psi^T(\xi) = (\psi(\xi), \xi_\ast^A) \quad \text{for} \quad \xi \in \mathfrak{g}. \quad (4.5)$$

where $\xi_\ast^A$ is the infinitesimal generator of $\xi$ with respect to $\Phi^\ast$.

Proof. Note that the map $(\Phi, T\Phi^\ast)$ is well defined. In fact, since $\Phi$ is an action by complete lifts, then $\Phi_\ast : A \to A$ is a Lie algebroid isomorphism for all $g \in G$. So, using (2.4) with $k = 0$, we have that

$$\rho(\Phi_g(\alpha_\ast)) = T_\alpha \phi_g(\rho(\alpha_\ast)) \quad \text{for all} \quad g \in G, \ x \in M \text{ and } \alpha_\ast \in A^\ast. \quad (4.6)$$

Moreover,

$$T_{\phi_g(\alpha_\ast)} \tau_x (T_{\alpha_\ast} \Phi_g^\ast(v_{\alpha_\ast})) = T_{\alpha_\ast} (\phi_g \circ \tau_x)(v_{\alpha_\ast}) = T_{\alpha_\ast} \phi_g(\rho(\alpha_\ast)) \quad (4.7)$$

for all $v_{\alpha_\ast} \in T_{\alpha_\ast} A^\ast$. Thus, from (4.6) and (4.7), we deduce that

$$T_{\phi_g(\alpha_\ast)} \tau_x (T_{\alpha_\ast} \Phi_g^\ast(v_{\alpha_\ast})) = \rho(\Phi_g(\alpha_\ast))$$

for all $(\alpha_\ast, v_{\alpha_\ast}) \in T_{\alpha_\ast} A^\ast$, that is,

$$(\Phi, T^\ast \Phi)(g, (\alpha_\ast, v_{\alpha_\ast})) \in T_{\phi_g(\alpha_\ast)} A^\ast.$$

Obviously, $(\Phi, T\Phi^\ast)$ is a free and proper action. We will now show that this action on $T^A A^\ast$ is by complete lifts. Firstly, note that the map $\psi^T : \mathfrak{g} \to \Gamma(T^A A^\ast)$ is well defined. In fact, since $\rho(\psi(\xi))$ is just the infinitesimal generator $\xi_M$ of $\xi$ with respect to the action $\phi : G \times M \to M$ and the projection $\tau_x : A^\ast \to M$ is equivariant, we have that

$$\rho(\psi(\xi)) = T_{\tau_x}(\xi_\ast^A) \quad \text{for all} \quad \xi \in \mathfrak{g}.$$

On the other hand, the infinitesimal generator $\xi_{T^A A^\ast} \in \mathfrak{X}(T^A A^\ast)$ of $\xi \in \mathfrak{g}$ with respect to the action $(\Phi, T\Phi^\ast)$ is the pair $(\xi_A, \xi_\ast^A)$, where $\xi_A$ is the infinitesimal generator of $\xi \in \mathfrak{g}$ with respect to $\Phi$ and $\xi_\ast^A$ is the complete lift of $\xi_\ast^A$. Moreover, the complete lift of $\psi^T(\xi)$ with respect to the Lie algebroid $T^A A^\ast$ is just $(\psi(\xi)^T, \xi_\ast^A)$. This is a consequence of the fact that $(\psi(\xi)^T, \xi_\ast^A) \in \mathfrak{X}(T^A A^\ast)$ is $T_{\tau_x} A^\ast$-projectable on $\rho_{T_{\tau_x} A^\ast}(\psi(\xi), \xi_\ast^A) = \xi_A$. and that, from (4.2), we deduce that

$$L_{(\psi(\xi)^T, \xi_\ast^A)}(\alpha, \beta) = \left(\mathcal{L}_{\psi(\xi)^T} \alpha, \mathcal{L}_{\xi_\ast^A} \beta\right)$$

for every projectable section $(\alpha, \beta)$ on $\Gamma((T^A A^\ast)^\ast)$. Here $\mathcal{L}$ is the standard Lie derivative.

Therefore, $(\Phi, T\Phi^\ast)$ is an action by complete lifts and consequently by automorphisms of Lie algebroids. Finally, a direct computation, using (4.3), proves that the action $(\Phi, T\Phi^\ast)$ preserves the Liouville section $\lambda_A$, i.e.

$$(\Phi, T\Phi^\ast)_g^\ast \lambda_A = \lambda_A \quad \text{for all} \quad g \in G.$$

Thus, using (2.4) and the fact that $(\Phi, T\Phi^\ast)_g$ is an automorphism of Lie algebroids, we conclude that $(\Phi, T\Phi^\ast)_g$ preserves the canonical symplectic-like section $\Omega_A$ of $T^A A^\ast$. \qed
The momentum map for the canonical action of $G$ on the Lie algebroid $T^A A^* \to A^*$. Denote by $J_A^* : A^* \to \mathfrak{g}^*$ the map given by

$$J_A^*(\alpha_x)(\xi) = \alpha_x(\psi(\xi)(x)) \quad \text{with} \quad x \in M, \quad \alpha_x \in A^*_x \quad \text{and} \quad \xi \in \mathfrak{g}.$$  

(4.8)

Then, we have the following result.

**Proposition 4.3.** The map $J_A^* : A^* \to \mathfrak{g}^*$ is an equivariant momentum map for the Poisson action $\Phi^* : G \times A^* \to A^*$.

**Proof.** From proposition 3.1, we have that if $\Pi_{A^*}$ is the linear Poisson structure on $A^*$ and $\hat{\psi}(\xi)$ is the linear function associated with the section $\psi(\xi) \in \Gamma(A)$, for each $\xi \in \mathfrak{g}$, the Hamiltonian vector field $\hat{\xi}$ is just $\hat{\psi}(\xi)$. Thus, $J_A^*$ is a momentum map for the Poisson action $\Phi^* : G \times A^* \to A^*$.

Now, we will prove that $J_A^*$ is equivariant, i.e.

$$J_A^* \circ \Phi^* = \text{Coad}_\mathfrak{g}^* \circ J_A^*.$$

Indeed, if $x \in M$ and $\alpha_x \in A^*_x$, then, from proposition 3.4, we have that

$$J_A^*(\Phi^* (\alpha_x))(\xi) = (\Phi^* (\alpha_x)) (\psi(\xi)(\phi_\xi(x))) = \alpha_x(\Phi^* (\psi(\xi)(\phi_\xi(x))))$$

$$= \alpha_x(\psi(Ad_{\phi_\xi}^G, \xi)(x)) = J_A^*(\alpha_x)(Ad_{\phi_\xi}^G, \xi)$$

$$= (\text{Coad}_\mathfrak{g}^*)(J_A^* (\alpha_x))(\xi)$$

for all $\xi \in \mathfrak{g}$. □

Now, using lemma 3.8, we have that the map

$$J_A^T : T^A A^* \to \mathfrak{g}^* \times \mathfrak{g}^*, \quad J_A^T(\alpha_x, v_{\alpha_x}) = (T_{\alpha_x} J_A^*(v_{\alpha_x}), J_A^*(\alpha_x))$$  

(4.9)

is equivariant with respect to the action $(\Phi, T\Phi^*)^T : TG \times T^A A^* \to T^A A^*$.

From the injectivity of $\psi_x$ (see remark 3.3), it follows that the restriction of $J_A^* : A^* \to \mathfrak{g}^*$ to $A^*_x$ is a linear epimorphism and therefore, for all $\alpha_x \in A^*_x$ the restriction of the tangent map $T_{\alpha_x} J_A^* : T_{\alpha_x} A^* \to T_{\alpha_x}(A^*_x) \cong \mathfrak{g}^*$ to $T_{\alpha_x} A^*_x$ is surjective. Thus, all the elements of $\mathfrak{g}^*$ are regular values of $J_A^*$ and

$$T_{\alpha_x} J_A^* \circ (\rho_{T_A^* A^*})_{\alpha_x} : T_{\alpha_x} A^* \to \mathfrak{g}^*$$

is surjective for all $\alpha_x \in A^*_x$. Note that

$$T_{\alpha_x} A^*_x = \ker T_{\alpha_x} \tau_x \subseteq (\rho_{T_A^* A^*})_{\alpha_x}(T_{\alpha_x} A^*).$$

In conclusion, if $\mu \in \mathfrak{g}^*$, then $J_{A^*}^{-1}(\mu)$ is a regular submanifold of $A^*$ and $(J_{A^*}^{-1})^{-1}(0, \mu)$ is a Lie subalgebroid of $T^A A^*$ over $J_{A^*}^{-1}(\mu)$ (see proposition 3.9). In fact, $(J_{A^*}^{-1})^{-1}(0, \mu)$ is just the prolongation $T^A (J_{A^*}^{-1}(\mu))$ of the Lie algebroid $A$ over the restriction $(\tau_x)_{(J_{A^*}^{-1})^{-1}(\mu)} : J_{A^*}^{-1}(\mu) \to M$ of $\tau_x : A^* \to M$ to the submanifold $J_{A^*}^{-1}(\mu)$. Note that $J_{A^*}^{-1}(\mu)$ is an affine subbundle of $A^*$ over $M$ and that $(\tau_x)_{(J_{A^*}^{-1})^{-1}(\mu)} : J_{A^*}^{-1}(\mu) \to M$ is the projection.
5. The reduction of the canonical cover of a fiberwise linear Poisson structure

Let \( (A, [\cdot, \cdot], \rho) \) be a Lie algebroid over the manifold \( M \) and \( \tau : A \to M \) the vector bundle projection. Suppose that \( \Phi : G \times A \to A \) is a free and proper action of a connected Lie group \( G \) by complete lifts with respect to the Lie algebra anti-morphism \( \psi : g \to A \). In the previous sections, we have shown that in this situation, we have a free and proper canonical action \( (\Phi, T\Phi^*) : G \times T^A A^* \to T^A A^* \) of the Lie group \( G \) on the symplectic-like Lie algebroid \( T^A A^* \) by complete lifts with respect to the Lie algebra anti-morphism \( \psi^T : g \to \Gamma(T^A A^*) \) given in (4.5). In addition, we have an equivariant momentum map \( J_{\lambda^*} : A^* \to g^* \) on \( A^* \) with respect to the left Poisson action \( \Phi^* : G \times A^* \to A^* \) of \( G \) on \( A^* \).

If \( \mu \) is an element of \( g^* \) then we obtain, in a natural way, a free and proper action \( (\Phi, T\Phi^*) : G_\mu \times T^A J_{\lambda^*}^{-1}(\mu) \to T^A J_{\lambda^*}^{-1}(\mu) \) of the isotropy group of \( \mu \) on the Lie algebroid \( T^A J_{\lambda^*}^{-1}(\mu) \) by restriction. Now, using theorem 3.11, we conclude that the reduced vector bundle

\[
(T^A A^*)_{\mu} = T^A J_{\lambda^*}^{-1}(\mu)/TG_{\mu} \to J_{\lambda^*}^{-1}(\mu)/G_{\mu}
\]

is a symplectic-like Lie algebroid with symplectic-like section \( \Omega_{\mu} \) characterized by

\[
\tilde{\pi}_\mu^* \Omega_{\mu} = \tau_{\mu}^* \Omega_{\lambda^*},
\]

where \( \tilde{\pi}_\mu : T^A J_{\lambda^*}^{-1}(\mu) \to (T^A A^*)_\mu \) is the canonical projection, \( \tau_{\mu} : T^A J_{\lambda^*}^{-1}(\mu) \to T^A A^* \) is the inclusion and \( \Omega_{\lambda^*} \) is the standard symplectic-like structure on \( T^A A^* \).

In what follows, we will describe this reduced Lie algebroid \( (T^A A^*)_{\mu} \). Firstly, we will discuss the case \( \mu = 0 \).

5.1. The case \( \mu = 0 \)

Note that, under this assumption, the isotropy group \( G_0 \) is just \( G \).

We will prove that the reduced symplectic-like Lie algebroid

\[
(T^A A^*)_0 = T^A J_{\lambda^*}^{-1}(0)/TG \to J_{\lambda^*}^{-1}(0)/G
\]

is the canonical cover of a fiberwise linear Poisson structure on the dual \( A_{\theta}^* \) of a certain Lie algebroid \( A_0/G \) over \( M/G \).

Description of the Lie algebroid \( A_0 \). The Lie algebroid \( A_0 \) over \( M/G \) is the space of orbits \( A/TG \) of the affine action of \( TG \) on \( A \) (see theorem 3.6). As we know (see the proof of theorem 3.6), if \( \tilde{\pi} : A \to A_0 = A/TG \) and \( \pi : M \to M/G \) are the canonical projections and \( (\llbracket \cdot, \cdot \rrbracket_{\lambda^*}, \rho_{\lambda^*}) \) is the Lie algebroid structure on \( A_0 \), then

\[
\llbracket X_0, Y_0 \rrbracket_{\lambda^*} \circ \pi = \tilde{\pi} (\llbracket X, Y \rrbracket), \quad \rho_{\lambda^*}(X_0) = T\pi (\rho(X))
\]

for \( X_0, Y_0 \in \Gamma(A_0) \) and \( X, Y \in \Gamma(A) \) satisfying

\[
X_0 \circ \pi = \tilde{\pi} \circ X, \quad Y_0 \circ \pi = \tilde{\pi} \circ Y.
\]

Note that with this structure, \( \tilde{\pi} : A \to A_0 \) is an epimorphism of Lie algebroids.

Now, we will prove that the vector bundle \( A_0 = A/TG \to M/G \) is isomorphic to

\[
(J_{\lambda^*}^{-1}(0)/G)^* \to M/G.
\]

In fact, one may easily test that the submanifold \( J_{\lambda^*}^{-1}(0) \) is just the annihilator \( (\psi(g))^0 \) of \( \psi(g) \). Therefore, the restriction \( r^0_\theta = \tau_{\lambda^*,\lambda^*}^{-1}(0) : J_{\lambda^*}^{-1}(0) \to M \) of \( \tau_\theta : A^* \to M \) to this submanifold is a vector bundle over \( M \). Moreover, a direct computation proves that this vector bundle is isomorphic to the dual vector bundle \( (A/\psi(g))^* \) of \( A/\psi(g) \) to \( M \).

Therefore, using the equivalences (3.13), we deduce that the three vector bundles

\[
A_0 = A/TG \to M/G, \quad (A/\psi(g))/G \to M/G, \quad (J_{\lambda^*}^{-1}(0)/G)^* \cong (J_{\lambda^*}^{-1}(0))^*/G \to M/G
\]
are isomorphic. Thus, we may induce isomorphic Lie algebroid structures on these vector bundles.

*The description of the Lie algebroid isomorphism between* \((T^A A^*)_0\) and \(T^A \mathfrak{a}_0^*\). In what follows, we identify \(A_0 = A / TG\) with \((J_{A^*}^{-1}(0)/G)^* \cong (J_{A^*}^{-1}(0))^*/G\). Under this identification, we denote by \(\varphi : A \to (J_{A^*}^{-1}(0)/G)^*\) the epimorphism of vector bundles corresponding to the quotient projection \(A \to A / TG\).

Let us consider the following epimorphism of vector bundles over \(\pi_0 : J_{A^*}^{-1}(0) \to A_0^* = J_{A^*}^{-1}(0)/G\)

\[
\varphi^T : T^A J_{A^*}^{-1}(0) \to T^A \mathfrak{a}_0^* \quad (a_x, v_{a_x}) \mapsto (\varphi(a_x), T_{a_x}\pi_0(v_{a_x})). \quad (5.2)
\]

Note that, using (5.1) and the fact that \(\bar{T}_0^0 \circ \pi_0 = \pi \circ T_0^0\), we have that \(\varphi^T\) is well defined. Here \(\bar{T}_0^0 : J_{A^*}^{-1}(0)/G \to M/G\) is the dual vector bundle of \((J_{A^*}^{-1}(0)/G)^* \to M/G\).

Now, since \(\varphi : A \to A_0\) is a Lie algebroid epimorphism, \(\varphi^T\) is also a Lie algebroid epimorphism. Furthermore, it is easy to prove, using that \(\varphi\) is \(TG\)-invariant, that \(\varphi^T\) is \(TG\)-invariant with respect to the action \((\Phi, T\Phi^*)^T\) of \(TG\) restricted to \(T^A J_{A^*}^{-1}(0)\). In fact,

\[
\varphi^T((\Phi, T\Phi^*)^T(\alpha_x, v_{a_x})) = (\varphi(\Phi_x(a_x)) + \psi(\Phi(\xi(\Phi_x(a_x))) + T_{a_x}\pi_0(T_{a_x}\Phi_x(v_{a_x} + \xi_x)))
\]

\[
= (\varphi(\alpha_x), T_{a_x}\pi_0(v_{a_x})) = \varphi^T(a_x, v_{a_x})
\]

for all \((g, \xi) \in G \times \mathfrak{g} \cong TG\) and \((a_x, v_{a_x}) \in T^A J_{A^*}^{-1}(0)\). Note that

\[
\varphi(\psi(\xi)) = \varphi(\Phi(e_x, \xi)) = 0.
\]

Thus, we have the following Lie algebroid epimorphism \(\tilde{\varphi}^T\) between \((T^A A^*)_0\) and \(T^A \mathfrak{a}_0^*\) over the identity of \(A_0^*:\)

\[
\begin{array}{ccc}
T^A J_{A^*}^{-1}(0) & \xrightarrow{\varphi^T} & T^A \mathfrak{a}_0^* \\
\Delta_0 & \xrightarrow{\tilde{\varphi}^T} & A_0^* \\
\end{array}
\]

In fact,

\[
\tilde{\varphi}^T[(a_x, v_{a_x})] = (\varphi(a_x), (T_{a_x}\pi_0)(v_{a_x})) \quad \text{for} \quad (a_x, v_{a_x}) \in T^A J_{A^*}^{-1}(0). \quad (5.4)
\]

Finally, we will prove that \(\tilde{\varphi}^T\) is an isomorphism, that is, \(\tilde{\varphi}^T\) is injective.

If \((e_x, v_{e_x}) \in \ker \varphi^T\), then \(e_x \in \ker \varphi = \psi_x(g)\) and \(v_{a_x}\) is a vertical vector with respect to \(\pi_0 : (J_{A^*})^{-1}(0) \to (J_{A^*})^{-1}(0)/G\). Then, there are \(\xi, \xi' \in \mathfrak{g}\) such that

\[
e_x = \psi_x(\xi) \quad \text{and} \quad v_{a_x} = \xi'_x(a_x).
\]

Then,

\[
\xi_M(x) = \rho(e_x) = T_{a_x}e'_x(v_{a_x}) = \xi'_M(x)
\]

and, since \(\phi : G \times M \to M\) is a free action, we conclude that \(\xi = \xi'\). Therefore,

\[
(e_x, v_{a_x}) = (\psi_x(\xi), \xi_x'(a_x)) = ((\Phi, T\Phi^*)^T(e_x, \xi, (0, 0))
\]

where \(e\) is the identity element of \(G\). Thus, \(\tilde{\varphi}^T\) is injective.

*The Lie algebroid isomorphism between* \((T^A A^*)_0\) and \(T^A \mathfrak{a}_0^*\) *is canonical*. We will see that \(\tilde{\varphi}^T\) is canonical, i.e.

\[
(\tilde{\varphi}^T)^* \Omega_{A_0} = \Omega_0, \quad (5.5)
\]

where \(\Omega_0\) is the reduced symplectic-like structure on \((T^A A^*)_0\) given in theorem 3.11 and \(\Omega_{A_0}\) is the canonical symplectic-like structure on \(T^A \mathfrak{a}_0^*\).
From (4.3), (5.2) and the definition of the morphism $\varphi : A \to A_0$, we obtain that

$$(\varphi^T)^*\lambda_{A_0} = \tilde{T}_0\lambda_A,$$

where $\lambda_{A_0}$ (respectively, $\lambda_A$) is the Liouville section of $A_0 \to M/G$ (respectively, of $A \to M$) and $\tilde{T}_0 : T^A J_A^{-1}(0) \to T^A \mathcal{A}^*$ is the inclusion.

Thus, since $\varphi^T$ and $\tilde{T}_0$ are Lie algebroid morphisms, we obtain that

$$(\varphi^T)^*\Omega_{A_0} = \tilde{T}_0^*\Omega_A.$$ 

On the other hand, if $\tilde{\pi}_0 : T^A J_A^{-1}(0) \to (T^A \mathcal{A}_0)^*$ is the canonical projection, it is clear that $\tilde{\varphi}^T \circ \tilde{\pi}_0 = \varphi^T$ which implies that

$$\tilde{\pi}_0^* (\tilde{\varphi}^T)^*\Omega_{A_0} = \tilde{T}_0^* \Omega_A$$

and, therefore,

$$(\tilde{\varphi}^T)^*\Omega_{A_0} = \Omega_0. \quad (5.6)$$

In the following theorem, we summarize the results obtained in the case $\mu = 0$.

**Theorem 5.1.** Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid on the manifold $M$ and $\Phi : G \times A \to A$ a free and proper action of a connected Lie group by complete lifts. Then, the reduced symplectic-like Lie algebroid

$$(T^A \mathcal{A}_0)^* = (T^A J_A^{-1}(0))/T^G J_A^{-1}(0)/G$$

is canonically isomorphic to the Lie algebroid $T_{H_0} \mathcal{A}_0^*$, equipped with the standard symplectic-like structure, where the Lie algebroid $A_0$ is the vector bundle $A_0 = A/T^G M \to M/G$

endowed with the quotient Lie algebroid structure characterized by (5.1).

### 5.2. The case $G_\mu = G$

Suppose that the assumptions of theorem 5.1 hold. Additionally, we consider a principal $G$-connection $A : TM \to g$ for the corresponding principal bundle $\pi : M \to M/G$. In such a case, we have a vector bundle morphism $A^i : A \to g$ given by

$$A^i(a_x) = A(\rho_x(a_x)), \quad \forall a_x \in A_x,$$

which satisfies the following properties:

(i) $A^i$ is equivariant with respect to $\Phi : G \times A \to A$ and the adjoint action, that is,

$$A^i(\Phi_x(a_x)) = Ad^g(\rho_x(A^i(a_x))), \quad \forall a_x \in A_x,$$

(ii) $A^i(\psi(\xi)(x)) = \xi$, for all $\xi \in g$ and $x \in M$.

Note that if $\pi : M \to M/G$ is the quotient projection, then we have

$$TQ = V\pi \oplus H \text{ and } A = \psi(g) \oplus H^A,$$

where $V\pi$ is the vertical bundle of $\pi$ and $H^A$ (respectively, $H$) is the vector bundle on $M$ whose fiber at $x \in M$ is the vector space

$$H^A_x = \{a_x \in A_x / A^i(a_x) = 0\} \quad \text{(respectively, } H_x = \{v_x \in T_xM / A(v_x) = 0\}).$$

Moreover, $H$ and $H^A$ are $G$-invariant vector bundles, that is,

$$H^A_{\phi_g(x)} = \Phi_x(H^A_x) \quad \text{and} \quad H_{\phi_g(x)} = T_x\phi_g(H_x), \quad \forall g \in G.$$

Now, if $\mu \in g^*$, we consider the section $a_\mu$ of $A^*$ given by

$$a_\mu(a_x) = \mu(A^i(a_x)).$$

25
with \( x \in M \) and \( a_x \in A_x \). This section has the following properties:

(i) \( \alpha_\mu(M) \subseteq J^{-1}_A(\mu) \). In fact,
\[
J_{\psi}(\alpha_\mu(x))(\xi) = \alpha_\mu(\psi(\xi)(x)) = \mu(A^A(\psi(\xi)(x))) = \mu(\xi)
\]
for all \( x \in M \) and \( \xi \in g \).

(ii) \( \Phi^*_g\alpha_\mu = \alpha_{\text{Corv}}\alpha_\mu \), for all \( g \in G \), which is a consequence from the equivariance properties of \( A^A \).

Thus, since \( G_\mu = G \), we deduce that \( \alpha_\mu \) is \( G \)-invariant, i.e.
\[
\Phi^*_g\alpha_\mu = \alpha_\mu. \tag{5.7}
\]

So, in what follows, we assume that there is a \( G \)-invariant 1-section \( \alpha_\mu \in \Gamma(A^*) \) of \( A^* \) with values in \( J^{-1}_A(\mu) \). Using (5.7), proposition 2.2 and the fact that the flow of \( \psi(\xi) \) is \( \{\Phi_{\exp(\xi)}\}_{\xi \in \mathbb{R}} \), we obtain that
\[
L^A_{\psi(\xi)}\alpha_\mu = 0. \tag{5.8}
\]

On the other hand,
\[
i_{\psi(\xi)}\alpha_\mu = \mu(A^A(\psi(\xi))) = \mu(\xi).
\]

Then,
\[
0 = L^A_{\psi(\xi)}\alpha_\mu = i_{\psi(\xi)}d^A\alpha_\mu. \tag{5.9}
\]

Denote \( \beta_\mu = d^A\alpha_\mu \). From (5.7) and since \( \Phi_g : A \to A \) is a Lie algebroid morphism we deduce that the 2-section \( \beta_\mu \) of \( A^* \) is \( G \)-invariant. Moreover, it satisfies \( i_{\psi(\xi)}\beta_\mu = 0 \) which implies that
\[
(\Phi^*_g(\psi(\xi)))^*\beta_\mu = \Phi^*_g\beta_\mu = \beta_\mu
\]
for all \( (g, \xi) \in G \times g \cong TG \).

Therefore, there exists a unique \( B_\mu \in \Gamma(\wedge^2 A^*_0) \) with the following property:
\[
\overline{\pi}^*B_\mu = \beta_\mu = d^A\alpha_\mu, \tag{5.10}
\]
where \( \overline{\pi} : A \to A_0 \) is the corresponding projection. It is clear that \( d^{A_0}B_\mu = 0 \).

The 2-section \( B_\mu \) of \( A^*_0 \) is said to be the magnetic term associated with \( \alpha_\mu \).

Now, we will prove that there is a Lie algebroid isomorphism, \( \Upsilon_{\alpha_\mu} : (\mathcal{T}^A A^*)_\mu \to T^{A_0}A^*_0 \), between the reduced Lie algebroid \( (\mathcal{T}^A A^*)_\mu \) and \( T^{A_0}A^*_0 \) such that the symplectic-like section \( \Omega_{\alpha_\mu} \) on \( T^{A_0}A^*_0 \) and the reduced symplectic-like section \( \Omega_\mu \) on \( (\mathcal{T}^A A^*)_\mu \) are related by the following formula:
\[
\Upsilon^*_{\alpha_\mu}(\Omega_{\alpha_\mu} - pr^{A_0}_1B_\mu) = \Omega_\mu.
\]

where \( pr_1 : T^{A_0}A^*_0 \to A_0 \) is the Lie algebroid morphism induced by the first projection.

The description of the Lie algebroid isomorphism \( \Upsilon_{\alpha_\mu} : (\mathcal{T}^A A^*)_\mu \to T^{A_0}A^*_0 \). Firstly, we will describe a Lie algebroid morphism between the reduced spaces \( (\mathcal{T}^A A^*)_\mu \) and \( (\mathcal{T}^A A^*)_0 \). Then, we may use theorem 5.1 in order to construct the isomorphism \( \Upsilon_{\alpha_\mu} \).

Using the fact that \( \alpha_\mu(M) \subseteq J^{-1}_A(\mu) \), we deduce that \( J^{-1}_A(\mu) \to M \) is an affine bundle on \( M \) such that
\[
J^{-1}_A(\mu) \cap A^*_\mu = \{ \beta_x \in A^*_\mu : \beta_x - \alpha_\mu(x) \in J^{-1}_A(0) \}
\]
for all \( x \in M \).

Now, we consider the affine bundle isomorphism
\[
\begin{array}{ccc}
J^{-1}_A(\mu) & \xrightarrow{sh_\mu} & J^{-1}_A(0) \\
M & \xrightarrow{Id} & M
\end{array}
\]
where \( sh_\mu(\beta_x) = \beta_x - \alpha_\mu(x) \) for all \( \beta_x \in J^{-1}_A(\mu) \cap A^*_\mu \).
From the $G$-invariance of $\alpha_\mu$, we deduce that $sh_\mu$ is equivariant with respect to the action $\Phi^* : G \times A^* \rightarrow A^*$, i.e.

$$(sh_\mu \circ \Phi^*)(\beta_x) = (\Phi^*_x \circ sh_\mu)(\beta_x)$$

for all $\beta_x \in A^*_x \cap J^{-1}_{A^*}(\mu)$ and $g \in G$. Moreover, one may induce a morphism of vector bundles

$$
\begin{array}{ccc}
T^A J^{-1}_{A^*}(\mu) & \overset{sh_\mu}{\longrightarrow} & T^A J^{-1}_{A^*}(0) \\
\tau_{T^A J^{-1}_{A^*}(\mu)} & & \tau_{T^A J^{-1}_{A^*}(0)} \\
J^{-1}_{A^*}(\mu) & \overset{sh_\mu}{\longrightarrow} & J^{-1}_{A^*}(0)
\end{array}
$$

where $T^A sh_\mu(\alpha_x, X_\beta) = (\alpha_x, T_\beta sh_\mu(X_\beta))$, with $\alpha_x, X_\beta \in T^A J^{-1}_{A^*}(\mu)$. Note that, since $\tau_{\ast T^A J^{-1}_{A^*}(0)} \circ sh_\mu = \tau_{\ast T^A J^{-1}_{A^*}(\mu)}$, $T^A sh_\mu$ is well defined. Furthermore, a direct proof shows that $T^A sh_\mu$ is an isomorphism of vector bundles. In fact, one can easily see that $T^A sh_\mu$ is a Lie algebroid isomorphism, taking into account that

$$
T^A sh_\mu([X, Y], [U, V]) = ([X, Y], [T sh_\mu \circ U \circ sh^{-1}_\mu, T sh_\mu \circ V \circ sh^{-1}_\mu]) \circ sh_\mu,
$$

$$
\rho_{T^A J^{-1}_{A^*}(0)}((T^A sh_\mu)(X, U)) = (T sh_\mu \circ \rho_{T^A J^{-1}_{A^*}(\mu)})(X, U)
$$

for all $X, Y \in \Gamma(A)$ and $U, V \in \mathfrak{X}(J^{-1}_{A^*}(\mu))$ which are $(\tau_\ast)_{T^A J^{-1}_{A^*}(\mu)}$-projectable on $\rho(X)$ and $\rho(Y)$, respectively.

Moreover, since $sh_\mu$ is equivariant, we deduce that $T^A sh_\mu$ is equivariant with respect to the action $(\Phi, T \Phi^*)^T$ of $G$ restricted to $T^A J^{-1}_{A^*}(\mu)$ and $T^A J^{-1}_{A^*}(0)$, respectively.

Thus, one induces a Lie algebroid isomorphism

$$
\begin{array}{ccc}
(T^A J^{-1}_{A^*}(\mu))/TG & \overset{T^A sh_\mu}{\longrightarrow} & (T^A J^{-1}_{A^*}(0))/TG \\
\tilde{T}_{T^A J^{-1}_{A^*}(\mu)} & & \tilde{T}_{T^A J^{-1}_{A^*}(0)} \\
J^{-1}_{A^*}(\mu)/G & \overset{sh_\mu}{\longrightarrow} & J^{-1}_{A^*}(0)/G
\end{array}
$$

Finally, the isomorphism $\Upsilon_{\alpha_\mu} : (T^A A^*)_\mu \rightarrow T^A A^*_\mu$ is defined as follows:

$$
\Upsilon_{\alpha_\mu} = \tilde{\varphi}^T \circ T^A sh_\mu,
$$

where $\tilde{\varphi}^T : (T^A A^*)_\mu \rightarrow T^A A^*_\mu$ is the Lie algebroid isomorphism defined by (5.4).

Relation between the symplectic-like structures on $(T^A A^*)_\mu$ and $T^A A^*_\mu$. Let $\lambda_A$ be the Liouville section on $T^A A^*$ and $t_0 : T^A(J^1_{A^*})^{-1}(0) \rightarrow T^A A^*$ (respectively, $t_\mu : T^A(J^1_{A^*})^{-1}(\mu) \rightarrow T^A A^*$) be the corresponding inclusion. Then,

$$
((T^A sh_\mu)^\ast (t_0^{\ast} \lambda_A))(\alpha_x, X_\beta) = (t_{\mu}^{\ast} \lambda_A)(\alpha_x, X_\beta) - \alpha_\mu(\alpha_x)
$$

for all $\beta_x \in J^{-1}_{A^*}(\mu)$ and $(\alpha_x, X_\beta) \in T^A J^{-1}_{A^*}(\mu)$.

On the other hand, if $p^0 : T^A J^{-1}_{A^*}(0) \rightarrow A$ is the Lie algebroid morphism induced by the first projection, we have that

$$
((p^0)^\ast T^A sh_\mu)^\ast \alpha_\mu(\alpha_x, X_\beta) = \alpha_\mu(\alpha_x).
$$
This implies that
\[
(T^A sh_\mu)^* (\iota_\mu^* \lambda_A + (pr_1)^* \alpha_\mu) = \iota_\mu^* \lambda_A
\]
and thus, from theorem 3.11, we deduce that
\[
(T^A sh_\mu)^* (\pi_0^* \Omega_0 - (pr_1)^* \beta_\mu) = \tilde{\pi}_\mu^* \Omega_\mu,
\]
where \( \Omega_0 \) (respectively, \( \Omega_\mu \)) is the symplectic-like structure on \((T^A J^{-1}_A(0))/TG\) (respectively, \((T^A J^{-1}_A(\mu))/TG\)) and \( \tilde{\pi}_0 : T^A J^{-1}_A(0) \to (T^A J^{-1}_A(0))/TG \) (respectively, \( \tilde{\pi}_\mu : T^A J^{-1}_A(\mu) \to (T^A J^{-1}_A(\mu))/TG \)) is the canonical projection.

Now, using the relations
\[
\tilde{\pi}_0 \circ T^A sh_\mu = T^A sh_\mu \circ \tilde{\pi}_\mu \quad \text{and} \quad \tilde{\pi} \circ pr_1^0 \circ T^A sh_\mu = pr_\lambda \circ \gamma_{\alpha_\mu} \circ \tilde{\pi}_\mu,
\]
and the facts
\[
(\tilde{\pi})^* \Omega_0 = \Omega_0 \quad \text{and} \quad \tilde{\pi}^* B_\mu = \beta_\mu,
\]
we conclude that (5.11) is equivalent to
\[
\tilde{\pi}_\mu^* (\gamma_{\alpha_\mu}^* \Omega_{\lambda_0} - \gamma_{\alpha_\mu}^* (pr_1^*(B_\mu))) = \tilde{\pi}_\mu^* \Omega_\mu.
\]
Therefore,
\[
\gamma_{\alpha_\mu}^* (\Omega_{\lambda_0} - pr_1^*(B_\mu)) = \Omega_\mu.
\]

The results obtained in this case may be summarized in the following theorem.

**Theorem 5.2.** Let \((A, [\cdot, \cdot], \rho)\) be a Lie algebroid on the manifold \(M\) and \(\Phi : G \times A \to A\) a free and proper action of a connected Lie group \(G\) by complete lifts. Suppose that we consider \(\mu \in g^*\) such that \(G = G_\mu\). Then, choosing any \(G\)-invariant section \(\alpha_\mu\) of \(A^*\) with values in \(J_A^* (\mu)\), there is a canonical Lie algebroid isomorphism
\[
\gamma_{\alpha_\mu} : ((T^A A^*)_\mu, \Omega_\mu) \to (T^A A^*_\mu, \Omega_{\lambda_0} - (pr_1)^* B_\mu)
\]
where \(A_0 = A/TG \to M/G\) endowed with the Lie structure characterized by (5.1), \(\Omega_{\lambda_0}\) is the canonical symplectic-like structure on \(T^A A^*_\mu\), \(pr_1 : T^A A^*_\mu \to A_0\) is the projection on the first factor and \(B_\mu \in \Gamma (\wedge^2 A^*_\mu)\) is the corresponding magnetic term associated with \(\alpha_\mu\) which is characterized by (5.10).

### 5.3. The general case

Let \((A, [\cdot, \cdot], \rho)\) be a Lie algebroid on the manifold \(M\) and \(\Phi : G \times A \to A\) a free and proper action of a connected Lie group \(G\) on \(A\) by complete lifts with respect to the Lie algebra anti-morphism \(\psi : g \to \Gamma (A)\).

Let \(\mu \in g^*\) and denote by \(g_\mu\) the isotropy algebra of \(\mu\). Then, the induced action \(\Phi : G_\mu \times A \to A\) is a free and proper action by complete lifts with respect to the restriction \(\psi : g_\mu \to \Gamma (A)\) of \(\psi\) to \(g_\mu\).

Now, denote by \(\tilde{\mu} \in g_\mu^*\) the restriction of \(\mu\) to \(g_\mu\) and by \(J^*_{\tilde{\mu}} : A^* \to g_\mu^*\) the map given by
\[
J^*_{\tilde{\mu}} = i^* \circ J_{A^*},
\]
where \(i^* : g^* \to g_\mu^*\) is the dual of the inclusion \(i : g_\mu \to g\). Then, \(J^*_{\tilde{\mu}}\) is the momentum map associated with the action of \(G_\mu\) on \(A\).
A direct computation proves that the isotropy group of \( \tilde{\mu} \in g_\mu \) with respect to the coadjoint action of \( G_\mu \), \((G_\mu)_\tilde{\mu}\), is just \( G_\mu \). Therefore, we are in the conditions of section 5.2 if we choose as the starting Lie group \( G_\mu \). Next, we choose a \( G_\mu \)-invariant section \( \alpha_\mu \in \Gamma(A^*) \) such that
\[
\alpha_\mu(M) \subset (J^\mu_A)^{-1}(\tilde{\mu}).
\]

This is always possible as we have shown in section 5.2. If \( A_{0,\mu} \) is the vector bundle \( A/TG_\mu \to M/G_\mu \) associated with the action \( \Phi^T : TG_\mu \times A \to A \), we denote by \( B_\mu \in \Gamma(\wedge^2 A_{0,\mu}^*) \) the corresponding magnetic term associated with \( \alpha_\mu \). Then, from theorem 5.2, we conclude that the reduced symplectic-like Lie algebroid
\[
(T^\mu_A)^\ast \tilde{\mu} = (T^\mu_A)^{-1}(\tilde{\mu})/TG_\mu \to J^\mu_\ast(\mu)/G_\mu
\]
is isomorphic to the symplectic-like Lie algebroid \( (T^A_{\tilde{\mu}})\tilde{\mu}^\ast, \Omega_{\tilde{\mu}} \), \( \Omega_{\tilde{\mu}} = pr^*_\tilde{\mu}(B_\mu) \), where \( \Omega_{\tilde{\mu}} \) is the canonical symplectic-like structure on \( T^A_{\tilde{\mu}} \) and \( pr_1 : T^A_{\tilde{\mu}} \to A_{0,\mu} \) is the projection on the first factor.

On the other hand, the inclusion \( i_{\mu,\tilde{\mu}} : J^\mu_\ast(\mu) \to (J^\mu_A)^{-1}(\tilde{\mu}) \) is \( G_\mu \)-invariant and induces a Lie algebroid \( TG_\mu \)-invariant monomorphism \( I : T^\lambda J^\mu_\ast(\mu) \to T^\lambda (J^\mu_A)^{-1}(\tilde{\mu}) \) over \( i_{\mu,\tilde{\mu}} \). Therefore, we have a Lie algebroid monomorphism \((I, i_{\mu,\tilde{\mu}}) : \)
\[
\begin{align*}
&T^\lambda J^\mu_\ast(\mu)/TG_\mu \\
\xrightarrow{\iota_{\mu,\tilde{\mu}}} &J^\mu_\ast(\mu)/G_\mu \quad \xrightarrow{\iota_{\mu,\tilde{\mu}}} (J^\mu_A)^{-1}(\tilde{\mu})/G_\mu
\end{align*}
\]
which is canonical with respect to \( \Omega_\mu \) and \( \Omega_{\tilde{\mu}} \) on the reduced spaces \( (T^\lambda J^\mu_\ast(\mu))/TG_\mu \) and \( (T^\lambda (J^\mu_A)^{-1}(\tilde{\mu}))/TG_\mu \), respectively.

Denote by \( \tilde{\iota}_{\mu} : T^\lambda (J^\mu_A)^{-1}(\tilde{\mu}) \to T^\lambda A^\ast \) and by \( \iota_{\mu} : T^\lambda (J^\mu_A)^{-1}(\mu) \to T^\lambda A^\ast \) the corresponding inclusions which are related by
\[
\tilde{\iota}_{\mu} = \iota_{\mu} \circ I.
\]

Now, if
\[
\begin{align*}
\pi_{\tilde{\mu}} : T^\lambda (J^\mu_\ast)^{-1}(\tilde{\mu}) &\to (T^\lambda (J^\mu_A)^{-1}(\tilde{\mu}))/TG_\mu \\
\pi_{\mu} : T^\lambda (J^\mu_A)^{-1}(\mu) &\to (T^\lambda (J^\mu_A)^{-1}(\mu))/TG_\mu
\end{align*}
\]
are the corresponding projections, we have that
\[
\pi^\mu_{\Omega_\mu} = i^\mu_\ast \Omega_A = I^\ast (i^\mu_\ast \Omega_A) = I^\ast (\pi^\mu_\ast \Omega_{\tilde{\mu}}).
\]
Then, using that \( \pi_{\mu} \circ I = \tilde{I} \circ \pi_{\mu} \), we conclude that
\[
\tilde{I} \Omega_{\tilde{\mu}} = \Omega_\mu.
\]

Therefore, \( \tilde{I} \) is a canonical Lie algebroid monomorphism. Thus, we have proved the main result of this section.

**Theorem 5.3.** Let \( (A, [\cdot, \cdot], \rho) \) be a Lie algebroid over the manifold \( M \) and \( \Phi : G \times A \to A \) a free and proper action of a connected Lie group by complete lifts. If \( \mu \in g^* \) and \( \tilde{\mu} \) is the restriction of \( \mu \) to \( g_\mu \), then, choosing a \( G_\mu \)-invariant section \( \alpha_\mu \) of \( A^* \) with values in \((J^\mu_A)^{-1}(\tilde{\mu})\), there exists a canonical embedding
\[
(T^\lambda A^*)_{\mu} \to T^{A_{0,\mu}}A_{0,\mu}^*
\]
from the reduced algebroid \((T^*A^*)\) equipped with the canonical reduced symplectic-like structure \(\Omega_{\mu}\) to the Lie algebroid \(T^{\alpha_{0,\mu}}A_{\nu,\mu}^*\) endowed with the symplectic-like structure

\[
\tilde{\Omega}_\mu = \Omega_{\alpha_{0,\mu}} - (pr_1)^*B_\mu.
\]

Moreover, this embedding is an isomorphism if and only if \(g = g_\mu\).

Here \(A_{\alpha_{0,\mu}}\) is the vector bundle

\[
A_{\alpha_{0,\mu}} = A/TG_\mu \rightarrow M/G_\mu.
\]

\(\Omega_{\alpha_{0,\mu}}\) is the canonical symplectic-like structure on \(T^{\alpha_{0,\mu}}A_{\nu,\mu}^*\), \(pr_1 : T^{\alpha_{0,\mu}}A_{\nu,\mu}^* \rightarrow A_{\alpha_{0,\mu}}\) is the projection on the first factor and \(B_\mu \in \Gamma(\wedge^2A_0^*)\) is the corresponding magnetic term associated with \(\alpha_\mu\) which is characterized by \((5.10)\).

**Examples 5.4.**

(i) If we apply the previous theorem to the particular case when \(A\) is the standard Lie algebroid \(TM \rightarrow M\) then we recover a classical result in cotangent bundle reduction theory (see [1, 12]).

(ii) For the case \(A = g \times TM\) from (ii) in examples 3.2, we have seen that the vector bundle \(T^*A^* \rightarrow A^*\) can be identified with \(g \times T (g^* \times TM) \rightarrow g^* \times T^*M\) (see example 4.1).

Moreover, the Lie algebroid \((g \times TM)/TG \rightarrow M/G\) is isomorphic to the Atiyah algebroid associated with the principal bundle \(\pi_M : M \rightarrow M/G\) (see \((3.15)\)).

Then, the reduced symplectic-like Lie algebroid \((T^*A^*)_0\), for the value \(\mu = 0 \in g^*\), is symplectically isomorphic to the canonical cover of the fiberwise linear Poisson structure of \((T^*M)/G\) induced by the Atiyah Lie algebroid \((TM)/G \rightarrow M/G\), i.e. \(T(T^*M)/G \rightarrow (T^*M)/G\). In fact, this last Lie algebroid is just the Atiyah algebroid associated with the principal bundle \(\pi_{T^*M} : T^*M \rightarrow (T^*M)/G\) (see \([13]\)) and its symplectic-like structure \(\Omega_{(T^*M)/G} \in \Gamma(\wedge^2(T^*(T^*M)/G))\) is the one induced by the \(G\)-invariant symplectic structure on \(T^*M\).

Now, we choose \(\mu \in g^*\) such that \(G = G_\mu\) and a \(G\)-invariant 1-form \(\alpha_\mu \in \Omega^1(M)\) on \(M\) such that \(\alpha_\mu(M) \subset J^{-1}(\mu)\), where \(J : T^*M \rightarrow g^*\) is the momentum map given as in \((1.1)\). Then, the reduced symplectic-like Lie algebroid \((T^*A^*)_\mu\) is symplectically isomorphic to the Atiyah algebroid associated with the principal bundle \(\pi_{T^*M} : T^*M \rightarrow (T^*M)/G\) endowed with the symplectic-like structure

\[
\Omega_{(T^*M)/G} - Y_\mu,
\]

where \(Y_\mu \in \Gamma(\wedge^2(T^*(M)/G))\) is the 2-section obtained from a magnetic term defined as follows.

We consider the epimorphism

\[
\tilde{\pi} : g \times TM \rightarrow TM/G, \quad \tilde{\pi}(\xi, v) = [v_\xi + \xi_M(x)].
\]

Then, we have that there exists \(B_\mu \in \Gamma(\wedge^2(T^*(M)/G))\) such that

\[
\tilde{\pi}^*B_\mu = (0, d\alpha_\mu).
\]

Finally, \(\gamma_\mu\) is just

\[
(T\tau_{T^*G}/G)^*B_\mu,
\]

where \(T\tau_{T^*G}/G : (T(T^*M))/G \rightarrow (TM)/G\) is the vector bundle induced by the equivariant tangent lift \(T\tau_{T^*M} : T(T^*M) \rightarrow TM\) of \(\tau_{T^*M} : T^*M \rightarrow M\).

We finish this paper with an application which is related with the reduction of non-autonomous Hamiltonian systems.
Example 5.5. Let \( p : M \to \mathbb{R} \) be a fibration. We denote by \( \tau_{V_p} : V_p \to M \) the vertical bundle associated with \( p \). Note that the sections of this vector bundle may be identified with the vector fields \( X \) on \( M \) such that \( \eta(X) = 0 \), where \( \eta \) is the exact 1-form \( p^*(dt) \) on \( M \), \( t \) being the standard coordinate on \( \mathbb{R} \).

This vector bundle admits, in a natural way, a Lie algebroid structure where the Lie bracket is the standard Lie bracket of vector fields and the anchor map is the inclusion of vertical vectors with respect to \( p \) into \( TM \).

Now, suppose that we additionally have a free and proper action \( \phi : G \times M \to M \) of a Lie group \( G \) on \( M \) which is fibered, i.e.

\[
p \circ \phi_g = p \quad \text{for all } g \in G.
\]

Then,

(i) the infinitesimal generators of this last action are vertical vector fields;
(ii) the tangent lifted action \( T\phi : G \times TM \to TM \) induces a free and proper action

\[
\Phi : G \times V_p \to V_p
\]

of \( G \) on the vertical vector bundle \( V_p \) of \( p \);
(iii) \( p \) induces a new fibration \( \tilde{p} : M/G \to \mathbb{R} \) on the quotient manifold \( M/G \).

Then, \( \Phi : G \times V_p \to V_p \) is an action by complete lifts with respect to the Lie algebra anti-morphism

\[
\psi : g \to V_p, \quad \psi(\xi) = \xi_M.
\]

Let \( \mu \) be an element of \( g^* \) and we denote by \( J_{V_p} : V^* p \to g^* \) the momentum map defined as in (4.8). Then, we have that the vector bundle \( T^M V_p(J_{V_p}^{-1}(\mu)) \to J_{V_p}^{-1}(\mu) \) may be identified in a natural way with the vertical bundle \( V_p(\mu) \to J_{V_p}^{-1}(\mu) \), where \( p_\mu : J_{V_p}^{-1}(\mu) \to \mathbb{R} \) is the fibration given by

\[
p_\mu = p \circ \tau_{J_{V_p}^{-1}(\mu)}.
\]

\[
\tau_{J_{V_p}^{-1}(\mu)} : J_{V_p}^{-1}(\mu) \to M
\]

being the corresponding projection. Under this identification the action \( (\Phi, T\Phi^*) : G_\mu \times V_p(\mu) \to V_p(\mu) \) given by (4.4) is described as follows. Consider the tangent lift \( T\Phi^* : G_\mu \times T(J_{V_p}^{-1}(\mu)) \to T(J_{V_p}^{-1}(\mu)) \) of the restricted dual action \( \Phi^* : G_\mu \times J_{V_p}^{-1}(\mu) \to J_{V_p}^{-1}(\mu) \). Since

\[
p_\mu \circ \Phi^* = p_\mu,
\]

we may induce an action of \( G_\mu \) on \( V_p(\mu) \) which is just \( (\Phi, T\Phi^*) \). Therefore, the action \( (\Phi, T\Phi^*)^T : T\mu \times V_p(\mu) \to V_p(\mu) \) is given by

\[
(\Phi, T\Phi^*)^T((g, \xi), v_\mu) = T_\mu(\Phi^*_g(v_\mu + \xi_\mu^*(\alpha_\mu))
\]

for all \( (g, \xi) \in G_\mu \times g_\mu \cong TG_\mu \) and \( \alpha_\mu \in V^* p \). Here, \( \xi_\mu^* \in \mathfrak{X}(V^* p) \) is the complete lift to \( V^* p \) of the infinitesimal generator of \( \xi \) with respect to the action \( \phi \).

Finally, from theorem 3.11, we conclude that the reduced vector bundle

\[
(V_p(\mu)/TG_\mu) \to J_{V_p}^{-1}(\mu)/G_\mu
\]

is a symplectic-like Lie algebroid.

If \( \mu = 0 \), then using theorem 5.1, we have that this symplectic-like Lie algebroid is isomorphic to \( T^{ha}A_0 \) where \( A_0 \) is the quotient vector bundle over \( M/G \) with total space \( V_p/TG \). We remark that the action of \( TG \cong G \times g \) on \( V_p \) is given by

\[
\Phi^T((g, \xi), v_\mu) = T_\mu(\phi_g(v_\mu + \xi_M(\mu))
\]
for \((g, \xi) \in G \times g\) and \(v_1 \in V_p\). In fact, the vector bundle \(A_0\) is isomorphic to the vertical bundle \(V\bar{p}\) with respect to the fibration \(\bar{p}: M/G \to \mathbb{R}\). The isomorphism is just
\[
V_p/TG \to V\bar{p}, \quad [v_1] \mapsto T\pi(v_1)
\]
where \(\pi: M \to M/G\) is the canonical projection. Therefore, \(T^{A_0}A^*_0\) may be identified with the vertical bundle \(V\bar{p} \to V^*\bar{p}\) with
\[
\bar{p} = \bar{\pi} \circ \tau_{V\bar{p}}: V^*\bar{p} \to \mathbb{R},
\]
where \(\tau_{V\bar{p}}: V^*\bar{p} \to M/G\) is the corresponding vector bundle projection. In conclusion, the reduced Lie algebroid \((Vp_0)/TG \to \tilde{J}_{Vp_0}(0)/G\) is canonically isomorphic to the Lie algebroid \(V\bar{p}\) on \(V^*\bar{p}\) with its standard symplectic-like structure.

Now, we consider \(\mu \in g^*\) such that \(G_\mu = G\). Let \(\alpha_\mu\) be a \(G\)-invariant 1-form on \(M\) such that
\[
\alpha_\mu(\xi_M) = \mu(\xi) \quad \text{for all } \xi \in g.
\]
Then, the restriction \(\alpha_\mu|_{Vp}\) of \(\alpha_\mu\) to the vertical bundle \(Vp\) of the fibration \(p: M \to \mathbb{R}\) determines a \(G\)-invariant section of \(V^*p\) with values in \(J_{V^*p}^{-1}(\mu)\).

Let \(\beta_\mu = d^{Vp}(\alpha_\mu|_{Vp})\). Equivalently, \(\beta_\mu\) is the restriction of \(d\alpha_\mu \in \Omega^2(M)\) to \(Vp \times Vp\). The magnetic term \(B_\mu\) associated with \(\alpha_\mu\) is the restriction to \(V\bar{p} \times V\bar{p}\) of the unique 2-form \(\bar{B}_\mu\) of \(M/G\) such that
\[
\pi^*\bar{B}_\mu = d\alpha_\mu,
\]
where \(\pi: M \to M/G\) is the quotient projection. Moreover, using theorem 5.2, we have that \((Vp_\mu)/TG\) is a symplectic-like Lie algebroid on \(J_{Vp}^{-1}(\mu)/G\) isomorphic to the Lie algebroid \(V\bar{p}\) endowed with the symplectic-like section
\[
\Omega_{V\bar{p}} = pr^1\bar{B}_\mu \in \Gamma(\wedge^2V^*\bar{p})
\]
with \(pr_1: V\bar{p} \to V\bar{p}\) the vector bundle morphism on \(\tau_{V\bar{p}}: V^*\bar{p} \to M/G\) given by the restriction to \(V\bar{p}\) of the tangent lift \(T\tau_{V\bar{p}}: T(V^*\bar{p}) \to T(M/G)\).

6. Conclusions and future work

In this paper, we have proved a reduction theorem for Lie algebroids with respect to a Lie group action by complete lifts. This result allows us to obtain a Lie algebroid version of the classic Marsden–Weinstein reduction theorem for symplectic manifolds. We remark that as for the usual Marsden–Weinstein reduction theorem, the presence of the Lie group is superfluous, and the infinitesimal action of its Lie algebra is sufficient, although we have chosen not to use this approach here.

Additionally, in this paper, the Marsden–Weinstein reduction process for symplectic-like Lie algebroids is applied to the particular case of the canonical cover of a fiberwise Poisson structure. It would be interesting to also obtain an analog version to the ‘bundle’ or ‘fibrating’ picture of cotangent bundle reduction in the setup of symplectic-like Lie algebroids, but this will be studied elsewhere.

It is also worth noticing that the classical Marsden–Weinstein reduction scheme not only explains how to obtain a reduced symplectic structure on a quotient manifold, but also shows that the reduced dynamics of a symmetric Hamiltonian function is again Hamiltonian with respect to this reduced symplectic structure. It is easy to prove that a similar phenomenon occurs for the reduction of symplectic-like algebroids by complete actions. In fact, under the same hypotheses as in theorem 3.11, if \(H: M \to \mathbb{R}\) is a \(G\)-invariant Hamiltonian function, then one can prove that the restriction to \(J^{-1}(\mu)\) of \(H\) is a \(G_\mu\)-invariant function, and thus, one can
induce a real smooth function \( H_\mu \) on \( J^{-1}(\mu)/G_\mu \). Moreover, the restriction of the Hamiltonian section \( \mathcal{H}_\mu^G \) to \( J^{-1}(\mu) \) is a section of the Lie algebroid \( (J^\mu(-)^1(0, \mu) \to J^{-1}(\mu)) \) which is \( (\tilde{\pi}_\mu, \pi_\mu) \)-projectable on the Hamiltonian section \( \mathcal{H}_\mu^G \). Thus, if \( \gamma : I \to M \) is a solution of Hamilton’s equations for \( H \) on the symplectic-like Lie algebroid \( A \to M \) passing through a point in \( J^{-1}(\mu) \), then the curve \( \gamma \) is contained in \( J^{-1}(\mu) \) and \( \pi_\mu \circ \gamma : I \to J^{-1}(\mu)/G_\mu \) is a solution of Hamilton’s equations for \( H_\mu \) on the reduced symplectic-like Lie algebroid \( A_\mu \to J^{-1}(\mu)/G_\mu \).

In view of the results of this paper, one could apply this process to the reduction of symmetric Hamiltonian systems on Poisson manifolds. We have postponed this study for a future work.

Acknowledgments

The authors have been partially supported by MEC (Spain) grants MTM2009-13383 and MTM2009-08166-E. The research of MR-O has also been partially supported by a Marie Curie Intra European Fellowship PIEF-GA-2008-220239 and a Marie Curie Reintegration grant PERG-GA-2010-27697. The research of IC-M and E-P has also been partially supported by the grants of the Canary government SOLSUBC20081000238 and ProID201000210. We also would like to thank D Iglesias, D Martín de Diego and E Martínez for their useful comments.

References

[1] Abraham R and Marsden J E 1987 Foundations of Mechanics 2nd edn (Reading, MA: Addison-Wesley)
[2] Bursztyn H, Cavalcanti G R and Gualtieri M 2007 Reduction of Courant algebroids and generalized complex structures Adv. Math. 211 726–65
[3] Cariñena J F, Nunes da Costa J M and Santos P 2005 Reduction of Lie algebroid structures Int. J. Geom. Methods Mod. Phys. 2 965–91
[4] Courant T J 1990 Dirac manifolds Trans. Am. Math. Soc. 319 631–61
[5] Grabowski J and Urbański P 1997 Tangent and cotangent lifts and graded Lie algebras associated with Lie algebroids Ann. Global Anal. Geom. 15 447–86
[6] Grabowski J and Urbański P 1997 Lie algebroids and Poisson–Nijenhuis structures Rep. Math. Phys. 40 195–208
[7] Fernandes R L 2002 Lie algebroids, holonomy and characteristic classes Adv. Math. 170 119–79
[8] Fernandes R L, Ortega J P and Ratiu T S 2009 The momentum map in Poisson geometry Am. J. Math. 131 1261–310
[9] Higgins P J and Mackenzie K 1990 Algebraic constructions in the category of Lie algebroids J. Algebra 129 194–230
[10] Iglesias D, Marrero J C, Martín de Diego D, Martínez E and Padrón E 2007 Reduction of symplectic Lie algebroids by a Lie subalgebroid and a symmetry Lie group Symmetry Integrability Geom.: Methods Appl. 3 049
[11] Kosmann-Schwarzbach Y 1995 Exact Gerstenhaber algebras and Lie bialgebroids. Geometric and algebraic structures in differential equations Acta Appl. Math. 41 153–65
[12] Kummer M 1991 On the construction of the reduced phase space of a Hamiltonian system with symmetry Indiana Univ. Math. J. 30 281–91 2
[13] de León M, Marrero J C and Martínez E 2005 Lagrangian submanifolds and dynamics on Lie algebroids J. Phys. A: Math. Gen. 38 R241–308
[14] Liu Z J, Weinstein A and Xu P 1997 Manin triples for Lie bialgebroids J. Diff. Geom. 45 547–74
[15] Mackenzie K 2005 General Theory of Lie Groupoids and Lie Algebroids (London Mathematical Society Lecture Note Series vol 213) (Cambridge: Cambridge University Press)
[16] Mackenzie K and Xu P 1994 Lie bialgebroids and Poisson groupoids Duke Math. J. 73 415–52
[17] Mackenzie K and Xu P 1998 Classical lifting processes and multiplicative vector fields Q. J. Math. 49 59–85
[18] Marsden J E, Misiolek G, Ortega J P, Perlmutter M and Ratiu T S 2007 Hamiltonian Reduction by Stages (Lecture Notes in Mathematics vol 1913) (Berlin: Springer)
[19] Marsden J E and Ratiu T 1986 Reduction of Poisson manifolds Lett. Math. Phys. 11 161–9

33
[20] Marsden J E and Weinstein A 1974 Reduction of symplectic manifolds with symmetry Rep. Math. Phys. 5 1 121–30
[21] Martínez E 2004 Classical field theory on Lie algebroids: multisymplectic formalism arXiv:math/0411352
[22] Martínez E 2011 private communication
[23] Ortega J P and Ratiu T S 2004 Momentum Maps and Hamiltonian Reduction (Progress in Mathematics vol 222) (Boston, MA: Birkhäuser Boston)
[24] Palais R S 1957 A global formulation of the Lie theory of transformation groups Mem. AMS 22
[25] Roggiero Ayala J P 2011 Lie algebroids over quotient spaces PhD Thesis Instituto Superior Tecnico, Lisbon
[26] Satzer J W 1977 Canonical reduction of mechanical systems invariant under Abelian group actions with an application to celestial mechanics Indiana Univ. Math. J. 26 951–76