Probabilistic Reasoning in the Description Logic $\mathcal{ALCP}$ with the Principle of Maximum Entropy

Rafael Peñaloza$^1$ and Nico Potyka$^2$

$^1$ Free University of Bozen-Bolzano, Italy
rafael.penaloza@unibz.it
$^2$ University of Osnabrück, Germany
npotyka@uni-osnabrueck.de

Abstract. A central question for knowledge representation is how to encode and handle uncertain knowledge adequately. We introduce the probabilistic description logic $\mathcal{ALCP}$ that is designed for representing context-dependent knowledge, where the actual context taking place is uncertain. $\mathcal{ALCP}$ allows the expression of logical dependencies on the domain and probabilistic dependencies on the possible contexts. In order to draw probabilistic conclusions, we employ the principle of maximum entropy. We provide reasoning algorithms for this logic, and show that it satisfies several desirable properties of probabilistic logics.

1 Introduction

A fundamental element of any intelligent application is storing and manipulating the knowledge from the application domain. Logic-based knowledge representation languages such as description logics (DLs) [1] provide a clear syntax and unambiguous semantics that guarantee the correctness of the results obtained. However, languages based on classical logic are ill-suited for handling the uncertainty inherent to many application domains. To overcome this limitation, various probabilistic logics have been investigated during the last three decades (e.g., [3,15,20]). In particular, several probabilistic DLs have been developed [18,19]. To handle probabilistic knowledge, many approaches require a complete definition of joint probability distributions (JPD) [5,6,8,16,25]. One approach to avoid a full JPD specification was proposed by Paris [22]: the user gives a partial specification through a set of probabilistic constraints and the partial knowledge is completed by means of the principle of maximum entropy.

In this paper we consider a new probabilistic extension of description logics based on the principle of maximum entropy. In our approach we group different axioms from a knowledge base together into so-called contexts, which are identified by a propositional formula. Intuitively, each context corresponds to a possible situation, in which the associated sub-KB is guaranteed to hold. Uncertainty is associated to the contexts through a set of probabilistic constraints, which are interpreted under the principle of maximum entropy.

To facilitate the understanding of our approach, we focus on the DL $\mathcal{ALC}$ [26] as a prototypical example of a knowledge representation language, and propositional probabilistic constraints as the framework for expressing uncertainty.
As reasoning service we consider subsumption relations between concepts given some partial knowledge of the current context. Since the knowledge in a knowledge base is typically incomplete, one cannot expect to obtain a precise probability for a given consequence. Instead, we compute a belief interval that describes all the probability degrees that can be associated to the consequence without contradiction. The lowest bound of the interval corresponds to a sceptical view, considering only the most fundamental models of the knowledge base. The upper bound, in contrast, reflects the credulous belief in which every context that is not explicitly removed is considered. In the worst-case, we get the trivial interval $[0, 1]$, in the best case, we get a point probability where the upper and lower bounds coincide. In some applications, it might be reasonable to consider only one of these bounds. For instance, if the probability interval that a treatment will cause heavy complications is $[0.01, 0.05]$, we might want to use the upper bound $0.05$. In contrast, when the probability interval that a treatment will be successful is $[0.7, 0.9]$, we might be more interested in the lower bound $0.7$.

The main contributions of this paper are the following:

- we define the new probabilistic description logic $ALCP$ that allows for a flexible description of axiomatic dependencies, and its reasoning problems (Section 3);
- we explain in detail how degrees of belief for the subsumption problem can be computed (Section 4); and
- we show that $ALCP$ satisfies several desirable properties of probabilistic logics (Section 5).

2 Maximum Entropy

We start by recalling the basic notions of probabilistic propositional logic and the principle of maximum entropy.

Let $L$ be a propositional language constructed over a finite signature $\text{sig}(L)$, i.e., a set of propositional variables, in the usual way. An $L$-interpretation $v$ is a truth assignment of the propositional variables in $\text{sig}(L)$. $\text{Int}(L)$ denotes the set of all $L$-interpretations. Satisfaction of a formula $\phi \in L$ by an $L$-interpretation $v \in \text{Int}(L)$ (denoted $v \models \phi$) is defined as usual. A probability distribution over $L$ is a function $P : \text{Int}(L) \rightarrow [0, 1]$ where $\sum_{v \in \text{Int}(L)} P(v) = 1$. Probability distributions are extended to arbitrary $L$-formulas $\phi$ by setting $P(\phi) = \sum_{v \models \phi} P(v)$.

**Definition 1 (probabilistic constraints, models).** Given the propositional language $L$, a probabilistic constraint (over $L$) is an expression of the form

$$c_0 + \sum_{i=1}^{k} c_i \cdot p(\phi_i) \geq 0$$

where $c_0, c_i \in \mathbb{R}$, and $\phi_i \in L$, $1 \leq i \leq k$. A probability distribution $P$ over $L$ is a model of the probabilistic constraint $c_0 + \sum_{i=1}^{k} c_i \cdot p(\phi_i) \geq 0$ if and only if $c_0 + \sum_{i=1}^{k} c_i \cdot P(\phi_i) \geq 0$. The distribution $P$ is a model of the set of probabilistic
constraints $R \models R$ iff it satisfies all the constraints in $R$. The set of all models of $R$ is denoted by $\text{Mod}(R)$. If $\text{Mod}(R) \neq \emptyset$, we say that $R$ is consistent.

Our probabilistic constraints can express the most common types of constraints considered in the literature of probabilistic logics. For instance, probabilistic conditionals $(\psi \mid \phi)[\ell,u]$ are satisfied iff $\ell \cdot P(\phi) \leq P(\psi \land \phi) \leq u \cdot P(\phi)$ \cite{17}. That is, the conditional is satisfied iff the conditional probability of $\psi$ given $\phi$ is between $\ell$ and $u$ whenever $P(\phi) > 0$. Sometimes $P(\phi) > 0$ is demanded, but strict inequalities are computationally difficult and the semantical differences are negligible in many cases, see \cite{24} for a thorough discussion. These conditions can be expressed in the form \cite{11} as follows

\begin{align*}
p(\psi \land \phi) - \ell \cdot p(\phi) & \geq 0, \quad \text{and} \\
u \cdot p(\phi) - p(\psi \land \phi) & \geq 0.
\end{align*}

Probabilistic constraints can also express more complex restrictions; for example, we can state that the probability that a bird cannot fly is at most one fourth of the probability that a bird flies through the constraint

\[ \frac{1}{4} p(\text{bird} \land \text{flies}) - p(\text{bird} \land \neg \text{flies}) \geq 0. \] (2)

To improve readability, we will often rewrite constraints in a more compact manner, using conditionals as in the first example, or e.g. rewriting (2) as \[ \frac{1}{4} p(\text{bird} \land \text{flies}) \geq p(\text{bird} \land \neg \text{flies}). \]

In general, consistent sets of probabilistic constraints have infinitely many models, and there is no obvious way to distinguish between them. One well-studied approach for dealing with this diversity is to focus on the model that maximizes the entropy

\[ H(P) = - \sum_{v \in \text{Int}(\mathcal{L})} P(v) \cdot \log P(v). \]

From an information-theoretic point of view, the maximum entropy (ME) distribution can be regarded as the most conservative one in the sense that it minimizes the information-theoretic distance (that is, the KL-divergence) to the uniform distribution among all probability distributions that satisfy our constraints. In particular, if there are no restrictions on the probability distributions considered, then the uniform distribution is the ME distribution, see, e.g., \cite{27} for a more detailed discussion of these issues. A complete characterization of maximum entropy for the purpose of uncertain reasoning can be found in \cite{24}.

**Definition 2 (ME-model).** Let $R$ be a consistent set of probabilistic constraints. The ME-model $P^\text{ME}_R$ of $R$ is the unique solution of the maximization problem $\arg \max_{P \models R} H(P)$.

Existence and uniqueness of $P^\text{ME}_R$ follows from the fact that $H$ is strictly concave and continuous, and that the probability distributions that satisfy $R$ form a
compact and convex set. \( P^\text{ME} \) is usually computed by deriving an unconstrained optimization problem by means of the Karush-Kuhn-Tucker conditions. The resulting problem can be solved, for instance, by (quasi-)Newton methods with cost \(|\text{Int}(L)|^3\), see, e.g., [21] for more details on these techniques.

3 The Probabilistic Description Logic \( \mathcal{ALCP} \)

\( \mathcal{ALCP} \) is a probabilistic extension of the classical description logic \( \mathcal{ALC} \) capable of expressing complex logical and probabilistic relations. As with classical DLs, the main building blocks in \( \mathcal{ALCP} \) are concepts. Syntactically, \( \mathcal{ALCP} \) concepts are constructed exactly as \( \mathcal{ALC} \) concepts. Given two disjoint sets \( N_C \) of concept names and \( N_R \) of role names, \( \mathcal{ALCP} \) concepts are built using the grammar rule:

\[
C ::= A \mid \neg C \mid C \sqcap C \mid \exists r.C,
\]

where \( A \in N_C \) and \( r \in N_R \). Note that we can derive disjunction, universal quantification and subsumption from these rules by using logical equivalences like \( C_1 \sqcup C_2 \equiv \neg (\neg C_1 \sqcap \neg C_2) \). The knowledge of the application domain is expressed through a finite set of axioms that restrict the way the different concepts and roles may be interpreted. To express both probabilistic and logical relationships, each axiom is annotated with a formula from \( L \) that intuitively expresses the context in which this axiom holds.

Definition 3 (KB). An \( L \)-restricted general concept inclusion (\( L \)-GCI) is of the form \( \langle C \sqsubseteq D : \kappa \rangle \) where \( C, D \) are \( \mathcal{ALCP} \) concepts and \( \kappa \) is an \( L \)-formula. An \( L \)-TBox is a finite set of \( L \)-GCIs. An \( \mathcal{ALCP} \) knowledge base (KB) over \( L \) is a pair \( \mathcal{K} = (R, T) \) where \( R \) is a set of probabilistic constraints and \( T \) is an \( L \)-TBox.

Example 4. Consider an application modeling beliefs about bacterial and viral infections using the concept names \( \text{strep} \) (streptococcal infection), \( \text{bac} \) (bacterial infection), \( \text{vir} \) (viral infection), \( \text{inf} \) (infection), and \( \text{ab} \) (antibiotic); and the role names \( \text{sf} \) (suffers from), and \( \text{suc} \) (successful treatment); and the propositional variables \( \text{res} \) (antibiotic resistance), and \( \text{h} \) (heavy use of antibiotics by patient). Define the \( L \)-TBox \( T_{\text{exa}} \) containing the \( L \)-GCIs:

\[
\langle \exists \text{sf}.\text{bac} \sqsubseteq \exists \text{suc}.\text{ab} : \neg \text{res} \neg \text{h} \rangle, \quad \langle \exists \text{sf}.\text{vir} \sqsubseteq \neg \exists \text{suc}.\text{ab} : \top \rangle, \quad \langle \text{strep} \sqsubseteq \text{bac} : \top \rangle, \\
\langle \exists \text{sf}.\text{bac} \sqsubseteq \neg \exists \text{suc}.\text{ab} : \text{res} \rangle, \quad \langle \text{bac} \sqsubseteq \text{inf} : \top \rangle, \quad \langle \text{vir} \sqsubseteq \text{inf} : \top \rangle,
\]

where \( \top \) is any \( L \)-tautology. For example, the first \( L \)-GCI states that a bacterial infection can be treated successfully with antibiotics if no antibiotic resistance is present and there was no heavy use of antibiotics; the second one states that viral infections can never be treated with antibiotics successfully. Consider additionally the set \( R \) containing the probabilistic constraints containing:

\[
(\text{res} | 0.05), \quad (\text{res} \mid \text{h} | 0.8).
\]

That is, the probability of an antibiotic resistance is 5% if no further information is given. However, if the patient used antibiotics heavily, the probability increases to 80%.
Notice that the probabilistic constraints, and hence the representation of the uncertainty in the knowledge, refer only to the propositional formulas that label the L-GCIs. In \( ALCP \), the uncertainty of the knowledge is handled through these propositional formulas as explained next.

A possible world interprets both the axiom language (i.e., the concept and role names) and the context language (the propositional variables). Intuitively, it describes a possible context (\( \mathcal{L} \)-interpretation) together with the relationships between concepts in that situation (\( ALCP \)-interpretation).

**Definition 5 (possible world).** A possible world is a triple \( \mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I}, v^\mathcal{I}) \) where \( \Delta^\mathcal{I} \) is a non-empty set (called the domain), \( v^\mathcal{I} \) is an \( \mathcal{L} \)-interpretation, and \( \mathcal{I} \) is an interpretation function that maps every concept name \( A \) to a set \( A^\mathcal{I} \subseteq \Delta^\mathcal{I} \) and every role name \( r \) to a binary relation \( r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I} \).

The interpretation function \( \mathcal{I} \) is extended to complex concepts as usual in DLs by letting \( (\neg C)^\mathcal{I} := \Delta^\mathcal{I} \setminus C^\mathcal{I} \); \( (\exists r.C)^\mathcal{I} := \{ d \in \Delta^\mathcal{I} \mid \exists e \in \Delta^\mathcal{I}.(d, e) \in r^\mathcal{I}, e \in C^\mathcal{I} \} \); and \( (C \cap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I} \). A possible world is a model of an \( \mathcal{L} \)-GCI if it satisfies the description logic constraint of the axiom whenever it satisfies the context.

**Definition 6 (model of TBox).** The possible world \( \mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I}, v^\mathcal{I}) \) is a model of the \( \mathcal{L} \)-GCI \( \langle C \sqsubseteq D : \kappa \rangle \), denoted as \( \mathcal{I} \models \langle C \sqsubseteq D : \kappa \rangle \), iff (i) \( v^\mathcal{I} \nvdash \kappa \), or (ii) \( C^\mathcal{I} \subseteq D^\mathcal{I} \). It is a model of the \( \mathcal{L} \)-TBox \( T \) if it is a model of all the \( \mathcal{L} \)-GCIs in \( T \).

The classical DL \( ALC \) is a special case of \( ALCP \) where all the axioms are annotated with an \( \mathcal{L} \)-tautology \( \top \). To preserve the syntax of classical DLs, we denote such \( \mathcal{L} \)-GCIs as \( C \sqsubseteq D \) instead of \( \langle C \sqsubseteq D : \top \rangle \). In this case, the condition (i) from Definition 6 cannot be satisfied, and hence a model is required to satisfy \( C^\mathcal{I} \subseteq D^\mathcal{I} \) for all \( \mathcal{L} \)-GCIs \( C \sqsubseteq D \) in the TBox. For a deeper introduction to classical \( ALC \), see [1].

According to our semantics, we only demand that the \( \mathcal{L} \)-GCIs are satisfied in some specific contexts. Thus, it is often useful to focus on the classical \( ALC \) TBox that contains the knowledge that holds in a particular situation. For a KB \( K = (R, T) \) and \( v \in \text{Int} (\mathcal{L}) \), the \( v \)-restricted TBox is the \( ALC \) TBox

\[ T_v := \{ C \sqsubseteq D \mid \langle C \sqsubseteq D : \kappa \rangle \in T, v \models \kappa \}. \]

The possible world \( \mathcal{I} \) satisfies \( T_v \) (\( \mathcal{I} \models T_v \)) if for all \( \mathcal{L} \)-GCIs \( C \sqsubseteq D \in T_v \) it holds that \( C^\mathcal{I} \subseteq D^\mathcal{I} \). In the following, we will often consider subsumption and strong non-subsumption between concepts w.r.t. a restricted TBox. We say that \( C \) is \( v \)-subsumed by \( D \) w.r.t. \( T_v \) (\( T_v \models C \sqsubseteq D \)) if for every \( \mathcal{I} \models T_v \) it holds that \( C^\mathcal{I} \subseteq D^\mathcal{I} \). Dually, \( C \) is \( v \)-strongly non-subsumed by \( D \) w.r.t. \( T_v \) (\( T_v \models C \nsubseteq D \)) if for every \( \mathcal{I} \models T_v \) it holds that \( C^\mathcal{I} \nsubseteq D^\mathcal{I} \). Notice that strong non-subsumption requires that the inclusion between axioms does not hold in any possible world satisfying \( T_v \). Hence, this condition is more strict than just negating the subsumption relation.

We now describe how the probabilistic constraints are handled in our logic. An \( ALCP \)-interpretation consists of a finite set of possible worlds and a probability function over these worlds.
Definition 7 (\textit{ALCP}-interpretation). An \textit{ALCP}-interpretation is a pair of the form $\mathcal{P} = (\mathcal{I}, P_\mathcal{I})$, where $\mathcal{I}$ is a non-empty, finite set of possible worlds and $P_\mathcal{I}$ is a probability distribution over $\mathcal{I}$.

Each \textit{ALCP}-interpretation induces a probability distribution over $\mathcal{L}$. The probability of a context can be obtained by adding the probabilities of all possible worlds in which this context holds.

Definition 8 (distribution induced by $\mathcal{P}$). Let $\mathcal{P} = (\mathcal{I}, P_\mathcal{I})$ be an \textit{ALCP}-interpretation. The probability distribution $P_\mathcal{P} : \text{Int}(\mathcal{L}) \rightarrow [0, 1]$ induced by $\mathcal{P}$ is defined by $P_\mathcal{P}(v) := \sum_{I \in \mathcal{I}, I | v = v} P_\mathcal{I}(I)$, where $I | v = v = \{(\Delta^I, \mathcal{I}, v^I) \in \mathcal{I} | v^I = v\}$.

As usual, reasoning is restricted to interpretations that satisfy the restrictions imposed by the knowledge base. In our case, we have to demand that the interpretation is consistent with both the classical and the probabilistic part of our knowledge base. That is, we consider only those possible worlds that satisfy both the terminological knowledge ($\mathcal{T}$) and the probabilistic constraints ($\mathcal{R}$).

Definition 9 (model). Let $\mathcal{P} = (\mathcal{I}, P_\mathcal{I})$ be an \textit{ALCP}-interpretation. $\mathcal{P}$ is consistent with the TBox $\mathcal{T}$ if every $I \in \mathcal{I}$ is a model of $\mathcal{T}$. $\mathcal{P}$ is consistent with the set of probabilistic constraints $\mathcal{R}$ iff $P_\mathcal{P} | v = \mathcal{R}$. The \textit{ALCP}-interpretation $\mathcal{P}$ is a model of the KB $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ iff it is consistent with both $\mathcal{T}$ and $\mathcal{R}$. As usual, a KB is consistent iff it has a model.

Notice that \textit{ALCP}-KBs can express both, logical and probabilistic dependencies between axioms. For instance, two $\mathcal{L}$-GCIs $\langle C_1 \sqsubseteq D_1 : \kappa_1 \rangle$ and $\langle C_2 \sqsubseteq D_2 : \kappa_2 \rangle$ where $\kappa_1 \Rightarrow \kappa_2$ express that whenever the first $\mathcal{L}$-GCI is satisfied, the second one must also hold. Similarly, the probabilistic dependencies between axioms are expressed via the probabilistic constraints of the labeling formulas.

We are interested in computing degrees of belief for subsumption relations between concepts. We define the conditional probability of a subsumption relation given a context with respect to a given \textit{ALCP}-interpretation following the usual notions of conditioning.

Definition 10 (probability of subsumption). Let $C, D$ be concepts, $\kappa$ a context and $\mathcal{P}$ an \textit{ALCP}-interpretation. The conditional probability of $C \sqsubseteq D$ given $\kappa$ with respect to $\mathcal{P}$ is

$$Pr_\mathcal{P}(C \sqsubseteq D | \kappa) := \frac{\sum_{I \in \mathcal{I}, I | v = \kappa, I | v = c \subseteq d} P_\mathcal{I}(I)}{\sum_{I \in \mathcal{I}, I | v = \kappa} P_\mathcal{I}(I)}.$$  \hspace{2cm} (3)

Notice that the denominator in (3) can be rewritten as

$$\sum_{I \in \mathcal{I}, I | v = \kappa} P_\mathcal{I}(I) = \sum_{v = \kappa} \sum_{I \in \mathcal{I}, I | v} P_\mathcal{I}(I) = \sum_{v = \kappa} P_\mathcal{P}(v) = P_\mathcal{P}(\kappa).$$

As usual, the conditional probability is only well-defined when $P_\mathcal{P}(\kappa) > 0$.

Recall that the set of probabilistic constraints $\mathcal{R}$ may be satisfied by an infinite class of probability distributions. In the spirit of maximum entropy reasoning, we consider only the most conservative ones in the sense that they induce the ME-model $P_{\mathcal{R}}^{\text{ME}}$ of $\mathcal{R}$. 

Definition 11 (ME-ALCP-model). An ALCP-model $P$ of $K$ is called an ME-ALCP-model of $K$ iff $P^R = P^R ME$. The set of all ME-ALCP-models of $K$ is denoted by $\text{Mod}_{ME}(K)$. $K$ is called ME-consistent iff $\text{Mod}_{ME}(K) \neq \emptyset$.

Note that ME-consistency is a strictly stronger notion of consistency. ME-consistent knowledge bases are always consistent, but the converse does not necessarily hold if the classical TBox obtained from $T$ by restricting to a context is inconsistent as we show in the following example.

Example 12. Let $\text{sig}(L) = \{x\}$ and $K = (R, T)$ be the KB with $R = \emptyset$ and $T = \{(A \sqcup \neg A \sqsubseteq A \sqcap \neg A : x)\}$. Since $A \sqcup \neg A \sqsubseteq A \sqcap \neg A$ is contradictorial, each ALCP-model of $K$ must satisfy $\neg x$. There certainly are such models, but in each such model $P$, $P^R(x) = 0$. However, since $R = \emptyset$, we have $P^R ME(x) = 0.5$ and hence $K$ has no ME-model.

ME-inconsistency rules out some undesired cases in which the whole knowledge base is consistent, but the TBox restricted to some context is inconsistent. The following theorem gives a simple characterization of ME-consistency: to verify ME-consistency of a KB, it suffices to check consistency of the TBoxes induced by the $L$-interpretations that have positive probability with respect to $P^R ME$. By the properties of the ME distribution, these are the interpretations that are not explicitly restricted to have zero probability through $R$.

Theorem 13. The KB $K = (R, T)$ is ME-consistent iff for every $v \in \text{Int}(L)$ such that $P^R ME(v) > 0$, $T_v$ is consistent.

For the rest of this paper we consider only ME-consistent KBs. Hence, whenever we speak of a KB $K$, we implicitly assume that $K$ has at least one ME-model.

We are interested in computing the probability of a subsumption relation w.r.t. a given KB $K$. Notice that, although we consider only one probability distribution $P^R ME$, there can still exist many different ME-models of $K$, which yield different probabilities for the same subsumption relation. One way to handle this is to consider the smallest and largest probabilities that can be consistently associated to this relation. We call them the sceptical and the credulous degrees of belief, respectively.

Definition 14 (degree of belief). Let $C, D$ be ALCP concepts, $\kappa$ a context, and $K = (R, T)$ an ALCP KB. The sceptical degree of belief of $C \sqsubseteq D$ given $\kappa$ w.r.t. $K$ is

$$B^S_K(C \sqsubseteq D \mid \kappa) := \inf_{P \in \text{Mod}_{ME}(K)} Pr_P(C \sqsubseteq D \mid \kappa).$$

The credulous degree of belief of $C \sqsubseteq D$ given $\kappa$ w.r.t. $K$ is

$$B^C_K(C \sqsubseteq D \mid \kappa) := \sup_{P \in \text{Mod}_{ME}(K)} Pr_P(C \sqsubseteq D \mid \kappa).$$

Example 15. Consider $K_{exa}$ from Example 4. If we ask for the degrees of belief that a patient who suffers from an infection can be successfully treated with
antibiotics, we obtain

\[ B^S_{\text{exa}}(\exists \text{sf.inf} \sqsubseteq \exists \text{succ.ab} \mid T) = 0, \]
\[ B^C_{\text{exa}}(\exists \text{sf.inf} \sqsubseteq \exists \text{succ.ab} \mid T) = 1. \]

These bounds are not very informative, but they are perfectly justified by our knowledge base since we do not know anything about the effectiveness of antibiotics with respect to infections in general. However, for a patient who suffers from a streptococcal infection we get

\[ B^S_{\text{exa}}(\exists \text{sf.strep} \sqsubseteq \exists \text{succ.ab} \mid T) = 0.9405, \]
\[ B^C_{\text{exa}}(\exists \text{sf.strep} \sqsubseteq \exists \text{succ.ab} \mid T) = 0.95. \]

If we know that this patient used antibiotics heavily in the past, then there is nothing in our knowledge base that guarantees the existence of a successful treatment. Hence, the degrees of belief become

\[ B^S_{\text{exa}}(\exists \text{sf.strep} \sqsubseteq \exists \text{succ.ab} \mid h) = 0 \]
\[ B^C_{\text{exa}}(\exists \text{sf.strep} \sqsubseteq \exists \text{succ.ab} \mid h) = 0.2. \]

Our definition of the sceptical degree of belief raises a philosophical question: should there be no difference between the degree of belief 0 and an infinitely small degree of belief? A dual question arises for the credulous degree of belief and the probability 1. However, as we show in the next section, the sceptical and credulous degrees of belief actually correspond to minimum and maximum rather than to infimum and supremum (see Corollary 30) so that these questions become vacuous. From the following theorem we can conclude that every intermediate degree can also be obtained by some model of the KB.

**Theorem 16 (Intermediate Value Theorem).** Let \( p_1 < p_2 \) and \( P_1 \) and \( P_2 \) be two ME-ALCP-models of the KB \( K = (R, T) \) such that \( \text{Pr}_{P_1}(C \sqsubseteq D \mid \kappa) = p_1 \) and \( \text{Pr}_{P_2}(C \sqsubseteq D \mid \kappa) = p_2 \). Then for each \( p \) between \( p_1 \) and \( p_2 \) there exists an ME-ALCP-model \( P \) of \( K \) such that \( \text{Pr}_P(C \sqsubseteq D \mid \kappa) = p \). 

As we will show in Corollary 30 both the sceptical degree \( B^S_K(C \sqsubseteq D \mid \kappa) \) and the credulous degree \( B^C_K(C \sqsubseteq D \mid \kappa) \) are in fact witnessed by some ME-models. Therefore it is meaningful to consider the whole interval of beliefs between \( B^S_K(C \sqsubseteq D \mid \kappa) \) and \( B^C_K(C \sqsubseteq D \mid \kappa) \).

**Definition 17 (belief interval).** Let \( C, D \) be \( \text{ALCP} \) concepts, \( \kappa \in L \) a context and \( K = (R, T) \) a \( \text{ALCP} \) KB. The belief interval for \( C \sqsubseteq D \) w.r.t. \( K \) given \( \kappa \) is

\[ B_K(C \sqsubseteq D \mid \kappa) := [B^S_K(C \sqsubseteq D \mid \kappa), B^C_K(C \sqsubseteq D \mid \kappa)]. \]
4 Computing Beliefs

In this section we show how to compute the belief interval. The first theorem states that the sceptical degree of belief for a subsumption relation can be computed by adding the probabilities of those $\mathcal{L}$-interpretations $w$ that entail this subsumption in the corresponding restricted TBox $T_w$.

**Theorem 18.** Let $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ be a KB, $C, D$ two concepts, and $\kappa$ a context such that $\Pr^\mathcal{ME}_\mathcal{R}(\kappa) > 0$. Then

$$B_s^\mathcal{K}(C \sqsubseteq D | \kappa) = \frac{\sum_{w \in \text{Int}(\mathcal{L}), T_w \models C \sqsubseteq D, w \models \kappa} P^\mathcal{ME}_\mathcal{R}(w)}{P^\mathcal{ME}_\mathcal{R}(\kappa)}.$$ 

Dually, the credulous degree of belief for a subsumption relation can be computed by removing all the situations in which this relation cannot possibly hold.

**Theorem 19.** Let $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ be a KB, $C, D$ two concepts, and $\kappa$ a context such that $\Pr^\mathcal{ME}_\mathcal{R}(\kappa) > 0$. Then

$$B_c^\mathcal{K}(C \sqsubseteq D | \kappa) = 1 - \frac{\sum_{w \in \text{Int}(\mathcal{L}), T_w \models C \not\sqsubseteq D, w \models \kappa} P^\mathcal{ME}_\mathcal{R}(w)}{P^\mathcal{ME}_\mathcal{R}(\kappa)}.$$ 

To prove these theorems, one can build two models of the KB $\mathcal{K}$, $\mathcal{P}$ and $\mathcal{Q}$ such that $\Pr^\mathcal{P}(C \sqsubseteq D | \kappa)$ and $\Pr^\mathcal{Q}(C \sqsubseteq D | \kappa)$ are those degrees expressed by Theorems 18 and 19, respectively. As a byproduct of these proofs, we obtain that the infimum and supremum that define the sceptical and the credulous degrees of belief actually correspond to minimum and maximum taken by some ME-models, yielding the following corollary.

**Corollary 20.** Let $\mathcal{K}$ be an ALCP KB, $C, D$ be two concepts, and $\kappa$ a context. There exist two ME-models $\mathcal{P}, \mathcal{Q}$ of $\mathcal{K}$ with $B_s^\mathcal{K}(C \sqsubseteq D | \kappa) = \Pr^\mathcal{P}(C \sqsubseteq D | \kappa)$ and $B_c^\mathcal{K}(C \sqsubseteq D | \kappa) = \Pr^\mathcal{Q}(C \sqsubseteq D | \kappa)$.

The direct consequence of Theorems 18 and 19 is that if we want to compute the belief interval for $C \sqsubseteq D$ given some context, it suffices to identify all $\mathcal{L}$-interpretations whose induced (classical) TBoxes entail the subsumption relation $C \sqsubseteq D$ (for the sceptical belief) or the strong non-subsumption $C \not\sqsubseteq D$ (for credulous belief). Recall that every set of propositional interpretations can be represented by a propositional formula. This motivates the following definition.

**Definition 21 (consequence formula).** An $\mathcal{L}$-formula $\phi$ is a consequence formula for $C \sqsubseteq D$ (respectively $C \not\sqsubseteq D$) w.r.t. the $\mathcal{L}$-TBox $\mathcal{T}$ if for every $w \in \text{Int}(\mathcal{L})$ it holds that $w \models \phi$ iff $T_w \models C \sqsubseteq D$ (respectively $T_w \models C \not\sqsubseteq D$).

If we are able to compute these consequence formulas, then the computation of the belief interval can be reduced to the evaluation of the probability of these formulas w.r.t. the ME-distribution satisfying $\mathcal{R}$.

**Theorem 22.** Let $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ be an ALCP KB, $\phi$ and $\psi$ be consequence formulas for $C \sqsubseteq D$ and $C \not\sqsubseteq D$ w.r.t. $\mathcal{T}$, respectively, and $\kappa$ a context. Then $B_s^\mathcal{K}(C \sqsubseteq D | \kappa) = P^\mathcal{ME}_\mathcal{R}(\phi | \kappa)$ and $B_c^\mathcal{K}(C \sqsubseteq D | \kappa) = 1 - P^\mathcal{ME}_\mathcal{R}(\psi | \kappa)$. 


Algorithm 1 Computing degrees of belief

Input: KB $\mathcal{K} = (\mathcal{R}, \mathcal{T})$, concepts $C, D$, context $\kappa$

Output: Belief degrees $(\mathcal{B}_s^\mathcal{K}(C \sqsubseteq D|\kappa), \mathcal{B}_c^\mathcal{K}(C \sqsubseteq D|\kappa))$

$\ell_s \leftarrow 0; \ell_c \leftarrow 0$

for all $v \in \text{Int}(\mathcal{L})$ do

if $v \models \kappa$ then

if $\mathcal{T}_v \models C \sqsubseteq D$ then

$\ell_s \leftarrow \ell_s + P_{\mathcal{ME}}(v)$

else if $\mathcal{T}_v \models C \not\sqsubseteq D$ then

$\ell_c \leftarrow \ell_c + P_{\mathcal{ME}}(v)$

end if

end if

return $(\ell_s/P_{\mathcal{ME}}(\kappa), 1 - \ell_c/P_{\mathcal{ME}}(\kappa))$

Example 23. In our running example, one can see that a consequence formula for $\exists sf.\text{strepl} \sqsubseteq \exists suc.\text{ab}$ is $\neg \text{res} \land \neg h$. Indeed, in order to deduce this consequence it is necessary to satisfy the first axiom of $\mathcal{T}_{\text{exa}}$, which is only guaranteed in the context $\neg \text{res} \land \neg h$. Similarly, $\text{res}$ is a consequence formula for $\exists sf.\text{strepl} \not\sqsubseteq \exists suc.\text{ab}$. Knowing both the consequence formulas and the ME-model, we can deduce

$\mathcal{B}_{\text{exa}}^\mathcal{K}(\exists sf.\text{strepl} \sqsubseteq \exists suc.\text{ab} | \top) = P_{\mathcal{ME}}(\neg \text{res} \land \neg h) = 0.9405$

and

$\mathcal{B}_{\text{exa}}^\mathcal{K}(\exists sf.\text{strepl} \sqsubseteq \exists suc.\text{ab} | h) = 1 - P_{\mathcal{ME}}(\text{res} | h) = 0.2$.

In particular, Theorem [22] implies that the belief interval can be computed in two phases. The first phase uses purely logical reasoning to compute the consequence formulas, while the second phase applies probabilistic inferences to compute the degrees of belief from these formulas. We now briefly explain how the consequence formulas can be computed.

Notice first that subsumption and non-subsumption are monotonic consequences in the sense of [2], that is, if an $\mathcal{ALC}$ TBox $\mathcal{T}$ entails the subsumption $C \sqsubseteq D$, then every superset of $\mathcal{T}$ also entails this consequence. Similarly, adding more axioms to a TBox entailing $C \not\sqsubseteq D$ does not remove this entailment. Moreover, the set of all $\mathcal{L}$-formulas (modulo logical equivalence) forms a distributive lattice ordered by generality, in which $\mathcal{L}$-interpretations are all the join prime elements. Thus, the consequence formulas from Definition [21] are in fact the so-called boundaries from [2]. Hence, they can be computed using any of the known boundary computation approaches.

Assuming that the number of contexts is small in comparison to the size of the TBox, it is better to compute the degrees of belief through a more direct approach following Theorems [18] and [19]. In order to compute $\mathcal{B}_s^\mathcal{K}(C \sqsubseteq D | \kappa)$ and $\mathcal{B}_c^\mathcal{K}(C \sqsubseteq D | \kappa)$, it suffices to enumerate all interpretations $v \in \text{Int}(\mathcal{L})$ and check whether $\mathcal{T}_v \models C \sqsubseteq D$ or $\mathcal{T}_v \models C \not\sqsubseteq D$, and $v \models \kappa$, or not (see Algorithm [1]). This approach requires $2^{\mu(\mathcal{L})}$ calls to a standard $\mathcal{ALC}$ reasoner, and each of these calls runs in exponential time on $|\mathcal{T}|$ [9]. Notice that this algorithm has an any-time behaviour: it is possible to stop its execution at any moment and obtain an approximation of the belief interval. Moreover, the longer the algorithm runs,
the better this approximation becomes. Thus, this method is adequate for a
system where finding good approximations efficiently may be more important
than computing the precise answers.

5 Properties

We now investigate some properties of probabilistic logics \cite{22}. First we show
that $\mathcal{ALCP}$ is language and representation invariant. Invariance is meant with
respect to logical objects. Language invariance means that just extending the
language without changing the knowledge base should not affect reasoning re-
sults. Representation invariance means that equivalent knowledge bases should
yield equal inference results. Notice that different notions of representation de-
pendence exist in the literature. For instance, in \cite{11} a very different notion is
considered, where the language and the knowledge base are changed simultane-
ously. This case is not covered by our notion of representation invariance. $\mathcal{ALCP}$
also satisfies an independence property; i.e., reasoning results about a part of the
language are not changed, when we add knowledge about an independent part
of the language. Finally, $\mathcal{ALCP}$ is continuous in the sense that minor changes in
the probabilistic knowledge expressed by a knowledge base cannot induce major
changes in the reasoning results.

Theorem 24 (Representation invariance). Let $K_i = (R_i, T_i)$, $i \in \{1, 2\}$, be
two KBs such that $\text{Mod}(R_1) = \text{Mod}(R_2)$ and $\text{Mod}(T_1) = \text{Mod}(T_2)$. Then for all
concepts $C, D$ and contexts $\kappa \in L$, $B_{K_1}(C \sqsubseteq D \mid \kappa) = B_{K_2}(C \sqsubseteq D \mid \kappa)$.

$\mathcal{ALCP}$ is not only representation invariant, but also language invariant. This
property is of computational interest, in particular in combination with inde-
pendence, that we investigate subsequently. To illustrate this, suppose that we
added knowledge about bone fractures in our medical example, which is inde-
pendent of the knowledge about infections. Independence guarantees that we
can ignore the knowledge about infections when answering queries about bone
fractures. In this way, we can decrease the size of the knowledge base. Language
invariance guarantees that we can also ignore the concepts, relations and proposi-
tional variables related to the infection domain. Thus, we can decrease the
size of the language. Exploiting both properties, the size of the computational
problems can sometimes be decreased significantly.

Theorem 25 (Language Invariance). Let $K_1, K_2$ be KBs over $\mathcal{L}^1, N^1_C, N^1_R$ and
$\mathcal{L}^2, N^2_C, N^2_R$, respectively. If $K_1 = K_2$, $\mathcal{L}^1 \subseteq \mathcal{L}^2, N^1_C \subseteq N^2_C$ and $N^1_R \subseteq N^2_R$, then for
all concepts $C, D \in N^1_C$ and contexts $\kappa \in \mathcal{L}^1$, it holds that
$B_{K_1}(C \sqsubseteq D \mid \kappa) = B_{K_2}(C \sqsubseteq D \mid \kappa)$.

For an $\mathcal{L}$-TBox $T$, we define the signature of $T$ to be the set $\text{sig}(T)$ of all
concept names and role names appearing in $T$. Likewise, $\text{sig}(R)$ is the set of all
propositional variables appearing in $R$. The signature of a KB $K = (R, T)$ is
$\text{sig}(K) := \text{sig}(R) \cup \text{sig}(T)$. 


Theorem 26 (Independence). Let $K_1, K_2$ be such that $\text{sig}(K_1) \cap \text{sig}(K_2) = \emptyset$, $C, D$ be two concepts, and $\kappa$ a context where $(\text{sig}(C) \cup \text{sig}(D) \cup \text{sig}(\kappa)) \cap \text{sig}(K_2) = \emptyset$. Then $B(C \sqsubseteq K_1 D \mid \kappa) = B(C \sqsubseteq K_1 \cup K_2 D \mid \kappa)$.

The last property we consider is continuity. One important practical feature of continuous probabilistic logics is that they guarantee a numerically stable behaviour. That is, minor rounding errors due to floating-point arithmetic will not result in major errors in the computed probabilities. As demonstrated by Paris in [22], measuring the difference between probabilistic knowledge bases is subtle and is best addressed by comparing knowledge bases extensionally; i.e., with respect to their model sets. To this end, Paris considered the Blaschke metric. Formally, the *Blaschke distance* $\|S_1, S_2\|_B$ between two convex sets $S_1, S_2$ is defined by

$$\inf\{\delta \in \mathbb{R} \mid \forall P_1 \in S_1 \exists P_2 \in S_2 : \|P_1, P_2\|_2 \leq \delta \quad \text{and} \quad \forall P_2 \in S_2 \exists P_1 \in S_1 : \|P_2, P_1\|_2 \leq \delta\}$$

Intuitively, $\|S_1, S_2\|_B$ is the smallest real number $d$ such that for each distribution in one of the sets, there is a probability distribution in the other that has distance at most $d$ to the former. We say that a sequence of knowledge bases $(K_i)$ converges to a knowledge base $K$ iff the classical part of each $K_i$ is equivalent to the classical part of $K$ and the probabilistic part converges to the probabilistic part of $K$. Our reasoning approach behaves indeed continuously with respect to this metric.

Theorem 27 (Continuity). Let $(K_i)$ be a convergent sequence of KBs with limit $K$ and $B_{K_i}(C \sqsubseteq D \mid \kappa) = [\ell_i, u_i]$. If $B_K(C \sqsubseteq D \mid \kappa) = [\ell, u]$, then $(\ell_i)$ converges to $\ell$ and $(u_i)$ converges to $u$ (with respect to the usual topology on $\mathbb{R}$).

6 Related Work

Relational probabilistic logical approaches can be roughly divided into those that consider probability distributions over the domain, those that consider probability distributions over possible worlds and those that combine both ideas [10]. Our framework belongs to the second group. Maximum entropy reasoning in propositional probabilistic logics has been discussed extensively, e.g., in [13, 22], and various extensions to first-order languages have been considered in recent years [3, 14, 15]. In these works, the domain is restricted to a finite number of constants or bounded in the limit. We circumvent the need to do so by combining a classical first-order logic with unbounded domain with a probabilistic logic with fixed domain.

Many probabilistic DLs have also been considered in the last decades [16, 18, 19]. Our approach is closest to Bayesian DLs [5, 6] and DISPONTE [25]. The greatest difference with the former lies in the fact that $\mathcal{ALCP}$ KBs do not require a complete specification of the probability distribution, but only a set of probabilistic constraints. Moreover, the previous formalisms consider only the
sceptical degree of belief, while we are interested in the full belief interval. In con-
trast to DISPONTE, \textit{ALCP} is capable of expressing both, logical and probabilistic
dependencies between the axioms in a KB; in addition, DISPONTE requires all uncer-
tainty degrees to be assigned as mutually independent point probabilities,
while \textit{ALCP} allows for a more flexible specification.

7 Conclusions

We have introduced the probabilistic DL \textit{ALCP}, which extends the classical DL
\textit{ALC} with the capability of expressing and reasoning about uncertain contextual
knowledge defined through the principle of maximum entropy. Effective reason-
ing methods were developed using the decoupling between the logical and the
probabilistic components of \textit{ALCP} KBs. We also studied the properties of this
logic in relation to other probabilistic logics.

We plan to extend this work in several directions. First, instead of considering
the ME-model, we could reason over all probability distributions that satisfy our
probabilistic constraints similar to \cite{12, 17, 20}. This will result in larger belief
intervals in general. A smaller interval is preferable since it corresponds to a
more precise degree of belief. However, when using all probability distributions
the size of the interval can be a good indicator for the variation of the possible
beliefs in our query with respect to the knowledge base.

In some applications it is also useful to allow more expressive propositional
or relational context languages like those proposed in \cite{4, 15, 23}. Similarly,
we can consider other DLs for our concept language. Indeed, \textit{ALC} was chosen
as a prototypical DL for studying the basic properties of our formalism. In-
cluding additional constructors into the formalism should be relatively simple.
In contrast, considering other reasoning problems beyond subsumption is less
straightforward. Recall, for instance, that if an \textit{ALCP} KB $K$ contains an in-
sistent context with positive probability, then $K$ has no models. It is thus unclear
how to handle the probability of consistency of a KB.

Practical reasoning with \textit{ALCP} can be currently performed by combining ex-
stisting ME-reasoners\footnote{3} with any \textit{ALC}-reasoner\footnote{4} according to Algorithm \ref{alg1}. Clearly,
such an approach can still be further optimized. We are working on combining
the classical and probabilistic reasoning parts in more sophisticated ways.

References

1. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F.
   (eds.): The Description Logic Handbook: Theory, Implementation, and Applica-
tions. Cambridge University Press, 2nd edn. (2007)
2. Baader, F., Knechtel, M., Peñaloza, R.: Context-dependent views to axioms and
   consequences of semantic web ontologies. J. of Web Semantics 12–13, 22–40 (2012)

\footnote{3}{https://www.fernuni-hagen.de/wbs/research/log4kr/}
\footnote{4}{http://owl.cs.manchester.ac.uk/tools/list-of-reasoners/}
3. Barnett, O., Paris, J.B.: Maximum entropy inference with quantified knowledge. Logic Journal of the IGPL 16(1), 85–98 (2008)
4. Beierle, C., Kern-Isberner, G., Finkhammer, M., Potyka, N.: Extending and completing probabilistic knowledge and beliefs without bias. KI 29(3), 255–262 (2015)
5. Ceylan, I.I., Peñaloza, R.: The Bayesian description logic BEL. In: Proc. of IJCAR 2014. LNCS, vol. 8562, pp. 480–494. Springer (2014)
6. d’Amato, C., Fanizzi, N., Lukasiewicz, T.: Tractable reasoning with Bayesian description logics. In: Proc. SUM 2008. LNCS, vol. 5291, pp. 146–159. Springer (2008)
7. De Bona, G., Cozman, F.G., Finger, M.: Towards classifying propositional probabilistic logics. J. of Applied Logic 12(3), 349–368 (2014)
8. Domingos, P.M., Lowd, D.: Markov Logic: An Interface Layer for Artificial Intelligence. Synthesis Lectures on Artificial Intelligence and Machine Learning, Morgan & Claypool Publishers (2009)
9. Donini, F.M., Massacci, F.: ExpTime tableaux for ALC. Artificial Intelligence 124(1), 87–138 (2000)
10. Halpern, J.Y.: An analysis of first-order logics of probability. Artificial Intelligence 46, 311–350 (1990)
11. Halpern, J.Y., Koller, D.: Representation dependence in probabilistic inference. JAIR pp. 319–356 (2004)
12. Hansen, P., Perron, S.: Merging the local and global approaches to probabilistic satisfiability. Intern. J. of Approx. Reasoning 47(2), 125 – 140 (2008)
13. Kern-Isberner, G.: Conditionals in nonmonotonic reasoning and belief revision. Springer, LNAI 2087 (2001)
14. Kern-Isberner, G., Thimm, M.: Novel semantical approaches to relational probabilistic conditionals. In: Proc. KR 2010. pp. 382–391. AAAI Press (2010)
15. Kern-Isberner, G., Lukasiewicz, T.: Combining probabilistic logic programming with the power of maximum entropy. Artif. Intell. 157(1-2), 139–202 (2004)
16. Klinov, P., Parsia, B.: A hybrid method for probabilistic satisfiability. In: Proc. CADE 2011. LNCS, vol. 6803, pp. 354–368. Springer (2011)
17. Lukasiewicz, T.: Probabilistic deduction with conditional constraints over basic events. JAIR 10, 380–391 (1999)
18. Lukasiewicz, T., Straccia, U.: Managing uncertainty and vagueness in description logics for the semantic web. J. of Web Semantics 6(4), 291–308 (2008)
19. Lutz, C., Schröder, L.: Probabilistic description logics for subjective uncertainty. In: Proc. KR 2010. AAAI Press (2010)
20. Nilsson, N.J.: Probabilistic logic. Artificial Intelligence 28, 71–88 (February 1986)
21. Nocedal, J., Wright, S.J.: Numerical Optimization. Springer, 2nd edn. (2006)
22. Paris, J.: The uncertain reasoner’s companion – A mathematical perspective. Cambridge University Press (1994)
23. Potyka, N.: Reasoning over linear probabilistic knowledge bases with priorities. In: Proc. SUM 2015. vol. 9310, pp. 121–136. Springer (2015)
24. Potyka, N.: Relationships between semantics for relational probabilistic conditional logics. In: Computational Models of Rationality. Essays dedicated to Gabriele Kern-Isberner, pp. 332–347. College Publications (2016)
25. Riguzzi, F., Bellodi, E., Lamma, E., Zese, R.: Epistemic and statistical probabilistic ontologies. In: Proc. URSW-12. vol. 900, pp. 3–14. CEUR-WS (2012)
26. Schmidt-Schauß, M., Smolka, G.: Attributive concept descriptions with complements. Artif. Intell. 48(1), 1–26 (1991)
27. Yeung, R.W.: Information theory and network coding. Springer Science & Business Media (2008)
Appendix: Proofs

**Theorem 13.** The KB $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ is ME-consistent iff for every $v \in \text{Int}(\mathcal{L})$ such that $P^{\text{ME}}_{\mathcal{R}}(v) > 0$, $\mathcal{T}_v$ is consistent.

**Proof.** For the “if” direction, let $v_1, \ldots, v_n \in \text{Int}(\mathcal{L})$ be all the $\mathcal{L}$-interpretations such that $P^{\text{ME}}_{\mathcal{R}}(v_i) > 0$, $1 \leq i \leq n$. Then, for every $i, 1 \leq i \leq n$ the induced TBox $\mathcal{T}_v_i$ has a classical model $\mathcal{I}_i = (\Delta^{I_i}, \mathcal{I}_i)$, by assumption. It is easy to verify that the $\mathcal{ALCP}$-interpretation $\mathcal{P} = (\mathcal{I}, P_\lambda)$ defined by

$$\mathcal{I} = \{\mathcal{I}_i = (\Delta^{I_i}, \mathcal{I}_i) \mid 1 \leq i \leq n\}$$

and $P_\lambda(\mathcal{I}_i) = P^{\text{ME}}_{\mathcal{R}}(v_i)$ for all $i, 1 \leq i \leq n$ is an ME-model of $\mathcal{K}$. Thus, $\mathcal{K}$ is consistent.

Conversely, let $\mathcal{P} = (\mathcal{I}, P_\lambda)$ be an ME-model of $\mathcal{K}$. Then, for every $v \in \text{Int}(\mathcal{L})$ with $P^{\text{ME}}_{\mathcal{R}}(v) > 0$ there is a possible world $(\Delta^I, \mathcal{I}, w^I)$ with $w^I = v$. Since $\mathcal{I}$ is a model of $\mathcal{T}$ and satisfies all contexts corresponding to the GCI in $\mathcal{T}_v$, it follows that $(\Delta^I, \mathcal{I}) \models \mathcal{T}_v$; thus, $\mathcal{T}_v$ must be consistent. $\square$

**Theorem 16 (Intermediate Value Theorem).** Let $p_1 < p_2$ and $\mathcal{P}_1$ and $\mathcal{P}_2$ be two ME-$\mathcal{ALCP}$-models of the KB $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ such that $Pr_{\mathcal{P}_1}(C \sqsubseteq D \mid \kappa) = p_1$ and $Pr_{\mathcal{P}_2}(C \sqsubseteq D \mid \kappa) = p_2$. Then for each $p$ between $p_1$ and $p_2$ there exists an ME-$\mathcal{ALCP}$-model $\mathcal{P}$ of $\mathcal{K}$ such that $Pr_{\mathcal{P}}(C \sqsubseteq D \mid \kappa) = p$.

**Proof.** Assume w.l.o.g. that $\mathcal{P}_i = (\mathcal{I}_i, P_i)$, $i = 1, 2$, are such that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$; if there exists some $\mathcal{I} \in \mathcal{I}_1 \cap \mathcal{I}_2$, it suffices to rename the elements in $\Delta^I$ in one of the probabilistic interpretations. Given $\lambda \in [0, 1]$, define $\mathcal{P}_\lambda = (\mathcal{I}_1 \cup \mathcal{I}_2, P_\lambda)$, where for every $\mathcal{I} \in \mathcal{I}_1 \cup \mathcal{I}_2$,

$$P_\lambda(\mathcal{I}) = \begin{cases} (1 - \lambda)P_1(\mathcal{I}) & \text{if } \mathcal{I} \in \mathcal{I}_1; \\ \lambda P_2(\mathcal{I}) & \text{otherwise.} \end{cases}$$

$\mathcal{P}_\lambda$ is consistent with $\mathcal{T}$ since $\mathcal{P}_1, \mathcal{P}_2$ are. We now show that $\mathcal{P}_\lambda$ induces the ME-model of $\mathcal{R}$, which implies that $\mathcal{P}_\lambda$ is an ME-$\mathcal{ALCP}$-model of $\mathcal{K}$. For all $v \in \text{Int}(\mathcal{L})$, we have

$$P^{\mathcal{P}_\lambda}(v) = \sum_{\mathcal{I} \in (\mathcal{I}_1 \cup \mathcal{I}_2) \mid v} P_\lambda(\mathcal{I})$$

$$= \sum_{\mathcal{I} \in \mathcal{I}_1 \mid v} P_\lambda(\mathcal{I}) + \sum_{\mathcal{I} \in \mathcal{I}_2 \mid v} P_\lambda(\mathcal{I})$$

$$= \sum_{\mathcal{I} \in \mathcal{I}_1 \mid v} (1 - \lambda) \cdot P_1(\mathcal{I}) + \sum_{\mathcal{I} \in \mathcal{I}_2 \mid v} \lambda \cdot P_2(\mathcal{I})$$

$$= (1 - \lambda) \cdot P^{\mathcal{P}_1}(v) + \lambda \cdot P^{\mathcal{P}_2}(v) = P^{\text{ME}}_{\mathcal{R}}(v),$$

where the last equation follows from $P^{\mathcal{P}_1} = P^{\mathcal{P}_2} = P^{\text{ME}}_{\mathcal{R}}$. Hence, each $\mathcal{P}_\lambda$ is indeed a ME-$\mathcal{ALCP}$-model of $\mathcal{K}$. For the probability of our subsumption relation,
we can derive similarly that \( P^\mathcal{L}_\lambda(\kappa)Pr_{\mathcal{P}_\lambda}(C \sqsubseteq D \mid \kappa) \) is equal to
\[
\sum_{I \in \mathcal{I}_1 \cup \mathcal{I}_2, I = \kappa, I \sqsubseteq \mathcal{C} \sqsubseteq D} P_\lambda(I) = \sum_{I \in \mathcal{I}_1, I = \kappa, I \sqsubseteq \mathcal{C} \sqsubseteq D} P_\lambda(I) + \sum_{I \in \mathcal{I}_2, I = \kappa, I \sqsubseteq \mathcal{C} \sqsubseteq D} P_\lambda(I) = P^\mathcal{L}_\lambda(\kappa)((1 - \lambda)p_1 + \lambda p_2).
\]

For every \( p \in [p_1, p_2] \) there exists a \( \lambda_p \in [0, 1] \) such that \( p = (1 - \lambda_p)p_1 + \lambda_p p_2 \). Using this value \( \lambda_p \) we obtain that \( Pr_{\mathcal{P}_\lambda}(C \sqsubseteq D \mid \kappa) = p \)

In order to prove Theorems 18 and 19 it is useful to consider a restricted class of \( \mathcal{ALCP} \)-interpretations in which each context is represented by at most one possible world. We call these interpretations \textit{pithy}.

\textbf{Definition 28 (pithy).} The \( \mathcal{ALCP} \)-interpretation \( \mathcal{P} = (\mathcal{I}, P_\mathcal{P}) \) is pithy if for every \( w \in \text{Int}(\mathcal{L}) \) there is at most one possible world \( (\Delta^\mathcal{L}, \mathcal{I}, v^\mathcal{I}) \in \mathcal{I} \) with \( v^\mathcal{I} = w \).

As the following lemma shows, pithy models are sufficient for computing the sceptical and credulous degrees of belief of a conditional subsumption relation and, by extension, the belief interval.

\textbf{Lemma 29.} Let \( \mathcal{K} \) be an \( \mathcal{ALCP} \) KB, \( C, D \) two concepts and \( \kappa \in \mathcal{L} \) such that \( P^\mathcal{ME}(\kappa) > 0 \). For every \( \mathcal{ALCP} \)-model \( \mathcal{P} \) of \( \mathcal{K} \) there exist pithy \( \mathcal{ALCP} \)-models \( \mathcal{Q}_1, \mathcal{Q}_2 \) of \( \mathcal{K} \) such that
\[
Pr_{\mathcal{Q}_1}(C \sqsubseteq D \mid \kappa) \leq Pr_{\mathcal{P}}(C \sqsubseteq D \mid \kappa) \leq Pr_{\mathcal{Q}_2}(C \sqsubseteq D \mid \kappa)
\]
and \( P^\mathcal{P} = P^\mathcal{Q}_1 = P^\mathcal{Q}_2 \).

\textit{Proof.} Let \( \mathcal{P} = (\mathcal{I}, P_\mathcal{P}) \). If \( \mathcal{P} \) is already pithy, then the result holds trivially. Otherwise, there must exist two possible worlds \( \mathcal{I}, \mathcal{J} \in \mathcal{I} \) such that \( v^\mathcal{I} = v^\mathcal{J} \).

There are four possible cases: (i) \( \mathcal{I} \sqsubseteq C \sqsubseteq D \) and \( \mathcal{J} \sqsubseteq C \sqsubseteq D \); (ii) \( \mathcal{I} \not\sqsubseteq C \sqsubseteq D \) and \( \mathcal{J} \sqsubseteq C \sqsubseteq D \); (iii) \( \mathcal{I} \sqsubseteq C \sqsubseteq D \) and \( \mathcal{J} \not\sqsubseteq C \sqsubseteq D \); and (iv) \( \mathcal{I} \not\sqsubseteq C \sqsubseteq D \) and \( \mathcal{J} \sqsubseteq C \sqsubseteq D \).

We construct a new model \( \mathcal{P}' \) by removing one of the possible worlds \( \mathcal{I}, \mathcal{J} \) and redistributing the probability according to these cases, as described next. For the first three cases, define \( \mathcal{H} := \mathcal{I} \setminus \{I\} \) and the probability distribution
\[
P_{\mathcal{H}}(\mathcal{H}) := \begin{cases} P_\mathcal{H}(\mathcal{H}) & \mathcal{H} \not= \mathcal{J} \\
P_\mathcal{H}(\mathcal{I}) + P_\mathcal{H}(\mathcal{J}) & \mathcal{H} = \mathcal{J} \end{cases}
\]
for all \( \mathcal{H} \in \mathcal{H} \). Then \( P^\mathcal{P} = P^\mathcal{P}' \) and \( \mathcal{P}' = (\mathcal{H}, P_\mathcal{P}') \) is still an ME-model of \( \mathcal{K} \).

Since the denominator in (3) is \( P^\mathcal{ME}(\kappa) \) independently of \( C \sqsubseteq D \), we have by construction that
\[
Pr_{\mathcal{P}'}(C \sqsubseteq D \mid \kappa) \leq Pr_{\mathcal{P}}(C \sqsubseteq D \mid \kappa).
\]
The case (iv) is symmetric to (iii), where the possible world \( \mathcal{J} \) is removed instead of \( \mathcal{I} \). Since \( \mathcal{H} \subset \mathcal{I} \), we can iteratively repeat this process until a pithy model \( \mathcal{Q}_1 \) is obtained.

\( \mathcal{Q}_2 \) can be constructed symmetrically. \( \Box \)
Notice that since $P^P = P^Q_1 = P^Q_2$, this lemma in particular implies that for each ME-model there exists a pithy ME-model that yields a smaller or equal probability for the subsumption relation, and dually one that yields a larger or equal probability. Note also that in each pithy ME-model, for all $w \in \text{Int}(\mathcal{L})$ with $P^\text{ME}_R(w) > 0$, there must be exactly one possible world $\mathcal{I}$ with $v^\mathcal{I} = w$ because otherwise $P_3$ could not be a probability distribution (the elementary events could not sum to 1). Moreover, since a pithy interpretation can contain at most $|\text{Int}(\mathcal{L})|$ possible worlds and the world corresponding to some $w \in \text{Int}(\mathcal{L})$ must have probability $P^\text{ME}_R(w)$, there is only a finite number of probabilities that pithy models can assign to a given subsumption relation. Hence, the infimum and supremum that define the sceptical and the credulous degrees of belief actually correspond to minimum and maximum taken by some pithy ME-models.

**Corollary 30.** Given an ALC\(\mathcal{P}\) KB $\mathcal{K}$, two concepts $C, D$ and a context $\kappa$, there exist two pithy ME-models $P, Q$ of $\mathcal{K}$ such that $B^P_K(C \sqsubseteq D \mid \kappa) = \Pr_P(C \sqsubseteq D \mid \kappa)$ and $B^Q_K(C \sqsubseteq D \mid \kappa) = \Pr_Q(C \sqsubseteq D \mid \kappa)$.

**Theorem 18.** Let $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ be a KB, $C, D$ two concepts, and $\kappa$ a context such that $P^\text{ME}_R(\kappa) > 0$. Then

$$B^P_K(C \sqsubseteq D \mid \kappa) = \frac{\sum_{w \in \text{Int}(\mathcal{L}), \mathcal{T}_w \models C \sqsubseteq D, w \models \kappa} P^\text{ME}_R(w) }{P^\text{ME}_R(\kappa)}.$$ 

**Proof.** For every $w \in \text{Int}(\mathcal{L})$, we construct an ALC\(\mathcal{P}\)-interpretation $\mathcal{I}_w$ as follows. If $\mathcal{T}_w \models C \sqsubseteq D$, then $\mathcal{I}_w$ is any model $(\Delta_w, \mathcal{I}_w, w)$ of $\mathcal{T}_w$; otherwise, $\mathcal{I}_w$ is any model $(\Delta_w, \mathcal{I}_w, w)$ of $\mathcal{T}_w$ that does not satisfy $C \sqsubseteq D$, which must exist by definition. Let now $P_\mathcal{K} = (\mathcal{I}, P_3)$ be the ALC\(\mathcal{P}\)-interpretation such that $\mathcal{I} = \{\mathcal{I}_w \mid w \in \text{Int}(\mathcal{L})\}$ and $P_3(\mathcal{I}_w) = P^\text{ME}_R(w)$ for all $w$. Then $P_\mathcal{K}$ is a model of $\mathcal{K}$. Moreover, it holds that

$$\Pr(P_\kappa(C \sqsubseteq D \mid \kappa) = \sum_{\mathcal{I}_w \models C \sqsubseteq D, w \models \kappa} P_3(\mathcal{I}_w) / P^\text{ME}_R(\kappa)$$

$$= \sum_{\mathcal{T}_w \models C \sqsubseteq D} P^\text{ME}_R(w) / P^\text{ME}_R(\kappa).$$

Thus, $P^\text{ME}_R(\kappa)B^P_K(C \sqsubseteq D \mid \kappa) \leq \sum_{\mathcal{T}_w \models C \sqsubseteq D, w \models \kappa} P^\text{ME}_R(w)$. If this inequality is strict, then w.l.o.g. there must exist a pithy probabilistic model $P = (\mathcal{J}, P_3)$ of $\mathcal{K}$ such that $\Pr_P(C \sqsubseteq D \mid \kappa) < \Pr_{P_\mathcal{K}}(C \sqsubseteq D \mid \kappa)$ (see Lemma 29). Hence for every $w \in \text{Int}(\mathcal{L})$ with $P^\text{ME}_R(w) > 0$ there exists exactly one $\mathcal{J}_w \in \mathcal{J}$ with $v^{\mathcal{J}_w} = w$. We thus have

$$\sum_{\mathcal{J}_w \models C \sqsubseteq D} P_3(\mathcal{J}_w) < \sum_{\mathcal{T}_w \models C \sqsubseteq D} P_3(\mathcal{I}_w).$$

Since $P_3(\mathcal{I}_w) = P_3(\mathcal{J}_w)$ for all $w$, then there must exist a valuation $v$ such that $\mathcal{T}_v \models C \sqsubseteq D$ but $\mathcal{J}_v \not\models C \sqsubseteq D$. Since $\mathcal{J}_v$ is a model of $\mathcal{T}_v$, it follows that $\mathcal{T}_v \not\models C \sqsubseteq D$. By construction, then we have that $\mathcal{T}_v \not\models C \sqsubseteq D$, which is a contradiction. \(\square\)
Theorem 19. Let $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ be a KB, $C, D$ two concepts, and $\kappa$ a context with $P_{\mathcal{R}}^{ME}(\kappa) > 0$. Then

$$B_{\mathcal{K}}^C(C \sqsubseteq D \mid \kappa) = 1 - \frac{\sum_{w \in \text{Int}(\mathcal{L}), w = \kappa = C} P_{\mathcal{R}}^{ME}(w)}{P_{\mathcal{R}}^{ME}(\kappa)}.$$ 

Proof. For every $w \in \text{Int}(\mathcal{L})$, construct an $\mathcal{ALCP}$-interpretation $\mathcal{I}_w$ as follows. If $\mathcal{T}_w \models C \nvdash D$, then $\mathcal{I}_w$ is any model $(\Delta^w, \mathcal{I}_w)$ of $\mathcal{T}_w$; otherwise, $\mathcal{I}_w$ is any model $(\Delta^w, \mathcal{I}_w)$ of $\mathcal{T}_w$ that satisfies $C \sqsubseteq D$. Let $\mathcal{P}_\mathcal{K} = (3, \mathcal{P}_3)$ be the $\mathcal{ALCP}$-interpretation with $\mathcal{I}_w = \{w \mid w \in \text{Int}(\mathcal{L})\}$ and $\mathcal{P}_3(\mathcal{I}_w) = P_{\mathcal{R}}^{ME}(w)$ for all $w$. Then $\mathcal{P}_\mathcal{K}$ is a model of $\mathcal{K}$. Moreover, it holds that

$$Pr_{\mathcal{P}_\mathcal{K}}(C \sqsubseteq D \mid \kappa) = \sum_{\mathcal{I}_w = C \sqsubseteq D, w = \kappa} P_{\mathcal{R}}(\mathcal{I}_w)/P_{\mathcal{R}}^{ME}(\kappa)$$

That is, $B_{\mathcal{K}}^C(C \sqsubseteq D \mid \kappa) = 1 - \sum_{\mathcal{I}_w = C \sqsubseteq D, w = \kappa} P_{\mathcal{R}}^{ME}(w)/P_{\mathcal{R}}^{ME}(\kappa)$. If this inequality is strict, then there exists a probabilistic model $\mathcal{P} = (3, \mathcal{P}_3)$ of $\mathcal{K}$ such that $Pr_{\mathcal{P}}(C \sqsubseteq D \mid \kappa) > Pr_{\mathcal{P}_\mathcal{K}}(C \sqsubseteq D \mid \kappa)$. By Lemma 29, we can assume w.l.o.g. that $\mathcal{P}$ is pithy. Hence for every $w \in \text{Int}(\mathcal{L})$ with $P_{\mathcal{R}}^{ME}(w) > 0$ there is exactly one $\mathcal{J}_w \in \mathcal{I}$ with $w^{\mathcal{J}_w} = w$, and thus,

$$\sum_{\mathcal{J}_w = C \sqsubseteq D} P_{\mathcal{P}_3}(\mathcal{J}_w) > \sum_{\mathcal{I}_w = C \sqsubseteq D} P_{\mathcal{P}}(\mathcal{I}_w).$$

Since $P_{\mathcal{P}_3}(\mathcal{I}_w) = P_{\mathcal{P}}(\mathcal{J}_w)$ for all $w$, there must exist some $v \in \text{Int}(\mathcal{L})$ such that $\mathcal{I}_v \nvdash C \sqsubseteq D$ but $\mathcal{J}_v \models C \sqsubseteq D$. As $\mathcal{J}_v$ is a model of $\mathcal{T}_v$, $\mathcal{I}_v \nvdash C \nvdash D$. By construction, we have that $\mathcal{I}_v \models C \sqsubseteq D$, which is a contradiction. 

Theorem 22. Let $\mathcal{K} = (\mathcal{R}, \mathcal{T})$ be an $\mathcal{ALCP}$ KB, $\phi$ and $\psi$ be consequence formulas for $C \sqsubseteq D$ and $C \nvdash D$ w.r.t. $\mathcal{T}$, respectively, and $\kappa$ a context. Then $B_{\mathcal{K}}^C(C \sqsubseteq D \mid \kappa) = P_{\mathcal{R}}^{ME}(\phi \mid \kappa)$ and $B_{\mathcal{K}}^C(C \sqsubseteq D \mid \kappa) = 1 - P_{\mathcal{R}}^{ME}(\psi \mid \kappa)$.

Proof. The result is a direct consequence of Definition 21 and Theorems 18 and 19. Indeed,

$$B_{\mathcal{K}}^C(C \sqsubseteq D \mid \kappa) = \sum_{\mathcal{I}_w = C \sqsubseteq D, w = \kappa} P_{\mathcal{R}}^{ME}(w)/P_{\mathcal{R}}^{ME}(\kappa)$$

The case of the credulous degree of belief is analogous.

Theorem 24 (Representation invariance). Let $\mathcal{K}_i = (\mathcal{R}_i, \mathcal{T}_i), i \in \{1, 2\}$, be two KBs such that $\text{Mod}(\mathcal{T}_1) = \text{Mod}(\mathcal{T}_2)$ and $\text{Mod}(\mathcal{T}_1) = \text{Mod}(\mathcal{T}_2)$. Then for all concepts $C, D$ and contexts $\kappa \in \mathcal{L}$, $B_{\mathcal{K}_1}(C \sqsubseteq D \mid \kappa) = B_{\mathcal{K}_2}(C \sqsubseteq D \mid \kappa)$. 

Proof. Let \( \mathcal{P} = \langle \mathcal{I}, \mathcal{P}_1 \rangle \) be an \( \mathcal{ALCP} \)-interpretation. Since \( \text{Mod}(\mathcal{I}_1) = \text{Mod}(\mathcal{I}_2) \), \( \mathcal{P} \) is consistent with \( \mathcal{I}_1 \) iff \( \mathcal{P} \) is consistent with \( \mathcal{I}_2 \). Since \( \text{Mod}(\mathcal{R}_1) = \text{Mod}(\mathcal{R}_2) \), \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) induce the same ME-model and \( \mathcal{P}_1 \) is (ME-)consistent with \( \mathcal{R} \) iff \( \mathcal{P}_2 \) is (ME-)consistent with \( \mathcal{R} \). Hence,

\[
B_{K_1}(C \subseteq D | \kappa) = \inf_{\mathcal{P} \in \text{Mod}_D(K_1)} Pr_\mathcal{P}(C \subseteq D | \kappa)
= \inf_{\mathcal{P} \in \text{Mod}_D(K_2)} Pr_\mathcal{P}(C \subseteq D | \kappa)
= B_{K_2}(C \subseteq D | \kappa)
\]

Analogously, we get that \( B_{K_1}^d(C \subseteq D | \kappa) = B_{K_2}^d(C \subseteq D | \kappa) \) and therefore \( B_{K_1}(C \subseteq D | \kappa) = B_{K_2}(C \subseteq D | \kappa) \).

**Theorem 25 (Language Invariance).** Let \( K_1, K_2 \) be KBs over \( L^1, N_1^C, N_1^K \) and \( L^2, N_2^C, N_2^K \), respectively. If \( K_1 = K_2, L^1 \subseteq L^2, N_1^C \subseteq N_2^C \) and \( N_1^K \subseteq N_2^K \), then for all concepts \( C, D \in N_2^C \) and contexts \( \kappa \in L^1 \), it holds that

\[
B_{K_1}(C \subseteq D | \kappa) = B_{K_2}(C \subseteq D | \kappa).
\]

**Proof.** It suffices to show that for every \( \mathcal{ALCP} \)-model \( \mathcal{P}_1 \) of \( K_1 \) there exists a \( \mathcal{ALCP} \)-model \( \mathcal{P}_2 \) of \( K_2 \) such that \( Pr_{\mathcal{P}_1}(C \subseteq D | \kappa) = Pr_{\mathcal{P}_2}(C \subseteq D | \kappa) \) and vice versa. Given an \( \mathcal{ALCP} \)-model \( \mathcal{P}_1 = \langle \mathcal{I}_1, \mathcal{P}_1 \rangle \) of \( K_1 \), we build \( \mathcal{P}_2 = \langle \mathcal{I}_2, \mathcal{P}_2 \rangle \) as follows. For each possible world \( \mathcal{I} \in \mathcal{I}_1 \) with probability \( p \), \( \mathcal{I}_2 \) contains a possible world \( \mathcal{I}' \) with probability \( p \) that extends \( \mathcal{I} \) assigning \text{false} \ to all new propositional variables and the empty set to all new role names and concept names. Since \( C, D, \kappa \) and \( K_2 \) depend only on \( \text{sig}(L^1), N_1^C, N_1^K \), \( \mathcal{P}_2 \) satisfies \( K_2 \) and \( Pr_{\mathcal{P}_1}(C \subseteq D | \kappa) = Pr_{\mathcal{P}_2}(C \subseteq D | \kappa) \) holds. Conversely, consider an \( \mathcal{ALCP} \)-model \( \mathcal{P}_2 = \langle \mathcal{I}_2, \mathcal{P}_2 \rangle \) of \( K_2 \). We obtain \( \mathcal{P}_1 \) from \( \mathcal{P}_2 \) by restricting the possible worlds in \( \mathcal{P}_2 \) to \( C, D, \kappa \). As before, it follows that \( \mathcal{P}_1 \) satisfies \( K_1 \) and \( Pr_{\mathcal{P}_1}(C \subseteq D | \kappa) = Pr_{\mathcal{P}_2}(C \subseteq D | \kappa) \).

In order to prove Independence, we need the following lemma. It states an independence property of ME-distributions over our context language.

**Lemma 31 (ME-independence).** Let \( \mathcal{R}_1, \mathcal{R}_2 \) be two finite sets of probability constraints such that \( \text{sig}(L_1) \cap \text{sig}(L_2) = \emptyset \), and let \( \mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2 \). Then \( P_{\mathcal{R}}^{ME} = P_{\mathcal{R}_1}^{ME} \cdot P_{\mathcal{R}_2}^{ME} \). In particular, for the marginal distributions of \( P_{\mathcal{R}}^{ME} \), we have \( P_{\mathcal{R}}^{ME}(v_i) = P_{\mathcal{R}_1}^{ME}(v_i) \) for all \( v_i \in \text{Int}(L_i), i \in \{1, 2\} \).

**Proof.** Since the signatures of \( L_i \) are disjoint, we can denote the valuations of the language over \( \text{sig}(L_1) \cup \text{sig}(L_2) \) by \( (v_1, v_2), v_i \in \text{sig}(L_i) \). Let us first consider the marginals of \( P = P_{1}^{ME} \cdot P_{2}^{ME} \). For all \( v_i \in \text{Int}(L_i) \), we have

\[
P(v_i) = \sum_{v_2 \in \text{Int}(L_i)} P(v_1, v_2) = P_{1}^{ME}(v_i) \sum_{v_2 \in \text{Int}(L_i)} P_{2}^{ME}(v_2)
= P_{1}^{ME}(v_i).
\]
Symmetrically, we can show that \( P(v_2) = P_{2}^{ME}(v_2) \). This means, in particular, that the marginals of \( P \) coincide with the corresponding maximum entropy solutions. Therefore,

\[
H(P) = \sum_{v_1} \sum_{v_2} P_{1}^{ME}(v_1) P_{2}^{ME}(v_2) \log(P_{1}^{ME}(v_1) \cdot P_{2}^{ME}(v_2))
\]

\[
= \sum_{v_2} P_{2}^{ME}(v_2) \sum_{v_1} P_{1}^{ME}(v_1) \log P_{1}^{ME}(v_1)
\]

\[
+ \sum_{v_1} P_{1}^{ME}(v_1) \sum_{v_2} P_{2}^{ME}(v_2) \log P_{2}^{ME}(v_2)
\]

\[
= H(P_{1}^{ME}) + H(P_{2}^{ME}).
\]

Using the independence bound for entropy (see, e.g., [27], Theorem 2.39), we have that

\[
H(P_{R}^{ME}) \leq H(P_{1}^{ME}) + H(P_{2}^{ME}) = H(P).
\]

Hence, it suffices to show that \( P_{1}^{ME} \cdot P_{2}^{ME} \) is indeed a model of \( R_1 \cup R_2 \). But this follows immediately from the facts that \( P_{i}^{ME} \) satisfies \( R_i \) and that the marginalization of \( P \) over one logic corresponds to the ME-distribution over the other.

\[\square\]

**Theorem 26 (Independence).** Let \( K_1, K_2 \) be s.t. \( \text{sig}(K_1) \cap \text{sig}(K_2) = \emptyset \), \( C, D \) be two concepts, and \( \kappa \) a context where \( (\text{sig}(C) \cup \text{sig}(D) \cup \text{sig}(\kappa)) \cap \text{sig}(K_2) = \emptyset \). Then \( B(C \sqsubseteq_{K_1} D \mid \kappa) = B(C \sqsubseteq_{K_1 \cup K_2} D \mid \kappa) \).

**Proof.** Let \( K_i = (R_i, T_i) \) for \( i \in \{1, 2\} \). Since the signatures of both KBs are disjoint, we will denote the valuations over the set of variables \( \text{sig}(R_1) \cup \text{sig}(R_2) \) as pairs \((w_1, w_2)\), where \( w_i \) is a valuation over \( \text{sig}(R_i) \). We know from Theorem 18 that

\[
P_{ME}(\kappa)B(C \subseteq_{K_1 \cup K_2} D \mid \kappa) = \sum_{(w_1, w_2)|\kappa} P_{R}^{ME}((w_1, w_2))
\]

\[
= \sum_{(w_1, w_2)|\kappa} P_{R}^{ME}((w_1, w_2)) \quad (4)
\]

\[
= \sum_{(w_1)|\kappa} P_{R}^{ME}(w_1) \quad (5)
\]

\[
= P_{R}^{ME}(\kappa)B(C \subseteq_{K_1} D \mid \kappa),
\]

where (4) follows from the monotonicity of subsumption in \( \mathcal{ALC} \) TBoxes, and (5) is a consequence of Lemma 31. \[\square\]

In order to prove Continuity, we start with a lemma that states continuity of ME-distributions over \( \mathcal{L} \). The proof is analogous to Paris’ proof of continuity of maximum entropy reasoning in his probabilistic logic [22], which is a sub-logic of our probabilistic logic over \( \mathcal{L} \).
Lemma 32 (ME-continuity). Let $\mathcal{R}$ be a set of probabilistic constraints and let $(\mathcal{R}_i)$ be a sequence of probabilistic constraints such that $(\text{Mod}(\mathcal{R}_i))$ converges to Mod($\mathcal{R}$). Then the sequence $(P_i)$ of ME-Models of $\mathcal{R}_i$ converges to the ME-model $P^\text{ME}_R$ of $\mathcal{R}$.

Proof. For brevity, let $M = \text{Mod}(\mathcal{R})$ and $M_i = \text{Mod}(\mathcal{R}_i)$. We show that for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|K_i, K\| < \delta$ implies that $\|P^\text{ME}_i, P^\text{ME}_R\|_1 < \epsilon$. Consider the set $S = \{P \in M \mid \|P, P^\text{ME}_R\|_1 \geq \frac{\epsilon}{2}\}$ of models of $\mathcal{R}$ that have at least distance $\frac{\epsilon}{2}$ to $P^\text{ME}_R$. By continuity of the euclidean distance and compactness of $M$, $S$ must be compact. Since the entropy function $H$ is continuous, the minimum $\nu = \min\{H(P^\text{ME}_R) - H(P) \mid P \in S\}$ does exist and $\nu > 0$ by unique maximality of $P^\text{ME}_R$. Since $H$ is defined on a compact set (the set of probability distributions over $\mathcal{L}$), $H$ is uniformly continuous. Therefore there exists a $\delta > 0$ such that for all distributions $P_1, P_2$ over $\mathcal{L}$, $\|P_1, P_2\|_1 < \delta$ implies that $H(P_1) - H(P_2) < \min\{\frac{\nu}{2}, \frac{\epsilon}{2}\}$. In particular, we can assume that $\delta < \frac{\epsilon}{2}$. Now, if $\|M_i, M\|_2 < \delta$, there is a $P_i \in M_i$ such that $\|P_i, P^\text{ME}_M\|_1 < \delta$ and a $P \in M$ such that $\|P^\text{ME}_i, P\|_1 < \delta$. Hence,

$$H(P^\text{ME}_R) < H(P_i) + \frac{\nu}{2} \leq H(P^\text{ME}_i) + \frac{\nu}{2},$$

and therefore $|H(P^\text{ME}_i) - H(P^\text{ME}_R)| < \frac{\epsilon}{2}$. In particular,

$$|H(P) - H(P^\text{ME}_R)| \leq |H(P) - H(P^\text{ME}_i)| + |H(P^\text{ME}_i) - H(P^\text{ME}_R)|$$

$$< \frac{\nu}{2} + \frac{\nu}{2} = \nu.$$

By definition of $\nu$, we can conclude that $P \in M\setminus S$ and therefore $\|P^\text{ME}_R, P\|_1 < \frac{\epsilon}{2}$. Hence,

$$\|P^\text{ME}_R, P^\text{ME}_i\|_1 \leq \|P^\text{ME}_R, P\|_1 + \|P, P^\text{ME}_i\|_1 < \frac{\epsilon}{2} + \delta < \epsilon.$$

$\square$

Theorem 27 (Continuity). Let $(K_i)$ be a convergent sequence of KBs with limit $K$ and $B_{K_i}(C \subseteq D \mid \kappa) = \{\ell, u_i\}$. If $B_K(C \subseteq D \mid \kappa) = \{\ell, u\}$, then $(\ell_i)$ converges to $\ell$ and $(u_i)$ converges to $u$ (with respect to the usual topology on $\mathcal{R}$).

Proof. Let $K = (\mathcal{R}, T)$ and $K_i = (\mathcal{R}_i, T_i)$. By assumption, $(\text{Mod}(\mathcal{R}_i))$ converges to Mod($\mathcal{R}$). Hence, Lemma 32 implies that the probability distributions induced by ME-models of $K_i$ converge to $P^\text{ME}_R$, which, in turn, is the probability distribution induced by all ME-models of $K$. Hence, infimum and supremum of the conditional probability of $C \subseteq D$ given $\kappa$ with respect to $K_i$ will converge to infimum and supremum with respect to $K$. $\square$