Implementation of optimality criteria in the design of communication networks

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Abstract. We consider two frequently encountered problems in the modeling of the communication network, related to the construction of a graph of the communication network that meets certain conditions. Heuristic algorithms for solving these problems are proposed. The question of uniqueness of the solution of the set tasks is investigated. A positive result of solving the problem was obtained and sufficient conditions of uniqueness were found. The research and development of appropriate software are of practical importance in the design of real communication networks.

1. Introduction

We shall consider the problems of building a communication network based on the existing disparate fragments of the communication network. Such tasks often arise when upgrading the network and are accompanied by a set of analytical studies to identify and achieve the required technical characteristics of the communication network. In the first case, we shall consider building a connected graph of the communication network based on the existing disconnected subgraphs of the network with the requirement to minimize construction costs, which corresponds to the minimum value of the sum of the edge lengths of the simulated graph. And in the second case, we need build a graph that has the specified information communication directions that meet the stability requirements. Both of these problems are of high practical significance in the field of communication network theory [1]. Requirements for the stability of the communication network are contained, for example, in [2]. See [3, 4] for approaches to estimating the stability of the communication network.

In fact, in the first case, the situation under consideration is the problem of constructing a spanning tree of minimal weight [5]. We shall solve this problem using a heuristic greedy algorithm. When constructing a mathematical model of a communication network, it is proposed to use an algorithmic module based on a greedy algorithm to solve this problem. Note that in general case, according to Kirchhoff's theorem [5], there can be more than one spanning tree in a connected graph. To find the skeleton of the minimum weight, the Kruskal and primal algorithms can be used [5]. In our case, the existing graph of the communication network needs to be completed to a graph for which all the newly completed edges would be elements of the spanning tree, which for the constructed graph is determined using the already mentioned algorithms. The following algorithms can be considered as adaptations of Kruskal's algorithm to the problem of building an optimal communication network with the introduction of new conditions for the uniqueness of the optimal solution. Adaptation is subject to the application of
the algorithm Kruskal to some complete graph built on the basis of the existing network graph connected.

We shall consider the formulation of optimization problems according to the classics [6–9] and others. According to the same references, we shall use the terms «acceptable solution», «goal», etc. (the main terms of graph theory are used according to even more classical works [10–12] and others.)

2. Building a communication network optimal by weight criterion

Let us consider building a communication network based on the specified initial communication network that meets the connectivity condition and the minimum total length of completed communication lines. (We shall call this problem “Task 1”.)

The problem statement looks like this. Let be \( G = < V, E > \) source graph of the communication network. We have an optimization problem of the following type:

\[
\sum_{i=1}^{N_G} \rho(r_i) \rightarrow \min \\
\forall d_i \in \tilde{D}, f(d_i) > 0, i \in [1, N^D] 
\]

Here \( r_i \) is building a line of communication, \( \rho(r_i) \) is the length of the communication line \( r_i \), \( \rho(r_i) \in R \) it is a function of the geographical distance between points with specified geographical coordinates determined by the incident vertices of the communication line \( r_i \), \( \tilde{D} = \{d_i\}_{i=1}^{N}\bar{D} \) is a set of all possible information communication directions \([4, 5]\), \( N^D \in N \) is a number of all possible communication directions \( (N^D = C_{N_i2}, N^D = (N_i2)! \times (2! (N_i2 - 2)!)^{-1}, d_i = (v_1^{(i)}, v_2^{(i)}) \) is the information direction of communication, where \( v_1^{(i)} \in V \) and \( v_2^{(i)} \in V \) form a matching pair, \( f(d_i) \) is reliability of the information direction of communication \( d_i \), \( f(d_i) \in [0, 1], f(d_i) \in R \). In other words, you need to build a connected graph for a given source graph with the minimum total length of the completed edges. The optimal solution of the problem is found using a greedy algorithm. Among pairs of graph vertex \( (v_1, v_2) \), \( v_1 \in V \), \( v_2 \notin V \), not yet included in the set of edges, such is chosen, the first one, vertex \( v_1 \) and \( v_2 \) in an already constructed graph belong to different connected components (i.e. \( f(d) = 0 \), where \( d \in \tilde{D}, d = (v_1, v_2) \)), and, the second one, among all such pairs, one is selected for which the value \( \rho(v_1, v_2) \) is minimal.

2.1 Algorithm for solving Task 1

The algorithm for solving Task 1 is as follows.

**Step 1.** For all edges of the original graph, put the weight coefficient equal to zero.

**Step 2.** We set the set of edges to be \( M := \emptyset \).

**Step 3.** We set the auxiliary set of vertices to \( V' = \{v_1\}, v_1 \in V \).

**Step 4.** We set \( V := V - V' \), i.e. exclude a vertex \( v_1 \in V \) from the set \( V \).

**Step 5.** If \( V = \emptyset \), then end. Else go to step 6.

**Step 6.** Choosing a pair of graph vertices \( (v_1, v_2) \) satisfying the condition:

1) \( v_1 \in V' \) and \( v_2 \in V 
2) \( (v_1, v_2) \in E 
3) v_1 \neq v_2 
4) \( (v_1, v_2) \notin M \)

If there exists a pair of graph vertices, then we assume

1) \( V := V - \{v_2\}. 
2) V' := V' \cup \{v_2\}. 
3) M := M \cup \{v_1, v_2\} \)
4) $E := E - \{(v_1, v_2)\}$

**Step 7.** Choosing a pair of graph vertices $(v_1, v_2)$ satisfying the condition:
1) $v_1 \in V'$ and $v_2 \in V$
2) $v_1 \neq v_2$
3) $(v_1, v_2) \notin M$
4) $\rho(v_1, v_2)$ is minimum among all $\rho(v_1, v_2)$ satisfying the condition $(v_1, v_2) \in E$

If there exists a pair of graph vertices, then we assume
1) $M := M \cup \{(v_1, v_2)\}$,
2) $E := E - \{(v_1, v_2)\}$
3) $V' := V - \{v_2\}$,
4) $V' := V' \cup \{v_2\}$.

**Step 8.** If $V = \emptyset$, then end, $M$ is the desired set of edges of the graph, which provides connectivity and the minimum total length. Else go to step 6.

The testing of this algorithm was carried out on random graphs, the generation aspects of which are described in [13,14].

2.2 The algorithmic complexity of the algorithm for solving task 1

**Statement 1.** The algorithmic complexity of the algorithm for solving problem 1 is $O(|V|^3 \times |E|)$.

**Proof.** To join the next vertex, you need to look through the list of all available edges of the original graph, therefore, the complexity will be proportional to $|E|$ (in fact, it is possible to optimize the enumeration of edges by viewing only those that do not yet connect the vertices in the graph already constructed). Consider the procedure for enumerating the vertices of a graph to attach another vertex. Let be $n$ a number of vertices in a graph and $i$ is a number of vertices already attached. To join the next $(i + 1)$-th vertex the analysis of pairwise vertices from sets of cardinalities $i$ and $n - i$. From here the number of pairs analyzed will be $\sum_{i=1}^{n} i(n - i)$. Convert this expression. We get

$$n \sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 = n^2(n + 1)/2 - n(n + 1)(2n + 1)/6 = n(n + 1)(n - 1)/6.$$  

Therefore, we have the cubic complexity of the power of the set of vertices. Given the proportionality of the algorithmic complexity of the cardinality of the set of edges of the graph $|E|$, we obtain the asymptotic $O(|V|^3 \times |E|)$.

2.3 Uniqueness condition for the optimal solution to Task 1

The following statement is proved, which gives the sufficiency of uniqueness of the optimal solution to Task 1.

**Statement 2.** If the distances between non-incident vertices of the graph are different, then this algorithm leads to the only optimal solution to Task 1.

Note that the difference in pairwise distances between the vertices of the graph is easily achieved due to the high resolution of the real number (for example, for the floating-point number format in the IEEE 754 standard, the possible range of numbers is from $4.94 \times 10^{-324}$ to $1.79 \times 10^{308}$), which represents the distance between the vertices and takes place on real-time communication networks.

2.4 The condition for invariant choosing the starting vertex of algorithm 1

**Statement 3.** In the case where the pairwise distances between all non-incident vertices of the initial graph of the communication network are different, algorithm 1 for solving Problem 1 is invariant with respect to the choice of the starting vertex.

We omit the proof of these statements here; it will be given in future publications.

3. Optimal construction of the graph of the communication network, providing a given level of stability for a fixed direction of communication

Consider another problem that often arises when designing / modeling a communication network. This task is associated not with the creation of connected graphs (more precisely, not so much with their
creation), but with the construction of a path from one vertex to another. In this case, it is desirable to obtain the optimal path (which, as a rule, will be called the smallest possible along the total length of the edges included in it). However, according to, for example, [10, 11] and others, even Dijkstra’s algorithm can accomplish this in a time proportional to \( N^3 \) (which, of course, is unacceptable, since the use of this algorithm after adding another edge in the graph will significantly increase the complexity of the algorithm and may require critical time consuming). Therefore, in solving this problem, apparently, only heuristic algorithms are needed. The authors have already used heuristic algorithms in solving problems described in [15,16].

Returning to the task of designing a communication network, we first formally describe the statement of the problem itself; note in advance that graph edges may appear in it, which cannot be included in the constructed path.

4. Construction of a path between two vertices of a graph with a minimum total length of edges

Let us note in advance, that the considered problem (we shall call it “Task 2") should be distinguished from the problem of finding the shortest path, since we have not a fixed graph, but a graph that must be modified by adding edges.

Let be \( V = \{v_1,v_2,...,v_n\} \) set of vertices of a (constructed) graph, and on the set \( V \), the coordinate functions \( x(v_i) \in R \) and \( y(v_i) \in R \), where \( i = 1,..n \). Moreover, there are no two different vertices \( v_i \in V \) and \( v_j \in V \), satisfying the condition \( x(v_i) = x(v_j) \) and \( y(v_i) = y(v_j) \), \( i \neq j \), \( i = 1,..n \), \( j = 1,..n \). That is, in other words, there are no two different vertices whose distance between them is zero.

We define the distance function between the vertices \( \rho(v_i,v_j) \), equal to the geographical distance between the vertex points \( v_i \) and \( v_j \). As a simple approximation, we can consider the Pythagorean theorem for determining the distance, i.e. \( \rho(v_i,v_j) = ((x(v_i) - x(v_j))^2 + (y(v_i) - y(v_j))^2)^{1/2} \). Here \( v_i \in V \) and \( v_j \in V \), \( i = 1,..n \), \( j = 1,..n \). Notice, that \( \rho(v_i,v_j) = \rho(v_j,v_i) \).

Let be \( v_1 \) and \( v_2 \) initial and final vertex of the constructed path. The object of construction (a valid solution) will be the set \( E = \{(v_1,v_2),(v_2,v_3),\ldots,(v_{m-1},v_m)\} \).

Let the restriction on the length of the completed edge be given \( L \). As a rule, this restriction is taken into account the mileage of the communication line to bypass (during construction) and is 130 km. Thus, we have an optimization problem:

\[
\sum_{i=1}^{m-1} \rho(v_i,v_{i+1}) \rightarrow \min
\]

Under restrictions: for elements of the set \( E = \{(v_1,v_2),(v_2,v_3),\ldots,(v_{m-1},v_m)\} \)

\[
v_i \in V, i = 1,..m; \forall (v_i,v_{i+1}), i = 1,..m - 1: \rho(v_i,v_{i+1}) < L
\]

As already noted, an exact algorithm is extremely undesirable here (especially since this is usually an auxiliary task, therefore, the algorithm itself for this task will be applied on the order of \( N \) times or more). The simplest heuristics that can be used to create algorithms for solving this problem are obvious: select a new vertex (and add an edge along it) “next to” some vertex of the graph we are considering.

However, this approach, when applied in practice (i.e., to real data arising in the problems considered by us), gave poor results: when searching for vertices that are close to the first given one (usually \( v_1 \) at the beginning of the work), but at the same time reducing the remaining distance to the second given one (usually \( v_2 \) at the beginning of the work), we often (in practice, in our tasks, in 20% or more, which is usually unacceptable) we find ourselves in a situation where it is not possible to choose such an “improving” vertex: all they are already included either in forbidden vertex set, or in the already constructed part of the path. A little help (but also not always) is a heuristic similar to the so-called “shaker” sorting ([9,11], etc.): if you select any of the two vertices between which the path is built, according to some additional preferences, as the starting one, then we reduce the probability of an unsuccessful outcome, but it still remains on the order of 10–15% of cases, which is also unacceptable. Therefore, in practice, it is proposed to apply the developed heuristic algorithm, similar to the halving
method. This procedure is recursion. As noted earlier, \( v_1 \) and \( v_2 \) initial and final vertex of the constructed path. Therefore, we call the procedure start as \( A2(v_1, v_2) \). We tentatively assume \( S = \{ v_1, v_2 \} \).

### 4.1 Algorithm for solving Task 2

The algorithm for solving Task 2 is as follows:

**Step 1.** If \( \rho(v_1, v_2) \leq L \), then end, return \( \{(v_1, v_2)\} \), else go to step 2.

**Step 2.** If for \( v_1 \) no vertex \( v_i \) exists, such as
\[
\rho(v_1, v_i) \leq L, i \neq 1, i \in \{1,2,\ldots n\}
\]
and for \( v_2 \) no vertex \( v_i \) exists, such as
\[
\rho(v_2, v_i) \leq L, i \neq 2, i \in \{1,2,\ldots n\},
\]
then end, return an empty set. Else go to step 3.

**Step 3.** Calculation of the optimal coordinates of the intermediate vertex as a half-sum of the corresponding coordinates of the vertices \( v_1 \) and \( v_2 \).

**Step 4.** Finding the closest vertex to this midpoint that does not belong to the set \( S \). If such a vertex exists, call it \( v_3 \) and go to step 5. Else finish, return an empty set.

**Step 5.** \( S := S \cup \{v_3\} \).

**Step 6.** If \( A2(v_1, v_3) \) and \( A2(v_2, v_3) \) successfully complete work, then end with value \( A2(v_1, v_3)\cup A2(v_2, v_3) \). Else go to step 4.

### 4.2 Properties of the algorithm for solving problem 2

The asymptotic behavior of the algorithmic complexity of algorithm 2 is not higher than \( O(n^2) \). Also, based on Statements 2 and 3, it is easy to show that for different distances between non-incident vertices of the original graph, algorithm 2 will give the only optimal solution. In addition, it should be noted that if we consider all possible communication directions for a communication network when setting requirements of non-zero stability, then the result of the algorithm 2 applied for each direction of communication with the given requirements will be identical to the result of the operation of algorithm 1. This algorithm 2 was used to build a communication network that meets the requirement for stability (reliability and survivability) for given information directions of communication with minimizing the cost of building new communication lines. Since the survivability of the communication information direction is ensured along with the reliability of the communication nodes, primarily backup routes, the task of constructing a communication network in terms of ensuring the survivability of the communication direction was just reduced to building a given number of paths between the corresponding nodes, provided that the total mileage of the completed communication lines is minimized, i.e., using the algorithm 2. At the same time, the reliability and survivability of the direction of communication defined by the vertices between which the shortest the path according to problem 2 was carried out on the basis of already available information about the stability of the direction of communication before the construction of the mentioned path, which in turn was an acceleration factor on the one hand, and, on the other hand, the assessment could be a criterion for stopping the procedure of constructing paths between given node nodes. The proof is practically obvious and follows from the probability theorem of the sum of events [17]. Namely if \( n - 1 \) paths have already been built and \( p_i \) is stability of the \( i \)-th path \( (i \in \{1,2,\ldots n - 1\}) \), then stability in this case was equal to

\[
P = \sum_{i=1}^{n-1} p_i - \sum_{i,j \in \{1,2,\ldots n-1\}} p_ip_j - \sum_{i,j,k \in \{1,2,\ldots n-1\}} p_ip_jp_k + \ldots + (-1)^{n-2} \prod_{i=1}^{n-1} p_i \quad (5)
\]

In the case of constructing another path, the stability will be equal to

\[
\hat{P} = P + p_n - Pp_n \quad (6)
\]

After mathematical transformations we get

\[
\hat{P} = \sum_{i=1}^{n} p_i - \sum_{i,j \in \{1,2,\ldots n\}} p_ip_j - \sum_{i,j,k \in \{1,2,\ldots n\}} p_ip_jp_k + \ldots + (-1)^{n-1} \prod_{i=1}^{n} p_i \quad (7)
\]

This in turn proves the possibility of calculating the stability of the communication direction after building a new path, based on the value of the stability index of this communication direction, calculated on the basis of existing paths. If it is necessary to obtain the most accurate estimate of the reliability of the given communication direction for the already constructed large dimension communication network,
it is recommended to use the algorithm described in detail in [18]. Concerning the task 2, it can also be proved that the greedy algorithm given will give the only optimal solution if all pairwise distances between vertices are different.

5. Conclusion

Thus, two problems are considered that arise when constructing a large dimension communication network with a minimum total mileage of communication lines being built, provided that the requirements for reliability and survivability are met. The first task solves the issue of reconnecting network connectivity. The second serves to build a communication network with the required reliability and survivability indicators. Moreover, in solving the second problem, the reliability and survivability of the communication direction, relative to which stability requirements are set, can be recalculated during the construction of paths between nodes taking into account a priori information about the reliability and survivability of the communication direction based on already constructed paths, which significantly speeds up the procedure control of stability indicators (due to the absence of the need to analyze the graph of the communication network in order to search for paths between two vertices at each iteration of constructing s). Algorithms are proposed for solving these optimization problems, which in turn were the basis for the development of appropriate software.

It should also be noted that the tasks under consideration can be complicated by the introduction of additional criteria for the effectiveness of the communication network, as well as the differentiation of communication nodes (which is a common practice in the design of communication networks), which are subject to various restrictions when forming the topology of the developed communication network. In this case, the proposed algorithms can be modified, which is the subject of further research.

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