From Reversible Programs to Univalent Universes and Back

Jacques Carette
McMaster University

Chao-Hong Chen Vikraman Choudhury Amr Sabry
Indiana University

Abstract

We establish a close connection between a reversible programming language based on type isomorphisms and a formally presented univalent universe. The correspondence relates combinators witnessing type isomorphisms in the programming language to paths in the univalent universe; and combinator optimizations in the programming language to 2-paths in the univalent universe. The result suggests a simple computational interpretation of paths and of univalence in terms of familiar programming constructs whenever the universe in question is computable.

1 Introduction

The proceedings of the 2012 Symposium on Principles of Programming Languages [1] included two apparently unrelated papers: Information Effects by James and Sabry and Canonicity for 2-dimensional type theory by Licata and Harper. The first paper, motivated by the physical nature of computation [23,26,29,5,13], proposed, among other results, a reversible language $\Pi$ in which every program is a type isomorphism. The second paper, motivated by the connections between homotopy theory and type theory [31,28], proposed a judgmental formulation of intensional dependent type theory with a twice-iterated identity type. During the presentations and ensuing discussions at the conference, it became apparent, at an intuitive and informal level, that the two papers had strong similarities. Formalizing the precise connection was far from obvious, however.

Here we report on a formal connection between appropriately formulated reversible languages on one hand and univalent universes on the other. In the next section, we give a rational reconstruction of the reversible programming language $\Pi$, focusing on a small “featherweight” fragment $\Pi_2$. In Sec. 3, we review basic homotopy type theory (HoTT) background leading to univalent fibrations which allow us to give formal presentations of “small” univalent universes. In Sec. 4 we define and establish
the basic properties of such a univalent subuniverse \( 0[2] \) which we prove in Sec. 5 as sound and complete with respect to the reversible language \( \Pi_2 \). Sec. 6 discusses the implications of our work and situates it into the broader context of the existing literature.

## 2 Reversible Programming Languages

Starting from the physical principle of “conservation of information” [16,14], James and Sabry [18] proposed a family of programming languages \( \Pi \) in which computation preserves information. Technically, computations are *type isomorphisms* which, at least in the case of finite types, clearly preserve entropy in the information-theoretic sense [18]. We illustrate the general flavor of the family of languages with some examples and then identify a “featherweight” version of \( \Pi \), called \( \Pi_2 \), to use in our formal development.

### 2.1 Examples

The examples below assume a representation of the type of booleans \( 2 \) as the disjoint union \( 1 \oplus 1 \) with the left injection representing \texttt{false} and the right injection representing \texttt{true}. Given an arbitrary reversible function \( f \) of type \( a \leftrightarrow_1 a \), we can build the reversible function \texttt{controlled} \( f \) that takes a pair of type \( 2 \otimes a \) and checks the incoming boolean; if it is false (i.e., we are in the left injection), the function behaves like the identity; otherwise the function applies \( f \) to the second argument. The incoming boolean is then reconstituted to maintain reversibility:

\[
\begin{align*}
\text{controlled} : \forall a. (a \leftrightarrow_1 a) & \rightarrow (2 \otimes a \leftrightarrow_1 2 \otimes a) \\
\text{controlled} f = 2 \otimes a & \quad \leftrightarrow_1 \langle \text{unfoldBool} \otimes \text{id} \rangle \\
(1 \oplus 1) \otimes a & \quad \leftrightarrow_1 \langle \text{distribute} \rangle \\
(1 \otimes a) \oplus (1 \otimes a) & \quad \leftrightarrow_1 \langle \text{id} \oplus (\text{id} \otimes f) \rangle \\
(1 \otimes a) \oplus (1 \otimes a) & \quad \leftrightarrow_1 \langle \text{factor} \rangle \\
(1 \otimes 1) \otimes a & \quad \leftrightarrow_1 \langle \text{foldBool} \otimes \text{id} \rangle \\
2 \otimes a &
\end{align*}
\]

The left column shows the sequence of types that are visited during the computation; the right column shows the names of the combinators\(^1\) that witness the corresponding type isomorphism. The code for \texttt{controlled} \( f \) provides constructive evidence (i.e., a program, a logic gate, or a hardware circuit) for an automorphism on \( 2 \otimes a \): it can be read top-down or bottom-up to go back and forth.

The \texttt{not} function below is a simple lifting of \texttt{swap} \(_+\) which swaps the left and right injections of a sum type. Using the \texttt{controlled} building block, we can build a controlled-not (\texttt{cnot}) gate and a controlled-controlled-not gate, also known as the \texttt{toffoli} gate. The latter gate is a universal function for combinational boolean circuits thus showing

\(^1\) We use names that are hopefully quite mnemonic; for the precise definitions of the combinators see the \( \Pi \)-papers [18,6,19,20,8] or the accompanying code at [https://git.io/v7WtW](https://git.io/v7WtW).
the expressiveness of the language:

\[
\begin{align*}
\text{not} & : 2 \leftrightarrow 1 \rightarrow 2 \\
\text{not} & = \text{unfoldBool} \odot_1 \text{swap} \odot_1 \text{foldBool} \\
\text{cnot} & : 2 \odot_1 2 \leftrightarrow 1 \odot_1 2 \\
\text{cnot} & = \text{controlled not} \\
\text{toffoli} & : 2 \odot_1 (2 \odot_1 2) \leftrightarrow 1 \odot_1 (2 \odot_1 2) \\
\text{toffoli} & = \text{controlled cnot}
\end{align*}
\]

While we wrote \text{controlled} in equational-reasoning style, \text{not} is written in the point-free combinator style. These are equivalent as \text{\leftrightarrow}_1 \langle - \rangle is defined in terms of the sequential composition combinator \odot_1.

As is customary in any semantic perspective on programming languages, we are interested in the question of when two programs are "equivalent." Consider the following six programs of type \(2 \leftrightarrow_1 2\):

\[
\begin{align*}
id_1, id_2, id_3, \text{not}_1, \text{not}_2, \text{not}_3 & : 2 \leftrightarrow_1 2 \\
id_1 & = \text{id} \odot_1 \text{id} \\
id_2 & = \text{not} \odot_1 \text{id} \odot_1 \text{not} \\
id_3 & = \text{uniti} \odot_1 \text{swapi} \odot_1 (\text{id} \odot_1 \text{id}) \odot_1 \text{swapi} \odot_1 \text{unitei} \\
\text{not}_1 & = \text{id} \odot_1 \text{not} \\
\text{not}_2 & = \text{not} \odot_1 \text{not} \odot_1 \text{not} \\
\text{not}_3 & = \text{uniti} \odot_1 \text{swapi} \odot_1 (\text{not} \odot_1 \text{id}) \odot_1 \text{swapi} \odot_1 \text{unitei}
\end{align*}
\]

The programs are all of the same type but this is clearly not a sufficient condition for "equivalence." Thinking extensionally, i.e., by looking at all possible input-output pairs, it is easy to verify that the six programs split into two classes: one consisting of the first three programs which are all equivalent to the identity function and the other consisting of the remaining three programs which all equivalent to boolean negation. In the context of \(\Pi\), we can provide evidence (i.e., a reversible program of type \(\leftrightarrow_2\) that manipulates lower level reversible programs of type \(\leftrightarrow_1\)) that can concretely identify programs in each equivalence class. We show such a level-2 program proving that \text{not}_3 is equivalent to \text{not}. For illustration, the program for \text{not}_3 is depicted in Fig. 1. We encourage the reader to map the steps below to manipulations...
on the diagram that would incrementally simplify it:

\[
\begin{align*}
\text{notOpt} & : \text{not}_1 \leftrightarrow \text{not}_2 \\
\text{notOpt} & = \text{unit} \odot_1 (\text{swap} \odot_1 ((\text{not} \odot_1 \text{id}) \odot_1 (\text{swap} \odot_1 \text{unite}_1))) \leftrightarrow_2 \text{id} \odot \text{assocLeft} \\
\text{unit}_1 & \odot_1 (\text{swap} \odot_1 ((\text{not} \odot_1 \text{id}) \odot_1 (\text{swap} \odot_1 \text{unite}_1))) \leftrightarrow_2 \text{id} \odot \text{swapLeft} \odot \text{id} \\
\text{unit}_1 & \odot_1 ((\text{id} \odot \text{not}) \odot_1 \text{swap} \odot_1 \text{unite}_1) \leftrightarrow_2 \text{id} \odot \text{assocRight} \\
\text{unit}_1 & \odot_1 ((\text{id} \odot \text{not}) \odot_1 (\text{swap} \odot_1 \text{unite}_1))) \leftrightarrow_2 \text{id} \odot \text{id} \odot \text{assocLeft} \\
\text{unit}_1 & \odot_1 ((\text{id} \odot \text{not}) \odot_1 (\text{swap} \odot_1 \text{unite}_1))) \leftrightarrow_2 \text{id} \odot \text{id} \odot \text{uniteLeft} \\
\text{unit}_1 & \odot_1 (\text{id} \odot \text{not} \odot_1 \text{unite}_1) \leftrightarrow_2 \text{assocLeft} \\
\text{unit}_1 & \odot_1 ((\text{id} \odot \text{not}) \odot_1 \text{unite}_1) \leftrightarrow_2 \text{unitLeft} \odot \text{id} \\
\text{not}_1 & \odot_1 (\text{unit} \odot_1 \text{unite}_1) \leftrightarrow_2 \text{assocRight} \\
\text{not}_1 & \odot_1 \text{id} \leftrightarrow_2 \text{id} \odot \text{leftInv} \\
\text{not}_1 & \leftrightarrow_2 \text{idRight} \\
\end{align*}
\]

It is worthwhile mentioning that the above derivation could also be drawn as one commutative diagram in an appropriate category, with each \( \leftrightarrow_2 \langle - \rangle \) as a 2-arrow (and representing a natural isomorphism). See Shulman’s draft book [27] for that interpretation.

### 2.2 A Small Reversible Language of Booleans: \( \Pi_2 \)

Having illustrated the general flavor of the \( \Pi \) family of languages, we present in full detail an Agda-based formalization of a small \( \Pi \)-based language which we will use to establish the connection to an explicit univalent universe. The language is the restriction of \( \Pi \) to the case of just one type \( 2 \):

\[
data\ 2 : \mathcal{U} \text{ where} \\
0_2, 1_2 : 2
\]

The syntax of \( \Pi_2 \) is given by the following four Agda definitions. The first definition \( \Pi_2 \) introduces the set of types of the language: this set contains just \( '2 \) which is a name for the type of booleans \( 2 \). The next three definitions introduce the programs (combinators) in the language stratified by levels. The level-1 programs of type \( \rightarrow_1 \) map between types; the level-2 programs of type \( \leftrightarrow_2 \) map between level-1 programs; and the level-3 programs of type \( \leftrightarrow_3 \) map between level-2 programs:

\[
data\ \Pi_2 : \mathcal{U} \text{ where} \\
'2 : \Pi_2
\]

\[
data\_\rightarrow_1_1 : \langle A \ B : \Pi_2 \rangle \rightarrow \mathcal{U} \text{ where} \\
\text{id} : \forall \{A\} \rightarrow A \rightarrow_1 A \\
\text{not} : '2 \leftrightarrow_1 '2 \\
\text{idl} : \forall \{A \ B\} \rightarrow (A \rightarrow_1 B) \rightarrow (B \rightarrow_1 A) \\
\text{idr} : \forall \{A \ B\} \rightarrow (A \rightarrow_1 B) \rightarrow (B \rightarrow_1 A) \\
\text{assoc} : \forall \{A \ B \ C\} \rightarrow (A \rightarrow_1 B) \rightarrow (B \rightarrow_1 C) \rightarrow (A \rightarrow_1 C)
\]

\[
data\_\leftrightarrow_2_2 : \forall \{A \ B\} \ (p \ q : A \rightarrow_1 B) \rightarrow \mathcal{U} \text{ where} \\
\text{id}_2 : \forall \{A \ B\} \ (p : A \rightarrow_1 B) \rightarrow p \rightarrow_2 p \\
\text{idl}_2 : \forall \{A \ B\} \ (p : A \rightarrow_1 B) \rightarrow \text{idl} \odot_1 p \rightarrow_2 p \\
\text{idr}_2 : \forall \{A \ B\} \ (p : A \rightarrow_1 B) \rightarrow \text{idr} \odot_1 p \rightarrow_2 p \\
\text{assoc}_2 : \forall \{A \ B \ C\} \ (p : A \rightarrow_1 B) \rightarrow (q : B \rightarrow_1 C) \rightarrow (r : C \rightarrow_1 D) \\
\rightarrow (p \odot_1 q) \odot_1 r \rightarrow_2 p \odot_1 (q \odot_1 r)
\]

\[
\_\odot_2_2 : \forall \{A \ B \ C\} \ (p \ q : A \rightarrow_1 B) \{r \ s : B \rightarrow_1 C\}
\]
What is particularly interesting, however, is that the collection of level-2 combinators This enables us to compute for any 2-combinator we have the following groups: (i) the first group contains the identity, inverses, and as well as a proof that the name of its canonical form, c is equivalent to its canonical form: c = 'id.

In the previous presentations of Π [6, 18, 8], the level-3 programs, consisting of just one trivial program 'trunc, were not made explicit. The much larger level-1 and level-2 programs of the full Π language [8] have been specialized to our small language. For the level-1 constructors, denoting reversible programs, type isomorphisms, permutations between finite sets, or equivalences depending on one’s favorite interpretation, we have two canonical programs 'id and 'not closed under inverses l₁ and sequential composition ⊗₁. For level-2 constructors, denoting reversible program transformations, coherence conditions on type isomorphisms, equivalences between permutations, or program optimizations depending on one’s favorite interpretation, we have the following groups: (i) the first group contains the identity, inverses, and sequential composition; (ii) the second group establishes the coherence laws for level-1 sequential composition (e.g., it is associative); and (iii) finally the third group includes general rules for inversions of level-1 constructors.

Each of the level-2 combinators of type p ↔ q is easily seen to establish an equivalence between level-1 programs p and q (as shown in previous work [8] and in Sec. 5). For example, composition of negation is equivalent to the identity:

What is particularly interesting, however, is that the collection of level-2 combinators above is complete in the sense that any equivalence between level-1 programs p and q can be proved using the level-2 combinators. Formally we have two canonical level-1 programs 'id and 'not and for any level-1 program p, we have evidence that either p ↔ 'id or p ↔ 'not.

To prove this, we introduce a type which encodes the knowledge of which level-1 programs are canonical. The type Which names the subset of ↔₁ which are canonical forms:

data Which : Λ where
  ID NOT : Which

refine : (w : Which) → '2 ↔₁ '2
refine ID = 'id
refine NOT = 'not

This enables us to compute for any 2-combinator c (the name of) its canonical form, as well as a proof that c is equivalent to its canonical form:

canonical : (c : '2 ↔₁ '2) → Σ[ c' : Which ] (c ↔₂ refine c')
canonical 'id = ID , 'id₂
We work in intensional type theory with one univalent universe \( \mathcal{U} \). As Ch. 4 of the HoTT book [28] details, this is problematic in the proof-relevant setting of HoTT. To ensure that a function \( f : A \rightarrow B \) is a quasi-inverse, if there is another function \( g : B \rightarrow A \) that acts as both a left and right inverse to \( f \):

\[
\begin{align*}
\text{is-qinv} & : \{ A : \mathcal{U} \} \rightarrow \{ f : A \rightarrow B \} \\
\text{is-qinv} & : \{ A \} \{ B \} f = \Sigma[ g : (B \rightarrow A) ] \Sigma[ \eta : g \circ f \sim id ] \Sigma[ \varepsilon : f \circ g \sim id ] (g \circ f \circ \eta \sim \varepsilon \circ f)
\end{align*}
\]

In general, for a given \( f \), there could be several unequal inhabitants of the type \( \text{is-qinv} f \). As Ch. 4 of the HoTT book [28] details, this is problematic in the proof-relevant setting of HoTT. To ensure that a function \( f \) can be an equivalence in at most one way, an additional coherence condition is added to quasi-inverses to define half adjoint equivalences:

\[
\begin{align*}
\text{is-hae} & : \{ A : \mathcal{U} \} \rightarrow \{ f : A \rightarrow B \} \\
\text{is-hae} & : \{ A \} \{ B \} f = \Sigma[ g : (B \rightarrow A) ] \Sigma[ \eta : g \circ f \sim id ] \Sigma[ \varepsilon : f \circ g \sim id ] (g \circ f \circ \eta \sim \varepsilon \circ f)
\end{align*}
\]

Using this latter notion, we can define a well-behaved notion of equivalences between two types:

\[
\begin{align*}
\text{is-equiv} & = \text{is-hae} \\
\approx & : \{ A : \mathcal{U} \} \rightarrow \{ B : \mathcal{U} \} \rightarrow \Sigma[ f : (A \rightarrow B) ] (\text{is-equiv } f)
\end{align*}
\]

It is straightforward to lift paths to equivalences as shown below:

\[
\begin{align*}
\text{ide} & : \{ A : \mathcal{U} \} \rightarrow A \approx A \\
\text{ide} & = \text{id}_A \circ \text{id}_A \circ \text{refl}_A \circ \text{refl}_A \\
\text{transport-equiv} & : \{ A : \mathcal{U} \} \{ P : A \rightarrow \mathcal{U} \} \rightarrow \{ a : A \} \rightarrow a \approx b \rightarrow P \approx P
\end{align*}
\]
id-to-equiv : {A B : U} → A ≃ B
id-to-equiv = transport-equiv id

Dually, univalence allows us to construct paths from equivalences. We postulate

univalence : (A B : U) → is-equiv (id-to-equiv {A} {B})

We also give a short form ua for getting a path from an equivalence, and prove some

computation rules for it:

ua : A ≃ B → A ≃ B
ua = pr₁ (univalence A B)

ua-β : id-to-equiv ° ua ∼ id
ua-β = pr₁ (pr₂ (pr₂ (univalence A B)))

ua-β₁ : transport id ° ua ∼ pr₁
ua-β₁ equiv = transport _ (ua-β₁ equiv) (ap pr₁)

ua-η : ua ° id-to-equiv ∼ id
ua-η = pr₁ (pr₂ (univalence A B))

3.2 Propositional Truncation

A type A is contractible (h-level 0 or (-2)-truncated), if it has a center of contraction,
and all other terms of A are connected to it by a path:

is-contr : (A : U) → U
is-contr A = Σ[ a : A ] Π[ b : A ] (a ≃ b)

As alluded to in the previous section, equivalences are contractible (4.2.13 in [28]):

is-equiv-is-contr : {A B : U} {f : A → B} → is-equiv f → is-contr (is-equiv f)

A type A is a proposition (h-level 1 or (-1)-truncated) if all pairs of terms of A are
connected by a path. Such a type can have at most one inhabitant; it is “contractible
if inhabited.” Finally, a type A is a set if for any two terms a and b of A, its type of
paths a ≃ b is a proposition:

is-prop : (A : U) → U
is-prop A = Π[ a : A ] Π[ b : A ] (a ≃ b)

is-set : (A : U) → U
is-set A = Π[ a : A ] Π[ b : A ] is-prop (a ≃ b)

Any type can be truncated to a proposition by freely adding paths. This is the
propositional truncation (or (-1)-truncation) which can be expressed as a higher
inductive type (HIT). The type constructor ∥_∥ takes a type A as a parameter; the
point constructor |_| coerces terms of type A to terms in the truncation; and the path
constructor ident identifies any two points in the truncation, making it a proposition.
We must do this as a postulate as Agda does not yet support HITs:

∥_∥ : (A : U) → U
Carette, Chen, Choudhury, and Sabry

Fig. 2. (left) Type family $P : A \rightarrow U$ as a fibration with total space $\Sigma_{x:A} P(x)$; (right) a path $x \equiv y$ in the base space induces an equivalence between the spaces (fibers) $P(x)$ and $P(y)$.

This makes $\|A\|$ the “free” proposition on any type $A$. The recursion principle (below) ensures that we can only eliminate a propositional truncation to a type that is a proposition:

$$\begin{align*}
&\text{ident : } \{A : U\} \to (a : A) \to \|A\| \\
&\text{rec-}\|\|- : [a : A] \to \|A\| \to P \\
&\text{rec-}\|\|-\beta : \Pi[a : A] \forall[\|\|a\|\| \to P] \to \|\|a\|\| \equiv \beta
\end{align*}$$

3.3 Type Families are Fibrations

As illustrated in Fig. 2, a type family $P$ over a type $A$ is a fibration with base space $A$, with every $x$ in $A$ inducing a fiber $P_x$, and with total space $\Sigma\{x : A\} P(x)$. \(^2\)

The path lifting property mapping a path in the base space to a path in the total space can be defined as follows:

$$\begin{align*}
&\text{lift : } \{A : U\} \{P : A \to U\} \{x \, y : A\} \to (u : P x) \to (p : x \equiv y) \to (x , u) \equiv (y , \text{transport} P p u) \\
&\text{lift u (refl x)} = \text{refl (x , u)}
\end{align*}$$

As illustrated in the figure below, the point $\text{transport} P p u$ is in the space $P y$. A path from that point to another point $v$ in $P y$ can be viewed as a virtual “path” between $u$ and $v$ that “lies over” $p$. Following Licata and Brunerie [24], we often use the syntax $u \equiv v[\{P \to P\}]$ for the path $\text{transport} P p u \equiv v$ to reinforce this perspective. In other words, the curved “path” between $u$ and $v$ below consists of first transporting $u$ to the space $P y$ along $p$ and then following the straight path in $P y$ to $v$.

\(^2\) In this and following figures, we color paths in blue and functions in red.
Given a fibration \( P \) and points \( x, y, u, \) and \( v \) as above, we have the following characterization of dependent paths in the total space:

\[
\text{module } _\{\{A : U\}\} \{\{P : A \to U\}\} \{\{x : A\}\} \{\{u : P x\}\} \{\{v : P y\}\} \text{ where}
\]

\[
\begin{align*}
\text{dpair} & : \Sigma[p : x == y] (u == v [P \downarrow p]) \to (x, u) == (y, v) \\
\text{dpair= } & : \text{refl} (x, u) = \text{refl} (x, u) \\
\text{dpair=β} & : (w : \Sigma[p : x == y] (u == v [P \downarrow p])) \to (\text{ap pr}_1 \circ \text{dpair=}) w == \text{pr}_1 w \\
\text{dpair=-e } & : (x, u) == (y, v) \to x == y \\
\text{dpair=-e } & = \text{ap pr}_1 \\
\end{align*}
\]

The first function builds a path in the total space given a path between \( u \) and \( v \) that lies over a path \( p \) in the base space; the second function is a computation rule for this path; and the third function eliminates a path in the total space to a path in the base space.

### 3.4 Univalent Fibrations

Univalent fibrations are defined by Kapulkin and Lumsdaine [21] in the simplicial set (sSet) model. In our context, a type family (fibration) \( P : A \to U \) is univalent if the map \( \text{transport-equiv} P \) defined in Sec. 3.1 is an equivalence, that is, if the space of paths in the base space is equivalent to the space of equivalences between the corresponding fibers. Fig. 2 (right) illustrates the situation: we know that for any fibration \( P \) that a path \( p \) in the base space induces via \( \text{transport-equiv} P p \) an equivalence between the fibers. For a fibration to be univalent, the reverse must also be true: every equivalence between the fibers must induce a path in the base space. Formally, we have the following definition:

\[
\begin{align*}
is\text{-univ-fib} : \{\{A : U\}\} \{\{P : A \to U\}\} \to U \\
is\text{-univ-fib} \{\{A\}\} P & = \forall (a b : A) \to \text{is-equiv} (\text{transport-equiv} P \{\{a\}\} \{\{b\}\}) \\
\end{align*}
\]

We note that the univalence axiom (for \( U \)) is a specialization of \( \text{is\text{-univ-fib}} \) to the identity fibration, \( \text{id} \). More generally, we can define universes à la Tarski by having a code \( \mathcal{U} \) for the universe and an interpretation function \( \mathcal{E} \) into \( U \). Such a presented universe is univalent if \( \mathcal{E} \) is a univalent fibration:
\[ \tilde{U} = \Sigma [ U : U ] (U \to U) \]
\[ \text{is-univalent} : \tilde{U} \to U \]
\[ \text{is-univalent} (U, El) = \text{is-univ-fib} El \]

As Christensen [9] explains, a type \( U \) is rarely the base of a univalent fibration. Yet, in that same paper, Christensen characterizes a class of types that is always the base of univalent fibrations. We explain this point and exploit it to build a custom univalent subuniverse in the next section.

4 The Subuniverse \( \tilde{U} [2] \)

We now have all the ingredients necessary to define the class of univalent subuniverses we are interested in. Given any type \( T \), we can build a propositional predicate that picks out from among all the types in the universe exactly those which are identified with \( T \). This lets us build up a “singleton” subuniverse of \( U \) as follows:

\[
\tilde{U} [T] : (T : U) \to \tilde{U} \]
\[ \tilde{U} [T] = U \cup \text{El} \]
\[ \text{where} \]
\[ U = \Sigma [ X : U ] \parallel X == T \parallel \]
\[ \text{El} = \text{pr}_1 \]

We will prove in this section and the next that choosing \( T \) to be \( 2 \) produces a universe that is sound and complete with respect to the language \( \Pi_2 \). The bulk of the argument consists of establishing that \( \tilde{U} [2] \) is a univalent universe. We focus on this argument in the first subsection. In the next two subsections, we use this result to characterize the points and paths in the type of codes for this universe. In Sec. 5 this characterization of points and paths will be shown to match the types and combinators of \( \Pi_2 \).

4.1 The Fibration \( \text{El}_2 \) is Univalent

The universe \( \tilde{U} [2] \) consists of a base space \( U[2] \) of the codes for the elements, and an interpretation function \( \text{El}_2 \), defined as follows:

\[ U[2] : U \]
\[ U[2] = \text{pr}_1 \tilde{U} [2] \]
\[ \text{El}_2 : \Sigma [ X : U ] \parallel X == 2 \parallel \]
\[ \text{El}_2 = \text{pr}_1 \]

The type family \( \text{El}_2 \) defines a fibration with base space \( U[2] \) as shown below:

Fiber 2 \[ \cong \] Fiber X

\[ (2, \text{refl } 2) \parallel (X, |p|) \]

Base Space \( U[2] = \Sigma [ X : U ] \parallel X == 2 \parallel \)
Our goal is to show that $\mathbb{E}^2$ is a univalent fibration. We establish this by chaining two equivalences. The first equivalence is a simple appeal to univalence in order to establish that $(X == 2) \simeq (X \simeq 2)$, i.e., our base space is equivalent to the space $\sum [X : U] \parallel X \simeq 2 \parallel$. We name this space $\text{BAut } 2$. Generally, $\text{BAut } T$ is the “classifying space” of all types that are (merely) equivalent to $T$. The second equivalence consists of proving that the first projection on $\text{BAut } T$ is in fact a univalent fibration, for all spaces with shape $\sum [X : U] \parallel X \simeq T \parallel$ for any type $T$. This is the lemma $\text{is-univ-fib-ElB}$ below whose original formulation is due to Christensen [9]:

$$\begin{align*}
\text{BAut} : (T : U) &\to U \\
\text{BAut } T &= \sum [X : U] \parallel X \simeq T \parallel \\
\text{ElB} : \{T : U\} &\to \text{BAut } T \to U \\
\text{ElB} &= \text{pr}_1 \\
\text{transport-equiv-ElB} : \{T : U\} &\to \text{transport-equiv ElB} (\text{dpair=-e } p) \\
\text{is-univ-fib-ElB} : \{T : U\} &\to \text{is-univ-fib ElB} (\text{dpair= (ua eqv, ident)}) \\
\eta : g \circ \text{transport-equiv ElB} \simeq \text{id} \\
\eta (\text{refl } _) &= \text{ap dpair=} (\text{dpair=} (\text{ua-η } \text{refl } _), \text{prop-is-set } (\lambda _ . \text{ident})) \\
\epsilon : \text{transport-equiv ElB} \circ g \simeq \text{id} \\
\epsilon \text{ eqv} &= \text{eqv=} (\text{transport-equiv-ElB} (\text{dpair=} (\text{ua eqv }, \text{ident}))) \\
&\text{ap (transport id) (dpair=-β (ua eqv, ident))} \\
&\text{ua-β} \text{ eqv } \\
\end{align*}$$

This establishes that $\mathbb{E}^2$ is a univalent fibration, giving us a characterization of paths in $U[2]$ in terms of equivalences on booleans which we exploit next.

### 4.2 The Base Space $U[2]$

The points in the base space $U[2]$ are all of the form $(X, |p|)$ where $p$ is of type $X == 2$. We evidently have a canonical point $2_0$:

$$\begin{align*}
2_0 : U[2] \\
2_0 &= (2, | \text{refl } 2 |)
\end{align*}$$

which directly corresponds to the boolean type in $\Pi_2$. We remind the reader that, by construction, $U[2]$ is path-connected. What remains is to characterize the 1-paths, 2-paths, and possibly higher paths in $U[2]$ and to relate them to the 1-combinators, 2-combinators, etc. in $\Pi_2$.

To conveniently refer to the paths in $U[2]$, we define the loop space on a (pointed) type, and show that the loop space on $\text{BAut } 2$ is equivalent to $2 \simeq 2$:

$$\begin{align*}
\Omega : \sum [T : U] T &\to U \\
\Omega (T, t_0) &= t_0 \simeq t_0 \\
\text{Aut} : \{T : U\} &\to U \\
\text{Aut } T &= T \simeq T \\
\text{b_0} : \{T : U\} &\to \text{BAut } T \\
\text{b_0 } \{T\} &= T, | \text{ide } T |
\end{align*}$$
The above result states that, in general, the loop space of the classifying space of a type $T$ is equivalent to the type of automorphisms of $T$. In particular, it follows that $\Omega(B\text{Aut}\ (2, 2_0) \simeq \text{Aut}\ 2)$ which reduces the problem of characterizing paths on $U[2]$ to the much simpler problem of characterizing automorphisms on the type of booleans. We now turn our attention to solving that problem.

### 4.3 Automorphisms on $2$

The type $2$ has two point constructors, and no path constructors, which means it has no non-trivial paths on its points except $\text{refl}$. In fact, we can prove in intensional type theory using large elimination, that the two constructors are disjoint. This is reflected in the absurd pattern when using dependent pattern matching in Agda. More generally, $2 \simeq 1 + 1$ and the disjoint union of two sets is a set:

$$0 \neq 1 \colon 0 \equiv 1 \to \bot$$

$$0 \neq 1, p = \text{transport} code p \ \text{tt}$$

where

$$\text{code} : 2 \to \bar{U}$$

$$\text{code} 0_2 = T$$

$$\text{code} 1_2 = \bot$$

Using $0 \neq 1$ and function extensionality (derivable from univalence) we can prove that there are exactly two different equivalences between $2$ and $2$. Furthermore, for any equivalence $f$, using the fact that $\text{is-equiv}\ f$ is a proposition, we can show that there are exactly two inhabitants of $2 \to 2$:

$$\text{id} \sim \text{not} : 2 \cong 2$$

$$\text{id} \sim = \text{id}, \text{qinv-is-hae} (\text{id}, \text{refl}, \text{refl})$$

$$\text{not} \sim = \text{not}, \text{qinv-is-hae} (\text{not}, (\lambda (0_2 \to \text{refl}\ 0_2 ; 1_2 \to \text{refl}\ 1_2)))$$

where

$$\text{not} : 2 \to 2$$

$$\text{not}\ 0_2 = 1_2$$

$$\text{not}\ 1_2 = 0_2$$

Here something very special happens: although in general the type formed by taking $n$ disjoint unions of $1$ has a space of automorphisms of size $n!$, in our case we have that $2$ and $1 \cong 2$ are of the same size. This combinatorial accident can actually be lifted to show that there is an equivalence between $2 \cong 2$ and $2$. By composing the chain of equivalences $\Omega (\bar{U}, 2_0) \simeq \Omega (B\text{Aut}(2), b_0) \simeq (2 \cong 2) \cong 2$ we obtain:

$$2 \cong 2_0 : 2 \cong (2_0 \equiv 2_0)$$

Thus there are only two distinct 1-loops in $U[2]$. Calling them $\text{id}_2$ and $\text{not}_2$, we obtain a decomposition:

$$\text{all-1-loops} : (p : 2_0 \equiv 2_0) \to (p \equiv \text{id}_2) + (p \equiv \text{not}_2)$$

that every loop in $U[2]$ is identifiable with either the identity or boolean negation.

For 2-loops in $U[2]$, the following analysis shows that they are identifiable with the trivial path. First, by applying the induction principle for disjoint unions, and path induction, we can prove $2$ is a set:

$$2 \equiv 2_0 : 2 \equiv (2_0 \equiv 2_0)$$

Thus there are only two distinct 1-loops in $U[2]$. Calling them $\text{id}_2$ and $\text{not}_2$, we obtain a decomposition:

$$\text{all-1-loops} : (p : 2_0 \equiv 2_0) \to (p \equiv \text{id}_2) + (p \equiv \text{not}_2)$$

that every loop in $U[2]$ is identifiable with either the identity or boolean negation.

For 2-loops in $U[2]$, the following analysis shows that they are identifiable with the trivial path. First, by applying the induction principle for disjoint unions, and path induction, we can prove $2$ is a set:
From this, we obtain that \( 2_0 \equiv 2_0 \) is also a set by using \( \text{ua} \) and \( \text{transport} \). This in turns shows the contractibility of 2-loops:

\[
\Omega 2_0\text{-is-set} : \text{is-set} \,(2_0 \equiv 2_0)
\]
\[
\Omega 2_0\text{-is-set} = \text{transport is-set} \,(\text{ua} \cong \Omega 2_0) \,2\text{-is-set}
\]

all-2-loops : \( \{ p : 2_0 \equiv 2_0 \} \rightarrow (\gamma : p \equiv p) \rightarrow \gamma \equiv \text{refl} \, p \)
all-2-loops \( \{ p \} \,\gamma = \Omega 2_0\text{-is-set} \, p \, p \, \gamma \,(\text{refl} \, p) \)

In the next section, we will use all-1-loops and all-2-loops as crucial ingredients for showing the correspondence between \( \text{U}[2] \) and \( \Pi_2 \).

Note that most of the results in this section are generic. However when we move beyond 2, the combinatorial explosion of the path space is such that explicit enumeration quickly becomes impractical, and other techniques will become necessary.

5 Correspondence between \( \text{U}[2] \) and \( \Pi_2 \)

Formalizing, in a precise sense, the connection between reversible functions in a programming language and paths in a univalent universe, as intuitive as it may seem, is rather subtle. Paths in HoTT come equipped with principles like the “contractibility of singletons”, “transport”, and “path induction” and none of these principles seem to have any direct counterpart in the world of reversible programming. We will however demonstrate how the semantics of an entire (but admittedly small) reversible programming language such as \( \Pi_2 \) can be captured by a specification as compact as \( \Sigma \,[ \, X : \text{U} \, || \, X \equiv 2 \, || \, \] . Our precise correspondence will consist of building mappings between \( \Pi_2 \) and \( \text{U}[2] \), for points, 1-paths, 2-paths, and 3-paths, such that each map is invertible up to the appropriate notion of equality. This gives a notion of soundness and completeness for each level.

5.1 Mappings

The mappings for points (level-0) are straightforward, as both \( \Pi_2 \) and \( \text{U}[2] \) are singletons:

\[
\begin{align*}
\llbracket - \rrbracket_0 : \Pi_2 & \rightarrow \text{U}[2] \\
\llbracket \cdot \rrbracket_0 & = 2_0 \\
\llbracket \cdot \rrbracket_0 & : \text{U}[2] \rightarrow \Pi_2 \\
\llbracket \cdot \rrbracket_0 & = 2_0
\end{align*}
\]

Level-1 is the first non-trivial level. To each syntactic combinator \( c : A \leftrightarrow_1 B \), we associate a path from \( \llbracket A \rrbracket_0 \) to \( \llbracket B \rrbracket_0 \) and vice-versa. The mapping from the univalent universe back to the syntax of the reversible language is only possible because we have a complete characterization of the paths in the universe (captured in the construction of all-1-loops in the previous section):

\[
\begin{align*}
\llbracket - \rrbracket_1 : \{ A \, B : \Pi_2 \} & \rightarrow A \leftrightarrow_1 B \rightarrow \llbracket A \rrbracket_0 = \llbracket B \rrbracket_0 \\
\llbracket \text{id} \rrbracket_1 & = \text{id} 2_0
\end{align*}
\]
At level-2, we know by the construction of all-2-loops in the previous section that all self-paths in the univalent universe are trivial. Nevertheless the mappings back and forth require quite a bit of (tedious) work. We show below a few cases of the mapping from 2-combinators to 2-paths and the full definition of the reverse mapping. In the first direction, it is a matter of using the necessary properties of paths in the univalent universe (e.g., each path has an inverse). These properties are proved by path induction. The reverse direction crucially relies again on the characterization of 1-loops and the fact that the identity equivalence and the equivalence that swaps the two booleans are distinct:

\[
\begin{align*}
\text{\texttt{\textcolor{red}{\textquotesingle not}}}_1 & = \text{\texttt{\textcolor{red}{\textquotesingle not}2}} \\
\text{\texttt{t}}_1 \cdot \text{\texttt{p}}_1 & = ! \ [ \text{\texttt{p}} ]_1 \\
\text{\texttt{p}}_1 \odot \text{\texttt{q}}_1 & = [ \text{\texttt{p}} ]_1 \ast [ \text{\texttt{q}} ]_1
\end{align*}
\]

\[
\text{\texttt{\textcolor{red}{\textquotesingle not}}}_1 : 2_0 \equiv 2_0 \rightarrow \text{\texttt{\textcolor{red}{\textquotesingle not}}}_1 : 2_0 \equiv 2_0
\]

\[\text{\textcolor{red}{\textquotesingle not}}_1 \text{ with all-1-loops } p\]

\[\text{\texttt{inl psd}} = \text{\texttt{id}}\]

\[\text{\texttt{inr pnot}} = \text{\texttt{\textcolor{red}{\textquotesingle not}}}
\]

For the final level-3, mapping from the univalent universe to \(\Pi_2\) is trivial as the latter has only one constructor at level-3. The other direction requires some involved reasoning in the univalent universe to construct the required 3-path:

\[
\text{\texttt{\textcolor{red}{\textquotesingle not}}}_3 : \{ p ^ q : 2_0 = 2_0 \} \rightarrow \text{\texttt{\textcolor{red}{\textquotesingle not}}}_3 : \{ q ^ p : 2_0 = 2_0 \}
\]

\[
\text{\texttt{\textcolor{red}{\textquotesingle not}}}_3 : \{ p ^ q : 2_0 = 2_0 \} \rightarrow \text{\texttt{\textcolor{red}{\textquotesingle not}}}_3 : \{ q ^ p : 2_0 = 2_0 \}
\]

\[\text{\texttt{\textcolor{red}{\textquotesingle not}}}_3 = \text{\texttt{\textcolor{red}{\textquotesingle not}3}}
\]
5.2 Coherence

It now remains to show that all these mapping are coherent with each other in the sense that each round trip produces a term that is identifiable with the original term, effectively showing soundness and completeness of the univalent universe with respect to \( \Pi_2 \). At level-0, this is trivial.

At level-1, soundness means that the mappings are inverses:

- any 1-combinator \( p \) mapped to a 1-path and back is 2-equivalent to itself, and
- there is always a 2-path between a 1-path \( p \) sent to a 1-combinator and back.

This is rather more succinct in code:

\[
\begin{align*}
&\text{completeness}_{1:1} : \{ p : \langle \tau \xrightarrow{1} \langle \tau \rangle \rangle \rightarrow \langle \tau \rangle_{1} \rightarrow p \in \Pi_{1} \text{ with canonical } p \mid \text{all-1-loops } [p]_{1} \\
&\text{ID}, \text{ p=id } \Rightarrow p = \text{ id}
\end{align*}
\]

They are also complete in the following sense:

- for any two 1-combinators which map to 1-paths which are related by a 2-path, the 1-combinators are related by a 2-combinator, and
- for any two 1-paths which map to 1-combinators which are related by a 2-combinator these are related by a 2-path.

Normally, completeness is a rather difficult result to prove. But in our case, the infrastructure from the previous section makes the proof immediate: For the first proof, the key is reversibility of the level-2 combinators using \( \lambda_2 \); for the second proof it is the reversibility of paths in the univalent universe that is critical:

\[
\begin{align*}
\text{completeness}_{1:1} & : \{ p : \langle \tau \xrightarrow{1} \langle \tau \rangle \rangle \rightarrow \langle \tau \rangle_{1} \rightarrow p \in \Pi_{1} \text{ with canonical } p \mid \text{all-1-loops } [p]_{1} \\
&\text{completeness}_{1:1}^{-1} : \{ q : \langle \tau \xrightarrow{1} \langle \tau \rangle \rangle \rightarrow \langle \tau \rangle_{1} \rightarrow q \in \Pi_{1} \text{ with canonical } q \mid \text{all-1-loops } [q]_{1} \\
&\text{completeness}_{1:1}^{-1}^{-1} : \{ p : \langle \tau \xrightarrow{1} \langle \tau \rangle \rangle \rightarrow \langle \tau \rangle_{1} \rightarrow p \in \Pi_{1} \text{ with canonical } p \mid \text{all-1-loops } [p]_{1} \\
&\text{completeness}_{1:1}^{-1}^{-1}^{-1} : \{ q : \langle \tau \xrightarrow{1} \langle \tau \rangle \rangle \rightarrow \langle \tau \rangle_{1} \rightarrow q \in \Pi_{1} \text{ with canonical } q \mid \text{all-1-loops } [q]_{1} \\
\end{align*}
\]

For level-2, the statements are informally quite similar (with all levels bumped up by one). For 2-combinators, the result is trivial. For the other direction starting from 2-paths in the univalent universe soundness is tricky to even state, mostly because the types involved in \( \langle \tau \rangle_{2} \rightarrow \langle \tau \rangle_{1} \) and \( \langle \tau \rangle_{1} \rightarrow \langle \tau \rangle_{2} \) are non-trivial. But enumeration of 1-loops reduces the complexity of the problem to “unwinding” complex expressions for identity paths:

\[
\begin{align*}
&\langle \tau \rangle_{2} : \{ p : \langle \tau \xrightarrow{1} \langle \tau \rangle \rangle \rightarrow \langle \tau \rangle_{1} \rightarrow p \in \Pi_{1} \text{ with canonical } p \mid \text{all-1-loops } [p]_{1} \\
&\langle \tau \rangle_{1} \rightarrow \langle \tau \rangle_{2} \quad \text{trunc}
\end{align*}
\]
Level-2 completeness offers no new difficulties:

\[
\begin{align*}
\text{completeness} & : \{ \sigma : \Pi \rightarrow \Omega \rightarrow \Omega \} \\
\text{completeness} & : \tau \leftrightsquigarrow \text{trunc}
\end{align*}
\]

\[
\begin{align*}
\text{completeness}^{-1} & : \{ \sigma : \Pi \rightarrow \Omega \rightarrow \Omega \} \\
\text{completeness}^{-1} & : \alpha \leftrightsquigarrow \text{trunc}
\end{align*}
\]

\[
\begin{align*}
\text{completeness}^{-1} & : \{ \sigma : \Pi \rightarrow \Omega \rightarrow \Omega \} \\
\text{completeness}^{-1} & : \alpha \leftrightsquigarrow \text{trunc}
\end{align*}
\]

6 Discussion and Related Work

Reversible Languages.

The practice of programming languages is replete with ad hoc instances of reversible computations: database transactions, mechanisms for data provenance, checkpoints, stack and exception traces, logs, backups, rollback recoveries, version control systems, reverse engineering, software transactional memories, continuations, backtracking search, and multiple-level “undo” features in commercial applications. In the early nineties, Baker [3,4] argued for a systematic, first-class, treatment of reversibility. But intensive research in full-fledged reversible models of computations and reversible programming languages was only sparked by the discovery of deep connections between physics and computation [23,26,29,5,13], and by the potential for efficient quantum computation [12].

The early developments of reversible programming languages started with a conventional programming language, e.g., an extended \(\lambda\)-calculus, and either

(i) extended the language with a history mechanism [30,22,17,10], or
(ii) imposed constraints on the control flow constructs to make them reversible [33].

More modern approaches recognize that reversible programming languages require a fresh approach and should be designed from first principles without the detour via conventional irreversible languages [32,25,2,11].

The \(\Pi\) Family of Languages

In previous work, Carette, Bowman, James, and Sabry [6,18,8] introduced the \(\Pi\) family of reversible languages based on type isomorphisms and commutative semiring identities. The fragment without recursive types is universal for reversible boolean circuits [18] and the extension with recursive types and trace operators [15] is a Turing-complete reversible language [18,6]. While at first sight, \(\Pi\) might appear ad hoc, it really arises naturally from an “extended” view of the Curry-Howard correspondence [8]: rather than looking at mere inhabitation as the main source of analogy between logic and computation, type equivalences becomes the source of analogy. This allows one to see an analogy between algebra and reversible computation. Furthermore, this works at multiple levels: that of 1-algebra (types form a semiring under isomorphism), but also 2-algebra (types and equivalences
form a weak Rig Groupoid). In other words, by taking “weak Rig Groupoid” as the starting semantics, one naturally gets $\Pi$ as the syntax for the language of proofs of isomorphisms – in the same way that many terms of the $\lambda$-calculus arise from Cartesian Closed Categories.

One can also flip this around, and use the $\lambda$-calculus as the internal language for Cartesian Closed Categories. However, as Shulman explains well in his draft book on approaching Categorical Logic via Type Theory [27], this works for many other kinds of categories. As we are interested in reversibility, it is most natural to look at Groupoids. Thus $\Pi_2$ represents the simplest non-trivial case of a (reversible) programming language distilled from such ideas.

What is more surprising is how this also turns out to be a sound and complete language for describing the univalent universe $\mathcal{U}[2]$.

The infinite real projective space $\mathbb{RP}^X$

Buchholtz and Rijke [7] use the “type of two element sets,” $\Sigma X : \mathcal{U} \mid X \equiv S^0 \parallel$, where $S^0$ is the 0-sphere, or the 0-iterated suspension of 2, that is, 2 itself. They construct the infinite real projective space $\mathbb{RP}^X$ by using universal covering spaces, and show that it is homotopy equivalent to the Eilenberg-Maclane space $K(\mathbb{Z}/2\mathbb{Z}, 1)$ which classifies all the 0-sphere bundles. Our reversible programming language is exactly the syntactic presentation of this classifying space. If we choose $S^1$ instead of $S^0$, we get the infinite complex projective space $\mathbb{CP}^X$, but it remains to investigate what kind of reversible programming language this would lead to.

If we consider the $\Pi$ language over all finite types, we conjecture that we should get a representation of $\prod_{n \in \mathbb{N}} K(S_n, 1)$ where $S_n$ is the symmetric group. The idea is that the $n^{th}$ homotopy group of an Eilenberg-Maclane space $K(G, n)$ is isomorphic to $G$ (and every other homotopy group is trivial). Thus, all necessary information about paths and equivalences between finite types is captured in this model.

Acknowledgement

We would like to thank Robert Rose for developing the model based on univalent fibrations, for extensive contributions to the code, and for numerous discussions.

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