Jacobi weights, fractional integration, and sharp Ulyanov inequalities

Polina Glazyrina and Sergey Tikhonov

Abstract. We consider functions $L^p$-integrable with Jacobi weights on $[-1, 1]$ and prove Hardy–Littlewood type inequalities for fractional integrals. As applications, we obtain the sharp $(L_p, L_q)$ Ulyanov-type inequalities for the Ditzian–Totik moduli of smoothness and the $K$-functionals of fractional order.

1. Introduction

The following $(L_p, L_q)$ inequalities of Ulyanov-type between moduli of smoothness of functions on $\mathbb{T}$ play an important role in approximation theory and functional analysis (see, e.g., [7, 13, 15]):

$$\omega^r(f, t)_q \leq C \left( \int_0^t (u^{-\sigma} \omega^r(f, u)_p)^{q_1} \frac{du}{u} \right)^{1/q_1},$$

(1.1)

where $r \in \mathbb{N}$, $0 < p \leq q \leq \infty$, $\sigma = \frac{1}{p} - \frac{1}{q}$, and $q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}$. Here the $r$-th moduli of smoothness of a function $f \in L_p(\mathbb{T})$ is given by

$$\omega^r(f, \delta)_p = \sup_{|h| \leq \delta} \|\Delta_h^r f(x)\|_{L^p(\mathbb{T})}, \quad 1 \leq p \leq \infty,$$

where

$$\Delta_h^r f(x) = \Delta_h^{r-1} (\Delta_h f(x)) \quad \text{and} \quad \Delta_h f(x) = f(x + h) - f(x).$$

Recently ([20, 23]) the sharp version of (1.1) was proved in the case $1 < p < q < \infty$:

$$\omega^r(f, t)_q \leq C \left( \int_0^t (u^{-\sigma} \omega^{r+\sigma}(f, u)_p)^{q_1} \frac{du}{u} \right)^{1/q},$$

(1.2)

where $\omega^r(f, u)_p$ is the moduli of smoothness of the (fractional) order $r > 0$. Moreover, it turned out that (1.2) also holds if $(p, q) = (1, \infty)$; see [21]. In this case $\sigma = 1$ and one can work with the classical (not necessary fractional) moduli of smoothness. On the other hand, (1.2) is not true ([21]) for $1 = p < q < \infty$ or $1 < p < q = \infty$.

In the present paper, we consider a nonperiodic case, namely $L_p$ spaces with Jacobi weights on an interval, and obtain inequalities similar to (1.2) for the fractional $K$-functionals and Ditzian–Totik moduli of smoothness. We start with notation.

Key words and phrases. Jacobi weights, Landau type inequalities, Hardy–Littlewood type inequalities, $K$-functionals, Ditzian–Totik moduli of smoothness, sharp Ulyanov inequality.

The first author was supported by the Ural Federal University development program with the financial support of young scientists. The second author was partially supported by MTM 2011-27637, 2009 SGR 1303, RFFI 13-01-00043, and NSH-979.2012.1.
Denote by \( w^{(a,b)}(x) = (1 - x)^a(1 + x)^b \), \( a, b > -1 \), the Jacobi weight on \([-1, 1]\). For \( 1 \leq p < \infty \), let \( L_p^{(a,b)} \) be the space of all functions \( f \) measurable on \([-1, 1]\) with the finite norm
\[
\|f\|_{p,(a,b)} = \left( \int_{-1}^{1} |f(x)|^p w^{(a,b)}(x) dx \right)^{1/p}.
\]
If \( a = b = 0 \), we write \( L_p = L_p^{(a,b)} \), \( \|\cdot\|_p = \|\cdot\|_{p,(0,0)} \). In the case \( p = \infty \), we set \( L_p^{(a,b)} := C[-1,1] \) and
\[
\|f\|_{\infty,(a,b)} = \|f\|_{\infty} = \max_{x \in [-1,1]} |f(x)|.
\]
For an arbitrary interval \([x_1, x_2] \), we set
\[
\|f\|_{L_p[x_1,x_2]} = \left( \int_{x_1}^{x_2} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty, \quad \|f\|_{L_\infty[x_1,x_2]} = \max_{x \in [x_1,x_2]} |f(x)|.
\]

For \( \alpha, \beta > -1 \), denote by \( \psi_k^{(\alpha,\beta)}(x) \), \( k = 0, 1, \ldots \), the system of Jacobi polynomials orthogonal on \([-1, 1]\) with the weight \( w^{(a,b)} \) and normalized by the condition
\[
\int_{-1}^{1} \left| \psi_k^{(\alpha,\beta)}(x) \right|^2 w^{(\alpha,\beta)}(x) dx = 1.
\]
The Jacobi polynomials are the eigenfunctions of the differential operator
\[
\mathcal{D} = \mathcal{D}_2^{(\alpha,\beta)} = \frac{-1}{w^{(\alpha,\beta)}(x)} \frac{d}{dx} w^{(\alpha,\beta)}(x)(1 - x^2) \frac{d}{dx},
\]
\[
\mathcal{D}\psi_k^{(\alpha,\beta)} = \left( \lambda_k^{(\alpha,\beta)} \right)^2 \psi_k^{(\alpha,\beta)}, \quad \lambda_k^{(\alpha,\beta)} = (k(k + \alpha + \beta + 1))^{1/2}.
\]
For a function \( f \in L_p^{(\alpha,\beta)} \), \( 1 \leq p \leq \infty \), the Fourier–Jacobi expansion is defined as follows:
\[
f(x) \sim \sum_{k=0}^{\infty} \widehat{f}_k^{(\alpha,\beta)} \psi_k^{(\alpha,\beta)}(x),
\]
where
\[
\widehat{f}_k^{(\alpha,\beta)} = \int_{-1}^{1} f(x) \psi_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx, \quad k = 0, 1, 2, \ldots
\]
Let \( \sigma > 0 \). If there exists a function \( g \in L_1^{(\alpha,\beta)} \) such that its Fourier–Jacobi expansion has the form
\[
g \sim \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha,\beta)} \right)^\sigma \widehat{g}_k^{(\alpha,\beta)} \psi_k^{(\alpha,\beta)}
\]
then we use the notation
\[
g = \mathcal{D}_\sigma^{(\alpha,\beta)} f
\]
and we call \( \mathcal{D}_\sigma^{(\alpha,\beta)} f \) the fractional derivative of order \( \sigma \) of the function \( f \). If there exists a function \( h \in L_1^{(\alpha,\beta)} \) such that its Fourier–Jacobi expansion has the form
\[
h \sim \widehat{h}_0^{(\alpha,\beta)} + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha,\beta)} \right)^{-\sigma} \widehat{h}_k^{(\alpha,\beta)} \psi_k^{(\alpha,\beta)},
\]
then we use the notation
\[
h = \mathcal{T}_\sigma^{(\alpha,\beta)} f
\]
and we call \( \mathcal{T}_\sigma^{(\alpha,\beta)} f \) the fractional integral of order \( \sigma \) of the function \( f \). Notice that \( \mathcal{T}_\sigma^{(\alpha,\beta)} \), \( \sigma > 0 \), is a bounded linear operator on \( L_1^{(\alpha,\beta)} \) (see, e.g., [3] Sec. 5, pp. 789–790).
The $K$-functional corresponding to the differential operator $D^{(\alpha, \beta)}$ and a real positive number $r$ is defined by

$$K^r(f, D^{(\alpha, \beta)}_r, t)_{p, (\alpha, \beta)} = \inf \left\{ \|f - g\|_{p, (\alpha, \beta)} + t^r \|D^{(\alpha, \beta)}_r g\|_{p, (\alpha, \beta)} : g \in W_{p, (\alpha, \beta)}^r \right\}$$  \hspace{1cm} (1.4)

(see [10] (1.9)), where $W_{p, (\alpha, \beta)}^r = \left\{ g : D^{(\alpha, \beta)}_r g \in L_p^{(\alpha, \beta)} \right\}$.

The main result of this paper is the following

**Theorem 1.** Let $1 < p < q < \infty$, $r > 0$, $\alpha \geq \beta > -1$, $\alpha \geq -1/2$. Suppose also that

$$\sigma = (2\alpha + 2) \left( \frac{1}{p} - \frac{1}{q} \right).$$

If $f \in L_p^{(\alpha, \beta)}$ and

$$\int_0^1 \left( u^{-\sigma} K^{r+\sigma} (f, D^{(\alpha, \beta)}_{r+\sigma}, u)_{p, (\alpha, \beta)} \right)^q \frac{du}{u} < \infty,$$

then $f \in L_q^{(\alpha, \beta)}$ and

$$K^r(f, D^{(\alpha, \beta)}_r, t)_{q, (\alpha, \beta)} \leq C \left( \int_0^1 \left( u^{-\sigma} K^{r+\sigma} (f, D^{(\alpha, \beta)}_{r+\sigma}, u)_{p, (\alpha, \beta)} \right)^q \frac{du}{u} \right)^{1/q}.$$

The rest of the paper is organized as follows. In Section 2 we obtain the key result to get sharp Ulyanov inequalities – the weighted inequalities of Hardy–Littlewood and Landau type for functions defined on the interval $[-1, 1]$. Section 3 contains the definition of fractional $K$-functionals with Jacobi weights and sharp Ulyanov inequalities for $K$-functionals (Theorem 3). In Section 4 analogous results for the Ditzian–Totik moduli of smoothness are obtained. Namely, we study a relationship between these moduli and the corresponding $K$-functionals and prove sharp Ulyanov inequalities for the Ditzian–Totik moduli in the case of $1 \leq p \leq q \leq \infty$ (Theorem 5). Theorem 1.

2. Inequalities for fractional integrals with Jacobi weights

2.1. Landau-type inequalities. We will need the following Hardy-type inequality (see, e.g., [5] and [19] Theorem 6.2, Example 6.8). We set $\frac{1}{q} := 0$ for $q = \infty$.

**Theorem A.** Let $1 \leq p \leq q \leq \infty$, $(p, q) \neq (\infty, \infty)$, $a > -\frac{1}{q}$, $\overline{\alpha} \in (0, \infty)$. Then the inequality

$$\|f(x)x^a\|_{L_q[0, \overline{\alpha}]} \leq C(p, q, a, \overline{\alpha}) \left\| f'(x)x^{a+\overline{\alpha}} \right\|_{L_p[0, \overline{\alpha}]}$$

holds for any locally absolutely continuous function $f$ on $(0, \overline{\alpha}]$ with the property $f(\overline{\alpha}) = 0$ if and only if $h \leq 1 - \left( \frac{1}{p} - \frac{1}{q} \right)$.

Let us mention that the quantity $C(p, q, a, \overline{\alpha})$ is nondecreasing with respect to $\overline{\alpha}$.

The following Landau–type inequality can be found in, e.g., [6] Ch. 2, Th. 5.6, p. 38.

**Theorem B.** For $1 \leq p \leq \infty$, $\ell \geq 2$, there is a constant $C(\ell)$ such that for all $r = 0, \ldots, \ell$ and any function $f$ with $f^{(\ell-1)}$ absolutely continuous on $[-\frac{1}{2}, \frac{1}{2}]$ and $f^{(\ell)} \in L_p \left[ -\frac{1}{2}, \frac{1}{2} \right]$ we have

$$\|f^{(r)}\|_{L_p\left[ -\frac{1}{2}, \frac{1}{2} \right]} \leq C(\ell) \left( \|f\|_{L_p\left[ -\frac{1}{2}, \frac{1}{2} \right]} + \|f^{(\ell)}\|_{L_p\left[ -\frac{1}{2}, \frac{1}{2} \right]} \right).$$

As a corollary of Theorem A and Theorem B we get
Lemma 1. Suppose that \( 1 \leq p \leq q \leq \infty, \ (p,q) \neq (\infty,\infty), \ a,b > -\frac{1}{q}, \ c,d > -\frac{1}{p}, \) \( r \) is a nonnegative integer, \( k \) is a positive integer, and
\[
h = k - \left( \frac{1}{p} - \frac{1}{q} \right).
\]
Then, there exists a constant \( C = C(p,q,a,b,c,d,r,k) \) such that for any function \( f \) with \( f^{(r+k-1)} \) absolutely continuous on \((-1,1)\) and \( f^{(r+k)} w^{(a+b+h)} \in L_p \) we have
\[
\left\| f^{(r)} w^{(a,b)} \right\|_q \leq C \left( \left\| f^{(c,d)} \right\|_p + \left\| f^{(r+k)} w^{(a+b+h)} \right\|_p \right).
\] (2.1)
Inequality (2.1) is sharp in the following sense. If \( a - c < r + \left( \frac{1}{p} - \frac{1}{q} \right) \), then for any \( \varepsilon > 0 \) there exists \( \{ f_n \} \subset C^{k+r}[-1,1] \) such that
\[
\left\| f_n^{(r)} w^{(a,b)} \right\|_q \cdot \left( \left\| f_n^{(c,d)} \right\|_1 + \left\| f_n^{(r+k)} w^{(a+b+\varepsilon,b+h)} \right\|_p \right)^{-1} \to \infty \quad \text{as} \quad n \to \infty.
\] (2.2)
The analogous statement also holds with respect to the parameter \( b \).

Proof of Lemma 1. It is enough to verify inequality (2.1) for \( k = 1 \). The proof in the general case is by induction on \( k \). Note that \( f^{(r)} \) is continuous on \([-\frac{1}{2}, \frac{1}{2}]\) by our assumption. We take \( \varpi \in [-\frac{1}{2}, \frac{1}{2}] \) such that
\[
\left| f^{(r)}(\varpi) \right| = \min \left\{ \left| f^{(r)}(x) \right| : x \in [-\frac{1}{2}, \frac{1}{2}] \right\}.
\]
Let \( g(x) = f^{(r)}(x) - f^{(r)}(\varpi) \), then
\[
\left\| f^{(r)} w^{(a,b)} \right\|_q \leq \left\| gw^{(a,b)} \right\|_q + \left\| f^{(r)}(\varpi) \right\| \left\| w^{(a,b)} \right\|_q
\leq \left\| gw^{(a,b)} \right\|_{L_\varpi[-1,\varpi]} + \left\| gw^{(a,b)} \right\|_{L_\varpi[\varpi,1]} + \left\| f^{(r)}(\varpi) \right\| \left\| w^{(a,b)} \right\|_{L_\varpi[-1,1]}.
\]
To estimate the first term, we apply Theorem A (for the interval \([-1,\varpi]\) instead of \([0,\varpi]\)) with \( h = 1 - \left( \frac{1}{p} - \frac{1}{q} \right) \):
\[
\left\| gw^{(a,b)} \right\|_{L_\varpi[-1,\varpi]} \leq 2^{|a|} \left\| g(x)(1 + x)^h \right\|_{L_\varpi[-1,\varpi]} \leq 2^{|a|} C \left\| g'(x)(1 + x)^h \right\|_{L_\varpi[-1,\varpi]}
\leq 2^{|a|+|a+h|} C \left\| g'(x)(1-x)^h (1 + x)^h \right\|_{L_\varpi[-1,\varpi]}
\leq 2^{|a|+|a+h|} C \left\| g'w^{(a+b+h)} \right\|_{L_\varpi[-1,1]} = 2^{|a|+|a+h|} C \left\| f^{(r+1)} w^{(a+b+h)} \right\|_{L_\varpi[-1,1]}.
\]
A similar estimate holds for \( \left\| gw^{(a,b)} \right\|_{L_\varpi[\varpi,1]} \) as well.

To estimate \( \left\| f^{(r)}(\varpi) \right\| \), we apply Theorem B
\[
\left\| f^{(r)}(\varpi) \right\| \leq \left\| f^{(r)} \right\|_{L_\varpi[-\frac{1}{2},\varpi]} \leq C \left( \left\| f \right\|_{L_\varpi[-\frac{1}{2},\varpi]} + \left\| f^{(r+1)} \right\|_{L_\varpi[-\frac{1}{2},\varpi]} \right)
\leq 2^{|c|+|d|+|a+h|+|b+h|} C \left( \left\| f^{(c,d)} \right\|_{L_\varpi[-1,1]} + \left\| f^{(r+1)} w^{(a+b+h)} \right\|_{L_\varpi[-1,1]} \right),
\]
where \( C \) depends only on \( r + 1 \). Thus, (2.1) follows.

Let us now show (2.2). Since for any \( 0 \leq \varepsilon \leq \varepsilon_2 \) the estimate
\[
w^{(a+b+\varepsilon_2,b+h)}(x) \leq 2^{\varepsilon_2} w^{(a+b+\varepsilon_1,b+h)}(x), \quad x \in [-1,1],
\]
holds, we can assume
\[
0 < \varepsilon \leq c - a + r + 1/p - 1/q.
\] (2.3)
For \( m > r + k \), consider the sequence of functions
\[
f_n(x) = \left( (x + 1/n - 1)_+ \right)^m, \quad x \in [-1, 1], \quad y_+ = \max\{y, 0\}.
\]
It is easy to verify that if \( \mu \geq 0 \) and \( \nu > -1/q \), then
\[
\left\| \left( (1/n - 1 + x)_+ \right)^\mu (1 - x)^\nu \right\|_q \lesssim \frac{1}{n^{\mu + \nu + 1/q}} \quad \text{as} \quad n \to \infty.
\]
Here \( A_n \asymp B_n \) as \( n \to \infty \) means that \( B_n/C \leq A_n \leq CB_n \) for some positive constant \( C \) and all \( n \). Using this, we get
\[
\left\| f_n w^{(c,d)} \right\|_p \lesssim \frac{1}{n^{m + c + 1/p}}, \quad \left\| f_n^{(r)} w^{(a,b)} \right\|_q \lesssim \frac{1}{n^{m - r + a + 1/q}},
\]
\[
\left\| f_n^{(r+k)} w^{(a+h+\varepsilon,b+h)} \right\|_p \lesssim \frac{1}{n^{m - r - k + a + h + \varepsilon + 1/p}} = \frac{1}{n^{m - r + a + \varepsilon + 1/q}}.
\]
Under assumption (2.3) we have
\[
\left\| f_n w^{(c,d)} \right\|_p + \left\| f_n^{(r+k)} w^{(a+h+\varepsilon,b+h)} \right\|_p \lesssim \frac{1}{n^{m - r + a + \varepsilon + 1/q}},
\]
and therefore,
\[
\frac{\left\| f_n^{(r)} w^{(a,b)} \right\|_q}{\left\| f_n w^{(c,d)} \right\|_p + \left\| f_n^{(r+k)} w^{(a+h+\varepsilon,b+h)} \right\|_p} \lesssim n^\varepsilon \quad \text{as} \quad n \to \infty,
\]
concluding the proof. \( \square \)

2.2. Hardy–Littlewood type inequalities. To prove Hardy–Littlewood type inequalities for the fractional integral \( I_{\sigma}^{(\alpha,\beta)} \), we will use the Muckenhoupt transplantation theorem \([18]\) Collorary 17.11, which is written in our notation as follows.

**Theorem C.** If \( 1 < p \leq q < \infty, \quad \alpha, \beta, \gamma, \delta > -1, \quad \alpha, \beta, \gamma, \delta > -1, \)
\[
s = \frac{1}{p} - \frac{1}{q},
\]
\[
\frac{\alpha}{\bar{\alpha}} = \frac{\alpha}{p} + \frac{\alpha - \gamma}{2} + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right), \quad \frac{\beta}{\bar{\beta}} = \frac{\beta}{p} + \frac{\beta - \delta}{2} + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right),
\]
the quantities \( \bar{\alpha} = (\alpha + 1)/p - \gamma \) and \( \bar{\beta} = (\beta + 1)/p - \delta \) are not positive integers, \( M = \max\{0, [\bar{\alpha}]\}, \quad N = \max\{0, [\bar{\beta}]\}, \quad f \in L^{(\bar{\alpha},\bar{\beta})}_p, \)
\[
\hat{f}_k^{(\bar{\alpha},\bar{\beta})} = 0, \quad 0 \leq k \leq M + N + 1,
\]
h is an integer, \( \nu_k \) has the form
\[
\nu_k = \sum_{j=0}^{J-1} c_j (k + 1)^{-s-j} + O \left( (k + 1)^{-s-J} \right)
\]
with \( J \geq \alpha + \beta + \gamma + \delta + 6 + 2M + 2N \) and \( 0 \leq \rho < 1 \), then
\[
T_\rho f(x) = \sum_{k=0}^{\infty} \rho^k \nu_k \hat{f}_k^{(\bar{\alpha},\bar{\beta})} \psi_{k+h}^{(\bar{\alpha},\bar{\beta})}(x)
\]
converges for every \( x \in (-1, 1), \)
\[
\|T_\rho f\|^{(\bar{\alpha},\bar{\beta})} \lesssim C \|f\|^{(\bar{\alpha},\bar{\beta})},
\]
where $C$ is independent of $\rho$ and $f$. Moreover, there is a function $Tf$ in $L^p_\alpha$ such that $T_\rho f$ converges to $Tf$ in $L^p_\alpha$ as $\rho \to 1^-$. If it is also assumed that $\alpha > 2\beta > -1$, $\alpha + 1 < (\alpha + 1)p$, $(b + 1) < (\beta + 1)p$, and $\max(0, b + 1) < b$

The next Hardy–Littlewood inequality is a simple corollary of Theorem [C]

**Corollary 1.** Let $1 < p < q < \infty$, $\alpha > b > -1$, $\alpha > \beta > -1$, $(\alpha + 1) < (\alpha + 1)p$, $(b + 1) < (\beta + 1)p$, and $\sigma \geq 1 - \frac{1}{p} - \frac{1}{q}$.

Let also $f \in L^p_\alpha$. Then there exists $C$ independent of $f$ such that

$$
\left\| T^{(\alpha, \beta)}_\sigma f \right\|_{q, (a, b)} \leq C \left\| f \right\|_{p, (a, b)}.
$$

In the special case $(\alpha, \beta) = (a, b)$, the Hardy–Littlewood inequality (2.4) was studied by Askey and Wainger [2 Sec. J] (see also [1]) and later by Bavinck and Trebels [3 Theorem 5.4], [4 Theorems 1 and 1']

**Theorem D ([2, 4]).** Let $1 < p < q < \infty$, $a > b > -1$, $a + b > -1$, and $\sigma \geq (2a + 2) \left( \frac{1}{p} - \frac{1}{q} \right)$.

If $f \in L^p_\alpha$, then $T^{(a, b)}_\sigma f \in L^q_\alpha$ and

$$
\left\| T^{(a, b)}_\sigma f \right\|_{q, (a, b)} \leq C(p, q, a, b) \left\| f \right\|_{p, (a, b)}.
$$

For $(\alpha, \beta) \neq (a, b)$ we have the following result.

**Theorem 2.** Let $1 < p < q < \infty$, $a > b > -1$, $a > -1/2$, $\alpha > \beta > -1$,

$$
p(\alpha - \beta) \leq 2(a - b) \leq q(\alpha - \beta),
$$

the quantities $A = (a + 1)/p - \alpha$ and $B = (b + 1)/p - \beta$ be not positive integers, and either $\alpha = a$, or $\alpha > a$ and $q > 2$, or $\alpha < a$ and $p < 2$. Let

$$
\sigma \geq (2a + 2) \left( \frac{1}{p} - \frac{1}{q} \right),
$$

$f \in L^p_\alpha \cap L^{(a, \beta)}_{1}$ and

$$
\widehat{T}^{(a, \beta)}_\sigma f_k = 0, \quad 0 \leq k \leq \max\{0, [A]\} + \max\{0, [B]\} - 1.
$$

Then there exists $C$ independent of $f$ such that

$$
\left\| T^{(a, \beta)}_\sigma f \right\|_{q, (a, b)} \leq C \left\| f \right\|_{p, (a, b)}.
$$

**Proof.** It is sufficient to prove this theorem for polynomials. Indeed, suppose that (2.8) holds for polynomials. Consider a sequence of polynomials $\{Q_m\}$ convergent to $f$ in $L^p_\alpha$ and $L^{(a, \beta)}_{1}$. Then $\{T^{(a, \beta)}_\sigma Q_m\}$ is a Cauchy sequence in $L^q_\alpha$ and it converges to some function $g$ in $L^q_\alpha$. Without loss of generality we can assume that $\{T^{(a, \beta)}_\sigma Q_m\}$ converges to $g$ a.e. on $[-1, 1]$. Since the operator $T^{(a, \beta)}_\sigma$ is continuous in $L^{(a, \beta)}_{1}$, the sequence $\{T^{(a, \beta)}_\sigma Q_m\}$ converges...
to $I_{\sigma}^{(\alpha, \beta)} f$ in $L_{1}^{(\alpha, \beta)}$. There is a subsequence $\{I_{\sigma}^{(\alpha, \beta)} Q_{m_j}\}$ convergent to $I_{\sigma}^{(\alpha, \beta)} f$ a.e. on $[-1, 1]$. Therefore, $g = I_{\sigma}^{(\alpha, \beta)} f$.

Let $f$ be a polynomial, i.e.,

$$f = \sum_{k=0}^{\infty} c_k \psi_k^{(\alpha, \beta)},$$

where $c_k = \hat{f}_k^{(\alpha, \beta)}$ and $c_k = 0$ for $k > \deg(f)$.

Case 1. Consider $\alpha \geq a, q \geq 2$. More precisely, under assumption of the theorem, the following relations are possible: $\alpha > a$ and $q > 2$ or $\alpha = a$ and $q \geq 2$.

Now, we define $\alpha_1$ and $p_1$. If $\alpha > a$, then we set

$$\alpha_1 = \frac{q\alpha - 2a}{q - 2},$$

and

$$\frac{\alpha_1}{p_1} = \frac{a}{p} + \frac{\alpha_1 - \alpha}{2} + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{p_1} \right).$$

In this case, we have

$$2\alpha_1 + 1 = \frac{2a + 1}{p} + \frac{2(\alpha - a)}{q - 2}$$

and

$$(2\alpha_1 + 2) \left( \frac{1}{p_1} - \frac{1}{q} \right) + \frac{1}{p} - \frac{1}{p_1} = (2a + 2) \left( \frac{1}{p} - \frac{1}{q} \right).$$

Notice that condition $\alpha > a$ implies that $\alpha_1 > \max\{a, \alpha, 0\}$ and $p < p_1 < q$.

If $\alpha = a$, then we set $\alpha_1 = a$, $p_1 = p$.

We divide the rest of the proof in Case 1 into three steps.

Step 1.1. We apply Theorem [C] with $(\overline{\alpha}, \overline{\beta}) = (p_1, p)$, $\sigma_1 = \frac{1}{2} - \frac{1}{p}$, and $\nu_k = \left( \lambda_k^{(\alpha_1, \alpha_1)} \right)^{1 - \sigma_1}$.

Then we have $\overline{\alpha}_1 = \overline{\alpha}_1$, $\sigma_k = \frac{1}{2} - \frac{1}{p}$, and

$$A = \frac{a + 1}{p} - \alpha, \quad B = \frac{b + 1}{p} - \beta.$$

Therefore, under condition (2.17) for any $\rho \in (0, 1)$, we obtain the inequality

$$\left\| c_0 + \sum_{k=1}^{\infty} \rho^k \left( \lambda_k^{(\alpha_1, \alpha_1)} \right)^{-\sigma_1} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{p_1, (\overline{\alpha}_1, \overline{\beta})} \leq C \| f \|_{p_1, (a, b)}.$$  

(2.11)

where $C$ is independent of $f$ and $\rho$. Since $f$ is a polynomial, the sum is finite, and we can rewrite (2.11) as

$$\left\| c_0 + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha_1, \alpha_1)} \right)^{-\sigma_1} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{p_1, (\overline{\alpha}_1, \overline{\beta})} \leq C \| f \|_{p_1, (a, b)}.$$  

(2.12)
Step 1.2. In view of \((2.6)\) and \((2.9)\), we have
\[
\sigma - \sigma_1 \geq (2\alpha_1 + 2) \left( \frac{1}{p_1} - \frac{1}{q} \right),
\]
we can apply Theorem D for the pair of spaces \(L_q^{(\alpha_1, \alpha_1)}\) and \(L_{p_1}^{(\alpha_1, \alpha_1)}\) to get
\[
\left\| c_0 + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha_1, \alpha_1)} \right)^{-\sigma} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{q,(\alpha_1, \alpha_1)} \leq C \left\| c_0 + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha_1, \alpha_1)} \right)^{-\sigma_1} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{p_1,(\alpha_1, \alpha_1)}.
\]

Step 1.3. We use Theorem C once again with \((\bar{q}, \bar{p}) = (q, q)\), \((\bar{\alpha}, \bar{\beta}) = (\alpha, \beta)\), \((\bar{\gamma}, \bar{d}) = (\alpha_1, \alpha_1)\), and
\[
\nu_k = \left( \lambda_k^{(\alpha, \beta)} / \lambda_k^{(\alpha_1, \alpha_1)} \right)^{-\sigma}.
\]
Then \(s = 0, \bar{\sigma} = a\),
\[
\bar{\theta} = \frac{\alpha_1 + 1}{q} - \alpha_1 = \alpha_1 \left( \frac{1}{q_1} - 1 \right) + \frac{1}{q} \leq -\frac{1}{2} \left( \frac{1}{q_1} - 1 \right) + \frac{1}{q} < 1, \quad [A] = [\bar{B}] = 0.
\]
We have
\[
\left\| c_0 + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha, \beta)} \right)^{-\sigma} c_k \psi_k^{(\alpha, \beta)} \right\|_{q,(\alpha_1, \alpha_1)} \leq C \left\| c_0 + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha_1, \alpha_1)} \right)^{-\sigma_1} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{q,(\alpha_1, \alpha_1)}.
\]
Relations \((2.5)\) and \((2.14)\) show that \(\bar{b} \leq b\), and hence,
\[
\left\| c_0 + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha, \beta)} \right)^{-\sigma} c_k \psi_k^{(\alpha, \beta)} \right\|_{q,(\alpha, \bar{b})} \leq 2b^{\bar{b}} \left\| c_0 + \sum_{k=1}^{\infty} \left( \lambda_k^{(\alpha, \beta)} \right)^{-\sigma} c_k \psi_k^{(\alpha, \beta)} \right\|_{q,(\alpha, \bar{b})}.
\]
Finally, combining \((2.12)\), \((2.13)\), and \((2.15)\), we obtain inequality \((2.8)\).

Case 2. Consider \(\alpha \leq a, p < 2\). More precisely, under assumption of the theorem, the following relations are possible: \(\alpha < a\) and \(p < 2\) or \(\alpha = a\) and \(p < 2\).

Now, we define \(\alpha_1\) and \(q_1\). If \(\alpha < a\), then we set
\[
\alpha_1 = \frac{2a - p\alpha}{2 - p},
\]
\[
a = \alpha_1 \frac{q_1}{q_1} + \frac{\alpha - \alpha_1}{2} + \frac{1}{2} \left( \frac{1}{q_1} - \frac{1}{q} \right).
\]
In this case, we have
\[
\frac{2a_1 + 1}{q_1} = \frac{2a + 1}{q} + \frac{2(a - \alpha)}{2 - p}
\]
and
\[
(2a_1 + 2) \left( \frac{1}{p} - \frac{1}{q_1} \right) + \frac{1}{q_1} - \frac{1}{q} = (2a + 2) \left( \frac{1}{p} - \frac{1}{q} \right).
\]
Notice that condition \(\alpha < a\) implies that \(\alpha_1 > \max\{a, \alpha, 0\}\) and \(p < q_1 < q\).

If \(\alpha = a\), then we set \(\alpha_1 = \alpha\), \(q_1 = q\).

We can argue similarly to the proof in Case 1 dividing the rest of the proof into three steps.
Step 2.1. We are going to use Theorem \(\Box\) with \((\overline{\gamma}, \overline{\nu}) = (p, p), (\overline{\alpha}, \overline{\beta}) = (\alpha_1, \alpha_1), (\overline{\gamma}, \overline{\delta}) = (\alpha, \beta), (\overline{c}, \overline{d}) = (a, b), h = 0, s = 0, \) and \(\nu_k = 1.\) Then \(\overline{a} = \alpha_1,\)
\[
\frac{\overline{\gamma}}{p} = b - \frac{\alpha_1 - \beta}{2} = \frac{\alpha_1}{p} - \frac{2(a - b) - p(\alpha - \beta)}{2p},
\]
(2.17)
\[
A = \frac{a + 1}{p} - \alpha, \quad B = \frac{b + 1}{p} - \beta.
\]
Therefore, under condition (2.17) for any \(\rho \in (0, 1),\) we obtain the inequality
\[
\left\| c_0 + \sum_{k=1}^{\infty} \rho^k c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{\rho, (\alpha_1, \overline{\delta})} \leq C \| f \|_{\rho, (a, b)},
\]
(2.18)
where \(C\) does not depend on \(f\) and \(\rho.\) Since \(f\) is a polynomial, the sum is finite. Taking into account (2.5) and (2.17), we conclude that \(\alpha_1 \geq \overline{b},\) and hence, and we can rewrite (2.18) as
\[
\left\| c_0 + \sum_{k=1}^{\infty} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{\rho, (\alpha_1, \alpha_1)} \leq C \| f \|_{\rho, (a, b)}.
\]
(2.19)

Step 2.2. Set \(\sigma_1 = \sigma - \left(\frac{1}{q_1} - \frac{1}{q}\right).\) In view of (2.6) and (2.16), we have
\[
\sigma_1 \geq (2\alpha_1 + 1) \left(\frac{1}{p} - \frac{1}{q_1}\right).
\]
We can apply Theorem \(\Box\) for the pair of spaces \(L_{q_1}^{(\alpha_1, \alpha_1)}\) and \(L_{p}^{(\alpha_1, \alpha_1)}\) to get
\[
\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1, \alpha_1)}\right)^{-\sigma_1} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{q_1, (\alpha_1, \alpha_1)} \leq C \left\| c_0 + \sum_{k=1}^{\infty} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{p, (\alpha_1, \alpha_1)}.
\]
(2.20)

Step 2.3. We use Theorem \(\Box\) once again with \((\overline{q}, \overline{\nu}) = (q, q_1), (\overline{\alpha}, \overline{\beta}) = (\alpha, \beta), (\overline{\gamma}, \overline{\delta}) = (\alpha_1, \alpha_1),\)
\(\overline{\overline{c}, \overline{d}) = (\alpha_1, \alpha_1),\) and
\[
\nu_k = \left(\lambda_k^{(\alpha, \beta)}\right)^{-(\sigma - \sigma_1)} \left(\lambda_k^{(\alpha_1, \alpha_1)}/\lambda_k^{(\alpha, \beta)}\right)^{\sigma_1}.
\]
Hence, \(s = \sigma - \sigma_1 = \frac{1}{q_1} - \frac{1}{q}, \) \(\overline{a} = a,\)
\[
\frac{\overline{b}}{q} = \frac{\alpha_1}{q_1} + \frac{\beta - \alpha_1}{2} + \frac{1}{2} \left(\frac{1}{q_1} - \frac{1}{q}\right) = \frac{b}{q} - \frac{q(\alpha - \beta) - 2(a - b)}{2q},
\]
(2.21)
and
\[
A = B = \frac{\alpha_1 + 1}{q_1} - \alpha_1 = \alpha_1 \left(\frac{1}{q_1} - 1\right) + \frac{1}{q_1} \leq -\frac{1}{2} \left(\frac{1}{q_1} - 1\right) + \frac{1}{q_1} < 1, \quad [A] = [B] = 0.
\]
We have
\[
\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha, \beta)}\right)^{-\sigma} c_k \psi_k^{(\alpha, \beta)} \right\|_{q_1, (\alpha, \overline{b})} \leq C \left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1, \alpha_1)}\right)^{-\sigma_1} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{q_1, (\alpha_1, \alpha_1)}.
\]
Taking into account (2.5) and (2.21), we see that \(\overline{b} \leq b,\) and hence,
\[
\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1, \alpha_1)}\right)^{-\sigma} c_k \psi_k^{(\alpha_1, \alpha_1)} \right\|_{q_1, (a, b)} \leq 2b - \overline{b} \left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha, \beta)}\right)^{-\sigma} c_k \psi_k^{(\alpha, \beta)} \right\|_{q_1, (a, \overline{b})}.
\]
(2.22)
Finally, combining (2.19), (2.20), and (2.22), we obtain inequality (2.8).
3. Ulyanov-type inequalities for $K$-functionals

Definitions and facts, given in this section and in the next one, are based on the books \cite{14,16}; see also \cite{8,10} and the recent survey \cite{11}.

In this section, we assume that $1 \leq p \leq \infty$, $a, b > -1$, $\alpha, \beta > -1$ and

$$\frac{a + 1}{p} - \alpha < 1, \quad \frac{b + 1}{p} - \beta < 1. \quad (3.1)$$

Then, since $L^{(a,b)}_p \subset L^{(a\beta)}_1$, the Fourier–Jacobi expansion \eqref{1.3} is well-defined for any $f \in L^{(a,b)}_p$.

Denote by $\Pi_n$ the set of all algebraic polynomials of degree at most $n$, $\Pi = \cup_{n \geq 0} \Pi_n$.

Let $P_{n,f} = P_n(f)_{p,(a,b)}$, $P_{n,f} \in \Pi_n$, be a near best polynomial approximant of a function $f \in L^{(a,b)}_p$, that is,

$$\|f - P_{n,f}\|_{p,(a,b)} \leq CE_n(f)_{p,(a,b)}, \quad E_n(f)_{p,(a,b)} = \inf \{\|f - P\|_{p,(a,b)} : P \in \Pi_n\}. \quad (3.2)$$

The $K$-functional corresponding to the differential operator $D^{(a,\beta)}$ and a real positive number $r$ is defined by

$$K^r(f, D^{(a,\beta)}_r, t)_{p,(a,b)} = \inf \left\{\|f - g\|_{p,(a,b)} + t^r\|D^{(a,\beta)}_r g\|_{p,(a,b)} : g \in W^{r,(a,\beta)}_{p,(a,b)} \right\} \quad (3.3)$$

(see \cite{10} (1.9)), where $W^{r,(a,\beta)}_{p,(a,b)} = \{g : g, D^{(a,\beta)}_r g \in L^{(a,b)}_p\}$. The following realization result holds:

$$K^r \left( f, D^{(a,\beta)}_r, 1/n \right)_{p,(a,b)} \leq \|f - P_{n,f}\|_{p,(a,b)} + n^{-r}\|D^{(a,\beta)}_r P_{n,f}\|_{p,(a,b)}, \quad 1 < p < \infty. \quad (3.4)$$

It is a corollary of Theorem 6.2 in \cite{10}. To apply this theorem, we have to show that the Cesàro operator $C^\ell_n$ given by

$$C^\ell_n(f) = \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \left(1 - \frac{k}{n+2}\right) \cdots \left(1 - \frac{k}{n+\ell}\right) \hat{f}_k \phi_k^{(a,\beta)}$$

is bounded in $L^{(a,b)}_p$ for some $\ell$. This fact is mentioned in \cite{8, Sec. 3]. Moreover, from \cite{18} Theorem 1.10, p. 4] (see also \cite{8, Theorem M}) it easily follows that the operator $C^\ell_n$ is bounded in $L^{(a,b)}_p$ for any

$$\ell \geq \max \left\{\frac{2(a+1)}{p} - \alpha - 1, \frac{2(b+1)}{p} - \beta - 1, \frac{2(a+1)}{p} - \alpha - \frac{1}{2} - \frac{1}{p}, \frac{2(b+1)}{p} - \beta - \frac{1}{2} - \frac{1}{p}, \frac{2(a - b) - (\alpha - \beta)}{p}\right\}. \quad (3.5)$$

Note that one can equivalently consider the boundedness of the Riesz means, see \cite{22} Theorem 3.19.

Now we formulate and prove the main result – Ulyanov type inequality for $K$-functionals with Jacobi weights. Theorem \cite{3} contains Theorem \cite{11} stated in Introduction, as a particular case.

**Theorem 3.** Let $1 < p < q < \infty$ and $r > 0$. Suppose that $\alpha, \beta > -1$, $a \geq b > -1$, $a \geq -1/2$, inequalities \ref{3.11} hold, and either $(\alpha, \beta) = (a, b)$, or

$$p(\alpha - \beta) \leq 2(a - b) \leq q(\alpha - \beta),$$

and $\alpha = a$, or $\alpha > a$, $q > 2$, or $\alpha < a$, $p < 2$.

Suppose also that

$$\sigma = (2a + 2) \left(\frac{1}{p} - \frac{1}{q}\right).$$
If \( f \in L_p^{(a,b)} \) and
\[
\int_0^1 \left( u^{-\sigma} K^{r+\sigma}(f, D_r^{(\alpha,\beta)}, u)_{p,(a,b)} \right)^q \frac{du}{u} < \infty,
\]
then \( f \in L_q^{(a,b)} \) and
\[
K^r(f, D_r^{(\alpha,\beta)}, 1/n)_{q,(a,b)} \leq C \left( \int_0^1 \left( u^{-\sigma} K^{r+\sigma}(f, D_r^{(\alpha,\beta)}, u)_{p,(a,b)} \right)^q \frac{du}{u} \right)^{1/q}, \tag{3.5}
\]

To complete the proof of (3.5), we have
\[
\text{Proof.} \text{ Using monotonicity properties of the } K\text{-functional, it is enough to verify inequality (3.5) for } t = 1/n, n \in \mathbb{N}. \text{ We have}
\[
K^r(f, D_r^{(\alpha,\beta)}, 1/n)_{q,(a,b)} \leq C \left( \| f - P_{n,f} \|_{q,(a,b)} + n^{-r} \| D_r^{(\alpha,\beta)} P_{n,f} \|_{q,(a,b)} \right), \tag{3.6}
\]
where \( P_{n,f} \) is given by (3.2). To estimate the first term, we apply [13 Theorem 4.1, (4.6)] to get
\[
\| f - P_{n,f} \|_{q,(a,b)} \leq C \left( \sum_{k=n}^{\infty} k^{q\sigma-1} \| f - P_k,f \|_{p,(a,b)}^{q} \right)^{1/q}.
\]
In view of the realization result (3.4), we obtain
\[
\| f - P_{n,f} \|_{q,(a,b)} \leq C \left( \sum_{k=n}^{\infty} k^{q\sigma-1} \| f - P_k,f \|_{p,(a,b)}^{q} \right)^{1/q}.
\]
\[
\leq C \left( \sum_{k=n}^{\infty} k^{q\sigma-1} K^{r+\sigma}(f, D_r^{(\alpha,\beta)}, 1/k)_{p,(a,b)}^{q} \right)^{1/q}.
\]
\[
\leq C \left( \int_0^1 \left( u^{-\sigma} K^{r+\sigma}(f, D_r^{(\alpha,\beta)}, u)_{p,(a,b)} \right)^q \frac{du}{u} \right)^{1/q}.
\]
To estimate the second term in (3.6), we use Theorem D or Theorem 2 depending on whether \( (\alpha, \beta) = (a, b) \) or \( (\alpha, \beta) \neq (a, b) \):
\[
n^{-r} \| D_r^{(\alpha,\beta)} P_{n,f} \|_{q,(a,b)} \leq C n^{r} n^{-(r+\sigma)} \| D_r^{(\alpha,\beta)} P_{n,f} \|_{p,(a,b)} \leq C n^{\sigma} K^{r+\sigma}(f, D_r^{(\alpha,\beta)}, 1/n)_{p,(a,b)}.
\]
To complete the proof of (3.5), we have
\[
n^{\sigma} K^{r+\sigma}(f, D_r^{(\alpha,\beta)}, 1/n)_{p,(a,b)} \leq C \left( \int_{1/2n}^{1/n} \left( u^{-\sigma} K^{r+\sigma}(f, D_r^{(\alpha,\beta)}, u)_{p,(a,b)} \right)^q \frac{du}{u} \right)^{1/q}.
\]

\[ \Box \]

4. Ulyanov-type inequalities for Ditzian–Totik moduli of smoothness

The (global) weighted modulus of smoothness of order \( r \geq 1 \) is given by
\[
\omega^r_{\nu}(f,t)_{p,(a,b)} = \Omega^r_{r}(f,t)_{p,(a,b)} + \inf_{P \in \Pi_{-1}} \|(f - P)w\|_{L_p[-1, -1+4k^2t^2]} + \inf_{P \in \Pi_{-1}} \|(f - P)w\|_{L_p[1-4k^2t^2, 1]},
\]
where \( w = (w^{(a,b)})^{1/p} \),
\[
\Omega_{x'}(f, t)_{p,(a,b)} \leq \sup_{0<h \leq t} \| \Delta_{h} f w \|_{L_{p}[-1+4k^{2}t^{2}, 1-4k^{2}t^{2}]}
\]
and
\[
\Delta_{h} f(x) = \sum_{i=0}^{r} (-1)^{i} \binom{r}{i} f \left( x + \frac{r-2i}{2} h(x) \right).
\]

Note that (see [16 (2.5.7)]) this definition is equivalent to the one given in [14] Chapter 6, Appendix B.

Let \( K_{x'}(f, t)_{p,(a,b)} \), \( r \in \mathbb{N} \), be the \( K \)-functional for the pair of spaces \( (L_{p}^{(a,b)}, \Omega_{x'}^{r}(f, t)_{p,(a,b)}) \),
where \( \Omega_{x'}^{r}(f, t)_{p,(a,b)} \) consists of functions \( g \in L_{p}^{(a,b)} \) such that \( g^{(r-1)} \in AC_{loc} \) and \( x'g^{(r)} \in L_{p}^{(a,b)} \) (see [14] (6.1.1)):
\[
K_{x'}(f, t)_{p,(a,b)} = \inf \left\{ \| f - g\|_{p,(a,b)} + t^{r} \| x'g^{(r)}\|_{p,(a,b)} : g \in \Omega_{x'}^{r}(f, t)_{p,(a,b)} \right\}.
\]

It is known that \( K_{x'}(f, t)_{p,(a,b)} \asymp \omega_{x'}(f, t)_{p,(a,b)} \) for \( a, b > 0 \); see [14] Theorem 6.1.1. Moreover, we have the following realization result:
\[
\omega_{x'}(f, t)_{p,(a,b)} \asymp \| f - P_{n,f} w_{p,(a,b)} + t^{r} \| x'P_{n,f}^{(r)}\|_{p,(a,b)}, \quad [1/t] = n.
\]

The proof of this equivalence (cf. [12]) is based on the Jackson-type inequality and the estimate of \( t^{r} \| x'\psi^{(r)}\|_{p,(a,b)} \) via \( \omega_{x'}(f, t)_{p,(a,b)} \) (the Nikolskii–Stechkin type inequality). The Jackson-type inequality was obtained in [14] Theorem 7.2.1 for the unweighted case and in [16] Sec. 2.5.2, (2.5.17) for the weighted case. The unweighted version of the Nikolskii–Stechkin type inequality was proved in [14] Theorem 7.3.1. This argument can be used to show the weighted version.

The relation between \( K \)-functionals [4.1] and [3.3] in the case when \( r \) is a positive integer follows from Corollary 2 below. Note that the case \((a, b) = (a, b)\) is due to Daï and Ditzian [8 Theorem 7.1] and is based on the Muckenhoupt transplantation theorem. We follow the idea of their proof and first obtain the following result.

**Theorem 4.** Let \( 1 < p < \infty \), \( r \) be a positive integer, and \( a, b, \alpha, \beta > -1 \) be such that (3.1) holds. Then there exists a constant \( C \) such that for any \( q \in \Pi \), we have
\[
\| \varphi^{q} Q^{(r)} \|_{p,(a,b)} \leq C \| D_{r}^{(\alpha,\beta)} Q \|_{p,(a,b)},
\]
\[
\| D_{r}^{(\alpha,\beta)} \left( Q - S_{r-1}^{(\alpha,\beta)} \right) \|_{p,(a,b)} \leq C \| \varphi^{q} Q^{(r)} \|_{p,(a,b)},
\]
where \( S_{r-1}^{(\alpha,\beta)} Q \) is the \((r-1)\)-th partial sum of the Fourier–Jacobi expansion of \( Q \), i.e.,
\[
S_{r-1}^{(\alpha,\beta)} Q = \sum_{k=0}^{r-1} \hat{Q}_{k}^{(\alpha,\beta)} \psi_{k}^{(\alpha,\beta)}.
\]

**Proof.** The proof of (4.3) and (4.4) is based on Theorem 3. Since \( \hat{Q}_{k}^{(\alpha,\beta)} = 0 \) starting from certain \( k \), we obtain
\[
D_{r}^{(\alpha,\beta)} Q = \sum_{k=1}^{\infty} \lambda_{k}^{(\alpha,\beta)} \hat{Q}_{k}^{(\alpha,\beta)} \psi_{k}^{(\alpha,\beta)} = \sum_{k=1}^{\infty} \lambda_{k}^{(\alpha,\beta)} \hat{Q}_{k+r}^{(\alpha,\beta)} \psi_{k+r}^{(\alpha,\beta)},
\]
\[
Q^{(r)} = \sum_{k=r}^{\infty} \lambda_{k} \hat{Q}_{k}^{(\alpha,\beta)} \psi_{k-r}^{(\alpha+r,\beta+r)} = \sum_{k=0}^{\infty} \lambda_{k+r} \hat{Q}_{k+r}^{(\alpha,\beta)} \psi_{k}^{(\alpha+r,\beta+r)},
\]
\[ \lambda_k = \lambda_k(\alpha, \beta, r) = \lambda_k^{(\alpha, \beta)} \cdot \lambda_k^{(\alpha + r - 1, \beta + r - 1)}. \]

To prove inequality (4.3), we apply Theorem C with \((\overline{\alpha}, \overline{\beta}) = (\alpha + r, \beta + r)\), \((\overline{\gamma}, \overline{\delta}) = (\alpha, \beta), (\overline{c}, \overline{d}) = (a, b), h = -r,\) and

\[ \nu_k = \lambda_k / \left(\lambda_k^{(\alpha, \beta)}\right)^r. \]

Then \(s = 0, (\overline{\alpha}, \overline{\beta}) = (a + pr/2, b + pr/2), A = (a + 1)/p - \alpha,\) and \(B = (b + 1)/p - \beta.\) On account of (3.1), we conclude that \(A < 1, B < 1,\) and therefore, all conditions of Theorem C are satisfied. Hence, we get

\[ \left\| \varphi^r Q^{(r)} \right\|_{p,(a,b)} = \left\| Q^{(r)} \right\|_{p,(a+pr/2,b+pr/2)} \leq C \left\| D^{(\alpha, \beta)} r \right\|_{p,(a,b)}. \]

Let us now obtain (4.4). We remark that \(g = D^{(\alpha, \beta)} r \left( Q - S_{r-1}^{(\alpha, \beta)} Q \right)\) is a polynomial and its Fourier–Jacobi coefficients satisfy \(\hat{g}_k^{(\alpha, \beta)} = 0\) for \(0 \leq k \leq r - 1.\) We apply Theorem C with \((\overline{\alpha}, \overline{\beta}) = (\alpha, \beta), (\overline{\gamma}, \overline{\delta}) = (\alpha + r, \beta + r), (\overline{c}, \overline{d}) = (a + pr/2, b + pr/2), h = r,\) and

\[ \nu_k = \left(\lambda_k^{(\alpha, \beta)}\right)^r / \lambda_k. \]

Then \(s = 0, (\overline{\alpha}, \overline{\beta}) = (a, b), A = (a + 1)/p - \alpha - r/2 < 1,\) and \(B = (b + 1)/p - \beta - r/2 < 1.\) Therefore, all conditions of Theorem C are satisfied, and we arrive at

\[ \left\| D^{(\alpha, \beta)} r \left( Q - S_{r-1}^{(\alpha, \beta)} Q \right) \right\|_{p,(a,b)} \leq C \left\| Q^{(r)} \right\|_{p,(a+pr/2,b+pr/2)} = C \left\| \varphi^r Q^{(r)} \right\|_{p,(a,b)}. \]

**Corollary 2.** Under assumptions of Theorem 4 there exists a constant \(C\) such that for any \(f \in L_p^{(a,b)}\) and \(t \in (0, t_0)\) we have

\[ K^r_{p}(f, t)_{p,(a,b)} \leq C K^r_{p}(f, D^{(\alpha, \beta)} r t)_{p,(a,b)} \] (4.5)

and

\[ K^r_{p}(f, D^{(\alpha, \beta)} r t)_{p,(a,b)} \leq C \left( K^r_{p}(f, t)_{p,(a,b)} + t^r \left\| f \right\|_{p,(a,b)} \right). \]

**Proof.** First, (4.3) and the realization result (4.2) yield that

\[ K^r_{p}(f, t)_{p,(a,b)} \leq \left\| f - P_{n,f} \right\|_{p,(a,b)} + t^r \left\| \varphi^r P_{n,f} \right\|_{p,(a,b)} \]

\[ \leq C \left( \left\| f - P_{n,f} \right\|_{p,(a,b)} + t^r \left\| D^{(\alpha, \beta)} r P_{n,f} \right\|_{p,(a,b)} \right) \leq C K^r_{p}(f, D^{(\alpha, \beta)} r t)_{p,(a,b)}, \]

which is (4.5).

Second, under condition (3.1), the operator \(A : \Pi \rightarrow \Pi_{r-1}\) given by

\[ A(Q) = D^{(\alpha, \beta)} r S_{r-1}^{(\alpha, \beta)} Q \]

is bounded in \(L_p^{(a,b)}\), i.e.,

\[ \left\| D^{(\alpha, \beta)} r S_{r-1}^{(\alpha, \beta)} Q \right\|_{p,(a,b)} \leq C(p, a, b, \alpha, \beta, r) \left\| Q \right\|_{p,(a,b)}. \] (4.6)

Using this, we obtain

\[ K^r_{p}(f, D^{(\alpha, \beta)} r t)_{p,(a,b)} \leq \left\| f - P_{n,f} \right\|_{p,(a,b)} + t^r \left\| D^{(\alpha, \beta)} r P_{n,f} \right\|_{p,(a,b)} \]

\[ \leq \left\| f - P_{n,f} \right\|_{p,(a,b)} + t^r \left\| D^{(\alpha, \beta)} r (P_{n,f} - S_{r-1}^{(\alpha, \beta)} P_{n,f}) \right\|_{p,(a,b)} + t^r \left\| D^{(\alpha, \beta)} r S_{r-1}^{(\alpha, \beta)} P_{n,f} \right\|_{p,(a,b)}. \]
Finally, (4.4) and (1.6) imply
\[
K^r(f, D_r^{(a,b)}, t)_{p,(a,b)} \leq C \left( \|f - P_{n,f}\|_{p,(a,b)} + t^{-r} \|\varphi_r P_{n,f}\|_{p,(a,b)} + t^r \|P_{n,f}\|_{p,(a,b)} \right) \\
\leq C \left( K^r_\varphi(f, t)_{p,(a,b)} + t^r \|f\|_{p,(a,b)} \right).
\]
\[
\square
\]

It is proved in [13, Theorem 11.2] that for \( f \in L_p, 0 < p < q \leq \infty \), and integer \( r \geq 1 \) the following Ulyanov-type inequality holds:
\[
\omega^r_\varphi(f, t)_q \leq C \left[ \int_0^t \left( u^{-\sigma} \omega^r_\varphi(f, u)_p \right)^{q_1} \frac{du}{u} \right]^{1/q_1},
\]
where \( q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty \end{cases} \) and \( \sigma = (2a + 2) \left( \frac{1}{p} - \frac{1}{q} \right) \). The next theorem refines this result.

**Theorem 5.** Let \( 1 \leq p < q \leq \infty \), \( a \geq b \geq 0 \), and \( r \) be a positive integer, and
\[
\sigma = (2a + 2) \left( \frac{1}{p} - \frac{1}{q} \right).
\]
Suppose that \( f \in L_p^{(a,b)} \) and
\[
\int_0^1 \left( u^{-\sigma} \omega^r_\varphi(f, u)_p \right)^{q_1} \frac{du}{u} < \infty.
\]
Then \( f \in L_q^{(a,b)} \) and
\[
\omega^r_\varphi(f, t)_{q,(a,b)} \leq C \left[ \int_0^t \left( u^{-\sigma} \omega^r_\varphi(f, u)_p \right)^{q_1} \frac{du}{u} \right]^{1/q_1} + C t^r E_{r-1}(f)_{p,(a,b)}, \tag{4.7}
\]
where
\[
q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty. \end{cases}
\]

**Remark.** (A). In particular, (4.7) implies
\[
\omega^r_\varphi(f, t)_q \leq C \left[ \int_0^t \left( u^{-1} \omega^{r+1}_\varphi(f, u)_p \right)^{q_1} \frac{du}{u} \right]^{1/q_1} + C t^r E_{r-1}(f)_p,
\]
when \( \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}, 1 \leq p < q \leq \infty \), and
\[
\omega^r_\varphi(f, t)_\infty \leq C \int_0^t u^{-2} \omega^{r+2}_\varphi(f, u) \frac{du}{u} + C t^r E_{r-1}(f)_1.
\]

(B). Corollary 2 shows that for \( 1 < p < q < \infty \) and positive integer \( \sigma \) Theorem 5 follows from Theorem 3.

**Proof.** The proof is similar to the proof of Theorem 3. The only substantial difference is that we use Lemma 1 instead of Theorem D and Theorem 2.

Using monotonicity properties of the moduli of smoothness, it is enough to verify inequality (4.7) for \( t = 1/n \), where \( n \) is a positive integer. Let \( P_{n,f} \) be defined by (3.2). Taking into account that \( \omega^r_\varphi(f, t)_{q,(a,b)} \asymp K^r_\varphi(f, t)_{q,(a,b)} \), we obtain
\[
\omega^r_\varphi(f, t)_{q,(a,b)} \leq C \left( \|f - P_{n,f}\|_{q,(a,b)} + n^{-r} \|\varphi_r P_{n,f}\|_{q,(a,b)} \right). \tag{4.8}
\]
To estimate the first term, we apply Theorem 4.1 from [13]. Assumption (4.3) of this theorem is exactly the Nikol’skii inequality

$$\|P_n\|_{q,(a,b)} \leq C n^{(2a+2)\left(\frac{1}{p} - \frac{1}{q}\right)} \|P_n\|_{p,(a,b)}, \quad P_n \in \Pi_n,$$

where $C = C(p, q, a, b)$, proved in [9] Theorem 4 (see also [17] Ch. 6, Theorem 1.8.4, 1.8.5)). Therefore, we have

$$\|f - P_n,f\|_{q,(a,b)} \leq C \left(\sum_{k=n}^{\infty} k^{q_1\sigma-1}\|f - P_{k,f}\|_{q_1, p,(a,b)}^q\right)^{1/q_1}.$$

Applying (1.2) and replacing the sum by the integral, we get

$$\|f - P_n,f\|_{q,(a,b)} \leq C \left(\int_0^R \left(u^{-\sigma} \omega^r_{\varphi} \left(f, u\right)_{p,(a,b)}\right)^{q_1} \frac{du}{u}\right)^{1/q_1}.$$

To estimate the second term in (4.8), we use Lemma [1]

$$\left\|\varphi^r P_n^{(r)}\right\|_{q,(a,b)} = \left\|\varphi^r (P_n - P_{r-1})^{(r)}\right\|_{q,(a,b)} \leq \left\|P_n - P_{r-1}\right\|_{p,(a,b)} + \left\|\varphi^{r+2\sigma} - \varphi P_n^{(r+\sigma)}\right\|_{p,(a,b)}.$$

Further we need the following two-weight inequality proved in [9] Theorem 4:

$$\left\|\varphi^{r+2\sigma} - \varphi P_n^{(r+\sigma)}\right\|_{p,(a,b)} \leq C n^{\sigma-\sigma} \left\|\varphi^{r+\sigma} P_n^{(r+\sigma)}\right\|_{p,(a,b)}.$$

Therefore, using monotonicity properties of moduli of smoothness, we get

$$n^{-r} \left\|\varphi^{r+2\sigma} - \varphi P_n^{(r+\sigma)}\right\|_{p,(a,b)} \leq C n^{\sigma} \omega_{\varphi}^{r+\sigma}(f, 1/n)_{p,(a,b)}$$

$$\leq C \left[\int_{1/2n}^{1/n} \left(u^{-\sigma} \omega_{\varphi}^{r+\sigma}(f, u)_{p,(a,b)}\right)^{q_1} \frac{du}{u}\right]^{1/q_1}. $$

To complete the proof we note that $\|P_n - P_{r-1}\|_{p,(a,b)} \leq 2E_{r-1}(f)_{p,(a,b)}$. □

**Acknowledgement.** The authors would like to thank F. Dai, Z. Ditzian, and G. Mastroianni for fruitful discussions and useful comments on the fractional $K$-functionals, and the referee for reading the paper carefully and several valuable comments.

**References**

[1] R. Askey and S. Wainger, On the behavior of special classes of ultraspherical expansions, I. J. Analyse Math., 15 (1965), 193–485.
[2] R. Askey and S. Wainger, A convolution structure for Jacobi series, Amer. J. Math., 91, no. 2 (1969), 463–485.
[3] H. Bavinck, A special class of Jacobi series and some applications, J. Math. Anal. Appl., 37 (1972), 767–797.
[4] H. Bavinck, W. Trebels, On $M^1_p$ multipliers for Jacobi expansions, Fourier analysis and approximation theory (Proc. Colloq., Budapest, 1976), Vol. I, Colloq. Math. Soc. János Bolyai, 19, North-Holland, Amsterdam-New York, 1978, 101–112.
[5] J. S. Bradley, Hardy inequalities with mixed norms, Canad. Math. Bull., 21 (1978), 405–408.
[6] R. DeVore, G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.
[7] R. DeVore, S. Riemenschneider, R. Sharpley, Weak interpolation in Banach spaces, J. Funct. Anal., 33 (1979), 58–94.
[8] F. Dai, Z. Ditzian, Littlewood-Paley theory and a sharp Marchaud inequality, Acta Sci. Math. (Szeged), 71 (2005), no. 1-2, 65–90.
[9] I. K. Daugavet, S. Z. Rafal'son, Certain inequalities of Markov–Nikolskii type for algebraic polynomials, Vestnik Leningrad. Univ., 1 (1972), 15–25.
[10] Z. Ditzian, Fractional derivatives and best approximation, Acta Math. Hungar., 81, no. 4 (1998), 323–348.
[11] Z. Ditzian, Polynomial approximation and $\omega^r(f, t)$ twenty years later, Surv. Approx. Theory, 3 (2007), 106–151.
[12] Z. Ditzian, V. H. Hristov, K. G. Ivanov, Moduli of smoothness and K-functionals in $L_p$, $0 < p < 1$, Constr. Approx., 11, no. 1 (1995), 67–83.
[13] Z. Ditzian, S. Tikhonov, Ul’yanov and Nikol’skii-type inequalities, J. Approx. Theory 133, no. 1 (2005), 100–133.
[14] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer, 1987.
[15] D. Haroske, H. Triebel, Embeddings of function spaces: a criterion in terms of differences, Compl. Var. Ell. Eq., 56, no. 10-11 (2011), 931–944.
[16] G. Mastroianni, G. Milovanović, Interpolation Processes. Basic Theory and Applications, Springer, Berlin, 2008.
[17] G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, World Scientific Publishing, NJ, 1994.
[18] B. Muckenhoupt, Transplantation theorems and multiplier theorems for Jacobi series, Mem. Amer. Math. Soc. 64, no. 356 (1986).
[19] B. Opic, A. Kufner, Hardy-type Inequalities, Longman Scientific & Technical, Harlow, 1990.
[20] B. Simonov, S. Tikhonov, Sharp Ul’yanov-type inequalities using fractional smoothness, Journal of Approx. Theory, 162, no. 9 (2010), 1654–1684.
[21] S. Tikhonov, Weak type inequalities for moduli of smoothness: the case of limit value parameters, J. Fourier Anal. Appl., 16, no. 4 (2010), 590–608.
[22] W. Trebels, Multipliers for $(C, \alpha)$-bounded Fourier expansions in Banach spaces and approximation theory, Lecture Notes in Mathematics, 329, Springer-Verlag, 1973.
[23] W. Trebels, Inequalities for moduli of smoothness versus embeddings of function spaces, Arch. Math., 94 (2010), 155–164.
[24] W. Trebels, U. Westphal, On Ulyanov inequalities in Banach spaces and semigroups of linear operators, J. Approx. Theory, 160, no. 1–2 (2009), 154–170.
[25] S. Wang, A generalized Ul’yanov type inequality on the sphere $S^{d-1}$, Acta Math. Sin., Chin. Ser. 54, no. 1 (2011), 115–124.

P. Yu. Glazyrina, Subdepartment of Mathematical Analysis and Theory of Functions, Ural Federal University, pr. Lenina 51, 620083 Ekaterinburg, Russia
E-mail address: polina.glazyrina@usu.ru

S. Tikhonov, ICREA and Centre de Recerca Matemàtica, Apartat 50 08193 Bellaterra, Barcelona, Spain
E-mail address: stikhonov@crm.cat