Almost normal operators mod Hilbert–Schmidt and the 
$K$-theory of the algebras $E\Lambda(\Omega)$

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Abstract. Is there a mod Hilbert–Schmidt analogue of the BDF-theorem, with the Pincus g-function playing the role of the index? We show that part of the question is about the $K$-theory of certain Banach algebras. These Banach algebras, related to Lipschitz functions and Dirichlet algebras have nice Banach-space duality properties. Moreover their corona algebras are $C^*$-algebras.

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1. Introduction

The BDF-theorem [7] classifies, up to unitary equivalence, the normal elements of the Calkin algebra, by the spectrum and the index of the resolvent. If the ideal of compact operators is replaced by the trace-class, for operators with trace-class self-commutator, the Pincus $g$-function ([8], [9]) is an $L^1$-function on $\mathbb{C}$ which extends the index of the essential resolvent. The $g$-function has been related to algebraic $K$-theory by L. G. Brown ([5], [6]) and in another direction, after work of J. W. Helton and R. Howe ([18]), the distribution to which the $g$-function gives rise, has been interpreted in terms of cyclic cohomology by A. Connes ([13]).

These developments around the $g$-function, were however not accompanied by a corresponding BDF-type result. In ([28], [26], [27]) we formulated conjectures about operators with trace-class self-commutator, an affirmative answer to which would fill this gap. Besides the initial evidence in favor of these conjectures, there was no further progress. The situation is roughly that the $g$-function viewed in the cyclic cohomology framework covers the index part and our work on Hilbert–Schmidt perturbations of normal operators ([25]) covers the part about trivial extensions, while the rest is wide open. The absence on the technical side of a normal dilation result which would correspond to the existence of inverses in Ext and which in

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the BDF context can be derived from the Choi–Effros completely positive lifting theorem, is a noted difficulty.

Our aim here is to decouple the normal dilation from the rest by introducing the algebras $E\Lambda(\Omega)$. In this way we are also able to bring $K$-theory to the study of this problem since we are led to the $K_0$-group of such an algebra.

The Banach *-algebras $E\Lambda(\Omega)$ are the natural framework to study operators with trace-class self-commutator which are obtained from compressions of normal operators to mod Hilbert–Schmidt reducing projections. Roughly $E\Lambda(\Omega)$, where $\Omega$ is a Borel subset of $\mathbb{C}$ is an algebra of operators in $L^2(\Omega, \lambda)$ with Hilbert–Schmidt commutators with the multiplication operators by Lipschitz functions, a construction reminiscent of Paschke-duality ([22]).

The algebras $E\Lambda(\Omega)$ have nice properties as Banach algebras. They resemble the Lipschitz algebras of [30], up to the use of a Hilbert–Schmidt norm instead of a uniform norm, which is a feature of the Dirichlet algebras of non-commutative potential theory ([1], [10], [11]). Actually the ideal $K\Lambda(\Omega)$ of compact operators in $E\Lambda(\Omega)$ is a Dirichlet algebra and we show that $E\Lambda(\Omega)$ can be viewed both as the algebra of multipliers or as the bidual of $K\Lambda(\Omega)$, when $\Omega$ is bounded. Since all this has the flavor of Banach algebra analogues of basic $C^*$-algebras, it is perhaps unexpected that the corona $E\Lambda(\Omega)/K\Lambda(\Omega)$ which is the analogue of the Calkin algebra is really a $C^*$-algebra. Note, however, that while the Dirichlet algebra $K\Lambda(\Omega)$ has the same simple $K$-theory as the algebra $K(H)$ of compact operators, the $K$-theory of $E\Lambda(\Omega)$ and hence of $E\Lambda(\Omega)/K\Lambda(\Omega)$, which interests us in connection with operators with trace-class self-commutator, is certainly richer.

On the technical side an essential ingredient is the existence of a bounded approximate unit consisting of projections for $K\Lambda(\Omega)$, which is a consequence of our work on norm-ideal perturbations of Hilbert-space operators ([25], [29]).

Concerning the relation of the operator theory problems to the $K$-theory of the algebras $E\Lambda(\Omega)$, we should point out that while the $K$-theory problem is so to speak the operator theory problem minus the dilation problem, actually certain outcomes of the $K$-theory problem could provide a negative answer to the dilation problem. If the $K$-theory of $E\Lambda(\Omega)$ exhibits some integrality property making $K_0$ less rich this would answer in the negative the dilation problem.

In addition to the first section, which is the introduction, the paper has five more sections.

Section 2 contains background material about the conjectures about almost normal operators modulo Hilbert–Schmidt. Details of certain connections between these problems, left out previously, are included for the reader’s convenience.

Section 3 introduces the algebras $E\Lambda(\Omega)$ and some of their basic properties. We also consider the ideal of compact operators $K\Lambda(\Omega)$ of $E\Lambda(\Omega)$ and the Banach algebra $E\Lambda(\Omega)_0$ which is the inductive limit of the $E\Lambda(\Omega)$ for bounded sets $\Omega$.

In section 4 we look at the $K$-theory of the Banach algebras considered. We show that the problem about a mod Hilbert–Schmidt BDF-type theorem for almost normal
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operators is equivalent to the normal dilation problem plus the problem whether the $K_0$-group of $E\Lambda(\Omega)$ is isomorphic via the Pincus $g$-function to the group $L^1_{\text{re}}(\mathcal{C}, \lambda)$ of real-valued $L^1$-functions with bounded support.

Section 5 returns to the algebras $K\Lambda(\Omega)$, $E\Lambda(\Omega)$ and $(E/K)\Lambda(\Omega)$ and gives results about duality, multipliers and the relation to $C^*$-algebras.

Section 6 contains concluding remarks in several directions: the action of bi-Lipschitz homeomorphisms on the algebras, the center of $\mathcal{C}/K\Lambda(\Omega)$, the relation to Dirichlet algebras and non-commutative potential theory, the possibility of similar constructions with other Schatten–von Neumann classes $C_p$ replacing the Hilbert–Schmidt class.

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2. Background

2.1. If $\mathcal{H}$ is a separable infinite-dimensional Hilbert space over $\mathbb{C}$, then $B(\mathcal{H})$ will denote the bounded operators on $\mathcal{H}$ and $C_p(\mathcal{H})$ the Schatten–von Neumann $p$-class. The $p$-norm $|\cdot|_p$ is $|T|_p = \text{Tr}(T^*T)^{p/2}$. In particular, $C_1(\mathcal{H})$ is the trace-class and $C_2(\mathcal{H})$ is the Hilbert–Schmidt class.

2.2. An operator $T \in B(\mathcal{H})$ is almost normal if its self-commutator $[T^*, T]$ is in $C_1(\mathcal{H})$. Equivalently, if $T = A + iB$ with $A = A^*$, $B = B^*$ then $[A, B] \in C_1$ since $2i[A, B] = [T^*, T]$. We shall denote by $\mathcal{AN}(\mathcal{H})$ the set of almost operators. Background material and references to the literature for many facts about operators with trace-class self-commutator can be found in the books [12], [21].

2.3. If $T = A + iB \in \mathcal{AN}(\mathcal{H})$ and if $Q, R \in \mathbb{C}[X, Y]$ are polynomials in two commuting indeterminates, then since $A, B$ the class of $A, B$ in $B(\mathcal{H})/C_1(\mathcal{H})$ commute, we shall also write $Q(A, B)$, $R(A, B)$ for elements in $B(\mathcal{H})$ so that $Q(A, B) = Q(A, B)$, $R(A, B) = R(A, B)$. Clearly these are only defined up to a $C_1$ perturbation. The Helton–Howe measure $P_T$ of $T = A + iB \in \mathcal{AN}(\mathcal{H})$ ([18]) is a compactly supported measure on $\mathbb{R}^2$ so that

$$\text{Tr}[Q(A, B), R(A, B)] = (2\pi i)^{-1} \int J(Q, R)dP_T$$

where

$$J(Q, R) = \frac{\partial Q}{\partial X} \frac{\partial R}{\partial Y} - \frac{\partial Q}{\partial Y} \frac{\partial R}{\partial X}.$$ 

Then supp $P_T \subset \sigma(T)$ and $P_T$ is absolutely continuous w.r.t. Lebesgue measure $\lambda$ and the Radon–Nikodym derivative $\frac{dP_T}{d\lambda} = g_T \in L^1(\mathbb{R}^2)$ is the Pincus principal function of $T$ (also called Pincus $g$-function).
2.4. Let $R_1^+(\mathcal{H}) = \{X \in B(\mathcal{H}) : X \text{ finite rank, } 0 \leq X \leq 1\}$, which is a directed ordered set. Then the obstruction to the existence of quasicentral approximate units relative to the Hilbert–Schmidt class ([25]) is

$$k_2(T_1, \ldots, T_n) = \liminf_{X \in R_1^+(\mathcal{H})} \max_{1 \leq j \leq n} ||T_j X||_2.$$ 

In [28] we showed that: if $T_1, T_2 \in AN(\mathcal{H})$, $k_2(T_1) = 0$ and $T_1 - T_2 \in C_2$, then $P_{T_1} = P_{T_2}$ (or equivalently $g_{T_1} = g_{T_2}$ a.e.).

2.5. We recall two of the conjectures about almost normal operators ([28] conjectures 3 and 4). Note that the second of these is a consequence of the first.

Conjecture 3 in [28]. If $T_1, T_2 \in AN(\mathcal{H})$ are so that $P_{T_1} = P_{T_2}$ then there is a normal operator $N \in B(\mathcal{H})$ and a unitary operator $U \in B(\mathcal{H} \oplus \mathcal{H})$ so that $T_1 \oplus N - U(T_2 \oplus N)U^* \in C_2$.

If true, this statement would represent a kind of BDF-theorem with $AN(\mathcal{H})$ and the Helton–Howe measure replacing the operators with compact self-commutator and respectively the index-data. Note also that the unitary equivalence is mod $C_2$ (not $C_1$).

Conjecture 4 in [28]. If $T \in AN(\mathcal{H})$ then there is $S \in AN(\mathcal{H})$ and a normal operator $M \in B(\mathcal{H} \oplus \mathcal{H})$ so that $T \oplus S - M \in C_2$.

This conjecture is an analogue of the existence of inverses in Ext in the analogue of the “Ext is a group” part of the BDF theorem. Note that the analogue of the results for trivial extensions (i.e., Weyl–von Neumann theorem part) is covered by our results in [25]. For the derivation of Conjecture 4 from Conjecture 3 one also uses the result of R. V. Carey and J. D. Pincus that every $L^1$-function is the $g$-function of some $T \in AN(\mathcal{H})$ (see 4.9).

2.6. We would like to remark that Conjectures 3 and 4 in [28] don’t bring the essential spectrum of the almost normal operators into the discussion. With consideration of the essential spectrum $\sigma_e(T)$, one might ask if $P_{T_1} = P_{T_2}$ and $\sigma_e(T_1) = \sigma_e(T_2)$ would imply $T_1 - UT_2U^* \in C_2$, form some unitary $U$, when $T_j \in AN(\mathcal{H})$, $j = 1, 2$.

We didn’t discuss the possibility of such a strengthening, because it seems to have to do also with phenomena of another kind involving perturbations of isolated points in $\sigma(T) \setminus \sigma_e(T)$.

2.7. A consequence of Conjecture 4 and hence also of Conjecture 3 is the following conjecture.

Conjecture 1 in [28]. If $T \in AN(\mathcal{H})$ then $k_2(T) = 0$.

The proof which was omitted in [28], involves using a result of [25], that $k_2(N) = 0$ for every normal operator $N$. Indeed, if Conjecture 4 holds for $T$, then $T \in AN(\mathcal{H})$ is unitarily equivalent mod $C_2$ to a compression $PN \mid PH$. 

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where \( P = P^* = P^2 \) is a projection, \( N \) is normal and \([P, N] \in C_2\). We infer that \( k_2(T) = k_2(PN \mid P\mathcal{H}) \). On the other hand \( k_2(N) = 0 \) implies there are \( X_n \in R^+_1(\mathcal{H}), X_n \uparrow I \) as \( n \to \infty \), so that \( \lim_{n \to \infty} \|[X_n, N]\|_2 = 0 \). If \( Y_n = PX_nP \) then \( Y_n \in R^+_1 \) and we have \( Y_n \uparrow P \) as \( n \to \infty \). We have

\[
||Y_n, PNP||_2 = ||P[X_n, PNP]P||_2 \leq ||P[X_n, NP]P||_2 + ||I - X_n, [P, N]P||_2.
\]

Since \([P, N]P \in C_2\) and \( I - X_n \downarrow 0 \) we have \( ||I - X_n, [P, N]P||_2 \to 0 \) as \( n \to \infty \).

On the other hand

\[
|P[X_n, NP]P|_2 \leq |P[I - X_n, N]P|_2 + |P[N, P](I - X_n)P||_2
\]

which converges to 0 as \( n \to \infty \). Thus, Conjecture 1 holds for \( T \), i.e., \( k_2(T) = 0 \).

2.8. We will also need to recall some of the results for normal operators which follow from [25]. Since \( k_2(N) = 0 \) for every normal operator \( N \), we can use the kind of non-commutative Weyl–von Neumann results in [25] to infer that: if \( N_1 \) and \( N_2 \) are normal operators on \( \mathcal{H} \) and \( \sigma(N_1) = \sigma(N_2) = \sigma_e(N_1) = \sigma_e(N_2) \) then there is a unitary operator \( U \) so that \( UN_1U^* - N_2 \in C_2 \) and \( |UN_1U^* - N_2|_2 < \varepsilon \) for a given \( \varepsilon > 0 \).

Also, if \( T \in AN(\mathcal{H}) \) and \( N \) is a normal operator with \( \sigma(N) = \sigma_e(T) \) then there is a unitary operator \( U : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} \) so that \( (T \oplus N)U - UT \) is Hilbert–Schmidt and \( |(T \oplus N)U - UT|_2 < \varepsilon \) for a given \( \varepsilon > 0 \).

3. The Banach Algebras \( E\Lambda(\Omega) \)

3.1. We shall define here the algebras \( E\Lambda(\Omega) \) and give a few of their basic properties.

If \( \Omega \subset \mathbb{C} \) is a Borel set and \( f \in L^\infty(\Omega, \lambda) \), with \( \lambda \) denoting Lebesgue measure, let \( M_f \) be the multiplication operator by \( f \) on \( L^2(\Omega, \lambda) \) and \( Df \) be the difference quotient

\[
Df(s, t) = \frac{f(s) - f(t)}{s - t} (s \neq t)
\]

which is the class up to null-sets of a Lebesgue-measurable function on \( \Omega \times \Omega \). Let further

\[
\Lambda(\Omega) = \{ f \in L^\infty(\Omega, \lambda) \mid Df \in L^\infty(\Omega \times \Omega, \lambda \otimes \lambda) \}
\]

be the subalgebra of essentially Lipschitz functions. If \( T \in B(L^2(\Omega, \lambda)) \) let \( L(T) \) be given by

\[
L(T) = \sup\{ ||[M_f, T]|_2 \mid f \in \Lambda(\Omega), ||Df||_\infty \leq 1 \}.
\]

We define \( E\Lambda(\Omega) \) to be the subalgebra of \( B(L^2(\Omega)) \)

\[
E\Lambda(\Omega) = \{ T \in B(L^2(\Omega, \lambda)) \mid L(T) < \infty \}.
\]
It is easily seen that $E\Lambda(\Omega)$ is a $*$-subalgebra of $B(L^2(\Omega, \Omega))$. Even more, $E\Lambda(\Omega)$ is an involutive Banach algebra with respect to the norm $||T|| = ||T|| + L(T)$ and the involution is isometric $||T|| = ||T^*||$. The proof is along standard lines and will be left to the reader.

3.2. If $\Omega$ is specified and $w \in \mathbb{C}$, let $(e(w))(z) = \exp(i \text{Re}(z \overline{w}))$ and let $U(w) = M_{e(w)}$, which is a unitary operator on $L^2(\Omega, \lambda)$. Also, if $\Omega$ is bounded, the multiplication operators by the functions which at $x + iy$ equal $x + iy$, $x, y$ will be denoted by $Z, X, Y$.

**Proposition 3.3.** If $T \in B(L^2(\Omega, \lambda))$ and

$$L_1(T) = \sup \{|w|^{-1}||T, U(w)||_2 \mid w \in \mathbb{C}\setminus\{0\}\}$$

then we have $L_1(T) \leq L(T) \leq 2L_1(T)$ and $||T||_1 = ||T|| + L_1(T)$ is an equivalent Banach algebra norm on $E\Lambda(\Omega)$.

If $\Omega$ is bounded then we have

$$L(T) = ||T, Z||_2.$$  

**Proof.** We first establish the assertions of the proposition in case $T \in C_2$. Then $T$ is given by a kernel $K \in L^2(\Omega \times \Omega, \lambda \otimes \lambda)$ and the kernel of $[M_f, T]$ is $(f(s) - f(t))K(s, t)$. The supremum of $C_2$-norms of $[M_f, T]$ over all $f$ with $\|Df\|_\infty \leq 1$ will then equal the $L^2$-norm of $(s - t)K(s, t)$, which for bounded $\Omega$ is the kernel of $[Z, T]$. On the other hand, if $f = e(w)|w|^{-1}$ we have $\|Df\|_\infty \leq 1$, so that $L_1(T) \leq L(T)$. Further, taking $w = ew_0$, for some $w_0$ with $|w_0| = 1$ and letting $\varepsilon \downarrow 0$, the supremum of $L^2$-norms of the corresponding $(f(s) - f(t))K(s, t)$ will be the $L^2$-norm of $\text{Re}(s - t)\overline{w}_0)K(s, t)$. The bound $L(T) \leq 2L_1(T)$ is then obtained taking for instance $w_0 = 1$ and $w_0 = i$.

To deal with general $T$, we first take up the assertion that $L(T) = ||Z, T||_2$ when $\Omega$ is bounded. Clearly it suffices to show that $L(T) \leq ||Z, T||_2$ the opposite inequality being obvious. By our results in [25], since $Z$ is a normal operator, there are finite rank projections $P_n \uparrow I$ so that $|[Z, P_n, Z]|_2 \to 0$ as $n \to \infty$. Then if $f$ is such that $\|Df\|_\infty \leq 1$, using the result for the Hilbert–Schmidt case, we have

$$||M_f, T||_2 \leq \limsup_{n \to \infty} ||M_f, P_n TP_n||_2 \leq \limsup_{n \to \infty} ||Z, P_n TP_n||_2 \leq \limsup_{n \to \infty} 2||Z, P_n||_2 ||T|| + ||Z, T||_2 = ||Z, T||_2.$$
To prove the assertion about $L_1(T)$ for unbounded $\Omega$ and general $T$, we proceed along similar lines, after showing that there exist finite rank projections $P_n \uparrow 1$ so that

$$\lim_{n \to \infty} \left( \sup_{w \in \mathbb{C} \setminus \{0\}} |w^{-1} U(w), P_n|_2 \right) = 0.$$ 

Let $\Omega_m = \{ z \in \Omega \mid m - 1 \leq |z| < m \}$ so that $\Omega$ is the disjoint union of the $\Omega_m$, $m \in \mathbb{N}$. On $L^2(\Omega_m, \lambda)$ we can find, by our result from [25], finite rank projections $P_{km}$ so that $P_{km} \uparrow I$ as $k \to \infty$ and $||P_{km}, Z||_2 \leq (k^2 m)^{-1}$. Observe that by the result about $|[Z, T]|_2$ we proved, this gives $L(P_{km}) \leq (k^2 m)^{-1}$. We then define the projection $P_m$ acting on $L^2(\Omega, \lambda) = L^2(\Omega_1, \lambda) \oplus L^2(\Omega_2, \lambda) \oplus \ldots$ to be $P_{m1} \oplus P_{m2} \oplus \ldots \oplus P_{mm} \oplus 0 \oplus 0 \oplus \ldots$ so that $P_m \uparrow I$ and $L(P_m) \leq L(P_{m1}) + \ldots + L(P_{mm}) \leq C_m^{-1}$. Since $\|Dw^{-1} e(w)\|_\infty \leq 1$ we have $||w^{-1} U(w), P_m||_2 \leq C_m^{-1}$ which clearly converges to zero as $m \to \infty$ uniformly for $w \in \mathbb{C} \setminus \{0\}$. We then have for $f \in \Lambda(\Omega)$ with $\|Df\|_\infty \leq 1$ and $T \in \mathcal{B}(L^2(\Omega, \lambda))$

$$||Mf, T||_2 \leq \limsup_{n \to \infty} ||Mf, P_n T P_n||_2
\leq \limsup_{n \to \infty} 2L_1(P_n T P_n)
\leq \limsup_{n \to \infty} (4L_1(P_n) \|T\| + 2L_1(T))
= 2L_1(T).$$

\[ \square \]

3.4. If $\Omega = \mathbb{C}$ the proposition provides a characterization of the algebra $E\Lambda(\Omega)$ which translates well after Fourier transform. Let $\mathcal{F} : L^2(\mathbb{C}, \lambda) \to L^2(\mathbb{C}, \lambda)$ be the unitary Fourier transform

$$(\mathcal{F} f)(w) = c \int_{\mathbb{C}} f(z) (e(-w))(z) d\lambda(z).$$

Then $\mathcal{F} U(w_0) = V(w_0) \mathcal{F}$ where $(V(w_0)g)(w) = g(w - w_0)$ and we have the following corollary.

**Corollary 3.5.** If

$$S, T \in \mathcal{B}(L^2(\mathbb{C}, \lambda))$$

and

$$M(S) = \sup\{|w_0|^{-1} |S - V(w_0)SV(w_0)^*|_2 \mid w_0 \in \mathbb{C} \setminus \{0\}\}$$

then we have

$$M(\mathcal{F} T \mathcal{F}^{-1}) = L_1(T) \text{ and } \mathcal{F} E\Lambda(\mathbb{C}) \mathcal{F}^{-1} = \{ S \in \mathcal{B}(L^2(\mathbb{C}, \lambda)) | M(S) < \infty \}.$$
3.6. If $\Omega_1 \subset \Omega_2$ let

$$i(\Omega_2, \Omega_1) : B(L^2(\Omega_2, \lambda)) \to B(L^2(\Omega_2, \lambda))$$

be the inclusion homomorphism defined by $i(\Omega_2, \Omega_1)(T) = T \oplus 0$ with respect to the decomposition $L^2(\Omega_2, \lambda) = L^2(\Omega_1, \lambda) \oplus L^2(\Omega_2 \setminus \Omega_1, \lambda)$. There is also a conditional expectation $\varepsilon(\Omega_1, \Omega_2) : B(L^2(\Omega_2, \lambda)) \to B(L^2(\Omega_1, \lambda))$.

$\varepsilon(\Omega_1, \Omega_2)(S) = M_{\chi_{\Omega_2}} SM_{\chi_{\Omega_2}} | L^2(\Omega_1, \lambda)$ where $\chi_{\Omega_1}$ is the indicator function of the subset $\Omega_1$ of $\Omega_2$. It is easily checked that the Banach algebras $E\Lambda(\Omega)$ behave well with respect to the $i(\Omega_2, \Omega_1)$ and $\varepsilon(\Omega_2, \Omega_1)$.

**Proposition 3.7.** If $\Omega_1 \subset \Omega_2$ then we have

$$i(\Omega_2, \Omega_1)(E\Lambda(\Omega_1)) \subset E\Lambda(\Omega_2)$$

and the inclusion is isometric with respect to the $\| \cdot \|$-norms and also with respect to the $\| \|_{1}$-norms and $L(\cdot)$ and $L_{1}(\cdot)$ are preserved. We also have $\varepsilon(\Omega_1, \Omega_2)(E\Lambda(\Omega_2)) = E\Lambda(\Omega_1)$ and $\varepsilon(\Omega_1, \Omega_2)$ is contractive both in the $\| \cdot \|$-norms and in the $\| \|_{1}$-norms and we have $\varepsilon(\Omega_1, \Omega_2)(i(\Omega_2, \Omega_1)(T)) = T$.

3.8. We define the Banach subalgebra $E\Lambda(\Omega_0) \subset E\Lambda(\Omega)$ to be the closure in $E\Lambda(\Omega)$ of $\bigcup \{ i(\Omega, \Omega_1)E\Lambda(\Omega_1) \mid \Omega_1 \subset \Omega, \Omega_1$ bounded Borel set $\}$. Equivalently $E\Lambda(\Omega_0)$ is the closure in $E\Lambda(\Omega)$ of $\bigcup_{r>0} i(\Omega, \Omega \cap rD)E\Lambda(\Omega \cap rD)$ where $D$ is the unit disk.

**Proposition 3.9.** $E\Lambda(\Omega_0)$ is an ideal in $E\Lambda(\Omega)$. If $X_{\Omega \cap rD}$ is the indicator function of $nD \cap \Omega$ as a subset of $\Omega$ and $M_n = M_{X_{\Omega \cap rD}}$ then $(M_n)_{n \geq 1}$ is an approximate unit of $E\Lambda(\Omega_0)$.

**Proof.** Since $\| M_n \| = \| M_n \| = 1$ and $M_n x = xM_n = x$ for any $x \in \bigcup \{ i(\Omega, \Omega_1)E\Lambda(\Omega_1) \mid \Omega_1 \subset \Omega, \Omega_1$ bounded Borel $\}$ as soon as $n$ is large enough, we clearly have that $(M_n)_{n \geq 1}$ is an approximate unit of $E\Lambda(\Omega_0)$.

To prove that $E\Lambda(\Omega_0)$ is a two-sided ideal in $E\Lambda(\Omega)$ it will suffice now to show that $TM_n \in E\Lambda(\Omega_0)$ and $M_n T \in E\Lambda(\Omega_0)$. Actually since we deal with involutive algebras it will suffice to show that $TM_n \in E\Lambda(\Omega_0)$ and this in turn reduces to checking that $\|(I - M_n)TM_m\| \to 0$ as $m \to +\infty$. It is easily seen that $L(T) < \infty$ implies $(I - M_{n+1})TM_n \in C_2$ and hence $\|(I - M_m)TM_n\| \leq \|(I - M_m)(I - M_{n+1})TM_n\| \to 0$ as $m \to +\infty$. Also if $K(z_1, z_2)$ is the kernel of $(I - M_{n+1})TM_n$ then $L((I - M_{n+1})TM_n) < \infty$ means $(z_1 - z_2)K(z_1, z_2)$ is in $L^2(\Omega \times \Omega, \lambda \otimes \lambda)$. Then if $m > n + 1$, $L((I - M_m)TM_n)$ is the $L^2$-norm of the kernel

$$(1 - X_{\Omega \cap rD})(z_1 - z_2)K(z_1, z_2)$$

which converges to zero as $m \to +\infty$. □
3.10. If $\Omega$ is bounded $C_2(L^2(\Omega, \lambda)) \subset E \Lambda(\Omega)$ and $\|X\| \leq (1 + d)|X|_2$ where $d$ is the diameter of $\Omega$ when $X \in C_2(L^2(\Omega, \lambda))$. If $\Omega$ is unbounded the $C_2 \Lambda(\Omega) = C_2(L^2(\Omega, \lambda)) \cap E \Lambda(\Omega)$ is only a subset of $C_2(L^2(\Omega, \lambda))$. Similarly $R \Lambda(\Omega)$ will denote $R(L^2(\Omega, \lambda)) \cap E \Lambda(\Omega)$ where $R(\mathcal{H})$ stands for the finite rank operator on $\mathcal{H}$. Remark also that if $L^2 \Lambda(\Omega)$ denotes functions $f \in L^2(\Omega, \lambda)$ so that $f(z)(1 + |z|) \in L^2$ then the linear span of $(\cdot, f)g$ is in $R \Lambda(\Omega)$ when $f, g \in L^2 \Lambda(\Omega)$. Note also that if $f \in L^\infty(\Omega, \lambda)$ then $\|M_f\| = \|M_f\|_1 = \|f\|_\infty = \|M_f\|$ since $L(M_f) = 0$ and $ML^\infty(\Omega) = \{M_f : f \in L^\infty(\Omega, \lambda)\} \subset E \Lambda(\Omega)$.

The following lemma records a consequence of the diagonalizability mod $C_2$ of normal operators, which appeared in the last part of the proof of Proposition 3.3.

**Lemma 3.11.** In $E \Lambda(\Omega)$ there are finite rank projections $P_n$, so that $P_n \uparrow 1$ and

$$
\lim_{n \to \infty} L(P_n) = 0.
$$

Moreover we have $P_n \in i(\Omega, \Omega \cap n\mathbb{D}) E \Lambda(\Omega \cap n\mathbb{D})$ and $[P_n, M_{X_{\Omega \cap m\mathbb{D}}}] = 0$ for all $m \in \mathbb{N}$.

We will also find it useful to have the following technical lemma when $\Omega$ is unbounded.

**Lemma 3.12.** Let $M_n = M_{X_n} \in ML^\infty(\Omega, \lambda)$ where $X_n$ is the indicator function of $\Omega \cap n\mathbb{D}$ as a subset of $\Omega$ and let $T \in E \Lambda(\Omega)$. Then we have $L(T - M_n TM_n) \to 0$ as $n \to \infty$.

**Proof.** If $\Omega_n = \Omega \cap n\mathbb{D}$, then we have $M_n TM_n = i(\Omega, \Omega_n)\mathbb{E}(\Omega_n, \Omega)(T)$. With $T_n$ denoting $\mathbb{E}(\Omega_n, \Omega)(T)$ and $X_n$ denoting $i(\Omega, \Omega_n)([Z, T_n])$ we have the following martingale properties. If $m \geq n$ then $M_n X_m M_n = X_n$ and $|X_n|_2 = L(T_n) \leq L(T)$. Hence, if $X$ is a weak limit of some subsequence of the $X_m$’s as $m \to \infty$ we will have $|X|_2 < \infty$ and $X_n = M_n X M_n$. Thus if $m \geq n$

$$
L(M_m TM_m - M_n TM_n) = L(\mathbb{E}(\Omega_m, \Omega)(M_m TM_m - M_n TM_n))
$$

$$
= \|\mathbb{E}(\Omega_m, \Omega)(M_m TM_m - M_n TM_n)\|_2
$$

$$
= |X_m - X_n|_2.
$$

Since $M_m TM_m$ converges weakly to $T$ and $X_m$ converges in 2-norm to $X$ as $m \to \infty$, we infer

$$
L(T - M_n TM_n) \leq \sup_{m \geq n} L(M_m TM_m - M_n TM_n) = \sup_{m \geq n} |X_m - X_n|_2 = |X_m - X_n|_2.
$$

The assertion of the lemma follows from

$$
|X - X_n|_2 = |X - M_n XM_n|_2 \to 0
$$
as $n \to \infty$.  \qed
3.13. We define $K\Lambda(\Omega) = \{ T \in E\Lambda(\Omega) \mid T \text{ compact} \}$. Clearly, $K\Lambda(\Omega)$ is a closed ideal in $E\Lambda(\Omega)$.

**Proposition 3.14.** The ideal $K\Lambda(\Omega)$ of $E\Lambda(\Omega)$ has an approximate unit $(P_n)_{n \geq 1}$ where $P_n$’s are self-adjoint projections with the properties outlined in Lemma 3.11. In particular $\bigcup_{n \geq 1} P_n B(L^2(\Omega, \lambda)) P_n$ is a dense subalgebra in $K\Lambda(\Omega)$ in $\| \cdot \|$-norm.

*Proof.* If $T \in K\Lambda(\Omega)$ then with the notation in Lemma 3.12 we actually have $\| T - M_n T M_n \| \to 0$ as $n \to \infty$ in view of the lemma and of the compactness of $T$ which gives $\| T - M_n T M_n \| \to 0$. In view of the involution, the proof reduces to showing that $\| T - P_m T \| \to 0$ as $m \to \infty$ where $P_m$ are the projections in Lemma 3.11 and $T \in K\Lambda(\Omega)$ satisfies $T = M_n T M_n$ for some fixed $n$.

Clearly $T$ being compact we have $\| T - P_m T \| \to 0$ as $m \to \infty$.

On the other hand if $0 \leq n$, $T - P_m T = i(\Omega, \Omega \cap n\mathbb{D})(T' - P_m' T')$ where $T' = \varepsilon(\Omega \cap n\mathbb{D}, \Omega)(T)$ satisfies $i(\Omega, \Omega \cap n\mathbb{D})(T') = T$ and $P_m' = \varepsilon(\Omega \cap n\mathbb{D}, \Omega)(P_m)$ is a projection. We have

$$L(T - P_m T) = L(T' - P_m' T')$$

$$= [Z, (I - P_m')T']_2$$

$$\leq L(I - P_m')\|T\| + \|I - P_m'[Z, T']\|_2 \to 0$$

since $L(P_m') \leq L(P_m) \to 0$ and $[Z, T'] \in C_2$, $P_m' \uparrow I$.

The remaining assertion follows from the fact that $P_n$ is an approximate unit once we remark that $P_n B(L^2(\Omega, \lambda)) P_n = P_n E\Lambda(\Omega) P_n = P_n K\Lambda(\Omega) P_n$ because $P_n = M_n P_n M_n$.

**Proposition 3.15.** The unit ball of $E\Lambda(\Omega)$ in $\| \cdot \|$-norm or $\| \cdot \|_1$-norm is closed in the weak operator topology and hence is weakly compact. Moreover, $E\Lambda(\Omega)$ is inverse-closed as a subalgebra of $B(L^2(\Omega))$ and also closed under $C^\infty$-functional calculus for normal elements. In particular if $T \in E\Lambda(\Omega)$ has bounded inverse and $T = V|T|$ is its polar decomposition, then $V, |T|$ are in $E\Lambda(\Omega)$.

The proof is an exercise along standard lines and will be omitted.

3.16. We shall denote by $(E/K)\Lambda(\Omega)$ the quotient-Banach algebra $E\Lambda(\Omega)/K\Lambda(\Omega)$ and by $p : E\Lambda(\Omega) \to (E/K)\Lambda(\Omega)$ the canonical surjection.

Remark also that we have $K\Lambda(\Omega) \subset E\Lambda(\Omega)$ since the dense subalgebra of $K\Lambda(\Omega)$ appearing in Proposition 3.14 is in $E\Lambda(\Omega)_0$. The quotient $E\Lambda(\Omega)_0/K\Lambda(\Omega)$ will also be denoted $(E_0/K)\Lambda(\Omega)$.

**Proposition 3.17.** Given $n \in \mathbb{N}$ there are $U_k \in E\Lambda(\Omega)$, $1 \leq k \leq n$, such that $U : L^2(\Omega, \lambda) \to L^2(\Omega, \lambda) \otimes \mathbb{C}^n$ defined by $U h = \sum_k U_k h \otimes e_k$ is a unitary operator. In particular we have $UE\Lambda(\Omega)U^* = E\Lambda(\Omega) \otimes M_n$ and $T \to UTU^*$.
is a spatial isomorphism of $E\Lambda(\Omega)$ and $E\Lambda(\Omega) \otimes \mathcal{M}_n$. Additionally we also have that

$$UE\Lambda(\mathbb{C})_0 U^* = E\Lambda(\mathbb{C})_0 \otimes \mathcal{M}_n.$$  

Proof. The existence of $U$ is a consequence of our results on normal operator mod $C_2$ ([25], 2.8). There will be some additional technicalities due to the fact that $\Omega$ may be unbounded. From ([25], 2.8) we get the existence of unitary operators $V_m : L^2(\Omega_m, \lambda) \to L^2(\Omega_m, \lambda) \otimes \mathbb{C}^n$, so that $|V_m Z - (Z \otimes I_n)V_m|_2 < 2^{-m-1}$, where $\Omega_m = \Omega \cap ((m + 1)\mathbb{D}\setminus m\mathbb{D})$. If $V_m h = \sum_k V_m k h \otimes e_k$, we have $||V_m Z||_2 < 2^{-m-1}$ and hence $L(U_k) < 1$ where $U_k = \bigoplus_{m \geq 0} V_m k$, so that if $U h = \sum_k U_k h \otimes e_k$ we will have that $U$ is unitary and $U_k \in E\Lambda(\Omega)$, $1 \leq k \leq n$.

It follows that $UE\Lambda(\Omega) U^* \subset E\Lambda(\Omega) \otimes \mathcal{M}_n$ and $U^* (E\Lambda(\Omega) \otimes \mathcal{M}_n) U \subset E\Lambda(\Omega)$, which implies that $UE\Lambda(\Omega) U^* = E\Lambda(\Omega) \otimes \mathcal{M}_n$ and that $T \to UTU^*$ is a spatial isomorphism of $E\Lambda(\Omega)$ and $E\Lambda(\Omega) \otimes \mathcal{M}_n$.

For the last assertion to be proved, note that the operator $U$ which we constructed, satisfies

$$U(i(\Omega, \Omega \cap n\mathbb{D})) E\Lambda(\Omega \cap n\mathbb{D})) U^* = (i(\Omega, \Omega \cap n\mathbb{D}) E\Lambda(\Omega \cap n\mathbb{D})) \otimes \mathcal{M}_n.$$  

The assertion then follows from the density of $\bigcup_{n \geq 1} i(\Omega, \Omega \cap n\mathbb{D}) E\Lambda(\Omega \cap n\mathbb{D})$ in $E\Lambda(\Omega)_0$. \hfill \Box

3.18. Along similar lines with 3.17 one can show that $E\Lambda(\Omega)$ is a huge algebra. For instance, since $Z$ and $Z \otimes I_{n\mathbb{H}}$ are unitarily equivalent mod $C_2$ and since $I \otimes \mathcal{B}(\mathcal{H})$ is in the commutant of $Z \otimes I_{n\mathbb{H}}$, ($\mathcal{H}$ a separable Hilbert space), one infers that $E\Lambda(\Omega)$ contains a subalgebra spatially isomorphic to $I \otimes \mathcal{B}(\mathcal{H})$.

In the remainder of this section we exhibit a few special operators which are in $E\Lambda(\Omega)$.

**Proposition 3.19.** Let $\Omega$ be a bounded open set and let $A^2(\Omega)$ be the Bergman space of square-integrable analytic functions. Assume moreover that the rational functions with poles in $\mathbb{C}\setminus\overline{\Omega}$ are dense in $A^2(\Omega)$. Then we have $P_{\Omega} \in E\Lambda(\Omega)$, where $P_{\Omega}$ is the orthogonal projection of $L^2(\Omega, \lambda)$ onto the subspace $A^2(\Omega)$.

Proof. This is a consequence of the Berger–Shaw inequality (see for instance [21, p. 128, Theorem 1.3]). Indeed $T = Z \mid A^2(\Omega)$ is a subnormal operator and the constant function $1$ is a rationally cyclic vector for $T$. The Berger–Shaw inequality then gives $\text{Tr}[T^*, T] < \infty$. With the simplified notation $P = P_{\Omega}$, we have

$$\text{Tr}[PZ^* P, PZP] < \infty.$$
Since \((I - P)ZP = 0\) and \([Z^*, Z] = 0\) this gives

\[ [PZ^*P, PZP] = PZ(I - P)Z^*P \]

and hence

\[ [P, Z] = PZ(I - P) \in \mathcal{C}_2. \]

\(\Box\)

### 3.20. The Hilbert-transform singular integral operator on \(\mathbb{C} (\mathbb{D}, \mathbb{D})\)

\[
Hf(\xi) = \lim_{\varepsilon \downarrow 0} \int_{|z-\xi|>\varepsilon} \frac{f(z)}{(\xi - z)^2} d\lambda(z)
\]

is a bounded operator on \(L^2(\mathbb{C}, \lambda)\) and hence also its compression \(H_{\Omega}\) to \(L^2(\Omega, d\lambda)\), where \(\Omega\) is bounded, is a bounded operator. Then also \(T_{\Omega} = [Z, H_{\Omega}]\) is a bounded operator and

\[
T_{\Omega}(f)(z) = \lim_{\varepsilon \downarrow 0} \int_{|z-\xi|>\varepsilon} \frac{f(z)}{\xi - z} d\lambda(z).
\]

We have \([Z, T_{\Omega}] = \langle \cdot, 1 \rangle 1\) where 1 denotes the constant function equal to 1. Since \([Z, T_{\Omega}]\) is rank one, we have \(T_{\Omega} \in E\Lambda(\Omega)\). Since \(z^{-1}\) is not in \(L^2(\mathbb{D}, \lambda)\), \(T_{\Omega}\) is not in \(\mathcal{C}_2\). It can be shown that \(T_{\Omega} \in \mathcal{C}_2^+\) (the ideal of compact operators with singular numbers \(s_n = O(n^{-1/2})\)). Also clearly the linear span of operators of the form \(M_f T_{\Omega} M_g\) gives operators \(K\) in \(E\Lambda(\Omega)\) which are in \(\mathcal{C}_2^+\) and the commutators of which \([Z, K]\) are dense in \(\mathcal{C}_2(L^2(\Omega, \lambda))\).

### 4. About the \(K\)-theory of \(E\Lambda(\Omega)\)

#### 4.1. Passing via almost normal operators, the Pincus \(g\)-function gives a homomorphism of the \(K_0\)-group of \(E\Lambda(\Omega)\) to \(L^1\)-functions. We shall prove that the Conjecture 3 about almost normal operators (see 2.5) implies that this homomorphism completely determines the group \(K_0(E\Lambda(\mathbb{C})_0)\). Conversely, assuming Conjecture 4, we will show that such a result about the \(K\)-theory of \(E\Lambda(\mathbb{C})_0\) implies Conjecture 3.

We begin with some technical facts.

**Lemma 4.2.** If \(F = F^2 \in E\Lambda(\Omega)\) and \(P = P^* = P^2 \in B(L^2(\Omega, \lambda))\) is the orthogonal projection onto \(F(L^2(\Omega, \lambda))\) then \(P \in E\Lambda(\Omega)\) and \(P\) and \(F\) have the same class in \(K_0\).

**Proof.** The orthogonal projection \(P\) is equal to \(\psi(FF^*)\) for some \(C^\infty\)-function \(\psi\). Hence \(P \in E\Lambda(\mathbb{C})\) is a consequence of Proposition 3.15 and \(tP + (1-t)F, t \in [0, 1]\) is a continuous path of projections, so \([P]_0 = [F]_0\). \(\Box\)
Lemma 4.3. Let $P \in E\Lambda(\Omega)$ be a self-adjoint projection, which is not finite rank and assume $\Omega$ is bounded. Then we have

$$PZP \in \mathcal{AN}(L^2(\Omega, \lambda)).$$

Proof. We have

$$[PZ^*P, PZP] = PZ(I - P)Z^*P - PZ^*(I - P)ZP \in C_1$$

since $(I - P)ZP = (I - P)[Z, P] \in C_2$ and $PZ(I - P) = [P, Z](I - P) \in C_2$. \hfill \Box

Proposition 4.4. Assume $\Omega$ is bounded. For every $\alpha \in K_0(E\Lambda(\Omega))$ there is a self-adjoint projection $P \in E\Lambda(\Omega)$, not of finite rank, so that $[P]_0 = \alpha$. The Pincus $g$-function $g_{PZP}$ depends only on $\alpha$ (i.e., not on the choice of $P$). Moreover, the map $K_0 \to L^1(\mathbb{C}, \lambda)$ which associates to a class $\alpha$ the $L^1$-function $g_{PZP}$ is a homomorphism.

Proof. The existence of unitary “Cuntz $n$-tuples” $U_1, \ldots, U_n$ in $E\Lambda(\Omega)$, which was shown in Proposition 3.17, implies that $[I]_0 = 0$ and that for a projection $Q \in M_n(E\Lambda(\Omega))$ there is a projection $P \in E\Lambda(\Omega)$ with $[P]_0 = [Q]_0$ so that $-[Q]_0 = [I - P]_0$. Hence $K_0(E\Lambda(\Omega))$ consists of classes of idempotents in $E\Lambda(\Omega)$ and these can be chosen to be self-adjoint by Lemma 4.2.

Again using Proposition 3.15 and Proposition 3.17 the fact that the map $\alpha \to g_{PZP}$ is a well-defined homomorphism is a consequence of the following two facts: a) if $P \in E\Lambda(\Omega)$ is a self-adjoint projection and $W \in E\Lambda(\Omega)$ is unitary, then $g_{(WPW^+ZWPW^+)} = g_{PZP}$ and b) if $P_1, P_2 \in E\Lambda(\Omega)$ are self-adjoint projections and $P_1P_2 = 0$, then $g_{P_1ZP_1} + g_{P_2ZP_2} = g_{(P_1 + P_2)Z(P_1 + P_2)}$.

To show that a) holds, remark that $g_{WPW^+ZWPW^+} = g_{PZP}$ by unitary equivalence and $PW^*ZWP - PZP \in C_2$. Moreover, in view of the argument in 2.7 we have $k_2(PZP) = 0$, $k_2(PW^*ZWP) = 0$ and we can then use 2.4 to get that $g_{PZP} = g_{WPW^+ZWP}$.

Assertion b) is proved by the same kind of combination of facts. By the argument of 2.7, we have

$$k_2(P_1ZP_1) = k_2(P_2ZP_2) = k_2((P_1 + P_2)Z(P_1 + P_2)) = 0.$$

We then remark that

$$P_1ZP_1 + P_2ZP_2 - (P_1 + P_2)Z(P_1 + P_2) \in C_2$$

and we can then use 2.4 to get

$$g_{(P_1 + P_2)Z(P_1 + P_2)} = g_{P_1ZP_1 + P_2ZP_2} = g_{P_1ZP_1} + g_{P_2ZP_2},$$

where we used the fact that

$$k_2(P_1ZP_1 + P_2ZP_2) = k_2(P_1ZP_1 \oplus P_2ZP_2) = 0.$$

\hfill \Box
4.5. The homomorphism \( K_0(E\Lambda(\Omega)) \to L^1_{rc}(\mathbb{C}, \lambda) \), constructed in Proposition 4.4, will be denoted by \( \Gamma(\Omega) \) or simply \( \Gamma \), when the bounded set \( \Omega \) is not in doubt (\( L^1_{rc}(\mathbb{C}, \lambda) \) being the \( L^1 \)-space of real-valued functions with compact support).

We shall also denote by \( \mathcal{A}\mathcal{N}\mathcal{D}(\mathcal{H}) \) the almost normal operators for which Conjecture 4 (see 2.5) holds. We shall call such almost-normal operators dilatable.

It is easily seen that this is equivalent to the fact that the almost-normal operator is a Hilbert–Schmidt perturbation of an almost-normal operator which is a compression \( PNP \) of a normal operator \( N \) by a projection \( P \) so that \( [P, N] \in \mathbb{C}_2 \).

In 2.7 we showed that if \( T \in \mathcal{A}\mathcal{N}\mathcal{D}(\mathcal{H}) \) then \( k_2(T) = 0 \). Next we will give a few simple facts about \( K \)-theory for some of the algebras related to \( E\Lambda(\Omega) \) and get some variants of the homomorphism \( \Gamma \).

4.6. If \( \Omega_1 \subset \Omega_2 \) are bounded Borel sets, then it is immediate from the construction of \( \Gamma \) that

\[
\Gamma(\Omega_2) \circ (i(\Omega_2; \Omega_1))_* = \Gamma(\Omega_1).
\]

In view of 3.8, \( E\Lambda(\mathbb{C})_0 \) is the inductive limit of the \( E\Lambda(\Omega) \) with \( \Omega \) bounded (the inclusion will be denoted \( i_0(\mathbb{C}, \Omega) \)). Then \( K_0(E\Lambda(\mathbb{C})_0) \) is the inductive limit of the \( K_0(E\Lambda(\Omega)) \), with bounded \( \Omega \), and there is a homomorphism

\[
\Gamma_\infty : K_0(E\Lambda(\mathbb{C})_0) \to L^1_{rc}(\mathbb{C}, \lambda)
\]

so that

\[
\Gamma_\infty \circ (i_0(\mathbb{C}, \Omega))_* = \Gamma(\Omega).
\]

**Lemma 4.7.** We have \( K_0(\mathcal{K}\Lambda(\Omega)) \cong \mathbb{Z}, K_1(\mathcal{K}\Lambda(\Omega)) = 0 \), for any \( \Omega \) (not of measure 0), the isomorphism for \( K_0 \) being given by the trace on \( B(L^2(\Omega, \lambda)) \). Moreover we have isomorphisms

\[
K_0(E\Lambda(\Omega)) \xrightarrow{p_*} K_0((E/\mathcal{K})\Lambda(\Omega))
\]

\[
K_0(E\Lambda(\mathbb{C})_0) \xrightarrow{p_*} K_0((E_0/\mathcal{K})\Lambda(\mathbb{C})).
\]

**Proof.** The assertions about the \( K \)-theory of \( \mathcal{K}\Lambda(\Omega) \) are a consequence of the last assertion in Proposition 3.14.

To get the isomorphisms between \( K_0 \)-groups of \( E\Lambda(\Omega) \) and \( (E/\mathcal{K})\Lambda(\Omega) \) and respectively \( E\Lambda(\mathbb{C})_0 \) and \( (E_0/\mathcal{K})\Lambda(\mathbb{C}) \) we use the 6-term \( K \)-theory exact sequences associated with

\[
0 \to \mathcal{K}\Lambda(\Omega) \to E\Lambda(\Omega) \to (E/\mathcal{K})\Lambda(\Omega) \to 0
\]

and

\[
0 \to \mathcal{K}\Lambda(\mathbb{C}) \to E\Lambda(\mathbb{C})_0 \to (E_0/\mathcal{K})\Lambda(\mathbb{C}) \to 0.
\]

Since \( K_1(\mathcal{K}\Lambda(\Omega)) = 0 \) we have that the homomorphisms \( p_* \) are surjective. The injectivity of the \( p_* \) means to show the connecting homomorphisms \( K^1 \to K^0 \).
are surjective. This is easily seen to be the case if we can prove \( E \Lambda (\Omega) \) and \( E \Lambda (\mathbb{C})_0 + \mathbb{C} I \) contain a Fredholm operator of index 1. If \( \Omega \) is bounded, there is a Fredholm operator of index 1, \( T \in B(L^2(\Omega, \lambda)) \) so that \([T, Z] \in C_2\). This in turn follows from the easily seen fact that \( Z \) is unitarily equivalent to \( Z \oplus \mu I_{n_0} + K \), where \( H \) is some infinite-dimensional Hilbert space, \( \mu \in \sigma(Z) \) and \( K \in C_2 \). For \( E \Lambda (\mathbb{C})_0 + \mathbb{C} I \) we can use the Fredholm operator \( T \in E \Lambda (\Omega) \) and consider \( T \oplus I_{L^2(\mathbb{C} \setminus \Omega, \lambda)} \in E \Lambda (\mathbb{C})_0 + \mathbb{C} I \).

4.8. In view of Lemma 4.7 we infer for bounded \( \Omega \) the existence of homomorphisms

\[
\tilde{\Gamma}(\Omega) : K_0((E/K) \Lambda(\Omega)) \to L^1_{rc}(\Omega, \lambda)
\]

and

\[
\tilde{\Gamma}_\infty : K_0((E_0/K) \Lambda(\mathbb{C})) \to L^1_{rc}(\mathbb{C}, \lambda)
\]

so that

\[
\tilde{\Gamma}(\Omega) \circ p_* = \Gamma(\Omega) \text{ and } \tilde{\Gamma}_\infty \circ p_* = \Gamma_\infty.
\]

**Fact 4.9.** The following assertions are equivalent.

(i) Conjecture 3 is true.

(ii) Conjecture 4 is true and \( \Gamma_\infty \) is an isomorphism.

(iii) Conjecture 4 is true and \( \Gamma_\infty \) is injective.

**Proof.** Since (ii) \( \Rightarrow \) (iii) it will be sufficient to show that (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (ii). Remark first that Conjecture 3 implies Conjecture 4. Indeed, if \( T \in \mathcal{AN}(\mathcal{H}) \) we can find \( S_1 \in \mathcal{AN}(\mathcal{H}) \) so that \( gS_1 = -gT \) (see [21] for instance). Then Conjecture 3 implies that there is a normal operator \( N_1 \) so that \( T \oplus S_1 \oplus N_1 - N \in C_2 \) where \( N \) is a normal operator. Thus we can take \( S = S_1 \oplus N_1 \) and then \( S \in \mathcal{AN} \) and \( T \oplus S \) is equal \( N \mod C_2 \), which is the assertion of Conjecture 4 for \( T \).

To show \( \Gamma_\infty \) is surjective consider \( g \in L^1_{rc}(\mathbb{C}, \lambda) \). By the work of Carey–Pincus there is \( T \in \mathcal{AN}(\mathcal{H}_1) \) so that \( gT = g \). By Conjecture 4 and the fact that it implies Conjecture 1 we see that \( T \) can be chosen to be \( QN \mid QH \) where \( N \) is a normal operator and \( Q \) an orthogonal projection, so that \([Q, N] \in C_2\). We may also assume \( \sigma(N) = n \mathbb{D} \) for some \( n \in \mathbb{N} \). Then by our results on normal operators \( \mod C_2 \), there is a unitary operator \( U : \mathcal{H} \to L^2(n \mathbb{D}, \lambda) \) so that \( ZU - UN \in C_2 \). Then taking \( P = UQU^* \), we have \( PZP - UQNQU^* \in C_2 \) and hence \( g_{PZP} = g_{QNO} = g \) so that \( \Gamma(n \mathbb{D})[P]_0 = g \). Clearly then \( \Gamma_\infty([i_0(\mathbb{C}, n \mathbb{D})(P)]_0) = g \).
To prove that assuming Conjecture 3 holds, $\Gamma_\infty$ is injective, let $\alpha \in K_0(E\Lambda(\mathbb{C})_0)$ be so that $\Gamma_\infty(\alpha) = 0$. Using 4.6 and Proposition 4.4 there is a self-adjoint projection $P \in E\Lambda(n\mathbb{D})$ for some $n \in \mathbb{N}$, so that $(i_0(\mathbb{C}, n\mathbb{D}))_0[P]_0 = \alpha$ and $\Gamma(n\mathbb{D})([P]_0) = \Gamma_\infty(\alpha) = 0$. Hence $g_{PZP} = 0$. Then Conjecture 3 gives that there is $m \geq n$ and there are normal operators $N$ and $N_1$ with $\sigma(N) = \sigma(N_1) = m\mathbb{D}$ so that

$$N - PZ \mid PL^2(n\mathbb{D}, \lambda) \oplus N_1 \in C_2.$$ 

Since we will use the operators $Z$ in $E\Lambda(n\mathbb{D})$ and $E\Lambda(m\mathbb{D})$ simultaneously, we shall denote them here by $Z_n$ and $Z_m$. Clearly, we may use a unitary equivalence and a $C_2$-perturbation to choose $N_1$. Similarly $N$ can be chosen unitarily equivalent to $Z_m$. Thus, we get a unitary operator

$$U : PL^2(n\mathbb{D}, \lambda) \oplus L^2(m\mathbb{D}, \lambda) \to L^2(m\mathbb{D}, \lambda)$$

so that $Z_mU - U(PZ_n | PL^2(n\mathbb{D}, \lambda) \oplus Z_m) \in C_2$. This means that $U$ gives rise to a partial isometry $W \in B(L^2(m\mathbb{D}, \lambda) \oplus L^2(m\mathbb{D}, \lambda))$ so that $W^*W = i(m\mathbb{D}, n\mathbb{D})(P) \oplus I$ and $WW^* = 0 \oplus I$ with the property that $[W, Z_m \oplus Z_m] \in C_2$. Then we have $W \in \mathcal{M}_2(E\Lambda(m\mathbb{D}))$. This gives $i(m\mathbb{D}, n\mathbb{D})_0[P]_0 + [I]_0 = [I]_0$ in $K_0(E\Lambda(m\mathbb{D}), \lambda)$, so that $[i(m\mathbb{D}, n\mathbb{D})(P)]_0 = 0$. But then we must have $\alpha = [i_0(\mathbb{C}, n\mathbb{D})(P)]_0 = i_0(\mathbb{C}, n\mathbb{D})_0[i(m\mathbb{D}, n\mathbb{D})(P)]_0 = 0$.

(iii) $\Rightarrow$ (i). Assume (iii) holds and let $T_1, T_2 \in \mathcal{A}N(\mathcal{H})$ with $gT_1 = gT_2$. Since Conjecture 4 is part of the assumption (iii) we have $T_1, T_2 \in \mathcal{A}N(\mathcal{H})$. This implies there are self-adjoint projection $P_1, P_2 \in E\Lambda(n\mathbb{D})$ for some $n \in \mathbb{N}$, so that $T_j$ is unitarily equivalent to a $C_2$-perturbation of $P_jZ \mid P_jL^2(n\mathbb{D}, \lambda)$, $j = 1, 2$. Moreover, we have $\Gamma(n\mathbb{D})[P_1]_0 = \Gamma(n\mathbb{D})[P_2]_0$ because $gT_1 = gT_2$. It follows that $\Gamma_\infty([i_0(\mathbb{C}, n\mathbb{D})(P_1)]_0) = \Gamma_\infty([i_0(\mathbb{C}, n\mathbb{D})(P_2)]_0)$ so that by (iii) we have $i_0(\mathbb{C}, n\mathbb{D})_0[P_1]_0 = i_0(\mathbb{C}, n\math{D})_0[P_2]_0$. Since $E\Lambda(\mathbb{C})_0$ is the inductive limit of the $E\Lambda(m\mathbb{D})$ we infer that $[i(m\mathbb{D}, n\mathbb{D})(P_1)]_0 = [i(m\mathbb{D}, n\mathbb{D})(P_2)]_0$ for some $m \geq n$. Hence there is a unitary equivalence in $\mathcal{M}_{p+q+1}(E\Lambda(n\mathbb{D}))$ between the $Q_j = i(m\mathbb{D}, n\mathbb{D})P_j \oplus I \oplus \cdots \oplus I \oplus 0 \oplus \cdots \oplus 0$, $j = 1, 2$ (there are $p$ summands $I$ and $q$ summands $0$). Indeed the equality of $K_0$-classes implies there is an invertible element intertwining $Q_1, Q_2$ and using Proposition 3.17 and Proposition 3.15 we can pass to the unitary in the polar decomposition of this invertible element of $\mathcal{M}_{p+q+1}(E\Lambda(n\mathbb{D}))$. This unitary will then commute with $Z \oplus \cdots \oplus Z$ modulo $C_2$ and hence will intertwine mod $C_2$ the compressions $Q_j(Z \oplus \cdots \oplus Z)Q_j$, $j = 1, 2$. These compressions are unitarily equivalent to

$$P_jZ \mid P_jL^2(n\mathbb{D}, \lambda) \oplus N_j$$

for some normal operators $N_j$, $j = 1, 2$. Thus $T_j \oplus N_j$, being unitarily equivalent mod $C_2$ to these compressions, will also be unitarily equivalent mod $C_2$, which proves (i) under the assumption (iii).
4.10. In view of Lemma 4.7 and of 4.8 we have that Fact 4.9 also holds with $\Gamma_\infty$ replaced by $\Gamma_\infty$.

5. Multipliers, Corona and Bidual of $KL(\Omega)$

5.1. We shall consider bounded multipliers $M(KL(\Omega))$, that is double centralizer pairs $(T', T'')$ of bounded linear maps $KL(\Omega) \to KL(\Omega)$ so that $T'(x)y = xT''(y)$.

**Proposition 5.2.** We have $M(KL(\Omega)) = E\Lambda(\Omega)$, that is, if $(T', T'') \in M(KL(\Omega))$, then there is $T \in E\Lambda(\Omega)$ so that $T'(x) = xT$ and $T''(x) = Tx$.

**Proof.** Let $(P_n)_{n \geq 1}$ be the approximate unit provided by Proposition 3.14 and define $K_n = T'(P_n)P_n = P_nT''(P_n)$. Clearly, the norms $\|K_n\|$ will be bounded by some constant $C$ and if $m > n$ we have

$$P_nK_mP_n = P_nT'(P_m)P_mP_n = P_nT'(P_m)P_n = P_nP_mT''(P_n) = P_nT''(P_n) = K_n.$$ 

Hence the weak limit $T$ of the $K_n$’s exists, and we shall have $P_nTP_n = K_n$. Also $L(T) \leq \sup_n (L(K_n) + 2\|T\|L(P_n)) < \infty$, so that $T \in E\Lambda(\Omega)$. Moreover, we have

$$T'(P_n) = w - \lim_{m \to \infty} T'(P_m)P_m$$

and similarly $T''(P_n) = TP_n$. This gives $P_nT''(x) = T'(P_n)x = P_nTx$ if $x \in KL(\Omega)$ and hence

$$T''(x) = \lim_{n \to \infty} P_nT''(x) = \lim_{n \to \infty} P_nTx = Tx.$$ 

Similarly $T'(x)P_n = xTP_m$ and $T(x) = \lim_{m \to \infty} T'(x)P_n = xT$. \qed

**Proposition 5.3.** The involutive Banach algebra $(E/\mathcal{K})\Lambda(\Omega)$ is a $\mathbb{C}^*$-algebra. Actually if $x \in E\Lambda(\Omega)$ the norm of $p(x)$ in $(E/\mathcal{K})\Lambda(\Omega)$ is equal to the norm of $x + \mathcal{K}$ in the Calkin algebra $\mathcal{B}/\mathcal{K}$. In particular $(E/\mathcal{K})\Lambda(\Omega)$ is isometrically isomorphic to a $\mathbb{C}^*$-subalgebra of $\mathcal{B}/\mathcal{K}$.

**Proof.** It is easily seen that all assertions follow from the equality of the norm of $p(x)$ with the norm of $x + \mathcal{K}$ in the Calkin algebra. This in turn will follow from the fact that with $(P_n)_{n \geq 1}$ denoting the approximate unit of $KL(\Omega)$ in Proposition 3.14

$$\lim_{n \to \infty} \|[I - P_n]x(I - P_n)\|$$
equals the Calkin norm of $x + \mathcal{K}$, if we will also show that
\[ \lim_{n \to \infty} L((I - P_n)x(I - P_n)) = 0. \]

In case $\Omega$ is bounded we indeed have
\[ L((I - P_n)x(I - P_n)) \leq \lim_{n \to \infty} (2\|x\| \|[I - P_n, z]\|_2 + |(I - P_n)z|((I - P_n)z)|_2 = 0. \]

In case $\Omega$ is unbounded we use Lemma 3.12 and write $x = x_0 + x_1$ where $x_0 = M_n x M_n$ with $m$ chosen so that $L(x_1) < \varepsilon$. We have
\[ \limsup_{n \to \infty} L((I - P_n)x_1(I - P_n)) \leq L(x_1) < \varepsilon \]
and since $\varepsilon > 0$ can be chosen arbitrarily small it will suffice to show that
\[ \limsup_{n \to \infty} L((I - P_n)x_0(I - P_n)) = 0. \]

This in turn can be seen as follows. Let $Z_k$ be the multiplication operator by $z(1 \wedge k|z|^{-1})$. Then for any $y \in E\Lambda(\Omega)$ we have
\[ L(y) = \limsup_{k \to \infty} \|[Z_k, y]\|_2. \]
Moreover if $k \geq m$, $[Z_k, x_0] = [Z_m, x_0]$. Hence
\[ L((I - P_n)x_0(I - P_n)) \leq 2\|x_0\|L((I - P_n) + |(I - P_n)z_m x_0(I - P_n)|_2 \to 0 \]
as $n \to \infty$. \hfill \Box

**Remark 5.4.** The $C^*$-algebra
\[ \{ p(M_f) \in (E/K)\Lambda(\Omega) \mid f \in C\Lambda(\Omega) \}, \]
where $C\Lambda(\Omega)$ denotes the norm closure of $\Lambda(\Omega)$ in $L^\infty(\Omega, \lambda)$, is in the center of $(E/K)\Lambda(\Omega)$.

Indeed, if $f \in \Lambda(\Omega)$ then $[M_f, x] \in C_2 \Lambda \subset K\Lambda(\Omega)$ if $x \in E\Lambda(\Omega)$ so that $p(M_f)$ is in the center of $(E/K)\Lambda(\Omega)$. Since $\|[M_f]\| = \|M_f\| = \|f\|_\infty$ if $f \in L^\infty(\Omega)$ and the center is clearly norm-closed in $(E/K)\Lambda(\Omega)$, the assertion follows.

**5.5.** We pass to describing the dual of $K\Lambda(\Omega)$ for bounded $\Omega$. Throughout $C_1$ and $C_2$ will stand for $C_1(L^2(\Omega, \lambda))$ and respectively $C_2(L^2(\Omega, \lambda))$.

**Proposition 5.6.** Assuming $\Omega$ is bounded, the dual of $K\Lambda(\Omega)$ can be identified isometrically with $(C_1 \times C_2) / \mathcal{N}$ where
\[ \mathcal{N} = \{ ([Z, H], H) \in C_1 \times C_2 \mid H \in C_2 \text{ with } [Z, H] \in C_1 \} \]
and the duality map $K\Lambda(\Omega) \times (C_1 \times C_2) \to \mathbb{C}$ is $(T, (x, y)) = \text{Tr}(Tx + [Z, T]y)$. 

Proof. Since \( T \rightarrow T \oplus [Z, T] \) identifies \( \mathcal{K}\Lambda(\Omega) \) isometrically with a closed subspace of \( \mathcal{K} \oplus \mathcal{C}_2 \) endowed with the norm \( \|K \oplus H\| = \|K\| + |H|_2 \), the dual of which is \( \mathcal{C}_1 \times \mathcal{C}_2 \), the proof will boil down to showing that \( \mathcal{N} \) is the annihilator of
\[
\{T \oplus [Z, T] \in \mathcal{K} \oplus \mathcal{C}_2 \mid T \in \mathcal{K}\Lambda(\Omega)\}.
\]
Since the set \( \mathcal{R} \) of finite rank operators is dense in \( \mathcal{K}\Lambda(\Omega) \), it will be sufficient to show that \( \mathcal{N} \) is the annihilator of
\[
\{R \oplus [Z, R] \in \mathcal{K} \oplus \mathcal{C}_2 \mid R \in \mathcal{R}\}.
\]
If \( R \in \mathcal{R} \) and \( (x, y) \in \mathcal{N} \) we have
\[
\text{Tr}(Rx + [Z, R]y) = \text{Tr}(R[Z, y] + [Z, R]y) = \text{Tr}([Z, Ry]) = 0.
\]
Conversely if \( (x, y) \in \mathcal{C}_1 \times \mathcal{C}_2 \) is such that
\[
\text{Tr}(Rx + [Z, R]y) = 0 \text{ for all } R \in \mathcal{R},
\]
then
\[
\text{Tr}(R(x - [Z, y])) = 0 \text{ for all } R \in \mathcal{R}
\]
and hence \( x = [Z, y] \), that is \( (x, y) \in \mathcal{N} \).

Lemma 5.7. Under the same assumptions and notations like in 5.6,
\[
\{([Z, R], R) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid R \in \mathcal{R}\}
\]
is dense in \( \mathcal{N} \).

Proof. Let \( (x, y) \in \mathcal{N} \), that is \( y \in \mathcal{C}_2 \) is such that \( x = [Z, y] \in \mathcal{C}_1 \). Let \( (P_n)_{n \geq 1} \) be self-adjoint projections of finite rank so that \( P_n \uparrow I \) and \( \|[P_n, Z]\|_2 \rightarrow 0 \). Then we have \( |yP_n - y|_2 \rightarrow 0 \) and also
\[
\|[Z, yP_n] - [Z, y]\|_1 = \|[Z, y]P_n + y[Z, P_n] - [Z, y]\|_1
\leq |y|_2||Z, P_n||_2 + ||Z, y||(I - P_n)||_1 \rightarrow 0
\]
as \( n \rightarrow \infty \).

Proposition 5.8. If \( \Omega \) is bounded, with the same notations as in Proposition 5.6, the dual of \( (\mathcal{C}_1 \times \mathcal{C}_2)/\mathcal{N} \) identifies with \( \mathcal{E}\Lambda(\Omega) \) via the duality map
\[
(T, (x, y)) \rightarrow \text{Tr}(Tx + [Z, T]y).
\]
In particular \( \mathcal{E}\Lambda(\Omega) \) identifies with the bidual of \( \mathcal{K}\Lambda(\Omega) \).
Proof. The dual of $(C_1 \times C_2)/\mathcal{N}$ is the orthogonal of $\mathcal{N}$ in $B \oplus C_2 = (C_1 \times C_2)^d$ (the usual duality based on the trace). Since Lemma 5.7 provides a dense subset of $\mathcal{N}$, it suffices to show that $\{T \oplus [Z, T] \in B \oplus C_2 \mid T \in E \Lambda(\Omega)\}$ is the orthogonal in $B \oplus C_2$ of $\{(Z, R), R) \in C_1 \times C_2 \mid R \in \mathcal{R}\}$. Indeed, if $T \oplus H \in B \oplus C_2$ is such that $\text{Tr}(T[Z, R] + HR) = 0$ for all $R \in \mathcal{R}$, then $\text{Tr}((-Z, T) + H)R) = 0$ for all $R \in \mathcal{R}$ and hence $H = [Z, T]$, which also implies $T \in E \Lambda(\Omega)$. Clearly, also if $T \in E \Lambda(\Omega)$ and $R \in \mathcal{R}$ we have

$$\text{Tr}(T[Z, R] + [Z, T]R) = \text{Tr}([-Z, T]R)) = 0.$$

\[Q.E.D.\]

6. Concluding Remarks

6.1. Isomorphisms induced by bi-Lipschitz map. Let $\Omega_1$ and $\Omega_2$ be Borel subsets of $\mathbb{C}$ and let $F : \Omega_1 \to \Omega_2$ be a map which is Lipschitz and has an inverse which is also Lipschitz (i.e., $F$ is bi-Lipschitz). Then if $\lambda_j$ is the restriction of Lebesgue measure to $\Omega_j$, the measures $F_*\lambda_1$ are $\lambda_2$ are mutually absolutely continuous with bounded Radon–Nikodym derivatives and the same holds for $(F^{-1})_*\lambda_2$ and $\lambda_1$ ([17]). This gives rise to a unitary operator

$$U(\Omega_2, \Omega_1)L^2(\Omega_1, \lambda_1) \to L^2(\Omega_2, \lambda_1)$$

which maps $f \in L^2(\Omega_1, \lambda_1)$ to $(f \circ F^{-1}) \cdot (dF_*\lambda_1/d\lambda_2)^{1/2}$. If $g \in L^\infty(\Omega_2, \lambda_2)$ then

$$U(\Omega_2, \Omega_1)^{-1} M_g U(\Omega_2, \Omega_1) = M_{g \circ F}.$$

The map $g \to g \circ F$ gives isomorphisms of $L^\infty(\Omega_2, \lambda_2)$ with $L^\infty(\Omega_1, \lambda_1)$ and of $\Lambda(\Omega_2)$ with $\Lambda(\Omega_1)$. Further $T \to U(\Omega_2, \Omega_1)^{-1}TU(\Omega_2, \Omega_1)$ is an isomorphism of $E \Lambda(\Omega_2)$ and $E \Lambda(\Omega_1)$. This is an isomorphism of Banach algebras with involution, which however is not isometric, since its norm depends on the Lipschitz constants of $F$ and $F^{-1}$. These isomorphisms preserve finite-rank operators and hence $\mathcal{K}(\Omega_2)$ is mapped onto $\mathcal{K}(\Omega_1)$. This in turn implies there is an induced $C^*$-algebra isomorphism of $(E/\mathcal{K})\Lambda(\Omega_2)$ with $(E/\mathcal{K})\Lambda(\Omega_1)$.

In particular the group of bi-Lipschitz homeomorphisms of a Borel set $\Omega$ has automorphic actions on $E \Lambda(\Omega)$ and $(E/\mathcal{K})\Lambda(\Omega)$.

6.2. In view of 5.4 it is a natural question to ask, what is the center of $(E/\mathcal{K})\Lambda(\Omega)$? This question which appeared in the preprint version of this paper has now been answered in [4]. In the case of bounded $\Omega$ the center is the $C^*$-algebra generated by $p(Z)$. Note that the answer to the Calkin-algebra analogue of this question, that is the determination of the center of the commutant of a separable commutative $C^*$-subalgebra of the Calkin algebra, is a particular case of our Calkin algebra bicommutant theorem ([24]).
6.3. $\mathcal{K}\Lambda(\Omega)$ as a Dirichlet algebra. The algebras $\mathcal{K}\Lambda(\Omega)$ are examples of Dirichlet algebras in the sense of non-commutative potential theory ([1], [10], [11]). The Dirichlet form can be described for instance via the construction of Dirichlet forms from derivations (Theorem 4.5 in [10] or Theorem 8.3 in [11]). This corresponds to working with the $C^*$-algebra of compact operators $\mathcal{K} = \mathcal{K}(L^2(\Omega, \lambda))$ and its trace $\text{Tr}$, which is densely defined, faithful, semifinite and lower semicontinuous.

The Hilbert space $\mathcal{H} = C_2 \oplus C_2$, where $C_2 = C_2(L^2(\Omega, \lambda))$ is a $\mathcal{K} - \mathcal{K}$-bimodule and $J(x \oplus y) = x^* \oplus y^*$ is an isometric antilinear involution of $\mathcal{H}$ exchanging the right and left actions of $\mathcal{K}$ on $\mathcal{H}$. Clearly $C_2$ identifies with $L^2(\mathcal{K}, \text{Tr})$ and there is an $L^2$-closable derivation $\partial$ of $\mathcal{K}\Lambda(\Omega) \cap C_2$ to $C_2$. The definition in case $\Omega$ is bounded, is $\partial a = [X, a] \oplus [Y, a]$. In general the definition can be given in terms of the kernel $K(z_1, z_2)$ of an element $a \in \mathcal{K}\Lambda(\Omega) \cap C_2$. Then the components of $\partial a$ have kernels $(x_1 - x_2)K(z_1, z_2)$ and respectively $(y_1 - y_2)K(z_1, z_2)$, which are square integrable since $a \in \mathcal{K}\Lambda(\Omega)$. Also clearly viewed the domain of definition of $\partial$ as part of $L^2(\mathcal{K}, \text{Tr})$, the map $\partial$ is $L^2$-closed. Moreover $\partial$ satisfies the symmetry condition $J\partial a = \partial a^*$. Then the Dirichlet form $\mathcal{E}$ which is obtained as the closure $\mathcal{E}[a] = \|\partial a\|_2^2$ is easily seen to be precisely square of the $L^2$-norm of $(z_1 - z_2)\mathcal{K}(z_1, z_2)$ which is the same as $(\mathcal{L}(a))^2$ defined for $a \in \mathcal{K}\Lambda(\Omega)$. The Markovian semigroup $T_t$ will then act on elements $a \in \mathcal{K}\Lambda(\Omega) \cap C_2$ which have kernels $K(z_1, z_2)$ as a multiplier which produces the element with kernel $e^{-t|z_1 - z_2|^2}K(z_1, z_2)$. In view of the Markovianity it is easy to see that $T_t$ extends to a semigroup of completely positive contraction on $\mathcal{K}\Lambda(\Omega)$, $\mathcal{E}\Lambda(\Omega)$ and also on $\mathcal{K}$ and $\mathcal{B}$. Moreover $T_t$ also induces a semigroup of completely positive contractions on $(\mathcal{E}/\mathcal{K})\Lambda(\Omega)$.

6.4. Replacing $C_2$ by some other $C_p$. One may wonder about the consequences of replacing the Hilbert–Schmidt class $C_2$ by some other $C_p$-class in the definition of $\mathcal{E}\Lambda(\Omega)$. This would mean to consider operators $T$ so that $[T, M_f] \in C_p$ for all $f \in \Lambda(\Omega)$ with $\|Df\|_\infty \leq 1$. The questions about $C_p$-perturbations of normal operators are still covered by our results ([25], [29]), however the passage of multiplication operators by Lipschitz functions would require the use of more difficult results on commutators and functional calculus, like those in [2].

6.5. Perhaps the study of the $K$-theory of the $\mathcal{E}\Lambda(\Omega)$ may benefit from more recent developments of bivariant $K$-theory beyond $C^*$-algebras (see [15]).

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