Solution of Potts-3 and Potts-$\infty$
Matrix Models with
the Equations of Motion Method

Gabrielle Bonnet

CEA/Saclay, Service de Physique Théorique
F-91191 Gif-sur-Yvette Cedex, France

Abstract

In this letter, we show how one can solve easily the Potts-3 + branching inter-
actions and Potts-$\infty$ matrix models, by the means of the equations of motion (loop
equations). We give an algebraic equation for the resolvents of these models, and
their scaling behaviour. This shows that the equations of motion can be a useful tool
for solving such models.
1 Introduction:

Random matrices are useful for a wide range of physical problems. In particular, they can be related to two-dimensional quantum gravity coupled to matter fields with a non-zero central charge $C \geq 1$. While $C \leq 1$ models are relatively well understood, the $C > 1$ domain remains almost totally unknown: there is a $C = 1$ "barrier". When studying $C \neq 0$ models, we are led to consider multi-matrix models [2, 3] which are often non-trivial. One class of difficult matrix models corresponds to the $q$-state Potts model (in short: Potts-$q$) on a random surface. This model is a $q$-matrix model where all the matrices are coupled to each other, thus making difficult the use of usual techniques such as the saddle point or the orthogonal polynomials method. Moreover, the $q \to 4$ limit corresponds to $C \to 1$; thus, by solving Potts-$q$ models, we shall gain a new understanding of the $C = 1$ barrier.

In this letter, we show that, contrary to what was previously thought, one can use the loop equations to solve the Potts-3 random matrix model, and we find that the resolvent (which generates many of the operators of the problem) obeys an algebraic equation that we write explicitely.

We also show that this method applies when one adds branching interactions (gluing of surfaces, also called "branched polymers") [4] and we derive the critical line of this extended model. The extension to the model with branching interactions and the study of its phase diagram is necessary to verify [4]'s conjecture about the $C = 1$ transition.

Finally, we apply the method to the Potts-$\infty$ matrix model, which corresponds to $C = \infty$.

As this work was approaching its completion, a paper appeared on the dilute Potts model [4], which partially overlaps our present work. In this article, the author also has an algebraic equation for the conventional Potts-3 model. Here, we go further as we consider the Potts-3 + branching interactions model. Moreover, his method is quite different: while he uses analytical considerations on the resolvents and large-$N$ techniques, we solve our model by the loop equations method, which can be extended to finite $N$ problems and is also more adapted to the use of renormalization group techniques [7, 8].

2 The Potts-3 + branching interactions model:

Let us define:

$$Z = \int d\Phi e^{-N^2 V(\Phi)}$$

$$V(\Phi) = g \frac{\text{tr}\Phi^3}{3N} + \psi\left(\frac{\text{tr}\Phi^2}{2N}, \frac{\text{tr}\Phi\delta\Phi\delta}{2N}\right)$$

$$\Phi = \begin{pmatrix} \Phi_1 & 0 & 0 \\ 0 & \Phi_2 & 0 \\ 0 & 0 & \Phi_3 \end{pmatrix}, \delta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \delta_{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
\[ \delta = \delta_1 + \delta_{-1}. \] We shall also use later the notation \( \delta_0 = \text{Id} \).

\( \Phi, \delta_0, \delta_1 \) and \( \delta_{-1} \) are \((3N) \times (3N)\) and \( \Phi_1, \Phi_2, \Phi_3 \) \( N \times N \) hermitean matrices. \( \psi \) is a general two-variable function, and will mainly appear through its partial derivatives \( U \) and \( c \) with respect to \( \Phi_2 \) and \( \frac{\delta \Phi_2}{N} \) respectively. If these are constants, then we recover the conventional Potts-3 model (no branching interactions). This model was given partial solution by J.M. Daul [9], by considering the analytic structure of the resolvents. He had its critical point and its associated critical exponent. He did not know, however, if the resolvent obeyed an algebraic equation. We shall give here the expression of this equation for the conventional and extended Potts-3 model. We also derive the critical line of the extended model and check it corresponds to Daul’s result in the particular case of the conventional model.

Let us note, for convenience:

\[ t_{i_1...i_n} = \frac{1}{3N} \langle \text{tr} \delta_{i_1} \Phi \delta_{i_2} \Phi ... \delta_{i_n} \Phi^k \rangle \] (4)

where \( i_1, ..., i_n \) can be +1, −1 or 0. This trace is non-zero if and only if \( i_1 + ... + i_n \equiv 0 \) (mod 3). \( \langle ... \rangle \) is the expectation value of \( ... \):

\[ \langle ... \rangle = \frac{1}{Z} \int d\Phi \langle ... \rangle e^{-N^2 V(\Phi)} \] (5)

A trace will be said to be “of degree \( m \)” if there are \( m \) matrices \( \Phi \) in it. For example, the above trace is of degree \( k + n - 1 \).

Let us now use the method of the equations of motion (or loop equations). If we make the infinitesimal change of variables in \( Z \):

\[ \Phi \rightarrow \Phi + \epsilon \delta_{i_1} \Phi \delta_{i_2} \Phi ... \delta_{i_n} \Phi \] (6)

with

\[ i_1 + i_2 + ... + i_n \equiv 0 \pmod{3} \]

then we obtain the expression of the general equations of motion:

\[ g t_{i_1...i_n} + U t_{i_1...i_n} + c (t_{(i_1+1)i_2...(i_n-1)} + t_{(i_1-1)i_2...(i_n+1)}) - \sum_{j=1}^{n-1} t_{i_1...i_j} t_{i_{j+1}...i_n} = 0 \] (7)

The first three terms come from the transformation of \( V(\Phi) \), and the last one, from the jacobian of the transformation.

Eq. (7) relates any expectation value of trace containing a quadratic term (i.e. a \( \Phi^2 \) term) to expectation values of traces of lower degrees. The problem is that we do not have any recursion relation for more general expectation values like \( t_{i_1...i_n} \) where all the \( i_k \neq 0 \). Moreover, when one wants to compute even a very simple trace : for example \( t_{\phi_n} \)
by using Eq. (7), one obtains, \( \left\lceil \frac{n}{2} \right\rceil \) steps later, a \( n - \left\lceil \frac{n}{2} \right\rceil \) degree complicated trace which does not contain quadratic terms any more. Thus, the recursion stops there. In fact, this problem can be overcome by a very simple idea: one uses the invariance of traces by cyclic permutations to get rid of the \( n + 1 \) degree term in Eq. (7). Then, one obtains relations between general traces, and it is thus possible to compute the expectation values of any trace in function of the first ones.

Let us see now how this idea applies to the computation of the resolvent. We denote:

\[
\omega_{i_1 i_2 \ldots i_n} = \frac{1}{3N} \langle \text{tr} \delta_{i_1} \Phi \delta_{i_2} \Phi \ldots \delta_{i_n} \frac{1}{z - \Phi} \rangle
\]

(8)

\( \omega_0 = \omega \) is the usual resolvent. Using the change of variables:

\[
\Phi \rightarrow \Phi + \epsilon \frac{1}{z - \Phi}
\]

(9)

we obtain the equation:

\[
z (U + gz) \omega - U - gz - g t_\Phi - \omega^2 + 2 c \omega_{1-1} = 0
\]

(10)

Similarly,

\[
\Phi \rightarrow \Phi + \epsilon \delta_1 \Phi \delta_{-1} \frac{1}{z - \Phi}
\]

(11)

yields:

\[
z (U + gz) \omega_{1-1} - (U + gz) t_\Phi - g t_{1-1} \Phi - \omega \omega_{1-1} + c \omega_{1 \ 0-1} + c \omega_{-1-1-1} = 0
\]

(12)

and, by the means of similar changes in variables, we have the equations:

\[
z (U + gz) \omega_{-1-1-1} - (U + gz) t_{1-1} \Phi - g t_{-1-1-1} \Phi + c \omega_{-1 \ 0-1-1} + c \omega_{1 \ 1-1-1} - \omega \omega_{-1-1-1} = 0
\]

(13)

\[
z (U + gz) \omega_{1 \ 1-1-1} - (U + gz) t_{-1-1} \Phi - g t_{1 \ 1-1} \Phi + c \omega_{1 \ 0 \ 1-1} + c \omega_{-1-1 \ 1-1-1} - \omega \omega_{1 \ 1-1-1} = 0
\]

(14)

These equations alone are not sufficient to compute \( \omega(z) \). Indeed, if we intend to calculate \( \omega(z) \), we generate the function \( \omega_{1-1}(z) \) (Eq. (10)). Then, in turn, we generate the function \( \omega_{-1-1-1}(z) \) (Eq. (12)) and so on.

As for \( \omega \) functions containing a 0 (i.e. a \( \Phi^2 \) term) such as \( \omega_{1 \ 0-1} \), they are easy to deal with: we know how to compute traces containing \( \Phi^2 \). \( \omega_{1 \ 0-1} = \frac{1}{3N} \text{tr} \delta_{i_1} \Phi^2 \delta_{-1} \frac{1}{z - \Phi} \), will be seen as \( \frac{1}{3N} \text{tr} \Phi^2 \delta_{-1} \frac{1}{z - \Phi} \delta_1 \). Then the change in variables:

\[
\Phi \rightarrow \Phi + \epsilon \delta_{-1} \frac{1}{z - \Phi} \delta_1
\]

(15)

yields (\( \omega_{1-1} = \omega_{-1 \ 1} \) for symmetry reasons)

\[
g \omega_{10-1} + (U + c) \omega_{1-1} + c z \omega - c = 0
\]

(16)
and similar changes in variables lead to the equations:

\[ g \omega_{-1} 0_{-1-1} + U \omega_{-1-1-1} + c \omega_{1} 0_{-1} + c z \omega_{1-1} - c t_\Phi = 0 \] (17)

\[ g \omega_{1} 0_{1-1-1} + U \omega_{1-1-1} + c z \omega_{-1-1-1} - c t_\Phi + c \omega_{-1} 0_{1-1} - t_\Phi \omega = 0 \] (18)

But, to compute \( \omega_{-1-1-1} \), as mentioned in the comments to Eq. (10), we have to substract two different changes in variables and use cyclicity of traces:

\[ \Phi \rightarrow \Phi + \epsilon [\Phi \delta_{-1} \Phi \delta_{-1} (z - \Phi)^{-1} \delta_{-1} \Phi - \Phi \delta_{-1} \Phi \delta_{-1} (z - \Phi)^{-1} \delta_{-1} \Phi^2] \] (19)

This equation, as we know how to compute \( (\omega_{-1-1} 1-1_{1} - \omega_{-1} 1-1_{1}) - \omega_{-1-1} 1_{1} + t_{\Phi} \omega_{1-1} = 0 \) (20)

Finally, as a result of these operations, we have:

\[ \omega_{-1-1} 1_{1-1-1} + \omega_{-1} 1_{1-1-1} = K(z) \] (22)

where \( K(z) \) only contains easy to compute \( \omega \) functions. We can then write an equation for \( \omega_{-1} 1_{1-1-1} \) and thus for \( \omega_{-1-1} 1_{1-1} \) which only involves \( \omega \) functions that we either already know or are able to compute similarly as was done during the two first steps of the procedure.

That way, our set of equations is closed, and we obtain a degree five algebraic equation for \( \omega(z) \). This expression applies to general expressions of \( U \) and \( c \). For the exact expression of this equation see Appendix A. The equation only contains four unknown parameters:

\[ t_\Phi = \frac{1}{N} \langle \text{tr} \Phi_1 \rangle, \ t_{1-1} = \frac{1}{N} \langle \text{tr} \Phi_1 \Phi_2 \rangle, \ t_{111} = \frac{1}{N} \langle \text{tr} \Phi_1 \Phi_2 \Phi_3 \rangle, \ t_{1-1-1} = \frac{1}{N} \langle \text{tr} \Phi_1 \Phi_2 \Phi_3 \rangle \] (23)

These parameters are also those that would be involved if we used the renormalisation group method \([10, 7, 8] \) to compute the Potts-3 model. The renormalization group flows would relate the conventional Potts-3 to the Potts-3 + branching interactions model, with arbitrary \( U \) and \( c \); but the presence of \( t_{111} \) shows us it would also be related to the dilute Potts-3 model, where one has a \( \frac{1}{N} \text{tr} (\Phi_1 + \Phi_2 + \Phi_3)^3 \) term. Finally, the \( t_{1-1-1} \) term shows
us it may also be related to more complicated quartic models.

We are now going to derive from our equation the critical behaviour and critical line of the model when \( U = 1 + h \text{tr}\Phi^2/6 \) and \( c \) is a constant. This is the most common type of extension of a matrix model to branching interactions. The values of the unknown parameters given in Eq. (23) are fixed by the physical constraint that the resolvent has only one physical cut which corresponds to the support of the eigenvalues of \( \Phi \). Then, one can study the critical behaviour of the model.

It is easy to look for the Potts critical line. Indeed, the scaling behaviour of the resolvent is then, if we denote the physical cut of \( \omega \) as \([a,b]\):

\[
\omega(z) \sim (z-a)^{\frac{1}{2}} \quad \text{when} \quad z \sim a \quad \text{and} \quad \omega(z) \sim (z-b)^{\frac{5}{6}} \quad \text{when} \quad z \sim b
\]

The corresponding exponent \( \gamma_s \) is \(-\frac{1}{5}\), which corresponds to the \( C = \frac{4}{5} \) central charge of the model.

Rather than looking for the resolvent for any values of the coupling constants, it is easier to search for the resolvent only on this critical line where the presence of the \( \frac{6}{5} \) exponent leads to simple conditions on the derivatives of the algebraic equation.

We obtain:

\[
\begin{align*}
105c^3 + 4g^2 &= 0 \\
2480625c^2(-1 - 4c + 43c^2) + 296100c(15 + 113c)h - 692968h^2 &= 0
\end{align*}
\]

(25)

Let us note here that, when \( h = 0 \) (no branching interactions) we recover the Potts-3 bicritical point which agrees with Daul’s result [9]:

\[
c = \frac{2 - \sqrt{47}}{43}, \quad g = \frac{\sqrt{105}}{2} \left( \frac{-3 + \sqrt{47}}{41 - \sqrt{47}} \right)^{\frac{3}{2}}
\]

Thus, we have shown that the resolvent for the model of Potts-3 plus branching interactions obeys a degree five algebraic equation. We have found the critical line and exponent of this extended model. This extends the results of Daul [9] who had only derived the position of the critical point and exponent of the conventional model. Finally, let us recall that, in a recent paper [3], P. Zinn-Justin obtains independently algebraic equations for similar problems. His method, though, does not involve loop equations, and is rather in the spirit of [3]. Moreover, it does not address the problem of branching interactions and thus overlaps our results only in the case of the conventional Potts-3 model.

3 The Potts-\(\infty\) model:

We are now going to briefly derive the solution for the Potts-\(\infty\) model, from the equations of motion point of view. The purpose of this part is mainly to show the efficiency of our method on this \(c = \infty\) model. This model was previously studied by Wexler in [11].
Let us denote $\Phi = \begin{pmatrix} \Phi_1 & & 0 \\ & \ddots & \\ 0 & & \Phi_q \end{pmatrix}$, and $X = \frac{\Phi_1 + \ldots + \Phi_q}{N} \otimes 1_{q \times q}$.

We shall define the Potts-$q$ partition function as

$$Z = \int d\Phi e^{-N V(\Phi)} \quad \text{where} \quad V(\Phi) = g \text{tr} \Phi^3 + U \text{tr} \Phi^2 + c \text{tr} \Phi X^2$$

($26$)

$V(\Phi)$ is of order $q$ when $q \to \infty$. First, let us use the equations of motion to relate

$$a(x) = \frac{1}{qN} \langle \text{tr} \frac{1}{x - \Phi} \rangle \quad \text{to} \quad b(y) = \frac{1}{qN} \langle \text{tr} \frac{1}{y - X} \rangle$$

($27$)

Let us also denote :

$$d(x, y) = \frac{1}{qN} \langle \text{tr} \frac{1}{x - \Phi} \frac{1}{y - X} \rangle$$

($28$)

$$\Phi \to \Phi + \epsilon \frac{1}{x - \Phi} \frac{1}{y - X}$$

yields

$$(x (gx + U) + cy - a(x) - \frac{b(y)}{q}) d(x, y) + g - gy b(y) - c a(x) - b(y) (gx + U) = 0$$

($29$)

We can get rid of $d(x, y)$ since, when $x (gx + U) + cy - a(x) - \frac{b(y)}{q} = 0$, $d(x, y)$ remains finite, thus $g - gy b(y) - c a(x) - b(y) (gx + U) = 0$. This is sufficient to relate $a(x)$ to $b(y)$. Moreover, the value of $b(y)$ is easy to compute when $q = \infty$.

Let us briefly summarize this computation : we calculate the value of $\langle \text{tr} X^n \rangle$ in the $q \to \infty$ limit.

First :

$$\langle \text{tr} X^n \rangle = \langle \text{tr} \Phi_1 \ldots \Phi_n \rangle + O(\frac{1}{q})$$

($30$)

(recall that all the $\Phi_i$ play the same role).

If we now separate the first $n$ matrices from the remaining $q - n$ (with $q \gg n$), and suppose there is a saddle point for the eigenvalues of $\frac{\Phi_{n+1} + \ldots + \Phi_q}{q}$, then this saddle point is (in the $q \to \infty$ limit) independent from the matrices $\Phi_1, \ldots, \Phi_n$. Then, in this limit, up to a change in variables : $\tilde{\Phi}_k = U \Phi_k U^{-1}$, we have $n$ independent matrices $\tilde{\Phi}_1 \ldots \tilde{\Phi}_n$. Each of them has the partition function

$$Z_{\Lambda C} = \int d\tilde{\Phi}_k e^{-N \left( g \text{tr} \tilde{\Phi}_k^3 + U \text{tr} \tilde{\Phi}_k^2 + c \text{tr} \tilde{\Phi}_k \Lambda C \right)}$$

($31$)

As $\text{tr} \Phi_1 \ldots \Phi_n = \text{tr} \tilde{\Phi}_1 \ldots \tilde{\Phi}_n$, we have $\langle \text{tr} X^n \rangle = \langle \text{tr} \tilde{\Phi}_1 \rangle_{\Lambda C} \ldots \langle \text{tr} \tilde{\Phi}_n \rangle_{\Lambda C}$ where $\langle \ldots \rangle_{\Lambda C}$ is the expectation value obtained with the partition function $Z_{\Lambda C}$ (cf Eq. (31)).

The matrices $\tilde{\Phi}_k$, $k = 1, \ldots n$ all play the same part, and $\langle \text{tr} X^n \rangle = \text{tr} \Lambda^n_{\Lambda C}$, thus

$$\text{tr} \Lambda^n_{\Lambda C} = \text{tr} \langle \tilde{\Phi}_1 \rangle^n$$

($32$)
This must give us $\Lambda_C$, provided we calculate $\langle \tilde{\Phi}_1 \rangle_{\Lambda_C}$ in function of $\Lambda_C$. This is a solvable problem, but it is much faster to note that

$$\Lambda_C = t_\Phi 1_{qN \times qN}$$

is solution. Thus, $\frac{1}{qN}(\text{tr}X^n) = (t_\Phi)^n$, and $b(y)$ is simply $(y - t_\Phi)^{-1}$.

This gives us immediately the solution for $a(x)$: it obeys a second order equation and reads:

$$a(x) = \frac{1}{2}(x(U + gx) + c_t_\Phi - \sqrt{(x(U + gx) + ct_\Phi)^2 - 4(U + gx + gt_\Phi)})$$

The Potts-$\infty$ plus branched polymers model is thus very similar to an ordinary pure gravity model. As previously, we compute the parameter $t_\Phi$ by imposing that the resolvent $a(x)$ has only one physical cut. The model is critical when $a(x)$ behaves as $(x - x_0)^{\frac{3}{2}}$, $x_0$ being a constant, and the critical point verifies (as in [11]):

$$g_c = \frac{1}{4\sqrt{2}} \quad \text{and} \quad c_c = -\frac{1}{2}$$

Let us note finally that the loop equations method used here is appropriate for the renormalization group method of [10, 7, 8].

4 Conclusion

In this letter, we have shown that it is possible to solve the Potts-3 and Potts-$\infty$ models on two-dimensional random lattices through the method of the equations of motion. We have obtained a closed set of loop equations for the Potts-3 model, which was thought to be impossible. We have shown that the Potts-3 resolvent obeys an order five equation, and this new knowledge opens the door to the calculation of expectation values of the operators of the model. We have extended the Potts-3 conventional model to Potts-3 plus branching interactions, and given the general algebraic equation and the Potts critical line of this model. Finally, we have shown our method also applies successfully to another Potts model: the Potts-$\infty$ model. We hope to generalize soon our method to more general Potts-$q$ models, in particular for large-$q$ Potts + branching interactions models.

A The equation for the Potts-3 resolvent:

Here is the degree five equation for the resolvent of this model, where $W(x)$ is related to $\omega(x)$ by $W(x) = \omega(x) - gx^2 - Ux$.

$$-24c^7 + 4c^4g^2 - 16t_{-1,0}c^5g^2 - 12t_{111,0}c^4g^3 - 4t_{1-1,0}c^2g^4 + 8t_{11-1,0}c^3g^4 + 68c^6gt_\Phi + 2c^5g^3t_\Phi + 3c^2g^4t_\Phi^2 + 60c^6U + 2c^3g^2U - 52t_{1-1,0}c^4g^2U + 20t_{111,0}c^3g^3U - 20c^5gt_\Phi U -$$
$$4\, c^2 \, g^3 \, t_\phi \, U - 36 \, c^5 \, U^2 - 3 \, c^2 \, g^2 \, U^2 + 36 \, t_{1-\phi} \, c^3 \, g^2 \, U^2 - 36 \, c^4 \, g \, t_\phi \, U^2 - 12 \, c^4 \, U^3 + 36 \, c^3 \, g \, t_\phi \, U^3 + 12 \, c^3 \, U^4 + 28 \, c^6 \, g \, x + 2 \, c^3 \, g^3 \, x - 12 \, t_{1-\phi} \, c^4 \, g^3 \, x + 8 \, t_{11-\phi} \, c^3 \, g^4 \, x - 36 \, c^5 \, g \, t_\phi \, x - 2 \, c^2 \, g^4 \, t_\phi \, x - 52 \, c^5 \, g \, U \, x - 4 \, c^2 \, g^3 \, U \, x + 36 \, t_{1-\phi} \, c^3 \, g^3 \, U \, x - 18 \, c^4 \, g^2 \, t_\phi \, U \, x + 2 \, c^4 \, g \, U^2 \, x + 54 \, c^3 \, g^2 \, t_\phi \, U^2 \, x + 22 \, c^3 \, g \, U^3 \, x - 24 \, c^5 \, g^2 \, x^2 - c^2 \, g^4 \, x^2 + 8 \, t_{1-\phi} \, c^4 \, g^3 \, x^2 + 2 \, c^4 \, g^3 \, t_\phi \, x^2 + 24 \, c^5 \, U \, x^2 + 20 \, c^4 \, g \, U^2 \, x^2 + 26 \, c^3 \, g^3 \, t_\phi \, U \, x^2 - 48 \, c^6 \, U^2 \, x^2 + 12 \, c^3 \, g^2 \, U^2 \, x^2 + 24 \, c^6 \, U^3 \, x^2 + 24 \, c^7 \, g \, U^3 \, x^3 + 6 \, c^4 \, g^3 \, x^3 + 4 \, c^3 \, g^4 \, t_\phi \, x^3 - 64 \, c^6 \, U \, x^3 + 2 \, c^3 \, g^3 \, U \, x^3 + 44 \, c^5 \, g \, U^2 \, x^3 - 16 \, c^6 \, g^2 \, U \, x^4 + 24 \, c^5 \, g^2 \, U \, x^4 + 4 \, c^5 \, g^3 \, x^5 + (24 \, c^5 \, g - 12 \, t_{1-\phi} \, c^3 \, g^3 + 8 \, t_{11-\phi} \, c^2 \, g^4 - 36 \, c^4 \, g^2 \, t_\phi - 2 \, c^4 \, g^4 \, t_\phi - 40 \, c^4 \, g \, U - 2 \, c \, g^3 \, U + 36 \, t_{1-\phi} \, c^2 \, g^3 \, U - 18 \, c^3 \, g^3 \, t_\phi \, U - 10 \, c^3 \, g \, U^2 + 54 \, c^2 \, g^2 \, t_\phi \, U^2 + 26 \, c^2 \, g \, U^3 + 24 \, c^5 \, x - 24 \, c^4 \, g^2 \, x - 2 \, c \, g^4 \, x + 16 \, t_{1-\phi} \, c^2 \, g^4 \, x + 4 \, c^3 \, g^3 \, t_\phi \, x - 36 \, c^6 \, U \, x + 4 \, c^3 \, g^2 \, U \, x + 52 \, c^2 \, g^4 \, t_\phi \, U \, x + 12 \, c^5 \, U^2 \, x + 36 \, c^6 \, g^2 \, U^2 \, x - 12 \, c^4 \, U^3 \, x + 12 \, c^3 \, U^4 \, x - 4 \, c^5 \, g \, x^2 + 14 \, c^3 \, g^3 \, x^2 + 12 \, c^2 \, g^4 \, t_\phi \, x^2 - 36 \, c^5 \, g \, U \, x^2 + 10 \, c^3 \, g^2 \, U \, x^2 + 30 \, c^4 \, g \, U^2 \, x^2 + 22 \, c^3 \, g^3 \, x^3 - 40 \, c^5 \, g^2 \, x^3 + 60 \, c^4 \, g^2 \, U \, x^3 + 12 \, c^3 \, g^2 \, U^2 \, x^3 + 18 \, c^4 \, g^3 \, x^4 + 2 \, c^3 \, g^3 \, U \, x^4 + 12 \, c^6 \, U \, x - 4 \, c^3 \, g^2 \, U \, x + 26 \, c \, g^3 \, t_\phi \, U - 12 \, c^4 \, U^2 + 18 \, c \, g^2 \, U^2 + 12 \, c^3 \, U^3 - 44 \, c^5 \, g \, x - 2 \, c^2 \, g^3 \, x + 12 \, c^2 \, g^4 \, t_\phi \, x + 30 \, c^4 \, U \, x + 20 \, c^2 \, g^3 \, U \, x + 26 \, c^2 \, g^3 \, x + 6 \, c^4 \, g^2 \, x^2 + 2 \, c \, g^4 \, x^2 + 12 \, c^3 \, g^2 \, U \, x^2 + 39 \, c^2 \, g^2 \, U^2 \, x^2 + 20 \, c^3 \, g^3 \, x^3 + 14 \, c^2 \, g^3 \, U \, x^3 + 2 \, c^3 \, g^3 \, x^4 \, U \, x^4 \, W(x) + (12 \, c^6 - 6 \, c^3 \, g^2 + 8 \, t_{1-\phi} \, c^4 \, g^4 + 2 \, c^3 \, g^3 \, t_\phi - 12 \, c^5 \, U - 4 \, c^2 \, g^2 \, U + 26 \, c \, g^3 \, t_\phi \, U - 12 \, c^4 \, U^2 + 18 \, c \, g^2 \, U^2 + 12 \, c^3 \, U^3 - 44 \, c^5 \, g \, x - 2 \, c^2 \, g^3 \, x + 12 \, c^2 \, g^4 \, t_\phi \, x + 30 \, c^4 \, U \, x + 20 \, c^2 \, g^3 \, U \, x + 26 \, c^2 \, g^3 \, x + 6 \, c^4 \, g^2 \, x^2 + 2 \, c \, g^4 \, x^2 + 12 \, c^3 \, g^2 \, U \, x^2 + 39 \, c^2 \, g^2 \, U^2 \, x^2 + 20 \, c^3 \, g^3 \, x^3 + 14 \, c^2 \, g^3 \, U \, x^3 + 2 \, c^3 \, g^3 \, x^4 \, U \, x^4 \, W(x) + ( -12 \, c^4 \, g - 2 \, c \, g^3 + 4 \, g^4 \, t_\phi - 10 \, c^3 \, g \, U + 4 \, g^3 \, U + 26 \, c \, g^3 \, U^2 + 20 \, c \, g^2 \, x + 4 \, g^4 \, x + 12 \, c^2 \, g^2 \, U \, x + 18 \, c^2 \, g^2 \, x^2 + 22 \, c \, g^3 \, U \, x^2 + 4 \, c^4 \, g^3 \, x^3 \, U \, x + 4 \, g^4 \, x^2 \, x^2 \, W(x) + 4 \, c^4 \, g^3 \, W(x)^3 = 0 $$

Note that $U$ and $c$ may depend, in the most general case, on $t_{1-\phi}$ and $t_{\phi^2}$, the latter being related to $t_\phi$ through the equation of motion: $g \, t_{\phi^2} + (U + 2c) \, t_\phi = 0$. In this article, we have computed explicitly the critical line for the particular case of $c$ constant and $U = 1 + \frac{1}{2} \, t_{\phi^2}$.

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