Some results on the Kampé de Fériet hypergeometric matrix function

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Abstract
In this paper, we obtain recursion formulas for the Kampé de Fériet hypergeometric matrix function. We also give finite and infinite summation formulas for the Kampé de Fériet hypergeometric matrix function.

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1 Introduction
Recursion formulas for the Appell function $F_2$ have been studied by Opps, Saad and Srivastava [7], followed by Wang [18], who presented the recursion relations for all Appell functions. The authors have carried out a systematic study of recursion formulas of multivariable hypergeometric functions including fourteen three variable Lauricella functions, three Srivastava’s triple hypergeometric functions and four $k$–variable Lauricella functions [13] and Exton’s triple hypergeometric functions [14]. The recursion formulas for the general triple hypergeometric function [17] were obtained in [15]. These results were unified and generalized in [13] for the three variable hypergeometric function. In [16], recursion formulas for the general Kampé de Fériet series and Srivastava and Daoust multivariable hypergeometric function were presented. In this paper, we begin a study of recursion formulas satisfied by one and two variable hypergeometric

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matrix functions. In particular, we present recursion formulas for the Gauss hypergeometric and Appell matrix functions.

The matrix theory is being recently used in theory of orthogonal polynomials and special functions. Special matrix functions appear in the literature related to Statistics [2], Lie theory [5] and more recently in connection with the matrix version of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families [8, 9, 10].

Let \( \mathbb{C}^{r\times r} \) be the vector space of \( r \) square matrices with complex entries. For any matrix \( A \in \mathbb{C}^{r\times r} \), its spectrum \( \sigma(A) \) is the set of eigenvalues of \( A \). If \( f(z) \) and \( g(z) \) are holomorphic functions of the complex variable \( z \), which are defined in an open set \( \Omega \) of the complex plane and \( A \in \mathbb{C}^{r\times r} \) with \( \sigma(A) \subset \Omega \), then from the properties of the matrix functional calculus [6], we have \( f(A)g(A) = g(A)f(A) \) if \( B \in \mathbb{C}^{r\times r} \) is a matrix for which \( \sigma(B) \subset \Omega \), and if \( AB = BA \) then \( f(A)g(B) = g(B)f(A) \). A square matrix \( A \) in \( \mathbb{C}^{r\times r} \) is said to be positive stable if \( \Re(\lambda) > 0 \) for all \( \lambda \in \sigma(A) \).

The reciprocal gamma function \( \Gamma^{-1}(z) = 1/\Gamma(z) \) is an entire function of the complex variable \( z \). The image of \( \Gamma^{-1}(z) \) acting on \( A \), denoted by \( \Gamma^{-1}(A) \), is a well defined matrix. If \( A + nI \) is invertible for all integers \( n \geq 0 \), then the reciprocal gamma function [12] is defined as \( \Gamma^{-1}(A) = (A)_n \Gamma^{-1}(A+nI) \), where \( (A)_n \) is the shifted factorial matrix function for \( A \in \mathbb{C}^{r\times r} \) given by [11]

\[
(A)_n = \begin{cases} 
I, & n = 0, \\
A(A+I) \cdots (A+(n-1)I), & n \geq 1.
\end{cases}
\]

\( I \) being the \( r \)-square identity matrix. If \( A \in \mathbb{C}^{r\times r} \) is a positive stable matrix and \( n \geq 1 \), then by [12] we have \( \Gamma(A) = \lim_{n \to \infty} (n-1)! (A)^{-1} n^A \).

The Gauss hypergeometric matrix function [11] is defined by

\[
_{2}F_{1}(A, B; C; x) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n (C)^{-1}}{n!} x^n, \tag{1.1}
\]

for matrices \( A, B \) and \( C \) in \( \mathbb{C}^{r\times r} \) such that \( C + kI \) is invertible for all \( k \geq 0 \) and \( |x| \leq 1 \).

The Appell matrix functions are defined as

\[
F_{1}(A, B, B'; C; x, y) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(B)m(B')_{n}(C)^{-1}}{m!n!} x^m y^n, \tag{1.2}
\]

\[
F_{2}(A, B, B'; C, C'; x, y) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(B)m(B')_{n}(C)_{m+n}(C')^{-1}}{m!n!} x^m y^n, \tag{1.3}
\]

\[
F_{3}(A, A', B, B'; C; x, y) = \sum_{m,n=0}^{\infty} \frac{(A)_{m}(A')_{n}(B)m(B')_{n}(C)_{m+n}}{m!n!} x^m y^n, \tag{1.4}
\]

\[
\]
\[
F(A, B; C, C'; x, y) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(B)_{m+n}(C)_{m+n}^{-1}(C')_{m+n}^{-1}}{m!n!} x^m y^n,
\]

(1.5)

where \( A, A', B, B', C, C' \) are positive stable matrices in \( \mathbb{C}^{r \times r} \) such that \( C + kI \) and \( C' + kI \) are invertible for all integers \( k \geq 0 \). For regions of convergence of (1.2)-(1.5), see [1, 3, 4].

The Kampé de Fériet hypergeometric matrix function is defined by [3, 4]

\[
F_{m_1, n_1; m_1'; n_1'}^{m_2, n_2; m_2'; n_2'} (A; B, C; D; E, F; x, y) = \sum_{m,n \geq 0} \prod_{i=1}^{m_1} (A_i)_{m+n} \prod_{i=1}^{n_1} (C_i)_{m+n} \prod_{i=1}^{m_2} (D_i)_{m+n}^{-1} \prod_{i=1}^{n_2} (E_i)_{m+n}^{-1} \prod_{i=1}^{m_2'} (F_i)_{m+n}^{-1} \frac{x^m y^n}{m!n!},
\]

(1.6)

where \( A \) abbreviates the sequence of matrices \( A_1, \ldots, A_{m_1} \), etc. and \( A_i, B_i, C_i, D_i, E_i, F_i \) are positive stable matrices in \( \mathbb{C}^{r \times r} \) such that \( D_i + kI, E_i + kI \), and \( F_i + kI \) are invertible for all integers \( k \geq 0 \).

Next, we recall the definition of derivative operator

\[
D_y f(y) = \lim_{h \to 0} \frac{f(y+h) - f(y)}{h},
\]

provided \( f \) is differentiable at \( y \). Also \( D_y^k f(y) = D_y (D_y^{k-1} f(y)) \), \( k = 0, 1, 2, \ldots \).

Throughout the paper, \( I \) denotes the identity matrix and \( s \) denotes a non-negative integer. Following abbreviated notations are used. We, for example, write

\[
A + sI = A_1 + sI, A_2 + sI, \ldots, A_{m_1} + sI
\]

\[
A' = A_1, A_2, \ldots, A_{i-1}, A_{i+1}, A_{i+2}, \ldots, A_{m_1}
\]

\[
A' + sI = A_1 + sI, A_2 + sI, \ldots, A_{i-1} + sI, A_{i+1} + sI, \ldots, A_{m_1} + sI.
\]

Also, we denote

\[
[A + kI]_s = \prod_{i=1}^{m_1} (A_i + kI)_s, \quad [A + kI]_s^{-1} = \prod_{i=1}^{m_1} (A_i + kI)_s^{-1}
\]

\[
[A' + kI]_s = \prod_{i=1, i \neq j}^{m_1} (A_i + kI)_s \quad [A' + kI]_s^{-1} = \prod_{i=1, i \neq j}^{m_1} (A_i + kI)_s^{-1}.
\]

2 Recursion formulas for the Kampé de Fériet hypergeometric matrix function

In this section, we obtain the recursion formulas for the Kampé de Fériet hypergeometric matrix function.
Theorem 2.1. Let $A_i + sI, i = 1, \ldots, m_1$ be invertible for all integers $s \geq 0$. Then the following recursion formula hold true for the Kampé de Fériet hypergeometric matrix function:

\[
F_{m_1; n_1, n_1'} \left( \begin{array}{c} A^i, A_i + sI : B, C \\ D^i : E, F \end{array} ; x, y \right)
\]

\[
= F_{m_1; n_1, n_1'} \left( \begin{array}{c} (A^i + 2I) : B, C \\ D^i : E, F \end{array} ; x, y \right)
\]

Furthermore, if $A_i - kI$ is invertible for integers $k \leq s$, then

\[
F_{m_1; n_1, n_1'} \left( \begin{array}{c} A^i, A_i + sI : B, C \\ D^i : E, F \end{array} ; x, y \right)
\]

\[
= F_{m_1; n_1, n_1'} \left( \begin{array}{c} (A^i + I) : B, C + I \\ D^i : E, F + I \end{array} ; x, y \right)
\]

where $A_i, B_i, C_i, D_i, E_i, F_i$ are positive stable matrices in $\mathbb{C}^{r \times r}$ such that $A_iA_j = A_jA_i$; $A_iB_j = B_jA_i$; $A_iC_j = C_jA_i$; $B_iC_j = C_jB_i$; $F_iE_j = E_jF_i$; $F_jD_i = D_jF_i$; $D_jE_i = E_jD_i$ and $D_i + kI, E_i + kI$ and $F_i + kI$ are invertible for all integers $k \geq 0$.

Proof. From the definition of the Kampé de Fériet hypergeometric matrix function (1.1) and the relation

\[
(A_i + I)_{m+n} = A_i^{-1}(A_i)_{m+n} (A_i + mI + nI)
\]

we get the following contiguous matrix relation:

\[
F_{m_1; n_1, n_1'} \left( \begin{array}{c} (A^i + I) : B, C \\ D^i : E, F \end{array} ; x, y \right)
\]

\[
= F_{m_1; n_1, n_1'} \left( \begin{array}{c} (A^i + 2I) : B, C \\ D^i : E, F \end{array} ; x, y \right)
\]

Replacing $A_i$ with $A_i + I$ in (2.3) and using (2.4), we have the following contiguous matrix relation:

\[
F_{m_1; n_1, n_1'} \left( \begin{array}{c} A^i, A_i + 2I : B, C \\ D^i : E, F \end{array} ; x, y \right)
\]

\[
= F_{m_1; n_1, n_1'} \left( \begin{array}{c} A^i + I, A_i + kI : B, C + I \\ D^i : E, F + I \end{array} ; x, y \right)
\]
Iterating this process $s$ times, we get (2.1). For the proof of (2.2) replace the matrix $A_i$ with $A_i - I$ in (2.4). As $A_i - I$ is invertible, we have

$$F_{m_1; n_1}^{m_1; n_1'} \left( A_i, A_i - I, B, C; D, E, F ; x, y \right) = F_{m_1; n_1}^{m_1; n_1'} \left( A_i, A_i - I, B + I, C + I; D + I, E + I, F + I ; x, y \right) \left[ D \right]^{-1} \left[ E \right]^{-1}$$

Iteratively, we get (2.2).

Using contiguous matrix relations (2.4) and (2.6), we get following forms of the recursion formulas for the Kampé de Fériet hypergeometric matrix function.

**Theorem 2.2.** Let $A_i + sI, i = 1, \ldots, m_1$ be invertible for all integers $s \geq 0$. Then the following recursion formula holds true for the Kampé de Fériet hypergeometric matrix function:

$$F_{m_1; n_1}^{m_1; n_1'} \left( A_i + sI, B, C; D, E, F ; x, y \right) = \sum_{k_1 + k_2 \leq s} \binom{s}{k_1, k_2} \left[A^s\right]_{k_1 + k_2} \left[B\right]_{k_1} \left[C\right]_{k_2} \times x^{k_1} y^{k_2} F_{m_1; n_1}^{m_1; n_1'} \left( A_i, B + k_1 I, C + k_2 I ; D + (k_1 + k_2) I, E + k_1 I, F + k_2 I ; x, y \right) \left[ D \right]^{-1}_{k_1 + k_2} \left[ E \right]^{-1}_{k_1} \left[ F \right]^{-1}_{k_2}$$

(2.7)

Furthermore, if $A_i - k I$ is invertible for integers $k \leq s$, then

$$F_{m_1; n_1}^{m_1; n_1'} \left( A_i - k I, B, C; D, E, F ; x, y \right) = \sum_{k_1 + k_2 \leq s} \binom{s}{k_1, k_2} \left[A^s\right]_{k_1 + k_2} \left[B\right]_{k_1} \left[C\right]_{k_2} \times (-x)^{k_1} (-y)^{k_2} F_{m_1; n_1}^{m_1; n_1'} \left( A_i - k I, B + k_1 I, C + k_2 I ; D + (k_1 + k_2) I, E + k_1 I, E + k_2 I ; x, y \right) \left[ D \right]^{-1}_{k_1 + k_2} \left[ E \right]^{-1}_{k_1} \left[ F \right]^{-1}_{k_2}$$

(2.8)

where $A_i, B_i, C_i, D_i, E_i, F_i$ are positive stable matrices in $C^{r \times r}$ such that $A_iA_j = A_j A_i; A_i B_i = B_i A_i; A_i C_j = C_j A_i; B_i C_j = C_j B_i; F_iE_j = E_j F_i; F_j D_i = D_i F_j; D_i E_j = E_j D_i$ and $D_i + k I, E_i + k I$ and $F_i + k I$ are invertible for all integers $k \geq 0$.

**Proof.** We prove (2.4) by applying mathematical induction on $s$. For $s = 1$, the result (2.4) is true by (2.4). Assuming (2.7) is true for $s = t$, that is,

$$F_{m_1; n_1}^{m_1; n_1'} \left( A_i, A_i + tI, B, C; D, E, F ; x, y \right) = \sum_{k_1 + k_2 \leq t} \binom{t}{k_1, k_2} \left[A^t\right]_{k_1 + k_2} \left[B\right]_{k_1} \left[C\right]_{k_2} \times x^{k_1} y^{k_2} F_{m_1; n_1}^{m_1; n_1'} \left( A_i + tI, B + k_1 I, C + k_2 I ; D + (k_1 + k_2) I, E + k_1 I, E + k_2 I ; x, y \right) \left[ D \right]^{-1}_{k_1 + k_2} \left[ E \right]^{-1}_{k_1} \left[ F \right]^{-1}_{k_2}$$

(2.9)
Replacing $A_i$ with $A_i + I$ in (2.9) and using the contiguous matrix relation (2.4) and simplifying, we get

$$F_{m_1; n_1; n'_2}^{m_2; n_2, n'_2} \begin{pmatrix} A', A_i + I; B, C; x, y \end{pmatrix}$$

$$= \sum_{k_1, k_2 \leq t} \binom{t}{k_1, k_2} [A^t]_{k_1 + k_2} [B]_{k_1} [C]_{k_2} x^{k_1} y^{k_2}$$

$$+ \sum_{k_1 + k_2 \leq t + 1} \binom{t}{k_1, k_2} [A^t]_{k_1 + k_2} [B]_{k_1} [C]_{k_2} x^{k_1} y^{k_2}$$

$$\times \sum_{k_1 + k_2 \leq t + 1} \binom{t}{k_1, k_2} [A^t]_{k_1 + k_2} [B]_{k_1} [C]_{k_2} x^{k_1} y^{k_2}$$

Apply the known relation \( \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \) and \( \binom{0}{k} = 0 \) (\( k > n \) or \( k < 0 \)), the above identity can be reduced to the following result:

$$F_{m_1; n_1; n'_2}^{m_2; n_2, n'_2} \begin{pmatrix} A', A_i + I; B, C; x, y \end{pmatrix}$$

$$= \sum_{k_1, k_2 \leq t + 1} \binom{t + 1}{k_1, k_2} [A^t]_{k_1 + k_2} [B]_{k_1} [C]_{k_2} x^{k_1} y^{k_2}$$

$$\times \sum_{k_1 + k_2 \leq t + 1} \binom{t}{k_1, k_2} [A^t]_{k_1 + k_2} [B]_{k_1} [C]_{k_2} x^{k_1} y^{k_2}$$

This establishes (2.7) for \( s = t + 1 \). Hence result (2.7) is true for all values of \( s \). The second recursion formula (2.8) is proved in a similar manner.

Now, we present the recursion formulas for matrix \( B_i, C_i \) of the Kampé de Fériet hypergeometric matrix function. We omit the proofs of the given below theorems.

**Theorem 2.3.** Let \( B_i + sI, i = 1, \ldots, n_1 \) be invertible for all integers \( s \geq 0 \). Then the following recursion formula holds true for the Kampé de Fériet hypergeometric matrix function:

$$F_{m_1; n_1; n'_2}^{m_2; n_2, n'_2} \begin{pmatrix} A; B_i + sI, C; x, y \end{pmatrix}$$

$$= F_{m_1; n_1; n'_2}^{m_2; n_2, n'_2} \begin{pmatrix} A; B, C; x, y \end{pmatrix}$$

$$+ x[A] [B^t] \sum_{k=1}^{s} F_{m_2; n_2, n'_2}^{m_1; n_1, n'_1} \begin{pmatrix} A + I; B_i + sI, C; x, y \end{pmatrix} [D]^{-1} [E]^{-1}.$$  (2.12)

Furthermore, if \( B_i - kI \) is invertible for integers \( k \leq s \), then

$$F_{m_1; n_1; n'_2}^{m_2; n_2, n'_2} \begin{pmatrix} A; B_i - sI, C; x, y \end{pmatrix}$$
$$= F^{m_1;n_1}_{m_2;n_2} \left( \begin{array}{c}
 A, B, C, D, E, F; x, y \end{array} \right)$$

$$- x[A][B]\sum_{k=0}^{s-1} F^{m_1;n_1}_{m_2;n_2} \left( \begin{array}{c}
 A + kI; B \times i, E; F; x, y \end{array} \right) [D]^{-1}[E]^{-1},$$

where $A_i, B_i, C_i, D_i, E_i, F_i$ are positive stable matrices in $\mathbb{C}^{r \times r}$ such that

$$A_i B_j = B_j A_i; B_i B_j = B_j B_i; F_j D_i = D_i F_j; D_i E_j = E_j D_i; F_i E_j = E_j F_i$$

and $D_i + kI, E_i + kI$ are invertible for all integers $k \geq 0$.

**Theorem 2.4.** Let $B_i + sI, i = 1, \ldots, n_1$ be invertible for all integers $s \geq 0$. Then the following recursion formula holds true for the Kampé de Fériet hypergeometric matrix function:

$$F^{m_1;n_1}_{m_2;n_2} \left( \begin{array}{c}
 A, B^i + sI, C; E, F; x, y \end{array} \right) = \sum_{k=0}^{s} \left( \begin{array}{c}
 s \end{array} \right) [A]_k [B^i]_k x^k$$

$$\times F^{m_1;n_1}_{m_2;n_2} \left( \begin{array}{c}
 A + kI; B^i + kI, C; E, F; x, y \end{array} \right) [D]^{-1}[E]^{-1},$$

Furthermore, if $B_i - kI$ is invertible for integers $k \leq s$, then

$$F^{m_1;n_1}_{m_2;n_2} \left( \begin{array}{c}
 A, B^i - sI, C; E, F; x, y \end{array} \right) = \sum_{k=0}^{s} \left( \begin{array}{c}
 s \end{array} \right) [A]_k [B^i]_k (-x)^k$$

$$\times F^{m_1;n_1}_{m_2;n_2} \left( \begin{array}{c}
 A + kI; B^i + kI, C; E, F; x, y \end{array} \right) [D]^{-1}[E]^{-1}.$$
where \( A_i, B_i, C_i, D_i, E_i, F_i \) are positive stable matrices in \( \mathbb{C}^{r \times r} \) such that
\[
A_i B_j = B_j A_i; \quad A_i C_j = C_j A_i; \quad B_i C_j = C_j B_i; \quad F_i E_j = E_j F_i; \quad D_i D_j = D_j D_i;
F_j D_i = D_i F_j; \quad D_i E_j = E_j D_i \quad \text{and} \quad D_i + kI, \ E_i + kI \ \text{and} \ F_i + kI \ \text{are invertible}
\]
for all integers \( k \geq 0 \).

Proof. Applying the definition of the Kampé de Fériet hypergeometric matrix function and the relation
\[
(D_i - I)^{-1} = (D_i - I)^{-1} = (D_i - I)^{-1} + n(D_i - I)^{-1},
\]
we obtain the following contiguous matrix relation:
\[
\begin{align*}
F_{m_1;n_1,n_1'}^{m_2;n_2,n_2'} \left( \frac{A}{D_i - I}; \frac{B,C}{E,F}; x,y \right) \\
= F_{m_1;n_1,n_1'}^{m_2;n_2,n_2'} \left( \frac{A:B,C}{D,E,F}; x,y \right) \\
+ x[A][B] \left[ F_{m_1;n_1,n_1'}^{m_2;n_2,n_2'} \left( \frac{A+B+C}{D+E+F}; x,y \right) (D_i - I)^{-1} \right] [D]^{-1} [E]^{-1} \\
+ y[A][C] \left[ F_{m_1;n_1,n_1'}^{m_2;n_2,n_2'} \left( \frac{A+B+C}{D+E+F}; x,y \right) (D_i - I)^{-1} \right] [D]^{-1} [F]^{-1}.
\end{align*}
\]
Using this contiguous matrix relation to the Kampé de Fériet hypergeometric matrix function with the matrix \( D_i - sI \) for \( s \) times, we get (2.16).

Next, we will present recursion formulas for the Kampé de Fériet hypergeometric matrix function \( E_i, F_i \). We omit the proof of the given below theorem.

**Theorem 2.6.** Let \( E_i - sI, i = 1, \ldots, n_2 \) be invertible for all integers \( s \geq 0 \). Then the following recursion formula holds true for the Kampé de Fériet hypergeometric matrix function:
\[
\begin{align*}
F_{m_1;n_1,n_1'}^{m_2;n_2,n_2'} \left( \frac{A}{D,E,F,E_i - sI}; \frac{B,C}{E,F}; x,y \right) \\
= F_{m_1;n_1,n_1'}^{m_2;n_2,n_2'} \left( \frac{A:B,C}{D,E,F}; x,y \right) \\
+ x[A][B] \left[ \sum_{k=1}^{s} F_{m_1;n_1,n_1'}^{m_2;n_2,n_2'} \left( \frac{A+B+C}{D+E+F,E_i - (k-1)sI}; x,y \right) (E_i - kI)^{-1} \right] [E_i]^{-1} [D]^{-1},
\end{align*}
\]
where \( A_i, B_i, C_i, D_i, E_i, F_i \) are positive stable matrices in \( \mathbb{C}^{r \times r} \) such that
\[
A_i B_j = B_j A_i; \quad E_i E_j = E_j E_i; \quad F_i E_j = E_j F_i; \quad F_j D_i = D_j F_i; \quad D_j E_i = E_i D_j \quad \text{and}
D_i + kI, \ E_i + kI \ \text{and} \ F_i + kI \ \text{are invertible}
\]
for all integers \( k \geq 0 \).

The recursion formulas for \( F_{m_2;n_2,n_2'}^{m_1;n_1,n_1'} \left( \frac{A,B,C}{D,E,F,F_i - sI}; x,y \right) \) are obtained by replacing \( B \leftrightarrow C, \ E \leftrightarrow F \) and \( x \leftrightarrow y \) in Theorem 2.6.

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3 Finite matrix summation formulas for the Kampé de Fériet hypergeometric matrix function by derivative operator

In this section, we obtain the finite matrix summation formulas for the Kampé de Fériet hypergeometric matrix function by derivative operator. The $p$-th derivative on $y$ of Kampé de Fériet hypergeometric matrix function is defined as follows:

\[
D^p_y \{ F_{m_1;n_1}^{m_2;n_2} \left( \frac{A;B,C}{D;E,F}; x, y \right) \} = [A]_p [C]_p \int F_{m_2,n_2}^{m_1,n_1} \left( \frac{A+pI;B,C+pI}{D+pI;E,F+pI}; x, y \right) [D]^{-1}_p [F]^{-1}_p,
\]

(3.1)

where $A_i$, $B_i$, $C_i$, $D_i$, $E_i$, $F_i$ are positive stable matrices in $\mathbb{C}^{r \times r}$ such that $A_i C_j = C_j A_i$; $B_i C_j = C_j B_i$; $C_i C_j = C_j C_i$; $D_i E_j = E_j D_i$; $D_i F_j = F_j D_i$ and $D_i + kI$, $E_i + kI$ and $F_i + kI$ are invertible for all integers $k \geq 0$.

By using the generalized Leibnitz formula

\[
D^p_y (f(y)g(y)) = \sum_{k=0}^{p} \binom{p}{k} D^{p-k} y D^k g(y)
\]

and (3.1), we derive the following finite matrix summation formulas of Kampé de Fériet hypergeometric matrix function.

**Theorem 3.1.** Let $A_i$, $B_i$, $C_i$, $D_i$, $E_i$, $F_i$ be positive stable matrices in $\mathbb{C}^{r \times r}$ such that $A_i C_j = C_j A_i$; $B_i C_j = C_j B_i$; $C_i C_j = C_j C_i$; $D_i E_j = E_j D_i$; $D_i F_j = F_j D_i$ and $D_i + kI$, $E_i + kI$ and $F_i + kI$ are invertible for all integers $k \geq 0$.

Then the following finite matrix summation formulas hold for Kampé de Fériet hypergeometric matrix function:

\[
\sum_{k=0}^{p} \binom{p}{k} [A]_k [C]_k y^k F_{m_2,n_2}^{m_1,n_1} \left( \frac{A+kI;B,C+kI}{D+kI;E,F+kI}; x, y \right) [D]^{-1}_k [F]^{-1}_k
\]

\[
= F_{m_1,n_1}^{m_2,n_2} \left( \frac{A;B,C}{D;E,F}; x, y \right).
\]

(3.2)

**Proof.** From definition of Kampé de Fériet hypergeometric matrix function and the generalized Leibnitz formula for differentiation of a product of two functions, we have

\[
D^p_y \{ y^{C_i+1} F_{m_2;n_2}^{m_1,n_1} \left( \frac{A;B,C}{D;E,F}; x, y \right) \}
\]

\[
= \sum_{k=0}^{p} \binom{p}{k} D^{p-k} y^{C_i+1} \int D^k \left( \frac{A;B,C}{D;E,F}; x, y \right) [D]^{-1}_k [F]^{-1}_k
\]

\[
= (C_i)_p y^{C_i+1} \sum_{k=0}^{p} \binom{p}{k} [A]_k [C]_k
\]

\[
x^k F_{m_2,n_2}^{m_1,n_1} \left( \frac{A+kI;B,C+kI}{D+kI;E,F+kI}; x, y \right) [D]^{-1}_k [F]^{-1}_k.
\]
where, using (3.1) and some simplification in the second equality. Next, we combine $y^{C_i+(p-1)I}$ with the variable $y$ in the Kampé de Fériet hypergeometric matrix function and apply the derivative operator $p$-times on $y$ to get the following result:

$$D_y^p\{y^{C_i+(p-1)I} F_{m_2,n_2}^{m_1,n_1}(A:B,C:D;E,F;x,y)\}$$

$$= \sum_{m,n=0}^{\infty} (C_i + nI)_p \prod_{i=1}^{m_1} (A_i)_{m+n} \prod_{i=1}^{n_1} (B_i)_n \prod_{i=1}^{m_2} (C_i)_n \prod_{i=1}^{n_2} (D_i)_{m+n} \prod_{i=1}^{m_2} (E_i)_{m+n} \prod_{i=1}^{n_2} (F_i)_{m+n}$$

$$\times \frac{x^m y^{C_i+(n-1)I}}{m!n!}$$

$$= (C_i)_p y^{C_i-I} F_{m_2,n_2}^{m_1,n_1}(A:B,C_i+pI,D;E,F;x,y).$$

Equating the above two relations leads to (3.2).

**Theorem 3.2.** Let $A_i, B_i, C_i, D_i, E_i, F_i$ be positive stable matrices in $C^r \times r$ such that $A_iC_i = C_iA_i$; $B_iC_i = C_iB_i$; $D_iE_i = E_iD_i$; $D_iF_i = F_iD_i$; $F_iF_j = F_jF_i$ and $D_i + kI, E_i + kI$ and $F_i + kI$ are invertible for all integers $k \geq 0$. Then the following finite matrix summation formulas of Kampé de Fériet hypergeometric matrix function hold true:

$$\sum_{k=0}^{p} \binom{p}{k} [A]_k [C]_k y^k F_{m_2,n_2}^{m_1,n_1}(A+B:C,D+E,F+kI;x,y) (F_i-pI)^{-1} [D]_{-1}^{-1} [F]^{-1}_k$$

$$= F_{m_2,n_2}^{m_1,n_1}(A:B,C,D,E,F;x,y).$$

(3.3)

where $(k-p)I$ is an invertible matrix for $0 \leq k \leq p$ and $i = 1, \ldots, n_2$.

**Proof.** Applying the derivative operator on $y^{F_i-1} F_{m_2,n_2}^{m_1,n_1}(A:B,C,D+E,F;x,y), p$-times, gives the formula in this theorem as explained in the proof of Theorem 3.1. We omit the details.

Applying derivative operator and some simple transformations, we can get the finite matrix summation formulas of Kampé de Fériet hypergeometric matrix function as follows.

**Theorem 3.3.** Let $A_i, B_i, C_i, D_i, E_i, F_i$ be positive stable matrices in $C^r \times r$ such that $A_iC_i = C_iA_i$; $B_iC_i = C_iB_i$; $D_iE_i = E_iD_i$; $D_iF_i = F_iD_i$ and $D_i + kI, E_i + kI$ and $F_i + kI$ are invertible for all integers $k \geq 0$. Then the following finite matrix summation formulas of Kampé de Fériet hypergeometric matrix function hold true:

$$\sum_{k=0}^{r} \binom{r}{k} (-1)^k F_{m_2,n_2}^{m_1,n_1}(A+B:C,D+E,F_i-kI,F_i;x,y) (F_i-F_i)((2-r)I-F_i)^{-1}$$

$$= (-1)^r [A]_r [C]_r y^r F_{m_2,n_2}^{m_1,n_1}(A+B:C,D+E,F+rI;x,y) (F_i-F_i)^{-1} [D]_{-1}^{-1} [F]^{-1}_r.$$  

(3.4)
where \((2 + k - r)I - F_i, F_i - kI\) and \(F_i + (k - 1)I\) is an invertible matrix for \(0 \leq k \leq r\) in \((3.4)\); \(F_i + rI\) is an invertible matrix in \((3.5)\); \(i = 1, \ldots , n'\).

**Proof:** We first prove identity \((3.4)\). From the definition of Kampé de Fériet hypergeometric matrix function and the generalized Leibnitz formula for differentiation of a product of two functions, we obtain the following result:

\[
D^r_y \left[ \left( \begin{array}{c} A : B, C \end{array} \right) \right] \left( \begin{array}{c} \frac{x}{y} \end{array} \right)
= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left( \begin{array}{c} A, B, C \end{array} \right) \left( \begin{array}{c} \frac{x}{y} \end{array} \right)^k \left( \begin{array}{c} \frac{x}{y} \end{array} \right)^{r-k} y^{r-k} \left( \begin{array}{c} \frac{x}{y} \end{array} \right)^{I_r(I-F_i)k(2I-F_i-rI)^{-1}y^{-r}}.
\]

Now using the derivative operator on Kampé de Fériet hypergeometric matrix function for \(r\)-times directly and equating with the above equality gives \((3.4)\) after some simplification. Next, applying the operator \(D^r_y\) on

\[
\left( \begin{array}{c} A : B, C \end{array} \right) \left( \begin{array}{c} \frac{x}{y} \end{array} \right) y^{r-rI} \times y^{r+I-rI},
\]

and proceeding as in the proof of \((3.4)\) gives the result \((3.5)\).

### 4 Infinite summation formulas for Kampé de Fériet hypergeometric matrix function

In this section, we will establish the infinite summation formulas of Kampé de Fériet hypergeometric matrix function.

**Theorem 4.1.** The following infinite summation formulas of Kampé de Fériet hypergeometric matrix function hold true:

\[
\sum_{k=0}^{\infty} \frac{(A_i)_k}{k!} t^k \left( \begin{array}{c} A_i + kI, A_i : B, C \end{array} \right) \left( \begin{array}{c} \frac{x}{1-t} \end{array} \right)^k \frac{y}{1-t},
\]

where \(i = 1, \ldots , m_1;\)

\[
\sum_{k=0}^{\infty} \frac{(B_i)_k}{k!} t^k \left( \begin{array}{c} B_i + kI, B_i : C \end{array} \right) \left( \begin{array}{c} \frac{x}{1-t} \end{array} \right)^k \frac{y}{1-t},
\]

and proceeding as in the proof of \((3.4)\) gives the result \((3.5)\).
Replacing and the transformation (4.1) hold true:

\[
\begin{align*}
\sum_{k,m,n=0}^{\infty} F_{m_1,n_1}^{n_2} \left( A; B, C; D+E, F; x, y \right) = \sum_{k=0}^{\infty} [A] k! [B] k! f_k,
\end{align*}
\]

where \( i = 1, \ldots, n_1 \).

**Proof:** We give a proof of identity (4.1). Applying the definition of Kampé de Fériet hypergeometric matrix function and transformation

\[
(A_i) (A_i + kI)_{m+n} = (A_i)_{m+n} (A_i + (m+n)I)_k,
\]

the left side of (4.1) can be expressed as

\[
\sum_{m,n=0}^{\infty} \, 1_F \left[ A; B; C; D+E, F; x, y \right] = \sum_{k=0}^{\infty} [A] k! [B] k! \frac{t^k}{k!}.
\]

Using the identity

\[
1_F \left[ A; B, C; D+E, F; x, y \right] = (1 - t)^{-A}.
\]

After some simplification, we get the right side of (4.1). This completes the proof of (4.1). Identity (4.2) are proved in a similar manner.

**Theorem 4.2.** Let \( A_i \; B_j = B_j A_i \); \( D_i E_j = E_j D_i \); \( D_i F_j = F_j D_i \); \( E_i F_j = F_j E_i \). Then infinite summation formulas of Kampé de Fériet hypergeometric matrix function hold true:

\[
F_{m_1,n_1}^{n_2} \left( A; B, C; D+E, F; x + t, y \right) = \sum_{k=0}^{\infty} [A] k! [B] k! \frac{t^k}{k!} \times F_{m_1,n_1}^{n_2} \left( A + kI; B + kI, C; D + kI; E, F; x + t, y \right) \frac{D} {D - A} k! \frac{E} {E - A} k!.
\]

**Proof:** From the definition Kampé de Fériet hypergeometric matrix function and the transformation \( (A_i) (A_i + kI)_{m+k} = (A_i)_{m+k} \), the right side of (4.4) can be written as

\[
\sum_{k,m,n=0}^{\infty} \prod_{i=1}^{m_1} (A_i)_{m+n+k} \prod_{i=1}^{n_1} (B_i)_{m+n} \prod_{i=1}^{n_1} (C_i)_{n+k} \prod_{i=1}^{n_1} (D_i)_{m+n+k} \prod_{i=1}^{n_2} (E_i)_{m+k} \prod_{i=1}^{n_2} (F_i)_{m+k} \frac{x^m y^n}{m! n! k!}.
\]

Replacing \( m + k \rightarrow l \) in the above result and after some simplification, we get

\[
\sum_{l,n=0}^{\infty} \prod_{i=1}^{m_1} (A_i)_{l+n} \prod_{i=1}^{n_1} (B_i)_{l+n} \prod_{i=1}^{n_1} (C_i)_n \prod_{i=1}^{n_1} (D_i)_{l+n} \prod_{i=1}^{n_2} (E_i)_l \prod_{i=1}^{n_2} (F_i)_l \frac{y^n}{l! n!} l \sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k
\]

Using relation

\[
\sum_{k=0}^{l} \binom{l}{k} x^k y^{l-k} = (x + y)^l
\]

in the inner summation. This completes the proof of (4.4).
Theorem 4.3. The following infinite summation formulas of Kampé de Fériet hypergeometric matrix function hold true:

$$
\sum_{k=0}^{\infty} \frac{(B_i)k}{k!} (-t)^k F_{m_2,n_2; n'_2}^{m_1,n_1; n'_1} \left( A_{:-kI,B';C}; \frac{1+t}{t} x, y \right) \\
= (1 + t)^{-B_1} F_{m_2,n_2; n'_2}^{m_1,n_1; n'_1} \left( A_{B,C}; \frac{1+t}{t} x, y \right), \tag{4.6}
$$

where $A_jB_i = B_iA_j$, $i = 1, \ldots, n_1$.

Proof: From the definition of Kampé de Fériet hypergeometric matrix function, the left side of (4.6) can be expressed as

$$
\sum_{k=0}^{\infty} \frac{(B_i)k}{m!n!k!} (-t)^k \prod_{i=1}^{m} (A_i)_{m+n} \prod_{j=1, j \neq i}^{n_1} (B_i)_m \prod_{i=1}^{n'_1} (C_i)_n \prod_{i=1}^{m_2} (D_i)_{m+n} \prod_{i=1}^{n_2} (E_i)^{m+n} \prod_{i=1}^{n'_2} (F_i)^{m+n-1} \\
= (1 + t)^{m}.
$$

Replacing $k = m + l$, changing the summation order and simplifying, we get

$$
\sum_{m,n=0}^{\infty} 1 F_0 \left[ \frac{B_i + mI}{-l}; -t \right] \prod_{i=1}^{m} (A_i)_{m+n} \prod_{i=1}^{n'_1} (C_i)_n \prod_{i=1}^{m_2} (D_i)_{m+n} \prod_{i=1}^{n_2} (E_i)^{m+n} \prod_{i=1}^{n'_2} (F_i)^{m+n-1} \frac{x^my^n}{m!n!} \\
= (1 + t)^{m}.
$$

Evaluating the inner $1 F_0$-series in the above equation by 4.3

$$
1 F_0 \left[ B_i + mI; -t \right] = (1 + t)^{-B_1 - mI}.
$$

and simplifying, we get the right side of this theorem. This completes the proof of this theorem.

Theorem 4.4. Let $A_i$, $B_i$, $C_i$, $D_i$, $E_i$, $F_i$ are positive stable matrices in $\mathbb{C}^{r \times r}$. Then infinite summation formulas of Kampé de Fériet hypergeometric matrix function hold true:

$$
\sum_{k=0}^{\infty} \frac{(B_i)k}{k!} \left( \frac{t+x}{t-1} \right)^k F_{m_2,n_2; n'_2}^{m_1,n_1; n'_1} \left( A_{:-kI,B';C}; \frac{1+t}{t} x, y \right) \\
= \left( \frac{1-x}{t+1} \right)^{B_1} F_{m_2,n_2; n'_2}^{m_1,n_1; n'_1} \left( A_{B,C}; \frac{1+t}{t} x, y \right), \tag{4.7}
$$

where $A_jB_i = B_iA_j$, $i = 1, \ldots, n_1$.

Proof: The proof of this theorem is similar to Theorem 4.3. We omit the details.

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