Minimum Degree, Independence Number and Pseudo \([2, b]\)-Factors in Graphs

Siham BEKKAI∗

USTHB, Faculty of Mathematics, PO Box 32 El-Alia Bab Ezzouar
16111 Algiers, Algeria

Abstract

A pseudo \([2, b]\)-factor of a graph \(G\) is a spanning subgraph in which each component \(C\) on at least three vertices verifies \(2 \leq d_C(x) \leq b\), for every vertex \(x\) in \(C\). The main contribution of this paper, is to give an upper bound to the number of components that are edges or vertices in a pseudo \([2, b]\)-factor of a graph \(G\). Given an integer \(b \geq 4\), we show that a graph \(G\) with minimum degree \(\delta\), independence number \(\alpha > \frac{b(\delta - 1)}{2}\) and without isolated vertices possesses a pseudo \([2, b]\)-factor with at most \(\alpha - \lceil \frac{b}{2}(\delta - 1) \rceil\) edges or vertices. This bound is sharp.

Key words: Pseudo \([2, b]\)-Factor; Independence Number; Minimum Degree.

1 Introduction

Throughout this paper, graphs are assumed to be finite and simple. For unexplained concepts and notations, the reader could refer to [2].

Given a graph \(G\), we let \(V(G)\) be its vertex set, \(E(G)\) its edge set and \(n\) its order. The neighborhood of a vertex \(x\) in \(G\) is denoted by \(N_G(x)\) and defined to be the set of vertices of \(G\) adjacent to \(x\); the cardinality of this set is called the degree of \(x\) in \(G\). For convenience, we denote by \(d(x)\) the degree of a vertex \(x\) in \(G\); by \(\delta\) the minimum degree of \(G\) and by \(\alpha\) its independence number. However, if \(H\) is a subgraph of \(G\) then we write \(d_H(x)\); \(\delta_H\) and \(\alpha(H)\) respectively for the degree of \(x\) in \(H\); the minimum degree and the

∗e-mail address: siham.bekkai@gmail.com
independence number of $H$. We denote by $d_G(x,y)$ the distance between $x$ and $y$ in the graph $G$.

A factor of $G$ is a spanning subgraph of $G$, that is a subgraph obtained by edge deletions only. If $S$ is the set of deleted edges, then this subgraph is denoted $G - S$. If $H$ is a subgraph of $G$, then $G - H$ stands for the subgraph induced by $V(G) - V(H)$ in $G$. By starting with a disjoint union of two graphs $G_1$ and $G_2$ and adding edges joining every vertex of $G_1$ to every vertex of $G_2$, we obtain the join of $G_1$ and $G_2$, denoted $G_1 + G_2$. For a positive integer $p$, the graph $pG$ consists of $p$ vertex-disjoint copies of $G$. In all what follows, we use disjoint to stand for vertex-disjoint.

In [1], we defined a pseudo 2-factor of a graph $G$ to be a factor each component of which is a cycle, an edge or a vertex. It can also be seen as a graph partition by a family of vertices, edges and cycles. Graph partition problems have been studied in lots of papers. They consist in partitioning the vertex set of $G$ by disjoint subgraphs chosen to have some specific properties. In [3], Enomoto listed a variety of results dealing with partitions into paths and cycles. The emphasis is generally on the existence of a given partition however, in our study of pseudo-factors, we take interest in the number of components that are edges or vertices in a pseudo-factor of $G$. In [1], we proved that every graph with minimum degree $\delta \geq 1$ and independence number $\alpha \geq \delta$ possesses a pseudo 2-factor with at most $\alpha - \delta + 1$ edges or vertices and that this bound is best possible. Motivated by the desire to know what happens in general cases, we define a pseudo $[a,b]$-factor (where $a$ and $b$ are two integers such that $b \geq a \geq 2$) as a factor of $G$ in which each component $C$ on at least three vertices verifies $a \leq d_C(x) \leq b$, for every $x \in C$. Clearly, a pseudo $[a,b]$-factor with no component that is an edge or a vertex is nothing but an $[a,b]$-factor. Surveys on factors and specifically $[a,b]$-factors and connected factors can be found in [6, 5]. In the present work, we study pseudo $[2,b]$-factors, we consider the case $b \geq 4$ and obtain an upper bound (in function of $\delta$, $\alpha$ and $b$) for the number of components that are edges or vertices in a pseudo $[2,b]$-factor of $G$. Note that, from a result by Kouider and Lonc ([4]), we deduce that if $\alpha \leq \frac{b(\delta - 1)}{2}$ then $G$ has a $[2,b]$-factor. Laying down the condition $\alpha > \frac{b(\delta - 1)}{2}$, the main result of this paper reads as follows:

**Theorem 1** Let $b$ be an integer such that $b \geq 4$ and $G$ a graph of minimum degree $\delta \geq 1$ and independence number $\alpha$ with $\alpha > \frac{b(\delta - 1)}{2}$. Then $G$ possesses a pseudo $[2,b]$-factor with at most $\alpha - \lfloor \frac{b}{2}(\delta - 1) \rfloor$ components that are edges or vertices.

The bound given in Theorem 1 is best possible. Indeed, let $b$ be an integer such that $b \geq 4$ and let $H$ be a nonempty set of vertices. The graph
\[ G = H + pK_2, \text{ where } p > \frac{b}{2}|H|, \text{ has minimum degree } \delta = |H| + 1 \text{ and independence number } \alpha = p. \text{ We can easily verify that } G \text{ possesses a pseudo } [2, b]\text{-factor with } \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor \text{ edges and we can not do better. Also, a simple example reaching the bound of Theorem 1 is a graph } G \text{ obtained by taking a graph } H \text{ on } n \text{ vertices in which every vertex is of degree between 2 and } b \ (b \geq 4), \text{ then taking } n \text{ additional independent vertices and joining exactly one isolated vertex to exactly one vertex of } H. \text{ The graph } G \text{ has minimum degree } \delta = 1, \text{ independence number } \alpha = n \text{ and can not do better.}

Combining Theorem 1 with the results of [1] and [4], we obtain

**Corollary 1** Let \( b \geq 2 \) be an integer such that \( b \neq 3 \). Let \( G \) be a graph of minimum degree \( \delta \) and independence number \( \alpha \) and without isolated vertices. Then \( G \) possesses a pseudo \([2, b]\)-factor with at most \( \max(0, \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor) \) edges or vertices.

## 2 Independence number, minimum degree and pseudo \([2, b]\)-factors

First of all, we put aside the case \( \delta = 1 \) for which we know that we have in \( G \) a pseudo \([2, b]\)-factor with at most \( \alpha \) edges or vertices. Indeed, if we regard a cycle as a component each vertex of which is of degree between 2 and \( b \), then we know that any graph \( G \) can be covered by at most \( \alpha \) cycles, edges or vertices (see for instance [7]). So the bound \( \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor \) holds for \( \delta = 1 \).

From now on, we assume that \( G \) has minimum degree \( \delta \geq 2 \). Let \( F \) be a subgraph of \( G \) such that \( 2 \leq d_F(x) \leq b \) for all \( x \in V(F) \). For the sake of simplifying the writing, such a subgraph \( F \) will be called a \([2, b]\)-subgraph of \( G \). Denote by \( D \) a smallest component of \( G - F \), set \( W = G - (D \cup F) \) and choose \( F \) in such a manner that:

\( (a) \) \( \alpha(G - F) \) is as small as possible;
\( (b) \) subject to \((a)\), the number of vertices of \( D \) is as small as possible;
\( (c) \) subject to \((a)\) and \((b)\), the number of vertices in \( F \) is as small as possible.

Notice that a subgraph \( F \) satisfying the conditions above exists since \( \delta \geq 2 \). Indeed, let us consider a longest path in \( G \) and let \( u \) be one of its endpoints. Let \( v \) be the farthest neighbor of \( u \) on this path and \( P_{uv} \) the segment of \( P \) joining \( u \) and \( v \). The cycle \( C \) formed by the path \( P_{uv} \) and the edge \( uv \) contains \( u \) and all its neighbors so \( \alpha(G - C) < \alpha \). Hence \( F \) is not empty.
We shall show the following theorem which yields Theorem [1]

**Theorem 2** Let \( b \) be an integer such that \( b \geq 4 \). Let \( G \) be a graph of minimum degree \( \delta \geq 2 \) and independence number \( \alpha \) such that \( \alpha > \frac{b(\delta-1)}{2} \). Then there exists a pseudo \([2,b]\)-factor of \( G \) such that \( F \) is the \([2,b]\)-subgraph of this pseudo \([2,b]\)-factor and \( F \) gives \( \alpha(G-F) \leq \alpha - \left\lfloor \frac{b}{2}(\delta-1) \right\rfloor \).

**Proof of Theorem 2** Let \( F \) be a \([2,b]\)-subgraph of \( G \) satisfying the conditions \((a), (b) \) and \((c)\). Denote by \( u_1, \ldots, u_m \) \((m \geq 1)\) the neighbors of \( D \) on \( F \) and by \( P_{ij} \) a path with internal vertices in \( D \) joining two vertices \( u_i \) and \( u_j \) with \( 1 \leq i, j \leq m \) and \( i \neq j \). The proof of Theorem 2 will be divided into several claims. The following one which will be intensively used reminds Lemma 1 in [1].

**Claim 1** Let \( F' \) be a \([2,b]\)-subgraph of \( G \) which contains the neighbors of \( D \) in \( F \) and at least one vertex of \( D \). Setting \( W' = G - (F' \cup D) \), we have \( \alpha(W') > \alpha(W) \).

**Proof of Claim 1** Set \( D' = D - F' \).
\(1\) If \( D' = \emptyset \) then by the choice of \( F \), we have \( \alpha(G-F) \leq \alpha(G-F') \). But \( \alpha(G-F) = \alpha(W) + \alpha(D) \geq \alpha(W) + 1 \) and \( \alpha(G-F') = \alpha(W') \), so \( \alpha(W) < \alpha(W') \).
\(2\) If \( D' \neq \emptyset \) then \( F' \) gives a component \( D' \) smaller than \( D \), so again by the choice of \( F \), we have \( \alpha(W') + \alpha(D) = \alpha(G-F) < \alpha(G-F') = \alpha(W') + \alpha(D') \). But as \( \alpha(D') \leq \alpha(D) \) then we obtain \( \alpha(W) < \alpha(W') \). \( \square \)

In the next claims, we try to learn more about the degrees in \( F \) of its vertices.

**Claim 2** For every \( i, 1 \leq i \leq m \), we have \( N_F(u_i) \cap \{u_1, \ldots, u_m\} = \emptyset \).

**Proof of Claim 2** Suppose that for some \( i, N_F(u_i) \cap \{u_1, \ldots, u_m\} \neq \emptyset \), then there exists a vertex \( u_j \) \((1 \leq j \leq m \) and \( j \neq i)\) such that \( u_iu_j \in E(F) \). Put \( e = u_iu_j \), then \((F-e) \cup P_{ij} \) is a \([2,b]\)-subgraph. Indeed, none of the vertices of \( F \) changes its degree in \((F-e) \cup P_{ij} \) and the internal vertices of \( P_{ij} \) are of degree 2. So taking \( F' = (F-e) \cup P_{ij} \) in Claim 1 we obtain \( \alpha(W) > \alpha(W) \), which is absurd. \( \square \)

**Claim 3** \( d_F(u_i) \leq b-1 \) for at most one vertex \( u_i, i = 1, \ldots, m \).

**Proof of Claim 3** Suppose to the contrary that there exist at least two distinct vertices \( u_k \) and \( u_l \) such that \( d_F(u_k) \leq b-1 \) and \( d_F(u_l) \leq b-1 \).
Then taking $F' = F \cup P_{kl}$ in Claim 5 (notice that in $F'$, $d_F(u_k)$ and $d_F(u_l)$ are at most $b$, and the internal vertices of $P_{kl}$ are of degree 2 in $F'$ so $F'$ is a $[2, b]$-subgraph of $G$), we obtain $\alpha(W) < \alpha(W)$ which is absurd. □

Let $S$ be the set of vertices $x$ in $\cup_{i=1}^{m} N_F(u_i)$ such that $x$ is a common neighbor of at least two vertices in $\{u_1, \ldots, u_m\}$. We have:

**Claim 4**

1. $d_F(x) \leq 3$ for every $x \in S$.
2. If $S$ contains a vertex $x$ such that $d_F(x) = 3$, then
   (a) For every $k$, $1 \leq k \leq m$, we have $d_F(u_k) = b$.
   (b) For every $y \in \cup_{i=1}^{m} N_F(u_i) - \{x\}$ we have $d_F(y) = 2$.

**Proof of Claim 4**

1. Suppose that $d_F(x) \geq 4$ for some $x \in S$. By definition, $x$ is the neighbor in $F$ of at least two vertices say $u_i$ and $u_j$ with $1 \leq i, j \leq m, i \neq j$. Put $e = xu_i$ and $e' = xu_j$. Then in $F' = (F - e - e') \cup P_{ij}$ only $x$ changes its degree but it remains at least 2. So $F'$ is a $[2, b]$-subgraph which leads to a contradiction by Claim 4.

2. Let $x$ be in $N_F(u_i) \cap N_F(u_j)$ ($1 \leq i, j \leq m$ and $i \neq j$) such that $d_F(x) = 3$. Suppose that there exists $u_k$ (which will be the only one by Claim 3) such that $d_F(u_k) \leq b - 1$, we can always assume that $k \neq i$. Then taking $F' = (F - e) \cup P_{ik}$, where $e = xu_i$, in Claim 4 gives a contradiction.

Furthermore, if we suppose that there exists $y \in N_F(u_k) - \{x\}$, with $1 \leq k \leq m$ (we can suppose without loss of generality that $k \neq i$) such that $d_F(y) \geq 3$. Then setting $e = xu_i$, $e' = yu_k$ and taking $F' = (F - e - e') \cup P_{ik}$ in Claim 4 gives a contradiction. Notice that $F'$ is a $[2, b]$-subgraph: indeed, only $x$ and $y$ lose 1 in their degree but they remain of degree at least 2 in $F'$ and the internal vertices of $P_{ik}$ are of degree 2 in $F'$. □

Claim 4 implies that $S$ is an independent set in $F$ and we will deduce later that it is also independent in $G$. But before that, we take a look at the neighbors of $\{u_1, \ldots, u_m\}$ which are not in $S$. For each $u_i$ ($i = 1, \ldots, m$), set $N^*_F(u_i) = \{x \in N_F(u_i); x \notin S\}$.

**Claim 5**

1. If there exist vertices $x$ in $\cup_{i=1}^{m} N^*_F(u_i)$ such that $d_F(x) \geq 3$, then these vertices are in the neighborhood of a same $u_k$, $1 \leq k \leq m$. 5
2. If there exist \( k, 1 \leq k \leq m \) such that \( d_F(u_k) \leq b - 1 \) and \( x \in \cup_{i=1}^{m} N_F(u_i) \) such that \( d_F(x) \geq 3 \), then \( x \in N^*_F(u_k) \).

**Proof of Claim 5**

1. Suppose that there exist \( x \in N^*_F(u_k) \) such that \( d_F(x) \geq 3 \), \( x' \in N^*_F(u_j) \) such that \( d_F(x') \geq 3 \) and \( 1 \leq j, k \leq m, j \neq k \). Then the subgraph \( (F-e-e') \cup P_{kj} \), where \( e = u_kx \) and \( e' = u_jx' \) is a \([2, b]\)-subgraph of \( G \). Taking \( F' = (F-e-e') \cup P_{kj} \) in Claim 1, we obtain a contradiction.

2. Suppose that there exist \( k, 1 \leq k \leq m \) such that \( d_F(u_k) \leq b - 1 \) and \( x \in \cup_{i=1}^{m} N_F(u_i) \) with \( d_F(x) \geq 3 \). By Claim 4(2), \( x \notin S \). Suppose that \( x \in N_F(u_i) \), with \( i \neq k \). Notice that the fact that \( d_F(u_k) \leq b - 1 \) forces \( d_F(u_i) \), by Claim 3, to be equal to \( b \). Taking \( F' = (F-e) \cup P_{ik} \), where \( e = xu_i \), in Claim 1 we obtain \( \alpha(W) > \alpha(W) \) which is absurd. \( \Box \)

Looking more closely at the structure of \( D \), we can say more about the degrees of the vertices in \( \cup_{i=1}^{m} N_F[u_i] \), where \( N_F[u_i] = N_F(u_i) \cup \{u_i\} \) is the closed neighborhood of \( u_i \). First, we remark that \( D \) has minimum degree at most 1.

**Remark 1** \( \delta_D \leq 1 \).

**Proof.** Suppose, by contradiction, that \( \delta_D \geq 2 \) then taking a longest path in \( D \) provides a cycle \( C \) which verifies \( \alpha(D-C) < \alpha(D) \). Put \( F' = F \cup C \), then \( F' \) is a \([2, b]\)-subgraph of \( G \). Moreover, \( \alpha(G-F') = \alpha(D-C) + \alpha(W) < \alpha(D) + \alpha(W) = \alpha(G-F) \) and this contradicts the choice of \( F \). \( \Box \)

Two cases are to consider, the case where \( D \) is a tree (a single vertex is a trivial tree) and the case where \( D \) contains a cycle. The following claim deals with this latter case.

**Claim 6** Suppose that \( D \) contains a cycle. Then

1. \( d_F(x) = 2 \) for all \( x \in \cup_{i=1}^{m} N_F(u_i) \).

2. \( d_F(u_i) = b \) for all \( i, 1 \leq i \leq m \).
Proof of Claim 6

1. Suppose that there exists a vertex \( x \in N_F(u_i) \) such that \( d_F(x) \geq 3 \) and let \( Q \) be an edge or a path with internal vertices in \( D - C \) joining \( u_i \) and \( C \). Then taking \( F' = (F - e) \cup Q \cup C \), where \( e = xu_i \), in Claim 1 gives a contradiction. Notice that \( d_F(x) \geq 2 \) and that \( u_i \) does not change its degree (nor do the other vertices of \( F \)) then \( F' \) is a \([2, b]\)-subgraph of \( G \).

2. Suppose that \( d_F(u_k) \leq b - 1 \) for some \( k, 1 \leq k \leq m \) and let \( Q \) be an edge or a path with internal vertices in \( D - C \) joining \( u_k \) and \( C \). Then, taking \( F' = F \cup Q \cup C \) in Claim 1 gives a contradiction. □

If \( D \) is a tree and \( \delta_D \neq 0 \), then \( D \) has at least two leaves, say \( x_0 \) and \( y_0 \). We relabel \( u_1, \ldots, u_m \), with \( m_1 \leq m \), the vertices in \( N_F(x_0) \cup N_F(y_0) \).

Claim 7 Suppose that \( D \) is a tree and that there exist two vertices \( x_0 \) and \( y_0 \) in \( D \) with \( d_D(x_0) = d_D(y_0) = 1 \) such that \( N_F(x_0) = N_F(y_0) \). Then

1. For all \( k, 1 \leq k \leq m_1 \), \( d_F(u_k) \geq b - 1 \).

2. If there exists a vertex \( x \in \bigcup_{i=1}^{m_1} N_F(u_i) \) such that \( d_F(x) \geq 3 \) then it is the only one.

3. If there exists a vertex \( u_k \) (with \( 1 \leq k \leq m_1 \)) such that \( d_F(u_k) = b - 1 \) then for every vertex \( x \in \bigcup_{i=1}^{m_1} N_F(u_i) \) we have \( d_F(x) = 2 \).

Proof of Claim 7 Let \( P \) be a path in \( D \) joining \( x_0 \) to \( y_0 \).

1. If there exists a vertex \( u_i \) (\( 1 \leq i \leq m_1 \)) such that \( d_F(u_i) \leq b - 2 \). Then taking \( F' = F \cup u_i x_0 P y_0 u_i \) in Claim 1 gives a contradiction.

2. Suppose that there exists a vertex \( x \in \bigcup_{i=1}^{m_1} N_F(u_i) \) such that \( d_F(x) \geq 3 \). If \( x \in S \) then Claim 1 gives what desired. If \( x \notin S \), then \( x \in N_F^*(u_k) \) for some \( 1 \leq k \leq m_1 \). Suppose that there exists \( y \in \bigcup_{i=1}^{m_1} N_F(u_i) \) such that \( d_F(y) \geq 3 \) so \( y \in N_F^*(u_j) \), for some \( 1 \leq j \leq m_1 \). By Claim 5(1), \( j = k \). Taking \( F' = (F - e - e') \cup u_k x_0 P y_0 u_k \), where \( e = xu_k \) and \( e' = yu_k \), in Claim 1 we obtain a contradiction.

3. Finally, suppose that there exists a vertex \( u_k \) (\( 1 \leq k \leq m_1 \)) such that \( d_F(u_k) = b - 1 \) and a vertex \( x \in \bigcup_{i=1}^{m_1} N_F(u_i) \) such that \( d_F(x) \geq 3 \). By Claim 5(2), \( x \in N_F^*(u_k) \). Put \( e = xu_k \). Then taking \( F' = (F - e) \cup u_k x_0 P y_0 \) in Claim 1 gives a contradiction. □
A path $I$ in $F$ with $V(I) \subset V(F)$, $E(I) \subset E(F)$ and such that every internal vertex $x$ of $I$ has $d_F(x) = 2$ is called an interval (or a segment) of $F$. We say that two disjoint intervals $I^{(1)}$ and $I^{(2)}$ in $F$ are path-independent if there exists no path internally disjoint from $F \cup D$ joining a vertex in $I^{(1)}$ to a vertex in $I^{(2)}$. We say that $t$ intervals $I^{(1)}, I^{(2)}, \ldots, I^{(t)}$ ($t \geq 2$) in $F$ are path-independent if they are pairwise path-independent. The following claim will be very useful. It is a shorter version of Lemma 2 in [1] with a short proof.

**Claim 8** Let $I^{(1)}, I^{(2)}, \ldots, I^{(t)}$ ($t \geq 2$) be $t$ disjoint intervals in $F$, containing no neighbor of $D$ and such that $\alpha(W \cup I^{(i)}) = \alpha(W)$ for every $i = 1, \ldots, t$. If $I^{(1)}, I^{(2)}, \ldots, I^{(t)}$ are path-independent, then $\alpha(D \cup W \cup I^{(1)} \cup I^{(2)} \cup \ldots \cup I^{(t)}) = \alpha(W \cup D)$.

**Proof of Claim**

Let $W_i$ be the union of components of $W$ with neighbors in $I^{(i)}$ ($i = 1, \ldots, t$). By hypothesis, the intervals $I^{(i)}$ are pairwise path-independent so $W_i \cap W_j = \emptyset$, for all $1 \leq i, j \leq t$, $i \neq j$. Hence $W' = \bigcup_{i=1}^{t} W_i, W_1 \cup I^{(1)}, \ldots, W_t \cup I^{(t)}$ form a partition of $W \cup I^{(1)} \cup \ldots \cup I^{(t)}$ and it follows that $\alpha(W \cup I^{(1)} \cup \ldots \cup I^{(t)}) = \alpha(W_1 \cup I^{(1)}) + \ldots + \alpha(W_t \cup I^{(t)}) + \alpha(W - \bigcup_{i=1}^{t} W_i)$. On the other hand, as $\alpha(W \cup I^{(i)}) = \alpha(W)$, for every $i = 1, \ldots, t$ then $\alpha(W_i \cup I^{(i)}) = \alpha(W_i)$. This yields $\alpha(W \cup I^{(1)} \cup \ldots \cup I^{(t)}) = \sum_{i=1}^{t} \alpha(W_i) + \alpha(W - \bigcup_{i=1}^{t} W_i) = \alpha(W)$. We finally get $\alpha(W \cup D \cup I^{(1)} \cup \ldots \cup I^{(t)}) = \alpha(W \cup D)$ because the intervals $I^{(i)}$ (with $i = 1, \ldots, t$) do contain no neighbor of $D$. \qed

Let $s$ be the vertex of $S$ (if it exists) such that $d_F(s) = 3$. We put $s$ aside before applying the procedure described hereafter. Provided always that $s$ exists, we set $N_F'(u_i) = N_F(u_i) - \{s\}$ if $s \in N_F(u_i)$ and $N_F'(u_i) = N_F(u_i)$ otherwise ($i = 1, \ldots, m$).

For $u_k$, $1 \leq k \leq m$, denote by $x^k_i$ ($i = 1, \ldots, |N_F'(u_k)|$) its neighbors that belong to $N_F'(u_k)$. Using this notation, we can have $x^k_i = x^k_j$, for some $1 \leq i, j \leq |N_F'(u_k)|$ and $1 \leq j \leq |N_F'(u_i)|$ ($1 \leq k, l \leq m, k \neq l$), in case $x^k_i \in N_F'(u_k) \cap N_F'(u_i) \subset S$.

From now on, let $m' = m_1$ if $D$ is a tree with at least two leaves and $m' = m$ otherwise. We choose the sense $u_k \rightarrow x^k_1$, as a sense of “orientation”. Let $u_k$ ($1 \leq k \leq m'$) be such that $d_F(u_k) = b$ and all its neighbors that are in $N_F'(u_k)$ are of degree 2 in $F$. Starting at $x^k_1$ and following the chosen orientation we go over from a vertex to its neighbor until meeting a vertex which we call $y^k_1$, such that $d_F(y^k_1) \geq 3$ or $y^k_{i+1} = u_j$ for some $j, 1 \leq j \leq m$ (where $y^k_{i+1}$ is the successor of $y^k_i$ following the chosen orientation). This gives an interval $x^k_1 \ldots y^k_1$ which we denote by $P^k_1 = [x^k_1, y^k_1]_F$. 8
We repeat the process using the other neighbors of \( u_k \) that are in \( N'_F(u_k) \). At the \( p^{th} \) step, we consider a vertex \( x^k_p \in X_p = N'_F(u_k) - (\cup_{i=1}^{p-1} V(P_i^k)) \) and construct a path \( P^k_p = x^k_p \ldots y^k_p \) containing \( x^k_p \) and such that \( d_F(y^k_p,+) \geq 3 \) or \( y^k_p = u_j \) for some \( j, 1 \leq j \leq m \). When \( X_r \) becomes empty at the \( r^{th} \) step \( (r \geq p) \), then we consider another vertex \( u_l \) \( (l \neq k) \). We choose as long as possible, \( u_l \) such that \( d_F(u_l) = b \) and its neighborhood that are in \( N'_F(u_l) \) are all of degree 2 in \( F \). We Choose a vertex in \( N'_F(u_l) - \cup_{i=1}^{r-1} V(P_i^k) \), and we do the same construction, until the vertices in \( N'_F(u_l) \) are all in \( (\cup_{i=1}^{r-1} V(P_i^k)) \cup (\cup_{i=1}^{r-1} V(P_i^k)) \). Denote by \( \mathcal{P} \) the set of paths obtained so far.

When it is no more possible to choose a vertex \( u_p, 1 \leq p \leq m' \), such that \( d_F(u_p) = b \) and with all its neighbors that are in \( N'_F(u_p) \) having degree 2 in \( F \), then we take the vertex \( u_q \) of degree at most \( b - 1 \) or having in its neighborhood \( N'_F(u_q) \) vertices of degree at least 3 in \( F \). Notice that \( u_q \) exists only if \( s \) does not (see Claim 4(2)) and if both a vertex \( u_q \) of degree at most \( b - 1 \) (which would be the only one by Claim 3) and vertices \( x^j_i \) of degree at least 3 exist, then these vertices are in the neighborhood of \( u_q \) (see Claim 5). Put \( N_q = \{ x^j_i \in N'_F(u_q) - V(\mathcal{P}) \text{ such that } d_F(x^j_i) = 2 \} \). Starting at a vertex \( x^j_i \in N_q \), we repeat the construction described above until \( N_q \) becomes empty. We update the set \( \mathcal{P} \) at each step.

By construction all the vertices of \( P^k_i \) are of degree 2 in \( F \) so \( V(P^k_i) \cap V(P^j_l) = \emptyset \) for every couple \( P^k_i, P^j_l \) of paths in \( \mathcal{P} \) (they are disjoint), moreover no vertex in \( P^k_i \) is adjacent in \( F \) to a vertex in \( P^j_l \), for all \( P^k_i, P^j_l \) in \( \mathcal{P} \).

We divide the set \( \mathcal{P} \) into three subsets, each containing the paths \( P^k_i = [x^k_i, y^k_i]_F \) of Type 1, Type 2 or Type 3, defined as follows:

**Type 1** If \( y^k_i = u_j \), with \( j \neq k \).

**Type 2** If \( y^k_i = u_j \), with \( j = k \).

**Type 3** If \( y^k_i \neq u_j \) for every \( j, 1 \leq j \leq m \).

For technical reasons, in case \( D \) is a trivial tree or a tree having no couple of leaves with the same neighborhood in \( F \), we stop the procedure described above when it remains no vertex \( u_p \) \( (1 \leq p \leq m') \) such that \( d_F(u_p) = b \), or when the remaining vertex \( u_p \) \( (1 \leq p \leq m') \) has in its neighborhood \( N'_F(u_p) \) a vertex of degree at least 3. We consider \( \mathcal{Q} \) the subset of \( \mathcal{P} \), of paths obtained till then. Let \( \mathcal{P}_1 = \mathcal{Q} \) in this case and \( \mathcal{P}_1 = \mathcal{P} \) in the others.

We show in what follows that the addition of a path of \( \mathcal{P}_1 \) to \( W \cup D \) augments \( \alpha(W \cup D) \) by at least 1.

**Claim 9** For each \( P^k_i \in \mathcal{P}_1 \), we have \( \alpha(W \cup D \cup P^k_i) > \alpha(W \cup D) \).
**Proof of Claim 9.** Let \( P_i^k \) be a path in \( \mathcal{P}_1 \).

1. If \( D \) contains a cycle, then taking \( F' = F - P_i^k \) gives what desired. Indeed, in this case all the vertices \( u_i \) are of degree \( b \) (by Claim 6), as \( b \geq 4 \) after the deletion of \( P_i^k \), the degree of the vertices \( u_i \) \((1 \leq i \leq m')\) remains at least 2. Moreover, by construction of \( P_i^k \), the degree of no vertex in \( F \) becomes smaller than 2, after deletion of \( P_i^k \). So \( F' \) is a \([2, b]\)-subgraph of \( G \). \( F' \) contradicts Condition (c) in the choice of \( F \) (because \( |V(F')| < |V(F)| \)) so \( \alpha(G - F') > \alpha(G - F) \) which yields \( \alpha(W \cup D \cup P_i^k) > \alpha(W \cup D) \).

2. If \( D \) is a tree possessing two vertices \( x_0 \) and \( y_0 \) of degree 1 in \( D \), having the same neighborhood in \( F \) \((N_F(x_0) = N_F(y_0))\). Then if \( P_i^k \) is of Type 1 or 3, then we reason as in (1) and we obtain what desired. If \( P_i^k \) is of Type 2, then (1) is no more efficient if \( d_F(u_k) = b - 1 \) (because the degree of \( u_k \) may become smaller than 2 when \( P_i^k \) is deleted). So we take \( F' = (F - P_i^k) \cup u_kx_0Py_0u_k \), where \( P \) is a path with internal vertices in \( D \) joining \( x_0 \) to \( y_0 \). The subgraph \( F' \) is a \([2, b]\)-subgraph of \( G \) (we have \( d_F(u_k) = d_F(u_k) \)) which gives by Claim 9 what desired.

3. In the other cases, as \( b \geq 4 \) and by the choice of the subset \( \mathcal{P}_1 \), the deletion of any path \( P_i^k \in \mathcal{P}_1 \), gives a \([2, b]\)-subgraph. Reasoning as in (1), we get what desired. \( \square \)

Notice that as \( D \) is independent from \( P_i^k \) (by construction) and from \( W \) then \( \alpha(W \cup D \cup P_i^k) = \alpha(W \cup P_i^k) + \alpha(D) \). Hence the conclusion in Claim 9 is equivalent to \( \alpha(W \cup P_i^k) > \alpha(W) \). For each path \( P_i^k = [x_i^k, y_i^k]_F \) in \( \mathcal{P}_1 \) and following the chosen orientation, let \( v_i^k \) be the first vertex of \( P_i^k \) such that \( \alpha(W \cup [x_i^k, v_i^k]_F) > \alpha(W) \). Notice that \( v_i^k \) is well defined by Claim 9. Denote by \( P_i^{k'} \) the interval \([x_i^k, v_i^k]_F \) of \( P_i^k \) and by \( \mathcal{P}' \) the set of the intervals \( P_i^{k'} \). In what follows, we take interest in the path-independence of the intervals of \( \mathcal{P}' \).

**Claim 10** Let \( P_i^{k'} \) and \( P_j^{l'} \) be two distinct intervals \([x_i^k, v_i^k]_F \) and \([x_j^l, v_j^l]_F \) in \( \mathcal{P}' \) such that \( 1 \leq k, l \leq m', k \neq l \). Then, \( P_i^{k'} \) and \( P_j^{l'} \) are path-independent.

**Proof of Claim 10.** By way of contradiction, suppose that there exist two vertices \( a_i^k \in P_i^{k'} \) and \( a_j^l \in P_j^{l'} \) such that \( a_i^k \) and \( a_j^l \) are joined by \( Q \) which is an edge in \( G \) or a path with internal vertices in \( W \). Choose \( a_i^k \) and \( a_j^l \) so as to minimize the sum \( d_F(x_i^k, a_i^k) + d_F(x_j^l, a_j^l) \). Recall that by construction \( xy \notin E(F) \) for every \( x \in P_i^{k'} \) and \( y \in P_j^{l'} \). The segments \([x_i^k, a_i^k]_F \) and \([x_j^l, a_j^l]_F \) verify the hypothesis of Claim 8. Indeed,
by the choice of \( a_i^k \) and \( a_j^l \), they are path-independent. Furthermore, as \([x_i^k, a_i^k]_F \subset [x_i^k, v_i^k]_F; [x_j^l, a_j^l]_F \subset [x_j^l, v_j^l]_F\) and by the choice of \( v_i^k \) and \( v_j^l \), we have \( \alpha(W \cup [x_i^k, a_i^k]_F) = \alpha(W) \) and \( \alpha(W \cup [x_j^l, a_j^l]_F) = \alpha(W) \). So by Claim 8 we obtain

\[
\alpha(W \cup [x_i^k, a_i^k]_F \cup [x_j^l, a_j^l]_F) = \alpha(W). \tag{*}
\]

Also, taking the \([2,b]\)-subgraph \( F' = (F - ([x_i^k, a_i^k]_F \cup [x_j^l, a_j^l]_F)) \cup Q \cup P_{kl} \), in Claim 1 gives \( \alpha((W - Q) \cup [x_i^k, a_i^k]_F \cup [x_j^l, a_j^l]_F) = \alpha(W) \). But as \( \alpha(W \cup [x_i^k, a_i^k]_F \cup [x_j^l, a_j^l]_F) \) is minimum and \( \alpha((W - Q) \cup [x_i^k, a_i^k]_F \cup [x_j^l, a_j^l]_F) \) hence we get \( \alpha(W \cup [x_i^k, a_i^k]_F \cup [x_j^l, a_j^l]_F) > \alpha(W) \) which contradicts (\(*\)). □

When \( k = l \) in the previous claim, then we consider the structure of \( D \). If \( D \) contains a cycle or \( D \) is a tree with two leaves \( x_0 \) and \( y_0 \) such that \( N_F(x_0) = N_F(y_0) \), then the following claim gives the path-independence of any couple of segments \( P_i^k \) and \( P_j^k \) in \( \mathcal{P}' \).

**Claim 11** Let \( P_i^k \) and \( P_j^k \) be two distinct segments of \( \mathcal{P}' \). Suppose that \( D \) contains a cycle or \( D \) is a tree with two leaves \( x_0 \) and \( y_0 \) such that \( N_F(x_0) = N_F(y_0) \). Then \( P_i^k \) and \( P_j^k \) are path-independent, for every \( k, 1 \leq k \leq m' \).

**Proof of Claim 11** Let \( P_i^k \) and \( P_j^k \) (with \( 1 \leq i, j \leq m' \), \( i \neq j \)) be two segments in \( \mathcal{P}' \). By way of contradiction, suppose that there is a path \( Q \) internally disjoint from \( F \) joining a vertex \( a_i^k \in P_i^k \) to a vertex \( a_j^k \in P_j^k \) and choose \( a_i^k \) and \( a_j^k \) so that the sum \( d_F(x_i^k, a_i^k) + d_F(x_j^k, a_j^k) \) is minimum. The segments \([x_i^k, a_i^k]_F \) and \([x_j^k, a_j^k]_F \) verify the hypothesis of Claim 8. Indeed, they are path-independent, by the choice of \( a_i^k \) and \( a_j^k \). Furthermore, as \([x_i^k, a_i^k]_F \subset [x_i^k, v_i^k]_F \) and \([x_j^k, a_j^k]_F \subset [x_j^k, v_j^k]_F \) and by the choice of \( v_i^k \) and \( v_j^k \), we have \( \alpha(W \cup [x_i^k, a_i^k]_F) = \alpha(W) \) and \( \alpha(W \cup [x_j^k, a_j^k]_F) = \alpha(W) \). So by Claim 8 we obtain

\[
\alpha(W \cup [x_i^k, a_i^k]_F \cup [x_j^k, a_j^k]_F) = \alpha(W). \tag{**}
\]

On the other hand, if \( D \) contains a cycle \( C \), then let \( Q' \) be a path with internal vertices in \( D \) joining \( u_k \) to a vertex on \( C \). If \( D \) is a tree with two leaves \( x_0 \) and \( y_0 \) such that \( N_F(x_0) = N_F(y_0) \). Then let \( P \) be a path with internal vertices in \( D \) joining \( x_0 \) to \( y_0 \). Taking \( F' = (F - ([x_i^k, a_i^k]_F \cup [x_j^k, a_j^k]_F)) \cup Q \cup Q' \cup C \) in the second one and using Claim 11, we obtain in both cases \( \alpha(W \cup [x_i^k, a_i^k]_F \cup [x_j^k, a_j^k]_F) \) which contradicts (**). □

Suppose now that \( D \) is either a trivial tree or \( D \) has no leaves with the same neighborhood in \( F \).

- If all couples of distinct segments \((P_i^k, P_j^k)\) \((k, 1 \leq k \leq m', 1 \leq i, j \leq |N_F(u_k)|)\) in \( \mathcal{P}' \) are path-independent then we have finished. It is par-
particularly the case if $s$ exists. Indeed, if we suppose to the contrary that there exist two distinct segments $P_i^k$ and $P_j^k$ ($k, 1 \leq k \leq m'$, $1 \leq i, j \leq |N_F(u_k)|$) in $\mathcal{P}'$ that are path-dependent, that is there is a path internally disjoint from $D \cup F$ joining a vertex in $a_i^k \in P_i^k$ to a vertex in $a_j^k \in P_j^k$. We choose these vertices so as to minimize the sum $d_F(x_i^k, a_i^k) + d_F(x_j^k, a_j^k)$. Reasoning as in the previous claims using Claim 8 and taking in Claim 1 $F' = (F - [x_i^k, a_i^k]_F \cup [x_j^k, a_j^k]_F) \cup P_{kr} - u_r, s$ where $u_r$ is a neighbor of $s$ such that $r \neq k$, we get a contradiction.

It is also the case if there exists a vertex $u_r$ (1 \leq r \leq m) that is of degree at most $b - 1$ in $F$ or that has in its neighborhood $N_F(u_r)$ a vertex $x$ such that $d_F(x) \geq 3$. Recall that in our case, this vertex is supposed to be put apart in the procedure we have used. So, if we suppose that there is a path internally disjoint from $D \cup F$ joining a vertex in $a_i^k \in P_i^k$ to a vertex in $a_j^k \in P_j^k$ ($k \neq r$). We choose these vertices so as to minimize the sum $d_F(x_i^k, a_i^k) + d_F(x_j^k, a_j^k)$. Here again, using Claim 8 and taking in Claim 1 $F' = (F - [x_i^k, a_i^k]_F \cup [x_j^k, a_j^k]_F) \cup P_{kr}$ if $d_F(u_r) \leq b - 1$ or $F' = (F - [x_i^k, a_i^k]_F \cup [x_j^k, a_j^k]_F) \cup P_{kr} - u_r, x$ if $d_F(u_r) = b$ and $d_F(x) \geq 3$ where $x \in N_F(u_r)$, we get a contradiction.

- If not, then this case is treated in following claim.

**Claim 12** Suppose that $D$ is a trivial tree or a tree with no leaves having the same neighborhood in $F$. Suppose moreover that there exist two distinct segments $P_i^k$ and $P_j^k$ ($k, 1 \leq k \leq m'$, $1 \leq i, j \leq |N_F(u_k)|$) in $\mathcal{P}'$ that are path-dependent. Then there exists no other couple of segments $(P_p^l, P_q^l)$ ($l \neq k, 1 \leq l \leq m'$, $1 \leq p, q \leq |N_F(u_l)|$, $p \neq q$) in $\mathcal{P}'$ that are path-dependent.

**Proof of Claim 12** The proof is basically the same as the previous. Let $a_i^k$ and $a_j^k$ be two vertices in $P_i^k$ and $P_j^k$ respectively that are joined by a path internally disjoint from $F \cup D$ and chosen so as to minimize the sum $d_F(x_i^k, a_i^k) + d_F(x_j^k, a_j^k)$. Suppose to the contrary that there exist two distinct segments $P_p^l, P_q^l$ ($l \neq k, 1 \leq l \leq m'$, $1 \leq p, q \leq |N_F(u_l)|$) and a path internally disjoint from $F \cup D$ joining a vertex $a_p^l \in P_p^l$ to a vertex $a_q^l \in P_q^l$ and choose these vertices in such a way that $d_F(x_p^l, a_p^l) + d_F(x_q^l, a_q^l)$ is minimum. Then using Claim 8 with four intervals and taking $F' = (F - ([x_i^k, a_i^k]_F \cup [x_j^k, a_j^k]_F) - ([x_p^l, a_p^l]_F \cup [x_q^l, a_q^l]_F)) \cup P_{kl}$ in Claim 1 yields a contradiction. As $k \neq l$, then Claim 10 guarantees the path-independence of the segments $P_r^k, P_t^l$ for $r \in \{i, j\}, t \in \{p, q\}$. □

By the claims above, we have that $\mathcal{P}'$ contains several segments that are path-independent. Furthermore, the following remark claims that an additional segment can be considered when needed, particularly when $S$ contains a vertex of degree 3.
Remark 2 If there exists a vertex $s \in S$ such that $d_F(s) = 3$ ($s$ is unique by Claim 4(2)). We consider two cases:

(i) If $s$ is in the neighborhood of three vertices $u_k, u_l, u_p$, with $1 \leq k, l, p \leq m$ and $k, l, p$ pairwise distinct. Then setting $P^* = \{s\}$ we have that $P^*$ is path-independent from any path in $\mathcal{P}'$ and $\alpha(W \cup P^*) > \alpha(W)$.

(ii) If $s$ is in the neighborhood of exactly two vertices, say $u_k$ and $u_l$, $k \neq l, 1 \leq l, k \leq m$. If furthermore $\mathcal{P}$ does not contain paths of Type 3, then there exists a path $P^*$ that is path-independent from any segment in $\mathcal{P}'$ included in a path of Type 1 or Type 2. Furthermore $\alpha(W \cup P^*) > \alpha(W)$.

Proof.

(i) First, taking $F' = (F - \{s\}) \cup P_{kl}$ in Claim 10 we obtain $\alpha(W \cup \{s\}) > \alpha(W)$. Of course, since by Claim 4(2) $d_F(u_i) = b$ for all $i = 1, \ldots, m$, then we have that $F'$ is a $[2, b]$-subgraph of $G$. As $s \in N_F(u_k) \cap N_F(u_l) \cap N_F(u_p) \subset N_F(u_k) \cap N_F(u_l)$ then we can write $\{s\} = P^*_1$ or $\{s\} = P^*_2$ as suitable to apply Claim 10 and show the path-independence of a vertex of degree 2 in $\mathcal{P}'$. Hence reasoning as in Claim 9, taking $F' = F - P$, we obtain $\alpha(W \cup P) > \alpha(W)$. Let $v$ be the first vertex of $P$ following the chosen orientation such that $\alpha(W \cup [s, v]_F) > \alpha(W)$. Setting $P^* = [s, v]_F$, we can show its path-independence with any segment in $\mathcal{P}'$ included in a path of Type 1 or Type 2, like in Claim 10.

(ii) Let us start from $s$ and go forward following the chosen orientation from a vertex of degree 2 in $F$ to a vertex of degree 2 in $F$, until coming across a vertex $y$ whose successor $y^+$ is of degree at least 3. We have that $y^+ \notin \{u_1, \ldots, u_m\}$, otherwise, going in the opposite direction, the segment $[y, s]_F$ (where $s$ is not taken) is a path of Type 3. Moreover, $y^+ \notin \bigcup_{i=1}^m N_F(u_i)$ because since $s$ exists then by Claim 4 every vertex in $\bigcup_{i=1}^m N_F(u_i)$ is of degree 2. So $y^+ \in V(F) - \bigcup_{i=1}^m N_F[u_i]$. The path $P = s \ldots y$ can be considered as a path deriving from $u_k$ ($P = P^*_1$) or deriving from $u_l$ ($P = P^*_2$) and hence reasoning as in Claim 9, taking $F' = F - P$, we obtain $\alpha(W \cup P) > \alpha(W)$. Let $v$ be the first vertex of $P$ following the chosen orientation such that $\alpha(W \cup [s, v]_F) > \alpha(W)$. Setting $P^* = [s, v]_F$, we can show its path-independence with any segment in $\mathcal{P}'$ included in a path of Type 1 or Type 2, like in Claim 10.

Finally, to count the number of pairwise path-independent segments in $\mathcal{P}'$, those whose independence is guaranteed by Claims 10, 11, and 12 we distinguish different cases according to the structure of $D$ and get in any case, at least $\left[ \frac{b(d-1)}{2} \right]$ (recall that $m' \geq \delta - 1$) path-independent segments, adding when necessary the path $P^*$ (in particular when $s$ exists). Notice that when $\mathcal{P}_3$ contains paths of Type 3, then in these paths one vertex in $\bigcup_{i=1}^m N_F(u_i)$ is used at once, so the bound $\left[ \frac{b(d-1)}{2} \right]$ holds, otherwise $P^*$ is added.

The segments in $\mathcal{P}' \cup \{P^*\}$, when added to $W \cup D$ augment $\alpha(W \cup D)$. Put $\mathcal{L} = \mathcal{P}' \cup \{P^*\}$. Recall that the segments of $\mathcal{L}$ are independent from $D$ by construction. For each $P \in \mathcal{L}$, let $W_P$ be the union of components of $W$.
that contain a neighbor of $P$.

We have that
\[
\alpha(W \cup \bigcup_{P \in \mathcal{L}} P) = \alpha(W - \bigcup_{P \in \mathcal{L}} W_P) + \sum_{P \in \mathcal{L}} \alpha(W_P \cup P)
\geq \alpha(W - \bigcup_{P \in \mathcal{L}} W_P) + \sum_{P \in \mathcal{L}} \alpha(W_P) + |\mathcal{L}|
\geq \alpha(W) + \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor.
\]
Hence
\[
\alpha = \alpha(G) = \alpha(W \cup D \cup F) \geq \alpha(W \cup D \cup \bigcup_{P \in \mathcal{L}} P)
\geq \alpha(D) + \alpha(W \cup \bigcup_{P \in \mathcal{L}} P)
\geq \alpha(D) + \alpha(W) + \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor
= \alpha(W \cup D) + \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor.
\]
So $\alpha(W \cup D) = \alpha(G - F) \leq \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ and the proof of Theorem 2 is achieved.

**Proof of Theorem 1.** Since by Theorem 2 $\alpha(G - F) \leq \alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$, then the subgraph of $G$ induced by $V(W \cup D) = V(G - F)$ can be covered by at most $\alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ cycles, edges or vertices (see for instance [7]). Denote by $\mathcal{E}$ the set of cycles, edges or vertices covering $G - F$. The graph $F \cup \mathcal{E}$ is a pseudo $[2, b]$-factor of $G$ with at most $\alpha - \left\lfloor \frac{b}{2}(\delta - 1) \right\rfloor$ edges or vertices. This completes the proof of Theorem 1.

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