SOLID-SET FUNCTIONS AND TOPOLOGICAL MEASURES ON
LOCALLY COMPACT SPACES

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ABSTRACT. A topological measure on a locally compact space is a set function on open and closed subsets which is finitely additive on the collection of open and compact sets, inner regular on open sets, and outer regular on closed sets. Almost all works devoted to topological measures, corresponding non-linear functionals, and their applications deal with compact spaces. The present paper is one in a series that investigates topological measures and corresponding non-linear functionals on locally compact spaces. Here we examine solid and semi-solid sets on a locally compact space. We then give a method of constructing topological measures from solid-set functions on a locally compact, connected, locally connected space. The paper gives examples of finite and infinite topological measures on locally compact, non-compact spaces and presents an easy way to generate topological measures on spaces whose one-point compactification has genus 0 from existing examples of topological measures on compact spaces.

1. INTRODUCTION

The study of topological measures (initially called quasi-measures) began with papers by J. F. Aarnes [1], [2], and [3]. There are now many papers devoted to topological measures and corresponding non-linear functionals; their application to symplectic topology has been studied in numerous papers (beginning with [11]) and a monograph ([12]).

To date, however, almost all these works deal with topological measures on compact spaces. In [4] J. F. Aarnes gives a definition of a topological measure on a locally compact space, presents a procedure for obtaining topological measures from solid set functions on a locally compact, connected, locally connected space, and constructs some examples. While [4] contains many interesting ideas, it is not entirely satisfactory. It contains incomplete proofs and sometimes asks the reader to adapt lengthy proofs from other papers to its subject matter. In

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addition, the approach in [4] makes heavy use of sets that are connected and co-connected (i.e. have connected complements). We do not think this is the right approach for the non-compact setting. For example, using these sets one may end up constructing trivial topological measures (see Example 6.2 in [4]). Finally, the paper has never been published in a refereed mainstream journal.

The construction technique employed by Aarnes for a compact space $X$ in [2] was later nicely simplified by D. J. Grubb, who used semi-solid sets in a compact space. Grubb presented his elegant construction in a series of lectures in 1998, but, unfortunately, never published it. Influenced by ideas of Aarnes and Grubb, we have developed an approach for constructing topological measures on locally compact spaces. Instead of sets that are connected and have connected complements we use sets that are connected and whose complement has finitely many bounded and unbounded components. Our approach allows us to extend a solid set function (see Definition 39) to a topological measure on $X$ when $X$ is a locally compact, connected, and locally connected space; the restriction of a topological measure to solid sets with compact closure is a solid set function that uniquely determines the topological measure. We obtain an easy way to construct topological measures on non-compact locally compact spaces whose one-point compactification has genus 0. (See [2], and section 9 for more information about genus.) Thus, we are able to produce a variety of topological measures on $\mathbb{R}^n$, half-spaces, punctured balls, etc..

The paper is organized as follows. In section 2 we give necessary topological preliminaries. In section 3 we study the structure of solid and semi-solid sets. In section 4 we give a definition and basic properties of topological measures on locally compact spaces, and in section 5 we do the same for solid-set functions. In section 6 on a locally compact, connected, and locally connected space we extend a solid-set function from bounded solid sets to compact connected and bounded semi-solid sets. In section 7 the extension is done to the finite unions of disjoint compact connected sets, and in section 8 the extension produces a topological measure that is uniquely defined by a solid set function (see Theorem 48 and Theorem 51). In section 9 we give examples and present an easy way (Theorem 64) to generate topological measures on locally compact, connected, and locally connected spaces whose one-point compactification has genus 0 from existing examples of topological measures on compact spaces. In this paper by a component of a set we always mean a connected component.
the closure of a set $E$ and by $\partial E$ the boundary of $E$. We denote by $\bigcup$ a union of disjoint sets.

**Definition 1.** A set $A \subseteq X$ is called bounded if $\overline{A}$ is compact. A set $A$ is solid if $A$ is connected, and $X \setminus A$ has only unbounded components. A set $A$ is semi-solid if $A$ is connected, and $X \setminus A$ has only finitely many components.

Several collections of sets will be used often. They include: $\mathcal{O}(X)$, the collection of open subsets of $X$; $\mathcal{C}(X)$, the collection of closed subsets of $X$; and $\mathcal{K}(X)$, the collection of compact subsets of $X$. $\mathcal{P}(X)$ is the power set of $X$.

Often we will work with open, compact or closed sets with particular properties. We use subscripts $c$, $s$ or $ss$ to indicate (open, compact, closed) sets that are, respectively, connected, solid, or semi-solid. For example, $\mathcal{O}_c(X)$ is the collection of open connected subsets of $X$, and $\mathcal{K}_s(X)$ is the collection of compact solid subsets of $X$.

Given any collection $\mathcal{E} \subseteq \mathcal{P}(X)$, we denote by $\mathcal{E}^*$ the subcollection of all bounded sets belonging to $\mathcal{E}$. For example, $\mathcal{A}^*(X) = \mathcal{K}(X) \cup \mathcal{O}^*(X)$ is the collection of compact and bounded open sets, and $\mathcal{A}^*_{ss}(X) = \mathcal{K}_{ss}(X) \cup \mathcal{O}^*_{ss}(X)$ is the collection of bounded open semi-solid and compact semi-solid sets. By $\mathcal{K}_0(X)$ we denote the collection of finite unions of disjoint compact connected sets.

**Definition 2.** A non-negative set function $\mu$ on a family of sets that includes compact sets is called compact-finite if $\mu(K) < \infty$ for each compact $K$. A non-negative set function is called simple if it only assumes values 0 and 1.

We consider set functions that are not identically $\infty$.

2. Preliminaries

This section contains necessary topological preliminaries. Some results in this section are known, but sometimes we give proofs for the reader’s convenience.

**Remark 3.** An easy application of compactness (see, for example, Corollary 3.1.5 in [8]) shows that

(i) If $K_\alpha \subseteq K \subseteq U$, where $U \in \mathcal{O}(X)$, $K, K_\alpha \in \mathcal{C}(X)$, and $K$ and at least one of $K_\alpha$ are compact, then there exists $\alpha_0$ such that $K_\alpha \subseteq U$ for all $\alpha \geq \alpha_0$. 


(ii) If $U\alpha \not\supset U$, $K \subseteq U$, where $K \in \mathcal{K}(X)$, $U$, $U\alpha \in \mathcal{O}(X)$ then there exists $\alpha_0$ such that $K \subseteq U\alpha$ for all $\alpha \geq \alpha_0$.

**Remark 4.**

(a) Suppose $X$ is connected, $U \in \mathcal{O}_c(X)$ and $F \in \mathcal{C}_c(X)$ are disjoint sets. If $U \cap F \neq \emptyset$ then $U \cup F$ is connected.

(b) If $X$ is locally compact and locally connected, for each $x \in X$ and each open set $U$ containing $x$, there is a connected open set $V$ such that $x \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact.

(c) If $V = \bigcup_{s \in S} V_s$ where $V$ and $V_s$ are open sets, then $\overline{V_s} \cap V_t = \emptyset$ for $s \neq t$. In particular, if $X$ is locally connected, and $V = \bigcup_{s \in S} V_s$ is a decomposition of an open set $V$ into connected components, then all components $V_s$ are open, and $\overline{V_s} \cap V_t = \emptyset$ for $s \neq t$.

**Lemma 5.** Let $U$ be an open connected subset of a locally compact and locally connected set $X$. Then for any $x, y \in U$ there is $V_{xy} \in \mathcal{O}_c^*(X)$ such that $x, y \in V_{xy} \subseteq \overline{V_{xy}} \subseteq U$.

**Proof.** Fix $x \in U$. Let

$$A = \{y \in U : \exists V_{xy} \in \mathcal{O}_c^*(X) \text{ such that } x, y \in V_{xy} \subseteq \overline{V_{xy}} \subseteq U\}.$$  

Clearly, $A$ is open, since if $y \in A$ then $V_{xy} \subseteq A$. The set $U \setminus A$ is also open, since if $y \in U \setminus A$ then by Remark 4 there exists $V \in \mathcal{O}_c^*(X)$ such that $y \in V \subseteq \overline{V} \subseteq U$. In fact, $V \subseteq U \setminus A$. (Otherwise, if $z \in V \cap A$ then $V_{xz} \cup V$ is a bounded open connected set with $x, y \in V_{xz} \cup V \subseteq \overline{V_{xz}} \cup \overline{V} \subseteq U$, i.e. $y \in A$.) Thus, $U \setminus A$ is also open. Since $x \in A$, we must have $A = U$. □

We would like to note the following fact. (See, for example, [7], Chapter XI, 6.2)

**Lemma 6.** Let $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$ in a locally compact space $X$. Then there exists a set $V \in \mathcal{O}_c^*(X)$ such that

$$K \subseteq V \subseteq \overline{V} \subseteq U.$$  

In the spirit of this result we can say more, given connectedness.

**Lemma 7.** Let $X$ be a locally compact, locally connected space, $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$. If either $K$ or $U$ is connected there exist a set $V \in \mathcal{O}_c^*(X)$ and a set $C \in \mathcal{K}(X)$ such that

$$K \subseteq V \subseteq C \subseteq U.$$  

One may take $C = \overline{V}$. 

Proof. Case 1: $K \in \mathcal{K} c(X)$. For each $x \in K$ by Remark 5 there is $V_x \in \mathcal{O}^*(X)$ such that $x \in V_x \subseteq \overline{V}_x \subseteq U$. By compactness of $K$, we may write $K \subseteq V_{x_1} \cup \ldots \cup V_{x_n}$. Since both $K$ and $V_{x_i}$ are connected and $x_i \in K \cap V_{x_i}$, $K \cup V_{x_i}$ is connected for each $i = 1, \ldots, n$. Hence,

$$V = \bigcup_{i=1}^n V_{x_i} = \bigcup_{i=1}^n (K \cup V_{x_i})$$

is a bounded open connected set for which

$$K \subseteq V \subseteq \overline{V} \subseteq \bigcup_{i=1}^n \overline{V}_{x_i} \subseteq U.$$

Take $C = \overline{V}$.

Case 2: $U \in \mathcal{O}_c(X)$. As in Case 1 we may find $V_1, \ldots, V_n \in \mathcal{O}^*(X)$ such that $K \subseteq V_1 \cup \ldots \cup V_n \subseteq \overline{V}_1 \cup \ldots \cup \overline{V}_n \subseteq U$.

Pick $x_i \in V_i$ for $i = 1, \ldots, n$. By Lemma 7 choose $W_i \in \mathcal{O}^*_c(X)$ with $x_1, x_i \in W_i \subseteq \overline{W}_i \subseteq U$ for $i = 2, \ldots, n$. Then the set $V_1 \cup W_i \cup V_i$ is connected for each $i = 2, \ldots, n$. Then

$$V = \bigcup_{i=1}^n V_{x_i} \cup \bigcup_{i=2}^n W_j = \bigcup_{i=2}^n (V_1 \cup W_i \cup V_i)$$

is open connected and

$$K \subseteq \bigcup_{i=1}^n V_{x_i} \subseteq V \subseteq \overline{V} \subseteq \bigcup_{i=1}^n \overline{V}_{x_i} \cup \bigcup_{i=2}^n \overline{W}_i \subseteq U.$$

Again, let $C = \overline{V}$. \qed

Lemma 8. Let $X$ be a locally compact, locally connected space. Suppose $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}(X)$. Then there exists $C \in \mathcal{K}_0(X)$ such that $K \subseteq C \subseteq U$.

Proof. Let $U = \bigcup_{i \in I'} U_i$ be the decomposition into connected components. Since $X$ is locally connected, each $U_i$ is open, and by compactness of $K$ there exists a finite set $I \subseteq I'$ such that $K \subseteq \bigcup_{i \in I} U_i$. Then $K \cap U_i = K \setminus \bigcup_{j \in I, j \neq i} U_j$ is a compact set. For each $i \in I$ by Lemma 7 choose $C_i \in \mathcal{K}_c(X)$ such that $K \cap U_i \subseteq C_i \subseteq U_i$. The set $C = \bigcup_{i \in I} C_i$ is the desired set. \qed

Lemma 9. Let $X$ be a connected, locally connected space. Let $A \in \mathcal{A}_c(X)$ and let $B$ be a component of $X \setminus A$. Then

(i) If $A$ is open then $B$ is closed and $\overline{A} \cap B \neq \emptyset$. 

(ii) If $A$ is closed then $B$ is open and $A \setminus B \neq \emptyset$. 

(iii) If $A$ is not a component of $X \setminus A$ then $B$ is not a component of $X \setminus A$. 

(iv) If $A$ is a component of $X \setminus A$ then $B$ is a component of $X \setminus A$. 

Proof. (i) If $A$ is open then $B$ is closed and $\overline{A} \cap B \neq \emptyset$. 

(ii) If $A$ is closed then $B$ is open and $A \setminus B \neq \emptyset$. 

(iii) If $A$ is not a component of $X \setminus A$ then $B$ is not a component of $X \setminus A$. 

(iv) If $A$ is a component of $X \setminus A$ then $B$ is a component of $X \setminus A$. 

Proof. The proofs for (ii) and (iii) are similar to (i) and (iv) respectively. \qed
(ii) If $A$ is closed then $B$ is open and $A \cap B \neq \emptyset$.

(iii) $A \sqcup \bigsqcup_{s \in S} B_s$ is connected for any family \{\(B_s\)\}_s \in S of components of $X \setminus A$.

(iv) $B$ is connected and co-connected.

Proof. The proof of the first two parts is not difficult. For the third part, observe that by Remark 4 $A \sqcup B$ is connected for each component $B$ of $X \setminus A$. To prove the last part, let $X \setminus A = \bigsqcup_{s \in S} B_s$ be a decomposition into connected components. For each $t \in S$ connected component $B_t$ is also co-connected, because

$$X \setminus B_t = A \sqcup \bigsqcup_{s \neq t} B_s$$

is a connected set by the previous part. □

**Lemma 10.** Let $X$ be a connected, locally connected space. Let $K \in \mathcal{K}(X)$, $K \subseteq U \in \mathcal{O}_c^*(X)$. Then at most a finite number of connected components of $X \setminus K$ are not contained in $U$.

Proof. Let $X \setminus K = \bigsqcup_{s \in S} W_s$ be the decomposition of $X \setminus K$ into connected components. Note that each component $W_s$ intersects $U$ since otherwise we would have $W_s \subseteq X \setminus U$, so $\overline{W_s} \subseteq X \setminus U$, so $\overline{W_s} \cap K = \emptyset$, which contradicts Lemma 9. Assume that there are infinitely many components of $X \setminus K$ that are not contained in $U$. Then we may choose components $W_i$, $i = 1, 2, \ldots$, such that $W_i \cap U \neq \emptyset$ and $W_i \cap (X \setminus U) \neq \emptyset$ for each $i$. Connectivity of $W_i$ implies that $W_i \cap \partial U \neq \emptyset$ for each $i$. Let $x_i \in W_i \cap \partial U$. By compactness of $\partial U$, let $x_0 \in \partial U$ be the limit point of $(x_i)$. Then $x_0 \in X \setminus U \subseteq X \setminus K = \bigsqcup_{s \in S} W_s$, i.e. $x_0 \in W_t$ for some $t \in S$. But then all but finitely many $x_i$ must also be in $W_t$, which is impossible, since $W_i \cap W_t = \emptyset$ for $t \neq i$. □

**Corollary 11.** Let $X$ be a connected, locally connected space. Let $K \in \mathcal{K}(X)$ and let $W$ be the union of bounded components of $X \setminus K$. Then $W \in \mathcal{O}_c^*(X)$.

Proof. By Lemma 7 pick $V \in \mathcal{O}_c^*(X)$ such that $K \subseteq V$. From Lemma 10 it follows that $W$ is bounded. By Lemma 9 $W$ is open. □

**Remark 12.** If $A \subseteq B$, $B \in \mathcal{O}_c^*(X)$ then $X \setminus B \subseteq X \setminus A$ and each unbounded component of $X \setminus B$ is contained in an unbounded component of $X \setminus A$.

**Lemma 13.** Let $X$ be a connected, locally connected space. Assume $A \subseteq B$, $B \in \mathcal{O}_c^*(X)$. Then each unbounded component of $X \setminus B$ is contained in an unbounded component of $X \setminus A$ and each unbounded component of $X \setminus A$ contains an unbounded component of $X \setminus B$. 

Proof. Suppose first that $A \subseteq K$, $K \in \mathcal{K}(X)$. The first assertion is Remark 12. Now let $E$ be an unbounded component of $X \setminus A$ which contains no unbounded components of $X \setminus K$. Then $E$ is contained in the union of $K$ and all bounded components of $X \setminus K$. By Corollary 11 this union is a bounded set, and so is $E$, which leads to a contradiction. Therefore, each unbounded component of $X \setminus A$ must contain an unbounded component of $X \setminus K$.

Now suppose $A \subseteq B$, $B \in \mathcal{A}^*(X)$. Choose $K \in \mathcal{K}(X)$ such that $A \subseteq B \subseteq K$. Let $E$ be an unbounded component of $X \setminus A$. By the previous part, $E$ contains an unbounded component $Y$ of $X \setminus K$. But $Y \subseteq G$ for some unbounded component $G$ of $X \setminus B$. Then $G \subseteq E$. □

Lemma 14. Let $X$ be locally compact, locally connected. Let $A \in \mathcal{A}^*(X)$. Then the number of unbounded components of $X \setminus A$ is finite.

Proof. Suppose first that $A \in \mathcal{K}(X)$. By Lemma 7 let $U \subseteq \mathcal{O}^*_c(X)$ be such that $A \subseteq U$. Then the assertion follows from Lemma 10. Now suppose that $A \in \mathcal{O}^*(X)$. Then $\overline{A} \in \mathcal{K}(X)$, so the number of unbounded components of $X \setminus \overline{A}$ is finite. From Lemma 13 it follows that the number of unbounded components of $X \setminus A$ is also finite, since it does not exceed the number of unbounded components of $X \setminus \overline{A}$. □

Lemma 15. Let $X$ be locally compact, connected, locally connected. Suppose $D \subseteq U$ where $D \in \mathcal{K}(X)$, $U \in \mathcal{O}^*(X)$. Let $C$ be the intersection of the union of bounded components of $X \setminus D$ with the union of bounded components of $X \setminus U$. Then $C$ is compact and $U \sqcup C$ is open.

Proof. Write

$$X \setminus D = V \sqcup W,$$

where $V$ is the union of bounded components of $X \setminus D$, and $W$ is the union of unbounded components of $X \setminus D$. Also write

$$X \setminus U = B \sqcup F,$$

where $B$ is the union of bounded components of $X \setminus U$, and $F$ is the union of unbounded components of $X \setminus U$. By Lemma 14 $F$ is a closed set. Let

$$C = V \cap B.$$

Clearly, $C$ and $U$ are disjoint. To see that $U \sqcup C$ is open, note first that $U \sqcup B = X \setminus F$ is an open set. Hence,

$$U \sqcup C = U \sqcup (V \cap B) = (U \cup V) \cap (U \sqcup B).$$
is also an open set. Now we shall show that $C$ is closed, i.e. that $X \setminus C$ is open. Note that $F \subseteq W$ by Remark 12. The set $W$ is open by Lemma 9. Now

$$X \setminus C = X \setminus (B \cap V) = (X \setminus B) \cup (X \setminus V) = (U \cup F) \cup (D \cup W) = (U \cup D) \cup (F \cup W) = U \cup W$$

is an open set. By Corollary 11 the set $C$ is bounded. □

3. Solid and semi-solid sets

**Remark 16.** Let $X$ be locally compact, locally connected. From Lemma 14 it follows that a bounded set $B$ is semi-solid if and only if the number of bounded components of $X \setminus B$ is finite. For a bounded solid set $A$ we have:

$$X \setminus A = \bigcup_{i=1}^{n} E_i$$

where $n \in \mathbb{N}$ and $E_i$'s are unbounded connected components.

**Lemma 17.** Let $X$ be locally compact, locally connected. If $A \in \mathcal{A}^*(X)$ then each bounded component of $X \setminus A$ is a solid bounded set.

**Proof.** Let

$$X \setminus A = \bigcup_{i \in I} B_i \cup \bigcup_{j \in J} D_j$$

be the decomposition of $X \setminus A$ into components, where $B_i$, $i \in I$ are bounded components, and $D_j$, $j \in J$ are unbounded ones. Pick a bounded component $B_k$. Then

$$X \setminus B_k = A \cup \bigcup_{i \neq k} B_i \cup \bigcup_{j \in J} D_j$$

Note that the set on the right hand side is connected by Lemma 9 and unbounded. Hence, $B_k$ is solid. □

A set $A \in \mathcal{A}^*_c(X)$ may not be solid. But we may make it solid by filling in the "holes" by adding to $A$ all bounded components of $X \setminus A$. More precisely, we have the following result.

**Lemma 18.** Let $X$ be locally compact, locally connected. For $A \in \mathcal{A}^*_c(X)$ let \{\(A_i\)\}_{i=1}^n be the unbounded components of $X \setminus A$ and \{\(B_s\)\}_{s \in S} be the bounded components of $X \setminus A$. Then the set \(\tilde{A} = A \cup \bigcup_{s \in S} B_s = X \setminus \bigcup_{i=1}^n A_i\) is solid.

**Proof.** The set $\tilde{A}$ is connected by Lemma 19 It is clear that $X \setminus \tilde{A}$ has only unbounded components. □
Definition 19. Let $X$ be locally compact, locally connected. For $A \in \mathcal{A}_c^*(X)$ let $\{A_i\}_{i=1}^n$ be the unbounded components of $X \setminus A$ and $\{B_s\}_{s \in S}$ be the bounded components of $X \setminus A$. We say that $\tilde{A} = A \sqcup \bigsqcup_{s \in S} B_s = X \setminus \bigsqcup_{i=1}^n A_i$ is a solid hull of $A$.

The next lemma gives some properties of solid hulls of connected sets that are bounded open or compact.

Lemma 20. Let $X$ be locally compact, connected, locally connected. Let $A, B \in \mathcal{A}_c^*(X)$.

(a1) If $A \subseteq B$ then $\tilde{A} \subseteq \tilde{B}$.
(a2) $\tilde{A}$ is a bounded solid set, $A \subseteq \tilde{A}$, and $A$ is solid iff $A = \tilde{A}$.
(a3) $\tilde{A} = \tilde{\tilde{A}}$.
(a4) If $A$ is open, then so is $\tilde{A}$. If $A$ is compact, then so is $\tilde{A}$.
(a5) If $A, B$ are disjoint bounded connected sets, then their solid hulls $\tilde{A}, \tilde{B}$ are either disjoint or one is properly contained in the other.

Proof. Part (a1) follows since each unbounded component of $X \setminus B$ is contained in an unbounded component of $X \setminus A$. If $A$ is compact, choose by Lemma 7 a set $U \in \mathcal{O}_c^*(X)$ that contains $A$. Since $\tilde{A}$ is a union of $A$ and bounded components of $X \setminus A$, applying Lemma 10 we see that $\tilde{A}$ is bounded. The rest of parts (a2) and (a3) is immediate. For part (a4), note that if $A$ is open (closed) then each of finitely many (by Lemma 14) unbounded components of $X \setminus A$ is closed (open) by Lemma 9. To prove part (a5) let $A, B \in \mathcal{A}_c^*(X)$ be disjoint. If $A \subseteq \tilde{B}$ then $\tilde{A} \subseteq \tilde{B}$ by parts (a1) and (a3). To prove that the inclusion is proper, assume to the contrary that $\tilde{A} = \tilde{B}$. If one of the sets $A, B$ is open and the other is closed, this equality means that $\tilde{A}$ is a proper clopen subset of $X$, which contradicts the connectivity of $X$. Suppose $A$ and $B$ are both closed (both open). Then it is easy to see that $\tilde{A} = E$, where $E$ is a bounded component of $X \setminus B$, an open (closed) set. Thus, $A$ is a proper clopen subset of $X$, which contradicts the connectivity of $X$. Therefore, $\tilde{A}$ is properly contained in $\tilde{B}$. Similarly, if $B \subseteq \tilde{A}$ then $\tilde{B} \subseteq \tilde{A}$, and the inclusion is proper. Suppose neither of the above discussed cases $A \subseteq \tilde{B}$ or $B \subseteq \tilde{A}$ occurs. Then by connectedness we must have:

$A \subseteq G, \ B \subseteq E$

where $G$ is an unbounded component of $X \setminus B$ and $E$ is an unbounded component of $X \setminus A$. Then $B \subseteq \tilde{B} \subseteq X \setminus G \subseteq X \setminus A$, i.e. $\tilde{B}$ is contained in a component of $X \setminus A$. Since $\tilde{B}$ is connected and $B \subseteq E$ we must have $\tilde{B} \subseteq E \subseteq X \setminus \tilde{A}$. \qed
Lemma 21. Let $X$ be locally compact, connected, locally connected. If $K \subseteq U$, $K \in \mathcal{K}(X)$, $U \in \mathcal{O}_s^*(X)$ then there exists $C \in \mathcal{K}_s(X)$ such that

$$K \subseteq C \subseteq U.$$ 

Proof. One may take $C$ to be the solid hull of the set $\overline{V}$ from Lemma 7. Then $C \subseteq U$ by Lemma 20. □

Lemma 22. Let $X$ be locally compact, connected, locally connected. Let $K \subseteq V$, $K \in \mathcal{K}_s(X)$, $V \in \mathcal{O}(X)$. Then there exists $W \in \mathcal{O}_s^*(X)$ such that

$$K \subseteq W \subseteq \overline{W} \subseteq V.$$ 

Proof. By Lemma 7 we may choose $U \in \mathcal{O}_c^*(X)$ such that

(1) $K \subseteq U \subseteq \overline{U} \subseteq V.$

Since $K \in \mathcal{K}_s(X)$ let

$$X \setminus K = \bigsqcup_{j=1}^n V_j$$

be the decomposition into connected components. Each $V_j$ is an unbounded open connected set. Since $X \setminus U \subseteq X \setminus K$, for each $j = 1, \ldots, n$ let $E_j$ be the union of all bounded components of $X \setminus U$ contained in $V_j$, and let $F_j$ be the union of finitely many by Lemma 14) unbounded components of $X \setminus U$ contained in $V_j$. By Lemma 9 each $F_j$ is closed. By Lemma 13 each $F_j$ is non-empty. Then by Lemma 9 non-empty set $F_j \cap U = \emptyset$. Set $B = \bigsqcup_{j=1}^n B_j$.

Then $B_j$ is connected because from (2) one sees that $D_j$ intersects every component comprising $F_j$. Thus, each $B_j$ is an unbounded closed connected set, $B_j \cap K = \emptyset$. Set

$$B = \bigcup_{j=1}^n B_j.$$
Then $B \cap K = \emptyset$. Now $K \subseteq X \setminus B$, so let $O$ be the connected component of $X \setminus B$ such that $K \subseteq O \subseteq X \setminus B$. Since $B = \bigcup_{j=1}^{n} B_j \subseteq X \setminus O$, $B$ is contained in the union of unbounded components of $X \setminus O$. Hence, each bounded component of $X \setminus O$ is disjoint from $B$, and so $\tilde{O} \subseteq X \setminus B$. Thus

$$K \subseteq O \subseteq \tilde{O} \subseteq X \setminus B \subseteq U.$$  

By (1) we see that $K \subseteq \tilde{O} \subseteq U \subseteq V$ and we may take $W = \tilde{O}$. \hfill \Box

**Remark 23.** The closure of a solid set need not be solid. For example, in the infinite strip $X = \mathbb{R} \times [0,1]$ the open set $U = ((1,3) \times (0,1)) \cup ((5,7) \times (0,1)) \cup ((2,6) \times (0.25,0.75))$ is solid, while its closure is not.

**Lemma 24.** Let $X$ be locally compact, connected, locally connected. Suppose $K \subseteq W$, $K \in \mathcal{K}(X)$, $W \in \mathcal{O}_{ss}(X)$. Then there exist $V \in \mathcal{O}_{ss}(X)$ and $D \in \mathcal{K}_{ss}(X)$ such that

$$K \subseteq V \subseteq D \subseteq W.$$  

**Proof.** By Lemma [7] choose $U \in \mathcal{O}_{c}(X)$ and $C \in \mathcal{K}(X)$ such that $K \subseteq U \subseteq C \subseteq W$.

Let $X \setminus W = \bigcup_{i=1}^{n} E_i$, $X \setminus C = \bigcup_{t \in T} V_t$, $X \setminus U = \bigcup_{s \in S} D_s$ be decompositions into connected components of $X \setminus W$, $X \setminus C$, $X \setminus U$ respectively. Then

$$\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{t \in T} V_t \subseteq \bigcup_{s \in S} D_s.$$  

Let $T_0 = \{ t \in T : V_t \text{ is unbounded} \}$. Let us index by $T'$ the family of all bounded components of $X \setminus C$ each of which contains a component of $X \setminus W$. So $\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{t \in T_0} V_t \cup \bigcup_{t \in T'} V_t$. Note that $T'$ is a finite index set. Now let us index by $S'$ the family of all bounded components of $X \setminus U$ each of which contains a component $V_t$ for some $t \in T'$. Note that $S'$ is a finite index set and

$$\bigcup_{t \in T'} V_t \subseteq \bigcup_{s \in S'} D_s.$$  

Consider

$$V = \tilde{U} \setminus \bigcup_{s \in S'} D_s.$$
Then $V$ is bounded. Also, $V$ is open. By Lemma 9 $V$ is connected. Since 
\[ X \setminus V = (X \setminus \tilde{U}) \sqcup \bigsqcup_{s \in S'} D_s \subseteq \bigsqcup_{s \in \mathcal{S}} D_s = X \setminus U \]
we see that $V \in \mathcal{O}^{*}_{ss}(X)$ (as the first equality indicates that $X \setminus V$ has finitely many components), and that $U \subseteq V$. Now consider 
\[ D = \tilde{C} \setminus \bigsqcup_{t \in T'} V_t. \]
Then $D$ is compact. By Lemma 9 $D$ is connected. We have 
\[ X \setminus D = (X \setminus \tilde{C}) \sqcup \bigsqcup_{t \in T'} V_t \subseteq (X \setminus \tilde{U}) \sqcup \bigsqcup_{s \in S'} D_s = X \setminus V, \]
so $X \setminus D$ has finitely many components, and $V \subseteq D$. Thus, $D \in \mathcal{K}_{ss}(X)$. Also, 
\[ X \setminus W = \bigsqcup_{i=1}^{n} E_i \subseteq \bigsqcup_{t \in T_0} V_t \sqcup V = (X \setminus \tilde{C}) \sqcup \bigsqcup_{t \in T'} V_t = X \setminus D. \]
Therefore, $D \subseteq W$. Then we have: 
\[ K \subseteq U \subseteq V \subseteq D \subseteq W, \]
where $V \in \mathcal{O}^{*}_{ss}(X)$ and $D \in \mathcal{K}_{ss}(X)$.

Let $V$ be an open subset of $X$ endowed with the subspace topology. Let $D \subseteq V$. By $\overline{D}^V$ we denote the closure of $D$ in $V$ with the subspace topology. As before, $\overline{D}$ stands for the closure of $D$ in $X$.

**Lemma 25.** Let $V \in \mathcal{O}(X)$, $D \subseteq V$. Suppose $V$ is endowed with the subspace topology.

a) If $D$ is bounded in $V$ with the subspace topology then $\overline{D}^V = \overline{D}$ and $\overline{D} \subseteq V$.

b) If $D$ is bounded in $X$ and $\overline{D} \subseteq V$ then $D$ is bounded in $V$.

**Proof.**

a) If $D$ is bounded in $V$ (with the subspace topology) then $\overline{D}^V$ is a compact subset of $V$, and so is a compact in $X$, hence, closed in $X$. That is, $\overline{D}^V = \overline{D}^V$. Since clearly $\overline{D}^V \subseteq \overline{D}$ and $D \subseteq \overline{D}^V$, we have: 
\[ \overline{D} \subseteq \overline{D}^V = \overline{D}^V \subseteq \overline{D}. \]

It follows that $\overline{D} = \overline{D}^V \subseteq V$.

b) Since $\overline{D}$ is compact in $X$ it is easy to see that $\overline{D}^V$ is compact in $V$.

\[ \square \]
Remark 26. Let $V \in O^*(X)$ be endowed with the subspace topology. From Lemma 25 we see that $D$ is bounded in $V$ iff $\overline{D} \subseteq V$. Hence, $D$ is unbounded in $V$ iff $\overline{D} \cap (X \setminus V) \neq \emptyset$.

The next two results give relations between being a solid set in a subspace of $X$ and being a solid set in $X$.

Lemma 27. Let $X$ be locally connected. Let $C \subseteq V$, $C \in \mathcal{S}(X)$, $V \in \mathcal{O}(X)$. Then $C \in \mathcal{S}(V)$, i.e. connected components of $V \setminus C$ are unbounded subsets of $V$.

Proof. Suppose $V \setminus C = \bigcup_{s \in S} V_s$ is the decomposition into connected components in $V$. Note that

$$X \setminus C = (X \setminus V) \cup (V \setminus C) = (X \setminus V) \cup \bigcup_{s \in S} V_s.$$

Assume that there exists $r \in S$ such that $V_r$ is bounded in $V$. By Lemma 25 $\overline{V_r} \cap (X \setminus V) = \emptyset$. Also, by Remark 4 $\overline{V_r} \cap V_s = \emptyset$ for each $s \neq r$. Thus, $\overline{V_r} \subseteq C \cup V_r$. Since $V_r \subseteq X \setminus C$ and $V_r$ is connected in $X$, assume that $V_r$ is contained in a component $U$ of $X \setminus C$. Then $V_r \subseteq U \cap \overline{V_r} \subseteq U \cap (C \cup V_r) \subseteq V_r$, so $U \cap \overline{V_r} \subseteq V_r$. Thus, $U = (U \cap \overline{V_r}) \cup (U \setminus \overline{V_r}) = V_r \cup (U \setminus \overline{V_r})$ is the disconnection of $U$, unless $U = V_r$. This shows that $U = V_r$ is a component of $X \setminus C$. But this is impossible, since $V_r$ is bounded and $C$ is solid. \qed

Lemma 28. Let $A \subseteq V$, $V \in \mathcal{O}^*(X)$. If $A \in \mathcal{A}_\mathcal{S}(V)$ then $A \in \mathcal{A}_\mathcal{S}^*(X)$.

Proof. If $A \in \mathcal{A}_\mathcal{S}(V)$ then $A$ is connected in $X$ and bounded in $X$. Since $V \in \mathcal{O}^*(X)$, we may write $X \setminus V = \bigcup_{i \in I} F_i$ where $F_i$ are unbounded connected components. Let $V \setminus A = \bigcup_{s \in S} E_s$ be the decomposition into connected components in $V$. Each $E_s$ is unbounded in $V$, i.e., $\overline{E_s} \cap (X \setminus V) \neq \emptyset$, hence, $\overline{E_s} \cap F_i \neq \emptyset$ for some $i \in I$. Let $I' = \{i \in I : F_i \cap \overline{E_s} \neq \emptyset \text{ for some } E_s\}$, and for $i \in I'$ let $S_i = \{s \in S : \overline{E_s} \cap F_i \neq \emptyset\}$. For $i \in I'$ the set $F_i \cup \bigcup_{s \in S_i} E_s$ is unbounded and connected. Since

$$X \setminus A = (X \setminus V) \cup (V \setminus A) = \bigcup_{i \in I'} (F_i \cup \bigcup_{s \in S_i} E_s) \cup \bigcup_{i \in I \setminus I'} F_i$$

is a disjoint union of unbounded connected sets, the proof is complete. \qed

Now we shall take a closer look at the structure of an open solid or semi-solid set that contains a closed solid or closed connected set.
Lemma 29. Let $X$ be locally compact, connected, locally connected. Let $C \subseteq V, C \in \mathscr{K}_s(X)$.

(i) Suppose $V \in \mathscr{O}_s(X)$. If $V \setminus C$ is connected then

$$V = C \sqcup W$$

where $W \in \mathscr{O}_{ss}(X)$.

If $V \setminus C$ is disconnected then

$$V = C \sqcup \bigcup_{i=1}^{n} V_i$$

where $V_i \in \mathscr{O}_{ss}(X)$, $i = 1, \ldots, n$.

(ii) Suppose $V \in \mathscr{O}_{ss}(X)$. Then

$$V = C \sqcup \bigcup_{i=1}^{n} V_i$$

where $V_i \in \mathscr{O}_{ss}(X)$, $i = 1, \ldots, n$.

Proof. (i) Suppose $V \in \mathscr{O}_s(X)$ and let

$$X \setminus V = \bigsqcup_{s \in S} F_s$$

be the decomposition into connected components, so $S$ is a finite index set and each $F_s$ is unbounded. If $V \setminus C$ is connected then taking $W = V \setminus C$ we see that

$$X \setminus W = X \setminus (V \setminus C) = (X \setminus V) \sqcup C = C \sqcup \bigsqcup_{s \in S} F_s$$

has finitely many components, i.e. $W \in \mathscr{O}_{ss}(X)$.

Now assume that $V \setminus C$ is not connected. By Lemma 27 and Remark 26 $C \in \mathcal{C}_s(V)$ and is bounded in $V$. The set $V \setminus C$ is also disconnected in $V$, so using Remark 16 let

$$V \setminus C = \bigsqcup_{i=1}^{n} V_i, \ n \geq 2$$

be the decomposition into connected (unbounded in $V$) components in $V$. Each $V_i$ is connected in $X$. To show that each $V_i \in \mathscr{O}_s(X)$ we only need to check that the components of $X \setminus V_i$ are unbounded. For simplicity, we shall show it for $V_i$. For $2 \leq j \leq n$ by Lemma 27 and Remark 26 $V_j$ intersects $X \setminus V$, hence, intersects some $F_s$. Let $S_1 = \{s \in S : F_s \cap \overline{V_j} \neq \emptyset \text{ for some } 2 \leq j \leq n\}$. By Remark 4 and Lemma 9 the set
\[
(\bigsqcup_{s \in S_1} F_s \sqcup C \sqcup \bigsqcup_{j=2}^n V_j) \text{ is connected. It is also unbounded. Now}
\]
\[
X \setminus V_1 = (X \setminus V) \cup (V \setminus V_1) = \bigcup_{s \in S} F_s \sqcup C \sqcup \bigsqcup_{j=2}^n V_j \bigcup \bigsqcup_{s \in S \setminus S_1} F_s
\]

Since \( X \setminus V_1 \) is the disjoint union of connected unbounded sets, it follows that \( V_1 \) is solid.

(ii) Suppose \( V \in \mathcal{O}_{ss}(X) \) and let \( \bigsqcup_{j=1}^k F_j \) be the components of \( X \setminus V \). By Lemma 27 and Remark 26 \( C \in \mathcal{C}_s(V) \) and is bounded in \( V \). Let
\[
V \setminus C = \bigsqcup_{i=1}^n V_i, \quad n \geq 1
\]

be the decomposition into connected components in \( V \) according to Remark 16. Each \( V_i \) is connected in \( X \), and to show that each \( V_i \in \mathcal{O}_{ss}(X) \) we only need to check that \( X \setminus V_i \) has finitely many components. For simplicity, we shall show it for \( V_1 \). We have:
\[
X \setminus V_1 = (X \setminus V) \cup (V \setminus V_1) = \bigcup_{j=1}^k F_j \cup C \cup \bigsqcup_{i \neq 1} V_i.
\]

Since \( X \setminus V_1 \) is a finite disjoint union of connected sets, the number of components of \( X \setminus V_1 \) is finite, so \( V_1 \in \mathcal{O}_{ss}(X) \).

\[
\square
\]

**Lemma 30.** Let \( X \) be locally compact, connected, locally connected. Suppose \( C \subseteq U, C \in \mathcal{K}(X), U \in \mathcal{O}_s^*(X) \). If \( U \setminus \tilde{C} \) is disconnected then
\[
U = C \sqcup \bigsqcup_{s \in S} V_s, \quad V_s \in \mathcal{O}_s^*(X).
\]

If \( U \setminus \tilde{C} \) is connected then
\[
U = C \sqcup \bigsqcup_{s \in S} V_s \sqcup W, \quad V_s \in \mathcal{O}_s^*(X), \quad W \in \mathcal{O}_s^*(X).
\]

**Proof.** Note first that \( \tilde{C} \in \mathcal{K}_s(X) \) and \( \tilde{C} \subseteq U \) by Lemma 20. Assume that \( U \setminus \tilde{C} \) is disconnected. By Lemma 29 we may write \( U = \tilde{C} \sqcup \bigsqcup_{i=1}^n U_i, \quad U_i \in \mathcal{O}_s^*(X) \). But \( \tilde{C} = C \sqcup \bigsqcup_{\alpha} V_\alpha \), where \( V_\alpha \) are bounded components of \( X \setminus C \), so by Lemma
each $V_\alpha \in \mathcal{O}_s^*(X)$. After reindexing, one may write

$$U = C \sqcup \bigsqcup_{s \in S} V_s, \quad V_s \in \mathcal{O}_s^*(X).$$

The proof for the case when $U \setminus \tilde{C}$ is connected follows similarly from Lemma 29. □

Lemma 31. Let $X$ be locally compact, connected, locally connected. Suppose that

$$V = \bigsqcup_{j=1}^m C_j \sqcup \bigsqcup_{t \in T} U_t$$

where $V \in \mathcal{O}_s^*(X)$, $C_j \in \mathcal{K}_s(X)$, $U_t \in \mathcal{O}_c^*(X)$. Then $T$ is finite.

Proof. The proof is by induction on $m$. Let $m = 1$. Using Lemma 29 we have

$$V \setminus C_1 = \bigsqcup_{i=1}^n V_i = \bigsqcup_{t \in T} U_t.$$

Since sets $V_i$ and $U_t$ are connected, $T$ must be finite. Now let $V = \bigsqcup_{j=1}^m C_j \sqcup \bigsqcup_{t \in T} U_t$ and assume that the result holds for any bounded open semi-solid set which contains less than $m$ compact solid sets. Using Lemma 29 we see that

$$V = C_1 \sqcup \bigsqcup_{i=1}^n V_i = C_1 \sqcup \bigsqcup_{j=2}^m C_j \sqcup \bigsqcup_{t \in T} U_t,$$

where $V_i \in \mathcal{O}_s^*(X)$. All involved sets are connected, so each set $V_i$ is the disjoint union of sets from the collection $\{C_2, \ldots, C_m, U_t, t \in T\}$. By the induction hypothesis each $V_i$ contains finitely many sets, and it follows that $T$ is finite. □

Lemma 32. Let $X$ be locally compact, connected, locally connected. If $A = \bigsqcup_{t \in T} A_t$, $A, A_t \in \mathcal{O}_s^*(X)$ with at most finitely many $A_t \in \mathcal{K}_s(X)$ then $T$ is finite.

Proof. Assume first that $A \in \mathcal{O}_s^*(X)$. If the cardinality $|T| > 1$ then there must be a compact solid set among $A_t$, and the result follows from Lemma 31. Assume now that $A \in \mathcal{K}_s(X)$ and write

$$A = \bigsqcup_{j=1}^m C_j \sqcup \bigsqcup_{t \in T} U_t,$$

where $C_j \in \mathcal{K}_s(X)$, $U_t \in \mathcal{O}_s^*(X)$. We need to show that $T$ is finite. By Lemma 22 choose $V \in \mathcal{O}_s^*(X)$ such that $A \subseteq V$. Then from Lemma 29 we may write
V \ A = \bigsqcup_{i=1}^{n} V_i, \text{ where } V_i \in \mathcal{O}^*_s(X). \text{ Then }
V = \bigsqcup_{j=1}^{m} C_j \sqcup \bigsqcup_{t \in T} U_t \sqcup \bigcup_{i=1}^{n} V_i,
and by Lemma 31 T is finite. \square

Remark 33. Lemma 14, Lemma 20, Lemma 22, and Lemma 27 are close to Lemmas 3.5, 3.6, 3.8, 3.9, and 4.2 in [4]. Lemma 15 is related to a part in the proof of Lemma 5.9 in [4]. The case ” V \ C is disconnected” in the first part of Lemma 29 is Lemma 4.3 in [4], and Lemma 32 is an expanded (to compact sets as well) version of Lemma 4.4 in [4]. In all instances our proofs are modified, expanded, or different, compared to the proofs in [4].

4. Definition and Basic Properties of Topological Measures on Locally Compact Spaces

Definition 34. A topological measure on X is a set function \( \mu : \mathcal{C}(X) \cup \mathcal{O}(X) \to [0, \infty] \) satisfying the following conditions:

(TM1) if \( A, B, A \sqcup B \in \mathcal{K}(X) \cup \mathcal{O}(X) \) then \( \mu(A \sqcup B) = \mu(A) + \mu(B) \);
(TM2) \( \mu(U) = \sup\{\mu(K) : K \in \mathcal{K}(X), K \subseteq U\} \) for \( U \in \mathcal{O}(X) \);
(TM3) \( \mu(F) = \inf\{\mu(U) : U \in \mathcal{O}(X), F \subseteq U\} \) for \( F \in \mathcal{C}(X) \).

Remark 35. It is important that in Definition 34 condition (TM1) holds for sets from \( \mathcal{K}(X) \cup \mathcal{O}(X) \). In fact, (TM1) fails on \( \mathcal{C}(X) \cup \mathcal{O}(X) \). See Example 62 or Example 63 below.

The following result gives some immediate properties of topological measures on locally compact spaces.

Lemma 36. The following is true for a topological measure:

(1) \( \mu \) is monotone, i.e. if \( A \subseteq B, A, B \in \mathcal{C}(X) \cup \mathcal{O}(X) \) then \( \mu(A) \leq \mu(B) \).
(2) If an increasing net \( U_s \nearrow U \), where \( U_s, U \in \mathcal{O}(X) \) then \( \mu(U_s) \nearrow \mu(U) \).
In particular, \( \mu \) is additive on \( \mathcal{O}(X) \).
(3) \( \mu(\emptyset) = 0 \).
(4) If \( V \sqcup K \subseteq U \), where \( U, V \in \mathcal{O}(X) \), \( K \in \mathcal{K}(X) \) then \( \mu(V) + \mu(K) \leq \mu(U) \).
(5) If \( \mu \) is compact-finite then \( \mu(A) < \infty \) for each \( A \in \mathcal{A}^*(X) \). \( \mu \) is finite (i.e. \( \mu(X) < \infty \)) iff \( \mu \) is real-valued.
If \( X \) is locally compact, locally connected then for any \( U \in \mathcal{O}(X) \)
\[
\mu(U) = \sup\{\mu(C) : C \in \mathcal{H}_0(X), C \subseteq U\}.
\]

If \( X \) is connected then
\[
\mu(X) = \sup\{\mu(K) : K \in \mathcal{K}(X)\}.
\]

If \( X \) is locally compact, connected, locally connected then also
\[
\mu(X) = \sup\{\mu(K) : K \in \mathcal{K}_s(X)\}.
\]

Proof. (t1) The monotonicity is immediate from Definition 34 if sets \( A \) and \( B \) are both open or both closed. It is also easy to show the monotonicity in the case when one of the sets is open and the other one is closed.

(t2) Suppose \( U_s \supset U, U_s, U \in \mathcal{O}(X) \). Let compact \( K \subseteq U \). By Remark 3 there is \( t \in S \) such that \( K \subseteq U_s \) for all \( s \geq t \). Then \( \mu(K) \leq \mu(U) \) for all \( s \geq t \), and we see from the inner regularity (whether \( \mu(U) < \infty \) or \( \mu(U) = \infty \)) that \( \mu(U_s) \not\supset \mu(U) \).

(t3) Easy to see since \( \mu \) is not identically \( \infty \).

(t4) Easy to see from part (TM2) of Definition 34.

(t5) If \( U \) is an open bounded set then \( \mu(U) \leq \mu(U) < \infty \). The second statement is obvious.

(t6) By Lemma 8 for arbitrary \( K \subseteq U, K \in \mathcal{C}(X), U \in \mathcal{O}(X) \) there is \( C \in \mathcal{C}_0(X) \) with \( K \subseteq C \subseteq U \). By monotonicity \( \mu(K) \leq \mu(C) \leq \mu(U) \).

Then
\[
\mu(U) = \sup\{\mu(K) : K \in \mathcal{C}(X), K \subseteq U\} 
\leq \sup\{\mu(C) : C \in \mathcal{C}_0(X), K \subseteq C \subseteq U\} \leq \mu(U)
\]

(t7) Follows from Lemma 7 and Lemma 20.

Proposition 37. Let \( X \) be locally compact. A set function \( \mu : \mathcal{O}(X) \cup \mathcal{C}(X) \to [0, \infty] \) satisfying (TM2) and (TM3) of Definition 34 also satisfies (TM1) if the following conditions hold:

(c1) \( \mu(U \cup V) = \mu(U) + \mu(V) \) for any disjoint open sets \( U, V \)

(c2) \( \mu(U) = \mu(K) + \mu(U \setminus K) \) whenever \( K \subseteq U, K \in \mathcal{H}(X), U \in \mathcal{O}(X) \).

Proof. Our proof is an expanded version of the proof of Proposition 2.2 in [13] where the result first appeared for compact-finite topological measures. Suppose that \( \mu \) is a set function satisfying (TM2), (TM3) as well as conditions (c1) and
We need to show that $\mu$ satisfies (TM1). $X$ is completely regular, so it is evident from (c1) and (TM3) that $\mu$ is finitely additive on $\mathcal{O}(X)$ and on $\mathcal{K}(X)$. Hence, we only need to check (TM1) in the situation when $A \in \mathcal{K}(X)$, $B \in \mathcal{O}(X)$, and $A \sqcup B$ is either compact or open. If $A \sqcup B$ is open then using condition (c2) we get:

$$\mu(A \sqcup B) = \mu((A \sqcup B) \setminus A) + \mu(A) = \mu(B) + \mu(A).$$

Now suppose $A \sqcup B \in \mathcal{K}(X)$. Note that (TM3) implies monotonicity of $\mu$ on $\mathcal{K}(X)$. Let $C \in \mathcal{K}(X)$, $C \subseteq B$. Then finite additivity and monotonicity of $\mu$ on $\mathcal{K}(X)$ gives:

$$\mu(A) + \mu(C) = \mu(A \sqcup C) \leq \mu(A \sqcup B).$$

By (TM2)

$$\mu(A) + \mu(B) \leq \mu(A \sqcup B).$$

Now we will show the opposite inequality. It is obvious if $\mu(A) = \infty$, so let $\mu(A) < \infty$, and for $\epsilon > 0$ pick $U \in \mathcal{O}(X)$ such that $A \subseteq U$ and $\mu(U) < \mu(A) + \epsilon$. Then compact set $A \sqcup B$ is contained in the open set $B \cup U$. Also, the compact set $(A \sqcup B) \setminus U = B \setminus U$ is contained in $B \cup U$, and $(B \cup U) \setminus (B \setminus U) = U$. Applying (TM2) and then condition (c2) we see that

$$\mu(A \sqcup B) \leq \mu(B \cup U) = \mu((B \cup U) \setminus (B \setminus U)) + \mu(B \setminus U)
= \mu(U) + \mu(B \setminus U) \leq \mu(U) + \mu(B)
\leq \mu(A) + \mu(B) + \epsilon$$

Thus,

$$\mu(A \sqcup B) \leq \mu(A) + \mu(B).$$

This finishes the proof. \[\square\]

**Remark 38.** The condition [c2] of Proposition 37, $\mu(U) = \mu(K) + \mu(U \setminus K)$ for $U$ open and $K$ compact, is a very useful one. Of course, any topological measure satisfies this condition. It is interesting to note that a similar condition regarding a bounded open subset of a closed set fails for topological measures, i.e.

$$\mu(F) = \mu(U) + \mu(F \setminus U)$$

where $F$ is closed and $U$ is open bounded, in general is not true, as Example 63 below shows.
5. SOLID SET FUNCTIONS

Our goal now is to extend a set function defined on a smaller collection of subsets of $X$ than $\mathcal{O}(X) \cup \mathcal{C}(X)$ to a topological measure on $X$. One such convenient collection is the collection of solid bounded open and solid compact sets, and the corresponding set function is a solid set function.

**Definition 39.** A function $\lambda : \mathcal{A}_s(X) \to [0, \infty)$ is a solid set function on $X$ if

(s1) whenever $\bigbigcup_{i=1}^{n} C_i \subseteq C, C, C_i \in \mathcal{K}_s(X)$, we have $\sum_{i=1}^{n} \lambda(C_i) \leq \lambda(C)$;

(s2) $\lambda(U) = \sup \{ \lambda(K) : K \subseteq U, K \in \mathcal{K}_s(X) \}$ for $U \in \mathcal{O}_s(X)$;

(s3) $\lambda(K) = \inf \{ \lambda(U) : K \subseteq U, U \in \mathcal{O}_s(X) \}$ for $K \in \mathcal{K}_s(X)$;

(s4) if $A = \bigbigcup_{i=1}^{n} A_i, A, A_i \in \mathcal{A}_s(X)$ then $\lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$.

**Lemma 40.** Let $X$ be locally compact, connected, locally connected. Suppose $\lambda$ is a solid set function on $X$. Then

(i) $\lambda(\emptyset) = 0$

(ii) if $\bigcup_{s \in S} A_s \subseteq A$, where $A_s, A \in \mathcal{A}_s(X)$, then $\sum_{s \in S} \lambda(A_s) \leq \lambda(A)$

**Proof.** From Definition 39 we see that $\lambda(\emptyset) = 0$. Now let $\bigcup_{s \in S} A_s \subseteq A$, where $A_s, A \in \mathcal{A}_s(X)$. Since $\sum_{s \in S} \lambda(A_s) = \sup \{ \sum_{s \in S'} \lambda(A_s) : S' \subseteq S, S'$ is finite $\}$, it is enough to assume that $S$ is finite. By regularity in Definition 39 we may take all sets $A_s$ to be disjoint compact solid. If also $A \in \mathcal{K}_s(X)$, the assertion is just part (s1) of Definition 39. If $A \in \mathcal{O}_s(X)$ then there exists $C \in \mathcal{K}_s(X)$ such that $\bigcup_{s \in S} A_s \subseteq C \subseteq A$ by Lemma 21. Now the assertion follows from parts (s1) and (s2) of Definition 39. \qed

6. EXTENSION TO $\mathcal{A}_s(X) \cup \mathcal{K}_c(X)$

We start with a solid set function $\lambda : \mathcal{A}_s(X) \to [0, \infty)$ on a locally compact, connected, locally connected space $X$. Our goal is to extend $\lambda$ to a topological measure on $X$. We shall do this in steps, each time extending the current set function to a new set function defined on a larger collection of sets.

**Definition 41.** Let $X$ be locally compact, connected, locally connected. For $A \in \mathcal{A}_s(X) \cup \mathcal{K}_c(X)$ define

$$\lambda_1(A) = \lambda(\tilde{A}) - \sum_{i \in I} \lambda(B_i),$$

where $\{B_i : i \in I\}$ is the family of bounded components of $X \setminus A$. 
By Lemma 17 each \( B_i \in \mathcal{A}_s^*(X) \). If \( A \in \mathcal{A}_s^*(X) \cup \mathcal{K}_c(X) \) then \( \bigcup_{i \in I} B_i \subseteq \tilde{A} \) and by Lemma 40
\[
\sum_{i \in I} \lambda(B_i) \leq \lambda(\tilde{A}).
\]

**Lemma 42.** The set function \( \lambda_1 : \mathcal{A}_s^*(X) \cup \mathcal{K}_c(X) \to [0, \infty) \) defined in Definition 41 satisfies the following properties:

(i) \( \lambda_1 \) is real-valued and \( \lambda_1 = \lambda \) on \( \mathcal{A}_s^*(X) \).

(ii) Suppose \( \bigcup_{i=1}^n A_i \cup \bigcup_{s \in S} B_s \subseteq A \), where \( A, A_i \in \mathcal{A}_s^*(X) \cup \mathcal{K}_c(X) \) and \( B_s \in \mathcal{A}_s^*(X) \). Then
\[
\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) \leq \lambda_1(A).
\]

In particular, if \( \bigcup_{i=1}^n C_i \subseteq C \) where \( C_i, C \in \mathcal{K}_c(X) \) then
\[
\sum_{i=1}^n \lambda_1(C_i) \leq \lambda_1(C)
\]

and if \( A \subseteq B, A, B \in \mathcal{A}_s^*(X) \cup \mathcal{K}_c(X) \) then
\[
\lambda_1(A) \leq \lambda_1(B).
\]

(iii) Suppose that \( \bigcup_{i=1}^n A_i \cup \bigcup_{s \in S} B_s = A \), where \( A, A_i \in \mathcal{A}_s^*(X) \cup \mathcal{K}_c(X) \) and \( B_s \in \mathcal{A}_s^*(X) \) with at most finitely many of \( B_s \in \mathcal{K}_c(X) \). Then
\[
\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) = \lambda_1(A).
\]

**Proof.**

(i) Obvious from Lemma 20, Definition 41 and Lemma 40.

(ii) Suppose that \( \bigcup_{i=1}^n A_i \cup \bigcup_{s \in S} B_s \subseteq A \), where \( A, A_i \in \mathcal{A}_s^*(X) \cup \mathcal{K}_c(X) \) and \( B_s \in \mathcal{A}_s^*(X) \). We may assume that \( A \in \mathcal{A}_s^*(X) \), since the inequality
\[
\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) \leq \lambda_1(A)
\]
is equivalent to
\[
\sum_{i=1}^n \lambda_1(A_i) + \sum_{s \in S} \lambda_1(B_s) + \sum_{t \in T} \lambda_1(D_t) \leq \lambda_1(\tilde{A}),
\]
where \( \{D_t : t \in T\} \) is the disjoint family of bounded components of \( X \setminus A \), and by Lemma 17 each \( D_t \in \mathcal{A}_s^*(X) \).

The proof is by induction on \( n \). For \( n = 0 \) the statement is Lemma 40. Suppose now \( n \geq 1 \) and assume the result is true for any disjoint collection (contained in a bounded solid set) of bounded semi-solid or
compact connected sets among which there are less than \( n \) non-solid sets. Assume now that we have \( n \) disjoint sets \( A_1, \ldots, A_n \) from the collection \( \mathcal{A}_s^* (X) \cup \mathcal{K}_c (X) \). Consider a partial order on \{\( A_1, A_2, \ldots, A_n \)\} where \( A_i \leq A_j \) iff \( \tilde{A}_i \subseteq \tilde{A}_j \). (See Lemma 20) Let \( A_1, \ldots, A_p \) where \( p \leq n \) be maximal elements in \{\( A_1, A_2, \ldots A_n \)\} with respect to this partial order.

For a maximal element \( A_k, k \in \{1, \ldots, p\} \) define the following index sets:

\[
I_k = \{i \in \{p+1, \ldots, n\} : A_i \text{ is contained in a bounded component of } X \setminus A_k\},
\]

\[
S_k = \{s \in S : B_s \text{ is contained in a bounded component of } X \setminus A_k\}.
\]

Let \( \{E_\alpha\}_{\alpha \in H} \) be the disjoint family of bounded components of \( X \setminus A_k \). Then we may say that

\[
I_k = \bigsqcup_{\alpha \in H} I_{k,\alpha}, \quad S_k = \bigsqcup_{\alpha \in H} S_{k,\alpha}
\]

where

\[
I_{k,\alpha} = \{i \in \{p+1, \ldots, n\} : A_i \subseteq E_\alpha\},
\]

\[
S_{k,\alpha} = \{s \in S : B_s \subseteq E_\alpha\}.
\]

The set \( I_k \) and each set \( I_{k,\alpha} \) has cardinality \(< n \). The set \( E_\alpha \) is a solid set according to Lemma 17 and

\[
\bigcup_{i \in I_{k,\alpha}} A_i \sqcup \bigcup_{s \in S_{k,\alpha}} B_s \subseteq E_\alpha.
\]

By induction hypothesis

\[
\sum_{i \in I_{k,\alpha}} \lambda_1 (A_i) + \sum_{s \in S_{k,\alpha}} \lambda_1 (B_s) \leq \lambda_1 (E_\alpha).
\]

It follows that

\[
\sum_{i \in I_k} \lambda_1 (A_i) + \sum_{s \in S_k} \lambda_1 (B_s) = \sum_{\alpha \in H} \left( \sum_{i \in I_{k,\alpha}} \lambda_1 (A_i) + \sum_{s \in S_{k,\alpha}} \lambda_1 (B_s) \right) \leq \sum_{\alpha \in H} \lambda_1 (E_\alpha).
\]

Then using part (i) and Definition 41 we have:

\[
\lambda_1 (A_k) + \sum_{i \in I_k} \lambda_1 (A_i) + \sum_{s \in S_k} \lambda_1 (B_s) \leq \lambda_1 (A_k) + \sum_{\alpha \in H} \lambda_1 (E_\alpha) = \lambda_1 (\tilde{A}_k).
\]

Notice that \( \tilde{A}_1, \ldots, \tilde{A}_p \), being the maximal elements, are all disjoint by part (a5) of Lemma 20. This also implies that the sets \( I_k, k = 1, \ldots, p \)
are disjoint (otherwise, if \( i \in I_k \) and also \( i \in I_m, \ 1 \leq k, m \leq p \) then \( \tilde{A}_k \cap \tilde{A}_m \neq \emptyset \)). Similarly, the sets \( S_k, \ k = 1, \ldots, p \) are also disjoint.

Consider the index set

\[
S' = S \setminus \bigcup_{k=1}^{p} S_k.
\]

Note that \( \{1, \ldots, n\} = \{1, \ldots, p\} \cup \bigcup_{k=1}^{p} I_k \). Indeed, if \( i \in \{1, \ldots, n\} \setminus \{1, \ldots, p\} \) we must have \( A_i \subseteq \tilde{A}_i \subseteq \tilde{A}_k \) for some maximal element \( A_k \) (where \( k \in \{1, \ldots, p\} \)), and since \( A_i \) and \( A_k \) are disjoint, \( A_i \) must be contained in a bounded component of \( A_k \), i.e. \( i \in I_k \). Now we have:

\[
\begin{align*}
\sum_{i=1}^{n} \lambda_1(A_i) + \sum_{s \in S} \lambda(B_s) & = \sum_{k=1}^{p} \left( \lambda_1(A_k) + \sum_{i \in I_k} \lambda_1(A_i) + \sum_{s \in S_k} \lambda(B_s) \right) + \sum_{s \in S'} \lambda(B_s) \\
& \leq \sum_{k=1}^{p} \lambda(\tilde{A}_k) + \sum_{s \in S'} \lambda(B_s) \\
& \leq \lambda(\tilde{A})
\end{align*}
\]

The first inequality is by formula (6), and for the last inequality we applied Lemma 40, since \( \{\tilde{A}_k\}_{k=1}^{p} \bigcup \{B_s\}_{s \in S'} \) is a collection of disjoint solid sets contained in the solid set \( A \).

(iii) The proof is almost identical to the proof of the previous part, and we keep the same notations. Again, we may assume that \( A \in \mathcal{A}^*_c(X) \), since the inequalities (3) and (4) become equalities. The proof is by induction on \( n \), and the case \( n = 0 \) is given by Lemma 32 and part (s4) of Definition 39. The inequalities in the induction step become equalities once one observes that (5) above becomes \( \bigcup_{i \in I_k, \alpha} A_i \sqcup \bigcup_{s \in S_k, \alpha} B_s = E_\alpha \) (note that \( \tilde{A}_k \subseteq A \), so \( E_\alpha \subseteq A \)). Since \( \bigcup_{k=1}^{p} \tilde{A}_k \sqcup \bigcup_{s \in S'} B_s = A \), the last inequality in the proof of the previous part becomes an equality by Lemma 32 and part (s4) of Definition 39.

\[\square\]

7. Extension to \( \mathcal{H}_0(X) \)

Our goal now is to extend the set function \( \lambda_1 \) to a set function \( \lambda_2 \) defined on \( \mathcal{H}_0(X) \). Recall that \( K \in \mathcal{H}_0(X) \) if \( K = \bigsqcup_{i=1}^{n} K_i \) where \( n \in \mathbb{N} \) and \( K_i \in \mathcal{H}_c(X) \) for \( i = 1, \ldots, n \).
Definition 43. For $K = \bigcup_{i=1}^{n} K_i$, where $K_i \in \mathcal{K}_c(X)$, let

$$\lambda_2(K) = \sum_{i=1}^{n} \lambda_1(K_i).$$

Lemma 44. The set function $\lambda_2$ from Definition 43 satisfies the following properties:

(i) $\lambda_2$ is real-valued, $\lambda_2 = \lambda_1$ on $\mathcal{K}_c(X)$ and $\lambda_2 = \lambda$ on $\mathcal{K}_s(X)$.

(ii) $\lambda_2$ is finitely additive on $\mathcal{K}_0(X)$.

(iii) $\lambda_2$ is monotone on $\mathcal{K}_0(X)$.

Proof. The first part easily follows from the definition of $\lambda_2$ and Lemma 42. The second part is obvious. To prove the third one, let $C \subseteq K$, where $C, K \in \mathcal{K}_0(X)$. Write $C = \bigcup_{i=1}^{n} C_i$, $K = \bigcup_{j=1}^{m} K_j$, where the sets $C_i (i = 1, \ldots, n)$ and $K_j (j = 1, \ldots, m)$ are compact connected. By connectivity, each $C_i$ is contained in one of the sets $K_j$. Consider index sets $I_j = \{ i : C_i \subseteq K_j \}$ for each $j = 1, \ldots, m$. By Lemma 42 we have $\sum_{i \in I_j} \lambda_1(C_i) \leq \lambda_1(K_j)$. Then

$$\lambda_2(C) = \sum_{i=1}^{n} \lambda_1(C_i) = \sum_{j=1}^{m} \sum_{i \in I_j} \lambda_1(C_i) \leq \sum_{j=1}^{m} \lambda_1(K_j) = \lambda_2(K).$$

8. Extension to $\mathcal{O}(X) \cup \mathcal{C}(X)$

We are now ready to extend the set function $\lambda_2$ to a set function $\mu$ defined on $\mathcal{O}(X) \cup \mathcal{C}(X)$.

Definition 45. For an open set $U$ define

$$\mu(U) = \sup \{ \lambda_2(K) : K \subseteq U, K \in \mathcal{K}_0(X) \},$$

and for a closed set $F$ let

$$\mu(F) = \inf \{ \mu(U) : F \subseteq U, U \in \mathcal{O}(X) \}. $$

Note that $\mu$ may assume $\infty$.

Lemma 46. The set function $\mu$ in Definition 45 satisfies the following properties:

(p1) $\mu$ is monotone, i.e. if $A \subseteq B$, $A, B \in \mathcal{O}(X) \cup \mathcal{C}(X)$ then $\mu(A) \leq \mu(B)$.

(p2) $\mu(A) < \infty$ for each $A \in \mathcal{K}^*(X)$, so $\mu$ is compact-finite.

(p3) $\mu \geq \lambda_2$ on $\mathcal{K}_0(X)$. 

(p4) Let $K \subseteq V, K \in \mathcal{K}(X), \ V \in \mathcal{O}(X)$. Then for any positive \( \epsilon \) there exists \( K_1 \in \mathcal{K}_0(X) \) such that \( K \subseteq K_1 \subseteq V \) and \( \mu(K_1) - \mu(K) < \epsilon \).

(p5) \( \mu = \lambda \) on \( \mathcal{K}_s^*(X) \).

(p6) \( \mu \) is finitely additive on open sets.

(p7) If \( G = F \cup K \), where \( G, F \in \mathcal{O}(X), \ K \in \mathcal{K}(X) \) then \( \mu(G) = \mu(F) + \mu(K) \). In particular, \( \mu \) is finitely additive on compact sets.

(p8) \( \mu \) is additive on \( \mathcal{O}(X) \), i.e. if \( V = \bigcup_{i \in I} V_i \), where \( V, V_i \in \mathcal{O}(X) \) for all \( i \in I \), then \( \mu(V) = \sum_{i \in I} \mu(V_i) \).

(p9) If \( G \cup V = F \) where \( G, F \in \mathcal{O}(X), \ V \in \mathcal{O}(X) \) then \( \mu(G) + \mu(V) \leq \mu(F) \).

(p10) If \( G \cup V \subseteq U \) where \( G \in \mathcal{O}(X), \ V, U \in \mathcal{O}(X) \) then \( \mu(G) + \mu(V) \leq \mu(U) \).

(p11) \( \mu = \lambda_1 \) on \( \mathcal{K}_c(X) \) and \( \mu = \lambda_2 \) on \( \mathcal{K}_0(X) \).

(p12) \( \mu(U) = \sup\{\mu(C) : C \subseteq U, C \in \mathcal{K}_c(X)\}, \ U \in \mathcal{O}(X) \).

Proof. (p1) It is obvious that \( \mu \) is monotone on open sets and on closed sets. Let \( V \in \mathcal{O}(X), \ F \in \mathcal{O}(X) \). The monotonicity in the case \( F \subseteq V \) is obvious. Suppose \( V \subseteq F \). For any open set \( U \) with \( F \subseteq U \) we have \( V \subseteq U \), so \( \mu(V) \leq \mu(U) \). Then taking infimum over sets \( U \) we obtain \( \mu(V) \leq \mu(F) \).

(p2) Let \( K \in \mathcal{K}(X) \). By Lemma\(^7\) choose \( V \in \mathcal{O}_s^*(X) \) and \( C \in \mathcal{K}_c(X) \) such that \( K \subseteq V \subseteq C \subseteq U \). For any \( D \in \mathcal{K}_0(X), D \subseteq V \) by Lemma\(^4\) we have \( \lambda_2(D) \leq \lambda_2(C) \), and \( \lambda_2(C) < \infty \). By Definition\(^4\) \( \mu(V) \leq \lambda_2(C) \), and then \( \mu(K) \leq \mu(V) \leq \lambda_2(C) < \infty \). Thus, \( \mu \) is compact-finite. If \( U \) is an open bounded set then \( \mu(U) \leq \mu(\overline{U}) < \infty \).

(p3) Let \( K \in \mathcal{K}_0(X) \). For any open set \( U \) containing \( K \) we have \( \mu(U) \geq \lambda_2(K) \) by the definition of \( \mu \). Then, again from the definition of \( \mu, \mu(K) \geq \lambda_2(K) \).

(p4) \( \mu(K) < \infty \), so by Definition\(^4\) find \( U \in \mathcal{O}(X) \) such that \( U \subseteq V, \mu(U) - \mu(K) < \epsilon \). Let \( U_1, \ldots, U_n \) be finitely many connected components of \( U \) that cover \( K \). By Lemma\(^7\) pick \( V_i \in \mathcal{O}_s^*(X) \) such that \( K \cap U_i \subseteq V_i \subseteq \overline{V_i} \subseteq U_i \) for \( i = 1, \ldots, n \). We may take \( K_1 = \bigcup_{i=1}^n \overline{V_i} \), for \( K_1 \subseteq V \) and

\[
\mu(K_1) - \mu(K) < \mu\left(\bigcup_{i=1}^n U_i\right) - \mu(K) \leq \mu(U) - \mu(K) < \epsilon.
\]

(p5) First we shall show that \( \mu = \lambda \) on \( \mathcal{O}_s^*(X) \). Let \( U \in \mathcal{O}_s^*(X) \), so by part (p2) \( \mu(U) < \infty \). By Definition\(^4\) given \( \epsilon > 0 \), choose \( K \in \mathcal{K}_0(X) \)
such that \( K \subseteq U \) and \( \mu(U) - \epsilon < \lambda_2(K) \). By Lemma \(21 \) there exists \( C \in \mathcal{K}_s(X) \) such that \( K \subseteq C \subseteq U \). Now using Lemma \(44 \) and Definition \(39 \) we have:

\[
\mu(U) - \epsilon < \lambda_2(K) \leq \lambda_2(C) \\
\leq \sup \{ \lambda_2(C) : C \subseteq U, C \in \mathcal{K}_s(X) \} \\
= \sup \{ \lambda(C) : C \subseteq U, C \in \mathcal{K}_s(X) \} = \lambda(U).
\]

Hence, \( \mu(U) \leq \lambda(U) \). For the opposite inequality, observe that by Lemma \(44 \) \( \lambda = \lambda_2 \) on \( \mathcal{K}_s(X) \), so by Definition \(39 \)

\[
\lambda(U) = \sup \{ \lambda(C) : C \subseteq U, C \in \mathcal{K}_s(X) \} \\
= \sup \{ \lambda_2(C) : C \subseteq U, C \in \mathcal{K}_s(X) \} \\
\leq \sup \{ \lambda_2(C) : C \subseteq U, C \in \mathcal{K}_0(X) \} = \mu(U).
\]

Therefore, \( \mu(U) = \lambda(U) \) for any \( U \in \mathcal{O}_s^*(X) \). Now we shall show that \( \mu = \lambda \) on \( \mathcal{K}_s(X) \). From part \( \text{(p3)} \) above and Lemma \(44 \) we have \( \mu \geq \lambda_2 = \lambda \) on \( \mathcal{K}_s(X) \). Since \( \mu = \lambda \) on \( \mathcal{O}_s^*(X) \), for \( C \in \mathcal{K}_s(X) \) we have by Definition \(39 \) and Defintion \(45 \)

\[
\lambda(C) = \inf \{ \lambda(U) : U \in \mathcal{O}_s^*(X), C \subseteq U \} \\
= \inf \{ \mu(U) : U \in \mathcal{O}_s^*(X), C \subseteq U \} \\
\geq \inf \{ \mu(U) : U \in \mathcal{O}(X), C \subseteq U \} = \mu(C)
\]

Therefore, \( \mu = \lambda \) on \( \mathcal{K}_s(X) \).

\(\text{(p6)}\) Let \( U_1, U_2 \in \mathcal{O}(X) \) be disjoint. For any \( C_1, C_2 \in \mathcal{K}_0(X) \) with \( C_i \subseteq U_i, \ i = 1, 2 \) we have by Lemma \(44 \) and Definition \(45 \)

\[
\lambda_2(C_1) + \lambda_2(C_2) = \lambda_2(C_1 \cup C_2) \leq \mu(U_1 \cup U_2).
\]

Then by Definition \(45 \) we obtain

\[
\mu(U_1) + \mu(U_2) \leq \mu(U_1 \cup U_2).
\]

For the converse inequality, note that given \( C \subseteq U_1 \cup U_2, \ C \in \mathcal{K}_0(X) \) we have \( C_i = C \cap U_i \in \mathcal{K}_0(X), \ i = 1, 2 \) (since each connected component of \( C \) must be contained either in \( U_1 \) or in \( U_2 \)) and \( C = C_1 \cup C_2 \). Then

\[
\lambda_2(C) = \lambda_2(C_1) + \lambda_2(C_2) \leq \mu(U_1) + \mu(U_2),
\]

giving

\[
\mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2).
\]
(p7) Let \( C_1, C_2 \) be a compact and a closed set that are disjoint. Given \( U \in \mathcal{O}(X) \), \( C_1 \sqcup C_2 \subseteq U \) we may find disjoint open sets \( U_1, U_2 \) such that
\[ U_1 \sqcup U_2 \subseteq U, \quad C_i \subseteq U_i, \quad i = 1, 2. \]
Then by parts [p6] and [p1]
\[ \mu(C_1) + \mu(C_2) \leq \mu(U_1) + \mu(U_2) = \mu(U_1 \sqcup U_2) \leq \mu(U), \]
so using Definition [45] we have
\[ \mu(C_1) + \mu(C_2) \leq \mu(C_1 \sqcup C_2). \]
For the converse inequality, observe that for any \( U_1, U_2 \in \mathcal{O}(X) \) such that \( C_i \subseteq U_i, \quad i = 1, 2 \) one may find disjoint open sets \( V_1, V_2 \) with \( C_i \subseteq V_i \subseteq U_i, \quad i = 1, 2. \) Then by parts [p6] and [p1]
\[ \mu(C_1 \sqcup C_2) \leq \mu(V_1 \sqcup V_2) = \mu(V_1) + \mu(V_2) \leq \mu(U_1) + \mu(U_2), \]
which gives by Definition [45]
\[ \mu(C_1 \sqcup C_2) \leq \mu(C_1) + \mu(C_2). \]

(p8) Let \( V = \bigsqcup_{i \in I} V_i \) with \( V, V_i \in \mathcal{O}(X)(X) \) for all \( i \in I \). By parts [p6] and [p1] for any finite \( I' \subseteq I \)
\[ \sum_{i \in I'} \mu(V_i) = \mu(\bigsqcup_{i \in I'} V_i) \leq \mu(V). \]
Then \( \sum_{i \in I} \mu(V_i) \leq \mu(V) \). To prove the opposite inequality, first assume that \( \mu(V) < \infty \). For \( \epsilon > 0 \) find a compact \( C \in \mathcal{K}_0(X) \) contained in \( V \) such that \( \mu(V) - \epsilon < \lambda_2(C) \). By compactness, \( C \subseteq \bigsqcup_{i \in I'} V_i \) for some finite subset \( I' \) of \( I \). Then \( C = \bigsqcup_{i \in I'} C_i \) where \( C_i = C \cap V_i \subseteq V_i \), and \( C_i \in \mathcal{K}_0(X) \) for each \( i \in I' \). By Lemma [44] and part [p3] we have:
\[ \mu(V) - \epsilon < \lambda_2(C) = \lambda_2(\bigsqcup_{i \in I'} C_i) = \sum_{i \in I'} \lambda_2(C_i) \leq \sum_{i \in I'} \mu(C_i) \leq \sum_{i \in I'} \mu(V_i) \]
Therefore, \( \mu(V) \leq \sum_{i \in I} \mu(V_i) \). This shows that \( \mu(V) = \sum_{i \in I} \mu(V_i) \) when \( \mu(V) < \infty \).
Now suppose $\mu(V) = \infty$. For $n \in \mathbb{N}$ find a compact $K \subseteq V$ such that $\mu(K) > n$. Choose a finite index set $I_n \subseteq I$ such that $K \subseteq \bigcup_{i \in I_n} V_i$. Then

$$\sum_{i \in I} \mu(V_i) \geq \sum_{i \in I_n} \mu(V_i) = \mu(\bigcup_{i \in I_n} V_i) \geq \mu(K) > n.$$ 

It follows that $\sum_{i \in I} \mu(V_i) = \infty = \mu(V)$.

(p9) It is enough to show the statement for the case $\mu(F) < \infty$. If $K \subseteq V$, $K \in \mathcal{K}_0(X)$ then $G \sqcup K \subseteq F$. By parts (p3), (p7) and (p1) $\mu(G) + \lambda_2(K) \leq \mu(G) + \mu(K) \leq \mu(F)$. Then $\mu(G) + \mu(V) \leq \mu(F)$.

(p10) It is enough to show the statement for the case $\mu(U) < \infty$. If $K \subseteq V$, $K \in \mathcal{K}_0(X)$ then $F = G \sqcup K \subseteq U$. By parts (p3), (p7) and Definition 45 $\mu(G) + \lambda_2(K) \leq \mu(G) + \mu(K) = \mu(F) \leq \mu(U)$. Then $\mu(G) + \mu(V) \leq \mu(U)$.

(p11) Let $C \in \mathcal{K}_{c}(X)$. According to Lemma 17 and Definition 19 write $\hat{C} \in \mathcal{K}_{c}(X)$ as $\hat{C} = C \sqcup \bigcup_{i \in I} U_i$ where $U_i \in \mathcal{O}_{s}^{*}(X)$ are the bounded components of $X \setminus C$. Given $\epsilon > 0$ choose by Definition 39 $V \in \mathcal{O}_{s}^{*}(X)$ such that $\hat{C} \subseteq V$ and $\lambda(V) - \lambda(\hat{C}) < \epsilon$. By parts (p8), (p9) and (p1)

$$\mu(C) + \sum_{i \in I} \mu(U_i) = \mu(C) + \mu(\bigcup_{i \in I} (U_i)) \leq \mu(\hat{C}) \leq \mu(V).$$

Then using part (p5) and Definition 41 we have:

$$\mu(C) \leq \mu(V) - \sum_{i \in I} \mu(U_i) = \lambda(V) - \sum_{i \in I} \lambda(U_i) \leq \lambda(\hat{C}) - \sum_{i \in I} \lambda(U_i) + \epsilon = \lambda_1(C) + \epsilon$$

Thus, $\mu(C) \leq \lambda_1(C)$. By part (p3) and Lemma 44 $\mu(C) \geq \lambda_2(C) = \lambda_1(C)$. So $\mu = \lambda_1$ on $\mathcal{K}_{c}(X)$. From part (p7) and Definition 43 we have $\mu = \lambda_2$ on $\mathcal{K}_0(X)$.

(p12) Using part (p3)

$$\mu(U) = \sup \{\lambda_2(C) : C \subseteq U, \ C \in \mathcal{K}_0(X)\} \leq \sup \{\mu(C) : C \subseteq U, \ C \in \mathcal{K}_0(X)\} \leq \sup \{\mu(C) : C \subseteq U, \ C \in \mathcal{K}(X)\}$$

For the converse inequality, given $C \subseteq U$, $U \in \mathcal{O}(X)$, $C \in \mathcal{K}(X)$ choose by Lemma 8 $K \in \mathcal{K}_0(X)$ with $C \subseteq K \subseteq U$. Then by parts (p1)
and \([\text{p11}]\) \(\mu(C) \leq \mu(K) = \lambda_2(K)\), so

\[
sup\{\mu(C) : C \subseteq U, C \in \mathcal{K}(X)\} \leq \sup\{\lambda_2(K) : K \subseteq U, K \in \mathcal{K}_0(X)\} = \mu(U).
\]

\[\square\]

**Lemma 47.** For the set function \(\mu\) in Definition[45]

\[
\mu(U) = \mu(K) + \mu(U \setminus K)
\]

whenever \(K \subseteq U, K \in \mathcal{K}(X), U \in \mathcal{O}(X)\).

**Proof.** We shall prove the statement in steps. Recall that \(\mu = \lambda_1\) on \(\mathcal{K}_c(X)\) and \(\mu = \lambda_2\) on \(\mathcal{K}_0(X)\) by part \([\text{p11}]\) of Lemma[46]

**STEP 1.** We shall show that

\[
\mu(U) = \mu(C) + \mu(U \setminus C)
\]

whenever \(C \subseteq U, U \in \mathcal{O}(X)\), \(K \in \mathcal{K}(X)\).

Let \(C = C_1 \cup C_2 \cup \ldots \cup C_n\), where each \(C_j \in \mathcal{K}_s(X)\). The proof is by induction on \(n\). Suppose \(n = 1\), i.e. \(C \in \mathcal{K}_s(X)\). By Lemma[29]

\[
U = C \cup \bigcap_{i=1}^n U_i
\]

where each \(U_i \in \mathcal{O}_{ss}(X)\). By Lemma[42]

\[
\mu(U) = \mu(C) + \sum_{i=1}^n \mu(U_i).
\]

Then

\[
\mu(U) - \mu(C) = \sum_{i=1}^n \mu(U_i) = \mu(U \setminus C),
\]

where the last equality follows from additivity of \(\mu\) on \(\mathcal{O}(X)\) in Lemma[46]

Suppose that result holds for all \(C \subseteq U, U \in \mathcal{O}_{ss}(X)\) where \(C\) is the disjoint union of less than \(n\) sets \(C_j \in \mathcal{K}_s(X)\). Now let \(C = C_1 \cup C_2 \cup \ldots \cup C_n\), where each \(C_j \in \mathcal{K}_s(X)\). By Lemma[29]

\[
U \setminus C_1 = \bigcap_{i=1}^n U_i
\]

where each \(U_i \in \mathcal{O}_{ss}(X)\). By connectivity each \(C_j, j = 2, \ldots, n\) is contained in one of the sets \(U_i\). For \(i = 1, \ldots, m\) let \(K_i\) be the disjoint union of those \(C_j, j \in \{2, \ldots, n\}\) that are contained in \(U_i\). Notice that each \(K_i\) is the union
of no more than $n - 1$ disjoint sets, and $\bigcup_{i=1}^{m} K_i = \bigcup_{j=2}^{n} C_j$. By induction hypothesis,

$$\mu(U_i) = \mu(U_i \setminus K_i) + \mu(K_i).$$

By finite additivity of $\mu$ on compact sets in Lemma 46

$$\mu(C) = \mu(\bigcup_{j=2}^{n} C_j) + \mu(C_1) = \mu(\bigcup_{i=1}^{m} K_i) + \mu(C_1) = \sum_{i=1}^{m} \mu(K_i) + \mu(C_1).$$

Also we have

$$U \setminus C = (U \setminus C_1) \setminus \bigcup_{j=2}^{n} C_j = (\bigcup_{i=1}^{m} U_i) \setminus (\bigcup_{i=1}^{m} K_i) = \bigcup_{i=1}^{m} (U_i \setminus K_i).$$

By the first part of the induction proof

$$\mu(U) = \mu(U \setminus C_1) + \mu(C_1).$$

Using (8), additivity of $\mu$ on $\mathcal{O}^*(X)$ in Lemma 46 (9), (10), and (11) we obtain:

$$\mu(U) = \mu(U \setminus C_1) + \mu(C_1)$$

$$= \mu(\bigcup_{i=1}^{m} U_i) + \mu(C_1)$$

$$= \sum_{i=1}^{m} \mu(U_i) + \mu(C_1)$$

$$= \sum_{i=1}^{m} \mu(U_i \setminus K_i) + \sum_{i=1}^{m} \mu(K_i) + \mu(C_1)$$

$$= \sum_{i=1}^{m} \mu(U_i \setminus K_i) + \mu(C)$$

$$= \mu(U \setminus C) + \mu(C)$$

STEP 2. We shall show that $\mu(U) = \mu(C) + \mu(U \setminus C)$ whenever $C \subseteq U$, $C \in \mathcal{K}_0(X)$, $U \in \mathcal{O}^*_s(X)$. Let $C = C_1 \sqcup C_2 \sqcup \ldots \sqcup C_n$, where each $C_i \in \mathcal{K}_c(X)$. The proof is by induction on $n$. Suppose $n = 1$, i.e. $C \in \mathcal{K}_c(X)$. By Lemma 30

$$U = C \sqcup W \sqcup \bigcup_{s \in S} V_s$$

where $V_s \in \mathcal{O}^*_s(X)$, $W \in \mathcal{O}^*_s(X)$ (W may be empty). By Lemma 42

$$\mu(U) = \mu(C) + \sum_{s \in S} \mu(V_s) + \mu(W).$$
Then
\[ \mu(U) - \mu(C) = \sum_{s \in S} \mu(V_s) + \mu(W) = \mu(U \setminus C), \]
where the last equality follows from additivity of \( \mu \) on \( \mathcal{O}(X) \) in Lemma \[46\] Suppose that the result holds for all \( C \subseteq U, U \in \mathcal{O}_s(X), C \in \mathcal{K}_c(X) \) where \( C \) is the disjoint union of less than \( n \) sets \( C_i \in \mathcal{K}_c(X) \). Now assume that \( C = C_1 \sqcup \ldots \sqcup C_n, C_i \in \mathcal{K}_c(X) \). As in the proof of Lemma \[42\] consider partial order on \( \{C_1, \ldots, C_n\} \) where \( C_i \leq C_j \) iff \( \tilde{C_i} \subseteq \tilde{C_j} \). Some parts of the argument here are as in the proof of Lemma \[42\]. Let \( C_1, \ldots, C_p, p \leq n \) be maximal elements in \( \{C_1, \ldots, C_n\} \) with respect to this partial order. Then \( \tilde{C_1}, \ldots, \tilde{C_p} \) are disjoint. This implies that the family
\[ \{W_s : s \in S\} = \bigsqcup_{k=1}^{p} \{ \text{bounded components of } X \setminus C_k \} \]
is a disjoint family of sets. Each \( W_s \in \mathcal{O}_s(X) \) by Lemma \[17\] and \( \bigsqcup_{s \in S} W_s \in \mathcal{O}(X) \), because \( \bigsqcup_{s \in S} W_s \subseteq U \). Let \( I = \{1, \ldots, n\} \setminus \{1, \ldots, p\} \). For each \( i \in I \) \( C_i \) is non-maximal element, and there exists \( k \in \{1, \ldots, p\} \) such that \( C_i \subseteq \tilde{C_k} \). In other words, each non-maximal set \( C_i, i \in I \) is contained in a bounded component of \( X \setminus C_k \) for some maximal element \( C_k \) (for some \( k \in \{1, \ldots, p\} \)), that is \( C_i \subseteq W_s \) for some \( s \in S \). Let \( S_1 \) be a finite subset of \( S \) such that for \( s \in S_1 \) the set \( W_s \) contains some \( C_i, i \in I \). Let \( S' = S \setminus S_1 \). For each \( s \in S_1 \) let \( C_s \) be the disjoint union of those sets \( C_i, i \in I \) that are contained in \( W_s \). Since \( |I| \leq n - 1 \), each \( C_s \) is the union of no more than \( n - 1 \) disjoint sets, and by induction hypothesis for each \( s \in S_1 \)
\[ \mu(W_s) = \mu(W_s \setminus C_s) + \mu(C_s). \]
(12)

Note also that
\[ \bigsqcup_{s \in S_1} C_s = \bigsqcup_{i \in I} C_i. \]
(13)
Then using Definition 19 and (13) we see that:

\[
\tilde{C}_1 \sqcup \ldots \sqcup \tilde{C}_p = C_1 \sqcup \ldots \sqcup C_p \sqcup \bigsqcup_{s \in S} W_s \\
= C_1 \sqcup \ldots \sqcup C_p \sqcup \bigsqcup_{s \in S_1} W_s \sqcup \bigsqcup_{s \in S'} W_s \\
= C_1 \sqcup \ldots \sqcup C_p \sqcup \bigsqcup_{i \in I} \bigsqcup_{s \in S_1} (W_s \setminus C_s) \sqcup \bigsqcup_{s \in S'} W_s \\
= C_1 \sqcup \ldots \sqcup C_n \sqcup \bigsqcup_{s \in S_1} (W_s \setminus C_s) \sqcup \bigsqcup_{s \in S'} W_s \\
= C_1 \sqcup \ldots \sqcup C_n \sqcup \bigsqcup_{i \in I} \bigsqcup_{s \in S_1} (W_s \setminus C_s) \sqcup \bigsqcup_{s \in S'} W_s \\
= C_1 \sqcup \ldots \sqcup C_n \sqcup \bigsqcup_{s \in S_1} (W_s \setminus C_s) \sqcup \bigsqcup_{s \in S'} W_s \\
= C_1 \sqcup \ldots \sqcup C_n \sqcup \bigsqcup_{i \in I} \bigsqcup_{s \in S_1} (W_s \setminus C_s) \sqcup \bigsqcup_{s \in S'} W_s \\
= C_1 \sqcup \ldots \sqcup C_n \sqcup \bigsqcup_{i \in I} \bigsqcup_{s \in S_1} (W_s \setminus C_s) \sqcup \bigsqcup_{s \in S'} W_s \\
= \mu(C_1 \sqcup \ldots \sqcup C_n) + \mu(W) = \mu(C) + \mu(W).
\]

We write

\[
\tilde{C}_1 \sqcup \ldots \sqcup \tilde{C}_p = C_1 \sqcup \ldots \sqcup C_n \sqcup W = C \sqcup W
\]

where

\[
W = \bigsqcup_{s \in S_1} (W_s \setminus C_s) \sqcup \bigsqcup_{s \in S'} W_s
\]

is an open bounded set (since \( W \subseteq \bigsqcup_{s \in S} W_s \subseteq U \)). Using Definition 19 and Definition 41, (12), (13), finite additivity of \( \mu \) on \( \mathcal{K}(X) \) and additivity of \( \mu \) on \( \mathcal{O}(X) \) in Lemma 46 we have:

\[
\mu(\bigsqcup_{k=1}^p \tilde{C}_k) = \sum_{k=1}^p \mu(C_k) + \sum_{s \in S} \mu(W_s)
\]

The sets \( U \setminus (C \sqcup W) = U \setminus (\tilde{C}_1 \sqcup \ldots \sqcup \tilde{C}_p) \) and \( W \) are disjoint open bounded sets whose union is \( U \setminus C \). Now using the result of Step 1, (14), just obtained equality \( \mu(\bigsqcup_{k=1}^p \tilde{C}_k) = \mu(W) + \mu(C) \), and additivity of \( \mu \) on \( \mathcal{O}(X) \) in Lemma
we have:

\[ \mu(U) = \mu(U \setminus \bigcup_{k=1}^{p} \tilde{C}_k) + \mu(\bigcup_{k=1}^{p} \tilde{C}_k) \]
\[ = \mu(U \setminus (C \sqcup W)) + \mu(W) + \mu(C) \]
\[ = \mu(U \setminus C) + \mu(C) \]

STEP 3. We shall show that \( \mu(U) = \mu(K) + \mu(U \setminus K) \) whenever \( K \subseteq U, K \in \mathcal{K}(X), U \in \mathcal{O}_c(X) \).

Using part \([p12]\) of Lemma \(46\) and Lemma \(7\) choose sets \( W \in \mathcal{O}_c(X) \) and \( D \in \mathcal{K}_c(X) \) such that

\[ K \subseteq W \subseteq D \subseteq U \text{ and } \mu(U) - \mu(W) < \epsilon. \]  

Let \( B \) be the union of bounded components of \( X \setminus U \) and let the open set \( V \) be the union of bounded components of \( X \setminus D \). Set

\[ C = B \cap V. \]

By Lemma \(15\) \( C \) is compact and \( U \sqcup C \) is open. The solid hull \( \tilde{D} = D \sqcup V \). Then by part \([p9]\) of Lemma \(46\) \( \mu(D) + \mu(V) \leq \mu(\tilde{D}) \). Note that by Lemma \(20\) \( V \subseteq \tilde{D} \subseteq \tilde{U} = U \sqcup B \). Then

\[ V \subseteq U \sqcup (B \cap V) = U \sqcup C. \]

It follows that

\[ K \sqcup C \subseteq D \sqcup V = \tilde{D} \subseteq U \sqcup C. \]

Since \( U \sqcup C \) is open, by Lemma \(22\) we may find \( W' \in \mathcal{O}_c(X) \) such that

\[ K \sqcup C \subseteq \tilde{D} \subseteq W' \subseteq U \sqcup C. \]

Then

\[ W' \setminus (K \sqcup C) \subseteq U \setminus K. \]

According to part \([p4]\) of Lemma \(46\) pick \( K_1 \in \mathcal{K}_0(X) \) such that

\[ K \sqcup C \subseteq K_1 \subseteq W' \text{ and } \mu(K_1) \leq \mu(K \sqcup C) + \epsilon. \]

By Step 2, \( \mu(W') = \mu(W' \setminus K_1) + \mu(K_1) \). Now using \(15\), Definition \(41\), \(16\), \(18\), \(17\), additivity on \( \mathcal{O}(X) \) and finite additivity of \( \mu \) on \( \mathcal{K}(X) \) in Lemma \(46\)
we have:

\[
\mu(U) - \epsilon < \mu(W) \leq \mu(D) = \mu(\tilde{D}) - \mu(V) \\
\leq \mu(\tilde{D}) - \mu(C) \\
\leq \mu(W') - \mu(C) \\
= \mu(W' \setminus K_1) + \mu(K_1) - \mu(C) \\
\leq \mu(W' \setminus (K \cup C)) + \mu(K \cup C) + \epsilon - \mu(C) \\
\leq \mu(U \setminus K) + \mu(K) + \mu(C) - \mu(C) + \epsilon \\
= \mu(U \setminus K) + \mu(K) + \epsilon
\]

It follows that \( \mu(U) \leq \mu(U \setminus K) + \mu(K) \). The opposite inequality is part (p10) of Lemma 46.

STEP 4. We shall show that \( \mu(U) = \mu(K) + \mu(U \setminus K) \) whenever \( K \subseteq U, K \in \mathcal{K}(X) \), \( U \in \mathcal{O}^*(X) \).

Let \( U = \bigsqcup_{i \in I} U_i \) be the decomposition of \( U \) into connected components, and let \( I' \) be a finite subset of \( I \) such that \( K \subseteq \bigsqcup_{i \in I'} U_i \). For each \( i \in I' \) the set \( K_i = K \cap U_i = K \setminus \bigsqcup_{j \in I \setminus i} U_j \in \mathcal{K}(X) \) and

\[
K = \bigsqcup_{i \in I'} K_i.
\]

By Step 3 we know that

\[
\mu(K_i) + \mu(U_i \setminus K_i) = \mu(U_i) \quad \text{for each} \quad i \in I'.
\]

Then using (19), finite additivity of \( \mu \) on \( \mathcal{K}(X) \) and additivity of \( \mu \) on \( \mathcal{O}(X) \) in Lemma 46 and (20) we have:

\[
\mu(K) + \mu(U \setminus K) = \mu(\bigsqcup_{i \in I'} K_i) + \mu(U \setminus \bigsqcup_{i \in I'} K_i) \\
= \sum_{i \in I'} \mu(K_i) + \mu(U \setminus \bigsqcup_{i \in I'} K_i) + \sum_{i \in I \setminus I'} \mu(U_i) \\
= \sum_{i \in I'} \mu(U_i) + \sum_{i \in I \setminus I'} \mu(U_i) = \sum_{i \in I} \mu(U_i) = \mu(U)
\]

STEP 5. We shall show that \( \mu(U) = \mu(K) + \mu(U \setminus K) \) whenever \( K \subseteq U, K \in \mathcal{K}(X) \), \( U \in \mathcal{O}(X) \).

First assume that \( \mu(U) < \infty \). Given \( \epsilon > 0 \) by Definition 45 find \( C \in \mathcal{K}(X) \) such that \( K \subseteq C \) and \( \mu(U) - \mu(C) < \epsilon \). Using Lemma 6 find \( V \in \mathcal{O}^*(X) \) such that

\[
K \subseteq C \subseteq V \subseteq U.
\]
By Step 4 \( \mu(V) = \mu(V \setminus K) + \mu(K) \). Then using monotonicity of \( \mu \) in Lemma 46 we see that

\[
\mu(C) \leq \mu(V) = \mu(V \setminus K) + \mu(K) \leq \mu(U \setminus K) + \mu(K).
\]

Then \( \mu(U) - \epsilon < \mu(C) \leq \mu(U \setminus K) + \mu(K) \). Therefore, \( \mu(U) \leq \mu(U \setminus K) + \mu(K) \).

The opposite inequality is part (p10) of Lemma 46. Therefore, if \( \mu(U) < \infty \) then

\[
\mu(U) = \mu(U \setminus K) + \mu(K).
\]

Now assume \( \mu(U) = \infty \). For \( n \in \mathbb{N} \) choose \( C \in \mathscr{K}(X) \) such that \( K \subseteq C \) and \( \mu(C) > n \). By Lemma 46 find \( V \in \mathscr{O}^*(X) \) such that

\[
K \subseteq C \subseteq V \subseteq U.
\]

Using again (21) we have:

\[
n < \mu(C) \leq \mu(V \setminus K) + \mu(K),
\]

i.e. \( n - \mu(K) \leq \mu(V \setminus K) \leq \mu(U \setminus K) \). Since \( \mu(K) \in \mathbb{R} \) by part (p2) of Lemma 46 it follows that \( \mu(U \setminus K) = \infty \), and \( \mu(U \setminus K) + \mu(K) = \mu(U) \). \( \square \)

**Theorem 48.** Let \( X \) be locally compact, connected, locally connected. A solid set function on \( X \) extends uniquely to a compact-finite topological measure on \( X \).

**Proof.** Definitions 41, 43 and 45 extend solid set function \( \lambda \) to a set function \( \mu \). We would like to show that \( \mu \) is a topological measure. Definition 45 and part (p12) of Lemma 46 show that \( \mu \) satisfies (TM2) and (TM3) of definition 34. Proposition 37, part (p6) of Lemma 46 and Lemma 47 show that \( \mu \) is a topological measure. By part (p2) of Lemma 46 \( \mu \) is compact-finite.

To show that the extension from a solid set function to a topological measure is unique suppose \( \nu \) is a topological measure on \( X \) such that \( \mu = \nu = \lambda \) on \( \mathscr{A}^*_s(X) \). If \( A \in \mathscr{K}_c(X) \) then by Definition 19 \( A = \tilde{A} \setminus (\bigcup_{s \in S} B_s) \), where \( \tilde{A}, B_s \in \mathscr{A}^*_s(X) \), so from Definition 41 it follows that \( \mu = \nu \) on \( \mathscr{K}_c(X) \), and, hence, on \( \mathscr{K}_0(X) \). From part (t6) of Lemma 36 it then follows that \( \mu = \nu \) on \( \mathscr{O}(X) \), so \( \mu = \nu \). \( \square \)

**Remark 49.** We will summarize the extension procedure for obtaining a topological measure \( \mu \) from a solid set function \( \lambda \) on a locally compact, connected, locally connected space. First, for a compact connected set \( C \) we have:

\[
\mu(C) = \lambda(\tilde{C}) - \sum_{i=1}^{n} \lambda(B_i),
\]
where \( \tilde{C} \) is the solid hull of \( C \) and \( B_i \) (open solid sets) are bounded components of \( X \setminus C \). For \( C \in \mathcal{K}_0(X) \), i.e. for a compact set which is the union of finitely many disjoint compact connected sets \( C_1, \ldots, C_n \), we have:

\[
\mu(C) = \sum_{i=1}^{n} \mu(C_i).
\]

For an open set \( U \) we have:

\[
\mu(U) = \sup \{ \mu(K) : K \subseteq U, K \in \mathcal{K}_0(X) \},
\]

and for a closed set \( F \) let

\[
\mu(F) = \inf \{ \mu(U) : F \subseteq U, U \in \mathcal{O}(X) \}.
\]

**Theorem 50.** If a solid set function \( \lambda \) is extended to a topological measure \( \mu \) then the following holds: if \( \lambda : \mathcal{A}_s^*(X) \rightarrow \{0, 1\} \) then \( \mu \) also assumes only values 0 and 1; if \( \sup \{ \lambda(K) : K \in \mathcal{K}_s(X) \} = M < \infty \) then \( \mu \) is finite and \( \mu(X) = M \).

**Proof.** Follows from Remark 49, part (p5) of Lemma 46, and part (t7) of Lemma 36.

**Theorem 51.** The restriction \( \lambda \) of a compact-finite topological measure \( \mu \) to \( \mathcal{A}_s^*(X) \) is a solid set function, and \( \mu \) is uniquely determined by \( \lambda \).

**Proof.** Let \( \lambda \) be the restriction of \( \mu \) to \( \mathcal{A}_s^*(X) \). Monotonicity of a topological measure (see Lemma 36) and (TM1) of Definition 34 show that \( \lambda \) satisfies conditions (s1) and (s4) of Definition 39. For \( U \in \mathcal{O}_s^*(X) \) by (TM2) let \( K \in \mathcal{K}(X) \) be such that \( \mu(U) - \mu(K) < \epsilon \) and by Lemma 21 we may assume that \( K \in \mathcal{K}_s(X) \). Part (s2) of Definition 39 follows. Part (s3) of Definition 39 follows from (TM3) and Lemma 22. Since \( \mu \) is compact-finite, \( \lambda \) is real-valued. Therefore, \( \lambda \) is a solid set function.

**Remark 52.** Lemma 40, Lemma 42, and Lemma 46 give us some additional properties of topological measures. For example, by part (p7) of Lemma 46, if a closed set \( F \) and a compact \( K \) are disjoint, then \( \mu(F \sqcup K) = \mu(F) + \mu(K) \).

9. **Examples**

When \( X \) is compact, a set is called solid if it and its complement are both connected. For a compact space \( X \) we define a certain topological characteristic, genus. See [2] for more information about genus \( g \) of the space. We are particularly interested in spaces with genus 0. One way to describe the “\( g = 0 \)”
condition is the following: if the union of two open solid sets in $X$ is the whole space, their intersection must be connected. (See [9].) Intuitively, $X$ does not have holes or loops. In the case where $X$ is locally path connected, $g = 0$ if the fundamental group $\pi_1(X)$ is finite (in particular, if $X$ is simply connected). Knudsen [10] was able to show that if $H^1(X) = 0$ then $g(X) = 0$, and in the case of CW-complexes the converse also holds.

The following two remarks for a compact space follow from results in [2]:

**Remark 53.** $g(X) = 0$ if and only if $X \setminus \bigcup_{i=1}^{n} C_i$ is connected for any finite disjoint family $\{C_i\}_{i=1}^{n}$ of closed solid sets.

**Remark 54.** If there is only one open (closed) solid set in a solid partition of $X$ (i.e. a partition of $X$ into a union of disjoint sets each of which is open solid or closed solid), then there is only one closed (open) solid set in this partition.

**Remark 55.** When $X$ is compact, a solid-set function on $X$ extends in a unique way to a topological measure on $X$. For precise definitions and extension procedure see [2].

The majority of existing examples of topological measures on compact spaces are given for spaces with genus 0. Here is one:

**Example 56** (Aarnes circle measure). Let $X$ be the unit square and $B$ be the boundary of $X$. Fix a point $p$ in $X \setminus B$. Define $\mu$ on solid sets as follows: $\mu(A) = 1$ if i) $B \subset A$ or ii) $p \in A$ and $A \cap B \neq \emptyset$. Otherwise, we let $\mu(A) = 0$. Then $\mu$ is a solid set function and, hence, extends to a topological measure on $X$. Note that $\mu$ is not a point mass. To demonstrate that $\mu$ is not a measure we shall show that $\mu$ is not subadditive. Let $A_1$ be a closed solid set consisting of two adjacent sides of $B$, $A_2$ be a closed solid set that is the other two adjacent sides of $B$, and $A_3 = X \setminus B$ be an open solid subset of $X$. Then $X = A_1 \cup A_2 \cup A_3$, $\mu(X) = 1$, but $\mu(A_1) + \mu(A_2) + \mu(A_3) = 0$.

The reason that the majority of existing examples of topological measures on compact spaces are given for spaces with genus 0 is the following. To obtain a topological measure it is enough to define a solid-set function. When a space has genus 0, in the definition of a solid-set function the hardest condition to verify, the irreducible partition condition, becomes easy to verify. When $X$ is locally compact, the hardest condition in Definition [39] to verify is the condition [54] that deals with solid partitions. But, as we shall see in this section, it turns out that this...
condition holds trivially for spaces whose one-point compactification has genus 0.

In this section we denote by $\hat{X}$ the one-point compactification of $X$.

**Lemma 57.** Let $X$ be locally compact and $\hat{X}$ be its one-point compactification. If $A \in \mathfrak{A}^s_*(X)$ then $A$ is solid in $\hat{X}$.

*Proof.* Since $A$ is connected in $X$, it is also connected in $\hat{X}$. Let $X \setminus A = \bigcup_{i=1}^n B_i$ be the decomposition into connected components. Each $B_i$ is an unbounded subset of $X$. We can write $\hat{X} \setminus A = \bigcup_{i=1}^n E_i$ where each $E_i = B_i \cup \{\infty\}$. It is easy to see that each $E_i$ is connected in $\hat{X}$. Thus, $\hat{X} \setminus A$ is connected, and so $A$ is solid in $\hat{X}$. □

**Lemma 58.** Let $X$ be a locally compact space whose one-point compactification $\hat{X}$ has genus 0. If $A \in \mathfrak{A}^s_*(X)$ then any solid partition of $A$ is the set $A$ itself.

*Proof.* Suppose first that $V \in \mathcal{O}^s_*(X)$ and its solid partition is given by

$$V = \bigsqcup_{i=1}^n C_i \sqcup \bigsqcup_{j=1}^m U_j$$

where each $C_i \in \mathcal{K}_s(X)$ and each $U_j \in \mathcal{O}^s_*(X)$. From Lemma 57 it follows that $\hat{X} \setminus V$ and each $C_i$ are closed solid sets in $\hat{X}$. Since $\hat{X}$ has genus 0, by Remark 53

$$\hat{X} \setminus ((\hat{X} \setminus V) \sqcup \bigsqcup_{i=1}^n C_i) = \bigsqcup_{j=1}^m U_j$$

must be connected in $\hat{X}$. It follows that $m = 1$ and we may write

$$V = \bigsqcup_{i=1}^n C_i \sqcup U_1.$$ 

Then $\{U_1, \hat{X} \setminus V, C_1, \ldots, C_n\}$ is a solid partition of $\hat{X}$, and it has only one open set. By Remark 54 this solid partition also has only one closed set in it, and it must be $\hat{X} \setminus V$. So each $C_i = \emptyset$, and the solid partition of $V$ is $V = U_1$, i.e. the set itself.

Now suppose that $C \in \mathcal{K}_s(X)$ and its solid partition is given by

$$C = \bigsqcup_{i=1}^n C_i \sqcup \bigsqcup_{j=1}^m U_j$$
where each $C_i \in \mathcal{K}_s(X)$ and each $U_j \in \mathcal{O}_s^*(X)$. Then $\{\hat{X} \setminus C, U_1, \ldots, U_m, C_1, \ldots, C_n\}$ is a solid partition of $\hat{X}$. Again by Remark 53

$$\hat{X} \setminus \bigcup_{i=1}^n C_i = (\hat{X} \setminus C) \sqcup U_1 \sqcup \ldots \sqcup U_m$$

must be connected in $\hat{X}$. It follows that $U_j = \emptyset$ for $j = 1, \ldots, m$. Then by connectivity of $C$ we see that the solid partition of $C$ must be the set itself. \qed

Remark 59. From Lemma 58 it follows that for any locally compact space whose one-point compactification has genus 0 the last condition of Definition 39 holds trivially. This is true, for example, for $X = \mathbb{R}^n$, half-plane in $\mathbb{R}^n$ with $n \geq 2$, or for a punctured ball in $\mathbb{R}^n$ with the relative topology.

Example 60. Lemma 58 may not be true for spaces whose one-point compactification has genus greater than 0. For example, let $X$ be an infinite strip $\mathbb{R} \times [0, 1]$ without the ball of radius $1/4$ centered at $(-1/2, 1/2)$, so $\hat{X}$ has genus greater than 0. It is easy to give an example of a solid partition of a bounded solid set (say, rectangle $[0, n] \times [0, 1]$ or $(0, n) \times [0, 1]$) which consists of $n$ solid sets (rectangles of the type $(i, i+1) \times [0, 1]$ or $[i, i+1] \times [0, 1]$) for any given odd $n \in \mathbb{N}$, $n > 1$.

We are ready to give examples of topological measures on locally compact spaces.

Example 61. Let $X$ be a locally compact space whose one-point compactification has genus 0. Let $\lambda$ be a real-valued topological measure on $X$ (or, more generally, a real-valued deficient topological measure on $X$; for definition and properties of deficient topological measures on locally compact spaces see [6]). Let $P$ be a set of two distinct points. For each $A \in \mathcal{A}_s^*(X)$ let $\nu(A) = 0$ if $\sharp A = 0$, $\nu(A) = \lambda(A)$ if $\sharp A = 1$, and $\nu(A) = 2\lambda(X)$ if $\sharp A = 2$, where $\sharp A$ is the number of points in $A \cap P$. We claim that $\nu$ is a solid set function. By Remark 59 we only need to check the first three conditions of Definition 39. The first one is easy to see. Using Lemma 21 and Lemma 22 it is easy to verify conditions (s2) and (s3) of Definition 39. The solid set function $\nu$ extends to a unique finite topological measure on $X$. Suppose, for example, that $\lambda$ is the Lebesgue measure on $X = \mathbb{R}^2$, the set $P$ consists of two points $p_1 = (0, 0)$ and $p_2 = (2, 0)$. Let $K_i$ be the closed ball of radius 1 centered at $p_i$ for $i = 1, 2$. Then $K_1, K_2$ and $C = K_1 \cup K_2$ are compact solid sets, $\nu(K_1) = \nu(K_2) = \pi$, $\nu(C) = 4\pi$. Since $\nu$ is not subadditive, $\nu$ is a topological measure that is not a measure.
The next two examples are adapted from Example 2.2 in [4] and are related to Example 56.

**Example 62.** Let $X$ be the unit disk on the plane with removed origin. $X$ is a locally compact Hausdorff space with respect to the relative topology. Any subset of $X$ whose closure in $\mathbb{R}^2$ contains the origin is unbounded in $X$. For $A \in \mathcal{S}^*(X)$ (since $A$ is also solid subset of the unit disk by Lemma 57) we define $\mu'(A) = \mu(A)$ where $\mu$ is the solid set function on the unit disk from Example 56. From Remark 59, Lemma 21, Lemma 22 and the fact that $\mu$ is a solid set function on $\hat{X}$ we see that $\mu'$ is a solid-set function on $X$. By Theorem 48 $\mu'$ extends uniquely to a topological measure on $X$, which we also call $\mu'$. Note that $\mu'$ is simple. We claim that $\mu'$ is not a measure. Let $U_1 = \{z \in X : \text{Im } z > 0\}$, $U_2 = \{z \in X : \text{Im } z < 0\}$ and $F = \{z \in X : \text{Im } z = 0\}$. Then $U_1, U_2$ are open (unbounded) in $X$ and $F$ is a closed (unbounded) set in $X$ consisting of two disjoint segments. Note that $X = F \cup U_1 \cup U_2$. Using Remark 49 we calculate $\mu'(F) = \mu'(U_1) = \mu'(U_2) = 0$. The boundary of the disk, $C$, is a compact connected set, $X \setminus C$ is unbounded in $X$, so $C \in \mathcal{K}(X)$. Since $\mu'(C) = 1$, we have $\mu'(X) = 1$. Thus, $\mu'$ is not subadditive, so it is not a measure.

This example also shows that on a locally compact space finite additivity of topological measures holds on $\mathcal{K}(X) \cup \mathcal{O}(X)$ by Definition 34, but fails on $\mathcal{E}(X) \cup \mathcal{O}(X)$. This is in contrast to topological measures on compact spaces, where finite additivity holds on $\mathcal{E}(X) \cup \mathcal{O}(X)$.

**Example 63.** Let $X = \mathbb{R}^2$, $l$ be a straight line and $p$ a point of $X$ not on the line $l$. For $A \in \mathcal{S}^*(X)$ define $\mu(A) = 1$ if $A \cap l \neq \emptyset$ and $p \in A$; otherwise, let $\mu(A) = 0$. Using Lemma 21 and Lemma 22 it is easy to verify the first three conditions of Definition 39. From Remark 59 it follows that $\mu$ is a solid set function on $X$. By Theorem 48 $\mu$ extends uniquely to a topological measure on $X$, which we also call $\mu$. Note that $\mu$ is simple. We claim that $\mu$ is not a measure. Let $F$ be the closed half-plane determined by $l$ which does not contain $p$. Then using Remark 49 we calculate $\mu(F) = \mu(X \setminus F) = 0$, and $\mu(X) = 1$. Failure of subadditivity shows that $\mu$ is not a measure.

The sets $F$ and $X \setminus F$ are both unbounded. Now take a bounded open disk $V$ around $p$ that does not intersect $l$. Then

$$X = V \cup (X \setminus V),$$

where $V \in \mathcal{O}^*(X)$, $\mu(V) = \mu(X \setminus V) = 0$, while $\mu(X) = 1$. 
This example also shows that on a locally compact space finite additivity of topological measures holds on $\mathcal{H}(X) \cup \mathcal{O}(X)$ by Definition 34 but fails on $\mathcal{G}(X) \cup \mathcal{O}(X)$. It fails even in the situation $X = V \sqcup F$, where $V$ is a bounded open set, and $F$ is a closed set.

The last two examples suggest that having a topological measure on $\hat{X}$ helps us to get a topological measure on $X$. In fact, we have the following result.

**Theorem 64.** Let $X$ be a locally compact, connected, locally connected space whose one-point compactification $\hat{X}$ has genus 0. Suppose $\nu$ is a solid set function on $\hat{X}$. For $A \in \mathcal{A}(X)$ define $\mu(A) = \nu(A)$. Then $\mu$ is a solid set function on $X$ and, thus, extends uniquely to a topological measure on $X$.  

**Proof.** Let $A \in \mathcal{A}(X)$. By Lemma 57, $A$ is a solid set in $\hat{X}$. Using Lemma 21, the fact that $\nu$ is a solid set function on $\hat{X}$, and that a bounded solid set does not contain $\infty$ it is easy to verify the first three conditions of Definition 39. By Remark 59 $\mu$ is a solid set function on $X$. \qed

Theorem 64 allows us to obtain a large variety of topological measures on a locally compact space from examples of topological measures on compact spaces.

**Example 65.** Let $X$ be a locally compact space whose one-point compactification $\hat{X}$ has genus 0. Let $n$ be a natural number. Let $P$ be the set of distinct $2n + 1$ points. For each $A \in \mathcal{A}(X)$ let $\nu(A) = i/n$ if $\#A = 2i$ or $2i + 1$, where $\#A$ is the number of points in $A \cap P$. When $X$ is compact, a set function defined in this way is a solid-set function (see Example 2.1 in [3], Examples 4.14 and 4.15 in [5]). By Theorem 64 $\nu$ is a solid-set function on $X$; it extends to a unique topological measure on $X$ that assumes values $0, 1/n, \ldots, 1$.

We conclude with an example of another infinite topological measure.

**Example 66.** Let $X = \mathbb{R}^n$ for any $n \geq 2$, and $\lambda$ be the Lebesgue measure on $X$. For $U \in \mathcal{O}(X)$ define $\mu(U) = 0$ if $0 \leq \lambda(U) \leq 1$ and $\mu(U) = \lambda(U)$ if $\lambda(U) > 1$. For $C \in \mathcal{K}(X)$ define $\mu(C) = 0$ if $0 \leq \lambda(C) < 1$ and $\mu(C) = \lambda(C)$ if $\lambda(C) \geq 1$. It is not hard to check the first three conditions of Definition 39. From Remark 59 it follows that $\mu$ is a solid set function on $X$. By Theorem 48 $\mu$ extends uniquely to a topological measure on $X$, which we also call $\mu$. Note that $\mu(X) = \infty$. $\mu$ is not subadditive, for we may cover a compact ball with Lebesgue measure greater than 1 by finitely many balls of Lebesgue measure less than 1. Hence, $\mu$ is not a measure.
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