Finite Heisenberg Groups and Seiberg Dualities
in Quiver Gauge Theories

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Abstract

A large class of quiver gauge theories admits the action of finite Heisenberg groups
of the form $\text{Heis}(\mathbb{Z}_q \times \mathbb{Z}_q)$. This Heisenberg group is generated by a manifest $\mathbb{Z}_q$ shift
symmetry acting on the quiver along with a second $\mathbb{Z}_q$ rephasing (clock) generator
acting on the links of the quiver. Under Seiberg duality, however, the action of the
shift generator is no longer manifest, as the dualized node has a different structure from
before. Nevertheless, we demonstrate that the $\mathbb{Z}_q$ shift generator acts naturally on the
space of all Seiberg dual phases of a given quiver. We then prove that the space of
Seiberg dual theories inherits the action of the original finite Heisenberg group, where
now the shift generator $\mathbb{Z}_q$ is a map among fields belonging to different Seiberg phases.
As examples, we explicitly consider the action of the Heisenberg group on Seiberg
phases for $\mathbb{C}^3/\mathbb{Z}_3$, $Y^{4,2}$ and $Y^{6,3}$ quiver.
1 Introduction

Generalizing an observation of [1], it was shown in [2] that for a class of $\mathbb{Z}_q$ orbifold quiver gauge theories with gauge group $SU(N)^p$, there is a set of discrete transformations $A$, $B$ and $C$ satisfying

$$A^q = B^q = C^q = 1, \quad AB = BAC. \quad (1.1)$$

Here the $A$ generator is inherited from the $\mathbb{Z}_q$ orbifold action, and corresponds to the manifest $\mathbb{Z}_q$ shift symmetry acting on the quiver. The $B$ generator may be thought of as a clock generator, which acts by rephasing the links of the quiver. This combination of clock and shift then generates the finite Heisenberg group as indicated above, where $C$ is essentially a uniform rephasing of all the links. These transformations satisfy three important properties: (i) they leave the superpotential invariant; (ii) they satisfy anomaly cancelation constraints for all $SU(N)$ gauge groups; and (iii) the above group relations are true up to elements in the center of the gauge group $SU(N)^p$, that is, up to gauge transformations.

The generators $A$, $B$ and $C$ can be interpreted, in the dual string theory, as operators counting the number of wrapped F-strings, D-strings and D3 branes respectively. Thus, the above Heisenberg group implies that the charges of D branes in the presence of Ramond-Ramond flux do not commute. In fact, an important motivation for our study is provided by recent investigations along these lines due to Belov, Moore and others [4, 5].

An extension of this structure was considered in [3] where nonconformal quiver gauge theories were considered. In this case, the Heisenberg group gets centrally extended as a result of having gauge groups with different ranks. In other words, the addition of fractional branes to the background induces a central extension of the Heisenberg group.

We believe the study of discrete symmetries of quiver gauge theories is interesting in its own right. As field theories, a natural question that arises in this context is the interplay between the existence of a finite Heisenberg group and Seiberg duality. Seiberg duality is an equivalence between two gauge theories [6], and has been extensively studied in the context of quiver gauge theories [7, 8, 9, 10].

In this paper we study the interplay between the existence of a finite Heisenberg group acting on orbifold quiver gauge theories and Seiberg duality. The generator $A$, realized as a shift symmetry, acts manifestly only on the symmetric phase of the theory. After Seiberg duality, most quivers lose this manifest shift symmetry associated with $A$. However, we demonstrate that this symmetry is naturally restored in the space of all Seiberg dual quivers.

Our main result is as follows: For a quiver gauge theory admitting the action of a finite Heisenberg group $\text{Heis}(\mathbb{Z}_q \times \mathbb{Z}_q)$ as above (1.1), there exists a similar finite Heisenberg group acting on the space of Seiberg dual theories to the quiver. Moreover, the action of the shift generator in the Heisenberg group maps fields among different Seiberg phases.

The paper is organized as follows. In section 2 we review some known aspects of quiver
gauge theories including the algorithm for performing Seiberg duality in quiver gauge theories and some of its properties. Also in section 2 we provide a constructive proof of the existence of a Heisenberg group acting on the spaces of Seiberg dual quivers. Section 3 contains several explicit examples of our construction; we consider $\mathbb{C}^3/\mathbb{Z}_3$, $Y^{4.2}$ and $Y^{6.3}$ quivers. We then conclude in Section 4.

2 Seiberg duality and discrete symmetries

In this section we begin with a discussion of generalities of Seiberg duality acting in quiver gauge theories. We then examine the implication of Seiberg duality on the $B$ and $C$ generators, which act by rephasings on the links. As discussed in [1, 2, 3], these abelian generators may be constructed by assigning discrete $U(1)$ charges to each node in the quiver consistent with anomaly cancellation. The anomaly cancelation requirement has a natural description in terms of the adjacency matrix, and we demonstrate how this carries over into the Seiberg dual phases as well. As a result, the $B$ and $C$ generators have a natural extension when acting on the space of Seiberg duals. By including the $A$ generator (now acting on the space of Seiberg duals), we are then able to prove that the space of Seiberg dual quivers naturally inherits the action of the finite Heisenberg group of the original (symmetric phase) quiver.

2.1 Seiberg duality on quivers

Seiberg duality has been discussed extensively in the context of quiver gauge theories. In this subsection we review its implementation and discuss some of its implications for quiver gauge theories. We consider only quiver gauge theories with gauge group $SU(N_i)$. In this case, a quiver gauge theory is completely defined by giving the rank of the gauge groups, the matter content and the superpotential. The matter content is conveniently encoded in the (antisymmetric) adjacency matrix denoted by $a_{MN}$.

A given Seiberg duality acts on a single node of the quiver. Thus, to state the rules of Seiberg duality, it is convenient to split the indices (labels for the nodes) according to: 0 to indicate the specific node under investigation; $i, j, \cdots$ to indicate neighboring nodes that have arrows pointing from the 0th to the $i$th node; $\tilde{i}, \tilde{j}, \cdots$ to denote neighboring nodes that have arrows pointing from the $j$th to the 0th node; and $a, b, \cdots$ to denote the remaining nodes, which we emphasize are not directly connected to the 0th node. Seiberg duality of
the 0th node is captured by the transformation

\[
\begin{align*}
\hat{a}_{0M} &= -a_{0M}, \\
\hat{a}_{ij} &= a_{ij}, \\
\hat{a}_{\tilde{i}\tilde{j}} &= a_{\tilde{i}\tilde{j}}, \\
\hat{a}_{aM} &= a_{aM}, \\
\hat{a}_{\tilde{i}j} &= a_{\tilde{i}j} - a_{0\tilde{i}}a_{0j}, \\
\hat{a}_{ji} &= a_{ji} + a_{0i}a_{0j},
\end{align*}
\]

(2.1)

where the caret denotes the new quantities after Seiberg duality. All other components follow from the antisymmetry of \(a\) and \(\hat{a}\). If we further make the assumption that any two nodes are only connected by edges with the same directionality, the adjacency matrix completely determines the field content and charges of the new theory. In addition, the rank of the 0th gauge group is now changed to be

\[
\hat{N}_0 = \sum_i (a_{0i}N_i) - N_0 = \sum_j (a_{0j}N_j) - N_0.
\]

(2.2)

Here we use the notation \(N_M\) to denote the rank of the gauge group at the \(M\)th node. Note that the second equality follows as a result of the original anomaly cancellation condition, and that the sum term above is easily understood as the effective number of flavors.

### 2.2 Discrete transformations and Seiberg Duality

The \(B\) and \(C\) operators both act by rephasings of the links of the quiver. A convenient manner to construct these operators is to assign discrete \(U(1)\) charges to the nodes, say charge \(n_K\) for the \(K\)th node. These charges are not arbitrary, but must satisfy the constraint of anomaly cancellation. (The superpotential constraint is automatically satisfied, since it corresponds to closed loops along the links.) Here we show that this anomaly cancellation constraint has a natural formulation in terms of the adjacency matrix \(a\). However, once we do this, it then becomes clear how to extend \(B\) and \(C\) to act on the Seiberg dual quivers. The result essentially follows by replacing the adjacency matrix \(a\) with the corresponding dual one \(\hat{a}\). This results in a map of the charges \(n_K\) to \(\hat{n}_K\) in a similar fashion as (2.2).

Before turning to the \(B\) and \(C\) rephasing operators, however, we first display the general feature of cubic anomaly cancelation for \(SU(N)^3\) and explicitly demonstrate that it is still met in the Seiberg dual phase. This calculation will then be extended to the discrete rephasing case in a straightforward manner. To begin, we note that the anomaly cancelation condition in the original Seiberg phase is given by

\[
a_{M0}N_0 + \sum_i (a_{Mi}N_i) + \sum_j (a_{Mj}N_j) \sum_a (a_{Ma}N_a) = 0.
\]

(2.3)
Given this condition, we wish to show that
\[ \hat{a}_{M0} \hat{N}_0 + \sum_i (\hat{a}_{Mi} \hat{N}_i) + \sum_j (\hat{a}_{Mj} \hat{N}_j) \sum_a (\hat{a}_{Ma} \hat{N}_a) = 0, \tag{2.4} \]
which is the equivalent statement in the Seiberg dual phase. The last summation factor in either of the above is unaffected by Seiberg duality of the 0\textsuperscript{th} node. Hence we will denote this term simply as \( \sum_a (\hat{a}_{Ma} N_a) = \Sigma \), because \( \hat{a}_{Ma} = a_{Ma} \). In fact, as a result of this, the anomaly condition is trivially met by all nodes except for the 0\textsuperscript{th} and those connected to it. Therefore, we will only consider the \( M = k \) and \( M = \tilde{k} \) terms. For \( M = k \), the left hand side of (2.4) becomes
\[
-\mathbf{a}_{k0} \left( \sum_i (\mathbf{a}_{0i} N_i) - N_0 \right) + \sum_i (\mathbf{a}_{ki} N_i) + \sum_j ((\mathbf{a}_{kj} - \mathbf{a}_{0k} \mathbf{a}_{j0}) N_j) + \Sigma \\
= \mathbf{a}_{k0} N_0 + \sum_i (\mathbf{a}_{ki} N_i) + \sum_j (\mathbf{a}_{kj} N_j) + \Sigma - \mathbf{a}_{k0} \sum_i (\mathbf{a}_{0i} N_i) - \sum_j (\mathbf{a}_{0k} \mathbf{a}_{j0} N_j) \\
= - \left( \mathbf{a}_{k0} \sum_i (\mathbf{a}_{0i} N_i) + \sum_j (\mathbf{a}_{k0} \mathbf{a}_{j0} N_j) \right) \\
= 0, \tag{2.5}
\]
where the terms are 0 as a result of the \( k \textsuperscript{th} \) and 0\textsuperscript{th} component of the original anomaly condition. This condition is likewise met by the \( M = 0 \) node because the only terms appearing simply flip sign. The \( \tilde{j} \) nodes follow in exactly the same manner as the above calculation because of relation (2.2). This shows that anomaly cancellation holds in the Seiberg dual phase, so long as it holds in the original phase.

We now turn to the discrete \( U(1) \) rephasings used to construct the \( B \) and \( C \) operators. Because these are abelian rephasings, we need to consider mixed anomalies, where the anomaly comes from the \( j^\mu SU(N)^2 \) triangle diagram where \( j^\mu \) is the conserved current of the \( U(1) \) that the rephasing is associated with. To accomplish the rephasing, we associate the phase \( \omega_K \) and charge \( n_K \) with the \( K \textsuperscript{th} \) node. A field represented by an arrow going from the \( K \textsuperscript{th} \) node to the \( L \textsuperscript{th} \) node gets rephased by \( \omega_K \omega_L^{-n_K} \), and we mean this component by component in the superfield.

We now consider an instanton number 1 at the \( M \textsuperscript{th} \) node, and so all fields represented by an arrow with either end on that node have a fermion zero mode (again, counting the other end of the arrow as an effective flavor symmetry). We follow the previous work [2, 3] and find that the general expression for the anomaly with this instanton number is
\[
(\omega_0^{n_0} \omega_M^{-n_M}) N_i a_M \prod_j ((\omega_j^{n_j} \omega_M^{-n_M}) N_j a_j M) \prod_j ((\omega_j^{n_j} \omega_M^{-n_M}) N_j a_j M) \prod_a ((\omega_a^{n_a} \omega_M^{-n_M}) N_a a_a M) = 1, \tag{2.6}
\]
where the equality is to be read as a requirement of the set of numbers \( n_K \) and phases \( \omega_K \). One can easily see that the factors of \( \omega_M \) cancel in this expression because the \( N_M \) are a zero
eigenvector of the adjacency matrix \( \mathbf{a} \), and follows because the effective number of arrows in and out are the same. The expression simplifies to

\[
(\omega_0^{n_0 N_0 a_{0M}} \prod_j (\omega_j^{n_j N_j a_{jM}}) \prod_j (\omega_j^{n_j N_j a_{jM}}) \prod_a (\omega_a^{n_a N_a a_{aM}}) = 1.
\] (2.7)

The above expression is general, and valid for any assignments of numbers and phases. As a further simplification, we wish to express all phases in terms of a single phase, \( \omega \). We note also that the phase \( \omega_K \) and the number \( n_K \) overparameterize exactly how each field is charged. We take that the phases are all \( e^{i\phi_K} \) where \( \phi_K \in (0 \cdots 2\pi] \), so that 1 is parameterized only by \( e^{2\pi i} \). One can vary \( \omega_K \) and \( n_K \) while leaving \( \omega_K^{n_K} \) fixed. Therefore, we find it useful to tune all of the \( \omega_K \) such that

\[
\omega_K = \omega^{1/N_K},
\] (2.8)

for some fixed \( \omega \) that is independent of \( K \), leaving all parametrization of the phases represented by the charges \( n_K \). Again, if \( \omega = 1 \) we parameterize this as \( \omega = e^{2\pi i} \) such that the roots above defining \( \omega_K \) are non trivial (we will return to this case in a moment). This greatly simplifies our expression, and we find

\[
(\omega_0^{n_0 a_{0M}} \prod_j (\omega_j^{n_j a_{jM}}) \prod_j (\omega_j^{n_j a_{jM}}) \prod_a (\omega_a^{n_a a_{aM}}) = 1
\]

\[
= \omega_0^{n_0 a_{0M} + \sum_j (n_j a_{jM}) + \sum_j (n_j a_{jM}) + \sum_a (n_a a_{aM}) = 1.
\] (2.9)

This, then, is the general expression for the anomaly when the \( M \)th node has an instanton number 1. We will abbreviate the exponential above as \( \hat{\mathbf{n}} \cdot \mathbf{a} \) for obvious reasons.

We therefore require that this condition is met by the numbers \( n_K \) for all values of \( M \) above. One may worry that this does not capture all possible instanton numbers. However, because the Pontryagin number for composite connections is additive, the basis of taking instanton number 1 for each node spans the space of all possible instanton numbers, and satisfying the anomaly is most restrictive for these individual basis vectors. One may see that the Pontryagin number is additive because the two triangle diagrams \( j^\mu SU_1(N_1)^2 \) and \( j^\mu SU_2(N_2)^2 \) exist (for a field charged under both of these gauge groups), but no cross term exists. Therefore, a fermionic field charged under both \( SU_1(N_1) \) and \( SU_2(N_2) \) will have \( J_1 \times N_2 + J_2 \times N_1 \) fermion zero modes, where \( J_i \) are the instanton numbers of the \( SU_i(N_i) \) gauge groups. The extra factors of \( N_i \) show up as a result of the trace over the gauge indices not “coupled to” in the triangle diagram.

A few words are now in order to discuss possible solutions to the above equations. If one picks the \( \hat{\mathbf{n}} \) to be a zero eigenvector of the adjacency matrix, there is no more condition on \( \omega \) and hence \( \omega \) is arbitrary. This is a full \( U(1) \) symmetry that is non-anomalous, and explains why the \( D \) and \( E \) operations of [4][3] were found to be simply \( \mathbb{Z}_p \) subgroups of these continuous \( U(1) \) factors. This also suggests how to take the rephasings of this kind through Seiberg duality: one reassigns \( \hat{\omega}_0 = \omega_0^{n_0/N_0} \sum_i a_{0i} n_i - N_0 \), and then reassign \( n_0 = \sum_i a_{0i} n_i - n_0 \).
(all other $n_M$ remain the same) and one again gets a $U(1)$ in the new Seiberg phase. One should note that the nodal charge of the fields remains unchanged in this case because the reassignment of the value of $n_0$ exactly cancels the change in the value of $\omega_0$ so that $\omega_0^{n_0} = \omega_0^{\hat{n}_0}$.

Let us now consider when the $n_M$ are integers, a case that matches the $B$ and $C$ operations of \cite{2, 3}. One can see that if one takes the $n_M$ such that $\text{GCD}\left(\{\overrightarrow{n} \cdot a\}_{M}\right) = \lambda$, \hspace{1cm} (2.10)
then we may simply require that $\omega^\lambda = 1$, \hspace{1cm} (2.11)
and we again have a symmetry. This time, however, the symmetry is not continuous, but is a $\lambda^{th}$ root of the center of the gauge group. If we label this rephasing as $Q$, we have that $Q^\lambda = 1$ up to the center of the gauge group. Certainly a class of such vectors is possible if $a$ has any (non zero) integer eigenvalues. Also note that the case $\lambda = 1$ corresponds to the $\omega = 1$ case previously mentioned. As stated, we parameterize this by $\omega = e^{2\pi i}$ such that the roots of \hspace{1cm} (2.8) are non trivial. This is easily seen to be a rephasing using the center of the gauge group, because the additional root of \hspace{1cm} (2.8) makes this an $N_i$ root of for a node with rank $N_i$. These rephasings are therefore gauge equivalent to 1. This also means that the integers $n_K$ in this case are only understood modulo $\lambda$, as one may shift the center of each gauge group independently.

Let us show how such symmetries map through Seiberg duality. First, in the original Seiberg phase, we assume that there is a solution to
\[ a \cdot \overrightarrow{n} \equiv 0 \pmod{\lambda}. \hspace{1cm} (2.12) \]
We therefore wish to find the new vector $\overrightarrow{n}'$ after Seiberg duality that satisfies
\[ \tilde{a} \cdot \overrightarrow{n}' \equiv 0 \pmod{\lambda}. \hspace{1cm} (2.13) \]
We again suppose that we only wish to change $\hat{n}_0 \neq n_0$, leaving all other integers alone $\hat{n}_M = n_M, M \neq 0$. We now note that again the $M = a$ components of the above equation are automatically satisfied: only the anomalies away from the node being dualized are affected. We now make an educated guess as how one transforms $n_0$. We guess that
\[ \hat{n}_0 = \sum_i (a_{0i}n_i) - n_0 \]
\[ = \sum_i (a_{0i}n_i) + \sum_a (a_{0a}n_a) - n_0 \]
\[ \equiv \sum_i (a_{j0}n_j) - n_0 \pmod{\lambda}, \hspace{1cm} (2.14) \]
where we have used $a_{0k} = 0$ and the original anomaly cancelation conditions. The calculation now goes through exactly as it did for the zero eigenvectors (2.5): all $=$ signs are simply replaced with $\equiv_{(\text{mod } \lambda)}$. Actually, one may expect this structure from the discussions of [11].

We now make one final comment. Seiberg duality of a given node is it’s own inverse, and in fact the rephasings discussed above transform into themselves up to the center of the gauge group. This is noted simply by the fact that after two Seiberg duals of one node, the phase associated with the node has gone from

\[ \omega_0 \rightarrow \omega_0 \sum_i a_{0i}N_i - N_0 \rightarrow \omega_0 \sum_i a_{0i}N_i - N_0 \sum_j a_{j0}N_j - N_0 \rightarrow \omega_0, \]

(2.16)

where in the second Seiberg duality we used that what we were calling “in” arrows are now called “out” arrows, hence the switch in the sum from indices with no tilde to ones with tilde. This is of course the same because the effective number of in and out arrows are the same, but this will be more important for the mapping of the numbers $n_M$. For this, we note that the number $n_0$ gets mapped as

\[ n_0 \rightarrow \sum_i a_{0i}n_i - n_0 \]
\[ \rightarrow \sum_j a_{j0}N_j - \left(\sum_i a_{0i}n_i - n_0\right) \]
\[ = n_0 + \sum_j a_{j0}N_j - \sum_i a_{0i}n_i \equiv n_0 \pmod{\lambda}. \]

(2.17)

In the last line, we have again used the fact that we are modding out by the center of the gauge group, which corresponds to taking the number $n_M$ as only being defined $\text{mod } \lambda$.

### 2.3 General proof of the existence of the Heisenberg Group

Given the natural mapping of discrete $U(1)$ charges under Seiberg duality found above, we now combine this with the results of [2, 3] to show that there is generically an action of the Heisenberg group on the space of Seiberg dual quivers. Here, we are assuming that there is a symmetric Seiberg phase with a natural shift symmetry, which we will call $A$ (see section 3 for some examples). A large class of these was examined in [2, 3] and a the action of a finite Heisenberg group was found. Assuming that there is a symmetric phase, and an action of a Heisenberg group on this phase, we will show the existence of an action of a Heisenberg group on the space of all Seiberg phases.

First, note that if there is an $A$ symmetry of the symmetric phase, this descends to the entire tree of possible Seiberg dual quivers, however mapping from one phase to another. This space was discussed in [9] where it was given the name of duality tree; each point represents a quiver. In particular the three-nod tree was shown to be related to Markov’s equation. We
will be careful to call the points on the duality tree points, which represent entire quivers, to distinguish them from the nodes of the quivers themselves which represent gauge groups. The action on the duality tree can be seen easily. Take that we have a symmetric phase with \( x \times y \) nodes, and there is a manifest \( \mathbb{Z}_x \) symmetry that permutes the \( M^{th} \) node to the \((M+x)\)th node. The entire tree is defined by taking an arbitrary number of Siebert dualities on any number of the nodes in any given order. We therefore label the point on the Markov tree that is \( \ell \) Seiberg dualities away from the symmetric phase by the Seiberg dualities it takes to get there: \( (s_1, s_2, s_3, \cdots s_\ell) \). Here, the integers \( s_i \) label which node of the quiver diagram is to be Seiberg dualized, i.e. \( s_i \in 1 \cdots x \times y \). There is an analogous quiver given by \( (s_1 + y, s_2 + x, s_3 + y, \cdots s_\ell + y) \equiv \vec{s} + \vec{y} \) with the same matter content, same gauge groups and couplings, simply with the labels changed. This, therefore, defines the action of the \( A \) operator on this quiver: it maps the two differently labeled field theoretic degrees of freedom into each other. So, if one wants to go from one branch to the other, one must apply a set of inverse Seiberg dualities to get back to the symmetric phase, apply the \( A \) operation (as many times as needed) to reorder the fields, and then apply the same set of Seiberg dualities. However, because the ordering of the fields have been switched, the Seiberg dualities are in fact being performed along a different branch of the duality tree. We emphasize here that this is much the same as the operation \( B \) where the rephasings are really on the order one gives the fields in, not their subscripts : \( A \) may have already rearranged them, or may not have. \( B \)'s matrix representation does not depend on this! So, when we say that “\( B \) is rephasing the field \( U_1 \) as \( u_1 \times U_1 \)”, we really mean that it rephases this way only when the fields are given in the canonical order \( (U_1, U_2, \cdots) \); it really rephases the first field in the list. \( A \) of course changes exactly which field this is. We think of the Seiberg dualities in the same way. We give the gauge groups and fields a canonical ordering which \( A \) shuffles. The \( S^{\vec{s}} \) simply Seiberg dualize according to \( \vec{s} \) in the order that the gauge groups are listed, not what their labels are, and so may or may not move out along a different branch, depending on whether an \( A \) is present or not.

We will make these comments more precise here. We have labeled the series of Seiberg dualities with a vector \( \vec{s} \), and we will call the series of Seiberg dualities \( S_{\vec{s}} \). The inverse is of course given by the same vector, simply with its entries reversed \( S_{\vec{s}}^{-1} = S_{\text{reverse order}(\vec{s})}^{-1} \). \( A \) is simply defined using the symmetric phase

\[
\hat{A}_{\vec{s}', \vec{s}} = \delta_{\vec{s}', \vec{s}} S_{\vec{s}} AS_{\vec{s}}^{-1}
\]  

(2.18) with all other entries for \( \vec{s} \) and \( \vec{s}' \) equal zero. One must be careful here to note that something new has actually happened. The \( S^{-1} \) that maps back to the symmetric phase and the \( S \) that maps one out are (as matrices) identical. However, because the fields on which they act have been shifted by \( A \) one is actually going out on a different branch of the duality tree. It is equivalent to going out on the one shifted by the \( \vec{y} \) vector. The rephasing
The above operators satisfy the Heisenberg group structure on the duality tree. Let us consider a general quiver in the tree \( Q_{\pi_1} \) and the action of \( \hat{A}, \hat{B} \) and \( \hat{C} \) on this quiver
\[
\hat{A}_{\pi_1} \hat{B}_{\pi_1} \hat{C}_{\pi_1} Q_{\pi_1} = \delta(\pi_1, \pi_1) S_{\pi} AB S_{\pi_1}^{-1} Q_{\pi_1},
\]
where we make a special note that there is only one non zero entry for each \( \pi \) and \( \pi' \) such that that the only implicit summation is over those \( \pi \) in the \( \delta \) symbols: the indices in the Seiberg duals are fixed. All of them are in fact exactly equal to \( \pi_1 \) because they each contain a Kronecker \( \delta \). It is again the action of \( A \) which is non trivial and mixes the fields, so exactly which gauge group the entries of \( \pi \) are referring to depend only on the order in which the gauge groups are listed, which is changed by the presence of an \( A \). In fact all of the relations found in the symmetric phase descend to the entire tree, and so we find
\[
\hat{A}\hat{B} = \hat{B}\hat{A}\hat{C}, \quad \hat{C} \text{ commutes with everything,} \quad \hat{A}^x = \hat{B}^x = \hat{C}^x = 1.
\]

This, then, is an action on the whole space of possible Seiberg dual theories, and the equals sign are read only up to the center of the gauge group. The ranks and possible gauge couplings are different in each Seiberg phase, and so we take that the gauge redundancies are modded out in each phase individually. There may be some understanding of this as some duality among the relevant Faddeev-Popov ghosts, however we satisfy ourselves here by simply noting that the extra factors are purely gauge.

## 3 Finite Heisenberg groups acting on the space of Seiberg dual quivers

### 3.1 \( \mathbb{C}^3 / \mathbb{Z}_3 \)

Let us first consider a simple example discussed originally in [11]. We take the gauge theory dual to the Maldacena limit of string theory on \( \mathbb{C}^3 / \mathbb{Z}_3 \) where the orbifold action is given...
by \((z_1, z_2, z_3) \rightarrow (\xi z_1, \xi z_2, \xi^{-2} z_3)\) where \(\xi\) is a cubic root of unity. The quiver diagram is represented in the center of the figure (1). Let the rank of each of the gauge group be \(N_c\). Performing Seiberg duality at the node doubles its rank and the node is represented by a red circle. Three different quivers can be obtained by performing duality at the 3 different nodes (See Figure 1). We will show in this section that there exists a Heisenberg group acting on the three Seiberg dual quivers.

We label the dual quivers on the top, left and right as T, L and R respectively. The A transformation permutes the nodes between different Seiberg dual quivers. Let it act as

\[
1L \rightarrow 2T \rightarrow 3R, \\
2L \rightarrow 3T \rightarrow 1R, \\
3L \rightarrow 1T \rightarrow 2R, \tag{3.1}
\]

where the number in the front denotes the node corresponding to the quiver represented by the alphabet next to it. B and C transformations are phase transformations of chiral fields. Let us denote the phase of field \(U_i\) by \(u_i\). These phases respect invariance of the

Figure 1: \(\mathbb{C}_3/\mathbb{Z}_3\) quivers. Each Seiberg dual does not possess a \(\mathbb{Z}_3\) symmetry. However, the space of Seiberg duals has a manifest shift symmetry generated by \(\mathbb{Z}_3\).
superpotential and anomaly cancelation. These relations for the phases in the left quiver are

\[ u_1 u_2 u_3 = 1, \]
\[ (u_1^3 u_2^3)^N = 1, \]
\[ (u_2^3 u_3^6)^N = 1, \]
\[ (u_3^6 u_1^2)^N = 1. \] (3.2)

They are further simplified to obtain

\[ u_3 = (u_1 u_2)^{-1} \quad (u_1 u_2)^{3N} = 1 \quad u_1^{6N} = u_2^{6N} = 1 \] (3.3)

Below we write a particular solution for the phase assignments. We write it in terms of \(3N\)-th root of unity denoted as \(\omega\) i.e. \(\omega^{3N} = 1\). The topmost row denote the subscript corresponding to the fields whose phases are written below in that column. The numbers in the rows corresponding to B and C denote the powers to which \(\omega\) is raised.

|   | 1 | 2 | 3 |
|---|---|---|---|
| B | L | 0 | 0 | 0 |
|   | T | 0 | 1 | -1 |
|   | R | 1 | 0 | -1 |
| C | L | -1 | 1 | 0 |
|   | T | 0 | -1 | 1 |
|   | R | 1 | 0 | -1 |

Let us make a general remark about a technical point. The generators of the Heisenberg group satisfy

\[ AB = BAC, \quad AC = CA, \quad BC = CB, \]
\[ A^q = B^q = C^q = 1, \] (3.6)

up to an element in center of the gauge group. These can be rewritten as

\[ C^{-1} A^{-1} B^{-1} AB = Z_1^c \]
\[ A^{-1} C^{-1} AC = Z_2^c \]
\[ B^{-1} C^{-1} BC = 1 \]
\[ A^q = 1 \]
\[ B^q = Z_3^c \]
\[ C^q = Z_4^c \] (3.7)
where $Z_i$'s are elements in center of the gauge group.

Thus, after explicitly constructing the elements $A, B$ and $C$, we need to present the central elements involved in the construction of the group. The particular elements in the center of the gauge group needed for the Heisenberg group can be written in terms of a set of 3 integers $(a_1, a_2, a_3)$. These three elements will stand for a rephasing at the nodes $(1,2,3)$ by $\omega^{3a_1/2}, \omega^{3a_2}, \omega^{3a_3}$ respectively for the case of the left quiver diagram. The factor of $1/2$ in the exponent of the phase associated with node 1 is due to the fact that the rank of the gauge group at node 1 is twice that of the others for the left quiver diagram. The needed elements of the center of the gauge group as denoted in equations (3.7) can then be written as

\begin{align*}
Z_{4L}^c : & a_2 = a_3 = 1, \quad a_1 = 0 \\
Z_{1T}^c : & a_1 = a_3 = -1, \quad a_2 = 0 \\
Z_{3R}^c : & a_1 = a_2 = 1, \quad a_3 = 0 \\
Z_{3T}^c = & Z_{1T}^c, \quad Z_{4T}^c = -Z_{1T}^c, \quad Z_{4R}^c = Z_{3R}^c \\
Z_{1L}^c = & Z_{1R}^c, \quad Z_{2L}^c = Z_{2T}^c = Z_{2R}^c = Z_{3L}^c = 1
\end{align*}

The center of the gauge group corresponds to the left, top and the right quiver diagrams depending on the extra subscript ($L$, $T$ or $R$) respectively.

### 3.2 Heisenberg group in $Y^{4,2}$

Consider the quiver diagram for $Y^{4,2}$ as shown in Fig. 2 with chiral fields $U_i, Y_i$ and $Z_i$.

It has a cyclic $\mathbb{Z}_2$- symmetry interchanging the nodes as $(15)(26)(37)(48)$. It is convenient to introduce the following notation, $U = [U_2, U_3, U_4, U_6, U_7, U_8]^T$, $Y = [Y_1, Y_3, Y_4, Y_5, Y_7, Y_8]^T$ and $Z = [Z_1, Z_6]^T$ to denote the sets of chiral fields. The $A$ transformation acts on chiral fields as

\[
\begin{align*}
AU & = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} U \\
AZ & = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Z \\
AY & = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} Y
\end{align*}
\]

(3.8)

where $I_3$ is a $3 \times 3$ matrix. We will try to find other symmetries which are related to phase changes of the chiral fields. Let the phases be represented by small case letters. The relations
Figure 2: $Y^{4,2}$ quiver diagram

among the phases arising due to invariance of the superpotential are

$$
\begin{align*}
z_1 u_2 y_3 u_8 &= 1, \\
u_2 u_3 y_4 &= 1, \\
u_3 u_4 y_5 &= 1, \\
u_4 z_5 u_6 y_7 &= 1, \\
u_6 u_7 y_8 &= 1, \\
u_7 u_8 y_1 &= 1.
\end{align*}
$$

These equations determine the action on $Y$.

$Y \rightarrow [(u_7 u_8)^{-1}, (z_1 u_2 u_8)^{-1}, (u_2 u_3)^{-1}, (u_3 u_4)^{-1}, (u_4 u_6 z_3)^{-1}, (u_6 u_7)^{-1}]^T Y$. There are further constraints on the phases due to anomalies. They are

$$
\begin{align*}
(u_8^2 y_1 z_1)^N &= 1, \\
(z_1 u_2 y_4)^N &= 1, \\
(u_2^2 y_3 u_5 u_3^2)^N &= 1, \\
(u_3^2 y_4 y_7 u_4^2)^N &= 1, \\
(u_4^2 y_5 z_3)^N &= 1, \\
(z_5 y_8 u_6^2)^N &= 1, \\
(u_6^2 y_7 y_1 u_7^2)^N &= 1, \\
(u_7^2 y y_3 u_8^2)^N &= 1.
\end{align*}
$$

(3.10)
These can be simplified to obtain

\[ u_2^{2N} = u_4^{2N} = u_6^{2N} = u_8^{2N}, \]
\[ u_2^N = u_6^N, \quad u_4^N = u_8^N, \]
\[ \left( \frac{u_3}{u_7} \right)^N = \left( \frac{u_2}{u_8} \right)^N, \]
\[ z_1^N = \left( \frac{u_3}{u_2} \right)^N, \quad z_5^N = \left( \frac{u_3}{u_4} \right)^N. \] (3.11)

Both the other generators of the Heisenberg group, namely \( B \) and \( C \), are phase changes which will act on \( U, Z \) and \( Y \) diagonally. They should also satisfy relations (3.11) and (3.9). A particular solution is

\[ B : z_1 = u_4 = u_7 = u_8 = 1 \quad u_2 = u_3 = \omega \quad z_5 = u_6 = \omega^{-1} \]
\[ C : u_2 = u_4 = u_6 = u_8 = 1 \quad z_1 = z_5 = \omega \quad u_3 = u_7 = \omega^{-1} \] (3.12)

Here, \( \omega^{2N} = 1 \). Thus, we have the action of \( \text{Heis}(\mathbb{Z}_2 \times \mathbb{Z}_2) \).

**Center of the gauge group**

The Heisenberg group is closed up to center of the gauge group. It will be useful to first give a quick look at the center. The center has 8 generators with 8 parameters (\( a_i \) say), one acting at each node. It changes the chiral field in its fundamental (anti-fundamental) representation by \( w^{\pm a_i} \). It is convenient to work with a new set of generators \( \alpha_i \), such that \( \alpha_i = a_i - a_{i+1} \) (\( \alpha_8 = a_8 - a_1 \)). However, there exists a relation between them.

The center acts on the chiral fields as

\[ U \rightarrow \text{diag}[\omega^{2a_2}, \omega^{2a_3}, \omega^{2a_4}, \omega^{2a_5}, \omega^{2a_7}, \omega^{2a_8}]U \]
\[ Y \rightarrow \text{diag}[\omega^{2a_1}, \omega^{2a_3}]Z \]
\[ Z \rightarrow \text{diag}[\omega^{-2a_2-2a_8}, \omega^{-2a_2-2a_8-2a_1}, \omega^{-2a_2-2a_3}, \omega^{-2a_3-2a_4}, \omega^{-2a_4-2a_6-2a_5}, \omega^{-2a_6-2a_7}]Y \] (3.13)

The generators \( \alpha_i \) satisfy a relation

\[ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0 \] (3.14)

For the transformations given in (3.12), the particular elements in the center are

\[ Z_1^c : \quad \alpha_2 = \alpha_3 = 1 \quad \alpha_5 = \alpha_6 = -1 \quad \alpha_1 = \alpha_4 = \alpha_7 = \alpha_8 = 0 \]
\[ Z_2^c : \quad \alpha_i = 0 \]
\[ Z_3^c = Z_1^c \]
\[ Z_4^c : \quad \alpha_1 = \alpha_5 = 1 \quad \alpha_3 = \alpha_7 = -1 \quad \alpha_2 = \alpha_4 = \alpha_6 = \alpha_8 = 0 \] (3.15)
3.2.1 Seiberg duals of $Y^{4,2}$

Toric duality in the case of $Y^{p,q}$ spaces shift the one of the singlet fields in the outside of the quiver diagram [12]. Therefore, we concentrate on two Seiberg duals phases of $Y^{4,2}$ in figure 3. They are obtained by Seiberg dualizing at nodes 2 and at node 6. We can think of A symmetry here as the one which takes node $i$ of the left quiver to a node $i+4$ ($i-4$ for $i > 4$) of the right quiver on the and vice-versa. B and C transformations are changes of phases of chiral fields, which are constrained. For the left quiver, the superpotential conditions are

$$
y_1u_7u_8 = 1, \quad y_3u_8u_1 = 1, \quad y_2u_1u_3 = 1, \quad y_5u_3z_2u_4 = 1, \quad y_7u_4z_5u_6 = 1, \quad y_8u_6u_7 = 1.
$$

(3.16)

The anomaly cancelation conditions are

$$
(u_8^2u_1^2y_1y_2)^N = 1, \quad (u_3^2y_2z_2)^N = 1, \quad (u_1^2u_3y_3y_5)^N = 1, \quad (u_4^2z_2y_7)^N = 1, \quad (u_4^2z_3y_5)^N = 1, \quad (u_7^2u_6y_8)^N = 1, \quad (u_7^2u_3y_8y_3)^N = 1.
$$

(3.17)

The former set helps to write the phases $y_i$ in terms of the other phases. The second set can be reduced to obtain

$$
u_1^2 = u_7^2, \quad u_4^N = u_8^N = \left(\frac{u_7}{z_2}\right)^N, \quad u_3^N = u_6^N = \left(\frac{u_1z_2}{u_7}\right)^N, \quad z_5^N = \left(\frac{u_1z_2}{u_7}\right)^N.
$$

(3.18)
A particular solution is to consider the phases
\[ u_1 = u_7 = 1 \quad u_3 = u_4 = u_6 = \omega \quad z_2 = z_5 = u_8 = \omega^{-1}. \] (3.19)

Next, we construct the phases for the transformations B and C acting on all the chiral fields in the two quivers. We present them in the table below, where the topmost row are the subscripts of \( u_i \) or \( z_i \). The numbers in other rows are the powers to which \( \omega \) is raised for the given transformation indicated on the left. The letters L and R in the second column refer to left and right quivers drawn in figure (3). We assume \( \omega^{2N} = 1 \).

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| B |   | -1 | 1 | 1 | -1 | 1 | 0 | -1 |
|   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| C |   | -1 | 1 | 1 | -1 | 1 | 0 | -1 |
|   | 0 | 1 | 0 | -1 | 0 | -1 | 1 | 1 |

(3.20)

The particular elements of the center of the gauge group needed to satisfy the Heisenberg algebra in this case can be written in notation used in section (3.2). We will use notation \( Z^c_i \) to denote various elements in the center and \( \alpha_i \) as its generator. We will also put a subscript L or R to denote to distinguish the action of the center on the two different quivers. They necessary \( Z^c_i \) are

\[ Z^c_{1L} = Z^c_{1R} = Z^c_{2L} = Z^c_{2R} = Z^c_{3R} = I \text{ (identity element)}, \]
\[ Z^c_{3L} : \quad \alpha_3 = \alpha_4 = \alpha_6 = 1 \quad \alpha_2 = \alpha_5 = \alpha_8 = -1 \quad \alpha_1 = \alpha_7 = 0, \]
\[ \alpha_2 = \alpha_7 = \alpha_8 = 1 \quad \alpha_1 = \alpha_4 = \alpha_6 = -1 \quad \alpha_3 = \alpha_5 = 0. \] (3.21)

### 3.3 \( Y^{6,3} \) quiver with manifest shift symmetry \( A \)

We look at a toric phase of \( Y^{6,3} \) quiver. It possesses \( A \) symmetry which act on nodes as \((1,5,9)\) \((2,6,10)\) \((3,7,11)\) \((4,8,12)\). The Heisenberg group for this case has been worked out in [2]. The superpotential and anomaly conditions are
Figure 4: Quiver gauge theory for $Y_{6,3}$ with a manifest $\mathbb{Z}_3$ shift symmetry.

\begin{align*}
  u_1 u_2 y_3 &= 1, & u_2 u_3 y_4 &= 1, \\
  u_3 z_4 u_5 y_6 &= 1, & u_5 u_6 y_7 &= 1, \\
  u_6 u_7 y_8 &= 1, & u_7 z_8 u_9 y_{10} &= 1, \\
  u_9 u_{10} y_{11} &= 1, & u_{10} u_{11} y_{12} &= 1, \\
  u_{11} z_{12} u_1 y_2 &= 1, & (u_1^2 y_3 z_{12})^N &= 1, & (u_1^2 u_2^2 y_2 y_4)^N &= 1, \\
  (u_2^2 u_3^2 y_3 y_6)^N &= 1, & (u_2^2 z_4 y_4)^N &= 1, \\
  (u_2^2 z_4 y_7)^N &= 1, & (u_3^2 u_5^2 y_6 y_8)^N &= 1, \\
  (u_5^2 u_7^2 y_7 y_{10})^N &= 1, & (u_7^2 z_8 y_8)^N &= 1, \\
  (z_8 u_9^2 y_{11})^N &= 1, & (u_9^2 u_{10}^2 y_{10} y_{12})^N &= 1, \\
  (u_{10}^2 u_{11}^2 y_{11} y_2)^N &= 1, & (u_{11}^2 z_{12} y_{12})^N &= 1.
\end{align*}

One particular solution is

\begin{align*}
  B : u_1 &= u_6 = u_7 = z_8 = 1 \\
  u_2 &= u_5 = u_{11} = z_{12} = \omega \\
  u_3 &= z_4 = u_9 = u_{10} = \omega^{-1} \\
  C : u_1 &= u_3 = z_4 = u_5 = u_7 = z_8 = u_9 = u_{11} = \omega^{-1} \\
  u_2 &= u_6 = u_{10} = \omega \\
  z_{12} &= \omega^5
\end{align*}

In order to write the center of the gauge group, we will use a similar notation as in section \ref{sec:4.2} for $Y^{4,2}$ case where $a_i$ for $i = 1...12$ denote the generator at each node. These generators are commutative and each of them changes the phase of an incoming(outgoing) quiver by
\( \omega^{-3a_i}(\omega^{+3a_i}) \). We take linear combinations of the generators as \( \alpha_i = a_i - a_{i+1}(\alpha_{12} = a_{12} - a_1) \). Then the elements in the center of the gauge group are

\[
Z_{c1}^\alpha : \alpha_5 = \alpha_{11} = 1, \quad \alpha_{10} = \alpha_{12} = -1 \quad \text{zero otherwise.}
\]

\[
Z_{c2}^\alpha : \alpha_{12} = 2 \quad \alpha_8 = -2 \quad \text{zero otherwise.}
\]

\[
Z_{c3}^\alpha : \alpha_2 = \alpha_5 = \alpha_{11} = \alpha_{12} = 1 \quad \alpha_3 = \alpha_4 = \alpha_9 = \alpha_{10} = -1 \quad \alpha_1 = \alpha_6 = \alpha_7 = \alpha_8 = 0
\]

\[
Z_{c4}^\alpha : \alpha_2 = \alpha_6 = \alpha_{10} = 1 \quad \alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{11} = -1 \quad \alpha_{12} = 5 \quad (3.25)
\]

3.4 Seiberg duals of \( Y^{6,3} \)

![Figure 5: Three Seiberg phases of \( Y_{6,3} \). Each phase lacks a \( \mathbb{Z}_3 \) shift symmetry but the symmetry is present when considering the space of three Seiberg phases.](image)

Now we look at certain Seiberg duals of \( Y^{6,3} \) quiver. Let us label the figures in figure on the top, left and right as \( T, L \) and \( R \). The A transformation acts by taking a node \( i \) of the top(left) quiver to a node \( i+4 \) (i-8 if \( i > 8 \)) of the right (top) quiver as well as a node \( i \) of the right quiver to a node \( i+4 \) (i-8 if \( i > 8 \)) of the left quiver. Let us consider the quiver diagram on the left and find the conditions for the phases associated with the B and
C transformations. Clearly, similar conditions can be written for the top and the right.

\[
y_1u_2z_1u_3 = 1, \\
y_6u_3z_4u_5 = 1, \\
y_7u_5u_6 = 1, \\
y_8u_6u_7 = 1, \\
y_10u_7z_8u_9 = 1, \\
y_{11}u_9u_{10} = 1, \\
y_{12}u_{10}u_{11} = 1, \\
y_2u_{11}u_{12} = 1, \\
y_1u_{12}u_2 = 1, \\
\text{(3.26)}
\]

\[
(y_1^2u_2^2) = 1, \\
(y_2^2u_3^2) = 1, \\
(y_3^2z_4^2) = 1, \\
(y_4^2u_5^2y_7^2) = 1, \\
(y_5^2z_7^2y_8^2) = 1, \\
(y_6^2u_9^2u_{10}^2u_{12}^2) = 1, \\
(y_7^2u_9^2u_{10}^2u_{12}^2) = 1, \\
(y_8^2u_{11}^2u_{12}^2u_5^2) = 1, \\
(y_{11}^2u_{12}^2u_7^2) = 1, \\
\text{(3.27)}
\]

Here the set (3.26) comes from superpotential invariance and the set (3.27) are the anomaly constraints. The first set allows \(y_i's\) to be solved in terms of other phases. The second set can be simplified to obtain

\[
u_3^N = \frac{u_2^N}{u_3^N}, \\
z_4^N = \left(\frac{y_1^2u_2^2}{u_3^2}\right)^N,
\]

\[
u_5^N = \left(\frac{u_3^2}{u_2^2}\right)^N, \\
u_6^N = \left(\frac{z_1u_2^2}{u_3^2}\right)^N,
\]

\[
u_7^N = \frac{u_2^N}{u_3^N}, \\
z_8^N = \left(\frac{z_1u_2^2}{u_2^2}\right)^N,
\]

\[
u_9^N = \frac{u_3^N}{u_3^N}, \\
u_{10}^N = \left(\frac{z_1u_2^2}{u_3^2}\right)^N,
\]

\[
u_{11}^N = \left(\frac{u_2^N}{u_3^N}\right)^N, \\
u_{12}^N = \left(\frac{z_1u_2^2}{u_3^2}\right)^N.
\]

(3.28)

For a particular solution, we will write B and C acting on \(u_1\) and \(z_i\) only. The numbers in the top row of the table below denote the subscripts of \(u_i\) and \(z_i\). The numbers in the rows corresponding to B and C denote the powers to which \(\omega\) is raised.

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|
| **B** |    |    |    |    |    |    |    |    |    |    |    |    |
| \(R\) | 1  | 0  | 1  | 0  | -1 | -1 | 0  | -1 | 1  | 0  | -1 | 1  |
| \(L\) | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| \(T\) | 1  | 1  | 0  | -1 | -1 | 0  | -1 | -1 | -1 | 0  | -1 | 0  |
| **C** |    |    |    |    |    |    |    |    |    |    |    |    |
| \(R\) | 1  | 0  | 1  | 0  | -1 | -1 | 0  | -1 | 1  | 0  | -1 | 1  |
| \(L\) | 1  | 0  | -1 | 1  | 0  | 1  | 0  | -1 | -1 | 0  | -1 | 1  |
| \(T\) | -1 | -1 | 0  | -1 | 1  | 0  | -1 | 1  | 1  | 0  | 1  | 0  |
We will write the center of gauge group in terms of generators $\alpha_i$ as defined in previous subsection. We reinterpret few of the generators as

For L quiver $\alpha_4 = a_4 - a_6$ $\alpha_5 = a_5 - a_7$ $\alpha_6 = a_6 - a_5$
For T quiver $\alpha_8 = a_8 - a_{10}$ $\alpha_9 = a_9 - a_{11}$ $\alpha_{10} = a_{10} - a_9$
For R quiver $\alpha_1 = a_1 - a_3$ $\alpha_2 = a_2 - a_1$ $\alpha_{12} = a_{12} - a_2$

The elements in the center of the gauge group needed for the algebra are

\[
Z_{1R}^c = Z_{1L}^c = Z_{2R}^c = Z_{2L}^c = Z_{2T}^c = Z_{3L}^c = I \quad \text{(Identity element)},
\]
\[
Z_{3R}^c : \quad \alpha_1 = \alpha_3 = \alpha_9 = \alpha_{12} = 1,
\]
\[
\alpha_5 = \alpha_6 = \alpha_8 = \alpha_{11} = -1,
\]
\[
\alpha_2 = \alpha_4 = \alpha_7 = \alpha_{10} = 0,
\]
\[
Z_{4L}^c : \quad \alpha_5 = \alpha_8 = \alpha_9 = \alpha_{11} = 1,
\]
\[
\alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = -1,
\]
\[
\alpha_3 = \alpha_6 = \alpha_{10} = \alpha_{12} = 0,
\]
\[
Z_{1T}^c : \quad \alpha_3 = \alpha_9 = \alpha_{10} = \alpha_{12} = 1,
\]
\[
\alpha_1 = \alpha_4 = \alpha_9 = \alpha_7 = -1,
\]
\[
\alpha_2 = \alpha_6 = \alpha_8 = \alpha_{11} = 0,
\]
\[
Z_{4R}^c = Z_{3R}^c \quad Z_{3T}^c = -Z_{4T}^c = Z_{1T}^c. \quad (3.30)
\]

4 Conclusion

We have demonstrated that the space of Seiberg duals to a given quiver gauge theory inherits the action of a finite Heisenberg group. An interesting new feature is that to consider the action of the Heisenberg group we are forced to enlarged the space from one quiver to the set of quiver gauge theories arising from performing Seiberg duality at various nodes.

It has been shown that different Seiberg phases are related to different toric phases [7, 8, 10]. It would, therefore, be nice to study the action of the Heisenberg group purely as a symmetry in the space of toric phases. This is particularly interesting in the space of quiver gauge theories which are Seiberg dual with different ranks of the gauge groups.

Assuming that there is a well-defined string theory dual for each Seiberg phase, our findings imply that charges of branes of one string theory are related to the charges of branes of a different theory. In a sense this provide a sort of stringy toric duality. A perhaps more speculative way of describing this situation is that string theory views different Seiberg duals as twisted sectors of a given theory. It would be interesting to explore these suggestions and we hope to return to some of these questions in the future.
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