An Independent Set Axiomatization of Symplectic Matroids

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1. Introduction

Symplectic matroids are rather esoteric objects, so they require more than the average amount of motivation. Thus we shall give a broader introduction than is strictly necessary for understanding our main result.

Symplectic matroids occur at the confluence of two streams in modern combinatorics: Coxeter groups and matroid theory. Recall that a Coxeter system is a pair \((W, S)\) where \(W\) is a group and \(S \subseteq W\) is a set of generators subject only to relations of the form

\[
(ss')^{m(s,s')} = 1,
\]

where \(m(s,s) = 1\) and \(m(s, s') = m(s', s) \geq 2\) if \(s \neq s'\). The group \(W\) is called a Coxeter group.

Perhaps the most famous examples of Coxeter groups are those that arise in Lie theory. There are exactly four infinite families of finite-dimensional complex simple Lie algebras (plus five exceptions that we will ignore here):

- \(A_n\) : special linear Lie algebras
- \(B_n\) : odd orthogonal Lie algebras
- \(C_n\) : symplectic Lie algebras
- \(D_n\) : even orthogonal Lie algebras

To each of these Lie algebras is associated a finite group called its Weyl group. The precise definition of a Weyl group is not important here; what is important is that it is always a Coxeter group. For example, the Weyl group of \(A_n\) is the symmetric group on \(n + 1\) letters, and it is not too difficult to verify directly that the symmetric group is a Coxeter group (let \(S\) be the set of all adjacent transpositions \((i, i + 1)\)).

Coxeter groups typically arise in combinatorics when some combinatorial concept is shown to be definable purely in terms of the symmetric group. Since the symmetric group is a Coxeter group, one can try replacing the term “symmetric group” in such a definition by an arbitrary Coxeter group to see if the result makes sense and is interesting. This simple tactic has turned out to be surprisingly fruitful, in part because it often reveals how techniques from Lie theory and other areas of mathematics can be brought to bear on combinatorial problems. A beautiful example of this is Cherednik’s proof of Macdonald’s inner product identities using double affine Hecke algebras. Roughly speaking, Macdonald formulated Coxeter group analogues of a classical combinatorial problem called the “Dyson conjecture,” and this paved the way for Cherednik’s discovery of an algebraic structure underlying the phenomena. For an exposition, see [9].

It is therefore natural to ask if matroids can be defined in terms of the symmetric group. If so, what do the Coxeter analogues of matroids look like? J. P. S. Kung [10,
11] seems to have been the first to ask and answer these questions. Later, I. M. Gelfand and V. V. Serganova [8] suggested a different definition of such analogues (which they called “WP-matroids”). In both cases the work has a strong geometric flavor, and one exciting possibility is that these “Coxeter matroids” may form the foundation for discrete symplectic and orthogonal geometry, in the same way that oriented matroids form the basis for MacPherson’s theory of combinatorial differentiable manifolds [13].

Before such a “combinatorial Erlanger program” can be carried out, however, many fundamental questions must first be answered. A basic fact about matroids is that they admit a wide variety of equivalent elementary axiomatizations: independent sets, circuit elimination, basis exchange, etc. But so far no analogous elementary axiomatizations for WP-matroids are known. Borovik, Gelfand and White [2] have made some progress in obtaining such elementary axiomatizations in the symplectic case, but their paper does not, for example, give an independent set axiomatization of symplectic matroids. Indeed, for some time it was suspected that finding such an axiomatization might be an intractable problem.

The main result of this paper is an independent set axiomatization for (Gelfand-Serganova) symplectic matroids. In addition to answering a very natural question, this result provides one of the simplest ways to date of explaining what a symplectic matroid is to someone with no background in matroids or Coxeter groups.

2. Definitions

The goal of this section is to give the definition of a symplectic matroid. The standard definition involves the Bruhat order on parabolic quotients of a Coxeter group, but in order to keep everything as simple as possible, we will take advantage of the results in [2] and define symplectic matroids in a way that requires no explicit mention of such concepts. Readers familiar with Coxeter groups who want the full story should see [15].

Let \( E_{\pm n} \) be the set \( \{\pm 1, \pm 2, \pm 3, \ldots, \pm n\} \). For brevity we will sometimes write the minus sign on top; e.g., we will write \( \bar{1} \) for \( -1 \). If \( w \) is a permutation of \( E_{\pm n} \) and \( B = \{b_1, b_2, \ldots, b_k\} \) is a subset of \( E_{\pm n} \), then we define

\[
wB \overset{\text{def}}{=} \{wb_1, wb_2, \ldots, wb_k\}.
\]

An important concept in Coxeter matroid theory is admissibility. If \( S \subseteq E_{\pm n} \), define

\[
\bar{S} \overset{\text{def}}{=} \{-s \mid s \in S\}.
\]

We say that \( S \) is admissible if \( S \cap \bar{S} = \emptyset \). A permutation \( w \) of \( E_{\pm n} \) is admissible if \( w(-x) = -wx \) for all \( x \in E_{\pm n} \). A total ordering \( \prec \) of the elements of \( E_{\pm n} \) is admissible if there exists an admissible permutation \( w \) such that \( x \prec y \) if and only if \( wx < wy \). (The reader may find it helpful to visualize an admissible ordering as a signed permutation \( \sigma \)
of \{1, 2, \ldots, n\} followed by the negative of the reversal of \(\sigma\), e.g., \(\bar{2}, 1, 3, 3, 1, 2\). If \(\prec\) is an admissible total ordering of \(E_{\pm n}\), then a map \(\omega : E_{\pm n} \to \mathbb{R}\) is said to be a weight function compatible with \(\prec\) if \(i \prec j\) implies \(\omega(i) \leq \omega(j)\).

One way of defining ordinary matroids involves the greedy algorithm [14, section 1.8]. This is the approach we shall take to symplectic matroids. Suppose we are given an admissible total ordering \(\prec\), a weight function \(\omega\) compatible with \(\prec\), and a collection \(B\) of subsets of \(E_{\pm n}\). Then we define the greedy solution of \(B\) to be the element \(B \in B\) that is constructed as follows: we begin with no elements in \(B\) and then we consider each element of \(E_{\pm n}\) in turn from the largest (relative to \(\prec\)) to the smallest, adding it to \(B\) unless doing so would make it impossible to end up with a member of \(B\) no matter which other elements of \(E_{\pm n}\) we subsequently add to \(B\).

For example, suppose \(n = 3\) and our admissible total ordering is the standard ordering. Let \(B = \{\{\bar{2}, 1, 3\}, \{\bar{2}, 1, 3\}\}\). We begin by putting 3 into \(B\), because there are certainly members of \(B\) containing 3. We next consider 2, but we can’t add 2 to \(B\), because if we do then regardless of what further numbers we add to \(B\), we can never produce a member of \(B\). (In other words, \(\{2, 3\}\) is not a subset of any member of \(B\).) Continuing in this way, we find that the greedy solution is \(\{\bar{2}, 1, 3\}\).

Finally, we say that the greedy solution \(B\) of \(B\) is optimal if \(\omega(B) \geq \omega(B')\) for all \(B' \in B\), where as usual \(\omega(B)\) denotes \(\sum_{b \in B} \omega(b)\). We can now define a symplectic matroid.

**Definition.** A symplectic matroid is a pair \((E_{\pm n}, B)\) where \(B\) is a nonempty family of equinumerous admissible subsets of \(E_{\pm n}\) with the property that for every admissible total ordering \(\prec\) of \(E_{\pm n}\) and every weight function compatible with \(\prec\), the greedy solution of \(B\) is optimal. The family \(B\) is called the family of bases of the symplectic matroid.

**Remark.** The equivalence of this definition of symplectic matroid with the usual definition is the content of [2, Theorem 16].

An example of a symplectic matroid is \((E_{\pm 3}, B)\) where \(B = \{1\bar{3}, 2\bar{3}, \bar{1}2, \bar{1}3\}\). Here \(1\bar{3}\) is to be understood as shorthand for the set \(\{1, \bar{3}\}\). Note that a symplectic matroid is not a matroid; it is an analogue of a matroid. (There is a sense in which ordinary matroids may be regarded as special cases of symplectic matroids, but this need not concern us here.)

### 3. The Main Result

If \((E_{\pm n}, B)\) is a symplectic matroid, we define its family \(\mathcal{I}\) of independent sets by

\[
\mathcal{I} \overset{\text{def}}{=} \{I \subseteq E_{\pm n} \mid I \subseteq B \text{ for some } B \in B\}.
\]

In the example of a symplectic matroid given in the last section, the family of independent sets is \(\mathcal{I} = \{\emptyset, 1, \bar{1}, 2, 3, 3\} \cup B\). Notice that we can recover \(B\) from \(\mathcal{I}\); the members of
\( \mathcal{B} \) are just the maximal members of \( \mathcal{I} \) with respect to inclusion. Thus, a characterization of \( \mathcal{I} \) could be used as an alternative definition or axiomatization of a symplectic matroid. This is precisely what the following theorem provides.

**Theorem 1.** A subset-closed family \( \mathcal{I} \) of admissible subsets of \( E_{\pm n} \) is the family of independent sets of a symplectic matroid if and only if it has the following property:

If \( I \) and \( J \) are members of \( \mathcal{I} \) such that \( |I| < |J| \) and such that for all \( y \in J \setminus I \), the set \( \{y\} \cup I \) is not in \( \mathcal{I} \), then \( I \cup J \) is inadmissible and there exists \( x \notin I \) such that both \( \{x\} \cup I \) and \( \{\bar{x}\} \cup I \setminus J \) are in \( \mathcal{I} \).

We remark that part of this theorem—the part about \( I \cup J \) being inadmissible—was previously known and is essentially Theorem 14 of [2]. Notice incidentally that the expression "\( \{\bar{x}\} \cup I \setminus J \)" looks ambiguous because it is not clear whether we take the union first or subtract first, but actually there is no ambiguity since the hypothesis prevents \( x \) from being an element of \( J \).

**Proof.** Unless otherwise specified, the terms "larger" and "smaller" in this proof refer to the admissible total ordering \( \prec \). The reader should visualize such an ordering by writing out the elements in order in a horizontal line, with the largest element first.

**Sufficiency.** Assume that \( \mathcal{I} \) has the stated property. Call a maximal (with respect to inclusion) member of \( \mathcal{I} \) a "basis" of \( \mathcal{I} \). All bases of \( \mathcal{I} \) are admissible, and the stated property of \( \mathcal{I} \) ensures that all bases of \( \mathcal{I} \) have the same number of elements. Let \( \mathcal{B} \) be the collection of bases of \( \mathcal{I} \). We now make the following claim, which we shall call (*).

(*) Let \( \prec \) be an admissible ordering. Let \( I \) be a set consisting of the first \( i \) elements of \( E_{\pm n} \) that are picked up by the greedy algorithm (for some \( i \geq 0 \)). Let \( J \) be a member of \( \mathcal{I} \) such that \( i < |J| \). Then the \((i+1)\)st element picked up by the greedy algorithm is no smaller than the smallest element of \( J \).

To see this, note first that if there exists \( y \in J \setminus I \) such that \( \{y\} \cup I \in \mathcal{I} \), then we are done, because then the greedy algorithm will pick up either \( y \) or something larger than \( y \), and \( y \) is trivially no smaller than the smallest element of \( J \). Otherwise, since \( \mathcal{I} \) has the stated property, there exists \( x \notin I \) such that \( \{x\} \cup I \) and \( \{\bar{x}\} \cup I \setminus J \) are both in \( \mathcal{I} \). Moreover, \( I \cup J \) is inadmissible, but each of \( I \) and \( J \) is admissible, so there exists \( z \in I \) such that \( \bar{z} \in J \). Choose the largest such \( z \). Then by the maximality of \( z \), the set \( S \) of elements of \( I \) that are larger than \( z \) is a subset of \( I \setminus J \), and therefore both \( S \cup \{x\} \) and \( S \cup \{\bar{x}\} \) are in \( \mathcal{I} \). Now \( \bar{x} \notin I \) (since \( \{x\} \cup I \) is in \( \mathcal{I} \) and is therefore admissible) and also \( x \notin I \), so neither \( x \) nor \( \bar{x} \) can be larger than \( z \)—otherwise, since both \( x \) and \( \bar{x} \) are "compatible" with \( S \), the greedy algorithm would have picked one of them (or some other element not in \( I \) that is even larger), and it didn’t. It follows that \( z \) appears before the “halfway point” (the point between the \( n \)th and the \((n+1)\)st elements in the ordering), and that \( x \) and \( \bar{x} \) both appear after \( z \) but before \( \bar{z} \). Then the \((i+1)\)st element picked up...
by the greedy algorithm must be no smaller than $x$, which is no smaller then $z$, which in turn is no smaller than the smallest element of $J$, since $z \in J$. This proves (*).

Now let $\prec$ be an admissible ordering and let $\omega$ be some weight function compatible with $\prec$. Let $B$ be the basis of $\mathcal{S}$ chosen by the greedy algorithm. We want to show that $B$ is optimal, so let $B'$ be another basis. We claim that for all $i > 0$, the $i$th element of $B$ is no smaller than the $i$th element of $B'$. For, given $i$, let $I$ be the set consisting of the largest $i - 1$ elements of $A$ and let $J$ be the set consisting of the largest $i$ elements of $B'$. Then $|I| < |J|$, so by (*), the $i$th element picked up by the greedy algorithm (i.e., the $i$th element of $B$) is no smaller than the smallest element (i.e., the $i$th element) of $J$. This proves the claim, which in turn shows that for all $i$, the weight of the $i$th element of $B$ is greater than or equal to the weight of the $i$th element of $B'$, so indeed $B$ is optimal.

**Necessity.** Suppose that $\mathcal{S}$ is the family of independent sets of a symplectic matroid, and let $I$ and $J$ be members of $\mathcal{S}$ such that $|I| < |J|$ and such that for all $y \in J \setminus I$, $\{y\} \cup I \notin \mathcal{S}$. Then, as already mentioned, [2, Theorem 14] implies that $I \cup J$ is inadmissible. The set $I \cup J$ may be partitioned into four disjoint sets $W$, $Y$, $Z$, and $\bar{Z}$, where $W$, $Y$, and $Z$ are defined as follows:

\[
W = I \setminus \bar{J} \\
Y = J \setminus (I \cup \bar{J}) \\
Z = I \cap \bar{J}
\]

In words, $Z$ is the subset of $I$ whose negatives are in $J$, $W$ is the rest of $I$, and $Y$ is what remains in $J$ after $W$, $Z$, and $\bar{Z}$ are removed.

Now let $X = E_{\pm n} \setminus (W \cup \bar{W} \cup Y \cup \bar{Y} \cup Z \cup \bar{Z})$. Define a “half” of $X$ to be a maximal (with respect to inclusion) admissible subset of $X$. Clearly, if $H$ is a half of $X$, then $H$ and $\bar{H}$ partition $X$ into two disjoint sets and $|H| = |\bar{H}|$. Define a “WXYZ ordering” to be an admissible ordering in which the elements of $W$ come first, then the elements of some half $H$ of $X$, then the elements of $Y$, and then the elements of $Z$. (This gives us half of $E_{\pm n}$, so the ordering of the rest of $E_{\pm n}$—namely $\bar{Z} \bar{Y} H \bar{W}$—is determined.)

Now suppose we are given a $WXYZ$ ordering with the weight function that equals one on $W$, $H$, $Y$, $Z$, and $\bar{Z}$ and equals zero after that. The greedy algorithm will begin by picking up the elements of $W$. We claim that some element of $H \cup Y$ must be picked up after that. For if not, the algorithm will pick up $Z$, since these are just the remaining elements of $I$. Then it will skip over $\bar{Z}$. This implies that the weight of the basis chosen will be $|I|$, but $J$ is contained in $W \cup Y \cup Z$ so the weight of $J$ is $|J| > |I|$, a contradiction.

The argument just given applies regardless of how the half $H$ of $X$ is chosen. Therefore the following set $S$ is nonempty:

\[
S = \{x \in X \mid \{x\} \cup W \in \mathcal{S} \text{ and } \{x\} \cup W \in \mathcal{S} \} \cup \{y \in Y \mid \{y\} \cup W \in F\}.
\]

(For if not, we could choose a half $H$ of $X$ such that for all $x \in H$, $\{x\} \cup W$ would not be in $\mathcal{S}$, and this would cause trouble for the greedy algorithm as just explained.)
Now construct an admissible ordering $\prec$ as follows. Begin with a $WXYZ$ ordering that minimizes the number of $x \in H$ such that $\{x\} \cup W \in \mathcal{I}$. Then reposition every element in $(H \cup Y) \cap S$ so that they now come after $Z$ (but before $\bar{Z}$). Finally, reposition the “mirror images” of the elements just moved to restore admissibility. For example, if the $WXYZ$ ordering were:

\[
\begin{align*}
    a & \succ b \succ c \succ d \succ e \succ f \succ g \succ \bar{g} \succ \bar{f} \succ \bar{e} \succ \bar{d} \succ \bar{c} \succ \bar{b} \succ \bar{a} \\
    W & \quad H \quad Y \quad Z
\end{align*}
\]

and $d$ and $e$ were in $S$ but $c$ and $f$ were not, then $\prec$ would be given by:

\[
\begin{align*}
    a & \succ b \succ c \succ f \succ g \succ d \succ e \succ \bar{e} \succ \bar{d} \succ \bar{g} \succ \bar{f} \succ \bar{c} \succ \bar{b} \succ \bar{a}.
\end{align*}
\]

Observe that by the minimality in the choice of $H$, the elements $x \in H \cup Y$ that are not repositioned have the property that $\{x\} \cup W \notin H$. Now give every element up to the end of $\bar{Z}$ weight one and give the rest of the elements weight zero. The greedy algorithm applied to this ordering will pick up the elements of $W$, and will skip over the elements of $H \cup Y$. Then it will pick up the elements of $Z$, since (as before) these are just the remaining elements of $I$. Now, as before, $J$ has greater weight than $I$, so the greedy algorithm must pick up another element before it reaches the end of $\bar{Z}$. It cannot pick up any element of $\bar{Z}$, so it must pick up one of the repositioned elements ($d$, $e$, $\bar{e}$, or $\bar{d}$ in the example above). Let $x$ be the first element so picked up. If $x \in X$, then we see that it satisfies the desired conditions (that both $\{x\} \cup I$ and $\{\bar{x}\} \cup I \setminus \bar{J}$ are in $\mathcal{I}$). Otherwise, $x$ cannot be in $Y$, because $Y \subseteq J$ and for no $x \in J \setminus I$ can we have $\{x\} \cup I \in \mathcal{I}$. So $x \in \bar{Y}$. In particular, $x \in \bar{J}$, so $\{\bar{x}\} \cup I \setminus \bar{J} = I$, which is trivially in $\mathcal{I}$. This completes the proof. 

4. **What Next?**

It would be nice to find an independent set axiomatization for all Coxeter matroids, not just symplectic ones. We might also hope to use Theorem 1 to obtain circuit elimination and basis exchange axioms for Coxeter matroids, since these axioms are closely related to independent set axioms in the ordinary matroid case.

It should also be fruitful to determine the precise connections between Gelfand-Serganova Coxeter matroids and all the other generalizations of matroids that exist in the literature. Following are some observations that were revealed by a quick literature search.

There is one special case of a symplectic matroid that has been rediscovered independently several times. It goes by different names: “Lagrangian matroid,” “symmetric matroid” [3], “$\Delta$-matroid” [3], and “pseudomatroid” [6]. All these concepts are equivalent, and Gelfand-Serganova symplectic matroids are strictly more general than all of them, as noted in [2]. In addition, there exists something called a “metroid” [7] which is almost equivalent to a $\Delta$-matroid, but technically it is a special case: metroids are $\Delta$-matroids.
that include the empty set as a feasible set. This is proved in [4]. Incidentally, the Mathematical Review 89a:05046 of [3] remarks that “symmetric matroid” is also used in the literature to refer to something completely different, but I have not been able to track down any instances of this other usage.

A concept that is earlier than any of the above is that of a “bimatroid” [10, 11]. In [7] it is shown that a bimatroid is a special case of a metroid. In [11], two concepts that are related to bimatroids are discussed: “orthogonal matroids” and “Pfaffian structures.” Orthogonal matroids are special cases of bimatroids and hence (confusingly) are special cases of Gelfand-Serganova symplectic matroids. Gelfand-Serganova orthogonal matroids (i.e., the case \( W = D_n \)) may also be viewed as special cases of Gelfand-Serganova symplectic matroids, but it is not immediately clear whether there is any direct connection between orthogonal matroids in the sense of [11] and orthogonal matroids in the sense of Gelfand-Serganova. To add to the confusion, sometimes Pfaffian structures are referred to as “symplectic matroids” because they are indeed symplectic analogues of matroids [11, 12] but it is not immediately clear what the precise relationship between them and the other concepts mentioned above is. One can get a Pfaffian structure out of a bimatroid, but they do not seem to be strictly equivalent, and thus a Pfaffian structure does not seem to be a special case of (say) a \( \Delta \)-matroid.

Finally, two other concepts that might be related to Coxeter matroids are “Coxeteroids” [1] and “multimatroids” [5]. The definition of a Coxeteroid is motivated by the observation that matroids and Coxeter groups both satisfy an “exchange condition.” A multimatroid is a certain generalization of a \( \Delta \)-matroid. In neither case is the exact connection with Coxeter matroids obvious.

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6. References

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