Sequential decoding for lossless streaming source coding with side information

Hari Palaiyanur, Student Member, Anant Sahai, Member

Abstract

The problem of lossless fixed-rate streaming coding of discrete memoryless sources with side information at the decoder is studied. A random time-varying tree-code is used to sequentially bin strings and a Stack Algorithm with a variable bias uses the side information to give a delay-universal coding system for lossless source coding with side information. The scheme is shown to give exponentially decaying probability of error with delay, with exponent equal to Gallager’s random coding exponent for sources with side information. The mean of the random variable of computation for the stack decoder is bounded, and conditions on the bias are given to guarantee a finite $\rho^{th}$ moment for $0 \leq \rho \leq 1$. Further, the problem is also studied in the case where there is a discrete memoryless channel between encoder and decoder. The same scheme is slightly modified to give a joint-source channel encoder and Stack Algorithm-based sequential decoder using side information. Again, by a suitable choice of bias, the probability of error decays exponentially with delay and the random variable of computation has a finite mean. Simulation results for several examples are given.

Index Terms

Data compression, side information, joint source-channel coding, sequential decoding, lossless source coding, Slepian-Wolf, error exponent, delay universal, stack algorithm, random variable of computation

I. INTRODUCTION

In this paper, we consider the problem of lossless source coding with side information shown in Figure 1. The seminal paper of Slepian and Wolf [1] was the first to give the achievable rate region for this problem, when the source consists of a pair of dependent random variables that are independent and identically distributed (IID) over time. A sequence of IID symbols is encoded and its compressed representation is given noiselessly to a decoder. The decoder also has access to side information that is correlated in a known way with the source. The side information generally permits the source to be compressed to a rate below its entropy and still recovered losslessly. If the source is $U$ and the side information $V$, then [1] showed that the conditional entropy, $H(U|V)$, is a sufficient rate to recover the $U$ with arbitrarily low probability of error.

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H. Palaiyanur and A. Sahai are with the Electrical Engineering Department at the University of California, Berkeley.
The currently known, robust methods of compression used in point-to-point lossless source coding generally employ variable length codes. Solutions such as Lempel-Ziv coding ([2], [3], [4]) and context-tree weighting [5] are also capable of efficiently compressing many sources with memory. Recently, these algorithms have been adapted to the ‘compression with side information’ problem when the side information is available to both the encoder and decoder. Cai, et al. [6] have shown how to modify the context-tree method to account for side information at the encoder. It is also possible to modify the Lempel-Ziv algorithms to account for side information at the encoder ([7], [8]).

The purpose of this paper, however, is to consider how to compress when the side information is available to the decoder only. This restriction disallows variable length codes as a generic solution. Variable length codes work because they assign short codewords to typical source strings and longer codewords to atypical strings. When the side information is available only to the decoder, the encoder cannot tell when the joint source is behaving atypically. As an example consider a binary equiprobable source $U$. Let $V$ be the output of $U$ passed through a binary symmetric channel with crossover probability $1/10$. Every $U$ source string of the same length occurs with equal probability, but clearly the side information allows the source to be compressed below 1 bit per symbol.

One approach around this problem is to use block codes such as LDPC codes to give a ‘structured’ binning of the source strings. The side information is then used at the decoder to distinguish amongst the source strings in the received bin. In the same mold, it is also possible to use turbo-codes as done by Aaron, Girod, et.al. ([9], [10]). Regardless of the type of code, lack of the side information at the encoder somehow necessitates a shift in ‘complexity’ from the encoder to the decoder.

The idea of shifting complexity from encoder to decoder in lossless source coding is not new. In [11], Hellman suggested using convolutional codes for joint source-channel coding in applications such as deep-space communications where computational effort at the encoder comes at a premium. Around the same time, papers of Koshelev [12] and Blizard [13] suggested using convolutional codes in conjunction with sequential decoders for the purposes of data compression and joint source-channel coding. These ideas extend naturally to the subject of this paper, lossless source coding and joint source-channel coding with side information available to the decoder only.

The approach of this paper is to use random, time-varying, infinite constraint length convolutional codes to
sequentially ‘bin’ an IID source and a Stack Algorithm sequential decoder to (almost) losslessly recover it. The
decoder has a variable ‘bias’ parameter, as in [14] by Jelinek, that allows for a tradeoff between probability of error
and moments of the random variable of computation associated with the sequential decoder. The proof techniques
are adaptations to source coding and joint source-channel coding of those of [14].

Table I shows the relation of this paper with some prior work. There are several lines of work in information
theory that our scheme is related to. As already mentioned, the main point of this paper is to extend the idea
of using convolutional encoding with sequential decoding for lossless source coding by modifying the decoder to
allow the use of side information.

In [12], Koshelev shows that there is a point-to-point source coding ‘cutoff rate’ for a stack-based sequential
decoding algorithm. That is, if the rate is larger than the cutoff rate, then the expected mean of computation
performed by the sequential decoder is finite. Work in the opposite direction by Arikan and Merhav [28] showed
that this cutoff rate is tight; if the rate is below the cutoff rate, the expected mean in computation is infinite.
Furthermore, [28] gives a lower bound to the cutoff rate for all moments of the random variable of computation,
not only the mean. Our result regarding computation parallels Koshelev’s, only with side information allowed at
the decoder. We give an upper bound to the ‘cutoff rate’ for moments in the interval [0, 1], of sequential decoding
for lossless source coding with side information at the decoder. When the side information is independent of
the source to be recovered, reducing to the point-to-point version of the problem, this cutoff rate coincides with that
of [28].

One interesting aspect of our scheme is its ‘anytime’ or delay-universal nature. By using an infinite constraint-
length convolutional code, it is possible to have a probability of error that goes to zero exponentially with delay.\footnote{Delay is defined as the difference between the decoding time and the time the symbol entered the encoder.}

| Block Codes       | Channel Coding | Source Coding | Source Coding with SI | Joint S-C Coding with SI |
|-------------------|----------------|---------------|-----------------------|--------------------------|
| Shannon [15]      | Shannon [15]   | Slepian and Wolf [1] | -                     | -                        |
| Block Error Exponent | Gallager [16]  | Csiszar and Körner [17] | Gallager [18]         | -                        |
| Convolutional and Tree Codes | Elias [19] | Hellman [11], Blizard [13] | -                     | -                        |
| Delay Error Exponent | Pinsker [20], Sahai [21] | Chang and Sahai [22] | Chang and Sahai [23] | Chang and Sahai [24] |
| Sequential Decoding | Jelinek [14] | Thm. 1        | Thm. 1                | Thm. 2                   |
| Delay Error Exponent | Jelinek [25] | Koshelev [12] | Thm. 3                | Thm. 4                   |
| Sequential Decoding | Jacobs and Savage [26] | Arikan and Berlekamp [27] | open | open |
| Computation Achievability | open | Merhav [28] | open | open |

**TABLE I**

Some references in data compression and channel coding and data compression with side information.
For certain problems in distributed control ([29], [30]), an exponentially decreasing probability of error is required to guarantee plant stability (in a moment sense). For these problems, the error exponent with delay determines the moments of the plant state that can be stabilized. The scheme presented in this paper, if there is a channel between encoder and the side-information aided decoder, achieves an error exponent with delay analogous to the point-to-point random block coding error exponent of Problem 5.16 of Gallager [16]. Recent work by Chang, et. al. ([31], [22], [23], [32]) has shown that in general, the best block error exponents are much lower than the best error exponents with delay achievable for problems of lossless source coding with and without side information.

The paper is organized as follows. In Section II-A we set up the problem of streaming source coding, perhaps with a noisy channel between encoder and decoder, and with side information available to the decoder. Then in Section II-B we give a description of the encoder and decoder. Next, in Section III we state the two main theorems. One theorem states the error exponent with delay for this scheme and the other theorem gives an upper bound to the asymptotic distribution of computation when using the stack algorithm. The next section gives some examples and simulation results showing that the proposed scheme can be implemented with non-prohibitive complexity. In the conclusion, we discuss some open questions left in this specific line of work and some future directions. Finally, in the appendix, we give proofs of the theorems of the text.

II. SEQUENTIAL DATA COMPRESSION WITH SIDE INFORMATION

A. Problem definition

The source is modelled as a sequence of IID random variables \((U_i, V_i), \ i \geq 1\), that take on values from a finite alphabet \(\mathcal{U} \times \mathcal{V}\). Each \((U_i, V_i)\) is drawn according to a probability mass function \(Q(u, v)\). With some abuse of notation, we will use \(Q(u)\), for \(u \in \mathcal{U}\), to denote the marginal probability \(\sum_{v \in \mathcal{V}} Q(u, v)\). Similarly, \(Q(v)\) will be the marginal \(\sum_{u \in \mathcal{U}} Q(u, v)\) for \(v \in \mathcal{V}\). Finally, \(Q(u|v) = Q(u, v)/Q(v)\) for \((u, v) \in \mathcal{U} \times \mathcal{V}\). Without loss of generality, assume \(Q(v) > 0, \ \forall \ v \in \mathcal{V}\). If \(U\) and \(V\) are independent, the point-to-point source coding problem is recovered.

Our goal is to code the \(U_i\) symbols causally into a fixed rate bit stream so that the symbols can be recovered losslessly by a decoder in the sense that a symbol \(U_i\) is recovered with probability 1 in the limit of large decoding delay. For reasons mentioned in the introduction, a truly fixed rate coding strategy that assigns the same number of bits to sequences of the same length will be pursued.

Figure 2 shows the setup of our ‘streaming’ source coding problem. At a discrete time instant \(n\), the encoder has access to the source realization up through time \(n\), which is denoted \(u^n_1\). Let the rate of the encoder be \(R\) bits per source symbol. The encoder at time \(n\) outputs \([nR] - [(n - 1)R]\) bits that are a function of \(u^n_1\). Based on the bits \(B_1^{[nR]}\) and the side information \(v^n_0\), the decoder at time \(n\) gives its estimate of the source symbols up through

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We will use \(z^j_i\) to denote the vector \((z_i, z_{i+1}, \ldots, z_j)\) if \(i \leq j\) and the null string if \(i < j\).
time $n$, denoted as $\hat{u}_1^n(n)$.

$$E_n : \mathcal{U}^n \rightarrow \{0, 1\}^{(nR) − (n−1)R}$$ (1)

$$B_{[(n−1)R]}^{[nR]}(u_1^n) = E_n(u_1^n)$$ (2)

$$D_n : \{0, 1\}^{[nR]} \times \mathcal{V}^n \rightarrow \mathcal{U}^n$$ (3)

$$\hat{u}_1^n(n) = D_n(B_{[(n−1)R]}^{[nR]}, v_1^n)$$ (4)

The only interesting values of $R$ lie in the interval $[H(U|V), \log_2(|\mathcal{U}|)]$ since we need a rate of at least the conditional entropy to losslessly encode the source, and if $R > \log_2(|\mathcal{U}|)$, we could just index the source sequences on a per-letter basis and losslessly recover them with no delay.

$$H(U|V) \triangleq \sum_{v \in \mathcal{V}} Q(v) \sum_{u \in \mathcal{U}} Q(u|v) \log_2 \frac{1}{Q(u|v)}$$ (5)

There are two measures of performance that we will evaluate. First is the tradeoff between probability of error and delay.

**Definition 1:** The probability of error with delay $d$, $P_e(d)$, is

$$P_e(d) \triangleq \sup_n P(\hat{u}_1^n(n + d) \neq u_1^n)$$ (6)

This probability is taken over the randomness in the source and any randomness that may be present in the encoder or decoder. The error exponent with delay, or reliability exponent $E(R)$ at the rate $R$ where the encoder/decoder operates is

$$E(R) \triangleq \lim_{d \rightarrow \infty} \inf \frac{1}{d} \log_2 P_e(d)$$ (7)

The second measure of performance lies in the random variable of computation. The motivation for developing sequential decoders has always been the opportunity to have a ‘nearly optimal’ decoder without exponentially growing complexity in block length or delay [33]. The amount of computation performed by our source decoder will be measured in the number of source sequences that are considered or compared against others.
Definition 2: If $u_n$ is the true source realization at time $n \geq 1$, the $i^{th}$ incorrect subtree, $C_i$, is

$$C_i \triangleq \{z^n_1 \in U^n : n \geq i, \ z_{i-1}^{i-1} = u_{i-1}^{i-1} \text{ and } z_i \neq u_i\}$$  \hspace{1cm} (8)$$

The $i^{th}$ random variable of computation, $N_i$, is the number of nodes in $C_i$ that are ever examined by the decoder.

The definition of $N_i$ is a bit vague for arbitrary decoders but becomes concrete for sequential decoders, because the defining property of sequential decoders is essentially that they examine paths in a tree or trellis structure one by one.\[3\]

B. A random binning scheme with a stack decoder

In this section, the encoder and decoder for the coding strategy of this paper is described. The encoder used is similar to the encoder used in the sequential source coding paper of [31]. The bit sequence is arrived at by the use of a random tree code, which can be implemented using a time-varying, infinite constraint length, random convolutional code. Figure 3 shows an example of such a code. We first envisage a uniform tree with $|U|$ branches emanating from each node. The branches are numbered $1, 2, \ldots, |U|$ to denote the extension of the parent sequence by one symbol from $U$. Hence, for all $k \geq 1$ there is a one-to-one correspondence between $|U|$-ary strings of length $k$ and nodes in the tree. These properties make clear that labelling the branches of the tree with an appropriate number of code bits would yield a tree encoding of the source: a sequential source code.

The sequential random binning scheme we use is an ensemble of tree codes, with every bit on every branch drawn identically and independently as Bernoulli $(1/2, 1/2)$ $(B(1/2))$ random variables. This means that if source sequences $u_n^1$ and $z_n^1$ are the same until time $n - d + 1$, i.e. $u_{n-d}^n = z_{n-d}^n$, but $u_{n-d+1} \neq z_{n-d+1}$, the probability that $u_n^1$ and $z_n^1$ are placed in the same ‘bin’ is $2^{-dR}$. This is because the last $dR$ bits of the codewords for $u_n^1$ and $z_n^1$ are drawn IID $B(1/2)$. We refer to the bits in the codewords of source sequences as ‘parities’ because we think of them as coming from a time-varying, infinite constraint length convolutional code.

Decoding will be done by a stack algorithm, and hence is also sequential. For explanations of the stack algorithm, refer to [34], [35], or [36]. The following is the specific stack algorithm used. We initialize the stack with the root node having a metric of 0.

1) Let $u_1^t$ denote the (partial) source sequence at the top of the stack. Remove $u_1^t$ from the stack and consider each of its $|U|$ extensions by one symbol from $U$, i.e. $(u_1^t, u), \forall u \in X$. Let $\tilde{u}_1^{t+1}$ be one of these extensions, and do the following for each of the extensions. If the parities of $\tilde{u}_1^{t+1}$ match the parities received, update the metric of $\tilde{u}_1^{t+1}$ and add it to the stack in a sorted way (highest metric on top). Otherwise discard $\tilde{u}_1^{t+1}$.

\[4\]

There is also some amount of ‘internal’ computation the decoder must do to determine the codewords of the source sequences. We assume an oracle gives the decoder any source codeword it wants at unit cost. This is somewhat significant in our random convolutional code implementation since the encoder’s output depends on all previous source symbols. This means that as time increases, there is an increasing complexity to determining the bits assigned to a source symbol.

Note that the parities of $\tilde{u}_1^{t+1}$ will match those received if and only if the label on the branch extending $u_1^t$ to $\tilde{u}_1^{t+1}$ matches the $R$ parities received in the last time step.
2) Let \( u_k^t \) denote the sequence on top of the stack after all the relevant extensions have been added. If the length of \( u_k^t \), \( k \), is up to the current time, declare \( u_k^t \) as the decoded source sequence so far. Otherwise repeat 1.

The metrics are updated in an additive manner, with the metric of \( \tilde{u}_{l+1}^t \) being \( \Gamma(\tilde{u}_{l+1}^t) = \Gamma(\tilde{u}_l^t) + \Gamma(\tilde{u}_{l+1}) \). For \((u, v) \in U \times V\), the metric for the source symbol \( u \) given side information \( v \) is \[
\Gamma(u) \triangleq G + \log_2(Q(u|v))
\]

The parameter \( G \) is the ‘bias’ and controls to a large extent the amount of searching through the tree the algorithm performs. The bias is used as a normalizer so that the true path through the tree has a metric that is slowly increasing in time, while false path metrics are dropped to \( -\infty \) by non-matching parities.

C. Joint source-channel coding with side information

Suppose there is a DMC between the encoder and the decoder. Let \( W \) be its probability transition matrix, from a finite input alphabet \( X \) to a finite output alphabet \( Y \). Assume there are \( \lambda > 0 \) channel uses per source symbol.

\[
\mathcal{E}_n : U^n \rightarrow X^{[n\lambda]-(n-1)\lambda}
\]

\[
x_{(n-1)\lambda}^{[n\lambda]}(u^n_i) = \mathcal{E}_n(u^n_i)
\]

\[
\mathcal{D}_n : Y^{[n\lambda]} \times V^n \rightarrow U^n
\]

\[
\tilde{u}_i^n(n) = \mathcal{D}_n(y_1^{[n\lambda]}, v_i^n)
\]

The random binning encoder and the stack decoder of the previous section changes only slightly. First, the encoding tree is restricted to having one channel symbol on each branch, rather than \( R \) bits. We will assume each channel input on the tree is drawn IID from a distribution \( \beta(x) \) on \( X \). Secondly, the stack decoder cannot discard paths based on parities anymore. So, if \( u, v, x_\lambda \), and \( y_\lambda \) are respectively the source symbol on a branch, side information symbol, channel inputs on the branch and the channel outputs received by the decoder, then the decoder assigns a metric of:

\[
\Gamma(u) = G + \log_2 \frac{Q(u|v)W(y_\lambda|x_\lambda)}{P(y_\lambda)}
\]

where \( P(y) \triangleq \sum_{x \in X} \beta(x)W(y|x) \) and \( P(y_\lambda) = \prod_{k=1}^{\lambda} P(y_k) \). The performance measures remain the same, with the error exponent at ‘rate’ \( \lambda \) being \( E(\lambda) = \lim \inf_{d \rightarrow \infty} -\frac{1}{d} \log_2 P_e(d) \).
Fig. 3. An example of a tree code for a source with ternary alphabet. Here the rate $R$ is one bit per source symbol.
III. Main Results

A. Functions of interest

We start with the definition of some functions that appear in the theorem statements. The following functions of the channel input distribution, $\beta(x)$, and channel transition probability matrix, $W(y|x)$, appear in [14].

$$E_0(\rho) \triangleq -\log_2 \left[ \sum_{y \in Y} P(y) \left( \sum_{x \in X} \beta(x) \left( \frac{W(y|x)}{P(y)} \right)^{1+\rho} \right) \right]$$

$$F(\rho) \triangleq -\log_2 \left[ \sum_{y \in Y} P(y) \left( \sum_{x \in X} \beta(x) \left( \frac{W(y|x)}{P(y)} \right)^{\rho} \right) \right]$$

$$G(\rho) \triangleq -\log_2 \left[ \sum_{y \in Y} P(y) \left( \sum_{x \in X} \beta(x) \left( \frac{W(y|x)}{P(y)} \right)^{1+\rho} \right) \right]$$

We define the following functions of the source distribution for $\rho \geq 0$. $E_{si}(\rho)$ can be found in [18] and the others are modifications of $E_{si}$.

$$E_{si}(\rho) \triangleq \log_2 \left[ \sum_{v \in V} Q(v) \left( \sum_{u \in U} Q(u|v)^{1+\rho} \right) \right]$$

$$F_{si}(\rho) \triangleq \log_2 \left[ \sum_{v \in V} Q(v) \left( \sum_{u \in U} Q(u|v)^{\rho} \right) \right]$$

$$G_{si}(\rho) \triangleq \log_2 \left[ \sum_{v \in V} Q(v) \left( \sum_{u \in U} Q(u|v)^{1+\rho} \right) \right]$$

If the side information is independent of $U$, then we get the simpler functions $E_s(\rho)$, $F_s(\rho)$ and $G_s(\rho)$ below.

$$E_s(\rho) \triangleq (1 + \rho) \log_2 \left[ \sum_{u \in U} Q(u)^{1+\rho} \right]$$

$$F_s(\rho) \triangleq \rho \log_2 \left[ \sum_{u \in U} Q(u)^{\rho} \right]$$

$$G_s(\rho) \triangleq \log_2 \left[ \sum_{u \in U} Q(u)^{1+\rho} \right]$$

B. Probability of error with delay

Theorem 1 (Error exponent with delay for source coding with side information): Suppose that the decoder has access to the side information and there is a noiseless rate $R$ binary channel between the encoder and decoder. Fix any $\epsilon > 0$ and let $\rho \in [0, 1]$. For the encoder/decoder of Section II-B, if the bias $G$ satisfies

$$G \leq \frac{1+\rho}{\rho} \left[ E_{si}(\rho) - F_{si}(\rho) \right]$$

then there is a constant $K_\epsilon < \infty$ so that

$$P_e(d) < K_\epsilon \exp \left( -d \left( \rho R - E_{si}(\rho) - \epsilon \right) \right)$$

Hence, with suitable choice of bias, the error exponent with delay can approach

$$E(R) = E_{r,si}(R) \triangleq \sup_{\rho \in [0,1]} \rho R - E_{si}(\rho)$$
If the side information is independent of the source \( U \), then \( E_{si}, F_{si} \) and \( G_{si} \) simplify to \( E_s, F_s \) and \( G_s \) respectively. So, in the case of straight point-to-point lossless source coding, we arrive at an source coding equivalent of Gallager’s random coding exponent:

\[
E_{r,pp}(R) \triangleq \max_{\rho \in [0,1]} \rho R - E_s(\rho) \tag{25}
\]

**Theorem 2 (Error exponent with delay for joint source-channel coding with side information):** Suppose there is a channel \( W \) between the encoder and the decoder and side information is available to the decoder. Fix any \( \epsilon > 0 \) and let \( \rho \in [0,1] \). For the encoder/decoder of Sections II-B and II-C if the bias \( G \) satisfies

\[
G \leq \frac{1 + \rho}{\rho} \left[ E_{si}(\rho) - F_{si}(\rho) - \lambda E_0(\rho) + \lambda F(\rho) \right] \tag{26}
\]

then, there is a constant \( K_\epsilon < \infty \) so that

\[
P_e(d) < K_\epsilon \exp_2 \left( -d \left( \lambda E_0(\rho) - E_{si}(\rho) - \epsilon \right) \right) \tag{27}
\]

Hence, with suitable choice of bias, the error exponent with delay can be

\[
E(\lambda) = E_{r, jscsi}(\lambda) \triangleq \sup_{\rho \in [0,1]} \lambda E_0(\rho) - E_{si}(\rho) \tag{28}
\]

By assuming the side information to be independent of the source, we once again have a scheme for joint source-channel coding. The error exponent achieved is the joint source-channel equivalent of Gallager’s random coding exponent ([16], Problem 5.16).

\[
E_{r, jsc}(\lambda) \triangleq \max_{\rho \in [0,1]} \lambda E_0(\rho) - E_s(\rho) \tag{29}
\]

The exponent of (29) is lower in general than the joint source-channel exponent of Csiszar [37].

**C. Random variable of computation**

**Theorem 3 (Computation of stack decoder with side information):** Suppose that the decoder has access to the side information and there is a rate \( R \) noiseless, binary channel between the encoder and decoder. Fix any \( \gamma \in [0,1] \). For the encoder/decoder of Section II-B if the bias \( G \) satisfies

\[
\frac{1 + \frac{\gamma}{\gamma} G_{si}(\gamma)}{\gamma} < G < \frac{1 + \frac{\gamma}{\gamma} \left[ \gamma R - F_{si}(\gamma) \right]}{\gamma} \tag{30}
\]

then the \( \gamma^{th} \) moment of computation is uniformly finite all for \( i \), i.e. \( \exists K < \infty \) such that \( \forall i, E[N_i^\gamma] < K \), if

\[
R > \frac{E_{si}(\gamma)}{\gamma} \tag{31}
\]

As a conclusion of the theorem, we show that the interval of viable bias values implicit in (30) is in fact non-empty if \( R > E_{si}(\gamma)/\gamma \).

By restricting to the point-to-point case, we see that the \( \gamma^{th} \) moment of computation, for \( \gamma \in [0,1] \), can be finite if

\[
R > \frac{E_s(\gamma)}{\gamma} \tag{32}
\]
This result has been known for $\gamma = 1$, due to Koshelev [12]. We conjecture that Theorem 3 remains true for $\gamma > 1$. This conjecture is supported by simulation, but unproven. It is established by using the results found in [28] that if $R < E_s(\gamma)/\gamma$, then $E[N^\gamma]$ cannot be uniformly bounded. Together, these results tell us that our stack decoder is doing as well as could be hoped for any sequential decoder in terms of the moments of computation for the point-to-point case.

**Theorem 4 (Computation of stack decoder for joint source-channel coding with side information):** Suppose there is a channel $W$ between the encoder and the decoder and side information is available to the decoder. Fix any $\gamma \in [0, 1]$. For the encoder/decoder of Sections II-B and II-C, if the bias $G$ satisfies

$$\frac{1 + \gamma}{\gamma} \left[ G_{st}(\gamma) - \lambda G(\gamma) \right] < G < \frac{1 + \gamma}{\gamma} \left[ \lambda F(\gamma) - F_{st}(\gamma) \right]$$

then the $\gamma^{th}$ moment of computation is uniformly finite all for $i$, i.e. $\exists K < \infty$ such that $\forall i$, $E[N^\gamma_i] < K$, if

$$\lambda E_0(\gamma) > E_{st}(\gamma)$$

Again, in the appendix, we show that if $E_0(\gamma) > E_{st}(\gamma)$, then the interval of acceptable bias values in (33) is non-empty.

By removing the side information, we see the condition needed for a finite $\gamma^{th}$ moment of computation, for $\gamma \in [0, 1]$ is:

$$\lambda E_0(\gamma) > E_s(\gamma)$$

The condition of (35) has a matching converse once again, which can be found in [28].

In section VI-G it is shown that the error exponent is positive when the bias is set as suggested in Theorems 3 and 4. Hence, the decoder is actually decoding correctly and the average computation is not finite simply because the Stack Algorithm is blindly following an incorrect path.

**D. Proof Outline**

The proofs are the source coding analog of the proofs of Theorems 2 and 3 of [14]. We give a proof outline for Theorem II for the point-to-point lossless source coding case, as the important ideas are all present without the excess notation.

Assume that $G \leq \frac{1 + \rho}{\rho} [E_s(\rho) - F_s(\rho)]$. We will show that for any $\epsilon > 0$, there is a $K_\epsilon < \infty$,

$$P_e(d) < K_\epsilon \exp \left( - d(\rho R - E_s(\rho) - \epsilon) \right)$$

We can assume $\rho R - E_s(\rho) - \epsilon > 0$; otherwise there is nothing to prove.

The error event of interest, $F_d$, is referred to in [14] as a failure event of depth $d$ and is defined in (37) and we will relate it to $P_e(d)$ at the end of the proof. Figure III-D shows paths that may lead to an error event of depth 3 occurring, i.e. $F_d$.

$$F_d \triangleq \left\{ \exists \tilde{u}_1, \tilde{u}_1 \neq u_1 \mid \Gamma(\tilde{u}_1^d) \geq \min_{1 \leq k \leq d} \Gamma(u_k^d) \text{ and parities of } \tilde{u}_1^d \text{ match } B_1^{dR} \right\}$$

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The event $F_d$ can be subdivided into events $F_{d,k}$ so that $F_d = \bigcup_{k=1}^d F_{d,k}$, where
\[
F_{d,k} \triangleq \left\{ \exists \, \tilde{u}_1^d, \tilde{u}_1 \neq u_1 \mid \Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^k) \text{ and parities of } \tilde{u}_1^d \text{ match } B_1^d \right\} \tag{38}
\]

Let $U_1^d$ be the random vector of the first $d$ source symbols and let $\tilde{u}_1^d$ be an arbitrary vector in $U_1^d$. By conditioning on the source sequence and applying the union bound, we get
\[
P(F_d) = \sum_{u_1^d \in X^d} Q(u_1^d) P(F_d | U_1^d = u_1^d) \tag{39}
\]
\[
\leq \sum_{k=1}^d \sum_{u_1^d} Q(u_1^d) P(F_{d,k} | U_1^d = u_1^d) \tag{40}
\]

Suppose $\tilde{u}_1^d$ is a false path that causes $F_{d,k}$ to occur. This means its parities match the received bits and its metric $\Gamma(\tilde{u}_1^d)$ is at least $\Gamma(u_1^k)$. Therefore,
\[
0 \leq \Gamma(\tilde{u}_1^d) - \Gamma(u_1^k) \tag{41}
\]
\[
= \sum_{i=1}^k \log_2 \left( \frac{Q(\tilde{u}_1^d)}{Q(u_1^i)} \right) + \sum_{i=k+1}^d \log_2 Q(\tilde{u}_1^d) + (d - k)G \tag{42}
\]

Now, denoting $1(\cdot)$ as the indicator function of its argument, and using a Gallager-style union bound, for $\rho \in [0,1]$, we have
\[
P(F_{d,k} | U_1^d = u_1^d) \leq E\left[ \left( \sum_{\tilde{u}_1^d : \tilde{u}_1 \neq u_1} 1(\tilde{u}_1^d \text{ causes } F_{d,k} \text{ to occur}) \right)^\rho \right] \tag{43}
\]
\[
\leq \left( \sum_{\tilde{u}_1^d : \tilde{u}_1 \neq u_1} E\left[ 1(\tilde{u}_1^d \text{ causes } F_{d,k} \text{ to occur}) \mid U_1^d = u_1^d \right] \right)^\rho \tag{44}
\]
\[
\triangleq \left( \sum_{\tilde{u}_1^d : \tilde{u}_1 \neq u_1} A_k(\tilde{u}_1^d, u_1^d) \right)^\rho \tag{45}
\]

Here (44) follows from Jensen’s inequality. Continuing with the bounding, we use the fact that the parity generation process is independent of everything else to get
\[
A_k(\tilde{u}_1^d, u_1^d) = E\left[ 1(\text{parities of } \tilde{u}_1^d \text{ match } B_1^{dR}) \cdot 1(\Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^k)) \mid U_1^d = u_1^d \right] \tag{46}
\]
\[
= E\left[ 1(\text{parities of } \tilde{u}_1^d \text{ match } B_1^{dR}) \right] E\left[ 1(\Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^k)) \mid U_1^d = u_1^d \right] \tag{47}
\]
\[
\leq \exp_2(-dR) \cdot \exp_2 \left( \frac{1}{1 + \rho} (\Gamma(\tilde{u}_1^d) - \Gamma(u_1^k)) \right) \tag{48}
\]

Substituting for $A_k(\tilde{u}_1^d, u_1^d)$, and removing the restriction that $\tilde{u}_1 \neq u_1$,
\[
P(F_{d,k} | u_1^d) \leq \left( \sum_{\tilde{u}_1^d} \exp_2(-dR) \exp_2 \left( \frac{1}{1 + \rho} \left[ \log_2 \left( \frac{Q(\tilde{u}_1^d)}{Q(u_1^d)} \right) + \log_2 Q(\tilde{u}_{k+1}^d) + (d - k)G \right] \right) \right)^\rho \tag{49}
\]
\[
= \exp_2 \left( -d\rho R + (d - k) \frac{\rho}{1 + \rho} G \right) \left( \sum_{\tilde{u}_1^d} \left( \frac{Q(\tilde{u}_1^d)}{Q(u_1^d)} \right)^\rho \right)^\rho \left( \sum_{\tilde{u}_{k+1}^d} Q(\tilde{u}_{k+1}^d) \right)^\rho \tag{50}
\]

\(^5\)Note that we need only pairwise independence of the parities along two paths.
Equation (50) follows from the standard algebra of interchanging sums and products. Finally, we are ready to complete the bound of $P(F_d)$.

$$P(F_d) \leq \sum_{k=1}^{d} 2^{-d \rho R + (d-k) \frac{\rho}{1 + \rho} G} \sum_{u_1^k} Q(u_1^k) \prod_{u_{k+1}^d} Q(u_{k+1}^d) \sum_{u_{k+1}^d} Q(u_{k+1}^d)$$

$$= \sum_{k=1}^{d} 2^{-d \rho R + (d-k) \frac{\rho}{1 + \rho} G} \sum_{u_1^k} Q(u_1^k) \prod_{u_{k+1}^d} Q(u_{k+1}^d) \sum_{u_{k+1}^d} Q(u_{k+1}^d)$$

We get (52) by noting that the $\tilde{u}$'s are just dummy variables and we are free to replace them with $u$'s. Next, we use the IID property of the source along with standard algebra to get to an exponential form. For example, we have

$$\left( \sum_{u_1^k} Q(u_1^k) \right)^{1+\rho} = \left( \sum_{u\in\mathcal{U}} Q(u) \right)^{k(1+\rho)}$$

$$= \exp \left( k E_s(\rho) \right)$$

Similarly,

$$\left( \sum_{u_1^k} Q(u_1^k) \right)^{\rho} = \left( \sum_{u\in\mathcal{U}} Q(u) \right)^{k\rho}$$

$$= \exp \left( k F_s(\rho) \right)$$

A bit more algebra and the condition on the bias gives:

$$P(F_d) \leq \sum_{k=1}^{d} \exp \left( -d \rho R + (d-k) \frac{\rho}{1 + \rho} G + k E_s(\rho) + (d-k) F_s(\rho) \right)$$

$$= \exp \left( d \left( \frac{\rho}{1 + \rho} G + F_s(\rho) - \rho R \right) \right) \sum_{k=1}^{d} \exp \left( k \left( E_s(\rho) - F_s(\rho) - \frac{\rho}{1 + \rho} G \right) \right)$$

$$\leq \exp \left( d \left( \frac{\rho}{1 + \rho} G + F_s(\rho) - \rho R \right) \right) \cdot d \exp \left( d \left( E_s(\rho) - F_s(\rho) - \frac{\rho}{1 + \rho} G \right) \right)$$

So, now we have for any $\epsilon > 0$,

$$P(F_d) \leq \tilde{K}_\epsilon \exp \left( -d (\rho R - E_s(\rho)) - \epsilon \right)$$

$$\tilde{K}_\epsilon \eqdef \max \left\{ d : \frac{\log_2 d}{d} \geq \epsilon \right\} < \infty$$

Note that $\tilde{K}_\epsilon < \infty$ and is independent of $d$ because $\log_2(d)/d$ goes to 0. Finally, we can prove the statement of the theorem. In order for a delay $d$ or greater error to occur it must be that $\tilde{u}_i(n+d) \neq u_i$ for some $1 \leq i \leq n$. 
Now, assuming the bias satisfies the required condition, we have

\[
P(\hat{u}_1^n(n + d) \neq u_1^n) = \sum_{k=0}^{n-1} P(\hat{u}_1^k(n + d) = u_1^k, \hat{u}_{k+1} \neq u_{k+1})
\]

(63)

\[
\leq \sum_{k=0}^{n-1} P(F_{d+n-k}) P(\hat{u}_1^k(n + d) = u_1^k)
\]

(64)

\[
\leq \sum_{k=0}^{\infty} P(F_{d+k})
\]

(65)

\[
\leq \sum_{k=0}^{\infty} \tilde{K} \epsilon 2^{-(d+k)(\rho R - E_s(\rho) - \epsilon)}
\]

(66)

\[
= 2^{-d(\rho R - E_s(\rho) - \epsilon)} \sum_{k=0}^{\infty} \tilde{K} \epsilon 2^{-k(\rho R - E_s(\rho) - \epsilon)}
\]

(67)

\[
= \tilde{K} \epsilon \exp(2^{-d(\rho R - E_s(\rho) - \epsilon)})
\]

(68)

The critical step is in (64), which says that if the decoded path and true path agree until time \(k\), the error event can be thought of as ‘rooted’ at time \(k+1\). Hence, we are reduced to the error event \(F_{d+n-k}\). The ideas used in the proof of the computation bound are essentially the same.
Fig. 4. A ternary tree. The true source sequence is shown above. The condition $\bar{u}_1 \neq u_1$ selects a portion of the tree containing paths that could potentially cause the error event $F_3$. 
IV. SIMULATIONS

The random time-varying encoder and stack decoder were simulated in software using a random number generator. The ‘experimental’ results are compared with the theory for verification. The probability of error with delay, $P_e(d)$, is the first quantity looked at experimentally. Since probability of error decays exponentially with delay, the logarithm of the probability of error decays linearly with delay. That is,

$$\log_2 P_e(d) \sim -E(R)d$$

The slope of the line on a log plot is thus the negative of the error exponent achieved by this scheme.

Further, if we assume that the moments of computation at any time are the same as the moments of computation in any incorrect subtree, we can compare the Pareto exponent of the simulation to the theory. This is done by comparing $\log_2 P(C \geq n)$ versus $\log_2 n$ on a graph, where $C$ is the number of computations performed at a time step. The fact that the distribution of computation is asymptotically Paretoian should yield that

$$\log_2 P(C \geq n) \sim -\gamma \log_2 n$$

where $\gamma$ is the Pareto exponent of computation.

A. Point to point

**Example 4.1:** We explore an example of point-to-point lossless source coding that will be comparable to the case when side information is available at the decoder only. The source $U_i$ is a sequence of IID $B(1/2)$ random bits. $V_i$ are generated by passing $U_i$ through a binary symmetric channel (BSC) with crossover probability $\epsilon = 1/10$. In this example, we consider the case when the side information is available at both the encoder and decoder. The situation is diagrammed in Figure 6. It is clear that since $V$ is available at both the encoder and decoder, compressing $U \oplus V$ is the same as compressing $U$. Figure 5 shows the relevant source coding functions for the error random variable $U \oplus V$. Since we are just encoding the noise, the rate must be at least $H(U|V) = H_b(\epsilon)$ where $H_b$ is the binary entropy function.

We experimentally estimate the error exponent with delay and Pareto exponent of computation. These are shown in Figures 7 and 8 respectively. Again, we see that we can achieve the random coding error exponent and the Pareto exponent guaranteed by theorem 3 holds. Since the bias value (0.7) is actually too high to guarantee achieving $E_{r,pp}(R)$ at rate $R = 0.7$, the error exponent in the experiment is somewhat surprising. However, we stress again that the fitting of a line to the curve is somewhat arbitrary and we cannot expect to have precise values of the slope beyond the first digit.

B. Side information

**Example 4.2:** We reuse the binary source example, where the side information is generated by passing the source bit through a BSC. The side information this time is only available at the decoder, as is shown in Figure 9. In this
Fig. 5. Functions of interest associated with a binary source with PMF (0.9, 0.1).

Fig. 6. An example of point-to-point source coding that can be compared to source coding with side information at the decoder. $U_i$ are Bernoulli (1/2) random bits, $V$ is $U$ passed through a BSC with crossover probability $\epsilon$. The encoder sequentially bins the error sequence $U \oplus V$.

In this case, the function $E_{si}(\rho)$ simplifies as below,

$$E_{si}(\rho) = \log_2 \sum_{v \in V} \left( \sum_{u \in U} Q(u, v)^{1/((1+\rho))} \right)^{1+\rho}$$  \hspace{1cm} (69)$$

$$= \log_2 \sum_{v \in V} Q(v) \left( \sum_{u \in U} Q(u|v)^{1/((1+\rho))} \right)^{1+\rho}$$  \hspace{1cm} (70)$$

$$= \log_2 \sum_{v=0}^{1} \frac{1}{2} \left( e^{1/(1+\rho)} + (1 - \epsilon)^{1/(1+\rho)} \right)^{1+\rho}$$  \hspace{1cm} (71)$$

$$= (1 + \rho) \log_2 \left( e^{1/(1+\rho)} + (1 - \epsilon)^{1/(1+\rho)} \right)$$  \hspace{1cm} (72)$$

This $E_{si}(\rho)$ is the same as the $E_s(\rho)$ function that appears if the side information $V$ is available at both the encoder and decoder, i.e. point-to-point coding of the error sequence. To compare to the case when $V$ is available...
TABLE II

| Error exponent with delay | Theoretical | Experimental |
|---------------------------|-------------|--------------|
|                           | 0.05        | ~ 0.06       |
| Pareto exponent of computation | ≥ 1         | ~ 1.2        |
| Conjectured Pareto exponent | 1.2         |              |

Comparison of theoretical and experimental performance in example 4.1.
at the encoder as well, we estimate the error exponent with delay and the Pareto exponent for computation through simulation in Figures 10 and 11 respectively. In this simulation, the rate is once again 0.7 bits per symbol, and the bias is 0.7. We see nearly identical values for the error exponent and Pareto exponent of the two examples, as we should.

V. Conclusion

In this paper, a scheme was described for the problem of joint source-channel coding with side information available only at the decoder. If the channel is noiseless, one immediately arrives at a scheme for (almost) lossless compression with side information at the decoder only. The coding is done in a ‘streaming’ manner in the sense that source symbols are encoded as they arrive. The encoder consists of an infinite constraint length random time-varying convolutional code, and the decoder is a Stack Algorithm sequential decoder with a variable ‘bias’ parameter.

Two performance measures were bounded for this system when coding IID sources over DMCs; probability of error with end-to-end delay and (average) computational effort of the decoder. We showed that various analogs of Gallager’s random coding error exponent could be achieved by suitable choice of bias. We also bounded the $\rho^{th}$ moment of computation for $0 \leq \rho \leq 1$. We thus established a lower bound for the cutoff rate for moments up to the mean for sequential decoding with side information. One would expect that a tweak to the analysis of [28], allowing for side information, would establish the matching upper bounds on the cutoff rate.
Following the work of Koshelev [12], it may be possible to even allow for finite memory Markov sources. Another important extension would be to consider two distributed encoders as in the paper of Slepian and Wolf [1]; the case when the side information $V$ is coded and required to be reconstructed. The scheme of Section II-B naturally allows for this by adding another tree code for the other source and modifying the metric update slightly. Simulation results have shown that the computation cost seems to be prohibitive except for high rates. Indeed, even the random coding exponents for correlated sources are generally much lower when both sources are coded [31]. Perhaps this is not surprising considering that the computational cutoff rate is closely tied to the ‘Gallager’ $E_s$ function indirectly through the random coding error exponent.

Fig. 10. Determining $E(R)$ for example 4.2.

Fig. 11. Determining the Pareto exponent for computation for example 4.2.
Fig. 12. Joint source-channel coding with side information available to the decoder.

 VI. APPENDIX - PROOFS

In this section we prove the theorems of the paper. First we show that the probability of error goes to zero exponentially with delay. This is done initially in the case when there is only a noiseless channel and the source is encoded at some rate $R$ bits per time unit. Then, we prove this for joint source-channel coding with side information when the source and the channel are ‘synchronized’ at one source symbol per channel use. Next, we prove the theorems regarding the random variable of computation. Again, we do this first in the case of source coding with side information, and then for joint source-channel coding with side information. Before diving into the proofs individually, we first examine the error events that show up.

Assume that $R$ is an integer so we need not worry about integer effects in the exposition, but the results hold for non-integer rates as well. Similarly, assume in the proof of Theorems 2 and 4 that $\lambda$ is an integer.

A. Error events

A source produces IID letters $(U_i, V_i)$ according to a joint distribution $Q(u, v)$ on a discrete alphabet $U \times V$. The $U_i$ are available to an encoder, and the $V_i$ are given to the decoder as side information. In the case of joint source-channel coding, there is a discrete memoryless channel with probability transition matrix $W(y|x)$ with finite input and output alphabets. We use the encoder and decoder of Section II-B. For joint source-channel coding, we assume there is one channel use for every source symbol. We denote vectors as $u_d, y_m, \ldots$ etc. We reserve the letters $u, x, y$ for the ‘true’ variables and $\tilde{u}, \tilde{x}, \tilde{y}$ for arbitrary ‘false’ variables.

The probability measure $P$ will refer to all randomness in the source as well as the randomly generated encoder. When no confusion arises, $Q$ will be applied to multiple symbols like $u_k$ with the meaning that $Q(u_k) = \prod_{l=1}^k Q(u_l)$.

The stack algorithm uses a metric, (implicitly a function of the side information, tree code and channel outputs if there are any), of $\Gamma(u) = \log(Q(u|v)W(y|x(u))/P(y)) + G$ for some bias $G \in \mathbb{R}$, where $P(y) = \sum_x \beta(x)W(y|x)$. If there is no channel, $\Gamma(u) = \log(Q(u|v)) + G$ if the parities of the sequence match the parities received by the decoder. Otherwise, we can set the metric for non-matching parities to be $-\infty$ to effectively drop them out of the computation.

7The appendix is lengthy and somewhat redundant for the convenience of the reviewer and will be trimmed for the final version.
8For a non-integer rate, the encoder outputs either $\lfloor R \rfloor$ or $\lceil R \rceil$ bits at every time instant. The integer effect is not important asymptotically, and for convenience we have used the integer assumption in proofs. In simulations, we have used non-integer rates.
stack. We now consider how the stack decoder could follow a false path. We say the stack decoder ‘visits’ a node if it computes a metric for that node.

Suppose the true source sequence is \((u_1^n, v_1^n)\) until time \(n\) and \(\tilde{u}_1^n\) is some other arbitrary source sequence. Viewed as paths through the encoding tree, \(u_1^n\) and \(\tilde{u}_1^n\) are the same if and only if they trace the same path from the root to depth \(n\) in the tree. Also, if they are not the same, there is some earliest point at which they diverge, call that time \(n - d + 1\). Equivalently, \(u_1^{n-d} = \tilde{u}_1^{n-d}\), but \(u_1^{n-d+1} \neq \tilde{u}_1^{n-d+1}\). Until time \(n - d\), because the stack decoder is a sequential decoder, the stack algorithm assigns \(u_1^{n-d}\) and \(\tilde{u}_1^{n-d}\) the same metric. In order for \(\tilde{u}_1^n\) to be the decoder’s estimated path at time \(n\), a necessary condition is:

\[
\Gamma(\tilde{u}_1^n) \geq \min_{n-d+1 \leq k \leq n} \Gamma(u_1^k) \tag{73}
\]

Noting that \(\Gamma(\tilde{u}_1^{n-d}) = \Gamma(u_1^{n-d})\), and the fact that the metric is additive, this reduces to:

\[
\Gamma(\tilde{u}_{n-d+1}^n) \geq \min_{1 \leq k \leq d} \Gamma(u_{n-d+1}^{n-d+k}) \tag{74}
\]

All randomness in the source, encoder/decoder, and channel is memoryless and stationary, so the probability of the above event occurring for some false \(\tilde{u}_{n-d+1}^n\) is the same as the probability of the event \(F_d\) defined below:

\[
F_d = \bigg\{ \exists \tilde{u}_1^d \in \mathcal{U}^d : \tilde{u}_1 \neq u_1, \ \Gamma(\tilde{u}_1^d) \geq \min_{1 \leq k \leq d} \Gamma(u_1^k) \bigg\} \tag{75}
\]

We call \(F_d\) the error event of depth \(d\). Figure [H-10] shows paths that may lead to an error event of depth \(3\) occurring, i.e. \(F_3\). We can further break up \(F_d\) into sub-events \(F_{d,k}\) so that:

\[
F_{d,k} = \bigg\{ \exists \tilde{u}_1^d \in \mathcal{U}^d : \tilde{u}_1 \neq u_1, \ \Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^k) \bigg\} \tag{76}
\]

\[
P(F_d) \leq \sum_{k=1}^d P(F_{d,k}) \tag{77}
\]

\[
P(F_{d,k}) = E \left[ 1 \left( \exists \tilde{u}_1^d \in \mathcal{U}^d : \tilde{u}_1 \neq u_1, \ \Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^k) \right) \right] \tag{78}
\]

\[
\leq E \left[ \left( \sum_{\tilde{u}_1^d \in \mathcal{U}^d, \tilde{u}_1 \neq u_1} 1(\Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^k)) \right)^\rho \right] \forall \rho \in [0, 1] \tag{79}
\]

Here \(1(\cdot)\) denotes the indicator function of its argument. The last line is in fact true for any \(\rho \geq 0\), but it is only useful in bounding if \(\rho \in [0, 1]\).

The probability of error with delay \(d\) at time \(n\) is \(P(\tilde{u}_{n-d}^n(n) \neq u_{n-d}^n, n\) is the decoder’s estimate of the source from time \(1\) to \(n - d\) produced at time \(n\). We will give an upper bound on the probability of error independent of \(n\) and depending only on \(d\), which is an upper bound on \(P_e(d)\).

If \(\tilde{u}_{n-d}^n(n) \neq u_{n-d}^n\), then there is some point at which they diverged, say \(n - d - l\). So \(\tilde{u}_{n-d-l}^n(n) = u_{n-d-l}^n\), but \(\tilde{u}_{n-d-l+1}^n(n) \neq u_{n-d-l+1}^n\). So the probability that a false decoded path and the true path diverged at time \(n - d - l\) is at most \(P(F_{d+l})\). Now we can use the union bound to get:

\[
P(\tilde{u}_{n-d}^n(n) \neq u_{n-d}^n) \leq \sum_{l=0}^{n-d} P(F_{d+l}) \tag{80}
\]
To get a bound independent of \( n \), we just set \( n \) to infinity and get

\[
P_e(d) \leq \sum_{l=0}^{\infty} P(F_{d+l})
\]  

(81)

As for the random variable of computation, we define a generic variable \( N \) below:

\[
N \triangleq \sum_{l=1}^{\infty} \sum_{\tilde{u}_1 \neq u_1} 1\left(\Gamma(\tilde{u}_1) \geq \min_{1 \leq k < \infty} \Gamma(u_1^k)\right)
\]  

(82)

\[
\leq \sum_{l=1}^{\infty} \sum_{\tilde{u}_1 \neq u_1} 1\left(\Gamma(\tilde{u}_1) \geq \min_{1 \leq k < \infty} \Gamma(u_1^k)\right)
\]  

(83)

By symmetry, it is clear that \( E[N^\gamma] = E[N_i^\gamma] \) for all \( i \geq 1 \) and any \( \gamma \geq 0 \). We want to find when \( E[N^\gamma] < \infty \).

By concavity, we have

\[
E[N^\gamma] \leq E\left[\left(\sum_{l=1}^{\infty} \sum_{\tilde{u}_1 \neq u_1} 1\left(\Gamma(\tilde{u}_1) \geq \min_{1 \leq k < \infty} \Gamma(u_1^k)\right)\right)^\gamma\right]
\]  

(84)

\[
\leq E\left[\left(\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\tilde{u}_1 \neq u_1} 1\left(\Gamma(\tilde{u}_1) \geq \Gamma(u_1^k)\right)\right)^\gamma\right]
\]  

(85)

\[
\leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} E\left[\left(\sum_{\tilde{u}_1 \neq u_1} 1\left(\Gamma(\tilde{u}_1) \geq \Gamma(u_1^k)\right)\right)^\gamma\right]
\]  

(86)

\[
= \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{l,k}
\]  

(87)

\[
A_{l,k} \triangleq E\left[\left(\sum_{\tilde{u}_1 \neq u_1} 1\left(\Gamma(\tilde{u}_1) \geq \Gamma(u_1^k)\right)\right)^\gamma\right]
\]  

(88)

Here are some further facts/definitions that are repeatedly used in the appendix:

1) The source and channel are memoryless. The parity generation process and channel input generation process are done IID for every branch/node.

2) Jensen’s inequality. If \( X \) is a random variable and \( f \) is a concave \( \cap \) function, \( E[f(X)] \leq f(E[X]) \). If \( \rho \in [0, 1] \), \( f(x) = x^\rho \) is concave \( \cap \).

3) By definition, for each \( y \in \mathcal{Y} \), \( P(y) = \sum_{x \in \mathcal{X}} \beta(x)W(y|x) \).

4) Definitions of the exponent functions \( E_{si}, E_0, \) etc. can be found in [III-A]

5) Sums and products of probabilities commute, and changing dummy variables can be used to simplify terms.

See Gallager [16], Chapter 5.

B. Probability of error - source coding with side information

**Theorem 5 (Restatement of Theorem 1):** Suppose that the decoder has access to the side information and there is a noiseless rate \( R \) binary channel between the encoder and decoder. Fix any \( \epsilon > 0 \) and let \( \rho \in [0, 1] \). For the

\[Sums of the form \sum u_1^k mean summing over all u_1^k \in \mathcal{T}^k. This is the meaning for all sums in the appendix, unless an additional condition such as \( \tilde{u}_1 \neq u_1 \) is explicitly stated.

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encoder/decoder of Section II-B, if the bias $G$ satisfies
\[ G \leq \frac{1 + \rho}{\rho} \left[ E_{si}(\rho) - F_{si}(\rho) \right] \tag{89} \]
then, there is a constant $K_\varepsilon < \infty$ so that
\[ P_e(d) < K_\varepsilon \exp \left( -d \left( \rho R - E_{si}(\rho) - \varepsilon \right) \right) \tag{90} \]
Hence, with suitable choice of bias, the error exponent with delay can be
\[ E(R) = E_{r,si}(R) \triangleq \sup_{\rho \in [0,1]} \rho R - E_{si}(\rho) \tag{91} \]

**Proof:** The letter $B$ will be used for the bits received by the decoder, which will be referred to as ‘parities’. We can specialize the event $F_d$ to this situation and write it as:
\[ F_d \triangleq \left\{ \exists \tilde{u}_1^d, \tilde{u}_1 \neq u_1 \mid \Gamma(\tilde{u}_1^d) \geq \min_{1 \leq k \leq d} \Gamma(u_1^k) \text{ and parities of } \tilde{u}_1^d \text{ match } B_1^{dR} \right\} \tag{92} \]
The event $F_d$ can be subdivided into events $F_{d,k}$ so that $F_d = \bigcup_{k=1}^{d} F_{d,k}$, where
\[ F_{d,k} \triangleq \left\{ \exists \tilde{u}_1^d, \tilde{u}_1 \neq u_1 \mid \Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^k) \text{ and parities of } \tilde{u}_1^d \text{ match } B_1^{dR} \right\} \tag{93} \]
Suppose $\tilde{u}_1^d$ is a false path that causes $F_{d,k}$ to occur. This means its parities match the received bits and its metric $\Gamma(\tilde{u}_1^d)$ is at least $\Gamma(u_1^k)$. Therefore,
\[ 0 \leq \Gamma(\tilde{u}_1^d) - \Gamma(u_1^k) \tag{94} \]
\[ = \sum_{l=1}^{d} \left( \log(Q(\tilde{u}_l|v_l)) + G \right) - \sum_{l=1}^{k} \left( \log(Q(u_l|v_l)) + G \right) \tag{95} \]
\[ = \sum_{l=1}^{k} \log \left( \frac{Q(\tilde{u}_l|v_l)}{Q(u_l|v_l)} \right) + \sum_{l=k+1}^{d} \log Q(\tilde{u}_l|v_l) + (d-k)G \tag{96} \]
Using a Gallager-style union bound, for $\rho \in [0,1]$, we have
\[ P(F_{d,k}) \leq E \left[ \left( \sum_{\tilde{u}_1^d: \tilde{u}_1 \neq x_1} 1(\tilde{u}_1^d \text{ causes } F_{d,k} \text{ to occur}) \right)^\rho \right] \tag{97} \]
\[ \leq \left( \sum_{\tilde{u}_1^d: \tilde{u}_1 \neq x_1} E \left[ 1(\tilde{u}_1^d \text{ causes } F_{d,k} \text{ to occur}) \right] \right)^\rho \tag{98} \]
Here, (a) is by Jensen’s inequality. By conditioning on the source sequence and applying the union bound, we
Continuing with the bounding, we use the fact that the parity generation process is independent of everything else to get

\[ P(F_d) = \sum_{u^d_1,v^d_1 \in V^d} Q(u^d_1,v^d_1) P(F_d|u^d_1,v^d_1) \]  

(99)

\[ \leq \sum_{u^d_1,v^d_1} Q(u^d_1,v^d_1) \prod_{k=1}^d P(F_{d,k}|u^d_1,v^d_1) \]  

(100)

\[ = \sum_{k=1}^d \sum_{u^d_1,v^d_1} Q(u^d_1,v^d_1) P(F_{d,k}|u^d_1,v^d_1) \]  

(101)

\[ \leq \sum_{k=1}^d \sum_{u^d_1,v^d_1} Q(u^d_1,v^d_1) \left( \sum_{\tilde{u}^d_1 \neq u_1} E \left[ 1(\tilde{u}^d_1 \text{ causes } F_{d,k} \text{ to occur} \mid u^d_1,v^d_1) \right] \right)^\rho \]  

(102)

\[ \Delta \sum_{k=1}^d \sum_{u^d_1,v^d_1} Q(u^d_1,v^d_1) \left( \sum_{\tilde{u}^d_1 \neq u_1} A_k(\tilde{u}^d_1,u^d_1,v^d_1) \right)^\rho \]  

(103)

\[ A_k(\tilde{u}^d_1,u^d_1,v^d_1) = E \left[ 1(\text{parities of } \tilde{u}^d_1 \text{ match } B^d_{1,k}) \cdot 1(\Gamma(\tilde{u}^d_1) \geq \Gamma(u^d_1)) \mid u^d_1,v^d_1 \right] \]  

(104)

\[ = E \left[ 1(\text{parities of } \tilde{u}^d_1 \text{ match } B^d_{1,k}) \right] E \left[ 1(\Gamma(\tilde{u}^d_1) \geq \Gamma(u^d_1)) \mid u^d_1,v^d_1 \right] \]  

(105)

\[ = 2^{-dR} E \left[ 1(\Gamma(\tilde{u}^d_1) \geq \Gamma(u^d_1)) \mid u^d_1,v^d_1 \right] \]  

(106)

\[ \leq \exp_2(-dR) \cdot \exp_2 \left( s(\Gamma(\tilde{u}^d_1) - \Gamma(u^d_1)) \right) \]  

(107)

\[ = \exp_2(-dR) \cdot \exp_2 \left( s \left[ \log_2 \frac{Q(\tilde{u}^d_1|v^d_1)}{Q(u^d_1|v^d_1)} + \log_2 Q(\tilde{v}^d_{k+1}|v^d_{k+1}) + (d-k)G \right] \right) \]  

(108)

for any \( s \geq 0 \)

Substituting for \( A_k(\tilde{u}^d_1,u^d_1,v^d_1) \), and removing the restriction that \( \tilde{u}_1 \neq u_1 \),

\[ P(F_{d,k}|u^d_1,v^d_1) \leq \left( \sum_{\tilde{u}_1^d} \exp_2(-dR) \exp_2 \left( s \left[ \log_2 \frac{Q(\tilde{u}^d_1|v^d_1)}{Q(u^d_1|v^d_1)} + \log_2 Q(\tilde{v}^d_{k+1}|v^d_{k+1}) + (d-k)G \right] \right) \right)^\rho \]  

(109)

\[ = \exp_2(-dR) + (d-k)sG \left( \sum_{\tilde{u}_1^d} \left( \frac{Q(\tilde{u}^d_1|v^d_1)}{Q(u^d_1|v^d_1)} \right)^s Q(\tilde{v}^d_{k+1}|v^d_{k+1})^s \right)^\rho \]  

(110)

\[ = \exp_2(-dR) + (d-k)sG \left( \sum_{\tilde{u}_1^d} \left( \frac{Q(\tilde{u}^d_1|v^d_1)}{Q(u^d_1|v^d_1)} \right)^s \left( \sum_{\tilde{a}_{k+1}} Q(\tilde{v}^d_{k+1}|v^d_{k+1})^s \right) \right)^\rho \]  

(111)

Relation (b) follows from the standard algebra of interchanging sums and products. Now, we substitute the last

\[ \text{Only pairwise independence of the parities along two disjoint paths is required.} \]
line into (103).

\[ P(F_d) \leq \sum_{k=1}^{d} 2^{-d(\rho+1)} \sum_{v_1^d} Q(v_1^d) \sum_{u_1^k} \sum_{u_1^d} Q(u_1|v_1^k) Q(u_1^d|v_1^d), \]

\[ \left( \sum_{\bar{u}_1^k} \frac{Q(\bar{u}_1^k|v_1^k)}{Q(u_1|v_1^k)} \right)^\rho \left( \sum_{\bar{u}_1^d} Q(\bar{u}_1^d|v_1^d) \right)^\rho \]

\[ = \sum_{k=1}^{d} 2^{-d(\rho+1)} \sum_{v_1^d} Q(v_1^d) \sum_{v_1^d} Q(v_1^d) \sum_{v_1^k} Q(u_1^k|v_1^k) \sum_{u_1^d} Q(u_1^d|v_1^d)^{1-s^k} \]

\[ \left( \sum_{\bar{u}_1^k} Q(\bar{u}_1^k|v_1^k)^s \right)^\rho \left( \sum_{\bar{u}_1^d} Q(\bar{u}_1^d|v_1^d)^s \right)^\rho \]

\[ = \sum_{k=1}^{d} 2^{-d(\rho+1)} \sum_{v_1^d} Q(v_1^d) \left( \sum_{u_1^d} Q(u_1^d|v_1^d)^{1-s^k} \right)^{1+\rho} \]

\[ \sum_{v_1^d} Q(v_1^d) \left( \sum_{u_1^d} Q(u_1^d|v_1^d)^{1-s^k} \right)^{1+\rho} \]

We get (c) by noting that the \( \bar{u} \)'s are just dummy variables and we are free to replace them with \( u \)'s and then setting \( s = \frac{1}{1+\rho} \). Next, we use the IID property of the source along with some algebra to get to an exponential form. For example, we have

\[ \sum_{v_1^d} Q(v_1^d) \left( \sum_{u_1^d} Q(u_1^d|v_1^d)^{1+\rho} \right)^{1+\rho} = \sum_{v_1^d} \sum_{u_1^d} Q(v_1^d) \left( \sum_{u_1^d} Q(u_1^d|v_1^d)^{1+\rho} \right)^{1+\rho} \]

\[ = \prod_{l=1}^{k} \sum_{u_l^d} Q(v_1^d) \left( \sum_{u_l^d} Q(u_l^d|v_1^d)^{1+\rho} \right)^{1+\rho} \]

\[ = \left[ \sum_{v_l^d} Q(v_l^d) \left( \sum_{u_l^d} Q(u_l^d|v_1^d)^{1+\rho} \right)^{1+\rho} \right]^k \]

Similarly,

\[ \sum_{v_1^d} Q(v_1^d) \left( \sum_{u_1^d} Q(u_1^d|v_1^d)^{1+\rho} \right)^\rho = \sum_{v_1^d} \sum_{u_1^d} Q(v_1^d) \left( \sum_{u_1^d} Q(u_1^d|v_1^d)^{1+\rho} \right)^\rho \]

\[ = \prod_{l=1}^{k} \sum_{u_l^d} Q(v_1^d) \left( \sum_{u_l^d} Q(u_l^d|v_1^d)^{1+\rho} \right)^\rho \]

\[ = \left[ \sum_{v_l^d} Q(v_l^d) \left( \sum_{u_l^d} Q(u_l^d|v_1^d)^{1+\rho} \right)^\rho \right]^k \]
Using the definitions of $E_{si}(\rho)$ and $F_{si}(\rho)$, we can rewrite the bound as:

$$P(F_d) \leq \sum_{k=1}^{d} \exp \left( -d\rho R + (d-k)\frac{\rho}{1+\rho} G + kE_{si}(\rho) + (d-k)F_{si}(\rho) \right)$$

$$= \exp \left( d\left( \frac{\rho}{1+\rho} G + F_{si}(\rho) - \rho R \right) \right) \sum_{k=1}^{d} \exp \left( k(E_{si}(\rho) - F_{si}(\rho) - \frac{\rho}{1+\rho} G) \right)$$

By the assumption of the theorem, (89), the following condition holds

$$E_{si}(\rho) - F_{si}(\rho) - \frac{\rho}{1+\rho} G \geq 0$$

Then, we can simplify the bound to

$$P(F_d) \leq \exp \left( d\left( \frac{\rho}{1+\rho} G + F_{si}(\rho) - \rho R \right) \right) \sum_{k=1}^{d} \exp \left( k(E_{si}(\rho) - F_{si}(\rho) - \frac{\rho}{1+\rho} G) \right)$$

$$\leq \exp \left( d\left( \frac{\rho}{1+\rho} G + F_{si}(\rho) - \rho R \right) \right) \cdot d \exp \left( d(E_{si}(\rho) - F_{si}(\rho) - \frac{\rho}{1+\rho} G) \right)$$

We get (d) from noting that the sum of the geometric series can be upper bounded by $d$ times the largest term.

Now this holds for all $\rho \in [0, 1]$, so

$$P(F_d) \leq \bar{K}_\epsilon \exp \left( -d(\rho R - E_{si}(\rho) - \epsilon) \right)$$

$$\bar{K}_\epsilon \triangleq \max \left\{ d : \frac{\log d}{d} \geq \epsilon \right\} < \infty$$

Note that $\bar{K}_\epsilon < \infty$ and is independent of $d$ because $\ln(d)/d$ goes to 0. We note that $E_{si}(\rho)$ is a differentiable function for all $\rho \geq 0$, with $E_{si}'(0) = H(U|V)$ (see [18]); that is, the slope at 0 is the conditional entropy of the source $U$ given the side information $V$. $E_{si}(\rho)$ is the source coding with side information coding analog to Gallager’s function $E_0(\rho)$. While Gallager’s function may be non-differentiable at points because it is the maximization of a function over probability distributions, $E_{si}(\rho)$ doesn’t suffer from this problem.

Now, assuming the bias satisfies the required condition, we have

$$P_e(d) \leq \sum_{k=0}^{\infty} P(F_{d+k})$$

$$\leq \sum_{k=0}^{\infty} \bar{K}_\epsilon 2^{-(d+k)(\rho R - E_{si}(\rho) - \epsilon)}$$

$$= \exp \left( -d(\rho R - E_{si}(\rho) - \epsilon) \right) \sum_{k=0}^{\infty} \bar{K}_\epsilon 2^{-k(\rho R - E_{si}(\rho) - \epsilon)}$$

Since we can choose $\epsilon$ arbitrarily small, the geometric series converges and we have

$$P_e(d) \leq \frac{\bar{K}_\epsilon}{1 - 2^{-(\rho R - E_{si}(\rho) - \epsilon)}} 2^{-d(\rho R - E_{si}(\rho) - \epsilon)}$$

$$= \bar{K}_\epsilon \exp \left( -d(\rho R - E_{si}(\rho) - \epsilon) \right)$$

This is true for all $\rho \in [0, 1]$, so $E(R) \geq E_{e,si}(R) = \sup_{\rho \in [0,1]} \rho R - E_{si}(\rho)$.
C. Probability of error - joint source channel coding with side information

**Theorem 6 (Restatement of Theorem [2]):** Suppose there is a channel $W$ between the encoder and the decoder and side information is available to the decoder. Fix any $\epsilon > 0$ and let $\rho \in [0,1]$. For the encoder/decoder of Sections II-B and II-C if the bias $G$ satisfies

$$G \leq \frac{1 + \rho}{\rho} \left[ E_{si}(\rho) - F_{si}(\rho) - \lambda E_{o}(\rho) + \lambda F(\rho) \right]$$

(135)

then, there is a constant $K_\epsilon < \infty$ so that

$$P_e(d) < K_\epsilon \exp_2 \left( -d \left( \lambda E_0(\rho) - E_{si}(\rho) - \epsilon \right) \right)$$

(136)

Hence, with suitable choice of bias, the error exponent with delay can be

$$E(\lambda) = E_{r, jscsi}(\lambda) \triangleq \sup_{\rho \in [0,1]} \lambda E_0(\rho) - E_{si}(\rho)$$

(137)

**Proof:** We will prove this for $\lambda = 1$ and then show how the proof changes for other $\lambda$. As in the previous proof, $P_e(d)$ can be bounded by $\sum_{i=0}^{\infty} P(F_{d+1})$, and $P(F_d)$ can be bounded by $\sum_{k=1}^{d} P(F_{d,k})$. So we start by bounding $P(F_{d,k})$. First condition on the true source sequence, channel inputs and channel outputs.

$$P(F_{d,k}) \leq \sum_{u_1^d, v_1^d, x_1^d, y_1^d} Q(u_1^d, v_1^d) R(x_1^d) W(y_1^d | x_1^d) \left( \sum_{\tilde{u}_1^d \in U^d, \tilde{u}_1 \neq u_1} 1(\Gamma(\tilde{u}_1) \geq \Gamma(u_1)) \right)^\rho \left| x_1^d, y_1^d, u_1^d, v_1^d \right|$$

(138)

$$\leq \sum_{u_1^d, v_1^d, x_1^d, y_1^d} Q(u_1^d, v_1^d) R(x_1^d) W(y_1^d | x_1^d) \left( \sum_{\tilde{u}_1^d \in U^d, \tilde{u}_1 \neq u_1} 1(\Gamma(\tilde{u}_1) \geq \Gamma(u_1)) \right)^\rho \left| x_1^d, y_1^d, u_1^d, v_1^d \right|^\rho \left( \sum_{\tilde{u}_1^d, \tilde{u}_1 \neq u_1} R(\tilde{u}_1) \left[ 1(\Gamma(\tilde{u}_1) \geq \Gamma(u_1)) \right] \left| x_1^d, y_1^d, u_1^d, v_1^d, \tilde{u}_1 \right| \right)^\rho$$

(139)

The last step is true by Jensen’s inequality. Now the only thing that is random in the expectation is the channel symbols used on the false $\tilde{u}$ paths.

$$P(F_{d,k}) \leq \sum_{u_1^d, v_1^d, x_1^d, y_1^d} Q(u_1^d, v_1^d) R(x_1^d) W(y_1^d | x_1^d) \left( \sum_{\tilde{u}_1^d \in U^d, \tilde{u}_1 \neq u_1} \sum_{\tilde{x}_1^d} R(\tilde{x}_1^d) \left[ 1(\Gamma(\tilde{u}_1) \geq \Gamma(u_1)) \right] \left| x_1^d, y_1^d, u_1^d, v_1^d, \tilde{x}_1^d \right| \right)^\rho$$

(140)

Now, we also have for all $s \geq 0$, $1(\Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^d)) \leq \exp_2(s(\Gamma(\tilde{u}_1^d) - \Gamma(u_1^d)))$. So,

$$\Gamma(\tilde{u}_1^d) - \Gamma(u_1^d) = \log_2 \frac{Q(\tilde{u}_1^d | \tilde{v}_1^d) W(\tilde{y}_1^d | \tilde{x}_1^d)}{P(\tilde{y}_1^d)} - \log_2 \frac{Q(u_1^d | \tilde{v}_1^d) W(\tilde{y}_1^d | \tilde{x}_1^d)}{P(\tilde{y}_1^d)} + (d - k)G$$

(142)

$$1(\Gamma(\tilde{u}_1^d) \geq \Gamma(u_1^d)) \leq \left( \frac{Q(\tilde{u}_1^d | \tilde{v}_1^d) W(\tilde{y}_1^d | \tilde{x}_1^d) W(\tilde{y}_{k+1}^d | \tilde{x}_{k+1}^d) Q(\tilde{u}_{k+1}^d | \tilde{v}_{k+1}^d)}{Q(u_1^d | \tilde{v}_1^d) W(\tilde{y}_1^d | \tilde{x}_1^d) P(\tilde{y}_{k+1}^d)} \right)^s 2^{s(d-k)G}$$

(143)
We can substitute this expression into the inequality for \( P(F_{d,k}) \).

\[
P(F_{d,k}) \leq \sum_{u_1^d, v_1^d} Q(v_1^d)Q(u_1^d | v_1^d)^{1-s_p} Q(v_{k+1}^d | v_{k+1}^d) R(x_1^d) W(y_1^d | x_1^d) P(y_{k+1}^d | x_{k+1}^d). \tag{144}
\]

\[
\leq \sum_{u_1^d, v_1^d} Q(v_1^d)Q(u_1^d | v_1^d)^{1-s_p} Q(v_{k+1}^d | v_{k+1}^d) W(y_1^d | x_{k+1}^d) S\left(\sum_{x_{k+1}^d} W(y_{k+1}^d | x_{k+1}^d) \frac{W(y_1^d | x_{k+1}^d)^s}{P(y_{k+1}^d)^s}\right)^{2 s (d-k) G} \tag{145}
\]

Now set \( s = 1/(1 + \rho) \).

\[
P(F_{d,k}) \leq \sum_{u_1^d, v_1^d} \left( Q(v_1^d)Q(u_1^d | v_1^d)^{1+s} Q(u_{k+1}^d | v_{k+1}^d) \right)

\left( \sum_{x_{k+1}^d} R(x_1^d) W(y_1^d | x_{k+1}^d) \sum_{u_{k+1}^d} Q(u_1^d | v_1^d)^{1+s} Q(u_{k+1}^d | v_{k+1}^d)^{1+s} \right)^{2 \frac{1}{1+s} (d-k) G} \tag{146}
\]

\[
\leq \left( \sum_{u_1^d, v_1^d} \left( Q(v_1^d)Q(u_1^d | v_1^d)^{1+s} Q(u_{k+1}^d | v_{k+1}^d) \right) \left( \sum_{u_{k+1}^d} Q(u_1^d | v_1^d)^{1+s} Q(u_{k+1}^d | v_{k+1}^d)^{1+s} \right)^{\rho} 2 \frac{1}{1+s} (d-k) G \right). \tag{147}
\]

To further reduce this expression, notice \( P(F_{d,k}) \leq A \cdot B \cdot 2 \frac{1}{1+s} (d-k) G \), where

\[
A \triangleq \sum_{u_1^d, v_1^d} \left( Q(v_1^d)Q(u_1^d | v_1^d)^{1+s} Q(u_{k+1}^d | v_{k+1}^d) \right) \left( \sum_{u_{k+1}^d} Q(u_1^d | v_1^d)^{1+s} Q(u_{k+1}^d | v_{k+1}^d)^{1+s} \right)^{\rho} \tag{148}
\]

\[
B \triangleq \sum_{x_1^d, y_1^d} R(x_1^d) W(y_1^d | x_1^d) \sum_{x_{k+1}^d} R(x_1^d) W(y_{k+1}^d | x_{k+1}^d) \sum_{u_{k+1}^d} Q(u_1^d | v_1^d)^{1+s} Q(u_{k+1}^d | v_{k+1}^d)^{1+s} \frac{W(y_{k+1}^d | x_{k+1}^d)^{1+s}}{P(y_{k+1}^d)^{1+s}} \tag{149}
\]

Now, we work on each term individually. \( A \) can be written in two parts, \( A = A_1 \cdot A_2 \), where \( A_1 \) is the term corresponding to the letters from time 1 to \( k \) and \( A_2 \) is the term corresponding to letters from time \( k + 1 \) to \( d \).
Explanations for steps are given after the equations.

\[ A_1 \triangleq \sum_{v_1^k} Q(v_1^k) \sum_{u_1^k} Q(u_1^k | v_1^k) \cdot \left( \sum_{u_2^k} Q(u_2^k | v_2^k) \right)^\rho \]  (150)

\[ \overset{(a)}{=} \sum_{v_1} \cdots \sum_{v_k} Q(v_1^k) \sum_{u_1} \cdots \sum_{u_k} \prod_{l=1}^k Q(u_l | v_l) \cdot \left( \sum_{u_1} \cdots \sum_{u_k} \prod_{j=1}^k Q(u_j | v_j) \right)^\rho \]  (151)

\[ \overset{(b)}{=} \sum_{v_1} \cdots \sum_{v_k} Q(v_1^k) \sum_{u_1} \cdots \sum_{u_k} \prod_{l=1}^k Q(u_l | v_l) \cdot \left( \prod_{j=1}^k \sum_{u_j} Q(u_j | v_j) \right)^\rho \]  (152)

\[ \overset{(c)}{=} \sum_{v_1} \cdots \sum_{v_k} Q(v_1^k) \left( \prod_{l=1}^k \sum_{u_l} Q(u_l | v_l) \right)^\rho \left( \prod_{j=1}^k \sum_{u_j} Q(u_j | v_j) \right)^\rho \]  (153)

\[ \overset{(d)}{=} \sum_{v_1} \cdots \sum_{v_k} Q(v_1^k) \prod_{l=1}^k \left( \sum_{u_l} Q(u_l | v_l) \right)^{1+\rho} \]  (154)

\[ \overset{(e)}{=} \sum_{v_1} \cdots \sum_{v_k} Q(v_1^k) \prod_{l=1}^k \left( \sum_{u_l} Q(u_l | v_l) \right)^{1+\rho} \]  (155)

\[ \overset{(f)}{=} \prod_{l=1}^k \left( \sum_{v_l} Q(v_l) \left( \sum_{u_l} Q(u_l | v_l) \right)^{1+\rho} \right) \]  (156)

\[ \overset{(g)}{=} \prod_{l=1}^k \left( \sum_{v_l} Q(v_l) \left( \sum_{u_l} Q(u_l | v_l) \right)^{1+\rho} \right) \]  (157)

\[ \overset{(h)}{=} \left( \sum_{v \in V} Q(v) \left( \sum_{u \in U} Q(u | v) \right)^{1+\rho} \right)^k \]  (158)

\[ = \exp_2 \left( k \log_2 \left[ \sum_{v \in V} Q(v) \left( \sum_{u \in U} Q(u | v) \right)^{1+\rho} \right] \right) \]  (159)

a) Memorylessness of source.

b) Sums and products commute.

c) Same as last step.

d) Replace dummy variables.

e) Combine common terms.

f) Memorylessness of source.

g) Commuting sum and product.

h) Dummy variable replacement, each of the \( k \) terms is the same.
Similarly, we work out $A_2$ below.

$$A_2 \triangleq \sum_{v^d_{k+1}} Q(v^d_{k+1}) \sum_{u^d_{k+1}} Q(u^d_{k+1}|v^d_{k+1}) \left( \sum_{\tilde{u}^d_{k+1}} Q(\tilde{u}^d_{k+1}) \right)^\rho$$

$$= \sum_{v^d_{k+1}} Q(v^d_{k+1}) \left( \sum_{u^d_{k+1}} Q(u^d_{k+1}|v^d_{k+1}) \right)^\rho$$

\[(a)\] \quad \sum_{v^d_{k+1}} Q(v^d_{k+1}) \cdot 1 \cdot \left( \sum_{\tilde{u}^d_{k+1}} Q(\tilde{u}^d_{k+1}) \right)^\rho

\[(b)\] \quad \sum_{v^d_{k+1}} Q(v^d_{k+1}) \left( \sum_{u^d_{k+1}} Q(u^d_{k+1}|v^d_{k+1}) \right)^\rho

\[(c)\] \quad \prod_{l=k+1}^d \sum_{v_l} Q(v_l) \left( \sum_{u_l} Q(u_l|v_l) \right)^\rho

\[(d)\] \quad \left( \sum_{v} Q(v) \left( \sum_{u} Q(u|v) \right)^\rho \right)^{d-k}

$$= \exp_2 \left( (d-k) \log_2 \left( \sum_{v \in \mathcal{V}} Q(v) \left( \sum_{u \in \mathcal{U}} Q(u|v) \right)^\rho \right) \right)$$

a) The sum of the probabilities in a conditional distribution is 1.
b) Replace dummy variable.
c) Memorylessness of source.
d) All $d - k$ terms in the product are the same.

Now use the definitions of $E_{sl}$ and $F_{sl}$ to write $A$ as:

$$A = A_1 \cdot A_2$$

$$= \exp_2 \left( k E_{sl}(\rho) + (d-k) F_{sl}(\rho) \right)$$

Analogously, we will write $B = B_1 \cdot B_2$ where $B_1$ is the product of terms concerning time 1 to $k$ and $B_2$ is the product of terms concerning time $k + 1$ to $d$.

$$B_1 \triangleq \sum_{y_i^1} \left( \sum_{x_i^1} R(x_i^1) W(y_i^1|x_i^1)^{1+\rho} \right) \left( \sum_{\tilde{x}_i^1} R(\tilde{x}_i^1) W(y_i^1|\tilde{x}_i^1)^{1+\rho} \right)^\rho$$

\[(a)\] \quad \sum_{y_i^1} \left( \sum_{x_i^1} R(x_i^1) W(y_i^1|x_i^1)^{1+\rho} \right)^{1+\rho}

\[(b)\] \quad \prod_{l=1}^k \sum_{y_l} \left( \sum_{x_l} R(x_l) W(y_l|x_l)^{1+\rho} \right)^{1+\rho}

\[(c)\] \quad \left( \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \beta(x) W(y|x)^{1+\rho} \right)^k

$$= \exp_2 \left( k \log_2 \left( \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \beta(x) W(y|x)^{1+\rho} \right)^{1+\rho} \right)$$

a) Replace dummy variables and combine common terms.
b) Use source memorylessness and commute products with sums.

c) All $k$ terms in the product are the same, replace the dummy variables.

Similarly for $B_2$,

$$B_2 = \sum_{y_{k+1}^d} \left( \sum_{x_{k+1}^d} R(x_{k+1}^d) W(y_{k+1}^d | x_{k+1}^d) \right) \left( \sum_{x_{k+1}^d} R(x_{k+1}^d) \left[ \frac{W(y_{k+1}^d | x_{k+1}^d)}{P(y_{k+1}^d)} \right]^{\frac{1}{1+\rho}} \right)^{\rho}$$

(174)

(a) $= \sum_{y_{k+1}^d} P(y_{k+1}^d) \left( \sum_{x_{k+1}^d} R(x_{k+1}^d) \left[ \frac{W(y_{k+1}^d | x_{k+1}^d)}{P(y_{k+1}^d)} \right]^{\frac{1}{1+\rho}} \right)^{\rho}$

(175)

(b) $= \sum_{y_{k+1}^d} P(y_{k+1}^d) \left( \sum_{x_{k+1}^d} R(x_{k+1}^d) \left[ \frac{W(y_{k+1}^d | x_{k+1}^d)}{P(y_{k+1}^d)} \right]^{\frac{1}{1+\rho}} \right)^{\rho}$

(176)

(c) $= \sum_{y_{k+1}^d} P(y_{k+1}^d)^{\frac{1}{1+\rho}} \left( \sum_{x_{k+1}^d} R(x_{k+1}^d) W(y_{k+1}^d | x_{k+1}^d)^{\frac{1}{1+\rho}} \right)^{\rho}$

(177)

(d) $= \prod_{t=k+1}^{d} \sum_{y_t} P(y_t)^{\frac{1}{1+\rho}} \left( \sum_{x_t} R(x_t) W(y_t | x_t)^{\frac{1}{1+\rho}} \right)^{\rho}$

(178)

(e) $= \left( \sum_{y} P(y)^{\frac{1}{1+\rho}} \left( \sum_{x} \beta(x) W(y | x)^{\frac{1}{1+\rho}} \right)^{\rho} \right)^{d-k}$

(179)

$$= \exp_2 \left( (d-k) \log_2 \left[ \sum_{y} P(y) \left( \sum_{x} \beta(x) \left[ \frac{W(y | x)}{P(y)} \right]^{\frac{1}{1+\rho}} \right)^{\rho} \right] \right)$$

(180)

a) Total probability: the sum in the first parentheses equals $P(y_{k+1}^d)$.

b) Replace dummy variables.

c) Move $P(y_{k+1}^d)$ out of second sum.

d) Memorylessness of channel, IID channel input generation and commute product with sums.

e) All $d-k$ terms are the same; replace dummy variables.

Use the definitions of $E_0$ and $F$ and substitute for $B_1$ and $B_2$ to get:

$$B = B_1 \cdot B_2$$

(181)

$$= \exp_2 \left( -kE_0(\rho) - (d-k)F(\rho) \right)$$

(182)
Finally, we can put everything together:

\[ P(F_{d,k}) \leq A \cdot B \]

\[ = \exp_2 \left( (d-k) \frac{\rho}{1+\rho} G + k E_{s_i}(\rho) + (d-k) F_{s_i}(\rho) - k E_0(\rho) - (d-k) F(\rho) \right) \]  

\[ P(F_d) \leq \sum_{k=1}^{d} P(F_{d,k}) \]

\[ \leq \sum_{k=1}^{d} \exp_2 \left( (d-k) \frac{\rho}{1+\rho} G + k E_{s_i}(\rho) + (d-k) F_{s_i}(\rho) - k E_0(\rho) - (d-k) F(\rho) \right) \]

\[ = \exp_2 \left( d \left[ \frac{\rho}{1+\rho} G + F_{s_i}(\rho) - F(\rho) \right] \right) \cdot \]

\[ \cdot \sum_{k=1}^{d} \exp_2 \left( k \left[ E_{s_i}(\rho) - F_{s_i}(\rho) - E_0(\rho) + F(\rho) - \frac{\rho}{1+\rho} G \right] \right) \]

Now suppose that \( \lambda \neq 1 \). The only thing that would change would be that instead of \( d \) channel inputs and outputs, there would be \( \lambda d \) channel inputs and outputs. The independence of the channel and source straightforwardly gives:

\[ P(F_{d,k}) \leq \exp_2 \left( (d-k) \frac{\rho}{1+\rho} G + k E_{s_i}(\rho) + (d-k) F_{s_i}(\rho) - k \lambda E_0(\rho) - (d-k) \lambda F(\rho) \right) \]

\[ P(F_d) \leq \exp_2 \left( d \left[ \frac{\rho}{1+\rho} G + F_{s_i}(\rho) - \lambda F(\rho) \right] \right) \cdot \]

\[ \cdot \sum_{k=1}^{d} \exp_2 \left( k \left[ E_{s_i}(\rho) - F_{s_i}(\rho) - \lambda E_0(\rho) + \lambda F(\rho) - \frac{\rho}{1+\rho} G \right] \right) \]

Now, we assume that \( G \leq \frac{1+\rho}{\rho} [E_{s_i}(\rho) - F_{s_i}(\rho) - \lambda E_0(\rho) + \lambda F(\rho)] \), so that the term in the exponential in the sum is positive. Then the total sum can be bounded by \( d \) times the \( d^{th} \) term in the sum.

\[ P(F_d) \leq d \exp \left( -d \left( \lambda E_0(\rho) - E_{s_i}(\rho) \right) \right) \]

The derivative at zero of \( E_0 \) is \( I(R,W) \) where

\[ I(R,W) = \sum_{x \in X, y \in Y} \beta(x) W(y|x) \log_2 \frac{\beta(x) W(y|x)}{\beta(x) P(y)} \]

and the derivative of \( E_{s_i} \) at zero is \( H(U|V) \), so if \( H(U|V) < \lambda I(R,W) \), there is some \( \rho \in (0,1] \) so that the difference \( \lambda E_0(\rho) - E_{s_i}(\rho) \) is strictly positive. The \( \rho \) can be optimized to give the source-channel random coding with side information exponent \( E_{r,sc}(\lambda) = \max_{\rho \in [0,1]} \lambda E_0(\rho) - E_{s_i}(\rho) \).

\[ D. \text{ Random variable of computation - source coding with side information} \]

\[ \textbf{Theorem 7 (Restatement of Theorem 3):} \] Suppose that the decoder has access to the side information and there is a rate \( R \) noiseless, binary channel between the encoder and decoder. Fix any \( \gamma \in [0,1] \). For the encoder/decoder of Section II-B if the bias \( G \) satisfies

\[ \frac{1+\gamma}{\gamma} G_{s_i}(\gamma) < G < \frac{1+\gamma}{\gamma} \left[ \gamma R - F_{s_i}(\gamma) \right] \]
then the \(\gamma^{th}\) moment of computation is uniformly finite all for \(i\), i.e. \(\exists K < \infty\) such that \(\forall i, E[N_i^\gamma] < K\), if

\[
R > \frac{E_{si}(\gamma)}{\gamma} 
\]  
(193)

**Proof:** Recall that

\[
E[N_\gamma] \leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{l,k} 
\]  
(194)

\[
A_{l,k} \triangleq E\left[ \left( \sum_{\tilde{u}_1^l : \tilde{u}_1 \neq u_1} 1(\Gamma(\tilde{u}_1^l) \geq \Gamma(u_1^k)) \right)^\gamma \right] 
\]  
(195)

From section VI-B(92), we already know that if \(l \geq k\),

\[
A_{l,k} \leq \exp_2 \left( (l - k) \frac{\gamma}{1 + \gamma} G + k E_{si}(\gamma) + (l - k) F_{si}(\gamma) - l \gamma R \right) 
\]  
(196)

If \(l \leq k\), we have

\[
A_{l,k} = \sum_{u_1^k, v_1^k} Q(u_1^k, v_1^k) E\left[ \left( \sum_{\tilde{u}_1^l : \tilde{u}_1 \neq u_1} 1(\Gamma(\tilde{u}_1^l) \geq \Gamma(u_1^k)) \right)^\gamma \right]_{u_1^k, v_1^k} 
\]  
(a)

\[
\leq \sum_{u_1^k, v_1^k} Q(u_1^k, v_1^k) \left( \sum_{\tilde{u}_1^l : \tilde{u}_1 \neq u_1} E\left[ 1(\Gamma(\tilde{u}_1^l) \geq \Gamma(u_1^k)) \right]_{u_1^k, v_1^k} \right)^\gamma 
\]  
(198)

\[
\leq \sum_{u_1^k, v_1^k} Q(u_1^k, v_1^k) \left( \sum_{\tilde{u}_1^l : \tilde{u}_1 \neq u_1} E\left[ 1(\text{parities of } \tilde{u}_1 \text{ match}) \exp_2 \left( \frac{(\Gamma(\tilde{u}_1^l) - \Gamma(u_1^k))}{1 + \gamma} \right) \right]_{u_1^k, v_1^k} \right)^\gamma 
\]  
(199)

(a) uses Jensen's inequality followed by linearity of conditional expectation. The parity generation process is independent on different branches of the encoding tree, and

\[
\exp_2(\Gamma(\tilde{u}_1) - \Gamma(u_1^k)) = \left( \frac{Q(\tilde{u}_1^l | v_1^l)}{Q(u_1^l | v_1^l)Q(u_{l+1}^k | v_{l+1}^k)} \right)^{2(l-k)G} 
\]  
(200)

so substituting gives

\[
A_{l,k} \leq \sum_{u_1^k, v_1^k} Q(u_1^k, v_1^k) \left( \sum_{\tilde{u}_1^l : \tilde{u}_1 \neq u_1} \left( \frac{Q(\tilde{u}_1^l | v_1^l)}{Q(u_1^l | v_1^l)Q(u_{l+1}^k | v_{l+1}^k)} \right)^{\frac{2(l-k)G}{1 + \gamma} - l R} \right)^\gamma 
\]  
(201)

\[
\leq \sum_{u_1^k, v_1^k} Q(u_1^k, v_1^k) \left( \sum_{\tilde{u}_1^l : \tilde{u}_1 \neq u_1} \left( \frac{Q(\tilde{u}_1^l | v_1^l)}{Q(u_1^l | v_1^l)Q(u_{l+1}^k | v_{l+1}^k)} \right)^{\frac{2(l-k)G}{1 + \gamma} - l R} \right)^\gamma 
\]  
(202)

\[
= \exp_2 \left( (l - k) \frac{\gamma}{1 + \gamma} G - l \gamma R \right) \left( \sum_{\tilde{u}_1^l} Q(\tilde{u}_1^l | v_1^l) \right)^{\frac{2(l-k)G}{1 + \gamma}} 
\]  
(203)

\[
C \triangleq \sum_{u_1^k, v_1^k} Q(u_1^k, v_1^k) \left( \sum_{\tilde{u}_1^l} Q(\tilde{u}_1^l | v_1^l) \right)^{\frac{2(l-k)G}{1 + \gamma}} 
\]  
(204)

\[
D \triangleq \sum_{u_{l+1}^k, v_{l+1}^k} Q(u_{l+1}^k, v_{l+1}^k) \left( \sum_{\tilde{u}_1^l} Q(\tilde{u}_1^l | v_1^l) \right)^{\frac{2(l-k)G}{1 + \gamma}} 
\]  
(205)
The terms corresponding to letters from time 1 to \( l \), \( C \) are the same as (160) in section VI-B so we have

\[
A_{l,k} \leq \exp \left( (l-k) \frac{\gamma}{1+\gamma} G - l\gamma R + lE_{si}(\gamma) \right) \cdot D
\]  
(206)

The term \( D \) can be simplified into an exponential form using \( G_{si} \):

\[
D \triangleq \sum_{u'_{l+1} v'_{l+1}} Q(u^k_{l+1})Q(u^k_{l+1}|v^k_{l+1})
\]
(207)

\[
= \sum_{v'_{l+1}} Q(v^k_{l+1}) \sum_{u'_{l+1}} Q(u^k_{l+1}|v^k_{l+1})
\]
(208)

\[
= \prod_{m=l+1}^k \sum_{v_m} Q(v_m) \sum_{u_m} Q(u_m|v_m)
\]
(209)

\[
= \left( \sum_{v \in V} Q(v) \sum_{u \in V} Q(u|v) \right)^{k-l}
\]
(210)

\[
= \exp \left( (k-l)G_{si}(\gamma) \right)
\]
(211)

So if \( k \geq l \), we have:

\[
A_{l,k} \leq \exp \left( (l-k) \frac{\gamma}{1+\gamma} G - \gamma R + lE_{si}(\gamma) + (k-l)G_{si}(\gamma) \right)
\]
(212)

Combining the bounds gives

\[
E[N^*] \leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{l,k}
\]
(213)

\[
= \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} A_{l,k} + \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} A_{l,k}
\]
(214)

\[
\leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{l,k} + \sum_{k=1}^{\infty} \sum_{l=k}^{\infty} A_{l,k}
\]
(215)

\[
\leq \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G - \gamma R + lE_{si}(\gamma) + (k-l)G_{si}(\gamma) \right)
\]

\[
+ \sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + kE_{si}(\gamma) + lE_{si}(\gamma) - \gamma R \right)
\]
(216)

\[
= \sum_{l=1}^{\infty} \exp \left( -l \left( \gamma R - E_{si}(\gamma) \right) \right) \sum_{k=l}^{\infty} \exp \left( -(k-l) \left( \frac{\gamma}{1+\gamma} G - G_{si}(\gamma) \right) \right) +
\]

\[
\sum_{k=1}^{\infty} \exp \left( -k \left( \gamma R - E_{si}(\gamma) \right) \right) \sum_{l=k}^{\infty} \exp \left( -(l-k) \left( -\frac{\gamma}{1+\gamma} G + F_{si}(\gamma) - \gamma R \right) \right)
\]
(217)

a) Substitute for \( A_{l,k} \).

b) Add and subtract \((l-k)\gamma R\) in the exponent of the second double sum.
The above sums converge if the following conditions are met:

\[
\gamma R > E_{ss}(\gamma)
\]
\[
\frac{\gamma}{1 + \gamma} G > G_{ss}(\gamma)
\]
\[
\frac{\gamma}{1 + \gamma} G < \gamma R - F_{ss}(\gamma)
\]

(218) \hspace{1cm} (219) \hspace{1cm} (220)

This concludes the proof assuming these conditions hold. To see that (219) and (220) can be satisfied by one choice of bias assuming (218), see section VI-F.

E. Random variable of computation - joint source channel coding with side information

**Theorem 8 (Restatement of Theorem 4):** Suppose there is a channel \( W \) between the encoder and the decoder and side information is available to the decoder. Fix any \( \gamma \in [0, 1] \). For the encoder/decoder of Sections II-B and II-C if the bias \( G \) satisfies

\[
\frac{1 + \gamma}{\gamma} [G_{ss}(\gamma) - \lambda G(\gamma)] < G < \frac{1 + \gamma}{\gamma} [\lambda F(\gamma) - F_{ss}(\gamma)]
\]

(221) then the \( \gamma \)-th moment of computation is uniformly finite all for \( i \), i.e. \( \exists K < \infty \) such that \( \forall i \), \( E[N_i^\gamma] < K \), if

\[
\lambda E_0(\gamma) > E_{ss}(\gamma)
\]

(222)

**Proof:** Again, we will show this for \( \lambda = 1 \) and at the end see how it changes for \( \lambda \neq 1 \). Recall that

\[
E[N_i^\gamma] \leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{l,k}
\]

(223)

\[
A_{l,k} \equiv E \left[ \left( \sum_{\tilde{u}_i \neq u_1} 1 \left( \Gamma(\tilde{u}_i) \geq \Gamma(u_1) \right) \right)^\gamma \right]
\]

(224)

From (138) in section VI-C we already know that if \( l \geq k \),

\[
A_{l,k} \leq \exp \left( (l - k) \frac{\gamma}{1 + \gamma} G + k E_{ss}(\gamma) + (l - k) F_{ss}(\gamma) - k E_0(\gamma) - (l - k) F(\gamma) \right)
\]

(225)

If \( l \leq k \), we have

\[
A_{l,k} = \sum_{u_1^k, v_1^k, x_1^k, y_1^k} Q(u_1^k, v_1^k) R(x_1^k) W(y_1^k|x_1^k) E \left[ \left( \sum_{\tilde{u}_1} 1 \left( \Gamma(\tilde{u}_1) \geq \Gamma(u_1) \right) \right)^\gamma \right]_{x_1^k, y_1^k, u_1^k, v_1^k}
\]

(226)

\[
\leq \sum_{u_1^k, v_1^k, x_1^k, y_1^k} Q(u_1^k, v_1^k) R(x_1^k) W(y_1^k|x_1^k) E \left[ \sum_{\tilde{u}_1} 1 \left( \Gamma(\tilde{u}_1) \geq \Gamma(u_1) \right) \right]_{x_1^k, y_1^k, u_1^k, v_1^k}^\gamma
\]

(227)

\[
\leq \sum_{u_1^k, v_1^k, x_1^k, y_1^k} Q(u_1^k, v_1^k) R(x_1^k) W(y_1^k|x_1^k) \left( \sum_{\tilde{u}_1} \exp \left( \frac{\Gamma(\tilde{u}_1) - \Gamma(u_1)}{1 + \gamma} \right) \right)_{x_1^k, y_1^k, u_1^k, v_1^k}^\gamma
\]

(228)

\[
= \sum_{u_1^k, v_1^k, x_1^k, y_1^k} Q(u_1^k, v_1^k) R(x_1^k) W(y_1^k|x_1^k) \left( \sum_{\tilde{u}_1} \exp \left( \frac{\Gamma(\tilde{u}_1) - \Gamma(u_1)}{1 + \gamma} \right) \right)_{x_1^k, y_1^k, u_1^k, v_1^k}^\gamma
\]

(229)

\[
\leq \sum_{u_1^k, v_1^k, x_1^k, y_1^k} Q(u_1^k, v_1^k) R(x_1^k) W(y_1^k|x_1^k) \left( \sum_{\tilde{u}_1} \sum_{\tilde{x}_1} \exp \left( \frac{\Gamma(\tilde{u}_1) - \Gamma(u_1)}{1 + \gamma} \right) \right)_{x_1^k, y_1^k, u_1^k, v_1^k}^\gamma
\]

(230)
a) Jensen’s inequality.

b) Linearity of expectation.

c) Conditioning on channel inputs along ‘false’ paths.

Now, we write out the term in the exponent to get:

\[
\exp \left( \frac{\Gamma(\tilde{u}_l) - \Gamma(u_k)}{1 + \gamma} \right) = \left( \frac{Q(u_{l+1}^l | v_{l+1}^l)W(y_{l+1}^l | \tilde{x}_{l+1}^l)P(y_{l+1}^k)}{Q(u_{l+1}^l | v_{l+1}^l)W(y_{l+1}^l | x_{l+1}^l)Q(u_{l+1}^k | v_{l+1}^k)W(y_{l+1}^k | x_{l+1}^k)} \right)^{1+\gamma} 2^{(l-k)mG} (231)
\]

So, substituting and merging terms gives:

\[
A_{l,k} \leq \sum_{u_{l+1}^k, v_{l+1}^k, x_{l+1}^k, y_{l+1}^k} Q(v_{l+1}^l | v_{l+1}^l)Q(u_{l+1}^k | v_{l+1}^k)Q(u_{l+1}^l | v_{l+1}^l) \frac{R(x_{l+1}^l)W(y_{l+1}^k | x_{l+1}^l) \Gamma}{(\sum_{\tilde{x}_{l+1}^l} R(\tilde{x}_{l+1}^l)W(y_{l+1}^l | \tilde{x}_{l+1}^l))^\gamma} 2^{(l-k)\frac{G}{1+\gamma}} (232)
\]

The terms corresponding to letters from time 1 to l are easily recognized to be the same as in the last sections, so we can extract them and get

\[
A_{l,k} \leq \exp \left( (l-k)\frac{\gamma}{1+\gamma}G + lE_{si}(\gamma) - lE_0(\gamma) \right) \left( \sum_{u_{l+1}^k, v_{l+1}^k} Q(v_{l+1}^k | v_{l+1}^k)Q(u_{l+1}^k | v_{l+1}^k) \frac{R(x_{l+1}^k)W(y_{l+1}^k | x_{l+1}^k) \Gamma}{(\sum_{\tilde{x}_{l+1}^k} R(\tilde{x}_{l+1}^k)W(y_{l+1}^k | \tilde{x}_{l+1}^k))^\gamma} \right) (233)
\]

\[
C \triangleq \sum_{u_{l+1}^k, v_{l+1}^k} Q(v_{l+1}^k | v_{l+1}^k)Q(u_{l+1}^k | v_{l+1}^k) \frac{R(x_{l+1}^k)W(y_{l+1}^k | x_{l+1}^k) \Gamma}{(\sum_{\tilde{x}_{l+1}^k} R(\tilde{x}_{l+1}^k)W(y_{l+1}^k | \tilde{x}_{l+1}^k))^\gamma} (234)
\]

\[
D \triangleq \sum_{x_{l+1}^k, y_{l+1}^k} R(x_{l+1}^k)W(y_{l+1}^k | x_{l+1}^k) \frac{\Gamma}{(\sum_{\tilde{x}_{l+1}^k} R(\tilde{x}_{l+1}^k)W(y_{l+1}^k | \tilde{x}_{l+1}^k))^\gamma} (235)
\]
Then, we work with $C$ and $D$ individually to get:

\[
C = \sum_{u_{i+1}, v_{i+1}} Q(u_{i+1})Q(v_{i+1} | v_{i+1}^k) \frac{1}{1+\gamma} \tag{237}
\]

\[
= \left( \sum_{v \in V} Q(v) \sum_{u \in U} Q(u | v) \frac{1}{1+\gamma} \right)^{k-l} \tag{238}
\]

\[
D = \sum_{x_{i+1}^k, y_{i+1}^k} R(x_{i+1}^k)W(y_{i+1}^k | x_{i+1}^k) \frac{1}{1+\gamma} P(y_{i+1}^k) \tag{239}
\]

\[
= \sum_{x_{i+1}^k, y_{i+1}^k} R(x_{i+1}^k) \left( \frac{W(y_{i+1}^k | x_{i+1}^k)}{P(y_{i+1}^k)} \right) \frac{1}{1+\gamma} P(y_{i+1}^k) \tag{240}
\]

\[
= \sum_{y_{i+1}^k} P(y_{i+1}^k) \sum_{x_{i+1}^k} R(x_{i+1}^k) \left( \frac{W(y_{i+1}^k | x_{i+1}^k)}{P(y_{i+1}^k)} \right) \tag{241}
\]

\[
= \left( \sum_{y \in Y} \beta(x) \left( \frac{W(y | x)}{P(y)} \right) \right) \frac{1}{1+\gamma} (k-l) \quad \tag{242}
\]

So finally for $k \geq l$, using the definitions of $G_{si}$ and $G$ gives

\[
A_{i,k} = \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + lE_{si}(\gamma) + (k-l)G_{si}(\gamma) - lE_0(\gamma) - (k-l)G(\gamma) \right) \tag{243}
\]

Now we split the double sum in the bound of $E[N^\gamma]$ and use the two cases of $l, k$ to get:

\[
E[N^\gamma] \leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{i,k} \tag{244}
\]

\[
= \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{i,k} + \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} A_{i,k} \tag{245}
\]

\[
\leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_{i,k} + \sum_{k=1}^{\infty} \sum_{l=k}^{\infty} A_{i,k} \tag{246}
\]

\[
\leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + lE_{si}(\gamma) + (k-l)G_{si}(\gamma) - lE_0(\gamma) - (k-l)G(\gamma) \right) +
\]

\[
\sum_{k=1}^{\infty} \sum_{l=k}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + kE_{si}(\gamma) + (l-k)F_{si}(\gamma) - kE_0(\gamma) - (l-k)F(\gamma) \right) \tag{247}
\]

\[
= \sum_{k=1}^{\infty} \exp \left( l(E_{si}(\gamma) - E_0(\gamma)) \right) \sum_{k=1}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + (k-l)G_{si}(\gamma) - (k-l)G(\gamma) \right)
\]

\[
+ \sum_{k=1}^{\infty} \exp \left( k(E_{si}(\gamma) - E_0(\gamma)) \right) \sum_{k=1}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + (l-k)F_{si}(\gamma) - (l-k)F(\gamma) \right) \tag{248}
\]

Now, if $\lambda \neq 1$, we would instead have

\[
E[N^\gamma] \leq \sum_{l=1}^{\infty} \exp \left( l(E_{si}(\gamma) - \lambda E_0(\gamma)) \right) \sum_{k=1}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + (k-l)G_{si}(\gamma) - \lambda(k-l)G(\gamma) \right) \tag{249}
\]

\[
+ \sum_{k=1}^{\infty} \exp \left( k(E_{si}(\gamma) - \lambda E_0(\gamma)) \right) \sum_{l=k}^{\infty} \exp \left( (l-k) \frac{\gamma}{1+\gamma} G + (l-k)F_{si}(\gamma) - \lambda(l-k)F(\gamma) \right)
\]
The above sums converge if the following conditions are met:

\[
E_{si}(\gamma) < \lambda E_{0}(\gamma) \quad (250)
\]

\[
\frac{\gamma}{1 + \gamma} G > G_{si}(\gamma) - \lambda G(\gamma) \quad (251)
\]

\[
\frac{\gamma}{1 + \gamma} F(\gamma) < F_{si}(\gamma) \quad (252)
\]

Condition (250) is effectively the requirement that the source coding computational cutoff rate for the \(\gamma^{th}\) moment is lower than the channel coding cutoff rate for the \(\gamma^{th}\) moment. This is needed in this case even though we are using joint source-channel coding. Conditions (251) and (252) combined require

\[
\frac{1 + \gamma}{\gamma} \left[ G_{si}(\gamma) - \lambda G(\gamma) \right] < G < \frac{1 + \gamma}{\gamma} \left[ \lambda F(\gamma) - F_{si}(\gamma) \right] \quad (253)
\]

\[\blacksquare\]

F. Showing the range of viable bias values is non-empty

Fix a \(\gamma \in [0, 1]\). For each \(v \in \mathcal{V}\) and \(y \in \mathcal{Y}\) define \(H(v)\) and \(J(y)\) as:

\[
H(v) \triangleq \sum_{u \in \mathcal{U}} Q(u|v)^{1+\gamma} \quad (254)
\]

\[
J(y) \triangleq \sum_{x \in \mathcal{X}} \beta(x) \left( \frac{W(y|x)}{P(y)} \right)^{\frac{1}{1+\gamma}} \quad (255)
\]

If we consider \(V\) to a random variable with distribution \(Q(v)\) on \(\mathcal{V}\) and \(Y\) to be a random variable with distribution \(P(y)\) on \(\mathcal{Y}\), then by definition we have the following relations:

\[
E_{si}(\gamma) = \log_2 E[ H(V)^{1+\gamma} ] \quad (256)
\]

\[
F_{si}(\gamma) = \log_2 E[ H(V)^{\gamma} ] \quad (257)
\]

\[
G_{si}(\gamma) = \log_2 E[ H(V) ] \quad (258)
\]

\[
E_{0}(\gamma) = - \log_2 E[ J(Y)^{1+\gamma} ] \quad (259)
\]

\[
F(\gamma) = - \log_2 E[ J(Y)^{\gamma} ] \quad (260)
\]

\[
G(\gamma) = - \log_2 E[ J(Y) ] \quad (261)
\]

By repeated use of Jensen’s inequality, since \(\gamma \in [0, 1]\), we also have

\[
E[H(V)^{1+\gamma}] \geq E[H(V)]E[H(V)]^{\gamma} \quad (262)
\]

\[
\geq E[H(V)]E[H(V)^{\gamma}] \quad (263)
\]

\[
E[J(Y)^{1+\gamma}] \geq E[J(Y)]E[J(Y)]^{\gamma} \quad (264)
\]

\[
\geq E[J(Y)]E[J(Y)^{\gamma}] \quad (265)
\]
Since $\log_2$ is a monotonically increasing function, this means:

\[
E_{si}(\gamma) \geq F_{si}(\gamma) + G_{si}(\gamma)
\]

(266)

\[
E_{0}(\gamma) \leq F(\gamma) + G(\gamma)
\]

(267)

Now, if $\gamma R > E_{si}(\gamma)$, then

\[
\gamma R - F_{si}(\gamma) - G_{si}(\gamma) > E_{si}(\gamma) - F_{si}(\gamma) - G_{si}(\gamma)
\]

(268)

\[
\geq 0
\]

(269)

Hence, \((\frac{1+\gamma}{\gamma}G_{si}(\gamma), \frac{1+\gamma}{\gamma}(\gamma R - F_{si}(\gamma)))\) is a non-empty open interval of bias values that give a finite $\gamma^{th}$ moment of computation if $\gamma R > E_{si}(\gamma)$ as shown in section VI-D.

For the joint source-channel case, we assume $\lambda E_{0}(\gamma) > E_{si}(\gamma)$. Then,

\[
\lambda F(\gamma) + \lambda G(\gamma) - F_{si}(\gamma) - G_{si}(\gamma) \geq \lambda E_{0}(\gamma) - E_{si}(\gamma)
\]

(270)

\[
> 0
\]

(271)

Hence, there is a non-empty open interval of allowable bias values in Theorem 4 if $\lambda E_{0}(\gamma) > E_{si}(\gamma)$.

G. Error exponent with bias set for computation

In this section, it is shown that if the bias can be set to achieve a $\gamma^{th}$ moment of computation while still allowing for a positive error exponent.

In the source coding with side information case, assume $\gamma R > E_{si}(\gamma)$, then we know (Thm. 1) that for all $\epsilon > 0$, there is a $K_\epsilon < \infty$ so that

\[
P_e(d) < K_\epsilon 2^{-d(\gamma R - E_{si}(\gamma) - \epsilon)}
\]

(272)

This is provided that the bias $G$ satisfies

\[
G \leq \frac{1+\gamma}{\gamma}[E_{si}(\gamma) - F_{si}(\gamma)]
\]

(273)

Also, from Thm. 3, the $\gamma^{th}$ moment of computation is finite provided

\[
\frac{1+\gamma}{\gamma}G_{si}(\gamma) < G < \frac{1+\gamma}{\gamma}(\gamma R - F_{si}(\gamma))
\]

(274)

Suppose the bias is set so that $G^* = \frac{1+\gamma}{\gamma}[E_{si}(\gamma) - F_{si}(\gamma)]$. Then there is a positive error exponent with delay. It is also true, however, that this choice of bias yields a finite $\gamma^{th}$ moment of computation. Since we assume $\gamma R > E_{si}(\gamma)$, it is immediate that $G^* < \frac{1+\gamma}{\gamma}(\gamma R - F_{si}(\gamma))$.

For the other inequality, we need that the log function is strictly concave $\cap$. This combined with the assumption that $U$ is not deterministic given $v \in \mathcal{V}$ for at least one $\mathcal{V}$ gives the strict inequality below:

\[
G_{si}(\gamma) + F_{si}(\gamma) < E_{si}(\gamma)
\]

(275)

Hence, $G^* > \frac{1+\gamma}{\gamma}G_{si}(\gamma)$ if the source $U$ is not deterministic given $v$ for at least one value of $v \in \mathcal{V}$.\footnote{If $U$ is conditionally deterministic given $v$ for all $v \in \mathcal{V}$, obviously the source coding with side information problem is not interesting as zero rate is needed.}
For the joint source-channel coding with side information case, an analogous line of reasoning gives that the choice \[ G^* = \frac{1+\gamma}{\gamma} \left[ E_{si}(\gamma) - F_{si}(\gamma) - \lambda E_0(\gamma) + \lambda F(\gamma) \right] \] gives a positive error exponent and finite \( \gamma^{th} \) moment of computation provided \( E_{si}(\gamma) < E_0(\gamma) \).

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