AN OBSTRUCTION RELATING LOCALLY FINITE POLYGONS TO TRANSLATION QUADRANGLES

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ABSTRACT. One of the most fundamental open problems in Incidence Geometry, posed by Tits in the 1960s, asks for the existence of so-called “locally finite generalized polygons” — that is, generalized polygons with “mixed parameters” (one being finite and the other not). In a more specialized context, another long-standing problem (from the 1990s) is as to whether the endomorphism ring of any translation generalized quadrangle is a skew field (the answer of which is known in the finite case). (The analogous problem for projective planes, and its positive solution, the “Bruck-Bose construction,” lies at the very base of the whole theory of translation planes.)

In this short note, we introduce a category, representing certain very specific embeddings of generalized polygons, which surprisingly controls the solution of both (apparently entirely unrelated) problems.

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1. INTRODUCTION

1.1. First problem — locally finite polygons. Consider a generalized $n$-gon $\Gamma$ with $s + 1$ points on each line and $t + 1$ lines through each point and let $s, t > 1$ (and note that $st$ is allowed to be infinite) — its incidence graph is a bipartite graph of diameter $n$ and girth $2n$. If $n$ is odd, then it is easy to show that $s = t$, see [22, 1.5.3]. If $n$ is even, though, there are examples where $s \neq t$, a most striking example being $n = 8$ in which case a theorem of Feit and Higman [7] implies that if $st$ is finite, $2st$ is a perfect square and so $s$ is never equal to $t$. If both $s$ and $t$
are finite, they are bounded by each other; to be more specific, $s \leq t^2 \leq s^4$ for $n = 4$ and $n = 8$ by results of Higman [9] (1975), and $s \leq t^3 \leq s^9$ for $n = 6$ by Haemers and Roos [8] (1981). These results can also be found in [22], §1.7.2. For other even values of $n$, $\Gamma$ cannot exist by a famous result of Feit and Higman which also appeared in their 1964 paper [7]. In loc. cit., several necessary divisibility conditions involving the parameters $s$ and $t$ of a generalized $n$-gon can be found (with $n \in \{3, 4, 6, 8\}$), given the existence of such an object with these parameters. An old and notorious question, first posed by Jacques Tits in the 1960s, represents the largest gap in our knowledge about parameters of generalized polygons:

“Do there exist locally finite generalized polygons?”

In other words, do there exist, up to duality, (thick) generalized polygons with a finite number of points incident with a line, and an infinite number of lines through a point?\(^1\) (Note that in Van Maldeghem’s book [22], such generalized polygons are called semi-finite.)

There is only a very short list of results on Tits’s question. All of them comprise the case $n = 4$.

- P. J. Cameron [4] showed in 1981 that if $n = 4$ and $s = 2$, then $t$ is finite.
- In [2] A. E. Brouwer shows the same thing for $n = 4$ and $s = 3$ and the proof is purely combinatorial (unlike a nonpublished but earlier proof of Kantor [15]).
- More recently, G. Cherlin used Model Theory (in [5]) to handle the generalized 4-gons with five points on a line.
- For other values of $n$ and $s$ (where $n$ of course is even), nothing is known without any extra assumptions.

Apart from the aforementioned results, there is only one other “general result” on parameters of generalized polygons (so without invoking additional structure through, e.g., the existence of certain substructures or the occurrence of certain group actions):

**Theorem 1.1** (Bruck and Ryser [3], 1949). If $\Gamma$ is a finite projective plane of order $m$, $m \in \mathbb{N}^\times$, and $m \equiv 1, 2 \mod 4$, then $m$ is the sum of two perfect squares.

1.2. **Second problem — endomorphisms of translation polygons.** In Incidence Geometry, one is often concerned with categories $\mathcal{C}$ of abelian groups such that if $A$ is an object of $\mathcal{C}$, and $\text{Hom}(A, A)$ is the endomorphism ring of $A$, where morphisms in $\mathcal{C}$ are group homomorphisms that preserve some specified substructure of the objects, then $\text{Hom}(A, A)$ is a skew field, and $A$ is a left or right vector space over $\text{Hom}(A, A)$. The reason of this interest is that such results lead to productive representations of associated geometrical objects (cf. the next paragraph).

One such classical example is the category $\mathcal{T}\mathcal{P}$ of translation groups of affine (or projective) translation planes, where morphisms are prescribed to preserve the plane attached to it, and all “points at infinity.” An early result in this theory is that, indeed, all $\text{Hom}(A, A)$s (which are called kernels) are skew fields, and this leads to the fact that one can represent the associated translation planes in a projective

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\(^1\)The aforementioned question can be found as Problem 5 in the “Ten Most Famous Open Problems” chapter of Van Maldeghem’s book [22], see also §10 of [17], etc.
space. This representation is the famous “Bruck-Bose” representation, and is arguably (?) the most fundamental tool to study translation planes.

A category which is related but much more difficult and mysterious, is that of translation groups of generalized quadrangles, say $T GQ$. In the same way as for planes one defines “kernels,” and again one aims at proving that all kernels are skew fields. So the challenge is, very roughly, the question whether all translation quadrangles can be embedded in projective space over a skew field.

In the finite case, where the answer was proved to be positive already in the 1980s, this embedding result became the most basic tool of the theory, such as for the planes, see [18].

More precisely, let $\Gamma^x$ be an infinite translation generalized quadrangle (TGQ) (see the next section for the definitions), with translation group $T$. Let $z$ be an affine point, that is, a point not collinear with $x$, and $\{U\}_z$ be the set of lines incident with $z$. If $V$ is any such line, put $v := \text{proj}_{V,x}$. Define $K$ as the set of endomorphisms of $T$ that map every $T_V$ into itself. Then $K$ is a ring (with multiplicative identity) without zero divisors [11], and for any such $V$, we have that $T, T_V$ and $T_v$ are left $K$-modules.

Conjecture 1.2 (Linearity for TGQs). $K$ is a skew field.

The property that $K$ is a skew field allows one to interpret $T, T_V$ and $T_v$ ($V \in \{U\}_z$) as vector spaces over $K$, so as to represent $\Gamma^x$ in projective space (over $K$), just as in the finite case — see [16, Chapter 8]. Throughout this paper, we will call TGQs for which the kernel is a skew field linear.

Remark 1.3. Note that both projective planes and generalized quadrangles are special cases of the class of generalized $n$-gons (respectively the case $n = 3$ and $n = 4$). “Translation generalized $n$-gons” only make sense for these two cases, though (cf. [22]).

Remark 1.4. For some special cases the aforementioned conjecture has been proved to be true; planar TGQs [12], TGQs with a strongly regular translation center [12] and of course finite TGQs [16] all satisfy the conjecture. Also, it has been shown in [13] that the more restricted “topological kernel” of a compact connected topological TGQ is a skew field.²

Remark 1.5. In [11] it is claimed (in Corollary 3.11) that the kernel of a TGQ always is a skew field. In the proof however, the author uses his Proposition 3.10 which states that any three distinct lines on the translation point, together with any affine point, generate a plane-like subGQ. This result is not true (even not in the finite case) — in fact, only a very restricted class of TGQs has this property. (Still, Theorem 3.11 of loc. cit. shows that if a TGQ does satisfy this property, it indeed is linear.) The paper [11] contains many other interesting results on infinite TGQs.

²Due to these rather restricted partial results, it is usually assumed that the kernel of a TGQ be a skew field.
1.3. The present paper. In this paper, it is our intention to show that if one curious property could be understood, these two seemingly (very) unrelated problems would follow. (In both problems, one wants to control the parameters of the polygon, but this is barely evidence for such connections to exist; see also [19] for an elaborate discussion.)

We will show that if there do not exist pointed generalized polygons $(\mathcal{U}, u)$, $(\mathcal{U}', u)$ (where $u$ is a point), together with an isomorphism $\gamma : \mathcal{U} \to \mathcal{U}'$ such that $\mathcal{U}'$ is ideally embedded in $\mathcal{U}$, and $u \in \mathcal{U}' \leq \mathcal{U}$ is fixed linewise by $\gamma$, then both problems have a positive answer (cf. Theorem 3.1 below).

As such, this property represents a “common geometrically divisor” of both problems.

2. Some definitions

2.1. In this paper, a generalized $n$-gon, $n \in \mathbb{N} \setminus \{0, 1, 2\}$, is a point-line incidence geometry $\Gamma = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ for which the following axioms are satisfied:

(i) $\Gamma$ contains no ordinary $k$-gon (as a subgeometry), for $2 \leq k < n$;
(ii) any two elements $x, y \in \mathcal{P} \cup \mathcal{B}$ are contained in some ordinary $n$-gon in $\Gamma$;
(iii) there exists an ordinary $(n+1)$-gon in $\Gamma$.

A generalized polygon (GP) is a generalized $n$-gon for some $n$, and $n$ is called the gonality of the GP. Note that projective planes are generalized 3-gons. Generalized 4-gons are also called generalized quadrangles. By (iii), generalized polygons have at least three points per line and three lines per point. So by this definition, we do not consider “thin” polygons.\footnote{For thin polygons, the questions under consideration obviously make no sense.} Note that points and lines play the same role; this is the principle of “duality”.

A pointed GP is a pair $(\Gamma, r)$ where $\Gamma$ is a GP and $r$ is either a point or a line in $\Gamma$.

It can be shown that generalized polygons have an order; there exist constants $s, t$ such that the number of points incident with a line is $s + 1$, and the number of lines incident with a point is $t + 1$, cf. [22, 1.5.3]. If the gonality is odd, then $s = t$, cf. loc. cit.

Remark 2.1. Generalized polygons were introduced by Tits in a famous work on triality [21] of 1959, in order to propose an axiomatic and combinatorial treatment for semisimple algebraic groups (including Chevalley groups and groups of Lie type) of relative rank 2.

2.2. A sub generalized polygon or subpolygon of a GP $\Gamma = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a GP $\Gamma' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ for which $\mathcal{P} \subseteq \mathcal{P}'$, $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{I}' \subseteq \mathcal{I}$. A subpolygon has the same gonality as its “ambient polygon” $\Gamma$. It is full if for any line $L$ of $\Gamma'$, we have that $xI'\mathcal{L}$ if and only if $xIL$. Dually, we define ideal subpolygons.
2.3. An automorphism of a generalized polygon \( \Gamma = (\mathcal{P}, \mathcal{B}, \mathbb{I}) \) is a bijection of \( \mathcal{P} \cup \mathcal{B} \) which preserves \( \mathcal{P}, \mathcal{B} \) and incidence. The full set of automorphisms of a GP forms a group in a natural way — the automorphism group of \( \Gamma \), denoted \( \text{Aut}(\Gamma) \). It is one of its most important invariants. If \( B \) is an automorphism group of a generalized polygon \( \Gamma = (\mathcal{P}, \mathcal{B}, \mathbb{I}) \), and \( R \) is a subset of \( \mathcal{P}, B \backslash \mathbb{I} \) is the subgroup of \( B \) fixing \( R \) pointwise (in this notation, a line is also considered to be a point set).

Morphisms between GPs are defined similarly as automorphisms, and if \( A \) and \( B \) are GPs, \( \text{End}(A, B) = \text{Hom}(A, B) \) denotes the set of all morphisms \( A \to B \). If \( A = (L, l) \) and \( B = (L', l') \) are pointed GPs, elements of \( \text{End}((L, l), (L', l')) \) map \( l \) to \( l' \). Also, if \( (L, l) = (L', l') \), \( \text{End}((L, l), (L', l')) \) is shortened to \( \text{End}(L, l) \), and \( \text{End}(L, [l]) \) denotes the subset of elements that fix \( l \) elementwise.

2.4. A translation generalized quadrangle (TGQ) \([16, 18]\) \( \Gamma^z \) is a generalized quadrangle for which there is an abelian automorphism group \( T \) that fixes each line incident with the point \( x \), while acting sharply transitively on the points not collinear with \( x \) (= the affine points “w.r.t. \( x \)).

Suppose \( z \) is an affine point (w.r.t. \( x \)). Let \( \mathcal{F} = \{T_M | M \mathbb{I} \} \) and \( \mathcal{F}^\ast = \{T_m | x \sim m \sim z \} \); if \( m \mathbb{I} M \), we denote \( T_m \) also by \( T_M \). Then for all \( L \mathbb{I} z \) we have:

- \( T_L \leq T_L^\ast \neq T_L \);
- \( T_LT_M = T \) for \( M \neq L \);
- \( T_A T_B \cap T_C = \{\text{id} \} \) for distinct lines \( A, B, C \).
- \( \{T_L^\ast/T_L \} \cup \{T_LT_M/T_L | M \neq L \} \) is a partition of \( T/T_L \).

Call \((T, \mathcal{F}, \mathcal{F}^\ast)\) a Kantor family of \( \Gamma^z \).

Conversely, from families \( \mathcal{F} \) and \( \mathcal{F}^\ast \) with these properties in an abelian group \( T \), one can construct a TGQ for which \((T, \mathcal{F}, \mathcal{F}^\ast)\) is a Kantor family, using a natural group coset geometry representation \([16, 18]\). If one starts from a TGQ as above, it is isomorphic to the reconstructed coset geometry (the isomorphism class of the geometry is independent of the chosen affine point \( z \)). This representation method was noted for a more general class of GQs (namely, for so-called “elation generalized quadrangles”) by Kantor \([14]\) in the finite case, and carries over without much change to the infinite case \([1]\).

3. Statement of main result

Define the category \( \mathcal{E} \) as follows: its objects are pairs \((X, \eta)\), where \( X = (\Gamma, x) \) is a pointed GP, and \( \eta \in \text{End}(\Gamma, [x]) \) is injective but not bijective. If \( X = ((\Gamma, x), \eta) \) and \( Y = ((\Gamma', x'), \eta') \) are objects in \( \mathcal{E} \), \( \text{Hom}(X, Y) \) consists of morphisms \( \beta : \Gamma \to \Gamma' \) sending \( x \) to \( x' \) (“pointed morphisms”) such that \( \beta \circ \eta = \eta' \circ \beta \).

Note that it follows that \( \Gamma^\eta \cong \Gamma \), that \( x \in \Gamma^\eta \), and that \( \Gamma^\eta \) is ideally embedded in \( \Gamma \). The isomorphism \( \eta \) describes the embedding, and fixes \( x \) lineewise.

**Theorem 3.1** (Main result). If \( \mathcal{E} \) is empty, then locally finite polygons do not exist, and all TGQs are linear.

In fact, only the “locally finite part” of \( \mathcal{E} \) and the part for which \( X = (\Gamma, x) \) “is” a TGQ matter for the proof.
Note that the morphism $\eta : (\Gamma, x) \mapsto (\Gamma, x)$ generated a sequence

\[ \cdots \to (\Gamma, x) \xrightarrow{\eta} (\Gamma, x) \xrightarrow{\eta} \cdots \]

of which the limits in both directions (that is, $\bigcup_{t \in \mathbb{Z}} \Gamma^{\eta t}$ and $\bigcap_{t \in \mathbb{Z}} \Gamma^{\eta t}$) are stable under $\eta$.

4. Obstruction for TGQs

4.1. Setting. $\Gamma = \Gamma^x = (\Gamma^x, T)$ is a TGQ with translation point $x$ and translation group $T$. We suppose that the number of points (and so also the number of lines) is not finite. The kernel $K$ is defined as above, and we suppose, by way of contradiction, that $K$ is not a skew field.

4.2. The GQs $\Gamma(\alpha, z)$. Let $\alpha \in \mathbb{K}^*$, and $z$ an affine point. To avoid trivialities, suppose that $\alpha$ is not a unit. Let $(T, T^\alpha, T^x)$ be the Kantor family in $T$ defined by $z$. It is easy to see that $(T^\alpha, T^\alpha, (T^x)^\alpha)$ (obvious notation) defines a Kantor family in $T^\alpha$. Moreover, the GQ $\Gamma(\alpha, z)$ defined by this Kantor family is thick (it has at least three points per line), and it is ideal. One can derive these properties easily from Lemma 4.1 below.

Throughout this section, we will fix $z$, so that we write $\Gamma(\alpha)$ instead of $\Gamma(\alpha, z)$.

4.3. Injectivity. The proof of the next lemma is essentially the same as in the finite case (which can be found in [16, Chapter 8]). We include its proof for the sake of convenience.

Lemma 4.1. Each element of $\mathbb{K}$ is injective.

Proof. Suppose that $\beta \in \mathbb{K}$ is such that $\ell^\beta_0 = \text{id}$ for some $\ell_0 \in T_0 \setminus \{\text{id}\}$, $T_0 \in \{T_U \mid U \in I\}$; then we must show that $\beta = 0$. (The choice of $T_0$ is arbitrary. If $\beta$ has a fixed point not in $\bigcup_{V \in \{U\}} V$, then it has a fixed point in each $V \setminus \{\text{id}\}$ as well.) Assume the contrary. Choose any element $\ell_i \in T_i \setminus \{\text{id}\}$, with $T_i \neq T_0$. Then the point $\ell_0 \ell_i$ is at distance two from $\text{id}$ in the collinearity graph of $\Gamma^x$. Since $\Gamma^x$ is thick there exist elements $\ell_i \in T_i \setminus \{\text{id}\}$ and $\ell_k \in T_k \setminus \{\text{id}\}$, $\{T_i, T_k\} \cap \{T_0, T_1\} = \emptyset$ and $T_i \neq T_k$, such that

\[ (2) \quad \ell_0 \ell_i = \ell_i \ell_k. \]

Letting $\beta$ act yields $\ell^\beta_i = \ell^\beta_i \ell^\beta_k$. First suppose that $\ell^\beta_i = \text{id}$; then $\ell^\beta_k = \text{id}$. Since $T_i \cap T_k = \{\text{id}\}$ we obtain that $\ell^\beta_i = \text{id}$. Analogously $\ell^\beta_k = \text{id}$ implies that $\ell^\beta_i = \text{id}$. Next suppose that neither $\ell^\beta_i$ nor $\ell^\beta_k$ equals $\text{id}$. In this case the line $T_i \ell^\beta_i$ of $\Gamma^x$ intersects the line $T_k$ in $\ell^\beta_k \neq \text{id}$ and intersects the line $T_i$ in $\ell^\beta_i \neq \text{id}$. Hence we have found a triangle in $\Gamma^x$, a contradiction. We conclude that $\ell^\beta_i = \text{id}$, and henceforth that $V^\beta = \text{id}$, for all $V \in \{T_U \mid U \in I\}$. By the connectedness of $\Gamma^x$ we know that $T = \langle V \mid V \in \{T_U \mid U \in I\} \rangle$, and hence it follows that $T^\beta = \text{id}$, that is, $\beta = 0$. ■

Corollary 4.2. For each $\alpha \in \mathbb{K} \setminus \{0\}$, $\Gamma(\alpha) \cong \Gamma$. ■
4.4. Proof of Theorem 3.1 for TGQs. Suppose $\Gamma$ is a TGQ with endomorphism ring $K$. Suppose $K$ is not a skew field; then there exists a $\zeta \in K$ which is not invertible. By definition of $K$ and Corollary 4.2, it follows that $((\Gamma, x), \zeta)$ is an object in $\mathcal{E}$.

5. Obstruction for locally finite polygons

Let $\Gamma$ be a generalized polygon. An ordered set $\mathcal{L}$ of lines is indiscernible if for any two increasing sequences $M_1, M_2, \ldots, M_n$ and $M'_1, M'_2, \ldots, M'_n$ (of the same length $n$) of lines of $\mathcal{L}$, there is an automorphism of $\Gamma$ mapping $M_i$ onto $M'_i$ for each $i$. It is indiscernible over $D$, if $D$ is a finite set of points and lines fixed by the automorphisms just described.

By combining the Compactness Theorem and Ramsey’s Theorem [10] (in a theory which has a model in which a given definable set is infinite), one can prove the following.

Theorem 5.1 (G. Cherlin [5]). Suppose there is an infinite locally finite generalized $n$-gon with finite lines. Then there is an infinite locally finite generalized $n$-gon $\Gamma$ containing an indiscernible sequence $\mathcal{L}$ of parallel (mutually skew) lines, of any specified order type. The sequence may be taken to be indiscernible over the set $D$ of all points incident with one fixed line $L$ of $\Gamma$.

Clearly, $D$ may supposed to be fixed pointwise in Theorem 5.1.

Remark 5.2. (i) Cherlin states the theorem only for generalized quadrangles, but the same statement also holds for “general” generalized polygons.

(ii) In Theorem 5.1 $\Gamma$ may supposed to be generated by $\mathcal{L} \cup \{L\}$ [6] (where $L$ is seen as a point set).

5.1. Setting. Let $\mathcal{L}$ be a fixed indiscernible set of (mutually skew) lines of $\Gamma$ over $L$ — here $\Gamma$ is generated by $\mathcal{L} \cup \{L\}$ (recall Remark 5.2(ii)). We suppose that $\mathcal{L}$ is (infinitely) countable.

If $S \subseteq \mathcal{L}$, by $\Gamma(S)$ we denote the full subpolygon generated by $S \cup \{L\}$ (since $L$ is fixed throughout, we do not specify $L$ in the notation). So $\Gamma = \Gamma(\mathcal{L})$.

Call a sequence of lines $N_{i_1}, N_{i_2}, \ldots, N_{i_l}$ in $\mathcal{L}$ increasing if $i_l < i_{l'}$ for $l < l'$.

Lemma 5.3. If $S \subseteq \mathcal{L}$ is a subset of $\mathcal{L}$, then $\Gamma(S)$ is disjoint from any line of $\mathcal{L} \setminus S$.

Proof. Immediate from indiscernibility. $lacksquare$

Let $\text{Sub}(\Gamma)$ be the set of subGPs of $\Gamma$ (including thin subGPs).

Lemma 5.4. The map $\Psi : 2^{\mathcal{L}} \mapsto \text{Sub}(\Gamma) : S \mapsto \varphi(S) = \Gamma(S)$ is an injection.

Proof. Immediate. $lacksquare$

The following lemma is folklore, and easy to prove. (In its statement, “countable” also comprises the finite case.)

Lemma 5.5. Let $\Delta$ be a generalized polygon, and $K$ be a subset of points of countable size. Let $\Delta(K)$ be the subpolygon of $\Delta$ generated by $K$. Then the number of points and lines of $\Delta(K)$ is countable.
Corollary 5.6. We have that \( t \) is countable, as is the number of points and lines of \( \Gamma \).

Proof. Since \( \Gamma \) is generated by \( \{L\} \cup \mathcal{L} \), it is generated by a countable number of points. By Lemma 5.5, the number of points and lines is countable. Clearly \( t \) also is.

5.2. The topology \( (\text{Sym}(X), \tau) \). Let \( X \) be the point set of \( \Gamma \) and let \( \text{Sym}(X) \) be the symmetric group on \( X \). We endow \( \text{Sym}(X) \) with the topology \( \tau \) of pointwise convergence — a subset of \( \text{Sym}(X) \) is closed if and only if it is the automorphism group of some first order structure. So \( \text{Aut}(\Gamma) \) is closed in \( \tau \). Note that a subset of \( \text{Sym}(X) \) is open if and only if it contains the elementwise stabilizer of some finite subset. The closed sets \( \mathcal{C} \) are characterized by the following property:

\[
(\gamma) \quad \text{Let } g \in \text{Sym}(X). \text{ Then } g \in \mathcal{C} \text{ if and only if for every finite subset } A \subseteq X \text{ there is a } g_A \in \mathcal{C} \text{ that agrees with } g \text{ on } A.
\]

In particular, \((\gamma)\) applies to \( C = \text{Aut}(\Gamma) \).

5.3. Automorphisms preserving \( \mathcal{L} \). We index \( \mathcal{L} \) by \( \mathbb{Q} \): \( \mathcal{L} = \{M_i\}_{i \in \mathbb{Q}} \), and let \( \mathbb{Q} \) be endowed with the natural (linear, dense) order \( \leq \). By assumption, \( \mathcal{L} \) is indiscernible over \( L \) w.r.t. \((\mathbb{Q}, \leq)\). Put \( \text{Aut}(\Gamma)|_{\mathcal{L}} =: A \). Let \( \alpha \) be any order preserving permutation of the index set \( \mathbb{Q} \). Let \( \phi : \mathbb{N} \rightarrow \mathbb{Q} \) be a bijection, and define, for each \( i \in \mathbb{N} \):

\[
\mathcal{L}_i = \{M_{\phi(j)}|0 \leq j \leq i\}.
\]

So \( \mathcal{L} = \bigcup_i \mathcal{L}_i \) and \( \mathcal{L}_i \subseteq \mathcal{L}_j \) for \( i \leq j \). For each \( k \in \mathbb{N} \), let \( A_k \) be the set of elements of \( A \) which have the same action as \( \alpha \) on \( \mathcal{L}_k \); it is not an empty set by indiscernibility and finiteness of \( \mathcal{L}_k \).

The next lemma is obvious.

Lemma 5.7. Let \( \Delta \) be a GP, and \( Y \) a generating set (of points, say). If \( \alpha \in \text{Aut}(\Delta) \), its action on \( \Delta \) is completely determined by its action on \( Y \).

(If \( \alpha' \in \text{Aut}(\Delta) \) would have the same action on \( Y \), \( \alpha \alpha'^{-1} \) fixes \( Y \) pointwise, so also the subGP generated by \( Y \). But this is \( \Delta \).)

Since \( \{L\} \cup \mathcal{L} \) generates \( \Gamma \), it is (by the previous lemma) clear that if \( \bigcap_k A_k =: \overline{A} \neq \emptyset \), its size is precisely \( 1 \) — that is, there is a unique automorphism (denoted \( \chi \)) of \( \Gamma \) fixing \( L \) pointwise and inducing \( \alpha \) on \( \mathcal{L} \). This is exactly what (i) of the following lemma says.

Lemma 5.8. (i) \( \overline{A} \neq \emptyset \), and so \( \chi \in \text{Aut}(\Gamma) \) is well-defined.

(ii) \( \chi \) stabilizes \( \mathcal{L} \cup L \).

Proof. (i) Take any \( k \in \mathbb{N} \); then the elements of \( A_k \) agree on \( \Gamma(\mathcal{L}_k) \) (that is, if \( \gamma, \epsilon \) are in \( A_k \), then \( \gamma \epsilon^{-1} = 1 |_{\mathcal{L}_k} \)). Clearly, the elements of \( A_k \) also agree on \( \Gamma(\mathcal{L}_k) \) with the elements of \( A_k \) if \( k' > k \). Now define \( \chi \) (inductively) by passing to the limit \( k \mapsto \infty \). Since \( \Gamma = \bigcup_j \Gamma(\mathcal{L}_j) \), \( \chi \) is a well-defined element of \( \text{Sym}(X) \) (where \( X \) is as in \( \text{5.2} \)). Consider any finite subset \( R \) of \( X \). Then there is some \( \ell \in \mathbb{N} \) for which \( R \subseteq \Gamma(\mathcal{L}_\ell) \). Any element \( \beta \in A_\ell \) coincides with \( \chi \) on \( R \). So since \( \text{Aut}(\Gamma) \) is closed, \( \chi \in A \).

(ii) is immediate.
(Note that if, for each \( i \in \mathbb{N} \), \( \chi_i \in A_i \) is chosen arbitrarily, \((\chi_j)\), converges to \( \chi \) in \((\text{Sym}(X), \tau)\).)

**Theorem 5.9** (Determination of \( A_L \)).

(i) \( A_L \) contains a subgroup \( \mathfrak{O} \) isomorphic to the complete group of order preserving permutations of \((\mathbb{Q}, \leq)\) acting naturally on \( \mathbb{Q} \).

(ii) \( A_L \) acts \( k \)-homogeneously on \( L \) for any \( k \in \mathbb{N}^\times \).

**Proof.** By Lemma 5.7, the kernel of the action of \( A_L \) on \( L \) is trivial. Hence (i) follows from Lemma 5.8.

(ii) follows from (i).

**Remark 5.10.** (a) It is not hard to show that \((A_L, L)\) is in fact isomorphic to the complete group of order preserving permutations of \((\mathbb{Q}, \leq)\) acting naturally on \( \mathbb{Q} \) (if \( A_L \) would properly contain the order preserving permutations of \((\mathbb{Q}, \leq)\), one can easily show to arrive at a contradiction). In that case, we can add to (ii) that the action is not 2-transitive.

(b) By (a), \( A_L \) cannot contain involutions.

**Corollary 5.11.** \( \text{Aut}(\Gamma) \) is uncountable. As a direct consequence, \( \Gamma \) is infinitely generated.

5.4. **Proof of Theorem 3.1 for locally finite polygons.** Now let \( I = (a, b) \) be an open interval in \( \mathbb{Q} \) with \( a < 0 < b \), and let \( \zeta \in \mathfrak{O} \) be a homothecy with center \( 0 \) and factor \( f < 1 \). Then \( \Gamma(I) \leq \Gamma(I) \neq \Gamma(I)^\leq \), and \( L \) is fixed pointwise by \( \zeta \). So \(((\Gamma(I), L), \zeta)\) is an object in \( \mathcal{E} \).

This concludes the proof of Theorem 3.1.

**Remark 5.12.** (a) Using the stability property referred to in §3, the author has recently proved that for a TQ of order \((s, t)\) to be nonlinear, \(s\) and \(t\) must be equal, and the characteristic of its kernel then must be 0.

(b) Currently we are trying to develop the automorphic theory set up in §5.
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