Beyond Squeezing à la Virasoro Algebra

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Abstract

The generalization of squeezing is realized in terms of the Virasoro algebra. The higher-order squeezing can be introduced through the higher-order time-dependent potential, in which the standard squeezing operator is generalized to higher-order Virasoro operators. We give a formula that describes the number of particles generated by the higher-order squeezing when a parameter specifying the degree of squeezing is small. The formula \cite{15} shows that the higher the order of squeezing becomes the larger the number of generated particles grows.
1 Introduction

Squeezing has received great attention in various fields such as quantum optics [1], cosmology [2] and quantum information [3].

In particular, particle generation is an important issue; in the time-dependent oscillator, the number of produced particles is increased by repeated application of squeezing.

Accordingly, it is expected that a generalization of squeezing can increase the number of the generated particles more effectively.

Braunstein and McLachlan [5] generalized the parametric amplification by producing k-photon correlation, and numerically showed the structure of phase space. There, the usual quadratic squeezing was extended to the cases with cubic, quartic or higher-order interactions. Next, the statistics of heterodyne and homodyne detection in cubic and quartic interactions were examined by Braunstein and Caves [6]. These studies have shown that the features of higher-order interactions are well reflected in the structure of the phase space. As a review, see [4].

In this paper, we investigate the generalized squeezing from the viewpoint of the Virasoro algebra.

The Virasoro algebra is an algebra used in string theory and represents symmetry including scale transformation [7] which is called the conformal symmetry. The symmetries of parallel translation and scale transformation make up SL(2, C) as a subalgebra of the Virasoro algebra.

Squeezing is a transformation of phase space, which preserves the area of the phase space of position and momentum. By the squeezing transformation, the position is scaled up (down) and the momentum is scaled down (up) in the opposite direction. Therefore, squeezing can be considered as a kind of scale transformation combined with parallel translation to form subalgebra of Virasoro algebra. This is the standpoint from which we construct a theory in this study.

The paper is organized as follows. First, in the next section, we introduce the Virasoro algebra with conformal symmetries. Next, in section 3, we examine the usual second-order squeezing in light of time-dependent oscillators.

In section 4, we investigate the N-th order squeezing and show that this satisfies the Virasoro algebra. In section 5, we discuss the particle generation of the N-th order squeezing. In section 6, we discuss the uncertainty relation of the N-th order squeezing. We examine the structure of the phase space of N-th order squeezing in section 7. The final section is devoted to discussion.

2 Virasoro algebra

In this section, we introduce the Virasoro algebra and see that it contains a scale transformation. Virasoro algebra is an algebra generated by generators $L_n$ that satisfy the following algebraic relation,
\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}, \]
where, \( c \) is called the central charge and commutes with any \( L_n \), \([c, L_n] = 0\).

\( L_{-1}, L_0 \) and \( L_{-1} \) satisfies
\[ [L_{\pm 1}, L_0] = \pm L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0 \]
which is \( SL(2, \mathbb{C}) \) subalgebra of Virasoro Algebra.

When \( c = 0 \), this algebra reduces to
\[ [L_n, L_m] = (n - m)L_{n+m}, \]
which is called Witt algebra (centerless Virasoro algebra) \([8]\).

The generator of this algebra \( L_n \) can be constructed by \( z \) and its differential operator \( \partial \) as follows:
\[ L_n = z^{n+1}\partial. \]

Then, we can understand the specific geometrical meaning of these algebras.

\( L_0 = z\partial \)
generates a scale transformation,
\[ e^{\theta L_0} f(z) = f(e^{\theta} z). \]

Next,
\( L_{-1} = \partial \)
generates a parallel transformation,
\[ e^{\theta L_{-1}} f(z) = f(z + \theta). \]

Finally,
\( L_1 = z^2\partial \)
generates a special conformal transformation,
\[ e^{\theta L_1} f(z) = f \left( \frac{z}{1 - \theta z} \right). \]

Now, the Virasoro algebra (or Witt algebra) contains scale transformation and parallel translation in it.
3 Squeezing and time-dependent oscillators

In this section, we introduce the usual second-order squeezing, and in particular, show that the squeezed state can be obtained from time-dependent oscillators.

As is well known, in a system that describes harmonic oscillators, squeezed states $|\theta\rangle$ are constructed by acting on the states with the following operator,

$$e^{\theta \hat{F}}, \quad \hat{F} = \frac{\hat{a}^2 - \hat{a}^{12}}{2},$$

where $\theta$ is a squeezing parameter, and $\hat{a}$ and $\hat{a}^\dagger$ are a creation and annihilation operator of the harmonic oscillator. The position and momentum operator are constructed as follows from $\hat{a}^\dagger$ and $\hat{a}$,

$$\left\{ \begin{array}{l}
\hat{x} = \sqrt{\frac{1}{2\omega_0}} (\hat{a} + \hat{a}^\dagger), \\
\hat{p} = i \sqrt{\frac{\omega_0}{2}} (\hat{a} - \hat{a}^\dagger). 
\end{array} \right.$$ 

We note that the generator $\hat{F}$ is rewritten into the following symmetric form using $\hat{x}$ and $\hat{p}$,

$$\hat{F} = i \frac{1}{2} (\hat{x}\hat{p} + \hat{p}\hat{x}).$$

Therefore, $\hat{x}$ and $\hat{p}$ are scaled by $\hat{F}$ as follows,

$$e^{\theta \hat{F}} \hat{x} e^{-\theta \hat{F}} = e^{\theta} \hat{x}, \quad e^{\theta \hat{F}} \hat{p} e^{-\theta \hat{F}} = e^{-\theta} \hat{p}.$$ 

The scaling factor of $\hat{x}$ and $\hat{p}$ are inverse with one another because $\hat{F}$ is symmetric while the commutation relation antisymmetric i.e. $[\hat{x}, \hat{p}] = -[\hat{p}, \hat{x}]$.

These relation reduce that the squeezed state preserves the minimum uncertainty relation,

$$\langle \theta | \Delta \hat{x}^2 | \theta \rangle \langle \theta | \Delta \hat{p}^2 | \theta \rangle = \frac{1}{4} \left| \langle \theta | [\hat{x}, \hat{p}] | \theta \rangle \right|^2 = \frac{1}{4},$$

where

$$\left\{ \begin{array}{l}
\Delta \hat{x} = \hat{x} - \langle \theta | \hat{x} | \theta \rangle = \hat{x} - e^{\theta} \langle 0 | \hat{x} | 0 \rangle, \\
\Delta \hat{p} = \hat{p} - \langle \theta | \hat{p} | \theta \rangle = \hat{p} - e^{-\theta} \langle 0 | \hat{p} | 0 \rangle, \\
\langle \theta | \Delta \hat{x}^2 | \theta \rangle = e^{2\theta} \left( \langle 0 | \hat{x}^2 | 0 \rangle - \langle 0 | \hat{x} | 0 \rangle^2 \right) = \frac{1}{2} e^{2\theta}, \\
\langle \theta | \Delta \hat{p}^2 | \theta \rangle = e^{-2\theta} \left( \langle 0 | \hat{p}^2 | 0 \rangle - \langle 0 | \hat{p} | 0 \rangle^2 \right) = \frac{1}{2} e^{-2\theta}.
\end{array} \right.$$ 

The squeezing operator can be rotated to any direction, namely,
\[ \bar{F}(\phi) = \frac{i}{2} (\hat{x}(\phi)\hat{p}(\phi) + \hat{p}(\phi)\hat{x}(\phi)) , \]

where

\[ \begin{align*}
\hat{x}(\phi) &= \hat{x}\cos\phi - \hat{p}\sin\phi, \\
\hat{p}(\phi) &= \hat{p}\cos\phi + \hat{x}\sin\phi.
\end{align*} \]

The time-dependent oscillators are described by,

\[ \hat{H} = \hat{H}_0 + \frac{1}{2}\omega(t)\hat{x}^2, \]

where

\[ \hat{H}_0 = \frac{\hat{p}^2}{2} + \frac{1}{2}\omega_0\hat{x}^2. \]

In the interaction picture, the Hamiltonian can be written as,

\[ \hat{H}_I = \frac{1}{2}\omega(t)\hat{x}_I^2, \]

where \[ \hat{x}_I = e^{i\hat{H}_0 t}\hat{x}e^{-i\hat{H}_0 t} = \hat{x}(\omega_0 t). \] The time evolution of the state can be described as follows,

\[ |0(t)\rangle = \hat{U}(t,0)|0\rangle, \quad \hat{U}(t,0) = Te^{\int_0^t dt\frac{1}{2}\omega(t)\hat{x}_I^2}, \]

where \( T \) is the time ordering operator.

If we divide the time evolution operator into the product of those in a small time lapse and write,

\[ \hat{U}(t,0)|0\rangle = \hat{U}(t,n\Delta t)\cdots\hat{U}(\Delta t,0)|0\rangle, \]

we can write the infinitesimal time evolution operator \( \hat{U}(t + \Delta t, t) \) becomes,

\[ \hat{U}((n+1)\Delta t, n\Delta t) = Te^{\left[-\frac{i}{2} \alpha(t)\hat{x}_I^2(t)\right]_{n\Delta t}^{(n+1)\Delta t} - \frac{i}{2} \int_{n\Delta t}^{(n+1)\Delta t} \alpha(t)\left(\hat{x}_I(t)\hat{p}_I(t) + \hat{p}_I(t)\hat{x}_I(t)\right) dt}, \]

where \( \alpha \) is given by \( \dot{\alpha}(t) = \omega(t) \). If \( \alpha(t) = \sum_n \delta(t - (n + \frac{1}{2})\Delta t) \), we obtain

\[ \hat{U}(t,0)|0\rangle = e^{\theta_n \bar{F}_n} \cdots e^{\theta_0 \bar{F}_0}|0\rangle, \]

where \( \theta_n = 2(n + \frac{1}{2})\Delta t \), \( \bar{F}_n = \bar{F}(2(n + \frac{1}{2})\Delta t) \). Then, in time-dependent oscillators, the time evolution of the state can be described as the repetition of the squeezing operations.

The above description is crucial in the generalization of the squeezing phenomenon in the following sections.

### 4 N-th order squeezing

In this section, N-th order squeezed state is represented by the Virasoro algebra, i.e. Witt algebra, as the time dependent anharmonic oscillator.
4.1 N-th order squeezing

We can construct the Virasoro algebra from the position and momentum operators in quantum mechanics\(^1\)
\[
\hat{L}_n \equiv -\frac{i}{2} (\hat{x}^{n+1} \hat{p} + \hat{p} \hat{x}^{n+1}).
\] (1)

\(\hat{L}_n\) satisfies
\[
[\hat{L}_n, \hat{L}_m] = (n - m) \hat{L}_{n+m}.
\]

This is the centerless Virasoro algebra (Witt algebra)\(^2\). Thus, \(\hat{L}_n\) represents a conformal transformation in \(x\) space. We note here that \(\hat{L}_0\) is a generator of the usual second-order squeezing transformation.

Similarly, the dual operator of \(\hat{L}_n\),
\[
\hat{\tilde{L}}_n \equiv \frac{i}{2} (\hat{p}^{n+1} \hat{x} + \hat{x} \hat{p}^{n+1})
\] (2)
also satisfies the Virasoro algebra,
\[
[\hat{\tilde{L}}_n, \hat{\tilde{L}}_m] = (n - m) \hat{\tilde{L}}_{n+m}.
\]

This shows that \(\hat{\tilde{L}}_n\) represents a conformal transformation in \(p\) space. If we express \(\hat{L}_n\) as a harmonic oscillator, it has the following form,
\[
\hat{L}_n = \frac{i}{2} \left( \frac{-i}{\sqrt{\omega_0}} \right) \left( \sqrt{\frac{1}{2\omega_0}} \right)^n (\hat{a} + \hat{a}^\dagger)^n(\hat{a} - \hat{a}^\dagger)^n .
\]

We call the state, obtained by applying the unitary operator \(\hat{L}_n\) to the vacuum, an “(Virasoro) N-th order squeezed state” and write it as follows,
\[
|\theta\rangle_n \equiv e^{\theta \hat{L}_n} |0\rangle.
\] (3)

Let us examine how \(\hat{x}\) and \(\hat{p}\) are transformed by the N-th order squeezing. First,
\[
[\hat{L}_n, \hat{x}] = -\hat{x}^{n+1}
\]
leads to the transformation of \(x\)
\[
S_n \hat{x} S_n^\dagger = \sum_{k=0}^{\infty} A_k (\theta \hat{x}^n)^k \hat{x},
\] (4)
\[
\begin{cases}
A_0 = 1, \\
A_1 = -1, \\
A_k = (-1)^k \frac{\prod_{j=1}^{k-1} (jn + 1)}{k!}, \quad k > 1,
\end{cases}
\]

\(1\)We take \(\hbar = 1\). Here we have tentatively assumed the operator ordering in defining (1). The more detailed analysis may give a non-vanishing central charge, but whether the analysis restores the violation of the minimum uncertainty relation for the N-th order squeezed state or not, is not clear at this moment. (See the Discussion.)

\(2\)Here the calculation has been performed using a position representation, \(p = -i \frac{\partial}{\partial x}\). This calculation can be extended to \(n \in \mathbb{C}\).
where $\hat{S}_n = e^{\theta \hat{L}_n}$. The sum can be estimated as
\[
S_n \hat{x} \hat{S}_n^\dagger = (1 + n(\theta \hat{x}^n))^{-\frac{1}{2}} \hat{x}.
\]  
(5)

If $n = 0$, this equation is reduced to the usual second-order squeezing,
\[
\hat{S}_0 \hat{x} \hat{S}_0^\dagger = e^{-\theta \hat{x}}.
\]

Next,
\[
[\hat{L}_n, \hat{p}] = i(1 + n) \hat{L}_{n-1},
\]
leads to the transformation of $\hat{p}$
\[
\hat{S}_n \hat{p} \hat{S}_n^\dagger = \sum_{k=0}^{\infty} B_k \theta^n \hat{L}_{kn-1},
\]
\[
\begin{align*}
B_0 &= 1, \\
B_1 &= (1 + n), \\
B_k &= \prod_{j=1}^{k} (1 - (j - 2)n) / k!.
\end{align*}
\]

This sum can also be calculated as
\[
\hat{S}_n \hat{p} \hat{S}_n^\dagger = (1 + n(\theta \hat{x}^n))^{\frac{1}{2} + 1} \hat{p} + \hat{p}(1 + n(\theta \hat{x}^n))^{\frac{1}{2} + 1}.
\]
(6)

If we take the commutative limit, that is, $\hat{x}$ and $\hat{p}$ commute in (7), these equations yield
\[
\begin{align*}
\hat{S}_n \hat{x} \hat{S}_n^\dagger &= (1 + n(\theta \hat{x}^n))^{-\frac{1}{2}} \hat{x}, \\
\hat{S}_n \hat{p} \hat{S}_n^\dagger &= (1 + n(\theta \hat{x}^n))^{\frac{1}{2} + 1} \hat{p}.
\end{align*}
\]
(8)

In this limit, the phase space volume is multiplied by $(1 + n(\theta \hat{x}^n))$ by $N$-th order squeezing.

Using the $\hat{x}$ and $\hat{p}$ operators rotated by angle $\phi$,
\[
\begin{align*}
\hat{x}(\phi) &= \hat{x} \cos \phi - \hat{p} \sin \phi, \\
\hat{p}(\phi) &= \hat{x} \sin \phi + \hat{p} \cos \phi,
\end{align*}
\]
the generalization of $\hat{L}_n$ in any direction, as in the usual second-order squeezing, is naturally defined by
\[
\hat{L}(\phi)_n = -\frac{i}{2} \left( \hat{x}^{n+1}(\phi) \hat{p}(\phi) + \hat{p}(\phi) \hat{x}^{n+1}(\phi) \right).
\]
(9)

Then, $\hat{L}(\phi)$ can be expanded in terms of $\hat{x}$ and $\hat{p}$ as follows:
\[
\hat{L}(\phi)_n = \sum_{n,m} A_{n,m} \hat{L}_{n,m},
\]
(10)

\[
\hat{L}_{n,m} = -\frac{i}{2} \left( \hat{x}^{m+1} \hat{p}^{n+1} + \hat{p}^{m+1} \hat{x}^{n+1} \right),
\]
(11)
where $A_{n,m}$ is some appropriate factor. All $\hat{L}(\phi)$ can be expanded as the linear combination of the $\hat{L}_{n,m}$, namely, the algebra generated by $\hat{L}_{n,m}$ is a generalization of the Virasoro algebra and is called $w_{\infty}$ algebra [9].

4.2 Time-dependent anharmonic oscillators

The time-dependent anharmonic oscillators are given by,

$$\hat{H} = \hat{H}_0 + \frac{1}{2} \lambda(t) \hat{x}^{n+2},$$

$$\hat{H}_0 = \frac{\hat{p}^2}{2} + \frac{1}{2} \omega_0 \hat{x}^2.$$

In the interaction picture, the Hamiltonian reads,

$$\hat{H}_I = \frac{1}{2} \omega(t) \hat{x}^{n+2}.$$  

In the same way as in the squeezing argument, the time evolution of the state can be written as follows,

$$|0(t)\rangle = \hat{U}(t,0)|0\rangle = e^{\theta_{n}\hat{L}_n(n)} \cdots e^{\theta_{0}\hat{L}_n(0)}|0\rangle,$$  

where $\theta_m = 2(m + \frac{1}{2})\Delta t$, $\hat{L}_n(m) = \hat{L}_n(2(m + \frac{1}{2})\Delta t)$.

Then, in time-dependent oscillators, the time evolution of the state can be obtained by the successive application of the N-th squeezing operators.

5 Particle production

Here we calculate the number of particle generated by N-th order squeezing. The expected number of particles in the N-th order squeezed state $|\theta\rangle$ is written as

$$n(\theta) = \langle 0|e^{-\theta\hat{L}_n}\hat{N}e^{\theta\hat{L}_n}|0\rangle = \langle 0|\hat{a}_\theta^\dagger \hat{a}_\theta|0\rangle,$$
where $\hat{a}_\theta$ and $\hat{a}_\theta^\dagger$ are

\[
\begin{align*}
\hat{a}_\theta &= e^{-\theta L_n} \hat{a} e^{\theta L_n}, \\
\hat{a}_\theta^\dagger &= e^{-\theta L_n} \hat{a}^\dagger e^{\theta L_n},
\end{align*}
\]

\[
\begin{align*}
\hat{a}_\theta &= \frac{1}{2} \hat{K}(n, \hat{x}) \left( (\cosh \hat{\Omega}(n, \hat{x})) \hat{a} + (\sinh \hat{\Omega}(n, \hat{x})) \hat{a}^\dagger \right) \\
&\quad + \frac{1}{2} \left( a \cosh \hat{\Omega}(n, \hat{x}) + \hat{a}^\dagger \sinh \hat{\Omega}(n, \hat{x}) \right) \hat{K}(n, \hat{x}), \\
\hat{a}_\theta^\dagger &= e^{-\theta L_n} \hat{a}^\dagger e^{\theta L_n}, \\
\hat{a}_\theta^\dagger &= \frac{1}{2} \hat{K}(n, \hat{x}) \left( (\cosh \hat{\Omega}(n, \hat{x})) \hat{a}^\dagger + (\sinh \hat{\Omega}(n, \hat{x})) \hat{a} \right) \\
&\quad + \frac{1}{2} \left( \hat{a}^\dagger \cosh \hat{\Omega}(n, \hat{x}) + \hat{a} \sinh \hat{\Omega}(n, \hat{x}) \right) \hat{K}(n, \hat{x}).
\end{align*}
\]

Here,

\[
\hat{\Omega}(n, \hat{x}) = \log(1 + n \theta \hat{x}^n) - \frac{1}{n - 1} \frac{1}{2}, \quad \hat{K}(n, \hat{x}) = (1 + n \theta \hat{x}^n)^{\frac{1}{2}}.
\]

These are a generalization of the Bogolyubov transformation. Using these equations, we obtain

\[
\langle 0 | \hat{a}^\dagger \hat{a} \theta | 0 \rangle = \langle 0 | (\sinh \log(1 + n(\theta \hat{x}^n)) - \frac{1}{n} \frac{1}{2})^2 \hat{a} \hat{a}^\dagger | 0 \rangle.
\]

As a result,

\[
n \langle \theta | \hat{N} | \theta \rangle_n \text{ is given by}
\]

\[
n \langle \theta | \hat{N} | \theta \rangle_n = A_0 \int dx e^{-x^2} \left( \sinh \log(1 + n(\theta \hat{x}^n)) - \frac{1}{n} \frac{1}{2} \right)^2, \]

where $A_0 = \left( \frac{\omega_0}{2} \right)^{1/4}$. If $n = 0$, $\langle \theta | \hat{N} | \theta \rangle_n$ reproduces the result of the usual second-order squeezing,

\[
\langle \theta | \hat{N} | \theta \rangle_0 = (\sinh \theta)^2.
\]

In the case of $n \neq 0$, if $\theta$ is small, we can expand $n \langle \theta | \hat{N} | \theta \rangle_n$ in $\theta$ and the following equation is obtained,

\[
n \langle \theta | \hat{N} | \theta \rangle_n \sim \frac{1}{4} \theta^2 (n + 2)^2 \Gamma \left( n + \frac{1}{2} \right) \\
- \frac{1}{8} \theta^3 ((-1)^n + 1) n(n + 2)^2 \Gamma \left( \frac{3n}{2} + 1 \right) + O(\theta^4).
\]

If the potential is $x^4$, namely for $n = 2$, $2 \langle \theta | \hat{N} | \theta \rangle_2$ reads

\[
2 \langle \theta | \hat{N} | \theta \rangle_2 = \frac{\sqrt{2 \pi e} e^\theta (\theta - 1) \text{erfc} \left( \frac{1}{\sqrt{2} \sqrt{\pi}} \right) + 2 \sqrt{\pi} \sqrt{\theta} (2 \theta + 1)(3 \theta - 1) + 1}{16 \theta^{3/2}},
\]

where $\text{erfc}(x)$ is called the complementary error function, given by
Figure 1: Particle number generation of N-th order squeezing. The horizontal axis represents squeezing parameter $\theta$ and the vertical axis represents particle number. The line of $n = 0$ cross that of $n = 2$ at $\theta = 1.39251$ and cross that of $n = 4$ at $\theta = 2.28158$.

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} \, dt = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!(2x)^{2n}}.$$  

If the potential is $x^6$, $n = 4$, $\langle \theta | \hat{N} | \theta \rangle_4$ becomes

$$\langle \theta | \hat{N} | \theta \rangle_4 = \frac{\sqrt{\pi}}{64\theta} 
\left(-10\pi\theta^{3/2} J_{-\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)^2
- 2\sqrt{2}\pi\theta^{1/2} J_{-\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right) \left(-30\theta^2 - 3\theta + 1\right) J_{\frac{1}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)\right)
- 2\sqrt{2}\pi\theta^{1/2} J_{-\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right) \left(1 - 15\theta\right) \sqrt{\theta} Y_{-\frac{1}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)
- 32\theta + 2\pi\theta \sqrt{\theta} (15\theta + 1) \left(J_{\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)\right)^2
+ 2\pi\theta \sqrt{\theta} (15\theta + 1) \left(J_{\frac{1}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)\right)^2
+ 12\pi\theta (5\theta + 1) J_{-\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right) J_{\frac{1}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)
+ \sqrt{2}\pi J_{-\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right) \left(2\sqrt{\theta} (15\theta + 1) J_{\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right) - Y_{-\frac{1}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)\right)
- \sqrt{2}\pi J_{-\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right) \left(2(1 - 15\theta)\theta J_{\frac{3}{4}} \left(\frac{1}{4\sqrt{\theta}}\right) - Y_{\frac{1}{4}} \left(\frac{1}{4\sqrt{\theta}}\right)\right)\right), \quad (20)$$

where $J_{a}$ and $Y_{a}$ are Bessel functions of the first and second kind.

The number of particles produced in quantum mechanics with having the time dependent $x^{n+2}$ potential for $n = 0, 2, 4$ is plotted in fig.1

This shows that for small $\theta$, the number of generated particles increase as the order of squeezing becomes higher.

10
The uncertainty relation of higher order

The usual second-order squeezed state satisfied the minimum uncertainty relation. We calculate the uncertainty relation in the case of the N-th order squeezing.

First, to determine \( n \langle \theta | \Delta x^2 | \theta \rangle = n(\theta | x^2 | \theta) - n(\theta | x | \theta)^2 \), we note

\[
\langle x_\theta | \theta \rangle = \langle x_\theta | x^2 \rangle \theta
\]

is an eigenstate of \( \hat{x} \) because of

\[
\langle x_\theta | \hat{x} \rangle = \langle x_\theta | e^{\theta \hat{L}} \rangle \hat{x} = \langle x_\theta | (1 + n(\theta \hat{x}^n))^{-\frac{n}{2}} \hat{x} e^{\theta \hat{L}} \rangle = \langle x_\theta | (1 + n(-\theta \hat{x}^n))^{-\frac{n}{2}} \hat{x}. \]

Then, we obtain

\[
\langle x_\theta \rangle n = \langle x_\theta | 0 \rangle = A_0 e^{-\frac{1}{2} \omega_0 (1 + n(\theta x^n))^{-\frac{n}{2}} x^2.} \] (22)

Using this vacuum state, we can calculate the uncertainty relation perturbatively in \( \theta \).

As a result, if \( n \) is even, we get

\[
n \langle \theta | \Delta \hat{p}^2 | \theta \rangle = |A_0|^2 \left( \frac{\sqrt{\pi}}{2 \omega_0^{3/2}} + 2 \Gamma \left( \frac{5}{2} + n \right) \omega_0^{-\frac{n}{2}} \theta + (3 + n) \Gamma \left( \frac{5}{2} + n \right) \omega_0^{-2-n \theta^2} \right) + O(\theta^3), \] (23)

while if \( n \) is odd, we have

\[
n \langle \theta | \Delta \hat{p}^2 | \theta \rangle = |A_0|^2 \left( \frac{\sqrt{\pi}}{2 \omega_0^{3/2}} + (3 + n) \Gamma \left( \frac{5}{2} + n \right) \omega_0^{-2-n \theta^2} \right) - 4 |A_0|^4 \Gamma(2 + \frac{n}{2})^2 \omega_0^{-2-n \theta^2} + O(\theta^3). \] (24)

In estimating \( n \langle \theta | \Delta \hat{p}^2 | \theta \rangle \), we will make use of \( \langle x_\theta \rangle_n \) and \( n \langle \theta | p | \theta \rangle_n = \int dx_n(x) x \frac{\partial}{\partial x} \langle x_\theta | \theta \rangle_n \).

As a result, if \( n \) is even, we get

\[
n \langle \theta | \Delta \hat{p}^2 | \theta \rangle = |A_0|^2 \left( \frac{\sqrt{\pi} \sqrt{\omega_0}}{2} + (1 + n) \Gamma \left( \frac{3}{2} + n \right) \omega_0^{\frac{n}{2}} \right),
\]

\[
\quad + |A_0|^2 \left( \frac{1}{2} (1 + n) \Gamma(3 + n) \omega_0^{\frac{n}{2} - n \theta^2} \right) + O(\theta^3), \] (25)

and if \( n \) is odd, we have

\[
n \langle \theta | \Delta \hat{p}^2 | \theta \rangle = |A_0|^2 \left( \frac{\sqrt{\pi} \sqrt{\omega_0}}{2} + \frac{1}{2} (1 + n) \Gamma(3 + n) \omega_0^{\frac{n}{2} - n \theta^2} \right) + O(\theta^3). \] (26)
From \( n(\theta|\Delta x^2|\theta)_n \) and \( n(\theta|\Delta p^2|\theta)_n \), the uncertainty relation of N-th order squeezing is given as follows. If \( n \) is even,

\[
\begin{align*}
    n(\theta|\Delta x^2|\theta)_n n(\theta|\Delta p^2|\theta)_n &= \frac{1}{4} - \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{n + 3}{2} \right) \omega_0 - \frac{\theta}{2} \\
    &+ \frac{1}{2\sqrt{\pi}} (n^2 + 5n + 5) \Gamma \left( \frac{n + 3}{2} \right) \omega_0 ^{-n} \theta ^2 \\
    &- \frac{2}{\pi} (n + 1) \Gamma \left( \frac{n + 3}{2} \right) \Gamma \left( \frac{n + 5}{2} \right) \omega_0 ^{-n} \theta ^2 + O(\theta^3). 
\end{align*}
\]

(27)

If \( n \) is odd,

\[
\begin{align*}
    n(\theta|\Delta x^2|\theta)_n n(\theta|\Delta p^2|\theta)_n &= \frac{1}{4} \\
    &- \frac{2}{\pi} \Gamma \left( \frac{n + 4}{2} \right) \omega_0 ^{-n} \theta ^2 \\
    &+ \frac{1}{2\sqrt{\pi}} (n^2 + 5n + 5) \Gamma \left( \frac{n + 3}{2} \right) \omega_0 ^{-n} \theta ^2 + O(\theta^3). 
\end{align*}
\]

(28)

As an example, if we take \( n = 2 \), the uncertainty relation is

\[
\begin{align*}
    2(\theta|\Delta x^2|\theta)_2 2(\theta|\Delta p^2|\theta)_2 &= \frac{1}{4} - \frac{3}{4} \frac{\theta}{\omega_0} + \frac{75}{8} \frac{\theta^2}{\omega_0^2} + O(\theta^3). 
\end{align*}
\]

(29)

The above results indicate that the minimum uncertainty relation is broken at the low order of perturbation in \( \theta \) in the N-th order squeezing.

### 7 Phase space of the higher order

We introduce a Husimi function \([10]\) to determine the structure of phase space. The Husimi function is defined by coherent representation,

\[ H(\alpha) = \langle \alpha|\hat{\rho}|\alpha \rangle, \]

where

\[ |\alpha \rangle = A_\alpha e^{\alpha \hat{a}} |0 \rangle, A_\alpha = e^{-|\alpha|^2}. \]

For the vacuum state, \( \hat{\rho} = |0\rangle \langle 0 | \), the Husimi function \( H_0(\alpha) \) is

\[ H_0(\alpha) = e^{-|\alpha|^2}. \]

To get the N-th order squeezed Husimi function \( H_{\theta_n}(\alpha) = \langle \tilde{\alpha}|\theta_n \rangle \langle \theta_n |\alpha \rangle \), we note

\[
\begin{align*}
    n(\theta|\alpha) &= \int \frac{d\beta d\tilde{\beta}}{\pi} e^{-\theta \mathcal{L}_n (0|\beta) \langle \tilde{\beta}| \alpha \rangle} e^{-\theta \mathcal{L}_n (0|\alpha) \langle \alpha| 0 \rangle}, \\
    &\end{align*}
\]

\[(30)\]
\[ L_n = \frac{1}{2} \left( \sqrt{\frac{\omega_0}{2}} \right)^{n+1} \left( (\alpha + \frac{\partial}{\partial \alpha} + \frac{\bar{\alpha}}{2})^{n+1} + (\alpha - \frac{\partial}{\partial \alpha} - \frac{\bar{\alpha}}{2}) (\alpha + \frac{\partial}{\partial \alpha} + \frac{\bar{\alpha}}{2})^{n+1} \right) \]

where we have used the following formula,

\[ \langle \beta | f(a, a^\dagger) | \alpha \rangle = f(\alpha, \frac{\partial}{\partial \alpha} + \frac{\bar{\alpha}}{2}) \langle \beta | \alpha \rangle. \]

Because \[ \langle 0 | \alpha \rangle = e^{-|\alpha|^2/2}, \]

the term \[ \frac{\partial}{\partial \alpha} + \frac{\bar{\alpha}}{2} \] vanishes and \[ n \langle \theta | \alpha \rangle \]

becomes

\[ n \langle \theta | \alpha \rangle = e^{-\theta L_n} e^{-|\alpha|^2/2}, \]

(31)

\[ L_n' = \sqrt{\frac{\omega_0}{2}} \left( \sqrt{\frac{1}{2\omega_0}} \right)^{n+1} e^{n+2}. \]

Therefore, the N-th order squeezed Husimi function is given by

\[ H_{\theta_n}(\alpha) = e^{-\theta \sqrt{\frac{\omega_0}{2}}} \left( \sqrt{\frac{1}{2\omega_0}} \right)^{n+1} (\alpha^{n+2} + \bar{\alpha}^{n+2}) e^{-|\alpha|^2/2}. \]

(32)

As a final result, the contour lines of phase space are depicted in fig. 2.

From the figure, we have found that the N-th order squeezing is narrowed in proportion to \((2 + n)\)-th square root.

8 Discussion

As the beyond squeezing effect, we have proposed the N-th order squeezing based on the Virasoro algebra, in which the N-th order squeezing is induced by the N-th order time-dependent potential. That is, the usual second-order squeezing operator is generalized to the higher-order Virasoro operators. The \( \hat{x} \) and \( \hat{p} \) are subject to local scale deformation in the opposite direction also in N-th order squeezing.

We have obtained a formula of the particle number produced by the N-th order squeezing, when the squeezing parameter \( \theta \) is small. The formula implies that the higher the order of the squeezing is, the larger the number of generated particle is.

It is interesting to note that the extension of N-th order squeezing in any direction is related to the \( W_\infty \)-algebra. The algebra of the transformation that preserves the area of phase space is called \( w_\infty \)-algebra, and the \( W_\infty \)-algebra is \footnote{This is called \( \text{sdiff}(\Sigma) \) where \( \Sigma \) is phase space of \((x, p)\).}
Figure 2: Behavior of Husimi function (32) in phase space for $n = 1, \ldots, 4$, where $n$ characterizes the power of the quantum-mechanical potential $x^{2+n}$. The squeezing parameter $\theta$ is set to $\theta = 0.5$, and the characteristic frequency $\omega_0$ of the free theory is set to $\omega_0 = 0.01$. The horizontal axis represents position $x$ and the vertical axis represents momentum $p$. 

(a) $n = 1$  

(b) $n = 2$  

(c) $n = 3$  

(d) $n = 4$
its the quantized version in which the area conservation is not always preserved. These algebras represent deep connections with integrable systems \[11\], and the relationship between Nth-order squeezing states and integrable systems need to be further studied.

In the context of string theory, a similar Virasoro algebra and \( W_{\infty} \)-algebra with \( x \) and \( p \) in the \( AdS_2/CFT_1 \) correspondence appears on the \( CFT_1 \) side \[12\]. Particles near the horizon of an extremal Reissner-Nordstrom black hole is described by conformal mechanics \[13\]. In \( AdS_2/CFT_1 \), the algebra of symmetry is the Virasoro algebra, and it is possible that understanding of the relationship between the Virasoro algebra in \( AdS_2/CFT_1 \) and that in the N-th order squeezing can elucidate the role of N-th order squeezing in black holes.

In the theory of reheating after inflation, the effect of squeezing by parametric resonance has been attracting attention in connection with the problem of baryogenesis \[14\], and it may be applied as an effective model for the N-th order squeezing.

The N-th order squeezing obtained in this discussion can be generalized to the two-mode Bogolyubov transformation or fermionic version. It is interesting to investigate the physics, which might bring the extensions of well-known theories such as Bardeen-Cooper-Schrieffer theory \[15\]. These studies will be the subject in the near future.

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