Identification of Finite Dimensional Linear Systems Driven by Lévy processes

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Lévy processes are widely used in financial mathematics, telecommunication, economics, queueing theory and natural sciences for modelling. A typical model is obtained by considering finite dimensional linear stochastic SISO systems driven by a Lévy process. In this paper we consider a discrete-time version of this model driven by the increments of a Lévy process, such a system will be called Lévy system. We focus on the problem of identifying the dynamics and the noise characteristics of such a Lévy system. The special feature of this problem is that the statistical description of the noise is given by the characteristic function (c.f.) of the driving noise not by its density function. As an alternative to the maximum likelihood (ML) method we develop and analyze a novel identification method by adapting the so-called empirical characteristic function method (ECF) originally devised for estimating parameters of c.f.-s from i.i.d. samples. Precise characterization of the errors of these estimators will be given, and their asymptotic covariance matrices will be obtained. We also demonstrate that the arguments implying asymptotic efficiency for the i.i.d. case can be adapted for the present case.

1 Introduction

Lévy processes are widely used to model phenomena arising in natural sciences, economics, financial mathematics, queueing theory and telecommunication. The classical model for modelling market dynamics, namely geometric Brownian motion, was proposed by Louis Bachelier. This model is still the accepted core model despite the fact that empirical studies revealed that its assumptions are not realistic. For example, since price movements are induced by transactions which can be unevenly distributed in real time, it would be more natural to use a time changed Brownian motion to model price dynamics. If the time change is defined by a gamma process, we obtain the so-called VG (shorthand for Variance Gamma) process. VG processes reproduce a number of stylized facts of real price processes, such as fat tails and large kurtosis. It can be shown that the above time changed Brownian process itself is a Lévy process. Extending the above construction novel price dynamics have been proposed by a variety of authors, called the geometric Lévy processes obtained by exponentiating a Lévy process.
The objective of this paper is to present a combination of advanced techniques in systems identification with a specific statistical technique, widely used in the context in finance, called the ECF (shorthand for empirical characteristic function) method. The ECF method was originally designed for i.i.d. samples and A. Feuerverger and P. McDunnogh [15] showed that it can be interpreted as the Fourier transform of an ML method. Several papers study the problem of identifying the noise characteristics of a linear system, but only a few pays attention to the problem of identifying the system parameters as well. Brockwell and Schlemm [27] consider the parametric estimation of the driving Lévy process of a multivariate continuous-time ARMA processes, but the identification of system parameters is out of the scope of their paper. Calder and Davis [23] discuss the M-estimators of ARMA processes with a given distribution on the noise process. The quasi-maximum likelihood estimation of multivariate Lévy-driven continuous-time ARMA processes is studied by Schlemm and Stelzer in [28], the method presented there identify the system parameters and the covariance structure of the noise process, but further characteristics of the driving noise are not estimated.

In this paper we present a three-stage identification method for single-input-single-output (SISO) that estimates both the system parameters and the characteristics of the noise process. We give the precise characterization of the estimation error as well. We adapt the ECF method for linear systems and demonstrate that our method can outperform standard system identification methods such as prediction error method or quasi maximum likelihood estimation method and that it is asymptotically efficient. In [32] the same problem is tackled. Two methods are proposed, the first is a two-step method that combines the prediction error method and the empirical characteristic function method for i.i.d. data, the second estimates the the system parameters and the noise parameters simultaneously. It is proved that the second method may estimate the system parameters in a more efficient way than the first one, still it does not give an efficient estimator. Moreover, the method presented in [32] is applicable only if the driving noise is a zero mean process.

A Lévy process \( (Z_t) \), \( t \in \mathbb{R} \) is a continuous-time stochastic process, which is much like a Wiener process: it is a stochastic process with stationary an independent increments, but discontinuities or jumps are allowed. For an excellent introduction to the theory of Lévy processes see [4].

A key building block in the theory of Lévy processes is the compound Poisson process, which is a Poisson process with random, independent and identically distributed jumps. Extending the notion of compound Poisson processes, a more general class Lévy processes is obtained via the formal expression

\[
Z_t = \int_0^t \int_{\mathbb{R}^1} xN(ds, dx),
\]

where \( N(dt, dx) \) is a time-homogeneous, space-time Poisson point-process, counting the number of jumps of size \( x \) at time \( t \). A simple and elegant introduction to Poisson point-processes in a general state-space is given in [34].
The interpretation of the above integral is fairly straightforward, but attention should be paid to some technical conditions, see below. The above process \((Z_t)\) is called a pure-jump Lévy process, indicating the fact that it is purely defined in terms of its jumps.

A basic technical tool in the theory of Poisson point-processes is the intensity of the process. The intensity of a Poisson point-process \(N(dt, dx)\) is formally defined by \(\mathbb{E}[N(dt, dx)]\), with \(\mathbb{E}\) denoting expectation. Due to time homogeneity, \(\mathbb{E}[N(dt, dx)]\) can be written as

\[
\mathbb{E}[N(dt, dx)] = dt \cdot \nu(dx),
\]

where \(\nu(dx)\) is the so-called Lévy-measure. Now the above representation of a pure-jump Lévy process given in (1) is mathematically rigorous if

\[
\int_{\mathbb{R}^d} \min(|x|, 1) \nu(dx) < \infty.
\]

The intuition behind this condition that small jumps have a finite contribution, and thus the fine structure of the sample paths is relatively smooth. In fact, it can be shown that the sample paths of \((Z_t)\) are of finite variation with probability 1. In the area of financial time series sample paths with finite variations are obtained for most indices, as supported by empirical evidence, see [7]. This phenomenon may be explained by the averaging effect when computing an index, such as SP500. We note in passing that condition (2) implies that for all finite \(t\)

\[
\mathbb{E} [ |Z_t| ] < \infty.
\]

For an explanation see the comments following 1.2 below.

It is easily seen that the characteristic function of \(Z_t\) can be written in the form

\[
\mathbb{E} \left[ e^{iuZ_t} \right] = e^{t\psi(u)}.
\]

Here \(\psi(u)\) is called the characteristic exponent. Note that the logarithm of the characteristic function is linear in \(t\), which is implied by the fact that \((Z_t)\) has independent and stationary increments.

The characteristic function plays a key role in the study of Lévy processes, because, unlike the density function of \(Z_t\), it typically has a closed form. The c.f. of a Lévy process is given by the celebrated Lévy-Khintchine formula, which, in the case of processes defined by (1), reduces to the following:

\[
\mathbb{E} \left[ e^{iuZ_t} \right] = \exp \left[ t \left( ibu + \int_{\mathbb{R}^d} (e^{iux} - 1) \nu(dx) \right) \right],
\]

where \(b\) is known as the drift coefficient.

1.1 Discrete-time Lévy-systems

A natural object for study is a linear stochastic system driven by a Lévy-process. Since the study of continuous-time systems would lead to a number of technical
difficulties, we restrict our attention to discrete-time, finite-dimensional linear stochastic systems driven by the increments of a Lévy-process:

\[ \Delta y = A(\theta^*, q^{-1}) \Delta Z, \]  

(4)

defined for the time range \(-\infty < n < +\infty\), where \( \Delta Z_n \) is the increment of a Lévy process \((Z_t)\) with \(-\infty < t < +\infty\), and \( Z_0 = 0 \), over an interval \([(n-1)h, nh)\), with \( h > 0 \) being a fixed sampling interval, and \(-\infty < n < +\infty\). The Lévy-measure of \( Z \) will be denoted by \( \nu(dx) = \nu(dx, \eta^*) \), where \( \eta^* \) denotes an unknown parameter-vector with a known range, say \( D_\eta \subset \mathbb{R}^r \). The operator \( A(\theta^*, q^{-1}) \) is a rational, stable (causal) and inverse stable function of the backward shift operator \( q^{-1} \), depending on some unknown parameter-vector \( \theta^* \), taking its values from some known set \( G_\theta \subset \mathbb{R}^p \). The observed output process is then \( \Delta y \). We call such systems briefly Lévy-systems.

The fundamental problem to be discussed in this paper is the efficient identification of both the systems and the noise parameters. The ML method would be appropriate in solving the full identification problem (i.e. estimating both \( \theta^* \) and \( \eta^* \)) along standard lines, if we knew the density function of \( \Delta Z_n \) is known, see \cite{25} and \cite{11}. Unfortunately, typically this is not the case with the mostly used Lévy processes.

Therefore we develop a new method, using a combination of the PE (prediction error) and an adapted version of the so-called ECF (empirical characteristic function) method, widely used in finance, to get a competitive alternative to the ML (maximum likelihood) method.

The ECF method was originally designed for i.i.d. samples. It has the remarkable property that under certain idealistic assumptions it is as efficient as the ML method. Certain extensions to dependent data are available in the literature at the cost of losing efficiency. Our main contribution is the development of a method for system identification using a suitably adapted ECF method, the efficiency of which is established solely relying on efficiency results for i.i.d. data.

Let us now describe a few additional technical details of our model. Let us assume that a state space representation in innovation form equation for this model is given by

\[ \Delta X_{n+1} = H(\theta^*) \Delta X_n + K(\theta^*) \Delta Z_n \]  

(5)

\[ \Delta Y_n = L(\theta^*) \Delta X_n + \Delta Z_n. \]  

(6)

Then stability and inverse stability of the system is then described by the following condition:

**Condition 1.1** It is assumed that the system matrices \( H(\theta) \) and \( H(\theta) - K(\theta) L(\theta) \) are stable for \( \theta \in G_\theta \).

To define the smooth dependence on \( \theta \) suppose that \( A(\theta, q^{-1}) \) is three-times continuously differentiable w.r.t. \( \theta \) for \( \theta \in D_\theta \). Let \( \mathcal{F} \) denote the natural filtration with \( \mathcal{F}_{n-1} = \sigma \{ \Delta Z_k : k \leq n - 1 \} \). The system \( \Delta y_n \) is certainly well-defined if \( \Delta Z \) satisfies the integrability condition \( \exists \). Namely, then \( \Delta y_n \) can be
written as a weighted sum of past values of $\Delta Z$, with exponentially decaying weights, converging in $L_1$. We will need the following technical condition:

**Condition 1.2** We assume that for all $q \geq 1$

$$
\int_{|x|\geq 1} |x|^q \nu(dx) < +\infty, \tag{7}
$$

and that $E[\Delta Z_n] = 0$.

It follows, see [4], that for $q \geq 1$ and for all $h \geq 0$, the $q$-th moments of the increments of $Z$ are finite:

$$
E[|\Delta Z_n|^q] < \infty. \tag{8}
$$

We note here that Condition 1.2 holds in our benchmark examples to be presented in the next Section.

Let $D_\theta$ and $D_\theta^*$ be compact domains such that

$$
\theta^* \in D_\theta^* \subset \text{int } D_\theta \quad \text{and} \quad D_\theta \subset G_\theta.
$$

The domains $D_\eta, D_\eta^*$ are defined analogously. Finally, let $\rho^* = (\theta^*, \eta^*)$ denote the joint parameter vector, and set

$$
D_\rho = D_\theta \times D_\eta, \quad D_\rho^* = D_\theta^* \times D_\eta^*, \quad G_\rho = G_\theta \times G_\eta.
$$

Before going into further details we present a few examples of Lévy processes used for modeling purposes.

## 2 Examples for widely used Lévy processes

Compound Poisson process is defined by a rate $\lambda$ and a jump size distribution $F$ via

$$
Z_t = \sum_{i=1}^{N_t} X_i,
$$

where $N_t$ is a Poisson process with rate $\lambda$, and $X_i$-s are i.i.d. random variables with distribution $F$. Compound Poisson processes are widely used for modeling in queueing theory. For example in [33] a generalized multi-server queue is used to model telecommunication networks. Among several properties of the model, customer arrivals, server failures and packet losses are modeled with compound Poisson processes.

To model the increments of the logarithm of a price process a wide range of geometric Lévy processes has been proposed by a variety of authors. Mandelbrot suggested to use $\alpha$-stable process to model the price dynamics of wool, see [17]. An $\alpha$-stable with $0 < \alpha < 2$ is defined via the Lévy measure

$$
\nu(dx) = C^- |x|^{-1-\alpha} 1_{x<0} dx + C^+ |x|^{-1-\alpha} 1_{x>0} dx.
$$
A recently widely studied class of Lévy processes is the CGMY process due to Carr, Geman, Madan and Yor [7]. It is obtained by setting $C^{-} = C^{+}$, and then, separately for $x > 0$ and $x < 0$, multiplying the Lévy-density of the original symmetric stable process with a decreasing exponential. The corresponding Lévy-measure, using standard parametrization, is of the form:

$$\nu(dx) = \frac{Ce^{-G|x|}}{|x|^{1+Y}}1_{x<0}dx + \frac{Ce^{-Mx}}{|x|^{1+Y}}1_{x>0}dx,$$

where $C, G, M > 0$, and $0 < Y < 2$. Intuitively, $C$ controls the level of activity, $G$ and $M$ together control skewness. Typically $G > M$ reflecting the fact that prices tend to increase rather than decrease. $Y$ controls the density of small jumps, i.e. the fine structure. For $Y < 1$ the integrability condition (2) is satisfied, thus corresponding Lévy process is of finite variation. The characteristic exponent $\psi(u)$ of the CGMY process is given by

$$CT((-Y)((M-\it{i}u)^Y - M^Y + (G + \it{i}u)^Y - G^Y),$$

where $\Gamma$ denotes the gamma-function. A more general class of tempered stable distributions is studied in [5]. Terdik and Woyczński obtains analytic formulas for the Rosiński measure of tempered processes.

Formally setting $Y = 0$ we get the Lévy density of the so-called Variance Gamma process (VG for short) that has been proposed by Madan, Carr and Chang [16]. The Variance Gamma (VG)-process is a time changed Brownian motion with drift, where the time change is a so-called gamma process, which is essentially the continuous time extension of the inverse of a Poisson process. It is a three-parameter class of processes, with explicit characteristic function and Lévy measure. Let $B_t(\theta, \sigma)$ be a Brownian motion with drift $\theta$ and volatility $\sigma$, i.e.:

$$B_t(\theta, \sigma) = \theta t + \sigma B_t,$$

where $(B_t)$ is a standard Brownian motion and let $\gamma_t(\mu, \nu)$ be a gamma process with mean $\mu$ and variance $\nu$, i.e. $\gamma_t(\mu, \nu)$ is a stochastic process with independent gamma distributed increments. The VG process $(X_t(\sigma, \nu, \theta))$ is defined as

$$X_t(\sigma, \nu, \theta) = B_{\gamma_t(1, \nu)}(\theta, \sigma).$$

Hence, the VG process is a time-changed Brownian motion. According to [16] $\theta$ controls the skewness and $\nu$ controls the kurtosis of the process. A possible other definition of the VG process is that $X_t(\sigma, \nu, \theta)$ can be written as the difference of two gamma processes.

The Lévy measure of a VG process can be obtained by first computing its characteristic function and then applying Lévy-Khintchine’s formula in the inverse direction. Thus we get:

$$\nu(dx) = \begin{cases} 
\frac{\mu^2}{\nu} \exp(-\frac{\mu^2}{2\nu}|x|) dx & \text{if } x < 0 \\
\frac{\nu^2}{\mu^2} \exp(-\frac{\mu^2}{2\nu} x) \frac{1}{x} dx & \text{if } x > 0,
\end{cases}$$
where the parameters $\mu_p, \nu_p, \mu_n, \nu_n$ are obtained in terms of the original parameters as follows:

$$
\mu_p = \frac{1}{2} \sqrt{\theta^2 + 2\sigma^2 \frac{\theta}{\nu}} + \frac{\theta}{2} \nu_p = \mu_p^2 \nu
$$

$$
\mu_n = \frac{1}{2} \sqrt{\theta^2 + 2\sigma^2 \frac{\theta}{\nu}} - \frac{\theta}{2} \nu_n = \mu_n^2 \nu
$$

From here we get the following remarkable property of VG processes: a VG process $X_t(\sigma, \nu, \theta)$ can be written as the difference of two gamma processes $(\gamma_p, t)$ and $(\gamma_n, t)$:

$$
X_t(\sigma, \nu, \theta) = \gamma_p(t, \mu_p, \nu_p) - \gamma_n(t, \mu_n, \nu_n).
$$

In particular, it follows that a VG process is of finite variation.

3 The empirical characteristic function method for i.i.d. data

We briefly describe the ECF method for i.i.d. samples, see [10]. The ECF method gives an efficient estimate of the unknown parameters of a given family of distributions [10]. A nice heuristic justification for this has been given by A. Feuerverger and P. McDunnogh in [15], showing that the equations defining the ECF method for i.i.d samples can be obtained as the Fourier transform of the likelihood equations.

Let $(r_1, r_2, \ldots, r_N)$ be i.i.d. observations, and let a closed form of the characteristic function $\varphi(u, \eta)$ be known, with $\eta$ being a $p$-dimensional parameter vector, and $u \in \mathbb{R}$. The true value of the parameter will be denoted by $\eta^*$. The idea is to estimate $\eta^*$ by a value of $\eta$ for which the characteristic function (cf) best matches the empirical characteristic function (ecf). The error for any fixed $u$ is defined as

$$
\overline{h}_N(u, \eta) = \frac{1}{N} \sum_{k=1}^{N} h_k(u, \eta),
$$

where $h_k(u, \theta)$ is the generalized moment function:

$$
h_k(u, \eta) = e^{iu \eta_k} - \varphi(u, \eta).
$$

An important property of the moment function is that

$$
\mathbb{E}[h_k(u, \eta^*)] = 0, \quad \text{for all } u,
$$

where $\eta^*$ denotes the true parameter. In the case of a finite set of moments evaluated, say at $u_1, \ldots, u_M$, with $M > p$, define

$$
\overline{h}_N(\eta) = (\overline{h}_N(u_1, \eta), \ldots, \overline{h}_N(u_M, \eta))^T,
$$
and its expectation
\[ g(\eta) = \mathbb{E} \left[ h_N(\eta) \right] . \]
Since \( g(\eta^*) = 0 \), we would obtain \( \eta^* \) by solving the over-determined system of equations
\[ g(\eta) = 0, \quad (9) \]
where \( \text{dim } g(\eta) = M > p \). Since \( g \) is not computable in practice we seek a least-square solution by minimizing the weighted quadratic cumulative error
\[ V_N(\eta) = |K^{-1}\overline{h}_N(\eta)|^2 \quad (10) \]
where \( K \) is an appropriate, \( m \times m \) weighing matrix.

Now we compute the asymptotic covariance matrix of the estimated parameter \( \hat{\eta}_N \). We denote the complex conjugate of a vector or matrix \( M \) with \( M^* \).

The gradient equation (\( p \) equations):
\[ \overline{h}_{\eta,N}(\eta)K^{-1}\overline{h}_N(\eta) = 0. \]
The corresponding approximating problem can be defined via
\[ GK^{-1}\overline{h}_N(\eta) = 0, \]
where \( G \) is the \( M \times p \) matrix \( G = g_\eta(\eta^*) \). The left hand side is considered as a new set of exactly \( p \) scores. The asymptotic gradient is
\[ g_\eta^*(\eta)K^{-1}g(\eta), \]
while its derivative at \( \eta^* \) (the Hessian of the asympt. cost) is
\[ R = g_\eta^*(\eta^*)K^{-1}g_\eta(\eta^*). \]
Then the Hessian of the asympt. cost is
\[ T = G^*K^{-1}G. \]
To get the asymptotic covariance of the new set of scores define the \( M \times M \) covariance matrix by
\[ C_{k,l} = \mathbb{E} \left[ h_n(u_k, \eta^*)h_n^*(u_l, \eta^*) \right]. \]
Note that we have
\[ C_{k,l} = \varphi(u_k - u_l, \eta^*) - \varphi(u_k, \eta^*)\varphi(-u_l, \eta^*). \]
Thus the asymptotic covariance of the new set of scores is
\[ S = G^*K^{-1}C^*K^{-1}G. \]
The asymptotic covariance of the estimator \( \hat{\eta}_N \) is then
\[ T^{-1}ST^{-1}, \]
or equivalently,

\[(G^* K^{-1} G)^{-1} G^* K^{-1} C K^{-1} G (G^* K^{-1} G)^{-1}.\]

It is easy to see that the optimal value of \(K\) is

\[K = C\]

eyielding the asymptotic covariance for \(\hat{\eta}_N\)

\[(G^* C^{-1} G)^{-1}.\]

### 4 Three-stage method

The main problem that we consider is the identification of Lévy systems, when both the system parameters and the noise parameters are unknown. In Section 9 we briefly sketch a method proposed in the literature, see [19], [20], for the statistical analysis of dependent data via the ECF method, and also point out the shortcoming of this approach.

Therefore we propose a completely different approach, combining the PE method and an adapted version of the ECF method. Our novel method a three-stage method, the first stage being a standard PE method for estimating the system parameters, taking into account only that the innovation process is i.i.d. with having moments of appropriate orders. Thus we get an estimate of the system parameters, say, \(\hat{\theta}_N\).

In the second stage, using a certainty equivalence argument, pretending that \(\hat{\theta}_N = \theta^*\), we estimate the innovation process by inverting the system using the estimated system parameters \(\hat{\theta}_N\). Then, the noise parameters \(\eta^*\) are estimated using the ECF method for i.i.d. sequences, resulting in an estimate \(\hat{\eta}_N\). These procedures will be briefly described in Section 5. Finally, in the third stage, once again using a certainty equivalence argument, pretending that \(\hat{\eta}_N = \eta^*\), we re-estimate the system parameters using a specific adaptation of the ECF method for systems-identification with i.i.d. innovation process, having a known characteristic function. This is the most original step in our procedure.

The analysis of the effects of the estimation errors of \(\hat{\theta}_N\) and \(\hat{\eta}_N\) on subsequent steps are based on moment estimates of the estimation errors. The latter can be obtained by extending the techniques of [22], and exploiting the fact that all finite moments of the innovation process are finite.

To set the stage for the final step of our procedure we briefly summarize a simple known result on the ML estimate for the identification of a linear stochastic system with i.i.d. innovation of known characteristics, more accurately of known probability density function, say \(f(., \eta^*)\), following [25]. In this case we can obtain the maximum likelihood estimate of the unknown system parameters \(\theta^*\) via solving

\[
\sum_{n=1}^{N} \frac{\partial}{\partial \theta} \log f (\varepsilon_n(\theta), \eta^*) = 0, \tag{11}
\]
where
\[
\varepsilon_n(\theta) = A^{-1}(\theta)\Delta y_n
\] (12)
is the estimated innovation process of the SISO system given under (4).

Then under certain technical conditions, such as the condition that \(\mathbb{E}[\Delta Z_n] = 0\), the asymptotic covariance matrix of the ML estimate is given by
\[
\Sigma_{ML} = \mu^{-1}(R_p^*)^{-1},
\] (13)
where
\[
\mu = \mathbb{E} \left[ \frac{\left( f'(\Delta Z_n, \eta^*) \right)^2}{f(\Delta Z_n, \eta^*)} \right],
\]
with \(f'\) being the derivative of \(f(., \eta^*)\) w.r.t its first variable, and \(R_p^*\) is defined in connection with the PE method. Since this method is efficient we have that \(\mu^{-1} \leq \sigma^2\), we note that the accuracy of the ML method can significantly surpass that of the PE method, i.e. we can have \(\mu^{-1} \ll \sigma^2\). Large difference between \(\mu^{-1}\) and \(\sigma^2\) can be achieved by taking the mixture of a mass like continuous pdf with and another continuous pdf. This \(\mu\) can be interpreted as a Fisher of location parameter estimate. This property of \(\mu\) will be discussed in more details in Section 8 and will be used in analyzing the efficiency of our proposed method in Section 7. The challenge we address in this paper if we can achieve the same accuracy in estimating \(\theta^*\) when we know the characteristic function of the innovation only rather than its p.d.f. The surprising answer is a yes, or rather an almost yes. Before going into details we briefly summarize the first two stages of our algorithm.

Hence, our proposed three-stage method can be summarized as follows
1. Estimate \(\theta^*\) by applying PE method, obtain \(\hat{\theta}_N\)
2. Invert the system with \(\theta = \hat{\theta}_N\), then estimate \(\eta^*\) by using the idea of ECF method for i.i.d. data and obtain \(\hat{\eta}_N\)
3. Re-estimate \(\theta^*\) by applying the ECF method for system identification to obtain an efficient estimate \(\hat{\theta}_N\) for the dynamics.

5 A summary of results on the PE method and the ECF method

In this section we briefly summarize the PE method for the case when the input noise has zero expectation, i.e. \(\mathbb{E}[\Delta Z_n] = 0\), and present a result that will be relevant later. Although general Lévy processes presented in Section 2 are not zero mean processes, by preprocessing our data, as is customary in classic time series analysis, we may achieve that \(\mathbb{E}[\Delta Z_n] = 0\). First, we define the estimated innovation process \(\varepsilon(\theta)\) as in the previous sections. The prediction
error estimator of parameters vector $\theta^*$ is then obtained by minimizing the cost function

$$V_{P,N}(\theta) = \frac{1}{2} \sum_{n=1}^{N} \varepsilon_n^2(\theta),$$

over $G_\theta$, see [31]. An alternative, more convenient definition of the PE estimator $\hat{\theta}_N$ is obtained by setting the gradient of the cost functions equal to zero, and considering the equations:

$$\frac{\partial}{\partial \theta} V_{P,N}(\theta, m) = \sum_{n=1}^{N} \varepsilon_{n\theta}(\theta) \varepsilon_n(\theta) = 0$$

The asymptotic cost function associated with the PE method is defined as

$$W_P(\theta) = \frac{1}{2} E \left[ \left( \varepsilon_{n\theta}(\theta)^T \varepsilon_{n\theta}(\theta) \right)^2 \right],$$

where $\varepsilon_{n\theta}(\theta)$ denotes the stationary solution of (12) when $-\infty < n < \infty$. (In general, the superscript $(s)$ will be used throughout this paper if the marked stochastic process is obtained by passing through a stationary process through an exponentially stable linear filter starting at $-\infty$, as opposed to initializing the filter at time 0 with some arbitrary initial condition, which is typically zero). We have

$$\frac{\partial}{\partial \theta} W_P(\theta^*) = 0$$

and

$$R^*_P := W_{P,\theta \theta}(\theta^*) = E \left[ \varepsilon_{n\theta}(\theta^*)^T \varepsilon_{n\theta}(\theta^*) \right].$$

Furthermore, $\theta = \theta^*$ is the unique solution of $W_{P,\theta}(\theta) = 0$ in $D_\theta$, see [4]. The asymptotic covariance matrix of the PE estimate of $\theta^*$ is given by

$$\Sigma_P = \sigma^2 \left( E \left[ \varepsilon_{n\theta}(\theta)^T \varepsilon_{n\theta}(\theta) \right] \right)^{-1}, \quad (14)$$

where $\sigma^2$ is the variance of $\Delta Z_1$. We will use this notation in a more general way:

**Definition 5.1** For a stochastic process $X_n$, and a function $f : \mathbb{Z} \rightarrow \mathbb{R}^+$ we say that

$$X_n = O_M(f(n))$$

if for all $q \geq 1$

$$\sup_n \frac{E^{1/q} |X_n|^q}{f(n)} < \infty$$

holds.

For the definition of $L$-mixing processes and for other corresponding definitions and theorems see the Appendix. Theorem 5.1, with minor variation, can be found in [22].
**Theorem 5.1** Under Conditions 1.1, 1.2 we have
\[
\hat{\theta}_N - \theta^* = -(R_F^*)^{-1} V_{P,N,\theta}(\theta^*, m^*) + r_N,
\]
with \( r_N = O_M(N^{-1}) \).

Direct consequence of this theorem is the following lemma, which can be proved with different methods as well.

**Lemma 5.1** Under Conditions 1.1, 1.2 we have
\[
\hat{\theta}_N - \theta^* = O_M(N^{-1/2}).
\]

### 6 The ECF method for i.i.d. data. Application for estimating the noise parameters.

The second, simplest problem is seemingly of mere technical interest, when we know the system parameters, but the noise parameters are unknown. In this case define and compute
\[
\varepsilon_n(\theta^*) = A^{-1}(\theta^*) \Delta y_n = A^{-1}(\theta^*) A(\theta^*) \Delta Z_n = \Delta Z_n,
\]
assuming, for the sake of simplicity, that \( \Delta Z_n = \varepsilon_n(\theta^*) = 0 \) for \( n \leq 0 \). After that we can apply the ECF method for i.i.d. samples to obtain the estimation of \( \eta^* \). An ideal score function for the ECF method to estimate \( \eta^* \) would be defined by
\[
h_{k,n}^{opt}(\theta^*, \eta^*) = e^{iu_k \varepsilon_n(\theta^*)} - \varphi(u_k, \eta^*).
\]
(15)

Since we are not given \( \theta^* \) we define an alternative, \( \theta \)-dependent score function via
\[
h_{k,n}(\theta, \eta) = e^{iu_k \varepsilon_n(\theta)} - \varphi(u_k, \eta),
\]
with a fix set of real numbers \( u_1, \ldots, u_M \), with \( NM \geq \dim \eta \). These are appropriate score functions since
\[
E \left[ h_{k,n}^{(s)}(\theta^*, \eta^*) \right] = 0.
\]

Define
\[
h_n(\theta, \eta) = (h_{1,n}(\theta, \eta), \ldots, h_{M,n}(\theta, \eta))^T
\]
and
\[
\bar{h}_N(\theta, \eta) = \sum_{n=1}^N h_n(\theta, \eta).
\]

The expectation of the score vector is denoted by
\[
g(\theta, \eta) = E \left[ \bar{h}_N(\theta, \eta) \right].
\]
For a fixed $\theta$ we proceed like in the i.i.d. case and obtain the $\theta$-dependent estimate $\hat{\eta}_N(\theta)$ of $\eta^*$ by finding a least squares solution to the over-determined system of equations

$$g(\theta, \eta) = \mathbb{E}[\tilde{h}_N(\theta, \eta)] = 0.$$ 

More precisely, define the $\theta$-dependent cost function

$$V_{E,N}(\theta, \eta) = \|K^{-1/2}h_{N}(\theta, \eta)\|^2,$$

where $K$ is a symmetric, positive definite weighting matrix and obtain $\hat{\eta}_N(\theta)$ as the solution of

$$V_{E,N,\eta}(\theta, \eta) = 0.$$

Again, define $G(\theta) = g_{\eta}(\theta, \eta^*)$, then the corresponding asymptotic equation reads as

$$G(\theta) = K^{-1}h_{N}(\theta, \eta).$$

Adapting the idea of the i.i.d. ECF one can easily show that the optimal choice of $K$ is $K = K(\theta) = C(\theta)$ with $C$ being an $M \times M$ matrix with entries

$$C_{k,l}(\theta) = \mathbb{E} \left[ h_{k,n}^*(\theta, \eta^*) h_{l,n}(\theta, \eta^*) \right].$$

Define the ($\theta$-dependent) asymptotic cost function as

$$W_{E}(\theta, \eta) = \mathbb{E} \left[ \|K^{-1/2}h_{n}(\theta, \eta)\|^2 \right].$$

For each the fixed $\theta$ define $\eta^*(\theta)$ such that

$$W_{E,\eta}(\theta, \eta^*(\theta)) = 0.$$

Let the Hessian of $W_E$ w.r.t. $\eta$ at $\eta = \eta^*(\theta)$ be denoted by

$$R^*_{E}(\theta) = W_{E,\eta\eta}(\theta, \eta^*(\theta)).$$

To formulate our result we need some technical conditions. Conditions 1 and 2 have been already presented in Section 1.1. Let $\rho$ be the joint parameter i.e. $\rho = (\theta, \eta)$. Let $D_\rho$ and $D^*_{\rho}$ be compact domains such that $\rho^* \in D^*_{\rho} \subset \text{int} D_\rho$ and $D_\rho \subset G_\rho$.

**Condition 6.1** For each $\theta \in D_\theta$ the equation $W_{E,\eta}(\theta, \eta) = 0$ have a unique solution in $D^*_{\eta}$.

**Lemma 6.1** Under Conditions 1.1, 1.2 and 6.1 we have $\hat{\eta}_N(\theta) - \eta^*(\theta) = O_M(N^{-1/2})$.

Our next result characterizes the estimation error of the ECF method for the noise parameter $\eta^*$.

**Theorem 6.1** Under Conditions 1.1, 1.2 and 6.1 we have

$$\hat{\eta}_N(\theta) - \eta^*(\theta) = -(R^*_{E}(\theta))^{-1}V_{E,\eta\eta}(\theta, \eta^*(\theta)) + O_M(N^{-1}).$$
The proofs of Lemma 6.1 and Theorem 6.1 are isomorphic to that of Lemma 5.1 and Theorem 5.1. Using this theorem with \( \theta = \hat{\theta}_N \), the estimation that we obtained by the PE method, we may conclude the following result.

**Theorem 6.2** Under Conditions 1.1, 1.2 and 6.1 we have

\[
\hat{\eta}_N - \eta^* = -(R^*_E(\theta^*))^{-1}V_{E,N}(\theta^*, \eta^*) + O_M(N^{-1}).
\]

The proof of the last result is obtained by the very same methods as Lemma 5.1 and Theorem 5.1, combining with \( \eta^*(\hat{\theta}_N) - \eta^*(\theta^*) = \eta^*(\hat{\theta}_N) - \eta^* = O_M(N^{-1/2}) \), and that

\[
\left\| W_{\eta N}^{-1}(\theta^*, \eta^*) - W_{\eta N}^{-1}(\hat{\theta}_N, \eta^*(\hat{\theta}_N)) \right\| = O_M(N^{-1/2}).
\]

7 Re-estimation of \( \theta^* \) by the ECF method for system identification. The ECF method for identifying the dynamics of a Lévy systems

If we were given the true value of \( \eta^* \) the score function would be

\[
h_{k,n}^{opt}(\theta) = \left( e^{i u_k \varepsilon_n(\theta)} - \varphi(u, \eta^*) \right) \varepsilon_n \theta(\theta).
\]

Since we are given only an estimation \( \hat{\theta}_N \) of \( \eta^* \) we will use the score function

\[
h_{k,n}(\theta) = \left( e^{i u_k \varepsilon_n(\theta)} - \varphi(u, \hat{\theta}_N) \right) \varepsilon_n \theta(\theta). \tag{16}
\]

In this section we analyze the identification of \( \theta^* \) with an arbitrary given noise characteristic \( \eta \). We prove consistency and we give the precise characterization of the estimation error. As we will see the same results remain valid if we work with \( h_{k,n}(\theta) \) instead of \( h_{k,n}^{opt}(\theta) \). The ECF method has been widely used in finance as an alternative to the ML method, assuming i.i.d. returns \([9], [10], [20]\). We adapt this technique to the problem of identifying a discrete-time Lévy system as described in \([4]\). In this section it is assumed that the characteristic function of the noise, or equivalently \( \eta^* \) is known. The problem we address is to identify the system dynamics specified by \( \theta^* \). Following the philosophy of the ECF method take a fix set \( u_i \)-s, \( 1 \leq i \leq M \). The first natural candidate for a score function would be

\[
f_{k,n}(\theta, \eta) = e^{i u_k \varepsilon_n(\theta)} - \varphi(u, \eta),
\]

see \([32]\). It turns out that the identification method that uses \( f \)-s as score functions does not give an efficient estimator. The score functions to be used following the basic idea of the ECF method and the instrumental variable method are defined as

\[
h_{k,n}(\theta, \eta) = \left( e^{i u_k \varepsilon_n(\theta)} - \varphi(u, \eta) \right) \varepsilon_n \theta(\theta) \tag{17}
\]
with $h_{k,n}(\theta)$ being $p \times 1$ vectors. While $h_{k,n}$ is the function that can be computed in practice, $h^{(s)}_{k,n}$ is easier to handle, because of its stationarity. These are indeed appropriate score functions, since we obviously have

$$
\mathbb{E} \left[ h^{(s)}_{k,n}(\theta^*, \eta^*) \right] = 0
$$

Fix a set of $u$-s: $u_1, \ldots, u_M$ and define $h_n(\theta) = (h_{1,n}^{T}(\theta), \ldots, h_{M,n}^{T}(\theta))^{T}$. Define the $p$-dimensional sample mean of the score vector as:

$$
\overline{h}_N(\theta) = \frac{1}{N} \sum_{n=1}^{N} h_n(\theta).
$$

Let $K > 0$ be a fixed symmetric, positive definite $Mp \times Mp$ weight matrix. Define the $Mp$-dimensional $g(\theta, \eta) = \mathbb{E} \left[ \overline{h}_N(\theta, \eta) \right]$. Note that $\theta = \theta^*$ is the solution of the over-determined set of non-linear algebraic equations

$$
g(\theta^*, \eta^*) = 0.
$$

Since $g$ is not computable we approximate it by $\overline{h}$ and we seek a least-square solution. Define the cost functions as

$$
V_N = V_N(\theta, \eta) = |K^{-1/2} \overline{h}_N(\theta, \eta)|^2,
$$

and obtain $\hat{\theta}_N(\eta)$ by solving

$$
V_{N\theta}(\theta, \eta) = 0.
$$

The system of equations in (19) is no longer over-determined because $\dim V_{N\theta} = p$. This gradient equation can be written as

$$
\overline{h}_{N\theta}(\theta, \eta)K^{-1} \overline{h}_N(\theta, \eta) = 0,
$$

and this $p$ equations can be considered as a set of new score functions. The corresponding asymptotic problem can be formulated as

$$
G(\theta)K^{-1} \overline{h}_N(\theta, \eta) = 0,
$$

with $G(\theta) = g_{\theta}(\theta, \eta^*)$.

The asymptotic cost function is defined by

$$
W(\theta, \eta) = \lim_{N \to \infty} \mathbb{E}[V_N(\theta, \eta)] = g^*(\theta, \eta)K^{-1} g(\theta, \eta).
$$

Let $\theta^*(\eta)$ denote the $\eta$ dependent the solution of the asymptotic equation

$$
W_\theta(\theta, \eta) = 0.
$$
Condition 7.1 For each \( \eta \in D_{\eta} \) the equation \( W_{\theta}(\theta, \eta) = 0 \) have a unique solution in \( D_{\eta} \).

Note that \( \theta^*(\eta) = \theta^* \) for each \( \eta \) holds, because \( \mathbb{E} [\varepsilon_{n\theta}(\theta^*)] = 0 \). The Hessian of \( W \) at \( \theta = \theta^*(\eta) = \theta^* \):

\[
R^*(\eta) = g^*_\theta(\theta^*, \eta)K^{-1}g_{\theta}(\theta^*, \eta).
\]

The following result, which can be proved using the reasoning seen in [22], is a martingale representation theorem for the \( \eta \)-dependent estimate of \( \theta^* \).

Theorem 7.1 Under Conditions 1.1, 1.2 and 7.1 we have

\[
\hat{\theta}_N(\eta) - \theta^* = -(R^*(\eta))^{-1}V_{N\theta}(\theta^*, \eta) + O_M(N^{-1}).
\]

Sketch of the proof: First, note that since \( \Delta y_n = \sum_{i=0}^{\tau} a_i(\theta^*) \Delta Z_i \), holds, \( \Delta y_n \) is a linear combination of \( L \)-mixing processes. Using the fact that a uniformly exponentially stable filter with \( L \)-mixing input produces a uniformly \( L \)-mixing output [24] we get that \( \Delta y_n \) is an \( L \)-mixing process. The innovation process and its derivatives with respect to \( \theta \) can be written as

\[
\varepsilon_n(\theta) = A^{-1}(\theta)\Delta y_n
\]

\[
\varepsilon_{n\theta}(\theta) = A_{\theta}^{-1}(\theta)\Delta y_n
\]

\[
\varepsilon_{n\theta\theta}(\theta) = A_{\theta\theta}^{-1}(\theta)\Delta y_n
\]

Again, since \( A^{-1}(\theta) \) and its derivative with respect to \( \theta \) are uniformly exponentially stable we conclude that the processes \( \varepsilon_n(\theta) \), \( \varepsilon_{n\theta}(\theta) \) and \( \varepsilon_{n\theta\theta}(\theta) \) are \( L \)-mixing uniformly in \( \theta \).

The next step is to show that for any given \( d > 0 \) the equation \( V_{N\theta}(\theta, \eta) = 0 \) has a unique solution in \( D_{\theta} \) and it is in the sphere \( S = \{ \theta : |\theta - \theta^*| < d \} \) with probability at least \( 1 - O(N^{-s}) \) for any \( s > 0 \), see Lemma 2.3. in [22].

We have

\[
0 = V_{N\theta} \left( \hat{\theta}_N, \eta \right) = V_{N\theta} \left( \theta^*, \eta \right) + \nabla V_{N\theta}(\eta) \left( \hat{\theta}_N - \theta^*, \eta \right),
\]

where

\[
\nabla V_{N\theta}(\eta) = \int_0^1 V_{N\theta}(\eta) \left( (1 - \lambda) \theta^* + \lambda \hat{\theta}_N, \eta \right) d\lambda.
\]

One may proceed like in the proof of Theorem 2.1. in [22] to conclude that

\[
\left\| \nabla^{-1}V_{N\theta}(\eta) - W_{\theta\theta}^{-1}(\theta^*, \eta) \right\| = O_M(N^{-1/2}).
\]

except from an event of probability \( O_M(N^{-s}) \) for any \( s > 0 \). Finally,

\[
\hat{\theta}_N - \theta^* = -\nabla^{-1}V_{N\theta}(\eta)V_{N\theta}(\theta^*, \eta) =
\]

\[
- \left( W_{\theta\theta}^{-1}(\theta^*, \eta) + O_M(N^{-1/2}) \right) V_{N\theta}(\theta^*, \eta) =
\]

\[
- W_{\theta\theta}^{-1}(\theta^*, \eta)V_{N\theta}(\theta^*, \eta) + O_M(N^{-1})
\]

16
holds, again except from an event of probability $O_M(N^{-s})$ for any $s > 0$, hence the last expression reads as

$$-(R^*(\eta)^{-1}V_{\theta\theta}(\theta^*, \eta) + O_M(N^{-1}).$$

By choosing $\eta$ to be equal to $\hat{\eta}_N$, the estimate of the noise that we obtained at the second step of the procedure, and using that $R^*$ and $V_{\theta\theta}$ is smooth enough in $\eta$ and that $\hat{\eta}_N - \eta^* = O_M(N^{-1/2})$ we obtain the following result.

**Theorem 7.2** Under Conditions 1.1, 1.2 and 7.1 we have

$$\hat{\theta}_N(\hat{\eta}_N) - \theta^* = -(R^*(\eta^*))^{-1}V_{\theta\theta}(\theta^*, \eta^*) + O_M(N^{-1}).$$

**8 Efficiency of the single term ECF method**

In this section we compute the asymptotic covariance of the estimator proposed in Section 7. Recall that $R^*_P = \mathbb{E} \left[ \varepsilon_n^{(s)}(\theta^*)\varepsilon_n^{(s)\prime}(\theta^*) \right]$, and define the $L \times L$ matrix $C$ with elements

$$C_{k,l} = \varphi(u_k - u_l, \eta^*) - \varphi(u_k, \eta^*)\varphi(-u_l, \eta^*).$$

**Theorem 8.1** Choosing $K = C \otimes R^*_P$, the inverse of the asymptotic covariance matrix of the estimator presented in Section 7 is

$$N \left( \psi^* C^{-1} \psi \right)^{-1} (R^*_P)^{-1},$$

where $\psi = (iu_1\varphi(u_1), \ldots, iu_M\varphi(u_M))^T$.

An essential property of $\hat{\eta}_N$ and $\hat{\theta}_N(\hat{\eta}_N)$ that they are asymptotically uncorrelated. This can be seen using direct calculation using the fact that $\mathbb{E} [\varepsilon_n(\theta^*)] = 0$. Using this observation and Theorem 8.1 we get that the covariance matrix of the estimator is

$$\text{Cov} \left( (\hat{\theta}_N - \theta^*)(\hat{\theta}_N - \theta^*)^\prime \right) =$$

$$(R^*)^{-1} \mathbb{E} [V_{\theta\theta}(\theta^*)V_{\theta\theta}(\theta^*)](R^*)^{-1} + O_M(N^{-2}).$$

To calculate the above expected value we first approximate $\overline{h}_\theta(\theta^*)$ with $g_\theta(\theta^*)$:

$$\mathbb{E} \left[ \overline{h}_\theta(\theta^*) K^{-1}\overline{h}(\theta)^\prime K^{-1}\overline{h}_\theta(\theta^*) \right] =$$

$$\mathbb{E} \left[ (g_\theta(\theta^*) + r_1) K^{-1}\overline{h}(\theta)^\prime K^{-1}(g_\theta(\theta^*) + r_1) \right],$$  \hspace{1cm} (24)
with \( r_1 = O_M^Q(N^{-1/2}) \). Now we calculate the covariance \( \mathbb{E} \left[ \mathbf{h}(\theta^*)' \mathbf{h}(\theta^*) \right] \)

\[
\mathbb{E} \left[ \mathbf{h}(\theta^*)' \mathbf{h}(\theta^*) \right] = \frac{1}{N} C \otimes R_P + r_2 = \frac{1}{N} K + r_2,
\]

where \( r_2 = O_M^Q(N^{-1}) \). Using this (24) reads as

\[
\frac{1}{N} Ng^*(\theta^*)(\theta^*)' + \frac{1}{N} N E \left[ r_1^*K^{-1}r_1 \right] + \frac{1}{N} N E \left[ r_3^*K^{-1}g^*(\theta^*) \right] + \frac{1}{N} N E \left[ r_2^*K^{-1}g^*(\theta^*) \right] = \frac{1}{N} Ng^*(\theta^*)(\theta^*)' + r_3,
\]

with \( r_3 = O_M^Q(N^{-3/2}) \), because \( g^*(\theta^*) \) is bounded. Hence, considering that \( g^*(\theta^*) = \psi^\ast \otimes R_P^* \), and using the mixed-product property and the inverse of a Kronecker product, reading as \((A \otimes B)(C \otimes D) = AC \otimes BD \) and \((A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \), the covariance can be calculated as follows:

\[
\text{Cov} \left( \hat{\theta}_N - \theta^*, \hat{\theta}_N - \theta^* \right)' = \frac{1}{N} ((\psi^* \otimes R_P^*)(C \otimes R_P^*)^{-1}(\psi^* \otimes R_P^*))^{-1} + r_3 = \frac{1}{N} ((\psi^*C^{-1}\psi) \otimes R_P^*)^{-1} + r_3 = \frac{1}{N} (\psi^*C^{-1}\psi)^{-1}(R_P^*)^{-1} + r_3,
\]

which concludes the proof.

### 8.1 Efficiency of the estimation procedure

Now we are ready to demonstrate that the proposed estimation method is asymptotically efficient. Use the full continuum of \( u \)-s and define \( K = C \) as an operator like in [10]

\[
(Cf)(s) = \int c(s,t)f(t)|\pi(t)|dt,
\]

with \( \pi \) being a probability measure on \( \mathbb{R} \), and

\[
c(s,t) = \mathbb{E} \left[ \mathbf{h}_{s,n}(\theta^*, \eta^*)(\theta^* \otimes \eta^*) \right],
\]

where the full continuum of \( u \)-s is defined via \( u_s = s \) for all \( s \in \mathbb{R} \).

Since \( \psi = (i\psi_1 \varphi(u_1), \ldots, i\psi_M \varphi(u_M))^T \) if \( M \) moment conditions is used, the asymptotic covariance matrix of the estimator with full continuum of \( u \)-s would be

\[
\left( ||i\psi \varphi(u, \eta^*)||^2 C \right)^{-1} (R_P^*)^{-1}.
\]
Note that in the above formula \( ||iu\varphi(u, \eta^*)||^2_C \) depends only on the noise characteristics and \( R^*_p \) depends on the derivative of the innovation process, hence on the parameters of the linear system. According to [13] asymptotic efficiency is reached if

\[
(\||iu\varphi(u, \eta^*)||^2_C)^{-1} = \mu,
\]

with

\[
\mu = \mathbb{E} \left[ \left( \frac{f'(\Delta Z_n, \eta^*)}{f(\Delta Z_n, \eta^*)} \right)^2 \right],
\]

where \( \mu \) was shown to be equal to the Fisher of the location parameter.

According to [10] for i.i.d. samples the ECF method with continuum \( u \)-s gives an asymptotically efficient estimate of an unknown parameter \( \lambda^* \) with asymptotic covariance

\[
(\||\varphi_\lambda(u, \lambda^*)||^2_C)^{-1}.
\]

Now we show that \((\||iu\varphi(u, \eta^*)||^2_C)^{-1}\) can be obtained as the asymptotic covariance of an efficient i.i.d. ECF method, thus the efficiency of our identification method for Lévy system follows. Consider the following identification problem: given a sequence of i.i.d. samples with distribution \( X + \lambda^* \), where \( \lambda^* \) is a location parameter to be estimated, and \( X \) is a random variable with known characteristic function. Let \( \varphi_{X+\lambda} \) denote the c.f. of \( X + \lambda \), then

\[
\frac{\partial}{\partial \lambda} \varphi_{X+\lambda}(u, \eta) = \frac{\partial}{\partial \lambda} \mathbb{E} \left[ e^{iu(X+\lambda)} \right] = \frac{\partial}{\partial \lambda} \left( e^{iu\lambda} \mathbb{E} \left[ e^{iuX} \right] \right) = iu\varphi_{X+\lambda}(u, \eta),
\]

thus the ECF method for i.i.d samples that estimates \( \lambda^* \) gives an asymptotic covariance

\[
(\||iu\varphi(u, \eta^*)||^2_C)^{-1},
\]

hence the efficiency follows.

9 Discussion

We briefly sketch the identification method that uses blocks of dependent data. Let us consider the parametric family of time series

\[
\Delta y_n(\theta, \eta) = A(\theta)\Delta Z_n(\eta),
\]

with \(-\infty < n < +\infty\). Note that for \((\theta, \eta) = (\theta^*, \eta^*)\) we recover our observed data, at least in a statistical sense.

The method proposed in the literature is based on the observation that, as an alternative to the joint probability density function, we can compute the joint characteristic function of blocks of unprocessed data, i.e. for blocks of the time series \((y_n)\). Indeed, fix a block length, say \( r \), and define the \( r \)-dimensional blocks

\[
\Delta Y^r_n(\theta, \eta) = (\Delta y_{n-1}(\theta, \eta), \ldots, \Delta y_{n-r}(\theta, \eta)).
\]
Then the characteristic function of $\Delta Y^r_r(\theta, \eta)$, with $u = (u_1, \ldots, u_r)^T$ being an arbitrary vector in $\mathbb{R}^r$, is given by

$$\varphi_n(u, \theta, \eta) = \prod_{k=1}^{\infty} \varphi_{\Delta Z(\eta)}(v_k(\theta)), \quad (28)$$

with $v_k(\theta) = \sum_{j=1}^{r} h_{k-j}(\theta) u_j$ and $h_l(\theta), \ l = 0, 1, \ldots$ denoting the impulse responses of the system $A(\theta)$.

Now the ECF method would be defined by fitting this theoretical characteristic function to the empirical characteristic function, obtained as the arithmetic mean of the individual scores

$$h_n(u, \theta, \eta) = e^{iu^T \Delta Y^r_r - \varphi_n(u, \theta, \eta)}.$$

Without going into further details we point out that the weakness of this approach is that the characteristic function $\varphi_n(u, \theta, \eta)$ is given in terms of an infinite product, and hence it is not clear how to use it in actual computations. Moreover, it is pointed out in the literature that the above ECF method for dynamic models may be less efficient than the ML method, see [8].

Furthermore, an interesting problem is to implement and analyze a recursive estimation method for the dynamics and noise characteristics of a Lévy system, this will be the subject of a forthcoming paper.

Our aim is to give an identification method for Wiener-Hammerstein models using the basic ideas of the ECF method.

A  

$L$-mixing processes

Let $\theta$ be a $d$-dimensional parameter vector.

**Definition A.1** We say that $x_n(\theta)$ is $M$-bounded if for all $q \geq 1$,

$$M_q(x) = \sup_{n>0, \theta \in D} \mathbb{E}^{1/q} |x_n(\theta)|^q < \infty$$

Define $\mathcal{F}_n = \sigma \{ e_i : i \leq n \}$ and $\mathcal{F}_n^+ = \sigma \{ e_i : i > n \}$ where $e_i$-s are i.i.d. random variables.

**Definition A.2** We say that a stochastic process $(x_n(\theta))$ is $L$-mixing with respect to $(\mathcal{F}_n, \mathcal{F}_n^+)$ uniformly in $\theta$ if it is $\mathcal{F}_n$ progressively measurable, $M$-bounded with any positive $r$ and

$$\gamma_q(r, x) = \sup_{n \geq r, \theta \in D} \mathbb{E}^{1/q} |x_n(\theta) - \mathbb{E} [x_n(\theta) | \mathcal{F}_n^+]|^q,$$

we have for any $q \geq 1$,

$$\Gamma_q(x) = \sum_{r=1}^{\infty} \gamma_q(r, x) < \infty.$$
Define
\[ \Delta x / \Delta^\alpha \theta = |x_n(\theta + h) - x_n(\theta)| / |h|^\alpha \]
for \( n \geq 0, \theta \neq \theta + h \in D \) with \( 0 < \alpha \leq 1 \).

**Definition A.3** We say that \( x_n(\theta) \) is \( M \)-Hölder continuous in \( \theta \) with exponent \( \alpha \) if the process \( \Delta x / \Delta^\alpha \theta \) is \( M \)-bounded.

**Theorem A.1** Let \( (u_n(\theta)) \) be an \( L \)-mixing uniformly in \( \theta \in D \) such that \( \mathbb{E}u_n(\theta) = 0 \) for all \( n \geq 0, \theta \in D \), and assume that \( \Delta u / \Delta \theta \) is also \( L \)-mixing uniformly in \( \theta, \theta + h \in D \). Then
\[
\sup_{\theta \in D_0} \left| \frac{1}{N} \sum_{n=1}^{N} u_n(\theta) \right| = O_M(N^{-1/2}) \quad (29)
\]

**Theorem A.2** Let \( D_0 \) and \( D \) be as above and let
\( W_\theta(\theta), \delta W_\theta(\theta), \theta \in D \subset \mathbb{R}^p \) be \( \mathbb{R}^p \)-valued continuously differentiable functions, let for some \( \theta^* \in D_0, W_\theta(\theta^*) = 0 \), and let \( W_{\theta\theta}(\theta^*) \) be nonsingular. Then for any \( d > 0 \) there exists positive numbers \( d', d'' \) such that
\[
|\delta W_\theta(\theta)| < d' \quad \text{and} \quad \|\delta W_{\theta\theta}(\theta)\| < d'' \quad (30)
\]
for all \( \theta \in D_0 \) implies that the equation \( W_\theta(\theta) + \delta W_\theta(\theta) = 0 \) has exactly one solution in a neighborhood of radius \( d \) of \( \theta^* \).

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