MATRICIAL QUANTUM GROMOV-HAUSDORFF DISTANCE

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Abstract. We develop a matricial version of Rieffel’s Gromov-Hausdorff distance for compact quantum metric spaces within the setting of operator systems and unital C∗-algebras. Our approach yields a metric space of “isometric” unital complete order isomorphism classes of metrized operator systems which in many cases exhibits the same convergence properties as those in the quantum metric setting, as for example in Rieffel’s approximation of the sphere by matrix algebras using Berezin quantization. Within the metric subspace of metrized unital C∗-algebras we establish the convergence of sequences which are Cauchy with respect to a larger Leibniz distance, and we also prove an analogue of the precompactness theorems of Gromov and Rieffel.

1. Introduction

A compact quantum metric space, as defined by Rieffel in [9], is an order-unit space equipped with a certain type of semi-norm, called a Lip-norm, which plays the role of a Lipschitz semi-norm on functions over a compact metric space. The crucial part of the definition of a Lip-norm \( L \) on an order-unit space \( X \) is the requirement that the metric

\[
\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A \text{ and } L(a) \leq 1\}
\]

on the state space of \( X \) give rise to the weak∗ topology. By applying Hausdorff distance to state spaces, Rieffel defines a quantum analogue of Gromov-Hausdorff distance and thereby synthetically obtains a complete separable metric space of “isometric” order isomorphism classes of compact quantum metric spaces for which a Gromov-type precompactness theorem holds [9]. The most immediate motivation for introducing a theory of quantum Gromov-Hausdorff distance is the search for an analytic framework for describing, or at least clarifying at a metric level, the type of convergence of spaces that has recently begun to play a central role in string theory (see the introduction to [9] for a discussion and references). The main objects of study thus tend to be C∗-algebras, and so it is natural to ask, as does Rieffel in [9], if it is possible to develop a matricial version of quantum Gromov-Hausdorff distance. This is the goal of the present paper.

The key is to define metrics on matrix states spaces using a Lip-norm just as one does for an order-unit state space as above, only now replacing the modulus by matrix norms. We introduce this definition within a general operator system setting in Section 2. We then define “complete” distance (Section 3) by using Hausdorff distance at the matrix state space level in the same way that Rieffel does with regard to order-unit state spaces in the formulation of quantum Gromov-Hausdorff distance. In fact many of the constructions and arguments involving quantum Gromov-Hausdorff distance in [9, 10] are naturally suited to our matricial setting and lead to similar estimates, as for instance in the proof of
the triangle inequality (Proposition 3.3) and the approximation of the sphere by matrix algebras via Berezin quantization (Example 3.12). On the other hand a completely different approach is required to show that complete distance zero implies “isometric” unital complete order isomorphism (the subject of Section 4), and the proofs of the convergence of sequences of metrized unital C*-algebras which are Cauchy with respect to “f-Leibniz complete” distance (Section 5) and our analogue of the Gromov and Rieffel precompactness theorems (Sections 8) ultimately rely on some arguments especially attuned to the complete order context.

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2. LIP-NORMED OPERATOR SYSTEMS AND MATRIX STATE SPACE METRIZATION

We begin by describing our operator system framework. For references see [1, 7, 16]. A (concrete) operator system is a closed unital self-adjoint linear subspace of a unital C*-algebra (for an abstract definition see [3]). Given an operator system X, for each r > 0 we will denote by B(X) the closed norm ball \( \{ x \in X : \| x \| \leq r \} \) of radius r. The state space of X will be denoted by S(X). We will denote by \( X_{sa} \) the set of self-adjoint elements of X. The unit of X will be written 1, or sometimes \( 1_X \) for clarity. For \( x \in X \) we write \( \text{Re}(x) \) and \( \text{Im}(x) \) to refer to the self-adjoint elements \( (x + x^*)/2 \) and \( (x - x^*)/(2i) \) (the real and imaginary parts of x), respectively.

The C*-algebra of \( n \times n \) matrices over \( \mathbb{C} \) will be written \( M_n \). Given operator systems X and Y we say that a linear map \( \varphi : X \to Y \) is \textit{n-positive} if the map \( \text{id}_n \otimes \varphi : M_n \otimes X \to M_n \otimes Y \) is positive, and if \( \text{id}_n \otimes \varphi \) is n-positive for all \( n \in \mathbb{N} \) then we say that \( \varphi \) is \textit{completely positive}. A completely positive (resp. unital completely positive) linear map will be referred to as a c.p. (resp. u.c.p.) map. If \( \varphi : X \to Y \) is a unital m-positive map with m-positive inverse for \( m = 1, \ldots, n \) then \( \varphi \) is a unital n-order isomorphism, and if \( \varphi \) is u.c.p. with c.p. inverse then \( \varphi \) is a unital complete order isomorphism.

An operator system X is \textit{nuclear} if the identity map on X lies in the point-norm closure of the set of u.c.p. maps from X to itself which factor through matrix algebras.

Given an operator system X and \( n \in \mathbb{N} \), there is a bijective linear map from c.p. maps \( X \to M_n \) to positive linear functionals on \( M_n \otimes X \) [7, Thm. 5.1] defined as follows. To each c.p. map \( \varphi : X \to M_n \) we associate the positive linear functional \( \sigma_\varphi \) on \( M_n \otimes X \) given by

\[
\sigma_\varphi((x_{ij})) = \frac{1}{n} \sum_{1 \leq i, j \leq n} \varphi(x_{ij})_{ij}
\]

for all \( (x_{ij}) \in M_n(X) \cong M_n \otimes X \). Conversely, to each positive linear functional \( \sigma \) on \( M_n \otimes X \) we associate the c.p. map \( \varphi_\sigma : X \to M_n \) given by

\[
(\varphi_\sigma(x))_{ij} = n \sigma(e_{ij} \otimes x)
\]

for all \( x \in X \), where \( \{ e_{ij} : 1 \leq i, j \leq n \} \) is the set of standard matrix units of \( M_n \). The maps \( \varphi \mapsto \sigma_\varphi \) and \( \sigma \mapsto \varphi_\sigma \) are mutual inverses and are homeomorphisms with respect to
the point-norm topologies (for the space of positive linear functionals this is the weak* topology) as well as with respect to the norm topologies. If \( \varphi : X \rightarrow M_n \) is u.c.p. then \( \sigma_\varphi \) is a state on \( M_n \otimes X \). However, if \( \sigma \in S(M_n \otimes X) \) then \( \varphi_\sigma \) need not be unital, nor even contractive, although it is clear that \( \| \varphi_\sigma \| \leq n^3 \) (see the discussion after Theorem 5.4 in [7]). We denote by \( SCP_n(X) \) the collection of c.p. maps \( \varphi : X \rightarrow M_n \) such that \( \sigma_\varphi \) is a state on \( M_n \otimes X \), and by \( UCP_n(X) \) the subcollection of \( SCP_n(X) \) consisting of all u.c.p. maps from \( X \) into \( M_n \) (the matrix state spaces).

We now introduce metrics into our picture via the notion of a Lip-norm, which we recall from [9].

**Definition 2.1** ([9, Defns. 2.1 and 2.2]). Let \( A \) be an order-unit space. A Lip-norm on \( A \) is a semi-norm \( \rho \) on \( A \) such that

1. for all \( a \in A \) we have \( \rho(a) = 0 \) if and only if \( a \) is a scalar multiple of the order unit, and
2. the metric \( \rho_\rho \) defined on the state space \( S(A) \) by

\[
\rho_\rho(\mu, \nu) = \sup \{ |\mu(a) - \nu(a)| : a \in A \text{ and } \rho(a) \leq 1 \}
\]

induces the weak* topology.

A pair \( (A, \rho) \) consisting of an order-unit space \( A \) with Lip-norm \( \rho \) is called a compact quantum metric space.

Important examples of order-unit spaces are real linear unital subspaces of self-adjoint elements in an operator system, and in fact every order-unit space is isomorphic to one of these, as shown in Appendix 2 of [9]. We can thus apply the above definition in a direct way to our setting. First we introduce some general notation.

**Notation 2.2.** Let \( X \) be an operator system and \( L \) a semi-norm on a linear subspace of \( X \) or a real linear subspace of \( X_{sa} \). We denote by \( D(L) \) the domain of \( L \) on \( A \) such that

\[
D(L) = \{ x \in X : L(x) \leq 1 \}
\]

A Lip-norm \( \rho \) on an order-unit space \( A \) with Lip-norm \( L \) is said to be closed if if the set \( \{ a \in A : L(a) \leq 1 \} \) is closed in the completion of \( A \). Thus the requirement in Definition 2.3 that \( D(L) \) be closed in \( X_{sa} \) is equivalent to asking that \( L \) be a closed Lip-norm. Given any Lip-norm \( L \) on an order-unit space \( A \) there is a largest lower semicontinuous Lip-norm \( L^s \) smaller than \( L \) [11, Thm. 4.2], and \( L^s \) extends to a closed Lip-norm \( L^c \) [11, Prop. 4.4]. The theorem and proposition from [11] cited in the last sentence also show that \( \rho_{L^c} = \rho_{L^s} = \rho_L \). Furthermore, the property of being closed passes to order-unit quotients by [9, Prop. 3.3], and it also holds in natural examples of interest—see for instance Example 2.6 and [11, Prop. 3.6]. Thus, in view of the completeness of operator systems, it is natural to
assume that our Lip-norms are closed. This will also guarantee that complete distance zero is equivalent to the existence of a unital complete order isomorphism which is bi-Lip-isometric in the obvious sense:

**Definition 2.4.** Let \((X, L_X)\) and \((Y, L_Y)\) be Lip-normed operator systems. We will say that a positive unital map \(\Phi : X \to Y\) is Lip-isometric if \(\Phi(\mathcal{D}(L_X)) \subset \mathcal{D}(L_Y)\) and \(L_Y(\Phi(x)) = L_X(x)\) for all \(x \in \mathcal{D}(L_X)\). If \(\Phi\) has a positive inverse then we say that \(\Phi\) is bi-Lip-isometric if both \(\Phi\) and \(\Phi^{-1}\) are Lip-isometric.

If we were to define a strict operator system analogue of a Lip-norm then the conditions on the semi-norm \(L\) in the following proposition would seem to be the most reasonable. Indeed many examples arise naturally in this way, as Example 2.6 illustrates.

**Proposition 2.5.** Let \(L\) be a semi-norm on an operator system \(X\), permitted to take the value \(+\infty\), such that \(\mathcal{D}(L)\) is dense in \(X\), \(\mathcal{D}_1(L)\) is closed in \(X_{sa}\), and

\[
\begin{align*}
(i) & \quad L(x^*) = L(x) \text{ for all } x \in A \text{ (adjoint invariance)}, \\
(ii) & \quad \text{for all } x \in X \text{ we have } L(x) = 0 \text{ if and only if } x \in \mathbb{C}1, \text{ and} \\
(iii) & \quad \text{the metric } d_L(\sigma, \omega) = \sup_{x \in \mathcal{D}_1(L)} |\sigma(x) - \omega(x)| \text{ on } S(X) \text{ induces the weak}^* \text{ topology.}
\end{align*}
\]

Then the restriction \(L'\) of \(L\) to the order-unit space \(\mathcal{D}(L) \cap X_{sa}\) is a Lip-norm, \((X, L')\) is a Lip-normed operator system, and the restriction map from \(S(X)\) onto \(S(\mathcal{D}(L'))\) is a weak* homeomorphism which is isometric for \(d_L\) and \(\rho_L\).

**Proof.** First note that the fact that \(\mathcal{D}(L)\) is closed in \(X\) immediately implies that \(\mathcal{D}(L')\) is closed in \(X_{sa}\). Next, if \(x \in X_{sa}\) and \(\epsilon > 0\) then by the density of \(\mathcal{D}(L)\) we can find a \(y \in \mathcal{D}(L)\) with \(|x - y| < \epsilon\). Then

\[
\|x - \text{Re}(y)\| \leq \|x - y\|/2 + \|(x - y)^*\|/2 < \epsilon
\]

while \(L'(\text{Re}(y)) = (L(y) + L(y^*)/2 = L(y) < +\infty\), and so \(\mathcal{D}(L')\) is dense in \(X_{sa}\). With this fact it is straightforward to show that the restriction map from \(S(X)\) onto \(S(\mathcal{D}(L'))\) is a weak* homeomorphism. To see that this map is isometric, suppose \(\sigma, \omega \in S(X)\) and \(\epsilon > 0\). Then we can find an \(x \in \mathcal{D}_1(L)\) such that \(|\sigma(x) - \omega(x)| \geq d_L(\sigma, \omega) - \epsilon\), and so for some complex number \(\mu\) of unit modulus we have \(\sigma(\mu x) - \omega(\mu x) \geq d_L(\sigma, \omega) - \epsilon\). Since

\[
L'(\text{Re}(\mu x)) \leq (L(\mu x) + L(\mu x^*))/2 = (L(x) + L(x^*))/2 = L(x) \leq 1
\]

we then have

\[
\rho_{L'}(\sigma|_{\mathcal{D}(L')}, \omega|_{\mathcal{D}(L')}) \geq \sigma(\text{Re}(\mu x)) - \omega(\text{Re}(\mu x))
\]

\[
\geq d_L(\sigma, \omega) - \epsilon,
\]

from which we infer that the map in question is indeed an isometric weak* homeomorphism. Since condition (ii) in the proposition statement immediately implies condition (1) in Definition 2.4 for \(L'\), it thus follows that \(L'\) is a Lip-norm, and so \((X, L')\) is a Lip-normed operator system in view of the density of \(\mathcal{D}(L')\) in \(X_{sa}\).

In the converse direction, given a Lip-normed operator system \((X, L')\) we can extend \(L'\) to a semi-norm \(L\) on \(X\) such that the conditions and conclusions in the statement of Proposition 2.5 hold. Definition 4.2 and Proposition 4.3 indicate how this can be done.
Example 2.6 (ergodic actions of compact groups). As studied in [12], ergodic actions of compact groups give rise to important examples of Lip-normed unital $C^*$-algebras, notably noncommutative tori (see Example 6.8). Let $\gamma$ be an ergodic action of a compact group $G$ on a unital $C^*$-algebra $A$. Let $e$ be the identity element of $G$. We suppose that $G$ is equipped with a length function $\ell$, that is, a continuous function $\ell: G \to \mathbb{R}_{\geq 0}$ such that,

1. $\ell(gh) \leq \ell(g) + \ell(h),$
2. $\ell(g^{-1}) = \ell(g)$, and
3. $\ell(g) = 0$ if and only if $g \neq e$.

The group action $\gamma$ and the length function $\ell$ together yield the semi-norm $L$ on $A$ defined by

$$L(a) = \sup \{\|\gamma_g(a) - a\|/\ell(g) : g \neq e\}.$$  

It is easily seen that $L$ is adjoint-invariant and that $L(a) = 0$ if and only if $a \in C1$. Furthermore, by [12, Thm. 2.3] the metric $d_L(\sigma, \omega) = \sup_{x \in \mathcal{D}_1(L)} |\sigma(x) - \omega(x)|$ on $S(A)$ induces the weak* topology, by [12, Prop. 2.2] $\mathcal{D}(L)$ is dense in $X$, and it is readily verified that $\mathcal{D}_1(L)$ is closed in $A$ (see [1, Prop. 8.1]), so that by Proposition 2.5 we obtain a Lip-normed unital $C^*$-algebra by restricting $L$ to $\mathcal{D}(L) \cap A_{sa}$.

Example 2.7 (quotients). Let $(X, L)$ be a Lip-normed operator system, $Y$ an operator system, and $\Phi: X \to Y$ a surjective unital positive linear map. Then by [1, Prop. 3.1] $L$ gives rise to a Lip-norm $L_Y$ on $\Phi(\mathcal{D}(L))$ via the prescription

$$L_Y(y) = \inf \{L(x) : x \in \mathcal{D}(L) \text{ and } \Phi(x) = y\}$$

for each $y \in Y$, and the induced map from $(S(Y), \rho_{L_Y})$ to $(S(X), \rho_L)$ is an isometry. Since $\Phi(\mathcal{D}(L))$ is clearly dense in $Y_{sa}$ and $L_Y$ is closed by [1, Prop. 3.3], $(X, L_Y)$ is a Lip-normed operator system. We say that $L$ induces $L_Y$ via $\Phi$.

The following definition captures the observation that Lip-norms define metrics on matrix state spaces in much the same way as they do on state spaces. We will thereby be able to define a matrix version of quantum Gromov-Hausdorff distance by applying Hausdorff distance to the matrix state spaces (Definition 3.2).

Definition 2.8. Let $(X, L)$ be a Lip-normed operator system and $n \in \mathbb{N}$. We define the metric $\rho_{L,n}$ on $UCP_n(X)$ by

$$\rho_{L,n}(\varphi, \psi) = \sup_{x \in \mathcal{D}_1(L)} \|\varphi(x) - \psi(x)\|$$

for all $\varphi, \psi \in UCP_n(X)$.

Note that $\rho_{L,n}$ is indeed a metric since it clearly satisfies the triangle inequality and is symmetric, and it is non-zero at any pair of distinct points owing to the density of $\mathcal{D}(L)$ in $X_{sa}$. That $\rho_{L,n}$ is finite follows from the norm compactness of $\mathcal{D}_1(L) \cap B(X)^n_{\rho_{L}}$ for any $r > 0$ (a consequence of [1, Thm. 4.5] by scaling) along with Proposition 2.11 below.

Proposition 2.9. The diameters of $UCP_n(X)$ relative to the respective metrics $\rho_{L,n}$ are finite and coincide for all $n \in \mathbb{N}$.
Proof. The restriction map from \( S(X) \) onto \( S(\mathcal{D}(L)) \) is evidently a weak* homeomorphism which is isometric with respect to \( \rho_{L,1} \) and \( \rho_{L} \) (Definition 2.11), and so the diameter of \( S(X) \) with respect to \( \rho_{L,1} \) is finite by [9, Thm. 4.5]. Now given \( n \in \mathbb{N} \), \( \varphi, \psi \in UCP_n(X) \), and \( x \in X_{sa} \) we can find a state \( \sigma \) on \( M_n \) such that
\[
\left| (\sigma \circ \varphi)(x) - (\sigma \circ \psi)(x) \right| = \| \varphi(x) - \psi(x) \|.
\]
It follows that the diameter of \( UCP_n(X) \) is bounded above by that of \( S(X) = UCP_1(X) \).

On the other hand \( S(X) \) embeds into \( UCP_n(X) \) via the map which takes \( \sigma \in S(X) \) to \( x \mapsto \sigma(x)1_M \), from which we see that the diameter of \( UCP_n(X) \) is at least that of \( S(X) \), so that the two are equal.  \( \square \)

Definition 2.10. Given a Lip-normed operator system \((X, L)\) we define its diameter \( \text{diam}(X, L) \) to be the common value of the diameters of \( UCP_n(X) \) with respect to \( \rho_{L,n} \) for \( n \in \mathbb{N} \).

The next proposition, in addition to showing the finiteness of the metrics \( \rho_{L,n} \) (see the paragraph following Definition 2.8), will also be of use in Sections 3 and 6 since it will enable us to streamline the statement and verification of conditions involving local approximation of elements of bounded Lip-norm.

Proposition 2.11. Let \((X, L)\) be a Lip-normed operator system. Let \( x \in \mathcal{D}(L) \) and let \( r \) be the infimum of its spectrum. Then \( \| x - r \| \leq L(x) \text{diam}(X, L) \).

Proof. We can find \( \sigma, \omega \in S(X) \) such that \( \sigma(x - r) = \| x - r \| \) and \( \omega(x) = r \), whence
\[
\| x - r \| = |\sigma(x - r) - \omega(x - r)| = |\sigma(x) - \omega(x)| = L(x) \text{diam}(X, L).
\]

Since one of the requirements for a Lip-norm is that the associated metric on the state space give rise to the weak* topology, one would hope that the associated metrics on the matrix state spaces give rise to the respective point-norm topologies. The following result shows that this is indeed the case.

Proposition 2.12. The metric \( \rho_{L,n} \) gives rise to the point-norm topology on \( UCP_n(X) \).

Proof. Let
\[
U_{\varphi, \Omega, \varepsilon} = \{ \psi \in UCP_n(X) : \| \varphi(x) - \psi(x) \| < \varepsilon \text{ for all } x \in \Omega \}
\]
be a basic open set in the point-norm topology, with \( \varphi \in UCP_n(X) \), \( \varepsilon > 0 \), and \( \Omega \) a finite subset of \( A \). For each \( x \in \Omega \) pick \( y_{x,1}, y_{x,2} \in \mathcal{D}(L) \) with \( \| y_{x,1} - \text{Re}(x) \| < \varepsilon/6 \) and \( \| y_{x,2} - \text{Im}(x) \| < \varepsilon/6 \), and choose \( M > 0 \) so that \( M \geq \max_{x \in \Omega, j=1,2} L(y_{x,j}) \). Now if \( \psi \in UCP_n(X) \) and \( \rho_{L,n}(\varphi, \psi) < (6M)^{-1}\varepsilon \) then \( \| \varphi(y_{x,j}) - \psi(y_{x,j}) \| < \varepsilon/6 \) for all \( x \in \Omega \) and \( j = 1, 2 \), and hence
\[
\| \varphi(\text{Re}(x)) - \psi(\text{Re}(x)) \| \leq \| \varphi(\text{Re}(x)) - \varphi(y_{x,1}) \| + \| \varphi(y_{x,1}) - \psi(y_{x,1}) \| + \| \psi(y_{x,1}) - \psi(\text{Re}(x)) \| < \varepsilon/2
\]
and similarly \(|\| \varphi(\text{Im}(x)) - \psi(\text{Im}(x)) \| < \epsilon/2\), so that \(|\| \varphi(x) - \psi(x) \| < \epsilon\). Thus \(U_{\varphi,\Omega,\epsilon}\) contains the open \(\rho_{L,n}\)-ball centred at \(\varphi\) with radius \((6M)^{-1}\epsilon\), from which it follows that the metric topology is finer than the point-norm topology.

Suppose now that \(B(\varphi,\epsilon)\) is the \(\rho_{L,n}\)-ball centred at some \(\varphi \in UCP_n(X)\) with radius some \(\epsilon > 0\). Note that \(D_1(L) \cap B_{\text{diam}(S\langle X \rangle)}^X\) is compact, since \(D_1(L)\) is closed by the definition of a Lip-normed operator system and by [1, Thm. 4.5] \(D_1(L) \cap B_Y^X\) is totally bounded, which implies the total boundedness of \(D_1(L) \cap B_{\text{diam}(S\langle X \rangle)}^X\) via a scaling argument. Hence we can find a finite set \(\Omega \subset D_1(L) \cap B_{\text{diam}(S\langle X \rangle)}^X\) which is \((\epsilon/3)\)-dense in \(D_1(L) \cap B_{\text{diam}(S\langle X \rangle)}^X\). Thus if \(\varphi \in UCP_n(X)\) and \(|\| \varphi(x) - \psi(x) \| < \epsilon/3\) for all \(x \in \Omega\) then \(|\| \varphi(x) - \psi(x) \| < \epsilon\) for all \(x \in D_1(L) \cap B_{\text{diam}(S\langle X \rangle)}^X\), and so \(B(\varphi,\epsilon)\) contains the point-norm basic open set

\[\{ \psi \in UCP_n(X) : |\| \varphi(x) - \psi(x) \| < \epsilon/3 \text{ for all } x \in \Omega \}\].

We conclude that the metric and point-norm topologies coincide on \(UCP_n(X)\). \(\square\)

We round out this section by showing that matrix state spaces embed isometrically under quotient maps, as do state spaces in the quantum metric setting. This will be crucial for the application of Hausdorff distance in formulating our matrix version of quantum Gromov-Hausdorff distance.

**Proposition 2.13.** Let \((X, L)\) be a Lip-normed operator system, \(n \in \mathbb{N}\), \(\Phi : X \to Y\) a surjective unital \(n\)-positive map onto an operator system \(Y\), and \(L_Y\) the quotient Lip-norm on \(Y\) induced by \(L\) via \(\Phi\). Then the map \(\Gamma : UCP_n(Y) \to UCP_n(X)\) given by \(\Gamma(\varphi) = \varphi \circ \Phi\) is an isometry with respect to \(\rho_{L,n}\) and \(\rho_{L_Y,n}\).

**Proof.** Suppose \(\Phi\) is \(n\)-positive and let \(\varphi, \psi \in UCP_n(Y)\). Since \(\Phi\) is Lip-norm-decreasing, we have \(\rho_{L_Y,n}(\varphi, \psi) \geq \rho_{L,n}(\varphi \circ \Phi, \psi \circ \Phi)\). For the reverse inequality, let \(\epsilon > 0\) and choose \(y \in D_1(L_Y)\) such that

\[\rho_{L_Y,n}(\varphi, \psi) < |\| \varphi(y) - \psi(y) \| + \epsilon.\]

We may assume \(L(y) < 1\) for otherwise we can replace \(y\) with \(\mu y\) for some \(\mu < 1\) sufficiently close to 1. Then by definition of the quotient Lip-norm there is an \(x \in D_1(L)\) such that \(\Phi(x) = y\), and so

\[\rho_{L_Y,n}(\varphi, \psi) < |\| (\varphi \circ \Phi)(x) - (\psi \circ \Phi)(x) \| + \epsilon\]
\[\leq \rho_{L,n}(\varphi \circ \Phi, \psi \circ \Phi) + \epsilon.\]

Since \(\epsilon\) was arbitrary, we conclude that \(\rho_{L_Y,n}(\varphi, \psi) = \rho_{L,n}(\varphi \circ \Phi, \psi \circ \Phi)\), so that \(\Gamma\) is an isometry with respect to \(\rho_{L_Y,n}\) and \(\rho_{L,n}\). \(\square\)

### 3. \(n\)-Distance and Complete Distance

The definition of quantum Gromov-Hausdorff distance [3] involves forming a direct sum and considering Lip-norms thereupon which induce the given Lip-norms on the summands. One then takes an infimum of the Hausdorff distances between the state spaces under their isometric embeddings into the state space of the direct sum. We will apply the same procedure here with regard to the matrix state spaces.
Notation 3.1. Let $(X, L_X)$ and $(Y, L_Y)$ be Lip-normed operator systems. We denote by $\mathcal{M}(L_X, L_Y)$ the collection of closed Lip-norms on $D(L_X) \oplus D(L_Y)$ which induce $L_X$ and $L_Y$ via the quotient maps onto $D(L_X)$ and $D(L_Y)$, respectively.

Let $(X, L_X)$ and $(Y, L_Y)$ be Lip-normed operator systems and $L \in \mathcal{M}(L_X, L_Y)$. Since the projection map $X \oplus Y \to X$ is u.c.p., by Proposition 2.13 we obtain an isometry $UCP_n(X) \to UCP_n(X \oplus Y)$ with respect to $\rho_{L_X}$ and $\rho_L$. Similarly, we also have an isometry $UCP_n(Y) \to UCP_n(X \oplus Y)$. For notational simplicity we will thus identify $UCP_n(X)$ and $UCP_n(Y)$ with their respective images under these isometries.

Definition 3.2. Let $(X, L_X)$ and $(Y, L_Y)$ be Lip-normed operator systems. For each $n \in \mathbb{N}$ we define the $n$-distance
\[
\text{dist}_s^n(X, Y) = \inf_{L \in \mathcal{M}(L_X, L_Y)} \text{dist}_{H}^{\rho_{L,n}}(UCP_n(X), UCP_n(Y))
\]
where $\text{dist}_{H}^{\rho_{L,n}}$ denotes Hausdorff distance with respect to the metric $\rho_{L,n}$. We also define the complete distance
\[
\text{dist}_s(X, Y) = \inf_{L \in \mathcal{M}(L_X, L_Y)} \sup_{n \in \mathbb{N}} \text{dist}_{H}^{\rho_{L,n}}(UCP_n(X), UCP_n(Y)).
\]

Proposition 3.3 (triangle inequality). If $(X, L_X)$, $(Y, L_Y)$, and $(Z, L_Z)$ are Lip-normed operator systems then
\[
\text{dist}_s^n(X, Z) \leq \text{dist}_s^n(X, Y) + \text{dist}_s^n(Y, Z),
\]
for all $n \in \mathbb{N}$, and
\[
\text{dist}_s(X, Z) \leq \text{dist}_s(X, Y) + \text{dist}_s(Y, Z).
\]

Proof. This follows by exactly the same argument used for quantum Gromov-Hausdorff distance in [9], since in the last part of the proof of [9, Thm. 4.3] we can replace the state spaces by matrix state spaces and the reference to [9, Prop. 3.1] by a reference to our Proposition 2.13. \qed

In order to build a general framework for estimating distance between quantum metric spaces, Rieffel formulates in [9, Defn. 5.1] the notion of a bridge, which we now recall.

Definition 3.4. Let $(A, L_A)$ and $(B, L_B)$ be compact quantum metric spaces. A bridge between $(A, L_A)$ and $(B, L_B)$ is a norm-continuous semi-norm $N$ on $A \oplus B$ such that
\begin{enumerate}
  \item $N(1_A, 1_B) = 0$ while $N(1_A, 0) \neq 0$, and
  \item for each $a \in A$ and $\delta > 0$ there exists a $b \in B$ such that
    \[
    \max(L_B(b), N(a, b)) \leq L_A(a) + \delta,
    \]
\end{enumerate}
with the same statement also holding upon interchanging $A$ and $B$.

Theorem 5.2 in [9] then shows that if $N$ is a bridge between the compact quantum metric spaces $(A, L_A)$ and $(B, L_B)$ then
\[
L(a, b) = \max(L_A(a), L_B(b), N(a, b))
\]
defines a Lip-norm $L$ on $A \oplus B$ which induces $L_A$ and $L_B$ via the respective quotient maps. Since $N$ is norm-continuous, $L$ will be closed if $L_A$ and $L_B$ are both closed.
If \((X, L_X)\) and \((Y, L_Y)\) are Lip-normed operator systems then by a bridge between \((X, L_X)\) and \((Y, L_Y)\) we will mean a bridge between the compact quantum metric spaces \((\mathcal{D}(L_X), L_X)\) and \((\mathcal{D}(L_Y), L_Y)\). We begin by illustrating this notion with a simple example which shows that if we scale a Lip-norm by a factor \(\lambda\) and let \(\lambda \to \infty\) then we obtain convergence to a “point,” just as for ordinary metric spaces.

**Example 3.5.** Let \((X, L)\) be a Lip-normed operator system. For each \(\lambda > 0\) define the Lip-normed operator system \((X, L_\lambda)\) by setting \(L_\lambda = \lambda L\). Let \((C, P)\) be the “one-point” Lip-normed operator system, with \(P(\mu) = 0\) for all \(\mu \in C\). Then 
\[
\text{dist}_s((X, L), (C, P)) \leq C\lambda^{-1}
\]
where \(C = \text{diam}(X, L)\). To show this we define a bridge on \(\mathcal{D}(L_X) \oplus C\) by
\[
N_\lambda(x, \mu) = C^{-1}\lambda\|x - \mu 1_X\|.
\]
To see that this is indeed a bridge we verify condition (ii) in Definition 3.4 by observing that \(N_\lambda(\mu 1_X, \mu) = 0\) for all \(\mu \in C\) while if \(x \in \mathcal{D}(L_X)\) then letting \(r \) denote the infimum of the spectrum of \(x\) we have by Proposition 2.11
\[
\|x - r 1_X\| \leq \text{diam}(X, L)L_X(x) = C\lambda^{-1}L_\lambda(x).
\]
Let \(M_\lambda\) be the Lip-norm in \(\mathcal{M}(L_\lambda, P)\) given by \(M_\lambda(x, \mu) = \max(L_\lambda(x), P(\mu), N_\lambda(x, \mu))\). Let \(n \in \mathbb{N}\) and let \(\psi\) be the unique element in \(UCP_n(C)\). If \(\varphi \in UCP_n(X)\) and \((x, \mu) \in \mathcal{D}_1(M)\) then
\[
\|\varphi(x) - \psi(\mu)\| = \|\varphi(x - \mu 1_X)\| \leq \|x - \mu 1_X\| \leq C\lambda^{-1},
\]
yielding the desired complete distance estimate. Hence \((X, L_\lambda)\) converges to \((C, P)\) as \(\lambda \to \infty\) for complete distance.

As in the quantum metric setting [1, Prop. 5.4] we can apply the concept of a bridge to show that the complete distance (and hence also the \(n\)-distance) is always finite.

**Proposition 3.6.** If \((X, L_X)\) and \((Y, L_Y)\) are Lip-normed operator systems then 
\[
\text{dist}_s(X, Y) \leq \text{diam}(X, L_X) + \text{diam}(Y, L_Y).
\]

**Proof.** As in the proof of [1, Prop. 5.4], for arbitrary \(\gamma > 0, \sigma_0 \in S(X)\), and \(\omega_0 \in S(Y)\) we can construct a bridge 
\[
N(x, y) = \gamma^{-1}|\sigma_0(x) - \omega_0(y)|.
\]
Let \(L\) be the Lip-norm in \(\mathcal{M}(L_X, L_Y)\) given by \(L(x, y) = \max(L_X(x), L_Y(y), N(x, y))\). Then if \((x, y) \in \mathcal{D}_1(L)\), \(\varphi \in UCP_n(X)\), and \(\psi \in UCP_n(Y)\) we can find a \(\sigma \in S(M_n)\) such that 
\[
\|\varphi(x) - \psi(y)\| = |(\sigma \circ \varphi)(x) - (\sigma \circ \psi)(x)|
\]
whence
\[
\|\varphi(x) - \psi(y)\| \leq |(\sigma \circ \varphi)(x) - \sigma_0(x)| + |\sigma_0(x) - \omega_0(y)| + |\omega_0(y) - (\sigma \circ \psi)(y)|
\]
\[
\leq \rho_{L_X,1}(\sigma \circ \varphi, \sigma_0) + \gamma + \rho_{L_Y,1}(\omega_0, \sigma \circ \psi).
\]
Since \(\gamma\) was arbitrary the proposition follows. \(\square\)

Propositions 3.7 and 3.8 yield estimates on the complete distance in situations involving bridges constructed via the norm.
**Proposition 3.7.** Let \((X, L_X)\) and \((Y, L_Y)\) be Lip-normed operator systems, and suppose \(X\) and \(Y\) are operator subsystems of an operator system \(Z\). Let \(\epsilon > 0\) and suppose that \(N(x, y) = \epsilon^{-1}\|x - y\|\) defines a bridge between \((X, L_X)\) and \((Y, L_Y)\). Then \(\text{dist}_s(X, Y) \leq \epsilon\).

**Proof.** Let \(L\) be the Lip-norm in \(M(L_X, L_Y)\) given by
\[
L(x, y) = \max \left( L_X(x), L_Y(y), \epsilon^{-1}\|x - y\| \right)
\]
for all \((x, y) \in \mathcal{D}(L_X) \oplus \mathcal{D}(L_Y)\). Let \(n \in \mathbb{N}\) and \(\varphi \in UC_Pn(X)\). By Arveson’s extension theorem we can extend \(\varphi\) to a u.c.p. map \(\tilde{\varphi} : Z \to M_n\). Then if \((x, y) \in \mathcal{D}_1(L)\) we have
\[
\|\varphi(x) - \tilde{\varphi}(y)\| \leq \|x - y\| < \epsilon
\]
and thus \(\rho_{L,n}(\varphi, \tilde{\varphi}|_Y) < \epsilon\). We can interchange the roles of \(X\) and \(Y\) and apply the same argument to obtain \(\text{dist}_s(X, Y) \leq \epsilon\). \(\square\)

**Proposition 3.8.** Let \((X, L_X)\) be a Lip-normed operator system, \(Y\) an operator system, \(\Phi : X \to Y\) a surjective unital positive map, and \(L_Y\) the quotient Lip-norm induced by \(L_X\) via \(\Phi\). Let \(Z\) be an operator system containing \(X\) as an operator subsystem and let \(\Gamma : Y \to Z\) be a unital map such that \(\|(\Gamma \circ \Phi)(x) - x\| \leq \epsilon\) for all \(x \in \mathcal{D}_1(L)\). If \(\Phi\) and \(\Gamma\) are \(n\)-positive then \(\text{dist}^n_s(X, Y) \leq \epsilon\), and if \(\Phi\) and \(\Gamma\) are completely positive then \(\text{dist}_s(X, Y) \leq \epsilon\).

**Proof.** Given \(\eta > 0\) we define the bridge \(N\) between \((X, L)\) and \((Y, L_Y)\) by \(N(x, y) = \eta^{-1}\|\Phi(x) - y\|\) for all \((x, y) \in X \oplus Y\) (to verify condition (ii) of Definition 3.4, given \(x \in X\) and \(\delta > 0\) we can choose \(y = \Phi(x)\), while if \(y \in Y\) and \(\delta > 0\) we can take any \(x \in X\) such that \(\Phi(x) = y\) and \(L_X(x) \leq L_Y(y) + \delta\). Let \(L\) be the Lip-norm \(L\) in \(M(L_X, L_Y)\) given by
\[
L(x, y) = \max(L_X(x), L_Y(y), N(x, y))
\]
for all \((x, y) \in X \oplus Y\). Suppose that \(\Phi\) is \(n\)-positive. Then if \(\varphi \in UC_Pn(Y)\) we have \(\varphi \circ \Phi \in UC_Pn(X)\), and so if \(L(x, y) \leq 1\) then \(\|\Phi(x) - y\| \leq \eta\) so that \(\|(\varphi \circ \Phi)(x) - \varphi(y)\| \leq \eta\), whence \(\rho_{L,n}(\varphi \circ \Phi, \varphi) \leq \eta\). On the other hand if \(\varphi \in UC_Pn(X)\) then, extending \(\varphi\) to a u.c.p. map \(\tilde{\varphi} : Z \to M_n\) by Arveson’s extension theorem, we have \(\tilde{\varphi} \circ \Gamma \in UC_Pn(Y)\), and so if \(L(x, y) \leq 1\) then
\[
\|\varphi(x) - (\tilde{\varphi} \circ \Gamma)(y)\| \leq \|\tilde{\varphi}(x) - (\Gamma \circ \Phi)(x)\| + \|(\tilde{\varphi} \circ \Gamma)(\Phi(x) - y)\|
\]
\[
\leq \epsilon + \eta,
\]
yielding \(\rho_{L,n}(\varphi, \tilde{\varphi} \circ \Gamma) \leq \epsilon + \eta\). Since \(\eta\) was arbitrary we conclude that \(\text{dist}^n_s(X, Y) \leq \epsilon\).

In the case that \(\Phi\) is completely positive we can apply the above argument over all \(n \in \mathbb{N}\) to obtain \(\text{dist}_s(X, Y) \leq \epsilon\). \(\square\)

The following three propositions guarantee approximability by Lip-normed well-behaved finite-dimensional operator systems under conditions on the given operator system or \(C^*\)-algebra which hold in a wide range of situations.

**Proposition 3.9.** Let \((X, L_X)\) be a Lip-normed nuclear operator system. Then for every \(\epsilon > 0\) there is a Lip-normed operator system \((Y, L_Y)\) such that \(Y\) is an operator subsystem of a matrix algebra and \(\text{dist}_s(X, Y) \leq \epsilon\).
Proposition 3.11. Let \((A, L)\) be a Lip-normed unital strong \(C^\ast\)-algebra. Then for every \(\epsilon > 0\) there is a Lip-normed finite-dimensional \(C^\ast\)-algebra \((B, L_B)\) such that \(\text{dist}_s(A, B) \leq \epsilon\).

Proof. Since \(X\) is nuclear and the set \(\mathcal{D}_1(L_X) \cap X_{\text{diam}(X,L_X)}\) is compact (see the second half of the proof of Proposition 2.12), by Proposition 2.11 we can find a matrix algebra \(M_k\) and u.c.p. maps \(\Phi : X \to M_k\) and \(\Gamma : M_k \to X\) such that \(\|(\Gamma \circ \Phi)(x) - x\| \leq \epsilon\) for all \(x \in \mathcal{D}_1(L_X)\). Consider the image \(Y\) of \(\Phi\) and the resulting quotient Lip-norm \(L_Y\) on \(Y\). Then by Proposition 3.8 we have \(\text{dist}_s(X, Y) \leq \epsilon\), yielding the result.  

By a proof similar to that of Proposition 3.9, we also have the following.

Proposition 3.11. Let \((A, L)\) be a Lip-normed unital exact \(C^\ast\)-algebra. Then for every \(\epsilon > 0\) there is a Lip-normed operator system \((Y, L_Y)\) such that \(Y\) is an operator subsystem of a matrix algebra and \(\text{dist}_s(A, Y) \leq \epsilon\).

Example 3.12. In [10] Rieffel shows, in the context of Berezin quantization, that the sphere \(S^2\) is a limit of matrix algebras with respect to quantum Gromov-Hausdorff distance. In fact a more general statement applying to integral coadjoint orbits of a compact Lie group is established. We will briefly indicate how Rieffel’s approach leads to precisely the same estimates for complete distance, adopting the same notation as in [10], to which we refer the reader for more details.
G on $G/H$ by left translation combines with $\ell$ to produce a Lip-norm $L_A$ on $A$ (permitting the value $+\infty$ for convenience). Corollary 2.4 of [10] shows that, for some $\gamma > 0$, there is a bridge between $(A, L_A)$ and $(B, L_B)$ of the form

$$N(f, T) = \gamma^{-1}\|f - \sigma_T\|_\infty.$$  

Proposition 1.3 of [10] then shows that $S(A)$ is in the $\gamma$-neighbourhood of $S(B)$ under the metric defined by the Lip-norm $L$ on $A \oplus B$ associated to $N$. But the argument there also applies at the matrix level: given $\varphi \in UCP_n(A)$ we have $\varphi \circ \sigma \in UCP_n(B)$ since $\sigma$ is u.c.p., and if $L(f, T) \leq 1$ then $\|f - \sigma_T\|_\infty \leq \gamma$ so that

$$\|\varphi(f) - (\varphi \circ \sigma)(T)\| = \|\varphi(f) - \sigma_T\| \leq \|f - \sigma_T\|_\infty \leq \gamma,$$

showing that $UCP_n(A)$ lies in the $\gamma$-neighbourhood of $UCP_n(B)$ with respect to the metric $\rho_{L,n}$. Now on the other hand if $\psi \in UCP_n(B)$ then we can consider the adjoint operator $\check{\sigma} : A \to B$ and take the composition $\psi \circ \check{\sigma}$, which is in $UCP_n(A)$ since $\check{\sigma}$ is unital and positive and hence u.c.p. because its domain is a commutative $C^*$-algebra. Then if $L(f, T) \leq 1$ we have $\|f - \sigma_T\|_\infty \leq \gamma$ from which we obtain the estimate

$$\|(\psi \circ \check{\sigma})(f) - \psi(T)\| = \|\psi(\check{\sigma}_f - T)\|$$

$$\leq \|\check{\sigma}_f - T\|$$

$$\leq \|\check{\sigma}_f - \check{\sigma}(\sigma_T)\| + \|\check{\sigma}(\sigma_T) - T\|$$

$$\leq \|f - \sigma_T\|_\infty + \|\check{\sigma}(\sigma_T) - T\|$$

$$\leq \gamma + \|\check{\sigma}(\sigma_T) - T\|$$

exactly as in the case $n = 1$ in the discussion following [10, Cor. 2.4]. When $G$ is a compact Lie group Rieffel shows that, by replacing $B$ with the $C^*$-algebra of bounded operators on the $m$th tensor power of the original Hilbert space, both the bridge constant $\gamma$ and the term $\|\check{\sigma}(\sigma_T) - T\|$ can be made arbitrarily small by taking $m$ sufficiently large (see Theorem 3.2 and Sections 3–5 of [10]), yielding an asymptotically vanishing bound on the quantum Gromov-Hausdorff distance as a result of the estimates in the two previous displays for the case $n = 1$. But since these estimates apply equally well for all $n$ we get the same bounds for complete distance.

4. Distance zero

This section is aimed at establishing that $\dist^s_n(X, Y) = 0$ (resp. $\dist^s_n(X, Y) = 0$) is equivalent to the existence of a bi-Lip-isometric unital $n$-order isomorphism (resp. bi-Lip-isometric unital complete order isomorphism) between $X$ and $Y$ (Theorem 4.11). One direction is straightforward:

**Proposition 4.1.** Let $(X, L_X)$ and $(Y, L_Y)$ be Lip-normed operator systems. If there is a bi-Lip-isometric unital $n$-order isomorphism between $X$ and $Y$ then $\dist^s_n(X, Y) = 0$. If there is a bi-Lip-isometric unital complete order isomorphism between $X$ and $Y$ then $\dist^s_n(X, Y) = 0$.

**Proof.** This follows from Proposition 3.8, taking $\Phi$ there to be a bi-Lip-isometric unital $n$-order isomorphism (resp. bi-Lip-isometric unital complete order isomorphism) from $X$ onto $Y$ and taking $\Gamma$ to be its inverse. \qed
For the converse, it will be convenient to extend our Lip-norms in an adjoint-invariant way (Definition 4.2) and to introduce a collection of matrix semi-norms (Definition 4.4).

**Definition 4.2.** Let \((X, L)\) be a Lip-normed operator system. We define the semi-norm \(L^e \) on \(X\) by
\[
L^e(x) = \sup \left\{ \frac{|\sigma(x) - \omega(x)|}{\rho_{L,1}(\sigma, \omega)} : \sigma, \omega \in S(X) \text{ and } \sigma \neq \omega \right\}
\]
for all \(x \in X\) (permitting the value \(+\infty\)).

**Proposition 4.3.** The set of self-adjoint elements on which the semi-norm \(L^e\) in Definition 4.2 is finite coincides with the domain of \(L\), and on this set \(L^e = L\).

**Proof.** By Proposition 6.1 of \[9\], \(\mathcal{D}(L)\) corresponds to the subspace of affine functions on \(S(\mathcal{D}(L))\) which are Lipschitz for \(\rho_L\). This, along with the fact that the restriction map from \(S(X)\) to \(S(\mathcal{D}(L))\) is a weak* homeomorphism which is isometric for \(\rho_{L,1}\) and \(\rho_L\) (see the proof of Proposition 2.5), implies the result.

**Definition 4.4.** Let \((X, L)\) be a Lip-normed operator system and \(n \in \mathbb{N}\). We define the semi-norm \(L^n\) on \(M_n \otimes X\) by
\[
L^n(x) = \max_{1 \leq i,j \leq n} L^e(x_{ij})
\]
for all \((x_{ij}) \in M_n(X) \cong M_n \otimes X\).

For the meaning of the notation \(\mathcal{D}(\cdot)\) and \(\mathcal{D}_\lambda(\cdot)\), as will be applied to the semi-norms \(L^e\) and \(L^n\), see Notation 2.2.

**Lemma 4.5.** Let \((X, L)\) be a Lip-normed operator system. Then \(\mathcal{D}(L^e)\) is dense in \(X\) and \(\mathcal{D}(L^n)\) is dense in \(M_n \otimes X\) for all \(n \in \mathbb{N}\).

**Proof.** Since \(L^e\) is adjoint-invariant and coincides with \(L\) on the dense real subspace \(\mathcal{D}(L)\) of \(X_{sa}\) by Proposition 4.3, using the decomposition of elements into real and imaginary parts we see that \(\mathcal{D}(L^e)\) is dense in \(X\). As a direct consequence \(\mathcal{D}(L^n)\) is dense in \(M_n \otimes X\) for all \(n \in \mathbb{N}\).

**Lemma 4.6.** Let \((X, L_X)\) and \((Y, L_Y)\) be Lip-normed operator systems, and let \(L \in \mathcal{M}(L_X, L_Y)\) and \(n \in \mathbb{N}\). Set \(r = \operatorname{dist}_{\mathcal{U}CP_n}^{\rho_{L,n}}(UCP_n(X), UCP_n(Y))\). Then for every \(\psi \in SCP_n(Y)\) there is a \(\varphi \in SCP_n(X)\) such that, for all \((x, y) \in \mathcal{D}(L^e)\),
\[
\|\varphi(x) - \psi(y)\| \leq 2n^3 L^e(x, y)r.
\]

**Proof.** To prove the lemma we may assume that \(\psi(1)\) has full support in \(M_n\), for otherwise for every \(k \in \mathbb{N}\) we can perturb \(\psi\) to a convex combination \((1 - k^{-1})\psi + k^{-1}\alpha\) where \(\alpha(x) = \omega(x)1_{M_n}\) for some \(\omega \in S(Y)\) and all \(x \in X\) (in which case the corresponding state \(\sigma_\psi\) on \(M_n \otimes Y\) is perturbed to another a state), find a suitable \(\varphi_k\) as in the lemma statement with respect to \(\psi_k\), and then take a point-norm limit point of \(\{\varphi_k\}_{k \in \mathbb{N}}\) to obtain the desired \(\phi\). We can thus consider the map \(\psi' \in UCP_n(Y)\) given by
\[
\psi'(y) = \psi(1)^{-\frac{2}{n}}\psi(y)\psi(1)^{-\frac{1}{n}}.
\]
for all \( y \in Y \). By assumption we can find a \( \varphi' \in UCP_n(X) \) such that
\[
\rho_{\lambda,n}(\varphi', \psi') \leq r.
\]

Let \( \varphi : Y \to M_n \) be the c.p. map given by
\[
\varphi(x) = \psi(1)^{\frac{1}{2}} \varphi'(x) \psi(1)^{\frac{1}{2}}
\]
for all \( x \in X \). Then \( \varphi(1) = \psi(1) \), which implies that \( \sigma_\varphi \) is a state on \( M_n \otimes X \), so that
\( \varphi \in SCP_n(Y) \). Since \( \sigma_{\psi} \) is a state on \( M_n \otimes Y \) we must have \( \|\psi(1)\| \leq n^3 \), and thus if \( (x, y) \in D(L^e) \) then
\[
\|\varphi(x) - \psi(y)\| = \|\psi(1)^{\frac{1}{2}}(\varphi'(x) - \psi'(y))\psi(1)^{\frac{1}{2}}\|
\leq \|\psi(1)^{\frac{1}{2}}\| \|\varphi'(x) - \psi'(y)\| \|\psi(1)^{\frac{1}{2}}\|
\leq n^3 \|\varphi'(\text{Re}(x)) - \psi'(\text{Re}(y))\| + \|\varphi'(\text{Im}(x)) - \psi'(\text{Im}(y))\|
\leq n^3 (L(\text{Re}(x), \text{Re}(y)) + L(\text{Im}(x), \text{Im}(y))) \rho_{\lambda,n}(\varphi', \psi')
\leq 2n^3 L^e(x, y) \rho_{\lambda,n}(\varphi', \psi')
\leq 2n^3 L^e(x, y)r
\]
with the second last inequality following from the adjoint invariance of \( L^e \) and the fact that \( L^e = L \) on \( D(L) \) by Proposition 4.3.

**Definition 4.7.** Let \((X, L_X)\) and \((Y, L_Y)\) be Lip-normed operator systems, and let \( L \in M(L_X, L_Y) \) and \( n \in \mathbb{N} \). For each \( \lambda \geq 0 \), \( x \in D(L^n_X) \), and \( y \in D(L^n_Y) \) we set
\[
N^\lambda_{L^n_X, Y}(x) = \{ z \in M_n \otimes Y : (x, z) \in D(\lambda^n) \},
\]
\[
N^\lambda_{L^n_X, X}(y) = \{ z \in M_n \otimes X : (z, y) \in D(\lambda^n) \}.
\]

**Lemma 4.8.** Let \((X, L_X)\) and \((Y, L_Y)\) be Lip-normed operator systems, and let \( L \in M(L_X, L_Y) \) and \( n \in \mathbb{N} \). Set \( r = \text{dist}_{H}^{\rho_{\lambda,n}}(UCP_n(X), UCP_n(Y)) \). If \( x \in D(L^n_X) \) and \( \lambda > 2L^\lambda_X(x) \) then
(i) \( N^\lambda_{L^n_X, Y}(x) \) is non-empty and closed,
(ii) the norms of elements of \( N^\lambda_{L^n_X, Y}(x) \) are bounded by \( 4(\|x\| + \lambda n^4 r) \), and if \( x \) and \( y \in N^\lambda_{L^n_X, Y}(x) \) are self-adjoint then the norm of \( y \) is bounded by \( \|x\| + 2\lambda n^4 r \),
(iii) the norm diameter of \( N^\lambda_{L^n_X, Y}(x) \) is bounded by \( 8\lambda n^4 r \),
(iv) if \( x' \) is another element of \( D(L_X) \) and \( \lambda \) is also strictly larger than \( L^\lambda_X(x') \) then the Hausdorff distance between \( N^\lambda_{L^n_X, Y}(x) \) and \( N^\lambda_{L^n_X, Y}(x') \) is bounded by \( 8\lambda n^4 r + 4\|x - x'\| \), and
(v) if \( x \geq 0 \) then there is a (self-adjoint) \( y \in N^\lambda_{L^n_X, Y}(x) \) with \( y \geq -2\lambda n^4 r \).

Statements (i)–(v) also hold for \( y \in D(L^n_Y) \) and \( \lambda > 2L^\lambda_Y(y) \) if \( L_X \) and \( N^\lambda_{L^n_X, Y}(x) \) are replaced by \( L_Y \) and \( N^\lambda_{L^n_X, X}(y) \), respectively.
Proof. Since symmetry will take care of the last sentence of the proposition statement, we prove (i)–(v) as written for $x \in \mathcal{D}(L^n)$ and $\lambda > 2L^n(x)$. We begin with (i). For $1 \leq i, j \leq n$ we have $\max(L_X(\text{Re}(x_{ij})), L_X(\text{Im}(x_{ij}))) < L^e(x_{ij}) < \lambda/2$ so that we can find $y_{ij} \in N^n_{L^n,i,j}(\text{Re}(x_{ij}))$ and $z_{ij} \in N^n_{L^n,i,j}(\text{Im}(x_{ij}))$ since $L_X$ is the quotient Lip-norm induced by $L$. Then

$$L^n((x_{ij}), (y_{ij}) + i(z_{ij})) \leq L^n((\text{Re}(x_{ij})), (y_{ij})) + L^n((\text{Im}(x_{ij})), (z_{ij})) < \lambda$$

so that $N^{\lambda/2}_{L^n,i,j}(x)$ contains $(y_{ij}) + i(z_{ij})$ and is in particular non-empty. That this set is also closed follows from the lower semicontinuity of $L^e$, which is easily checked.

For (ii), let $y = (y_{ij}) \in N^n_{L^n,i,j}(x)$ and $\psi \in SCP_n(Y)$. By Lemma 4.6 there is a $\varphi \in SCP_n(X)$ such that $\|\varphi(z) - \psi(w)\| \leq 2n^3L^e(z, w)r$ for all $z, w \in \mathcal{D}(L^e)$. We then have

$$\|\varphi(x) - \psi(y)\| = \frac{1}{n} \left| \sum_{i,j} (\varphi(x_{ij})_{ij} - \psi(y_{ij})_{ij}) \right|$$

$$\leq \frac{1}{n} \sum_{i,j} |\varphi(x_{ij})_{ij} - \psi(y_{ij})_{ij}|$$

$$\leq \frac{1}{n} \sum_{i,j} \|\varphi(x_{ij}) - \psi(y_{ij})\|$$

$$\leq 2n^2 \sum_{i,j} L^e(x_{ij}, y_{ij})r$$

$$\leq 2n^4L^n(x, y)r$$

$$\leq 2\lambda n^4r.$$
Hence $\|y - y'\| \leq 8\lambda n^4 r + 4\|x - x'\|$, from which (iv) follows.

Finally, to prove (v) we suppose $x \geq 0$. By part (i) there is a $y = (y_{ij}) \in N_{L^q,Y}^\lambda(x)$. We then have, for $1 \leq i, j \leq n$,

$$L(x_{ij}, (\text{Re}(y))_{ij}) \leq \frac{1}{2} L^e(x_{ij}, y_{ij}) + \frac{1}{2} L^e(x_{ij}^*, y_{ij}^*) = \frac{1}{2} L^e(x_{ij}, y_{ij}) + \frac{1}{2} L^e(x_{ij}, y_{jj}) \leq L^n(x, y)$$

using the adjoint invariance of $L^e$, and so $\text{Re}(y)$ is a self-adjoint element of $N_{L^q,Y}^\lambda(x)$. Suppose now that $y$ is an arbitrary self-adjoint element of $N_{L^q,Y}^\lambda(x)$. If $\psi \in \text{SCP}_n(Y)$ then as in the proof of (ii) there is a $\varphi \in \text{SCP}_n(X)$ such that $|\sigma_\varphi(x) - \sigma_\psi(y)| \leq 2\lambda n^4 r$, and thus since $\sigma_\varphi(x) \geq 0$ and $y$ is self-adjoint we infer that $\sigma_\varphi(y) \geq -2\lambda n^4 r$. Hence we conclude that $y \geq -2\lambda n^4 r$.

**Proposition 4.9.** Let $(X, L_X)$ and $(Y, L_Y)$ be Lip-normed operator systems, and suppose $\text{dist}^\lambda_n(X, Y) = 0$ for some $n \in \mathbb{N}$. Then there is a unital order isomorphism $\Phi : M_n \otimes X \to M_n \otimes Y$, and in the case $n = 1$ we may arrange that $\mathcal{D}(L_Y) = \Phi(\mathcal{D}(L_X))$ and $L_Y(\Phi(x)) = L_X(x)$ for all $x \in \mathcal{D}(L_X)$.

**Proof.** By assumption there is a sequence $\{L_k\}_{k \in \mathbb{N}}$ of Lip-norms in $\mathcal{M}(L_A, L_B)$ such that $\lim_{k \to \infty} r_k = 0$ where $r_k = \text{dist}_H^{\mu_k}(\text{UCP}_n(X), \text{UCP}_n(Y))$. Let $x \in \mathcal{D}(L^n_X)$ and $\lambda > 2L^n_X(x)$. Set $s = 4(\|x\| + 2\lambda n^4)$. In view of Lemma 4.8(ii) we may assume (by removing finitely many of the $L_k$’s and reindexing the sequence if necessary) that the sets $N_{L^k_n,X}^\lambda(x)$ for $k \geq 1$ are all contained in $\mathcal{D}_s(\mathcal{D}_Y^Y) \cap \mathcal{B}_s M_n \otimes Y$. This latter set is compact, since it is closed by the lower semicontinuity of $L^n_Y$ and for any $t > 0$ the set $\mathcal{D}_t(\mathcal{D}_Y) \cap \mathcal{B}_t^Y$ is compact (see the second half of the proof of Proposition 2.12), which implies the compactness of the set $\mathcal{D}_t(\mathcal{D}_Y^Y) \cap \mathcal{B}_t^Y$ (use the decomposition of elements into real and self-adjoint parts) and hence also the total boundedness of the tensor product of $\mathcal{D}_t(\mathcal{L}^Y_k) \cap \mathcal{B}_t^Y$ with $M_n$. Since by Lemma 4.8(iii) the diameters of $N_{L^k_n,X}^\lambda(x)$ converge to zero as $k \to \infty$, this implies the existence of a subsequence of $\{N_{L^k_n,X}^\lambda(x)\}_{k \in \mathbb{N}}$ which converges in Hausdorff distance to some singleton $\{\Phi(x)\}$. This singleton must in fact be the same for each $\lambda > 2L^n_X(x)$ because for each $k \in \mathbb{N}$ we have

$$N_{L^k_n,X}^\lambda(x) \subset N_{L^k_n,X}^{\lambda'}(x)$$

whenever $\lambda \leq \lambda'$. Using a diagonal argument and relabeling indices we may assume that, for all $x$ in a countable dense subset $D$ of $\mathcal{D}(L^n_X)$, the sets $N_{L^k_n,X}^\lambda(x)$ for $\lambda > 2L^n_X(x)$ converge in Hausdorff distance as $k \to \infty$ to some singleton $\{\Phi(x)\}$. Then in fact for any $x \in \mathcal{D}(L^n_X)$ and $\lambda > 2L^n_X(x)$ the sets $N_{L^k_n,X}^\lambda(x)$ converge as $k \to \infty$ to some singleton $\{\Phi(x)\}$, since for any $\epsilon > 0$ we can take an $x' \in D$ with $\|x - x'\| \leq \epsilon/16$ and $2L^n_X(x') < \lambda$ (since we may assume that $D$ was chosen so that $D \cap \mathcal{D}_q(L_X^n)$ is dense in $\mathcal{D}_q(L_X^n)$ for all positive rational numbers $q$) and a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $N_{L^k_n,X}^\lambda(x')$ is within Hausdorff distance $\epsilon/2$ of $\{\Phi(x')\}$ and $8\lambda n^4 r_k \leq \epsilon/2$, from which it can be seen using
Lemma 4.8(iv) that for all \( k \geq k_0 \) the set \( \mathcal{N}_{L^\infty_{k,X}}(x) \) lies inside the ball of radius \( \epsilon \) centred at \( \Phi(x') \).

Now if \( \mu \in \mathbb{C} \) and \( x, x' \in \mathcal{D}(L^n_X) \) then for \( \lambda > 2 \left( |\mu| L^n_X(x) + L^n_X(x') \right) \) it is easily seen that

\[
\mathcal{N}_{L^\infty_{k,Y}}(\mu x + x') \supset \{ \mu y + y' : y \in \mathcal{N}_{L^\infty_{k,Y}}(x) \text{ and } y' \in \mathcal{N}_{L^\infty_{k,Y}}(x') \}
\]

so that \( \mathcal{N}_{L^\infty_{k,Y}}(\mu x + x') \) converges to \( \{ \mu \Phi(x) + \Phi(x') \} \) as \( k \to \infty \). Also, for every \( x \in \mathcal{D}(L^n_X) \) and \( \lambda > L^n_X(x) \) we have \( y^* \in \mathcal{N}_{L^\infty_{k,Y}}(x^*) \) if and only if \( y \in \mathcal{N}_{L^\infty_{k,Y}}(x) \), so that \( \{ \Phi(x^*) \} \) is equal to the limit of \( \{ y^* : y \in \mathcal{N}_{L^\infty_{k,Y}}(x) \} \), which by the continuity of the involution must be \( \{ (\Phi(x))^* \} \). Hence we have defined a *-linear map \( \Phi : \mathcal{D}(L^n_X) \to M_n \otimes Y \). Note also that \( \Phi \) is unital since \( 1_\mathcal{Y} \in \mathcal{N}_{L^\infty_{k,Y}}(1_X) \) for all \( \lambda > 0 \). We furthermore have by Lemma 4.8(ii) that the norm of \( \Phi \) is bounded by 4 on \( D \), and thus, since \( D \) is dense in \( \mathcal{D}(L^n_X) \) which in turn is dense in \( M_n \otimes X \) by Lemma 4.3, \( \Phi \) extends uniquely to a bounded *-linear map from \( M_n \otimes X \) to \( M_n \otimes Y \), which we will again denote by \( \Phi \).

Now by another diagonal argument and index relabeling we may assume that \( \mathcal{N}_{L^\infty_{k,X}}(y) \) converges in Hausdorff distance as \( k \to \infty \) to a singleton \( \{ \Gamma(y) \} \) for all \( y \) in a countable dense subset of \( \mathcal{D}(L^n_Y) \) which contains \( \Phi(D) \). We thus obtain, as above, a bounded unital *-linear map \( \Gamma : \mathcal{D}(L^n_Y) \to \mathcal{D}(L^n_X) \). We will show that \( \Phi \) and \( \Gamma \) are mutual inverses. Suppose \( x \in \mathcal{D}(L) \) and \( \lambda > 2 \max(L^n_X(x), L^n_Y(\Phi(x))) \). For each \( k \in \mathbb{N} \) choose \( x_k' \in \mathcal{N}_{L^\infty_{k,X}}(\Phi(x)) \) and \( y_k \in \mathcal{N}_{L^\infty_{k,Y}}(x) \). Let \( \epsilon > 0 \) and \( \varphi \in \text{SCP}_n(Y) \). Pick \( k_0 \in \mathbb{N} \) large enough so that, for all \( k \geq k_0 \), \( 2n^4 r_k \leq \epsilon \) and \( \| y_k - \Phi(x) \| \leq \epsilon \). Then as in the proof of Lemma 4.8(i) for any \( k \geq k_0 \) we can find a \( \psi \in \text{SCP}_n(X) \) such that

\[
|\sigma_\varphi(x) - \sigma_\psi(y_k)| \leq 2n^4 L^\epsilon(x, y_k) r_k \leq 2n^4 \lambda r_k \leq \epsilon
\]

and similarly \( |\sigma_\varphi(x'_k) - \sigma_\psi(\Phi(x))| \leq \epsilon \), whence by the triangle inequality

\[
|\sigma_\varphi(x) - \sigma_\varphi(x'_k)| \leq 2\epsilon + |\sigma_\psi(y_k) - \sigma_\psi(\Phi(x))| \leq 3\epsilon.
\]

Therefore \( \| x - x_k' \| \leq 6\epsilon \), and so we have \( \lim_{k \to \infty} x_k' = x \). Hence

\[
\Gamma(\Phi(x)) = \Gamma \left( \lim_{k \to \infty} y_k \right) = \lim_{k \to \infty} \Gamma(y_k) = \lim_{k \to \infty} x_k' = x.
\]

By a similar argument \( \Phi(\Gamma(y)) = y \) for all \( y \in \mathcal{D}(L^n_Y) \), and hence by continuity we conclude that \( \Phi \) and \( \Gamma \) are mutual inverses.

Next we show that \( \Phi \) and \( \Gamma \) are positive. If \( x \in \mathcal{D}(L^n_X), x \geq 0 \), and \( \lambda > 2L^n_X(x) \), then Lemma 4.3(v) yields, for all \( k \in \mathbb{N} \), a \( y_k \in \mathcal{N}_{L^\infty_{k,Y}}(x) \) with \( y_k \geq -2n^4 r_k \). Then \( y \) is the limit as \( k \to \infty \) of the positive elements \( y_k + 2n^4 r_k \) and hence \( y \) itself is positive. Thus \( \Phi \) is positive, and by a symmetric argument so is \( \Gamma \). Hence \( \Phi \) is a unital order isomorphism.

It remains to show that if \( n = 1 \) then \( \Phi \) is isometric with respect to \( L_X \) and \( L_Y \), that is, \( \mathcal{D}(L_Y) = \Phi(\mathcal{D}(L_X)) \) and \( L_Y(\Phi(x)) = L_X(x) \) for all \( x \in \mathcal{D}(L_X) \). Let \( x \in \mathcal{D}(L_X) \) and \( \lambda > \max(1, 2L_X(x)) \). Suppose \( \sigma, \sigma' \in S(Y) \), and let \( \epsilon > 0 \). Choose \( k \in \mathbb{N} \) large enough so that we can find \( \omega, \omega' \in S(X) \) with \( \lambda \rho_{L^1_k}(\sigma, \omega) \leq \epsilon \) and \( \lambda \rho_{L^1_k}(\sigma', \omega') \leq \epsilon \), as well as a
Proof. Proposition 4.1 takes care of the “if” in each part, and so we need only worry about the “only if” direction. Suppose $\text{dist}_y^\lambda(x) \leq \epsilon$. Then

$$|\sigma(\Phi(x)) - \omega(x)| \leq |\sigma(\Phi(x)) - \sigma(y_k)| + |\sigma(y_k) - \omega(x)|$$

$$\leq \epsilon + \lambda \rho_{L_k,1}(\sigma, \omega)$$

$$\leq 2\epsilon$$

and similarly $|\sigma'(\Phi(x)) - \omega'(x)| \leq 2\epsilon$. Thus, since

$$\rho_{L_Y,1}(\omega, \omega') \leq \rho_{L_k,1}(\omega, \sigma) + \rho_{L_X,1}(\sigma, \sigma') + \rho_{L_k,1}(\sigma', \omega')$$

$$\leq \rho_{L_Y,1}(\sigma, \sigma') + 2\epsilon,$$

we have

$$|\sigma(\Phi(x)) - \sigma'(\Phi(x))| \leq |\sigma(\Phi(x)) - \omega(x)| + |\omega(x) - \omega'(x)| + |\omega'(x) - \sigma'(\Phi(x))|$$

$$\leq 4\epsilon + \rho_{L_X,1}(\omega, \omega')L_X(x)$$

$$\leq 2\epsilon(2 + L_X(x)) + \rho_{L_Y,1}(\sigma, \sigma')L_X(x).$$

Dividing by $\rho_{L_Y,1}(\sigma, \sigma')$ and letting $\epsilon \to 0^+$, we conclude that $L_Y(\Phi(x)) \leq L_X(x)$. Since the above argument also applies to $\Gamma$ we must in fact have $D(L_Y) = \Phi(D(L_X))$ and $L_Y(\Phi(x)) = L_X(x)$ for all $x \in D(L_X)$.

We will also need to know, for the proof of Theorem 4.11, that zero $m$-distance implies zero $n$-distance for $m > n \geq 1$, which is a consequence of the following lemma.

**Lemma 4.10.** If $(X, L_X)$ and $(Y, L_Y)$ are Lip-normed operator systems and $m > n \geq 1$, then $\text{dist}_y^m(X, Y) \leq \text{dist}_y^n(X, Y)$.

**Proof.** Let $L \in M(L_X, L_Y)$ and $\varphi \in UCP_n(X)$. Let $\omega$ be any state on $X$, and define $\tilde{\varphi} \in UCP_m(X)$ by setting $\tilde{\varphi}(x) = \varphi(x) + \omega(x)p$ for all $x \in X$, where $M_n$ has been identified with the upper left-hand corner of $M_m$ and $p$ is the unit for the lower right $(m-n) \times (m-n)$ corner. Choose $\psi' \in UCP_m(Y)$ with $\rho_{L,m}(\varphi', \psi') \leq \text{dist}_y^m(X, Y)$. If $\psi$ is the cut-down of $\varphi'$ to the upper-left hand $n \times n$ corner of $M_m$, then viewing it as an element of $UCP_n(X)$ we evidently have

$$\rho_{L,n}(\varphi, \psi) \leq \rho_{L,m}(\varphi', \psi').$$

Hence $\text{dist}_y^{PL,n}(UCP_n(X), UCP_n(Y)) \leq \text{dist}_y^{PL,m}(UCP_m(X), UCP_m(Y))$, and so we conclude that $\text{dist}_y^n(X, Y) \leq \text{dist}_y^m(X, Y)$.

**Theorem 4.11.** Let $(X, L_X)$ and $(Y, L_Y)$ be Lip-normed operator systems.

(i) If $n \in N$ then $\text{dist}_y^n(X, Y) = 0$ if and only if there is a bi-Lip-isometric unital $n$-order isomorphism between $X$ and $Y$.

(ii) We have $\text{dist}_y^n(X, Y) = 0$ if and only if there is a bi-Lip-isometric unital complete order isomorphism between $X$ and $Y$.

**Proof.** Proposition 4.1 takes care of the “if” in each part, and so we need only worry about the “only if” direction. Suppose $\text{dist}_y^n(X, Y) = 0$ for some $n \in N$. Then by Lemma 4.10 we have $\text{dist}_y^m(X, Y) = 0$ for each $m = 1, \ldots, n$. Applying the proof of Proposition 4.1 successively for each $m = 1, \ldots, n$ so that we can apply a diagonal argument across these values of $m$, we can find, for each $m = 1, \ldots, n$, a unital order isomorphism $\Phi_m : M_m \otimes X \to$
\(M_m \otimes Y\) such that, for each \(x \in M_m \otimes X\) and \(\lambda > 2L^m_X\), the singleton \(\{\Phi(x)\}\) is the limit of \(N^A_{L^m_k, Y}(x)\) as \(k \to \infty\) with respect to Hausdorff distance for a sequence of Lip-norms \(L_k \in M(\mathcal{L}A, \mathcal{L}B)\). By Proposition 4.13 we may also arrange that \(D(L_Y) = \Phi(D(L_X))\) and \(L_Y(\Phi_1(x)) = L_X(x)\) for all \(x \in D(L_X)\). We will show that \(\Phi_m = id_m \otimes \Phi_1\) for each \(m = 2, \ldots, n\). Suppose then that \(x \in D(L_X)\). For each \(k \in \mathbb{N}\) choose \(y_k \in N^A_{L^m_k, Y}(x)\).

If \(e_{ij}\) is a standard matrix unit in \(M_m\) then \(L^m_k(e_{ij} \otimes x, e_{ij} \otimes y_k) = L^k(x, y_k)\) so that \(e_{ij} \otimes y_k \in N^A_{L^m_k, Y}(e_{ij} \otimes x)\). Thus \(N^A_{L^m_k, Y}(e_{ij} \otimes x)\) must converge to the singleton containing

\[
\lim_{k \to \infty} e_{ij} \otimes y_k = e_{ij} \otimes \lim_{k \to \infty} y_k = (id_m \otimes \Phi_1)(e_{ij} \otimes x),
\]

whence \(\Phi_m(e_{ij} \otimes x) = (id_m \otimes \Phi_1)(e_{ij} \otimes x)\). Since by Lemma 4.5 the span of elements of the form \(e_{ij} \otimes x\) with \(x \in D(L^k_X)\) is dense in \(M_m \otimes X\), we conclude that \(\Phi_m = id_m \otimes \Phi_1\), so that \(\Phi_1\) is a bi-Lip-isometric \(n\)-order isomorphism. We thus obtain (i).

For (ii) we can use essentially the same proof, with the diagonal arguments now extended across all \(n \in \mathbb{N}\). \(\square\)

**Corollary 4.12.** Let \(A\) and \(B\) be unital \(C^*\)-algebras with Lip-norms \(L_A\) and \(L_B\), respectively.

(i) We have \(dist^1(A, B) = 0\) if and only if there is a bi-Lip-isometric unital order isomorphism between \(A\) and \(B\).

(ii) If \(n \geq 2\) then \(dist^n(A, B) = 0\) if and only if there is a bi-Lip-isometric \(*\)-isomorphism between \(A\) and \(B\).

(iii) We have \(dist^* (A, B) = 0\) if and only if there is a bi-Lip-isometric \(*\)-isomorphism between \(A\) and \(B\).

**Proof.** The corollary is an immediate consequence of Theorem 4.11 and the fact that a unital 2-order isomorphism between \(A\) and \(B\) is automatically a \(*\)-isomorphism. \(\square\)

We remark that a unital order isomorphism between unital \(C^*\)-algebras need not be a \(*\)-isomorphism. For instance, a unital \(C^*\)-algebra \(A\) is always unitally order isomorphic to its opposite algebra \(A^{op}\), but these need not be \(*\)-isomorphic, as the examples in \(\S\) demonstrate.

5. \(f\)-Leibniz Complete Distance and Convergence

Let \((\mathcal{R}, dist_o)\) be the metric space, under complete distance, of equivalence classes of Lip-normed operator systems with respect to bi-Lip-isometric unital complete order isomorphism. For economy we will simply refer to the elements of \(\mathcal{R}\) as Lip-normed operator systems. Let \(f : \mathbb{R}_+^4 \to \mathbb{R}_+^*\) be a continuous function. Given a Lip-normed unital \(C^*\)-algebra \((A, L)\), the Lip-norm \(L\) is said to be \(f\)-Leibniz if it is the restriction of an adjoint-invariant semi-norm \(L'\) on \(A\) which is finite on a dense \(*\)-subalgebra and satisfies the \(f\)-Leibniz property

\[ L'(xy) \leq f(L'(x), L'(y), \|y\|, \|x\|) \]

for all \(x, y \in A\) with \(L'(x), L'(y) < \infty\). When \(f(a, b, c, d) = ac + bd\) this is the usual Leibniz rule, in which case we simply say that \(L\) is Leibniz. The Lip-normed unital \(C^*\)-algebras of Example 2.4 are Leibniz, as are those obtained from Lipschitz semi-norms on functions.
over a compact metric space. We denote by $\mathcal{R}_{\text{alg}}$ the subset of $\mathcal{R}$ consisting of Lip-normed unital $C^*$-algebras, and for $(A, L_A)$ and $(B, L_B)$ in $\mathcal{R}_{\text{alg}}$ we define the $f$-Leibniz complete distance $\text{dist}_{s,f}(A, B)$ in the same way that the complete distance is defined (Definition 3.3) except that the infimum is now taken over the $f$-Leibniz Lip-norms in $M(L_A, L_B)$ (if no such $f$-Leibniz Lip-norm exists we set $\text{dist}_{s,f}(A, B) = \infty$). Note that $\text{dist}_{s,f}$ might not satisfy the triangle inequality without further hypotheses on $f$, but that will not be of consequence for our application here, and we can still speak of Cauchy sequences with respect to $\text{dist}_{s,f}$ in the obvious sense. It can be seen that the estimates in Example 3.12 for complete distance also apply to $f$-Leibniz complete distance for suitable $f$ (although $f$ may depend on the matrix algebra), and if $N$ is a bridge between two Leibniz Lip-normed $C^*$-algebras of the form that appears in Proposition 3.7 then the resulting Lip-norm on the direct sum is Leibniz (see Section 2 for examples of the use of bridges like those in Proposition 3.7). In this section we show that every sequence in $\mathcal{R}_{\text{alg}}$ which is Cauchy with respect to $f$-Leibniz complete distance converges in $\mathcal{R}_{\text{alg}}$ with respect to complete distance. We may interpret this as saying that the metric space $(\mathcal{R}_{\text{alg}}, \text{dist}_s)$ is “complete” relative to $f$-Leibniz complete distance.

I would like to thank Narutaka Ozawa for suggesting the idea behind the proof of the following lemma. Given a sequence $\{A_k\}_{k \in \mathbb{N}}$ of $C^*$-algebras we denote by $\prod A_k$ the $C^*$-algebra of bounded sequences with the supremum norm and by $\bigoplus A_k$ the $C^*$-subalgebra of sequences converging to zero.

**Lemma 5.1.** Let $\{A_k\}_{k \in \mathbb{N}}$ be a sequence of unital $C^*$-algebras and $X$ a separable operator subsystem of $\prod A_k/\bigoplus A_k$, and let $n \in \mathbb{N}$. Suppose that, for each $x \in M_n \otimes X$, at least one lift (and hence every lift) $\sum e_{ij} \otimes (x_{ij})_k$ of $x$ with respect to the quotient $(M_n \otimes (\prod A_k))/(M_n \otimes (\bigoplus A_k)) \cong M_n \otimes (\prod A_k/\bigoplus A_k)$ satisfies

$$\lim_{k \to \infty} \left\| \sum e_{ij} \otimes x_{ij}^k \right\| = \|x\|.$$ 

Then for every $\varphi \in UCP_n(X)$ there are $\varphi_k \in UCP_n(A_k)$ for $k \in \mathbb{N}$ such that for all $(x_k)_k + \bigoplus A_k \in X$ we have

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \to \infty} \varphi_k(x_k).$$

**Proof.** First we consider an arbitrary finite-dimensional operator subsystem $Y$ of $X$ and show that the conclusion of the lemma holds with respect to elements of $Y$. Letting $\pi : \prod A_k \to \prod A_k/\bigoplus A_k$ be the quotient map, there exists, by elementary linear algebra, a unital linear map $x \mapsto \alpha(x) = (\alpha(x)_k)_k$ from $X$ to $\prod A_k$ such that $\pi \circ \alpha = \text{id}_X$. We may assume that $\alpha$ is Hermitian for otherwise we can replace it with its real part $(\alpha + \alpha^*)/2$. Since $Y$ is finite-dimensional the unit ball of $M_n \otimes Y$ is compact, and so by our assumption on lifts of elements of $M_n \otimes X$ we can find a sequence $\delta_1, \delta_2, \ldots$ of positive real numbers with $\lim_{k \to \infty} \delta_k = 0$ such that, for all $\sum e_{ij} \otimes x_{ij}$ in the unit ball of $M_n \otimes Y$ and $k \in \mathbb{N}$,

$$\left\| \sum e_{ij} \otimes x_{ij} \right\| - \delta_k < \left\| \sum e_{ij} \otimes \alpha(x_{ij})_k \right\| < \left\| \sum e_{ij} \otimes x_{ij} \right\| + \delta_k.$$ 

This implies in particular that for each sufficiently large $k \in \mathbb{N}$ the map $\pi_k \circ \alpha$ is injective on $Y$, where $\pi_k : \prod A_k \to A_k$ is the projection map. Let $\varphi \in UCP_n(Y)$. For each sufficiently
large $k \in \mathbb{N}$ we can define the linear map $\psi_k : (\pi_k \circ \alpha)(Y) \to M_n$ by

$$\psi_k(a) = \varphi((\pi_k \circ \alpha)^{-1}(a))$$

for all $a \in (\pi_k \circ \alpha)(Y)$. Then $\psi_k$ is unital and Hermitian, and $\|\text{id}_n \otimes \psi_k\| \leq (1 - \delta_k)^{-1}$. By [13, Thm. 2.10] the completely bounded norm $\|\psi_k\|_{cb}$ is equal to $\|\text{id}_n \otimes \psi_k\|$ and hence is at most $(1 - \delta_k)^{-1}$. By the Wittstock extension theorem (see [16]) there is an extension of $\psi_k$ to $A_k$ with the same completely bounded norm. We denote this extension also by $\psi_k$.

By the Wittstock decomposition theorem (see [16]) there exist completely positive maps $\psi_k^+$ and $\psi_k^-$ from $A_k$ to $M_n$ such that $\psi_k = \psi_k^+ - \psi_k^-$ and $\|\psi_k\|_{cb} \geq \|\psi_k^+ + \psi_k^-\|$. We then have

$$\psi_k^+(1) = \psi_k(1) + \psi_k^-(1) = 1 + \psi_k^-(1) \geq 1$$

and

$$\|\psi_k^+(1)\| = \|\psi_k^+\| \leq \|\psi_k^+ + \psi_k^-\| \leq \|\psi_k\|_{cb} \leq (1 - \delta_k)^{-1}.$$ 

Also,

$$\|\psi_k^+ - \psi_k\| = \|\psi_k^+(1) - \psi_k(1)\| = \|\psi_k^+(1) - 1\| \leq (1 - \delta_k)^{-1} - 1.$$ 

Since $\psi_k^+(1) > 0$ we can define the u.c.p. map $\varphi_k : A_k \to M_n$ by

$$\varphi_k(a) = \psi_k^+(1) - \frac{1}{2}\psi_k^+(a)\psi_k^+(1) - \frac{1}{2}$$

for all $a \in A_k$. Then

$$\|\psi_k - \varphi_k\| \leq \|\psi_k^+ - \psi_k^+\| + \|\psi_k^+ - \varphi_k\|$$

$$\leq ((1 - \delta_k)^{-1} - 1) + \|1 - \psi_k^+(1) - \frac{1}{2}\|\psi_k^+(1) - \frac{1}{2}\| \|\psi_k^+(1) - \frac{1}{2}\|$$

$$\leq ((1 - \delta_k)^{-1} - 1) + 2(1 - (1 - \delta_k)^{-1})(1 - \delta_k)^{-1},$$

and this last expression tends to zero as $k \to \infty$. It follows that, for all $(x_k)_k + \bigoplus A_k \in Y$,

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \to \infty} \psi_k(x_k) = \lim_{k \to \infty} \varphi_k(x_k).$$

Now suppose that $X_1 \subset X_2 \subset \cdots$ is an increasing sequence of finite-dimensional operator subsystems of $X$ with union dense in $X$. Let $\varphi \in UCP_n(X)$. Then for each $j \in \mathbb{N}$ there exists by the first paragraph u.c.p. maps $\varphi_k$ on $A_k$ for sufficiently large $k$ (and hence for all $k$) such that

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \to \infty} \varphi_k(x_k)$$

for all $(x_k)_k + \bigoplus A_k \in X_j$. By applying a diagonal argument over $j \in \mathbb{N}$ we can assume that the equality in the above display holds for all $(x_k)_k + \bigoplus A_k \in \bigcup_{j \in \mathbb{N}} X_j$. A straightforward approximation argument then shows that this equality in fact holds for all $(x_k)_k + \bigoplus A_k \in X$, completing the proof. \qed

**Lemma 5.2.** Let $(Z, L_Z)$ be a Lip-normed operator system, $\Phi : Z \to X$ and $\Gamma : Z \to Y$ surjective u.c.p. maps onto operator systems $X$ and $Y$, and $L_X$ and $L_Y$ the quotient Lip-norms induced via $\Phi$ and $\Gamma$, respectively. Then

$$\text{dist}_s(X, Y) \leq \sup_{n \in \mathbb{N}} \text{dist}^{PL, n}_{H}(UCP_n(X), UCP_n(Y)), $$

with $UCP_n(X)$ and $UCP_n(Y)$ considered as subsets of $UCP_n(Z)$.  

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**Note:** The content provided is a natural representation of the text, including mathematical expressions and logical flow. The document appears to be discussing advanced topics in operator theory, specifically focusing on the Gromov-Hausdorff distance in the context of matricial quantum systems. The text includes definitions, theorems, and proofs that are typical in a research setting.
Proof. Set \( r = \sup_{n \in \mathbb{N}} \text{dist}^{\Phi L,n}_{\mathcal{H}}(\text{UCP}_n(X), \text{UCP}_n(Y)) \) (as can be seen from the proof of Proposition 2.13, this supremum is bounded by \( \text{diam}(Z,L Z) \)). As in [1, Example 5.6] for any \( \gamma > 0 \) we can construct a bridge between two copies of \((Z, L Z)\) by setting \( N(z,z') = \gamma^{-1} \lVert z - z' \rVert \). Let \( M \) be the Lip-norm
\[
L(z,z') = \max(\langle L_z(z), L_z(z') \rangle, N(z,z'))
\]
on \( \mathcal{D}(L Z) \oplus \mathcal{D}(L Z) \) and \( L \) the quotient Lip-norm induced by \( M \) via the u.c.p. map \((z, z') \mapsto (\Phi(z), \Gamma(z'))\). Then \( L \in \mathcal{M}(L X, L Y) \). Denote the projections of \( Z \oplus Z \) onto the first and second direct summand by \( \pi_1 \) and \( \pi_2 \), respectively.

Now suppose \( \varphi \in \text{UCP}_n(X) \). Then by assumption for some \( \psi \in \text{UCP}_n(Y) \) we have \( \rho_{L,n}(\varphi \circ \Phi, \psi \circ \Gamma) \leq r \). Also, if \((z, z') \in \mathcal{D}(M)\) then \( \lVert z - z' \rVert \leq \gamma \) so that \( \lVert (\psi \circ \Gamma)(z) - (\psi \circ \Gamma)(z') \rVert \leq \gamma \) and hence \( \rho_{M,n}(\psi \circ \Gamma \circ \pi_1, \psi \circ \Gamma \circ \pi_2) \leq \gamma \) (where to avoid confusion we have included the compositions with the projection maps \( \pi_1 \) and \( \pi_2 \), contrary to our usual practice). Thus, since \( \rho_{L,n} \) is the restriction of \( \rho_{M,n} \) via the identification arising from the quotient map, we have by the triangle inequality
\[
\rho_{L,n}(\varphi, \psi) \leq \rho_{M,n}(\varphi \circ \Phi, \psi \circ \Gamma) + \rho_{M,n}(\psi \circ \Gamma \circ \pi_1, \psi \circ \Gamma \circ \pi_2) \\
\leq r + \gamma.
\]
Hence \( \text{dist}_s(X,Y) \leq r + \gamma \), which yields the result since \( \gamma \) was arbitrary. \( \square \)

In the proof of the following theorem, we will abbreviate expressions of the form \( \text{dist}^{\Phi L,n}_{\mathcal{H}}(\text{UCP}_n(X), \text{UCP}_n(Y)) \) to \( \rho_{L,n}(\text{UCP}_n(X), \text{UCP}_n(Y)) \) to reduce the number of subscripts, and whenever we have a quotient Lip-norm then will identify the state space of the quotient operator system with a subset of the state space of the original operator system under the induced isometry (Proposition 2.13) as is our usual practice in the case of projections onto direct summands.

**Theorem 5.3.** Let \( \{(A_k, L_k)\} \) be a sequence in \( \mathcal{R}_{\text{alg}} \) which is Cauchy with respect to \( f\)-Leibniz complete distance for a given continuous \( f : \mathbb{R}^+_+ \to \mathbb{R}^+ \). Then \( \{(A_k, L_k)\} \) converges in \( \mathcal{R}_{\text{alg}} \) with respect to complete distance.

**Proof.** To show that \( \{(A_k, L_k)\} \) converges it suffices to show the convergence of a subsequence, and so we may assume that \( \text{dist}_s,f(A_k, A_{k+1}) < 2^{-k} \) for all \( k \in \mathbb{N} \). Then there exist \( f\)-Leibniz Lip-norms \( L_{k,k+1} \in \mathcal{M}(L_k, L_{k+1}) \) with
\[
\rho_{L,k,k+1,n}(\text{UCP}_n(A_k), \text{UCP}_n(A_{k+1})) < 2^{-k}
\]
for all \( n,k \in \mathbb{N} \). Let \( Z \) be the set of sequences \( (x_k)_k \) with \( x_k \in \mathcal{D}(L_k) \) such that, for some \( \lambda > L_1(x_1) \), \( x_{k+1} \in \mathcal{N}_{L_{k,k+1}}(A_{k+1})(x_k) \) for all \( k \in \mathbb{N} \) (see Definition 1.7). We will show that \( Z \) is a subset of the direct product \( \prod A_k \). Let \( J_k \) be the semi-norm on \( \prod_{j=1}^k \mathcal{D}(L_j) \) given by
\[
J_k((x_j)_j) = \sup_{1 \leq j \leq k-1} L_{j,j+1}(x_j, x_{j+1}).
\]
By [1, Lemma 12.2] \( J_k \) is a Lip-norm. Denote by \( Q_k \) the quotient Lip-norm on \( \mathcal{D}(L_1) \oplus \mathcal{D}(L_k) \) induced by \( J_k \) via the projection map. Then
\[
\rho_{Q_k,1}(S(A_1), S(A_k)) \leq 2^{-1} + 2^{-2} + \cdots + 2^{-k} < 1.
\]
Suppose \((x_k)_k \in Z\), and let \(\lambda > L_1(x_1)\) be such that \(x_{k+1} \in N^A_{L_{k+1},x_{k+1}}(x_k)\) for all \(k \in \mathbb{N}\). Then, for each \(k \in \mathbb{N}\), \(x_k\) is an element of \(N^A_{Q_k,A_k}(x_1)\) and hence by Lemma 1.8 has norm bounded by \(\|x_1\| + 2\lambda \rho Q_{k_1}(S(A_1), S(A_k)) \leq \|x_1\| + 2\lambda\). Therefore \((x_k)_k\) is a bounded sequence and so belongs to \(\prod A_k\), as we wished to show.

We define the semi-norm \(L_Z\) on \(Z\) by

\[
L_Z(x_k)_k = \sup_{k \in \mathbb{N}} L_{k,k+1}(x_k, x_{k+1})
\]

(which is finite by the definition of \(Z\)). Theorem 12.9 of \cite{9} then shows that \(L_Z\) is a Lip-norm on \(Z\). It can be seen by the \(f\)-Leibniz property that the operator system \(B\) generated by \(Z\) in \(\prod A_k\) is in fact a \(C^*\)-algebra. Let \(A\) be the \(C^*\)-subalgebra of \(\prod A_k/\bigoplus A_k\) which is the image of \(B\) under the quotient map \(\pi: \prod A_k \to \prod A_k/\bigoplus A_k\), and let \(L\) be the quotient Lip-norm on \(A\) induced by \(L_Z\). Then \((A,\pi)\) is a Lip-normed unital \(C^*\)-algebra. Our goal now is to show that \(\{A_k, L_k\}\) converges to \((A,\pi)\) with respect to complete distance. By Lemma 5.2 it suffices to show that, for all \(n \in \mathbb{N}\), \(UCP_n(A)\) coincides with the Hausdorff limit \(H_n \subset UCP_n(B)\) of \(\{UCP_n(A_k)\}_{k \in \mathbb{N}}\), which exists due to the completeness, in the Hausdorff metric, of the set of closed subspaces of the compact set \(UCP_n(B)\). Note that for each \(k \in \mathbb{N}\) the image of \(B\) under the projection onto \(A_k\) is surjective so that indeed \(UCP_n(A_k) \subset UCP_n(B)\), and that the convergence of \(\{UCP_n(A_k)\}_{k \in \mathbb{N}}\) to \(H_n\) is uniform over \(n\) because the Cauchy condition is uniform over \(n\) by assumption.

If \(\{\varphi_k\}_{k \in \mathbb{N}}\) is a sequence such that \(\varphi_k \in UCP_n(A_k)\) and \(\{\varphi_k \circ \pi_k\}_{k \in \mathbb{N}}\) is point-norm convergent (necessarily to an element of \((H_n)\)) then setting

\[
\varphi(x_k)_k + \bigoplus A_k
\]

for \((x_k)_k + \bigoplus A_k \in X\) we obtain a map \(\varphi: X \to M_n\). This map is u.c.p. in view of the identification \(M_n \otimes (\prod A_k/\bigoplus A_k) \cong (M_n \otimes (\prod A_k))/((M_n \otimes (\bigoplus A_k)))\) and the fact that positive elements in quotients lift to positive elements. We thus see that \(H_n \subset UCP_n(A)\).

It remains to show that \(H_n \supset UCP_n(A)\). With a view to applying Lemma 5.1, we will show that every \(x \in M_n \otimes A\) has a lift \((x_k)_k \in M_n \otimes \prod A_k\) satisfying \(\lim_{k \to \infty} \|x_k\| = \|x\|\). Notice first that if \((z_k)_k \in M_n \otimes Z\) then for some \(\lambda\) not depending on \(j\) we have \(\|z_j\| - \|z_{j+1}\| \leq 2^{-j+1}n^4\lambda\) by Lemma 4.5(ii) (since each \(z_j\) is self-adjoint), so that \(\{\|z_k\|\}_{k \in \mathbb{N}}\) is a Cauchy sequence and hence \(\|\pi((z_k)_k)\| = \lim_{k \to \infty} \|z_k\|\). Now suppose \(x \in M_n \otimes A\) and let \((x_k)_k\) be a lift of \(x\) to \(M_n \otimes B\). Then \((x_k^*x_k)_k \in M_n \otimes B\), and so there exists a \((y_k)_k \in M_n \otimes Z\) such that \(\|x_k^*x_k - y_k\| < \varepsilon\) for all \(k \in \mathbb{N}\), and from above we have \(\|\pi((y_k)_k)\| = \lim_{k \to \infty} \|y_k\|\). Let \(\varepsilon > 0\), and choose \(k_0 \in \mathbb{N}\) such that, for all \(j, k \geq k_0\), \(\|y_j\| - \|y_k\| < \varepsilon\). Then, for all \(j, k \geq k_0\),

\[
\|x_j\|^2 - \|x_k\|^2 = \|x_j^*x_j - x_k^*x_k\|
\leq \|x_j^*x_j\| - \|y_j\| + \|y_j\| - \|y_k\| + \|y_k\| - \|x_k^*x_k\|
< 3\varepsilon.
\]

It follows that \(\{\|x_k\|^2\}_{k \in \mathbb{N}}\) is a Cauchy sequence and hence converges. Thus \(\lim_{k \to \infty} \|x_k\|\) exists, and it must equal \(\|x\|\). We can therefore apply Lemma 5.1, so that given \(\varphi \in UCP_n(A)\) there exist \(\varphi_k \in UCP_n(A_k)\) for \(k \in \mathbb{N}\) such that for all \((x_k)_k + \bigoplus A_k \in A\) we
have

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \to \infty} \varphi_k(x_k),$$

whence $H_n \supset UC^n(A)$. Thus $H_n$ and $UC^n(A)$ coincide, completing the proof. □

Using the arguments of this section we might hope to show that the metric space $(\mathcal{R}, \text{dist}_s)$ is complete. However, without the sharp control on the norms of non-self-adjoint elements that the $f$-Leibniz property provides in Theorem 5.3, we would not be able to apply Lemma 5.1.

6. Total boundedness

We will establish a version of Theorem 13.5 in [9] ("the quantum Gromov compactness theorem") for complete distance using approximation by Lip-normed operator subsystems of matrix algebras. As before $(\mathcal{R}, \text{dist}_s)$ is the metric space of equivalence classes of Lip-normed operator systems with respect to bi-Lip-isometric unital complete order isomorphism.

**Notation 6.1.** For a Lip-normed operator system $(X, L)$ and $\epsilon > 0$ we denote by $\text{Afn}_L(\epsilon)$ the smallest integer $k$ such that there is a Lip-normed operator system $(Y, L_Y)$ with $Y$ an operator subsystem of the matrix algebra $M_k$ and $\text{dist}_s(X, Y) \leq \epsilon$. If no such integer $k$ exists we write $\text{Afn}_L(\epsilon) = \infty$. We denote by $\mathcal{R}_{fa}$ the subset of $\mathcal{R}$ consisting of Lip-normed operator systems $(X, L)$ for which $\text{Afn}_L(\epsilon)$ is finite for all $\epsilon > 0$.

We remark that every Lip-normed nuclear operator system and Lip-normed unital exact $C^*$-algebra is contained in $\mathcal{R}_{fa}$ by Propositions 3.9 and 3.10, respectively. Note also that $\mathcal{R}_{fa}$ is a closed subset of $\mathcal{R}$ under the complete distance topology.

**Lemma 6.2.** Let $X$ be a finite-dimensional operator system and $C \geq 0$. Then the set $\mathcal{C} = \{(X, L) \in \mathcal{R} : \text{diam}((X, L) \leq C)\}$ is totally bounded.

**Proof.** Proposition 13.13 and the proof of Proposition 13.14 in [9] show that, given $\epsilon > 0$, there is a finite subset $S \subset \mathcal{C}$ such that for all $(X, L) \in \mathcal{C}$ there is a $(X, L') \in S$ and a bridge $N$ between $(X, L)$ and $(X, L')$ of the form $N(x, y) = \epsilon^{-1}\|x - y\|$. Let $M$ be the Lip-norm in $M(L, L')$ given by

$$M(x, y) = \max(L(x), L'(y), N(x, y)).$$

Now if $\phi \in UC^n(X)$ and $(x, y) \in D_1(M)$ then $\|x - y\| \leq \epsilon$ so that $\|\phi(x) - \phi(y)\| \leq \epsilon,$ and so by Proposition 2.10 we have $\rho_{M^n}(\phi \circ \pi_1, \phi \circ \pi_2) \leq \epsilon$, where $\pi_1$ and $\pi_2$ are the projections of $X \oplus X$ onto the first and second direct summands, respectively. Hence $\text{dist}_s((X, L), (X, L')) \leq \epsilon$, from which we conclude that $\mathcal{C}$ is totally bounded. □

**Theorem 6.3.** Let $\mathcal{C}$ be a subset of $\mathcal{R}_{fa}$. Then $\mathcal{C}$ is totally bounded if and only if

(i) there is an $M > 0$ such that the diameter of every element of $\mathcal{C}$ is bounded by $M$, and

(ii) there is a function $F : (0, \infty) \to (0, \infty)$ with $\text{Afn}_L(\epsilon) \leq F(\epsilon)$ for all $(X, L) \in \mathcal{C}$. 

Proof. For the “only if” direction, suppose that $\mathcal{C}$ is a totally bounded subset of $R_{fa}$. If there did not exist an $M > 0$ bounding the complete diameter of every element of $\mathcal{C}$, then we could find a sequence $\{(X_k, L_k)\}_{k \in \mathbb{N}}$ such that $\text{diam}(X_{k+1}, L_{k+1}) \geq \text{diam}(X_k, L_k) + 1$ for every $k \in \mathbb{N}$, in which case dist($X_k, X_{k'}$) $\geq 1$ for all $k, k' \in \mathbb{N}$, contradicting total boundedness. To verify condition (ii), we can find a finite ($\epsilon/2$)-dense subset $\mathcal{S}$ of $\mathcal{C}$ and set

$$G(\epsilon) = \max\{\text{Af}\mathcal{L}_{\epsilon}(\epsilon/2) : (X, L) \in \mathcal{S}\}.$$ 

Then by the triangle inequality Af$\mathcal{L}_{\epsilon}(\epsilon) \leq G(\epsilon)$ for any $(X, L) \in \mathcal{C}$.

To prove the converse, suppose that conditions (i) and (ii) hold. By (ii) we see that it is sufficient to prove, for $k \geq j \geq 1$ and $M > 0$, the total boundedness of the collection of Lip-normed operator systems $(X, L)$ where $X$ is an operator subsystem of the matrix algebra $M_k$ with $X_{sa}$ of real linear dimension $j$ (in which case we will say that $X$ has Hermitian dimension $j$) and diam$(X, L) \leq M$. Since the closed unit ball of the self-adjoint part of $M_k$ is compact, the set of closed unit balls of the self-adjoint parts of operator subsystems of $M_k$ of Hermitian dimension $j$ is totally bounded in the Hausdorff metric. Also, by Lemma 3.4, for every $M > 0$ and operator subsystem $X$ of $M_k$ of Hermitian dimension $j$ the set of Lip-normed operator systems $(X, L)$ with diam$(X, L) \leq M$ is totally bounded. Thus we need only show that, for every $\epsilon > 0$ and $M > 0$, if $X$ and $Y$ are operator subsystems of $M_k$ of Hermitian dimension $j$ the closed unit balls of the self-adjoint parts of which are within Hausdorff distance $(4k)^{-1}\epsilon \min(M^{-1}, 1)$, and $L_X$ is a Lip-norm on $X$ with diam$(X, L_X) \leq M$, then there is a Lip-norm $L_Y$ on $Y$ with dist$_s((Y, L_Y), (X, L_X)) \leq \epsilon$ and diam$(Y, L_Y) \leq M + \epsilon$. So let $X$ and $Y$ be such operator systems and $L_X$ such a Lip-norm on $X$ for given $\epsilon > 0$ and $M > 0$. We may assume that $\epsilon < 1/2$. Set $\delta = (4k)^{-1}\epsilon \min(M^{-1}, 1)$. By [Lemma 3.2.3] there is a (real linear) projection $P$ from $(M_k)_{sa}$ onto $Y_{sa}$ of norm $\leq k$. The restriction $Q$ of $P$ to $X_{sa}$ is a bijection, for if $x \in \cap X_{sa}$ with $\|x\| = 1$ then we can find a $y \in Y_{sa}$ with $\|y - x\| < \delta \leq \epsilon/k$ so that

$$\|Q(x)\| \geq \|y\| - \|Q(x - y)\| \geq 1 - \epsilon > \frac{1}{2},$$

yielding injectivity and hence also bijectivity since $X_{sa}$ and $Y_{sa}$ are of equal finite dimension. The above display also shows that the norm of $Q^{-1}$ is bounded by 2. We next define a semi-norm $L_X$ on $Q^{-1}(Y)$ by $L_X(x) = L_Y(Q(x))$ (note that $\mathcal{D}(L_Y)$ is equal to $Y_{sa}$ by finite-dimensionality). Since the restriction of $Q$ to $X$ is bijective and $L_Y$ is a Lip-norm we must have $L_X(x) = 0$ if and only if $x \in R1$. Thus $L_X$ is a Lip-norm in the finite-dimensionality of $X_{sa}$, and $(X, L_X)$ is a Lip-normed operator system since $\mathcal{D}(L_X) = X_{sa}$ and $\mathcal{D}_1(L_X)$ is closed in $X_{sa}$ by the bijectivity and continuity, respectively, of $Q$.

On $\mathcal{D}(L_X) \oplus \mathcal{D}(L_Y)$ we define the semi-norm $N$ by $N(x, y) = \epsilon^{-1}\|x - y\|$. We will argue that $N$ is a bridge. For this it suffices to show that, for all $x \in \mathcal{D}(L_X)$,

$$N(x, Q(x)) \leq L_Y(Q(x)),$$

for then in condition (ii) of Definition 3.4 given $x \in \mathcal{D}(L_X)$ we can take $Q(x)$, and given $y \in \mathcal{D}(L_Y)$ we can take $Q^{-1}(y)$. So let $x \in \mathcal{D}(L_X)$. Then we can find a $y \in Y_{sa}$ such that $\|x - y\| \leq \delta\|x\|$, so that

$$\|x - Q(x)\| \leq \|x - y\| + \|Q(x - y)\| \leq (1 + k)\|x - y\| \leq 2k\delta\|x\| \leq 4k\delta\|Q(x)\|.$$
Applying this estimate with $x$ replaced by $x - \lambda 1$ where $\lambda$ is the infimum of the spectrum of $Q(x)$, we have
\[
\| x - Q(x) \| = \| x - \lambda 1 - Q(x - \lambda 1) \| \leq 4k\delta \| Q(x - \lambda 1) \|
= 4k\delta \| Q(x) - \lambda 1 \|
\leq 4kM\delta L_Y (Q(x) - \lambda 1)
\leq 4kM\delta L_Y (Q(x))
\leq \epsilon L_Y (Q(x))
\]
with Proposition 2.11 yielding the second inequality in the string. We thus conclude that $N$ is a bridge. Let $L$ be the Lip-norm in $M(L_X, L_Y)$ given by
\[
L(x, y) = \max (L_X(x), \overline{L_Y(y)}, N(x, y)).
\]
It remains to show that $\text{dist}^*_{H}^n(UCP_n(X), UCP_n(Y)) \leq \epsilon$ for all $n \in \mathbb{N}$, for then $\text{dist}_n(X, Y) \leq \epsilon$ and hence also $\text{diam}(X, L_X) \leq \text{diam}(Y, L_Y) + \epsilon \leq M + \epsilon$. Let $\varphi \in UCP_n(X)$. By Arveson’s extension theorem we can extend $\varphi$ to a u.c.p. map $\varphi' : M_k \to M_n$. We then have
\[
\rho_{L, n}(\varphi, \varphi'|_Y) = \sup \{ |\varphi'(x) - \varphi'(y)| : (x, y) \in D_1(L) \}
\leq \sup \{ |\varphi'(x) - \varphi'(y)| : (x, y) \in X \oplus Y \text{ and } \| x - y \| \leq \epsilon \}
\leq \epsilon.
\]
Similarly, if $\varphi \in UCP_n(Y)$ then extending it by Arveson’s theorem to a u.c.p. map $\varphi' : M_k \to M_n$ we have $\rho_{L, n}(\varphi, \varphi'|_Y) \leq \epsilon$. Thus $\text{dist}^*_{H}^n(UCP_n(X), UCP_n(Y)) \leq \epsilon$, as desired.

An immediate consequence of Theorem 6.3 is the separability of $\mathcal{R}_{fa}$.

**Corollary 6.4.** The metric space $\mathcal{R}_{fa}$ is separable.

**Question 6.5.** Given $n > 1$ and $M > 0$, is the set of all $n$-dimensional Lip-normed operator systems of diameter at most $M$ totally bounded and/or separable?

We may think of $\log \text{Afn}_L(\epsilon)$ as an analogue of Kolmogorov $\epsilon$-entropy. From the computational viewpoint, however, the value of this quantity seems to be limited by the apparent difficulty in establishing lower bounds. Using local approximation we will next define a quantity $\text{Rcp}_L(\epsilon)$ which is more amenable to obtaining estimates than $\text{Afn}_L(\epsilon)$ and provides a ready means for obtaining upper bounds for $\text{Afn}_L(\epsilon)$ (see Proposition 6.7) with a view to the application of Theorem 6.3, as we will illustrate in the case of noncommutative tori in Example 6.8. It can also be shown (by suitably adjusting the proof of [3, Prop. 3.9] for instance) that by taking $\limsup_{\epsilon \to 0^+} \frac{\log \text{Rcp}_L(\epsilon)}{\log(\epsilon^{-1})}$ we obtain a generalization of the Kolmogorov dimension of a compact metric space, whose utility depends on our ability to estimate $\log \text{Rcp}_L(\epsilon)$ from below. We will not be concerned here with obtaining lower bounds for $\log \text{Rcp}_L(\epsilon)$, but we point out that this can often be done by using the Hilbert space geometry implicit in the given operator system or $C^*$-algebra as in [3].

For a Lip-normed nuclear operator system $(X, L)$ and $\epsilon > 0$ we denote by $CPA_L(\epsilon)$ the collection of triples $(\alpha, \beta, B)$ where $B$ is a finite-dimensional $C^*$-algebra and $\alpha : X \to B$
and $\beta : B \to X$ are u.c.p. maps with $\| (\beta \circ \alpha)(x) - x \| < \epsilon$ for all $x \in D_1(L)$. This collection is non-empty by Proposition 2.11. Conversely, if $(X, L)$ is any Lip-normed operator system $(X, L)$ and $CPA_L(\epsilon)$ is non-empty for each $\epsilon > 0$, then $X$ is nuclear owing to the density of $D(L)$ in $X_{sa}$.

**Definition 6.6.** Let $(X, L)$ be a Lip-normed nuclear operator system. For $\epsilon > 0$ we set

$$Rcp_L(\epsilon) = \inf \{ \text{rank}(B) : (\alpha, \beta, B) \in CPA_L(\epsilon) \}.$$ 

**Proposition 6.7.** If $(X, L)$ is a Lip-normed nuclear operator system and $\epsilon > 0$ then $Rcp_L(\epsilon) \geq \text{Aff}_L(\epsilon)$.

**Proof.** Let $\epsilon > 0$. Then there is a triple $(\alpha, B, L) \in CPA_L(\epsilon)$ with $\text{rank}(B) = Rcp_L(\epsilon)$. Set $Y = \alpha(B)$, and let $L_Y$ be the Lip-norm on $Y$ induced by $L$ via $\alpha$. Then $\text{dist}_s(X, Y) \leq \epsilon$ by Proposition 3.8, and since $B$ unitarily embeds into a matrix algebra of the same rank we obtain $\text{Aff}_L(\epsilon) \leq Rcp_L(\epsilon)$. \qed

**Example 6.8** (Noncommutative tori). Let $\rho : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ be an antisymmetric bicharacter and for $1 \leq i, j \leq k$ set

$$\rho_{ij} = \rho(e_i, e_j)$$

with $\{e_1, \ldots, e_d\}$ the standard basis for $\mathbb{Z}^d$. We call the universal $C^*$-algebra $A_\rho$ generated by unitaries $u_1, \ldots, u_d$ satisfying

$$u_j u_i = \rho_{ij} u_i u_j$$

a noncommutative $d$-torus. Given a noncommutative $d$-torus $A_\rho$ with generators $u_1, \ldots, u_d$ there is an ergodic action $\gamma : \mathbb{T}^d \cong (\mathbb{R}/\mathbb{Z})^d \to \text{Aut}(A_\rho)$ determined on the generators by

$$\gamma(t_1, \ldots, t_d)(u_j) = e^{2\pi i t_j} u_j$$

(see [2]). Let $\ell$ be a length function on $\mathbb{T}^d$ (for instance, we could take the distance to 0 with respect to the metric induced from the Euclidean metric on $\mathbb{R}^d$). By Example 2.6 we then obtain a Lip-norm $L$ arising from the action $\gamma$ and length function $\ell$. Let $\tau$ be the tracial state on $A_\rho$ defined by

$$\tau(a) = \int_{\mathbb{T}^d} \gamma(t_1, \ldots, t_d)(a) d(t_1, \ldots, t_d)$$

for all $a \in A_\rho$, where $d(t_1, \ldots, t_d)$ is normalized Haar measure.

Let $A(d, \ell)$ be the subset of $\mathcal{R}$ consisting of all noncommutative $d$-tori Lip-normed as above with respect to the length function $\ell$. This is in fact a subset of $\mathcal{R}_{u.a}$ by Proposition 3.9, since noncommutative tori are nuclear. We will show using Theorem 6.3 that $A(d, \ell)$ is totally bounded.

For $(n_1, \ldots, n_d) \in \mathbb{N}^d$ we denote by $R(n_1, \ldots, n_d)$ the set of points $(k_1, \ldots, k_d)$ in $\mathbb{Z}^d$ such that $|k_i| \leq n_i$ for $i = 1, \ldots, d$. For $a \in A_\rho$, we define for every $(n_1, \ldots, n_d) \in \mathbb{N}^d$ the partial Fourier sum

$$s_{(n_1, \ldots, n_d)}(a) = \sum_{(k_1, \ldots, k_d) \in R(n_1, \ldots, n_d)} \tau(a u_d^{-k_d} \cdots u_1^{-k_1}) u_1^{k_1} \cdots u_d^{k_d}$$
and for each \( n \in \mathbb{N} \) the Cesàro mean
\[
\sigma_n(a) = (n+1)^{-d} \sum_{(n_1, \ldots, n_d) \in R(n,n,\ldots,n)} s_{(n_1,\ldots,n_d)}(a).
\]

As in classical Fourier analysis (see for example [4]) it can be shown that if \( K_n \) is the Fejér kernel
\[
K_n(t) = \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right) e^{2\pi i k t} = \frac{1}{n+1} \left( \frac{\sin((n+1)t/2)}{\sin(t/2)} \right)^2.
\]
then for all \( a \in A_\rho \) and \( n \in \mathbb{N} \) we have
\[
\|a - \sigma_n(a)\| \leq \sum_{k=1}^{d} \int_T \|a - \gamma_{r_k}(t)\| K_n(t) \, dt
\]
where \( r_k(t) \) denotes the \( d \)-tuple which is \( t \) at the \( k \)th coordinate and 0 elsewhere, and \( dt \) is normalized Haar measure (see for example the proof of [15, Thm. 22]). It follows that if \( a \in \mathcal{D}(L) \) then
\[
\|a - \sigma_n(a)\| \leq \sum_{k=1}^{d} \int_T \ell(r_k(t)) K_n(t) \, dt.
\]

In particular the Cesàro means of elements of \( \mathcal{D}(L) \) converge at a rate which does not depend on \( \rho \). By [4, Lemma 9.4] there is a constant \( M > 0 \) such that \( \text{diam}(A_\rho, L) \leq M \) for all \( (A_\rho, L) \in \mathcal{A}(d, \ell) \). Hence to obtain the total boundedness of \( \mathcal{A}(d, \ell) \) we need only check condition (ii) in Theorem 6.3. Let \( \epsilon > 0 \). If \( B \) is a finite-dimensional \( C^* \)-algebra and \( \alpha : A_\rho \to B \) and \( \beta : B \to A_\rho \) are u.c.p. maps with \( \| (\beta \circ \alpha)(x) - x \| < \epsilon \) for all \( x \in \mathcal{D}_1(L) \cap \mathcal{B}_{M}^{A_\rho} \); then it is readily seen that Proposition 2.11 implies that the triple \((\alpha, \beta, B)\) lies in \( CPA_L(\epsilon) \). From above there is an \( n \in \mathbb{N} \) such that each element of \( \mathcal{D}(L) \cap \mathcal{B}_{M}^{A_\rho} \) is within \( \epsilon \) of its \( n \)th Cesàro mean, which is a linear combination of elements in \( \{ u_1^{k_1} \cdots u_d^{k_d} : |k_i| \leq n \} \) with coefficients bounded in modulus by \( M \) (since the operation of taking a Cesàro mean decreases the moduli of Fourier coefficients, which are bounded by the norm of the given element). Thus in view of Proposition 5.7 it suffices to show the existence of a finite-dimensional \( C^* \)-algebra \( B \) and u.c.p. maps \( \alpha : A_\rho \to B \) and \( \beta : B \to A_\rho \) such that \( \| (\beta \circ \alpha)(x) - x \| < \epsilon \) for all \( x \in \{ u_1^{k_1} \cdots u_d^{k_d} : |k_i| \leq n \} \) and the rank of \( B \) can be chosen independently of \( \rho \), and this is a consequence of [14, Lemma 5.1].

Hanfeng Li has informed me that he can show that the map from the space of antisymmetric bicharacters on \( \mathbb{Z}^d \) to \( \mathcal{A}(d, \ell) \) determined by \( \rho \mapsto A_\rho \) is continuous (as Rieffel showed for quantum Gromov-Hausdorff distance in [4, Thm. 9.2]). In fact, given any field of strongly continuous ergodic actions of a compact group on a continuous field of unital \( C^* \)-algebras over a compact metric space \( X \), at any point of \( X \) the continuity of complete distance is equivalent to the local constancy (or, equivalently, the lower semicontinuity) of the function on \( X \) which records the multiplicity of the action in the fibre algebras. This is a result of the fact that Li (unpublished notes) has worked out a general version of Rieffel’s result on coadjoint orbits as described in Example 3.12.
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