Abstract

We consider neutrino oscillations in vacuum in the framework of quantum field theory in which neutrino production and detection processes are part of a single Feynman diagram and the corresponding cross section is computed in the standard way, \textit{i.e.} with final states represented by plane waves. We use assumptions which are realized in actual experiments and concentrate on the detection process. Moreover, we also allow for a finite time interval of length $\tau$ during which the detector records neutrino events. In this context we are motivated by accelerator-neutrino oscillation experiments where data taking is synchronized in time with the proton spill time of the accelerator. Given the final momenta and the direction of neutrino propagation, we find that in the oscillation amplitude—\textit{for all practical purposes}—the neutrino energy $Q$ is fixed, apart from an interval of order $2\pi\hbar/\tau$ around a mean energy $\bar{Q}$; this is an expression of energy non-conservation or the time-energy uncertainty relation in the detection process due to $\tau < \infty$. We derive in excellent approximation that in the amplitude the oscillation effect originates from massive neutrinos with the same energy $\bar{Q}$, \textit{i.e.} oscillations take place in space without any decoherence effect, while the remaining integration over $Q$ in the interval of order $2\pi\hbar/\tau$ around $\bar{Q}$ results in a time-correlation function expressing the time delay between neutrino production and detection.
1 Introduction

Neutrino oscillations have been proposed about 60 years ago \cite{1,2} and are at present firmly established by a host of experiments \cite{3}. However, the theory of neutrino oscillations is still not a fully closed subject. It is true that no thorough examination of the derivation of the standard neutrino oscillation probabilities \cite{4,5,6,7,8,9} has ever cast doubts on their validity in realistic experiments—for reviews see e.g. \cite{10,11,12,13,14,15}, but nevertheless some subtle issues, albeit of rather theoretical nature, remain to be clarified—for a recent paper see \cite{16}. There are still two theoretical frameworks, which are kind of competing, namely the quantum-mechanical wave-packet approach \cite{17,18,19,20,21,22,23} and the quantum-field-theoretical approach \cite{24,25,26,27,28,29,30}. In \cite{30} a comparison between the two approaches is carried out. In this paper we stick to the quantum-field-theory (QFT) framework because we think that it provides a more realistic interface between theory and experiment. This framework suggests to consider neutrino production and detection as a compound process \cite{24,25,31,32} included in a single Feynman diagram in which neutrinos propagating from the source to the detector are represented as inner lines.

The present paper can be considered as continuation of \cite{15,26,28}. (Here we do not take into account matter effects \cite{33,34,35,36}.) We revisit the QFT derivation of neutrino oscillations for two reasons:

i. In \cite{15,26,28} we have made the unrealistic assumption that the detector particle is in an energy eigenstate. In the present work we allow for an energy spread of the order of the thermal motion of the detector particle.\footnote{If different energies of the detector-particle state are not correlated, as advocated in \cite{37,38,39}, then interference of states with different energies is not observable and the integration over the energy of the detector particle happens in the cross section, not in the amplitude. If this were the case, then, regarding the oscillation amplitude, this would effectively amount to the assumption of an energy eigenstate of the detector particle. Since we are not sure about this issue, we consider the more general setting of admitting an energy spread of the detector particle in the oscillation amplitude.}

ii. One could be tempted to believe that neutrinos oscillate in time in accelerator experiments because, in order to reduce background, in such experiments beam-induced neutrino events are correlated in time with the proton spill time of the accelerator \cite{40,41,42,43}. Therefore, we simulate this situation by building in a finite time interval $\tau$ in the QFT oscillation amplitude in which the detector particle is ready for recording a neutrino. In practice, $\tau$ is between 1 and 10 $\mu$s.

We employ a twofold strategy to tackle the QFT of neutrino oscillations. On the one hand, we stress the necessity of using assumptions in our theoretical considerations that are realized in actual experiments. On the other hand, we focus on the neutrino detection process since our analysis will show that this is the essential process concerning coherence in the neutrino oscillation amplitude, as demonstrated earlier in \cite{44,45}. At any rate, it will turn out that, using this strategy, neutrino oscillations follow naturally from QFT. This includes the standard way of computing cross sections, i.e. with plane waves in the final states. Though for definiteness we will consider a model process for neutrino production and detection, our results will be of general validity.
We emphasize that in the present paper we are not concerned about decoherence effects in the oscillation probabilities originating from imperfect determination of the neutrino energy. Our subject is a possibly more fundamental decoherence effect in the oscillation amplitude which could not be overcome even with perfect knowledge of the energies and momenta of all particles in the final state of the neutrino detection process.

The plan of the paper is as follows. In section 2, in order to fix some notation, we write down the Hamiltonian in the $V-A$-theory and in section 3 we introduce our model process and define its associated 4-momenta; moreover, we define the wave functions $\psi_S$ and $\psi_D$ of the neutrino source and detector particle, respectively, and discuss the functions in time which allow us to incorporate the finite lifetime of the source particle and the finite time interval for the neutrino flavour measurement in the detector. After having written down the full amplitude for the compound neutrino production–detection process in section 4 and discussed the most suitable order of integrations in section 5, we study in detail in section 6 the relevant energies and the approximate energy conservation in the detection process. Section 7 is devoted to the asymptotic limit of large distance between source and detector and section 8 contains the discussion of the last remaining integration in the variable $Q$ which at this stage can be considered as the neutrino energy. We demonstrate in section 9 that neutrino oscillations naturally follow from the QFT formalism. Finally, we summarize and present our conclusions in section 10. Some mathematical tools, used in the body of the paper, are presented in the appendix.

2 Hamiltonian

For simplicity, we assume that the average energy of the neutrino beam is well below the mass of the $W^\pm$ boson. Therefore, in neutrino production and detection we are allowed to use the $V-A$ Hamiltonian

$$H_{V-A} = \frac{G_F}{\sqrt{2}} (V^\mu - A^\mu) (V_\mu - A_\mu)$$

with

$$V^\mu = \bar{u}\gamma^\mu U_{CKM}d + \bar{\nu}\gamma^\mu U_{PMNS}^\dagger \ell, \quad A^\mu = \bar{u}\gamma^\mu \gamma_5 U_{CKM}d + \bar{\nu}\gamma^\mu \gamma_5 U_{PMNS}^\dagger \ell,$$

where $U_{CKM}$ is the Cabibbo–Kobayashi–Maskawa quark mixing matrix [46] and $U_{PMNS}$ the Pontecorvo–Maki–Nakagawa–Sakata lepton mixing matrix [47]. We can write the $V-A$ current as

$$V^\mu - A^\mu = j^\mu_{\text{had}} + j^\mu_{\text{lep}}$$

with

$$j^\mu_{\text{had}} = \bar{u}\gamma^\mu (\mathbb{1} - \gamma_5) U_{CKM}d, \quad \text{and} \quad j^\mu_{\text{lep}} = \bar{\nu}\gamma^\mu (\mathbb{1} - \gamma_5) U_{PMNS}^\dagger \ell$$

being the hadronic and the leptonic current, respectively. Furthermore, we use the notation

$$U_{PMNS} = (U_{\alpha j}) \quad \text{with} \quad \alpha = e, \mu, \tau \text{ and } j = 1, 2, 3$$

when we refer to elements of the PMNS matrix. The symbol $\nu$ denotes the vector of the three neutrino mass eigenfields $\nu_j$. In an analogous way we employ the symbols $u, d$ and $\ell$ for up-quarks, down-quarks and charged leptons, respectively.
3 Example and model process for neutrino production and detection

We assume that $\beta$-decay of the nucleus $N = (Z, A)$ into the daughter nucleus $N' = (Z + 1, A)$, i.e.

$$N(p) \to N'(p') + e^-(p_e') + \bar{\nu}_e(q),$$

constitutes the neutrino production process. The 4-momenta are indicated in parentheses. For neutrino detection, we assume the inverse $\beta$-decay

$$\bar{\nu}_e(q) + p(k) \to n(k') + e^+(k_e'),$$

where the neutron and the positron are measured in coincidence.

We remind the reader that we consider neutrino source and detection processes as a single compound process. Therefore, the flavour neutrino $\bar{\nu}_e$ indicates that the production vertex is connected to the detection vertex by neutrino propagators with (virtual) momentum $q$ and mass $m_j$.

Though we have already fixed the notation of our 4-momenta, we have to take into account that we will frequently need energies and spatial momenta separately. The 4-momenta of $N$ and the proton are decomposed as

$$p = \begin{pmatrix} E_S \\ \vec{p} \end{pmatrix} \quad \text{and} \quad k = \begin{pmatrix} E_D \\ \vec{k} \end{pmatrix},$$

respectively, where

$$E_S = \sqrt{m_N^2 + \vec{p}^2} \quad \text{and} \quad E_D = \sqrt{m_p^2 + \vec{k}^2}.$$  \hfill (9)

Furthermore,

$$p' = \begin{pmatrix} E'_S \\ \vec{p}' \end{pmatrix}, \quad p'_e = \begin{pmatrix} E'_{eS} \\ \vec{p}'_e \end{pmatrix}, \quad k' = \begin{pmatrix} E'_D \\ \vec{k}' \end{pmatrix}, \quad k'_e = \begin{pmatrix} E'_{eD} \\ \vec{k}'_e \end{pmatrix}$$  \hfill (10)

with

$$E'_S = \sqrt{m_N'^2 + (\vec{p}')^2}, \quad E'_{eS} = \sqrt{m_e^2 + (\vec{p}'_e)^2},$$

$$E'_D = \sqrt{m_n^2 + (\vec{k}')^2}, \quad E'_{eD} = \sqrt{m_e^2 + (\vec{k}'_e)^2}.$$  \hfill (11)

Note that all momenta associated with the neutrino source process have the letter $p$, while those associated with the detection process have $k$. All final momenta are primed, in contrast to the unprimed incoming ones. Energies associated with the source process carry the subscript $S$, while those associated with the detection process have $D$.

The state of the source particle $N$ is assumed to be smeared out in momentum space by $\psi_S(\vec{p})$ and that of the detector particle $n$ by $\psi_D(\vec{k})$. Moreover, we assume that the functional forms of both $\psi_S(\vec{p})$ and $\psi_D(\vec{k})$ are such that they correspond to a localization of source and detector particle around $\vec{x} = \vec{0}$ in coordinate space. The respective shifts
to the space-time locations of the source and detection process will be performed in the
compound amplitude $\mathcal{A}$ in the next section.

Since $N$ decays we introduce the time dependence

$$\Theta(t - t_S) \exp \left( -\Gamma(t - t_S)/2\gamma \right) \quad (12)$$

for the neutrino source, where $\Theta$ is the Heaviside function, $\Gamma$ the decay width and

$$\gamma = E_S/m_N. \quad (13)$$

Concerning the detector, we assume that it measures in a time interval of length $\tau$ centered
around the time $t_M$ ($t_M > t_S$). Thus we introduce the function

$$\Theta_\tau(t - t_M) = \begin{cases} 1 & \text{for } |t - t_M| < \frac{\tau}{2}, \\ 0 & \text{for } |t - t_M| > \frac{\tau}{2}, \end{cases} \quad (14)$$

to model the time-dependent data-taking in the detector. We have to distinguish between
the time $t_M$, the “measurement time,” and the time $t_D$ which is the time when the state
of the detector particle is described by $\psi_D$. Since all

$$t_D \in [t_M - \tau/2, t_M + \tau/2] \quad (15)$$

are equivalent, we finally have to average over $t_D$ in the cross section of the compound
process.

We stress that the functions of equations (12) and (14) break translation invariance in
time. Therefore, energy conservation in the production and detection processes can only
hold approximately.

## 4 Amplitude

We introduce, for the relevant expectation values of the hadronic currents, the notation

$$\frac{G_F}{\sqrt{2}} \langle N'(p') | j^\mu_{\text{had}}(x) | N(p) \rangle \equiv J^\mu_{h,S}(p', p) \, e^{i(p'-p) \cdot x}, \quad (16)$$

$$\frac{G_F}{\sqrt{2}} \langle n(k') | (j^\mu_{\text{had}}(x))^\dagger | p(k) \rangle \equiv J^\mu_{h,D}(k', k) \, e^{i(k'-k) \cdot x}. \quad (17)$$

In these expressions all particles are energy-momentum eigenstates. The functions $\psi_S(\vec{p})$
and $\psi_D(\vec{k})$ will be taken into account below in the amplitude $\mathcal{A}$ of the compound process.

In order to formulate that the source particle is located at $\vec{x}_S$ at the time $t_S$ and that
the detector particle is located at $\vec{x}_D$ at the time $t_D$, it is convenient to define the 4-vectors

$$x_S = \left( \begin{array}{c} t_S \\ \vec{x}_S \end{array} \right) \quad \text{and} \quad x_D = \left( \begin{array}{c} t_D \\ \vec{x}_D \end{array} \right). \quad (18)$$

This has some similarity with the introduction of the “time-dependent propagator” in [48]. However,
in that paper the authors eventually set $\tau = 0$ and use plane waves also for the neutrino source and
detector particles. Finally, their conclusions seem to differ considerably from ours.
In lowest order the compound amplitude $A$ has two vertices, one for the source process associated with the space-time variable $x_1$, and one for the detection process associated with $x_2$.

Now we are in a position to put together the compound amplitude:

$$A = -i \sum_{j=1}^{3} \int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4} e^{-iq(x_1-x_2)}$$

$$\times \int d^3p \psi_S(\vec{p}) \exp \left[-ip \cdot (x_1 - x_S) - \Gamma(t_1 - t_S)/(2\gamma)\right] \Theta(t_1 - t_S)$$

$$\times e^{i(p' + p'_e) \cdot x_1} J_{h,S}(p', p)$$

$$\times \int d^3k \psi_D(\vec{k}) \exp \left[-ik \cdot (x_2 - x_D)\right] \Theta(t_2 - t_M)$$

$$\times e^{i(k' + k'_e) \cdot x_2} J_{h,D}(k', k)$$

$$\times U_{ej} U_{ej}^* \bar{u}_e(p'_e) \gamma_\lambda (1 - \gamma_5) \frac{q + m_j}{q^2 - m_j^2 + i\epsilon} \gamma_\rho (1 - \gamma_5) v_e(k'_e).$$

In the last line, $u_e$ and $v_e$ are the 4-spinors of the electron and positron, respectively. The shift of the source and detector particles from $x = 0$ to $x_S$ and $x_D$, respectively, has been accomplished by the replacements

$$e^{-ip \cdot x_1} \rightarrow e^{-ip \cdot (x_1 - x_S)} \quad \text{and} \quad e^{-ip \cdot x_2} \rightarrow e^{-ip \cdot (x_2 - x_D)},$$

cf. equations (16) and (17). Therefore, $\psi_S(\vec{p})$ and $\psi_D(\vec{k})$ refer to wave functions in momentum space at times $t_S$ and $t_D$, respectively.

5 Integrations

To proceed further in a transparent way, the order of the integrations is crucial:

1. $\int d^3 x_1$ and $\int d^3 x_2$,
2. $\int d^3 p$ and $\int d^3 k$,
3. $\int dt_1$ and $\int dt_2$,
4. $\int d^3 q$ in the asymptotic limit $L \equiv |\vec{x}_D - \vec{x}_S| \rightarrow \infty$,
5. $\int dq^0$ in an approximation to be discussed.

While the first three integrations are elementary, the last two can only be performed in suitable approximations and will be treated in separate sections.

The first integration leads to $\delta$-functions:

$$(2\pi)^6 \delta(\vec{p} + \vec{q} - \vec{p}' - \vec{p}'_e) \delta(\vec{k} - \vec{q} - \vec{k}' - \vec{k}'_e).$$

(21)
Therefore, the second integration amounts to the replacements
\[ \vec{p} \rightarrow \tilde{\vec{p}} \equiv \vec{p}' + \vec{q}, \quad \vec{k} \rightarrow \tilde{\vec{k}} \equiv \vec{k}' + \vec{q} \] (22)
everywhere in \( \mathcal{A} \). In order to keep track of these replacements in the energies as well, we define
\[ \tilde{E}_S = \sqrt{m_N^2 + (\tilde{\vec{p}})^2}, \quad \tilde{E}_D = \sqrt{m_p^2 + (\tilde{\vec{k}})^2}, \quad \tilde{\gamma} = \tilde{E}_S/m_N, \] (23)
and
\[ \tilde{\vec{p}} = \left( \frac{\tilde{E}_S}{\vec{p}} \right), \quad \tilde{\vec{k}} = \left( \frac{\tilde{E}_D}{\vec{k}} \right). \] (24)
The time integrations can be performed explicitly as well. In summary, after three integration steps the amplitude has the form
\[
\mathcal{A} = -i(2\pi)^3 \times e^{i(p'+p_e')x_S} \times e^{i(k'+k_e')x_D} \times \sum_{j=1}^{3} \int q^4 q e^{iq(x_D-x_S)}
\times \psi_S(\tilde{\vec{p}}) J_{h,S}^\lambda(p',\tilde{\vec{p}}) \times \psi_D(\tilde{\vec{k}}) J_{h,D}^\rho(k',\tilde{\vec{k}})
\times \frac{1}{\iota \left( q^0 + \Delta \tilde{E}_S \right) + \Gamma/(2\tilde{\gamma})} \times \frac{\sin \left( \frac{\pi}{2} \left( q^0 - \Delta \tilde{E}_D \right) \right)}{\pi(q^0 - \Delta \tilde{E}_D)} \frac{e^{i(q^0 - \Delta \tilde{E}_D)(t_M-t_D)}}{e^{i\gamma^2 / q^2 - m_j^2 + i\epsilon \kappa_5 (1 - \gamma_5) v_e(k_e')}}. \] (25a)
(25b)
(25c)
(25d)
For simplicity of notation we have introduced the energy differences
\[ \Delta \tilde{E}_S = \tilde{E}_S - E'_S - E'_{eS} \quad \text{and} \quad \Delta \tilde{E}_D = \tilde{E}_D - E'_D - E'_{eD}. \] (26)

6 The detection process

Before we proceed further with the final integrations, it is useful consider the detection process in more detail. This will help us to understand the relevant approximations.

Approximate energy conservation in neutrino detection: The amplitude \( \mathcal{A} \) depends on the measurement interval \( \tau \)—cf. equation (25c). In the limit of infinitely long measurement time we obtain exact energy conservation in the measurement reaction, i.e.
\[
\lim_{\tau \to \infty} \frac{\sin \left( \frac{\pi}{2} (q^0 - \Delta \tilde{E}_D) \right)}{\pi(q^0 - \Delta \tilde{E}_D)} = \delta \left( q^0 - \Delta \tilde{E}_D \right). \] (27)
The symbol “\( \delta \)” in this equation denotes the one-dimensional delta function.
In accelerator experiments, $\tau$ is not infinitely long but of the order of microseconds and energy will only approximately be conserved in the detection reaction.\(^3\) Inserting $\hbar$ into the sine function, the approximate range of $q^0$ is determined by the inequality
\[
\frac{\tau}{2\hbar} |q^0 - \Delta \tilde{E}_D| \lesssim \pi, \tag{28}
\]
where the right-hand side indicates the first zero of the sine function.\(^4\) In all numerical considerations involving $\tau$ we will set $\tau = 10^{-6}$ s, which is in the right ballpark used in experiments \([40, 41, 42, 43]\). Then, with $\hbar \simeq 6.582 \times 10^{-22}$ MeV we obtain
\[
|q^0 - \Delta \tilde{E}_D| \lesssim \frac{2\pi \hbar}{\tau} \simeq 4 \times 10^{-9}$ eV. \tag{29}
\]
We conclude that—with a realistic value of $\tau$—approximate energy conservation in the detection reaction is much better than any energy measurement accuracy in particle scattering.

**Sign and order of magnitude of $q^0$:** Let us recall realistic neutrino-detection reactions. One possibility is a general inverse $\beta$-decay with a stable detector particle, where the charged lepton in the final state uniquely indicates the neutrino flavour. The other one is elastic neutrino–electron scattering, without unique neutrino-flavour determination. Moreover, in practice, the detector particle in these reactions is at rest. Of course, it is at rest only for all practical purposes because in a fluid it undergoes thermal motion and in a solid body it oscillates around its equilibrium position.

In the case of a general inverse $\beta$-decay, stability of the detector particle means, in particular, that it cannot have $\beta$-decay. Therefore, neglecting the small neutrino masses, the mass of the detector particle is smaller than the sum over the masses in the final state of the detection process, \(i.e.\) there is an energy threshold to overcome. Usually it is necessary to observe the track of the final charged lepton in the detector, then, in addition, the kinetic energy of the charged lepton must be above a certain minimum.

In our model detection reaction $\bar{\nu}_e + p \rightarrow e^+ + n$ this threshold is given by the well-known value $m_n + m_e - m_p \simeq 1.8$ MeV. Since here the positron and the neutron are measured in coincidence, no minimum kinetic energy of the positron is required.

In elastic neutrino-electron scattering as the detection reaction the kinetic energy of the electron in the final state must be at least several 100 keV, much larger than the energy of thermal motion, in order to have reasonable background rejection.

Putting these rather trivial points together, we conclude that $\Delta \tilde{E}_D$ of equation \((26)\) or any analogon thereof corresponding to other detection reactions must be negative and its absolute value of the order of several 100 keV or more. For definiteness we set
\[
-\Delta \tilde{E}_D \gtrsim 0.5$ MeV, \tag{30}
\]
\(^3\)It appears strange that selecting detector events from a specific time interval introduces a tiny energy non-conservation, however, this can be conceived as an expression of the time-energy uncertainty relation.

\(^4\)More realistically, we should take some multiple of $\pi$. We will do this later in section \([9]\) where we discuss an approximate integral over $Q$.\]
which we will use in the following for numerical estimates. Consequently, because of the
approximate energy conservation expressed by equation (29), we find that only values of
$q^0$ which fulfill

$$q^0 < 0 \quad \text{and} \quad -q^0 \gtrsim 0.5 \text{MeV}$$

(31)
can contribute significantly to $A$. This will be relevant in the next section.

7 Integration over $d^3q$ and asymptotic limit $L \to \infty$

The integration $\int d^3q$ cannot be performed exactly, but anyway we need the result only
in the asymptotic limit $L = |\vec{x}_D - \vec{x}_S| \to \infty$. For this purpose we use the theorem proven
in [26] that is reproduced in the present paper in appendix A as theorem 1.

In order to simplify our notation, we define, in view of equation (31), the positive
quantity

$$Q = -q^0.$$  (32)

Then, the theorem tells us that, in the asymptotic limit $L \to \infty$, the dominant contribu-
tions to $A$, equation (25), require $Q^2 > m^2_j \forall j$ and drop as $1/L$, while subdominant
terms diminish at least as $1/L^{3/2}$. In order to conveniently formulate the dominant terms
as given by the theorem, we define

$$\vec{q}_j = \sqrt{Q^2 - m^2_j} \vec{\ell} \quad \text{and} \quad q_j = \left( \frac{Q}{\vec{q}_j} \right)$$  (33)

with

$$\vec{\ell} = \vec{x}_D - \vec{x}_S \quad \text{and} \quad \ell = \frac{\vec{L}}{L}.$$  (34)

Note that the 4-momentum vectors $q_j$ are on-mass-shell:

$$q_j^2 = m^2_j.$$  (35)

Denoting the amplitude in the asymptotic limit by $A_\infty$ and invoking once more theorem 1,
we obtain the result

$$A_\infty = -i(2\pi)^3 \times e^{i(p' + p) \cdot x_S} \times e^{i(k' + k) \cdot x_D} \times \left( -\frac{2\pi^2}{L} \right)$$

$$\times \sum_{j=1}^3 \int dQ \Theta (Q^2 - m^2_j) e^{-iQ(t_D - t_S) + i\sqrt{Q^2 - m^2_j} L}$$

$$\times \psi_S(\vec{p}) J^\lambda_{h,S}(p', \vec{p}) \bigg|_{q=-q_j} \times \psi_D(\vec{k}) J^\rho_{h,D}(k', \vec{k}) \bigg|_{q=-q_j}$$

$$\times \frac{i}{Q - \Delta E_S + i\Gamma/(2\gamma)} \bigg|_{q=-q_j} \times \left( \frac{\pi \Theta(Q + \Delta E_D)}{\pi \Theta(Q + \Delta E_D)} \right)$$

$$\times e^{-i(Q + \Delta E_D)(t_M - t_D)} \bigg|_{q=-q_j}$$

$$\times U_{e_j}U^*_{e_j} \bar{u}_e(p'_e) \gamma^\lambda (1 - \gamma_5) \left( \gamma_q + m_j \right) \gamma^\rho (1 - \gamma_5) v_e(k'_e).$$  (36a, 36b, 36c, 36d)

\footnote{In general, for antineutrino detection one finds $q^0 < 0$ while $q^0 > 0$ holds for neutrino detection.}
In this asymptotic limit the intermediate virtual neutrinos become on-mass-shell, which allows us to use the decomposition
\[ q_j - m_j = \sum_s v(q_j, s) \bar{v}(q_j, s), \tag{37} \]
where \( s \) denotes the two independent spin states of the antineutrino. As a consequence, equation (36) can be written as
\[
\mathcal{A}_\infty = \frac{1}{L} \sum_{j=1}^3 U_{e j} U_{e j}^{*} \int dQ \Theta \left( Q^2 - m_j^2 \right) e^{-iQ(t_D - t_S) + i\sqrt{Q^2 - m_j^2} L} \times \mathcal{A}_S(p', p'_e, q_j) \mathcal{A}_D(k', k'_e, q_j) \times \sin \left( \frac{\pi}{2} (Q + \Delta E_D) \right) e^{-i(Q + \Delta E_D)(t_M - t_D)} \bigg|_{q=-q_j}. \tag{38a}
\]

The amplitudes \( \mathcal{A}_S(p, p'_e, q_j) \) and \( \mathcal{A}_D(k, k'_e, q_j) \) refer to the neutrino source and detection reaction, respectively. These amplitudes are unique up to factors which play no role in the following discussion:
\[
\mathcal{A}_S(p', p'_e, q_j) \propto \psi_S(p) \gamma_\lambda \left( 1 - \gamma_5 \right) v(q_j, +), \tag{39a}
\]
\[
\mathcal{A}_D(k', k'_e, q_j) \propto \psi_D(k) \gamma_\rho \left( 1 - \gamma_5 \right) e(q_j, +) \gamma_\rho \left( 1 - \gamma_5 \right) v_e(k'_e). \tag{39b}
\]
In the 4-spinor \( v(q_j, +) \) we have indicated the positive helicity of the antineutrino.

In view of the discussion in the previous section, in equation (38) the integration over negative \( Q \) is completely negligible. In addition, only values of \( Q \) with \( Q \gg m_j \) are relevant. Moreover, equation (33) suggests to consider \( Q \) simply as the neutrino energy.

Finally, it is expedient to discuss the condition for the applicability of the asymptotic limit given by the theorem of [26]. Inserting \( \hbar c \), this condition reads
\[
\frac{QL}{\hbar c} \gg 2\pi. \tag{40}
\]
Taking as an example \( Q = 0.5 \) MeV and \( L = 300 \) km, we find
\[
\frac{QL}{\hbar c} \simeq 0.8 \times 10^{18}. \tag{41}
\]
Thus for all realistic situations the asymptotic limit is justified.

8 Integration over \( dQ \)

Estimating the widths of \( \psi_D \) and \( \psi_S \): These wave functions describe the momentum distributions of the neutrino-dectort and source particles, respectively. Since in our discussion we put more weight on the neutrino-detection reaction than on the neutrino-source
reaction, we consider first the detector particle, which in our model reaction is the proton. As stated in the introduction, we assume that its non-relativistic thermal motion, \textit{i.e.} the mean value $\langle E_{\text{kin},D} \rangle$ of its kinetic energy, gives the correct ballpark estimate of its energy spread. Therefore, the width of $\psi_D$ denoted by $\sigma_D$ is estimated in the following way:

$$\langle E_{\text{kin},D} \rangle = \frac{3}{2} kT \Rightarrow \sigma_D \sim \sqrt{2m_p \langle E_{\text{kin},D} \rangle} = \sqrt{3m_p kT}. \quad (42)$$

Inserting $kT \simeq (1/40)$ eV for room temperature, we obtain the order-of-magnitude estimate

$$\sigma_D \sim 10 \text{ keV.} \quad (43)$$

Since in our model reaction the neutrino comes from some $\beta$-decay of a nucleus at rest, $\sigma_S$, the width of $\psi_S$, could be an order of magnitude larger. If the neutrino source, for instance charged pions, decays in flight, we would assume that $\sigma_S$ is much larger. Whatever the real value of $\sigma_S$ is, in the following only $\sigma_S \gtrsim s_D \quad (44)$ will be relevant.

**The limit $\tau \to \infty$:** As discussed in section 6, in this limit we have exact energy conservation in the detection process:

$$Q + \Delta \tilde{E}_D \bigg|_{q=-q_j} = 0 \quad \text{with} \quad \Delta \tilde{E}_D \bigg|_{q=-q_j} = \tilde{E}_D \bigg|_{q=-q_j} - E'_D - E'_{eD}. \quad (45)$$

This equation depends on the neutrino mass $m_j$. Thus, for every $j = 1, 2, 3$ there will be a separate solution $Q_j$ of equation (45). Knowing that $Q \gtrsim 0.5 \text{ MeV}$, we are allowed to perform an expansion in $m_j^2$:

$$\tilde{E}_D \bigg|_{q=-q_j} = \left[ m_p^2 + \left( \vec{k}' + \vec{k}_e - \sqrt{Q^2 - m_j^2 \vec{\ell}} \right)^2 \right]^{1/2} \quad (46a)$$

$$\simeq \left[ m_p^2 + \left( \vec{k}' + \vec{k}_e - Q \vec{\ell} \right)^2 \right]^{1/2} + \frac{m_j^2}{2m_p Q} \vec{\ell} \cdot \left( \vec{k}' + \vec{k}_e - Q \vec{\ell} \right). \quad (46b)$$

Equation (46b) immediately leads to the approximate solution $Q_j$ of equation (45) given by

$$Q_j = \bar{Q} - \frac{m_j^2}{2m_p Q} \vec{\ell} \cdot \vec{k} \quad \text{with} \quad \vec{k} = \vec{k}' - \vec{k}_e - \bar{Q} \vec{\ell}, \quad (47)$$

where $\bar{Q}$ is the solution of

$$\bar{Q} + \left[ m_p^2 + \left( \vec{k} \right)^2 \right]^{1/2} - E'_D - E'_{eD} = 0. \quad (48)$$

Note that, in the approximation $m_j = 0$, $\vec{k}$ is the momentum of the detector particle picked out by the momentum configuration at hand, defined by $\vec{\ell}$, $\bar{Q}$, $\vec{k}'$ and $\vec{k}_e$—cf. equation (22). Therefore,

$$\left| \vec{k} \right| \lesssim \sigma_D. \quad (49)$$
Let us estimate the order of magnitude of the term proportional to \( m_j^2 \) in equation (47). Taking into account equation (49), \( Q \gtrsim 0.5 \text{ MeV} \) and \( m_j \lesssim 0.1 \text{ eV} \) \(^2\), this term is of order \( 10^{-13} \text{ eV} \) or smaller, for the detector particle being a proton.\(^6\) Thus, for all practical purposes, we can set \( m_j = 0 \) in equation (47) and use \( Q_j = \bar{Q} \) for \( j = 1, 2, 3 \) in the context of neutrino oscillations.

In effect, with \( m_j = 0 \) we infer from equation (45) that, with fixed energies of the final particles in the detection process, the detector particle has a definite energy. This is exactly what has been assumed in [26, 28], while here we have demonstrated that this is correct with excellent accuracy.

As a side remark and anticipating neutrino oscillations, using the \( m_j \)-dependent \( Q_j \) of equation (47) in \( A_{\infty} \) of equation (38), we would introduce tiny, in practice irrelevant oscillations in time, i.e. in \( t_D - t_S \), in addition to those in space, i.e. in \( L \).

Finally, we discuss the delta function of equation (27), referring to energy conservation in the detection process, in the approximation \( m_j = 0 \). For this purpose we define the 4-momentum

\[
\bar{q}(Q) = Q \left( \frac{1}{\ell} \right),
\]

which is obtained from \( q_j \) by setting \( m_j = 0 \). Then, linear expansion in \( Q - \bar{Q} \) of the energy conservation relation gives

\[
Q + \Delta \tilde{E}_D \bigg|_{q=-\bar{q}(Q)} = (1 - \beta)(Q - \bar{Q})
\]

with

\[
\beta = - \frac{\partial \tilde{E}_D}{\partial Q} \bigg|_{q=-\bar{q}(Q)} = \frac{\vec{k} \cdot \vec{\ell}}{\tilde{E}_D} \quad \text{and} \quad \tilde{E}_D = \tilde{E}_D \bigg|_{q=-\bar{q}(Q)}.
\]

Therefore, in the approximation \( m_j = 0 \), the delta function of equation (27) reads

\[
\delta \left( Q + \Delta \tilde{E}_D \bigg|_{q=-\bar{q}(Q)} \right) = \frac{1}{1 - \beta} \delta (Q - \bar{Q}).
\]

Evidently, for a given momentum configuration, \( \beta \) is the projection of the velocity of the detector particle onto the direction \( \vec{\ell} \) in units of \( c \), the velocity of light. Since the detector particle is at rest apart from thermal motion, we have \( |\beta| \ll 1 \). In the case of the proton, with equations (43) and (49) we find \( |\beta| \lesssim 10^{-5} \).

**Finite \( \tau \) and the relevant range of \( Q \):** In this case the neutrino energy is not fixed. However we will have approximate energy conservation in the detection process—cf. section 6—expressed as

\[
\left| Q + \Delta \tilde{E}_D \bigg|_{q=-q_j} \right| \lesssim \frac{2\pi \hbar}{\tau}
\]

\(^6\)If the detector particle is an electron, this upper limit would be larger by a factor of 2000, but the mass correction would still be negligible.
Note that the neutrino energy $Q$ is not only restricted by the $\tau$-dependent factor in equation (38c) but also by its occurrence in $\psi_S$ and $\psi_D$ in $A_S$ of equation (39a) and in $A_D$ of equation (39b), respectively. (Its role in the exponents of equation (38) will be discussed in the next section, when we treat neutrino oscillations.) However, numerically the width $\sigma_D$ is much larger than $(2\pi h)/\tau$—compare equation (29) with equation (43). Therefore, we have

$$\sigma_D \gg \frac{2\pi h}{\tau}. \quad (55)$$

Finally, taking into account equation (44), we find that indeed equation (54) gives the strongest restriction on $Q$.

In the further discussion of equation (54) we set $m_j = 0$. According to equation (29), the relevant integration interval of the variable $Q$ extends, for all practical purposes, only over a tiny interval of a length $\Delta Q$ not much larger than $10^{-8}$ eV. As stressed in section 6, the reason is that the $\tau$-dependent factor in equation (38c) is almost a $\delta$-function. Therefore, we decompose $Q$ as

$$Q = \bar{Q} + \delta Q, \quad (56)$$

where $\bar{Q}$ is the mean value of the relevant range of $Q$. Thus we have

$$-\frac{1}{2} \Delta Q \lesssim \delta Q \lesssim \frac{1}{2} \Delta Q \quad \text{and} \quad \Delta Q \sim \frac{2\pi h}{\tau}. \quad (57)$$

The mean energy $\bar{Q}$ is determined by equation (48).

**Computation of $\bar{Q}$:** Let us now compute $\bar{Q}$ approximately by expanding $\bar{Q}$ in powers of $1/m_p$ as

$$\bar{Q} = Q_0 + \Delta_1 + \Delta_2 + \Delta_3 + \ldots \quad \text{with} \quad Q_0 = E'_D + E'_{eD} - m_p. \quad (58)$$

This expansion originates in an expansion of the square root in equation (48), which makes sense because of equation (49) and $\sigma_D^2/m_p^2 \sim 10^{-10}$. The lowest order $Q_0$ corresponds to a proton at rest without thermal motion. Defining

$$\vec{q}_0 = \vec{k}' + \vec{k}'_{eD} - \bar{Q}_0 \vec{\ell}, \quad (59)$$

we find

$$\Delta_1 = -\frac{\vec{q}_0^2}{2m_p}, \quad \Delta_2 = -\frac{\vec{q}_0^2}{2m_p^2} \vec{\ell} \cdot \vec{q}_0, \quad \Delta_3 = -\frac{\vec{q}_0^2}{2m_p^3} \left( \vec{\ell} \cdot \vec{q}_0 \right)^2. \quad (60)$$

The order of $\Delta_3$ is given by $\sigma_D^4/m_p^3 \sim 10^{-11}$ eV.

**An approximation for $A_\infty$:** The length $\Delta Q$ of the integration interval for $Q$ is very small. So it is quite conceivable that, when $Q$ is in this interval, the $Q$-dependence of the amplitudes $A_S$ and $A_D$ will be totally negligible because $\Delta Q$ is much smaller than all relevant energies in these amplitudes. This statement requires, in particular, that the relevant scales on which the wave functions $\psi_S$ and $\psi_D$ vary, are much larger than $\Delta Q$.
But this is a natural assumption due to \( \sigma_S \gtrsim \sigma_D \gg \Delta Q \). Likewise, we can also neglect the neutrino masses in these amplitudes. Therefore, we can make the replacements

\[
q_i \rightarrow \bar{q} (\bar{Q}) = \bar{Q} \left( \frac{1}{\ell} \right)
\]

in \( \mathcal{A}_S \) and \( \mathcal{A}_D \). Denoting the resulting \( Q \)-independent amplitudes by \( \bar{\mathcal{A}}_S \) and \( \bar{\mathcal{A}}_D \), we extract them from the integral in equation (38). Finally, switching from the integration variable \( Q \) to \( \delta Q \)—cf. equation (56), we obtain the approximation

\[
\mathcal{A}_\infty \simeq \frac{1}{L} \bar{\mathcal{A}}_S \bar{\mathcal{A}}_D \sum_{j=1}^{3} U_{ej} U_{ej}^* \int_{-\Delta Q/2}^{\Delta Q/2} d\delta Q \\
\exp \left[ i \left( -(Q + \delta Q) (t_D - t_S) + \sqrt{(Q + \delta Q)^2 - m_j^2 L} \right) \right] \\
\times \left. \frac{\sin \left( \frac{\pi}{2} (Q + \Delta \tilde{E}_D) \bar{Q} \right)}{\pi (Q + \Delta \tilde{E}_D) } \right|_{q=-\bar{q}(\bar{Q}+\delta Q)}
\]

The discussion of neutrino oscillations in the next section will be based on this form of \( \mathcal{A}_\infty \).

9 Neutrino oscillations

In this section, for the sake of clarity, we do not use natural units but insert \( \hbar \) and \( c \) in the appropriate places.

**The oscillation phase:** Denoting for each \( j = 1, 2, 3 \) the collection of phases appearing in the exponents of equation (62) by \( \mathcal{E}_j \), we have

\[
\mathcal{E}_j = \frac{1}{\hbar} \left[ -(Q + \delta Q) (t_D - t_S) + \sqrt{(Q + \delta Q)^2 / c^2 - (m_j c)^2 L} \right] \\
- \left( Q + \Delta \tilde{E}_D \right) (t_M - t_D) \right|_{q=-\bar{q}(\bar{Q}+\delta Q)}.
\]

Clearly, neutrino masses must not be neglected in these phases because there the effect of \( m_j^2 \) is amplified by the huge macroscopic length \( L \), which is the cause of observable neutrino oscillations. However, in view of \( m_j \ll \bar{Q} \), \( |\delta Q| \ll \bar{Q} \) and \( |\delta Q| \ll m_p \), we perform an expansion in \( 1/\bar{Q} \) in equation (63a) until \((1/\bar{Q})^3\) and in equation (63b) until \((\delta Q)^2\):

\[
\mathcal{E}_j \simeq -\frac{\bar{Q}}{\hbar} \left( t_D - t_S - \frac{L}{c} \right) - \frac{(m_j c^2)^2 L}{2hc\bar{Q}} - \frac{(m_j c^2)^4 L}{8hc\bar{Q}^3} \\
- \frac{\delta Q}{\hbar} \left[ t_D - t_S - \frac{L}{c} - \frac{(m_j c^2)^2 L}{2c\bar{Q}^2} \left( 1 - \frac{\delta Q}{\bar{Q}} \right) \right]
\]
\[-\frac{\delta Q}{\hbar} (t_M - t_D) \left[ 1 - \beta + \frac{\delta Q}{2\bar{E}_D} (1 - \beta^2) \right]. \quad (64c)\]

Note that, in an expansion of \( \mathcal{E}_j \) in \( 1/\bar{Q} \), terms \( \frac{\bar{Q}}{\hbar c} \left( \frac{\delta Q}{\bar{Q}} \right)^n \) with \( n = 2, 3, 4, \ldots, \) i.e. without powers of \( m_j^2 \), cannot occur. This can be inferred from setting \( m_j = 0 \) in the square root of \( \mathcal{E}_j \). The term linear in \( \delta Q \) in equation (64c) has already been displayed in equation (51), whereas the definitions of \( \beta \) and \( \bar{E}_D \) can be found in equation (52). The expansion in equation (64) contains the dominant terms, which we will take into account in the following, plus corrections to the dominant terms, which will be neglected but allow an error estimation.

Obviously, the first term on the right-hand side of equation (64a) drops out in the cross section and thus has no physical effect. The second term in equation (64a) gives rise to neutrino oscillations with oscillation lengths proportional to inverse mass-squared differences \( m_i^2 - m_j^2 \), while the third term is a correction to the second one but irrelevant in practice.

**Approximations:** To proceed with the amplitude of equation (62), we make the following approximations:

i. We replace the phase by its expanded form, equation (64).

ii. As announced above, we simplify \( \mathcal{E}_j \) further by dropping the third term in equation (64a) and by neglecting the terms quadratic in \( \delta Q \) in equations (64b) and (64c).

iii. Finally, in the integral of equation (62) we take the limit \( \Delta Q \to \infty \).

In summary, with these approximations, equation (62) reads

\[
A_\infty \simeq \frac{1}{L} \bar{A}_S \bar{A}_D \\
\times \sum_{j=1}^{3} U_{ej} U_{ej}^* \int_{-\infty}^{\infty} \delta Q \exp \left[ -i\frac{\delta Q}{\hbar} (T - \Delta t_j) \right] \times \frac{\sin \left( \frac{\pi}{2\hbar} (1 - \beta) \delta Q \right)}{\pi (1 - \beta) \delta Q}, \quad (65a) \\
\times \exp \left[ -i \left( \frac{Q}{\hbar} \left( t_D - t_S - \frac{L}{c} \right) + \frac{(m_j c^2)^2 L}{2\hbar c Q^2} \right) \right], \quad (65b)\]

where we have defined

\[ T = t_M - t_S - \frac{L}{c} + \beta (t_D - t_M) \quad \text{and} \quad \Delta t_j = \frac{(m_j c^2)^2 L}{2c Q^2}. \quad (66)\]

**Integration over \( \delta Q \):** The integral in equation (65) can now be performed exactly by using theorem 2 of the appendix. The result is

\[
A_\infty \simeq \frac{1}{L(1 - \beta)} \bar{A}_S \bar{A}_D \sum_{j=1}^{3} U_{ej} U_{ej}^* \Theta_r \left( \frac{T - \Delta t_j}{1 - \beta} \right), \quad (67a)\]
\[ \times \exp \left[ -i \left( \frac{Q}{\hbar} \left( t_D - t_S - \frac{L}{c} \right) + \frac{(m_j c^2 L)}{2 \hbar c Q} \right) \right]. \tag{67b} \]

Now we turn to a discussion of the quality of our approximations in the integration over \( \delta Q \). Let us reconsider in equation (65) finite boundaries \( \pm \Delta Q/2 \) of the integral. We know that the width \( \sigma_D \) of \( \psi_D \) provides a kind of cut-off for the \( Q \)-integration. Moreover, for \( Q \sim \sigma_D \) the specific form of \( \psi_D \), which we do not really know, would come into play. Therefore, a \( \Delta Q \) fulfilling \( 2\pi \hbar / \tau \ll \Delta Q \ll \sigma_D \) would give an excellent approximation of the integral without the need to know \( \psi_D \). With \( \sigma_D \sim 10 \text{ keV} \) and \( 2\pi \hbar / \tau \sim 4 \times 10^{-9} \text{ eV} \) for \( \tau = 10^{-6} \text{ s} \) there is ample space for such a \( \Delta Q \). However, in order to be able to compute the integral over \( d\delta Q \) explicitly, we have taken the limit \( \Delta Q \to \infty \). Therefore, we have to check which error we make by taking this limit. As discussed in the appendix, the integration over \( \delta Q \) amounts to two integrals of the form

\[ I \equiv \int_{-\Delta Q/2}^{\Delta Q/2} d\delta Q \frac{\sin \left( \frac{\delta Q}{2\pi} \right)}{2\pi \delta Q} \text{ with } t = \frac{T - \Delta t_j}{1 - \beta} \pm \frac{\tau}{2}. \tag{68} \]

Denoting by \( 1/f \) the error we make in \( I \) by extending the boundaries to infinity and using equation (A.11), we find

\[ \frac{1}{f} \gtrsim \frac{8\hbar}{\pi t \Delta Q} \Rightarrow \Delta Q \gtrsim 1.7 \times 10^{-9} \text{ eV} \times f \times 10^{-6} \text{ s}. \tag{69} \]

We see that it is possible to have \( \Delta Q \ll \sigma_D \) even for very small errors like \( 1/f = 10^{-6} \), provided \( t \) is not too small. However, for \( t \to 0 \), the boundary \( \Delta Q \) approaches \( \sigma_D \) and, for very small regions around the jump discontinuities of \( \Theta \tau \), equation (67a) might not be a good approximation.

**Physical interpretation of the result for \( A_\infty \):** Clearly, the phases in equation (67b) lead to neutrino oscillations.

So we turn to the physical interpretation of \( \Theta \tau \) in equation (67a). We note that the time interval \([t_M - \tau /2, t_M + \tau /2]\) when the detector takes data is specified and, therefore, equation (67a) makes a statement about the time \( t_S \) when the neutrino is produced. That statement is to be derived from

\[ -\frac{\tau}{2} < \frac{T - \Delta t_j}{1 - \beta} < \frac{\tau}{2}. \tag{70} \]

Actually, this equation leads to three intervals, one for each massive neutrino:

\[ t_M + \beta (t_D - t_M) - \frac{L}{c} - \Delta t_j - \frac{\tau}{2} (1 - \beta) < t_S < t_M + \beta (t_D - t_M) - \frac{L}{c} - \Delta t_j + \frac{\tau}{2} (1 - \beta). \tag{71} \]

Now it is interesting to ask the question whether this abstract QFT result has a simple physical interpretation.

Firstly we consider the quantity \( \Delta t_j \). This is the only term in equation (71) which contains a neutrino mass \( m_j \). It is easy to see that an ultrarelativistic particle with mass \( m_j \) covers the distance \( L \) in the time \( L/c + \Delta t_j \). Therefore, \( \Delta t_j \) is the time delay due
to the neutrino velocity being slightly smaller than the velocity of light. In other words, given the time when it is measured, it must be produced earlier than a massless particle by the time $\Delta t_j$.

Secondly we demonstrate that in equation (71) the remaining terms have a nice classical interpretation as well. It is sufficient to consider a one-dimensional setting. We know that the detector particle is at the location $x_D$ at time $t_D$ and moves with velocity $v = \beta c - c f$. In addition, having already taken into account $\Delta t_j$, the neutrino propagates with velocity $c$ and is located at $x_S$ where it is produced at time $t_S$. Therefore, we have the trajectories

$$x_D(t) = (t - t_D)v + x_D \quad \text{and} \quad x_\nu(t) = (t - t_S)c + x_S$$

(72)

for the detector particle and the neutrino, respectively. We furthermore stipulate that the neutrino is measured at time $t'_M$. Classically, $t'_M$ is determined by the equation

$$x_\nu(t'_M) = x_D(t'_M) \Rightarrow t'_M = \frac{1}{1 - \beta} \left( t_S - \beta t_D + \frac{L}{c} \right),$$

(73)

where we have set $L = x_D - x_S$. Since $t'_M$ must be inside the measurement interval, we obtain the inequalities

$$t_M - \frac{\tau}{2} < \frac{1}{1 - \beta} \left( t_S - \beta t_D + \frac{L}{c} \right) < t_M + \frac{\tau}{2},$$

(74)

which are equivalent to those of equation (71) without $\Delta t_j$.

In a nutshell, we have thus found that the $\Theta_\tau$-term in equation (67), albeit derived from QFT, can be nicely interpreted in the classical sense. This strengthens our confidence in the QFT model presented here.

Note that in principle the occurrence of $\Delta t_j$ in equation (67a) is a source of decoherence in the amplitude, if $\tau \lesssim \Delta t_j$. However, in practice this source of decoherence is irrelevant. To give a numerical example, we set $\bar{Q} = 0.5 \text{ MeV}$, $m_j = 0.1 \text{ eV}$ and $L = 300 \text{ km}$. Then $\Delta t_j \approx 2 \times 10^{-17} \text{ s}$ and for all practical purposes we have $\tau \gg \Delta t_j$.

**Averaging over $t_D$:** As already stated in section 3 it is necessary to average in the cross section over the time $t_D$—see equation (15). In the following we neglect $\Delta t_j$. Since $(\Theta_\tau)^2 = \Theta_\tau$, this amounts to

$$\frac{1}{\tau} \int_{t_M - \tau/2}^{t_M + \tau/2} dt_D \Theta_\tau \left( \frac{T}{1 - \beta} \right) \equiv F_\beta(T'/\tau) \quad \text{with} \quad T' = t_M - t_S - L/c.$$  

(75)

Because of the rectangular form of $\Theta_\tau$ we obtain the trapezoid form of $F_\beta(T'/\tau)$ displayed in figure 1. Since $|\beta| \ll 1$ this averaging has no practical effect.

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7 In the wave-packet picture this corresponds to the decoherence due to the separation of the wave packets pertaining to neutrinos with different masses moving therefore with slightly different velocities.

8 Here we have assumed that the tiny intervals of $T$ around $\pm (1 - \beta)\tau/2$, where $\Theta_\tau \left( \frac{T}{1 - \beta} \right)$ is possibly not a good description of the time-dependence of the amplitude, does not affect the averaging.
Figure 1: The function $F_\beta(T'/\tau)$ plotted against $\xi = T'/\tau$ for $\beta = 0.1$ (left plot) and $\beta = -0.1$ (right plot). Note that this value of $\beta$ is grossly exaggerated for the purpose of giving a visible effect in the figure.

**Final approximation for the amplitude:**  Now we take into account that for all practical purposes we can neglect both $\Delta t_j$ and $\beta$, as we have demonstrated above. Removing in addition the unphysical part of the phase in equation (67), and denoting the thus obtained approximate amplitude by $A'_\infty$, we arrive at the exceedingly simple formula

$$A'_\infty = \frac{1}{L} \bar{A}_S \bar{A}_D \sum_{j=1}^{3} U_{ej} U_{ej}^* \Theta_\tau \left( t_M - t_S - \frac{L}{c} \right) \exp \left( -i \frac{(m_j c^2)^2 L}{2 \hbar c Q} \right). \quad (76)$$

This is the main result of our paper.

### 10 Summary and conclusions

**Summary of assumptions:**

1. The process of neutrino production and detection is considered as one compound process, with the neutrino source located around $\vec{x}_S$ and detection around $\vec{x}_D$ such that the two locations are separated by a macroscopic distance $L = |\vec{x}_D - \vec{x}_S|$.

2. Neutrino oscillation probabilities are to be derived from the cross section of the compound process in the standard way, *i.e.* with all final particles in production and detection represented by plane waves.

3. The detector particle is at rest apart from thermal motion; it is assumed that the latter gives the correct order of magnitude of its energy spread.\(^9\)

4. For detecting a neutrino event and separating it from the background an energy of at least several $100\text{keV}$ has to be deposited in the detector.

\(^9\)Detector particles are bound in atoms and molecules and neighbouring bound states will influence the thermal behaviour of the detector particle.
Concerning items 1 and 2 we have studied the amplitude leading to the cross section in the asymptotic limit $L \to \infty$. To do so we have taken advantage of a theorem proven in [26]. As for item 2 this is our essential assumption and agrees with the computation of cross sections in particle-physics textbooks. We think that item 3 contains a reasonable assumption for realistic experiments, and the same applies to item 4.

**What is new in the present paper:** In [15, 26, 28] it has been assumed that the detector particle is in an energy eigenstate. In the present paper we have relaxed this assumption and we have allowed for a finite time interval $\tau$ for the measurement in the neutrino detector, as practised in oscillation experiments with neutrino sources generated by accelerators.

**The limit $\tau \to \infty$:** Nevertheless in this limit we obtain for all practical purposes the same result as in [15, 26, 28]:

i) For each neutrino mass $m_j$, the neutrino energy $Q_j$, equation (47), is completely fixed by the detection process. However, the dependence of $Q_j$ on $m_j$, which leads to a tiny amount of neutrino oscillations in time in addition to those in space, is totally negligible and for all practical purposes we have a common neutrino energy $\bar{Q}$, equation (48), leading to oscillations in space only.

ii) There is no decoherence effect in the amplitude $A_\infty$. Therefore, the only source of decoherence is the incomplete knowledge of the final momenta in the detection cross section.

Let us repeat the logic leading to statements i) and ii). In the limit $\tau \to \infty$ we obtain a $\delta$-function in energy for the detection process—see equation (27)—and in the asymptotic limit $L \to \infty$ the direction of the neutrino momentum is fixed by $\vec{\ell}$ of equation (34), which leaves, neglecting $m_j$, only $\bar{Q}$ to be determined by equation (48) in terms of final momenta. The summation over the final momenta occurs in the cross section—in the standard way of computing it in particle physics; therefore, final momenta (and energies) have no bearing on the coherence issue in the amplitude we are concerned with in our paper.

**Finite $\tau$:** If $\tau$ is not infinitely large, energy conservation in the detection process is not exact but this has no bearing on the utmost negligibility of $m_j$ in the kinematics of this process. Now the neutrino energy $Q$ can vary within an interval of approximate length $\Delta Q \sim 2\pi\hbar/\tau$ around a mean value $\bar{Q}$ determined by equation (48). Note that $\Delta Q$ is an expression of the time–energy uncertainty principle due to the finite measurement time interval $\tau$. According to the QFT formalism we still have to perform the integration over $Q$ in the interval $[\bar{Q} - \Delta Q/2, \bar{Q} + \Delta Q/2]$, the last integration in computing the compound amplitude of neutrino production and detection. And exactly this integration leads to the time correlation between neutrino source and neutrino detection, expressed by the function $\Theta_\tau$ in $A_\infty$, equation (67), that one would expect on physical grounds.

Furthermore, neutrino oscillations still happen with the same energy $\bar{Q}$ as before when the limit $\tau \to \infty$ was considered, without any decoherence for realistic $\tau$—see,
however, the discussion on $\tau$ and $\Delta t_j$ in the previous section. We emphasize that the QFT formalism naturally leads to oscillations in space but not in time, which confirms the qualitative analysis in [37, 38, 39].

Finally, we emphasize that our analysis corroborates the importance of the neutrino detection process for questions concerning coherence in the neutrino oscillation amplitude [44, 45].

**Conclusions:** In summary, we have demonstrated that QFT naturally leads to the standard neutrino oscillations in vacuum and the time correlation between neutrino production and detection, provided we take into account experimental conditions and compute the cross section of the compound neutrino production–detection process in the standard way, *i.e.* using planes waves for the final states. Although for definiteness we have assumed specific reactions for neutrino production and detection, our conclusions are general. In effect, we have found that neutrino oscillations take place in space with a single energy $\bar{Q}$. This is in contradiction to the wave-packet picture but we think in our framework this conclusion is compelling. Of course this contradiction is only a matter of theoretical interest and not of practical relevance.

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10Interpreting $L/c$ as the time of flight and defining momenta $P_j = \bar{Q}/c - (m_j c^2)^2/(2c\bar{Q})$, one could rewrite the phases in equation 67b such that oscillations occur in time with different momenta, but this would be an unnecessary and artificial reformulation without physical content, because the true time information is contained in $\Theta_\tau$.

11In the wave packet picture oscillations take place in both space and time.
A Appendix

Firstly, we recapitulate the theorem proven in [26], which we need in the context of the integration over $d^3q$.

**Theorem 1** Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a three times continuously differentiable function of the variable $\vec{q}$ such that $\Phi$ itself and all its first and second derivatives decrease at least like $1/|\vec{q}|^2$ for $|\vec{q}| \to \infty$, $A$ a real number and

$$J(\vec{L}) \equiv \int d^3q \, \Phi(\vec{q}) \, e^{-i\vec{q} \cdot \vec{L}} \, \frac{1}{A - |\vec{q}|^2 + i\varepsilon}. \quad (A.1)$$

Then in asymptotic the limit $L \equiv |\vec{L}| \to \infty$ one obtains, for $A > 0$,

$$J(\vec{L}) = -\frac{2\pi^2}{L} \phi\left(-\sqrt{A} \frac{\vec{L}}{L}\right) \, e^{i\sqrt{A}L} + O(L^{-3/2}), \quad (A.2)$$

whereas for $A < 0$ the integral decreases like $L^{-2}$.

Secondly, we compute a simple integral, used in the body of the paper for the integration over $dq^0$.

**Theorem 2** Let $\tau$ be positive and $T \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} du \, \exp(iTu) \frac{\sin\left(\frac{1}{2}\tau u\right)}{\pi u} = \begin{cases} 1 & \text{for } |T| < \frac{1}{2}\tau, \\ 0 & \text{for } |T| > \frac{1}{2}\tau. \end{cases} \quad (A.3)$$

**Proof:** We take the function $\Theta_\tau(t)$ defined in equation (14) and compute its Fourier transform

$$\tilde{\Theta}_\tau(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-iut} \Theta_\tau(t) = \sqrt{\frac{2}{\pi}} \frac{\sin\left(\frac{1}{2}\tau u\right)}{u}. \quad (A.4)$$

Application of the inverse Fourier transform then leads to

$$\Theta_\tau(T) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \, e^{iT\tau} \tilde{\Theta}_\tau(u) = \int_{-\infty}^{\infty} du \, e^{iT\tau} \frac{\sin\left(\frac{1}{2}\tau u\right)}{\pi u}, \quad (A.5)$$

which is the desired result.

Actually, equation (A.3) is not only true in the sense of the Fourier transform on $L^2(\mathbb{R})$, it also holds pointwise. No special precautions have to be taken for the convergence of the improper integral because both integrals

$$\lim_{a \to \infty} \int_{-a}^{0} du \, \exp(iTu) \frac{\sin\left(\frac{1}{2}\tau u\right)}{\pi u} \quad \text{and} \quad \lim_{b \to \infty} \int_{0}^{b} du \, \exp(iTu) \frac{\sin\left(\frac{1}{2}\tau u\right)}{\pi u} \quad (A.6)$$

exist for $|T| \neq \tau/2$. However, if $|T| = \tau/2$, then both integrals in equation (A.6) diverge. A simple remedy, which also makes sense for $|T| = \tau/2$, is given by

$$\lim_{a \to \infty} \int_{-a}^{a} du \, \exp(iTu) \frac{\sin\left(\frac{1}{2}\tau u\right)}{\pi u} = \ldots$$

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\[
\lim_{a \to \infty} \int_{-a}^{a} \frac{du}{2\pi u} \left[ \sin \left( (T + \tau/2)u \right) - \sin \left( (T - \tau/2)u \right) \right] = \begin{cases} 
1 & \text{for } |T| < \frac{1}{2} \tau, \\
\frac{1}{2} & \text{for } |T| = \frac{1}{2} \tau, \\
0 & \text{for } |T| > \frac{1}{2} \tau.
\end{cases} \tag{A.7}
\]

Let us now estimate the rate of convergence of the two integrals in equation (A.7). First we observe that for all \( r \in \mathbb{N} \) the inequalities
\[
-\frac{2}{\pi} \left( \frac{1}{2r-1} - \frac{1}{2r+1} \right) \left( \int_{-\infty}^{\infty} \frac{dy \sin y}{y} \right) < 0 \tag{A.8}
\]
hold due to \( 1/y \) being monotonously decreasing and \( \int_{0}^{\pi} dy \sin y = 2 \). Then, for
\[
t = |T \pm \tau/2| \tag{A.9}
\]
and setting
\[
ta = (2r - 1)\pi \quad \text{and} \quad y = tu \tag{A.10}
\]
we obtain
\[
-\frac{2}{\pi ta} < \left( \int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{du}{2\pi u} \sin tu = \left( \int_{-\infty}^{-ta} + \int_{ta}^{\infty} \right) \frac{dy}{2\pi y} \sin y < 0. \tag{A.11}
\]
We see that with \( t \to 0 \) the convergence of the integral worsens. This reflects the discontinuity in \( t \):
\[
\int_{-\infty}^{\infty} \frac{du}{2\pi u} \sin tu = \begin{cases} 
1/2 & \text{for } t > 0, \\
0 & \text{for } t = 0.
\end{cases} \tag{A.12}
\]
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