A modified Möbius $\mu$-function

RASA STEUDING, JÖRN STEUDING, LÁSZLÓ TÓTH

Rend. Circ. Mat. Palermo 60 (2011), 13–21

Abstract

We investigate a modified Möbius $\mu$-function which is related to an infinite product of shifted Riemann zeta-functions. We prove conditional and unconditional upper and lower bounds for its summatory function, and, finally, we discuss relations with Riemann’s hypothesis.

Mathematics Subject Classification: 11A25, 11N37, 11M26

Key Words and Phrases: Möbius $\mu$-function, infinitary convolution, Riemann zeta-function, Riemann hypothesis

1 Introduction and prehistory

The classical Möbius $\mu$-function is defined by $\mu(1) = 1$, $\mu(n) = 0$ if $n$ has a quadratic divisor $\neq 1$, and $\mu(n) = (-1)^r$ if $n$ is the product of $r$ distinct primes. It is easily seen that $\mu(n)$ is multiplicative and appears as coefficients of the Dirichlet series representation of the reciprocal of the Riemann zeta-function:

$$\zeta(s)^{-1} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

both representations being valid for $\sigma > 1$, where $s = \sigma + it$ with $i := \sqrt{-1}$ is a complex variable. Riemann’s famous open hypothesis on the non-vanishing of $\zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$ is known to be equivalent to the estimate

$$M(x) := \sum_{n \leq x} \mu(n) \ll x^{1/2+\epsilon}$$

for any positive $\epsilon$. Odlyzko & te Riele [9] disproved the original Mertens hypothesis [8], that is $|M(x)| < x^{1/2}$, by showing

$$\liminf_{x \to \infty} \frac{M(x)}{x^{1/2}} < -1.009 \quad \text{and} \quad \limsup_{x \to \infty} \frac{M(x)}{x^{1/2}} > 1.06;$$
for more details see Titchmarsh [14] (incl. the notes to §14).

In this note we are concerned with asymptotic properties of a modified Möbius function which is defined as the multiplicative arithmetical function $\mu_\infty$ given by

$$\mu_\infty(p^r) = (-1)^{|B(\nu)|},$$

(1)

for any prime power $p^r, \nu \in \mathbb{N}$, where $|B(\nu)|$ is the number of nonzero terms in the binary representation of the integer $\nu$, i.e., $\nu = \sum_{j \in B(\nu)} 2^j$. Here $\mu_\infty(p) = \mu_\infty(p^2) = -1, \mu_\infty(p^3) = 1, \mu_\infty(p^4) = -1, \mu_\infty(p^5) = \mu_\infty(p^6) = 1, \mu_\infty(p^7) = -1$, etc. This arithmetical function was introduced by Cohen and Hagis [2] and it is an interesting function for several reasons.

First of all, $\mu_\infty$ is the inverse of the function constant 1 under the infinitary convolution given by

$$\mu_\infty \ast \ast g(n) = \sum_{d | n} f(d)g(n/d),$$

(2)

where the sum is over the so called infinitary divisors of $n$, which are defined in the following way. The infinitary divisors of the integer $n = \prod p^r > 1$ are number 1 and the products of prime power divisors of $n$ of the form $p^{2j}$, where $j \in B(\nu)$ with the notation of above. By convention, $1 |_\infty 1$. The term “infinitary” is justified by an equivalent definition given by Cohen [1]. By a curious property, $p^b |_\infty p^a$ holds if and only if the binomial coefficient $\binom{a}{b}$ is odd.

On the other hand, the function $\mu_\infty$ is identical with the function denoted by $\mu^{**}$, which is the inverse of the function constant 1 under the bi-unitary convolution defined by

$$\mu_\infty \ast \ast g(n) = \sum_{d | n} f(d)g(n/d),$$

(3)

where $(a, b)_\ast \ast$ denotes the greatest common unitary divisor of $a$ and $b$. Recall that $\delta$ is said to be a unitary divisor of $k$ if $e$ divides $k$ with greatest common divisor $(e, k/e) = 1$. The sum in (3) is over the so called bi-unitary divisors of $n$. The bi-unitary divisors of a prime power $p^a$ ($a \geq 1$) are all divisors $p^b$ with $b = 0, 1, 2, \ldots, a$, except $p^{a/2}$ for $a$ even. The concept of bi-unitary divisor is due to Suryanarayana [11], while properties of the bi-unitary convolution are given by Haukkananen [4].

Note that both the infinitary and bi-unitary convolutions are commutative. The infinitary convolution is associative, however the bi-unitary convolution is not associative. The identity with respect to both convolutions is the function $\delta$ given by $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. Furthermore, $f$ has an inverse under each convolutions if and only if $f(1) \neq 0$. If $f$ and $g$ are multiplicative, then $f \times_\infty g$ and $f \times_\ast \ast g$ are also multiplicative. Moreover, if $f$ is a non-zero multiplicative function, then their inverses under each convolutions are also multiplicative, cf. [2, 4, 5].

Besides $\mu_\infty$ Cohen and Hagis [2] also investigated the functions $\tau_\infty(n)$ and $\sigma_\infty(n)$, denoting the number and the sum of the infinitary divisors of $n$, proving asymptotic formulae for the summatory functions of $\tau_\infty(n)$ and $\sigma_\infty(n)$. Asymptotic formulae for the corresponding bi-unitary functions $\tau^{**}(n)$ and $\sigma^{**}(n)$ were established in papers [11, 12, 13]. All of these functions are multiplicative.

These asymptotic formulae may be compared with those involving the classical divisor function $\tau(n)$ and the sum-of-divisors function $\sigma(n)$. In this note we shall prove several results concerning the summatory function of $\mu_\infty$ similar to those for the classical Möbius function.
2 Main results

The analytic method has proved to be a rather powerful approach to study the classical Möbius \( \mu \)-function. We shall mimic this approach and prove first a representation for the generating Dirichlet series:

**Theorem 1.** For \( \sigma > 1 \),

\[
m(s) := \sum_{n=1}^{\infty} \frac{\mu_\infty(n)}{n^s} = \prod_{j=0}^{\infty} \zeta(2^j s)^{-1}.
\]

**Proof.** Expanding into an Euler product and using (1),

\[
\sum_{n=1}^{\infty} \frac{\mu_\infty(n)}{n^s} = \prod_p \sum_{\nu=0}^{\infty} \frac{\mu_\infty(p^\nu)}{p^{\nu s}} = \prod_p \sum_{\nu=0}^{\infty} \left(1 - \frac{1}{p^{2\nu}}\right) = \prod_{j=0}^{\infty} \zeta(2^j s)^{-1}.
\]

The theorem is proved. 

We are mainly interested in the asymptotic behaviour of the summatory function of the modified Möbius \( \mu \)-function,

\[
\mathcal{M}(x) := \sum_{n\leq x} \mu_\infty(n),
\]

as \( x \to \infty \). Our first theorem is unconditional:

**Theorem 2.** There exists a positive constant \( c \) such that

\[
(4) \quad \mathcal{M}(x) = O \left( x \exp \left( -c (\log x)^{3/5} (\log \log x)^{-1/5} \right) \right),
\]

and, for any positive \( \epsilon \),

\[
\mathcal{M}(x) = \Omega(x^{\beta-\epsilon}),
\]

where \( \beta \) is the supremum over all real parts of \( \zeta \)-zeros (hence \( \frac{1}{2} \leq \beta \leq 1 \)).

Here \( f = \Omega(g) \) denotes the negation of \( f = o(g) \). The proof of the first assertion follows along the lines of the proof of the prime number theorem; the second statement is rather similar to the so-called Mertens conjecture or how the size of the summatory function of the Möbius \( \mu \)-function is related to the zeros of the zeta-function.

**Proof.** We start with the big-Oh estimate. By Perron’s formula, for \( c > 1 \),

\[
\mathcal{M}(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} m(s) \frac{x^s}{s} ds + \mathcal{E},
\]

where

\[
(5) \quad \mathcal{E} = O \left( \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|\mu_\infty(n)|}{n^c} + \frac{x \log x}{T} \right).
\]
We shall move the path of integration to the left. Korobov [7] and Vinogradov [15] (independently) proved
\[
\zeta(s) \neq 0 \quad \text{in} \quad \sigma \geq 1 - C (\log |t| + 3)^{-2/3} (\log \log(|t| + 3))^{-1/3},
\]
where \( C \) is some positive absolute constant; moreover, in the same region the estimate
\[
(6) \quad \zeta(\sigma + it)^{-1} \ll (\log |t| + 3)^{2/3} (\log \log(|t| + 3))^{1/3}
\]
holds. The first complete proof due to Richert appeared in Walfisz [16] (see also [6], §12). Denote the rectangular contour with vertices \( c \pm iT, 1 - \Delta \pm iT \) by \( C \), where \( \Delta := \frac{C}{2} (\log |T| + 3)^{-2/3} (\log \log(|T| + 3))^{-1/3} \). Then there are no \( \zeta \)-zeros on or in the interior of \( C \), and we deduce from Cauchy’s theorem
\[
\mathcal{M}(x) = \frac{1}{2\pi i} \left\{ \int_{c-iT}^{1-\Delta-iT} + m \int_{1-\Delta-iT}^{1-\Delta+iT} + \int_{1-\Delta+iT}^{c+iT} \right\} \mathfrak{m}(s) \frac{x^s}{s} \, ds + \mathcal{E}.
\]
In order to bound the appearing integrals we note for \( \sigma > 1 \)
\[
(7) \quad \frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |\zeta(s)| \leq \zeta(\sigma);
\]
these inequalities follow factorwise from the Euler product representation of the zeta-function. For any non-negative integer \( J \) let
\[
(8) \quad \mathfrak{m}(s) = N_J(s) \prod_{0 \leq j < J} \zeta(2^j s)^{-1}, \quad \text{where} \quad N_J(s) := \prod_{j \geq J} \zeta(2^j s)^{-1}.
\]
In view of (7), for \( \sigma > 2^{-J} \),
\[
(9) \quad |N_J(s)| \leq \prod_{j \geq J} \frac{\zeta(2^j \sigma)}{\zeta(2^j + 1 \sigma)} = \zeta(2^J \sigma),
\]
hence the function \( N_J(s) \) is bounded for \( \sigma > 2^{-J} \). Thus we find via (6)
\[
\int_{1-\Delta-iT}^{c+iT} \mathfrak{m}(s) \frac{x^s}{s} \, ds = \int_{1-\Delta+iT}^{c+iT} \zeta(s)^{-1} N_1(s) \frac{x^s}{s} \, ds \ll \frac{x^c}{T \Delta},
\]
and similarly,
\[
\int_{1-\Delta-iT}^{1-\Delta+iT} \mathfrak{m}(s) \frac{x^s}{s} \, ds \ll x^{1-\Delta} \int_0^T |\zeta(\sigma + it)|^{-1} \frac{dt}{1 + |t|} \ll x^{1-\Delta} \frac{\log T}{\Delta}.
\]
Collecting together, we arrive at
\[
(10) \quad \mathcal{M}(x) \ll \frac{x^c}{T \Delta} + x^{1-\Delta} \frac{\log T}{\Delta} + \mathcal{E}.
\]
With \( c = 1 + (\log x)^{-1} \) we have \( \sum \mu_\infty(n)n^{-c} \ll \zeta(c) \ll \log x \) in the estimate for \( \mathcal{E} \); choosing \( T \) such that \( T \log T = x^{\Delta} \), we obtain (4).

Now we prove the big-Omega result. For \( \sigma > 1 \), we find by partial summation
\[
(11) \quad \sum_{n>x} \frac{\mu_\infty(n)}{n^s} = -\frac{\mathcal{M}(x)}{x^s} + s \int_x^\infty \mathcal{M}(u) u^{-s-1} \, du.
\]
Assuming \( M(x) = o(x^\alpha) \) for some positive \( \alpha \), as \( x \to \infty \), the right hand-side converges for \( \sigma > \alpha \). Hence, \( m(s) \) has a convergent Dirichlet series representation for \( \sigma > \alpha \), and thus defines an analytic function in this half-plane. For \( \alpha < \beta \) this contradicts the poles of \( m(s) \) at the nontrivial zeros of \( \zeta(s) \) on the critical line (see [14], §10.2). The theorem is proved. 

An alternative proof is based on the function

\[
F(s) := \sum_{n=1}^{\infty} \frac{\mu(n) - \mu_\infty(n)}{n^s} = \zeta(s)^{-1} \left( 1 - \prod_{j=1}^{\infty} \zeta(2^j s)^{-1} \right);
\]

the analytic behaviour of \( F(s) \) implies (via Perron’s formula) that its summatory function \( \sum_{n \leq x} (\mu(n) - \mu_\infty(n)) = M(x) - \Im(x) \) is small, from which one deduces the estimates of Theorem 2 by corresponding ones for \( \Im(x) \).

Next, we shall prove an explicit formula for \( \Im(x) \) subject to the truth of the Riemann hypothesis.

**Theorem 3.** The Riemann hypothesis is true if and only if \( \Im(x) \ll x^{1/2 + \epsilon} \). Moreover, if the Riemann hypothesis is true, then

\[
\Im(x) = \sum_{0 \leq j < J} \sum_{\rho \in \mathbb{R}, |\gamma| < T} \frac{x^{\rho/2}}{|\rho/2|^c} c_j(\rho) + o(x^{1-j})
\]

with some non-zero constants \( c_j(\rho) \) if all zeros are simple (otherwise a modified formula holds with \( c_j \) being polynomials in \( \log x \) according to the multiplicities of \( \rho \)), and

\[
\Im(x) \ll x^{1/2} \exp \left( (\log x)^{1/2} (\log \log x)^{14} \right);
\]

furthermore, the line \( \sigma = 0 \) is a natural boundary for \( m(s) \).

It follows from (12) that \( \Im(x) = O(x^{1/2}) \). In view of Theorem 2 we deduce that the latter bound holds also unconditionally.

**Proof.** If the Riemann hypothesis is true, then

\[
\zeta(\sigma + it)^{-1} \ll t^\epsilon \quad \text{for} \quad \sigma > \frac{1}{2}
\]

and all positive \( \epsilon \) as \( t \to \infty \) (see [14], §14.2); moreover, for any real interval of length one, there exists a real number \( t \) from this interval such that the latter estimate holds also for \( s = \frac{1}{2} + it \) (see [14], §14.16). Incorporating this bound in place of (6), we get instead of estimate (10)

\[
\Im(x) \ll x^{c T^\epsilon - 1} + x^{1-\delta T^\epsilon} + \frac{x \log x}{T}
\]

with any \( \delta > \frac{1}{2} \); now choosing \( c \) as in the previous proof and \( T \) such that \( T = x^{1/2 + \epsilon} \) the desired bound follows.

If \( \Im(x) \ll x^{1/2 + \epsilon} \), then we deduce from (11) that

\[
m(s) = \sum_{n=1}^{\infty} \frac{\mu_\infty(n)}{n^s} = s \int_{1}^{\infty} \Im(u) u^{-s-1} du
\]

is convergent and hence analytic for \( \sigma > 1 \), which implies the non-vanishing of the zeta-function in this half-plane.
For the sake of simplicity, we assume besides the Riemann hypothesis that all \( \zeta \)-zeros are simple. Similarly to the proof of Theorem 2, by the calculus of residues, for \( \delta := 3 \cdot 2^{-J-2} \),

\[
\mathfrak{M}(x) = \frac{1}{2\pi i} \left\{ \int_{c-iT}^{\delta-iT} + \int_{\delta+iT}^{c+iT} + \int_{\delta-iT}^{\delta+iT} \right\} m(s) \frac{x^s}{s} ds + \mathcal{E} + \Sigma,
\]

where \( \mathcal{E} \) is the error term bounded in (5), \( \Sigma \) is the sum of residues, and the parameter \( T \) is chosen such that \( T \neq 2^{-J} \gamma \) for all ordinates \( \gamma \) of \( \zeta \)-zeros and \( 0 \leq j < J \). All residues arise from zeros \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \); hence

\[
\text{Res}_{s=2^{-j}\rho} m(s) \frac{x^s}{s} = \lim_{s \to 2^{-j}\rho} (s - 2^{-j}\rho) \zeta(2^{-j}s)^{-1} N_j(s) \prod_{0 \leq j < J, \gamma \neq j} \zeta(2^{-j}s)^{-1} \frac{x^s}{s} = c_j(\gamma) \frac{x^{2^{-j}\rho}}{2^{-j}\rho}
\]

with some non-zero constant \( c_j(\gamma) \). Summing up over all zeros \( \rho = \frac{1}{2} + i\gamma \) with \( |\gamma| \) for all \( 0 \leq j < J \) yields an expression for \( \Sigma \) which constitutes the main term of the formula (12). In order to bound the integrals we recall that \( N_J(s) \ll 1 \) for \( \sigma > 2^{-J} \) (hence on \( \sigma = \delta \)) by (9). Under assumption of the Riemann hypothesis we have, besides (14),

\[
\zeta(\sigma + it)^{-1} \ll t^{\sigma-1/2+\epsilon} \quad \text{for} \quad \sigma < \frac{1}{2};
\]

this follows easily from (14) by use of the functional equation and Stirling’s formula; hence

\[
\prod_{0 \leq j < J} \zeta(2^j s)^{-1} \ll t^{(2^j - 1)\sigma - J/2 + \epsilon}
\]

as \( t \to +\infty \). In view of (8) we find

\[
\int_{\delta-iT}^{\delta+iT} m(s) \frac{x^s}{s} ds = \int_{\delta-iT}^{\delta+iT} N_{J+1}(s) \zeta(2^j s)^{-1} \prod_{0 \leq j < J} \zeta(2^j s)^{-1} \frac{x^s}{s} ds \ll x^\delta T^{(2^j - 1)\delta - J/2 + \epsilon}
\]

and

\[
\int_{\delta-iT}^{c+iT} m(s) \frac{x^s}{s} ds \ll x T^{\epsilon - 1}.
\]

Collecting together and choosing \( c \) as in the previous proof, we arrive at

\[
\mathfrak{M}(x) - \Sigma \ll \frac{x \log x}{T} + x T^{\epsilon - 1} + x^{\delta} T^{(2^j - 1)\delta - J/2 + \epsilon}.
\]

Since \( (2^j - 1)\delta - J/2 < 0 \), the right-hand side is \( = o(x^{2^{-j}}) \) which proves (12).

Estimate (13) is a consequence of recent work of Soundararajan [10] who obtained for the summatory function of the ordinary Môbius function \( \mu(n) \) via Perron’s formula

\[
\sum_{n \leq x} \mu(n) = \frac{1}{2\pi i} \int_{c-i[x]}^{c+i[x]} \frac{x^s}{s\zeta(s)} ds + O(\log x).
\]
the estimate on the right-hand side of (13) by contour integration to the right of the critical line \( \sigma = \frac{1}{2} \); since \( \mathfrak{M}(x) \) is also bounded by the above integral for \( \sigma > \frac{1}{2} \), we may adopt his bound for our case too.

For the assertion that there is no meromorphic continuation beyond the imaginary axis it suffices to show that in any neighbourhood of any point \( it \) with large imaginary part \( t > 0 \) there exists a pole of \( m(s) \). Given any \( \epsilon > 0 \), we have to find a nontrivial zero \( \rho = \beta + i\gamma \) of \( \zeta(s) \) such that

\[
|it - 2^{-j}(\beta + i\gamma)| < \epsilon
\]

for some positive integer \( j \). Since for any sufficiently large \( j \) we have \( 0 < 2^{-j}\beta \leq \frac{\epsilon}{2} \), we have to find a zero \( \rho = \beta + i\gamma \) satisfying

(15) \[ |2^j t - \gamma| < 2^{j-1}\epsilon. \]

By the Riemann–von Mangoldt formula with an error term under assumption of the truth of the Riemann hypothesis,

\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O \left( \frac{\log T}{\log \log T} \right)
\]

(see [14], §14.13), we find for the number of zeros satisfying condition (15) the estimate

\[
N(2^j t + 2^{-j-1}\epsilon) - N(2^j t - 2^{j-1}\epsilon) \geq \frac{2^j \epsilon}{2\pi} \log \frac{2^j t - 2^{j-1}\epsilon}{2\pi e} + O \left( \frac{\log(2^j t)}{\log \log(2^j t)} \right).
\]

Setting \( 2^{1-j} = \epsilon \), this leads to

\[
N(2t/\epsilon + 1) - N(2t/\epsilon - 1) \geq \frac{1}{\pi} \log(2t/\epsilon) + o(\log(2t/\epsilon)),
\]

which is positive for sufficiently large \( t \). Hence, there is a singularity in any neighbourhood of almost any arbitrary point \( it \), and thus the imaginary axis is a natural boundary for \( m(s) \). The theorem is proved. •

Formula (12) is similar to the explicit formula for the summatory function of the classical Möbius \( \mu \)-function:

\[
M(x) = \sum_{n \leq x} \mu(n) = \sum_{0 < \gamma < T} \frac{x^\rho}{\rho \zeta'(\rho)} + \text{error}(x, T),
\]

which is valid under assumption of the Riemann hypothesis and the so-called essential simplicity hypothesis that all \( \zeta \)-zeros are simple, resp. with obvious modifications if there are multiple zeros (see [14], §14.27).

## 3 Heuristics

Finally, we discuss some related heuristics based on an old idea due to Denjoy [3] for the classical \( \mu \)-function which give support for Riemann’s hypothesis. Whereas Denjoy argued for the Möbius \( \mu \)-function we consider the modified function \( \mu_{\infty}(n) \). Assume that \( \{X_n\} \) is a sequence of random variables with distribution

\[
P(X_n = +1) = P(X_n = -1) = \frac{1}{2}.
\]
Define $S_0 = 0$ and $S_n = \sum_{j=1}^{n} X_j$, then $\{S_n\}$ is a symmetrical random walk in $\mathbb{Z}^2$ with starting point at 0. A simple application of Chebyshev’s inequality yields, for any positive $c$,

$$
P\{|S_n| \geq cn^{1/2}\} \leq \frac{1}{2c^2},$$

which shows that large values for $S_n$ are rare events. By the theorem of Moivre-Laplace this can be made more precise. It follows that

$$
\lim_{n \to \infty} P \left\{ |S_n| < cn^{1/2} \right\} = \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} \exp \left( -\frac{1}{2}x^2 \right) \, dx.
$$

Since the right-hand side above tends to 1 as $c \to \infty$, we obtain

$$
\lim_{n \to \infty} P \left\{ |S_n| \ll n^{1/2+\epsilon} \right\} = 1
$$

for every $\epsilon > 0$. We observe that this might be regarded as a model for the value-distribution of the modified Möbius function $\mu_\infty(n)$. (However, for the classical Möbius function $\mu(n)$ one has to exclude the squarefull integers $n$ since for those values $\mu(n) = 0$.) The law of the iterated logarithm would even give the stronger estimate

$$
\lim_{n \to \infty} P \left\{ |S_n| \ll (n \log \log n)^{1/2} \right\} = 1,
$$

which suggests for $\mathfrak{M}(x)$ the upper bound $(x \log \log x)^{1/2}$. This estimate is pretty close to the $\mu_\infty$-variant of the so-called weak Mertens hypothesis:

$$
\int_{1}^{X} \left( \frac{\mathfrak{M}(x)}{x} \right)^2 \, dx \ll \log X.
$$

The latter bound implies the Riemann hypothesis and the essential simplicity hypothesis; the proof follows exactly the same argument as in the case of the classical Möbius function (see [14], §14.29) since the generating Dirichlet series of $\mu$ and $\mu_\infty$ differ only by the factor $\prod_{j \geq 1} \zeta(2^j s)^{-1}$ which has no deeper influence.

References

[1] G.L. Cohen, On an integer’s infinitary divisors, *Math. Comp.* 54 (1990), 395–411.

[2] G.L. Cohen, P. Hagis, Jr., Arithmetic functions associated with the infinitary divisors of an integer, *Internat. J. Math. Math. Sci.* 16 (1993), 373–383.

[3] A. Denjoy, L’Hypothèse de Riemann sur la distribution des zéros de $\zeta(s)$, reliée à la théorie des probabilités, *Comptes Rendus Acad. Sci. Paris* 192 (1931), 656–658.

[4] P. Haukkanen, Basic properties of the bi-unitary convolution and the semi-unitary convolution, *Indian J. Math.* 40 (1998), 305–315.

[5] P. Haukkanen, On the k-ary convolution of arithmetical functions, *Fibonacci Quart.* 38 (2000), 440–445.

[6] A. Ivič, *The theory of the Riemann zeta-function with applications*, John Wiley & Sons, New York, 1985.

[7] N.M. Korobov, Estimates of trigonometric sums and their applications, *Uspehi Mat. Nauk* 13 (1958), 185–192 (Russian).
[8] F. MERTENS, Über eine zahlentheoretische Funktion, *Sem.ber. Kais. Akad. Wiss. Wien* 106 (1897), 761–830.

[9] A.M. ODLYZKO, H.J.J. TE RIELE, Disproof of the Mertens conjecture, *J. Reine Angew. Math.* 367 (1985), 138–160.

[10] K. SOUNDRARAJAN, Partial sums of the Möbius function, *J. Reine Angew. Math.* 631 (2009), 141–152.

[11] D. SURYANARAYANA, The number of bi-unitary divisors of an integer, in *The theory of arithmetic functions* (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1971), pp. 273–282, Lecture Notes in Math., Vol. 251, Springer, 1972.

[12] D. SURYANARAYANA, R. SITA RAMA CHANDRA RAO, The number of bi-unitary divisors of an integer, II, *J. Indian Math. Soc.* 39 (1975), 261–280.

[13] D. SURYANARAYANA, M. V. SUBBARAO, Arithmetical functions associated with the bi-unitary \( k \)-ary divisors of an integer, *Indian J. Math.* 22 (1980), 281–298.

[14] E.C. TITCHMARSH, *The theory of the Riemann zeta-function*, Oxford University Press, 1986, 2nd ed., revised by D.R. Heath-Brown.

[15] I.M. VINOGRADOV, A new estimate for the function \( \zeta(1 + it) \), *Izv. Akad. Nauk SSSR, Ser. Mat.* 22 (1958), 161–164 (Russian).

[16] A. WALFISZ, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, VEB Deutscher Verlag der Wissenschaften, 1963.