HYPERELLiptIC JACOBIANS WITHOUT COMPLEX MULTIPLICATION AND STEINBERG REPRESENTATIONS IN POSITIVE CHARACTERISTIC

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Abstract. In his previous papers [22, 23, 25] the author proved that in characteristic \( \neq 2 \) the Jacobian \( J(C) \) of a hyperelliptic curve \( C : y^2 = f(x) \) has only trivial endomorphisms over an algebraic closure \( K_\alpha \) of the ground field \( K \) if the Galois group \( \text{Gal}(f) \) of the irreducible polynomial \( f(x) \in K[x] \) is either the symmetric group \( S_n \) or the alternating group \( A_n \). Here \( n \geq 9 \) is the degree of \( f \). The goal of this paper is to extend this result to the case of certain “smaller” doubly transitive Galois groups. Namely, we treat the infinite series of Galois groups \( S_k(x) \) and \( U_2(x) \) of the irreducible polynomial \( f(x) \) with degree \( m \) greater than \( 2^m - 1 \). We refer the reader to [16, 17, 12, 13, 15, 11, 22, 23, 25, 26] for a discussion of known results about, and examples of, hyperelliptic jacobians, Endomorphisms of abelian varieties, Steinberg representations.

1. Introduction

Let \( K \) be a field and \( K_\alpha \) its algebraic closure. Assuming that \( \text{char}(K) = 0 \), the author [22] proved that the Jacobian \( J(C) = J(C_f) \) of a hyperelliptic curve

\[
C = C_f : y^2 = f(x)
\]

has only trivial endomorphisms over \( K_\alpha \) if the Galois group \( \text{Gal}(f) \) of the irreducible polynomial \( f \in K[x] \) is “very big”. Namely, if \( n = \deg(f) \geq 5 \) and \( \text{Gal}(f) \) is either the symmetric group \( S_n \) or the alternating group \( A_n \), then the ring \( \text{End}(J(C_f)) \) coincides with \( \mathbb{Z} \). Later the author [24, 27] proved that \( \text{End}(J(C_f)) = \mathbb{Z} \) for infinite series \( \text{Gal}(f) = L_2(2^m) := \text{PSL}_2(\mathbb{F}_{2^m}) \) and \( n = 2^m + 1 \) (with \( \dim(J(C_f)) = 2^{m-1} \)), \( \text{Gal}(f) = \text{Suzuki group} \text{Sz}(2^{2m+1}) = 2B_2(2^{2m+1}) \) and \( n = 2(2^m+1) + 1 \) (with \( \dim(J(C_f)) = 2^{2m+1} \)), \( \text{Gal}(f) = U_3(2^m) := \text{PSU}_3(\mathbb{F}_{2^m}) \) and \( n = 2^{3m} + 1 \) (with \( \dim(J(C_f)) = 2^{3m-1} \)). We refer the reader to [16, 17, 12, 13, 15, 11, 22, 23, 25, 26] for a discussion of known results about, and examples of, hyperelliptic jacobians without complex multiplication.

When \( \text{char}(K) > 2 \), the author [25] proved that \( \text{End}(J(C_f)) = \mathbb{Z} \) if \( n \geq 9 \) and \( \text{Gal}(f) = S_n \) or \( A_n \). The aim of the present paper is to extend this result to the case of already mentioned series of doubly transitive Galois groups \( L_2(2^m), \text{Sz}(2^{2m+1}) \) and \( U_3(2^m) \). Notice that it is known [24, 27] that in this case either \( \text{End}(J(C)) = \mathbb{Z} \) or \( J(C) \) is a supersingular abelian variety and the real problem is how to prove that \( J(C) \) is not supersingular.

Remark 1.1. The groups \( L_2(2^m), U_3(2^m) \) (with \( m \geq 2 \)) and \( \text{Sz}(2^{2m+1}) \) constitute an interesting important class of finite simple groups called simple Bender groups [2, Chapter 2]. Our main theorem deals with all these groups except \( L_2(4) \cong A_5 \).
2. Main result

Throughout this paper we assume that $K$ is a field of prime characteristic $p$ different from 2. We fix its algebraic closure $K_a$ and write $\text{Gal}(K)$ for the absolute Galois group $\text{Aut}(K_a/K)$.

**Theorem 2.1.** Let $K$ be a field with $p = \text{char}(K) > 2$, $K_a$ its algebraic closure, $f(x) \in K[x]$ an irreducible separable polynomial of degree $n \geq 9$. Let us assume that $n$ and the Galois group $\text{Gal}(f)$ of $f$ enjoy one of the following properties:

(i) There exists a positive integer $m \geq 3$ such that $n = 2^m + 1$ and $\text{Gal}(f)$ contains a subgroup isomorphic to $L_2(2^m)$;

(ii) There exists a positive integer $m \geq 1$ such that $n = 2^{2(2m+1)} + 1$ and $\text{Gal}(f)$ contains a subgroup isomorphic to $S_3(2^{2m+1})$;

(iii) There exists a positive integer $m \geq 2$ such that $n = 2^{3m} + 1$ and $\text{Gal}(f)$ contains a subgroup isomorphic to $U_3(2^m)$.

Let $C_f$ be the hyperelliptic curve $y^2 = f(x)$. Let $J(C_f)$ be its jacobian, $\text{End}(J(C_f))$ the ring of $K_a$-endomorphisms of $J(C_f)$. Then $\text{End}(J(C_f)) = \mathbb{Z}$.

**Remark 2.2.** Replacing $K$ by a suitable finite separable extension, we may assume in the course of the proof of Theorem 2.1 that $\text{Gal}(f) = L_2(2^m), S_3(2^{2m+1})$ or $U_3(2^m)$ respectively. Taking into account that all these groups are simple non-abelian and replacing $K$ by its abelian extension obtained by adjoining to $K$ all 2-power roots of unity, we may also assume that $K$ contains all 2-power roots of unity.

As was already pointed out, in light of results of [24, 27] and Remark 2.2, our Theorem 2.1 is an immediate corollary of the following auxiliary statement.

**Theorem 2.3.** Suppose $n$ is an odd integer. Suppose $K$ is a field, $\text{char}(K) = p \neq 2$ and $K$ contains all 2-power roots of unity. Suppose that $f(x) \in K[x]$ is a separable polynomial of degree $n$, whose Galois group $\text{Gal}(f)$ enjoys one of the following properties:

(i) There exists a positive integer $m \geq 3$ such that $n = 2^m + 1$ and $\text{Gal}(f) = L_2(2^m)$;

(ii) There exists a positive integer $m$ such that $n = 2^{2(2m+1)} + 1$ and $\text{Gal}(f) = S_3(2^{2m+1})$;

(iii) There exists a positive integer $m \geq 2$ such that $n = 2^{3m} + 1$ and $\text{Gal}(f) = U_3(2^m)$.

Let $C$ be the hyperelliptic curve $y^2 = f(x)$ of genus $g = \frac{n-1}{2}$ over $K$ and let $J(C)$ be the jacobian of $C$.

Then $J(C)$ is not a supersingular abelian variety.

**Example 2.4.** Let $k$ be an algebraically closed field of characteristic 7. Let $K = k(z)$ be the field of rational functions in variable $z$ with constant field $k$. We write $\overline{k(z)}$ for an algebraic closure of $k(z)$. According to Abhyankar [11], the Galois group of the polynomial

$$x^9 - zx^7 + 1 \in k(z)[x] = K[x]$$

is $L_2(8)$ (see also [20, §3.3, Remarque 2(a)]). Hence the four-dimensional jacobian of the hyperelliptic curve $y^2 = x^9 - zx^7 + 1$ has no nontrivial endomorphisms over $\overline{k(z)}$.

We prove Theorem 2.3 in Section 4.
3. Permutation groups, permutation modules and very simple representations

Let $B$ be a finite set consisting of $n \geq 5$ elements. We write $\text{Perm}(B)$ for the group of permutations of $B$. A choice of ordering on $B$ gives rise to an isomorphism $\text{Perm}(B) \cong S_n$. Let $G$ be a subgroup of $\text{Perm}(B)$. For each $b \in B$ we write $G_b$ for the stabilizer of $b$ in $G$; it is a subgroup of $G$.

**Remark 3.1.** Assume that the action of $G$ on $B$ is transitive. It is well-known that each $G_b$ is of index $n$ in $G$ and all the $G_b$'s are conjugate in $G$. Each conjugate of $G_b$ in $G$ is the stabilizer of a point of $B$. In addition, one may identify the $G$-set $B$ with the set of cosets $G/G_b$ with the standard action by $G$.

Let $F$ be a field. We write $F^B$ for the $n$-dimensional $F$-vector space of maps $h : B \to F$. The space $F^B$ is provided with a natural action of $\text{Perm}(B)$ defined as follows. Each $s \in \text{Perm}(B)$ sends a map $h : B \to F$ into $sh : b \mapsto h(s^{-1}(b))$. The permutation module $F^B$ contains the $\text{Perm}(B)$-stable hyperplane

$$(F^B)^0 = \{ h : B \to F \mid \sum_{b \in B} h(b) = 0 \}$$

and the $\text{Perm}(B)$-invariant line $F \cdot 1_B$ where $1_B$ is the constant function 1.

Clearly, $(F^B)^0$ contains $F \cdot 1_B$ if and only if $\text{char}(F)$ divides $n$. If this is not the case then there is a $\text{Perm}(B)$-invariant splitting $F^B = (F^B)^0 \oplus F \cdot 1_B$.

Clearly, $F^B$ and $(F^B)^0$ carry natural structures of $G$-modules. Their characters depend only on the characteristic of $F$.

Let us consider the case of $F = Q$. Then the character of $Q^B$ is called the permutation character of $B$. Let us denote by $\chi = \chi_B : G \to Q$ the character of $(Q^B)^0$. Clearly, $1 + \chi$ is the permutation character of $B$. It is also clear that the representation of $G$ in $(Q^B)^0$ is orthogonal.

Now, let us consider the case of $F = F_2$. If $n$ is even then let us define the $\text{Perm}(B)$-module $Q_B := (F_2^B)^0 / (F_2 \cdot 1_B)$. If $n$ is odd then let us put $Q_B := (F_2^B)^0$.

**Remark 3.2.** Clearly, $Q_B$ is a faithful $G$-module (recall that $n \geq 5$). If $n$ is odd then $\dim_{F_2} Q_B = n - 1$ and one may view the $Q[G]$-module $(Q^B)^0$ as a lifting to characteristic zero of the $F_2[G]$-module $(Q^B)^0$. If $n$ is even then $\dim_{F_2} Q_B = n - 2$.

Let $G^{(2)}$ be the set of 2-regular elements of $G$. Clearly, the Brauer character of the $G$-module $F_2^B$ coincides with the restriction of $1 + \chi_B$ to $G^{(2)}$. This implies easily that the Brauer character of the $G$-module $(F_2^B)^0$ coincides with the restriction of $\chi_B$ to $G^{(2)}$.

**Remark 3.3.** Let us denote by $\phi_B = \phi$ the Brauer character of the $G$-module $Q_B$. One may easily check that $\phi_B$ coincides with the restriction of $\chi_B$ to $G^{(2)}$ if $n$ is odd and with the restriction of $\chi_B - 1$ to $G^{(2)}$ if $n$ is even.

We refer to [24] [23] [25] for a discussion of the following definition.

**Definition 3.4.** Let $V$ be a vector space over a field $F$, let $G$ be a group and $\rho : G \to \text{Aut}_F(V)$ a linear representation of $G$ in $V$. We say that the $G$-module $V$ is very simple if it enjoys the following property:

If $R \subset \text{End}_F(V)$ is an $F$-subalgebra containing the identity operator $\text{Id}$ such that

$$\rho(\sigma) R \rho(\sigma)^{-1} \subset R \quad \forall \sigma \in G$$

then either $R = F \cdot \text{Id}$ or $R = \text{End}_F(V)$. 

Remarks 3.5.  
(i) If $G'$ is a subgroup of $G$ and the $G'$-module $V$ is very simple then obviously the $G$-module $V$ is also very simple.
(ii) A very simple module is absolutely simple (see [24] Remark 2.2(ii)).
(iii) Clearly, the $G$-module $V$ is very simple if and only if the corresponding $\rho(G)$-module $V$ is very simple. This implies easily that if $H \to G$ is a surjective group homomorphism then the $G$-module $V$ is very simple if and only if the corresponding $H$-module $V$ is very simple.
(iv) Let $G'$ be a normal subgroup of $G$. If $V$ is a very simple $G$-module then either $\rho(G') \subset \text{Aut}_k(V)$ consists of scalars (i.e., lies in $k \cdot \text{Id}$) or the $G'$-module $V$ is absolutely simple. See [24] Remark 5.2(iv)].
(v) Suppose $F$ is a discrete valuation field with valuation ring $O_F$, maximal ideal $m_F$ and residue field $k = O_F/m_F$. Suppose $V_F$ a finite-dimensional $F$-vector space, $\rho_F : G \to \text{Aut}_F(V_F)$ a $F$-linear representation of $G$. Suppose $T$ is a $G$-stable $O_F$-lattice in $V_F$ and the corresponding $k|G|$-module $T/m_F T$ is isomorphic to $V$. Assume that the $G$-module $V$ is very simple. Then the $G$-module $V_F$ is also very simple. See [24] Remark 5.2(vii).

Theorem 3.6. Suppose one of the following conditions holds:

(i) There exists a positive integer $m \geq 3$ such that $n = 2^m + 1$ and $G \cong L_2(2^m)$;
(ii) There exists a positive integer $m$ such that $n = 2^{2(2^m+1)} + 1$ and $G \cong S_2(2^{2m+1})$;
(iii) There exists a positive integer $m \geq 2$ such that $n = 2^m + 1$ and $G \cong U_3(2^m)$.

Then the $G$-module $Q_B$ is very simple.

Proof. See Th. 7.10 and Th. 7.11 of [24] and Th. 4.4 of [27].

The following statement provides a criterion for a representation over $F_2$ to be very simple.

Theorem 3.7. Suppose that a positive integer $N > 1$ and a group $H$ enjoy the following properties:

- $H$ does not contain a subgroup of index dividing $N$ except $H$ itself.
- Let $N = ab$ be a factorization of $N$ into a product of two positive integers $a > 1$ and $b > 1$. Then either there does not exist an absolutely simple $F_2[H]$-module of $F_2$-dimension $a$ or there does not exist an absolutely simple $F_2[H]$-module of $F_2$-dimension $b$.

Then each absolutely simple $F_2[H]$-module of $F_2$-dimension $N$ is very simple.

Proof. This is Corollary 4.12 of [24].

Theorem 3.8. Suppose that there exist a positive integer $m \geq 2$ and an odd power prime $q$ such that $n = \frac{2^m - 1}{q - 1}$. Suppose $G$ is a subgroup of $S_n$ and contains a subgroup isomorphic to $L_m(q)$. Then the $G$-module $Q_B$ is very simple.

Proof. In light of Remark 3.5(i), we may assume that $G = L_m(q)$.

Assume that $(m, q) \neq (4, 3)$. Then the assertion of Theorem 3.8 follows easily from [25] Remark 4.4 and Corollary 5.4. (The proof in [25] was based on a result of Guralnick [3].)

So, we may assume that $m = 4, q = 3$. We have $n = \#(B) = 40$ and $\text{dim}_{F_2}(Q_B) = 38$. According to the Atlas [4] pp. 68-69, $G = L_4(3)$ has two
conjugacy classes of maximal subgroups of index 40. All other maximal subgroups have index greater than 40. Therefore all subgroups of $G$ (except $G$ itself) have index greater than $39 > 38$. This implies that each action of $G$ on $B$ is transitive. The corresponding permutation character (in notations of [11]) coincides (in both cases) with $1 + \chi_4$, i.e., $\chi = \chi_4$. Since 40 is even, we need to consider the restriction of $\chi - 1$ to the set of 2-regular elements of $G$ and this restriction coincides with the absolutely irreducible Brauer character $\phi_4$ (in notations of [11], p. 165). In particular, the corresponding $G$-module $Q_B$ is absolutely simple. It follows from the Table on p. 165 of [11] that all absolutely irreducible representations of $G$ in characteristic 2 have dimension which is not a strict divisor of 38. Combining this observation with the absence of subgroups in $G$ of index $\leq 38$, we conclude, thanks to Theorem 3.7, that $Q_B$ is very simple. This ends the proof of Theorem 3.8. $\Box$

4. Proof of Theorem 2.3

So, we assume that $K$ contains all 2-power roots of unity, $f(x) \in K[x]$ is an irreducible separable polynomial of degree $n = 2g + 1$ or $2g + 2$ and $n \geq 5$. Therefore the jacobian $J(C)$ of the hyperelliptic curve $C : y^2 = f(x)$ is a $g$-dimensional abelian variety defined over $K$. The group $J(C)_2$ of its points of order 2 is a $2g$-dimensional $F_2$-vector space provided with the natural action of $\text{Gal}(K)$. It is well-known (see for instance [24 Sect. 5]) that the image of $\text{Gal}(K)$ in $\text{Aut}(J(C)_2)$ is canonically isomorphic to $\text{Gal}(f)$. Let us recall a well-known explicit description of the $\text{Gal}(f)$-module $J(C)_2$ (see, for instance, [17] or [24]). Let $\mathfrak{R} \subset K_n$ be the $n$-element set of roots of $f$. We view $G = \text{Gal}(f)$ as a certain subgroup of the group $\text{Perm}(\mathfrak{R}) \cong S_n$ of all permutations of $\mathfrak{R}$. If we put $B = \mathfrak{R}$ then the $2g$-dimensional $F_2$-vector space $Q_B = (F_2^{3r})^0$ carries the natural structure of faithful $G$-module. It is well-known that the natural homomorphism $\text{Gal}(K) \to \text{Aut}_{F_2}(J(C)_2)$ factors through the canonical surjection $\text{Gal}(K) \to \text{Gal}(f) = G$ and the $G$-modules $J(C)_2$ and $(F_2^{3r})^0$ are isomorphic [16] [24].

We deduce Theorem 2.3 from the following assertion.

**Lemma 4.1.** Let $F$ be a field, whose characteristic is not 2 and assume that $F$ contains all 2-power roots of unity. Let $F_a$ be an algebraic closure of $F$. Let $g$ be a positive integer and $G$ be a finite simple non-abelian group enjoying the following properties:

(a) Let $2^r$ be the largest power of 2 that divides the order of $G$. Then either $2^r$ divides $2g$ or $G = L_4(3)$, $g = 19$.

(b) Either the Schur multiplier of $G$ is a group of odd order or $G = Sz(8), g = 32$.

Suppose that $X$ is a $g$-dimensional abelian variety over $F$ such that the image of $\text{Gal}(F)$ in $\text{Aut}(X_2)$ is isomorphic to $G$ and the corresponding faithful representation $ho : G \to \text{Aut}(X_2) \cong \text{GL}(2g, F_2)$

enjoys the following properties:

(c) The representation $\rho$ is very simple; in particular, it is absolutely irreducible;

(d) If $(G, g) \neq (L_4(3), 19)$ then one may lift $\rho$ to an orthogonal representation of $G$ in characteristic zero.

Then the ring $\text{End}(X)$ of all $F_a$-endomorphisms of $X$ coincides with $Z$. 
Lemma 4.1 will be proven in the next Section.

Proof of Theorem 2.3. Clearly, $n \geq 9$ is odd. Let us put

$$G = \text{Gal}(f), g = \frac{n - 1}{2}, F = K, X = J(C).$$

Notice that $n - 1$ is a power of 2. We already observed that the image of $\text{Gal}(K)$ in $\text{Aut}(J(C)_2)$ is isomorphic to $\text{Gal}(f)$. Let us recall a well-known explicit description of the $\text{Gal}(f)$-module $J(C)_2$ (see, for instance, [17] or [24]). Let $\mathfrak{R} \subset K$ be the $n$-element set of roots of $f$. We view $G = \text{Gal}(f)$ as a certain subgroup of the group $\text{Perm}(\mathfrak{R}) \cong S_n$ of all permutations of $\mathfrak{R}$. It is known (see for instance [24]) that under the assumptions of Theorem 2.3 $G$ is a doubly transitive permutation group. If we put $B = \mathfrak{R}$ then the $(n - 1)$-dimensional $\mathbf{F}_2$-vector space $Q_B = (\mathbf{F}_2^{\mathfrak{R}})^0$ carries the natural structure of faithful $G$-module.

It is well-known that the natural homomorphism $\text{Gal}(K) \to \text{Aut}_{\mathbf{F}_2}(J(C)_2)$ factors through the canonical surjection $\text{Gal}(K) \twoheadrightarrow \text{Gal}(f) = G$ and the $G$-modules $J(C)_2$ and $(\mathbf{F}_2^{\mathfrak{R}})^0$ are isomorphic [10].

Now consider the case of $F = \mathbf{Q}$. Recall that the $G$-module $(\mathbf{Q}^{\mathfrak{R}})^0$ is orthogonal.

Clearly, the $\mathbf{Q}[G]$-module $(\mathbf{Q}^{\mathfrak{R}})^0$ is a lifting to characteristic zero of the $\mathbf{F}_2[G]$-module $Q_B$. This proves that the condition (d) of Lemma 4.1 holds true.

On the other hand, it follows from Theorem 3.6 that the $G = \text{Gal}(f)$-module $(\mathbf{F}_2^{\mathfrak{R}})^0 = J(C)_2$ is very simple. (It is actually a Steinberg representation of $G$ which explains the title of this paper.) This proves that the condition (c) of Lemma 4.1 holds true. The validity of the condition (a) follows from the known formulas for the orders of the simple groups involved [7, p. 8]. (In fact, $2g = n - 1$ does coincide with the largest power of 2 dividing the order of $G$.) It follows from the Table in [6] §4.15A that the order of the Schur multiplier of $G$ is an odd number except the case $G = \text{Sz}(32), n = 65, g = 32$. So, we may apply Lemma 4.1 to $X = J(C)$ and conclude that $J(C)$ is not supersingular.

\[\Box\]

5. ABELIAN VARIETIES WITHOUT COMPLEX MULTIPLICATION

We keep all the notations and assumptions of Lemma 4.1. We write $T_2(X)$ for the 2-adic Tate module of $X$ and

$$\rho_{2,X} : \text{Gal}(F) \to \text{Aut}_{\mathbf{Z}_2}(T_2(X))$$

for the corresponding 2-adic representation. It is well-known that $T_2(X)$ is a free $\mathbf{Z}_2$-module of rank $2\dim(X) = 2g$ and $X_2 = T_2(X)/2T_2(X)$ (the equality of Galois modules). Let us put

$$H = \rho_{2,X}(\text{Gal}(F)) \subset \text{Aut}_{\mathbf{Z}_2}(T_2(X)).$$

Clearly, the natural homomorphism $\tilde{\rho}_{2,X} : \text{Gal}(F) \to \text{Aut}(X_2)$ defining the Galois action on the points of order 2 is the composition of $\rho_{2,X}$ and (surjective) reduction map modulo 2

$$\text{Aut}_{\mathbf{Z}_2}(T_2(X)) \twoheadrightarrow \text{Aut}(X_2).$$

This gives us a natural (continuous) surjection

$$\pi : H \twoheadrightarrow \tilde{\rho}_{2,X}(\text{Gal}(F)) \cong G,$$

whose kernel consists of elements of $1 + 2\text{End}_{\mathbf{Z}_2}(T_2(X))$. By 4.1(c), the $G$-module $X_2$ is very simple. By Remark 3.5(iii), the $H$-module $X_2$ is also very simple. Here
the structure of $H$-module is defined on $X_2$ via

$$H \subset \text{Aut}_{\mathbb{Z}_2}(T_2(X)) \to \text{Aut}(X_2).$$

Clearly, the $\mathbb{Q}_2[H]$-module $V_2(X)$ is a lifting to characteristic zero of the very simple $\mathbb{F}_2[H]$-module $X_2$ and therefore is also very simple, thanks to Remark 8.6(v). Here $V_2(X) = T_2(X) \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$ is the $\mathbb{Q}_2$-Tate module of $X$ containing $T_2(X)$ as a $H$-stable $\mathbb{Z}_2$-lattice. In particular, the $\mathbb{Q}_2[H]$-module $V_2(X)$ is absolutely simple.

The choice of polarization on $X$ gives rise to a non-degenerate alternating bilinear form (Riemann form) \[\mathbf{18}\]

$$e : V_2(X) \times V_2(X) \to \mathbb{Q}_2(1) \cong \mathbb{Q}_2.$$ Since $F$ contains all 2-power roots of unity, $e$ is $\text{Gal}(F)$-invariant and therefore is $H$-invariant. In particular,

$$H \subset \text{Sp}(V_2(X), e) \subset \text{SL}(V_2(X)).$$

Here $\text{Sp}(V_2(X), e)$ is the symplectic group attached to $e$. In particular, the $H$-module $V_2(X)$ is symplectic.

There exists a finite Galois extension $L$ of $F$ such that all endomorphisms of $X$ are defined over $L$. Clearly, $\text{Gal}(L) = \text{Gal}(F_a/L)$ is an open normal subgroup of finite index in $\text{Gal}(F)$ and

$$H' = \rho_{2,X}(\text{Gal}(L)) \subset \text{Aut}_{\mathbb{Z}_2}(T_2(X)) \subset \text{Aut}_{\mathbb{Q}_2}(V_2(X)))$$

is an open normal subgroup of finite index in $H$. We write $\text{End}^0(X)$ for the $\mathbb{Q}$-algebra $\text{End}(X) \otimes \mathbb{Q}$ of endomorphisms of $X$.

Recall \[\mathbf{18}\] that the natural map $\text{End}^0(X) \otimes \mathbb{Q} \mathbb{Q}_2 \to \text{End}_{\mathbb{Q}_2}V_2(X)$ is an embedding, whose image lies in the centralizer

$$\text{End}_{\text{Gal}(L)}V_2(X) = \text{End}_{H'}V_2(X).$$

Since the $H$-module $V_2(X)$ is very simple and $H'$ is normal in $H$, we conclude, thanks to Remark 8.6(iv) that either the $H'$-module $V_2(X)$ is absolutely simple or $H'$ consists of scalars. In the former case $\text{End}_{H'}V_2(X) = \mathbb{Q}_2$ and therefore

$$\text{End}^0(X) \otimes \mathbb{Q} \mathbb{Q}_2 \cong \mathbb{Q}_2.$$ This implies easily that $\text{End}(X) = \mathbb{Z}$ and we are done. So, let us assume that $H'$ consists of scalars. We are going to arrive to a contradiction that proves the Lemma.

Since $H' \subset H$ consists of symplectic automorphisms of $V_2(X)$, either $H = \{1\}$ or $H = \{-1\}$. In particular, $H'$ is always finite. Since $H'$ is a subgroup of finite index in $H$, the group $H$ is finite. In particular, the kernel of the reduction map modulo 2

$$\text{Aut}_{\mathbb{Z}_2}T_2(X) \supset H \to G \subset \text{Aut}(X_2)$$

consists of periodic elements and, thanks to Minkowski-Serre Lemma \[\mathbf{21}\], $Z := \ker(H \to G)$ has exponent 1 or 2. This implies that $Z$ is commutative.

Since $Z$ is normal in $H$ and the $H$-module $V_2(X)$ is very simple, either $Z$ consists of scalars or the $Z$-module $V_2(X)$ is absolutely simple, thanks to Remark 8.6(iv). On the other hand, since $Z$ is commutative and $\dim_{\mathbb{Q}_2}(V_2(X)) = 2g > 1$, the $Z$-module $V_2(X)$ is not absolutely simple. Hence $Z$ consists of scalars. Since $Z \subset H$ consists of symplectic automorphisms of $V_2(X)$, either $Z = \{1\}$ or $Z = \{-1\}$. In other words, either $H = G$ or $H$ is a central double cover of $G$. As we have already seen, $V_2(X)$ is an absolutely irreducible symplectic $2g$-dimensional representation of
However, if $G = \text{Sz}(8)$ then, according to the Atlas [41 p. 28], $H$ does not have an absolutely irreducible symplectic 64-dimensional representation in characteristic zero. Therefore $G$ is not isomorphic to $\text{Sz}(8)$. Also, if $G = L_4(3)$ then, according to the Atlas [41 pp. 68–69], $H$ does not have an absolutely irreducible 38-dimensional representation in characteristic zero. Therefore $G$ is not isomorphic to $L_4(3)$ and, by §4.1(b), the Schur multiplier of $G$ has odd order. This implies that $H$ is a trivial central extension of $G$, i.e., either $Z = \{1\}$ and $H = G$ or $Z = \{\pm 1\}$ and $H = G \times \{\pm 1\}$. In both cases one may view $V_2(X)$ as absolutely simple symplectic $\mathbb{Q}_2[G]$-module and also as a lifting of the $\mathbb{F}_2[G]$-module $X_2$ to characteristic zero. The condition §4.1(b) means that the representation of $G$ in $V_2(X)$ satisfies the condition of classical Brauer-Nesbitt theorem (§3 Theorem 1; see also [5] §62, [10] p. 249) and therefore has defect 0 [19] §16.4. This implies that its reduction, i.e., the $\mathbb{F}_2[G]$-module $X_2$, is projective and all liftings of $X_2$ to characteristic zero must be isomorphic [19] §14.4. Since, by §4.1(d), there exists an orthogonal lifting of $X_2$, we conclude that the $G$-module $V_2(X)$ must be also orthogonal. But $V_2(X)$ is symplectic absolutely simple and therefore cannot be orthogonal. This gives us the desired contradiction.

6. Complements to [25]

Theorem 6.1. Let $K$ be a field with $\text{char}(K) \neq 2$, $K_a$ its algebraic closure, $f(x) \in K[x]$ an irreducible separable polynomial of degree $n$. Let us assume that $n$ and the Galois group $\text{Gal}(f)$ of $f$ enjoy the following properties:

(i) There exist a positive integer $m > 2$ and an odd power prime $q$ such that $n = \frac{q^m - 1}{q - 1}$.

(ii) $\text{Gal}(f)$ contains a subgroup isomorphic to $L_m(q)$.

Let $C_f$ be the hyperelliptic curve $y^2 = f(x)$. Let $J(C_f)$ be its jacobian, $\text{End}(J(C_f))$ the ring of $K_a$-endomorphisms of $J(C_f)$. Then $\text{End}(J(C_f)) = \mathbb{Z}$.

Proof. When $(q, m) \neq (3, 4)$, the assertion of Theorem 6.1 is proven in [25]. (The case $(q, m) = (3, 4)$ was treated in [26] under an additional assumption that $\text{char}(K) = 0$.) So, in the course of the proof we may assume that $(q, m) = (3, 4)$ and therefore $n = 40$, $g = 19$. Replacing $K$ by a suitable separable extension, we may assume that $G := \text{Gal}(f) = L_4(3)$. Taking into account that $L_4(3)$ is simple non-abelian and replacing $K$ by its abelian extension obtained by adjoining to $K$ all 2-power roots of unity, we may also assume that $K$ contains all 2-power roots of unity.

Let $\mathfrak{M}$ be the set of roots of $f$. It follows from Theorem 3.8 that the $G = \text{Gal}(f)$-module $Q_{\mathfrak{M}}$ is very simple. It follows from Corollary 5.3 of [24] that either $\text{End}(J(C_f)) = \mathbb{Z}$ or $\text{char}(K) > 2$ and $J(C_f)$ is a supersingular abelian variety. Now the result follows from Lemma 4.4. $\square$

Example 6.2. Suppose $p$ is an odd prime, $q > 1$ is a power of $p$, $m > 2$ is an even integer. Let us put $n = (q^m - 1)/(q - 1)$. Suppose $k$ is an algebraically closed field of characteristic $p$ and $K = k(z)$ is the field of rational functions. The Galois group of $x^n + zx + 1$ over $K$ is $L_m(q)$ and the Galois group of $x^n + x + z$ over $K$ is $\text{PGL}_m(\mathbb{F}_q)$ [2] p. 1643. Therefore the jacobians of the hyperelliptic curves $y^2 = x^m + zx + 1$ and $y^2 = x^m + x + z$ have no nontrivial endomorphisms over an algebraic closure $K_a$ of $K$. It follows from Proposition 4.5 and Example 4.2(iv) of [29] that these jacobians are not isogenous over $K_a$ if $m$ and $q - 1$ are not relatively prime.
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