Sharp upper bounds on the $k$-independence number in regular graphs

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Abstract

The $k$-independence number of a graph $G$ is the maximum size of a set of vertices at pairwise distance greater than $k$. In this paper, for each positive integer $k \geq 2$ and $r \geq 3$, we prove sharp upper bounds for the $k$-independence number in an $n$-vertex $r$-regular graph.

Keywords: $k$-independence number, independence number, chromatic number, $k$-distance chromatic number, regular graphs

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1 Introduction

Throughout this paper, all graphs are simple, undirected, and finite. For a nonnegative integer $k$, a $k$-independent set in a graph $G$ is a vertex set $S \subseteq V(G)$ such that the distance between any two vertices in $S$ is bigger than $k$. Note that the 0-independent set is $V(G)$ and a 1-independent set is an independent set. The $k$-independence number of a graph $G$, written $\alpha_k(G)$, is the maximum size of a $k$-independent set in $G$.

It is known that $\alpha_1(G) \geq \frac{n}{\chi(G)}$, where $\chi(G)$ is the chromatic number of a graph $G$. Similarly, by finding the $k$-distance chromatic number of $G$, we can find a lower bound for $\alpha_k(G)$. It will be discussed in Section 4. Other graph parameters such as the average distance [4], injective chromatic number [6], packing chromatic number [5], and strong
chromatic index \[9\] are also directly related to the \( k \)-independence number. Lower bounds on the corresponding distance or packing chromatic number can be given by finding upper bounds on the \( k \)-independence number. Alon and Mohar \[2\] asked the extremal value for the distance chromatic number in graphs of a given girth and degree.

Firby and Haviland \[4\] proved an upper bound for \( \alpha_k(G) \) in an \( n \)-vertex connected graph. We give a proof for the upper bound below because with a similar idea, we prove Theorem 3.2 which is one of the main results in this paper.

**Theorem 1.1.** (\[4\]) If \( G \) is an \( n \)-vertex connected graph with \( \text{diam}(G) \geq k+1 \), and \( G \not\cong K_n \), then

\[
2 \leq \alpha_k(G) \leq \begin{cases} 
\frac{2n}{k+2}, & \text{if } k \text{ is odd}, \\
\frac{2n-2}{k+1}, & \text{if } k \text{ is even}.
\end{cases}
\]

Furthermore, bounds are sharp.

**Proof.** Assume \( S_k \subset V(G) \) is a \( k \)-set of \( G \), if any pair of vertices in \( S_k \) has distance at least \( k \). Because \( \text{diam}(G) \geq k+1 \), there are two vertices \( u, v \in V(G) \), such that \( d_G(u, v) = k+1 \), thus \( u, v \in S(k+1) \). By the definition of \( \alpha_k(G) \), we get that \( \alpha_k(G) = \max |S(k+1)| \geq 2 \).

For the upper bounds, we can prove them by induction.

If \( k \) is odd. When \( k = 1 \), it’s trivial that \( \alpha_k(G) = n \). When \( k = 3 \), \( \forall u, v \in S_3 \), \( d_G(u, v) \geq 3 \), which means \( N(v) \cap N(u) = \emptyset \). Therefore, we have \( |N(S_3)| \geq |S_3| \). While \( |N(S_3)|+|S_3| = n \), so we get that \( |S_3| \leq \frac{n}{2} \). If \( G \) is a comb, then \( \alpha_2(G) = \frac{n}{2} \). For \( S(k+1) \), \( \forall u, v \in S(k+1) \), \( d_G(u, v) \geq k+1 \) and \( N(v) \cap N(u) = \emptyset \), we can get that \( |N(S(k+1))| \geq |S(k+1)| \) and \( N(S(k+1)) \) is a \( (k-1) \)-set, which means \( |N(S(k+1))| \leq \frac{2(n-|S(k+1)|)}{k} \). Therefore, we have \( \alpha_k(G) = \max |S(k+1)| \leq \frac{2n}{k+2} \). If \( G \) is a subdivision of a comb \( H \), which is constructed by replacing each pendant edge of \( H \) with a path of length \( \frac{k+1}{2} \), then \( \alpha_k(G) = \frac{2n}{k+2} \).

If \( k \) is even. When \( k = 2 \), \( \forall u, v \in S_2 \), \( d_G(u, v) \geq 2 \), we have \( |S_2| \leq n-1 \). If \( G \) is a star \( K_{1,n-1} \), then \( \alpha_1(G) = n-1 \). For \( S(k+1) \), \( \forall u, v \in S(k+1) \), \( d_G(u, v) \geq k+1 \) and \( N(v) \cap N(u) = \emptyset \), we can get that \( |N(S(k+1))| \geq |S(k+1)| \) and \( N(S(k+1)) \) is a \( (k-1) \)-set, which means \( |N(S(k+1))| \leq \frac{2(n-|S(k+1)|)}{k-1} \). Therefore, we have \( \alpha_k(G) = \max |S(k+1)| \leq \frac{2(n-1)}{k+1} \). If \( G \) is a subdivision of a star \( K_{1,\frac{2(n-1)}{k+1}} \), which is constructed by replacing each edge of \( K_{1,\frac{2(n-1)}{k+1}} \) with a path of length \( \frac{k+2}{2} \), then \( \alpha_k(G) = \frac{2n-2}{k+1} \).

In 2000, Kong and Zhao \[7\] showed that for every \( k \geq 2 \), determining \( \alpha_k(G) \) is NP-complete for general graphs. They also showed that this problem remains NP-hard for regular bipartite graphs when \( k \in \{2, 3, 4\} \) \[8\]. It is well-known that for an \( n \)-vertex \( r \)-regular graph \( G \), we have \( \alpha_1(G) \leq \frac{n}{2} \). Also, for \( k = 2 \), we have \( \alpha_2(G) \leq \frac{n}{d+1} \) because for any pair of two vertices \( u, v \) in a 2-independent set, we have \( N(u) \cap N(v) = \emptyset \). For each fixed
integer, \( k \geq 2 \) and \( r \geq 3 \), Beis, Duckworth, and Zito \cite{beis2016sharp} proved upper bounds for \( \alpha_k(G) \) in random \( r \)-regular graphs.

In this paper, for all positive integers \( k \) and \( r \geq 3 \), we prove sharp upper bounds for \( \alpha_k(G) \) in an \( n \)-vertex \( r \)-regular graph. In Section 2, for all positive integers \( k \) and \( r \geq 3 \), we provide infinitely many \( r \)-regular graphs with \( \alpha_k(G) \) attaining the sharp upper bounds. In Section 3, the proof for the bounds will be given. In Section 4, we provide some questions.

For undefined terms, see West \cite{west2015introduction}.

## 2 Construction

In this section, we construct \( n \)-vertex \( r \)-regular graphs with the \( k \)-independence numbers achieving equalities in the upper bounds in Theorem \cite{beis2016sharp}

**Definition 2.1.** Assume that \( k = 6l - 4 \) for \( l \in N \).

Let \( V_1 = \{v_{11}, \ldots, v_{1r}\} \) such that for each \( i \in [r] \), the degree of \( v_{1i} \) is \( r \), \( N[v_{1i}] \) induces a copy of \( K_{r+1} - K_2 \), and for each \( i \neq j \in [r] \), \( v_{1i} \) is not adjacent to \( v_{1j} \) and \( N[v_{1i}] \cap N[v_{1j}] = \emptyset \).

Let \( V_2 = \{N(v_{11}), \ldots, N(v_{1r})\} \) such that for each \( i \neq j \in [r] \), \( N(v_{1i}) \) is not adjacent to \( N(v_{1j}) \), and for each \( i \in [r] \) and \( h \in \{1, 2, 3\} \), \( v_{h1}^i \in N(v_{1i}) \) and \( v_{h1}^i \) is not adjacent to \( v_{h2}^i \).

For a positive integer \( 1 \leq x \leq l - 1 \), let \( V_{3x} = \{v_{(3x+1)i}, \ldots, v_{(3x)r}\} \) such that for each \( i \in [r] \), \( v_{(3x)i}^h \) is adjacent to \( v_{(3x-1)i}^h \) for \( h \in \{1, 2\} \), \( N[v_{(3x)i}] \setminus v_{(3x-1)i}^h \) induces a copy of \( K_{r-1} \), and for each \( i \neq j \in [r] \), \( v_{(3x)i}^h \) is not adjacent to \( v_{(3x)j}^h \) and \( N[v_{(3x)i}] \cap N[v_{(3x)j}] = \emptyset \).

Let \( V_{3x+1} = \{N(v_{(3x)i}) \setminus v_{(3x-1)i}^1, \ldots, N(v_{(3x)i}) \setminus v_{(3x-1)i}^r\} \) such that for each \( i \neq j \in [r-1] \), \( N[v_{(3x)i}] \setminus v_{(3x-1)i}^j \) is not adjacent to \( N[v_{(3x)j}] \setminus v_{(3x-1)j}^j \).

Let \( V_{3x+2} = \{v_{(3x+2)i}^1, \ldots, v_{(3x+2)i}^r\} \), \( h \in \{1, 2\} \) such that for each \( i \in [r] \), \( v_{(3x+2)i}^1 \) is adjacent to \( v_{(3x+1)i}^2 \) and \( v_{(3x+2)i}^h \) is adjacent to all vertices in \( N[v_{(3x)i}] \setminus v_{(3x-1)i}^h \) for each \( i \neq j \in [r] \), \( v_{(3x+2)i}^h \) is not adjacent to \( v_{(3x+2)j}^h \) except for \( x = l - 1 \).

Let \( H_{r,k}^{1} \) be the resulting graph, let \( G_{r,k,t}^1 = tH_{r,k}^{1} \).

**Definition 2.2.** Assume that \( r \) is odd and \( k = 6l - 4 \) for \( l \in N \).

Let \( V_1 = \{v_{11}, \ldots, v_{1r}\} \) such that for each \( i \in [r] \), the degree of \( v_{1i} \) is \( r \), \( N[v_{1i}] \) induces a copy of \( K_{r+1} - \frac{r-1}{2} K_2 \), and for each \( i \neq j \in [r] \), \( v_{1i} \) is not adjacent to \( v_{1j} \) and \( N[v_{1i}] \cap N[v_{1j}] = \emptyset \).

Let \( V_2 = \{N(v_{11}), \ldots, N(v_{1r})\} \) such that for each \( i \neq j \in [r] \), \( N(v_{1i}) \) is not adjacent to \( N(v_{1j}) \), and for each \( i \in [r] \) and \( h \in \{1, 2\} \), \( v_{21}^i, v_{2h}^i \in N(v_{1i}) \) and \( v_{2h}^i \) is not adjacent to \( v_{22}^i \).

For a positive integer \( 1 \leq x \leq l - 1 \), let \( V_{3x} = \{v_{(3x)i}, \ldots, v_{(3x)r}\} \) such that for each \( i \in [r] \), \( v_{(3x)i} \) is adjacent to \( v_{(3x-1)i} \) for all \( h \in [r-1] \) and for each \( i \neq j \in [r] \), \( v_{(3x)i} \) is not adjacent to \( v_{(3x)j} \).
Let $V_{3x+1} = \{v_{(3x+1)1}, \ldots, v_{(3x+1)r}\}$ such that for each $i \in [r]$, $v_{(3x+1)i}$ is adjacent to $v_{(3x)i}$; $N[v_{(3x+1)i}] \setminus v_{(3x)i}$ induces a copy of $K_r$ and for each $i \neq j \in [r]$, $N[v_{(3x+1)i}] \cap N[v_{(3x+1)j}] = \emptyset$.

Let $V_{3x+2} = \{N(v_{(3x+1)1}) \setminus v_{(3x)1}, \ldots, N(v_{(3x+1)r}) \setminus v_{(3x)r}\}$ such that for each $i \in [r]$, $N(v_{(3x+1)i}) \setminus v_{(3x)i}$ induces a copy of $K_{r-1}$.

Let $H^2_{r,k}$ be the resulting graph, let $G^2_{r,k,t} = tH^2_{r,k}$.

**Definition 2.3.** Assume that $r$ is even and $k = 6l - 4$ for $l \in N$.

Let $V_1 = \{v_{11}, \ldots, v_{1r}\}$ such that for each $i \in [r]$, the degree of $v_{1i}$ is $r$, $N[v_{1i}]$ induces a copy of $K_{r+1} = \frac{r-2}{2}K_2$, and for each $i \neq j \in [r]$, $v_{1i}$ is not adjacent to $v_{1j}$ and $N[v_{1i}] \cap N[v_{1j}] = \emptyset$.

Let $V_2 = \{N(v_{11}), \ldots, N(v_{1r})\}$ such that for each $i \neq j \in [r]$, $N(v_{1i})$ is not adjacent to $N(v_{1j})$, and for each $i \in [r]$ and $h \in \lceil \frac{r-2}{2} \rceil$, $v_{2h}, v_{2h}^h \in N(v_{1i})$ and $v_{2h}^h$ is not adjacent to $v_{2h}$.

For a positive integer $1 \leq x \leq l - 1$, let $V_{3x} = \{v_{(3x)1}, \ldots, v_{(3x)r}\}$ such that for each $i \in [r]$, $v_{(3x)i}$ is adjacent to $v_{(3x-1)i}$ for all $h \in [r-2]$ and for each $i \neq j \in [r]$, $v_{(3x)i}$ is not adjacent to $v_{(3x)j}$.

Let $V_{3x+1} = \{v_{(3x+1)1}^h, \ldots, v_{(3x+1)r}^h\}$, $h \in \{1, 2\}$ such that for each $i \in [r]$, $v_{(3x+1)i}^h$ is adjacent to $v_{(3x)i}$, $v_{(3x+1)i}^h$ is adjacent to $v_{(3x+1)i}^2$, $N[v_{(3x+1)i}^h] \setminus v_{(3x)i}$ induces a copy of $K_{r-1}$, and for each $i \neq j \in [r]$, $N[v_{(3x+1)i}^h] \cap N[v_{(3x+1)j}] = \emptyset$, $v_{(3x+1)i}^h$ is not adjacent to $v_{(3x+1)j}$.

Let $V_{3x+2} = \{v_{(3x+2)1}^h, \ldots, v_{(3x+2)r}^h\}$, $h \in [r-2]$ such that for each $i \in [r]$, $v_{(3x+2)i}^h$ is adjacent to $v_{(3x+1)i}^h$ and $v_{(3x+2)i}^2$ induces a copy of $K_{r-2}$, and for each $i \neq j \in [r]$, $v_{(3x+2)i}^h$ is not adjacent to $v_{(3x+2)j}^h$, except for $x = l - 1$. 
Let $H^3_{r,k}$ be the resulting graph, let $G^3_{r,k,t} = tH^3_{r,k}$.

**Proposition 2.4.** Let $l \in \mathbb{N}$, for $r \geq 3$, $k = 6l - 4$ and $t \geq 1$, the graph $G^i_{r,k,t}$, $i \in \{1, 2, 3\}$ in the definition is $r$-regular and $\alpha_k(G^i_{r,k,t}) = \frac{n}{l(r+1)}$, where $n = trl(r + 1)$. 

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Definition 2.5. Assume that \( k = 6l - 3 \) for \( l \in N \).

Use the same definition of \( V_1, V_2, V_{3x}, V_{3x+1}, V_{3x+2}, 1 \leq x \leq l - 1 \) in \( H_{r,k}^1 \), and in \( V_{3x+2} \), for each \( i \neq j \in [r], v^h_{(3x+2)i} \) is not adjacent to \( v^h_{(3x+2)j} \) for all \( x \).

Let \( V_{3l} = \{v_{(3l)1}, v_{(3l)2}\} \) such that for each \( i \in [r], v_{(3l)1} \) is adjacent to \( v_{3l-1}^1 \) and \( v_{(3l)2} \) is adjacent to \( v_{(3l-1)}^2 \).

Let \( H_{r,k}^4 \) be the resulting graph, let \( G_{r,k,t}^4 = tH_{r,k}^4 \).

![Figure 4: The graph \( H_{r,k}^4 \)](image)

**Proposition 2.6.** Let \( l \in N \), for \( r \geq 3, k = 6l - 3 \) and \( t \geq 1 \), the graph \( G_{r,k,t}^4 \) in the definition is \( r \)-regular and \( \alpha_k(G_{r,k,t}^4) = \frac{n}{r(r+1)+2} \), where \( n = tlr(r+1) + 2t \).

Definition 2.7. Assume that \( k = 6l - 2 \) for \( l \in N \).

Use the same definition of \( V_1, V_2, V_{3x}, 1 \leq x \leq l \) and \( V_{3x+1}, V_{3x+2}, 1 \leq x \leq l - 1 \) in \( H_{r,k}^1 \). While in \( V_{3x} \), for each \( i \neq j \in [r], v_{(3x)i} \) is not adjacent to \( v_{(3x)j} \) except for \( x = l \) and in \( V_{3x+2} \), for each \( i \neq j \in [r], v^h_{(3x+2)i} \) is not adjacent to \( v^h_{(3x+2)j} \) for all \( x \).

Let \( H_{r,k}^5 \) be the resulting graph, let \( G_{r,k,t}^5 = tH_{r,k}^5 \).

**Proposition 2.8.** Let \( l \in N \), for \( r \geq 3, k = 6l - 2 \) and \( t \geq 1 \), the graph \( G_{r,k,t}^5 \) in the definition is \( r \)-regular and \( \alpha_k(G_{r,k,t}^5) = \frac{n}{r(r+1)+1} \), where \( n = tlr(r+1) + tr \).

Definition 2.9. Assume that \( r \) is odd and \( k = 6l - 1 \) for \( l \in N \).
Use the same definition of $V_1$, $V_2$, $V_{3x}$, $1 \leq x \leq l$ and $V_{3x+1}$, $V_{3x+2}$, $1 \leq x \leq l - 1$ in $H_{r,k}^2$.

Let $V_{3l+1} = \{v_{(3l+1)1}\}$ such that for each $i \in [r]$, $v_{(3l+1)1}$ is adjacent to $v_{(3l)i}$.

Let $H_{r,k}^6$ be the resulting graph and $G_{r,k,t}^6 = tH_{r,k}^6$.

Figure 5: The graph $H_{r,k}^5$

Figure 6: The graph $H_{r,k}^6$
Definition 2.10. Assume that \( r \) is even and \( k = 6l - 1 \) for \( l \in \mathbb{N} \).

Use the same definition of \( V_1, V_2, V_{3x}, 1 \leq x \leq l \) and \( V_{3x+1}, V_{3x+2}, 1 \leq x \leq l - 1 \) in \( H_{r,k}^3 \).

Let \( V_{3x+1} = \{ v_{(3l+1)1}, v_{(3l+1)2} \} \) such that for each \( i \in [r] \), \( v_{(3l+1)1} \) is adjacent to \( v_{(3l)i} \) and \( v_{(3l+1)2} \) is also adjacent to \( v_{(3l)i} \).

Let \( H_{r,k}^7 \) be the resulting graph and \( G_{r,k,t}^7 = tH_{r,k}^7 \).

Proposition 2.11. Let \( l \in \mathbb{N}, k = 6l - 1 \) and \( t \geq 1 \).

For odd \( r \geq 3 \), the graph \( G_{r,k,t}^6 \) in the definition is \( r \)-regular and \( \alpha_k(G_{r,k,t}^6) = \frac{rn}{(lr+1)(r+1)} \), where \( n = t(lr+1)(r+1) \).

For even \( r \geq 4 \), the graph \( G_{r,k,t}^7 \) in the definition is \( r \)-regular and \( \alpha_k(G_{r,k,t}^7) = \frac{rn}{(lr+1)(r+1)+1} \), where \( n = t(lr+1)(r+1)+t \).

Definition 2.12. Assume that \( r \) is odd and \( k = 6l \) for \( l \in \mathbb{N} \).

Use the same definition of \( V_1, V_2, V_{3x}, 1 \leq x \leq l \) and \( V_{3x+2}, 1 \leq x \leq l - 1 \) in \( H_{r,k}^2 \). While in \( V_{3x+1} \), for each \( i \neq j \in [r] \), \( v_{(3x)i} \) is not adjacent to \( v_{(3x)j} \) except for \( x = l \).

Let \( H_{r,k}^8 \) be the resulting graph and \( G_{r,k,t}^8 = tH_{r,k}^8 \).

Definition 2.13. Assume that \( r \) is even and \( k = 6l \) for \( l \in \mathbb{N} \).
Use the same definition of $V_1, V_2, V_{3x}, V_{3x+1}, 1 \leq x \leq l$ and $V_{3x+2}, 1 \leq x \leq l-1$ in $H_{r,k}^3$. While in $V_{3x+1}$, for each $i \neq j \in [r]$, $h \in \{1, 2\}$, $v_{(3x+1)i}^h$ is not adjacent to $v_{(3x+1)j}^h$ except for $x = l$.

Let $H_{r,k}^9$ be the resulting graph and $G_{r,k,t}^9 = tH_{r,k}^9$.

Proposition 2.14. Let $l \in N$, $k = 6l$ and $t \geq 1$. 
For odd $r \geq 3$, the graph $G_{r,k,t}^8$ in the definition is $r$-regular and $\alpha_k(G_{r,k,t}^8) = \frac{n}{l(r+1)+2}$, where $n = tl(r+1) + 2tr$.

For even $r \geq 4$, the graph $G_{r,k,t}^9$ in the definition is $r$-regular and $\alpha_k(G_{r,k,t}^9) = \frac{n}{l(r+1)+3}$, where $n = tl(r+1) + 3tr$.

**Definition 2.15.** Assume that $r$ is odd and $k = 6l + 1$ for $l \in \mathbb{N}$.

Use the same definition of $V_1, V_2, V_{3x}, V_{3x+1}, 1 \leq x \leq l$ and $V_{3x+2}, 1 \leq x \leq l-1$ in $H_{r,k}^2$.

Let $V_{3l+2} = \{v_{(3l+2)}1, \ldots, v_{(3l+2)(r-1)}\}$ such that for each $i \in [r-1]$ and $j \in [r]$, $v_{(3l+2)i}$ is adjacent to $v_{(3l+1)j}$.

Let $H_{r,k}^{10}$ be the resulting graph and $G_{r,k,t}^{10} = tH_{r,k}^{10}$.

![Figure 10: The graph $H_{r,k}^{10}$](image)

**Definition 2.16.** Assume that $r$ is even and $k = 6l + 1$ for $l \in \mathbb{N}$.

Use the same definition of $V_1, V_2, V_{3x}, V_{3x+1}, 1 \leq x \leq l$ and $V_{3x+2}, 1 \leq x \leq l-1$ in $H_{r,k}^3$.

Let $V_{3l+2} = \{v_{(3l+2)}1, \ldots, v_{(3l+2)(r-2)}\}$ such that for each $i \in [r-2]$ and $j \in [r]$, $v_{(3l+2)i}$ is adjacent to $v_{(3l+1)j}$.

Let $H_{r,k}^{11}$ be the resulting graph and $G_{r,k,t}^{11} = tH_{r,k}^{11}$.

**Proposition 2.17.** Let $l \in \mathbb{N}$, $k = 6l + 1$ and $t \geq 1$.

For odd $r \geq 3$, the graph $G_{r,k,t}^{10}$ in the definition is $r$-regular and $\alpha_k(G_{r,k,t}^{10}) = \frac{rn}{(l+3)(r+1)-4}$, where $n = t(lr+3)(r+1) - 4t$. 

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For even \( r \geq 3 \), the graph \( G_{r,k,t}^{11} \) in the definition is \( r \)-regular and \( \alpha_k(G_{r,k,t}^{11}) = \frac{rn}{(lr+4)(r+1)}-6 \), where \( n = t(lr+4)(r+1) - 6t \).

### 3 Sharp Upper Bounds

In this section, we prove the sharp upper bounds for \( \alpha_k(G) \) in an \( n \)-vertex \( r \)-regular graph \( G \). First we investigate the relevant properties of \( k \)-independent set of \( G \).

Let \( N^i(S) \) to present the subsequent neighborhood of \( N^{i-1}(S) \), i.e., \( N^i(S) = N(N^{i-1}(S)) \setminus (N^{i-2}(S) \cup N^{i-1}(S)) \). Note that \( N^0(S) = S, N^1(S) = N(S), N^2(S) = N(N(S)) \setminus (S \cup N^1(S)) \), etc.

**Lemma 3.1.** Let \( G \) be an \( n \)-vertex \( r \)-regular graph and let \( S \) be a \( k \)-independent set of \( G \). If \( N^{i-1}(S), N^i(S), N^{i+1}(S) \) are three consecutive sets of \( G \) as defined, where \( 3 \leq i \leq \frac{k}{2} - 1 \), then we have \( |N^{i-1}(S)| + |N^i(S)| + |N^{i+1}(S)| \geq (r+1)|S| \).

**Proof.** Let \( v_j, v_h \in S, j, h \in \{1, 2, \ldots, |S|\} \), we have \( N^i(v_j) \cap N^i(v_h) = \emptyset \), where \( 0 \leq i \leq \frac{k}{2} - 1 \), thus \( N^i(v_j) \geq 1 \) and \( N^i(S) \geq |S| \).

Assume \( |N^i(v_j)| = a \), we have \( d_{N^i(v_j)}(v) \leq a - 1, \forall v \in N^i(v_j) \). For \( G \) is \( r \)-regular, so \( d_{N^{i-1}(v_j) \cup N^{i+1}(v_j)}(v) \geq r - a + 1 \). Then we have \( |N^{i-1}(v_j)| + |N^i(v_j)| + |N^{i+1}(v_j)| \geq r + 1 \), thus \( |N^{i-1}(S)| + |N^i(S)| + |N^{i+1}(S)| \geq (r+1)|S| \). \( \Box \)

Here we give an example when \( a = 5, r = 7 \) in Figure 12.
The lemmas allow us to prove the main result of this section.

**Theorem 3.2.** Let $G$ be an $n$-vertex $r$-regular graph, $c$ be the upper bound of $\alpha_k(G)$, i.e., $\alpha_k(G) \leq c$. Then

$$c = \begin{cases} 
\frac{n}{2} & \text{if } k = 1 \\
\frac{n}{l(r+1)} & \text{if } k = 6l - 4 \\
\frac{n}{r(r+1)+2} & \text{if } k = 6l - 3 \\
\frac{n}{l(r+1)+1} & \text{if } k = 6l - 2 \\
\frac{r}{(r+1)(r+1)+1} & \text{if } k = 6l - 1, \text{ } r \text{ is odd} \\
\frac{r}{(r+1)(r+1)+2} & \text{if } k = 6l - 1, \text{ } r \text{ is even} \\
\frac{n}{l(r+1)+3} & \text{if } k = 6l, \text{ } r \text{ is odd} \\
\frac{r}{l(r+1)+3} & \text{if } k = 6l, \text{ } r \text{ is even} \\
\frac{r}{(r+3)(r+1)+4} & \text{if } k = 6l + 1, \text{ } r \text{ is odd} \\
\frac{r}{(r+4)(r+1)+6} & \text{if } k = 6l + 1, \text{ } r \text{ is even}
\end{cases}$$

(1)

Where $l \in \mathbb{N}^+$. The bounds are tight with the constructions we mention before.

**Proof.** Let $S$ be the $k$-independent set of $G$, then $\forall v_j, v_h \in S$, $d(v_j, v_h) \geq k + 1$.

**Case 1** $k = 1$

Then for all $v_j, v_h \in S$, $d(v_j, v_h) \geq 2$. In $N^1(S)$, different parts can share vertices, i.e., $N^1(v_j) \cap N^1(v_h) \neq \emptyset$. Then we have $N^1(S) \geq |S|$ and $|S| + |N^1(S)| \leq n$, so $|S| \leq \frac{n}{2}$ with the equality achieved for $G = G_{\frac{n}{2}, \frac{n}{2}}$.

**Case 2** $k = 6l - 4$

Figure 12: Example $a = 5$, $r = 7$
Then for all \( v_j, v_h \in S \), \( d(v_j, v_h) \geq 6l - 3 \). For the shortest distance \( 6l - 3 \) is odd, so none of the parts can share vertices in the tail set \( N^x(S) \), but there are edges between different parts, i.e., \( N^x(v_j) \cap N^x(v_h) = \emptyset \) and \( N^x(v_j) \sim N^x(v_h) \).

Assume \( P \) is the shortest path between \( v_j \) and \( v_h \), so \( |P| \geq 6l - 3 \). We can get at least one edge from the tail set, and two path \( P_1, P_2 \) respectively from \( N^0(v_j) \to N^x(v_j) \) and \( N^0(v_h) \to N^x(v_h) \), and \( |P_1| + |P_2| = 2x \geq |P| - 1 \), i.e., \( x \geq 3l - 2 \).

By Lemma we have \( N^i(v_j) \cap N^i(v_h) = \emptyset \), where \( 0 \leq i \leq \frac{k}{2} \), and \( N^0(S) = |S| \), \( N^1(S) = r|S| \), \( |N^{i-1}(S)| + |N^i(S)| + |N^{i+1}(S)| \geq \left((r + 1)|S|\right) \), \( 3 \leq i \leq \frac{k}{2} - 1 \). Assume the three consecutive sets \( N^{i-1}(S), N^i(S), N^{i+1}(S) \) as a unit.

Between \( N^0(S) \) and \( N^1(S) \), we can get one edge and whenever a unit appears, we can get a three length path. As \( 3l - 2 = 3(l - 1) + 1 \), we have \( |P| \geq 6l - 2 \). We can get two path \( P_1, P_2 \) respectively from \( N^0(v_j) \to N^x(v_j) \) and \( N^0(v_h) \to N^x(v_h) \), and \( |P_1| + |P_2| = 2x \geq |P| \), i.e., \( x \geq 3l - 1 \).

As \( 3l - 1 = 3(l - 1) + 1 \), and between \( N^{x-1}(S) \) and \( N^x(S) \) we can get one edge, so we have \( l - 1 \) units from \( N^2(S) \to N^{3l-2}(S) \), and \( |N^{3l-1}(S)| \geq \frac{2|S|}{r} \). Then \( |S| + r|S| + (l - 1)(r + 1)|S| + \frac{2|S|}{r} \leq n \), i.e., \( |S| \leq \frac{rn}{l(r+1)+2} \). The equality can be achieved when use the construction of \( G^4_{r,k,t} \), \( i \in \{1, 2, 3\} \).

**Case 3** \( k = 6l - 3 \)

Then for all \( v_j, v_h \in S \), \( d(v_j, v_h) \geq 6l - 2 \). For the shortest distance \( 6l - 2 \) is even, so in the tail set \( N^x(S) \), different parts can share vertices, i.e., \( N^x(v_j) \cap N^x(v_h) \neq \emptyset \).

Assume \( P \) is the shortest path between \( v_j \) and \( v_h \), so \( |P| \geq 6l - 2 \). We can get two path \( P_1, P_2 \) respectively from \( N^0(v_j) \to N^x(v_j) \) and \( N^0(v_h) \to N^x(v_h) \), and \( |P_1| + |P_2| = 2x \geq |P| \), i.e., \( x \geq 3l - 1 \).

As \( 3l - 1 = 3(l - 1) + 1 \), and between \( N^{x-1}(S) \) and \( N^x(S) \) we can get one edge, so we have \( l - 1 \) units from \( N^2(S) \to N^{3l-2}(S) \), and \( |N^{3l-1}(S)| \geq \frac{2|S|}{r} \). Then \( |S| + r|S| + (l - 1)(r + 1)|S| + \frac{2|S|}{r} \leq n \), i.e., \( |S| \leq \frac{rn}{l(r+1)+2} \). The equality can be achieved when use the construction of \( G^4_{r,k,t} \).

**Case 4** \( k = 6l - 2 \)

Then for all \( v_j, v_h \in S \), \( d(v_j, v_h) \geq 6l - 1 \). For the shortest distance \( 6l - 1 \) is odd, so \( N^x(v_j) \cap N^x(v_h) = \emptyset \) and \( N^x(v_j) \sim N^x(v_h) \), where \( N^x(S) \) is the tail set.

Similarly, we can get \( x \geq 3l - 1 \). As \( 3l - 1 = 3(l - 1) + 2 \), we can get two length path from \( N^0(S) \to N^2(S) \) and \( l - 1 \) units from \( N^3(S) \to N^{3l-1}(S) \). For \( |N^2(S)| \geq |S| \), we have \( |S| + r|S| + |S| + (l - 1)(r + 1)|S| \leq n \), i.e., \( |S| \leq \frac{rn}{(r+1)(r+1)} \). The equality can be achieved when use the construction of \( G^5_{r,k,t} \).

**Case 5** \( k = 6l - 1 \)

Then for all \( v_j, v_h \in S \), \( d(v_j, v_h) \geq 6l \). For the shortest distance \( 6l \) is even, so \( N^x(v_j) \cap N^x(v_h) \neq \emptyset \), where \( N^x(S) \) is the tail set.

Similarly, we can get \( x \geq 3l \). As \( 3l = 3(l - 1) + 2 + 1 \), we can get one edge between \( N^{x-1}(S) \) and \( N^x(S) \), a two length path from \( N^0(S) \to N^2(S) \), and \( l - 1 \) units from \( N^3(S) \to N^{3l-1}(S) \).

If \( r \) is odd, we have \( |N^{3l}(S)| \geq \frac{|S|}{r} \), then \( |S| + r|S| + (l - 1)(r + 1)|S| + \frac{|S|}{r} \leq n \), i.e., \( |S| \leq \frac{rn}{(r+1)(r+1)} \). The equality can be achieved when use the construction of \( G^6_{r,k,t} \).
If \( r \) is even, we have \( |N^{3l}(S)| \geq \frac{2|S|}{r} \), then \( |S| + r|S| + |S| + (l-1)(r+1)|S| + \frac{2|S|}{r} \leq n \), i.e., \( |S| \leq \frac{rn}{(r+2)(r+1)} \). The equality can be achieved when use the construction of \( G_{r,k,t}^{7} \).

Case 6 \( k = 6l \)

Then for all \( v_j, v_h \in S \), \( d(v_j, v_h) \geq 6l + 1 \). For the shortest distance \( 6l + 1 \) is odd, so \( N^x(v_j) \cap N^x(v_h) = \emptyset \) and \( N^x(v_j) \sim N^x(v_h) \), where \( N^x(S) \) is the tail set.

Similarly, we can get \( x \geq 3l \). As \( 3l = 3(l-1)+2+1 \), we can get one edge between \( N^{3l-1}(S) \) and \( N^{3l}(S) \), a two length path from \( N^0(S) \rightarrow N^2(S) \), and \( l-1 \) units from \( N^3(S) \rightarrow N^{3l-1}(S) \).

If \( r \) is odd, we have \( |N^{3l}(S)| \geq |S| \), then \( |S| + r|S| + |S| + (l-1)(r+1)|S| + |S| \leq n \), i.e., \( |S| \leq \frac{n}{(r+1)^{2}} \). The equality can be achieved when use the construction of \( G_{r,k,t}^{8} \).

If \( r \) is even, we have \( |N^{3l}(S)| \geq 2|S| \), then \( |S| + r|S| + |S| + (l-1)(r+1)|S| + 2|S| \leq n \), i.e., \( |S| \leq \frac{rn}{(r+3)(r+1)} \). The equality can be achieved when use the construction of \( G_{r,k,t}^{9} \).

Case 7 \( k = 6l + 1 \)

Then for all \( v_j, v_h \in S \), \( d(v_j, v_h) \geq 6l + 2 \). For the shortest distance \( 6l + 2 \) is even, so \( N^x(v_j) \cap N^x(v_h) \neq \emptyset \), where \( N^x(S) \) is the tail set.

Similarly, we can get \( x \geq 3l + 1 \). As \( 3l + 1 = 3(l-1) + 2 + 1 + 1 \), we can get one edge between \( N^{x-1}(S) \) and \( N^x(S) \), one edge between \( N^{3l-1}(S) \) and \( N^{3l}(S) \), a two length path from \( N^0(S) \rightarrow N^3(S) \), and \( l-1 \) units from \( N^3(S) \rightarrow N^{3l-1}(S) \).

If \( r \) is odd, we have \( |N^{3l}(S)| \geq |S| \) and \( |N^{3l+1}(S)| \geq \frac{(r-1)|S|}{r} \), then \( |S| + r|S| + |S| + (l-1)(r+1)|S| + |S| + \frac{(r-1)|S|}{r} \leq n \), i.e., \( |S| \leq \frac{rn}{(r+3)(r+1)} \). The equality can be achieved when use the construction of \( G_{r,k,t}^{10} \).

If \( r \) is even, we have \( |N^{3l}(S)| \geq 2|S| \) and \( |N^{3l+1}(S)| \geq \frac{(r-2)|S|}{r} \), then \( |S| + r|S| + |S| + (l-1)(r+1)|S| + 2|S| + \frac{(r-2)|S|}{r} \leq n \), i.e., \( |S| \leq \frac{rn}{(r+4)(r+1)-6} \). The equality can be achieved when use the construction of \( G_{r,k,t}^{11} \). \( \square \)

4 Questions

Aida, Cioab˘a, and Tait \[1]\] obtained two spectral upper bounds for the k-independence number of a graph. They constructed graphs that attain equality for their first bound and showed that their second bound compares favorably to previous bounds on the k-independence number. We may ask whether given an independence number, there is an upper or lower bound for the spectral radius (the largest eigenvalue of a graph) in an \( n \)-vertex regular graph.

**Question 4.1.** Given a positive integer \( t \), what is the best lower bound for the spectral radius in an \( n \)-vertex \( r \)-regular graph to guarantee that \( \alpha_k(G) \geq t + 1 \)?

If \( r \geq 3 \), \( G \) is an \( n \)-vertex \( r \)-regular graph, which is not a complete graph, then \( \alpha_1(G) \geq \frac{n}{\chi(G)} \geq \frac{n}{r} \) by Brooks’ Theorem. For \( k \geq 2 \), it is natural to ask a lower bound for \( \alpha_k(G) \) in an \( n \)-vertex \( r \)-regular graph.
**Question 4.2.** For \( r \geq 3 \), what is the best lower bound for \( \alpha_k(G) \) in an \( n \)-vertex \( r \)-regular graph?

The \( k \)-th power of the graph \( G \), denoted by \( G^k \), is a graph on the same vertex set as \( G \) such that two vertices are adjacent in \( G^k \) if and only if their distance in \( G \) is at most \( k \). The \( k \)-distance \( t \)-coloring, also called distance \( (k, t) \)-coloring, is a \( k \)-coloring of the graph \( G^k \) (that is, any two vertices within distance \( k \) in \( G \) receive different colors). The \( k \)-distance chromatic number of \( G \), written \( \chi_k(G) \), is exactly the chromatic number of \( G^k \). It is easy to see that \( \chi(G) = \chi_1(G) \leq \chi_k(G) = \chi(G^k) \).

It was noted by Skupień that the well-known Brooks’ theorem can provide the following upper bound:

\[
\chi_k(G) \leq 1 + \Delta(G^k) \leq 1 + \Delta \sum_{i=1}^{k} (\Delta - 1)^{k-1} = 1 + \Delta \frac{(\Delta - 1)^{k-1} - 1}{\Delta - 2},
\]

for \( \Delta \geq 3 \). Let \( M =: 1 + \Delta \frac{(\Delta - 1)^{k-1} - 1}{\Delta - 2} \). Consider a \((k, \chi_k(G))\)-coloring. Let \( V_i \) be the vertex set with the color \( i \) for \( i \in [\chi_k(G)] \). Then we have \( \chi_k(G) \alpha_k(G) \geq n \). Thus for \( r \geq 3 \), if \( G \) is an \( n \)-vertex \( r \)-regular graph, then we have \( \alpha_k(G) \geq \frac{n}{\chi_k(G)} \geq \frac{n}{M} \). Since equality in inequality (2) holds only when \( G \) is a Moore graph, the lower bound is not tight. Thus, we might be interested in answering Question 4.2.

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