Quantum mechanical laws impose that arbitrary quantum states cannot be perfectly cloned or amplified without introducing some unavoidable noise in the process [1,2]. This is a consequence of the linearity and unitary evolution of quantum mechanics and guarantees against unphysical situations such as the violation of the Heisenberg uncertainty principle or the superluminal exchange of information [3,4]. In any deterministic linear amplifier noise is unavoidably added in the process and any input pure state results in a mixed output one.

This has profound implications also from a practical point of view in the frame of quantum information processing and quantum metrology. For example, it greatly limits the possibility of restoring the information carried by some fragile quantum state by amplifying it after it has been degraded in a lossy channel. Or, it can forbid to distinguish among different parameter values if they are encoded in partially overlapping quantum states.

As an illustrative example let us consider the case that some quantum information (or classical parameter value) is encoded in the complex amplitude $\alpha$ of a coherent state $|\alpha\rangle$. If the state amplitude is made too small (generally by losses) then the strong overlap between different states can make it impossible to correctly distinguish among them. Simply amplifying the states would not solve the problem because it would also amplify the quantum fluctuations of the coherent states, thus in fact increasing their overlap and making the situation even worse (see Fig. 1).

A solution to this problem would be provided by an ideal noiseless amplifier of coherent states of light whose action can be mathematically described as

$$|\alpha\rangle \rightarrow |g\alpha\rangle,$$

where $g > 1$ is the amplification gain. Referring to the above example, a sufficient noiseless amplification of partially-overlapped coherent states would allow one to make them exactly distinguishable.

The transformation (1) is unphysical, but can be implemented probabilistically in an approximate way. Ralph and Lund [5] recently proposed a scheme based on the application of multiple quantum-scissors blocks [6,7] to non-deterministically amplify the low-amplitude portions obtained by splitting a coherent state before their coherent recombination in an interferometric setup. Although the complete scheme is almost impossible to realize with current technologies, the functioning of its quantum-scissors core element has been recently demonstrated by two experimental groups [8,9].

Here we follow a completely different route, based on a combination of photon addition and subtraction, and
show that the performances of this approach are far superior, both in terms of higher effective amplification, and of higher fidelity of the final states to the ideal target coherent state $|\alpha\rangle$.

Addition and subtraction of single photons are the result of the application of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ to an arbitrary state of light. Depending on the ordering of such operations, a transformation $\hat{a}\hat{a}^\dagger$ or $\hat{a}^\dagger\hat{a}$ can be applied to the initial state. Sequences and coherent superpositions of such quantum operators have been recently demonstrated experimentally \[13\]. Making a coherent linear combination of these two operations with suitable weights one can obtain

$$\hat{G} = (g-2)\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger = (g-1)\hat{n} + 1, \quad (2)$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the photon number operator. As shown in Ref. \[12\], the operation (2) is a good approximation of the ideal noiseless amplification process \[11\] for weak coherent states.

The performance of the approximate amplifier (2) can be quantified by its effective gain and fidelity. For an input coherent state $|\alpha\rangle$, the un-normalized output state of the amplifier reads $\hat{G}|\alpha\rangle$. We define the effective amplification gain $g_{\text{eff}}$ as the ratio of the mean values of annihilation operator $\hat{a}$ for the output state $\hat{G}|\alpha\rangle$ and input state $|\alpha\rangle$. Since $\langle \alpha|\hat{a}|\alpha\rangle = \alpha$ we have

$$g_{\text{eff}} = \frac{1}{\alpha} \frac{\langle \alpha|\hat{G}\hat{a}\hat{G}|\alpha\rangle}{\langle \alpha|\hat{G}^\dagger\hat{G}|\alpha\rangle}. \quad (3)$$

On inserting the expression for the operator $\hat{G}$ into Eq. (3) we obtain after some algebra

$$g_{\text{eff}} = 1 + \frac{(g-1)\left[1 + (g-1)|\alpha|^2\right]}{1 + (g^2-1)|\alpha|^2 + (g-1)^2|\alpha|^4}. \quad (4)$$

The fidelity of the amplifier is defined as normalized overlap of the output state $\hat{G}|\alpha\rangle$ with the ideal target coherent state $|\alpha\rangle$,

$$F = \frac{|\langle \alpha|\hat{G}|\alpha\rangle|^2}{|\langle \alpha|\hat{G}^\dagger\hat{G}|\alpha\rangle|^2}. \quad (5)$$

A straightforward calculation yields

$$F = \frac{\left[1 + g(g-1)|\alpha|^2\right]^2 e^{-(g-1)^2}|\alpha|^2}{1 + (g^2-1)|\alpha|^2 + (g-1)^2|\alpha|^4}. \quad (6)$$

Of particular interest is the nominal gain $g = 2$. In this case the formula for $\hat{G}$ simplifies, as one term in the superposition (2) vanishes and we obtain

$$\hat{G}_{g=2} = \hat{a}\hat{a}^\dagger. \quad (7)$$

Application of such transformation for noiseless amplification has been originally discussed in Ref. [13], and its action is evident if applied to a weak coherent state approximately described as $|\alpha\rangle = |0\rangle + \alpha|1\rangle$: one gets $\hat{a}\hat{a}^\dagger(|0\rangle + \alpha|1\rangle) \rightarrow \hat{a}(1) + \sqrt{2}\alpha|2\rangle \rightarrow |0\rangle + 2\alpha|1\rangle$, i.e., a doubling of the coherent state amplitude. The advantage of the transformation (4) is that its experimental implementation does not require interferometric stability unlike the general case of $g \neq 2$.

The experiment is based on a unique and versatile setup for implementing creation and annihilation operators that has been recently used to arbitrarily engineer quantum light states and test fundamental quantum mechanical rules \[10, 11, 14, 15\]. The addition of a single photon to an arbitrary light state is obtained by conditional stimulated parametric down-conversion in a non-linear crystal. The photon addition in the output signal mode is heralded by the detection (by an on/off photodetector $D_a$) of a single photon in the idler down-conversion channel. On the other hand, single-photon subtraction is implemented by conditionally attenuating a state by detecting (by an on/off photodetector $D_s$) a single photon reflected from a high-transmissivity beam-splitter (BS). By placing the parametric down-converter and the beam-splitter in series along the path of a traveling coherent state, one can herald the application of the $\hat{a}\hat{a}^\dagger$ operator by looking for coincident detections from $D_a$ and $D_s$, as shown in Fig. 2. The low parametric gain and the low reflectivity of BS (set to about 5 % for these measurements) guarantee that the experimental scheme is a very faithful implementation of the ideal operator sequence.

![FIG. 2: Schematic experimental setup. Two blocks for conditional single-photon addition and subtraction are placed in the path of a coherent state. A coincident click (C) from the two on/off photodetectors heralds the successful realization of the $\hat{a}\hat{a}^\dagger$ operator sequence and the probabilistic noiseless amplification of the input coherent state. High-frequency, time-domain, balanced homodyne detection is then used for a full reconstruction of the involved quantum states.

The main light source is a mode-locked Ti:sapphire laser producing 1.5 ps pulses at 786 nm and with a repetition rate of about 82 MHz. Most of the laser emission is frequency-doubled to become the pump for the down-conversion process. An attenuated portion of the laser emission is used as the coherent field $|\alpha\rangle$, which is injected along the signal mode of the down-converter (a]
type-I, $\beta$—barium borate crystal) and eventually crosses a variable-reflectivity beam-splitter (BS, a half-wave plate and polarizing beam-splitter combination) before being mixed in a 50-50 beam-splitter with another portion of the original laser field serving as the local oscillator (LO) for balanced time-domain homodyne detection. Differently from previous experiments that only involved single-click heralding or phase-independent states, here a particular care has to be taken in order to perform phase-sensitive homodyne measurements triggered at a relatively low rate (ranging from about 20 cps for low $|\alpha|$ values to about 70 cps for $|\alpha| = 1$). An active stabilization of the relative phase between the signal state and the LO has been implemented to this purpose by using the DC component of the homodyne current as a control signal in a feedback loop.

Quadrature measurements for the amplified $\hat{a}\hat{a}^\dagger|\alpha\rangle$ state are obtained by time integration of the pulsed homodyne signal synchronous to a coincident $D_\alpha - D_\alpha^\dagger$ click. The next pulses (not coincident with any trigger event) are also analyzed in order to acquire homodyne data for the corresponding un-amplified $|\alpha\rangle$ input state. Finally, quadrature measurements of the vacuum state obtained by blocking the signal beam are also acquired for normalization. An absolute calibration of the input coherent state amplitude $|\alpha\rangle$ is obtained by comparing the rate of stimulated photon addition events to spontaneous ones.

The experimental estimation of the effective gain $g_{\text{eff}}$ is simply obtained by measuring the ratio of the mean values of the amplitude quadratures for the output and the input states. This is done by locking the relative phase between the coherent state $|\alpha\rangle$ and the local oscillator to an interference maximum (or minimum). Interestingly enough, since these two quantities are measured with the same homodyne detector, they suffer identical losses, therefore detection inefficiency factors out in their ratio. The experimental effective gain is plotted in Fig. 3A as a function of $|\alpha|$, together with that calculated for $g = 2$. For low values of $|\alpha|$ the effective gain is very close to the ideal value of 2, corresponding to an effective intensity amplification of $\approx 4$.

About $10^5$ quadrature measurements distributed in 11 values of the LO phase in the $[0, \pi]$ interval are then acquired to perform a quantum tomographic reconstruction of the states based on an iterative max-likelihood algorithm [16, 17]. The experimental fidelity of the amplified state to the target state is calculated by comparing the amplified state to a coherent state of double amplitude $|2\alpha\rangle$ (obtained by halving the amplitude attenuation experienced by the portion of the laser emission injected in the down-converter crystal), through their reconstructed density matrix elements. Experimental fidelity values corresponding to three different amplitudes of the input coherent state are plotted together with the calculated curves in Fig. 3B. We find a very good agreement with the expected behavior and a high-fidelity ($F > 90\%$) operation of our noiseless amplifier is preserved up to input coherent state amplitudes $|\alpha| \lesssim 0.65$, corresponding to $g_{\text{eff}} \approx 1.6$. The very high fidelity of our noiseless amplifier is also evident in the little distortions experienced by the Wigner functions of the amplified states, whose contour plots are also shown in the right panels of Fig. 3.

The noise properties of the amplifier may be succinctly characterized by measuring the variances of the amplitude and phase quadratures ($x_{\text{amp}}$ and $p_{\text{amp}}$) of the amplified coherent state. The results are plotted in Fig. 3C, where one can see that the variances lie far below the value of $2g_{\text{eff}}^2 - 1$ shot-noise units corresponding to the best deterministic linear amplifier. From the knowledge of the quadrature fluctuations and the effective amplification gain we can also determine the equivalent input noise of the amplifier [8, 18, 19],

$$N_{\text{eq}} = \frac{\langle (\Delta x_{\text{amp}})^2 \rangle}{g_{\text{eff}}^2} - \langle (\Delta x_{\text{in}})^2 \rangle.$$  

A direct calculation reveals that the approximate noiseless amplification $|\alpha\rangle \rightarrow a\hat{a}^\dagger|\alpha\rangle$ exhibits negative $N_{\text{eq}}$ for all $\alpha$. Experimentally, we find that our amplifier indeed achieves $N_{\text{eq}} < -0.48$ for all considered coherent state amplitudes $|\alpha| \leq 1.4$. 

FIG. 3: Dependence of the (A) effective gain and (B) final state fidelity vs. input state amplitude $|\alpha|$ for a nominal gain $g = 2$. Red solid curves are calculated for the addition/subtraction scheme; blue dashed curves are for the quantum-scissors method; square dots indicate experimental data. (C) Measured variances (corrected for the detection efficiency $\eta = 0.6$) of the amplitude and phase quadratures of the amplified coherent state and the corresponding (blue solid) curve for the best deterministic amplifier. The right panels show contour plots of the reconstructed Wigner functions for three amplified coherent states of different amplitudes.
It is now quite instructive to compare the performances of the amplifier based on the combination of photon addition and subtraction to those of other schemes of noiseless amplification. In the experiments based on quantum scissors [8, 9], the state is truncated at Fock state $|1\rangle$, whose weight is increased so as to emulate the amplification. An output state of such amplifier corresponding to the input coherent state $|\alpha\rangle$ is thus given by

$$|\psi(\alpha)\rangle = \frac{1}{\sqrt{1 + g^2|\alpha|^2}}(0 + g|\alpha|1).$$  \hspace{1cm} (8)

The effective gain and fidelity of amplifier based on quantum scissors can be defined similarly as above, only the output state $\hat{G}|\alpha\rangle$ has to be replaced with $|\psi(\alpha)\rangle$,

$$g_{\text{eff, QS}} = \frac{1}{\alpha}\langle\psi(\alpha)|\hat{a}|\psi(\alpha)\rangle, \quad F_{\text{QS}} = |\langle g\alpha|\psi(\alpha)\rangle|^2. \hspace{1cm} (9)$$

On inserting the state (8) into these formulas we obtain

$$g_{\text{eff, QS}} = \frac{g}{1 + g^2|\alpha|^2}, \hspace{1cm} (10)$$

and

$$F_{\text{QS}} = (1 + g^2|\alpha|^2)e^{-g^2|\alpha|^2}. \hspace{1cm} (11)$$

For $g = 2$, we obtain from Eqs. (4), (9), (10) and (11) the following expressions,

$$g_{\text{eff}} = 1 + \frac{1 + |\alpha|^2}{1 + 3|\alpha|^2 + |\alpha|^4}, \quad g_{\text{eff, QS}} = \frac{2}{1 + 4|\alpha|^2}, \hspace{1cm} (12)$$

and

$$F = \frac{(1 + 2|\alpha|^2)e^{-|\alpha|^2}}{1 + 3|\alpha|^2 + |\alpha|^4}, \quad F_{\text{QS}} = (1 + 4|\alpha|^2)e^{-4|\alpha|^2}. \hspace{1cm} (13)$$

The effective gain and fidelity of the amplifier based on quantum scissors for a nominal gain $g = 2$ are also plotted in Figs. 3A and 3B. Both quantities decrease with increasing $|\alpha|$ but the amplifier based on the combined photon addition and subtraction greatly outperforms the one based on quantum scissors, also in terms of equivalent input noise. In fact, the scissors-based amplifier can even lead to effective attenuation, $g_{\text{eff}} < 1$, because the very crude state truncation becomes the dominant effect as soon as the condition $|g\alpha|^2 \ll 1$ is not satisfied. The fidelity of the amplification achieved with our approach is also much better than the one achievable by the scheme based on thermal noise addition and single-photon subtraction proposed in [33]. This scheme is unavoidably limited in its performance and can exhibit high fidelity only in the regime where the effective gain drops very quickly with increasing $|\alpha|$. Although still able to conditionally improve phase estimation, the thermal addition has the detrimental side-effect of significantly reducing the purity of the amplified state, whereas our scheme is in principle able to preserve the unit purity of the input coherent states.

Among other applications, noiseless amplification can enhance the performance of state-discrimination and phase-estimation schemes. In particular, consider a protocol where a unitary transformation $|\alpha\rangle \rightarrow |\alpha e^{i\theta}\rangle$ imprints information about phase shift $\theta$ onto the phase quadrature $p$ that is measured by a balanced homodyne detector. We have $\langle p \rangle = 2|\alpha|\sin \theta$, and for small $\theta$ we may construct an estimator $\hat{\theta}_{\text{est}} = \frac{\langle p \rangle}{2|\alpha|}$ whose variance is inversely proportional to the total mean number of photons in the probe coherent state, $V(\hat{\theta}_{\text{est}}) = \frac{1}{g^2|\alpha|^2}$, which is the well-known standard quantum limit \[20, 21\]. If the coherent state is noiselessly amplified before detection, the variance of $\hat{\theta}_{\text{est}}$ is conditionally reduced by a factor $R_V = g_{\text{eff}}^2((\Delta p_{\text{amp}})^2)/(\Delta p_{\text{in}}^2)$. For a perfect noiseless amplifier one gets $R_V = g_{\text{eff}}^2$. Experimental values of $R_V = 0.45, 0.64, 0.76$ for $|\alpha| = 0.4, 0.7, 1.0$, respectively, indicate the clear improvement in phase estimation achieved with the present scheme. The state-discrimination and phase-estimation ability of our amplifier is further illustrated in Fig. 4, where the Wigner function of an incoherent mixture of two coherent states with the same amplitude $|\alpha| = 1.0$ and a $\pi/2$ phase offset is shown before and after noiseless amplification by the photon addition and subtraction scheme. The effect of our high-fidelity noiseless amplifier is that of allowing a clear discrimination and a much better phase estimation for the states that were almost totally overlapped before amplification.

We anticipate numerous applications of the demonstrated noiseless amplifier in quantum information processing and quantum metrology. It can compensate for losses in quantum communication schemes and can

![FIG. 4: Experimental Wigner functions for an incoherent mixture of $|\alpha\rangle$ and $|\alpha\rangle$ before (front) and after (rear) amplification, with $|\alpha| = 1.0$. The equal-weight incoherent mixtures are simulated by summing the experimentally-reconstructed Wigner functions and those obtained by imposing them a $\pi/2$ phase offset.](image-url)
be used to distill and concentrate entanglement\cite{5,12}. Since it preserves quantum coherence it could be used for breeding small cat-like states of the form $|\alpha\rangle \pm |\alpha\rangle$. As clearly shown above, it can improve the performance of phase-estimation schemes \cite{13} and enable high-fidelity probabilistic cloning and discrimination of coherent states. Moreover, a fully tunable amplification gain can be achieved with an extended interferometric version of the present setup that can also emulate Kerr nonlinearity\cite{12}. The present approach to high-fidelity noiseless amplification, largely outperforming concurrent schemes, will certainly represent an essential tool for the emerging quantum technologies.

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