Compatible Poisson Structures
of Toda Type Discrete Hierarchy

HENRIK ARATYN\textsuperscript{1} AND KLAUS BERING\textsuperscript{2}

Department of Physics
University of Illinois at Chicago
845 W. Taylor St.
Chicago, Illinois 60607-7059

Abstract

An algebra isomorphism between algebras of matrices and difference operators is used to investigate the discrete integrable hierarchy. We find local and non-local families of $R$-matrix solutions to the modified Yang-Baxter equation. The three $R$-theoretic Poisson structures and the Suris quadratic bracket are derived. The resulting family of bi-Poisson structures include a seminal discrete bi-Poisson structure of Kupershmidt at a special value.

Keywords: Integrable Systems, Classical R-Matrix, Discrete Toda Lattice, Compatible Poisson Brackets.

PACS number(s): 02.10.Sp, 02.10.Tq, 03.20.+i, 04.60.Nc, 11.30.-j, 11.30.Na.

1 Introduction: A Matrix Formulation of the Lattice Hierarchy

A popular framework for dealing with infinite lattice systems is the differential-difference calculus \cite{Pi} based on a Lax difference operator

\[ L = \sum_{k \in \mathbb{Z}} u_k(x) \Delta^k, \] (1.1)

where $\Delta$ is the translation operator $\Delta^k f(x) = f(x+k) \Delta^k$, or equivalently $[\Delta, x] = \Delta$. In these cases, the dynamics of the field $u_k(x)$ are governed by a Lax equation

\[ \frac{\partial L}{\partial t_n} = [P_+ (L^n), L], \quad n \geq 0, \] (1.2)

where $P_+$ is a projection-like operator, whose detailed form is discussed in Section 1.2. It is one of the hallmarks of the discrete hierarchy that the dynamical field $u_k(x)$ only interacts

\textsuperscript{1}E-mail:aratyn@uic.edu
\textsuperscript{2}E-mail:bering@uic.edu
with itself in points that belong to the same (affine) integer lattice \( \mathbb{Z} + x \). Therefore, if one assumes that the Hamiltonians \( \mathcal{P}_+(L^n) \) do not have explicit \( x \)-dependence, one may ignore the fractional part of the space coordinate \( x \), as it only labels isomorphic non-interacting substructures of the discrete hierarchy. In other words, it is legitimate to consider the space variable \( x \) to live on the set \( \mathbb{Z} \) of integer numbers rather than the whole continuous space \( \mathbb{R} \). This trivial fact reduces the discrete hierarchy to an infinite, but countable, matrix problem. The matrix picture becomes even clearer if one writes the dynamical field \( u_k(x) \) as a matrix \( u_{ij}, i, j \in \mathbb{Z} \):

\[
u_k(x) = u_{x,x+k} = u(x, x+k) \quad \text{or} \quad u_{ij} = u_{j-i}(i) .
\]

It is convenient to call \( u_k(x) \equiv u_{x,x+k} \) a link from a lattice point \( x \in \mathbb{Z} \) to a lattice point \( x + k \in \mathbb{Z} \), and to call the integer \( k \in \mathbb{Z} \) the (signed) length of the link. Translating the difference operator \( L \) into a matrix \( u = (u_{ij}) \) turns out to be very fruitful, partially because this provides a better geometric and algebraic understanding. For illustrative purposes, consider the seminal bi-Poisson structure of Ref. \([1]\), here extended to both positive and negative link lengths

\[
\{u_n(x), u_m(y)\}_{K1} = \frac{1}{2}\varepsilon(n-\frac{1}{2}) + \varepsilon(m-\frac{1}{2}) [u_{n+m}(x) \delta_{x+n,y} - u_{n+m}(y) \delta_{x+y+m}] ,
\]

\[
\{u_n(x), u_m(y)\}_{K2} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \varepsilon(k) + \varepsilon(k+m-n) u_{n-k}(y) u_{m+k}(x) \delta_{x+k,y}
+ \frac{1}{2} u_n(x) u_m(y) \sum_{k \in \mathbb{Z}} \varepsilon(k) + \varepsilon(k+m-n)
- \varepsilon(k+m-\frac{1}{2}) - \varepsilon(k-n+\frac{1}{2}) \delta_{x+k,y} , \quad n, m \in \mathbb{Z} ,
\]

and where \( \varepsilon \) denotes the sign function. Notice that the Kupershmidt Poisson brackets \( \{\cdot, \cdot\}_{K1} \) and \( \{\cdot, \cdot\}_{K2} \) are non-local in the \( x \) and \( y \) coordinates. It is quite elaborate to verify by brute force that \( \{\cdot, \cdot\}_{K1} \) and \( \{\cdot, \cdot\}_{K2} \) are a bi-Poisson structure, not to mention identifying allowed deformations of this. We shall see that the non-locality is an artifact of the difference operator language, and that the matrix picture, combined with classical \( R \)-matrix theory, provides an elegant and effective formalism to deal with such bi-Poisson structures.

The outline of the paper is as follows. We continue this Section \([1]\) with describing the connection between infinite matrices and difference operators. This provides a useful connection between the Lax formulations of the two pictures. In Section \([2]\) we revisit classical \( R \)-matrix theory for the matrix formulation, where in particular the conditions for integrability are given in detail. We provide local and non-local classes of solutions to these conditions, that are useful for model building. In Section \([3]\) the Lax equations of motion are recast as the Hamiltonian equations of motion by finding viable Poisson brackets, that gives rise to bi-Poisson structures. It is natural in this connection to briefly review the three \( R \)-theoretic Poisson brackets and the Suris quadratic bracket construction \([2]\). As a new result, we show that the Suris bracket \( \{\cdot, \cdot\}_{s2} \) decomposes into the second \( R \)-theoretic bracket \( \{\cdot, \cdot\}_{r2} \) and a piece \( \{\cdot, \cdot\}_{\Omega2} \), that plays no role for on-shell dynamics. Our main results for the bracket structure are given in eq. \([3.21]-[3.23]\) and eq. \([3.36a]-[3.38]\). In Appendix \([A]\) we translate
our Poisson bracket results into the notation of Kupershmidt \[1\] for comparison. From a $R$-theoretic perspective, it is remarkable that already the original discrete hierarchy of Kupershmidt induces the generalized construction of Suris. Finally in Appendix $\text{B}$ we give a $2 \times 2$ dimensional example that illustrates aspects of the Suris theory.

1.1 An Algebra Isomorphism

We consider an algebra isomorphism from the associative algebra $\mathcal{A}$ of infinite-dimensional matrices (to be specified below) to a certain sub-algebra $\mathcal{A}_\Delta$ of difference operators,

$$\mathcal{A} \ni u \mapsto L = \sum_{k \in \mathbb{Z}} u_{x,x+k} \Delta^k \in \mathcal{A}_\Delta .$$

The algebra isomorphism maps matrix multiplication into composition of difference operators:

$$\sum_{k \in \mathbb{Z}} (uv)_{x,x+k} \Delta^k = \left( \sum_{k_1 \in \mathbb{Z}} u_{x,x+k_1} \Delta^{k_1} \right) \left( \sum_{k_2 \in \mathbb{Z}} v_{x,x+k_2} \Delta^{k_2} \right) , \quad u, v \in \mathcal{A} .$$

Geometrically, it is a useful fact that the total length $k = k_1 + k_2$ of links is preserved under the matrix multiplication/composition.

As an easy application, the matrix description provides a pictorial understanding for the coefficient functions $(L^n)_k (x)$ of the higher powers of the Lax operator

$$L^n = \sum_{k \in \mathbb{Z}} \sum_{k_1, \ldots, k_n \in \mathbb{Z}} u_{x,x+k_1} \Delta^{k_1} \cdots u_{x,x+k_n} \Delta^{k_n} = \sum_{k \in \mathbb{Z}} (L^n)_k (x) \Delta^k , \quad n \geq 0 .$$

They are

$$(L^n)_k (x) = \sum_{k_1, \ldots, k_n \in \mathbb{Z}} \prod_{r=1}^n u \left( x + \sum_{i=1}^{r-1} k_i, x + \sum_{i=1}^r k_i \right) = (u^n)_{x,x+k} ,$$

or in words, the $k$’th coefficient of the $n$’th power of the Lax operator corresponds to $n$ consecutive links with the two free ends a distance $k$ apart. Therefore,

$$L^n = \sum_{k \in \mathbb{Z}} (u^n)_{x,x+k} \Delta^k , \quad n \geq 0 .$$

A difference operator, that has all the $x$’s appearing to the left of all $\Delta$’s, is called a normal ordered difference operator.$^3$

Let us now define the matrix algebra $\mathcal{A}$ itself. For simplicity, let $\mathcal{A}$ be the algebra of matrices with only finitely many non-zero entries

$$\mathcal{A} := \{ u = (u_{ij}) | u_{ij} = 0 \text{ for } |i| + |j| \gg 0 \} .$$

$^3$Ref. $\text{II}$ uses anti-normal ordering, with all the $\Delta$’s appearing to the left of all $x$’s.
This choice ensures that the matrix multiplication and the matrix trace (tr) are well-defined operations.4

### 1.2 Lax Formulation

In this Section, we return to the Lax equation (1.2) and implement the corresponding matrix formulation, which appears naturally in the theory of classical $R$-matrices (cf. the next Section 2). The Lax equation for $L$ can be written in several ways

$$
\frac{\partial L}{\partial t_n} = [\mathcal{P}_+(L^n), L] = [L, \mathcal{P}_-(L^n)] = \frac{1}{2}[\mathcal{R}(L^n), L], \quad L \in \mathcal{A}_\Delta, \quad n \geq 0, \quad (1.11)
$$

for the operators $\mathcal{P}_\pm$ and $\mathcal{R}$ related through

$$
\mathcal{P}_+ + \mathcal{P}_- = 1 \quad \text{and} \quad \mathcal{R} = \mathcal{P}_+ - \mathcal{P}_-. \quad (1.12)
$$

Although we are going to consider different examples of the operator triples $(\mathcal{P}_+, \mathcal{P}_-, \mathcal{R})$, we will always assume that the three operators $\mathcal{P}_+, \mathcal{P}_-$ and $\mathcal{R}$ are interlocked via the above two relations (1.12). Hence it is always enough to specify one of them. As an example, consider Hamiltonians of the form

$$
\mathcal{P}_+(L^n) := (L^n)_{\geq 0} + \frac{\nu - 1}{2}(L^n)_0, \quad n \geq 0, \quad (1.13)
$$

where $\nu$ is a constant parameter \[3\], which is related to a choice of operator ordering prescription. The choice $\nu = 1$ leads to the standard hierarchy with the Hamiltonians given by $(L^n)_{\geq 0}$, while $\nu = -1$ leads to the so-called modified hierarchy with the Hamiltonians $(L^n)_{\geq 1}$.

To implement the matrix program, one seeks the matrix counterparts $R, P_\pm : \mathcal{A} \rightarrow \mathcal{A}$ of the operators $\mathcal{R}, \mathcal{P}_\pm : \mathcal{A}_\Delta \rightarrow \mathcal{A}_\Delta$. They satisfy

$$
P_+ + P_- = 1 \quad \text{and} \quad R = P_+ - P_. \quad (1.14)
$$

as well. Inspired by eq. (1.9), we claim that the sought-for identification is provided by

$$
\mathcal{R}(L^n) = \sum_{k \in \mathbb{Z}} (R(u^n))_{x,x+k} \Delta^k, \\
\mathcal{P}_\pm(L^n) = \sum_{k \in \mathbb{Z}} (P_\pm(u^n))_{x,x+k} \Delta^k, \quad n \geq 0. \quad (1.15)
$$

It may look discouraging that (for instance) the $P_+$ operator does not act on the $\Delta$-part at all, but only on the $u$-part, as one usually counts the power $k$ in the $\Delta^k$-factor to determine

---

4 If one forgets the associative matrix multiplication structure, but keeps the Lie commutator operation, $\mathcal{A}$ is often referred to as the Lie algebra $\text{gl}(\infty)$ in the mathematical literature. Also we stress that the above $\mathcal{A}$ has no identity matrix and no invertible matrices in this infinite dimensional case. At a few places in the paper we refer to invertible matrices and to make rigorous sense of this, $\mathcal{A}$ should be chosen as a Banach algebra with an algebra norm. It is out of scope to provide details here.

4
the action of $\mathcal{P}_+$ (cf. eq. (1.13)). However, one should recall that $k$ is also available in the $u$-part as a link length.

One can now lift the Lax equation (1.11) to the corresponding matrix algebra $A$:

$$\frac{\partial u}{\partial t_n} = [P_+(u^n), u] = [u, P_-(u^n)] = \frac{1}{2}[R(u^n), u], \quad u \in A, \quad n \geq 0.$$  

(1.16)

In fact, the equivalence of the Lax eq. (1.11) for the difference operator $L \in A_\Delta$ and the Lax eq. (1.16) for the matrix $u \in A$ follows straightforwardly from the algebra isomorphism (1.6) and the prescription of eq. (1.15).

Let us work out the corresponding matrix maps $R, P_\pm : A \to A$ of the example eq. (1.13) given above. To this end, we need to introduce some notation. Let $e_{ij} \in A$ denote an elementary matrix such that $(e_{ij})_{kl} = \delta_{i,k} \delta_{j,l}$. As is well-known, the $e_{ij}$'s constitute a standard basis for the matrix algebra $A$, and hence a generic algebra element $u$ can be decomposed as $u = \sum_{i,j \in \mathbb{Z}} u_{ij} e_{ij} \in A$. Similarly, one only have to determine the linear maps $R, P_\pm : A \to A$ on the basis $e_{ij}$. The conversion prescription of eq. (1.15) is satisfied if one let

$$R(e_{ij}) = \varepsilon_\nu(j-i) e_{ij},$$
$$P_+(e_{ij}) = \theta_\nu(j-i) e_{ij},$$
$$P_-(e_{ij}) = \theta_\nu(i-j) e_{ij},$$

(1.17)

where we have defined a sign function $\varepsilon_\nu(x)$ as

$$\varepsilon_\nu(x) := \begin{cases} 
1 & \text{for } x > 0 \\
\nu & \text{for } x = 0 \\
-1 & \text{for } x < 0 
\end{cases}$$

(1.18)

and a corresponding step function

$$\theta_\nu(x) := \frac{1}{2}[1 + \varepsilon_\nu(x)] = \begin{cases} 
1 & \text{for } x > 0 \\
\frac{1 + \nu}{2} & \text{for } x = 0 \\
0 & \text{for } x < 0 
\end{cases}$$

(1.19)

2 $R$-Matrix Formalism

The classical $R$-matrix theory provides a universal method for constructing three compatible Poisson structures and infinitely many commuting charges for a wide class of integrable models [4]. A classical $R$-matrix is by definition a linear map $R : A \to A$ such that the $R$ bracket

$$[u, v]_R := \frac{1}{2}[R(u), v] + \frac{1}{2}[u, R(v)]$$

(2.1a)

$$= [P_+(u), P_+(v)] - [P_-(u), P_-(v)],$$

(2.1b)

5See also e.g. [2, 5, 6, 7, 8, 9]. In [10] an extension of the $R$-formalism to the fermionic Toda model is given.
is a Lie-bracket, \textit{i.e.} it satisfies the Jacobi identity. A sufficient condition for the Jacobi identity is provided by the modified Yang-Baxter equation $Y_{B\alpha}(u,v) = 0$, where the modified Yang-Baxter operator is given by

$$Y_{B\alpha}(u,v) := [R(u), R(v)] - 2R[u, v]_R + \alpha[u, v]. \quad (2.2)$$

As we already have seen in the previous Section, it is convenient to define projection-like operators $P_{\pm} : A \to A$:

$$P_{\pm} := \frac{1}{2}(1 \pm R), \quad R = P_+ - P_- . \quad (2.3)$$

We emphasize that in general $\text{Im}(P_+)$ and $\text{Im}(P_-)$ do not form a direct sum, and $P_{\pm}$ are not necessarily idempotent operators.

It is instructive to see how the integrable model arises. Consider an \textit{abelian} subalgebra $A_0 \subseteq A$. (Usually we simply consider the infinite hierarchy $A_0 \supseteq \{ u^n \mid n = 1, 2, \ldots \}$ generated by a single algebra element $u \in A$.) The dynamical flow

$$\delta_v u = [P_+(v), u] = [u, P_-(v)] = \frac{1}{2}[R(v), u] , \quad u, v \in A_0 , \quad (2.4)$$

is generated by a Hamiltonian $P_+(v)$. After some straightforward algebra, the commutator of two flows reads

$$[\delta_w, \delta_v] u = \frac{1}{4}[N(v,w), u] , \quad u, v, w \in A_0 , \quad (2.5)$$

where we have defined the Nijenhuis tensor \cite{11}

$$\frac{1}{4}N(u,v) = \begin{array}{c} N_+(u,v) + N_-(u,v) \\ = [P_+(u), P_+(v)] + P_\pm[u, v]_R \end{array} \quad (2.6a)$$

and the chiral Nijenhuis tensors

$$N_{\pm}(u,v) := P_\pm[P_+(u), P_\pm(v)] . \quad (2.7)$$

The Nijenhuis tensor is equal to the modified Yang-Baxter operator

$$N(u,v) = [u,v] + [R(u), R(v)] - 2R[u, v]_R = Y_{B1}(u,v) . \quad (2.8)$$

The flows $\delta_n$, $n \geq 0$, commute for an integrable system. From eq. (2.5), a sufficient integrability condition is provided by the modified Yang-Baxter equation

$$Y_{B1}(u,v) \equiv N(u,v) = 0 , \quad (2.9)$$

which our examples in Sections 2.1, 2.2 will satisfy. More generally, integrability is guaranteed if there exists a linear operator $B : A \to A$, such that

$$N(u,v) = B[u, v] , \quad (2.10)$$
as can easily be checked from eq. (2.5). When we assume a vanishing Nijenhuis tensor $N = 0$, it follows from eq. (2.6b), that the $\pm P_\pm$ operators are Lie-algebra homomorphisms $(\mathcal{A}, [\cdot, \cdot]_R) \to (\mathcal{A}, [\cdot, \cdot])$, and in particular that the images $\text{Im}(P_\pm)$ are two Lie sub-algebras:

$$[\text{Im}(P_\pm), \text{Im}(P_\pm)] \subseteq \text{Im}(P_\pm).$$  \hspace{1cm} (2.11)

The vanishing of the chiral Nijenhuis tensors

$$N_+ = 0 \quad \text{and} \quad N_- = 0 \hspace{1cm} (2.12)$$

implies the vanishing of the Nijenhuis tensor $N = YB_1 = 0$ (cf. eq. (2.6a)). The opposite statement is not true. The local and non-local examples in the next Sections 2.1-2.2 will meet the stronger condition (2.12).

Both the vanishing Nijenhuis tensor condition (2.9) and the vanishing chiral Nijenhuis tensor condition (2.12) are stable under conjugation of the $R$-matrix $R'(u) = aR(a^{-1}ua)a^{-1}$ with an invertible algebra element $a$. The conjugation procedure can be used to generate new $R$-solutions from old $R$-solutions, although we will not pursue this here (cf. footnote 4).

### 2.1 A Class of Local $R$-matrix Solutions

Here we propose a class of local solutions $R, P_\pm : \mathcal{A} \to \mathcal{A}$ to the condition eq. (2.12) that is parametrized by an arbitrary function $\nu : \mathbb{Z} \to \mathbb{C}$ and that generalizes (1.17). It is given by

$$R(e_{ij}) = \mathcal{E}_\nu(j, i) \ e_{ij}$$

$$P_+(e_{ij}) = \Theta_\nu(j, i) \ e_{ij}$$

$$P_-(e_{ij}) = \Theta_{-\nu}(i, j) \ e_{ij},$$  \hspace{1cm} (2.13)

where we have defined a generalized sign function (involving now $\nu = \nu(x)$ being a local function)

$$\mathcal{E}_\nu(x, y) := \begin{cases} 1 & \text{for } x > y \\ \nu(x) & \text{for } x = y \\ -1 & \text{for } x < y \end{cases} \hspace{1cm} (2.14)$$

and a corresponding step function

$$\Theta_\nu(x, y) := \frac{1}{2} [1 + \mathcal{E}_\nu(x, y)] = \begin{cases} 1 & \text{for } x > y \\ \frac{1 + \nu(x)}{2} & \text{for } x = y \\ 0 & \text{for } x < y \end{cases} \hspace{1cm} (2.15)$$

The choice of $\nu : \mathbb{Z} \to \mathbb{C}$ is related to a ($x$-local) choice of operator ordering prescription. Here we refer to a $R$-solution as being local if the standard basis $e_{ij}$ diagonalizes the $R$-matrix. (We emphasize that a local solution usually becomes non-local in terms of the difference operator fields $u_k(x)$.) We prove in the next Section 2.2 that the chiral Nijenhuis tensors vanish (cf. eq. (2.12)), so that $R$ satisfies the modified Yang-Baxter equation $YB_1(R) = 0$. 

7
The operators $P_+$ and $P_-$ from eq. (2.13) “project” onto (weakly) upper or (weakly) lower triangular matrices, respectively, but they may share diagonal matrices unless $\nu(x) = \pm 1$:

$$\text{Im}(P_+) \cap \text{Im}(P_-) \subseteq \{0\} \Leftrightarrow \text{Im}(\nu) \subseteq \{\pm 1\}. \quad (2.16)$$

Thus in general, the sub-algebras $\text{Im}(P_+)$ and $\text{Im}(P_-)$ do not form a direct sum. Similarly, the operators $P_+$ and $P_-$ are idempotent ($P_+^2 = P_+$) if and only if the $R$-matrix is an involution ($R^2 = 1$), which holds precisely when $\nu(x) = \pm 1$, as expressed by

$$4 P_+ P_- = 1 - R^2 = (1 - \nu^2) \delta, \quad (2.17)$$

and where $\delta : A \rightarrow A$ projects onto the diagonal matrices

$$\delta(e_{ij}) := \delta_{i,j} e_{ij}, \quad \delta^2 = \delta, \quad R \delta = \nu \delta = \delta R. \quad (2.18)$$

### 2.2 A Class of Non-Local $R$-Matrix Solutions

There is a non-local generalization \cite{4} of the solutions in (2.13) that reads as

$$R(e_{ij}) = \varepsilon(j-i) e_{ij} + \delta_{i,j} \sum_{m \in \mathbb{Z}} \nu_{i,m} e_{mm}$$

$$P_+(e_{ij}) = \theta(j-i) e_{ij} + \frac{1}{2} \delta_{i,j} \sum_{m \in \mathbb{Z}} \nu_{i,m} e_{mm} \quad (2.19)$$

$$P_-(e_{ij}) = \theta(i-j) e_{ij} - \frac{1}{2} \delta_{i,j} \sum_{m \in \mathbb{Z}} \nu_{i,m} e_{mm}.$$

The diagonal case $\nu_{i,j} = \nu_i \delta_{i,j}$ corresponds to the previous local solution. We claim that the non-local $R$-matrix possesses vanishing chiral Nijenhuis tensors (cf. eq. (2.12)). To prove this, it is enough to consider $N_\pm(e_{ij}, e_{kl})$ for two basis elements $e_{ij}$ and $e_{kl}$. $P_+$ and $P_-$ “project” onto links with weakly positive and weakly negative link length, respectively. The Lie-bracket preserves the total link length. So to give a non-zero contribution to $N_\pm(e_{ij}, e_{kl})$ both entries $e_{ij}$ and $e_{kl}$ have to be zero-length links. But the zero-length links are nothing but the diagonal matrices and those commute trivially.

In the non-local case, the sub-algebras $\text{Im}(P_+)$ and $\text{Im}(P_-)$ form a direct sum if and only if the matrix $\nu_{i,j}$ is an involution:

$$\text{Im}(P_+) \cap \text{Im}(P_-) \subseteq \{0\} \Leftrightarrow \nu^2 = 1. \quad (2.20)$$

### 2.3 The $R$-Bracket

For the local and non-local solutions (2.13) and (2.19), the $R$-matrix $R = R^{(0)} + R^{(1)}$ is a linear function of $\nu$, where the superscript “(0)” and “(1)” refer to the power of $\nu$. The $R$-bracket $[\cdot, \cdot]_R$ inherits this linear $\nu$-dependence, and can be split accordingly

$$[\cdot, \cdot]_R = [\cdot, \cdot]^{(0)}_R + [\cdot, \cdot]^{(1)}_R \quad (2.21)$$
into two mutually compatible Lie-brackets $[\cdot, \cdot]^{(0)}_R$ and $[\cdot, \cdot]^{(1)}_R$. By definition, the $R$-bracket is a Lie pencil in $\nu$. We have

$$[e_{ij}, e_{kl}]_R = \frac{1}{2} [\mathcal{E}_\nu(j, i) + \mathcal{E}_\nu(l, k)] (\delta_{j,k} e_{i,l} - \delta_{i,l} e_{k,j})
= [\Theta_\nu(j, i) \Theta_\nu(l, k) - \Theta_\nu(i, j) \Theta_\nu(k, l)] (\delta_{j,k} e_{i,l} - \delta_{i,l} e_{k,j})$$
(2.22)

for the local $R$-matrix (2.13). This is a $\text{gl}(\infty)$ Lie algebra $[e_{ij}, e_{kl}] = \delta_{j,k} e_{i,l} - \delta_{i,l} e_{k,j}$ with a 2-cocycle-like prefactor. For the non-local $R$-matrix (2.19), only the first-order contribution in $\nu$ is changed. It reads

$$[e_{ij}, e_{kl}]^{(1)}_R = \frac{1}{2} \delta_{i,j} (\nu_{j,k} - \nu_{i,l}) e_{kl} + \frac{1}{2} \delta_{k,l} (\nu_{k,j} - \nu_{l,i}) e_{ij} .$$
(2.23)

It is a curious fact that links $e_{mm}$ of zero-length can never be produced in a $R$-bracket $[\cdot, \cdot]_R$ of the local or non-local type (cf. eqs. (2.13) and (2.19)). Specifically,

$$\text{tr} (e_{mm}[u, v]_R) = 0 .$$
(2.24)

In contrast, the same does not hold for the standard $\text{gl}(\infty)$ Lie-bracket $[\cdot, \cdot]$, where for instance $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}$ yields two zero-length links if $i \neq j$.

### 2.4 Equations of Motion and Time Evolution

It is interesting to write out the equations of motion in coordinates. If we insert the local solutions (2.13) for $R$, $P_+$ and $P_-$ into the Lax eq. (1.16), we get

$$\frac{\partial u_{ij}}{\partial t} = \frac{1}{2} \sum_{k \in \mathbb{Z}} [\mathcal{E}_\nu(k, i) (u^n)_{ik} u_{kj} - u_{ik} \mathcal{E}_\nu(j, k) (u^n)_{kj}]$$
(2.25a)

$$= \sum_{k \in \mathbb{Z}} [\Theta_\nu(k, i) (u^n)_{ik} u_{kj} - u_{ik} \Theta_\nu(j, k) (u^n)_{kj}]$$
(2.25b)

$$= \sum_{k \in \mathbb{Z}} [u_{ik} \Theta_\nu(k, j) (u^n)_{kj} - \Theta_\nu(i, k) (u^n)_{ik} u_{kj}] ,$$
(2.25c)

respectively. The non-local generalization of eq. (2.25a) reads

$$\frac{\partial u_{ij}}{\partial t} = \frac{1}{2} \sum_{k \in \mathbb{Z}} [\varepsilon(k - i) (u^n)_{ik} u_{kj} - u_{ik} \varepsilon(j - k) (u^n)_{kj}]$$
(2.26)

$$+ \frac{1}{2} u_{ij} \sum_{k \in \mathbb{Z}} (\nu_{k,i} - \nu_{k,j})(u^n)_{kk} .$$

The non-local generalizations of eqs. (2.25b) and (2.25c) are similar.
2.5 Lattice Truncations

We next address the question whether we can constrain the dynamical fields $u_k(x)$ without violating the equations of motion? As is easily seen in the matrix formalism, it is consistent with the equations of motion \[ \text{(2.26)} \] to deploy “rectangular” type of truncations of the matrix algebra $A$ to a sub-algebra

$$ A_I := \{ u = (u_{ij}) \in A \mid u_{ij} \neq 0 \Rightarrow i, j \in I \} \text{ (2.27)} $$

for some index-set $I \subseteq \mathbb{Z}$. From a difference operator perspective, it is natural to consider “diagonal” type of truncations. Here, we discuss two “diagonal” type of truncations that are often used in applications:

- **Truncation of the link length from below:** It is consistent with the equations of motion to consider a truncated model with

$$ \forall k < N : \ u_k(x) \equiv 0. \text{ (2.28)} $$

To prove this, we note that the first (second) term on the rhs. of \[ \text{(2.25a)} \] has an index variable $k \geq i\ (k \leq j)$, so that the left hand side $\partial u_{ij}/\partial t_n$ depends at least linearly on a link $u_{kj}\ (u_{ik})$ with a signed length $j - k \leq j - i\ (k - i \leq j - i)$. So if the field $u_{ij}$ (and its fellow fields with less or equal link length) are annihilated at some time $(t_1, t_2, \ldots)$, the equations of motions \[ \text{(2.25b)} \] cannot undo that for other times.

- **Truncation of the link length from above:** A similar examination of eq. \[ \text{(2.25c)} \] shows that there is also a consistent truncation from above:

$$ \forall k > M : \ u_k(x) \equiv 0. \text{ (2.29)} $$

The two truncation schemes are also consistent with the non-local solution \[ \text{(2.26)} \], because when considering the left hand side $\partial u_{ij}/\partial t_n$, the non-local terms are always hidden behind at least one power of $u_{ij}$. By invoking both of the above truncation schemes, we get models with only a finite number of different fields $u_M(x), \ldots, u_N(x)$ with link lengths between $M$ and $N$; all located inside an infinite universal enveloping construction. This fact makes the discrete hierarchy highly accessible for applications.

3 Poisson Brackets

Before we proceed with constructing Poisson brackets, we need to introduce a few standard notions to fix the notation. A non-degenerate bilinear form

$$ \langle u, v \rangle = \text{tr}(uv) = \langle v, u \rangle \text{ (3.1)} $$

is inherited from the matrix trace (tr). Note that a bi-linear form $\langle \cdot, \cdot \rangle$, in contrast to a sesqui-linear form, has no internal transposition (or Hermitian conjugate for that matter). This is mainly to ensure that the bilinear form is invariant/associative:

$$ \langle u, [v, w] \rangle = \langle [u, v], w \rangle. \text{ (3.2)} $$
The non-degenerate bilinear form gives rise to an identification of the algebra $\mathcal{A}$ with the set $\mathcal{A}^*$ of linear functionals on $\mathcal{A}$. For a linear operator $R : \mathcal{A} \to \mathcal{A}$, the dual operator $R^* : \mathcal{A}^* \to \mathcal{A}^*$ becomes identified with the transposed operator

$$\langle v, R(u) \rangle = \langle R^*(v), u \rangle .$$

(3.3)

One may always decompose an operator $R$ in symmetric and skew-symmetric parts

$$R_\pm := \frac{R \pm R^*}{2} .$$

(3.4)

Let us note for later that the non-local $R$-solution (2.19) from Section 2.2 decomposes as

$$R_-(e_{ij}) = \varepsilon(j-i) e_{ij} + \frac{1}{2} \delta_{i,j} \sum_{m \in \mathbb{Z}} \nu_{[i,m]} e_{mm}$$

(3.5)

$$R_+(e_{ij}) = \frac{1}{2} \delta_{i,j} \sum_{m \in \mathbb{Z}} \nu_{[i,m]} e_{mm} .$$

(3.6)

Notice that the skew-symmetric part $R_-$ in this case is again a $R$-matrix, with vanishing chiral Nijenhuis tensors (cf. eq. (2.12)).

The adjoint action $\text{ad}(u) : \mathcal{A} \to \mathcal{A}$ is defined as $\text{ad}(u)v := [u, v]$. Because of the invariant/associative property of the bilinear form, the coadjoint action (from right) $\text{ad}^*(u) : \mathcal{A}^* \to \mathcal{A}^*$ is identified with minus the adjoint action

$$\langle \text{ad}^*(u)v, w \rangle = \langle v, \text{ad}(u)w \rangle = \langle v, [u, w] \rangle = - \langle [u, v], w \rangle = - \langle \text{ad}(u)v, w \rangle .$$

(3.7)

The gradient $\nabla f$ of a function $f = f(u)$ on the dual space $u \in \mathcal{A}^*$ can be defined implicitly via the infinitesimal variational formula

$$\delta f = \langle \delta u , \nabla f \rangle .$$

(3.8)

Explicitly, the gradient is

$$\nabla = \sum_{i,j \in \mathbb{Z}} e_{ij} \frac{\partial}{\partial u_{ji}} .$$

(3.9)

Notice the $i \leftrightarrow j$ transposition of indices in the above formula.

### 3.1 Conserved Charges

As is well-known, a hallmark of an integrable system is an infinity of conserved charges. In the discrete hierarchy, the charge densities are defined as

$$h_n(x) = \frac{(L^n)_0 (x)}{n} = \frac{(u^n)_{x,x}}{n} , \quad n > 0 ,$$

(3.10)
and the charges themselves are defined as
\[
H_n = \sum_{x \in \mathbb{Z}} h_n(x) = \frac{1}{n} \text{tr}(u^n), \quad n > 0 ,
\] (3.11)

where the trace in the last equation can be thought of as the sum of all closed loops build out of \(n\) consecutive links. It follows directly from the Lax eq. (1.16) and from the cyclicity of the matrix trace \(\text{tr}[u, v] = 0\), that the charges \(H_n, n > 0\), are conserved in time. Also, one observes easily that
\[
(L^n)_k(x) = (u^n)_{x,x+k} = \frac{\partial H_{n+1}}{\partial u_{x+k,x}},
\] (3.12)
or, equivalently,
\[
\nabla H_{n+1} = u^n.
\] (3.13)
The above facts are slightly more elaborate to establish purely within the difference operator approach (cf. for instance Theorem III.2.7 of Ref. [1]).

One may rewrite [4, 8] the Lax eq. (1.16) as a standard Hamiltonian equation on the dual space \(\mathcal{A}^*\)
\[
\frac{\partial u}{\partial t_n} = \frac{1}{2} [R \nabla H_{n+1}, u] = -\frac{1}{2} \text{ad}^* (R \nabla H_{n+1}) u, \quad u \in \mathcal{A}^*. \tag{3.14}
\]

It is useful to seek for Poisson bracket structures \(\{\cdot, \cdot\}_p, p = 1, 2, \ldots, n + 1\), such that the time evolutions can be reproduced as Hamiltonian flows with the conserved charges \(H_{n+2-p}\) acting as generators
\[
\{f(u), H_{n+2-p}\}_p \overset{?}{=} \frac{\partial f}{\partial t_n} \overset{3.8}{=} \left( \frac{\partial u}{\partial t_n}, \nabla f \right) \overset{1.16}{=} \frac{1}{2} \left( [R(u^n), u], \nabla f \right). \tag{3.15}
\]

In other words, find Poisson brackets such that the Lax eq. (1.16) can be written as the Hamiltonian eq. (3.15). This will ensure the Lenard relations
\[
\{ \cdot, H_{n+1}\}_p = \{ \cdot, H_{n}\}_{p+1}, \quad n, p > 0 . \tag{3.16}
\]

Note that the \(H_n\)'s play a double role as both Hamiltonian generators and as conserved quantities. This forces them to mutually “commute” in a Poisson sense:
\[
\{ H_n, H_m \}_p = 0, \quad n, m, p > 0 . \tag{3.17}
\]

For the local and the non-local solution discussed in Sections 2.1-2.2 each Poisson bracket \(\{\cdot, \cdot\}_p\) will be a linear function of \(\nu\)-parameter
\[
\{\cdot, \cdot\}_p = \{\cdot, \cdot\}^{(0)}_p + \{\cdot, \cdot\}^{(1)}_p \tag{3.18}
\]
of two mutually compatible Poisson brackets \(\{\cdot, \cdot\}^{(0)}_p\) and \(\{\cdot, \cdot\}^{(1)}_p\). This is sometimes referred to as a Poisson pencil in \(\nu\).
3.2 1st Poisson Bracket

We now derive the first Poisson bracket from the $R$-matrix formalism. From eq. (3.15) and the Lax eq. (1.16), we get

$$\{f, H_{n+1}\} = \partial f \partial t_n = \langle \partial u \partial t_n, \nabla f \rangle$$

$$= \frac{1}{2} \langle [R(u^n), \nabla f] \rangle + \frac{1}{2} \langle [u^n, R(\nabla f)] \rangle$$

$$= \frac{1}{2} \langle u, [\nabla f, R(u^n)] \rangle + \frac{1}{2} \langle u, [R(\nabla f), u^n] \rangle$$

$$= \langle u, [\nabla f, \nabla H_{n+1}] \rangle ,$$

(3.19)

where the second term (which is identically zero) was added to achieve the required skewsymmetry. We immediately recognize the first $R$-theoretic Poisson structure

$$\{f, g\}_R := \langle u, [\nabla f, \nabla g]_R \rangle .$$

(3.20)

For the local $R$-matrix solution (2.13), this leads to

$$\{u_{ij}, u_{kl}\}_R = \langle u, [e_{ji}, e_{lk}]_R \rangle = \frac{1}{2} [\mathcal{E}_\nu(i, j) + \mathcal{E}_\nu(k, l)] (\delta_{i,l} u_{kj} - \delta_{j,k} u_{il}) .$$

(3.21)

The local bracket $\{\cdot, \cdot\}_R$ has a pictorial interpretation as a concatenation of two “incoming” links with a 2-cocycle-like prefactor. For a non-local $\nu$, the first-order contribution in $\nu$ is

$$\{u_{ij}, u_{kl}\}_R^{(1)} = \frac{1}{2} \delta_{i,j} (\nu_{i,l} - \nu_{j,k}) u_{kl} + \frac{1}{2} \delta_{k,l} (\nu_{i,l} - \nu_{k,j}) u_{ij} .$$

(3.22)

The bracket operation $\{\cdot, \cdot\}_R^{(1)}$ preserves at least one of the “incoming” links $u_{ij}$ and $u_{kl}$ (up to an overall factor). To have a non-vanishing “outgoing” link, say $u_{ij}$, the other “incoming” link $u_{kl}$ should have zero link length, and it should “interact at a distance” with an endpoint of the first “incoming” link. The interaction at a distance is mediated through a non-vanishing, non-local matrix element $\nu_{n,m}$. In the difference operator language, the local bracket transforms into

$$\{u_n(x), u_m(y)\}_R = \frac{1}{2} [\varepsilon(n) - \nu_x \delta_{n,0} + \varepsilon(n) - \varepsilon(m) - \nu_y \delta_{m,0}]$$

$$\times [u_{n+m}(x) \delta_{x+n,y} - u_{n+m}(y) \delta_{x,y+m}] .$$

(3.23)

3.3 Higher Poisson Brackets

We now generalize the method of Section 3.2 to the higher brackets. Postponing the question of the Jacobi identity, let us tentatively write down skewsymmetric candidates for the $(p +
1)th bracket structure \( \{ \cdot, \cdot \}_{p+1,r} \), labeled by an integer \( r = 0, 1, \ldots, p \). From the eq. (3.15) and the Lax eq. (1.16), we get

\[
\{ f, H_{n+1-p} \}_{p+1,r} = \frac{\partial f}{\partial t_n} = \langle \frac{\partial u}{\partial t_n}, \nabla f \rangle
\]

\[
= \frac{1}{2} \langle [R(u^n), u], \nabla f \rangle + \frac{1}{2} \langle [u^{n-p}, u], R(u^r (\nabla f) u^{p-r}) \rangle
\]

\[
= \frac{1}{2} \langle u, [\nabla f, R(u^r u^{n-p} u^{p-r})] \rangle + \frac{1}{2} \langle u, [R(u^r (\nabla f) u^{p-r}), u^{n-p}] \rangle
\]

\[
= \frac{1}{2} \langle u, [\nabla f, R(u^r (\nabla H_{n+1-p}) u^{p-r})] \rangle + \frac{1}{2} \langle u, [R(u^r (\nabla f) u^{p-r}), \nabla H_{n+1-p}] \rangle ,
\]

where the second term (which is identically zero) was added to achieve the required skew-symmetry. In this way we obtain the \( r \)'th candidate for the \((p+1)\)'th \( R \)-matrix Poisson structure

\[
\{ f, g \}_{p+1,r} := \frac{1}{2} \langle u, [\nabla f, R(u^r (\nabla g) u^{p-r})] \rangle + \frac{1}{2} \langle u, [R(u^r (\nabla f) u^{p-r}), \nabla g] \rangle .
\]

The eq. (3.24) is an inhomogeneous linear equation in the Poisson structure, with source terms generated from the Lax eq. (1.16). We may not have properly identified possible homogeneous Poisson bracket parts \( \{ \cdot, \cdot \}_{p+1} \) that commute with all the charges \( H_n, n > 0 \). Besides these homogeneous contributions, the potential bracket candidates are convex linear combinations of the basis brackets \( \{ \cdot, \cdot \}_{p+1,r}, r = 0, 1, \ldots, p \). Again we stress that most of brackets are going to be discarded, as they will not satisfy the Jacobi identity.

### 3.4 2nd Poisson Bracket

As we saw in the last Section, the potential bracket candidates for a quadratic bracket include the convex linear combinations of the two basis elements \( \{ \cdot, \cdot \}_{2,0} \) and \( \{ \cdot, \cdot \}_{2,1} \). The 2nd \( R \)-theoretic Poisson structure [5] turns out to be the symmetric average

\[
\{ f, g \}_{R2} := \frac{1}{2} \{ f, g \}_{2,0} + \frac{1}{2} \{ f, g \}_{2,1} = \frac{1}{4} \langle u, [\nabla f, R\{ u, \nabla g \} \} ] \rangle - (f \leftrightarrow g) ,
\]

where \( \{ u, v \}_+ := uv + vu \). If the Jacobi identity holds, one may show that \( \{ \cdot, \cdot \}_{R2} \) is always compatible with the first \( R \)-theoretic bracket \( \{ \cdot, \cdot \}_{R1} \). This follows by shifting \( u \rightarrow u + \lambda 1 \) in eq. (3.26), because the shifted 2nd Poisson bracket

\[
\{ f, g \}_{R2} (u+\lambda 1) = \{ f, g \}_{R2} (u) + \lambda \{ f, g \}_{R1} (u)
\]

can be re-interpreted as a Poisson pencil between the two brackets. (It is enough to let \( f \) and \( g \) be linear functions of \( u \), so that \( \nabla f \) and \( \nabla g \) are \( u \)-independent, which simplifies the above argument.) Moreover, one may prove [5] that sufficient conditions for the Jacobi identity for the \( \{ \cdot, \cdot \}_{R2} \) bracket are, that \( R \) and \( R_- \) satisfy the modified Yang-Baxter equations \( YB_\alpha(R) = 0 \) and \( YB_\alpha(R_-) = 0 \) with the same parameter \( \alpha \). This indeed is the case for the local and non-local solutions (cf. eqs. (2.13) and (2.19)).
A generalization of the quadratic bracket is due to Suris [2]. He defines a bracket

\[ 2\{ f, g \}_S = \langle A_1((\nabla f) u), (\nabla g) u \rangle - \langle A_2(u \nabla f), u \nabla g \rangle + \langle S_1(u \nabla f), (\nabla g) u \rangle - \langle S_2((\nabla f) u), u \nabla g \rangle, \]

where \( A_1, A_2, S_1, \) and \( S_2 \) are linear maps \( A \rightarrow A \) satisfying \( A_i^* = -A_i \) and \( S_i^* = S_i \). We assume everywhere in Section 3 that

\[ R = A_1 + S_1 = A_2 + S_2, \]

and that both \( A_1 \) and \( A_2 \) satisfy the modified Yang-Baxter equation \( YB_\alpha(A_i) = 0, \ i = 1, 2 \).

With the above assumptions one can show that the two Suris conditions

\[ 2S_1[u, v]_{A_2} = [S_1(u), S_1(v)] \]
\[ 2S_2[u, v]_{A_1} = [S_2(u), S_2(v)] \]

are sufficient for the Jacobi identity to hold. Also, they imply the modified Yang-Baxter equation \( YB_\alpha(R) = 0 \), and that \( \{ \cdot, \cdot \}_R \) and \( \{ \cdot, \cdot \}_S \) are compatible Poisson brackets.

Note that the opposite does not hold, i.e. that \( YB_\alpha(R) = 0 \) does not necessarily imply the Suris conditions (3.30). We give a counterexample in Appendix B. Also, the 2nd \( R \)-theoretic Poisson structure \( \{ \cdot, \cdot \}_R \) can be seen as a special case of the Suris construction if one let \( A_1 = A_2 = R_\pm \) and \( S_1 = S_2 = R_\mp \), because in this case the Suris condition \( 2R_+[u, v]_{R_-} = [R_+(u), R_+(v)] \) does follow from the modified Yang-Baxter equations \( YB_\alpha(R) = 0 \) and \( YB_\alpha(R_-) = 0 \).

It is known that a compatible quadratic Poisson structure for the discrete hierarchy is not unique [2], although a full classification of ambiguities is still an open problem. Here, we give a family of solutions that can be described using the quadratic Suris bracket. To this end, define a skewsymmetric linear map \( \Omega : A \rightarrow A \)

\[ \Omega := \frac{A_1 - A_2}{2} = \frac{S_2 - S_1}{2} = -\Omega^*, \]

where we used eq. (3.29) in the second equality. One may decompose the Suris variables entirely in terms of \( R \) and \( \Omega \):

\[ A_1 = R_- + \Omega, \quad A_2 = R_- - \Omega, \quad S_1 = R_+ - \Omega, \quad S_2 = R_+ + \Omega, \]

as well as the Suris bracket itself

\[ \{ \cdot, \cdot \}_S = \{ \cdot, \cdot \}_R + \{ \cdot, \cdot \}_\Omega, \]

where

\[ \{ f, g \}_\Omega := \frac{1}{2}(\Omega [u, \nabla f], [u, \nabla g]). \]

The structure \( \{ \cdot, \cdot \}_\Omega \) is not required to satisfy the Jacobi identity, and hence it is not necessarily a Poisson bracket, although this turns out to be the case for our example below.
A sufficient condition for this to happen is given by the Yang-Baxter equation \( YB_0(\Omega) = 0 \). In general, the structure \( \{ \cdot, \cdot \}_\Omega \) does not contribute to the Hamiltonian eq. (3.15), because \( \{ H_n, \cdot \}_\Omega = 0, n > 0 \). Hence, even for the more general Suris bracket \( \{ \cdot, \cdot \}_S \), the dynamics is governed by the \( \{ \cdot, \cdot \}_R \) bracket alone.

We claim that the non-local \( R \)-solution (2.19), together with the choice

\[
\Omega(\epsilon_{ij}) = \delta_{i,j} \sum_{m \in \mathbb{Z}} \omega_{i,m} \epsilon_{mm},
\]

(3.35)

for some skewsymmetric matrix \( \omega_{i,j} = -\omega_{j,i} \), meets all the conditions of the Suris construction. The proofs are very similar to the discussion given in Section 2.2. First of all, both \( A_1 \) and \( A_2 \) are of the same form as the non-local \( R \)-solution (2.19), and therefore they too have vanishing chiral Nijenhuis tensors \( \mathcal{N}_\pm(A_i) = 0 \), and hence \( YB_1(A_i) = 0 \). Secondly, both sides of the Suris conditions (3.30) vanish. For instance, the lhs. is of the form \( S_i(w) \), where \( w = 2[u,v]A_i \). Because of the special form of the two \( S_i \) maps, \( i = 1,2 \), only diagonal parts of \( w \) could potentially contribute. On the other hand, diagonal parts of \( [u,v]A_i \) do not exist according to eq. (2.21). So the lhs. is zero. The rhs. is zero, because both \( S_i(u) \) and \( S_i(v) \) are diagonal matrices, and hence commute.

Let us write down the Suris quadratic bracket \( \{ \cdot, \cdot \}_S = \{ \cdot, \cdot \}_S^{(0)} + \{ \cdot, \cdot \}_S^{(1)} \) in detail

\[
\{ u_{ij}, u_{kl} \}_S^{(0)} = \frac{1}{2} \left[ \epsilon(k-i) + \epsilon(l-j) \right] u_{il} u_{kj},
\]

(3.36a)

\[
\{ u_{ij}, u_{kl} \}_S^{(1)} = \omega_{ij,kl} u_{ij} u_{kl},
\]

(3.36b)

where

\[
\omega_{ij,kl} := \frac{1}{4} \left( \nu_{[i,j]} - \nu_{[j,i]} + \nu_{[k,l]} - \nu_{[l,k]} \right) + \frac{1}{2} \left( \omega_{j,i} + \omega_{i,k} + \omega_{k,j} \right) = -\omega_{kl,ij}.
\]

(3.37)

For simplicity, we have collected all the \( \omega_{i,j} \)-terms inside the \( \{ \cdot, \cdot \}_S^{(1)} \)-part. (Strictly speaking, this represents a minor abuse of notation, because \( \omega_{i,j} \) does not need to be first order in \( \nu \).) Figuratively speaking, the Suris bracket \( \{ \cdot, \cdot \}_S \) consists of two parts \( \{ \cdot, \cdot \}_S^{(1)} \) and \( \{ \cdot, \cdot \}_S^{(0)} \) that represent elastic and inelastic scattering of two “incoming” links, respectively. In other words, the first order bracket \( \{ \cdot, \cdot \}_S^{(1)} \) preserves the two “incoming” links \( u_{ij} \) and \( u_{kl} \), while the two “incoming” links exchange a pair of endpoints in the zero order bracket \( \{ \cdot, \cdot \}_S^{(0)} \).

If we restrict the \( \Omega \)-contribution to be the form \( \omega_{i,j} = \omega_{i-j} = -\omega_{j-i} \), the quadratic bracket reads

\[
\{ u_n(x), u_m(y) \}_S^{(0)} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left[ \epsilon(k) + \epsilon(k+m-n) \right] u_{n-k}(y) u_{m+k}(x) \delta_{x+k,y}
\]

\[
\{ u_n(x), u_m(y) \}_S^{(1)} = \frac{1}{2} u_n(x) u_m(y) \sum_{k \in \mathbb{Z}} \left( \nu_x \delta_{k,-m} - \nu_y \delta_{k,n} - \omega_{k+m-n} + \omega_{k+m} - \omega_{k-n} \right) \delta_{x+k,y}
\]

(3.38)

in the difference operator language.
3.5 3rd Poisson Bracket

The potential bracket candidates for a cubic Poisson bracket are convex linear combinations of the three basis elements $\{\cdot, \cdot\}_{3,0}$, $\{\cdot, \cdot\}_{3,1}$ and $\{\cdot, \cdot\}_{3,2}$. The 3rd $R$-theoretic bracket \[5, 6\] turns out to be given entirely by the candidate $\{\cdot, \cdot\}_{3,1}$:

$$\{f, g\}_{R3} := \{f, g\}_{3,1} = \frac{1}{2} \langle u, [\nabla f, R(u (\nabla g) u)] \rangle + \frac{1}{2} \langle u, [R(u (\nabla f) u), \nabla g] \rangle , \quad (3.39)$$

One may show \[5, 6\] that the three $R$-theoretic brackets $\{\cdot, \cdot\}_{R1}$, $\{\cdot, \cdot\}_{R2}$ and $\{\cdot, \cdot\}_{R3}$ are compatible Poisson structures if both $R$ and $R_-$ satisfy the modified Yang-Baxter equation $\text{YB}_{\alpha}(R) = 0$ and $\text{YB}_{\alpha}(R_-) = 0$ with the same parameter $\alpha$. In general, the third bracket $\{\cdot, \cdot\}_{R3}$ is not compatible with the Suris quadratic bracket $\{\cdot, \cdot\}_{S2}$.

We derive

$$\{u_{ij}, u_{kl}\}_{R3} = \frac{1}{2} \sum_m [\mathcal{E}_\nu(m, i) + \mathcal{E}_\nu(l, m)] u_{il} u_{km} u_{mj} - \frac{1}{2} \sum_m [\mathcal{E}_\nu(j, m) + \mathcal{E}_\nu(m, k)] u_{im} u_{ml} u_{kj} \quad (3.40)$$

for the non-local $R$-solution \[2.19\]. Its local first order $\nu$ terms are

$$\{u_{ij}, u_{kl}\}_{R3}^{(1)} = \frac{1}{2} u_{ij} [u_{ki} \nu_i u_{it} - u_{kj} \nu_j u_{jl}] - [(i, j) \leftrightarrow (k, l)] , \quad (3.41)$$

while the non-local first order $\nu$ terms read

$$\{u_{ij}, u_{kl}\}_{R3}^{(1)} = \frac{1}{2} u_{ij} \sum_{m \in \mathbb{Z}} u_{km} (\nu_{m,i} - \nu_{m,j}) u_{ml} - [(i, j) \leftrightarrow (k, l)] . \quad (3.42)$$

There is an interesting duality between the 1st and the 3rd bracket, which (formally) facilitates the proof of the Jacobi identity for the third bracket. Following Oevel and Ragnisco \[5\], one notices that matrix inversion $u \mapsto u^{-1}$ maps the first and third bracket into each other up to an overall minus sign. In detail, consider linear functionals $f(u) = \langle a, u \rangle$ and $g(u) = \langle b, u \rangle$ for some constant algebra elements $a, b \in \mathcal{A}$. Then for invertible $u$'s

$$\nabla f = a , \quad \nabla g = b , \quad \nabla(f(u^{-1})) = -u^{-1} au^{-1} \quad \text{and} \quad \nabla(g(u^{-1})) = -u^{-1} bu^{-1} . \quad (3.43)\quad (3.44)$$

It follows from the definitions \[3.20\] and \[3.39\] that

$$\{f(u^{-1}), g(u^{-1})\}_{R3}(u) = \frac{1}{2} \langle [u, u^{-1} au^{-1}], R(b) \rangle - (a \leftrightarrow b)$$

$$= \frac{1}{2} \langle [a, u^{-1}], R(b) \rangle - (a \leftrightarrow b)$$

$$= - \{f, g\}_{R3}(u^{-1}) . \quad (3.45)$$
This provides a proof of the Jacobi identity within the matrix group of invertible matrices (cf. footnote [4]). Similarly, one notices that the 2nd bracket is self-dual under \( u \mapsto u^{-1} \) up to an overall minus sign

\[
\{ f(u^{-1}), g(u^{-1}) \}_R^2(u) = \frac{1}{2} \langle [u, u^{-1} a u^{-1}], R\{u, u^{-1} b u^{-1}\} \rangle_+ - (a \leftrightarrow b)
\]

\[
= \frac{1}{2} \langle [a, u^{-1}], R\{b, u^{-1}\} \rangle_+ - (a \leftrightarrow b) 
\]

\[
= - \{ f, g \}_R^2(u^{-1}). \tag{3.46}
\]

Interestingly, the structure \( \{ \cdot, \cdot \}_\Omega^2 \) is also self-dual under \( u \mapsto u^{-1} \), but with an overall plus sign

\[
\{ f(u^{-1}), g(u^{-1}) \}_\Omega^2(u) = \frac{1}{2} \langle \Omega [u, u^{-1} a u^{-1}], [u, u^{-1} b u^{-1}] \rangle - (a \leftrightarrow b)
\]

\[
= \frac{1}{2} \langle \Omega [a, u^{-1}], [b, u^{-1}] \rangle - (a \leftrightarrow b) 
\]

\[
= + \{ f, g \}_\Omega^2(u^{-1}), \tag{3.47}
\]

so different parts of the Suris bracket \( \{ \cdot, \cdot \}_S^2 \) has different transformation properties under duality. Of course, one may claim the Suris bracket is self-dual under \( u \mapsto u^{-1} \) if one simultaneously changes the sign of \( \Omega \), or equivalently, one simultaneously exchanges \( A_1 \leftrightarrow A_2 \) and \( S_1 \leftrightarrow S_2 \).

ACKNOWLEDGMENT: This work has been partially supported by DOE grant DOE-ER-40173.

\[\text{A} \quad \text{The Kupershmidt Bi-Poisson Structure}\]

Here we translate our results into the \( q_k(x) \) fields used by Kupershmidt [1]. He uses an anti-normal ordered Lax operator of the form

\[
L = \sum_{k \in \mathbb{Z}} \Delta^{-k} q_k(x). \tag{A.1}
\]

Comparing with eq. (1.1), one derives the translation formula

\[
q_k(x) = u_{k+x,x} = u_{-k}(k+x). \tag{A.2}
\]

We may facilitates the \( u_k(x) \leftrightarrow q_k(x) \) translation of the Poisson structures by the following observation. First of all, from the matrix definition \( u_k(x) = u_{x,x+k} \), one notices that a change of variables \( u_k(x) \leftrightarrow q_k(x) \) corresponds to a transposition of the link matrix \( u_{ij} \leftrightarrow u_{ji} \). Secondly, notice that the \( \{ \cdot, \cdot \}_R^1 \) bracket (3.21) and (3.22) and the \( \{ \cdot, \cdot \}_S^2 \) bracket (3.36a) and (3.36b) are invariant under a transposition \( u \mapsto u^T \) of the \( u \)-matrix combined with a
change of the sign $\nu \to -\nu$ of the $\nu$-matrix. Hence the $u_k(x) \leftrightarrow q_k(x)$ translation simply amounts to a change of the sign of $\nu$. The $\{\cdot, \cdot\}_{R1}$ bracket (3.23) and the $\{\cdot, \cdot\}_{S2}$ bracket (3.38) become

$$\{q_n(x), q_m(y)\}_{R1} = \frac{1}{2} \left[ \varepsilon(n) + \nu_x \delta_{n,0} + \varepsilon(m) + \nu_y \delta_{m,0} \right] \times [q_{n+m}(x) \delta_{x+n,y} - q_{n+m}(y) \delta_{x,y+m}] ,$$

$$\{q_n(x), q_m(y)\}_{S2}^{(0)} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left[ \varepsilon(k) + \varepsilon(k+m-n) \right] q_{n-k}(y) q_{m+k}(x) \delta_{x+k,y} ,$$

$$\{q_n(x), q_m(y)\}_{S2}^{(1)} = \frac{1}{2} q_n(x) q_m(y) \sum_{k \in \mathbb{Z}} \left[ -\nu_x \delta_{k,-m} + \nu_y \delta_{k,n} - \omega_{k+m-n} + \omega_{k+m} - \omega_k + \omega_{k-n} \right] \delta_{x+k,y} , \quad n, m \in \mathbb{Z} .$$

(A.3)

In the original model of Ref. [1], the fields corresponding to positive link lengths are constrained

$$q_{-1}(x) \simeq 1 \quad \text{and} \quad \forall k \leq -2 : q_k(x) \simeq 0 ,$$

so that the Lax operator reads

$$L = \Delta + \sum_{k \geq 0} \Delta^{-k} q_k(x) .$$

(A.5)

The constraints have to be consistent with the equations of motion, written either as Hamiltonian equations (3.15) or as Lax equations (1.16) – with or without use of Poisson brackets, respectively. Previously in Section 2.5, we saw that the constraints $\forall k \leq -2 : q_k(x) \simeq 0$ are consistent with the Lax formulation. Also, it is easy to check from the bi-Poisson structure eq. [A.3] that the constraints $\forall k \leq -2 : q_k(x) \simeq 0$ decouple from the theory in the Hamiltonian sense, i.e. that the Hamiltonian vectorfields $\{q_k(x), \cdot\} \simeq 0$ vanish for both brackets when $k \leq -2$. On the other hand, the constraint $q_{-1}(x) \simeq 1$ induces non-trivial conditions on the model. From the Lax eq. (1.11) using $\mathcal{P}_-$ and the expansion eq. (1.14), one derives

$$\frac{\partial q_{-1}(x)}{\partial t_n} = \frac{1-\nu(x)}{2} q_{-1}(x) [(L^n)_0 (x) - (L^n)_0 (x-1)] ,$$

(A.6)

so consistency requires $\nu = 1$. Moreover, in the Hamiltonian formulation, where one imposes that the field $q_{-1}(x) \equiv 1$ “Poisson commutes” with the other fields $q_n(x), n \geq 0$, one is lead to the choice $\nu = 1$ and $\omega_k = kc - \varepsilon(k) = -\omega_{-k}$ with some immaterial constant $c \in \mathbb{C}$. (Again we stress that the on-shell dynamics are not affected by the $\{\cdot, \cdot\}_{\Omega2}$ contributions.) With this choice, the brackets (A.3) simplify to

$$\{q_n(x), q_m(y)\}_{K1} = q_{n+m}(x) \delta_{x+n,y} - q_{n+m}(y) \delta_{x,y+m} ,$$

$$\{q_n(x), q_m(y)\}_{K2}^{(0)} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left[ \varepsilon(k) + \varepsilon(k+m-n) \right] q_{n-k}(y) q_{m+k}(x) \delta_{x+k,y} ,$$

$$\{q_n(x), q_m(y)\}_{K2}^{(1)} = \frac{1}{2} q_n(x) q_m(y) \sum_{k \in \mathbb{Z}} \left[ \varepsilon(k) + \varepsilon(k+m-n) - \varepsilon(k+m+\frac{1}{2}) - \varepsilon(k-n-\frac{1}{2}) \right] \delta_{x+k,y} , \quad n, m \geq 0 ,$$

(A.7)
which agree with the formula (III.3.4) and the formula (III.4.15a-c) in Ref. [1].

B  Example: Mat$_{2 \times 2}(\mathbb{C})$

Here we give a counterexample, that shows that $Y_{\alpha}(R) = Y_{\alpha}(A_1) = Y_{\alpha}(A_2) = 0$, taken together with the relation $R = A_i + S_i$, does not necessarily imply the Suris conditions (3.30).

Consider the 4-dimensional associative algebra $\mathcal{A} = \text{Mat}_{2 \times 2}(\mathbb{C}) \cong \mathbb{C}^4$. A convenient basis is given by the $2 \times 2$ unit-matrix $\sigma_4 \equiv 1$ and the three Pauli $\sigma_i$ matrices, $i = 1, 2, 3$, which satisfy the relation

$$\sigma_i \sigma_j = \delta_{ij} 1 + \alpha \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k , \quad i, j = 1, 2, 3 , \quad \text{(B.1)}$$

where $\epsilon_{ijk}$ is the 3-dimensional Levi-Civita symbol. A non-degenerate, associative/invariant bilinear form is inherited from the matrix trace (tr):

$$\langle \sigma_\mu, \sigma_\nu \rangle = \text{tr} (\sigma_\mu \sigma_\nu) = 2 \delta_{\mu,\nu} , \quad \mu, \nu = 1, 2, 3, 4 . \quad \text{(B.2)}$$

We now search for $R$-matrix solutions to the modified Yang-Baxter equation $Y_{\alpha}(R) = 0$, where $\alpha \in \mathbb{C}$ is a given fixed complex number. Let us consider a linear injective map $\Phi : \mathbb{C}^3 \ni \vec{r} = (r_1, r_2, r_3) \mapsto R \in \text{End}(\mathcal{A}) , \quad \text{(B.3)}$

that maps a complex rotation vector $\vec{r}$ into its rotation matrix $R$

$$R(\sigma_i) := i \sum_{j,k=1}^{3} \epsilon_{ijk} r_j \sigma_k , \quad i = 1, 2, 3 , \quad \text{(B.4)}$$

$$R(1) := 0 .$$

In other words, $R$ rotates the basis of $\sigma_i$ matrices, $i = 1, 2, 3$, around the rotation axis $\vec{r}$. Note that $R = -R^*$ is skewsymmetric. The $R$-bracket reads

$$[\sigma_i, \sigma_j]_R = \sigma_i [r_j] , \quad i, j = 1, 2, 3 , \quad \text{(B.5)}$$

$$[1, \cdot ]_R = 0 .$$

We claim that $Y_{\vec{r}, \vec{r}}(R) = 0$, i.e. that

$$2R[\sigma_\mu, \sigma_\nu]_R = [R(\sigma_\mu), R(\sigma_\nu)] + \vec{r} \cdot \vec{r} [\sigma_\mu, \sigma_\nu] , \quad \mu, \nu = 1, 2, 3, 4 , \quad \text{(B.6)}$$

where $\vec{r} \cdot \vec{r} := \sum_{i=1}^{3} r_i^2$ is a “bilinear” length-square, i.e. without a complex conjugation. Technically, since the $\sigma_4$-sector is trivial, the eq. (B.6) reduces to a “dual” Yang-Baxter identity

$$\sum_{j,k=1}^{3} \epsilon_{ijk} (R(\sigma_j), R(\sigma_k)) - 2R \left[ \sigma_j, \sigma_k \right]_R + \vec{r} \cdot \vec{r} \left[ \sigma_j, \sigma_k \right] = 0 , \quad i = 1, 2, 3 , \quad \text{(B.7)}$$
which is easy to verify by direct calculation.

Now let us apply this fact to a specific example. Define five rotation vectors

\[ \vec{r} = (1, i, \sqrt{\alpha}) , \quad \vec{s}_1 = (1, i, 0) = -\vec{s}_2 . \]  

(B.8)

and

\[ \vec{a}_1 = \vec{r} - \vec{s}_1 = (0, 0, \sqrt{\alpha}) , \quad \vec{a}_2 = \vec{r} - \vec{s}_2 = (2, 2i, \sqrt{\alpha}) . \]  

(B.9)

The corresponding five rotation matrices are skewsymmetric

\[ A_i = \Phi(\vec{a}_i) = -A_i^* , \]
\[ S_1 = \Phi(\vec{s}_1) = \Phi(-\vec{s}_2) = -\Phi(\vec{s}_2) = -S_2 = S_2^* , \]  

(B.10)

\[ R = \Phi(\vec{r}) = \Phi(\vec{a}_i + \vec{s}_i) = \Phi(\vec{a}_i) + \Phi(\vec{s}_i) = A_i + S_i , \quad i = 1, 2 , \]

and they each satisfy a (modified) Yang-Baxter equation

\[ Y_{B\alpha}(R) = 0 \]
\[ Y_{B\alpha}(A_i) = 0 \]
\[ Y_{B0}(S_i) = 0 \]  

(i = 1, 2). (B.11)

Using \( Y_{B0}(S_1) = 0 \), one may reduce the Suris operator

\[ (u, v) \mapsto 2S_1[u, v]_A - [S_1(u), S_1(v)] = 2S_1[u, v]_R \]  

(B.12)

to only one term (cf. eq. (3.30)). Therefore, the “dual” Suris condition simplifies to

\[ \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} S_1[\sigma_j, \sigma_k]_R = i \sum_{j=1}^{3} r_j s_{1,j} [\sigma_i] = -i \vec{r} \cdot \vec{\sigma} s_{1,i} \neq 0 , \quad i = 1, 2 . \]  

(B.13)

Thus the Suris conditions (3.30) are not met, despite eq. (B.11).

It is however generally valid that the two Suris conditions eq. (3.30), the modified Yang-Baxter equation \( Y_{B\alpha}(R) = 0 \), taken together with the relation \( R = A_i + S_i \), imply the two modified Yang-Baxter equations \( Y_{B\alpha}(A_i) = 0 \) for the skewsymmetric maps \( A_i, i = 1, 2 \) (cf. Theorem 2 in Ref. [3]).

References

[1] B. Kupershmidt, “Discrete Lax Equations and Differential-Difference Calculus”, Asterisque 123 (1985), pp.1–212.

[2] Yu.B. Suris, “On the Bi-Hamiltonian Structure of Toda and Relativistic Toda Lattices”, Phys. Lett. A 180, 419 (1993); “Nonlocal Quadratic Poisson Algebras, Monodromy Map, and Bogoyavlensky Lattices” Journal of Mathematical Physics 38 4179 (1997); “The Problem of Integrable Discretization : Hamiltonian Approach”, Progress in Mathematics 219, (Birkhäuser, Basel, 2003).
[3] W. Oevel, “Poisson Brackets for Integrable Lattice Systems”, in “Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman”, A.S. Fokas and I.M. Gelfand (eds.), (Birkhäuser, Boston, 1996), pp.261–283.

[4] M.A. Semenov-Tian-Shansky, “What is a Classical R-Matrix”, Funct. Anal. Appl. 17 (1983), pp.259-272; Reyman A.G. and Semenov-Tian-Shansky M.A., “Group Theoretical Methods in the Theory of Finite Dimensional Integrable Systems”, in: “Encyclopaedia of Mathematical Science, V.16: Dynamical Systems VII”, (Springer, New York, 1994), pp.116-225.

[5] W. Oevel and O. Ragnisco, “R-Matrices and Higher Poisson Brackets for Integrable Systems”, Physica A 161 (1989) 181.

[6] Li L.-C. and Parmentier S., “Nonlinear Poisson Structures and r-Matrices”, Commun. Math. Phys. 125, 545 (1989).

[7] H. Aratyn, E. Nissimov, S. Pacheva and I. Vaisburd, “R-Matrix Formulation of KP Hierarchies and their Gauge Equivalence,” Phys. Lett. B 294, 167 (1992), arXiv:hep-th/9209006

[8] J. Harnad and M.-A. Wisse, “Moment Maps to Loop Algebras, Classical R-Matrix and Integrable Systems”, in: “Quantum Groups Integrable Models and Statistical Systems”, J. Letourneux and L. Vinet (eds.), (World Scientific, Singapore, 1993), pp.105–117, arXiv:hep-th/9301104

[9] M. Blaszak, “R-Matrix Approach to Multi-Hamiltonian Lax Dynamics”, Reports on Mathematical Physics 40 (1997) 395.

[10] V.V. Gribanov, V.G. Kadyshevsky, A.S. Sorin, “Generalized Fermionic Discrete Toda Hierarchy”, arXiv:nlin.SI/0311030

[11] S. Okubo and A. Das, “A Systematic Study of the Toda Lattice”, Ann. Phys. 190 (1989) 215.