Bimaximal mixing from the leptonic new texture for triangular mass matrices

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Abstract

An analysis of the leptonic texture for the new triangular mass matrices has been carried out. In particular, it is shown that both bimaximal and nearly bimaximal solutions for solar and atmospheric neutrino anomalies can be generated within this pattern. We have also derived exact and compact parametrization of the leptonic mixing matrix in terms of the lepton masses and the parameters $\alpha, \beta'$ and $\delta$. A consistency with the CHOOZ reactor result for $V_{m13}$ and a smallness of the Jarlskog’s invariant parameter are obtained.

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1 Introduction:

One of the most important challenging tasks of present and future experiments is to establish whether or not neutrinos have possible rest masses. A clear evidence of non–vanishing masses for neutrinos influences various research areas from particle physics, as well as from astrophysics and cosmology. Massive neutrinos are regarded as the best candidate for hot dark matter and would play a profound role in the formation and stability of our universe structure.

At present, the recent SuperKamiokande Collaboration [1] observation has provided a strong evidence for neutrino oscillations with large mixing as well as non–zero neutrino masses. Similar indications in favor of neutrino oscillations comes from the results of the atmospheric neutrino experiments (Kamiokande [2], IBM [3], Soudan 2 [4] and MACRO [5]). Another sign for this proposal comes also from the solar neutrino experiments (Homestake [6], Kamiokande [7], GALLEX [8], SAGE [9] and SuperKamiokande [10]).

Let us mention that there is one more possible indication in favor of neutrino oscillation, from the Laboratory experiment by the Liquid Scintillation Neutrino Detector (LSND) [11] at Los Alamos, but preliminary data from KARMEN [12] failed to reproduce this evidence. Here we adopt a conservative approach by not taking into account the possibility alluded by the LSND data, waiting for its confirmation by other experiments.

A solution to the solar and atmospheric neutrino problem with neutrino oscillations requires the existence of two different scales of neutrino mass–squared differences, which corresponds to the existence of three massive neutrinos. These three massive neutrinos are mixings of three flavor neutrinos whose existence is known from the measurements of the invisible width of the $Z$ boson done by LEP experiments [13].

In the solar neutrino experiments three possible solutions have been proposed through the matter enhanced neutrino oscillation (i.e. MSW solution [14]) if $\Delta m^2_\odot \simeq 5 \times 10^{-6} eV^2$ and $\sin^2 2\theta_\odot \simeq 6 \times 10^{-3}$ (small angle case), or $\Delta m^2_\odot \simeq 2 \times 10^{-5} eV^2$ and $\sin^2 2\theta_\odot \simeq 0.76$ (large angle case) and through the long–distance vacuum neutrino oscillation called the ”just so” vacuum oscillation $\Delta m^2_\odot \simeq 8 \times 10^{-11} eV^2$ and $\sin^2 2\theta_\odot \simeq 0.75$.

On the other hand, since the CHOOZ experiment [15] excludes oscillation of $\nu_\mu \rightarrow \nu_e$ with a large mixing angle for $\Delta m^2_{atm} \geq 9 \times 10^{-4} eV^2$, the atmospheric $\nu_\mu$ deficit is explained by the maximal mixing between $\nu_\mu$ and $\nu_\tau$.

Phenomenological analysis favor two solutions for the solar and atmospheric neu-
trino problem,

* large $\nu_\mu - -\nu_\tau$ mixing for the atmospheric anomalies and matter enhanced (MSW) small mixing angle oscillations for solar neutrinos.

* vacuum oscillations and bimaximal or nearly bimaximal mixing of three light neutrinos.

In the minimal Standard Model based on left–handed two component neutrino fields and no right–handed neutrino fields in the Lagrangian, neutrino are two component massless particles. A simplest extension of the Standard Model with massive neutrinos is obtained by adding right–handed neutrino field $\nu_R$ per family with the $SU(2)_L \otimes U(1)_Y$ quantum numbers (0, 0). Neutrinos acquire then Dirac masses by analogy with the quarks and charged leptons. Neutrino mass eigenstates are then different from the weak eigenstates, leading to neutrino mixing and the violation of family lepton numbers.

The Yukawa interaction for three generations is parametrized in terms of $3 \times 3$ matrices which contains a large amount of free parameters compared to the physical measurables one.

To overcome such a freedom, extra symmetries (ansätze) are introduced to cast the fermion mass matrices in some particular form \cite{16, 17, 18, 19}. In particular, a general classification and analysis of symmetric or hermitian mass matrices having textures zeroes consistent with the measured values of the fermions masses and mixing angles has been carried out in \cite{19}.

For non–hermitian mass matrices, Branco et al. \cite{22} have shown that for three generations, fermion masses with textures zeroes can just be obtained by redefining the fermion fields in a special weak basis transformation which has no observable consequences.

An another class of fermion masses patterns arises in the framework of Marseille–Mainz noncommutative geometry model \cite{20}, namely triangular mass matrices \cite{21, 25}. They are typical for reducible but indecomposable representations of graded Lie algebras.

It has been shown recently \cite{23, 24} that it is possible to express these triangular mass matrices in a economic and concise way with a minimum set of parameters, through a specific weak basis transformation.

Indeed, these textures involve 5 complex numbers instead of 6, which means that one extra parameter is either zero or dependent of the others. A connection between these mass matrices and the so–called Nearest–Neighbor Interactions was established in \cite{24}. More details will be presented elsewhere \cite{26}.
In this article, we would like to examine if such a type of mass matrix can be suitable for lepton sector and explore a simple form of triangular neutrino mass matrix \( T_\nu \), which contributes to the \( \nu_\mu - \nu_\tau \) mixing and a charged triangular lepton mass matrix \( T_l \),

\[
T_\nu = \begin{pmatrix}
\alpha' & 0 & 0 \\
0 & \beta' & 0 \\
0 & k'_2 & \gamma'
\end{pmatrix}
\]

where \( \alpha', \beta', \gamma' \) and \( k'_2 \) are positive parameters and

\[
T_l = \begin{pmatrix}
\alpha & 0 & 0 \\
k_1 e^{i\phi_1} & \beta & 0 \\
0 & k_2 e^{i\phi_r} & \gamma
\end{pmatrix} = P^\dagger T'_l P
\]

with

\[
T'_l = \begin{pmatrix}
\alpha & 0 & 0 \\
k_1 & \beta & 0 \\
0 & k_2 & \gamma
\end{pmatrix}, \quad P = \begin{pmatrix}
e^{i\phi_1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\phi_r}
\end{pmatrix}
\]

where \( \alpha, \beta, \gamma, k_1 \) and \( k_2 \) are all real positive.

The characteristic feature of this type of lepton mass matrices is that the contribution to \( \nu_e - \nu_\mu \) mixing comes not from the neutrino mass matrix \( T_\nu \) but from the charged lepton \( T_l \) and the contribution to \( \nu_\mu - \nu_\tau \) mixing angle comes from both \( T_\nu \) and \( T_l \). Therefore, we can obtain a large \( \nu_\mu - \nu_\tau \) mixing angle from \( T_\nu \) by taking the small contribution from \( T_l \). It will be shown that this parametrization is a good candidate pattern to describe the bimaximal and nearly bimaximal mixing between \( \nu_e - \nu_\mu \) and \( \nu_\mu - \nu_\tau \).

Our paper is organized as follows. In section 2 we review how to transform triangular mass matrices into new triangular forms through a weak basis transformation. In particular leptonic mass matrices (1) and (2) are obtained when choosing a specific weak basis. In section 3 we generate the bimaximal and nearly bimaximal mixing from these patterns and reconstruct the mass matrices corresponding to this bimaximal solution. In section 4 we end up with some conclusions.

## 2 New triangular mass matrix applied to leptons:

Here we adopt the usual attitude in considering that the nonidentity of the mass eigenstates and weak interaction states leads to the weak mixing of fermionic states
with equal charges but different flavors which means that the weak mixing matrix relevant for the charged current interactions is obtained from the mass matrices. Moreover, the flavor structure of the Yukawa interactions is not constrained by any symmetry but the charged current interactions depend only on the left handed fermion fields. Thus, there is much freedom in defining a weak basis for the fermions where the full information contained in any nonsingular mass matrix \( M = T U \), can be recasted by means of a triangular mass matrix \( T \).

This type of matrix is obtained from the freedom in choosing the right handed basis through (unobservable) unitary transformation \( U \) and from the fact that the charged current weak interactions involve left–handed fields only. It has been shown in [21] that the triangular mass matrix corresponds to the classification of the fermion families in reducible but indecomposable representations.

In what follows we consider the mass and the weak charged current Lagrangian terms for Dirac neutrinos and charged leptons, \( L = \bar{\nu}_R T_\nu \nu_R + \bar{l}_R T_l l_R + g \bar{\nu}_L M^+ i L + h.c. \) (4) where \( T_\nu \) and \( T_l \) are lower triangular mass matrices,

\[
T_\nu = \begin{pmatrix}
\alpha' & 0 & 0 \\
\beta' & 0 & 0 \\
k_1' e^{i\phi_1} & k_2' e^{i\phi_2} & \gamma'
\end{pmatrix}, \quad T_l = \begin{pmatrix}
\alpha & 0 & 0 \\
\beta & 0 & 0 \\
k_1 e^{i\phi_1} & k_3 e^{i\phi_3} & \gamma
\end{pmatrix}
\] (5)

As it is well known, the physics is invariant if the following transformations \( \nu_R \rightarrow V_\nu \nu_R, l_R \rightarrow V_l l_R, \nu_L \rightarrow U \nu_L \) and \( l_L \rightarrow U l_L \) are performed on the right and left handed lepton fields where the matrices \( U, V_\nu \) and \( V_l \) are unitary. This means that all sets of mass matrices related to each others through,

\[
T_\nu' = U^\dagger T_\nu V_\nu, \quad T_l' = U^\dagger T_l V_l
\] (6)

give rise to the same masses and mixings.

In particular, it has been shown recently [23, 24] that for any arbitrary triangular \( 3 \times 3 \) mass matrices \( T_\nu \) and \( T_l \), we can always find weak basis for lepton fields such that the hermitian \( (T_\nu' T_\nu')_{ij} = (T_l' T_l')_{ij} = 0 \) for fixed \( i \) and \( j \) such that \( i \neq j \) where the new triangular mass matrices \( T_\nu' \) and \( T_l' \) are obtained from the above unitary transformations.

We have also supplied a classification of all possible textures that contain the minimal set of parameters and written the original non–hermitian mass matrix in terms of the triangular mass matrix elements making therefore a bridge and a close connection between the Nearest– Neighbor Interactions (NNI) and the new triangular forms [24].
We will be concerned here, with those textures corresponding to vanishing matrix elements $(1, 3)$,

\[
\begin{aligned}
(T'_{\nu}T'^{\dagger}_{\nu})_{13} &= (U^\dagger T_{\nu}T'^{\dagger}_{\nu}U)_{13} = 0 \\
(T'_{l}T'^{\dagger}_{l})_{13} &= (U^\dagger T_{l}T'^{\dagger}_{l}U)_{13} = 0
\end{aligned}
\]  

(7)

where in this basis the new triangular mass matrices $T'_{\nu}$ and $T'_{l}$ are given as follows, see [23, 24],

\[
T'_{\nu} = \begin{pmatrix}
\alpha' \\
\beta' \\
k_1' e^{i\phi_1'} \\
k_2' e^{i\phi_2'}
\end{pmatrix},
T'_{l} = \begin{pmatrix}
\alpha \\
\beta \\
k_1 e^{i\phi_1} \\
k_2 e^{i\phi_2}
\end{pmatrix}
\]  

(8)

To construct the appropriate unitary matrix $U$ that gives this requirement, we have first obtained $U_{j1}$ which is the eigenvectors of the matrix $(T_{\nu}T'^{\dagger}_{\nu} + k T_{l}T'^{\dagger}_{l})$ with $\lambda$ as eigenvalue,

\[
(T_{\nu}T'^{\dagger}_{\nu} + k T_{l}T'^{\dagger}_{l})_{ji} U_{i1} = \lambda U_{j1}
\]  

(9)

where $k$ is a complex parameter expressing the way the neutral and charged leptons are correlated. From $U_{j1}$, the whole unitary matrix $U$ can be recovered.

We can rewrite this for arbitrary complex $k$ as:

\[
k = \frac{\alpha' k_1'}{\alpha k_1} e^{i(\phi_1 - \phi_1')}, \quad \lambda = \alpha'^2 + k \alpha^2
\]  

(10)

In particular the condition,

\[
k_1' = 0
\]  

(11)

accomodates the atmospheric neutrino solution which corresponds to a choice of a specific weak basis.

The above mass matrices with (11) can be rewritten as:

\[
T'_{\nu} = \begin{pmatrix}
\alpha' \\
\beta' \\
k_1' e^{i\phi_1'} \\
k_2' e^{i\phi_2'}
\end{pmatrix} = P'^{\dagger}_{\nu} T'_{\nu} P_{\nu}, \quad T'_{l} = \begin{pmatrix}
\alpha \\
\beta \\
k_1 e^{i\phi_1} \\
k_2 e^{i\phi_2}
\end{pmatrix} = P'^{\dagger}_{l} T'_{l} P_{l}
\]  

(12)

where $T'_{\nu,l}$ are real mass matrices and $P_{\nu,l}$ are diagonal mass matrices given by,

\[
T'_{\nu} = \begin{pmatrix}
\alpha' \\
\beta' \\
k_1' \\
k_2'
\end{pmatrix}, \quad P_{\nu} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\phi_2'}
\end{pmatrix},
\]

\[
T'_{l} = \begin{pmatrix}
\alpha \\
\beta \\
k_1 \\
k_2
\end{pmatrix}, \quad P_{l} = \begin{pmatrix}
e^{i\phi_1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\phi_2}
\end{pmatrix}
\]  

(13)
A particularly remarkable feature of this texture is that the contribution to $\nu_e - \nu_\mu$ mixing comes from the charged lepton mass matrix $T'_l$ which can be taken large in order to describe the "just so" vacuum oscillation solution for the solar neutrino problem whereas the neutrino $T'_\nu$ and the charged lepton $T'_l$ mass matrices contribute to $\nu_\mu - \nu_\tau$ mixing. To explain the atmospheric and solar neutrino anomalies, it suffices to take a large mixing angle between the second and third generations from the neutrino sector and large mixing between the first and second generations from the charged lepton sector.

Next, we make the following observation. By using the freedom in choosing the right handed fields for both sectors, as well as the left handed fields through the neutral diagonal phase matrix $P_\nu$,

$$\nu_{L,R} \rightarrow P_\nu \nu_{L,R} \quad l_{L,R} \rightarrow P_\nu l_{L,R}$$

it leads to a real mass matrix for the neutral sector,

$$T_\nu = T'_\nu = \begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \beta' & 0 \\ 0 & k'_2 & \gamma' \end{pmatrix}$$

and a complex mass matrix for the charged sector,

$$T_l = (P_1P_\nu^\dagger)T'_l(P_1P_\nu^\dagger) = \begin{pmatrix} \alpha & 0 & 0 \\ k_1e^{i\phi_1} & \beta & 0 \\ 0 & k_2e^{i\phi_2} & \gamma \end{pmatrix}$$

These lepton mass matrices have 11 parameters, i.e. 4 real moduli for neutral sector and 5 real moduli with two phases $\phi_1$ and $\phi_2 = \phi_2 - \phi'_2$ for the charged sector.

Our purpose is to study the relations between the physical quantities and eleven parameters for the neutral $T_\nu$ and charged $T_l$ lepton mass matrices. Introducing the three neutral lepton mass eigenvalues $D_\nu = (m_{\nu_e}, -m_{\nu_\mu}, m_{\nu_\tau})$ for $T_\nu$ as input parameters, $T_\nu$ will be parametrized by one free parameter. Indeed, the characteristic polynomial for the neutral symmetric matrix $T_\nu T_\nu^\dagger$ has the following relations between the squared masses and the parameters,

$$m_{\nu_e}^2 + m_{\nu_\mu}^2 + m_{\nu_\tau}^2 = \alpha'^2 + \beta'^2 + \gamma'^2 + k'^2$$

$$m_{\nu_e}^2 m_{\nu_\mu}^2 + m_{\nu_e}^2 m_{\nu_\tau}^2 + m_{\nu_\mu}^2 m_{\nu_\tau}^2 = \alpha'^2\beta'^2 + \alpha'^2\gamma'^2 + \beta'^2\gamma'^2 + \alpha'^2k'^2$$

$$m_{\nu_e}^2 m_{\nu_\mu}^2 m_{\nu_\tau}^2 = \alpha'^2\beta'^2\gamma'^2$$

When solved for $\alpha', \gamma'$ and $k'_2$, these relations give:

$$\alpha'^2 = m_{\nu_e}^2$$
\[ \gamma'^2 = \frac{m_{\nu_e}^2 m_{\nu_e}^2}{\beta'^2} \]
\[ k'^2_2 = -\frac{(\beta'^2 - m_{\nu_e}^2)(\beta'^2 - m_{\nu_e}^2)}{\beta'^2} \]  
\[ (18) \]

The neutrino mass matrix can be written in terms of the free parameter \( \beta' \) as:

\[ T_\nu = \begin{pmatrix} m_{\nu_e} & 0 & 0 \\ 0 & \beta' & 0 \\ 0 & \sqrt{(\beta'^2 - m_{\nu_e}^2)(m_{\nu_e}^2 - \beta'^2)} & m_{\nu_\mu} m_{\nu_e} \end{pmatrix} \]  
\[ (19) \]

\( T_\nu T_\nu^T \) is diagonalized by an orthogonal matrix \( O_\nu \),

\[ O_\nu^T T_\nu T_\nu^T O_\nu = D_\nu^2 \]  
\[ (20) \]

where \( O_\nu \) is given by:

\[ O_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_\nu & \sin \theta_\nu \\ 0 & -\sin \theta_\nu & \cos \theta_\nu \end{pmatrix} \]  
\[ (21) \]

with

\[ \tan^2 \theta_\nu = \frac{\beta'^2 - m_{\nu_e}^2}{m_{\nu_e}^2 - \beta'^2} \]  
\[ (22) \]

In the same way, \( T_l \) is also parametrized by two real moduli and two phases as free parameters by introducing the three charged lepton mass eigenvalues \( D_l = (m_e, -m_\mu, m_\tau) \) as input parameters. Indeed, we have:

\[ m_e^2 + m_\mu^2 + m_\tau^2 = \alpha^2 + \beta^2 + \gamma^2 + k_1^2 + k_2^2 \]
\[ m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 = \alpha^2 \beta^2 + \alpha^2 \gamma^2 + \beta^2 \gamma^2 + \alpha^2 k_2^2 + \gamma^2 k_1^2 + k_1 k_2 \]
\[ m_\mu^2 m_\tau^2 = \alpha^2 \beta^2 \gamma^2 \]  
\[ (23) \]

When solved, we get:

\[ k_1^2 = \frac{1}{2} \left( m_e^2 + m_\mu^2 + m_\tau^2 - 2\alpha^2 - \beta^2 \right) \left( 1 - \sqrt{1 - 4 \frac{(\alpha^2 - m_e^2)(\alpha^2 - m_\mu^2)(\alpha^2 - m_\tau^2)}{\alpha^2 (m_e^2 + m_\mu^2 + m_\tau^2 - 2\alpha^2 - \beta^2)}} \right) \]
\[ k_2^2 = \frac{1}{2} \left( m_e^2 + m_\mu^2 + m_\tau^2 - \beta^2 - 2\gamma^2 \right) \left( 1 - \sqrt{1 - 4 \frac{(\gamma^2 - m_e^2)(\gamma^2 - m_\mu^2)(\gamma^2 - m_\tau^2)}{\gamma^2 (m_e^2 + m_\mu^2 + m_\tau^2 - \beta^2 - 2\gamma^2)}} \right) \]
\[ \gamma^2 = \frac{m_e^2 m_\mu^2 m_\tau^2}{\alpha^2 \beta^2} \]  
\[ (24) \]
The charged lepton hermitian matrix \( T_l T_l^\dagger \) is diagonalized by a unitary matrix \( U_L^l = P.O_l \) and \( P = P_l P_l^\dagger \) is a diagonal phase matrix,

\[
U_L^l T_l T_l^\dagger U_L^l = O_l^T T_l^T O_l = D_l^2
\]  

(25)

Here \( O_l \) is an orthogonal matrix which can be written in terms of its eigenvectors,

\[
O_l = \begin{pmatrix}
x_e & x_\mu & x_\tau 
y_e & y_\mu & y_\tau 
z_e & z_\mu & z_\tau 
\end{pmatrix}
\]  

(26)

with

\[
\begin{pmatrix}
x_i 
y_i 
z_i 
\end{pmatrix} = \frac{1}{f_i} \begin{pmatrix}
\alpha \beta k_1 k_2 
-\beta k_2 (\alpha^2 - m_i^2) 
(\alpha^2 - m_i^2)(\beta^2 - m_i^2) - m_i^2 k_1^2 
\end{pmatrix}
\]  

(27)

and the normalization factors \( f_i \) are given as follows,

\[
f_i^2 = [(\alpha^2 - m_i^2)(\beta^2 - m_i^2) - m_i^2 k_1^2] \prod_{j,j \neq i} (m_i^2 - m_j^2), \text{ for } i, j = e, \mu, \tau
\]  

(28)

The lepton flavor mixing matrix \( V_m \) connecting the flavor eigenstates to the mass eigenstates can be expressed as a product of the two matrices \( O_\nu \) and \( U_L^l = P.O_l \) that diagonalize the neutrinos and charged lepton mass matrices respectively,

\[
V_m = O_\nu^T.P.O_l
\]  

(29)

3 Bimaximal mixing from the new triangular lepton mass matrices:

The interpretation of SuperKamiokande data is compatible with maximal mixing between the atmospheric \( \nu_\mu \) and \( \nu_\tau \) and the ” just so ” vacuum oscillation solution of the solar neutrino problem. The combination of these two possibilities gives the bimaximal mixing matrix,

\[
V_m \mid_{bi} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 
0 & 0 & 1
\end{pmatrix}
\]  

(30)

Such a neutrino mixing pattern has received a great deal of attention [27, 28]. In particular, this can be reconstructed from the product of two orthogonal matrices,

\[
V_m = \begin{pmatrix}
1 & 0 & 0 
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} 
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix}
\cos \theta_l & \sin \theta_l & 0 
-\sin \theta_l & \cos \theta_l & 0 
0 & 0 & 1
\end{pmatrix} = O_\nu^T.O_l
\]  

(31)
where $O_\nu$ corresponds here to the maximal mixing between the second and the third generations for neutrinos and $O_l$ is parametrized in terms of the mixing angle $\theta_l$ expressing the mixing between the first and second generations for the charged leptons. In this case, the large angle solar neutrino solution is obtained for $\cos \theta_l = -\sin \theta_l = \frac{1}{\sqrt{2}}$.

We return now to the analysis of our lepton mass matrices. In accordance with the atmospheric neutrino experiments data, the maximal mixing angle between the second and the third generations for neutrino is realized in our neutrino mass matrix pattern for $\theta_\nu = \frac{\pi}{4}$ which implies that,

$$\beta'^2 = \frac{m_{\nu_\mu}^2 + m_{\nu_\tau}^2}{2} \tag{32}$$

The neutrino mass matrix that reproduces this maximal mixing is given by:

$$T_\nu = \begin{pmatrix}
    m_{\nu_e} & 0 & 0 \\
    0 & \sqrt{\frac{m_{\nu_\tau}^2 + m_{\nu_\mu}^2}{2}} & 0 \\
    0 & \frac{\sqrt{2}}{\sqrt{m_{\nu_\mu}^2 + m_{\nu_\tau}^2}} & m_{\nu_\mu} m_{\nu_\tau}
\end{pmatrix} \tag{33}$$

We want now to determine the charged lepton mass matrix corresponding to the "just so" solution for the solar neutrino problem. A typical matrix $O_l$ consistent with this vacuum solution is:

$$O_l = \begin{pmatrix}
    O_{11} \approx \cos \theta_l & O_{12} \approx \sin \theta_l & O_{13} \\
    O_{21} \approx -\sin \theta_l & O_{22} \approx \cos \theta_l & O_{23} \\
    O_{31} & O_{32} & O_{33}
\end{pmatrix} \tag{34}$$

where $|O_{13}|, |O_{23}|, |O_{31}|$ and $|O_{32}|$ are all small parameters and $O_{33} \approx 1$. Note that this pattern implies a nearly maximal mixing between the first and second generations.

This structure is obtained simply by considering $\gamma^2 = m_{\tau}^2(1 - \delta^2)$ where $\delta \ll$ dimensionless parameter and expanding (27) and (28) to the first order in $\delta$ by using (23) and (24).

With respect to this analysis, the mixing matrix $O_l$ reads in terms of the charged lepton masses and the parameters $\alpha$ and $\delta$ as:

$$O_l = \begin{pmatrix}
    \sqrt{\frac{\Delta m_{\nu_e}^2}{\Delta m_{\mu_e}^2}}(1 + c_{ee}\delta) & \sqrt{\frac{\Delta m_{\nu_\tau}^2}{\Delta m_{\mu_e}^2}}(1 - c_{e\mu}\delta) & c_{e\tau}\delta \\
    -\sqrt{\frac{\Delta m_{\mu_e}^2}{\Delta m_{\mu_e}^2}}(1 + c_{\mu e}\delta) & \sqrt{\frac{\Delta m_{\nu_\tau}^2}{\Delta m_{\mu_e}^2}}(1 - c_{\mu \mu}\delta) & c_{\mu \tau}\delta \\
    c_{\tau e}\delta & -c_{\tau \mu}\delta & \sqrt{\frac{\Delta m_{\nu_\tau}^2}{\Delta m_{\tau e}^2}}(1 - c_{\tau \tau}\delta)
\end{pmatrix} + O(\delta^2) \tag{35}$$
The small dimensionless parameters $c_{ij}$ are given as:

\[
\begin{align*}
    c_{ee} &= \frac{m^2_e m^2_\mu \Delta m^2_{\mu e}}{2 \Delta m^2_{\tau e} (\alpha^2 \Delta m^2_{\tau \mu} + m^2_e \Delta m^2_{\mu \alpha})} \\
    c_{\mu e} &= \frac{m^2_e m^2_\mu \Delta m^2_{\mu e}}{2 \Delta m^2_{\tau \mu} (\alpha^2 \Delta m^2_{\tau \mu} + m^2_e \Delta m^2_{\mu \alpha})} \\
    c_{\tau e} &= \frac{m_e m_\mu}{2 \Delta m^2_{\tau \mu} (\alpha^2 \Delta m^2_{\tau \mu} + m^2_e \Delta m^2_{\mu \alpha})} \\
    c_{e\mu} &= \frac{m_e m_\mu}{2 \Delta m^2_{\tau \mu} \Delta m^2_{\mu \alpha}} \\
    c_{\tau \mu} &= \frac{m_e m_\mu \Delta m^2_{\mu \alpha}}{2 \Delta m^2_{\tau e} (\alpha^2 \Delta m^2_{\tau \mu} + m^2_e \Delta m^2_{\mu \alpha})} \\
    c_{e\tau} &= 0 \\
    c_{\mu \tau} &= 0 \\
    c_{\tau \tau} &= \frac{m^2_e m^2_\mu (m^2_\tau - \alpha^2) + (m^2_\mu - \alpha^2) (\alpha^2 - m^2_e)}{2 (m^2_\tau - m^2_\mu) (m^2_\tau - m^2_e) (\alpha^2 \Delta m^2_{\tau \mu} + m^2_e \Delta m^2_{\mu \alpha})}
\end{align*}
\]

(36)

where we have used the following notation for the mass squared differences $\Delta m^2_{ij} = m^2_i - m^2_j$, $\Delta m^2_{\alpha i} = \alpha^2 - m^2_i$ and $\Delta m^2_{\mu \alpha} = m^2_\mu - \alpha^2$ with $i = e, \mu, \tau$.

In the limit $\delta = 0$ with $m_\tau \gg m_\mu, m_e$ (i.e. $k_2 = 0$), we obtain the matrix $O_l$,

\[
O_l = \begin{pmatrix}
    \sqrt{\frac{\Delta m^2_{\mu \alpha}}{\Delta m^2_{\mu e}}} & \sqrt{\frac{\Delta m^2_{\mu \alpha}}{\Delta m^2_{\mu e}}} & 0 \\
    -\sqrt{\frac{\Delta m^2_{\mu e}}{\Delta m^2_{\mu \alpha}}} & \sqrt{\frac{\Delta m^2_{\mu e}}{\Delta m^2_{\mu \alpha}}} & 0 \\
    0 & 0 & 1
\end{pmatrix}
\]

(37)

which corresponds to just the mixing between the first and second generations and a consistency with the large angle solution for the solar neutrino anomalie imply:

\[
\alpha^2 = \frac{m^2_\mu + m^2_e}{2}
\]

(38)
The charged lepton mass matrix for this maximal solution is:

$$T'_l = \begin{pmatrix}
\sqrt{m_e^2 + m_\mu^2} & 0 & 0 \\
\frac{m_\mu^2 - m_e^2}{\sqrt{2(m_e^2 + m_\mu^2)}} & \sqrt{2m_\mu m_e} & 0 \\
0 & \frac{m_e^2 + m_\mu^2}{\sqrt{2(m_e^2 + m_\mu^2)}} & m_\tau
\end{pmatrix}$$ (39)

and in terms of the phase $\phi_1$, this can be written as:

$$T_l = \begin{pmatrix}
\sqrt{m_e^2 + m_\mu^2} & 0 & 0 \\
\frac{m_\mu^2 - m_e^2}{\sqrt{2(m_e^2 + m_\mu^2)}} & e^{i\phi_1} \frac{\sqrt{2m_\mu m_e}}{\sqrt{m_e^2 + m_\mu^2}} & 0 \\
0 & 0 & m_\tau
\end{pmatrix}$$ (40)

The above lepton mass matrices (33) and (39) or (40) permit us as desired to obtain a bimaximal scenario for solar and atmospheric neutrino. They lead automatically to a vanishing $V_{m13}$ mixing matrix element which makes $\nu_e - \nu_\mu$ and $\nu_\mu - \nu_\tau$ oscillations to be effectively a two–channel problem.

From our texture, we obtain for the leptonic mixing matrix $V_m$ ( see Appendix ),

$$V_m = \begin{pmatrix}
0.707 + 10^{-10}\delta & 0.707 - 10^{-10}\delta & 7 \times 10^{-7}\delta \\
-0.5 - 0.0002 \delta & 0.5 + 0.0002 \delta & -0.706 + 5.7 \times 10^{-8}\delta \\
-0.5 + 0.0002 \delta & 0.5 - 0.0002 \delta & 0.706 - 5.3 \times 10^{-8}\delta
\end{pmatrix}$$ (41)

where we have ignored the phases and used the maximal solutions (32), (38) and the following charged lepton masses at the $Z$–boson mass scale [29],

$$m_e(m_Z) = 0.487 MeV; \quad m_\mu(m_Z) = 102.7 MeV; \quad m_\tau(m_Z) = 1746.5 \pm 0.3 MeV$$ (42)

This result is in a good agreement with the nonvanishing but small $V_{m13}$ coming from the CHOOZ long–baseline neutrino oscillation experiment that measures $\overline{\nu}_e$ disappearence.

Moreover, the Jarlskog’s parameter $J$, which is related to CP violation, is given by,

$$J = \text{Im} \left( V_{m12}V_{m13}^*V_{m22}^*V_{m23} \right)$$

$$= \frac{\delta}{4} \frac{\Delta m^2_{\mu}}{\Delta m^2_{\tau e}} c_\tau \sin 2\theta_\nu \sin 2\theta_1 \sin f_\tau + O(\delta^2)$$

$$\approx 10^{-7} \delta$$ (43)

showing the smallness of the CP violation phenomena in the leptonic sector.
4 Conclusion:

In summary, we have investigated an extension of the Standard Model with right-handed Dirac neutrino per family by using the new textures for fermionic triangular mass matrices. We have proposed new triangular mass matrices of the type (8) as the most general description of the leptonic sector since the charged current weak interactions involve only left–handed fields and the physics is invariant under the weak transformations (6).

In particular, the patterns (1) and (2) can accommodate the bimaximal and nearly bimaximal solutions for atmospheric and solar anomalies. We have arrived also at compact formulae for the leptonic mixing matrix in terms of the masses and the parameters $\alpha, \beta'$ and $\delta$.

Moreover, we have found that they are consistent with the CHOOZ reactor data and account for the smallness of the CP violation in the leptonic sector.

Instead of considering Dirac neutrinos, it is possible to carry out a similar analysis for Majorana neutrinos where an extensive use of the new triangular mass matrices is taken. Here the Majorana symmetric mass matrix is $M_\nu = T_\nu M_\nu^{-1} T_\nu^T$ and the charged lepton mass matrix $T_l$ is given by (2). This study is under investigation [30].

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Appendix

For sake of completeness we give here the expressions for the mixing matrix $V_m$ in terms of the lepton masses, $\alpha, \beta'$ and $\delta$,

$$V_{mij} = O_{\nu 1i} O_{l 1j} e^{i\phi_1} + O_{\nu 2i} O_{l 2j} + O_{\nu 3i} O_{l 3j} e^{-i\phi_1}$$

Hence,

$$V_{m11} = \sqrt{\frac{\Delta m^{2}_{\mu\alpha}}{\Delta m^{2}_{\mu e}} (1 + c_{ee} \delta)} e^{i\phi_1}$$

$$V_{m12} = \sqrt{\frac{\Delta m^{2}_{\mu e}}{\Delta m^{2}_{\mu e}} (1 - c_{e\mu} \delta)} e^{i\phi_1}$$
\[
V_{m_{13}} = c_{\tau \tau} \delta e^{i\phi_1}
\]
\[
V_{m_{21}} = -\sqrt{\frac{\Delta m_{\nu e}^2}{\Delta m_{\nu \mu}^2}} \left(1 + c_{\mu e} \delta \right) - \sqrt{\frac{\Delta m_{\nu \mu}^2}{\Delta m_{\nu \nu}} c_{\tau \mu} \delta e^{-i\phi_\nu}
\]
\[
V_{m_{22}} = \sqrt{\frac{\Delta m_{\nu e}^2}{\Delta m_{\nu \mu}^2}} \left(1 - c_{\mu \mu} \delta \right) + \sqrt{\frac{\Delta m_{\nu \mu}^2}{\Delta m_{\nu \nu}} c_{\tau \mu} \delta e^{-i\phi_\nu}
\]
\[
V_{m_{23}} = \sqrt{\frac{\Delta m_{\nu e}^2}{\Delta m_{\nu \mu}^2}} c_{\mu \tau} \delta - \sqrt{\frac{\Delta m_{\nu \mu}^2}{\Delta m_{\nu \nu}} \left(1 - c_{\tau \tau} \delta \right) e^{-i\phi_\nu}
\]
\[
V_{m_{31}} = -\sqrt{\frac{\Delta m_{\nu e}^2}{\Delta m_{\nu \mu}^2}} \left(1 + c_{\mu e} \delta \right) + \sqrt{\frac{\Delta m_{\nu \mu}^2}{\Delta m_{\nu \nu}} c_{\tau \mu} \delta e^{-i\phi_\nu}
\]
\[
V_{m_{32}} = \sqrt{\frac{\Delta m_{\nu e}^2}{\Delta m_{\nu \mu}^2}} \left(1 - c_{\mu \mu} \delta \right) - \sqrt{\frac{\Delta m_{\nu \mu}^2}{\Delta m_{\nu \nu}} \left(1 - c_{\tau \tau} \delta \right) e^{-i\phi_\nu}
\]
\[
V_{m_{33}} = \sqrt{\frac{\Delta m_{\nu e}^2}{\Delta m_{\nu \mu}^2}} c_{\mu \tau} \delta + \sqrt{\frac{\Delta m_{\nu \mu}^2}{\Delta m_{\nu \nu}} \left(1 - c_{\tau \tau} \delta \right) e^{-i\phi_\nu}
\]

In addition, the matrix elements of the charged mass matrix reconstructed in terms of lepton masses, \(\alpha\) and \(\delta\) are:

\[
\beta = \frac{m_e m_\mu}{\alpha} \left(1 + \frac{\delta^2}{2}\right) + O\left(\delta^4\right)
\]
\[
\gamma = m_\tau \left(1 + \frac{\delta^2}{2}\right) + O\left(\delta^4\right)
\]
\[
k_1 = \sqrt{\frac{\Delta m_{\tau \mu}^2}{\alpha}} + \frac{m_e^2 m_\mu^2 \Delta m_{\mu \alpha}^2}{2\alpha(\alpha^2 \Delta m_{\tau \mu}^2 + m_e^2 \Delta m_{\mu \alpha}^2)} \delta^2 + O\left(\delta^3\right)
\]
\[
k_2 = \alpha \sqrt{\frac{\Delta m_{\tau \mu}^2}{\alpha^2 \Delta m_{\tau \mu}^2 + m_e^2 \Delta m_{\mu \alpha}^2}} \delta + O\left(\delta^3\right)
\]

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