Maximum Sustainable Yield (MSY) value in predator-prey model with holling type ii functional response

F Ilahi* and A Widiana

1Department of Mathematics, UIN Sunan Gunung Djati, Bandung, Indonesia
2Department of Biology, UIN Sunan Gunung Djati, Bandung, Indonesia

*Corresponding author’s email: fadilah.ilahi@uinsgd.ac.id

Abstract. Maximum Sustainable Yield (MSY) is a management exploitation that aims to maximize the catch but still maintain its sustainability. In this article, the authors try to build a model which have been modified from basic Lotka-Volterra model with addition logistic function to prey population and Holling Type II functional response to predator population as well as harvesting of both population. The equilibrium points and the stability are done here through the linearization. Dynamical simulations are shown to illustrate the dynamics of the two populations. Based on the results of analysis and dynamical simulation, the MSY value can be achieved when the harvesting efforts does not exceed or equal the MSY value.

1. Introduction

Maximum sustainable yield (MSY) is a simple management to manage fish resources by considering that more exploited resources cause loss of productivity. Schaefer (1954) was the first to introduce MSY to fisheries in a single species following the logistical laws of growth and proportional cro yields. Legovic et al. (2010) proposed MSY by taking into account species interactions, such as prey-predator relationships [1,2].

MSY aims to maintain the size of the fish population at the maximum point when the fish population is in a balanced condition between the populations of fish harvested and fish populations that are not harvested [3,4]. The level of catching or harvesting of fish can provide economic benefits for the community, and the process of harvesting fish must be according to needs, namely by not over-exploiting fish, it allows fish populations to remain productive [5]. Harvesting on the concept of maximum sustainable yield (MSY) is that fish populations that are not harvested grow and develop to replace part of the fish harvested, besides that the growth rate, survival rate and reproductive rate will increase when harvesting reduces density, so it will produce a surplus of biomass that can be harvested in the long run. If not, then sustainable harvesting is not possible [6–8].

Predator-prey systems are one type of system in which the population of two species is combined. Increasing the number of predators will cause a decrease in the number of prey, but the number predator will increase as well as the number of prey [9,10]. However, the level of predation is very high and it cause the extinction of prey species in a short time [5,11]. Therefore to prevent extinction of prey population, it is necessary to involved Holling type II interaction which assumes that every species in the population has its own carrying capacity [12-15]. In this paper, MSY will be discussed in multispecies systems and prey-predator systems are considered in accordance with the concept.
2. Methods
In this part, we will construct the mathematical model based on some assumptions as follow: 1) There are prey and predator species in closed populations, 2) The population studied is the fish population. Prey species are categorized as small fish species while predator species are categorized as large fish species, 3) Carrying capacity in the growth of prey and predator species, 4) Prey populations will increase due to growth in prey populations by following logistical growth and will decrease due to predation following the Holling type II response, 5) Predator populations will increase due to predation and will decrease by natural death because there is no prey as their food, 6) The predator spends its time on two activities, namely searching for prey and handling prey which consists of chasing, preying and digesting, 7) The prey and predator population will decrease when harvesting occurs.

Table 1. List of variables and parameters.

| Notation | Description |
|----------|-------------|
|  $x$     | Prey population |
|  $y$     | Predator population |
|  $r$     | Intrinsic growth of prey population |
|  $k$     | Carrying capacity |
|  $a$     | The maximum constant rate of prey consumption (predation level) |
|  $c$     | Saturation level of predator |
|  $b$     | The constant rate of conversion of prey biomass into predatory biomass |
|  $m$     | The death rate of the predator population due to the absence of prey |
|  $e$     | The amount of effort taken in the harvesting process |

Based on the assumptions, the interaction diagram that for the model is as follows:

Figure 1. Interaction diagram of mathematical model.

Mathematical model that constructed from the interaction diagram above described as a system below:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{axy}{cy + x} - ex \quad (1)$$

$$\frac{dy}{dt} = \frac{abxy}{cy + x} - my - ey \quad (2)$$

The analysis will start with the equilibrium point, coexistence condition, and the stability of the system. The equilibrium point will be obtained when every variables are constant, such as

$$\frac{dx}{dt} = \frac{dy}{dt} = 0$$

The system has two equilibrium points, namely

$$E_1 = \left(\frac{k(r - e)}{r}, 0\right) \quad (3)$$
\[ E_2 = \left( \frac{k(bcr - bce - ab + m + e)}{bcr}, \frac{k(ab - m - e)(bcr - bce - ab + m + e)}{bc^2r(m + e)} \right) \]  

The point \( E_1 \) is exist under condition \( r > e \), while \( E_2 \) is coexistence under one of these conditions as follow:

(H1) : \( bc < 1 \) and \( e < ab - m \)

(H2) : \( bc > 1, r > (ab - m) \) and \( e < ab - m \)

(H3) : \( bc > 1, r < (ab - m) \) and \( e < \left( \frac{bcr - ab + m}{bc - 1} \right) \)

The stability of the system will be determine by eigen value from Jacobian matrix. The general Jacobian matrix will be presented as

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix} = \begin{pmatrix}
r - \frac{2rx}{k} & - \frac{acy^2}{(cy + x)^2} - e & - \frac{ax^2}{(cy + x)^2} \\
\frac{bacy^2}{(cy + x)^2} & \frac{abx^2}{(cy + x)^2} & - m - e
\end{pmatrix}
\]

For the stability in \( E_1 \) the Jacobian matrix will be formed by

\[
J_1 = \begin{pmatrix}
e - r & - a \\
0 & ab - e - m
\end{pmatrix}
\]

Suppose \( \lambda \) is an eigenvalue. A system is said to be stable if the eigenvalues \( \lambda \) are real and negative. The eigenvalues from \( J_1 \) are

1. \( \lambda_1 < 0 \Leftrightarrow e < r \)
2. \( \lambda_2 < 0 \Leftrightarrow ab - m < e \)

By obtaining those eigenvalues, therefore the equilibrium point of \( E_1 \) is stable.

The stability in \( E_2 \) determined by the eigenvalues from this Jacobian matrix as follow:

\[
J_2 = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

\[
\det \begin{pmatrix}
\lambda - A & - B \\
- C & \lambda - D
\end{pmatrix} = 0
\]

\[
\Leftrightarrow \lambda^2 + \lambda p_1 + p_2 = 0
\]

with

\[-(A + D) = p_1 \]

\[AD - BC = p_2\]

From the equation (6) we will obtained two eigenvalues such as

\[
\lambda_1 = \frac{-p_1 + \sqrt{p_1^2 - 4p_2}}{2}
\]

\[
\lambda_2 = \frac{-p_1 - \sqrt{p_1^2 - 4p_2}}{2}
\]
In order to $E_2$ to be stable, it follows the condition that $p_1$ and $p_2$ must be positive. With some algebra operations, $p_1$ and $p_2$ will eventually lead to the condition as follows:

1. $p_1 > 0$
   \[\frac{ab^2cm + ab^2cr - a^2b^2 - bce^2 - 2bcem - bcm^2 + e^2 + 2em + m^2}{ab^2c} > 0\]
   \[\iff c > \frac{ab}{b(r+m)} \text{ and } bc < 1\]

2. $p_2 > 0$
   \[\frac{-ab^2ce^2 - ab^2cem + ab^2cer + ab^2emr - a^2b^2e - a^2b^2m + bce^3 + 2bce^2m - bce^2r + bcem^2 + 2bcemr - bcm^2r + 2abe^2 + 4abem + 2abm^2 - e^3 - 3em^2 - m^3}{b^2ac} > 0\]
   \[\iff e < ab - m \text{ and } e < \frac{bcr - ab + m}{bc - 1}\]

Therefore, the equilibrium point $E_2$ will be stable if it meets the following conditions:
\[c > \frac{ab}{b(r+m)}, bc < 1 \text{ and } e < (ab - m)\]

The formula and all the conditions will be investigate through the dynamical simulation. This following data is going to be simulate to make sure that the mathematical analysis that have been done is reasonable. The initial condition for the simulation are $x(0) = 20$, $y(0) = 1$, and $t = 0..200$ unit of time.

**Table 2.** Values of parameters for dynamical simulation.

| Notation | Value of parameters | Simulation I | Simulation II |
|----------|--------------------|--------------|---------------|
| $r$      | 2                  | 0.6          | 0.6           |
| $k$      | 100                | 100          | 100           |
| $a$      | 0.9                | 0.9          | 0.9           |
| $c$      | 1                  | 1.5          | 1.5           |
| $b$      | 0.8                | 0.8          | 0.8           |
| $m$      | 0.1                | 0.1          | 0.1           |

**Figure 2.** Dynamical simulation of the predator-prey system.
The prey population experienced a very significant increase from $t = 0$ to $t = 5$ per unit of time, the initial prey population numbered 20 increased to 78 due to the number of predator populations still following the prey population decreased at $t = 7$ per time and constant with $x = 65$ starting at $t = 71$. The predator population also experienced a very significant increase from the point $t = 0$ to $t = 45$ per unit of time, the initial predator population numbered 1 increased to a total of 28 due to the interaction with prey and the constant predator population with $y = 28$ starting from $t = 50$.

So it can be concluded that the prey population will be stable to reach the maximum limit with a number of 65, the prey population decreases due to predation and harvesting. As well as a stable predator population and reaching a maximum limit of 28, the predator population is affected by the number of prey populations due to saturation, natural death and harvesting.

3. Results and Discussion

After determining the equilibrium points, coexistence condition and stability for the system, now we will determine the value of maximum sustainable yield (MSY) and its policies. Yield function for combination harvesting of prey and predator [12-17], namely:

$$Y(e) = e \left( \frac{k(bcr - bce - ab + m + e)}{bcr} + \frac{k(ab - m - e)(bcr - bce - ab + m + e)}{bc^2r(m + e)} \right)$$

The policy of MSY concept of fishing with the same harvesting effort in both populations can be seen through the following figure

![Figure 3](image_url). The relationship between harvesting and the value of MSY.

Simulation I shows that predator biomass decreases and prey biomass increases as well as harvesting efforts increase. Both species can coexist when $e < 0.6$ and predator biomass becomes zero when $e = 0.62$. MSY function $Y(e)$ is achieved when harvesting is half of the biotic potential of prey, while predators experience extinction. So, it can be concluded that the MSY level will not be achieved with positive abundance of both populations and stability only for prey biomass.

Simulation II shows that the two populations between prey and predator were simultaneously extinct when harvesting $e = 0.5$. Biomass from both populations decreases when harvesting efforts increase. The MSY curve is concave down with respect to harvesting and the MSY function $Y(e)$ is reached when $e = 0.12$. So it can be concluded that MSY can occur when $e = 0.12$, which causes both populations to be protected from extinction.
4. Conclusion

According to the result of analysis and dynamical simulation, there are several factors that influenced the MSY value to be maximum but it still remain safe for both population. The first factor is the constant rate of conversion of prey biomass into predatory biomass \(b\), and the second one is saturation level of predator \(c\). The multiplication between those parameters must be less than zero \((bc < 1)\). When \((bc > 1)\), the MSY value will decrease as well as prey and predator population will get extinct.

Furthermore, the parameters \(e\) and \(c\) are very influential on the dynamics of prey populations. For parameters \(e\) the increasing rate of harvesting efforts carried out by humans towards prey and predators, the prey population also increases because harvesting is also carried out on predators. As well as for parameters \(c\) increasing saturation level of predator, prey population also increases due to reduced prey population that is eaten.

5. References

[1] Legović T, Klanjšček J and Geček S 2010 Maximum sustainable yield and species extinction in ecosystems Ecological Modelling 221 12 1569-1574
[2] Legović T and Geček S 2010 Impact of maximum sustainable yield on independent populations Ecological Modelling 221 17 2108-2111
[3] Geček S and Legović T 2012 Impact of maximum sustainable yield on competitive community Journal of theoretical biology 307 96-103
[4] Legović T and Geček S 2012 Impact of maximum sustainable yield on mutualistic communities Ecological Modelling 230 63-72
[5] Lan G, Fu Y, Wei C and Zhang S 2018 Dynamical analysis of a ratio-dependent predator–prey model with Holling III type functional response and nonlinear harvesting in a random environment Advances in Difference Equations 2018 1 198
[6] Ghosh B, Kar T K and Legovic T 2014 Relationship between exploitation, oscillation, MSY and extinction Mathematical biosciences 256 1-9
[7] Ghosh B, Kar T K and Legović T 2014 Sustainability of exploited ecologically interdependent species Population ecology 56 3 527-537
[8] Paul P and Kar T K 2016 Impacts of invasive species on the sustainable use of native exploited species Ecological Modelling 340 106-115
[9] Avelino P P, Bazeia D, Losano L, Menezes J and de Oliveira B F 2018 Spatial patterns and biodiversity in off-lattice simulations of a cyclic three-species Lotka-Volterra model EPL (Europhysics Letters) 121 4 48003
[10] Dannemann T, Boyer D and Miramontes O 2018 Lévy flight movements prevent extinctions and maximize population abundances in fragile Lotka–Volterra systems Proceedings of the National Academy of Sciences 115 15 3794-3799
[11] Hu D and Cao H 2017 Stability and bifurcation analysis in a predator–prey system with Michaelis–Menten type predator harvesting. Nonlinear Analysis: Real World Applications 33 58-82
[12] Paul P, Kar T K and Ghorai A 2018 Impact of marine reserve on maximum sustainable yield in a traditional prey-predator system Communications in Nonlinear Science and Numerical Simulation 54 34-49
[13] Sen M, Simha A and Raha S 2018 Adaptive Control Based Harvesting Strategy for a Predator–Prey Dynamical System Acta biotheoretica 1-21
[14] Pal D, Mahapatra G S and Samanta G P 2018 New approach for stability and bifurcation analysis on predator–prey harvesting model for interval biological parameters with time delays Computational and Applied Mathematics 37 3 3145-3171
[15] Liu X and Huang Q 2018 The dynamics of a harvested predator-prey system with Holling type IV functional response Biosystems
Acknowledgments
This article is the result of the research from the cluster “Penelitian Pembinaan dan Peningkatan Kapasitas” arranged by LP2M UIN Sunan Gunung Djati, and this is fully funded by BOPTN UIN Sunan Gunung Djati year 2018.