Study of a toy model and its relation to the Hubbard model with infinite range hopping

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October 7, 2018

Abstract

A toy model of strongly correlated fermions is studied using Green function and functional integration methods. The model exhibits a metal-insulator transition as the interaction is varied. In the case of unrestricted hopping is established an equivalence of the model with the Hubbard model with infinite range hopping. The generalization to the case with $N$ components is made.

1 Introduction

The simplest model to describe strong electron correlations is the Hubbard model \cite{Hubbard}. Unfortunately, the only exact solution available corresponds to the

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one dimensional case \[2\]. Thus, to understand its physical properties we have to relay upon approximate solutions which in many instances are highly non-trivial. Another alternative is to formulate exactly soluble toy models which share some properties with realistic many body models. The importance of the toy models lies in the fact that many interesting properties of complicated strongly correlated electron systems are simulated in a comprehensive way.

A simple toy model was proposed by Hatsugai and Kohmoto \[3\] (to be referred from now on as the 'HK model'). This model is exactly solvable in a very simple way and have the same atomic and band limits as the Hubbard model. Also, as Hatsugai and Kohmotto have shown, the model describes a metal-insulator transition (MIT). This MIT scenario of the HK model was studied from a scaling point of view by Continentino \[4\] who also formulated a boson version of the HK model. A closely related toy model was also discussed by Baskaran \[5\]. There is also an attractive version of the HK model which was proposed recently by Mattis and Bedersky \[6\].

Another different toy model consisting by a Hubbard model with an infinite range hopping has been solved exactly \[7\]. This model has the same atomic limit as the Hubbard model but does not have obviously the same band limit since hopping is unrestricted. As shown in ref. \[7\], this model is an insulator at half-filling for any on site Coulombian repulsion \( U > 0 \).

In this work we discuss the HK model using Green function and integral functional methods. It is shown that the HK model and the Hubbard model with infinite range hopping are equivalent in the thermodynamic limit if the hopping in the HK model is assumed to be unrestricted. We also solve exactly the \( N \) component HK model for any finite \( N \). This solution is obtained by an exact evaluation of the functional integral representation of the model. From this solution one obtains also the exact solution of the \( N \) component Hubbard model with infinite range hopping. For the sake of completeness, we also discuss a lattice fermion model where the interaction, rather than the hopping is of infinite range.

2 The HK model

The HK model is described by the Hamiltonian:
\[ H = \sum_{\vec{k}} \sum_{\sigma} [\epsilon(\vec{k}) - \mu] n_{\vec{k}\sigma} + U \sum_{\vec{k}} n_{\vec{k}\uparrow} n_{\vec{k}\downarrow} \] (2.1)

where \( n_{\vec{k}\sigma} \equiv c_{\vec{k}\sigma}^\dagger c_{\vec{k}\sigma} \) and \( \epsilon(\vec{k}) = -2t \sum_{i=1}^{d} \cos k_i \). \( \mu \) is the chemical potential. The dispersion \( \epsilon(\vec{k}) \) corresponds to a nearest neighbour hopping on a hypercubic lattice in \( d \) dimensions. Note that, in contrast to the Hubbard model, the Coulombian repulsion is local in \( \vec{k} \)-space rather than in real space. Since the Hamiltonian is local in \( \vec{k} \)-space and the interacting part commutes with the non-interacting one, it is straightforward to solve it. For instance, the exact free energy density is

\[ f = -\frac{1}{L \beta} \sum_{\vec{k}} \ln \left\{ 1 + 2 e^{\beta[\mu - \epsilon(\vec{k})]} + e^{\beta[2\mu - 2\epsilon(\vec{k}) - U]} \right\} \] (2.2)

where \( \beta = 1/T \), \( T \) being the temperature, and \( L \) is the number of lattice sites. Note that in the zero bandwidth limit we obtain the free energy density of the atomic limit of the Hubbard model. Also, in the limit \( U = 0 \) we get the band limit.

The Matsubara Green function is defined by \( G_{\sigma}(\vec{k}, \tau) = -\langle T_{\vec{k}\sigma}(\tau) c_{\vec{k}\sigma}^\dagger(0) \rangle \). In frequency representation it is given straightforwardly by

\[ G_{\sigma}(\vec{k}, \omega_n) = \frac{1 - n_{\vec{k}\sigma}}{i \omega_n + \mu - \epsilon(\vec{k})} + \frac{n_{\vec{k}\sigma}}{i \omega_n + \mu - \epsilon(\vec{k}) - U}, \] (2.3)

where \( \omega_n = (2n + 1)\pi T \), \( n \in \mathbb{Z} \), and

\[ n_{\vec{k}\sigma} = \frac{e^{\beta[\mu - \epsilon(\vec{k})]} + e^{\beta[2\mu - 2\epsilon(\vec{k}) - U]}}{1 + 2 e^{\beta[\mu - \epsilon(\vec{k})]} + e^{\beta[2\mu - 2\epsilon(\vec{k}) - U]}}. \] (2.4)

Note that we are assuming a paramagnetic phase, that is, \( n_{\vec{k}\uparrow} = n_{\vec{k}\downarrow} \). At \( T = 0 \) Eq.(2.4) assumes the form of a step function:

\[ n_{\vec{k}\sigma}(T = 0) = \theta(\mu_0 - \epsilon(\vec{k})) \frac{1}{2} \theta(U - |\epsilon(\vec{k}) - \mu_0|) + \theta(|\epsilon(\vec{k}) - \mu_0| - U)], \] (2.5)

where \( \mu_0 \) is the chemical potential at \( T = 0 \). \( \theta(x) \) is the usual Heaviside function. This occupation number has two discontinuous jumps. An immediate consequence of this is that Luttinger’s theorem does not hold. Note
that the exact Green function for the HK model share the same atomic and band limits with the Hubbard model.

As in the case of the Hubbard model, a half-filling condition is obtained by setting \( \mu = U/2 \). At half-filling and \( T = 0 \) we have that the Green function becomes

\[
G_{\sigma}(\vec{k}, \omega) = \frac{i\omega}{[i\omega - \epsilon(\vec{k})]^2 - \frac{U^2}{4}}.
\]  

(2.6)

In above, \( \omega \) is not more a Matsubara frequency but an Euclidian frequency, that is, we still have an analytic continuation to imaginary time but in zero temperature. Thus, the frequency \( \omega \) is continuous and not discrete.

From the poles of Eq.(2.6) we obtain two bands in straight analogy with the case of the Hubbard model where we have the so called lower Hubbard band and the upper Hubbard band. The poles gives the energy bands:

\[
E_- = \epsilon(\vec{k}) - \frac{U}{2}
\]  

(2.7)

\[
E_+ = \epsilon(\vec{k}) + \frac{U}{2}
\]  

(2.8)

Note that the gap has a size which depends on \( U \). If the two bands are separated by a nonzero gap we have an insulator. The value of \( U \) which gives a zero gap is determined by demanding that the top of the lower band coincides with the bottom of the lower one. Thus we find that the critical value of \( U \) that signals a MIT is given by \( U_c = 4td = W \). Thus for \( U < U_c \) the system is metallic. Note that in the HK model we have a MIT at any dimension while in the case of the Hubbard model it is well known that in one dimension no MIT exists at half-filling [2].

It is also interesting to study the behavior of the spin susceptibility. This is given by the response function:

\[
\chi(\vec{k}, \nu_n) = -\mu_B^2 \Pi(\vec{k}, \nu_n)
\]  

(2.9)

where \( \mu_B \) is the Bohr magneton and \( \Pi(\vec{k}, \nu_n) \) is the polarization Green function which is given by:

\[
\Pi(\vec{k}, \nu_n) = \frac{1}{L/\beta} \sum_{\vec{q}} \sum_{\omega_n} G(\vec{k} + \vec{q}, \nu_n + \omega_n)G(\vec{q}, \omega_n).
\]  

(2.10)
where \( \nu_n = 2n\pi T \), \( n \in \mathbb{Z} \), is a Bose Matsubara frequency. We are omitting spin indices for simplicity. The Matsubara sum is over a fermion frequency. We will use the Green functions at half-filling. We evaluate the susceptibility at zero temperature in the static limit and at the nesting wave vector \( \vec{Q} = (\pi, ..., \pi) \). Performing the Matsubara sum and taking the zero temperature limit gives

\[
\chi(\vec{Q},0) = -\frac{\mu^2}{4} \int_{1BZ} \frac{d^d q}{(2\pi)^d} \left( \frac{\theta(U/2 + \epsilon(\vec{q})) - \theta(U/2 - \epsilon(\vec{q}))}{2\epsilon(\vec{q})} \right)
+ \frac{\theta(U/2 + \epsilon(\vec{q})) - \theta(-\epsilon(\vec{q}) - U/2)}{2\epsilon(\vec{q}) + U} \frac{\theta(\epsilon(\vec{q}) - U/2) - \theta(U/2 - \epsilon(\vec{q}))}{2\epsilon(\vec{q}) - U}
+ \frac{\theta(\epsilon(\vec{q}) - U/2) - \theta(-\epsilon(\vec{q}) - U/2)}{2\epsilon(\vec{q})}
\]

(2.11)

When \( U = U_c = W \) the integral of the second term in Eq. (2.11) is divergent for \( \vec{q} = 0 \) ("infrared divergence") while the integral of the third term is divergent for \( \vec{q} = \vec{Q} \) ("ultraviolet divergence").

Let us evaluate explicitly the spin susceptibility. If we use a square density of states, \( \rho(\omega) = (1/W)\theta(W/2 - \omega)\theta(\omega + W/2) \), and consider \( U > U_c \) we get

\[
\chi(\vec{Q},0) = -\frac{\mu^2}{4W} \ln \left( \frac{U + U_c}{U - U_c} \right)
\]

(2.12)

Thus we have in fact that the susceptibility diverges as \( U \) approaches \( U_c \). Note that we do not have a simple power law behavior for the susceptibility. This should be contrasted with other descriptions of the MIT. For example, the Brinkman and Rice transition \[8\] gives a power law behavior for the static susceptibility, which is calculated for a zero value of the reciprocal lattice vector rather than in the nesting wave vector.

The analysis of this section allows us to conclude that the MIT in the HK model is a second order transition with mass gap \( \Delta = U - U_c \). Note that \( \Delta \) plays a role similar to the \( r_0 \) coupling associated to the quadratic part in the Ginsburg-Landau Hamiltonian. Remember that in a Ginsburg-Landau theory it is assumed that \( r_0 \approx T - T_c \) where \( T_c \) is the critical temperature associated to the second order transition. Here we have \( U \) rather than \( T \).
3 The Hubbard model with infinite range hopping

The kinetic energy of the HK model is identical to the one of the Hubbard model. Thus it is natural to ask about the differences of behavior between the HK model and the Hubbard model as we manipulate or change the hopping matrix elements. In this section we will consider the HK model with infinite range hopping (IRH). The IRH Hubbard model was already studied and solved exactly [7]. We will solve exactly the IRH HK model and show that the solution is the same in the thermodynamic limit, to that of the IRH Hubbard model. This establishes the equivalence between the IRH HK model and the IRH Hubbard model. In the case of infinite range hopping it does not matter if the electrons interact locally in \( \vec{k} \)-space or in real space. In both situations the solution is the same.

When the hopping is of infinite range the dispersion is given by \( \epsilon(\vec{k}) = -Lt\delta_{\vec{k}, 0} \) [7]. The exact free energy density for the IRH HK model is obtained by substituting this dispersion in the exact expression Eq.(2.2) and taking the thermodynamic limit. It is readily obtained that

\[
f_{IRH} = -2t - \frac{1}{\beta} \ln[1 + 2e^{\beta\mu} + e^{\beta(2\mu-U)}],
\]

which is the same expression that corresponds to the free energy density of the IRH Hubbard model [7].

Also, by substituting the IRH dispersion in Eq.(2.3) and taking the thermodynamic limit one obtains the exact Green function for the IRH HK model:

\[
G_{IRH}(\vec{k}, \omega_n) = \frac{1 - \delta_{\vec{k}, 0}}{Z_0} \left( \frac{1 + e^{\beta\mu}}{i\omega_n + \mu} + \frac{e^{\beta\mu} + e^{\beta(2\mu-U)}}{i\omega_n + \mu - U} \right),
\]

\( Z_0 \) being the atomic partition function per site. Since the free energy of the IRH HK model coincides with that of the IRH Hubbard model, the above expression gives the exact Green function of the IRH Hubbard model. It turns out that the two models have the same perturbative expansion in the thermodynamic limit. Note that the \( \vec{k} = 0 \) mode does not propagate while all the others \( \vec{k} \) modes have \( \vec{k} \) independent propagators. Note also that the \( \vec{k} \neq 0 \) modes have their propagators identical to the atomic limit Green function.
This behavior is manifested also in the expression for the free energy, Eq. (3.1), where we have a term given by the atomic solution corresponding to the modes $\vec{k} \neq 0$ and a term $-2t$ corresponding to the $\vec{k} = 0$ mode. This means that in the IRH regime the $\vec{k} = 0$ and $\vec{k} \neq 0$ modes separate.

From Eq. (3.2) one obtains the electron density:

$$n = \frac{2}{L^2} \sum \sum e^{-i\omega_n} \delta_{IRH}(\vec{k}, \omega_n)$$

$$= \frac{2[e^{\beta \mu} + e^{\beta(2\mu-U)}]}{1 + 2e^{\beta \mu} + e^{\beta(2\mu-U)}}$$

(3.3)

The above relation can also be obtained directly from the expression for the free energy as $n = -(\partial f_{IRH}/\partial \mu)_\beta$. Solving Eq. (3.3) for $\mu$ one obtains

$$\mu = U + \beta^{-1} \ln \left[ \sqrt{(1-n)^2 + n(2-n)e^{-\beta U}} - (1-n) \right]$$

(3.4)

in agreement with reference [7].

From Eq. (3.2) we obtain the density of doubly occupied sites $\bar{d}$:

$$\bar{d}(\beta, U, \mu) = \frac{e^{\beta(2\mu-U)}}{1 + 2e^{\beta \mu} + e^{\beta(2\mu-U)}}$$

(3.5)

The above expression in function of $\mu$ is far more compact than that one obtained in ref. [7] which has been written as a function of $n$. When $\mu = U/2$, which corresponds to $n = 1$, one has

$$\bar{d} = \frac{1}{2(e^{\frac{U}{T}} + 1)}$$

(3.6)

which again agrees with ref. [7].

4 The HK model with N components

In section 2 we solved exactly the HK model with $N = 2s + 1 = 2$ components ($s$ is the spin). In this section we show that this model is also exactly soluble for an arbitrary number of components.
Let us define the $N$ component spinors:

$$\psi^\alpha \equiv \begin{pmatrix} \psi^\alpha_1 \\ \cdot \\ \cdot \\ \psi^\alpha_N \end{pmatrix}, \quad (4.1)$$

$$\bar{\psi}^\alpha \equiv \begin{pmatrix} \psi^\dagger_1 \\ \cdot \\ \cdot \\ \psi^\dagger_N \end{pmatrix}. \quad (4.2)$$

Each component of the expression above is a Grassmann field. We will evaluate the exact partition function by writing it as an functional integral over the fields defined above. The partition function is given by

$$Z = \int \prod_k D\psi^\alpha_k D\bar{\psi}^\alpha_k e^{-S[\psi^\alpha_k, \bar{\psi}^\alpha_k]} \quad (4.3)$$

where the action $S$ is

$$S[\psi^\alpha_k, \bar{\psi}^\alpha_k] = \int_0^\beta d\tau \sum_k [\bar{\psi}^\alpha_k (\partial_\tau - \mu - U/2 + \epsilon(\vec{k})) \psi^\alpha_k + (U/2) \bar{\psi}^\alpha_k \psi^\alpha_k]^2, \quad (4.4)$$

where it is understood that the fields are time dependent (Matsubara time) and that they satisfy antiperiodic boundary conditions in Matsubara time.

We eliminate the quartic term by means of a Hubbard-Stratonovich transformation, getting the new action,

$$S' = \int_0^\beta d\tau \sum_k [\bar{\psi}^\alpha_k (\partial_\tau - \mu - U/2 + \epsilon(\vec{k}) - iU \phi_k) \psi^\alpha_k + (U/2) \phi_k^2] \quad (4.5)$$

where $\phi_k$ is an auxiliary Bose field. Since the new action is quadratic in the Fermi fields, it is straightforward integrate out these and obtain the following effective action:

$$S_{eff} = -N \ln \det[\partial_\tau - \mu - U/2 + \epsilon(\vec{k}) - iU \phi_k] + \frac{U}{2} \int_0^\beta d\tau \sum_k \phi_k^2 \quad (4.6)$$
We can evaluate exactly the determinant appearing in the above equation by solving the differential equation:

\[
[\partial_\tau - \mu - U/2 + \epsilon(\vec{k}) - iU \phi_\vec{k}(\tau)]f_n(\vec{k}, \tau) = \alpha_n(\vec{k})f_n(\vec{k}, \tau) \tag{4.7}
\]

with \(f_n\) satisfying the antiperiodic boundary condition \(f_n(\vec{k}, 0) = -f_n(\vec{k}, \beta)\). The Eq.(4.7) is easy to solve and the solution is given by

\[
f_n(\vec{k}, \tau) = c \exp \left( \int^\tau_0 d\tau' [\mu + U/2 - \epsilon(\vec{k}) + iU \phi_\vec{k}(\tau') + \alpha_n(\vec{k})] \right), \tag{4.8}
\]

c being an arbitrary constant. By applying the antiperiodic boundary condition to this solution we get

\[
\alpha_n(\vec{k}) = -i\omega_n - \mu - U/2 + \epsilon(\vec{k}) - i(U/\beta) \int^\beta_0 d\tau \phi_\vec{k}(\tau) \tag{4.9}
\]

where \(\omega_n\) is a Fermi Matsubara frequency. Thus, the determinant which appears in the effective action is given by the product of the \(\alpha_n\)'s. Therefore,

\[
S_{eff} = -N \sum_{n=-\infty}^{+\infty} \sum_{\vec{k}} \ln[-i\omega_n - \mu - U/2 + \epsilon(\vec{k}) - i(U/\beta) \int^\beta_0 d\tau \phi_\vec{k}(\tau)] + \frac{U}{2} \int^\beta_0 d\tau \phi_\vec{k}(\tau) \tag{4.10}
\]

The Matsubara sum in Eq.(4.10) is easily done by standard methods. The partition function is then written in the form:

\[
Z = \int \prod_{\vec{k}} D\phi_\vec{k} \left\{ 1 + e^{\beta[\mu + U/2 - \epsilon(\vec{k})] + \int^\beta_0 d\tau \phi_\vec{k}(\tau)} \right\}^N e^{-\frac{U}{2} \int^\beta_0 d\tau \phi_\vec{k}^2(\tau)} \tag{4.11}
\]

By expanding the term between braces and performing the Gaussian integrals one obtains the exact expression for the partition function of the \(N\) component HK model:

\[
Z = \prod_{\vec{k}} \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} e^{n\beta[\mu - \epsilon(\vec{k})]} e^{\beta^2 n(n-1)U/2} \tag{4.12}
\]

From Eq.(4.12) one obtains the free energy density:
\[ f = -\frac{1}{L\beta} \sum_{\vec{k}} \ln \left\{ \frac{N!}{\sum_{n=0}^{N} n!(N-n)!} e^{n\beta[\mu-\epsilon(\vec{k})]} e^{\beta n(n-1)U} \right\}. \] (4.13)

Note that in the case of \( N = 2 \) the result given in Eq.(2.2) is recovered. Also, in the zero bandwidth limit one obtains the atomic limit of a Hubbard model with \( N \) components.

It is worth to emphasize that the above result is exact for any finite \( N \).

We can also look for the solution at large \( N \). The limit \( N \to \infty \) of Eq.(4.13) is not obvious because of the factorials. However, we can obtain the large \( N \) solution by a saddle point evaluation of the functional integral with the effective action (4.10). In order to perform this saddle point evaluation it is useful to make the rescaling \( U \to \frac{U}{2}, \phi_{\vec{k}} \to N\phi_{\vec{k}} \). In this way we get that \( S_{\text{eff}} = N S_{\text{eff}} \) where \( S_{\text{eff}} \) is \( N \) independent (note that the term \( U/2 \) inside the logarithm is rescaled to \( U/(2N) \) which is zero at large \( N \)). In this case the saddle point solution corresponds to the exact solution when \( N \to \infty \). The large \( N \) solution is found to be just the Hartree-Fock approximation for the HK model.

From the exact expression (4.13) it is possible to obtain the exact free energy density for the \( N \) component IRH Hubbard model. It is given by

\[ f_{\text{IRH}}(N) = -Nt - \frac{1}{\beta} \ln \left[ \frac{N!}{\sum_{n=0}^{N} n!(N-n)!} e^{n\beta[\mu+\epsilon(\vec{k})]} e^{\beta n(n-1)U} \right]. \] (4.14)

5 The infinite range interaction case

Let us write the interaction of the HK model in the real space representation. This is achieved by Fourier transformation:

\[ c_{\vec{k}\sigma} = \frac{1}{\sqrt{L}} \sum_{i} \exp (-i\vec{k} \cdot \vec{R}_i)c_{i\sigma}. \] (5.1)

It is readily obtained that the interaction part is given by

\[ H_1 = \frac{U}{L} \sum_{ijkl} \delta_{i+k,j+l} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger c_{k\uparrow} c_{l\downarrow}. \] (5.2)
The above interaction is obviously of infinite range. We note that it is not necessary to consider only interaction between opposite spins. Generalizing the above interaction in a such a way as to include interaction between like spins correspond to a shift in the chemical potential. In this situation we have a particular case of the HK model which consist of a model with Hamiltonian given by

$$H = -t \sum_{<i,j>} \sum_{\sigma} c_i^\dagger \sigma c_i \sigma + h.c. - \mu \sum_i \sum_{\sigma} n_{i\sigma} + \frac{U}{2L} \sum_{i,j} n_i n_j, \quad (5.3)$$

where $n_i = \sum_{\sigma} n_{i\sigma}$. Since the interaction is the same for every pair of sites in the lattice we can write $\sum_{i,j} n_i n_j = (\sum_i n_i)^2$. It is straightforward to write the above Hamiltonian in $\vec{k}$-space as

$$H = \sum_{\vec{k}} \sum_{\sigma} [\epsilon(\vec{k}) - \mu] n_{\vec{k}\sigma} + \frac{U}{2L} (\sum_{\vec{k}} n_{\vec{k}})^2. \quad (5.4)$$

We write the partition function in the same way as in the last section, that is, by going to an integral functional representation, the fermionic fields being substituted by Grassmann fields. As before, the quartic term in the action is eliminated by applying a Hubbard-Stratonovich transformation. We obtain the following action involving an auxiliary Bose field $\phi$:

$$S = \int_0^\beta d\tau \left\{ \sum_{\vec{k}} \sum_{\sigma} \overline{\psi}_{\vec{k}\sigma} [\partial_\tau - \mu + \epsilon(\vec{k}) - iU\phi] \psi_{\vec{k}\sigma} + \frac{U}{2} \phi^2 \right\}. \quad (5.5)$$

By integrating the fermions and carrying the calculations in a way similar to the one in the preceeding section, we get an effective action $S_{eff} = L\overline{S}_{eff}$, where $\overline{S}_{eff}$ is given by

$$\overline{S}_{eff} = -2 \int_{-\infty}^\infty d\epsilon \rho_0(\epsilon) \sum_n \ln[-i\omega_n - \mu + \epsilon - \frac{U}{\beta} \int_0^\beta d\tau \phi(\tau)] + \frac{U}{2} \int_0^\beta d\tau \phi^2(\tau), \quad (5.6)$$

where $\rho_0(\epsilon)$ is the bare density of states. Therefore, we write the partition function in the following form:

$$Z = N \int D\phi e^{-L\overline{S}_{eff}[\phi]}, \quad (5.7)$$
where $N$ is a normalization factor. From Eq. (5.7) we see that in the thermodynamic limit the saddle point of the action corresponds to the exact solution of the problem. The saddle point is given by an imaginary number in the repulsive case: $\phi_0 = i n_0$, $n_0$ being a real number in the interval $[0, 2]$. In the attractive case the saddle point is a real number because the resulting Hubbard-Stratonovich transformation will not involve the imaginary unity $i$ multiplying the linear term in $\phi$ as in the preceding equations. The saddle point equation is given by

$$n_0 = \frac{2}{\beta} \sum_n \int_{-\infty}^{\infty} d\epsilon \frac{\rho_0(\epsilon)}{i \omega_n + \mu - \epsilon - Un_0}. \tag{5.8}$$

Eq. (5.8) is just a Hartree-Fock approximation. This means that in the case of an infinite range interaction of the form described in Eq. (5.3) the Hartree-Fock approximation is the exact solution of the problem. Note that in contrast to the case of the HK model, we have a Fermi liquid in the present situation.

6 Conclusion

Strongly correlated electron systems are very important in today condensed matter physics. Unfortunately, even the simplest models developed in this field are very complicated to handle. For this reason, many important features cannot be understood exactly. For example, the MIT in the Hubbard model can be studied only approximately. The exactly soluble one dimensional Hubbard model does not exhibit a MIT at half-filling. However, the MIT in the Hubbard model can be simulated through the study of exactly solvable toy models like the one we have discussed in this paper, the HK model. We discussed the HK model within the framework of the Green function and functional integration formalism. The metal-insulator transition at half-filling is easily characterized from the structure of the poles of the Green function. The exact Green function is used for evaluate the static susceptibility at the nesting wave vector at zero temperature. It was found that the susceptibility diverges as $U$ approaches the critical value $U_c$, signaling a localization transition. This also happens in the Brinkman-Rice approximation in the Hubbard model. However, in the Brinkman-Rice case the static susceptibility is evaluated at zero wave vector rather than in the nesting vec-
tor. Also, in the case treated here we do not have a power law behavior of the susceptibility.

Another interesting result is that the HK model with infinite range hopping is equivalent to the Hubbard model with infinite range hopping, giving the same physical information. For example, the free energy of both models coincide. Actually, this is an expected result once the IRH Hubbard model separate the $\vec{k} = 0$ modes of the $\vec{k} \neq 0$ modes and the HK model merges the exact atomic and band limits of the Hubbard model. Thus, we have shown that the HK model shares one more property with the Hubbard model. In the limit of unrestricted hopping the two models are equivalent.

Finally, the HK model admits a further generalization. It can be solved exactly even in the case of an arbitrary number of components.

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