MEAN-SQUARE ALMOST AUTOMORPHIC SOLUTIONS FOR
STOCHASTIC DIFFERENTIAL EQUATIONS
WITH HYPERBOLICITY

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Abstract. In the setting of mean-square exponential dichotomies, we study
the existence and uniqueness of mean-square almost automorphic solutions of
non-autonomous linear and nonlinear stochastic differential equations.

1. Introduction. In order to generalize the almost periodicity, the notion of al-
mmost automorphy was introduced by Bochner [2] in 1962. As shown in [10], a
continuous function f defined on a Banach space X is said to be almost automor-
phic if any sequence {t′n} of real numbers contains a subsequence {tn} such that
the limit
\[ g(t) := \lim_{n \to \infty} f(t + t_n) \]
is well defined and
\[ \lim_{n \to \infty} g(t - t_n) = f(t) \]
for all \( t \in \mathbb{R} \). Veech [32] presented an example which is almost automorphic but not
almost periodic.

During the last few decades, the notion of almost automorphy was used in the
study of differential equations. In 1981 Johnson [17] constructed a first-order scalar
almost periodic equation which has an almost automorphic solution but no almost
periodic solutions. In the monograph [28], Shen and Yi gave an example in which
the general dynamics produced by almost periodic differential equations is almost

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automorphic but has no almost periodicity. Some recent results on almost automor-
phy was introduced for stochastic processes and the existence, uniqueness and
behaviors are essentially deterministic with the stochasticity built into or hidden in
the time-dependent state spaces. In [6, 14], the concept of mean-square almost au-
tomorphy was introduced for stochastic processes and the existence, uniqueness and
asymptotic stability of mean-square almost automorphic solutions were established
for some linear and nonlinear stochastic differential equations.

In this paper we investigate mean-square almost automorphic solutions for non-
autonomous stochastic differential equations with hyperbolicity. In the setting that
the equation
\[ dx(t) = A(t)x(t)dt + G(t)x(t)d\omega(t), \quad t \in \mathbb{R}, \]
where \( A(t) = (A_{ij}(t))_{n \times n} \) and \( G(t) = (G_{ij}(t))_{n \times n} \) are both almost automorphic
matrices, admits a mean-square exponential dichotomy, we discuss the existence of
a unique mean-square almost automorphic solution for the non-autonomous linear
SDE
\[ dx(t) = (A(t)x(t) + f(t))dt + (G(t)x(t) + g(t))d\omega(t), \quad t \in \mathbb{R}, \]
where \( f, g \) are both mean-square almost automorphic vector-valued progresses.
Moreover, we also prove the existence and uniqueness of the mean-square almost
automorphic solution for the nonlinear SDE
\[ dx(t) = (A(t)x(t) + f(t, x(t)))dt + (G(t)x(t) + g(t, x(t)))d\omega(t), \quad t \in \mathbb{R}. \]

This paper is organized as follows. In Section 2, we present some preliminary
results. In Section 3, we establish the existence and uniqueness of mean-square almost
automorphic solutions for linear SDE (2) and nonlinear SDE (3). In the
proof of our main results, we will use the definitions and properties of product
integration for SDEs, which will be presented in Appendix for the convenience of
readers.

2. Preliminaries. Throughout this paper we use \( A^T \) to denote the transpose of a
matrix or a vector \( A \), and let \( \omega(t) = (\omega_1(t), \ldots, \omega_n(t))^T \) be an \( n \)-dimensional Browni-
nian motion. Let \( \| \cdot \| \) denote the Euclidean norm in \( \mathbb{R}^n \) or operator norm in \( \mathbb{R}^{n \times m} \).
Let \( L^p([a, b]; \mathbb{R}^{n \times m}) \) be the space of all measurable functions \( f \) such that
\( \int_a^b \| f(t) \|^p dt < \infty \) and \( C(I; \mathbb{R}^{n \times m}) \) be the set of all continuous \( \mathbb{R}^{n \times m} \)-valued functions defined on an interval \( I \). In addition, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) denote a probability space, where \( \Omega \) is the sample
space, \( \mathcal{F} \) is its Borel \( \sigma \)-algebra, \( \mathbb{P} \) is the Wiener measure. Let \( L^2(\Omega, \mathbb{R}^n) \) be the
family of all \( \mathbb{R}^n \)-valued random variables \( x \) such that \( \mathbb{E}\| x \|^2 := \int_\Omega \| x \|^2 d\mathbb{P} < \infty \), and set
\[ \| x \|_2 := \left( \mathbb{E}\| x \|^2 \right)^{1/2} = \left( \int_\Omega \| x \|^2 d\mathbb{P} \right)^{1/2}. \]
As defined in [6, 14], a stochastic process \( x : \mathbb{R} \to L^2(\Omega, \mathbb{R}^n) \) is said to be \textit{stochastically continuous} if
\[
\lim_{t \to s} \mathbb{E}\|x(t) - x(s)\|^2 = 0, \quad \forall s \in \mathbb{R},
\]
and a stochastically continuous stochastic process \( x \) is said to be \textit{mean-square almost automorphic} if for every sequence \( \{t_n\} \) of real numbers there exists a subsequence \( \{t'_n\} \) and a stochastic process \( y : \mathbb{R} \to L^2(\Omega, \mathbb{R}^n) \) such that
\[
\lim_{n \to \infty} \mathbb{E}\|x(t + t'_n) - y(t)\|^2 = 0, \quad \lim_{n \to \infty} \mathbb{E}\|y(t - t'_n) - x(t)\|^2 = 0
\]
for \( t \in \mathbb{R} \). We use \( AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)) \) to denote the set of all such stochastic processes. Moreover, it was proved in [14, Theorem 2.4] that \( AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)) \) is a Banach space equipped with the norm
\[
\|x\|_\infty := \sup_{t \in \mathbb{R}} \|x(t)\|_2 = \sup_{t \in \mathbb{R}} (\mathbb{E}\|x(t)\|^2)^{1/2}.
\]
Thus the following properties are obvious: (1) \( \lambda f + \mu g \in AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)) \) for every scalar \( \lambda, \mu \) and \( f, g \in AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)) \); (2) if \( f \in AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)) \), then \( \|f\|_\infty \leq M < \infty \) and so \( f \) is bounded in \( L^2(\Omega, \mathbb{R}^n) \); (3) if \( f_n \in AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)) \) and \( f_n \to f \) uniformly in \( \mathbb{R} \), then \( f \in AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)) \).

**Lemma 2.1.** [14, Theorem 2.6] Let \( f : \mathbb{R} \times L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n) \) be a mean-square almost automorphic function in \( t \in \mathbb{R} \) for each \( x \in L^2(\Omega, \mathbb{R}^n) \). Assume that there exists a constant \( L > 0 \), independent of \( t \), such that for all \( x, y \in L^2(\Omega, \mathbb{R}^n) \),
\[
\mathbb{E}\|f(t, x) - f(t, y)\|^2 \leq L \mathbb{E}\|x - y\|^2, \quad \forall t \in \mathbb{R}.
\]
Then for any mean-square almost automorphic function \( x \in L^2(\Omega, \mathbb{R}^n) \), the function \( F : \mathbb{R} \to L^2(\Omega, \mathbb{R}^n) \) defined by \( F(t) := f(t, x(t)) \) is mean-square almost automorphic.

Recall that a solution \( x \) of a stochastic differential equation
\[
dx(t) = f(t, x(t))dt + g(t, x(t))d\omega(t)
\]
on a finite subinterval \([a, b]\) is said to be \textit{unique} if any other solution \( \tilde{x} \) is indistinguishable from \( x \), that is,
\[
\mathbb{P}\{t \in [a, b] : x(t) = \tilde{x}(t)\} = 1,
\]
and if the assumption of existence and uniqueness theorem holds on every finite subinterval of \( \mathbb{R} \), then (4) has a unique solution \( x \) on the entire interval \(( -\infty, \infty) \).

Finally, we recall the definition of mean-square exponential dichotomy. Let \( \Phi(t) \) be a fundamental matrix of (1). By [22, Theorem 3.2.4], \( \Phi(t) \) is invertible with probability 1 for all \( t \) in any finite subinterval of \(( -\infty, \infty) \). We say that (1) admits a \textit{mean-square exponential dichotomy} if there exist positive constants \( M \) and \( \alpha \) and projections \( P(t) \) such that
\[
\Phi(t) \Phi^{-1}(s) P(s) = P(t) \Phi(t) \Phi^{-1}(s), \quad \forall t \geq s,
\]
and
\[
\mathbb{E}\|P(t) \Phi(t) \Phi^{-1}(s)\|^2 \leq M e^{-\alpha(t-s)}, \quad \forall t \geq s, \tag{5}
\]
\[
\mathbb{E}\|Q(t) \Phi(t) \Phi^{-1}(s)\|^2 \leq M e^{-\alpha(s-t)}, \quad \forall t \leq s, \tag{6}
\]
where \( Q(t) = \text{Id} - P(t) \). The notion of mean-square exponential dichotomies was introduced by Stanzhyts’kyi [30] and Stoica [31].

We remark here that the classical notion of exponential dichotomy was first introduced for ordinary differential equations by Perron in [26], and plays an important
role in the study of dynamical behaviors of differential equations, particularly in what concerns the study of stable and unstable invariant manifolds, and therefore has attracted much attention during the last few decades. We refer to the books [9, 24] for details and further references related to exponential dichotomies.

3. Main results. In this section, we establish the existence and uniqueness of mean-square almost automorphic solutions of nonautonomous linear SDE (2) and nonlinear SDE (3) when the linear equation (1) admits a mean-square exponential dichotomy.

Lemma 3.1. Suppose that \( A, G \) are almost automorphic and (1) admits a mean-square exponential dichotomy. Assume further that \( f, g \) are mean-square almost automorphic functions. Then (2) has a bounded solution in \( L^2(\Omega; \mathbb{R}^n) \), which is given by

\[
x(t) = \int_{-\infty}^{t} \Phi(t) \Phi^{-1}(s) P(s) g(s) d\omega(s) + \int_{-\infty}^{t} \Phi(t) \Phi^{-1}(s) P(s) [f(s) - G(s) g(s)] ds
- \int_{t}^{\infty} \Phi(t) \Phi^{-1}(s) Q(s) g(s) d\omega(s)
- \int_{t}^{\infty} \Phi(t) \Phi^{-1}(s) Q(s) [f(s) - G(s) g(s)] ds.
\] (7)

Proof. Since \( \Phi(t) \) is a fundamental matrix of (1), we know that

\[
d\Phi(t) = A(t) \Phi(t) dt + G(t) \Phi(t) d\omega(t).
\] (8)

Set

\[
\xi(t) = \int_{-\infty}^{t} \Phi^{-1}(s) P(s) g(s) d\omega(s) + \int_{-\infty}^{t} \Phi^{-1}(s) P(s) [f(s) - G(s) g(s)] ds
- \int_{t}^{\infty} \Phi^{-1}(s) Q(s) g(s) d\omega(s)
- \int_{t}^{\infty} \Phi^{-1}(s) Q(s) [f(s) - G(s) g(s)] ds.
\]

Consequently,

\[
d\xi(t) = \Phi^{-1}(t) [f(t) - G(t) g(t)] dt + \Phi^{-1}(t) g(t) d\omega(t).
\] (9)

Let \( x(t) = \Phi(t) \xi(t) \). Using (8), (9) and the Itô Product Rule (see e.g. [22]), we obtain

\[
dx(t) = d\Phi(t) \xi(t) + \Phi(t) d\xi(t) + G(t) \Phi(t) \Phi^{-1}(t) g(t) dt
= [A(t) x(t) + f(t)] dt + [G(t) x(t) + g(t)] d\omega(t),
\]

which means that (7) is the solution of (2).

It remains to prove that \( x \) is bounded in \( L^2(\Omega; \mathbb{R}^n) \). Since \( G(t) \) is an almost automorphic function, there exists a constant \( \gamma_1 > 0 \) such that

\[
\|G(t)\| \leq \gamma_1, \quad \forall t \in \mathbb{R}.
\] (10)

Similarly, for \( f, g \in AA(\mathbb{R}; L^2(\Omega; \mathbb{R}^n)) \) there exists a constant \( \gamma_2 > 0 \) such that

\[
\mathbb{E}\|f(t)\|^2 + \mathbb{E}\|g(t)\|^2 \leq \gamma_2, \quad \forall t \in \mathbb{R},
\] (11)

where \( a \vee b \) denotes the maximum of \( a \) and \( b \). It follows from the elementary inequality

\[
\left\| \sum_{k=1}^{m} a_k \right\|^2 \leq m \sum_{k=1}^{m} \|a_k\|^2,
\] (12)
that
\[ E\|x(t)\|^2 \]
\[ = E\left| \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)g(s)ds + \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)[f(s) - G(s)g(s)]ds \right|^2 \]
\[ - \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)g(s)ds - \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)[f(s) - G(s)g(s)]ds \right|^2 \]
\[ \leq 4E\left\| \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)g(s)ds \right\|^2 + 4E\left\| \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)g(s)ds \right\|^2 \]
\[ + 4E\left\| \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)[f(s) - G(s)g(s)]ds \right\|^2 \]

By using the boundedness properties (11) and the inequalities (5), we first evaluate the first term of the right-hand side by Itô isometry property of stochastic integrals as follows:
\[ E\left\| \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)g(s)ds \right\|^2 \leq E\int_{-\infty}^{t} \|P(t)\Phi(t)\Phi^{-1}(s)g(s)\|^2 ds \]
\[ \leq M\int_{-\infty}^{t} e^{-\alpha(t-s)}\|g(s)\|^2 ds \leq M\frac{\gamma_2}{\alpha}. \]

Similarly, by using the boundedness properties (11), the inequalities (6), and the isometry property of Itô integral, one can deduce the second term as follows:
\[ E\left\| \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)g(s)ds \right\|^2 \leq M\int_{t}^{\infty} e^{-\alpha(s-t)}\|g(s)\|^2 ds \leq M\frac{\gamma_2}{\alpha}. \]

As to the third term, it follows from \( E\|x\| \leq \sqrt{E\|x\|^2}\), Cauchy-Schwarz inequality, the boundedness properties (10)-(11), and the inequalities (5) that
\[ E\left\| \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)[f(s) - G(s)g(s)]ds \right\|^2 \]
\[ = E\left\| \int_{-\infty}^{t} (P(t)\Phi(t)\Phi^{-1}(s))^\frac{1}{2} (P(t)\Phi(t)\Phi^{-1}(s))^\frac{1}{2} [f(s) - G(s)g(s)]ds \right\|^2 \]
\[ \leq \left( \int_{-\infty}^{t} E\|P(t)\Phi(t)\Phi^{-1}(s)\|^2 ds \right) \left( \int_{-\infty}^{t} E\|P(t)\Phi(t)\Phi^{-1}(s)\| E[f(s) - G(s)g(s)]^2 ds \right) \]
\[ \leq 2M\gamma_2(\gamma_1^2 + 1) \left( \int_{-\infty}^{t} e^{-\frac{\alpha}{2}(t-s)} ds \right)^2 \]
\[ \leq \frac{8M\gamma_2(\gamma_1^2 + 1)}{\alpha^2}. \]

Following the same idea in the proof of the third term, one can prove the last term as follows:
\[ E \left( \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)[f(s) - G(s)g(s)]ds \right)^2 \leq \frac{8M\gamma_2(\gamma_1^2 + 1)}{\alpha^2}. \]
Therefore,
\[ \mathbb{E}\|x(t)\|^2 \leq \frac{8M\gamma_2(\alpha + 8(\gamma_1^2 + 1))}{\alpha^2}, \]
which means that \( x \) is bounded in \( L^2(\Omega, \mathbb{R}^n) \), and this completes the proof.

Following Dollard and Friedman’s idea \[12\], our next lemma shows that the fundamental matrix \( \Phi \) of (1) can be represented by the product integration. Such a representation was given by Gill and Johansen \[15\] and Slavík \[29\] for ordinary differential equations.

**Lemma 3.2.** Let \( A, G \in C(\mathbb{R}; \mathbb{R}^{n \times n}) \). Then the principal matrix solution \( \Phi(t) \) of the system (1) at the initial time \( t_0 \) can be represented as
\[ \Phi(t) = \prod_{t_0}^t e^{(A(s)ds + G(s)d\omega(s))}, \] (13)
where \( \prod \) is the notation of product integration.

**Proof.** The proof as well as the definition and properties of product integration of SDE (1) will be given in Appendix A.

**Lemma 3.3.** Assume that system (1) admits a mean-square exponential dichotomy, and \( A, G \in C(\mathbb{R}; \mathbb{R}^{n \times n}) \) are almost automorphic matrices. Then the projections \( P(t), Q(t) \) in (5) and (6) are mean-square almost automorphic.

**Proof.** First since \( A, G \in C(\mathbb{R}; \mathbb{R}^{n \times n}) \) are almost automorphic, we know that for every sequence \( \{t'_n\} \) of real numbers there exists a subsequence \( \{t_n\} \) such that for some functions \( \tilde{A}(t) \) and \( \tilde{G}(t) \),
\[ \lim_{n \to \infty} A(t + t_n) = \tilde{A}(t), \quad \lim_{n \to \infty} \tilde{A}(t - t_n) = A(t), \]
and
\[ \lim_{n \to \infty} G(t + t_n) = \tilde{G}(t), \quad \lim_{n \to \infty} \tilde{G}(t - t_n) = G(t) \]
for each \( t \in \mathbb{R} \).

It follows from Lemma 3.2 that
\[ \Phi(t) = \prod_{t_0}^t e^{(A(s)ds + G(s)d\omega(s))} \]
is the principal matrix solution \( \Phi(t) \) of the system (1) at the initial time \( t_0 \). Then, from the Definition 4.7 we know that
\[ \Phi^{-1}(t) = \left( \prod_{t_0}^t e^{(A(s)ds + G(s)d\omega(s))} \right)^{-1} = \prod_{t_0}^t e^{(A(s)ds + G(s)d\omega(s))}. \]

Let \( P(t) = \Phi(t)P\Phi^{-1}(t) \) with \( P \) is a projection such that \( P^2 = P \). It is easy to verify that \( P(t) \) is mean-square bounded and stochastically continuous in \( \mathbb{R} \). Hence, we can suppose that for every sequence \( \{t'_n\} \) of real numbers, there exists a subsequence \( \{t_n\} \) such that
\[ \lim_{n \to \infty} \mathbb{E}\|P(t + t_n) - \tilde{P}(t)\|^2 = 0 \]
for each \( t \in \mathbb{R} \). In addition, it follows from Theorem 4.8 that for every sequence \( \{t'_n\} \) of real numbers there exists a subsequence \( \{t_n\} \) such that
\[ P(t + t_n)\Phi(t + t_n)\Phi^{-1}(s + t_n) \]
Similarly, for every sequence \( \{t_n\} \) of real numbers, there exists a subsequence \( \{t_{n_k}\} \) such that
\[
\lim_{n \to \infty} \mathbb{E}[\|P(t + t_{n_k})\Psi(1 + t_{n_k}) \Psi^{-1}(s + t_{n_k}) - \hat{P}(t)\Psi^{-1}(s)\|^2] = 0.
\]
for \( t \in \mathbb{R} \). Likewise, for every sequence \( \{t_n^*\} \) of real numbers, there exists a subsequence \( \{t_{n_k}^*\} \) such that
\[
\lim_{n \to \infty} \mathbb{E}[\|Q(t + t_{n_k} - t_{n_k}^*)\Phi(t + t_{n_k} - t_{n_k}^*) \Phi^{-1}(s + t_{n_k} - t_{n_k}^*) - (\hat{P}_t - \hat{\hat{P}}(t))\Phi(t)\Phi^{-1}(s)\|^2] = 0.
\]
Combining the above arguments and inequalities (5)-(6), we have
\[
\begin{aligned}
\mathbb{E}[\|\hat{P}(t)\Phi(t)\Phi^{-1}(s)\|^2] &\leq 2Me^{-\alpha(t-s)}, & \forall t \geq s, \\
\mathbb{E}[\|(\hat{P}_t - \hat{\hat{P}}(t))\Phi(t)\Phi^{-1}(s)\|^2] &\leq 2Me^{-\alpha(s-t)}, & \forall t \leq s.
\end{aligned}
\]
It follows from the same method as in [9, pp. 19] that \( \hat{P}(t) \) is indistinguishable from \( P(t) \). Now the proof is finished. \( \square \)

**Theorem 3.4.** Assume that (1) admits a mean-square exponential dichotomy, \( A, G \in C(\mathbb{R}; \mathbb{R}^{2n}) \) are almost automorphic, and \( f, g \in AA(\mathbb{R}; L^2(\Omega; \mathbb{R}^n)). \) Then (2) has a unique bounded mean-square almost automorphic solution in \( L^2(\Omega, \mathbb{R}^n) \).

**Proof.** By Lemma 3.1, we know that the function
\[
x(t) = \int_{-\infty}^t \Phi(t)\Phi^{-1}(s)P(s)g(s)ds + \int_{-\infty}^t \Phi(t)\Phi^{-1}(s)P(s)[f(s) - G(s)g(s)]ds
\]
is a bounded solution in $L^2(\Omega, \mathbb{R}^n)$ of (2).

Now we show that $x(t)$ is a mean-square almost automorphic stochastic process. Since $f$ and $g$ are mean-square almost automorphic functions, i.e., for every sequence $\{t'_n\}$ of real numbers, there exists a subsequence $\{t_n\}$ such that

$$
\lim_{n \to \infty} \mathbb{E}\|f(t + t_n) - \tilde{f}(t)\|^2 = 0, \quad \lim_{n \to \infty} \mathbb{E}\|f(t - t_n) - f(t)\|^2 = 0,
$$

and

$$
\lim_{n \to \infty} \mathbb{E}\|g(t + t_n) - \tilde{g}(t)\|^2 = 0, \quad \lim_{n \to \infty} \mathbb{E}\|g(t - t_n) - g(t)\|^2 = 0
$$

for each $t \in \mathbb{R}$. In addition, it follows from Lemma 3.3 that projections $P(t)$ is mean-square automorphy, that is, for every sequence $\{t'_n\}$ of real numbers, there exists a subsequence $\{t_n\}$ such that

$$
\lim_{n \to \infty} \mathbb{E}\|P(t + t_n) - \tilde{P}(t)\|^2 = 0, \quad \lim_{n \to \infty} \mathbb{E}\|P(t - t_n) - P(t)\|^2 = 0
$$

for any $t \in \mathbb{R}$. Let

$$
M_1(t) := \int_{-\infty}^{t} \Phi(t)\Phi^{-1}(s)P(s)g(s)d\omega(s),
$$

$$
\tilde{M}_1(t) := \int_{-\infty}^{t} \Psi(t)\Psi^{-1}(s)\tilde{P}(s)\tilde{g}(s)d\omega(s),
$$

where $\Psi(t)$ is given by Lemma 3.3. Thus,

$$
\mathbb{E}\|M_1(t + t_n) - \tilde{M}_1(t)\|^2
\leq 2\mathbb{E}\left\|\int_{-\infty}^{t} \Phi(t + t_n)\Phi^{-1}(s + t_n)P(s + t_n)g(s + t_n)d\omega(s) - \int_{-\infty}^{t} \Psi(t)\Psi^{-1}(s)\tilde{P}(s)\tilde{g}(s)d\omega(s)\right\|^2
$$

$$
+ 2\mathbb{E}\left\|\int_{-\infty}^{t} \Phi(t + t_n)\Phi^{-1}(s + t_n)P(s + t_n)(g(s + t_n) - \tilde{g}(s))d\omega(s)\right\|^2
$$

$$
+ 2\mathbb{E}\left\|\int_{-\infty}^{t} \Psi(t)\Psi^{-1}(s)\tilde{P}(s)\tilde{g}(s)d\omega(s)\right\|^2
$$

(14)

where

$$
I_1 = \int_{-\infty}^{t} \mathbb{E}\|\Phi(t + t_n)\Phi^{-1}(s + t_n)P(s + t_n)\|^2\mathbb{E}\|(g(s + t_n) - \tilde{g}(s))\|^2ds,
$$

and

$$
I_2 = \int_{-\infty}^{t} \mathbb{E}\|\Phi(t + t_n)\Phi^{-1}(s + t_n)P(s + t_n) - \Psi(t)\Psi^{-1}(s)\tilde{P}(s)\\|^2\mathbb{E}\|\tilde{g}(s)\|^2ds.
$$

By Lemma 3.3, taking limit $n \to \infty$ in (14), we obtain

$$
\lim_{n \to \infty} \mathbb{E}\|M_1(t + t_n) - \tilde{M}_1(t)\|^2 = 0, \quad t \in \mathbb{R}.
$$
Similarly, one can prove
\[ \lim_{n \to \infty} \mathbb{E} \| \tilde{M}_1(t - t_n) - M_1(t) \|^2 = 0, \quad t \in \mathbb{R}. \]

Now, let
\[ N_1(t) = \int_t^{\infty} \Phi(t) \Phi^{-1}(s) Q(s) g(s) d\omega(s), \]
\[ M_2(t) = \int_t^{-\infty} \Phi(t) \Phi^{-1}(s) P(s) [f(s) - G(s) g(s)] ds, \]
\[ N_2(t) = \int_t^{\infty} \Phi(t) \Phi^{-1}(s) Q(s) [f(s) - G(s) g(s)] ds, \]
and
\[ \tilde{N}_1(t) = \int_t^{\infty} \Psi(t) \Psi^{-1}(s) \tilde{Q}(s) \tilde{g}(s) d\omega(s), \]
\[ \tilde{M}_2(t) = \int_t^{-\infty} \Psi(t) \Psi^{-1}(s) \tilde{P}(s) [\tilde{f}(s) - \tilde{G}(s) \tilde{g}(s)] ds, \]
\[ \tilde{N}_2(t) = \int_t^{\infty} \Psi(t) \Psi^{-1}(s) \tilde{Q}(s) [\tilde{f}(s) - \tilde{G}(s) \tilde{g}(s)] ds. \]

Clearly, the proof above is also valid for proving that \( N_1(t) \) is mean-square automorphy, that is, for every sequence \( \{t'_n\} \) of real numbers, there exists a subsequence \( \{t_n\} \) such that
\[ \lim_{n \to \infty} \mathbb{E} \| N_1(t + t_n) - \tilde{N}_1(t) \|^2 = 0, \quad \lim_{n \to \infty} \mathbb{E} \| \tilde{N}_1(t - t_n) - N_1(t) \|^2 = 0. \]

As to the second term \( M_2(t) \), it follows from Cauchy-Schwarz inequality, the boundedness properties (10), elementary inequality (12), and the inequalities (5) that
\[ \mathbb{E} \| M_2(t + t_n) - \tilde{M}_2(t) \|^2 \]
\[ = \mathbb{E} \left\| \int_{-\infty}^{t} \Phi(t + t_n) \Phi^{-1}(s + t_n) P(s + t_n) [f(s + t_n) - G(s + t_n) g(s + t_n)] ds \right\|^2 \]
\[ - \int_{-\infty}^{t} \Psi(t) \Psi^{-1}(s) \tilde{P}(s) [\tilde{f}(s) - \tilde{G}(s) \tilde{g}(s)] ds \| \right\|^2 \]
\[ \leq 3 \mathbb{E} \left\| \int_{-\infty}^{t} \Phi(t + t_n) \Phi^{-1}(s + t_n) P(s + t_n) [f(s + t_n) - f(s)] ds \right\|^2 \]
\[ + 3 \gamma_1^2 \mathbb{E} \left\| \int_{-\infty}^{t} \Phi(t + t_n) \Phi^{-1}(s + t_n) P(s + t_n) [g(s + t_n) - \tilde{g}(s)] ds \right\|^2 \]
\[ + 3 \mathbb{E} \left\| \int_{-\infty}^{t} \Phi(t + t_n) \Phi^{-1}(s + t_n) P(s + t_n) [G(s + t_n) - \tilde{G}(s)] ds \right\|^2 \]
\[ \leq 3 \left( \int_{-\infty}^{t} \mathbb{E} \| \Phi(t + t_n) \Phi^{-1}(s + t_n) P(s + t_n) \|^2 ds \right) \left( \int_{-\infty}^{t} \mathbb{E} \| f(s + t_n) - f(s) \|^2 ds \right) \]
\[ + 3 \gamma_1^2 \left( \int_{-\infty}^{t} \mathbb{E} \| \Phi(t + t_n) \Phi^{-1}(s + t_n) P(s + t_n) \|^2 ds \right) \left( \int_{-\infty}^{t} \mathbb{E} \| g(s + t_n) - \tilde{g}(s) \|^2 ds \right) \]
\[ + 3 \left( \int_{-\infty}^{t} \mathbb{E} \| \Phi(t + t_n) \Phi^{-1}(s + t_n) P(s + t_n) - \Psi(t) \Psi^{-1}(s) \tilde{P}(s) \|^2 ds \right) \]
\[ \times \left( \int_{-\infty}^{t} \mathbb{E} \| \tilde{f}(s) - \tilde{G}(s) \tilde{g}(s) \|^2 ds \right) \]
By Lemma 3.3, one can prove that for every sequence \( \{t'_n\} \) of real numbers, there exists a subsequence \( \{t_n\} \) such that
\[
\lim_{n \to \infty} \mathbb{E}\|M_2(t + t_n) - \tilde{M}_2(t)\|^2 = 0, \quad t \in \mathbb{R},
\]
and
\[
\lim_{n \to \infty} \mathbb{E}\|\tilde{M}_2(t - t_n) - M_2(t)\|^2 = 0, \quad t \in \mathbb{R}.
\]
Repeating the same argument as above, one can prove that for every sequence \( \{t'_n\} \) of real numbers there exists a subsequence \( \{t_n\} \) such that
\[
\lim_{n \to \infty} \mathbb{E}\|N_2(t + t_n) - \tilde{N}_2(t)\|^2 = 0, \quad \lim_{n \to \infty} \mathbb{E}\|\tilde{N}_2(t - t_n) - N_2(t)\|^2 = 0
\]
for each \( t \in \mathbb{R} \). Now we obtain
\[
\lim_{n \to \infty} \mathbb{E}\|x(t + t_n) - \tilde{x}(t)\|^2 = 0, \quad \lim_{n \to \infty} \mathbb{E}\|\tilde{x}(t - t_n) - x(t)\|^2 = 0, \quad t \in \mathbb{R},
\]
where \( \tilde{x} = \tilde{M}_1 + \tilde{M}_2 - \tilde{N}_1 - \tilde{N}_2 \). Thus we have proved that \( x \) is a mean-square almost automorphic solution of (2) associated with the initial conditions \( x(t_0) = x_0 \) and \( y(t_0) = y_0 \), respectively. Let \( u(t) = x(t) - y(t) \). Then
\[
u(t) = \Phi(t)\Phi^{-1}(t_0)(x_0 - y_0),
\]
which is a mean-square almost automorphic solution of (1). Let
\[
\Gamma_1(t) = P(t)\Phi(t)\Phi^{-1}(t_0), \quad \Gamma_2(t) = Q(t)\Phi(t)\Phi^{-1}(t_0).
\]
Since \( P(t) \) and \( u(t) \) are both mean-square automorphic, it follows that for every sequence \( \{t'_n\} \) of real numbers there exists a subsequence \( \{t_n\} \) such that
\[
\lim_{n \to \infty} \mathbb{E}\|\Gamma_1(t + t_n) - \tilde{\Gamma}_1(t)\|^2 = 0
\]
and
\[
\lim_{n \to \infty} \mathbb{E}\|\tilde{\Gamma}_1(t - t_n) - \Gamma_1(t)\|^2 = 0
\]
for each \( t \in \mathbb{R} \). Since \( \Gamma_1(t) = P(t)\Phi(t)\Phi^{-1}(t_0) \) satisfies mean-square exponential dichotomy (5), it follows from (15) that \( \mathbb{E}\|\Gamma_1(t)\|^2 = 0 \) for any \( t \in \mathbb{R} \) as \( t_n \to +\infty \). Hence, it follows from (16) that \( \mathbb{E}\|\Gamma_1(t)\|^2 = 0 \) for any \( t \in \mathbb{R} \). Likewise, we can prove that \( \mathbb{E}\|\Gamma_2(t)\|^2 = 0 \) for any \( t \in \mathbb{R} \). Thus the mean-square almost automorphic solution \( u(t) \) satisfies \( \mathbb{E}\|u(t)\|^2 = \mathbb{E}\|(P(t) + Q(t))\Phi(t)\Phi^{-1}(t_0)(x_0 - y_0)\|^2 \equiv 0 \), that is, \( x \) is indistinguishable from \( y \). This completes the proof.

**Remark 1.** The existence and uniqueness of mean-square almost automorphic (mild) solutions was studied by Chang, Zhao and N’Guérékata [6] for non-autonomous SDEs under the following condition that, for every sequence \( \{t'_n\} \) of real
numbers and for every $\varepsilon > 0$, there exists a subsequence $\{t_n\}$ and $N \in \mathbb{N}$ such that

$$
\begin{align*}
\|\Phi(t + t_n)\Phi^{-1}(s + t_n) - \Phi(t)\Phi^{-1}(s)\| &< \varepsilon e^{-\delta(t-s)} \\
\|\Phi(t - t_n)\Phi^{-1}(s - t_n) - \Phi(t)\Phi^{-1}(s)\| &< \varepsilon e^{-\delta(t-s)}
\end{align*}
$$

(17)

for all $n > N$ and $t \geq s$. Later, in [8], Chen and Lin replaced the assumption by mean-square bi-almost automorphic condition to deal with the existence of some stochastic evolution equations. In fact, the above conditions can be expressed in an exact way by using product integration. More precisely, condition (17) and the assumption of mean-square bi-almost automorphic can be directly derived by using Lemma 3.3.

**Theorem 3.5.** Assume that (1) admits a mean-square exponential dichotomy, and $A, G \in C(\mathbb{R}; \mathbb{R}^{n \times n})$ are almost automorphic. Suppose further that $f$ and $g$ are both mean-square almost automorphic processes in $t \in \mathbb{R}$ for every $x \in L^2(\Omega, \mathbb{R}^n)$, and satisfy the Lipschitz condition in mean-square for all $x, y \in L^2(\Omega, \mathbb{R}^n)$ and $t \in \mathbb{R}$, i.e.,

$$
\begin{align*}
\mathbb{E}\|f(t, x) - f(t, y)\|^2 &\leq L \mathbb{E}\|x - y\|^2, \\
\mathbb{E}\|g(t, x) - g(t, y)\|^2 &\leq L' \mathbb{E}\|x - y\|^2
\end{align*}
$$

(18)

with constants $L, L' > 0$. Then (3) has a unique mean-square almost automorphic solution $x(t)$ provided

$$
\frac{12M(\alpha L' + 4L + 4\gamma^2L')}{\alpha^2} < 1.
$$

**Proof.** An argument similar to the one used in Lemma 3.1 shows that a mean-square almost automorphic solution of (3) is given by

$$
\begin{align*}
x(t) &= \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)g(s, x(s))d\omega(s) \\
&\quad + \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)[f(s, x(s)) - G(s)g(s, x(s))]ds \\
&\quad - \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)g(s, x(s))d\omega(s) \\
&\quad - \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)[f(s, x(s)) - G(s)g(s, x(s))]ds.
\end{align*}
$$

(19)

Now let the operator $T: L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)$ be defined as

$$
\begin{align*}
(Tx)(t) &= \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)g(s, x(s))d\omega(s) \\
&\quad + \int_{-\infty}^{t} P(t)\Phi(t)\Phi^{-1}(s)[f(s, x(s)) - G(s)g(s, x(s))]ds \\
&\quad - \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)g(s, x(s))d\omega(s) \\
&\quad - \int_{t}^{\infty} Q(t)\Phi(t)\Phi^{-1}(s)[f(s, x(s)) - G(s)g(s, x(s))]ds.
\end{align*}
$$

Now we show that $T$ is a contraction on $AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n)))$. For $x, y \in AA(\mathbb{R}; L^2(\Omega, \mathbb{R}^n))$ and each $t \in \mathbb{R}$, we have

$$
\mathbb{E}\|T(x)(t) - (Ty)(t)\|^2
$$
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\[
\begin{align*}
&\leq 6E \left\| \int_{-\infty}^{t} P(t) \Phi(t) \Phi^{-1}(s)(g(s, x(s)) - g(s, y(s)))d\omega(s) \right\|^2 \\
&+ 6E \left\| \int_{-\infty}^{t} P(t) \Phi(t) \Phi^{-1}(s)(f(s, x(s)) - f(s, y(s)))ds \right\|^2 \\
&+ 6\gamma^2 E \left\| \int_{t}^{\infty} Q(t) \Phi(t) \Phi^{-1}(s)(g(s, x(s)) - g(s, y(s)))d\omega(s) \right\|^2 \\
&+ 6E \left\| \int_{t}^{\infty} Q(t) \Phi(t) \Phi^{-1}(s)(f(s, x(s)) - f(s, y(s)))ds \right\|^2 \\
&+ 6\gamma^2 E \left\| \int_{t}^{\infty} Q(t) \Phi(t) \Phi^{-1}(s)(g(s, x(s)) - g(s, y(s)))d\omega(s) \right\|^2 \\
&+ 6\gamma^2 E \left\| \int_{t}^{\infty} Q(t) \Phi(t) \Phi^{-1}(s)(f(s, x(s)) - f(s, y(s)))ds \right\|^2.
\end{align*}
\]

(20)

Since $g$ satisfies Lipschitz condition (18), by using Itô isometry property and inequalities (5), the first term of right-hand side in (20) can be deduced as follows:

\[
E \left\| \int_{-\infty}^{t} P(t) \Phi(t) \Phi^{-1}(s)(g(s, x(s)) - g(s, y(s)))d\omega(s) \right\|^2 \\
= \int_{-\infty}^{t} E \| P(t) \Phi(t) \Phi^{-1}(s) \|^2 E \| g(s, x(s)) - g(s, y(s)) \|^2 ds \\
\leq L'M \int_{-\infty}^{t} e^{-\alpha(t-s)} E \| x - y \|^2 ds \\
\leq \frac{L'M}{\alpha} \sup_{s \in \mathbb{R}} E \| x(s) - y(s) \|^2.
\]

As to the second term in (20), it follows from $E \| x \| \leq \sqrt{E \| x \|^2}$, Cauchy-Schwarz inequality, and the inequalities (5) that

\[
\begin{align*}
&\leq \left( \int_{-\infty}^{t} E \| P(t) \Phi(t) \Phi^{-1}(s) \| ds \right) \\
&\times \left( \int_{-\infty}^{t} E \| P(t) \Phi(t) \Phi^{-1}(s) \| E \| f(s, x(s)) - f(s, y(s)) \|^2 ds \right) \\
&\leq LM \left( \int_{-\infty}^{t} e^{-\frac{\alpha}{2}(t-s)} ds \right)^2 \sup_{s \in \mathbb{R}} E \| x(s) - y(s) \|^2 \\
&\leq \frac{4LM}{\alpha^2} \sup_{s \in \mathbb{R}} E \| x(s) - y(s) \|^2.
\end{align*}
\]

Clearly, the proof above is also valid for proving other terms in the right-hand side in (20). Thus, we have

\[
E \| (Tx)(t) - (Ty)(t) \|^2
\]
Example 1. Consider the following equation

$$dx(t) = -ax(t)dt + \sigma x(t)dt, 
\begin{align*}
\begin{cases}
dx(t) = -ax(t)dt + \sigma x(t)dt, \\
dy(t) = ay(t)dt + \sigma y(t)dt,
\end{cases}
\end{align*}$$

(22)

with initial data $(x(0), y(0))$, where $a$, $\sigma$ are constants satisfying $a > 0$ and $\sigma^2 < 2a$. The solution of (22) is given by

$$x(t) = x(0) \exp \left[ \left( -a - \frac{\sigma^2}{2} \right) t + \sigma \omega(t) \right],$$
$$y(t) = y(0) \exp \left[ \left( a - \frac{\sigma^2}{2} \right) t + \sigma \omega(t) \right].$$

Thus it is easy to verify that

$$\begin{align*}
\mathbb{E}\|x(t)\|^2 &\leq e^{(-2a+\sigma^2)(t-s)}\mathbb{E}\|x(s)\|^2, \\
\mathbb{E}\|y(t)\|^2 &\leq e^{-(2a+\sigma^2)(s-t)}\mathbb{E}\|y(s)\|^2,
\end{align*}$$

for $t \geq s$. The proof is completed. 

$\square$

Hence, we have

$$\|(Tx)(t) - (Ty)(t)\|_2^2 \leq \theta \sup_{s \in \mathbb{R}} \|x(s) - y(s)\|_2^2$$

with

$$\theta = \frac{12M(\alpha L' + 4L + 4\gamma_1^2 L')}{\alpha^2} \in (0, 1).$$

It follows from (21) and

$$\sup_{s \in \mathbb{R}} \|x(s) - y(s)\|_2^2 \leq \left( \sup_{s \in \mathbb{R}} \|x(s) - y(s)\|_2 \right)^2$$

that

$$\|T(t)x(t) - T(t)y(t)\|_2 \leq \sqrt{\theta}\|x(s) - y(s)\|_\infty.$$
and therefore (22) admits a mean-square exponential dichotomy. Let \( f(t) \) and \( g(t) \) be mean-square almost automorphic functions. Then the system
\[
\begin{align*}
\frac{dx(t)}{dt} &= (-ax(t) + f(t))dt + (\sigma x(t) + g(t))d\omega(t) \\
\frac{dy(t)}{dt} &= (ax(t) + f(t))dt + (\sigma x(t) + g(t))d\omega(t)
\end{align*}
\]
has a unique mean-square almost automorphic solution due to Theorem 3.4. In addition, assume that \( f(t, x) \) and \( g(t, x) \) are mean-square almost automorphic functions and satisfy the Lipschitz condition (18) with the condition \( 16L + 8(1 + 2\gamma^2 L') < 2a - \sigma^2 \). Then the system
\[
\begin{align*}
\frac{dx(t)}{dt} &= (-ax(t) + f(t, x))dt + (\sigma x(t) + g(t, x))d\omega(t) \\
\frac{dy(t)}{dt} &= (ax(t) + f(t, x))dt + (\sigma x(t) + g(t, x))d\omega(t)
\end{align*}
\]
has a unique mean-square almost automorphic solution by using Theorem 3.5.

4. Product integration for SDEs. The concept of product integration was introduced by Volterra [33] and then developed by Schlesinger [27] and Masani [23]. See Slavík [29] for a systematical presentation. The original idea of product integration is very similar in spirit to the procedures for finding numerical solutions to
\[
\frac{dx(t)}{dt} = A(t)x(t). \tag{23}
\]
An approximate solution of (23) can be obtained by using the Euler tangent-line method, which is based on the observation that the approximative value of (23) at the point \( t_0 + \Delta t \) is
\[
x(t_0 + \Delta t) \approx x(t_0)e^{A(t_0)\Delta t}
\]
for \( \Delta t \) sufficiently small. Proceeding in this manner, we can obtain the following approximative value
\[
x(t) \approx x(t_0)e^{A(t_0)\Delta t} \cdots e^{A(t_0)\Delta t}.
\]
Such a fact has been proved provided \( A(t) \) is continuous in \( \mathbb{R} \) (see, e.g., [29]). Thus, if \( \Delta t \to 0 \), the calculation given above can converge to a value, and this value will be denoted by the symbol
\[
\prod_{t_0}^{t} e^{A(t)\, dt} \cdot x(t_0),
\]
which is a solution of (23) with the initial condition \( x(t_0) \). We refer the reader to Dollard and Friedman [12], Gill and Johansen [15] and Slavík [29] for the definitions of product integration to (23).

**Definition 4.1.** A function (process) \( A(t) \) is called a step function (step process) if there exists a partition \( P := \{a = t_0 < t_1 < \cdots < t_m = b\} \) such that
\[
A(t) \equiv A_k \quad \text{for} \quad t_k \leq t \leq t_{k+1} \quad (k = 0, \ldots, m - 1).
\]
Denote by \( L^0([a, b]; \mathbb{R}) \) (or \( \mathcal{L}_0^0([a, b]; \mathbb{R}) \)) the family of all such functions (processes).

**Definition 4.2.** Let \( A, G \in L^0([a, b]; \mathbb{R}) \) (or, \( A, G \in \mathcal{L}_0([a, b]; \mathbb{R}) \)) be step functions (or step processes) mentioned above with the same partition \( P \) (if not, we can make it). Define the product integral with respect to the Brownian motion \( \omega(t) \) by
\[
F_{A,G}(a, b) = \prod_{k=0}^{m-1} e^{(A_k(t_{k+1} - t_k) + G_k(\omega_{t_{k+1}} - \omega_{t_k}))}.
\]
Lemma 4.3. Let $A, G \in L^0([a, b]; \mathbb{R})$ (or, $A, G \in L^0([a, b]; \mathbb{R})$) with the same partition $P$. Then

(i): $F_{A,G}(a, a) = 1$.

(ii): $F_{A,G}(a, t)$ is continuous on $[a, b]$ and satisfies the integral equation

$$F_{A,G}(a, t) = 1 + \int_a^t A(\tau)F_{A,G}(a, \tau)d\tau + \int_a^t G(\tau)F_{A,G}(a, \tau)d\omega(\tau), \ a.s. \quad (24)$$

where the last integral in (24) is Itô integral.

(iii):

$$E\|\log F_{A,G}(a, b)\|^2 = E\left(\int_a^b A(t)dt\right)^2 + E\int_a^b \|G(t)\|^2dt,$$

when $A, G \in L^0([a, b]; \mathbb{R})$.

(iv):

$$E\|\log F_{A,G}(a, b)\|^2 = \left(\int_a^b A(t)dt\right)^2 + \int_a^b \|G(t)\|^2dt,$$

when $A, G \in L^0([a, b]; \mathbb{R})$.

Proof. Part (i) is obvious. Note that

$$F_{A,G}(a, t) = e^{A_0(t-t_0)+G_0(\omega_t-\omega_{t_0})}, \quad t \in [t_0, t_1],$$

$$e^{A_1(t-t_1)+G_1(\omega_t-\omega_{t_1})}e^{A_0(t_1-t_0)+G_0(\omega_{t_1}-\omega_{t_0})}, \quad t \in [t_1, t_2],$$

$$\ldots$$

$$e^{A_{m-1}(t-t_{m-1})+G_{m-1}(\omega_t-\omega_{t_{m-1}})} \cdots e^{A_0(t_1-t_0)+G_0(\omega_{t_1}-\omega_{t_0})}, \quad t \in [t_{m-1}, t_m],$$

and so forth, one can use the induction to show that $F_{A,G}(a, t)x_0$ is a solution of the scalar, linear Itô stochastic differential equation

$$dx = A(t)x(t)dt + G(t)x(t)d\omega(t)$$

with the initial condition $x(a) = x_0$ except at the division points, and one can easily verify that $F_{A,G}(a, t)$ is continuous on $[a, b]$. Thus one can obtain (24) and (ii) is proved.

Now we prove (iii). Note that $\omega_{t_{j+1}} - \omega_{t_j}$ is independent of $G_jG_j(\omega_{t_{j+1}} - \omega_{t_i})$ if $i < j$. Thus,

$$E\|\log F_{A,G}(a, b)\|^2 = \sum_{0 \leq i, j \leq m-1} \left(E[A_iA_j(t_{i+1} - t_i)(t_{j+1} - t_j)] + E[(A_iG_j + G_iA_j)(t_{i+1} - t_i)]ight)$$

$$\omega_{t_{j+1}} - \omega_{t_j}) + E[G_iG_j(\omega_{t_{i+1}} - \omega_{t_i})(\omega_{t_{j+1}} - \omega_{t_j})]$$

$$= \sum_{0 \leq i, j \leq m-1} E[A_iA_j(t_{i+1} - t_i)(t_{j+1} - t_j)] + \sum_{0 \leq i \leq m-1} E[G_i^2(\omega_{t_{i+1}} - \omega_{t_i})^2]$$

$$= \sum_{0 \leq i, j \leq m-1} E[A_iA_j(t_{i+1} - t_i)(t_{j+1} - t_j)] + \sum_{0 \leq i \leq m-1} E[G_i^2(t_{i+1} - t_i)]$$

$$= E\left(\int_a^b A(t)dt\right)^2 + E\int_a^b \|G(t)\|^2dt,$$
when \( A, G \in \mathcal{L}^0([a, b]; \mathbb{R}) \). Obviously, when \( A, G \in L^0([a, b]; \mathbb{R}) \), we obtain (25), as required. \( \square \)

**Lemma 4.4.** [22, Lemma 1.5.6] For any \( G \in L^2([a, b]; \mathbb{R}) \), there exists a sequence of step functions \( G^n(t) \) such that

\[
\lim_{\mu(P) \to 0} a \int_{a}^{b} \|G(t) - G^n(t)\|^2 dt = 0,
\]

(26)

where \( \mu(P) \) denotes the length of the longest subinterval of the partition \( P \).

Similarly, we can prove the following lemma

**Lemma 4.5.** For any \( A \in L^1([a, b]; \mathbb{R}) \), there exists a sequence \( \{A^n(t)\} \) of step functions such that

\[
\lim_{\mu(P) \to 0} \left( \int_{a}^{b} (A(t) - A^n(t)) dt \right)^2 = 0.
\]

(27)

Proceeding in the same way as in (25), for any step functions \( A^n, A^m, G^n, G^m \in L^0([a, b]; \mathbb{R}) \), we have

\[
\mathbb{E}[\|\log F_{A^n, G^n}(a, b) - \log F_{A^m, G^m}(a, b)\|^2] = \left( \int_{a}^{b} (A^n(t) - A^m(t)) dt \right)^2 + \int_{a}^{b} \|G^n(t) - G^m(t)\|^2 dt \to 0 \quad \text{as} \quad \mu(P) \to 0.
\]

That is to say that \( \log F_{A^n, G^n}(a, b) \) is a Cauchy sequence in \( L^2(\Omega, \mathbb{R}) \). So the limit exists and we define the limit as the product integral with respect to the Brownian motion \( \omega(t) \). This leads to the following definition.

**Definition 4.6.** Let \( A, G \in C([a, b]; \mathbb{R}) \). Then the product integral is defined by

\[
\prod_{a}^{t} e^{(A(\tau)d\tau + G(\tau)d\omega(\tau))} = \lim_{\mu(P) \to 0} \prod_{a}^{t} e^{(A^n(\tau)d\tau + G^n(\tau)d\omega(\tau))},
\]

where \( G^n \) and \( A^n \) are sequences of step functions such that (26), (27) hold respectively.

Now let \( A, G \in C([a, b]; \mathbb{R}^{n \times n}) \), by using matrix notation, we can define the multi-dimensional product integral as follows

\[
F_{A,G}(a, t)
\]

\[
= \prod_{a}^{t} \exp(A(\tau)d\tau + G(\tau)d\omega(\tau))
\]

\[
= \prod_{a}^{t} \exp \left( \begin{pmatrix}
A_{11}(\tau) & \cdots & A_{1n}(\tau) \\
\vdots & \ddots & \vdots \\
A_{n1}(\tau) & \cdots & A_{nn}(\tau)
\end{pmatrix}
\right) d\tau + \begin{pmatrix}
G_{11}(\tau) & \cdots & G_{1n}(\tau) \\
\vdots & \ddots & \vdots \\
G_{n1}(\tau) & \cdots & G_{nn}(\tau)
\end{pmatrix} d\omega(\tau)
\]

\[
= \lim_{\mu(P) \to 0} \prod_{a}^{t} \exp((A^n(\tau)d\tau + G^n(\tau)d\omega(\tau))),
\]

where \( A^n = (A^n_{ij})_{n \times n} \) and \( G^n = (G^n_{ij})_{n \times n} \) are sequences of matrices, with \( A^n_{ij} \) and \( G^n_{ij} \) are step functions.

We now establish some basic properties of the product integration.
Proof of Lemma 3.2. Let $A^n(t) = (A^n_{ij}(t))_{n \times n}$ and $G^n(t) = (G^n_{ij}(t))_{n \times n}$ be sequences of matrices with $A^n_{ij}(t)$ and $G^n_{ij}(t)$ are step functions which converge to $A(t), G(t)$ in $L^1$ sense and $L^2$ sense respectively. Following the same idea in Lemma 4.3, we have

$$F_{A^n,G^n}(a,t) = I d + \int_a^t A^n(\tau)dF_{A^n,G^n}(a,\tau)d\tau + \int_a^t G^n(\tau)dF_{A^n,G^n}(a,\tau)d\omega(\tau).$$  (28)

Taking the limit of (28) as $\mu(P) \to 0$, it follows from the definition of the solution of Itô stochastic differential equations (see e.g., [13, Chapter 5]) that $F_{A,G}(a,t)x_0$ is the solution of (1) with the initial condition $x(a) = x_0$.

Now let $x_k$ denote the $k$-th column of $F_{A,G}(a,t)$, i.e.,

$$x_k = \prod_{a}^{t} e^{(A(t)dt+G(t)d\omega(t))} \cdot e_k,$$

where $e_k$ is the $k$-th vector from the canonical basis of $\mathbb{R}^n$. From the analysis above, one can see that the vector functions $\{x_k\}_{k=1}^n$ are all the solutions of (1) with the initial condition $x_k(a) = e_k$. Thus the system of functions $\{x_k\}_{k=1}^n$ is linearly independent and represents a fundamental set of solutions of the system (1). Thus it follows from the above results and (28) that (13) is the principal matrix solution of the system (1) at the initial time $t_0$, and this completes the proof of Lemma 3.2.

Note that $F_{A,G}(a,t)$ is nonsingular almost surely for all $t \in [a,b]$, thus we can give the following definition

**Definition 4.7.** Let $A(t), G(t) \in C([a,b]; \mathbb{R}^{n \times n})$, and let $a_0, a_1 \in [a,b]$ such that $a_0 < a_1$. Then $\prod_{a_0}^{a_1} e^{(A(t)dt+G(t)d\omega(t))}$ can be defined by

$$\prod_{a_0}^{a_1} e^{(A(t)dt+G(t)d\omega(t))} = \left( \prod_{a_0}^{a_1} e^{(A(t)dt+G(t)d\omega(t))} \right)^{-1}. $$  (29)

**Theorem 4.8.** Let $A(t), G(t) \in C([a,b]; \mathbb{R}^{n \times n})$, and let $a_0, a_1, a_2 \in [a,b]$. Then

$$\prod_{a_0}^{a_2} e^{(A(t)dt+G(t)d\omega(t))} = \prod_{a_0}^{a_1} e^{(A(t)dt+G(t)d\omega(t))} \prod_{a_1}^{a_2} e^{(A(t)dt+G(t)d\omega(t))}. $$  (30)

**Proof.** Note that (29) in Definition 4.7, one can set $a_0 \leq a_1 \leq a_2$. The left hand side of (30) can be written as

$$\prod_{a_0}^{a_2} e^{(A(t)dt+G(t)d\omega(t))} = \lim_{\mu(P) \to 0} \prod_{a_0}^{a_2} e^{(A^n(t)dt+G^n(t)d\omega(t))},$$

where $P$ is a partition of $[a_0,a_2]$, and $A^n$ and $G^n$ are sequences of step functions. Without loss of generality, we can always choose partition $P$ such that $a_1 \in P$. Thus it is obvious to know that (30) holds, and this completes the proof.

**Theorem 4.9.** Let $\{A_n\}_{n=1}^\infty \in L^1([a,b]; \mathbb{R}^{n \times n})$ and $\{G_n\}_{n=1}^\infty \in L^2([a,b]; \mathbb{R}^{n \times n})$ be sequences of integrable functions that satisfy

1. $A_n(t) \to A(t), G_n(t) \to G(t)$ as $n \to \infty$ for each $t \in [a,b]$,

2. there exist integrable functions $\hat{A} \in L^1([a,b]; \mathbb{R})$ and $\hat{G} \in L^2([a,b]; \mathbb{R})$ such that

$$\|A_n(t)\| \leq \hat{A}(t), \|G_n(t)\| \leq \hat{G}(t),$$

for all $t \in [a,b]$, $\forall n \in \mathbb{N}$. 


Then
\[
\prod_{a}^{b} e^{(A(\tau)d\tau + G(\tau)d\omega(\tau))} = \lim_{n \to \infty} \prod_{a}^{b} e^{(A_n(\tau)d\tau + G_n(\tau)d\omega(\tau))}.
\]

**Proof.** The proof follows from Lebesgue’s dominated convergence theorem and the fact that an integrable function is also a product integrable function (see, e.g., [23, Theorem 16.3, pp. 169]). We omit the details here. \qed

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**REFERENCES**

[1] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, New York, 1974.
[2] S. Bochner, A new approach to almost periodicity, *Proc. Natl. Acad. Sci., USA*, 48 (1962), 2039–2043.
[3] J. Campos and M. Tarallo, Almost automorphic linear dynamics by Favard theory, *J. Differential Equations*, 256 (2014), 1350–1367.
[4] T. Caraballo and D. Cheban, Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard’s separation condition I, *J. Differential Equations*, 246 (2009), 108–128.
[5] T. Caraballo and D. Cheban, Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard’s separation condition II, *J. Differential Equations*, 246 (2009), 1164–1186.
[6] K. Chang, Z. Zhao and G. M. N’Guërekata, Square-mean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces, *Comput. Math. Appl.*, 61 (2011), 384–391.
[7] Z. Chen and W. Lin, Square-mean pseudo almost automorphic process and its application to stochastic evolution equations, *J. Funct. Anal.*, 261 (2011), 69–89.
[8] Z. Chen and W. Lin, Square-mean weighted pseudo almost automorphic solutions for non-autonomous stochastic evolution equations, *J. Math. Pures Appl.*, 100 (2013), 476–504.
[9] W. A. Coppel, *Dichotomy in Stability Theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York/Berlin, 1978.
[10] T. Diagana, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer, New York, 2013.
[11] H. Ding, W. Long and G. M. N’Guërekata, Almost automorphic solutions of nonautonomous evolution equations, *Nonlinear Anal.*, 70 (2009), 4158–4164.
[12] J. D. Dollard and C. N. Friedman, *Product Integration with Applications to Differential Equations*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1979.
[13] L. C. Evans, *An Introduction to Stochastic Differential Equations*, American Mathematical Society, Providence, RI, 2013.
[14] M. Fu and Z. Liu, Square-mean almost automorphic solutions for some stochastic differential equations, *Proc. Amer. Math. Soc.*, 138 (2010), 3689–3701.
[15] R. D. Gill and S. Johansen, A survey of product integration with a view toward application in survival analysis, *Ann. Stat.*, 18 (1990), 1501–1555.
[16] D. J. Higham, Mean-square and asymptotic stability of the stochastic theta method, *SIAM J. Numerical Anal.*, 38 (2000), 755–769.
[17] R. A. Johnson, A linear, almost periodic equation with an almost automorphic solution, *Proc. Amer. Math. Soc.*, 82 (1981), 199–205.
[18] P. E. Kloeden and T. Lorenz, Mean-square random dynamical systems, *J. Differential Equations*, 253 (2012), 1422–1438.
[19] A. G. Ladde and G. S. Ladde, *An Introduction to Differential Equations: Stochastic Modeling, Methods, and Analysis*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.
[20] Z. Liu and K. Sun, Almost automorphic solutions for stochastic differential equations driven by Lévy noise, *J. Funct. Anal.*, 226 (2014), 1115–1149.
[21] C. Lizama and J. G. Mesquita, Almost automorphic solutions of non-autonomous difference equations, *J. Math. Anal. Appl.*, 407 (2013), 339–349.
[22] X. Mao, *Stochastic Differential Equations and Applications*, Second edition. Horwood Publishing Limited, Chichester, 2008.

[23] P. R. Masani, *Multiplicative Riemann integration in normed rings*, Trans. Amer. Math. Soc., **61** (1947), 147–192.

[24] J. Massera and J. Schäffer, *Linear Differential Equations and Function Spaces*, in: *Pure and Applied Mathematics*, vol. 21, Academic Press, 1966.

[25] G. M. N’Guérékata, *Topics in Almost Automorphy*, Springer, New York, Boston, Dordrecht, London, Moscow, 2005.

[26] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z., **32** (1930), 703–728.

[27] L. Schlesinger, *Neue Grundlagen für einen infinitesimalkalkül der Matrizen*, Math. Zeit., **33** (1931), 33–61.

[28] W. Shen and Y. Yi, *Almost automorphic and almost periodic dynamics in skew-product semiflows*, Mem. Amer. Math. Soc., **136** (1998), x+93 pp.

[29] A. Slavík, *Product Integration, its History and Applications*, Matfyzpress, Prague, 2007.

[30] O. M. Stanzhyts’kyi, *Investigation of exponential dichotomy of Itô stochastic systems by using quadratic forms*, Ukr. Mat. Zh., **53** (2001), 1545–1555.

[31] D. Stoica, *Uniform exponential dichotomy of stochastic cocycles*, Stochastic Process. Appl., **120** (2010), 1920–1928.

[32] W. A. Veech, *On a theorem of Bochner*, Ann. of Math., **86** (1967), 117–137.

[33] V. Volterra, *Sulle equazioni differenziali lineari*, Rendiconti Accademia dei Lincei, **4** (1887), 393–396.

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