Logarithmically Improved Serrin’s Criteria for Navier-Stokes Equations

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Abstract

In this paper we prove the logarithmically improved Serrin’s criteria to the three-dimensional incompressible Navier-Stokes equations.

Keywords: Navier-Stokes equations, Serrin’s Criterion, global regularity.

1 Introduction

As is well-known that the three-dimensional incompressible Navier-Stokes equations in $\mathbb{R}^3$ take the form

\begin{equation}
\begin{aligned}
\begin{cases}
    u_t + u \cdot \nabla u + \nabla q = \mu \Delta u, \\
    \nabla \cdot u = 0,
\end{cases}
\end{aligned}
\end{equation}

where $u = (u_1, u_2, u_3)^T$ is the velocity of the flows, $q$ is the scalar pressure and $\mu$ is the viscosity of the fluid which is the inverse of the Reynolds number. We refer the reader to \cite{[10, 18, 5, 11]} for general descriptions of Euler and Navier-Stokes equations.

Despite a great deal of efforts by mathematicians and physicists, the question of whether a solution of the 3D incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial data with finite energy is still one of the most outstanding mathematical open problems \cite{[7]}. In the absence of a well-posedness theory, the development of blowup/non-blowup criteria is of major importance for both theoretical and practical purposes. There has been a lot of progress on Euler and Navier-Stokes equations along this direction. For example, by the well-known Serrin’s

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criteria \cite{12,14,15,16} for Navier-Stokes equations, any Leray-Hopf weak solution $u$ is smooth for $0 \leq t \leq T$ provided that
\[
\int_0^T \|u(t, \cdot)\|_{L^p}^s \, dt < \infty
\]
holds for any pair of constants $(p, s)$ with $\frac{2}{p} + \frac{2}{s} \leq 1$ and $3 < p \leq \infty$. In fact, the highly nontrivial case of $p = 3$ is also true, which was only proved recently by Iskauriaza, Seregin and Sverak \cite{8}. By the celebrated Beale-Kato-Majda’s criterion \cite{1}, if
\[
\int_0^T \|\nabla \times u(t, \cdot)\|_{L^\infty} \, dt < \infty,
\]
where $\omega = \nabla \times u$ is the vorticity, then $u$ is smooth at time $T$. The above Beale-Kato-Majda’s criterion was slightly improved by Kozono-Taniuchi \cite{9} by replacing $\|\omega(t, \cdot)\|_{L^\infty}$ as $\|\omega(t, \cdot)\|_{BMO}$. Let us also cite a result by Constantin and Fefferman \cite{6} who gave a condition involving only the direction of the vorticity and a condition involving the lower bound of the pressure by Seregin and Sverak \cite{13}.

Our purpose of this paper is to establish the logarithmically improved Serrin’s criterion for three-dimensional Navier-Stokes equations.

**Theorem 1.1.** Suppose that $u_0 \in H^2(\mathbb{R}^3)$ and $u$ is a smooth solution to the incompressible Navier-Stokes equations (1.1) with the initial data $u_0(x)$ for $0 \leq t < T$. Then $u$ is smooth at time $t = T$ provided that
\[
\int_0^T \frac{\|u(t, \cdot)\|_{L^p}^s}{1 + \ln \left( e + \|u(t, \cdot)\|_{L^\infty} \right)} \, dt < \infty \tag{1.2}
\]
for any pair of constants $(p, s)$ with $\frac{2}{p} + \frac{2}{s} \leq 1$ and $3 < p \leq \infty$.

**Remark 1.2.** Serrin’s criteria were first obtained by Prodi \cite{12} and Serrin \cite{14} in the whole space case. The local version was established by Serrin \cite{15} when $\frac{2}{p} + \frac{2}{s} < 1$ and Struwe when $\frac{2}{p} + \frac{2}{s} = 1$. For simplicity, we just focus on the whole space case and do not pursue the bounded domain case. Moreover, in Theorem 1.2 we just assume that $u$ is a smooth solution for $0 \leq t < T$. However, the conclusion in Theorem 1.2 is still true for Leray-Hopf weak solutions, which can be proved by the same smoothing technique as in Struwe \cite{16}.

**Remark 1.3.** We remark here that recently Chan and Vasseur \cite{3} proved a logarithmically improved Serrin’s criteria under assuming that
\[
\int_0^T \int_{\mathbb{R}^3} \frac{|u(t, x)|^5}{\ln \left( e + |u(t, x)| \right)} \, dx \, dt < \infty.
\]
Noting that
\[
\int_0^T \frac{\|u(t, \cdot)\|_{L^5}^5}{1 + \ln \left( e + \|u(t, \cdot)\|_{L^\infty} \right)} \, dt \leq \int_0^T \int_{\mathbb{R}^3} \frac{|u(t, x)|^5}{\ln \left( e + |u(t, x)| \right)} \, dx \, dt,
\]
we find that our result covers the result in [3] by letting $p = s = 5$ in (1.2). Moreover, Chan and Vasseur use De Giorgi’s method, while our proof is just based on energy method and is much simpler. After the completion of the paper, we learned that the authors in [20] have independently proved results similar to those in Theorem 1.1 in the framework of multiplier space.

2 Logarithmically Improved Serrin’s Criterion for Navier-Stokes Equations

In this section, we establish the logarithmically improved Serrin’s criterion for Navier-Stokes Equations and prove Theorem 1.1.

First of all, for any smooth solution $u$ to the three-dimensional Euler and Navier-Stokes equations, one has the well-known energy law:

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 = 0.$$  \hspace{1cm} (2.1)

Next, let us apply $\nabla^3$ to (1.1) and then take the $L^2$ inner product of the resulting equations with $\nabla^3 u$. The standard energy estimate gives

$$\frac{1}{2} \frac{d}{dt} \|\nabla^3 u\|_{L^2}^2 + \mu \|\nabla^4 u\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla^3 u \nabla^3 q dx - \int_{\mathbb{R}^3} \nabla^3 u \nabla^3 (u \cdot \nabla u) dx.$$  \hspace{1cm} (2.2)

Noting the incompressible constraint $\nabla \cdot u = 0$ and using integration by parts, one has

$$\frac{1}{2} \frac{d}{dt} \|\nabla^2 u\|_{L^2}^2 + \mu \|\nabla^3 u\|_{L^2}^2 = -2 \int_{\mathbb{R}^3} \nabla^2 u (\nabla u \cdot \nabla u) dx - \int_{\mathbb{R}^3} \nabla^2 u \nabla^2 u \cdot \nabla u dx \leq 5 \|u\|_{L^p} \|\nabla^2 u\|_{L^{2p/2}} \|\nabla^3 u\|_{L^2},$$

for $3 < p \leq \infty$, where we used Hölder inequality and integration by parts in the last inequality. Since $3 < p \leq \infty$, one has $2 \leq \frac{2p}{p-2} < 6$. Consequently, by the following standard multiplicative inequality

$$\|\nabla^2 u\|_{L^{2p/2}}^2 \leq C \|\nabla^2 u\|_{L^2}^{1-\frac{2}{p}} \|\nabla^3 u\|_{L^2}^{\frac{2}{p}}$$

we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla^2 u\|_{L^2}^2 + \mu \|\nabla^3 u\|_{L^2}^2 \leq C \|u\|_{L^p} \|\nabla^2 u\|_{L^2} \|\nabla^3 u\|_{L^2} + \mu \frac{1}{2} \|\nabla^3 u\|_{L^2}^2 \leq C(p, \mu) \|u\|_{L^p} \|\nabla^2 u\|_{L^2} + \mu \frac{1}{2} \|\nabla^3 u\|_{L^2}^2 \leq C(p, \mu) \frac{\|u\|_{L^p}^{2p/3}}{1 + \ln (e + \|u\|_{L^\infty})} [1 + \ln (e + \|u\|_{L^\infty})] \|\nabla^2 u\|_{L^2} + \mu \frac{1}{2} \|\nabla^3 u\|_{L^2}^2.$$
Noting (2.1), we derive from (2.3) that
\[
\frac{d}{dt} \| \nabla^2 u \|_{L^2}^2 + \mu \| \nabla^3 u \|_{L^2}^2 \leq 2C(p, \mu) \| u \|_{L^p}^{2p} \left[ 1 + \ln \left( e + \| u \|_{L^\infty} \right) \right] \| \nabla^2 u \|_{L^2}^2. \tag{2.4}
\]

Then using Gronwall’s inequality, we have
\[
1 + \ln \left( e + \| \nabla^2 u(t, \cdot) \|_{L^2}^2 \right) \leq \left[ 1 + \ln \left( e + \| \nabla^2 u_0 \|_{L^2}^2 \right) \right] \times \exp \left\{ 2C(p, \mu) \int_0^t \frac{\| u(\tau, \cdot) \|_{L^p}^{2p}}{1 + \ln \left( e + \| u \|_{L^\infty} \right)} d\tau \right\}, \tag{2.5}
\]
which gives a finite bound for \( \| \nabla^2 u(T, \cdot) \|_{L^2}^2 \) provided that
\[
\int_0^T \frac{\| u(\tau, \cdot) \|_{L^p}^{2p/3}}{1 + \ln \left( e + \| u \|_{L^\infty} \right)} d\tau
\]
is finite.

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