Twist Deformation of $l$-Conformal Galilei Hopf Algebra

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Abstract

The six Abelian twist-deformations of $l$-conformal Galilei Hopf algebra are considered. The corresponding twisted space-times are derived as well.

1 Introduction

The idea to use noncommutative coordinates is quite old - it goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, however, there were found new formal arguments based mainly on Quantum Gravity [2], [3] and String Theory models [4], [5], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature.

Presently, it is well known, that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries, one can distinguish three types of quantum spaces [6], [7] (for details see also [8]):

i) Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [9]-[11]

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},
\]

ii) Lie-algebraic modification of classical space-time [11]-[14]

\[
[\tilde{x}_\mu, \tilde{x}_\nu] = i\theta^\rho_{\mu\nu}\tilde{x}_\rho,
\]
and

**iii)** Quadratic deformation of Minkowski and Galilei spaces [11], [14]-[16]

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta^\rho_{\mu\nu} \hat{x}_\rho \hat{x}_\tau, \]

with coefficients \(\theta_{\mu\nu}, \theta^\rho_{\mu\nu}\) and \(\theta^\rho_{\mu\nu}\) being constants.

Besides, it has been demonstrated in [8], that in the case of so-called N-enlarged Newton-Hooke Hopf algebras \(U_0^{(N)}(NH_\pm)\) the twist deformation provides the new space-time noncommutativity of the form\(^1\)

\[ [t, x_i] = 0, \quad [x_i, x_j] = if_{\pm}\left(\frac{t}{\tau}\right) \theta_{ij}(x), \]

with time-dependent functions \(f_+(\frac{t}{\tau}) = f\left(\sinh\left(\frac{t}{\tau}\right), \cosh\left(\frac{t}{\tau}\right)\right)\), \(f_-(\frac{t}{\tau}) = f\left(\sin\left(\frac{t}{\tau}\right), \cos\left(\frac{t}{\tau}\right)\right)\), \(\theta_{ij}(x) \sim \theta_{ij} = \text{const}\) or \(\theta_{ij}(x) \sim \theta_{ij}^k x_k\) and \(\tau\) denoting the time scale parameter - the cosmological constant. Moreover, the different relations between all mentioned above quantum spaces (1), (2), (3) and (4)) have been summarized in paper [8].

It should be noted that the described above classification can be supplemented by the proper deformations of so-called \(l\)-conformal Galilei Hopf structure \(U_0^{(l)}(G)\) provided in [17]. In general, a conformal extension of the Galilei Hopf algebra is parametrized by a positive half integer \(l\), which justifies the term \(l\)-conformal Galilei Hopf structure. Particularly, the instance of \(l = \frac{1}{2}\) (well-known in the literature as the Schroedinger algebra) has been the focus of most studies (for a review see e.g. [18]). Besides, motivated by current investigation of the nonrelativistic version of the AdS/CFT correspondence, the interest in conformal Galilei Hopf algebras with \(l > \frac{1}{2}\) is growing rapidly in the last time [19]-[24].

In this article we investigate the Abelian twist deformations of \(U_0^{(l)}(G)\) Hopf structure which provide (as we shall see for a moment) six types of space-time noncommutativity (see formulas (28)-(34)). The motivations for such a kind of studies are manyfold. First of all, we construct explicitly the deformation of quite general (known) Hopf algebra at nonrelativistic level. Secondly, we get the completely new quantum (twist-deformed) space-times associated with

\[ x_0 = ct. \]

\(^2\)The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [9], [10]), for quantum N-enlarged Newton-Hooke Hopf algebras.
The paper is organized as follows. In first section we recall the basic facts concerning $l$-conformal Galilei Hopf algebra $U^{(l)}(G)$. The second section is devoted to its twist deformations and to the derivation of corresponding quantum space-times. The final remarks are provided in the last section.

2 $l$-conformal Galilei Hopf algebra $U^{(l)}(G)$

In this section we recall basic facts associated with the $l$-conformal Galilei Hopf algebra $U^{(l)}(G)$ provided in article [17]. Hence, the $l$-conformal Galilei Hopf algebra includes the generators of time translations, dilatations, special conformal transformations, spatial rotations, spatial translations, Galilei boosts and accelerations. Denoting the mentioned generators by $H, D, K, M_{ij}$ and $G^{(n)}_i$, respectively, where $i = 1, \ldots, d$ is a spatial index and $n = 0, 1, \ldots, 2l$, one can write the following algebraic relations:

\[ [M_{ij}, M_{kl}] = -i(\delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il}) , \]
\[ [M_{ij}, G^{(n)}_k] = -i(\delta_{ik}G^{(n)}_j - \delta_{jk}G^{(n)}_i) , \]
\[ [K, G^{(n)}_i] = i(n - 2l)G^{(n+1)}_i , \quad [D, G^{(n)}_i] = i(n - l)G^{(n)}_i , \]
\[ [H, K] = 2iD , \quad [D, K] = iK , \]
\[ [H, D] = iH , \quad [H, G^{(n)}_i] = inG^{(n-1)}_i , \]

as well as coalgebraic

\[ \Delta_0(a) = a \otimes 1 + 1 \otimes a , \quad S_0(a) = -a , \]

sectors. It should be also observed that operators $H, D$ and $K$ form so$(2,1)$ subalgebra, which is the conformal algebra in one dimension. Besides, one can notice that the instances of $n = 0$ and $n = 1$ in $G^{(n)}_i$ correspond to the spatial translations and Galilei boosts respectively, while the higher values of index $n$ are linked to the accelerations. Finally, it is easy to check that all above
generators are represented on the classical space of functions as follows [25]

\[ M_{ij} \triangleright f(t, \vec{x}) = i \left(x_i \partial_j - x_j \partial_i\right) f(t, \vec{x}) , \quad (11) \]
\[ H \triangleright f(t, \vec{x}) = i \partial_t f(t, \vec{x}) , \quad (12) \]
\[ G_i^{(n)} \triangleright f(t, \vec{x}) = i t^n \partial_i f(t, \vec{x}) , \quad (13) \]
\[ D \triangleright f(t, \vec{x}) = i \left(t \partial_t + lx^i \partial_i\right) f(t, \vec{x}) , \quad (14) \]
\[ K \triangleright f(t, \vec{x}) = i \left(t^2 \partial_t + 2ltx^i \partial_i\right) f(t, \vec{x}) . \quad (15) \]

Of course, for \( D = K = 0 \) and \( n = 0, 1 \) we reproduce well-known (ordinary) Galilei Hopf structure \( \mathcal{U}_0(\mathcal{G}) \).

3 Twist deformations of \( l \)-conformal Galilei Hopf algebra and the corresponding quantum space-times

Let us now turn to the twist deformations of the Hopf structure described in previous section. First of all, in accordance with Drinfeld twist procedure [26]-[28], the algebraic sector of twisted \( l \)-conformal Galilei Hopf algebra \( \mathcal{U}_\alpha^{(l)}(\mathcal{G}) \) remains undeformed (see (5)-(9)), while the coproducts and antipodes transform as follows (see formula (10))

\[ \Delta_0(a) \rightarrow \Delta_\alpha(a) = \mathcal{F}_\alpha \circ \Delta_0(a) \circ \mathcal{F}_\alpha^{-1} , \quad S_\alpha(a) = u_\alpha S_0(a) u_\alpha^{-1} , \quad (16) \]

with \( u_\alpha = \sum f(1) S_0(f(2)) \) (we use Sweedler’s notation \( \mathcal{F}_\alpha = \sum f(1) \otimes f(2) \)).

Besides, it should be noted, that the twist factor \( \mathcal{F}_\alpha \in \mathcal{U}_\alpha^{(l)}(\mathcal{G}) \otimes \mathcal{U}_\alpha^{(l)}(\mathcal{G}) \) satisfies the classical cocycle condition

\[ \mathcal{F}_{\alpha 12} \cdot (\Delta_0 \otimes 1) \mathcal{F}_\alpha = \mathcal{F}_{\alpha 23} \cdot (1 \otimes \Delta_0) \mathcal{F}_\alpha , \quad (17) \]

and the normalization condition

\[ (\epsilon \otimes 1) \mathcal{F}_\alpha = (1 \otimes \epsilon) \mathcal{F}_\alpha = 1 , \quad (18) \]

with \( \mathcal{F}_{\alpha 12} = \mathcal{F}_\alpha \otimes 1 \) and \( \mathcal{F}_{\alpha 23} = 1 \otimes \mathcal{F}_\alpha \).

It is well known, that the twisted algebra \( \mathcal{U}_\alpha^{(l)}(\mathcal{G}) \) can be described in terms of so-called classical \( r \)-matrix \( r \in \mathcal{U}_\alpha^{(l)}(\mathcal{G}) \otimes \mathcal{U}_\alpha^{(l)}(\mathcal{G}) \), which satisfies the classical Yang-Baxter equation (CYBE)

\[ [[ r_\alpha, r_\alpha ]] = [ r_{\alpha 12} + r_{\alpha 13} + r_{\alpha 23} ] = 0 , \quad (19) \]

where symbol \([ [ \cdot, \cdot ], \cdot \cdot ]\) denotes the Schouten bracket and for \( r = \sum_i a_i \otimes b_i \)

\[ r_{12} = \sum_i a_i \otimes b_i \otimes 1 , \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i , \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i . \]
In this article we consider six types of Abelian twist deformation of \(l\)-conformal Galilei Hopf algebra, described by the following \(r\)-matrices\(^3\):

\[
\begin{align*}
r_1 &= \frac{1}{2} \alpha_{ij} G^{(n)}_i \wedge G^{(m)}_i \quad [\alpha_{ij} = -\alpha_{ji}] , \\
r_2 &= \alpha_2 G^{(n)}_i \wedge M_{kl} \quad [i, k, l - \text{fixed}, i \neq k, l] , \\
r_3 &= \alpha_3 H \wedge M_{kl} , \\
r_4 &= \alpha_4 K \wedge M_{kl} , \\
r_5 &= \alpha_5 K \wedge G^{(n)}_i \quad [n = 2l] , \\
r_6 &= \alpha_6 D \wedge M_{kl} ,
\end{align*}
\]

where \(\alpha_{ij}, \alpha_2, \alpha_3, \ldots \alpha_6\) denote the deformation parameters. Due to the Abelian character of the above carriers (all of them contain the mutually commuting elements of the algebra), the corresponding twist factors can be obtained in a standard way \([26]-[28]\), i.e. they take the form

\[
\mathcal{F}_a = \exp (ir_a) \quad ; \quad a = 1, 2, \ldots, 6 .
\] (26)

The corresponding quantum space-times are defined as the representation spaces (Hopf modules) for \(l\)-conformal Galilei Hopf algebra \(U^{(l)}_\alpha (G)\), with action of the generators \(M_{ij}, H, G^{(n)}_i, K\) and \(D\) given by (11)-(15) (see e.g. \([9],[10]\)). Besides, the \(\ast\)-multiplication of arbitrary two functions covariant under \(U^{(l)}_\alpha (G)\) is defined as follows

\[
f(t, \bar{x}) \ast_a g(t, \bar{x}) := \omega \circ ((\mathcal{F}_a)^{-1} \triangleright f(t, \bar{x}) \otimes g(t, \bar{x})) ,
\] (27)

where symbol \(\mathcal{F}_a\) denotes the twist factors (see (26)) and \(\omega \circ (a \otimes b) = a \cdot b\). Consequently, we get

1. \([t, x_a]_{*1} = 0 , [x_a, x_b]_{*1} = i\alpha_{ij} l^{n+m} (\delta_{ai} \delta_{bj} - \delta_{aj} \delta_{bi}) ,
\] (28)

2. \([t, x_a]_{*2} = 0 \]

\([x_a, x_b]_{*2} = 2i\alpha_2 l^n [\delta_{ia}(x_k \delta_{bl} - x_l \delta_{bk}) - \delta_{ib}(x_k \delta_{al} - x_l \delta_{ak})] ,
\] (29)

3. \([t, x_a]_{*3} = 2i\alpha_3 (x_k \delta_{la} - x_l \delta_{ka}) , [x_a, x_b]_{*3} = 0 ,
\] (30)

4. \([t, x_a]_{*4} = 2i\alpha_4 l^2 (x_k \delta_{la} - x_l \delta_{ka}) ,
\] (31)

\([x_a, x_b]_{*4} = 4i\alpha_4 l t [x_a(x_k \delta_{bl} - x_l \delta_{bk}) - x_b(x_k \delta_{al} - x_l \delta_{ak})] ,
\]

\(^{3}a \wedge b = a \otimes b - b \otimes a.\)
5. 

\[ [t, x_a]_{\tau_5} = 2i\alpha_5 t^{2(1+l)} \delta_{ia} , \tag{32} \]

\[ [x_a, x_b]_{\tau_5} = 4i\alpha_5 l t^{2l+1}(x_a\delta_{bi} - x_b\delta_{ai}) , \tag{33} \]

and

6. 

\[ [t, x_a]_{\tau_6} = 2i\alpha_6 t(x_k\delta_{ia} - x_l\delta_{ka}) , \tag{34} \]

\[ [x_a, x_b]_{\tau_6} = 2i\alpha_6 l [x_a(x_k\delta_{bl} - x_l\delta_{bk}) - x_b(x_k\delta_{al} - x_l\delta_{ak})] , \]

respectively. It should be noted that three first spaces are the same as in the case of so-called twisted $N$-enlarged Galilei algebra [8], while the remaining ones correspond to the conformal sector of considered in present article Hopf structure. Obviously, for deformation parameters $\alpha_{ij}^{1}$ and $\alpha_{2}, \ldots, \alpha_{6}$ approaching zero the above quantum space-times become classical.

4 Final remarks

In this article we consider six Abelian twist-deformations of $l$-conformal Galilei Hopf algebra $\mathcal{U}_l^{(l)}(\mathcal{G})$. The corresponding twisted space-times are derived as well. It should be noted, however, that present studies can be extended in various ways. First of all, one can find the dual Hopf structures $\mathcal{D}_a^{(l)}(\mathcal{G})$ with the use of FRT procedure [29] or by canonical quantization of the corresponding Poisson-Lie structures [30]. Besides, as it was already mentioned in Introduction, one should ask about the basic dynamical models corresponding to the $l$-conformal space-times (28)-(34). Finally, one can also consider more complicated (non-Abelian) twist deformations of $l$-conformal Hopf algebras, i.e. one can find the twisted coproducts, corresponding noncommutative space-times and dual Hopf structures. Such problems are now under consideration.

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