Existence for stochastic 2D Euler equations with positive $H^{-1}$ vorticity

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Abstract

We prove the existence of non-negative measure- and $H^{-1}$-valued vorticity solutions to the stochastic 2D Euler equations with transport vorticity noise, starting from any non-negative vortex sheet. This extends the result by Delort [Del91] to the stochastic case.

1 Introduction

In this paper we consider the two-dimensional (2D) stochastic Euler equations, in vorticity form, with transport noise, namely, on $[0,T] \times \mathbb{T}^2$,

$$\partial_t \xi + u \cdot \nabla \xi + \sum_k \sigma_k \cdot \nabla \xi \circ W^k = 0,$$

$$u = K * \xi. \quad (1.1)$$

This equation represents the motion of an incompressible fluid in a periodic domain, perturbed by a noise of transport-type. We show in our main result, Theorem 3.1, the weak existence (weak in both the probabilistic and the analytic sense) of certain measure-valued solutions for this equation: more precisely, for any initial datum $\xi_0$ which is a non-negative measure and in $H^{-1}$, there exists a weak, measure- and $H^{-1}$-valued solution to (1.1). This result extends the existence result by Delort [Del91] to the stochastic case.

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The deterministic (incompressible, $d$-dimensional) Euler equations
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= 0, \\
\text{div} u &= 0,
\end{align*}
describe the evolution of the velocity $v(t, x) \in \mathbb{R}^d$ and the pressure $p(t, x) \in \mathbb{R}$ of an incompressible fluid; for this discussion, for simplicity, we assume $x$ in $\mathbb{T}^d$, the $d$-dimensional torus. The local (in time) well-posedness for smooth solutions has been established since [EM70], a general description of Euler equations can be found for example in [MP94], [Lio96], [MB02]. In the two-dimensional case, the Euler equations are formally equivalent to a non-linear transport equation, namely the vorticity equation
\begin{align*}
\partial_t \xi + u \cdot \nabla \xi &= 0, \\
u &= K \ast \xi.
\end{align*}
(1.2)
Here $\xi(t, x) := \text{curl} u(t, x) = \partial_{x_1} u_2(t, x) - \partial_{x_2} u_1(t, x)$ is the scalar vorticity, which expresses how fast the fluid rotates around a point $x$, and the kernel $K : \mathbb{T}^2 \to \mathbb{R}^2$ is given by $K = \nabla G$, where $G : \mathbb{T}^2 \to \mathbb{R}$ is the Green function of the Laplacian on $\mathbb{T}^2$, restricted on the $L^2$ functions with zero mean. Global well-posedness among essentially bounded solutions has been proved in [Wo33] and [Jud63], see also [Yud95] for an extension to almost bounded functions and [MP94, Section 2.3] for a different proof using flows. Beyond bounded solutions, a global existence result has been given in [DM87] for vorticity in $L^p(\mathbb{T}^2)$, with $1 < p < \infty$. The case that we are interested in here is when the vorticity is measure-valued. One of the main results in this context is a global existence result by Delort [Del91], where the vorticity has a distinguished sign and is in $H^{-1}(\mathbb{T}^2)$: precisely for any non-negative measure $H^{-1}(\mathbb{T}^2)$ initial datum $\xi_0$, there exists a non-negative measure- and $H^{-1}(\mathbb{T}^2)$-valued solution. This includes the case of an initial vortex sheet, that is when $\xi_0$ is concentrated on a line. Later papers by Schochet [Sch95] and Poupaud [Pou02] gave a somehow clearer argument which we will use mostly here. A more general existence result, where the vorticity is the sum of a non-negative measure $H^{-1}(\mathbb{T}^2)$ and an $L^p(\mathbb{T}^2)$ function, for $p \geq 1$, has been given in [VW93]. The study of such irregular vorticity solutions has several motivations; it represents physically relevant situations, where the vorticity is concentrated on sets with zero Lebesgue measure (see e.g. [MP94, Chapter 6]); it is also related to possible anomalous energy dissipation (to our knowledge, energy conservation or dissipation remains an open problem for
Delort solutions, see [CFLS16] for energy conservation for unbounded vorticity) and to boundary layers (see e.g. [Cho78]), though we will not explore these aspects here.

Before passing to the stochastic case, we give a short idea of the proof of Delort’s result. The strategy passes through an usual compactness and convergence argument: take a sequence of approximants, show a priori uniform bounds, derive the compactness, show the stability of the equation in the limit. The main problem comes from the stability, in particular from the stability of the non-linear term. The main point by Delort (and later Schchet) is that the non-linear term is continuous among non-negative, or more-generally bounded from below, $H^{-1}(T^2)$ measures, with respect to the weak topologies (the precise topologies will be given later). Hence it suffices to show preservation of the non-negativity, or at least a bound from below, and uniform a priori bounds on the total variation norm and the $H^{-1}(T^2)$ norm of a solution $\xi$. The transport and divergence-free nature of the vorticity equation (1.2) implies easily the non-negativity and the a priori bound on the total variation norm. The $H^{-1}(T^2)$ norm of $\xi$ is equivalent to the $L^2(T^2)$ norm of $u = K * \xi$, that is the energy, so the a priori $H^{-1}(T^2)$ bound is equivalent to an a priori bound on the energy of $u$, which is also classical and available. This allows to have compactness and stability as required.

Concerning the stochastic case, Euler equations with noise have been investigated in a large amount of papers, here we only give a review of a small selection of them. Many works take additive or multiplicative noise, but dependent on $u$ and not on its gradient. The first works are [BF99], which shows the global well-posedness of strong solutions, with additive noise, for bounded vorticity in a 2D domain (though adaptedness is not proven there), [Bes99] and [BP01], which show the global existence of martingale solutions, with multiplicative noise, for $L^2(T^2)$ vorticity in 2D. We also mention [MV00] for a geometric approach in the case of finite-dimensional additive noise and [CC99] for an approach via nonstandard analysis. The works [Kim09], [GHV14] prove local strong well-posedness, with additive and, for the second work, multiplicative noise, in general dimension for smooth solutions. The second paper [GHV14] shows also global well-posedness among smooth solutions in 2D for additive and linear multiplicative noise.

The transport noise is considered first in [Yok14], [SY14], [CT15], where the transport term is put on the velocity and not on the vorticity; these works show the global existence of martingale solutions for $L^2(T^2)$ vorticity in 2D. The model (1.1) is considered in [BFM16], which proves global strong
well-posedness for bounded vorticity on the two-dimensional torus. In the papers [FL19b] and [FL19a], the authors consider also the same model, with very irregular vorticity (not in $H^{-1}(\mathbb{T}^2)$), and they show an existence result putting a measure on initial conditions which is invariant for the dynamics or almost invariant (that is, it is absolutely continuous with respect to the invariant measure). The analogue of this model in 3D, where a stochastic advection term also appears, is considered in [CFH19], which shows the local strong well-posedness and a Beale-Kato-Majda criterion for non-explosion. We also point out the work [FGP11], which considers (1.1) with initial mass concentrated in a finite number of point vortices and shows a regularization by noise phenomenon, precisely that, with full probability, no collapse of vortices happens with a certain noise, while it does happen without noise.

There are several reasons why to use transport noise. The addition of a transport noise preserves the transport structure of the vorticity equation: at a formal level, a solution $\xi(t, x, \omega)$ follows the characteristics of the associated SDE, that is $\xi(t, X(t, x, \omega), \omega) = \xi(0, x)$, where $X(t, x, \omega)$ is the stochastic flow solution to

$$dX(t) = u(t, X(t)) dt + \sum_k \sigma_k(X(t)) \circ dW^k(t).$$

This fact follows from the Itô formula and here the Stratonovich noise is essential, because the Itô formula for this noise works as the chain rule, without second-order corrections. Furthermore, there is a derivation of models with stochastic transport term in [Hol15], [DH18], there are applications using the transport noise to model uncertainties, e.g. [CCH+18], also the linear stochastic transport equation has been used as a toy model for turbulence, see e.g. [Gaw08], though we will not go into these directions here. The main feature of interest here is the fact that the transport noise preserves at least formally the $L^\infty(\mathbb{T}^2)$ norm of the solution and also, when the coefficients $\sigma_k$ are divergence-free as here, the $L^p(\mathbb{T}^2)$ norm for any $1 \leq p \leq \infty$ (this can be seen for example via a priori bounds for the $L^p$ norm). As a consequence, the transport noise preserves the total mass of the vorticity and its positivity (if $\xi_0 \geq 0$, then also $\xi_t \geq 0$), two properties that are crucial to apply the argument by Schochet. This is not the case for a generic additive or multiplicative noise as considered in the above mentioned papers. Indeed the additive noise would not guarantee positivity of the solution. We should say though that some multiplicative noises may still be used for our purposes: for example one can take a linear multiplicative noise in $\xi$, $+\sigma_k \xi \circ \dot{W}^k$ this
gives preservation of the positivity and may also give some uniform bound on the mass (for this example, the Itô noise may work as well). Moreover, what we really need to have Delort’s result is some uniform $\mathcal{M}(\mathbb{T}^2)$ bound of the vorticity and some uniform $L^\infty$ bound, or even $L^p$ bound in the line of the more general result in [WW93], on the negative part of the vorticity; such uniform bounds may hold also for an additive noise, though anyway the arguments for the proof would be more complicated.

We remark here that the stochastic vorticity equation (1.1) preserves in particular the enstrophy, that is the $L^2(\mathbb{T}^2)$ norm of the solution $\xi$, but not the energy, that is the $L^2(\mathbb{T}^2)$ norm of the velocity $u = K * \xi$. Indeed, the equation for the velocity is formally

$$
\partial_t u + (u \cdot \nabla) u + \sum_k (\sigma_k \cdot \nabla + D\sigma_k) u \circ \dot{W}^k = -\nabla p - \gamma,
$$

(1.3)

with the velocity $u(t, x, \omega)$, the pressure $p(t, x, \omega)$ and also the constant (in space) $\gamma(t, \omega)$ are unknown, $\gamma$ here is needed to keep $u$ with zero mean. In this equation for $u$, the zero order term $(D\sigma_k) u \circ \dot{W}^k$ appears and causes the velocity not to be preserved anymore.

Let us also mention that the transport noise has been used to show regularization by noise phenomena for the transport equations, though this is mostly limited to the linear case (see [FGP10], [FF13], [BFGM14], [FMN14] as examples among many others), while the extension of [FGP11] to the case of a more general, measure-valued vorticity meets relevant difficulties; in other nonlinear hyperbolic cases, only a few results are available (e.g. [GM18], [DFV14]).

Before passing to our case, we point out two peculiarities of the stochastic case in the strategy that one uses to get martingale solutions, see for example [BP01]. Analogously to the deterministic case, the strategy passes through an usual tightness and convergence argument: take a sequence of approximants, show a priori uniform bounds for suitable moments, derive the tightness, show the stability of the equation in the limit. We point out two facts, which we will use in our proof as well. Firstly, to derive tightness, say, in $C([0, T]; X)$, where $X$ is a functional space on $\mathbb{T}^2$, one needs a uniform bound on the marginals and also a uniform bound in $C^\alpha([0, T]; Y)$, where $Y$ is another functional space, typically a negative-order Sobolev space, containing $X$. The latter is the stochastic counterpart of the Aubin-Lions lemma, introduced since at
least $[FG95]$. Secondly, for the stability, one option is to exploit Skorokhod representation theorem to pass from convergence in law to a.s. convergence. Now we come back to our contribution. Our main result Theorem 3.1 extends the result by Delort $[Del91]$ in the stochastic setting with transport noise. To our knowledge, no such extension has been studied before in the stochastic setting. Our proof combines the argument by Schochet $[Sch95]$ to deal with the nonlinear term and the arguments by e.g. $[BP01]$ to deal with the stochasticity, as explained before; more details are given in Section 3.

Two comments are in place. Firstly, as mentioned, the transport noise gives preservation of mass and of positivity and thus allows to extend easily the Schochet argument concerning these aspects. On the contrary, the additive noise would fail at this point, while some multiplicative noise may still work and the additive noise may work with a more complicated argument.

Secondly, the Schochet argument does not immediately goes through for the uniform $L^2(T^2)$ bound on the velocity $u$: as we have seen, the equation for the velocity (1.3) does not preserve the energy. To handle this problem, we assume a certain structure of the $\sigma_k$ so that a uniform $L^2(T^2)$ bound on $u$ can be still obtained. These assumptions allow still to deal with relevant cases, as for example “locally isotropic” covariance matrices, see the discussion in Section 2.2.

We make one last comment on an alternative possible strategy. In $[BFM16$, Section 7], we give an alternative proof of well-posedness among bounded solutions, which is based on the Doss-Sussmann transformation: if $\psi$ is the stochastic flow solutions to

$$d\psi_t = \sum_k \sigma_k(\psi_t) \circ dW^k_t,$$

then $\tilde{\xi}(t, x, \omega) := \xi(t, \psi(t, x, \omega), \omega)$ satisfies a random Euler-type PDE, precisely

$$\partial_t \tilde{\xi} + \tilde{u} \cdot \nabla \tilde{\xi} = 0,$$

$$\tilde{u}(t, x, \omega) = D\psi^{-1}(t, x, \omega) \int_{T^2} K(\psi(t, x, \omega) - \psi(t, y, \omega))\tilde{\xi}(t, y, \omega)dy,$$  \hspace{1cm} (1.4)

where $\psi^{-1}$ is the inverse flow. This is a nonlinear transport equation, where the kernel $K$ has been replaced by the above random kernel. In the case that $\sigma_k = \epsilon_k 1_{k \in \{1, 2\}}$, where $\epsilon_k$ is the canonical basis on $\mathbb{R}^2$, the PDE (1.4) is exactly the Euler vorticity equation: indeed in this case the stochastic Euler
equations correspond simply to a random shift in Lagrangian coordinates. In
the general case however, this is not the Euler vorticity equation, but it enjoys
similar properties since the random kernel has similar regularity properties,
the well-posedness among bounded solutions can be derived applying
the deterministic arguments to the random PDE (1.4). One may then wonder
if a similar argument is possible here, namely using Schochet’s arguments for
the random PDE (1.4). In this context however, there are two problems at
least. Firstly, even if we could just apply Delort’s or Schochet’s result to
(1.4), at \( \omega \) fixed, this would only give the existence of a measure-valued and
\( H^{-1} \)-valued solution to (1.4) at \( \omega \) fixed, with no adaptedness property; so a
compactness argument would probably be needed anyway. Secondly, from
(1.4) it is not immediately clear how to get a uniform \( H^{-1}(T^2) \) bound on \( \tilde{\xi} \).
Moreover this type of arguments à la Doss-Sussmann do not work intrinsically
on the stochastic PDE. Let us remark anyway that the argument may still
give some hints: for example, it would be interesting to see if, using the
equation for \( \tilde{u} \) instead of the equation for \( u \), one can get a uniform \( H^{-1}(T^2) \)
bound without the additional assumptions on \( \sigma \) that are needed here. We
leave this point for future research.

2 The setting

2.1 Notation

We recall some notation frequently used in the paper. We use the letters \( t, x, \omega \)
for a generic element in \([0, T], T^2, \Omega \) resp.; the coordinates of \( x \) are indicated
with \((x^1, x^2)\), while the partial derivatives are denoted by \( \partial_{x^1}, \partial_{x^2} \). Unless
differently specified, the derivatives \( \nabla, D, \Delta \) are indended with respect to the
space \( x \). The Sobolev spaces are denoted with \( W^{s,p} \) or, for \( p = 2 \), \( H^s = W^{s,2} \).
For a map \( f : T^2 \to \mathbb{R}^2 \), recall that \( Df = (\partial_{x^j} f^i) \) and \( \nabla f = (Df)^T \).
For functional spaces, we often put the input variables \((t, x, \omega)\) as subscripts:
for example, the spaces \( L^2(T^2) \), \( C([0, T]) \), is denoted in short as \( L^2_t \), \( C_t \) resp.;
as another example, the space \( C([0, T]; (H^{-4}(T^2), w)) \) of continuous functions
on \([0, T] \), which take values in \( H^{-4}(T^2) \) with the weak topology, is denoted
in short as \( C_t(H^{-4}_x, w) \). The symbol \( f * g \) stands for the convolution in the
space variable (that is on \( T^2 \) ) between two functions or distributions \( f \) and
\( g \) on \( T^2 \).

The space \( \mathcal{M}_x = \mathcal{M}(T^2) \) is the space of finite signed Radon measures on \( T^2 \);
it is a Banach space, endowed with the total variation norm $\| \cdot \|_{M_x}$ and is the dual of the space $C_x = C(T^2)$ of the space of continuous functions on $T^2$; the notation $\langle \mu, \varphi \rangle$ will be used to denote the duality product between a measure $\mu$ in $M_x$ and a function $\varphi$ in $C_x$. The closed ball of center 0 and radius $M$ on $M_x$ is denoted by $M_{x,M}$ (the radius here refers to the strong norm $\| \cdot \|_{M_x}$). Following the notation above, the space $C([0,T];(M_{x,M}, w^*))$ of continuous functions on $[0,T]$, which take values in the closed ball of radius $M$ of $M_x$ with the weak-* topology, is denoted in short as $C_t(M_{x,M}, w^*)$. We will also use $M_{x,+}$ for the set of non-negative finite Radon measures on $T^2$ and $M_{x,M,+,no-\text{atom}}$ for the set of non-negative non-atomic Radon measures with total mass $\leq M$, $M_{x,y}$ for the space of finite Radon measures on $T^{2\times 2}$.

For $x = (x^1, x^2)$ in $\mathbb{R}^2$, we call $x^\perp := (-x^2, x^1)$. Similarly, for a function $f : T^2 \to \mathbb{R}$, we call $\nabla f := (\nabla f)^\perp = (-\partial x^2 f, \partial x^1 f)$.

Given a probability space $(\Omega, \mathcal{A}, P)$ with a filtration $(\mathcal{F}_t)$, the symbol $\mathcal{P}$ will be used for the progressively measurable $\sigma$-algebra associated with $(\mathcal{F}_t)$ (not the predictable $\sigma$-algebra).

The letter $C$ will be used for constants which may change from one line to another.

### 2.2 Assumptions on the noise

Here we give the assumptions on the noise coefficients $\sigma_k$:

**Condition 2.1.** We assume that:

- The vector fields $\sigma_k : T^2 \to \mathbb{R}^2$, $k \in \mathbb{N}$, are of class $C^1_x$, divergence-free and satisfy

\[ \sum_k \| \sigma_k \|_{C^1_x}^2 < \infty. \]

In particular, there exists a continuous function $a : T^2 \times T^2 \to \mathbb{R}^{2\times 2}$, called infinitesimal covariance matrix, such that

\[ \sum_k \sigma_k(x) \sigma_k(y)^T = a(x, y), \]

with convergence in $C_{x,y}$ (that is uniformly in $(x, y)$). Moreover $a$ is differentiable in $x$ and in $y$ and

\[ \sum_k \partial_x \sigma_k(x) \sigma_k(y)^T = \partial_x a(x, y), \quad i = 1, 2, \]
with convergence in \( C_{x,y} \), and analogously for the derivative in \( y \).

- The function \( a \) defined above satisfies, for some \( c \geq 0 \),
  \[ a(x,x) = cI_2, \]
  where \( I_2 \) is the \( 2 \times 2 \) identity matrix.

- The function \( a \) defined above satisfies
  \[ \partial_y a(x,y) |_{y=x=0} = 0, \quad i = 1, 2, \quad \forall x \in \mathbb{T}^2. \]

The second assumption reads
\[ \sum_k \sigma^i_k(x)\sigma^j_k(x) = c\delta_{ij}, \quad \forall x, \quad \forall i, j = 1, 2. \]

The third assumption reads
\[ \sum_k \sigma^i_k(x)\partial_{x_j}\sigma^h_k(x) = \partial_y a(x,y) |_{y=x=0}, \quad \forall x, \quad \forall i, j, h = 1, 2. \]

The second assumption allows to simplify the Itô-Stratonovich correction, because it cancels the first order term \( \frac{1}{2} \sum_k (\sigma_k \cdot \nabla_k)\sigma_k \cdot \nabla\xi \) in the correction and it makes the second order term \( \sum_k \text{Tr}[\sigma_k\sigma_k^T D^2\xi] \) a constant coefficient operator: indeed we have, for every \( j = 1, 2 \),
\[ \sum_k (\sigma_k(x) \cdot \nabla)\sigma^j_k(x) = \sum_k \sum_i \partial_{x_i}[\sigma^j_k\sigma^i_k](x) = \sum_i \partial_{x_i}a^{ij}(\cdot, \cdot)(x) = 0, \quad \sum_k \text{tr}[\sigma_k(x)\sigma_k(x)^T D^2\xi(x)] = c\Delta\xi(x). \]

Hence the Itô-Stratonovich correction reads formally
\[ \sum_k [\sigma_k \cdot \nabla\xi(x), W^k]_t = -\int_0^t c\Delta\xi(x)dr \]
and the vorticity equation (1.1) reads formally in Itô form
\[ \partial_t \xi + u \cdot \nabla\xi + \sum_k \sigma_k \cdot \nabla\xi \dot{W}^k = \frac{1}{2} c\Delta\xi, \quad u = K \ast \xi, \quad (2.1) \]
These assumptions are stronger than the ones in [BFM16] for well-posedness of bounded vorticity solutions. Precisely, in [BFM16] only the first two assumptions are made (the second in a slightly weaker form) and the second assumption is not essential (a first-order Itô correction can be treated via lengthly computations). Here on the contrary, it seems unclear from the proof whether the second and third assumptions can be removed. Indeed these two assumptions guarantee that the equation for the velocity $u$ has no first order term other than the transport one, which in turn implies a key energy bound of solution (that is the $L^2$ norm of $u$).

There is a relevant class of example of non trivial (that is, non constant) $\sigma_k$ satisfying Condition 2.1:

**Example 2.2.** Let $\beta > 3$ and define

$$\sigma_k(x) = (\cos(k \cdot x) + \sin(k \cdot x)) \frac{k_\perp}{|k|^{\beta}}, \quad k \in \mathbb{Z}^2 \setminus \{0\}.$$

Since $\beta > 0$, we infer that $\sum_k \|\sigma_k\|_{L^2_x}^2 < \infty$ and we can calculate that

$$a(x, y) = \sum_{k \in \mathbb{Z}^2, k \neq 0} \cos(k \cdot (x - y)) \frac{k_\perp (k_\perp)^T}{|k|^{2\beta}} =: a(x - y),$$

$$a(x, x) = 2 \sum_{k \in \mathbb{Z}^2, k_1 \geq 0, k_2 > 0} \frac{1}{|k|^{2\beta - 2}} I_2.$$

Note that, in this example, the infinitesimal covariance matrix $a$ is translation-invariant (that is $a(x, y) = a(x - y)$) and even (that is $a(x) = a(-x)$). More in general, if $a$ is translation-invariant and even, then it satisfies the third assumption in Condition 2.1. Indeed

$$\partial_y a(x, y) \big|_{y=x} = -\partial_z a(z) \big|_{z=0} = 0.$$

We could *morally* include the class of isotropic infinitesimal covariance matrices in our setting. On $\mathbb{R}^2$, an infinitesimal covariance matrix $a$ is called isotropic if it is translation- and rotation-invariant, that is $a(x, y) = a(x - y)$ for all $(x, y)$ and $R^T a(Rx) R = a(x)$ for every rotation matrix $R$, and $a(0) = cI$ for some $c > 0$; rotation invariance implies that $a$ is even (take $Rx = -x$), hence a sufficiently regular isotropic matrix $a$ satisfies the Condition 2.1 (precisely, for a regular one can find $\sigma_k$ regular satisfying Condition 2.1).
The isotropic condition on \( a \) means morally that the noise \( \sum_k \sigma_k(x) \circ \dot{W}_t \) is Gaussian, white in time, coloured and isotropic in space. Such a class has been considered in the mathematical and physical literature, see e.g. [BH86], [Gaw08], even without the nonlinear term: for example, the same type of noise, but irregular in space, provides a simplified model for the study of passive scalars in a turbulent motion (see [Gaw08], [FGV01], [LJR02]). Strictly speaking we are not allowed here to take an isotropic matrix as infinitesimal covariance matrix \( a \) here, for the simple reason that the torus itself (considered as \([-1, 1]^2 \) with periodic boundary conditions) is not rotation-invariant. However, one may still take a translation-invariant and rotation-invariant on a neighborhood of the diagonal \( \{x = y\} \). Moreover the torus setting here is taken to avoid technicalities at infinity, but we believe a similar construction, including isotropic vector fields, would go through also in the full space case.

2.3 The nonlinear term

We focus now on the definition and properties of the nonlinear term \( \langle \xi, u^\xi \cdot \nabla \varphi \rangle \). For this, we introduce some notation. The space \( \mathcal{M}_x = \mathcal{M}(\mathbb{T}^2) \) is the space of finite signed Radon measures on \( \mathbb{T}^2 \); it is a Banach space, endowed with the total variation norm \( \|\cdot\|_{\mathcal{M}_x} \), and is the dual of the space \( C_x = C(\mathbb{T}^2) \) of the space of continuous functions on \( \mathbb{T}^2 \). We endow \( \mathcal{M}_x \) with the Borel \( \sigma \)-algebra generated by the weak-* topology. The notation \( \langle f, g \rangle \) denotes the \( L^2 \) duality product between two functions or distributions on \( \mathbb{T}^2 \); in particular, \( \langle \mu, \varphi \rangle \) will be used to denote the duality product between a measure \( \mu \) in \( \mathcal{M}_x \) and a function \( \varphi \) in \( C_x \).

For \( \xi \) in \( L_x^{4/3} \), \( u^\xi \) is in \( W_x^{1,4/3} \) by Lemma [C31] and so in \( L_x^4 \) by Sobolev embedding; hence the product \( \xi u^\xi \) is in \( L_x^1 \) and the nonlinear term makes sense for regular \( \varphi \). However, if \( \xi \) is not in \( L_x^{4/3} \), the product \( \xi u^\xi \) is not defined in general. To overcome this difficulty, we use the following Lemma, due to Schochet [Sch95] Lemma 3.2 and discussion thereafter. For \( \varphi \) in \( C_x^2 \), we define the function \( F_{\varphi} : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}^{2 \times 2} \) by

\[
F_{\varphi}(x, y) := \frac{1}{2} K(x - y) \cdot (\nabla \varphi(x) - \nabla \varphi(y)) 1_{x \neq y}.
\]

**Lemma 2.3.** The following hold:

- For every \( \varphi \) in \( C_x^2 \), \( F_{\varphi} \) is bounded everywhere by \( C \|\varphi\|_{C_x^2} \) and continuous outside the diagonal \( \{(x, y) \mid x = y\} \).
For any measure $\xi$ in $\mathcal{M}_x$, the formula
\[
\langle N(\xi), \varphi \rangle := \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} F_\varphi(x, y) \xi(dx) \xi(dy), \quad \varphi \in C^\infty_x,
\]
defines a distribution in $H^{-4}_x$ and we have
\[
\|N(\xi)\|_{H^{-4}_x} \leq C \|\xi\|_{\mathcal{M}_x}^2.
\]

The map $N : \mathcal{M}_x \to H^{-4}_x$ is Borel, where $\mathcal{M}_x$ is endowed with the Borel $\sigma$-algebra generated by the weak-* topology.

The map $N$ coincides with the nonlinear term in equation (1.1) for $\xi$ in $L^{4/3}_x$: precisely, for $\xi$ in $L^{4/3}_x$,
\[
\langle \xi, u^{\xi} \cdot \nabla \varphi \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} F_\varphi(x, y) \xi(dx) \xi(dy). \quad (2.2)
\]

For every $M > 0$, the map $N$, restricted on the set $\mathcal{M}_{x,M,+}$ of non-negative non-atomic measures on $\mathbb{T}^2$ with total mass $\leq M$, is continuous (that is $\xi \mapsto \langle N(\xi), \varphi \rangle$ is continuous for every $\varphi$ in $H^4$); hence $N$ is the only continuous extension of the nonlinear term within $\mathcal{M}_{x,M,+}$.

Thanks to this Lemma, we use the right-hand side of (2.2) as the definition for the nonlinear term in (1.1). Note that, by Lemma 4.18 stated later, the set $\mathcal{M}_{x,M,+}$ includes the case of $\xi$ in $\mathcal{M}_{x,+} \cap H^{-1}_x$ we are interested in here. The last assertion of Lemma 2.3 is a consequence of Lemma 4.16, while the other assertions are proved in the Appendix.

### 2.4 Definition of measure-valued solutions

Now we can give the definition of a measure-valued solution to the stochastic vorticity equation. Again we give some notation. For any fixed $M > 0$, $\mathcal{M}_{x,M}$ is the set of all finite signed Radon measures $\mu$ on $\mathbb{T}^2$ with total variation $\|\mu\|_{\mathcal{M}_x} \leq M$. We consider $\mathcal{M}_{x,M}$ endowed with the weak-* topology induced by $\mathcal{M}_x$, which makes it a compact Polish space (by Banach-Alaoglu theorem and by e.g. [Bre11, Theorem 3.28]), and with the Borel $\sigma$-algebra generated by this topology. Given a filtration $(\mathcal{F}_t)_t$, we call $\mathcal{P}$ the associated progressive $\sigma$-algebra (not the predictable $\sigma$-algebra). A cylindrical Brownian motion on $(\mathcal{F}_t)_t$ is a sequence $W = (W^k)_k$ of independent $(\mathcal{F}_t)_t$-Brownian motions.
Definition 2.4. Fix $M > 0$. Assume that the vector fields $\sigma_k$ satisfy the Condition (2.1). A weak distributional $\mathcal{M}_{x,M}$-valued solution to the vorticity equation (1.1) is an object $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, P, (W^k)_k, \xi)$, where $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, P, (W^k)_k)$ is a cylindrical Brownian motion with the usual assumptions, $\xi : [0,T] \times \Omega \to \mathcal{M}_{x,M}$ is $P$ Borel measurable (with $P$ the progressive $\sigma$-algebra associated with $(\mathcal{F}_t)_t$) and it holds

$$\xi_t = \xi_0 - \int_0^t N(\xi_r)dr - \sum_k \int_0^t \sigma_k \cdot \nabla \xi_r dW^k_r + \frac{1}{2} \int_0^t c\Delta \xi_r dr, \quad \forall t, \quad P-a.s.$$ (2.3)

(the $P$-exceptional set being independent of $t$), as equality in $H^{-4}_x$. This definition is the rigorous formulation of (1.1) in the Itô form (2.1), under the Condition (2.1) on $\sigma_k$.

Lemmas 2.3 and B.1 imply that $\xi$ and the integrands in (2.3) are $P$ Borel measurable as $H^{-4}_x$-valued maps, moreover

$$\mathbb{E} \sum_k \int_0^T \|\sigma_k \cdot \nabla \xi_r\|_{H^{-4}_x}^2 dr \leq M^2 \sum_k \|\sigma_k\|_{C_x}^2. \quad (2.4)$$

Hence the deterministic integrals and the stochastic Itô integral make sense resp. as Bochner and stochastic integrals in $H^{-4}_x$.

Moreover, Lemma B.2 shows that (2.3) is equivalent to the formulation with test functions, namely, for every $\varphi$ in $C_c^\infty$,

$$\langle \xi_t, \varphi \rangle = \langle \xi_0, \varphi \rangle + \int_0^t \langle N(\xi_r), \varphi \rangle dr$$

$$+ \sum_k \int_0^t \langle \xi_r, \sigma_k \cdot \nabla \varphi \rangle dW^k_r$$

$$+ \frac{1}{2} \int_0^t \langle \xi_r, c\Delta \varphi \rangle dr, \quad \text{for every } t, \quad P-a.s., \quad (2.5)$$

(the $P$-exceptional set being independent of $t$).

We sometimes say that $\xi$ is an $L^p_x$-valued solution if has finite $L^p_{t,\omega}(L^p_x)$ norm for some $1 \leq m \leq \infty$, where we identify a measure with its density if the density exists. Similarly for $H^{-1}_x$-valued solutions.
3 Global existence for vorticity in $\mathcal{M}_{x,+} \cap H^{-1}_x$

We give the main result of this paper. Here $\mathcal{M}_{x,+}, \mathcal{M}_{x,M,+}$ are the subsets of $\mathcal{M}_x$ resp. of non-negative measures and of non-negative measures with total variation $\leq M$.

**Theorem 3.1.** Assume the Condition 2.1 on $\sigma_k$, fix $M > 0$. For every $\xi_0$ in $\mathcal{M}_{x,M,+} \cap H^{-1}_x$, there exists a weak distributional $\mathcal{M}_{x,M}$-valued solution $(\Omega, A, (F_t)_{t \in \mathbb{R}}, P, (W^k)_{k \in \mathbb{N}}, \xi)$ to the vorticity equation (1.1), with $\xi$ in $C_t(\mathcal{M}_{x,M,+}, w^*) \cap L^2_{t,\omega}(H^{-1}_x)$.

**Remark 3.2.** Note that, if $\xi$ is a solution and $\alpha$ is a real constant (in space, time and $\Omega$), then $\xi + \alpha$ is also a solution (this follows from $K^\ast \alpha = 0$). Hence the result can be generalized as follows: for every $\xi_0$ in $\mathcal{M}_{x,M} \cap H^{-1}_x$ with negative part bounded by a constant $\alpha$, then there exists a weak distributional $\mathcal{M}_{x,M}$-valued solution $\xi$ to (1.1), in $C_t(\mathcal{M}_{x,M,+}, w^*) \cap L^2_{t,\omega}(H^{-1}_x)$, with negative part bounded by $\alpha$ for all $t$, $P$-a.s.

This is relevant in particular because, if we start from a velocity $u$, then $\text{curl}[u]$ cannot be non-negative (unless $\text{curl}[u] = 0$), but it can be bounded from below.

The strategy of the proof goes as follows:

**Compactness argument:** We take $\xi^\varepsilon$ solutions with regular bounded initial datum $\xi_0^\varepsilon$ and we show that $\xi^\varepsilon$ are tight via suitable a priori bounds. The steps are:

1. Prove the non-negativity and a uniform $L^\infty_{t,\omega}(\mathcal{M}_{x,+})$ bound on $\xi^\varepsilon$: This follows from the conservation of non-negativity and from the conservation of mass for the vorticity equation (1.1).

2. Prove a uniform $L^2_{t,\omega}(H^{-1}_x)$ bound on $\xi^\varepsilon$: Since the $L^2_{t,\omega}(H^{-1}_x)$ norm of $\xi$ is equivalent to the $L^2_{t,\omega}(L^2_x)$ norm of the velocity $u^\varepsilon := u^\xi$, we write the equation (4.2) for $u^\xi$ and prove a uniform energy bound on $u^\xi$. Note that, oppositely to the deterministic case, the energy (the $L^2$ norm of $u^\xi$) is not preserved, due to the additional term $(\nabla \sigma_k) \cdot u \circ W^k$ in the equation for the velocity $u^\xi$. To prove the energy bound, the assumptions on $\sigma_k$ play a crucial role.

3. Prove a uniform $L^2_{\omega}(C^\alpha_t(H^{-4}_x))$ bound on $\xi^\varepsilon$, for $\alpha < 1/2$: This follows from the Lipschitz bounds in time on the deterministic integrals in the
vorticity equation (1.1) and the Hölder bound in time on the stochastic integral as $H^{-4}$-valued object.

4. Show tightness in $C_t(M_{x,M}, w^*) \cap (L^2_t(H^{-1}_x), w)$ (where $w^*$ refers to the weak-* topology on $M_{x,M}$ and $w$ refers to the weak topology on $L^2_t(H^{-1}_x)$): This follows from the previous uniform bounds. Actually, one could have tightness simply in $C_t(M_{x,M}, w^*)$, without using the $L^2_t(H^{-1}_x)$ bound, but this will be useful in the convergence part.

**Convergence argument:** We show that any limit point $\xi$ of $\xi^\epsilon$ solves the vorticity equation (1.1). The steps are:

1. Pass to an a.s. convergence: By Skorokhod-Jacubowski theorem we have, up to subsequences, an a.s. convergence on a larger probability space of $\xi^\epsilon$ to some $\xi$ in $C_t(M_{x,M}, w^*) \cap (L^2_t(H^{-1}_x), w)$.

2. Show that the a.s. limit $\xi$ of any subsequence satisfies (2.3): For the nonlinear term, we use the Schochet approach and in particular: the continuity of the nonlinear term among non-negative non-atomic measures as in Lemma 4.16 and the fact that $H^{-1}_x$ measures are non-atomic. For the stochastic term, we use the approximation of the stochastic integral via Riemann sums as in [BGJ13].

The main result will follow from Lemma 4.14 which will show that the limit $\xi$ of any subsequence satisfies (2.3).

## 4 Proof of the main result

### 4.1 A priori bounds

We fix $M > 0$ and the initial condition $\xi_0$ in $M_{x,M,+} \cap H^{-1}_x$. We take $\xi^\epsilon$ to be the $L^\infty_{t,x,\omega}$ solution to the stochastic vorticity equation (2.3), with initial condition $\xi_0^\epsilon = \xi_0 * \rho_\epsilon$, where $\rho_\epsilon$ are standard mollifiers on $\mathbb{T}^2$ (precisely, we take a $C^\infty_c$ non-negative even function on $\mathbb{R}^2$, we define $\rho_\epsilon = \epsilon^{-2} \rho(\epsilon^{-1} \cdot)$ and make it periodic). The existence (and the uniqueness) of such $L^\infty_{t,x,\omega}$ solution is proved in [BFM16]. Precisely, for any $\epsilon > 0$, [BFM16, Theorem 2.14] implies the existence and the strong uniqueness of the stochastic flow $\Phi^\epsilon = \Phi^\epsilon(t,x,\omega)$, Lebesgue-measure-preserving and continuous with
respect to \((t, x)\), solution to the SDE

\[
d\Phi^\varepsilon(t, x) = \int_{T^2} K(\Phi^\varepsilon(t, x) - \Phi^\varepsilon(t, y))\xi^\varepsilon_0(y)dydt + \sum_k \sigma_k(\Phi^\varepsilon(t, x))dW^k_t.
\]

As a consequence of \cite[Proposition 5.1]{BFM16}, if we define, for every \(t\),

\[
\xi^\varepsilon_t(\omega) = (\Phi(t, \cdot, \omega))_\# \xi^\varepsilon_0
\]

(4.1)

(the image measure of \(\xi^\varepsilon_0\) under \(\Phi(t, \cdot, \omega)\)), then, for a.e. \(\omega\), for every \(t\) \(\xi^\varepsilon_t(\omega)\) admits a density with respect to the Lebesgue measure, this density is in \(L^\infty_{t,\omega,x}\) and, for every \(\varphi\) in \(C^\infty_x\), \((\xi_t, \varphi)\) is progressively measurable and satisfies (2.5). It follows, see Remark \ref{r3.3}, that, up to taking an indistinguishable version, such \(\xi^\varepsilon\) is a weak distributional \(M_{x,M^\varepsilon}\)-valued solution in the sense of Definition \ref{d2.4}, for some \(M_{x^\varepsilon} \geq \|\xi^\varepsilon_0\|_{M_x}\).

### 4.1.1 Bound in \(L^\infty_{t,\omega}(M_x)\)

We start with a uniform \(L^\infty_{t,\omega}(M_x)\) bound:

**Lemma 4.1.** For every \(\varepsilon > 0\) fixed, for a.e. \(\omega\), we have

\[
\sup_{t \in [0,T]} \|\xi^\varepsilon_t\|_{M_x} \leq \|\xi^\varepsilon_0\|_{M_x} \leq M.
\]

In particular, up to taking an indistinguishable version, \(\xi^\varepsilon\) is a \(M_{x,M^\varepsilon}\)-valued solution (in the sense of Definition \ref{d2.4}). Moreover, for a.e. \(\omega\), we have: for every \(t\), \(\xi^\varepsilon_t \geq 0\) (that is, it is a non-negative measure).

**Proof.** The non-negativity follows directly from the fact that \(\xi^\varepsilon_0\) and so \(\xi^\varepsilon\) are non-negative and from the representation formula (4.1). Concerning the bound, this also follows from the representation formula (4.1), but we prefer giving a PDE (short) proof. Using \(\varphi \equiv 1\) as test function in (2.5), we get, for a.e. \(\omega\): for every \(t\),

\[
\int_{T^2} \xi^\varepsilon_t dx = \int_{T^2} \xi^\varepsilon_0 dx.
\]

Since \(\xi^\varepsilon_t \geq 0\), we get that \(\|\xi^\varepsilon_t\|_{L^\infty_{t,\omega}(M_x)} \leq \|\xi^\varepsilon_0\|_{M_x}\) for every \(t\), for a.e. \(\omega\). Defining \(\xi^\varepsilon = 0\) outside the exceptional set where the bound is not satisfied, we get that \(\xi^\varepsilon\) is in \(C^t(M_{x,M})\) and so is a \(M_{x,M}\)-valued solution. \(\square\)
Remark 4.2. In the proof of the previous Lemma, we used the fact that $\xi^\varepsilon$ is positive (and so $\int_T^2 \xi^\varepsilon$ is the $M_x$ norm of $\xi^\varepsilon$), but this is not essential. Indeed, the vorticity equation is a transport equation with divergence-free velocity field, therefore the mass is conserved at least under suitable regularity assumption on the velocity, which are satisfied here.

4.1.2 Equation for the velocity and for its energy

In view of a uniform $H^{-1}_x$ bound on $\xi^\varepsilon$, we will: 1) get an equation for the velocity $u^\varepsilon = u^\varepsilon = K \ast \xi^\varepsilon$, then 2) get an equation for the energy of $u^\varepsilon$, that is the $L^2_x$ norm of $u^\varepsilon$, then 3) conclude a uniform $L^2_p$ bound on $u^\varepsilon$; by Lemma C.1 this bound is equivalent to a uniform $H^{-1}_x$ bound on $\xi^\varepsilon$, up to the space average to $\xi^\varepsilon$.

Here we consider a solution $\xi$ to the vorticity equation (2.3), with sufficiently integrability to include the (bounded) approximants $\xi^\varepsilon$. As we have seen in the introduction (formula (1.3))

$$\partial_t u + (u \cdot \nabla)u + \sum_k (\sigma_k \cdot \nabla + (D\sigma_k)^T)u \circ \dot{W}^k = -\nabla p - \gamma,$$

$$\text{div} u = 0,$$

where $p : [0, T] \times T^2 \times \Omega \to \mathbb{R}$ and $\gamma : [0, T] \times \Omega \to \mathbb{R}^2$ are unknown (and random). The rigorous result is as follows.

Lemma 4.3. Assume Condition 2.1 on $\sigma_k$. Let $\xi$ be a $M_{x,M}$-valued distributional solution to the stochastic vorticity equation, assume that $\xi$ is also in $L^p_{t,\omega}(L^p_x)$ for some $2 < p < \infty$ and define $u = K \ast \xi$. Then $u$ is in $L^p_{t,\omega}(W^{1,p}_x)$ and is a distributional solution to the stochastic Euler equation, that is it holds

$$u_t = u_0 - \int_0^t \Pi[(u_r \cdot \nabla)u_r]dr - \sum_k \int_0^t \Pi[\sigma_k \cdot \nabla u_r + (D\sigma_k)^T u_r]dW^k + \frac{1}{2} \int_0^t c\Delta u_r dr, \quad \text{for every } t, \quad P - \text{a.s.},$$

as equality among $H^{-1}_x$-valued processes, where $\Pi$ is the Leray projector on the divergence-free zero-mean $H^{-1}_x$ distributions.
The proof is essentially based on the following formal equality, for $v : \mathbb{T}^2 \to \mathbb{R}^2$ divergence-free and $w : \mathbb{T}^2 \to \mathbb{R}^2$:

$$\text{curl}[v \cdot \nabla w + (Dv)^T w] = v \cdot \nabla \text{curl}[w].$$

Using this equality, one can formally pass from the velocity equation (4.2) to the vorticity equation (1.1). The rigorous proof is given in the Appendix.

From the equation of the velocity we get the equation for the expected valued of the energy:

**Lemma 4.4.** Under the assumptions of Lemma 4.3, we have

$$E\|u_t\|_{L_x^2}^2 = E\|u_0\|_{L_x^2}^2 - 2E \int_0^t \int_{\mathbb{T}^2} u \cdot \Pi[(u \cdot \nabla)u] dx dr$$

$$- E \int_0^t \int_{\mathbb{T}^2} c|\nabla u|^2 dx dr$$

$$+ E \int_0^t \sum_k \int_{\mathbb{T}^2} |\Pi[(\sigma_k \cdot \nabla + (D\sigma_k)^T)u]|^2 dx dr,$$

for every $t$. (4.4)

One can get this equation formally from (4.2) by applying the Itô formula to $\|u\|_{L_x^2}^2$. However this is not possible rigorously, because the rigorous equation (4.3) holds in $H_x^{-1}$ and the square of the $L_x^2$ norm is not continuous on $H_x^{-1}$. The rigorous proof of (4.4) is based on a regularization argument and is postponed to the appendix.

4.1.3 Bound in $L_{t, \omega}^2(H_x^{-1})$

Now we give a uniform $L_{t, \omega}^2(H_x^{-1})$ bound on the approximants $\xi^\varepsilon$. This bound is not essential for the compactness argument, but it is essential for the convergence argument, as we will see.

**Lemma 4.5.** It holds

$$E\|\xi_t^\varepsilon\|_{H_x^{-1}}^2 \leq C(\|\xi_0\|_{H_x^{-1}}^2 + \|\xi_0\|_{M_x}^2),$$

for every $t$. In the proof of this lemma, we use crucially the assumptions 2.1 on $\sigma_k$. We recall that $u^\varepsilon = u^\varepsilon = K * \xi^\varepsilon$. 18
Proof. By Remark \[D.6\] the $H^{-1}_x$ norm is a Borel function on $(\mathcal{M}, w_*)$, therefore $\|\xi^\varepsilon\|_{H^{-1}_x}$ and $\|u^\varepsilon\|_{L^2_x}$ are progressively measurable and the expectations of their moments make sense. By Lemma \[C.1\] applied to $\xi^\varepsilon_t - \int \xi^\varepsilon(y) dy$, we have, for every $t$,

$$
\|\xi^\varepsilon_t\|_{H^{-1}_x} \leq C \|u^\varepsilon_t\|_{L^2} + \left| \int_{T^2} \xi^\varepsilon_t dx \right|
$$

$$
\|u^\varepsilon_t\|_{L^2} \leq C \|\xi^\varepsilon_t\|_{H^{-1}_x}.
$$

By Lemma \[4.1\] the $L^1$ norm of $\xi^\varepsilon$ is uniformly bounded by $\|\xi_0\|_{\mathcal{M}_x}$. Hence it is enough to show, for every $t$,

$$
\mathbb{E}\|u^\varepsilon_t\|_{L^2}^2 \leq C \|u_0\|_{L^2}^2.
$$

We will show the above bound using the velocity equation (4.3). We start with (4.4) applied to $u^\varepsilon$. Since $u^\varepsilon$ is divergence-free, the nonlinear term in (4.4) vanishes: indeed

$$
\int_{T^2} u^\varepsilon \cdot \Pi[(u^\varepsilon \cdot \nabla)u^\varepsilon] dx = \int_{T^2} \Pi[u^\varepsilon] \cdot (u^\varepsilon \cdot \nabla)u^\varepsilon dx
$$

$$
= \int_{T^2} u^\varepsilon \cdot (u^\varepsilon \cdot \nabla)u^\varepsilon dx = 0.
$$

For the term with $\sigma_k$, since $\Pi$ is a projector in $L^2_x$, we have

$$
\sum_k \int_{T^2} \|\Pi[(\sigma_k \cdot \nabla + (D\sigma_k)^T)u^\varepsilon]\|^2 dx \leq \sum_k \int_{T^2} |(\sigma_k \cdot \nabla + (D\sigma_k)^T)u^\varepsilon|^2 dx
$$

$$
= \int_{T^2} \left[ \sum_k |\sigma_k \cdot \nabla u^\varepsilon|^2 + \sum_k |(D\sigma_k)^T u^\varepsilon|^2 + 2 \sum_{i,j,h} \sigma^i_k \partial_x \sigma^j_k (u^\varepsilon)^h \partial_x (u^\varepsilon)^j \right] dx.
$$

Now we use the assumptions \[2.1\] precisely that $\sum_i \sigma^i_k \sigma^j_k = c \delta_{ij}$ and that $\sum_k \sigma^i_k \partial_x \sigma^j_k = 0$ for all $i, j, h$, with uniform (with respect to $x$) convergence in the series over $k$: we get

$$
\sum_k \int_{T^2} \|\Pi[(\sigma_k \cdot \nabla + (D\sigma_k)^T)u^\varepsilon]\|^2 dx \leq \int_{T^2} c |\nabla u^\varepsilon|^2 dx + \sum_k \|\sigma_k\|_{C^2} \int_{T^2} |u^\varepsilon|^2 dx.
$$

Putting all together, we obtain for every $t$

$$
\mathbb{E}\|u^\varepsilon_t\|_{L^2}^2 \leq \mathbb{E}\|u_0\|_{L^2}^2 + \sum_k \|\sigma_k\|_{C^2} \int_0^t \mathbb{E}\|u^\varepsilon_r\|_{L^2}^2 dr.
$$
We apply Gronwall lemma to $E\|u^\varepsilon_t\|_{L^2_x}^2$: we conclude, for every $t$,

$$E\|u^\varepsilon_t\|_{L^2_x}^2 \leq E\|u^\varepsilon_0\|_{L^2_x}^2 \exp\left[t \sum_k \|\sigma_k\|_{C^1}\right],$$

which implies the desired bound since $u^\varepsilon_0$ is deterministic and $\|u^\varepsilon_0\|_{L^2_x} \leq C\|u_0\|_{L^2_x}$. The proof is complete. $\square$

### 4.1.4 Bound in $L^m_{\omega}(C_t^\alpha(H_x^{-4}))$

Now we prove a uniform $L^m_{\omega}(C_t^\alpha(H_x^{-4}))$ bound, for $m \geq 2$:

**Lemma 4.6.** Fix $2 \leq m < \infty$. For every $0 < \alpha < 1/2$, we have

$$E\|\xi^\varepsilon\|_{C_t^\alpha(H_x^{-4})}^m \leq C(\|\xi_0\|_{M_x}^{2m} + \|\xi_0\|_{M_x}^m), \quad \text{for every } t.$$  

**Proof.** Note that, by Remark [D.6], $\omega \mapsto \|\xi\|_{C_t^\alpha(H_x^{-4})}$ is measurable. Using the equation (2.3), we get, for every $t$,

$$E\|\xi^\varepsilon_t - \xi^\varepsilon_s\|_{H_x^{-4}}^m \leq CE\left\|\int_s^t N(\xi^\varepsilon_r)\,dr\right\|_{H_x^{-4}}^m + C\epsilon^mE\left\|\sum_k \int_s^t \sigma_k \cdot \nabla \xi^\varepsilon_r W^k\right\|_{H_x^{-4}}^m + C\epsilon^mE\left\|\int_s^t \Delta \xi^\varepsilon_r\,dr\right\|_{H_x^{-4}}^m.$$  

By Lemma [2.3] we have for the nonlinear term

$$E\left\|\int_s^t N(\xi^\varepsilon_r)\,dr\right\|_{H_x^{-4}}^m \leq C(t - s)^m\|\xi^\varepsilon\|_{L^\infty_{t,\omega}(M_x)}^{2m}.$$  

For the stochastic integral, the Burkholder-Davis-Gundy inequality and Lemma
B.1 gives

\[
\mathbb{E} \left\| \sum_k \int_s^t \sigma_k \cdot \nabla \xi^\varepsilon \circ dW^k \right\|_{H_x^{-4}}^m \\
\leq C \mathbb{E} \left( \sum_k \int_s^t \|\sigma_k \cdot \nabla \xi^\varepsilon\|_{H_x^{-4}}^m \right)^{m/2} \\
\leq C(t-s)^{m/2} \left( \sum_k \|\sigma_k\|_{C_x}^2 \right)^{m/2} \|\xi^\varepsilon\|_{L^\infty_t(L^m_x)}^m.
\]

Finally for the second order term, again Lemma B.1 gives

\[
\mathbb{E} \left\| \int_s^t \Delta \xi^\varepsilon \, dr \right\|_{H_x^{-4}}^m \leq C \|\xi^\varepsilon\|_{L^\infty_t(L^m_x)}^m.
\]

We put all together and we recall the a priori bound on \( \|\xi^\varepsilon\|_{L^\infty_t(L^m_x)} \) in Lemma 4.1, we obtain

\[
\mathbb{E} \|\xi^\varepsilon_t - \xi^\varepsilon_s\|_{H_x^{-4}}^m \leq C(t-s)^{m/2}(\|\xi_0\|_{L^m_x}^m + \|\xi_0\|_{L^m_x}^{2m}),
\]

where the constant \( C \) depends on \( \sum_k \|\sigma_k\|_{B_x}^2 \) and on \( c \). By the Kolmogorov criterion (or the Sobolev embedding in \( t \)), recalling that \( \xi^\varepsilon \) is already continuous as \( H_x^{-4} \)-valued process, we get, for every \( 0 < \alpha < 1/2 \),

\[
\mathbb{E} \|\xi^\varepsilon\|_{C^\alpha_t(L^m_x)}^m \leq C(\|\xi_0\|_{L^m_x}^m + \|\xi_0\|_{L^m_x}^{2m}).
\]

The proof is complete. \( \Box \)

4.2 Tightness

In this section we prove the tightness of \( \xi^\varepsilon \) on \( C_t(M_{x,M}, w^*) \cap (L^2_t(H_x^{-1}), w) \) (recall that \( M > 0 \) is fixed such that \( \|\xi_0\|_{M_x} \leq M \)).

We recall that \( (M_{x,M}, w^*) \) is metrizable with the distance \( d_{M_{x,M}}(\mu, \nu) = \sum_j 2^{-j} |\langle \mu - \nu, \varphi_j \rangle| \), therefore \( C_t(M_{x,M}, w^*) \) is metrizable as well: see Remark 4.9 with \( X = C_x \). Here the space \( C_t(M_x, w^*) \cap (L^2_t(H_x^{-1}), w) \) is defined as the subspace of \( C_t(M_x, w^*) \) whose paths have finite \( L^2_t(H_x^{-1}) \) norm. On this subspace, the topology is induced by the \( C_t(M_x, w^*) \) topology and the \( (L^2_t(H_x^{-1}), w) \) norm, that is, any open set in \( C_t(M_x, w^*) \cap (L^2_t(H_x^{-1}), w) \) is
the union of a $C_t(\mathcal{M}_x, w^*)$-open set and a $(L_t^2(H_x^{-1}), w)$-open set. The Borel
σ-algebra on $C_t(\mathcal{M}_x, w^*) \cap (L_t^2(H_x^{-1}), w)$ is then generated by the Borel σ
algebras related to the $C_t(\mathcal{M}_x, w^*)$ topology and the $(L_t^2(H_x^{-1}), w)$ topology.
Actually, the Borel σ-algebra generated by $(L_t^2(H_x^{-1}), w)$ coincides with the
Borel σ-algebra generated by the strong topology on $L_t^2(H_x^{-1})$, by Lemma D.1.

Note that, by Lemmas 4.1 and 4.5, for any $\epsilon > 0$, $\xi^\epsilon_t$ takes values in $\mathcal{M}_{x,M} \cap H_x^{-1}$ for every $t$, $P$-a.s. and so for every $\omega$ up to taking an indistinguishable version. Hence, by Lemma B.4, $\xi$ is measurable as $C_t(\mathcal{M}_{x,M}, w^*)$-map and as $(L_t^2(H_x^{-1}), w)$-valued map (both spaces being endowed with their Borel
σ-algebras), that is $\xi^\epsilon_t$ is a random variable with values in $C_t(\mathcal{M}_{x,M}, w^*) \cap (L_t^2(H_x^{-1}), w)$.

**Lemma 4.7.** Fix $M > 0$ such that $\|\xi_0\|_{\mathcal{M}_x} \leq M$. The family $(\xi^\epsilon_t)_{\epsilon}$ is tight
on $C_t(\mathcal{M}_{x,M}, w^*) \cap (L_t^2(H_x^{-1}), w)$.

We start with a generalization of [BM13, Lemma 3.1]. The latter is a refined
version of the compactness argument in [FG95], which can be seen as a stochastic version of the Aubin-Lions lemma. Given a Banach space $X$, we call $B_M^X$ the closed ball in $X$ of radius $M$.

**Lemma 4.8.** Let $X, Y$ be separable Banach spaces with $Y$ densely embedded in $X$. Then, for every $M \geq 0$, $\alpha > 0$, $a \geq 0$, the set

$$A_a = \{z \in C_t(B_M^{X^*}, w^*) \mid \|z\|_{C_t^2(Y^*)} \leq a\}$$

is compact in $C_t(B_M^{X^*}, w^*)$.

**Remark 4.9.** For the proof, we recall the following facts. First, the ball $B_M^{X^*}$
derived with the weak-* topology is metrizable with the distance $d_{B_M^{X^*}}(w, w') = \sum_j 2^{-j}|\langle w - w', \varphi_j \rangle|$, where $(\varphi_j)_j$ is a dense sequence in $B_1^X$, see [Bre11, Theorem 3.28]. Hence the set $C_t(B_M^{X^*}, w^*)$ is metrizable with the distance

$$d(z, z') = \sup_{t \in [0,T]} \sum_j 2^{-j}|\langle z - z', \varphi_j \rangle|, \quad z, z' \in C_t(B_M^{X^*}, w^*). \quad (4.5)$$

Moreover, given a sequence $(z^n)_n$ and $z$ in $C_t(B_M^{X^*}, w^*)$ and a set $D$, dense
in $X$, the following three conditions are equivalent:

- $(z^n)_n$ converges to $z$ in $C_t(B_M^{X^*}, w^*)$;
• for every $\varphi$ in $X$, $\langle z^n, \varphi \rangle$ converges uniformly (that is, in $C_t$) to $\langle z, \varphi \rangle$;

• for every $\varphi$ in $D$, $\langle z^n, \varphi \rangle$ converges uniformly (that is, in $C_t$) to $\langle z, \varphi \rangle$.

The equivalence between the first two conditions can be seen using the distance (4.5). The equivalence between the last two points can be seen approximating a generic $\varphi$ in $X$ with elements in $D$ and using the uniform bound $\|z_n\|_{X^*} \leq M$.

Proof. Since $C_t(B_{M}^{X^*}, w^*)$ is metrizable, compactness is equivalent to sequential compactness. Let $(z^n)_n$ be a sequence in $A_n$, we have to find a subsequence $(z^{n_k})_k$ which converges in $C_t(B_{M}^{X^*}, w^*)$ to an element of $A_n$.

For fixed $t$ in $Q \cap [0, T]$, $(z^n_t)_n$ is a sequence in $B_{M}^{X^*}$, hence, by Banach-Alaoglu theorem, there exists a subsequence $(z^{n_k}_t)_k$ converging weakly-* to an element $\tilde{z}_t$ in $B_{M}^{X^*}$. By a diagonal procedure, we can make the sequence $(n_k)_k$ independent of $t$ in $Q \cap [0, T]$.

On the other side, let $D$ be a countable dense set in $Y$, and so in $X$. The fact that $z^n$ are equicontinuous and equibounded in $Y^*$ implies that, for every $\varphi$ in $D$, the functions $t \mapsto \langle z^{nk}_t, \varphi \rangle$ are equicontinuous and equibounded, their $C^\alpha$ norm being bounded by $a\|\varphi\|_Y$. Hence, by Ascoli-Arzelà theorem, there exists a subsequence converging in $C_t$ to some element $t \mapsto f^\varphi_t$, which also satisfies $\|f^\varphi_t\|_{C_t^\alpha} \leq a\|\varphi\|_Y$. By a diagonal procedure, we can choose the subsequence independent of $\varphi$ in $D$. With a small abuse of notation, we continue using $n_k$ for this subsequence. Then, for all $t$ in $Q \cap [0, T]$, for all $\varphi$ in $D$, $\langle \tilde{z}_t, \varphi \rangle = f^\varphi_t$.

Fix $t$ in $[0, T]$ and let $(t_j)_j$ be a sequence in $Q \cap [0, T]$ converging to $t$. The sequence $(\tilde{z}_{t_j})_j$ is in $B_{M}^{X^*}$, so, up to subsequences, it converges weakly-* to an element $z_t$ in $B_{M}^{X^*}$. On the other hand, for all $\varphi$ in $D$, by continuity of $t \mapsto f^\varphi_t$, we must have $\langle z_t, \varphi \rangle = f^\varphi_t$.

The map $t \mapsto \langle z_t, \varphi \rangle = f^\varphi_t$ is continuous for every $\varphi$ in $D$, and actually for every $\varphi$ in $X$, by an approximation argument. Hence $z$ is in $C_t(B_{M}^{X^*}, w^*)$.

Moreover

$$\|z_t - z_s\| = \sup_{\varphi \in D, \|\varphi\|_Y \leq 1} |\langle z_t - z_s, \varphi \rangle|$$

$$= \sup_{\varphi \in D, \|\varphi\|_Y \leq 1} |f^\varphi_t - f^\varphi_s| \leq a|t - s|^\alpha,$$

and similarly for $\|\dot{z}_t\|_Y$ alone. Hence $\|z\|_{C_t^\alpha(Y^*)} \leq a$ and so $z$ is in $A_n$.

Finally, for every $\varphi$ in $D$, $(z^{nk}, \varphi)$ converges uniformly to $f^\varphi = \langle z, \varphi \rangle$, therefore, by Remark 4.3, $z^{nk}$ converges to $z$ in $C_t((B_{M}^{X^*}, w^*))$. The proof is complete.
As a consequence of the previous Lemma and the Banach-Alaoglu theorem, we get the following:

**Lemma 4.10.** For every $M \geq 0$, $\alpha > 0$, for every $a, b \geq 0$, the set

\[ A_{a,b} = \{ \mu \in C_t(\mathcal{M}_{x,M}, w^*) \cap L_t^m(H_x^{-1}) \mid \|\mu\|_{C_t^p(H_x^{-1})} \leq a, \|\mu\|_{L_t^m(H_x^{-1})} \leq b \} \]

is metrizable and compact in $C_t(\mathcal{M}_{x,M}, w^*) \cap (L_t^m(H_x^{-1}), w)$.

**Proof.** Since the topologies on $C_t(\mathcal{M}_{x,M}, w^*)$ and on the closed ball of radius $b$ in $(L_t^m(H_x^{-1}), w^*)$ are metrizable, $A_{a,b}$ is metrizable as well and the compactness is equivalent to the sequential compactness.

Let $(\mu^n)_n$ be a sequence in $A_{a,b}$. By the previous Lemma, applied to $X = C_x$ and $Y = H_x^1$, there exists a sub-subsequence $(\mu^{n_k})_k$ converging to some $\mu$ in $C_t(\mathcal{M}_{x,M}, w^*)$ with $\|\mu\|_{C_t^p(H_x^{-1})} \leq a$. On the other hand, by the Banach-Alaoglu theorem, there exists a subsequence, which we can assume $(\mu^{n_k})_k$ up to relabelling, converging to some $\nu$ in $(L_t^m(H_x^{-1}), w)$ with $\|\nu\|_{L_t^m(H_x^{-1})} \leq b$.

Using these two limits, for every $g$ in $C_t$ and every $\varphi$ in $C_x^1$, we have

\[ \int_0^T g(t)\langle \mu_t, \varphi \rangle dt = \int_0^T g(t)\langle \nu_t, \varphi \rangle dt. \]

Hence $\mu = \nu$ and so $\mu$ is the limit in $A_{a,b}$ of the subsequence $(\mu^{n_k})_k$. The proof is complete. \( \square \)

We are ready to prove tightness of $\xi^\varepsilon$.

**Proof of Lemma 4.7.** As we have seen at the beginning of this section, by Lemmas 4.1 and 4.5, for any $\varepsilon > 0$, $\xi^\varepsilon$ is, up to an indistinguishable version, a $C_t(\mathcal{M}_x, w^*) \cap (L_t^2(H_x^{-1}), w)$-valued random variable. Lemma 4.10 ensures that the set $A_{a,b}$ defined in that Lemma is metrizable and compact in $C_t(\mathcal{M}_x, w^*) \cap (L_t^2(H_x^{-1}), w)$. The Markov inequality gives

\[
P\{\xi^\varepsilon \notin A_{a,b}\} \leq P\{\|\xi^\varepsilon\|_{C_t^p(H_x^{-1})} > a\} + P\{\|\xi^\varepsilon\|_{L_t^m(H_x^{-1})} > b\} \leq a^{-m}E\|\xi^\varepsilon\|_{C_t^p(H_x^{-1})}^m + b^{-m}E\|\xi^\varepsilon\|_{L_t^m(H_x^{-1})}^m.
\]

By Lemmas 4.5 and 4.6, the right-hand side above can be made arbitrarily small, uniformly in $\varepsilon$, taking $a$ and $b$ large enough. The tightness is proved. \( \square \)

As a consequence, we have actually:
Corollary 4.11. The family \( (\xi^\varepsilon, W)_\varepsilon \) (where \( W = (W^k)_k \)) is tight on the space \( \chi := [C_t(\mathcal{M}_{x,M}, w*) \cap (L_t^m(\mathbb{H}^{-1})_j), \mathbb{C}_t^2] \times \mathbb{C}_t^2 \).

Proof. The tightness of \((\xi^\varepsilon, W)\) follows easily from the tightness of the marginals. \( \square \)

4.3 Convergence

We can apply the Skorohod-Jakubowski representation theorem, see \cite{Jak97}, to the family \((\xi^\varepsilon, W)_\varepsilon\) and the space \( \chi = [C_t(\mathcal{M}_{x,M}, w*) \cap (L_t^m(\mathbb{H}^{-1})_j), \mathbb{C}_t^2] \times \mathbb{C}_t^2 \). Indeed the family \((\xi^\varepsilon, W)_\varepsilon\) is tight by Corollary 4.11 and the space \( \chi \) satisfies the assumption (10) in \cite{Jak97}: for given sequences \((t_i)_i\) dense in \([0, T]\), \((\varphi_j)_j\) dense in \( C_\varepsilon \), the maps \( f_{i,j} \) and \( g_{i,k} \), defined on \( \chi \) by \( f_{i,j}(\mu, \gamma) = \langle \mu_t, \varphi_j \rangle \) and \( g_{i,k}(\mu, \gamma) = \arctan(\gamma^k_t) \), form a sequence of continuous, uniformly bounded maps separating points in \( \chi \).

Hence, by the Skorohod-Jakubowski representation theorem, there exist an infinitesimal sequence \((\varepsilon_j)_j\), a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, P)\), a \( \chi \)-valued sequence \((\tilde{\xi}^j, \tilde{W}^j)_j\) and a \( \chi \)-valued random variable \((\tilde{\xi}, \tilde{W})\) such that \((\tilde{\xi}^j, \tilde{W}^j)\) has the same law of \((\xi^\varepsilon, W)\) and \((\tilde{\xi}, \tilde{W})\) converges to \((\xi, W)\) a.s. in \( C_t(\mathcal{M}_{x,M}, w*) \cap (L_t^m(\mathbb{H}^{-1})_j), w) \). For notation, we use \( \tilde{W}^j\) and \( \tilde{W}^k \) for the \( k \)-th component of the \( \mathbb{C}_t^2\)-valued random variables \( \tilde{W}^j \) and \( \tilde{W} \).

Call \( \tilde{\mathcal{F}}^0 \) the filtration generated by \( \tilde{\xi}, \tilde{W} \) and the \( P \)-null sets on \( (\tilde{\Omega}, \tilde{\mathcal{F}}, P) \) and call \( \tilde{F}_t = \cap_{s>t} \tilde{\mathcal{F}}^0_s \). Similarly, call \( \tilde{\mathcal{F}}^0_j \) the filtration generated by \( \tilde{\xi}^j, \tilde{W}^j \) and the \( P \)-null sets on \( (\tilde{\Omega}, \tilde{\mathcal{F}}, P) \) and call \( \tilde{\mathcal{F}}^j_t = \cap_{s>t} \tilde{\mathcal{F}}^0_{s,j} \).

Lemma 4.12. The filtration \((\tilde{\mathcal{F}}_t)_t\) is complete and right-continuous and \( \tilde{W} \) is a cylindrical Brownian motion with respect to it. Moreover \( \tilde{\xi} \) is an \((\mathcal{M}_{x,M}, w*)\)-valued \((\tilde{\mathcal{F}}_t)_t\)-progressively measurable process. Similarly for \((\tilde{\mathcal{F}}^j_t)_t\) and \( \tilde{\xi}^j \), for each \( j \).

The proof of this Lemma is simple but technical and postponed to the appendix.

Each copy \( \tilde{\xi}^j \) of the approximant \( \xi^\varepsilon \) is a solution to the stochastic vorticity equation:

Lemma 4.13. For each fixed \( j \), the object \((\tilde{\Omega}, \tilde{\mathcal{F}}^j_t, \tilde{P}, \tilde{W}^j, \tilde{\xi}^j)\) is a \( \mathcal{M}_{x,M} \)-valued solution to the vorticity equation \((\ref{eq:vorticity})\) with the initial condition \( \tilde{\xi}^j_0 = \xi^\varepsilon_0 \) \( P \)-a.s. Moreover Lemmas 4.3 and 4.6 hold for \( \tilde{\xi}^j \) in place of \( \xi^\varepsilon \) and, \( P \)-a.s., \( \tilde{\xi}^j \) is non-negative for every \( t \).

Also the proof of this Lemma is technical and postponed to the appendix.
4.4 Limiting equation

Now we show that \( \tilde{\xi} \) satisfies the vorticity equation with \( \tilde{W} \) as Brownian motion. With this Lemma, Theorem 3.1 is proved.

**Lemma 4.14.** The object \((\tilde{\Omega}, \tilde{A}, (\tilde{F}_t)_t, \tilde{P}, \tilde{W}, \tilde{\xi})\) is a \( \mathcal{M}_{x,M} \)-valued solution to the vorticity equation (1.1), which is also in \( C_t(\mathcal{M}_{x,M}, w^*) \cap (L^2_t(H^{-1}_x), w) \).

To prove Lemma 4.14 we will show (2.5) for \((\tilde{\xi}, \tilde{W})\) for every test function \( \varphi \) in \( C^\infty_x \). By Lemma 4.13, for each \( j \), \((\tilde{\xi}^j, \tilde{W}^j)\) satisfies (2.5) for every \( \varphi \) in \( C^\infty_x \). Hence it is enough to pass to the \( \tilde{P} \)-a.s. limit, as \( j \to \infty \), in each term of (2.5) for \((\tilde{\xi}^j, \tilde{W}^j)\), possibly choosing a subsequence, for every \( t \) and every \( \varphi \) in \( C^\infty_x \). We fix \( t \) in \([0, T]\) and \( \varphi \) in \( C^\infty_x \).

We start with the deterministic linear terms:

\[
\langle \tilde{\xi}^j_t, \varphi \rangle, \langle \tilde{\xi}^j_0, \varphi \rangle \text{ and } \int_0^t \langle \tilde{\xi}^j_r, c \Delta \varphi \rangle dr
\]

converge \( P \)-a.s. to the corresponding terms without the superscript \( j \), thanks to the convergence of \( \tilde{\xi}^j \) to \( \tilde{\xi} \) in \( C_t(\mathcal{M}_{x,M}, w^*) \).

4.4.1 The nonlinear term

Concerning the nonlinear term, we recall Lemma 2.3 and we follow the Schochet argument, see Schochet [Sch95] and Poupaud [Pou02, Section 2]. The first main ingredient for the convergence is the following:

**Lemma 4.15.** Fix \( M > 0 \). For every \( \varphi \) in \( C^2_x \), the map \( \mu \mapsto \langle N(\mu), \varphi \rangle \) is continuous on the subset \( \mathcal{M}_{x,M,+} \text{-no-atom} \) of \( \mathcal{M}_{x,M} \) of non-negative non-atomic measures with total mass bounded by \( M \), endowed with the weak-* topology.

We use the following result, a version of the classical Portmanteau theorem (which deals with probability measures rather than non-negative measures):

**Lemma 4.16.** Let \( X \) be a compact metric space. Assume that \((\nu_k)_k\) is a sequence of non-negative bounded measures and converges to \( \nu \) in \((\mathcal{M}(X), w^*)\). Let \( F \) be a closed set in \( X \) with \( \nu(F) = 0 \) and let \( \psi : X \to \mathbb{R} \) be a bounded Borel function, continuous on \( X \setminus F \). Then the sequence \((\langle \nu_k, \psi \rangle)_k\) converges to \( \langle \nu, \psi \rangle \).
Proof. Let $\varepsilon > 0$, we have to prove that $|\langle \nu^k - \nu, \psi \rangle| < C\varepsilon$ for $k$ large enough, for some constant $C$. The fact that $\nu(F) = 0$ implies the existence of $\delta > 0$ such that $\nu(\bar{B}(F, \delta)) < \varepsilon$, where $\bar{B}(F, \delta) := \{x \in X \mid d(x, F) \leq \delta \}$. As the function $1_{\bar{B}(F, \delta)}$ is upper semi-continuous and $(\nu^k)_k$ converges weakly-* to $\nu$, there exists $k$ such that $\nu^k(\bar{B}(F, \delta)) < \varepsilon$ for all $k \geq k$. By Urysohn lemma, there exists a continuous function $\rho$ with $0 \leq \rho \leq 1$, $\rho = 1$ on $F$ and $\rho = 0$ on $\bar{B}(F, \delta)^c$; it is easy to see that $\psi(1 - \rho)$ is then continuous on all $X$. Now we split

$$|\langle \nu^k - \nu, \psi \rangle| \leq |\langle \nu^k, \psi \rho \rangle| + |\langle \nu^k - \nu, \psi(1 - \rho) \rangle| + |\langle \nu, \psi \rho \rangle|.$$  

(4.6)

For the first term in the right-hand side, we have

$$|\langle \nu^k, \psi \rho \rangle| \leq \sup_k |\nu^k(\bar{B}(F, \delta))| \sup_X |\psi| = \sup_k \nu^k(\bar{B}(F, \delta)) \sup_X |\psi| \leq \varepsilon \sup_X |\psi|. \hspace{1cm} (4.7)$$

The same inequality holds for the third term in the right-hand side of (4.6). Finally, the second term in (4.6) is bounded by $\varepsilon$ provided $k$ is large enough, by weak-* convergence of $(\nu^k)_k$. The proof is complete. $\square$

**Remark 4.17.** It is only in (4.7) in the proof of Lemma 4.16 that we need to use that the process $\xi$ takes values in non-negative measures.

**Proof of Lemma 4.15.** We have to show that, for every sequence $(\mu^n)_n$ converging to $\xi$ in $\mathcal{M}_{x,M,+}$, $(N(\mu^n), \varphi)$ converges to $(N(\mu), \varphi)$. By Lemma D.5 in the Appendix, $(\mu^n \otimes \mu^n)_n$ converges weakly-* to $\xi \otimes \xi$. Moreover, since $\mu$ has no atoms, then $\mu \otimes \mu$ gives no mass to the diagonal $D = \{(x, y) \mid x = y\}$: indeed, by the Fubini theorem,

$$(\mu \otimes \mu)(D) = \int_{\mathbb{T}^2} \mu(dx) \int_{\{x\}} \mu(dy) = 0.$$  

We are now in a position to apply Lemma 4.16 to the sequence $(\mu^n \otimes \mu^n)_n$, with the state space $X = (\mathbb{T}^2)^2$, with $F = D$ and with $\psi = F_\varphi$, which is continuous outside the diagonal $D$: we get that

$$\langle \mu^n \otimes \mu^n, F_\varphi \rangle \to \langle \mu \otimes \mu, F_\varphi \rangle,$$

which is exactly the desired convergence. The proof is complete. $\square$
The second ingredient for the convergence of the nonlinear term is the following:

**Lemma 4.18.** Let \( \mu \) be in \( \mathcal{M}_x \cap H^{-1}_x \). Then \( \mu \) has no atoms.

**Proof.** Fix \( x_0 \) in \( \mathbb{T}^2 \), we have to prove that \( \mu(\{x_0\}) = 0 \). Let \( \rho : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function with \( 0 \leq \rho \leq 1 \), supported on \( B_1(0) \) (the ball centered at 0 with radius 1) and with \( \rho(x) = 1 \) if and only if \( x = 0 \). For \( n \) positive integer, call \( \rho_n(x) = \rho(n(x - x_0)) \) and take its periodic version on \( \mathbb{T}^2 \), which, with small abuse of notation, we continue calling \( \rho_n \). Now \( (\rho_n)_n \) is a nonincreasing sequence which converges pointwise to 1\( \{x_0\} \), so \( (\langle \mu, \rho_n \rangle)_n \) converges to \( \mu(\{x_0\}) \).

On the other hand \( |\langle \mu, \rho_n \rangle| \leq \|\mu\|_{H^{-1}_x} \|\rho_n\|_{H^1_x} \). For the \( H^1_x \) norm of \( \rho_n \), we have

\[
\|\nabla \rho_n\|_{L^2_x}^2 = \int_{\mathbb{R}^2} n^2 |\nabla \rho(nx)|^2 \, dx = \int_{B_{1/n}(0)} |\nabla \rho(y)|^2 \, dy
\]

and so \( \|\nabla \rho_n\|_{L^2} \) converges to 0 as \( n \to \infty \). In a similar and easier way one sees that \( \|\rho_n\|_{L^2} \) converges to 0. So \( \|\rho_n\|_{H^1} \) tends to 0. Hence \( (\langle \mu, \rho_n \rangle)_n \) tends also to 0 and therefore \( \mu(\{x_0\}) = 0 \). \( \square \)

**Remark 4.19.** It is only for the previous Lemma that we need to use that the process \( \xi \) is \( H^{-1}_x \)-valued.

We are now able to conclude the convergence of the nonlinear term in (2.3). Fix \( \omega \) in a full measure set such that \( (\hat{\xi}^j)_j \) converges in \( C_t(\mathcal{M}_x, M, w^*) \). Since \( \hat{\xi} \) belongs to \( L^2_t(H^{-1}_x) \), \( \xi_r \) belongs to \( H^{-1}_x \) for all \( r \) in a full measure set \( S \) of \( [0,T] \). In particular, by Lemma 4.18, \( \xi_r \) has no atoms. Hence, for all \( r \) in \( S \), Lemma 4.18 implies the convergence of

\[
\langle N(\xi^j), \varphi \rangle = \langle \xi^j \otimes \xi^j, F_{\varphi} \rangle
\]
towards the same term without \( j \). By the dominated convergence theorem (in \( r \)), its time integral converges as well. This proves convergence for the nonlinear term.
4.4.2 The stochastic integral

It remains to prove convergence of the stochastic term. We follow the strategy in Brzezniak-Goldys-Jegaraj. We use the notation \( t_i = 2^{-l_i} \),

\[
Y_{j,k}(t) = \langle \bar{\xi}_t^j, \sigma_k \cdot \nabla \varphi_t \rangle,
\]

\[
Y_{j,k}^{K,l}(t) = 1_{k \leq K} \sum_i Y_{j,k}(t_i^l) 1_{[t_i^l, t_i^{l+1}])(t)
\]

and similarly without \( j \). Finally we call \( \rho_{j,k} \) the modulus of continuity of \( Y_{j,k} \), namely

\[
\rho_{j,k}(a) = \sup_{|t-s| \leq a} |Y_{j,k}(t) - Y_{j,k}(s)|
\]

and similarly without \( j \). Note that \( \rho_{j,k} \) and \( \rho_k \) are \( \tilde{\mathcal{F}}_T \)-measurable on \( \tilde{\Omega} \), since the above supremum can be restricted to rational times \( t, s \). We split

\[
\left| \sum_k \int_0^t (\bar{\xi}_t^j, \sigma_k \cdot \nabla \varphi)d\tilde{W}^{(j),k} - \int_0^t (\bar{\xi}_t, \sigma_k \cdot \nabla \varphi)d\tilde{W}^k \right| \\
\leq \left| \sum_k \int_0^t (Y_{j,k} - Y_{j,k}^{K,l})d\tilde{W}^{(j),k} \right| \\
+ \sum_k \int_0^t Y_{j,k}^{K,l}d\tilde{W}^{(j),k} - \int_0^t Y_{K,l}^{K,l}d\tilde{W}^k \\
+ \sum_k \int_0^t (Y_k - Y_{K,l}^{K,l})d\tilde{W}^k \\
=: T1 + T2 + T3.
\]

Concerning the first addend \( T1 \), we have

\[
ET1^2 = \sum_k E \int_0^t |Y_{j,k} - Y_{j,k}^{K,l}|^2 dr.
\]
In order to have uniform estimates with respect to \( j \), we want to use the convergence in \( C_t \) of \( Y^{j,k} \) to \( Y^k \). For this, we split again the right-hand side:

\[
\sum_k E \int_0^t |Y_{j,k} - Y_{j,k}^{K,l}|^2 dr 
\leq C \sum_k E \int_0^t |Y_{j,k}^{K,l} - Y_k^{K,l}|^2 dr + C \sum_k E \int_0^t |Y_k - Y_k^{K,l}|^2 dr + C \sum_k E \int_0^t |Y_{j,k} - Y_k|^2 dr 
=: T_{11} + T_{12} + T_{13}.
\]

For \( T_{11} \), we have

\[
T_{11} = C \sum_{k \leq K} E \int_0^t |Y_{j,k}^{K,l} - Y_k^{K,l}|^2 dr 
\leq C \sum_{k \leq K} E \sup_r |Y_{j,k}(r) - Y_k(r)|^2
\]

For \( T_{13} \), we have similarly

\[
T_{13} = C \sum_{k \leq K} E \int_0^t |Y_{j,k} - Y_k|^2 dr + C \sum_{k > K} E \int_0^t |Y_{j,k} - Y_k|^2 dr 
\leq C \sum_{k \leq K} E \sup_r |Y_{j,k}(r) - Y_k(r)|^2 + C \sum_{k > K} \|\sigma_k\|_{C_x}^2
\]

where we have used that \( \sup_r |Y_{j,k}(r)| \leq C \|\sigma_k\|_{C_x}^2 \) (the constant \( C \) here being dependent on \( M \), the upper bound of \( \|\xi_j\|_{A_x} \) and \( \varphi \)). For \( T_{12} \), we have

\[
T_{12} = C \sum_{k \leq K} E \int_0^t |Y_k - Y_k^{K,l}|^2 dr + C \sum_{k > K} E \int_0^t |Y_k|^2 dr 
\leq C \sum_{k \leq K} E \rho_k (2^{-l})^2 + C \sum_{k > K} \|\sigma_k\|_{C_x}^2.
\]

This complete the bound for \( T_1 \). Concerning the term \( T_3 \), we have

\[
ET_3^2 = \sum_k E \int_0^t |Y_k - Y_k^{K,l}|^2 dr,
\]

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which is the term \( T_{12} \) up to a multiplicative constant and can therefore bounded as \( T_{12} \). Finally, we note that the term \( T_2 \) can be written as

\[
T_2 = \left| \sum_{k \leq K} \sum_i [Y_{j,k}(t_i^l)(W_{t_{i+1}^l}^{(j),k} - W_{t_i^l}^{(j),k}) - Y_k(t_i^l)(W_{t_{i+1}^l}^k - W_{t_i^l}^k)] \right|
\]

Putting all together, we find

\[
E \left| \sum_k \int_0^t \langle \tilde{\xi}^j, \sigma_k \cdot \nabla \varphi \rangle d \tilde{W}^{(j),k} - \int_0^t \langle \tilde{\xi}, \sigma_k \cdot \nabla \varphi \rangle d \tilde{W}^k \right|^2 \leq C \left( \sum_{k \leq K} E \rho_k(2^{-l})^2 + \sum_{k > K} \|\sigma_k\|_{C_\infty}^2 \right)
+ C \sum_{k \leq K} E \sup_r |Y_{j,k}(r) - Y_k(r)|^2
+ CE \left( \sum_{k \leq K} \sum_i [Y_{j,k}(t_i^l)(W_{t_{i+1}^l}^{(j),k} - W_{t_i^l}^{(j),k}) - Y_k(t_i^l)(W_{t_{i+1}^l}^k - W_{t_i^l}^k)] \right)^2.
\]

We first choose \( K \) such that \( \sum_{k > K} \|\sigma_k\|_{C_\infty}^2 < \varepsilon \). For each \( k \), since \( Y_k \) is a continuous function, \( \rho_k(2^{-l}) \) converges to 0 as \( l \to \infty \) \( \tilde{P} \)-a.s.. Moreover \( Y_k \) is also essentially bounded, therefore, by dominated convergence theorem, \( E \rho_k(2^{-l})^2 \) also converges to 0. Hence, for \( K \) fixed as before, we can choose \( l \) such that

\[
\sum_{k \leq K} E \rho_k(2^{-l})^2 < \varepsilon.
\]

Again for each \( k \), due to the convergence of \( \tilde{\xi}^j \) in \( C_k(M_{x,M},w^*) \), \( \sup_r |Y_{j,k}(r) - Y_k(r)|^2 \) converges to 0 as \( j \to \infty \) \( \tilde{P} \)-a.s.. Moreover \( Y_{j,k} \) are bounded uniformly in \( j \), therefore, by dominated convergence theorem, \( E \sup_r |Y_{j,k}(r) - Y_k(r)|^2 \) also converges to 0. Hence, for \( K \) fixed as before, we can choose \( \bar{j} \) such that, for every \( j \geq \bar{j} \),

\[
\sum_{k \leq K} E \sup_r |Y_{j,k}(r) - Y_k(r)|^2 < \varepsilon.
\]

Finally, for \( K \), \( l \) fixed as before, the term

\[
\sum_{k \leq K} \sum_i [Y_{j,k}(t_i^l)(W_{t_{i+1}^l}^{(j),k} - W_{t_i^l}^{(j),k}) - Y_k(t_i^l)(W_{t_{i+1}^l}^k - W_{t_i^l}^k)]
\]
converges to 0 as \( j \to \infty \) \( \tilde{P} \)-a.s.; therefore, by dominated convergence theorem, also its second moment converges to 0. Hence, for \( K, l \) fixed as before, we can choose a new \( \tilde{j} \) such that, for every \( j \geq \tilde{j} \),

\[
E \left| \sum_{k \leq K} \sum_{i} |Y_{j,k}(t_i^j)(W_{t_{i+1}^j}^{(j),k} - W_{t_i^j}^{(j),k}) - Y_{k}(t_i^j)(W_{t_{i+1}^k}^{k} - W_{t_i^k}^{k})| \right|^2 < \varepsilon.
\]

This proves that the stochastic term in (2.3) converges in \( L^2 \) norm, and so \( \tilde{P} \)-a.s. up to subsequences.

We have proved that all the terms in (2.5) passes to the \( \tilde{P} \)-a.s. limit, up to subsequences, and therefore \( \xi \) is a solution to (2.5), so to (1.1) with \( \tilde{W} \) as Brownian motion. The proof of Lemma 4.14 is complete.

Appendices

A On the nonlinear term in Euler equations

Proof of Lemma 2.3. This Lemma is essentially due to Schochet [Sch95, Lemma 3.2 and discussion thereafter], we use here the interpretation of Poupaud [Pou02, Section 2].

- Since \( K \) is smooth outside the diagonal \( \{x = y\} \), \( F_\varphi \) is smooth outside the diagonal. Recall by Lemma C.1 we have \( |K(x - y)| \leq C|x - y|^{-1} \). Therefore, using that \( \nabla \varphi \) is Lipschitz, we get

\[
|F_\varphi(x, y)| \leq \frac{1}{2}|K(x - y)||D^2 \varphi|_c |x - y| \leq C\|\varphi\|_{C^2_x},
\]

which gives the bound on \( F_\varphi \).

- For every \( \xi \) in \( \mathcal{M}_x \), for every \( \varphi \) in \( C^2_x \), \( F_\varphi \) is bounded and so \( \langle N(\xi), \varphi \rangle \) is well-defined and

\[
|\langle N(\xi), \varphi \rangle| = \left| \int \int F_\varphi(x, y)\xi(dx)\xi(dy) \right| \leq C\|\varphi\|_{C^2_x}\|\xi\|_{\mathcal{M}_x} \leq C\|\varphi\|_{H^4} \|\xi\|_{\mathcal{M}_x}^2,
\]

where we used the Sobolev embedding in the last inequality. In particular \( N(\xi) \) is a well-defined linear bounded functional on \( H^4 \).
The map $\mathcal{M}_x \ni \xi \mapsto \xi \otimes \xi \in \mathcal{M}_{x,y}$ is Borel (with respect to the Borel $\sigma$-algebras generated by the weak-* topologies on $\mathcal{M}_x$ and $\mathcal{M}_{x,y}$), by Lemma D.5. Also, for every $\varphi$ in $C^2_x$, the map

$$\mathcal{M}_{x,y} \ni \mu \mapsto \langle \mu, F_\varphi \rangle \in \mathbb{R}$$

is Borel by Lemma D.4. Therefore, for every $\varphi$ in $C^2_x$, $\mathcal{M}_x \ni \xi \mapsto \langle N(\xi), \varphi \rangle \in \mathbb{R}$ is Borel, that is $N$ is weakly-* Borel as an $H^{-4}_x$-valued map. Since $H^{-4}_x$ is a separable reflexive space, $N$ is Borel by Lemma D.1.

Recall that $K$ is odd by Lemma C.1, therefore

$$\int \xi(x)u(x) \cdot \nabla \varphi(x)dx = \int \int \xi(x)\xi(y)K(x-y) \cdot \nabla \varphi(x)dxdy$$

$$= -\int \int \xi(x)\xi(y)K(y-x) \cdot \nabla \varphi(x)dxdy$$

$$= -\int \int \xi(x)\xi(y)K(x-y) \cdot \nabla \varphi(y)dxdy,$$

where in the last equality we swapped $x$ and $y$. Hence

$$\int \xi(x)u(x) \cdot \nabla \varphi(x)dx$$

$$= \frac{1}{2} \int \int \xi(x)\xi(y)K(x-y) \cdot (\nabla \varphi(x) - \nabla \varphi(y))dxdy.$$

Continuity of $N$ on $\mathcal{M}_{x,M,+,-atom}$ follows from Lemma 4.16.

### B Technical lemmas

**Lemma B.1.** Assume Condition 2.1 on $\sigma_k$. Then the maps

\begin{align*}
\mathcal{M}_x \ni \mu &\mapsto \mu \in H^{-4}_x, \\
\mathcal{M}_x \ni \mu &\mapsto \sigma_k \cdot \nabla \mu \in H^{-4}_x, \quad k \in \mathbb{N}, \\
\mathcal{M}_x \ni \mu &\mapsto \Delta \mu \in H^{-4}_x
\end{align*}

(B.1)
are linear norm-to-norm and weak-* to weak continuous, in particular Borel (where $M_x$ is endowed with the weak-* topology). Moreover we have the bounds

$$\|\mu\|_{H^{-4}_x} + \|\Delta \mu\|_{H^{-4}_x} \leq C\|\mu\|_{M_x},$$

$$\|\sigma_k \cdot \nabla \mu\|_{H^{-4}_x} \leq C\|\sigma_k\|_{C_x}\|\mu\|_{M_x}.$$ 

**Proof.** The maps in (B.1), tested against a test function $\varphi$, read formally

$$\mu \mapsto \langle \mu, \varphi \rangle,$$

$$\mu \mapsto -\langle \mu, \sigma_k \cdot \nabla \varphi \rangle,$$

$$\mu \mapsto \langle \mu, \Delta \varphi \rangle.$$ 

Now, by Sobolev embedding, for any $\varphi$ in $H^{4}_x$, the functions $\varphi, \sigma_k \cdot \nabla \varphi, \Delta \varphi$ are continuous with

$$\|\varphi\|_{C_x} \leq C\|\varphi\|_{H^4_x},$$

$$\|\sigma_k \cdot \nabla \varphi\|_{C_x} \leq \|\sigma_k\|_{C_x}\|\nabla \varphi\|_{C_x} \leq C\|\sigma_k\|_{C_x}\|\nabla \varphi\|_{H^4_x},$$

$$\|\Delta \varphi\|_{C_x} \leq C\|\varphi\|_{H^4_x}.$$ 

Hence the maps in (B.1) are weak-* to weak continuous. Taking the supremum over $\varphi$ in the unit ball of $H^{4}_x$, we get also the norm-to-norm continuity and the desired bounds. 

**Lemma B.2.** Assume Condition 2.1 on $\sigma_k$, fix $M > 0$. For any object $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, P, (W^k)_k, \xi)$, where $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, P, (W^k)_k)$ is a cylindrical Brownian motion with the usual assumptions and $\xi : [0, T] \times \Omega \to M_{x,M}$ is $P$ Borel measurable, then (2.3) holds if and only if (2.5) holds for every $\varphi$ in $C^{\infty}_x$.

**Proof.** Assume that (2.3) holds, fix $\varphi$ in $C^{\infty}_x$. Then (2.5) follows by applying the linear continuous functional $\langle \cdot, \varphi \rangle$ to (2.3) and exchanging the functional with the integrals.

Conversely, assume that (2.5) holds for every $\varphi$ in $C^{\infty}_x$. Lemma B.1 implies that $\xi$ and all the integrands in (2.3) are progressively measurable as $H^{-4}_x$ processes and that the deterministic and stochastic integrals are well-defined (see (2.4)). Now we have for every test function $\varphi$ in $C^{\infty}_x$, by (2.5) exchanging
\[ \langle \xi_t, \varphi \rangle = \langle \xi_0, \varphi \rangle + \langle \int_0^t N(\xi_r)dr, \varphi \rangle \]
\[ \quad - \langle \sum_k \int_0^t \sigma_k \cdot \nabla \xi_r \rangle dW^k_r, \varphi \rangle \]
\[ \quad + \frac{1}{2} \langle \int_0^t c \Delta \xi_dr, \varphi \rangle, \quad \text{for every } t, \ P - \text{a.s.,} \]

that is (2.3) tested against \( \varphi \), where the \( P \)-exceptional set can depend on \( \varphi \).

Taking \( \varphi \) in a countable set of \( C^\infty \times \), dense in \( H^4_x \), we deduce (2.3) in \( H^{-4}_x \). The proof is complete.

Remark B.3. Let \( \xi^\varepsilon(\omega) = (\Phi(t, \cdot, \omega))_\# \xi^\varepsilon_0 \) be defined as at the beginning of Section 4.1. For a.e. \( \omega \), for every \( t \), \( \xi^\varepsilon \) lies in \( M_{x,M^\varepsilon} \), where \( M^\varepsilon \geq \| \xi^\varepsilon_0 \|_M \).

Moreover, \( \langle \xi_t, \varphi \rangle \) is progressively measurable and continuous (because it satisfies (2.5)) for every \( \varphi \) in \( C^\infty_x \), hence, by density of \( C^\infty_x \) in \( C_x \), for every \( \varphi \) in \( C_x \). Hence, by Lemma D.3 and Remark D.3, up to redefining \( \xi^\varepsilon = 0 \) on the \( P \)-exceptional set where \( \xi^\varepsilon \) is not in \( M_{x,M^\varepsilon} \) for some \( t \), \( \xi^\varepsilon \) is \( P \)-Borel as \( M_x \)-valued map and satisfies (2.5) for every \( \varphi \) in \( C^\infty_x \). By Lemma B.2, \( \xi^\varepsilon \) is a \( M_{x,M^\varepsilon} \)-valued solution in the sense of Definition 2.4.

Proof of Lemma 4.3. First part: identities with curl. We start proving that, for \( v : \mathbb{T}^2 \to \mathbb{R}^2 \) regular and divergence-free and \( w : \mathbb{T}^2 \to \mathbb{R}^2 \) regular,

\[ \text{curl} [v \cdot \nabla w + (Dv)^T w] = v \cdot \nabla \text{curl} [w]. \quad (B.2) \]

Indeed, we have (in the following, we omit the sum symbols over \( i \) and \( j \))

\[ \text{curl} [v \cdot \nabla w + (\nabla v)^T \cdot w] - v \cdot \nabla \text{curl} [w]. \]
\[ = \partial_x v^i \partial_x w^2 - \partial_x v^i \partial_x w^1 + \partial_x v^j \partial x^1 w^j - \partial_x v^j \partial x^1 w^j \]
\[ = \partial_x v^1 \partial_x w^2 - \partial_x v^2 \partial x^1 w^1 + \partial_x v^2 \partial x^1 w^2 - \partial_x v^1 \partial x^2 w^1 \]
\[ = \text{div} [v] \text{curl} [w] = 0. \]

By Lemma C.1, (B.2) implies that

\[ \Pi [v \cdot \nabla w + (Dv)^T w] = K * [v \cdot \nabla \text{curl} [w]], \]

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where Π is the Leray projector in $L^2$ on the divergence-free zero-mean functions. For $v : \mathbb{T}^2 \to \mathbb{R}^2$ regular and divergence-free and $\xi : \mathbb{T}^2 \to \mathbb{R}$ regular, we can apply the above formula to $w = K*\xi$: we get

$$\Pi[v \cdot \nabla K*\xi + (Dv)^T K*\xi] = K*[v \cdot \nabla \xi];$$

we used that, by Lemma C.1, $\text{curl}[w] = \xi - \gamma$, where $\gamma$ is a real number (precisely, the space average of $\xi$), and the contribution of $\gamma$ in the right-hand side is zero. The equality (B.3), intended as equality in $H^{-1}_x$, holds also for general $\xi$ in $L^p_x$ with $2 < p < \infty$ and $v$ divergence-free and in $W^{1,p}_x$ (in particular continuous by the Sobolev embedding): indeed, we can approximate $\xi$ and $v$ in $L^p_x$ and $W^{1,p}_x$ resp. with regular $\xi^n$ and regular divergence-free $v^n$ and we use that, by Lemma C.1, $K*\xi^n$ converge to $K*\xi$ in $W^{1,p}_x$ and so in $L^\infty_x$ by Sobolev embedding, so we can get (B.3) tested against any test function in $C^\infty_x$.

**Second part**: conclusion. Since $\xi$ is in $L^p_{t,\omega}(L^p_x)$, $u$ is in $L^p_{t,\omega}(W^{1,p}_x)$ by Lemma C.1. We note also that, for $\xi$ in $L^p_{t,\omega}(L^p_x)$, the nonlinear term can be written as $u \cdot \nabla \xi$, by Lemma 2.3, and the equation (2.3) holds actually in $H^{-2}_x$: indeed, as one can prove testing against $H^3_x$ functions and using the density of $H^3$ in $H^2$, $\xi$ and all the integrands of (2.3) takes values in $H^{-2}_x$ and are progressively measurable as $H^{-2}_x$-valued processes (and their deterministic and stochastic $H^{-2}_x$-valued integrals coincide with the $H^{-3}_x$-valued integrals). Now we apply to (2.3) the operator

$$K* \cdot : H^{-2}_x \to H^{-1}_x,$$

which is linear and bounded by Lemma C.1. By the first part of this proof, we get, for a.e. $\omega$, as equality in $H^{-1}_x$: for every $t$,

$$u_t = u_0 - \int_0^t \Pi[u_r \cdot \nabla u_r + (Du)^T u_r]dr$$

$$- \sum_k \int_0^t \Pi[\sigma_k \cdot \nabla u_r + (D\sigma)^T u_r]dW^k_r$$

$$+ \frac{1}{2} \int_0^t c\Delta u_r dr$$

Now we note that $(Du)^T u_r = \nabla[|u_r|^2]/2$ and so its Leray projection is zero. Hence we get (4.3).
Proof of 4.4. For any \( \delta > 0 \), we call 
\[ R^\delta = \rho_\delta \ast \cdot : H_x^{-1} \to L_x^2 \]
the linear bounded operator given by the convolution with \( \rho_\delta \), where \( (\rho_\delta)_\delta \) is a standard family of mollifiers on \( \mathbb{T}^2 \) (precisely, \( \rho_1 \) is a non-negative, even \( C^\infty_c \) function on \( \mathbb{R}^2 \), \( \rho_\delta = \delta^{-2} \rho_1(\delta^{-1} \cdot) \) and is defined on the torus by periodicity); \( R^\delta \) is extended componentwise to vector fields. Note that \( R^\delta f \to f \) in \( L_x^2 \), resp. \( H_x^{-1} \) for every \( f \) in \( L_x^2 \), resp. \( H_x^{-1} \). We apply \( R^\delta \) to (4.3): we get
\[
R^\delta u_t = R^\delta u_0 - \int_0^t R^\delta \Pi [u_r \cdot \nabla u_r] dr \\
- \int_0^t R^\delta \Pi [\sigma_k \cdot \nabla u_r + (D\sigma_k)^T u_r] dW_r^k \\
+ \frac{1}{2} \int_0^t c R^\delta \Delta u_r dr,
\]
as equality for every \( t \) among \( L_x^2 \)-valued processes. Now we can apply the Itô formula (from [DPZ14, Theorem 4.32]) to the square of the \( L_x^2 \) norm, which is \( C^2 \) on \( L_x^2 \) with uniformly continuous derivatives on bounded subsets of \( L_x^2 \). We get, for every \( t \),
\[
\| R^\delta u_t \|_{L_x^2}^2 = \| R^\delta u_0 \|_{L_x^2}^2 - 2 \int_0^t \langle R^\delta u_r, R^\delta \Pi [u_r \cdot \nabla u_r] \rangle dr \\
- 2 \int_0^t \langle R^\delta u_r, R^\delta \Pi [\sigma_k \cdot \nabla u_r + (D\sigma_k)^T u_r] \rangle dW_r^k \\
+ \int_0^t \langle R^\delta, c R^\delta \Delta u_r \rangle dr + \sum_k \int_0^t \| R^\delta \Pi [\sigma_k \cdot \nabla u_r + (D\sigma_k)^T u_r] \|_{L_x^2}^2 dr.
\]
Since \( R^\delta \) is linear bounded also on \( L_x^p \) and \( W_x^{1,p} \), \( R^\delta u \) is in \( W_x^{1,p} \) and so it has finite \( L_{t,\omega}^p(L_x^\infty) \) norm, therefore
\[
\sum_k \mathbb{E} \int_0^T | \langle R^\delta u_r, R^\delta \Pi [\sigma_k \cdot \nabla u_r + (D\sigma_k)^T u_r] \rangle |^2 dr \\
\leq C \| u \|_{L_x^2(L_x^\infty)}^2 \left( \sum_k \| \sigma_k \|_{C^1_x} \right) \| u \|_{L_{t,\omega}^2(W_x^{1,2})}^2.
\]
Hence the stochastic integral is a martingale with zero mean. Similarly the integrands in the deterministic integrals have finite $L^1_{t,\omega}$ norm and we can take expectation: we get

$$E\|R\delta u_t\|_{L^2_x}^2 = E\|R\delta u_0\|_{L^2_x}^2 - 2E \int_0^t \langle R\delta u_r, R\delta \Pi[u_r \cdot \nabla u_r] \rangle dr - E \int_0^t cE\|R\delta \nabla u_r\|_{L^2_x}^2 dr + E \int_0^t \sum_k \|R\delta \Pi[\sigma_k \cdot \nabla u_r + (D\sigma_k)^T u_r]\|_{L^2_x}^2 dr,$$

where we have used integration by parts and that $R\delta$ commutes with $\Delta$ and $\nabla$. Finally, we note that $R\delta f \to f$ in $L^2_x$ for every $f \in L^2_x$. We exploit this fact for $f = u_r$, $f = \Pi[u_r \cdot \nabla u_r]$, $f = \sigma_k \cdot \nabla u_r + (D\sigma_k)^T u_r$, and $f = \nabla u_r$, and use the dominated convergence theorem in $r$ and $\omega$ and $k$, to pass $\delta \to 0$ and obtain (4.4). The proof is complete. \hfill $\square$

**Lemma B.4.** Let $X$ be a closed convex subset of a topological vector space, endowed with its Borel $\sigma$-algebra, assume that $X$ is also a Polish space. Let $\zeta : [0, T] \times \Omega \to X$ be a $\mathcal{B}([0, T]) \times \mathcal{A}$ Borel measurable map.

- The set $C_t(X)$ is a Polish space and, if, for every $\omega$, $t \mapsto \zeta_t$ is in $C_t(X)$, then $\omega \mapsto \zeta(\cdot, \omega)$ is $\mathcal{A}$ Borel measurable as $C_t(X)$-valued map.

- If $X$ is a separable reflexive Banach space and, for every $\omega$, $t \mapsto \zeta_t$ is in $L^2_t(X)$ (more precisely, has finite $L^2_t(X)$ norm), then $\omega \mapsto \zeta(\cdot, \omega)$ (more precisely, its equivalence class) is $\mathcal{A}$ Borel measurable as $L^2_t(X)$-valued map.

**Proof.** For the first point, the fact that $C_t(X)$ is a Polish space is well known. Moreover the Borel $\sigma$-algebra $\mathcal{B}(C_t(X))$ on $C_t(X)$ is generated by the evaluation maps $\pi_t(\gamma) = \gamma_t$: indeed, $\mathcal{B}(C_t(X))$ is generated by the maps

$$\gamma \mapsto d(\gamma(t), g(t)), \quad t \in [0, T] \cap \mathbb{Q}, \quad g \in C_t(X),$$

with $d$ distance on $X$, and these maps are measurable in the $\sigma$-algebra generated by the evaluation maps (because they are composition of the evaluation maps and a Borel function on $X$). Now, for every $t$, the map $\pi_t(\zeta) = \zeta_t$ is $\mathcal{A}$ Borel, by the Fubini theorem, hence, if $\zeta$ is $C_t(X)$-valued, then it is $\mathcal{A}$ Borel measurable as $C_t(X)$-valued map.

For the second point, we note that, by Lemma [D.1], it is enough to show that $\zeta$ is weakly progressively measurable. Since the dual of $L^2_t(X)$ is $L^2_t(X^*)$ (see
\[ \omega \mapsto \int_0^T \langle \zeta(t, \omega), \varphi(t) \rangle_{X^*, X} \, dt \]

is measurable. But this follows from Fubini theorem. The proof is complete.

\[ \square \]

**Proof of Lemma 4.12** Call \( \tilde{F}_t^0 = \sigma\{\tilde{\xi}_s, \tilde{W}_s \mid 0 \leq s \leq t\} \) the filtration generated by \( \tilde{\xi} \) and \( \tilde{W} \). Clearly \( \tilde{W} \) and \( \tilde{\xi} \) are adapted to \( \tilde{F}_t^0 \). We claim that \( \tilde{W} \) is a cylindrical Brownian motion with respect to \( \tilde{F}_t^0 \). Indeed, \( \tilde{W} \) is a cylindrical Brownian motion with respect to its natural filtration, as a.s. limit of cylindrical Brownian motions. Moreover, for every \( 0 \leq s_1 \leq \ldots \leq s_h \leq s < t \), \( \tilde{W}_t^{(j)} - \tilde{W}_s^{(j)} \) is independent of \( (\tilde{\xi}_{s_1}^{(j)}, \tilde{W}_{s_1}^{(j)}, \ldots, \tilde{\xi}_{s_h}^{(j)}, \tilde{W}_{s_h}^{(j)}) \), therefore \( \tilde{W}_t - \tilde{W}_s \) is independent of \( (\tilde{\xi}_{s_1}, \tilde{W}_{s_1}, \ldots, \tilde{\xi}_{s_h}, \tilde{W}_{s_h}) \). This proves our claim.

Recall that \( \tilde{F}_t \) is the filtration generated by \( \tilde{F}_t^0 \) and the \( \tilde{P} \)-null sets on \((\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})\) and that \( \tilde{F}_t = \cap_{s > t} \tilde{F}_s^0 \). We argue as in the proof of [Bas11, Proposition 2.5, Point 1] (note that the proof is valid for any filtration making \( \tilde{W} \) Brownian motions) and we get that the filtration \( (\tilde{F}_t)_t \) is complete and right-continuous and \( \tilde{W} \) is still a cylindrical Brownian motion with respect to it. Finally \( \tilde{\xi} \) is an \( (\mathcal{M}_{x,M}, w_*) \)-valued \( (\tilde{F}_t)_t \)-adapted and continuous process, hence also progressively measurable.

In a similar (and easier) way, one gets the result for \( (\tilde{F}_t^j)_t \), \( \tilde{W}^{(j)} \) and \( \tilde{\xi}^j \), for each \( j \).

\[ \square \]

**Proof of Lemma 4.13** Let us fix \( j \). We have to verify equation (2.3) for \( (\xi^j, \tilde{W}^{(j)}) \) for every \( \varphi \in C_\infty \). The idea is taken by [BGJ13, Section 5]: it is enough to verify that, for every \( \varphi \) in \( C_\infty \), for every \( t \), the random variables

\[ Z_t := \langle \xi^j, \varphi_t \rangle - \langle \xi^j, \varphi \rangle - \int_0^t \langle N(\xi_r^j), \varphi \rangle \, dr \]

\[ - \sum_k \int_0^t \langle \xi_r^j, \sigma_k \cdot \nabla \varphi \rangle \, dW_r^k - \frac{1}{2} \int_0^t \langle \xi_r^j, c \Delta \varphi \rangle \, dr \]  

and \( \tilde{Z}_t \), obtained as in (B.4) replacing \( (\xi^j, \tilde{W}) \) with \( (\tilde{\xi}^j, \tilde{W}^{(j)}) \), have the same law. We fix \( t \) and \( \varphi \) in \( C_\infty \). By Lemma 2.3 and Lemma B.1 all the terms in (B.4) but the nonlinear term and the stochastic integral are Borel functions of \( \xi^j \) with respect to the \( C_t(\mathcal{M}_{x,M}, w_*) \) topology. Concerning the stochastic
integral we use an approximation argument. For every positive integers $K$ and $N$, calling $t^N_i = 2^{-Ni}$ for $i$ integer, the map

$$C_t(\mathcal{M}_{x,M}, w^*) \times C^N_t \ni (\xi, W) \mapsto \sum_{k=1}^K \sum_{i, t^N_{i+1} \leq t} \langle \xi^N_{t^N_i}, \sigma_k \cdot \nabla \varphi t^N_i \rangle (W_{t^N_{i+1}} - W_{t^N_i})$$

is a continuous, in particular Borel function. By the continuity of $t \mapsto \langle \xi_t, \sigma_k \cdot \nabla \varphi_t \rangle$ for every $k$, for a.e. $\omega$, and by the square-summability of $\|\sigma_k\|_{C_2}$, we get via the dominated convergence theorem that, as $(N, K) \to \infty$,

$$\sum_k E \int_0^T |\langle \xi_t, \sigma_k \cdot \nabla \varphi_t \rangle - 1_{k \leq K} \sum_i \langle \xi^N_{t^N_i}, \sigma_k \cdot \nabla \varphi t^N_i \rangle 1_{[t^N_{i+1}, t^N_i)}(t)|^2 \, dt \to 0,$$

so by the Itô isometry we obtain that, as $(N, K) \to \infty$,

$$\sum_{k=1}^K \sum_{i, t^N_{i+1} \leq t} \langle \xi^N_{t^N_i}, \sigma_k \cdot \nabla \varphi t^N_i \rangle (W_{t^N_{i+1}} - W_{t^N_i}) \to \sum_k \int_0^t \langle \xi^N_r, \sigma_k \cdot \nabla \varphi_r \rangle dW^k_r \text{ in } L^2_\omega.$$

Similarly for $(\tilde{\xi}^j, \tilde{W}^{(j)})$ (with convergence in $L^2_\omega$). We conclude that

$$Z_t = F_t(\xi^j) + L^2 - \lim_{N,K} G_{N,K,t}(\xi^j, W),$$

$$Z_t = F_t(\tilde{\xi}^j) + L^2 - \lim_{N,K} G_{N,K,t}(\tilde{\xi}^j, \tilde{W}^{(j)})$$

for some Borel maps $F_t$ and $G_{N,K,t}$. Since $(\xi^j, W)$ and $(\tilde{\xi}^j, \tilde{W}^{(j)})$ have the same law, also $Z_t$ and $\tilde{Z}_t$ have the same law. Since, $\mathbb{P}$-a.s., $Z_t = 0$ for every $t$, also, $\mathbb{P}$-a.s., $\tilde{Z}_t = 0$ for every $t$, and so, by Lemma [B.2] (\(\tilde{\Omega}, \tilde{\mathcal{A}}, (\tilde{F}_t^j), \tilde{P}, \tilde{W}^{(j)}, \tilde{\xi}^j) \) solves (\[\Box\]).

Concerning Lemmas [4.3] and [4.6] for any integer $h$, as a consequence of Remark [D.6] the maps

$$C_t(\mathcal{M}_{x,M}, w^*) \ni \xi \mapsto (\|\xi_t\|_{H^h^2})_t \in C_t,$$

$$C_t(\mathcal{M}_{x,M}, w^*) \ni \xi \mapsto \|\xi\|_{C^h_t(H^h_2)} \in \mathbb{R}$$

are Borel. Hence $\|\xi^j_t\|_{H^h_2}$ and $\|\tilde{\xi}^j_t\|_{H^h_2}$ have the same laws (as $C_t$-valued random variables) and so Lemma [4.3] holds for $\tilde{\xi}^j$. Similarly $\|\tilde{\xi}^j_t\|_{H^h_2}$ and $\|\tilde{\xi}^j_t\|_{H^h_2}$ have the same laws and so Lemma [4.6] holds for $\tilde{\xi}^j$.

Finally, concerning non-negativity, we note that the set $\{\xi_t \geq 0, \forall t\}$ is Borel in $C_t(\mathcal{M}_{x,M}, w^*)$, because it can be written as $\langle \xi_t, \varphi \rangle \geq 0$ for all rational $t$ and all $\varphi$ in a countable dense set in $C_2$. Since $\xi^j_t$ is concentrated on $\{\xi_t \geq 0, \forall t\}$, also $\tilde{\xi}^j$ is concentrated on this set. The proof is complete. \(\square\)
C The torus and the Green function

We consider the torus $\mathbb{T}^2$ as the two-dimensional manifold obtained from $[-1, 1]^2$ identifying the opposite sides; we call $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ the quotient map. A continuous ($C_x$) function is understood here as a continuous periodic function on $\mathbb{R}^2$, with period 2 on both $x_1$ and $x_2$ directions, and can be identified with a continuous function on the torus $\mathbb{T}^2$. For $s$ positive integer, a $C^s_x$ function on $\mathbb{T}^2$ is a $C^s$ periodic function on $\mathbb{R}^2$ (with period 2). Similarly, for $s$ positive integer and $1 \leq p \leq \infty$, a $W^{s,p}_x$ function on $\mathbb{T}^2$ is a $W^{s,p}_{loc}$ periodic function on $\mathbb{R}^2$ (with period 2). One can also define a Riemannian structure on the torus via the quotient map $\pi$ so that $\pi$ is a local isometry; the local isometry implies that the gradient, the covariant derivatives etc transform naturally, moreover the $C^s$ and $W^{s,p}$ spaces defined via the Riemannian structure coincide with the corresponding spaces of periodic functions as defined above.

The space of distribution $D_x'$ on $\mathbb{T}^2$ is understood as the dual space of $C^\infty$ periodic functions on $\mathbb{R}^2$; the spaces of functions can be identified with subspaces of distribution via the $L^2$ scalar product $\langle f, g \rangle = \int_{[-1,1]^2} f(x)g(x)dx$.

The space of measures $\mathcal{M}_x$ is the space of distributions on $\mathbb{T}^2$ which are continuous (precisely, can be extended continuously) on $C_x$; the space $\mathcal{M}_x$ can be identified with the space of finite signed Radon measures on $\mathbb{T}^2$ and with the quotient space of finite signed Radon measures on $[-1, 1]^2$ under the map $\pi$, via the $L^2_x$ scalar product:

$$\langle f, \mu \rangle = \int_{[-1,1]^2} f(x)\mu(dx), \quad \forall f \in C_x.$$

For $s$ positive integer and $1 < p < \infty$, calling $p'$ the conjugate exponent of $p$, the space $W^{-s,p'}$ is the space of distributions on $\mathbb{T}^2$ which can be continuously extended to $W^{s,p}$.

The convolution on the torus is understood as

$$f * g(x) = \int_{[-1,1]^2} f(y)g(x-y)dy$$

for $f, g$ periodic functions on $\mathbb{R}^2$.

We recall here some standard facts on the Green function $G$ of the Laplacian on the zero-mean functions, that is

$$\Delta G(\cdot, y) = \delta_y, \quad \forall y \in \mathbb{T}^2.$$
Lemma C.1. The following facts hold:

1. The Green function $G$ is translation invariant (that is $G(x, y) = G(x - y)$), even, regular outside 0, with $-C^{-1} \log |x| \leq G(x) \leq -C \log |x|$ in a neighborhood of 0.

2. The kernel $K$ is divergence-free (in the distributional sense), odd, regular outside 0, with $C^{-1} |x|^{-1} \leq |K(x)| \leq C |x|^{-1}$ in a neighborhood of 0.

3. Let $\xi$ be a distribution on $\mathbb{T}^2$ with zero mean, define $u = K \ast \xi$. Then $\text{div} u = 0$ and $\xi = \text{curl} u$.

4. Let $u$ be a vector-valued distribution on $\mathbb{T}^2$ with zero mean and with $\text{div} u = 0$, define $\xi = \text{curl} u$. Then $u = K \ast \xi$.

5. Let $u$ be a vector-valued distribution on $\mathbb{T}^2$, define $\xi = \text{curl} u$. Then $\Pi u = K \ast \xi$, where $\Pi$ is the Leray projector on zero-mean divergence-free distributions.

6. Let $\xi$ be a distribution on $\mathbb{T}^2$ with zero mean, define $u = K \ast \xi$. For any $1 < p < \infty$, for any integer $s$, $\xi$ is in $W^{s,p}_x$ if and only if $u$ is in $W^{s+1,p}_x$ and it holds

$$C^{-1} \|\xi\|_{W^{s,p}_x} \leq \|u\|_{W^{s+1,p}_x} \leq C \|\xi\|_{W^{s,p}_x}.$$  

For the proof, we recall the following facts:

- Any distribution $f$ on $\mathbb{T}^2$ can be written in Fourier series as $f = \sum_k a_k e^{ik \cdot x}$ (the convergence being when tested against a smooth periodic function), see [Tri83, Section 9] and [Tri78, Section 4.11.1].

- For any integer $s$ and any $1 < p < \infty$, the Sobolev space $W^{s,p}_x$ can also be written in terms of Fourier series, that is

$$W^{s,p}_x = \{ f \in \mathcal{D}'_x \mid \tilde{f}^s := \sum_k a_k (1 + |k|^2)^{s/2} e^{ik \cdot x} \in L^p_x \},$$  

with $\|\tilde{f}^s\|_{L^p_x}$ as equivalent norm, see [Tri78, Section 4.11.1]. This fact is well-known for $s \geq 0$. We give a sketch of the proof for $s < 0$ for completeness. We have to show that the above right-hand side is the
dual space of $W_x^{-s,p'}$: indeed, for every distributions $f$ continuous on $W_x^{-s,p'}$, it holds

$$|\langle \tilde{\varphi}^{-s}, \tilde{f}^s \rangle| = |\langle \varphi, f \rangle| \leq C'\|\varphi\|_{W_x^{-s,p'}} \leq C'\|\tilde{\varphi}^{-s}\|_{L^p_x}, \forall \varphi \in W_x^{-s,p'},$$

hence $\tilde{f}^s$ belongs to $L^p_x$, so $f$ belongs to the right-hand side of (C.1).

- **Regularity theory:** The Laplacian operator $\Delta$, intended in the sense of distribution, acts multiplying each Fourier coefficient $a_k$ by $|k|^2$. In particular, it is invertible on the subspace of zero-mean distributions and its inverse acts multiplying each Fourier coefficient $a_k$ by $|k|^{-2}1_{k\neq0}$. It follows that the inverse $\Delta^{-1}$ of the Laplacian (on zero-mean distributions) maps $W^{s,p}$ into $W^{s+2,p}$, for any integer $s$ and any $1 < p < \infty$.

- **Hodge decomposition:** if $f$ is a $\mathbb{R}^2$-valued distributions with $\text{div} f = 0$ and $\text{curl} f = 0$, then $f$ is a constant: indeed, if $a_k$ are the Fourier coefficients of $f$, we have $a_k \cdot k = 0$ and $a_k \cdot k^\perp = 0$ for every $k$, therefore $a_k = 0$ for every $k \neq 0$.

**Proof of Lemma C.1**

1. The fact that $G$ is translation-invariant is due to the translation invariant property of the torus: if $\varphi$ is periodic and zero-mean and solves $\Delta \varphi = \delta_0$ in the distributional sense, then $\varphi(x-y)$ is still periodic and zero-mean and solves $\Delta \varphi = \delta_y$. For the even and regularity property and the bounds, see e.g. [BFM16, Proposition B.1] and references therein.

2. The fact that $K$ is divergence-free, odd and regular outside 0 is a consequence of the definition of $K$ and the properties of $G$. For the bounds, see again [BFM16, Proposition B.1].

3. The fact that $u$ is divergence-free follows from the same property of $K$. Call $\psi = (-\Delta)^{-1}\xi = -G \ast \xi$. Then $u = -\nabla^\perp \psi$ and so

$$\text{curl} u = \partial_{x_1} u^2 - \partial_{x_2} u^1 = -\Delta \psi = \xi,$$

where all the computations are intended using test functions.

4. Call $\tilde{u} = K \ast \xi$. We deduce from the previous points that $\text{curl}(u - \tilde{u}) = 0$ and that $\text{div}(u - \tilde{u}) = 0$. From this we conclude that $u - \tilde{u}$ is a constant, therefore is $= 0$ as both functions have zero mean.
5. This follows from the previous point, applied to $\Pi u$ in place of $u$.

6. If $u$ is in $W^{s+1,p}$ then $\xi = \text{curl} u$ is in $W^{s,p}$. Conversely, if $\xi$ is in $W^{s,p}$, then $\psi = (-\Delta)^{-1}\xi$ is in $W^{s+2,p}$ and so $u = -\nabla^{-1}\psi$ is in $W^{s+1,p}$. 

\[\square\]

D Measurability

We include here various standard concepts and results about measurability. We recall the definition of strong, weak, weak-* and Borel measurability for a Banach-space valued map. We are given a $\sigma$-finite measure space $(E, \mathcal{E}, \mu)$, a Banach space $V$ and a function $f : E \to V$:

- we say that $f$ is strongly measurable if it is the pointwise (everywhere) limit of a sequence of $V$-valued simple measurable functions (i.e. of the form $\sum_{i=1}^N v_i 1_{A_i}$ for $A_i$ in $\mathcal{E}$ and $v_i$ in $V$);
- we say that $f$ is weakly measurable if, for every $\varphi$ in $V^*$, $x \mapsto \langle f(x), \varphi \rangle_{V,V^*}$ is measurable;
- if $V = U^*$ is the dual space of a Banach space $U$, we say that $f$ is weakly-* measurable if, for every $\varphi$ in $U$, $x \mapsto \langle f(x), \varphi \rangle_{V,U}$ is measurable;
- we say that $f$ is resp. strongly Borel, weakly Borel, weakly-* Borel measurable if, for every open set $A$ in $V$ resp. in the strong, weak, weak-* topology, $f^{-1}(A)$ is in $\mathcal{E}$. We omit strongly/weakly/weakly-* when clear.

The following result is morally Pettis measurability theorem. The present version is a consequence of cite[Chapter I Propositions 1.9 and 1.10]VTC87.

**Lemma D.1.** Assume that $V$ is a separable Banach space. Then the notions of strong measurability, weak measurability, strongly Borel measurability and weakly Borel measurability coincide. They also coincide with weak-* measurability and weakly-* Borel measurability if in addition $V$ is reflexive.

We prove here a statement concerning weak-* and weakly-* Borel measurability, which applies in particular to $\mathcal{M}_x = (C_x)^*$. We call $\bar{B}_R$ the closed centered ball in $V$ of radius $R$ (in the strong topology).
Lemma D.2. Assume that \( V = U^* \) is the dual space of a separable Banach space \( U \). Then the notions of weak-* measurability and of weakly-* Borel measurability coincide. Moreover, for any sequence \((\varphi_k)_k\) dense in the unit centered ball of \( U \), the Borel \( \sigma \)-algebra associated to the weak-* topology is generated by
\[
\langle \cdot, \varphi_k \rangle.
\]

Remark D.3. We recall that, if \((E, \mathcal{E})\) is a measurable space, \( \mathcal{I} \) generates the \( \sigma \)-algebra \( \mathcal{E} \) and \( F \) is a subset of \( E \), then the \( \sigma \)-algebra \( \mathcal{E}|_F = \{ A \cap F \mid A \in \mathcal{E} \} \) on \( F \) is the \( \sigma \)-algebra generated on \( F \) by \( \mathcal{I}|_F = \{ I \cap F \mid I \in \mathcal{I} \} \). In particular, the Borel \( \sigma \)-algebra restricted to a subset \( F \) is the Borel \( \sigma \)-algebra on \( F \) (with the topology restricted on \( F \)) and the previous statement can be extended to subsets of \( U \).

Proof. We fix the sequence \((\varphi_k)_k\). We call \( \mathcal{B} \) the Borel \( \sigma \)-algebra associated to the weak-* topology and \( \mathcal{C} \) the \( \sigma \)-algebra generated by the maps \( \langle \cdot, \varphi \rangle \) for \( \varphi \) in \( U \). Since \( \varphi_k \) are dense in the unit centered ball of \( U \), \( \mathcal{C} \) is generated by the maps \( \langle \cdot, \varphi_k \rangle \). We will show that \( \mathcal{B} = \mathcal{C} \), this implies both statements in the lemma. Since the maps \( \langle \cdot, \varphi \rangle \), for \( \varphi \) in \( U \), are continuous in the weak-* topology, \( \mathcal{C} \subseteq \mathcal{B} \). For the converse inclusion, it is enough to show that, for any \( R > 0 \), for any open set \( A \) in the weak-* topology, the sets \( \bar{B}_R \) and \( A \cap \bar{B}_R \) are in \( \mathcal{C} \), where \( \bar{B}_R \) is the closed centered ball in \( V \) of radius \( R \) (in the strong topology). By separability of \( U \), we can fix a sequence \((\varphi_k)_k\) which is dense in the unit centered ball of \( U \). For any \( R > 0 \), the ball \( \bar{B}_R \) is in \( \mathcal{C} \) because the strong norm on \( V \) is \( \mathcal{C} \)-measurable: indeed it can be written as
\[
\|v\| = \sup_k |\langle v, \varphi_k \rangle|.
\]

We recall that the weak-* topology, restricted on \( \bar{B}_R \) is separable and metrizable (see [Bre11, Theorem 3.28] for metrizability, separability follows by compactness), with the distance
\[
d(v, v') = \sum_k 2^{-k} |\langle v - v', \varphi_k \rangle|.
\]

Now, for every \( v \) in \( \bar{B}_R \), \( d(v, \cdot) \) is \( \mathcal{C} \)-measurable, hence any open ball with respect to \( d \) is in \( \mathcal{C} \). Moreover, for any open set \( A \) in the weak-* topology, \( A \cap \bar{B}_R \) can be written as countable union of open balls with respect to \( d \), hence \( A \cap \bar{B}_R \) is in \( \mathcal{C} \). The proof is complete. \( \square \)
Now we give a measurability property of the testing against bounded, but not necessarily continuous maps. Here, given a Polish space $X$, $\mathcal{M}(X)$ is the set of finite Radon measures on $X$.

**Lemma D.4.** Let $F : X \to \mathbb{R}$ be a bounded Borel function on a compact metric space $X$ (in particular $X = \mathbb{T}^2$). Then the map

$$
\Psi_F : \mathcal{M}(X) \ni \mu \mapsto \int_X F(x)\mu(dx) \in \mathbb{R}
$$

is Borel with respect to the weak-* topology on $\mathcal{M}(X)$.

**Proof.** If $F$ is continuous, then also $\Psi_F$ is continuous in the weak-* topology, in particular weakly-* Borel. If $F = 1_A$ is the indicator of an open set $A$ in $\mathbb{T}^2$, then $1_A$ is the pointwise (everywhere) non-decreasing limit on $\mathbb{T}^2$ of continuous functions $F_n$; so, by the dominated convergence theorem, $\Psi_{1_A}$ is the pointwise limit of $\Psi_{F_n}$ and so it is also weakly-* Borel. For the case of general $F$, we use the monotone class theorem. We consider the set $W$ of Borel functions $F$ on $X$ such that $\Psi_F$ is weakly-* Borel. Then $W$ contains the indicators of all the open sets, it is a vector space and it is stable under monotone non-decreasing convergence: indeed, if $(F_n)_n$ is a non-decreasing sequence in $W$ converging pointwise to $F$, then, by the dominated convergence theorem, $\Psi_F$ is the pointwise limit of $\Psi_{F_n}$, in particular weakly-* Borel, and so $F$ belongs also to $W$. Then, by the monotone class theorem, $W$ contains all bounded Borel functions $F$ on $\mathbb{T}^2$, which gives the result. \hfill $\Box$

We recall a classical fact for the product of measures. For a compact metric space $X$, we call $\mathcal{M}(X)$ the set of finite Radon measures on $X$, dual to the space $C(X)$ of continuous function on $X$, and, for $M > 0$, $\mathcal{M}_M(X)$ the closed centered ball on $\mathcal{M}(X)$ of radius $M$.

**Lemma D.5.** For any compact metric space $X$, the map $G : \mathcal{M}(X) \ni \mu \mapsto \mu \otimes \mu \in \mathcal{M}(X \times X)$ is Borel with respect to the weak-* topologies. Moreover, for any $M > 0$, the map $G$, restricted on $\mathcal{M}_M(X)$ with values in $\mathcal{M}_{M^2}(X \times X)$, is continuous with respect to the weak-* topologies.

**Proof.** For $M > 0$, we call $G_M : \mathcal{M}_M(X) \to \mathcal{M}_{M^2}(X \times X)$ the map $G$ restricted on $\mathcal{M}_M(X)$ with values in $\mathcal{M}_{M^2}(X \times X)$. We start showing the continuity of $G_M$. By metrizability of $\mathcal{M}(X)$ and $\mathcal{M}_{M^2}(X \times X)$, it is enough to show that, if $(\mu^n)_n$ is a sequence in $\mathcal{M}(X)$ converging weakly-*
to $\mu$, then $(\mu^n \otimes \mu^n)_n$ converges weakly-$*$ to $\mu \otimes \mu$. For every two continuous functions $\varphi, \psi$ on $X$, we have

$$\langle \varphi \otimes \psi, \mu^n \otimes \mu^n \rangle = \langle \varphi, \mu^n \rangle \langle \psi, \mu^n \rangle \to \langle \varphi \otimes \psi, \mu \otimes \mu \rangle.$$ 

Now the set of all linear combinations of $\varphi \otimes \psi$ for all continuous functions $\varphi, \psi$ is a subalgebra of $C(X \times X)$ which separates point, hence, by the Stone-Weierstrass theorem, it is dense in $C(X \times X)$. Then, for any $\phi$ continuous function on $X \times X$, by a standard approximation argument on $\phi$ we get that $(\langle \phi, \mu^n \otimes \mu^n \rangle)_n$ converges to $\langle \phi, \mu \otimes \mu \rangle$. This shows continuity of the map $G$ restricted to $\mathcal{M}_M(X)$.

For Borel measurability on the full space, take any open set $A$ in $\mathcal{M}(X \times X)$, then $G^{-1}(A)$ is the non-decreasing union of $G^{-1}_M(A \cap \mathcal{M}_M(X \times X))$ for $M$ in $\mathbb{N}$. By continuity of $G_M$, $G^{-1}_M(A \cap \mathcal{M}_M(X \times X))$ is open, hence Borel, in $\mathcal{M}_M(X)$. Moreover $\mathcal{M}_M(X)$ is itself a Borel set in $\mathcal{M}(X)$: indeed the closed centered ball $\overline{B}_R$ in a dual space $V = U^*$ is Borel, as shown in the proof of Lemma [D.2] So $G^{-1}_M(A \cap \mathcal{M}_M(X \times X))$ is Borel in $\mathcal{M}(X)$. Therefore $A$ is Borel in $\mathcal{M}(X)$. The proof is complete.

We conclude on measurability of the $H^h$ norms:

**Remark D.6.** For any fixed integer $h$, the $H^h_x$ norm can be written as supremum of $|\langle \cdot, \varphi \rangle|$ over a set $D$ of $\varphi$ in $C_x$, with $D$ countable and dense in $H^h_x$. Therefore the $H^h_x$ norm is a lower semi-continuous function and Borel function on $(\mathcal{M}_x, w^*)$.

The $C^\alpha_t(\mathcal{H}^h_x)$ norm can be written as

$$\|f\|_{C^\alpha_t(\mathcal{H}^h_x)} = \sup_{t \in \mathbb{Q} \cap [0,T]} \|f_t\|_{\mathcal{H}^h_x} + \sup_{s,t \in \mathbb{Q} \cap [0,T], s < t} \|f_t - f_s\|_{\mathcal{H}^h_x}|t - s|^\alpha$$

(note the supremum over a countable set of times). Therefore, for any fixed $M > 0$, the $C^\alpha_t(\mathcal{H}^h_x)$ norm is a lower semi-continuous function, in particular a Borel function, on $C_t(\mathcal{M}_{x,M}, w^*)$.

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References

[Bas11] Richard F. Bass. Stochastic processes, volume 33 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2011.

[Bes99] Hakima Bessaih. Martingale solutions for stochastic Euler equations. Stochastic Anal. Appl., 17(5):713–725, 1999.

[BF99] Hakima Bessaih and Franco Flandoli. 2-D Euler equation perturbed by noise. NoDEA Nonlinear Differential Equations Appl., 6(1):35–54, 1999.

[BFGM14] Lisa Beck, Franco Flandoli, Massimiliano Gubinelli, and Mario Maurelli. Stochastic odes and stochastic linear pdes with critical drift: regularity, duality and uniqueness, 2014.

[BFM16] Zdzisław Brzeźniak, Franco Flandoli, and Mario Maurelli. Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity. Arch. Ration. Mech. Anal., 221(1):107–142, 2016.

[BGJ13] Zdzislaw Brzeźniak, Beniamin Goldys, and Terence Jegaraj. Weak solutions of a stochastic Landau-Lifshitz-Gilbert equation. Appl. Math. Res. Express. AMRX, (1):1–33, 2013.

[BH86] Peter Baxendale and Theodore E. Harris. Isotropic stochastic flows. Ann. Probab., 14(4):1155–1179, 1986.

[BM13] Zdzisław Brzeźniak and Elżbieta Motyl. Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D domains. J. Differential Equations, 254(4):1627–1685, 2013.

[BP01] Zdzisław Brzeźniak and Szymon Peszat. Stochastic two dimensional Euler equations. Ann. Probab., 29(4):1796–1832, 2001.

[Bre11] Haim Brezis. Functional analysis, Sobolev spaces and partial differential equations. Universitext. Springer, New York, 2011.

[CC99] Marek Capiński and Nigel J. Cutland. Stochastic Euler equations on the torus. Ann. Appl. Probab., 9(3):688–705, 1999.
[CCH+18] Colin Cotter, Dan Crisan, Darryl D. Holm, Wei Pan, and Igor Shevchenko. Modelling uncertainty using circulation-preserving stochastic transport noise in a 2-layer quasi-geostrophic model, 2018.

[CFH19] Dan Crisan, Franco Flandoli, and Darryl D. Holm. Solution Properties of a 3D Stochastic Euler Fluid Equation. J. Nonlinear Sci., 29(3):813–870, 2019.

[CFLS16] A. Cheskidov, M. C. Lopes Filho, H. J. Nussenzveig Lopes, and R. Shvydkoy. Energy conservation in two-dimensional incompressible ideal fluids. Comm. Math. Phys., 348(1):129–143, 2016.

[Cho78] Alexandre Joel Chorin. Vortex sheet approximation of boundary layers. J. Computational Phys., 27(3):428–442, 1978.

[CT15] Ana Bela Cruzeiro and Iván Torrecilla. On a 2D stochastic Euler equation of transport type: existence and geometric formulation. Stoch. Dyn., 15(1):1450012, 2015.

[Del91] Jean-Marc Delort. Existence de nappes de tourbillon en dimension deux. J. Amer. Math. Soc., 4(3):553–586, 1991.

[DFV14] François Delarue, Franco Flandoli, and Dario Vincenzi. Noise prevents collapse of Vlasov-Poisson point charges. Comm. Pure Appl. Math., 67(10):1700–1736, 2014.

[DH18] Theodore D. Drivas and Darryl D Holm. Circulation and energy theorem preserving stochastic fluids, 2018.

[DM87] Ronald J. DiPerna and Andrew J. Majda. Concentrations in regularizations for 2-D incompressible flow. Comm. Pure Appl. Math., 40(3):301–345, 1987.

[DPZ14] Giuseppe Da Prato and Jerzy Zabczyk. Stochastic equations in infinite dimensions, volume 152 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, second edition, 2014.

[DU77] J. Diestel and J. J. Uhl, Jr. Vector measures. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.
[EM70] David G. Ebin and Jerrold Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. of Math. (2), 92:102–163, 1970.

[FF13] E. Fedrizzi and F. Flandoli. Noise prevents singularities in linear transport equations. J. Funct. Anal., 264(6):1329–1354, 2013.

[FG95] Franco Flandoli and Dariusz Gątarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. Probab. Theory Related Fields, 102(3):367–391, 1995.

[FGP10] F. Flandoli, M. Gubinelli, and E. Priola. Well-posedness of the transport equation by stochastic perturbation. Invent. Math., 180(1):1–53, 2010.

[FGP11] F. Flandoli, M. Gubinelli, and E. Priola. Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations. Stochastic Process. Appl., 121(7):1445–1463, 2011.

[FGV01] G. Falkovich, K. Gawędzki, and M. Vergassola. Particles and fields in fluid turbulence. Rev. Modern Phys., 73(4):913–975, 2001.

[FL19a] Franco Flandoli and Dejun Luo. Kolmogorov equations associated to the stochastic two dimensional Euler equations. SIAM J. Math. Anal., 51(3):1761–1791, 2019.

[FL19b] Franco Flandoli and Dejun Luo. $\rho$-white noise solution to 2D stochastic Euler equations. to appear on Probab. Theory Related Fields, 2019.

[FMN14] Franco Flandoli, Mario Maurelli, and Mikhail Neklyudov. Noise prevents infinite stretching of the passive field in a stochastic vector advection equation. J. Math. Fluid Mech., 16(4):805–822, 2014.

[Gaw08] Krzysztof Gawędzki. Stochastic processes in turbulent transport, 2008.

[GHV14] Nathan E. Glatt-Holtz and Vlad C. Vicol. Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise. Ann. Probab., 42(1):80–145, 2014.
[GM18] Benjamin Gess and Mario Maurelli. Well-posedness by noise for scalar conservation laws. Comm. Partial Differential Equations, 43(12):1702–1736, 2018.

[Hol15] Darryl D. Holm. Variational principles for stochastic fluid dynamics. Proc. A., 471(2176):20140963, 19, 2015.

[Jak97] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. Teor. Veroyatnost. i Primenen., 42(1):209–216, 1997.

[Jud63] V. I. Judovič. Non-stationary flows of an ideal incompressible fluid. Ž. Vyčisl. Mat. i Mat. Fiz., 3:1032–1066, 1963.

[Kim09] Jong Uhn Kim. Existence of a local smooth solution in probability to the stochastic Euler equations in $\mathbb{R}^d$. J. Funct. Anal., 256(11):3660–3687, 2009.

[Lio96] Pierre-Louis Lions. Mathematical topics in fluid mechanics. Vol. 1, volume 3 of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[LJR02] Yves Le Jan and Olivier Raimond. Integration of Brownian vector fields. Ann. Probab., 30(2):826–873, 2002.

[MB02] Andrew J. Majda and Andrea L. Bertozzi. Vorticity and incompressible flow, volume 27 of Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[MP94] Carlo Marchioro and Mario Pulvirenti. Mathematical theory of incompressible nonviscous fluids, volume 96 of Applied Mathematical Sciences. Springer-Verlag, New York, 1994.

[MV00] R. Mikulevicius and G. Valiukevicius. On stochastic Euler equation in $\mathbb{R}^d$. Electron. J. Probab., 5:no. 6, 20, 2000.

[Pou02] Frédéric Poupaud. Diagonal defect measures, adhesion dynamics and Euler equation. Methods Appl. Anal., 9(4):533–561, 2002.
[Sch95] Steven Schochet. The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation. Comm. Partial Differential Equations, 20(5-6):1077–1104, 1995.

[SY14] Wilhelm Stannat and Satoshi Yokoyama. Weak solutions of non coercive stochastic Navier-Stokes equations in $\mathbb{R}^2$. Aust. J. Math. Anal. Appl., 11(1):Art. 17, 19, 2014.

[Tri78] Hans Triebel. Interpolation theory, function spaces, differential operators, volume 18 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-New York, 1978.

[Tri83] Hans Triebel. Theory of function spaces, volume 78 of Monographs in Mathematics. Birkhäuser Verlag, Basel, 1983.

[VW93] Italo Vecchi and Si Jue Wu. On $L^1$-vorticity for 2-D incompressible flow. Manuscripta Math., 78(4):403–412, 1993.

[Wol33] W. Wolibner. Un théorème sur l’existence du mouvement plan d’un fluide parfait, homogène, incompressible, pendant un temps infiniment long. Math. Z., 37(1):698–726, 1933.

[Yok14] Satoshi Yokoyama. Construction of weak solutions of a certain stochastic Navier-Stokes equation. Stochastics, 86(4):573–593, 2014.

[Yud95] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. Math. Res. Lett., 2(1):27–38, 1995.