Poor man’s scaling: XYZ Coqblin–Schrieffer model revisited

Eugene Kogan\textsuperscript{1,2} and Zheng Shi\textsuperscript{3}\textsuperscript{*}

\textsuperscript{1}Jack and Pearl Resnick Institute, Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
\textsuperscript{2}Max-Planck-Institut für Physik komplexer Systeme, Dresden 01187, Germany
\textsuperscript{3}Dahlem Center for Complex Quantum Systems and Physics Department, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

We derive the condition for the algebraic renormalizability of the Hamiltonian for a particular case of the general model describing a quantum impurity embedded into an itinerant electron gas. We show that the XYZ Coqblin–Schrieffer model introduced by one of us earlier is renormalizable in this sense, write down the poor man’s scaling equations for the model and analyze the renormalization group flows in the cases of both constant and pseudogap densities of states.

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I. INTRODUCTION

In the celebrated Kondo problem, a seemingly innocuous magnetic impurity coupled to a band of itinerant electrons gives rise to an infrared logarithmic divergence in not only resistivity but also almost all thermodynamic and kinetic properties.\textsuperscript{1–10} To interpret this logarithmic divergence, Anderson\textsuperscript{11} proposed the idea of the so-called poor man’s scaling: the effects of high-energy excitations can be absorbed into renormalized coupling constants at low energies. It was later realized that similar physics is found in many more complicated impurity models with internal degrees of freedom. Among these is the Coqblin–Schrieffer (CS) model motivated by the orbital degeneracy of transition metal ions with unfilled $d$ or $f$ shell.\textsuperscript{12–15} The CS model has recently attracted renewed interest in various contexts.\textsuperscript{6–9}

The study of spin anisotropy\textsuperscript{16–19} has yielded rich results as an offshoot of the original isotropic Kondo problem. Following works on the spin-anisotropic Kondo model,\textsuperscript{16–19} one of the authors introduced anisotropic CS models and derived the poor man’s scaling equations for these models.\textsuperscript{4,10–12} In this work we return to the consideration of what we previously called the XYZ CS model; the treatment is simplified but more detailed than before, and an error in the previously obtained scaling equations is corrected. We also take into account a possible power-law energy dependence of the density of states of itinerant electrons at the Fermi energy (i.e. a pseudogap density of states),\textsuperscript{14,18–24} which can arise in semimetals, nodal superconductors as well as one-dimensional interacting systems.

The rest of the paper is constructed as follows. In Section I we review the condition for the algebraic renormalizability of the Hamiltonian for the general impurity model of a quantum impurity embedded into an itinerant electron gas. Such renormalizability allows to describe the renormalization group flow by the scaling equations of a few coupling constants. In Appendix A we prove that the XYZ CS model satisfies this condition. In Section II we discuss the solutions to the poor man’s scaling equations for the XYZ Kondo model, which is the $N = 2$ special case of the XYZ CS model. We consider both constant and pseudogap densities of states for the itinerant electrons, and plot the corresponding weak-coupling flow diagrams. In Section IV we present and analyze the poor man’s scaling equations for the XYZ CS model in the more general case $N > 2$, again for both constant and pseudogap densities of states. We conclude in Section V.

II. ALGEBRAIC RENORMALIZABILITY

The quantum impurity that we consider is coupled to conduction electrons and described by the Hamiltonian\textsuperscript{21–23}

\begin{equation}
H = \sum_{k\alpha} \epsilon_cr^+_k\epsilon_{k}\alpha + \sum_{k,k',\alpha\beta,ab} V_{\beta\alpha,ba}X_{ba}r^+_k\epsilon_{k'}\alpha\beta_{\bar{c}} \epsilon_{k}\bar{c}, \tag{1}\end{equation}

where $c^+_k\alpha$ creates a conduction electron with wave vector $k$, channel $\alpha$, and energy $\epsilon_k$. The Hubbard $X$-operator is defined as $X_{ba} = |b\rangle\langle a|$, where $|a\rangle, |b\rangle$ are the impurity states.

We are interested in the case where $V_{\beta\alpha,ba}$ can be written as a sum of direct products of Hermitian matrices $\{G^p\}$ and $\{\Gamma^p\}$, which act respectively in impurity and channel Hilbert spaces. Suppressing all impurity and channel indices, this means

\begin{equation}V = 2 \sum_{pp'} c_{pp'} G^p \otimes \Gamma^{p'}\tag{2}\end{equation}

We assume that $\{G^p\}$ is closed with respect to commutation and generates a Lie algebra $g$, and $\{\Gamma^p\}$ generates an algebra isomorphic to (and henceforth will not be distinguished from) $g$.

While studying the physics in the vicinity of the Fermi energy, we must account for the virtual transitions from and to electron states at higher energies. In the poor man’s scaling formalism\textsuperscript{4}, one reduces the semi-bandwidth of the conduction electrons from $D$ to $D - \delta D$, $\delta D < 0$ is infinitesimal, discarding the electronic states in the energy intervals $(D - \delta D, D)$ and $(-D - \delta D, D)$; however, virtual transitions through
these states are retained in the form of a modified coupling constant $V$, such that the impurity scattering matrix elements are the same at low energies. The coupling $V$ is therefore renormalized as the energy scale $D$ is reduced. In the lowest order perturbation theory (one loop approximation), the scaling equation has the form

$$\frac{dc^{ij}_{pp'}}{d\ln \Lambda} = - \sum_{st's't'} f^p_{st} f^{p'}_{s't'} G_{ss'} c_{st'} \Gamma^i_{tt'},$$ \tag{3}$$

where $\Lambda = D/D_0$, $D_0$ is the initial semi-bandwidth, and $f^p_{st}$ are the structure constants of the Lie algebra $g$.

In the previous publications of one of the authors, the issue of algebraic renormalizability of the interaction Eq. (2) was addressed in the framework of poor man’s CS model in Sec. IV, let us start from the analysis of the CS model will be motivated and discussed in detail in the remainder of this paper.

### III. XYZ KONDO MODEL

#### A. From spins to Hubbard operators

To motivate and explain our treatment of the XYZ CS model in Sec. IV, let us start from the analysis of the spin-anisotropic Kondo model

$$H = \sum_{kk\alpha\beta} c^\dagger_{k\alpha} c_{k\beta} + \sum_{k\alpha\beta} J_{ij} S_i^j \sigma^j_{\alpha\beta} \Gamma^{i}_{k\alpha} c_{k\beta},$$ \tag{7}$$

where $S^x, S^y, S^z$ are the impurity spin operators, $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices, $J_{ij}$ is the anisotropic exchange coupling matrix, and summation with respect to any repeated Cartesian index is implied. The Hamiltonian Eq. (7) is renormalizable when $J_{ij} = J_i \delta_{ij}$ is diagonal. The corresponding scaling equations are:

$$\frac{dJ_x}{d\ln \Lambda} = -2J_y J_z,$$
$$\frac{dJ_y}{d\ln \Lambda} = -2J_x J_z,$$
$$\frac{dJ_z}{d\ln \Lambda} = -2J_x J_y.$$

(Here and further on we take the constant density of states of the itinerant electrons to be equal to 1).

The interaction in Eq. (7) can be written down alternatively using Hubbard $X$-operators

$$V = J_S \left( X_+ c^+_i c_i + X_- c^+_i c_i \right) + J_A \left( X_+ c^+_i c_i + X_- c^+_i c_i \right) + J_z \left( X_+ c^+_i c_i + X_- c^+_i c_i \right) + \frac{1}{2} J_z \left( X_+ + X_- \right) \left( c^+_i c_i + c^+_i c_i \right),$$ \tag{9}$$

where $J_S = (J_x + J_y)/2$ and $J_A = (J_x - J_y)/2$. (To avoid cluttering we omit the wave vector indices.) Thus the scaling equation Eq. (8) can be written down as

$$\frac{dJ_S}{d\ln \Lambda} = -2J_S J_z,$$
$$\frac{dJ_A}{d\ln \Lambda} = 2J_A J_z,$$
$$\frac{dJ_z}{d\ln \Lambda} = -2(J_A^2 - J_S^2).$$ \tag{10}$$

(Of course, Eq. (10) can also be obtained from Eq. (8).)

When $J_A = 0$, we return to the previously well-studied $XXZ$ Kondo model,

$$\frac{dJ_S}{d\ln \Lambda} = -2J_S J_z, \frac{dJ_z}{d\ln \Lambda} = -2J_A^2.$$ \tag{11}$$

These scaling equations exhibit Kosterlitz-Thouless (KT) physics: initial parameters satisfying $0 < |J_S| < -J_z$ lead to a flow towards the fixed line $J_S = 0$, $J_z < 0$; otherwise, either $J_z > 0$ or $0 < -J_S < |J_S|$ results in a flow to strong coupling $|J_S| \to \infty$ and $J_z \to \infty$. The separatrix between the two regimes is $|J_S| = -J_z$.
flow trajectories in the vicinity of the line of fixed points $J_S = J_A = 0$. Recalling our analysis in the $J_A = 0$ and $J_S = 0$ cases, we find this fix line is semistable: $|J_S|$ is a relevant perturbation and $|J_A|$ is irrelevant for $J_z > 0$, and the situation is reversed for $J_z < 0$. Any trajectories near this line flow away from it if either $J_S \neq 0$ or $J_A \neq 0$, and it is natural to assume they flow to strong coupling.

An additional line of fixed points is given by $J_S = J_A$ and $J_z = 0$. In fact, a little more algebra shows that one phase boundary is given by a plane $J_z + J_S - J_A = 0$ which contains this fixed line:

$$\frac{d}{d \ln \Lambda} (J_z + J_S - J_A) = -2 (J_z + J_S - J_A) (J_S + J_A).$$

(12)

By assumption $J_S \geq 0$ and $J_A \geq 0$, so $J_S + J_A > 0$ and any deviation from the plane $J_z + J_S - J_A = 0$ is relevant. The most “economic” assumption for the flow diagram is that these deviations flow to one of the strong coupling phases: either $J_S \to \infty$ and $J_z \to \infty$, or $J_A \to \infty$ and $J_z \to -\infty$.

Our picture is confirmed by the numerical flow diagram Fig. 1, which we plot for $J_S \geq 0$ and $J_A \geq 0$ by numerically integrating the weak-coupling scaling equation Eq. (10). Scaling flows in the various limiting cases we have considered are highlighted.

Symmetry considerations lead us to the conclusion that there are four strong-coupling phases in total: $J_S \to \pm \infty$, $J_A \to \pm \infty$, $J_z \to -\infty$. These are separated by six phase boundaries: $J_S = 0$, $J_A = 0$, $J_A \pm J_S = 0$ and $J_z = 0$. Not too surprisingly, in terms of the original coupling constants $J_x$, $J_y$ and $J_z$, the phase boundaries are simply $J_x = \pm J_y$, $J_y = \pm J_z$ and $J_z = \pm J_x$, where we recover $U(1)$ spin rotation symmetry.

### B. Pseudogap density of states

The flow diagram of the XYZ Kondo model is much more interesting when the itinerant electrons have a (local) density of states with a power-law dependence upon the energy because of, for instance, the electron dispersion:

$$\rho(\epsilon) = C|\epsilon|^r, \text{ if } |\epsilon| < D.$$  

(13)

This model was considered by us previously but only the flow diagram of the XXZ case ($J_x = J_y$) was discussed. In this section we clarify the full 3d flow diagram as a special case of the XYZ CS model.

The scaling equation in the appropriate units (for details see Refs. 14-16) is

$$\frac{d J_x}{d \ln \Lambda} = r J_x - 2 J_y J_z$$
$$\frac{d J_y}{d \ln \Lambda} = r J_y - 2 J_x J_z$$
$$\frac{d J_z}{d \ln \Lambda} = r J_z - 2 J_x J_y.$$  

(14)

Fig. 1 has a trivial fixed point $J_x = J_y = J_z = 0$ corresponding to a decoupled impurity spin, and four non-trivial fixed points

$$|J_x| = |J_y| = |J_z| = \frac{r}{2}; \quad J_x J_y J_z = \frac{r^3}{8},$$  

(15)

describing a finite isotropic Heisenberg exchange. Each non-trivial fixed point belongs to a critical surface which separates one of the strong-coupling phases from the decoupled phase controlled by the trivial fixed point.
FIG. 2: Elliptic cones Eq. (17) containing flow trajectories of the pseudogap Kondo model. Red, green, blue, yellow and magenta surfaces have $\beta = 0$, $\alpha = 3\beta$, $\alpha = \beta$, $\alpha = \beta/3$ and $\alpha = 0$, respectively.

From Eq. (14), we find

$$\frac{d}{d \ln \Lambda} \left[ \alpha J_x^2 + \beta J_y^2 - (\alpha + \beta)J_z^2 \right] = 2r \left[ \alpha J_x^2 + \beta J_y^2 - (\alpha + \beta)J_z^2 \right],$$

for arbitrary $\alpha$ and $\beta$. If the right hand side vanishes for the bare coupling constants, it remains zero along the scaling flow. Each flow line thus lies completely on the elliptic cone

$$\alpha J_x^2 + \beta J_y^2 - (\alpha + \beta)J_z^2 = 0,$$

where $\alpha$ and $\beta$ are now determined by the bare coupling constants. This family of elliptic cones foliates the phase space, and all touch each other along the isotropic lines $|J_x| = |J_y| = |J_z|$, as shown in Fig. 2.

For the degenerate cases $\alpha = 0$ or $\beta = 0$ the elliptic cone becomes a pair of planes. The flow trajectories on such planes were presented in our previous publication [14]. Here, in Fig. 3 we show the trajectories on one of the cones, which nevertheless constitute a general representation of the flow diagram because the behaviors of different cones are rather similar.

IV. XYZ COQBLIN-SCHRIEFFER MODEL

In this section, we turn our attention to the XYZ CS model with an arbitrary number of channels $N$, thereby generalizing our $N = 2$ results in Section [11].

FIG. 3: Typical flow trajectories of the pseudogap Kondo model with $r = 0.1$ on the surface of a cone Eq. (17); we have chosen $\alpha = \beta$. The trivial fixed point at the origin is painted in black, and the non-trivial finite-coupling fixed points in red. The isotropic orange flow trajectories are shared by all elliptic cones; the green trajectories flow towards weak coupling, the blue ones towards strong coupling, and the red trajectories lie on phase boundaries.

A. Constant density of states

The CS model with full $SU(N)$ symmetry is represented by the Hamiltonian

$$H = \sum_m \epsilon c_m^\dagger c_m + J \sum_{mm'} X_{mm'} c_m^\dagger c_{m'}$$

$$-(J/N) \sum_{mm'} X_{mm'} c_m^\dagger c_{m'},$$

where the quantum number $m, m' = 1, \ldots, N$. The scaling equation for the Hamiltonian Eq. (18) has the form

$$\frac{dJ}{d \ln \Lambda} = -NJ^2.$$  

For $N = 2$ the model coincides with the spin-isotropic Kondo model.

The anisotropic Kondo model represented in terms of Hubbard $X$-operators Eq. (9) has motivated one of us [16] to consider the algebraic renormalizability of the anisotropic generalization of the CS model (or the "XYZ CS model") for arbitrary $N$, Eq. (6). As proved in Appendix A, Eq. (6) is renormalizable, and the scaling equa-
tions are\textsuperscript{1}
\[
\frac{dJ_S}{d\ln \Lambda} = -(N-2)J_S^2 - 2J_SZ,
\]
\[
\frac{dJ_A}{d\ln \Lambda} = (N-2)J_A^2 + 2J_AZ,
\]
\[
\frac{dJ_Z}{d\ln \Lambda} = -N(J_S^2 - J_A^2);
\]
or equivalently, in terms of $J_x$, $J_y$ and $J_z$,
\[
\frac{dJ_x}{d\ln \Lambda} = -(N-2)J_xJ_y - 2J_yJ_z,
\]
\[
\frac{dJ_y}{d\ln \Lambda} = -\frac{N-2}{2}(J_x^2 + J_y^2) - 2J_xJ_z,
\]
\[
\frac{dJ_z}{d\ln \Lambda} = -NJ_xJ_y.
\]

The XXZ CS model introduced in the previous publication\textsuperscript{19} is a particular case of the interaction Eq. (19). Its scaling equation is recovered by setting $J_A = 0$ (i.e. $J_x = J_y$) in Eq. (20). In the fully isotropic case $J_x = J_y = J_z$ we recover the scaling equation for the original CS model Eq. (19).

We turn back to analyze the flow diagram for Eq. (20) in the $N > 2$ case. It should first be recognized that, owing to the quadratic terms proportional to $N - 2$, $J_S \to -J_S$ and $J_A \to -J_A$ are no longer symmetries of the scaling equations, and taking $J_A = 0$ or $J_S = 0$ no longer brings us back to the familiar equations for the anisotropic Kondo model. Nevertheless, the KT physics remains in action; it is only the separatrices that are different from before, namely $J_S = J_A < 0$ and $J_S = -(N/2)J_A < 0$ (when $J_A = 0$) as well as $J_S = J_A > 0$ and $J_S = -(N/2)J_A < 0$ (when $J_S = 0$). For instance, provided $J_A = 0$, systems whose initial parameters are located between the two separatrices (i.e. $J_S < 0$, $J_A < J_S < -(2/N)J_A$) will flow to one of the $J_S = 0$, $J_S < 0$ fixed points, and other systems flow to strong coupling $J_S \to \pm \infty$, $J_z \to \infty$ depending on the initial sign of $J_S$.

Similar to the spin-anisotropic Kondo model with $N = 2$, Eq. (20) indicates that the XYZ CS model has a line of fixed points $J_S = J_A = 0$ for any $N$. Since all $N$-dependent terms are quadratic in $J_S$ or $J_A$, linearization again tells us that these fixed points are semistable, with $|J_S|$ relevant and $J_A$ irrelevant for $J_A > 0$, and $|J_A|$ relevant and $J_A$ irrelevant for $J_S < 0$.

A further distinction from the Kondo case turns up when we consider other lines of fixed points. For $N > 2$, the line $J_S = -J_A$, $J_z = 0$ is not a fixed line any more, but describes two flow trajectories instead: the scaling equations become $dJ_S/d\ln \Lambda = -(N-2)J_S^2$ in this case, so $J_S < 0$ flows to weak coupling and $J_S > 0$ flows to strong coupling. On the other hand, the $N = 2$ fixed line $J_S = J_A$, $J_z = 0$ is replaced by $J_z = -(N/2-1)J_S = -(N/2-1)J_A$, which does not lie on the $J_S-J_A$ plane for $N > 2$.

The phase boundaries again contain the separatrices and the fixed lines. For instance, one interesting linear combination of the coupling constants is
\[
\frac{d}{d\ln \Lambda} \left( \frac{N}{2} J_S - J_A + J_z \right)
= - (NJ_S + 2J_A) \left( \frac{N}{2} J_S - J_A + J_z \right).
\]

This suggests when $J_S > 0$ and $J_A > 0$, the plane $(N/2)J_S - J_A + J_z = 0$ is a phase boundary since any infinitesimal deviation from it is a relevant perturbation. Note that this phase boundary contains the separatrix $J_z = -(N/2)J_S < 0$ and $J_A = 0$, the separatrix $J_z = J_A > 0$ and $J_S > 0$, in addition to the fixed line $J_z = -(N/2-1)J_S = -(N/2-1)J_A$. We can find three other phase boundaries in a similar fashion: $J_S + J_A = 0$ ($J_S < 0$ and $J_A > 0$), $J_S - (N/2)J_A = J_z = 0$ ($J_S < 0$ and $J_A < 0$), $(N/2)(J_S + J_A) + J_z = 0$ ($J_S > 0$ and $J_A < 0$), $J_S = 0$ ($J_A > 0$), and $J_A = 0$ ($J_S < 0$). By analogy with the $N = 2$ case, these six phase boundaries divide the parameter space into four strong-coupling phases: $J_S \to \pm \infty$, $J_z \to \infty$; $J_A \to \pm \infty$, $J_z \to -\infty$. We illustrate our findings in the flow diagram Fig. 4 for the $N = 3$ case.

**B. Pseudogap density of states**

When the density of states takes a power-law form Eq. (13), the scaling equation for the XYZ CS model becomes
\[
\frac{dJ_S}{d\ln \Lambda} = rJ_S - (N-2)J_S^2 - 2J_SZ,
\]
\[
\frac{dJ_A}{d\ln \Lambda} = rJ_A + (N-2)J_A^2 + 2J_AZ,
\]
\[
\frac{dJ_z}{d\ln \Lambda} = rJ_S - N(J_S^2 - J_A^2).
\]

We plot the corresponding flow diagram in Fig. 5.

There is first and foremost a trivial decoupled fixed point $J_S = J_A = J_z = 0$ (black). In addition, several nontrivial fixed points exist where one of the coupling constants vanishes: $(J_S, J_A, J_z) = (r/N, 0, r/N)$ (red), $(-r/2, 0, N r/4)$ (red), $(0, -r/N, -r/N)$ (green), $(0, r/2, -N r/4)$ (green) and $(r/(N-2), -r/(N-2), 0)$ (magenta). The four $J_A = 0$ and $J_S = 0$ fixed points are critical points with two stable directions, one in-plane and the other out-of-plane. Therefore, these four fixed points belong to the phase boundaries which separate the strong-coupling phases from the decoupled phase; these phase boundaries intersect the $J_A = 0$ ($J_S = 0$) plane at

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\textsuperscript{1} A similar equation was obtained in the previous publication of one of the authors (Eq. (46) of Ref.\textsuperscript{10}), but due to an elementary algebra mistake, the analog of Eq. (21) (and its special case for $N = 3$ in Ref.\textsuperscript{17}) contained an error.
FIG. 4: Numerical 3D flow diagram of the \( XYZ \) CS model with \( N = 3 \), calculated from the weak-coupling scaling equation Eq. (20). We show two different perspectives, because the symmetries \( J_S \rightarrow - J_S \) and \( J_A \rightarrow - J_A \) are lost in contrast to the \( N = 2 \) Kondo case. The red trajectories lie on the \( J_A = 0 \) plane and the green ones lie on the \( J_S = 0 \) plane; the KT separatrices are no longer reflection-symmetric with respect to the \( J_S = 0 \) or \( J_A = 0 \) planes. The gray line is the fixed line \( J_z = -(N/2-1)J_A = -(N/2-1)J_A \), and the blue trajectories satisfy \( J_S = -J_A, J_z = 0 \). The remaining colored trajectories are either within one of the phase boundaries given in the text or within a plane intersecting a phase boundary. All four strong-coupling phases are shown: \( J_S \rightarrow \pm \infty \) and \( J_z \rightarrow \infty \); \( J_A \rightarrow \pm \infty \) and \( J_z \rightarrow -\infty \).

As shown in Fig. 5 all of the following straight lines between the trivial fixed point and the nontrivial ones are valid scaling trajectories: \( J_S = J_z, J_A = 0 \) and \( J_z = -NJ_S/2, J_A = 0 \) (orange), \( J_A = J_z, J_S = 0 \) and \( J_z = -NJ_A/2, J_S = 0 \) (brown) and \( J_S = -J_A, J_z = 0 \) (magenta). Each of these lines is divided into three segments by the trivial, stable fixed point and the nontrivial, unstable fixed point.

Based on our observations, as with the \( N = 2 \) case, we expect the phase space for the \( N > 2 \) pseudogap \( XYZ \) CS model to be divided into five phases, namely a weak-coupling phase controlled by the trivial fixed point \( J_S = J_A = J_z = 0 \) and four strong-coupling phases \( J_S \rightarrow \pm \infty, J_z \rightarrow \infty \) and \( J_A \rightarrow \pm \infty, J_z \rightarrow -\infty \). Unfortunately, the linear terms in Eq. (23) make it a difficult task in general to determine the phase boundaries or scaling invariants analytically.

V. CONCLUSIONS

In this work, we have studied the algebraic renormalizability of a particular case of the quantum impurity model in an itinerant electron gas in the weak-coupling regime. Our theory is applied to the \( XYZ \) Coqblin–Schrieffer model introduced by one of us earlier. We
write down the poor man’s scaling equations under constant and pseudogap densities of states, and discuss their solutions for both the $N = 2$ case (the anisotropic XYZ Kondo model) and the $N > 2$ case in detail. The corresponding 3D weak-coupling flow diagrams are presented.

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Appendix A: Algebraic renormalizability of the XYZ CS model

The commutation relations necessary for writing down scaling equation for the interaction Eq. (6) are

$$[c_m^\dagger, c_{m'} c_{m''}, c_{m'''}] = \delta_{m m'} c_{m'}^\dagger c_{m''} - \delta_{m' m''} c_{m'}^\dagger c_{m'''} c_m. \quad \text{(A1)}$$

(and the same for the Hubbard operators). Substituting the generators into the L.H.S. of Eq. (5) we get:

for the $J_3^z$ terms

$$\sum_{m \neq m', m''} \left[ X_{m m'}, X_{m' m''} \right] [c_{m'}^\dagger, c_{m''}, c_{m'''}]$$

$$= \sum_{m \neq m', m''} \left( X_{m m'} \delta_{m' m''} - X_{m' m''} \delta_{m m''} \right)$$

$$\cdot \left( c_{m'}^\dagger c_{m''} \delta_{m m''} - c_{m'}^\dagger c_{m''} \delta_{m' m''} \right)$$

$$= 2 \left\{ (N-2) \sum_{m \neq m'} X_{m m'} [c_{m'}^\dagger, c_{m''}, c_m] \right\}$$

$$+ (N-1) \sum_m X_{m m} c_m - \sum_{m m'} X_{m m'} c_m c_{m'} \right\}$$

$$= 2 \left\{ (N-2) \sum_{m \neq m'} X_{m m'} [c_{m'}^\dagger, c_{m''}, c_m] \right\}$$

$$+ N \sum_m X_{m m} c_m - \sum_{m m'} X_{m m'} c_m c_{m'} \right\}; \quad \text{(A2)}$$

for the $J_3^z$ terms

$$\sum_{m \neq m', m''} \left[ X_{m m'}, X_{m' m''} \right] [c_{m'}^\dagger, c_{m''}, c_{m'''}]$$

$$= \sum_{m \neq m', m''} \left( X_{m m'} \delta_{m' m''} - X_{m' m''} \delta_{m m''} \right)$$

$$\cdot \left( c_{m'}^\dagger c_{m''} \delta_{m m''} - c_{m'}^\dagger c_{m''} \delta_{m' m''} \right)$$

$$= -2 \sum_m X_{m m} c_m; \quad \text{(A4)}$$

for the $J_A J_z$ terms

$$\sum_{m \neq m', m''} \left[ X_{m m'}, X_{m' m''} \right] [c_{m'}^\dagger, c_{m''}, c_{m'''}]$$

$$= \sum_{m \neq m', m''} \left( X_{m m'} \delta_{m' m''} - X_{m' m''} \delta_{m m''} \right)$$

$$\cdot \left( c_{m'}^\dagger c_{m''} \delta_{m m''} - c_{m'}^\dagger c_{m''} \delta_{m' m''} \right)$$

$$= 2 \sum_m X_{m m} c_m; \quad \text{(A5)}$$

* Electronic address: Eugene.Kogan@biu.ac.il
† Electronic address: zheng.sh@sfsu-berlin.de
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