Computation of confidence intervals for Poisson processes

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Abstract

We present an algorithm which allows a fast numerical computation of Feldman-Cousins confidence intervals for Poisson processes, even when the number of background events is relatively large. This algorithm incorporates an appropriate treatment of the singularities that arise as a consequence of the discreteness of the variable.

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1 Introduction

In physics there are many situations where the outcome of an experiment is a positive integer number with a Poisson distribution. This is the case for instance of the number of events of a certain type produced in high energy collisions. The statistical analysis of these processes is a difficult task when the result obtained is in the limit of the sensitivity of the experiment. In general, the number of events $n_0$ obtained in an experiment consists of background events with known mean $b$ and signal events, whose mean $\mu$ is the quantity that we want to determine. The problem arises when the number of events obtained $n_0$ is significantly lower than the background expected $b$. This happens in some experiments on neutrino oscillations, for instance in the KARMEN 2 experiment [1].

Usually, after performing an experiment, one decides whether to give the results on the unknown parameter $\mu$ in the form of a central confidence interval or an upper bound. This decision (called ‘flip-flopping’) is based on the data and, as has been shown by Feldman and Cousins [2], introduces a bias that may cause that the intervals cover the true value $\mu$ with a smaller frequency than the stated confidence level. To solve this and other problems, they introduce a new ordering

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principle that unifies the treatment of central confidence intervals and upper limits. This is possible because the Neyman construction of confidence intervals allows the choice of the ordering principle with which the intervals are constructed. Typical choices lead to the construction of either central intervals or upper confidence limits. The choice of Feldman and Cousins gives intervals that are two-sided or upper limits depending on the result of the experiment and not on the choice of the experimentalist. These intervals avoid the undercoverage caused by ‘flip-flopping’ and are non-empty in all cases. Some variants of their method have been also proposed.

To consider the Feldman-Cousins confidence intervals as an alternative to standard intervals, in practice one needs to calculate these intervals for arbitrary $n_0$ and $b$. The Tables provided in Ref. [2], for $n_0 \leq 20$, $b \leq 15$, are sufficient for small luminosity/statistics experiments, but for higher luminosities in general $b$ and $n_0$ are larger. One possibility is to extend these Tables using the same systematic computational method of Ref. [2], whose speed is not optimized and consumes a lot of time. More convenient is to develop a program which takes $n_0$ and $b$ as inputs and gives as output the confidence interval, requiring a minimal number of calculations. This is what is done here. The program can be used either directly to compute the confidence interval for given $n_0$ and $b$ or in conjunction with other routines. This is especially useful, for instance to calculate expected limits from rare high energy processes for different values of the center of mass energy or the collider luminosity, which is the case that we were primarily interested in.

In the following we introduce a procedure to compute the Feldman-Cousins intervals in an efficient way for arbitrary $n_0$ and $b$, in principle only limited by the machine precision. In Section 2 we review Neyman’s construction of the confidence intervals for a Poisson variable, emphasizing some points that simplify the numerical calculation. Section 3 is more technical and devoted to explain in depth how to translate this method for the computer calculation. In Section 4 we present our results. The FORTRAN implementation of the algorithm is given in the Appendix. Other implementations in C and Mathematica [6] (about 100 times slower than the FORTRAN version) can be obtained from the author.

2 Construction of the confidence intervals

The probability to observe $n$ events in a Poisson process consisting of signal events with unknown mean $\mu$ and background events with known mean $b$ is given by the formula

$$P(n \mid \mu; b) = \frac{(\mu + b)^n}{n!}e^{-(\mu+b)},$$

with $n, \mu, b \geq 0$, and $n$ restricted to integer values. The construction of the confidence intervals on the unknown variable $\mu$ follows Neyman’s method of the confidence belts.

The first step in this procedure is to construct, for a fixed value of $b$ and for different values of $\mu$, the confidence intervals $[n_1(\mu; b), n_2(\mu; b)]$ such that the probability to obtain a result between
$n_1$ and $n_2$ is greater or equal than $\alpha$, the confidence level (C. L.),

$$P(n \in [n_1, n_2] \mid \mu; b) = \sum_{n=n_1}^{n_2} P(n \mid \mu; b) \geq \alpha.$$  \hspace{1cm} (2)

It is worth to note that for the more general case of a continuous variable $x$ the intervals $[x_1(\mu; b), x_2(\mu; b)]$ satisfy $P(n \in [x_1, x_2] \mid \mu; b) = \alpha$. For a discrete variable $n$ it is not possible to obtain the exact equality, and to avoid undercoverage it is replaced by the inequality in Eq. (2).

The choice of the intervals $[n_1, n_2]$ is not unique, and determines the type of confidence intervals on $\mu$ that are constructed. The most common choices are $n_2 = \infty$, $P(n \leq n_1 \mid \mu; b) \leq 1 - \alpha$, which gives upper confidence bounds, and $P(n \geq n_2 \mid \mu; b) \leq (1 - \alpha)/2$ which leads to central confidence intervals. The prescription of Ref. [2] is based on a likelihood ratio $R$, constructed as follows.

1. For any values of $b$ and $n$, one considers which value of $\mu$ would maximize the probability $P(n \mid \mu; b)$. It is straightforward to find that for $n \geq 1$, $P(n \mid \mu; b)$ considered as a function of $\mu$ grows for $\mu < n - b$, has a maximum at $\mu = n - b$ and decreases for $\mu > n - b$. As $\mu$ is restricted to lie in the positive real axis, if $n \geq b$ the maximum is $\mu = n - b$, otherwise the maximum is $\mu = 0$. In the case $n = 0$ the maximum is also $\mu = 0$, so we define

$$\mu_{\text{best}}(n; b) = \max\{0, n - b\} \hspace{1cm} (3)$$

as the value which maximizes $P(n \mid \mu; b)$.

2. Then, for any value of $\mu$ we consider the quantity $R(n; \mu, b)$ defined as

$$R(n; \mu, b) = \frac{P(n \mid \mu; b)}{P(n \mid \mu_{\text{best}}(n; b); b)} \leq 1,$$ \hspace{1cm} (4)

on which the Feldman-Cousins ordering principle is based. To construct the interval $[n_1(\mu; b), n_2(\mu; b)]$, for each value of $\mu$ (and fixed $b$) one takes values of $n$ with decreasing $R(n; \mu, b)$, summing up their probabilities $P(n \mid \mu; b)$ until the total equals or exceeds the C. L. desired $\alpha$. Thus the interval $[n_1, n_2]$ is the set of values of $n$ necessary to satisfy the inequality in Eq. (2), taken with the largest $R(n; \mu, b)$.

The simplicity of this prescription allows a fast computer implementation. Instead of generating a large table of values for $n$ and taking those with the largest $R$, we can directly find these values and add them successively to construct the interval. (This is difficult to do with the more involved prescriptions of Refs. [4, 5].) For this purpose we will examine the behaviour of $R$. If we consider

$$R(x; \mu, b) = \frac{(\mu + b)x^{-(\mu+b)}}{(\mu_{\text{best}}(x; b) + b)x^{-(\mu_{\text{best}}(x; b)+b)}}$$ \hspace{1cm} (5)
as a function of the continuous variable $x$, we can look for its maximum. Let us first consider $\mu \neq 0$, $b \neq 0$. For $x$ sufficiently small, $\mu_{\text{best}}(x; b) = 0$ and $R(x; \mu, b) = (1 + \mu/b)^xe^{-\mu}$ is increasing. For $x \geq b$, $\mu_{\text{best}}(x; b) = x - b$ and $R(x; \mu, b) = (e(\mu + b)/x)^xe^{-(\mu+b)}$ grows for $x < \mu + b$, falls for $x > \mu + b$ and has a local maximum at $x = \mu + b$, which is then the global maximum. This is also true for $\mu \neq 0$, $b = 0$. For $\mu = 0$, $b \neq 0$ and $x < b$, $\mu_{\text{best}}(x; b) = 0$ and $R(x; 0, b) = 1$, its maximum possible value. For $x \geq b$, $R(x; 0, b) = (b/x)^xe^{-b}$ decreases with $x$. Hence the maximum is still $x = \mu + b$, although not unique. The only remaining case with $\mu = 0$, $b = 0$ in which the Poisson distribution is singular must be treated separately.

![Figure 1: Confidence belt for $b = 3$, with $\alpha = 0.9$. The vertical line is drawn for $n_0 = 8$.](image)

With this method, and for different values of $\mu$ ($b$ is fixed), one calculates the confidence intervals $[n_1(\mu; b), n_2(\mu; b)]$ obtaining a confidence belt like the one showed in Fig. 1 for $b = 3$ and a C. L. of 0.9. To find the confidence interval $[\mu_1, \mu_2]$ for a particular experimental value $n_0$, one draws a vertical line at $n = n_0$ and finds the maximum and minimum values of $\mu$ for which the line intersects the confidence belt. In Fig. 1 we observe that $\mu_1$ is the smallest $\mu$ such that $n_2(\mu; b) = n_0$, whereas $\mu_2$ is the largest $\mu$ for which $n_1(\mu; b) = n_0$.

### 3 Implementation

Let us explain how the method described in Section 2 is made suitable for the evaluation in a computer. For the calculations we use the FORTRAN version of the program compiled with fort77 under Linux on a Pentium III-450 (compiled with g77 the program runs about 25% slower), and for the plots we also use the Mathematica version.
The first problem in the practical realization of the Neyman construction is that, for large \( b, n \) or \( \mu \), the factors of the Poisson probability formula in Eq. (1) can overflow (or underflow) the computer capacity in intermediate calculations. The factor \((\mu + b)^n\) may be very large, for instance \(54^{178}\) is larger than the biggest double precision real number in the FORTRAN compiler used, approximately \(10^{308}\). However, the exponential factor in Eq. (1) compensates for it in the final result. For \(\mu + b + n > 230\) we evaluate \(P\) using the expression

\[
P(n | \mu; b) = e^{-\mu} \prod_{i=1}^{n} \left(\frac{\mu + b}{i}\right) e^{-b},
\]

with the product calculated factor by factor. This extends the allowed size of the parameters of our program, with the disadvantage of a larger computing time. For \(\mu + b + n \leq 230\) the factor \((\mu + b)^n\) is not too large, and can be directly calculated. In this case the factorials up to \(170! \sim 10^{307}\) are calculated at the beginning of the main program and stored in the array \texttt{fact} to save time, whereas for \(n > 170\) the expression of the Poisson formula is divided by \(\prod_{i=1}^{n} i\) factor by factor.

The quantity \(R\) in Eq. (4) is a ratio of probabilities and can be computed without any problem cancelling out the common factors and defining a function \(R\). (Defining \(\mu_{\text{best}}\) as \texttt{DIM(FLOAT(n),b)} instead of \texttt{MAX(0d0,FLOAT(n)-b)} as we do would not have improved the speed significantly.)

The core of the algorithm is the subroutine \texttt{NRANGE}, used to calculate the confidence intervals \([n_1(\mu; b), n_2(\mu; b)]\) for arbitrary \(\mu\) and \(b\). Its arguments are the variables \(r\mu\) \((\mu)\), \(b\) and the confidence level desired \(CL\). The output \(n1\) and \(n2\) is given in a \texttt{COMMON} block, together with a variable \(CLac\), the C.L. finally achieved (in general it is greater than \(CL\)) which is useful for other purposes. The discussion in the last Section simplifies the implementation of the algorithm considerably, because we have found that the values of \(n\) that maximize \(R(n; \mu, b)\) concentrate around \(\mu + b\). This improves the speed by an order of magnitude for large values of the parameters, since we do not need to calculate a large table \((n, P(n), R(n))\) and sort it. Instead, we know the maximum \(R\) is one of the two integers nearest to \(\mu + b\). We begin with \(n1=\text{INT}(r\mu+b), n2=n1+1\). If \(R(n1,r\mu,b)\) is larger than \(R(n2,r\mu,b)\) we take \(n1\), decrease \(n1\) and add \(P(n1,r\mu,b)\) to \(CLac\). Otherwise, we take \(n2\), increase \(n2\) and add \(P(n2,r\mu,b)\) to \(CLac\). Repeating this until \(CLac\) is greater than \(CL\) and taking into account that \(n1\) must be greater than zero we obtain the desired interval \([n1, n2]\). The singular case \(\mu = b = 0\) is treated separately. With the \texttt{Mathematica} version of this subroutine we can plot confidence belts like that in Fig. 1.

The calculation of the confidence interval \([\mu_1, \mu_2]\) is done using two functions \texttt{RMU1(n,b,CL)} and \texttt{RMU2(n,b,CL)}, where \(n\) is the experimental number of events. We discuss them in turn.

As we see in Fig. 1, the lower limit \(\mu_1\) is the minimum value of \(\mu\) such that \(n_2(\mu; b) = n_0\). Within our framework, the calculation is done looking for the minimum \(r\mu\) such that \(n2\) calculated with \texttt{NRANGE(rmu,b,CL)} equals \(n\). The search is done with the bisection method. Starting with the limits \(r\mu_{\text{min}}=0d0, r\mu_{\text{max}}=\text{FLOAT}(n)-b+1d0\) (\texttt{RMU1} must be between these two values) we calculate the midpoint of the interval, \(r\mu_{\text{med}}\), and \texttt{NRANGE(rmu_{med},b,CL)}. If \(n2\) is
greater or equal than \( n \), we move \( \text{rmumax} \) to \( \text{rmumed} \), otherwise we move \( \text{rmumin} \) to \( \text{rmumed} \). This is repeated until the length of the interval is smaller than the desired precision \( \delta \), which we take as the maximum of 0.01 and 0.0005 times the background \( b \). We summarize the algorithm in Fig. 2.

Figure 2: Flux diagram for the calculation of \( \text{RMU1} \).

In principle the calculation of \( \text{RMU2} \) would follow an analogous procedure. However, in this case we find an extra problem. Except for a few cases with small \( b \), the confidence belt is not as simple as in Fig. 1 but displays a more elaborated structure as can be seen in the example of Fig. 3.

The fact that \( n_1(\mu; b) \) is not a monotonic function of \( \mu \) is due to the discreteness of \( n \). This causes that the set of \( \mu \) values for which the vertical line at \( n = n_0 \) intersects the belt is not connected for \( n_0 = 0 - 4 \). The effect is relevant since the upper limit \( \mu_2 \) is defined as the largest \( \mu \) for which \( n_1(\mu; b) = n_0 \). Thus a modification of the algorithm is required not to miss the small wedges in the function.

For \( n \geq b \) the behaviour is as expected and we can use the same algorithm as for \( \mu_1 \). We have checked values of \( b \) between 0 and 50 and have found that for \( n \geq b \) the function \( n_1(\mu; b) \) does not have any singularity, so in this case we can safely adapt the routine \( \text{RMU1} \). We look for the maximum \( \text{rmu} \) such that \( n_1 \) calculated with \( \text{NRANGE}(\text{rmu}, b, \text{CL}) \) equals \( n \). The search is again done with the bisection method. We start with the initial values \( \text{rmumin} = \text{MAX}(0d0, \text{FLOAT}(n) - b) \), \( \text{rmumax} = 3d0 \cdot \text{SQRT}(\text{FLOAT}(n) + b + 1d0) \). (If \( \text{rmumax} \) is not sufficiently large, we increment it in steps of \( \text{SQRT}(\text{FLOAT}(n) + b + 1d0) \).) We calculate the midpoint of the interval, \( \text{rmumed} \), and calculate \( \text{NRANGE}(\text{rmumed}, b, \text{CL}) \). If \( n_1 \) is lower or equal to \( n \), we move \( \text{rmumin} \) to \( \text{rmumed} \), otherwise we...
move \textit{rmumax} to \textit{rmumed}. This is repeated until the length of the interval is smaller than the desired precision \textit{delta}.

For $n < b$ we sample the interval for possible singularities, which consumes more time. This is done in three iterations with increasing number of points. To minimize the length of the interval and optimize the density of the sampling, we take \textit{rmumin} = \text{MAX}(0.0, \text{FLOAT}(n) - b) and increase it in steps of 1 while $n_1$ is lower or equal to $n$. As the upper limit we take \textit{rmumax} = $3.0 \times \text{SQRT}(\text{FLOAT}(n) + b + 1.0)$, sufficiently high so that the initial interval contains all the singularities for a C. L. of 0.99 or less.

In the first step we divide the interval between \textit{rmumin} and \textit{rmumax} in 10 parts and check if any of the points selected has $n_1$ lower or equal to $n$. If it is so, we change \textit{rmumin} to the largest of them (this is always safe) and start again with this new \textit{rmumin}. This first sampling with a small number of points finds wedges like those for $n = 1, 3$ in Fig. 3 and saves a lot of computing time. The narrow wedges at $n = 0, 2$ require more dense samplings.

If the points calculated have $n_1$ greater than $n$, we check the upper half of the interval for singularities. The second iteration divides the upper half in 20 parts and checks 19 points. If it finds any singular point with $n_1$ lower or equal to $n$, it changes \textit{rmumin} and starts again at the first step. If not, the third iteration divides the upper half in 500 parts. If a singular point is found, it changes \textit{rmumin} and starts the first step. If not, unless some kind of singular behavior is found (in which case a fourth sampling with 5000 points is performed) it is assumed that there do not exist singularities and \textit{rmumax} is changed.

An additional speed improvement is implemented: if in the second or third iterations the

Figure 3: Confidence belt for $b = 7$, with $\alpha = 0.95$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\end{figure}
density of points is sufficiently high (the points are closer than `stepmin`), the number of points is decreased and no more iterations are performed. The flux diagram of `RMU2` is shown in Fig. 4.

![Flux diagram for the calculation of RMU2.](image)

**Figure 4:** Flux diagram for the calculation of `RMU2`.

To simplify changing the parameters of this routine, the number of iterations and their respective number of points are stored in the variables `maxit` and `maxdivs`. The calculation for `n ≥ b` is done with the same function with `maxdivs(1)=2` and only the first iteration.

One may notice that some upper limits obtained with `RMU2` are different from those quoted in Ref. [2]. This is again a consequence of the discreteness of `n`. The upper limit `μ_2` for `n_0` fixed is not always a decreasing function of `b` (dotted lines in Fig. 3). This behaviour is corrected in Ref. [2] forcing the function to be nonincreasing, calculating the upper limit `μ_2` from `b = 25` to `b = 0` in steps of −0.001. The corrected value is then the maximum of `μ_2(n_0, x)` for `x ≥ b`. Some people, however, find this ad hoc correction questionable [4]. At any rate we could also follow this procedure getting the same values of Ref. [2] and the solid line in Fig. 4. This requires a very long calculation (25000 different values of `b`), for instance the time to calculate the `n_0 = 0` line is 34 m 27 s.

Of course, to obtain `μ_2` for a particular `b` it is not necessary to calculate the whole interval `[b, 20]` and it is enough to consider approximately `[b, b + 1]`. For this purpose we use the function

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Figure 5: Dependence of $\mu_2$ on the expected background $b$ for fixed $n_0$. The solid and dotted lines are obtained with and without correction, respectively.

RMU2c. This routine examines the behaviour of RMU2 in the interval $[b, b + 1]$ and corrects the value if necessary. The adjacent maxima that can be seen in Fig. 5 are found with the simple golden section search of Ref. [7]. (Other more sophisticated methods offer no advantage since the function does not seem to be differentiable at the maximum.) The initial bracketing of the maximum is very delicate as can be also seen in this Figure. We do it examining RMU2($n, b_1, CL$) taking $b_1$ with increments of 0.1 until RMU2 begins to grow, then in increments of 0.05 until it begins to decrease. Then the golden section method is applied to find the maximum with a precision of 0.001. This maximum is then compared to the value at $b$ to take the largest value.

This method again brings a substantial speed improvement over the blind computation in steps of $-0.001$ in $b$, as we will see in next Section, Examining the behaviour of $\mu_2$ we have also found that the correction is not necessary in general for $n > b/2$ and the function RMU2 could be used directly. There are however some exceptions, for instance $n = 10, b = 14$ with a C. L. of 0.95. To be conservative, we will only use RMU2 when $n \geq b$.

4 Results

To obtain our results we use the same precision that is used in Ref. [2], a minimum step $\text{stepmin}$ of 0.005 in $\mu$ and an accuracy of 0.01 in the upper and lower limits of the confidence intervals.
To calculate the limits in their Tables II–IX including the singular cases it is enough to consider in the third iteration $\text{maxdivs}(3)=100$ for confidence levels of 68.7%, 90% and 95%, and $\text{maxdivs}(3)=300$ for a C. L. of 99%. For better comparison we use $\text{maxdivs}(3)=500$ as we do in the rest of the calculations to ensure that all singularities are found. The running time is summarized in Table 1.

| C. L. | $t_1$ (s) | $t_2$ (s) |
|-------|-----------|-----------|
| 68.7% | 4.1       | 1 m 9.0 s |
| 90%   | 6.1       | 1 m 28.6 s|
| 95%   | 6.8       | 1 m 36.8 s|
| 99%   | 7.9       | 1 m 57.4 s|

Table 1: Time spent in the calculation of the confidence intervals for $n_0 \leq 20$ and $b \leq 15$, with RMU2 ($t_1$) and with RMU2c ($t_2$).

To check if our algorithm in fact handles large numbers efficiently we measure the time spent to calculate the upper limit $\mu_2$ with RMU2 for $n_0 = b$ between 0 and 200, obtaining the solid line in Fig. 6. We can also use RMU2c forcing the program to look for unexisting spurious maxima in $b$ (and hence also for singularities with $\text{maxdivs}(3)=500$) obtaining the dotted line.

![Figure 6: Time required to calculate the upper limit $\mu_2$ with the function RMU2 ($t_1$) and with RMU2c forcing to look for adjacent maxima ($t_2$) as explained in the text.](image-url)
It is amazing to observe that the computing time not only does not grow quickly with $b$ as it could be expected, but remains almost constant for $b < 75$. This is achieved with (i) a fast algorithm to find the singularities if they exist, (ii) the optimization of \texttt{NRANGE} to calculate only the data really needed, and (iii) the calculation of the factorials up to 170! at the beginning of the program. For $b \geq 75$ $\texttt{rmu} + b + \texttt{FLOAT}(n)$ is sometimes larger than 230 and the time required begins to grow linearly with $b$ after the gap between 75 and 100, as can be observed in the Figure.

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A Appendix

\begin{verbatim}
PROGRAM DEMO
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION FACT(0:170)
COMMON /range/ n1,n2,CLac
COMMON /factorial/ FACT
DATA CL /0.99d0/
FACT(0)=1d0 ! FACT initialization
DO i=1,170
   FACT(i)=FACT(i-1)*DFLOAT(i)
ENDDO

C CALCULATION OF TABLES VIII, IX OF REF[2]

DO b=0d0,4d0,0.5d0
   DO n=0,20
      PRINT 100,n,b,RMU1(n,b,CL),RMU2c(n,b,CL)
   ENDDO
ENDDO
DO b=5d0,15d0,1d0
   DO n=0,20
      PRINT 100,n,b,RMU1(n,b,CL),RMU2c(n,b,CL)
   ENDDO
\end{verbatim}
DOUBLE PRECISION FUNCTION P(n, rmu, b)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION FACT(0:170)
COMMON /factorial/ FACT
IF ((rmu .EQ. 0d0) .AND. (b .EQ. 0d0)) THEN ! These lines
   P=0 ! are not
   IF (n .EQ. 0) P=1 ! needed if P is
      RETURN ! only called
ENDIF ! from NRANGE
IF (rmu+b+FLOAT(n) .LE. 230d0) THEN
   P=(rmu+b)**n*EXP(-rmu-b)
   IF (n .LE. 170) THEN
      P=P/FACT(n)
   ELSE
      P=P/FACT(170) ! This is not normally used because for
      DO i=171,n ! n > 170 rmu+b+FLOAT(n) will be larger
         P=P/FLOAT(i) ! than 230
      ENDDO
   ENDIF
ELSE
   P=EXP(-rmu)
   DO i=1,n
      P=P*(rmu+b)/FLOAT(i)
   ENDDO
   P=P*EXP(-b)
ENDIF
RETURN
END

DOUBLE PRECISION FUNCTION R(n, rmu, b)
IMPLICIT REAL*8 (A-H,O-Z)
IF (n .LT. 0) THEN
   R=0d0
   RETURN
ENDIF
RETURN
END
RETURN
ENDIF
R=EXP(MAX(0d0,FLOAT(n)-b)-rmu)
IF (n .GT. 0) R=R*((rmu+b)/(MAX(0d0,FLOAT(n)-b)+b))**n
RETURN
END

SUBROUTINE NRANGE(rmu,b,CL)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON /range/ n1,n2,CLac ! CLac for future use
IF ((rmu .EQ. 0d0) .AND. (b .EQ. 0d0)) THEN
  n1=0d0 ! Special case
  n2=0d0
  RETURN
ENDIF
n1=INT(rmu+b) ! The maximum R is between
n2=n1+1 ! these values
r1=R(n1,rmu,b)
r2=R(n2,rmu,b)
CLac=0d0
DO WHILE ((CLac .LT. CL) .AND. (n1 .GE. 0))
  IF (r1 .GT. r2) THEN
    CLac=CLac+P(n1,rmu,b)
    n1=n1-1
    r1=R(n1,rmu,b)
  ELSE
    CLac=CLac+P(n2,rmu,b)
    n2=n2+1
    r2=R(n2,rmu,b)
  ENDIF
ENDDO
DO WHILE (CLac .LT. CL) ! No need to calculate R
  CLac=CLac+P(n2,rmu,b)
  n2=n2+1
ENDDO
n1=n1+1
n2=n2-1
RETURN
END
DOUBLE PRECISION FUNCTION RMU1(n,b,CL)  
IMPLICIT REAL*8 (A-H,O-Z)  
COMMON /range/ n1,n2,CLac  
CALL N RANGE(0d0,b,CL)  
IF (n2 .GE. n) THEN  
   RMU1=0d0  
   RETURN  
ENDIF  
RMU1=0d0  
rmumax=FLOAT(n)-b+1d0  
delta=MAX(0.01d0,0.0005d0*b)  
DO WHILE ((rmumax-rmumin) .GE. delta)  
   rmumed=(rmumin+rmumax)/2d0  
   CALL N RANGE(rmumed,b,CL)  
   IF (n2 .GE. n) THEN  
      rmumax=rmumed  
   ELSE  
      rmumin=rmumed  
   ENDIF  
ENDDO  
RMU1=(rmumin+rmumax)/2d0  
RETURN  
END  

DOUBLE PRECISION FUNCTION RMU2(n,b,CL)  
IMPLICIT REAL*8 (A-H,O-Z)  
DIMENSION maxdivs(4)  
LOGICAL safe,safenow,sing,changemin  
COMMON /range/ n1,n2,CLac  
DATA maxdivs /10,20,500,5000/  
maxit=4  
stepmin=0.005d0  
rmumin=MAX(0d0,FLOAT(n)-b)  
CALL N RANGE(rmumin,b,CL)  
IF (FLOAT(n) .LT. b) THEN  
   DO WHILE (n1 .LE. n)  
      rmumin=rmumin+1d0  
   ENDW  
   RETURN  
END
CALL NRANGE(rmumin,b,CL)
ENDDO
rmumin=rmumin-1d0
safe=.FALSE.
ELSE
   maxdivs(1)=2 ! Use bisection method when
   safe=.TRUE. ! there aren't sing.
ENDIF
rmumax=3d0*SQRT(FLOAT(n)+b+1d0) ! Large enough for most purposes
CALL NRANGE(rmumax,b,CL)
DO WHILE (n1 .LE. n) ! If not, increase it
   rmumax=rmumax+SQRT(FLOAT(n)+b+1d0)
   CALL NRANGE(rmumax,b,CL)
ENDDO
delta=MAX(0.01d0,0.0005d0*b)
DO WHILE ((rmumax-rmumin) .GE. delta)
   step=(rmumax-rmumin)/FLOAT(maxdivs(1))
   rmumin2=rmumin
   DO i=1,maxdivs(1)-1
      CALL NRANGE(rmumin+FLOAT(i)*step,b,CL)
      IF (n1 .LE. n) rmumin2=rmumin+FLOAT(i)*step
   ENDDO
   IF (rmumin2 .GT. rmumin) THEN
      rmumin=rmumin2 ! New rmumin -> change it
   ELSE
      safenow=safe ! Have to look for singularities
      sing=.FALSE. ! if they may exist
      changemin=.FALSE.
      it=2
      DO WHILE ((safenow .EQ. .FALSE.) .AND. (it .LE. maxit))
         ndivs=maxdivs(it)
         step=(rmumax-rmumin)/FLOAT(2*ndivs)
         IF (step .LT. stepmin) THEN ! step is small
            ndivs=INT((rmumax-rmumin)/(2*stepmin))+1 ! enough and this
            step=(rmumax-rmumin)/FLOAT(2*ndivs) ! will be the
            safenow=.TRUE. ! last iteration
         ENDIF
      CALL NRANGE((rmumin+rmumax)/2d0,b,CL)
      n_prev=n1
      DO i=1,ndivs-1

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CALL N RANGE(rmumin+FLOAT(i+ndivs)*step,b,CL)
IF (n1 .LE. n) THEN
   rmumin2=rmumin+FLOAT(i+ndivs)*step    ! New rmumin
   changemin=.TRUE.
ENDIF
IF (n1 .LT. n_prev) sing=.TRUE.
n_prev=n1
ENDDO
IF (changemin .EQ. .TRUE.) THEN
   rmumin=rmumin2    ! Change rmumin
   safenow=.TRUE.    ! Exit loop
ELSE
   IF ((sing .EQ. .FALSE.) .AND. (it .EQ. maxit-1)) THEN
      safenow=.TRUE.    ! Enough iterations
   ENDIF
ENDIF
ENDDO
it=it+1    ! Next iteration
ENDDO
IF (changemin .EQ. .FALSE.) rmumax=(rmumin+rmumax)/2d0
ENDIF
ENDDO
RMU2=(rmumin+rmumax)/2d0
RETURN
END

DOUBLE PRECISION FUNCTION RMU2c(n,b,CL)
IMPLICIT REAL*8 (A-H,O-Z)
LOGICAL sing
PARAMETER (R=0.61803399d0,C=1d0-R)
RMU2c=RMU2(n,b,CL)
IF (n .GE. INT(b)) RETURN    ! Do not need correction
step1=0.1d0    ! Go downhill in steps of 0.1
step2=0.05d0    ! Go uphill in steps of 0.05
deltab=0.001d0    ! Final precision in b
b1max=b+1d0    ! Look for maximum up to b+1
sing=.FALSE.

C Bracket the maximum, if any
b1=b
a1=RMU2c
DO WHILE ((b1 .LE. b1max) .AND. (sing .EQ. .FALSE.)) ! Go downhill
    a_next=RMU2(n,b1+step1,CL)
    IF (a_next .GT. a1) THEN
        sing=.TRUE.
    ELSE
        a1=a_next
        b1=b1+step1
    ENDIF
ENDDO
IF (sing .EQ. .FALSE.) RETURN ! RMU2 is always decreasing
b2=b1+step1-step2
a2=a_next
DO WHILE (a_next .GE. a2) ! Go uphill
    a2=a_next
    b2=b2+step2
    a_next=RMU2(n,b2+step2,CL)
ENDDO
b4=b2+step2
a4=a_next
IF (RMU2c .GT. a2+0.05d0) RETURN ! This maximum will not be larger
IF (b4-b2 .GT. b2-b1) THEN
    b3=b2+C*(b4-b2)
    a3=RMU2(n,b3,CL)
ELSE
    b3=b2
    a3=a2
    b2=b3-C*(b3-b1)
    a2=RMU2(n,b2,CL)
ENDIF
C Find the maximum
DO WHILE (b4-b1 .GE. deltab)
    IF (a3 .GT. a2) THEN
        b1=b2
        b2=b3
        b3=R*b2+C*b4
        a1=a2
    ELSE
...
a2=a3
a3=RMU2(n,b3,CL)
ELSE
b4=b3
b3=b2
b2=R*b3+C*b1
a4=a3
a3=a2
a2=RMU2(n,b2,CL)
ENDIF
ENDDO
RMU2c=MAX(RMU2c,a2,a3)
RETURN
END

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