Finite generation of the log canonical ring for 3-folds in char p

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Abstract. We prove that the log canonical ring of a klt pair of dimension 3 with $\mathbb{Q}$-boundary over an algebraically closed field of characteristic $p > 5$ is finitely generated. In the process we prove log abundance for such pairs in the case $\kappa = 2$.

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1. Introduction

We will work over an algebraically closed field $k$.

The existence of good log minimal models for klt threefold pairs in char $p > 5$ is proven in [2] when $K_X + B$ is big and generalised in [3] to the case where only $B$ is big. For klt threefold pairs with arbitrary boundary this would follow from [2] and the log abundance conjecture.

Conjecture 1.1 (Log abundance). Let $(X, B)$ be a klt threefold pair over a field of char $p > 5$ such that $K_X + B$ is nef. Then $K_X + B$ is semi-ample.

Log abundance is proved for log canonical threefolds in characteristic zero over a sequence of papers: of Miyaoka [12], [13] and [14], Kawamata [6] and Keel, Matsuki and McKernan [9]. In positive characteristic it is still open. A corollary of log abundance is the following, which we prove here.

Theorem 1.2. Let $(X, B)$ be a klt threefold pair over a field of char $p > 5$ with $\mathbb{Q}$-boundary $B$. Then the log-canonical ring $R(K_X + B) = \bigoplus H^0([m(K_X + B)])$ is finitely generated.

Finite generation can be proved in characteristic zero by using a canonical bundle formula to reduce to lower dimension (see [5]). This seems harder to
obtain in positive characteristic due to inseparability and wild ramification. Instead we prove finite generation via a special case of Conjecture 1.1:

**Theorem 1.3.** Let \((X, B)\) be a klt threefold pair over a field of char \(p > 5\) with \(\mathbb{Q}\)-boundary \(B\), such that \(\kappa(X, K_X + B) = 2\) and \(K_X + B\) is nef. Then \(K_X + B\) is semi-ample.

The proof follows that used in the proof of log abundance in [10]. We run suitable LMMPs with scaling to reduce to the case where the EWM map for \(K_X + B\) has equidimensional fibres. We then show that the image of the EWM map is a projective variety by comparing it to a subvariety of a Chow variety, show that \(\mathbb{Q}\)-factoriality is preserved under such a morphism, and finally deduce that \(K_X + B\) is the pullback of a divisor which must therefore be ample.

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2. Preliminaries

2.1. **Algebraic spaces.** We need to know that some standard results about morphisms of schemes also hold for maps of algebraic spaces. These are:

**Lemma 2.1** (Stein Factorisation). Let \(S\) be a scheme, and \(f : X \to Y\) be a proper morphism of algebraic spaces over \(S\) with \(Y\) locally Noetherian. Then \(f\) can be factorised into a map with connected geometric fibres followed by a finite map.

*Proof.* [[15], A018]

**Lemma 2.2** (Zariski’s Main Theorem I). Let \(f : X \to Y\) be a quasi-finite and separated map of algebraic spaces over a scheme \(S\), such that \(Y\) is quasi-compact and quasi-separated. Then there exists a factorisation \(X \to T \to Y\) so that \(X \to T\) is a quasi-compact open immersion and \(T \to Y\) is finite.

*Proof.* [[15], 05W7]

For a birational map of algebraic spaces \(X \dasharrow Y\) we can define the total transform of a subspace in exactly the same way as we would for schemes, it is \(p_2(p_1^{-1}(Z))\) where \(p_1\) and \(p_2\) are the projections from the graph of the map.

**Corollary 2.3** (Zariski’s Main Theorem II). Let \(f : X \dasharrow Y\) be a birational map of proper algebraic spaces. If \(P\) is a fundamental point of \(f\) then its total transform \(f(P)\) is connected and has positive dimension.

*Proof.* This can be deduced from Lemma 2.1 and Lemma 2.2 applied to the projection from the graph of the birational map.

2.2. **Chow varieties.** Inseparable field extensions cause complications to arise in the theory of Chow varieties, which mean the results are weaker in positive characteristic. This subsection summarizes the results which still hold. The subject is treated in [11].
Definition 2.4 (Algebraic cycle) An $n$-dimensional algebraic cycle in a scheme $X$ over $k$ is a formal linear combination of $n$-dimensional reduced and irreducible subschemes of $X$ over $k$.

Definition 2.5 (Well defined family of algebraic cycles, [[11], I.3.10]) A well defined family of algebraic cycles in $X$ consists of a reduced base scheme $Z$ and a closed subscheme $U$ of $X \times_k Z$ together with the projection morphism $g : U \to Z$ such that:

- $U = \sum m_i[U_i]$ is an algebraic cycle
- $g$ is proper
- Every component of $U$ maps onto an irreducible component of $Z$, and every fibre is either $n$-dimensional or empty
- A final technical condition must be satisfied: we omit description of it as it is automatically satisfied when the base is normal.

There are two additional conditions on families of algebraic cycles which appear only in positive characteristic. They are automatically satisfied in characteristic zero.

Definition 2.6 ([[11], I.4.7]) A family of algebraic cycles satisfies the Chow-field condition if for every $z \in Z$, the intersection of all fields of definition of the cycle corresponding to $z$ is equal to $k(z)$.

A family of algebraic cycles satisfies the field of definition condition if for every $z \in Z$, the cycle corresponding to $z$ is defined over $k(z)$.

If the base of a family is normal, that family satisfies the Chow-field condition. These conditions lead to three attempts to define the Chow functor:

Definition 2.7 ([[11], I.4.11]) Suppose $X$ is a scheme over a field $k$. For every $k$-scheme $Z$ define the following sets. Note that only the last is a functor.

\[
Chow_{big}(X)(Z) = \left\{ \text{Well defined proper algebraic families of non-negative cycles of } X \times Z/k \right\}
\]

\[
Chow(X)(Z) = \left\{ \text{Well defined proper algebraic families of non-negative cycles of } X \times Z/k \text{ satisfying the Chow field condition} \right\}
\]

\[
Chow_{small}(X)(Z) = \left\{ \text{Well defined proper algebraic families of non-negative cycles of } X \times Z/k \text{ satisfying the field of definition condition} \right\}
\]

$Chow_{big}(X)(Z)$ and $Chow(X)(Z)$ agree when $Z$ is normal. $Chow_{small}$ is a functor, but $Chow$ is only a partial functor as pullbacks do not always remain within it.

Theorem 2.8 ([[11], I.4.13]). There is a scheme $Chow_{d,d'}(X)$ which coarsely represents both $Chow_{d,d'}(X)$ and $Chow_{small}(X)$

Although $Chow$ is not a functor, it has a universal family in most cases.
Theorem 2.9 ([11],I.4.14). Let $X$ be a projective scheme over a field $k$ and $V$ an irreducible component of $\text{Chow}_{d,d'}(X)$ of positive dimension. Then there exists a universal family $U \in \text{Chow}_{d,d'}(X)(V)$.

2.3. Reduction maps.

Theorem 2.10. Let $X$ be a variety over an uncountable field, and $L$ a nef $\mathbb{R}$-divisor. There exists a rational map $f : X \to Z$, called the nef reduction map, with the following properties:

- $f$ is proper over an open subset $U$ of $Z$
- $L|_F \equiv 0$ on very general fibres over $U$
- A curve $C$ through a very general point $x \in X$ satisfies $C \cdot L = 0$ if and only if $C$ is contracted by $f$

The existence of the nef reduction map was proven for nef line bundles in characteristic zero in [1] and it was noted that the same proof applies in the more general situation above in [4]. When the nef reduction map exists, the dimension of its image is called the nef dimension of $L$, and is denoted $n(X, L)$. It satisfies $\kappa(X, L) \leq \nu(X, L) \leq n(X, L)$.

Definition 2.11 ([8, 0.4.1]) A nef line bundle $L$ on a scheme $X$ is Endowed With Map (EW) if there is a proper map $f : X \to Z$ to an algebraic space such that $f$ contracts a subvariety $Y$ if and only if $L|_Y$ is not big. We may always assume that such a map has geometrically connected fibres.

2.4. Pullbacks. Here we quote some results relating contractions and nef divisors.

Lemma 2.12. Let $f : X \to Z$ be a projective contraction between normal quasi-projective varieties over $k$ and $L$ a nef $\mathbb{R}$-divisor on $X$ such that $L|_F \sim_\mathbb{R} 0$ where $F$ is the generic fibre of $f$. Assume $\dim Z \leq 3$ if $k$ has char $p > 0$. Then there exist a diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\phi} & X \\
\downarrow{f'} & & \downarrow{f} \\
Z' & \xrightarrow{\psi} & Z
\end{array}
$$

with $\phi, \psi$ projective birational, and an $\mathbb{R}$-Cartier divisor $D$ on $Z'$ such that $\phi^*L \sim_\mathbb{R} f'^*D$. Moreover, if $Z$ is $\mathbb{Q}$-factorial, then we can take $X' = X$ and $Z' = Z$.

Proof. [3, 5.6]. See also [7, 2.1].

Lemma 2.13. Let $X$ be a normal projective variety over an uncountable $k$ of char $p > 0$. Suppose $L$ is a nef $\mathbb{Q}$-divisor on $X$ with equal Kodaira and nef dimensions $\kappa(L) = n(L) \leq 2$. Then $L$ is endowed with a map $X \to V$ to a proper algebraic space $V$ of dimension equal to $\kappa(L)$.

Proof. [3, 7.2]
3. Equidimensional morphisms

Definition 3.1 (Pullback of a Weil divisor) Let $X$ and $Y$ be normal varieties and $f : X \to Y$ a morphism with equidimensional fibres. For a Weil divisor $D$ on $Y$, its pullback to $X$, denoted $D_X$, is defined as follows. First let $Y_0$ be the smooth locus of $Y$ and $X_0$ be its pre-image by $f$. $D|_{Y_0}$ is Cartier, so the pullback $f^*|_{X_0}(D|_{Y_0})$ is a well defined Cartier divisor on $X_0$. The complement of $X_0$ in $X$ has codimension at least 2, and so this pullback extends uniquely to a Weil divisor $D_X$ on $X$.

Proposition 3.2. Let $X$ and $Y$ be normal varieties over $k$ and $f : X \to Y$ a morphism with equidimensional fibres. Suppose $X$ is $\mathbb{Q}$-factorial. Then $Y$ is also $\mathbb{Q}$-factorial.

Proof. Let $D$ be an irreducible Weil divisor on $Y$. Given a closed point $y \in Y$, let $F_1, ..., F_n$ be the irreducible components of the fibre $F$ over $y$. Choose points $x_i \in F_i$ so that $x_i \notin F_j$ for all $j \neq i$. We wish to find $\phi$ so that $D_X$ is $\text{div}(\phi)$ on an open subvariety $U$ containing $x_1, ..., x_n$. As we are interested in showing that some multiple of $D$ is Cartier we are free to replace $D$ with a multiple. Whenever we do this it will be assumed that we also replace $D_X$ and any rational functions currently defining them.

Replacing $D$ with a multiple there is a function $\phi$ defining $D_X$ in a neighbourhood of $x_1$. We may assume for induction that $\phi$ defines $D_X$ in a neighbourhood of $\{x_1, ..., x_{i-1}\}$. Suppose there is a component $E$ of $\text{Supp}(\text{div}(\phi))$ which passes through $x_i$ and has different coefficients as a component of $\text{div}(\phi)$ and $D_X$. By assumption this component cannot pass through any of $\{x_1, ..., x_{i-1}\}$. As $X$ is $\mathbb{Q}$-factorial there is a function $\theta$ which defines some multiple of $E$ in a neighbourhood of $x_i$. By replacing $D$ with a multiple and $\theta$ by an appropriate power we may assume $D - \text{div}(\phi) - \text{div}(\theta)$ has zero $E$ coefficient. We may write $\text{div}(\theta) = kE + E'$ where $E \notin \text{Supp}(E')$. Some multiple of $E'$ is Cartier so after replacing $D$ and $\theta$ by a further multiple we can find $\psi$ such that $E' = A_1 - A_2 - \text{div}(\psi)$, where $A_1$ and $A_2$ are very ample divisors whose support does not contain any of $x_1, ..., x_i$. Now $\text{div}(\phi \cdot \theta \cdot \psi)$ and $D_X$ agree in one more component than $\text{div}(\phi)$ and $D_X$ did, and $\text{div}(\phi \cdot \theta \cdot \psi) = \text{div}(\phi)$ in neighbourhoods of $x_1, ..., x_{i-1}$. Thus by Noetherian induction we may replace $\phi$ so that $\text{div}(\phi)$ is equal to $D_X$ at $\{x_1, ..., x_i\}$. By induction there is $\phi$ which defines $D_X$ on an open subvariety containing each of $x_1, ..., x_n$. In particular $\phi$ defines $D_X$ in a neighbourhood containing a dense open subset of the fibre over $y$.

Now intersect $X$ with general hyperplanes to obtain a morphism $f|_Z : Z \to Y$ from normal variety $Z$ which is finite over a neighbourhood of $y$. As the hyperplanes were general we may shrink $Z$ in a neighbourhood of $f^{-1}(y)$ to assume that $D_Z = \text{div}(\phi)$ and $f|_Z$ is finite.

As the problem is local we may assume $Z = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ respectively. We have $f : \text{Spec}(A) \to \text{Spec}(B)$, which will be induced by a finite algebra map $\psi : B \to A$. We may shrink further to assume $\phi$ is an element of $A$. It then defines a $B$-linear map $\tilde{\phi} : A \to A$ by multiplication.
Define the norm $N(\phi)$ of $\phi$ to be the determinant of this linear map, which is an element of $B$. We now claim that some multiple of $D$ is defined by $N(\phi)$.

Let $P \in \text{Spec}(B)$, and choose $Q \in \text{Spec}(A)$ such that $f(Q) = P$ (so that $\psi^{-1}Q = P$ as ideals). The $B/P$ linear map $\tilde{\phi}_q : A/Q \to A/Q$ induced by multiplication by $\phi$ has determinant equal to $N(\phi)(\text{mod } P)$. Therefore if $N(\phi) \in P$ then $\det(\tilde{\phi}) = 0$ so there is some $0 \neq s' \in A/Q$ such that $\tilde{\phi}_q(s) = 0$. This lifts to $s \in A\backslash Q$ such that $\phi \cdot s \in Q$. As $Q$ is prime this implies $\phi \in Q$. Conversely if $\phi \in Q$ then $\tilde{\phi}_q$ is the zero map and so $N(\phi) \in P$. Thus $\text{div}(N(\phi))$ must be supported on $D$ and hence is equal to some multiple of $D$ as $D$ is irreducible.

Thus we have shown that $D$ is $\mathbb{Q}$-Cartier in a neighbourhood of the point $y$, and $y$ was arbitrary so $D$ is $\mathbb{Q}$-Cartier.

\[\square\]

4. Proof of main results

Proof of 1.3. Firstly we can replace $X$ by a small crepant $\mathbb{Q}$-factorialisation by [2, 1.6]. Furthermore we may extend the base field to assume it is uncountable.

Lemma 4.1. Let $X$ be a normal projective variety of dimension $n$ over an uncountable field with nef $\mathbb{Q}$-Cartier divisor $D$ such that $\kappa(X, D) = n - 1$. Then $n(X, D) = n - 1$.

Proof. As $\kappa = n - 1$ there is a positive integer $m$ such that $mD$ is Cartier and $\phi := \phi|_{mD} : X \dashrightarrow \mathbb{P}^N$ has $n - 1$ dimensional image $Y$. Sections $D' \in |mD|$ correspond to hyperplanes in $\mathbb{P}^N$ under this embedding. Given a general point $y \in Y$, such that $f$ is defined on all of the fibre over $y$, let $H_1, \ldots, H_{n-1}$ be hyperplanes intersecting only at $y$. Let $D_1, \ldots, D_{n-1}$ be the corresponding elements of $|mD|$. As $D$ is nef but not not big, $D^n = 0$. $D_1 \cdot \ldots \cdot D_{n-1}$ is an effective 1-cycle supported on the closure of the fibre over $y$, and by the nefness of $D$, $D \cdot C = 0$ for each curve in this fibre. We have covered an open subset of $X$ with $D$-trivial curves and so $n(X, D) \leq n - 1$. The result follows from the inequality $\kappa(X, D) \leq n(X, D)$. \[\square\]

We may now apply Theorem 2.10 to obtain an almost proper nef reduction map $f : X \dashrightarrow Z$ to a smooth surface $Z$. Also Lemma 2.13 implies that $K_X + B$ is EWM to a proper algebraic space $V$.

As $\kappa(K_X + B) = 2$ we may choose $M \geq 0$ such that $K_X + B \sim_{\mathbb{Q}} M$. $M$ is numerically trivial on a very general fibre of $f$ so it must be vertical over $Z$. Therefore by Lemma 2.12 there is birational $\phi : W \to X$, contraction $f : W \to Z$ and big divisor $D$ on $Z$ such that $\phi^*(K_X + B) \sim_{\mathbb{Q}} f^*D$. We may replace $W$ and $Z$ with higher models in order to assume $W \to Z$ is flat. Although they may now have bad singularities, replacing $D$ with its pullback still gives a well defined Cartier divisor on $Z$ satisfying the above relation.

The next two lemmas use similar arguments to those in the characteristic zero proof of abundance in [10, Ch. 15].
Lemma 4.2. There is a sequence of \( K_X + B \)-trivial flips and divisorial contractions leading to a model \((X', B')\) for \((X, B)\) which is also EWM to \(V\), and so that no divisor on \(X'\) is contracted to a point on \(V\).

Proof. We have the following set-up:

\[
\begin{array}{c}
W \\
\phi \\
\downarrow \phi \\
X \\
\downarrow f \\
\phi(X) = V \\
\end{array}
\]

As \( D \) is big we may change \( D \) up to \( \mathbb{Q} \)-linear equivalence so that \( D = A + E \) where \( A \) is ample and effective, \( E \) is effective and the two share no components. Also there is an effective \( \mathbb{Q} \)-divisor \( M \) such that \( M \sim_\mathbb{Q} K_X + B \) and \( \phi^* M = f^* D \).

Suppose we are not already in the situation we require, so there is some reduced irreducible Weil divisor \( F \) on \( X \) contracted to a point by \( g \), so every curve on \( F \) is \( K_X + B \)-trivial. Let \( F_W \) be the birational transform of \( F \) on \( W \). As \( f \) is flat, \( f(F_W) = \Gamma \) is 1-dimensional. By construction \( \Gamma \cdot D = 0 \). \( A \) was ample so \( \Gamma \cdot A > 0 \) hence \( \Gamma \cdot E < 0 \) and so \( \Gamma \) is a component of \( \operatorname{Supp}(E) \). Therefore \( \operatorname{Supp}(f^* E) \subset \operatorname{Supp}(\phi^* M) \) contains \( F_W \), so \( F \) is contained in \( \operatorname{Supp}(M) \).

Let \( \Gamma_W \) be a general curve in \( F_W \) which is surjective to \( \Gamma \), in particular \( \Gamma_W \) should be contained in no component of \( \operatorname{Supp}(f^* D) \) besides \( F_W \). The projection formula gives \( \Gamma_W \cdot \phi^* M = 0 \), \( \Gamma_W \cdot f^* E < 0 \) and \( \Gamma_W \cdot f^* A > 0 \). Thus there is some component \( A_W \) of \( \operatorname{Supp}(f^* A) \) such that \( \Gamma_W \cap A_W \neq \emptyset \). \( A_W \) is not contracted over \( X \) because every divisor which is exceptional over \( X \) is contained in \( \operatorname{Supp}(f^* E) \). \( \operatorname{Supp}(E) \cap \operatorname{Supp}(A) \) contains no divisor so as \( f \) is flat \( \operatorname{Supp}(f^* E) \cap \operatorname{Supp}(f^* A) \) also contains no divisor.

The image of \( \Gamma_W \) on \( X \), \( \Gamma_X \), is 1-dimensional as the general curves of which \( \Gamma_W \) is one cover \( F_W \) and \( F_W \) is not contracted over \( X \). We know \( \Gamma_X \cdot M = 0 \), and if \( A_X \) is the birational transform of \( A_W \) then \( \Gamma_X \cdot A_X > 0 \). Therefore we must have \( \Gamma_X \cdot F < 0 \).

Run a \( K_X + B + \epsilon F \)-MMP with scaling of some ample divisor for \( \epsilon \) sufficiently small, which terminates by [3, 1.6]. We claim that every step is \( K_X + B \)-trivial. This holds for the first step because any curve with \( K_X + B + \epsilon F < 0 \) is contained in \( F \) and so is contracted over \( V \). This implies that \( K_X + B \) is the pullback of a \( \mathbb{Q} \)-Cartier divisor on the contracted variety, and hence so is the new \( K_{X'} + B' \) after the contraction or flip. Hence \( K_{X'} + B' \) is also nef, and is still EWM to \( V \). The birational transform of \( F \) is still contracted over \( V \). Thus the same argument applies inductively to each step.

The LMMP terminates on some model \((X', B')\), and \( K_{X'} + B' \) is EWM to \( V \). The divisor \( F \) must have been contracted during the LMMP, as otherwise its birational transform \( F' \) would be covered by \( K_{X'} + B' + \epsilon F' \)-negative curves.

After repeating this procedure finitely many times we arrive at the model described in the statement. \( \square \)

Lemma 4.3. \( V \) is a projective variety.
Proof. Let \((X, B)\) be the pair constructed in Lemma 4.2, so \(K_X + B\) is nef and EWM with associated map \(g : X \to V\) which contracts no divisor to a point. As \(X\) is reduced we may assume \(V\) carries its reduced induced structure. Finally we can replace \(V\) by its normalisation.

There is a smooth open subvariety \(U\) of \(V\) such that \(V \setminus U\) has codimension 2, for which \(g_U : X_U = g^{-1}(U) \to U\) is a morphism of varieties. As \(V\) is reduced, \(g_U\) is a well defined family of algebraic cycles. It is therefore an element of \(\text{Chow}^{\text{big}}(U)\). As \(U\) is normal it is also an element of \(\text{Chow}(U)\), which is enough to ensure that there is a morphism \(U \to \text{Chow}(X)\) by Theorem 2.8. \(U\) is separated and \(\text{Chow}(X)\) is proper, therefore \(U \to \text{Chow}(X)\) is separated.

As each point of \(U\) represents a cycle supported on a different locus of \(X\), the image of \(U\) is contained in a component \(C\) of \(\text{Chow}(X)\) of positive dimension. By Theorem 2.9 there is a universal family \(h : U \to C\). Let \(Z\) be the closure of the image of \(U\) in \(\text{Chow}(X)\) with reduced scheme structure, and \(X'\) the inverse image of \(Z\) in \(U\). The morphism \(X' \to Z\) has geometrically connected fibres.

The morphism \(U \to Z\) is quasi-finite as every fibre over a \(k\) point contains either 1 or 0 points. Apply Zariski’s Main Theorem 2.2 to this morphism, to factorise it as \(U \to \hat{Z} \to Z\) where \(U \to \hat{Z}\) is an open immersion and \(\hat{Z} \to Z\) is finite. Let \(\hat{X} = X \times_k \hat{Z}\). Finally let \(Z^\nu\) be the normalisation of \(\hat{Z}\) and \(X^\nu = Z^\nu \times_{\hat{Z}} \hat{X}\). We get an open immersion \(U \to Z^\nu\) and \(X^\nu \to Z^\nu\) with 1-dimensional fibres and purely 3-dimensional \(X^\nu\).

The universal family \(U\) is a subvariety of \(X \times_k C\), so comes with a natural projection to \(X\). By composition we get a natural projection \(\pi : X^\nu \to X\).

\[
\begin{array}{cccccccc}
X & \to & U & \to & X' & \to & X^\nu & \to & X \\
\downarrow & & \downarrow h & & \downarrow h' & & \downarrow h^\nu & & \downarrow g \\
C & \to & Z & \to & Z^\nu & \to & U & \to & V
\end{array}
\]

We have a birational map between \(Z^\nu\) and \(V\), and we apply Zariski’s Main Theorem 2.3 to show this is an isomorphism.

Firstly as \(V \setminus U\) is of codimension at most 2, \(Z^\nu \to V\) has no fundamental points.

Suppose \(P\) is a fundamental point of \(V \dashrightarrow Z^\nu\). Let \(\Gamma\) be a general curve through \(P\). We may assume \(\Gamma\) passes through no other point of \(V \setminus U\). Let \(\Gamma^\nu\) be its birational transform on \(Z^\nu\). By choosing \(\Gamma\) to be general we determine that \(\Gamma^\nu\) intersects the total transform of \(P\) at a general finite set of points of the total transform of \(P\).

We can pull back the universal family from \(X^\nu \to Z^\nu\) to \(Y^\nu \to \Gamma^\nu\). \(Y^\nu\) is a geometrically connected 2-dimensional cycle, and all but a 1-dimensional subvariety is over \(U\). Let \(Y\) be the pre-image of \(\Gamma\) by \(g\).

For dimensional reasons, the image of \(Y^\nu\) on \(X\) is supported on the closure of the image of \(X_U \cap Y^\nu\), and this is equal to \(\overline{X_U} = g^{-1}(\Gamma) = Y\). The image of \(X_U \cap Y^\nu\) is supported precisely on the union of the supports of the cycles corresponding to the points of \(\Gamma_U\). The supports of these cycles cover all of \(Y\) except for \(g^{-1}(P)\). We know that the image of the projection from \(Y^\nu\) to \(X\)
must be supported on all of $Y$. Therefore given some component of the support of $g^{-1}(P)$, it must be contained in the support of some cycle represented by a point $Q \in \Gamma^v \setminus \Gamma_U$. Suppose that the cycle represented by $Q$ also contains some other $1$-dimensional subscheme of $Y$ not contained in $g^{-1}(P)$. This is impossible as that subscheme could not be contained in any $2$-dimensional component of $Y^v$ (as each of the cycles near it are disjoint from one another but supposedly intersect this cycle). Thus one of our general points is supported only on the support of $g^{-1}(P)$. But there are only countably many cycles supported there and uncountably many points of the total transform of $P$, and we have seen that a general finite collection of these points contains at least one point from the countable set. This implies that the total transform of $P$ cannot be a curve and so $V \cong Z^v$.

\[ \square \]

$X$ is $\mathbb{Q}$-factorial so by Proposition 3.2, $V$ is $\mathbb{Q}$-factorial and so by Lemma 2.12 $K_X + B \sim g^*D$ for some big and nef divisor $D$. $K_X + B$ is EWM with map $g$, so $D$ is big and strictly nef on a surface. Therefore Kodaira’s lemma and the Nakai-Moishezon criterion imply that $D$ is ample.

\[ \square \]

Proof of 1.2. The case $\kappa = 3$ is proved in [2, 1.3], $\kappa = 2$ follows from Theorem 1.3, $\kappa = 1$ from Lemma 4.4 below, and $\kappa = 0$ and $\kappa = -\infty$ are obvious.

Lemma 4.4. Let $X$ be a projective normal variety and $L$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ with $\kappa(X, L) = 1$. Then the section ring $\oplus_k H^0(X, [kL])$ is finitely generated.

Proof. It is enough to show that $R = \oplus_k H^0(X, [kL])$ is finitely generated for any $a$, so we may freely replace $L$ with a higher power. By doing so we may assume $L$ is effective and Cartier, and that $H^0(X, L)$ contains an element linearly independent from $1$.

There are natural strict inclusions

$$H^0(X, \mathcal{O}_X) \subset H^0(X, L) \subset H^0(X, 2L) \subset \ldots$$

$\mathcal{B}^0 = \{1\}$ is a $k$ basis for $H^0(X, \mathcal{O}_X)$. Let $\mathcal{B}^0 = \{1_L\} \subset H^0(X, L)$, and let $\mathcal{B}^1$ be a collection of functions completing $\mathcal{B}^0$ to a basis of $H^0(X, L)$. Fix some $x \in \mathcal{B}^1$. Inductively we may assume $\mathcal{B}_i = \bigcup_{j=0}^i \mathcal{B}_j^i$ is a basis of $H^0(X, iL)$, let $\mathcal{B}_{i+1}^j$ be the vectors in $1_L \cdot \mathcal{B}_j^i$ for each $j \in \{0, \ldots, i\}$, and take $\mathcal{B}_{i+1}^j$ to complete $\bigcup_{j=0}^i \mathcal{B}_{i+1}^j$ to a basis of $H^0(X, (i + 1)L)$. We claim that we may choose $\mathcal{B}_{i+1}^i$ to contain $x \cdot \mathcal{B}^i$. This is equivalent to saying that

$$\text{Span}(\{x \cdot \mathcal{B}_i^j\}) \cap \text{Span}(\{1_L \cdot \mathcal{B}_i\}) = 0$$

and $x \cdot \mathcal{B}_i^j$ is linearly independent. For the first assertion, there can be no such non-zero element by comparing the order of the zero at $L$ implied by membership of the two subspaces. The second assertion follows from the linear independence of $\mathcal{B}_i^i$ and that $K(X)$ is a domain.

Multiplication by $1 \in \mathcal{B}^0$ induces the natural injection among these basis vectors. By construction, $x \in \mathcal{B}_1^1$ also induces an injection among them. In
particular we see that $|B_k| - |B_{k-1}| \leq |B_{k+1}| - |B_k|$, and so the sequence of integers $|B_k| - |B_{k-1}|$ is increasing.

The condition $\kappa(L) = 1$ implies that by possibly replacing $L$ with a higher multiple there are constants $A$ and $B$ such that

$$Ak \leq H^0(X, kL) = \sum_{j=0}^{k} (|B_j| - |B_{j-1}|) \leq Bk$$

This implies that the sequence stabilises at an integer at most $B$, and so the injections $x : B_k^k \to B_{k+1}^{k+1}$ are bijections for $k$ large enough.

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