Calabi-Yau structure and Bargmann type transformation on the Cayley projective plane

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Abstract. Our purpose is to show the existence of a Calabi-Yau structure on the punctured cotangent bundle \( T_0^* (P^2 \mathbb{O}) \) of the Cayley projective plane \( P^2 \mathbb{O} \) and to construct a Bargmann type transformation from a space of holomorphic functions on \( T_0^* (P^2 \mathbb{O}) \) to \( L_2 \)-space on \( P^2 \mathbb{O} \). The space of holomorphic functions corresponds to the Fock space in the case of the original Bargmann transformation. A Kähler structure on \( T_0^* (P^2 \mathbb{O}) \) was given by identifying it with a quadrics in the complex space \( \mathbb{C}^{27} \setminus \{ 0 \} \) and the natural symplectic form of the cotangent bundle \( T_0^* (P^2 \mathbb{O}) \) is expressed as a Kähler form. Our construction of the transformation is the pairing of polarizations, one is the natural Lagrangian foliation given by the projection map \( q : T_0^* (P^2 \mathbb{O}) \to P^2 \mathbb{O} \) and another is the polarization given by the Kähler structure.

The transformation gives a quantization of the geodesic flow in terms of one parameter group of elliptic Fourier integral operators whose canonical relations are defined by the graph of the geodesic flow action at each time. It turns out that for the Cayley projective plane the results are not same with other cases of the original Bargmann transformation for Euclidean space, spheres and other projective spaces.

1. Introduction

The fundamental and historical problem in the quantization theory will be how to assign a function on a phase space to an operator acting on the space of quantum states and the assignment satisfies some algebraic condition, like a Lie algebra homomorphism. The phase space appearing in the theory has a structure, a symplectic structure. There are many theory relating with this problem. One method is the theory of deformation quantization. Also there is the opposite theory, an assignment of operators to functions, from an operator to a function. In the (pseudo)differential operator theory and Fourier integral operator theory, the basic assignment of operators to their principal symbol (and sub-principal symbol) is a fundamental isomorphism between the spaces of operators and functions on the phase space modulo lower order classes.

The famous transformation, called Bargmann transformation was introduced in [Ba] and gives one aspect of the quantization of the unitary representation. The method to construct such a transformation is given by the pairing of two polarizations, real polarization and complex polarization, on \( \mathbb{C}^n \) interpreted as \( \mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n \), complex space and fiber space by Lagrangian fibers \( \pi : \mathbb{C}^n \to \mathbb{R}^n \). Under precise treatments of this method it was given a similar operator for the case of the sphere in [Ra2], in [FY] for the complex projective space and for the quaternion projective spaces in [Fu1].

Among the projective spaces the Cayley projective plane \( P^2 \mathbb{O} \) is the exceptional one and our purpose in this paper is to show that we can also construct such an operator for this manifold in the same method. This case will be one of the non-trivial examples to which we can apply this method, ”pairing of two polarizations” ([Ra1], [Hi1], [Hi2], [Fu1], [FY]).

In the paper [Fu2] a Kähler structure on its punctured cotangent bundle \( T_0^* (P^2 \mathbb{O}) \) was constructed by embedding it into the complex space \( \mathbb{C}^{27} \setminus \{ 0 \} \) as an intersection of null sets of several quadric polynomials, which gives the realization of the natural symplectic form as a Kähler form. Here we show the holomorphic triviality of the canonical line bundle of this complex manifold by giving a nowhere vanishing global holomorphic 16-form explicitly.

There are several study of the existence of Kähler structure on the (punctured) cotangent bundle of a certain class of manifolds, like [Ra1], [Sz1], [Sz2], [Koi], [Li], [FT], also see [Be], [So] in relation with a special property of the geodesic flow, \( SC(\mathbb{C}) \)-manifolds.

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The classical Bargmann transformation gives a correspondence between monomials on \( \mathbb{C}^n \) and Hermite functions on \( \mathbb{R}^n \), which are the eigenfunctions of the harmonic oscillator and this facts were applied to various problems, especially to Töplitz operator theory (there are so many, but here I just cite one book [BS]). Also there are many precise treatments and modifications of this transformation (for examples, [Hi1], and recent works in [Ch1], [Ch2]).

For our case we show the restrictions of monomials defined on \( \mathbb{C}^{27} \setminus \{0\} \) to the embedded punctured cotangent bundle \( T_0^*(P^2\Omega) \) are mapped to eigenfunctions of the Laplacian on \( P^2\Omega \).

This paper is organized as follows. In §2, we explain a realization of quaternion and octonion number fields, \( \mathbb{H} \) and \( \mathbb{O} \), in a complex matrix algebra. Multiplication law in the octonion is interpreted in the two \( 2 \times 2 \)-complex matrix algebra \( \mathbb{C}(2) \times \mathbb{C}(2) \).

In §3, we introduce the Jordan algebra \( J(3) \) of \( 3 \times 3 \) octonion matrices. Cayley projective plane \( P^2\Omega \) is realized in this Jordan algebra. Following an earlier result in [Fu2] we explain the embedding of the punctured cotangent bundle \( T_0^*(P^2\Omega) \) of the Cayley projective plane into the complexified Jordan algebra \( \mathbb{C} \otimes_{\text{R}} J(3) =: J(3)^{\mathbb{C}} \) of \( 3 \times 3 \) complexified octonion matrices:

\[
\tau_{\mathbb{O}} : T^*(P^2\Omega) \rightarrow J(3)^{\mathbb{C}}.
\]

We denote the image \( \tau_{\mathbb{O}}(T^*(P^2\Omega)) = \mathbb{X}_\mathbb{O} \). Also we state that the natural symplectic form \( \omega^{P^2\Omega} \) is a Kähler form.

In §4, using the defining equations of the punctured cotangent bundle of the Cayley projective plane embedded in the complex Jordan algebra \( J(3)^{\mathbb{C}} \), we give an open covering by complex coordinates neighborhoods and show by an elementary way that the canonical line bundle of the complex structure is holomorphically trivial by explicitly constructing a nowhere vanishing holomorphic global section (we put it \( \Omega_\mathbb{O} \), that is, a 16-degree holomorphic differential form which coincides with the restriction of a smooth 16-degree differential form on the whole complexified Jordan algebra \( J(3)^{\mathbb{C}} \).

In §5, we resume a basic fact on symplectic manifolds with integral symplectic form and a method of the geometric quantization. Here we consider two types of typical polarizations (real and positive complex). Then we apply the method to our case (= \( T_0^*(P^2\Omega) \)) and give a Bargmann type transformation in the form of a fiber integration on the punctured cotangent bundle \( T_0^*(P^2\Omega) \) to the base space \( P^2\Omega \).

In §6, first we show the nowhere vanishing holomorphic global section \( \Omega_\mathbb{O} \) constructed in §4 is \( F_4 \)-invariant. Incidentally, we determine the product \( \Omega_\mathbb{O} \wedge \Omega_{\mathbb{O}} \) in terms of the Liouville volume form \( dV_{T^*(P^2\Omega)} := \frac{1}{16!} \left( \omega^{P^2\Omega} \right)^{16} \) of the cotangent bundle \( T^*(P^2\Omega) \).

Also we introduce a class of subspaces consisting of holomorphic functions on \( \mathbb{X}_\mathbb{O} \) satisfying some \( L_2 \) conditions. These will correspond to the Fock space in the Euclidean case.

In §7, we determine the exterior product of the Riemann volume form pull-backed to the cotangent bundle \( T_0^*(P^2\Omega) \) and the nowhere vanishing global holomorphic section \( \Omega_\mathbb{O} \) in terms of the Liouville volume form. For this purpose we fix a local coordinates at a point in \( P^2\Omega \) which is also used in the section §9.

In §8, we discuss invariant polynomials and a similar feature to harmonic polynomials with respect to the natural representation of the group \( F_4 \) to the Jordan algebra \( J(3) \) and its extension to the polynomial algebra. Then, based on a general theorem in [He] (also [HL] and [Kos]) we state the eigenfunction decomposition of \( L_2 \) space of \( P^2\Omega \).

In §9, based on the data obtained until §8 we discuss our Bargmann type transformation is a bounded operator, or isomorphism or unbounded according to the Hilbert space structures in the Fock-like space. Some cases says there are quantum states in \( L_2(P^2\Omega) \) which are approximated by classical phenomena, but can not be observed directly by classical mechanical way.

Finally in §10 we mention that our Fock-like spaces have the reproducing kernel and a relation with the geodesic flow action.
2. Representations of quaternion and octonion algebras by complex matrix algebras

First, we fix a representation of quaternion numbers $h = h_01 + h_1i + h_2j + h_3k$ (1, i, j, k) are standard basis of the quaternion number field $\mathbb{H}$ and $h_i \in \mathbb{R}$ as a $2 \times 2$ complex matrix in the following way:

$$\rho_{\mathbb{H}} : \mathbb{H} \ni h \mapsto \begin{pmatrix} h_0 + h_1\sqrt{1} & h_2 + h_3\sqrt{1} \\ -h_2 + h_3\sqrt{1} & h_0 - h_1\sqrt{1} \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \in \mathbb{C}(2),$$

where we understand that quaternions $h_01 + h_1i$ and $h_2 + h_3i \in \mathbb{H}$ are complex numbers $\lambda = h_0 + \sqrt{1}h_1$ and $\mu = h_2 + h_3\sqrt{1} \in \mathbb{C}$ respectively. Hence by this representation the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ is isomorphic to the “algebra” of the whole $2 \times 2$ complex matrix algebra $\mathbb{C}(2)$ (we put $z_i = x_i + \sqrt{-1}y_i \in \mathbb{C}$):

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \ni h = z_01 + z_1i + z_2j + z_3k \mapsto \begin{pmatrix} z_0 + \sqrt{-1}z_1 & z_2 + \sqrt{-1}z_3 \\ -z_2 + \sqrt{-1}z_3 & z_0 - \sqrt{-1}z_1 \end{pmatrix} \in \mathbb{C}(2).$$

We denote this map also by $\rho_{\mathbb{H}}$ and the inverse map is

$$\rho_{\mathbb{H}}^{-1} : \mathbb{C}(2) \ni A = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \rho_{\mathbb{H}}^{-1}(A) = \frac{z_1 + z_4}{2}1 + \frac{z_1 - z_4}{2}i + \frac{z_2 - z_3}{2}j + \frac{z_2 + z_3}{2}\sqrt{-1}k.$$

For $h = h_01 + h_1i + h_2j + h_3k \in \mathbb{H}$ (or $\in \mathbb{C} \otimes \mathbb{H}$), we denote its conjugation by $\theta(h) = h_01 - h_1i - h_2j - h_3k$, then for $\rho_{\mathbb{H}}(h) = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$, $\rho_{\mathbb{H}}(\theta(h)) = \begin{pmatrix} w_4 & -w_2 \\ -w_3 & w_1 \end{pmatrix}$ and the product $\rho_{\mathbb{H}}(\theta(h))\rho_{\mathbb{H}}(h) = \rho_{\mathbb{H}}(h)\rho_{\mathbb{H}}(\theta(h)) = (w_1w_4 - w_2w_3) \cdot Id = det_{\mathbb{H}}(h) \cdot Id$, where Id is the $2 \times 2$ identity matrix.

Let $\{e_i\}_{i=0}^3$ be the standard basis of the octonion number field $\mathbb{O}$ such that $e_0$ is the basis of the center. We identify $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ with the basis $\{1, i, j, k\}$ of the quaternion number field. By the multiplication law $e_ie_4 = e_{i+4}$ ($i = 0, 1, 2, 3$) we express an (complexified) octonion number $x = \sum x_ie_i$ as the sum of two quaternion numbers:

$$x = \sum_{i=0}^3 x_ie_i + \sum_{i=0}^3 x_{i+4}e_i = a + b \cdot e_4 \in \mathbb{H} \oplus \mathbb{H}e_4 \text{ or } \mathbb{C} \otimes \mathbb{H} \oplus \mathbb{C} \otimes \mathbb{Re}_4.$$

The complexification $\mathbb{C} \otimes \mathbb{O}$ is identified as

$$\mathbb{C} \otimes \mathbb{O} \cong \mathbb{C}(2) \oplus \mathbb{C}(2)e_4$$

through the map $\rho_{\mathbb{O} \otimes} = \rho_{\mathbb{O}}$.

We define the conjugation operation in $\mathbb{O}$ (and also in $\mathbb{C} \otimes \mathbb{O}$) with the same notation $\theta$ for the quaternion case as

$$\theta : h = \sum h_ie_i \mapsto \theta(h) = h_01 - \sum_{i=1}^7 h_ie_i.$$

The conjugation $\theta$ is interpreted in the matrix representation through the representation $\rho_0$ as

$$\theta : \mathbb{C}(2) \oplus \mathbb{C}(2)e_4 \ni Z + We_4 = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}e_4$$

$$\mapsto \theta(Z + We_4) = \theta(Z) - We_4 = \begin{pmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{pmatrix} - \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}e_4.$$

**Remark 1.** The multiplication law of the octonions in the matrix form is given in (4.4).

**Remark 2.** We use the conjugation $\overline{z} = x - \sqrt{-1}y$ only for the complex number $z = x + \sqrt{-1}y$ and do not use the operation $\theta$ for the conjugate of complex numbers to avoid confusion. So, for a complex octonion number $z = \sum \{z\}e_i$, $\{z\} \in \mathbb{C}$, we mean $\overline{z} = \sum \{\overline{z}\}e_i$ and it holds $\theta(\overline{z}) = \overline{\theta(z)}$. Also for an octonion matrix $A = \begin{pmatrix} z_{ij} \end{pmatrix}$ we mean $\overline{A} := \begin{pmatrix} \overline{z_{ij}} \end{pmatrix}$ and $\theta(A) := \begin{pmatrix} \theta(z_{ij}) \end{pmatrix}$. 
3. Cayley projective plane and its punctured cotangent bundle

In this section, we refer [SV], [Mu] and [Yo] for all the necessary facts on the exceptional group $F_4$ and the Cayley projective plane.

Let $\mathcal{J}(3)$ be a subspace of the $3 \times 3$ octonion matrices:

$$\mathcal{J}(3) = \left\{ \begin{pmatrix} t_1 & z & \theta(y) \\ \theta(z) & t_2 & x \\ y & \theta(x) & t_3 \end{pmatrix} \mid x, y, z \in \mathbb{O}, t_i \in \mathbb{R} \right\}.$$ 

We introduce a product in $\mathcal{J}(3)$, called a “Jordan product”, by

$$\mathcal{J}(3) \times \mathcal{J}(3) \ni (A, B) \mapsto A \circ B := \frac{AB + BA}{2} \in \mathcal{J}(3).$$

It is called an exceptional Jordan algebra and of 27-dimensional over $\mathbb{R}$. Then the group of $\mathbb{R}$-linear algebra automorphisms is the exceptional Lie group $F_4$:

$$F_4 := \{ g \in GL(\mathcal{J}(3)) \cong GL(27, \mathbb{R}) \mid g(A \circ B) = g(A) \circ g(B), \ g(Id) = Id, \ A, B \in \mathcal{J}(3) \}.$$ 

There are various characterizations for the group $F_4$ (see for examples, [Yo], [SV]).

The complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{J}(3) =: \mathcal{J}(3)^C$ consists of $3 \times 3$ matrices with components of the complexified octonions of the form:

$$\mathcal{J}(3)^C = \left\{ \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \mid x, y, z \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}, \xi_i \in \mathbb{C} \right\}$$

and is an exceptional Jordan algebra over $\mathbb{C}$ of the complex dimension 27. The group of $\mathbb{C}$-linear algebra automorphisms is the complex Lie group $F_4^C$:

$$F_4^C := \{ g \in GL(\mathcal{J}(3))^C \cong GL(27, \mathbb{C}) \mid g(A \circ B) = g(A) \circ g(B), \ g(Id) = Id, \ A, B \in \mathcal{J}(3)^C \}.$$ 

We may regard $F_4 \subset F_4^C$ in a natural way.

**Definition 3.1.** The Cayley projective plane $P^2 \mathbb{O}$ is defined as

$$P^2 \mathbb{O} = \{ X \in \mathcal{J}(3) \mid X^2 = X, \ \text{tr}(X) = 1 \}.$$ 

It is known that the group $F_4$ acts on $P^2 \mathbb{O}$ in two point homogeneous way.

Let $X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in P^2 \mathbb{O}$, then it is known that the stationary subgroup of the point $X_1$ in $F_4$ is isomorphic to $Spin(9)$ and $F_4 \ni g \mapsto g \cdot X_1$ gives an isomorphism:

$$F_4/Spin(9) \cong P^2 \mathbb{O}.$$ 

For $X = \begin{pmatrix} \xi_1 & x_3 & \theta(x_2) \\ \theta(x_3) & \xi_2 & x_1 \\ x_2 & \theta(x_1) & \xi_3 \end{pmatrix}, \ Y = \begin{pmatrix} \eta_1 & y_3 & \theta(y_2) \\ \theta(y_3) & \eta_2 & y_1 \\ y_2 & \theta(y_1) & \eta_3 \end{pmatrix} \in \mathcal{J}(3)$, we define their inner product by

$$\langle X, Y \rangle_{\mathcal{J}(3)} := \text{tr}(X \circ Y) = \sum_{i=1}^3 \xi_i \eta_i + 2 \langle x_i, y_i \rangle_{\mathbb{R}^8},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^8}$ denotes the standard Euclidean inner product of $x_i$ and $y_i \in \mathbb{O} \cong \mathbb{R}^8$.

This inner product has a property

$$\langle X \circ Y, Z \rangle_{\mathcal{J}(3)} = \langle X, Y \circ Z \rangle_{\mathcal{J}(3)}, \ X, Y, Z \in \mathcal{J}(3).$$
In particular, since the trace function $\mathcal{J}(3) \ni A \mapsto \text{tr} (A)$ is invariant under the $F_4$ action, that is
\begin{equation}
(3.5) \quad \text{tr} (g \cdot A) = \text{tr} (A), \; g \in F_4, \; A \in \mathcal{J}(3),
\end{equation}
this inner product is invariant under the action by $F_4$ (hence $F_4$ can be seen as $F_4 \subset SO(27)$):
\begin{equation}
(3.6) \quad \langle g \cdot A, g \cdot B \rangle_{\mathcal{J}(3)} = \text{tr} (g \cdot A \circ g \cdot B) = \text{tr} (A \circ B) = \langle A, B \rangle_{\mathcal{J}(3)}.
\end{equation}

The tangent bundle $T(P^2\mathbb{O})$ is identified with a subspace in $\mathcal{J}(3) \times \mathcal{J}(3)$ such that
\begin{equation*}
T(P^2\mathbb{O}) = \left\{ (X, Y) \in \mathcal{J}(3) \times \mathcal{J}(3) \mid X \in P^2\mathbb{O}, \; X \circ Y = \frac{1}{2} Y \right\}.
\end{equation*}

We consider the Riemannian metric $g^{P^2\mathbb{O}}$ on the manifold $P^2\mathbb{O}$ being induced from the inner product in $\mathcal{J}(3)$: $g_X^{P^2\mathbb{O}}(Y_1, Y_2) := \langle Y_1, Y_2 \rangle_{\mathcal{J}(3)}$. $Y_1, Y_2 \in T_X(P^2\mathbb{O})$.

Using this metric, hereafter we identify the tangent bundle $T(P^2\mathbb{O})$ and the cotangent bundle $T^*(P^2\mathbb{O})$.

Let $Y_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_{X_1}(P^2\mathbb{O})$. The stationary subgroup at the point $(X_1, Y_1) \in T(P^2\mathbb{O})$ is known as being isomorphic to $\text{Spin}(7)$ and the two point homogeneity of the action by $F_4$ gives us the isomorphism $F_4/\text{Spin}(7) \cong S(P^2\mathbb{O})$, the unit (co)tangent sphere bundle of $P^2\mathbb{O}$.

The inner product on $\mathcal{J}(3)$, $(\cdot, \cdot)_{\mathcal{J}(3)}$, is extended to the complexification $\mathcal{J}(3)^c$ as a complex bi-linear form in a natural way, which we denote by $(\cdot, \cdot)_{\mathcal{J}(3)^c}$. Then the extension as the Hermitian inner product on the complexification $\mathcal{J}(3)^c$ is given by $(A, B)_{\mathcal{J}(3)^c}$, $A, B \in \mathcal{J}(3)^c$ (see Remark 2 for the matrix $B$).

We will denote the norm of $a \in \mathbb{O}$ by $|a| = \sqrt{(a, a)_{\mathcal{J}(3)^c}}$ and by $||X|| = \sqrt{(X, X)_{\mathcal{J}(3)^c}}$ the norm of $X \in \mathcal{J}(3)$, respectively. Also with the same way for elements $a \in \mathbb{C} \otimes \mathbb{R} \mathbb{O}$ and $A \in \mathcal{J}(3)^c$, we denote their norms.

The punctured cotangent bundle $T^*(P^2\mathbb{O})\setminus \{0\} =: T^*_0(P^2\mathbb{O})$ is realized as a subspace in $\mathcal{J}(3)^c$ with the following form:

**Theorem 3.2 ([Fu2]).** Let $X_\mathbb{O}$ be a complex quadric in $\mathcal{J}(3)^c$:
\begin{equation}
(3.7) \quad X_\mathbb{O} = \left\{ A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \mid x, y, z \in \mathbb{C} \otimes \mathbb{R} \mathbb{O}, \; \xi_i \in \mathbb{C}, \; A^2 = 0, \; A \neq 0 \right\}.
\end{equation}

Then the correspondence between $T^*_0(P^2\mathbb{O})$ and $X_\mathbb{O}$ is given by
\begin{equation}
(3.8) \quad \tau_0 : T^*_0(P^2\mathbb{O}) \cong T_0(P^2\mathbb{O}) \ni (X, Y) \mapsto \tau_0(X, Y) = 1 \otimes (||Y||^2 X - Y^2) + \sqrt{-1} \otimes \frac{||Y||Y}{\sqrt{2}}.
\end{equation}

Then
**Theorem 3.3 ([Fu2]).**
\begin{equation}
(3.9) \quad \tau_0^* \left( \sqrt{-2} \partial \partial ||A||^{1/2} \right) = \omega^{P^2\mathbb{O}},
\end{equation}
where we denote by $\omega^{P^2\mathbb{O}}$ the natural symplectic form on the cotangent bundle $T^*(P^2\mathbb{O})$.

The inverse $\tau_0^{-1}$ is given by
\begin{equation*}
\tau_0^{-1} : X_\mathbb{O} \ni A \mapsto (X, Y) = (X(A), Y(A)) \in \mathcal{J}(3) \times \mathcal{J}(3),
\end{equation*}
\begin{equation}
\begin{aligned}
X(A) &= \frac{1}{2||A||} \cdot (A + \overline{A}) + \frac{\Lambda^2 \overline{A}}{||A||^2}, \\
Y(A) &= -\frac{\sqrt{-1}}{\sqrt{2}} \cdot ||A||^{-1/2} (A - \overline{A}).
\end{aligned}
\end{equation}

4. Complex coordinate neighborhoods and Calabi-Yau structure

We denote the holomorphic part of the complexified cotangent bundle $T^* \mathcal{X}_\mathcal{O} \otimes \mathbb{C}$ by $T^* \mathcal{X}_\mathcal{O}^\mathbb{C}$ (and likewise $T^* (\mathcal{X}_\mathcal{O})^\mathbb{C}$ is the anti-holomorphic subbundle).

In this section we show that the canonical line bundle \( \bigwedge^6 T^* \mathcal{X}_\mathcal{O}^\mathbb{C} \) is holomorphically trivial by explicitly constructing a nowhere vanishing global holomorphic section (Theorem 4.7). For this purpose we consider an open covering by explicit coordinates neighborhoods and show that the Jacobians of the coordinates transformations is a coboundary form of \( \mathbb{C}^* \)-valued zero form.

The condition \( A^2 = 0 \) in (3.7) is expressed in the following six equations in terms of octonions:

\begin{align}
(\xi_3 + \xi_2)x + \theta(yz) &= 0, \\
(\xi_1 + \xi_3)y + \theta(zx) &= 0, \\
(\xi_2 + \xi_1)z + \theta(xy) &= 0,
\end{align}

The condition \( 0 \neq A \in \mathcal{X}_\mathcal{O} \) is equivalent to one of the components \( x, y, \) or \( z \) being non zero.

**Proposition 4.1.**

\[ \mathcal{X}_\mathcal{O} \ni A, \text{ then } \text{tr}(A) = \xi_1 + \xi_2 + \xi_3 = 0. \]

This property does not appear in an explicit form in (4.1) and (4.2) but plays an important role in §8. Although it is proved in [Fu2], we give an elementary proof based on the permitted regulations in the octonion.

**Proof.** Since the associativity

\[ a \cdot \theta(b) = a \theta(b) \]

holds, by multiplying \( z \) from the left to the equality \( (\xi_3 + \xi_2)x + \theta(yz) = 0 \) it holds the equality:

\[ z \cdot (\xi_3 + \xi_2)x + z \cdot \theta(z) = (\xi_3 + \xi_2)zx + z\theta(x) - (\xi_3 + \xi_2)(\xi_1 + \xi_4) \theta(y) + z\theta(z) \cdot \theta(y) = 0. \]

Hence if we assume \( y \neq 0 \)

\[ (\xi_3 + \xi_2)(\xi_1 + \xi_3) = z\theta(z) \]

and by the same way

\[ (\xi_2 + \xi_1)(\xi_3 + \xi_1) = \theta(x)x. \]

These imply that

\[ (\xi_3 + \xi_2)(\xi_1 + \xi_3) + (\xi_2 + \xi_1)(\xi_3 + \xi_1) + \xi_2^2 = (\xi_1 + \xi_2 + \xi_3)^2 = 0. \]

and we have

\[ \xi_1 + \xi_2 + \xi_3 = 0. \]

From the arguments above the same holds for other cases of \( x \neq 0 \) or \( z \neq 0. \]

**Remark 3.** The property above can be seen easily, if we use the transitivity of the action of the group \( F_4 \) on the (co)tangent sphere bundle.

Also from the definition of the map \( \tau_\mathcal{O} \), \( \text{tr}(A) = \text{tr}(\tau_\mathcal{O}(X,Y)) = 0 \) is equivalent to \( \text{tr}(Y) = 0. \)
Here we mention the following fact, which is a special case described in Proposition 8.20.

**Lemma 4.2.** Assume that a linear function \( f : \mathcal{J}(3)^C \to \mathbb{C} \)

\[
f(A) = 2 \sum_{i=0}^{7} (a_i \{z\}_i + b_i \{y\}_i + c_i \{x\}_i) + \sum_{i=1}^{3} \alpha_i \xi_i
\]

vanishes on \(X_0\) (see Remark 2 for the notation). Then \( f \) is a constant multiple of the trace function \( A \mapsto \text{tr}(A), \: A \in \mathcal{J}(3)^C\).

**Proof.** Put \( a = \sum a_i e_i, \: b = \sum b_i e_i, \: c = \sum c_i e_i \in \mathbb{C} \otimes \mathbb{R} \mathbb{O} \) and \( B = \begin{pmatrix} \alpha_1 & a & \theta(b) \\ \theta(a) & \alpha_2 & c \\ b & \theta(c) & \alpha_3 \end{pmatrix} \in \mathcal{J}(3)^C\).

Then

\[
f(A) = \text{tr}(A \circ B) := f_B(A).
\]

Let \( Y = \begin{pmatrix} 0 & z & \theta(y) \\ \theta(z) & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \in T_{X_1}(\mathbf{P}^2 \mathbb{O}), \) where \( X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \) \( z = \sum z_i e_i, \: y = \sum y_i e_i \in \mathbb{O}, \) then

\[
\tau_0(X_1, Y) = \begin{pmatrix} |z|^2 + |y|^2 & 0 & 0 \\ 0 & -|z|^2 - \theta(yz) \\ 0 & -yz & -|y|^2 \end{pmatrix} + \sqrt{|z|^2 + |y|^2} \begin{pmatrix} 0 & \sqrt{-1} z & \sqrt{-1} \theta(y) \\ \sqrt{-1} \theta(z) & 0 & 0 \\ \sqrt{-1} y & 0 & 0 \end{pmatrix} \in X_0.
\]

Let \( y = 0 \) and \( A = \tau_0(X_1, Y) \). Then, from the assumption, we have

\[
f_B(A) = \text{tr}(B \circ A) = (|z|^2)(\alpha_1 - \alpha_2) + 2 \sum \sqrt{-1}|z| z_i a_i = 0,
\]

for any \( \pm z_i \in \mathbb{R} \). Hence \( \alpha_1 = \alpha_2 \) and also \( a_i = 0 \) for \( i = 0, \cdots, 7 \). Likewise we have \( \alpha_1 = \alpha_3 \) and \( b_i = 0 \) for \( i = 0, \cdots, 7 \).

Then we may assume

\[
f_B(\tau_0(X_1, Y)) = 2(c, \theta(yz))^{\mathbb{R}^*} = 0 \text{ for any } y, z \in \mathbb{O}.
\]

Hence \( c_i = 0 \) for \( i = 0, \cdots, 7 \), which shows our assertion, that is \( a = b = c = 0, \: \alpha := \alpha_1 = \alpha_2 = \alpha_3 \) and

\[
f_B(A) = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 = \alpha \cdot \text{tr}(A).
\]

\( \Box \)

**Corollary 4.3.** The space spanned by \( X_0 := [X_0] \) is a 26-dimensional complex linear subspace in \( \mathcal{J}(3)^C \) and is equal to \( \text{tr}^{-1}(0) \).

Let \( z, y, x \in \mathbb{C} \otimes \mathbb{R} \mathbb{O} \) and put

\[
\rho_0(z) = Z + W e_4 = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} e_4,
\]

(4.3)

\[
\rho_0(y) = Y + V e_4 = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} + \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} e_4,
\]

\[
\rho_0(x) = X + U e_4 = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} e_4,
\]

where \( z_i, w_i, v_i, x_i, u_i \in \mathbb{C} \).
Then the conditions (4.1) and (4.2) are rewritten in terms of the matrices \( Z, Y, V, X, U \) as
\[
\begin{aligned}
\xi_1(\theta(X) - Ue_4) &= (Y + Ve_4)(Z + We_4) = YZ - \theta(W)V + (WY + V\theta(Z))e_4, \\
\xi_2(\theta(Y) - Ve_4) &= (Z + We_4)(X + Ue_4) = ZX - \theta(U)W + (UZ + W\theta(X))e_4, \\
\xi_3(\theta(Z) - We_4) &= (X + Ue_4)(Y + Ve_4) = XY - \theta(V)U + (VX + U\theta(Y))e_4,
\end{aligned}
\]

\[
\begin{aligned}
\xi_1^2 + \det Z + \det W + \det Y + \det V &= 0, \\
\xi_2^2 + \det Z + \det W + \det X + \det U &= 0, \\
\xi_3^3 + \det Y + \det V + \det X + \det U &= 0.
\end{aligned}
\]

Hereafter (until §7), we denote the matrix \( A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \in \mathcal{J}(3)^C \) in the form of a vector \( \xi_1, \xi_2, \xi_3, z_1, \ldots, z_4, w_1, \ldots, w_4, y_1, \ldots, y_4, v_1, \ldots, v_4, x_1, \ldots, x_4, u_1, \ldots, u_4 \) \( \in \mathbb{C}^{27} \), using the components given in (4.3) by the map \( \rho_0 \).

The conditions for matrices in \( \mathcal{X}_0 \) require that at least one of the off-diagonal components in the matrix \( A \) is non-zero. Hence, for example, we assume that there is at least one component in the matrix \( \rho_0(z) = Z + We_4 = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} e_4 \), say \( z_1 \neq 0 \) and put \( O_{z_1} = \{ A \in \mathcal{X}_0 \mid z_1 \neq 0 \} \). Also we define other open subsets \( \{ O_{z_1}, O_{w_1}, O_{y_1}, O_{v_1}, O_{x_1}, O_{u_1} \}_{i=1}^4 \) in a same way like \( O_{z_1} \). Then we have

**Proposition 4.4.** The 24 subsets
\[
\{ O_{z_1}, O_{w_1}, O_{y_1}, O_{v_1}, O_{x_1}, O_{u_1} \}_{i=1}^4 =: U_0
\]
are all open coordinate neighborhoods and totally is an open covering of \( \mathcal{X}_0 \).

**Proof.** We give a local coordinates on \( O_{z_1} \). Other cases will be given by the same way.

From the equations in (4.4) we select 5 equations expressed in \( 2 \times 2 \) complex matrices including the complex variable \( z_1 \) and from the equations in (4.5) we select one equation also including the complex variable \( z_1 \):
\[
\begin{align}
\xi_1 & \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} w_4 & -w_2 \\ -w_3 & w_1 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}, \\
-\xi_1 & \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \begin{pmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{pmatrix}, \\
-\xi_2 & \begin{pmatrix} y_4 & y_2 \\ -y_3 & y_1 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} u_4 & -u_2 \\ -u_3 & u_1 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}, \\
-\xi_2 & \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}, \\
-\xi_3 & \begin{pmatrix} z_4 & -z_2 \\ -z_3 & z_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} v_4 & -v_2 \\ -v_3 & v_1 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \\
-\xi_3 & \begin{pmatrix} x_4 & x_2 \\ -x_3 & x_1 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} v_4 & -v_2 \\ -v_3 & v_1 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.
\end{align}
\]

\[
\begin{aligned}
\xi_2^2 + z_1 z_4 - z_2 z_3 + w_1 w_4 - w_2 w_3 + x_1 x_4 - x_2 x_3 + u_1 u_4 - u_2 u_3 &= 0.
\end{aligned}
\]
From these we can select 10 equations including the variable $z_1$:

\[
\begin{align*}
    f_1 &= -\xi_2y_4 + z_1x_1 + z_2x_3 - (u_4w_1 - u_2w_3) = 0, \\
    f_2 &= \xi_2y_2 + z_1x_2 + z_2x_4 - (u_4w_2 - u_2w_4) = 0, \\
    f_3 &= \xi_2v_1 + u_1z_1 + u_2z_3 + (w_1x_4 - w_2x_3) = 0, \\
    f_4 &= \xi_2v_3 + u_3z_1 + u_4z_3 + (w_3x_4 - w_4x_3) = 0, \\
    f_5 &= -\xi_2z_1 + y_2z_3 - (u_4v_1 - w_2v_3) = 0, \\
    f_6 &= \xi_2z_3 + y_2z_2 + (w_2v_1 + w_1v_3) = 0, \\
    f_7 &= \xi_2u_2 - v_1z_2 + v_2z_1 + w_1y_2 + w_2y_4 = 0, \\
    f_8 &= \xi_2u_4 - v_1z_2 + v_4z_1 + w_3y_2 + w_4y_4 = 0, \\
    f_9 &= -\xi_3z_1 + x_3y_2 + (x_4y_4 - (v_3u_2 + v_1u_4) = 0, \\
    f_{10} &= \xi_3^2 + z_1z_4 - z_2z_3 + w_1u_4 - w_2u_4 + w_3x_4 - x_2x_3 + u_1u_4 - u_2u_3 = 0.
\end{align*}
\]

(4.9)

The 10 variables

\[(x_1, x_2, u_1, u_3, y_1, y_3, v_2, v_4, \xi_3, z_4)\]

are coefficients of the variable $z_1$, and can be solved in terms of the remaining 17 variables easily.

In fact, with one more additional equation

\[(4.10) \quad f_{11} = \xi_1 + \xi_2 + \xi_3 = 0,
\]

we can solve the 11 variables

\[(x_1, x_2, u_1, u_3, y_1, y_3, v_2, v_4, z_4, \xi_3, \xi_1)\]

in terms of the remaining 16 variables

\[(x_3, x_4, u_2, u_4, y_2, y_4, v_1, v_3, z_1, z_2, z_3, w_1, w_2, w_3, w_4, \xi_2)\]

in which, except $z_1 \neq 0$ other variables can take any values in $\mathbb{C}$.

Here, if we choose the equation

\[f_{10} = \xi_1^2 + z_1z_4 - z_2z_3 + w_1u_4 - w_2u_4 + y_1y_4 - y_3y_3 + v_1u_4 - v_2u_3 = 0,
\]

instead of the tenth equation $f_{10}$ in (4.9)( = first equation in (4.5)), then the variable $\xi_1$ should be chosen as an independent variable.

In any choice (in $O_{\tilde{z}_1}$ case, $\xi_1$ or $\xi_2$) once we fix them (here we choose as above), and denote by $P_{z_1}$ the projection map

\[
P_{z_1} : O_{\tilde{z}_2} \ni (\xi_1, \xi_2, \xi_3, z_1, \ldots, z_n, w_1, \ldots, w_4, y_1, \ldots, y_4, v_1, \ldots, v_4, x_1, \ldots, x_n, u_1, \ldots, u_4)
\]

\[= (a_1, \ldots, a_{27}) \mapsto (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4, y_2, y_4, v_1, v_3, x_3, x_4, u_2, u_4)
\]

\[= (a_2, a_4, a_5, a_6, a_8, a_9, a_{10}, a_{11}, a_{13}, a_{15}, a_{16}, a_{18}, a_{22}, a_{23}, a_{25}, a_{27}) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^{14}.
\]

Then, the pair $(O_{z_1}, P_{z_1})$ is a local coordinates neighborhood (note that dim$_{\mathbb{C}} X_0 = 16$).

In any case in $U_0$, once we fix independent variables, then we denote the dependent variables as $x_1(*)$, $x_2(*)$, $\ldots$, $(or \ a_1(*)$, $a_2(*)$, $\ldots)$, etc., where $*$ means the independent variables.

**Corollary 4.5.** Each coordinate neighborhood $O_{\xi}$ in $U_0$ is dense in $X_0$. Hence any number of intersections of open sets in $U_0$ is also open dense.

**Proof.** It will be enough to show the case $O_{x_1}$. So, let $A \subset X_0 \setminus O_{x_1}$. Assume, say $A \subset O_{x_2}$, then the subset $z_1 = 0$ is defined by an rational equation: $z_1 = \frac{\xi_2x_4 - z_2x_3 + y_2y_4 - y_3y_3}{x_1} = 0$. Hence the subset $z_1 = 0$ must be at most codimension 1 in $X_0$.

**Proposition 4.6.** Let $O_{u_i}$ and $O_{u_j}$ be any of two open coordinate neighborhoods in $U_0 = \{O_{u_i}\}_{i=1}^{24}$. 
Then the Jacobian \( J_{a_j, a_i} = \det d\left( P_{a_j} \circ P_{a_i}^{-1} \right) \) of the coordinate transformation \( P_{a_j} \circ P_{a_i}^{-1} \) is given by

\[
J_{a_j, a_i} = \left( \frac{a_j}{a_i} \right)^5 \text{ on } P_{a_i}(O_{a_i} \cap O_{a_i}).
\]

**Proof.** Let \( \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and define a map

\[
\tilde{\sigma} : \mathbb{C}(2) \ni S \mapsto \tilde{\sigma}(S) := \sigma \cdot S \cdot \sigma \in \mathbb{C}(2),
\]

then \( \theta(\tilde{\sigma}(S)) = \tilde{\sigma}(\theta(S)) \). This property of \( \tilde{\sigma} \) naturally induces an automorphism of \( \mathcal{J}(3)^{\mathbb{C}} \), which we denote by the same notation \( \tilde{\sigma} : \mathcal{J}(3)^{\mathbb{C}} \to \mathcal{J}(3)^{\mathbb{C}} \).

By Proposition 4.4, it will be enough to determine the Jacobian \( J_{z_1, a_i} \) for the cases of

\[
O_{z_1} \cap O_{a_1} = O_{a_i} \cap O_{a_i}, \quad \text{for } i \geq 5.
\]

Furthermore, by the symmetries of the components \( x, y, z \) and the automorphism \( \tilde{\sigma} \) explained above we see that it is enough to determine them for the 5 cases

\[
O_{z_1} \cap O_{z_2}, O_{z_1} \cap O_{z_3}, O_{z_1} \cap O_{z_4}, O_{z_1} \cap O_{z_5}, O_{z_1} \cap O_{z_2}, O_{z_2} \cap O_{z_1}.
\]

All the determinants can be done by the basic way of the calculation of the determinants. So we show two cases \( O_{z_1} \cap O_{z_2} \) and \( O_{z_1} \cap O_{z_5} \), how they look like.

**[I]** \( O_{z_1} \cap O_{z_2} = O_{a_i} \cap O_{a_i} \) case: For this case we consider the coordinate transformation \( P_{z_2} \circ P_{z_1}^{-1} \), which is given by the correspondence:

\[
(\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, w_4, y_2, y_1, v_1, v_3, x_3, x_4, u_2, u_4) \mapsto (\xi_2, z_1, z_2, z_4, w_1, w_2, w_3, w_4, y_2, y_4, v_2, v_4, x_1, x_2, u_2, u_4)
\]

where the coordinates \( (\xi_2, z_1, z_2, z_4, w_1, w_2, w_3, w_4, y_1, y_2, y_4, v_1, v_4, x_1, x_2, u_2, u_4) \) are given by the rational functions:

\[
\begin{align*}
\xi_2 &= \frac{x_2 y_4 - z_2 y_3 + w_4 w_3 - u_4 w_2}{z_2}, \\
y_2 &= y_2, \quad y_4 = y_4, \\
v_2 &= -\xi_2 y_2 + v_1 z_2 - z_2 y_3 + w_4, \\
v_4 &= -\xi_2 y_4 + v_3 z_2 - z_2 y_3 - w_4 w_2, \\
z_1 &= z_1, \quad z_2 = z_2, \\
z_4 &= -\xi_2 z_2 + x_3 - w_1 + w_2 w_3 - x_3 x_4 - x_3 x_2 + u_1 u_4 + u_2 u_3, \\
w_1 &= w_1, \quad w_2 = w_2, \\
w_3 &= w_3, \quad w_4 = w_4, \\
\xi_2 &= \xi_2.
\end{align*}
\]

We change the orderings of the coordinates in \( P_{z_1}(O_{z_1}) \) with “even” permutations as

\[
(\xi_2, z_1, z_2, w_1, w_2, w_3, w_4, y_2, y_4, u_2, u_4, z_1, v_1, v_3, x_3, x_4)
\]

and \( P_{z_2}(O_{z_2}) \) as

\[
(\xi_2, z_1, z_2, w_1, w_2, w_3, w_4, y_2, y_4, u_2, u_4, z_1, v_2, v_4, x_1, x_2).
\]

Then the Jacobi matrix is of the form that

\[
\begin{pmatrix}
Id_{11} & C \\
0_{5,11} & D
\end{pmatrix},
\]

where \( Id_{11} \) is a 11 \times 11 identity matrix, \( 0_{5,11} \) is a 5 \times 11 zero matrix and \( D \) is given by

\[
\begin{pmatrix}
\frac{2z_1}{z_2} & 0 & 0 & 0 & 0 \\
0 & \frac{2z_1}{z_2} & 0 & 0 & 0 \\
0 & 0 & \frac{2z_1}{z_2} & 0 & 0 \\
x_2 & \ast & \ast & \ast & \ast \\
-x_1 & \ast & \ast & \ast & \ast
\end{pmatrix}
\]
functions: where the coordinates \((x)\) which is given by the correspondence:

\[
J_{z_2,x_1} = \det D = \left(\frac{z_2}{z_1}\right)^5.
\]

**II** \(O_{w_1} \cap O_{z_1} = O_{a_6} \cap O_{a_4}\) case: For this case we consider the coordinate transformation \(P_{w_1} \circ P_{z_1}^{-1}\), which is given by the correspondence:

\[
(\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_1, v_1, v_2, x_1, x_2, x_3, u_1, u_2) \\
\mapsto (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_1, v_1, v_2, x_1, x_3, u_1, u_2),
\]

where the coordinates \((x_1, x_3, u_1, u_2, y_1, y_2, v_1, v_2, z_1, z_2, z_3, z_4, w_1, w_2, w_3, \xi_2)\) are given by the rational functions:

\[
\begin{align*}
  x_1 &= \frac{\xi_2 y_4 - z_2 x_3 + u_4 w_1 - u_3 w_3}{z_1}, & x_3 &= x_3, & u_1 &= -\frac{\xi_2 v_1 - u_2 z_3 - w_1 x_4 + w_2 x_3}{z_1}, & u_2 &= w_2, \\
  y_3 &= -\frac{\xi_2 x_3 - y_2 z_3 + w_3 x_1 + u_3 v_3}{z_1}, & y_4 &= y_4, & v_1 &= v_1, & v_2 &= -\frac{\xi_2 y_2 + u_1 z_2 - w_1 y_2 - w_3 y_3}{z_1}, \\
  z_1 &= z_1, & z_2 &= z_2, & z_3 &= z_3, & z_4 &= -\frac{\xi_2 z_1 - z_2 x_3 - w_2 x_1 + w_3 x_2 + x_2 x_3 - w_1 y_4 + w_2 y_3}{z_1}, \\
  w_1 &= w_1, & w_2 &= w_2, & w_3 &= w_3, & \xi_2 &= \xi_2.
\end{align*}
\]

We change the orderings of the coordinates in \(P_{z_1}(O_{z_1})\) by the “odd” permutation as

\[
(\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_1, y_2, v_1, v_2, x_1, x_4, u_1, u_2) \\
\mapsto (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_1, y_2, v_1, v_2, x_1, x_4, u_1, u_2)
\]

and \(P_{w_1}(O_{w_1})\) by the even permutation as

\[
(\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_1, y_2, v_1, v_2, x_1, x_3, u_1, u_2) \\
\mapsto (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_1, y_2, v_1, v_2, x_1, x_3, u_1, u_2)
\]

Then the Jacobi matrix is of the form that

\[
\begin{pmatrix}
  I_{d_{11}} C' \\
  0_{5,11} D'
\end{pmatrix},
\]

where the matrix \(D'\) is given by

\[
D' = \begin{pmatrix}
-\frac{w_1}{z_1} & 0 & 0 & 0 & 0 \\
0 & -\frac{w_1}{z_1} & 0 & 0 & 0 \\
\frac{w_1}{z_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{w_1}{z_1} & 0 \\
0 & 0 & 0 & 0 & \frac{w_1}{z_1}
\end{pmatrix}.
\]

The matrix \(C'\) and components \(*\) are given by some functions. Hence the Jacobian \(J_{w_1,z_1}\) is

\[
J_{w_1,z_1} = \det D' = -\left(\frac{w_1}{z_1}\right)^5.
\]

From the above Proposition 4.6 we can see that the \(\mathbb{C}^*\)-valued 1-cocycle defined by \(\{J_{a_i,a_j}\}_{a_i,a_j \in \{z_1,\ldots, w_4\}}\) is the coboundary of \(\mathbb{C}^*\)-valued 0-cochain \(\{h_i = \frac{1}{a_i}\}\), we have
The set of holomorphic sections

\[ \left\{ h_{z_i} = \frac{1}{z_i^2}, h_{w_i} = \frac{1}{w_i^2}, h_{y_i} = \frac{1}{y_i^2}, h_{x_i} = \frac{1}{x_i^2}, h_{u_i} = \frac{1}{u_i^2} \right\}, \]

each function is defined on the open coordinate neighborhood \( O_{z_i}, O_{w_i}, \) and so on, together define a nowhere vanishing global holomorphic section \( \Omega_0 \) of the canonical line bundle \( \wedge^1 T^*(\mathcal{X}_0) \).

Here the above notations \( h_{z_i}, h_{w_i}, h_{y_i}, h_{x_i}, h_{u_i} \) abbreviate the 16-degree local holomorphic differential forms, for example, \( h_{x_i} \) abbreviates

\[ h_{x_i} \text{ d}z_1 \wedge d z_2 \wedge d z_3 \wedge d w_1 \wedge d w_2 \wedge d w_3 \wedge d y_1 \wedge d y_2 \wedge d y_3 \wedge d x_1 \wedge d x_2 \wedge d x_3 \wedge d u_1 \wedge d u_2 \wedge d u_3. \]

Remark 4. As in the case for the sphere, the nowhere vanishing global holomorphic 16-form \( \Omega_0 \) coincides with the restriction of a smooth 16-form \( \tilde{\Omega}_0 \) defined on the whole space \( \mathcal{X}^{(3)} \) and there is a smooth 11-form \( n \) on \( \mathcal{X}^{(3)} \) with the property that

\[ \tilde{\Omega}_0 \wedge n = d a_1 \wedge a_2 \wedge a_3 \cdots \wedge a_{27}. \]

For the description of these smooth forms we need a troublesome preparation for the coordinates choices and we do not use the forms later so that we omit the construction.

We mention that since the transition function of the canonical line bundle on \( \mathcal{X}_0 \) is invariant under the multiplication by non-zero complex numbers, it is a pull-back of a complex line bundle on the quotient space \( \mathcal{X}_0 := C^* \mathcal{X}_0. \) More precisely

**Proposition 4.8.**

1. Interpreting the calculations above in terms of the homogeneous coordinates we see that the canonical line bundle \( \mathcal{K}^{\mathcal{X}_0} = \wedge^{16} T^*(\mathcal{X}_0) \) of the quotient space \( \mathcal{X}_0 \) is isomorphic to \( \bigotimes^3 \mathcal{L}^* \mathcal{X}_0, \) where \( \mathcal{L} \) is the tautological line bundle on the projective space \( \mathbb{P}^{26} \mathbb{C}, \mathcal{L} \subset \mathbb{P}^{26} \mathbb{C} \times \mathbb{C}^{27}. \)

2. Let \( \mathcal{V} \) be the kernel of the projection map \( \pi : \mathcal{X}_0 \rightarrow \mathcal{X}_0, \)

\[ \mathcal{V} := \ker d\pi \subset T(\mathcal{X}_0), \]

which can be seen naturally as a complex line bundle trivialized by the holomorphic vector field corresponding to the dilation action

\[ \mathcal{X}_0 \ni A \mapsto t \cdot A \in \mathcal{X}_0. \]

In this sense we denote it by \( \mathcal{V}_C. \) Then by the exact sequence

\[ \{0\} \rightarrow \pi^*(T^*(\mathcal{X}_0)^*) \rightarrow T^*(\mathcal{X}_0)^* \rightarrow \mathcal{V}_C^* \rightarrow \{0\} \]

we know that the canonical line bundle \( \mathcal{K}^{\mathcal{X}_0} = \pi^*(\mathcal{K}^{\mathcal{X}_0}) \otimes \mathcal{V}_C^* \) is holomorphically trivial, since \( \pi^*(\mathcal{L}) \) is holomorphically trivial.

5. Symplectic manifolds and polarizations

In this section we review an aspect of a geometric quantization theory in a restricted framework fitting only to our purpose. In the subsections \$5.3\$ and \$5.4\$ and in section \$6\$ we explain how the framework is adapted to our case.

5.1. Integral symplectic manifold

Let \((M, \omega^M)\) be a symplectic manifold with the symplectic form \(\omega^M.\) In this paper we assume that

\[ [\text{In1}] \text{ the map } H^2(M, \mathbb{Z}) \rightarrow H^2_{dR}(M, \mathbb{R}) \text{ is injective, or the group } H^2(M, \mathbb{Z}) \text{ has no torsion and,} \]

\[ [\text{In2}] \text{ the de Rham cohomology class } [\omega^M] \text{ of the symplectic form } \omega^M \text{ is in this image.} \]

Then the complex line bundle \( \mathbb{L}_{\omega^M} := L \in H^1(M, \mathbb{C}^*) \cong H^2(M, \mathbb{Z}) \) corresponding to the cohomology class \([\omega^M] \) is unique (of course, up to isomorphism). The first condition is satisfied, for example if \( M \) is simply connected and our case \( M = \mathcal{X}_0 \) satisfies both of these conditions trivially, since \( H^2(\mathcal{X}_0, \mathbb{Z}) = \{0\}. \)
Under these assumptions, the unique complex line bundle $L$ has the canonically defined connection $\nabla$, which is defined as follows:

Let $\{U_i\}$ be an open covering of $M$ with several “good” properties required in the arguments below (it is always possible for manifolds). Then there are one-forms $\{f_i\}$, each of which is defined on $U_i$ and $df_i = \omega^M$. Then the correction of smooth functions $\{c_{ij}\}$ defined by $dc_{ij} = f_j - f_i$ on $U_i \cap U_j$ satisfy that $c_{jk} - c_{ik} + c_{ij}$ takes integers on $U_i \cap U_j \cap U_k$, and the transition functions $\{g_{ij} = e^{2\pi \sqrt{-1} c_{ij}}\}$ defines the line bundle $\pi : \mathbb{L} \to M$.

The connection $\nabla$ on $\mathbb{L}$ is defined as

$$\nabla_X(s_i) = 2\pi \sqrt{-1}(f_i, X)s_i \text{ on } U_i, \ (X \text{ is a vector field})$$

where $s_i$ is a nowhere vanishing section on $U_i$ identifying $U_i \times \mathbb{C}$ and $\pi^{-1}(U_i) \subset \mathbb{L}$ in such a way that

$$U_i \times \mathbb{C} \ni (x, z) \mapsto z \cdot s_i(x) \in \pi^{-1}(U_i).$$

Here $(f_i, X)$ denotes the pairing of a one-form $f_i$ and a tangent vector $X$.

If we choose all the functions $c_{ij}$ being real valued, we may regard that the line bundle $\mathbb{L}$ is equipped with an Hermitian inner product, which we denote by $\langle \cdot, \cdot \rangle^\mathbb{L}$ at $x \in M$. Hereafter we assume that the line bundle $\mathbb{L}$ is equipped with such an Hermitian inner product.

We may regard that the space $C^\infty(M)$ is a Lie algebra by the Poisson bracket $\{f, g\} := \omega^M(H_f, H_g)$, where $H_f$ denotes the Hamilton vector field with the Hamiltonian $f$ defined by the equality $(df, \bullet) = \omega^M(H_f, \bullet)$. The space $\Gamma(\mathbb{L}, M)$ is a central object in the quantization theory. There is a basic fact that the correspondence from $g \in C^\infty(M)$ to the operator $T_g$, assignment of a function to an operator,

$$T_g : \Gamma(\mathbb{L}, M) \ni s \mapsto \nabla_{H_g}(s) + 2\pi \sqrt{-1} g \cdot s$$

is a Lie algebra homomorphism, $[T_g, T_h] = T(g, h)$, and it is the main theme in the quantization theory how to assign a function on a phase space to an operator on the configuration space.

### 5.2. Real and complex polarizations

Let $(M, \omega^M)$ be a symplectic manifold with the symplectic form $\omega^M$ $(\dim M = 2n)$. The skew-symmetric bi-linear form $\omega^M_p$ at each point $p \in M$ is naturally extended to the complexification $T(M) \otimes \mathbb{C} = T(M)^\mathbb{C}$ as the skew-symmetric complex bi-linear form which we denote with the same notation.

Let $F$ be a subbundle of the complex fiber dimension $n$ in $T(M)^\mathbb{C}$ satisfying the properties that

1. $F$ is maximal isotropic with respect to the skew-symmetric bi-linear form $\omega^M$,
2. $F$ is integral, that is $F \cap \mathcal{F}$ has constant rank and $F, F + \mathcal{F}$ is closed under bracket operation of vector fields taking values in these subbundles.

In this paper we only treat two extreme cases,

(P1) $F = \overline{F}$, and

(P2) $F + \overline{F} = T(M)^\mathbb{C}$.

First one is the complexification of a Lagrangian foliation $L \subset T(M)$, $F = L \otimes \mathbb{C}$ and we call it a real polarization. The second case is called a complex polarization.

If there is a polarization satisfying the second condition $F + \overline{F} = T(M)^\mathbb{C}$, then $M$ has a almost complex structure $J$ and the subbundle $F$ is identified with $(0,1)$-vectors in $T(M)^\mathbb{C}$ (anti-complex subbundle). The integrability condition implies that $M$ becomes a complex manifold. When we put

$$g(\alpha, \beta) := \omega^M(J(\alpha), \beta), \ \alpha, \beta \text{ vector fields on } M,$$

then $g$ is a non-singular symmetric bi-linear form on $T(M)$ and moreover it defines a Hermitian form on $T(M)^\mathbb{C}$. Under the condition that the form $g$ is positive definite, then it is equivalent that $M$ has a Kähler structure. We call such a polarization a positive polarization.

Hence it is equivalent that if there is a positive complex polarization on the symplectic manifold $M$,
then $M$ is a Kähler manifold and the symplectic form $\omega^M$ is a Kähler form. Also real polarization is always positive.

In this paper we consider two polarizations on the space $X_0$, one is the real polarization $F$ naturally defined on the cotangent bundle and a Kähler polarization (= positive complex polarization) $G$ described in (3.7), (3.8) and (3.9).

5.3. Hilbert space structure on the spaces of polarized sections

Now let $M$ be a symplectic manifold satisfying the conditions $[\text{In1}]$ and $[\text{In2}]$ as in the subsection §5.1 and fix a line bundle $L$ corresponding to the cohomology class $[\omega^M]$ with the connection $\nabla$ and the Hermitian inner product explained in the above subsections and assume that there is a polarization $F$ on $M$.

Let $U$ be an open subset in $M$. We introduce a space $C_F(U) \subset C^\infty(U)$ by

$$C_F(U) = \{ h \in C^\infty(U) \mid X(h) = 0, \forall X \in \Gamma(F, U), \text{vector fields taking values in } F \}$$

and a subspace $\Gamma_F(L, U)$ of smooth sections in $\Gamma(L, U)$ by

$$\Gamma_F(L, U) = \{ s \in \Gamma(L, U) \mid \nabla_X(s) = 0, \forall X \in \Gamma(F, U) \}. $$

Let $U$ be an open subset such that there is an one-form $\theta$ on $U$ satisfying

$$d\theta = \omega^M, \text{ and } \langle \theta, X \rangle = 0 \text{ for vectors } X \in F.$$ 

Although it is not canonical, we may locally identify the spaces $C_F(U)$ and $\Gamma_F(L, U)$ by fixing a nowhere vanishing section $s : U \to L$ with the property that $\nabla_X(s) = 0$ for $X \in F$ in such a way that

$$C_F(U) \ni \varphi \mapsto \varphi \cdot s \in \Gamma_F(L, U).$$

Then under this identification, the connection $\nabla$ is

$$\nabla_X(\varphi \cdot s) = X(\varphi) \cdot s + 2\pi \sqrt{-1} \varphi \cdot \langle \theta, X \rangle \cdot s = X(\varphi) \cdot s,$$

for vector field $X$ taking values in $F$.

If $F$ is a real polarization, then the function space $C_F(U)$ consists of such functions that are constant along each leaf $\cap U$ of the Lagrangian foliation, and if $F$ is a complex polarization, then $C_F(U)$ consists of holomorphic functions on $U$.

We call these sections $\in \Gamma_F(L, U)$ “polarized sections” (with respect to a polarization $F$) and are the main objects in the geometric quantization theory. We may regard, according to the polarization, that they express quantum states in the real polarization case and that they express good classical observables in the complex polarization. The above identification indicates the local nature of the polarized sections according to the polarization.

One basic problem is to introduce an inner product on the space $\Gamma_F(L, M)$ of $L$-valued polarized sections and a related space (which will be explained later) in a reasonable way (or without additional assumptions) to make it a (pre-)Hilbert space and the most interesting problem is to see a transformation from one space of polarized sections $\Gamma_G(L, M)$ (by a polarization $G$) to another space $\Gamma_F(L, M)$ of polarized sections by another polarization $F$.

We discuss two cases according to the polarizations (real and positive complex) how we introduce an inner product below in [RP] (real polarization) and in [CP] (complex polarization).

Under our assumptions we work only on density, (partial) half density, or (partial)1/4-density spaces. The meaning of “partial” will be explained in Remark 5.

[RP] Let $F$ be a real polarization. In this paper, for avoiding unnecessary generality, we assume more strongly that

(RP1) there is a submersion to an orientable manifold $N$,

$$\Phi : M \longrightarrow N$$

whose fibers are connected Lagrangian submanifolds.
So, the real polarization $F$ is defined as the kernel $F = \text{Ker} \, d\Phi$ of a surjective submersion $\Phi : M \to N$ and the functions in $C^r_F(M)$ are naturally descended to the base space $N$, that is $\Phi^* (C^\infty(N)) = C^r_F(M)$.

Let $\alpha, \beta \in \Gamma_F(\mathbb{L}, M)$, then by the equality

$0 = (\nabla_X(\alpha), \beta) + (\alpha, \nabla_X(\beta)) = X(\langle \alpha, \beta \rangle), \text{ for } X \in F,$

the function $(\alpha, \beta)^L$ is constant on each fiber. Hence it can be naturally identified with a function on the base manifold $N$. For such functions we need not integrate along the leaves and it will be enough to consider the integration to the transversal direction of the leaves. This is realized by the integration on $N$ the base manifold $\text{Ker} \, F$.

We consider the integration to the transversal direction of the leaves. This is realized by the integration on $N$ the base manifold $\text{Ker} \, F$.

Instead of the space $\Gamma_F(\mathbb{L}, M)$, we consider $\mathbb{L}$-valued polarized (or exactly to say, we call horizontal and partial) half-densities $\varphi \in \Gamma_F(\bigwedge^\max F^0, M)$ and/or horizontal and partial $\frac{1}{4}$-densities $\varphi \in \Gamma_F(\bigwedge^\max F^0, M)$, where $F^0$ is the annihilator of $F$,

$F^0 = \{ \xi \in T^*(M) \mid \langle \xi, X \rangle = 0, \forall X \in F \}.$

We can introduce a (partial) connection $\nabla_X(\xi) := i_X(d\xi)$ on $\bigwedge^\max F^0 = \bigwedge^\max (d\Phi)^*(\Phi^*(T^*(N)))$, where $\xi$ is a differential form in $\Gamma(\bigwedge^\max F^0, M), X \in F$ and $i_X$ denotes the interior product with a tangent vector $X \in F$.

Note that

$i_X(d\xi) = i_X \circ d\xi \in \Gamma\left(\bigwedge^\max F^0, M\right), \text{ for } X \in F \text{ and } \xi \in \Gamma\left(\bigwedge^\max F^0, M\right).$

Since $i_X(\xi) = 0$ for $\xi \in \Gamma\left(\bigwedge^\max F^0, M\right)$ by $X \in F$

$\nabla_X(f \cdot \xi) = i_X \circ (d(f \cdot \xi)) = i_X \circ (df \wedge \xi + f \cdot d\xi) = X(f) \cdot \xi - df \wedge i_X(\xi) + f \cdot i_X(d\xi)$

$= X(f) \xi + f \cdot \nabla_X(f \cdot \xi), \text{ for } X \in F \text{ and } f \in C^\infty(M),$
differential form and density (volume form) line bundle) and we consider the square root bundle $\sqrt{\Lambda^\text{max} F^0}$.

Sections in $\Gamma_F(\Lambda^\text{max} F^0, M)$ or $\Gamma_F(\sqrt{\Lambda^\text{max} F^0}, M)$ are not the half densities or 1/4-densities, since $\Lambda^\text{max} T^*(M) \equiv \Lambda^\text{max} F^* \otimes \Lambda^\text{max} F^0 = \text{trivial bundle}$ given by the Liouville volume form. So we should call the sections in $\Gamma_F(\Lambda^\text{max} F^0, M)$ or in $\Gamma_F(\sqrt{\Lambda^\text{max} F^0}, M)$ polarized “partial” half density or “partial” 1/4-density.

Differential forms $\mu \in \Gamma_F(\Lambda^\text{max} F^0, M)$ is descended to densities $\mu_* \in \Gamma(\Lambda^\text{max} T^*(N), N)$ (highest degree differential form) on the base manifold $N$, that is there is a unique highest degree differential form $\mu_* \in \Gamma(\Lambda^\text{max} T^*(N), N)$ such that $\Phi^*(\mu_*) = \mu$ by the isomorphism (5.1), and then we can integrate $\mu_*$ on $N$. Hence we have a natural linear form

$$I_N : \Gamma_F(\Lambda^\text{max} F^0, M) \ni \mu \longmapsto I_N(\mu) := \int_N \mu_* \in \mathbb{C}.$$

If we denote the inverse map of $\Phi^*$ of (5.1) by $\Phi_*$, then

$$\int_N \mu_* = \int_N \Phi_*(\mu).$$

In turn, we consider the square root bundle $\sqrt{\Lambda^\text{max} F^0}$, which can be seen as a partial 1/4-density bundle on $M$. Then we can also introduce a partial connection $\triangledown_X^{1/2}$ on the line bundle $\sqrt{\Lambda^\text{max} F^0}$ and as well it is defined also on the line bundle $L \otimes \sqrt{\Lambda^\text{max} F^0}$. Hence we consider “$L$-valued polarized (or horizontal) partial 1/4-densities” $\alpha \otimes \eta \in \Gamma_F(L \otimes \sqrt{\Lambda^\text{max} F^0}, M)$ and define their product by making use of the Hermitian inner product on $L$ with the formula

$$\Gamma_F(L \otimes \sqrt{\Lambda^\text{max} F^0}, M) \times \Gamma_F(L \otimes \sqrt{\Lambda^\text{max} F^0}, M) \to \Gamma_F(\Lambda^\text{max} F^0, M)$$

$$(\alpha \otimes \mu, \beta \otimes \nu) \longmapsto \langle \alpha, \beta \rangle_L : \mu \otimes \nu \in \Gamma_F(\Lambda^\text{max} F^0, M).$$

The resulting horizontal partial half density $\langle \alpha, \beta \rangle_L : \mu \otimes \nu \in \Gamma_F(\Lambda^\text{max} F^0, M)$, is identified with a density on $N$. Hence we can define a pairing (or an inner product) for the sections in $\Gamma_F(L \otimes \sqrt{\Lambda^\text{max} F^0}, M)$ by the integration of the corresponding density on $N$ in a natural way,

$$\Gamma_F(L \otimes \sqrt{\Lambda^\text{max} F^0}, M) \times \Gamma_F(L \otimes \sqrt{\Lambda^\text{max} F^0}, M) \to \mathbb{C},$$

$$(a \otimes \mu, b \otimes \nu) \longmapsto I_N(\Phi_*([a, b]^L \cdot \mu \otimes \nu)) = \int_N \Phi_*([a, b]^L \cdot \mu \otimes \nu).$$

For the real polarization $\mathcal{F}$ on our space $X_\mathcal{D}$, that is, through (3.8) $\mathcal{F}$ is defined as the kernel of the differential of the projection map $q : T_0 P^2 \mathcal{O} \to P^2 \mathcal{O}$, first we trivialize the line bundle $L$ by a nowhere vanishing polarized section $s_0 \in \Gamma_F(L, X_\mathcal{D})$ with $\langle s_0, s_0 \rangle^L \equiv 1$. We call this trivialization of the line bundle $L$ a “unitary trivialization”.

Next, let $dv_{P^2 \mathcal{O}}$ be the Riemann volume form on $P^2 \mathcal{O}$. We consider the square root

$$\sqrt{q \circ (\tau_0)^{-1} \ast (dv_{P^2 \mathcal{O}})} = \{q \circ (\tau_0)^{-1} \ast (dv_{P^2 \mathcal{O}}) \in \Gamma_F(\Lambda^\text{max} F^0, X_\mathcal{D}) \},$$

and identify a $L$-valued polarized partial 1/4-density $\xi \otimes \mu \in \Gamma_F(L \otimes \sqrt{\Lambda^\text{max} F^0}, X_\mathcal{D})$ with $f \cdot s_0 \otimes$
引进一个内积在空间 $\Gamma_T X$ 上。在 (5.4) 密度上，$\Gamma_G (\mathbb{L} \otimes \bigwedge^{7!} \mathcal{F}_0, \mathcal{X}_G)$ (在取完成之后)。

[CP] 令 $G$ 为一个复正交 polarization 在 $M$ 上，其对称形式 $\omega^M$ 表达为一个 Kähler 形式：

$$\sqrt{-1} \partial \bar{\partial} \phi = \omega^M.$$ 

线束 $\mathbb{L}$ 对应到的 cohomology class $[\omega^M]$ 装配了一个 Hermitian 内积 $(\cdot , \cdot )^L$ 如 5.1 所解释。

两个 section $a, b \in \Gamma_G (\mathbb{L}, M)$ 的内积是关于 $M$ 上的一个函数，并且可以被整合到 Liouville 体积形式 $dV_M := \frac{1}{m!} (-1)^{n(n-1)/2} \{ \omega^M \}^n$ (dim $M = 2n$)。因此，我们可以引入内积在空间 $\Gamma_G (\mathbb{L}, M)$ 上的内积，我们不依赖于任何其他附加假设。

我们也可以引入内积在空间 $\mathbb{L}$ 上的“polarized” sections 的 canonical line bundle $K^G$ 作为 polarization $G$。

在 $\mathbb{L}$ 上的 canonical line bundle $K^G = \bigwedge^{\max} T^* (M)^G$ 是 line bundle 的最高 degree exterior product of the holomorphic part $T^* (M)^G$ 的 complexified cotangent bundle $T^* (M)^G$ ((1, 0) type cotangent vectors)，是 complex polarization $G$ 的 annihilator (0, 1 tangent vectors)，如 $F^0$ 对于 real polarization $F$。Canonical line bundle 的内积可以认为是 half densities (complex valued half density) 的 isomorphism $K^G \otimes \overline{K^G} = \bigwedge^\max T^* (M)^G$. 我们可以引入 partial connection $\nabla^G_X$ ($X \in G$) 作为 complex polarization $G$ 的 same way as for the real polarization. 然后我们考虑 space $\Gamma_G (\mathbb{L} \otimes K^G, M)$ 的 “$\mathbb{L}$-valued polarized sections of the canonical line bundle” 以及使用 Hermitian 内积 on $\mathbb{L}$ 我们有 highest degree differential form

$$\langle a \otimes \mu , b \otimes \nu \rangle = \langle a, b \rangle^L \cdot \mu \wedge \nu \in \Gamma \left( \bigwedge^{\max} T^* (M)^G , M \right),$$

where $a, b \in \Gamma_G (\mathbb{L}, M)$ and $\mu, \nu \in \Gamma_G (K^G, M)$. The quantity $\mu \wedge \nu$ can be seen as a (complex valued) density on $M$. Hence we have an intrinsic (pre-)Hilbert space structure on the space $\Gamma_G (\mathbb{L} \otimes K^G, M)$.

For the complex polarization $\mathcal{G}$ 我们在空间 $\mathcal{X}_G$ 上使用一种又叫 Calabi-Yau structure on $\mathcal{X}_G$ to identify the space $\Gamma_G (\mathbb{L} \otimes K^G, \mathcal{X}_G)$ with the space $C^G (\mathcal{X}_G)$ of holomorphic functions on $\mathcal{X}_G$ by the correspondence

$$\Gamma : C^G (\mathcal{X}_G) \ni h \longmapsto \gamma (h) = h \cdot t_0 \otimes \Omega_0 \in \Gamma_G (\mathbb{L} \otimes K^G, \mathcal{X}_G).$$

The existence of the nowhere vanishing holomorphic 16-form $\Omega_0$ on $\mathcal{X}_G$ was proved in Proposition (4.7) and $t_0$ is taken for trivializing the line bundle $\mathbb{L}$ satisfying the property $\nabla^G_X (t_0) = 0$.

We call a trivialization of the line bundle $\mathbb{L}$ by the section $t_0$ a "holomorphic trivialization". We will determine the relation of the sections $s_0$ and $t_0$, $t_0 = g_0 s_0$ in the subsection § 6.1.

### 5.4. Pairing of polarizations and a Bargmann type transformation

First, we recall the fiber integration. Let $\phi : M \to N$ be a differentiable map between two manifolds.

Let $\sigma \in \Gamma \left( \bigwedge^p T^* (M), M \right)$ be a differential form with the degree $p \geq \dim M - \dim N := d$. For $g \in \Gamma \left( \bigwedge^q T^* (N), N \right)$ with compact support satisfying $q = m - p = \dim M - p \geq 0$ (we denote the space of sections with compact support by $\Gamma_0 (\ast , \ast )$). We assume

$$\int_M |\sigma \wedge \phi^*(g)| < +\infty$$
for any \( g \in \Gamma_0(\hat{\mathcal{L}}T^*(N)) \) and define a linear functional
\[
g \mapsto \int_M \sigma \wedge \phi^*(g),
\]
which is understood as a distribution on the space \( \Gamma_0(\hat{\mathcal{L}}T^*(N)) \). We denote this distribution by \( \phi_*(\sigma) \) and express as
\[
\phi_*(\sigma)(g) := \int_N \phi_*(\sigma) \wedge g = \int_M \sigma \wedge \phi^*(g).
\]
(5.5)

If \( \phi \) is a submersion, then \( \phi_*(\sigma) \) is a smooth differential form of degree \( p - d \).

In the last subsection we introduced inner products on the spaces \( \Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big) \) and \( \Gamma_G\Big(\mathcal{L} \otimes \bigwedge^{\max} T^*(M)^G, M\Big) = \Gamma_G\Big(\mathcal{L} \otimes K^G, M\Big) \) for a real polarization \( F \) satisfying the condition (RP1) and a positive complex polarization \( G \) on an integral symplectic manifold \( M \). Our main purpose is to construct a transformation
\[
\mathfrak{B} : \Gamma_G\Big(\mathcal{L} \otimes K^G, M\Big) \rightarrow \Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big)
\]
(5.6)
or it may be understood as a sesqui-linear form on
\[
\Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big) \times \Gamma_G\Big(\mathcal{L} \otimes K^G, M\Big) \xrightarrow{\mathfrak{B}} \Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big) \rightarrow \mathbb{C}.
\]
(5.7)

For the sections \( (\alpha \otimes \mu, \beta \otimes \nu) \in \Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big) \times \Gamma_G\Big(\mathcal{L} \otimes K^G, M\Big) \) their product
\[
(\alpha, \beta)^L \cdot |\mu \otimes \nu|
\]
(\( | \star | \) means a section of \( |K^G \otimes \bigwedge^{\max} F^0| \)) is a partial 3/4-density on \( M \) and so we need some modification to integrate it, since there are no manifold of the dimension 3/4 \( \times \dim M \).

Since we identify the half density space \( \Gamma\Big(\bigwedge^{\max} T^*(N), N\Big) \) with a \( L_2 \)-space by fixing a Riemann volume form \( dv_N \), we define a sesqui-linear form
\[
\Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big) \times \Gamma_G\Big(\mathcal{L} \otimes K^G, M\Big) \rightarrow \mathbb{C}
\]
by
\[
\Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big) \times \Gamma_G\Big(\mathcal{L} \otimes K^G, M\Big) \ni (\alpha \otimes \mu, \beta \otimes \nu)
\rightarrow \int_M (\alpha, \beta)^L \cdot \Phi^*(f_\mu dv_N) \wedge \nu,
\]
(5.8)
where we can put \( \mu = \Phi^*(f_\mu)\sqrt{\Phi^*(dv_N)} \) with a function \( f_\mu \in C^\infty(N) \), that is we multiply the partial 1/4-density \( \sqrt{\Phi^*(dv_N)} \) to the partial 1/4-density \( \nu = \Phi^*(f_\nu)\sqrt{\Phi^*(dv_N)} \), then \( \sqrt{\Phi^*(dv_N)} \otimes \Phi^*(dv_N) \otimes \mu \) is a (complex valued) highest degree differential form (or can be thought as a density) on \( M \) and we can define a sesqui-linear form.

Once we have a sesqui-linear form
\[
P : \Gamma_F\Big(\mathcal{L} \otimes \bigwedge^{\max} F^0, M\Big) \times \Gamma_G\Big(\mathcal{L} \otimes K^G, M\Big) \rightarrow \mathbb{C}
\]

it is rewritten as

\[ P(\alpha \otimes \mu, \beta \otimes \nu) = \sum_{i} I_N(\Phi_*(\alpha, \alpha_i)^L \cdot \mu \otimes \mu_i), \]

where we put \( \mathfrak{B}(\beta \otimes \nu) = \sum \alpha_i \otimes \mu_i. \)

In our space \( T^*O(\mathcal{X}_O) \cong \mathcal{X}_O \) we have two polarizations \( \mathcal{F} \) (real) and \( \mathcal{G} \) (Kähler) and we identify the spaces \( C_G(\mathcal{X}_O) \) and \( \Gamma_G(\mathcal{L} \otimes K^G, \mathcal{X}_O) \) by (5.4). The inner product on the space \( C_G(\mathcal{X}_O) \) induced by the map \( \gamma \) will be explicitly described in (6.9) at the end of \( \S 6.2 \) in terms of the Liouville volume form. There we will introduce a parameter family of inner products on the space \( \Gamma_G(\mathcal{L} \otimes K^G, \mathcal{X}_O). \)

We recall the sections \( s_0 \) and \( t_0 \) and describe our Bargmann type transformation including the quantity \((s_0, t_0)^L.\)

Let \( \theta^{P^2O} \) be the canonical one-form on the cotangent bundle \( T^*(P^2O) \), then \( d\theta^{P^2O} = \omega^{P^2O} \) and for any \( X \in \mathcal{F}, \ theta^{P^2O}(X) = 0 \). So let \( s_0 \) be a nowhere vanishing polarized (with respect to the real polarization \( \mathcal{F} \)) global section of \( \mathcal{L} \) defining a trivialization \( \mathcal{X}_O \times \mathbb{C} \cong \mathcal{L} \) by the correspondence

\[(5.9) \quad \mathcal{X}_O \times \mathbb{C} \ni (A, z) \longmapsto z \cdot s_0(A) \in \mathcal{L}, \]

with \( <s_0, s_0>^L = 1.\)

Also by the relation

\[ \tau_0^* \left( -2 \frac{\partial}{\partial \theta} |A|^1/2 \right) = \omega^{P^2O}, \]

given in Theorem (3.3), we take a (complex) one-form

\[ \theta_G = \sqrt{-2} \frac{\omega}{\theta} |A|^{1/2}, \]

then \( d\tau_G^*(\theta_G) = \omega^{P^2O} \) and \( \theta_G(X) = 0 \) for \( X \in \mathcal{G} \), since \( X \) is a \((0, 1)\) tangent vector.

Then we can trivialize the line bundle \( \mathcal{L} \) by making use of a nowhere vanishing global section \( t_0 \) in a similar way to (5.9).

Using the identifications (5.4) and the correspondence

\[ C^\infty(P^2O) \ni g \longmapsto \{q \circ \tau_0^{-1}\}^*(g) \cdot s_0 \otimes \sqrt{\{q \circ \tau_0^{-1}\}^*(dv_{P^2O})} \]

\[ \longmapsto \{q \circ \tau_0^{-1}\}^*(g) \cdot s_0 \otimes \{q \circ \tau_0^{-1}\}^*(dv_{P^2O}) \]

the integral (5.8) is rewritten as

\[(5.10) \quad \int_{\mathcal{X}_O} \{q \circ \tau_0^{-1}\}^*(g) \cdot \overline{\tau(\mathcal{L})} \cdot (s_0, t_0)^L \cdot \{q \circ \tau_0^{-1}\}^*(dv_{P^2O}) \wedge \overline{\Omega_0}, \]

and it is also expressed in terms of the fiber integration as follows:

\[(5.11) \quad \int_{\mathcal{X}_O} \{q \circ \tau_0^{-1}\}^*(g) \cdot \overline{\tau(\mathcal{L})} \cdot (s_0, t_0)^L \cdot \{q \circ \tau_0^{-1}\}^*(dv_{P^2O}) \wedge \overline{\Omega_0} \]

\[= \int_{P^2O} g \cdot \{q \circ \tau_0^{-1}\}^* ((s_0, t_0)^L \cdot \overline{\tau(\mathcal{L})}) \cdot dv_{P^2O}. \]

Then the Bargmann type transformation

\[ \mathfrak{B} : C_G(\mathcal{X}_O) \rightarrow C^\infty(P^2O), \ C_G \ni h \longmapsto \mathfrak{B}(h), \]

is defined as

\[(5.12) \quad \mathfrak{B}(h) = \{q \circ \tau_0^{-1}\}^* (h \cdot (t_0, s_0)^L \Omega_0). \]
Hence we can express the integral (5.11) as
\[
\int_{p^2\mathcal{O}} g \cdot \mathcal{B}(h) \, dv_{p^2\mathcal{O}}.
\]

Remark 6. The section \( s_0 \) is free of \( U(1) \)-multiple and \( t_0 \) is of free from a constant \( \in \mathbb{C}^* \).

6. Bargmann type transformation

For expressing the Bargmann type transformation explicitly and to determine its \( L_2 \) continuity, we need to know the function \( \langle s_0, t_0 \rangle^L = g_0 \), and relations of \( \Omega_0 \wedge \overline{\Omega_0} \) and \( \{ q \circ \tau_0^{-1} \}^* (dv_{p^2\mathcal{O}}) \wedge \overline{\Omega_0} \) with the Liouville volume form \( dV_{\tau_r(P^2\mathcal{O})} \) explicitly. In this section we determine them.

6.1. Holomorphic trivialization and unitary trivialization

The relation of the sections \( s_0 \) and \( t_0 \) is given by a function \( g_0 = \langle s_0, t_0 \rangle^L \), that is
\[
(6.1) \quad t_0 = g_0 \cdot s_0.
\]

The function \( g_0 \) satisfies an equation
\[
\nabla_X(t_0) = 2\pi \sqrt{-1} (\sqrt{-2} \partial ||A||^{1/2}, X)g_0 \cdot s_0
\]
\[
= \nabla_X(g_0s_0) = X(g_0)s_0 + 2\pi \sqrt{-1}g_0 \cdot (\theta^{p^2\mathcal{O}}, X)s_0,
\]
and we have an equation for the function \( g_0 \):
\[
(6.2) \quad 2\pi \sqrt{-1} \cdot \left( \tau_0^* \left( \sqrt{-1} \sqrt{2} \partial ||A||^{1/2} \right) - \theta^{p^2\mathcal{O}} \right) g_0 = dg_0.
\]

Put \( g_0 = e^{2\pi \sqrt{-1} \lambda} \), then the equation (6.2) reduces to the equation
\[
(6.3) \quad d\lambda = \tau_0^* \left( \sqrt{2} \sqrt{-1} \partial ||A||^{1/2} \right) - \theta^{p^2\mathcal{O}}.
\]

To get a solution \( \lambda \) we need to consider the real and imaginary parts in the formula
\[
\sqrt{2} \sqrt{-1} \partial ||A||^{1/2}
\]
separately. So, put
\[
\tau_0^* \left( \sqrt{2} \sqrt{-1} \partial ||A||^{1/2} \right) := a + \sqrt{-1}b
\]
with real and imaginary parts of the one-form \( \tau_0^* \left( \sqrt{2} \sqrt{-1} \partial ||A||^{1/2} \right) \) on \( \mathcal{J}(3) \times \mathcal{J}(3) \). Then
\[
d(d\lambda) = d(\tau_0^* \left( \sqrt{2} \sqrt{-1} \partial ||A||^{1/2} \right) - \theta^{p^2\mathcal{O}}) = 0
\]
implies that there are real valued functions \( \lambda_{Re} \) and \( \lambda_{Im} \) such that
\[
a - \theta^{p^2\mathcal{O}} = d\lambda_{Re}, \text{ and}
\]
\[
d\lambda_{Im} = b.
\]

The problem to solve the equation (6.3) reduces to find explicitly the functions \( \lambda_{Re} \) and \( \lambda_{Im} \).

Let \((X, Y) \in T^*_0(P^2\mathcal{O}) \subset \mathcal{J}(3) \times \mathcal{J}(3) \). Here again we remark that we are identifying the cotangent space \( T^*_X(P^2\mathcal{O}) \) and the tangent space \( T_X(P^2\mathcal{O}) \) by the Riemannian metric defined by \((Y_1, Y_2)_X^{P^2\mathcal{O}} := \text{tr} \left( Y_1 \circ Y_2 \right) \) for \( Y_i \in T_X(P^2\mathcal{O}) \isom \mathcal{J}(3), \) that is for \( Y_i \in T_X(P^2\mathcal{O}) \subset \mathcal{J}(3), \) \( i = 1, 2, \)
\[
Y_1 = \left( \begin{array}{ccc}
\epsilon_1 & u_3 & \theta(u_2) \\
\theta(u_3) & \epsilon_2 & u_1 \\
u_2 & \theta(u_1) & \epsilon_3
\end{array} \right), \quad Y_2 = \left( \begin{array}{ccc}
\eta_1 & v_3 & \theta(v_2) \\
\theta(v_3) & \eta_2 & v_1 \\
v_2 & \theta(v_1) & \eta_3
\end{array} \right), \quad \epsilon_i, \eta_i \in \mathbb{R}, u_i, v_i \in \mathbb{C} \isom \mathbb{R}^8,
\]
(6.4) \[(Y_1, Y_2)^{P_2\Omega}_X := \text{tr}(Y_1 \circ Y_2) = \sum \epsilon_1 \eta_i + 2 \sum (u_i, v_i)^{2a}.\]

Based on this expression, by the notation \((Y, dX)\) for
\[
X = \begin{pmatrix} 
\xi_1 & x_3 & \theta(x_2) \\
\theta(x_3) & \xi_2 & x_1 \\
x_2 & \theta(x_1) & \xi_3 
\end{pmatrix} \in \mathcal{J}(3), \quad Y = \begin{pmatrix} 
\epsilon_1 & u_3 & \theta(u_2) \\
\theta(u_3) & \epsilon_2 & u_1 \\
u_2 & \theta(u_1) & \epsilon_3 
\end{pmatrix} \in \mathcal{J}(3),
\]

we mean the canonical one-form
\[(Y, dX) := \sum \epsilon_i d\xi_i + 2 \sum \{u_i, d\{x_i\} + \{u_2, d\{x_2\} + \{u_3, d\{x_3\},
\]
on \(T^*(\mathcal{J}(3)) \equiv \mathcal{J}(3) \times (\mathcal{J}(3))^* \equiv \mathcal{J}(3) \times (\mathcal{J}(3))\), or its restriction to \(T^*(P^2\Omega)\), that is, in the inner product expression \((6.4)\) we understand as \(\eta_i\) and \({v_k}\), \((k = 1, 2, 3, i = 0, \cdots, 7)\) are replaced by the differentials \(d\xi_i\) and \(d\{x_k\}\) of the corresponding components in \(X \in \mathcal{J}(3)\), respectively.

Also for \(A \in \mathcal{J}(3)^C\) and an one form \(B\) on \(\mathcal{J}(3)^C\) we express the complex one form \((A, dB)\) in the same way.

Let \((X, Y) \in T(P^2\Omega) \equiv T^*(P^2\Omega)\) and put \(A = \tau_0(X, Y) = 1 \otimes (||Y||^2X - Y^2) + \sqrt{-1} \otimes \frac{||Y||^4}{2}\), then

**Proposition 6.1.** (see [Fu2])

\[
\frac{1}{2}||Y||^4 = ||a||^2 = ||b||^2, \quad ||A||^2 = ||a||^2 + ||b||^2 = ||Y||^4, \quad \text{and} \quad (da, a) = ||Y||^2(Y, dY) = (db, b).
\]

In the expression
\[
\tau^*(dA, A) = \tau^*(dA) = (a - \sqrt{-1}b, da + \sqrt{-1}db) = d||A||^2
\]
and it is proved in [Fu2] (page 179) that
\[
(dY, Y \circ Y) = 0.
\]

Hence
\[
\tau_0^*(\sqrt{2} \sqrt{-1}d||A||^{1/2} - \theta^{P^2\Omega}) = \sqrt{2} \sqrt{-1} \frac{1}{4||Y||^2} \{\sqrt{-2} \cdot (dY, X) - 2||Y||^2(Y, dY) - \theta^{P^2\Omega}
\]
\[
= -(dY, X) - (Y, dX) + \sqrt{-1} \frac{1}{\sqrt{2}||Y||}(Y, dY) = \frac{\sqrt{-1}}{\sqrt{2}} d||Y||,
\]
since \(d(X, Y) = (dX, Y) + (Y, dX) = d\text{tr}(X \circ Y) = 0\) for \((X, Y) \in T^*(P^2\Omega)\). Hence finally we may choose the solutions \(\lambda_{Re}\) and \(\lambda_{Im}\) with
\[
\lambda_{Re} \equiv 0 \text{ and } \lambda_{Im} = \frac{1}{\sqrt{2}} ||Y||.
\]

Hence

**Proposition 6.2.**

\[
g_0 = e^{-\sqrt{2\pi} ||Y||}, \text{ or it is expressed on } \mathcal{X}_0 \text{ as } g_0 = e^{-\sqrt{2\pi} ||A||^{1/2}}.
\]

Now we have
\[
\mathcal{B} : C_0(\mathcal{X}_0) \to C^\infty(\mathcal{P}^2\Omega),
\]
\[
\mathcal{B}(h) = \{q \circ (\tau_0)^{-1}\}_\ast(h \cdot (t_0, s_0)^{1/2} \Omega_0) = \{q \circ (\tau_0)^{-1}\}_\ast(h \cdot e^{-\sqrt{2\pi} ||A||^{1/2}} \Omega_0).
\]
The solution $\lambda_{Re}$ can be an arbitrary real constant. However the absolute value $|g_0|$ does not depend on the chosen constant $\lambda_{Re}$.

6.2. Fock-like space

We show

**Proposition 6.3.** The nowhere vanishing global holomorphic section $\Omega_\mathfrak{O}$ of the canonical line bundle $K^G$ is $F_4$-invariant.

**Proof.** Let $\alpha \in F_4$. The action of $\alpha$ on $\mathfrak{X}_\mathfrak{O}$ is naturally defined from the action on $P^2\mathfrak{O}$ and the action is holomorphic. We denote it with the same notation $\alpha : \mathfrak{X}_\mathfrak{O} \to \mathfrak{X}_\mathfrak{O}$.

We can put $\alpha^*(\Omega_\mathfrak{O}) = K_\alpha \cdot \Omega_\mathfrak{O}$ with a nowhere vanishing holomorphic function $K_\alpha = K_\alpha(A)$. Then

$$\alpha^*(\Omega_\mathfrak{O}) \cap \alpha^*(\Omega_\mathfrak{O}) = \alpha^*(\Omega_\mathfrak{O} \cap \Omega_\mathfrak{O}) = |K_\alpha|^2 \cdot \Omega_\mathfrak{O} \cap \Omega_\mathfrak{O}.$$ 

We can express

$$\Omega_\mathfrak{O} \cap \Omega_\mathfrak{O} = D \cdot \frac{1}{16!} \{\tau_0^{-1}\}^* \left(\omega^{P^2\mathfrak{O}}\right)^{16}$$

by the Liouville volume form $\frac{1}{16!} \left(\omega^{P^2\mathfrak{O}}\right)^{16}$ and a function $D = D(A)$ on $\mathfrak{X}_\mathfrak{O}$. Hence

$$\alpha^*(\Omega_\mathfrak{O} \cap \Omega_\mathfrak{O}) = \alpha^*(D) \cdot \frac{1}{16!} \{\tau_0^{-1}\}^* \left(\omega^{P^2\mathfrak{O}}\right)^{16},$$

since the action by $\alpha$ on $\mathfrak{X}_\mathfrak{O}$ is symplectic. Hence

$$\alpha^*(D) = |K_\alpha|^2 \cdot D.$$ 

By comparing the behaviours of $\Omega_\mathfrak{O}$ and the Liouville volume form $dV_{P^2\mathfrak{O}}$ under the dilation action by positive numbers:

$$T_t : \mathfrak{X}_\mathfrak{O} \to \mathfrak{X}_\mathfrak{O}, \ A \to t \cdot A,$$

we can see on the coordinate neighborhood $O_{z_1}$

$$T_t^*(\Omega_\mathfrak{O} \cap \Omega_\mathfrak{O}) = \frac{1}{(tz_1)^5} d(tz_1) \wedge \cdots \wedge d(tz_2) \wedge \frac{1}{(t^2)^5} d(tz_1) \wedge \cdots \wedge d(tz_2)$$

$$= t^{22} \frac{1}{z_1^5} dz_1 \wedge \cdots \wedge d\xi_2 \wedge \frac{1}{z_1^5} d\xi_1 \wedge \cdots \wedge d\xi_2$$

$$= t^{22} \cdot D(A) \cdot \frac{1}{16!} \{\tau_0^{-1}\}^* \left(\omega^{P^2\mathfrak{O}}\right)^{16} = D(t \cdot A) \cdot t^{14} \frac{1}{16!} \{\tau_0^{-1}\}^* \left(\omega^{P^2\mathfrak{O}}\right)^{16}.$$ 

Hence

$$D(t \cdot A) = t^{14} \cdot D(A).$$

Note that the action $T_t$ on $T_0^*(P^2\mathfrak{O})$ defined via the map $\tau_0$ is

$$(6.6) \quad \tau_0^{-1} \circ T_t \circ \tau_0 : T_0^*(P^2\mathfrak{O}) \ni (X, Y) \mapsto (X, \sqrt{t}Y) \in T_0^*(P^2\mathfrak{O}).$$

Then since $||\alpha(A)|| = ||A||$

$$\alpha^*(D)(A) = D(\alpha(A)) = D \left(||\alpha(A)|| \cdot \frac{\alpha(A)}{||\alpha(A)||}\right)$$

$$= ||A||^{14} \cdot D \left(\frac{\alpha(A)}{||\alpha(A)||}\right) = ||A||^{14} \cdot |K_\alpha(A)|^2 D \left(\frac{A}{||A||}\right).$$
hence
\[ (6.7) \quad D \left( \frac{\alpha(A)}{||\alpha(A)||} \right) = |K_\alpha(A)|^2 D \left( \frac{A}{||A||} \right). \]

This equality implies that the function $K_\alpha$ is bounded on $\mathcal{X}_D$. Especially, if we consider it on the coordinate open subset $O_{z_1} \cong \mathbb{C}^* \times \mathbb{C}^{15}$ ($z_1 \neq 0$), then it can be extended to a holomorphic function on $\mathbb{C} \times \mathbb{C}^{15} \supset O_{z_1}$ and is bounded there. Hence the function $K_\alpha$ is a constant function on the whole space $\mathcal{X}_D$.

Then by the property
\[ K_{\alpha, \beta} = K_\alpha \cdot K_\beta, \alpha, \beta \in F_4, \]

$F_4 \ni \alpha \mapsto K_\alpha$ is a one-dimensional representation of the compact simply connected group $F_4$, so that we have not only $|K_\alpha| \equiv 1$ for any $\alpha \in F_4$, but also it must hold always $K_\alpha \equiv 1$. This implies $\Omega_D$ is $F_4$-invariant. \hfill \Box

**Corollary 6.4.** Since the action of $F_4$ on $S(\mathcal{X}_D) = \{ A \in \mathcal{X}_D \mid ||A|| = 1 \}$ is transitive, the function is of the form $D(A) = C_1 \times ||A||^4$ with the constant $C_1 = 2^{26}$. Especially we have
\[ (6.8) \quad \tau_0^* (\Omega_D \wedge \pi_D^0) (X, Y) = 2^{26} ||Y||^{12} \frac{1}{16!} (\omega^{j = 0})^{16}. \]

**Proof.** It is enough to determine the constant $C_1$.

Following the expression (4.6) of the matrix $A \in \mathcal{J}(3)^C$ we denote
\[ A = \left( \begin{array}{ccc} \xi_1 & c^l + c'' e_4 & \theta(b' + b'' e_4) \\ \xi_2 & a' + a'' e_4 & \xi_3 \end{array} \right) \in \mathcal{J}(3)^C \]

by
\[ (\xi_1, \xi_2, \xi_3, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4, y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4), \]

where
\begin{align*}
\rho_{\mathbb{H}}(c') + \rho_{\mathbb{H}}(c'') e_4 &= \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} e_4, \\
\rho_{\mathbb{H}}(b') + \rho_{\mathbb{H}}(b'') e_4 &= \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} + \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} e_4, \\
\rho_{\mathbb{H}}(a') + \rho_{\mathbb{H}}(a'') e_4 &= \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} e_4.
\end{align*}

The correspondence between $c = c' + c'' e_4 = \sum \{c_i\}, e_i$ (and also $b = b' + b'' e_4 = \sum \{b_i\}, e_i, a = a' + a'' e_4 = \sum \{a_i\}, e_i)$, and the components $\{z_i, w_i\}$ is given in (2.2) and (2.3).

By a simple calculation we have
\[ ||A||^2 = \sum_{i=1}^3 |\xi_i|^2 + 2 \sum_{i=1}^4 |a'|^2 + |a''|^2 + |b'|^2 + |b''|^2 + |c'|^2 + |c''|^2 \]
\[ = \sum_{i=1}^3 |\xi_i|^2 + \sum_{i=1}^4 |z_i|^2 + |w_i|^2 + |y_i|^2 + |v_i|^2 + |x_i|^2 + |u_i|^2. \]

Then we rewrite for $A \in O_{z_1}$
\[ A \leftrightarrow (\xi_1, \xi_2, \xi_3, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4, y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4)) \]
as
\[ (x_3, x_4, u_2, u_4, y_2, y_4, v_1, v_3, z_1, z_2, z_3, w_1, w_2, w_3, w_4, \xi_2 : x_1, x_2, u_1, u_3, y_1, y_3, v_2, v_4, z_4, \xi_1, \xi_3) \]
\(= (s_1, \ldots, s_{16}; s_{17}, \ldots, s_{27}),\)

that is, the first 16 coordinates give local coordinates on \(O_2\), and the remaining coordinates \((s_{17}, \ldots, s_{27})\) are rational functions of the coordinates \((s_1, \ldots, s_{16})\), that is \(s_j = s_j(s_1, \ldots, s_{16}), j \geq 17\) (especially \(s_9 = z_1\) and the explicit form of each \(s_j\) for \(j > 16\) is given in (4.16)).

In particular, we see from the explicit form of these functions at the point \(A = A(z_1) = A(0, \ldots, z_1, 0, \ldots, 0)\)

\[s_j(A(z_1)) = s_j(0, \ldots, 0, 0, s_9, 0, \ldots, 0) = 0, \quad 17 \leq j \leq 27,\]

and for \(i \leq 16, j \geq 17\)

\[\frac{\partial s_j}{\partial s_i}(A(z_1)) = 0.\]

On \(O_2\), it holds

\[\Omega_2 \cap \Omega_{27} = \frac{1}{|z_1|^{10}} d\xi_1 \wedge d\xi_4 \wedge \cdots \wedge d\xi_9 \wedge d\eta_1 \wedge \cdots \wedge d\eta_4 \wedge d\xi_2 \]

\[= \frac{1}{|s_9|^{10}} ds_1 \wedge \cdots \wedge ds_{16} \wedge d\eta_1 \wedge \cdots \wedge d\eta_4 = D(A) \cdot \frac{1}{16!} (\tau_0^{-1})^*(\omega^{P^2_0})^{16}.\]

We calculate the right hand side at the point \(A(z_1) = (0, \ldots, 0, z_1, 0, \ldots, 0, \ldots, 0) \in O_2\), using the expression of \(\omega^{P^2_0}\) (see Theorem 3.2):

\[\omega^{P^2_0} = \{\tau_0\}^*(\sqrt{-2\partial\partial|A|^{1/2}}).\]

Then

\[\partial\partial|A|^{1/2} = \frac{1}{4} \cdot \partial\left( \sum_{i=1}^{27} |s_i|^2 \right)^{-3/4} \cdot \sum s_i ds_i \]

\[= -\frac{3}{16} \sum_{i=1}^{27} |s_i|^2 \sum s_j ds_j \wedge \sum_{i=1}^{27} s_i ds_i + \frac{1}{4} \sum_{i=1}^{27} |s_i|^2 \sum_{i=1}^{27} ds_i \wedge ds_i.\]

Here we evaluate it at the point \(A(z_1)\), then it is given by

\[-\frac{3}{16} |s_9|^{-3/2} \cdot |s_9|^2 ds_9 \wedge ds_9 + \frac{1}{4} |s_9|^{-3/2} \sum_{i=1}^{16} ds_i \wedge ds_i \]

\[= \frac{1}{16} |s_9|^{-3/2} ds_9 \wedge ds_9 + \frac{1}{4} |s_9|^{-3/2} \sum_{1 \leq i \leq 8 \text{ and } 10 \leq i \leq 16} ds_i \wedge ds_i.\]

Hence

\[(\omega^{P^2_0})^{16} = (\sqrt{-2\partial\partial|A|^{1/2}})^{16} = 16! \cdot \frac{1}{16} \cdot \frac{1}{4^{15}} (\sqrt{-2})^{16} \cdot \frac{1}{|s_9|^{3/2 \times 16}} ds_1 \wedge ds_1 \wedge \cdots \wedge ds_{16} \wedge ds_{16}\]

\[= 16! \cdot \frac{1}{2^{10}} \cdot \frac{1}{|s_9|^2} ds_1 \wedge ds_1 \wedge \cdots \wedge ds_{16} \wedge ds_{16}\]

at the point \(A(z_1) = A(0, \ldots, 0, s_9, 0, \ldots, 0, 0, \ldots, 0)\) \((s_9 = z_1)\). Consequently we have

\[\Omega_2 \cap \Omega_{27}|_{A(z_1)} = \Omega_2 \cap \Omega_{27}|_{A(z_1)} = D \left( \frac{s_9}{|s_9|} \right) \cdot |s_9|^{14} \cdot \frac{1}{2^{10}} |s_9|^{-24} ds_1 \wedge ds_1 \wedge \cdots \wedge ds_{16} \wedge ds_{16}\]

and the constant \(C_1\) is

\[C_1 = 2^{26}, \quad D(A) = 2^{26} |A|^{14}, \quad \Omega_2 \cap \Omega_{27} = 2^{26} |A|^{14} \frac{1}{16!} (\tau_0^{-1})^*(\omega^{P^2_0})^{16}.\]
Let \( \mathbb{C}[\mathcal{F}(3)^{\mathbb{C}}] = \sum \mathcal{P}_k[\mathcal{F}(3)^{\mathbb{C}}] \) be the algebra of polynomials (and of polynomial functions) on \( \mathcal{F}(3)^{\mathbb{C}} \) with the 27 complex variables \((\xi_1, \xi_2, \xi_3, z_1, w_1, y_1, v_1, x_1, u_1) \) \( i = 1, \ldots, 4 \) and \( \mathcal{P}_k \) is a subspace of degree \( k \) homogeneous polynomials.

In the following, we use the notation \( \mathcal{P}_k[\mathbb{X}_0] \) for the space of the restrictions of degree \( k \) homogeneous polynomials to \( \mathbb{X}_0 \) and \( \mathbb{C}[\mathbb{X}_0] := \sum \mathcal{P}_k[\mathbb{X}_0] \).

Recall the correspondence
\[
\gamma : \mathbb{C}[\mathbb{X}_0] = \sum \mathcal{P}_k[\mathbb{X}_0] \ni p \mapsto \gamma(p) = p \cdot t_0 \otimes \Omega_0 \in \Gamma_\mathbb{C}(\mathbb{L} \otimes K^\mathbb{C}, \mathbb{X}_0).
\]

We define a one-parameter family of inner products \( \{(*, *)_\varepsilon\}_{\varepsilon \in \mathbb{R}} \) on the space \( \Gamma_\mathbb{C}(\mathbb{L} \otimes K^\mathbb{C}, \mathbb{X}_0) \) by the following way that
\[
\Gamma_\mathbb{C}(\mathbb{L} \otimes K^\mathbb{C}, \mathbb{X}_0) \times \Gamma_\mathbb{C}(\mathbb{L} \otimes K^\mathbb{C}, \mathbb{X}_0) \ni (h \cdot t_0 \otimes \Omega_0, g \cdot t_0 \otimes \Omega_0) \mapsto \int_{\mathbb{X}_0} h \cdot \overline{g} \cdot (t_0, t_0)^\mathbb{L} \cdot ||A||^\varepsilon \cdot \Omega_0 \bigwedge \Omega_0
\]
\[
= 2^{26} \int_{\mathbb{X}_0} h \cdot \overline{g} \cdot e^{-2\pi ||A||^{1/2}} \cdot ||A||^{14 + \varepsilon} \cdot \{\tau_0^{-1}\}^*(dV_{P^2}) = (h, g)_\varepsilon,
\]
then through the map \( \gamma \) we also consider a one-parameter family of inner products on the space \( \mathbb{C}[\mathbb{X}_0] \).

According to the values of \( \varepsilon \) and \( k \), the integral (6.9) for functions \( f, g \in \mathcal{P}_k[\mathbb{X}_0] \) need not be finite. In fact, for \( k > -11 - \varepsilon/2 \) the integral (6.9) converges. This can be seen by making use of Corollary 6.4. Hence we define

**Definition 6.5.** We denote by \( \mathfrak{g}_\varepsilon \) the completion of the space \( \sum_{k > -11 - \varepsilon/2} \mathcal{P}_k[\mathbb{X}_0] \) with respect to the integral (6.9) and the remaining finite dimensional space \( \sum_{k \leq -11 - \varepsilon/2} \mathcal{P}_k[\mathbb{X}_0] \) with a suitable inner product.

7. Pairing with the Riemann volume form

Let \( dV_{p^2} \) be the Riemann volume form on \( P^2 \mathbb{O} \). The purpose in this section is to show

**Proposition 7.1.**
\[
\{q \circ \tau_0^{-1}\}^*(dV_{p^2}) = C_{RC}(A) \cdot ||A||^2 \left( \frac{1}{16!} \left( \omega_{p^2} \right)^{16} \right)(A).
\]
\[
\int_{\mathbb{X}_0} h \cdot \overline{g} \cdot e^{-2\pi ||A||^{1/2}} \cdot ||A||^{14 + \varepsilon} \cdot \{\tau_0^{-1}\}^*(dV_{P^2}) = (h, g)_\varepsilon,
\]

7.1. A local coordinates

For the determination of the constant \( C_{RC}(A/||A||) \) we choose a local coordinates around the point \( X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in P^2 \mathbb{O} \).

The condition \( X^2 = X \) for \( X \in P^2 \mathbb{O} \subset \mathcal{F}(3) \) is expressed as

\[
\begin{cases}
(t_1 + t_2)a + \theta(b) = a, & (t_1 + t_2)b + \theta(c) = b, & (t_2 + t_1)c + \theta(ab) = c, \\
(t_1^2 + \theta(c) + \theta(b))b = t_1, & t_2^2 + \theta(c) + a\theta(a) = t_2, & t_3^2 + \theta(a) + b\theta(b) = t_3, & \text{and} & \text{tr} X = t_1 + t_2 + t_3 = 1.
\end{cases}
\]
where \(a, b, c \in \mathbb{O}, t_i \in \mathbb{R}\). Using the last equation in (7.3), first 6 conditions are rewritten in the forms of

\[
\begin{cases}
t_1 a = \theta(bc), & t_2 b = \theta(ca), & t_3 c = \theta(ab), \\
(t_1 - 1/2)^2 + c\theta(c) + \theta(b)c = (t_1 - 1/2)^2 + |c|^2 + |b|^2 = 1/4, \\
(t_2 - 1/2)^2 + \theta(c)c + a\theta(a) = (t_2 - 1/2)^2 + |c|^2 + |a|^2 = 1/4, \\
(t_3 - 1/2)^2 + \theta(x)x + b\theta(b) = (t_3 - 1/2)^2 + |a|^2 + |b|^2 = 1/4.
\end{cases}
\]  

(7.4)

Let

\[
W_1 = \left\{(b, c) \in \mathbb{O}^2 \mid |c|^2 + |b|^2 < \frac{1}{8}\right\}.
\]

Then we can solve the equations (7.4) in the following order:

First, we solve the fourth equation in (7.4) with respect to \(t_1\) under the condition \(|c|^2 + |b|^2 < \frac{1}{8}\) with the solution

\[t_1 = \frac{1}{2} + \sqrt{\frac{1}{4} - |b|^2 - |c|^2} > \frac{1}{2},\]

Then the component \(a\) is given by \((b, c)\) by the first equation in (7.4) as

\[a = \frac{\theta(bc)}{t_1}.
\]

This solution \(a\) satisfies the inequality:

\[|a| = \frac{|bc|}{t_1} < 2 \cdot \frac{|b|^2 + |c|^2}{2} < \frac{1}{8}.
\]

With these we can solve the variable \(t_2\) in the fifth equation in (7.4) with the solution

\[t_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - |c|^2 - |a|^2},\]

where \(|c|^2 + |a|^2 < \frac{1}{8} + \frac{1}{8} < \frac{1}{4}\) implies that \(t_2 < \frac{1}{2}\).

Now, with these solutions expressed in terms of the variables \((b, c) \in W_1\) we define a map

\[
\mathcal{M} : W_1 \ni (b, c) \mapsto X = \begin{pmatrix} t_1 & c & \theta(b) \\ \theta(c) & t_2 & a \\ b & \theta(a) & 1 - t_1 - t_2 \end{pmatrix} \in P^2 \mathbb{O}.
\]

(7.6)

Then the matrix \(\mathcal{M}(b, c)\) satisfies the condition (7.4), so that we can choose components \((b, c)\) as a local coordinates around the point \(X_1\). We denote by \(\tilde{W}_1 = \mathcal{M}(W_1)\). The point \(X_1\) corresponds to \((0, 0) \in W_1\).

**Lemma 7.2.** In terms of the local coordinates

\[
(b, c) = \left(\sum_{i=0}^7 \{b\}_i e_i, \sum_{i=0}^7 \{c\}_i e_i\right)
\]

introduced above around the point \(X_1\), the Riemann volume form \(dv_{P^2 \mathbb{O}}\) at the point \(X_1\) is

\[
dv_{P^2 \mathbb{O}}(0, 0) = d\{b\}_0 \wedge \cdots \wedge d\{b\}_7 \wedge d\{c\}_0 \wedge \cdots \wedge d\{c\}_7.
\]

(7.7)

**Proof.** We can see this by

\[
d\mathcal{M}_{(0,0)} \left(\frac{\partial}{\partial \{b\}_0}\right)_{X_1} = \left(\frac{\partial}{\partial \{b\}_0}\right)_{X_1} + \sum_{i=1}^3 \frac{\partial t_i(0,0)}{\partial \{b\}_0} \left(\frac{\partial}{\partial t_i}\right)_{X_1} + \sum_{i=0}^7 \frac{\partial \{a\}_i(0,0)}{\partial \{b\}_0} \left(\frac{\partial}{\partial \{a\}_i}\right)_{X_1} = \left(\frac{\partial d\{b\}}{\partial \{b\}_0}\right)_{X_1},
\]
where we know

\[ \frac{\partial t_1(0,0)}{\partial b_0} = \frac{-b_0}{\sqrt{1 - |b|^2 - |c|^2}}, \]

\[ \frac{\partial t_2(0,0)}{\partial b_0} = \frac{-2b_0 - 2\sum_{i=0}^{3} \alpha_i}{2\sqrt{1 - |b|^2 - |c|^2}}, \]

\[ = 0, \text{ etc.,} \]

since \( a(0,0) = \sum \alpha_i e_i = 0 \). Other derivatives are also

\[ \frac{\partial t_1}{\partial (b_i)} |_{(0,0)} = 0, \frac{\partial t_i}{\partial (c_j)} |_{(0,0)} = 0, \frac{\partial (a_i)}{\partial (b_j)} |_{(0,0)} = 0, \frac{\partial (a_j)}{\partial (c_i)} |_{(0,0)} = 0. \]

Hence

\[ dM_{(0,0)} \left( \frac{\partial}{\partial (b_i)} \right) = \left( \frac{\partial}{\partial (b_i)} \right) x_i, \quad dM_{(0,0)} \left( \frac{\partial}{\partial (c_i)} \right) = \left( \frac{\partial}{\partial (c_i)} \right) x_i. \]

Then the metric tensor \( g_{ij} \) with respect to the coordinates \((b, c)\) at the point \((b, c) = (0, 0)\) is \( g_{ij} = \delta_{ij} \). \( \square \)

7.2. Explicit determination of the pairing with the Riemann volume form

Let

\[ A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \in X_0, \text{ where } \xi_i \in \mathbb{C}, z, y, x \in \mathbb{C} \otimes \mathcal{O}. \]

Put \( \tau_0^{-1}(A) = (X(A), Y(A)) \), then

\[ X(A) = A + \frac{\bar{z}}{2||A||} + A \circ \frac{\bar{z}}{||A||^2} \quad (\text{see (3.10)}) \]

\[ = \begin{pmatrix} \xi_1 + \frac{\bar{z}}{2||A||} + \frac{1}{2||A||^2} \left( \xi_1 \bar{z} + |z|^2 + |y|^2 \right) \\ \frac{\theta(z) \bar{z}}{2||A||} + \frac{\xi_2 \bar{z}}{2||A||^2} + \frac{1}{2||A||^2} \left( \xi_2 |z|^2 + |x|^2 \right) \\ \frac{y \bar{z}}{2||A||} + \frac{\xi_3 \bar{z}}{2||A||^2} + \frac{1}{2||A||^2} \left( \xi_3 |z|^2 + |x|^2 + |y|^2 \right) \end{pmatrix}. \]

From the above expression of \( \tau_0^{-1}(A) = (X(A), Y(A)) \) we consider two components of the matrix \( X(A) \in P^2 \mathcal{O} \) for \( A \in U_{z_1} \): \( c = z + \bar{z}, b = y + \bar{y} \)

\[ e = x + \bar{x}, \quad b = y + \bar{y}, \quad \text{such that } X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

Take a point \( A_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in U_{z_1} \), then \( q \circ \tau_0^{-1}(A_1) = X_1 \). On the other hand the point \( A_1 \in X_0 \) corresponds to the matrices

\[ A_1 \leftrightarrow (\xi_1, \xi_2, \xi_3, Z, W, Y, V, X, U) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

(see the matrix representation (4.3) of the octonions and vector representation (4.6) of elements in \( J(3) \)).

So we consider points \( A \in U_{z_1} \) around a point

\[ P_{z_1}(A) = (\xi_2, z_1, z_2, z_3, w_1, w_2, w_3, y_2, y_4, v_1, v_3, x_3, x_4, u_2, u_4) \]

\[ = (-1, \sqrt{-1}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0). \]

By the explicit expression (4.16) of the other dependent variables \((\xi_1, \xi_3, y_1, y_3, v_2, v_1, x_1, x_2, u_1, u_2)\) in the matrix representation \((\xi_1, \xi_2, \xi_3, Z, W, Y, V, X, U)\) of \( A_1 \) is \((0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).\)

For avoiding the confusion of the expression of octonion and its matrix expression by the map \( \rho_0 \), recall the correspondence (2.2) and (2.3)).
Now we determine the differentials modulo anti-holomorphic differentials

\[
\{q \circ \tau_0^{-1}\}^* (dc) = \sum_{i=0}^7 \{q \circ \tau_0^{-1}\}^* (d(c)_{i}) \otimes e_i, \quad \text{and}
\]

\[
\{q \circ \tau_0^{-1}\}^* (db) = \sum_{i=0}^7 \{q \circ \tau_0^{-1}\}^* (d(b)_{i}) \otimes e_i,
\]

at the point \(A_1\).

Each component of \(b\) and \(c\) is given by

\[
\{c\}_i = \frac{\{z\}_i + \{\bar{z}\}_i}{2|A||} + \frac{-\xi_3\{\bar{z}\}_i - \bar{\xi}_3\{z\}_i + \{\theta(zy + x\bar{y})\}_i}{2|\Omega|^2},
\]

and

\[
\{b\}_i = \frac{\{y\}_i + \{\bar{y}\}_i}{2|A||} + \frac{-\xi_2\{\bar{y}\}_i - \bar{\xi}_2\{y\}_i + \{\theta(zx + y\bar{x})\}_i}{2|\Omega|^2}.
\]

The pull-back \(\{q \circ \tau_0^{-1}\}^* (dv_{P_{2\Omega}})\) is expressed as

\[
\{q \circ \tau_0^{-1}\}^* (dv_{P_{2\Omega}}) = \sum_{i=0}^{16} \Sigma_i, \quad \text{with} \quad \Sigma_i \in \Gamma \left( \bigwedge \text{tensor}^i (\Omega) \otimes \bigwedge \text{tensor}^{i-1} (\Omega) \right).
\]

In particular,

\[
\Sigma_i \wedge \overline{\Sigma}_i = 0 \text{ for } i \geq 1, \quad \text{and} \quad \Sigma_j = \Sigma_{16-j}.
\]

Hence for the determination of the constant \(C_{RC}(Y/\|Y\|)\), it is enough to consider the terms consisting of holomorphic differentials

\[d\xi_2, d\xi_3, d\xi_4, d\xi_5, d\xi_6, d\xi_7, d\xi_8, d\xi_9, d\xi_{10}, d\xi_{11}, d\xi_{12}, d\xi_{13}, d\xi_{14}, d\xi_{15}, d\xi_{16} \]

and may ignore the anti-holomorphic differentials \(d\xi_4, d\xi_5, d\xi_6, d\xi_7\), etc. so that in the expression of equalities below we denote them as \(* \equiv \ast\), which means both sides coincide modulo anti-holomorphic differentials.

Here we gather up relations of the holomorphic differentials of dependent variables by independent variables at the point \(A_1\). See (4.16) for the explicit expression of each variable \(\xi_1, \xi_3, \cdots, x_2, x_1, x_2, u_1, u_2\) in terms of independent variables \(\xi_2, z_1, \cdots, x_3, x_4, u_2, u_4\).

All the equalities in the Lemmas blow hold at the point \(A_1\).

**Lemma 7.3.**

\[\|A_1\| = 2, \quad d|A||_2^2 \equiv \{\pi d\pi_z + \pi d\pi_z + \pi \xi_1 d\xi_1 + \pi \xi_2 d\xi_2\} \equiv -2d\xi_2,\]

\[z_4(A_1) = \sqrt{-1}, \quad d\xi_4|_{A_1} = -d\xi_2 - 2\sqrt{-1}d\xi_2, \quad \xi_3(A_1) = 0, \quad d\xi_3|_{A_1} = 0, \quad \xi_3|_{A_1} = -d\xi_2,\]

\[dx_1|_{A_1} = \sqrt{-1}dx_1, \quad dx_1|_{A_1} = -\sqrt{-1}dx_1, \quad dy_1|_{A_1} = -\sqrt{-1}dy_1, \quad dx_2|_{A_1} = -\sqrt{-1}dx_2,\]

\[dy_2|_{A_1} = \sqrt{-1}dy_2, \quad dw_1|_{A_1} = -\sqrt{-1}dw_1, \quad dw_3|_{A_1} = -\sqrt{-1}dw_3, \quad du_2|_{A_1} = -\sqrt{-1}du_2,\]

\[dw_4|_{A_1} = -\sqrt{-1}dw_4, \quad dw_5|_{A_1} = -\sqrt{-1}dw_5, \quad dw_6|_{A_1} = -\sqrt{-1}dw_6, \quad dw_7|_{A_1} = -\sqrt{-1}dw_7, \quad dw_8|_{A_1} = -\sqrt{-1}dw_8, \quad dw_9|_{A_1} = -\sqrt{-1}dw_9, \quad dw_{10}|_{A_1} = -\sqrt{-1}dw_{10}.\]

**Lemma 7.4.**

\[d(c)_{i|A_1} \equiv \frac{d\{z\}_i}{2\|A_1\|^2} = \frac{\{z\}_i + \{\overline{z}\}_i}{2\|A_1\|^2}, d\xi_2,\]

and for each \(i = 0, \cdots, 7\)

\[d(c)_{0|A_1} \equiv \frac{-\sqrt{-1}d\xi_2}{2^2}, \quad d(c)_{1|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_1}{2^2}, \quad d(c)_{2|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_3}{2^2}, \quad d(c)_{3|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}},\]

\[d(c)_{4|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}}, \quad d(c)_{5|A_1} \equiv \frac{dw_1 - dw_4}{2^3\sqrt{-1}}, \quad d(c)_{6|A_1} \equiv \frac{dw_1 - dw_3}{2^3}, \quad d(c)_{7|A_1} \equiv \frac{dw_2 + dw_3}{2^3\sqrt{-1}},\]

\[d(c)_{0|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_1}{2^2}, \quad d(c)_{1|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_3}{2^2}, \quad d(c)_{2|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}}, \quad d(c)_{3|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}},\]

\[d(c)_{4|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}}, \quad d(c)_{5|A_1} \equiv \frac{dw_1 - dw_4}{2^3\sqrt{-1}}, \quad d(c)_{6|A_1} \equiv \frac{dw_1 - dw_3}{2^3}, \quad d(c)_{7|A_1} \equiv \frac{dw_2 + dw_3}{2^3\sqrt{-1}},\]

\[d(c)_{0|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_1}{2^2}, \quad d(c)_{1|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_3}{2^2}, \quad d(c)_{2|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}}, \quad d(c)_{3|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}},\]

\[d(c)_{4|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}}, \quad d(c)_{5|A_1} \equiv \frac{dw_1 - dw_4}{2^3\sqrt{-1}}, \quad d(c)_{6|A_1} \equiv \frac{dw_1 - dw_3}{2^3}, \quad d(c)_{7|A_1} \equiv \frac{dw_2 + dw_3}{2^3\sqrt{-1}},\]

\[d(c)_{0|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_1}{2^2}, \quad d(c)_{1|A_1} \equiv \frac{d\xi_2 - \sqrt{-1}dz_3}{2^2}, \quad d(c)_{2|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}}, \quad d(c)_{3|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}},\]

\[d(c)_{4|A_1} \equiv \frac{d\xi_2 + dz_3}{2^2\sqrt{-1}}, \quad d(c)_{5|A_1} \equiv \frac{dw_1 - dw_4}{2^3\sqrt{-1}}, \quad d(c)_{6|A_1} \equiv \frac{dw_1 - dw_3}{2^3}, \quad d(c)_{7|A_1} \equiv \frac{dw_2 + dw_3}{2^3\sqrt{-1}},\]
where we can ignore the term \( \{ z \} \), since \( \{ x \} \) and \( \{ \xi \} \) are zero and \( d\{ \xi \} = \sum_{\alpha, \beta, \gamma = 0} \{ \xi \} \alpha d\{ x \} \beta = \{ \xi \} d\{ x \} \alpha, \) and for \( i = 1, \cdots, 7 \),

\[
\begin{align*}
d\{ b \}_{1, A_i} &\equiv \frac{dx_1 - \sqrt{-1}dy_1}{2^2}, \\
d\{ b \}_{2, A_i} &\equiv \frac{dy_2 - \sqrt{-1}dx_3}{2^2}, \\
d\{ b \}_{3, A_i} &\equiv \frac{dx_3 - \sqrt{-1}Idx_4}{2^2}, \\
d\{ b \}_{4, A_i} &\equiv \frac{dv_1 + \sqrt{-1}dv_3}{2^2}, \\
d\{ b \}_{5, A_i} &\equiv \frac{dv_4 + \sqrt{-1}dv_1}{2^2}, \\
d\{ b \}_{6, A_i} &\equiv \frac{-dv_2 + \sqrt{-1}dv_2}{2^2}, \\
d\{ b \}_{7, A_i} &\equiv \frac{dv_2 - \sqrt{-1}dv_3}{2^2}.
\end{align*}
\]

Based on these data

**Proposition 7.5.** At the point \( A_i \), the holomorphic component of the pull-back \( \{ q \circ \tau_0^{-1} \}^* (dv_{F_0}) \) is equal to

\[
\begin{align*}
\{ q \circ \tau_0^{-1} \}^* (dv_{F_0}) &\equiv \{ q \circ \tau_0^{-1} \}^* (d\{ x \})_0 \wedge \cdots \wedge d\{ x \}_7 \wedge d\{ b \}_0 \wedge \cdots \wedge d\{ b \}_7 |_{A_i} \\
&= \frac{dx_1 - \sqrt{-1}dy_1}{2^2} \wedge \frac{dy_2 - \sqrt{-1}dx_3}{2^2} \wedge \frac{dx_3 - \sqrt{-1}Idx_4}{2^2} \wedge \frac{dv_1 + \sqrt{-1}dv_3}{2^2} \wedge \frac{dv_4 + \sqrt{-1}dv_1}{2^2} \wedge \frac{dv_2 - \sqrt{-1}dv_3}{2^2} \\
&\wedge \frac{dv_2 - \sqrt{-1}dv_3}{2^3} \wedge \frac{dx_2 - \sqrt{-1}dx_2}{2^3} \wedge \frac{dx_4 - \sqrt{-1}dx_4}{2^2} \wedge \frac{dx_4 - \sqrt{-1}dy_4}{2^2} \\
&\wedge \frac{dy_2 - \sqrt{-1}dx_2}{2^2} \wedge \frac{dx_3 - \sqrt{-1}dx_3}{2^2} \wedge \frac{dv_3 + \sqrt{-1}dv_4}{2^2} \wedge \frac{dv_3 + \sqrt{-1}dv_1}{2^2} \\
&\wedge \frac{dv_2 - \sqrt{-1}dv_3}{2^2} \wedge \frac{dx_3 - \sqrt{-1}dx_3}{2^2} \wedge \frac{dx_4 - \sqrt{-1}dx_4}{2^2} \wedge \frac{dx_4 - \sqrt{-1}dy_4}{2^2} \\
&= \frac{1}{2^{21}} \cdot \frac{dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_2 \wedge dx_4 \wedge dx_2 \wedge dy_1 \wedge dy_4 \wedge dx_3 \wedge dz_1 \wedge dz_2 \wedge dz_3 \wedge dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4 \wedge d\xi_2}{2^{21} \sqrt{-1}}.
\end{align*}
\]

Hence we can derive the relation (7.1) in Proposition 7.1.

By the formula (7.1) we have an expression of the Bargmann type transformation (5.12).

**Corollary 7.6.**

\[
\begin{align*}
\mathcal{B}(h)(X) : dv_{F_0}(X) &= \{ q \circ \tau_0^{-1} \} \circ \left( h \cdot g_0 \cdot \{ q \circ \tau_0^{-1} \} \circ (dv_{F_0}) \right) \wedge (1) \\
&= \{ q \circ \tau_0^{-1} \} \circ \left( h \cdot g_0 \cdot 2^6 \cdot \{ A \}^3 \cdot \frac{1}{16} \{ \tau_0^{-1} \} \circ \left( \omega_{P^2 O} \omega_{P^2 O} \right) \right) \\
&= 2^6 \cdot q_0 \left( \left( h(X, \ast) \right) \cdot e^{-\sqrt{2} \pi ||x||^2} \ast ||x||^6 \cdot dv_{T \ast}(P^2 O)(X, \ast) \right).
\end{align*}
\]

8. Invariant polynomials and harmonic polynomials on the Jordan algebra \( J(3) \)

In this section we describe invariant polynomials on \( J(3) \) and commuting differential operators with constant coefficients under the action by the automorphism group \( F_4 \) of the Jordan algebra \( J(3) \) (see [He] and [HL] for the framework here and [Yo] for necessary properties of \( F_4 \) in relation with \( P^2 O \)).

8.1. Correspondence between polynomials and differential operators with constant coefficients

Let

\[
\mathbb{R}^N \times \mathbb{R}^N \ni (x, \xi) = (x_1, \ldots, x_N, \xi_1, \ldots, \xi_N) \mapsto \langle x, \xi \rangle = \sum x_i \xi_i \in \mathbb{R},
\]

be the standard non-degenerate symmetric bi-linear form. We also use the same notation for its extension to the complex bi-linear form defined on \( \mathbb{C}^N \times \mathbb{C}^N \).

Let \( \alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{ 0 \})^N \) be a multi-index. Differential operators \( D_x \) with constant
(complex) coefficients are expressed in the form

\[ D = D_x = \sum_{|\alpha| \leq k} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \sum a_\alpha D^\alpha_x, \]

where \( a_\alpha \in \mathbb{C} \) and \( D^\alpha_x := \frac{\partial^{\alpha_1 + \cdots + \alpha_N}}{\partial x^\alpha} = \frac{\partial^{\alpha_1} \cdots \partial x^{\alpha_N}}{\partial x^\alpha}. \)

Let \( D = \sum a_\alpha D^\alpha_x \) be a constant coefficient partial differential operator defined on \( \mathbb{R}^N \), then by the relation

\[ (8.1) \]

\[ e^{-(x, \xi)} D_x(e^{(x, \xi)})(x) = \sum a_\alpha \xi^\alpha := Q^D(\xi), \]

where \( \xi^\alpha = \xi^1 \alpha_1 \cdots \xi^N \alpha_N \), the correspondence \( D \leftrightarrow Q^D(\xi) \) is bijective (\( Q^D \) is the symbol of the differential operator \( D \)), that is, the algebra \( \mathbb{C}[\mathbb{R}^N] = \mathbb{C}[x_1, \ldots, x_N] = \sum_{k=0}^\infty D_k[x_1, \ldots, x_N] \) of (complex coefficient) polynomials on \( \mathbb{R}^N \) and the algebra \( D[x_1, \ldots, x_N] = \sum_{k=0}^\infty D_k[x_1, \ldots, x_N] \) of linear differential operators with constant (complex) coefficients are isomorphic. Here we denote by \( D_k[x_1, \ldots, x_N] \) the subspace of homogeneous polynomials of degree \( k \) and by \( D_k = D_k[x_1, \ldots, x_N] \) the subspace consisting of homogeneous differential operators with constant coefficients of order \( k \).

We will denote the differential operator corresponding to a polynomial \( Q \in \mathbb{C}[x_1, \ldots, x_N] \) by \( D_Q \).

Let \( g \in \text{GL}(N, \mathbb{R}) \) and define \( \mathcal{P}_g : \mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C}[x_1, \ldots, x_N] \) an algebra isomorphism in the natural way:

\[ Q = Q(x) = \sum a_\alpha \cdot x^\alpha \mapsto \mathcal{P}_g(Q)(x) = Q(g^{-1}(x)) = \sum a_\alpha \cdot (g^{-1}(x))^\alpha, \]

where \( g^{-1} = \left\{ g^{-1}_{1,i} \right\}_{1,j} \) and \( (g^{-1}(x))^\alpha = \left( \sum_i g^{-1}_{1,i} x_i \right)^{\alpha_1} \cdots \left( \sum_i g^{-1}_{N,i} x_i \right)^{\alpha_N}. \)

The following relation will be seen easily.

**Lemma 8.1.** Let \( g \in \text{GL}(N, \mathbb{R}) \) and \( D \) a linear differential operator with constant coefficients. Then, \( \mathcal{P}_g \circ D = D \circ \mathcal{P}_g \) on the space of the whole polynomial functions, if and only if \( Q^D(\xi) = Q^{D\circ \mathcal{P}_g}(\xi) \).

Next, we introduce a Hermitian inner product \( \langle \cdot, \cdot \rangle \) on the space of polynomials \( \mathbb{C}[x_1, \ldots, x_N] \) by the following way:

we fix the coordinates \( (x_1, \ldots, x_N) \in \mathbb{R}^N \) and define the inner product between monomials \( x^\alpha \) and \( x^\beta \) by

\[ \langle x^\alpha, x^\beta \rangle := \alpha_1! \cdots \alpha_N! \cdot \delta_{\alpha_1, \beta_1} \cdots \delta_{\alpha_N, \beta_N} := \alpha! \cdot \delta_{\alpha, \beta}. \]

The inner product on the space \( \mathbb{C}[x_1, \ldots, x_N] \) introduced above is used only in this section.

Then the following properties will be seen easily too.

**Lemma 8.2.** Let \( D = \sum a_\alpha D^\alpha_x \) be a differential operator with constant coefficients. Then for any polynomial \( Q = \sum C_\beta x^\beta \)

\[ \langle Q^D, Q \rangle = \langle \sum a_\alpha x^\alpha, \sum C_\beta x^\beta \rangle = \sum_\alpha \alpha! \cdot \delta_{\alpha, \beta} \cdot a_\alpha \cdot C_\beta \]

\[ = D(\overline{Q})(0) =: \langle D, Q \rangle = \langle (D_Q, \overline{P}) \rangle, \]

where we replace the variables \( \xi_i \) to \( x_i \). In particular, if the order of the differential operator \( D \) and the degree of a polynomial \( Q \) coincides, then

\[ D(\overline{Q})(x) = D(\overline{Q})(0), \]

and the spaces \( \mathcal{P}_k \) and \( \mathcal{P}_l \) are always orthogonal, if \( k \neq l \).
It holds a kind of associativity:

\[(D_1 \circ D_2, Q) = D_1 \circ D_2(Q)(0) = \langle Q^{D_1}, D_2(Q) \rangle = \langle [Q^{D_1}, D_2(Q)], D_1, D_2(Q) \rangle.\]

The equation (8.3) can be understood that the Hermitian inner product we introduced is a pairing between the space \(D[x_1, \ldots, x_N]\) of differential operators with constant coefficients and the space of polynomials, especially by this pairing the space \(D(\mathbb{R}^N)\) is identified with the (restricted) dual space \(D(\mathbb{R}^N) \cong \bigoplus_{k=0}^{\infty} \mathcal{P}_k[x_1, \ldots, x_N]^\ast\) of \(\mathbb{C}[x_1, \ldots, x_N]\). With respect to the action of \(g \in \text{GL}(N, \mathbb{R})\) on \(\mathbb{C}[x_1, \ldots, x_N]\) the dual action of \(g\) on \(D(\mathbb{R}^N)\) is

\[D(\mathbb{R}^N) \ni D \mapsto \mathcal{P}_g^{-1} \circ D \circ \mathcal{P}_g =: \mathcal{P}_g^*(D)\]

and satisfies the relation

\[(\mathcal{P}_g^*(D), f) = (D, \mathcal{P}_g(f)), \quad D \in \mathcal{D}[x_1, \ldots, x_N], \quad f \in \mathbb{C}[x_1, \ldots, x_N].\]

**8.2. Trace function and invariant polynomials**

We recall two important properties Theorems 8.3 and 8.4 on the action of the group \(F_4\) on \(\mathcal{J}(3)\). Also the properties (3.5), (3.6) should be reminded in this section (see [SV] and [Yo]).

**Theorem 8.3.** For any \(A \in \mathcal{J}(3)\), there exists an element \(\alpha \in F_4\) such that

\[\alpha(A) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix},\]

where the triple of quantities \(\{\xi_i\}\) depends only on \(A\) and does not depend on such an element \(\alpha \in F_4\) which sends \(A\) to a diagonal matrix in \(\mathcal{J}(3)\).

**Theorem 8.4.** The representation of \(F_4\) to \(\mathcal{J}(3)\) is decomposed into two mutually orthogonal irreducible subspaces, that is

\[\mathcal{J}(3) = \mathcal{J}_0(3) \oplus \mathbb{R} \cdot \text{Id},\]

where \(\mathcal{J}_0(3) = \{ A \in \mathcal{J}(3) \mid \text{tr}(A) = 0 \}\) and \(\text{Id}\) is the \(3 \times 3\) identity matrix which is the fixed point in \(\mathcal{J}(3)\) under the action of \(F_4\).

It holds the same decomposition in the complexified Jordan algebra \(\mathcal{J}(3)^\mathbb{C}\) by the action of the complex group \(F_4^\mathbb{C}\).

In this section we express

\[\mathcal{J}(3) \ni A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \leftrightarrow (z_0, \ldots, z_7, y_0, \ldots, y_7, x_0, \ldots, x_7, \xi_1, \xi_2, \xi_3)\]

with the coefficients of

\[z = \sum \{x\}_i e_i = \sum z_i e_i, \quad y = \sum \{y\}_i e_i = \sum y_i e_i, \quad x = \sum \{x\}_i e_i = \sum x_i e_i\]

and do not use the notation \(\{z\}_i, \{y\}_i, \{x\}_i\) (see (4.6)). We also denote these coordinates as

\[(z_0, \ldots, z_7, y_0, \ldots, y_7, x_0, \ldots, x_7, \xi_1, \xi_2, \xi_3) = (s_1, \ldots, s_{24}, s_{25}, s_{26}, s_{27})\]

or

\[(z_0, \ldots, z_7, y_0, \ldots, y_7, x_0, \ldots, x_7, \xi_1, \xi_2, \xi_3) = (s_1, \ldots, s_{24}, \xi_1, \xi_2, \xi_3).\]

We denote the (complex valued) polynomial algebra over \(\mathcal{J}(3)\) by \(\mathbb{C}[\mathcal{J}(3)]\) with the independent
variables \{z_i, \ldots, x_i, \xi_1, \xi_2, \xi_3\} and also regard it as the algebra of polynomial functions. It is equipped with a Hermitian inner product explained in the preceding subsection § 8.1.

Then, we can identify by the isometric way the space \( P_1[J(3)] \equiv (J(3)^C)^* \) with the space \( J(3)^C \) through the correspondence

\[
J(3)^C \ni A \mapsto h_A \in P_1[J(3)], \quad h_A(X) = \text{tr} (X \circ A) = (X, A)^{J(3)^C}.
\]

The action of the group \( F_4 \) is extended to the space \( \mathbb{C}[J(3)] \) as denoted in §8.1:

\[
\mathbb{C}[J(3)] \ni Q \mapsto (P_g(Q)(X) := Q(g^{-1}(X))
\]

and the extended action leaves the degree of the polynomials and the inner product.

**Definition 8.5.** We denote a subspace in each \( P_k[J(3)] \) by \( I_k \) consisting of invariant polynomials under the extended action of the group \( F_4 \) and put \( I = I_{F_4} = \sum_{k \geq 0} I_k \), the algebra of invariant polynomials under the action of the Lie group \( F_4 \) on \( J(3) \).

By the property (3.4), the functions \( J(3) \ni A \mapsto \text{tr} (A^k) := T_k(A) \) is well-defined and are invariant polynomials (of course, these are also well defined on \( J(3)^C \)). Then,

**Proposition 8.6.** All the invariant polynomials in \( P_k[J(3)] \) are given by the linear sums of polynomials of the products

\[
T_1^{i_1} \cdot T_2^{i_2} \cdot T_3^{i_3}
\]

under the condition that \( i_1 + 2i_2 + 3i_3 = k \) \((0 \leq i_1, i_2, i_3 \leq k)\) and

\[
\dim \mathbb{C} I_k = \text{number of the solutions } (i_1, i_2, i_3) \text{ under the condition } i_1 + 2i_2 + 3i_3 = k \text{ } \text{ } \text{(8.11)}
\]

Proof. Let \( f \in I_k \) be an invariant polynomial. Then by the property (8.7) in Theorem 8.3 and the invariance of the trace function (3.5), \( f(A) = f(\alpha(A)) = f \begin{pmatrix}
\xi_1 & 0 & 0 \\
0 & \xi_2 & 0 \\
0 & 0 & \xi_3
\end{pmatrix} \) depend only on the triple \( \{\xi_1\}_{i=1}^3 \) which appears when it is expressed as a diagonal matrix given in the above Theorem 8.3.

Let \( \sigma_1 : J(3) \rightarrow J(3) \) be a permutation defined by

\[
\sigma_1 : J(3) \ni A \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in J(3).
\]

Likewise we can define other two permutations \( \sigma_2 \) and \( \sigma_3 \) among the quantities \( \{\xi_i\} \) by the matrices

\[
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \text{ respectively. These are elements in } F_4 \text{ and satisfy } f(\sigma_i(A)) = f(A).
\]

Hence the invariant polynomial ring \( I = I_{F_4} = \sum_{k \geq 0} I_k \in \mathbb{C}[J(3)] \) is generated by three elementary symmetric polynomials

\[
\xi_1 + \xi_2 + \xi_3, \quad \xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1 \quad \text{and} \quad \xi_1 \xi_2 \xi_3.
\]

This is equivalent to say that the subalgebra of invariant polynomials of positive degree \( I_+ := \sum_{k \geq 1} I_k \) (i.e., without constant terms) is generated by three invariant polynomials

\[
T_1(A) = \text{tr} (A) = \sum_{i=1}^3 \xi_i, \quad T_2(A) = \text{tr} (A^2) = 2 \sum_{i=0}^7 (z_i^2 + y_i^2 + x_i^2) + \sum_{i=1}^3 \xi_i^2 = ||A||^2.
\]
The last formula (8.11) is given by solving the equation $i_1 + 2 \cdot i_2 + 3 \cdot i_3 = k$ (see Appendix). 

In the proof above we used a property of the multiplication low in the octonion $\mathbb{O}$:

\[
\Re(x \cdot yz) = \Re(y \cdot zx) = \Re(z \cdot xy), \quad \Re(zx \cdot y) = \Re(\theta(y) \cdot \theta(zx)) = \Re(y \cdot zx) \quad \text{and similar identities.}
\]

Next, we mention (see Lemma 8.1 and Theorem 8.3)

**Proposition 8.7.** The invariant polynomial ring $I = \sum I_k$ and differential operators with constant coefficients commuting with the $F_4$ action are isomorphic. Especially, the differential operators corresponding to the generators $T_1, T_2$ and $T_3$ of the invariant polynomial ring are

\[
T_1 = e^{-(x, \xi)} L(e^{(x, \xi)}) (x) \longleftrightarrow L = L(z, y, x, \xi_1, \xi_2, \xi_3) := \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial \xi_3},
\]

\[
T_2 \longleftrightarrow -\Delta := 2 \sum_{i=0}^7 \frac{\partial^2}{\partial z_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial x_i^2},
\]

\[
T_3 \longleftrightarrow \Gamma := \sum \frac{\partial^3}{\partial \xi_1^3} + 3 \left( \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) \circ \sum_{i=0}^7 \frac{\partial^2}{\partial z_i^2} + 3 \left( \frac{\partial}{\partial \xi_3} + \frac{\partial}{\partial \xi_1} \right) \circ \sum_{i=0}^7 \frac{\partial^2}{\partial y_i^2}
\]

\[+ 3 \left( \frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial \xi_3} \right) \circ \sum_{i=0}^7 \frac{\partial^2}{\partial x_i^2} + 6 \sum_{i,j,k=0}^7 \frac{\partial^3}{\partial x_i \partial y_j \partial z_k}.
\]

The second operator is the Laplacian on the Euclidean space $\mathbb{R}^{27} \cong J(3)$.

The last term of the operator $\Gamma$ consists of $8^3$ partial differential operators of the form $\frac{\partial^3}{\partial x_i \partial y_j \partial z_k}$ with suitable signs.

We define an $F_4$-invariant subspace $H_k$ in $\mathcal{P}_k[J(3)]$ inductively and call polynomials therein “Cayley harmonic polynomial”.

**Definition 8.8.**

0. $H_0$ is the space of the constant functions $= \mathcal{P}_0$.

1. $H_1 = \{ \text{the linear functions: } \sum_{i=0}^7 (a_i z_i + b_i y_i + c_i x_i) + \sum_{i=1}^3 d_i \xi_i \text{ with } \sum d_i = 0 \}$,
   this space is isomorphic to $\{ B \in J(3)^7 \mid \text{tr} (B) = 0 \}$,
   and we have an orthogonal decomposition $\mathcal{P}_1 = H_1 \oplus H_0 I_1$.

\[\vdots\]

k. $H_k$ is the orthogonal complement of the space $\left( \sum_{i=0}^{k-1} H_i \cdot I_{k-1} \right)$ taken in $\mathcal{P}_k$.

\[\mathcal{P}_k[J(3)] = H_k \oplus \sum_{i=1}^k H_{k-i} \cdot I_i, \quad (8.13)\]
The subspace $H_k$ can be seen as the space corresponding to the space of harmonic polynomials for the case of $SO(n)$ acting on $\mathbb{R}^n$. In fact

**Proposition 8.9.** Let $\mathcal{S}_k$ be

$$\mathcal{S}_k := \{ Q \in \mathcal{P}_k[J(3)] \mid L(Q) = 0, \Delta(Q) = 0, \Gamma(Q) = 0 \}.$$ 

Then $\mathcal{S}_k = H_k$.

Before proving this Proposition we show a

**Lemma 8.10.** For each $k$ the space

$$I_k + H_1 \cdot I_{k-1} + \cdots + H_{k-1} \cdot I_1 = I_k + \mathcal{P}_1 \cdot I_{k-1} + \cdots + \mathcal{P}_{k-1} \cdot I_1.$$ 

The right hand side is not a direct sum, while the left hand side is a direct sum which will be proved later after several preparations.

**Proof.** It is apparent

$$\sum_{i=0}^{k-1} H_i \cdot I_{k-i} \subset \sum \mathcal{P}_i \cdot I_{k-i}.$$ 

Since $I_2 \supset I_1 \cdot I_1$, 

$$H_1 \cdot I_1 + I_2 \supset (H_1 \cdot I_1 + I_1 \cdot I_1) + I_2 = \mathcal{P}_1 \cdot I_1 + I_2.$$ 

Hence, $H_1 \cdot I_1 + I_2 = \mathcal{P}_1 \cdot I_1 + I_2$.

Assume

$$\sum_{i=0}^{j-1} H_i \cdot I_{j-i} = \sum_{i=0}^{j-1} \mathcal{P}_i \cdot I_{j-i}, \text{ for } j \leq k.$$ 

Then using the property $I_j \supset I_a \cdot I_b$ for $a + b = j$, we can show inductively

$$\sum_{i=0}^{j-1} H_i \cdot I_{j-i} \supset \sum_{i=0}^{j-1} \mathcal{P}_i \cdot I_{j-i}, \text{ for } j \leq k.$$ 

\[\square\]

**Proof of the Proposition 8.9.**

The symbols of operators $L$, $\Delta$ and $\Gamma$ are $T_1$, $T_2$ and $T_3$, respectively. By the relation (8.5) we have

\[
\langle Q, \mathcal{P}_{k-1} T_1 \rangle = \langle L(Q), \mathcal{P}_{k-1} \rangle, \quad \langle Q, \mathcal{P}_{k-2} T_2 \rangle = \langle \Delta(Q), \mathcal{P}_{k-2} \rangle,
\]

\[
\langle Q, \mathcal{P}_{k-3} T_3 \rangle = \langle \Gamma(Q), \mathcal{P}_{k-3} \rangle.
\]

Then, from above Lemma 8.10 the coincidence of $\mathcal{S}_k$ and $H_k$ will be apparent.

\[\square\]

**Lemma 8.11.**

$$L(T_1) = 3, \quad L(T_2) = 2T_1, \quad L(T_3) = 3T_2,$$

$$\Delta(T_2) = 198, \quad \Delta(T_3) = 198T_1, \quad \Gamma(T_3) = 562.$$ 

**Remark.** Invariant polynomials above are not necessarily orthogonal. For example

$$\langle T_2, T_1^2 \rangle = L^2(T_2)(0) = L(L(T_2))(0) = 2L(T_1)(0) = 6.$$
Next we show that the sum (8.14) is a direct sum as mentioned in the above Lemma 8.10 (summed up in Proposition 8.17).

By definition it is enough to show the sum

\[ H_{k-1} \cdot I_1 + \cdots + H_1 \cdot I_{k-1} + I_k \]

is a direct sum. For this purpose we prepare several lemmas.

**Lemma 8.12.** The map \( L : I_k \to I_{k-1} \) is surjective for all \( k = 1, 2, \cdots \).

**Proof.** Let \( t : I_k \to I_{k+1} \) be a map defined by

\[ t(T) = T_1 \cdot T, \]

then \( t \) is injective. In fact, if there is an element \( T \in I_k \) satisfying

\[ L \circ t(T) = L(T_1 \cdot T) = 3T + T_1 \cdot L(T) = 0, \]

then again we have

\[ 3L(T) + 3L(T) + T_1 \cdot L^2(T) = 0 \text{ and } 3T - \frac{1}{6} T_1^2 \cdot L^2(T) = 0. \]

By iterating this procedure we have

\[ T = 0. \]

Hence the map \( L \circ t \) is injective, which means that the map \( L : I_{k+1} \to I_k \) is already surjective (in fact isomorphic) on \( t(I_k) \). \( \square \)

Based on the equality (8.14) and the Lemma below (8.14), we can construct an orthogonal basis of the space \( I_k \) inductively \( \{ \varphi_k(i) \}_{i=1}^{\dim I_k} \) in the following way:

**Definition 8.13.**

\[ I_1 = \{ \{ \varphi_1(1) = T_1 \} \}, \]
\[ I_2 = \{ \{ \varphi_2(1) = T_2 = T_1 \cdot \varphi_1(1), \ \varphi_2(2) = T_2 - 1/3T_1^2 \} \}, \]

where \( \varphi_2(2) \) is taken to be orthogonal to \( \varphi_2(1) \) and equivalently \( L(\varphi_2(2)) = 0, \)
\[ I_3 = \{ \{ \varphi_3(1) = T_3 = T_1 \varphi_2(1), \ \varphi_3(2) = T_3 \varphi_2(2), \ \varphi_3(3) = T_3 - T_2T_1 + 2/9T_1^3 \} \}, \]

where \( \varphi_3(3) \) is taken to be orthogonal to \( \varphi_3(1) \) and \( \varphi_3(2) \), which is also taken to satisfy \( L(\varphi_3(3)) = 0 \) and is determined uniquely up to constant multiples,
\[ I_4 = \{ \{ \varphi_4(1) = T_4 = T_1 \varphi_3(1), \ \varphi_4(2) = T_4 \varphi_3(2), \ \varphi_4(3) = T_4 \varphi_3(3) \} \]
\[ \varphi_4(4) = T_4^2 - 2/3T_2T_1^2 + 1/9T_1^4 \}, \]

where \( \varphi_4(4) \) is taken to satisfy \( L(\varphi_4(4)) = 0 \)
and is equal to \( \varphi_4(4) = \varphi_2(2)^2. \)

Likewise we can continue the construction in such a way that if \( \{ \varphi_k(i) \}_{i=1}^{\dim I_k} \) is constructed as above for \( k = 1, 2, 3, 4, \) then we define for \( k \geq 5 \)

\[ \varphi_{k+1}(i) = T_1 \varphi_k(i) \text{ for } i = 1, \cdots, \dim I_k \text{ and for } j = 1, \cdots, \dim I_{k+1} - \dim I_k, \]
\[ \varphi_{k+1}(\dim I_k + j) \text{ is chosen as being orthogonal to all } \varphi_{k+1}(i), i = 1, \cdots, \dim I_k + j - 1. \]

The orthogonality condition \( \langle \varphi_{k+1}(\dim I_k + j), T_1 \varphi_k(i) \rangle = 0 \) implies that \( L(\varphi_{k+1}(\dim I_k + j)) = 0. \)

**Lemma 8.14.** The construction is guaranteed by the property that if \( f \) and \( g \in I_k \) is orthogonal and \( L(g) = 0, \) then \( T_1 \cdot f \) and \( T_1 \cdot g \) is orthogonal, since

\[ \langle T_1 f, T_1 g \rangle = \langle f, L(T_1 g) \rangle = 3\langle f, g \rangle = 0. \]
Lemma 8.15. Put $\mathcal{N}_k = \{ T \in I_k \mid L(T) = 0 \}$, then

$$\dim \mathcal{N}_{k+1} = \dim I_{k+1} - \dim I_k,$$

and is equal to the number of the non-negative integer solutions $(a, b)$ of the equation

$$2a + 3b = k + 1. \tag{8.15}$$

Proof. Put $\varphi_2(2) = T_2 - 1/3T_1^2 := \varphi_2$ and $\varphi_3(3) = T_3 - T_2T_1 + 2/9T_1^3 := \varphi_3$. Then both of these are irreducible polynomials, since they are not decomposed into lower degree polynomials even on the subspace $z = y = x = 0$.

By $L(\varphi_2 \cdot \varphi_3) = 0$, products of any powers of these two polynomials are in the kernel of the map $L$. So corresponding to the non-negative integer solutions $(a, b)$ of (8.15) we have a basis of the kernel $\mathcal{N}_{k+1}$.

Lemma 8.16. For any $j$ and $\ell$

$$\dim H_j \cdot I_\ell = \dim H_j \cdot I_\ell.$$

Proof. It will be apparent if $\dim I_\ell = 1$.

We prove the property by induction and we show that the natural map $H_j \otimes \mathcal{N}_\ell \rightarrow H_j \cdot \mathcal{N}_\ell$ is isomorphic. So we assume for $\forall \ell \leq k$ and any $j \geq 0$ it holds the isomorphism

$$H_j \otimes \mathcal{N}_\ell \cong H_j \cdot \mathcal{N}_\ell. \tag{8.16}$$

Let

$$\sum_{(a, b) \text{ runs through the solutions of (8.15)}} h_{a, b} \cdot \varphi_2^a \cdot \varphi_3^b = 0, \ h_{a, b} \in H_j. \tag{8.17}$$

Let $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ be all the solutions of (8.15):

$$2a_i + 3b_i = k + 1,$$

Assume $a_1 > a_2 > \cdots > a_n$, then $b_1 < b_2 < \cdots < b_n$. Then the we can assume the expression (8.17) has one of the following two forms:

$$\begin{align*}
\text{[1]} : & \quad \text{if } a_n > 0, \varphi_2^{a_n} \cdot p + \varphi_3^{b_n} \cdot h_{a_n, b_n} = 0, \text{ or} \\
\text{[2]} : & \quad \text{if } a_n = 0, \varphi_2^{a_n-1} \cdot p + \varphi_3^{b_n} \cdot h_{a_n, b_n} = 0.
\end{align*} \tag{8.18}$$

In any case the polynomial $\varphi_2$ does not divide the polynomial $\varphi_3$, so that we may put $h_{a_n, b_n} = \varphi_2 \cdot Q$ with a polynomial $Q \in \mathcal{P}_{j-2}$. Then by the equality (8.14) the polynomial $Q \cdot \varphi_2 \in H_{j-1} \cdot I_1 + H_{j-2} \cdot I_2 + \cdots + H_1 \cdot I_{j-1} + I_j$. On the other hand $Q \cdot \varphi_3 = h_{a_n, b_n} \in H_j$. Hence by the definition of the space $H_j$ which is orthogonal complement of the space $H_{j-1} \cdot I_1 + H_{j-2} \cdot I_2 + \cdots + H_1 \cdot I_{j-1} + I_j$, hence $h_{a_n, b_n} = 0$ and also $p = 0$. By iterating the arguments we see that in the expression

$$\sum_{(a_i, b_i) \text{ runs through the solutions of (8.15)}} h_{a_i, b_i} \cdot \varphi_2^{a_i} \cdot \varphi_3^{b_i},$$

all the coefficient polynomials $h_{a_i, b_i}$ must be zero.

Finally we see from the sequences

$$\begin{array}{ccc}
\{0\} & \rightarrow & H_j \otimes \mathcal{N}_{k+1} \\
& \downarrow & \quad \downarrow \\
\{0\} & \rightarrow & H_j \cdot \mathcal{N}_{k+1}
\end{array}$$

\[\xrightarrow{\text{inclusion}}\]

$$\begin{array}{ccc}
\{0\} & \xrightarrow{\text{inclusion}} & H_j \cdot I_{k+1} \\
& \downarrow & \quad \downarrow \\
\{0\} & \rightarrow & H_j \cdot I_k
\end{array}$$

$$\begin{array}{ccc}
\{0\} & \rightarrow & H_j \otimes I_{k+1} \\
\xrightarrow{\text{Id} \otimes L} & \rightarrow & H_j \otimes I_k \\
& \downarrow & \quad \downarrow \\
\{0\} & \rightarrow & H_j \cdot I_{k+1} \\
\xrightarrow{\text{inclusion}} & \rightarrow & H_j \cdot I_k
\end{array}$$
two spaces
\[ H_j \otimes I_{k+1} \cong H_j \cdot I_{k+1} \]
are isomorphic. \(\square\)

**Proposition 8.17.** For each \( k \), the sum \( H_k + H_{k-1} \cdot I_1 + \cdots + H_1 \cdot I_{k-1} + I_k \) is a direct sum.

**Proof.** First we remark that the sums \( \mathcal{P}_1 = H_1 + I_1 \) and \( \mathcal{P}_2 = H_2 + H_1 \cdot I_1 + I_2 \) are orthogonal sums. The first one is included in the definition and the second one is shown as

\[ \langle h_1 T_1, T_2 \rangle = \langle (h_1, L(T_2)) \rangle = \langle (h_1, 2T_1) \rangle = \langle L(h_1), 2T_1 \rangle = 0, \text{ where } h_1 \in H_1. \]

Then we assume that the sum
\[ H_j + H_{j-1} \cdot I_1 + \cdots + H_1 \cdot I_{j-1} + I_j \]
are direct sums for \( j \leq k \).

We express
\[ T \in H_k \cdot I_1 + H_{k-1} \cdot I_2 + \cdots + H_1 \cdot I_k + I_{k+1} \]
as
\[ T = h_k \cdot T_1 + \sum_{i=1}^{\dim I_k} h_{k-1}(i) \cdot \varphi_2(i) + \cdots + \sum_{i=1}^{\dim I_k} h_1(i) \varphi_k(i) + \sum_{i=1}^{\dim I_{k+1}} h_0(i) \varphi_{k+1}(i) = 0, \]
where \( h_j(i) \in H_j \) and \( \varphi_j(i) \) are the basis polynomials of \( I_j \) constructed in the Definition 8.13. Then by the induction hypothesis, \( L(T) = 0 \) implies

\[ h_k = 0, h_{k-1}(1) \varphi_2 = 0, h_{k-2}(1) \varphi_3(1) + h_{k-2}(2) \varphi_3(2) = 0, \ldots, \sum_{i=1}^{\dim I_k} h_0(i) \varphi_k(i) = 0, \]
that is, the coefficient polynomials \( h_j(i) \) of the basis included in the orthogonal complement of \( \mathcal{N}_j \) are zero.

Hence it will be enough to show

\[ (8.20) \quad h_{k-1}(2) \varphi_2 + h_{k-2}(3) \varphi_3 + h_{k-3}(4) \varphi_4(4) + \cdots + \sum_{i=\dim I_{k-1}+1}^{\dim I_k} h_1(i) \varphi_k(i) + \sum_{i=\dim I_{k+1}}^{\dim I_k} h_0(i) \varphi_{k+1}(i) = 0 \]
implies all the coefficient polynomials \( h_j(i) = 0 \) and constants \( h_0(i) = 0 \) (see the proof of Lemma 8.14 for the polynomials \( \varphi_2 \) and \( \varphi_3 \)). As in the proof of the Lemma 8.16, the equation (8.20) can be rewritten as

\[ (8.21) \quad \varphi_2 \cdot P = -\varphi_3 \cdot Q \]
where the polynomial \( P = h_{k-1}(2) + \cdots \) is the sum of all the terms including some power \(( \geq 0)\) of \( \varphi_2 \) and \( Q = g_1 + g_2 \varphi_3 + \cdots \) (especially \( g_1 = h_{k-2}(3) \in H_{k-2} \)) is a polynomial of the polynomial \( \varphi_3 \) with the coefficient polynomials \( g_i \in H_j \) with the degree of \( g_i = k + 1 - 3i \). Since \( \varphi_2 \) does not divide \( \varphi_3 \), \( Q \) must be divided by \( \varphi_2 \), that is we have

\[ \varphi_2 \cdot Q_1 = Q = g_1 + g_2 \varphi_3 + \cdots, \]
where \( Q_1 \in \mathcal{P}_{k-4} \). Hence by Lemma 8.10

\[ Q = g_1 + g_2 \varphi_3 + \cdots \in H_{k-3} I_1 + H_{k-4} I_2 + \cdots + I_{k-2}, \]
which implies that \( g_1 = 0 \). Hence we can rewrite (8.21) as

\[ \varphi_2 \cdot P = -\varphi_3^2 \cdot Q_2 = -\varphi_3^2(g_2 + \cdots). \]
By iterating the same arguments as above we see that $Q = 0$ and $P = 0$. Hence we obtain $h_{k-1}(2) = 0$ and
\[
h_{k-3}(4)\varphi_4(4) + \cdots + \sum_{i = \dim I_{k-1} + 1}^{\dim I_k} h_1(i)\varphi_k(i) + \sum_{i = \dim I_k + 1}^{\dim I_{k+1}} h_0(i)\varphi_{k+1}(i) = 0.
\]
Again by the same argument for the above polynomial we obtain $h_{k-3}(4) = 0$. In a similar manner, we can find $h_3(i) = 0$ and $h_0(i) = 0$, which proves our assertion. □

We put $H := \sum_{k \geq 0} H_k$, and denote by $I_4(\mathcal{J}(3)^C) = \sum_{k \geq 0} I_k(\mathcal{J}(3)^C)$ invariant polynomial functions extended to the complexification $\mathcal{J}(3)^C$ in the natural way.

Since the function taking the trace $A \mapsto \text{tr} (A)$ is linear and $A \mapsto A^k$ is an operation inside the Jordan algebra $\mathcal{J}(3)$ (according to the definition of products) and also its complexification $\mathcal{J}(3)^C$, the extensions of the invariant polynomials $T_i$ to $\mathcal{J}(3)^C$ coincide with the trace functions on the complexification $\mathcal{J}(3)^C$:
\[
\mathcal{J}(3)^C \ni A \mapsto \text{tr} (A), A^2 \mapsto \text{tr} (A^2) \text{ and } A^3 \mapsto \text{tr} (A^3).
\]

Let $N_{\mathcal{J}(3)^C}$ be the common null set (other than zero) of the invariant polynomial functions (with respect to $F_4$ action) naturally extended to the complexified space $\mathcal{J}(3)^C$:
\[
N_{\mathcal{J}(3)^C} := \left\{ A = \begin{pmatrix} \xi_1 & z & \theta(y) \\ \theta(z) & \xi_2 & x \\ y & \theta(x) & \xi_3 \end{pmatrix} \in \mathcal{J}(3)^C \mid A \neq 0, T_1(A) = T_2(A) = T_3(A) = 0 \right\}.
\]

Remark 9. Let $A \in N_{\mathcal{J}(3)^C}$. Then at least one of the three components $z, y, x$ does not vanish. Since if $A \in N_{\mathcal{J}(3)^C}$ and assume $z = y = x = 0$, then $T_1(A) = \sum \xi_i = 0$, $T_2(A) = \sum \xi_i^2 = 0$ and $T_3(A) = \sum \xi_i^3 = 0$. Hence these imply that $\xi_i = 0$ too.

By the Proposition 4.1

Proposition 8.18. $X_\mathcal{O} = \tau_0(T_0^*(P^2\mathcal{O})) \subset N_{\mathcal{J}(3)^C}$ and the non-singular part of the space $N_{\mathcal{J}(3)^C}$ has dim $N_{\mathcal{J}(3)^C} = 24$.

Proof. Let $A \in X_\mathcal{O}$. Then $T_1(A) = \eta_1 + \eta_2 + \eta_3 = 0$ (Proposition 4.1), and $A^2 = 0$ implies $T_2(A) = 0$ and $T_3(A) = 0$ trivially. Hence $X_\mathcal{O} = \tau_0(T_0^*(P^2\mathcal{O})) \subset N_{\mathcal{J}(3)^C}$.

The second assertion is seen by noticing that at the points $z = y = x = 0$ the three differentials
d$T_1$, $dT_2$, $dT_3$
are linearly independent. □

Let $A \in \mathcal{J}(3)^C$ and consider the functions of the form
\[
(8.22) \quad \mathcal{J}(3)^C \ni X \mapsto \text{tr} (X \circ A) =: h_A(X).
\]
Since $X_\mathcal{O}$ is $F_4$ invariant, the nontrivial subspace in $H_1$ linearly spanned by the functions
\[
h_A : \mathcal{J}(3)^C \ni X \mapsto \text{tr} (X \circ A) \quad (A \in \mathcal{J}(3)^C, \text{ tr} (A) = 0),
\]
is an invariant subspace in $H_1$. Here note that $\text{tr} (g(X) \circ A) = \text{tr} (X \circ \theta(g(A)))$ for $g \in F_4$ and $\text{tr} (\theta(g(A))) = \text{tr} (A) = 0$.

However the representation of the group $F_4$ to $H_1$ is irreducible (Theorem 8.4), the space $H_1$ must be spanned by these functions. Also the same holds that the subspace in $H_1$ linearly spanned by the functions
\[
h_A : \mathcal{J}(3)^C \ni X \mapsto \text{tr} (X \circ A) \quad (A \in N_{\mathcal{J}(3)^C}),
\]
coincides with $H_1$ (see Lemma 4.2 and Corollary 4.3). These facts imply
**Proposition 8.19.** All the point in \( N_{\mathcal{J}}(3)^C \) can be expressed as a linear sum of points in \( \mathcal{X}_O \) and this fact implies that the space \( N_{\mathcal{J}}(3)^C \) is path-wise-connected.

**Proof.** Since any linear function \( \mathcal{J}(3)^C \ni X \mapsto \text{tr}(X \circ A) = h_A(X) \) with \( A \in N_{\mathcal{J}}(3)^C \) is a linear sum of functions of the form \( \text{tr}(X \circ B_i) = h_{B_i}(X) \) with \( B_i \in \mathcal{X}_O \),

\[
\text{tr}(X \circ A) = \sum c_i \text{tr}(X \circ B_i) \quad \text{on} \quad \mathcal{J}(3),
\]

where \( B_i \in \mathcal{X}_O \),

\[
A = \sum c_i B_i \quad \text{with} \quad B_i \in \mathcal{X}_O.
\]

Let \( A \) and \( A' \) be in \( N_{\mathcal{J}}(3)^C \). Assume \( A = \sum c_i B_i \) and \( A' = \sum c'_i B'_i \) where \( B_i, B'_i \in N_{\mathcal{J}}(3)^C \). Then the second assertion is proved by connecting points \( B_i \) and \( B'_i \) suitably in \( \mathcal{X}_O \). After that, since \( N_{\mathcal{J}}(3)^C \) is a cone, we can connect the points \( \sum c_i B'_i \) and \( \sum c'_i B'_i \) with the segment

\[
\hat{\alpha}(t) := \sum ((1-t)c_i B'_i + t c'_i B'_i) \quad (t \in [0,1]).
\]

Hence \( N_{\mathcal{J}}(3)^C \) is path-wise-connected.

In general, the space \( H_k \) of “Cayley-harmonic polynomials” can be orthogonal sum of two subspaces \( H_k^{(1)} \) and \( H_k^{(2)} \). \( H_k^{(1)} \) is the subspace linearly spanned by the powers \( \text{tr}(X \circ A)^k \) with \( A \in N_{\mathcal{J}}(3)^C \) and \( H_k^{(2)} \) is the orthogonal complement of \( H_k^{(1)} \) in \( H_k \). The orthogonality is equivalent to the property that Cayley-harmonic functions in \( H_k^{(2)} \) are vanishing on the subset \( N_{\mathcal{J}}(3)^C \) (see [He]).

In our case the second subspace \( H_k^{(2)} \) is always \( \{0\} \), that is,

**Proposition 8.20.**

\[
H_k^{(2)} = H_k \bigcap \sqrt{I_+(\mathcal{J}(3)^C)} = H_k \bigcap I_+(\mathcal{J}(3)^C) = \{0\}.
\]

**Proof.** The first equality is a consequence of Hilbert Nullstellensatz and the irreducibility of \( N_{\mathcal{J}}(3)^C \) implies the second equality.

We see the latter one by the following observation that the equation \( T_1(A) = 0 \) is linear so that if we replace the variable \( x \) by \( -x \), then the space \( N_{\mathcal{J}}(3)^C \) can be seen as a subset defined by \( T_2(A) = 0 \) in the quadrics \( Q_2 = \{ A \in \mathbb{C}^2 \mid T_2(A) = 0 \} \) and the polynomial \( T_3 \) restricted on the space \( z = y = x = 0 \) is irreducible even modulo \( T_2 \). i.e., there are no decomposition such that \( T_3(A) = \xi_1^2 \xi_2 + \xi_1 \xi_2^2 = (a \xi_1 + b \xi_2)(a \xi_1^2 + b \xi_1 \xi_2 + c \xi_2^2) \) on \( \xi_1^2 + \xi_2^2 + \xi_1 \xi_2 = 0 \). Hence the space \( N_{\mathcal{J}}(3)^C \) must be irreducible and we have

\[
\sqrt{I_+(\mathcal{J}(3)^C)} = I_+(\mathcal{J}(3)^C).
\]

In fact, our space \( N_{\mathcal{J}}(3)^C \) is an irreducible algebraic manifold and a complete intersection. In particular, there are points in \( N_{\mathcal{J}}(3)^C \) at which the differentials \( dT_1, dT_2, dT_3 \) are linearly independent (see the Lemma 4 on page 345 [Kos] for these aspects).

Especially, as a corollary of Proposition 8.19 we have

**Proposition 8.21.** The representation of \( F_4 \) to the space \( H_k = H_k^{(1)} \) is irreducible for each \( k \).

**Proof.** Since \( \mathcal{X}_O \) is connected, if the space \( H_k \) is decomposed into two invariant subspaces, \( H_k = G_1 \oplus G_2 \), then they are orthogonal. Consequently, according to this decomposition the space \( \mathcal{X}_O \) must be separated into two non intersecting closed subsets and this is a contradiction.

Hence each \( H_k \) must be irreducible under the action by the group \( F_4 \).

Since the functions in the invariant polynomials \( I_k \) are constant on the manifold \( P^2 \mathbb{C} \), by restricting polynomial functions in \( \mathcal{P}_k[\mathcal{J}(3)] \) to \( P^2 \mathbb{C} \) the decompositions \( \mathcal{P}_k[\mathcal{J}(3)] = H_k + I_1 H_{k-1} + \cdots + I_k \) for each
Since any smooth function on \( P^2 \) can be extended to a smooth function on an open neighborhood of \( P^2 \) and the Weierstrass approximation theorem guarantees that any smooth function can be approximated in the \( C^\infty \)–topology by polynomials. Hence the space \( \sum_{k=0}^\infty H_k|_{P^2} \) is dense in \( C^\infty(P^2) \).

Before interpreting the decomposition stated in Theorem 8.22 in the framework of the Peter-Weyl theorem for a symmetric space of our case \( P^2 \) we remark about the Riemannian metric on \( P^2 \).

**Proposition 8.23.** The Cayley projective plane \( P^2(\mathbb{D}) \cong F_4/\text{Spin}(9) \) is an irreducible Riemannian symmetric space, that is, the stationary subgroup \( \text{Spin}(9) \) acts irreducibly on the tangent space \( T_X, P^2(\mathbb{D}) \).

By Schur’s lemma this implies that \( P^2(\mathbb{D}) \) has an essentially unique \( F_4 \)-invariant Riemannian metric. Thus, \( (\cdot, \cdot)_{P^2(\mathbb{D})} \) coincides with the metric on \( P^2(\mathbb{D}) \) induced from the Killing form of the Lie algebra of \( F_4 \) up to a constant factor.

Let \( \Phi_k : H_k \otimes H_k^* \rightarrow C^\infty(F_4) \) be a map defined by

\[
H_k \otimes H_k^* \ni h \otimes \varphi \mapsto \Phi_k(h \otimes \varphi)(g) = \varphi(P_{g^{-1}}(h)), \quad g \in F_4,
\]

then the Peter-Weyl theorem says that the image of the map \( \Phi_k \) is the direct sum of the \( \dim H_k \) number of the spaces, all of which are isomorphic to \( H_k \).

Recall we explained the identification (3.2) of the quotient space \( F_4/\text{Spin}(9) \) with \( P^2 \) induced from the correspondence \( F_4 \ni g \mapsto g(X_1) \in P^2 \).

If we consider a subspace \( H_k^*|_{\text{Spin}(9)} \) consisting of linear forms in \( H_k^* \) which are invariant under the action by \( \text{Spin}(9) \), then the functions in

\[
\Phi_k(H_k \otimes H_k^*|_{\text{Spin}(9)}),
\]

are \( \text{Spin}(9) \) invariant, so that it can be descended naturally to functions on \( F_4/\text{Spin}(9) \cong P^2 \subset J(3) \).

For \( X \in J(3) \) we denote the linear form \( J_X \in H_k^* \)

\[
H_k \ni h \mapsto J_X(h) = h(X),
\]

that is, this is an evaluation at \( X \in J(3) \). In particular, we take a linear form \( J_{X_1} \in H_k^*|_{\text{Spin}(9)} \), then it can be written as

\[
J_{X_1}(P_{g^{-1}}(h)) = P_{g^{-1}}(h)(X_1) = h(g(X_1)).
\]

Hence through the identification \( F_4/\text{Spin}(9) \cong P^2 \) the function \( J_{X_1}(P_{g^{-1}}(h)) \) is the restriction of the original polynomial function \( h \in H_k \) to \( P^2 \). Then we have

\[
\sum_{k=0}^\infty \Phi_k(H_k \otimes \{J_{X_1}\}) = \sum_{k=0}^\infty H_k|_{P^2}.
\]

Since \( \dim H_{k+1} > \dim H_k \) (see Appendix) and the space \( \sum_{k=0}^\infty H_k|_{P^2} \) is already dense in \( C^\infty(P^2) \), a fundamental theorem on compact symmetric spaces gives us

**Proposition 8.24.** Each irreducible representation of the group \( F_4 \) appears in \( C^\infty(P^2) \) with multiplicity one as in the above way and incidentally \( \dim H_k^*|_{\text{Spin}(9)} = 1 \). Moreover by the Proposition 8.23
we can see that this decomposition is the eigenspace decomposition of the Laplacian on \( P^2 \Omega \).

The dimension of the space \( H_k^*_{\text{Spin}(9)} \) is always one and the linear form \( J_{X_1} \) can be seen as a base vector of the space \( H_k^*_{\text{Spin}(9)} \) for any \( k \).

9. Inverse of Bargmann type transformation

In this section, based on the data obtained until §8 we consider the boundedness and invertibility of our Bargmann type transformation

\[
\mathfrak{B} : \sum \mathcal{P}_k[X_0] \longrightarrow C^\infty(P^2 \Omega)
\]

with respect to the parameter family of the inner products \( \{(\ast \ast)_\varepsilon \}_{-22<\varepsilon} \) on the space \( \sum \mathcal{P}_k[X_0] \) defined in (6.9). It has a dense image from \( \mathcal{P}_k[X_0] \) always for a possible value of the parameter \( \varepsilon \), but unlike the cases of spheres and other projective spaces (see [Ra2], [Fu1], [FY]), it need not be an isomorphism when \( \varepsilon = 0 \). This means in cases of the values of the parameter \( \varepsilon > -47/2 \) there are quantum states in \( L_2(P^2 \Omega) \) which can not be seen by classical observables.

9.1. Inverse transformation

Let \( \mathcal{A}_k \) be a transformation defined by

\[
\mathcal{A}_k : H_k \ni \varphi \longmapsto \int_{P^2 \Omega} \varphi(X) \cdot (\text{tr}(X \circ A))^k \, dv_{P^2 \Omega}(X) \in \mathcal{P}_k[X_0].
\]

and

\[
\mathcal{A}_k^* : \mathcal{P}_k[X_0] \ni h \longmapsto \int_{X_0} h(A) \cdot (\text{tr}(X \circ A))^k \, dv_{P^2 \Omega}(X) = \mathcal{A}_k^* (h) = \mathcal{A}_k (\varphi). \]

The correspondence by \( \gamma \) is defined in (5.4).

**Proposition 9.1.** For any inner product defined on the space \( \mathcal{P}_k[X_0] \) depending on the value of the parameter \( \varepsilon \), the operator \( \mathcal{A}_k \) is a constant multiple of a unitary operator.

**Proof.** For \( \varphi \in H_k \) the inner product

\[
(\mathcal{A}_k(\varphi), \mathcal{A}_k(\varphi)) \varepsilon
\]

is expressed as

\[
(\mathcal{A}_k(\varphi), \mathcal{A}_k(\varphi)) \varepsilon = \int_{X_0} \int_{P^2 \Omega} \varphi(X) \cdot (\text{tr}(X \circ A))^k \, dv_{P^2 \Omega}(X) \cdot e^{-2\sqrt{2\pi||A||erez}} \cdot ||A||^{erez} \cdot \Omega \wedge \overline{\Omega}
\]

\[
= \int_{P^2 \Omega} \int_{P^2 \Omega} \left( \int_{X_0} (\text{tr}(X \circ A))^k \cdot e^{-2\sqrt{2\pi||A||erez}} \cdot ||A||^{erez} \cdot \Omega \wedge \overline{\Omega} \times \varphi(\vec{X}) \right) \, dv_{P^2 \Omega}(X) \, dv_{P^2 \Omega}(\vec{X})
\]

Here we consider the operator \( \mathcal{B}_k \)

\[
\mathcal{B}_k : \mathcal{P}_k[X_0] \ni h \longmapsto \int_{X_0} h(A) \cdot (\text{tr}(X \circ A))^k e^{-2\sqrt{2\pi||A||erez}} \cdot ||A||^{erez} \cdot \Omega(A) \wedge \overline{\Omega}(A) \in H_k.
\]

Since \( H_k \) consists of linear sums of functions of the form \( (\text{tr}(X \circ A))^k \) by arbitrary \( A \in X_0 \) (see Proposition 8.20), we see that \( \mathcal{B}_k(h) \in H_k \). Then the inner product (9.3) is understood as

\[
(\mathcal{A}_k(\varphi), \mathcal{A}_k(\varphi)) \varepsilon = (\mathcal{B}_k \circ \mathcal{A}_k(\varphi), \varphi)^{P^2 \Omega}.
\]

Then the operator \( \mathcal{B}_k \circ \mathcal{A}_k \) commutes with the \( F_d \) action on \( H_k \). Hence it must be a constant multiple of
identity operator (we denote this constant by \( b_k \)) so that the kernel function defined by the integral

\[
L_k(X, \tilde{X}) := \int_{X_0} \left( \text{tr}(\tilde{X} \circ A) \right)^k \left( \text{tr}(X \circ A) \right)^k \cdot e^{-2\sqrt{2}\pi ||A||^{1/2}} \cdot ||A||^r \cdot \Omega_5(A) \wedge \overline{\Omega_5(A)}
\]

must satisfies the invariance:

(9.5) \( L_k(g(X), \tilde{X}) = L_k(X, g^{-1}(\tilde{X})) \), for \( g \in \mathcal{F}_4, X, \tilde{X} \in P^2\mathcal{O} \).

Then the constant \( b_k \) is given by

(9.6) Trace of the operator \( \mathcal{B}_k \circ \mathcal{A}_k = \int_{P^2\mathcal{O}} L_k(X, X)dv_{P^2\mathcal{O}} = b_k \dim H_k \).

and the integral \( \int_{P^2\mathcal{O}} L_k(X, X)dv_{P^2\mathcal{O}} \) is given by

\[
\int_{P^2\mathcal{O}} L_k(X, X)dv_{P^2\mathcal{O}} \equiv L_k(X, X) \cdot \text{Vol}(P^2\mathcal{O}),
\]

since by the invariance (9.5) the function \( L_k(X, X) \) is a constant function and apparently is non-zero.

Now we know \( \mathcal{A}_k \) is injective and so

\[
\dim H_k \leq \dim \mathcal{P}[X_0].
\]

On the other hand, degree \( k \) polynomials generated by the invariant polynomials which are naturally extended to the complexification \( \mathcal{J}(3)^C \), that is, the polynomials

\[
\sum_{i=0}^{k} \mathcal{P}_{k-i}[\mathcal{J}(3)^C] \cdot I_{k-i} = \sum_{i=0}^{k} H_{k-i} \cdot I_i
\]

(see Lemma 8.10) are all vanishing on the manifold \( X_0 \) so that

\[
\dim \mathcal{P}[X_0] \leq \dim \mathcal{P}[\mathcal{J}(3)^C] = \sum_{i=0}^{k} \dim H_{k-i} \cdot \dim I_i,
\]

(see Proposition 8.17).

Hence the operator \( \mathcal{A}_k \) is also surjective to the space \( \mathcal{P}_k[X_0] \). Consequently, the operator \( \mathcal{B}_k \) is a constant multiple of a unitary operator and hence so is also \( \mathcal{A}_k \). \( \square \)

Next, we determine the concrete value of the constant \( b_k \):

**Proposition 9.2.**

\[
\text{Vol}(P^2\mathcal{O})L_k(X, X) = b_k \cdot \dim H_k = 2^{26} \cdot \text{Vol}(P^2\mathcal{O}) \cdot \text{Vol}(S(P^2\mathcal{O})) \cdot \frac{\Gamma(4k + 44 + 2\varepsilon)}{2^{2k+66+3\pi(4k+44+2\varepsilon)},}
\]

where the constant \( \text{Vol}(S(P^2\mathcal{O})) \) is the volume of the unit cotangent sphere bundle \( S(P^2\mathcal{O}) \) of \( P^2\mathcal{O} \) with respect to the volume form

\[
d\sigma_{S(P^2\mathcal{O})} := \frac{1}{161} \cdot b^{P^2\mathcal{O}} \wedge \left( \omega^{P^2\mathcal{O}} \right)^{15} |S(P^2\mathcal{O})|.
\]

**Proof.** Since \( L_k(X, X) = \int_{X_0} \left| \text{tr}(X \circ A) \right|^{2k} \cdot e^{-2\sqrt{2}\pi ||A||^{1/2}} \cdot ||A||^r \cdot \Omega_5(A) \wedge \overline{\Omega_5(A)} \) does not depend on the point \( X \in P^2\mathcal{O} \), we have

\[
L_k(X, X) = \int_{X_0} \left| \text{tr}(X \circ A) \right|^{2k} \cdot e^{-2\sqrt{2}\pi ||A||^{1/2}} \cdot ||A||^r \cdot \Omega_5(A) \wedge \overline{\Omega_5(A)}
\]
where \( dv_{F_4} \) is the normalized Haar measure on \( F_4 \).

The function

\[
\int_{F_4} \left| \operatorname{tr} \left( X \circ g \left( \frac{A}{|A|} \right) \right) \right|^{2k} dv_{F_4}(g)
\]

(9.8)
does not depend neither on \( X \in P^2\Omega \) nor on \( A \in X_\Omega \), since the trace function \( A \mapsto \operatorname{tr}(A) \) is \( F_4 \)-invariant, the group \( F_4 \) acts both on the spaces \( P^2\Omega \) and the cotangent sphere bundle \( S(P^2\Omega) \cong S(X_\Omega) \) transitively and the Haar measure \( dv_{F_4} \) is bi-invariant.

Let \((X,Y) \in T_0^+ (P^2\Omega) \). Put \( A_g(X,Y) := g(\tau_0(X,Y)) \), then

\[
g(\tau_0(X,Y)) = g \left( ||Y||^2 X - Y^2 + \sqrt{-1} \otimes \frac{||Y||}{\sqrt{2}} Y \right) = g(||Y||^2 g(X) - g(Y)^2 + \sqrt{-1} \otimes \frac{||g(Y)||}{\sqrt{2}} g(Y).
\]

Hence

\[
\tau_0^{-1}(A_g(X,Y)) = (X(A_g(X,Y)), Y(A_g(X,Y)) = (g(X), g(Y)) \in T_0^+ (P^2\Omega).
\]

The integral (9.8) is expressed as

\[
\frac{1}{||A_g(X,Y)||^{2k}} \int_{F_4} \left| \operatorname{tr} X(A_g(X,Y)) \circ A_g(X,Y) \right|^{2k} dv_{F_4}(g)
\]

\[
= \frac{1}{||Y||^{4k}} \int_{F_4} \left| \operatorname{tr} g(X) \circ \left( ||g(Y)||^2 g(X) - g(Y)^2 + \sqrt{-1} \otimes \frac{||g(Y)||}{\sqrt{2}} g(Y) \right) \right|^{2k} dv_{F_4}(g)
\]

\[
= \frac{1}{||Y||^{4k}} \int_{F_4} \left( \frac{1}{2} ||g(Y)||^2 \right)^{2k} dv_{F_4} = \frac{1}{2^{2k}},
\]

since

\[
g(X)^2 = g(X), \quad \operatorname{tr} g(X) = 1, \quad g(X) \circ g(Y) = \frac{1}{2} g(Y)
\]

and we used the property

\[
\operatorname{tr} (X \circ Y) \circ Z = \operatorname{tr} X \circ (Y \circ Z).
\]

Now the integral (9.7) is

\[
\int_{X_\Omega} \int_{F_4} \left| \operatorname{tr} X \circ g(A) \right|^{2k} dv_{F_4}(g) \cdot e^{-2\sqrt{2}\pi ||A||^{1/2}} ||A||^\varepsilon \cdot \Omega_\Omega(A) \wedge \overline{\Omega_\Omega(A)}
\]

\[
= \frac{1}{2^{2k}} ||A||^{2k+\varepsilon} \cdot e^{-2\sqrt{2}\pi ||A||^{1/2}} \cdot \Omega_\Omega(A) \wedge \overline{\Omega_\Omega(A)}
\]

(9.9)

\[
= \frac{2^{26}}{2^{2k}} \int_{T_0^+ (P^2\Omega)} ||Y||^{4k+28+2\varepsilon} \cdot e^{-2\sqrt{2}\pi ||Y||} \cdot dV_{T^+ (P^2\Omega)},
\]

where we used the relation (6.8). Then according to the decomposition of the space \( T_0^+ (P^2\Omega) \cong \mathbb{R}_+ \times S(P^2\Omega) \), we can decompose the Liouville volume form \( dV_{T^+ (P^2\Omega)} \) as

\[
dV_{T^+ (P^2\Omega)} = t^{15} dt \wedge d\sigma_{S(P^2\Omega)},
\]

where \( d\sigma_{S(P^2\Omega)} \) is the volume form on the unit cotangent sphere bundle \( S(P^2\Omega) \). Finally we have the
integral (9.9) as
\[
\frac{2^{26}}{2^{2k}} \int_{S(P^2)} |Y|^{|4k+28+2e|} e^{-2\sqrt{2\pi}|Y|} \, dV_{P^2} = \frac{2^{26}}{2^{2k}} \int_{S(P^2)} dS_{P^2} \int_0^\infty t^{4k+28+2e} e^{-2\sqrt{2\pi}t} \, t^{15} \, dt
\]
\[
= \frac{2^{26}}{2^{2k}} \cdot \text{Vol}(S(P^2)) \cdot \frac{1}{2^{20} \pi^{44}} \cdot \text{Vol}(S(P^2)) \cdot \frac{\Gamma(4k+44+2e)}{2^{88+3\pi^{4k+2e}}},
\]
and
\[
(9.10) \quad b_k = \frac{\text{Vol}(P^2) \cdot \text{Vol}(S(P^2))}{2^{40+3\pi^{44}+2e}} \cdot \frac{\Gamma(4k+44+2e)}{2^{86} \pi^{4k} \dim H_k}.
\]

**Proposition 9.3.** Since both of the transformations \( A_k \) and the restriction of the transformation \( \mathcal{B} \) to the space \( \mathcal{P}_k[\mathcal{X}_0] \) (for short we denote it by \( T_k := \mathcal{B}|_{\mathcal{P}_k[\mathcal{X}_0]} \)) commute with \( F_4 \) action and the representation of \( F_4 \) on \( H_k \) is irreducible (see (8.24)), the composition \( T_k \circ A_k \) on \( H_{k+1} \) is equal to \( a_k \text{Id} \), where the constant \( a_k \) is given by
\[
a_k = 2^{6} \cdot \text{Vol}(S^{15}) \cdot \text{Vol}(P^2) \cdot \frac{\Gamma(2k+22)}{2^{2k+11} \cdot \pi^{2k+22} \dim H_k} \cdot \frac{\Gamma(2k+22)}{2^{3k+11} \cdot \pi^{2k+22} \dim H_k}.
\]

**Proof.** Let \( f \in H_k \) then by Corollary 7.6
\[
T_k \left( A_k(f) \right)(X) = \int_{P^2} f(\tilde{X}) \cdot \left\{ \text{tr} (\tilde{X} \circ \tau_0(X, *)) \right\}^k \cdot e^{-\sqrt{2\pi}||*||} \cdot dV_{P^2} (\tilde{X}) \cdot dV_{P^2} (X, *)
\]
\[
= \int_{P^2} f(\tilde{X}) \cdot \left\{ \text{tr} (\tilde{X} \circ \tau_0(X, *)) \right\}^k \cdot e^{-\sqrt{2\pi}||*||} \cdot ||*||^6 \cdot dV_{P^2} (\tilde{X}) \cdot dV_{P^2} (X, *)
\]
\[
= \int_{P^2} f(\tilde{X}) K_k(\tilde{X}, X) dV_{P^2} (\tilde{X}) \cdot dV_{P^2} (X),
\]
where we put the fiber integral as
\[
K_k(\tilde{X}, X) = \left\{ \text{tr} (\tilde{X} \circ \tau_0(X, *)) \right\}^k \cdot e^{-\sqrt{2\pi}||*||} \cdot ||*||^6 \cdot dV_{P^2} (\tilde{X}, X, *)
\]
The kernel function \( K_k(\tilde{X}, X) \) satisfies the property similar to the kernel function \( L_k(\tilde{X}, X) \):
\[
K_k(g \cdot \tilde{X}, X) = K_k(\tilde{X}, g^{-1}(X)).
\]
Then by this property (9.12) that \( K_k(X, X) \) is constant and we have
\[
\text{tr} (T_k \circ A_k) = a_k \cdot \dim H_k = 2^{6} \int_{P^2} K_k(X, X) dV_{P^2} (X) = 2^{6} \cdot K_k(X, X) \cdot \text{Vol}(P^2).\]
Since \( \text{tr}(X \circ \tau_0(X, Y)) = 1/2||Y||^2 \),
\[
\text{tr} (T_k \circ A_k) = a_k \cdot \dim H_k = 2^{6} \int_{P^2} K_k(X, X) dV_{P^2} (X) = 2^{6} \cdot K_k(X, X) \cdot \text{Vol}(P^2).\]
If we choose a point \( X = X_1 \), then the above fiber integral is expressed as
\[
(1/2)^k \cdot \text{Vol}(P^2(X_1, *))
\]
By the relation of the Poincaré polynomials (9.6) that

\[ \dim_{\mathbb{C}} \]

where we express the integral using the local coordinates on \( \widetilde{W}_1 \) (see Subsection 7.1) around the fiber \( q^{-1}(X_1) \) and the dual coordinates \( (X, Y) = (b, c, \beta, \gamma) \mapsto \sum_i \beta_i b_i + \gamma_i c_i \in T_X(\mathcal{W}_1) \). Then the integral (9.13) over the point \( X_1 \) is

\[
(1/2)^k \int_{q^{-1}(X_1)} ||Y||^{2k+6} e^{-\sqrt{2\pi} ||Y||} d\beta_0 \wedge \cdots \wedge d\beta_7 \wedge d\gamma_0 \wedge \cdots \wedge d\gamma_7
\]

\[
= (1/2)^k \int_{\mathbb{R}^{16}} \left( \sum \beta_i^2 + \gamma_i^2 \right)^{k+3} e^{-\sqrt{2\pi} \sqrt{\sum (\beta_i^2 + \gamma_i^2)}} d\beta_0 \cdots d\beta_7 d\gamma_0 \cdots d\gamma_7 = \frac{\Gamma(2k+22)}{2^{2k+11} \cdot \pi^{2k+22}} \cdot \text{Vol}(S^{15}).
\]

\[
\square
\]

**Corollary 9.4.** The operator norm \( ||\mathfrak{B}^{-1}_{p_k[x_0]}|| \) is given by

\[
||\mathfrak{B}^{-1}_{p_k[x_0]}|| = \frac{\sqrt{b_k}}{a_k} = \sqrt{\frac{\text{Vol}(P^2(2)) \cdot \text{Vol}(F(S^{15})) \cdot \text{Vol}(P^2(2)) \cdot 1}{2^{2k+11} \cdot \pi^{2k+22} \cdot 2^{8k} \cdot \text{dim} H_k}} =: C(\varepsilon) \cdot N(k),
\]

where \( C(\varepsilon) \) includes only \( \varepsilon \) and \( N(k) \) is a function of \( k \) and

\[
N(k)^2 = \frac{2^{4k} \cdot \text{dim} H_k \cdot 1}{2^{8k} \cdot \Gamma(2k+22)^2}.
\]

**Proof.** From \( B_k \circ A_k = b_k Id \) and \( T_k \circ A_k = a_k Id \) we have \( T_k = (a_k/b_k) B_k \). It follows from (9.4) and (9.6) that \( A_k/\sqrt{b_k} \) is a unitary operator. Hence we have

\[
||\mathfrak{B}^{-1}_{p_k[x_0]}|| = ||T_k^{-1}|| = \frac{b_k}{a_k} \cdot \frac{1}{\sqrt{b_k}} = \frac{\sqrt{b_k}}{a_k}.
\]

\[
\square
\]

It is enough to see (9.15) for the behavior of the norm (9.14) when \( k \to \infty \) and for this purpose we mention two properties of the Gamma function.

**Lemma 9.5.**

\[
\lim_{k \to \infty} \frac{\Gamma(k + \alpha_1) \cdots \Gamma(k + \alpha_\varepsilon)}{\Gamma(k + \beta_1) \cdots \Gamma(k + \beta_\varepsilon)} = \begin{cases} +\infty, & \text{if } \sum \alpha_i > \sum \beta_i, \\ 1, & \text{if } \sum \alpha_i = \sum \beta_i, \\ 0, & \text{if } \sum \alpha_i < \sum \beta_i. \end{cases}
\]

**Lemma 9.6.**

\[
\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}} \cdot \prod_{j=0}^{n-1} \Gamma \left( z + \frac{j}{n} \right).
\]

Then by Lemma 9.6

\[
N(k)^2 = \frac{2^{44+4\varepsilon}}{\sqrt{2\pi}} \cdot \text{dim} H_k \cdot \prod_{j=0}^{3} \frac{\Gamma(k + 11 + \varepsilon/2 + j/4)}{\Gamma(k + 11)^2 \cdot \Gamma(k + 11 + 1/2)^2}.
\]

By the relation of the Poincaré polynomials \( PP(t) = PH(t) \cdot PI(t) \) (see (A.2)), the dimension of \( H_k \) is given as

\[
\dim H_k = 24 + k - 1 C_k + 2 \cdot 24 + k - 2 C_{k-1} + 2 \cdot 24 + k - 3 C_{k-2} + 24 + k - 4 C_{k-3}.
\]
Hence (9.16) is
\begin{align*}
N(k)^2 &= \frac{2^{44+4}e^{\sqrt{2}\pi}}{\sqrt{2\pi}} \left( \frac{\Gamma(24 + k)}{\Gamma(k + 1)} + 2 \frac{\Gamma(23 + k)}{\Gamma(k)} + 2 \frac{\Gamma(22 + k)}{\Gamma(k - 1)} + \frac{\Gamma(21 + k)}{\Gamma(k - 2)} \right) \\
&\quad \times \prod_{j=0}^{k} \frac{\Gamma(k + 11 + \varepsilon/2 + j/4)}{\Gamma(k + 11)^2 \cdot \Gamma(k + 11 + 1/2)^2}.
\end{align*}

Hence we have

**Theorem 9.7.** (1) Let \( \varepsilon = -\frac{47}{4} \), then the Bargmann type transformation
\[ \mathfrak{B} : \mathfrak{F}^{-47/4} \rightarrow L_2(P^2O, dv_{P^2O}) \]
is an isomorphism, although it is not unitary.

(2) If \( \varepsilon > -\frac{47}{4} \), then the Bargmann type transformation is bounded with the dense image, but not an isomorphism between the spaces \( \mathfrak{F}_\varepsilon \) and \( L_2(P^2, dv_{P^2}) \).

(3) If \( -22 < \varepsilon < -\frac{47}{4} \), then the inverse of the Bargmann type transformation
\[ \mathfrak{B}^{-1} : L_2(P^2, dv_{P^2}) \rightarrow \mathfrak{F}_\varepsilon \]
is bounded, while the Bargmann type transformation can not be extended to the whole Fock-like space \( \mathfrak{F}_\varepsilon \).

(4) Let \( \varepsilon \leq -22 \). Then, for such a \( k \) that \( 2k + 22 + \varepsilon \leq 0 \), the integral (6.9) does not converge, although the Bargmann type transformation is defined for such polynomials. Hence by defining an inner product on the finite dimensional space \( \sum_{2k + 22 + \varepsilon \leq 0} P_k[X_0] \) in a suitable way, the Bargmann type transformation behave in the same way as the case of (3) (see Definition 6.5).

**Proof.** (1) By Lemma 9.5, when \( \varepsilon = -\frac{47}{4} \), \( \lim_{k \to \infty} \sqrt{b_k}/a_k \) and \( \lim_{k \to \infty} a_k/\sqrt{b_k} \) are non-zero finite. Hence \( \mathfrak{B} \) gives an isomorphism between \( \mathfrak{F}^{-47/4} \) and \( L_2(P^2O, dv_{P^2O}) \).

(2) For \( \varepsilon > -\frac{47}{4} \), \( \lim_{k \to \infty} a_k/\sqrt{b_k} = 0 \). Hence \( \mathfrak{B} \) can be extended to the whole Fock-like space \( \mathfrak{F}_\varepsilon \) with dense image. Although, \( \mathfrak{B} \) is injective on \( \mathfrak{F}_\varepsilon \), \( \mathfrak{B}^{-1} \) is not bounded.

(3) The opposite situation to (2) case holds.

(4) For \( 0 \leq k \leq -11 - \varepsilon/2 \), we define an inner product on the finite dimensional space \( \sum_{2k + 22 + \varepsilon \leq 0} P_k[X_0] \) in a suitable way and \( \lim_{k \to \infty} \sqrt{b_k}/a_k = 0 \) so that the same assertion with (3) holds.

**Remark 10.** The result in the above theorem differs from the original Bargmann transformation and other cases of the spheres, complex projective spaces and quaternion projective spaces for which the Bargmann type transformations are always isomorphisms ([Ba], [Ra2], [Fu1], [FY]) without a modification factor in the weight for defining an inner product in the Fock-like space.

10. Some additional results

10.1. Reproducing kernel of the Fock-like space \( \mathfrak{F}_\varepsilon \)

As an application of the explicit determination of the constant \( b_k \) we show our Fock-like space \( \mathfrak{F}_\varepsilon \) has the reproducing kernel.

Since the operator \( A_k \) is an isomorphism from \( H_k \) to \( P_k[X_0] \) and the operator \( B_k \circ A_k \equiv b_k \), the composition \( A_k \circ B_k \equiv b_k \) too. The kernel function (we put it as \( R_k(A, B), (A, B) \in X_0 \times X_0 \)) of the composition
\[ A_k \circ B_k = \frac{b_k}{b_k} \]
which is the identity operator on \( P_k[X_0] \), is expressed as
\[ R_k(A, B) = \int_{P^2O} (\text{tr } X \circ A)^k (\text{tr } X \circ \mathfrak{B})^k dv_{P^2O} \cdot e^{-2\sqrt{2\pi}((||A||^{1/2} + ||B||^{1/2})(||A|| \cdot ||B||)^{14+\varepsilon})/b_k} \]
\[ \mathcal{H} \text{ence} \]

Hence the sum

\[ R(A, B) := \sum_{k=0}^{\infty} R_k(A, B) \]

is estimated as

\[
\frac{1}{b_k} \left| \int_{P^2(\mathbb{C})} \left( \frac{\text{tr} (X \circ A/||A||)^k \cdot \text{tr} (X \circ \overline{B}/||B||)^k \cdot \text{d}v_{P^2(\mathbb{C})} \cdot e^{-2\sqrt{2}(||A||^{1/2} + ||B||^{1/2})(||A|| \cdot ||B||)^{k+14+\varepsilon}}}{1} \right) \right|
\]

\[
\leq \frac{\text{Vol}(P^2(\mathbb{C}))^{2^{40+3\varepsilon}2^{44+2\varepsilon}}}{\text{Vol}(S(P^2(\mathbb{C})))} \cdot e^{-2\sqrt{2}(||A||^{1/2} + ||B||^{1/2})(||A|| \cdot ||B||)^{14+\varepsilon}} \cdot \sum_{k=0}^{\infty} \frac{2^{8k}2^{4k} \cdot (||A|| ||B||)^k}{\Gamma(4k+4+2\varepsilon)} \times \left( \frac{\Gamma(24+k)}{\Gamma(k+1)} + 2 \frac{\Gamma(23+k)}{\Gamma(k)} + 2 \frac{\Gamma(22+k)}{\Gamma(k-1)} + \frac{\Gamma(21+k)}{\Gamma(k-2)} \right).}
\]

This inequality implies that the series converges locally uniformly on the space \( \mathcal{X}_0 \times \mathcal{X}_0 \) and the function \( R(A, B) \) is holomorphic there. So \( R(A, B) \) is the reproducing kernel of the Hilbert space \( \mathcal{F}_\varepsilon \) \((\varepsilon < -22)\).

**10.2. Geodesic flow and eigenspaces of Laplacian on \( P^2(\mathbb{C}) \)**

Let \( \phi_t \) \((t \in \mathbb{R})\) be an action on \( \mathcal{X}_0 \) defined by

\[ \mathcal{X}_0 \ni A \mapsto \phi_t(A) = e^{2\sqrt{-1}t} \cdot A. \]

Then this is an interpretation of the geodesic flow action onto the space \( \mathcal{X}_0 \) through the map \( \tau_0 \).

Let \( p \in \mathcal{P}_k[\mathcal{X}_0] \). Then

\[
\phi_t^*(p \cdot t_0 \otimes \Omega_0)(A) = e^{2\sqrt{-1}t(11+k)} \cdot p(A) \cdot t_0(A) \otimes \Omega_0(A).
\]

Let \( p \in \mathcal{P}_k[\mathcal{X}_0] \) and \( q \in \mathcal{P}_t[\mathcal{X}_0] \) with \( k \neq \ell \), then

**LEMMA 10.1.**

\[
(p, q)_\varepsilon = \int_{\mathcal{X}_0} p \cdot \overline{q} \cdot g_0^2 \cdot ||A||^\varepsilon \cdot \Omega_0 \wedge \overline{\Omega_0} = 0.
\]

**PROOF.** The transformation \( \phi_t^* \) on \( \Gamma^G(\mathcal{L} \otimes K^\varepsilon, \mathcal{X}_0) \) is unitary, hence

\[
(\phi_t^*(p), \phi_t^*(q))_\varepsilon = (p, q)_\varepsilon \text{ for any } t \in \mathbb{R}.
\]

On the other hand

\[
\phi_t^*(p \cdot \overline{q} \cdot (t_0, t_0)^{\overline{1/2}} \Omega_0 \wedge \overline{\Omega_0}) = e^{2\sqrt{-1}(k-\ell)t} \cdot (p \cdot \overline{q} \cdot (t_0, t_0)^{1/2} \Omega_0 \wedge \overline{\Omega_0}).
\]

Hence \( (p, q)_\varepsilon = 0. \)

Let \( \Delta^{P^2(\mathbb{C})} \) be the Laplacian on \( P^2(\mathbb{C}) \). Then

**PROPOSITION 10.2.** The geodesic flow action on \( \mathcal{X}_0 \) and the action given by the one parameter group

\[ \left\{ e^{2\sqrt{-1}t \cdot \sqrt{\Delta^{P^2(\mathbb{C})}} + 121} \right\} \]

of unitary transformations consisting of the Fourier integral operators commute through the Bargmann type transformation.

**PROOF.** This is shown based on the data that the \( k \)-th eigenvalue of the Laplacian \( \Delta^{P^2(\mathbb{C})} \) is given by
\[ k^2 + 22k \] and the Bargmann type transformation on each subspace \( P_k[\mathbb{X}_0] \) maps to \( H_k \) which coincides with the \( k \)-th eigenspace of the Laplacian (Propositions 8.23, 8.24). □

**Remark 11.** Finally we mention that in a forthcoming paper the reproducing kernel above will be made clear to relate with a differential equation satisfied by some hypergeometric functions and also a Töplitz operator theory on \( P^2[\mathbb{O}] \) will be discussed.

### A. Appendix: Generating functions of Poincaré series

In this Appendix we consider the generating functions of the Poincaré series of

1. the polynomial algebra: \( PP(t) = \sum \dim P_k t^k \),
2. the algebra of invariant polynomials: \( PI(t) = \sum \dim I_k t^k \) and
3. the space of the Cayley harmonic polynomials: \( PH(t) = \sum \dim H_k t^k \),

and prove the inequality:

\[
\text{(A.1)} \quad \dim H_{k+1} > \dim H_k.
\]

In fact, these formal power series converge for \( |t| < 1 \), which will be seen by explicitly determining their generating functions.

The generating function \( PI(t) \) of the Poincaré series of the dimensions of invariant polynomials \( I = \sum I_k \) is determined as

\[
PI(t) = \sum_{k=0}^{\infty} \dim I_k t^k = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/3]} \left( \left\lfloor \frac{k-3l}{2} \right\rfloor + 1 \right) t^k = \sum_{k=0}^{\infty} \sum_{i_1+2i_2+3i_3=k, i_1,i_2,i_3\in\mathbb{N}_0} t^k
\]

\[
= \sum_{(i_1,i_2,i_3)\in\mathbb{N}_0\times\mathbb{N}_0\times\mathbb{N}_0} \frac{1}{1-t} \frac{1}{1-t^2} \frac{1}{1-t^3}.
\]

The generating function \( PP(t) \) of the polynomial algebra \( \mathbb{C}[s_1, \ldots, s_N] = \sum P_k \) is given by

\[
PP(t) = \sum \dim P_k t^k = \sum_{k=0}^{\infty} \sum_{N+k-1} C_k t^k
\]

\[
= \sum_{(r_1,r_2,\ldots,r_N)\in\mathbb{N}_0^N} t^{r_1+r_2+\cdots+r_N} = \left( \frac{1}{1-t} \right)^N, \text{ in which } N = 27 \text{ for our case.}
\]

Let \( PH(t) \) be the generating function of the Poincaré series of the dimensions of Cayley harmonic polynomials, then by Lemma 8.16 and Proposition 8.17

\[
\text{(A.2)} \quad PP(t) = PH(t) \cdot PI(t)
\]

and we have

\[
PH(t) = \left( \frac{1}{1-t} \right)^{24} \cdot (1+t)(1+t+t^2)
\]

\[
\text{(A.3)} \quad = \sum_{k=0}^{\infty} 24+k-1C_k t^k \cdot (1+2t+2t^2+t^3) = \sum_{k=0}^{\infty} \dim H_k t^k.
\]

Then

**Proposition A.1.** \( \dim H_k < \dim H_{k+1} \).

This can be proved by the following elementary fact:

**Lemma A.2.** Let \( f(t) = \sum a_k t^k \) and \( g(t) = \sum b_k t^k \) be formal power series with positive coefficients
and satisfies the condition that

\[ f_n < b_n. \]

Then the coefficients of the product formal power series \( f \cdot g \) is increasing.

**Proof.** Since the \( n \)-th coefficient \( c_n \) of the product \( f \cdot g \) is

\[ c_n = \sum_{i=0}^{n} a_{n-i} b_i, \]

\[ c_{n+1} - c_n = a_{n+1} b_0 + a_n (b_1 - b_0) + \cdots + a_0 (b_{n+1} - b_n). \]

In the above expression, each term is non-negative by assumption so that \( c_{n+1} - c_n \geq 0 \). In addition if \( \{b_n\} \) is strictly increasing, then \( \{c_n\} \) is also strictly increasing at least one of the coefficient being \( a_k > 0 \).

\[ \square \]

**Proof of Proposition A.1.**

In our case, all the coefficients of the polynomial \((1 + t)(1 + t + t^2) = 1 + 2t + 2t^2 + t^3\) are positive and the coefficients of the power series expansion of the factor \( \left( \frac{1}{1-t} \right)^{24} \) are positive and strictly increasing. In fact, the \( k \)-th coefficient of the power series expansion of the function \( \left( \frac{1}{1-t} \right)^{24} \) is \( 24 + k \cdot C_k \) and strictly increasing, since it is a generating function of the Poincaré power series of the polynomial algebra \( \mathbb{C}[s_1, \ldots, s_{24}] \) of 24 variables. Hence the assertion for our power series \( PH(t) = (1 + 2t + 2t^2 + t^3) \cdot \left( \frac{1}{1-t} \right)^{24} \) is proved.

\[ \square \]

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