BAHADUR EFFICIENCY OF EDF BASED NORMALITY TESTS WHEN PARAMETERS ARE ESTIMATED

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In the present paper, some well-known tests based on empirical distribution functions (EDF) with estimated parameters for testing composite normality hypothesis are revisited, and some new results on asymptotic properties are provided. In particular, the approximate Bahadur slopes are obtained in the case of close alternatives for the EDF-based tests as well as the likelihood ratio test. The local approximate efficiencies are calculated for several close alternatives. The obtained results could serve as a benchmark for evaluation of the quality of recent and future normality tests. Bibliography: 35 titles.

1. Introduction

For testing the goodness-of-fit (GOF) null hypothesis that the sample is taken from a fully specified continuous distribution $F_0$, the tests predominantly used in practice are those based on some distance between the empirical distribution function (EDF) $F_n$ and $F_0$.

The most widely used EDF-based test is the well-known Kolmogorov–Smirnov test [17] with statistic $D_n = \sup_x |F_n(x) - F_0(x)|$ based on the $L^\infty$ distance. Other popular tests include the Cramer–von Mises [9] and Anderson–Darling [1] tests based on the weighted $L^2$ distance between $F_n$ and $F_0$. Different variations of these test statistics exist. Watson proposed the centered versions of the Kolmogorov–Smirnov [34] (see also [10, 11]) and the Cramer–von Mises [33] tests. Other variants were proposed by Kuiper [18] and Khmaladze [16], among others.

The properties of EDF-based tests are well-known. All these tests are distribution-free under the null hypothesis that makes them omnibus GOF tests applicable regardless of $F_0$. Their asymptotic distributions follow from the limiting process of $F_n(t) - F_0(t)$ when $n \to \infty$, which is the Brownian bridge. Large deviations of these statistics are available in [23].

However, more often than not, we would like to test a composite GOF null hypothesis that the sample comes from a family of distributions $F_0(x; \theta)$ indexed by a finite-dimensional parameter $\theta$. In this scenario, we need to estimate $\theta$ in order to apply the EDF-based tests. The problem is that the tests are no longer distribution-free, and their distribution depends on $F_0$ and $\theta$.

In case of location-scale families, it can be easily shown that the distribution does not depend on the location and scale parameters, but only on $F_0$. Therefore, in this case we can consider GOF tests for particular null location-scale families of distributions such as normal, exponential, logistic, Cauchy, etc.

The modified EDF-based tests have been proposed and/or their properties investigated by Durbin [8], Kac, Kiefer and Wolfowitz [15], Lilliefors [19,20], Sukhatme [31], and the asymptotic theory have been studied by Durbin [12] and Stephens [30], among others.

A popular tool for asymptotic comparison of tests is the Bahadur asymptotic efficiency. One of the advantages over other types of efficiencies is that it is more convenient when the

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asymptotic distributions are not normal. A comprehensive review of the Bahadur efficiencies of EDF-tests for the simple null hypothesis is available in [23].

The calculation of Bahadur efficiency is heavily dependent on the large deviation function of the test statistic, which is not available for the statistics with estimated parameters. An approach in this direction was done by Arcones [2] for the case of Kolmogorov–Smirnov normality test (also known as Lilliefors normality test), however, only upper and lower estimates for large deviations were obtained in a very complicated form. The only test for which the Bahadur efficiencies were calculated is the Kolmogorov–Smirnov exponentiality test [27]. There the corresponding large deviations were obtained using particular convenient properties of the exponential distribution.

When large deviations are unavailable, a common way out is to use the so-called approximate Bahadur efficiency. Instead of the large deviations, its calculation requires only the tail behaviour of the asymptotic distribution. The quality of approximation has been shown to be good locally and for some statistics (e.g., $U$-statistics [25,26] and their supremum [22,24]), exact and approximate Bahadur efficiency locally coincide.

In the present paper, we compare EDF-based tests in terms of approximate Bahadur efficiency when testing the null normality hypothesis with both parameters unknown. In Sec. 2, we present the test statistics and their asymptotic behavior, and in Sec. 3, we calculate the efficiencies.

2. TEST STATISTICS

Consider now the case of testing normality, i.e., the null hypothesis is $H_0 : F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$, where $\Phi$ is a standard normal distribution function, and unknown parameters $\mu$ and $\sigma$ are the mean and standard deviation.

The tests we consider are all based on difference

$$\Delta_n(t; \mu, \sigma) = F_n(t) - \Phi\left(\frac{t - \hat{\mu}}{\hat{\sigma}}\right);$$

- the Kolmogorov–Smirnov normality test with statistic
  $$D_n = \sup_{t \in \mathbb{R}} |\Delta_n(t; \mu, \sigma)|;$$
  (1)

- the Cramer–von Mises normality test
  $$\omega_n^2 = \int_{-\infty}^{\infty} \Delta_n^2(t; \mu, \sigma) d\Phi\left(\frac{t - \hat{\mu}}{\hat{\sigma}}\right);$$
  (2)

- the Anderson–Darling normality test
  $$A_n^2 = \int_{-\infty}^{\infty} \frac{\Delta_n^2(t; \mu, \sigma)}{\Phi\left(\frac{t - \hat{\mu}}{\hat{\sigma}}\right) \left(1 - \Phi\left(\frac{t - \hat{\mu}}{\hat{\sigma}}\right)\right)} d\Phi\left(\frac{t - \hat{\mu}}{\hat{\sigma}}\right);$$
  (3)

- the Watson–Darling variation of the Kolmogorov–Smirnov normality test
  $$G_n = \sup_{t \in \mathbb{R}} |\Delta_n(t; \mu, \sigma) - \int_{-\infty}^{\infty} \Delta_n(z; \mu, \sigma) d\Phi\left(\frac{z - \hat{\mu}}{\hat{\sigma}}\right) dz|;$$
  (4)

- the Watson variation of the Cramer–von Mises normality test
  $$U_n^2 = \int_{-\infty}^{\infty} \left(\Delta_n(t; \mu, \sigma) - \int_{-\infty}^{\infty} \Delta_n(z; \mu, \sigma) d\Phi\left(\frac{z - \hat{\mu}}{\hat{\sigma}}\right) dz\right)^2 d\Phi\left(\frac{t - \hat{\mu}}{\hat{\sigma}}\right) dt,$$
  (5)
where $\hat{\mu} = \tilde{X}_n$ and $\hat{\sigma}^2 = S^2$ are the maximum likelihood estimators of $\mu$ and $\sigma^2$. To describe the asymptotic distribution of the test statistics, we define the following empirical processes:

$$
\eta_n(x; \mu, \sigma^2) = F_n(\mu + \sigma x) - \Phi(x);
$$

$$
\xi_n(x; \mu, \sigma^2) = F_n(\mu + \sigma x) - \Phi(x) - \int_{-\infty}^{\infty} (F_n(\mu + \sigma z) - \Phi(z))\varphi(z) \, dz.
$$

Then, our statistics can be represented as

$$
D_n = \sup_{x \in \mathbb{R}} |\eta_n(x; \hat{\mu}, \hat{\sigma}^2)|;
$$

$$
\omega_n^2 = \int_{-\infty}^{\infty} \eta_n^2(x; \hat{\mu}, \hat{\sigma}^2) \varphi(x) \, dx;
$$

$$
A_n^2 = \int_{-\infty}^{\infty} \frac{\eta_n^2(x; \hat{\mu}, \hat{\sigma}^2)}{\Phi(x)(1 - \Phi(x))} \varphi(x) \, dx;
$$

$$
G_n = \sup_{x \in \mathbb{R}} |\xi_n(x; \hat{\mu}, \hat{\sigma}^2)|;
$$

$$
U_n^2 = \int_{-\infty}^{\infty} \xi_n^2(x; \hat{\mu}, \hat{\sigma}^2) \varphi(x) \, dx.
$$

It can be easily shown that all statistics are location and scale free under the null hypothesis of normality. Therefore, in what follows we assume that true parameters are $\mu_0 = 0$ and $\sigma_0 = 1$.

### 2.1. Asymptotic behaviour

**Theorem 2.1.** Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from normal $\mathcal{N}(0,1)$. Then the empirical processes $\sqrt{n}\eta_n(x; \hat{\mu}, \hat{\sigma}^2)$ and $\sqrt{n}\xi_n(x; \hat{\mu}, \hat{\sigma}^2)$ converge weakly in $D(\mathbb{R})$ to centered Gaussian processes $\eta(x)$ and $\xi(x)$ whose covariance functions are respectively equal to

$$
K_\eta(x, y) = \Phi(\min(x, y)) - \Phi(x)\Phi(y) - \varphi(x)\varphi(y) - \frac{1}{2}xy\varphi(x)\varphi(y),
$$

$$
K_\xi(x, y) = \Phi(\min(x, y)) - \Phi(x)\Phi(y) + \frac{1}{2}\Phi(x)(1 - \Phi(x)) + \frac{1}{2}\Phi(y)(1 - \Phi(y))
$$

$$
+ \frac{1}{2\sqrt{\pi}}(\varphi(x) + \varphi(y)) - \varphi(x)\varphi(y) - \frac{1}{2}xy\varphi(x)\varphi(y) + \frac{1}{12} - \frac{1}{4\pi}.
$$

**Proof.** For a fixed $x$, from [28] we have the following representation:

$$
\sqrt{n}\eta_n(x; \hat{\mu}, \hat{\sigma}^2) = \sqrt{n}\eta_n(x; 0, 1) + \sqrt{n}\hat{\mu} \cdot \frac{\partial}{\partial \mu} \mathbb{E} \left[ I\{X_1 < \mu + \sigma x\} - \Phi(x) \right] \bigg|_{\mu=0, \sigma^2=1}
$$

$$
+ \sqrt{n}(\sigma^2 - 1) \cdot \frac{\partial}{\partial \sigma^2} \mathbb{E} \left[ I\{X_1 < \mu + \sigma x\} - \Phi(x) \right] \bigg|_{\mu=0, \sigma^2=1} + o_P(1)
$$

$$
= \sqrt{n}\eta_n(x; 0, 1) + \varphi(x) \cdot \sqrt{n}\hat{\mu} + \frac{x}{2} \varphi(x) \cdot \sqrt{n}(\sigma^2 - 1) + o_P(1).
$$

It is straightforward to show from the multivariate central limit theorem that the finite dimensional distributions are asymptotically normal.

The tightness of this process follows from the tightness property of the first summand (see [5, Chap. 3]). The remaining components are just deterministic continuous functions of $x$ multiplied by a random variable, and as such tight in $C(\mathbb{R})$. 

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Taking into account the Bahadur representation of the estimator for $\sigma^2$,

$$\hat{\sigma}^2 - 1 = \frac{1}{n^2} \sum_{i,j} \frac{(X_i - X_j)^2}{2} - 1 = \frac{2}{n} \sum_i X_i^2 - \frac{1}{2} + o_P(1),$$

we obtain that the covariance function is

$$K_\eta(x, y) = K_0(x, y) + \varphi(y)E[I\{X < x\}X] + \frac{y\varphi(y)}{2}E[I\{X < x\}(X^2 - 1)] + \varphi(x)\varphi(y) + \frac{xy\varphi(x)\varphi(y)}{4} = K_0(x, y) - \varphi(y) \cdot \varphi(x) - \frac{y\varphi(y)}{2} \cdot x\varphi(x) - \varphi(x) \cdot \varphi(y) - \frac{1}{2}xy\varphi(x)\varphi(y),$$

where

$$K_0(x, y) = \Phi(\min(x, y)) - \Phi(x)\Phi(y)$$

is the covariance function of the limiting process $\eta;x, 0, 1\}$. The same arguments for convergence of $\eta_n$ hold for the empirical process $\xi_n$, too, following its representation as

$$\sqrt{n}\xi_n(x; \hat{\mu}, \hat{\sigma}^2) = \sqrt{n}\eta_n(x; 0, 1) + \frac{1}{2} - \frac{1}{n} \sum_{i=1}^n \Phi(-X_i) + \sqrt{\hat{\sigma}^2} \cdot \varphi(x) \cdot \sqrt{n}\hat{\mu}\mu$$

$$\times \frac{\partial}{\partial \mu} E[I\{X_1 < \mu + \sigma x\} - \Phi(x) - \Phi\left(\frac{\mu - X_i}{\sigma}\right)]_{\mu = 0, \sigma^2 = 1} + \sqrt{n}(\hat{\sigma}^2 - 1)$$

$$\times \frac{\partial}{\partial \sigma^2} E[I\{X_1 < \mu + \sigma x\} - \Phi(x) - \Phi\left(\frac{\mu - X_i}{\sigma}\right)]_{\mu = 0, \sigma^2 = 1} + o_P(1)$$

$$= \sqrt{n}\eta_n(x; 0, 1) + \frac{1}{n} \sum_{i=1}^n \Phi(X_i) - \frac{1}{2} + (\varphi(x) - \frac{1}{2\sqrt{\pi}}) \cdot \sqrt{n}\hat{\mu}\mu$$

$$+ \frac{x}{2}\varphi(x) \cdot \sqrt{n}(\hat{\sigma}^2 - 1) + o_P(1),$$

while its covariance function is

$$K_\xi(x, y) = K_0(x, y) + \frac{1}{2\sqrt{\pi}}(\phi(x) + \phi(y) - \frac{1}{\sqrt{\pi}} - \phi(x)\phi(y) - \frac{1}{2\sqrt{\pi}})$$

$$- \phi(y)\phi(x) - \frac{1}{2\sqrt{\pi}} + (\phi(x) - \frac{1}{2\sqrt{\pi}})(\phi(y) - \frac{1}{2\sqrt{\pi}}) - \frac{1}{2}y\phi(y)x\varphi(x)$$

$$- \frac{1}{2}x\phi(x)y\varphi(y) + \frac{1}{2}xy\phi(x)\varphi(y) + \frac{\Phi(x)}{2} + \frac{\Phi(y)}{2} + \frac{1}{12}$$

$$- \frac{1}{2}(1 - (1 - \Phi(x))^2) - \frac{1}{2}(1 - (1 - \Phi(y))^2)$$

$$= K_0(x, y) + \frac{1}{2}\Phi(x)(1 - \Phi(x)) + \frac{1}{2}\Phi(y)(1 - \Phi(y))$$

$$+ \frac{1}{2\sqrt{\pi}}(\varphi(x) + \varphi(y)) - \varphi(x)\varphi(y) - \frac{1}{2}xy\varphi(x)\varphi(y) + \frac{1}{12} - \frac{1}{4\pi}. \quad \Box$$
The limiting distributions of EDF based test statistics are given in the following corollary.

**Corollary 2.1.** Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from normal $N(0, 1)$. Then we have

\[
\sqrt{n}D_n \xrightarrow{d} \sup_{t \in \mathbb{R}} |\eta(t)|,
\]

\[
n\omega_n^2 \xrightarrow{d} \sum_{i=1}^{\infty} \lambda_i Z_i^2,
\]

\[
nA_n^2 \xrightarrow{d} \sum_{i=1}^{\infty} \nu_i Z_i^2,
\]

\[
\sqrt{n}G_n \xrightarrow{d} \sup_{t \in \mathbb{R}} |\xi(t)|,
\]

\[
nU_n^2 \xrightarrow{d} \sum_{i=1}^{\infty} \zeta_i Z_i^2,
\]

where $Z_i$ are i.i.d. standard normal random variables, and $\{\lambda_i\}, \{\nu_i\}$ and $\{\zeta_i\}$ are sequences of eigenvalues of integral operators $W, A$ and $U$ defined by

\[
Wq(x) = \int_{-\infty}^{\infty} K_\eta(x,y)q(y)\varphi(y) \, dy,
\]

\[
Aq(x) = \int_{-\infty}^{\infty} \frac{K_\eta(x,y)}{\Phi(x)(1-\Phi(x))\Phi(y)(1-\Phi(y))} q(y)\varphi(y) \, dy,
\]

and

\[
Uq(x) = \int_{-\infty}^{\infty} K_\xi(x,y)q(y)\varphi(y) \, dy,
\]

respectively.

The convergence holds for statistics $D_n$ and $G_n$ from the continuous mapping theorem, while for statistics $\omega_n^2, A_n, \text{ and } U_n$, the proof follows from continuous mapping theorem, Mercer’s theorem and Karhunen–Loeve decomposition of a Gaussian process (see, e.g., [13]).

### 3. Approximate Bahadur efficiency

Let $\mathcal{G} = \{G(x; \theta)\}$ be a family of distribution functions (DF’s) with densities $g(x; \theta)$ such that $G(x; \theta)$ is normal only for $\theta = 0$. We assume that the DF’s from the class $\mathcal{G}$ satisfy the regularity conditions from [25, Assumptions WD].

Assume that $T_n = T_n(X_1, \ldots, X_n)$ is a sequence of test statistics where the null hypothesis $H_0 : \theta \in \Theta_0$ is rejected for $T_n > t_0$. Let the sequence of DF’s of the test statistic $T_n$ converge in distribution to a nondegenerate DF $F$. Additionally, assume that

$$
\log(1 - F(t)) = -\frac{a_T t^2}{2}(1 + o(1)), \quad t \to \infty,
$$

and the limit in probability under the alternative

$$
\lim_{n \to \infty} T_n / \sqrt{n} = b_T(\theta) > 0
$$

exists for $\theta \in \Theta_1$. 

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The approximate relative Bahadur efficiency with respect to another test statistic \( V_n = V_n(X_1, \ldots, X_n) \) is defined as
\[
e^*_{T,V}(\theta) = \frac{c^*_T(\theta)}{c^*_V(\theta)},
\]
where
\[
c^*_T(\theta) = a_T^2(\theta)
\]
is the Bahadur approximate slope of \( T_n \). This is a measure of a test efficiency proposed by Bahadur in [3].

When studying the asymptotic efficiency, it is of interest to see the performance of tests for alternatives close to the null distribution. We define the local approximate Bahadur efficiency for such alternatives by
\[
e^*_{T,V} = \lim_{\theta \to 0} \frac{c^*_T(\theta)}{c^*_V(\theta)}.
\]
The local approximate efficiency often coincides with the exact one.

Here we calculate the approximate relative Bahadur efficiency against some common close alternatives with respect to the likelihood ratio test (LRT). The LRT has proven to be the optimal test in terms of the exact Bahadur efficiency, and is frequently used as a benchmark for comparison.

3.1. Local Bahadur slope of the LRT for normality. In [4], it was shown that the local exact Bahadur slope of LR test is equal to \( 2K(\theta) \), where \( K(\theta) \) is the Kullback–Leibler distance from the alternative distribution indexed by \( \theta \) to the family of null distributions. In the case of the null normality hypothesis, it is equal to
\[
K(\theta) = \inf_{\mu,\sigma} E_{\theta} \log \frac{g(X,\theta)}{\frac{1}{\sigma} \varphi\left(\frac{X-\mu}{\sigma}\right)} = \inf_{\mu,\sigma} \int_{-\infty}^{\infty} \log \frac{g(x,\theta)}{\frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right)} g(x;\theta) \, dx,
\]
where \( \varphi(x) \) is the standard normal density. In the case of close alternatives \( g(x;\theta) \), its behaviour is given in the following theorem.

**Theorem 3.1.** For a given density \( g(x;\theta) \) from \( \mathcal{G} \), it holds
\[
2K(\theta) = \left( \int_{-\infty}^{\infty} \frac{(g'_\theta(x;0))^2}{g(x;0)} \, dx - \frac{1}{\sigma^2_0} \left( \int_{-\infty}^{\infty} xg'_\theta(x;0) \, dx \right)^2 \right) - \frac{1}{2\sigma^4_0} \left( \int_{-\infty}^{\infty} (x-\mu_0)^2 g'_\theta(x;0) \, dx \right)^2 \cdot \theta^2 + o(\theta^2),
\]
where \( \mu_0 \) and \( \sigma^2_0 \) are parameters of normal distribution \( g(x;0) \).

**Proof.** The infimum in (11) is reached for
\[
\mu(\theta) = \int_{-\infty}^{\infty} xg(x;\theta) \, dx
\]
and
\[
\sigma^2(\theta) = \int_{-\infty}^{\infty} (x-\mu(\theta))^2 g(x;\theta) \, dx.
\]
It is straightforward that $\mu(0) = \mu_0$, $\sigma^2(0) = \sigma_0^2$, as well as

$$
\mu'(0) = \int_{-\infty}^{\infty} x g'_\theta(x; 0) \, dx \tag{15}
$$

$$
\mu''(0) = \int_{-\infty}^{\infty} x g''_\theta(x; 0) \, dx \tag{16}
$$

$$
(\sigma^2)'(0) = \int_{-\infty}^{\infty} (x - \mu_0)^2 g'_\theta(x; 0) \, dx \tag{17}
$$

$$
(\sigma^2)''(0) = -2 \left( \int_{-\infty}^{\infty} x g'_\theta(x; 0) \, dx \right)^2 + \int_{-\infty}^{\infty} (x - \mu_0)^2 g''_\theta(x; 0) \, dx. \tag{18}
$$

Differentiating $K(\theta)$ along $\theta$ with the help of expressions (15)–(18), we obtain that $K'(0) = 0$ and $K''(0)$ are equal to the right-hand side of (12). Expanding $K(\theta)$ in the Maclaurin series, we complete the proof. \□

The alternatives from $\mathcal{G}$ satisfy the conditions from [29], and, hence, the local approximate slope of LRT also has representations (12).

### 3.2. Local Bahadur slopes of the EDF based tests

**Theorem 3.2.** For the statistics $D_n$, $\omega^2_n$, $A^2_n$, $G_n$ and $U^2_n$, and alternative density $g(x, \theta) \in \mathcal{G}$, the Bahadur approximate slopes are

$$
c_D(\theta) = \frac{1}{\sup_x K_\theta(x, x)} \left( \sup_x \left| g^*(x) \right| \right)^2 \cdot \theta^2 + o(\theta^2);
$$

$$
c_{\omega^2}(\theta) = \frac{1}{\lambda_1} \int_{-\infty}^{\infty} \left( g^*(x) \right)^2 \varphi(x) \, dx \cdot \theta^2 + o(\theta^2);
$$

$$
c_{A^2}(\theta) = \frac{1}{\nu_1} \int_{-\infty}^{\infty} \frac{\left( g^*(x) \right)^2}{\Phi(x)(1 - \Phi(x))} \varphi(x) \, dx \cdot \theta^2 + o(\theta^2);
$$

$$
c_{G}(\theta) = \sup_{x \in \mathbb{R}} \left| g^*(x) \right| \int_{-\infty}^{\infty} \left( g^*(u) \right) \varphi(u) \, du + o(\theta^2);
$$

$$
c_{U^2}(\theta) = \int_{-\infty}^{\infty} \left( g^*(x) \right)^2 \varphi(x) \, dx + o(\theta^2),
$$

respectively, where $\lambda_1$, $\nu_1$ and $\zeta_1$ are the largest eigenvalues of operators $\mathcal{W}$, $\mathcal{A}$ and $\mathcal{U}$ defined in (6)–(8), and

$$
g^*(x) = G'_\theta(x; 0) + g(x; 0)(\mu'(0) + x\sigma'(0)).$$
Proof. For each $x \in \mathbb{R}$, using the law of large numbers for U-statistics with estimated parameters [14], the limit in probability of $\eta_n(x, \hat{\mu}, \hat{\sigma}^2)$ is

$$B(x, \theta) = G(\mu(\theta) + \sigma(\theta)x, \theta) - \Phi(x) = \int_{-\infty}^{\mu(\theta)+\sigma(\theta)x} g(u, \theta) \, du - \Phi(x).$$

Further we have that

$$B'_{\theta}(x, \theta) = g(\mu(\theta) + \sigma(\theta)x, \theta)(\mu'(\theta) + \sigma'(\theta)x) + \int_{-\infty}^{\mu(\theta)+\sigma(\theta)x} g'_{\theta}(u, \theta) \, du.$$  

When $\theta = 0$, the expression above is equal to

$$B'_{\theta}(x, 0) = g(x, 0)(\mu'(0) + \sigma'(0)x) + G'_{\theta}(x; 0).$$

Hence, we obtain

$$B(x, \theta) = g^*(x) \cdot \theta + o(\theta), \quad \theta \to 0.$$  

Following [32, Chap. 19], the limits in $P_0$ probability of statistics $D_n$, $W_n$ and $A_n$ are then

$$b_D(\theta) = \sup_x |g^*(x)| \cdot \theta + o(\theta);$$

$$b_\omega(\theta) = \int_{-\infty}^{\infty} (g^*(x))^2 \varphi(x) \, dx \cdot \theta^2 + o(\theta^2);$$

$$b_A(\theta) = \int_{-\infty}^{\infty} \frac{(g^*(x))^2}{\Phi(x)(1 - \Phi(x))} \varphi(x) \, dx \cdot \theta^2 + o(\theta^2).$$

Analogously, using the process $\xi_n(x; \hat{\mu}, \hat{\sigma}^2)$, we obtain that the limits in probability of $G_n$ and $U_n^2$ are

$$b_G(\theta) = \sup_x |g^*(x) - \int_{-\infty}^{\infty} (g^*(u)) \varphi(u) \, du| \cdot \theta + o(\theta);$$

$$b_{U^2}(\theta) = \int_{-\infty}^{\infty} \left( g^*(x) - \int_{-\infty}^{\infty} (g^*(u)) \varphi(u) \, du \right)^2 \varphi(x) \, dx \cdot \theta^2 + o(\theta^2).$$

The tail behaviour of the supremum of a Gaussian process follows from [21], and the constant $a_T$ from (9) is equal to the supremum on the diagonal of the covariance function. Therefore, we get $a_D = \sup_t K_0(t, t)$ in the case of $D_n$, and $a_G = \sup_t K(t, t)$ in the case of $G_n$.

For the integral type statistic $\omega^2_n$, using the result of Zolotarev [35], we have that the logarithmic tail behaviour of $\tilde{\omega}^2 = \sqrt{n} \omega^2_n$ is

$$\log(1 - F_{\tilde{\omega}^2}(x)) = -\frac{x^2}{2\lambda_1} + o(x^2), \quad x \to \infty,$$

and hence, $\tilde{a}_{\tilde{\omega}^2} = \frac{1}{\lambda_1}$, where $\lambda_1$ is the largest eigenvalue of the integral operator $W$ defined in (6). Analogously, we get $\tilde{a}_{A^2} = \frac{1}{\nu_1}$ and $\tilde{a}_{U^2} = \frac{1}{\zeta_1}$ for statistics $A^2$ and $U^2$. □
### 3.3. Calculation of efficiencies.

The close alternatives we consider here are

- a Lehmann alternative with density
  \[ g_1(x; \theta) = (1 + \theta) \Phi(\theta)(x)\varphi(x); \]
- the first Ley–Paindaveine alternative with density
  \[ g_2(x; \theta) = \varphi(x)e^{-\theta(1-\Phi(x))}(1 + \theta\Phi(x)); \]
- the second Ley–Paindaveine alternative with density
  \[ g_3(x; \theta) = \varphi(x)(1 - \theta\pi \cos(\pi\Phi(x)); \]
- a contamination alternative (with \( N(\mu, \sigma^2) \)) alternative with density
  \[ g_4(x; \theta) = (1 - \theta)\varphi(x) + \frac{\theta}{\sigma}\varphi(x - \frac{\mu}{\sigma}). \]

To calculate the efficiency, one needs to find the largest eigenvalues \( \lambda_1, \nu_1 \) and \( \zeta_1 \) from Corollary 2.1. Since we cannot obtain them analytically, we use the approximation method from [6] (see also [7]).

The values of efficiencies are presented in Table 3.3. We can see that the integral tests are more efficient than the supremum ones. Among them, the Anderson–Darling test is the best one for almost all considered alternatives. Additionally, the Watson-type modifications of Kolmogorov–Smirnov and Cramer–von Mises tests are less efficient than the original versions.

These results can serve as a benchmark for evaluation of the quality of recent and future normality tests.

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