OPTIMAL DECAY RATE OF SOLUTIONS FOR NONLINEAR KLEIN-GORDON SYSTEMS OF CRITICAL TYPE

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ABSTRACT. We consider the decay rate of solutions to nonlinear Klein-Gordon systems with a critical type nonlinearity. We will specify the optimal decay rate for a specific class of Klein-Gordon systems containing the dissipative nonlinearities. It will turn out that the decay rate which is previously found in some models is optimal.

1. INTRODUCTION

In this article, we consider the following system of nonlinear Klein-Gordon equations:

\[(\Box + m_j^2)u_j = F_j(u, \partial_t u, Du), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad j = 1, \cdots, N,\]

where \(\Box = \partial_t^2 - \Delta\), \(d\) is a positive integer, \(u = (u_j)_{1 \leq j \leq N}\) is an \(\mathbb{R}^N\)-valued unknown function, and

\[Du := (\partial_i u_j)_{1 \leq i \leq d, 1 \leq j \leq N}\]

with \(\partial_k = \partial_{x_k}(k = 1, \cdots, d)\). The masses \(m_j\) are positive constants. We assume that the nonlinearity \(F = (F_j)_{1 \leq j \leq N}\) is of critical order, that is, it satisfies

\[|F(\zeta, \eta, \Theta)| \leq C (|\zeta|^2 + |\eta|^2 + |\Theta|_{F_r}^{2+2d})^{\frac{1}{2}(1 + \frac{2}{d})}\]

for \((\zeta, \eta, \Theta) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N \times d}\). Here \(|\cdot|_{F_r}\) denotes the Frobenius norm,

\[|\Theta|_{F_r}^2 = \sum_{i=1}^N \sum_{j=1}^d |\theta_{ij}|^2\]

of a matrix \(\Theta = (\theta_{ij})_{1 \leq i \leq N, 1 \leq j \leq d}\). We note that (1.1) includes a single Klein-Gordon equation as the special case \(N = 1\). Here we are interested in the decay rate of solutions to (1.1). Our main goal is to give an upper bound on the decay rate of solutions to a class of Klein-Gordon equations and/or systems. In particular, it will turn out that, in some specific models, a known decay rate is an optimal one.

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There is a number of previous results on the decay rate and asymptotic behavior of Klein-Gordon equations and systems. It is known that the order of the nonlinearity in (1.2) is a critical order regarding the long-time asymptotic behavior.

Let us introduce some previous results in the single equation case. First, a solution of a single linear Klein-Gordon equation decays like
\[ \|u(t)\|_{L^\infty} = O(t^{-\frac{d}{2}}). \]
As for the single equation with the power type nonlinearity \( F = |u|^p u \), there exists a solution which approaches to a free solution as \( t \to \infty \) if \( p > \frac{2}{d} \) (see [9, 10, 14, 15]). Remark that the asymptotics holds in \( L^\infty \) in some cases and that the decay rate of a solution is the same as (1.3) in such cases. On the other hand, if \( 0 < p \leq \frac{2}{d} \), Glassey [3] and Matsumura [12] show that there is no non-trivial solution which asymptotically approach to a free solution. Our case \( p = \frac{2}{d} \) is critical in this sense.

When the nonlinearity is of critical order, there are several possibilities on the asymptotic behavior of solutions and it depends on the shape of the nonlinearity. For instance, as for the equation
\[ (\Box + 1)u = F(u, \partial_t u, \partial_x u), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \]
which is the case of \( d = 1 \) and \( N = 1 \) in (1.1), the following is known. Georgiev-Yordanov [2] consider the cubic nonlinearity \( F = u^3 \) and prove that there is a global solution which decays like \( O(t^{-\frac{1}{2}}) \) in \( L^\infty \) but does not behave like a free solution. In this case, it is known that the asymptotic behavior of the solution is a modified scattering (see [11, 13]). On the other hand, Moriyama [13] and Katayama [5] shows that there are some equations such that a solution behaves like a free solution (for example, an equation with \( F = 3uu_x^2 - 3uu_x^2 - u^3 \)).

Not only the asymptotic behavior but also the decay rate is different from (1.3) in some cases. Indeed, if we take \( F = -(\partial_t u)^3 \), a logarithmic decay
\[ \|u(t)\|_{L^\infty} = O(t^{-\frac{1}{2}}(\log t)^{-\frac{1}{2}}) \]
is obtained by Sunagawa [20]. Note that the decay rate of the solution is faster than a free solution due to the influence of the nonlinearity. This kind of phenomenon is due to so-called nonlinear dissipation.

Next we consider the system case. In general, the behavior of the solution is expected to become richer in the system case. The asymptotic behavior in the single case described above is also found in the system case. First, Sunagawa [16] and Katayama-Ozawa-Sunagawa [6] find some conditions on the nonlinearity \( F \) and mass \( m_j \) which ensure the existence of an asymptotically free solution for \( d = 1, 2 \). At the same time, Sunagawa [17] shows that there exists a solution which decays like \( O(t^{-\frac{1}{2}}) \) in \( L^\infty \), the same decay as a free solution, but does not behave like a free solution and exhibits a modified-scattering type behavior. Masaki-Segata-Uriya [11] show modified scattering in the complex-valued case for \( d = 1, 2 \). Remark that the single complex-valued equation is equivalent to a system of real-valued system (see also [19]). In the above cases, the decay rate of the solutions is the same as that of (1.3).

Kim-Sunagawa [7] show that if we consider the nonlinearity
\[ F_j = \mu_1 |u|^2 u_j - \mu_2 |\partial_t u|^2 \partial_t u_j \quad (\mu_1 \in \mathbb{R}, \mu_2 > 0, j = 1, 2, \ldots, N), \]
then the corresponding model admits a solution which is compactly-supported and decays like
\[ \|u(t)\|_{L^\infty} = O(t^{-\frac{1}{2}}(\log t)^{-\frac{1}{2}}) \]
under an appropriate condition on the initial data. Moreover, the solution \( u \) satisfies
\[
\|u(t, \cdot)\|_{L^\infty} + \|\partial_t u(t, \cdot)\|_{L^\infty} + \|\partial_x u(t, \cdot)\|_{L^\infty} \leq C t^{-\frac{d}{2}} (\log t)^{-\frac{1}{2}}
\]
in this case. We note that this nonlinearity is a typical example which has the nonlinear dissipation effect. They give a condition on nonlinearity for which there exists a solution satisfying (1.6).

The ratio of the mass coefficients \( m_j \) also matters in the system case. [16] gave such an example. It is shown that, for one fixed nonlinearity, the corresponding solution blows up in finite time for some choices of \( m_j \), and the global solution exists for other choices. According to [18], the system
\[
\begin{cases}
(\Box + m_1^2) u_1 = 0 \\
(\Box + m_2^2) u_2 = u_3
\end{cases}
\]
is an example for which the solution behaves differently from a solution of a single equation. The second component \( u_2 \) of a solution decays no faster than \( t^{-\frac{d}{2}} (\log t)^{\frac{1}{2}} \) in \( L^\infty \).

In this article, we show that a class of the nonlinear Klein-Gordon equations/system can not admit a (good) solution decaying faster than the decay rate (1.6). This in particular shows that the rate (1.6) is optimal for a class of nonlinearity including (1.4). Recently, Kita [8] studies the optimal decay rate of solution to a dissipative nonlinear Schrödinger equation. Our approach is inspired by his argument.

1.1. Main results. Let us state our main theorem. The assumptions on a nonlinearity is as follows:

Assumption 1.1. The nonlinearity \( F \) satisfies the estimate (1.2). Moreover, we have uniqueness of a classical solution to the initial value problem of the equation (1.1) with \( F \), that is, if \( u_1 \) and \( u_2 \) are two classical solutions and if \( u_1(t_0) = u_2(t_0) \) holds for some \( t_0 \) then we have \( u_1 \equiv u_2 \).

We say a solution is forward-global if the solution exists on \([T, \infty)\) for some \( T \in \mathbb{R} \). Now we state our main theorem.

Theorem 1.1. Suppose that the nonlinearity \( F \) satisfies Assumption 1.1. Let \( u \) be a forward-global classical solution to (1.1) with the nonlinearity \( F \). Assume that there exist constants \( C > 0 \) and \( T_1 > 0 \) such that
\[
(1.7) \quad \|u(t, \cdot)\|_{L^\infty} + \|D u(t, \cdot)\|_{L^\infty} \leq C t^{-\frac{d}{2}} (\log t)^{-\frac{1}{2}}
\]
holds for all \( t \geq T_1 \). Also assume that for any \( \varepsilon > 0 \) there exists \( T_2 \geq T_1 \) such that
\[
(1.8) \quad \|\partial_t u(t, \cdot)\|_{L^\infty} \leq \varepsilon t^{-\frac{d}{2}} (\log t)^{-\frac{1}{2}}
\]
holds for all \( t \geq T_2 \). Then \( u \) is identically zero.

Remark 1.1. In Theorem 1.1 we assume that the existence of a forward-global solution of (1.1). We note that the existence of a forward-global solution is not trivial. In fact, even in the simple case of \( d = 1 \) and \( N = 1 \), small data global existence is not obvious (for instance, see [5]). A sufficient condition for small data global existence is given in [1].

Remark 1.2. We would emphasize that it is not necessary to assume that the solution \( u \) is compactly supported.

We consider a classical solution in Theorem 1.1. However, by combining with an appropriate well-posedness result, we can show the same conclusion as Theorem 1.1 for other
class of solutions. As such an example, we give a result for $H^2_x \times H_x^1$-solutions, which is our second result. In the following, we assume $d = 1$ and $m_1 = \cdots = m_N$ for simplicity.

Before stating the result, let us introduce the notion of an $H^2_x \times H_x^1$-solution.

**Definition 1.1.** We say that a function $u$ is an $H^2_x \times H_x^1$-solution on an interval $I \subset \mathbb{R}$ to (1.1) if $u \in C(I; (H^2_x(\mathbb{R}))^N) \cap C^1_t(I; (H_x^1(\mathbb{R}))^N)$ and $u$ obeys the Duhamel formula:

$$
\begin{pmatrix}
  u_j(t) \\
  \partial_t u_j(t)
\end{pmatrix}
= 
\begin{pmatrix}
  \cos \langle \partial_x \rangle (t - t_0) & \langle \partial_x \rangle^{-1} \sin \langle \partial_x \rangle (t - t_0) \\
  -\langle \partial_x \rangle \sin \langle \partial_x \rangle (t - t_0) & \cos \langle \partial_x \rangle (t - t_0)
\end{pmatrix}
\begin{pmatrix}
  u_j(t_0) \\
  \partial_t u_j(t_0)
\end{pmatrix}
+ \int_{t_0}^t \begin{pmatrix}
  \langle \partial_x \rangle^{-1} \sin \langle \partial_x \rangle (t - s) \\
  \cos \langle \partial_x \rangle (t - s)
\end{pmatrix}
F_j(u(s), \partial_t u(s), \partial_x u(s)) \, ds
$$

(1.9)

for $t_0, t \in I$.

We state the assumption on nonlinearities.

**Assumption 1.2.** The nonlinearity $F$ satisfies

$$
|\partial_{\zeta, \eta, \theta}^\alpha F(\zeta, \eta, \theta)| \leq C \left( |\zeta|^2 + |\eta|^2 + |\theta|^2 \right)^{3/2 - |\alpha|}
$$

(1.10)

for $0 \leq |\alpha| \leq 3$.

By a standard argument, one can show that if the nonlinearity $F$ satisfies the above assumption then the local well-posedness of the initial value problem of (1.1) holds in $H^2_x \times H_x^1$. Furthermore, the equation admits a classical solution for smooth data.

The following is our second result.

**Theorem 1.2.** Suppose that the nonlinearity $F$ satisfies Assumption 1.2. Let $u$ be a forward-global $H^2_x \times H_x^1$-solution to (1.1). If $u$ satisfies (1.7) and (1.8) then $u$ is identically zero.

It is clear that Kim-Sunagawa’s model (1.4) satisfies Assumption 1.2. The above theorem shows that the decay rate (1.6) is optimal for this model.

## 2. Proof of Theorem 1.1

In the next section, we prove Theorem 1.1. A key ingredient is a **localized linear energy**: For a classical solution $u$, $t \in \mathbb{R}$, and $y \in \mathbb{R}^d$,

$$
E_{t,y}(u(t)) := \sum_{j=1}^N \int_{|x-y| \leq t} \left\{ |\nabla u_j(t, x)|^2 + |\partial_t u_j(t, x)|^2 + m_j^2 |u_j(t, x)|^2 \right\} \, dx.
$$

(2.1)

Remark that the quantity is well-defined for a classical solution even when it does not decay near the spatial infinity.

**Proof.** Define the localized linear energy $E_{t,y}$ for any $y \in \mathbb{R}$ as in (2.1). Then, it follows that

$$
\frac{d}{dt} E_{t,y}(u(t)) = 2 \sum_{j=1}^N \int_{|x-y| \leq t} \left\{ (\nabla u_j) \cdot (\partial_t \nabla u_j) + (\partial_t u_j)(\partial_t^2 u_j) + (m_j^2 u_j)(\partial_t u_j) \right\} \, dx
+ \sum_{j=1}^N \int_{|x-y| = t} \left\{ |\nabla u_j(t, x)|^2 + |\partial_t u_j(t, x)|^2 + m_j^2 |u_j(t, x)|^2 \right\} \, dx.
$$
We calculate the first term above. By (1.7), we deduce
\[
\int_{|x-y|\leq t} \left\{(\nabla u_j) \cdot (\partial_t \nabla u_j) + (\partial_t u_j)(\partial_t^2 u_j) + (m_j^2 u_j)(\partial_t u_j)\right\} \, dx
\]
\[
= \int_{|x-y|\leq t} (\partial_t u_j) \left( \partial_t^2 - \Delta + m_j^2 \right) u_j \, dx + \int_{|x-y|=t} \partial_t u_j \nabla u_j \cdot \frac{x}{|x|} \, dx
\]
\[
= \int_{|x-y|\leq t} (\partial_t u_j) F_j(u, \partial_t u, Du) \, dx + \int_{|x-y|=t} \partial_t u_j \nabla u_j \cdot \frac{x}{|x|} \, dx
\]
\[
\geq -C \int_{|x-y|\leq t} (\partial_t u_j) \left( |u|^2 + |\partial_t u|^2 + |Du|_{F_r}^2 \right)^{\frac{1}{2}(1+\frac{2}{d})} \, dx
\]
\[
+ \int_{|x-y|=t} \partial_t u_j \nabla u_j \cdot \frac{x}{|x|} \, dx.
\]
Using the assumption (1.7) and (1.8), we have
\[
-C \int_{|x-y|\leq t} (\partial_t u_j) \left( |u|^2 + |\partial_t u|^2 + |Du|_{F_r}^2 \right)^{\frac{1}{2}(1+\frac{2}{d})} \, dx
\]
\[
\geq -C \int_{|x-y|\leq t} \left( |u|^2 + |\partial_t u|^2 + |Du|_{F_r}^2 \right)^{\frac{1}{2}(1+\frac{2}{d})} \, dx
\]
\[
\geq -C \varepsilon t^{-1} (\log t)^{-1} E_{t,y}(u(t))
\]
when \( d = 1, 2 \). Similarly, one has
\[
-C \int_{|x-y|\leq t} (\partial_t u_j) \left( |u|^2 + |\partial_t u|^2 + |Du|_{F_r}^2 \right)^{\frac{1}{2}(1+\frac{2}{d})} \, dx
\]
\[
\geq -C \int_{|x-y|\leq t} \left( |u|^2 + |\partial_t u|^2 + |Du|_{F_r}^2 \right)^{\frac{1}{2}(1+\frac{2}{d})} \, dx
\]
\[
\geq -C \varepsilon t^{-\frac{2}{d}} t^{-1} (\log t)^{-1} E_{t,y}(u(t))
\]
when \( d \geq 3 \). Combining these estimates, we obtain
\[
\frac{d}{dt} E_{t,y}(u(t)) \geq -C \varepsilon t^{-1} (\log t)^{-1} E_{t,y}(u(t))
\]
\[
+ \sum_{j=1}^{N} \int_{|x-y|=t} \left\{ m_j^2 |u_j|^2 + |\nabla u_j|^2 + |\partial_t u_j|^2 + 2\partial_t u_j \nabla u_j \cdot \frac{x}{|x|} \right\} \, dx
\]
\[
\geq -C \varepsilon t^{-1} (\log t)^{-1} E_{t,y}(u(t))
\]
\[
+ \sum_{j=1}^{N} \int_{|x-y|=t} \left\{ m_j^2 |u_j|^2 + \left| \frac{\partial_t u_j x}{|x|} + \nabla u_j \right|^2 \right\} \, dx.
\]
Therefore we reach to the estimate
\[
(2.2) \quad \frac{d}{dt} E_{t,y}(u(t)) \geq -C \varepsilon t^{-1} (\log t)^{-1} E_{t,y}(u(t)).
\]
Fix \( \delta \in (0, d) \). By using (2.2), we deduce that
\[
\frac{d}{dt} \left( (\log t)^{\delta} E_{t,y}(u(t)) \right) = \delta t^{-1} (\log t)^{\delta-1} E_{t,y}(u(t)) + (\log t)^{\delta} \frac{d}{dt} E_{t,y}(u(t))
\]
\[
\geq \delta t^{-1} (\log t)^{\delta-1} E_{t,y}(u(t)) - C \varepsilon t^{-1} (\log t)^{\delta-1} E_{t,y}(u(t))
\]
\[
\geq 0
\]
for \( \varepsilon \) sufficiently small and \( t \geq T_2(\varepsilon) \). Integrating this with respect to time, we have
\[
(\log t_1)^\delta E_{t_1,y}(u(t_1)) \leq (\log t_2)^\delta E_{t_2,y}(u(t_2))
\]
for \( T_2 \leq t_1 \leq t_2 \).

On the other hand, by using (1.7) and (1.8), we deduce
\[
E_{t,y}(u(t)) \leq C(\|u\|_{L^\infty} + \|\partial_t u\|_{L^\infty} + \|Du|_{F}\|_{L^\infty})^2 \sum_{j=1}^{N} \int_{|x-y| \leq t} \, dx
\]
\[
\leq C(\log t)^{-d}
\]
for \( t \geq T_2 \). Thus, we obtain
\[
(\log t_1)^\delta E_{t_1,y}(u(t_1)) \leq (\log t_2)^{\delta-d}.
\]
Since \( \delta < d \), the right-hand side converges to zero as \( t_2 \) tends to infinity, showing that \( E_{t_1,y}(u(t_1)) = 0 \). One then sees that
\[
u(t_1, x) = 0, \quad \partial_t u(t_1, x) = 0
\]
for \( |x-y| < t_1 \). Since \( t_1 \) is independent on \( y \), this is true for any choice of \( y \in \mathbb{R}^d \). Thus, we obtain
\[
(2.3) \quad u(t_1, x) = 0, \quad \partial_t u(t_1, x) = 0
\]
for all \( x \in \mathbb{R}^d \). By the assumption (1.2), the zero solution is also a forward-global classical solutions satisfying (2.3). Hence, we obtain the desired conclusion by the uniqueness of a classical solution. \( \square \)

3. PROOF OF THEOREM 1.2

Proof. We set the localized linear energy \( E_{t,y} \) as in (2.1). Pick sequences \( \{u_{0,n}\} \subset H_2^3(\mathbb{R})^N \) and \( \{u_{1,n}\} \subset H_2^2(\mathbb{R})^N \) so that
\[
\|u(0) - u_{0,n}(0)\|_{H_2^1} + \|\partial_t u(0) - \partial_t u_{1,n}(0)\|_{H_2^1} \rightarrow 0
\]
as \( n \rightarrow \infty \). By Assumption 1.2 there exists a classical solution \( u_n(t) \) to (1.1) with data \( (u_n(0), \partial_t u_n(0)) = (u_{0,n}, u_{1,n}) \). Then, a standard blowup criterion and continuous dependence show that
\[
(3.1) \quad \|u_n - u\|_{L^\infty((0,\tau);H_2^1)} + \|\partial_t u_n - \partial_t u\|_{L^\infty((0,\tau);H_2^1)} \rightarrow 0
\]
as \( n \rightarrow \infty \) for all \( \tau > 0 \).

Let \( \varepsilon > 0 \) to be chosen later. Fix \( \tau > 2T_2(\varepsilon) \). By (3.1), we deduce that there exists a constant \( N_0 = N_0(\varepsilon, \tau) \in \mathbb{N}_{\geq 0} \) such that
\[
\|\partial_t u_n(t)\|_{L^\infty} \leq \|\partial_t u_n(t) - \partial_t u(t)\|_{L^\infty} + \|\partial_t u(t)\|_{L^\infty}
\]
\[
\leq C\|\partial_t u_n(t) - \partial_t u(t)\|_{H_2^1} + \|\partial_t u(t)\|_{L^\infty}
\]
\[
\leq C\varepsilon t^{-\frac{1}{2}}(\log t)^{-\frac{1}{4}}
\]
for all \( n > N_0 \) and \( t \in (2T_2, \tau) \). In the same way, we also deduce that there exists \( N_1 \geq N_0 \) such that
\[
\|u_n(t)\|_{L^\infty} + \|\partial_x u_n(t)\|_{L^\infty} \leq C\|u_n(t) - u(t)\|_{H_2^1} + \|\partial_x u(t)\|_{L^\infty} + \|u(t)\|_{L^\infty}
\]
\[
\leq Ct^{-\frac{1}{2}}(\log t)^{-\frac{1}{4}}
\]
for all \( n > N_1 \) and \( t \in (2T_2, \tau) \).
The rest of the proof is almost the same as that of Theorem 1.1. By arguments similar to the proof of (2.2), we obtain
\[
\frac{d}{dt} E_{t,y}(u_n(t)) \geq -C \varepsilon t^{-1} (\log t)^{-1} E_{t,y}(u_n(t))
\]
for \( t \in (2T, \tau) \) and \( n > N_1 \). Similarly,
\[
E_{t,y}(u_n(t)) \leq C (\log t)^{-d}
\]
holds for \( t \in (2T, \tau) \) and \( n > N_1 \). We now fix \( \delta \in (0, d) \) and choose \( \varepsilon > 0 \) so small that we have
\[
(\log 2T)^{\delta} E_{2T,y}(u_n(2T)) \leq (\log \tau)^{\delta} E_{\tau,y}(u_n(\tau)) \leq (\log \tau)^{\delta-d}
\]
for all \( n > N_1(\varepsilon, \tau) \). We pass to the limit \( n \to \infty \) to obtain
\[
(\log 2T)^{\delta} E_{2T,y}(u(2T)) \leq (\log \tau)^{\delta-d}.
\]
Since \( \tau > 2T(\varepsilon) \) is arbitrary, we obtain
\[
(\log 2T)^{\delta} E_{2T,y}(u(2T)) = 0
\]
by letting \( \tau \to \infty \). Since this is true for all \( y \in \mathbb{R}^d \), we have \( u(2T) = \partial_t u(2T) = 0 \). By the uniqueness of an \( H^1_x \times H^1_y \)-solution, we conclude that \( u \) is identically zero. \[\square\]

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References

[1] Jean-Marc Delort, *Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi linéaire à données petites en dimension 1*, Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 1, 1–61. MR1833089
[2] Vladimir Georgiev and Borislav Yordanov, *Asymptotic behaviour of the one-dimensional klein–gordon equation with a cubic nonlinearity*, preprint (1996).
[3] Robert T. Glassey, *On the asymptotic behavior of nonlinear wave equations*, Trans. Amer. Math. Soc. 182 (1973), 187–200. MR330782
[4] Nakao Hayashi and Pavel I. Naumkin, *The initial value problem for the cubic nonlinear Klein-Gordon equation*, Z. Angew. Math. Phys. 59 (2008), no. 6, 1002–1028. MR2457221
[5] Soichiro Katayama, *A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension*, J. Math. Kyoto Univ. 39 (1999), no. 2, 203–213. MR1709289
[6] Soichiro Katayama, Tohru Ozawa, and Hideaki Sunagawa, *A note on the null condition for quadratic nonlinear Klein-Gordon systems in two space dimensions*, Comm. Pure Appl. Math. 65 (2012), no. 9, 1285–1302. MR2954616
[7] Donghyun Kim and Hideaki Sunagawa, *Remarks on decay of small solutions to systems of Klein-Gordon equations with dissipative nonlinearities*, Nonlinear Anal. 97 (2014), 94–105. MR3146374
[8] Naoyasu Kita, *Optimal decay rate of solutions to 1d Schrödinger equation with cubic dissipative nonlinearity*, Journal of Applied Science and Engineering A 1 (2019), no. 1, 15–18.
[9] S. Klainerman and Gustavo Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math. 36 (1983), no. 1, 133–141. MR680085
[10] Sergiu Klainerman, *Long-time behavior of solutions to nonlinear evolution equations*, Arch. Rational Mech. Anal. 78 (1982), no. 1, 73–98. MR654553
[11] Satoshi Masaki, Junichi Segata, and Kota Uriya, *Long range scattering for the complex-valued klein-gordon equation with quadratic nonlinearity in two dimensions*, arXiv preprint arXiv:1810.02158 (2018).
[12] Akitaka Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci. 12 (1976/77), no. 1, 169–189. MR0420031
[13] Kazunori Moriyama, *Normal forms and global existence of solutions to a class of cubic nonlinear Klein-Gordon equations in one space dimension*, Differential Integral Equations 10 (1997), no. 3, 499–520. MR1744859

[14] Jalal Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Differential Equations 46 (1982), no. 3, 409–425. MR681231

[15] , *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. 38 (1985), no. 5, 685–696. MR803256

[16] Hideaki Sunagawa, *On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension*, J. Differential Equations 192 (2003), no. 2, 308–325. MR1990843

[17] , *A note on the large time asymptotics for a system of Klein-Gordon equations*, Hokkaido Math. J. 33 (2004), no. 2, 457–472. MR2073010

[18] , *Large time asymptotics of solutions to nonlinear Klein-Gordon systems*, Osaka J. Math. 42 (2005), no. 1, 65–83. MR2130963

[19] , *Remarks on the asymptotic behavior of the cubic nonlinear Klein-Gordon equations in one space dimension*, Differential Integral Equations 18 (2005), no. 5, 481–494. MR2136975

[20] , *Large time behavior of solutions to the Klein-Gordon equation with nonlinear dissipative terms*, J. Math. Soc. Japan 58 (2006), no. 2, 379–400. MR2228565