Reconstructing $f(R)$ gravity from the spectral index

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Recent cosmological observations are in good agreement with the scalar spectral index $n_s$ with $n_s - 1 \approx -2/N$, where $N$ is the number of e-foldings. In previous work [T. Chiba, Prog. Theor. Exp. Phys. 2015, 073E02 (2015)], the reconstruction of the inflaton potential for a given $n_s$ was studied, and it was found that, for $n_s - 1 = -2/N$, the potential takes the form of either an $\alpha$-attractor model or a chaotic inflation model with $\phi^2$ to the leading order in the slow-roll approximation. Here we consider the reconstruction of the $f(R)$ gravity model for a given $n_s$ both in the Einstein frame and in the Jordan frame. We find that for $n_s - 1 = -2/N$ (or more general $n_s - 1 = -p/N$), $f(R)$ is given parametrically and is found to asymptote to $R^2$ for large $R$. This behavior is generic as long as the scalar potential is of slow-roll type.

1. Introduction

The latest Planck data [1] are in good agreement with the scalar spectral index $n_s$ with $n_s - 1 \approx -2/N$, where $N$ is the number of e-foldings. The quadratic chaotic inflation model [2], Starobinsky model [3], Higgs inflation with nonminimal coupling [4,5], and the $\alpha$-attractor model connecting them with one parameter “$\alpha$” [6–8] are typical examples that predict such a relation. Are there any other inflation models predicting such a relation? With this motivation, in Ref. [9], we studied such an inverse problem: we reconstructed $V(\phi)$ from a given $n_s(N)$ and found that, for $n_s - 1 = -2/N$, $V(\phi)$ is either $\tanh^2(\gamma \phi/2)$ (“T-model”) [6–8] or $\phi^2$ (chaotic inflation) to the leading order in the slow-roll approximation.

This paper is a continuation of that project: reconstruct $f(R)$ from a given $n_s(N)$. Since $f(R)$ gravity in vacuum is equivalent to a scalar field coupled to Einstein gravity via a conformal transformation [10–14] and the spectral index is invariant under the conformal transformation [15,16], the problem is very simple: convert the reconstructed $V(\phi)$ into $f(R)$. We provide such a procedure in Sect. 2. We find that as long as the scalar potential is of slow-roll type $f(R)$ is approximated by $R^2$. In fact, for $n_s - 1 = -2/N$ (or more general $n_s - 1 = -p/N$), $f(R)$ only asymptotes to $R^2$ for large $R$ irrespective of $V(\phi)$. In Sect. 3, we also provide the procedure to reconstruct $f(R)$ without relying on the Einstein frame. Section 4 is devoted to the summary.

2. $f(R)$ from $n_s(N)$: Analysis in the Einstein frame

In this section, we explain the method to reconstruct $f(R)$ for a given $n_s(N)$ using the equivalent action in the Einstein frame.

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We study the action (Jordan frame action) that is given by

\[ S = \int \sqrt{-g} d^4x \frac{1}{2\kappa^2} f(R), \]

(1)

where \( f(R) \) is a function of the Ricci scalar \( R \) and \( \kappa^2 = 8\pi G \).

In order to determine \( f(R) \), we utilize the equivalence of \( f(R) \) gravity with the Einstein–scalar system [10–14], for which we already have reconstructed possible shapes of the potential for \( n_s - 1 = -p/N \) [9]. To show the equivalence, first we note that Eq. (1) is equivalent to the action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} f(\psi) + \frac{f_\psi}{2\kappa^2} (R - \psi) \right], \]

(2)

where \( f_\psi = df/d\psi \). The variation with respect to the auxiliary field \( \psi \) gives \( \psi = R \) if \( f_\psi \neq 0 \) and the action (2) reduces to \( f(R) \) action (1) on-shell. Then, by the conformal transformation \( g_{\mu\nu}^E = g_{\mu\nu}^F / f_\psi \) and by introducing \( \kappa\phi = \sqrt{3/2} \ln f_\psi \), the action (2) can be rewritten as (the so-called Einstein frame action)

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R_E - \frac{1}{2} (\nabla_E \phi)^2 - V(\phi) \right], \]

(3)

where the quantities with the subscript \( E \) denote that defined by the metric \( g_{\mu\nu}^E \) and \( \phi \) and \( V(\phi) \) are given using \( \psi = R \) by

\[ \kappa \phi = \sqrt{\frac{3}{2}} \ln f_R \equiv \sqrt{\frac{3}{2}} \ln F, \]

(4)

\[ V(\phi) = \frac{F R - f}{2\kappa^2 F^2}. \]

(5)

Therefore, once \( f(R) \) is given, \( V(\phi) \) is determined by the above relation. The relation can be converted. Namely, once \( V(\phi) \) is given, \( f(R) \) is determined by [17,18]

\[ R = e^{\sqrt{3/2} \kappa \phi} \left( \sqrt{6} \kappa V, \phi + 4\kappa^2 V \right), \]

(6)

\[ f(R) = e^{2\sqrt{3/2} \kappa \phi} \left( \sqrt{6} \kappa V, \phi + 2\kappa^2 V \right), \]

(7)

where \( V, \phi = dV/d\phi \). In the following, we set \( \kappa = 1 \).

We note one important consequence of the relations (6) and (7). Namely, if \( V, \phi \) is negligible compared with \( V \), then \( R \approx 4e^{\sqrt{3/2} \kappa \phi} V = 4FV \) and \( f \simeq 2F^2 V \simeq \frac{1}{2} RF \) and hence by solving this differential equation it follows that \( f \) is approximately proportional to \( R^2 \) (see Ref. [18] for a similar observation).\(^1\)

Therefore, \( R^2 \) gravity is quite common in slow-roll inflation.

2.1. \( n_s - 1 = -2/N \)

In our previous paper, we found that \( n_s \) can be written in terms of the derivative with respect to the e-folding number \( N \) as \( n_s - 1 = (\ln(V_N/V^2))_N \) and \( d\phi/dN = \sqrt{V_N/V} \), where \( V_N = dV/dN \). Therefore, for a given \( n_s(N) \), we can reconstruct \( V(\phi) \).

\(^1\) This explains the results in Ref. [19] where it is found numerically that in the slow-roll regime \( f(R) \propto R^{2-\delta} \) and \( \delta \) decreases as the tensor-to-scalar ratio \( r \) decreases. In fact, \( r = 8(V,\phi/V)^2 \).
Fig. 1. The power index of $f(R)$ as a function of $\phi$. The solid curves are for the $\alpha$-attractor model with $\gamma = 5, \sqrt{2}/3, 1/5$ from top to bottom. The dotted curve is for a quadratic potential.

In particular, for $n_s - 1 = -2/N$, the potential is found to be either $V(\phi) = \frac{1}{2}m^2\phi^2$ (chaotic inflation) or $V(\phi) = V_0 \tanh^2(\gamma \phi/2)$ ($\alpha$-attractor model (T-model)) \cite{9}. So, using the above relations (6) and (7), we immediately obtain the corresponding $f(R)$.

2.1.1. Chaotic inflation

For $V(\phi) = \frac{1}{2}m^2\phi^2$, from Eqs. (6) and (7), we obtain $f(R)$ parametrically in terms of $\phi$:

$$R = m^2e^{\sqrt{2/3}\phi} \left( \sqrt{6} + 2\phi^2 \right) = 3m^2F(\ln F + (\ln F)^2),$$

$$f = m^2e^{2\sqrt{2/3}\phi} \left( \sqrt{6} + \phi^2 \right) = \frac{3}{2}m^2F^2(2 \ln F + (\ln F)^2).$$

(8)  

(9)

For large $R$, $R \simeq 2m^2e^{\sqrt{2/3}\phi}\phi^2$ and $f \simeq m^2e^{2\sqrt{2/3}\phi}\phi^2 \simeq \frac{1}{2}RF$ and hence $f \propto R^2$, as noted above. For small $R$ (or small $\phi$), $f \simeq \sqrt{6}m^2\phi \simeq R$ and the Einstein gravity is recovered. In fact, the power index of the functional form of $f(R)$ is calculated by

$$\frac{d \ln f}{d \ln R} = \frac{\sqrt{6} + 2\phi}{\sqrt{6} + \phi}. \quad (10)$$

Hence, since $\phi \sim m^{-1} \sim 10^6$ at the beginning of the chaotic inflation, the power index is very close to 2 and $f \propto R^2$, and for small $\phi$ the index approaches unity. Since $\phi \simeq 2\sqrt{N}$ for large $N$, the index at the observationally relevant scale deviates from 2 (see Fig. 1).

2.1.2. $\alpha$-attractor

For $V(\phi) = V_0 \tanh^2(\gamma \phi/2)$,\footnote{Here we have fixed an integration constant so that $V(0) = 0$. Note that the potential is only accurate for large $\gamma \phi$ since the slow-roll approximation is used to reconstruct $V(\phi)$.}

$$R = V_0e^{\sqrt{2/3}\phi} \left( \sqrt{6}\gamma \sech^2(\gamma \phi/2) \tanh(\gamma \phi/2) + 4 \tanh^2(\gamma \phi/2) \right),$$

$$f = V_0e^{2\sqrt{2/3}\phi} \left( \sqrt{6}\gamma \sech^2(\gamma \phi/2) \tanh(\gamma \phi/2) + 2 \tanh^2(\gamma \phi/2) \right).$$

(11)  

(12)
Fig. 2. The power index of $f(R)$ as a function of $\phi$. Left: $p = 1/2$ (solid), $p = 1$ (logarithmic: dashed), $p = 4$ (chaotic: dotted). Right: $p = 3$ for the first case potential in Eq. (16).

For large $R$,

$$R \simeq V_0 e^{\sqrt{2}/3 \phi} (4 + (4\sqrt{6} \gamma - 16)e^{-\gamma \phi}) = 4V_0 F(1 + (\sqrt{6} \gamma - 4)F^{-\gamma \sqrt{2}/3}),$$

(13)

$$f \simeq V_0 e^{2\sqrt{2}/3 \phi} (2 + (4\sqrt{6} \gamma - 8)e^{-\gamma \phi}) = 2V_0 F^2(1 + (2\sqrt{6} \gamma - 4)F^{-\gamma \sqrt{2}/3}).$$

(14)

So, for large $R$, $f \simeq R^2/(8V_0)$ and $R^2$ gravity is approached. The Starobinsky model corresponds to $V(\phi) = V_0 (1 - e^{-\sqrt{2}/3 \phi})^2$ and can be approximated by the T-model with $\gamma = \sqrt{2}/3$ for large $\phi$. The power index of the functional form of $f(R)$ is given by

$$\frac{d \ln f}{d \ln R} = \frac{4 + (\sqrt{6} \gamma - 4)e^{-\gamma \phi}}{2 + (\sqrt{6} \gamma - 2)e^{-\gamma \phi}}.$$

(15)

The index is shown in Fig. 1 for $\gamma = 5, \sqrt{2}/3, 1/5$ from top to bottom.

2.2. $n_s - 1 = -p/N$

For $n_s - 1 = -p/N$ ($p \neq 2$ and $p > 0$ are assumed), the reconstructed potential is [9]

$$V(\phi) = \begin{cases} 
V_0 - V_1 \left((1 - \frac{p}{2}) \phi\right)^{2(p-1)/(p-2)} & (p \neq 1) \\
V_0 + V_1 \ln \phi & (p = 1) \\
\lambda \phi^{2(p-1)} & (p > 1)
\end{cases}$$

(16)

where in the first case $\phi \leq 0(\geq 0)$ for $p > 2(< 2)$. $f(R)$ is constructed using Eqs. (6) and (7). In the left-hand panel of Fig. 2, the power index of $f(R)$ as a function of $\phi$ is shown for $p = 1/2$ (solid), $p = 1$ (logarithmic: dashed), $p = 4$ (chaotic: dotted). In the right-hand panel, the index for $p = 3$ for the first case potential in Eq. (16) is shown. We have assumed $V_0 = V_1$. We find that in all cases $f(R)$ approaches $R^2$.

It is interesting to note that although in terms of the scalar field potential we have a wide variety of the functional form of $V(\phi)$: power-law, exponential, and logarithmic, in terms of $f(R)$, for the same $n_s(N)$, the functional form is very limited: it only asymptotes to $R^2$ for large $R$.

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1 Setting $\gamma = \sqrt{2}/3$ in Eqs. (13) and (14) gives $f \simeq R^2/(8V_0) + 2R + 8V_0$. The last two terms are different from the Starobinsky model ($f = R^2/(8V_0) + R$), which comes from neglecting the higher-order terms in Eqs. (13) and (14).
3. Analysis in the Jordan frame

Finally, for completeness, we provide the analysis in the Jordan frame, in which the action is given by Eq. (1).

The equations of motion in a flat Friedmann universe in vacuum are given by

\[ F H^2 = \frac{1}{6} (FR - f) - H \dot{F}, \]  
\[ -2FH = F' - H \dot{F}, \]  

where \( F = df / dR \) and the dot denotes the derivative with respect to the cosmic time \( t \), and \( R = 6(2H^2 + \dot{H}) \).

We introduce the following slow-roll parameters \([20,21]\):

\[ \epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\dot{F}}{2HF}, \quad \epsilon_3 = \frac{\ddot{F}}{HF}. \]  

We assume that these parameters are small in the slow-roll approximation. In terms of these parameters, the scalar spectral index \( n_s \) is given by \([20,21]\)

\[ n_s - 1 = 3 - 2\sqrt{\frac{1}{4} + \frac{(1 + \epsilon_1 - \epsilon_2 + \epsilon_3)(2 - \epsilon_2 + \epsilon_3)}{(1 - \epsilon_1)^2}} \]
\[ \simeq -4\epsilon_1 + 2\epsilon_2 - 2\epsilon_3 - 4\epsilon_1^2 + 2\epsilon_1\epsilon_2 - 2\epsilon_1\epsilon_3. \]  

Note that the expression in the first line is exact as long as the slow-roll parameters can be regarded as constants. In the second line the expression is expanded up to the second order in the slow-roll parameters. We also note that from Eq. (18) we have

\[ \epsilon_1 = -\epsilon_2(1 - \epsilon_3). \]  

The e-folding number \( N \), which measures the amount of inflationary expansion from a particular time \( t \) until the end of inflation \( t_{\text{end}} \), is defined by

\[ N = \int_t^{t_{\text{end}}} H dt. \]  

We assume that \( N \) is large (say \( N \sim O(10) \sim O(10^2) \)) under the slow-roll approximation. In terms of \( N \), from \( dN = -H dt \), the slow-roll parameters are rewritten as

\[ \epsilon_1 = \frac{H_N}{H}, \quad \epsilon_2 = -\frac{F_N}{2F}, \quad \epsilon_3 = -\frac{H_N}{H} - \frac{F_{NN}}{F_N} = -\epsilon_1 - \frac{F_{NN}}{F_N}, \]  

where the subscript \( N \) denotes the derivative with respect to \( N \). Now we introduce a bookkeeping rule to assign the order of smallness to the quantities in the slow-roll parameters according to the number of derivatives with respect to \( N \):

\[ \frac{F_N}{F} \sim O(1), \quad \frac{F_{NN}}{F_N} \sim O(1), \quad \frac{F_{NN}}{F} \sim O(2). \]  

Then, up to \( O(2) \), the spectral index is given by

\[ n_s - 1 = -2\frac{F_N}{F} + 2\frac{F_{NN}}{F_N} - \frac{3}{2}\frac{F_N^2}{F^2}. \]
3.1. \( n_s - 1 = -2/N \)

As an example, let us consider the case of \( n_s - 1 = -2/N \). The analysis for \( n_s - 1 = -p/N \) is similar.

3.1.1. \( O(1) \)

Up to \( O(1) \), the scalar spectral index in Eq. (25) is given by

\[
n_s - 1 = -2 \frac{F_N}{F} + 2 \frac{F_{NN}}{F_N} = 2 (\ln F_N/F)_N = -\frac{2}{N}.
\]

(26)

Then, Eq. (26) is easily integrated to give

\[
F(N) = \alpha N^\beta,
\]

(27)

where \( \alpha \) and \( \beta \) are positive constants.\(^4\)

Then, we use Eq. (17) as a differential equation for \( f(R) \). Since \( \dot{F} = -\beta HN^{\beta - 1} = -\beta (F/\alpha)^{\beta - 1}/\beta \) and \( R = 6H^2(2 - \epsilon_1) \), Eq. (17) becomes up to \( O(1) \)

\[
f = \frac{1}{2}RF + \frac{1}{8}(4 - \alpha)\beta R(F/\alpha)^{1-1/\beta}.
\]

(28)

This type of equation is known as d’Alembert’s differential equation, and the general solution is given by

\[
R = A \exp \left( \int \frac{1}{2} \left( \frac{4 - \alpha)(\beta - 1)\beta}{8\alpha} (F/\alpha)^{-1/\beta} \right) dF \right),
\]

(29)

where \( A \) is an integration constant. Up to \( O(1) \), the result is

\[
R = AF \left( 1 - \frac{(4 - \alpha)(2\beta - 1)\beta}{4\alpha} (F/\alpha)^{-1/\beta} \right).
\]

(30)

Putting this into Eq. (28), \( f(R) \) is parametrically given by

\[
f = \frac{1}{2}AF^2 \left( 1 - \frac{(4 - \alpha)(\beta - 1)\beta}{2\alpha} (F/\alpha)^{-1/\beta} \right).
\]

(31)

We find that by setting \( 1/\beta \to \gamma \sqrt{3/2} \) these solutions (30) and (31) agree with Eqs. (13) and (14).\(^5\)

3.1.2. \( O(2) \)

However, there can be a case where the \( O(1) \) terms in Eq. (25) cancel and only give a higher-order term. For example, let us consider \( F_N/F = s/N^q \) where \( 0 < q < 1 \). Then, the \( O(1) \) terms in Eq. (25) are

\[
-2 \frac{F_N}{F} + 2 \frac{F_{NN}}{F_N} = -\frac{2q}{N}.
\]

(32)

Therefore, we need the \( O(2) \) terms to calculate \( n_s \). Equation (25) up to \( O(2) \) becomes

\[
n_s - 1 = -\frac{2q}{N} - \frac{3s^2}{2N^2q} = -\frac{2}{N}.
\]

(33)

\(^4\) \( \beta > 0 \) is understood from \( \epsilon_1 \simeq -\epsilon_2 = \beta/(2N) \).

\(^5\) Note that \( V(\phi) \) in Eqs. (13) and (14) should contain an integration constant corresponding to the shift of \( \phi \), which has been fixed there.
Since the $O(2)$ term is proportional to $N^{-2q}$, $q = 1/2$ is required. Then, from Eq. (33), we find $s = \sqrt{2/3}$ and $F$ is determined as

$$F(N) = C \exp \left( \frac{2\sqrt{2N}}{3} \right).$$

(34)

where $C$ is a constant. Equation (17) becomes up to $O(1)$

$$f = \frac{1}{2}RF + \frac{1}{2} \ln F/C.$$  

(35)

The solution is given by

$$R = A \exp \left( \int \frac{1}{2} + \frac{1}{2} \ln F/C - \frac{F}{2} \frac{dF}{2 \ln F/C} \right).$$

(36)

Up to $O(1)$, the solution is

$$R = AF (\ln F/C)^2,$$

(37)

and $f(R)$ is given by

$$f = \frac{1}{2} AF^2 (\ln F/C)^2.$$  

(38)

These solutions (37) and (38) agree with Eqs. (8) and (9) for large $R$.

### 4. Summary

In this paper, motivated by the relation $n_s - 1 \simeq -2/N$ indicated by recent cosmological observations, we derived $f(R)$ (the Lagrangian density of $f(R)$ gravity) from $n_s(N)$ in the slow-roll approximation. We introduced two approaches to the problem. The first approach is to utilize the equivalence of $f(R)$ gravity with the Einstein–scalar system and to determine $f(R)$ from the scalar field potential $V(\phi)$, which is already known [9]. The second approach is to derive $f(R)$ directly from $n_s(N)$. In the first approach, we found that, if $V_{\phi}$ is negligible compared with $V$, then $f(R)$ is approximated by $R^2$. $R^2$ gravity is quite common in slow-roll inflation.

For $n_s - 1 = -2/N$, we found that $f(R)$ is determined parametrically in terms of either $\phi$ (Einstein frame case) or $F = f_R$ (Jordan frame case). The results of the two approaches agree. The reconstructed $f(R)$ has a common feature: $f(R) \propto R^2$ for large $R$. The results for $n_s - 1 = -p/N$ are similar. In order to recover general relativity at the present time, $f(R)$ is required to satisfy $f(R) \simeq R$ at small $R$. Therefore, for the same $n_s$, a rather restricted functional form of $f(R)$ is allowed, although a wide variety of functional forms of $V(\phi)$ is possible.

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6 The corrections due to the time variation of the slow-roll parameters are found to be of even higher order, $O(3)$. 
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