Low-regularity Seiberg-Witten moduli spaces on manifolds with boundary

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Abstract

For a compact spin$^c$ manifold $X$ with boundary $b_1(\partial X) = 0$, we consider moduli spaces of solutions to the Seiberg-Witten equations in a generalized double Coulomb slice in $W^{1,2}$ Sobolev regularity. We prove they are Hilbert manifolds, prove denseness and "semi-infinite-dimensionality" properties of the restriction to $\partial X$, and establish a gluing theorem.

To achieve these, we prove a general regularity theorem and a strong unique continuation principle for Dirac operators, and smoothness of a restriction map to configurations of higher regularity on the interior, all of which are of independent interest.

Contents

1 Introduction 2

2 Analytical preparation 6
  2.1 A regularity theorem for Dirac operators 6
  2.2 A strong UCP for Dirac operators 9

3 Seiberg-Witten moduli spaces in split Coulomb slice 10
  3.1 Coulomb slice on 3-manifolds 10
  3.2 Split Coulomb slice on 4-manifolds 12
  3.3 Restriction to the boundary and twisting 21
  3.4 Seiberg-Witten moduli in split Coulomb slice 25

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1 Introduction

This article is motivated by an effort to provide a new framework for constructing so-called Floer theories, which are used to construct invariants of low-dimensional manifolds, knots, as well as Lagrangians in symplectic manifolds. Originally, Floer implanted ideas from finite-dimensional Morse theory (cf. [Sch93, Wit82]) to the study of functions on infinite-dimensional spaces (often called functionals on configuration spaces), leading to the resolution of the Arnold conjecture in symplectic topology [Flo87, Flo88]. Inspired by Donaldson’s proof of the diagonalization theorem for 4-manifolds [Don83], Floer used these ideas to construct homological invariants of 3-manifolds, aiming to establish gluing formulas for Donaldson’s invariants. An analogous construction for knots in 3-manifolds has been carried out by Kronheimer and Mrowka and plays the key role in the proof that Khovanov homology detects the unknot [KM11]. An analogue for the Seiberg-Witten equations (in place of the anti-self-duality (ASD) equations of Donaldson’s theory) is the central ingredient of Manolescu’s disproof of the triangulation conjecture in dimensions $\geq 5$ [Man13].

While the idea of using Morse theory in infinite-dimensional settings dates back to Atiyah and Bott [AB83], Floer’s novelty was dealing with functionals having critical points of infinite index. They key observation was that the unstable manifolds of critical points were of finite index with respect to a certain subbundle of the tangent space, allowing one to define finite relative indices of critical points and eventually leading to a computation of "middle-dimensional" homology groups of the space. As noted by Atiyah [Ati88] (cf. [CJS95]), Floer theory was, from the very beginning, understood as describing the behavior of so-called semi-infinite-dimensional cycles.

As in Morse homology (cf. [Sch93]), one needs to overcome analytical difficulties to even define Floer theories and prove they are well-defined and independent of choices –
one needs to establish regularity and compactness of moduli spaces of trajectories between critical points. Moreover, Morse theory is generally not well-suited for equivariant constructions since one in general cannot guarantee regularity of the moduli spaces without breaking the symmetry coming from a group action. To address these problems and to allow general constructions of equivariant (cf. [DS19, KM07, Lin18, Mil19]) and generalized Floer homologies (cf. [Man14, Lin15, AB21]), Lipyanskiy [Lip08] introduced a framework for using such semi-infinite-dimensional cycles as a tool for defining Floer theories and the author has further developed these methods in [Suw20]. This article contains results required to rigorously define the relative invariants of 4-manifolds with boundary and maps induced by cobordisms in this construction, using the Seiberg-Witten equations. Moreover, the key result is a gluing theorem which is fundamental for establishing functoriality of the cobordism maps.

**Results.** Consider a 4-manifold $X$ with boundary a nonempty collection of rational homology spheres (i.e., $b_1(\partial X) = 0$) and a spin$^c$ structure $\hat{s}$ on $X$. We study the moduli space of solutions to the Seiberg-Witten equations for pairs $(A, \Phi)$ of a spin$^c$ connection $A$ on $X$ and a section $\Phi$ of the spinor bundle $S^+$ over $X$ associated to the spin$^c$ structure $\hat{s}$:

\[
\begin{align*}
\frac{1}{2} F^+_{A^v} - \rho^{-1}((\Phi\Phi^*)_0) &= 0, \\
\mathcal{D}^+_{A} \Phi &= 0.
\end{align*}
\]  

(Definition 3.4). The split Coulomb slice with respect to a reference connection $A_0$ is given by the equations

\[
\begin{align*}
d^*(A - A_0) &= 0, \\
d^*({\iota_{\partial X}^*}(A - A_0)) &= 0,
\end{align*}
\]  

(where $\iota_{\partial X} : \partial X \hookrightarrow X$ denotes the inclusion) together with a condition restricting $A - A_0$ to a subset of codimension $b_0(\partial X) - 1$ (Definition 3.15). This additional condition depends on a choice of a gauge splitting $s$ (Definition 3.11). The moduli space $\mathcal{M}^+_s(X, \hat{s})$ (Definition 3.31) is the quotient of the space of $L^2_1$-solutions to the Seiberg-Witten equations in this split Coulomb slice by the (discrete) action of the split gauge group (Definition 3.18); this gauge group preserves the split Coulomb slice as well as the set of solutions to the Seiberg-Witten equations. There is also a residual action of $S^1$ on $\mathcal{M}^+_s(X, \hat{s})$ given by multiplication of the spinor component, $\Phi \mapsto z\Phi$, by complex numbers in the unit circle $z \in S^1 \subset \mathbb{C}$.

Firstly, we choose a gauge twisting $\tau$ (Definition 3.24) and define the ($S^1$-equivariant) twisted restriction map $R_\tau : \mathcal{M}^+_s(X, \hat{s}) \to \mathcal{C}_{cc}(\partial X, s)$, taking values in the configurations on $\partial X$ in the Coulomb slice (Definition 3.2). The gauge splittings and gauge twistings we introduce generalize the double Coulomb slice used in [Lip08, Kha]...
and twistings utilized in [KLS18]. We prove regularity, denseness and “semi-infinite-dimensionality” of the Seiberg-Witten moduli spaces:

**Semi-infinite-dimensionality Theorem.** The moduli spaces \( \mathcal{M}_s^0(X, \mathfrak{s}) \) are Hilbert manifolds. The differential of the twisted restriction map \( R_\tau: \mathcal{M}_s^0(X, \mathfrak{s}) \to C_{cc}(\partial X, \mathfrak{s}) \) decomposes into \( \Pi^\pm DR_\tau: TM^0_s(X, \mathfrak{s}) \to H^\pm(\partial X, \mathfrak{s}) \) which is Fredholm and \( \Pi^\pm DR_\tau: TM^0_s(X, \mathfrak{s}) \to H^\pm(\partial X, \mathfrak{s}) \) which is compact.

Moreover, if \( b_0(\partial X) > 1 \), then for any connected component \( Y_0 \subset \partial X \) the restriction \( R_\tau|_{Y_0} \) to \( Y_0 \) has dense differential.

The maps \( \Pi^\pm \) come from a decomposition \( C_{cc}(\partial X, \mathfrak{s}) = H^+(\partial X, \mathfrak{s}) \oplus H^-(\partial X, \mathfrak{s}) \) according to the eigenvalues of the operator \((d, \mathcal{D}_{\mathcal{B}_0})\), also called a polarization (Definition 3.3).

The proof utilizes the Atiyah-Patodi-Singer boundary value problem for an extended linearized Seiberg-Witten operator (Definition 4.1). Then we prove its properties transfer to the restriction to the Coulomb slice.

Our low regularity setting requires us to prove a regularity theorem (Regularity Theorem) for an operator of the form \( D = D_0 + K: L^1_1 \to L^2 \), where \( D_0 \) is a Dirac operator and \( K \) is any compact operator, making it a result of independent interest. We also prove a strong version of the unique continuation principle for \( D \) (Strong UCP Theorem).

Secondly, we prove a gluing theorem for a composite cobordism. Assume \( X \) splits as \( X = X_1 \cup_Y X_2 \) along a rational homology sphere \( Y \). If the auxiliary data of gauge splittings and gauge twistings are compatible in a suitable sense (see Proposition 3.29 and Proposition 5.5), then the moduli space \( \mathcal{M}_s^0(X, \mathfrak{s}) \) can be recovered from the fiber product of \( \mathcal{M}_s^0(X_1, \mathfrak{s}) \) and \( \mathcal{M}_s^0(X_2, \mathfrak{s}) \) over the configuration space of \( Y \), in a way compatible with the twisted restriction maps:

**Gluing Theorem.** Assume \( s_Z \) and \((s_{1,Z}, s_{2,Z})\) are compatible. Then there is an \( S^1\)-equivariant diffeomorphism \( F: \mathcal{M}_s^0(X, \mathfrak{s}) \to \mathcal{M}_s^0(X_1, \mathfrak{s}_1) \times_Y \mathcal{M}_s^0(X_2, \mathfrak{s}_2) \) such that \( R_\tau \) is \( S^1\)-equivariantly homotopic to \((R_{\tau_1} \times_Y R_{\tau_2}) \circ F \).

The proof uses the following fact of independent interest. We show that the restriction map of solutions in \( \mathcal{M}_s^0(X, \mathfrak{s}) \) to a submanifold in the interior of \( X \) is smooth as a map into a configuration space of higher regularity (Theorem 5.2). While it is easy to prove that its image lies in the space of smooth configurations (Lemma 5.1), the proof of smoothness of this map is non-trivial.

**Applications.** The Semi-infinite-dimensionality Theorem together with compactness of moduli spaces proved in [KM07] (with minor modifications to account for the double Coulomb slice instead of the Coulomb-Neumann slice used in [KM07]) show that the maps \( R_\tau: \mathcal{M}_s^0(X, \mathfrak{s}) \to C_{cc}(Y, \mathfrak{s}) \) are semi-infinite-dimensional cycles in \( C_{cc}(\partial X, \mathfrak{s}) \)
as defined in [Lip08, Suw20], establishing the existence of relative Seiberg-Witten invariants of $X$. If the boundary $\partial X$ is connected, these do not depend on the choice of an integral splitting (Lemma 3.30). This also implies that cobordisms $\partial W = -Y_1 \cup Y_2$ induce correspondences (defined in [Lip08, Suw20]) between the configuration spaces over $Y_1$ and $Y_2$.

Our methods apply to perturbed equations as well, which we did not include for the sake of simplicity. Varying the metrics and perturbations gives cobordisms between the relevant moduli spaces on $X$, and changing the reference connections on $X$ gives isomorphic moduli spaces with homotopic restriction maps. This means that the relative invariant of $X$, up to a suitable cobordism relation, does not depend on the choices of perturbations and metric.

Crucially, the Gluing Theorem says that the correspondence induced by a composite cobordism is (homotopic to) the composition of the respective correspondences, proving the theory comes with a TQFT-like structure. We hope to prove that this theory recovers a non-equivariant version of monopole Floer homology $\tilde{HM}^*$ and describe a construction supposed to recover all of the flavors of monopole Floer homology (for rational homology spheres) in an upcoming work.

Applications of Seiberg-Witten-Floer theory suggest including the case $b_1(\partial X) > 0$ or allowing $X$ to have cylindrical or conical ends (like in [KM97]) with certain asymptotic conditions on the configurations. The methods presented here should suffice to prove regularity and semi-infinite-dimensionality of the corresponding moduli spaces. However, to establish the semi-infinite-dimensional theory in full one needs to carefully select a slice for the gauge action and deal with compactness issues which we hope to address in further work.

Finally, the methods presented here should be applicable (with appropriate adjustments) for defining semi-infinite-dimensional Floer theories in other contexts, e.g., in the Yang-Mills-Floer theory.

**Organization.** In section 2 we prove the regularity theorem for a Dirac operator with compact perturbation, the Regularity Theorem, and a strong unique continuation principle for a Dirac operator with a low-regularity potential, the Strong UCP Theorem.

In section 3 we introduce the basic notions of Seiberg-Witten theory on 3- and 4-manifolds. For a chosen gauge splitting the split Coulomb slice is defined, and for a gauge twisting a twisted restriction map to the boundary is introduced. The choices of splittings and twistings are shown to be equivalent to a choice of an integral splitting, one which does not require any twisting.

Section 4 provides proofs of some of the key properties of the moduli spaces: regularity, semi-infinite-dimensionality and denseness of the restriction map (Semi-infinite-dimensionality Theorem).
Finally, section 5 contains the proof of the Gluing Theorem, showing that moduli on a composite cobordism are a fiber product of the moduli on its components.

**Notations.** In this article we use the notation $L^p_k$ for Sobolev spaces of regularity $k$, in accordance with the literature in gauge theory and Floer theory in low dimensional topology. These are often denoted by $W^{k,p}$ or $H^{k,p}$ or, when $p = 2$, simply by $H^k$.

All manifolds are assumed to be smooth, submanifolds to be smoothly embedded, and manifolds with boundary to have smooth boundary.

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Most results of this paper have been part of the author’s Ph.D. thesis [Suw20], although the proofs have been revised and some results have been generalized. In particular, [Suw20] does not consider 4-manifolds with more than two boundary components, or the general split Coulomb slice on them.

## 2 Analytical preparation

We prove two results which are fundamental for the surjectivity proofs in section 4.

The first one, the **Regularity Theorem**, is a regularity theorem for operators of the form $D = D_0 + K : L^2_1 \to L^2$, where $D_0$ is a first-order elliptic operator and $K$ is a compact operator. In our applications $K$ is a multiplication by an $L^2_1$-configuration on a 4-manifold and thus it does not factor as a map $L^2_1 \to L^2_1$. If it did, elliptic regularity would immediately imply the regularity result. The novelty here is the general form of the perturbation $K$ which is only assumed to be compact as a map $L^2_1 \to L^2$.

The second result, the **Strong UCP Theorem**, is a strong unique continuation principle in a similar low regularity setting. This can be understood as a strengthening of the unique continuation results of [KM07, Section 7].

### 2.1 A regularity theorem for Dirac operators

Let $X$ be a smooth Riemannian manifold with asymptotically cylindrical ends (without boundary). Let $D_0$ be an elliptic operator $D_0 : C^\infty(X; E) \to C^\infty(X; F)$ of order 1 which is asymptotically cylindrical on the ends of $X$. Let $K : L^2_1(X; E) \to L^2(X; F)$ be a compact operator which has a compact formal adjoint $K^* : L^2_1(X; E) \to L^2(X; F)$.
and which extends to a continuous operator $K : L^2(X;E) \to L^2_{-l}(X;F)$ for some $l$. Consider the operator

$$D = D_0 + K : L^2_1(X;E) \to L^2(X;F).$$

The aim of this subsection is to prove the following regularity theorem:

**Regularity Theorem.** Assume that $v \in L^2(X;E)$ satisfies $Dv = h$ for some $h \in L^2(X;F)$. Then $v \in L^2_1(X;E)$. Moreover, if $v,h \in L^2_{1,\text{loc}}(X;E)$ instead, then $v \in L^2_{1,\text{loc}}(X;E)$.

In the course of the proof we will make use of the following simple lemma (cf. Lipyanskiy [Lip08, Lemma 44]):

**Lemma 2.1.** Suppose the sequence of Hilbert space operators $\{A_i : V \to W\}$ is uniformly bounded and weakly convergent to $A$ in the sense that for any $v \in V$ we have $A_i(v) \to A(v)$.

If $K : W \to U$ is compact, then $A_i \circ K \to A \circ K$.

We also need the following version of the Gårding inequality, proven in [Shu92, Appendix 1, Lemma 1.4]:

**Proposition 2.2 (cylindrical Gårding inequality).** For every $s,t \in \mathbb{R}$ there exists $C > 0$ such that for any $u \in C^\infty_0(X;E)$

$$\|u\|_{L^2_{s+1}} \leq C(\|D_0u\|_{L^2_s} + \|u\|_{L^2_t}).$$

With these in hand, we are ready to prove the Regularity Theorem.

**Proof (Proof of Regularity Theorem).** By considering

$$\begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & K^* \\ K & 0 \end{pmatrix} : L^2_1(X;E \oplus F) \to L^2(X;E \oplus F)$$

where $D_0^*$ is the formal adjoint of $D_0$ we may assume, without loss of generality, that $E = F$ and $D_0$ is formally self-adjoint. Since $X$ has bounded geometry and $D_0$ is uniformly elliptic, Proposition 4.1 in [Shu92, Section 1] implies that the minimal and maximal operators $D_0^{\text{min}}$ and $D_0^{\text{max}}$ of $D_0 : L^2(X;E) \to L^2(X;E)$ coincide, and their domains are equal to $L^2_1(X;E)$. Thus $D_0$ is essentially self-adjoint.

Consider the equation

$$(D_0 + ic)\psi + K\psi = h + icv \quad (1)$$

for large $c \in \mathbb{R}$. Certainly, $\psi = v \in L^2(X;E)$ solves the equation. We will prove that for large $c$ equation (1) has a unique solution $\psi$ in $L^2_1(X;E)$, which is also unique in $L^2(X;E)$, so that $v = \psi \in L^2_1(X;E)$, as wished.
Firstly, we want to prove $D_0 + ic : L^2_1(X; E) \to L^2(X; E)$ is invertible. Since the spectrum of $D_0$ is real, $(D_0 + ic)^{-1} : L^2(X; E) \to L^2(X; E)$ exists and is bounded by $\frac{1}{|c|}$ (cf. [Lan93, Chapter XIX, Theorem 2.4]). Therefore by Proposition 2.2 we have the first inequality:
\[
\| (D_0 + ic)^{-1} u \|_{L^2_1} \leq \| D_0 (D_0 + ic)^{-1} u \|_{L^2} + \| (D_0 + ic)^{-1} u \|_{L^2} \\
\leq \| u - ic (D_0 + ic)^{-1} u \|_{L^2} + (1/|c|) \| u \|_{L^2} \\
\leq (2 + 1/|c|) \| u \|_{L^2}
\]
and therefore $(D_0 + ic)^{-1}$ is a bounded operator $L^2(X; E) \to L^2_1(X; E)$.

Secondly, we want to prove weak convergence of $(D_0 + ic)^{-1}$ to $0$ as $|c| \to \infty$, i.e., that for each $v \in L^2(X; E)$ we have $(D_0 + ic)^{-1}(v) \to 0$ in $L^2_1(X; E)$ as $|c| \to \infty$. The spectral theorem for unbounded self-adjoint operators [Lan93, Chapter XIX, Theorem 2.7] provides us with an orthogonal decomposition $L^2(X; E) = \bigoplus_n H_n$ such that the restriction $D_n = D_0|_{H_n} : H_n \to H_n$ is a bounded operator (in $L^2$-norm on $H_n$) and $D_0 = \bigoplus_n D_n$. In particular, $H_n \subset \text{Dom}(D_0) = L^2_1(X; E)$. For each $v_n \in H_n$ we thus have, by Proposition 2.2,
\[
\| v_n \|_{L^2_1} \leq C(\| D_0 v_n \|_{L^2} + \| v_n \|_{L^2}) \leq C(C_n + 1) \| v_n \|_{L^2} = C_n^\prime \| v_n \|_{L^2},
\]
where $C_n = \| D_n \|_{L^2}$. Therefore
\[
\| (D_0 + ic)^{-1} v_n \|_{L^2_1} \leq C_n^\prime \| (D_0 + ic)^{-1} v_n \|_{L^2} \leq \frac{C_n^\prime}{|c|} \| v_n \|_{L^2} \xrightarrow{|c| \to \infty} 0
\]
which proves that for any $v \in L^2(X; E)$, $(D_0 + ic)^{-1} v \xrightarrow{|c| \to \infty} 0$ in $L^2_1(X; E)$, i.e., $(D_0 + ic)^{-1}$ converges weakly to $0$, as wished.

We proceed to proving existence and uniqueness of solutions to (1) in $L^2_1(X; E)$ for large $|c|$. Using Lemma 2.1 we conclude that $T = (D_0 + ic)^{-1} K : L^2_1(X; E) \to L^2_1(X; E)$ converges strongly to $0$. Thus, for large $|c|$, the operator $\text{Id} + (D_0 + ic)^{-1} K : L^2_1(X; E) \to L^2_1(X; E)$ is invertible. Composing with $(D_0 + ic)$ we conclude that $D_0 + ic + K = D + ic : L^2_1(X; E) \to L^2(X; E)$ is invertible for large $|c|$. Existence and uniqueness of $\psi \in L^2_1(X; E)$ solving (1) follows.

To conclude the first part of the proof we need to establish uniqueness of solutions to (1) in $L^2(X; E)$. We have that $(D_0 + ic + K)(\psi - v) = 0$. This implies that $\psi - v$ is perpendicular in $L^2(X; E)$ to the image of $D_0^* - ic + K^* = D_0 - ic + K : L^2_1(X; E) \to L^2(X; E)$. However, we already established that for large $|c|$ the latter operator is invertible and it follows that $\psi - v = 0$; thus $v = \psi \in L^2_1(X; E)$.
Finally, it remains to consider the more general case \( v, h \in L^2_{\text{loc}} \). For any compact \( A \subset X \) we can take a compactly supported bump function \( \rho : X \to [0,1] \) with \( \rho|_A = 1 \) and define \( v' = \rho v \in L^2(X; E) \). Then \( Dv' = h' \) for some \( h' \in L^2(X; E) \) and therefore \( v' \in L^2(X; E) \) as proven above, so \( v|_A \in L^2(A; E|_A) \). Repeating the argument for every compact \( A \subset X \) shows that \( v \in L^2_{1,\text{loc}}(X; E) \).

\[ \square \]

### 2.2 A strong UCP for Dirac operators

Let \( M \) be a connected Riemannian manifold and \( S \) a real (resp. complex) Dirac bundle over it (e.g., the real (resp. complex) spinor bundle associated to a spin\(^c\)-structure on \( M \)) with connection \( A \). Denote by \( \mathcal{D}_A : \Gamma(S) \to \Gamma(S) \) the corresponding Dirac operator. Let \( V \in L^n(M; \mathbb{R}) \) be a potential. Here we prove the following unique continuation theorem for spinors and potentials of low regularity.

**Strong UCP Theorem.** The differential inequality

\[
|\mathcal{D}_A \Phi| \leq V|\Phi|
\]

has unique continuation property in \( L^{\frac{2n+2}{n+2}}_{1,\text{loc}}(M; S) \).

The version of this theorem for \( d + d^* \) instead of \( \mathcal{D}_A \) has been proven in [Wol92, Theorem 2]:

**Theorem 2.3 (Wolff, [Wol92, Theorem 2]).** Suppose \( M \) is an \( n \)-dimensional manifold, \( n \geq 3, p = \frac{2n}{n+2} \), and \( \omega \in L^p_{1,\text{loc}}(\Omega^*(M)) \) such that \( |d\omega| + |d^*\omega| \leq V|\omega| \) with \( V \in L^n_{\text{loc}}(M) \). Then if \( \omega \) vanishes on an open set, it vanishes identically.

It thus suffices to reduce our problem to Wolff’s result, following the idea of [Man94]. Since the proof in [Man94] does not explain the reduction rigorously, we describe the procedure below.

**Proof.** The problem is local, thus we need only to consider the case of \( M \) being \( \mathbb{R}^n \) with some metric \( g \). If \( A_0 \) is the flat connection on \( S \), \( \mathcal{D}_A - \mathcal{D}_A_0 \) is a smooth operator of order 0, so (again using locality) we can assume that \( A \) is flat. Moreover, by contractibility of \( \mathbb{R}^n \), we can decompose \( S \) into irreducible components, and irreducible components must be isomorphic to the real (resp. complex) spinor bundle \( \tilde{S} \) (resp. \( \tilde{S}_C \)) associated to the unique spin structure. Thus we have reduced the problem to the case where \( S = \tilde{S} \otimes \mathbb{R}^k \) (resp. \( S = \tilde{S} \otimes \mathbb{C}^k \)) for some \( k \).

The real (resp. complex) spinor bundle \( \tilde{S} \) (resp. \( \tilde{S}_C \)) embeds into \( \mathcal{C}_\ell_n(\mathbb{R}^n) \) (resp. \( \mathcal{C}_\ell_n(\mathbb{C}^n) \)). Furthermore, [LM89, Theorem 5.12] implies that the Dirac operator on the Clifford bundle \( \mathcal{C}_\ell_n(\mathbb{R}^n) \) (resp. \( \mathcal{C}_\ell_n(\mathbb{C}^n) = \mathcal{C}_\ell(\mathbb{R}^n) \otimes \mathbb{C} \)) is equivalent to \( d + d^* \) on \( \Lambda^*(\mathbb{R}^n) \) (resp. \( \Lambda^*(\mathbb{R}^n) \otimes \mathbb{C} \)) via the canonical isomorphism \( \mathcal{C}_\ell(\mathbb{R}^n) \simeq \Lambda^*(\mathbb{R}^n) \). Thus, \( \mathcal{D}_{A_0} \) is equivalent to \( (d + d^*) \otimes 1_{\mathbb{R}^k} \Lambda^*(\mathbb{R}^n) \otimes \mathbb{R}^k \to \Lambda^*(\mathbb{R}^n) \otimes \mathbb{R}^k \) (resp. \( (d + d^*) \otimes 1_{\mathcal{C}_\ell} \Lambda^*(\mathbb{R}^n) \otimes \mathbb{C}^k \) for some \( k \).
The proof of Theorem 2.3 goes through for differential forms with coefficients in \( \mathbb{R}^k \) (resp. \( \mathbb{C}^k \)), establishing the unique continuation property for \((d + d^*) \otimes \mathbb{R}^k \) (resp. \((d + d^*) \otimes 1_{\mathbb{C}^k}\)). This finishes the proof.

In the article we will use the following special case:

**Corollary 2.4 (UCP for Dirac operators in 4d).** In the setting of the Strong UCP Theorem, assume \( n = 4 \) and \( V \in L^2_1(M; \text{Aut}(S)) \). Then any solution \( \Phi \in L^2_{1,\text{loc}}(M; S) \) to

\[
\mathcal{D}_A \Phi + V \Phi = 0
\]

which is zero on some open set is identically zero.

### 3 Seiberg-Witten moduli spaces in split Coulomb slice

In this section we introduce the moduli spaces of the Seiberg-Witten equations on a Riemannian 4-manifold with boundary a collection of rational homology spheres, together with the restriction maps to the boundary. For 3-manifolds, we introduce the Coulomb slice and its polarization, a decomposition of the tangent space into a sum of two infinite-dimensional subspaces. For 4-manifolds, we introduce the double Coulomb slice and what we call the split Coulomb slice together with the split gauge group. Since the restriction maps are generally not invariant with respect to the split gauge group, we need to introduce appropriate twisted restriction maps as well.

The split gauge fixing is a key novel element that generalizes the gauge slice introduced by Khandhawit [Kha] and Lipianskiy [Lip08]. It simplifies the proof of the Gluing Theorem letting us to reduce it to the case of untwisted restriction maps.

#### 3.1 Coulomb slice on 3-manifolds

We begin by introducing polarizations on the Seiberg-Witten configuration space in Coulomb gauge on an oriented rational homology sphere \( Y \) and collections of such.

Let \( g \) be a Riemannian metric and \( s \) be a spin\(^c\) structure on \( Y \). Denote by \( S_Y \) the associated spinor bundle and choose a smooth spin\(^c\) connection \( B_0 \). The Seiberg-Witten configuration space on \( Y \) is the space

\[
\mathcal{C}(Y,s) = (B_0,0) + L^2_{1/2}(i\Omega^1(Y) \oplus \Gamma(S_Y))
\]

consisting of pairs \((B, \Psi)\) of a spin\(^c\) connection and a spinor on \( Y \).

In Seiberg-Witten theory one investigates the Chern-Simons-Dirac functional \( \mathcal{L} \)

\[
\mathcal{L}(B, \Psi) = -\frac{1}{8} \int_Y (B^t - B^0_0) \wedge (F_{B^t} + F_{B^0}) + \frac{1}{2} \int_Y \langle \mathcal{D}_B \Psi, \Psi \rangle.
\]
on the configuration space. The gauge group \(G(Y) = L^2_{3/2}(Y; S^1)\) acts on \(C(Y, \mathfrak{s})\) via 
\[ u(A, \Phi) = (A - u^{-1}du, u\Phi) \] where \(u \in G(Y)\), leaving \(\mathcal{L}\) invariant. If one used spaces of higher regularity, one could work with the quotient of the configuration space by the action of the gauge group. However, in the low regularity setting the action of \(G(Y)\) on the spinors is not continuous. Because of that (and in applications concerning the Seiberg-Witten stable homotopy type, cf. [KLS18]), it is preferable to take the Coulomb slice as the model for the quotient by the identity component of the gauge group. Indeed, for \(Y\) a rational homology sphere the Hodge decomposition gives the \(L^2\)-orthogonal decomposition

\[ \Omega^1(Y) = \Omega^1_C(Y) \oplus \Omega^1_{CC}(Y) \]

where \(\Omega^1_C(Y) = \{b \in \Omega^1(Y) | d^*b = 0\} = d^*(\Omega^2(Y))\) is the space of (smooth) co-closed forms and \(\Omega^1_{CC}(Y) = \{b \in \Omega^1(Y) | db = 0\} = d(\Omega^0(Y))\) is the space of (smooth) closed forms. Denote by \(\Pi_d\) the projection \(\Pi_d : L^2_{1/2}(i\Omega^1(Y)) \to L^2_{1/2}(i\Omega^1_C(Y)) = d \left( L^2_{3/2}(\Omega^0(Y)) \right) \) along \(L^2_{1/2}(i\Omega^1_{CC}(Y))\).

**Lemma 3.1 (gauge fixing in 3d).** On \(Y\), there is a continuous choice of based and contractible gauge transformations putting forms in the Coulomb slice, i.e., a homomorphism

\[ L^2_{1/2}(i\Omega^1(Y)) \to G^{e, o}(Y) \]

\[ a \mapsto u_a \]

such that \(a - u_a^{-1}du_a = (1 - \Pi_d)a \in L^2_{1/2}(i\Omega^1_C(Y))\), where \(G^{e, o}(Y) = \left\{ e^f | f \in L^2_{3/2}(i\Omega^0(Y)), \int_Y f = 0 \right\} \subseteq G(Y)\). For each \(a\), there is exactly one such \(u_a\).

**Proof.** Denote \(\Omega^0_0(Y) = \{ f \in \Omega^0(Y) | \int_Y f = 0 \}\). The exterior derivative \(d : L^2_{3/2}(\Omega^0_0(Y)) \to L^2_{1/2}(i\Omega^1_C(Y))\) has inverse \(G_d\). Denote by \(\Pi_d\) the orthogonal projection \(\Omega^1(Y) \to d(\Omega^0(Y))\). We take \(u_a = e^{G_d \Pi_d a}\) which has the desired properties. Uniqueness follows from the fact that \(df = 0\) and \(\int_Y f = 0\) imply \(f = 0\).

Moreover, for \(Y\) a rational homology sphere we have \(G(Y) = G^e(Y) = \left\{ e^f | f \in L^2_{3/2}(Y; i\mathbb{R}) \right\}\).

Indeed, taking \(\tilde{u} = u - u^{-1}du\) gives us \(\tilde{u}\) with \(d(\tilde{u}^{-1}d\tilde{u}) = 0\), which implies \(d\tilde{u} = 0\) and thus \(\tilde{u}\) is constant. It follows that there are bijections

\[ C_{CC}(Y, \mathfrak{s}) \leftrightarrow \frac{(C(Y, \mathfrak{s}) \oplus G^{e, o}(Y))}{G^{e, o}(Y)} , \]

\[ (C_{CC}(Y, \mathfrak{s})) / S^1 \leftrightarrow \frac{(C(Y, \mathfrak{s}))}{G(Y)} , \]

justifying the restriction to the Coulomb slice:

\[ \begin{align*}
C_{CC}(Y, \mathfrak{s}) & \leftrightarrow \frac{(C(Y, \mathfrak{s}) \oplus G^{e, o}(Y))}{G^{e, o}(Y)} , \\
(C_{CC}(Y, \mathfrak{s})) / S^1 & \leftrightarrow \frac{(C(Y, \mathfrak{s}))}{G(Y)} ,
\end{align*} \]

11
Definition 3.2 (Coulomb slice). The Coulomb slice on $Y$ with respect to the reference connection $B_0$ is the space of configurations

$$C_{\text{CC}}(Y, \mathfrak{s}) = (B_0, 0) + L^2_{1/2}(i\Omega^1_{\text{CC}}(Y) \oplus \Gamma(S_Y)) \subset C(Y, \mathfrak{s}).$$

The following subspaces are crucial to the analysis of Atiyah-Patodi-Singer boundary value problem for the Seiberg-Witten equations on 4-manifolds with boundary $Y$.

Definition 3.3 (polarization on the Coulomb slice). We define $H^+(Y, \mathfrak{s})$ (resp. $H^-(Y, \mathfrak{s})$) to be the closure of the span of positive (resp. nonpositive) eigenvalues of $(\ast d) \oplus D_{B_0} : L^2_{1/2}(i\Omega^1_{\text{CC}}(Y) \oplus \Gamma(S_Y)) \to L^2_{1/2}(i\Omega^1_{\text{CC}}(Y) \oplus \Gamma(S_Y))$.

We denote by $\Pi^\pm : L^2_{1/2}(i\Omega^1_{\text{CC}}(Y) \oplus \Gamma(S_Y)) \to H^\pm(Y, \mathfrak{s})$ the projection onto $H^\pm(Y, \mathfrak{s})$ along $H^\mp(Y, \mathfrak{s})$.

One of our goals is to prove that the moduli spaces of solutions to the Seiberg-Witten equations on $X$ are, in a precise sense, comparable to the negative subspace $H^-(Y, \mathfrak{s})$ via the restriction map (cf. Semi-infinite-dimensionality Theorem).

Finally, if $Y = \bigsqcup Y_i$ is a disjoint sum of oriented rational homology spheres $Y_i$ then we define

$$C(Y, \mathfrak{s}) = \prod_i C(Y_i, \mathfrak{s}_i), \quad C_{\text{CC}}(Y, \mathfrak{s}) = \prod_i C_{\text{CC}}(Y_i, \mathfrak{s}_i),$$

$$\mathcal{L}(\prod_i b_i, \prod_i \Psi_i) = \sum_i \mathcal{L}(b_i, \Psi_i) \quad H^\pm(Y, \mathfrak{s}) = \prod_i H^\pm(Y_i, \mathfrak{s}_i).$$

Moreover, note that there are a natural identifications between configuration spaces for $Y$ and oppositely oriented $-Y$. For a spin$^c$ structure $\mathfrak{s}$ with its spinor bundle $S_Y$ there is the conjugate spin$^c$ structure $\overline{S}_Y$ determined by the conjugate bundle $\overline{S}_Y$, and the anti-linear isomorphism $S_Y \simeq \overline{S}_Y$ induces natural affine isomorphisms

$$C(Y, \mathfrak{s}) \simeq C(-Y, \overline{\mathfrak{s}}) \quad C_{\text{CC}}(Y, \mathfrak{s}) \simeq C_{\text{CC}}(-Y, \overline{\mathfrak{s}}) \quad H^\pm(Y, \mathfrak{s}) \simeq H^\mp(-Y, \overline{\mathfrak{s}}).$$

3.2 Split Coulomb slice on 4-manifolds

We turn our attention to the Seiberg-Witten equations and gauge fixings for configurations on a connected oriented 4-manifold $X$ with nonempty boundary $\partial X \neq \emptyset$ satisfying $b_1(\partial X) = 0$, oriented using the outward normal. As explained by Khandawit [Kha], the most convenient slice for these is a kind of a double Coulomb slice (which was already used by Lipianskiy [Lip08]), which imposes both coclosedness of the connection 1-form on both $X$ and $\partial X$, as well as an auxiliary condition near $\partial X$. We drop
this auxiliary condition from the definition of the double Coulomb slice and instead introduce the split Coulomb slice which generalizes the constructions of Khandhawit and Lipyanskiy. This allows one to choose a gauge fixing which do not require twisting or ones that are more geometric in nature, depending on one's needs. Indeed, twisting is necessary in Khandhawit’s and Lipyanskiy’s gauge fixing, in which the restriction map may not commute with the residual gauge group action.

We begin by introducing the Seiberg-Witten equations on 4-manifolds.

**Definition 3.4 (Seiberg-Witten equations).** The Seiberg-Witten map is defined by

\[
SW : C(X, \hat{s}) \to L^2(i\Omega^+(X) \oplus \Gamma(S_X^-)),
\]

\[
SW(A, \Phi) = \left( \frac{1}{2} F^+_A - \rho^{-1}((\Phi \Phi^*)_0), \mathcal{D}^+_A \Phi \right),
\]

where \( A^t \) denotes the connection induced by \( A \) on \( \det(S_X^+) \) and \( F^+_A \) denotes the self-dual part of its curvature, according to the splitting \( \Lambda^2(X) = \Lambda^+(X) \oplus \Lambda^-(X) \) by the eigenspaces of the Hodge star \( * \).

The Seiberg-Witten equations are the equations given by \( SW(A, \Phi) = 0 \), that is,

\[
\begin{align*}
\frac{1}{2} F^+_A - \rho^{-1}((\Phi \Phi^*)_0) &= 0, \\
\mathcal{D}^+_A \Phi &= 0.
\end{align*}
\]

Note that continuity and smoothness of the map \( SW \) follow from Theorem A.1 and the fact that continuous multilinear maps on Banach spaces are smooth.

These equations are equivariant with respect to the action of the gauge group \( G(X) = L^2_2(X; S^1) \). As is easily seen, the solution set is invariant under this action.

**Lemma 3.5 (gauge group action on a 4-manifold).** The gauge group \( G(X) \) acts on \( C(X, \hat{s}) \) via \( u(A, \Phi) = (A - u^{-1} du, u \Phi) \) where \( u \in G(X) \). Moreover, \( SW(A, \Phi) = 0 \) if and only if \( SW(u(A, \Phi)) = 0 \).

Note that this action is not continuous since the multiplication \( L^2_2(X) \times L^2_1(X) \to L^2_1(X) \) is not continuous. It is well-defined since \( G(X) \subset L^\infty(X) \cap L^2_2(X) \) and the multiplication \((L^2_2(X) \cap L^\infty(X)) \times L^2_1(X) \to L^2_1(X) \) is continuous.

In order to prove that the moduli spaces of solutions are manifolds we need to investigate the differential of \( SW \).

\[
D_{(A, \Phi)} SW : L^2_1(i\Omega^1(X) \oplus \Gamma(S_X^+)) \to L^2(i\Omega^+(X) \oplus \Gamma(S_X^-)),
\]

\[
D_{(A, \Phi)} SW(a, \phi) = (a^+ a, \mathcal{D}^+_A a) + (-\rho^{-1}(\phi \Phi^* + \Phi \phi^*)_0, \rho(a) \Phi + \rho(A^\delta) \phi).
\]
Similarly, at \((e, A, \Phi)\) the differential of the gauge group action is:

\[
T\mathcal{G}(X) = L^2_2(X; \mathbb{R}) \to L^2_2(i\Omega^1(X) \oplus \Gamma(S^+_X)),
\]

\[
f \mapsto (-df, f\Phi).
\]

As in dimension 3, we can fix gauge using the Coulomb condition, i.e., require that the 1-form is coclosed. Adding the same condition on the boundary \(\partial X\) ensures that the restriction to the boundary lies in the previously defined Coulomb slice (cf. Definition 3.2):

**Definition 3.6 (double Coulomb slice).** We define the double Coulomb slice:

\[
\Omega^{1}_{\text{CC}}(X) = \{a \in \Omega^1(X) | d^* a = 0, d^* (\iota_{\partial X}^* a) = 0\}.
\]

The gauge group preserving it is called the harmonic gauge group:

\[
\mathcal{G}^h(X) = \{u : X \to S^1 | u^{-1} du \in L^2_1(i\Omega^1_{\text{CC}}(X))\}.
\]

While the action of the full gauge group \(\mathcal{G}(X)\) is not continuous, the action of \(\mathcal{G}^h(X)\) is, which will be proven in Lemma 3.9.

To define the split Coulomb slice we need to first understand the harmonic gauge group and its relation to harmonic functions and forms on \(X\). Notice that we have a well-defined homomorphism

\[
\delta : \mathcal{G}(X) \longrightarrow L^2_1(i\Omega^1(X))
\]

\[
u \longmapsto \delta(u) = u^{-1} du
\]

which, restricted to harmonic gauge transformations, induces

\[
\delta : \mathcal{G}^h(X) \rightarrow i\mathcal{H}^1_D(X)
\]

where

\[
\mathcal{H}^1_D(X) = \{a \in \Omega^1(X) | da = 0, d^* a = 0, \iota_{\partial X}^* a = 0\}
\]

is the space of harmonic 1-forms with Dirichlet boundary conditions. Note that \(\delta\) (both on \(\mathcal{G}(X)\) and on \(\mathcal{G}^h(X)\)) is an inclusion modulo \(S^1\), i.e., \(\ker \delta = S^1\), the group of constant gauge transformations.

On the other hand, the exponential map

\[
\exp : L^2_2(i\Omega^0(X)) \longrightarrow \mathcal{G}(X)
\]

\[
f \longmapsto e^f
\]

restricted to the space of doubly harmonic functions

\[
\mathcal{H}(X) = \{f \in \Omega^0(X) | \Delta f = 0, \Delta(f|_{\partial X}) = 0\}
\]

14
yields a homomorphism
\[ \exp : i\mathcal{H}(X) \to \mathcal{G}^h(X) \]
since the conditions \( \Delta f = 0 \) and \( \Delta(f|_{\partial X}) = 0 \) for an imaginary-valued function \( f \) are equivalent to \( df \in L^2_1(i\Omega^1_{CC}(X)) \). Importantly, the composition \( \delta \circ \exp : L^2_1(i\Omega^0(\mathcal{P}(X))) \to L^2_1(i\Omega^1(X)) \) is exactly the differential \( f \mapsto df \).

Denote the image of this exponential map by \( \mathcal{G}^{h,e}(X) = \exp(i\mathcal{H}(X)) \). As the next proposition explains, \( \mathcal{G}^{h,e}(X) \) is the identity component of \( \mathcal{G}^h(X) \). Thus, our goal will be to find a gauge fixing that dispenses with the action of this identity component, saving only the action by \( S^1 \), the constant elements.

**Proposition 3.7 (sequence of harmonic gauge groups).** The following sequence is exact:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{G}^{h,e}(X) & \longrightarrow & \mathcal{G}^h(X) & \longrightarrow & \pi_0\mathcal{G}^h(X) & \longrightarrow & 0.
\end{array}
\]  
(3)

The identity component \( \mathcal{G}^{h,e}(X) \) is isomorphic to \( S^1 \times \mathbb{R}^{b_0(\partial X)-1} \) and the group of components \( \pi_0\mathcal{G}^h(X) \) is naturally isomorphic to \( H^1(X;\mathbb{Z}) \).

**Remark 3.8.** Recall that \( H^1(X;\mathbb{Z}) \cong \text{Hom}(\pi_1(X),\mathbb{Z}) \) has no torsion.

**Proof.** Crucial to the understanding of the gauge group is the homomorphism (2). Hodge theory provides an identification \( H^1_D(X) \cong H^1(X,\partial X;\mathbb{R}) \). Thus, we will consider \( \delta \) as a map \( \delta : \mathcal{G}^h(X) \to H^1(X,\partial X;\mathbb{R}) \) with kernel \( S^1 \). This will be used to establish a map of horizontal short exact sequences

\[
\begin{array}{ccccccc}
& & S^1 & \longrightarrow & S^1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}^{h,e}(X) & \longrightarrow & \mathcal{G}^h(X) & \longrightarrow & \pi_0\mathcal{G}^h(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow^{\delta} & & \downarrow & & \\
0 & \longrightarrow & H^0(\partial X;\mathbb{R})/H^0(X;\mathbb{R}) & \longrightarrow & H^1(X,\partial X;\mathbb{R}) & \longrightarrow & H^1(X;\mathbb{R})/H^1(X;\mathbb{Z}) \cong H^1(X;\mathbb{Z}) \big/ H^1(X;\mathbb{Z})
\end{array}
\]  
(4)

where the vertical sequences are also exact, as will be shown in the course of the proof.

Firstly, we prove that \( \pi_0\mathcal{G}^h(X) \cong H^1(X;\mathbb{Z}) \). Notice that for any closed loop \( \gamma \subset X \) the period of \( \delta(u) \), i.e., the integral \( \int_\gamma \delta(u) \), is an integer multiple of \( 2\pi i \); and it is zero whenever \( \gamma \) is contractible. (In fact it is the obstruction to lifting \( u|_{\gamma} : \gamma \to S^1 \) to a map \( u|_{\gamma} : \gamma \to \mathbb{R} \).) This way any \( u \in \mathcal{G}^h(X) \) determines an element \( [u] \in H^1(X;2\pi i\mathbb{Z}) \) and we get a homomorphism \( \mathcal{G}^h(X) \to H^1(X;2\pi i\mathbb{Z}) \). Since the periods (having values in
\(2\pi i\mathbb{Z})\) do not change under homotopy, this descends to a map \(\pi_0\mathcal{G}^h(X) \to H^1(X; 2\pi i\mathbb{Z})\) and from its construction it follows that it coincides with the composition

\[
\mathcal{G}^h(X) \xrightarrow{\delta} H^1(X, \partial X; i\mathbb{R}) \to H^1(X; i\mathbb{R})
\]

which has image in \(H^1(X; 2\pi i\mathbb{Z})\). It remains to notice that any element in \(x \in H^1(X; 2\pi i\mathbb{Z})\) can be lifted to an element \(\tilde{x} = H^1(X, \partial X; i\mathbb{R}) \simeq iH_1^B(X)\) and then integrated along curves to obtain an element \(u_\tilde{x} \in \mathcal{G}^h(X)\) mapping to \(x\) (see (6)). After dividing 1-forms by \(2\pi i\) we obtain a natural isomorphism \(\pi_0\mathcal{G}^h(X) \simeq H^1(X; \mathbb{Z})\), as wished.

Moreover, this shows that the kernel \(K\) of the map \(\mathcal{G}^h(X) \to H^1(X; 2\pi i\mathbb{Z})\) maps via \(\delta\) to the kernel of \(H^1(X, \partial X; i\mathbb{R}) \to H^1(X; i\mathbb{R})\). We thus get the map

\[
\delta|_K : K \to \text{im}(H^0(\partial X; i\mathbb{R}))
\]

to the image of \(H^0(\partial X; i\mathbb{R})\) in \(H^1(X, \partial X; i\mathbb{R})\). The map \(\delta|_K\) itself has kernel \(S^1\).

With this in mind, we turn our focus to \(\mathcal{G}^{h,e}(X) = \exp(\mathcal{H}(X))\). Since any harmonic function \(f\) on \(X\) is determined by its restriction \(f|_{\partial X}\) to \(\partial X\), and harmonic functions on \(\partial X\) are locally constant, we have an isomorphism

\[
H^0(\partial X; \mathbb{R}) \simeq \mathcal{H}(X)
\]

and we will denote any element \(g\) in \(H^0(\partial X; \mathbb{R})\) by \(f|_{\partial X}\) for the unique \(f \in \mathcal{H}(X)\) such that \(f|_{\partial X} = g\). We obtain a canonically defined surjection

\[
\exp : H^0(\partial X; i\mathbb{R}) \longrightarrow \mathcal{G}^{h,e}(X)
\]

\[
\exp : H^0(\partial X; i\mathbb{R}) \xrightarrow{\exp} \mathcal{G}^{h,e}(X) \xrightarrow{\delta} H^1(X, \partial X; i\mathbb{R})
\]

with kernel generated by restrictions of \(H^0(X; 2\pi i\mathbb{Z})\). After dividing by \(2\pi i\) we get an isomorphism \(\mathcal{G}^{h,e}(X) \simeq H^0(\partial X; \mathbb{R})/H^0(X; \mathbb{Z}) \simeq S^1 \times (H^0(\partial X; \mathbb{R})/H^0(X; \mathbb{R}))\), as wished.

Finally, recall that the composition

\[
H^0(\partial X; i\mathbb{R}) \simeq \mathcal{H}(X) \xrightarrow{\exp} \mathcal{G}^{h,e}(X) \xrightarrow{\delta} H^1(X, \partial X; i\mathbb{R})
\]

is given by \(f|_{\partial X} \mapsto df\) and therefore, by Hodge theory, represents the boundary map in the exact sequence

\[
0 \to H^0(X; \mathbb{R}) \to H^0(\partial X; \mathbb{R}) \to H^1(X, \partial X; \mathbb{R}) \to H^1(X; \mathbb{R}) = 0
\]
Thus $\delta(G^{h,e}(X)) = \text{im}(H^0(\partial X; i\mathbb{R})) = \delta(K)$. Since ker $\delta|_{G^{h,e}(X)} = S^1 = \text{ker}\delta|_K$ and $G^{h,e}(X) \subseteq K$, we obtain that $G^{h,e}(X) = K$, i.e., the sequence (3) is exact, as wished.

\textbf{Lemma 3.9 (}$G^h$ acts continuously\textbf{).} The action of $G^h(X)$ on $C(X, \hat{s})$ is smooth.

\textit{Proof.} By Proposition 3.7 it suffices prove that the action of $G^{h,e}(X)$ is smooth. Further, by Theorem A.1 it suffices to prove that $G^{h,e}(X) \subseteq L^2_3(X; S^1)$ and that this injection is continuous. Since $f \in i\mathcal{H}(X)$ and $\mathcal{H}(X)$ is finite-dimensional, there are constants $C, C'$ such that $\|\exp(f)\|_{L^2_3} \leq C\|f\|_{L^2_3} \leq C'C\|f\|_{L^2_3}$, finishing the proof.

Let us compare different splittings. If $s, s'$ are two different splittings, then for any $[u] \in \pi_0G^h(X)$ we have $s(s')^{-1} \in G^{h,e}(X)$. Therefore any two gauge splittings differ by a homomorphism $\pi_0G^h(X) \rightarrow G^{h,e}(X)$.

Our goal is to reduce the gauge group action to the action of $S^1$ and the action of a chosen lift of $\pi_0G^h(X)$ to $G^h(X)$. Precisely, we will consider splittings $s : \pi_0G^h(X) \rightarrow G^h(X)$ of (3). In order to choose the gauge fixing we need to understand that a gauge splitting induces another splitting on the level of homology.

\textbf{Proposition 3.10 (homological splitting).} Any splitting $s : \pi_0G^h(X) \rightarrow G^h(X)$ of (3) induces a splitting $s^H : H^1(X; \mathbb{R}) \rightarrow H^1(X, \partial X; \mathbb{R})$ of the exact sequence

$$0 \rightarrow H^0(\partial X; \mathbb{R})/H^0(X; \mathbb{R}) \rightarrow H^1(X, \partial X; \mathbb{R}) \rightarrow H^1(X; \mathbb{R}) \rightarrow 0.$$  \hfill (5)

\textit{Proof.} Composing $H^1(X; \mathbb{Z}) \xrightarrow{2\pi i} \pi_0G^h(X) \xrightarrow{s} G^h(X) \xrightarrow{\delta} H^1(X, \partial X; \mathbb{R})$ we get a linear map which by linearity uniquely extends to a section $s_H : H^1(X; \mathbb{R}) \rightarrow H^1(X, \partial X; \mathbb{R})$ of the aforementioned exact sequence.

\textbf{Definition 3.11 (gauge splitting).} By a gauge splitting we call a splitting $s : \pi_0G^h(X) \rightarrow G^h(X)$ of the exact sequence (3). We denote by $s^H : H^1(X; \mathbb{R}) \rightarrow H^1(X, \partial X; \mathbb{R})$ the associated homological splitting.

We clarify the relationship between gauge splittings and homological splittings.

\textbf{Proposition 3.12 (gauge splittings from homological splittings).} Let $\sigma : H^1(X; \mathbb{R}) \rightarrow H^1(X, \partial X; \mathbb{R})$ be a homological splitting, i.e., a splitting of (5). Then up to action of $S^1$ there exists a unique gauge splitting $s$ such that $\sigma = s^H$.

\textit{Proof.} For existence, choose $x_0 \in X$ and consider the map

$$I_{x_0} : \{\eta \in H^1_D(X) \mid [\eta] \in H^1(X; 2\pi i\mathbb{Z})\} \rightarrow G^h(X),$$

$$\eta \mapsto \left(I_{x_0}(\eta)(x) = \exp\left(\int_{x_0}^x \eta\right)\right).$$  \hfill (6)

Notice that $\delta(I_{x_0}(\eta)) = \eta$. Therefore taking $s([u]) = I_{x_0}(\sigma([\delta(u)]))$ we get a gauge
splitting \( s : \pi_0 \mathcal{G}^h(X) \to \mathcal{G}^h(X) \) with \( \sigma = s^H \).

The uniqueness up to action of \( S^1 \) follows from the exactness of the middle vertical sequence in (4).

In order to find the appropriate gauge fixing we need the following analogue of Proposition 3.7 for 1-forms.

**Lemma 3.13 (decomposing 1-forms).**
\( \Omega^1(X) = \Omega^1_{CC}(X) + d(\Omega^0(X)) \) and \( \Omega^1_{CC}(X) \cap d(\Omega^0(X)) = d(\mathcal{H}(X)) \).

**Proof.** This follows from the proof of [Kha, Proposition 2.2]. (Note that our definition of \( \Omega^1_{CC}(X) \) differs from Khandhawit’s, which we denote by \( \Omega^1_{\perp}(X) \) (cf. Definition 3.21).)

In particular, we can decompose 1-forms as
\[
\Omega^1(X) = \Omega^1_{CC}(X) \oplus d(\Omega^0_{\partial}(X))
\]
where
\[
\Omega^0_{\partial}(X) = \left\{ f \in \Omega^0(X) \middle| \int_{Y_i} f = 0 \text{ for each component } Y_i \subset \partial X \right\}.
\]

Denoting by \( \Pi_{CC} \) the projection onto \( \Omega^1_{CC}(X) \) along \( d(\Omega^0_{\partial}(X)) \) we obtain the following analog of Lemma 3.1.

**Lemma 3.14 (Coulomb gauge fixing in 4d).**

There is a unique homomorphism
\[
L^2_1(i\Omega^1(X)) \to \mathcal{G}^{e,0}(X) = \exp(L^2_2(i\Omega^0_{\partial}(X)))
\]
\[a \mapsto u^CC_a\]
such that
\[
a - (u^CC_a)^{-1} du^CC_a = \Pi_{CC} a \in L^2_1(i\Omega^1_{CC}(X)).
\]

**Proof.** The projection \( (1 - \Pi_{CC}) \) on \( \Omega^1(X) \) has image in \( d(\Omega^0_{\partial}(X)) \) and that \( d \) is injective on \( \Omega^0_{\partial}(X) \). Therefore there is a unique homomorphism \( L^2_1(i\Omega^1(X)) \to L^2_2(i\Omega^0_{\partial}(X)) \) sending \( a \) to the unique \( f^CC_a \in \Omega^0_{\partial}(X) \) such that \( a - df^CC_a \in \Omega^1(X) \). Then we take \( u^CC_a = \exp(f^CC_a) \).

We can further decompose
\[
\Omega^1_{CC}(X) = \left( \Omega^1_{CC}(X) \cap (\mathcal{H}^1_D(X))^\perp \right) \oplus \mathcal{H}^1_D(X).
\]

A homological splitting \( s^H \) provides a decomposition
\[
\mathcal{H}^1_D(X) = s^H(H^1(X; \mathbb{R})) \oplus d(\mathcal{H}(X))
\]
which is an analogue of (3) for $\mathcal{H}_D^1(X)$. With these in hand, we are ready to define the split Coulomb slice.

**Definition 3.15 (split Coulomb slice).** Let $s$ be a gauge splitting and $s^H$ its associated homological splitting. The **split Coulomb slice** is

$$
\Omega^1_s(X) = \{a \in \Omega^1_{CC}(X) | a \in (\Omega^1_{CC}(X) \cap (\mathcal{H}_D^1(X))^\perp) \oplus s^H(H^1(X; i\mathbb{R}))\}.
$$

In particular, we have that

$$
\Omega^1(X) = \Omega^1_s(X) \oplus d(\Omega^0(X))
$$

and parallel to Lemma 3.14 and Lemma 3.1 we can use the projection $\Pi_s$ onto the first factor along the second one to obtain:

**Lemma 3.16 (split gauge fixing in 4d).** There is a unique homomorphism

$$
L^2_t(i\Omega^1(X)) \to \mathcal{G}^e, o(X)
$$

such that

$$
a - (u^s_a)^{-1} du^s_a = \Pi_s a \in L^2_t(i\Omega^1_s(X)).
$$

**Proof.** The projection $(1 - \Pi_s)$ on $\Omega^1(X)$ has image in $d(\Omega^0(X))$ and $d$ is injective on $\Omega^0_0(X)$. Therefore there is a unique homomorphism $L^2_t(i\Omega^1(X)) \to L^2_t(i\Omega^0_0(X))$ sending $a$ to the unique $f^s_a \in \Omega^0_0(X)$ such that $a - df^s_a \in \Omega^1(X)$. Then we take $u^s_a = \exp(f^s_a)$. □

**Remark 3.17 (continuous gauge fixing within double Coulomb slice).** If we only consider $a \in \Omega^1_{CC}(X)$, then the above map has image in $\mathcal{G}^h(X)$, which is finite-dimensional. Since the latter gauge group acts continuously on the configuration space by Lemma 3.9, we conclude that putting $(A, \Phi) \in \mathcal{C}_{CC}(X, \hat{s})$ into split Coulomb slice $\mathcal{C}_s(X, \hat{s})$ can be done continuously with respect to $(A, \Phi)$.

The gauge group acting on this split Coulomb slice is the product of $S^1$ and the split gauge group:

**Definition 3.18 (split harmonic gauge group).** Let $s$ be a gauge splitting. The **split gauge group** is defined to be

$$
\mathcal{G}^{h, o}_s(X) = s(\pi_0 \mathcal{G}^h(X)).
$$

**Lemma 3.19 (split gauge group preserves the split Coulomb slice).** For $u \in \mathcal{G}^{h, o}_s(X)$ we have $u^{-1} du \in L^2_t(i\Omega^1_s(X))$.

Conversely, if $u^{-1} du \in L^2_t(i\Omega^1_s(X))$, then for some $z \in S^1$ we have $zu \in \mathcal{G}^{h, o}_s(X)$. 19
Proof. One direction follows directly from the definition of $s^H$: if $u = s([u])$, then 
$s^H([\delta(u)]) = \delta(u) = u^{-1}du \in \mathcal{H}_D^1(X) \simeq H^1(X, \partial X; i\mathbb{R})$, so $u^{-1}du \in \text{im } s^H \subset L_1^2(i\Omega^1_D(X))$.

The other direction follows by chasing arrows in the diagram (4). □

The circle $\circ$ in the superscript indicates that the only constant gauge transformation contained in $\mathcal{G}^{h,\circ}_s(X)$ is the identity. This way we do not forget the $S^1$-action when taking the quotient by the split gauge group.

We want to compare different split slices together with the split gauge group actions. Choose two splittings $s, s'$. These determine a map $s' \cdot s^{-1} : \pi_0\mathcal{G}^h(X) \to \mathcal{G}^{h,e}(X)$. Viewing $\pi_0\mathcal{G}^h(X)$ as a sublattice $\pi_0\mathcal{G}^h(X) \simeq H^1(X; 2\pi i\mathbb{Z}) \subset H^1(X; i\mathbb{R})$, let $\nu : H^1(X; i\mathbb{R}) \to \mathcal{G}^{h,e}(X)$ be any homomorphism extending $s' \cdot s^{-1}$. Define

$$F_{\nu} : C_s(X, \hat{s}) \longrightarrow C_s(X, \hat{s}) \quad (A, \Phi) \longmapsto \nu(\Pi_{\text{im } s^H}(A - A_0)) \cdot (A, \Phi).$$

where $\Pi_{\text{im } s^H} : i\Omega^1_{CC}(X) \to s^H(H^1(X; i\mathbb{R}))$ is the projection along $i(\Omega^1_{CC}(X) \cap \mathcal{H}_D^1(X)) \oplus id(\mathcal{H}(X))$ (cf. (7), (8)).

**Proposition 3.20 (equivalence of slices).** The map $F_{\nu}$ is well-defined, a diffeomorphism, equivariant with respect to the action of $\pi_0\mathcal{G}^h(X) \simeq \mathcal{G}^{h,\circ}_s(X) \simeq \mathcal{G}^{h,\circ}_s(X)$.

Proof. Firstly, we need to show that the image of $F_{\nu}$ actually lies in $C_s(X, \hat{s})$. Equivalently, we want to show that

$$A_{\nu} : \Omega^1_{CC}(X) \longrightarrow \Omega^1_{CC}(X) \quad a \longmapsto \nu(\Pi_{\text{im } s^H} a) \cdot a = a + \delta(\nu(\Pi_{\text{im } s^H} a))$$

maps $\Omega^1_s(X)$ to $\Omega^1_{s'}(X)$. We have $\delta(\mathcal{G}^{h,e}(X)) = d(\mathcal{H}(X))$ and, moreover, $\delta \circ \nu$ is a homomorphism, thus a linear map $H^1(X; i\mathbb{R}) \to d(\mathcal{H}(X))$. What follows is that $A_{\nu}$ is a linear map. Now $A_{\nu}$ restricted to $\Omega^1_{CC}(X) \cap (\mathcal{H}_D^1(X))^\perp$ is identity by definition, so it suffices to show $A_{\nu}(\text{im } s^H) \subset \text{im } (s')^H$. Furthermore, $H^1(X; i\mathbb{R})$ is spanned by $[\delta(\pi_0\mathcal{G}^h(X))]$ and therefore $s^H(H^1(X; i\mathbb{R}))$ is spanned by $\delta(\mathcal{G}^{h,\circ}_s(X))$, so it suffices to show $A_{\nu}(\delta(\mathcal{G}^{h,\circ}_s(X))) \subset \text{im } (s')^H$. But we defined $\nu$ so that for any $u \in \mathcal{G}^{h,\circ}_s(X)$ $\nu(\delta(u)) \cdot u \in \mathcal{G}^{h,\circ}_s(X)$, which implies $A_{\nu}(\delta(u)) = \delta(\nu(\delta(u)) \cdot u) \in \text{im } (s')^H$, as wished.

The map $F_{\nu}$ is smooth because the map $\nu$ is smooth and the action of the finite-dimensional $\mathcal{G}^h(X)$ on $C(X, \hat{s})$ is smooth.

It is invertible because $F^{-1}_{1/\nu}$ is its inverse. Indeed, since $\text{im } \nu \subset \mathcal{G}^{h,e}(X)$, we have that $\delta \nu \in d(\mathcal{H}(X))$, so $\Pi_{\text{im } s^H}(\delta \nu) \equiv 0$. Therefore

$$\Pi_{\text{im } s^H}(\nu(\Pi_{\text{im } s^H}(A - A_0))A - A_0) = \Pi_{\text{im } s^H}(A - A_0),$$

where $\Pi_{\text{im } s^H}$ is the projection along $i(\Omega^1_{CC}(X) \cap \mathcal{H}_D^1(X)) \oplus id(\mathcal{H}(X))$.
\[
(\nu(\Pi_{im,s} H (A - A_0)) A - A_0))^{-1} = (\nu(\Pi_{im,s} H (A - A_0)))^{-1}
\]
and \( F_{\nu} \circ F_{\nu} = \text{id} \) follows. \qed

Finally, we discuss the gauge slice used by Lipyanskiy \cite{Lip08} and Khandhawit \cite{Kha,KLS18}. They require that \( a \in \Omega^1_{CC}(X) \) and that for each component \( Y_i \subset \partial X \) we have \( \int_{Y_i} \iota^*(sa) = 0 \). Using Stokes' theorem one can show that for \( a \in \Omega^1_{CC}(X) \) this integral condition is equivalent to the condition that \( \int_X df \wedge sa = 0 \) for any \( f \in \mathcal{H}(X) \). This fits into our setup perfectly, since there is exactly one homological splitting \( s^H_\perp \) such that \( \text{im } s^H_\perp = \mathcal{H}^1_D(X) \cap (d(\mathcal{H}(X)))^\perp \).

**Definition 3.21 (orthogonal splitting).** We call \( s^H_\perp \) the **orthogonal homological splitting**. We say that a splitting \( s \) is a **orthogonal splitting** if \( s = s^H_\perp \).

### 3.3 Restriction to the boundary and twisting

Unless \( \partial X \) is connected, we are not guaranteed that the restriction to the boundary is invariant under the action of the split gauge group \( G^h_s(X) \). If it happens to be invariant for some \( s \), we call such \( s \) an **integral splitting**. For a general \( s \), we introduce and prove the existence of **twistings** of the restriction map, making it invariant under the action of \( G^h_s(X) \) action even for non-integral \( s \). As mentioned before, integral splittings are utilized in the proof of the **Gluing Theorem**, while non-integral splittings may be more convenient in other contexts (e.g., in constructions of \cite{Lip08,Kha,KLS18}).

We start by defining the restriction maps for an embedding \( \iota_Y : Y \hookrightarrow X \) of an oriented 3-manifold \( Y \). Denote by \( s \) the restriction to \( Y \) of the spin\(^c\) structure \( \hat{s} \) on \( X \). We get canonical identifications \( S^+_X|_Y \simeq S_Y \). Assuming \( Y \) is a geodesic codimension-1 submanifold of \( X \), the spin\(^c\) connection \( A_0 \) induces a spin\(^c\) connection \( B_0 \) on \( Y \) by simple restriction: \( \nabla B_0 = \iota_Y^* \nabla A_0 \). Let \( a \in L^2_Y(i\Omega^1(Y)) \), \( A \in A_0 + L^2_Y(i\Omega^1(Y)) \), \( \Phi \in L^2_Y(\Gamma(S_X^Y)) \) and \( u \in \mathcal{G}(X) \). We define the restrictions:

\[
R(a) = \iota_Y^*(a) \in L^{2}(i\Omega^1(Y)),
R(A) = B_0 + \iota_Y^*(A - A_0) \in B_0 + L^{2}_{1/2}(i\Omega^1(Y)),
R(\Phi) = \Phi|_Y \in L^{2}_{1/2}(\Gamma(S_Y)),
R(u) = u|_Y \in \mathcal{G}(Y).
\]

Integral splittings are the ones for which restriction maps are invariant under the split gauge group.
Definition 3.22 (integral splitting). We call a gauge splitting $s$ integral if for each $u \in G_{s,h}(X)$ we have $u|_{\partial X} \equiv 1$.

Equivalently, $s$ is integral if the composition $\pi_0 G^h(X) \overset{s}{\rightarrow} G^h(X) \overset{R}{\rightarrow} G^h(\partial X) \simeq (S^1)\pi_0(\partial X)$ is trivial.

The integrality of $s$ is closely connected to the integrality of $s^H$.

Proposition 3.23 (homological classification of integral splittings). If $s$ is integral, then $s^H(H^1(X;\mathbb{Z})) \subset H^1(X,\partial X;\mathbb{Z})$, i.e., $s^H$ is integral as well.

Given any integral homological splitting $\sigma$ there exists a unique integral splitting $s$ such that $\sigma = s^H$.

Proof. Assume $s$ is integral. Choose $y_0 \in \partial X$ and consider the map $I_{y_0}$ defined in (6).

We know $s$ and $I_{y_0} \circ s^H \circ [\delta]$ differ by action of $S^1$, but since both are equal to 1 at $y_0$, this implies that for any $y \in \partial X$ and any embedded curve $\gamma : [0,1] \rightarrow X$ with $\gamma(0) = y_0$ and $\gamma(1) = y$ we have that $\exp \left( \int_{y_0}^y s^H([\delta(u)]) \right) = 1$ and thus $\int_{y_0}^y s^H([\delta(u)]) \in 2\pi i \mathbb{Z}$. This proves that $s^H$ is integral.

Similarly, if $\sigma$ is integral, then $s = I_{y_0} \circ \sigma \circ [\delta]$ satisfies that $\sigma = s^H$ and $s(y) = \exp \left( \int_{y_0}^y s^H([\delta(u)]) \right) = 1$ for any $y \in \partial X$. \qed

To find a consistent way of twisting the boundary values of 1-forms we consider ways to “undo” the action of $G_{s,h}(X)$ on the boundary “in a linear fashion”.

Definition 3.24 (gauge twisting). We call a homomorphism $\tau : H^1(X; i\mathbb{R}) \rightarrow G^h(\partial X) \simeq (S^1)\pi_0(\partial X)$ a gauge twisting for $s$ if the composition $\pi_0 G^h(X) \simeq H^1(X; 2\pi i \mathbb{Z}) \hookrightarrow H^1(X; i\mathbb{R}) \overset{\tau}{\rightarrow} G^h(\partial X)$ agrees with the action of the split gauge group on the boundary, $R \circ s : \pi_0 G^h(X) \rightarrow G^h(\partial X)$.

Continuous homomorphisms from a vector space to $S^1$ correspond to linear functionals on the vector space. Thus, every such twisting $\tau$ comes from a linear map $d\tau : H^1(X; i\mathbb{R}) \rightarrow H^0(\partial X; i\mathbb{R})$ and $\tau = \exp \circ (d\tau)$. We utilize it to prove the existence of gauge twistings for $s$, and one could use it to classify all possible gauge twistings for $s$. Actually, every such homomorphism $\tau$ is a gauge twisting for some $s$, but we do not use this fact in this article.

Lemma 3.25 (existence of gauge twistings). For a given gauge splitting $s$ there exists a gauge twisting $\tau$. 

22
Proof. Since $\pi_0 \mathcal{G}^h(X)$ is free, we can lift the map $R \circ s : H^1(X; 2\pi i \mathbb{Z}) \simeq \pi_0 \mathcal{G}^h(X) \to \mathcal{G}^h(\partial X) \simeq (S^1)^{\pi_0(X)}$ to a homomorphism $\tilde{\tau} : H^1(X; 2\pi i \mathbb{Z}) \to H^0(\partial X; i \mathbb{R})$:

$$H^0(\partial X; i \mathbb{R}) \xrightarrow{\exp} \pi_0 \mathcal{G}^h(X) \xrightarrow{R \circ s} (S^1)^{\pi_0(\partial X)},$$

and this extends to a map $\tilde{\tau} : H^1(X; i \mathbb{R}) \to H^0(\partial X; i \mathbb{R})$ by linearity. Taking $\tau = \exp \circ \tilde{\tau} : H^1(X; i \mathbb{R}) \to (S^1)^{\pi_0(\partial X)}$ gives a gauge twisting for $s$. \qed

With $\tau$ in hand, there is a way of defining a twisting on the whole Coulomb slice, enabling us to finally define the twisted restriction maps.

**Definition 3.26 (twisted restriction map).** We define the Coulomb slice twisting $\tau_{CC} : L^2_1(i \Omega^1_{CC}(X)) \to \mathcal{G}^h(\partial X)$ associated to $\tau$ to be the composition

$$L^2_1(i \Omega^1_{CC}(X)) \xrightarrow{\Pi_{L^2_1}} \mathcal{H}_{D}(X) \simeq H^1(X, \partial X; \mathbb{R}) \xrightarrow{\iota_X} H^1(X; \mathbb{R}) \xrightarrow{\tau} (S^1)^{\pi_0(\partial X)} \simeq \mathcal{G}^h(\partial X).$$

We define the twisted restriction map

$$R_{\tau} : \mathcal{C}_{CC}(X, \hat{s}) \to \mathcal{C}_{CC}(\partial X, \hat{s})$$

by the formula $R_{\tau}(A, \Phi) = (R(A), \tau_{CC}(A - A_0)R(\Phi)).$

**Remark 3.27.** What is of importance for defining the twisted restriction maps is the map $\tau_{CC} : i \Omega^1_{CC}(X) \to (S^1)^{\pi_0(\partial X)}$. The extension of $\tau_{CC}$ to the whole of $i \Omega^1_{CC}(X)$ is artificial: it does not undo the action of $\mathcal{G}^{h,e}(X)$ on the boundary as one might expect.

With more work, including a choice of a based gauge group $\mathcal{G}^h_0(X) \subset \mathcal{G}^h(X)$ (such that $\mathcal{G}^h(X)/\mathcal{G}^h_0(X) \simeq S^1$) and a more general twisting, one could work with the full $i \Omega^1_{CC}(X)$ and then quotient by the action of $\mathcal{G}^h_0(X)$. However, this would introduce unnecessary complications.

These twisted restriction maps are indeed invariant under $\mathcal{G}^{h,o}_s(X)$.

**Lemma 3.28 (twisted restriction map is invariant under split gauge group).** Let $\tau$ be a gauge twisting for $s$. For any $(A, \Phi) \in \mathcal{C}_{CC}(X, \hat{s})$ (resp. $(a, \phi) \in L^2_1(i \Omega^1_{CC}(X) \oplus \Gamma(S^1_{X}))$) and $u \in \mathcal{G}^{h,o}_s(X)$ we have

$$R_{\tau}(u(A, \Phi)) = R_{\tau}(A, \Phi)$$

(resp. $R_{\tau}(u(a, \phi)) = R_{\tau}(a, \phi)$).
Proof. Since \( u \in G^h(X) \), we have \( \iota_{\partial X}^*(u^{-1} du) = 0 \), so \( R(A - u^{-1} du) = R(A) \).

It remains to prove
\[
\tau_{CC}(A - A_0 - u^{-1} du)R(u\Phi) = \tau_{CC}(A - A_0)R(\Phi)
\]
but that is equivalent to
\[
(\tau_{CC}(-u^{-1} du)R(u))\tau_{CC}(A - A_0)R(\Phi) = \tau_{CC}(A - A_0)R(\Phi)
\]
so it suffices to prove \( \tau_{CC}(u^{-1} du) = R(u) \). Since \( s \) splits (3), we have \( u = s([u]) \), where \([u] \in \pi_0 G^h(X)\) is the homotopy class of \( u \). So we have to prove \( \tau_{CC}(u^{-1} du) = R \circ s([u]) \). This follows directly from Definition 3.24 of the twistings and Definition 3.26 of the twisted restriction map, since \( u^{-1} du \in H^1_D(X) \) and the isomorphism \( \pi_0 G^h(X) \simeq H^1(X; 2\pi i \mathbb{Z}) \) is given by \([u] \mapsto [u^{-1} du] \).

We conclude these sections by showing that choosing \( \tau \) is essentially equivalent to choosing an integral splitting. In general, one can restrict themselves to considering integral splittings without any twisting at all.

**Proposition 3.29 (twistings are integral splittings).** Let \( \tau \) be a twisting for \( s \). Then there is an integral splitting \( s_\mathbb{Z} \) and an equivariant diffeomorphism
\[
F_{s,\tau} : C_s(X, \hat{s}) \to C_{s_\mathbb{Z}}(X, \hat{s})
\]
such that \( R \circ F_{s,\tau} = R_{\tau} \).

**Proof.** Every function \( f \in \mathcal{H}(X) \) is determined by its restriction to the boundary \( f|_{\partial X} \), which is locally constant. We thus have the exact sequence
\[
0 \to H^0(\partial X; 2\pi i \mathbb{Z}) \to H^0(\partial X; i\mathbb{R}) \simeq i\mathcal{H}(X) \xrightarrow{f \mapsto \exp(f|_{\partial X})} G^h(\partial X) \to 0
\]
as well as
\[
0 \to H^0(X; 2\pi i \mathbb{Z}) \to i\mathcal{H}(X) \xrightarrow{\exp} G^{h,e}(X) \to 0
\]
and from these two it follows that
\[
0 \to H^0(\partial X; 2\pi i \mathbb{Z})/H^0(X; 2\pi i \mathbb{Z}) \to G^{h,e}(X) \xrightarrow{\iota_{\partial X}} G^h(\partial X) \to 0
\]
is exact. Since the group to the left is discrete it follows that there exists a unique lift

24
\[ \tilde{\tau} \text{ of } \tau \text{ to } G^{h,e}(X): \]
\[ \begin{array}{ccc}
G^{h,e}(X) \xrightarrow{\tilde{\tau}} & H^1(X; i\mathbb{R}) \xrightarrow{\tau} & G^h(\partial X)
\end{array} \]

We define
\[ s_Z([u]) = (\tilde{\tau}([u]))^{-1} \cdot s([u]) \]
for any \([u] \in \pi_0G(X) \simeq H^1(X; 2\pi i\mathbb{Z}),\] and
\[ F_{s,\tau} = F_{\tilde{\tau}}^{-1} \]
using the construction of \(F_\nu\) of Proposition 3.20. This gives an equivariant diffeomorphism from \(C_s(X, \hat{s})\) to \(C_{s_Z}(X, \hat{s}).\)

The equality \(R \circ F_{s,\tau} = R_{\tau}\) follows from the construction. \(\square\)

Even though the spaces \(C_{s_Z}(X, \hat{s})\) and \(C_{s_2}(X, \hat{s})\) are equivariantly diffeomorphic by Proposition 3.20, the corresponding restriction maps differ by a twist. Thus, \(a \text{ priori}\) we cannot get rid of the choice of a splitting. However, this is not relevant to most of the applications because for connected boundary there is no choice to make.

**Lemma 3.30 (uniqueness of integral splittings).** If \(\partial X \neq \emptyset\) is connected, there exists exactly one integral splitting \(s_Z.\)

**Proof.** In this case, the restriction map \(G^{h,e}(X) \to G^h(\partial X)\) is an isomorphism (cf. (10)). Therefore for each element \(\pi_0G(X) \simeq H^1(X; 2\pi i\mathbb{Z})\) there exists exactly one representative \(u \in G^h(X)\) such that \(u|_{\partial X} = 1.\) \(\square\)

### 3.4 Seiberg-Witten moduli in split Coulomb slice

We conclude this section by defining the **Seiberg-Witten moduli spaces**, the main object of study of this article. We also prove they only depend on the choice of \(s_Z\) associated to \(s\) and \(\tau.\)

Thanks to Lemma 3.19, we can define the following.

**Definition 3.31 (moduli spaces on 4-manifolds with boundary).** We define the moduli spaces in split slice:

\[ \tilde{M}_s^0(X, \hat{s}) = \{ (A, \Phi) \in C_s(X, \hat{s}) | \text{SW}(A, \Phi) = 0 \}, \]
\[ M_s^0(X, \hat{s}) = \tilde{M}_s^0(X, \hat{s}) / G_s^{h,\circ}(X). \]
We also define a version of the moduli space using the full double Coulomb slice,

\[ \widetilde{M}_{CC}^\circ(X, \hat{s}) = \{(A, \Phi) \in C_{CC}(X, \hat{s}) | \text{SW}(A, \Phi) = 0\}, \]

which will be utilized in some of the proofs.

From Lemma 3.28 it follows that

**Corollary 3.32.** There is a well-defined restriction map

\[ R_\tau : M^\circ_s(X, \hat{s}) \to C_c(\partial X, \hat{s}). \]

A direct consequence of Proposition 3.29 is

**Corollary 3.33 (dependence on twistings).** Given s and \( \tau \), there is an integral splitting \( s_\mathbb{Z} \) and a diffeomorphism

\[ F_{s, \tau} : M^\circ_s(X, \hat{s}) \to M^\circ_{s_\mathbb{Z}}(X, \hat{s}) \]

such that \( R \circ F_{s, \tau} = R_\tau \).

### 4 Properties of moduli spaces

In this section we prove that (Semi-infinite-dimensionality Theorem):

- the moduli spaces of solutions to the Seiberg-Witten equations on \( X \) are Hilbert manifolds,
- the restriction map to the boundary is “semi-infinite”, i.e., Fredholm in the negative direction and compact in the positive direction,
- if \( \partial X \) is disconnected, restriction to a single boundary component has dense differential.

This is done by analyzing the properties of the linearized Seiberg-Witten operator \( D_{SW} \). We start by investigating an extended version of this operator, \( \tilde{D}_{SW} \). The reason is that the standard Atiyah-Patodi-Singer boundary value problem as well as the elliptic theory developed in section 2 can be directly applied to the study of \( \tilde{D}_{SW} \). Our understanding of the gauge action (subsection 3.2) will allow us to transfer these properties to \( D_{SW} \).

#### 4.1 Extended linearized SW operator

Here we apply the Atiyah-Patodi-Singer boundary value problem to an extended version of the linearized Seiberg-Witten operator, \( \tilde{D}_{SW} \). The properties we prove are the direct analogues of the properties of \( D_{SW} \) which are proved in the next section.
Definition 4.1 (extended linearized SW operator). We define the extended linearized Seiberg-Witten operator

$$\tilde{D}_{SW}(A, \Phi) : L^2_{\Omega^1}(X) \oplus \Gamma(S^+_X)$$
$$\rightarrow L^2(X; i\mathbb{R}) \oplus L^2(i\Omega^+(X) \oplus \Gamma(S^-_X))$$

by adding a component related to the linearization of gauge action:

$$\tilde{D}_{SW}(A, \Phi)(a, \phi) = (d^*a, D_{SW}(A, \Phi)(a, \phi)).$$

In order to study the Atiyah-Patodi-Singer boundary value problem we need to introduce the appropriate operator on the boundary and consider its Calderón projector. Denote $Y = \partial X$ and define

$$\tilde{L} : i\Omega^1(Y) \oplus \Gamma(S_Y) \oplus i\Omega^0(Y) \rightarrow i\Omega^1(Y) \oplus \Gamma(S_Y) \oplus i\Omega^0(Y),$$
$$\tilde{L}(b, \psi, c) = (\star db - dc, D_0 \psi, -d^*b).$$

This is a first-order self-adjoint elliptic operator. Denote by $\tilde{H}^+(Y, s)$ (resp. $\tilde{H}^-(Y, s)$) the closure of the span of positive (resp. nonpositive) eigenspaces of $\tilde{L}$ in $L^2_{\Omega^1}(i\Omega^1(Y) \oplus \Gamma(S_Y) \oplus i\Omega^0(Y))$, and by $\tilde{\Pi}^\pm$ the projection onto $\tilde{H}^+(Y, s)$ along $\tilde{H}^-(Y, s)$. The proof of the following proposition follows a standard argument; we briefly recall it to set up the stage for the proofs in the rest of this section.

Proposition 4.2 (semi-infinite-dimensionality of $\tilde{D}_{SW}$).

The operator

$$\tilde{D}_{SW}(A, \Phi) \oplus \tilde{\Pi}^- R : L^2_{\Omega^1}(X) \oplus \Gamma(S^+_X)$$
$$\rightarrow L^2(X; i\mathbb{R}) \oplus L^2(i\Omega^+(X) \oplus \Gamma(S^-_X)) \oplus \tilde{H}^-(Y, s)$$

is Fredholm of index

$$2 \text{ind}_{\mathbb{C}} D_0 + b_1(X) - b_1^+(X) - b_1(Y) - 1. \quad (12)$$

Moreover, the positive part of the restriction map from the kernel of $\tilde{D}_{SW}$, $\tilde{\Pi}^+ R : \ker(\tilde{D}_{SW}(A, \Phi)) \rightarrow \tilde{H}^+(Y, s)$, is compact.

Proof. We can write $\tilde{D}_{SW}(A, \Phi) = \tilde{D} + \tilde{K}$ where

$$\tilde{D}(a, \phi) = (d^+a, D_0 \phi, d^*a)$$

and

$$\tilde{K}(a, \phi) = (0, -\rho^{-1}(\phi \Phi^* + \Phi^* a)_{\mathbb{C}}, \rho(a) \Phi + \rho(A^\delta) \phi).$$
As explained in [Kha, Proposition 3.1], applying the Atiyah-Patodi-Singer boundary value problem [KM07, Theorem 17.1.3] to $\tilde{D}$ proves that $\tilde{D} \oplus \tilde{\Pi}R$ is Fredholm with index equal to (12). Furthermore, [KM07, Theorem 17.1.3] implies that for any bounded sequence $(u_i) \subset L^2_1(i\Omega^1(X) \oplus \Gamma(S_X^+))$ such that $(\tilde{D}(u_i))$ is Cauchy, the sequence $(\tilde{\Pi}^+Ru_i)$ is precompact.

The operator $\tilde{K}$ is compact by Theorem A.1. Since $\tilde{D}\tilde{SW}_{(A,\Phi)} = \tilde{D} + \tilde{K}$, thus $\tilde{D}\tilde{SW}_{(A,\Phi)}$ is Fredholm with the same index as $\tilde{D}$. Moreover, since $\tilde{K}$ is compact, for any sequence $(u_i) \subset \ker(\tilde{D}\tilde{SW}_{(A,\Phi)})$ we can choose a subsequence such that the sequence of $\tilde{D}(u_i) = -\tilde{K}(u_i)$ is convergent, thus Cauchy. By what we proved in the previous paragraph, the sequence $(\tilde{\Pi}^+Ru_i)$ is precompact. This shows that $\tilde{\Pi}^+R : \ker(\tilde{D}\tilde{SW}_{(A,\Phi)}) \to \tilde{H}^+(Y,s)$ is compact. \hfill \textsquare

The proof of surjectivity utilizes both of the analytical results of section 2 (cf. [Lip08, Theorem 2]).

**Proposition 4.3 (surjectivity of $\tilde{D}\tilde{SW}$).** The operator $\tilde{D}\tilde{SW}_{(A,\Phi)}$ is surjective.

**Proof.** Assume $\tilde{D}\tilde{SW}_{(A,\Phi)}$ is not surjective. Proposition 4.2 implies its image is closed, so there is $0 \neq \tilde{v} \in \mathcal{V}(X,\hat{s}) \oplus L^2(i\Omega^0(X))$ orthogonal to $\text{im}\, \tilde{D}\tilde{SW}$. Recall that $\tilde{K}$ is a certain multiplication by $p = (A - A_0, \Phi) \in L^2_1(i\Omega^1(X) \oplus \Gamma(S_X^+))$. Let $X^* = X \cup ([0,\infty) \times Y)$ with cylindrical metric on the end, and extend the spinor bundle $S_X$ to $S_{X^*}$ which is cylindrical on ends. Extend $p$ to $p^* \in L^2_1(i\Omega^1(X^*) \oplus \Gamma(S_{X^*}^+))$ in an arbitrary way and $v$ to $\tilde{v}^* \in \mathcal{V}(X^*,\hat{s}) \oplus L^2(i\Omega^0(X^*))$ by zero on $[0,\infty) \times Y$. We have

$$\langle \tilde{D}^*\tilde{v}^*, (\tilde{D} + \tilde{K})\tilde{v}^* \rangle_{L^2_1(X^*)} = \langle \tilde{v}^*, \tilde{D}\tilde{SW}_{(A,\Phi)}(w|X) \rangle_{L^2_1(X)} = 0$$

for any $w \in \mathcal{T}\mathcal{C}(X^*,\hat{s})$. Therefore $\tilde{v}^*$ is a weak solution to $\tilde{D}^*\tilde{v}^* + \tilde{K}^*\tilde{v}^* = 0$ where $\tilde{D}^*, \tilde{K}^*$ are formal adjoints of $\tilde{D}, \tilde{K}$, respectively. The map $\tilde{K}^* : L^2_1 \rightarrow L^2$ is compact by Theorem A.1. Thus from the Regularity Theorem it follows that $\tilde{v}^* \in L^2_1(i\Omega^+(X^*) \oplus \Gamma(S_{X^*}^-) \oplus i\Omega^0(X^*))$ and it is a solution to $(\tilde{D}^* + \tilde{K}^*)\tilde{v}^* = 0$. Furthermore, Corollary 2.4 implies that $\tilde{v}^* = 0$ and therefore $v = 0$. Thus, by contradiction, we have proved that $D\tilde{SW}$ is surjective. \hfill \textsquare

Finally, we focus on the density of the restriction map from the kernel of $D\tilde{SW}$ to one boundary component. The proof of this proposition utilizes some of the ideas we have just seen (cf. [Lip08, Lemma 5]).
**Proposition 4.4 (density of moduli on one boundary component).** Assume the boundary $Y$ has at least two connected components and let $Y_0 \subset Y$ be any one of these components. Then the restriction

$$R : \ker \widetilde{DSW}_{(A,\Phi)} \rightarrow L^2_{1/2}(i\Omega^1(Y_0) \oplus \Gamma(S_{Y_0}))$$

is dense.

**Proof.** Assume, by contradiction, that it is not dense and choose a nonzero element $v \in L^2_{1/2}(i\Omega^1(Y_0) \oplus \Gamma(S_{Y_0}))$ which is $L^2_{1/2}$-perpendicular to its image. Since $\widetilde{DSW}_{(A,\Phi)}$ is surjective by Proposition 4.3, the map

$$\widetilde{DSW}_{(A,\Phi)} \oplus \Pi_v R : L^2_1(i\Omega^1(X) \oplus \Gamma(S_X^+)) \rightarrow L^2(X; i\mathbb{R}) \oplus L^2(i\Omega^+(X) \oplus \Gamma(S_X^-)) \oplus Cv$$

has finite-dimensional cokernel, where $\Pi_v$ is the $L^2_{1/2}$-projection onto $v \in L^2_{1/2}(i\Omega^1(Y_0) \oplus \Gamma(S_{Y_0}))$.

From the definition of $v$ it follows that $\widetilde{DSW}_{(A,\Phi)} \oplus \Pi_v R$ is not surjective, since otherwise there would be an element $w \in L^2_1(i\Omega^1(X) \oplus \Gamma(S_X^+))$ which solves $\widetilde{DSW}_{(A,\Phi)}(w) = 0$ such that $\Pi_v(R(w)) \neq 0$. Therefore, we can pick an element $(a, v)$ which is orthogonal to the image of $\widetilde{DSW}_{(A,\Phi)} \oplus \Pi_v R$, where $a \in L^2(X; i\mathbb{R}) \oplus L^2(i\Omega^+(X) \oplus \Gamma(S_X^-))$.

As in the proof of Proposition 4.3, attach a cylindrical end along $Y$ to get $X^* = X \cup [0, \infty) \times Y$. Extend $a$ to $a^*$ by 0 on $[0, \infty) \times Y$. Since $\widetilde{K}$ is a certain multiplication by $p = (A - A_0, \Phi) \in L^2_1(i\Omega^1(X) \oplus \Gamma(S_X^+))$, extend $p$ to $p^* \in L^2_1(i\Omega^1(X^*) \oplus \Gamma(S_X^-))$ in an arbitrary way to get $\widetilde{K}$ defined on $X^*$. By Theorem A.1, $\widetilde{K}$ is a compact operator $L^2_1 \rightarrow L^2$.

Since $(a, v) \perp \text{im}(\widetilde{DSW}_{(A,\Phi)} \oplus \Pi_v R)$, we get that $a^*$ is a weak solution to $(\widetilde{D}^* + \widetilde{K}^*)a^* = 0$ on the interior of $X^*_1 = X^* \setminus ([0, \infty) \times Y_0)$. Take any compact set $C \subset X^*_1$ and a smooth bump function $\eta : X^* \rightarrow [0, 1]$ such that $\eta|_C = 1$ and supp $\eta \subset X^*_1$. It follows that $(\widetilde{D}^* + \widetilde{K}^*)(\eta a) = \rho^*(d\eta)|a \in L^2(X^*)$. Therefore by the Regularity Theorem we get that $\eta a \in L^2(X^*)$, and in particular $a \in L^2_1(\widetilde{C})$. Varying $C$ we obtain $a^* \in L^2_1,_{\text{loc}}(X^*_1)$. Since $a^* \equiv 0$ on $[0, \infty) \times (Y \setminus Y_0) \subset X^*_1$, Corollary 2.4 implies that $a^* = 0$ on $X^*_1$, so $a = 0$ on $X$.

This is a contradiction since we can extend $v \in L^2_{1/2}(i\Omega^1(Y_0) \oplus \Gamma(S_{Y_0}))$ to $\tilde{v} \in L^2_1(i\Omega^1(X) \oplus \Gamma(S_X^+))$ such that $R(\tilde{v}) = v$. Then

$$0 = \langle \langle \widetilde{DSW}_{(A,\Phi)}(\tilde{v}), \Pi_v R\tilde{v}, (a, v) \rangle \rangle_{L^2_{1/2}}$$

$$= \langle \langle \widetilde{DSW}_{(A,\Phi)}(\tilde{v}), v, (0, v) \rangle \rangle_{L^2_{1/2}}$$

$$= 0 + \langle v, v \rangle_{L^2_{1/2}}$$

29
implying $\|v\|_{L^2_{1/2}} = 0$, which contradicts the assumption that $v \neq 0$. \hfill \Box

4.2 Regularity and semi-infinite-dimensionality of moduli spaces

We turn our focus to the operator $D_{SW}$. The main difficulty in transferring the results of subsection 4.1 from $\tilde{D}_{SW}$ to $D_{SW}$ is the presence of the Coulomb condition on both $X$ and $\partial X$. The split gauge condition introduces an additional twist to the story.

A key fact is that the differential of the gauge group action at $e$ (cf. Lemma 3.5) preserves the kernel of $D_{SW}$ at a solution $(A, \Phi)$.

Lemma 4.5. Assume $\hat{D}_A \Phi = 0$. Then for any $f \in L^2_2(i\Omega^0(X))$ we have $D_{SW}(A, \Phi)(df, -f\Phi) = 0$.

Proof. We compute:

$$(\hat{D} + \hat{K})(df, -f\Phi)$$

$$= (\rho^{-1}(f\Phi^* + \Phi(f\Phi)^*)_0, \rho(df)\Phi - \hat{D}_A_0(f\Phi) - \rho(A^\delta)f\Phi)$$

$$= (\rho^{-1}(f\Phi^* - \Phi f\Phi^*)_0, \rho(df)\Phi - \rho(df)\Phi - f(\hat{D}_A_0 \Phi + \rho(A^\delta)\Phi))$$

$$= (0, -f \hat{D}_A \Phi)$$

$$= (0, 0),$$

which finishes the proof. \hfill \Box

Following the idea of [Kha, Proposition 3.1], we deduce the semi-infinite-dimensionality and compute the index of $D_{SW}$ from Proposition 4.2. These methods will be utilized to prove further results in this section, too.

Proposition 4.6 (semi-infinite-dimensionality of $D_{SW}$). The operator

$$D_{(A, \Phi)}SW \oplus \Pi^- R : L^2_1(i\Omega^1(X) \oplus \Gamma(S^+_X))$$

$$\rightarrow L^2(i\Omega^+ + \Gamma(S^-_X)) \oplus H^{-}(Y, s)$$

is Fredholm of index

$$2 \text{ind}_C + b_1(X) - b^+(X) - b_1(Y).$$

Moreover, the restriction $\Pi^+ R : \ker(D_{(A, \Phi)}SW) \rightarrow H^+(Y, s)$ is compact, where $\ker(D_{(A, \Phi)}SW) \subset L^2_1(i\Omega^1(X) \oplus \Gamma(S^+_X))$. 

30
Proof. Firstly, we compare the respective polarizations. Recall that the decomposition of \( L^2_{1/2}(i\Omega^1(Y) \oplus \Gamma(S_Y)) \) into \( \tilde{H}^+(Y, s) \oplus \tilde{H}^-(Y, s) \) is given by the eigenspaces of \( \tilde{L} \).

Decomposing

\[
i\Omega^1(Y) \oplus \Gamma(S_Y) \oplus i\Omega^0(Y) = (i\Omega^1_C(Y) \oplus \Gamma(S_Y)) \oplus (\Omega^1_C(Y) \oplus i\Omega^0(Y))
\]

we see that \( \tilde{L} \) decomposes as \( \begin{pmatrix} *d & 0 & 0 \\ 0 & \hat{D}_{B_0} & -d^s \\ -d^s & 0 & 0 \end{pmatrix} \). Denote by \( \Pi^\pm \) the spectral projections of \( \begin{pmatrix} 0 & -d^s \\ -d^s & 0 \end{pmatrix} \) in \( L^2_{1/2}(i\Omega^1_C(Y) \oplus i\Omega^0(Y)) \). It follows that \( \tilde{\Pi}^\pm = \Pi^\pm \oplus \Pi^\pm \).

This is enough to prove the statement about compactness. Indeed, the map \( \Pi^+ R : \ker(D_{(A,\Phi)} SW) \to H^+(Y, s) \) is just the composition

\[
\ker(D_{(A,\Phi)} SW) \hookrightarrow \ker(D_{(A,\Phi)} SW) \xrightarrow{\tilde{D}SW_{(A,\Phi)}} \tilde{H}^+(Y, s) \xrightarrow{\Pi^+} H^+(Y, s)
\]

where the map in the middle is compact by Proposition 4.2.

For Fredholmness, denote by

\[
\Pi_C : L^2_{1/2}(\Omega^1_C(Y) \oplus \Omega^0(Y)) \to L^2_{1/2}(-\Omega^1_C(Y) \oplus \Omega^0(Y))
\]

the projection onto \( L^2_{1/2}(\Omega^1_C(Y)) \oplus \mathbb{R}_{\pi_0} \) (where \( \mathbb{R}_{\pi_0} \subset \Omega^0(Y) \) is the space of locally constant functions) along \( \{0\} \oplus L^2_{1/2}(\Omega^0_0(Y)) \) (where \( \Omega^0_0(Y) = \{ f \in \Omega^0(Y) | \forall i \int_Y f = 0 \} \)). This projection can be used to define the split Coulomb slice for the orthogonal splitting \( s_\perp \) (Definition 3.21). Precisely, we have

\[
L^2_{1}(i\Omega^1_{s_\perp} (X) \oplus \Gamma(S^+_X)) = \ker(d^s, \Pi_C).
\]

Khandhawit [Kha, Proposition 3.1] proves that \( \im \Pi^\perp \) and \( \ker \Pi_C \) are complementary, and then [KM07, Proposition 17.2.6] implies that

\[
\tilde{D}SW_{(A,\Phi)} \oplus (\Pi^\perp R) \oplus (\Pi_C R) : L^2_{1}(i\Omega^1(X) \oplus \Gamma(S^+_X)) \xrightarrow{(15)} L^2_{1}(i\Omega^1(X) \oplus \Gamma(S^+_X)) \oplus L^2_{1/2}(\Omega^0_0(Y)) \oplus H^-(Y, s)
\]

is Fredholm since (11) is Fredholm. Since \( \Pi_C|_{\im \Pi^\perp} \) is an isomorphism onto \( \im \Pi_C \), therefore the proof of [KM07, Proposition 17.2.6] implies that the index of (15) is the same as the index of (11).

The following lemma is a simple exercise in Fredholm theory.

**Lemma 4.7.** Let \( (F, G) : H \to A \oplus B \) be Fredholm. Then \( \tilde{F} = F|_{\ker G} : \tilde{H} = \ker G \to A \) is Fredholm and has index equal to \( \text{ind}((F, G)) + \dim \coker G \).
It implies that the operator $D_{(A,\Phi)}SW\oplus(\Pi^-R)$ is Fredholm as a map $\ker(\ast d\oplus(\Pi CR)) = L_1^2(i\Omega^1_- (X) \oplus \Gamma(S^+_X)) \rightarrow L^2(i\Omega^+ (X) \oplus \Gamma(S^-_X)) \oplus H^-(Y,\mathfrak{s})$ and has index

$$\text{ind}(\tilde{D}\text{SW}_{(A,\Phi)} \oplus (\Pi^- R) \oplus (\Pi CR)) + \dim \text{coker}(\ast d \oplus (\Pi CR))$$

$$= \left[ 2 \text{ind}_C \mathcal{D}_{\alpha}^+ - b_0(X) + b_1(Y) - b_1(Y) - b_1(Y) \right] + b_0(X)$$

$$= 2 \text{ind}_C \mathcal{D}_{\alpha}^+ + b_1(X) - b_1(Y) - b_1(Y).$$

Thus, we have proven the Proposition for a particular splitting, $s = s_\perp$.

For any splitting $s$, the inclusion $\Omega^1_0(X) \rightarrow \Omega^1_{CC}(X)$ is of codimension $\dim(H^1(X, Y; \mathbb{R})) - \dim(H^1(X; \mathbb{R})) = b_0(Y) - 1$. Therefore, the split double Coulomb slice $L_1^2(i\Omega^1_0 (X) \oplus \Gamma(S^+_X))$ is a finite-dimensional subspace of the “full” double Coulomb slice $L_2^2(i\Omega^1(X) \oplus \Gamma(S^+_X))$.

Therefore the Fredholmness of (13) for $s = s_\perp$ implies the Fredholmness of

$$D_{(A,\Phi)}SW\oplus\Pi^- R : L_1^2(i\Omega^1_0 (X) \oplus \Gamma(S^+_X)) \rightarrow L^2(i\Omega^+ (X) \oplus \Gamma(S^-_X)) \oplus H^-(Y,\mathfrak{s})$$

and this, in turn, implies the Fredholmness of (13) for any splitting $s$. Moreover, the index of (16) is equal to $b_0(Y) - 1$ plus (14) for any splitting $s$, finishing the proof. \qed

We turn to deducing the surjectivity of $D\text{SW}$ from the surjectivity of $\tilde{D}\text{SW}$.

**Proposition 4.8 (surjectivity of $D\text{SW}$).** For any gauge splitting $s$, the differential

$$D_{(A,\Phi)}SW : L_1^2(i\Omega^1_0 (X) \oplus \Gamma(S^+_X)) \rightarrow L^2(i\Omega^+ (X) \oplus \Gamma(S^-_X))$$

is surjective.

**Proof.** We will prove a stronger statement, that this extended differential together with the exact part of the restriction to the boundary

$$(\tilde{D}\text{SW}_{(A,\Phi)}, \Pi_d R, \Pi_{V_s}) = (D\text{SW}_{(A,\Phi)}, d^*, \Pi_d R, \Pi_{V_s}) :$$

$$L_1^2(i\Omega^1 (X) \oplus \Gamma(S^+_X)) \rightarrow L^2(i\Omega^+ (X) \oplus \Gamma(S^-_X)) \oplus L^2(i\Omega^0(X)) \oplus L^2_{1/2}(i\Omega^1_- (Y)) \oplus V_s$$

is surjective, where $\Pi_d : L_1^2(i\Omega^1 (Y)) \rightarrow L^2_{1/2}(i\Omega^1_- (Y))$ is the projection along $L^2_{1/2}(i\Omega^1_- CC(Y))$ and $\Pi_{V_s} : \Omega^1(X) \rightarrow V_s = (\text{im } s)^\perp$ is the orthogonal projection. The Proposition will follow since $L^2_{1/2}(i\Omega^1_- (X) \oplus \Gamma(S^+_X)) = \ker(d^*, \Pi_d R, \Pi_{V_s})$. Proposition 4.3 implies that $\tilde{D}\text{SW}_{(A,\Phi)} : L_1^2(i\Omega^1 (X) \oplus \Gamma(S^+_X)) \rightarrow L^2(i\Omega^+ (X) \oplus \Gamma(S^-_X))$ is surjective. To prove surjectivity of (17) it thus remains to prove that $\Pi_d R : \ker(\tilde{D}\text{SW}_{(A,\Phi)}) \rightarrow L^2_{1/2}(i\Omega^1_- (Y))$ is surjective and that $\Pi_{V_s} : \ker(\tilde{D}\text{SW}_{(A,\Phi)}, \Pi_d R) \rightarrow V_s$ is surjective.

32
We prove that $\Pi_d R \big|_{\ker \overline{D\Sigma W}_{(A, \Phi)}}$ is surjective. Take any $g \in L^2_{1/2}(i\Omega^0(Y))$ representing a given element $dg \in L^2_{1/2}(i\Omega^1_C(Y))$. Take the unique $f \in L^2_{1/2}(i\Omega^0(X))$ such that $\Delta f = 0$ and $f|_Y = g$. Then $\Pi_d R(df, -f \Phi) = df|_Y = dg$. The required surjectivity follows since $(df, -f \Phi) \in \ker D\Sigma W_{(A, \Phi)}$, which follows from $d^* df = \Delta f = 0$ and Lemma 4.5.

Similarly we prove $\Pi_c \big|_{\ker \overline{D\Sigma W}_{(A, \Phi)}, \Pi_d R}$ is surjective. By (8) and definition of $V_s$, the orthogonal projection $d(\mathcal{H}(X)) \rightarrow V_s$ is an isomorphism and therefore for any $v \in V_s$ there is $f \in \mathcal{H}(X)$ such that $\Pi_c (df) = v$. Moreover $\overline{D\Sigma W}(df, -f \Phi) = 0$ and $\Pi_d R(df, -f \Phi) = 0$, as wished. \hfill \Box

Finally, we prove the density of the restriction map to a connected component of $Y$.

**Proposition 4.9 (density of moduli on one boundary component).** The restriction

$$R : \ker (D_{(A, \Phi)} \Sigma W) \rightarrow L^2_{1/2}(i\Omega^1_C(Y_0) \oplus \Gamma(S_{Y_0}))$$

to a connected component $Y_0 \subset Y$ is dense, where we consider $\ker (D_{(A, \Phi)} \Sigma W) \subset L^2_{1/2}(i\Omega^1(X) \oplus \Gamma(S^+_X))$.

**Proof.** Take any $(b, \psi) \in L^2_{1/2}(i\Omega^1_C(Y_0) \oplus \Gamma(S_{Y_0})) \subset L^2_{1/2}(i\Omega^1(Y_0) \oplus \Gamma(S_{Y_0}))$. It follows from Proposition 4.4 that we can take a sequence $(\tilde{a}_k, \phi_k) \in \ker \overline{D\Sigma W}_{(A, \Phi)}$ such that $R_{Y_0}(\tilde{a}_k, \phi_k) \rightarrow (b, \psi)$ in $L^2_{1/2}(i\Omega^1_C(Y_0) \oplus \Gamma(S_{Y_0}))$. Using the decomposition (9) we can write $\tilde{a}_k = a_k + df_k$ for $f_k \in i\Omega^0(X)$ and $a_k \in \Omega^1(X)$. Since $Y_0$ is connected, we can change $f_k$ by a constant to obtain $\int_{Y_0} f_k = 0$. Decomposing $R_{Y_0}(\tilde{a}_k) = R_{Y_0}(a_k) + R_{Y_0}(df_k)$ we get that $R_{Y_0}(a_k) \rightarrow b$ and $R_{Y_0}(df_k) \rightarrow 0$. This together with $\int_{Y_0} f_k = 0$ implies that $f_k|_{Y_0} \rightarrow 0$ and therefore

$$(b, \psi) = \lim_{k \rightarrow \infty} R_{Y_0}(\tilde{a}_k, \phi_k) = \lim_{k \rightarrow \infty} R_{Y_0}(a_k, \phi_k + if_k \Phi)$$

which finishes the proof because $a_k \in L^2_{1/2}(i\Omega^1(X))$ and $D \Sigma W(a_k, \phi_k + if_k \Phi) = D \Sigma W(a_k + idf_k, \phi_k) = 0$ by Lemma 4.5. \hfill \Box

The results of Proposition 4.6, Proposition 4.8 and Proposition 4.9 can be summarized as follows.

**Semi-infinite-dimensionality Theorem.** The moduli spaces $\mathcal{M}_c^\circ(X, \hat{s})$ are Hilbert manifolds. The differential of the twisted restriction map $R_\tau : \mathcal{M}_c^\circ(X, \hat{s}) \rightarrow C_c(\partial X, \hat{s})$ decomposes into $\Pi^- DR_\tau : TM^c_\circ(X, \hat{s}) \rightarrow H^-(\partial X, \hat{s})$ which is Fredholm and $\Pi^+ DR_\tau : TM^c_\circ(X, \hat{s}) \rightarrow H^+(\partial X, \hat{s})$ which is compact.

Moreover, if $b_0(\partial X) > 1$, then for any connected component $Y_0 \subset \partial X$ the restriction $R_{\tau, Y_0}$ to $Y_0$ has dense differential.
5 Gluing along a boundary component

This section is devoted to the proof of the main result of this article, the Gluing Theorem, which relates the moduli spaces of solutions on $X_1$, $X_2$ and $X = X_1 \cup Y \cup X_2$, where $Y$ is a rational homology sphere, oriented as a component of the boundary of $X_1$. Under the identification $C_{cc}(Y, s) \simeq C_{cc}(-Y, \mathfrak{s})$ there are $S^1$-equivariant twisted restriction maps $R_\tau, Y : \mathcal{M}_{s_1}^C(X_1, \mathfrak{s}_1) \to C_{cc}(Y, \mathfrak{s})$, where $\mathfrak{s} = \mathfrak{s}|_{Y}$. One can expect the fiber product

$$\mathcal{M}_{s_1}^C(X_1, \mathfrak{s}_1) \times_Y \mathcal{M}_{s_2}^C(X_2, \mathfrak{s}_2) = \{(A_1, \Phi_1, A_2, \Phi_2) \in \mathcal{M}_{s_1}^C(X_1, \mathfrak{s}_1) \times \mathcal{M}_{s_2}^C(X_2, \mathfrak{s}_2) | R_{\tau_1, Y}(A_1, \Phi_1) = R_{\tau_2, Y}(A_2, \Phi_2)\}$$

to be diffeomorphic to $\mathcal{M}_{s}^C(X, \mathfrak{s})$, and this turns out to be true. One would also like to have this map intertwine the twisted restriction maps to $\partial X$, but this is a bit too much: the splittings and twistings $(s, \tau), (s_1, \tau_1), (s_2, \tau_2)$, need to enjoy certain compatibility, and even then the restriction maps may not match on the nose but need to be homotoped to each other. This reflects the fact that we did not quotient by the action of $S^1$ on the configuration spaces.

The proof utilizes the following fact which is of independent interest. Let $X' \subset \mathcal{X}$ be a submanifold, the closure of which is contained in the interior $\mathcal{X}$. Then the restriction map from $\mathcal{M}_{cc}^C(X, \mathfrak{s})$ to $L_k^2$-configurations on $X'$ is well-defined and smooth for any $k \geq 0$. Well-definedness follows from a standard argument, but proving the smoothness of this map turns out to be a surprisingly delicate task which we tackle in subsection 5.1.

The same strategy should work to prove smoothness of restriction maps to interior submanifolds for other types of moduli spaces appearing in gauge theory, e.g., for the space of anti-self-dual connections on $G \hookrightarrow P \to X$. The key is the ellipticity of the equations together with the gauge fixing.

5.1 Smoothness of restrictions

Assume $X' \subset \mathcal{X}$ is a submanifold with closure contained in $\mathcal{X}$. The goal is to show (cf. Theorem 5.2) that the restriction map $R : \mathcal{M}_{cc}^C(X, \mathfrak{s}) \to \mathcal{C}_k(X', \mathfrak{s}|_{X'})$ is smooth for any $k$, where $\mathcal{C}_k(X', \mathfrak{s}|_{X'}) = (A_0|_{X'}, 0) + L_k^2(i\Omega^1(X') \oplus \Gamma(S^+_X))$. We restrict ourselves to the case $k = 2$, but the same strategy may be used iteratively, bootstrapping the result to any $k$, if needed.

Due to Theorem A.2 we may assume, without loss of generality, that $X'$ is of codimension 0. Since we require the closure of $X'$ to be contained in the interior $\mathcal{X}$, we may as well assume that $X'$ is a closed submanifold.
The following fundamental fact shows that any element in the image of the restriction map is itself a smooth configuration.

**Lemma 5.1 (interior smoothness of solutions).** [KM07, Lemma 5.1.5] Every \( \gamma \in \tilde{\mathcal{M}}_{CC}(X, \hat{s}) \) is smooth on \( X \).

We are ready to prove the main theorem of this subsection. Note that the surjectivity assumption is satisfied whenever \( \partial X \neq \emptyset \) due to Proposition 4.8.

**Theorem 5.2 (restriction is smooth on solution sets).** Assume that for any \((A, \Phi) \in \tilde{\mathcal{M}}_{CC}(X, \hat{s})\) the operator \( DS\overline{W}(A, \Phi) \) is surjective. Then the restriction map \( R : \tilde{\mathcal{M}}_{CC}(X, \hat{s}) \to \mathcal{C}_k(X' | X') \) is smooth.

**Proof.** Choose a compact codimension-0 submanifold \( X'' \subset X \) such that \( X' \subset X'' \). Let us introduce an intermediate space \( L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X)) \) defined as the completion of \( i\Omega^1_{CC}(X) \oplus \Gamma(S_X) \) with respect to the norm \( \|v\|_{2,X''} = \sqrt{\|\tilde{D}v\|_{L^2_2(X'')}^2 + \|v\|_{L^2_2(X)}^2} \), so that \( L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X)) \) is a Hilbert space. Define

\[
\tilde{\mathcal{M}}_{CC,X''}(X, \hat{s}) = \tilde{\mathcal{M}}_{CC}(X, \hat{s}) \cap ((A_0, 0) + L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X))),
\]

the set of solutions to the Seiberg-Witten equations in the corresponding configuration space. Since \( \tilde{D} \) is elliptic and \( \tilde{D}v = (\tilde{D}v, 0) \), by Theorem A.3 the restriction map

\[
L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X)) \to L^2_{2}(i\Omega^1(X') \oplus \Gamma(S_X))
\]

is continuous linear (thus smooth). It thus suffices to prove that the moduli space \( \tilde{\mathcal{M}}_{CC,X''}(X, \hat{s}) \) is a smooth submanifold of the configuration space \( (A_0, 0) + L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X)) \) and that the identity map \( \text{Id}_2 : \tilde{\mathcal{M}}_{CC}(X, \hat{s}) \to \tilde{\mathcal{M}}_{CC,X''}(X, \hat{s}) \) is well-defined, continuous and smooth.

Firstly, it is well-defined by Lemma 5.1.

Secondly, we prove that \( \tilde{\mathcal{M}}_{CC,X''}(X, \hat{s}) \) is a smooth submanifold of \( (A_0, 0) + L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X)) \). Define \( L^2_{1,X''}(i\Omega^+X \oplus \Gamma(S_X)) \) to be the Hilbert space obtained as the completion of \( i\Omega^+(X) \oplus \Gamma(S_X) \) with respect to the norm \( \|v\|_{L^2_2(X) \cup L^2_2(X')} = \sqrt{\|v\|_{L^2_2(X)}^2 + \|v\|_{L^2_2(X')}^2} \).

By the Implicit Function Theorem, it suffices to prove the following Lemma.

**Lemma 5.3.** The differential

\[
DS\overline{W}(A, \Phi) : L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X)) \to L^2_{1,X''}(i\Omega^+(X) \oplus \Gamma(S_X))
\]

is surjective at each \((A, \Phi) \in \tilde{\mathcal{M}}_{CC,X''}(X, \hat{s})\).
Proof. We assumed that the operator $D\text{SW}_{(A,\Phi)}$ is surjective. Choose any $w \in L^2_{1,X''}(i\Omega^+(X) \oplus \Gamma(S^+_X))$ and pick a solution $v \in L^2_1(i\Omega^1_{CC}(X) \oplus \Gamma(S^+_X))$ to $D\text{SW}_{(A,\Phi)}(v) = w$. Then $\hat{D}v|_{X''} = -\hat{K}v|_{X''} + w|_{X''}$ is in $L^2_1(X'')$ since $w|_{X''} \in L^2_1(X'')$, $v|_{X''} \in L^2_1(X'')$ and $\hat{K}|_{X''}$ is a certain multiplication by $(A - A_0, \Phi)|_{X''}$, which is smooth by Lemma 5.1. 

Finally, it remains to prove that the identity map
\[
\tilde{\mathcal{M}}^0_{CC}(X, \hat{s}) \to \tilde{\mathcal{M}}^0_{CC,X''}(X, \hat{s})
\]
is smooth. We start by identifying the tangent spaces at $p = (A, \Phi)$; we have
\[
T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s}) = \ker D\text{SW}_p,
\]
\[
T_p\tilde{\mathcal{M}}^0_{CC,X''}(X, \hat{s}) = (\ker D\text{SW}_p) \cap L^2_{2,X''}(i\Omega^1_{CC}(X) \oplus \Gamma(S_X)).
\]

**Lemma 5.4 (isometry of the tangent spaces).** The identity map $T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s}) \to T_p\tilde{\mathcal{M}}^0_{CC,X''}(X, \hat{s})$ is well-defined, and an isometry.

**Proof.** Take any $v \in T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s})$; in particular, $D\text{SW}_{(A,\Phi)}(v) = 0$. Well-definedness follows since $\hat{D}v|_{X''} = -\hat{K}v|_{X''}$ and as before, $\hat{K}|_{X''}$ is a multiplication by a smooth configuration on $X''$. This also implies
\[
\|v\|^2_{L^2_1(X)} \leq \|v\|^2_{2,X''} = \|v\|^2_{L^2_1(X)} + \|\hat{D}v\|^2_{L^2_1(X'')}
\]
\[
= \|v\|^2_{L^2_1(X)} + \|\hat{K}v\|^2_{L^2_1(X'')}
\]
\[
\leq \|v\|^2_{L^2_1(X)} + C_p\|v\|^2_{L^2_1(X'')}
\]
\[
\leq (1 + C_p)\|v\|^2_{L^2_1(X)}
\]
which proves this map is an isometry. 

While the $L^2(X)$ norm is not complete on $L^2_1(i\Omega^1_{CC}(X) \oplus \Gamma(S^+_X))$, the $L^2(X)$-orthogonal complement $H$ to $T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s})$ is a closed subspace of $L^2_1(i\Omega^1_{CC}(X) \oplus \Gamma(S^+_X))$ such that $H + T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s}) = L^2_1(i\Omega^1_{CC}(X) \oplus \Gamma(S^+_X))$ and $H \cap T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s}) = \{0\}$, thus by the open mapping theorem
\[
L^2_1(i\Omega^1_{CC}(X) \oplus \Gamma(S^+_X)) = T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s}) \oplus H.
\]
By the Implicit Function Theorem there is a neighborhood $U$ of $p$ such that the affine projection
\[
U \cap \tilde{\mathcal{M}}^0_{CC}(X, \hat{s}) \to U \cap \left((A_0, 0) + T_p\tilde{\mathcal{M}}^0_{CC}(X, \hat{s})\right)
\]
along $H$ is a diffeomorphism. Similarly, the Implicit Function Theorem implies that there is a neighborhood $V$ of $p$ such that the affine projection

$$U \cap \hat{\mathcal{M}}^o_{CC,X'}(X,\hat{s}) \rightarrow U \cap \left( (A_0,0) + T_p\hat{\mathcal{M}}^o_{CC,X'}(X,\hat{s}) \right)$$

along $H' = H \cap L^2_{2,X'}(\Omega^1_{CC}(X) \oplus \Gamma(S_X))$ is a diffeomorphism since $H'$ is the $L^2(X)$-orthogonal complement to $T_p\hat{\mathcal{M}}^o_{CC,X'}(X,\hat{s})$. Thus the identity map $\hat{\mathcal{M}}^o_{CC}(X,\hat{s}) \rightarrow \hat{\mathcal{M}}^o_{CC,X'}(X,\hat{s})$ near $p$ factors as

$$U \cap \hat{\mathcal{M}}^o_{CC}(X,\hat{s}) \rightarrow U \cap \left( (A_0,0) + T_p\hat{\mathcal{M}}^o_{CC}(X,\hat{s}) \right) \xrightarrow{id} V \cap \left( (A_0,0) + T_p\hat{\mathcal{M}}^o_{CC,X'}(X,\hat{s}) \right) \rightarrow V \cap \hat{\mathcal{M}}^o_{CC,X'}(X,\hat{s})$$

where the middle identity map is smooth by Lemma 5.4 and the two other maps are smooth since they are parametrizations coming from the Implicit Function Theorem, as we just showed. \qed

### 5.2 Proof of the gluing theorem

We are ready to prove the gluing theorem.

Let $X = X_1 \cup_Y X_2$ with $Y$ connected and $b_1(\partial X_i) = 0$. Let $\hat{s}$ be a spin$^c$ structure and $A_0$ be a reference spin$^c$ connection on $X$. Let restrictions of $\hat{s}$ be the spin$^c$ structures used to define configuration spaces and the restrictions of $A_0$ to be the reference connections. Denote $Y_1 = \partial X_1 \setminus Y$, $Y_2 = \partial X_2 \setminus Y$, $\hat{s}_i = \hat{s}|_{X_i}$. Fix gauge splittings $s, s_1, s_2$ and twistings $\tau, \tau_1, \tau_2$ on $X, X_1, X_2$.

Denote by $s_Z, s_{1,Z}, s_{2,Z}$ the associated integral splittings given by Proposition 3.29. We say they are compatible if $s_Z$ corresponds to $(s_{1,Z}, s_{2,Z})$ under the following identification.

**Proposition 5.5 (integral splittings on a composite cobordism).** There is a canonical identification between the set of integral splittings on $X$ and the set of pairs of integral splittings on $X_1$ and $X_2$.

**Proof.** Choose an integral splitting $s$. Take any $a \in \mathcal{H}_D^1(X)$. Denote by $\tilde{a}_i$ its restriction to $X_i$. For each $i$ there is a unique $f_i \in L^2_{2}(|0^1(X_i)|)$ such that $\Delta f_i = 0$, $f_i|_Y = G_du^*_r(a)$, and $f_i|_{Y_i} = 0$. The resulting $a_i = \tilde{a}_i - df_i$ is in $\mathcal{H}_D^1(X_i)$, thus we get a map

$$R_H : \mathcal{H}_D^1(X) \rightarrow \mathcal{H}_D^1(X_1) \times \mathcal{H}_D^1(X_2)$$

sending $a$ to $(a_1, a_2)$. Note that $\mathcal{H}_D^1(X_i) \subset \Omega^1_{CC}(X_i)$ and therefore $R_H$ coincides with doing the gauge fixing of Lemma 3.16 on both components, i.e., $a \mapsto (\Pi_{CC}(a|_{X_1}), \Pi_{CC}(a|_{X_2}))$.
The cohomology class \([a] \in H^1(X; \mathbb{R})\) restricts to \((\tilde{a}_1, \tilde{a}_2) = ([a_1], [a_2]) \in H^1(X_1; \mathbb{R}) \times H^1(X_2; \mathbb{R})\). It follows that the composition

\[
H^1(X_1; \mathbb{R}) \times H^1(X_2; \mathbb{R}) \xrightarrow{\iota_*} H^1(X; \mathbb{R}) \xrightarrow{s^H} \mathcal{H}_D(X) \to \mathcal{H}_D(X_1) \times \mathcal{H}_D(X_2) \to H^1(X_1; \mathbb{R}) \times H^1(X_2; \mathbb{R})
\]

is the identity, where \(\iota_*\) is the inverse of the restriction map \(H^1(X; \mathbb{R}) \to H^1(X_1; \mathbb{R}) \times H^1(X_2; \mathbb{R})\) (invertible by Mayer-Vietoris). We can thus choose \((s^H_1, s^H_2)\) to be the composition \(R_H \circ s^H \circ \iota_*\). Since \(s^H\) was integral, thus \(s^H_i\) are integral and by Proposition 3.23 there exist unique integral splittings \(s_1, s_2\) inducing \(s^H_1, s^H_2\).

On the other hand, given integral splittings \(s_1, s_2\) on \(X_1, X_2\), we can choose \(s^H\) to be \((R_H)^{-1} \circ (s^H_1, s^H_2) \circ (\iota_*)^{-1}\) and by Proposition 3.23 there is a unique integral splitting \(s\) inducing \(s^H\).

**Proposition 4.8** guarantees that whenever \(\partial X \neq \emptyset\), the moduli \(\mathcal{M}_0(X, \hat{s})\) is a smooth Hilbert manifold, and the same follows for \(\mathcal{M}^c_0(X_i, \hat{s})\). We want to include the case when \(X\) is a closed manifold, when it is well-known that to achieve surjectivity one, in general, needs to perturb the metric on \(X\) or perturb the Seiberg-Witten equations. Therefore, we assume that for any \((A, \Phi) \in \text{SW}^{-1}(0)\) the operator \(D \text{SW}_{(A, \Phi)}\) is onto.

**Remark 5.6.** The careful reader may notice that we do not assume any transversality of the maps \(R_{\tau_1, Y}\) and thus the fiber product may not \textit{a priori} be a manifold. That it is a manifold follows from the proof of the theorem. What is not proven here, but may be useful in other contexts, is that the transversality is indeed equivalent to \(D \text{SW}_{(A, \Phi)}\) being surjective for all \((A, \Phi) \in \mathcal{M}_0(X, \hat{s})\).

**Gluing Theorem.** Assume \(s_Z\) and \((s_{1,Z}, s_{2,Z})\) are compatible. Then there is an \(S^1\)-equivariant diffeomorphism \(F : \mathcal{M}_0^c(X, \hat{s}) \to \mathcal{M}_0^c(X, \hat{s}_1) \times_Y \mathcal{M}_0^c(X_2, \hat{s}_2)\) such that \(R_\tau\) is \(S^1\)-equivariantly homotopic to \((R_{\tau_1} \times_Y R_{\tau_2}) \circ F\).

**Proof.** By Corollary 3.33 we can assume, without loss of generality, that \(s = s_Z\) and \(s_i = s_{i,Z}\), as well as \(\tau \equiv 1\) and \(\tau_i \equiv 1\). We thus drop \(\tau\)'s from the notation entirely.

The plan is as follows. We will construct a map

\[
F : \mathcal{M}_0^c(X, \hat{s}) \to \mathcal{M}_0^c(X_1, \hat{s}_1) \times \mathcal{M}_0^c(X_2, \hat{s}_2),
\]

and a homotopy

\[
H : \mathcal{M}_0^c(X, \hat{s}) \to (\mathcal{C}_c(Y_1, \hat{s})) \times (\mathcal{C}_c(Y_2, \hat{s}))
\]

between \(R\) and \((R_{Y_1} \times R_{Y}) \circ F\). Then we will prove that \(F\) intertwines the actions of \(G_{s}^{h, c}(X)\) and \(G_{s_1}^{h, c}(X_1) \times G_{s_2}^{h, c}(X_2)\), and that \(H\) is \(G_{s}^{h, c}(X)\)-invariant; thus, both \(F\) and \(H\)
descend to $\mathcal{M}_s^0(X, \hat{s})$. Furthermore, we will prove $F$ and $H$ are continuous and smooth, and that $F$ has image in the fiber product. Finally, we will show that $F$ is a smooth immersion onto $\bar{\mathcal{M}}_s^0(X_1, \hat{s}_1) \times_Y \bar{\mathcal{M}}_s^0(X_2, \hat{s}_2)$. The $S^1$-equivariance of $F$ and $H$ will be apparent from the construction.

**Step 1.** We start by constructing $F$, proving its smoothness and that its image lies in the fiber product. Take $(A, \Phi) \in \mathcal{M}_s^0(X, \hat{s})$. We would like to simply restrict it to the components $X_i$ and then put into split Coulomb slice. By Lemma 3.16 modulo $S^1$ there a unique way of doing that using a contractible gauge transformation. Here we make a different choice than in Lemma 3.16, requiring $\int_Y f = 0$ instead of $\int_X f = 0$. Precisely, for $a \in L_1^2(i\Omega^1(X_i))$ choose $\tilde{u}_a^s = e^{f_a}$ such that $f_a \in L_2^2(i\Omega^0(X_i))$, $\int_Y f_a = 0$ and $a - (\tilde{u}_a^s)^{-1} \partial \tilde{u}_a^s \in L_1^2(i\Omega^1(X_i))$. Define

$$F(A, \Phi) = \left( \tilde{u}_{(A-A_0)|X_1}^s (A, \Phi)|_{X_1}, \tilde{u}_{(A-A_0)|X_2}^s (A, \Phi)|_{X_2} \right).$$

Since $A-A_0$ was in double Coulomb slice, thus $(A-A_0)|X_1$ (resp. $(A-A_0)|X_2$) is already coclosed on $X_1$ (resp. $X_2$) and on $Y_1$ (resp. $Y_2$), moreover $(A-A_0)|X_1$ and $(A-A_0)|X_2$ agree on $Y$. Denoting the restriction to $Y$ by $b_{(A, \Phi)} = \iota_Y^*(A-A_0)$, Theorem 5.2 together with Theorem A.2 imply that $b_{(A, \Phi)}$ is smooth and depends smoothly on $(A, \Phi) \in \mathcal{M}_s^0(X, \hat{s})$ as an element of $L_3^2/2$. Notice that $f_i = f_{(A-A_0)|X_i}$ can be decomposed as $f_i = f_i^0 + f_i^s$, where $f_i^0$ are the unique solutions to

$$f_i^0|_{Y_i} = 0, \quad f_i|_{Y} = g, \quad \Delta f_i = 0, \quad (18)$$

where $g = G_{\partial} \Pi_{\partial} b_{(A, \Phi)}$ depends smoothly on $(A, \Phi)$ as an element of $L_2^2/2$. Moreover, the map $g \mapsto f_i^0$ is linear and continuous as a map $L^2_{s+1/2} \rightarrow L^2_{s+1}$ for $s \geq 0$, so $f_i^0$ depend smoothly on $(A, \Phi)$ as elements of $L_2^2$. Furthermore, since $a|_{X_i} - df_i^0 \in L_2^2(i\Omega^1_{CC}(X_i))$, $f_i^s$ are the unique elements of $\mathcal{H}_{1/2}^2(X_i)$ such that $a|_{X_i} - df_i^0 - df_i^s \in L_2^2(i\Omega^1_{CC}(X_i))$ and $f_i|_{Y_i} = 0$. By Remark 3.17, $f_i^s$ depend continuously on $a|_{X_i} - df_i^0 \in L_2^2$. Which proves that $f_i \in L_3^2$ depend continuously and linearly on $g \in L_5^2/2$, thus depend smoothly on $g$, so they depend smoothly on $(A, \Phi)$.

This establishes the well-definedness and smoothness of $F$. That its image lies in the fiber product follows directly from the construction.

**Step 2.** We proceed to constructing $H$. By (18), the functions $f_i$ are locally constant on $Y_i$. We also have

$$R_{Y_i}(F(A, \Phi)) = e^{f_i|_{Y_i}} \cdot R_{Y_i}(A, \Phi)$$

39
by the construction of $F$. Thus we can define

$$H(A, \Phi, t) = \left( e^{t\tilde{f}_1}|_{Y_1}, e^{t\tilde{f}_2}|_{Y_2} \right) R(A, \Phi)$$

which at $t = 0$ coincides with $R(A, \Phi)$ and at $t = 1$ coincides with $(R_{Y_1}, R_{Y_2}) \circ F(A, \Phi)$.

**Step 3.** We now investigate the equivariance of $F$ under the actions of gauge groups. Let $u \in \mathcal{G}^{h,\alpha}_s(X)$. From Lemma 3.16 it follows that there is exactly one contractible \( \bar{u}_i = e^{\tilde{f}_i} \in \mathcal{G}^c(X_i) \) which puts $-u^{-1}du$ into $L^2_i(i\Omega^1_i(X_i))$ with $f_{Y_i} \tilde{f}_i = 0$. Equivalently, this is the unique \( u_i = e^{\tilde{f}_i} \in \mathcal{G}^c(X_i) \) with $f_{Y_i} \tilde{f}_i = 0$ such that $\bar{u}_i u_i|_{X_i} \in \mathcal{G}^{h,\alpha}_s(X_i)$. Define $u_i = \tilde{u}_i u_i|_{X_i}$. Since $\bar{u}_i$ is contractible, thus $[i_{X_i}^* (u^{-1}du)] = [u_i^{-1}du_i]$ in $H^1(X_i; 2\pi i \mathbb{Z})$. Therefore the map $u \mapsto (u_1, u_2)$ provides the canonical isomorphism

$$\mathcal{G}^{h,\alpha}_s(X) \simeq \mathcal{G}^{h,\alpha}_s(X_1) \times \mathcal{G}^{h,\alpha}_s(X_2) \quad (19)$$

which agrees with the isomorphism coming from the Mayer-Vietoris sequence $H^1(X; \mathbb{Z}) \simeq H^1(X_1; \mathbb{Z}) \times H^1(X_2; \mathbb{Z})$. From the construction of $F(A, \Phi)$ it follows that $(u|_{X_1}^{-1}, u|_{X_2}^{-1})F(u(A, \Phi))$ differs from $F(A, \Phi)$ exactly by the factor of $(e^{\tilde{f}_1}, e^{\tilde{f}_2})$. Thus $(u_1^{-1}, u_2^{-1})F(u(A, \Phi)) = F(A, \Phi)$. This proves that $F$ commutes with the gauge group action as identified in (19).

**Step 4.** We prove the invariance of $H$ under $\mathcal{G}^{h,\alpha}_s(X)$. By the construction of $H$, and since $u|_{\partial X} = 1$ for $u \in \mathcal{G}^{h,\alpha}_s(X)$, we have $H(u(A, \Phi), t) = (e^{\tilde{f}_1}|_{Y_1}, e^{\tilde{f}_2}|_{Y_2})H(A, \Phi, t)$ where $\tilde{f}_i$ are as in the previous paragraph. Since $R_H(\mathcal{H}^H_{\mathcal{D}}(X)) \in (\text{im} s_1^H) \times (\text{im} s_2^H)$ we get that $R_H(u^{-1}du)$ is already in the split Coulomb slice on $X_1$ and $X_2$. Moreover, $a_i = i_{X_i}^*(u^{-1}du)$ is coclosed on $X_i$ and $Y_i$. Since there are functions $\hat{f}_i$ satisfying

$$\Delta \hat{f}_1 = 0, \quad \hat{f}_1|_{Y_1} = 0, \quad \hat{f}_1|_{Y} = G_{a_1}i^* a, $$

$$\Delta \hat{f}_2 = 0, \quad \hat{f}_2|_{Y_2} = 0, \quad \hat{f}_2|_{Y} = G_{a_2}i^* a, $$

the uniqueness of $\tilde{f}_i$ implies $\tilde{f}_i = \hat{f}_i$ and therefore $\tilde{f}_i|_{Y_i} = 0$. Thus $H(u(A, \Phi), t) = H(A, \Phi, t)$, as wished.

**Step 5.** We show that $F$ is bijective onto the fiber product, following the argument in [Lip08]. Let $(A_i, \Phi_i) \in \tilde{\mathcal{M}}^\alpha(A_i, \tilde{s})$ such that $R_Y(A_1, \Phi_1) = R_Y(A_2, \Phi_2)$. These would give a configuration on $X$ if the normal components of connections $A_1$ and $A_2$ agreed on $Y$. Let $h_1 dt$ and $h_2 dt$ be the $dt$-components of $(A_1 - A_0)|_Y$ and $(A_2 - A_0)|_Y$. We want to find harmonic functions $f_i \in L^2_2(X_i; i \mathbb{R})$ such that

$$f_1|_Y = f_2|_Y,$$

$$\partial_t f_1|_Y + h_1 = \partial_t f_2|_Y + h_2,$$
\[ f_1|_{Y_1} = 0, \ f_2|_{Y_2} = 0. \]

Take a tubular neighborhood \([-\varepsilon, \varepsilon] \times Y \subset X\) of \(Y\). Let \(\{\phi_\lambda\}\) be an eigenbasis for \(\Delta_Y\) and write \(h_2 - h_1 = \sum \lambda \phi_\lambda\). Since \(h_2 - h_1 \in L^2_{1/2}\), therefore \(\sum \lambda \lambda^{1/2}|c_\lambda|^2 < \infty\) and thus the following are well-defined as elements of \(L^2_2([[-\varepsilon, \varepsilon] \times Y; i\mathbb{R})]):

\[
\begin{align*}
g_1 &= \frac{1}{2} \sum c_\lambda \lambda^{-1/2} e^{\lambda^{1/2}i} \varphi_\lambda, \\
g_2 &= \frac{1}{2} \sum c_\lambda \lambda^{-1/2} e^{-\lambda^{1/2}i} \varphi_\lambda,
\end{align*}
\]

which satisfies \(\partial_t g_1|_Y - \partial_t g_2|_Y = h_1 - h_0\). Let \(\rho \in C^\infty(X; \mathbb{R})\) be a bump function supported in \([-\varepsilon, \varepsilon] \times Y\) which is identically 1 in a neighborhood of \(Y\). The configurations \(e^{\rho g_1}(A_1, \Phi_1)\) and \(e^{\rho g_2}(A_2, \Phi_2)\) patch to give a \(L^2_1\) configuration \((A', \Phi')\), but this is not necessarily in the Coulomb slice because \(\rho f^l\) are not necessarily harmonic. Take \(f \in L^2_2(X; i\mathbb{R})\) such that

\[ f|_{Y_1} = 0, \ f|_{Y_2} = 0, \ \Delta f = -\Delta(\rho g_1) - \Delta(\rho g_2). \]

Denote \((A'', \Phi'') = e^f(A', \Phi') \in \tilde{\mathcal{M}}\mathcal{L}\mathcal{C}(X, \tilde{\mathfrak{s}})).\) Finally, by Remark 3.17 we can continuously deform \((A'', \Phi'')\) to a configuration \((A, \Phi)\) such that \(F(A, \Phi) = ((A_1, \Phi_1), (A_2, \Phi_2))\).

**Step 6.** We need to prove that \(F^{-1}\) constructed in the previous step is continuous. Notice \((g_1, g_2)\) as elements of \(L^2_2\) depend continuously on \(h_1 - h_0 \in L^2_{1/2}\) which in turn depends continuously on \(A_1 - A_0\) and \(A_2 - A_0 \in L^2_{1/2}\). Moreover, \(f \in L^2_{2}\) depends continuously on \((g_1, g_2) \in L^2_2\). If the multiplication \(L^2_2 \times L^2_1 \rightarrow L^2_1\) was continuous on 4-manifolds then we would have shown that the map \(((A_1, \Phi_1), (A_2, \Phi_2)) \rightarrow (A, \Phi)\) which we constructed is continuous. Since it is not true, we need to show that \(e^{\rho g_1} \Phi_1 \in L^2_1\) depends continuously on the initial configurations.

We will prove it depends continuously on \(h_2 - h_1 \in L^2_{1/2}\) and \(\Phi_1 \in L^2_1\). Let \((A'_1, \Phi'_1)\) and \((A'_2, \Phi'_2)\) be another choice of configurations, and denote the corresponding harmonic functions on \([-\varepsilon, 0] \times Y\) and \([0, \varepsilon] \times Y\) by \(g'_1, g'_2\). Then

\[
\|e^{g_1} \Phi_1 - e^{g'_1} \Phi'_1\|_{L^2_1} \leq \|e^{g'_1}(\Phi_1 - \Phi'_1)\|_{L^2_1} + \|(e^{g_1} - e^{g'_1}) \Phi_1\|_{L^2_1} \\
\leq C(\|e^{g'_1}\|_{L^\infty} + \|e^{g'_1}\|_{L^2_1})\|\Phi_1 - \Phi'_1\|_{L^2_1} \\
+ C\|e^{g_1} - e^{g'_1}\|_{L^2_1}\|\Phi_1\|_{L^2_1} \\
+ C\|e^{g_1} - e^{g'_1}\|_{L^\infty([-\varepsilon, -\delta] \times Y)}\|\Phi_1\|_{L^2_1([-\varepsilon, -\delta] \times Y)} \\
+ C\|e^{g_1} - e^{g'_1}\|_{L^\infty([-\delta, 0] \times Y)}\|\Phi_1\|_{L^2_1([-\delta, 0] \times Y)}
\]

for any \(\delta\). One can choose \(\delta\) to have \(\|\Phi_1\|_{L^2_1([-\delta, 0] \times Y)}\) as small as one wants while
\|e^{g_1} - e^{g'_1}\|_{L^\infty([-\delta,0] \times Y)} \leq 2. Moreover, we have

\|e^{g_1} - e^{g'_1}\|_{L^\infty([-\varepsilon,-\delta] \times Y)} \leq \|g_1 - g'_1\|_{L^\infty([-\varepsilon,-\delta] \times Y)} \leq C\|(h_1 - h_2) - (h'_1 - h'_2)\|_{L_{1/2}^2}

via a direct computation (or by interior regularity estimates following from Theorem A.3). This finishes the proof that the inverse map is continuous.

**Step 7.** Finally, we prove that the differential of \(F\) is invertible. Assume this is not the case, so that there exists \((A, \Phi) \in \tilde{\mathcal{M}}_s(X, \hat{s})\) such that for any \(\varepsilon > 0\) there is \((A', \Phi') \in \tilde{\mathcal{M}}_s(X, \hat{s})\) such that \(0 < D = \|F(A, \Phi) - F(A', \Phi')\|_{L_1^2} < 1\) and \(\|(A - A', \Phi - \Phi')\|_{L_1^2} \leq D\varepsilon\). We get that \(\|g - g'\|_{L_{5/2}^2} \leq C_{5/2}D\varepsilon\) and thus

\[\|f_i - f'_i\|_{L_{5/2}^3} \leq C\|g - g'\|_{L_{5/2}^2} \leq CC_{5/2}D\varepsilon.\]

From this and Theorem A.1 it follows that \(D = \|F(A, \Phi) - F(A', \Phi')\|_{L_1^2} \leq C''\|(A - A', \Phi - \Phi')\|_{L_1^2} \leq C''D\varepsilon\) for some \(C''\) depending on \(\|\Phi\|_{L_1^2}\). Choosing \(\varepsilon = \frac{1}{1+C''}\) gives the desired contradiction. \(\Box\)

**A Appendix**

This section presents some standard analytical results which are used repeatedly in the article. We recall the Gårding inequality, Sobolev multiplication and trace theorems and the Implicit Function Theorem.

**Theorem A.1 (Sobolev multiplication theorem).** Let \(M\) be a manifold with compact boundary and cylindrical ends.

Assume \(k, l \geq m\) and \(1/p + 1/q \geq 1/r\) for \(p, q, r \in (1, \infty)\). Then the multiplication

\[L_p^k(M) \times L_q^l(M) \to L_m^r(M)\]

is continuous if any of these hold:

- \((k - n/p) + (l - n/q) \geq m - n/r\) and both \(k - n/p < 0, l - n/q < 0\),
- \(\min(k - n/p, l - n/q) \geq m - n/r\) and either \(k - n/p > 0\) or \(l - n/q > 0\),
- \(\min(k - n/p, l - n/q) > m - n/r\) and either \(k - n/p = 0\) or \(l - n/q = 0\).

What is more, whenever it is continuous, it restricts to a compact map on \(\{f\} \times L_p^k(M) \to L_m^r(M)\) provided that \(l > m\) and \(l - n/q > m - n/r\).
Proof. We can reduce to the case of a manifold without boundary by considering a double of $M$, or by attaching cylindrical ends along $\partial M$.

To obtain continuity, combine [Pal68, Theorem 9.6] for compact manifolds together with [KM07, Theorem 13.2.2] for an infinite cylinder.

For the compactness statement on a compact manifold, for each sequence $g_i L^q_k(M) \to g$ take any sequence $f_j \in C^\infty(M)$ such that $\|f - f_j\|_{L^k(M)} \leq \frac{1}{2^j}$. By continuity of multiplication there is a constant $C$ such that

$$\|(f - f_j)g_i\|_{L^r_k(M)} \leq C \|f - f_j\|_{L^k(M)} \leq C \frac{1}{2^j}.$$  

Set $a_{0,i} = i$ and define inductively $a_{j,i}$ in the following manner. Take $a_{j,i} = a_{j-1,i}$ whenever $i < j$. Since $f_j \in C^\infty(M)$, the sequence $f_j g_i$ is bounded in $L^q_k(M)$ and by Sobolev embedding is precompact in $L^r_k(M)$. Thus, we can choose $(a_{j,i})_i$ to be a subsequence of $(a_{j-1,i})_i$ satisfying

$$\|f_j g_{a_{j,j}} - f_j g_{a_{j,i}}\|_{L^r_k(M)} \leq \frac{1}{2^j}$$

for any $i \geq j$. We finally get, for $i \geq 0$,

$$\|f g_{a_{j,j}} - f g_{a_{j,i}}\|_{L^r_k(M)} \leq \|(f - f_j)(g_{a_{j,j}}-g_{a_{j,i}})\|_{L^r_k(M)}$$

$$+ \|f_j (g_{a_{j,j}} - g_{a_{j,i}})\|_{L^r_k(M)}$$

$$\leq \frac{2C}{2^j} + \frac{1}{2^j}$$

making $f g_{a_{j,j}}$ a Cauchy sequence in $L^r_k(M)$, as wished.

Compactness for a cylinder is proved in [KM07, Theorem 13.2.2].

**Theorem A.2 (Sobolev trace theorem).** Suppose $N \subset M$ is a closed compact $n-j$-dimensional smooth submanifold, $1 \leq p < \infty$, and $k,l \geq 0$ with $k - j/p \geq l > 0$. Then the restriction map extends to continuous

$$L^p_k(M) \to L^p_l(N).$$

Proof. See [Pal68, Theorem 9.3].

We will make use of the Implicit Function Theorem in the following form (cf. [Lan93]).
**Implicit Function Theorem.** Suppose $A, B$ are Hilbert spaces and $F : A \to B$ is a smooth map such that the derivative $D_p F$ at $p$ is surjective and that its kernel splits with $C$ as a complementary subspace, $A = \ker(D_p F) \oplus C$. Let $\Pi : A \to \ker(D_p F)$ denote the projection onto the kernel along $C$. Then there are open neighborhoods $U \subset A$ of $p$, $V \subset B$ of $F(p)$ and $W \subset \ker(D_p F)$ of 0 and a smooth diffeomorphism $G : V \times W \to U$ such that

$$V \times W \xrightarrow{G} U \xrightarrow{(F, \Pi)} B \oplus \ker(D_p F)$$

is the identity map (i.e., $G = F^{-1}$).

In particular, the projection

$$\Pi : F^{-1}(0) \cap U \to \ker(D_p F)$$

along $C$ is a local diffeomorphism.

We will also utilize the Gårding inequality.

**Theorem A.3 (Gårding inequality).** Let $D$ be a first-order elliptic operator with smooth coefficients on a compact manifold $M$ (possibly with boundary) and $M' \subset M$ be open with compact closure. Then there is a constant $C$ such that for any $\gamma \in L^p_{k+1}(M)$ we have

$$\|\gamma\|_{L^p_{k+1}(M')} \leq C(\|D\gamma\|_{L^p_k(M)} + \|\gamma\|_{L^p_k(M)}).$$

**Proof.** This follows from [Shu92, Appendix 1, Lemma 1.4] by extending $D$ to the cylindrical-end manifold $M^* = M \cup (\partial M) \times [0, \infty)$. Indeed, taking a smooth bump function $\rho$ such that $\rho|_{M'} = 1$ and $\rho|_{M^* \setminus M} = 0$ we get

$$\|\gamma\|_{L^p_{k+1}(M')} \leq \|\rho\gamma\|_{L^p_{k+1}(M^*)} \leq C(\|D(\rho\gamma)\|_{L^p_k(M^*)} + \|\rho\gamma\|_{L^p_k(M^*)}) \leq CC\gamma(\|D(\gamma)\|_{L^p_k(M)} + \|\gamma\|_{L^p_k(M)}).$$

where the middle inequality follows from [Shu92].

**Remark A.4.** The author’s understanding is that without the use of twistings there is (in general) no choice of $s, s_1, s_2$ making $F$ commute with the restriction maps on the nose. This problem does not show up in the construction of monopole Floer homology since there one quotients the moduli and configuration spaces by $S^1$ after blowing up.

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