ON DIRECT SUMMANDS OF HOMOLOGICAL FUNCTORS ON LENGTH CATEGORIES

ALEX MARTSINKOVSKY

Abstract. We show that direct summands of certain homological functors arising from bifunctors with a fixed argument in an abelian category are again of that form whenever the fixed argument has finite length.

1. Introduction

This note concerns the conjecture of M. Auslander that a direct summand of a covariant Ext-functor is again of that form. More precisely, suppose $A$ is an abelian category with enough projectives, $A$ is an object of $A$, and $F$ is a direct summand of the functor $\text{Ext}^1(A, -)$. Then the question is whether $F$ is of the form $\text{Ext}^1(B, -)$ for some object $B$ of $A$. The motivation for this problem comes from Auslander’s foundational work [1] on coherent functors from abelian categories to the category of abelian groups. It is an immediate consequence of Yoneda’s lemma and the left-exactness of the Hom functor that each coherent functor gives rise (non-uniquely) to an exact sequence $X \to Y \to Z \to 0$ in the original abelian category. Specializing to such exact sequences which are short exact, one is naturally led to consider the corresponding subcategory $A_0$ of coherent functors. Thus the objects of this subcategory are coherent functors whose projective resolutions are images of short exact sequences in $A$ under the Yoneda embedding. An immediate example of such a functor is $\text{Ext}^1(A, -)$.

As Auslander showed [1] Prop. 4.3], if the answer to the above question is in the affirmative for any $A \in A$ and any $F$, then the functors of the form $\text{Ext}^1(B, -)$ are the only injectives in $A_0$. He also showed [1] Prop. 4.7] that the answer is positive assuming that $A$ is of finite projective dimension. P. Freyd [6] showed the same in the case $A$ has countable sums. In [2] Auslander undertook a more systematic study of this problem and gave a unifying proof of the above two results. In addition, he provided a detailed analysis in the case when $A$ was a category of modules over a ring. Among other things, he showed that, in general, a direct summand of the functor $\text{Ext}^1(A, -)$, where $A$ is a finitely generated module over a ring $R$, need not be of the form $\text{Ext}^1(B, -)$ for some finitely generated $R$-module $B$, even when $R$ is noetherian.

In this note, we return to general abelian categories and show that the answer to the above question is still positive if $A$ has finite length. In fact, this result is proved in greater generality – it holds for any additive functor of the form $G(A, -)$ (resp., $G(-, A)$), provided the natural transformations between two such functors come from the morphisms of their first (resp., second) arguments, see Th. [3] for details.

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A preliminary result, establishing the Fitting lemma for objects of finite length in an abelian category, is recalled in the next section. In the section following the proof of Th. 3, we apply this theorem to the functors covariant Ext, covariant Hom modulo projectives, and Tor.

2. Preliminaries

This section contains preliminary results, all of which are well-known. For the convenience of the reader, I have included their proofs in the case of a general abelian category, since I could not find them in the literature.

Let \( \mathcal{A} \) be an abelian category and \( A \) an object of \( \mathcal{A} \). Recall that the intersection of two subobjects of \( A \) always exists and can be defined as the pullback of the corresponding monomorphisms [3 Prop. 4.2.3]. Furthermore, the union of two subobjects of \( A \) always exists and can be defined as the pushout over their intersection [4 Prop. 1.7.4]. In particular, for any endomorphism \( f : A \to A \), the intersection \( \text{Ker} f^n \cap \text{Im} f^n \) and the union \( \text{Ker} f^n + \text{Im} f^n \) are defined as subobjects of \( A \) for any natural \( n \).

The commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A \\
\downarrow{f^2} & & \downarrow{f} \\
A & \xrightarrow{f} & A
\end{array}
\]

shows that \( \text{Ker} f \) is a subobject of \( \text{Ker} f^2 \) and, likewise, \( \text{Ker} f^n \) is a subobject of \( \text{Ker} f^{n+1} \). It also shows that \( f \) induces an epimorphism \( \text{Im} f \to \text{Im} f^2 \) and, likewise, an epimorphism \( \text{Im} f^n \to \text{Im} f^{n+1} \).

The commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f^2} & A \\
\downarrow{f} & & \downarrow{f} \\
A & \xrightarrow{f} & A
\end{array}
\]

shows that \( \text{Im} f^2 \) is a subobject of \( \text{Im} f \) and, likewise, \( \text{Im} f^{n+1} \) is a subobject of \( \text{Im} f^n \).

The object \( A \) is said to be of finite length if it has both DCC and ACC. This is equivalent to saying that there exists a finite chain of subobjects

\[
0 = A_0 \subset A_1 \subset \ldots \subset A_s = A
\]

whose successive quotients are simple objects.

**Lemma 1.** Let

\[
\begin{array}{ccc}
0 & \xrightarrow{a} & A & \xrightarrow{b} & B & \xrightarrow{r} & C & \xrightarrow{t} & 0 \\
& & 0 & \xrightarrow{c} & D & \xrightarrow{d} & E & \xrightarrow{t} & 0
\end{array}
\]

be a commutative diagram in \( \mathcal{A} \) with exact rows. Then \( (c, d, E) \) is a pushout of \( (a, b) \).
Proof. Let \((x, y, Z)\) be a pushout of \((a, b)\). We then have a commutative diagram

where the horizontal and the two diagonal sequences of solid arrows are short exact. Since \(da = cb\), the universal property of the pushout yields a map \(u : Z \to E\) such that \(uy = d\) and \(ux = c\). We now claim that \(s = tu\). First, notice that \(ra = tcb\) because both morphisms are zero. Hence, by the universal property of the pushouts, there exists a unique morphism \(q : Z \to C\) such that \(qy = r\) and \(qx = tc\). But both morphisms \(s\) and \(tu\) have this property, and therefore \(s = tu\). Now the commutativity of the triangles \(DZE\) and \(BZE\), one can easily show now that \((c, d, E)\) is the desired pushout. □

Lemma 2. In the above notation, let \(f : A \to A\) be an endomorphism.

1. If \(A\) has ACC, then \(0 \simeq \ker f^n \cap \im f^n\) for all \(n\) large enough. If \(f\) is an epimorphism, then it is an isomorphism.

2. If \(A\) has DCC, then the inclusion of \(\ker f^n + \im f^n\) in \(A\) is an isomorphism for all \(n\) large enough. If \(f\) is a monomorphism, then it is an isomorphism.

3. (The Fitting Lemma) If \(A\) is of finite length, then \(A \simeq \ker f^n \bigoplus \im f^n\) for all \(n\) large enough. The morphism \(f\) is nilpotent on \(\ker f^n\) for all \(n\) and is an isomorphism on \(\im f^n\) for all \(n\) large enough. Also, \(f\) is an isomorphism whenever it is a monomorphism or an epimorphism.

Proof. (1) Since \(A\) has ACC, the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker f^n & \rightarrow & A & \rightarrow & f^n & A & \rightarrow & \coker f^n & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \ker f^{n+1} & \rightarrow & A & \rightarrow & f^{n+1} & A & \rightarrow & \coker f^{n+1} & \rightarrow & 0
\end{array}
\]
shows that the leftmost vertical morphism is an isomorphism for all \( n \geq p \) for some \( p \). By the snake lemma\(^1\), the vertical morphism in the middle is also an isomorphism. The right-hand side of the diagram shows that this map is induced by \( f \). In other words, the restriction of \( f \) to \( \text{Im} f^n \) is an isomorphism for all \( n \geq p \). This implies that \( f \) is an isomorphism whenever it is an epimorphism. Since the images of powers of \( f \) form a descending chain, the restriction of any power of \( f \) to \( \text{Im} f^n \) is also an isomorphism for all \( n \geq p \). In particular, this is true for \( f^n \). On the other hand, the restriction of \( f^n \) to \( \text{Ker} f^n \) is zero. It now follows immediately that \( \text{Ker} f^n \cap \text{Im} f^n \cong 0 \).

(2) Since \( A \) has DCC, the inclusion \( \text{Im} f^{n+1} \to \text{Im} f^n \) is an isomorphism for all \( n \geq p \) for some \( p \). Therefore the inclusion \( \text{Im} f^{2n} \to \text{Im} f^n \) is also an isomorphism. Thus in the commutative diagram

\[
\begin{array}{ccc}
0 & \to & K \\
\downarrow & & \downarrow f^n \\
0 & \to & \text{Ker} f^n \\
\downarrow & & \downarrow f^n \\
& & \text{A} \\
\downarrow & & \downarrow f^n \\
& & \text{Im} f^n \\
\downarrow & & \downarrow f^n \\
& & 0
\end{array}
\]

with exact rows, the rightmost vertical map is an isomorphism. It now follows from Lemma\(^1\) that the inclusion of \( \text{Ker} f^n + \text{Im} f^n \) in \( A \) is an isomorphism. In particular, when \( f \) is a monomorphism, it is also an epimorphism, and hence an isomorphism.

(3) The union \( \text{Ker} f^n + \text{Im} f^n \) is the image of the morphism

\[
q : \text{Ker} f^n \coprod \text{Im} f^n \to A
\]

induced by the inclusions of the summands in \( A \). By (2), \( q \) is an epimorphism. The kernel of \( q \) is isomorphic (as an object, but not necessarily as a subobject) to \( \text{Ker} f^n \cap \text{Im} f^n \)\(^4\) p. 27, which is isomorphic to 0 by (1). Since \( q \) is an epimorphism and a monomorphism, it is an isomorphism. The rest has already been established. \( \Box \)

3. Direct summands of functors

Suppose \( A \) is an abelian category and \( G : A^{op} \times A \to \text{Ab} \) an additive (covariant) bifunctor with values in abelian groups. We make a further assumption that for any object \( A \) the map

\[
(A, A) \to \text{Nat}(G(A, -), G(A, -)) : f \mapsto G(f, -),
\]

sending a morphism to the corresponding natural transformation, is a surjection.

**Theorem 3.** Under the above assumptions, let \( A \) be an object of \( A \) of finite length and \( F \) a direct summand of the functor \( G(A, -) \). Then there is a subobject \( B \) of \( A \) such that \( F \cong G(B, -) \).

**Proof.** Let

\[
\begin{array}{ccc}
F & \to & G(A, -) \\
\pi & \to & F
\end{array}
\]

be a factorization of the identity map. Then the composition

\[
G(A, -) \xrightarrow{\pi} F \xrightarrow{\iota} G(A, -)
\]

\(^1\)For a proof of the snake lemma in a general abelian category, see [5].
is an idempotent endomorphism of $G(A, -)$ and, by the assumption on $G$, $\iota \tau = G(f, -)$ for some endomorphism $f : A \to A$. If $f$ is an epimorphism in $\mathcal{A}$, then by Lemma 2, it is an isomorphism. In such a case, since both $\iota \tau$ and $1 = \pi \iota$ are isomorphisms, $F$ and $G(A, -)$ become isomorphic, and we are done. Thus we may assume that $f : A \to A$ is not an epimorphism.

For an epi-mono factorization of $f$ (in $\mathcal{A}$)

$$A \xrightarrow{\alpha} \text{Im } f \xrightarrow{\beta} A$$

through its image, we have $G(f, -) = G(\alpha, -)G(\beta, -)$. Postcomposing $\iota \tau = G(f, -)$ with $\iota$, we have $\iota = G(f, -)\iota$. Setting $\delta := G(\beta, -)\iota$, we have a commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\delta} & G(A, -) \\
\downarrow{\iota} & & \downarrow{\iota} \\
G(\alpha, -) & \xrightarrow{G(\iota, -)} & G(A, -) \\
\downarrow{G(\beta, -)} & & \downarrow{G(\iota, -)} \\
G(\text{Im } f, -) & \xrightarrow{G(\iota, -)} & G(A, -)
\end{array}
$$

Since $G(\alpha, -)\delta = \iota$ and $\iota$ is a split monomorphism, the same is true for $\delta$. This shows that $F$ is a direct summand of $G(A_1, -)$, where $A_1 := \text{Im } f$ is a proper subobject of $A$ and at the same time an image of $A$ under an endomorphism. Repeating the foregoing argument with $A_1$ in place of $A$, etc., we have a descending chain $A \supset A_1 \supset A_2 \supset \ldots$ such that each term is the image of an endomorphism of the previous term and $F$ is a direct summand of each $G(A_i, -)$. Since this chain must stabilize, the last endomorphism $A_n \to A_n$ must be surjective, hence an isomorphism. But then, as we saw in the beginning of the proof, $F \simeq G(A_n, -)$. $\square$

Recall that $\mathcal{A}$ is called a length category if all of its objects have finite length.

**Corollary 4.** Suppose $\mathcal{A}$ is an abelian length category and $G : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \text{Ab}$ an additive (covariant) bifunctor with values in abelian groups. Suppose that for any object $A$ any natural transformation $G(A, -) \to G(A, -)$ is induced by a suitable morphism $A \to A$. If $F$ is a direct summand of $G(A, -)$, then there is a subobject $B$ of $A$ such that $F \simeq G(B, -)$. $\square$

**Remark 5.** The above theorem and corollary remain true, with obvious modifications, for the functor $G(A, -)$ if $G$ is an additive (covariant) functor on $\mathcal{A} \times \mathcal{A}$.

**4. Direct summands of some homological functors**

In this section, we shall give some immediate applications of Th. 3. We begin with the covariant Ext functor. The relevant result needed in this case concerns natural transformations between covariant Ext functors. It is due to Hilton-Rees and says that such transformations are in one-to-one, arrow-reversing correspondence with projective equivalence classes of morphisms in the corresponding contravariant arguments. This result means that, by assigning to each object the corresponding covariant Ext functor, one has a full contravariant embedding of the category modulo projectives in the abelian category of additive functors on the original category with values in abelian groups, where the morphisms between such
functors are natural transformations. We shall now give a short and straightforward proof of this result.

**Theorem 6** (Hilton-Rees). Assume that the abelian category $A$ has enough projectives and let $A$ and $B$ be objects of $A$. Then the correspondence

$$(B, A) \rightarrow (\text{Ext}^1(A, -), \text{Ext}^1(B, -)) : f \mapsto \text{Ext}^1(f, -)$$

induces an isomorphism

$$(B, A) \rightarrow \text{Next}^{1,1}(A, B)$$

between the group $(B, A)$ of morphisms from $B$ to $A$ modulo the morphisms factoring through projectives and the group $\text{Next}^{1,1}(A, B)$ of natural transformations from $\text{Ext}^1(A, -)$ to $\text{Ext}^1(B, -)$.

**Proof.** An exact sequence

$0 \rightarrow \Omega A \rightarrow P \rightarrow A \rightarrow 0$

with $P$ projective gives rise to an exact sequence of functors

$0 \rightarrow (A, -) \rightarrow (P, -) \rightarrow (\Omega A, -) \rightarrow \text{Ext}^1(A, -) \rightarrow 0.$

Taking functor morphisms into $\text{Ext}^1(B, -)$ and applying Yoneda’s lemma yields a commutative diagram with exact top row

$$0 \rightarrow \text{Next}^{1,1}(A, B) \rightarrow ((\Omega A, -), \text{Ext}^1(B, -)) \rightarrow ((P, -), \text{Ext}^1(B, -))$$

$$\downarrow \cong \quad \downarrow \cong$$

$$\text{Ext}^1(B, \Omega A) \rightarrow \text{Ext}^1(B, P)$$

The horizontal morphism on the bottom is part of a long exact sequence and therefore its kernel is isomorphic to $\text{Coker}((B, P) \rightarrow (B, A))$, which is $(B, A)$. Finally, $\text{Ext}^1(f, -) \in \text{Next}^{1,1}(A, B)$ is mapped by the horizontal homomorphism to the right multiplication by $\theta f$. Under the Yoneda isomorphism, this transformation is sent to $\theta f$, which is also the image of $f \in (B, A)$ in $\text{Ext}^1(B, \Omega A)$. Thus, the inverse of the constructed isomorphism is precisely $[f] \mapsto \text{Ext}^1(f, -)$. \hfill $\square$

The above proof also works for natural transformations between contravariant $\text{Ext}$-functors. In this case however, one needs to replace $\text{Hom}$ modulo projectives by $\text{Hom}$ modulo injectives, denoted by $(A, B)$. We thus have the Hilton-Rees theorem for the contravariant $\text{Extension}$ functor.

**Theorem 7.** Assume that the abelian category $A$ has enough injectives and let $A$ and $B$ be objects of $A$. Then the correspondence

$$(A, B) \rightarrow (\text{Ext}^1(-, A), \text{Ext}^1(-, B)) : f \mapsto \text{Ext}^1(-, f)$$

induces an isomorphism

$$(A, B) \rightarrow \text{Next}(A, B)^{1,1}$$

between the group $(A, B)$ of morphisms from $A$ to $B$ modulo the morphisms factoring through injectives and the group $\text{Next}(A, B)^{1,1}$ of natural transformations from $\text{Ext}^1(-, A)$ to $\text{Ext}^1(-, B)$. \hfill $\square$

Combining Th. 6 with Th. 3 we have
Proposition 8. Let $A$ be an object of finite length in an abelian category $A$ with enough projectives. If $F$ is a direct summand of the functor $\operatorname{Ext}^1(A, -)$, then there is a subobject $B$ of $A$ such that $F \simeq \operatorname{Ext}^1(B, -)$. □

In the case when the endomorphism ring of $A$ is artinian as an abelian group, the assumption that $A$ have enough projective can be dropped. To this end, recall the following result of Oort [8, p. 561]:

Theorem 9. Let $A$ be an abelian category all of whose objects are artinian. If $\lambda : \operatorname{Ext}^1(B, -) \to \operatorname{Ext}^1(A, -)$ is a natural map and $\operatorname{Hom}(A, B)$ is an artinian group, then there exists a morphism $\alpha \in \operatorname{Hom}(A, B)$ which induces $\lambda$. □

Combining this result with Th. 3, we have

Corollary 10. Suppose $A$ is an abelian length category and and $A$ is an object whose endomorphism ring is artinian as an abelian group. If $F$ is a direct summand of the functor $\operatorname{Ext}^1(A, -)$, then there is a subobject $B$ of $A$ such that $F \simeq \operatorname{Ext}^1(B, -)$. □

Combining Th. 7 with Th. 8 we have

Proposition 11. Let $A$ be an object of finite length in an abelian category $A$ with enough injectives. If $F$ is a direct summand of the functor $\operatorname{Ext}^1(-, A)$, then there is a subobject $B$ of $A$ such that $F \simeq \operatorname{Ext}^1(-, B)$. □

Next, we look at direct summands of the functor $\operatorname{Tor}^1(A, -)$. For that, we fix a ring $\Lambda$ and view $\operatorname{Tor}^1_{\Lambda}(-, -)$ as a bifunctor $\text{f.p.} (\text{mod-}\Lambda) \times \Lambda\text{-mod} \to \text{Ab}$, whose first argument is taken from the category of finitely presented right $\Lambda$-modules. It is well-known (and is not difficult to prove) that, for any finitely presented right $\Lambda$-module $A$, we have a functor isomorphism $\operatorname{Tor}^1(A, -) \simeq (\widetilde{\operatorname{Tr}} A, -)$, where $\widetilde{\operatorname{Tr}} A$ denotes the transpose of $A$. If $A'$ is another finitely presented right $\Lambda$-module, then, by Yoneda’s lemma applied to the category of modules modulo projectives,

$$\operatorname{Nat}(\operatorname{Tor}^1(A, -), \operatorname{Tor}^1(A', -)) \simeq ((\widetilde{\operatorname{Tr}} A, -), (\widetilde{\operatorname{Tr}} A', -)) \simeq (\operatorname{Tr} A', \operatorname{Tr} A),$$

But the transpose, viewed as a functor on the category of finitely presented modules modulo projectives, is a duality, and therefore the latter is isomorphic to $(A, A')$. We now have, in view of Remark 5, the following

Proposition 13. Let $A$ be a finitely presented right $\Lambda$-module of finite length. If $F$ is a direct summand of the functor $\operatorname{Tor}^1(A, -)$, then there is a finitely presented submodule $B$ of $A$ (automatically of finite length) such that $F \simeq \operatorname{Tor}^1(B, -)$. □
Proof. The proof of Theorem \ref{theo:module-problem} produces a module $B = A_n$, where
\[ A = A_0 \supset A_1 \supset \ldots \supset A_n = B \]
is a sequence of submodules of $A$ such that each $A_{i+1}$ is a homomorphic image of $A_i$. We cannot immediately use this theorem, because the category of finitely presented modules does not, in general, have kernels and is not therefore abelian. But the proof would still work if we show that each $A_i$ is finitely presented. Since for each $i$ the kernel of the epimorphism $A_i \to A_{i+1}$, being a submodule of a module of finite length, is finitely generated, $A_{i+1}$ is finitely presented whenever $A_i$ is. But $A = A_0$ is finitely presented, and an induction argument finishes the proof. \hfill \Box

Remark 14. The above result is also true, with obvious modifications, for left finitely presented modules.

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