MULTILINEAR ALGEBRA FOR MINIMUM STORAGE REGENRATING CODES

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ABSTRACT. An \((n, k, d, \alpha)-\text{MSR}\) (minimum storage regeneration) code is a set of \(n\) nodes used to store a file. For a file of total size \(k\alpha\), each node stores \(\alpha\) symbols, any \(k\) nodes recover the file, and any \(d\) nodes can repair any other node via each sending out \(\alpha/(d - k + 1)\) symbols.

In this work, we explore various ways to re-express the infamous product-matrix construction using skew-symmetric matrices, polynomials, symmetric algebras, and exterior algebras. We then introduce a multilinear algebra foundation to produce \((n, k, (k-1)t, (k-1)t))\-MSR codes for general \(t\geq 2\). At the \(t=2\) end, they include the product-matrix construction as a special case. At the \(t=k\) end, we recover determinant codes of mode \(m=k\); further restriction to \(n=k+1\) makes it identical to the layered code at the MSR point. Our codes' sub-packetization level—\(\alpha\)—is independent of \(n\) and small. It is less than \(L^2(d-k+1)\), where \(L\) is Alrabiah–Guruswami’s lower bound on \(\alpha\). Furthermore, it is less than other MSR codes’ \(\alpha\) for a subset of practical parameters. We offer hints on how our code repairs multiple failures at once.

1. INTRODUCTION

Distributed storage systems emerge as a nontraditional coding problem where the user gains and loses by multiples of a chunk of symbols called node. The user wants to decode the original message by connecting to (only) a fraction of nodes. Moreover, nodes are actively checking for failures and are restored when a node failure is detected. This motivates the following definition:

**Definition 1.** [DGW+10, WD09, RSKR09] An \((n, k, d, \alpha, \beta, M)\)-regenerating code is a collection of \(n\) nodes used to store an \(M\)-symbol file. The storage is configured such that (a) each node stores \(\alpha\) symbols; (b) any \(k\) nodes contain sufficient information to recover the file; and (c) any \(d\) nodes can repair any other failing node by each sending out \(\beta\) symbols.

In terms of random variables and entropies [Dun14, (4)–(6)], [Tia14, Definition 1], a file \(\Phi\) is a (random) vector in \(\mathbb{F}^M\), where \(\mathbb{F}\) is the working alphabet. Each node stores a vector \(W_h \in \mathbb{F}^\alpha\) depending on \(\Phi\), where \(h \in [n] := \{1, 2, \ldots, n\}\) is the node index. That means \(H(W_h | \Phi) = 0\) for all \(h \in [n]\). Any \(k\) vectors (any \(k\) nodes) suffice to recover the file \(\Phi\), so

\[H(\Phi | W_{h_1}, W_{h_2}, \ldots, W_{h_k}) = 0\]

for arbitrary distinct indices \(h_1, h_2, \ldots, h_k \in [n]\). The actual procedure that recovers \(\Phi\) from \(W_{h_1}, W_{h_2}, \ldots, W_{h_k}\) is called the downloading scheme or the data recovery scenario.
When, say, the $f$th node fails for some $f \in [n]$, a subset $\mathcal{H} \subseteq [n] \setminus \{f\}$ of $d$ nodes will be asked to help. A helper node with index $h \in \mathcal{H}$ sends a vector $S^H_{h \to f} \in F^\beta$ to repair the failing one. That means $H(S^H_{h \to f} \mid W_h) = 0$ for all $f \in [n]$ and all $h \in \mathcal{H} \subseteq [n] \setminus \{f\}$. The content of the failing node can be derived from the help messages. To rephrase it,

$$H(W_f \mid S^H_{h_1 \to f}, S^H_{h_2 \to f}, \ldots, S^H_{h_d \to f}) = 0$$

for arbitrary distinct indices $f, h_1, h_2, \ldots, h_d \in [n]$ and $\mathcal{H} := \{h_1, h_2, \ldots, h_d\}$. The actual procedure that recovers $W_f$ from $S^H_{h_1 \to f}, S^H_{h_2 \to f}, \ldots, S^H_{h_d \to f}$ is called the repairing scenario.

This definition immediately poses a dilemma: In order to store files more efficiently, node contents should share very little mutual information. But then, repairing a node becomes more difficult as it is hard to find relations among vectors sharing little mutual information. The quantity $\beta$ is referred to as the repair bandwidth as it represents the required bandwidth of the network (from a helper to the failure). Another interpretation is that, when the code is linear, $d\beta/\alpha$ is the average length of the parity check equations used to compute symbols in the $f$th node.

From here researches split into two paths. The first path characterizes the homogeneous trade-off among $\alpha$, $\beta$, and $M$. Here, ratios $\alpha/M$ and $\beta/M$ are used to measure the normalized node size and bandwidth, respectively. An illustrative trade-off between $\alpha/M$ and $\beta/M$ is plotted in Figure 1. It has $(k, d) = (3, 3)$ and arbitrary $n \geq 4$. The inner bound and the outer bound meet in this case, i.e., existing codes achieve the theoretically best trade-off. In general, however, the two bounds disagree; more works are needed to close the gap. For the latest results on the achievable side, see [RSKR11, SRKR12a, SRKR12b, TSA15, SSK15, GEC14, EM16, EM16, EM19, DL19] and references therein. See [Duu14, PK15, Tia15, SPK16, EMT15, MT15, HJ16, Dmu19] for the latest results on the unfeasible side. Together they summarize existing works on the first path.

In a trade-off plot such as Figure 1, the lower right solid point is called the $M_{BR}$ (minimum bandwidth regeneration) point since it minimizes $\beta/M$. The upper left solid point is called the $M_{SR}$ (minimum storage regeneration) point because it minimizes $\alpha/M$. Both $M_{BR}$ and $M_{SR}$ points are of particular interest for their extremity as well as the fact that existing codes achieve the cut-set bound for all parameters. Between the two, the $M_{SR}$ point attracts notable attention as it strengthens the $M_\delta S$ property through asking for the optimal repair bandwidth. Regenerating codes aiming for the $M_{SR}$ point are what constitute the second path. On this path, Definition 1 takes a simpler form.

**Definition 2.** [WD09, RSKR09] An $(n, k, d, \alpha, \alpha/(d-k+1), \alpha)$-regenerating code is called an $(n, k, d, \alpha)$-$M_{SR}$ code. The parameter $\alpha$ is called the sub-packetization level.

Remark: Historically, an $(n, k, d, \alpha)$-$M_{SR}$ code is first an $[n, k]$-$M_{\delta S}$ code over $F^\alpha$ and then equipped with the repairing property. Why $\beta = \alpha/(d-k+1)$ is the least possible repair bandwidth when $\alpha = M$ is not hard to see. Here we adapt the argument of information flow from [DGW+10].

Say we want the file and download the first $k-1$ nodes $W_1, W_2, \ldots, W_{k-1}$. Instead of downloading one more node, we pretend that the $n$th node fails and ask the first $d$ nodes to repair. We know the help messages from the first $k-1$
The $\alpha/M$-to-$\beta/M$ trade-off when $(k, d) = (3, 3)$ and $n \geq d + 1 = 4$ is arbitrary. It is settled in the sense that: (a) every solid point is achieved by some existing regenerating codes; (b) every segment is achieved by the space-sharing technique; and (c) every point outside (exclusively) the segments is provably not achievable owing to some carefully-crafted inequalities of Shannon type. (Neither axis starts from 0.)

Nodes $S_{d}^{[1]} \rightarrow S_{d}^{[2]} \rightarrow \ldots \rightarrow S_{d}^{[k-1]} \rightarrow n$ because they can be derived from the node contents $W_1, W_2, \ldots, W_{k-1}$, respectively. What is new are the other help messages $S_{d}^{[k]} \rightarrow S_{d}^{[k+1]} \rightarrow \ldots \rightarrow S_{d}^{[d]} \rightarrow n$. Together we have $(k-1)\alpha + (d-k+1)\beta$ symbols. From here we can reconstruct the $n$th node. Since we now have the full contents of $k$ nodes, we comprehend the file $\Phi$. In virtue of the conservation law of information, $(k-1)\alpha + (d-k+1)\beta \geq M$. Since $M$ is fixed to be $ka$, we obtain $\beta \geq \alpha/(d-k+1)$.

This type of argument is what cut-set bounds refer to.

Having the cut-set bounds in mind, works aiming at the MsR point either stick to $(\beta, M) := (\alpha/(d-k+1), ka)$ or, less frequently, require proximity. It is then reasonable to ask, What is the minimal sub-packetization level $\alpha$ a code can achieve? A series of works [GTC14, BK18, AG19] pursue the answer from below; the best known lower bound on $\alpha$ is the following.

**Theorem 3.** [AG19, Theorem 1] For any $(n, k, d, \alpha)$-MsR code,

$$\alpha \geq \exp\left(\frac{k-1}{4(d-k+1)}\right).$$

(Remark: It is later discovered that their bound is not valid when $k = d$. See Appendix [E] for more details.)

Other works pursued the minimal $\alpha$ through inventing new codes. During this period, some appealing properties are defined and fulfilled. One instance is to fix $n = d + 1$ and ask for the so-called optimal-access property, where every helper disk reads and transfers $\beta = \alpha/(d-k+1)$ symbols without forming linear combinations. The best result in this paradigm is clay code [VRP + 18]. Another instance is to relax the restriction $n = d + 1$ and show that $n - d$ can be arbitrarily large at the expense of increasing $\alpha$ [RKV16]. Yet another branch is to refine Reed–Solomon codes over large fields: refinement here means that a help message is not a whole symbol in the big field, but a fraction of it. See [CYB20] for the latest update on which of...
the aforementioned nice properties about the repairment of Reed–Solomon codes are enabled. There are also works that focus on repairing multiple failing nodes at once. This is further branched into two models—one model allows failing nodes to help each other while the other prohibits [CJM+13, SH13, YB19]. Lastly, we remark that some works proposed that since the exactness in $\alpha = \beta(d - k + 1)$ creates too much burden ($\alpha$ being exponential in $n$ etc.), one considers relaxing it "by $\epsilon$" [GLJ18]. In doing so, the sub-packetization level $\alpha$ grows logarithmically in $n$. This save is, colloquially, doubly-exponential in $n$.

In this paper, we fall back to the classical Definition 2 where nodes fail one at a time, no access property is considered, and the overall code is not Reed–Solomon in itself. We first review the well-known product-matrix construction. The product-matrix code, originated from [RSK11], paved the path of MSR codes and accumulates a decent amount of interests, for both simplicity and a sub-packetization level as low as $\alpha = k - 1$. Despite of the popularity, we have not encountered any code that specializes to product-matrix code.

Later when we were working on [DLW20], we found that multilinear algebra is the right language to describe certain regenerating codes. We attempt and succeed in describing product-matrix compactly in terms of multilinear algebra. We present this description after a brief algebra review. The description further leads to a natural extension of product-matrix codes, which is the main contribution of this paper.

**Theorem 4 (main theorem).** Let $n, k, d,$ and $t$ be integers such that $n - 1 \geq d \geq k \geq 2$ and $d \leq t(d - k + 1)$. Let $\alpha := (t-1)^{(d-k+1)}$. If $\alpha \leq 3003$, then there exists an $(n, k, d, \alpha)$-MSR code over some sufficiently large field.

We name it *Atrahasis code* after the fictional character who survived a seven-day flood in an Akkadian epic recorded on clay tablets.

Two proofs of the main theorem are found in section 6. As will be clarified later, both proofs depend on whether a certain determinant is non-vanishing. We precomputed all cases under $\alpha \leq 3003$, and found no counterexample. We believe that this determinant is nonzero for all $\alpha$.

**Conjecture 5.** Theorem 4 holds for all $\alpha$.

1.1. **Paradigms comparison.** A comparison is made in Table 1. From top to bottom: product-matrix at the MP point [RSK11, section IV]; and then at the MSR point [ibid, section V]; clay code family [SAK15, YB17, VRP+18]; attempts of [GFV17, RKV16] to separate $n$ from $d + 1$; this work extending the product-matrix approach; refinement of Reed–Solomon codes [GWT*17, YB19, CYP20]; $\epsilon$-MSR code relaxing the cut-set bound [RTGE17, GLJ18]; layered code [TSA+15]; determinant code [EM19a]; and cascade code [EM19b]. From left to right, whether the code: achieves the MSR point; aims for points between MSR and MP; achieves the MP point; achieves the cut-set bound; allows $d > k$ besides the $d = k$ case; allows $n > d + 1$ besides the $n = d + 1$ case; and has the optimal-access property. The last two columns list the expected sub-packetization level $\alpha$ and the working field size $|F|$. “Alon” means that the only general bound on field size comes from the combinatorial Nullstellensatz [Alon99]. See section 7 for detailed bounds on and instances of $|F|$.
Table 1. A comparison about what parameters each paradigm is interested in: \( r = d - k + 1 \) and \( t = \lceil d/(d - k + 1) \rceil \). See section 1.1 for details. See Figures 3 to 9 starting from page 32 for details about \( \alpha \).

| code      | M\$R | M\$R | cut  | \( d > k \) | \( n > d + 1 \) | I/O | \( \alpha \approx \) | \( |F| \approx \) |
|-----------|------|------|------|-------------|-------------|-----|----------------|-------------|
| prod-mat@B | \times | \times | \bigcirc | \bigcirc | \bigcirc | \times | \( d \) | \( n \) |
| prod-mat@S | \bigcirc | \times | \times | \bigcirc | \bigcirc | \times | \( k - 1 \) | \( n \) |
| clay      | \bigcirc | \times | \times | \bigcirc | \bigcirc | \bigcirc | \( r^{n/r} \) | \( n \) |
| GFV17     | \bigcirc | \times | \times | \bigcirc | \bigcirc | \times | \( r^{k(n-k)} \) | Alon |
| RKV16     | \bigcirc | \times | \times | \bigcirc | \bigcirc | \bigcirc | \( r^{n/r} \) | Alon |
| Atrahasis | \bigcirc | \times | \times | \bigcirc | \bigcirc | \bigcirc | \( \binom{k-1}{t-1} \) | Alon |
| refined-RS| \bigcirc | \times | \times | \bigcirc | \bigcirc | \bigcirc | \( n^n \) | \( n - 1 \) |
| \( \epsilon \)-M\$R | \bigcirc | \times | \times | \( \epsilon \) | \bigcirc | \bigcirc | \log n | \( O(n) \) |
| layered   | \bigcirc | \bigcirc | \bigcirc | \times | \times | \bigcirc | \( \binom{k}{k/2} \) | 2 |
| determinant| \bigcirc | \bigcirc | \bigcirc | \bigcirc | \times | \bigcirc | \( \binom{k}{k/2} \) | \( n \) |
| cascade   | \bigcirc | \bigcirc | \bigcirc | \times | \bigcirc | \bigcirc | \( r^k \) | \( n \) |

1.2. Shortening fills gaps. Throughout existing works, it is common to see that the code construction is given for a sparse family of parameters, but that does not mean the code only applies to a small range of situations. This is because there is a way to tune the parameters of an MSR code. More precisely, we have a lemma.

**Lemma 6** (shortening an M\$R-code). Given an \( (n,k,d,\alpha) \)-M\$R code, that is, an \( (n,k,d,\alpha/(d-k+1),ka) \)-regenerating code, there exists an \( (n-1,k-1,d-1,\alpha,(k-1)\alpha) \)-M\$R code, that is, an \( (n-1,k-1,d-1,\alpha,\alpha/(d-k+1),(k-1)\alpha) \)-regenerating code, over the same alphabet.

**Proof.** The key idea is to constrain that the \( n \)th node stores constant contents. For instance, let \( 0 \in F \) be a symbol in the working alphabet. Then we set \( W_n = [0 \ 0 \ \cdots \ 0] \in F^\alpha \).

For number of nodes (\( n \)): Since we don’t need any storage to keep an all-zero vector \( W_n \), we retire the \( n \)th node. Now there are \( n - 1 \) nodes left.

For the node size (\( \alpha \)): Since the first \( n - 1 \) nodes stores what they used to, the node size remains the same; the old \( \alpha \) is the new \( \alpha \).

For the file size (\( M \)): Consider the encoding functions of the last \( k \) nodes (including the \( n \)th) as a whole \( W_{n-k+1}^n : \mathbb{F}^M \to (\mathbb{F}^\alpha)^k \). Since \( M = ka \) in the old M\$R code, \( W_{n-k+1}^n \) is a bijection. Since we then fix \( W_n \), the file can only take values in the preimage

\[ \{ \Phi \in \mathbb{F}^M : \text{last } \alpha \text{ components of } W_{n-k+1}^n(\Phi) = W_n \} \subseteq \mathbb{F}^M. \]

So the preimage is of cardinality \( |\mathbb{F}|^{(k-1)\alpha} \). This leads to the new file size \( (k-1)\alpha \), which is the “new \( k \)” multiplied by the “new \( \alpha \)”.
For downloading scheme \((k)\): We want that any \(k - 1\) from the first \(n - 1\) nodes recover the file. This is possible because whenever we download \(k - 1\) nodes, we remember setting the \(n\)th node all-zero. This means that we know the content of \(k\) nodes in the old \(\text{M\&R}\) code. By definition, any \(k\) nodes recover the file in the old \(\text{M\&R}\) code. So any \(k - 1\) nodes recover the file in the new \(\text{M\&R}\) code.

For repair bandwidth \((\beta)\): The remaining nodes execute the repairing scheme as usual, so \(\beta = \alpha/(d - k + 1)\) remains the same, which is also the “new \(\alpha\)” divided by the “new \((d - k + 1)\)”.

For number of helpers \((d)\): Since all nodes know \(W_n\), any failing node will ask for \(d - 1\) helpers and simulate how the \(n\)th node could have helped. Since this means that the failing node has \(d\) help messages (one derived from \(W_n\)), it can repair itself. So \(d - 1\) is the new \(d\).

This technique is called shortening as it mimics the shortening of linear block codes. It bears the same meaning as in the title of [Duu19]. The technique can be applied iteratively.

**Lemma 7** (shortening saturation). Let \(n, k, d, \alpha, \delta\) be positive integers. Given an \((n + \delta, k + \delta, d + \delta, \alpha)\)-\(\text{M\&R}\) code. There exists an \((n, k, d, \alpha)\)-\(\text{M\&R}\) code over the same alphabet.

Note that \(d - k + 1\) is invariant under successive shortening. The main functionality of shortening is to reduce our main theorem to a task of composing a sparse family of \(\text{M\&R}\) codes. More precisely, the following theorem and Lemma 7 imply Theorem 4.

**Theorem 8** (primitive step). Fix integers \(n, k, \) and \(d\) such that \(n - 1 \geq d \geq k \geq 2\). Assume \(t := d/(d - k + 1)\) is an integer and \(\alpha = \binom{t(d-k+1)}{t-1} \leq 3003\). Then there exists an \((n, k, d, \alpha)\)-\(\text{M\&R}\) code over some sufficiently large field.

Section 6 in its entirety serves as the proof of Theorem 8 modulo the field size part. Section 7 completes the field size part.

### 1.3. Organization

Section 2 reviews the product-matrix code at the \(\text{M\&R}\) point. Section 3 prepares some algebra definitions for our paraphrase and generalization of product-matrix. Section 4 paraphrases the product-matrix framework in terms of multilinear algebra. Section 5 states an explicit \((9, 5, 6, 6)\)-\(\text{M\&R}\) code and then moves on to \((n, k; 3(k-1)/2, \alpha)\)-\(\text{M\&R}\) codes as a nontrivial example and a bridge to the general result. Section 6 proves Theorem 8 modulo the field size part. Section 7 handles the field size part. Appendix A analyzes the performance of Atrahasis code when two nodes fail at once. Appendix B compares existing codes numerically.

### 2. Product-Matrix at \(\text{M\&R}\)

In this section, we review the classical idea of product-matrix at the \(\text{M\&R}\) point [RSK11 section V]. The construction consists of two parts: a \(d = 2(k - 1)\) \(\text{M\&R}\) code and a stretching to \(d > 2(k - 1)\). The precise statement of the former is below.

**Theorem 9** (primitive product-matrix). [RSK11 section V] Let \(n - 1 \geq d = 2(k - 1) \geq 2\). There exists an \((n, k, d, k - 1)\)-\(\text{M\&R}\) code over any field \(\mathbb{F}\) such that \(|\{a^{k-1} : a \in \mathbb{F}\}| \geq n\).

This, with Lemma 7 immediately implies the following.
Proposition 10 (stretched product-matrix). [RSK11] section V.C] Let \( n - 1 \geq d \geq 2(k - 1) \geq 2 \). There exists an \((n, k, d - k + 1)\)-M\(\mathcal{S}\)R code over any field \(\mathbb{F}\) such that \(|\{a^{k-1} : a \in \mathbb{F}\}| \geq n + d - 2(k - 1)\).

We brief the proof of Theorem 9 in the rest of this section. How to generalize product-matrix to \(d < 2(k - 1)\) cases remains open since [RSK11] was published. This region is usually referred to as the high-rate region in literature. Our main contribution in section [6] answers the question positively.

2.1. The primitive construction. Assume \( n - 1 \geq d = 2(k - 1) \geq 2 \). Hereby we recite the \((n, k, d, k)\)-M\(\mathcal{S}\)R code construction of product-matrix. To specify this and every other code construction, we go over four steps: file format and \(M\) (closely related to \(n\)), node configuration and \(\alpha\), downloading scheme (closely related to \(k\)), and repairing scheme and \(\beta\) (closely related to \(d\)). Follow the subsubsection titles.

2.1.1. File format and \(M\). Let \(\mathbb{F}\) be a field of order \(n\) or greater. Over \(\mathbb{F}\), let

(file format)

\(S_1, S_2 \in \mathbb{F}^{(k-1) \times (k-1)}\)

be two \((k - 1)\)-by-\((k - 1)\) symmetric matrices. We use \((S_1, S_2)\) to pre-encode the file. That is to say, since each symmetric matrix has \((k - 1)k/2\) free entries, they jointly represent a file of size \(M := (k - 1)k\) symbols.

2.1.2. Node configuration and \(\alpha\). For each \(h \in [n]\), the \(h\)th node selects a scalar \(\xi_h \in \mathbb{F}\) and a (row) vector \(y_h \in \mathbb{F}^{k-1}\). The node then stores the vector

(node content)

\[y_h^\top S_1 + \xi_h y_h^\top S_2 \in \mathbb{F}^{k-1}.

That means, each node stores \(\alpha := k - 1\) symbols. For downloading and repairing, we put some requirements on the selection of \(\xi_h\)'s and \(y_h\)'s.

Axiom 11. The selection of \(\xi_h\)'s and \(y_h\)'s shall meet the following three Mp\(\mathcal{S}\) requirements.

(Mp\(\mathcal{S}x\)) All \(\xi_h\) are distinct.

(Mp\(\mathcal{S}y\)) Any \(k - 1\) many \(y_h\)'s span \(\mathbb{F}^{k-1}\). That is, \(\text{span}(y_h^1, y_h^2, \ldots, y_h^{k-1}) = \mathbb{F}^{k-1}\) for all distinct indices \(h_1, h_2, \ldots, h_{k-1} \in [n]\).

(Mp\(\mathcal{S}d\)) Any \(d\) concatenated vectors \([y_h^1, \xi_1 y_h^1], [y_h^2, \xi_2 y_h^2], \ldots, [y_h^d, \xi_d y_h^d]\) span \(\mathbb{F}^d\). That is to say, \(\text{span}((y_h^1, \xi_1 y_h^1), (y_h^2, \xi_2 y_h^2), \ldots, (y_h^d, \xi_d y_h^d)) = \mathbb{F}^d\) for all distinct indices \(h_1, h_2, \ldots, h_d \in [n]\).

Section 2.3 breaks down how to find \(\xi_h\)'s and \(y_h\)'s based on Reed–Solomon codes.

2.1.3. Downloading scheme. We now explain why any \(k\) nodes recover the file in the format of \((S_1, S_2)\). In doing so, observe that \(y_h^\top S_1 (y_j^\top)^\top\) behaves like a bi-linear form in \(y_h^\top\) and \(y_j^\top\). Furthermore, it is symmetric because \(S_1\) is—\(y_h^\top S_1 (y_j^\top)^\top = y_j^\top S_1 (y_h^\top)^\top\).

Proposition 12. Let \(S_1, S_2\) be symmetric but unknown. Let \(\xi_h\)'s and \(y_h\)'s satisfy (Mp\(\mathcal{S}x\)) and (Mp\(\mathcal{S}y\)). Then \(k\) many \(y_h^\top S_1 + \xi_h y_h^\top S_2\) uniquely determine \(S_1, S_2\).

Proof. Due to the symmetry possessed by the node configuration, it suffices to check if the first \(k\) nodes recover the file \((S_1, S_2)\). Fix any distinct indices \(i, j \in [k]\). We download the vector \(y_i^\top S_1 + \xi_i y_i^\top S_2\), so we can deduce the scalar \((y_i^\top S_1 + \xi_i y_i^\top S_2)(y_j^\top)^\top\), which happens to be \(y_i^\top S_1 (y_j^\top)^\top + \xi_i y_i^\top S_2 (y_j^\top)^\top\). Similarly, we download the vector
$y^*_j S_1 + \xi_j y^*_j S_2$, so we can deduce the scalar $(y^*_j S_1 + \xi_j y^*_j S_2)(y^*_j)^T$, which happens to be $y^*_j S_1(y^*_j)^T + \xi_j y^*_j S_2(y^*_j)^T$ by symmetry. Hence we can now decouple the values

\[
\begin{bmatrix}
y^*_i S_1(y^*_j)^T + \xi_i y^*_i S_2(y^*_j)^T \\
y^*_i S_1(y^*_j)^T + \xi_j y^*_i S_2(y^*_j)^T
\end{bmatrix} = \begin{bmatrix} 1 & \xi_i \\
1 & \xi_j \end{bmatrix} \begin{bmatrix} y^*_i S_1(y^*_j)^T \\
y^*_i S_2(y^*_j)^T \end{bmatrix}.
\]

The square matrix above is invertible because (MPSX) reads $\xi_i \neq \xi_j$. So we can deduce (separate/isolate) the value of $y^*_i S_1(y^*_j)^T$. This leads to an oracle that outputs the value of $y^*_i S_1(y^*_j)^T$ for any distinct $i, j \in [k]$. We now call the oracle for a fixed $i$ and arbitrary $j \in [k] \setminus \{i\}$. Owing to (MPSX), $y^*_j$ for $j \in [k] \setminus \{i\}$ span $F^{k-1}$, so we can recover $y^*_i S_1$ as a vector. Now we vary $i$, and conclude that we can recover $S_1$ as a matrix. For $S_2$, repeat the same procedure after getting the decoupled value $y^*_i S_2(y^*_j)^T$. This procedure that recovers both $S_1$ and $S_2$ witnesses the claim that a file can be recovered from any $k$ nodes.

2.1.4. Repairing scheme and $\beta$. Let $f \in [n]$ be the index of a failing node. Let $\mathcal{H} \subseteq [n] \setminus \{f\}$ be the $d$ helper nodes that are going to transmit help messages. For every $h \in \mathcal{H}$, the $h$th node will transmit

(help message) $(y^*_h S_1 + \xi_h y^*_h S_2)(y^*_f)^T \in F$

to the $f$th node. The left parentheses enclose the content of the $h$th node. This message is a 1-by-1 scalar so $\beta = 1$. Now we verify that the failing node can repair its content after receiving $d$ many help messages.

**Proposition 13.** Let $S_1, S_2$ be symmetric but unknown. Let $\xi_h$s and $y^*_h$s satisfy (MPSX). Then $d$ many $(y^*_h S_1 + \xi_h y^*_h S_2)(y^*_f)^T$ uniquely determine $y^*_f S_1 + \xi_f y^*_f S_2$.

**Proof.** Without loss of generality, assume that the first $d$ nodes are helping and that $f > d$. Then what the failing node receives can be rewritten as

$$(y^*_h S_1 + \xi_h y^*_h S_2)(y^*_f)^T = \begin{bmatrix} y^*_h & \xi_h y^*_h \end{bmatrix} \begin{bmatrix} S_1(y^*_f)^T \\
S_2(y^*_f)^T \end{bmatrix}$$

for $h \in [d]$. The right-hand side is the product of a 1-by-$d$ vector with a $d$-by-1 vector (recall $d = 2(k-1)$). (MPSX) reads that $[y^*_1, \xi_1 y^*_1, \ldots, y^*_d, \xi_d y^*_d]$ span $F^d$—i.e., they form an invertible matrix. Hence the failing node can reproduce $S_1(y^*_f)^T$ and $S_2(y^*_f)^T$. Now it remains to compute the linear combination $S_1(y^*_f)^T + S_2(y^*_f)^T \xi_f = (y^*_f S_1 + \xi_f y^*_f S_2)^T$ in order to restore the content $y^*_f S_1 + \xi_f y^*_f S_2$. □

This concludes the $(n, k, 2(k-1), k-1)$-MgR code specification needed to prove Theorem 9 modulo field size. Before addressing field size in section 2.3, we offer an alternative construction for the same region of parameters.

2.2. The skew construction. One straightforward variant of the previous subsection is that the symmetric matrices $S_1, S_2$ of dimensions $(k-1) \times (k-1)$ can be replaced by skew-symmetric matrices $A_1, A_2$ of dimensions $k \times k$. Firstly we observe that this does not change the file size; it is still $M = k(k-1)$. Next we lengthen “$y^*_h$” such that “$y^*_h A_1$” and other products make sense. Although it now seems like the node size should be $k$, we claim that we can still form an MgR code with the exact same parameters as before. That is, an $(n, k, d, k-1)$-MgR code for all $n-1 \geq d = 2(k-1) \geq 2$. We elaborate the specification in the rest of this subsection.
2.2.1. File format and $M$. Let $F$ be a field of order $n$ or greater. Over $F$, let

(file format) \[ A_1, A_2 \in F^{k \times k} \]

be two $k$-by-$k$ skew-symmetric matrices when char $F \neq 2$. When char $F = 2$, they have zeros on the diagonal. They are matrices such that $wA_1w^T = wA_2w^T = 0$ for all $w \in F^k$. We use $(A_1, A_2)$ to pre-encode the file. Since each matrix has $k(k-1)/2$ free entries, they jointly represent a file of size $M = k(k-1)$ symbols.

2.2.2. Node configuration and $h$. For each $h \in [n]$, let the $h$th node select a scalar $\xi_h \in F$ and a nonzero (row) vector $w_h^s \in F^k$. Then it stores

(node content) \[ w_h^s A_1 + \xi_h w_h^s A_2 \in F^k. \]

It looks like the node needs to store $k$ symbols but $k-1$ symbols suffice. This is because \((w_h^s A_1 + \xi_h w_h^s A_2) (w_h^s)^T = 0\)—the vector to be stored lies in a codimension-1 subspace. We now have $\alpha := k - 1$. For downloading and repairing to work, we assign some requirements on the selection of $\xi_h$s and $w_h^s$s (cf. Axiom 11).

Axiom 14. The selection of $\xi_h$s and $w_h^s$s shall comply with the following three MDS requirements:

(MPSx) All $\xi_h$ are distinct. (Same as in Axiom 11)
(MPSw) Any $k$ many $w_h^s$s span $F^k$. That is, span\(<w_1^s, \ldots, w_h^s> = F^k$ for all distinct $h_1, \ldots, h_k \in [n]$. (Dimension changed accordingly.)
(MPSq) Any $d$ concatenated vectors $[w_h^s, \xi_h w_h^s]$ span $F^{2k}/\langle[w_f^s, 0], [0, w_f^s]\rangle$. That is, span\(<[w_1^s, \xi_1 w_1^s], \ldots, [w_d^s, \xi_d w_d^s], [w_f^s, 0], [0, w_f^s]\rangle = F^{2k}$ for all distinct $f, h_1, \ldots, h_d \in [n]$. (Dimension changed accordingly.)

Section 2.3 deals with how to find $\xi_h$s and $w_h^s$s.

2.2.3. Downloading scheme. Notice that $w_i^s A_1 (w_i^s)^T = -w_i^s A_2 (w_i^s)^T$ and, in particular, $w_i^s A_1 (w_i^s)^T = 0$. We are to verify that any $k$ node contents recover the file $(A_1, A_2)$.

Proposition 15. Let $A_1$, $A_2$ be skew-symmetric matrices with zero diagonal and with unknown elements off the diagonal. Let $\xi_h$s and $w_h^s$s satisfy (MPSx) and (MPSw). Then $k$ many $w_h^s A_1 + \xi_h w_h^s A_2$ uniquely determine $A_1, A_2$.

Proof. On account of the symmetry, it suffices to demonstrate how to recover the file from the first $k$ nodes. Fix any distinct indices $i, j \in [n]$. We deduce the scalar \((w_i^s A_1 + \xi_i w_i^s A_2)(w_j^s)^T = w_i^s A_1 (w_j^s)^T + \xi_i w_i^s A_2 (w_j^s)^T\) from what we download from the $i$th node. We deduce the scalar \((w_j^s A_1 + \xi_j w_j^s A_2)(w_i^s)^T = -w_i^s A_1 (w_j^s)^T - \xi_j w_i^s A_2 (w_i^s)^T\) from what is downloaded form the $j$th node. Now decouple.

\[
\begin{bmatrix}
  w_i^s A_1 (w_j^s)^T + \xi_i w_i^s A_2 (w_j^s)^T \\
  -w_i^s A_1 (w_i^s)^T - \xi_j w_i^s A_2 (w_j^s)^T
\end{bmatrix}
= \begin{bmatrix}
  1 & \xi_i \\
  -1 & -\xi_j
\end{bmatrix}
\begin{bmatrix}
  w_i^s A_1 (w_j^s)^T \\
  w_i^s A_2 (w_j^s)^T
\end{bmatrix}
\]

By (MPSx), the square matrix is invertible. Hence we can isolate the value of $w_i^s A_1 (w_j^s)^T$. We now have $w_i^s A_1 (w_j^s)^T$ for a fixed $i$ and various $j \in [k] \setminus \{i\}$. Besides, we know $w_i^s A_1 (w_i^s)^T$ (which is 0). On grounds of (MPSw), we have collected the products of $w_i^s A_1$ with a basis of $F^k$, which leads to the recovery of $w_i^s A_1$ as a vector. Then we vary $i$ to rebuild $A_1$ as a matrix. For $A_2$, repeat the same procedure with $w_i^s A_2 (w_j^s)^T$. □
2.2.4. Repairing scheme and $\beta$. Let $f$ be the index of a failing node. Let $H \subseteq [n] \setminus \{f\}$ be the $d$ helper nodes that will transmit help messages. For every $h \in H$, the $h$th node transmits
\[(w_h^* S_1 + \xi_h w_h^* S_2)(w_f^*)^\top \in \mathbb{F}\]
to the $f$th node. This is a 1-by-1 scalar so $\beta := 1$. Next, we justify that the failing node can repair after gathering $d$ help messages.

**Proposition 16.** Let $A_1, A_2$ be skew-symmetric matrices with zero diagonal and with unknown entries off the diagonal. Let $\xi$, $w$, and $w_\ast s$ satisfy (MPSq). Then $d$ many $(w_h^* A_1 + \xi_h w_h^* A_2)(w_f^*)^\top$ (granted that $h \neq f$) uniquely determine $w_f^* A_1 + \xi_f w_f^* A_2$.

**Proof.** By virtue of the symmetry, we assume $H = [d]$ and $f > d$. Then the failing node rewrites what it receives:
\[
(w_h^* A_1 + \xi_h w_h^* A_2)(w_f^*)^\top = [w_h^* \xi_h w_h^*] [A_1(w_f^*)^\top A_2(w_f^*)^\top]
\]
for all $h \in [d]$. Other than that, the $f$th node knows
\[
0 = [0 \ w_f^*] [A_1(w_f^*)^\top A_2(w_f^*)^\top] = [w_f^* 0] [A_1(w_f^*)^\top A_2(w_f^*)^\top]
\]
as part of the code construction. So it knows the product of
\[
[A_1(w_f^*)^\top A_2(w_f^*)^\top] \in \mathbb{F}^{2k \times 1}
\]
with vectors $[w_1^* \xi_A w_1^*], \ldots, [w_d^* \xi_A w_d^*], [w_f^* 0], \text{ and } [w_f^* 0]$. Those vectors span $\mathbb{F}^{2k}$ by (MPSq), so the failing node can infer $A_1(w_f^*)^\top$ (the transpose of $w_f^* A_1$) and $A_2(w_f^*)^\top$ (the transpose of $w_f^* A_2$). Thus it infers the original node content $w_f^* A_1 + \xi_f w_f^* A_2$.

This concludes the alternative $(n, k, 2(k-1), k-1)$-M$\Sigma$R code construction. Next we address how to select $\xi_h$s, $y_h$s, and $w_h$'s.

2.3. Selecting $\xi$, $y$, and $w$. [RSK11] suggested using Reed–Solomon codes. Here are the details.

**Lemma 17.** Let $a_1, a_2, \ldots, a_n \in \mathbb{F}$ be such that $a_1^{k-1}, a_2^{k-1}, \ldots, a_n^{k-1}$ are all distinct. For each $h \in [n]$ let $\xi_h := a_h^{k-1} \in \mathbb{F}$ and $y_h := [1 \ a_h \ \cdots \ \ a_h^{k-2}] \in \mathbb{F}^{k-1}$. Then Axiom [17] is satisfied.

**Proof.** (Remark: Since $|\{a^{k-1} : a \in \mathbb{F}\}| \geq n$, the existence of $a_1, a_2, \ldots, a_n$ is a non-problem.) First, (MPSq) is satisfied because the $(k-1)$th powers of the points are all distinct. Next, (MPSq) is satisfied because $y_h$’s are (transposes of) distinct column vectors of a Reed–Solomon code; and Reed–Solomon codes are MPS codes. Lastly, (MPSq) is satisfied because $[y_h^* \xi_A y_h^*] = [1 \ a_h \ \cdots \ a_h^{d-1}]$ is again a column of a Reed–Solomon code. This concludes the field size part of Theorem 9. A similar idea is used to fulfill Axiom [14].

**Lemma 18.** Let $a_1, a_2, \ldots, a_n \in \mathbb{F}$ be such that $a_1^{k-1}, a_2^{k-1}, \ldots, a_n^{k-1}$ are all distinct. For each $h \in [n]$, let $\xi_h := a_h^{k-1} \in \mathbb{F}$ and $w_h := [1 \ a_h \ \cdots \ a_h^{k-1}] \in \mathbb{F}^k$. Then Axiom [17] holds.
Proof. $(\text{MPSX})$ and $(\text{MPSW})$ hold for the same reason $(\text{MPSX})$ and $(\text{MPSY})$ in the previous lemma do. For $(\text{MPSq})$, it suffices to check that this $2k$-by-$2k$ matrix is invertible. To do so, we attempt to eliminate shaded entries using row operations. For each $i = k, k - 1, \ldots, 2$ (notice the order), subtract $a_n$ times the $(i - 1)$th row from the $i$th row. We arrive at:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
(a_1 - a_n)a_1 & (a_2 - a_n)a_2 & \cdots & (a_d - a_n)a_d \\
\vdots & \vdots & & \vdots \\
(a_1 - a_n)a_1^{k-1} & (a_2 - a_n)a_2^{k-1} & \cdots & (a_d - a_n)a_d^{k-1} \\
\vdots & \vdots & & \vdots \\
a_1^{d} & a_2^{d} & \cdots & a_d^{d} \\
\end{bmatrix}
\]

For each $i = k, k - 1, \ldots, 2$, subtract $a_n$ times the $(k+i-1)$th row from the $(k+i)$th row. We reach:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
(a_1 - a_n)a_1 & (a_2 - a_n)a_2 & \cdots & (a_d - a_n)a_d \\
\vdots & \vdots & & \vdots \\
(a_1 - a_n)a_1^{k-1} & (a_2 - a_n)a_2^{k-1} & \cdots & (a_d - a_n)a_d^{k-1} \\
\vdots & \vdots & & \vdots \\
a_1^{k} & a_2^{k} & \cdots & a_d^{k} \\
\vdots & \vdots & & \vdots \\
a_1^{d} & a_2^{d} & \cdots & a_d^{d} \\
\end{bmatrix}
\]

Eliminate the first and the $(k+1)$th rows using the last two columns. Rescale all but the last two columns. Then we are left with a Vandermonde minor. □

What we were doing here looks like—and in fact is—shortening a Reed–Solomon code to from a generalized Reed–Solomon code. We knew the matrix is invertible because the latter code is MpS.

2.4. A polynomial shorthand. As Reed–Solomon codes admit polynomial descriptions, so do codes built upon Reed–Solomon codes. Here is a concise paraphrase of the primitive construction paired with Reed–Solomon vectors in terms of polynomials.
2.4.1. File format and $M$. Let $F[y, y']_{k-2}$ be the set of symmetric polynomials of bi-degree $(k-2, k-2)$ or less. To put it another way, $F[y, y']_{k-2}$ is a vector space over $F$ spanned by $1$, $y + y'$, $yy'$, $y^2 + y'^2$, $y^2y' + yy'^2$, $y^3 + y'^3$, . . ., $y^{k-2}y'^{k-2}$. One can identify the coefficient of $y^{i-1}y'^{j-1}$ with the $(i,j)$th entry of a $(k-1)$-by-$(k-1)$ symmetric matrix. Let $s_1(y, y'), s_2(y, y') \in F[y, y']_{k-2}$. Then the coefficients of $s_1(y, y'), s_2(y, y')$ carry a file of size $M = k(k-1)$.

2.4.2. Node configuration and $\alpha$. For each $h \in [n]$, the $h$th node stores

\[ s_1(a_h, y') + a_h^{k-1} s_2(a_h, y') \in F[y'] \]

as a polynomial in $y'$. This univariate polynomial has degree $k - 2$ or less, so $\alpha = k - 1$.

2.4.3. Downloading scheme. Say we download the first $k$ nodes. Fix distinct $i, j \in [k]$. We can specialize $s_1(a_i, y') + a_i^{k-1} s_2(a_i, y')$ to $s_1(a_i, a_j) + a_i^{k-1} s_2(a_i, a_j) \in F$. So we can specialize $s_1(a_j, y') + a_j^{k-1} s_2(a_j, y')$ to $s_1(a_j, a_i) + a_j^{k-1} s_2(a_j, a_i) = s_1(a_i, a_j) + a_i^{k-1} s_2(a_i, a_j) \in F$. Now we possess two evaluations of the polynomial $s_1(a_i, a_j) + x s_2(a_i, a_j) \in F[x]$, at $x = a_i^{k-1}$ and at $x = a_j^{k-1}$. Therefore, we can recover the constant term $s_1(a_i, a_j)$ and the linear term $s_2(a_i, a_j)$. Repeat this for all $i \neq j$, then we can recover $s_1$ and $s_2$ as we have sufficiently many evaluations.

2.4.4. Repairing scheme and $\beta$. When the $f$th node fails, the $h$th node sends

\[ s_1(a_h, a_f) + a_h^{k-1} s_2(a_h, a_f) \in F \]

to the $f$th node for every $h \in H$. This is a field element so $\beta = 1$.

Now consider $s_1(y, a_f) + y^{k-1} s_2(y, a_f) \in F[y]$ as a polynomial in $y$ of degree $d$ or less. Then the help messages are evaluations of these polynomials at $d$ distinct points. Therefore, the failing node can learn $s_1(y, a_f)$ (the lower degree part) and $s_2(y, a_f)$ (the higher degree part). And it determines $s_1(y, a_f) + a_f^{k-1} s_2(y, a_f)$.

We end this section with a remark that a similar description can be carried out with anti-symmetric polynomials.

3. Algebra Background

This section gives self-contained definitions of tensor, symmetric, and exterior algebras that will be used in our construction. Contents of this section can be found in standard textbooks. To skip, proceed to section 4 on page 17.

Let $F$ be a field. Our framework measures information in $F$-symbols so the finiteness of $F$ is not mandatory. However, finite fields—especially those with characteristic 2—are usually assumed for applications (distributed storage). On the other hand, a crucial part of the construction implies that the field must have sufficiently many elements; we elaborate the implication later in section 7.

Let $U, V, W$ be finite dimensional vector spaces over $F$. Elements of $U$ are denoted by $u$ with or without proper subscripts, elements of $V$ by $v$, and element of $W$ by $w$. For brevity, we call vector spaces spaces.

The dual space of $U$, denoted by $U^\vee$, is the space consisting of all linear transformations from $U$ to $F$. We call elements of $U^\vee$ functionals to distinguish them from elements of $U$, which we call vectors. Since $U$ is of finite dimension, $U$ and $U^\vee$ share the same dimension. Furthermore, $(U^\vee)^\vee$ is isomorphic to $U$ canonically—a vector $u \in U$ gives rise to a map from $U^\vee$ to $F$ by mapping a functional $\phi \in U^\vee$ to $\phi(u) \in F$. It turns out that linear transformations defined in this way exhaust all
possible linear transformations from $U^\vee$ to $\mathbb{F}$. The field element $\phi(u) \in \mathbb{F}$ is called the evaluation of $\phi$ at $u$. The action that takes a functional $\phi \in U^\vee$ as the input and returns $\phi(u) \in \mathbb{F}$ is called evaluating $\phi$ at $u$ or simply evaluating at $u$. For any subspace $V \subseteq U$, the restriction of $\phi$ to $V$ is a functional from $V$ to $\mathbb{F}$ that evaluates $v \in V \subseteq U$ to $\phi(v)$. This restriction is denoted by $\phi \mid V$. The corresponding action is called restricting $\phi$ to $V$, or simply restricting to $V$.

A crucial part of our construction involves evaluations of a functional $\phi \in U^\vee$ at a list of vectors $u_1, u_2, u_3, \ldots \in U$. Interesting things happen when these vectors share some linear relations. For instance, if we want to evaluate $\phi \in U^\vee$ at $u_1$, $u_2$, and $u_1 - 3u_2$, then we can also evaluate at only the first two vectors $u_1, u_2$ and compute the third evaluation by linearity $\phi(u_1 - 3u_2) = \phi(u_1) - 3\phi(u_2)$. From an information theoretic perspective, the information content of $\phi(u_1), \phi(u_2)$, and $\phi(u_1 - 3u_2)$ is no more than that of $\phi(u_1)$ and $\phi(u_2)$. More generally, if $V$ is a subspace of $U$ and we want to know the restriction $\phi \mid V$, it suffices to choose a basis of $V$ (any basis) and evaluate at each vector in the basis. For all intents and purposes, which basis is used does not affect the properties of the codes; only the size of the basis $\dim V$ matters.

### 3.1. Tensors and tensor products.

Let $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_d \in U$ form a basis of $U$ of dimension $d$. Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_l \in V$ form a basis of $V$ of dimension $l$. Denoted by $U \otimes V$, the tensor product of $U$ and $V$ is the space that consists of formal sums of the form

$$\sum_{ij} a_{ij} \bar{u}_i \otimes \bar{v}_j. \tag{1}$$

Here $a_{ij} \in \mathbb{F}$, and each $\bar{u}_i \otimes \bar{v}_j$ is an unbreakable, free variable whose sole purpose is to carry its coefficient. The addition is term-wise:

$$\sum_{ij} a_{ij} \bar{u}_i \otimes \bar{v}_j + \sum_{ij} b_{ij} \bar{u}_i \otimes \bar{v}_j := \sum_{ij} (a_{ij} + b_{ij}) \bar{u}_i \otimes \bar{v}_j. \tag{2}$$

The scalar multiplication is distributive:

$$c \cdot \sum_{ij} a_{ij} \bar{u}_i \otimes \bar{v}_j := \sum_{ij} (ca_{ij}) \bar{u}_i \otimes \bar{v}_j$$

for any $c \in \mathbb{F}$. The dimension is $\dim(U \otimes V) = \dim(U) \cdot \dim(V) = dl$.

It is quite obvious that we could have put $a_{ij}$ into a $d$-by-$l$ array and define $U \otimes V$ to be the space of arrays (matrices). However, doing so prevents us from seeing the greater picture: we may pretend that the character “$\otimes$” is an infixed binary operator from $U \oplus V$ to $U \otimes V$ that sends

$$(u, v) = \left( \sum_i a_i \bar{u}_i, \sum_j b_j \bar{v}_j \right) \in U \oplus V,$$

where $a_i, b_j \in \mathbb{F}$, to

$$u \otimes v := \sum_{ij} (a_i b_j) \bar{u}_i \otimes \bar{v}_j \in U \otimes V. \tag{2}$$

This map is bi-linear in the sense that it is linear in $u$, meaning

$$(u + cu') \otimes v = \sum_{ij} (a_i b_j + ca'_i b_j) \bar{u}_i \otimes \bar{v}_j = u \otimes v + cu' \otimes v,$$
and linear in \( v \), meaning
\[
  u \otimes (v + cv') = \sum_{ij} (a_i b_j + ca_i b_j') \bar{u}_i \otimes \bar{v}_j = u \otimes v + cu \otimes v'.
\]

But it is not linear in both, meaning \((u + cu') \otimes (v + cv') \neq (u \otimes v) + cu' \otimes v'\) in general. Once we give \( u \otimes v \)—the juxtaposition of “\( \otimes \)" with arbitrary vectors—an interpretation, describing an element of \( U \otimes V \) can be done by summing a finite list of \( u_i \otimes v_i \) where these \( u_i \) and \( v_i \) are not necessarily the same vectors as \( \bar{u}_i \) and \( \bar{v}_i \).

We then treat \( U \otimes V \) as a collection of formal sums of the form \( \sum a_i u_i \otimes v_i \), subject to the bi-linearity relation, where \( u_i \in U \) and \( v_i \in V \) are arbitrary vectors. The addition of formal sums is done by adding the coefficients of the matched \((u_i \otimes v_i)\)-terms and leaving unmatched terms intact. For example \((2u_1 \otimes v_1 + u_2 \otimes 7v_2) + (−u_2 \otimes v_2 + u_3 \otimes 8v_3)\) is equal to \((2u_1 \otimes v_1 + 6u_2 \otimes v_2 + 8u_3 \otimes v_3)\). This is the basis-free definition of \( U \otimes V \). A corollary is that no matter which basis we choose in formula (1) we will end up defining the vector space structure on \( U \otimes V \), up to isomorphism.

We call an element of \( U \otimes V \) a tensor to distinguish it from vectors, elements of plainer spaces like \( U, V, W \). The fact that \(-u_1 \otimes v_1 - u_2 \otimes v_2 + u_1 \otimes v_2 + u_2 \otimes v_1\) and \( u_2 \otimes (v_3 + v_1) - u_1 \otimes (v_1 - v_2)\) along with \((-u_1 + u_2) \otimes v_1 + (u_1 - u_2) \otimes v_2\) as well as \((u_2 - u_1) \otimes (v_1 - v_2)\) describe the same tensor inspires a question, What is the least amount of “\( \otimes \)" required to describe a tensor? In the tensor product of only two spaces, this question boils down to decomposing a matrix \([a_{ij}]_{ij}\) into a product \( CR\) of a \( d \)-by-\( r \) matrix \( C\) and an \( r \)-by-\( l \) matrix \( R\) with the least possible \( r\).

(Remark: When \( r \) reaches the minimum, columns of \( C\) are a basis of the column space of \([a_{ij}]_{ij}\); rows of \( R\) are a basis of the row space.) The number \( r\) is called the rank of a tensor, which resembles the rank of a matrix. When \( r = 1\), the tensor is of the form \( au \otimes v\) for \( a \in \mathbb{F} \) and \((u, v) \in U \oplus V\). This is called a rank-1 tensor or a simple tensor.

The new tensor notation defined in formula (2) possesses more convenience than formula (1). Consider again the tensor product \( U \otimes V\). We interpret \( u \otimes v \) as the collection of tensors of the form \( \sum a_i u_i \otimes v_i\), that is, the formal sums where the “\( U\)-component” is always \( u\). We interpret \( U \otimes v \) as the collection of tensors of the form \( \sum a_i u_i \otimes v\). If \( W\) is a subspace of \( U\), then we interpret \( W \otimes v \) as the collection of tensors where the “\( U\)-component” is always from \( W\). It is easy to check that \( u \otimes V, U \otimes v, \) and \( W \otimes V\) are all subspaces of \( U \otimes V\).

The tensor notation generalizes to combinations of three or more spaces. Let \( U \) and \( V \) have bases \( \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_d \) and \( \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_l \), respectively. Let \( W \) be a \( k\)-dimensional space with basis \( \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k\). It is not hard to imagine that \( U \otimes (V \otimes W), (U \otimes V) \otimes W,\) and any other similar combination all give the same vector space structure. It is common to unify them as \( U \otimes V \otimes W\), a space consisting of formal sums of the form
\[
\sum_{hij} a_{hij} \bar{u}_h \otimes \bar{v}_i \otimes \bar{w}_j.
\]

The addition is term-wise. The scalar multiplication is distributive. The dimension is \( \dim(U \otimes V \otimes W) = \dim(U) \cdot \dim(V) \cdot \dim(W) = dlk\). Similar to formula (2), we interpret
\[
  u \otimes v \otimes w = \left( \sum_h a_h \bar{u}_h \right) \otimes \left( \sum_i b_i \bar{v}_i \right) \otimes \left( \sum_j c_j \bar{w}_j \right).
\]
where \((u, v, w) \in U \oplus V \oplus W\) and \(a_h, b_i, c_j \in \mathbb{F}\), as
\[
\sum_{hij} (a_h b_i c_j) \bar{u}_h \otimes \bar{v}_i \otimes \bar{w}_j \in U \otimes V \otimes W.
\]

It is tri-linear in the sense that \((u + cu') \otimes v \otimes w = u \otimes v \otimes w + cu' \otimes v \otimes w\) and \(u \otimes (v + cu') \otimes w = u \otimes v \otimes w + cu \otimes v' \otimes w\) along with \(u \otimes v \otimes (w + cw') = u \otimes v \otimes w + cu \otimes v \otimes w'\). This again gives us a versatile way to describe tensors in \(U \otimes V \otimes W\), namely by formal sums of the form
\[
\sum_i a_i u_i \otimes v_i \otimes w_i.
\]

We can ask again what is the least possible length of formal sums that describe a certain tensor, and call this number its \textit{rank}. And then we can talk about whether a tensor is of rank one; a rank-1 tensor is of the form \(au \otimes v \otimes w\). In a general tensor product of three or more spaces, computing the rank or determining whether a tensor is of rank one is difficult. But all we need is that every tensor is the sum of several rank-1 tensors, i.e., rank-1 tensors span the whole space. As a consequence, we can describe a linear transformation from a tensor space by describing the image of every rank-1 tensor.

The dual of a tensor product is the tensor product of duals, i.e., \((U \otimes V \otimes W)^\vee\) is isomorphic to \(U^\vee \otimes V^\vee \otimes W^\vee\). Let \(\phi \in (U \otimes V \otimes W)^\vee\) be a functional. (We do not have a word to distinguish “plain” functionals in \(U^\vee, V^\vee,\) or \(W^\vee\) and tensor-flavored functionals in \((U \otimes V \otimes W)^\vee\).) Every tensor is a sum of rank-1 tensors and \(\phi\) is linear, so describing \(\phi\) is equivalent to describing \(\phi\)’s evaluations at rank-1 tensors.

### 3.2. Tensor power, symmetric power, and exterior power

Let \(T^0V\) be \(\mathbb{F}\); let \(T^1V\) be \(V\); and let \(T^pV\) be a product \(V \otimes V \otimes \cdots \otimes V\) of \(p\) many \(V\). This is called the \textit{pth tensor power} of \(V\). Some authors write \(V^{\otimes p}\). Let \(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_l\) form a basis of \(V\). Tensors in \(T^pV\) are of the form
\[
(3) \quad \sum a_{i_1 i_2 \cdots i_p} \bar{v}_{i_1} \otimes \bar{v}_{i_2} \otimes \cdots \otimes \bar{v}_{i_p}
\]
where \(a_{i_1 i_2 \cdots i_p} \in \mathbb{F}\) and the summation is over \(i_1, i_2, \ldots, i_p \in [l]\). Here \([l] := \{1, 2, \ldots, l\}\). Same as before, we allow arbitrary vectors to build-up rank-1 tensors before summing them. Thus a tensor in \(T^pV\) can be described by a sum of the form
\[
\sum_i a_i v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_p}
\]
where \(a_i \in \mathbb{F}\) and \(v_{i_j} \in V\) are arbitrary vectors. The addition is done via matching rank-1 tensors. The scalar multiplication is distributive. The dimension is \(\dim(T^pV) = \dim(V)^p = l^p\). To avoid confusion, it is worth noting that \(v_1 \otimes v_2\) is in general \textit{not} equal to \(v_2 \otimes v_1\) unless \(v_1\) is a multiple of \(v_2\) or \(v_2 = 0\).

Let \(Y\) be an \(m\)-dimensional space over \(\mathbb{F}\). Let \(S^0Y\) be \(\mathbb{F}\); let \(S^1Y\) be \(Y\). Let \(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m\) form a basis of \(Y\). Let \(S^qY\) be the space consisting of formal sums of the form
\[
\sum_{i_1 i_2 \cdots i_q} a_{i_1 i_2 \cdots i_q} \bar{y}_{i_1} \otimes \bar{y}_{i_2} \otimes \cdots \otimes \bar{y}_{i_q}
\]
where the summation is over all \(1 \leq i_1 \leq i_2 \leq \cdots \leq i_q \leq m\) and each \(\bar{y}_{i_1} \otimes \bar{y}_{i_2} \otimes \cdots \otimes \bar{y}_{i_q}\) is an unbreakable, free variable. This is called the \textit{qth symmetric power} of \(Y\).

The addition is term-wise. The scalar multiplication is distributive. The dimension is \(\dim(S^qY) = \binom{\dim(Y) + q - 1}{q} = \binom{m + q - 1}{q}\). When \(m \leq 0\) or \(q < 0\), the summation is
empty, so the space is a singleton $\mathbb{F}^0 = \{0\}$. The space becomes interesting after we define the *symmetric-multiplication*

$$\Theta: T^q Y \longrightarrow S^q Y$$

that sends $\bar{y}_{j_1} \otimes \bar{y}_{j_2} \otimes \cdots \otimes \bar{y}_{j_q}$ to

$$\bar{y}_{i_1} \otimes \bar{y}_{i_2} \otimes \cdots \otimes \bar{y}_{i_q}$$

where $i_1 \leq i_2 \leq \cdots \leq i_q$ is the sorted copy of indices $j_1, j_2, \ldots, j_q$. For a sum of many $\bar{y}_{j_1} \otimes \bar{y}_{j_2} \otimes \cdots \otimes \bar{y}_{j_q}$ like formula $(3)$, $\Theta$ applies to each summand and the images are added together. This makes $\Theta$ a linear transformation.

We call elements of $S^q Y$ *tensors*. The symmetric-multiplication $\Theta$ allows us to describe tensors in $S^q Y$ more concisely: We interpret

$$y_1 \otimes y_2 \otimes \cdots \otimes y_q$$

as

$$\Theta(y_1 \otimes y_2 \otimes \cdots \otimes y_q) \in S^q Y$$

where $y_1, y_2, \ldots, y_q \in Y$. Then we can use arbitrary vectors in $Y$ to describe tensors in $S^q Y$—what make up $S^q Y$ are formal sums of rank-1 tensors of the form $\sum a_i y_{i_1} \otimes y_{i_2} \otimes \cdots \otimes y_{i_q}$ where $a_i \in \mathbb{F}$ and $y_{ij} \in Y$ are arbitrary vectors. The addition is done via matching rank-1 tensors. The scalar multiplication is distributive. This syntax has the following two infamous characterizations.

**Multilinearity:** it is linear in every “$y$”, meaning that

$$y_1 \circ \cdots \circ (y_i + c y'_i) \circ \cdots \circ y_q$$

is equal to

$$(y_1 \circ \cdots \circ y_1 \circ \cdots \circ y_q) + c(y_1 \circ \cdots \circ y'_1 \circ \cdots \circ y_q).$$

**Commutativity:** swapping two $y$’s does nothing,

$$y_1 \circ \cdots \circ y_i \circ \cdots \circ y_j \circ \cdots \circ y_q = y_1 \circ \cdots \circ y_j \circ \cdots \circ y_i \circ \cdots \circ y_q.$$  

The proof is routine and omitted. Note that tensors in $T^q V$ are also multilinear in the same sense—$v_1 \circ \cdots \circ (v_i + c v'_i) \circ \cdots \circ v_q$ is equal to $(v_1 \circ \cdots \circ v_1 \circ \cdots \circ v_q) + c(v_1 \circ \cdots \circ v'_1 \circ \cdots \circ v_q)$.

Consider the symmetric square $S^2 Y$. We interpret $y \circ Y$ as the collection of tensors of the form $\sum a_i y \circ y_i$, that is, the formal sums where the first component is always $y$. We interpret $Y \circ y$ as the collection of tensors of the form $\sum a_i y_i \circ y$, which is the same subset as $y \circ Y$. For higher symmetric powers, one can interpret $y_1 \circ y_2 \circ Y$, $y_1 \circ Y \circ y_2$, $Y \circ y_1 \circ Y \circ y_2$, etc. similarly. It is easy to verify that they are all subspaces of $S^q Y$ for the obvious choices of $q$. In particular, $Y \circ Y \circ \cdots \circ Y = S^q Y$.

Let $\Lambda^0 W$ be $\mathbb{F}$; let $\Lambda^1 W$ be $W$. Let $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k$ form a basis of $W$. Let $\Lambda^q W$ be the space consisting of formal sums of the form

$$\sum a_{i_1 i_2 \cdots i_q} \bar{w}_{i_1} \wedge \bar{w}_{i_2} \wedge \cdots \wedge \bar{w}_{i_q}$$

where the summation is over all $1 \leq i_1 < i_2 < \cdots < i_q \leq k$ and each $\bar{w}_{i_1} \wedge \bar{w}_{i_2} \wedge \cdots \wedge \bar{w}_{i_q}$ is an unbreakable, free variable. This is called the *qth exterior power of W*. The addition is term-wise. The scalar multiplication is distributive. The dimension is $\binom{k}{q}$. When $q < 0$ or $q > k$, the summation is empty, so the space is a singleton $\mathbb{F}^0 = \{0\}$. The space becomes interesting after we define the *wedge-multiplication*

$$\Delta: T^q W \longrightarrow \Lambda^q W$$
that sends \( \bar{w}_{j_1} \otimes \bar{w}_{j_2} \otimes \cdots \otimes \bar{w}_{j_q} \) to
\[
\begin{cases}
0 \in \Lambda^q W & \text{if some indices coincide}, \\
(-1)^\sigma \bar{w}_{i_1} \wedge \bar{w}_{i_2} \wedge \cdots \wedge \bar{w}_{i_q} & \text{otherwise},
\end{cases}
\]
where \( i_1 < i_2 < \cdots < i_q \) is the sorted copy of indices \( j_1, j_2, \ldots, j_q \), and \( \sigma \) is the number of swaps used to sort. The parity of \( \sigma \) is commonly called the parity of the permutation that sends \( i_1 \) to \( j_1 \), sends \( i_2 \) to \( j_2 \), et seq. For a sum of many \( \bar{w}_{j_1} \otimes \bar{w}_{j_2} \otimes \cdots \otimes \bar{w}_{j_q} \) like formula (3), \( \Delta \) applies to each summand and the images are added together. This makes \( \Delta \) a linear transformation.

Elements of \( \Lambda^q W \) are sometimes called multi-vectors. We still call them tensors. The wedge-multiplication \( \Delta \) allows us to describe tensors in \( \Lambda^q W \) more concisely: We interpret \( w_1 \wedge w_2 \wedge \cdots \wedge w_q \) as
\[
\Delta(w_1 \otimes w_2 \otimes \cdots \otimes w_q) \in \Lambda^q W
\]
where \( w_1, w_2, \ldots, w_q \in W \). Then we can use arbitrary vectors in \( W \) to describe tensors in \( \Lambda^q W \)—what make up \( \Lambda^q W \) are formal sums of rank-1 tensors of the form \( \sum a_i w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_q} \) where \( a_i \in \mathbb{F} \) and \( w_{i_j} \in W \) are arbitrary vectors. The addition is done via matching rank-1 tensors. The scalar multiplication is distributive. The dimension is \( \dim(\Lambda^q W) = \binom{\dim(W)}{q} \). This syntax has the following two infamous characterizations.

**Multilinearity:** It is linear in every “\( w \)”, meaning that
\[
w_1 \wedge \cdots \wedge (w_i + cw_i') \wedge \cdots \wedge w_q = w_1 \wedge \cdots \wedge w_i \wedge \cdots \wedge w_q + c(w_1 \wedge \cdots \wedge w_i' \wedge \cdots \wedge w_q).
\]

**Anti-commutativity:** \( w_1 \wedge \cdots \wedge w_i \wedge \cdots \wedge w_j \wedge \cdots \wedge w_q = 0 \) if \( w_i = w_j \).

This implies that swapping two \( w \)'s causes a sign change,
\[
w_1 \wedge \cdots \wedge w_i \wedge \cdots \wedge w_j \wedge \cdots \wedge w_q = -w_1 \wedge \cdots \wedge w_j \wedge \cdots \wedge w_i \wedge \cdots \wedge w_q.
\]

The proof is routine and omitted.

Consider the wedge square \( \Lambda^2 W \). We interpret \( w \wedge W \) as the collection of tensors of the form \( \sum a_i w_i \wedge w_i \), that is, the formal sums where the first component is always \( w \). We interpret \( W \wedge w \) as the collection of tensors of the form \( \sum a_i w_i \wedge w \), which is the same subset as \( w \wedge W \). For higher exterior powers, one can interpret \( w_1 \wedge w_2 \wedge W, w_1 \wedge W \wedge w_2, w_1 \wedge W \wedge W \wedge w_2, \) etc. similarly. It is easy to verify that they are all subspaces of \( \Lambda^q W \) for the obvious choices of \( q \). In particular, \( W \wedge W \wedge \cdots \wedge W = \Lambda^q W \).

### 4. MsR Product-Matrix in Algebra

The purpose of this section is to introduce the multilinear algebra foundation to the classical constructions such that it leads to natural generalizations. Usage of multilinear algebra in the context of distributed storage dates back to a conference presentation [DL17].

#### 4.1. The symmetric translation.
This subsection translates the primitive construction in section 2.1 into the multilinear algebra language reviewed in the last section. Recall \( n - 1 \geq d = 2(k - 1) \geq 2 \).
4.1.1. File format and $M$. Let $X$ be $\mathbb{F}^2$. Let $Y$ be $\mathbb{F}^{k-1}$. Let the file be represented by a linear transformation

\[ \phi: X \otimes S^2 Y \rightarrow \mathbb{F}. \]

The file size $M$ is the dimension of $X \otimes S^2 Y$, which is $2 \cdot (k-1)k/2$.

4.1.2. Node configuration and $\alpha$. For each $h \in [n]$, the $h$th node selects star vectors $x_h^* \in X$ and $y_h^* \in Y$. And then the node stores the restriction

\[ \phi \mid x_h^* \otimes y_h^* \circ Y \subseteq (x_h^* \otimes y_h^* \circ Y)^\vee. \]

The node size $\alpha$ is the dimension of $x_h^* \otimes y_h^* \circ Y$, which is $k-1$. The axioms are translated as well.

**Axiom 19.** The selection of the star vectors should conform to the following three $\mathcal{M}$PS properties.

- **(MPSx2)** Any two $x_h^*$s span $X$. That is to say, $\text{span}(x_h^*, x_{h_2}) = X$ for all distinct $h_1, h_2 \in [n]$.

- **(MPSy2)** Any $k-1$ many $y_h^*$s span $Y$. That is, $\text{span}(y_h^*, y_{h_2}, \ldots, y_{h_{k-1}}) = Y$ for all distinct $h_1, h_2, \ldots, h_{k-1} \in [n]$.

- **(MPSyd2)** Any $d$ many $x_h^* \otimes y_h^*$ span $X \otimes Y$. That is, $\text{span}(x_h^*, y_h^* \otimes y_{h_2}, \ldots, x_h^* \otimes y_{h_d} \otimes y_{h_d}) = X \otimes Y$ for all distinct $h_1, \ldots, h_d \in [n]$.

Note that **(MPSy2)** coincides with **(MPSy)**. Notice a potential identification $x_h^* := [1, \xi_h]$. Then $x_h^* \otimes y_h^*$ corresponds to $[y_h^*, \xi_h x_h^*]$.

4.1.3. Downloading scheme. The downloading scheme boils down to whether $k$ restrictions $\phi \mid x_h^* \otimes y_h^* \circ Y$ recover the file $\phi$. It is equivalent to this.

**Proposition 20.** Assume **(MPSx2)** and **(MPSy2)**, then a total of $k$ $x_h^* \otimes y_h^* \circ Y$ span $X \otimes S^2 Y$.

**Sketch.** Consider the first $k$ many $x_h^* \otimes y_h^* \circ Y$s. Fix distinct $i, j \in [k]$. Then $x_i^* \otimes y_i^* \circ Y$ contains $x_i^* \otimes y_i^* \circ y_j^*$ and $x_j^* \otimes y_j^* \circ Y$ contains $x_j^* \otimes y_j^* \circ y_i^*$. So in the span of them is span$\langle x_i^*, x_j^* \rangle \otimes y_i^* \circ y_j^*$. By **(MPSx2)**, the latter is $X \otimes y_i^* \circ y_j^*$. Vary $j$ over $[k] \setminus \{i\}$, then they span $X \otimes y_i^* \circ Y$ by **(MPSy2)**. Vary $i$ over $[k]$, they span $X \otimes Y \circ Y$.

4.1.4. Repairing scheme and $\beta$. When the $f$th node fails, the $h$th node, for each $h \in \mathcal{H}$, sends

\[ \text{(help message)} \quad \phi(x_h^* \otimes y_h^* \circ y_f^*) \in \mathbb{F} \]

to the failing node. It is an evaluation so $\beta = 1$. Whether or not the failing recovers from the help messages reduces to whether $\phi(x_h^* \otimes y_h^* \circ y_f^*)$, a total of $d$ of them, determine $\phi \mid x_f^* \otimes y_f^* \circ Y$. An equivalent statement is here.

**Proposition 21.** Assume **(MPSyd2)**, then a total of $d$ $x_h^* \otimes y_h^* \circ y_f^*$ span $x_f^* \otimes y_f^* \circ Y$ for any $f$.

**Sketch.** **(MPSyd2)** reads $d$ $x_h^* \otimes y_h^*$ span $X \otimes Y$. So $d$ $x_h^* \otimes y_h^* \circ y_f^*$ span $X \otimes Y \circ y_f^*$. The latter contains $x_f^* \otimes y_f^* \circ Y$.

4.2. The skew translation. This subsection translates the skew construction in section 2.2 to the multilinear algebra language.
4.2.1. File format and $M$. Let $X$ be $\mathbb{F}^2$. Let $W$ be $\mathbb{F}^k$. Let the file $\phi$ be a linear transformation

(file format) \[ \phi: X \otimes \Lambda^2 W \rightarrow \mathbb{F}. \]

The file size $M$ is $\dim(X \otimes \Lambda^2 W) = k(k - 1)$.

4.2.2. Node configuration and $\alpha$. For each $h \in [n]$, the $h$th node selects star vectors $x_h^* \in X$ and $w_h^* \in W$. And then the node stores

(node content) \[ \phi | x_h^* \otimes w_h^* \wedge W \in (x_h^* \otimes w_h^* \wedge W)^\vee. \]

The node size $\alpha$ is thus $k - 1$, because $w_h^* \wedge w_h^*$ vanishes. We do not forget translating the axiom.

**Axiom 22.** The selection of the star vectors should fulfill the following three MDS properties.

1. \[(\text{MPSx2})\] Any two $x_h^*$ span $X$.
2. \[(\text{MPSw2})\] Any $k$ many $w_h^*$ span $W$. That is, $\text{span}(w_{h_1}^*, w_{h_2}^*, \ldots, w_{h_k}^*) = W$ for all distinct $h_1, h_2, \ldots, h_k \in [n]$.
3. \[(\text{MPSq2})\] Any $d$ many $x_h^* \otimes w_h^*$ span $X \otimes (W/(w_f^*))$ for every other index $f$. That is, $\text{span}(x_{h_1}^* \otimes w_{h_1}^*, \ldots, x_{h_d}^* \otimes w_{h_d}^*) + X \otimes w_f^* = X \otimes W$ for all distinct $f, h_1, \ldots, h_d \in [n]$.

Note that (MPSx2) coincides with the one in Axiom 19 and (MPSw2) coincides with (MPSw). Notice the potential identification $x_h^* := [1 \, \xi_h]$. Then $x_h^* \otimes w_h^*$ corresponds to $[w_h^* \, \xi_h w_h^*]$, and $X \otimes w_f^*$ to $\text{span}( [w_f^* \, 0] , [0 \, w_f^*] )$.

4.2.3. Downloading scheme. The downloading scheme can be summarized by the following proposition, proof of which is omitted for now. But it is a special case of the general theorem.

**Proposition 23.** With (MPSx2) and (MPSw2) assumed, $k \, x_h^* \otimes w_h^* \wedge W$ span $X \otimes \Lambda^2 W$.

4.2.4. Repairing scheme and $\beta$. The repairing scheme can be summarized by the following proposition, proof of which is omitted for now. It is a special case of the general theorem.

**Proposition 24.** With (MPSq2) assumed, $d \, x_h^* \otimes w_h^* \wedge w_f^*$ span $X \otimes W \wedge w_f^*$ for any other $f$.

4.3. Relation to the polynomial construction. The key is to identify $Y := \mathbb{F}^{k-1}$ with $\mathbb{F}^2[y]_{k-2}$. Then replace $S^2 Y$ with $S^2(\mathbb{F}^2[y]_{k-2})$. One also identifies $X := \mathbb{F}^2$ with $\mathbb{F} \oplus \mathbb{F}^k$. Then $X \otimes Y \cong \mathbb{F}^2[y]_{2k-3}$ as vector spaces. It is possible as well to translate the skew construction into polynomials. Identify $W := \mathbb{F}^{k}$ with $\mathbb{F}[w]_{k-1}$.

5. **Bridge to High-Rate Codes**

Recall the product-matrix mechanism provides MsR codes with $d = 2(k - 1)$ and Lemma 7 enables $d > 2(k - 1)$. It remains open whether there are high-rate codes (meaning $d < 2(k - 1)$ in this context) that share a similar, if not the same, design.

The subsequent two subsections mean to motivate a universal construction by giving an explicit $(9, 5, 6, 6)$-MsR code and then its moderate generalization to all $d = (3/2)(k - 1)$ cases.
5.1. **Explicit (9, 5, 6, 6)-MgR code.** Here is an explicit, ready-to-use (9, 5, 6, 6)-MgR code.

5.1.1. **File format and M.** Let $\mathbb{F}$ be of order 16; it could be realized by the quotient ring $\mathbb{F}_2[z]/(z^4 + z + 1)$. Let $X$ be $\mathbb{F}^3$. Let $Y$ be $\mathbb{F}^3$. (They play distinct roles and should not be identified.) Let $S^3Y$ be the symmetric cube. Let the file $\phi$ be any linear transformation

(file format) \[ \phi: X \otimes S^3Y \rightarrow \mathbb{F}. \]

The file size $M$ is the dimension of $X \otimes S^3Y$. Here $\dim(X) = 3$ and $\dim(S^3Y) = \binom{9}{3} = 10$, so $M = 30$.

5.1.2. **Node configuration and $\alpha$.** Let $a_1, a_2, \ldots, a_9$ be $0, z^3, z^6, z^{-3}, z^{-6}, z^{-1}, z^{-2}, z^{-4}, z^{-8}$, respectively. For each $h \in [9]$, the $h$th node selects star vectors $x_h^\ast := [1 \ a_h^2 \ a_h^6] \in X$ and $y_h^\ast := [1 \ a_h \ a_h^3]^\ast \in Y$. And then the node stores the restriction

(node content) \[ \phi \upharpoonright x_h^\ast \otimes y_h^\ast \in (x_h^\ast \otimes y_h^\ast \otimes S^3Y)^\ast. \]

The node size $\alpha$ is the dimension of $S^3Y$, so $\alpha = 6$.

5.1.3. **Downloading scheme.** It happens that any five $x_h^\ast \otimes y_h^\ast \otimes S^3Y$ span $X \otimes S^3Y$, hence any five node contents recover the file $\phi$.

5.1.4. **Repairing scheme and $\beta$.** Say the $f$th node fails and the first six nodes are commanded to fix it. For each $h \in [6]$, the $h$th node sends

(help message) \[ \phi \upharpoonright x_h^\ast \otimes y_h^\ast \otimes Y \otimes y_f^\ast \in (x_h^\ast \otimes y_h^\ast \otimes Y \otimes y_f^\ast)^\ast. \]

The repair bandwidth $\beta$ is thus $\dim(Y) = 3$. It happens that any six $x_h^\ast \otimes y_h^\ast \otimes Y \otimes y_f^\ast$ span $x_f^\ast \otimes y_f^\ast \otimes S^3Y$, hence the repairing scheme works. The specification of the $(9, 5, 6, 6)$-MgR code ends here. (This is indeed MgR because $\beta = 3 = \alpha/(d-k+1)$.)

5.2. **Warm-up $(n, k, (3/2)(k-1))-\text{MgR code.}$** In this subsection, we portray a $d = (3/2)(k-1)$ construction as a bridge to the general construction in section [6].

The first nontrivial $(k, d)$ pair in this vein is $(5, 6)$, the parameters used in the last subsection. The upcomers are $(7, 9)$ followed by $(9, 12)$ as well as $(11, 15)$. Claims in this subsection will not be proven as their general counterparts in section [8] come with proofs.

5.2.1. **File format and $M$.** Let $\mathbb{F}$ be a field. Let $X$ still be $\mathbb{F}^3$. Let $Y$ be $\mathbb{F}^{k-2}$. (In general, contrary to the previous subsection, $\dim(Y)$ need not be equal to $\dim(X)$.) Let $S^3Y$ be the symmetric cube. Let the file $\phi$ be any linear transformation

(file format) \[ \phi: X \otimes S^3Y \rightarrow \mathbb{F}. \]

The file size $M$ is the dimension of $X \otimes S^3Y$. Here $\dim(X) = 3$ and $\dim(S^3Y) = \binom{k}{3}$, so $M = (k-2)(k-1)k/2$. 
5.2.2. Node configuration and $\alpha$. For each $h \in [n]$, the $h$th node selects star vectors $x_h^* \in X$ and $y_h^* \in Y$. And then the node stores the restriction (node content)

$$\phi \mid x_h^* \otimes y_h^* \otimes S^2 Y \in (x_h^* \otimes y_h^* \otimes S^2 Y)^\vee.$$ 

The node size $\alpha$ is the dimension of $S^2 Y$, so $\alpha = (k-1)(k-2)/2$. The selection of the star vectors shall meet three MFD requirements.

**Axiom 25.** The selection of the star vectors are such that:

- (MpSx3) Any three $x_h^*$'s span $X$. That is, $\text{span}(x_{h_1}^*, x_{h_2}^*, x_{h_3}^*) = X$ for all distinct $h_1, h_2, h_3 \in [n]$.
- (MpSy3) Any $k - 2$ many $y_h^*$'s span $Y$. That is, $\text{span}(y_{h_1}^*, \ldots, y_{h_{k-2}}^*) = Y$ for all distinct $h_1, \ldots, h_{k-2} \in [n]$.
- (MpSd3) Any $d$ many $x_h^* \otimes y_h^* \otimes Y$ span $X \otimes S^2 Y$. That is, $x_{h_1}^* \otimes y_{h_1}^* \otimes Y + \cdots + x_{h_d}^* \otimes y_{h_d}^* \otimes Y = X \otimes S^2 Y$ for all distinct $h_1, \ldots, h_d \in [n]$.

It is unclear whether there are easy ways to generate star vectors. There are some heuristics that suggest hopeful patterns; accordingly we found some working instances by brute force. See section 7.2.

5.2.3. Downloading scheme. Say the first $k$ nodes are retrieved. For any distinct indices $h, i, j \in [k]$, we extract $\phi(x_h^* \otimes y_h^* \otimes y_i^* \otimes y_j^*)$, $\phi(x_h^* \otimes y_h^* \otimes y_j^* \otimes y_i^*)$, and $\phi(x_h^* \otimes y_h^* \otimes y_i^* \otimes y_j^*)$ from the $h$th, the $i$th, and the $j$th nodes, respectively. From here we learn $\phi \mid X \otimes y_h^* \otimes y_i^* \otimes y_j^*$ by (MpSx3). Next, we learn $\phi \mid X \otimes y_h^* \otimes y_i^* \otimes Y$ by applying (MpSy3) to various $j \in [k] \setminus \{h, i\}$. Once done, we vary $i$ to study $\phi \mid X \otimes y_h^* \otimes Y \otimes Y$ by (MpSy3). The latter then helps us reestablish $\phi \mid X \otimes Y \otimes Y \otimes Y$, which is the file per se.

5.2.4. Repairing scheme and $\beta$. Say the $f$th node fails and the first $d$ nodes are commanded to fix it. For each $h \in [d]$, the $h$th node sends (help message)

$$\phi \mid x_h^* \otimes y_h^* \otimes Y \otimes y_j^* \in (x_h^* \otimes y_h^* \otimes Y \otimes y_j^*)^\vee.$$ 

The repair bandwidth $\beta$ is thus $\dim(Y) = k - 2$. By (MpSd3), the failing node learns $\phi \mid X \otimes Y \otimes y_j^*$ from the help messages. And then the node specializes it to $\phi \mid x_h^* \otimes Y \otimes y_j^*$, which is $\phi \mid x_h^* \otimes y_j^* \otimes Y \otimes Y$. One can see here that the validity of the repairing scheme depends entirely on whether (MpSd3), the third MFD axiom, holds. The subtlety is how to design star vectors.

We close this section with a remark that codes defined in this subsection are special cases of the general code in section 6.1. Axiom 25, for instance, is a special case of Axiom 27. See Table 2 for complete relations among all constructions.

6. Atrahasis Code

We specify and verify the general Atrahasis code in this section. That will prove Theorem 5. Recall that the parameters we are interested in are integers $n, k, d, t$ such that $n - 1 \geq d \geq k \geq 2$ and $t = d/(d - k + 1)$. We invite readers to organize parameters in this form.

**Proposition 26.** Asterisks are unimportant place holders. The matrix

$$\begin{bmatrix}
d - k + 1 & k - 1 & d & \alpha & \alpha \log_2 |\mathbb{F}| \\
1 & t - 1 & t & \beta & \beta \log_2 |\mathbb{F}| \\
* & * & kd & M & M \log_2 |\mathbb{F}|
\end{bmatrix}$$

is of rank one.
Table 2. Product-matrix constructions (top left cell), its accents (first row), and their generalizations (other rows). Omitted entries are omitted to reduce repetition, not because they are not possible. Note that polynomials come in two flavors (symmetric and anti-symmetric/skew-symmetric/alternating/exterior). But we only talk about the symmetric ones.

| t | symmetric | exterior | polynomial |
|---|-----------|----------|------------|
| product-matrix | 2 | section 2.1 | section 2.2 | section 2.4 |
| multilinear algebra | 2 | section 4.1 | section 4.2 | – |
| | 3 | section 5.2 | – | section 7.3 |
| | ≥ 2 | section 6.1 | section 6.2 | – |

Proof. Trivial. □

For Theorem 8, we provide two proofs. One utilizes symmetric power and the other leans on exterior power.

6.1. Symmetric power proof of Theorem 8

6.1.1. File format and $M$. Let $X$ be $\mathbb{F}^t$. Let $Y$ be $\mathbb{F}^{k-t+1}$. Let $S^t Y$ be the $t$th symmetric power of $Y$. Let the file $\phi$ be encoded as a linear transformation $(\text{file format})$

$$
\phi: X \otimes S^t Y \rightarrow \mathbb{F}.
$$

This arbitrary map in $(X \otimes S^t Y)^\vee$ is able to carry $\dim(X \otimes S^t Y)$ symbols. Here $\dim(X) = t$ and $\dim(S^t Y) = \binom{k}{t}$. Therefore, $M = t \binom{k}{t} = k \binom{k-1}{t-1}$.

6.1.2. Node configuration and $\alpha$. Let $[n] := \{1, 2, \ldots, n\}$ represent the set of nodes. For each $h \in [n]$, the $h$th node selects two star vectors: $x_h^* \in X$ and $y_h^* \in Y$. Next, the $h$th node stores a restriction of the file $(\text{node content})$

$$
\phi \mid x_h^* \otimes y_h^* \otimes S^{t-1} Y \in (x_h^* \otimes y_h^* \otimes S^{t-1} Y)^\vee.
$$

This restriction is a linear transformation from the domain $x_h^* \otimes y_h^* \otimes S^{t-1} Y$. As a consequence, it can be fully recorded by $\dim(x_h^* \otimes y_h^* \otimes S^{t-1} Y)$ symbols. This quantity coincides with $\dim(S^{t-1} Y)$, which is $\binom{k-1}{t-1}$. In summary, the sub-packetization level is $\alpha = \binom{k-1}{t-1}$.

For the downloading scheme and repairing scheme later in the proof, the selection of star vectors $x_h^*$s and $y_h^*$s are not arbitrary. There are several conditions they need to fulfill.

Axiom 27. Assume that the selection of the star vectors fulfills the following three MpS conditions.

(MpSxt) Any $t$ many $x_h^*$s span $X$. Namely, $\text{span}(x_{h_1}^*, \ldots, x_{h_t}^*) = X$ for all distinct indices $h_1, \ldots, h_t \in [n]$.

(MpSyt) Any $k - t + 1$ many $y_h^*$s span $Y$. To wit, $\text{span}(y_{h_1}^*, \ldots, y_{h_{k-t+1}}^*) = Y$ for all distinct indices $h_1, \ldots, h_{k-t+1} \in [n]$.
\( (\text{MpSdt}) \) Any \( d \) many \( x_h^* \otimes y_h^* \otimes S^{t-2}Y \) span \( X \otimes S^{t-1}Y \). More specifically, \( x_{h_1}^* \otimes y_{h_1}^* \otimes S^{t-2}Y + \cdots + x_{h_d}^* \otimes y_{h_d}^* \otimes S^{t-2}Y \) is \( X \otimes S^{t-1}Y \) for all distinct indices \( h_1, \ldots, h_d \in [n] \).

This axiom generalizes Axiom 25. As commented there, it is unclear how such star vectors can be found easily. The existence of star vectors, on the other hand, is guaranteed by Alon’s combinatorial Nullstellensatz. That being said, we have no control over the upper bound on field size other than Alon’s. (Bounds from DeMillo–Lipton–Schwartz–Zippel, if not coincident, are looser.) See section 7.1 for more details.

6.1.3. Downloading scheme. To verify that downloading any \( k \) nodes suffices to recover the whole file \( \phi \), let \( \mathcal{K} \subseteq [n] \) be the indices of downloaded nodes, \( |\mathcal{K}| = k \). We now possess complete knowledge of \( \phi \mid x_h^* \otimes y_h^* \otimes S^{t-1}Y \) for all \( h \in \mathcal{K} \). Provided that \( \phi \) is linear, we recover the restriction to the span

\[
\phi \mid \sum_{h \in \mathcal{K}} x_h^* \otimes y_h^* \otimes S^{t-1}Y.
\]

Whether or not this is \( \phi \) per se depends on whether or not the span is the original domain, \( X \otimes S^tY \).

**Proposition 28.** Take \( (\text{MpSxt}) \) and \( (\text{MpSy}) \) as granted. For any \( k \)-element subset \( \mathcal{K} \subseteq [n] \),

\[
\sum_{h \in \mathcal{K}} x_h^* \otimes y_h^* \otimes S^{t-1}Y = X \otimes S^tY.
\]

**Proof.** Let \( i_1, i_2, \ldots, i_t \in \mathcal{K} \) be \( t \) distinct indices. Then \( \sum_{h \in \mathcal{K}} x_h^* \otimes y_h^* \otimes S^{t-1}Y \) contains the following \( t \) tensors

\[
x_{i_1}^* \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_t}^*,
x_{i_2}^* \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_t}^*,
x_{i_3}^* \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_t}^*,
\]

\[ \vdots \]

\[
x_{i_{t-1}}^* \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes \cdots \otimes y_{i_{t-1}}^* \otimes y_{i_t}^* = x_{i_1}^* \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_t}^*,
\]

\[ \vdots \]

\[
x_{i_t}^* \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_{t-1}}^* = x_{i_1}^* \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_t}^*.
\]

A vector under a wide hat is missing from the product. Note that the right column consists of \( \otimes y_{i_1}^* \otimes \cdots \otimes y_{i_t}^* \) led by \( x_{i_1}^* \) and \( x_{i_2}^* \), and all the way up to \( x_{i_t}^* \). By the distributive law, tensors in the right column span

\[
\text{span} (x_{i_1}^*, x_{i_2}^*, \ldots, x_{i_t}^*) \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_t}^*.
\]

Invoking \( (\text{MpSxt}) \), this subspace is

\[
X \otimes y_{i_1}^* \otimes y_{i_2}^* \otimes y_{i_3}^* \otimes \cdots \otimes y_{i_t}^*.
\]

Let \( i_t \) vary over \( \mathcal{K} \setminus \{i_1, \ldots, i_{t-1}\} \) (all possible indices such that all subscripts are distinct). Then these subspaces sum to

\[
X \otimes y_{i_1}^* \otimes \cdots \otimes y_{i_{t-1}}^* \otimes \text{span}(y_{i_t}^* : i_t \in \mathcal{K} \setminus \{i_1, \ldots, i_{t-1}\}).
\]
According to (MPSy), the span can be replaced by $Y$. Thus $\sum_{h \in \mathcal{K}} x_h^i \otimes y_h^i \otimes S_t^{-1}Y$ contains
\[ X \otimes y_h^i \otimes \cdots \otimes y_{i-1}^i \otimes Y \]
for any $i_1, \ldots, i_{t-1} \in \mathcal{K}$. Now we replicate the same procedure to replace $y_{i-1}^i$ by $Y$, and then replace $y_{i-2}^i$ by $Y$. In the end, we show that $\sum_{h \in \mathcal{K}} x_h^i \otimes y_h^i \otimes S_t^{-1}Y$ contains
\[ X \otimes Y \otimes \cdots \otimes Y, \]
which is the domain of $\phi$. \hfill \Box

6.1.4. Repairing scheme and $\beta$. Let $f \in [n]$ be the index that points to the failing node. Let $\mathcal{H} \subseteq [n] \setminus \{f\}$ be the $d$ indices, $|\mathcal{H}| = d$, that point to the helper nodes. When the $f$th node fails, each helper node $h \in \mathcal{H}$ sends the restriction
\[
(\text{help message}) \quad \phi \mid x_h^i \otimes y_h^i \otimes S_t^{-2}Y \otimes y_f^i \in (x_h^i \otimes y_h^i \otimes S_t^{-2}Y \otimes y_f^i)^\vee
\]
to the former. The $h$th node knows what to send because the help message is a further restriction (to a smaller subspace) of its node content. In particular, $x_h^i \otimes y_h^i \otimes S_t^{-2}Y \otimes y_f^i \subseteq x_h^i \otimes y_h^i \otimes S_t^{-1}Y$. In sending the help message, the helper node needs to transmit $\dim(x_h^i \otimes y_h^i \otimes S_t^{-2}Y \otimes y_f^i)$ symbols. This is $\dim(S_t^{-2}Y)$, or $S_{t-2}Y$ for short. So the repair bandwidth is $\beta = \binom{k-2}{t-2}$. Now the help messages are sent.

Upon the reception of help messages, the failing node recalls its original content if the corresponding subspaces span its domain. More precisely, it relies on the following containment.

**Proposition 29.** Take (MPSy) as granted. For any $d$-subset $\mathcal{H} \subseteq [n] \setminus \{f\}$,
\[
\sum_{h \in \mathcal{H}} x_h^i \otimes y_h^i \otimes S_t^{-2}Y \otimes y_f^i \supseteq x_f^i \otimes y_f^i \otimes S_t^{-1}Y.
\]

**Proof.** Specialize (MPSy) at $\mathcal{H}$. We obtain
\[
\sum_{h \in \mathcal{H}} x_h^i \otimes y_h^i \otimes S_t^{-2}Y = X \otimes S_t^{-1}Y.
\]
Citing the distributive law, we further deduce that
\[
\sum_{h \in \mathcal{H}} x_h^i \otimes y_h^i \otimes S_t^{-2}Y \otimes y_f^i = X \otimes S_t^{-1}Y \otimes y_f^i.
\]
The right-hand side is $X \otimes y_f^i \otimes S_t^{-1}Y$; the latter clearly contains a subspace $x_f^i \otimes y_f^i \otimes S_t^{-1}Y$. And we are done proving. \hfill \Box

Theorem 8’s proof is now complete up to Axiom 27 (which is closely related to the field size part of the theorem statement). We defer that part until section 6.1. One also sees that this subsection specializes to section 6.1 when $t = 2$, and to section 5.2 when $t = 3$. The rest of this section is an alternative proof of Theorem 8 utilizing exterior power.

6.2. Exterior power proof of Theorem 8

6.2.1. File format and $M$. Let $X$ be $\mathbb{F}^t$. Let $W = \mathbb{F}^k$. Let $\Lambda^t W$ be the $t$th exterior power. Let the file $\phi$ be any linear transformation
\[
(\text{file format}) \quad \phi: X \otimes \Lambda^t W \rightarrow \mathbb{F}.
\]
The file size is thus the dimension of $X \otimes \Lambda^t W$, which means $M = t{k \choose i} = k{k-1 \choose i-1}$. 
6.2.2. Node configuration and $\alpha$. For each $h \in [n]$, the $h$th node selects star vectors $x_h^* \in X$ and $w_h^* \in W$. And then the node stores the restriction

$$(\text{node content}) \quad \phi | x_h^* \otimes w_h^* \wedge \Lambda^{t-1}W.$$  

The node size $\alpha$ is the dimension of this subspace, which is $\dim(w_h^* \wedge \Lambda^{t-1}W)$. Notice that we do not automatically equal it to $\dim(\Lambda^{t-1}W)$. This is because $w_h^* \wedge$ will eliminate a tensor whenever its $\Lambda^{t-1}$-fragment is a multiple of $w_h^*$, viz. $w_h^* \wedge w_h^* \wedge \omega = 0$ for all $\omega \in \Lambda^{t-2}W$. To rephrase it, the $\Lambda^{t-1}$-W-fragment contributes, and only contributes, tensors “up to $w_h^*$”.

**Lemma 30.** Let $w \in W$, then $w \wedge \Lambda^{t-1}W \cong \Lambda^{t-1}(W/\langle w \rangle)$ as a vector space.

**Proof.** We claim the desired linear isomorphism

$$w \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_{t-1} \longmapsto (w_1 + \langle w \rangle) \wedge (w_2 + \langle w \rangle) \wedge \cdots \wedge (w_{t-1} + \langle w \rangle).$$

One can confirm that this map is well-defined, linear, injective, and surjective. □

With the lemma, we argue that $\alpha = \dim(w_h^* \wedge \Lambda^{t-1}W) = \dim(\Lambda^{t-1}(W/\langle w_h^* \rangle)) = \dim(\Lambda^{t-1}(\mathbb{F}^{k-1}))$. So the node size is indeed $\alpha = \binom{k-1}{t-1}$. Next, we state the axioms concerning the star vectors.

**Axiom 31.** The selection of the star vectors satisfies the following three MDS conditions.

- (MDSxt) Any $t$ many $x_h^*$s span $X$. That is, span$\langle x_{h_1}^*, \ldots, x_{h_t}^* \rangle = X$ for all distinct $h_1, \ldots, h_t \in [n]$.
- (MDSwt) Any $k$ many $w_h^*$s span $W$. That is, span$\langle w_{h_1}^*, \ldots, w_{h_k}^* \rangle = W$ for all distinct $h_1, \ldots, h_k \in [n]$.
- (MDSqt) Any $d$ many $x_h^* \otimes w_h^* \wedge \Lambda^{t-2}W$ span $X \otimes \Lambda^{t-1}(W/\langle w_f^* \rangle)$ for every other index $f$. That is to say, $x_{h_1}^* \otimes w_{h_1}^* \wedge \Lambda^{t-2}W + \cdots + x_{h_d}^* \otimes w_{h_d}^* \wedge \Lambda^{t-2}W + X \otimes w_f^* \wedge \Lambda^{t-2}W$ is $X \otimes \Lambda^{t-1}W$ for all distinct $f, h_1, \ldots, h_d \in [n]$.

(MDSxt) coincides with the one in Axiom 27. Axiom 31 is a generalization of Axiom 14. Remarks under Axioms 25 and 27 (that we do not have efficient algorithm to generate star vectors) also apply here. See section 7 for how we overcome this.

6.2.3. Downloading scheme. Whether or not any $k$ node contents recover the file $\phi$ is equivalent to whether any $k$ corresponding domains span $\phi$’s. We end up relying on this proposition.

**Proposition 32.** Assume (MDSxt) and (MDSwt). Let $K \subseteq [n]$ be a $k$-subset. Then

$$\sum_{h \in K} x_h^* \otimes w_h^* \wedge \Lambda^{t-1}W = X \otimes \Lambda^tW.$$  

**Sketch.** Similar strategy to Proposition 28. First, obtain $X \otimes w_1^* \wedge w_2^* \wedge \cdots \wedge w_n^*$. And then replace lower letter $w$’s by capital $W$, one after another. In doing so, use the free knowledge $w \wedge w = 0$. □
6.2.4. Repairing scheme and \( \beta \). The \( h \)-th node, for each helper index, sends to the \( f \)-th node, the restriction
\[
\phi \mid x_h^* \otimes w_h^* \land \Lambda^{t-2} W \land w_f^*.
\]
Subspace \( x_h^* \otimes w_h^* \land \Lambda^{t-2} W \land w_f^* \) is contained in the node domain \( x_h^* \otimes w_h^* \land \Lambda^{t-1} W \).
Subspace \( x_h^* \otimes w_h^* \land \Lambda^{t-2} W \land w_f^* \) has dimension \( \dim(w_h^* \land \Lambda^{t-2} W \land w_f^*) \).
Invoking Lemma 30 twice, we can write \( w_h^* \land \Lambda^{t-2} W \land w_f^* \equiv \Lambda^{t-2}(W/\text{span}(w_h^*)) \land w_f^* \equiv \Lambda^{t-2}(W/\text{span}(w_h^*, w_f^*)) \). Hence the dimension is \( \dim(W/\text{span}(w_h^*, w_f^*)) \) choose \( t-2 \), that will lead to \( \beta = \binom{k-2}{t-2} \). The effectiveness of repairing is handled below.

**Proposition 33.** Assume \( (\text{MDS}^{\text{xt}}) \). Let \( f \in [n] \) and let \( H \subseteq [n] \setminus \{ f \} \) be such that \( |H| = d \). Then
\[
\sum_{h \in H} x_h^* \otimes w_h^* \land \Lambda^{t-2} W \land w_f^* \supseteq x_f^* \otimes W \land \Lambda^{t-1} W.
\]

**Proof.** Multiply \( (\text{MDS}^{\text{qt}}) \) by \( w_f^* \) from the right. Replace \( X \) by \( x_f^* \). \( \square \)

This finishes the proof of Theorem 8 modulo field size for the second time. One also sees that this subsection specializes to section 4.2 when \( t = 2 \). In the next section, we deal with the elephant in the room.

7. Star Selection and Field Size

We left open how nodes select star vectors such that \( (\text{MPS}^{\text{xt}}), (\text{MPS}^{\text{yt}}), \) and \( (\text{MPS}^{\text{dt}}) \) in section 6.2 hold. And then in section 6.1 we assume \( (\text{MPS}^{\text{xt}}), (\text{MPS}^{\text{yt}}), \) and \( (\text{MPS}^{\text{dt}}) \) without specifying how. Nor did we disclose how to fulfill \( (\text{MPS}^{\text{xt}}), (\text{MPS}^{\text{yt}}), \) and \( (\text{MPS}^{\text{dt}}) \) in section 6.2. In this section, we propose two approaches. One is an existence bound (as commented below Axiom 27). The other is by brute force.

7.1. A loose bound. Recall N. Alon’s combinatorial Nullstellensatz.

**Lemma 34.** [Alon99, Theorem 1.2] Let \( F \) be a field. Let \( t_1, \ldots, t_n \) be nonnegative integers. Let \( f(x_1, \ldots, x_n) \) be a polynomial over \( F \) in \( n \) variables. Suppose \( \deg f = t_1 + \cdots + t_n \) and the coefficient of \( x_1^{t_1} \cdots x_n^{t_n} \) in \( f \) is nonzero. Let \( S_1, \ldots, S_n \subseteq F \) be any subsets with \( |S_i| > t_i \) for all \( i \in [n] \). Then \( f(s_1, \ldots, s_n) \neq 0 \) for some \( s_1 \in S_1, s_2 \in S_2, \) and all the way up to \( s_n \in S_n \).

A common use of the combinatorial Nullstellensatz is to insert variables into a square matrix that is presumed to be invertible. Imagine its determinant being a multivariate polynomial. If this polynomial is nonzero, one can find a top total-degree monomial within. Its degree will be the \( t_1, \ldots, t_n \) in the statement and the lower bounds on the sizes of \( S_1, \ldots, S_n \). Subsets \( S_1, \ldots, S_n \) are usually assumed to be the field \( F \) itself so \( t_1, \ldots, t_n \) serve as lower bounds on the field size.

Frequently it is the case that all we need is a finite bound, so we do not have to keep track of \( t \)-s. In such circumstances, Alon’s theorem reads: A nontrivial polynomial has a nonzero evaluation. Notice its elementary converse—nonzero evaluation implies nonzero polynomial.

Now what we demand is the existence of star vectors \( x_1^*, x_2^*, \ldots, x_n^*, y_1^*, \ldots, y_n^* \) that satisfy \( (\text{MPS}^{\text{xt}}), (\text{MPS}^{\text{yt}}), \) and \( (\text{MPS}^{\text{dt}}) \). Take \( (\text{MPS}^{\text{xt}}) \) as an example. Whether or not any \( t \) vectors among \( x_1^*, \ldots, x_n^* \in X \) span \( X \) is equivalent to whether any \( t \) vectors form a \( t \)-by-\( t \) matrix with a nonzero determinant. Let \( f(x_1^*, \ldots, x_t^*) \) be
the determinant written as a polynomial in the coordinates of \(x^*_1, \ldots, x^*_k\). Then we want to show

\[
\prod_{i_1, \ldots, i_k} f(x^*_{i_1}, \ldots, x^*_{i_k}) \neq 0
\]

as a polynomial. This is true because plugging in Reed–Solomon columns results in a nonzero evaluation. Similarly, for \((\text{MDSyt})\), let \(g(y^*_1, \ldots, y^*_{k-t+1})\) be the determinant in terms of \(y\)'s. Then

\[
\prod_{i_1, \ldots, i_{k-t+1}} g(y^*_{i_1}, \ldots, y^*_{i_{k-t+1}})
\]

is, again, not a zero polynomial due to Reed–Solomon codes.

Up to this point, it remains to show that

\[
\prod_{i_1, \ldots, i_d} h(x^*_{i_1}, y^*_{i_1}, \ldots, x^*_{i_d}, y^*_{i_d})
\]

is nonzero, where \(h\) is the determinant corresponding to \((\text{MDSdt})\). This one is hard, because we do not know any code that guarantees \(not\) to evaluate \(h\) to zero. Nonetheless, there is a shenanigan to overcome small cases.

**Algorithm 35.** We executed the following for all \(\alpha \leq 3003\) cases.

1. Let \(\mathbf{F}\) be a finite field of small prime order. For instance \(|\mathbf{F}| = 127\).
2. Let \(x^*_1, \ldots, x^*_d \in \mathbf{F}^n\) and \(y^*_1, \ldots, y^*_d \in \mathbf{F}^{k-t+1}\) be random vectors of the prescribed lengths drawn from any ensemble.
3. Select a basis \(\bar{\eta}_1, \ldots, \bar{\eta}_{(k-t-2)}\) of \(S^{n-2}\mathbf{F}^{k-t+1}\). Select for \(S^{t-1}\mathbf{F}^{k-t+1}\), too.
   (The standard ones in section \([3]\) are preferred.)
4. Expand \(x^*_i \otimes y^*_i \otimes \bar{\eta}_j\) for all \(i \in [d]\) and all \(j \in \left(\binom{k-t-2}{t-2}\right)\) as very long vectors in \(\mathbf{F}^{d\beta}\), and stack them to form a \(d\beta\)-by-\(d\beta\) matrix.
5. Compute the determinant \(h(x^*_1, y^*_1, \ldots, x^*_d, y^*_d) \in \mathbf{F}'\) of the matrix. If it is nonzero, then \(h\) has a nonzero evaluation and hence is nonzero. We declare a pass. Otherwise redraw random vectors and start over.

Remarks: All \(\alpha \leq 3003\) cases passed; some did require a second run as the determinant vanished in the first run. Computing over a finite field \(\mathbf{F}'\) in place of \(\mathbb{Q}\) (or floating numbers) is essential because the arithmetic is exact and fast. The sole purpose of the field \(\mathbf{F}'\) is to witness \(h \neq 0\) over \(\mathbb{Z}\), so it does not have to be the same field \(\mathbf{F}\) we define the actual code over. A smaller field causes a faster computation with a lower pass rate. The result of Algorithm 35 can be summarized as follows.

**Proposition 36.** For all \(\alpha \leq 3003\) cases, the determinant \(h(x^*_1, y^*_1, \ldots, x^*_d, y^*_d)\) is not the zero polynomial over \(\mathbb{Z}\).

Proposition 36 and Lemma 34 jointly imply Axiom 27 for all \(\alpha \leq 3003\) cases, which completes the proof of Theorem 8 on the basis of section 6.1. And we are done proving our main theorem if readers are satisfied with \(\alpha \leq 3003\). Otherwise, here is a conditional result.

**Proposition 37.** If, for some \(k, d, t\), the determinant \(h(x^*_1, y^*_1, \ldots, x^*_d, y^*_d)\) is not the zero polynomial in the coordinates of \(x^*_1, y^*_1, \ldots, x^*_d, y^*_d\), then an \((n, k, d, \alpha)\)-MSR code exists over some sufficiently large field. In particular, if \(h\) is never a zero polynomial, then Conjecture 7 holds.
Table 3. Parameter tuples with known instances found by brute force (jointly with some clever heuristics). Omitted entries inherit values from upper neighbors. One sees that the field size grows exponentially in the node number.

| $t$ | $|F|$ | $n$ | $k$ | $d$ | $\alpha$ | $\beta$ | $M$ |
|-----|------|-----|-----|-----|---------|--------|-----|
| 2   | $O(n)$ | $> d$ | $k$ | $2(k-1)$ | $k-1$ | 1 | $k(k-1)$ |
| 3   | 16   | 9   | 5   | 6   | 6       | 3       | 30  |
|     | 256  | 13  | .   | .   | .       | .       | .   |
|     | 2048 | 17  | .   | .   | .       | .       | .   |
|     | 32   | 11  | 7   | 9   | 15      | 5       | 105 |
|     | 256  | 13  | .   | .   | .       | .       | .   |
|     | 1024 | 15  | .   | .   | .       | .       | .   |
|     | 32   | 16  | 9   | 12  | 28      | 7       | 252 |
|     | 64   | 16  | 11  | 15  | 45      | 9       | 495 |
|     | 64   | 19  | 13  | 18  | 66      | 11      | 858 |
| 4   | 32   | 10  | 7   | 8   | 20      | 10      | 140 |
|     | 256  | 12  | .   | .   | .       | .       | .   |
|     | 64   | 13  | 10  | 12  | 84      | 28      | 840 |
|     | 128  | 14  | .   | .   | .       | .       | .   |
|     | 512  | 15  | .   | .   | .       | .       | .   |
| 5   | 128  | 11  | 9   | 10  | 70      | 35      | 630 |
|     | $n$  | $> k$ | .   | .   | .       | .       | .   |

Remark: Sometimes, in place of the combinatorial Nullstellensatz, the DeMillo–Lipton–Schwartz–Zippel lemma is cited. The lemma reads: Let $S$ be a finite subset of a field $F$. Let $f(x_1, \ldots, x_n)$ be a degree-$t$ polynomial over $F$. Select $s_1, \ldots, s_n \in S$ independently, uniformly at random. Then $f(s_1, \ldots, s_n) = 0$ with probability at most $t/|S|$. This lemma gives a strictly worse bound on the field size since $t$ is the “$l^1$-degree”, while the Nullstellensatz deals with the “$l^\infty$-degree”.

7.2. Brute force. Throughout Axioms [11 14 25 27] and [31] the first two conditions are always easy to fulfill. One queries the list of $[n, t]$-MDS codes over a chosen field and let $x^*_h$ be the columns of the generator matrix of a chosen code. Similarly, one chooses an $[n, k]$ (or $[n, k-t+1]$)-MDS code and let $w^*_h$ (or $y^*_h$) be the columns of its generator matrix. However, that does not say anything about whether the third, be it (MDSdt) or (MDSqt), is met. To demonstrate our strategy for generating practical codes, pretend that we want to build an $(n, 5, 6, 6)$-MSR code.

Algorithm 38. Here is what we did.

1. Let $F$ be of order 16; it could be realized by $F_{16} := F_2[z]/(z^4 + z + 1)$.
2. Each node chooses a unique point $a_h \in F$.
3. The $x$-vectors are of the form $x^*_h := [1 \ a^2_h \ a^6_h]$.
4. The $y$-vectors are of the form $y^*_h := [1 \ a_h \ a^3_h]$. 
(5) Enumerate and assert \((\text{MPSxt})\), \((\text{MPSyt})\), and \((\text{MPSdt})\).

The largest pool of points we can find is \(\{0, z^3, z^6, z^{-3}, z^{-6}, z^{-1}, z^{-2}, z^{-4}, z^{-8}\}\), as is posed in section 5.1. Therefore, we announce that there exist an \((9, 5, 6, 6)\)-MSR code over \(\mathbb{F}_{16}\). On the basis of this example, variables of this ensemble are as follows.

- \(n\) the number of nodes depends on how many points can be added to the point pool before \((\text{MPSxt})\), \((\text{MPSyt})\), or \((\text{MPSdt})\) breaks.
- One should try a larger field in order to find a larger collection of points.
- One can try a different pattern for \(x\)-vectors, for instance \([1 a^2 x_3]^{\top}\) and \([1 a^3 a_1 x_3]^{\top}\).
- One may try a different pattern for \(y\)-vectors.
- One may try to fulfill \((\text{MPSxt})\), \((\text{MPSwt})\), and \((\text{MPSqt})\) instead, i.e., the skew version.

It is unclear at this stage what is the best practice to find the point pool. So far brute force works better than any heuristics alone. We devoted some computing resources and the results are listed in Table 3.

7.3. General polynomial shorthand. Depending on how star vectors are selected, it is possible to further simplify the code description. For instance, for the example code in section 7.2 one can identify \(\phi | \bar{x}_1 \otimes S^3 Y, \phi | \bar{x}_2 \otimes S^3 Y, \phi | \bar{x}_3 \otimes S^3 Y\) with symmetric polynomials \(s_1(y, y', y''), s_2(y, y', y''), s_3(y, y', y'') \in \mathbb{F}[y, y', y'']_3\) of tri-degree at most \((3, 3, 3)\) without quadratic terms. Here \(\bar{x}_1, \bar{x}_2, \bar{x}_3\) are a basis of \(X\). Then the node content becomes the specialization \((s_1 + a^2 s_2 + a^3 s_3)(a_h, y', y'')\).

The help message becomes \((s_1 + a^2 s_2 + a^3 s_3)(a_h, y', y'')\).

8. Discussion

[RSKR09] section IV] argued that regenerating codes at the MPR point must specialize to their proposal. Subsequently, the product-matrix construction at the MPR point must be a direct generalization of the former proposal. This phenomenon is seen anew when determinant code generalizes layered code. The \([RSKR09]\)-product-matrix pair and the layered–determinant pair overlap at the \(k = d\) MPR point.

On the other side, at the MSR point, the product matrix is not succeeded by any more general code until now. We propose Atrahasis codes as a general code whose symmetric version includes the product matrix. Coincidentally, the exterior version of Atrahasis intersects the layered–determinant pair at the MSR point. This inspires us to wonder whether the symmetric and exterior versions are in fact the same construction.

Among Table 3 there is a row highlighted. It is a \((14, 10, 12, 84)\)-MSR code over the field of order 128. It is, in particular, a \([14, 10]\)-MPS code over the alphabet \(\mathbb{F}_{128}^{84}\) that detects four errors or corrects two. It has parameters

\[
\begin{bmatrix}
1 & d - k + 1 & 4 & 3 & 84 & 388 \\
3 & d - k & 4 & 12 & 336 & 2352 \\
\end{bmatrix}
\begin{bmatrix}
1 & 840 & 5880 \\
\end{bmatrix}.
\]
Figure 2. To the left: centralized model. Healthy nodes send help messages to an agent. The total bandwidth $\gamma_{ce}(2, 6)$ is the number of symbols passing solid lines. To the right: cooperative model. Healthy nodes send help messages directly to the failing nodes, while the latter can help each other. The total bandwidth $\gamma_{co}(2, 6)$ is the number of symbols passing solid lines.

For comparison, the improved Hadoop Distributed File System \cite{DH17} is a $[14, 10]$-MPS code over the field of order 256. It has parameters

\[
\begin{bmatrix}
1 & d-k+1 & \alpha & \alpha \log_2 |F| \\
* & d & \ast & d\beta \log_2 |F| \\
k & \ast & M & M \log_2 |F|
\end{bmatrix}
= \begin{bmatrix}
1 & 4 & 1 & 8 \\
* & 13 & \ast & 54 \\
10 & \ast & 10 & 80
\end{bmatrix}.
\]

Note that the latter matrix is not of rank one because Hadoop is not an M$\delta$R code to begin with. One sees that the $(14, 10, 12, 84)$-Atrahasis has a huge sub-packetization. But when it comes to homogeneous measures, such as $d\beta/M$, our $2352/5880 = 40\%$ is much better than $54/80 = 67.5\%$.

Appendix A. Bonus Property: Repairing Two Nodes at Once

In general, failures separate in time. But there may be circumstances where multiple nodes fail at once. Definitions \cite{01} and \cite{02} do not cover the case when there are more nodes to be repaired. What Definitions \cite{01} and \cite{02} guarantee is that, A, so far as there are $k$ healthy nodes left, the file is safe. B, if there are $d$ healthy nodes left, one may call the repairing protocol for each and every failing node. This does not capture how efficient the repairing can be done. For that, two definitions are made in \cite{CJM+13} \cite{SH13}, and related in \cite{YB19}.

**Definition 39.** Let there be $c$ failing nodes, and $d$ nodes are to help. The centralized (total) bandwidth $\gamma_{ce}(c, d)$ is the total number of symbols the $d$ helper nodes send to a central agent, who will repair the failings after gathering all help messages. The cooperative (total) bandwidth $\gamma_{co}(c, d)$ is how many symbols are sent over the network, from a helper node or a failing one, that contribute to repairing.

See Figure 2 for illustration. Note that we do not normalize the total bandwidth by the number of helping, or failing, nodes. One reason is that it is unclear what the denominator should be. When there is one failing node, $\gamma_{co}(1, d) = \gamma_{co}(1, d) = d\beta$. We now demonstrate how a $t = 3$ Atrahasis code from section 5.2 repairs two
failing nodes. Recall the parameters \((d, \alpha, \beta, M) = \left(\frac{3(k-1)}{2}, \frac{(k-1)}{2}, k - 2, k\alpha\right)\) and definitions \(X := \mathbb{F}^3\) and \(Y := \mathbb{F}^{k-2}\).

**A.1. The \(t = 3\) Atrahasis code under centralized model.** Say the \(f\)th and \(g\)th nodes fail and the first \(d\) nodes will help through an agent (centralized model). To the \(f\)th node, the \(h\)th healthy node wants to send the restriction \(\phi \mid x_h^* \otimes y_h^* \odot Y \setminus y_f^*\); to the \(g\)th it wants to send \(\phi \mid x_h^* \odot y_h^* \odot Y \setminus y_g^*\). Since the agent will take care of the redistribution, the \(h\)th will simply send \(\phi \mid x_h^* \odot y_h^* \odot Y \setminus (y_f^*, y_g^*)\) to the agent. The dimension of the subspace \(x_h^* \otimes y_h^* \odot Y \setminus (y_f^*, y_g^*)\), \(2k - 5\), is the number of symbols being sent. Multiplied by the number of helper nodes,

\[
d(2k - 5) = \frac{3}{2}(k - 1)(2k - 5) = 3k^2 - \frac{21}{2}k + \frac{15}{2}
\]

is the total bandwidth \(\gamma_{ce}(2, d)\) if done in this way. According to [YB19], the least possible bandwidth as a multiple of \(\alpha\) is

\[
\frac{2d}{d + 2 - k} \cdot \alpha = \frac{3(k-1)}{3(k-1)/2 + 2 - k} \cdot \frac{(k-1)(k-2)}{2} = 3k^2 - 15k + O(1).
\]

We are \(9k/2\) away from the optimal value. Remark: [YB19] achieve the optimal bandwidth \(2d\alpha/(d + 2 - k)\) for a significantly greater \(\alpha\).

In fact, Atrahasis code can do better. Imagine that when the agent receives the help messages, it will first reconstruct the \(f\)th node. Once done, the virtual \(f\)th node in agent’s memory is able to send help message \(\phi \mid x_f^* \otimes y_f^* \odot Y \setminus y_f^*\). In other words, one failing node becomes the helper of the other failing, so the second (\(g\)th) node requires \(d - 1\), not \(d\), helpers. Therefore, the helper nodes should and will send the following messages: For \(1 \leq h \leq d - 1\), the \(h\)th node sends \(\phi \mid x_h^* \otimes y_h^* \odot Y \setminus (y_f^*, y_g^*)\) to the agent, while the \(d\)th node sends \(\phi \mid x_h^* \odot y_h^* \odot Y \setminus y_f^*\). Then the reduced \(\gamma_{ce}(2, d)\) is

\[
(2k - 5)(d - 1) + 1(k - 2) = 3k^2 - \frac{23}{2}k + \frac{21}{2}.
\]

In actuality, Atrahasis can do even better. Recall (in section 5.2) that the \(f\)th node learns \(\phi \mid X \otimes Y \odot Y \setminus y_f^*\) from the help messages before it restricts to its usual content. For two failing nodes, the agent needs, and only needs, to learn \(\phi \mid X \otimes Y \odot Y \setminus (y_f^*, y_g^*)\). This subspace has dimension \(3(k - 2)^2\). Regardless of which node should send what, we claim our final bandwidth

\[
\gamma_{ce}(2, d) = 3(k - 2)^2 = 3k^2 - 12k + 12.
\]

This is \(3k\) away from the optimality.

**Appendix B. Benchmarks**

In Figures 4 to 9, we list some \(\alpha\) for small parameters from this and various other works. Compare them with lower bounds in Figure 3. See also Tables 1 and 3. Note that Atrahasis’s \(\alpha\) does not depend on \(n\) while other works either stick to \(n = d + 1\) or have \(\alpha \to \infty\) as \(n - d \to \infty\). In the following manner our sub-packetization level is polynomial in the lower bound (Theorem 3).

**Theorem 40.** Let \(t = d/(d - k + 1)\). Let \(\alpha = \binom{k - 1}{t - 1}\). Then

\[
\alpha \leq 2^{k - 1}, (k - 1)^{t - 1}.
\]
Figure 3. Lower bounds of the sub-packetization level $\alpha$ from [AG19, Theorem 1]. The horizontal axis is $k$. The vertical axis is $d - k + 1$ as $d < k$ cases are meaningless (and $d - (k - 1) = d/t = (k-1)/(t-1) = \alpha/\beta$ bears more semantics than $d-k$ does). Omit $k=1$ for triviality. It is later discovered that their bound is not valid when $k = d$.

Moreover, if $d, k, t, \alpha$ go to infinity with $d - k + 1 < C$ bounded, then

$$\alpha < \exp\left(\frac{k - 1}{4(d - k + 1)}\right)^{4C\log 2}.$$ 

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Figure 5. [SAK15] gave M\textsubscript{SR} codes with parameter quadruple \((n, k, d, \alpha) = (sq, n - q, n - 1, q^s)\) where \(q\) is a prime power. Some \(\alpha\)'s are put in boxes. Shortening applies but is omitted from this and the remaining figures. The shaded area is where their \(\alpha\) falls below ours.

Figure 6. [RKV16] extended [SAK15]'s result via providing M\textsubscript{SR} codes with parameters \((n, k, d, \alpha) = (sq, n - q - m, n - 1 - m, q^s)\). In other words, they allows \(n > d + 1\). We plot the \(\alpha\) when \(m = 1\) (i.e., when \(n = d + 2\)); and shade area where their \(\alpha\) falls below ours. Note that for any fixed \(k, d\) such that \(k < d\), their \(\alpha\) exceeds ours as \(n - d\) increases.

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Figure 7. [WTB16] contributed M\$SR codes with parameters $(n, k, d, \alpha) = (k + r, (r + 1)m, n - 1, r^m)$. They only enforce the optimal repair bandwidth for systematic nodes. This results in the least possible $\alpha$ among all M\$SR-related codes we have seen.

Figure 8. [GFV17] gave codes with parameters $(n, k, d, \alpha) = (k + r, k, k + \rho + 1, \rho^{k (r + 1)})$. They only enforce the optimal repair bandwidth for systematic nodes. Note that their $\alpha$ depends on $n$ beyond $k, d$. When $n = d + 1$, it coincides with Figure 5. We display the $n = d + 2$ case here. It was remarked that their $\alpha$ could be optimized further but we decided to print the very $\alpha$ given therein. Note that [GFV17] is chronologically before [RKV16].

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