Densely homogeneous fuzzy spaces

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ABSTRACT
We extend the concept of being densely homogeneous to include fuzzy topological spaces. We prove that our extension is a good extension in the sense of Lowen. We prove that a-cut topological space \((X, \mathcal{I}_a)\) of a DH fuzzy topological space \((X, \mathcal{I})\) is DH in general only for \(a = 0\).

Keywords:
Cut topologies
Densely homogeneous
Fuzzy CDH
Good extension

1. INTRODUCTION
As defined in [1], the notion of a fuzzy set in a set \(X\) is a function from \(X\) into the closed interval \([0,1]\). Accordingly, Chang [2] introduced the notion of a fuzzy topological space on a non-empty set \(X\) as a collection of fuzzy sets on \(X\), closed under arbitrary suprema and finite infima and containing the constant fuzzy sets \(0\) and \(1\). Mathematicians extended many topological concepts to include fuzzy topological spaces such as: separation axioms, connectedness, compactness and metrizability. Several fuzzy homogeneity concepts were discussed in [3-11]. A separable topological space \((X, \tau)\) is countable dense homogeneous (CDH) [12] if given any two countable dense subsets \(A\) and \(B\) of \((X, \tau)\) there is a homeomorphism \(f: (X, \tau) \rightarrow (X, \tau)\) such that \(f(A) = B\). It is known that CDH and DH topological concepts are independent. The study of DH topological spaces is continued in [22-28] and other papers. As a main goal of the present work we will show how the definition of DH topological spaces can be modified in order to define a good extension of it in fuzzy topological spaces. We will give relationships between CDH and DH fuzzy.
Throughout this paper, if $X$ is a set, then $|X| = \text{Card } X$ will denote its cardinality. We write $\mathbb{Q}$ (resp. $\mathbb{N}$) to denote the set of all rational numbers (resp. natural numbers). The closure of a fuzzy set $\lambda$ in a fuzzy topological space $(X, \mathcal{S})$ will be denoted by $\mathcal{C}(A)$. Associated with a given topological space $(X, \tau)$ and arbitrary subset $A$ of $X$, we denote the relative topology on $A$ by $\tau_A$, the closure of $A$ by $\mathcal{C}(A)$ and the boundary of $A$ by $\text{Bd}(A)$. Topological spaces as well as we will deal with cut topological space.

### 2. PRELIMINARIES

In this paper we shall follow the notations and definitions of [29] and [30]. If $(X, \tau)$ is a topological space, then the class of all lower semi-continuous functions from $(X, \tau)$ to $([0,1], \tau_0)$, where $\tau_0$ is the usual Euclidean topology on $[0,1]$, is a fuzzy topology on $X$. This fuzzy topology is denoted by $\omega(\tau)$. The following definitions and propositions will be used in the sequel:

**Definition 2.1.** [9] Let $X$ be a non-empty set, $A$ be a non-empty subset of $X$ and $P$ be a collection of fuzzy points in $X$. Then
- $\mathbb{Q}(A)$ will denote the set $\mathbb{Q}(A) = \{ x_r: x_r \text{ is a fuzzy point with } x \in A \text{ and } r \in \mathbb{Q} \cap (0,1) \}$.
- The support of $P$, denoted by $S(P)$, is defined by $S(P) = \{ x: x_a \in P \text{ for some } a \}$.

**Definition 2.2.** [21] A subset $A$ of a topological space $(X, \tau)$ is called a $\sigma$-discrete set if it is the union of countably many sets, each with the relative topology, being a discrete topological space.

**Definition 2.3.** [21] A topological space $(X, \tau)$ is called densely homogeneous (DH) iff
- $X$ has a $\sigma$-discrete dense subset.
- If $A$ and $B$ are two $\sigma$-discrete dense subsets of $X$, then there is a homeomorphism $h: (X, \tau) \rightarrow (X, \tau)$ such that $h(A) = B$.

**Definition 2.4.** [31] Associated with a given fuzzy topological space $(X, \mathcal{S})$ and arbitrary subset $M$ of $X$, we define the induced fuzzy topology on $M$ or the relative topology on $M$ by $\mathcal{S}_M = \{ \lambda | M: \lambda \in \mathcal{S} \}$.

**Definition 2.5.** [9] A fuzzy topological space $(X, \mathcal{S})$ is said to be semi-discrete iff for any $x \in X$, there exists a fuzzy point or a fuzzy crisp point $x_a$ for some $a$ with $x_a \in \mathcal{S}$. **Definition 2.6.** [32] Let $(X, \mathcal{S})$ be a fuzzy topological space and let $P$ be a collection of fuzzy points of $X$. Then $P$ is said to be
- Dense(I) if for every non-zero fuzzy open set $\lambda$ there exists $p \in P$ such that $p \in \lambda$.
- Dense(II) if $\mathcal{C}(\bigcup_{p \in P} p) = 1$.

**Definition 2.7.** [9] A fuzzy topological space $(X, \mathcal{S})$ is called separable iff there exists a countable dense(I) collection of fuzzy points of $X$. **Definition 2.8.** [33] A property $\mathcal{P}_f$ of a fuzzy topological space is said to be a good extension of the property $\mathcal{P}$ in classical topology iff whenever the fuzzy topological space is topologically generated, say by $(X, \tau)$, then $(X, \omega(\tau))$ has property $\mathcal{P}_f$ iff $(X, \tau)$ has property $\mathcal{P}$.

**Definition 2.9.** [34] Let $(X, \mathcal{S})$ be a fuzzy topological space and $a \in [0,1]$. The topology $\{ \lambda^{-1}(a,1]: \lambda \in \mathcal{S} \}$ on $X$ is called an $a$-cut topological space of $(X, \mathcal{S})$ and will be denoted by $\mathcal{S}_a$. The topological space $(X, \mathcal{S}_a)$ will be called an $a$-cut topological space of $(X, \mathcal{S})$. **Definition 2.10.** [9] A fuzzy topological space $(X, \mathcal{S})$ is said to be countable dense homogeneous; denoted CDH; iff
- $(X, \mathcal{S})$ is separable.
- If $P$ and $W$ are two countable dense(I) collections of fuzzy points of $X$, then there is a fuzzy homeomorphism $h: (X, \mathcal{S}) \rightarrow (X, \mathcal{S})$ such that $h(S(P)) = S(W)$.

**Proposition 2.11.** [9] Let $(X, \mathcal{S})$ be a fuzzy topological space and let $P$ be a collection of fuzzy points of $X$. Then we have the following
- If $P$ is dense(I), then $\mathcal{C}(S(P))$ is dense(II).
- If $P$ is dense(II), then $\mathcal{C}(S(P))$ is dense(I).

**Proposition 2.12.** [9] Let $(X, \tau)$ be a topological space, $A \subseteq X$, and $P$ be a collection of fuzzy points of $X$. Then we have the following
- If $A$ is dense in $(X, \tau)$, then $\mathcal{C}(A)$ is dense(I) in $(X, \omega(\tau))$.
- If $P$ is dense(I) in $(X, \omega(\tau))$, then $S(P)$ is dense in $(X, \tau)$.

**Proposition 2.13.** [35] Let $(X_1, \tau_1)$ and $(Y, \tau_2)$ be two topological spaces. Then $f: (X_1, \tau_1) \rightarrow (Y, \tau_2)$ is continuous iff $f: (X, \omega(\tau_1)) \rightarrow (Y, \omega(\tau_2))$ is fuzzy continuous. **Proposition 2.14.** [9] Let $f: (X, \mathcal{S}_1) \rightarrow (Y, \mathcal{S}_2)$ be a fuzzy homeomorphism map and $P$ be a collection of fuzzy points of $X$. Then we have the following

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3. **DH FUZZY TOPOLOGICAL SPACES**

In this section, we will define DH fuzzy topological spaces. We will prove that our new concept is a fuzzy topological property and a good extension of DH topological property in the sense of Lowen.

**Definition 3.1.** A collection $P$ of fuzzy points of a fuzzy topological space $(X, \mathcal{S})$ is said to be
- $\sigma$-semi-discrete iff $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $(A_n, \mathcal{S}_{A_n})$ is semi-discrete for all $n \in \mathbb{N}$.
- $\sigma$-semi-discrete dense (I) iff $P$ is $\sigma$-semi-discrete and $P$ is dense (I).
- $\sigma$-semi-discrete dense (II) iff $P$ is $\sigma$-semi-discrete and $P$ is dense (II).

**Definition 3.2.** A fuzzy topological space $(X, \mathcal{S})$ is said to be densely homogeneous (DH) iff
- $(X, \mathcal{S})$ has a $\sigma$-semi-discrete dense(I) collection of fuzzy points.
- If $P$ is a $\sigma$-semi-discrete dense(I) collection of fuzzy points of $(X, \mathcal{S})$, then there is a fuzzy homeomorphism $h: (X, \mathcal{S}) \rightarrow (X, \mathcal{S})$ such that $h(S(P)) = S(W)$.

**Lemma 3.3.** Let $(X, \mathcal{S})$ be a fuzzy topological space and $P$ be a $\sigma$-semi-discrete collection of fuzzy points of $X$. Then $\mathcal{S}(P)$ is a $\sigma$-semi-discrete collection of fuzzy points of $(X, \mathcal{S})$. Proof. It is easy to see that $h(S(P)) = S(\mathcal{S}(P))$ and hence the result is obvious. Theorem 3.4. A fuzzy topological space $(X, \mathcal{S})$ is DH iff
- $(X, \mathcal{S})$ has a $\sigma$-semi-discrete dense(II) collection of fuzzy points.
- If $P$ and $W$ are two $\sigma$-semi-discrete dense (II) collections of fuzzy points of $(X, \mathcal{S})$, then there is a fuzzy homeomorphism $h: (X, \mathcal{S}) \rightarrow (X, \mathcal{S})$ such that $h(S(P)) = S(W)$.

**Proposition 2.11.** Let $(X, \mathcal{S})$ be a $\sigma$-semi-discrete dense (I) collection of fuzzy points $P$. By Proposition 2.11 (i), $\mathcal{S}(S(P))$ is dense (I) and by Lemma 3.3, $\mathcal{S}(S(P))$ is $\sigma$-semi-discrete. Let $P$ and $W$ be any two $\sigma$-semi-discrete dense (II) collections of fuzzy points of $(X, \mathcal{S})$. Then by Proposition 2.11 (ii) and Lemma 3.3, $\mathcal{S}(S(P))$ and $\mathcal{S}(S(W))$ are both $\sigma$-semi-discrete dense (I) collections of fuzzy points of $(X, \mathcal{S})$. Then there is a fuzzy homeomorphism $h: (X, \mathcal{S}) \rightarrow (X, \mathcal{S})$ such that $h(S(P)) = S(W)$. Thus, $h(S(P)) = S(W)$. The proof of the other direction of this theorem is similar to the above one. Lemma 3.5. Let $(X, \tau)$ be a topological space. Let $A$ be a non-empty subset of $X$ and $P$ be a collection of fuzzy points of $X$. Then
- $\tau_A$ is the discrete topology iff $(A, \omega(\tau)_A)$ is semi-discrete.
- If $A$ is $\sigma$-discrete in $(X, \tau)$, then $\mathcal{S}(A)$ is $\sigma$-semi-discrete in $(X, \omega(\tau))$.
- If $P$ is $\sigma$-semi-discrete dense in $(X, \omega(\tau))$, then $\mathcal{S}(P)$ is $\sigma$-discrete in $(X, \tau)$.

**Proof.** (i) Suppose that $\tau_A$ is the discrete topology and let $x \in A$. Then there exists $U \in \tau$ such that $\{x\} = U \cap A$. So, $x \in x_A = x_{U \cap A} = x_{\{x\}} \in \omega(\tau)_A$. But clearly $\omega(\tau)_A$ is the crisp point with support $x$. Conversely, suppose that $(A, \omega(\tau)_A)$ is a semi-discrete fuzzy topological space and let $x \in A$. Then there exists a fuzzy point or a fuzzy crisp point $x_a$ such that $x_a \in \omega(\tau)_A$. Choose $\lambda \in \omega(\tau)$ such that $x_a = \lambda \cap x_A$. Thus, $\{x\} = x^{-1}(0,1] \cap A$ and hence $\{x\} \in \tau_A$.

(ii) Since $A$ is $\sigma$-discrete in $(X, \tau)$, then $A = \bigcup_{n=1}^{\infty} A_n$ with $\tau_{A_n}$ is the discrete topology for all $n \in \mathbb{N}$. So, by part (i) $(A_n, \omega(\tau)_{A_n})$ is semi-discrete for all $n$. Since $S(\mathcal{S}(A)) = A$, then $\mathcal{S}(S(P))$ is $\sigma$-semi-discrete in $(X, \omega(\tau))$. (iii) Since $P$ is $\sigma$-semi-discrete in $(X, \omega(\tau))$, then $\mathcal{S}(P) = \bigcup_{n=1}^{\infty} A_n$ with $(A_n, \omega(\tau)_{A_n})$ is semi-discrete for all $n \in \mathbb{N}$. So, by part (i) $\tau_{A_n}$ is the discrete topology for all $n$. Therefore, $\mathcal{S}(P)$ is $\sigma$-discrete in $(X, \tau)$.

**Theorem 3.6.** Let $(X, \tau)$ be a topological space. Then $(X, \tau)$ is DH iff $(X, \omega(\tau))$ is DH. Proof. Suppose that $(X, \tau)$ is DH. Then $(X, \tau)$ has a $\sigma$-discrete dense subset $A$. By Lemma 3.5 (ii) and
Proposition 2.12 (i), $\mathcal{Q}(A)$ is $\sigma$-semi-discrete dense (I) in $(X, \omega(\tau))$. Let $P$ and $W$ be two $\sigma$-semi-discrete dense(I) collections of fuzzy points of $(X, \omega(\tau))$. Then by Lemma 3.5 (iii) and Proposition 2.12 (ii), $S(P)$ and $S(W)$ are both $\sigma$-discrete dense subsets of $(X, \tau)$. Thus, there is a homeomorphism $h: (X, \tau) \to (X, \tau)$ such that $h(S(P)) = S(W)$. Proposition 2.13 ends the proof of this direction.

Conversely if $(X, \omega(\tau))$ is DH, then $(X, \omega(\tau))$ has a $\sigma$-discrete dense (I) collection of fuzzy points $P$. By Lemma 3.5 (iii) and Proposition 2.12 (ii), $S(P)$ is $\sigma$-discrete dense in $(X, \tau)$. Let $A$ and $B$ be two $\sigma$-discrete dense subsets of $(X, \tau)$. Then by Lemma 3.5 (ii) and Proposition 2.12 (i), $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are both $\sigma$-semi-discrete dense (I) collections of fuzzy points of $(X, \omega(\tau))$. Thus, there is a fuzzy homeomorphism $h: (X, \omega(\tau)) \to (X, \omega(\tau))$ such that $h(S(\mathcal{Q}(A))) = S(\mathcal{Q}(B))$. So, $h(A) = B$. Proposition 2.13 ends the proof of this direction.

Corollary 3.7. DH in fuzzy topological spaces is a good extension of DH in topological spaces. Recall that a property $\mathcal{P}$ of fuzzy topological spaces is called a fuzzy topological property if whenever $(X, \mathcal{S})$ possesses $\mathcal{P}$ and $h: (X, \mathcal{S}) \to (Y, \mathcal{S}')$ is a fuzzy homeomorphism, then $(Y, \mathcal{S}')$ possesses $\mathcal{P}$. Lemma 3.8. Let $f: X \to Y$ be a bijection map. Then

- For any two fuzzy sets $\lambda, \mu$ in $X$, $f(\lambda \cap \mu) = f(\lambda) \cap f(\mu)$.
- For any $A \subseteq X$, $f(X_A) = X_{f(A)}$.

Proof. Straightforward. Lemma 3.9. Let $f: (X, \mathcal{S}_1) \to (Y, \mathcal{S}_2)$ be a fuzzy homeomorphism. Let $A$ be a non-empty subset of $X$ and $P$ be a collection of fuzzy points of $X$. Then

- If $(A, (\mathcal{S}_1)_A)$ is semi-discrete, then $(f(A), (\mathcal{S}_2)_{f(A)})$ is semi-discrete.
- If $P$ is $\sigma$-semi-discrete, then $f(P)$ is $\sigma$-semi-discrete.

Proof. (i) Let $y \in f(A)$. Say $y = f(x)$ for some $x \in A$. Since $(A, (\mathcal{S}_1)_A)$ is semi-discrete, there exists $r \in (0, 1]$ such that $x_r \in (\mathcal{S}_1)_A$. Choose $\lambda \in \mathcal{S}_1$ such that $x_r = \lambda \cap X_A$. Then by Lemma 3.8, $y_r = (f(x)r) = f(x) = f(\lambda \cap X_A) = f(\lambda) \cap f(X_A) = f(\lambda) \cap X_{f(A)}$. Since $f$ is fuzzy open, it follows that $y_r \in (\mathcal{S}_2)_{f(A)}$. ii) Since $P$ is $\sigma$-semi-discrete, $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $(A_n, (\mathcal{S}_1)_{A_n})$ is semi-discrete for all $n \in \mathbb{N}$. By Proposition 2.14 (i), $S(f(P)) = f(S(P)) = f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$. Also, by (i) we have $(f(A_n), (\mathcal{S}_2)_{f(A_n)})$ is semi-discrete for all $n \in \mathbb{N}$. It follows that $f(P)$ is $\sigma$-semi-discrete.

Theorem 3.10. In fuzzy topological spaces, "Being "DH" is a fuzzy topological property. Proof. Assume $(X, \mathcal{S})$ is a DH fuzzy topological space and let $f: (X, \mathcal{S}_1) \to (Y, \mathcal{S}_2)$ be a fuzzy homeomorphism where $(Y, \mathcal{S}_2)$ is a fuzzy topological space. Choose a $\sigma$-semi-discrete dense(I) collection of fuzzy points $P$ of $(X, \mathcal{S}_1)$. According to Lemma 3.9 (ii) and Proposition 2.14 (ii), $f(P)$ will be $\sigma$-semi-discrete dense(I) in $(Y, \mathcal{S}_2)$. Let $P$ and $W$ be any two $\sigma$-semi-discrete dense(I) collections of fuzzy points of $(Y, \mathcal{S}_2)$. Then by Lemma 3.9 (ii) and Proposition 2.14 (ii), $f^{-1}(P)$ and $f^{-1}(W)$ are two $\sigma$-semi-discrete dense(I) collections of fuzzy points of $(X, \mathcal{S}_1)$. Since $(X, \mathcal{S}_1)$ is DH, there is a fuzzy homeomorphism $h: (X, \mathcal{S}_1) \to (X, \mathcal{S}_1)$ such that $h(S(f^{-1}(P))) = S(f^{-1}(W))$. Define $g: (Y, \mathcal{S}_2) \to (Y, \mathcal{S}_2)$ by $g = f \circ h \circ f^{-1}$. Then $g$ is a fuzzy homeomorphism. Using Proposition 2.14 (i), we can see that $g(S(P)) = S(W)$.

4. RELATIONSHIPS BETWEEN DH AND CDH FUZZY TOPOLOGICAL SPACES

In this section, we will give some relationships between DH and CDH fuzzy topological spaces.

The following useful lemma follows easily: Lemma 4.1. Let $(X, \mathcal{S})$ be a fuzzy topological space and $P$ be a collection of fuzzy points of $X$ with $S(P)$ is countable and non-empty. Then $P$ is $\sigma$-semi-discrete.

Theorem 4.2. Let $(X, \mathcal{S})$ be a fuzzy topological space for which $X$ is countable. Then $(X, \mathcal{S})$ is DH iff $(X, \mathcal{S})$ is semi-discrete. Proof. Since the result is obvious when $|X| = 1$, we will assume that $|X| > 1$. Suppose that $(X, \mathcal{S})$ is DH and assume on the contrary that $(X, \mathcal{S})$ is not semi-discrete. Then there exists $y \in X$ such that $y \notin \mathcal{S}$ for all $0 < a \leq 1$. Set $P = Q(X)$ and $W = Q(X \setminus \{y\})$. It is not difficult to see that $P$ and $W$ are dense (I). Also, by Lemma 4.1, $P$ and $W$ are $\sigma$-semi-discrete. So there is a fuzzy homeomorphism $h: (X, \mathcal{S}) \to (X, \mathcal{S})$ such that $h(S(P)) = S(W)$, therefore, $h(X) = X \setminus \{y\}$ which is a contradiction since $h$ is an onto map.

Conversely, suppose that $(X, \mathcal{S})$ is semi-discrete. Then by Proposition 2.15 (ii), $(X, \mathcal{S})$ is separable. Choose a countable dense (I) collection of fuzzy points $P$. Then $S(P)$ is countable and by Lemma 4.1, $P$ is $\sigma$-semi-discrete. Let $P$ and $W$ be any two $\sigma$-semi-discrete dense(I) collections of fuzzy points. Then by Proposition 2.15 (i), $S(P) = S(W) = X$ and the identity fuzzy map completes the proof. Corollary 4.3. Let $(X, \mathcal{S})$ be a fuzzy topological space for which $X$ is countable. Then $(X, \mathcal{S})$ is CDH iff $(X, \mathcal{S})$ is DH. Proof. Follows from Proposition 2.16 and Theorem 4.2. Theorem 4.4. If $(X, \mathcal{S})$ is separable and DH fuzzy topological space, then $(X, \mathcal{S})$ is CDH. Proof. Follows from the definitions and Lemma 4.1.

Recall that a fuzzy topological space $(X, \mathcal{S})$ is hereditarily separable if every subspace of $(X, \mathcal{S})$ is separable. Recall that a fuzzy topological space is second countable if it has a countable base. It is well known that second countable fuzzy topological spaces are hereditarily separable. Lemma 4.5. If $(X, \mathcal{S})$ is a hereditarily separable fuzzy topological space and $P$ is a $\sigma$-semi-discrete collection of fuzzy points of $(X, \mathcal{S})$, Densely homogeneous fuzzy spaces (Samer Al Ghour)
then $S(P)$ is countable. Proof. Since $P$ is $\sigma$-semi-discrete, then $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $(A_n, \mathcal{J}_A)$ is semi-discrete for all $n \in \mathbb{N}$. Since $(X, \mathcal{J})$ is hereditarily separable, then for each $n \in \mathbb{N}, (A_n, \mathcal{J}_A)$ is separable and by Proposition 2.15 (ii) it follows that $A_n$ is countable. Thus, $S(P)$ is countable.

Theorem 4.6. If $(X, \mathcal{J})$ is hereditarily separable and CDH fuzzy topological space, then $(X, \mathcal{J})$ is DH. Proof. Since $(X, \mathcal{J})$ is hereditarily separable, then it is separable. So, there exists a countable dense (I) collection of fuzzy points $P$ and by Lemma 4.1, $P$ is $\sigma$-semi-discrete. Let $P$ and $W$ be two $\sigma$-semi-discrete dense(I) collections of fuzzy points. Then by Lemma 4.5, $S(P)$ and $S(W)$ are countable. By Proposition 2.11, $Q(S(P))$ and $Q(S(W))$ are countable dense(I). Since $(X, \mathcal{J})$ is CDH, there is a fuzzy homeomorphism $h: (X, \mathcal{J}) \rightarrow (X, \mathcal{J})$ such that $h(S(P)) = h(S(Q(S(P)))) = S(Q(S(W))) = S(W)$. Corollary 4.7. Let $(X, \mathcal{J})$ be a hereditarily separable fuzzy topological space. Then $(X, \mathcal{J})$ is CDH iff $(X, \mathcal{J})$ is DH. Proof. Follows from Theorems 4.4 and 4.6. Corollary 4.8. Let $(X, \mathcal{J}_A)$ be a second countable fuzzy topological space. Then $(X, \mathcal{J})$ is CDH iff $(X, \mathcal{J})$ is DH.

5. CUT TOPOLOGICAL SPACES
In this section we will mainly show that a-cut topological space $(X, \mathcal{J}_a)$ of a fuzzy topological $(X, \mathcal{J})$ is DH in general only if $a = 0$. Lemma 5.1. Let $(X, \mathcal{J})$ be a fuzzy topological space. Let $B$ be non-empty subset of $X$ and let $P$ be a collection of fuzzy points of $X$. Then
- $(B, \mathcal{J}_B)$ is semi-discrete iff $(\mathcal{J}_B)_{B}$ is the discrete topology on $B$.
- If $B$ is $\sigma$-discrete in $(X, \mathcal{J}_P)$, then $Q(B)$ is $\sigma$-semi-discrete in $(X, \mathcal{J})$.
- If $P$ is $\sigma$-semi-discrete in $(X, \mathcal{J})$, then $S(P)$ is $\sigma$-discrete in $(X, \mathcal{J}_{\mathcal{J}})$.

Proof. (i) Suppose that $(B, \mathcal{J}_B)$ is semi-discrete and let $x \in B$. Then there exists a fuzzy point or a fuzzy crisp point $x_\alpha$ for some $\alpha \in \mathcal{B}$. Choose $\lambda \in \mathcal{J}$ such that $x_\alpha = \lambda \cap X_B$. Then $(\mathcal{J}_\mathcal{J})_{B}$ is the discrete topology on $B$ and let $x \in B$. Then there exists $\lambda \in \mathcal{J}$ such that $(x) = (\mathcal{J}_\mathcal{J})_{B}$ and $\lambda \cap X_B$ is the fuzzy or crisp point $x_\alpha(\mathcal{J})$, on the other hand, $\lambda \cap X_B \in \mathcal{B}$. Since $B$ is $\sigma$-discrete in $(X, \mathcal{J}_P)$, then $\bigcup_{n=1}^{\infty} B_n$ with $(\mathcal{J}_B)_{B}$ is the discrete topology for all $n \in \mathcal{N}$. By (i), $(\mathcal{J}_B)_{B}$ is semi-discrete for all $n \in \mathcal{N}$. Since $S(Q(B)) = B$, then $Q(B)$ is $\sigma$-discrete in $(X, \mathcal{J})$. (ii) Let $P$ be a $\sigma$-semi-discrete in $(X, \mathcal{J})$, then $S(P) = \bigcup_{n=1}^{\infty} A_n$ with $(\mathcal{J}_B)_{B}$ is semi-discrete for all $n \in \mathcal{N}$. By (i), $(\mathcal{J}_B)_{B}$ is the discrete topology on $A_n$ for all $n \in \mathcal{N}$. It follows that $S(P)$ is $\sigma$-discrete in $(X, \mathcal{J})$. Theorem 5.2. If $(X, \mathcal{J})$ is a DH fuzzy topological space, then $(X, \mathcal{J}_A)$ is DH.

Proposition 5.3. Let $(X, \mathcal{J})$ be a topological space with $X$ is countable. Then the following are equivalent:
- $(X, \mathcal{J})$ is CDH.
- $\tau$ is the discrete topology on $X$.
- $(X, \mathcal{J})$ is DH.

Theorem 5.4. Let $X$ be a countable set and let $(X, \mathcal{J})$ be a fuzzy topological space. Then the following are equivalent:
- $(X, \mathcal{J})$ is DH.
- $(X, \mathcal{J})$ is CDH.
- $(X, \mathcal{J}_A)$ is DH.
- $(X, \mathcal{J}_A)$ is CDH.

Proof. Follows from Theorem 4.2, Lemma 5.1 (i) and Proposition 5.3. In fact if $a > 0$, then $(X, \mathcal{J})$ being DH does not imply, in general, that $(X, \mathcal{J}_A)$ is DH. This will be explained in the following counterexample: Example 5.5. For fixed $0 < a < 1$, let $X = \{x, y\}$ and define $\mathcal{J} = \{\emptyset, x_{a/2} \cup y_{a/2} \cup x_{a/2} \cup y_{a/2}\}$. It is clear that $(X, \mathcal{J})$ is semi-discrete and so by Theorem 4.2, it is DH. On the other hand, since $\mathcal{J}_A = \{\emptyset, X\}$, $(X, \mathcal{J}_A)$ is not DH.
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