Energy Distribution of a Schwarzschild Black Hole in a Magnetic Universe

I. Radinschi*
”Gh. Asachi” Technical University, Iasi, Romania

October 25, 2018

Abstract

We obtain the energy distribution of a Schwarzschild black hole in a magnetic universe in the Tolman prescription.

PACS: 04. 20.-q; 04. 70.-s
Keywords: energy, Schwarzschild black hole

1 INTRODUCTION

The localization of energy is a long-standing problem in the theory of general relativity. Numerous attempts have been made for a solution.

Virbhadra and his collaborators investigated the problem of the energy-momentum localization by using the energy-momentum complexes. The results obtained for several particular space-times (the Kerr-Newman, the Einstein-Rosen and the Bonnor-Vaidya) lead to the conclusion that different energy-momentum complexes give the same energy distribution for a given space-time [1]-[6]. Aguirregabiria, Chamorro and Virbhadra [7] showed that several energy-momentum complexes coincide for any Kerr-Schild class metric. Xulu obtained interesting results about the energy distribution of a charged dilaton black hole [8] and about the energy associated with a Schwarzschild black hole in a magnetic universe [9]. Also, recently, Xulu [10] obtained the total energy of a model of universe based on the Bianchi I type

*jessica@etc.tuiasi.ro
metric. The author calculated the energy distribution of a dyonic dilaton black hole [11] and the energy of the Bianchi type I solution [12]. Also, we obtained the energy distribution in a static spherically symmetric nonsingular black hole space-time [13]. Recently, Virbhadra [14] shows that different energy-momentum complexes give the same and reasonable results for many space-times.

Xulu [9] obtained the energy associated with a Schwarzschild black hole in a magnetic universe. Melvin’s magnetic universe [15] is a solution of the Einstein-Maxwell equations corresponding to a collection of parallel magnetic lines of force held together by mutual gravitation. The physical structure of the magnetic universe and its dynamical behavior was studied by Thorne. Ernst [16] obtained axially symmetric exact solution to the Einstein-Maxwell equations representing a Schwarzschild black hole immersed in Melvin’s uniform magnetic universe.

The purpose of this paper is to compute the energy distribution for the Ernst space-time by using the Tolman prescription. We use the geometrized units ($G = 1$, $c = 1$) and follow the convention that the Latin indices run from 0 to 3.

2 ENERGY IN TOLMAN’S PRESCRIPTION

It is interesting to evaluate the energy distribution of a magnetic black hole. We know that the Einstein-Maxwell equations are

\[ R_{i}^{k} - \frac{1}{2} g_{i}^{k} R = 8\pi T_{i}^{k}, \]

\[ F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \]

\[ \frac{1}{\sqrt{-g}} (\sqrt{-g} F^{ik})_{,k} = 4\pi J^{i}. \]

The energy-momentum of the electromagnetic field is given by

\[ T_{i}^{k} = \frac{1}{4\pi} (-F_{im} F^{km} + \frac{1}{4} g_{i}^{k} F_{mn} F^{mn}). \]

Ernst [16] obtained the axially symmetric electrovac solution ($J^{i} = 0$) to the equations (1), (2) and (3) that corresponds to a Schwarzschild black hole in Melvin’s magnetic universe.
The metric is given by

\[ ds^2 = \Delta^2[(1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2d\theta^2] - \Delta^{-2}r^2\sin^2\theta d\varphi^2. \quad (5) \]

The Cartan components of the magnetic field are given by

\[ H_r = \Delta^{-2}B_0\cos\theta, \]
\[ H_\theta = -\Delta^{-2}B_0(1 - \frac{2M}{r})^{\frac{1}{2}}\sin\theta, \quad (6) \]

where

\[ \Delta = 1 + \frac{1}{4}B_0^2r^2\sin^2\theta \quad (7) \]

and \( M \) and \( B_0 \) are constants. We note that the Ernst solution is a black hole solution and \( r = 2M \) is the event horizon.

The Tolman energy-momentum complex [17] is given by

\[ \Upsilon^k_i = \frac{1}{8\pi}U_i^{kl}, \quad (8) \]

where \( \Upsilon^0_\theta \) and \( \Upsilon^0_\alpha \) are the energy and momentum components.

\[ U_i^{kl} = \sqrt{-g}(-g^{ik}V^l_j + \frac{1}{2}g^{ik}_{\ jl}g^{pm}V_{pm}), \quad (9) \]

with

\[ V_{lk} = -\Gamma^i_{jk} + \frac{1}{2}g_{ij}\Gamma^m_{mk} + \frac{1}{2}g^m_{ij}\Gamma^n_{nj}. \quad (10) \]

Also, the energy-momentum complex \( \Upsilon^k_i \) also satisfies the local conservation laws

\[ \frac{\partial \Upsilon^k_i}{\partial x^k} = 0. \quad (11) \]

The energy and momentum in Tolman prescription are given by

\[ P_i = \iiint \Upsilon^0_i dx^1dx^2dx^3. \quad (12) \]

Using Gauss’s theorem we obtain

\[ P_i = \frac{1}{8\pi} \int \int U^0_i n_\alpha dS, \quad (13) \]
where \( n_\alpha = (\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}) \) are the components of a normal vector over an infinitesimal surface element \( dS = r^2 \sin \theta d\theta d\phi \).

The Tolman energy-momentum complex gives the correct result if the calculations are carried out in quasi-Cartesian coordinates. We transform the line element (1) to quasi-Cartesian coordinates \( t, x, y, z \) according to

\[
\begin{align*}
r &= (x^2 + y^2 + z^2)^{1/2}, \\
\theta &= \cos^{-1}(\frac{z}{\sqrt{x^2 + y^2 + z^2}}), \\
\varphi &= \tan^{-1}(y/x).
\end{align*}
\]

(14)

The metric (1) becomes [9]

\[
ds^2 = \Delta^2(1 - \frac{2M}{r}) dt^2 - \left[ \Delta^2(\frac{ax^2}{r^2}) + \Delta^{-2}(\frac{y^2}{x^2 + y^2}) \right] dx^2 - \left[ \Delta^2(\frac{ay^2}{r^2}) + \Delta^{-2} \right] dy^2 - \Delta^2 \left[ \frac{2Mz^2}{r^2(r - 2M)} \right] dz^2 - \Delta^2 \left[ \frac{4Mxz}{r^2(r - 2M)} \right] dxdz - \Delta^2 \left[ \frac{4Myz}{r^2(r - 2M)} \right] dydz,
\]

where

\[
a = \frac{2M}{r - 2M} + \frac{r^2}{x^2 + y^2}. \quad (16)
\]

The components of the Tolman energy-momentum complex are calculated with the Maple GR Tensor II Release 1.50.

Because the components of the \( U_{ki}^{\alpha} \) are too many we give only those which are involved in the calculation of the energy

\[
\begin{align*}
U_{01}^{01} &= \frac{2Mx}{r} + \frac{\Delta^4 - 1}{2} \frac{x}{x^2 + y^2}, \\
U_{02}^{02} &= \frac{2My}{r} + \frac{\Delta^4 - 1}{2} \frac{y}{x^2 + y^2}, \\
U_{03}^{03} &= \frac{2Mz}{r^3}.
\end{align*}
\]

(17)

Using (12), (17) and applying (13) we obtain the energy distribution for the Ernst space-time

\[
E(r) = M + \frac{1}{8\pi} \int_{\theta=0}^{\theta} \int_{\varphi=0}^{2\pi} \frac{(\Delta^4 - 1)}{2} r^2 \sin \theta d\theta d\varphi.
\]

(18)
We replace in (18) the value of \( \Delta \) from (7) and consider the values of \( G \) and \( c \) and we have

\[
E(r) = Mc^2 + \frac{1}{6} B_0^2 r^3 + \frac{1}{20} \frac{G}{c^4} B_0^4 r^5 + \frac{1}{140} \frac{G^2}{c^8} B_0^6 r^7 + \frac{1}{2520} \frac{G^3}{c^{12}} B_0^8 r^9.
\]  

(19)

The relation (19) can be also written

\[
E(r) = Mc^2 + \frac{1}{8\pi} \iiint B_0^2 dV + \frac{1}{20} \frac{G}{c^4} B_0^4 r^5 + \frac{1}{140} \frac{G^2}{c^8} B_0^6 r^7 + \frac{1}{2520} \frac{G^3}{c^{12}} B_0^8 r^9.
\]  

(20)

In the expression of the energy distribution the first term represents the rest mass-energy of the Schwarzschild black hole, the second is the special relativistic value for the energy of the uniform magnetic field and the other terms that remain in the expression are due to the general relativistic effect.

3 DISCUSSION

The main purpose of the present paper is to show that the problem of the localization of energy in relativity can be solved by using the energy-momentum complexes. The Bondi opinion [18] is that a nonlocalizable form of energy is not admissible in relativity so its location can in principle be found. Some interesting results which have been found recently [7]-[13], support the idea that the several energy-momentum complexes can give the same and acceptable result for a given space-time. Also, in his recent paper Virbhadra [14] emphasized that though the energy-momentum complexes are non-tensors under general coordinate transformations, the local conservation laws with them hold in all coordinate systems. Chang, Nester and Chen [19] showed that the energy-momentum complexes are actually quasilocal and legitimate expressions for the energy-momentum.

We calculated the energy distribution for the Ernst space-time in the Tolman prescription and find the same result as the result obtained by S. S. Xulu [9] in the Einstein prescription. The energy increases because of the presence of the magnetic field.

References

[1] K. S. Virbhadra, Phys. Rev. D41 (1990) 1086.
[2] K. S. Virbhadra, *Phys. Rev. D* **42** (1990) 2919.

[3] F. I. Cooperstock and S. A. Richardson, in *Proc. 4th Canadian Conf. on General Relativity and Relativistic Astrophysics* (World Scientific, Singapore, 1991).

[4] N. Rosen and K. S. Virbhadra, *Gen. Rel. Grav.* **25** (1993) 429.

[5] K. S. Virbhadra, *Pramana-J. Phys.* **45** (1995) 215.

[6] A. Chamorro and K. S. Virbhadra, *Pramana-J. Phys.* **45** (1995) 181.

[7] J. M. Aguirregabiria, A. Chamorro and K. S. Virbhadra, *Gen. Rel. Grav.* **28** (1996) 1393.

[8] S. S. Xulu, *Int. J. Mod. Phys.* **D7** (1998) 773.

[9] S. S. Xulu, *Int. J. Mod. Phys.* **A**, in press

[10] S. S. Xulu, *gr-qc/9910015*

[11] I. Radinschi, *Acta Physica Slovaca*, **49**(5) (1999) 789.

[12] I. Radinschi, *Acta Physica Slovaca*, in press

[13] I. Radinschi, *Modern Physics Letters A*,(15), Nos. 11&12 (2000) 803.

[14] K. S. Virbhadra, *Phys. Rev. D* **60** (1999) 104041.

[15] M. A Melvin, *Phys. Lett.* **8** (1964) 65.

[16] F. J. Ernst, *J. Math. Phys.* **15** (1974) 1409.

[17] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, London, 1934, 227)

[18] H. Bondi, *Proc. R. Soc. London A* **427** (1990) 249.

[19] Chia-Chen Chang, J. M. Nester and Chiang-Mei Chen, *Phys. Rev. Lett.* **83** (1999) 1897.