Differential Geometry

Log-concavity of complexity one Hamiltonian torus actions

Log-concavité des actions toriques hamiltoniennes de complexité un

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1. Introduction

In statistical physics, the relation $S(E) = k \log W(E)$ is called Boltzmann’s principle where $W$ is the number of states with given values of macroscopic parameters $E$ (like energy, temperature, ...), $k$ is the Boltzmann constant, and $S$ is the entropy of the system, which measures the degree of disorder in the system. For the additive values $E$, it is well known that the entropy is always a concave function. (See [9] for more details.) In a symplectic setting, consider a Hamiltonian $G$-manifold $(M, \omega)$ with the moment map $\mu: M \rightarrow g^*$. The Liouville measure $m_L$ is defined by

$$m_L(U) := \int_U \frac{\omega^n}{n!}$$

for any open set $U \subset M$. Then the push-forward measure $m_{DH} := \mu_* m_L$, called the Duistermaat–Heckman measure, can be regarded as a measure on $g^*$ such that for any Borel subset $B \subset g^*$, $m_{DH}(B) = \int_{\mu^{-1}(B)} \frac{\omega^n}{n!}$ tells us that how many states of our system have momenta in $B$. By the Duistermaat–Heckman theorem [2], $m_{DH}$ can be expressed in terms of the density function $DH(\xi)$ with respect to the Lebesgue measure on $g^*$. Therefore the concavity of the entropy of a given periodic Hamiltonian system on $(M, \omega)$ can be interpreted as the log-concavity of $DH(\xi)$ on the image of $\mu$. A. Okounkov [10]
proved that the density function of the Duistermaat–Heckman measure is log-concave on the image of the moment map for the maximal torus action, when \((M, \omega)\) is the co-adjoint orbit of some classical Lie groups. In [3], W. Graham showed that the log-concavity of the density function of the Duistermaat–Heckman measure also holds for any Kähler manifold admitting a holomorphic Hamiltonian torus action. V. Ginzberg and A. Knutson conjectured independently that the log-concavity holds for any Hamiltonian G-manifolds, but this turns out to be false in general, shown by Y. Karshon [5]. Further related works can be found in [7] and [1].

As noted in [5] and [3], log-concavity holds for Hamiltonian toric (i.e. complexity zero) actions, and Y. Lin dealt with the log-concavity of complexity two Hamiltonian torus actions in [7]. However, there is no result on the log-concavity of complexity one Hamiltonian torus action. This is why we restrict our interest to complexity one. From now on, we assume that \(M\) is an open \(m\)-face (resp. \((n-m)-face\) of the x-ray, \(\xi\)). Let \(\xi \subseteq M\) be the set of points whose stabilizers are \(T_n\)-actions. By relabeling, we can assume that the \(T_n\)'s are connected and the stabilizer of points in \(M\) is \(T_n\)-dimensional. Our interest is mainly in open dense in its corresponding reduced symplectic form on \(M\).

Now, we define the x-ray of our action. Let \(T_1, \ldots, T_N\) be the subgroups of \(T_n\) which occur as stabilizers of points in \(M\). Let \(M_i\) be the set of points whose stabilizers are \(T_i\). By relabeling, we can assume that the \(M_i\)'s are connected and the stabilizer of points in \(M_i\) is \(T_i\). Then, \(M = M_i\) is a disjoint union of \(M_i\)'s. Also, it is well known that \(M_i\) is open dense in its closure and is just a component of the fixed set \(M\). Let \(M_i\) be the set of \(M_i\)'s. Then, the \(x\)-ray of \((M^{\infty}, \omega, \mu)\) is defined as the set of \(\mu(M_i)\)'s. Here, we recall a basic lemma:

**Lemma 2.1.** (See [4, Theorem 3.6].) Let \(h\) be the Lie algebra of \(T_i\). Then \(\mu(M_i)\) is locally of the form \(x + h^\perp\) for some \(x \in t^\ast\).

By this lemma, \(\dim_{\mathbb{R}} \mu(M_i) = m\) for \((n-1-m)\)-dimensional \(T_i\). Each image \(\mu(M_i)\) (resp. \(\mu(M_i)\)) is called an \(m\)-face (resp. an open \(m\)-face) of the x-ray if \(M_i\) is \((n-1-m)\)-dimensional. Our interest is mainly in open \((n-2)\)-faces of the x-ray, i.e. codimension one in \(t^\ast\). Fig. 1 is an example of \(x\)-ray with \(n = 3\) where thick lines are \((n-2)\)-faces. Now, we can prove the main theorem.

**Proof of Theorem 1.1.** When \(n = 2\), we obtain a proof by [6, Lemma 2.19]. So, we assume \(n \geq 3\). Pick arbitrary two points \(x_0, x_1\) in the image of \(\mu\). We should show that

\[
t \log(DH(x_1)) + (1-t) \log(DH(x_0)) \leq \log(DH(tx_1 + (1-t)x_0))
\]

(1)
for each \( t \in [0, 1] \). Put \( x_t = tx_1 + (1 - t)x_0 \).

Let us fix a decomposition \( T = S^1 \times \cdots \times S^1 \). By the decomposition, we identify \( t \) with \( \mathbb{R}^{n-1} \), and \( t \) carries the usual Riemannian metric \( \langle \cdot, \cdot \rangle_0 \) which is a bi-invariant metric. This metric gives the isomorphism

\[
\iota : t \mapsto t^* , \quad X \mapsto \langle X, X \rangle_0.
\]

For a small \( \epsilon > 0 \), pick two regular values \( \xi_1 \) in the ball \( B(x_t, \epsilon) \) for \( i = 0, 1 \) which satisfy the following two conditions:

i. \( \xi_1 - \xi_0 \in t(\mathbb{Q}^{n-1}) \),

ii. the line \( L \) containing \( \xi_0, \xi_1 \) in \( t^* \) meets each open \( m \)-face transversely for \( m = 1, \ldots, n - 2 \).

Transversality guarantees that the line does not meet any open \( m \)-face for \( m \leq n - 3 \). Put

\[
\xi_t = t\xi_1 + (1 - t)\xi_0 \quad \text{and} \quad X = t^{-1}(\xi_1 - \xi_0).
\]

Let \( t \subset t \) be the one-dimensional subalgebra spanned by \( X \). By i., \( t \) becomes a Lie algebra of a circle subgroup of \( T \), call it \( K \). Let \( t' \) be the orthogonal complement of \( t \) in \( t \). Again by i., \( t' \) becomes a Lie subgroup of an \((n - 2)\)-dimensional subtorus of \( T \). Let

\[
p : t^* \to \mathbb{R} = \langle t' \rangle
\]

be the orthogonal projection along \( t^* = \iota(t') \). If we put \( \mu' = p \circ \mu \), then \( \mu' : M \to t^* \) is a moment map of the restricted \( T' \)-action on \( M \). Put \( \xi' = \iota(\xi_t) \) for \( t \in [0, 1] \).

We want to show that \( \xi' \) is a regular value of \( \mu' \). For this, we show that each point \( x \in \mu'^{-1}(\xi') \) is a regular point of \( \mu' \). By ii. and Lemma 2.1, stabilizer \( T_x \) is finite or one-dimensional. If \( T_x \) is finite, then \( x \) is a regular point of \( \mu \) so that it is also a regular point of \( \mu' \). If \( T_x \) is one-dimensional, then \( \mu(x) \) is a point of an open \((n - 2)\)-face \( \mu(M_i) \) such that \( x \in M_i \).

Let \( h \) be the Lie algebra of \( T_i = T_x \). By Lemma 2.1, \( p(\mu(M_i)) = p(h^+) \), and the kernel \( t \) of \( p \) is not contained in \( h^+ \) by transversality. So, \( p(h^+) \) is the whole \( t^* \) because \( \dim h^+ = \dim t^* \), and this means that \( x \) is a regular point of \( \mu' \). Therefore, we have shown that \( \xi' \) is a regular value of \( \mu' \).

Since \( \xi' \) is a regular value, the preimage \( \mu'^{-1}(\xi') \) is a manifold and \( T' \) acts almost freely on it, i.e. stabilizers are finite. So, if we denote by \( M_{\xi'} \) the symplectic reduction \( \mu'^{-1}(\xi')/T' \), then it becomes a symplectic orbifold carrying the induced symplectic \( T'/T' \)-action. We can observe that the image of \( \mu'^{-1}(\xi') \) through \( \mu \) is the thick dashed line in Fig. 1. Since \( K/(K \cap T') \cong T/T' \), we will regard \( K/(K \cap T') \) and \( t \) as \( T/T' \) and its Lie algebra, respectively. The map \( \mu_X := \langle \cdot, X \rangle \) induces a map on \( M_{\xi'} \) by \( T \)-invariance of \( \mu \), call it just \( \mu_X \) where \( \langle \cdot, t \times t' \mapsto \mathbb{R} \) is the evaluation pairing. Then, we can observe that \( \mu_X \) is a Hamiltonian of the \( K/(K \cap T') \)-action on \( M_{\xi'} \), and that \( M_{\xi'} \) is symplectomorphic to the symplectic reduction of \( M_{\xi'} \) at the regular value \( \langle \xi_t, X \rangle \) with respect to \( \mu_X \). If we denote by \( DH_X \) the Duistermaat–Heckman function of \( \mu_X : M_{\xi'} \to \mathbb{R} \), then we have \( DH_X(\xi_t) = DH_X(\langle \xi_t, X \rangle) \) for \( t \in [0, 1] \). Since \( M_{\xi'} \) is a four-dimensional symplectic orbifold with Hamiltonian circle action, \( DH_X \) is log-concave by Lemma 2.2 below. Since \( x_t \) and \( \xi_t \) are sufficiently close and \( DH \) is continuous by [2], we can show (1) by log-concavity of \( DH_X \). \( \square \)

**Lemma 2.2.** Let \((N, \sigma)\) be a closed four-dimensional Hamiltonian \( S^1 \)-orbifold. Then the density function of the Duistermaat–Heckman measure is log-concave.

**Proof.** Let \( \phi : N \to \mathbb{R} \) be a moment map. Then the density function \( DH : \text{Im} \phi \to \mathbb{R}_{\geq 0} \) of the Duistermaat–Heckman measure is given by

\[
DH(t) = \int_{N_t} \sigma_t.
\]

for any regular value \( t \in \text{Im} \phi \). Let \((a, b) \subset \text{Im} \phi \) be an open interval consisting of regular values of \( \phi \) and fix \( t_0 \in (a, b) \). By the Duistermaat–Heckman theorem [2], \([\sigma_t] - [\sigma_{t_0}] = -e(t - t_0)\) for any \( t \in (a, b) \), where \( e \) is the Euler class of the \( S^1 \)-fibration \( \phi^{-1}(t_0) \to \phi^{-1}(t_0)/S^1 \). Therefore

\[
DH'(t) = -\int_{N_t} e
\]

and

\[
DH''(t) = 0
\]

for any \( t \in (a, b) \). Note that \( DH(t) \) is log-concave on \((a, b)\) if and only if it satisfies \( DH(t) \cdot DH''(t) - DH'(t)^2 \leq 0 \) for all \( t \in (a, b) \). Hence \( DH(t) \) is log-concave on any open intervals consisting of regular values.
Let \( c \) be any interior critical value of \( \phi \) in \( \text{Im} \phi \). Then it is enough to show that the jump in the derivative of \( \log DH' \) is negative at \( c \). First, we will show that the jump of the value \( DH'(t) = -\int_{\mathcal{N}_t} e \) is negative at \( c \). Choose a small \( \epsilon > 0 \) such that \( (c - \epsilon, c + \epsilon) \) does not contain a critical value except for \( c \). Let \( \mathcal{N}_c \) be a symplectic cut of \( \phi^{-1}[c - \epsilon, c + \epsilon] \) along the extremum so that \( \mathcal{N}_c \) becomes a closed Hamiltonian \( S^1 \)-orbifold whose maximum is the reduced space \( M_{c+\epsilon} \) and the minimum is \( M_{c-\epsilon} \). Using the Atiyah–Bott–Berline–Vergne localization formula for orbifolds [8], we have

\[
0 = \int_{\mathcal{N}_c} 1 = \sum_{p \in N^S \cap \phi^{-1}(c)} \frac{1}{d_p} \frac{1}{p_1 p_2 \lambda^2} + \int_{M_{c-\epsilon}} \frac{1}{\lambda + \epsilon} + \int_{M_{c+\epsilon}} \frac{1}{-\lambda - \epsilon},
\]

which is equivalent to

\[
0 = \sum_{p \in N^S \cap \phi^{-1}(c)} \frac{1}{p_1 p_2} = \int_{\mathcal{N}_{c-\epsilon}} e^- - \int_{\mathcal{N}_{c+\epsilon}} e^+,
\]

where \( d_p \) is the order of the local group of \( p \), \( p_1 \) and \( p_2 \) are the weights of the tangential \( S^1 \)-representation on \( T_p N \), and \( e^- (e^+ \text{ respectively}) \) is the Euler class of \( \phi^{-1}(c - \epsilon) (\phi^{-1}(c + \epsilon) \text{ respectively}) \). Since \( c \) is in the interior of \( \text{Im} \phi \), we have \( p_1 p_2 < 0 \) for any \( p \in N^S \cap \phi^{-1}(c) \). Hence the jump of \( DH'(t) = -\int_{\mathcal{N}_t} e \) is negative at \( c \), which implies that the jump of \( \log DH(t)' = \frac{DH(t)}{DH(t)} \) is negative at \( c \) (by continuity of \( DH(t) \)). This finishes the proof. \( \square \)

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