REPRESENTABILITY THEOREMS, UP TO HOMOTOPY

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Abstract. We prove two representability theorems, up to homotopy, for presheaves taking values in a closed symmetric combinatorial model category $V$. The first theorem resembles the Freyd representability theorem, the second theorem is closer to the Brown representability theorem. As an application we discuss a recognition principle for mapping spaces.

1. Introduction

It is a classical question in homotopy theory whether for a given space $X \in \mathcal{T}_{\text{op}}$ there exists a space $Y \in \mathcal{T}_{\text{op}}$ such that $X \simeq \text{hom}(S^1, Y)$. Several solutions to this question have emerged, beginning with Sugawara’s work in [26]. The approach proposed by Stasheff, [25], Boardman and Vogt, [5], and May, [20] are known nowadays as operadic, while Segal’s loop space machine, [24], is closer to Lawvere’s algebraic theory. Later on, the canonical delooping machine by Badzioch, Chung, and Voronov, [2], provided a simplicial algebraic theory $\mathcal{T}_n$, $n \geq 0$ allowing for the recognition of an $n$-fold loop spaces as a homotopy algebras over $\mathcal{T}_n$.

Is there a similar recognition principle for mapping spaces of the form $X = \text{hom}(A, Y)$ for $A \in \mathcal{T}_{\text{op}}$ other than $S^n$? For $A = S^2 \vee S^3$, there is no simplicial algebraic theory $\mathcal{T}$ such that all spaces of the form $X = \text{hom}(S^2 \vee S^3, Y) = \Omega^2 Y \times \Omega^3 Y$ are homotopy algebras over $\mathcal{T}$, [2, p. 2]. More generally, if a space $A$ has rational homology in more than one non-zero dimension, there is no simplicial algebraic theory $\mathcal{T}$ such that all the spaces of the form $X = \text{hom}(A, Y)$ would have the structure of homotopy algebras over $\mathcal{T}$, [3].

In this paper we suggest to use a bigger category than an algebraic theory for the recognition of arbitrary mapping spaces, up to homotopy. This category will be closed under arbitrary homotopy colimits, unlike an algebraic theory which is closed only under finite coproducts. The minimal subcategory of spaces containing $A$ and closed under the homotopy colimits is denoted $C(A)$. This approach was first introduced by Badzioch, the second author, and Dorabiala in [1], where an attempt to limit the homotopy colimits involved was made. We choose a different approach and consider the functors defined on the large subcategory of spaces $C(A)$. Given a space $X \in \mathcal{T}_{\text{op}}$, suppose there exists a functor $F: C(A) \rightarrow \mathcal{T}_{\text{op}}$ taking homotopy colimits to homotopy limits and satisfying $F(A) \simeq X$. The question of whether there exists a space $Y$ such that $X \simeq \text{hom}(A, Y)$ for some $Y$ is equivalent to the question of representability of the functor $F$, up to homotopy.

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Theorems about representability of functors in mathematics are naturally divided into two main types: Freyd and Brown representability theorems. In both cases some exactness condition for the functor under consideration is necessary. Theorems of Freyd type use a set-theoretical assumption about the functor, for example the solution set condition or accessibility, [14, 3, Ex. G,J], [17, 4.84], [22, 1.3]. Theorems of Brown type use set-theoretical assumptions about the domain category, such as the existence of sufficiently many compact objects, [6], [21], [18], [13].

In this paper we address the question of representability of functors, up to homotopy, taking values in a closed symmetric combinatorial model category. After the technical preliminaries in Section 2, we prove the Freyd version of the representability theorem, up to homotopy, in Section 3. The solution set condition is replaced by the requirement that the functor be small. Theorem 3.1 generalizes theorem [17, 4.84] to functors defined on a $\mathcal{V}$-model category, instead on a $\mathcal{V}$-category. On the other hand, this is a generalization of a theorem published by the first author on the representability of small contravariant functors from spaces to spaces, [8].

Brown representability, up to homotopy, is proved in Section 4. The set-theoretical condition concerning the domain category is local presentability. In other words, we show that for any $\mathcal{V}$-presheaf $H$ defined on a combinatorial $\mathcal{V}$-model category $\mathcal{M}$ and taking homotopy colimits to homotopy limits, there is a fibrant object $Y \in \mathcal{M}$ and a natural transformation $h \cdot H(\cdot) \rightarrow \text{hom}(\cdot, Y)$, which is a weak equivalence for every cofibrant $X \in \mathcal{M}$. A similar theorem for functors taking values in simplicial sets was proved by Jardine in [16]. However, the conditions required for the Brown representability, up to homotopy, to hold are formulated for the homotopy category of the model category and do not allow for an easy verification in an arbitrary combinatorial model category.

In Section 5 we provide an example of a non-small presheaf, defined on a non-combinatorial model category, which is not representable up to homotopy, This shows that representability theorems are not tautological.

In the last Section 6 we interpret the Brown representability, up to homotopy, as a recognition principle for mapping spaces for an arbitrary space $A$ (instead of $S^n$).

2. (Model) Categorical preliminaries

For every closed symmetric monoidal combinatorial model category $\mathcal{V}$ and a $\mathcal{V}$-model category $\mathcal{M}$ (not necessarily combinatorial) we can consider the category of small presheaves $\mathcal{V}^{\mathcal{M}^{\text{op}}}$. This is a $\mathcal{V}$-category of functors, which are left Kan extensions from small subcategory of $\mathcal{M}$. $\mathcal{V}^{\mathcal{M}^{\text{op}}}$ is cocomplete by [17, 5.34]. Since $\mathcal{V}$ is a combinatorial model category, it is in particular locally presentable. Therefore, the category of of small presheaves $\mathcal{V}^{\mathcal{M}^{\text{op}}}$ is also complete by [10].

A natural transformation $f : F \rightarrow G$ in $\mathcal{V}^{\mathcal{M}^{\text{op}}}_0$ is a cofibrant-projective weak equivalence (resp., a cofibrant-projective fibration) if for all cofibrant $M \in \text{cat}\mathcal{M}$ the induced map $f_M : F(M) \rightarrow G(M)$ is a weak equivalence (resp., fibration). The notion of a projective weak equivalence (resp., a projective fibration) in $\mathcal{V}^{\mathcal{M}^{\text{op}}}_0$ is a particular case of the cofibrant-projective analog, when all objects of $\mathcal{M}$ are cofibrant, e.g., for the trivial model structure on $\mathcal{M}$. If (cofibrant-)projective fibrations and weak equivalences give rise to a model structure on the category of small presheaves, then this model structure is called (cofibrant-)projective.
First of all would like to establish the existence of the cofibrant-projective model structure on $\mathcal{V}_0^{M^\text{op}}$. For a simplicial model category $\mathcal{M}$ and $\mathcal{V} = \mathcal{S}$ this theorem was proven in [9, 2.8]; for a combinatorial model category $M^\text{op}$ this theorem was proven in [4, 3.6]. But the case of contravariant small functors from a combinatorial model category to $\mathcal{V}$ is still uncovered by the previous results.

**Condition 2.1.** Every trivial fibrations in $\mathcal{V}$ is an effective epimorphism in $\mathcal{V}_0$ (cf., [23, II, p. 4.1]).

Basic examples of categories satisfying this condition are (pointed) simplicial sets, spectra, chain complexes.

**Theorem 2.2.** Let $\mathcal{V}$ be closed symmetric monoidal model category satisfying condition [2, 7] and let $\mathcal{M}$ be a $\mathcal{V}$-model category. The category of small functors $\mathcal{V}^{M^\text{op}}$ may be equipped with the cofibrant-projective model structure.

**Proof.** Let $I$ and $J$ be the classes of generating cofibrations and generating trivial cofibrations in $\mathcal{V}$, and let $\mathcal{M}_{\text{cof}}$ denote the subcategory of cofibrant objects of $\mathcal{M}$.

The cofibrant-projective model structure on the category of small presheaves $\mathcal{V}^{M^\text{op}}$ is generated by the following classes of maps.

$$J = \{ R_M \otimes i | I \ni i: A \to B, M \in \mathcal{M}_{\text{cof}} \}$$

$$\mathcal{J} = \{ R_M \otimes j | J \ni j: U \cong V, M \in \mathcal{M}_{\text{cof}} \}$$

It suffices to verify that these two classes of maps admit the generalized small object argument [7]. More specifically, we need to show that $J$ and $\mathcal{J}$ are locally small. In other words, for any map $f: X \to Y$ we need to find a set $W$ of maps in $J$-cof (resp., $\mathcal{J}$-cof), such that every morphism of maps $R_M \otimes i \to f$ factors through an element in $W$. By adjunction, it is sufficient to find a set of cofibrant objects $U$, such that every map $R_M \to X \times_X Y^B$ would factor through an element of $U$.

Let $F = X^A \times_Y Y^B$. As any small functor, $F: M^\text{op} \to \mathcal{V}$ is a left Kan extension from a small full subcategory $\mathcal{D}$ of $\mathcal{M}$, hence a weighted colimit of representable functors, which, in turn, may be viewed as a coequalizer, by the dual of [17, 3.68].

$$F = \int^D R_D \otimes FD = \text{coeq} \left( \coprod_{f: D' \to D} R_{D'} \otimes FD \supseteq \coprod_D R_D \otimes FD \right).$$

Therefore, every $\mathcal{V}$-natural transformation $R_A \to F$ in $(\mathcal{V}^{M^\text{op}})_0$ factors through $R_D \otimes FD$ for some $D \in \mathcal{D}$ by the weak Yoneda lemma for $\mathcal{V}$-categories. Unfortunately, $R_D \otimes FD$ is not necessarily $J$-cofibrant. Fortunately we can find an $\mathcal{L}$-cofibrant object $U$ with a factorization $R_A \to U \to R_D \otimes FD$ for every cofibrant $A \in \mathcal{M}$.

Let $q: \tilde{D} \to D$ be a cofibrant replacement in $\mathcal{M}$, then put $U = R_{\tilde{D}} \otimes \tilde{D}$ with the map $U \to R_D \otimes FD$ composed of $R_{\tilde{D}} \otimes q$ and $\text{hom}(\cdot, q) \otimes \tilde{D}$. It suffices to show that the induced map $\text{hom}(R_A, U) \to \text{hom}(R_A, R_D \otimes FD)$ is epimorphism. This map factors, in turn, as a composition of two maps $\text{hom}(A, \tilde{D}) \otimes \tilde{D} \to \text{hom}(A, D) \otimes D$ each of which is a tensoring of an effective epimorphism with an object of $\mathcal{V}$. Since a map is an effective epimorphism iff it is a coequaliser of some pair of parallel maps (see, e.g., the dual of [15, 10.9.4]). Hence, this is a composition of two effective epimorphisms, which is an epimorphism. In particular, is $S$ is a unit.
of \( \mathcal{V} \), then \( \hom_{\mathcal{V}}(S, \hom(R_A, U)) \to \hom_{\mathcal{V}}(S, \hom(R_A, R_D \otimes F D)) \) is a surjection of sets.

**Definition 2.3.** Consider the following classes of maps in \( \mathcal{V}^{\mathcal{M}^{op}} \).

\[
\mathcal{F}_0 = \left\{ \emptyset = \hocolim_{\emptyset} \emptyset \to R_{\hocolim_{\emptyset} \emptyset} = R_\emptyset \right\}
\]

\[
\mathcal{F}_1 = \{ R_X \otimes A \to R_{X \otimes A} | X \in \mathcal{M}, A \in \mathcal{V} - \text{cofibrant objects} \}
\]

The map \( R_X \otimes A \to R_{X \otimes A} \) is the unit of the adjunction \( Y : \mathcal{M} \rightleftarrows \mathcal{V}^{\mathcal{M}^{op}} : - \otimes \text{Id}_M \).

\[
\mathcal{F}_2 = \left\{ \begin{array}{c}
R_A \to R_B \\
R_C
\end{array} \right\} \quad \begin{array}{c}
A \to B \\
C \to D
\end{array} \quad \text{- homotopy pushout of cofibrant objects in } \mathcal{M}
\]

\[
\mathcal{F}_3 = \left\{ \begin{array}{c}
\hocolim_{k<\kappa}(R_{A_0} \to \ldots \to R_{A_k} \to R_{A_k+1} \to \ldots) \\
R_{\text{colim}_{k<\kappa} A_k}
\end{array} \right\} \quad \begin{array}{c}
A_0 \hookrightarrow \ldots \hookrightarrow A_k \hookrightarrow A_{k+1} \hookrightarrow \ldots \\
A_k \text{ cofibrant for all } k < \kappa
\end{array}
\]

Then let us define \( \mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \).

**Proposition 2.4.** Let \( \mathcal{M} \) be a \( \mathcal{V} \)-model category, such that the category of small functors \( \mathcal{V}^{\mathcal{M}^{op}} \) may be equipped with the cofibrant projective model structure. Let \( F \in \mathcal{V}^{\mathcal{M}^{op}} \) be a cofibrant-projectively fibrant small functor taking weighted homotopy colimits of cofibrant objects to homotopy limits in \( \mathcal{V} \). Recall the class \( \mathcal{F} \) of maps from Definition 2.3, then the functor \( F \) is \( \mathcal{F} \)-local.

**Proof.** Follows from Yoneda’s lemma and the fact that the colimit of a sequence of cofibrations of cofibrant objects is a homotopy colimit.

**Remark 2.5.** We have only included in the class \( \mathcal{F} \) the morphisms which are required for the proof of the inverse implication: \( \mathcal{F} \)-local functors are equivalent to the representable functors, see Theorem 3.1. In some situation the class \( \mathcal{F} \) of maps may be reduced even further. For example, if \( \mathcal{V} = \mathcal{S} \), then the subclass \( \mathcal{F}_1 \) of maps is redundant, since every weighted homotopy colimit in a simplicial category can be expressed in terms of the classical homotopy colimits, cf. [8, Lemma 3.1]. If \( \mathcal{V} = \mathcal{M} = \mathcal{S} \), then \( \mathcal{F}_1 \) is also not needed due to Spanier-Whitehead duality, cf. [1] Lemma 7.2. But in general weighted homotopy colimits may not be expressed in terms of the classical homotopy colimits (homotopy colimits with contractible weight), as discussed by Lukáš Vokřínek, [27].

3. FREDERICK REPRESENTABILITY THEOREM, UP TO HOMOTOPY

**Theorem 3.1.** Let \( \mathcal{V} \) is a closed symmetric monoidal model category satisfying condition [24]. Suppose, in addition, that the domains of the generating cofibrations of \( \mathcal{V} \) are cofibrant. Let \( \mathcal{M} \) be a \( \mathcal{V} \)-model category, then the category \( \mathcal{V}^{\mathcal{M}^{op}} \) of small \( \mathcal{V} \)-presheaves on \( \mathcal{M} \) may be equipped with the cofibrant-projective model structure. A small functor \( F \) is cofibrant-projectively weakly equivalent to a representable functor if and only if it takes homotopy colimits of cofibrant objects to homotopy limits.
Proof. Let $\lambda$ be a regular cardinal such that $\mathcal{V}$ is a $\lambda$-combinatorial model category. Let $\mathcal{I}$ and $\mathcal{J}$ be its sets of generating cofibrations and trivial cofibrations, respectively. We assume that the domains of the maps in $\mathcal{I}$ are cofibrant.

Given a small functor $F \in \mathcal{V}^{\mathcal{M}^{\text{op}}}$, taking homotopy colimits to homotopy limits, consider a cofibrant replacement $\tilde{F} \to F$ in the cofibrant-projective model structure. Then there is a $\lambda$-sequence $F_0 \to F_1 \to \cdots \to F_k \to F_{k+1} \to \cdots \tilde{F}$ such that $F_0(M) = \emptyset$ for all $M \in \mathcal{M}$, $\tilde{F} = \text{colim} F_k$, and $F_{k+1}$ is obtained from $F_k$ as a pushout:

\[
\begin{array}{ccc}
R_M \otimes A & \to & F_k \\
\downarrow & & \downarrow \\
R_M \otimes B & \to & F_{k+1}
\end{array}
\]

where $M \in \mathcal{M}$ is cofibrant, and $(A \leftrightarrow B) \in \mathcal{I}$, $A, B \in \mathcal{V}$ are also cofibrant by assumption.

Our proof will proceed by induction. Recall the class $\mathcal{F}$ of maps from Definition 2.3, then the fibrant replacement of the given functor $F$ is $\mathcal{F}$-local by Proposition 2.4.

Notice that $F_0 = \emptyset$ is $\mathcal{F}$-equivalent to $R_\emptyset = R_{\text{hocolim}_\emptyset \emptyset}$, since the map $\emptyset \to R_\emptyset$ is in $\mathcal{F}_0 \subset \mathcal{F}$.

Suppose for induction that $F_k$ is $\mathcal{F}$-equivalent to a representable functor $R_{X_k}$, where $X_k$ is a fibrant and cofibrant object of $\mathcal{M}$, then there is a commutative diagram

\[
\begin{array}{ccc}
R_M \otimes A & \to & R_{X_k} \\
\downarrow & \downarrow & \downarrow \\
F_k & \to & R_{X_k} \\
\downarrow & \downarrow & \downarrow \\
R_M \otimes B & \to & F_{k+1}
\end{array}
\]

where the upper horizontal arrow is induced by the universal property of the unit of the adjunction, and $X_k'$ is a fibrant and cofibrant object of $\mathcal{M}$ obtained as a middle term of the factorization $M \otimes A \leftrightarrow X_k' \to X_k$. Since $F_k$ is a cofibrant functor, there exists a lift $F_k \to R_{X_k'}$ which is an $\mathcal{F}$-equivalence by the 2-out-of-3 property and does not violate the commutativity if the above diagram, since the upper slanted arrow $R_M \otimes A \to R_{X_k'}$ is also a natural map induced by the universal property of the unit of adjunction $\varepsilon$. 

Let us put $P = M \otimes B \coprod_{M \otimes A} X'_k$. Then we obtain a commutative diagram

![Diagram](https://example.com/diagram.png)

in which all the solid slanted arrows are $\mathcal{F}$-equivalences and the inner square is a homotopy pushout. Hence, the homotopy pushout of the outer square is $\mathcal{F}$-equivalent to $R_{X_{k+1}}$, therefore the dashed arrow is also an $\mathcal{F}$-equivalence. Finally, we put $X_{k+1}$ to be the middle element in the factorization $X'_k \hookrightarrow X_{k+1} \twoheadrightarrow P$, then $R_P \to R_P$ is also an $\mathcal{F}$-equivalence, and so is the lift (by the 2-out-of-3) $F_{k+1} \to R_{X_{k+1}}$, which exists since $F_{k+1}$ cofibrant.

And if $\kappa$ is a limit ordinal, then $F_\kappa = \text{colim}_{k<\kappa} F_k$. Since this is the colimit of a sequence of cofibrations of cofibrant functors, $\text{colim}_{k<\kappa} F_k \simeq \text{hocolim}_{k<\kappa} F_k$. But $F_k$ is $\mathcal{F}$-equivalent to $R_{X_k}$. Moreover, by construction, there is a sequence of cofibrations $X_0 \hookrightarrow \ldots \hookrightarrow X_k \hookrightarrow X_{k+1} \hookrightarrow \ldots$, hence $\text{hocolim} R_{X_k}$ is $\mathcal{F}$-equivalent to $R_{\text{colim} X_k}$. Additionally, if $\kappa$ is not large enough to ensure that $\text{colim}_{k<\kappa} X_k$ is fibrant, we can consider the fibrant replacement $\text{colim}_{k<\kappa} X_k \Rightarrow \text{colim}_{k<\kappa} X_k = X_\kappa$. Combining these facts together we conclude that $F_\kappa$ is $\mathcal{F}$-equivalent to a representable functor $R_{X_\kappa}$, which is represented in a fibrant and cofibrant object.

For $\kappa$ large enough we obtain $F = F_\kappa$, hence $F$ is $\mathcal{F}$-equivalent to a functor represented in a fibrant and cofibrant object. But $F$ is an $\mathcal{F}$-local functor by Proposition 2.4 and so is $R_{X_\kappa}$ for every $\kappa$. Hence, $F \simeq R_{X_\kappa}$ for some $\kappa$, since an $\mathcal{F}$-equivalence of $\mathcal{F}$-local functors is a weak equivalence. $\square$

The formal category theory dual of the above theorem is the following statement.

**Theorem 3.2.** Let $\mathcal{M}$ be a $\mathcal{V}$-model category, suppose the category $\mathcal{V}^\mathcal{M}$ of small $\mathcal{V}$-presheaves on $\mathcal{M}$ may be equipped with the fibrant-projective model structure. Then a small functor is fibrant-projectively weakly equivalent to a representable functor if and only if it takes homotopy limits of fibrant objects to homotopy limits.

4. **Brown representability theorem, up to homotopy**

In this section we prove a homotopy version of Brown representability theorem for contravariant functors from a locally presentable $\mathcal{V}$-model category $\mathcal{M}$ to $\mathcal{V}$, taking homotopy colimits to homotopy limits. Notice that our proof does not use explicitly the presence of compact objects, as most of the known proves in this field do, but rather we show directly that a contravariant homotopy functor from $\mathcal{M}$ to $\mathcal{V}$ is cofibrant projectively weakly equivalent to a small functor, and than use the Freyd representability theorem, up to homotopy.
Lemma 4.1. Let $\mathcal{M}$ be a combinatorial $\mathcal{V}$-category. Then any functor $H: \mathcal{M}^{\text{op}} \to \mathcal{V}$ taking homotopy colimits of cofibrant objects to homotopy limits is cofibrant-projectively weakly equivalent to a small homotopy functor $F \in \mathcal{V}^{\mathcal{M}^{\text{op}}}$.

Proof. Suppose $\mathcal{M}$ is a $\lambda$-combinatorial model category, for some cardinal $\lambda$ such that the weak equivalences are a $\lambda$-accessible subcategory of the category of maps of $\mathcal{M}$.

Consider a functor $H: \mathcal{M}^{\text{op}} \to \mathcal{V}$ taking homotopy colimits of cofibrant objects to homotopy limits. Then the natural map

$$H \to F = \lim_{i: \mathcal{M}_\lambda \to \mathcal{M}} i^*H$$

is a weak equivalence in the cofibrant-projective model category, because both functors take homotopy colimits to homotopy limits and every cofibrant object of $\mathcal{M}$ is a ($\lambda$-filtered) homotopy colimit of $\lambda$-presentable cofibrant objects, [19, Cor. 5.1], on which the two functors coincide.

But we can interpret the right Kan extension as a weighted inverse limit of representable functors

$$F = \lim_{i: \mathcal{M}_\lambda \to \mathcal{M}} i^*H = \int_{M \in \mathcal{M}_\lambda} R^H_M = \{i^*H, i^*Y\},$$

Where $Y: \mathcal{M} \to \mathcal{V}^{\mathcal{M}^{\text{op}}}$ is the Yoneda embedding. Then by the theorem of Day and Lack, [10], $F$ is small as an inverse limit of small functors.

Remark 4.2. When we speak about sufficiently large filtered colimits in a combinatorial model category, they turn out to be homotopy colimits, no matter if the objects participating in them are cofibrant or not. Therefore the above lemma admits a brief formulation: any functor taking homotopy colimits to homotopy limits is levelwise weakly equivalent to a small functor.

Theorem 4.3. Let $\mathcal{M}$ be a combinatorial $\mathcal{V}$-category. Then any functor $H: \mathcal{M}^{\text{op}} \to \mathcal{V}$ taking homotopy colimits of cofibrant objects to homotopy limits is cofibrant-projectively weakly equivalent to a representable functor.

Proof. Follows from Theorem 3.1 by Lemma 4.1.

5. Counter-example

We have proved so far two representability theorems, up to homotopy. The first is of Freyd type, i.e., some set theoretical conditions are required to be satisfied by the functor in question. The second one is of Brown type, i.e., some set theoretical assumptions apply to the domain category. But what happens if we withdraw all set theoretical assumptions both from the domain category and from the functor? Are exactness conditions alone able to ensure representability, up to homotopy?

Mac Lane’s classical example of a functor $B: \text{Grp} \to \text{Set}$, which assigns to each group $G$ the set of all homomorphisms from the free product of a large collection of non-isomorphic simple groups to $G$, is an example of a (strictly) non representable functor. Notice that neither $B$ satisfies the solution set condition, nor the category $B^{\text{op}}$ is locally presentable. Perhaps representability up to homotopy is less demanding and would persist without any conditions?
Our example is similar in nature to Mac Lane’s example, but it has also another predecessor. E. Dror Farjoun gave an example of a failure of Brown representability for the generalized Bredon cohomology. [11]

Consider the closed symmetric combinatorial model category of spaces $S$ consider the category of small functors $M = S^S$. Consider a non-small functor $B: S \to S$, e.g., $B = \text{hom}(\text{hom}(\cdot, S^0), S^0)$. Notice, that $B \notin M$, since $B$ is not an accessible functors, as all small functors are. Then $H = \text{hom}(-, B): M \to S$ is well defined, since for any small functor $F \in M$, there exists a small subcategory $i: A \hookrightarrow S$, such that $F = \text{Lan}_i i^* F$, hence $H(F) = \text{hom}(F, B) = \text{hom}_{S^A}(i^* F, i^* B) \in S$.

We have defined a functor $H : M^{op} \to S$ taking homotopy colimits of cofibrant objects to homotopy limits, but non-representable, even up to homotopy, otherwise there would exist a small fibrant functor $F$ taking homotopy limits. In other words, every mapping space $X$ admits $k$-ary operations, i.e., maps $X \times \ldots \times X \to X$ satisfying a long list of higher associativity conditions.

For a space $A$ more general then $S^1$ it is insufficient to consider the structure given by the maps $X \times \ldots \times X \to X$, $[3]$. We will consider, instead, the structure given by the mutual interrelations of all possible homotopy inverse limits of $X$, the same structure that would be created if $X$ indeed was equivalent to $\text{hom}(A, Y)$ for some $Y$. Indeed, consider the subcategory of $A$-cellular spaces $C(A) \subset \mathcal{T}_{op_*}$. This is the minimal subcategory containing $A$ and closed under the homotopy colimits, then any homotopy colimit of a diagram involving $A$ is transferred by the functor $\text{hom}(-, Y)$ into a homotopy limit of an opposite diagram involving $X$. In other words, every mapping space $X$ is equipped with a functor $F_X : C(A) \to \mathcal{T}_{op_*}$. Moreover, this functor has a very nice property: it takes homotopy colimits into homotopy limits.

Our goal is to show the converse statement: if for a given $X \in \mathcal{T}_{op_*}$ there exists a functor $F : C(A)^{op} \to \mathcal{T}_{op_*}$ taking homotopy colimits to homotopy limits and satisfying $F(A) \simeq X$, then this functor is weakly equivalent to a representable functor, i.e., there is a space $Y$ such that $F(-) \simeq \text{hom}(-, Y)$.

**Theorem 6.1.** For any cofibrant $A \in \mathcal{T}_{op_*}$ and any $X \in \mathcal{T}_{op_*}$, there is an object $Y \in \mathcal{T}_{op_*}$ satisfying $X \simeq \text{hom}(A, Y)$ if and only if there exists a functor $F : C(A)^{op} \to \mathcal{T}_{op_*}$ taking homotopy colimits to homotopy limits and satisfying $F(A) \simeq X$. 

Proof. Necessity of the condition is clear. We will prove the sufficiency now.

Consider the right Bousfield localization of $\text{Top}_A$ with respect to $A$. By [15, 5.1.1(3)] we obtain a cofibrantly generated model structure, which is also combinatorial, since we have chosen to work with a locally presentable model of topological spaces. We denote the new model category by $\text{Top}_A^A$. Then $C(A)$ is the subcategory of cofibrant objects of $\text{Top}_A^A$. We denote the inclusion functor by $i: C(A) \to \text{Top}_A^A$.

Given a functor $F: C(A) \to \text{Top}_A$ taking homotopy colimits to homotopy limits and satisfying $F(A) = X$, consider the functor $H = \text{Lan}_i F$. Then $H: \text{Top}_A^A \to \text{Top}_A$ satisfies the condition of Theorem 4.3, hence there exists $Y \in \text{Top}_A^A$ such that $H(\cdot) \simeq \text{hom}(\cdot, Y)$, in particular, $H(A) = F(A) \simeq X \simeq \text{hom}(A, Y)$.

□

References

[1] B. Badzioch, D. Blanc, and W. Doria. Recognizing mapping spaces. J. Pure Appl. Algebra, 218(1):181–196, 2014.
[2] B. Badzioch, K. Chung, and A. A. Voronov. The canonical delooping machine. J. Pure Appl. Algebra, 208(2):531–540, 2007.
[3] B. Badzioch and W. Doria. A note on localizations of mapping spaces. Israel J. Math., 177:441–444, 2010.
[4] G. Biedermann and B. Chorny. Duality and small functors. Algebr. Geom. Topol., 15(5):2609–2657, 2015.
[5] J. M. Boardman and R. M. Vogt. Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.
[6] E. H. Brown, Jr. Cohomology theories. Ann. of Math. (2), 75:467–484, 1962.
[7] B. Chorny. A generalization of Quillen’s small object argument. Journal of Pure and Applied Algebra, 204:568–583, 2006.
[8] B. Chorny. Brown representability for space-valued functors. Israel J. Math., 194(2):767–791, 2013.
[9] B. Chorny. Homotopy theory of relative simplicial presheaves. Israel J. Math., 205(1):471–484, 2015.
[10] B. Day and S. Lack. Small limits of functors. Journal of Pure and Applied Algebra, 210:651–663, 2007.
[11] E. Dror Farjoun. Homotopy and homology of diagrams of spaces. In Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math. 1286, pages 93–134. Springer, Berlin, 1987.
[12] L. Fajstrup and J. Rosicky. A convenient category for directed homotopy. Theory and Applications of Categories, 21(1):7–20, 2008.
[13] J. Franke. Brown representability theorem for triangulated categories. Topology, 40(4):667–680, 2001.
[14] P. Freyd. Abelian categories. Harper and Row, New York, 1964.
[15] P. S. Hirschhorn. Model categories and their localizations, volume 99 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.
[16] J. F. Jardine. Representability theorems for presheaves of spectra. J. Pure Appl. Algebra, 215(1):77–88, 2011.
[17] G. M. Kelly. Basic concepts of enriched category theory, volume 64 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1982.
[18] H. Krause. A brown representability theorem via coherent functors. Topology, 41:853–861, 2002.
[19] M. Makkai, J. Rosický, and L. Vokřínek. On a fat small object argument. Adv. Math., 254:49–68, 2014.
[20] J. P. May. The geometry of iterated loop spaces. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.
[21] A. Neeman. Triangulated categories, volume 148 of Annals of Mathematics Studies. Princeton University Press, 2001.
[22] A. Neeman. Brown representability follows from rosicky’s theorem. Journal of Topology, 2:262–276, 2009.
[23] D. G. Quillen. *Homotopical Algebra*. Lecture Notes in Math. 43. Springer-Verlag, Berlin, 1967.

[24] G. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[25] J. D. Stasheff. Homotopy associativity of $H$-spaces, I, II. *Trans. Amer. Math. Soc.* 108 (1963), 275-292; *ibid.*, 108:293–312, 1963.

[26] M. Sugawara. $h$-spaces and spaces of loops. *Mathematical Journal of Okayama University*, 5:5–11, 1956.

[27] L. Vokřínek. Homotopy weighted colimits. Preprint, 2012.

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