Stress field solution to anisotropic plate with mode II crack based on pan-complex function

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Abstract. The crack-tip stress problem to the anisotropic materials was mainly discussed under the shear loading model. For the sake of solving the stress boundary problem, the pan-complex function method was achieved in the first place. The elastic theory of composite materials has been clearly reviewed in order to determine the material physical parameters. Equilibrium equations and deformable compatibility condition were introduced on the basis of plane elastic mechanics. The key to the settlement of the boundary question must lie in solving the partial derivative equation and the axial transformation of the pan-complex variables. The reasonable stress functions were chosen in order to meet the needs of shear loading states. The pan-complex function and the polar coordinate replace approach are utilized with the aim of solving the stress boundary problems for the anisotropic plate. The formulas of the stress components near the crack-tip region were finally determined by the real functions. The method in this article may be much important to the fracture analysis of composite materials.

1. Introduction

The composite materials or anisotropic materials must be very important really in application to many engineering structures nowadays. The general fracture theories talk about the crack problems of commonly used materials in great detail and have been very successful in ordinary utilization [1-3]. Nevertheless, the fracture mechanics of composite materials may be paid more and more attention to many structures [4, 5]. It is well known that the composite materials have the anisotropic character evidently. Some books have introduced the fracture idea into the failure analysis of composite materials. The method of fracture mechanics must be of great importance to the stress calculation and strength assessment of composite structure. The utilization of pan-complex variable functions may provide a valid tool for solving the stress boundary problems [6-8]. It is reported in some articles that the complete solutions of crack-tip stress fields have been discussed in all details for orthotropic materials. Typical stress boundary problems for cracked orthotropic plate have been solved by using the complex variable method. Nevertheless the detailed solutions for anisotropic materials have not been obtained thoroughly. Thus it is necessary to find a new solution for the stress fields in cracked anisotropic plate. Especially, the study of the mode II loading problem as shown in Figure 1 must be full of actual significance for the fracture research of anisotropic materials. So the present work will be mainly to solve the stress field problems for the cracked anisotropic plate. By constructing the new stress function and the transformation of pan-complex variables, the solution of the stress boundary problem can be obtained with satisfaction.
2. Pan-complex function and elastic equations

Certain configurations of cracks and loadings must be discussed via local stress fields at the crack tip. For composite materials, the stress-field problems can be solved by using pan-complex analytic function approach. The method is explained as below.

2.1. Pan-complex function and partial derivations

The solution of some partial differential equations must be with the aid of the complex variables. Therefore, the pan-complex variable \( w \) and its conjugate \( \bar{w} \) have to be introduced. They are defined by the following:

\[
\begin{align*}
  w &= x + qy = x + gy + i hy, \\
  \bar{w} &= x + \bar{q}y = x + gy - i hy
\end{align*}
\]

Where, \( q = g + ih \), \( \bar{q} = g - ih \), and \( i = \sqrt{-1} \). Both \( g \) and \( h \) are real arbitrary constants, and also \( h > 0 \) is the need for convenience. The complex numbers \((q, \bar{q})\) are of the following relations:

\[
q + \bar{q} = 2g, \quad q - \bar{q} = 2hi, \quad q\bar{q} = g^2 + h^2
\]

The pan-complex variables \((w, \bar{w})\) have the following property:

\[
w\bar{w} = (x + qy)(x + \bar{q}y) = (x + gy)^2 + h^2 y^2 = x^2 + 2gxy + g^2 y^2 + h^2 y^2
\]

Hence, the norm of the pan-complex variables can be determined by:

\[
\|w\| = \|\bar{w}\| = \sqrt{(x + gy)^2 + h^2 y^2}
\]

Suppose the real function \( F(x, y) \) to be expressed by the complex variables, that is \( F(w, \bar{w}) \). Then the partial derivations of \( F \) with \( x \) or \( y \) can be transformed into:

\[
\frac{\partial F(w, \bar{w})}{\partial x} = \frac{\partial F}{\partial w} \frac{\partial w}{\partial x} + \frac{\partial F}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial x} = \frac{\partial F}{\partial w} + \frac{\partial F}{\partial \bar{w}}
\]

\[
\frac{\partial F(w, \bar{w})}{\partial y} = \frac{\partial F}{\partial w} \frac{\partial w}{\partial y} + \frac{\partial F}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial y} = q \frac{\partial F}{\partial w} + \bar{q} \frac{\partial F}{\partial \bar{w}}
\]

And
According to the reference axes along the crack with length of $2a$, as shown in Figure 1, the pan-complex variables can be expressed as:

\[
\begin{align*}
    w &= x + gy + ihy = a + r \cos \theta + gr \sin \theta + ihr \sin \theta \\
    \bar{w} &= x + gy - ihy = a + r \cos \theta + gr \sin \theta - ihr \sin \theta
\end{align*}
\]

In order to simplify the complex functions later on, it is necessary for the trigonometric function expression to be transformed as below form:

\[
\begin{align*}
    \cos \theta + g \sin \theta &= J \cos \beta, \quad h \sin \theta = J \sin \beta
\end{align*}
\]

Where, $J > 0$ is restricted. Hence the following relations can be given:

\[
\begin{align*}
    \cos \theta + g \sin \theta + ih \sin \theta &= J (\cos \beta + i \sin \beta) = Je^{i\beta} \\
    J &= \sqrt{(\cos \theta + g \sin \theta)^2 + (h \sin \theta)^2} \\
    \tan \beta &= \frac{h \sin \theta}{\cos \theta + g \sin \theta}
\end{align*}
\]

Evidently, the new variables $(J, \beta)$ are relative to both parameters $(g, h)$. The variable $J$ has the meaning of norm, and angle $\beta$ varies with $\theta$ regularly. For example, the parameters are selected as $(g = -0.5, \ h = 0.9)$ and $(g = 0.3, \ h = 1.2)$. Two set data of the new variables $(J, \beta)$ with $\theta$ can be calculated, and the curves in Figure 2 may show the specific property. The new concept of the variable replacement in above relations will be of very significance in the future applications.

![Figure 2. Relation curves of $\beta - \theta$ and $J - \theta$.](image)
2.2. Two-dimensional elastic equations

The plane stress problems of anisotropic materials are common in engineering structure, so the stress fields must be solved firstly. Generally, the body forces can not be included, and the two-dimensional equilibrium equations are written out as below:

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad , \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \]  \hspace{0.5cm} (8)

In order to satisfy above two equilibrium equations, the stress function \( F = F(x, y) \) for the plane stress components can be introduced usually as follows:

\[ \sigma_x = \frac{\partial^2 F}{\partial y^2} , \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} , \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \]  \hspace{0.5cm} (9)

The plane deformation of the materials can be indicated by three strains (\( \varepsilon_x, \varepsilon_y, \gamma_{xy} \)). Two-dimensional compatibility condition of elastic deformation must be satisfied by:

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \]  \hspace{0.5cm} (10)

The relationship of the strains with the plane stresses ought to be determined. For the plane stress state, the constitutive equations of the anisotropic materials may be written in terms of compliances:

\[
\begin{align*}
\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy} \\
\varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy} \\
\gamma_{xy} &= a_{66}\sigma_y + a_{26}\sigma_y + a_{66}\tau_{xy}
\end{align*}
\]  \hspace{0.5cm} (11)

Obviously, there are six independent compliance constants in common case. Upon the substitution of the stress expression (9) in above strain equation, and again into the compatibility equation (10), the governing equation of the stress function can be obtained as follows:

\[ \frac{\partial^4 F}{\partial y^4} + A_1 \frac{\partial^4 F}{\partial x \partial y^3} + A_2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + A_3 \frac{\partial^4 F}{\partial x^3 \partial y} + A_4 \frac{\partial^4 F}{\partial x^4} = 0 \]  \hspace{0.5cm} (12)

In which

\[ A_1 = -\frac{2a_{16}}{a_{11}} , \quad A_2 = \frac{a_{66} + 2a_{12}}{a_{11}} , \quad A_3 = -\frac{2a_{26}}{a_{11}} , \quad A_4 = \frac{a_{22}}{a_{11}} \]

Notice that the key point may come to a target of looking for the proper stress function \( F \) in terms of stress boundary conditions and to be suitable for the governing equation (12).

3. Solution for centre crack plate

For the sake of solving the crack problem as shown in Figure 1, the stress function \( F \) may be selected by the following form:

\[ F = B\Psi(w) + B\Phi(\bar{w}) + c_0(w - \bar{w})^2 \]  \hspace{0.5cm} (13)

On the basis of above equations (4) and (9), the stress components can be expressed by the complex stress function, those are given by:
\[
\begin{align*}
\sigma_x &= \frac{\partial^2 F}{\partial y^2} = q^2 B \Psi'' + \overline{q}^2 \overline{B} \overline{\Psi}'' - 8h^2 c_0 = 2 \text{Re}(q^2 B \Phi) - 8h^2 c_0 \\
\sigma_y &= \frac{\partial^2 F}{\partial x^2} = B \Psi'' + \overline{B} \overline{\Psi}'' = 2 \text{Re}(B \Phi) \\
\tau_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} = -(q B \Psi'' + \overline{q} \overline{B} \overline{\Psi}'' = -2 \text{Re}(q B \Phi)
\end{align*}
\]

Where, \( \Phi = \Phi(w) = \Psi''(w) \). With the aid of the expression (4) on the partial derivations, and substituting the stress function formula (13) into the governing equation (12), then we obtain that:

\[
(q^4 + A_q q^3 + A_2 q^2 + A_q q + A_4) B \Phi'' + (\overline{q}^4 + A_\overline{q} \overline{q}^3 + A_2 \overline{q}^2 + A_\overline{q} \overline{q} + A_4) \overline{B} \overline{\Phi}'' = 0
\]

In view of this, it can bring about the following characteristic equations:

\[
\begin{align*}
q^4 + A_q q^3 + A_2 q^2 + A_q q + A_4 &= 0 \\
\overline{q}^4 + A_\overline{q} \overline{q}^3 + A_2 \overline{q}^2 + A_\overline{q} \overline{q} + A_4 &= 0
\end{align*}
\]

The roots of the fourth power equations can be written as:

\[
q_1 = g_1 + i h_1 \ , \quad q_2 = g_2 + i h_2 \ , \quad q_3 = \overline{g}_1 = g_1 - i h_1 \ , \quad q_4 = \overline{g}_2 = g_2 - i h_2
\]

Where, the real numbers \( h_1 \) and \( h_2 \) are positive, and suppose that: \( h_1 > h_2 > 0 \). Therefore, two pan-complex variables \( w_1 \) and \( w_2 \) can be expressed by:

\[
w_1 = x + q_1 y = x + g_1 y + i h_1 y \ , \quad w_2 = x + q_2 y = x + g_2 y + i h_2 y
\]

For anisotropic materials with the crack as shown in Figure 1, two pan-complex functions \( \{ \Phi_1(w_1), \Phi_2(w_2) \} \) should be selected as follows:

\[
\Phi_1 = \sqrt{\frac{w_1^2}{w_1^2 - a^2}} , \quad \Phi_2 = \sqrt{\frac{w_2^2}{w_2^2 - a^2}}
\]

The stress components can be determined in terms of two pan-complex functions \( \Phi_1 \) and \( \Phi_2 \), the stress equation (14) can be transformed into:

\[
\begin{align*}
\sigma_x &= 2 \text{Re}(q_1^2 B_1 \Phi_1 + q_2^2 B_2 \Phi_2) + T \\
\sigma_y &= 2 \text{Re}(B_1 \Phi_1 + B_2 \Phi_2) \\
\tau_{xy} &= -2 \text{Re}(q_1 B_1 \Phi_1 + q_2 B_2 \Phi_2)
\end{align*}
\]

In which \( T \) is arbitrary constant. The uncertain constants in above equations must be determined afterwards in terms of stress boundary conditions.

For the sake of solving the stress boundary problem as shown in Figure 1, the stress boundary conditions must be determined as follows:

\[
\begin{align*}
\sigma_x &= \sigma_y = 0 \ , \ \tau_{xy} = \tau \quad \text{at} \quad x^2 + y^2 \to \infty \\
\sigma_y &= 0 \ , \ \tau_{xy} = 0 \quad \text{at} \quad x^2 < a^2 \ \text{and} \ y = 0 \ (\text{crack surface})
\end{align*}
\]
Thus it can be seen that the following relations are obtained:
\[ 2 \text{Re}(q_1^2 B_1 + q_2^2 B_2) + T = 0, \quad \text{Re}(B_1 + B_2) = 0, \quad -2 \text{Re}(q_1 B_1 + q_2 B_2) = \tau \]
\[ \text{Im}(B_1 + B_2) = 0, \quad \text{Im}(q_1 B_1 + q_2 B_2) = 0 \]

The solution can be given by combining five equations, and the constants are solved as follows:

\[ B_1 = \frac{\tau - g_2 - g_1 + i(h_1 - h_2)}{2(g_1 - g_2)^2 + (h_1 - h_2)^2} \]
\[ B_2 = -B_1, \quad T = \tau(g_1 + g_2) \quad (20) \]

To take the advantage of the previous results, the stress components can be determined by:

\[ \sigma_x = \tau \text{Re}[\frac{g_2 - g_1 + i(h_1 - h_2)}{(g_1 - g_2)^2 + (h_1 - h_2)^2} (q_1 \sqrt{\frac{w_1^2}{w_1^2 - a^2}} - q_2 \sqrt{\frac{w_2^2}{w_2^2 - a^2}})] + T \]
\[ \sigma_y = \tau \text{Re}[\frac{g_2 - g_1 + i(h_1 - h_2)}{(g_1 - g_2)^2 + (h_1 - h_2)^2} (\sqrt{\frac{w_1^2}{w_1^2 - a^2}} - \sqrt{\frac{w_2^2}{w_2^2 - a^2}})] \]
\[ \tau_{xy} = -\tau \text{Re}[\frac{g_2 - g_1 + i(h_1 - h_2)}{(g_1 - g_2)^2 + (h_1 - h_2)^2} (q_1 \sqrt{\frac{w_1^2}{w_1^2 - a^2}} - q_2 \sqrt{\frac{w_2^2}{w_2^2 - a^2}})] \quad (21) \]

In the neighborhood of the crack tip \((r \ll a)\) as shown in Figure 1, the pan-complex functions can be transformed by using the polar coordinate system \((x = a + r \cos \theta, \quad y = r \sin \theta)\). Notice that:

\[ w + a \approx 2a, \quad \frac{w^2}{w^2 - a^2} \approx \frac{a + 1}{2a} = \frac{1}{2r} \left( \cos \theta + g \sin \theta + 2i \beta \sin \beta \right) \]

Therefore, the pan-complex functions can be changed into:

\[ \Phi_1 = \sqrt{\frac{w_1^2}{w_1^2 - a^2}} = \frac{a e^{-\beta_1/2}}{2r \sqrt{J_1}} = \frac{a}{2r \sqrt{J_1}} (\cos \frac{\beta_1}{2} - i \sin \frac{\beta_1}{2}) \]
\[ \Phi_2 = \sqrt{\frac{w_2^2}{w_2^2 - a^2}} = \frac{a e^{-\beta_2/2}}{2r \sqrt{J_2}} = \frac{a}{2r \sqrt{J_2}} (\cos \frac{\beta_2}{2} - i \sin \frac{\beta_2}{2}) \quad (22) \]

In which,

\[ \beta_1 = \arctan \frac{h_1 \sin \theta}{\cos \theta + g_1 \sin \theta}, \quad J_1 = \frac{(\cos \theta + g_1 \sin \theta)^2 + (h_1 \sin \theta)^2}{(\cos \theta + h_1 \sin \theta)^2 + (h_1 \sin \theta)^2} \]
\[ \beta_2 = \arctan \frac{h_2 \sin \theta}{\cos \theta + g_2 \sin \theta}, \quad J_2 = \frac{(\cos \theta + g_2 \sin \theta)^2 + (h_2 \sin \theta)^2}{(\cos \theta + h_2 \sin \theta)^2 + (h_2 \sin \theta)^2} \]

Then the stresses at near crack-tip can be simplified as:
\[
\sigma_x = \frac{\tau}{D_{12}} \sqrt{\frac{a}{2r}} \left( \frac{D_1}{\sqrt{J_1}} \cos \frac{\beta_1}{2} + \frac{D_5}{\sqrt{J_2}} \cos \frac{\beta_2}{2} + \frac{D_6}{\sqrt{J_1}} \sin \frac{\beta_1}{2} + \frac{D_7}{\sqrt{J_2}} \sin \frac{\beta_2}{2} \right) + \tau (g_1 + g_2) \\
\sigma_y = \frac{\tau}{D_{12}} \sqrt{\frac{a}{2r}} \left( \frac{D_1}{\sqrt{J_1}} \cos \frac{\beta_1}{2} + \frac{D_5}{\sqrt{J_2}} \cos \frac{\beta_2}{2} + \frac{D_6}{\sqrt{J_1}} \sin \frac{\beta_1}{2} + \frac{D_7}{\sqrt{J_2}} \sin \frac{\beta_2}{2} \right) \\
\tau_{xy} = \frac{\tau}{D_{12}} \sqrt{\frac{a}{2r}} \left( \frac{D_1}{\sqrt{J_1}} \cos \frac{\beta_1}{2} + \frac{D_5}{\sqrt{J_2}} \cos \frac{\beta_2}{2} + \frac{D_6}{\sqrt{J_1}} \sin \frac{\beta_1}{2} - \frac{D_7}{\sqrt{J_2}} \sin \frac{\beta_2}{2} \right) 
\]

(23)

Where,

\[D_1 = g_1^2 - g_1 g_2 + h_1^2 - h_1 h_2, \quad D_2 = g_2^2 - g_1 g_2 + h_2^2 - h_1 h_2,\]
\[D_{12} = D_1 + D_2 = (g_1 - g_2)^2 + (h_1 - h_2)^2, \quad D_3 = g_1 h_2 - g_2 h_1\]
\[D_4 = (g_2 - g_1) g_1^2 - (g_1 + g_2) h_1^2 + 2 g_1 h_1 h_2, \quad D_5 = (g_1 - g_2) g_2^2 - (g_1 + g_2) h_2^2 + 2 g_2 h_1 h_2\]
\[D_6 = (h_2 - h_1) h_1^2 - (h_1 + h_2) g_1^2 + 2 g_1 g_2 h_1, \quad D_7 = (h_1 - h_2) h_2^2 - (h_1 + h_2) g_2^2 + 2 g_1 g_2 h_2\]

The constants are in relation to one another as below:

\[D_4 + D_5 = -(g_1 + g_2) [(g_1 - g_2)^2 + (h_1 - h_2)^2] = -(g_1 + g_2) D_{12}\]
\[D_6 + D_7 = -(h_1 + h_2) [(h_1 - h_2)^2 + (g_1 - g_2)^2] = -(h_1 + h_2) D_{12}\]

It is obvious that the effective value of the stress fields will tend to infinity at the crack tip \((r \to 0)\), this is so-called singular stress fields. Finally, the solution of the stress fields for the anisotropic materials has been determined by the basic real functions.

**Acknowledgments**

This work was supported by the Natural Science Foundation of China (NSFC Grant No: 51475372).

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