THE ENTIRE FACE IRREGULARITY STRENGTH OF
A BOOK WITH POLYGONAL PAGES

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Abstract

A face irregular entire labeling is introduced by Baca et al. recently, as a modification of the well-known vertex irregular and edge irregular total labeling of graphs and the idea of the entire colouring of plane graph. A face irregular entire \(k\)-labeling \(\lambda: V \cup E \cup F \rightarrow \{1, 2, \ldots, k\}\) of a 2-connected plane graph \(G = (V, E, F)\) is a labeling of vertices, edges, and faces of \(G\) such that for any two different faces \(f\) and \(g\), their weights \(w_f(f)\) and \(w_g(f)\) are distinct. The minimum \(k\) for which a plane graph \(G\) has a face irregular entire \(k\)-labeling is called the entire face irregularity strength of \(G\), denoted by \(\text{ef}\(\text{s}(G)\)).

This paper deals with the entire face irregularity strength of a book with \(m\) \(n\)-polygonal pages, where embedded in a plane as a closed book with \(n\) -sided external face.

Keywords and phrases: Book, entire face irregularity strength, face irregular entire \(k\)-labeling, plane graph, polygonal page.

NILAI KETAKTERATURAN SELURUH MUKA
GRAF BUKU SEGI BANYAK

Abstrak

Pelabelan tak teratur seluruh muka diperkenalkan oleh Baca et al. baru-baru ini, sebagai suatu modifikasi atas pelabelan total tak teratur titik dan tak teratur sisi suatu graf serta ide tentang pewarnaan lengkap pada graf bidang. Pelabelan \(k\)-tak teratur seluruh muka \(\lambda: V \cup E \cup F \rightarrow \{1, 2, \ldots, k\}\) dari suatu graf bidang 2-connected \(G = (V, E, F)\) adalah suatu pelabelan seluruh titik, sisi, dan muka internal dari \(G\) sedemikian sehingga untuk sebarang dua muka \(f\) dan \(g\) berbeda, bobot muka \(w_f(f)\) dan \(w_g(f)\) juga berbeda. Bilangan bulat terkecil \(k\) sedemikian sehingga suatu graf bidang \(G\) memiliki suatu pelabelan \(k\)-tak teratur seluruh muka disebut nilai ketakteraturan seluruh muka dari \(G\), dinotasikan oleh \(\text{efs}(G)\).

Kami menentukan nilai eksak dari nilai ketakteraturan seluruh muka graf buku segi-\(n\), dimana pada bidang datar dapat digambarkan seperti suatu buku tertutup.

Kata Kunci: Graf bidang, graf buku segi-\(n\), nilai ketakteraturan seluruh muka, pelabelan lengkap \(k\)-tak teratur muka.

1. Introduction

Let \(G\) be a finite, simple, undirected graph with vertex set \(V(G)\) and edge set \(E(G)\). A total labeling of \(G\) is a mapping that sends \(V \cup E\) to a set of numbers (usually positive or nonnegative integers). According to the condition defined in a total labeling, there are many types of total labeling have been investigated.

Baca, Jendrol, Miller, and Ryan in [1] introduced a vertex irregular and edge irregular total labeling of graphs. For any total labeling \(f: V \cup E \rightarrow \{1, 2, \ldots, k\}\), the weight of a vertex \(v\) and the weight of an edge \(e = xy\) are defined by \(w(v) = f(v) + \sum_{u \in E} f(uv)\) and \(w(xy) = f(x) + f(y) + f(xy)\), respectively. If all the vertex weights are distinct, then \(f\) is called a vertex irregular total \(k\)-labeling, and if all the edge weights are distinct, then \(f\) is called an edge irregular total \(k\)-labeling. The minimum value of \(k\) for which there exist a vertex (an edge) irregular total labeling \(f: V \cup E \rightarrow \{1, 2, \ldots, k\}\) is called the total vertex (edge) irregularity.
strength of $G$ and is denoted by $tvS(G)$ ($tes(G)$), respectively. There are several bounds and exact values of $tvS$ and $tes$ were determined for different types of graphs given in [1] and listed in [2].

Furthermore, Ivanco and Jendrol in [3] posed a conjecture that for arbitrary graph $G$ different from $K_5$ and maximum degree $\Delta(G)$,

$$tes(G) = \max \left\{ \frac{|E(G)| + 2}{3}, \frac{\Delta(G) + 1}{2} \right\}.$$ 

Combining previous conditions on irregular total labeling, Marzuki et al. [4] defined a totally irregular total labeling. A total $k$-labeling $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ of $G$ is called a totally irregular total $k$-labeling if for any pair of vertices $x$ and $y$, their weights $w(x)$ and $w(y)$ are distinct and for any pair of edges $x_1x_2$ and $y_1y_2$, their weights $w(x_1x_2)$ and $w(y_1y_2)$ are distinct. The minimum $k$ for which a graph $G$ has totally irregular total labeling, is called total irregularity strength of $G$, denoted by $ts(G)$. They have proved that for every graph $G$,

$$ts(G) \geq \max\{tes(G), tvS(G)\}$$  \hspace{1cm} (6)

Several upper bounds and exact values of $ts$ were determined for different types of graphs given in [4], [5], [6], and [7].

Motivated by this graphs invariants, Baca et al. in [8] studied irregular labeling of a plane graph by labeling vertices, edges, and faces then considering the weights of faces. They defined a face irregular entire labeling.

A 2-connected plane graph $G = (V, E, F)$ is a particular drawing of planar graph on the Euclidean plane where every face is bound by a cycle. Let $G = (V, E, F)$ be a plane graph.

A labeling $\lambda : V \cup E \cup F \rightarrow \{1, 2, \ldots, k\}$ is called a face irregular entire $k$-labeling of the plane graph $G$ if for any two distinct faces $f$ and $g$ of $G$, their weights $w_\lambda(f)$ and $w_\lambda(g)$ are distinct. The minimum $k$ for which a plane graph $G$ has a face irregular entire $k$-labeling is called the entire face irregularity strength of $G$, denoted by $efS(G)$. The weight of a face $f$ under the labeling $\lambda$ is the sum of labels carried by that face and the edges and vertices of its boundary. They also provided the boundaries of $efS(G)$.

**Theorema A.** Let $G = (V, E, F)$ be a 2-connected plane graph $G$ with $n_i$ $i$-sided faces, $i \geq 3$. Let $a = \min\{i|n_i \neq 0\}$ and $b = \max\{i|n_i \neq 0\}$. Then

$$\left\lfloor \frac{2a + n_3 + n_4 + \cdots + n_b}{2b + 1} \right\rfloor \leq efS(G) \leq \max \{n_i|3 \leq i \leq b\}.$$ 

For $n_b = 1$, they gave the lower bound as follow

**Theorema B.** Let $G = (V, E, F)$ be a 2-connected plane graph $G$ with $n_i$ $i$-sided faces, $i \geq 3$. Let $a = \min\{i|n_i \neq 0\}$, $b = \max\{i|n_i \neq 0\}$, $n_b = 1$ and $c = \max\{i|n_i \neq 0, i < b\}$. Then

$$efS(G) \geq \left\lfloor \frac{2a + |F| - 1}{2c + 1} \right\rfloor.$$ 

Moreover, by considering the maximum degree of a 2-connected plane graph $G$, they obtained the following theorem.

**Theorem C.** Let $G = (V, E, F)$ be a 2-connected plane graph $G$ with maximum degree $\Delta$. Let $x$ be a vertex of degree $\Delta$ and let the smallest (and biggest) face incident with $x$ be an $a$-sided (and a $b$-sided) face, respectively. Then

$$efS(G) \geq \left\lfloor \frac{2a + \Delta - 1}{2b} \right\rfloor.$$ 

They proved that Theorem B is tight for Ladder graph $L_n$, $n \geq 3$, and its variation and Theorem C is tight for wheel graph $W_n$, $n \geq 3$. In this paper, we determine the exact value of $efS$ of a book with $m$ $n$-polygonal pages which is greater than the lower bound given in Theorem A - C.
2. Main Results

Considering Theorem C, \( efs(W_n) \), and a condition where every face of a plane graph shares common vertices or edges, our first result provide a lower bound of the entire face irregularity strength of a graph with this condition. This can be considered as generalization of Theorem A, B, and C.

**Lemma 2.1.** Let \( G = (V, E, F) \) be a 2-connected plane graph with \( n_i \) \( i \)-sided faces, \( i \geq 3 \). Let \( a = \min\{i \mid n_i \neq 0\} \), \( b = \max\{i \mid n_i \neq 0\} \), \( c = \max\{i \mid n_i \neq 0, i < b\} \), and \( d \) be the number of common labels of vertices and edges which have bounded every face of \( G \). Then

\[
efs(G) \geq \begin{cases} \left\lceil \frac{2a + |F| - d - 1}{2c - d + 1} \right\rceil, & \text{for } n_b = 1, \\ \left\lceil \frac{2a + |F| - d}{2b - d + 1} \right\rceil, & \text{otherwise.} \end{cases}
\]

**Proof.** Let \( \lambda : V \cup E \cup F \to \{1, 2, \ldots, k\} \) be a face irregular entire \( k \)-labeling of 2-connected plane graph \( G = (V, E, F) \) with \( efs(G) = k \). Our first proof is for \( n_b \neq 1 \). By Theorem A, the minimum face-weight is at least \( 2a + 1 \) and the maximum face-weight is at least \( 2a + |F| \). Since \( G \) is 2-connected, each face of \( G \) is a cycle. It implies that every face might be bounded by common vertices and edges.

Let \( d \) be the number of common labels of vertices and edges which have bounded every face of \( G \) and \( D \) be the sum of all common labels. Then the face-weights \( w_\lambda(f_1), w_\lambda(f_2), \ldots, w_\lambda(f_{|F|}) \) are all distinct and each of them contains \( D \), implies the variation of face-weights is depend on \( 2a - d + 2 \leq i \leq 2b - d + 1 \) labels.

Without adding \( D \), the maximum sum of a face label and all vertices and edges-labels surrounding it is at least \( 2a + |F| - d \). This is the sum of at most \( 2b - d + 1 \) labels. Thus, we have \( efs(G) \geq \left\lceil \frac{2a + |F| - d}{2b - d + 1} \right\rceil \).

For \( n_b = 1 \), it is a direct consequence from Theorem B with the same reason as in the result above. \( \blacksquare \)

This lower bound is tight for ladder graphs and its variation and wheels given in [8].

A book with \( mn \)-polygonal pages \( B_{mn} \), \( m \geq 1, n \geq 3 \), is a plane graph obtained from \( m \)-copies of cycle \( C_n \) that share a common edge. There are many ways drawing \( B_{mn} \) for which the external face of \( B_{mn} \) can be an \( n \)-sided face or a \((2n-2)\)-sided face.

By considering that topologically, \( B_{mn} \) can be drawn on a plane as a closed book such that \( B_{mn} \) has an \( n \)-sided external face, an \( n \)-sided internal face, and \( m - 1 \) number of \((2n-2)\)-sided internal faces, the entire face irregularity strength of \( B_{mn} \) is provided in the next theorem.

**Theorem 2.2.** For \( B_{mn} \), \( m \geq 1, n \geq 3 \), be a book with \( m \) \( n \)-polygonal pages whose an \( n \)-sided external face, an \( n \)-sided internal face, and \( m - 1 \) \((2n-2)\)-sided internal faces, we have

\[
efs(B_{mn}) = \begin{cases} 2, & \text{for } m \in \{1, 2\}; \\ \left\lceil \frac{4n + m - 7}{4n - 5} \right\rceil, & \text{otherwise.} \end{cases}
\]

**Proof.** Let \( B_{mn}, m \geq 1, n \geq 3 \), be a 2-connected plane graph. For \( m \in \{1, 2\} \), by Lemma 2.1, we have \( efs(B_{mn}) \geq 2 \). Labeling the \( n \)-sided external face by label 2 and all the rests by label 1, then all face-weights are distinct. Thus, \( efs(B_{mn}) = 2 \).

Now for \( m > 2 \), let \( z = efs(B_{mn}) \). Since every internal face of \( B_{mn} \) shares 2 common vertices, \( a = n \), \( b = 2n - 2 \), and \( n_b > 1 \), by Lemma 2.1, we have \( z \geq \frac{2a + |F|-2}{2b-1} = \frac{2n+m-1}{4n-5} \). Consider that \( z = \frac{2n+m-1}{4n-5} \) is not valid, since for \( m \leq 2n - 4 \), the maximum label is 1.

Moreover, since \( B_{mn} \) has at least 2 face-weights which are contributed by the same number of labels, there must be 2 faces of the same weight. Then the divisor must be at least \( 4n - 4 \). Thus we have \( z \geq \frac{4n+m-7}{4n-5} \).
Next, to show that $z$ is an upper bound for entire face irregularity strength of $B_m^n$, let $B_m^n$, $m \geq 1$, $n \geq 3$, be the 2-connected plane graph with an $n$-sided internal face $f_{int}^n$, $m - 1$ $(2n - 2)$-sided internal faces and an external $n$-sided face $f_{ext}^n$.

Let $m_1 = \left\lfloor \frac{m}{2} \right\rfloor$ and $m_2 = m - m_1$. Our goal is to have $m_1$ distinct even face-weights and $m_2$ distinct odd face-weights such that $m (2n - 2)$-sided face-weights are distinct and form an arithmetic progression.

Let $z = \left\lfloor \frac{4n + m - 7}{4n - 5} \right\rfloor$. It can be seen that $B_m^n$ has $m$ different paths of length $(n - 1)$. Next, we divide $m_1$ paths into $S = \left\lceil \frac{m_1}{4n - 5} \right\rceil$ parts, where part $s$-th consists of $(4n - 5)$ paths, for $1 \leq s \leq S - 1$, and part $S$-th consists of $r_1 = m_1 - (S - 1)(4n - 5)$ paths. Also, we divide $m_2$ paths into $T = \left\lceil \frac{m_2 + 1}{4n - 5} \right\rceil$ parts, where the first part consists of $(4n - 5)$ paths, part $t$-th consists of $(4n - 5)$ paths, for $2 \leq t \leq T - 1$, and part $T$-th consists of $r_2 = m_2 - (T - 1)(4n - 5)$ paths.

Let

\[
V(B_m^n) = \{x, y, u(s)^{2j}, u(S)^{2j}, v(t)^{2j} | 1 \leq s \leq S - 1, 1 \leq t \leq T - 1, 1 \leq i \leq 4n - 5, 1 \leq j \leq n - 2, 1 \leq k \leq r_1, 1 \leq l \leq r_2 \};
\]

\[
E(B_m^n) = \{xy\} \cup \{u(s)_i^j = x u(s)_i^j, u(s)^{2j-1} = u(s)_i^{2j-2} u(s)_i^{2j}, u(s)_i^{2n-3} = u(s)_i^{2n-4} y, 1 \leq s \leq S - 1, 1 \leq i \leq 4n - 5, 2 \leq j \leq n - 2 \} \cup \{u(S)_i^j = x u(S)_i^j, u(S)^{2j-1} = u(S)_i^{2j-2} u(S)_i^{2j}, u(S)_i^{2n-3} = u(S)_i^{2n-4} y, 1 \leq i \leq r_1, 2 \leq j \leq n - 2 \} \cup \{v(t)_i^j = x v(t)_i^j, v(t)^{2j-1} = v(t)_i^{2j-2} v(t)_i^{2j}, v(t)_i^{2n-3} = v(t)_i^{2n-4} y, 1 \leq t \leq T, 1 \leq i \leq 4n - 5, 2 \leq j \leq n - 2 \} \cup \{v(T)_i^j = x v(T)_i^j, v(T)^{2j-1} = v(T)_i^{2j-2} v(T)_i^{2j}, v(T)_i^{2n-3} = v(T)_i^{2n-4} y, 1 \leq i \leq r_2, 2 \leq j \leq n - 2 \};
\]

\[
F(B_m^n) = \{f_{ext}^n, f_{int}^n, u(s)_i^{2n-2}, u(S)_i^{2n-2}, v(t)_i^{2n-2} \neq v(1)_1^{2n-2}, v(T)_i^{2n-2} | 1 \leq s \leq S - 1, 1 \leq t \leq T - 1, 1 \leq i \leq 4n - 5, 1 \leq k \leq r_1, 1 \leq l \leq r_2 \};
\]

Where $f_{ext}^n$ is bounded by cycle $x v(1)_1^2 v(1)_2^{2n-4} y x$;

\[
f_{int}^n = \text{bounded by cycle } xu(1)_1^2 u(1)_1^2 \cdots u(1)_1^{2n-4} y x;
\]

$u(s)_i^{2n-2}$ is bounded by cycle $x u(s)_i^2 u(s)_i^4 \cdots u(s)^{2n-4} y u(s)^{2n-4} u(s)^{2n-6} \cdots u(s)^{2} x$ for $1 \leq i \leq S, i \neq r_1$;

$u(S)_r_{1}^{2n-2}$ is bounded by cycle $x u(S)_r_{1}^2 u(S)_r_{1}^4 \cdots u(S)^{2n-4} y u(T)^{2n-4} y u(T)^{2n-6} \cdots u(T)_r_{1} x$; and

$v(t)_i^{2n-2}$ is bounded by cycle $x v(t)_i^2 v(t)_i^4 \cdots v(t)^{2n-4} y v(t)^{2n-4} y v(t)^{2n-6} \cdots v(t)_i^{2n-4} x$ for $1 \leq i \leq T, i \neq r_2$;

Our notations above imply that, without losing generality, for $v(t)_i^j$, we let $2 \leq i \leq 4n - 5$ for $t = 1$. It means that there is no vertex or edge or face $v(1)_1$.

Now, we divide our labeling of $B_m^n$ into 2 cases as follows:

**Case 1. For odd** $m$ **with** $2 \leq r_2 \leq 2n - 1$ **or even** $m$;**

Define an entire $k$-labeling $\lambda : V \cup E \cup F \rightarrow \{1, 2, \ldots, k\}$ of $B_m^n$ as follows.

\[
\lambda(x) = \lambda(y) = \lambda(xy) = \lambda(f_{ext}^n) = 1;
\]

\[
\lambda(f_{int}^n) = 2;
\]
Hence, we propose the following open problem.

Note that our result in Theorem 2.2 show that the labeling \( \lambda \) is a face irregular entire \( z \)-labeling. Then we have evaluate the face-weights set \( \{ w(f_{\text{ext}}^n), w(f_{\text{int}}^n), w(u(s))^i_{2^n-2}, w(v(t))^i_{2^n-2} \} \) as follows.

\[
\lambda(u(s))^i_{2^n-2} = \begin{cases} 
2s - 1 & \text{for } 2 \leq s \leq S, 1 \leq i \leq \min(r_1, 2n - 2) \text{ and } 1 \leq j \leq 2n - i - 1 \\
2s & \text{for } 2 \leq s \leq S, 1 \leq i \leq \min(r_1, 2n - 2) \text{ and } 2n - i \leq j \leq 2n - 2 \\
2s & \text{for } 2 \leq s \leq S, 2n - 1 \leq i \leq \min(r_1, 4n - 5) \text{ and } 1 \leq j \leq 2n - 2 - \frac{i - 2n + 2}{2} - 2 \\
2s + 1 & \text{for } 2 \leq s \leq S, 2n - 1 \leq i \leq \min(r_1, 4n - 5) \text{ and } 2n - 2 - \frac{i - 2n + 2}{2} - 1 \leq j \leq 2n - 2 \\
2t - 1, & \text{for } 1 \leq t \leq T, 1 \leq i \leq \min(r_2, 2n - 2) \text{ and } 1 \leq j \leq 2n - i - 2 \\
2t, & \text{for } 1 \leq t \leq T, 1 \leq i \leq \min(r_2, 2n - 2) \text{ and } 2n - i - 1 \leq j \leq 2n - 3 \\
2t, & \text{for } 1 \leq t \leq T, 2n - 1 \leq i \leq \min(r_2, 4n - 5) \text{ and } 1 \leq j \leq 2n - 2 - \frac{i - 2n + 2}{2} - 3 \\
2t - 1, & \text{for } 1 \leq t \leq T, 2n - i \leq i \leq \min(r_2, 2n - 1) \text{ and } j = 2n - 2 \\
2t, & \text{for } 1 \leq t \leq T - 1, 2n - 1 \leq i \leq 4n - 5 \text{ and } j = 2n - 2 \\
2t, & \text{for } t = T, 2n - 1 \leq i \leq \min(r_2 - 1, 4n - 6) \text{ and } j = 2n - 2 \\
\end{cases}
\]

\[
\lambda(v(t))^i_{2^n-2} = \begin{cases} 
2t + 1, & \text{for } 1 \leq t \leq T, 2n - 1 \leq i \leq \min(r_2, 4n - 5) \text{ and } 2n - 2 - \frac{i - 2n + 2}{2} - 2 \leq j \leq 2n - 3 \\
2t - 2, & \text{for } 1 \leq t \leq T, i = 1 \text{ and } j = 2n - 2 \\
2t - 1, & \text{for } 1 \leq t \leq T, 2 \leq i \leq \min(r_2, 2n - 1) \text{ and } j = 2n - 2 \\
2t, & \text{for } 1 \leq t \leq T - 1, 2n - 1 \leq i \leq 4n - 5 \text{ and } j = 2n - 2 \\
2t, & \text{for } t = T, 2n - 1 \leq i \leq \min(r_2 - 1, 4n - 6) \text{ and } j = 2n - 2 \\
\end{cases}
\]

Case 2. For odd \( m \) with \( r_2 = 1 \) or \( 2n \leq r_2 \leq 4n - 5 \):

Define an entire \( k \)-labeling \( \lambda^* : V \cup E \cup F \to \{1, 2, \ldots, k\} \) of \( B_m^n \) as follows.

\[
\lambda^*(x) = \lambda^*(y) = \lambda^*(xy) = \lambda^*(f_{\text{ext}}^n) = 1; \\
\lambda^*(f_{\text{int}}^n) = 2; \\
\lambda^*(u(s))^i = \lambda(u(s))^i; \\
\lambda^*(v(t))^i = \begin{cases} 
2T - 2, & \text{for } r_2 = 1, t = T, i = 1, j = 1 \\
2T - 1, & \text{for } r_2 = 1, t = T - 1, i = 4n - 5, j = 2n - 2 \\
\lambda(v(t))^i + 1, & \text{for } r_2 \text{ odd, } 2n \leq r_2 \leq 4n - 5, t = T, i = r_2, j = 1 \\
\lambda(v(t))^i - 1, & \text{for } r_2 \text{ odd, } 2n \leq r_2 \leq 4n - 5, t = T, i = r_2 - 1, j = 2n - 2 \\
\lambda(v(t))^i - 1, & \text{for } r_2 \text{ even, } 2n \leq r_2 \leq 4n - 5, t = T, i = r_2 - 1, j = 2n - 3 \\
\lambda(v(t))^i + 1, & \text{for } r_2 \text{ even, } 2n \leq r_2 \leq 4n - 5, t = T, i = r_2 - 1, j = 2n - 2 \\
\lambda(v(t))^i, & \text{for otherwise.} \\
\end{cases}
\]

It is easy to check that the labeling \( \lambda \) is an entire \( z \)-labeling. Then we have evaluate the face-weights set \( \{ w(f_{\text{ext}}^n), w(f_{\text{int}}^n), w(u(s))^i_{2^n-2}, w(v(t))^i_{2^n-2} \} \) as follows.

\[
w(f_{\text{ext}}^n) = 2n + 1; \\
w(f_{\text{int}}^n) = 2n + 2; \\
w(u(s))^i_{2^n-2} = \begin{cases} 
(2s - 1)(4n - 5) + 2i + 2, & \text{for } 1 \leq s \leq S - 1, 1 \leq i \leq 4n - 5, s = S - 1, i = r_1; \\
(2s - 1)(4n - 5) + 2i, & \text{for } 1 \leq s \leq S - 1, 1 \leq i \leq r_1; \\
(2s - 1)(4n - 5) + 2r_1, & \text{for even } m, s = S - 1, i = r_1; \\
(2s - 1)(4n - 5) + 2r_1 - 1, & \text{for odd } m, s = S - 1, i = r_1; \\
\end{cases}
\]

\[
w(v(t))^i_{2^n-2} = \begin{cases} 
(2t - 1)(4n - 5) + 2i + 2, & \text{for } 1 \leq t \leq T - 1, 1 \leq i \leq 4n - 5, t = T, i = 1; \\
(2t - 1)(4n - 5) + 2i + 1, & \text{for } 1 \leq t \leq T - 1, 1 \leq i \leq 4n - 5, t = T, i = 1; \\
(2t - 1)(4n - 5) + 2i + 1, & \text{for } t = T, 1 \leq i \leq 2n - 1. \\
\end{cases}
\]

Since all face-weights are distinct, then \( \lambda \) is a face irregular entire \( z \)-labeling of \( B_m^n \) where \( m \) is odd with \( 2 \leq r_2 \leq 2n - 1 \) or \( m \) is even; and \( \lambda^* \) is a face irregular entire \( z \)-labeling of \( B_m^n \) where \( m \) is odd with \( r_2 = 1 \) or \( 2n \leq r_2 \leq 4n - 5 \). Thus, \( z = \frac{4n + m - 7}{4n - 5} \) is the entire face irregularity strength of \( B_m^n \).

Note that our result in Theorem 2.2 show that the \( \text{ef}s(B_m^n) \) is greater than the lower bound in Lemma 2.1.

Hence, we propose the following open problem.
Open Problems

1. Find a class of graph which satisfy a condition where the lower bound in Lemma 2.1 is sharp;
2. Generalize the lower bound for any condition.

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