Network Revenue Management with Limited Switches: Known and Unknown Demand Distributions

David Simchi-Levi, Yunzong Xu, Jinglong Zhao
Institute for Data, Systems and Society, Massachusetts Institute of Technology, Cambridge, MA 02139
dslevi@mit.edu, yxu@mit.edu, jinglong@mit.edu

Our work is motivated by a common business constraint in the retail industry. While retailers respect the advantages of dynamic pricing, they must limit the number of price changes to be within some range due to various practical reasons.

We study the classical price-based network revenue management problem, where a retailer has finite initial inventory of multiple resources to sell over a finite time horizon. We consider both known and unknown distribution settings, and derive policies that have the best-possible asymptotic performance in both settings. Our results suggest an intrinsic difference between the expected revenue associated with how many switches are allowed: in the distributionally-known setting, a resource-dependent amount of switching budgets is required to achieve sublinear regret; and in the distributionally-unknown setting, a resource-dependent piecewise-constant function that characterize the degree of sublinear regret.

Our results are the first to exhibit a separation in regrets between inventory constrained and unconstrained problems. Practically, retailers may benefit from our study in designing their budgets of price switches.

1. Introduction

We consider the classical network revenue management problem. A firm has finite inventory of multiple resources to sell over a finite time horizon. The starting inventory is unreplenishable and exogenously given. The firm can control its sales through sequential decisions on the offered prices, which come from a discrete set of candidates. Its objective is to maximize the cumulated revenue.

We consider two settings when demand is either distributionally-known, or distributionally-unknown; in both cases we assume demand to be stochastic, independent and time homogeneous (stochastic IID demand) over the time horizon. Such settings are well studied in the literature. In the distributionally-known case, see Gallego and Van Ryzin (1997), Jasin (2014); in the distributionally-unknown case, see Besbes and Zeevi (2012), Ferreira et al. (2018). The literature has also studied other settings assuming different demand models, where demand could be adversarial (Mehta et al. 2005, Ball and Queyranne 2009), could be correlated over time (Araman and Caldentey 2009, Truong and Wang 2019), and could be time non-homogeneous (Adelman 2007, Topaloglu 2009, Erdelyi and Topaloglu 2011). But all are beyond the scope of this paper.
In the distributionally-known setting, the firm must trade-off between revenue-centric decisions which maximize immediate expected revenue irrespective of inventory constraints, and inventory-centric decisions which maximize the yield from the remaining inventory. Revenue-centric decisions tend to be myopic and favor the most popular items, while inventory-centric decisions tend to be conservative and favor the highly stocked items. Intuitively, the optimal policy alternates between revenue-centric and inventory-centric decisions based on the remaining inventory and time periods.

On top of that, in the distributionally-unknown setting, the firm must also trade-off between exploitation decisions which utilize the learned information to maximize the expected revenue as if it was in the distributionally-known setting, and exploration decisions which discover the demand distributions of the less certain actions, regardless of how rewarding an action is. Exploitation decisions tend to favor the more rewarding items (with respect to inventory constraints), while exploration decisions tend to favor the less discovered items. Intuitively, the optimal policy alternates between exploitation and exploration decisions based on the information learned from the realized demands.

In both settings, the optimal policy has to adjust its decisions and instantaneously switch between actions over the time horizon. However, not all retailers have the infrastructure to query the realized demand in real-time, to adjust its decisions instantaneously, or to switch between actions as freely as possible, because changing the posted prices is too costly for many retailers (Levy et al. 1998, Zbaracki et al. 2004), and frequent price changes may confuse the customers (Jørgensen et al. 2003). A common practice for many firms is that they restrict the number of price changes to be as few as possible; see Netessine (2006), Chen et al. (2015), Cheung et al. (2017), Chen and Chao (2019), Simchi-Levi and Xu (2019), Perakis and Singhvi (2019).

Motivated by this problem, we analyze the asymptotic performance of policies with limited switches on the network revenue management problem with known and unknown demand distribution. In both settings, we show there is an intrinsic difference between the expected revenue earned associated with how many switches are allowed, which further depends on the number of resources.

1.1. Models Considered
We consider the time horizon to consist of a discrete number of time periods. We model each product as having discrete “prices points” at which it could be sold. This captures situations where fixed price points have been pre-determined by market standards, e.g. a common menu of prices that end in $9.99: $69.99, $79.99, $99.99. There is quite a difference here from the papers that model each product as having a continuous price range; see Jasin (2014) for distributionally-known setting, and Wang et al. (2014), Chen and Shi (2019), Li and Zheng (2019) for distributionally-unknown setting. Our work requires completely different analytical techniques.
We model the business constraints of limited price changes as a hard constraint. The retailer is initially endowed with a constant number of switching budgets, to change the posted price (or price vector) from one to another. For example, if there is only one product, a sequence of prices ($79.99, $89.99, $79.99, $89.99) uses two distinct prices, and makes three price changes.

We will consider the following two demand models one by one, where the design and analysis of effective policies to the second model are based on understandings of the first model.

1. **Distributionally known** demand (Section 2): In each time period, a stochastic, independent and time homogeneous demand is triggered from the prices we post.

2. **Distributionally unknown** demand (Section 3): We only know that the demand is independent and time homogeneous, but we do not have any prior knowledge about the distributions.

We have to learn the distributions over time.

For both settings, we can generalize our results to the “Bandits with Knapsacks” model (Badanidiyuru et al. 2013), which allows the per-unit revenue to be random, and further allows for any correlated distributions of the consumption quantity and the revenue.

1.1.1. **Asymptotic regime.** We focus on the asymptotic performance of policies. We introduce our results in both settings using linear scaling as our asymptotic regime, which is omnipresent in the literature. In the distributionally-known case, see Gallego and Van Ryzin (1997), Liu and Van Ryzin (2008), Jasin (2014), Bumpensanti and Wang (2018); in the distributionally-unknown case, see Besbes and Zeevi (2012), Ferreira et al. (2018), Chen and Shi (2019). Specifically, we assume both the time horizon and all the initial inventory levels are scaled by the same factor $\kappa$, while demand distributions in all periods remain the same.

There is a second asymptotic regime in the literature that does not require linear scaling. Instead, it only requires the minimum quantity of any resource (and time periods) to go to infinity. In the distributionally-known case, see Alaei et al. (2012), Ma et al. (2018); in the distributionally-unknown case, see Badanidiyuru et al. (2013).

1.1.2. **Optimality Notion.** We adopt the revenue loss as our optimality notion. In the distributionally-known setting, revenue loss is the gap of expected revenue between our proposed policy and the optimal policy with infinite switching budgets. The optimal policy with infinite switching budgets could be solved using a dynamic program, yet due to curse of dimensionality people deem it hard to solve (Gallego and Van Ryzin 1997). Revenue loss is one common metric to quantify the optimality gap, among the rich literature of different metrics; see Adelman (2007), Jasin (2014), Goldberg and Chen (2018).

In the distributionally-unknown setting, revenue loss is often referred to as “regret”, which is the gap of expected revenue between our proposed policy and the optimal clairvoyant policy with the
same level of switching budgets. The optimal clairvoyant policy is endowed with perfect knowledge of the distributions, but not the exact realizations. And it may have to switch between actions, because even the optimal policy in the distributionally-known setting may have to switch between many actions; see Badanidiyuru et al. (2013). This connection is how the design and analysis of the distributionally-known model helps with the distributionally-unknown one.

Specifically, the minimal (distribution-dependent) regret in the distributionally-known setting can be viewed as a measure of “the price of limited switches” (i.e., the revenue loss due to limiting switches), while the minimax (distribution-free) regret in the distributionally-unknown setting can be viewed as a measure of “the price of learning” (i.e., the revenue loss due to not knowing the demand distributions).

1.2. Main Results
We provide full characterization of the revenue loss under two demand models, endowed with different levels of switching budgets.

In the distributionally-known setting (Section 2), there exists a critical switching budget above which we show the revenue loss is in the order of $\Theta(\sqrt{T})$; and below which the loss is in the order of $\Theta(T)$. Specifically, the classical static policy by directly implementing the DLP solution suggests an $O(\sqrt{T})$ revenue loss – and this is the critical number of price changes to achieve sublinear revenue loss; if we must have one less price change, we show an instance dependent lower bound that any policy must incur an $\Omega(T)$ revenue loss. Combining the above results, we show that there is an intrinsic gap on expected revenue loss, if one more critical price change is allowed. See Figure 1.

![Figure 1](image-url)

Figure 1 The intrinsic gap on expected revenue loss. The vertical axis only shows the order, not the constants.
In the distributionally unknown setting (Section 3), we show matching upper and lower bounds in the order of $\tilde{\Theta}(T^{2-2^{-\frac{1}{\nu(s,m)}}})$, where $\nu(m,s) = \left\lfloor \frac{s-m-1}{K-1} \right\rfloor$. Here $s$ stands for the switching budget, $m$ for the number of resources, and $K$ for the number of feasible price vectors. Specifically, we propose an $S-$Switch Balanced Exploration with Static Exploitation algorithm that achieves an $\tilde{O}(T^{2-2^{-\frac{1}{\nu(s,m)}}})$ revenue loss; and we show a lower bound that any policy must incur an $\tilde{\Omega}(T^{2-2^{-\frac{1}{\nu(s,m)}}})$ revenue loss in the worst case distributions. Our result generalizes the result in Simchi-Levi and Xu (2019), which is a special case when $m = 0$, i.e. no inventory constraints.

In the distributionally unknown setting, our results reveal an interesting separation between the online learning problems with inventory constraints and without inventory constraints. Curiosity has been aroused around this separation for long, that the regret bounds in both the classical multi-armed bandit problems and the bandits with knapsacks problems are in the order of $\tilde{O}(\sqrt{T})$. Yet people believe that the problem with inventory constraints is “harder” than the unconstrained counterparts. Our work shows that the constrained and unconstrained problems are indeed different, and characterize the dependency on the number of inventory constraints. If we fix all the other problem primitives unchanged and only add in one more inventory constraint, then the regret bounds is going to be larger or equal to before, illustrating that inventory constraints indeed increase the revenue loss – they make the problem “harder”.

We establish equivalence results of limited switching budgets and limited adaptivity, in the context of network revenue management problems. Limited switching budgets, as we have introduced in Section 1.1, is the hard constraint that one cannot change prices more than a fixed number of times. Limited adaptivity, as originally introduced in Dean et al. (2005, 2008) for stochastic packing and stochastic knapsack problems, is the hard constraint that one cannot collect feedbacks and adapt to the feedbacks more than a fixed number of times. We can think of full adaptivity as the ability to collect immediate feedbacks, or to query the status of the system; and limited adaptivity corresponds to collect batched and delayed feedbacks. An algorithm with limited switching budgets can keep track of the demands in each period, yet constrained on price changes; while an algorithm with limited adaptivity can prescribe a trajectory of different prices with unlimited changes, yet this has to be done without knowing the status of the system. We show that when either ability (switching budgets or adaptivity) is constrained, we do not need the other ability more than necessary.

1.3. Other Related Literature

There are two streams of network revenue management problems, the quantity-based problem where customers first arrive and then seller makes accept / reject decisions, and the price-based

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$\tilde{O}(\cdot)$ notation stands for the big-O notation $O(\cdot)$ up to several logarithmic factors.
problem where seller first announces a price and then customers makes purchase decisions. For more discussion, see Talluri and Van Ryzin (2006), Gallego et al. (2018). Quantity-based and price-based problems are very closely related. When there is only one single resource, these two problems are equivalent (Maglaras and Meissner 2006).

There is a rich literature in re-optimization techniques in network revenue management, featuring in frequent or infrequent adjustments in the action space (depending on if the problem is quantity- or price-based). Re-optimization techniques choose careful moments in time to make adjustments (e.g. to change prices in the price-based context), to better respond to the randomness, and to achieve a smaller expected revenue loss. Jasin (2014), in the context of price-based network revenue management, requires $O(\log T)$ number of price changes to achieve a $O(\log T)$ order of expected revenue loss, compared to the DLP upper bound. For more discussion of the re-optimization techniques, as well as the selection of different upper bounds, see Reiman and Wang (2008), Secomandi (2008), Chen and Homem-de Mello (2010), Ciocan and Farias (2012), Jasin and Kumar (2012), Bumpensanti and Wang (2018) for quantity-based re-optimization methods; see Jasin (2014), Chen et al. (2015) for price-based re-optimization methods; see Chen and Farias (2013), Vera et al. (2019) for re-optimization methods on single-resource dynamic pricing problems; see Arlotto and Xie (2018), Arlotto and Gurvich (2019), Vera and Banerjee (2019) for re-optimization methods on multi-secretary problems and multiple constraint knapsack problems, which are directly related to the network revenue management problem.

There are several ways to model the business constraint of limited switching budgets. Literature have studied quite a few models of infrequent price changes, which are similar to the switching budgets we consider in our paper. In the distributionally-known setting, Netessine (2006) considers a single-product deterministic demand model, and focuses on the impact of a limited number of price changes. Chen et al. (2015) considers the same network revenue management model as we do, and focuses on asynchronously changing the prices of a small subset of products. In the distributionally unknown setting, Cheung et al. (2017) considers a dynamic pricing model where the demand function is unknown but belongs to a known finite set, and a pricing policy makes $O(\log \log \cdots \log T)$ number of price changes. This number is very small, yet not a constant. Chen and Chao (2019) studies a multi-period stochastic joint inventory replenishment and pricing problem with unknown demand and limited price changes. Simchi-Levi and Xu (2019) considers the stochastic multi-armed bandit problem with a general switching constraint. All of the above three papers adopt the idea of switching budgets as this paper does. Yet none of the above paper considers the existence of non-replenishable inventory constraints.

To the best of our knowledge, this paper is the first to characterize the exact dependency of expected revenue loss on limited switching budgets, in the price-based network revenue management
setup, with and without the knowledge of demand distributions. In particularly, results in this paper generalize and unify several results in the following papers: Gallego and Van Ryzin (1994), Besbes and Zeevi (2012), Badanidiyuru et al. (2013), Ferreira et al. (2018), Ma et al. (2018), Simchi-Levi and Xu (2019).

1.4. Mathematical Notations

Throughout this paper, let \( \mathbb{R} \) be the set of real numbers, and \( \mathbb{R}_+ \) the set of non-negative real numbers. Let \( \mathbb{N} \) be the set of positive integers. \( \forall N \in \mathbb{N}, \) let \( [N] = \{1, 2, ..., N\} \) denote the set of integers that are no more than \( N \). We use bold font letters for vectors, where we do not explicitly indicate how large the dimension is. Let \( p^T \) be the transpose of any vector \( p \). For any vector \( x \in \mathbb{R}^K \), let \( \|x\|_0 = \sum_{k \in [K]} 1_{x_k \neq 0} \) be the \( L_0 \) norm of \( x \), i.e. the number of non-zero elements in vector \( x \). For any real number \( x \in \mathbb{R} \), let \( x^+ = \max\{x, 0\} \) be the non-negative part of \( x \). For any positive real number \( x \in \mathbb{R}_+ \), let \( \lfloor x \rfloor \) be the largest integer that is smaller or equal to \( x \); for any non-positive real number \( x \in \mathbb{R} - \mathbb{R}_+ \), let \( \lfloor x \rfloor = 0 \).

2. Known Demand Distributions

2.1. Definition of Problems

We study the classical network revenue management problem. Let there be discrete, finite time horizon with \( T \) periods. Time starts from period 1 and ends in period \( T \). Let there be \( n \) different products generated by \( m \) different resources, each resource endowed with finite initial inventory \( B_i, \forall i \in [m] \). Let \( A = [a_{ij}]_{i \in [m], j \in [n]} \) be the consumption matrix. Each entry \( a_{ij} \in \mathbb{R}_+ \) stands for the amount of inventory \( i \in [m] \) used, if one unit of product \( j \in [n] \) is sold. Each column \( A^j \) stands for the consumption vector of each resource. Each column contains at least one nonzero entry.

In each period \( t \), the firm can offer a price for each product from a finite set of \( K \) price vectors, which we denote using \( \{p_1, p_2, ..., p_K\} \). Here \( p_k = (p_{1,k}, p_{2,k}, ..., p_{n,k}) \) are the prices for products \( 1, ..., n \), under price vector \( k \in K \). All the price vectors are fixed and given. Let \( p_{\text{max}} = \max_{j,k} p_{j,k} \) be the maximum price. Given price \( p_k \), the demand for each product \( j \in [n] \) conforms a given Bernoulli distribution \( Q_{j,k} := Q_j(p_k) \in \{0, 1\} \). Denote \( q_{j,k} := E[Q_{j,k}] \) as the expected demand. For each unit of demand generated for product \( j \in [n] \) under price vector \( p_k, \forall k \in [K] \), the firm generates \( p_{j,k} \) units of revenue by depleting \( a_{ij} \) units of each inventory \( i \in [m] \). If no demand is generated, all the remaining inventory is carried over into the next period. The selling process stops at the first time \( \tau \) when the total cumulative consumption of any resource exceeds its initial inventory.

This modeling framework gives us the flexibility to have two identical consumption vectors sold at two different prices. We simply treat them as different products, i.e. in the airline context, main cabin vs. basic economy.
In Section 1 we have addressed the business constraint that a firm is often prevented from changing the price vectors too many times. Throughout the horizon, we can change the posted price vectors for no more than $S$ times. We treat $S$ as a fixed constant. Intuitively, when $S$ is small we are more constrained, and when $S$ is large we are close to the classical network revenue management problem studied in the literature. Our objective is to maximize the total expected revenue in the face of limited switches.

We adopt the general notation $\pi : \mathbb{R}^n \times [T] \rightarrow [n]$ to denote any policy with the full information about stochastic distributions, that suggests which price vector to use given the remaining periods $T$ and remaining inventory $B$. For any $s \in \mathbb{N}$, let $\Pi[s]$ be the set of policies that changes prices for no more than $s$ times. For any $s, s' \in \mathbb{N}$ such that $s \leq s'$, $\Pi[s] \subseteq \Pi[s']$. Let $\Pi := \lim_{s \rightarrow +\infty} \Pi[s]$ be the set of policies with infinite switching budgets (no switch constraints). Let $\text{Rev}(\pi)$ be the expected revenue that policy $\pi$ generates in expectation. Let $\pi^*$ be the optimal dynamic policy with infinite switching budgets.

We adopt linear scaling as our asymptotic regime. We assume all the problem parameters $\mathcal{I} = (m, n, K, p, Q, A)$ are fixed. We assume $T$ and $B_j, \forall j \in [m]$ are at the same comparable order, and larger than either one of $m, n, K$.

2.2. Overview of Results

We summarize our results, as well as some closely related results from literature, in Figure 2.

![Figure 2](image)

**Figure 2** Summary of results in the distributionally-known setting. Results in the literature are on the left side; our results are on the right side.

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For any problem instance $I = (m, n, K, p, Q, A)$, literature have studied the following deterministic linear program (DLP).

$$J_{DLP} = \max_{\{x_k\}_k \in [K]} \sum_{k \in [K]} \sum_{j \in [n]} p_{j,k} q_{j,k} x_k$$

s.t. \hspace{1cm} \sum_{k \in [K]} \sum_{j \in [n]} a_{ij} q_{j,k} x_k \leq B_i \hspace{1cm} \forall \ i \in [m] \tag{2}\]

\hspace{1cm} \sum_{k \in [K]} x_k \leq T \tag{3} \]

\hspace{1cm} x_k \geq 0 \hspace{1cm} \forall \ k \in [K] \tag{4} \]

It is well known that the above DLP serves as an upper bound to any policy $\pi \in \Pi$ ("DP OPT" as in Figure 2), even an optimal policy with infinite switching budgets. The gap between the best policy and the DLP upper bound is in the order of $\Theta(\sqrt{T})$.

We explain the $\Omega(\sqrt{T})$ lower bound in Figure 2. It is a worst case lower bound from Bumpensanti and Wang (2018), Ma et al. (2018). Both results from literature require the feasible price set to have only one price (vector). Since there is only one price, the optimal dynamic programming policy is essentially a fixed, zero-switch policy, which is a special case when DSL = 1. It remains an interesting open question how small could the gap be, between the DLP upper bound and any policy endowed with more than $(DSL - 1)$ switching budgets.

2.3. Lower Bounds

We start with a closer look of the DLP optimal solutions. Let the set of optimal solutions from the DLP be $X^* = \arg \max_{x \in \mathbb{R}^K} \{1, 2, 3, 4\}$ are satisfied}. Let $DSL = \min\{\|x\|_0 | x \in X^* \}$ be the least number of non-zero variables of any optimal solution, namely, the Dimension of Sparsest LP–solutions. Let $DSL = \arg \min\{\|x\|_0 | x \in X^* \}$ be the set of such solutions. For any $x^* \in DSL$, let $K(x^*) = \{k \in [K] | x^*_k \neq 0 \} \subseteq [K]$ be the subset of dimensions that are non-zero in $x^*$.

**Theorem 1.** For any problem instance $I = (m, n, K, p, Q, A)$, there is an associated DSL number. Any policy $\pi \in \Pi[DLS - 2]$ earns an expected revenue

$$\mathbb{E}[\text{Rev}(\pi)] \leq J_{DLP} - \Omega(T)$$

**Proof of Theorem 1.** For any problem instance $I = (m, n, K, p, Q, A)$. Any policy $\pi \in \Pi[DLS - 2]$ only selects no more than $(DSL - 1)$ many price vectors. $\forall l \in [DSL - 1]$, let $p_{o_l}$ be the $l^{th}$ price that policy $\pi$ selects; let $\tau_{o_l}$ be the total number of periods that price $p_{o_l}$ is offered, under policy $\pi$. Notice that both $a_l$ and $\tau_{o_l}$ are random, i.e. which price vector to offer and for how long each price vector is offered they both depend on the random trajectory.
Now denote $Y_{j,a_i}$ as the random amount of product $j$ sold, during the periods that price vector $a_i$ is offered. We know $\mathbb{E}[Y_{j,a_i} | \tau_{a_i}] = \tau_{a_i} q_{j,a_i}$. Here $\tau_{a_i}$ is a random amount, so we cannot directly use concentration inequalities. But we can adapt the trick from Slivkins (2019). Suppose there was a tape of length $T$ for each product $j$ and each price vector $a_i$, with each cell independently sampled from the distribution of $Q_{j,a_i}$. This tape serves as a coupling of the random reward: in each period $t$ if price vector $a_i$ is offered, we simply generate a demand from the $t^{th}$ cell of the tape. Now we can use Hoeffding inequality:

$$\forall a_i, \forall j, \forall \tau_{a_i}, \Pr\left( |Y_{j,a_i} - \tau_{a_i} q_{j,a_i}| \leq \sqrt{3\tau_{a_i} \ln T} \right) \geq 1 - \frac{2}{T^6}$$

Denote the following event $E$:

$$\forall a_i, \forall j, \forall \tau_{a_i}, |Y_{j,a_i} - \tau_{a_i} q_{j,a_i}| \leq \sqrt{3\tau_{a_i} \ln T}. \quad (5)$$

Using a union bound we have:

$$\Pr\left( \forall a_i, \forall j, \forall \tau_{a_i}, |Y_{j,a_i} - \tau_{a_i} q_{j,a_i}| \leq \sqrt{3\tau_{a_i} \ln T} \right) \geq 1 - \frac{2}{T^3}$$

because $n, m, K$ are all less than $T$. The happening of such event means that the realized demands are close to the expected demands, suggesting that LP is a good proxy of any policy $\pi \in \Pi[\text{DSL} - 2]$.

Now if we focus on the usage of any price vector $a_i$, the total expected revenue is $\sum_{j \in [n]} Y_{j,a_i} p_{j,a_i}$. Then the total expected revenue generated during the entire horizon can be upper bounded by

$$\sum_{l \in [\text{DSL} - 1]} \sum_{j \in [n]} Y_{j,a_i} p_{j,a_i} \leq \sum_{l \in [\text{DSL} - 1]} \sum_{j \in [n]} (q_{j,a_i} r_{a_i} + \sqrt{3T \ln T}) p_{j,a_i} \leq \left( \sum_{l \in [\text{DSL} - 1]} \sum_{j \in [n]} q_{j,a_i} r_{a_i} p_{j,a_i} \right) + n^2 \sqrt{3T \ln T},$$

where the last inequality is because $\text{DSL} \leq (n + 1)$. On the other hand, the consumption of inventory $i$ must not violate the inventory constraint.

$$\sum_{l \in [\text{DSL} - 1]} \sum_{j \in [n]} Y_{j,a_i} a_{ij} \leq B_i.$$

Lower bounding $Y_{j,a_i}$ by $q_{j,a_i} r_{a_i} - \sqrt{3T \ln T}$ we have

$$\sum_{l \in [\text{DSL} - 1]} \sum_{j \in [n]} Y_{j,a_i} a_{ij} \leq B_i + \sum_{l \in [\text{DSL} - 1]} \sum_{j \in [n]} \sqrt{3T \ln T} \leq B_i + n^2 \sqrt{3T \ln T}.$$

Now construct the following LP:

$$J^{\text{Perturbed}} = \max_{\{x_k\}_{k \in [K]}} \sum_{k \in [K]} \sum_{j \in [n]} p_{j,k} q_{j,k} x_k$$

s.t. $\sum_{k \in [K]} \sum_{j \in [n]} a_{ij} q_{j,k} x_k \leq B_i + n^2 \sqrt{3T \ln T} \quad \forall i \in [m]$

$$\sum_{k \in [K]} x_k \leq T$$

$$x_k \geq 0 \quad \forall k \in [K]$$
If we let $T$ and $B$ to scale linearly by $\kappa$, then the perturbed term $n^2\sqrt{3T\ln T}$ is negligible, and so the optimal solution is scaled by $\kappa$. This means that if we restrict ourselves to a solution that uses no more than $(\text{DSL} - 1)$ non-zero variables, we incur a linear loss, i.e. $(\sum_{l \in [\text{DSL} - 1]} \sum_{j \in [n]} q_{j,a_l} r_{a_l p_j, a_l}) = (1 - \Omega(1)) J^{\text{perturbed}}$.

Since from each solution $x^*$ of this perturbed LP, we can find a corresponding discounted solution $x^* \cdot (1 + n^2\sqrt{3T\ln T})$ that is feasible to the DLP. This suggests that $J^{\text{perturbed}} \leq J^{\text{DLP}} \cdot (1 + n^2\sqrt{3T\ln T})$, because DLP is a maximization problem.

Putting all together, and conditional on event $E$ that happens with probability at least $1 - \frac{2}{\pi^2}$,

$$
E[\text{Rev}(\pi) | E] \leq (1 - \Omega(1)) J^{\text{perturbed}} + n^2\sqrt{3T\ln T} p_{\text{max}} \\
\leq (1 - \Omega(1)) J^{\text{DLP}} \cdot (1 + \frac{n^2\sqrt{3T\ln T}}{B_{\text{min}}}) + n^2\sqrt{3T\ln T} p_{\text{max}} \\
\leq J^{\text{DLP}} - \Omega(T)
$$

which suggests that $E[\text{Rev}(\pi)] \leq J^{\text{DLP}} - \Omega(T)$. □

**Proposition 1** (Proposition 4, Bumpensanti and Wang (2018); Lemma 5, Ma et al. (2018)). There exists a problem instance $I_0 = (m, n, K, p, Q, A)$, such that any policy $\pi \in \Pi$ earns an expected revenue $E[\text{Rev}(\pi)] \leq J^{\text{DLP}} - \Omega(\sqrt{T})$.

### 2.4. Upper Bounds

We use the following static control policy, which is well known in the literature (Gallego and Van Ryzin 1997, Maglaras and Meissner 2006, Ma et al. 2018).

**Definition 1.** Any $\pi \in \Pi[\text{DSL} - 1]$ policy induced by DLP:

1. For any problem instance $I = (m, n, K, p, Q, A)$, solve the DLP as defined by (1), (2), (3), and (4). Find an optimal solution with the least number of non-zero variables, $x^* \in \text{DSL}$.
2. Arbitrarily choose any permutation $\sigma : [\text{DSL}] \rightarrow K(x^*)$ from all $(\text{DSL})!$ possibilities.
3. Set the price vector to be $p_{\sigma(1)}$ for the first $x_{\sigma(1)}$ periods, then $p_{\sigma(2)}$ for the next $x_{\sigma(2)}$ periods, ..., and finally $p_{\sigma(\text{DSL})}$ for the last $x_{\sigma(\text{DSL})}$ period.

We explain the second step permutation. Suppose $K(x^*) = \{1, 3, 4\}$. In this case, DSL = 3 and there are 6 permutations. There are 6 possible policies as suggested in Definition 1. Some of these policies have better performance than others. But in an asymptotic regime, they all have sublinear regret, compared to the DLP upper bound. This result is not surprising in the literature. Cooper (2002), Liu and Van Ryzin (2008) have proved similar results in similar but different settings.

We assume that $x_k, \forall k \in [K]$ are integers, because rounding issues incur a revenue loss of at most $(m \cdot \max_k p_k^T q_k)$, which is negligible in an asymptotic regime.
Theorem 2. Any policy $\pi$ as defined in Definition 1 earns an expected revenue

$$\mathbb{E}[\text{Rev}(\pi)] \geq J^{\text{DLP}} - \sqrt{\frac{1}{2} m \sqrt{nT}}$$

Proof of Theorem 2. For any policy as defined in Definition 1, let $x^*$ be the associated optimal solution. Let $Y_{j,k} = \sum_{i=1}^{x^*_j} Q_{j,k}(\xi_l)$ be the total number of product $j$ sold under policy $\pi$, if there were no inventory constraints. We use $\xi_l$ to stand for the randomness of each Bernoulli random variables. $Y_{j,k}$ is the sum of $x^*_j$ random variables, thus itself is a random variable.

DLP suggests an upper bound of expected revenue under policy $\pi$, which is taken as if there were no inventory constraints. When there are no inventory constraints, whichever permutation we take is irrelevant – all the permutations achieve the same expected revenue. To evaluate policy $\pi$, we focus on how many units or each resource are truncated due to inventory constraints.

$$\mathbb{E}[\text{Rev}(\pi)] \geq J^{\text{DLP}} - \mathbb{E} \left[ \sum_{i \in [m]} \left( \sum_{j \in [n]} \sum_{k \in [K]} a_{ij} Y_{j,k} - B_i \right)^+ \right] \cdot p_{\text{max}}$$

$$\geq J^{\text{DLP}} - \mathbb{E} \left[ \left( \sum_{j \in [n]} \sum_{k \in [K]} a_{ij} Y_{j,k} - B_i \right)^+ \right] \cdot p_{\text{max}}$$

(6)

For each unit of resource $i \in [m]$ that is truncated, at most $p_{\text{max}}$ units of revenue are lost. It suffices to upper bound the truncated units.

Note that $\mathbb{E}\left[ \sum_{j \in [n]} \sum_{k \in [K]} a_{ij} Y_{j,k} \right] \leq B_i$, due to (2); and that $\sum_{j \in [n]} \sum_{k \in [K]} a_{ij} Y_{j,k}$ is a sum of at most $(nT)$ independent Bernoulli random variables, due to (3).

$$\mathbb{E} \left[ \left( \sum_{j \in [n]} \sum_{k \in [K]} a_{ij} Y_{j,k} - B_i \right)^+ \right] = \int_0^{+\infty} \Pr \left( \sum_{j \in [n]} \sum_{k \in [K]} a_{ij} Y_{j,k} \geq B_i + u \right) du$$

$$\leq \int_0^{+\infty} e^{-\frac{2u^2}{\pi nT}} du$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-\frac{2u^2}{\pi nT}} du$$

$$= \sqrt{\frac{\pi nT}{8}} \leq \sqrt{\frac{nT}{2}}$$

(7)

where the first equality is due to non-negativity, and the first inequality is Hoeffding’s inequality.

Note that in the last line, $\pi \approx 3.14$ is the ratio of a circle’s circumference to its diameter, not to be confused with policy $\pi$.

Plugging (7) into (6), we finish the proof. □
3. Unknown Demand Distributions

3.1. Definition of Problems

We keep studying the model described in Section 2.1 in the distributionally-unknown setting. Given price $p_k$, the demand for each product $j \in [n]$ is an unknown Bernoulli random variable, $Q_{j,k} := Q_j(p_k) \in \{0,1\}$. That is, the expected value $q_{j,k} := \mathbb{E}[Q_{j,k}]$ is unknown and has to be sequentially learned over time. The main trade-off we would like to address in this setting is between exploration and exploitation.

Let $\phi$ denote any non-anticipating learning policy, and $\phi_t \in [k]$ denote the price vector chosen by policy $\phi$ at round $t \in [T]$. More formally, $\phi_t$ establishes a probability kernel acting from the space of historical actions and observations to the space of actions at round $t$. Let $P^\phi_Q$ and $\mathbb{E}_Q^\phi$ be the probability measure and expectation induced by policy $\phi$ and latent distributions $Q = (Q_{j,k})_{j \in [n], k \in [K]}$. For any $s \in \mathbb{N}$, let $\Phi[s]$ be the set of policies that changes prices for no more than $s$ times almost surely for all $Q$. For any $s, s' \in \mathbb{N}$ such that $s \leq s'$, $\Phi[s] \subseteq \Phi[s']$. Let $\Phi := \lim_{s \to +\infty} \Phi[s]$ be the set of policies with infinite switching budgets (no switching constraints). Let $\text{Rev}_Q(\pi^*_s) = \sup_{\pi \in \Pi[s]} \text{Rev}_Q(\pi)$ be the optimal expected revenue that a clairvoyant policy $\pi \in \Pi[s]$ (with the knowledge of $Q$) generates when the underlying distribution is $Q$. The performance of a learning policy is measured against $\text{Rev}_Q(\pi^*_s)$. Specifically, we define the $s$-switch regret of a learning policy $\phi \in \Phi[s]$ as the worst-case difference between the expected performance of the optimal clairvoyant policy and the expected performance of policy $\phi$: $R^s_T = \sup_{Q} \{|\text{Rev}_Q(\pi^*_s) - \text{Rev}_Q(\phi)|\}$, where $\text{Rev}_Q(\phi)$ denotes the expected revenue of $\phi$ when the underlying distribution is $Q$.

3.2. Upper Bounds

We first propose a policy that provides an upper bound on regret. Our policy, called the $S$-Switch Balanced Exploration with Static Exploitation (SS-BESE) policy, is described in Algorithm 1. The policy presents a novel and non-trivial combination of the techniques from the SS-SE policy proposed in Simchi-Levi and Xu (2019), the Balanced Exploration policy proposed in Badanidiyuru et al. (2013), and the Static Control policy defined in Definition 1.

We show that the SS-BESE policy is indeed an $S$-switch policy and establish the following upper bound on its regret.

**Theorem 3.** Let $\phi$ be the SS-BESE policy, then $\phi \in \Phi[s]$ and

$$R^s_T = O(T^{2-\frac{1}{2\nu(m)}(s,m)}),$$

where $\nu(m,s) = \left\lfloor \frac{s-m-1}{K-1} \right\rfloor$.

We overlook the rounding issues in step 3 of Algorithm 1, which are easy to fix in regret analysis.
Algorithm 1: S-Switch Balanced Exploration with Static Exploitation (SS-BESE)

Input: $K, T, s, m, n, A$

Partition: Calculate $\nu(s, m) = \left\lfloor \frac{s-m-1}{K-1} \right\rfloor$.

Divide the entire time horizon $1, \ldots, T$ into $\nu(s, m) + 1$ intervals: $(t_0 : t_1], (t_1 : t_2], \ldots, (t_{\nu(s, m)} : t_{\nu(s, m)+1}]$, where the endpoints are defined by $t_0 = 0$ and

$$t_i = \left\lfloor \frac{k^1 - 2 - 2^{-(i-1)}}{2 - 2^{-\nu(s, m)}} T \frac{2 - 2^{-(i-1)}}{2 - 2^{-\nu(s, m)}} \right\rfloor, \quad \forall i = 1, \ldots, \nu(s, m) + 1.$$

Initialization: Let $a_0$ be a random price vector in $\{p_1, p_2, \ldots, p_K\}$.

Policy:

1: for $l = 1, \ldots, \nu(s, m)$ do

2: Recompute the set $\Delta_l$ of “potentially perfect distributions” $D$ over price vectors, according to the definition of $\Delta_l$ in [Badanidiyuru et al. (2013)].

3: For each $k \in [K]$, pick any distribution $D_{l,k} \in \Delta_l$ such that $D(k) \geq \frac{1}{2} \max_{D' \in \Delta_l} D'(k)$. Draw $\frac{t_l - t_{l-1}}{|A_l|}$ independent samples from $D_{l,k} \in \Delta_l$. Denote the realized total number of $p_i$ by $N_i^l(i) (i \in [K])$.

4: Calculate $N_i(l) = \sum_{k=1}^{K} N_i^l(k)$ for all $i \in [K]$.

5: Let $a_{t_{l-1}+1} = a_{t_{l-1}}$. Starting from this action, choose each price vector $p_i$ for $N_i(i)$ consecutive rounds, $i \in [K]$. Mark the last chosen price vector in round $t_l$ as $a_{t_l}$.

6: Stop once time horizon is met or one of the resources is exhausted.

7: end for

8: In the last interval, calculate $\tilde{Q}$ based on the empirical averages of samples, which is an unbiased estimator of $Q$. Using $\tilde{Q}$ as an input of the true demand distributions, solve the DLP defined in Section 2.1. Adopt a static control policy that satisfies Definition 1 (see Section 2.4) that starts from $a_{t_{\nu(s, m)}}$ for the last interval.

3.3. Lower Bounds

We prove a matching lower bound for the worst-case regret.

**Theorem 4.** There exists a class of problem instances $\mathcal{I} = \{(m, n, K, p, Q, A)|Q \in \{0,1\}^n\}$ such that for all $\phi \in \Phi[s]$,

$$R^\phi_s(T) = \Omega(T \frac{1}{2 - 2^{-\nu(s, m)}}),$$

where $\nu(m, s) = \left\lfloor \frac{s-m-1}{K-1} \right\rfloor$. 

3.4. Implications
As the results in Section 3 indicate, there exists a class of online network revenue management instances with \( m \) resources such that the minimax \( s \)-switch regret is of the order \( \tilde{\Theta}\left(T^{2-\frac{1}{2\ln m}}\right) \). This suggests that given a fixed switching budget, the increase of the number of resources, \( m \), may result in an increase of the order of the minimax regret of our problem. To the best of our knowledge, this is the first kind of results that explicitly characterize how the dimension of the inventory constraints make an online learning problem “harder” (in the sense that it increases the necessary number of switches for a certain regret bound). While both the classical multi-armed bandit problem and the bandits with knapsacks / online network revenue management problem \( \text{(Besbes and Zeevi 2012)} \) \( \text{(Badanidiyuru et al. 2013)} \) \( \text{(Ferreira et al. 2018)} \) essentially exhibit the same regret rate \( \tilde{\Theta}(\sqrt{T}) \), where the order of \( T \) is not affected by the dimension of the inventory constraints, our results indicate that this is not the case when there is an additional switching constraint.

References
Adelman D (2007) Dynamic bid prices in revenue management. Operations Research 55(4):647–661.
Alaei S, Hajiaghayi M, Liaghat V (2012) Online prophet-inequality matching with applications to ad allocation. Proceedings of the 13th ACM Conference on Electronic Commerce, 18–35 (ACM).
Araman VF, Caldentey R (2009) Dynamic pricing for nonperishable products with demand learning. Operations research 57(5):1169–1188.
Arlotto A, Gurvich I (2019) Uniformly bounded regret in the multisecretary problem. Stochastic Systems 9(3):231–260.
Arlotto A, Xie X (2018) Logarithmic regret in the dynamic and stochastic knapsack problem. arXiv preprint arXiv:1809.02016 .
Badanidiyuru A, Kleinberg R, Slivkins A (2013) Bandits with knapsacks. 2013 IEEE 54th Annual Symposium on Foundations of Computer Science, 207–216 (IEEE).
Ball MO, Queyranne M (2009) Toward robust revenue management: Competitive analysis of online booking. Operations Research 57(4):950–963.
Besbes O, Zeevi A (2012) Blind network revenue management. Operations research 60(6):1537–1550.
Bumpensanti P, Wang H (2018) A re-solving heuristic with uniformly bounded loss for network revenue management. arXiv preprint arXiv:1802.06192 .
Chen B, Chao X (2019) Parametric demand learning with limited price explorations in a backlog stochastic inventory system. IISE Transactions 1–9.
Chen L, Homem-de Mello T (2010) Re-solving stochastic programming models for airline revenue management. Annals of Operations Research 177(1):91–114.
Chen Q, Jasin S, Duenyas I (2015) Real-time dynamic pricing with minimal and flexible price adjustment. *Management Science* 62(8):2437–2455.

Chen Y, Farias VF (2013) Simple policies for dynamic pricing with imperfect forecasts. *Operations Research* 61(3):612–624.

Chen Y, Shi C (2019) Network revenue management with online inverse batch gradient descent method. *Available at SSRN*.

Cheung WC, Simchi-Levi D, Wang H (2017) Dynamic pricing and demand learning with limited price experimentation. *Operations Research* 65(6):1722–1731.

Ciocan DF, Farias V (2012) Model predictive control for dynamic resource allocation. *Mathematics of Operations Research* 37(3):501–525.

Cooper WL (2002) Asymptotic behavior of an allocation policy for revenue management. *Operations Research* 50(4):720–727.

Dean BC, Goemans MX, Vondrák J (2005) Adaptivity and approximation for stochastic packing problems. *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, 395–404 (Society for Industrial and Applied Mathematics).

Dean BC, Goemans MX, Vondrák J (2008) Approximating the stochastic knapsack problem: The benefit of adaptivity. *Mathematics of Operations Research* 33(4):945–964.

Erdelyi A, Topaloglu H (2011) Using decomposition methods to solve pricing problems in network revenue management. *Journal of Revenue and Pricing Management* 10(4):325–343.

Ferreira KJ, Simchi-Levi D, Wang H (2018) Online network revenue management using thompson sampling. *Operations research* 66(6):1586–1602.

Gallego G, Topaloglu H, et al. (2018) *Revenue management and pricing analytics* (Springer).

Gallego G, Van Ryzin G (1994) Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management science* 40(8):999–1020.

Gallego G, Van Ryzin G (1997) A multiproduct dynamic pricing problem and its applications to network yield management. *Operations research* 45(1):24–41.

Goldberg DA, Chen Y (2018) Beating the curse of dimensionality in options pricing and optimal stopping. *arXiv preprint arXiv:1807.02227*.

Jasin S (2014) Reoptimization and self-adjusting price control for network revenue management. *Operations Research* 62(5):1168–1178.

Jasin S, Kumar S (2012) A re-solving heuristic with bounded revenue loss for network revenue management with customer choice. *Mathematics of Operations Research* 37(2):313–345.

Jørgensen S, Taboubi S, Zaccour G (2003) Retail promotions with negative brand image effects: Is cooperation possible? *European Journal of Operational Research* 150(2):395–405.
Levy D, Dutta S, Bergen M, Venable R (1998) Price adjustment at multiproduct retailers. *Managerial and decision economics* 19(2):81–120.

Li X, Zheng Z (2019) Dynamic pricing with external information and inventory constraint. *Available at SSRN* .

Liu Q, Van Ryzin G (2008) On the choice-based linear programming model for network revenue management. *Manufacturing & Service Operations Management* 10(2):288–310.

Ma W, Simchi-Levi D, Zhao J (2018) Dynamic pricing under a static calendar. *Available at SSRN 3251015* .

Maglaras C, Meissner J (2006) Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing & Service Operations Management* 8(2):136–148.

Mehta A, Saberi A, Vazirani U, Vazirani V (2005) Adwords and generalized on-line matching. *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS’05)*, 264–273 (IEEE).

Netessine S (2006) Dynamic pricing of inventory/capacity with infrequent price changes. *European Journal of Operational Research* 174(1):553–580.

Perakis G, Singhvi D (2019) Dynamic pricing with unknown non-parametric demand and limited price changes. *Available at SSRN 3336949* .

Reiman MI, Wang Q (2008) An asymptotically optimal policy for a quantity-based network revenue management problem. *Mathematics of Operations Research* 33(2):257–282.

Secomandi N (2008) An analysis of the control-algorithm re-solving issue in inventory and revenue management. *Manufacturing & Service Operations Management* 10(3):468–483.

Simchi-Levi D, Xu Y (2019) Phase transitions and cyclic phenomena in bandits with switching constraints. *Available at SSRN 3380783* .

Slivkins A (2019) Introduction to multi-armed bandits. *arXiv preprint arXiv:1904.07272* .

Talluri KT, Van Ryzin GJ (2006) *The theory and practice of revenue management*, volume 68 (Springer Science & Business Media).

Topaloglu H (2009) Using lagrangian relaxation to compute capacity-dependent bid prices in network revenue management. *Operations Research* 57(3):637–649.

Truong VA, Wang X (2019) Prophet inequality with correlated arrival probabilities, with application to two sided matchings. *arXiv preprint arXiv:1901.02552* .

Vera A, Banerjee S (2019) The bayesian prophet: A low-regret framework for online decision making. *Abstracts of the 2019 SIGMETRICS/Performance Joint International Conference on Measurement and Modeling of Computer Systems*, 81–82 (ACM).

Vera A, Banerjee S, Gurvich I (2019) Online allocation and pricing: Constant regret via bellman inequalities. *arXiv preprint arXiv:1906.06361* .
Wang Z, Deng S, Ye Y (2014) Close the gaps: A learning-while-doing algorithm for single-product revenue management problems. *Operations Research* 62(2):318–331.

Zbaracki MJ, Ritson M, Levy D, Dutta S, Bergen M (2004) Managerial and customer costs of price adjustment: direct evidence from industrial markets. *Review of Economics and statistics* 86(2):514–533.