Deformed Weitzenböck Connections and Teleparallel Gravity

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We study conditions on a generic connection written in terms of first-order derivatives of the vielbein in order to obtain (possible) equivalent theories to Einstein Gravity. We derive the equations of motion for these theories which are based on the new connections. We recover the Teleparallel Gravity equations of motion as a particular case. The analysis of this work might be useful to Double Field Theory to find other connections determined in terms of the physical fields.

I. INTRODUCTION

Teleparallel Gravity (TG or TEGR) is an alternative formulation to Einstein Gravity (EG) [1–3]. It is based on parallelizable manifolds equipped with the Weitzenböck connection (the connection of the parallelization) rather than the Levi-Civita connection. One of the key aspects of the theory is that the Weitzenböck connection furnishes a null Riemann curvature tensor, and since this connection is metric-compatible, the dynamics of the theory is based only on the torsion. The Weitzenböck connection (the connection of the parallelization) on parallelizable manifolds equipped with the Weitzenböck connection furnishes a null Riemann curvature tensor, and since this connection is metric-compatible, the dynamics of the theory is based only on the torsion. The Weitzenböck connection is of the form \( W_{\mu\nu}^\rho = \partial_\mu e_\nu^\rho - \partial_\nu e_\mu^\rho \), where \( \mu \) and \( \rho \) are curved and flat indices respectively, and \( e_\mu^\rho \) is the vielbein. The vielbein is treated as the fundamental field in the theory and the metric is interpreted as a byproduct coming from it. The name of TG comes from the fact that there is a notion of absolute parallelism between vectors located at different spacetime points. Indeed, one way to see this is to notice that the vielbein field is covariantly constant with respect to the Weitzenböck connection in a coordinate basis. This means that we can compare the flat components of a vector field defined at different spacetime points just like we would do in a global basis of flat spacetime.

TG has been found to have applications in cosmology by studying theories called \( f(T) \)-Theories [4–10], where \( T \) stands for scalar combinations built out of the torsion. In general, the local Lorentz symmetry group of these theories is broken and only a remnant subgroup of it survives [11–13]. It is easy to see that spacetime tensors built out of the Weitzenböck connection (for instance the torsion \( T_{\mu\nu}^\rho \)) will not always transform as scalars under local Lorentz transformations. In fact, the Weitzenböck connection itself, in a coordinate basis, does not transform like a scalar under local Lorentz transformations but under global ones. Global transformations bring the immediate worry about extra degrees of freedom coming from the vielbein, since in \( D = 4 \), six out of the 16 components of the vielbein are gauged away by local Lorentz transformations. However, since the TG action turns out to be equal to the Einstein-Hilbert action, up to a boundary term, the equations of motion of TG are equivalent to the Einstein equations and thus the theory enjoys local Lorentz symmetry.

The TG idea has also been applied to Double Field Theory (DFT) [14–16], which is a string-inspired field theory with manifest T-duality symmetry. In DFT, many geometric notions like diffeomorphisms, covariant derivatives (connections) and curvature tensors arise in a natural and generalized way. In fact, its geometric structure is closely related to generalized geometry [17]. However, in contrast to General Relativity, the DFT version of the Levi-Civita connection has not all of its components fully determined in terms of the physical fields, but only some projections of the connection are. This also implies that the DFT curvature tensor is not fully determined in terms of the physical fields [18]. In an attempt to determine the connection in terms of the physical fields, the TG version of DFT was considered [19]. The connection was set equal to the Weyl connection. The resulting theory, based on this connection, reproduces the DFT action (up to a boundary term) and the dynamics is based on the (generalized) torsion. Similarly as in TG, the DFT action possesses a local double Lorentz symmetry but the generalized torsion transforms only under global double Lorentz transformations.

Motivated by this global-local mechanism of TG, and inspired by the necessity of DFT to consider other connections determined in terms of the physical fields of the theory, we will try to obtain equivalent theories to EG defined by other connections rather than the Levi-Civita or the Weitzenböck connection. These connections should be determined in terms of the vielbein and its derivative. Although these connections will transform under global Lorentz transformations, the resulting theories will possess local Lorentz symmetry. Our starting point will be the most general connection written in terms of first-order derivatives of the vielbein:

\[
\Gamma_{\mu\nu}^\rho = W_{\mu\nu}^\rho - a_1 T_{\nu}^\rho \sigma \sigma - b_1 T_{\nu}^\rho \sigma + b_2 T_{\mu}^\rho \nu - c_1 g_{\mu\nu} \Gamma_{\sigma}^\rho - d_1 \delta_{\mu}^\rho \Gamma_{\nu}^\sigma - d_2 \delta_{\nu}^\rho \Gamma_{\mu}^\sigma. \tag{1.1}
\]

\[\]
Where $T_{\mu \nu}^\rho = 2W_{[\mu \nu]}^\rho$ is the torsion of the Weitzenböck connection and $g_{\mu \nu}$ is the spacetime metric. This connection is parametrized by six real parameters and it does not have any symmetry requirement imposed on its indices so it will yield, generically, torsion and non-metricity. In a coordinate basis, this connection transforms in a proper way under general coordinate transformations and is a scalar under global Lorentz transformations. This means that for generic coefficients, the local Lorentz symmetry is broken (see, however, below equation (II.11)).

The rest of the paper is organized as follows. In section (II) we will introduce the notation used in the paper and show how to obtain different actions equivalent to the Einstein-Hilbert action for connections with generic features. We will not talk about matter coupling so the theories obtained from the generic connections will be equivalent to EG in vacuum. In section (III) we will find constraints on the parameters of (I.I). The parameters that satisfy these constraints will give the desired connections that allow for actions equivalent, up to a boundary term, to the Einstein-Hilbert action and we will analyze particular values of these parameters. In section (IV) we will obtain the equations of motion for these theories based on the new connections. In section (V) we summarize our work.

II. NOTATION AND IDEA

We will work in $D = 4$ spacetime dimensions, $\mu, \nu, \ldots$ are curved indices and $a, b, \ldots$ denote flat indices. Our fundamental field is the vielbein $e_a^\mu$ and we define the metric as a byproduct $g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu$. Here $\eta_{ab}$ is the constant Minkowski metric. The vielbein satisfies $e_a^\mu e^b_\mu = \delta^b_a$ and $e^a_\mu e^b_\nu = \delta^b_\mu$ where we raise and lower curved and flat indices with the metric $g_{\mu \nu}$ and $\eta_{ab}$ respectively. We define the flat derivative $D_a = e_a^\mu \partial_{\mu}$. It is useful to introduce the following quantities:

$$\Omega_{ab}^c = D_a e^b_\rho e^c_\rho, \quad \tau_{ab}^c = 2\Omega_{[ab]}^c. \quad (II.1)$$

Here $\tau_{ab}^c$ are the anholonomy coefficients. The Riemann curvature tensor is defined as follows:

$$R_{\mu \rho \sigma}^\lambda = \partial_\sigma R_{\mu \rho}^\lambda - \partial_\rho R_{\mu \sigma}^\lambda - \Gamma_{\rho \sigma}^\lambda R_{\mu \lambda}^\rho + \Gamma_{\rho \lambda}^\sigma R_{\mu \sigma}^\lambda. \quad (II.2)$$

For generic affine connections the Riemann tensor is only antisymmetric in the first two indices. In a metric-affine spacetime the affine connection admits a general decomposition of the form:

$$\Gamma_{\mu \nu}^\sigma = \{\sigma_{\mu \nu}\} + K_{\mu \nu}^\sigma + L_{\mu \nu}^\sigma, \quad (III.3)$$

where the Christoffel symbols, contorsion tensor and the $L_{\mu \nu}^\sigma$ tensor are given respectively by:

$$\{\sigma_{\mu \nu}\} = \frac{1}{2} g^{\sigma \tau} (\partial_\mu g_{\nu \tau} - \partial_\nu g_{\mu \tau} + \partial_\tau g_{\mu \nu}), \quad (IV.4)$$

$$K_{\mu \nu}^\sigma = \frac{1}{2} g^{\sigma \tau} (T_{\mu \tau}^\rho g_{\rho \nu} - T_{\nu \tau}^\rho g_{\rho \mu} - T_{\mu \nu}^\rho g_{\rho \tau}), \quad (II.5)$$

$$L_{\mu \nu}^\sigma = -\frac{1}{2} g^{\sigma \tau} (\nabla_\mu g_{\nu \tau} - \nabla_\nu g_{\mu \tau} + \nabla_\tau g_{\mu \nu}). \quad (II.6)$$

Here, the torsion is $T_{\mu \nu}^\sigma = 2\Gamma_{[\mu \nu]}^\sigma$ and the covariant derivative is defined as $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma_{\mu \nu}^\tau V_\tau$. The combination $Q_{\mu \rho \nu} = \nabla_\mu g_{\rho \nu}$ is known as the non-metricity tensor. When we plug the above decomposition in (II.2) we obtain:

$$R_{\mu \rho \sigma}^\lambda (\Gamma) = R_{\mu \rho \sigma}^\lambda (\{\}) + \{\partial_\mu^\lambda \chi_{C_{\sigma \rho}} \} + 2 \{\nabla_\mu C_{\nu \rho}^\sigma - 2C_{\mu [\rho]}^\sigma C_{\nu \lambda}^\rho \}, \quad (III.7)$$

where $C_{\mu \sigma}^\rho = K_{\mu \sigma}^\rho + L_{\mu \sigma}^\rho$ and $\nabla_\mu$ is referred to the Levi-Civita connection. The Ricci tensor and Ricci scalar are defined as

$$R_{\mu \rho} = R_{\mu \rho \nu}^\nu, \quad R = g^{\mu \nu} R_{\mu \nu}. \quad (III.8)$$

The decomposition (III.7) on the Ricci scalar yields:

$$R(\Gamma) = R(\{\}) + \{\nabla_\mu C_{\nu \rho}^\sigma - \nabla_\nu C_{\mu \rho}^\sigma \} + C_{\mu \sigma \rho}^\lambda C_{\nu \lambda}^\rho - C_{\sigma \lambda \rho}^\mu C_{\mu \lambda}^\rho \gamma. \quad (III.9)$$

The Levi-Civita connection is torsionless and metric-compatible, which means that $T_{\mu \nu}^\sigma = 0$ and the metric can pass through the covariant derivative.

The relation between the components of the affine connection written in a holonomic basis and an anholonomic one is of the form:

$$\Gamma_{\mu \nu}^\sigma = W_{\mu \nu}^\rho + e_{\nu \rho}^\sigma e^\rho_\sigma, \quad (III.10)$$

For instance, if $\Gamma_{\mu \nu}^\rho$ is the Levi-Civita connection then $\omega_{\mu a}^b$ is the usual spin connection (III.12). For simplicity, we will refer to $\omega_{\mu a}^b$ as a gauge connection. By inserting (III.1) in (III.10) and using $W_{\mu \nu}^\rho = -e_{\rho \sigma}^\mu e_{\sigma}^\nu e^\sigma_\rho \Omega_{ab}^c$ the gauge connection gets the following form:

$$w_{ab}^c = a_1 \tau_{ab}^c - b_1 \tau_{ab}^c - b_2 \tau_{ba}^c + c_1 \eta_{ab} \tau_d^2 + d_1 \partial_a \tau_d^c + d_2 \partial_b \tau_d^c. \quad (III.11)$$
The above gauge connection transforms as a scalar under diffeomorphisms. However, for generic coefficients, it does not transform like a gauge connection under local gauge transformations, but only global ones. In fact, if we force this connection to transform under local Lorentz transformations we see that the only possibility is with $a_1 = b_1 = b_2 = -1/2$, $c_1 = d_1 = d_2 = 0$. This is precisely the coefficients for the Levi-Civita spin connection:

$$
\left\{ w_{ab}^c = \frac{1}{2} \tau_{ab}^c + \frac{1}{2} \tau^{ab}_d w_{dc}^c. \right. \quad (\text{II.12})
$$

In planar indices, the Riemann tensor \((\text{II.2})\) takes the form:

$$
R_{abc}^d = D_a w_{bc}^d - D_b w_{ac}^d - w_{ac}^e w_{be}^d + w_{bc}^e w_{ae}^d - \tau_{ab}^c w_{ec}^d. \quad (\text{II.13})
$$

Similarly as before, the Ricci tensor and the Ricci scalar are of the form $R_{ab} = R_{abc}^c$ and $R = \eta^{ab} R_{ab}$ respectively.

The idea is to use the connection \((\text{II.11})\) (or equivalently \((\text{III.1})\)) and find conditions on the coefficients $(a_1, b_1, b_2, c_1, d_1, d_2)$ such that the Ricci scalar for this connection vanishes. In this way, we obtain an equality between the usual Einstein-Hilbert action and an action with $C$-terms (up to a boundary term). Indeed, consider the Einstein-Hilbert action with the scalar curvature \((\text{II.9})\):

$$
\int dx^4 \sqrt{-g} R(\Gamma) = \int dx^4 \sqrt{-g} \left( R(\{\}) + C_{\mu\lambda}^\sigma C_{\sigma}^{\mu\lambda} - C_{\sigma}^\lambda C_{\mu}^{\mu\lambda} \right). \quad (\text{II.14})
$$

We have dropped out the covariant derivative terms since they form a total derivative. By setting $R(\Gamma) = 0$ in \((\text{II.14})\) we simply obtain:

$$
\int dx^4 \sqrt{-g} R(\{\}) = - \int dx^4 \sqrt{-g} \left( C_{\mu\lambda}^\sigma C_{\sigma}^{\mu\lambda} - C_{\sigma}^\lambda C_{\mu}^{\mu\lambda} \right). \quad (\text{II.15})
$$

We stress again that the above equality should be understood up to a boundary term. In the second line of \((\text{II.15})\), the tensor $C_{\mu\rho}^\sigma$ depends on the general connection $\Gamma$. This means that connections that furnish a null Ricci scalar can be used to yield an action equivalent to the Einstein-Hilbert action by using its torsion and/or non-metricity. This is indeed the case for the usual TG, where the right hand side of \((\text{II.15})\) reproduces exactly the teleparallel action with $C_{\mu\rho}^\sigma(W) = K_{\mu\rho}^\sigma(W)$ (see section \((\text{III.4})\)).

The reader might be worried about the fact that we are considering global Lorentz transformations. But all of the connections we will obtain reproduce theories which possess local gauge transformations as established by the action \((\text{II.13})\), reducing in this way the number of degrees of freedom of the vielbein. This is the same mechanism as in teleparallel gravity, where the affine connection is chosen to be the Weitzenböck connection $W_{\mu\nu}^\rho$ which is invariant only under global Lorentz transformations but the resulting theory turns out to have local Lorentz symmetry. For simplicity we will mostly work with flat indices. The usual Einstein-Hilbert action with Levi-Civita connection after a partial integration is of the form:

$$
\int dx^4 \sqrt{-g} R(\{\}) = \int dx^4 \sqrt{-g} \left( - \tau_{ac}^c \tau_{ab}^b + \frac{1}{2} \tau_{\rho a}^b \tau_{\rho b}^a + \frac{1}{4} \tau_{a b}^c \tau_{a b}^c \right), \quad (\text{II.16})
$$

where we have used \((\text{II.13})\) and \((\text{II.12})\). Equation \((\text{II.15})\) then gets the form:

$$
\int dx^4 \sqrt{-g} \left( - \tau_{ac}^c \tau_{ab}^b + \frac{1}{2} \tau_{\rho a}^b \tau_{\rho b}^a + \frac{1}{4} \tau_{a b}^c \tau_{a b}^c \right) = - \int dx^4 \sqrt{-g} \left( C_{ab}^c C_{c}^{ab} - C_{cb}^c C_{a}^{ab} \right). \quad (\text{II.17})
$$

III. EQUATIONS FOR COEFFICIENTS

As mentioned before we would like to find conditions on the coefficients of the general connection \((\text{II.11})\) such that the Ricci scalar of this connection vanishes. We plug \((\text{II.11})\) in the Riemann tensor \((\text{II.13})\) and contract indices to obtain the Ricci scalar. The result is the following:

$$
R(w) = D_a \tau_{b}^{ab} \left( a_1 + 2 b_2 + 3 c_1 + 3 d_1 + b_1 \right) + \tau_{abc} \tau_{abc} \left( a_1 b_1 - b_1 b_1 - a_1 b_2 - b_2 \right) + \tau_{abc} \left( - a_1 b_1 + a_1 b_2 - a_1 b_1 - b_1 b_1 \right) + \tau_{a b}^c \tau_{a b}^c \left( - a_1 b_1 - b_1 b_2 + 2 b_1 c_1 - 4 b_1 d_1 - a_1 b_2 - b_2 b_2 + 2 b_2 c_1 - 2 b_2 d_1 - a_1 c_1 + 3 c_1 c_1 - 12 c_1 d_1 + 2 a_1 d_1 + 3 d_1 d_1 + c_1 + d_1 \right). \quad (\text{III.1})
$$

All of the terms are independent of each other so in order to obtain a null Ricci scalar we must set each parentheses
to zero. This defines a set of four quadratic equations for five variables \((a_1, b_1, b_2, c_1, d_1)\). Note that the coefficient \(d_2\) has dropped out from (III.1) but does not drop out from the Riemann tensor nor the Ricci tensor. The reason is the following. It is known that a Riemann tensor with a generic connection is invariant under:

\[
\Gamma_{\mu\nu}^{\rho} \rightarrow \Gamma_{\mu\nu}^{\rho} + \delta_{\nu}^{\rho} \partial_{\mu} \phi,
\]

where \(\phi\) is a scalar. However, the Ricci scalar built from this generic Riemann tensor is invariant under a relaxed version of the above equation:

\[
\Gamma_{\mu\nu}^{\rho} \rightarrow \Gamma_{\mu\nu}^{\rho} + A_{\mu} \delta_{\nu}^{\rho}.
\]

This is due to a contraction between the metric and \(\partial_{\mu} A_{\nu}\). We can see that the term with coefficient \(d_2\) in (III.1) has exactly the same form as in (III.3). We will come back to this point later.

We might have set to zero the Riemann tensor or the Ricci tensor in order to obtain equations for the coefficients. Nevertheless, we analyze the vanishing of the Ricci scalar because the solutions for this one includes the cases for a null Riemann and Ricci-flat tensors. Therefore, teleparallel gravity must show up in general, a non-vanishing curvature. To simplify the analysis we consider particular cases in the following subsections.

### A. Metric-Compatible case

In this subsection we restrict to the case of metric-compatible connections, i.e.,

\[
D_{a\ell} = w_{a\ell} = 0 \quad \Rightarrow \quad w_{a(bc)} = 0. \quad \text{(III.4)}
\]

The above condition implies

\[
a_1 = b_1, \quad c_1 = d_1, \quad d_2 = 0, \quad \text{(III.5)}
\]

The set of four equations derived from (III.1) together with conditions (III.5) yield only four solutions which will be referred to as cases 1 to 4. These are respectively:

\[
a_1 = -1, \quad b_2 = -1, \quad c_1 = 2/3, \quad \text{ (III.6)}
\]

\[
a_1 = -2/3, \quad b_2 = -4/3, \quad c_1 = 2/3, \quad \text{ (III.7)}
\]

\[
a_1 = -1/3, \quad b_2 = 1/3, \quad c_1 = 0, \quad \text{ (III.8)}
\]

\[
a_1 = 0, \quad b_2 = 0, \quad c_1 = 0, \quad \text{ (III.9)}
\]

Replacing cases 1 to 4 in the general connection (I.1) we obtain the following connections, respectively:

\[
\begin{align*}
(1) & \quad \Gamma_{\mu\nu}^{\rho} = W_{\mu\nu}^{\rho} + (W) T_{\nu\mu}^{\rho} - T_{\rho\mu}^{\rho} - T_{\rho\nu}^{\rho} - 2 \frac{2}{3} g_{\mu\nu} T_{\rho\sigma}^{\rho\sigma} - 2 \frac{1}{3} g_{\mu\nu} T_{\rho\sigma}^{\rho\sigma}, \\
(2) & \quad \Gamma_{\mu\nu}^{\rho} = W_{\mu\nu}^{\rho} + 2 \frac{2}{3} T_{\nu\mu}^{\rho} - 2 \frac{1}{3} T_{\rho\mu}^{\rho} - 4 \frac{1}{3} T_{\rho\nu}^{\rho}, \\
(3) & \quad \Gamma_{\mu\nu}^{\rho} = W_{\mu\nu}^{\rho} + 1 \frac{1}{3} T_{\nu\mu}^{\rho} - 1 \frac{1}{3} T_{\rho\mu}^{\rho} + 1 \frac{1}{3} T_{\rho\nu}^{\rho}, \\
(4) & \quad \Gamma_{\mu\nu}^{\rho} = W_{\mu\nu}^{\rho}.
\end{align*}
\]

Equivalently, we use (I.11) with the above coefficients. As expected, one of the solutions (eq. (III.13)) is the Weitzenböck connection.

The above connections (III.10)-(III.13) are metric compatible (the non-metricity tensor vanishes) and present non-vanishing torsion. Thus, only the contorsion tensor \(K_{abc}\) will contribute to (II.17). The contorsion tensor for the metric-compatible case is equal to:

\[
K_{abc} = C_{abc} = w_{abc}^{(i)} - w_{(i)} abc.
\]

It is straightforward to check that (III.17) is satisfied for the four cases when plugging (III.14) in (III.17). That is:

\[
\begin{align*}
(1) & \quad K_{abc}^{(i)} = \frac{1}{2} P_{abc}^{(i)} - K_{abc}^{(i)} a b c = - \frac{1}{2} P_{abc}^{(i)} - \frac{1}{4} P_{abc}^{(i)} + \tau_{a b}^{(i)} c, \\
(2) & \quad K_{abc}^{(i)} = \frac{1}{2} P_{abc}^{(i)} - K_{abc}^{(i)} a b c = - \frac{1}{2} P_{abc}^{(i)} + \tau_{a b}^{(i)} c, \\
(3) & \quad K_{abc}^{(i)} = \frac{1}{2} P_{abc}^{(i)} - K_{abc}^{(i)} a b c = - \frac{1}{2} P_{abc}^{(i)} + \tau_{a b}^{(i)} c, \\
(4) & \quad K_{abc}^{(i)} = \frac{1}{2} P_{abc}^{(i)} - K_{abc}^{(i)} a b c = - \frac{1}{2} P_{abc}^{(i)} + \tau_{a b}^{(i)} c.
\end{align*}
\]

with \(i = 1, \ldots, 4\) labeling cases 1 to 4. We want to remark that the above connections not only yield non-vanishing torsion but also non-vanishing Riemann curvature and Ricci tensors. The only exception being case 4, i.e. the Weitzenböck case \(\Gamma_{\mu\nu}^{\rho} = W_{\mu\nu}^{\rho}\) (only has non-vanishing torsion).

### B. Non-metricity case

We begin studying the simplest case of torsionless non-metric connection. The condition is:

\[
T_{\mu\nu}^{\sigma} = 0 \quad \Rightarrow \quad \tau_{ab}^{c} = w_{ab}^{c} - w_{ba}^{c},
\]

\[
\text{(III.16)}
\]
which implies that the coefficients must satisfy
\begin{align}
1 + 2a_1 &= 0 \\
b_1 - b_2 &= 0 \\
d_2 - d_1 &= 0.
\end{align}

However, equations (III.17) together with the set of four quadratic equations derived from (III.1) yield no solution. There have been other attempts for constructing equivalent theories to EG based purely on non-metricity. For instance in [20] a vanishing $\Gamma^{\mu\nu\rho}$ was considered, allowing the theory to be described only in terms of the non-metricity tensor $Q_{\mu
u\rho} = \partial_{\mu}g_{\nu\rho}$. This is equivalent as choosing $w_{\alpha\beta} = e_{\alpha}^{\mu}e_{\beta}^{\nu}e_{\gamma}^{\rho}W_{\mu\nu\rho}$. We believe that if we had allowed an extra arbitrary constant factor in front of $W_{\mu\nu\rho}$ in (I.1), then our procedure would have also yielded this gauge connection as a particular result. However, the transformation properties of this connection under diffeomorphisms would have been compromised.

As said before there are infinite solutions to our set of four equations presenting, in a generic way, non-metricity and torsion. Just to analyze a particular case we study the condition for a Weyl’s space:
\begin{equation}
\nabla_{\mu}g_{\nu\rho} = -2A_{\mu}g_{\nu\rho} \Rightarrow w_{\alpha(bc)} = A_{\alpha}\eta_{bc}.
\end{equation}

Where $A_{\mu}$ is a one-form field (Weyl’s vector field). Since we want a connection only in terms of derivatives of the vielbein the only possibility is $A_{\mu} = \alpha\tau^{\mu}_{\alpha}$ for some real constant $\alpha$. This implies the conditions
\begin{align}
a_1 &= b_1, \quad c_1 = d_1, \quad \alpha = d_2.
\end{align}

These conditions on the coefficients are the same as the previous case (III.15) except we have a non-trivial condition on $d_2$. Since $d_2$ does not show up in (III.1), the solutions to the quadratic equations plus (III.19) are the same as the previous case but with an extra term, i.e.:
\begin{align}
^{(i)}\Gamma^{\mu\nu\rho} &= (W)_{\mu\nu\rho} \Gamma^{(W)} - \alpha T^{\mu\nu\rho}_{\sigma\delta\eta} ,
\end{align}

where the dots represent quadratic terms. From the above derivative terms we see that the only possibility for the curvature tensor to vanish is that all of the coefficients must be set equal to zero, implying that the Weitzenböck connection is the only connection written in terms of derivatives of the vielbein that makes the curvature tensor to vanish. So, if the connections we have obtained so far were gauge-transformed (under local Lorentz transformations) of the Weitzenböck connection, it would imply a vanishing curvature tensor for these connections. As we have stated in the previous examples, this is not the case. More generally, different set of coefficients live in different Lorentz gauge orbits.

On the other hand, we have mentioned about transformations that leave the Riemann tensor or Ricci scalar invariant, (III.12) and (III.13) respectively. From a geometric point of view, the connections are meant to be defined up to a "gauge transformation" of the form (III.12) as long as the new connection enjoys the same index symmetry properties as the old one. For instance, the Levi-Civita connection is symmetric in its two lower indices and if we apply the transformation (III.12) the symmetry in the two indices will be broken resulting in a new connection with torsion and non-metricity. The connections obtained in this work do not have a particular definite symmetry on their indices, and thus, transformations of the form (III.12) might be considered. However, none of our connections are related in this sense, since there is no combination of vielbein that produces a non-trivial scalar as in the last term of (III.12). The only transformation relevant for us is (III.13). In this case there are connections that are related as we have explicitly shown in (III.20). Despite of this, the geometric properties de-

IV. GAUGE REDUNDANCY AND DEFORMED WEITZENBÖCK CONNECTIONS

One might be worry about the fact that the different connections we are obtaining are, in some sense, gauge artifacts of the Weitzenböck connection. First of all, we would like to adopt the point of view that a gauge transformation in this context should be understood as a local transformation performed on the vielbein (since the vielbein is our physical field). We want to stress again that only when taking global Lorentz transformations, the connection $\Gamma^{\mu\nu\rho}$ and tensors built out of it are well defined, in the sense that they transform as a scalar under Lorentz transformations. And only when considering actions or equations of motion the local Lorentz symmetry is restored back. This is completely analogous to the usual teleparallel gravity case. Thus, if we consider global Lorentz transformations it is clearly not possible to gauge-transform the different connections into the Weitzenböck connection. Moreover, if we allowed for local Lorentz transformations it is not possible neither. Take for instance the curvature tensor (II.13) with the general connection (III.1):
defined by $\tilde{\Gamma}$ are different from $\Gamma$: Clearly, $\tilde{\Gamma}$ yields non-trivial non-metricity while $\Gamma$ does not. Also, the form of the Riemann tensor for these connections are different since the $d_2$ term does not decouple from the Riemann tensor (see (V.11)).

Having clarified the gauge redundancy issue, the interpretation we are giving to the new connections is that they are deformed versions of the Weitzenböck connection \[21\]. Given two connections $\Gamma_1$ and $\Gamma_2$ we can always relate them by a tensor. Take $\Gamma_1 = W + w_1$ and $\Gamma_2 = W + w_2$ as in the decomposition (1.10). Since $w$ is a tensor under diffeomorphisms, we obtain $\Gamma_1 = \Gamma_2 + (w_1 - w_2)$. Thus, $\Gamma_1$ and $\Gamma_2$ are related by a tensor and we say $\Gamma_1$ is a deformed version of $\Gamma_2$. In this work, what we did is to find those $w$ parametrized by (1.11) such that $\Gamma_1 = W + w$ is a deformation of $\Gamma_2 = W$ and reproduce the same action (or equations of motion) for the vielbein.

V. EQUATIONS OF MOTION

In this section we will obtain the equations of motion for the vielbein in terms of quantities associated to the connections we have obtained in the previous sections. As a warm-up, we first obtain the equations of motion in the teleparallel case (i.e. $w^{ab} = 0$) as it is usually done in the literature. However, we will apply a more general procedure in (V.B) that will include the Weitzenböck case as a particular case.

A. Weitzenböck case

We are interested in the right hand side of (1.15) using planar indices and the Weitzenböck connection $w^{ab} = 0$. For metric-compatible connections the non-metricity tensor vanishes, thus the only contribution to the action comes from the contorsion tensor, i.e. $C^{ab} = K^{ab}$. Expanding in terms of the torsion we get:

\[
\frac{1}{2} (T^{ab} - K^{ab}) = -\frac{1}{2} T^{abc} T_{abc} - \frac{1}{2} T^{abc} T_{abc} = 0.
\]

Where we recall that $T^{abc} = -\tau^{abc}$. Since the Riemann tensor associated to the Weitzenböck connection vanishes to find equations of motion in terms of only $T^{abc}$. The right hand side of (1.15) thus gives:

\[
\int dx^4 e \left( -\frac{1}{4} T^{abc} T_{abc} - \frac{1}{2} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{abc} \right) = \int dx^4 e \left( -\frac{1}{2} T^{abc} T_{abc} \right).
\]

Here $e = \sqrt{-g}$ and in the last line we have introduced the tensor $\hat{T}^{abc}$ known as the superpotential

\[
\hat{T}^{abc} = K_{cba} + 2 T_{abc} + \frac{1}{2} T^{cde} T_{cde} = 0.
\]

The superpotential satisfies $\hat{T}^{abc} = 2 \hat{T}^{[ab}{}_c]$. Due to this property the variation of the lagrangian is easier to perform:

\[
\delta e \left( -\frac{1}{2} e \hat{T}^{abc} T_{abc} \right) = \frac{1}{2} e e_{d\mu} e^d e^{\mu} \hat{T}^{abc} T_{abc} - \frac{1}{2} e \hat{T}^{abc} \delta e T_{abc}.
\]

where

\[
\delta e \hat{T}^{abc} T_{abc} = e \hat{T}^{abc} \left( 2 \delta_k e_{\mu} e_{\mu} \hat{\Omega}^{cb} + \tau_{ab} e_{\mu} e_{\mu} \delta_k e_{\mu} \right) + 2 \delta e_{\mu} e_{\mu} (V.5)
\]

Since the last term of (V.3) is antisymmetrized on indices $(a, b)$ we can use:

\[
2 D_{[a} A_{b]} = 2 \hat{D}_{[a} A_{b]} + 2 (V.6)
\]

where $D_{[a} A_{b]} = D_{a} A_{b} - (V.6)$. When substituting (V.6) in (V.3) we see that the second term in the right-hand side of (V.6) cancels the $\tau$ term in the first line of (V.5) since $2 w^{[ab} = \tau^{[ab}$. We proceed to use the product rule:

\[
\delta e \hat{T}^{abc} (V.1) = 2 \hat{D}_{[a} \delta e_{\mu} e_{\mu} (V.7)
\]

The first term on the right-hand side of (V.7) yields a total derivative and the last line of (V.7) comes from expanding $\hat{D}_{[a} e_{\mu}$. Putting all together we have:

\[
\delta e \left( -\frac{1}{2} e \hat{T}^{abc} T_{abc} \right) = \frac{1}{2} e e_{d\mu} e^d e^{\mu} \hat{T}^{abc} T_{abc} + 2 e e_{a\mu} e_{\mu} (V.8)
\]

up to a total derivative. The $\Omega$ term of (V.7) combines with the $\Omega$ term of (V.5) to form the $\tau$ term in the second line of (V.8). The equations of motion are simply:

\[
(1.15)
\]

\[
2 \hat{D}_c \hat{T}^{[cd} + 2 \hat{T}^{[cd} (V.9)
\]
Here we have used \( T^{(4)}_{ab} = -\tau_{ab} \) and \( R^{(4)}_{ab} = \varepsilon^{(4)}_{ab} \).

**B. Generic case**

In the previous subsection we obtained the equations of motion by varying the right hand side of (II.15). This is our goal here too. However, the variation procedure of the preceding subsection depends heavily on the form of the torsion with respect to the vielbein and if we perform that procedure for the new connections it is not easy to see how to rearrange terms in meaningful quantities.

In the Weitzenböck case, the equations of motion are fully described in terms of the torsion since there is no curvature. In the case of the new connections, all of them have non-zero curvature and we expect the Ricci tensor to appear in the equations of motion. Therefore, we will follow here a different route to derive the equations of motion for our generic connections, and in particular, it will yield the equations of motion for the Weitzenböck case (V.9).

We start with the following action,

\[
S = \int eR(\Gamma) dx^4,
\]

(\text{V.10})

where \( R(\Gamma) = g^{\mu\nu} R_{\mu\nu} + \rho(\Gamma) \). We are assuming that \( \Gamma \) is written in terms of first-order derivatives of the vielbein. We now vary the action with respect to the vielbein,

\[
\delta S = \int \delta eR(\Gamma) dx^4
+ \int e \delta g^{\mu\nu} R_{\mu\nu} + \rho(\Gamma) + \int e \delta g^{\mu\nu} \delta R_{\mu\nu} + \rho(\Gamma).
\]

(V.11)

The first term of (\text{V.11}) is as usual,

\[
- \int e \epsilon_{\alpha\beta} R(\Gamma) \delta e^{\alpha\beta}.
\]

(V.12)

The second term of (\text{V.11}) gives:

\[
2 \int e R_{\mu\nu} \rho(\Gamma) e^{\alpha\beta} \epsilon_{\alpha\beta}.
\]

(V.13)

The third term of (\text{V.11}) can be decomposed as follows. We use the decomposition for \( \Gamma \) as given by (\text{II.3}) implying a decomposition for the Riemann tensor (\text{II.7}). Thus, we obtain:

\[
\int e g^{\mu\nu} \delta (R_{\mu\nu} + \rho(\Gamma)) + \int e g^{\mu\nu} \delta R_{\mu\nu} + \rho(\Gamma).
\]

(V.14)

The variation of the first term of (\text{V.14}) is obtained by varying (\text{II.2}) with respect to \( \Gamma \) with \( \Gamma \) being the Levi-Civita connection. The result is:

\[
\int e g^{\mu\nu} \nabla^\rho [\delta \Gamma]_{\alpha\beta} = 0.
\]

(V.15)

Which yields zero, since it is a total derivative. The second and third term of (\text{V.14}) can be rewritten as follows:

\[
\int e g^{\mu\nu} \delta (2 \nabla [\rho C_{\rho\nu} + 2 C_{[\rho\nu]}]) =
\]

\[
\delta \left( \int e g^{\mu\nu} (2 \nabla [\rho C_{\rho\nu} + 2 C_{[\rho\nu]}]) - \int e \delta g^{\mu\nu} (2 \nabla [\rho C_{\rho\nu}]) \right)
- \int e \delta g^{\mu\nu} (2 \nabla [\rho C_{\rho\nu}]),
\]

(V.16)

and

\[
\int e g^{\mu\nu} \delta (2 C_{[\rho\nu]} + 2 C_{\rho\nu}) =
\]

\[
\delta \left( \int e g^{\mu\nu} (2 C_{[\rho\nu]} + 2 C_{\rho\nu}) - \int e \delta g^{\mu\nu} (2 C_{[\rho\nu]} + 2 C_{\rho\nu}) \right)
- \int e \delta g^{\mu\nu} (2 C_{[\rho\nu]} + 2 C_{\rho\nu}).
\]

(V.17)

Putting all together

\[
\delta S = \int e \left( g_{\mu\nu} \left( -R(\Gamma) + 2 \nabla [\rho C_{\rho\nu} + 2 C_{[\rho\nu]}] \right) + 2 R_{[\rho\nu]} + 2 \nabla [\rho C_{\rho\nu}] + 2 C_{[\rho\nu]} + 2 C_{\rho\nu} \right) e^{\alpha\beta} \delta e_{\alpha\beta} +
\]

\[
\int e \left( 2 C_{[\rho\nu]} + 2 C_{\rho\nu} \right).
\]

(V.18)

Since we are dealing with \( \Gamma \)'s such that the Ricci scalar vanishes then the action (\text{V.10}) vanishes. This implies that \( \delta S = 0 \) off-shell. Therefore, the the right hand side of (\text{V.18}) must vanish identically. On the other hand, the last term of (\text{V.18}) is equal to the variation of the right hand side of (\text{II.15}) and we need to impose this variation to vanish in order to obtain the equations of motion. So, the resulting (equivalent) equations of motion are:

\[
2 R_{[\rho\nu]} + 2 \nabla [\rho C_{\rho\nu}] + 2 C_{[\rho\nu]} + 2 C_{\rho\nu} = 0,
\]

(V.19)

where we have used \( R(\Gamma) = 0 \). Equivalently, in flat indices the equations of motion are:

\[
2 R_{[a|c]|b} + \eta_{ab} [2 \nabla [C_{a|c]} + 2 C_{[a|c]} + 2 C_{[a|c]}] = 0,
\]

(V.19)

where we have used \( R(\Gamma) = 0 \). Equivalently, in flat indices the equations of motion are:

\[
2 R_{[a|c]|b} + \eta_{ab} [2 \nabla [C_{a|c]} + 2 C_{[a|c]} + 2 C_{[a|c]}] = 0.
\]

(V.19)
Equations (V.19) and (V.20) are the equations of motion for the general case (I.1) and (II.11) respectively. In the case of metric-compatible connections only the contorsion tensor contributes to $C_{ab}^d$. In this case, and after some manipulations, equation (V.20) gets the form:

$$2R_{(a|b)}^c(w) + (1)\hat{T}_{ab}^c + 2\hat{T}_{cd}^e T_{ab}^e - K_{cd}^e$$

$$- \frac{1}{2} \hat{T}_{cd}^e \hat{T}_{ab}^c + 2D_{[a} T_{b|c]}^e + T_{[b|d]c} T_{a]d}^c + T_{[b|d]c} T_{a]d}^c = 0,$$  \hspace{1cm} (V.21)

where $\hat{T}_{ab}^c$ was defined in (V.3). It is easy to see that (V.21) reduces to (V.9) for the Weitzenböck case, $w_{ab}^c = 0$, $R_{abc}^d(w) = 0$, $T_{ab}^c = -\tau_{ab}^c$, with the help of the following identity:

$$(1)\hat{D}_{a}^c \tau_{ab}^c + 2(1)\hat{D}_{[a} T_{b|c]}^c - (1) \hat{T}_{cd}^e \hat{T}_{ab}^e + (1) \hat{T}_{cd}^e \hat{T}_{ab}^e + \tau_{ab}^c \tau_{cd}^d = 0,$$  \hspace{1cm} (V.22)

As a final remark, we have verified that when plugging the connections (III.10)-(III.13) inside equations (V.21) reproduce the same equations of motion for the viebien as in teleparallel gravity or Einstein gravity.

VI. SUMMARY

We have analyzed conditions on a generic connection (I.1) in order to reproduce equivalent formulations to Einstein Gravity. We have found in section (III A) that there are only four metric-compatible connections that satisfy the conditions, one of them being the Weitzenböck connection. In this sense we are obtaining Teleparallel Gravity as a particular case. For a generic case that includes both torsion and non-metricity, there seems to be an infinite amount of solutions. In section (III B) we analyzed a particular case of a Weyl space with Weyl’s vector being proportional to $T^\mu_{\sigma} (W)$ (the vector part of the Weitzenböck torsion). The connections (III.20) have torsion and non-metricity and are related to the metric-compatible ones (III.10)-(III.13) through the transformation (III.3). Our generic connections (except for the Weitzenböck one) present non-vanishing curvature, torsion and non-metricity so the dynamics of the theories, defined by these connections, is represented by a mixture of those quantities. This can be seen in the equations of motion (V.19) or (V.20) where the Ricci tensor, torsion and non-metricity enters in a non-trivial way.

In our approach, we have the local Lorentz group broken (except for actions and equations of motion) and the interpretation we are giving is that the new connections are deformed versions of the Weitzenböck connection defined over parallelizable spacetimes. Of course, in this work we just analyzed the equivalence with Einstein Gravity in vacuum (no source terms coupled). So, the theories defined here, using different connections, are candidates for a complete equivalence to Einstein Gravity when sources are added. In the case of the Weitzenböck connection, there is a complete equivalence with Einstein Gravity when sources are included (see [22]). We leave for future work a complete discussion of matter coupling, interpretation of the gravitational force acting on particles and geodesics.

As a final remark, we believe our procedure here could be applied to DFT in order to find other connections determined in terms of the physical fields. However, notions of contorsion and non-metricity should be defined first in order to find a suitable decomposition as in (III.3). We also leave this issue for future work.

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