EXCEPTIONAL COSMETIC SURGERIES ON $S^3$

HUYGENS C. RAVELOMANANA

Abstract. This paper concerns the truly or purely cosmetic surgery conjecture. We give a survey on exceptional surgeries and cosmetic surgeries. We prove that the slope of an exceptional truly cosmetic surgery on a hyperbolic knot in $S^3$ must be $\pm 1$ and the surgery must be toroidal but not Seifert fibred. As consequence we show that there are no exceptional truly cosmetic surgeries on certain types of hyperbolic knot in $S^3$. We also give some properties of Heegaard Floer correction terms and torsion invariants for exceptional cosmetic surgeries on $S^3$.

1. Introduction

Dehn surgery is an essential tool in 3-manifold topology. Cosmetic surgery addresses the question: when do two surgeries along the same knot, but with distinct slopes, produce the same manifold? Such a situation does happen, but observations suggest that for generic knots and 3-manifolds this should be very rare. Let $Y$ be an oriented 3-manifold and $K$ a knot in $Y$. Let $Y_K(\alpha)$ and $Y_K(\beta)$ be the result of two Dehn filling on $K$ with two distinct slopes $\alpha$ and $\beta$. If $Y_K(\alpha) \cong Y_K(\beta)$ as oriented manifold we say that the two surgeries are truly (or purely) cosmetic. The following conjecture was proposed by Gordon in [7, Conjecture 6.1] and is stated as conjecture (A) in problem 1.81 of Kirby list of problems in low-dimensional topology [14].

Conjecture 1.1 (Cosmetic surgery conjecture). Let $M$ be a compact connected oriented irreducible 3-manifold with torus boundary and which is not a solid torus. Let $\alpha$ and $\beta$ be two inequivalent slopes on $\partial M$. If $M(\alpha) \cong M(\beta)$, then the homeomorphism is orientation-reversing. Equivalently, two surgeries on inequivalent slopes are never truly cosmetic.

Here two slopes on $\partial M$ are called equivalent if there exists an orientation-preserving homeomorphism of $M$ which takes one to the other.

For the trivial knot, it is known that there can be infinitely many distinct surgeries which give the same output. Mathieu, in [17], gives an infinite family of distinct Dehn surgeries on a trefoil knot in $S^3$ which give homeomorphic manifolds. In [1] Bleiler, Hodgson and Weeks described an oriented hyperbolic 3-manifold with torus boundary having two distinct Dehn fillings which give two oppositely oriented copies of the lens space $L(49, 18)$. 
In [2] Boyer and Lines proved that \( \Delta''_K(1) \) must vanish in order to have truly cosmetic surgery.

**Proposition 1.2** (Boyer and Lines). Let \( K \) be a knot in an \( \mathbb{Z} \)-homology sphere \( Y \). If \( \Delta''_K(1) \neq 0 \), then there is no orientation preserving homeomorphism between \( Y_K(r) \) and \( Y_K(s) \) if \( r \neq s \).

Recently, with help of Heegaard Floer theory and Casson invariant, new criteria for cosmetic surgeries on knots in \( S^3 \), and more generally knots in \( L \)-space homology spheres, have been established by Yi and Zhongtao Wu in [21].

**Theorem 1.3** (Yi Ni and Zhongtao Wu). Suppose \( K \) is a nontrivial knot in \( S^3 \), \( r, s \in \mathbb{Q} \cup \{ \infty \} \) are two distinct slopes such that \( S_K(r) \) is homeomorphic to \( S_K(s) \) as oriented manifolds. Then \( r, s \) satisfy that

1. \( r = -s \);
2. suppose \( r = p/q \), where \( p \), \( q \) are coprime integers, then: \( q^2 \equiv -1 \pmod{p} \);
3. \( \tau(K) = 0 \), where \( \tau \) is the concordance invariant defined by Ozsváth-Szabó and Rasmussen.

Using Ni and Wu’s result combined with the progress made on exceptional surgeries on \( S^3 \) and the result of Boyer and Lines we provide the following new property of cosmetic surgery on hyperbolic knot in \( S^3 \).

**Theorem 1.4.** Let \( K \) be a hyperbolic knot in \( S^3 \), and \( r, s \in \mathbb{Q} \cup \{ \infty \} \) two distinct exceptional slopes on \( \partial N(K) \). If \( S_K(r) \) is homeomorphic to \( S_K(s) \) as oriented manifolds, then the surgery must be toroidal and non-Seifert fibred, moreover \( \{ r, s \} = \{ +1, -1 \} \).

From this we deduce that there are no exceptional truly cosmetic surgeries on some classical families of knots.

**Corollary 1.5.** There are no truly cosmetic surgeries on non-trivial algebraic knot in \( S^3 \).

**Corollary 1.6.** There are no exceptional truly cosmetic surgeries on an alternating hyperbolic knot in \( S^3 \).

**Corollary 1.7.** There are no exceptional truly cosmetic surgeries on arborescent knots in \( S^3 \).

We have also the following properties of the Heegaard Floer correction term and the Alexander polynomial.
Corollary 5.10. If a hyperbolic knot $K \subset S^3$ admits an exceptional truly cosmetic surgery then the Heegaard Floer correction term of any $1/n$ ($n \in \mathbb{Z}$) surgery on $K$ satisfies
\[ d(S^3_K(1/n)) = 0. \]

Corollary 5.9. If a 3-manifold $Y$ is the result of an exceptional truly cosmetic surgery on a hyperbolic knot $K$ in $S^3$ then:
\[ |t_0(K)| + 2 \sum_{i=1}^{n} |t_i(K)| \leq \text{rank}HF_{\text{red}}(Y), \]
where the number $t_i(K)$ for $i \in \mathbb{Z}$ is the torsion invariant of the Alexander polynomial $\Delta_K(T)$ of $K$ and $n$ is the degree of $\Delta_K(T)$.

Organization. This paper is organized as follow. In section 2 we give some background and survey exceptional surgeries, cosmetic surgeries. In section 3 we survey some result on cosmetic surgeries obtained using Heegaard Floer theory and give the proof of Theorem 1.4. In section 4 we give examples of family of knots in $S^3$ which do not admit exceptional truly cosmetic surgeries. In section 5 we enumerate some miscellaneous properties of Heegaard Floer invariant for exceptional truly cosmetic surgeries on $S^3$.

Acknowledgment. I would like to thank my supervisor Steven Boyer for his support and for suggesting the cosmetic surgery problem. This work was carried out while the author was a graduate student at UQÀM and CIRGET in Montréal.

2. Exceptional cosmetic surgeries

2.1. Topological background.

2.1.1. Distance between slopes. Let $M$ be a compact, connected, oriented 3-manifold and let $T \subset \partial M$ be a torus. The distance, denoted $\Delta(\alpha, \beta)$, between two slopes $\alpha$ and $\beta$ on $T$ is their minimal geometric intersection number. That is
\[ \Delta(\alpha, \beta) = \min \{\sharp C_1 \cap C_2 : C_1, C_2 \text{ simple closed curve representing } \alpha \text{ and } \beta \text{ respectively}\} \]
The distance has the following straightforward properties:
\begin{itemize}
  \item $\Delta(\alpha, \beta) = |\alpha \cdot \beta|.$
  \item $\Delta(\alpha, \beta) = 0$ iff $\alpha = \beta.$
  \item $\Delta(\alpha, \beta) = 1$ iff $\{\alpha, \beta\}$ form a basis of $H_1(\partial M; \mathbb{Z}).$
  \item If we fix a basis $\{\mu, \lambda\}$ of $H_1(T; \mathbb{Z})$, then for $\alpha = p\mu + q\lambda$ and $\beta = p'\mu + q'\lambda$
\end{itemize}
\[ \Delta(\alpha, \beta) = |pq' - qp'|. \]
When \( \partial M \) consist of a single torus there is a formula relating the order of the first homology of the filled manifold to the distance of the filling slope from the rational longitude.

**Lemma 2.1** (Watson [35]). Let \( \alpha \) be a slope on \( \partial M \). There is a constant \( c_M \) such that

\[
|H_1(M(\alpha); \mathbb{Z})| = c_M \Delta(\alpha, \lambda_M).
\]

If we denote \( i : \partial M \to M \) the natural inclusion then the constant \( c_M \) is the quantity

\[
c_M = |\text{Tor}(H_1(M; \mathbb{Z}))| \text{ ord}(i_* \lambda_M),
\]

where \( \text{ord}(i_* \lambda_M) \) is the order of \( i_* \lambda_M \) in the homology of \( M \).

**2.1.2. Surgery on a link.** Assume that \( Y \) is an integer homology sphere. Let \( L = K_1 \cup \cdots \cup K_m \) be a link in \( Y \). Each component of \( L \) has a canonical longitude therefore every surgery on \( L \) can be described by an \( m \)-tuple \((p_1/q_1, \ldots, p_m/q_m)\) of elements in \( \mathbb{Q} \cup \{\infty\} \).

By a *framed link* we mean the data of the link \( L \) with such an \( m \)-tuple. The \( m \)-tuple itself will be called the framing of the link. A framed link will be denoted by calligraphic letter, like \( \mathcal{L} \). We will write \( Y(\mathcal{L}) \) for the result of a Dehn surgery on a framed link \( \mathcal{L} \).

The *framing matrix* of a framed link \( \mathcal{L} \) in \( Y \) is the matrix \( F(\mathcal{L}) \) defined by

\[
F(\mathcal{L})_{ij} = \begin{cases} 
q_j \ 	ext{lk}(K_i, K_j) & \text{if } i \neq j \\
p_i & \text{if } i = j
\end{cases}
\]

where \( \text{lk}(, ,) \) denotes the linking number. The framing matrix gives a presentation for \( H_1(Y(\mathcal{L}); \mathbb{Z}) \), in particular

\[
|\det (F(\mathcal{L}))| = |H_1(Y(\mathcal{L}); \mathbb{Z})|.
\]

For the case of a 2-component link, the framing matrix has the form

\[
F(\mathcal{L}) = \begin{pmatrix}
p_1 & q_2 \ 	ext{lk}(K_1, K_2) \\
q_1 \ 	ext{lk}(K_2, K_1) & p_2
\end{pmatrix}
\]

For more details we refer to [33].

**2.2. Exceptional surgeries.** A compact, connected orientable 3-manifold \( M \) will be called *irreducible* if every properly embedded 2-sphere in \( M \) bounds a 3-ball. Otherwise \( M \) will be called *reducible*. It will be called *boundary irreducible* if every simple closed curve on \( \partial M \) which bounds a disk in \( M \) bounds a disk in \( \partial M \), and otherwise *boundary reducible*. All embedded surfaces in a 3-manifold we will be considering will be bicollared if not stated otherwise. From now on we will use the following definition.
**Definition 2.2.** A properly embedded non-empty surface $F$ in a compact, orientable 3-manifold $M$ is said to be *essential* if it is a 2-sphere which does not bound a 3-ball or if it has the following properties:

1. $F$ has no 2-sphere component,
2. the inclusion morphism $\pi_1(F_i) \to \pi_1(M)$ is injective for every component $F_i$ of $F$,
3. no component of $F$ is parallel into $\partial M$.

Let $F \subset M$ be a properly embedded surface with boundary and $T$ be a torus component of $\partial M$. Each component of $\partial F \cap T$ is a simple closed curve on $T$ and they all determine the same slope. A slope $r$ on $T$ is called *boundary slope* if it is the slope of a boundary component of an essential surface in $M$. If the corresponding surface is a punctured torus then the slope will also be called a *toroidal slope*.

If all the components of $\partial M$ are tori or $\partial M$ is empty, $M$ is said to be *hyperbolic* if its interior admits a complete finite volume Riemannian metric of constant sectional curvature $-1$. If $M$ is hyperbolic then it is irreducible, boundary irreducible, contains no essential tori or annuli and is not a Seifert fibred manifold. Thurston’s hyperbolization theorem implies that the last statement is an equivalence. A hyperbolic structure on $M$ is unique up to isometry by the Mostow-Prasad rigidity theorem.

Fix $M$ a hyperbolic 3-manifold with $\partial M$ a union of tori. In this section we will discuss Dehn filling of $M$. Let $T$ be a component of $\partial M$. By studying metric completions of incomplete “hyperbolic” 3-manifolds, W. Thurston discovered that except for a finite number of slopes all the Dehn fillings of $M$ along $T$ give hyperbolic manifolds.

**Theorem 2.3** (Thurston, [34]). Let $M$ be a compact connected oriented 3-manifold with boundary a union of tori. Let $T$ be a component of $\partial M$. If $\text{int}(M)$ admits a complete finite volume hyperbolic structure, for all but finitely many slopes $\alpha$ on $T$, $M(\alpha)$ is hyperbolic and the core of the Dehn filling is isotopic to the unique shortest geodesic in this manifold.

Let’s consider the set $E(M, T)$ of non-hyperbolic slope on $T$. A slope in $E(M, T)$ is called an *exceptional slope*. By the above theorem it is a finite set, and one goal of Dehn filling theory is to understand this set of slopes. One of the main “techniques” in this study is to find a bound on the distance $\Delta(r,s)$ between two exceptional slopes $r$ and $s$.

**Theorem 2.4** (Lackenby-Meyerhoff, [16]). Let $M$ be a compact orientable 3-manifold with boundary a torus, and with interior admitting a complete finite-volume hyperbolic structure. If $r$ and $s$ are exceptional slopes on $\partial M$, then their intersection number $\Delta(r,s)$ is at most 8.
This bounds is achieved by the figure-8 exterior, indeed
\[ E(\text{figure-8 exterior}) = \{ \infty, 0, \pm 1, \pm 2, \pm 3, \pm 4 \}. \]

It was conjectured by Gordon that the distance of two exceptional slopes is less than 5 for almost all hyperbolic 3-manifold with torus boundary.

**Conjecture 2.5.** Let \( M \) be an hyperbolic 3-manifold with boundary a torus. If \( \alpha \) and \( \beta \) are two exceptional slopes on \( \partial M \), then \( \Delta(\alpha, \beta) \leq 5 \) unless \( M \) is one of \( W(1), W(2), W(-5/2), \) or \( W(-5) \), see figure 1.

![Figure 1. W(1), W(2), W(-5/2), W(-5)](image)

The conjecture is known to be true if the two slopes are both toroidal [6].

For non-toroidal exceptional surgeries there are three principal results.

**Theorem 2.6** (Cyclic surgery theorem, Culler-Gordon-Luecke-Shalen[5]). Let \( M \) be a compact, oriented, irreducible 3-manifold which is not a Seifert fibred space. Assume that \( \partial M \) is a torus and let \( r, s \) be two slopes on \( \partial M \). If \( \pi_1(M(r)) \) and \( \pi_1(M(s)) \) are cyclic, then \( \Delta(r, s) \leq 1 \).

**Theorem 2.7** (Finite surgery theorem, Boyer-Zhang [3]). Let \( M \) be a compact orientable hyperbolic 3-manifold with torus boundary. If \( r, s \) are two slopes on \( \partial M \) such that \( \pi_1(M(r)) \) and \( \pi_1(M(s)) \) are finite, then \( \Delta(r, s) \leq 3 \).

**Theorem 2.8** (Gordon-Luecke, [10]). Let \( M \) be a compact orientable irreducible 3-manifold with torus boundary. If \( r, s \) are two slopes on \( \partial M \) such that \( M(r) \) and \( M(s) \) are both reducible, then \( \Delta(r, s) \leq 1 \).

We summarize all the results about the bounds on \( \Delta(r, s) \) for \( r, s \in E(M) \) in table 2.2.

The list of knots in \( S^3 \) which admit pair of toroidal slopes at distance 4 or more is also known by work of Gordon [6] and Gordon and Ying-Qing Wu in [12].

**Theorem 2.9** (Gordon and Ying-Qing Wu). A knot \( K \) in \( S^3 \) is hyperbolic and admits two toroidal surgeries \( S^3_K(r_1), S^3_K(r_2) \) with \( \Delta(r_1, r_2) \geq 4 \) if and only if \( (K, r_1, r_2) \) is equivalent to one of the following, where \( n \) is an integer.
### Table 1. Distance table.

|                | reducible | cyclic | finite | toroidal | small Seifert |
|----------------|-----------|--------|--------|----------|---------------|
| reducible      | 1         | 1      | 1      | 3        | 4             |
| cyclic         |           | 1      | 2      | 8        | 8             |
| finite         |           |        | 3      | 8        | 8             |
| toroidal       |           |        |        | 8        | 8             |
| small Seifert  |           |        |        |          | 8             |

(1) $K = L_1(n)$, $r_1 = 0$, $r_2 = 4$.

(2) $K = L_2(n)$, $r_1 = 2 - 9n$, $r_2 = -2 - 9n$.

(3) $K = L_3(n)$, $r_1 = -9 - 25n$, $r_2 = -(13/2) - 25n$.

(4) $K$ is the Figure 8 knot, $r_1 = 4$, $r_2 = -4$.

The knots $L_1(n)$, $L_2(n)$ and $L_3(n)$ are the knots obtained from the right components of the links $L_1$, $L_2$, $L_3$ in Figure 2 after $1/n$-surgery on the left components. In the particular case where $\Delta(r_1, r_2) = 4$, then $K = L_1(n)$, $r_1 = 0$, $r_2 = 4$; or $K = L_2(n)$, $r_1 = 2 - 9n$, $r_2 = -2 - 9n$.

![Figure 1](image1.png)  
(1)  
![Figure 2](image2.png)  
(2)  
![Figure 3](image3.png)  
(3)

In term of the slope of the toroidal surgery there is a bound on the denominator $q$ of a toroidal slope $p/q$.

**Theorem 2.10** (Gordon-Luecke, [11]). Let $K$ be a hyperbolic knot in $S^3$ and suppose that $S^3_K(p/q)$ contains an essential torus. Then $|q| \leq 2$.

For the case where the slope is non-integral we have a complete understanding of toroidal surgeries which is given by the following theorem.

**Theorem 2.11** (Gordon and Luecke, [9]). Let $K$ be a hyperbolic knot in $S^3$ that admits a non-integral surgery containing an incompressible torus. Then $K$ is one of the Eudave-Muñoz knots $k(l, m, n, p)$ and the surgery is the corresponding half-integral surgery.
2.3. Cosmetic surgery.

Definition 2.12. Two Dehn fillings $M(\alpha)$ and $M(\beta)$, where $\alpha \neq \beta$, are called cosmetic if there is a homeomorphism $h: M(\alpha) \to M(\beta)$. They are called truly cosmetic if $h$ can be chosen to be orientation-preserving. We also call two Dehn surgeries cosmetic (resp. truly cosmetic) if the corresponding Dehn fillings are cosmetic (resp. truly cosmetic).

Example 2.13. Here are some examples of cosmetic filling for two distinct slopes.

- If $K$ is an amphichiral knot in $S^3$ and $M = S^3 \setminus N(K)$, then $M(\alpha)$ is orientation reversing homeomorphic to $M(-\alpha)$.
- It was shown by Mathieu [17] that if $M$ is the complement of the trefoil knot in $S^3$ then we have an infinite family of pairs of distinct slopes which give homeomorphic manifolds. Precisely, for any positive integer $k$,

$$M\left(\frac{18k + 9}{3k + 1}\right) \cong -M\left(\frac{18k + 9}{3k + 2}\right).$$

These Dehn filling manifolds are Seifert fibred with normalized Seifert invariants $(0; k-3/2; (2,1), (3,1), (3,2))$. Such manifolds do not admit orientation-reversing homeomorphisms. Therefore the fillings are not truly cosmetic.
- If $M$ is the complement of the unknot in $S^3$, $M$ is a solid torus, then the Dehn filling manifolds are lens spaces and

$$M(p/q_1) \cong +M(p/q_2) \text{ iff } q_2 \equiv q_1^{\pm 1} \text{ [mod] } p,$$

for pairs of relatively prime integers $(p, q_1)$ and $(p, q_2)$.

For the first and the third examples one can find a homeomorphism of $M$ which takes one slope to the other.

For the case $b_1(Y) > 0$ and the core of the Dehn filling is homotopically trivial in $Y$ the following result was proved by Lackenby.

Theorem 2.14 (Lackenby, [15]). Let $Y$ be a compact oriented 3-manifold with $H_1(Y, \mathbb{Q}) \neq 0$. Let $K$ be a homotopically trivial knot in $Y$, such that $M = Y \setminus N(K)$ is irreducible and atoroidal. Let $M(p/q)$ be the Dehn filling along $K$ with slope $p/q$. Then there is a natural number $C(Y, K)$ which depends only on $Y$ and $K$ such that, if $|q| > C(Y, K)$ then $M(p/q)$ is orientation-preserving homeomorphic to $M(p'/q')$ iff $p/q = p'/q'$.

The assumption that $K$ is homotopically trivial can be dropped and replaced by $K$ homologically trivial and $Y$ reducible or $K$ having finite order in $\pi_1(Y)$ [15]. Taut sutured manifold theory is used to construct the bound $C(Y, K)$.

Theorem 2.15 (Wu, [39]). Let $r$ and $r'$ be two distinct rational numbers with $rr' > 0$, let $K$ be a non-trivial knot in an $L$-space $\mathbb{Z}$-homology sphere $Y$ and let $M = Y \setminus N(K)$. Then $M(r) \ncong M(r')$.\n
Yi Ni has also studied cosmetic surgeries for manifolds $Y$ with $b_1(Y) > 0$. For this he used the Thurston norm with Heegaard Floer homology.

**Theorem 2.16** (Yi Ni [20]). Suppose $Y$ is a closed 3–manifold with $b_1(Y) > 0$. Let $K$ be a null-homologous knot in $Y$, so that the inclusion map $Y - K \to Y$ induces an isomorphism $H_2(Y - K) \cong H_2(Y)$ and we can identify $H_2(Y)$ with $H_2(Y - K)$. Suppose $r \in \mathbb{Q} \cup \{\infty\}$ and let $Y_K(r)$ be the manifold obtained by $r$–surgery on $K$. Suppose $(Y, K)$ satisfies that

$$x_Y(h) < x_{Y - K}(h),$$

where $x_M$ is the Thurston norm in $M$. If two rational numbers $r, s$ satisfy that $Y_K(r) \cong \pm Y_K(s)$, then $r = \pm s$.

We can replace the assumption on the Thurston norm with another condition to obtain the following.

**Theorem 2.17** (Yi Ni [20]). Suppose $Y$ is a closed 3–manifold with $b_1(Y) > 0$. Suppose $K$ is a null-homologous knot in $Y$. Suppose $x_Y \equiv 0$, while the restriction of $x_{Y - K}$ on $H_2(Y)$ is nonzero. Then we have the same conclusion as Theorem 2.16. Namely, if two rational numbers $r, s$ satisfy that $Y_K(r) \cong \pm Y_K(s)$, then $r = \pm s$.

We will be mainly interested in truly cosmetic surgery along hyperbolic knots $K$ in a rational homology sphere $Y$. By Theorem 2.3, $Y_K(r)$ is hyperbolic for all except a finite number of slopes $r$ on $\partial N(K)$. Let $r$ and $s$ be such hyperbolic slopes. Assume $Y_K(r)$ is homeomorphic to $Y_K(s)$. Then by Mostow rigidity there is an isometry $h$ between $Y_K(r)$ and $Y_K(s)$. This isometry takes the unique shortest geodesic in $Y_K(r)$ to the unique shortest geodesic in $Y_K(s)$. Apart from a finite number of slopes, the shortest geodesic is isotopic to the core of the Dehn filling, and if this is true for the slopes $r$ and $s$ we can assume that $h$ takes the core of the Dehn filling in $Y_K(r)$ to the core of the Dehn filling $Y_K(s)$. Therefore $h$ takes the meridian $r$ to the meridian $s$. In particular $h$ restricts to a homeomorphism of $Y_K$ which takes $r$ to $s$. Moreover a homeomorphism of a one-cusped orientable hyperbolic 3-manifold which changes the slope of some peripheral curve has to be orientation reversing. Therefore the two slopes $r$ and $s$ are not equivalent. One can then deduce the following, see [1].

**Proposition 2.18** (Bleiler-Hodgson-Weeks,[1]). Let $M$ be a compact connected oriented hyperbolic 3-manifold with boundary a torus. Let $r$ and $s$ be distinct slopes on $\partial M$, such that $M(r)$ (resp. $M(s)$) is hyperbolic and the core of the Dehn filling solid torus is isotopic to the shortest geodesic in $M(r)$ (resp. $M(s)$), which we assume is unique. If $M(r)$ is homeomorphic to $M(s)$, then there is an orientation-reversing homeomorphism of $M$ which takes $r$ to $s$ but no orientation preserving one. In particular, apart from a
finite number of slopes, there are no truly cosmetic filling of $M$ with two inequivalent slopes.

For cosmetic filling on a complete finite volume hyperbolic 3-manifold $M$, the remaining cases are then:

- One of the Dehn filling manifolds has a hyperbolic structure but the core of the Dehn filling is not isotopic to the shortest geodesic.
- The Dehn filling manifold is not hyperbolic.

The second possibility is the case of exceptional filling. We will focus on this last situation, that is cosmetic surgeries or filling which are also exceptional.

Using Lemma 2.1, we can deduce the following two preliminary lemmas on cosmetic filling. Let $M$ be a compact, connected, oriented hyperbolic manifold with boundary a torus and assume $\partial_1(M) = 1$. Fix a canonical basis $\{\mu, \lambda_M\}$ for $H_1(\partial M)$, where $\lambda_M$ is the rational longitude.

**Lemma 2.19.** Let $p/q$ and $p/q'$ be exceptional slopes such that $0 < p$ and $q < q'$. If $M(p/q)$ and $M(p/q')$ are homeomorphic then we must be in one of the following cases:

(a) $p = 1$ and $|q - q'| \leq 8$.
(b) $p \in \{7, 5\}$ and $q' = q + 1$.
(c) $p \in \{4, 3\}$ and $q' \in \{q + 1, q + 2\}$.
(d) $p = 2$ and $q' \in \{q + 2, q + 4\}$.

**Proof.** We have the bound $\Delta(p/q, p/q') = |pq' - qp| = p|q - q'| \leq 8$, so $p \leq 8$. If $p = 1$ then $|q - q'| \leq 8$. If $p \in \{8, 7, 6, 5\}$ then $|q - q'| \leq 1$ and $q' = q + 1$. On the other hand $p$ and $q$ (resp. $p$ and $q'$) must be relatively prime, thus since one of $q$ and $q + 1$ is even and $p$ cannot be 6 or 8. Similarly if $p \in \{4, 3\}$ then $|q - q'| \leq 2$ and $q' \in \{q + 1, q + 2\}$. If $p = 2$ then $|q - q'| \leq 4$ and $q' \in \{q + 1, q + 2, q + 3, q + 4\}$ but we must have $q \equiv q' \pmod{2}$ so $q' \in \{q + 2, q + 4\}$. 

For the case of reducible or cyclic filling we have the following lemma.

**Lemma 2.20.** Assume the hypothesis of Lemma 2.19. If $M(p/q)$ is cyclic or reducible and is homeomorphic to $M(p/q')$ then $p = 1$ and $q' = q + 1$.

**Proof.** The distance between two reducible slopes or two cyclic slopes is at most one, so $\Delta(p/q, p/q') = |pq' - qp| = p|q' - q| \leq 1$. It follows that $p = 1$ and $q' = q + 1$. 

3. Cosmetic Surgery on $S^3$
3.1. Results from Heegaard Floer theory. Recall that knot Floer homology associates to a null-homologous knot $K$ a $\mathbb{Z} \oplus \mathbb{Z}$–filtered $\mathbb{Z}[U]$-complex $CFK^\infty(Y, K)$, generated over $\mathbb{Z}$ by $(T_\alpha \cap T_\beta) \times (\mathbb{Z} \oplus \mathbb{Z})$ equipped with a function $F : (T_\alpha \cap T_\beta) \times (\mathbb{Z} \oplus \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}$ with the property that $F(U \cdot \{x; i, j\}) = (i - 1, j - 1)$ and $F(y; i', j') \leq F([x; i, j])$ for all $y$ having nonzero coefficient in $\partial x$. The Euler characteristic of this homology is also the Alexander polynomial of the knot $K$. From more details on the subject we refer to [[24], [25], [26], [29], [30], [31]]

When $K$ is a knot in $S^3$ admitting an $L$-space surgery, the following characterization of $\widehat{HFK}(Y, K)$ will be useful. It was proved in [[28] theorem 1.2].

**Proposition 3.1.** Let $K$ be a knot in $S^3$. If there is a rational number $r$ for which $Y_r(K)$ is an $L$-space, then there is an increasing sequence of integers $n_k < \ldots < n_1$ with the property that $n_i = -n_{k-i}$, and $\widehat{HFK}(K, j) = 0$ unless $j = n_i$ for some $i$, in which case $\widehat{HFK}(K, j) \cong \mathbb{Z}$.

An immediate corollary [[39] Corollary 3.8] is a simplified expression for the Alexander polynomials of such knots.

**Corollary 3.2.** Let $K$ be a knot that admits an $L$-space surgery. Then the Alexander polynomial of $K$ has the form

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j} (T^{m_j} + T^{-n_j})$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < \ldots < n_k$.

The following proposition, a variant of a result by Zhongtao Wu and Yi Ni, is one of the main ingredients for the proof of Theorem 1.4. More precisely it implies that the cosmetic surgery cannot be Seifert fibred.

**Proposition 3.3** (Yi Ni and Zhongtao Wu, [21]). Let $p, q > 0$ be two coprime integers. If there is an orientation preserving homeomorphism between $S^3_K(p/q)$ and $S^3_K(-p/q)$ then

$$\sum_{s \in \text{Spin}^c(S^3_K(p/q))} \chi(\text{HF}_\text{red}(S^3_K(p/q), s)) = 0.$$ 

The next proposition due to P. Ozsváth and Z. Szabó will be essential for excluding the possibility of a rational homology 3-sphere Seifert fibred cosmetic surgery.

**Proposition 3.4** (P. Ozsváth and Z. Szabó, [23]). Let $Y$ be a rational homology 3-sphere Seifert fibred space. Then for one of the orientations of $Y$, $HF_\text{red}(Y)$ is supported in even degree.
**Corollary 3.5.** There are no truly cosmetic surgeries on a non-trivial knot in $S^3$ which yields a rational homology sphere Seifert fibred space.

**Proof.** Let $K$ be a non-trivial knot in $S^3$. Let us suppose that there is an orientation preserving homeomorphism between $S^3_K(r)$ and $S^3_K(-r)$, by Proposition 3.3

$$\sum_{s \in \text{Spin}^c(S^3_K(r))} \chi(HF_{\text{red}}(S^3_K(r), s)) = 0.$$  

On the other hand by Proposition 3.4, we can assume $HF_{\text{red}}(S^3_K(r))$ is supported in even degree so

$$\sum_{s \in \text{Spin}^c(S^3_K(r))} \chi(HF_{\text{red}}(S^3_K(r), s)) = \pm \text{rank } HF_{\text{red}}(S^3_K(r)).$$

Therefore we must have $HF_{\text{red}}(S^3_K(r)) = 0$ in which case $S^3_K(r)$ is an $L$-space. In particular the knot $K$ admit an $L$-space surgery. Then $K$ admits an integral $L$-space surgery and by Corollary 3.2 the knot Floer homology satisfies: there is an increasing sequence of integers $n_{-k} < \ldots < n_k$ with the property that $n_i = -n_{-i}$, and $\widehat{HFK}(K, j) = 0$ unless $j = n_i$ for some $i$, in which case $\widehat{HFK}(K, j) \cong \mathbb{Z}$. This implies that the Alexander polynomial of $K$ has the form

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^k (-1)^{k-j}(T^{n_j} + T^{-n_j}),$$

for some increasing sequence of positive integers $0 < n_1 < n_2 < \ldots < n_k$.

If $\Delta_K(T) = 1$, then $\widehat{HFK}(K, 0) = \mathbb{Z}$, and $\widehat{HFK}(K, j) = 0$ for any other $j$. Hence $g(K) = 0$ and $K$ is the unknot, which we have excluded.

Thus $\Delta_K(T) \neq 1$ and by a straightforward computation

$$\Delta''_K(1) = 2 \sum_{j=1}^k (-1)^{k-j} n_j^2.$$  

Then the fact that $0 < n_1 < n_2 < \ldots < n_k$ implies $\Delta''_K(1) \neq 0$. Using Proposition 1.2, $K$ does not admit a truly cosmetic surgery.

□

### 3.2. Proof of Theorem 1.4.

**Lemma 3.6.** Let $K$ be a hyperbolic knot in $S^3$, and $r, r' \in \mathbb{Q} \cup \{\infty\}$ two distinct exceptional slopes on $\partial N(K)$. If $S_K(r)$ is homeomorphic to $S_K(r')$ as oriented manifolds, then $r$ and $r'$ are in the following table
\begin{tabular}{|c|c|c|c|c|}
\hline
  \( r \) & 2 & 1 & 1/2 & 1/3 & 1/4 \\
\hline
  \( r' \) & -2 & -1 & -1/2 & -1/3 & -1/4 \\
\hline
\end{tabular}

Proof. Write \( r = p/q \) and \( r' = p/q' \). By Yi Ni and Zhongtao Wu \( r = -r' \) so \( q = -q' \), then \( \Delta(r, r') = p|q - q'| = 2p|q| \leq 8 \) i.e \( p|q| \leq 4 \). Therefore \( p \in \{1, 2, 3, 4\} \). If \( p = 1 \) then \( |q| \leq 4 \) and we have the case \( r \in \{1, 1/2, 1/3, 1/4\} \). If \( p = 2 \) then \( |q| \leq 2 \), since \( q \) is odd we have \( |q| = 1 \) so \( r = 2 \). Now we need to exclude the case \( p \in \{3, 4\} \).

An orientation preserving homeomorphism \( f : M(p/q) \to M(p/q') \) induces an isomorphism \( f_* : H_1(M(p/q)) \to H_1(M(p/q')) \). Since \( H_1(M(p/q)) = \mathbb{Z}/p\mathbb{Z} \) is generated by the class \([\mu]_q\) of the meridian,

\[
  f_* [\mu]_q = [\mu]_{q'} u,
\]

for some unit \( u \in (\mathbb{Z}/p\mathbb{Z})^* \).

Let’s recall that the linking pairing of \( M(p/q) \) is a non-degenerate bilinear form

\[
  lk_q : \text{Tor}(H_1(M(p/q))) \otimes \text{Tor}(H_1(M(p/q))) \to \mathbb{Q}/\mathbb{Z},
\]

which is defined via some intersection count. One can check that

\[
  lk_q([\mu]_q, [\mu]_q) = -q/p.
\]

To see this, let \( D \) be a meridian disk for the surgery torus such that \( p\mu + q\lambda = \partial D \), in \( M(p/q) \). Since \( Y \) is a \( \mathbb{Z} \)-homology sphere \( \lambda_M = \partial\Sigma \) for some surface \( \Sigma \). Then \( p\mu = \partial D - q\partial\Sigma = \partial(D - q\Sigma) \) and by definition

\[
 lk_q([\mu]_q, [\mu]_q) = \frac{(D - q\Sigma) \cdot \mu}{p} \quad [\text{mod } \mathbb{Z}]
\]

where the dot “\( \cdot \)” denotes the intersection number. Now \( \mu \) can be pushed off of \( D \) so \( D \cdot \mu = 0 \), and \( \partial\Sigma = \lambda_M \Sigma \) so \( \Sigma \cdot \mu = 1 \). Therefore

\[
  lk_q([\mu]_q, [\mu]_q) = -\frac{q}{p} \quad [\text{mod } \mathbb{Z}].
\]

The map \( f \) induces an isomorphism between the linking pairing of \( M(p/q) \) and \( M(p/q') \) since it preserves oriented intersection number. Therefore

\[
  lk_q([\mu]_q, [\mu]_q) = lk_q(f_* [\mu]_q, f_* [\mu]_q) \quad [\text{mod } \mathbb{Z}]
  = lk_q([\mu]_{q'}, u, [\mu]_{q'} u) \quad [\text{mod } \mathbb{Z}]
  = lk_q([\mu]_{q'}, [\mu]_{q'}) u^2 \quad [\text{mod } \mathbb{Z}].
\]

Thus

\[
  -\frac{q}{p} \equiv -\frac{q'}{p} u^2 \quad [\text{mod } \mathbb{Z}], \quad \text{i.e. } q \equiv q' u^2 \quad [\text{mod } p].
\]
We apply this congruence to the case \( p \in \{4,3\} \). For \( p = 4 \) (resp. \( p = 3 \)), \( u \in \{1,3\} \) (resp. \( u \in \{1,2\} \)). Therefore \( u^2 = 1 \) and \( q \equiv q' \pmod{4} \), but \( q' \in \{q+1,q+2\} \) by Lemma 2.19 which is not possible. Thus \( p \notin \{3,4\} \).

\[ \square \]

In this lemma it is essential that \( \text{int}(M) \) has a complete finite volume hyperbolic structure since the bound is on the diameter of \( E(M) \). Thus the examples given in [17] do not fall into these categories.

For a hyperbolic knot \( K \) in \( S^3 \), if we take in account the type of manifold obtained after surgery we have the following lemma.

**Lemma 3.7.** There are no truly cosmetic surgeries on hyperbolic knot in \( S^3 \) which yields a reducible manifold.

**Proof.** If \( K \subset S^3 \) is a hyperbolic knot and \( r,r' \) are two reducible slopes on \( \mathcal{N}(K) \) then \( \Delta(r,r') \leq 1 \) by Theorem 2.8. However by Lemma 3.6, the distance between two cosmetic slopes is at least two. This is not possible. \( \square \)

3.2.1. **Proof of Theorem 1.4.** We can now finish the proof of Theorem 1.4. By Lemma 3.7 and Corollary 3.5 \( S_K(r) \) is irreducible and non-Seifert fibred. Therefore since the manifold is not hyperbolic it contains an essential torus. On the other hand, by Theorem 2.10, if \( S_K(r) \) is toroidal and \( r = p/q \) then \( |q| \leq 2 \). Therefore using the distance table 2.2 and table of Lemma 3.6 we can deduce that: \( r = 2 \) and \( s = -2 \), or \( r = 1 \) and \( s = -1 \), or \( r = 1/2 \) and \( s = -1/2 \).

By Theorem 2.11, if there is a non-integral slope on \( \partial(S^3 \setminus \mathcal{N}(K)) \) which gives a toroidal manifold then \( K \) is one of the Eudave-Muñoz knots \( k(l,m,n,p) \) and the surgery is the corresponding half-integral surgery. Therefore there is at most one slope which can give an essential torus. Thus there is no non-integral cosmetic slope which give toroidal manifold. This excludes the case \( r = 1/2 \) and \( s = -1/2 \).

Now we have either \( r = 1 \) or \( r = 2 \). If \( r = 2 \) then \( \Delta(2,-2) = 4 \) and Theorem 2.9 gives the complete list of all hyperbolic knots in \( S^3 \) with two toroidal slopes \( r_1 \) and \( r_2 \) at distance 4. Precisely, there is an integer \( n \) and an homeomorphism of \( S^3 \) which send the triple \( (K,r_1,r_2) \) to \( (L_1(n);0,4), \ n \neq 0,1 \) or \( (L_2(n);2-9n,-2-9n), \ n \neq 0, \pm 1 \). Where \( L_i(n), i = 1,2 \) denotes the knot obtained from the right component of the link \( L_i, \ i = 1,2 \) (see figure below) after \( 1/n \) surgery on the left component.
The manifold obtained after surgery is then $S^3(L_2(n); 2 - 9n)$ or $S^3(L_2(n); -2 - 9n)$ or $S^3(L_1(n); 0)$ or $S^3(L_2(n); 4)$. Therefore we can check that

\[ |2 - 9n| = |H_1\left(S^3(L_2(n); 2 - 9n)\right)| \neq |H_1\left(S^3(L_2(n); -2 - 9n)\right)| = |2 + 9n| \]

\[ 0 = |H_1\left(S^3(L_1(n); 0)\right)| \neq |H_1\left(S^3(L_2(n); 4)\right)| = 4. \]

Since $n \neq 0$, this completes the proof of Theorem 1.4. □

4. COSMETIC SURGERY ON SOME SPECIAL CLASSES OF KNOTS

As consequences of Theorem 1.4, let us give some results about cosmetic surgeries along algebraic knots, alternating knots and arborescent knots in $S^3$.

4.1. Algebraic knots. An algebraic knot $K \subset S^3$ is the link of an irreducible complex plane curve singularity that is, $K$ is the transversal intersection

\[ K = \{ f = 0 \} \cap S^3 \]

where $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is an irreducible holomorphic function, and $S^3 = \{ z \in \mathbb{C}^2 : ||z|| = \epsilon \}$ for $\epsilon > 0$ sufficiently small. The natural orientations of $S^3$ and of the regular part of $\{ f = 0 \}$ induces a natural orientation on $K$. When $\{ f = 0 \}$ is not smooth at the origin, $K$ is not the unknot. The Heegaard Floer homology of the result of a surgery on an algebraic knot has been computed by A. Némethi in [19].

Proposition 4.1 (A. Némethi [19]). Let $K \subset S^3$ be an algebraic knot, $p, q > 0$ two coprime integers and $Y = -S^3_K(-p/q)$. Then $HF_{\text{red}}(Y)$ is supported in even degree.

This leads to the following corollary.

Corollary 1.5. There are no truly cosmetic surgeries on non-trivial algebraic knot in $S^3$.

Proof. The proof is similar to the proof of Corollary 3.5 since the $HF_{\text{red}}$ of the resulting manifold is supported in even degree by Proposition 4.1. □
4.2. Alternating knots. Combining Theorem 1.4 with work of Kazuhiro Ichihara and Hidetoshi Masai, we have the following result for alternating knot in $S^3$.

**Corollary 1.6.** There are no exceptional truly cosmetic surgeries on an alternating hyperbolic knot in $S^3$.

*Proof.* The corollary is a consequence of the classification of exceptional surgeries on alternating knots which was done by Kazuhiro Ichihara and Hidetoshi Masai [13] combined with Theorem 1.4. By this classification if an alternating hyperbolic knot in $S^3$ admits an exceptional surgery with slope $r$, then either:

- $K$ is a twist knot $K[2n, \pm 2]$ for $n \neq 0$ (which includes the figure-8),
- $K$ is a two bridge knot $K[a,b]$ with $|a|, |b| > 2$ and $r = 0$ if both $a, b$ are even and $r = 2b$ if $a$ is odd and $b$ is even,
- $K$ is a pretzel knot $P(a, b, c)$ with $a, b, c \neq 0, \pm 1$ and $r = 0$ if $a, b, c$ are all odd and $r = 2(b + c)$ if $a$ is even and $b, c$ are odd. Moreover $S^3_K(r)$ is toroidal but not Seifert fibred.

The Alexander polynomial of a twist knot $K[2n, \pm 2]$ is $\Delta_K(t) = 2n + 1 - n(t + t^{-1})$, so $\Delta'_K(1) = -2n \neq 0$. Therefore by Proposition 1.2 we cannot have truly cosmetic surgery for the first case. For the last two cases $r \neq \pm 1$, so these exceptional slopes cannot be truly cosmetic slopes by Theorem 1.4. \qed

4.3. Arborescent knot. Another class of knots in $S^3$ is the class of arborescent knots. Let’s recall that a Montesinos tangle is a sum of several rational tangles and an arborescent tangle is one that can be obtained by summing several Montesinos tangles together in an arbitrary way. An arborescent knot or link is a knot or link obtained by joining the endpoints of the arcs in an arborescent tangle by two arcs.

**Corollary 1.7.** There are no exceptional truly cosmetic surgeries on arborescent knots in $S^3$.

*Proof.* In light of Theorem 1.4 we only have to check that $\pm 1$ surgery on an arborescent knot do not yield two homeomorphic toroidal manifold. Following Ying-Qing Yu, there are three types of arborescent knot: type I, type II and type III. By Theorem 3.6 of [36], every non-trivial surgery on a type III arborescent knot gives a hyperbolic manifold, therefore we shall focus on type II and type I arborescent knots. For type I, they are Montesinos knots with length at most 3, which again split as 2-bridge knots and Montesinos knots of length 3. The 2-bridge knots which admit toroidal surgery are given in Theorem 1.1 of [4] and they are among the knots in Corollary 1.6, hence they do not admit truly cosmetic surgery. The case of Montesinos knots of length 3 is dealt with in
precisely if $K$ is a Montesinos knot of length 3 and $\delta$ is a slope on $\partial N(K)$, then $S^3_{K}(\delta)$ is toroidal if and only if, following notation in [37], $(K, \delta)$ is equivalent to one of

- $K = K(1/q_1, 1/q_2, 1/q_3)$, $q_i$ odd, $|q_i| > 1$, $\delta = 0$.
- $K = K(1/q_1, 1/q_2, 1/q_3)$, $q_1$ even, $q_2, q_3$ odd, $|q_i| > 1$, $\delta = 2(q_2 + q_3)$.
- $K = K(-1/2, 1/3, 1/(6 + 1/n))$, $n \neq 0$, $-1$, $\delta = 16$ if $n$ is odd, and 0 if $n$ is even.
- $K = K(-1/3, -1/(3 + 1/n), 2/3)$, $n \neq 0$, $-1$, $\delta = -12$ when $n$ is odd, and $\delta = 4$ when $n$ is even.
- $K = K(-1/2, 1/5, 1/(3 + 1/n))$, $n$ even, and $n \neq 0$, $\delta = 5 - 2n$.
- $K = K(-1/2, 1/3, 1/(5 + 1/n))$, $n$ even, and $n \neq 0$, $\delta = 1 - 2n$.
- $K = K(-1/(2 + 1/n), 1/3, 1/3)$, $n$ odd, $n \neq -1$, $\delta = 2n$.
- $K = K(-1/2, 1/3, 1/(3 + 1/n))$, $n$ even, $n \neq 0$, $\delta = 2 - 2n$.
- $K = K(-1/2, 2/5, 1/9)$, $\delta = 15$.
- $K = K(-1/2, 2/5, 1/7)$, $\delta = 12$.
- $K = K(-1/2, 1/3, 1/7)$, $\delta = 37/2$.
- $K = K(-2/3, 1/3, 1/4)$, $\delta = 13$.
- $K = K(-1/3, 1/3, 1/7)$, $\delta = 1$.

After checking this list for knots which are listed more than once we notice that there are at most three toroidal slopes. A knot $K(t_1, t_2, t_3)$ admits exactly two toroidal surgeries if and only if it is equivalent to one of the following 5 knots:

- $K(-1/2, 1/3, 2/11)$, $\delta = 0$ and $-3$;
- $K(-1/3, 1/3, 1/3)$, $\delta = 0$ and 2;
- $K(-1/3, 1/3, 1/7)$, $\delta = 0$ and 1;
- $K(-2/3, 1/3, 1/4)$, $\delta = 12$ and 13;
- $K(-1/3, -2/5, 2/3)$, $\delta = 4$ and 6.

and it admits exactly three toroidal surgeries if and only if it is the figure-8 or the knot $K(-1/2, 1/3, 1/7)$. No pair of toroidal slopes corresponding to these knots admitting more that one toroidal slope are ±1 in the standard Seifert framing so this excludes the possibility of truly cosmetic surgery.

The remaining case is that of type II arborescent knots. There are knots that have a Conway sphere cutting it into two Montesinos tangles of type $T(r_i, 1/2)$, $i = 1, 2$ where $r_i \in \mathbb{Q} \cup \{\infty\}$. According to Theorem 1.1 of [38], there are three distinct knots $K_1, K_2, K_3$ such that an arborescent knot $K \in S^3$ admits an exceptional slope $\delta$ if and only if $(K, \delta)$ is isotopic to $(K_1, 3)$, $(K_2, 0)$, $(K_3, -3)$, in which case the slope is toroidal. Therefore since there is exactly one slope for each knot, there are no truly cosmetic surgery.
5. Some properties of Heegaard Floer invariants.

5.1. The correction term. From the absolute $\mathbb{Q}$-grading we can derive a new numerical invariant for rational homology three-spheres equipped with Spin$^c$ structures.

**Definition 5.1** (P. Ozsváth and Z. Szabó, [22]). Let $(Y, s)$ be a rational homology three-sphere equipped with a Spin$^c$ structure. The correction term $d(Y, s)$ is the minimal $\mathbb{Q}$-grading of any non-torsion element in the image of $HF^\infty(Y, s)$ in $HF^+(Y, s)$, i.e

$$d(Y, s) = \min \{ \tilde{\text{gr}}(\pi_*(x)) \mid x \in HF^\infty(Y, s) \}$$

where $\pi_* : HF^\infty(Y, s) \to HF^+(Y, s)$ is the map in the long exact sequence

$$\cdots \to HF^-(Y, s) \xrightarrow{i_*} HF^\infty(Y, s) \xrightarrow{\pi_*} HF^+(Y, s) \to \cdots$$

There is another interpretation of $d(Y, s)$ using the reduced homology $HF_{\text{red}}(Y, s)$. By definition of $HF_{\text{red}}(Y, s)$, we have the isomorphism:

$$HF^+(Y, s) \cong \frac{\mathbb{Z}[U, U^{-1}]}{U \mathbb{Z}[U]} \oplus HF_{\text{red}}(Y, s),$$

then $d(Y, s)$ is the $\mathbb{Q}$-grading of the lowest degree generator of $\mathbb{Z}[U, U^{-1}]/U \mathbb{Z}[U]$.

For the 3-sphere, the homologies $HF^\infty(S^3)$ are all supported in degree zero so $d(S^3) = 0$.

For the case of a 3-manifold $Y_0$ with $H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}$, there is a unique Spin$^c$ structure $s_0$ such that $c_1(s_0) = 0$. We can then define two correction terms as follow.

**Definition 5.2** (P. Ozsváth and Z. Szabó, [22]). We define the correction terms $d_{+1/2}(Y_0)$, resp. $d_{-1/2}(Y_0)$, to be the minimal $\mathbb{Q}$-grading of any non-torsion element in the image of $HF^\infty(Y_0, s_0)$ in $HF^+(Y_0, s_0)$ with grading $+1/2$ resp. $-1/2$ modulo 2.

**Proposition 5.3** (P. Ozsváth and Z. Szabó, [22]). Let $H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}$. Then,

1. $d_{1/2}(Y_0) - 1 \leq d_{-1/2}(Y_0)$.
2. $d_{\pm 1/2}(Y_0, s) = d_{\pm 1/2}(Y_0, s')$.
3. $d_{\pm 1/2}(Y_0, s) = -d_{\mp 1/2}(-Y_0, s)$.

**Proof.** See [22], section 4.2. \qed

For integral homology spheres there is only one Spin$^c$ structure so there is a unique correction term. The following proposition gives a relationship between correction terms for integral homology spheres and the correction terms for zero-surgeries on knots they contain.
**Proposition 5.4** (P. Ozsváth and Z. Szabó, [22]). Let $K \subset Y$ be a knot in an integral homology three-sphere and let $Y_1$ be the manifold obtained by +1 surgery on $K$. Then,

$$d(Y) - \frac{1}{2} \leq d_{-1/2}(Y_0), \quad \text{and} \quad d_{+1/2}(Y_0) - \frac{1}{2} \leq d(Y_1).$$

**Proof.** See [22] section 4.2. \qed

In light of Proposition 5.4, the correction terms for homology $S^1 \times S^2$ can be used to give obstructions for obtaining a given three-manifold as zero-surgery on a knot in the three-sphere. Specifically, since $d(S^3) = 0$, we see that if $Y_0$ is obtained as zero-surgery on a knot in $S^3$, then

$$-\frac{1}{2} \leq d_{-1/2}(Y_0).$$

Moreover, by reflecting the knot we also obtain the bound

$$d_{1/2}(Y_0) \leq \frac{1}{2}.$$

**Proposition 5.5** (P. Ozsváth and Z. Szabó, [22]). Let $K \subset S^3$ be an oriented knot in the three-sphere. Then,

$$d_{1/2}(S^3_K(0)) - \frac{1}{2} = d(S^3_K(1))$$

$$d(S^3_K(-1)) - \frac{1}{2} = d_{-1/2}(S^3_K(0))$$

**Proof.** This is a direct consequence of the surgery long exact sequence, in view of the structure of $HF^+(S^3)$. \qed

Ozsváth and Z. Szabó also proved the following result concerning the correction term for $1/n$-surgery.

**Proposition 5.6** (P. Ozsváth and Z. Szabó, [22]). Let $K \subset Y$ be a knot in an integral homology three-sphere. Then, we have the following inequalities (where here $n$ is any positive integer):

$$d_{1/2}(Y_K(0)) - \frac{1}{2} \leq d(Y_K(1/(n + 1))) \leq d(Y_K(1/n)) \leq d(Y)$$

$$d(Y) \leq d(Y_K(-1/n)) \leq d(Y_K(-1/(n + 1))) \leq d_{-1/2}(Y_K(0)) + \frac{1}{2}.$$

**Proof.** See [22] Corollary 9.14. \qed
5.2. **Torsion invariant of the Alexander polynomial.** Here are some results relating the torsion of the Alexander polynomial of a knot in an integer homology sphere to +1-surgery. In what follows, \( Y \) will be an oriented integer homology sphere. We define

\[
N(Y) := \text{rank} \text{HF}_{\text{red}}(Y)
\]

**Definition 5.7.** Let \( K \) be a knot in a rational homology sphere \( Y \) and let its normalized Alexander polynomial be

\[
\Delta_K(T) = a_0 + \sum_{i=1}^d a_i (T^i + T^{-i}).
\]

We define the \( i \)-th torsion invariant of the Alexander polynomial to be

\[
t_i = \sum_{j=1}^d j a_{|i|+j}.
\]

**Theorem 5.8** (P. Ozsváth and Z. Szabó, [22]). Let \( Y \) be an integral homology three-sphere and \( K \subset Y \) be a knot. Then there is a bound:

\[
|t_0(K)| + 2 \sum_{i=1}^d |t_i(K)| \leq \text{rank} \text{HF}_{\text{red}}(Y) + \frac{d(Y)}{2} + \text{rank} \text{HF}_{\text{red}}(Y_K(1)) - \frac{d(Y_K(1))}{2},
\]

**Proof.** See [22] Theorem 6.1. \( \square \)

5.3. **Correction term and torsion invariants for exceptional cosmetic surgeries.** Theorem 1.4 also induces the following results about correction terms.

**Corollary 5.9.** If a 3-manifold \( Y \) is the result of an exceptional truly cosmetic surgery on a hyperbolic knot \( K \) in \( S^3 \) then:

\[
|t_0(K)| + 2 \sum_{i=1}^n |t_i(K)| \leq \text{rank} \text{HF}_{\text{red}}(Y),
\]

where the number \( t_i(K) \) for \( i \in \mathbb{Z} \) is the torsion invariant of the Alexander polynomial \( \Delta_K(T) \) of \( K \) and \( n \) is the degree of \( \Delta_K(T) \).

This lower bound is strictly positive since not all the torsion \( t_i(K) \) are zero.

**Proof.** Let \( K \subset S^3 \) be a hyperbolic knot such that there is an orientation preserving homeomorphism between \( S^3_K(r) \) and \( S^3_K(r') \) for two distinct rational numbers \( r \) and \( r' \). Let \( Y = S^3_K(r) \), by Theorem 1.4 we can assume \( r = +1 \). By Theorem 5.8 we have the inequality:

\[
|t_0(K)| + 2 \sum_{i=1}^n |t_i(K)| \leq \text{rank} \text{HF}_{\text{red}}(S^3) + \frac{d(S^3)}{2} + \text{rank} \text{HF}_{\text{red}}(Y) - \frac{d(Y)}{2},
\]
where the number $t_i(K)$ for $i \in \mathbb{Z}$ is the torsion invariant of the Alexander polynomial $\Delta_K(T)$ of $K$ and $n$ is the degree of $\Delta_K(T)$. On the other hand we also have the identity:

$$\lambda(Y) = \chi(HF_{\text{red}}(Y)) - \frac{d(Y)}{2}$$

from [32], where $\lambda$ stand for the Casson invariant. We also know that $\text{rank}HF_{\text{red}}(S^3) = d(S^3) = 0$. Now by the surgery formula for Casson invariant

$$\lambda(Y) = \lambda(S^3) + \lambda(L(1,1)) + \Delta''_K(1) = \Delta''_K(1)$$

and by Proposition 1.2 $\Delta''_K(1) = 0$, thus $\lambda(Y) = 0$. By Proposition 3.3 $\chi(HF_{\text{red}}(Y)) = 0$, hence $d(Y) = 0$. This proves the desired result. □

**Corollary 5.10.** If a hyperbolic knot $K \subset S^3$ admits an exceptional truly cosmetic surgery then the Heegaard Floer correction term of any $1/n$ ($n \in \mathbb{Z}$) surgery on $K$ satisfies

$$d(S^3_K(1/n)) = 0.$$ 

**Proof.** Let $K$ be as in proof of Corollary 5.9. Let $d_{1/2}(S^3_K(0))$ and $d_{-1/2}(S^3_K(0))$ be the two correction terms for the 0-surgery along $K$. Let $n$ be a positive integer, by Proposition 5.6,

$$d_{1/2}(S^3_K(0)) - \frac{1}{2} \leq d(S^3_K(1/(n + 1))) \leq d(S^3_K(1/n)) \leq d(S^3) = 0$$

By Proposition 5.5 we have

$$d_{1/2}(S^3_K(0)) - \frac{1}{2} = d(S^3_K(+1)).$$

By the proof of Corollary 5.9 $d(S^3_K(+1)) = 0$, this completes the proof. □

**References**

[1] S. A. Bleiler, C. D. Hodgson, J. R. Weeks, *Cosmetic surgery on knots*, Geometry and Topology Monogr. 2, 23-24. 12 p, 1999.

[2] S. Boyer and D. Lines, *Surgery formulae for Casson’s invariant and extensions to homology lens spaces*, J. Reine Angew. Math. 405 (1990), 181-220.

[3] S. Boyer and X. Zhang, *A proof of the finite filling conjecture*, J. Differential Geom. 59 (2001), no. 1, 87–176.

[4] Mark Brittenham and Ying-Qing Wu, *The classification of exceptional Dehn surgeries on 2-bridge knots*, Comm. Anal. Geom. 9 (2001), no. 1, 97–113.

[5] M. Culler, C. MCA. Gordon, J. Luecke and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math., 125 (1987),237-300.

[6] C. McA. Gordon, *Boundary slopes of punctured tori in 3-manifolds*, Trans. Amer. Math. Soc. 350 (1998), no. 5, 1713–1790.

[7] C. McA. Gordon, *Dehn surgery on knots*, in: Proceedings of the International Congress of Mathematicians vol. I, II (Kyoto 1990), Mathematical Society of Japan, Tokyo (1991), 631–642.

[8] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. 2 (1989), 371–415.
[9] C. McA. Gordon and J. Luecke, Non-integral toroidal Dehn surgeries, Comm. Anal. Geom. 12 (2004), 417-485.
[10] C. McA. Gordon and J. Luecke, Reducible manifolds and Dehn surgery, Topology 35 (1996) 385–410.
[11] C. McA. Gordon and J. Luecke, Surgeries producing essential tori I, Comm. Anal. Geom. 3 (1995), 597-644.
[12] C. McA. Gordon and Ying-Qing Wu, Toroidal Dehn fillings on hyperbolic 3-manifolds, Mem. Amer. Math. Soc. 194 (2008), no. 909.
[13] Kazuhiro Ichihara and Hidetoshi Masai, Exceptional surgeries on alternation knots, arXiv:math.GT/1310.3472.
[14] R. Kirby, Problems in low-dimensional topology, AMS/IP Studies in Advanced Mathematics, volume 2, part 2, Amer. Math. Soc. (1997).
[15] M. Lackenby, Dehn surgery on knots in 3-manifolds, J. Amer. Math. Soc. 10 (1997) 835–864.
[16] M. Lackenby and R. Meyerhoff, The maximal number of exceptional Dehn surgeries, Invent. Math. 191 (2013), no. 2, 341–382.
[17] Y. Mathieu, Closed 3-manifolds unchanged by Dehn surgery, Journal of Knot Theory and Its Ramifications, Vol. I No. 3 (1992) 279-296.
[18] Y. Mathieu, Sur les nœuds qui ne sont pas déterminés par leur complément et problèmes de chirurgie dans les variétés de dimension 3, PhD thesis, Marseille (1990).
[19] A. Némethi, On the floer homology of $S^3_{-p/q}(K)$, published as part of ‘Graded roots and singularities’, in Singularities in geometry and topology, World Sci. Publ., Hackensack, NJ, 2007, 394–463.
[20] Yi Ni, Thurston norm and cosmetic surgeries, in: Low-dimensional and Symplectic Topology, Proceedings of Symposia in Pure Mathematics. No.82. (2011), pp. 53-63.
[21] Yi Ni and Zhongtao Wu, Cosmetic surgeries on knots in $S^3$, to appear in J. reine angew. Math..
[22] P. Ozsváth and Z. Szabó, Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary, Adv. Math., 173:179–261, 2003. math.SG/0110170.
[23] P. Ozsváth and Z. Szabó, On the Floer homology of plumbed three-manifolds, Geom. Topol. 7 (2003), 185–224 (electronic).
[24] P. Ozsváth and Z. Szabó, Holomorphic disks and topological invariants for closed three-manifolds, Ann. of Math. 159(3) (2004), 1027–1158.
[25] P. Ozsváth and Z. Szabó, Holomorphic disks and 3-manifold invariants: properties and applications, Ann. of Math. 159 (2004) 1159–1245.
[26] P. Ozsváth and Z. Szabó, Holomorphic disks and knot invariants, Adv. Math. 186(1) (2004), 58–116.
[27] P. Ozsváth and Z. Szabó, Holomorphic disks and genus bounds, Geom. Topol. 8 (2004), 311–334.
[28] P. Ozsváth and Z. Szabó, On knot Floer homology and lens space surgeries, Topology 44 (2005) 1281-1300.
[29] P. Ozsváth and Z. Szabó, Introduction to Heegaard Floer theory, Clay Math. Proc. Volume 5 (2006), 3-28.
[30] P. Ozsváth and Z. Szabó, Lectures on Heegaard Floer homology, Clay Math. Proc. Volume 5 (2006), 29-70.
[31] J. Rasmussen, Floer homology and knot complement, PhD thesis, Harvard University, 2003.
[32] R. Rustamov, Surgery formula for the renormalized Euler characteristic of Heegaard Floer homology, preprint (2004), available at arXiv:math.GT/0409294.
[33] N. Saveliev, *Invariants for homology spheres*, Encyclopedia of Mathematical Sciences, Springer 2002.

[34] W. P. Thurston, *The geometry and topology of 3-manifolds*, Princeton Univ. Math. Dept. (1979).

[35] L. Watson, *Involutions on 3-manifolds and Khovanov homology*, PhD thesis, Université du Québec à Montréal, 2009.

[36] Ying-Qing Wu, *Dehn surgery on arborescent knots*, J. Diff. Geom. 42 (1996), 171-197.

[37] Ying-Qing Wu, *The classification of toroidal Dehn surgeries on Montesinos knots*, Comm. Anal. Geom. 19 (2011), no. 2, 305–345.

[38] Ying-Qing Wu, *Exceptional surgery on large arborescent knots*, Pacific J. Math. 252 (2011), no. 1, 219–243.

[39] Zhongtao Wu, *Cosmetic surgery in L-space homology spheres*, Geometry and Topology 15 (2011) 1157-1168.

Huygens C. Ravelomanana, CIRGET-UQAM
P. O. Box 8888, Centre-ville, Montréal, Qc, H3C 3P8, Canada
e-mail: huygens@cirget.ca