Mean-square convergence rates of implicit Milstein type methods for SDEs with non-Lipschitz coefficients

Xiaojie Wang

Abstract
A class of implicit Milstein type methods is introduced and analyzed in the present article for stochastic differential equations (SDEs) with non-globally Lipschitz drift and diffusion coefficients. By incorporating a pair of method parameters $\theta, \eta \in [0, 1]$ into both the drift and diffusion parts, the new schemes are indeed a kind of drift-diffusion double implicit methods. Within a general framework, we offer upper mean-square error bounds for the proposed schemes, based on certain error terms only getting involved with the exact solution processes. Such error bounds help us to easily analyze mean-square convergence rates of the schemes, without relying on a priori high-order moment estimates of numerical approximations. Putting further globally polynomial growth condition, we successfully recover the expected mean-square convergence rate of order one for the considered schemes with $\theta \in \left[\frac{1}{2}, 1\right], \eta \in [0, 1]$. Also, some of the proposed schemes are applied to solve three SDE models evolving in the positive domain $(0, \infty)$. More specifically, the particular drift-diffusion implicit Milstein method ($\theta = \eta = 1$) is utilized to approximate the Heston $\frac{3}{2}$-volatility model and the stochastic Lotka-Volterra competition model. The semi-implicit Milstein method ($\theta = 1, \eta = 0$) is used to solve the Ait-Sahalia interest rate model. Thanks to the previously obtained error bounds, we reveal the optimal mean-square convergence rate of the positivity preserving schemes under more relaxed conditions, compared with existing relevant results in the literature. Numerical examples are also reported to confirm the previous findings.

Keywords
Stochastic differential equations · Implicit Milstein type methods · Mean-square convergence rates · Heston $\frac{3}{2}$-volatility model · Ait-Sahalia interest rate model · Stochastic lotka-volterra competition model · Positivity preserving schemes
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1 Introduction

Stochastic differential equations (SDEs) find applications in a wide range of scientific areas such as finance, chemistry, biology, engineering and many other branches of science. In general, analytical solutions to nonlinear SDEs are not available and development and analysis of numerical methods for simulation of SDEs are of significant interest in practice. To analyze the numerical approximations, a global Lipschitz condition is often imposed on the coefficient functions of SDEs [30, 43]. Nevertheless, SDEs arising from applications rarely obey such a traditional but restrictive condition. Notable examples of SDEs with non-globally Lipschitz continuous coefficients include numerous models such as the $\frac{3}{2}$-volatility model [19, 33],

$$dX_t = X_t(\mu - \alpha X_t)dt + \beta X_t^{3/2} dW_t, \quad X_0 = x_0 > 0, \quad \mu, \alpha, \beta > 0, \quad (1.1)$$

and the Ait-Sahalia interest rate model [1],

$$dX_t = (\alpha_{-1} X_t^{-1} - \alpha_0 + \alpha_1 X_t - \alpha_2 X_t^{\kappa}) dt + \sigma X_t^\rho dW_t, \quad X_0 = x_0 > 0, \quad (1.2)$$

from mathematical finance, where $\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \sigma > 0$ are positive constants and $\kappa > 1, \rho > 1$. Evidently, coefficients of these two models violate the global Lipschitz condition. As already shown in [25], the popularly used Euler-Maruyama method produces divergent numerical approximations when used to solve a large class of SDEs with super-linearly growing coefficients, such as Eq. 1.1 and Eq. 1.2. Therefore special care must be taken to design and analyze convergent numerical schemes in the absence of the Lipschitz regularity of coefficients. Recent years have witnessed a prosper growth of relevant works devoted to the numerical analysis of SDEs under non-globally Lipschitz conditions, with an emphasis on analyzing implicit schemes [2–5, 10, 20–22, 39, 40, 44, 52, 55, 57], and devising explicit methods based on modifications of traditionally explicit schemes [9, 12, 13, 18, 23, 24, 26–29, 32, 37, 38, 46, 47, 49–51, 56], to just mention a few. Although explicit methods such as tamed methods [26, 47] and truncated schemes [18, 37], computationally more efficient than implicit ones for one time step, are able to well tackle non-stiff SDEs with super-linearly growing coefficients, they usually face a severe stepsize restriction due to stability issues when used to solve stiff SDE systems [43]. Moreover, explicit time stepping schemes like tamed methods, similarly to the classical explicit Euler/Milstein schemes, are usually not positivity preserving when applied to approximate financial models whose solutions naturally remain positive (see, e.g., [8, 21, 48]).

In this article we are concerned with implicit Milstein schemes for mean-square approximations of Itô SDEs with non-globally Lipschitz continuous coefficients, in the form of

$$dX_t = f(X_t) dt + g(X_t) dW_t, \quad t \in (0, T], \quad X_0 = x_0, \quad (1.3)$$
where \( W : [0, T] \times \Omega \rightarrow \mathbb{R}^m \) stands for the \( \mathbb{R}^m \)-valued standard Brownian motion, \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) the drift coefficient function, and \( g : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m} \) the diffusion coefficient function. Mean-square approximations are of particular importance for the computation of statistical quantities of the solution process of Eq. 1.3 through computationally efficient multilevel Monte Carlo (MLMC) methods [15]. Recall that Milstein-type schemes achieve a higher mean-square convergence rate than the Euler-type schemes and can be combined with the MLMC approach to reduce computational costs further [14–16]. In the literature, various Milstein type methods [5–7, 18, 21, 28, 31, 32, 34, 51, 53, 56] have been studied and the present work proposes a class of implicit Milstein-type schemes and establish a mean-square convergence theory for the new schemes. On a uniform mesh constructed over \([0, T]\) with a uniform time step-size \( h = \frac{T}{N}, N \in \mathbb{N} \), we develop a family of double implicit Milstein-type methods with a pair of method parameters \((\theta, \eta)\) for Eq. 1.3 as follows:

\[
Y_{n+1} = Y_n + \theta f(Y_{n+1})h + (1 - \theta) f(Y_n)h + g(Y_n)\Delta W_n + \sum_{j_1, j_2=1}^{m} \mathcal{L}^{j_1} g_{j_2}(Y_n)I^{j_1, j_2+1}_{j_1, j_2} + \frac{\eta}{2} \sum_{j=1}^{m} \mathcal{L}^{j} g_j(Y_n)h - \frac{\eta}{2} \sum_{j=1}^{m} \mathcal{L}^{j} g_j(Y_{n+1})h, \quad Y_0 = X_0, \tag{1.4}
\]

where \( \theta, \eta \in [0, 1], \Delta W_n := W_{n+1} - W_n, n \in \{0, 1, 2, \ldots, N-1\} \) and \( \mathcal{L}^{j_1} g_{j_2}, I^{j_1, j_2+1}_{j_1, j_2} \) are precisely defined by Eq. 2.3. When \( d = m = 1 \), the schemes Eq. 1.4 coincide with the proposed ones in [21], where the authors used the positivity preserving schemes to solve the \( \frac{3}{2} \)-volatility model Eq. 1.1 and proved its strong convergence with no convergence rate revealed. After assigning \( \eta = 0 \), the proposed scheme reduces to the classical \( \theta \)-Milstein method, which has been studied in [7, 30, 57]. But in the regime of possibly super-linearly growing diffusion coefficients \( g \), the strong convergence rate of the \( \theta \)-Milstein method is, up to the best of our knowledge, still an open problem. This paper shall fill the gap.

Also, we mention that an order reduction would be caused due to additional costs of approximating multiple stochastic integrals \( I^{j_1, j_2+1}_{j_1, j_2} \) when the multi-dimensional SDEs are driven by non-commutative noise. As clarified in [45, section 7], the effective order of Milstein methods should be \( \frac{2}{3} \) in the case of non-commutative noise when the multiple stochastic integrals are efficiently approximated. We refer to the simulation method proposed by Wiktorsson [54] and see also [17] for implementation issues. Compared to the order 0.5 strong Euler-type schemes, which attains the effective order 0.5, there is still a significantly improved convergence for the Milstein methods in the non-commutative noise setting. Furthermore, we mention that the application of fully implicit (stochastically implicit) methods are unavoidable for stiff systems where the stochastic part plays an essential role (see [43, pp.33] and [42] for detailed comments and an illustrative example). By incorporating a pair of method parameters \( \theta, \eta \in [0, 1] \) into the drift and diffusion parts, here we construct a kind of fully implicit methods for general multi-dimension SDE systems with non-Lipschitz coefficients. Finally, we point out that proving the expected convergence rate of the proposed schemes for SDEs in non-Lipschitz settings, especially for the above two financial models, is
highly non-trivial and remains an unsolved problem. The present work aims to fill these gaps by successfully establishing a first order of mean-square convergence for the scheme Eq. 1.4 in different settings, covering the two aforementioned financial models.

By formulating certain generalized monotonicity conditions in a domain $D \subset \mathbb{R}^d$ (Assumption 3.1), we develop an easy and novel approach to derive upper mean-square error bounds for the proposed schemes, which only get involved with the exact solution processes (Theorem 3.3). The framework is broad and covers the two aforementioned SDE financial models. Such error bounds are powerful as they help us to easily analyze mean-square convergence rates of the schemes, without relying on a priori high-order moment estimates of numerical approximations. Putting further globally polynomial growth and coercivity conditions in $\mathbb{R}^d$ (Assumption 4.1), we utilize the derived upper error bound to successfully identify a mean-square convergence rate of order one for the schemes Eq. 1.4 solving general SDEs Eq. 1.3 (see Theorem 4.2, Corollaries 4.3, 4.4).

Later in Section 5, we turn our attention to two scalar SDE models Eqs. 1.1 and 1.2 arising in mathematical finance and a stochastic Lotka-Volterra (LV) competitive model Eq. 5.21 from ecology. Since the considered models evolve in the positive domain $D = (0, \infty)$, instead of the whole space $\mathbb{R}$, the convergence theory developed in Section 4 cannot be applied in this situation. In order to address such issues, we apply two particular schemes covered by Eq. 1.4 to approximate these specific models, which are capable of preserving positivity of the continuous models. More precisely, the drift-diffusion double implicit Milstein method with parameters $\theta = \eta = 1$ is utilized to approximate the Heston $\frac{3}{2}$-volatility model Eq. 1.1 and the stochastic LV competitive model Eq. 5.21, resulting in a recurrence of a quadratic equation with an explicit solution. And the semi-implicit Milstein method with a pair of parameters $\theta = 1, \eta = 0$ is used to solve the Ait-Sahalia interest rate model Eq. 1.2 in both a standard and a critical regime. Both schemes are able to preserve positivity of the underlying models and their mean-square convergence rates are carefully analyzed. With the aid of the previously obtained error bounds, we prove a first order of mean-square convergence for both schemes under mild assumptions for the first time, which fills the gap left by [21, 48]. Compared with existing relevant results for first order schemes, more relaxed conditions are put here. Specifically, the drift-diffusion double implicit Milstein scheme is shown to achieve a mean-square convergence rate of order one when used to solve the Heston $\frac{3}{2}$-volatility model Eq. 1.1 with model parameters obeying $\frac{\sigma^2}{\beta^2} \geq \frac{5}{2}$ (Theorem 5.2). Also, the semi-implicit Milstein method is proved to retain a mean-square convergence rate of order one, when solving the Ait-Sahalia interest rate model Eq. 1.2, for full model parameters in the standard regime $\kappa + 1 > 2\rho$ (Theorem 5.10) and for model parameters obeying $\frac{\sigma^2}{\alpha^2} \geq 2\kappa - \frac{3}{2}$ and $\frac{\alpha^2}{\sigma^2} > \frac{\kappa+1}{2\sqrt{2}}$ in the general critical case $\kappa + 1 = 2\rho$ (Theorem 5.13).

Recall that a kind of Lamperti-backward Euler method was proposed and analyzed in [44] for a class of scalar SDEs defined in a domain, covering the above two financial models. There a mean-square convergence rate of order one was proved for the scheme applied to the $\frac{3}{2}$-volatility model with parameters satisfying $\frac{\sigma^2}{\beta^2} \geq 5$ (see [44, Proposition 3.2]). Also, the scheme used to approximate the Ait-Sahalia interest
rate model owns a first mean-square convergence order for full model parameters in the case $\kappa + 1 > 2\rho$ and for parameters obeying $\frac{\alpha^2}{\sigma^2} > 5$ in a special critical case $\kappa = 2$, $\rho = 1.5$ (see Propositions 3.5, 3.6 from [44]). Unlike the Lamperti transformed scheme introduced in [44], we propose and analyze the implicit Milstein-type schemes applied to SDEs directly. From the above discussions, one can easily detect that our convergence results improve relevant ones in [44]. On the one hand, we prove the expected convergence rate for the $\frac{1}{2}$-volatility model on the condition $\frac{\alpha^2}{\rho^2} \geq \frac{5}{2}$, also improving the restriction $\frac{\alpha^2}{\rho^2} \geq 5$ required in [44]. On the other hand, our approach is able to treat the Ait-Sahalia model in the general critical case $\kappa + 1 = 2\rho$, with a first mean-square convergence order identified under conditions $\frac{\alpha^2}{\sigma^2} > 2\kappa - \frac{3}{2}$ and $\frac{\alpha^2}{\sigma^2} > \frac{\kappa + 1}{2\sqrt{2}}$, which is, as far as we know, missing in the literature. For the special critical case $\kappa = 2$, $\rho = 1.5$ studied in [44], the restriction $\frac{\alpha^2}{\sigma^2} > 5$ there is moderately relaxed to $\frac{\alpha^2}{\sigma^2} > \frac{5}{2}$ here.

To conclude, the main contributions of the article are summarized as follows: (i) a family of double implicit Milstein-type schemes is introduced for multi-dimension SDE systems with non-Lipschitz coefficients; (ii) a novel approach of the error analysis is developed to recover the mean-square convergence rate of order one for the schemes, which fills several gaps in the literature; (iii) the optimal mean-square convergence rate of the positivity preserving schemes applied to two financial models is obtained for the first time and more relaxed conditions are required, compared with existing relevant results for first order schemes in the literature. Therefore, this work can justify an efficient Multilevel Monte Carlo method [15] for SDEs with non-globally Lipschitz coefficients including the above models.

The remainder of this article is structured as follows. In the forthcoming section, a setting is formulated and a family of new Milstein-type schemes are introduced. Upper mean-square error bounds of the proposed schemes are then elaborated in Section 3. Equipped with the obtained error bounds, mean-square convergence rates of the schemes are analyzed in Section 4 for a general class of SDEs, under further globally polynomial growth conditions. Additionally, applications of the error bounds to two schemes for several SDE models in practice are examined in Section 5, with an optimal convergence rate revealed. Further, some numerical tests are provided to confirm the theoretical findings and a brief conclusion is made at the end of the article.

2 SDEs and the proposed schemes

Throughout this paper, we use $\mathbb{N}$ to denote the set of all positive integers and let $d, m \in \mathbb{N}, T \in (0, \infty)$ be given. Let $\| \cdot \|$ and $\langle \cdot , \cdot \rangle$ denote the Euclidean norm and the inner product of vectors in $\mathbb{R}^d$, respectively. Adopting the same notation as the vector norm, we denote $\| A \| := \sqrt{\text{trace}(A^T A)}$ as the trace norm of a matrix $A \in \mathbb{R}^{d \times m}$. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$, we use $\mathbb{E}$ to mean the expectation.
and \( L^r(\Omega; \mathbb{R}^d), r \geq 1 \), to denote the family of \( \mathbb{R}^d \)-valued random variables \( \xi \) satisfying \( \mathbb{E}[\|\xi\|^r] < \infty \). Let us consider the following SDEs of Itô type:

\[
\begin{aligned}
    dX_t &= f(X_t) \, dt + g(X_t) \, dW_t, \quad t \in (0, T], \\
    X_0 &= x_0,
\end{aligned}
\]

(2.1)

where \( f: \mathbb{R}^d \to \mathbb{R}^d \) is the drift coefficient function, and \( g: \mathbb{R}^d \to \mathbb{R}^{d \times m} \) is the diffusion coefficient function, frequently written as \( g = (g_{i,j})_{d \times m} = (g_1, g_2, \ldots, g_m) \) for \( g_{i,j}: \mathbb{R}^d \to \mathbb{R} \) and \( g_{j}: \mathbb{R}^d \to \mathbb{R}^d, i \in \{1, 2, \ldots, d\}, j \in \{1, 2, \ldots, m\} \). Moreover, \( W: [0, T] \times \Omega \to \mathbb{R}^m \) stands for the \( \mathbb{R}^m \)-valued standard Brownian motions with respect to \( \{\mathcal{F}_t\}_{t \in [0,T]} \) and the initial data \( X_0: \Omega \to \mathbb{R}^d \) is assumed to be \( \mathcal{F}_0 \)-measurable.

In general, the system of SDEs Eq. 2.1 does not have a closed-form solution. In order to approximate Eq. 2.1, we construct a uniform mesh on \([0, T]\) with \( h = \frac{T}{N} \) being the stepsize, for any \( N \in \mathbb{N} \). On the uniform mesh, we propose a family of double implicit Milstein methods with a pair of method parameters \((\theta, \eta)\), given by

\[
Y_{n+1} = Y_n + \theta f(Y_{n+1})h + (1 - \theta)f(Y_n)h + g(Y_n)\Delta W_n + \sum_{j_1, j_2=1}^{m} \mathcal{L}^{j_1} g_{j_2}(Y_n) I_{j_1, j_2}^{n, n+1}
\]

\[
+ \frac{\eta}{2} \sum_{j=1}^{m} \mathcal{L}^{j} g(Y_n)h - \frac{\eta}{2} \sum_{j=1}^{m} \mathcal{L}^{j} g(Y_{n+1})h, \quad Y_0 = X_0,
\]

(2.2)

where \( \theta, \eta \in [0, 1], \Delta W_n := W_{n+1} - W_n, n \in \{0, 1, 2, \ldots, N - 1\} \), and

\[
\mathcal{L}^{j_1} := \sum_{k=1}^{d} g_{k,j_1} \frac{\partial}{\partial x^k}, \quad I_{j_1, j_2}^{n, n+1} := \int_{I_n}^{I_{n+1}} \int_{I_n}^{I_{n+1}} W_{s_1} \, dW_{s_2}, \quad j_1, j_2 \in \{1, 2, \ldots, m\}.
\]

(2.3)

In the following we use \( \frac{\partial \phi}{\partial x} \) to denote the Jacobian matrix of the vector function \( \phi: \mathbb{R}^d \to \mathbb{R}^d \) and one can observe that, for \( g_{j_2}: \mathbb{R}^d \to \mathbb{R}^d, j_2 \in \{1, 2, \ldots, m\} \),

\[
\mathcal{L}^{j_1} g_{j_2}(x) = \sum_{k=1}^{d} g_{k,j_1} \frac{\partial g_{j_2}(x)}{\partial x^k} = \frac{\partial g_{j_2}(x)}{\partial x^{j_1}}(x) g_{j_1}(x), \quad x \in \mathbb{R}^d.
\]

(2.4)

By incorporating a pair of method parameters \( \theta, \eta \in [0, 1] \) into the drift and diffusion coefficients, the newly proposed schemes are implicitly defined when \( \theta + \eta \neq 0 \) and their well-posedness will be discussed later. Taking \( \eta = 0 \) in Eq. 2.2, the above double implicit Milstein methods Eq. 2.2 reduce to the classic \( \theta \) Milstein methods [30], which
are drift implicit and given by

\[ Y_{n+1} = Y_n + \theta f(Y_{n+1})h + (1 - \theta) f(Y_n)h + g(Y_n)\Delta W_n \]
\[ + \sum_{j_1, j_2=1}^{m} \mathcal{L}^{j_1} g_{j_2}(Y_n) I_{j_1, j_2}^{n, n+1}, \quad Y_0 = X_0. \]  

(2.5)

In general, a straightforward introduction of implicitness into approximations of the diffusion term containing random variables suffers from unbounded numerical approximations with positive probability, see [43, Chapter 1.3.4] for clarifications. When the diffusion coefficient \( g \) fulfills the so-called commutativity condition, namely,

\[ \mathcal{L}^{j_1} g_{j_2} = \mathcal{L}^{j_2} g_{j_1}, \quad j_1, j_2 \in \{1, ..., m\}, \]  

(2.6)

by recalling

\[ I_{j_1, j_2}^{n, n+1} + I_{j_2, j_1}^{n, n+1} = \Delta W_{n}^{j_1} \Delta W_{n}^{j_2}, \quad j_1, j_2 \in \{1, ..., m\}, j_1 \neq j_2 \]  

(2.7)

and

\[ I_{j}^{n, n+1} = \frac{1}{2} (|\Delta W_{n}^{j}|^2 - h), \quad j \in \{1, ..., m\}, \]  

(2.8)

one can recast the proposed double implicit Milstein method Eq. 2.2 as

\[ Y_{n+1} = Y_n + \theta f(Y_{n+1})h + (1 - \theta) f(Y_n)h + g(Y_n)\Delta W_n \]
\[ + \frac{1}{2} \sum_{j_1, j_2=1}^{m} \mathcal{L}^{j_1} g_{j_2}(Y_n) \Delta W_{n}^{j_1} \Delta W_{n}^{j_2} \]
\[ - \frac{(1-n)}{2} \sum_{j=1}^{m} \mathcal{L}^{j} g_{j}(Y_n)h - \frac{n}{2} \sum_{j=1}^{m} \mathcal{L}^{j} g_{j}(Y_{n+1})h, \quad Y_0 = X_0. \]  

(2.9)

Here an implicit approximation is introduced with an additional method parameter \( \eta \in [0, 1] \) only in the last term that does not contain any random variable. In [7], such schemes were applied to scalar linear SDEs with several multiplicative noise terms \( m > 1 \) and their mean-square stability properties were studied. In particular, the commutativity condition Eq. 2.6 is fulfilled when \( m = 1 \) and the newly proposed schemes Eq. 2.2 (or Eq. 2.9 equivalently) applied to the scalar SDEs \( d = m = 1 \) reduce to

\[ Y_{n+1} = Y_n + \theta f(Y_{n+1})h + (1 - \theta) f(Y_n)h + g(Y_n)\Delta W_n + \frac{1}{2} g'(Y_n)\Delta W_n^2 \]
\[ - \frac{(1-n)}{2} g'(Y_n)h - \frac{n}{2} g'(Y_{n+1})h, \quad Y_0 = X_0. \]  

(2.10)

Such schemes have been examined in [21], where the authors recovered the strong convergence rate only under globally Lipschitz conditions. Moreover, the authors used
Eq. 2.10 to solve the $3/2$-volatility model Eq. 1.1 and proved its strong convergence with no convergence rate revealed. Roughly speaking, the main difficulty of recovering the convergence rate is caused by the super-linearly growing diffusion coefficients of SDEs. In the literature, a lot of researchers [3–5, 13, 18, 23, 24, 32, 34, 38–40, 47, 49, 50, 52, 56] attempt to analyze strong approximations of SDEs with super-linearly growing diffusion coefficients. However, the strong convergence rate of the classical $\theta$-Milstein method in the regime of possibly super-linearly growing diffusion coefficients is, up to the best of our knowledge, still an open problem. The present article aims to establish a mean-square convergence theory for the generalized $\theta$-Milstein schemes Eq. 1.4 within a general framework, which fills several gaps in the literature and provides improved convergence results for computational finance. Finally, it is worthwhile to emphasize that the newly proposed double implicit Milstein methods Eq. 2.2 do not require the commutativity condition Eq. 2.6 and thus work for non-commutative noise driven SDEs.

3 Upper mean-square error bounds for the schemes

The aim of the present section is to derive upper mean-square error bounds of the implicit Milstein type methods for SDEs taking values in a domain $D \subset \mathbb{R}^d$, which will help us to easily analyze the mean-square convergence rate of the schemes later. To this end, we set up a general framework by making two key assumptions as follows.

**Assumption 3.1** [Generalized monotonicity conditions in a domain] Assume that the diffusion coefficients $g_j : D \to \mathbb{R}^d$, $j \in \{1, 2, \ldots, m\}$ are differentiable in a domain $D \subset \mathbb{R}^d$ and that the drift coefficient $f : D \to \mathbb{R}^d$ and the diffusion coefficient $g = (g_1, g_2, \ldots, g_m) : D \to \mathbb{R}^{d \times m}$ of SDEs Eq. 2.1 satisfy certain monotonicity conditions in $D \subset \mathbb{R}^d$. More accurately, for method parameters $\theta, \eta \in [0, 1]$ there exist constants $q \in (2, \infty)$, $\xi \in (1, \infty)$, $L_1, L_2 \in [0, \infty)$ and $h_0 \in (0, T]$ such that, $\forall x, y \in D$, $h = \frac{T}{N} \in (0, h_0)$,

$$2 \langle x - y, f(x) - f(y) \rangle + (q - 1) \|g(x) - g(y)\|^2 + \frac{q}{2} h \sum_{j_1, j_2=1}^m \|\mathcal{L}^{j_1} g_{j_2}(x) - \mathcal{L}^{j_1} g_{j_2}(y)\|^2 \leq L_1 \|x - y\|^2 \quad (3.1)$$

$$+ \eta h \left( \sum_{j=1}^m [\mathcal{L}^{j_1} g_j(x) - \mathcal{L}^{j_1} g_j(y)], f(x) - f(y) \right) + (1 - 2\theta) h \|f(x) - f(y)\|^2 \leq L_1 \|x - y\|^2, \quad (3.2)$$

$$\left\langle x - y, \theta [f(x) - f(y)] - \frac{q}{2} \sum_{j=1}^m [\mathcal{L}^{j_1} g_j(x) - \mathcal{L}^{j_1} g_j(y)] \right\rangle \leq L_2 \|x - y\|^2. \quad (3.3)$$

Conditions in Assumption 3.1 are crucial to the error analysis for the proposed schemes and are called generalized monotonicity conditions in a domain $D$. When $\eta = 0$, the implicit methods Eq. 2.2 reduce to the classic $\theta$ Milstein methods Eq. 2.5 and the above two conditions are satisfied as $\theta \in \left[ \frac{1}{2}, 1 \right]$, $g$, $\mathcal{L}^{j_1} g_{j_2}$, $j_1, j_2 \in \{1, 2, \ldots, m\}$.
satisfy the globally Lipschitz condition

\[ \|g(x) - g(y)\|^2 + \sum_{j_1, j_2 = 1}^m \| \mathcal{L}^{j_1} g_{j_2}(x) - \mathcal{L}^{j_1} g_{j_2}(y) \|^2 \leq L \| x - y \|^2, \quad \forall x, y \in D, \]  

(3.3)

and \( f \) obeys the monotonicity condition

\[ \langle x - y, f(x) - f(y) \rangle \leq L \| x - y \|^2, \quad \forall x, y \in D. \]  

(3.4)

Such a global monotonicity condition Eq. 3.4 is frequently used in the literature, to ensure the well-posedness of drift-implicit methods and to derive their strong convergence rates. When the diffusion \( g \) is not globally Lipschitz, which is the case for the aforementioned models Eqs. 1.1, 1.2, things become much more involved. As one can see later, Assumptions 3.6 and 3.8 below provide sufficient conditions that imply Assumption 3.1 and allow for non-globally Lipschitz diffusion coefficient. Since Assumption 3.1 alone does not suffice to guarantee the well-posedness of SDEs and the considered schemes in the domain \( D \), we additionally require the following assumptions.

**Assumption 3.2** (Well-posedness of SDEs and schemes) Assume \( \sum_{j_1, j_2 = 1}^m \mathbb{E}[\| \mathcal{L}^{j_1} g_{j_2}(X_0) \|^2] < \infty \) and SDE Eq. 2.1 possesses a unique \( \{ \mathcal{F}_t \}_{t \in [0, T]} \)-adapted \( D \)-valued global solution with continuous sample paths, \( X: [0, T] \times \Omega \to D \subseteq \mathbb{R}^d \), satisfying \( \sup_{s \in [0, T]} \mathbb{E}[\| X_s \|^2] + \sup_{s \in [0, T]} \mathbb{E}[\| f(X_s) \|^2] < \infty \). Moreover, for \( \theta, \eta \in [0, 1] \) specified in Assumption 3.1 suppose the proposed scheme Eq. 2.2 admits a unique \( \{ \mathcal{F}_{n \eta} \}_{n=0}^N \)-adapted solution \( \{ Y_n \}_{n=0}^N \), taking values in the domain \( D \).

We mention that Assumption 3.2 is necessary but not strict. For example, by taking \( D \) to be the whole space \( \mathbb{R}^d \), i.e., \( D = \mathbb{R}^d \), Assumptions 3.1 and Assumption 4.1 below together suffice to imply Assumption 3.2. In addition, some models in practice taking values in \( D = (0, \infty) \) are also given in Section 5 to satisfy the above assumptions. Under the above two assumptions, we are able to formulate the following main result of this section that offers upper mean-square error bounds for the underlying schemes.

**Theorem 3.3** (Upper mean-square error bounds) Let Assumptions 3.1, 3.2 hold with \( \theta \in [\frac{1}{2}, 1] \) and \( 2L_2 \eta \leq \nu \) for some \( \nu \in (0, 1) \). Let \( \{ X_t \}_{t \in [0, T]} \) and \( \{ Y_n \}_{0 \leq n \leq N} \) be solutions to Eqs. 2.1 and 2.2, respectively. Then there exists a uniform constant \( C \) such that, for any \( n \in \{1, 2, ..., N\}, N \in \mathbb{N} \),

\[ \mathbb{E}[\| X_{t_n} - Y_n \|^2] \leq C \left( \sum_{i=1}^n \mathbb{E}[\| R_i \|^2] + \frac{1}{\eta} \sum_{i=1}^n \mathbb{E}[\| \mathbb{E}(R_i | \mathcal{F}_{i-1}) \|^2] \right), \]  

(3.5)
where we denote

\[ R_i := \theta \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_i}) \, ds + (1 - \theta) \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_{i-1}}) \, ds + \int_{t_{i-1}}^{t_i} g(X_s) - g(X_{t_{i-1}}) \, dW_s \]

\[ - \sum_{j_1, j_2 = 1}^{m} \mathcal{L}^{j_1} g_{j_2}(X_{t_{i-1}}) I_{j_1, j_2}^{t_{i-1}, t_{i+1}} + \frac{\eta h}{4} \sum_{j = 1}^{m} \left[ \mathcal{L}^j g_j(X_{t_i}) - \mathcal{L}^j g_j(X_{t_{i-1}}) \right], \quad i \in \{1, 2, \ldots, n\} \quad (3.6) \]

Throughout this paper, by \( C \) we denote a generic deterministic positive constant, which might vary for each appearance but is independent of the time stepsize \( h = \frac{T}{N} > 0, \quad N \in \mathbb{N} \). It is interesting to observe that the term \( R_i, i \in \{1, 2, \ldots, N\}, \quad N \in \mathbb{N} \) defined by Eq. 3.6 only gets involved with the exact solutions to SDEs. Such error bounds can be used to analyze mean-square convergence rates of the schemes without relying on a priori high-order moment estimates of numerical approximations. The proof of Theorem 3.3 is postponed, which requires the following two lemmas.

**Lemma 3.4** Let Assumptions 3.1, 3.2 hold and let \( R_i, i \in \{1, 2, \ldots, N\}, \quad N \in \mathbb{N} \) be defined by Eq. 3.6. Then it holds

\[ \mathbb{E}[\|R_i\|^2] < \infty, \quad \forall i \in \{1, 2, \ldots, N\}, \quad N \in \mathbb{N}. \quad (3.7) \]

**Proof** of Lemma 3.4. In light of Eq. 3.1, one can show, \( \forall x, y \in D \),

\[ (q - 1)\|g(x) - g(y)\|^2 + \frac{\eta h}{4} \sum_{j_1, j_2 = 1}^{m} \|\mathcal{L}^{j_1} g_{j_2}(x) - \mathcal{L}^{j_1} g_{j_2}(y)\|^2 \]

\[ \leq L_1 \|x - y\|^2 - 2\{x - y, \quad f(x) - f(y)\} - (1 - 2\theta)h\|f(x) - f(y)\|^2 \]

\[ -\eta h \left( \sum_{j = 1}^{m} \left[ \mathcal{L}^j g_j(x) - \mathcal{L}^j g_j(y) \right], \quad f(x) - f(y) \right) \]

\[ \leq (L_1 + 1)\|x - y\|^2 + \frac{\eta h}{4} \sum_{j = 1}^{m} \|\mathcal{L}^j g_j(x) - \mathcal{L}^j g_j(y)\|^2 \]

\[ + \frac{\theta + (m + \theta)h}{\epsilon} \|f(x) - f(y)\|^2, \]

and thus, \( \forall x, y \in D \),

\[ (q - 1)\|g(x) - g(y)\|^2 + \frac{\eta h}{4} \sum_{j_1, j_2 = 1}^{m} \|\mathcal{L}^{j_1} g_{j_2}(x) - \mathcal{L}^{j_1} g_{j_2}(y)\|^2 \]

\[ \leq (L_1 + 1)\|x - y\|^2 + \frac{\theta + (m + \theta)h}{\epsilon} \|f(x) - f(y)\|^2. \quad (3.9) \]
Combining this with Assumption 3.2 guarantees, for any $i \in \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$ and $s \in [t_{i-1}, t_i]$,

\begin{align}
\mathbb{E}[\|g(X_s) - g(X_{t_{i-1}})\|^2] &< \infty, \tag{3.10} \\
& \quad +2h \sum_{j_1, j_2=1}^{m} \mathbb{E}[\|\mathcal{L}^{j_1} g_{j_2}(X_{t_{i-1}}) - \mathcal{L}^{j_1} g_{j_2}(X_0)\|^2] < \infty. \tag{3.11}
\end{align}

This in turn implies, for any $i \in \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$ and $s \in [t_{i-1}, t_i]$,

\begin{align}
& \mathbb{E}\left[\left\|\int_{t_{i-1}}^{t_i} g(X_s) - g(X_{t_{i-1}}) \, dW_s\right\|^2\right] = \int_{t_{i-1}}^{t_i} \mathbb{E}[\|g(X_s) - g(X_{t_{i-1}})\|^2] \, ds < \infty, \\
& \mathbb{E}\left[\left\|\sum_{j_1, j_2=1}^{m} \mathcal{L}^{j_1} g_{j_2}(X_{t_{i-1}})I_{j_1, j_2}^{h,k+1}\right\|^2\right] = \frac{h^2}{2} \sum_{j_1, j_2=1}^{m} \mathbb{E}[\|\mathcal{L}^{j_1} g_{j_2}(X_{t_{i-1}})\|^2] < \infty, \\
& \mathbb{E}\left[\left\|\frac{\eta}{2} \sum_{j=1}^{m} \mathcal{L}^j g(X_{t_i})h - \frac{\eta}{2} \sum_{j=1}^{m} \mathcal{L}^j g(X_{t_{i-1}})h\right\|^2\right] < \infty. \tag{3.12}
\end{align}

The desired assertion follows, by taking Eq. 3.12 and the assumption $\sup_{s \in [0, T]} \mathbb{E}[\|f(X_s)\|^2] < \infty$ into account. \hfill \square

Based on the boundedness of $\mathbb{E}[\|R_i\|^2]$, $i \in \{1, 2, \ldots, N\}$, one can arrive at the subsequent moment bounds.

**Lemma 3.5** Let $\theta \in (0, 1]$, $\eta \in [0, 1]$, $2L_2h \leq \nu$ for some $\nu \in (0, 1)$ and let Assumptions 3.1, 3.2 hold. Then it holds for all $k \in \{0, 1, 2, \ldots, N\}$, $N \in \mathbb{N}$ that

\begin{align}
\mathbb{E}\left[\left\|e_k - \theta \Delta f_k^{X,Y} h + \frac{\eta}{2} \sum_{j=1}^{m} \Delta (\mathcal{L}^j g_k)^{X,Y} h\right\|^2\right] < \infty, \quad & \mathbb{E}[\|e_k\|^2] < \infty, \nonumber \\
\mathbb{E}[\|\Delta f_k^{X,Y}\|^2] < \infty, \quad & \mathbb{E}[\|\Delta g_k^{X,Y}\|^2] < \infty, \quad \sum_{j_1, j_2=1}^{m} \mathbb{E}[\|\Delta (\mathcal{L}^{j_1} g_{j_2})^{X,Y}\|^2] < \infty. \tag{3.13}
\end{align}

where for any $k \in \{0, 1, \ldots, N\}$, $j_1, j_2 \in \{1, 2, \ldots, m\}$ we denote

\begin{align}
e_k & := X_{t_k} - Y_k, \quad \Delta f_k^{X,Y} := f(X_{t_k}) - f(Y_k), \quad \Delta g_k^{X,Y} := g(X_{t_k}) - g(Y_k), \\
\Delta (\mathcal{L}^{j_1} g_{j_2})^{X,Y} & := \mathcal{L}^{j_1} g_{j_2}(X_{t_k}) - \mathcal{L}^{j_1} g_{j_2}(Y_k). \tag{3.14}
\end{align}
Proof of Lemma 3.5. We first note that, for any $k \in \{0, 1, \ldots, N-1\}$, $j_1, j_2 \in \{1, 2, \ldots, m\}$,

$$X_{t_{k+1}} = X_{t_k} + \int_{t_k}^{t_{k+1}} f(X_s) \, ds + \int_{t_k}^{t_{k+1}} g(X_s) \, dW_s$$

$$= X_{t_k} + \theta f(X_{t_{k+1}}) h + (1 - \theta) f(X_{t_k}) h + g(X_{t_k}) \Delta W_k$$

$$+ \sum_{j_1, j_2=1}^m \mathcal{L}^j \left( g_{j_1}(X_{t_k}) I_{j_1,j_2}^{t_{k}, t_{k+1}} + \frac{n}{2} \sum_{j=1}^m \mathcal{L}^j g_j(X_{t_k}) h - \frac{n}{2} \sum_{j=1}^m \mathcal{L}^j g_j(X_{t_{k+1}}) h + R_{k+1} \right). \tag{3.15}$$

where $R_{k+1}$ is defined by Eq. 3.6. Using the short-hand notation Eq. 3.14, we subtract Eq. 2.2 from Eq. 3.15 to get

$$e_{k+1} = e_k + \theta \Delta f_{k+1} X_{t_k} h + (1 - \theta) \Delta f_k X_{t_k} h + \Delta g_{k} X_{t_k} \Delta W_k + \sum_{j_1, j_2=1}^m \Delta (\mathcal{L}^j g_{j_2})_{k}^{X,Y} I_{j_1,j_2}^{t_{k}, t_{k+1}}$$

$$+ \frac{n}{2} \sum_{j=1}^m \Delta (\mathcal{L}^j g_{j})_{k}^{X,Y} h - \frac{n}{2} \sum_{j=1}^m \Delta (\mathcal{L}^j g_{j})_{k+1}^{X,Y} h + R_{k+1}, \quad k \in \{0, 1, \ldots, N-1\}. \tag{3.16}$$

Denoting further

$$\mathcal{J}_{k}^{X,Y} := e_k - \theta \Delta f_{k} X_{t_k} h + \frac{n}{2} \sum_{j=1}^m \Delta (\mathcal{L}^j g_{j})_{k}^{X,Y} h, \quad k \in \{0, 1, \ldots, N\}, \tag{3.17}$$

one can recast Eq. 3.16 as

$$\mathcal{J}_{k+1}^{X,Y} = \mathcal{J}_{k}^{X,Y} + \Delta f_{k+1} X_{t_k} h + \Delta g_{k} X_{t_k} \Delta W_k$$

$$+ \sum_{j_1, j_2=1}^m \Delta (\mathcal{L}^j g_{j_2})_{k}^{X,Y} I_{j_1,j_2}^{t_{k}, t_{k+1}} + R_{k+1}. \tag{3.18}$$

Squaring both sides of the above equality yields

$$\| \mathcal{J}_{k+1}^{X,Y} \|^2 = \| \mathcal{J}_{k}^{X,Y} + \Delta f_{k+1} X_{t_k} h + \Delta g_{k} X_{t_k} \Delta W_k + \sum_{j_1, j_2=1}^m \Delta (\mathcal{L}^j g_{j_2})_{k}^{X,Y} I_{j_1,j_2}^{t_{k}, t_{k+1}} + R_{k+1} \|^2$$

$$= \| \mathcal{J}_{k}^{X,Y} \|^2 + h^2 \| \Delta f_{k+1} \|^2 + \| \Delta g_{k} \Delta W_k \|^2.$$
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\[ + \left\| \sum_{j_1, j_2=1}^{m} \Delta(L^{j_1}g_{j_2})^X_k Y^k I^{h,k+1}_{j_1,j_2} \right\|^2 + \left\| R_{k+1} \right\|^2 \\
+ 2h \langle \mathcal{J}^X_k, \Delta f^X_k \rangle + 2(\mathcal{J}^X_k, \Delta g^X_k \Delta W_k) \\
+ 2 \left\{ \mathcal{J}^X_k, \sum_{j_1, j_2=1}^{m} \Delta(L^{j_1}g_{j_2})^X_k Y^k I^{h,k+1}_{j_1,j_2} \right\} \\
+ 2(\mathcal{J}^X_k, R_{k+1}) + 2h(\Delta f^X_k, \Delta g^X_k \Delta W_k) \\
+ 2h \left\{ \Delta f^X_k, R_{k+1} \right\} + 2 \left\{ \Delta g^X_k \Delta W_k, \sum_{j_1, j_2=1}^{m} \Delta(L^{j_1}g_{j_2})^X_k Y^k I^{h,k+1}_{j_1,j_2} \right\} \\
+ 2(\Delta g^X_k \Delta W_k, R_{k+1}) + 2 \left\{ \sum_{j_1, j_2=1}^{m} \Delta(L^{j_1}g_{j_2})^X_k Y^k I^{h,k+1}_{j_1,j_2}, R_{k+1} \right\}. \tag{3.19} \]

With this at hand, we first prove \( \mathbb{E}[\|\mathcal{J}^X_k\|^2] < \infty \) for all \( k \in \{0, 1, \ldots, N\} \) based on an induction argument. Noting that \( X_0 = Y_0 \) we thus have \( \mathbb{E}[\|\mathcal{J}^X_0\|^2] = 0 \). We assume \( \mathbb{E}[\|\mathcal{J}^X_k\|^2] < \infty \) for some \( k \in \{0, 1, \ldots, N-1\} \), which together with Eq. 3.2 implies

\[ \infty > \mathbb{E}[\|\mathcal{J}^X_k\|^2] = \mathbb{E}\left[ e_k - \theta \Delta f^X_k h + \frac{h}{2} \sum_{j=1}^{m} \Delta(L^j g_j(X)_k)^X_k \right] \\
= \mathbb{E}[\|e_k\|^2] + h^2 \mathbb{E}\left[ \frac{1}{2} \sum_{j=1}^{m} \Delta(L^j g_j(X)_k)^X_k - \theta \Delta f^X_k \right] \\
- 2\theta h \mathbb{E}[e_k, \Delta f^X_k] + \eta h \mathbb{E}\left[ e_k, \sum_{j=1}^{m} \Delta(L^j g_j(X)_k)^X_k \right] \geq (1 - 2hL_2) \mathbb{E}[\|e_k\|^2] + h^2 \mathbb{E}\left[ \frac{1}{2} \sum_{j=1}^{m} \Delta(L^j g_j(X)_k)^X_k - \theta \Delta f^X_k \right]^2. \tag{3.20} \]

Therefore, for \( 2hL_2 \leq \nu < 1, \theta \in (0, 1) \) and for some \( k \in \{0, 1, 2, \ldots, N-1\} \) it holds

\[ \mathbb{E}[\|e_k\|^2] < \infty, \quad \mathbb{E}\left[ \frac{1}{2} \sum_{j=1}^{m} \Delta(L^j g_j(X)_k)^X_k - \theta \Delta f^X_k \right]^2 \leq \infty. \tag{3.21} \]
This along with the generalized monotonicity condition Eq. 3.1 shows, for some \( k \in \{0, 1, 2, \ldots, N - 1\}, \)

\[
(q - 1) \mathbb{E}[\| \Delta g_k^{X,Y} \|^2] + \frac{2}{m} h \sum_{j_1,j_2=1}^{m} \mathbb{E}[\| \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} \|^2]
\leq L_1 \mathbb{E}[\| e_k \|^2] - 2 \mathbb{E}[\langle \mathcal{J}_k^{X,Y}, \Delta f_k^{X,Y} \rangle] - h \mathbb{E}[\| \Delta f_k^{X,Y} \|^2]
\leq L_1 \mathbb{E}[\| e_k \|^2] + \frac{1}{m} \mathbb{E}[\| \mathcal{J}_k^{X,Y} \|^2] < \infty.
\] (3.22)

In view of Eqs. 3.17, 3.21, 3.22 and the assumption \( \theta > 0 \), one can easily see

\[
\mathbb{E}[\| \Delta g_k^{X,Y} \|^2] < \infty, \sum_{j_1,j_2=1}^{m} \mathbb{E}[\| \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} \|^2] < \infty, \mathbb{E}[\| \Delta f_k^{X,Y} \|^2] < \infty
\] (3.23)

for some \( k \in \{0, 1, 2, \ldots, N - 1\} \). These bounded moments suffice to ensure, for some \( k \in \{0, 1, 2, \ldots, N - 1\}, \)

\[
\mathbb{E}[\| \sum_{j_1,j_2=1}^{m} \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} I_{j_1,j_2}^{l_1,l_2} \|^2] = \frac{h^2}{2} \sum_{j_1,j_2=1}^{m} \mathbb{E}[\| \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} \|^2] < \infty,
\]

\[
\mathbb{E}[\| \Delta g_k^{X,Y} \Delta W_k \|^2] = h \mathbb{E}[\| \Delta g_k^{X,Y} \|^2] < \infty,
\] (3.24)

and

\[
\mathbb{E}\left[\langle \mathcal{J}_k^{X,Y}, \sum_{j_1,j_2=1}^{m} \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} I_{j_1,j_2}^{l_1,l_2+1} \rangle\right] = 0,
\]

\[
\mathbb{E}\left[\langle \Delta f_k^{X,Y}, \sum_{j_1,j_2=1}^{m} \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} I_{j_1,j_2}^{l_1,l_2+1} \rangle\right] = 0, \quad \mathbb{E}[\langle \Delta f_k^{X,Y}, \Delta g_k^{X,Y} \Delta W_k \rangle] = 0 \quad (3.25)
\]

\[
\mathbb{E}\left[\langle \mathcal{J}_k^{X,Y}, \sum_{j_1,j_2=1}^{m} \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} I_{j_1,j_2}^{l_1,l_2+1} \rangle\right] = 0, \quad \mathbb{E}[\langle \mathcal{J}_k^{X,Y}, \Delta g_k^{X,Y} \Delta W_k \rangle] = 0.
\]

Equipped with these estimates and taking expectations on both sides of Eq. 3.19, one can derive

\[
\mathbb{E}[\| \mathcal{J}_{k+1}^{X,Y} \|^2] = \mathbb{E}[\| \mathcal{J}_k^{X,Y} \|^2] + h^2 \mathbb{E}[\| \Delta f_k^{X,Y} \|^2] + h \mathbb{E}[\| \Delta g_k^{X,Y} \|^2]
\]

\[
+ \frac{h^2}{2} \sum_{j_1,j_2=1}^{m} \mathbb{E}[\| \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} \|^2] + \mathbb{E}[\| R_{k+1} \|^2] + 2h \mathbb{E}[\langle \mathcal{J}_k^{X,Y}, \Delta f_k^{X,Y} \rangle]
\]

\[
+2 \mathbb{E}[\langle \mathcal{J}_k^{X,Y}, R_{k+1} \rangle] + 2h \mathbb{E}[\langle \Delta f_k^{X,Y}, R_{k+1} \rangle] + 2 \mathbb{E}[\langle \Delta g_k^{X,Y} \Delta W_k, R_{k+1} \rangle] \quad (3.26)
\]

\[
+2 \mathbb{E}\left[\sum_{j_1,j_2=1}^{m} \Delta (\mathcal{L}^{j_1} g_{j_2})_k^{X,Y} I_{j_1,j_2}^{l_1,l_2+1}, R_{k+1}\right].
\]
Owing to the assumption that $\mathbb{E}[\| J^X_k \|^2] < \infty$ for some $k \in \{0, 1, 2, ..., N - 1\}$ and its consequence Eq. 3.23 as well as Eq. 3.7, one can use the Cauchy-Schwarz inequality to infer

$$\mathbb{E}[\| J_{k+1}^X \|^2] \leq 3\mathbb{E}[\| J_k^X \|^2] + 3h^2\mathbb{E}[\| \Delta f_k^X \|^2] + 2h\mathbb{E}[\| \Delta g_k^X \|^2]$$

$$+ h^2 \sum_{j_1, j_2=1}^m \mathbb{E}[\| (L^{j_1} g^{j_2})_k^X \|^2] + 5\mathbb{E}[\| R_{k+1} \|^2] < \infty. \quad (3.27)$$

Based on the induction argument, the assertion $\mathbb{E}[\| J_k^X \|^2] < \infty$ holds for all $k \in \{0, 1, 2, \ldots, N\}$. Following the same lines as used in Eqs. 3.20-3.23, the boundedness of $\mathbb{E}[\| J_n^X \|^2]$ for all $n \in \{0, 1, 2, \ldots, N\}$ ensures that

$$\mathbb{E}[\| e_k \|^2] < \infty, \quad \mathbb{E}[\| \Delta g_k^X \|^2] < \infty, \quad \sum_{j_1, j_2=1}^m \mathbb{E}[\| (L^{j_1} g^{j_2})_k^X \|^2]$$

$$< \infty, \quad \mathbb{E}[\| \Delta f_k^X \|^2] < \infty \quad (3.28)$$

hold for all $k \in \{0, 1, 2, \ldots, N\}$. The desired assertion are thus justified. \hfill \Box

Before proceeding further, we point out that the moment bounds in Eq. 3.13, depending on $N$, are not proved to be uniformly bounded with respect to $N$. This means that the moment bounds might depend on the step number $N$. However, such moment bounds are enough for the subsequent error analysis, which does not rely on the precise uniform moment bounds of the numerical approximations. Now we are well prepared to prove Theorem 3.3.

**Proof** of Theorem 3.3. Recalling $J_k^X := e_k - \Delta f_k^X h + \frac{\eta}{2} \sum_{j=1}^m \Delta (L^j g^j)_k^X h$ and using its consequence $\Delta f_k^X h = \frac{\eta}{2} (e_k - J_k^X + \frac{\eta}{2} \sum_{j=1}^m \Delta (L^j g^j)_k^X h)$ and Eq. 3.13, we derive from Eq. 3.26 that, for any $k \in \{0, 1, 2, \ldots, N - 1\}$,

$$\mathbb{E}[\| J_{k+1}^X \|^2] = \mathbb{E}[\| J_k^X \|^2] + (1 - 2\eta)h^2\mathbb{E}[\| \Delta f_k^X \|^2] + h\mathbb{E}[\| \Delta g_k^X \|^2]$$

$$+ \frac{h^2}{2} \sum_{j_1, j_2=1}^m \mathbb{E}[\| (L^{j_1} g^{j_2})_k^X \|^2] + \mathbb{E}[\| R_{k+1} \|^2] + 2h\mathbb{E}[\langle e_k, \Delta f_k^X \rangle]$$

$$+ \eta h^2\mathbb{E}[\langle \sum_{j=1}^m (L^j g^j)_k^X, \Delta f_k^X \rangle] + 2\mathbb{E}[\langle J_k^X, R_{k+1} \rangle] + \frac{\eta}{2} \mathbb{E}[\langle e_k, R_{k+1} \rangle]$$

$$- \frac{2}{\eta} \mathbb{E}[\langle J_k^X, R_{k+1} \rangle] + \eta \mathbb{E}[\langle \sum_{j=1}^m (L^j g^j)_k^X, R_{k+1} \rangle]$$

$$+ 2\mathbb{E}[\langle \Delta g_k^X, \Delta W_k \rangle, R_{k+1}] + 2\mathbb{E}[\left\langle \sum_{j_1, j_2=1}^m (L^{j_1} g^{j_2})_k^X I_{j_1, j_2}^{k, N}, R_{k+1} \right\rangle]$$
\[
\begin{align*}
&= \mathbb{E}[\|\mathcal{J}_k^{X,Y}\|^2] + (1 - 2\theta)h^2\mathbb{E}[\|\Delta f_k^{X,Y}\|^2] + h\mathbb{E}[\|\Delta g_k^{X,Y}\|^2] \\
&\quad + \frac{h^2}{2} \sum_{j_1, j_2=1}^{m} \mathbb{E}\left[\|\Delta(\mathcal{L}^{j_1} g_{j_2})^{X,Y}\|^2\right] + \mathbb{E}[\|R_{k+1}\|^2] + 2h\mathbb{E}[\langle e_k, \Delta f_k^{X,Y} \rangle] \\
&\quad + \eta h^2 \mathbb{E}\left[\sum_{j=1}^{m} \Delta(\mathcal{L}^j g_k)^{X,Y}, \Delta f_k^{X,Y}\right] + \frac{2\theta}{\theta} E[\mathbb{E}(\mathcal{J}_k^{X,Y}, \mathbb{E}(R_{k+1} | \mathcal{F}_k))] \\
&\quad + \frac{2}{\theta} \mathbb{E}[\langle e_k, \mathbb{E}(R_{k+1} | \mathcal{F}_k) \rangle] + \frac{nh}{\theta} \mathbb{E}\left[\sum_{j=1}^{m} \Delta(\mathcal{L}^j g_k)^{X,Y}, R_{k+1}\right] \\
&\quad + 2E[\langle \Delta g_k^{X,Y} \Delta W_k, R_{k+1}\rangle] + 2\mathbb{E}\left[\sum_{j_1, j_2=1}^{m} \Delta(\mathcal{L}^{j_1} g_{j_2})^{X,Y} I_{j_1, j_2}^{k, k+1}, R_{k+1}\right]. \tag{3.29}
\end{align*}
\]

Using the Cauchy-Schwarz inequality and the Young inequality gives

\[
\begin{align*}
\frac{2\theta}{\theta} E[\mathbb{E}(\mathcal{J}_k^{X,Y}, \mathbb{E}(R_{k+1} | \mathcal{F}_k))] &\leq h\mathbb{E}[\|\mathcal{J}_k^{X,Y}\|^2] + \frac{(\theta - 1)^2}{\theta} h^2 \mathbb{E}[\|\mathbb{E}(R_{k+1} | \mathcal{F}_k)\|^2], \\
\frac{2}{\theta} \mathbb{E}[\langle e_k, \mathbb{E}(R_{k+1} | \mathcal{F}_k) \rangle] &\leq h\mathbb{E}[\|e_k\|^2] + \frac{1}{\theta} \mathbb{E}[\|\mathbb{E}(R_{k+1} | \mathcal{F}_k)\|^2], \\
\frac{nh}{\theta} \mathbb{E}\left[\sum_{j=1}^{m} \Delta(\mathcal{L}^j g_k)^{X,Y}, R_{k+1}\right] &\leq \frac{\theta - 1}{\theta} h^2 \sum_{j=1}^{m} \mathbb{E}[\|\Delta(\mathcal{L}^j g_k)^{X,Y}\|^2] \\
&\quad + \frac{n\theta}{(\theta - 1)^2} \mathbb{E}[\|R_{k+1}\|^2], \\
2E[\langle \Delta g_k^{X,Y} \Delta W_k, R_{k+1}\rangle] &\leq (q - 2) h\mathbb{E}[\|\Delta g_k^{X,Y}\|^2] + \frac{1}{q^2} \mathbb{E}[\|R_{k+1}\|^2], \\
2\mathbb{E}\left[\sum_{j_1, j_2=1}^{m} \Delta(\mathcal{L}^{j_1} g_{j_2})^{X,Y} I_{j_1, j_2}^{k, k+1}, R_{k+1}\right] &\leq \frac{\theta - 1}{\theta} h^2 \sum_{j_1, j_2=1}^{m} \mathbb{E}[\|\Delta(\mathcal{L}^{j_1} g_{j_2})^{X,Y}\|^2] + \frac{2}{q^2} \mathbb{E}[\|R_{k+1}\|^2]. \tag{3.30}
\end{align*}
\]

Taking these estimates into consideration and recalling Eq. 3.1 yield

\[
\begin{align*}
\mathbb{E}[\|\mathcal{J}_k^{X,Y}\|^2] &\leq (1 + h)\mathbb{E}[\|\mathcal{J}_k^{X,Y}\|^2] + (1 - 2\theta)h^2\mathbb{E}[\|\Delta f_k^{X,Y}\|^2] + h(q - 1)\mathbb{E}[\|\Delta g_k^{X,Y}\|^2] \\
&\quad + \frac{\theta h^2}{2} \sum_{j_1, j_2=1}^{m} \mathbb{E}[\|\Delta(\mathcal{L}^{j_1} g_{j_2})^{X,Y}\|^2] + \eta h^2 \mathbb{E}\left[\sum_{j=1}^{m} \Delta(\mathcal{L}^j g_k)^{X,Y}, \Delta f_k^{X,Y}\right] \\
&\quad + \frac{(q - 1)}{q^2} \mathbb{E}[\|e_k\|^2] + \frac{\theta^2 - 2\theta + 2}{\theta} \mathbb{E}[\|\mathbb{E}(R_{k+1} | \mathcal{F}_k)\|^2] + h\mathbb{E}[\langle e_k, \Delta f_k^{X,Y} \rangle] \\
&\quad + \frac{1}{\theta} \mathbb{E}[\|e_k\|^2] + \frac{\theta^2 - 2\theta + 2}{\theta^2} \mathbb{E}[\|\mathbb{E}(R_{k+1} | \mathcal{F}_k)\|^2]. \tag{3.31}
\end{align*}
\]
By iteration and observing $J_0^{X,Y} = e_0 - \theta \Delta J_0^{X,Y} h + \frac{\eta}{2} h \sum_{j=1}^{m} \Delta (L^j g_j)_0^{X,Y} = 0$ we deduce

$$
\mathbb{E}[\|J_{k+1}^{X,Y}\|^2] \leq (1 + L_1)h \sum_{i=0}^{k} (1 + h)^{(k-i)} \mathbb{E}[\|e_i\|^2]
+ \left( \frac{q-1}{q-2} + \frac{\eta^2 m}{(q-1) \sigma^2} + \frac{2}{q-1} \right) \sum_{i=0}^{k} (1 + h)^{(k-i)} \mathbb{E}[\|R_{i+1}\|^2]
+ \frac{\sigma^2 - 2 \eta^2 + 2}{\sigma^2 h} \sum_{i=0}^{k} (1 + h)^{(k-i)} \mathbb{E}[\|R_{i+1}\|_{\mathcal{F}_i}]^2.
$$

Additionally, the assumption Eq. 3.2 ensures

$$
\mathbb{E}[\|J_{k+1}^{X,Y}\|^2] = \mathbb{E}\left[ \left\| e_{k+1} - \theta \Delta J_{k+1}^{X,Y} h + \frac{\eta}{2} \sum_{j=1}^{m} \Delta (L^j g_j)_{k+1}^{X,Y} \right\|^2 \right]
= \mathbb{E}[\|e_{k+1}\|^2] + \mathbb{E}\left[ \left\| \theta h \Delta J_{k+1}^{X,Y} - \frac{\eta}{2} \sum_{j=1}^{m} \Delta (L^j g_j)_{k+1}^{X,Y} \right\|^2 \right]
- 2h \mathbb{E}\left[ \left( e_{k+1}, \theta \Delta J_{k+1}^{X,Y} - \frac{\eta}{2} \sum_{j=1}^{m} \Delta (L^j g_j)_{k+1}^{X,Y} \right) \right]
\geq (1 - 2L_2 h) \mathbb{E}[\|e_{k+1}\|^2].
$$

Inserting this into Eq. 3.32 yields

$$
(1 - 2L_2 h) \mathbb{E}[\|e_{k+1}\|^2] \leq (1 + L_1) h e^T \sum_{i=0}^{k} \mathbb{E}[\|e_i\|^2] + \left( \frac{q-1}{q-2} + \frac{\eta^2 m}{(q-1) \sigma^2} + \frac{2}{q-1} \right) e^T \sum_{i=0}^{k} \mathbb{E}[\|R_{i+1}\|^2]
+ \frac{\sigma^2 - 2 \eta^2 + 2}{\sigma^2 h} e^T \sum_{i=0}^{k} \mathbb{E}[\|R_{i+1}\|_{\mathcal{F}_i}]^2.
$$

Owing to $2L_2 h \leq \nu < 1$ by assumption and bearing the moment bounds Eq. 3.13 in mind, one can apply Gronwall’s inequality to acquire the desired assertion. □

It is worthwhile to point out that, conditions in Assumption 3.1 are not difficult to be fulfilled. For instance, the following assumption suffices to imply Assumption 3.1.

**Assumption 3.6** Assume that the diffusion coefficients $g_j : \mathbb{R}^d \to \mathbb{R}^d$, $j \in \{1, 2, ..., m\}$ are differentiable in a domain $D \subset \mathbb{R}^d$. There exist constants $q \in (2, \infty)$,
\( \zeta \in (0, \infty) \) and \( L_3 \in [0, \infty) \) such that, for all \( x, y \in D, h \in (0, 2\zeta) \), the drift and diffusion coefficients of SDEs Eq. 2.1 obey
\[
2(x - y, f(x) - f(y)) + (q - 1)\|g(x) - g(y)\|^2 \\
+ \zeta \sum_{j_1,j_2=1}^{m} \|\mathcal{L}^{j_1}g_{j_2}(x) - \mathcal{L}^{j_1}g_{j_2}(y)\|^2 \leq L_3 \|x - y\|^2. \tag{3.35}
\]

We mention that such a condition was also used in [5, Theorem 2.3] for the backward Milstein method (\( \theta = 1, \eta = 0 \)) and \( D = \mathbb{R}^d \). It is not difficult to check that, when the above condition Eq. 3.35 holds, all conditions in Assumption 3.1 are satisfied with \( \theta \in [\frac{1}{2}, 1], \eta = 0, L_1 = L_3 \) and \( L_2 = \frac{\theta L_3}{2} \). As a direct consequence of Theorem 3.3, we get the following corollary.

**Corollary 3.7** Let Assumptions 3.2, 3.6 be fulfilled with \( \theta L_3 h \leq \nu \) for some \( \nu \in (0, 1) \) and \( \theta \in [\frac{1}{2}, 1], \eta = 0 \). Let \( \{X_t\}_{t \in [0,T]} \) and \( \{Y_n\}_{0 \leq n \leq N} \) be solutions to SDEs Eq. 2.1 and the semi-implicit Milstein method Eq. 2.5, respectively. Then the mean-square error upper bound Eq. 3.5 holds.

Observe that the condition Eq. 3.35 would impose a strict restriction on the polynomial growth of the diffusion coefficient, which excludes practical models such as the \( \frac{3}{2} \)-volatility model Eq. 1.1 and the Ait Sahalia model Eq. 1.2. This can be remedied by utilizing the following assumption.

**Assumption 3.8** Assume that the diffusion coefficients \( g_j : D \to \mathbb{R}^d, j \in \{1, 2, ..., m\} \) are differentiable in the domain \( D \subset \mathbb{R}^d \). For method parameters \( \theta \in (\frac{1}{2}, 1], \eta \in [0, 1], \) there exist constants \( q \in (2, \infty), \varrho \in (1, \infty) \) and \( L_4, L_5, L_6 \in [0, \infty) \) such that, for all \( x, y \in D \), the drift and diffusion coefficients of SDEs Eq. 2.1 obey
\[
2(x - y, f(x) - f(y)) + (q - 1)\|g(x) - g(y)\|^2 \leq L_4 \|x - y\|^2, \\
\frac{q}{2} \sum_{j_1,j_2=1}^{m} \|\mathcal{L}^{j_1}g_{j_2}(x) - \mathcal{L}^{j_1}g_{j_2}(y)\|^2 + \eta \sum_{j=1}^{m} \|\mathcal{L}^j g_j(x) - \mathcal{L}^j g_j(y)\| \leq L_5 \|x - y\|^2, \\
2(x - y, f(x) - f(y)) + \frac{q}{2} \sum_{j=1}^{m} \|\mathcal{L}^j g_j(x) - \mathcal{L}^j g_j(y)\| \leq L_6 \|x - y\|^2. \tag{3.36}
\]

One can straightforwardly verify that Assumption 3.8 implies Assumption 3.1 and one gets the following corollary, as a direct consequence of Theorem 3.3.

**Corollary 3.9** Let Assumptions 3.2, 3.8 hold with \( \theta \in (\frac{1}{2}, 1], \eta \in [0, 1] \) and \( 2L_6 h \leq \nu \) for some \( \nu \in (0, 1) \). Let \( \{X_t\}_{t \in [0,T]} \) and \( \{Y_n\}_{0 \leq n \leq N} \) be solutions to SDEs Eq. 2.1 and the double implicit Milstein method Eq. 2.2, respectively. Then the mean-square error upper bound Eq. 3.5 holds.

In Section 5, we will show that the above two financial models and their numerical schemes fulfill Assumption 3.8 and one can thus rely on Corollary 3.9 to obtain the
desired convergence rate. Before closing this section, we would like to mention that, the previously obtained mean-square error bound Eq. 3.5 is powerful as it helps us to easily analyze mean-square convergence rates of the schemes, without relying on a priori high-order moment estimates of numerical approximations. This will be seen in the forthcoming two sections, where we shall use the error bounds to recover the expected mean-square convergence rates of the proposed schemes in various circumstances.

4 Mean-square convergence rates under globally polynomial growth conditions

Equipped with the previously derived upper mean-square error bounds, the present section aims to identify the expected mean-square convergence rate of the underlying schemes Eq. 2.2 for SDEs in the whole space $\mathbb{R}^d$ under further globally polynomial assumptions. To this end, we make the following globally polynomial growth and coercivity conditions on the drift and diffusion coefficients.

**Assumption 4.1** [Globally polynomial growth and coercivity conditions in $\mathbb{R}^d$]
Assume both the drift coefficient $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the diffusion coefficients $g_j: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $j \in \{1, 2, \ldots, m\}$ of SDEs Eq. 2.1 are continuously differentiable in $\mathbb{R}^d$, and there exist some positive constants $\gamma \in [1, \infty)$ and $p^* \in [6\gamma - 4, \infty)$ such that,

$$\left\langle x, f(x) \right\rangle + \frac{p^*-1}{2} \|g(x)\|^2 \leq C (1 + \|x\|^2), \quad \forall x \in \mathbb{R}^d, \tag{4.1}$$

$$\left\| \left( \frac{\partial f}{\partial x}(x) - \frac{\partial f}{\partial x}(\bar{x}) \right)y \right\| \leq C (1 + \|x\| + \|\bar{x}\|)^{\gamma-2} \|x - \bar{x}\| \cdot \|y\|, \quad \forall x, \bar{x}, y \in \mathbb{R}^d, \tag{4.2}$$

$$\left\| \left( \frac{\partial g_j}{\partial x}(x) - \frac{\partial g_j}{\partial x}(\bar{x}) \right)y \right\| \leq C (1 + \|x\| + \|\bar{x}\|)^{\gamma-3} \|x - \bar{x}\|^2 \cdot \|y\|^2, \quad \forall x, \bar{x}, y \in \mathbb{R}^d, \quad j \in \{1, 2, \ldots, m\}. \tag{4.3}$$

Additionally we assume that the vector functions $\eta L^j g_j: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\eta \in [0, 1]$, $j \in \{1, 2, \ldots, m\}$ are continuously differentiable and

$$\eta \left\| \left( \frac{\partial (L^j g_j)}{\partial x}(x) - \frac{\partial (L^j g_j)}{\partial x}(\bar{x}) \right)y \right\| \leq C (1 + \|x\| + \|\bar{x}\|)^{\gamma-2} \|x - \bar{x}\| \cdot \|y\|, \quad \forall x, \bar{x}, y \in \mathbb{R}^d. \tag{4.4}$$

Moreover, the initial data $X_0$ is supposed to be $\mathcal{F}_0$-adapted, satisfying

$$\|X_0\|_{L^{p^*}(\Omega; \mathbb{R}^d)} < \infty. \tag{4.5}$$

Recall that we use $\frac{\partial \phi}{\partial x}$ to denote the Jacobian matrix of a vector function $\phi: \mathbb{R}^d \rightarrow \mathbb{R}^d$. We mention that the condition Eq. 4.1 is usually called a coercivity condition, which is a classical one in the literature to guarantee that the exact solution has finite $p^*$-th moments, i.e., $\sup_{t \in [0, T]} \|X_t\|_{L^{p^*}(\Omega; \mathbb{R}^d)} < \infty$. The remaining conditions Eq. 4.2-Eq. 4.4 are a kind of polynomial growth conditions, which have been also used in [5, 32] to carry out the error analysis of Milstein type methods. In Section 4.2, we present...
a system of SDEs that fulfill the above conditions. Note that the condition Eq. 4.2 immediately implies

\[ \| \frac{\partial f}{\partial x} (x,y) \| \leq C (1 + \| x \|)^{\gamma - 1} \| y \|, \quad \forall x, y \in \mathbb{R}^d, \quad (4.6) \]

which in turn implies

\[ \| f(x) - f(\tilde{x}) \| \leq C (1 + \| x \| + \| \tilde{x} \|)^{\gamma - 1} \| x - \tilde{x} \|, \quad \forall x, \tilde{x} \in \mathbb{R}^d, \quad (4.7) \]
\[ \| \tilde{f}(x) \| \leq C (1 + \| x \|)^\gamma, \quad \forall x \in \mathbb{R}^d. \quad (4.8) \]

Likewise, the assumption Eq. 4.3 ensures

\[ \| \frac{\partial g}{\partial x} (x,y) \|^2 \leq C (1 + \| x \|)^{\gamma - 1} \| y \|^2, \quad \forall x, y \in \mathbb{R}^d, \quad j \in \{1, 2, \ldots, m\}, \quad (4.9) \]

and therefore

\[ \| g_j(x) - g_j(\tilde{x}) \|^2 \leq C (1 + \| x \| + \| \tilde{x} \|)^{\gamma - 1} \| x - \tilde{x} \|^2, \quad \forall x, \tilde{x} \in \mathbb{R}^d, \quad j \in \{1, 2, \ldots, m\}. \quad (4.10) \]

This in turn gives

\[ \| g(x) \|^2 \leq C (1 + \| x \|)^{\gamma + 1}, \quad \forall x \in \mathbb{R}^d. \quad (4.11) \]

Similarly as above, the assumption Eq. 4.4 promises

\[ \eta \| \frac{\partial L_j^j g_j}{\partial x} (x,y) \| \leq C (1 + \| x \|)^{\gamma - 1} \| y \|, \quad \forall x, y \in \mathbb{R}^d, \quad (4.12) \]

and hence

\[ \eta \| L_j^j g_j(x) - L_j^j g_j(\tilde{x}) \| \leq C (1 + \| x \| + \| \tilde{x} \|)^{\gamma - 1} \| x - \tilde{x} \|, \quad \forall x, \tilde{x} \in \mathbb{R}^d, \]
\[ \eta \| L_j^j g_j(x) \| \leq C (1 + \| x \|)^\gamma, \quad \forall x \in \mathbb{R}^d. \quad (4.13) \]

Further, Assumption 4.1 together with Assumption 3.1 in $D = \mathbb{R}^d$ suffices to guarantee Assumption 3.2 holds in $D = \mathbb{R}^d$. More formally, under these assumptions, the SDE Eq. 2.1 possesses a unique adapted solution with continuous sample paths, $X : [0, T] \times \Omega \rightarrow \mathbb{R}^d$, satisfying

\[ \sup_{t \in [0,T]} \| X_t \|_{L^p(\Omega; \mathbb{R}^d)} \leq C \left( 1 + \| X_0 \|_{L^p(\Omega; \mathbb{R}^d)} \right) < \infty, \quad p \in [2, p^*], \quad (4.14) \]

and thus $\sup_{s \in [0,T]} \mathbb{E}[\| X_s \|^2] + \sup_{s \in [0,T]} \mathbb{E}[\| f(X_s) \|^2] + \sum_{j_1, j_2=1}^m \mathbb{E}[\| L_j^{j_1} g_{j_2}(X_0) \|^2] < \infty$, where $p^* \in [6\gamma - 4, \infty)$ comes from Assumption 4.1. Further, the condition Eq. 3.2 in $\mathbb{R}^d$ from Assumption 3.1 ensures that the implicit Milstein type
Mean-square convergence rates of implicit methods Eq. 2.2 are well-defined in $\mathbb{R}^d$. Thanks to Assumption 4.1 as well as the above implications, one can straightforwardly show

$$\|X_{t_1} - X_{t_2}\|_{L^4(\Omega; \mathbb{R}^d)} \leq C \left( 1 + \sup_{t \in [0, T]} \|X_t\|_{L^\gamma(\Omega; \mathbb{R}^d)} \right) |t_1 - t_2|^{\frac{1}{2}},$$

$$\forall \delta \in [1, \frac{p^*}{\gamma}], \ t_1, t_2 \in [0, T].$$

(4.15)

4.1 Analysis of the mean-square convergence rate

We are now ready to give the main result of this section that reveals the optimal mean-square convergence rate of the considered schemes under Assumptions 3.1, 4.1.

**Theorem 4.2** (Mean-square convergence rates of the schemes) Let coefficients of SDEs Eq. 2.1 and method parameters of the schemes Eq. 2.2 obey Assumption 3.1 in the whole space $D = \mathbb{R}^d$. Let Assumption 4.1 be fulfilled and let the step-size $h = \frac{T}{N} \in (0, \frac{1}{2\theta L^2})$ with $\theta \in [\frac{1}{2}, 1], N \in \mathbb{N}$. Then SDEs Eq. 2.1 and the schemes Eq. 2.2 admit unique adapted solutions in $\mathbb{R}^d$, denoted by $X_t \mid_{t \in [0, T]}$ and $Y_n \mid_{n \in \mathbb{N}}$, respectively. Furthermore, there exists a constant $C > 0$, independent of $N \in \mathbb{N}$, such that, for any $N \in \mathbb{N}$,

$$\sup_{0 \leq n \leq N} \|X_n - Y_n\|_{L^2(\Omega; \mathbb{R}^d)} \leq C \left( 1 + \|X_0\|_{L^\max(2\gamma, 3\gamma-2, 6\gamma-4)}(\Omega; \mathbb{R}^d) \right) h.$$  

(4.16)

**Proof** of Theorem 4.2. The above discussion reminds us that all conditions in Assumptions 3.1, 3.2 hold in $D = \mathbb{R}^d$. Therefore, Theorem 3.3 is applicable here and we only need to properly estimate two error terms $\mathbb{E}[||R_i||^2]$ and $\mathbb{E}[||\mathbb{E}(R_i | \mathcal{F}_{i-1})||^2]$ before arriving at the expected mean-square convergence rate. Recalling the definition of $\{R_i\}_{0 \leq i \leq N}$ given by Eq. 3.6 and using a triangle inequality yield

$$\|R_i\|_{L^2(\Omega; \mathbb{R}^d)} \leq \theta \left\| \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_{i-1}}) \, ds \right\|_{L^2(\Omega; \mathbb{R}^d)} + (1 - \theta) \left\| \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_{i-1}}) \, ds \right\|_{L^2(\Omega; \mathbb{R}^d)}$$

$$+ \left\| \int_{t_{i-1}}^{t_i} g(X_s) - g(X_{t_{i-1}}) \, dW_s - \sum_{j_1, j_2 = 1}^{m} \mathcal{L}^{j_1} g_{j_2}(X_{t_{i-1}}) I_{j_1, j_2}^{i-1, i} \right\|_{L^2(\Omega; \mathbb{R}^d)}$$

$$+ \left\| \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}^j g_j(X_{t_{i-1}}) h - \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}^j g_j(X_{t_{i-1}}) h \right\|_{L^2(\Omega; \mathbb{R}^d)}$$

$$=: I_1 + I_2 + I_3 + I_4.$$  

(4.17)
Next we handle the first term in (4.17) and the second term can be treated similarly. Using the Hölder inequality, Eq. 4.7, Eqs. 4.14 and 4.15 shows

\[
\mathbb{I}_1 \leq \theta \int_{t_i}^{t_i} \| f(x_s) - f(x_{ti}) \|_{L^2(\Omega; \mathbb{R}^d)} \, ds \\
\leq C \int_{t_i}^{t_i} \left( 1 + \| x_s \|_{L^{2\gamma-2}(\Omega; \mathbb{R}^d)}^{\gamma-1} + \| x_{ti} \|_{L^{2\gamma-2}(\Omega; \mathbb{R}^d)}^{\gamma-1} \right) \| x_s - x_{ti} \|_{L^{\left( \gamma - 2 \right)/\gamma}(\Omega; \mathbb{R}^d)} \, ds \\
\leq C \left( 1 + \sup_{t \in [0,T]} \| x_t \|_{L^{2\gamma-2}(\Omega; \mathbb{R}^d)}^{\gamma-1} \right) h^{\frac{3}{2}} . \tag{4.18}
\]

In the same way, one can also obtain

\[
\mathbb{I}_2 \leq C \left( 1 + \sup_{t \in [0,T]} \| x_t \|_{L^{2\gamma-2}(\Omega; \mathbb{R}^d)}^{\gamma-1} \right) h^{\frac{3}{2}} . \tag{4.19}
\]

Before coming to the estimate of \( \mathbb{I}_3 \), we note that, for any differentiable functions \( \phi: \mathbb{R}^d \to \mathbb{R}^d \),

\[
\phi(x_t) - \phi(x_s) = \frac{\partial \phi}{\partial x}(x_s)(x_t - x_s) + \mathcal{R}(x_s, x_t) \\
= \frac{\partial \phi}{\partial x}(x_s) \left( \int_s^t f(x_\xi) \, d\xi + \int_s^t g(x_\xi) \, dW_\xi \right) + \mathcal{R}(x_s, x_t), \quad s < t , \tag{4.20}
\]

where for short we denote

\[
\mathcal{R}(x_s, x_t) := \int_0^1 \left[ \frac{\partial \phi}{\partial x}(x_s + r(x_t - x_s)) - \frac{\partial \phi}{\partial x}(x_s) \right] (x_t - x_s) \, dr . \tag{4.21}
\]

As a direct consequence of Eqs. 2.4 and 4.20, one can show

\[
\int_{t_{i-1}}^{t_i} g(x_s) - g(x_{t_{i-1}}) \, dW_s - \sum_{j_1, j_2=1}^m \mathcal{L}^{j_1} g^{j_2}(x_{t_{i-1}}) \mathcal{I}^{j_1, j_2}_{t_{i-1}, t_i} \\
= \sum_{j_2=1}^m \int_{t_{i-1}}^{t_i} \left[ g^{j_2}(x_s) - g^{j_2}(x_{t_{i-1}}) - \sum_{j_1=1}^m \mathcal{L}^{j_1} g^{j_2}(x_{t_{i-1}})(W_s^{j_1} - W_{t_{i-1}}^{j_1}) \right] dW_s^{j_2} \\
= \sum_{j_2=1}^m \int_{t_{i-1}}^{t_i} \left[ g^{j_2}(x_s) - g^{j_2}(x_{t_{i-1}}) - \sum_{j_1=1}^m \frac{d g^{j_2}}{d x}(X_{t_{i-1}})g^{j_1}(x_{t_{i-1}})(W_s^{j_1} - W_{t_{i-1}}^{j_1}) \right] dW_s^{j_2} \\
= \sum_{j=1}^m \int_{t_{i-1}}^{t_i} \left[ \frac{d g^{j_2}}{d x}(X_{t_{i-1}}) \left( \int_{t_{i-1}}^s f(x_\xi) \, d\xi + \int_{t_{i-1}}^s g(x_\xi) - g(x_{t_{i-1}}) \right) dW_\xi \right] + \mathcal{R}^{g_j}(x_{t_{i-1}}, x_s) \, dW_s^{j_2} . \tag{4.22}
\]
Bearing this in mind, one can utilize the Itô isometry to obtain

\[
\|I_3\|^2 = \sum_{j=1}^{m} \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \left\| g_{j_2}(X_s) - g_{j_2}(X_{t_{i-1}}) - \sum_{j_1=1}^{m} \mathcal{L}^{j_1} g_{j_2}(X_{t_{i-1}})(W_{s}^{j_1} - W_{t_{i-1}}^{j_1}) \right\|^2 \right] ds
\]

\[
= \sum_{j=1}^{m} \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \left\| \frac{\partial g_j}{\partial x}(X_{t_{i-1}}) \left( \int_{t_{i-1}}^{s} f(X_\xi) \, d\xi + \int_{t_{i-1}}^{s} [g(X_\xi) - g(X_{t_{i-1}})] dW_\xi \right) \right\|^2 \right] ds
\]

\[
+ R_{g_j}(X_{t_{i-1}}, X_s) \right\|^2 \right] ds
\]

\[
\leq 3 \sum_{j=1}^{m} \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \left\| \frac{\partial g_j}{\partial x}(X_{t_{i-1}}) \int_{t_{i-1}}^{s} f(X_\xi) \, d\xi \right\|^2 \right] ds
\]

\[
+ 3 \sum_{j=1}^{m} \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \left\| R_{g_j}(X_{t_{i-1}}, X_s) \right\|^2 \right] ds
\]

\[
+ 3 \sum_{j=1}^{m} \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \left\| \frac{\partial g_j}{\partial x}(X_{t_{i-1}}) \int_{t_{i-1}}^{s} g(X_\xi) - g(X_{t_{i-1}}) \, dW_\xi \right\|^2 \right] ds. \tag{4.23}
\]

In the following we cope with the above three items separately. Thanks to Eq. 4.8, Eq. 4.9 and the Hölder inequality, we first get

\[
\left\| \frac{\partial g_j}{\partial x}(X_{t_{i-1}}) \int_{t_{i-1}}^{t_i} f(X_\xi) \, d\xi \right\|_{L^2(\Omega; \mathbb{R}^d)} \leq C \int_{t_{i-1}}^{t_i} \left\| f(X_{t_{i-1}}) \right\|_{L^2(\Omega; \mathbb{R})} \, d\xi
\]

\[
\leq C \int_{t_{i-1}}^{t_i} \left\| f(X_{t_{i-1}}) \right\|_{L^2(\Omega; \mathbb{R})} \, d\xi
\]

\[
\leq \frac{\varepsilon_j}{\varepsilon_j} \left( 1 + \|X_{t_{i-1}}\| \right)^{\frac{\gamma_j}{\varepsilon_j}} \left( 1 + \|X_{t_{i-1}}\| \right)^{\gamma_j} \|X_{t_{i-1}}\| \, d\xi
\]

\[
\leq C \lambda \left( 1 + \sup_{t \in [0, T_1]} \|X_t\|_{L^2(\Omega; \mathbb{R}^d)} \right). \tag{4.24}
\]

Again, using the Itô isometry, the Hölder inequality, Eq. 4.9, Eq. 4.10 and Eq. 4.15 yields

\[
\mathbb{E} \left[ \left\| \frac{\partial g_j}{\partial x}(X_{t_{i-1}}) \int_{t_{i-1}}^{s} g(X_\xi) - g(X_{t_{i-1}}) \, dW_\xi \right\|^2 \right]
\]

\[
= \sum_{l=1}^{m} \int_{t_{i-1}}^{s} \left\| \frac{\partial g_j}{\partial x}(X_{t_{i-1}}) \left[ g_l(X_\xi) - g_l(X_{t_{i-1}}) \right] \right\|_{L^2(\Omega; \mathbb{R}^d)} \, d\xi
\]

\[
\leq C \sum_{l=1}^{m} \int_{t_{i-1}}^{s} \left( 1 + \|X_{t_{i-1}}\| \right)^{\frac{\gamma_l}{\varepsilon_l}} \left\| g_l(X_\xi) - g_l(X_{t_{i-1}}) \right\|_{L^2(\Omega; \mathbb{R})} \, d\xi
\]
\[ \leq C \int_{t_{i-1}}^{s} \left( 1 + \|X_{t_i-1}\| \right)^{\frac{\gamma - 1}{\gamma}} \left( 1 + \|X_{\xi}\| + \|X_{t_{i-1}}\| \right)^{\frac{\gamma - 1}{\gamma}} \|X_{\xi} - X_{t_{i-1}}\|^2_{L^2(\Omega;\mathbb{R})} \, d\xi \]
\[ \leq C \int_{t_{i-1}}^{s} \left( \|X_{\xi}\| + \|X_{t_{i-1}}\| \right)^{\gamma - 1} \|X_{\xi} - X_{t_{i-1}}\|^2_{L^2(\Omega;\mathbb{R})} \, d\xi \]
\[ \leq C h^2 \left( 1 + \sup_{t \in [0, T]} \|X_t\|^{\frac{4 \gamma - 2}{2 \gamma - 2}}_{L^{4 \gamma - 2}(\Omega; \mathbb{R}^d)} \right). \tag{4.25} \]

In light of Eqs. 4.3, 4.14 and 4.15, one can further use the Hölder inequality to acquire

\[ \|R_{g_j}(X_{t_{i-1}}, X_s)\|_{L^2(\Omega; \mathbb{R}^d)} \]
\[ \leq \int_0^1 \left\| \left[ \frac{\partial g_j}{\partial x} (X_{t_{i-1}} + r (X_s - X_{t_{i-1}})) \right] - \frac{\partial g_j}{\partial x} (X_{t_{i-1}}) \right\|_{L^2(\Omega; \mathbb{R}^d)} \, dr \]
\[ \leq C \int_0^1 \left( 1 + \|r X_s + (1 - r) X_{t_{i-1}}\| + \|X_{t_{i-1}}\| \right)^{\gamma - 3} \|X_s - X_{t_{i-1}}\|^2_{L^2(\Omega; \mathbb{R})} \, dr \]
\[ \leq C h \left( 1 + \sup_{t \in [0, T]} \|X_t\|^{\frac{4 \gamma - 3}{2 \gamma - 3}}_{L^{4 \gamma - 3}(\Omega; \mathbb{R}^d)} \right). \tag{4.26} \]

Plugging the above three estimates Eqs. 4.24-4.26 into 4.23 gives

\[ \mathbb{I}_3 \leq C h^2 \left( 1 + \sup_{t \in [0, T]} \|X_t\|^{\frac{4 \gamma - 3}{2 \gamma - 3}}_{L^{4 \gamma - 3}(\Omega; \mathbb{R}^d)} \right). \tag{4.27} \]

With regard to \( \mathbb{I}_4 \), we utilize Eq. 4.13-Eq. 4.15 and the Hölder inequality to obtain

\[ \mathbb{I}_4 \leq \frac{nh}{2} \sum_{j=1}^m \left\| \mathcal{L}^j g_j(X_{t_i}) - \mathcal{L}^j g_j(X_{t_{i-1}}) \right\|_{L^2(\Omega; \mathbb{R}^d)} \]
\[ \leq C h \left( 1 + \|X_{t_i}\| + \|X_{t_{i-1}}\| \right)^{\gamma - 1} \|X_{t_i} - X_{t_{i-1}}\|^2_{L^2(\Omega; \mathbb{R})} \]
\[ \leq C h^3 \left( 1 + \sup_{t \in [0, T]} \|X_t\|^{2 \gamma - 1}_{L^{4 \gamma - 2}(\Omega; \mathbb{R}^d)} \right). \tag{4.28} \]

Putting all the above estimates together we derive from Eq. 4.17 that

\[ \|R_i\|_{L^2(\Omega; \mathbb{R}^d)} \leq C h^2 \left( 1 + \sup_{t \in [0, T]} \|X_t\|^{\frac{4 \gamma - 3}{2 \gamma - 3}}_{L^{4 \gamma - 3}(\Omega; \mathbb{R}^d)} \right). \tag{4.29} \]
Noting that the stochastic integral vanishes under the conditional expectation, one can, similarly as in Eq. 4.17, infer that

\[ \|E(R_t \mid F_{t_{i-1}})\|_{L^2(\Omega; \mathbb{R}^d)} \leq \theta \left\| \mathbb{E}\left( \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_i}) \mathrm{d}s \mid F_{t_{i-1}} \right) \right\|_{L^2(\Omega; \mathbb{R}^d)} 
+ (1 - \theta) \left\| \mathbb{E}\left( \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_i}) \mathrm{d}s \mid F_{t_{i-1}} \right) \right\|_{L^2(\Omega; \mathbb{R}^d)} 
+ \frac{m}{2} \sum_{j=1}^{m} \left\| \mathbb{E}\left[ [L^j g_j(X_{t_i}) - L^j g_j(X_{t_{i-1}})] \right] \mid F_{t_{i-1}} \right\|_{L^2(\Omega; \mathbb{R}^d)} \]

\[ =: \mathbb{I}_5 + \mathbb{I}_6 + \mathbb{I}_7. \] (4.30)

In order to estimate \( \mathbb{I}_5 \), we first note that

\[ \mathbb{E}\left( \int_{t_{i-1}}^{t_i} \frac{\partial f}{\partial x}(X_s) \int_s^{t_i} g(X_\xi) \mathrm{d}W_\xi \mid F_{t_{i-1}} \right) = \int_{t_{i-1}}^{t_i} \mathbb{E}\left( \int_s^{t_i} \frac{\partial f}{\partial x}(X_s) g(X_\xi) \mathrm{d}W_\xi \mid F_{t_{i-1}} \right) \mathrm{d}s 
= \int_{t_{i-1}}^{t_i} \mathbb{E}\left( \mathbb{E}\left( \int_s^{t_i} \frac{\partial f}{\partial x}(X_s) g(X_\xi) \mathrm{d}W_\xi \mid F_s \right) \mid F_{t_{i-1}} \right) \mathrm{d}s 
= 0. \] (4.31)

Using this and Eq. 4.20 with \( \phi = f \) ensures

\[ \mathbb{E}\left( \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_i}) \mathrm{d}s \mid F_{t_{i-1}} \right) = \mathbb{E}\left( \int_{t_{i-1}}^{t_i} \left[ \frac{\partial f}{\partial x}(X_s) \int_s^{t_i} f(X_\xi) \mathrm{d}\xi + \mathcal{R}_f(X_s, X_{t_i}) \right] \mathrm{d}s \mid F_{t_{i-1}} \right), \] (4.32)

and thus

\[ \mathbb{I}_5 = \theta \left\| \mathbb{E}\left( \int_{t_{i-1}}^{t_i} \left[ \frac{\partial f}{\partial x}(X_s) \int_s^{t_i} f(X_\xi) \mathrm{d}\xi + \mathcal{R}_f(X_s, X_{t_i}) \right] \mathrm{d}s \mid F_{t_{i-1}} \right) \right\|_{L^2(\Omega; \mathbb{R}^d)} \]
\[ \leq \theta \left\| \int_{t_{i-1}}^{t_i} \frac{\partial f}{\partial x}(X_s) \int_s^{t_i} f(X_\xi) \mathrm{d}\xi + \mathcal{R}_f(X_s, X_{t_i}) \right\|_{L^2(\Omega; \mathbb{R}^d)} \mathrm{d}s 
\leq \theta \int_{t_{i-1}}^{t_i} \left\| \frac{\partial f}{\partial x}(X_s) f(X_\xi) \right\|_{L^2(\Omega; \mathbb{R}^d)} \mathrm{d}\xi \mathrm{d}s + \theta \int_{t_{i-1}}^{t_i} \left\| \mathcal{R}_f(X_s, X_{t_i}) \right\|_{L^2(\Omega; \mathbb{R}^d)} \mathrm{d}s. \] (4.33)

where the Jensen inequality was used for the second step. Here we employ Eq. 4.6, Eq. 4.8 and the Hölder inequality to show

\[ \left\| \frac{\partial f}{\partial x}(X_s) f(X_\xi) \right\|_{L^2(\Omega; \mathbb{R}^d)} \leq C \left\| (1 + \|X_s\|)^{\gamma-1} (1 + \|X_\xi\|)^\gamma \right\|_{L^2(\Omega; \mathbb{R})} \]
\[ \leq C (1 + \sup_{t \in [0,T]} \|X_t\|)^\gamma \|X_t\|^{2\gamma-2} \] (4.34)
and employ Eqs. 4.2, 4.15 and the Hölder inequality to arrive at

$$\| \mathcal{R}_f (X_s, X_t) \|_{L^2(\Omega; \mathbb{R}^d)} \leq \int_0^1 \left\| \frac{\partial f}{\partial x} (X_s + r (X_t - X_s)) - \frac{\partial f}{\partial x} (X_t - X_s) \right\|_{L^2(\Omega; \mathbb{R}^d)} dr$$

$$\leq C \int_0^1 \left( 1 + \| r X_t + (1 - r) X_s \| + \| X_s \| \| X_t - X_s \|^2 \right) \left\| \mathcal{H}_f (X_t) \right\|_{L^2(\Omega; \mathbb{R}^d)} dr$$

$$\leq C h \left( 1 + \sup_{t \in [0, T]} \| X_t \|_{L^2(\Omega; \mathbb{R}^d)} \right).$$

(4.35)

Inserting Eqs. 4.34 and 4.35 into Eq. 4.33 implies

$$\| \mathbb{E} \|_{L^2(\Omega; \mathbb{R}^d)} \leq C h^2 \left( 1 + \sup_{t \in [0, T]} \| X_t \|_{L^2(\Omega; \mathbb{R}^d)} \right).$$

(4.36)

The estimates of \( \| \mathbb{E} \|_{L^2(\Omega; \mathbb{R}^d)} \) are similar and one can also get

$$\| \mathbb{E} \|_{L^2(\Omega; \mathbb{R}^d)} \leq C h^2 \left( 1 + \sup_{t \in [0, T]} \| X_t \|_{L^2(\Omega; \mathbb{R}^d)} \right).$$

(4.37)

Therefore, from Eq. 4.30 it immediately follows that

$$\| \mathbb{E} (R_i | \mathcal{F}_{t_{i-1}}) \|_{L^2(\Omega; \mathbb{R}^d)} \leq C h^2 \left( 1 + \sup_{t \in [0, T]} \| X_t \|_{L^2(\Omega; \mathbb{R}^d)} \right).$$

(4.38)

In view of Theorem 3.3 and Eq. 4.14, we validate the desired assertion Eq. 4.16.

As already mentioned at the end of Section 3, Theorem 3.3 still holds when Assumption 3.1 is replaced by Assumption 3.6 or Assumption 3.8. Therefore, the following two corollaries follow directly from Corollaries 3.7, 3.9.

**Corollary 4.3** Let Assumption 3.6 be fulfilled with \( D = \mathbb{R}^d \) and let \( \theta L_3 h \leq \nu \) for some \( \theta \in [\frac{1}{2}, 1] \), \( \nu \in (0, 1) \). Let conditions in Assumption 4.1 be all satisfied. Then SDEs Eq. 2.1 and the semi-implicit Milstein methods Eq. 2.5 admit unique adapted solutions in \( \mathbb{R}^d \) and Eq. 4.16 holds, namely, the schemes Eq. 2.5 retain a mean-square convergence rate of order one.

**Corollary 4.4** Let Assumption 3.8 be fulfilled with \( D = \mathbb{R}^d \) and let conditions in Assumption 4.1 be all satisfied. Let \( \theta L_6 h \leq \nu \) for some \( \theta \in (\frac{1}{2}, 1] \), \( \nu \in [0, 1] \), and for some \( \nu \in (0, 1) \). Then SDEs Eq. 2.1 and the proposed schemes Eq. 2.2 admit unique adapted solutions in \( \mathbb{R}^d \) and Eq. 4.16 holds, namely, the schemes Eq. 2.2 retain a mean-square convergence rate of order one.

### 4.2 An example with numerical simulations

In this subsection, we aim to give an example SDE that satisfies Assumptions 3.1, 4.1. To this end, let us first consider the following semi-linear stochastic partial
Mean-square convergence rates of implicit differential equation (SPDE) [35, 36]:

\[
\begin{aligned}
    &\frac{d}{dt}u(t, x) = \left[ \frac{d^2}{dx^2} u(t, x) + u(t, x) - u^3(t, x) \right] dt + g(u(t, x)) dW_t, \quad t \in (0, T], \ x \in (0, 1), \\
    &u(0, x) = u(t, 1) = 0, \\
    &u(0, x) = u_0(x),
\end{aligned}
\]  

(4.39)

where \( g : \mathbb{R} \to \mathbb{R} \) and \( W : [0, T] \times \Omega \to \mathbb{R} \) is the real-valued standard Brownian motion. Such an SPDE is usually termed as the stochastic Allen-Cahn equation. Next we want to spatially discretize the above SPDE to obtain an SDE system. On the interval \([0, 1]\) we construct a uniform mesh with stepsize \( \Delta x := \frac{1}{K} \) and denote \( x_i = i \Delta x, \ i = 1, 2, ..., K - 1 \). Discretizing the SPDE Eq. 4.39 spatially by a finite difference method yields a system of SDEs:

\[
\begin{aligned}
    &dX_t = [A_t X_t + F(X_t)] dt + G(X_t) dW_t, \quad t \in (0, T], \ X_0 = x_0,
\end{aligned}
\]  

(4.40)

where \( X_t = (X_1, X_2, \ldots, X_{K-1})^T := (u(t, x_1), u(t, x_2), \ldots, u(t, x_{K-1}))^T \), \( A_t \in \mathbb{R}^{(K-1) \times (K-1)} \), \( x_0 = (u_0(x_1), u_0(x_2), ..., u_0(x_{K-1}))^T \) and

\[
A_t = K^2 \begin{bmatrix}
    -2 & 1 & 0 & \cdots & 0 & 0 \\
    1 & -2 & 1 & \cdots & 0 & 0 \\
    0 & 1 & -2 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & -2 & 1 \\
    0 & 0 & 0 & \cdots & 1 & -2
\end{bmatrix}, \quad F(X) = \begin{bmatrix}
    X_1 - (X_1)^3 \\
    X_2 - (X_2)^3 \\
    \vdots \\
    X_{K-1} - (X_{K-1})^3
\end{bmatrix}, \quad G(X) = \begin{bmatrix}
    g(X_1) \\
    g(X_2) \\
    \vdots \\
    g(X_{K-1})
\end{bmatrix}.
\]

We do not consider the error caused by the spatial discretization but focus on the temporal discretization of the SDE system Eq. 4.40, done by the semi-implicit Milstein method (\( \theta = 1, \eta = 0 \)). Moreover, we assume \( g \in C^3_b(\mathbb{R}, \mathbb{R}) \), i.e., \( g \) is three times differentiable with derivatives bounded. It is easy to check all conditions in Assumption 4.1 are fulfilled in \( D = \mathbb{R}^{K-1} \) with \( \gamma = 3 \) and for any \( p^* \geq 14 \). By setting \( \theta = 1, \eta = 0 \), conditions Eqs. 3.1, 3.2 in Assumption 3.1 are also both satisfied in \( D = \mathbb{R}^{K-1} \). Therefore, Theorem 4.2 is applicable, with the first convergence rate obtained for the semi-implicit Milstein method. Since the SDE system has commutative noise, the Milstein type methods do not involve the Levy area [30, 43] and can be implemented as easily as the Euler type methods.

In what follows we set \( g(u) = \sin(u) + 1 \) and \( u_0(x) = 1 \) and do some numerical experiments. In Fig. 1, we plot mean-square errors of the semi-implicit Milstein method (\( \theta = 1, \eta = 0 \)) for the SDE system Eq. 4.40 with \( K = 4 \). There one can observe a convergence rate of order one, as the step-sizes shrink. Here and below numerical approximations are performed using six different stepsizes \( h = 2^{-i}, i = 2, 3, \ldots, 7 \). The “exact” solution is identified as the numerical one using a fine stepsize \( h_{\text{exact}} = 2^{-12} \) and the expectations are approximated by computing averages over \( 10^4 \) samples. For comparison, we also discretize Eq. 4.40 by the tamed Milstein method for non-Lipschitz SDEs [32, 51]. Tables 1, 2, and 3 provide mean-square approximation errors of these two methods for three cases \( K = 4, 8, 16 \). Clearly, the tamed Milstein method gives satisfactory results in the low dimension case \( K = 4 \) when the time stepsize
is small, i.e., $h = 2^{-5}, 2^{-6}, 2^{-7}$. As the dimension $K$ increases ($K = 8, 16$), the tamed Milstein method gives large errors and the approximations become unreliable for even small stepsizes. However, the semi-implicit Milstein method performs much better, even in high dimension case $K = 16$. This happens because the eigenvalues $\{\lambda_i\}_{i=1}^{K-1}$ of $A$ are $\lambda_i = -4K^2 \sin^2(i\pi/2K) < 0$ and the problem Eq. 4.40 turns to be a very stiff system [43] as $K$ increases. As a kind of explicit method, the tamed Milstein method applied to solve stiff system, faces severe time step-size reduction due to the stability issue. On the contrary, the semi-implicit Milstein method has excellent stability property and is well suited for such stiff system.

**5 Convergence rates of positivity preserving schemes for SDEs with non-globally Lipschitz coefficients**

In the present section, we turn our attention to the aforementioned scalar SDE models Eqs. 1.1 and 1.2 arising from mathematical finance. Unlike general SDEs studied in the

![Mean-square errors of semi-implicit Milstein method](image)

**Fig. 1** Mean-square convergence rate of the semi-implicit Milstein method for Eq. 4.40 ($K = 4$)
previous section, the considered models do not evolve in the whole space \( \mathbb{R} \), but only in the positive domain \( D = (0, \infty) \). This thus makes the convergence theory developed in the previous section not applicable in this situation. Moreover, preservation of positivity is usually a desirable modeling property and positivity of the approximation is, in many cases, necessary in order for the numerical scheme to be well defined (see, e.g., Eqs. 5.2 and 5.30 below). However, numerical schemes are, in general, not able to preserve positivity. For example, the classical Euler-Maruyama method fails to preserve positivity for any scalar SDE [8]. In this section we choose two particular schemes from Eq. 2.10 to approximate three models, which are capable of preserving positivity of the continuous models. By means of the previously obtained error bound, we carefully analyze the expected mean-square convergence rate of the resulting numerical approximations.

### 5.1 The double implicit Milstein scheme for the Heston \( \frac{3}{2} \)-volatility model

As the first considered financial model, let us look at the Heston \( \frac{3}{2} \)-volatility model [19, 33]:

\[
dX_t = X_t(\mu - \alpha X_t)dt + \beta X_t^{3/2} dW(t), \quad X_0 = x_0 > 0, \quad \mu, \alpha, \beta > 0, \quad t > 0, \quad (5.1)
\]

---

**Table 1** Mean-square approximation errors for two schemes (\( K = 4 \))

| Stepsizes \( h \) | Semi-implicit Milstein | Tamed Milstein |
|-------------------|------------------------|---------------|
| \( h = 2^{-2} \)  | 0.228228472003678      | 1.334521881473836 |
| \( h = 2^{-3} \)  | 0.142671496841737      | 0.669681337348534  |
| \( h = 2^{-4} \)  | 0.092138829109993      | 0.30484585687944   |
| \( h = 2^{-5} \)  | 0.050402455908956      | 0.104293855220492  |
| \( h = 2^{-6} \)  | 0.026477850950294      | 0.044846913728710  |
| \( h = 2^{-7} \)  | 0.014040231850694      | 0.023917375308279  |

---

**Table 2** Mean-square approximation errors for two schemes (\( K = 8 \))

| Stepsizes \( h \) | Semi-implicit Milstein | Tamed Milstein |
|-------------------|------------------------|---------------|
| \( h = 2^{-2} \)  | 0.337954132405219      | 3.127906338055271  |
| \( h = 2^{-3} \)  | 0.215927776446030      | 2.264234907688349  |
| \( h = 2^{-4} \)  | 0.143858604122065      | 1.393606951102787  |
| \( h = 2^{-5} \)  | 0.082829804151649      | 0.792573908322782  |
| \( h = 2^{-6} \)  | 0.045812280417151      | 0.375592176659368  |
| \( h = 2^{-7} \)  | 0.025766349691283      | 0.060932654697185  |
Table 3 Mean-square approximation errors for two schemes ($K = 16$)

| Stepsizes $h$ | Semi-implicit Milstein | Tamed Milstein |
|---------------|-------------------------|----------------|
| $h = 2^{-2}$  | 0.483493085665317      | 5.551900376818316 |
| $h = 2^{-3}$  | 0.310800712759207      | 4.923675828972162 |
| $h = 2^{-4}$  | 0.209040389203629      | 4.088007760382119 |
| $h = 2^{-5}$  | 0.122832739545349      | 3.113087309657318 |
| $h = 2^{-6}$  | 0.070290414827683      | 2.028019061710102 |
| $h = 2^{-7}$  | 0.041888961361398      | 1.128188804503420 |

which can be viewed as an inverse of a Cox-Ingersoll-Ross (CIR) process [44]. Such an equation is also used for modelling term structure dynamics [11]. Recently, some researchers [9, 21, 44] proposed and analyzed different positivity-preserving numerical schemes for strong approximations of the $\frac{3}{2}$-process. Similarly to [21], we choose a particular double implicit Milstein scheme Eq. 2.10 with $\theta = \eta = 1$ to approximate the above $\frac{3}{2}$-process. Furthermore, we attempt to prove the expected convergence rate for the scheme, which is missing in [21].

Given $T \in (0, \infty)$ and $N \in \mathbb{N}$, one can construct a uniform mesh on the interval $[0, T]$ with the uniform stepsize $h = \frac{T}{N}$. Based on the uniform mesh, we apply the drift-diffusion double implicit Milstein scheme Eq. 2.10 with $\theta = \eta = 1$ to the model Eq. 5.1, resulting in, for $n \in \{0, 1, 2, ..., N - 1\}$,

$$Y_{n+1} = Y_n + Y_{n+1} (\mu - \alpha Y_{n+1}) h + \beta Y_n^{3/2} \Delta W_n + \frac{3\beta^2}{4} Y_n^2 |\Delta W_n|^2 - \frac{3\beta^2}{4} Y_{n+1}^2 h, \quad Y_0 = X_0,$$

(5.2)

which is a quadratic equation and has a unique positive solution explicitly given by $Y_0 = X_0$ and

$$Y_{n+1} = \left( \frac{\sqrt{(1 - \mu h)^2 + 4h(\alpha + \frac{3}{4} \beta^2)(Y_n + \beta |Y_n|^{3/2} \Delta W_n + \frac{3}{4} \beta^2 |Y_n|^2 |\Delta W_n|^2})}{2 \alpha h + \frac{3}{2} \beta^2 h} \right) > 0,$$

(5.3)

given that $Y_n > 0, n \in \{0, 1, 2, ..., N - 1\}$. We mention that no additional restriction is put on the stepsize $h > 0$ to ensure the positivity of the above approximations. In order to carry out the error analysis for the scheme using Theorem 3.3, we should first justify all conditions required in Assumptions 3.1, 3.2, which are clarified in the forthcoming lemma.

**Lemma 5.1** Let $\mu, \alpha, \beta > 0, X_0 > 0$. Then the Heston $\frac{3}{2}$ volatility model Eq. 5.1 has a unique global solution in $(0, \infty)$ and the scheme Eq. 5.2 produces unique positivity preserving approximations given by Eq. 5.3. When $\alpha > \frac{3}{2} \beta^2$, the SDE model Eq. 5.1 and the scheme Eq. 5.2 obey Assumptions 3.1, 3.2 in the domain $D = (0, \infty)$ for some $2 < q < 1 + \frac{8\alpha}{9\beta^2}$.
Proof of Lemma 5.1. The well-posedness of the considered model Eq. 5.1 and the scheme Eq. 5.2 in the positive domain \((0, \infty)\) can be found in \([21, 44]\). It remains to validate the other conditions in Assumptions 3.1, 3.2. For brevity, we denote the drift and diffusion coefficients of SDE Eq. 5.1 by

\[
f(x) := x(\mu - \alpha x), \quad g(x) := \beta x^2, \quad x \in \mathbb{R}_+.
\] (5.4)

As a result, \(g'(x)g(x) = \frac{3}{2} \beta^2 x^2, \ x \in \mathbb{R}_+\) and one can find a positive constant \(\bar{c} > 0\) such that

\[
\mathbb{E}(x, y, h) := \frac{\varrho}{2} \|g'g(x) - g'(y)\|^2 + \eta h (g'g(x) - g'(y), f(x) - f(y) - h\|f(x) - f(y)\|^2
\]

\[
= \frac{\varrho}{2} \beta^4 h(x^2 - y^2)^2 + \frac{3}{2} \beta^2 \eta h(x^2 - y^2)(x - y) - \frac{3}{2} \beta^2 \eta \alpha h(x^2 - y^2)^2
\]

\[-h [\mu^2(x - y)^2 - 2\mu \alpha (x - y)(x^2 - y^2) + \alpha^2(x^2 - y^2)^2]
\]

\[
= \left\{ \frac{\varrho}{2} \beta^4 - \frac{3}{2} \beta^2 \alpha - \alpha^2 \right\} (x + y)^2 + \left[ \frac{3}{2} \beta^2 \mu + 2 \mu \alpha \right] (y + x) - \mu^2 \left( x - y \right)^2 h
\]

\[
\leq \bar{c}(x - y)^2 h, \quad \forall \ x, y \in \mathbb{R}_+, (5.5)
\]

where we used the facts that \(\eta = 1\) and that \(\frac{\varrho}{2} \beta^4 - \frac{3}{2} \beta^2 \alpha - \alpha^2 < 0\) for some \(\varrho > 1\) since \(\alpha > \frac{3}{2} \beta^2\) by assumption. Further, we take some \(2 < q < 1 + \frac{8\alpha}{\varrho \beta^2}\) to promise \(\frac{\varrho}{4} (q - 1) \beta^2 - 2\alpha \leq 0\) and hence

\[
2\langle x - y, f(x) - f(y) \rangle + (q - 1) \|g(x) - g(y)\|^2 + \mathbb{E}(x, y, h)
\]

\[
= 2 \mu |x - y|^2 - 2\alpha(x^2 - y^2)(x - y) + (q - 1) \beta^2 (x^2 - y^2)^2 + \mathbb{E}(x, y, h)
\]

\[
\leq 2 \mu |x - y|^2 + \left[ \frac{\varrho}{4} (q - 1) \beta^2 - 2\alpha \right] (x + y)(x - y)^2 + \bar{c}(x - y)^2 h
\]

\[
\leq (2\mu + \bar{c} T) |x - y|^2, \quad \forall \ x, y \in \mathbb{R}_+, (5.6)
\]

which means the condition Eq. 3.1 in Assumption 3.1 is fulfilled. Now we validate Eq. 3.2 as follows:

\[
\{x - y, f(x) - f(y) - \frac{1}{2} [g'g(x) - g'(y)]\}
\]

\[
= \mu |x - y|^2 - \alpha (x^2 - y^2)(x - y) - \frac{3}{2} \beta^2 (x^2 - y^2)(x - y)
\]

\[
\leq \mu |x - y|^2, \quad \forall \ x, y \in \mathbb{R}_+. (5.7)
\]

Next we note that for any \(p^* \leq 4\)

\[
\{x, f(x)\} + \frac{p^* - 1}{2} \|g(x)\|^2 = \mu x^2 - (\alpha - \frac{p^* - 1}{2} \beta^2) x^3 \leq \mu x^2, \quad \forall \ x \in \mathbb{R}_+, (5.8)
\]

where the assumption \(\alpha > \frac{3}{2} \beta^2\) was again used. This assures \(\sup_{t \in [0, T]} \|X_t\|_{L^4(\Omega, \mathbb{R}_+)} < \infty\) and thus \(\sup_{s \in [0, T]} \mathbb{E}[\|X_s\|^2] + \sup_{s \in [0, T]} \mathbb{E}[\|f(X_s)\|^2] < \infty\), as required in Assumption 3.2. \(\square\)

Now we are able to apply Theorem 3.3 to deduce the convergence rate of the numerical scheme.
Theorem 5.2 Let $X_0 > 0$ and let $\mu, \alpha, \beta > 0$ satisfy $\alpha \geq \frac{5}{2}\beta^2$. Let $\{X_t\}_{t \in [0,T]}$ and $\{Y_n\}_{0 \leq n \leq N}$ be uniquely given by Eqs. 5.1 and 5.3, respectively. Let $h \in (0, \frac{1}{2\mu})$. Then there exists a constant $C > 0$, independent of $N \in \mathbb{N}$, such that

$$\sup_{0 \leq n \leq N} \|Y_n - X_{t_n}\|_{L^2(\Omega; \mathbb{R})} \leq Ch.$$  \hspace{1cm} (5.9)

Proof of Theorem 5.2. As already clarified in the proof of Lemma 5.1, the considered model and the scheme obey Assumptions 3.1, 3.2 in the domain $D = (0, \infty)$. Therefore, Theorem 3.3 is applicable here and it remains to estimate two error terms $R_i$ and $R_i^t$, $i \in \{1, 2, ..., N\}$, before attaining the convergence rate. First of all, we recall $f(x) := x(\mu - \alpha x) - \beta x^2$, $x \in \mathbb{R}_+$. Following the notation used in Theorem 3.3, one can easily see

$$\|R_i\|_{L^2(\Omega; \mathbb{R})} \leq \left\| \int_{t_i}^{t_{i-1}} f(X_s) - f(X_{t_i}) \, ds \right\|_{L^2(\Omega; \mathbb{R})} + \frac{\alpha}{2} \left\| g' g(X_{t_i}) - g'(X_{t_{i-1}}) \right\|_{L^2(\Omega; \mathbb{R})}$$

$$+ \left\| \int_{t_i}^{t_{i-1}} \left[ g(X_s) - g(X_{t_i}) - g'(X_{t_i}) (W_s - W_{t_i}) \right] \, dW_s \right\|_{L^2(\Omega; \mathbb{R})} =: I_1 + I_2 + I_3. \hspace{1cm} (5.10)$$

Applying the Itô formula to the quadratic polynomial $f(x) = x(\mu - \alpha x), x \in \mathbb{R}_+$ yields

$$f(X_t) - f(X_s) = \int_s^t \left[ f'(X_r) f(X_r) + \frac{1}{2} f''(X_r) g^2(X_r) \right] \, dr + \int_s^t f'(X_r) g(X_r) \, dW_r, \hspace{1cm} (5.11)$$

and thus

$$I_1 \leq \int_{t_i}^{t_{i-1}} \int_s^t \left\| f'(X_r) f(X_r) + \frac{1}{2} f''(X_r) g^2(X_r) \right\|_{L^2(\Omega; \mathbb{R})} \, dr \, ds$$

$$+ \int_{t_i}^{t_{i-1}} \left( \int_s^t \left\| f(X_r) g(X_r) \right\|^2 \, dr \right)^{\frac{1}{2}} \, ds$$

$$\leq Ch^{\frac{3}{2}} \left( 1 + \sup_{s \in [0,T]} \|X_s\|^3_{L^6(\Omega; \mathbb{R})} \right), \hspace{1cm} (5.12)$$

where one used the Itô isometry and computed that $f'(x) f(x) = (\mu - \alpha x)(\mu - 2\alpha x)$, $f''(x) g^2(x) = -2\alpha \beta^2 x^3$ and $f'(x) g(x) = \beta (\mu - 2\alpha x) x^{\frac{3}{2}}$, $x \in \mathbb{R}_+$. Since $g'(x) g(x) = \frac{3}{2} \beta^2 x^2$, $x \in \mathbb{R}_+$ is also a quadratic polynomial, one can repeat the same lines as above to arrive at

$$I_2 \leq Ch^{\frac{3}{2}} \left( 1 + \sup_{s \in [0,T]} \|X_s\|^3_{L^6(\Omega; \mathbb{R})} \right). \hspace{1cm} (5.13)$$
Also, applying the Itô formula to \( g(x) = \beta x^\frac{3}{2} \) and \( g'(x)g(x) = \frac{3}{2} \beta^2 x^2 \), \( x \in \mathbb{R}_+ \), using the Itô isometry and considering Eq. 5.13, one can show

\[
|I_3|^2 = \int_{t_{i-1}}^{t_i} \| g(X_s) - g(X_{t_{i-1}}) - g'(X_{t_{i-1}})(W_s - W_{t_{i-1}}) \|^2_{L^2(\Omega; \mathbb{R})} ds \\
= \int_{t_{i-1}}^{t_i} \left[ \int_{t_{i-1}}^{s} \left[ g'(X_r) f(X_r) + \frac{1}{2} g''(X_r) g^2(X_r) \right] dr \\
+ \int_{t_{i-1}}^{s} \left[ g'(X_r) g(X_{t_{i-1}}) - g'(X_{t_{i-1}}) g(X_r) \right] dW_r \right]^2_{L^2(\Omega; \mathbb{R})} ds \\
\leq 2h \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \mathbb{E} \left[ \| g'(X_r) f(X_r) + \frac{1}{2} g''(X_r) g^2(X_r) \|^2 \right] dr \\
+ 2 \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \mathbb{E} \left[ \| g'(X_r) g(X_{t_{i-1}}) - g'(X_{t_{i-1}}) g(X_r) \|^2 \right] dr \\
\leq C h^2 \left( 1 + \sup_{s \in [0,T]} \| X_s \|^6_{L^6(\Omega; \mathbb{R})} \right),
\] (5.14)

where we computed that \( g'(x) f(x) = \frac{3}{2} \beta x^\frac{3}{2} (\mu - \alpha x), g''(x) g^2(x) = \frac{3}{4} \beta^3 x^3, x \in \mathbb{R}_+ \). Gathering the above three estimates together, we derive from Eq. 5.10 that

\[
\| R_i \|^2_{L^2(\Omega; \mathbb{R})} \leq C h^2 \left( 1 + \sup_{s \in [0,T]} \| X_s \|^3_{L^6(\Omega; \mathbb{R})} \right).
\] (5.15)

At the moment it remains to bound \( \mathbb{E}(R_i | \mathcal{F}_{t_{i-1}}) \| L^2(\Omega; \mathbb{R}^d) \), which, similarly to Eq. 4.30, can be decomposed into two terms by a triangle inequality:

\[
\mathbb{E}(R_i | \mathcal{F}_{t_{i-1}}) \| L^2(\Omega; \mathbb{R}) \leq \left\| \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \left( f(X_s) - f(X_{t_i}) \right) | \mathcal{F}_{t_{i-1}} \right] ds \right\|_{L^2(\Omega; \mathbb{R})} \\
+ \frac{h}{2} \left\| \mathbb{E} \left[ \left( g'(X_s) g(X_{t_{i-1}}) - g'(X_{t_{i-1}}) g(X_s) \right) | \mathcal{F}_{t_{i-1}} \right] \right\|_{L^2(\Omega; \mathbb{R})}.
\] (5.16)

Keeping Eq. 5.11 in mind, recalling that the Itô integral vanishes under the conditional expectation (see Eq. 4.31 for clarification) and utilizing the Jensen inequality, we derive

\[
\left\| \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \left( f(X_s) - f(X_{t_i}) \right) | \mathcal{F}_{t_{i-1}} \right] ds \right\|_{L^2(\Omega; \mathbb{R})} \\
= \left\| \int_{t_{i-1}}^{t_i} \int_{t_{i}}^{s} \mathbb{E} \left[ \left( f'(X_r) f(X_r) + \frac{1}{2} f''(X_r) g^2(X_r) \right) | \mathcal{F}_{t_{i-1}} \right] dr ds \right\|_{L^2(\Omega; \mathbb{R})} \\
\leq \int_{t_{i-1}}^{t_i} \int_{t_{i}}^{s} \left\| f'(X_r) f(X_r) + \frac{1}{2} f''(X_r) g^2(X_r) \right\|_{L^2(\Omega; \mathbb{R})} dr ds \\
\leq C h^2 \left( 1 + \sup_{s \in [0,T]} \| X_s \|^3_{L^6(\Omega; \mathbb{R})} \right).
\] (5.17)
Following the same arguments as before, one can derive
\[ \frac{h}{2} \left\| \mathbb{E} \left[ (g' g(X_t) - g' g(X_{t-1})) \bigg| \mathcal{F}_{t-1} \right] \right\|_{L^2(\Omega; \mathbb{R})} \leq C h^2 \left( 1 + \sup_{s \in [0, T]} \| X_s \|_{L^6(\Omega; \mathbb{R})}^3 \right). \] (5.18)

Plugging these two estimates into Eq. 5.16 results in
\[ \| \mathbb{E}(R_t | \mathcal{F}_{t-1}) \|_{L^2(\Omega; \mathbb{R})} \leq C h^2 \left( 1 + \sup_{s \in [0, T]} \| X_s \|_{L^6(\Omega; \mathbb{R})}^3 \right). \] (5.19)

Analogously to Eq. 5.8, the assumption \( \alpha \geq \frac{5}{2} \beta^2 \) ensures
\[ \sup_{t \in [0, T]} \| X_t \|_{L^6(\Omega; \mathbb{R})} < \infty. \] (5.20)

Thanks to Eq. 5.20 and Theorem 3.3, the assertion Eq. 5.9 follows based on Eqs. 5.15 and 5.19. \( \square \)

**Remark 5.3** Recall that strong convergence of the implicit Milstein scheme Eq. 5.2 for the \( \frac{3}{2} \) process was analyzed by Higham et al. [21], with no convergence rates recovered. Later in [44], with the aid of the Lamperti transformation, Neuenkirch and Szpruch [44] proposed a Lamperti transformed backward Euler method for a class of scalar SDEs in a domain including the \( \frac{3}{2} \) process as a special case. There a mean-square convergence rate of order 1 was proved for the Lamperti-backward Euler method solving the \( \frac{3}{2} \) process when the model parameters obey \( \frac{\alpha}{\beta^2} > 5 \) (see Propositions 3.2 from [44]). In this work we turn to the implicit Milstein scheme Eq. 5.2, covered by Eq. 2.2 and also studied in [21], and successfully prove a mean-square convergence rate of order 1 for the scheme on the condition \( \frac{\alpha}{\beta^2} > \frac{5}{2} \). This not only fills the gap left by [21], but also significantly relaxes the restriction put on the model parameters as required in [44].

### 5.2 The double implicit Milstein scheme for the stochastic Lotka-Volterra competition model

In this subsection, we consider the scalar stochastic Lotka-Volterra (LV) competitive model [41]
\[ dX_t = [b X_t - a X_t^2] \, dt + \sigma X_t \, dW_t, \quad X_0 = x_0 > 0 \] (5.21)

for a single species, where individuals within the species are competitive and \( b, a, \sigma \) are all positive numbers. The well-posedness of the model in the positive domain \((0, \infty)\) is known in the paper [41], where a positivity-preserving scheme is proposed, but with no convergence rate revealed. On the uniform mesh, we apply the double implicit
Milstein scheme Eq. 2.10 with $\theta = \eta = 1$ to numerically solve the model Eq. 5.21 as follows:

$$Y_{n+1} = Y_n + (bY_n - aY_n^2)h + \sigma Y_n \Delta W_n + \frac{1}{2} \sigma^2 Y_n^2 \Delta W_n^2 - \frac{1}{2} \sigma^2 Y_{n+1} h, \quad Y_0 = X_0.$$  (5.22)

Obviously, it is a quadratic equation and has a unique positive solution:

$$Y_{n+1} = \frac{-b + \sqrt{(b - a)h + \frac{3}{2} \sigma^2 h^2 + 4aY_n^2 \sigma^2 \Delta W_n^2}}{2ah} > 0.$$  (5.23)

given that $Y_n > 0$, $n \in \{0, 1, 2, \ldots, N - 1\}$. We highlight that no additional restriction is put on the stepsize $h > 0$ to ensure the positivity of the above approximations. Also, one can easily verify that Assumptions 3.1, 3.2 are both fulfilled in the domain $D = (0, \infty)$.

**Lemma 5.4** Let $b, a, \sigma > 0$, $X_0 > 0$. Then the stochastic LV competitive model Eq. 5.21 has a unique global solution in $(0, \infty)$ satisfying

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t|^p \right] < \infty, \quad \forall p \geq 2. \quad (5.24)$$

Moreover, the scheme Eq. 5.22 produces unique positivity preserving approximations given by Eq. 5.23. The SDE model Eq. 5.21 and the scheme Eq. 5.22 obey Assumptions 3.1, 3.2 in the domain $D = (0, \infty)$.

**Proof** of Lemma 5.4. The well-posedness of the model in the positive domain $(0, \infty)$ is known in [41] and the moment bound Eq. 5.24 comes from [41, Lemma 2.2]. As discussed above, the scheme Eq. 5.22 has a unique positive solution given by Eq. 5.23. Consequently, all conditions in Assumption 3.2 are satisfied with $D = (0, \infty)$. Now it remains to validate conditions in Assumption 3.1. For brevity, we denote the drift and diffusion coefficients of SDE Eq. 5.1 by

$$f(x) := bx - ax^2, \quad g(x) := \sigma x, \quad x > 0.$$  (5.25)

By setting $\theta = \eta = 1$, the conditions Eqs. 3.1, 3.2 in Assumption 3.1 reduce to

$$2 \langle x - y, f(x) - f(y) \rangle + (q - 1) \|g(x) - g(y)\|^2$$

$$+ \frac{9}{2} h \|g'(x)g(x) - g'(y)g(y)\|^2$$

$$\leq L_1 \|x - y\|^2,$$

$$\{x - y, [f(x) - f(y)] - \frac{1}{2} [g'(x)g(x) - g'(y)g(y)]\}$$

$$\leq L_2 \|x - y\|^2, \quad \forall x, y \in (0, \infty). \quad (5.26)$$

Note that the diffusion $g$ is a linear function and $g'(x)g(x) = \sigma^2 x$, $x \in \mathbb{R}_+$. is also linear. Further, note that $f'(x) = b - 2ax \leq b$, $\forall x \in (0, \infty)$. These facts ensure that conditions Eqs. 5.26-5.27 are both satisfied, which validates Assumption 3.1. □
Thanks to Lemma 5.4 and similarly to the proof of Theorem 5.2, we are now able to apply Theorem 3.3 to deduce the convergence rate of the numerical scheme Eq. 5.22.

**Theorem 5.5** Let \( b, a, \sigma > 0, X_0 > 0 \). Let \( \{X_t\}_{t \in [0,T]} \) and \( \{Y_n\}_{0 \leq n \leq N} \) be uniquely given by Eqs. 5.21 and 5.22, respectively. For \( h > 0 \) satisfying \((2b - \sigma^2)h < 1\), there exists a constant \( C > 0 \), independent of \( N \in \mathbb{N} \), such that

\[
\sup_{0 \leq n \leq N} \|Y_n - X_{t_n}\|_{L^2(\Omega; \mathbb{R})} \leq Ch.
\] (5.28)

By estimating \( \mathbb{E}[\| R_i \|^2] \) and \( \mathbb{E}[\| \mathbb{E}(R_i[F_{t_i}^{-1}]) \|^2] \), \( i \in \{1, 2, ..., N\} \), the proof of Theorem 5.5 is similar to that of Theorem 5.2 and omitted here. Different from Theorem 5.2 for the Heston-\( \frac{3}{2} \) volatility model, no further restriction is put on the parameters of the model Eq. 5.21 because the diffusion coefficient is linear and all required conditions are satisfied for full parameters \( b, a, \sigma > 0 \).

### 5.3 The semi-implicit Milstein scheme for the Ait-Sahalia-type interest rate model

The next SDE financial model that we aim to numerically investigate is the generalized Ait-Sahalia-type interest rate model [1], described by

\[
\text{d} X_t = (\alpha_{-1} X_{t-}^{-1} - \alpha_0 + \alpha_1 X_t - \alpha_2 X_t^\kappa) \, \text{d} t + \sigma X_t^\rho \, \text{d} W_t, \quad t > 0, \ X_0 = x_0 > 0,
\] (5.29)

where \( \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \sigma > 0 \) are positive constants and \( \kappa > 1, \rho > 1 \). Compared with the previous financial model Eq. 5.1, a complication in Eq. 5.29 is due to the drift containing a term \( \alpha_{-1} X_{t-}^{-1} \) that does not behave well near the origin. The well-posedness of the model Eq. 5.29 has been already shown in [48, Theorem 2.1] and we repeat it as follows.

**Proposition 5.6** Let \( \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \sigma > 0 \) be positive constants and \( \kappa > 1, \rho > 1 \). Given any initial data \( X_0 = x_0 > 0 \), there exists a unique, positive global solution \( \{X_t\}_{t \geq 0} \) to Eq. 5.29.

Recently, such a model has been numerically studied by many authors [9, 44, 48, 52], with an emphasis on introducing and analyzing various positivity preserving strong approximation schemes (see Remark 5.14 for more details). Different from numerical schemes introduced in [9, 44, 48, 52], we apply the newly proposed Milstein scheme to the model Eq. 5.29 with \( \kappa + 1 \geq 2\rho \), covering both the standard regime \( \kappa + 1 > 2\rho \) and the critical regime \( \kappa + 1 = 2\rho \), and successfully recover the expected mean-square convergence rate, by use of the previously obtained error bounds. Given a uniform mesh on the interval \([0, T]\) with the uniform stepsize \( h = \frac{T}{N}, N \in \mathbb{N}, T \in (0, \infty) \), we apply the proposed Milstein type scheme Eq. 2.2 with \( \theta = 1, \eta = 0 \) (called the semi-implicit Milstein method) to the above model Eq. 5.29 and obtain numerical approximations, given by \( Y_0 = X_0 \) and

\[
Y_{n+1} = Y_n + h[\alpha_{-1} Y_{n+1}^{-1} - \alpha_0 + \alpha_1 Y_{n+1} - \alpha_2 Y_{n+1}^\kappa]
+ \sigma Y_n^\rho \Delta W_n + \frac{1}{2} \rho \sigma^2 Y_{n+1}^{2\rho-1}(|\Delta W_n|^2 - h), \quad n \in \{0, 1, 2, ..., N - 1\}.
\] (5.30)
The next lemma concerns the well-posedness of the scheme Eq. 5.30, which can be easily checked based on the observation that the drift coefficient function satisfies a monotonicity condition (consult [48, Lemma 3.1] and Eq. 5.38).

**Lemma 5.7** Let conditions in Proposition 5.6 be all satisfied. For \( h \in (0, \frac{1}{\sigma_1}] \), the semi-implicit Milstein scheme Eq. 5.30 is well-defined in the sense that it admits a unique solution, preserving positivity of the underlying model Eq. 5.29.

For simplicity of notation in the following analysis, we update the definitions of functions \( f, g \) in Section 5.1 and denote the coefficients of SDE Eq. 5.29 by

\[
    f(x) := \alpha_1 x^{-1} - \alpha_0 + \alpha_1 x - \alpha_2 x^\kappa, \quad g(x) := \sigma x^\rho, \quad x \in \mathbb{R}_+.
\]

(5.31)

It is easy to check that

\[
    g'(x)g(x) = \rho \sigma^2 x^{2\rho - 1}, \quad f(x) - f(y) = \left( -\frac{\alpha_1}{x^\kappa} + \alpha_1 - \alpha_2 \frac{x^\kappa - y^\kappa}{x - y} \right)(x - y).
\]

(5.32)

In addition, for \( t \in [1, \infty) \) we introduce a function \( z_t: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) defined by

\[
    z_t(x, y) := \frac{x^{1 - \rho} - y^{1 - \rho}}{x - y}, \quad x, y \in \mathbb{R}_+.
\]

(5.33)

In the following error analysis for the numerical approximations, we cope with the standard case \( \kappa + 1 > 2\rho \) and the critical case \( \kappa + 1 = 2\rho \) separately, since different cases own different model properties.

### 5.3.1 The standard case \( \kappa + 1 > 2\rho \)

At first, we focus on the standard case \( \kappa + 1 > 2\rho \) and recall a lemma concerning (inverse) moment bounds of the solution to Eq. 5.29, quoted from [48, Lemma 2.1].

**Lemma 5.8** Let conditions in Proposition 5.6 be fulfilled with \( \kappa + 1 > 2\rho \) and let \( \{X_t\}_{t \geq 0} \) be the unique solution to Eq. 5.29. Then for any \( p \geq 2 \) it holds that

\[
    \sup_{t \in [0, \infty)} \mathbb{E}[|X_t|^p] < \infty, \quad \sup_{t \in [0, \infty)} \mathbb{E}[|X_t|^{-p}] < \infty.
\]

(5.34)

In order to achieve the mean-square convergence rate of the scheme by means of Theorem 3.3, we need to check all conditions required in Assumptions 3.1, 3.2, which are clarified in the forthcoming lemma.

**Lemma 5.9** Let conditions in Proposition 5.6 be all fulfilled with \( \kappa + 1 > 2\rho \) and let \( h \in (0, \frac{1}{\sigma_1}] \). Then the Ait-Sahalia model Eq. 5.29 and the scheme Eq. 5.30 obey Assumptions 3.1, 3.2 in the domain \( D = (0, \infty) \).

**Proof** of Lemma 5.9. Note first that the well-posedness of the model and the scheme in \( D = (0, \infty) \) has been proven in Proposition 5.6 and Lemma 5.7. It remains to confirm the other conditions. We first claim that, for any \( c > 0 \) there exists \( a_0 \in [0, \infty) \) such that \( z_{2\rho-1} \leq cz_\kappa + a_0 \), where we recall that \( z_t \) is defined by Eq. 5.33. Clearly,
\(z_{2\rho-1}(x, y) > 0\) and \(z_k(x, y) > 0\) for all \(x, y > 0\). Without loss of generality, we assume \(x > y > 0\). Since \(\kappa - 1 > 2\rho - 2\), for any \(c > 0\) one can find \(a_0 \in [0, \infty)\) such that \(\sup_{\nu>0} \left( (2\rho - 1)v^{2\rho-2} - ck\nu^{k-1} \right) \leq a_0\). As a consequence,

\[
(x - y)(z_{2\rho-1} - cz_k) = x^{2\rho-1} - y^{2\rho-1} - c(x^k - y^k)
\]

\[
= (x - y)\int_0^1 [(2\rho - 1)(y + \xi(x-y))^{2\rho-2} - c\kappa(y + \xi(x-y))^{k-1}] d\xi
\]

\[
\leq a_0(x-y), \quad \forall x > y > 0.
\]  

(5.35)

The claim is thus validated. So one can choose \(c < \frac{\sqrt{a_0}}{\sqrt[4]{\rho \sigma^2}}\) for some \(q > 1\) such that \(\frac{q}{2} \rho^2 \sigma^4 c^2 - \alpha_2^2 < 0\) and thus

\[
\frac{q}{2} \rho^2 \sigma^4 \|g'(x)g(x) - g'(y)g(y)\|^2 - \|f(x) - f(y)\|^2
\]

\[
= \left(\frac{q}{2} \rho^2 \sigma^4 \left(\frac{x^{2\rho-1} - y^{2\rho-1}}{x-y}\right)^2 - \left(\frac{q-1}{x-y} - \alpha_1 + \alpha_2 x^k y^k\right)^2\right)(x - y)^2
\]

\[
= h\left[\frac{q}{2} \rho^2 \sigma^4 z_{2\rho-1} - \left(\frac{q-1}{x-y} - \alpha_1 + \alpha_2 z_k\right)^2\right](x - y)^2
\]

\[
= h\left[\frac{q}{2} \rho^2 \sigma^4 z_{2\rho-1} - \left(\frac{q-1}{x-y} - \alpha_1 - \alpha_2 z_k\right)^2\right](x - y)^2
\]

\[
\leq h\left[\frac{q}{2} \rho^2 \sigma^4 z_{2\rho-1} - \alpha_2 z_k^2 + 2\alpha_1 \alpha_2 z_k\right](x - y)^2
\]

\[
= h\left[\frac{q}{2} \rho^2 \sigma^4 \|g'(x)g(x) - g'(y)g(y)\|^2 - \|f(x) - f(y)\|^2\right](x - y)^2
\]

\[
\leq Ch(x - y)^2, \quad \forall x, y \in \mathbb{R}_+.
\]  

(5.36)

Furthermore, one can readily compute that, for any \(\kappa + 1 > 2\rho\) and for some \(q > 2\),

\[
\sup_{x>0} \left( f'(x) + \frac{q-1}{2} g'(x)^2 \right) = \sup_{x>0} \left( -\alpha_{-1} x^{-2} + \alpha_1 - \alpha_2 \kappa x^{k-1} + \frac{(q-1)\rho^2 \sigma^2}{2}\right)
\]

\[
=: L < \infty.
\]  

(5.37)

This implies that

\[
\langle x - y, f(x) - f(y) \rangle + \frac{q-1}{2} \|g(x) - g(y)\|^2
\]

\[
= \int_0^1 f'(y + \xi(x-y))d\xi \cdot (x - y)^2 + \frac{q-1}{2} \left[\int_0^1 g'(y + \xi(x-y))d\xi \right]^2 \cdot (x - y)^2
\]

\[
\leq \int_0^1 \left[ f'(y + \xi(x-y)) + \frac{q-1}{2} g'(y + \xi(x-y))^2 \right] d\xi \cdot (x - y)^2
\]

\[
\leq L(x - y)^2, \quad \forall x, y \in \mathbb{R}_+.
\]  

(5.38)

Gathering Eqs. 5.36 and 5.38 together, the condition Eq. 3.1 is hence justified in the domain \(D = (0, \infty)\) with \(\theta = 1, \eta = 0\). From Eq. 5.38, one can assert that Eq. 3.2
is satisfied in $D = (0, \infty)$ with $\theta = 1, \eta = 0, L_2 = \alpha_1$. Thus all conditions in Assumption 3.1 fulfilled in the domain $D = (0, \infty)$. Assumption 3.2 follows by taking Proposition 5.6, Lemmas 5.7, 5.8 sideration.

At the moment, we are well prepared to carry out the error analysis for the numerical approximations with the help of Theorem 3.3.

**Theorem 5.10** Let $\{X_t\}_{t \in [0,T]}$ and $\{Y_n\}_{0 \leq n \leq N}$ be solutions to Eqs 5.29 and 5.30, respectively. Let $q \in (2, \infty)$, $p \in (1, \infty)$, let $\alpha_{-1, 1}, \alpha_0, \alpha_1, \alpha_2, \sigma > 0$, let $\kappa > 1, \rho > 1$ obey $\kappa + 1 > 2p$, and let $h \in (0, \frac{1}{2\alpha_1})$. Then there exists a constant $C > 0$, independent of $N \in \mathbb{N}$, such that

$$\sup_{1 \leq n \leq N} \|X_{tn} - Y_n\|_{L^2(\Omega; \mathbb{R})} \leq Ch. \quad (5.39)$$

**Proof** of Theorem 5.10. As implied by Lemma 5.9, all conditions in Assumptions 3.1, 3.2 are fulfilled in $D = (0, \infty)$. Based on Theorem 3.3, one just needs to properly estimate $\|R_i\|_{L^2(\Omega; \mathbb{R})}$ and $\|\mathbb{E}(R_i | F_{t_{i-1}})\|_{L^2(\Omega; \mathbb{R})}$, $i \in \{1, 2, \ldots, N\}$. Following the notation used in Eqs 3.6 and 5.31, we first split the estimate of $\|R_i\|_{L^2(\Omega; \mathbb{R})}$ as follows:

$$\|R_i\|_{L^2(\Omega; \mathbb{R})} \leq \left\| \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_i}) \, ds \right\|_{L^2(\Omega; \mathbb{R})} + \left\| \int_{t_{i-1}}^{t_i} g(X_s) - g(X_{t_{i-1}}) - g'(X_{t_{i-1}})(W_s - W_{t_{i-1}}) \, dW_s \right\|_{L^2(\Omega; \mathbb{R})} =: I_4 + I_5. \quad (5.40)$$

Repeating the same arguments as used in Eq. 5.12, we apply the Itô formula to $f(x) = \alpha_{-1}x^{-1} - \alpha_0 + \alpha_1x - \alpha_2x^2$, $x \in \mathbb{R}_+$ and use Lemma 5.8 to derive

$$I_4 \leq \int_{t_{i-1}}^{t_i} \|f(X_s) - f(X_{t_i})\|_{L^2(\Omega; \mathbb{R})} \, ds$$

$$\leq Ch^\frac{3}{2} \left( 1 + \sup_{s \in [0, T]} \|X_s\|^2_{L^2(\Omega; \mathbb{R})} + \sup_{s \in [0, T]} \|X_s^{-1}\|^3_{L^6(\Omega; \mathbb{R})} \right)$$

$$\leq Ch^\frac{3}{2}. \quad (5.41)$$

Similarly to Eq. 5.14, by means of the Itô isometry and the Itô formula applied to $g(x) = \sigma x^\rho$ and $g'(x)g(x) = \rho x^{2-2\rho} - 1$, $x \in \mathbb{R}_+$ one can show

$$|I_5|^2 \leq 2h \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \mathbb{E}[\|g'(X_r)f(X_r) + \frac{1}{2}g''(X_r)g^2(X_r)\|^2] \, dr \, ds + 2\int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \mathbb{E}[\|g'(X_r) - g'(X_{t_{i-1}})\|^2] \, dr \, ds$$

$$\leq Ch^3. \quad (5.42)$$
where the (inverse) moment bounds in Lemma 5.8 were also used for the last step. Inserting Eq. 5.41 and Eq. 5.42 into Eq. 5.40 implies

$$
\|R_i\|_{L^2(\Omega;\mathbb{R})} \leq C h^{\frac{3}{2}}.
$$

(5.43)

In the same spirit of Eq. 5.17, we rely on the use of the Itô formula applied to $f(x) = \alpha_{-1}x^{-1} - \alpha_0 + \alpha_1 x - \alpha_2 x^2$ to show

$$
\begin{align*}
\|\mathbb{E}(R_i|\mathcal{F}_{t_{i-1}})\|_{L^2(\Omega;\mathbb{R}^d)} &\leq \left\| \int_{t_{i-1}}^{t_i} \mathbb{E} \left( [f(X_s) - f(X_{t_i})] \right) \mathcal{F}_{t_{i-1}} \right\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq \int_{t_{i-1}}^{t_i} \left\| f'(X_r)f(X_r) + \frac{1}{2} f''(X_r)g^2(X_r) \right\|_{L^2(\Omega;\mathbb{R})} \, dr \, ds \\
&\leq C h^2,
\end{align*}
$$

(5.44)

where we recalled that the Itô integral vanishes under the conditional expectation and also used the Jensen inequality and Lemma 5.8. Armed with these two estimates, one can apply Theorem 3.3 to arrive at the desired assertion. \(\square\)

### 5.3.2 The critical case $\kappa + 1 = 2\rho$

In what follows we turn to the general critical case $\kappa + 1 = 2\rho$ and present first a lemma concerning (inverse) moment bounds of the solution process, which can be proved by following the same lines in the proof of Lemma 5.8 (cf. [48, Lemma 2.1]).

**Lemma 5.11** Let conditions in Proposition 5.6 be all fulfilled with $\kappa + 1 = 2\rho$ and let $\{X_t\}_{t \geq 0}$ be the unique solution to Eq. 5.29. Then we have, for any $2 \leq p_1 \leq \frac{\sigma^2 + 2\alpha_2}{\sigma^2}$ and for any $p_2 \geq 2$,

$$
\sup_{t \in [0, \infty)} \mathbb{E}[|X_t|^{p_1}] < \infty, \quad \sup_{t \in [0, \infty)} \mathbb{E}[|X_t|^{-p_2}] < \infty.
$$

(5.45)

For the purpose of analyzing the convergence rate of the numerical approximations, we validate all conditions of Assumptions 3.1, 3.2 in the next lemma, which is required by Theorem 3.3.

**Lemma 5.12** Let conditions in Proposition 5.6 be all fulfilled with $\kappa + 1 = 2\rho$ and let $h \in (0, \frac{1}{\alpha_1}]$. Let the model parameters obey $\frac{\alpha_2}{\sigma^2} \geq 2\kappa - \frac{3}{2}$ and $\frac{\alpha_2}{\sigma^2} > \frac{k+1}{2\sqrt{2}}$. Then the SDE model Eq. 5.29 and the scheme Eq. 5.30 satisfy Assumptions 3.1, 3.2 in the domain $D = (0, \infty)$.

**Proof** of Lemma 5.12. Recall that the well-posedness of the model and the scheme in $D = (0, \infty)$ has been proven in Proposition 5.6 and Lemma 5.7. It remains to verify the other conditions. Thanks to the assumptions $\kappa + 1 = 2\rho$ and $\frac{\alpha_2}{\sigma^2} > \frac{k+1}{2\sqrt{2}}$, one can
find \( \rho > 1 \) such that \( \alpha_2^2 > \frac{\rho^2}{2} \sigma^4 = \frac{\rho}{2} \kappa (\kappa + 2) \sigma^4 \) and thus

\[
\begin{align*}
&h \frac{\rho}{2} \| g'(x) - g'(y) \|^2 - h \| f(x) - f(y) \|^2 \\
&= h \left( \frac{\rho^2}{2} \sigma^4 \frac{\alpha_2^2}{2} - \left( \frac{\alpha_1}{2} - \alpha_1 + \alpha_2 \alpha_2 \right)^2 \right) (x - y)^2 \\
&= h \left( \frac{\rho^2}{2} \sigma^4 \frac{\alpha_2^2}{2} - \left( \frac{\alpha_1}{2} - \alpha_1 \right)^2 - \alpha_2 \alpha_2 \left( \frac{\alpha_1}{2} - \alpha_1 \right) \right) (x - y)^2 \\
&\leq h \left( \frac{\rho^2}{2} \sigma^4 - \alpha_2 \alpha_2 \left( \frac{\alpha_1}{2} - \alpha_1 \right) \right) (x - y)^2 \\
&\leq C h (x - y)^2, \quad \forall x, y \in \mathbb{R}_+.
\end{align*}
\] (5.46)

Noting \( \kappa + 1 = 2 \rho \) again, one can deduce from Eq. 5.37 that

\[
\begin{align*}
&\sup_{x > 0} \left( f'(x) + \frac{q-1}{2} \| g'(x) \|^2 \right) \leq \sup_{x > 0} \left( \alpha_1 \left( \alpha_2 \kappa - \frac{(q-1)\sigma^2}{2} \kappa \right) \right), \quad \forall x \in \mathbb{R}_+.
\end{align*}
\] (5.47)

Since \( \frac{\sigma_2^2}{\sigma^2} \geq 2 \kappa - \frac{3}{2} = \frac{\kappa + 1}{2} > \frac{1}{8} (\kappa + 2 + \frac{1}{2}) \) for \( \kappa > 1 \), one can find \( q > 2 \) such that \( \frac{\sigma_2^2}{\sigma^2} \geq \frac{q}{q-1} (\kappa + 2 + \frac{1}{2}) \), i.e., \( \alpha_2 \kappa - \frac{(q-1)\sigma^2}{2} = \alpha_2 \kappa - \frac{(q-1)\sigma^2}{2} (\kappa + 1) \geq 0 \) in Eq. 5.47, and thus, similarly to Eq. 5.38,

\[
(x - y, f(x) - f(y)) + \frac{q-1}{2} \| g(x) - g(y) \|^2 \leq \alpha_1 (x - y)^2, \quad \forall x, y \in \mathbb{R}_+.
\] (5.48)

Combining this with Eq. 5.46 ensures that the condition Eq. 3.1 is fulfilled in \( D = (0, \infty) \) with \( \theta = 1, \eta = 0 \). The condition Eq. 3.2 follows from Eq. 5.48 directly. Finally, since \( \frac{\sigma_2^2}{\sigma^2} \geq 4 \kappa - 2 > 2 \kappa \) by assumption \( \frac{\sigma_2^2}{\sigma^2} \geq 2 \kappa - \frac{3}{2}, \kappa > 1 \), in view of Lemma 5.11 one can infer

\[
\begin{align*}
&\sup_{t \in [0, T]} \| f_t \|_{L^2(\Omega; \mathbb{R})} < \infty \quad \text{and} \\
&\sup_{t \in [0, T]} \| f_t \|_{L^2(\Omega; \mathbb{R})} \leq \sup_{t \in [0, T]} \left( \alpha_1 \| f_t \|_{L^2(\Omega; \mathbb{R})} \right) + \alpha_2 \| f_t \| = C h (x - y)^2, \quad \forall x, y \in \mathbb{R}_+.
\end{align*}
\] (5.49)

Therefore, all conditions in Assumptions 3.1, 3.2 are confirmed in the domain \( D = (0, \infty) \).

Now we are in a position to derive the convergence order with the aid of Theorem 3.3.

**Theorem 5.13** Let \( \{ X_t \}_{t \in [0, T]} \) and \( \{ Y_n \}_{0 \leq n \leq N} \) be solutions to Eqs. 5.29 and 5.30, respectively. Let conditions in Proposition 5.6 be all fulfilled with \( \kappa + 1 = 2 \rho \) and let \( h \in \left( 0, \frac{1}{\alpha_1} \right) \). Let the model parameters \( \alpha_2, \alpha_1, \alpha_2, \sigma > 0, \kappa > 1, \rho > 1 \) obey

\[
\frac{\alpha_2^2}{\sigma^2} \geq 2 \kappa - \frac{3}{2} \quad \text{and} \quad \frac{\alpha_2^2}{\sigma^2} > \frac{\kappa + 1}{2 \sqrt{2}}.
\]

Then there exists a constant \( C > 0 \), independent of \( N \in \mathbb{N} \), such that

\[
\sup_{1 \leq n \leq N} \| X_t - Y_n \|_{L^2(\Omega; \mathbb{R})} \leq C h.
\] (5.50)
Proof of Theorem 5.13. As already verified in Lemma 5.12, all conditions in Assumptions 3.1, 3.2 are fulfilled in $D = (0, \infty)$. Based on Theorem 3.3, one only needs to properly estimate $\|R_i\|_{L^2(\Omega; \mathbb{R})}$ and $\|\mathbb{E}(R_i|\mathcal{F}_{t_{i-1}})\|_{L^2(\Omega; \mathbb{R})}$. Similarly as above, we split the error term $\|R_i\|_{L^2(\Omega; \mathbb{R})}$ into two parts:

$$
\|R_i\|_{L^2(\Omega; \mathbb{R})} \leq \left\| \int_{t_{i-1}}^{t_i} f(X_s) - f(X_{t_i}) \, ds \right\|_{L^2(\Omega; \mathbb{R})} + \left\| \int_{t_{i-1}}^{t_i} g(X_s) - g(X_{t_{i-1}}) - g'(X_{t_{i-1}})(W_s - W_{t_{i-1}}) \, dW_s \right\|_{L^2(\Omega; \mathbb{R})}
=: I_6 + I_7,
$$

(5.51)

where the coefficients $f, g$ are defined by Eq. 5.31. The Itô formula applied to $f(x) = \alpha_{-1} x^{-1} - \alpha_0 + \alpha_1 x - \alpha_2 x^\kappa, x \in \mathbb{R}_+$ gives

$$
I_6 \leq \int_{t_{i-1}}^{t_i} \|f(X_s) - f(X_{t_i})\|_{L^2(\Omega; \mathbb{R})} \, ds \\
\leq Ch^\frac{3}{2} \left( 1 + \sup_{s \in [0,T]} \|X_s\|_{L^{2\kappa-2}(\Omega; \mathbb{R})}^{2\kappa-1} + \sup_{s \in [0,T]} \|X_s\|_{L^\kappa(\Omega; \mathbb{R})}^{-3} \right).
$$

(5.52)

Following the same lines as in Eq. 5.42, one can similarly show

$$
|I_7| \leq Ch^\frac{3}{2} \left( 1 + \sup_{s \in [0,T]} \mathbb{E}[|X_s|^{4\kappa-2}] + \sup_{s \in [0,T]} \mathbb{E}[|X_s|^{-\gamma}] \right).
$$

(5.53)

where we set $1_{\{\rho < 2\}} = 1$ for $\rho < 2$ and $1_{\{\rho < 2\}} = 0$ for $\rho \geq 2$. Since $\frac{\sigma_2^2 + 2\alpha_2}{\sigma_2^2} \geq 4\kappa - 2$ by the assumption $\frac{\sigma_2^2}{\sigma_2^2} \geq 2\kappa - \frac{3}{2}$, we can plug these two estimates into Eq. 5.51 and use Lemma 5.11 to get

$$
\|R_i\|_{L^2(\Omega; \mathbb{R})} \leq Ch^\frac{3}{2}.
$$

(5.54)

Moreover, similarly to Eq. 5.44, applying the Itô formula to $f(x) = \alpha_{-1} x^{-1} - \alpha_0 + \alpha_1 x - \alpha_2 x^\kappa$ and noting the Itô integral vanishes under the conditional expectation we deduce

$$
\|\mathbb{E}(R_i|\mathcal{F}_{t_{i-1}})\|_{L^2(\Omega; \mathbb{R}^d)} \leq \int_{t_{i-1}}^{t_i} \mathbb{E}\left(\|f(X_s) - f(X_{t_i})\|_{L^2(\Omega; \mathbb{R})}^2\right) \, ds \leq \int_{t_{i-1}}^{t_i} \|f'(X_r)f(X_r) + \frac{1}{2} f''(X_r)g^2(X_r)\|_{L^2(\Omega; \mathbb{R})} \, dr \, ds \\
\leq Ch^\frac{3}{2} \left( 1 + \sup_{s \in [0,T]} \|X_s\|_{L^{2\kappa-2}(\Omega; \mathbb{R})}^{2\kappa-1} + \sup_{s \in [0,T]} \|X_s\|_{L^\kappa(\Omega; \mathbb{R})}^{-3} \right).
$$

(5.55)

In light of Lemma 5.11 and with the help of Theorem 3.3, one can obtain the assertion Eq. 5.50.

\[ \square \]
Remark 5.14 Recall that Szpruch et al. [48] examined the backward Euler method for the Ait-Sahalia model Eq. 5.29 and proved its strong convergence only when $\kappa + 1 > 2\rho$, but without revealing a rate of convergence. Very recently, the authors of [52] fill the gap by identifying the expected mean-square convergence rate of order $\frac{1}{2}$ for stochastic theta methods applied to the Ait-Sahalia model under conditions $\kappa + 1 \geq 2\rho$. In 2014, a kind of Lamperti-backward Euler method was introduced in [44] for the Ait-Sahalia model, with a mean-square convergence rate of order 1 identified for the full parameter range in the general standard case $\kappa + 1 > 2\rho$ and for a particular critical case $\kappa = 2$, $\rho = 1.5$ when $\frac{\alpha^2}{\sigma^2} > 5$ (see Propositions 3.5, 3.6 from [44]). As shown above, we apply the semi-implicit Milstein method Eq. 5.30 to the Ait-Sahalia model, which is able to treat both the general standard case and a more general critical case $\kappa + 1 = 2\rho$ for any $\kappa$, $\rho > 1$. Moreover, we prove a mean-square convergence rate of order 1 for the full parameter range in the general standard case and for parameters satisfying $\frac{\alpha^2}{\sigma^2} > 2\kappa - \frac{3}{2}$ and $\frac{\alpha^2}{\sigma^2} > \frac{\kappa + 1}{2\sqrt{2}}$ in the general critical case. For the special critical case $\kappa = 2$, $\rho = 1.5$, the restriction on parameters reduces into $\frac{\alpha^2}{\sigma^2} > \frac{5}{2}$, which is moderately more relaxed than $\frac{\alpha^2}{\sigma^2} > 5$ as required in [44].

5.4 Numerical tests

The aim of this subsection is to illustrate the above theoretical findings by providing several numerical examples. Two different schemes covered by Eq. 2.2 are utilized to simulate the two previously studied financial models. The resulting mean-square approximation errors are computed at the endpoint $T = 1$ and the desired expectations are approximated by averages over 10,000 samples. Moreover, the “exact” solutions are identified as numerical ones using a fine stepsize $h_{\text{exact}} = 2^{-12}$.

![Fig. 2](image-url) One-path simulations of the drift-diffusion double implicit Milstein method for the Heston $\frac{3}{2}$-volatility model (Left) and the stochastic LV model (Right)
As the first example, let us first look at the following SDE,

\[ dX_t = X_t (\mu - \alpha X_t) \, dt + (\beta X_t^{3/2} + \sigma X_t) \, dW(t), \quad X_0 = 1, \quad t \in (0, 1]. \tag{5.56} \]

When \( \sigma = 0 \) and \( \beta = 0 \), the considered SDE Eq. 5.56 reduces to the Heston \( \frac{3}{2} \)-volatility model Eq. 5.1 and the stochastic LV competitive model Eq. 5.21, respectively. We choose the parameters \((\mu, \alpha, \beta, \sigma) = (2, \frac{5}{2}, 1, 0)\) such that \( \alpha \geq \frac{5}{2} \beta^2 \) for the \( \frac{3}{2} \)-model Eq. 5.1 and \((\mu, \alpha, \beta, \sigma) = (2, 1, 0, 1)\) for the stochastic LV competitive model Eq. 5.21. By taking \( \theta = \eta = 1 \), we discrete these two models by the drift-diffusion double implicit Milstein method Eq. 2.2, which is explicitly solvable here (see Eqs. 5.3 and 5.23). In the following simulations, the expectations are approximated by computing averages over \( 10^4 \) samples and the “exact” solutions are identified as approximations using a fine stepsize \( h_{\text{exact}} = 2^{-12} \). It turns out that the resulting numerical approximations always remain positive for all \( 10^4 \) paths. In Fig. 2, we present one-path simulations of the drift-diffusion double implicit Milstein method for the Heston \( \frac{3}{2} \)-volatility model (Left) and the stochastic LV model (Right), which are shown to be positive. To test the mean-square convergence rates, we depict in Fig. 3 mean-square approximation errors \( e_h \) against six different stepsizes \( h = 2^{-i}, i = 4, 5, \ldots, 9 \) on a log-log scale. Also, two reference lines of slope 1 and \( \frac{1}{2} \) are given there. From Fig. 3 one can easily detect that the approximation errors decrease at a slope close to 1 when stepsizes shrink, coinciding with the predicted convergence order obtained in Theorems 5.2 and 5.5. Suppose that the approximation errors \( e_h \) obey a power law relation \( e_h = Ch^\delta \) for \( C, \delta > 0 \), so that \( \log e_h = \log C + \delta \log h \). Then we do a least squares power law fit for \( \delta \) and get the value 0.9923 for the rate \( \delta \) with residual of 0.0719. Again, this confirms the expected convergence rate in Theorems 5.2 and 5.5.

As the second example model, we look at the Ait-Sahalia interest rate model, given by

\[ dX_t = (\alpha_{-1} X_t^{-1} - \alpha_0 + \alpha_1 X_t - \alpha_2 X_t^\rho) \, dt + \sigma X_t^\rho \, dW_t, \quad X_0 = 1, \quad t \in (0, 1]. \tag{5.57} \]

Let us consider both the standard case \( \kappa + 1 > 2\rho \) and the critical case \( \kappa + 1 = 2\rho \), by taking two sets of model parameters:

- Case I: \( \kappa = 4, \rho = 2, \alpha_{-1} = \frac{3}{2}, \alpha_0 = 2, \alpha_1 = 1, \alpha_2 = 1, \sigma = 1; \)
- Case II: \( \kappa = 3, \rho = 2, \alpha_{-1} = \frac{3}{2}, \alpha_0 = 2, \alpha_1 = 1, \alpha_2 = \frac{9}{2}, \sigma = 1. \)

It is easy to check that Case I corresponds to the standard case and Case II corresponds to the critical case \( \kappa + 1 = 2\rho \) satisfying \( \frac{\alpha_0}{\alpha_2} \geq \frac{\kappa + 1}{\kappa - \frac{3}{2}} \) and \( \frac{\alpha_0}{\alpha_2} > \frac{\kappa + 1}{2\sqrt{2}} \). The semi-implicit Milstein scheme Eq. 5.30 is used to simulate the model Eq. 5.57 for these two cases. As shown in Fig. 4, the mean-square approximation error lines have slopes close to 1 for both cases. A least squares fit produces a rate 0.9798 with residual of 0.0929 for Case I and a rate 1.0129 with residual of 0.0968 for Case II. Hence, numerical results are consistent with strong order of convergence equal to one, as already revealed in Theorems 5.10 and 5.13.
Mean-square convergence rates of implicit stepsizes

Fig. 3 Mean-square convergence rates of the drift-diffusion double implicit Milstein method for the Heston $\frac{3}{2}$-volatility model (Left) and the stochastic LV model (Right)

Mean-square errors for approximations of the Ait-Sahalia model: Case I

Fig. 4 Mean-square convergence rates of the semi-implicit Milstein method Eq. 5.30 for the Ait-Sahalia interest rate model (Left for Case I and right for Case II)

6 Conclusion

The present work introduces a family of implicit Milstein type methods for strong approximations of stochastic differential equations (SDEs) with non-globally Lipschitz drift and diffusion coefficients. An easy and direct approach of the error analysis is developed to recover the expected mean-square convergence rate of order one for
the proposed schemes. In particular, the optimal convergence rate of the positivity preserving schemes applied to three models in practice is obtained for the first time and more relaxed conditions are required, compared with existing results for first order schemes in the literature. In the future, we attempt to identify the general $L^p$ rate of convergence with $p \geq 2$ for the schemes, which is highly non-trivial.

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** Declarations**

**Conflict of interest** The author declares no competing interests.

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