Stationary Phase Method in Discrete Wigner Functions and Classical Simulation of Quantum Circuits

Lucas Kocia  
National Institute of Standards and Technology, Gaithersburg, Maryland, 20899, U.S.A.

Peter Love  
Department of Physics, Tufts University, Medford, Massachusetts 02155, U.S.A.

We apply the periodized stationary phase method to discrete Wigner functions of systems with odd prime dimension using results from $p$-adic number theory. We derive the Wigner-Weyl-Moyal (WWM) formalism with higher order $\hbar$ corrections representing contextual corrections to non-contextual Clifford operations. We apply this formalism to a subset of unitaries that include diagonal gates such as the $\pi/2$ gates. We characterize the stationary phase critical points as a quantum resource injecting contextuality and show that this resource allows for the replacement of the $p^n$ points that represent $t$ magic state Wigner functions on $p$-dimensional qudits by $\leq p^j$ points. We find that the $\pi/8$ gate introduces the smallest higher order $\hbar$ correction possible, requiring the lowest number of additional critical points compared to the Clifford gates. We then establish a relationship between the stabilizer rank of states and the number of critical points necessary to treat them in the WWM formalism. This allows us to exploit the stabilizer rank decomposition of two qutrit $\pi/8$ gates to develop a classical strong simulation of a single qutrit marginal on $t$ qutrit $\pi/8$ gates that are followed by Clifford evolution, and show that this only requires calculating $3^{t+1}$ critical points corresponding to Gauss sums. This outperforms the best alternative qutrit algorithm (based on Wigner negativity and scaling as $\sim 3^{9/8}$ for $10^{-2}$ precision) for any number of $\pi/8$ gates to full precision.

I. INTRODUCTION

There has been much recent interest in efficient algorithms for the classical simulation of quantum circuits. As the age of noisy intermediate scale quantum (NISQ) computers approaches, we are reaching the limit of quantum circuits that can be validated by classical computers [1].

There have been many recent algorithms developed [2–8] for qubits. There are comparably fewer for higher-dimensional qudits [9,10] and part of the goal of this paper is to address this deficit. More significantly, this paper aims to bridge the gap between strong simulation qubit methods, which favor stabilizer rank representation of magic states, and odd-dimensional qudit methods, which currently favor Wigner negativity. We introduce a novel qutrit algorithm that is more efficient than the current state-of-the-art algorithm based on Wigner negativity and is directly reliant on stabilizer rank decomposition, through the perspective of higher order $h$ corrections introduced by a discrete and periodized form of the stationary phase method. Many physicists are aware of the stationary phase method through its energy-dependant form in the continuous case, where it is often called the WKB approximation [11–15].

Quantum circuits treated in terms of a path integral formalism require terms at orders greater than $\hbar^0$ for them to attain quantum universality [16–18]. In the continuous case, this corresponds to requiring non-Gaussianity as a resource. In the discrete case, this non-Gaussianity is often presented in terms of the negativity of Wigner functions or contextuality [19,20]. The resource theory of contextuality in finite-dimensional systems has been studied recently [21,22].

In the present paper we develop a method to use contextuality as a resource in finite-dimensional systems using the discrete version of the method of stationary phase. We introduce a “periodized” form of the stationary phase method to discrete Wigner functions with odd prime dimension using results from $p$-adic number theory and explain how it differs from the continuous version of the stationary phase method. This allows us to derive the Wigner-Weyl-Moyal formalism with asymptotically decreasing order $h$ corrections to a subset of unitaries that include the $T$-gate and $\pi/8$ gates. These serve as contextual corrections to non-contextual Clifford operations.

Fundamentally, the arithmetic we will be employing is very simple: we will split sums over exponentials, $I$, into local sums $I_{x^j}$:

\[
I = \sum_{x \in (\mathbb{Z}/p^m\mathbb{Z})^n} \exp \left[ \frac{2\pi i}{p^m} S(x) \right] \\
= \sum_{\bar{x} \in (\mathbb{Z}/p^j\mathbb{Z})^n} \left\{ \sum_{x \mod p^j = \bar{x}} \exp \left[ \frac{2\pi i}{p^m} S(x) \right] \right\} \\
\equiv \sum_{\bar{x} \in (\mathbb{Z}/p^j\mathbb{Z})^n} I_{\bar{x}},
\]

where $j < m$. Such a splitting done naively produces a sum over $p^j$ terms each containing a sum over $p^{m-j}$ terms, with no reduction in the cost of the original sum of $p^m$ terms. However, we will see that we can obtain closed forms for the local sums, thereby reducing the number of terms to just $p^j$. 

Related approaches include recent proposals to use non-Gaussianity as a resource in continuous (infinite dimensional) quantum mechanics, especially optics \cite{23,24} as well as the well-established traditional field dealing with semiclassical propagators in continuous systems \cite{25,26}.

This paper’s new theoretical contributions to the field can be organized into three main points:

1. The paper introduces a formal extension of the Wigner-Weyl Moyal (WWM) formalism for discrete odd prime dimensions to higher order \( \hbar \) corrections through uniformization.

2. This extension establishes a new relationship between stabilizer rank of states, Wigner phase space dimension and a bound on number of critical points.

3. This result offers a more efficient alternative to negativity as the vehicle of non-contextuality in implementations of gate extensions to the Clifford gates with low magic state stabilizer ranks.

As a result of these theoretical advancements, we obtain a new method for classical simulation of quantum circuits involving Clifford+\( \frac{\pi}{8} \) gates for qutrits that scales better than the leading existing method and is useful for intermediate-sized circuits.

This paper is organized as follows: in Section II we review the stationary phase method in continuous systems. In Section III we briefly introduce aspects of the WWM formalism and develop its quantum channel representation of gates. We introduce closed-form results for quadratic Gauss sums in Section IV which allows us to demonstrate that Clifford unitaries correspond to single quadratic Gauss sums and so are free operators in our WWM resource theory. This motivates us to introduce the stationary phase method for discrete odd prime-dimensional systems in Section V and to introduce non-free operations through the perspective of higher than quadratic Gauss sum uniformization. Section VI compares and characterizes this discrete stationary phase method with its continuous analog. The periodized stationary phase method is then used to evaluate the qutrit \( \frac{\pi}{8} \) gate magic state in Section VII producing an expression with \( 3^t \) critical points for \( t \) states. We then establish a relationship between stabilizer rank, Wigner phase space dimension and number of critical points in Section VIII and define them as our measure of contextuality or non-free resource. These results allow us to leverage the optimal stabilizer decomposition of pairs of \( \frac{\pi}{8} \) gate magic states in Section IX and produce an expression with \( 3^t \) critical points for \( t \) states. We discuss future directions of study in Section X and close the paper with some concluding remarks in Section XI.

II. STATIONARY PHASE IN CONTINUOUS SYSTEMS

The stationary phase method is an exact method of reexpressing the integral of an exponential of a function in terms of a sum of contributions from critical points:

\[
\int_{-\infty}^{\infty} dx e^{i\mathcal{S}[x]} = \sum_j \int \mathcal{D}[x_j] e^{\frac{i}{\hbar} \mathcal{S}[x_j] + \frac{1}{2} \delta \mathcal{S}[x_j] + \ldots},
\]

where generally \( \mathcal{S}[x_j] \) is a functional over \( x_j \), and the sum is indexed by critical “trajectories” \( j \) defined by

\[
\frac{\delta \mathcal{S}[x]}{\delta x} \bigg|_{x=x_j} = 0,
\]

and we have chosen to exercise our freedom to factor out the term \( \frac{1}{\hbar} \) from \( \mathcal{S} \), for later clarity. These contributions from critical points correspond to Gaussian integrals, or Fresnel integrals in the general case, which can be analytically evaluated, or higher-order uniformizations corresponding to two or more critical points. In general, the higher order contributions asymptotically decrease in significance. When \( \mathcal{S} \) is the classical action of a particle, Eq. [3] defines the critical points as classical trajectories parametrized by time that satisfy the Hamiltonian associated with \( \mathcal{S} \). (In this case, \( \mathcal{S} \) is functional of \( x \) because it is the integral of the Lagrangian over time, the latter of which is a function of \( x \) and \( \dot{x} \)).

Terminating Eq. [3]’s expansion of \( \mathcal{S}[x] \) at second order corresponds to making a first order approximation in \( \hbar \):

\[
\int_{-\infty}^{\infty} dx e^{i\mathcal{S}[x]} = \sum_j \left[ \frac{-\frac{\partial^2 \mathcal{S}_j}{\partial x_j \partial x_j'}}{2\pi i \hbar} \right]^{1/2} e^{i \mathcal{S}(x_j, x'_j) / \hbar} + \mathcal{O}(\hbar^2),
\]

where the sum is over all classical paths that satisfy the boundary conditions \( \frac{\partial \mathcal{S}}{\partial x_j} = \frac{\partial \mathcal{S}}{\partial x_j'} = 0 \).

Here we will be interested in double-ended propagators, or quantum channels, and so discuss the stationary phase approximation in reference to two integrals. Given functions \( \mathbf{x}(x'') \) and \( \mathbf{x}'(x'''') \), the propagator between them can be written as

\[
U(\mathbf{x}, \mathbf{x}', S) = \int_{-\infty}^{\infty} d\mathbf{x}'' \int_{-\infty}^{\infty} d\mathbf{x}''' e^{i S(\mathbf{x}'', \mathbf{x}'''; x'') / \hbar} \mathbf{x}'(\mathbf{x'''}))
\]

\[
= \sum_j \left| \frac{\partial^2 S}{\partial \mathbf{x} \partial \mathbf{x}'} \right|^{1/2} e^{i S_j(\mathbf{x}, \mathbf{x}'') / \hbar} + \mathcal{O}(\hbar^2)
\]

where the action \( S \) is assumed to be a polynomial in \( \mathbb{C}[x_1'', \ldots, x_n'', x_1''', \ldots, x_m'''] \) and \( S_j \) corresponds to the action expanded around a critical point of \( S \)—a “local” term—and is a quadratic polynomial in \( \mathbb{C}[x_1', \ldots, x_n', x_1''', \ldots, x_m'''] \).

We have purposely been general with our choice of representation \( x \) and \( x' \). If one chooses to represent the
propagator in terms of initial and final position, \(|q\rangle\) and \(|q'|\) respectively, then the above equation becomes

\[
U(q, q', S) = \int_{-\infty}^{\infty} dq'' \int_{-\infty}^{\infty} dq''' e^{iS(q'' q''' + qt'/t'')} / \hbar (q''' | q')
\]

and the first term is often called the Van Vleck propagator or, more precisely, the Van Vleck-Morette-Gutzwiller propagator [28-30].

Instead of this position representation, we will employ the “center-chord” representation because it is more suitable for discrete Hilbert spaces [31]. This representation produces the Wigner-Weyl-Moyal (WWM) formalism and allows for stationary phase expansions around the centers \(x\) and \(x'\). We briefly develop aspects of this formalism that are pertinent to gate concatenation in the next section.

### III. DISCRETE WIGNER FUNCTIONS: WIGER-WEYL-MOYAL FORMALISM

The discrete Wigner formulation of quantum mechanics (the discrete WWM formalism) is equivalent to the matrix and path integral representations that are more commonly known [32, 33]. A complete description of the odd-dimensional WWM formalism can be found elsewhere [31, 33, 34]. Here we introduce the basic formalism of Weyl symbols of multiple gates, which will be necessary in order to develop a propagator-like treatment.

#### A. Preliminaries

We consider the case of \(d\) odd. We will later restrict this further to \(d = p^h\) for \(p\) odd prime and \(h > 0\), not to be confused with the reduced Planck’s constant. We set \(\hbar = \frac{d}{2^h}\).

Consistent with [31, 33, 34], we define the symplectic matrix

\[
\mathcal{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},
\]

for \(I_n\) the \(n\)-dimensional identity.

The Wigner function of a state \(\hat{\rho}\) can be written

\[
\rho_x(x, x_q) = \text{Tr} \left( d^{-n} \sum_{\lambda \in (\mathbb{Z}/d\mathbb{Z})^n} d\lambda \exp^{i \mathcal{J} \lambda^T J x e^{-\pi x \lambda_p \lambda_q \hat{X} \hat{Y} \hat{\lambda}_p \hat{\lambda}_q \hat{\rho}} \right),
\]

where \(\hat{X}\) and \(\hat{Z}\) are the corresponding generalized \(d\)-dimensional Pauli operators.

The generalized symplectic matrix for \(n\) degrees of freedom can be defined through the introduction of the matrix \(\mathcal{H}_n\):

\[
\mathcal{J}_n \mathcal{H}_n^{-1} = \mathcal{H}_n^{-1} \mathcal{J}_n = \begin{pmatrix} 0 & J & -J & \cdots \\ -J & 0 & J & \cdots \\ J & -J & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\]

The symplectic area for an odd number of centers is defined to be

\[
\Delta_{2n+1}(x, x_1, \ldots, x_{2n}) = (x_1 - x, \ldots, x_{2n} - x) \mathcal{H}_n^{-1} \mathcal{J}_n (x_1 - x, \ldots, x_{2n} - x).
\]

For instance,

\[
\Delta_3(x, x_1, x_2) = 2x^T \mathcal{J} (x_1 - x_2) + 2x_1^T \mathcal{J} x_2,
\]

and

\[
\Delta_5(x, x_1, x_2, x_3, x_4) = (x_1 - x, x_2 - x, x_3 - x, x_4 - x)^T \mathcal{H}_4^{-1} \mathcal{J}_4 (x_1 - x, x_2 - x, x_3 - x, x_4 - x)
\]

\[
= 2x_2^T \mathcal{J} (x_3 - x_1) - 2x_1^T \mathcal{J} (x_3 + x_1) + 2x_4^T \mathcal{J} x_2 + 2x_1^T \mathcal{J} x_2 + 2x_4^T \mathcal{J} (x_3 + x_1 - x_4 - x_2) + 2x_5^T \mathcal{J} x_1.
\]

The Weyl symbol of a product of an even \(2n\) number of operators \(A_{2n}, \ldots, A_1\) has a phase determined by this

\[
\frac{1}{d^{2nN}} \sum_{x_1, \ldots, x_{2n}} A_{2n}(x_{2n}) \cdots A_1(x_1) \times \exp \left[ \frac{i}{\hbar} \Delta_{2n+1}(x, x_1, \ldots, x_{2n}) \right].
\]
Notice here the convention of associating a factor of $d^{-N}$ to every sum over $x_i$, a 2N-vector corresponding to a conjugate pair of N-dimensional momenta and position: $x \equiv (x_p, x_q)$. This is a convention that we will continue to use throughout this paper. For instance, we expect

$$d^{-N} \sum_x A(x) = 1,$$  \hspace{1cm} (13)

for any quantum gate $\hat{A} = \hat{U}$ or density matrix state $\hat{A} = \hat{\rho}$. We can extend the definition given in Eq. 12 to an odd number $(2n-1)$ of products by considering the Weyl symbols of 2n products of operators and setting $\hat{A}_{2n} = \hat{I}$, which has the corresponding Weyl symbol $I(x) = 1$. This leaves the $x_{2n}$ variable alone in the argument of the exponential, free to be summed over to produce a Kronecker delta function. The remaining exponentiated terms are defined to be $\Delta_{2n}(x, x_1, \ldots, x_{2n-1})$. Therefore, this approach is also a way to extend the definition of the symplectic area given in Eq. 2 to include an even number of centers.

As an example that will be useful later, consider the Weyl symbol of the product of the four operators $I A_3 A_2 A_1$:

$$A_3 A_2 A_1(x) = \frac{1}{d^{2N}} \sum_{x_3, x_2, x_1 \in (\mathbb{Z}/d\mathbb{Z})^{2N}} A_3(x_3) A_2(x_2) A_1(x_1)$$ \hspace{1cm} (14)

$$\times \delta(x_3 - x_2 + x_1 - x) \exp \left[ \frac{i}{\hbar} \Delta_4(x, x_1, x_2, x_3) \right],$$

where

$$\Delta_4(x, x_1, x_2, x_3) = 2x_2^T J x_3 - 2x_1^T J x_3$$ \hspace{1cm} (15)

$$+ 2x_1^T J x_2 + 2x_1^T J (x_1 + x_3 - x_2).$$

Notice that the prefactor fell by $d^{2N}$ compared to Eq. 12 since that is the dimension of the vector inside the Kronecker delta function. This agrees with our established standard of a factor of $d^{-N}$ for every sum over $x_i$ since after the Kronecker delta function is summed over, the resulting expression involves only a double sum over phase space and so we expect two factors of $d^{-N}$.

With the WWM formalism for the Weyl symbols of products of operators thus established, we move on to consider quantum channels, which are double-ended propagators.

### B. Discrete Weyl Double-Ended Propagator

In general, past treatment of the quantum propagation of a state $\hat{\rho}$ by a unitary $\hat{U}$ in the WWM formalism have dealt with Eq. 12 for $2m = 4$ all together, setting $\hat{A}_4 = \hat{I}$, $\hat{A}_2 = \hat{\rho}'$, and $\hat{A}_3 = \hat{U}^\dagger \equiv \hat{U}$. However, such a treatment intimately ties the propagator to the initial and final states, $\hat{\rho}'$ and $\hat{\rho}$ respectively, with intermediate phases $\Delta_5$, and as a result does not produce a self-contained double-ended propagator (a quantum channel) that acts on states by integration without any additional functions or phases, as in Eq. 3. A double-ended propagator form is often both more familiar and more useful for generality.

This is straightforward to remedy by defining $UU^*(x, x')$ to be the Weyl symbol of such a quantum channel, which contains Eq. 12 for $2m = 4$, with $A_4(x_4) = 1$ and $x_4$ summed away, and $A_2(x_2) = \hat{\rho}'$ and its sum over $x_2$ not taken, and $A_3(x) = A_1^\dagger(x) = U(x)$. Relabelled the variables of summation, this produces:

$$UU^*(x, x') = \frac{1}{d^N} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/d\mathbb{Z})^{2N}} U(x_1) U^*(x_2)$$ \hspace{1cm} (16)

$$\times \exp \left[ \frac{2\pi i}{d} \Delta_5(x_3, x, x_1, x', x_2) \right],$$

where we also used the identity $\Delta_5(x, x_1, x_2, x_3) = \Delta_5(x_3, x, x_1, x', x_2)$. This is the same as using Eq. 14 for $A_2(x_2) = \hat{\rho}'(x_2)$ and not taking the sum over $x_2$.

Notice in Eq. 16 the inclusion of the full phase $\Delta_5$ in the expression. This allows us to rewrite Eq. 12 in the familiar simpler form of a quantum channel:

$$(U \rho U^\dagger)(x) = \frac{1}{d^{2N}} \sum_{x' \in (\mathbb{Z}/d\mathbb{Z})^{2N}} UU^*(x, x') \rho(x').$$  \hspace{1cm} (17)

Our choice of normalization in Eq. 16 allows us to continue the standard of including a factor of $d^{-N}$ to every sum over a pair of conjugate phase space degrees of freedom in Eq. 17. This means that $d^{-2N} \sum_{x, x' \in (\mathbb{Z}/d\mathbb{Z})^{2N}} UU^*(x, x') = d^{2N}$ while $d^{-N} \sum_{x \in (\mathbb{Z}/d\mathbb{Z})^{2N}} UU^\dagger(x) = d^N = Tr(\hat{U} \hat{U}^\dagger)$.

The notation $UU^*(x, x')$ for the Weyl symbol of this propagator is perhaps clunky, but we hope that Eq. 16 motivates why we chose to denote it the way we did. To avoid any confusion, we point out that $UU^*(x, x') \neq UU^\dagger(x) = 1$; $UU^\dagger(x)$ is the Weyl symbol of $\hat{U} \hat{U}^\dagger = \hat{I}$ and so is a function of only one variable, $x$ (although it turns out to be independent of it); $UU^*(x, x')$ can most appropriately be associated with the Weyl symbol of the superoperator $\circ \hat{U} \bullet U^\dagger$, where $\circ$ and $\bullet$ denote the operators the superoperator acts on, and so is a function of two variables, denoted $x$ and $x'$.

As added incentive, $UU^*(x, x')$ is naively a simpler function to deal with than the Weyl symbol of a unitary operator, $U(x)$. $UU^*(x, x')$ is real-valued just like the Weyl symbols ($\rho(x)$) of density functions (Wigner functions), while $U(x)$ is generally complex-valued. Also, $UU^*(x, x')$ resembles traditional propagators more closely in that it can be said to take states from $x'$ to $x$, whereas $U(x)$ requires pairing with $U^*(x)$ and summation over intermediate values with the appropriate phase as in Eq. 14 to act on a state.

For Clifford gates $\hat{U}$, the associated Weyl symbol $U(x)$ is a non-negative map; $U(x)$ takes non-negative states
to non-negative states. This is clearer for $UU^*(\mathbf{x}, \mathbf{x}')$, which actually becomes a non-negative real function: $UU^*(\mathbf{x}, \mathbf{x}') \geq 0$. There is a similar simplification that occurs when $\hat{\rho}$ is a stabilizer state: the Wigner function $\rho(\mathbf{x})$ is a non-negative function for stabilizer states. Since Clifford gates take stabilizer states to stabilizer states, classical simulation of stabilizer states under Clifford gate evolution using this discrete Wigner formulation can be accomplished in polynomial time and has been shown to be non-contextual. This is really just a restatement of the Gottesmann-Knill theorem [31].

However, Clifford gates and stabilizer states do not allow for universal quantum computation (even with Pauli measurements). The extension of this set by $\frac{p}{q}$ gates or magic states to allow for universal quantum computation introduces contextuality into $UU^*(\mathbf{x}, \mathbf{x}')$ or $\rho(\mathbf{x})$, respectively, which means that they can now also have negative values. This negativity is equivalent to contextuality [33–38]. Since the contextual part of a classical simulation is solely responsible for its transition from polynomial to exponential scaling, it is useful to treat contextuality as an operational resource [39] that can be added to a polynomially-efficient classical backbone to allow it to simulate a quantum process appropriately. The goal of any classical simulation of such a contextuality injection scheme should be a classical computation cost that scales as efficiently as possible with the contextuality present.

Previous attempts at developing a framework for classical algorithms for simulation of qudit quantum circuits that leverage contextuality have done so indirectly by using negativity [9]. Recently, there have also been some more direct approaches [22]. The relationship between contextuality and higher orders of $h$ in the WWM formalism has been recently established [16–18]. Therefore, a more natural (and historically successful, if continuous systems are included) approach that remains unexplored is to use a semiclassical approach, wherein contextual corrections to non-contextual backbone are added with higher order $h$ expansions using the stationary phase method. Previous derivation of WWM in odd dimensions [31] [33] [34] have not been able to accomplish this because they begin by using results from the continuous case of the WWM formalism, where the stationary phase method is able to be applied in its traditional form, and then “discretize” and “periodize”. This does not allow for the derivation of higher order $h$ corrections because anharmonic trajectories cannot be obtained by “periodizing” to a discrete Weyl phase space grid [31].

Here we are able to develop a method that includes such higher order terms for discrete odd prime-dimensional systems. We never appeal to the continuous case and instead use a periodized version of Taylor’s theorem to develop the stationary phase method in the discrete setting. The resultant method offers an alternative opportunity for using contextuality efficiently as a resource in the classical simulation of quantum circuits.

IV. GAUSS SUMS

We now restrict to $d = p^h$ for $p$ odd prime and $h \in \mathbb{Z}$. We refer to the $p$-adic integers $\mathbb{Z}_p$ and $p$-adic numbers $\mathbb{Q}_p$, and the valuation of their elements $x$ as $\nu_p(x)$. See Appendix A for a brief introduction to this terminology.

Consider, as in [40] [41], the Gauss sum

$$G_h(A, v) = p^{nh/2} \sum_{x \in (\mathbb{Z}/p^h\mathbb{Z})^n} \exp \left[ \frac{2\pi i}{p^h} \left( \mathbf{x}^T \frac{A}{2} \mathbf{x} \right) \right] \times \exp \left[ \frac{2\pi i}{p^h} v^T \mathbf{x} \right],$$

and where for $h \leq 0$ $G_h(A, v) = p^{nh/2}$. We note that \( \frac{A}{2} \equiv A \times 2^{-1} \in \mathbb{Q}_p \), i.e., it should be evaluated in $\mathbb{Q}_p$.

The result of evaluating this sum can be summarized in the following list [41]:

**Proposition 1 Gauss Sums**

(a) If $\det A \neq 0$ and $h$ is large enough that $A' = p^h A^{-1}$ has entries in $\mathbb{Z}_p$. Then $G_h(A, 0)$ is $p^{\nu_p(\det A)}$ times a root of unity.

(b) If $\exists u \in \mathbb{Z}_p^n$ such that $v = Au$ then $G_h(A, v) = G_h(A) \exp \left( -\frac{\pi i}{p^h} u^T Au \right)$. Otherwise, $G_h(A, v) = 0$.

(c) Either $G_1(A, v) = 0$ or it is $p^{(h-1)/2}$ times a root of unity as in (a).

One of the simplest examples of the utility of this Proposition in the WWM formalism is given in the next Section where Clifford unitaries are considered.

A. Gauss Sums on Clifford Gates

By virtue of the fact that the Weyl symbol of a unitary gate is in $\mathbb{C}$, any unitary gate’s Weyl symbol can always be written in the form

$$U(\mathbf{x}) = \exp \left[ \frac{2\pi i}{p^h} S(\mathbf{x}) \right],$$

where $S(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ and $\sum_{\mathbf{x}} U(\mathbf{x}) = 1$. Given $\mathbf{x} \in (\mathbb{Z}/d\mathbb{Z})^n$ we can make this more precise by restricting $S(\mathbf{x})$ to be a polynomial in the ring $\mathbb{C}[\mathbb{Z}/d\mathbb{Z}]^n$ where $\deg(S) < d$ (without loss of generality). We call $S(\mathbf{x})$ the center-generating action or just simply the action due to its semiclassical role [34].

From Eq. [19] we see that the quantum channel associated with this gate can be written as:
Clifford gates have actions that fall into a subring of \( \mathbb{C}[(\mathbb{Z}/d\mathbb{Z})^{n}] \): \( S(x) \in \mathbb{Z}/d\mathbb{Z}[(\mathbb{Z}/d\mathbb{Z})^{n}] \) where \( \deg(S) \leq 2 \) \[1\]. We can write the action for a Clifford gate as:

\[
S(x) = x^T B x + \alpha^T \mathcal{J}^T x,
\]

(21)

where \( B \) is a \( 2n \times 2n \) symmetric matrix with elements in \( \mathbb{Z}/d\mathbb{Z} \).

The results on Gauss sums from Section \[IV\] are explicitly suited for evaluating the Weyl symbols associated with Clifford gates, since the associated actions of Clifford gates have degree less than or equal to two (are maximally quadratic). Let us evaluate \( UU^*(x, x') \) from this perspective:

\[
U(x_1)U^*(x_2) = \sum_{x_3 \in (\mathbb{Z}/d\mathbb{Z})^{2N}} U(x_1)U^*(x_2)
\]

\[
\times \exp \left[ \frac{2\pi i}{p} \Delta_5(x_3, x_1, x_1', x_2) \right],
\]

(22)

where \( U(x) = \exp \left[ \frac{2\pi i}{p} \left( x^T B x + \alpha^T \mathcal{J}^T x \right) \right] \).

We define the Hessian

\[
A = -2 \begin{pmatrix} -B \mod p & \mathcal{J} \\ -\mathcal{J} & B \mod p \end{pmatrix},
\]

(23)

the vector

\[
v(x, x', x_3) = \mathcal{J} \left( \frac{2(x_3 - x + x')}{} + (\alpha \mod p) \right)
\]

(24)

and the scalar

\[
c(x, x', x_3) = 2 \left[ x^T \mathcal{J} x + x_3^T \mathcal{J}(x + x') \right],
\]

(25)

so that the sum can be rewritten to make use of Proposition \[1b\]:

\[
UU^*(x, x') = \sum_{x_3, x_2, x_1 \in (\mathbb{Z}/p\mathbb{Z})^{2N}} \exp \left\{ \frac{\pi i}{p} \left[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2v^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2c \right] \right\}
\]

(26)

\[
= \sum_{x_3 \in (\mathbb{Z}/p\mathbb{Z})^{2N}} \exp \left[ -\frac{\pi i}{p} (u^T A u - 2c) \right] G_1(A, 0),
\]

(27)

where \( u = A^{-1} v \), for \[22\]

\[
A^{-1} = -\frac{1}{2} \begin{pmatrix} -I & B^{-1} \mathcal{J} \\ B^{-1} \mathcal{J} & I \end{pmatrix} \left( B - \mathcal{J} B^{-1} \mathcal{J} \right)^{-1} \left( B - \mathcal{J} B^{-1} \mathcal{J} \right)^{-1} \begin{pmatrix} -I & \mathcal{J} B^{-1} \\ -\mathcal{J} B^{-1} & I \end{pmatrix},
\]

(28)

where we assume \( \det B \neq 0 \). We drop the arguments on \( u, v, \) and \( c \) and such subsequent terms for conciseness. More details can be found in Appendix \[3\].

Since \( B \) can be expressed with entries in \( \mathbb{Z}/p\mathbb{Z} \) it follows that \( (B - \mathcal{J} B^{-1} \mathcal{J})^{-1} \in \mathbb{Z}_p \) and so \( u \in \mathbb{Z}_p \). Thus, \( G_1(A, 0) \neq 0 \forall x, x', x_3 \in (\mathbb{Z}/p\mathbb{Z})^{2N} \) by Proposition \[1b\]. Gathering powers of \( x_3 \), we define the Hessian

\[
A' = 0,
\]

(29)

the vector

\[
2v^T = (2x' + (\alpha \mod p)) \mathcal{J} (\mathcal{J} B - I)^{-1} + 4(x + x')^T \mathcal{J},
\]

(30)
and we define the scalar
\[ 2c' = 4x' T [4\beta J - 4\beta JB^{-1} J - J] x + 8\alpha T \beta J [1 - B^{-1} J] x \] (31)
for \( \beta = -\frac{1}{2} J^T (B - JB^{-1} J)^{-1} \).

This allows us to rewrite (see Appendix [B])
\[UU^*(x, x') = G_1(A, 0) \sum_{x_3 \in (\mathbb{Z}/p\mathbb{Z})^{2N}} \exp \left[ -\frac{\pi i}{p} \left( x_3^T A' x_3 + 2v'^T x_3 + 2c' \right) \right]. \] (32)

We can therefore conclude, for values of \( x' \) and \( x \) that satisfy \( x = \mathcal{M} (x' + \frac{\alpha}{2}) + \frac{\alpha}{2} \), \( \mathcal{M} \equiv (1 + JB)^{-1}(1 - JB) = (1 - JB)(1 + JB)^{-1} \).

By Proposition [1], since \( A' = 0 \), the sum over \( x_3 \) is non-zero if and only if \( v' = 0 \) (see Appendix [B]):
\[ x = \mathcal{M} \left( x' + \frac{\alpha}{2} \right) + \frac{\alpha}{2}. \] (34)

Or, in other words, we are presented with the usual plane wave sum identity for a Kronecker delta function.

We can therefore conclude, for values of \( x' \) and \( x \) that satisfy \( x = \mathcal{M} (x' + \frac{\alpha}{2}) + \frac{\alpha}{2} \), \( \mathcal{M} \equiv (1 + JB)^{-1}(1 - JB) = (1 - JB)(1 + JB)^{-1} \).

There is only one solution \( x \) given \( x' \), and so it follows that \( d^{-2N} \sum_{x, x' \in (\mathbb{Z}/p\mathbb{Z})^{2N}} \mathcal{M} (x, x') = d^{2N} \) as expected. The Clifford gate symplectically transforms Wigner functions by point-to-point permutation.

Note that this approach is no longer suitable if \( S(x) \) has powers higher than quadratic, or, if even though it is quadratic, \( B \) and \( \alpha \) do not have elements in \( \mathbb{Z} \). This is because the Gauss sum results from Proposition [1] cannot apply in these cases.

Unlike Clifford unitaries, non-Clifford unitary gates generally have center-generating actions that are polynomials with coefficients in \( \mathbb{C} \) (Eq. [19]) and so cannot be treated in the same manner above as Proposition [1] does not hold. However, we will see in Section [VII] that there exists a subset of non-Clifford gates that have actions with coefficients in a smaller field that is in closer to the integer field the Clifford gate actions lie in than the complex field: the rational \( \mathbb{Q} \) field. We will find that instead of an expression in terms of a single Gauss sum (an identity for a Kronecker delta function) as for the Clifford gates, a sum of Gauss sums (or their higher power uniformizations) can express this subset of non-Clifford gates, and for this the Gauss sum results must be generalized by the stationary phase method.

V. STATIONARY PHASE

Given a polynomial \( S(x) \in \mathbb{Q}_p[x_1, \ldots, x_n] \) such that \( \frac{\partial S(x)}{\partial x} \) has coefficients in \( \mathbb{Z}_p \), we can express the sum, \( \mathcal{I} \), in terms of local terms \( \mathcal{I}_x \):
\[ \mathcal{I} = \sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^n} \exp \left[ \frac{2\pi i}{p^m} S(x) \right] \] (35)
\[ = \sum_{\bar{x} \in (\mathbb{Z}/p^n\mathbb{Z})^n} \left\{ \sum_{x = \bar{x} \pmod{p^m}} \exp \left[ \frac{2\pi i}{p^m} S(x) \right] \right\} \equiv \sum_{\bar{x} \in (\mathbb{Z}/p^n\mathbb{Z})^n} \mathcal{I}_{\bar{x}}, \]
where \( j < m \).

Making use of the polynomial version of Taylor’s theorem [13],
\[ S(a + p^j x) = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial x^\alpha} S(x)|_{x=a} \cdot p^{|\alpha|} x^\alpha, \] (36)
where the sum is over the components of \( 0 \leq \alpha_i \in \mathbb{Z}^n \) such that \( |\alpha| = \prod_i (\alpha_i!) \), Fisher proved the following Lemma [11]:

Lemma 1 (Fisher1) Given positive integers \( j \) and \( k \), a polynomial \( S(x) \in \mathbb{Q}_p[x_1, \ldots, x_n] \), and a point \( a \in \mathbb{Z}_p^n \), let \( S_a^{(\leq k)}(x) \) denote the Taylor polynomial of degree \( k \) for \( S(a + x) \). Assume that \( p^j S(x) \) and the partial derivatives have coefficients in \( \mathbb{Z}_p \), for some integer \( t \), and let
\[ \epsilon = \epsilon_k = \min \{ t, \lfloor \log k / \log p \rfloor \}. \] (37)

Then \( S(a + p^j x) = S_a^{(\leq \epsilon)}(p^j x) \mod p^{k-\epsilon} \) as polynomials in \( x \).

We express the quadratic Taylor polynomial of \( S(x) \) at \( a \) as \( S_a^{(\leq 2)}(x) = S(a) + \nabla S(a) \cdot x + \frac{1}{2} x \cdot H_a \cdot x \), where
\[ H_a = \left. \frac{\partial^2 S}{\partial x_i \partial x_j} \right|_{x=a} \] is the Hessian matrix of \( f \) at \( a \).

We proceed as follows:

Theorem 1 (Fisher2) Let \( S(x) \in \mathbb{Q}_p[x_1, \ldots, x_n] \) such that \( \nabla S(x) = (\frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n}) \) has coefficients in \( \mathbb{Z}_p \). Define \( \epsilon_k \) as in Eq. [17]. Given integers \( m \geq j \geq 1 \) and a point \( a \in \mathbb{Z}_p^n \) with reduction \( \bar{a} \in (\mathbb{Z}/p^m\mathbb{Z})^n \), define
\[ \mathcal{I}_a = \mathcal{I}_a(f; \mathbb{Z}/p^m\mathbb{Z}) \] as in Eq. [33]. Then:

(a) The local sum \( \mathcal{I}_a \) vanishes unless \( \nabla S(a) \equiv 0 \mod p^{\min(j, m-j)} \).
(b) If \( m \leq 3j - \epsilon_3 \) and \( \nabla S(a) \equiv 0 \mod p^\min(j, m - j) \), then
\[
I_\alpha = p^{nm/2}e^{2\pi i S(a)/p^n} G_{m-2j}(H_\alpha p^{-j} \nabla S(a)). \tag{38}
\]

(c) Assume that \( \det(H_\alpha) \neq 0 \), and choose an integer \( h \) such that \( p^h H_\alpha^{-1} \) has coefficients in \( \mathbb{Z}_p \). Assume that \( j > h \) and \( 2j + h \leq m \leq 3j - \epsilon_3 \). Then the following are equivalent:

(i) \( I_\alpha \neq 0 \),
(ii) \( \nabla S(a) \in p^j H_{\alpha} \cdot \mathbb{Z}_p^n \), i.e., \( p^j H_{\alpha} \nabla S(a) \),
(iii) \( \exists \alpha \in \mathbb{Z}_p^n : \nabla S(a) = 0 \) in \( \mathbb{Z}_p^n \) and \( \alpha \equiv a \mod p^j \), and \( I_\alpha = p^{nm/2}e^{2\pi i S(a)/p^n} G_{m}(H_\alpha, 0) \).

As an example consider \( S(x) = x^3 + 2x^2 \) over \( \mathbb{Z}/3^2 \mathbb{Z} \):
\[
I = \sum_{x \in \mathbb{Z}/3^2 \mathbb{Z}} \exp \left[ \frac{2\pi i}{3} (x^3 + 2x^2) \right]. \tag{39}
\]
We reexpress this as a sum of local terms \( I_x \):
\[
I = \sum_{x \in \mathbb{Z}/3^2 \mathbb{Z}} I_x, \tag{40}
\]
where
\[
I_x = \sum_{x \in \mathbb{Z}/3^2 \mathbb{Z}} \exp \left[ \frac{2\pi i}{3} (x^3 + 2x^2) \right]. \tag{41}
\]

Since \( S(x) \in \mathbb{Z}_p \) it follows that \( t = 0 \) and so \( \epsilon = 0 \).

\[ 0 \mod 3 = \nabla S(x) = 3x^2 + 4x \] has solution \( x = 0 \). Therefore, by Theorem 1(a),
\[
I = I_\alpha. \tag{42}
\]
Furthermore, by Theorem 1(b), since \( 2 = m \leq 3j - \epsilon_3 = 3 \), it follows that
\[
I = I_\alpha = 3e^{2\pi i S(0)/3^2} = 3. \tag{43}
\]
Therefore, due to Theorem 1 we are able to simplify the sum to just one term. This serves as a small illustration of the usefulness of the stationary phase method.

A. Diagonal Unitary Example

Diagonal gates with rational eigenvalues can always be written as \( \hat{U} = \exp \left[ \frac{2\pi i}{p^k} S(\hat{q}) \right] \) where \( S(\hat{q}) \in \mathbb{Q}[\hat{q}] \). Hence, their Weyl symbols are \( U(x_q) = \exp \left[ \frac{2\pi i}{p^k} S(x_q) \right] \).

The first non-trivial example of such a diagonal non-Clifford gate we can consider is a generalization of the example considered at the end of the last section—a gate with a cubic action with coefficients in \( \mathbb{Z} \):
\[
S_9(x) = Cx_9^3 + Bx_9^2 + \alpha \mathcal{J}^T x. \tag{44}
\]
Note that the generalized \( \frac{2}{3} \)-gates are of this form for qudits \( \mathbb{Z}_N \). However, here we consider \( S_9 \) as a polynomial over \( \mathbb{Z}/3^2 \mathbb{Z} \) so that its Weyl symbol is \( U_9(x) \equiv \exp \left( \frac{2\pi i}{3^2} S_9(x) \right) \).

It is easy enough to verify that this corresponds to a unitary operator (see Appendix C):
\[
(U_9 U_9^*) (x) = \left( \frac{1}{3^2} \right)^2 \sum_{x', x'' \in (\mathbb{Z}/3^2 \mathbb{Z})^2} U_9(x') U_9^*(x') \exp \left[ \frac{2\pi i}{3^2} \Delta_3 (x, x', x'') \right] \tag{45}
\]
\[
= \frac{1}{3^4} \sum_{x', x'' \in (\mathbb{Z}/3^2 \mathbb{Z})^2} \exp \left\{ \frac{2\pi i}{3^2} \left[ C(-x_9'^3 + x_9'^3) + B(-x_9'^2 + x_9'^2) + \alpha \mathcal{J}(x' - x'') + 2 \mathcal{J}^T (x' - x'') + 2x'^T \mathcal{J} x'' \right] \right\}
\]
\[
= 1.
\]

So now consider the double-ended propagator corresponding to \( S_9 \):
\[
U_9 U_9^* (x, x') = \frac{1}{3^2} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/3^2 \mathbb{Z})^2} U(x_1) U^*(x_2) \exp \left[ \frac{2\pi i}{p} \Delta_3 (x_3, x_1, x', x_2) \right] \tag{46}
\]
\[
= \frac{1}{3^2} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/3^2 \mathbb{Z})^2} \exp \left\{ \frac{2\pi i}{3^2} \left[ S_9(x_1) - S_9(x_2) - \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^T \left( \begin{array}{cc} 0 & \mathcal{J} \\ -\mathcal{J} & 0 \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + 2 \left( \begin{array}{c} x_3 - x + x' \\ x_3 - x - x' \end{array} \right)^T \mathcal{J} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + c \right] \right\},
\]
where we remind ourselves that
\[
c = 2 \left[ x'^T \mathcal{J} x + x_9^T \mathcal{J} (x + x') \right]. \tag{47}
\]
the quadratic Gauss sum and Kronecker delta function identities (see Appendix C):

\[
U_9 U^*_9 (\mathbf{x}, \mathbf{x}') = \begin{cases} 
9 \sum_{x_{2q} \in \mathbb{Z}/3^2} \exp \left\{ \frac{2\pi i}{3} \left[ C' x_{2q}^3 + B' x_{2q}^2 + \alpha' x_{2q} + c' \right] \right\} & \text{for } x_q = (-\alpha_q + x'_q) \mod 3^2 \\
0 & \text{otherwise.}
\end{cases}
\]

where

\[
C' = -2C,
\]

\[
B' = -3C(2x'_q - \alpha_q),
\]

\[
\alpha' = (2x'_p - 2x_p - 2\alpha_p + (2x'_q - \alpha_q)(2B - 6Cx'_q + 3C\alpha_q)),
\]

and

\[
c' = -(2x'_q - \alpha_q)(-x'_p + x_p + \alpha_p + (2x'_q - \alpha_q)(-B + C(2x'_q - \alpha_q)).
\]

This leaves a single sum over \(x_{2q}\) of an exponentiated cubic polynomial \(S(x_{2q}) = C' x_{2q}^3 + B' x_{2q}^2 + \alpha' x_{2q} + c'\) with coefficients in \(\mathbb{Z}\). We proceed to find the exponential argument's zeros \(\{\tilde{x}_{2q}\}\), the stationary phase points:

\[
\frac{\partial S}{\partial x_{2q}} \bigg|_{x_{2q} = \tilde{x}_{2q}} = 0 = 3C' \tilde{x}_{2q}^2 + 2B' \tilde{x}_{2q} + \alpha',
\]

\[
\Leftrightarrow \tilde{x}_{2q} = \left[ -2B' \pm \left( 4B'^2 - 12C' \alpha' \right)^{1/2} \right] (6C')^{-1},
\]

where all the arithmetic operations are understood to be taken \(\mod 3^2\).

Therefore, by Theorem 1(a) and (b),

\[
U_9 U^*_9 (\mathbf{x}, \mathbf{x}') = 9 \left\{ \sum_{x_{2q} \in \mathbb{Z}/3^2} \sum_{x_{2q} \in \mathbb{Z}/3^2} \exp \left[ \frac{2\pi i}{3^2} S(x_{2q}) \right] \right\} \quad \text{for } x_q = (-\alpha_q + x'_q) \mod 3^2,
\]

\[
\frac{\partial S(x_{2q})}{\partial x_{2q}} \mod 3 = [66 + 12x_{2q}(4 + x_{2q})] \mod 3 = 0 \forall x_{2q}.
\]

Therefore, \(\tilde{x}_{2q}\) takes all values \(\mod 3\) and so

\[
U_9 U^*_9 ((0,0), (1,4)) = 9 \left\{ \exp \left[ \frac{2\pi i}{3^2} S(0) \right] + \exp \left[ \frac{2\pi i}{3^2} S(1) \right] + \exp \left[ \frac{2\pi i}{3^2} S(2) \right] \right\},
\]

\[
\frac{\partial S(x_{2q})}{\partial x_{2q}} \mod 3 = [68 + 12x_{2q}(4 + x_{2q})] \mod 3 \neq 0 \forall x_{2q},
\]

and so \(U_9 U^*_9 ((0,0), (2,4)) = 0\).

This example illustrates the usefulness of this periodized stationary phase method for evaluating non-Clifford gate Weyl symbols and its potential for simplifying or reducing the full sum.

For general diagonal gates, the coefficients in their action polynomials with fall in \(\mathbb{Q}\) instead of \(\mathbb{Z}\). We will develop a technique in Section VII that will show how to change the domain of summation from \(\mathbb{Z}/p^h\mathbb{Z}\) to \(\mathbb{Z}/p^h\mathbb{Z}\), where \(h' > h\), such that the resultant equivalent action has coefficients in \(\mathbb{Z}\) thereby allowing for its treatment by the method of periodized stationary phase developed
VI. CHARACTERIZING THIS “PERIODIZED” STATIONARY PHASE METHOD

A. Periodized Stationary Phase

The stationary phase method in the continuous case can be described as representing an integral over a continuous domain by a set of discrete points \( \{ \tilde{x}_i \} \). These are the stationary points of the phase of the integral. The phase is expanded by Taylor’s theorem at these discrete points \( \{ \tilde{x}_i \} \), and if the expansion is truncated at quadratic order, then Gaussian integrals result.

The notion of the stationary phase method differs when there is no longer a proper Euclidean metric to define continuous distances or areas for the integrand’s domain. Similarly, the traditional version of Taylor’s theorem does not hold. Instead, we must use the “periodized” version of Taylor’s theorem \(^{14}\) in a “periodized” stationary phase approximation. In this version, the points \( \{ \tilde{x}_i \} \) where the phase’s derivative is zero correspond to critical points where the “periodized” phase is stationary, i.e., where the phase, taken every \( x_i \) points, slows down so that it ceases cancelling its opposing contributions around the unit circle with equal weights.

In the discrete case we consequently look for stationarity along equally-spaced periodic intervals that span the whole summand, instead of at particular points of the integrand; stationary phase points no longer correspond to points in the integrand at which an expansion to second order in the action is made producing Gaussian contributions, but instead correspond to “reduced” points representing periodic intervals of the summand at which an expansion to second order in the action is made producing quadratic Gauss sum contributions.

B. Uniformization

According to Theorem \( \text{[1]} \), the discrete stationary phase method drops the size of summation by a factor of \( p^{m-\lceil m/3 \rceil} \) and changes the summand to a sum of quadratic Gauss sums with closed-form solutions multiplied by phases. However, the stationary phase method can also decrease the domain by a larger multiple of \( p \), but the resultant sum is over terms that are themselves sums with higher than quadratic order; there are no longer quadratic Gauss sums.

To prove part (b) of Theorem \( \text{[1]} \), the result we will most rely on moving forward, Fisher appealed to Lemma \( \text{[1]} \) to show that if \( m \leq kj - \epsilon_k = 3j - \epsilon_3 \) and \( \nabla S(a) \equiv 0( \mod p^{\min(j,m-j)} ) \), then for any \( x \in (\mathbb{Z}/p^{m-j}\mathbb{Z})^n \),

\[
S(a + p^j x) = S(a) + p^j \nabla S(a) \cdot x + \frac{1}{2} p^{2j} H_a(x) + \frac{1}{6} p^{-3j} (\nabla^3 S)^3(x^3)/p^{m-3j},
\]

This implies

\[
I_{a} = e^{2\pi i S(a+p^j x)/p^m} \sum_{x \in (\mathbb{Z}/p^{m-j}\mathbb{Z})^n} e^{2\pi i [p^{-j} \nabla S(a) x + \frac{1}{6} p^{2j} H_a(x)]/p^{m-2j}},
\]

which can be rewritten in terms of the Gauss sum notation as above in Theorem \( \text{[1]} \) (b).

This trivially generalizes to higher order polynomials:

**Theorem 2 (Generalization of Fisher2 (b))** Let \( S(x) \in \mathbb{Q}_p[x_1, \ldots, x_n] \) such that \( \nabla S(x) = (\frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n}) \) has coefficients in \( \mathbb{Z}_p \). Define \( e_k \) as in Eq. \( \text{[37]} \). Given integers \( m \geq j \geq 1 \) and a point \( a \in \mathbb{Z}_p^n \) with reduction \( \bar{a} \in (\mathbb{Z}/p^m\mathbb{Z})^n \), define \( I_{\bar{a}} = I_{\bar{a}}(f; \mathbb{Z}/p^m\mathbb{Z}) \) as in Eq. \( \text{[35]} \). Then: if \( m \leq kj - \epsilon_k \) and \( \nabla S(a) \equiv 0 \mod p^{\min(j,m-j)} \), then

\[
I_{\bar{a}} = e^{2\pi i S(a+p^j x)/p^m} \sum_{x \in (\mathbb{Z}/p^{m-j}\mathbb{Z})^n} e^{2\pi i [\nabla^k S(a) x]/p^{m-j}}.
\]

**Proof** The same proof as Fisher’s can be used here. \( \blacksquare \)

In particular, for \( k = 4 \),

\[
I_{\bar{a}} = e^{2\pi i S(a)/p^m} \sum_{x} e^{2\pi i [p^{-j} \nabla S(a) x + \frac{1}{6} p^{2j} H_a(x)]/p^{m-3j}},
\]

where

\[
A_n = \sum_{i,j,k} \frac{\partial^3 S(x)}{\partial x_i \partial x_j \partial x_k} \bigg|_{x=a} x_i x_j x_k.
\]

Completing the cube for \( n = 1 \) by setting \( x' = (\nabla^3 S)^\frac{1}{3} (x + p^{-j} \nabla^2 S/\nabla S) \) this can be rewritten as:

\[
I_{\bar{a}} = e^{2\pi i (S(a)+\Gamma)/p^m} \sum_{x} e^{2\pi i [\nabla x' - x'^3]/p^{m-3j}},
\]

for

\[
\Gamma = -p^{-3j} \nabla S(a) (\nabla^2 S)/(\nabla^3 S) + \frac{1}{2} p^{-3j} (\nabla^2 S)^3/(\nabla^3 S)
\]

\[
- \frac{1}{6} p^{-3j} (\nabla^2 S)^3/(\nabla^3 S)^2,
\]

\[
\Xi = (\nabla^3 S)^{-\frac{1}{3}} \left[ p^{-2j} \nabla S(a) - p^{-2j} (\nabla^2 S)^2/(\nabla^3 S) + \frac{1}{2} p^{-2j} (\nabla^2 S)^2/(\nabla^3 S) \right].
\]

(Note that \( (\nabla^3 S)^\frac{1}{3} \) is well-defined for prime dimension.)

This is a discrete sum analog to the Airy function \( \text{[16]} \). Similarly, for \( k = 5 \), setting \( x' = (\nabla^4 S)^{\frac{1}{5}} (x + (p^{-j} \nabla^3 S/\nabla^4 S)) \) for \( n = 1 \) allows the equation to be rewritten as:

\[
S_{\bar{a}} = e^{2\pi i (S(a)+\Gamma)/p^m} \sum_{x'} e^{2\pi i [\nabla x' - x'^2 + x'^4]/p^{m-4j}},
\]
This is a discrete sum analog to the *Pearcey integral* [47].

Higher order instances can be similarly developed leading to discrete sum analogs to integrals familiar in catastrophe theory [48].

For \( n \) degrees of freedom with a domain of summation of \( z/p^m \mathbb{Z} \) (as might be produced by, for instance, \( n \) \( p^m \)-dimensional qudits), the naive numerical summation involves a sum over \( p^m \) terms. Using the Gauss sum \( k = 3 \) simplification or appropriate uniformization level for \( k > 3 \), this sum can be reduced to a sum over \( p^m \) terms involving Gauss, Airy, Pearcey etc. sums, which number \( \prod_j p^{m-j} \) and can be pre-computed and stored for use during the summation. These terms can perhaps also be approximated numerically instead of tabulated, thereby eschewing the exponential cost in storage. This is the current approach with Airy functions and Pearcey integrals in the continuous case in computation, for instance [49].

We note that this uniformization to include non-Clifford gates within the WWM formalism was not possible under the previous derivation of the formalism in terms of powers of \( \hbar \) [31] [33] [34]. This is because that derivation took the WWM formalism from the continuous case, where the stationary phase method is able to be applied in its traditional form, and then “discretized” and “periodized” the final propagator. As a result, the propagator could only be written to order \( \hbar \), which just includes Clifford propagation, and higher order corrections could not be derived because anharmonic trajectories cannot be obtained by “periodizing” to a discrete Weyl phase space grid. In this way, this paper finally accomplishes this extension to higher orders of \( \hbar \) by instead using this different “periodized” stationary phase method, thereby solving this old problem for the first time. Just as in the old derivation, it finds that Clifford propagation is captured at order \( \hbar^0 \), but unlike the old derivation it is able to formally ascribe a power of \( \hbar \) required to include (diagonal and their Clifford transformations) non-Clifford gates in a formal Taylor series, as discussed in Section VII A.

### VII. QUTRIT \( \pi/8 \) GATE

Here we show how to change the domain of summation for a gate from \( \mathbb{Z}/p^h \mathbb{Z} \) to \( \mathbb{Z}/p^{h'} \mathbb{Z} \), where \( h' > h \), such that the resultant equivalent action will go from having coefficients in \( \mathbb{Q} \) to \( \mathbb{Z} \). This is a useful technique for general diagonal gates, since the coefficients in their action polynomials with generally fall in \( \mathbb{Q} \) instead of \( \mathbb{Z} \). Reexpressing their Weyl symbols in this way will allow for their evaluation by the method of stationary phase. Though we will only show this for the qutrit \( \pi/8 \) gate as an example, this technique can be equivalently applied for all diagonal gates with rational coefficients, as as well as their Clifford transformations, which only symplectically permute the action.

The \( T \)-gate is a non-Clifford single qudit gate. It extends the Clifford gate set to allow for universal quantum computation [44]. Generalizations of the \( T \)-gate to qudits are frequently called \( \pi/8 \) gates. They differ for dimension \( p = 3 \) and \( p > 3 \) [44] [45].

For \( p > 3 \), with \( \omega = e^{2\pi i/p} \),

\[
U_v = \sum_{j=1}^{p-1} \omega^{vk} |k\rangle \langle k|
\]

where

\[
v_k = \frac{1}{12} k(\gamma' + k(6z' + (2k - 3)\gamma')) + k\epsilon',
\]

for \( \gamma', z', \epsilon' \in \mathbb{Z}/p\mathbb{Z} \). For \( p = 3 \), with \( \zeta = e^{2\pi i/9} \),

\[
U_v = \sum_{j=1}^{2} \zeta^{v_k} |k\rangle \langle k|
\]

where

\[
v = (0, 6z' + 2\gamma' + 3\epsilon', 6z' + \gamma' + 6\epsilon') \mod 9.
\]

Despite these differences, these \( \pi/8 \) gates are all diagonal with entries that are rational powers of \( e^{2\pi i/p} \). This means that their Weyl symbol actions \( S_{\pi/8}(x) \notin \mathbb{Z}[x] \) but lie in \( \mathbb{Z}[x] \) when the dimension \( p^h \) of the system is increased to \( p^{h'} \).

To demonstrate this we examine a particular \( \pi/8 \) gate for qutrits and its corresponding magic state.

Let \( z' = 1 \), \( \gamma' = 2 \) and \( \epsilon' = 0 \) so that

\[
U_v(0, 1, 8) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta & 0 \\
0 & 0 & \zeta^8
\end{pmatrix}.
\]

The corresponding Weyl symbol is

\[
U_{\pi/8}(x_q) = \exp \left[ -\frac{2\pi i}{3} \frac{2}{3} x_q + 3x_q^2 \right]
\]

\[
= \exp \left[ \frac{2\pi i}{3} S_{\pi/8}(x_q) \right],
\]
for \( x_q \in \mathbb{Z}/3^2\mathbb{Z} \) and \( x_q \in \mathbb{Z}/3\mathbb{Z} \).

\( S_{\pi/8}(x) \notin \mathbb{Z}[x] \) but it does lie in \( \mathbb{Z}[x] \) when the domain is squared. We want to consider the sum over \( x_q \) of \( U_{\pi/8}(x) = U_{\pi/8}(x_q) \) and reexpress it in terms of an exponential such that it is a sum over the larger space \( \mathbb{Z}/3^2\mathbb{Z} \) of a polynomial with integer coefficients. Hence, we want to find \( \alpha, \beta \in \mathbb{Z} \) s.t.

\[
\alpha x_q^{2x^3} + \beta x_q^3 \mod 3^2 = -2 \left( 3(x_q \mod 3)^2 + (x_q \mod 3) \right) \mod 3^2. \tag{77}
\]

Substituting in \( x_q = 1 \) and \( x_q = 3^2 - 1 = -1( \mod 3^2) \) we find the system of equations

\[
4 \times (-2) \mod 3^2 = \alpha + \beta \mod 3^2, \tag{78}
\]

and

\[
14 \times (-2) \mod 3^2 = \alpha - \beta \mod 3^2, \tag{79}
\]

which has the solution \( \alpha = 0 \) and \( \beta = 1 \).

Therefore,

\[
\sum_{x_q \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left[ \frac{2\pi i}{3} \left( -\frac{2}{3} \right) (x_q + 3x_q^2) \right] = \frac{1}{3} \sum_{x_q \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} x_q^3 \right] = \frac{1}{3} \sum_{x_q \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} S'_{\pi/8}(x) \right].
\]

We now consider

\[
U_{\pi/8}^* U_{\pi/8}(x, x') = \frac{1}{3^2} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/3\mathbb{Z})^2} U_{\pi/8}(x_1) U_{\pi/8}(x_2) \exp \left[ \frac{2\pi i}{3} \Delta_5(x_3, x, x_1, x', x_2) \right], \tag{82}
\]

\[
= \frac{1}{3^2} \sum_{x_1, x_2 \in (\mathbb{Z}/3\mathbb{Z})^2} \exp \left[ \frac{2\pi i}{3^2} \left( S'_{\pi/8}(x_1) - S'_{\pi/8}(x_2) \right) \right]
\times \exp \left[ \frac{2\pi i}{3^2} \left( \begin{array}{c} -3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \end{array} \begin{bmatrix} 0 & \mathcal{J} \\ -\mathcal{J} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \times 3 \begin{bmatrix} x_3 - x + x' \\ x_3 - x - x' \end{bmatrix}^T \mathcal{J} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3c \right) \right],
\]

where we remind ourselves again that

\[
c = 2 \left[ x'^T \mathcal{J} x + x^T \mathcal{J} (x + x') \right]. \tag{83}
\]

Simplifying, we find (see Appendix D):
\[ U_{\pi/s}U^*_{\pi/s}(x', x') = \frac{1}{3} \sum_{x_2q \in \mathbb{Z}/3\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ (-x_2q + x_2q + x_3' - x_2q) \right] \right\} \]

\[ \times \exp \left\{ \frac{2\pi i}{3^2} \left\{ 2(x_2p - x_2p)x_2q + (x_p + 3x_p'x_2q' - (3x_2p + x_2p')x_2q) \right\} \right\} \]

\[ \times \delta \left[ 2 \times 3 (x_q - x_q') \mod 3^2 \right] \]

\[ = \begin{cases} \frac{1}{3} \sum_{x_2q \in \mathbb{Z}/3\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} S''(x_2q, x_q, x_q') \right] & \text{if } x_q' = x_q \mod 3, \\ 0 & \text{otherwise} \end{cases} \] (84)

Therefore, by Theorem 1(a) and (b), for \( x_q' = x_q \mod 3 \) (and \( j = 1 \),

\[ U_{\pi/s}U^*_{\pi/s}(x, x') = \frac{1}{3} \sum_{x_2q \in \mathbb{Z}/3\mathbb{Z}} \left\{ \sum_{x_2q' \in \mathbb{Z}/3\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} S''(x_2q) \right] \right\} \]

\[ = \frac{1}{3} \sum_{\{x_2q\} \mod 3} \exp \left[ \frac{2\pi i}{3^2} S''(\tilde{x}_2q) \right] G_0(H_{\tilde{x}_2q}, 3^{-1} \nabla S''(\tilde{x}_2q)), \] (85)

where \( G_0 \) is defined in Eq. 18.

We note that Eq. 85 is 3-periodic in all of its arguments, and so its simplification by Theorem 1 is a special case: its sums can simply be restricted to be over \( \mathbb{Z}/3\mathbb{Z} \) with the appropriate power of 3 added to compensate. It is easy to find that Eq. 86 is precisely this equation, since the preceding phase \( \exp \left[ \frac{2\pi i}{3^2} S''(\tilde{x}_2q) \right] \) is the only non-trivial part when \( G_0 = 1 \) and the domain of summation is appropriately reduced.

As an example, consider \( x_q = x_q' = 0 \) and \( x_p = x_p' = 1 \):

\[ \frac{\partial S''(x_2q, 0, 1, 1, 1)}{\partial x_2q} = 6x_2q, \] (88)

and we consider the critical points \( \tilde{x}_2q \) when the above is equal to 0 (mod 3). Hence, \( \tilde{x}_2q = \{0, 1, 2\} \) and so

\[ U_{\pi/s}U^*_{\pi/s}(1, 0, 1, 0) = \frac{1}{3} \left\{ \exp \left[ \frac{2\pi i}{3^2} S''(0) \right] G_0(H_0, 3^{-1} \nabla S''(0)) \right\} \]

\[ + \exp \left[ \frac{2\pi i}{3^2} S''(1) \right] G_0(H_1, 3^{-1} \nabla S''(1)) \]

\[ + \exp \left[ \frac{2\pi i}{3^2} S''(2) \right] G_0(H_2, 3^{-1} \nabla S''(2)) \right\} \}

\[ = \left\{ \exp \left[ \frac{2\pi i}{3^2} 0 \right] + \exp \left[ \frac{2\pi i}{3^2} 2 \right] + \exp \left[ \frac{2\pi i}{3^2} 16 \right] \right\}. \] (89)

The Gaussians above all are equivalent to \( G_0(0, 0) = 1 \). Therefore, as we can see, we are really just performing the same sum as in Eq. 85 but over the smaller domain \( \mathbb{Z}/3\mathbb{Z} \) and compensating by multiplying in the correct powers of 3.

We are interested in the magic state of this gate, which corresponds to it acting on \( H|0\rangle = |p = 0\rangle \). The Wigner function of \( |p = 0\rangle \) is \( \rho'(x) \equiv \frac{1}{3} \delta_{x_p, 0} \). Hence (see Appendix D),
\[ \rho_{\pi/8}(x) = \sum_{x'} U_{\pi/8} U_{\pi/8}^* (x, x') \rho'(x') = \begin{cases} \frac{1}{3^2} \sum_{x_2q \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} S_{\pi/8}(x_2q, x_q, x_p, x'_p = 0) \right] & \text{if } x'_q = x_q \pmod{3}, \\ 0 & \text{otherwise} \end{cases} \]

If we let \( S(x_p, x_q) \) be the phase, we note that \( \partial S / \partial x_q = 0 \pmod{3} \) and \( \nabla^2 S \equiv H = 0 \pmod{3^0} \forall x_p, x_q \). Hence, evaluation at any phase space point, or linear combination thereof, requires summation over all three reduced phase space points \( \tilde{x}_2q \in \mathbb{Z}/3^2\mathbb{Z} \) since they are all critical points.

Again, we note that Eq. 90 is \( 3^2 \)-periodic in all of its arguments, and so its simplification by Theorem 1 is again a particularly simple special case where its sums can simply be restricted to be over \( \mathbb{Z}/3\mathbb{Z} \) with the appropriate power of 3 added to compensate.

\[ \frac{1}{3^2} \sum_{x_2q \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} (x_2q + 2x_q) \right] - x_2q + 2 \times 3(x_2q - x_q) \right] \}. \quad (90) \]

A. Application

We consider as an example, computation of the qutrit circuit outcome:

\[ P = \text{Tr} \left[ |0\rangle \langle 0| \hat{U}_C \hat{U}_{\pi/8} \hat{H}^\otimes k |0\rangle \otimes^n \right]. \quad (91) \]

This corresponds to the probability of the outcome \(|0\rangle \langle 0|\) after \( k \) qutrit magic states are acted on by a random Clifford circuit \( U_C \) on \( n \geq k \) total qutrits.

A recent study introduced a method to sample this distribution using Monte Carlo methods on Wigner functions [11]. Results were demonstrated for one to ten qutrit states, that required \( 10^7 \) to \( 10^8 \) samples, respectively, to attain precision \( (P_{\text{simplified}} - P) < 10^{-2} \) with 95\% confidence [11].

Before the Clifford gates are applied, in the Wigner picture, the circuit can be described by \( k \) products of Eq. 90 and \( (n-k) \) products of \( \delta_{x_q,0} \). The random Clifford gate can be described as an affine transformation by a symplectic matrix \( \mathcal{M} \) and vector \( \alpha \) [16] [50], as described in Section IV A such that

\[ \begin{pmatrix} x_p' \\ x_q' \end{pmatrix} = \mathcal{M} \begin{pmatrix} x_p \\ x_q \end{pmatrix} + \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}. \quad (92) \]

The form of the \( \mathcal{M} \) and \( \alpha \) for the Clifford gates are given in [16]. The final contraction to \(|0\rangle \langle 0|\) corresponds to a sum over all \( n \) of the \( x_p \) and \( x_q \) except for the first degree of freedom, which is only considered at \( x_q = 0 \) (and all \( x_p \)).

We note that this is a strong simulation algorithm. Therefore, it is interesting to compare the number of terms produced by the stationary phase method that must be summed over in this strong simulation with the number of samples that must be taken in the prior strong simulation by Pashayan et al. However, a direct comparison of the two on an equal footing is a bit blurred by the fact that the latter only reports results based on a Monte Carlo sampling of their terms, producing a result correct only up to a precision \( \epsilon = 0.01 \). Their explicit evaluation would produce a result correct up to precision \( \epsilon = 0.0 \) and would likely scale far worse (we estimate at least \( 3^{2t} \) for a naive evaluation of the \( 2t \) dimensional Wigner function, see Fig. II). Monte Carlo calculation of finite sums almost always reduce the number of terms that must be evaluated for intermediate to large sums, and we assume that is the case with Pashayan et al.’s results. Thus we will proceed to compare the two methods with the caveat that Monte Carlo sampling of the sums produced by the stationary phase method will likely also improve its performance for intermediate to large values of \( t \).

For \( k \) magic states in a 100-qutrit system, we have \( k \) one-dimensional sums (integrals) over \( x_{q_1} \) and 100 sums over all the \( x_p \) and 99 sums over all the \( x_{q_2} \) except \( x_{q_1} \), resulting in a total of \(( k + 199 \) sums. This can be concisely written by letting \( x = (x_{p_1}, \ldots, x_{p_{100}}, x_{q_1}, \ldots, x_{q_{100}}) \equiv (x_p, x_q) \) and \( \rho_{\pi/8}(x_q) \equiv \rho_{\pi/8}(x_p, x_q) \) such that

\[ P = \sum_{x' \in \mathcal{D}} \left[ \prod_{i=1}^{100} \rho_{\pi/8}(x'_i) \prod_{j=1}^{101} \delta(x'_{q_i}) \right], \quad (93) \]

for

\[ \mathcal{D} = \left\{ x' \left| (\mathcal{M}^{-1} (x' - \frac{\alpha}{2}) - \frac{\alpha}{2} \right) \right. \pmod{3^2} = 0 \right\}, \quad (94) \]

where \( (\mathcal{M}^{-1} (x' - \frac{\alpha}{2}) - \frac{\alpha}{2} \pmod{3^2})_i \) for \( i \in \{1, \ldots, 200\} \). Notice that we have effectively acted with \( \hat{U}_C \) on the bra \( \langle 0| \) in Eq. 91 to produce the restriction \( (\mathcal{M}^{-1} (x' - \frac{\alpha}{2}) - \frac{\alpha}{2} \pmod{3^2})_i = 0 \) in Eq. 93 as this simplifies our subsequent analysis, although this is not necessary, as we argue later. We also transformed \( \sum_{x' \in D} \delta(x' - \frac{\alpha}{2}) \to \sum_{x'} \) since the sum is taken modulo \( 3^2 \) and this is a symplectic transformation, i.e., this transformation merely permutes the order of summation and summation is invariant to order. There are 199 sums in Eq. 93 and the \( \rho_{\pi/8} \) contain \( k \) more, for a total of \(( k + 199 \) \).

The restriction on the sum over \( x' \) to be in \( D \), defined in Eq. 94, can be treated as an additional Kronecker delta function modulo \( 3^2 \), \( \delta(\mathcal{M}^{-1} (x' - \frac{\alpha}{2}) - \frac{\alpha}{2})_{101} \), multiplying the full sum over \( x' \).

We can begin reducing the number of sums by first summing out all \( \delta(x'_{q_1}) \) in Eq. 93, which will replace all
We have seen that a $\pi/8$ gate (and magic state) can be captured with only $p$ critical points. This is commensurate with its role as the “minimally” non-Clifford gate $\chi$ and that its direct products have the smallest stabilizer rank $\chi$. We will now see that this relationship between minimal stabilizer rank and this small number of critical points is not a coincidence. Note however, that this is in contrast to the amount of negativity present in this magic state, as its Wigner function has the largest possible negativity $\kappa$. This suggests that negativity is not the most efficient way to introduce “magic” or non-contextuality in a practical algorithm, and indeed we find this to be the case here.

In the stationary phase method applied to the infinite dimensional (continuous) case, the critical points correspond to intersections between Gaussian manifolds, the continuous generalization of stabilizer states $\kappa$. However, in the “periodized” stationary phase method examined here, we cannot expect the same relationship to hold. Nevertheless, a relationship can still be established between the number of stabilizer states that a state can be expressed in terms of—its stabilizer rank—and the number of critical points necessary to represent that state:

**Theorem 3 (Stabilizer Rank and Critical Points)**

Given a state with stabilizer rank $n$, the number of its
critical points is at most \((p^2)^\left\lceil\frac{m}{p}\right\rceil\) in the space spanned by the stabilizers. It is at most \(p^\left\lceil\frac{m}{p}\right\rceil\) if the state is diagonal in this new space.

**Proof** We are given a state on \(n\) \(p\)-dimensional qudits that has stabilizer rank \(m\) (i.e. can be expressed in terms of \(m\) stabilizer states). The \(m\) stabilizer states define a subspace of dimension \(\leq m\). We want to express this subspace in terms of a tensor product of subspaces of dimension \(\leq p\). We can accomplish this by taking \(\left\lceil\frac{m}{p}\right\rceil\) sets of \(p\) stabilizer states and have each set define a subspace of dimension \(\leq p\). The tensor product of all of these define the subspace of dimension \(\leq m\). Each subspace of dimension \(\leq p\) can now be identified with a \(p\)-dimensional qudit.

By Theorem 2, the corresponding Weyl phase space in each of these degrees of freedom can always be reduced to at most \(p^2\) critical points. If the state reexpressed in the new degree of freedom is diagonal (i.e. its Weyl symbol only depends on \(x_q\) or \(x_p\) in the \(i\)th degree of freedom), then this reduces to \(p\) critical points. Hence, the total number of critical points is bounded from above by the product of the number of critical points in each of these subspaces, resulting in \((p^2)^\left\lceil\frac{m}{p}\right\rceil\) (\(p^\left\lceil\frac{m}{p}\right\rceil\) if the state is diagonal in each subspace) critical points.

**IX. TWO QUTRIT \(\pi/8\) GATE MAGIC STATES**

A single qubit \(T\)-gate magic state consists of at least two stabilizer states. The two qubit magic state is somewhat remarkable in that it can also be written in terms of only two stabilizer states:

\[
|A^{\otimes 2}\rangle = \frac{1}{2}(|00\rangle + i|11\rangle) + \frac{e^{i\pi/4}}{2}(|01\rangle + |10\rangle),
\]

where each line consists of a stabilizer state.

In the qutrit case, a similar pattern holds. A single qutrit \(T\)-gate magic state consists of at least three stabilizer states. Moreover, a two qutrit magic state can also be written in terms of only three stabilizer states:

\[
\hat{U}^{\otimes 2}_{\pi/8} \hat{F}^{\otimes 2}_{\pi/8} |00\rangle = \frac{1}{3}(|00\rangle + |12\rangle + |21\rangle) + \frac{e^{-2i\pi/9}}{3}(|02\rangle + |20\rangle + e^{2i\pi/3}|11\rangle) + \frac{e^{2i\pi/9}}{3}(|01\rangle + |10\rangle + e^{-2i\pi/3}|22\rangle),
\]

where again each line consists of a stabilizer state (\(\hat{F}\) is the Hadamard gate or discrete Fourier transform).

Qubit strong simulation algorithms based on stabilizer rank have been able to leverage this fact to halve their exponential scaling of terms \([5,8]\). Indeed, the performance can be improved slightly more by using the fact that six qubit \(T\)-gate magic states can be written in terms of only seven stabilizer states \([4]\).

The dependence of the number of critical points on the intermediate values of \(x_q\) for the \(\pi/8\) gate magic state, means that generally the number of critical points scales exponentially with number of \(\pi/8\) gates. However, if this stabilizer rank for two qutrit \(\pi/8\) gate magic states can be used, the exponent is reduced by a factor of 2.
From a dimensional perspective, it seems that the same sort of improvement should be possible with two $\pi/8$ gate magic state Weyl symbols from Eq. (90) the qutrit $\pi/8$-gate magic state in Eq. (90) is written in a space with dimension $3^2$ instead of the expected 3 in order for its action to have coefficients in $\mathbb{Z}$. Hence, it is dimensionally possible to “join” two magic states, which are embedded in a $3^2$-dimensional space but are actually 3-dimensional, into a single $3^2$ dimensional space. This turns out to be possible by appealing to Theorem 3.

The three stabilizer states in Eq. (90) correspond to the Wigner functions $\delta(x_{q_1} \oplus x_{q_2})\delta(x_{p_1} \oplus x_{p_2})$, $\delta(x_{q_1} \oplus x_{q_2} \oplus 1)\delta(x_{p_1} \oplus x_{p_2} \oplus x_{q_1} \oplus x_{q_2} \oplus 2)$, and $\delta(x_{q_1} \oplus x_{q_2} \oplus 2)\delta(x_{q_1} \oplus x_{p_2} \oplus 2x_{q_1} \oplus 2)$, respectively, where the circles around the addition and subtraction operators indicate modulo 3 arithmetic. As a result, by Theorem 3 we can rotate in $x_{q_1}$, $x_{q_2}$ space by 90 degrees by setting $X_{2q} = 2^{-1}(x_{2q_1} + x_{2q_2})$ and $x_{2q} = 2^{-1}(x_{q_1} - x_{q_2})$ to restrict to their subspace and gain a reduction in the number of critical points (see Appendix D):

$$
\rho_{\pi/8}^{\otimes 2}(x_1, x_2) = \frac{1}{3^2} \sum_{x_{2q_1} \in \mathbb{Z}/3^2 \mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ (-x_{2q_1} + 2x_{q_1})^3 - x_{2q_1}^3 + 2 \times 3(x_{2q_1} - x_{q_1})x_{p_1} \right] \right\}
$$

$$
\times \frac{1}{3^2} \sum_{x_{2q_2} \in \mathbb{Z}/3^2 \mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ (-x_{2q_2} + 2x_{q_2})^3 - x_{2q_2}^3 + 2 \times 3(x_{2q_2} - x_{q_2})x_{p_2} \right] \right\}
$$

$$
= \frac{1}{3^2} \sum_{x_{2q_1}, x_{2q_2} \in \mathbb{Z}/3^2 \mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ -500(X_{2q_1}^3 + 3X_{2q_1}x_{2q_2}^2) + 150(X_{2q_1} + x_{2q_1})x_{q_1} - 60(X_{2q_1} + x_{2q_1})x_{q_1}^2 + 8x_{q_1}^3 \right] \right\}
$$

$$
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 2 \times 3(5(X_{2q_1} + x_{2q_1}) - x_{q_1})x_{p_1} \right] \right\}
$$

$$
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 150(X_{2q_1} - x_{2q_1})^2x_{q_1} - 60(X_{2q_1} - x_{2q_1})x_{q_1}^2 + 8x_{q_1}^3 \right] \right\}
$$

$$
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 2 \times 3(5(X_{2q_1} - x_{2q_1}) - x_{q_1})x_{p_1} \right] \right\},
$$

where $X_{2q} = x_{2q_1} + x_{2q_2}$ and $x_{2q} = x_{2q_1} - x_{2q_2}$ so that $x_{2q_1} = 2^{-1}(X_{2q} + x_{2q}) = 5(X_{2q} + x_{2q})$ and $x_{2q_2} = 2^{-1}(X_{2q} - x_{2q}) = 5(X_{2q} - x_{2q})$ ($2^{-1}$ is equal to 5 in the $\mathbb{Z}/3^2\mathbb{Z}$ ring).

We then see that for each of the three values that $X_{2q}$ takes in the sum, the exponent is a quadratic polynomial for $x_{2q}$ with coefficients in $\mathbb{Z}/3^2\mathbb{Z}$ and so is a Gauss sum. Thus, the end result is a sum of three Gauss sums, exactly the same number of Gauss sums as we found for a single $\pi/8$ gate in Section VII.

We further see that the $x_{qj}$ are in qubit terms and $x_{pj}$ are all linear terms just as before, and therefore the same arguments for the scaling of the marginal trace hold here. Thus the number of critical points is reduced to $3^{2+1}$ so that the cost of simulation of the example in Section VII is $O(3^{2+1})$. This can be seen by the lowest black curve in Figure I.

This performance is better than the Monte Carlo algorithm of Pashayan et al. (9) and is achieved for an algorithm with no Monte Carlo sampling error. Use of Monte Carlo would further improve this scaling. It is interesting to note that weak simulation of qudits by the method of (10) scales as $3^{0.32t}$ and so it is perhaps possible that Monte Carlo could improve this strong simulation algorithm to be more efficient than weak simulation.

X. FUTURE DIRECTIONS

One of the central pillars of the stationary phase method used here is that non-contextual operations are efficiently classically simulable. This means that the Weyl symbols of Clifford gates reduce to a Gauss sum and affect simple symplectic transformations in Weyl phase space. Furthermore, stabilizer states have non-negative Wigner functions that characterize functions of affine subspaces of the discrete phase space.

For a similar approach to work for qubits, the same operations must be non-contextual in order to be able to be free resources. This requires the WWM formalism to be extended from two generators, $p$ and $q$, to three generators that become Grassmann elements (17). The resultant Grassmann algebra cannot be treated over disjoint states in phase space (13), such as we have done here, and so a Grassmann calculus must be used. It would be interesting to see if such the stationary phase method can be applied to the Grassmann algebra and produce a similar treatment of qubit $\pi/8$ gates.

Another interesting direction for future study regards the mathematical relationship between $\pi/8$ gate magic state stabilizer rank and the number of critical points in exponentiated multidimensional polynomials. The improvement in scaling found in Section XIX found by rotating by 90 degrees in the $q_{2q_1}q_{2q_2}$ plane is really due
to the reduction in the cubic power of a polynomial with respect to one degree when it is reexpressed as an in-
separable polynomial with the second degree of freedom. 

Theorem 3 strongly suggests that a similar simplification 
holds for higher numbers of $\pi/8$ gate magic states that 
are known to have lower stabilizer ranks [3]. The form of 
this relationship may be helpful in finding such lower sta-
bilizer ranks for even higher numbers of $\pi/8$ gate magic 
states than are currently know, as well as establishing 
concrete bounds.

A related interesting question regards approximate sta-
bilizer rank of magic states instead of their exact stabil-
izer rank. This is sufficient for weak simulation and fre-
quently leads to a more efficient algorithm since we can 
get away with introducing error in the probability distri-
bution we are sampling that is supposed to represent the 
extact probability distribution. A weak simulation result 
for qudits has recently been developed [10]. However, it 
would be interesting to see if there is a similar result to 
Theorem 3 that deals with approximate stabilizer rank 
and if there is some such “approximate” analog to the 
discrete stationary phase method that is useful for weak 
simulation.

XI. CONCLUSION

This paper established the stationary phase method 
as a way to understand the order $h^0$ non-contextual Cliff-
ford subtheory in the WWM formalism, producing single 
Gauss sums, and higher order $h$ contextual extensions of 
the subtheory, in terms of uniformizations—higher or-
der order sums—that can be reexpressed in terms of a sum 
over critical points or Gauss sums. This firmly tracks 
with the same relationships that exist in the continu-
ous infinite-dimensional Hilbert space treatment of Gaus-
sianity and non-Gaussianity, even though the stationary 
phase method introduced here for qudit systems differs 
in that it is a “periodized” stationary phase.

We discussed these differences and similarities between 
the continuous and discrete case. This involved compar-
ing this measure of non-contextuality to negativity. We 
found that the usage of higher order $h$ uniformizations 
through the stationary phase method is more efficient 
than using negativity for $\pi/8$ gate magic states, at least in 
the manner that has so far been tried. This also seems to 
have been noticed in the discrete community, which has 
turned to favor stabilizer state decomposition of magic 
states [5, 8]. By relating the stabilizer rank of magic 
states to the number of critical points necessary to treat 
them, this paper falls in line with this latter approach in 
treating non-Clifford gates.

We found that we are able to calculate a single qutrit 
marginal from a system consisting of $t \pi/8$ gate magic 
states that are then evolved under Clifford gates, with 
a sum consisting of $3^t+1$ critical points corresponding to 
closed-form Gauss sums when the magic states are kept 
separable. This scaling improves to $3^{2t+1}$ when pairs of 
magic states are rotated into each other in accord with 
the optimal two-qutrit $\pi/8$ gate magic state stabilizer 
rank. We showed that the latter scaling improves upon 
the current state-of-the-art.

All of this taken together establishes the usefulness of 
contextuality for practical application of classical simu-
lation of qudit quantum algorithms through the venue of 
semiclassical higher order corrections in $h$ accomplished by 
the stationary phase method.

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Appendix A: p-adic Numbers

We define the p-adic order or p-adic valuation, \( \nu_p(n) \), of a non-zero integer \( n \) to be the highest exponent \( \nu \) such that \( p^{\nu} \) divides \( n \) and set \( \nu_p(0) = \infty \).

p-adic numbers \( \mathbb{Q}_p \) can be written as:

\[
\sum_{i=k}^{\infty} c_i p^i, \quad (A1)
\]

for \( c_i \in \{0,1,2,\ldots,(p-1)\} \) and \( k \) is an integer. p-adic integers, \( \mathbb{Z}_p \), are p-adic numbers where \( c_i = 0 \) for all \( i < 0 \).

Let us define the absolute value of a p-adic number \( n \), \( |n|_p \), to be the inverse of its valuation taken to the \( p \)th power: \( |n|_p = \frac{1}{p^{\nu_p(n)}} \). Hence, we can define the metric \( |n - m|_p \) to denote the distance between two p-adic numbers \( m \) and \( n \). Notice that this means that \( m \) and \( n \) are “close” together if their distance is a large power of \( p \), which is the opposite expected from the Euclidean metric. As a result, this p-adic formalism presents an alternative way to complete the rational numbers to the real numbers \( \mathbb{R} \); completing the rationals with respect to the p-adic metric, \( (\mathbb{Q},|\cdot|_p) \), produces the p-adic numbers.

The results presented in this paper come from p-adic number theory. However, it should be clear from this brief presentation that positive integers and positive rational numbers with terminating base \( p \) expansions will have terminating p-adic expressions (Eq. A1) that are identical to their base \( p \) expansions. Since these are primarily the cases we will be concerned with in this paper, a more thorough understanding of p-adic theory is not really strictly necessary.

Appendix B: Gauss Sums on Clifford Gates

In preparation of the final sum over \( x_3 \), we expand out the exponent’s phase:

\[
u^T A u - 2c = v^T (A^{-1})^T A A^{-1} v - 2c = \left( \begin{array}{c} 2(x_3 - x + x') + (\alpha \mod p) \\ 2(x_3 - x - x') + (\alpha \mod p) \end{array} \right)^T J^T \begin{pmatrix} 1 \ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \ 1 \end{pmatrix} \begin{pmatrix} -I \ JB^{-1} \end{pmatrix} \begin{pmatrix} 2(x_3 - x + x') + (\alpha \mod p) \\ 2(x_3 - x - x') + (\alpha \mod p) \end{pmatrix}
\]

\[
= \left\{ \begin{array}{c} 2(x_3 - x + x') + (\alpha \mod p) \right\} \begin{pmatrix} \frac{1}{2} \ 1 \end{pmatrix} \begin{pmatrix} -I \ JB^{-1} \end{pmatrix} \begin{pmatrix} 2(x_3 - x + x') + (\alpha \mod p) \\ 2(x_3 - x - x') + (\alpha \mod p) \end{pmatrix}
\]

\[
+ [2(x_3 - x - x') + (\alpha \mod p)] \begin{pmatrix} \frac{1}{2} \ 1 \end{pmatrix} \begin{pmatrix} -I \ JB^{-1} \end{pmatrix} \begin{pmatrix} 2(x_3 - x + x') + (\alpha \mod p) \\ 2(x_3 - x - x') + (\alpha \mod p) \end{pmatrix}
\]

\[
+ \left\{ \begin{array}{c} 2(x_3 - x + x') + (\alpha \mod p) \right\} \begin{pmatrix} \frac{1}{2} \ 1 \end{pmatrix} \begin{pmatrix} -I \ JB^{-1} \end{pmatrix} \begin{pmatrix} 2(x_3 - x + x') + (\alpha \mod p) \\ 2(x_3 - x - x') + (\alpha \mod p) \end{pmatrix}
\]

\[
- 4[x^T J x + x_3^3 J(x + x')].
\]

Gathering powers of \( x_3 \), we define the Hessian

\[
A' = 4J^T \frac{1}{2} (B - JB^{-1}J)^{-1} (-I - B^{-1}J) J (B2)
\]

\[
4J^T \frac{1}{2} (B - JB^{-1}J)^{-1} (B^{-1}J + I) J
\]

\[
= 0,
\]
\[2v^T = \left\{ 2J^T \frac{-1}{2} (B - J B^{-1} J)^{-1} (-I - J B^{-1} - J B^{-1} - I) J (2x' + (\alpha \mod p)) \right\}^T \]

\[= (2x' + (\alpha \mod p))^T J (-2I - 2J B^{-1})^T (B - J B^{-1} J)^{-1} (-2I + 2J B^{-1}) J 2 + 4(x + x')^T J \]

By Proposition 1b, since \( B, B^{-1} \) and \( (B - J B^{-1} J)^{-1} \) are symmetric, \( J^{-1} = J^T = -J \), and

\[B - J B^{-1} J = (\mp J B^{-1} J)^{-1} (I + J B^{-1} J)^{-1} (\mp J B^{-1} J)^{-1}, \tag{B4}\]

and we define the scalar

\[2c' = (2(-x + x') + \alpha)^T J^T (B - J B^{-1} J)^{-1} \frac{-1}{2} (-I) J (2(-x + x') + \alpha) \]

\[+ (2(-x' - \alpha))^T J^T (B - J B^{-1} J)^{-1} \frac{-1}{2} (-J B^{-1}) J (2(-x + x') + \alpha) \]

\[+ (2(-x + \alpha))^T J^T (B - J B^{-1} J)^{-1} \frac{-1}{2} (J B^{-1}) J (2(-x - x') - \alpha) \]

\[+ (2(-x - x'))^T J^T (B - J B^{-1} J)^{-1} \frac{-1}{2} (I) J (2(-x - x') - \alpha) - 4x'^T J x \]

\[= 2[4x'^T \beta J x + 2x'^T \beta J x + 4x'^T \beta J x + 2x'^T \beta J x + 4x'^T \beta J x - 2\alpha^T \beta J B^{-1} J x + 4x'^T \beta J B^{-1} J x \]

\[+ 4 x'^T \beta J B^{-1} J x + 2x'^T \beta J B^{-1} J \alpha] - 4 x'^T J x \]

\[= 2[8x'^T \beta J x + 4x'^T \beta J x + 4x'^T \beta J B^{-1} J x + 4x'^T \beta J B^{-1} J x + 4x'^T \beta J B^{-1} J x] - 4 x'^T J x \]

\[= x'^T [16 \beta J - 16 \beta J B^{-1} J - 4 J x + \alpha^T [8 \beta J - 8 \beta J B^{-1} J] x \]

\[= 4x'^T [4 \beta J - 4 \beta J B^{-1} J - J x + 8 \alpha^T \beta J B^{-1} J] x \]

\[= 4x'^T (J^T (B - J B^{-1} J)^{-1} \frac{-1}{2} J - 4J^T (B - J B^{-1} J)^{-1} \frac{-1}{2} J B^{-1} J - J)((1 + J B^{-1} J)(1 - J B)(x' + \alpha \alpha^T) + \frac{\alpha^T}{2}) \]

\[+ 8 \alpha^T J^T (B - J B^{-1} J)^{-1} \frac{-1}{2} J [1 - B^{-1} J]((1 + J B^{-1} J)(1 - J B)(x' + \alpha^T) + \frac{\alpha}{2}) \]

for \( \beta = -\frac{1}{2} J^T (B - J B^{-1} J)^{-1}. \)

By Proposition 4, since \( A' = 0 \), the sum over \( x_3 \) is non-zero if and only if \( v' = 0 \):

\[x^T = (2x' + \alpha)^T J (J B^{-1} J)^{-1} J - x'^T \]

\[= x'^T J (2 + (J B - 1) J B^{-1} J)^{-1} J \]

\[+ \alpha^T \frac{1}{2} J (1 + J B - 1 - J B)(J B^{-1} J)^{-1} 1 - J B) - J)((1 + J B^{-1} J)(1 - J B)(x' + \alpha \alpha^T) + \frac{\alpha^T}{2}) \]

\[\Rightarrow x = (-J M^{-1} J)^T (x' + \alpha \alpha^T) + \frac{\alpha^T}{2} \]

\[= -J (-J M \alpha J)\alpha (x' + \alpha \alpha^T) + \frac{\alpha}{2} \]

\[= M (x' + \alpha \alpha^T) + \frac{\alpha}{2}. \tag{B7}\]
Thus we find that the sum over $x_3$ is non-zero i.f.f. $x = \mathcal{M}(x' + \frac{\Lambda}{2}) + \frac{\Lambda}{2}$ and for these values the phase $2c'$ is equal to zero too. We can see this last fact by examining the quadratic, linear and constant parts of $2c'$ w.r.t. $x'$ separately.

---

**Quadratic terms in $x'$:**

$$4x^T J (B - JB^{-1}J)^{-1} J - 2J(B - JB^{-1}J)^{-1}JB^{-1}J - J(1 + JB)^{-1}(1 - JB)x'$$  \hspace{1cm} (B8)

$$= 4x^T J (B - JB^{-1}J)^{-1} J(2 - 2B^{-1}J + J(B - JB^{-1}J))(1 + JB)^{-1}(1 - JB)x'$$

$$= 4x^T J (B - JB^{-1}J)^{-1} J(2 - 2B^{-1}J + JB + B^{-1}J)(1 + JB)^{-1}(1 - JB)x'$$

$$= 4x^T J (B - JB^{-1}J)^{-1} J(1 - B^{-1}J)(1 + JB)(1 + JB)^{-1}(1 - JB)x'$$

$$= 4x^T J(-JB + 1)^{-1}(-B^{-1}J + 1)J^{-1}J(1 - B^{-1}J)(1 - JB)x'$$

$$= 4x^T J x'$$

$$= 0.$$  \hspace{1cm} (B9)

Using the results from the same simplification from the work on the quadratic part to simplify the first two terms in the linear part, we find:

$$2x^T J \alpha + 2x^T J(B - JB^{-1}J)^{-1} J(1 - B^{-1}J)(1 + JB)\alpha + 4\alpha^T J(B - JB^{-1}J)^{-1} J(1 - B^{-1}J)(1 + JB)^{-1}x'$$

$$= 2x^T J \alpha + 2x^T J(1 - JB)^{-1}(-B^{-1}J + 1)^{-1}(1 - B^{-1}J)(1 + JB)\alpha$$

$$+ 4\alpha^T J(1 - JB)^{-1}(-B^{-1}J + 1)^{-1}(1 - B^{-1}J)(1 + JB)(1 + JB)^{-1}x'$$

$$= 2x^T J(1 - JB)^{-1}(1 - JB + 1 + JB)\alpha + 4\alpha^T J(1 + JB)^{-1}x'$$

$$= 4x^T J(1 - JB)^{-1} \alpha + 4\alpha^T J(1 + JB)^{-1}x'$$

$$= 4x^T J(1 - JB)^{-1} \alpha - ((1 + JB)^{-1}J)\alpha$$

$$= 4x^T J - (J - JB)\alpha - ((J + JB)\alpha)$$

$$= 4x^T J - (J - JB)\alpha + (J - JB)\alpha$$

$$= 4x^T J\alpha$$

$$= 0.$$  \hspace{1cm} (B10)

**Constant terms:**

$$2\alpha^T J(B - JB^{-1}J)^{-1} J(1 - B^{-1}J)((1 + JB)^{-1}(1 - JB) + 1)\alpha$$

$$= 2\alpha^T J(1 - JB)^{-1}(-B^{-1}J + 1)^{-1}(1 - B^{-1}J)((1 + JB)^{-1}(1 - JB) + 1)\alpha$$

$$= 2\alpha^T J(1 - JB)^{-1}(1 - JB + 1 + JB)\alpha$$

$$= 4\alpha^T J(1 - JB)^{-1}(1 + JB)^{-1}\alpha$$

$$= 4\alpha^T J(1 - JB)\alpha$$

$$= -4\alpha^T (JBJB)\alpha$$

Since $((JBJB)^{-1})^T = -(JBJB)^{-1}$, it follows that

$$-4\alpha^T (JBJB)^{-1}\alpha = 4\alpha^T (JBJB)^{-1}\alpha = 0.$$  \hspace{1cm} (B12)

---

**Appendix C: First Non-Trivial Example**

In particular, for these values of $x$ and $x'$, $U^*U(x, x') = d_2$ since the phase $2c'$ is equal to zero and $G_1(0) = d_2$ for a $4N$ dimensional summation variable $(x_1, x_2)$. 

\[(U_9 U_9^*) (x) = \left( \frac{1}{3^2} \right)^2 \sum_{x', x'' \in (\mathbb{Z}/3^2\mathbb{Z})^2} U_9(x''') U_9^*(x') \exp \left( \frac{2\pi i}{3^2} \Delta_3(x, x', x'') \right) \]
\[= \frac{1}{3^2} \sum_{x', x'' \in (\mathbb{Z}/3^2\mathbb{Z})^2} \exp \left\{ \frac{2\pi i}{3^2} \left[ C(-x'_q^3 + x'_q^n) + B(-x'_q^2 + x'_q^2) + \alpha \mathcal{J}(x' - x'') + 2x'^T \mathcal{J}(x' - x'') + 2x''^T \mathcal{J} x'' \right] \right\} \]
\[= \frac{1}{3^2} \sum_{x', x'' \in (\mathbb{Z}/3^2\mathbb{Z})^2} \exp \left\{ \frac{2\pi i}{3^2} \left[ C(-x'_q^3 + x'_q^n) + B(-x'_q^2 + x'_q^2) - \alpha_p(x'_q - x''_q) - 2x_p(x'_q - x''_q) \right] \right\} \]
\[\times \exp \left\{ \frac{2\pi i}{3^2} \left[ -2x_q - \alpha_q + 2x'_q x''_p + (\alpha_q + 2x_q - 2x'_q) x''_p \right] \right\} \]
\[= \sum_{x'_q, x''_q \in (\mathbb{Z}/3^2\mathbb{Z})} \exp \left\{ \frac{2\pi i}{3^2} \left[ C(-x'_q^3 + x'_q^n) + B(-x'_q^2 + x'_q^2) - \alpha_p(x'_q - x''_q) - 2x_p(x'_q - x''_q) \right] \right\} \]
\[\times \delta \left[ (2x'_q - 2x_q - \alpha_q) \mod 3^2 \right] \delta \left[ (-2x''_q + 2x_q + \alpha_q) \mod 3^2 \right] \]
\[= \sum_{x'_q, x''_q \in (\mathbb{Z}/3^2\mathbb{Z})} \exp \left\{ \frac{2\pi i}{3^2} \left[ C(-x'_q^3 + x'_q^n) + B(-x'_q^2 + x'_q^2) - \alpha_p(x'_q - x''_q) - 2x_p(x'_q - x''_q) \right] \right\} \]
\[\times \delta \left[ (x'_q - x''_q) \mod 3^2 \right] \delta \left[ (2x'_q + 2x''_q - 4x_q - 2\alpha_q) \mod 3^2 \right] \]
\[= 1. \]

So we now consider

\[U_9 U_9^*(x, x') = \frac{1}{9^2} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/3^2\mathbb{Z})^{2N}} U(x_1) U^*(x_2) \exp \left[ \frac{2\pi i}{p} \Delta_5(x_3, x, x_1, x', x_2) \right] \]
\[= \frac{1}{9^2} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/3^2\mathbb{Z})^2} \exp \left\{ \frac{2\pi i}{3^2} \left[ S_9(x_1) - S_9(x_2) - \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^T \left( \begin{array}{cc} 0 & \mathcal{J} \\ -\mathcal{J} & 0 \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + 2 \left( \begin{array}{c} x_3 - x + x' \\ x_3 - x - x' \end{array} \right)^T \mathcal{J} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + c \right) \right\}, \]

where we remind ourselves that

\[c = 2 \left[ x'^T \mathcal{J} x + x''^T \mathcal{J} (x + x') \right]. \tag{C3} \]
\[ U_9 U_9^* (x, x') = \frac{1}{q^2} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/3\mathbb{Z})^3} \exp \left\{ \frac{2\pi i}{3^2} \left[ C x_{1q}^3 + B x_{1q}^2 - C x_{2q}^3 - B x_{2q}^2 + c \right] \right\}, \] (C4)
\[ \times \exp \left\{ \frac{2\pi i}{3^2} \left[ \alpha^T J (x_1 - x_2) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2 \left( \frac{x_3 - x + x'}{x_3 - x - x'} \right)^T J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \right\} \]
\[ = \frac{1}{q^2} \sum_{x_1, x_2, x_3 \in (\mathbb{Z}/3\mathbb{Z})^3} \exp \left\{ \frac{2\pi i}{3^2} \left[ C x_{1q}^3 + B x_{1q}^2 - C x_{2q}^3 - B x_{2q}^2 + c \right] \right\}, \]
\[ \times \exp \left\{ \frac{2\pi i}{3^2} \left[ \alpha_p (x_{1q} - x_{2q}) - 2(x_{3p} - x_p + x'_p) x_{1q} - 2(x_{3p} - x_p - x'_p) x_{2q} \right] \right\} \]
\[ \times \exp \left\{ \frac{2\pi i}{3^2} \left[ -\alpha_q + 2(x_{2q} + x_{3q} - x_q + x'_q) \right] x_{1p} \right\} \exp \left\{ \frac{2\pi i}{3^2} \left[ \alpha_q + 2(-x_{1q} + x_{3q} - x_q - x'_q) \right] x_{2p} \right\} \]
\[ = \sum_{x_{1q}, x_{2q} \in \mathbb{Z}/3\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ C x_{1q}^3 + B x_{1q}^2 - C x_{2q}^3 - B x_{2q}^2 + c \right] \right\}, \]
\[ \times \exp \left\{ \frac{2\pi i}{3^2} \left[ \alpha_p (x_{1q} - x_{2q}) - 2(x_{3p} - x_p + x'_p) x_{1q} - 2(x_{3p} - x_p - x'_p) x_{2q} \right] \right\} \]
\[ \times \delta \left\{ \left[ \alpha_q - 2(x_{2q} + x_{3q} - x_q + x'_q) \right] \mod 3^2 \right\} \delta \left\{ \left[ -\alpha_q - 2(-x_{1q} + x_{3q} - x_q - x'_q) \right] \mod 3^2 \right\} \]
\[ = \sum_{x_{1q}, x_{2q} \in \mathbb{Z}/3\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ C x_{1q}^3 + B x_{1q}^2 - C x_{2q}^3 - B x_{2q}^2 + c \right] \right\}, \]
\[ \times \exp \left\{ \frac{2\pi i}{3^2} \left[ \alpha_p (x_{1q} - x_{2q}) - 2(x_{3p} - x_p + x'_p) x_{1q} - 2(x_{3p} - x_p - x'_p) x_{2q} \right] \right\} \]
\[ \times \delta \left\{ \left[ -2(x_{2q} + 2x_{3q} - 2x_q - x_{1q}) \right] \mod 3^2 \right\} \delta \left\{ \left[ 2(\alpha_q - x_{2q} - x_{1q} - 2x'_q) \right] \mod 3^2 \right\} \]
We replace the $x_{3q}$ sum and sum away $x_{3p}$:

$$U_g U_3^r(x,x') = \sum_{x_{1q},x_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \sum_{x_3 \in (\mathbb{Z}/3^2\mathbb{Z})^2} \exp \left\{ \frac{2\pi i}{3^2} \left[ Cx_{1q}^3 + Bx_{1q}^2 - Cx_{2q}^3 - Bx_{2q}^2 + 2(x^T \mathbf{J} x - x_{3p}(x_q + x'_q)) \right] \right\},$$

\[
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ a_p(x_{1q} - x_{2q}) - 2(x_{3p} - x_p + x'_p)x_{1q} - 2(x_{3p} - x_p - x'_p)x_{2q} \right] \right\} \\
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ \alpha_p(x_{1q} - x_{2q}) - 2(x_{3p} - x_p + x'_p)x_{1q} - 2(x_{3p} - x_p - x'_p)x_{2q} \right] \right\} \\
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 2x_q(x_p + x'_p) - (x_{2q} - x_{1q})(x_p + x'_p) \right] \right\} \delta \left\{ \left[ 2(\alpha_q - x_{2q} - x_{1q} - 2x'_q) \right] \mod 3^2 \right\} \\
= \sum_{x_{1q},x_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ Cx_{1q}^3 + Bx_{1q}^2 - Cx_{2q}^3 - Bx_{2q}^2 + 2(x^T \mathbf{J} x) \right] \right\},$$

\[
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ a_p(x_{1q} - x_{2q}) - 2(-x_p + x'_p)x_{1q} - 2(-x_p - x'_p)x_{2q} + 2x_q(x_p + x'_p) - (x_{2q} - x_{1q})(x_p + x'_p) \right] \right\} \\
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 2(-x_q - x'_q - x_{2q} - x_{1q})x_{3p} \right] \right\} \delta \left\{ \left[ 2(\alpha_q - x_{2q} - x_{1q} - 2x'_q) \right] \mod 3^2 \right\} \\
= 9 \sum_{x_{1q},x_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ Cx_{1q}^3 + Bx_{1q}^2 - Cx_{2q}^3 - Bx_{2q}^2 + 2(x^T \mathbf{J} x) \right] \right\},$$

\[
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ a_p(x_{1q} - x_{2q}) - 2(-x_p + x'_p)x_{1q} - 2(-x_p - x'_p)x_{2q} + 2x_q(x_p + x'_p) - (x_{2q} - x_{1q})(x_p + x'_p) \right] \right\} \\
\times \delta \left\{ \left[ 2(-x_q - x'_q - x_{2q} - x_{1q}) \right] \mod 3^2 \right\} \delta \left\{ \left[ 2(\alpha_q - x_{2q} - x_{1q} - 2x'_q) \right] \mod 3^2 \right\} \\
= 9 \sum_{x_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ C(-x_q - x'_q - x_{2q})^3 + B(-x_q - x'_q - x_{2q})^2 - Cx_{2q}^3 - Bx_{2q}^2 + 2(x^T \mathbf{J} x) + 2x_q(x_p + x'_p) \right] \right\},$$

\[
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ a_p(-x_q - x'_q - 2x_{2q}) - 2(-x_p + x'_p)(-x_q - x'_q - 2x_{2q}) + 2x_q(-x_p + x'_p)x_{2q} + 2x_{2q} + x_q + x'_q)(x_p + x'_p) \right] \right\} \\
\times \delta \left\{ \left[ 2(\alpha_q + x_{2q} - x'_q) \right] \mod 3^2 \right\}, \\
= \left\{ \begin{array}{ll} 9 \sum_{x_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ C'x_{2q}^3 + B'x_{2q}^2 + \alpha' x_{2q} + c' \right] \right\} & \text{for } x_q = (-\alpha_q + x'_q) \mod 3^2 \\
0 & \text{otherwise.} \end{array} \right. \]

where

$$C' = -2C,$$  \hspace{1cm} (C6)

$$B' = -3C(2x'_q - \alpha_q),$$  \hspace{1cm} (C7)

$$\alpha' = (2x'_p - 2x_p - 2\alpha_p + (2x'_q - \alpha_q)(2B - 6C x'_q + 3C\alpha_q)), \hspace{1cm} (C8)$$

and

$$c' = -(2x'_p - \alpha_p)(-x'_p + x_p + \alpha_p + (2x'_q - \alpha_q)(-B + C(2x'_q - \alpha_q))).$$ \hspace{1cm} (C9)

where in the last step we could have replaced $x_{2q}$ instead of $x_{1q}$ with similar results.

**Appendix D: Qutrit π/8 Gate**
\[ U_{\pi/8} U_{\pi/8}^* (x, x') = \frac{1}{3^2} \sum_{x_1, x_2, x_3 \in \mathbb{Z}/3\mathbb{Z}}^n U_{\pi/8} (x_1) U_{\pi/8}^* (x_2) \exp \left[ \frac{2\pi i}{3} \Delta_5 (x_3, x_1, x_1', x_2) \right], \]  
\[ = \frac{1}{3^2} \sum_{x_1, x_2, x_3 \in \mathbb{Z}/3\mathbb{Z}} \exp \left[ \frac{2\pi i}{3} S_{\pi/8} (x_1) \right] \exp \left[ -\frac{2\pi i}{3} S_{\pi/8} (x_2) \right] \times \exp \left\{ \frac{2\pi i}{3} \left[ -\left( x_1 \right)^T \left[ \begin{array}{c} \mathcal{J} \\ -\mathcal{J} \end{array} \right] \left( x_1 \right) - 2 \left( x_3 - x + x' \right)^T \mathcal{J} \left( x_2 \right) + c \right] \right\}, \]  
\[ = \frac{1}{3^2} \sum_{x_1, x_2, x_3 \in \mathbb{Z}/3\mathbb{Z}} \exp \left[ \frac{2\pi i}{3} \left[ S_{\pi/8}' (x_1) - S_{\pi/8}' (x_2) \right] \right] \times \exp \left\{ \frac{2\pi i}{3} \left[ -3 \left( x_1 \right)^T \left[ \begin{array}{c} \mathcal{J} \\ -\mathcal{J} \end{array} \right] \left( x_1 \right) - 2 \times 3 \left( x_3 - x + x' \right)^T \mathcal{J} \left( x_2 \right) + 3c \right] \right\}, \]  

where we remind ourselves again that  
\[ c = 2 \left[ x^{T} \mathcal{J} x + x'^{T} \mathcal{J} (x + x') \right]. \]

We first sum away the linear monomials in \( x_{1p} \) and \( x_{2p} \) to produce Kronecker delta functions:
We are interested in the magic state of this gate, which corresponds to it acting on $|p = 0\rangle$. The Wigner function of $|p = 0\rangle$ is $\rho'(x) = \frac{1}{2} \delta_{x,p}$. Hence,
\[ \rho_{\pi/8}(x) \equiv \sum_{x'} U_{\pi/8}^* U_{\pi/8}(x, x') \rho'(x') = \begin{cases} \frac{1}{3^2} \sum_{x_2 \in \mathbb{Z}/3^2 \mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} S_{\pi/8}^w(x_2 q, x_q, x_p, x'_p = 0) \right] & \text{if } x'_q = x_q \text{ (mod 3)}, \\ 0 & \text{otherwise} \end{cases} \]

\[ = \frac{1}{3^2} \sum_{x_2 \in \mathbb{Z}/3^2 \mathbb{Z}} \sum_{x'_q \in \mathbb{Z}/3^2 \mathbb{Z}} \times \exp \left\{ \frac{2\pi i}{3^2} \left[ (-x_2 q + x_q + x'_q)^3 \right] \right\} \times \exp \left\{ \frac{2\pi i}{3^2} \left[ -x_2^3 \right] \right\} \times \exp \left\{ \frac{2\pi i}{3^2} 3 \left[ (2x_2 q + x'_q - 3x_q) x_p \right] \right\} \delta \left[ 2 \times 3 \left( x_q - x'_q \right) \text{ (mod } 3^2) \right] \]

\[ = \frac{1}{3^2} \sum_{x_2 \in \mathbb{Z}/3^2 \mathbb{Z}} \times \exp \left\{ \frac{2\pi i}{3^2} \left[ (-x_2 q + 2x_q)^3 \right] \right\} \times \exp \left\{ \frac{2\pi i}{3^2} \left[ -x_2^3 \right] \right\} \times \exp \left\{ \frac{2\pi i}{3^2} 2 \times 3 \left[ (x_2 q - x_q) x_p \right] \right\} . \]
\[
\rho_{\pi/8}^2(x_1, x_2) = \frac{1}{32} \sum_{x_{2q_1} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ (-x_{2q_1} + 2x_{q_1})^3 - x_{2q_1}^3 + 2 \times 3(x_{2q_1} - x_{q_1})x_{p_1} \right] \right\} \\
\times \frac{1}{3^2} \sum_{x_{2q_2} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ (-x_{2q_2} + 2x_{q_2})^3 - x_{2q_2}^3 + 2 \times 3(x_{2q_2} - x_{q_2})x_{p_2} \right] \right\} \\
= \frac{1}{3^4} \sum_{X_{2q}, \sigma_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ (-5(X_{2q} + \sigma_{2q}) + 2x_{q_1})^3 - (5(X_{2q} + \sigma_{2q}))^3 + 2 \times 3(5(X_{2q} + \sigma_{2q}) - x_{q_1})x_{p_1} \right] \right\} \\
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ (-5(X_{2q} - \sigma_{2q}) + 2x_{q_2})^3 - (5(X_{2q} - \sigma_{2q}))^3 + 2 \times 3(5(X_{2q} - \sigma_{2q}) - x_{q_2})x_{p_2} \right] \right\} \\
= \frac{1}{3^4} \sum_{X_{2q}, \sigma_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left\{ \frac{2\pi i}{3^2} \left[ -500(X_{2q}^3 + 3X_{2q}\sigma_{2q}^2) + 150(X_{2q} + \sigma_{2q})^2x_{q_1} - 60(X_{2q} + \sigma_{2q})x_{q_1}^2 + 8x_{q_1}^3 \right] \right\} \\
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 2 \times 3(5(X_{2q} + \sigma_{2q}) - x_{q_1})x_{p_1} \right] \right\} \\
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 150(X_{2q} - \sigma_{2q})^2x_{q_2} - 60(X_{2q} - \sigma_{2q})x_{q_2}^2 + 8x_{q_2}^3 \right] \right\} \\
\times \exp \left\{ \frac{2\pi i}{3^2} \left[ 2 \times 3(5(X_{2q} - \sigma_{2q}) - x_{q_2})x_{p_2} \right] \right\} ,
\]

where \( X_{2q} = x_{2q_1} + x_{2q_2} \) and \( \sigma_{2q} = x_{2q_1} - x_{2q_2} \) so that \( x_{2q_1} = 2^{-1}(X_{2q} + \sigma_{2q}) = 5(X_{2q} + \sigma_{2q}) \) and \( x_{2q_2} = 2^{-1}(X_{2q} - \sigma_{2q}) = 5(X_{2q} - \sigma_{2q}) \) (2^{-1} is equal to 5 in the \( \mathbb{Z}/3^2\mathbb{Z} \) ring).

We used the following identity in the second step:

\[
\sum_{x_{2q_1}, x_{2q_2} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} f(x_{2q_1}, x_{2q_2}) \right] \quad \text{(E2)}
\]

\[= \sum_{X_{2q}=3^2-1,3^2, \ldots , 2(3^2-1)} \sum_{\sigma_{2q}=-|X_{2q}-3^2-1|, -|X_{2q}-2(3^2-1)|+2, \ldots , |X_{2q}-2(3^2-1)|} \times \exp \left[ \frac{2\pi i}{3^2} f(5(X_{2q} + \sigma_{2q}), 5(X_{2q} - \sigma_{2q})) \right] \]

\[= \sum_{X_{2q}, \sigma_{2q} \in \mathbb{Z}/3^2\mathbb{Z}} \exp \left[ \frac{2\pi i}{3^2} f(5(X_{2q} + \sigma_{2q}), 5(X_{2q} - \sigma_{2q})) \right] ,
\]

if \( f(x_{2q_1}, x_{2q_2}) \) is a polynomial with coefficients in \( \mathbb{Z}/3^2\mathbb{Z} \).

In the last step we used the fact that the summand is 3-periodic in both \( X_{2q} \) and \( \sigma_{2q} \) (since it was 3-periodic in \( x_{2q_1} \) and \( x_{2q_2} \), and reduced to the fundamental domain \( \mathbb{Z}/3\mathbb{Z} \). (As we discussed before, for \( X_{2q} \), which is a cubic function in the exponent, this is a special case of Theorem [1]).