Adiabatic hydrodynamic modes in dielectric environment in random electric field

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Dielectric is considered in the electric field that has equal to zero the first moment and different from zero the second moment of strength in an equilibrium. Equations of ideal hydrodynamics are obtained in such field for the case of the neglect of dissipative effects. A new variable - second moment of electric field strength is included in Euler equation. Temporal equation for this variable is obtained on the basis of Maxwell equations in the hydrodynamic approximation. Adiabatic one-dimensional waves of small amplitude are studied in this system. Proceed from the theoretical estimation of the intracrystalline field in an ionic crystal the good consent of the obtained numeral values of transversal velocity of this wave with transversal velocity of sound for isotropic crystals of alkaline haloids is found.

KEYWORDS: electrohydrodynamics, isotropic second moment of electric field strength, one-dimensional plane wave, sound mode, transversal velocity of sound, alkaline haloid.

Introduction

We will produce description of dielectric, as a continuous environment by the local quantities of density $\rho$, pressure $P$ and mass velocity $\vec{v}$. Except these cleanly hydrodynamic quantities, usually in the electrohydrodynamics a state of environment is characterized by strength of electric field $\vec{E}$ (see [1] and the literature over brought there). We will consider a not piezoelectric isotropic crystal. In a crystal there is the microscopic electric field that is a random quantity with equal to zero the first moment of strength $\vec{E}$. We will suppose that it is present the different from zero second moment of electric field strength. As shown in [2,3], the external magnetic field with the indicated statistical properties provides the shear resiliency of conducting liquid at small oscillations. That is why interesting to study the small long-wave oscillations of liquid dielectric in the random electric field with the purpose of establishing a possible connection with the shear resiliency of the random intracrystalline field. Quadratic on an electric field the Maxwell stress tensor is included in standard Euler equation [1], for that we will build temporal equation in the electrohydrodynamic approximation taking into account statistical properties of the field.
1 Equations of ideal electrohydrodynamics in the random field

We will write standard equations of ideal electrohydrodynamics, ignoring all dissipative effects [1]. Continuity equation

\[ \partial_t \rho + \text{Div} \rho \vec{v} = 0 \]  

(1)

and Euler equation

\[ \partial_t (\rho v_i) + \partial_k \pi_{ik} = 0, \]  

(2)

denotations are here entered for a spatial derivative \( \partial/\partial x_k = \partial_k \) and for the momentum stream density tensor

\[ \pi_{ik} = \rho v_i v_k + P \delta_{ik} - \left( \langle E_i E_k \rangle - \langle E^2 \rangle \delta_{ik} / 2 \right) \varepsilon / 4 \pi. \]

(3)

Here \( \varepsilon \) is relative permittivity that we will consider as a constant value, that allows to ignore striction pressure.

But if we consider fully ionized plasma or ionic liquid, electric polarization corresponds to a rearrangement of the bound electrons in the material, which creates an additional charge density, known as the bound charge density. Then, all charge: ionic \( \sigma \) and bound \( \sigma_b \), must be included to Euler equation, and instead using dielectric induction \( \text{Div} \vec{D} = \varepsilon \text{Div} \vec{E} = 4 \pi \sigma \), we must use

\[ \text{Div} \vec{E} = 4 \pi (\sigma + \sigma_b). \]

That gives instead (4)

\[ \pi_{ik} = \rho v_i v_k + P \delta_{ik} - \left( \langle E_i E_k \rangle - \langle E^2 \rangle \delta_{ik} / 2 \right) / 4 \pi. \]

(4)

As we see, the Maxwell stress tensor that is fully determined by the second moment of electric field \( \langle E_i E_k \rangle \) is included in Euler equation. Brackets mean averaging on the random phases of electric field (see [4] p. 439). We will study an isotropic dielectric, then the equilibrium value of the field correlation is such:

\[ \langle E_i E_m \rangle_0 = \langle E^2 \rangle_0 \delta_{lm} / 3 = \text{const}. \]

(5)

For closing of the system of equations it is needed to build equation for this new variable. It is done in analogy to the case of magnetic hydrodynamics in the random external magnetic field [2,3]. We will write down Maxwell equation for electric field as random quantity

\[ \partial \varepsilon \vec{E} / \partial t = c \text{Rot} \vec{B} - 4 \pi q \vec{v}. \]

(6)

Here \( q \) is a local density of charge. Because of electroneutrality it has an equilibrium value \( q_0 = 0 \). We will be interested in one-dimensional linear adiabatic oscillations in this system. Temporal equation for a central moment \( \langle E_i E_k \rangle \) we will obtain from (6), multiplying on \( E_k \) in the same spatio-temporal point and making symmetrization [2,3]. We will take into account that the equilibrium field \( \vec{E} \) exists in the immobile system, that is why, according to Galilean transformations [5], \( \vec{B}' = \vec{B} + [\vec{v}, \vec{E}] \varepsilon / c \).

Then, we can write down

\[ \partial \varepsilon \vec{E}_i / \partial t = c \text{Rot}_i \left( \vec{B} + [\vec{v}, \vec{E}] \varepsilon / c \right) E_k + c \text{Rot}_k \left( \vec{B} + [\vec{v}, \vec{E}] \varepsilon / c \right) E_i - 4 \pi q (v_i E_k + v_k E_i). \]

(7)

We consider the fields of velocity and magnetic induction uncorrelated with electric one. After averaging on random phases (see, for example, [4] p. 439) and using (5) we have the linearized equation
\[ \partial \langle E_i E_k \rangle / \partial t = (\partial_k v_i + \partial_i v_k - 2 \partial_i \delta_{ik} \langle E_i^2 \rangle / 3, \] (8)

Thermal fluctuations we ignored. The entropy deviation is included in the momentum stream density tensor \([1] \). We will consider an adiabatic process, then after linearization we have for pressure \( P = (\partial P/\partial \rho)_s \rho \) and \([2] \) looks as

\[ \partial_t v_i + \partial_k \left\{ \delta_{ik} v_s^2 \rho - (\langle E_i E_k \rangle - \langle E_i E_l \rangle \delta_{ik} / 2) / 4\pi \rho_0 \right\} = 0. \] (9)

Here \( \rho_0 \) is an equilibrium value of mass density and \( v_s^2 = (\partial P/\partial \rho)_s / \rho_0 \). From \([1] \) after linearization we have

\[ \partial_t \rho + \rho_0 \text{div} \vec{v} = 0. \] (10)

2 Adiabatic one-dimension waves of small amplitude

We will consider one-dimensional waves, along direction of distribution we will direct a coordinate axis \( z \). Let all electrohydrodynamic quantities depend only on \( z \) and \( t \). Equation \([3] \) is symmetric on tensor indexes \( i \) and \( k \) that is why it contains 6 equations for the components of symmetric tensor \( \langle E_i E_k \rangle \). Equation \( \partial_t \langle E_x E_y \rangle = 0 \) has a trivial solution \( \langle E_x E_y \rangle = \text{const} \) and moves away from other system. Other 9 equations \([3]-[10] \) it comfortably to present in a matrix form

\[ \partial_t \Psi_\alpha + Z_{\alpha \beta} \partial_z \Psi_\beta = 0. \] (11)

A vector of state is here entered

\[ \Psi = (\rho, v_x, v_y, v_z, \langle E_x E_x \rangle, \langle E_x E_y \rangle, \langle E_x E_z \rangle, \langle E_y E_y \rangle, \langle E_y E_z \rangle, \langle E_z E_z \rangle) \] (12)

and a matrix with next nonzero components

\[ Z_{14} = \rho_0, Z_{26} = Z_{38} = -2Z_{45} = -2Z_{47} = 2Z_{49} = -1/4\pi \rho_0, \]

\[ Z_{41} = v_s^2 \rho_0, Z_{54} = Z_{74} = -2Z_{62} = -2Z_{83} = 2 \langle E_0^2 \rangle / 3. \] (13)

In a plane one-dimensional wave dependence of vector of the state on a coordinate and time looks as \([6] \) (p.49-55)

\[ \Psi_\alpha = A_\alpha \exp (ikz - i\omega t). \] (14)

Substitution of \( \Psi_\alpha \) in \( \Psi_\beta = VA_\alpha, \) gives

\[ Z_{\alpha \beta} A_\beta = VA_\alpha, \] (15)

where \( V = \omega/k \) is phase velocity of wave, \( A_\alpha \) is a right eigenvector of matrix \( Z \). Solving equation \( \Psi_\alpha = VA_\alpha \) in standard way we find the eigenvalues \( V \) of matrix \( Z \)

\[ V = \left\{ 0, 0, 0, -\sqrt{\langle E_0^2 \rangle / 12\pi \rho_0}, \sqrt{\langle E_0^2 \rangle / 12\pi \rho_0}, -\sqrt{\langle E_0^2 \rangle / 12\pi \rho_0}, \right. \]

\[ \left. \sqrt{\langle E_0^2 \rangle / 12\pi \rho_0}, -\sqrt{v_s^2 + \langle E_0^3 \rangle / 6\pi \rho_0}, \sqrt{v_s^2 + \langle E_0^3 \rangle / 6\pi \rho_0} \right\}, \] (16)
and also eigenvectors. Because an eigenvector is determine within a scalar multiplier, each of the found waves is characterized one independent parameter as that we will choose amplitude of the second central moment of the field. Non-spreading perturbations of one of diagonal elements of the second central moment of the field tensor and density correspond values \( V = 0 \):

\[
A_1 = \{1/8\pi v_s^2, 0, 0, 0, 0, 0, 0, 1\}, \\
A_2 = \{-1/8\pi v_s^2, 0, 0, 0, 0, 0, 1, 0\}, \\
A_3 = \{-1/8\pi v_s^2, 0, 0, 1, 0, 0, 0, 0\}.
\]

Fourth and fifth eigenvalues correspond the mode of transversal velocity oscillations along an axis \( x \) and component \( \langle E_x E_z \rangle \):

\[
A_4 = \left\{ 0, \sqrt{3/4\pi \langle E_0^2 \rangle} \rho_0, 0, 0, 0, 1, 0, 0 \right\}, \\
A_5 = \left\{ 0, -\sqrt{3/4\pi \langle E_0^2 \rangle} \rho_0, 0, 0, 0, 1, 0, 0 \right\}.
\]

Sixth and seventh eigenvalues correspond the mode of transversal velocity oscillations along an axis \( y \) and component \( \langle E_y E_z \rangle \):

\[
A_6 = \left\{ 0, 0, \sqrt{3/4\pi \langle E_0^2 \rangle} \rho_0, 0, 0, 0, 1, 0 \right\}, \\
A_7 = \left\{ 0, 0, -\sqrt{3/4\pi \langle E_0^2 \rangle} \rho_0, 0, 0, 0, 1, 0 \right\}.
\]

Last two eigenvalues correspond the mode of longitudinal velocity oscillations along an axis \( z \), mass density and diagonal components \( \langle E_x E_x \rangle \) and \( \langle E_y E_y \rangle \):

\[
A_8 = \left\{ 3\rho_0/2 \langle E_0^2 \rangle, 0, 0, -3v_s^2 + \langle E_0^2 \rangle /6\pi\rho_0/2 \langle E_0^2 \rangle, 1, 0, 1, 0 \right\}, \\
A_9 = \left\{ 3\rho_0/2 \langle E_0^2 \rangle, 0, 0, 3v_s^2 + \langle E_0^2 \rangle /6\pi\rho_0/2 \langle E_0^2 \rangle, 1, 0, 1, 0 \right\}.
\]

As it is obvious from (16)-(20), there are two transversal modes of oscillations with velocity

\[
v_t = \sqrt{\langle E_0^2 \rangle /12\pi\rho_0}
\]

and one longitudinal with velocity \( v_l = \sqrt{v_s^2 + 2v_t^2} \). In this sense a situation is completely analogical to the sound modes in an isotropic solid (see \[2\] and \[7\] p. 124-128). If \( \langle E_0^2 \rangle = 0 \), than we have ordinary sound in a liquid: \( v_t = v_s \) and \( v_l = 0 \).

### 3 Velocity of transversal sound in alkaline haloids

We will apply an expounded theory to the low-frequency long-wave sound oscillations in an isotropic ionic crystal. We will study alkaline haloids. As is generally known \[8,9\], here exactly electric interaction is decisive. Electrostatic potential operating on a point ion is such (see \[9\] (2.55)):

\[
\varphi = -Ae/r,
\]

(22)
here $r$ is distance between the nearest neighbours in a grate, $A$ is the Madelung constant which is depending on crystal structure. We will take the structure of NaCl, for that $A = 1.746$ (see [9] p. 65). Obviously, in a crystal there is the microscopic electric field that is a random quantity with the first moment of electric field strength $\vec{E}$ is equal to zero. Exchange field is of the same order. We will suppose that the second moment corresponds to (5). Characteristic strength of intracrystalline electric field proceed from (22) is such $|\vec{E}| \sim |\text{Grad}\varphi|$, and its square can be estimated as

$$\langle E_0^2 \rangle = \langle E_l E_l \rangle_0 \sim 4\pi Se^2/r^4. \quad (23)$$

A constant $S$ we can find from comparison with experimental data once for NaCl structure. We use the tabular values of the bulk modulus B ([8] table 3.5) for the calculation of transversal velocity of sound. It easily to obtain for the Young’s modulus $A$ expression through the bulk modulus $B$ and Poisson’s coefficient $\mu$ (see [7] (5,9)) $E = 3(1-2\mu)B$, then in mechanical approach (see [7] (22.4)) transversal velocity of sound is such

$$v^{\text{tab}}_t = \sqrt{\frac{3(1-2\mu)B}{2\rho(1+\mu)}}. \quad (24)$$

On the other hand, according to the formula (21), it is possible to estimate the same velocity, using electrohydrodynamic approach and estimation (23). The density of crystals is taken from [10], low-frequency relative permittivity is taken from [8] (table 5.1). The values of Poisson’s coefficient are taken from [11]. Then $S \approx 1.226$. Values of relative permittivity, marked with the apostrophe ', are taken from [9] (table 2.2). $r$ is taken from [8] (table 3.5). All data are at normal conditions.

As we see in table 1, values of transversal velocity that it is necessary to equate at a velocity of sound, calculated on the obtained formula (21), even at so rough estimation of equilibrium correlation value of the electrostatic field according to a formula (23), surprisingly well coincide with calculated, proceed from these mechanical data of crystals, on a formula (24).

## 4 Conclusions

- Thus, the evolution of dielectric with a constant relative permittivity is studied in approximation of ideal hydrodynamics. Electric field as a random quantity is characterized by a nonzero equilibrium value of the second central moment of electric field strength. The linear system of equations is obtained for the mass density, velocity and tensor of the second central moment of electric field strength.

- Adiabatic modes in this system are studied. Found two transversal with consilient phase velocities and longitudinal sound modes. Converting in zero the equilibrium value of the second central moment of electric field strength transversal modes disappear, and a longitudinal mode passes to the ordinary sound in a liquid.

- The estimation of characteristic equilibrium value of the second central moment of electric field strength for an ionic crystal is produced, that showed good accordance of the values calculated proceed from expounded an electrohydrodynamic theory of transversal sound velocity to the values that is given by an ordinary mechanical theory.
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Table 1: Velocity of transversal sound in some alkaline haloids

| crystal | $B \times 10^{11} \text{dyn/sm}^3$ | $\mu$ | $\rho_0 \text{g/sm}^3$ | $r \text{ Å}$ | $v_{tab}^{\text{lab}} \times 10^5 \text{sm/s}$ | $v_t \times 10^5 \text{sm/s}$ |
|---------|----------------------------------|--------|------------------------|-------------|---------------------------------|-----------------|
| LiF     | 6.71                             | 0.214  | 2.64                   | 2.014       | 4.24                            | 4.66            |
| LiCl    | 2.98                             | 0.245  | 2.07                   | 2.570       | 2.97                            | 3.23            |
| LiBr    | 2.38                             | 0.256  | 3.46                   | 2.751       | 2.00                            | 2.18            |
| NaF     | 4.65                             | 0.234  | 2.56                   | 2.317       | 3.43                            | 3.57            |
| NaCl    | 2.40                             | 0.243  | 2.17                   | 2.820       | 2.62                            | 2.62            |
| NaBr    | 1.99                             | 0.270  | 3.21                   | 2.989       | 1.84                            | 1.92            |
| KF      | 3.05                             | 0.274  | 2.48                   | 2.674       | 2.56                            | 2.72            |
| KCl     | 1.74                             | 0.259  | 1.98                   | 3.147       | 2.25                            | 2.20            |
| KBr     | 1.48                             | 0.283  | 2.75                   | 3.298       | 1.65                            | 1.70            |
| KI      | 1.17                             | 0.265  | 3.13                   | 3.533       | 1.44                            | 1.39            |
| RbF     | 2.62                             | 0.276  | 2.88                   | 2.815       | 2.19                            | 2.28            |
| RbCl    | 1.56                             | 0.268  | 2.76                   | 3.291       | 1.76                            | 1.71            |
| RbBr    | 1.30                             | 0.267  | 2.78                   | 3.445       | 1.61                            | 1.55            |
| RbI     | 1.06                             | 0.309  | 3.56                   | 3.671       | 1.14                            | 1.21            |