Two-phase compressible/incompressible Navier–Stokes system with inflow-outflow boundary conditions

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Abstract

We prove the existence of a weak solution to the compressible Navier–Stokes system with singular pressure that explodes when density achieves its congestion level. This is a quantity whose initial value evolves according to the transport equation. We then prove that the “stiff pressure” limit gives rise to the two-phase compressible/incompressible system with congestion constraint describing the free interface. We prescribe of the velocity at the boundary and the value of density at the inflow part of the boundary of a general bounded $C^2$ domain. There are no restrictions on the size of the boundary conditions.

Keywords: Compressible/incompressible Navier–Stokes system, inhomogeneous boundary conditions, weak solutions, renormalized continuity equation, stiff pressure limit

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1 Introduction

The fluid-type equations are often used as macroscopic models for collective dynamics. In the present paper we are particularly interested in a system that has been analysed in [8, 7] as a model the motion of big crowds. It is a two phase compressible-incompressible model describing the evolution of some averaged macroscopic quantities describing the crowd: the velocity \( u \), the density \( \varrho \) and the congestion density \( \varrho^* \). The latter describes the preferences of the individuals, or their physical dimensions, that restrict the neighbours from being too close to each other (from penetrating each other). It is set for each individual in the crowd at the initial time, and simply transported along with the flow. This means that when the density of the crowd \( \varrho \) reaches this constraining value \( \varrho^* \), the crowd behaves like the incompressible fluid. When the density \( \varrho \) is strictly less than \( \varrho^* \) the crowd behaves like compressible fluid, except that the particles move freely as there is no contact between them. The behaviour of the crowd is described by either incompressible or compressible Navier-Stokes equations on the moving subdomains separated by the interphase described by the relation:

\[
\pi(\varrho^* - \varrho) = 0. \tag{1.1}
\]

Here \( \pi \) is the unknown “pressure”, or rather, the Lagrangian multiplier associated with incompressibility condition satisfied by the velocity. Relation (1.1) states that \( \pi \) appears only on the subdomain with congestion. For \( \varrho < \varrho^* \), on the other hand, \( \pi \) is equal to 0.

The similar free boundary problem was already analysed by Lions and Masmoudi in [16] for \( \varrho^* = 1 \). The authors showed that the two-phase system can be approximated by purely compressible Navier-Stokes equations with the pressure \( \pi_n(\varrho) \approx \varrho^{\gamma_n} \) and \( \gamma_n \to \infty \). The same kind of limit passage was also investigated later on for the PDE models of tumor growth [22, 23]. In the current paper we will focus on another approximation of the unknown pressure \( \pi_\varepsilon \approx \varepsilon(\varrho^* - \varrho) \), which has some benefits from the numerical perspective, see [7, 6]. Similar forms and asymptotic limits of the singular pressure appear in the models of traffic models [2, 1, 3], collective dynamics [6], or granular flow [18, 20].

All of the previous analytical results for the derivation of the two-phase compressible-incompressible system were obtained either for the whole space case, or for the bounded domain with zero Dirichlet boundary condition. In this paper, we want to extend the analysis to the setting where the inflow and outflow of the crowd is allowed, making the model suitable to describe various evacuation scenarios. Some numerical simulations for the hyperbolic version of such model were already performed in [7].

Our starting point is the following system of equations in \((0, T) \times \Omega\):

\[
\begin{align*}
\partial_t \varrho + \text{div}_x(\varrho u) &= 0, \quad (1.2a) \\
\partial_t (\varrho u) + \text{div}_x(\varrho u \otimes u) + \nabla_x \pi_\varepsilon \left( \frac{\varrho}{\varrho^*} \right) - \text{div}_x S(\nabla_x u) &= \varrho(w - u), \quad (1.2b) \\
\partial_t \varrho^* + u \cdot \nabla_x \varrho^* &= 0, \quad (1.2c)
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^d, d = 2, 3 \) is a bounded domain of class \( C^2 \), \( \varrho = \varrho(t, x) \) is the unknown mass density, \( u = u(t, x) \) is the unknown velocity, \( \varrho^* = \varrho^*(t, x) \) is the unknown congestion density, \((t, x) \in (0, T) \times \Omega \equiv Q_T, \varepsilon > 0, \) and \( w = w(x) \) is given.
The stress tensor $S$ will be specified below (as stress tensor for compressible Newtonian fluid). The pressure $\pi$ depends on the ratio of densities $\frac{\varrho}{\varrho^*}$. It models a constraint on the density of the fluid $\varrho$ that cannot exceed the value $\varrho^*$. More precisely, we take

$$\pi_\varepsilon\left(\frac{\varrho}{\varrho^*}\right) = \varepsilon \frac{\left(\frac{\varrho}{\varrho^*}\right)^\alpha}{\left(1 - \frac{\varrho}{\varrho^*}\right)^\beta}$$

with some $\alpha > 1$ and $\beta > \frac{3}{2}$ if $d = 3$ and $\beta > 2$ if $d = 3$. Note that the pressure fulfills

$$\pi(0) = 0, \quad \pi'(z) > 0 \quad \text{for } 0 < z < 1.$$  

(1.4)

This is similar to the so-called hard sphere pressure considered, e.g., in [5], that can be viewed as a special case of system (1.2) with $\varrho^* = \text{const.}$ Similarly as in [5] we can relax the assumption on $\pi$ for $\varrho < \varrho^*$ by addition of non-monotone pressure that vanishes for $\varrho \to \varrho^*$. To avoid unnecessary technicalities, we skip this point in this paper. We consider our system together with initial conditions

$$\varrho(0) = \varrho_0, \quad \varrho u(0) = \varrho_0 u_0, \quad \varrho^*(0) = \varrho^*_0,$$

(1.5)

and boundary conditions

$$u|_{\partial \Omega} = u_B, \quad \varrho|_{\Gamma_{in}} = \varrho_B, \quad \varrho^*|_{\Gamma_{in}} = \varrho^*_B,$$

(1.6)

where

$$\Gamma_{in} = \left\{ x \in \partial \Omega \mid u_B \cdot n < 0 \right\}, \quad \Gamma_{out} = \left\{ x \in \partial \Omega \mid u_B \cdot n \geq 0 \right\}$$

(1.7)

(we include to the outflow part of the boundary also the part where the normal velocity component is zero).

Note that system (1.2a)–(1.2c) reminds system studied (for homogeneous boundary conditions for the velocity) in [17]; there, the role of the congestion density was played by the entropy. Using a similar idea as in the above mentioned paper we introduce a new quantity $Z := \frac{\varrho}{\varrho^*}$; then the new unknown function $Z$ satisfies (at least formally, for smooth solutions) the continuity equation and we obtain the following system

$$\partial_t \varrho + \text{div}_x(\varrho u) = 0,$$

(1.8a)

$$\partial_t (\varrho u) + \text{div}_x(\varrho u \otimes u) + \nabla_x \pi_\varepsilon(Z) - \text{div}_x S(\nabla_x u) = \varrho(w - u),$$

(1.8b)

$$\partial_t Z + \text{div}_x(Zu) = 0$$

(1.8c)

with initial

$$\varrho(0) = \varrho_0, \quad \varrho u(0) = \varrho_0 u_0, \quad Z(0) = Z_0 := \frac{\varrho_0}{\varrho^*_0},$$

(1.9)

and boundary conditions

$$u|_{\partial \Omega} = u_B, \quad \varrho|_{\Gamma_{in}} = \varrho_B, \quad Z|_{\Gamma_{in}} = Z_B := \frac{\varrho_B}{\varrho^*_B},$$

(1.10)
Note, however, that by standard techniques we can get certain "better" information on the "density" only from the pressure term, therefore, as in [17], we will consider a certain interplay of the initial and boundary conditions for \( \rho \) and \( \rho^* \) which leads to the fact that the boundary and initial conditions for \( \rho \) are controlled by the initial and boundary conditions for \( Z \) (see (2.7)). Furthermore, we also have that the initial and boundary conditions for \( Z \) belong to the interval \((0,1)\).

Combining the approach from [17] with [5] we will be able to show that under certain additional technical assumptions problem (1.8a)–(1.8c) with initial (1.9) and boundary conditions (1.10) possesses a weak solution defined below. We even slightly improve the result from [5] in the sense that we may include for global-in-time existence result the case when the velocity flux is zero, see Remark 4.2. Next, it is possible to show that also (1.2a)–(1.2c) with initial (1.5) and boundary conditions (1.6) has a solution: the approach is based on suitable renormalization which allows us to return back to the unknown function \( \rho^* \). On the other hand, we are more interested in the limit passage \( \varepsilon \to 0^+ \); we perform the limit passage in the formulation with the unknown function \( Z \) and later return back to the formulation with the function \( \rho^* \). Here, we follow ideas from [8] or [21]. We will show that with \( \varepsilon \to 0^+ \) the weak solutions to system (1.2) converge in some sense to the weak solution of the target system

\[
\partial_t \rho + \text{div}_x (\rho u) = 0, \quad (1.11a)
\]
\[
\partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x \pi - \text{div}_x S(\nabla_x u) = \rho (w - u), \quad (1.11b)
\]
\[
\partial_t \rho^* + u \cdot \nabla_x \rho^* = 0, \quad (1.11c)
\]
\[
0 \leq \rho \leq \rho^*, \quad (1.11d)
\]
\[
\text{div}_x u = 0 \text{ in } \{ \rho = \rho^* \}, \quad (1.11e)
\]
\[
\pi \geq 0 \text{ in } \{ \rho = \rho^* \}, \quad \pi = 0 \text{ in } \{ \rho < \rho^* \}. \quad (1.11f)
\]

We call this system a free boundary two-phase compressible/incompressible system. To justify this name note that when \( \rho = \rho^* \), i.e., when the density achieves its maximal value, due to condition (1.11e) the system behaves like inhomogeneous incompressible Navier-Stokes equations. When on the other hand \( \rho < \rho^* \), the system behaves like compressible pressureless Navier-Stokes equations with time-space variable upper bound for the density. This is one of the novelties here, as before we always had to consider the background pressure in the barotropic form.

One of the inspirations for this work was the numerical paper [7], in which the inviscid variant of system (1.2) was considered with "do nothing" boundary condition for the velocity at the outflow part of the boundary. Even though this condition is often used in numerics, it is not very suitable for analysis, in particular for global-in-time type solutions for large data.

Even though we combine approaches from several other papers, the result itself is new and requires nontrivial extensions of previously known techniques and results.
2 Main result

In what follows, we formulate main results of the paper. Before doing so, we state the main assumptions on the data of our problem. Concerning the given field $w$, we consider (for simplicity)

$$w \in L^\infty(Q_T; R^d).$$

Further, the stress tensor $S$ is characteristic for the Newtonian fluid and it is given by

$$S(\nabla_x u) = \mu \left( \nabla_x u + \nabla^t_x u \right) + \lambda \text{div}_x u I, \quad \mu > 0, \quad \lambda \geq 0.$$  

The pressure $\pi_\varepsilon(\cdot)$ has the form (1.3) with $\alpha > 1$ and $\beta > \frac{5}{2}$.

Similarly as in [5], we consider the following regularity assumptions

$$\varrho_B, \varrho^*_B \in C(\Gamma_{in}), \quad u_B \in C^2(\partial \Omega; R^d), \quad \int_{\partial \Omega} u_B \cdot n \, dS_x dt \geq 0.$$  

Furthermore, we assume that

$$0 < \varrho_0 < \varrho^*_0, \quad \text{a.e. in } \Omega, \quad \varrho^*_0 \in L^\infty(\Omega),$$

$$\int_\Omega H_\varepsilon \left( \frac{\varrho_0}{\varrho^*_0} \right) \, dx < \infty, \quad \text{ess inf}_\Omega \varrho_0 > 0, \quad \text{ess inf}_\Omega (\varrho^*_0 - \varrho_0) > 0,$$

$$u_0 \in L^2(\Omega; R^d),$$

where

$$H_\varepsilon(z) = z \int_0^z \pi_\varepsilon(s) \frac{1}{s^2} \, ds.$$  

For the boundary data,

$$0 < \varrho_B < \varrho^*_B, \quad \text{a.e. on } \Gamma_{in}, \quad \text{ess inf}_{\Gamma_{in}} \varrho_B > 0, \quad \text{ess inf}_{\Gamma_{in}} (\varrho^*_B - \varrho_B) > 0.$$  

Note that these assumptions yield that the initial energy

$$E_0 := \int_\Omega \left( \frac{1}{2} \varrho_0 |u_0|^2 + H_\varepsilon \left( \frac{\varrho_0}{\varrho^*_0} \right) \right) \, dx < +\infty$$

as well as that there are positive constants $c_*$ and $c^*$ such that

$$c_* \leq \frac{1}{\varrho^*_0} \leq c^*, \quad c_* \leq \frac{1}{\varrho^*_B} \leq c^* \quad \text{a.e.}$$

Whence, rewriting our problem to the form (1.8a)–(1.10), we immediately have that

$$c_* \varrho_0 \leq Z_0 \leq c^* \varrho_0 \quad \text{a.e. in } \Omega, \quad c_* \varrho_B \leq Z_B \leq c^* \varrho_B \quad \text{a.e. on } \Gamma_{in}.$$  

Let us now introduce the definition of a weak solution to problem (1.8a)–(1.10).
Definition 2.1. We say that \((\rho, u, Z)\) is a bounded energy weak solution of problem (1.8a)-(1.10) on a time interval \((0, T)\) if the following five conditions are satisfied.

1. The triple of functions \((\rho, u, Z)\) fulfills:

\[
0 \leq c_* \rho \leq Z \leq c^* \rho \text{ a.e. in } Q_T, \quad \text{for } 0 < c_* \leq c^* < \infty,
\]

\[
0 \leq Z < 1 \text{ a.e. in } (0, T) \times \Omega, \quad \pi_\varepsilon(Z) \in L^1(0, T; L^1(\Omega)) \quad (2.8)
\]

\[
u \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d)), \quad u|_{I \times \partial \Omega} = u_B.
\]

2. The function \(\rho \in C^{\text{weak}}([0, T], L^1(\Omega))\) satisfies the integral identity

\[
\int_\Omega \rho(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \rho_0(\cdot) \varphi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left( \rho \partial_t \varphi + \rho u \cdot \nabla_x \varphi \right) \, dx \, dt - \int_0^\tau \int_{\Gamma_{\text{in}}} \rho_B u_B \cdot n \varphi \, dS_x \, dt \quad (2.9)
\]

for any \(\tau \in [0, T]\) and \(\varphi \in C^1([0, T] \times (\Omega \cup \Gamma_{\text{in}}))\).

3. The function \(\rho u \in C^{\text{weak}}([0, T], L^1(\Omega; \mathbb{R}^d))\) satisfies the integral identity

\[
\int_\Omega \rho u(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx - \int_\Omega \rho_0 u_0(\cdot) \cdot \varphi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left( \rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla_x \varphi + \pi_\varepsilon(Z) \text{div}_x \varphi - S(\nabla_x u) : \nabla_x \varphi \right) \, dx \, dt \quad (2.10)
\]

\[
+ \int_0^\tau \int_\Omega \rho (w - u) \cdot \varphi \, dx \, dt
\]

for any \(\tau \in [0, T]\) and any \(\varphi \in C^1([0, T] \times \Omega; \mathbb{R}^d)\).

4. The function \(Z \in C^{\text{weak}}([0, T], L^1(\Omega))\) satisfies the integral identity

\[
\int_\Omega Z(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega Z_0(\cdot) \varphi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left( Z \partial_t \varphi + Z u \cdot \nabla_x \varphi \right) \, dx \, dt - \int_0^\tau \int_{\Gamma_{\text{in}}} Z_B u_B \cdot n \varphi \, dS_x \, dt \quad (2.11)
\]

for any \(\tau \in [0, T]\) and \(\varphi \in C^1([0, T] \times (\Omega \cup \Gamma_{\text{in}}))\).

5. There is a Lipschitz extension \(u_\infty \in W^{1,\infty}(\Omega; \mathbb{R}^d)\) of the vector field \(u_B\) such that the following
The energy inequality holds
\[ \int_{\Omega} \left( \frac{1}{2} \rho |u - u_\infty|^2 + H_\varepsilon(Z) \right)(\tau) \, dx + \int_0^\tau \int_{\Omega} \mathcal{S}(\nabla_x(u - u_\infty)) : \nabla_x(u - u_\infty) \, dx \, dt \]
\[ + \int_0^\tau \int_{\Omega} \pi_\varepsilon(Z) \text{div}_x u_\infty \, dx \, dt \]
\[ \leq \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0 - u_\infty|^2 + H_\varepsilon(Z_0) \right) \, dx - \int_0^\tau \int_{\Omega} \rho u \cdot \nabla_x u_\infty \cdot (u - u_\infty) \, dx \, dt \]
\[ - \int_0^\tau \int_{\Omega} \mathcal{S}(\nabla_x u_\infty) : \nabla_x (u - u_\infty) \, dx \, dt - \int_0^\tau \int_{\Gamma_{in}} H_\varepsilon(Z_B) u_B \cdot \mathbf{n} \, dS_x \, dt \]
\[ + \int_0^\tau \int_{\Omega} \rho (w - u) \cdot (u - u_\infty) \, dx \, dt \]
for a.a. \( \tau \in (0, T) \).

**Remark 2.2.** The continuity equations (1.8a) and (1.8c) give rise to
\[ \int_{\Omega} X(\tau) \, dx \leq \int_{\Omega} X_0 \, dx - \int_0^\tau \int_{\Gamma_{in}} X_B u_B \cdot \mathbf{n} \, dS_x \, dt \] (2.13)
for all \( X = \rho, Z, \) and \( \tau \in [0, T] \). It can be obtained by taking in (2.9) and (2.11) test functions \( \varphi = \varphi_\eta \), where
\[ \varphi_\eta(x) = \begin{cases} 
1 & \text{if dist}(x, \Gamma_{out}) > \eta \\
\frac{1}{\eta} \text{dist}(x, \Gamma_{out}) & \text{if dist}(x, \Gamma_{out}) \leq \eta,
\end{cases} \]
and then by letting \( \eta \to 0^+ \).

**Definition 2.3.** We call \((Z, u)\) a renormalized solution of the continuity equation (1.8c) provided:
- \( Z \in L^\infty(0, T; L^\infty(\Omega)) \), and \( u \in L^2(0, T; W^{1,2}(\Omega, R^3)) \),
- \((Z, u)\) satisfies the weak formulation of the continuity equation (2.11),
- for any \( b \in C^1[0, 1], b(Z) \in C_{\text{weak}}([0, T]; L^1(\Omega)) \) the weak formulation of the renormalized equation is satisfied, i.e.,
\[ \int_{\Omega} b(Z)(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_{\Omega} b(Z_0)(\cdot) \varphi(0, \cdot) \, dx \]
\[ = \int_0^\tau \int_{\Omega} \left( b(Z) \partial_t \varphi + b(Z) u \cdot \nabla_x \varphi - \varphi (b'(Z) Z - b(Z)) \text{div}_x u \right) \, dx \, dt \] (2.14)
\[ - \int_0^\tau \int_{\Gamma_{in}} b(Z_B) u_B \cdot \mathbf{n} \varphi \, dS_x \, dt \]
for any \( \varphi \in C^1_c([0, T] \times (\Omega \cup \Gamma_{in})) \).
A weak solution to problem (1.8a)–(1.10) satisfying in addition renormalized continuity equations (2.14) for both \((Z, u)\) and \((\rho, u)\) is called a renormalized weak solution.

**Remark 2.4.** Note that due to \([4, Lemma 3.1]\), and since we know that since both \(\rho\) and \(Z\) are essentially bounded functions and \(u \in L^2(0, T; W^{1,2}(\Omega; R^d))\), any weak solution in the sense of Definition 2.1 is in fact a renormalized weak solution, provided \(\Gamma_{in}\) is a \(C^2\) open \((d - 1)\) dimensional manifold.

In the proof that follows we will need that the boundary data for the velocity, \(u_B\), can be extended to the whole \(\Omega\) in such a way that the extension is sufficiently smooth and its divergence is non-negative. This follows from the following result (see \([5, Lemmata 5.1, 5.2, and 5.3]\))

**Lemma 2.5.** Let \(V \in C^2(\partial \Omega; R^d)\) be a vector field on the boundary \(\partial \Omega\) of a bounded \(C^2\) domain \(\Omega\). Let

\[
\int_{\partial \Omega} V \cdot n \, dS_x \geq 0.
\]

Then there exist a vector field

\[
V_\infty \in W^{2,q}(\Omega; R^d), \quad 1 \leq q < \infty, \quad \text{div}_x V_\infty \geq 0 \quad a.e. \text{ in } \Omega
\]

verifying \(V_\infty|_{\partial \Omega} = V\).

If in addition

\[
\int_{\partial \Omega} V \cdot n \, dS_x = K > 0,
\]

then the extension \(V_\infty\) satisfy

\[
\text{div}_x V_\infty \geq 0 \quad a.e. \text{ in } \Omega, \quad \text{and} \quad \text{ess inf}_\Theta (\text{div}_x V_\infty) \geq C > 0,
\]

where \(\Theta\) is an open subset satisfying \(\bar{\Theta} \subset \Omega\).

Our first result is a global-in-time existence theorem for solutions defined above.

**Theorem 2.6.** Let \(\Omega \subset R^d, d = 2, 3\), be a bounded domain of class \(C^2\) such that \(\Gamma_{in}\) is an open \(C^2\) \((d - 1)\) dimensional manifold. Let \(\varepsilon > 0, T > 0\). Under the assumptions (2.1)–(2.6), the problem (1.8)–(1.10) admits at least one bounded energy weak solution \((\rho, u, Z)\) on \((0, T)\) in the sense of Definition 2.1. Moreover \((\rho, u, Z)\) is a renormalized solution in the sense of Definition 2.3.

Next we consider solutions to problem (1.2)–(1.7) with (2.2).

**Definition 2.7.** We say that \((\rho, u, \rho^*)\) is a bounded energy weak solution of problem (1.2)–(1.7) with (2.2) on a time interval \((0, T)\) if the following five conditions are satisfied.
1. The triple of functions belongs fulfills:

\[
0 \leq \varrho < \varrho^* \text{ a.e. in } Q_T
\]
\[
c_* \leq \frac{1}{\varrho^*} \leq c^* \text{ a.e. in } (0, T) \times \Omega, \quad \pi_\vartheta \left( \frac{\varrho}{\varrho^*} \right) \in L^1(0, T; L^1(\Omega)) \tag{2.17}
\]
\[
u \in L^2(0, T; W^{1,2}(\Omega; R^d)), \quad u|_{t=0} = u_B.
\]

2. The function \( \varrho \in C_{\text{weak}}([0, T], L^1(\Omega)) \) satisfies the integral identity

\[
\int_\Omega \varrho(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \varrho_0(\cdot) \varphi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left( \varrho \partial_t \varphi + \varrho u \cdot \nabla x \varphi \right) \, dx \, dt - \int_0^\tau \int_{\Gamma_{in}} \varrho_B u_B \cdot \nu \varphi \, dS_x \, dt \tag{2.18}
\]

for any \( \tau \in [0, T] \) and \( \varphi \in C^1_c([0, T] \times (\Omega \cup \Gamma_{in})). \)

3. The function \( \varrho u \in C_{\text{weak}}([0, T], L^1(\Omega; R^d)) \) satisfies the integral identity

\[
\int_\Omega \varrho u(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx - \int_\Omega \varrho_0 u_0(\cdot) \varphi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left( \varrho u \cdot \partial_t \varphi + \varrho u \otimes u \cdot \nabla x \varphi + \pi_\vartheta \left( \frac{\varrho}{\varrho^*} \right) \text{div}_x \varphi - S(\nabla x u : \nabla x \varphi) \right) \, dx \, dt + \int_0^\tau \int_\Omega \varrho (w - u) \cdot \varphi \, dx \, dt \tag{2.19}
\]

for any \( \tau \in [0, T] \) and any \( \varphi \in C^1_c([0, T] \times \Omega; R^d). \)

4. The function \( \varrho^* \in C_{\text{weak}}([0, T], L^1(\Omega)) \) satisfies the integral identity

\[
\int_\Omega \varrho^*(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \varrho_0^*(\cdot) \varphi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left( \varrho^* \partial_t \varphi + \varrho^* \text{div}_x (\varphi u) \right) \, dx \, dt - \int_0^\tau \int_{\Gamma_{in}} \varrho^*_B u_B \cdot \nu \varphi \, dS_x \, dt \tag{2.20}
\]

for any \( \tau \in [0, T] \) and \( \varphi \in C^1_c([0, T] \times (\Omega \cup \Gamma_{in})). \)

5. There is a Lipschitz extension \( u_\infty \in W^{1,\infty}(\Omega; R^d) \) of the vector field \( u_B \) such that the following
energy inequality holds

\[
\int_{\Omega} \left( \frac{1}{2} \rho |u - u_\infty|^2 + H_\epsilon \left( \frac{\rho}{\rho^*} \right) \right) \, dx + \int_0^T \int_{\Omega} S(\nabla_x (u - u_\infty)) : \nabla_x (u - u_\infty) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \pi_\epsilon \left( \frac{\rho}{\rho^*} \right) \text{div}_x u_\infty \, dx \, dt \\
\leq \int_{\Omega} \left( \frac{1}{2} \rho_0 |u_0 - u_\infty|^2 + H_\epsilon \left( \frac{\rho_0}{\rho^*} \right) \right) \, dx - \int_0^T \int_{\Omega} \rho u \cdot \nabla_x u_\infty \cdot (u - u_\infty) \, dx \, dt \\
- \int_0^T \int_{\Omega} S(\nabla_x u_\infty) : \nabla_x (u - u_\infty) \, dx \, dt + \int_0^T \int_{\Omega} H_\epsilon \left( \frac{\rho_B}{\rho^*_B} \right) u_B \cdot n \, dS_x \, dt \\
+ \int_0^T \int_{\Gamma_{in}} \rho (w - u) \cdot (u - u_\infty) \, dx \, dt
\] (2.21)

for a.a. \( \tau \in (0, T) \).

Bounded energy weak solution to problem (1.2)–(1.7) with (2.2) satisfying in addition renormalized continuity equation (2.14) for \((\rho, u)\) is called a renormalized weak solution.

We have the following result.

**Theorem 2.8.** Let \( \Omega \subset R^d, d = 2, 3, \) be a bounded domain of class \( C^2 \) such that \( \Gamma_{in} \) is an open \( C^2 \) \( d - 1 \) dimensional manifold. Let \( \varepsilon > 0, T > 0 \). Under the assumptions (2.1)–(2.6) the problem (1.2)–(1.7) with (2.2) admits at least one renormalized bounded energy weak solution \((\rho, u, \rho^*)\) on \((0, T)\) in the sense of Definition 2.7.

Our next results concern the limit passage \( \varepsilon \to 0 \). We will show that when \( \varepsilon \to 0 \) the weak solutions from the previous section approximate weak solutions to the system (1.11) defined below.

**Definition 2.9.** A quadruple \((\rho, u, \rho^*, \pi)\) is called a global finite energy weak solution to (1.11), (2.2), with the initial data (1.5), (2.4), and the boundary conditions (1.6), (2.3) if for any \( T > 0 \):

(i) There holds:

\[
0 \leq \rho \leq \rho^* \quad \text{a.e. in } (0, T) \times \Omega, \\
\text{div}_x u = 0 \quad \text{a.e. in } \{\rho = \rho^*\}, \\
(\rho^* - \rho) \pi = 0,
\] (2.22)

and

\[
\rho \in C_w([0, T]; L^\infty(\Omega)), \\
\rho^* \in C_w([0, T]; L^\infty(\Omega)), \\
u \in L^2(0, T; W^{1,2}(\Omega, R^d)), \quad |\rho u|^2 \in L^\infty(0, T; L^1(\Omega)), \\
\pi \in M^+((0, T) \times \Omega).
\]
(ii) For any $0 \leq \tau \leq T$, equations (1.11a), (1.11b), (1.11c) are satisfied in the weak sense, more precisely:
- the continuity equation:

$$
\int_\Omega \varrho(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \varrho_0(\cdot) \varphi(0, \cdot) \, dx \\
= \int_0^\tau \int_\Omega \left( \varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right) \, dx \, dt - \int_0^\tau \int_{\Gamma_{in}} \varrho_B \mathbf{u}_B \cdot \mathbf{n} \varphi \, dS_x \, dt
$$

holds for any $\tau \in [0, T]$ and $\varphi \in C^1([0, T] \times (\Omega \cup \Gamma_{in}))$.
- the momentum equation:

$$
\int_\Omega \varrho \mathbf{u}(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx - \int_\Omega \varrho_0 \mathbf{u}_0(\cdot) \varphi(0, \cdot) \, dx \\
= \int_0^\tau \int_\Omega \left( \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \pi \text{div}_x \varphi - S(\nabla_x \mathbf{u}) : \nabla_x \varphi \right) \, dx \, dt \\
+ \int_0^\tau \int_\Omega \varrho (\mathbf{w} - \mathbf{u}) \cdot \varphi \, dx \, dt
$$

holds for any $\tau \in [0, T]$ and any $\varphi \in C^1([0, T] \times \Omega; R^d)$,
- the transport equation for $\varrho^*:

$$
\int_\Omega \varrho^*(\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_\Omega \varrho^*_0(\cdot) \varphi(0, \cdot) \, dx \\
= \int_0^\tau \int_\Omega \left( \varrho^* \partial_t \varphi + \varrho^* \text{div}_x(\varphi \mathbf{u}) \right) \, dx \, dt - \int_0^\tau \int_{\Gamma_{in}} \varrho^*_B \mathbf{u}_B \cdot \mathbf{n} \varphi \, dS_x \, dt
$$

for any $\tau \in [0, T]$ and $\varphi \in C^1([0, T] \times (\Omega \cup \Gamma_{in}))$.

(iii) There is a Lipschitz extension $\mathbf{u}_\infty \in W^{1,\infty}(\Omega; R^d)$ of the vector field $\mathbf{u}_B$ such that the following energy inequality holds

$$
\int_\Omega \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_\infty|^2(\tau) \, dx + \int_0^\tau \int_\Omega S(\nabla_x (\mathbf{u} - \mathbf{u}_\infty)) : \nabla_x (\mathbf{u} - \mathbf{u}_\infty) \, dx \, dt + \int_0^\tau \int_\Omega \pi \text{div}_x \mathbf{u}_\infty \, dx \, dt \\
\leq \int_\Omega \frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{u}_\infty|^2 \, dx - \int_0^\tau \int_\Omega \varrho \mathbf{u} \cdot \nabla_x \mathbf{u}_\infty \cdot (\mathbf{u} - \mathbf{u}_\infty) \, dx \, dt \\
- \int_0^\tau \int_\Omega S(\nabla_x \mathbf{u}_\infty) : \nabla_x (\mathbf{u} - \mathbf{u}_\infty) \, dx \, dt + \int_0^\tau \int_\Omega \varrho (\mathbf{w} - \mathbf{u}) \cdot (\mathbf{u} - \mathbf{u}_\infty) \, dx \, dt
$$

for a.a. $\tau \in (0, T)$.

**Remark 2.10.** In the above definition all the terms must make sense, in particular, $\pi$ is not only a measure, but it is sufficiently regular so that the condition (2.23) makes sense.
Our main theorem in this parts reads as follows.

**Theorem 2.11.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain of class $C^2$ such that $\Gamma_{in}$ is an open $C^2$ $d-1$ dimensional manifold. Let $T > 0$, and let assumptions (2.1)–(2.6) be satisfied. If, in addition either

$$\int_{\partial \Omega} \mathbf{u}_B \cdot \mathbf{n} \, dS_x = K > 0,$$

or

$$\int_{\Omega} Z_0 \, dx + T \int_{\Gamma_{in}} Z_B |\mathbf{u}_B \cdot \mathbf{n}| \, dS_x < |\Omega|,$$

then the problem (1.11), with (2.2) admits at least one renormalized bounded energy weak solution $(\varrho, \mathbf{u}, \varrho^*, \pi)$ on $(0, T)$ in the sense of Definition 2.9.

The paper is organized as follows. In Section 3 we present the approximate scheme starting from the level of truncations of singular pressure at the level described by parameter $\delta$. Later on, in Section 4 we obtain the uniform estimates with respect to $\delta$ and pass to the limit. As an outcome of this section we prove the Theorem 2.6 and also Theorem 2.8. In Section 5 we recall uniform estimates with respect to $\varepsilon$ and perform the limit passage $\varepsilon \to 0$ and conclude the proof of Theorem 2.11.

## 3 Approximate solution

The purpose of this section is to construct approximate solutions to system (1.8a)–(1.10). We are not going to explain the details of the whole procedure but only to summarize it and to explain how can the existing literature be employed. We approximate the singular pressure in system (1.8) by the truncation

$$\pi_\delta(Z) = \begin{cases} 
\pi_\varepsilon(Z) & \text{if } Z \in [0, 1 - \delta] \\
\pi_\varepsilon(1 - \delta) + \varepsilon(Z - 1 + \delta)^\gamma_+ & \text{if } Z \in (1 - \delta, \infty),
\end{cases}$$

where the exponent $\gamma$ has to be chosen sufficiently large in order to obtain sufficient estimates, in particular we need $\gamma > d$. This truncation allows to combine the arguments from [17, 4] in order to construct the solutions to system (1.8) with $\pi_\varepsilon$ replaced by $\pi_\delta$. This consists of regularising both equations for $\varrho$ and $Z$ by adding small viscosity term, we consider

$$\partial_t \varrho - \eta \Delta_x \varrho + \text{div}_x(\varrho \mathbf{u}) = 0,$$

$$\varrho(0, x) = \varrho_0(x), \; (-\eta \nabla_x \varrho + \varrho \mathbf{u}) \cdot \mathbf{n}|_{x \in \partial \Omega} = \begin{cases} \rho_B \mathbf{u}_B \cdot \mathbf{n} & \text{if } |\mathbf{u}_B \cdot \mathbf{n}|(x) \leq 0, \; x \in \partial \Omega, \\
\rho_B \mathbf{u}_B \cdot \mathbf{n} & \text{if } |\mathbf{u}_B \cdot \mathbf{n}|(x) > 0, \; x \in \partial \Omega,
\end{cases}$$

$$\partial_t (\varrho \mathbf{u}) + \text{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \pi_\delta(Z) = \text{div}_x \mathbb{S}(\nabla_x \mathbf{u}) - \eta \nabla_x \varrho \cdot \nabla_x \mathbf{u} + \eta \text{div}_x \left( |\nabla_x (\mathbf{u} - \mathbf{u}_\infty)|^2 \nabla_x (\mathbf{u} - \mathbf{u}_\infty) \right).$$

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\begin{align}
\mathbf{u}(0, x) &= \mathbf{u}_0(x), \quad \mathbf{u}|_{t=0} = \mathbf{u}_B, \\
\partial_t Z - \eta \Delta_x Z + \text{div}_x(Z \mathbf{u}) &= 0, \\
Z(0, x) &= Z_0(x), \quad (-\eta \nabla_x Z + Z \mathbf{u}) \cdot \mathbf{n}|_{t=0} = \\
&= \begin{cases} 
Z_B \mathbf{u}_B \cdot \mathbf{n} & \text{if } [\mathbf{u}_B \cdot \mathbf{n}](x) \leq 0, \ x \in \partial \Omega, \\
Z \mathbf{u}_B \cdot \mathbf{n} & \text{if } [\mathbf{u}_B \cdot \mathbf{n}](x) > 0, \ x \in \partial \Omega,
\end{cases}
\end{align}

with positive parameters \( \varepsilon > 0, \ \delta > 0, \ \eta > 0 \). The solution to this system is obtained by means of Galerkin approximation of the momentum equation and Banach Fixed point theorem for existence of unique local in time solutions. We are not going to repeat all these details here, the details of this procedure can be found in [4] for the case with one continuity equation and it can be combined with the ideas and techniques from [17], where two continuity equations were considered, exactly in the same setting as here: one quantity is included into the pressure, the other into the momentum. Let us only notice that when \( \mathbf{u} \) is replaced by it’s Galerkin approximation \( \mathbf{u}_n \), then it is still possible to prove the comparison principle between \( \rho \) and \( Z \), similarly to the above mentioned paper. Indeed, taking \( c_\star, c^\star \) as in (2.8) we may write

\begin{align}
\partial_t (Z - c_\star \rho) - \eta \Delta_x (Z - c_\star \rho) + \text{div}_x (\mathbf{u}_n(Z - c_\star \rho)) &= 0, \\
\partial_t (c^\star \rho - Z) - \eta \Delta_x (c^\star \rho - Z) + \text{div}_x (\mathbf{u}_n(c^\star \rho - Z)) &= 0,
\end{align}

with the corresponding boundary conditions. Therefore, exactly as in Lemma 4.3 from [4] we show that since both equations have non-negative initial conditions, it is easy to see that also the solutions are non-negative and due to the uniqueness of solutions we deduce that

\[ (Z - c_\star \rho)(t, x) \geq \inf_{x \in \Omega}(Z_0 - c_\star \rho_0)e^{-Kt}, \quad (c^\star \rho - Z)(t, x) \geq \inf_{x \in \Omega}(c^\star \rho_0 - Z_0)e^{-Kt}, \]

as well as \( \rho(t, x) \geq \inf_{x \in \Omega} e^{-Kt} \), where

\[ K = \|\text{div} \mathbf{u}_n\|_{L^\infty(Q_T)}. \]

Using again the assumptions on the initial data \[2.7\], we therefore obtain

\[ 0 < c_\star \rho \leq Z \leq c^\star \rho \text{ a.e. in } Q_T. \]

With these inequalities in place we can let \( n \to \infty \) and \( \eta \to 0 \) in order to obtain the weak solutions to system \[1.8\] with \( \pi_\varepsilon \) replaced by \( \pi_\delta \). Note, however, that in the limit process we do not control the \( L^\infty \) norm of \( \text{div} \mathbf{u} \) which finally leads to

\[ 0 \leq c_\star \rho \leq Z \leq c^\star \rho \text{ a.e. in } Q_T. \]

All other steps are more or less standard except for the fact that the solution is renormalized in the sense of Definition \[2.3\]. This fact, however, follows directly from [4, Lemma 3.1] and will return to the procedure when letting \( \varepsilon \to 0^+ \).

The existence result is summarised in the following theorem.
Theorem 3.1. Let \( \Omega \subset \mathbb{R}^d, d = 2, 3, \) be a bounded domain of class \( C^2 \) such that \( \Gamma_{in} \) is an open \( C^2 \) \( d-1 \) dimensional manifold. Let \( \delta > 0, \varepsilon > 0 \) and \( T > 0. \) Under the assumptions (2.1) – (2.6) the problem (1.10) with the pressure \( \pi_{\varepsilon} \) replaced by \( \pi_{\delta} \) admits at least one renormalized bounded energy weak solution \( (\varrho_{\delta}, u_{\delta}, Z_{\delta}) \), i.e.

1. The triple \( (\varrho_{\delta}, u_{\delta}, Z_{\delta}) \) belongs to the following functional space:

\[
\varrho_{\delta}, Z_{\delta} \in L^\infty(0, T; L^2(\Omega)), \quad 0 \leq c_* \varrho_{\delta} \leq Z_{\delta} \leq c^* \varrho_{\delta} \quad \text{a.e. in } (0, T) \times \Omega, \\
u_{\delta} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d)), \quad u_{\delta}|_{T \times \partial \Omega} = u_B. 
\] (3.8)

2. The function \( \varrho_{\delta} \in C_{\text{weak}}([0, T], L^2(\Omega)) \) satisfies the integral identity

\[
\int_{\Omega} \varrho_{\delta}(\tau, \cdot)\varphi(\tau, \cdot) \, dx - \int_{\Omega} \varrho_0(\cdot)\varphi(0, \cdot) \, dx \\
= \int_0^T \int_{\Omega} \left( \varrho_{\delta}\partial_t \varphi + \varrho_{\delta} u_{\delta} \cdot \nabla_x \varphi \right) \, dx \, dt - \int_0^T \int_{\Gamma_{in}} \varrho_B u_B \cdot n \varphi \, dS_x \, dt
\] (3.9)

for any \( \tau \in [0, T] \) and \( \varphi \in C^1_c([0, T] \times (\Omega \cup \Gamma_{in})) \). In particular,

\[
\int_\Omega \varrho_{\delta}(\tau, \cdot) \, dx \leq \int_\Omega \varrho_0 \, dx - \int_0^T \int_{\Gamma_{in}} \varrho_B u_B \cdot n \, dS_x \, dt. 
\] (3.10)

3. The renormalized continuity equation

\[
\int_{\Omega} b(\varrho_{\delta})(\tau, \cdot)\varphi(\tau, \cdot) \, dx - \int_{\Omega} b(\varrho_0)(\cdot)\varphi(0, \cdot) \, dx \\
= \int_0^T \int_{\Omega} \left( b(\varrho_{\delta})\partial_t \varphi + b(\varrho_{\delta}) u_{\delta} \cdot \nabla_x \varphi + (b(\varrho_{\delta}) - b(\varrho_B)) u_{\delta} \cdot \nabla_x u_{\delta} \right) \, dx \, dt
\] (3.11)

holds for any \( b \in C[0, \infty) \) with \( b' \in C_c[0, \infty), \tau \in [0, T], \) and \( \varphi \in C^1_c([0, T] \times (\Omega \cup \Gamma_{in})). \)

4. The function \( \varrho_{\delta} u_{\delta} \in C_{\text{weak}}([0, T], L^2(\mathbb{R}^d)) \) satisfies the integral identity

\[
\int_{\Omega} \varrho_{\delta} u_{\delta}(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx - \int_{\Omega} \varrho_0 u_0(\cdot) \cdot \varphi(0, \cdot) \, dx \\
= \int_0^T \int_{\Omega} \left( \varrho_{\delta} u_{\delta} \cdot \partial_t \varphi + \varrho_{\delta} u_{\delta} \otimes u_{\delta} : \nabla_x \varphi + \pi_{\delta}(Z_{\delta}) \nabla_x \varphi - \mathcal{S}(\nabla_x u_{\delta}) : \nabla_x \varphi + \varrho_{\delta}(w - u_{\delta}) \cdot \varphi \right) \, dx \, dt
\] (3.12)

for any \( \tau \in [0, T] \) and \( \varphi \in C^1_c([0, T] \times \Omega; \mathbb{R}^d). \)
5. The function $Z_\delta \in C_{\text{weak}}([0,T], L^\gamma(\Omega))$ satisfies the integral equalities (3.9) – (3.11) with $\phi_0$ replaced by $Z_\delta$, $\phi_\beta$ replaced by $Z_B$, and $\phi_\varepsilon$ replaced by $Z_0$.

6. The energy inequality

$$
\int_\Omega \left( \frac{1}{2} \phi_0 |u_\delta - u_\infty|^2 + H_\delta(Z_\delta) \right) (\tau) \, dx + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x (u_\delta - u_\infty)) : \nabla_x (u_\delta - u_\infty) \, dx \, dt \\
\leq \int_\Omega \left( \frac{1}{2} \phi_0 |u_0 - u_\infty|^2 + H_\delta(Z_0) \right) \, dx - \int_0^\tau \int_\Omega \pi_\delta(Z_\delta) \text{div}_x u_\infty \, dx \, dt \\
- \int_0^\tau \int_\Omega g_\delta u_\delta \cdot \nabla_x u_\infty \cdot (u_\delta - u_\infty) \, dx \, dt - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x u_\infty) : \nabla_x (u_\delta - u_\infty) \, dx \, dt \\
- \int_0^\tau \int_{\Gamma_\infty} H_\delta(Z_B) u_B \cdot n \, dS_x \, dt + \int_0^\tau \int_\Omega g_\delta (w - u_\delta) \cdot (u_\delta - u_\infty) \, dx \, dt
$$

(3.13)

holds for a.a. $\tau \in (0,T)$ and the extension $u_\infty \in W^{1,\infty}(\Omega; R^d)$ of $u_B$ discussed above. In (3.13), the function $H_\delta(z)$ is defined by

$$
H_\delta(z) = z \int_0^z \frac{\pi_\delta(s)}{s^2} \, ds.
$$

(3.14)

# 4 Uniform estimates with respect to $\delta$ and limit $\delta \to 0$

In this section we prove our first main result, Theorem 2.6. For this reason we will deduce some a-priori estimates that are uniform with respect to $\delta$ with fixed positive $\varepsilon$. Then we will improve these estimates to show that the approximation of the pressure is in fact uniformly integrable. This, together with Lions compactness argument for the density sequence will allow us to identify the limit of the pressure term, and hence the whole system.

## 4.1 Uniform estimates

We start by deriving uniform estimates for the triple $(\phi_\delta, u_\delta, Z_\delta)$ constructed in Theorem 3.1. Note that due to Lemma 2.5, we have $\int_\Omega \pi_\delta(Z_\delta(t, \cdot)) \text{div}_x u_\infty \, dx \geq 0$ at all time levels, and so, following [4] Section 4.3.3] we can show that the energy inequality (3.13) in combination with the conservation of mass (3.10) yields

$$
\|H_\delta(Z_\delta)\|_{L^\infty(0,T;L^1(\Omega))} \leq \text{c(data),}
$$

$$
\|\phi_\delta u_\delta\|_{L^\infty(0,T;L^1(\Omega))} \leq \text{c(data),}
$$

$$
\|u_\delta\|_{L^2(0,T;W^{1,2}(\Omega))} \leq \text{c(data).}
$$

(4.1)

From these estimates, using (3.14), it follows that

$$
\|Z_\delta\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq \text{c(data),}
$$

(4.2)

and hence, due to (3.8)

$$
\|\phi_\delta\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq \text{c(data).}
$$

(4.3)
Moreover
\[ \text{ess sup}_{t \in (0,T)} \int_{\Omega} \tilde{H}_\delta(Z_\delta(t,x)) \, dx \leq c(\text{data}), \]  
(4.4)
where
\[ \tilde{H}_\delta(z) = \begin{cases} 
\varepsilon(1 - z)^{-(\beta-1)} & \text{if } z \in [0, 1 - \delta], \\
\varepsilon\delta^{-(\beta-1)} + \varepsilon\delta^{-\beta}(z - 1 + \delta) & \text{if } z \in (1 - \delta, \infty). 
\end{cases} \]  
(4.5)

This fact follows from the form of the pressure \( \pi_\delta \) and the energy \( H_\delta \), where \( \tilde{H}_\delta(Z_\delta) \) contains the most singular terms in \( \delta \) for \( \delta \to 0^+ \). By virtue of (4.1) and (4.3) together with (3.8)
\[ \| \rho_\delta u_\delta \|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))} + \| Z_\delta u_\delta \|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))} \leq c(\text{data}), \]  
(4.6)

### 4.2 Limit in the continuity equation and boundedness of density

We deduce from the estimates (4.1)–(4.3) that
\[ \begin{array}{l}
\rho_\delta \to \rho \quad \text{in } L^2(0,T;W^{1,2}(\Omega)), \\
Z_\delta \to^* Z \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)), \\
\rho_\delta \to^* \rho \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)),
\end{array} \]  
(4.7)

at least for a subsequence. We also deduce from the continuity equation (3.9) and its version for \( Z_\delta \), thanks to (4.6) and (3.8), that the sequences of functions \( t \mapsto \int_\Omega \rho_\delta \phi \, dx \), and \( t \mapsto \int_\Omega Z_\delta \phi \, dx \), \( \phi \in C^1_c(\Omega) \), are equi-continuous. Therefore, by the Arzelà–Ascoli theorem and separability of \( L^{\gamma'}(\Omega) \), we get
\[ \begin{array}{l}
\rho_\delta \to \rho \quad \text{in } C_{\text{weak}}(0,T;L^\gamma(\Omega)), \\
Z_\delta \to Z \quad \text{in } C_{\text{weak}}(0,T;L^\gamma(\Omega)).
\end{array} \]  
(4.8)

Both of the sequences converge strongly in \( L^2(0,T;W^{-1,2}(\Omega)) \) due to compact embedding \( L^\gamma(\Omega) \hookrightarrow W^{-1,2}(\Omega) \). In particular, this implies that
\[ \begin{array}{l}
\rho_\delta u_\delta \to \rho u \quad \text{in } L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega)), \\
Z_\delta u_\delta \to Z u \quad \text{in } L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega)).
\end{array} \]  

This enables us to pass to the limit in the weak formulation (3.9) so that we get the identity
\[ \begin{align*}
&\int_{\Omega} \rho(\tau,\cdot) \phi(\tau,\cdot) \, dx - \int_{\Omega} \rho_0(\cdot) \phi(0,\cdot) \, dx \\
= &\int_0^\tau \int_{\Omega} \left( \rho \partial_t + \rho u \cdot \nabla x \phi \right) \, dx dt - \int_0^\tau \int_{\Gamma_{in}} \rho_B u_B \cdot n \phi \, dS_x dt
\end{align*} \]  
(4.9)
and its analogue for $Z$. Both hold for any $\tau \in [0, T]$ and $\varphi \in C^1_c([0, T] \times (\Omega \cup \Gamma_{in}))$.

To conclude this subsection, we deduce from (4.4) that

$$0 \leq Z(t, x) < 1 \text{ a.e. in } \Omega.$$  \hfill (4.10)

Indeed, for any fixed sufficiently small $\delta^* > 0$, we have

$$\int_{\Omega} \tilde{H}_{\delta^*}(Z(t)) \, dx \leq \liminf_{\delta \to 0} \int_{\Omega} \tilde{H}_{\delta}(Z_\delta(t)) \, dx \leq \liminf_{\delta \to 0} \int_{\Omega} \tilde{H}_{\delta}(Z_\delta(t)) \, dx \leq c(\text{data})$$  \hfill (4.11)

for almost all $t \in (0, T)$, where the first inequality is a consequence of convexity of function $H_{\delta^*}(\cdot)$ on $[0, 1 - \delta)$ as well its linearity in $Z_\delta$ in the remaining part, second inequality follows from monotonicity of the map $\delta \mapsto \tilde{H}_{\delta}(Z)$ in a small right neighbourhood of 0, and the third inequality follows from (4.4).

Next, as $H_{\delta}(\cdot)$ is globally Lipschitz, using the continuity equation we deduce that $Z \in C([0, T]; L^1(\Omega))$ and then $\tilde{H}_{\delta^*}(Z) \in C([0, T]; L^1(\Omega))$. Therefore formula (4.11) implies

$$\int_{\Omega} \tilde{H}_{\delta^*}(Z(t)) \, dx \leq c(\text{data}) \quad \text{for all } t \in [0, T],$$  \hfill (4.12)

and uniformly in $\delta^*$. This implies that $Z \leq 1$. Finally letting $\delta^* \to 0$ in (4.12), recalling (4.5), we obtain

$$\int_{\Omega} (1 - Z)^{-(\beta-1)} \, dx \leq c(\text{data}) \quad \text{for all } t \in [0, T],$$

which yields (4.10).

### 4.3 Uniform integrability of pressure

In order to pass to the limit in the weak formulation of the momentum equation (3.12), we have to improve estimates for pressure. So far, we do not even know whether the pressure is uniformly integrable in $\delta$. In this section we are going to prove it.

A general tool to obtain these estimates is the following Bogovskii lemma (see, e.g., [14] or [11, Theorem 10.11]).

**Lemma 4.1.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then there exists a linear operator

$$\mathcal{B} : \left\{ f \in C^\infty_c(\Omega; \mathbb{R}^d) : \int_{\Omega} f \, dx = 0 \right\} \mapsto C^\infty_c(\Omega; \mathbb{R}^d)$$

satisfying the following three properties.

1. For all $f \in C^\infty_c(\Omega; \mathbb{R}^d)$ satisfying $\int_{\Omega} f \, dx = 0$

$$\text{div}\mathcal{B}[f] = f.$$
2. Let \( \mathcal{L}^p(\Omega) := \{ f \in L^p(\Omega) \mid \int_\Omega f \, dx = 0 \} \). Then the operator \( \mathcal{B} \) extends to a bounded linear operator from \( \mathcal{L}^p(\Omega) \) to \( W^{1,p}(\Omega) \) for any \( 1 < p < \infty \). In other words, for each \( 1 < p < \infty \) there is \( c(p) > 0 \) such that for all \( f \in \mathcal{L}^p(\Omega) \)

\[
\| \mathcal{B}[f] \|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leq c(p) \| f \|_{L^p(\Omega)}.
\]

3. If \( f = \nabla g \) for some \( g \in L^q(\Omega), 1 < q < \infty \) with \( g \cdot n|_{\partial\Omega} = 0 \) in the sense of normal traces, then there is \( c(q) > 0 \) such that

\[
\| \mathcal{B}[f] \|_{L^q(\Omega;\mathbb{R}^3)} \leq c(q) \| g \|_{L^q(\Omega;\mathbb{R}^3)}
\]

for all \( g \) with the above properties.

We employ this lemma to construct suitable test functions for the momentum equation. Note that by standard density argument we can extend the class of test functions in (3.12) to certain \( W^{1,q} \)-functions with zero trace in \( \Gamma_{\text{out}} \). Our test function will be a suitable test function due to estimates performed below.

We use in (3.12) the following test function

\[
\varphi = \eta(t) \mathcal{B} \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right),
\]

where \( \eta \in C^1_c(0,T) \) and \( \psi \in C^\infty_c(\Omega), 0 \leq \eta, \psi \leq 1 \). Then we have

\[
\int_0^T \int_\Omega \eta \pi_\delta(Z_\delta) \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right) \, dx \, dt = \sum_{i=1}^6 I_i,
\]

where

\[
I_1 = -\int_0^T \partial_t \eta \int_\Omega \varphi_\delta \mathcal{B} \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right) \, dx \, dt,
\]

\[
I_2 = \int_0^T \eta \int_\Omega \varphi_\delta \mathcal{B} \left( \nabla_x (Z_\delta \mathcal{B} \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right) \right) \, dx \, dt,
\]

\[
I_3 = -\int_0^T \eta \int_\Omega \varphi_\delta \mathcal{B} \left( \nabla_x \mathcal{B} \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right) \right) \, dx \, dt,
\]

\[
I_4 = -\int_0^T \eta \int_\Omega \varphi_\delta \mathcal{B} \left( \nabla_x \mathcal{B} \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right) \right) \, dx \, dt,
\]

\[
I_5 = \int_0^T \eta \int_\Omega \varphi_\delta \mathcal{B} \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right) \, dx \, dt,
\]

\[
I_6 = \int_0^T \eta \int_\Omega \varphi_\delta \mathcal{B} \left( \psi Z_\delta - \frac{\psi}{\int_\Omega \psi \, dx} \int_\Omega \psi Z_\delta \, dx \right) \, dx \, dt.
\]
Clearly, $|\sum_{i=1}^{6} I_i| \leq C$ with $C$ independent of $\delta$ by estimates (4.1)–(4.3) and we end up with

$$
\int_{0}^{T} \int_{\Omega} \eta \pi_{\delta}(Z_{\delta}) \left( \psi Z_{\delta} - \frac{\psi}{\int_{\Omega} \psi \, dx} \int_{\Omega} \psi Z_{\delta} \, dx \right) \, dx \, dt \leq C. \tag{4.15}
$$

We now choose $\psi \in C_{c}^{\infty}(\Omega)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $K$ some compact set $K \subset \subset \Omega$. For the fixed set $K$ and any such $\psi$ as above we further denote

$$
M_{\delta,K} = \max_{t \in [0,T]} \int_{\Omega} Z_{\delta} \psi \, dx.
$$

We claim that for each $K \subset \subset \Omega$ there exists $\delta_0 > 0$ and $\lambda > 1$ such that for any $\delta < \delta_0$ it holds

$$
\lambda M_{\delta,K} < \int_{\Omega} \psi \, dx. \tag{4.16}
$$

This follows from several facts shown before. The form of $\tilde{H}_{\delta}(Z_{\delta})$, see (4.5), yields that for any positive (sufficiently small) $\delta$ we have on the set $Z_{\delta} > 1 - \delta$ (recall that $\varepsilon$ is fixed at this moment)

$$
\tilde{H}_{\delta}(Z_{\delta}) > \frac{C}{\delta^{\beta}}. \tag{4.17}
$$

Since $\beta > \frac{5}{2}$ (if $d = 3$; note that it is enough to have $\beta > 2$ which is the case if $d = 2$) and due to the $L^{\infty}(0, T; L^1(\Omega))$ bound of $\tilde{H}_{\delta}(Z_{\delta})$ we see that for arbitrarily small $\theta > 0$ there exists $\delta_\theta$ such that

$$
\sup_{\delta < \delta_\theta} \sup_{t \in [0,T]} |x \in \Omega : Z_{\delta}(t, x) > 1 - \delta| < \theta.
$$

Furthermore, for $\theta > 0$, sufficiently small, the $L^{\infty}(0, T; L^1(\Omega))$ bound of $\tilde{H}_{\delta}(Z_{\delta})$ implies that we may take $\delta_\theta = \theta^2$. Therefore we have (we assume $\delta \leq \delta_\theta$):

$$
\max_{t \in [0,T]} \int_{\Omega} Z_{\delta} \psi \, dx = \sup_{t \in [0,T]} \int_{\{Z_{\delta}(t,x) > 1 - \delta\}} Z_{\delta} \psi \, dx + \sup_{t \in [0,T]} \int_{\{Z_{\delta}(t,x) \leq 1 - \delta\}} Z_{\delta} \psi \, dx
$$

$$
\leq \theta^{\frac{1}{\gamma}} \|Z_{\delta}\|_{L^{\infty}(0, T ; L^{\gamma}(\Omega))} + (1 - \delta_\theta) \int_{\Omega} \psi \, dx
$$

$$
\leq C \theta^{\frac{1}{\gamma}} + (1 - \delta_\theta) \int_{\Omega} \psi \, dx.
$$

Thus, taking $\theta$ possibly even smaller, we may achieve by taking $\gamma$ sufficiently large that $C \theta^{\frac{1}{\gamma}} < \frac{\delta_\theta}{2} \int_{\Omega} \psi \, dx$ which leads to

$$
\max_{t \in [0,T]} \int_{\Omega} Z_{\delta} \psi \, dx \leq \left( 1 - \frac{\delta_\theta}{2} \right) \int_{\Omega} \psi \, dx. \tag{4.18}
$$

Thus, for $\lambda := \frac{1}{1 - \frac{\delta_\theta}{2}}$ we showed (4.16).
We return to inequality (4.15). Clearly, for the set \( O_1 := \{(t, x) \in (0, T) \times \Omega : Z_\delta(t, x) < \frac{\lambda M_{K, \delta}}{\int_{\Omega} \psi \, dx} < 1\} \) we have

\[
\left| \int \int_{O_1} \eta \pi_\delta(Z_\delta) \left( \psi Z_\delta - \frac{\psi}{\int_{\Omega} \psi \, dx} \int_{\Omega} Z_\delta \psi \, dx \right) \, dx \, dt \right| \leq \left| \int \int_{O_1} \eta \pi_\delta \left( \frac{\lambda M_{K, \delta}}{\int_{\Omega} \psi \, dx} \right) \left( \psi Z_\delta - \frac{\psi}{\int_{\Omega} \psi \, dx} \int_{\Omega} Z_\delta \psi \, dx \right) \, dx \, dt \right| \leq C T.
\]

Next

\[
\int \int_{(0,T) \times \Omega \setminus O_1} \eta \pi_\delta(Z_\delta) \psi Z_\delta \, dx \, dt \leq C + \int \int_{(0,T) \times \Omega \setminus O_1} \eta \pi_\delta(Z_\delta) \left( \frac{\psi}{\int_{\Omega} \psi \, dx} \int_{\Omega} Z_\delta \psi \, dx \right) \, dx \, dt \leq C + \frac{1}{\lambda} \int \int_{(0,T) \times \Omega \setminus O_1} \eta \pi_\delta(Z_\delta) Z_\delta \, dx \, dt.
\]

The computations above imply that for any \( K \subset \subset \Omega \) and corresponding \( \psi \in C^\infty_c(\Omega) \) there exists \( C = C(K) \) such that

\[
\int_{0}^{T} \int_{\Omega} \eta \psi \pi_\delta(Z_\delta) Z_\delta \, dx \, dt \leq C(K). \tag{4.19}
\]

Whence we also have

\[
\int \int_{\{Z_\delta \leq 1 - \delta\}} \eta \psi \varepsilon (1 - Z_\delta)^{-\beta} \, dx \, dt \leq C(K), \tag{4.20}
\]

where \( \eta \in C^\infty_c(0, T) \).

**Remark 4.2.** Note that this estimate was obtained without assuming short time interval for zero velocity flux as it was the case in [5]. Therefore, from this point of view our paper even improves the result in the above cited paper in the sense that global in time solution exists provided

\[
\int_{\partial \Omega} u_B \cdot n \, dS \geq 0.
\]

The case of negative flux, however, remains an interesting open problem.

### 4.4 Equi-integrability of pressure

In order to show equi-integrability of the sequence \( \pi_\delta(Z_\delta) \), we shall use the renormalized continuity equation (3.11) with \( \varrho_\delta \) replaced by \( Z_\delta \). We fix the same cut-off functions \( \eta \) in the time variable and \( \psi \) in the spatial variables as in the previous section and \( 0 \leq \psi \in C^1_c(\Omega) \) and consider the following test function

\[
\varphi = \eta(t) \mathcal{B}(\psi b(Z_\delta) - \alpha_\delta) \quad \text{where} \quad \alpha_\delta = \frac{1}{|\Omega|} \int_{\Omega} b(Z_\delta) \, dx
\]
with
\[ b(Z) = \begin{cases} 
-\ln(1/2) & \text{if } Z \in [0, 1/2], \\
-\ln(1 - Z) & \text{if } Z \in (1/2, 1 - \delta), \\
-\ln \delta & \text{if } Z \in [1 - \delta, \infty). 
\end{cases} \]

We note that
\[ b'(Z) = \frac{1}{1 - Z} 1_{(1/2,1-\delta)}(Z), \]

where, as above, \(1_E(Z)\) denotes the characteristic function of a set \(E\). In view of (4.2), (4.4), and (4.20), we notice also that for any \(1 \leq p < \infty\), and any compact \(K \subset \Omega\),

\[
\begin{align*}
\|b(Z_\delta)\|_{L^\infty(0,T;L^p(K))} & \leq c(\text{data}, K, p), \\
\|\eta Z_\delta b'(Z_\delta) - b(Z_\delta)\|_{L^p((0,T) \times K)} & \leq c(\text{data}, K, \eta), \\
\|Z_\delta b'(Z_\delta) - b(Z_\delta)\|_{L^\infty(0,T;L^{3/2-1}(K))} & \leq c(\text{data}, K),
\end{align*}
\]

where \(\eta\) is as above. We test the momentum equation (3.12) by \(\varphi\) to obtain the following identity

\[
\int_0^T \eta \int_\Omega \psi \varepsilon \pi \delta(Z_\delta)b(Z_\delta) \, dx \, dt = \sum_{i=1}^8 I_i,
\]

where
\[
\begin{align*}
I_1 &= \frac{1}{|\Omega|} \int_0^T \eta(t) \int_\Omega \psi b(Z_\delta) \, dx \int \varepsilon \pi \delta(Z_\delta) \, dx \, dt, \\
I_2 &= -\int_0^T \partial_t \eta \int_\Omega g_\delta u_\delta \cdot \mathcal{B}(\psi b(Z_\delta) - \alpha_\delta) \, dx \, dt, \\
I_3 &= \int_0^T \eta \int_\Omega g_\delta u_\delta \cdot \mathcal{B}\left(\text{div}_x(\psi b(Z_\delta) u_\delta)\right) \, dx \, dt, \\
I_4 &= -\int_0^T \eta \int_\Omega g_\delta u_\delta \cdot \mathcal{B}\left(b(Z_\delta) u_\delta \cdot \nabla_x \psi - \frac{1}{|\Omega|} \int_{\Omega} b(Z_\delta) u_\delta \cdot \nabla_x \psi \, dx\right) \, dx \, dt, \\
I_5 &= \int_0^T \eta \int_\Omega g_\delta u_\delta \cdot \mathcal{B}\left[\psi \left(Z_\delta b'(Z_\delta) - b(Z_\delta)\right)\text{div}_x u_\delta - \frac{1}{|\Omega|} \int_{\Omega} \psi \left(Z_\delta b'(Z_\delta) - b(Z_\delta)\right)\text{div}_x u_\delta \, dx\right] \, dx \, dt, \\
I_6 &= \int_0^T \eta \int_\Omega \mathcal{S}(\nabla_x u_\delta) : \nabla_x \mathcal{B}(\psi b(Z_\delta) - \alpha_\delta) \, dx \, dt, \\
I_7 &= -\int_0^T \eta \int_\Omega g_\delta u_\delta \otimes u_\delta : \nabla_x \mathcal{B}(\psi b(Z_\delta) - \alpha_\delta) \, dx \, dt, \\
I_8 &= -\int_0^T \eta \int_\Omega g_\delta (w - u_\delta) \, dx \cdot \mathcal{B}(\psi b(Z_\delta) - \alpha_\delta) \, dt.
\]
The above calculation involves integration by parts and the renormalized equation (3.11) for unknown
\(Z_\delta\). The function \(b\) is clearly admissible in the renormalized continuity equation. We verify, using the
approach of [10], estimates (4.1)–(4.4), and (4.21), that for any \(\beta > 5/2\) there is \(\gamma > 3/2\) (sufficiently
large - \(\gamma \to \infty\) as \(\beta \to \frac{5}{2}+\)) such that absolute values of \(I_1, \ldots, I_8\) are bounded above by some positive
constants. The most severe constraints on the values of \(\beta\) and \(\gamma\) within these calculations are imposed
in estimating the term \(|I_5|\). Note that in case \(d = 2\) the same approach as in [10] can be used to see
that the most singular term can be estimated for \(\beta > 2\). Effectuating this process, we obtain that for
any compact set \(K \subset \Omega\),
\[
\|\eta \pi(\delta) b(Z_\delta)\|_{L^1((0,T) \times K)} \leq c(\text{data, } K, \eta).
\] (4.22)
Consequently, for fixed \(\varepsilon > 0\), the sequence \(\pi(\delta)\) is equi-integrable in \(L^1(Q)\) for any \(Q \Subset (0, T) \times \Omega\)
and
\[
\pi(\delta) \to \pi(Z) \quad \text{in } L^1(J \times K)
\] (4.23)
for any compact set \(J \times K \subset (0, T) \times \Omega\) at least for a chosen subsequence (not relabeled).

### 4.5 Momentum equation

With the help of (4.1), (4.3), and (4.22) employed in the momentum equation (3.12), we verify equicon-
tinuity of the sequence \(t \mapsto \int_\Omega \varphi(t, \cdot) \varphi(\cdot) \, dx\) in \(C[0, T]\) for any \(\varphi \in C^1_c(\Omega)\). Therefore, we may use
the Arzelà–Ascoli theorem in combination with (4.6) the separability of \(L^{2\gamma/(\gamma+1)}(\Omega)\) to show that
\[
\varphi_\delta \to \varphi \quad \text{in } C_{\text{weak}}([0, T]; L^{2\gamma/(\gamma+1)}(\Omega)).
\] (4.24)
Consequently, the imbedding \(L^{2\gamma/(\gamma+1)}(\Omega) \hookrightarrow \hookrightarrow W^{-1,2}(\Omega)\) (for \(\gamma > \frac{3}{2}\)) in combination with the weak convergence of \(u_n\) in \(L^2(0, T; W^{1,2}(\Omega))\) implies
\[
\varphi_\delta \to \varphi \quad \text{in } L^2(0, T; L^{6\gamma/4\gamma+3}(\Omega)).
\] (4.25)
Thus, letting \(\delta \to 0\) in weak formulation of (3.12) while using (4.7), (4.8), (4.23), and (4.25) we
obtain that for any \(\tau \in [0, T]\) and \(\varphi \in C^1_c([0, T] \times \Omega; R^d)\),
\[
\int_\Omega \varphi(t, \cdot) \cdot \varphi(t, \cdot) \, dx - \int_\Omega (\varphi_0 u_0)(\cdot) \cdot \varphi(0, \cdot) \, dx = \int_\Omega \left(\varphi \otimes u : \nabla \varphi + \varepsilon \pi(Z) \text{div} \varphi\right) \, dx
\]
\[
- \int_0^T \int_\Omega (\nabla \varphi \cdot \nabla \varphi) \, dx + \int_0^T \int_\Omega (\varphi(w - u) \cdot \varphi) \, dx dt.
\] (4.26)

The proof of Theorem 2.6 is therefore complete if we show that
\[
\pi(Z) = \pi\varepsilon(Z),
\] (4.27)
which amounts, in fact, to show that the sequence \(Z_\delta\) converges almost everywhere in \(Q_T\).
4.6 Strong convergence of $Z_\delta$

We denote by $\nabla_x \Delta^{-1}$ the pseudodifferential operator of the Fourier symbol $\frac{i\xi}{|\xi|^2}$ and by $\mathcal{R}$ the Riesz transform of the Fourier symbol $\frac{\xi \otimes \xi}{|\xi|^2}$. Following Lions [15] with modified in [12], we shall use the test function

$$\varphi(t,x) = \eta(t) \psi(x) \nabla_x \Delta^{-1}(Z_\delta \psi), \quad \eta \in C^1_c(0,T), \quad \psi \in C^1_c(\Omega)$$

in the approximate momentum equation (3.12) and the test function

$$\varphi(t,x) = \eta(t) \psi(x) \nabla_x \Delta^{-1}(Z \psi), \quad \eta \in C^1_c(0,T), \quad \psi \in C^1_c(\Omega)$$

in the limit momentum equation (4.26), subtract the resulting identities, and then perform the limit $\delta \to 0$. These calculations are laborious but nowadays standard. One can find details e.g. in [12, Lemma 3.2], [19], [9] or [11, Chapter 3]) to obtain the following identity

$$\int_0^T \int_{\Omega} \eta \psi^2 (\pi(Z) Z - (2\mu + \lambda) \text{div}_x u Z) \, dx \, dt$$

$$- \int_0^T \int_{\Omega} \eta \psi^2 (\pi(Z) Z - (2\mu + \lambda) Z \text{div}_x u) \, dx \, dt$$

$$= \int_0^T \eta \int_{\Omega} \psi^2 u \cdot \left( Z \mathcal{R} \cdot (\varrho u) - \varrho u \cdot \mathcal{R}(Z) \right) \, dx \, dt$$

$$- \lim_{\delta \to 0} \int_0^T \int_{\Omega} \eta \int_{\Omega} \psi^2 u_\delta \cdot \left( Z_\delta \mathcal{R} \cdot (\varrho_\delta u_\delta) - \varrho_\delta u_\delta \cdot \mathcal{R}(Z_\delta) \right) \, dx \, dt. \quad (4.28)$$

In (4.28) and in the sequel the overlined quantities $\overline{b(Z, u)}$, resp. $\overline{b(Z)}$ denote $L^1(Q_T)$-weak limits of sequences $b(Z_\delta, u_\delta)$ resp. $b(Z_\delta)$ (or $b_\delta(Z_\delta)$ if this is the case).

The most non-trivial moment in this process is to show that the right-hand side of this identity vanishes. The details of this calculation and reasoning can be found in [12, Lemma 3.2], [9], [19], or [11, Chapter 3]. Consequently,

$$\overline{(\pi(Z) Z - \pi(Z) Z)} = (2\mu + \lambda) \overline{(Z \text{div}_x u - Z \text{div}_x u)}, \quad (4.29)$$

and so

$$(2\mu + \lambda) \int_0^T \int_{\Omega} \left( Z \text{div} u - Z \text{div} u \right) \, dx \, dt = \int_0^T \int_{\Omega} \left( \pi(Z) Z - \pi(Z) Z \right) \, dx \, dt \leq 0, \quad (4.30)$$

due to the fact that $\pi(Z)$ is increasing.

The next (and the last) step in the proof follows closely Section 4.5 in [4]. Note, that this procedure does not depend on the momentum equation anymore, therefore it will hold also for the limit passage $\varepsilon \to 0$.

Since both $(Z_\delta, u_\delta)$ and $(Z, u)$ satisfy the renormalized continuity equation (3.11), we obtain, in particular, that

$$\int_\Omega \overline{L(Z(\tau))} \varphi \, dx - \int_\Omega L(Z_0) \varphi(0, x) \, dx$$
\[
\begin{align*}
&= \int_0^\tau \int_\Omega \left( L(Z) \partial_t \varphi + L(Z) u \cdot \nabla_x \varphi - \varphi Z \text{div}_x u \right) \, dx \, dt + \int_{\Gamma_{\text{in}}} \int_0^\tau L(Z_B) u_B \cdot n \varphi \, dS_x \, dt, \\
\end{align*}
\]

and

\[
\begin{align*}
&= \int_0^\tau \int_\Omega \left( L(Z) \partial_t \varphi + L(Z) u \cdot \nabla_x \varphi - \varphi Z \text{div}_x u \right) \, dx \, dt + \int_{\Gamma_{\text{in}}} \int_0^\tau L(Z_B) u_B \cdot n \varphi \, dS_x \, dt \\
\end{align*}
\]

where \( L(Z) = \frac{Z}{Z} \), and \( \varphi \in C^1_c([0, T] \times (\Omega \cup \Gamma_{\text{in}})) \). Subtracting these inequalities, we obtain

\[
\begin{align*}
&= \int_0^\tau \int_\Omega \left( L(Z) - L(Z) \right)(\tau, x) \, dx \, dt \\
&= \int_0^\tau \int_\Omega \left( L(Z) - L(Z) \right) \left( u \cdot \nabla_x \varphi + \partial_t \varphi \right) \, dx \, dt - \int_0^\tau \int_\Omega \varphi \left( Z \text{div}_x u - Z \text{div}_x u \right) \, dx \, dt.
\end{align*}
\]

Hence, by virtue of (4.30) and using also in particular the function \( \varphi \) independent of time, we get

\[
\begin{align*}
&= \int_0^\tau \int_\Omega \left( Z \log Z - Z \log Z \right)(\tau, x) \, dx \, dt \\
&= \int_0^\tau \int_\Omega \left( Z \log Z - Z \log Z \right) \left( u \cdot \nabla_x \varphi + \partial_t \varphi \right) \, dx \, dt - \int_0^\tau \int_\Omega \varphi \left( Z \text{div}_x u - Z \text{div}_x u \right) \, dx \, dt \leq 0 \tag{4.31}
\end{align*}
\]

for any \( \tau \in [0, T] \) and \( \varphi \in C^1_c(\Omega \cup \Gamma_{\text{in}}) \) with \( \varphi \geq 0 \). To show that the above formula for \( \varphi(x) \rightarrow 1 \) gives

\[
\begin{align*}
&= \int_0^\tau \int_\Omega \left( Z \log Z - Z \log Z \right)(\tau, \cdot) \, dx \leq 0, \tag{4.32}
\end{align*}
\]

we can follow step by step the procedure described in Section 4.7 in [5]. On the other hand, we have

\[
Z \log Z - Z \log Z \geq 0 \quad \text{a.e. in } Q_T
\]

since \( Z \log Z \) is convex. Thus, (4.32) yields

\[
Z \log Z = Z \log Z \quad \text{a.e. in } Q_T,
\]

and so

\[
Z_\delta \rightarrow Z \quad \text{a.e. in } Q_T \text{ and in } L^p(Q_T) \text{ for } 1 \leq p < \infty, \tag{4.33}
\]

cf. e.g. [11, Theorem 10.20]. We deduce from (4.33) and (4.22) that for any compact \( K \subset \Omega \),

\[
\pi_\delta(Z_\delta) \rightarrow \pi(Z) \text{ a.e. in } Q_T \text{ and in } L^1((0, T) \times K). \tag{4.34}
\]

In particular, we have \( \overline{\pi(Z)} = \pi(Z) \) in equation (4.26), note, however, that this information is restricted solely to \( Z \) and does not imply the strong convergence of \( \varrho_\delta \).
4.7 Energy inequality

We first integrate (3.13) over $0 < \tau_1 < \tau_2 < T$ to obtain that

$$
\int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{1}{2} \phi_0 |u_0 - u_\infty|^2 + H_\delta(Z_\delta) \right) (\tau, \cdot) \, dx \, d\tau + \int_{\tau_1}^{\tau_2} \int_{\Omega} S(\nabla_x (u_\delta - u_\infty)) \cdot \nabla_x (u_\delta - u_\infty) \, dx \, d\tau \\
\leq \int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{1}{2} \phi_0 |u_0 - u_\infty|^2 + H_\delta(Z_0) \right) \, dx \, d\tau - \int_{\tau_1}^{\tau_2} \int_{\Omega} \pi_\delta(Z_\delta) \text{div}_x u_\infty \, dx \, d\tau \\
- \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} \phi_\delta u_\delta \cdot \nabla_x u_\infty \cdot (u_\delta - u_\infty) \, dx \, d\tau \, d\tau - \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} S(\nabla_x u_\infty) : \nabla_x (u_\delta - u_\infty) \, dx \, d\tau \, d\tau \\
- \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Gamma_{in}} H_\delta(Z_B) u_B \cdot n \, dS_x \, d\tau \, d\tau + \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} \phi(w - u_\delta) \cdot (u_\delta - u_\infty) \, dx \, d\tau \, d\tau.
$$

(4.35)

Now, we can use the convergences established in Section 4.2 and in (4.33) and (4.34) at the right-hand side and the same convergences in combination with the lower weak semi-continuity of convex functionals at the left-hand side (see e.g. [11, Theorem 10.20]). Note, however, that we must be slightly careful with the pressure term, where the convergence is only local. We therefore get

$$
\int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{1}{2} \phi |u - u_\infty|^2 + H_\varepsilon(Z) \right) (\tau, \cdot) \, dx \, d\tau + \int_{\tau_1}^{\tau_2} \int_{\Omega} S(\nabla_x (u - u_\infty)) : \nabla_x (u - u_\infty) \, dx \, d\tau \\
+ \int_{\tau_1}^{\tau_2} \int_0^{\tau - \alpha} \int_{\Omega} \pi_\varepsilon(Z) \text{div}_x u_\infty \, dx \, d\tau \, d\tau \leq \int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{1}{2} \phi_0 |u_0 - u_\infty|^2 + H_\varepsilon(Z_0) \right) \, dx \, d\tau \\
- \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} \phi u \cdot \nabla_x u_\infty \cdot (u - u_\infty) \, dx \, d\tau \, d\tau - \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} S(\nabla_x u_\infty) : \nabla_x (u - u_\infty) \, dx \, d\tau \, d\tau \\
- \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Gamma_{in}} H_\varepsilon(Z_B) u_B \cdot n \, dS_x \, d\tau \, d\tau + \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} \phi(w - u) \cdot (u - u_\infty) \, dx \, d\tau \, d\tau.
$$

(4.36)

Since the inequality holds for any $\alpha > 0$, sufficiently small, and any $K$ compact subset of $\Omega$, we easily obtain at the end

$$
\int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{1}{2} \phi |u - u_\infty|^2 + H_\varepsilon(Z) \right) (\tau, \cdot) \, dx \, d\tau + \int_{\tau_1}^{\tau_2} \int_{\Omega} S(\nabla_x (u - u_\infty)) : \nabla_x (u - u_\infty) \, dx \, d\tau \\
\leq \int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{1}{2} \phi_0 |u_0 - u_\infty|^2 + H_\varepsilon(Z_0) \right) \, dx \, d\tau - \int_{\tau_1}^{\tau_2} \int_{\Omega} \pi_\varepsilon(Z) \text{div}_x u_\infty \, dx \, d\tau \\
- \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} \phi u \cdot \nabla_x u_\infty \cdot (u - u_\infty) \, dx \, d\tau \, d\tau - \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} S(\nabla_x u_\infty) : \nabla_x (u - u_\infty) \, dx \, d\tau \, d\tau \\
- \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Gamma_{in}} H_\varepsilon(Z_B) u_B \cdot n \, dS_x \, d\tau \, d\tau + \int_{\tau_1}^{\tau_2} \int_0^\tau \int_{\Omega} \phi(w - u) \cdot (u - u_\infty) \, dx \, d\tau \, d\tau.
$$

(4.37)
4.8 Renormalized continuity equation

In this section following [4] we can generalize the DiPerna-Lions theory for continuity equation with nonhomogenous boundary data. Recall that due to (4.7) \( \varrho \in L^2(0,T;L^\gamma(\Omega)) \) and \( u \in L^2(0,T;W^{1,2}(\Omega)) \), and in particular \( \gamma > 2 \). Due to [4, Lemma 3.1] we may formulate the following result:

**Lemma 4.3.** Let \( \Omega \subset \mathbb{R}^d \), \( d = 2,3 \), be bounded domain of class \( C^2 \) such that \( \Gamma_{in} \) is an open \( C^2 \) \( d-1 \) dimensional manifold. Let \( \varrho_B \) and \( u_B \) satisfy assumptions (2.3). Let \( \varrho \in L^2(0,T;L^\gamma(\Omega)) \) with \( \gamma > 2 \) and \( u \in L^2(0,T;W^{1,2}(\Omega)) \) satisfy the continuity equation in a weak sense, as in (4.9).

Then \( (\varrho, u) \) is also a renormalized solution of the continuity equation (4.9), namely it verifies (2.14) with \( \varrho \) instead of \( B \).

For the detailed proof see [4, Section 3.1]. Let us present here just a sketch of it. The main difficulty is to reconstruct the the boundary term on \( \Gamma_{in} \). To this end the set \( \Omega \cup \Gamma_{in} \) is extended by properly chosen open (nonempty) set \( \tilde{\Omega}_h \) being in neighborhood of \( \Gamma_{in} \) (one can think about it as an thin pillow attached to \( \Gamma_{in} \) outside of \( \Omega \)). It is defined as follows:

\[
\tilde{U}_h^+(\Gamma_{in}) = \{ x \in U_h^+(\Gamma_{in}) \mid x = X(s,x_0) \text{ for a certain } x_0 \in \Gamma_{in} \text{ and } 0 < s < h \}
\]

where

\[
U_h^+(\Gamma_{in}) := \{ x_0 + zn(x_0) \mid 0 < z < h, x_0 \in \Gamma, n \} \cap (\mathbb{R}^d \setminus \Omega)
\]

and

\[
X'(s,x_0) = -\tilde{u}_B(X(s,x_0)), \quad X(0) = x_0 \in U_h^+(\Gamma_{in}) \cup \Gamma_{in} \text{ for } s > 0, \quad X(s,x_0) \in U_h^+(\Gamma_{in})
\]

with

\[
\tilde{u}_B(x) = u_B(x_0), \quad x = x_0 + zn(x_0) \in U_h^+(\Gamma_{in})
\]

The set \( \tilde{U}_h^+(\Gamma_{in}) \) is nonempty and open, for details see [4, Section 3.1]. Here the regularity of \( \Gamma_{in} \) is used. Moreover proper extension \( \tilde{u}_B \) and \( \tilde{\varrho}_B \) of \( u_B \) and \( \varrho_B \) on \( \tilde{U}_h^+(\Gamma_{in}) \) is constructed such that \( \tilde{u}_B \in C^1(\tilde{U}_h^+(\Gamma_{in})), \tilde{\varrho}_B \in W^{1,\infty}(\tilde{U}_h^+(\Gamma_{in})) \), and

\[
\text{div}_x(\tilde{\varrho}_B \tilde{u}_B) = 0 \text{ in } \tilde{U}_h^+(\Gamma_{in}), \quad \tilde{\varrho}_B|\Gamma_{in} = \varrho_B, \quad \tilde{u}_B|\Gamma_{in} = u_B. \tag{4.39}
\]

Then extension of \( (\varrho, u) \) to the set \( \Omega_h := \Omega \cup \Gamma_{in} \cup \tilde{U}_h^+(\Gamma_{in}) \), where \( (\varrho, u)(t,x) = (\tilde{\varrho}_B, \tilde{u}_B) \) on \( \tilde{U}_h^+(\Gamma_{in}) \), satisfies continuity equation in the sense of distributions on \( \Omega_h \) and \( \varrho \in L^2(0,T;L^2(\Omega_h)) \) and \( u \in L^2(0,T;W^{1,2}(\Omega_h)) \). Then by classical DiPerna and Lions arguments with Friedrichs lemma provide that

\[
\int_{\Omega_h} b(\varrho)(\tau,\cdot) \varphi(\tau,\cdot) \, dx - \int_{\Omega_h} b(\varrho_0)(\cdot) \varphi(0,\cdot) \, dx
\]

\[
= \int_0^T \int_{\Omega_h} \left( b(\varrho) \partial_t \varphi + b(\varrho) u \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho) \varrho) \text{div}_x u \right) \, dx \, dt \tag{4.40}
\]

holds for any \( b \in C([0,\infty)) \) with \( b' \in C_c([0,\infty)), \tau \in [0,T], \text{ and } \varphi \in C^1_c([0,T] \times \Omega_h). \)
In order to find boundary term $\int_0^T \int_{\Gamma_B} b(\varrho_B) u_B \cdot n \varphi \ dS_z \ dt$ we write

$$
\int_{\Omega_h} b(\varrho) u \cdot \nabla_x \varphi \ dx = \int_{\Omega} b(\varrho) u \cdot \nabla_x \varphi \ dx + \int_{\tilde{U}_h^+(\Gamma_{in})} b(\varrho) u \cdot \nabla_x \varphi \ dx.
$$

By (4.39) and integration by parts

$$
\int_{\tilde{U}_h^+(\Gamma_{in})} b(\varrho) u \cdot \nabla_x \varphi \ dx = \int_{\Gamma_{in}} b(\varrho_B) u_B \cdot n \varphi \ dx + \int_{\tilde{U}_h^+(\Gamma_{in})} (\tilde{\varrho}_B b(\tilde{\varrho}_B) - b(\tilde{\varrho}_B) \text{div}_x \tilde{u}_B) \ dx.
$$

Inserting two above identities to (4.40), letting $h \to 0$, recalling regularity of $(\tilde{\varrho}_B, \tilde{u}_B)$ we obtain desired conclusion of Lemma 4.3.

Furthermore, since (4.9) is satisfied also for $Z$ instead of $\varrho$, due to (4.7) and by arguments of Lemma 4.3 $(Z, u)$ satisfies (2.14). That finishes the proof of Theorem 2.6. □

4.9 Recovery of the system in terms of $(\varrho, u, \varrho^*)$

Our aim now is to prove that solution $(\varrho, u, Z)$ can be identified with the solution $(\varrho, u, \varrho^*)$ to the problem (1.2). Namely, we need to show the existence of $\varrho^*$ satisfying Definition 2.7. To this end we will use combination of arguments from [4] Section 3.1, and from [8] Section 4. First note that since $Z_0 > 0$ we have

$$
\frac{\varrho_0}{Z_0} = \varrho_0^*.
$$

Moreover, recall that we already know that $\varrho$ and $Z$ satisfy renormalized continuity equations.

As in previous section, let us construct set $\tilde{U}_h^+(\Gamma_{in})$ and let us extend continuity equations for $(\varrho, u)$ and $(Z, u)$ on $\Omega_h := \Omega \cup \Gamma_{in} \cup \tilde{U}_h^+(\Gamma_{in})$. In particular extension $\tilde{u}_B, \tilde{\varrho}_B$, and $\tilde{Z}_B$ of $u_B, \varrho_B$, and $Z_B$ on $\tilde{U}_h^+(\Gamma_{in})$ is, such that $\tilde{u}_B \in C^1(\tilde{U}_h^+(\Gamma_{in}))$ and $\tilde{\varrho}_B, \tilde{Z}_B \in W^{1,\infty}(\tilde{U}_h^+(\Gamma_{in}))$, and

$$
\text{div}_x(\tilde{\varrho}_B \tilde{u}_B) = 0 \text{ in } \tilde{U}_h^+(\Gamma_{in}), \quad \tilde{\varrho}_B|_{\Gamma_{in}} = \varrho_B, \quad \tilde{u}_B|_{\Gamma_{in}} = u_B,
$$

$$
\text{div}_x(\tilde{Z}_B \tilde{u}_B) = 0 \text{ in } \tilde{U}_h^+(\Gamma_{in}), \quad \tilde{Z}_B|_{\Gamma_{in}} = Z_B.
$$

(4.41)

Then extensions of $(\varrho, u)$ and $(Z, u)$ to the set $\Omega_h := \Omega \cup \Gamma_{in} \cup \tilde{U}_h^+(\Gamma_{in})$, where $(\varrho, u)(t, x) = (\tilde{\varrho}_B, \tilde{u}_B)$ and $(Z, u)(t, x) = (\tilde{Z}_B, \tilde{u}_B)$ on $\tilde{U}_h^+(\Gamma_{in})$ satisfy continuity equations in the sense of distributions on $\Omega_h$.

Applying convolution with a standard family of regularizing kernels we obtain the regularized functions $[\varrho]_\omega, [Z]_\omega$ which satisfy

$$
\partial_t [\varrho]_\omega + \text{div}_x([\varrho]_\omega u) = R_1^\omega \text{ a.e. in } (0, T) \times \Omega_{\omega,h}
$$

(4.42)

$$
\partial_t [Z]_\omega + \text{div}_x([Z]_\omega u) = R_2^\omega \text{ a.e. in } (0, T) \times \Omega_{\omega,h}
$$

(4.43)

where

$$
\Omega_{\omega,h} := \{ x \in \Omega_h \mid \text{dist}(x, \partial \Omega_h) > \omega \}.
$$

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Due to Friedrichs commutator lemma, see e.g. [11, Lemma 10.12], we find that
\[ R^1_\omega \to 0 \text{ and } R^2_\omega \to 0 \text{ in } L^1_{\text{loc}}((0, T) \times \Omega_h) \text{ as } \omega \to 0. \]

Let us now multiply (4.42) by \( \frac{1}{|Z|_{\omega + \lambda}} \), and (4.43) by \(-\frac{|\phi| + \lambda \phi_0^*}{(|Z|_{\omega + \lambda})^2} \), with \( \lambda > 0 \). Then after some algebraic manipulations we find that
\[
\partial_t \left( \frac{|\phi| + \lambda \phi_0^*}{|Z|_{\omega + \lambda}} \right) + \text{div}_x \left( \left( \frac{|\phi| + \lambda \phi_0^*}{|Z|_{\omega + \lambda}} \right) u \right) - \left( \frac{|\phi| + \lambda \phi_0^*}{|Z|_{\omega + \lambda}} \right) \text{div}_x u \]
\[= R^1_\omega \left( \frac{|\phi| + \lambda \phi_0^*}{|Z|_{\omega + \lambda}} \right) \text{ div}_x C \Omega_h \quad \text{a.e. in } (0, T) \times \Omega_{\omega, h}. \]

Testing above by \( \varphi \in C^1_c([0, T] \times \Omega_h) \), after integration by parts and after passing with \( \omega \to 0 \) we obtain that
\[
\int_{\Omega_h} \left( \frac{\rho + \lambda \phi_0^*}{Z + \lambda} \right) (\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_{\Omega_h} \left( \frac{\rho_0 + \lambda \phi_0^*}{Z_0 + \lambda} \right) (\cdot) \varphi(0, \cdot) \, dx
\]
\[= \int_0^T \int_{\Omega_h} \left( \left( \frac{\rho + \lambda \phi_0^*}{Z + \lambda} \right) \partial_t \varphi + \left( \frac{\rho + \lambda \phi_0^*}{Z + \lambda} \right) u \cdot \nabla_x \varphi + \left( \frac{\rho + \lambda \phi_0^*}{Z + \lambda} \right) \text{div}_x u \right) \, dx \, dt \quad (4.44)\]
for any \( \varphi \in C^1_c([0, T] \times \Omega_h) \).

Next we distinguish two cases:

Case 1. For \( Z = 0 \), due to (2.8) we notice that that \( \rho = 0 \) and therefore \( \frac{\rho + \lambda \phi_0^*}{Z + \lambda} = \phi_0^* \) and \( \frac{\rho + \lambda \phi_0^*}{(Z + \lambda)^2} + \frac{\lambda \phi_0^*}{Z + \lambda} = \phi_0^* \), then (4.44) becomes trivial.

Case 2. For \( Z > 0 \), we find that \( \frac{\rho + \lambda \phi_0^*}{Z + \lambda} \leq \max \{ \phi_0^*, \frac{1}{Z} \} \). Since \( \rho + \lambda \phi_0^* \) converges strongly to \( \rho \) as \( \lambda \to 0 \), as well as \( Z + \lambda \) converges strongly to \( Z \) as \( \lambda \to 0 \), we can pass with \( \lambda \to 0 \) in (4.44) using Legesgue’s Dominated convergence theorem, to obtain that
\[
\int_{\Omega_h} \frac{\rho}{Z} (\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_{\Omega_h} \frac{\rho_0}{Z_0} (\cdot) \varphi(0, \cdot) \, dx
\]
\[= \int_0^T \int_{\Omega_h} \left( \frac{\rho}{Z} \partial_t \varphi + \frac{\rho}{Z} u \cdot \nabla_x \varphi + \frac{\rho}{Z} \text{div}_x u \right) \, dx \, dt \quad (4.45)\]
holds for any \( \varphi \in C^1_c([0, T] \times \Omega_h) \). By the same steps as in the proof of Lemma 4.3 we find that after passing with \( h \to 0 \) we obtain
\[
\int_{\Omega} \frac{\rho}{Z} (\tau, \cdot) \varphi(\tau, \cdot) \, dx - \int_{\Omega} \frac{\rho_0}{Z_0} (\cdot) \varphi(0, \cdot) \, dx
\]
\[= \int_0^T \int_{\Omega} \left( \frac{\rho}{Z} \partial_t \varphi + \frac{\rho}{Z} u \cdot \nabla_x \varphi + \frac{\rho}{Z} \text{div}_x u \right) \, dx \, dt - \int_0^T \int_{\Gamma_B} \frac{\rho_B}{Z_B} u_B \cdot n \varphi \, dS_x dt \quad (4.46)\]

Obviously, \( \rho^* \) defined as \( \begin{cases} \frac{1}{Z} \end{cases} \) satisfies \( \rho^* \in \{ \min \left\{ \frac{1}{Z}, \phi_0^* \right\}, \max \left\{ \frac{1}{Z}, \phi_0^* \right\} \} \) a.e. in \((0, T) \times \Omega\), and thus \( Z = \frac{\rho}{\rho^*} \) a.e. in \((0, T) \times \Omega\).

Finally we can conclude that Theorem 2.8 is proven. \( \square \)
5 Passage to the limit $\varepsilon \to 0$

The purpose of this section is to perform the limit $\varepsilon \to 0$ in the auxiliary system \((1.8-1.10)\) to prove the Theorem 2.11. From now on, by \(\{\varrho_\varepsilon, Z_\varepsilon, u_\varepsilon\}_{\varepsilon > 0}\) we denote the sequence of solutions obtained in the previous section.

5.1 Convergence following from the uniform estimates

The energy inequality established in the previous section, gives rise to the following estimates that are uniform with respect to $\varepsilon$:

\[
\sup_{t \in [0,T]} \left( \|\sqrt{\varrho_\varepsilon} u_\varepsilon(t)\|_{L^2(\Omega)} + \|H(Z_\varepsilon)(t)\|_{L^1(\Omega)} \right) \leq C,
\]

\[
\int_0^T \|u_\varepsilon\|_{W^{1,2}(\Omega,\mathbb{R}^3)}^2 \, dt \leq C. \tag{5.1}
\]

From here and from the construction we also deduce that

\[0 \leq Z_\varepsilon < 1, \quad \text{a.e. in } Q_T, \quad 0 \leq c_\ast \varrho_\varepsilon \leq Z_\varepsilon \leq c_\ast \varrho_\varepsilon, \tag{5.2}\]

in particular both sequences $Z_\varepsilon, \varrho_\varepsilon$ are uniformly bounded in $L^p(Q_T)$ for any $p \leq \infty$. Therefore, up to the subsequence, we have

\[u_\varepsilon \to u \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3)), \]

\[Z_\varepsilon \to Z \quad \text{in } C_{\text{weak}}([0,T];L^p(\Omega)), \tag{5.3}\]

\[\varrho_\varepsilon \to \varrho \quad \text{in } C_{\text{weak}}([0,T];L^p(\Omega)), \]

for any $p$ finite. In addition to that, the limiting $Z$ and $\varrho$ will satisfy

\[0 \leq Z \leq 1, \quad \text{a.e. in } Q_T, \quad 0 \leq c_\ast \varrho \leq Z \leq c_\ast \varrho. \tag{5.4}\]

To obtain the uniform $L^1$ bound for the pressure we can follow one of two strategies:

(i) for the zero flux, i.e. for $\int_{\partial \Omega} u_B \cdot n \, dS_x = 0$, we use our smallness assumption

\[\int_\Omega Z_0 \, dx + T \int_{\Gamma,n} Z_B |u_B \cdot n| \, dS_x < |\Omega|, \tag{5.5}\]

to see that it implies the condition \((4.16)\) from the level of uniform in $\delta$ estimates, and we repeat the argument from the previous section.

(ii) for the positive flux, thanks to the energy estimate \((4.37)\) and the property \((2.16)\) we already control the $L^1$ norm of $\pi_\varepsilon$ uniformly on the subset $\mathcal{O}$. In order to control the pressure on the complementary part, we repeat the Bogovskii type estimate with the test function:

\[\psi = \eta(t) \mathcal{B}(\xi), \tag{5.6}\]
where

\[
\xi = \begin{cases} 
1 & \text{in } \omega \setminus \mathcal{O}, \\
-\frac{|\Omega \setminus \mathcal{O}|}{|\mathcal{O}|} & \text{otherwise.}
\end{cases}
\]  

(5.7)

Note that since \( \mathcal{O} \) is an open subset of \( \Omega \), the test function \( \psi \) is well defined and it belongs to \( W^{1,p}_0(\Omega) \) for any \( p < \infty \), so it is an admissible test function.

Similarly as in the previous limit passage we can in fact show that the local-in-time pressure bound holds true with \( \eta = 1 \). However, we do not have anymore the equi-integrability of the pressure as on the previous level of approximation. Therefore, the convergence in the sense of measures is the most we can hope for, we have

\[
\pi_\varepsilon(Z_\varepsilon) \rightarrow \pi \text{ weakly in } \mathcal{M}^+([0,T] \times K),
Z_\varepsilon \pi_\varepsilon(Z_\varepsilon) \rightarrow \pi_1 \text{ weakly in } \mathcal{M}^+([0,T] \times K).
\]  

(5.8)

The limiting momentum equation therefore reads

\[
\partial_t (\varepsilon u) + \text{div}_x (\varepsilon u \otimes u) + \nabla_x \pi - \text{div}_x \mathbb{S}(\nabla_x u) = 0,
\]

in the sense of distributions.

Already at this point we can identify the second limit in (5.8) using the explicit form of the pressure (1.3). We have

\[
Z_\varepsilon \pi_\varepsilon(Z_\varepsilon) = \pi_\varepsilon(Z_\varepsilon) - \varepsilon \frac{1}{(1 - Z_\varepsilon)^{\beta-1}},
\]  

(5.9)

thus letting \( \varepsilon \rightarrow 0 \) and observing that by (5.1), the last term converges to zero strongly, we obtain the relation

\[
\pi_1 = \pi
\]  

(5.10)

in the sense of distributions. The thing that remains to be shown in order to deduce the constraint \( (1 - Z)\pi = 0 \) is that we have \( \pi_1 = Z\pi \) in some sense (note that on the l.h.s. we have multiplication of measure by the \( L^\infty \) function only). The recovery of the constraint requires stronger information about the convergence of \( Z_\varepsilon \), and additional information about the regularity of \( \pi \) and \( Z \).

5.2 Strong convergence of \( Z_\varepsilon \)

We first need to show that we have a variant of effective viscous flux equality. It can be derived by testing the approximate momentum equation by the inverse divergence operator \( \psi \eta \nabla_x \Delta^{-1}[1_{\Omega}Z_\varepsilon] \) and by testing the momentum equation by \( \psi \eta \nabla_x \Delta^{-1}[1_{\Omega}Z] \), and by comparison of the limits, for any \( \psi \in C^\infty_c((0,T)) \) and \( \eta \in C^\infty_c(\Omega) \). Note that already at this stage we need to justify what the product \( \pi Z \) means, i.e.,
whether \( \eta \psi \nabla_x \Delta^{-1}[1_\Omega Z] \) is regular enough to be used as a test function in the limiting momentum equation. To justify this step we first write weak formulation of the limiting momentum equation

\[
\langle \pi, \text{div}_x \xi \rangle_{(\mathcal{A}(Q_T), C(Q_T))} = \int_0^T \int_\Omega S(\nabla_x u) : \nabla_x \xi \, dx \, dt - \int_0^T \int_\Omega g_\xi \cdot \partial_t \xi \, dx \, dt - \int_0^T \int_\Omega g(u) \otimes u : \nabla_x \xi \, dx \, dt - \int_0^T \int_\Omega \varrho(w - u) \cdot \xi \, dx \, dt
\] (5.11)

that is satisfied for all \( \xi \in C^1_\xi(Q_T) \). From now on we will treat this formula as a definition of \( \int_0^T \int_\Omega \pi \text{div}_x \xi \, dx \, dt \). Let us check that the r.h.s. of (5.11) makes sense for \( \xi \) from much wider class, it is enough that

\[
\nabla_x \xi \in L^2(0, T; L^2(\Omega; R^{d \times d})), \quad \nabla_x \xi \in L^{5/2}(0, T; L^{5/2}(\Omega; R^{d \times d})), \quad \partial_t \xi \in L^1(0, T; L^2(\Omega; R^d)).
\]

The second property is a consequence of simple interpolation property

\[
\| \nabla \varrho u \|^2_{L^{5/3}(Q_T)} \leq \| \nabla \varrho u \|^2_{L^\infty(0, T; L^1(\Omega))} \| \varrho \|^2_{L^1(0, T; L^3(\Omega))}.
\]

So, if we take

\[
\xi \in W^{1,5/2}_0(Q_T; R^d),
\] (5.12)

the r.h.s. of (5.11) will be well defined. Note that our \( C^1_\xi(Q_T) \) functions are dense in \( W^{1,5/2}_0(Q_T) \).

Let us check that \( \xi := \psi \phi \nabla_x \Delta^{-1}[1_\Omega Z] \) has these properties. First, note that the Riesz operator

\[
\mathcal{A} = \nabla_x \Delta^{-1} : L^p(R^d) \to D^{1,p}(R^d; R^d)
\]

(homogeneous Sobolev space) is a continuous linear operator and we have that

\[
\| \nabla_x \mathcal{A}[v] \|_{L^p(R^d; R^d)} \leq C(p) \| v \|_{L^p(R^d)},
\]

for any \( 1 < p < \infty \). Therefore

\[
\| \nabla_x \xi \|_{L^\infty(0, T; L^p(R^d; R^d))} = \| \nabla_x \left( \psi \phi \nabla_x \Delta^{-1}[1_\Omega Z] \right) \|_{L^\infty(0, T; L^p(R^d; R^d))} \leq C(p, \psi, \phi)(1 + \| 1_\Omega Z \|_{L^\infty(0, T; L^p(R^d))}) \leq C,
\]

for any \( p < \infty \). Next, using the continuity equation we get that

\[
\partial_t \xi = \partial_t \left( \psi \phi \nabla_x \Delta^{-1}[1_\Omega Z] \right) = \phi \partial_t \psi \nabla_x \Delta^{-1}[1_\Omega Z] + \psi \phi \nabla_x \Delta^{-1}[1_\Omega \partial_t Z] = \phi \partial_t \psi \nabla_x \Delta^{-1}[1_\Omega Z] - \phi \psi \nabla_x \Delta^{-1}[\text{div}_x(1_\Omega Z u)].
\]

Using the properties of the double Riesz transform we obtain

\[
\| \partial_t \xi \|_{L^p(0, T; L^p(\Omega))} \leq C(p, \psi, \phi)(1 + \| Z u \|_{L^p(0, T; L^p(\Omega))}) \leq C,
\] (5.13)
for some $p > 5/2$. Whence, we have shown that $\xi = \psi \phi \nabla_x \Delta^{-1}[1 \Omega Z]$ is indeed in the class $(5.12)$.

Now, note that

$$\text{div}_x \xi = \psi \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1 \Omega Z] + \psi \phi Z,$$

so we can define

$$\langle \pi, \phi \psi Z \rangle_{(\mathcal{M}(Q_T), C(Q_T))} := \langle \pi, \text{div}_x \xi \rangle_{(\mathcal{M}(Q_T), C(Q_T))} - \langle \pi, \psi \nabla_x \phi \cdot \nabla_x \Delta^{-1}[1 \Omega Z] \rangle_{(\mathcal{M}(Q_T), C(Q_T))}. \quad (5.14)$$

This means that $\langle \pi, \phi \psi Z \rangle_{(\mathcal{M}(Q_T), C(Q_T))}$ is well defined iff $\langle \pi, \phi \psi \nabla_x \Delta^{-1}[1 \Omega Z] \rangle_{(\mathcal{M}(Q_T), C(Q_T))}$ is well defined. For that we need $\nabla_x \Delta^{-1}[1 \Omega Z]$ to be at least $C(Q_T)$, but this is true, as $Z \in C^{\text{weak}}(0, T; L^p(\Omega))$ for sufficiently large $p$. In particular, for $p > d$, using the Morrey inequality, we get $\nabla_x \Delta^{-1}[1 \Omega Z] \in C([0, T] \times \Omega; R^d)$.

Taking the above into account, we have

$$\lim_{\varepsilon \to 0^+} \int_0^T \int_\Omega \psi \phi \left( \pi_\varepsilon(Z_\varepsilon) - (\lambda + 2\mu)\text{div}_x u_\varepsilon \right) Z_\varepsilon \, dx \, dt$$

$$= \langle \pi, \phi \psi Z \rangle_{(\mathcal{M}(Q_T), C(Q_T))} - (\lambda + 2\mu) \int_0^T \int_\Omega \psi \phi \text{div}_x u Z \, dx \, dt. \quad (5.15)$$

From $(5.15)$ it follows that

$$\langle \pi, \phi \psi Z \rangle_{(\mathcal{M}(Q_T), C(Q_T))} - (\lambda + 2\mu) \int_0^T \int_\Omega \psi \phi \left( Z \text{div}_x u - Z \text{div}_x u \right) \, dx \, dt$$

$$= \langle \pi_1, \phi \psi \rangle_{(\mathcal{M}(Q_T), C(Q_T))} - \langle \pi, \phi \psi Z \rangle_{(\mathcal{M}(Q_T), C(Q_T))}$$

$$= \langle \pi, \phi \psi (1 - Z) \rangle_{(\mathcal{M}(Q_T), C(Q_T))} \geq 0 \quad (5.16)$$

where we have used subsequently $(5.10)$, and the limit of $(5.2)$. Since both pairs $(Z_\varepsilon, u_\varepsilon)$ and $(Z, u)$ satisfy the renormalized continuity equation, we can use the renormalization in the form $b(z) = z \log z$ to justify that

$$Z_\varepsilon \to Z \quad \text{strongly in } L^p((0, T) \times \Omega), \quad \forall p < \infty.$$

The proof is identical as for the limit passage $\delta \to 0$.

Note, however, that similarly as in the previous section this property is not transferred to the sequence $\varrho_\varepsilon$; for which we only have $(5.3)$. Nevertheless, using this information and formula $(5.16)$ we can justify that

$$\langle \pi, \phi \psi (1 - Z) \rangle_{(\mathcal{M}(Q_T), C(Q_T))} = 0.$$

Note that at this stage we can repeat the arguments from Section 4.9 in order to come back to the solution in terms of $\varrho, u, \varrho^*$. Indeed, note that the proof is based only on the properties of the renormalized continuity equations for $\varrho$ and $Z$, on the boundedness of $Z$ and $\varrho$, and on the regularity of $u$ which are the same as on the previous level of approximation.

Having this, justification of condition $(2.22)$ amounts to repetition of proof of [21, Lemma 4], see also [16, Lemma 2.1]. The proof of Theorem 2.11 is thus complete. □
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