GENERALIZATION OF A RAMANUJAN IDENTITY

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Abstract. The Euler product for the Landau–Ramanujan constant could have motivated a curious identity by Ramanujan that appears in his notebooks two times. This observation involves a square root and the first four prime numbers of the form $4n + 3$, i.e., 3, 7, 11, 19. Berndt asks whether Ramanujan’s identity is an isolated result, or if there are other identities of this type. With this work we would like to give a possible answer to Berndt’s question.

1. Introduction

Let $B(x)$ denote the number of positive integers not exceeding $x$ that can be expressed as a sum of two squares. Landau [6], [7, pp. 641–669] in 1908 showed that

$$B(x) \sim K \frac{x}{\sqrt{\log x}}$$

as $x \to \infty$,

where $K$ is a constant. Independently, Ramanujan in his first letter to Hardy [2, pp. 52 and 60–62], [3, p. 24], [5, p. xxiv] in 1913 stated the following. The number of numbers greater than $A$ and less than $x$ that can be expressed as a sum of two squares is

$$K \int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x),$$

where $K = 0.764\ldots$ and $\theta(x)$ is very small when compared with the previous integral. This statement also appears in Ramanujan’s second and third notebooks [8, pp. 307 and 363]. Since then the quantity $K$ has been known as the Landau–Ramanujan constant [4, pp. 98–104]. An exact formula for $K$ is given by its Euler product expansion

$$K = \frac{1}{\sqrt{2}} \prod_p \left( \frac{1}{1 - 1/p^2} \right)^{1/2},$$

where $p$ runs through the primes of the form $4n + 3$. This could have motivated the following observation that appears in Ramanujan’s notebooks [8, pp. 309 and 363] two times:

$$\sqrt{2 \left( 1 - \frac{1}{3^2} \right) \left( 1 - \frac{1}{7^2} \right) \left( 1 - \frac{1}{11^2} \right) \left( 1 - \frac{1}{19^2} \right)} = \left( 1 + \frac{1}{7} \right) \left( 1 + \frac{1}{11} \right) \left( 1 + \frac{1}{19} \right).$$

(1)

“It may be somewhat interesting to note,” Ramanujan wrote about (1) in one of his letters to Hardy [3, p. 177], referring to the fact that the squared numbers on the left-hand side are the first four prime numbers of the form $4n + 3$. We examine the identity (1) that can be found in Berndt’s book [2, p. 20] and also in the Andrews–Berndt book [1] pp. 410–411] with only a brief discussion.

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2. Generalization of Ramanujan’s identity

Berndt [2, p. 20, Entry 6] asks whether Ramanujan’s identity in (1) is an isolated result, or if there are other identities of this type. The result of this section is a possible answer to Berndt’s question.

**Lemma 2.1.** Let \( n \geq m \geq 1 \) be integers, and let \( (a_k) \) be a sequence of real numbers such that \( a_k \neq 0,1 \) for all \( k = m, \ldots, n \) and \(-1 < a_\ell < 0\) for an even number of \( \ell \in \{m, \ldots, n\} \) indices. Then

\[
\left( \prod_{k=m}^{n} \frac{a_k + 1}{a_k - 1} \right) \left( \prod_{k=m}^{n} \left( 1 - \frac{1}{a_k^2} \right) \right) = \prod_{k=m}^{n} \left( 1 + \frac{1}{a_k^2} \right).
\]  

(2)

**Proof.** Because of the condition on the elements of \( (a_k) \), the expression under the square root and the right-hand side are nonnegative. Both sides of (2) are equal to zero if and only if \( a_k = -1 \) for some \( k \in \{m, \ldots, n\} \). Suppose that \( a_k \neq -1 \) for all \( k = m, \ldots, n \). Note that for a given real number \( a \neq 0, \pm 1 \), we have

\[
\frac{a + 1}{a - 1} = \frac{1 + \frac{1}{a}}{1 - \frac{1}{a}} = \frac{\left( 1 + \frac{1}{a} \right)^2}{1 - \frac{1}{a^2}}.
\]  

(3)

By using this observation, straightforward arithmetic gives the result. \( \square \)

Henceforth we use Lemma 2.1 to deduce such identities of the form of (2), which have a closed-form expression for the product \( \prod_{k=m}^{n} \frac{a_k + 1}{a_k - 1} \).

**Theorem 2.2** (Generalization of Ramanujan’s identity). Let \( a \in (-\infty, -2/3) \cup (-1/2, -1/3] \cup ((-1/6, \infty) \setminus \{0, 1\}) \) be a real number. Then

\[
\sqrt{\frac{a + 1}{a - 1} \left( 1 - \frac{1}{a^2} \right) \left( 1 - \frac{1}{(2a + 1)^2} \right) \left( 1 - \frac{1}{(3a + 2)^2} \right) \left( 1 - \frac{1}{(6a + 1)^2} \right)} = \left( 1 + \frac{1}{2a + 1} \right) \left( 1 + \frac{1}{3a + 2} \right) \left( 1 + \frac{1}{6a + 1} \right).
\]  

(4)

By substituting \( a = 3 \) into (4), we arrive at Ramanujan’s identity (1).

**Proof.** Suppose that \( a \neq -1/3 \). We can use Lemma 2.1 with \( (a_k) = (2a + 1, 3a + 2, 6a + 1) \). For the first product of (2), we find that

\[
\prod_{k=1}^{3} \frac{a_k + 1}{a_k - 1} = \frac{(2a + 1) + 1}{(2a + 1) - 1} \cdot \frac{(3a + 2) + 1}{(3a + 2) - 1} \cdot \frac{(6a + 1) + 1}{(6a + 1) - 1} = \frac{(a + 1)^2}{a^2}.
\]

On the other hand, if we suppose that \( a \neq -1 \), by using (3), we have

\[
\frac{a + 1}{a - 1} \left( 1 - \frac{1}{a^2} \right) = \frac{\left( 1 + \frac{1}{a} \right)^2}{1 - \frac{1}{a^2}} \left( 1 - \frac{1}{a^2} \right) = \frac{(a + 1)^2}{a^2}.
\]

Since both sides of (4) are equal to zero if and only if \( a = -1 \) or \( a = -1/3 \), the proof is complete. \( \square \)
Remark 2.3 (Alternative form). It is clear from the proof of Theorem 2.2 that the following identity holds. Let \( a \in (-\infty, -2/3) \cup (-1/2, -1/3] \cup ((-1/6, \infty) \setminus \{0\}) \) be a real number. Then

\[
\frac{a+1}{a} \cdot \sqrt{\left(1 - \frac{1}{(2a+1)^2}\right) \left(1 - \frac{1}{(3a+2)^2}\right) \left(1 - \frac{1}{(6a+1)^2}\right)} = \left(1 + \frac{1}{2a+1}\right) \left(1 + \frac{1}{3a+2}\right) \left(1 + \frac{1}{6a+1}\right). \tag{5}
\]

We can derive similar identities by using Lemma 2.1 with the sequence \((a_k) = (2a+1, 3a+1, 6a+5)\) or with \((a_k) = (2a+1, 4a+1, 4a+3)\) with a suitable condition on \(a\). By substituting \(a = 1\) into (5), we find that

\[
2 \cdot \sqrt{\left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{5^2}\right) \left(1 - \frac{1}{7^2}\right)} = \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7}\right). \tag{6}
\]

We generalize (6) for further odd denominators in (9) of Theorem 3.1.

3. Identities with telescoping products

In this section we use Lemma 2.1 with appropriate \((a_k)\) sequences, for which the product \(\prod_{k=m}^{n} \frac{a_{k+1}}{a_k} \) has a telescoping property.

**Theorem 3.1.** Let \(n\) and \(m\) be integers. For \(n \geq m \geq 2\), we have

\[
\sqrt{\frac{n(n+1)}{m(m-1)}} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{k^2}\right) = \prod_{k=m}^{n} \left(1 + \frac{1}{k}\right). \tag{7}
\]

For \(n \geq m \geq 1\), we have

\[
\sqrt{\frac{2n+1}{2m-1}} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{(2k)^2}\right) = \prod_{k=m}^{n} \left(1 + \frac{1}{2k}\right) \tag{8}
\]

and

\[
\sqrt{\frac{n+1}{m}} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{(2k+1)^2}\right) = \prod_{k=m}^{n} \left(1 + \frac{1}{2k+1}\right). \tag{9}
\]

Note that (7) gives \((n+1)/m\), which appears under the square root in (9).

**Proof.** In order to prove (7), according to Lemma 2.1 we have to show the closed-form of a telescoping product. We find that

\[
\prod_{k=m}^{n} \frac{k+1}{k} = \frac{m+1}{m-1} \cdot \frac{m+2}{m} \cdot \frac{m+3}{m+1} \cdot \cdots \cdot \frac{n-1}{n-3} \cdot \frac{n}{n-2} \cdot \frac{n+1}{n-1} = \frac{n(n+1)}{m(m-1)}.
\]

The proofs of (8) and (9) are analogous. \(\square\)
Berndt’s question may have other interesting answers. It would be worth examining Lemma 2.1 further with various \((a_k)\) sequences. In the following theorem, we use \((a_k) = (k^3)\).

**Theorem 3.2.** Let \(n \geq m \geq 2\) be integers. Then

\[
\sqrt{\frac{m(m-1)+1}{m(m-1)}} \cdot \frac{n(n+1)}{n(n+1)+1} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{k^6}\right) = \prod_{k=m}^{n} \left(1 + \frac{1}{k^3}\right).
\]

**Proof.** According to Lemma 2.1, we have to deduce the following.

\[
\prod_{k=m}^{n} k^3 - 1 = \frac{m^3 + 1}{(n+1)^3 + 1} \cdot \prod_{k=m}^{n} \frac{(k+1)^3 + 1}{k^3 - 1} = \frac{m^3 + 1}{(n+1)^3 + 1} \cdot \prod_{k=m}^{n} \frac{k+2}{k-1}
\]

\[
= \frac{m^3 + 1}{(n+1)^3 + 1} \cdot \frac{n(n+1)(n+2)}{(m-1)m(m+1)} = \frac{m(m-1)+1}{m(m-1)} \cdot \frac{n(n+1)}{n(n+1)+1}. \quad \square
\]

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