Asymptotic Analysis of Boundary Layer Correctors and Applications

Daniel Onofrei and Bogdan Vernescu

August 6, 2010

Abstract

In this paper we extend the ideas presented in Onofrei and Vernescu [Asymptotic Analysis, 54, 2007, 103-123] and introduce suitable second order boundary layer correctors, to study the $H^1$-norm error estimate for the classical problem in homogenization. Previous second order boundary layer results assume either smooth enough coefficients (which is equivalent to assuming smooth enough correctors $\chi_j, \chi_{ij} \in W^{1,\infty}$), or smooth homogenized solution $u_0$, to obtain an estimate of order $O(\epsilon^2)$. For this we use the periodic unfolding method developed by Cioranescu, Damlamian and Griso [C. R. Acad. Sci. Paris, Ser. I 335, 2002, 99-104]. We prove that in two dimensions, for nonsmooth coefficients and general data, one obtains an estimate of order $O(\epsilon^2)$. In three dimensions the same estimate is obtained assuming $\chi_j, \chi_{ij} \in W^{1,p}$, with $p > 3$. We also discuss how our results extend, in the case of nonsmooth coefficients, the convergence proof for the finite element multiscale method proposed by T.Hou et al. [J. of Comp. Phys., 134, 1997, 169-189] and the first order corrector analysis for the first eigenvalue of a composite media obtained by Vogelius et al.[Proc. Royal Soc. Edinburgh, 127A, 1997, 1263-1299].

1 Introduction

This paper is dedicated to the study of error estimates for the classical problem in homogenization using suitable boundary layer correctors. Let $\Omega \subset \mathbb{R}^N$, denote a bounded convex polyhedron or a convex bounded domain with a sufficiently smooth boundary. Consider also the unit cube $Y = (0,1)^N$. It is well known that for $A \in L^\infty(Y)^{N \times N}$, $Y$-periodic with $m|\xi|^2 \leq A_{ij}(y)\xi_i\xi_j \leq M|\xi|^2$, for any $\xi \in \mathbb{R}^N$, the solu-
tions of
\[
\begin{aligned}
-\nabla \cdot (A(x)\nabla u_\epsilon(x)) &= f \quad \text{in } \Omega \\
\quad u_\epsilon &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]
(1)
have the property that (see [21], [14], [3], [4]),
\[u_\epsilon \rightharpoonup u_0 \quad \text{in } H^1_0(\Omega)\]
where \(u_0\) verifies
\[
\begin{aligned}
-\nabla \cdot (A^{\text{hom}}\nabla u_0(x)) &= f \quad \text{in } \Omega \\
\quad u_0 &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]
(2)
with
\[A_{ij}^{\text{hom}} = M_Y \left( A_{ij}(y) + A_{ik}(y) \frac{\partial \chi_j}{\partial y_k} \right)\]
(3)
where \(M_Y(\cdot) = \frac{1}{|Y|} \int_Y \cdot dy\) and \(\chi_j \in W^1_{\text{per}}(Y) = \{ \chi \in H^1_{\text{per}}(Y) | M_Y(\chi) = 0 \}\) are the solutions of the local problem
\[-\nabla_y \cdot (A(y)(\nabla \chi_j + e_j)) = 0\]
(4)
Here \(e_j\) represent the canonical basis in \(\mathbb{R}^N\).

In this paper, \(\nabla\) and \((\nabla \cdot)\) will denote the full gradient and divergence operators respectively, and with \(\nabla_x, (\nabla_x \cdot)\) and \(\nabla_y, (\nabla_y \cdot)\) will denote the gradient and the divergence in the slow and fast variable respectively.

**Remark 1.1.** Throughout this paper, we will denote by \(\Phi\) the continuous extension of a given function \(\Phi \in W^{p,m}(\Omega)\) with \(p, m \in \mathbb{Z}\), to the space \(W^{p,m}(\mathbb{R}^N)\). With minimal assumption on the smoothness of \(\Omega\) a stable extension operator can be constructed (see [22], Ch. VI, 3.1).

The formal asymptotic expansion corresponding to the above results can be written as
\[u_\epsilon(x) = u_0(x) + \epsilon w_1(x, \frac{x}{\epsilon}) + \ldots\]
where
\[w_1(x, \frac{x}{\epsilon}) = \chi_j \left( \frac{x}{\epsilon} \right) \frac{\partial u_0}{\partial x_j}\]
(5)
We make the observation that the Einstein summation convention will be used and that the letter \(C\) will denote a constant independent of any other parameter, unless otherwise specified.

A classical result (see [21], [14], [17], [3]), states that with additional regularity assumptions on the local problem solutions \(\chi_j\) or on \(u_0\) one has
\[||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon})||_{H^1(\Omega)} \leq C \epsilon^{\frac{1}{2}}\]
(6)
Without any additional assumptions a similar result has been recently proved by G. Griso in [10], using the Periodic Unfolding method developed in [3], i.e.,

\[ \|u_\varepsilon(\cdot) - u_0(\cdot) - \epsilon \chi_j(\cdot) Q_\varepsilon(\frac{\partial u_0}{\partial x_j})\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}\|u_0\|_{H^2(\Omega)} \tag{7} \]

with

\[ x \in \bar{\Omega}_\varepsilon, \quad Q_\varepsilon(\phi)(x) = \sum_{i_1, \ldots, i_N} M_{Y}^\varepsilon(\phi)(\epsilon \xi + \epsilon i) \tilde{x}_{1,\xi}^{i_1} \cdots \tilde{x}_{N,\xi}^{i_N}, \quad \xi = \left[ \frac{x}{\varepsilon} \right] \]

for \( \phi \in L^2(\Omega) \), \( i = (i_1, \ldots, i_N) \in \{0, 1\}^N \) and

\[ \tilde{x}_{k,\xi}^{i_k} = \begin{cases} \frac{x_k - \xi_k}{\epsilon} & \text{if } i_k = 1 \\ \frac{x_k - \xi_k}{\epsilon} - 1 & \text{if } i_k = 0 \end{cases} \quad x \in \epsilon(\xi + Y) \]

where \( M_{Y}^\varepsilon(\phi) = \frac{1}{\epsilon^N} \int_{\xi + \epsilon Y} \phi(y) dy \) and \( \bar{\Omega}_\varepsilon = \bigcup_{\xi \in \mathbb{Z}^N} \{ \epsilon \xi + \epsilon Y; (\epsilon \xi + \epsilon Y) \cap \Omega \neq \emptyset \} \).

In order to improve the error estimates in (9) boundary layer terms have been introduced as solutions to

\[ - \nabla \cdot (A(\frac{x}{\varepsilon}) \nabla \theta_\varepsilon) = 0 \quad \text{in } \Omega, \quad \theta_\varepsilon = w_1(x, \frac{x}{\varepsilon}) \quad \text{on } \partial \Omega \tag{8} \]

Assuming \( A \in C^\infty (Y) \), \( Y \)-periodic matrix and a sufficiently smooth homogenized solution \( u_0 \) it has been proven in [4] (see also [17]) that

\[ \|u_\varepsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \cdot) + \theta_\varepsilon(\cdot)\|_{H^1(\Omega)} \leq C \epsilon \tag{9} \]

\[ \|u_\varepsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \cdot) + \theta_\varepsilon(\cdot)\|_{L^2(\Omega)} \leq C \epsilon^2. \tag{10} \]

In [18], Moskow and Vogelius proved the above estimates assuming \( A \in C^\infty (Y) \), \( Y \)-periodic matrix and \( u_0 \in H^2(\Omega) \) or \( u_0 \in H^3(\Omega) \) for (9) or (10) respectively. Inequality (9) is proved in [11] for the case when \( A \in L^\infty (Y) \) and \( u_0 \in W^{2,\infty}(\Omega) \).

In [25], Sarkis and Versieux showed that the estimates (9) and respectively (10) still holds in a more general setting, when one has \( u_0 \in W^{2,p}(\Omega) \), \( \chi_j \in W^{1,q}_{\text{per}}(Y) \) for (9), and \( u_0 \in W^{3,p}(\Omega) \), \( \chi_j \in W^{1,q}_{\text{per}}(Y) \) for (10), where, in both cases, \( p > N \) and \( q > N \) satisfy \( \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} \).

In [25] the constants in the right hand side of (9) and (10) are proportional to \( \|u_0\|_{W^{2,p}(\Omega)} \) and \( \|u_0\|_{W^{3,p}(\Omega)} \) respectively.

In order to improve the error estimate in (9) and (10) one needs to consider the second order boundary layer corrector, \( \varphi_\varepsilon \) defined as the solution of,

\[ - \nabla \cdot (A(\frac{x}{\varepsilon}) \nabla \varphi_\varepsilon) = 0 \quad \text{in } \Omega, \quad \varphi_\varepsilon(x) = \chi_{ij}(\frac{x}{\varepsilon}) \frac{\partial^2 u_0}{\partial x_i \partial x_j} \quad \text{on } \partial \Omega \tag{11} \]
where \( \chi_{ij} \in W_{per}(Y) \) are solution of the following local problems,

\[
\nabla_y \cdot (A \nabla_y \chi_{ij}) = b_{ij} + A_{ij}^{\text{hom}}
\]

(12)

with \( A_{ij}^{\text{hom}} \) defined by (2), \( M_Y(b_{ij}(y)) = -A_{ij}^{\text{hom}} \), and \( b_{ij} = -A_{ij} \cdot A_{ik} \frac{\partial \chi_j}{\partial y_k} + \epsilon_\partial \partial_y^k (A_{ik} \chi_j) \).

For the case when \( u_0 \in W^{3,\infty}(\Omega) \) and \( \chi_{ij} \in W^{1,\infty}(Y) \), with the help of \( \phi_\epsilon \) defined in (11), Alaire and Amar proved in [1] the following result

\[
||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon \theta_\epsilon(\cdot) - \epsilon^2 \chi_{ij}(\cdot) \frac{\partial^2 u_0}{\partial x_i \partial x_j}||_{H^1(\Omega)} \leq C \epsilon^2 ||u_0||_{W^{3,\infty}(\Omega)}
\]

(13)

This result shows that with the help of the second order correctors one can essentially improve the order of the estimate (9). In the general case of nonsmooth periodic coefficients, \( A \in L^\infty(Y) \), and \( u_0 \in H^2(\Omega) \), inspired by Griso’s idea, we proved in [20]

\[
||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon \chi_{ij}(\cdot) Q_\epsilon \left( \frac{\partial u_0}{\partial x_j} \right) + \epsilon \theta_\epsilon(\cdot)||_{H^1(\Omega)} \leq C \epsilon ||u_0||_{H^2(\Omega)}
\]

(14)

with \( \beta_\epsilon \) defined by

\[
- \nabla \cdot (A(\cdot) \frac{x}{\epsilon}) \nabla \beta_\epsilon = 0 \quad \text{in} \quad \Omega, \quad \beta_\epsilon = u_1(x, \frac{x}{\epsilon}) \quad \text{on} \quad \partial \Omega
\]

(15)

where \( u_1(x, \frac{x}{\epsilon}) = \chi_{ij}(\cdot) Q_\epsilon \left( \frac{\partial u_0}{\partial x_j} \right). \)

When \( u_0 \in W^{3,p}(\Omega) \) with \( p > N \) we also proved in [20] that

\[
||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon \chi_{ij}(\cdot) \frac{\partial u_0}{\partial x_j} + \epsilon \theta_\epsilon(\cdot)||_{L^2(\Omega)} \leq C \epsilon^2 ||u_0||_{W^{3,p}(\Omega)}.
\]

(16)

In this paper, we present a refinement of (13) for the case of nonsmooth coefficients and general data. To do this we start by describing the asymptotic behavior of \( \phi_\epsilon \) with respect to \( \epsilon \).

The key difference between the case of smooth coefficients, and the nonsmooth case discussed in the present paper is that in the former, by means of the maximum principle or Avellaneda’s compactness results (see [2]), it can be proved that the second order boundary layer corrector \( \phi_\epsilon \) is bounded in \( L^2(\Omega) \) and is of order \( O \left( \frac{1}{\sqrt{\epsilon}} \right) \) in \( H^1(\Omega) \), while in the latter one cannot use the aforementioned techniques to describe the asymptotic behavior of \( \phi_\epsilon \) in \( L^2(\Omega) \) or \( H^1(\Omega) \). Moreover one can see that \( \phi_\epsilon \) is not bounded in \( L^2(\Omega) \) in general (see [2]), and
therefore one needs to carefully address the question of the asymptotic behavior of \( \varphi_\varepsilon \) with respect to \( \varepsilon \).

First, we can easily observe that \( \varepsilon \varphi_\varepsilon \) can be interpreted as the solution of an elliptic problem with variable periodic coefficients and with weakly convergent data in \( H^{-1}(\Omega) \). For this class of problems a result of Tartar, \[24\] (see also \[6\]) implies

\[
\varepsilon \varphi_\varepsilon \rightharpoonup 0 \quad \text{in} \quad H^1(\Omega)
\]

As a consequence of Lemma 2.2 we obtain that for \( u_0 \in H^3(\Omega) \) and \( \chi, \chi_{ij} \in W_{1,p}^{1,p}(Y) \), for some \( p > N \), we have

\[
||\varepsilon \varphi_\varepsilon||_{H^1(\Omega)} \leq C \varepsilon \frac{1}{2} ||u_0||_{H^3(\Omega)} \tag{17}
\]

Using (17) we are able to prove that for \( u_0 \in H^3(\Omega) \) and \( \chi, \chi_{ij} \in W_{1,p}^{1,p} \) with \( p > N \) we have

\[
||u(\cdot) - u_0(\cdot) - \varepsilon \chi_{ij}(\cdot) \frac{\partial u_0}{\partial x_j} + \varepsilon \chi_{ij}(\cdot) \frac{\partial^2 u_0}{\partial x_i \partial x_j}||_{H^1(\Omega)} \leq C \varepsilon \frac{1}{2} ||u_0||_{H^3(\Omega)} \tag{18}
\]

Remark 3.5 states that in two dimensions due to a Meyer type regularity for the solutions of the cell problems, \( \chi, \chi_{ij} \), estimate (18) holds only assuming \( u_0 \in H^3(\Omega) \).

In Section 2.4 we use (18) to extend the results in \[18\] to the case of nonsmooth coefficients. Namely, in two dimensions Moskow and Vogelius (see \[18\]) considered the Dirichlet spectral problem associated to \[1\]

\[
\begin{cases}
- \nabla \cdot (A(x) \nabla u_\varepsilon(x)) = \lambda u_\varepsilon & \text{in} \ \Omega \\
u_\varepsilon = 0 & \text{on} \ \partial \Omega
\end{cases}
\tag{19}
\]

The eigenvalues of (19) form an increasing sequence of positive numbers, i.e.,

\[
0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots
\]

and it is well known that we have \( \lambda_j \rightarrow \lambda_j \) as \( \varepsilon \rightarrow 0 \) for any \( j \geq 0 \) where

\[
0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots
\]

are the Dirichlet eigenvalues of the homogenized operator, i.e.,

\[
\begin{cases}
- \nabla \cdot (A_{\text{hom}} \nabla u(x)) = \lambda u & \text{in} \ \Omega \\
u = 0 & \text{on} \ \partial \Omega
\end{cases}
\tag{20}
\]

For \( A \in C^\infty(Y) \), \( Y \)-periodic, and assuming that the eigenfunctions of (20) belong to \( H^{2+r}(\Omega) \), with \( r > 0 \), Moskow and Vogelius analysed in \[18\], the first corrector of the homogenized eigenvalue of (20) and proved that (See Thm. 3.6), up to a subsequence,

\[
\frac{\lambda - \lambda_j}{\varepsilon} \rightarrow \lambda \int_\Omega \theta_* \varphi \, dx \tag{21}
\]
where $\theta_\epsilon$ is a weak limit of $\theta_\epsilon$ in $L^2(\Omega)$, and $u$ is the normal eigenvector associated to the eigenvalue $\lambda$.

Using (18) we show that the result obtained in [18] for the first corrector of the homogenized eigenvalue holds true in the general case of nonsmooth periodic coefficients $A \in L^\infty_{per}(\mathcal{Y})$.

2 A Fundamental Result

In this section we analyze the asymptotic behavior with respect to $\epsilon$ of the solutions to a certain class of elliptic problems with highly oscillating coefficients and boundary data. The main result is stated in Proposition 2.2 but we will first present a technical Lemma which will be useful in what follows,

**Lemma 2.1.** Let $\Phi$ be such that $\Phi \in W^{1,p}_{per}(\mathcal{Y})$ with $p > N$, and let $\psi \in H^1(\Omega)$. Then we have

$$
\int_{\Omega} |\nabla_y \Phi(\frac{x}{\epsilon})|^2 (\psi(x) - M^\epsilon_Y(\psi)(x))^2 dx \leq C\epsilon^2 \|\Phi\|^2_{W^{1,p}(\mathcal{Y})} \|\psi\|^2_{H^1(\Omega)}
$$

(22)

**Proof.** Let $\tilde{\Omega}_\epsilon \subset \mathbb{R}^N$ be the smallest union of integer translates of $\epsilon Y$ that cover $\Omega$, i.e.

$$
\tilde{\Omega}_\epsilon = \bigcup_{\xi \in Z_\epsilon} (\xi \epsilon + \epsilon Y)
$$

where

$$
Z_\epsilon = \{\xi \in \mathbb{Z}^N, (\xi \epsilon + \epsilon Y) \cap \Omega \neq \emptyset\}
$$

We start by recalling that there exists a linear and continuous extension operator $\mathcal{P} : H^1(\Omega) \rightarrow H^1(\tilde{\Omega}_\epsilon)$, with the continuity constant independent of $\epsilon$ (see [19] for details). In the rest of this section, without having to specify it every time, every function in $H^1(\Omega)$ will be extended through $\mathcal{P}$ to $H^1(\tilde{\Omega}_\epsilon)$. Next we proceed with the proof of the Lemma. We have

$$
\int_{\Omega} |\nabla_y \Phi(\frac{x}{\epsilon})|^2 (\psi(x) - M^\epsilon_Y(\psi)(x))^2 dx \leq \int_{\tilde{\Omega}_\epsilon} |\nabla_y \Phi(\frac{x}{\epsilon})|^2 (\psi(x) - M^\epsilon_Y(\psi)(x))^2 dx
$$

$$
\leq \sum_{\xi \in Z_\epsilon} \int_{\xi \epsilon + \epsilon Y} |\nabla_y \Phi(\frac{x}{\epsilon})|^2 (\psi(x) - M^\epsilon_Y(\psi)(x))^2 dx
$$

$$
\leq \sum_{\xi \in Z_\epsilon} \epsilon^N \int_{\mathcal{Y}} |\nabla_y \Phi|^2 (\psi(\xi \epsilon + \epsilon y) - M^\epsilon_Y(\psi)(\xi \epsilon + \epsilon y))^2 dy
$$

(23)
Let $\psi(\xi + \epsilon y) = z_\xi(y)$. Using this in (23) we obtain,

\[
\int_\Omega |\nabla_y \Phi(\frac{x}{\epsilon})|^2 (\psi(x) - M^\epsilon_y(\psi)(x))^2 \, dx \leq 
\]

\[
\leq \sum_{\xi \in Z} \epsilon^N \int_Y |\nabla_y \Phi|^2 \left( z_\xi(y) - \frac{1}{|Y|} \int_Y z_\xi(s) \, ds \right)^2 \, dy \leq 
\]

\[
\leq \sum_{\xi \in Z} \epsilon^N \|\Phi\|^2_{W^{1,p}(Y)} \left\| z_\xi - \frac{1}{|Y|} \int_Y z_\xi(s) \, ds \right\|_{L^{\frac{2p}{2p-2}}(Y)}^2
\]

(24)

Note that $\nabla_y z_\xi = \epsilon \nabla_x \psi(\xi + \epsilon y)$. Then, using this one can easily observe that (81) in the Appendix, together with the Poincare-Wirtinger inequality, implies

\[
\left\| z_\xi - \frac{1}{|Y|} \int_Y z_\xi(s) \, ds \right\|_{L^{\frac{2p}{2p-2}}(Y)} \leq c_p \|\nabla_y z_\xi\|_{L^2(Y)}
\]

(25)

From (25) in (24) we have

\[
\int_\Omega |\nabla_y \Phi(\frac{x}{\epsilon})|^2 (\psi(x) - M^\epsilon_y(\psi)(x))^2 \, dx \leq 
\]

\[
\leq c_p \epsilon^N \|\Phi\|^2_{W^{1,p}(Y)} \sum_{\xi \in Z} \|\nabla_y z_\xi\|^2_{L^2(Y)}
\]

\[
= c_p \epsilon^{N+2} \|\Phi\|^2_{W^{1,p}(Y)} \sum_{\xi \in Z} \int_Y (\nabla_x \psi(\xi + \epsilon y))^2 \, dy = 
\]

\[
= c_p \epsilon^2 \|\Phi\|^2_{W^{1,p}(Y)} \sum_{\xi \in Z} \int_{\xi + \epsilon \mathbb{Y}} |\nabla_x \psi|^2 \, dx \leq 
\]

\[
\leq C \epsilon^2 \|\Phi\|^2_{W^{1,p}(Y)} \|\psi\|^2_{H^1(\Omega)}
\]

(26)

where $C$ depends on $p$ only. So the statement of the Lemma is proved.

Proposition 2.2. Let $\Omega \subset \mathbb{R}^N$ be bounded and either of class $C^{1,1}$ or convex. Consider the following problem,

\[
\begin{cases}
- \nabla \cdot (A(\frac{x}{\epsilon}) \nabla y_\epsilon) = h & \text{in } \Omega \\
y_\epsilon = g_\epsilon & \text{on } \partial \Omega
\end{cases}
\]

(27)

where $h \in L^2(\Omega)$, the coefficient matrix $A$ satisfies the hypothesis of the first section, and we have that there exists $\phi_\epsilon \in W^{1,p}_{\text{per}}(Y)$ with $p > N$, and $z_\epsilon$ a bounded sequence in $H^1(\Omega)$ such that

\[
g_\epsilon(x) = \epsilon \phi_\epsilon(\frac{x}{\epsilon}) z_\epsilon(x) \ a.e. \ \Omega.
\]

(28)
Then there exists \( y_\ast \in H^1_0(\Omega) \) such that
\[
y_\varepsilon \rightharpoonup y_\ast \text{ in } H^1(\Omega)
\] (29)
and \( y_\ast \) satisfies
\[
\begin{aligned}
\nabla \cdot (A^{\text{hom}} \nabla y_\ast) &= h \text{ in } \Omega, \\
y_\ast &= 0 \text{ on } \partial \Omega
\end{aligned}
\] (30)
and \( A^{\text{hom}} \) is the classical homogenized matrix defined in (3). Moreover we have
\[
\| y_\varepsilon - y_\ast - \varepsilon \chi_j(\frac{x}{\varepsilon}) Q_\varepsilon (\frac{\partial y_\ast}{\partial x_j}) \|_{H^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}} \left( 1 + || y_\ast ||_{H^2(\Omega)} \right)
\] (31)
where \( \chi_j \in W_{\text{per}}(Y) \) are defined in (4), \( Q_\varepsilon \) is defined in (7) and \( C \) depends only on \( p \).

**Proof.** To prove (29) and (30) Tartar’s result concerning problems with weakly converging data in \( H^{-1} \) could be used. We prefer to present here a different proof based on the periodic unfolding method developed in [5], which will also imply (31). First observe that the solution of (27) satisfy,
\[
y_\varepsilon = y_\varepsilon^{(1)} + y_\varepsilon^{(2)} + y_\varepsilon^{(3)}
\] (32)
where \( y_\varepsilon^{(1)}, y_\varepsilon^{(2)}, y_\varepsilon^{(3)} \) satisfy respectively,
\[
\begin{aligned}
-\nabla \cdot (A(\frac{x}{\varepsilon}) \nabla y_\varepsilon^{(1)}) &= h \text{ in } \Omega, \\
y_\varepsilon^{(1)} &= 0 \text{ on } \partial \Omega
\end{aligned}
\] (33)
\[
\begin{aligned}
-\nabla \cdot (A(\frac{x}{\varepsilon}) \nabla y_\varepsilon^{(2)}) &= 0 \text{ in } \Omega, \\
y_\varepsilon^{(2)} &= \varepsilon \Phi_\ast (\frac{x}{\varepsilon}) Q_\varepsilon (z_\varepsilon) \text{ on } \partial \Omega
\end{aligned}
\] (34)
\[
\begin{aligned}
-\nabla \cdot (A(\frac{x}{\varepsilon}) \nabla y_\varepsilon^{(3)}) &= 0 \text{ in } \Omega, \\
y_\varepsilon^{(3)} &= \varepsilon \Phi_\ast (\frac{x}{\varepsilon}) (z_\varepsilon - Q_\varepsilon (z_\varepsilon)) \text{ on } \partial \Omega
\end{aligned}
\] (35)
First note that from Theorem 4.1 in [10], stated here in (7), we have
\[
\| y_\varepsilon^{(1)}(x) - y_\ast(x) - \varepsilon \chi_j(\frac{x}{\varepsilon}) Q_\varepsilon (\frac{\partial y_\ast}{\partial x_j}) \|_{H^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}} || y_\ast ||_{H^2(\Omega)}.
\] (36)
From (10) (see the two estimates before Theorem 4.1 there), by using an interpolation inequality, we immediately arrive at,
\[
\| y_\varepsilon^{(2)} \|_{H^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}} || \Phi_\ast ||_{H^1(Y)} || z_\varepsilon ||_{L^2(\Omega)}
\] (37)
Finally, for $y^{(3)}_\varepsilon$ we obtain,
\[
\|y^{(3)}_\varepsilon\|_{H^1(\Omega)} \leq C\|\varepsilon \Phi_\varepsilon(\frac{z_\varepsilon - Q_\varepsilon(z_\varepsilon)}{\varepsilon})\|_{H^1(\Omega)} = \\
C\|\varepsilon \Phi_\varepsilon(\frac{z_\varepsilon - Q_\varepsilon(z_\varepsilon)}{\varepsilon})\|_{L^2(\Omega)} + C\|\nabla y \Phi_\varepsilon(\frac{z_\varepsilon - Q_\varepsilon(z_\varepsilon)}{\varepsilon})\|_{L^2(\Omega)} + \\
+C\|\varepsilon \Phi_\varepsilon(\frac{z_\varepsilon - Q_\varepsilon(z_\varepsilon)}{\varepsilon})\|_{L^2(\Omega)} + \varepsilon \|\Phi_\varepsilon\|_{W^{1,p}(\Omega)}\|z_\varepsilon\|_{H^1(\Omega)} \\
\leq C\|\nabla y \Phi_\varepsilon(\frac{z_\varepsilon - M_\varepsilon z_\varepsilon}{\varepsilon})\|_{L^2(\Omega)} + C\|\nabla_\varepsilon \Phi_\varepsilon(\frac{z_\varepsilon - M_\varepsilon z_\varepsilon}{\varepsilon})\|_{L^2(\Omega)} + \\
+C\|\Phi_\varepsilon\|_{W^{1,p}(\Omega)}\|z_\varepsilon\|_{H^1(\Omega)} = C\|\Phi_\varepsilon\|_{W^{1,p}(\Omega)}\|z_\varepsilon\|_{H^1(\Omega)}
\]
where $C$ depends only on $p$ and where we used triangle inequality in the fourth line above and we used Lemma 2.1 and (A.3) of the Appendix, respectively, to estimate the first and the second terms in the fifth line. From (30), (31), (33) in (32) we obtain the statement of the Proposition.

3 Boundary layer error estimates

In this section, for the case of $L^\infty$ coefficients, with the only assumptions that $\chi_j, \chi_{ij} \in W_{\text{per}}^{1,p}(Y)$ for some $p > N$ and $u_0 \in H^3(\Omega)$ we show that the left hand side of (13) is of order $\varepsilon^{\frac{3}{2}}$. Indeed we have,

**Theorem 3.1.** Let $A \in L^\infty(Y)$ and $u_0 \in H^3(\Omega)$. If there exists $p > N$ such that $\chi_j, \chi_{ij} \in W_{\text{per}}^{1,p}(Y)$ then we have

\[
\left\|u_\varepsilon(\cdot) - u_0(\cdot) - \varepsilon w_1(\cdot, \varepsilon) + \varepsilon^2 \chi_j(\frac{\partial^2 u_0}{\partial x_i \partial x_j}) \right\|_{H^1(\Omega)} \leq \\
\leq C\varepsilon^{\frac{3}{2}}\|u_0\|_{H^3(\Omega)}.
\]

**Proof.** As we did before, for the sake of simplicity, we will assume $N = 3$ the two dimensional case being similar. We will also assume for the moment that the coefficients are smooth enough, as in Appendix, Section B relations (72). For any $i, j \in \{1, 2, 3\}$ let $\chi_i^n \in W_{\text{per}}(Y)$ be the solutions of

\[
\nabla_y \cdot (A_n \nabla_y \chi_i^n) = b_{ij}^n - M_Y(b_{ij}^n)
\]

where

\[
b_{ij}^n = -A_{ij} - A_{ik} \frac{\partial \chi_i^n}{\partial y_k} - \frac{\partial}{\partial y_k} (A_{ik} \chi_j^n)
\]

and

\[
\|b_{ij}^n\|_{H^1(\Omega)} \leq C\|\chi_i^n\|_{H^3(\Omega)}.
\]
and $M_Y(.)$ is the average on $Y$. From Corollary B.8

$$|\nabla_y \chi_{ij}^n|_{L^2(Y)} < C \text{ and } \chi_{ij}^n \rightarrow \chi_{ij} \text{ in } W_{\text{per}}(Y), \forall i, j \in \{1, \ldots, N\}$$

where

$$\int_Y A(y) \nabla_y \chi_{ij} \cdot \nabla_y \psi dy = (b_{ij} - M_Y(b_{ij}), \psi)_{(W_{\text{per}}(Y))(W_{\text{per}}(Y))^*)}$$

for any $\psi \in W_{\text{per}}(Y)$ and with

$$b_{ij} = -A_{ij} - A_{ik} \frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik} \chi_j).$$

We define

$$u_2^n(x, y) = \chi_{ij}^n(y) \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)$$

and

$$(v_2^n(x, y))_k = A^n_{ki}(y) \chi_{ij}^n(y) \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + A^n_{ik}(y) \frac{\partial \chi_{ij}^n}{\partial y_k} \frac{\partial^2 u_0}{\partial x_j \partial x_i} \quad (40)$$

Following the same ideas as in [18] we can show that $\nabla_x \cdot M_Y(v_2^n) = 0$. Let

$$R_{ki}^l = M_Y(A^n_{kli} + A^n_{ki} \frac{\partial \chi_i^n}{\partial y_l}).$$

$$(C^n(y))_{ij} = A^n_{ij}(y) + A^n_{ik}(y) \frac{\partial \chi^n_{ij}}{\partial y_k}$$

$$A^n_{hom} = M_Y(C^n(y))$$

Consider $\alpha_{ij}^n \in [L^2(Y)]^3$ defined by,

$$\alpha_{ij}^n = \left( \begin{array}{c} A^n_{1i} \chi^n_{ij} + A^n_{1i} \frac{\partial \chi^n_{ij}}{\partial y_0} - R_{1i}^1 \\ A^n_{2i} \chi^n_{ij} + A^n_{2i} \frac{\partial \chi^n_{ij}}{\partial y_0} - R_{2i}^1 \\ A^n_{3i} \chi^n_{ij} + A^n_{3i} \frac{\partial \chi^n_{ij}}{\partial y_0} - R_{3i}^1 \end{array} \right) + \beta_{ij}^n$$

with

$$\beta_{1j}^n = (0, -\phi_{3j}^n, \phi_{2j}^n)^T$$

$$\beta_{2j}^n = (\phi_{3j}^n, 0, -\phi_{1j}^n)^T \text{ for } j \in \{1, 2, 3\}$$

$$\beta_{3j}^n = (-\phi_{2j}^n, \phi_{1j}^n, 0)^T$$

where $T$ denotes the transpose. The functions $\phi_{ij}^n \in W_{\text{per}}(Y)$ were defined in [20], as solutions of

$$\text{curl}_y \phi_{ij}^n = B^n_{ij} \quad \text{and} \quad \text{div}_y \phi_{ij}^n = 0; \quad (41)$$
where \( B^n(y) = C^n(y) - A_{n}^{\text{hom}} \) and \( B^n_i \) denotes the vector \( B^n_i = (B^n_{ij})_{i} \in [L^2_{\text{per}}(Y)]^N \). It was observed in [20] that for every \( 1 \in \{1, 2, ..., N\} \)

\[
\phi^n_i \to \phi_i \text{ in } [W_{\text{per}}(Y)]^N \text{ where } \text{curl}_Y \phi_i = B_i \text{ and } \text{div}_Y \phi_i = 0 \quad (42)
\]

The conditions on \( \chi_j, \chi_{ij} \) and Remark 3.11 in [9] imply that \( \|\phi_i\|_{W^{1,p}(Y)} < C \). Next, using the symmetry of the matrix \( A \) we observe that the vectors \( \alpha^n_{ij} \) defined above are divergence free with zero average over \( Y \).

This implies that there exists \( \psi^n_{ij} \in [W_{\text{per}}(Y)]^3 \), (see Theorem 3.4, [9] adapted for the periodic case) so that

\[
\text{curl}_Y \psi^n_{ij} = \alpha^n_{ij} \text{ and } \text{div}_Y \psi^n_{ij} = 0 \text{ for any } i, j \in \{1, 2, 3\} \quad (43)
\]

By using \( (42), \text{Corollary 3.4 Corollary 3.8 in the definition of } \alpha^n_{ij} \text{ above, we have,}

\[
\alpha^n_{ij} \to \alpha_{ij} \text{ in } [L^2(Y)]^3 \quad (44)
\]

where the form of \( \alpha_{ij} \) is identical with that of \( \alpha^n_{ij} \) and can be obviously obtained from \( (44) \). Using the above convergence result and Theorem 3.9 from [9] adapted to the periodic case, we obtain that

\[
\psi^n_{ij} \to \psi_{ij} \text{ in } W_{\text{per}}(Y) \text{ for any } i, j \in \{1, 2, 3\}
\]

and \( \psi_{ij} \) satisfy

\[
\text{curl}_Y \psi_{ij} = \alpha_{ij} \text{ and } \text{div}_Y \psi_{ij} = 0 \text{ for } i, j \in \{1, 2, 3\} \quad (45)
\]

The hypothesis on \( \chi_j \) and \( \chi_{ij} \) implies that \( \alpha_{ij} \) defined at \( (44) \) belongs to the space \([L^p(Y)]^3\) and for all pairs \((i, j)\) with \( i, j \in \{1, 2, 3\} \) we have

\[
\|\alpha_{ij}\|_{[L^p(Y)]^3} \leq C(\|\beta_{ij}\|_{[L^p(Y)]^3} + \|\chi_j\|_{L^p(Y)} + \|\chi_{ij}\|_{W^{1,r}(Y)}) \leq C \quad (46)
\]

Inequality \( (46) \) and Remark 3.11 in [9] imply that

\[
\|\psi_{ij}\|_{[W^{1,r}(Y)]^3} \leq C \text{ for } i, j \in \{1, 2, 3\} \quad (47)
\]

Define \( p(x, y) = \psi_{ij}(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j}(x) \) and \( v_2(x, y) = \text{curl}_x p(x, y) \). We can see that \( p \in H^1(\Omega, H^1_{\text{per}}(Y)) \) and \( v_2 \in L^2(\Omega, H^1_{\text{per}}(Y)) \). Obviously we have that \( \nabla_x \cdot v_2 = 0 \) in the sense of distributions (see [18]). Next, using \( (40) \) we observe that \( \nabla_x \cdot M_Y(v_*) = 0 \) where \( v_* \) is such that

\[
v^n_* \to v_* \text{ weakly in } L^2(\Omega, L^2_{\text{per}}(Y))
\]

We have that

\[
(v_*(x, y))_k = A_{ki}(y) \chi_j(y) \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + A_{ki}(y) \frac{\partial \chi_{ij}}{\partial y_l} \frac{\partial^2 u_0}{\partial x_j \partial x_i}
\]
Using this and the fact that
\[ \int_{\Omega \times Y} (\nabla_y \cdot v_2) \Phi(x, y) dxdy = \int_{\Omega \times Y} (\nabla_y \cdot \text{curl}_x p(x, y)) \Phi(x, y) dxdy = -\int_{\Omega \times Y} (\nabla_x \cdot \text{curl}_y p(x, y)) \Phi(x, y) dxdy \]
for any smooth function \( \Phi \in D(\Omega; D(Y)) \), one can immediately see that
\[ \nabla_y \cdot v_2 = -\nabla_x \cdot v_* \] (48)
in the sense of distributions. Let \( p^n(x, y) = \psi^n(y) \frac{\partial^2 u_0}{\partial x_j \partial x_j}(x) \) and \( v^n_2(x, y) = \text{curl}_x p^n(x, y) \). Consider \( \psi^n_\epsilon \) and \( \xi^n_\epsilon \) defined as follows,
\[
\psi^n_\epsilon(x) = u^n_\epsilon(x) - u_0(x) - \epsilon w^n_1(x, \frac{x}{\epsilon}) - \epsilon^2 u^n_2(x, \frac{x}{\epsilon}) \]
\[ \xi^n_\epsilon(x) = A^n(\frac{x}{\epsilon}) \nabla u^n_\epsilon - r^n_0(x, \frac{x}{\epsilon}) - \epsilon v^n_1(x, \frac{x}{\epsilon}) - \epsilon^2 v^n_2(x, \frac{x}{\epsilon}) \] (49)
Note that
\[ A^n(\frac{x}{\epsilon}) \nabla \psi^n_\epsilon(x) - \xi^n_\epsilon(x) = \epsilon^2 (v^n_2(x, \frac{x}{\epsilon}) - A^n(\frac{x}{\epsilon}) \nabla_x u^n_2(x, \frac{x}{\epsilon})) \] (50)
We have
**Lemma 3.2.**

(i) \( \| \psi^n_\epsilon \|_{W^{1,1}(\Omega)} < C \) and \( \| \xi^n_\epsilon \|_{L^1(\Omega)} < C \)
and there exists \( \psi_\epsilon \in W^{1,1}(\Omega) \) and \( \xi_\epsilon \in L^1(\Omega) \) such that
\( \psi^n_\epsilon \rightharpoonup \psi_\epsilon \), \( \nabla \psi^n_\epsilon \rightharpoonup \nabla \psi_\epsilon \), \( \xi^n_\epsilon \rightharpoonup \xi_\epsilon \), weakly-* in the sense of measures.

Also we have
\[ \psi_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon w_1(x, \frac{x}{\epsilon}) - \epsilon^2 u_2(x, \frac{x}{\epsilon}) \]
\[ \xi_\epsilon(x) = A(\frac{x}{\epsilon}) \nabla u_\epsilon - r_0(x, \frac{x}{\epsilon}) - \epsilon v_1(x, \frac{x}{\epsilon}) - \epsilon^2 v_2(x, \frac{x}{\epsilon}) \]
(ii) Moreover, \( \xi_\epsilon \in L^2(\Omega) \), \( \psi_\epsilon \in H^1(\Omega) \) and we have
\[ A(\frac{x}{\epsilon}) \nabla \psi_\epsilon(x) - \xi_\epsilon(x) = \epsilon^2 (v_2(x, \frac{x}{\epsilon}) - A(\frac{x}{\epsilon}) \nabla_x u_2(x, \frac{x}{\epsilon})) \] (52)
with
\[ \nabla \cdot \xi_\epsilon(x) = 0 \] (53)
in the sense of distributions.
Proof. Using the fact that, for any \( i, j \in \{1, 2, 3\} \), \( \chi^n_j, \chi^n_{ij} \in W_{\text{per}}(Y) \) and \( \psi^n_{ij} \in [W_{\text{per}}(Y)]^3 \) are bounded functions in these spaces, from the definition one can immediately see that
\[
\|\psi^n\|_{W^{1,1}(\Omega)} < C \quad \text{and} \quad \|\xi^n\|_{L^1(\Omega)} < C.
\]
Recall that
\[
\chi^n_j \rightharpoonup \chi_j, \chi^n_{ij} \rightharpoonup \chi_{ij} \quad \text{in} \quad W_{\text{per}}(Y) \quad \text{and} \quad \psi^n_{ij} \rightharpoonup \psi_{ij} \quad \text{in} \quad [W_{\text{per}}(Y)]^3.
\]
Using the above convergence results and the Appendix the statement (i) in Lemma 3.2 follows immediately. Observe that \( \chi_j, \chi_{ij} \in W^{1,p}_\text{per}(Y) \), with \( p > 3 \) imply
\[
\psi \in H^1(\Omega) \quad (54)
\]
To prove (54) it is enough to see that
\[
|\langle u_2(x, \frac{x}{\epsilon}), \nabla \psi \rangle|_{H^1(\Omega)} \leq \epsilon^2 |\langle \chi_{ij}, L^\infty(Y) \rangle| |u_0|_{H^2(\Omega)} + \\
+ \epsilon |\langle \chi_{ij}, W^{1,p}(Y) \rangle| |u_0|_{H^3(\Omega)} + \epsilon^2 |\langle \chi_{ij}, L^\infty(Y) \rangle| |u_0|_{H^3(\Omega)}
\]
the rest of the necessary estimates being trivial. Similarly, from the definition of \( r_0, v_2 \), and \( v_2 \) and the hypothesis \( \chi_j, \chi_{ij} \in W^{1,p}(Y) \), with \( p > 3 \) we see that \( \xi \in L^2(\Omega) \). Next note that we immediately have
\[
A^n(\frac{x}{\epsilon}) \nabla \psi^n \rightharpoonup A(\frac{x}{\epsilon}) \nabla \psi \quad \text{weakly-* in the sense of measures} \quad (55)
\]
Relation (52) follows immediately from (51), (55) the relations (72) of Section B in the Appendix and a limit argument based on the convergence results obtained at (i). Recall that in the smooth case it is known from (18) that
\[
\nabla \cdot \xi^n = 0
\]
This is equivalent to
\[
\int_\Omega \xi^n \nabla \Phi(x)dx = 0 \quad \text{for any} \quad \Phi \in \mathcal{D}(\Omega)
\]
Using the fact that \( \xi \in L^2(\Omega) \), and that we have
\[
\xi^n \rightharpoonup \xi \quad \text{weakly-* in the sense of measures}
\]
we obtain (53). We make the remark that a different proof for (53) can be found in (25).

We can make the observation that \( \chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y) \), with \( p > 3 \), implies \( \psi_{ij} \in W^{1,p}_{\text{per}}(Y) \). Using this we obtain,
\[
|\langle \nabla u_2(x, \frac{x}{\epsilon}), \nabla \psi \rangle|_{L^2(\Omega)} \leq \|\chi_{ij}\|_{L^\infty(Y)} \|\nabla \frac{\partial^2 u_0}{\partial x_j \partial x_i} \|_{L^2(\Omega)} \leq \\
\leq \|\chi_{ij}\|_{W^{1,p}(Y)} |u_0|_{H^3(\Omega)} \leq \\
\leq C |u_0|_{H^3(\Omega)} \quad (56)
\]
13
\[ \|v(x, x/\varepsilon)\|_{L^2(\Omega)} \leq C \|\psi_{ij}\|_{L^\infty(\Gamma)} \left\| \nabla_x \frac{\partial^2 u_0}{\partial x_i \partial x_j} \right\|_{L^2(\Omega)} \leq \]
\[ \leq C \sum_{i,j} \|\psi_{ij}\|_{W^{1,p}(\Gamma)} \|u_0\|_{H^3(\Omega)} \leq \]
\[ \leq C \|u_0\|_{H^3(\Omega)} \quad (57) \]

where in (57) above we used (47). Similarly as in [18] using (56), (57) in (48), we arrive at
\[ \|A(\frac{x}{\varepsilon}) \nabla \psi_{\varepsilon}(x) - \xi_{\varepsilon}(x)\|_{L^2(\Omega)} \leq C \varepsilon^2 \|u_0\|_{H^3(\Omega)} \]

Consider the second boundary layer \( \varphi_{\varepsilon} \) defined as solution of
\[ \nabla \cdot (A(\frac{x}{\varepsilon}) \nabla \varphi_{\varepsilon}) = 0 \quad \text{in } \Omega, \quad \varphi_{\varepsilon} = u_2(x, x/\varepsilon) \quad \text{on } \partial \Omega \quad (58) \]

Using (54) and similar arguments as in [18] we obtain that
\[ \|u_\varepsilon(x) - u_0(x) - \varepsilon w_1(x, \xi) + \varepsilon \theta_\varepsilon(x) - \varepsilon^2 u_2(x, \xi) + \varepsilon^2 \varphi_{\varepsilon}\|_{H^1(\Omega)} \leq \]
\[ \leq C \varepsilon^2 \|u_0\|_{H^3(\Omega)} \quad (59) \]

Next we make the observation that without any further regularity assumption on \( u_0 \) or on the matrix of coefficients \( A \) one cannot make use of neither Avellaneda compactness result nor the maximum principle to obtain a \( L^2 \) or \( H^1 \) bound for \( \varphi_{\varepsilon} \). In fact in [2] it is presented an example where a solution of (58) would blow up in the \( L^2 \) norm. Although the unboundedness of \( \varphi_{\varepsilon} \) in \( L^2 \) we can still make the observation that using a result due to Luc Tartar [24] (see also [3], Section 8.5) concerning the limit analysis of the classical homogenization problem in the case of weakly convergent data in \( H^{-1}(\Omega) \) together with a few elementary computations we can obtain that
\[ \varepsilon \varphi_{\varepsilon} \overset{\text{weak}}{\rightharpoonup} 0 \quad \text{in } H^1(\Omega) \]

Then applying Proposition 2.2 with \( h = 0 \), \( y_\varepsilon = \varepsilon \varphi_{\varepsilon} \), \( \phi_\varepsilon(y) = \chi_{ij}(y) \),
\[ z_\varepsilon(x) = z(x) = \frac{\partial^2 u_0}{\partial x_i \partial x_j} \] we obtain that
\[ \|\varepsilon \varphi_{\varepsilon}\|_{H^1(\Omega)} \leq C \varepsilon^2 \|u_0\|_{H^3(\Omega)} \quad (60) \]

Using (60) in (59) we have
\[ \|u_\varepsilon(x) - u_0(x) - \varepsilon w_1(x, \xi) + \varepsilon \theta_\varepsilon(x) - \varepsilon^2 \chi_{ij}(\frac{x}{\varepsilon}) \frac{\partial^2 u_0}{\partial x_i \partial x_j}\|_{H^1(\Omega)} \leq \]
\[ \leq C \varepsilon^2 \|u_0\|_{H^3(\Omega)} \quad (61) \]

and this concludes the proof of Theorem 3.1
Remark 3.3. Following similar arguments as in the above Theorem, we can adapt the result in Theorem 3.2 (the convex case) in \[11\] and obtain
\[
\|\epsilon \varphi_\epsilon\|_{L^2(\Omega)} \leq C\|u_0\|_{H^3(\Omega)}
\]
(62)

Remark 3.4. It has been shown in \[22\] that the assumptions \(\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)\) for some \(p > N\) are implied by the conditions that the BMO semi-norm norm of the coefficients matrix \(a\) is small enough (see \[22\] for the precise statement). In a different work by M. Vogelius and Y.Y. Lin \[16\], it has been shown that one can have \(\chi_j, \chi_{ij} \in W^{1,\infty}_{\text{per}}(Y)\) in the case of piecewise discontinuous matrix of coefficients when the discontinuities occur on certain smooth interfaces (see \[16\] for the precise statement). It is clear that the lack of smoothness in the matrix \(A\) and the fact that we only assume \(u_0 \in H^3(\Omega)\) would not allow one to use neither Avellaneda compactness principle nor the maximum principle to obtain bounds for \(\varphi_\epsilon\) in \(L^2\) or \(H^1\).

Remark 3.5. For \(N = 2\) we could use a Meyers type regularity result and prove that there exists \(p > 2\) such that \(\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)\). Therefore Theorem \[3.1\] holds true in this case in the very general conditions that \(u_0 \in H^3(\Omega)\) and \(A \in L^\infty(Y)\).

4 A natural extra term in the first order corrector to the homogenized eigenvalue of a periodic composite medium

In this section we analyze the Dirichlet eigenvalues of an elliptic operator corresponding to a composite medium with periodic microstructure. This problem was initially studied in \[18\], for the case of \(C^\infty\) coefficients. We generalize their result to the case of \(L^\infty\) coefficients. We will first state a simple consequence of Theorem \[3.1\] which will play a fundamental role further in our analysis.

Corollary 4.1. Let \(\Omega \subset \mathbb{R}^2\) be a bounded, convex curvilinear polygon of class \(C^\infty\). Let \(A \in L^\infty(Y)\) and \(u_0 \in H^{2+r}(\Omega)\) with \(r > 0\). Then there exists a constant \(C_\epsilon\) independent of \(u_0\) and \(\epsilon\) such that
\[
\|u_\epsilon(.) - u_0(.) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon \theta_\epsilon(.)\|_{L^2(\Omega)} \leq C_\epsilon \epsilon^{1+r}\|u_0\|_{H^{2+r}(\Omega)}
\]

Proof. From Theorem 2.1 in \[20\], if \(u_0 \in H^2(\Omega)\), we have
\[
\|u_\epsilon(.) - u_0(.) - \epsilon w_1(\cdot, \frac{\cdot}{\epsilon}) + \epsilon \theta_\epsilon(.)\|_{H^1(\Omega)} \leq C\epsilon\|u_0\|_{H^2(\Omega)}
\]
(63)

Indeed note that using Remark 3.5 and the properties of \(Q_\epsilon\) we have that
\[
\|\nabla w_1(\cdot, \frac{\cdot}{\epsilon}) - \nabla u_1(\cdot, \frac{\cdot}{\epsilon})\|_{L^2(\Omega)} < \epsilon\|\chi_j\|_{W^{1,p}(Y)}\|u_0\|_{H^1(\Omega)} < C
\]
(64)
where \( w_1 \) and \( u_1 \) are defined at (5) and (15), respectively. Also, using
the definition of \( \theta_\epsilon \) and \( \beta_\epsilon \), we have
\[
||\nabla \theta_\epsilon - \nabla \beta_\epsilon||_{L^2(\Omega)} < ||w_1(\cdot, \cdot) - u_1(\cdot, \cdot)||_{H^1(\Omega)} < C
\]
Using the last two inequalities in (64) we obtain (63). Next we may
see that, for \( u_0 \in H^3(\Omega) \), Theorem 3.1 immediately implies that
\[
||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1(\cdot, \cdot)\|_{L^2(\Omega)} \leq C \epsilon^2 \|u_0\|_{H^3(\Omega)} \quad (65)
\]
From (63) and (65) together with an interpolation argument (see Theorem 2.4 in [18]), we prove the statement of the Corollary.

Next we will state the spectral problem and recall briefly the result
obtained in [18]. On the domain \( \Omega \subset \mathbb{R}^2 \), we consider the spectral
problem (19) associated with operator \( L_\epsilon \), i.e.,
\[
\begin{aligned}
L_\epsilon v_\epsilon = -\nabla \cdot (A(\frac{x}{\epsilon}) \nabla v_\epsilon(x)) &= \lambda_\epsilon v_\epsilon \quad \text{in } \Omega \\
v_\epsilon &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\quad (66)
\]
If we consider the eigenvalue problem for the operator \( L \doteq -\text{div}(A^{\text{hom}} \nabla) \)
with \( A^{\text{hom}} \) defined at (2), i.e.,
\[
\begin{aligned}
Lv = \lambda v \quad \text{in } \Omega \\
v = 0 \quad \text{on } \partial \Omega
\end{aligned}
\quad (67)
\]
then it is well known that for \( \lambda \) simple eigenvalue of (67), for each \( \epsilon \)
small enough, there exists \( \lambda_\epsilon \), an eigenvalue of (66) such that
\[
\lambda_\epsilon \to \lambda
\]
For any \( f \in L^2(\Omega) \), we define \( T_\epsilon f = u_\epsilon \) where \( u_\epsilon \in H^1_0(\Omega) \) is the
solution of \( L_\epsilon u_\epsilon = f \) in \( \Omega \), and similarly \( T f = u_0 \) with \( u_0 \in H^1_0(\Omega) \)
solution of \( Lu_0 = f \). \( T_\epsilon \) and \( T \) are compact and self adjoint operators
from \( L^2(\Omega) \) into \( L^2(\Omega) \). Moreover \( T_\epsilon \to T \) pointwise.
It can be seen that \( \mu_\epsilon = \frac{1}{\lambda_\epsilon} \) are the eigenvalues of \( T_\epsilon \) and \( \mu_k = \frac{1}{\lambda_k} \)
are the eigenvalues of \( T \). From the definition of \( T_\epsilon \) and \( T \), the
eigenvectors corresponding to \( \mu_\epsilon \) and respectively \( \mu_k \) are the same as
the eigenvectors of \( L_\epsilon \) and \( L \) corresponding to \( \lambda_\epsilon \) and respectively \( \lambda_k \).
It is proved in [18] that if \( \Omega \) is a bounded convex domain or bounded
with a \( C^{2,\beta} \) boundary we have that
\[
|\lambda - \lambda_\epsilon| \leq C \epsilon
\quad (68)
\]
16
for \( \epsilon \) sufficiently small. Moreover in the case of a smooth matrix of coefficients \( a \), and for the eigenvectors of \( L \) in \( H^{2+r}(\Omega) \), for some \( r > 0 \), using [9] and [10] and a result of Osborne [19], they obtain that

\[
\lambda^{n} - \lambda = \epsilon_n \lambda \int_{\Omega} \theta_{\epsilon_n} v dx + O(\epsilon_n^{1+r})
\]

for any sequence \( \epsilon_n \to 0 \) and \( \theta_{\epsilon} \) defined by

\[
- \nabla \cdot (A(\frac{x}{\epsilon}) \nabla \theta_{\epsilon}) = 0 \text{ in } \Omega, \quad \theta_{\epsilon} = \chi_{j}(\frac{x}{\epsilon}) \frac{\partial v}{\partial x_j} \text{ on } \partial \Omega
\]  

(69)

From the Corollary 4.1 we obtain that the result of Moskow and Vogelius (see Theorem 3.6) remains true in the general case of nonsmooth coefficients, i.e.,

**Theorem 4.2.** In the hypothesis of Corrolary 4.1 if \( \lambda_* \) is the limit of the sequence \( \frac{(\lambda^{n} - \lambda)}{\epsilon_n} \) (as \( \epsilon_n \to 0 \)) then there exists a function \( \theta_* \), weak limit point of the sequence \( \theta_{\epsilon_n} \) in \( L^2(\Omega) \), so that

\[
\lambda_* = \lambda \int_{\Omega} \theta_* v dx
\]

Conversely, if \( \theta_* \) is a weak limit point of the sequence \( \theta_{\epsilon_n} \) in \( L^2(\Omega) \) the there exists a sequence \( \epsilon_n \to 0 \) such that

\[
\frac{(\lambda^{n} - \lambda)}{\epsilon_n} \to \lambda \int_{\Omega} \theta_* v dx
\]

In the end we make the observation that the case when \( \lambda \) is a multiple eigenvalue can be treated similarly as in [18] (see Remark 3.7).

**Appendices**

In the following appendices we will present the proofs for some of the results used in the previous Sections not included in the main body of the chapter for the sake of clarity of the exposition.

**A Definition and Properties of the Unfolding Operator**

Let \( \Xi_{\epsilon} = \{ \xi \in \mathbb{Z}^N; (\epsilon \xi + \epsilon Y) \cap \Omega \neq \emptyset \} \) and define

\[
\tilde{\Omega}_{\epsilon} = \bigcup_{\xi \in \Xi_{\epsilon}} (\epsilon \xi + \epsilon Y)
\]  

(70)
Let us also consider $H^1_{\text{per}}(Y)$ to be the closure of $C^\infty_{\text{per}}(Y)$ in the $H^1$ norm, where $C^\infty_{\text{per}}(Y)$ is the subset of $C^\infty(\mathbb{R}^N)$ of $Y$-periodic functions, and
\[ W_{\text{per}}(Y) = \left\{ v \in H^1_{\text{per}}(Y)/\mathbb{R} : \frac{1}{|Y|} \int_Y vdy = 0 \right\} \]
(see [6] for properties).

Next, similarly as in [5],[7], if we have a periodical net on $\mathbb{R}^N$ with period $Y$, by analogy with the one-dimensional case, to each $x \in \mathbb{R}^N$ we can associate its integer part, $\lfloor x \rfloor_Y$, such that $x - \lfloor x \rfloor_Y \in Y$ and its fractional part respectively, i.e., $\{x\}_Y = x - \lfloor x \rfloor_Y$. Therefore we have:
\[ x = \epsilon \left\{ \frac{x}{\epsilon} \right\}_Y + \epsilon \left\{ \frac{x}{\epsilon} \right\}_Y \quad \text{for any} \quad x \in \mathbb{R}^N. \]

We will recall in the following the definition of the Unfolding Operator as it have been introduced in [5](see also [7]), and review a few of its principal properties. Let the unfolding operator be defined as $T_\epsilon : L^2(\tilde{\Omega}_\epsilon) \to L^2(\tilde{\Omega}_\epsilon \times Y)$ with
\[ T_\epsilon(\phi)(x, y) = \phi(\epsilon \left\{ \frac{x}{\epsilon} \right\}_Y + \epsilon y) \quad \text{for all} \quad \phi \in L^2(\tilde{\Omega}_\epsilon) \]
We have (see [5]):

**Theorem A.1.** For any $v, w \in L^2(\Omega)$ we have

1. $T_\epsilon(vw) = T_\epsilon(v)T_\epsilon(w)$
2. $\nabla_y(T_\epsilon(u)) = \epsilon T_\epsilon(\nabla_x u)$ where $u \in H^1(\Omega)$
3. $\int_\Omega udx = \frac{1}{|Y|} \int_{\tilde{\Omega}_\epsilon \times Y} T_\epsilon(u)dxdy$
4. $\left| \int_\Omega udx - \int_{\tilde{\Omega}_\epsilon \times Y} T_\epsilon(u)dxdy \right| < |u|_{L^1(\{x \in \tilde{\Omega}_\epsilon \times \partial\tilde{\Omega} \})}$
5. $T_\epsilon(\psi) \to \psi$ uniformly on $\Omega \times Y$ for any $\psi \in C(\Omega)$
6. $T_\epsilon(w) \to w$ strongly in $L^2(\Omega \times Y)$
7. Let $\{w_\epsilon\} \subset L^2(\Omega \times Y)$ such that $w_\epsilon \to w$ in $L^2(\Omega)$. Then
\[ T_\epsilon(w_\epsilon) \to w \text{ in } L^2(\Omega \times Y) \]
8. Let $w_\epsilon \to w$ in $H^1(\Omega)$. Then there exists a subsequence and $\hat{w} \in L^2(\Omega; H^1_{\text{per}}(Y))$ such that:
   a) $T_\epsilon(w_\epsilon) \to w$ in $L^2(\Omega; H^1(Y))$
   b) $T_\epsilon(\nabla w_\epsilon) \to \nabla_x w + \nabla_y \hat{w}$ in $L^2(\Omega \times Y)$
Another important property of the Unfolding Operator it is presented in the next Theorem due to Damlamian and Griso, see [10].

**Theorem A.2.** For any \( w \in H^1(\Omega) \) there exists \( \tilde{w}_\varepsilon \in L^2(\Omega, H^1_{per}(Y)) \) such that

\[
\begin{align*}
\|\tilde{w}_\varepsilon\|_{L^2(\Omega, H^1_{per}(Y))} &\leq C\|\nabla_x w\|_{L^2(\Omega)}^N \\
\|T(\nabla_x w) - \nabla_x w - \nabla_y \tilde{w}_\varepsilon\|_{L^2(Y, H^{-1}(\Omega))} &\leq C\varepsilon\|\nabla_x w\|_{L^2(\Omega)}^N
\end{align*}
\]

(71)

where \( C \) only depends on \( N \) and \( \Omega \).

Next present some interesting technical results obtained in [10], which are used in Section 4. Define \( \rho_\varepsilon(\cdot) = \inf\{\frac{\rho(\cdot)}{\varepsilon}, 1\} \) where \( \rho(x) = \text{dist}(x, \partial \Omega) \). Define also \( \hat{\Omega}_\varepsilon = \{x \in \Omega ; \rho(x) < \varepsilon\} \) and for any \( \phi \in L^2(\Omega) \) consider \( M^\varepsilon_\psi(\phi)(x) = \frac{1}{|Y|} \int_Y T(\phi)(x, y)dy \). Let \( v \in H^1(\Omega) \) be arbitrarily fixed, and the regularization \( Q_\varepsilon \) defined at (7). Then (see Griso [10], for the proofs)

**Proposition A.3.** We have

1. \( \|\nabla_x \rho_\varepsilon\|_{L^\infty(\Omega)} = \|\nabla_x \rho_\varepsilon\|_{L^\infty(\hat{\Omega}_\varepsilon)} = \varepsilon^{-1} \)

2. \( \|(1-\rho_\varepsilon)v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\hat{\Omega}_\varepsilon)} \leq C\varepsilon^\frac{1}{2} \|v\|_{H^1(\Omega)} \) for any \( v \in H^1(\Omega) \)

3. \( \|\nabla_x v\|_{L^2(\hat{\Omega}_\varepsilon)} \leq C\varepsilon^\frac{1}{2} \|v\|_{H^1(\Omega)} \Rightarrow \|Q_\varepsilon(\nabla_x v)\|_{L^2(\hat{\Omega}_\varepsilon)} + \|M^\varepsilon_\psi(\nabla_x v)\|_{L^2(\hat{\Omega}_\varepsilon)} \leq C\varepsilon^\frac{1}{2} \|v\|_{H^2(\Omega)} \) for any \( v \in H^2(\Omega) \).

4. \( \|\psi(\frac{\cdot}{\varepsilon})\|_{L^2(\hat{\Omega}_\varepsilon)} + \|\nabla_y \psi(\frac{\cdot}{\varepsilon})\|_{L^2(\hat{\Omega}_\varepsilon)} \leq C\varepsilon^\frac{1}{2} \|\psi\|_{H^1(Y)} \) for every \( \psi \in H^1_{per}(Y) \)

5. \( \|M^\varepsilon_\psi(v)\|_{L^2(\hat{\Omega}_\varepsilon)} \leq \|v\|_{L^2(\hat{\Omega}_\varepsilon)} \) for any \( v \in L^2(\hat{\Omega}_\varepsilon) \)

6. \( \begin{align*}
\|v - T(\nabla_x v)\|_{L^2(\Omega, Y)} &\leq C\varepsilon\|\nabla v\|_{L^2(\Omega)}^N \\
\|Q_\varepsilon(\nabla_x v) - M^\varepsilon_\psi(v)\|_{L^2(\Omega)} &\leq C\varepsilon\|\nabla v\|_{L^2(\Omega)}^N
\end{align*} \)

for any \( v \in H^1(\Omega) \)

7. \( \|Q_\varepsilon(\psi(\frac{\cdot}{\varepsilon}))\|_{L^2(\Omega)} \leq C\|\psi\|_{L^2(\hat{\Omega}_\varepsilon, Y)} \|\psi\|_{L^2(Y)} \) for any \( v \in L^2(\hat{\Omega}_\varepsilon, Y) \) and \( \psi \in L^2(Y) \)

### B Convergence results and the smoothing argument

Let \( m_n \in C^\infty \) be the standard mollifying sequence, i.e., \( 0 < m_n \leq 1, \int_{\mathbb{R}^N} m_n dz = 1, \text{supp}(m_n) \subset B(0, \frac{1}{n}) \). Define \( A^n(y) = (m_n * A)(y) \), where \( a \) has been defined in the Introduction (see [11]). We have:

\[ 1.A^n - Y \text{ periodic matrix} \]
2. $|A^n|_{L^\infty} < |A|_{L^\infty}$

3. $A^n \to A$ in $L^p$ for any $p \in (1, \infty)$ \hspace{1cm} (72)

From (72) we have that $c|\xi|^2 \leq A^n_{ij}(y)\xi_i\xi_j \leq C|\xi|^2 \forall \xi \in \mathbb{R}^N$. Define

$$(A^h_{nm})_{ij} = M_Y(A^n_{ij}(y) + A^n_{ik}(y)\frac{\partial \chi^n_j}{\partial y_k}) \hspace{1cm} (73)$$

where $M_Y(\cdot) = \frac{1}{|Y|} \int_Y \cdot \, dy$ and $\chi^n_j \in W_{per}(Y)$ are the solutions of the local problem

$$-\nabla_y \cdot (A(y)(\nabla \chi^n_j + e_j)) = 0 \hspace{1cm} (74)$$

Next we present a few important convergence results needed in the smoothing argument developed in the previous Sections.

**Lemma B.1.** Let $f_n, f \in H^{-1}(\Omega)$ with $f_n \rightharpoonup f$ in $H^{-1}(\Omega)$ and let $b^n, b \in L^\infty(\Omega)$, with

$$c|\xi|^2 \leq b^n_{ij}(y)\xi_i\xi_j \leq C|\xi|^2$$

$$c|\xi|^2 \leq b_{ij}(y)\xi_i\xi_j \leq C|\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and

$$b^n \to b \text{ in } L^2(\Omega)$$

Consider $\zeta_n \in H^1_0(\Omega)$ the solution of

$$\int_{\Omega} b^n(x)\nabla \zeta_n \nabla \psi dx = \int_{\Omega} f_n \psi dx$$

for any $\psi \in H^1_0(\Omega)$. Then we have

$$\zeta_n \rightharpoonup \zeta \text{ in } H^1_0(\Omega)$$

and $\zeta$ verifies

$$\int_{\Omega} b(x)\nabla \zeta \nabla \psi dx = \int_{\Omega} f \psi dx \text{ for any } \psi \in H^1_0(\Omega).$$

**Proof.** Immediately can be observed that

$$||\zeta_n||_{H^1_0(\Omega)} \leq C$$

and therefore there exists $\zeta$ such that on a subsequence still denoted by $n$ we have

$$\zeta_n \rightharpoonup \zeta \text{ in } H^1_0(\Omega) \hspace{1cm} (75)$$

For any smooth $\psi \in H^1_0(\Omega)$ easily it can be seen that

$$\int_{\Omega} b^n(x)\nabla \zeta_n \nabla \psi dx \to \int_{\Omega} b(x)\nabla \zeta \nabla \psi dx$$

and this implies the statement of the Lemma. Due to the uniqueness of $\varphi$ one can see that the limit (75) holds on the entire sequence. \qed
Remark B.2. Using similar arguments it can be proved that the results of Lemma B.1 hold true if we replace the Dirichlet boundary conditions with periodic boundary conditions.

Corollary B.3. Let $u^n_\epsilon \in H^1_0(\Omega)$ be the solution of
\[
\begin{cases}
-\nabla \cdot (A_\epsilon^n(x)\nabla u^n_\epsilon) = f & \text{in } \Omega \\
u^n_\epsilon = 0 & \text{on } \partial \Omega
\end{cases}
\]
We then have $u^n_\epsilon \rightharpoonup u_\epsilon$ in $H^1_0(\Omega)$ where $u_\epsilon$ verifies
\[
\begin{cases}
-\nabla \cdot (A_\epsilon(x)\nabla u_\epsilon) = f & \text{in } \Omega \\
u_\epsilon = 0 & \text{on } \partial \Omega
\end{cases}
\]

Proof. Using (72) we have that $A_\epsilon^n(x) \rightharpoonup A_\epsilon(x)$ in $L^2(\Omega)$ and the statement follows immediately from Remark B.2.

Corollary B.4. for $j \in \{1, \ldots, N\}$, let $\chi^n_j \in \mathcal{W}_{\text{per}}(Y)$ be the solution of
\[
-\nabla_y \cdot (A^n_\epsilon(y)(\nabla \chi^n_j + e_j)) = 0
\]
where $\{e_j\}_j$ denotes the canonical basis of $\mathbb{R}^N$. Then we have $\chi^n_j \rightharpoonup \chi_j$ in $W_{\text{per}}(Y)$ where $\chi_j \in \mathcal{W}_{\text{per}}(Y)$ verifies
\[
-\nabla_y \cdot (A(y)(\nabla \chi_j + e_j)) = 0
\]

Proof. From (72) we obtain
\[
\frac{\partial}{\partial y_i} A^n_\epsilon_j(y) \rightharpoonup \frac{\partial}{\partial y_i} A_j(y) \text{ in } (\mathcal{W}_{\text{per}}(Y))^\prime
\]
The statement of the Remark follows then immediately from Remark B.2.

Proposition B.5. Let $v \in [H^1(\Omega)]^N$ be arbitrarily fixed and for every $j \in \{1, \ldots, N\}$, let $\chi_j \in \mathcal{W}_{\text{per}}(Y)$ be defined as in (74), and $\chi^n_j \in \mathcal{W}_{\text{per}}(Y)$, for $j \in \{1, \ldots, N\}$, to be the solutions of (76). Define $h^n(x, \frac{x}{\epsilon}) = \chi^n_j(\frac{x}{\epsilon})v_j$, $h(x, \frac{x}{\epsilon}) = \chi_j(\frac{x}{\epsilon})v_j$, $g^n(x, \frac{x}{\epsilon}) = \chi^n_j(\frac{x}{\epsilon})Q_\epsilon(v_j)$, $g(x, \frac{x}{\epsilon}) = \chi_j(\frac{x}{\epsilon})Q_\epsilon(v_j)$. We have that
1. $g^n \rightharpoonup g$ in $H^1(\Omega)$
2. If $v \in [W^{1,p}(\Omega)]^N$, $p > N$, then $h^n \rightharpoonup h$ in $H^1(\Omega)$
Proof. First note that applying Corollary B.4 to the sequence \( \{ \chi_n^j \} \) we have
\[
\chi_j \xrightarrow{n} \chi_j \text{ in } W_{\text{per}}(Y) \quad (77)
\]
Next we have
\[
\|g^n(x, \frac{x}{\epsilon})\|^2_{H^1(\Omega)} = \int_\Omega \left( \chi_j^n(\frac{x}{\epsilon}) Q_x(v_j) \right)^2 dx + \frac{1}{\epsilon^2} \int_\Omega (\nabla_y \chi_j^n(\frac{x}{\epsilon}) Q_x(v_j))^2 dx + \int_\Omega (\frac{x}{\epsilon} \nabla_x Q_x(v_j))^2 dx \quad (78)
\]
and
\[
\|h^n(x, \frac{x}{\epsilon})\|^2_{H^1(\Omega)} = \int_\Omega \left( \chi_j^n(\frac{x}{\epsilon}) v_j \right)^2 dx + \frac{1}{\epsilon^2} \int_\Omega (\nabla_y \chi_j^n(\frac{x}{\epsilon}) v_j)^2 dx + \int_\Omega (\frac{x}{\epsilon} \nabla_x v_j)^2 dx \quad (79)
\]
For the first convergence in Theorem B.5 we use that
\[
\|\chi_j^n(\frac{x}{\epsilon}) Q_x(v_j)\|_{H^1(\Omega)} \leq C \|\chi_j^n\|_{W_{\text{per}}(Y)} \quad (80)
\]
We can see that (78) imply that
\[
\|g^n(x, \frac{x}{\epsilon}) - g(x, \frac{x}{\epsilon})\|^2_{L^2(\Omega)} = \int_\Omega \left( \chi_j^n(\frac{x}{\epsilon}) - \chi_j(\frac{x}{\epsilon}) \right)^2 (Q_x(v_j))^2 dx
\]
and using (80) we obtain the desired result.
For the second convergence result in Theorem B.5 we will recall now a very important inequality (see [15], Chp. 2) to be used for our estimates. For any \( p > N \) we have
\[
\|\phi\|_{L^\frac{2p}{p-2}(\Omega)} \leq c(p)(\|\phi\|_{L^2(\Omega)} + \|\nabla \phi\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} ^{1-\frac{2}{p}}) \quad (81)
\]
for any \( \phi \in H^1(\Omega) \) and where \( c(p) \) is a constant which depends only on \( q, N, \Omega \). Then, for \( v \in [W^{1,p}(\Omega)]^N \) with \( p > N \), using (77), the Sobolev embedding \( W^{1,p}(\Omega) \subset L^\infty(\Omega) \) and (81) in (79) we obtain
\[
\|h^n(x, \frac{x}{\epsilon})\|^2_{H^1(\Omega)} < C
\]
where the constant \( C \) above does not depend on \( n \).
Next we can easily observe that
\[
\|h^n(x, \frac{x}{\epsilon}) - h(x, \frac{x}{\epsilon})\|^2_{L^2(\Omega)} = \int_\Omega \left( \chi_j^n(\frac{x}{\epsilon}) - \chi_j(\frac{x}{\epsilon}) \right)^2 (v_j)^2 dx
\]
and in either of the above cases, (77) and a few simple manipulations imply that
\[
h^n(x, \frac{x}{\epsilon}) \xrightarrow{n} h(x, \frac{x}{\epsilon}) \text{ in } L^2(\Omega)
\]
This together with the bound on the sequence \( \{h^n(x, \frac{x}{\epsilon})\} \) implies the statement of the Corollary. \( \square \)
The two convergence results in the next Corollary will follow immediately from Proposition B.4.

**Corollary B.6.** Let 
\[ w_n^1(x, \frac{x}{\epsilon}) = \chi_n^j(x) \partial u_0 \partial x_j \]
and 
\[ u_n^1(x, \frac{x}{\epsilon}) = \chi_n^j(x) Q_x(\partial u_0 \partial x_j). \]

Then we have
1. If \( u_0 \in W^{3,p}(\Omega) \) for \( p > N \),
   \[ w_n^1 \rightharpoonup w_1 \text{ in } H^1(\Omega) \]
2. If \( u_0 \in H^2(\Omega) \),
   \[ u_n^1 \rightharpoonup u_1 \text{ in } H^1(\Omega) \]

**Corollary B.7.** Let \( \theta^n_\epsilon \) be the solution of
\[ - \nabla \cdot (A^n(x, \frac{x}{\epsilon}) \nabla \theta^n_\epsilon) = 0 \text{ in } \Omega, \theta^n_\epsilon = w_1^1(x, \frac{x}{\epsilon}) \text{ on } \partial \Omega \quad (82) \]
and \( \beta^n_\epsilon \) be the solution of
\[ - \nabla \cdot (A^n(x, \frac{x}{\epsilon}) \nabla \beta^n_\epsilon) = 0 \text{ in } \Omega, \beta^n_\epsilon = u_1^1(x, \frac{x}{\epsilon}) \text{ on } \partial \Omega \quad (83) \]

We have that
(i) if \( u_0 \in W^{3,p}(\Omega), p > N \), then
   \[ \theta^n_\epsilon \rightharpoonup \theta_\epsilon \text{ in } H^1(\Omega) \]
(ii) if \( u_0 \in H^2(\Omega) \), then
   \[ \beta^n_\epsilon \rightharpoonup \beta_\epsilon \text{ in } H^1(\Omega) \]

where \( \theta_\epsilon \) and \( \beta_\epsilon \) satisfies
\[ - \nabla \cdot (A(x, \frac{x}{\epsilon}) \nabla \theta_\epsilon) = 0 \text{ in } \Omega, \theta_\epsilon = w_1(x, \frac{x}{\epsilon}) \text{ on } \partial \Omega \quad (84) \]
and
\[ - \nabla \cdot (A(x, \frac{x}{\epsilon}) \nabla \beta_\epsilon) = 0 \text{ in } \Omega, \beta_\epsilon = u_1(x, \frac{x}{\epsilon}) \text{ on } \partial \Omega \quad (85) \]

**Proof.** Using Corollary B.6 and a few simple arguments one can simply show that
\[ - \nabla \cdot (A(x, \frac{x}{\epsilon}) \nabla w_1^1(x, \frac{x}{\epsilon})) \rightharpoonup - \nabla \cdot (A(x, \frac{x}{\epsilon}) \nabla w_1(x, \frac{x}{\epsilon})) \text{ in } H^{-1}(\Omega) \]
and
\[ - \nabla \cdot (A(x, \frac{x}{\epsilon}) \nabla u_1^1(x, \frac{x}{\epsilon})) \rightharpoonup - \nabla \cdot (A(x, \frac{x}{\epsilon}) \nabla u_1(x, \frac{x}{\epsilon})) \text{ in } H^{-1}(\Omega) \]

Homogenizing the data in the problems (82) and (83) and using Corollary B.6 and Lemma B.1 the statement follows immediately. 
\[ \square \]
Corollary B.8. For any $i,j \in \{1,\ldots,N\}$ let $\chi^n_{ij} \in W_{per}(Y)$ be the solutions of:
\[
\nabla_y \cdot (A^n \nabla_y \chi^n_{ij}) = b^n_{ij} - M_Y(b^n_{ij})
\]
(86)
where
\[
b^n_{ij} = -A^{ij} - A^n_{ik} \frac{\partial \chi^n_j}{\partial y_k} - \frac{\partial}{\partial y_k} (A^n_{ik} \chi^n_j)
\]
and $M_Y(.)$ is the average on $Y$.
Then we have
\[
\chi^n_{ij} \rightharpoonup \chi_{ij} \text{ in } W_{per}(Y) \text{ for any } i,j \in \{1,\ldots,N\}
\]
where $\chi_{ij}$ satisfies
\[
\int_Y A(y) \nabla_y \chi_{ij} \nabla_y \psi dy = (b_{ij} - M_Y(b_{ij}), \psi)_{(W_{per}(Y)', W_{per}(Y))}
\]
(87)
for any $\psi \in W_{per}(Y)$ and with
\[
b_{ij} = -A^{ij} - A_{ik} \frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k} (A_{ik} \chi_j).
\]
Proof. For any $\psi \in W_{per}(Y)$, we have that,
\[
\int_Y (b^n_{ij} - M_Y(b^n_{ij}) \psi dy = \int_Y (-A^{ij} - A^n_{ik} \frac{\partial \chi^n_j}{\partial y_k}) \psi dy + (A^{hom}_n)_{ij} \int_Y \psi dy + \int_Y A^n_{ki} \chi^n_j \frac{\partial \psi}{\partial y_k} dy
\]
(88)
where we have used that $M_Y(b^n_{ij}) = -(A^{hom}_n)_{ij}$ (see [18]).
Using (72), (77), and simple manipulations we can prove that
\[
A^n_{ik} \frac{\partial \chi^n_j}{\partial y_k} \rightarrow A_{ik} \frac{\partial \chi_j}{\partial y_k} \text{ in } L^2(Y)
\]
(89)
and
\[
A^n_{ik} \chi^n_j \rightarrow A_{ik} \chi_j \text{ in } L^2(Y)
\]
(90)
From (58), (72) and (88) we have that
\[
(A^{hom}_n)_{ij} \rightarrow A_{ij}^{hom}
\]
(91)
Finally using (72), (77), (58) and (90) in (88) we obtain that
\[
b^n_{ij} - M_Y(b^n_{ij}) \rightharpoonup b_{ij} - M_Y(b_{ij}) \text{ in } (W_{per}(Y))'
\]
This and Remark B.2 complete the proof of the statement.

Remark B.9. We can easily observe that we have
\[
A^n_{ij} \chi^n_{ij} \rightharpoonup A_{ij} \chi_{ij}, A^n_{ij} \frac{\partial \chi^n_{ij}}{\partial y_k} \rightharpoonup A_{ij} \frac{\partial \chi_{ij}}{\partial y_k} \text{ weakly in } W_{per}(Y)
\]
References

[1] G. Allaire and M. Amar, *Boundary layer tails in periodic homogenization*. ESAIM: Control, Optimization and Calc. of Variations, May, Vol. 4 (1999), 209-243.

[2] M. Avellaneda and F.-H. Lin, *Homogenization of elliptic problems with Lp boundary data*. Appl. Math. Optim., 15 (1987), 93-107.

[3] Bakhvalov N.S., Panasenko G.P. *Homogenization: Averaging processes in periodic media*. Nauka, Moscow, 1984 (in Russian); English transl., Kluwer, Dordrecht/Boston/London, 1989.

[4] A. Bensoussan, J.-L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*. North Holland, Amsterdam, 1978.

[5] D. Cioranescu, A. Damlamian, G. Griso, *Periodic unfolding and homogenization*. C. R. Acad. Sci. Paris, Ser. I 335 (2002), 99-104.

[6] D. Cioranescu and P. Donato, *An Introduction to Homogenization*. Oxford University Press, 1999.

[7] A. Damlamian, *An elementary introduction to periodic unfolding*. Proc. of the Narvik Conference 2004, A. Damlamian, D. Lukkassen, A. Meidell, A. Piatnitski editors, Gakuto Int. Series, Math. Sci. App. vol. 24, Gakkotosho (2006), 119–136.

[8] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[9] V. Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations* Springer-Verlag, 1986.

[10] G. Griso, *Error estimate and unfolding for periodic homogenization*. Asymptotic Analysis, 40 (2004), 269-286.

[11] G. Griso, *Interior Error Estimate for Periodic Homogenization*. Analysis and Applications, 4, 1 (2006), 61-80.

[12] P. Grisvard, *Elliptic problems in nonsmooth domains*. London, Pitman, 1985.

[13] T.Y. Hou and X.H. Wu, *A multi-scale finite element method for elliptic problems in composite materials and porous media*. J. of Comp. Phys., 134 (1997), 169-189.

[14] V. V. Jikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag, Berlin, 1994.

[15] O. A. Ladyzhenskaya and N. N. Ural’tseva, *Linear and quasilinear elliptic equations*. Academic Press, New York, 1968.
[16] Y. Y. Li and M. Vogelius, *Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients*. Arch. rational mech. anal., 153 (2000), 91-151.

[17] J.L. Lions, *Some methods in the mathematical analysis of systems and their controls*. Science Press, Beijing, Gordon and Breach, New York 1981.

[18] S. Moskow and M. Vogelius, *First-order corrections to the homogenised eigenvalues of a periodic composite medium. A convergence proof*, Proceedings of the Royal Society of Edinburgh, 127A (1997), 1263-1299.

[19] J. E. Osborn, *Spectral Approximation for Compact Operators*. Mathematics of Computation, vol. 29, No. 131 (1975), 712-725.

[20] D. Onofrei and B. Vernescu, *Error estimates in periodic homogenization with non-smooth coefficients*. Asymptotic Analysis, 54 (2007), 103-123.

[21] E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*. Springer Verlag, Berlin 1980.

[22] Sun-Sig Byun, *Elliptic Equations with BMO coefficients in Lipschitz domains*. Tran. of AMS, vol. 357, Number 3 (2004), 1025-1046.

[23] E. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1971.

[24] L. Tartar, *Cours Peccot au College de France*, 1977.

[25] H. Versieux and M. Sarkis, *Convergence analysis for the numerical boundary corrector for elliptic equations with rapidly oscillating coefficients* SIAM J. Num. An., 46, (2008), 545-576.

[26] H.F. Weinberger, *Variational Methods for Eigenvalue Approximation*. SIAM, Philadelphia, 1974.