LOCAL NEWFORMS FOR GENERIC REPRESENTATIONS OF UNRAMIFIED $U_{2n+1}$ AND RANKIN-SELBERG INTEGRALS

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Abstract. Recently Atobe-Oi-Yasuda established the newform theory for irreducible tempered generic representations of unramified $U_{2n+1}$ over non-archimedean local fields. In this paper we extend their result to every irreducible generic representations and compute the dimensions of the spaces of oldforms. We also compute the Rankin-Selberg integrals attached to newforms and oldforms under a natural assumption on the $\gamma$-factors defined by the Rankin-Selberg integrals.

1. Introduction

Newforms have their roots in the classical theory of Atkin-Lehner for modular forms. In that setting, newforms are cusp forms which are simultaneously eigenfunctions of all Hecke operators. As a consequence, their Fourier coefficients satisfy strong recurrence relations, and their $L$-functions are well behaved. On the other hand, oldforms are cusp forms originated from newforms of lower level via certain level raising procedures. Probably the first definition and attempt to study newforms in the local $p$-adic setting dates back to Casselman’s work ([Cas73]) on $GL_2$. His result singles out, in every irreducible generic (complex) representation of $GL_2$, a unique (up to scalars) non-zero vector. These vectors are called local newforms as they appeared as local components of adelizations of modular newforms. Consequently, local newforms can be used to study automorphic forms on $GL_2$ as well as their applications. They also play indispensable roles for the ramification theory of representations of $p$-adic $GL_2$.

Casselman’s result was subsequently extended to irreducible generic representations of $p$-adic $GL_r$ by Jacquet-Piatetski-Shapiro-Shalika ([JPSS81], see also [Jac12], [Mat13]). Their method was based on the Rankin-Selberg integrals constructed in [JPSS83]. Then built on the result of Jacquet et al, Reeder ([Ree91]) studied oldforms for generic representations of $GL_r$; he showed that oldforms can be produced from a newform via level raising operators and obtained the dimensions of the spaces of oldforms. Recently, Humphries gave alternative characterizations of local newforms of generic representations of $GL_r$ in terms of $K$-types ([Hum22]); Atobe-Kondo-Yasuda established the theory of local newforms for every irreducible representation of $p$-adic $GL_r$ in [AKY21].

For other $p$-adic classical groups, Roberts-Schmidt developed the theory of local newforms and oldforms for what they called paramodular representations of $PGSp_4$ ([RS07]). These include all irreducible generic representations and some of non-generic ones. For generic representations, they in addition computed the zeta (cf. [Nov79]) integrals attached to newforms and oldforms (see also [Che22]). By adopting the arguments in [RS07], Miyachi obtained the theory of local newforms and oldforms for generic representations of unramified $U_{2,1}$ in the series of papers ([Miy13b], [Miy13a], [Miy13c], [Miy18]), in which he also computed the zeta integrals (cf. [GPS83]) attached to newforms.

1.1. Results of Atobe-Oi-Yasuda. In a recent preprint [AOY22], Atobe-Oi-Yasuda extended Miyachi’s result and established the theory of local newforms for irreducible tempered generic representations of unramified $U_{2n+1}$ over non-archimedean local fields. To state their results, let $F$ be a finite field extension of $\mathbb{Q}_p$ with $p > 2$ and $E$ be the unramified quadratic field extension of $F$. Denote by $x \mapsto \bar{x}$ the action of the non-trivial element in the Galois group $Gal(E/F)$ on $E$. This action and its notation extend naturally to matrices with entries in $E$. Let $\mathfrak{O}_F$ (resp. $\mathfrak{o}_F$) be the valuation ring of $F$ (resp. $E$), $\mathfrak{p}_F$ (resp. $\mathfrak{p}_E$) be its maximal ideal and $q_F = |\mathfrak{O}_F/\mathfrak{p}_F|$ (resp. $q_E = |\mathfrak{o}_E/\mathfrak{p}_E| = q^2$). Fix an element $\delta \in \mathfrak{O}_E^*$ with $\bar{\delta} = -\delta$ and an additive character $\psi_F$ of $F$.
that is trivial on $\mathfrak{o}_E$ but not on $p_E^{-1}$. Put $\psi_E(x) = \psi_F\left(\frac{x - x_E}{2E}\right)$ for $x \in E$. Then $\psi_E$ defines an additive character of $E$ which is trivial on $F$ and $\mathfrak{o}_E$ but not on $p_E^{-1}$.

Let $J_N \in \text{GL}_N(E)$ be the element defined inductively by

$$J_1 = (1) \quad \text{and} \quad J_N = \begin{pmatrix} J_{N-1} \\
1 \end{pmatrix}.$$ 

Then one has the unitary group $U_{2n+1}(F) \subset \text{GL}_{2n+1}(E)$ defined by

$$U_{2n+1}(F) = \{ g \in \text{GL}_{2n+1}(E) \mid \tilde{g} J_{2n+1} g = J_{2n+1}\}.$$ 

For an integer $m \geq 0$, let $K_{n,m} \subset U_{2n+1}(F)$ be the open compact subgroup given by

$$K_{n,m} = 1 + \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & p_E^{-m} \\
\mathfrak{p}_E^n & 1 + \mathfrak{p}_E^m & \mathfrak{o}_E \\
\mathfrak{p}_E^n & \mathfrak{p}_E^m & \mathfrak{o}_E \end{pmatrix} \cap U_{2n+1}(F).$$

These $K_{n,m}$ are the open compact subgroups considered in [AOY22]. Notice that the matrices used to defined the unitary groups in this paper and in [AOY22] are different. However, the associated unitary groups are conjugate by a diagonal element in $\text{GL}_{2n+1}(\mathfrak{o}_E)$; hence $K_{n,m}$ remains unchanged for all $m$.

Let $\pi$ be an irreducible complex representation of $U_{2n+1}(F)$. Then by results of Mok ([Mok15]) and Gan-Gross-Prasad ([GGP12] Theorem 8.1), one can attached to $\pi$ an $(2n + 1)$-dimensional conjugate-orthogonal complex representation $\phi_{\pi}$ of the Weil-Deligne group of $E$. Let $\epsilon(s, \phi_{\pi}, \psi_E)$ denotes the $\epsilon$-factor attached to $\phi_{\pi}$ and $\psi_E$ (cf. [1579]). Then since $\psi_E$ is unramified and trivial on $F$, it can be written as

$$\epsilon(s, \phi_{\pi}, \psi_E) = q_E^{-a_{\pi}(s - 1/2)}$$

for some integer $a_{\pi} \geq 0$ (cf. [GGP12] Proposition 5.2 (2)). Now their results can be stated as follows.

**Theorem 1.1** (Atobe-Oi-Yasuda). Let $(\pi, \psi_{\pi})$ be an irreducible tempered representation of $U_{2n+1}(F)$.

1. If $\pi$ is non-generic, then $\psi_{\pi}^{K_{n,m}} = 0$ for all $m$.

2. If $\pi$ is generic, then

$$\dim_{\mathbb{C}} \psi_{\pi}^{K_{n,m}} = \begin{cases} 0 & \text{if } m < a_{\pi}, \\
1 & \text{if } m = a_{\pi}. \end{cases}$$

Moreover, if $m > a_{\pi}$, then $\psi_{\pi}^{K_{n,m}} \neq 0$.

Let us briefly point out the key ingredients in their ingenious proof. To prove Theorem 1.1 (1), they applied the local Gan-Gross-Prasad conjecture ([GGP12]) established by Beuzart-Plessis ([BPT4], [BPT5], [BPT6]) and the following property of $K_{n,m}$. If we put

$$R_{n,m} = U_{2n}(F) \cap K_{n,m}$$

then $R_{n,m}$ is a hyperspecial maximal compact subgroup of $U_{2n}(F)$, up to conjugate an element in the diagonal torus of $U_{2n}(F)$. Here we identify the unramified $U_{2n}(F)$ with a subgroup of $U_{2n+1}(F)$ fixing an anisotropic vector. To prove Theorem 1.1 (2), they applied the endoscopy character relation established by Mok ([Mok15]) as well as the theory of local newforms established by Jacquet et al ([JPSSS1]). For this, they proved an analogue of the fundamental lemma for the open compact subgroups $K_{n,m}$.

**1.2. Main results of this paper.** In this paper we further develop the theory of local newforms of unramified $U_{2n+1}$ initiated by Atobe-Oi-Yasuda. Our goal is twofold. First, we extend their results from tempered generic representations to every generic representations and compute the dimensions of the spaces of oldforms. Second, we compute the Rankin-Selberg integrals (cf. [BAS09]) attached to newforms and oldforms under a natural assumption on the $\gamma$-factors defined by the Rankin-Selberg integrals.
1.2.1. Dimension formula. Our first result is the following dimension formula for every generic representations.

**Theorem 1.2.** Let \((\pi, \mathcal{V}_\pi)\) be an irreducible generic representation of \(U_{2n+1}(F)\). Then we have\(^1\)

\[
\dim_{\mathbb{C}} \mathcal{V}_{\pi}^{K_{n,m}} = \left( \frac{m-a_\pi}{2} \right) + n
\]

for every integer \(m \geq 0\).

Let us make a few remarks:

1. Following the usual convention, non-zero elements in the line \(\mathcal{V}_{\pi}^{K_{n,m}}\) are called \textit{newforms} of \(\pi\); elements in \(\mathcal{V}_{\pi}^{K_{n,m}}\) for \(m > a_\pi\) are called \textit{oldforms} of \(\pi\).

2. The proof of Theorem 1.2 consists of two steps. The first step is to obtain \((\mathbf{1.2})\) when \(\pi\) is tempered, which follow from the results of Reeder ([Ree91 Theorem 1]) and Atobe-Oi-Yasada ([AOY22 Theorem 1.2]). The next step is to extend \((\mathbf{1.2})\) to non-tempered \(\pi\); such \(\pi\) is isomorphic to a full induced representation of \(U_{2n+1}(F)\) by the standard module conjecture (cf. [CS98, Mui01]). Therefore, our task is to compute the double coset decompositions of \(U_{2n+1}(F)\) by maximal parabolic subgroups and \(K_{n,m}\). Similar computations already appeared in the works of Roberts-Schmidt ([RS07 Chapter 5]) and Miyachi ([Miy13c]); our approach mimics theirs.

3. Equation \((\mathbf{1.2})\) is compatible with results of Miyauchi in [Miy13b] and [Miy13c]. Consequently, by combining Theorem 1.2 with the results of Miyauchi in [Miy13a Theorem 4.3] and [Miy18], one sees immediately that the \(\varepsilon\)-factors defined by the Rankin-Selberg integrals and the associated Weil-Deligne representation agree.

4. Finally, we point out that similar reduction procedures can be used to reduce the newform conjecture for \(p\)-adic \(SO_{2n+1}\) proposed by Gross ([Gro15]) and Tsai ([Tsa13, Tsa16]) from generic representations to tempered generic representations.

1.2.2. Rankin-Selberg integrals. As mentioned in [AOY22], their proof does not involve Rankin-Selberg integrals; hence the relation between Rankin-Selberg integrals and newforms remains unclear. On the other hand, from results in the literature, newforms usually serve as "test vectors" for certain Rankin-Selberg integrals, in the sense that if one computes the Rankin-Selberg integral attached to a newform, then one obtains the \(L\)-factor attached to the generic representation (cf. [JPSS81, RS07, Miy18, Che22]). In view of these, another goal of this paper is to investigate this relation (under a natural assumption). The Rankin-Selberg integrals considered here are those attached to generic representations of \(U_{2n+1} \times \text{Res}_{E/F} GL_{r}\) where \(\text{Res}_{E/F}\) stands for the Weil restriction from \(E\) to \(F\). These local integrals are indeed coming from their global counterparts, which were first studied by Tamir in [Tam91] when \(r = n\), following the idea of Gelbart and Piatetski-Shapiro ([GPSR87 Part B]). His results were later extended to the case \(r < n\) by Ben-Artzi and Soudry in [BAS09], and to the case \(r > n\) by Ginzburg-Rallis-Soudry in [GRS11], whose local integrals were further studied by Morimoto-Soudry in [MS20].

To describe our second result, let \((\tau, \mathcal{V}_\tau)\) be an unramified representation of \(\text{Res}_{E/F} GL_{r}(F) = GL_{r}(E)\) that is an induced representation of \(Langlands type\). This includes all (classes of) irreducible unramified generic representations of \(GL_{r}(E)\), but also contains reducible ones. Moreover, it has the following features:

(i) the space \(\mathcal{V}_{\tau}^{GL_{r}(E)}\) is one-dimensional; (ii) \(\tau\) admits a unique non-zero Whittaker functional \(\Lambda_{\tau,\psi}\) which is non-trivial on \(\mathcal{V}_{\tau}^{GL_{r}(E)}\); (iii) the unique irreducible quotient \(J(\tau)\) of \(\tau\) is unramified. Because of (i) and (ii), we can fix a basis \(v_\tau\) of \(\mathcal{V}_{\tau}^{GL_{r}(E)}\) with \(\Lambda_{\tau,\psi}(v_\tau) = 1\). Now let \(s\) be a complex number. Then by inflating the representation \(\tau \otimes \text{det}_{F}^{s-1/2}\) to a parabolic subgroup of unramified \(U_{2r}(F)\) of Siegel type, we get a normalized induced representation \(\rho_{\tau,s}\) of \(U_{2r}(F)\), whose underlying space \(I_{1}(\tau,s)\) consisting of smooth \(\psi_{\tau}\)-valued functions on \(U_{2r}(F)\) satisfying the usual rule. The Rankin-Selberg integral \(\Psi_{\tau,s}(v \otimes \xi_{s})\) is attached to \(v \in \mathcal{V}_{\tau}\) and \(\xi_{s} \in I_{1}(\tau,s)\). By applying the normalized intertwining operator on \(I_{1}(\tau,s)\) together with the \(r\)-factor one result of certain Hom space, one gets the \(\gamma\)-factor \(\gamma_{d}(s,\pi \tau,\psi_{F})\) depending on the choice of \(d\) and \(\psi_{F}\) defined by these local integrals. In this paper, we assume the following.

\(^{1}\)Here we understand that \(\binom{a}{b} = 0\) when \(a < b\).
Assumption 1.3. Suppose that $1 \leq r \leq n$. Then
\[
\gamma(\tau) = \gamma(s, \phi, \tau, \psi) = \gamma(s, \phi, \phi, \tau, \psi)
\]
where $\phi, \phi$ is the $L$-parameter of $J(\tau)$ under the local Langlands correspondence of $GL_r(E)$.

Assume $1 \leq r \leq n$ from now on. Then $U_2(F)$ can be regarded as a subgroup of $U_{2n+1}(F)$ via a natural embedding. For each $m \geq 0$, let us put $R_{r,m} = U_2(F) \cap K_{n,m}$. Then $R_{r,m}$ satisfies the similar property as $R_{n,m}$ mentioned in the previous subsection. In particular, the space $I_r(\tau,s)_{R_{r,m}}$ is one-dimensional and admits a generator $\xi^{m}_{\tau,s}$ with $\xi^{m}_{\tau,s}(I_{2r}) = v_{\tau}$. Now our second result can be stated as follow.

Theorem 1.4. Let $(\pi, V_{\pi})$ be an irreducible generic representation of $U_{2n+1}(F)$ and fix a non-zero Whittaker functional $\Lambda_{\pi, \psi_E}$ on $V_{\pi}$ as well as a basis $v_{\pi}$ of $V_{\pi}^{K_{n,n}}$. Then under the Assumption 1.3,
\[
\Psi_{n,r}(v_{\pi} \otimes \xi^{m}_{\pi}) = \frac{L(s, \phi, \phi, \tau, \psi)}{L(2s, \phi, \phi, \tau, \psi)} \cdot \Lambda_{\pi, \psi_E}(v_{\pi})
\]
provided that the Haar measures are suitably chosen (cf. [7.7]). Here $\Lambda_{\pi, \psi_E}$ stands for the Asai representation (cf. [GRS11], p.20-p.21). Moreover, if $\pi$ is tempered, then $\Lambda_{\pi, \psi_E}(v_{\pi})$ is non-zero.

Again, let us make a few remarks:

1. Here we only compute the Rankin-Selberg integrals when $\tau$ is unramified because if $\tau$ is irreducible and generic, but not unramified, then we can show that (without the Assumption 1.3) $\Psi_{n,r}(v \otimes \xi_{\pi}) = 0$ for every $v \in V_{\pi}^{K_{n,n}}$ and $\xi_{\pi} \in I_{r}(\tau,s)$ (cf. [Che22] Lemma 4.3).

2. The proof of (1.3) is resemblance to that of [Che22] Equation (1.7)], due to the similar nature and technical closeness of the constructions. A more interesting but difficult question is to show that $\Lambda_{\pi, \psi_E}(v_{\pi})$ is non-zero. For this, one way is to prove that the associated Whittaker functions of newforms or non-zero oldforms are non-trivial on the diagonal. In the literature, this is usually done by using the $P_{n+1}$-theory, where $P_{n+1}$ is the mirobolic subgroup of $GL_{n+1}$ (cf. [LPSS1], [RS07], [Miy13], [Tsa13]). Here we take a different approach which utilizes the Gan-Gross-Prasad conjecture as mentioned before. In fact, this is inspired by the proof of [AOY22] Theorem 1.1 (1)] which is also the reason for the temperedness assumption. However, the advantage of our proof lies in its simplicity.

3. In this paper, we also compute the Rankin-Selberg integrals attached to oldforms under the Assumption 1.3. Notice that the formula (1.2) is obtained via an abstract method; hence is not suitable for the computations. In order to compute these integrals, we construct conjectural bases of oldforms via level raising operators coming from elements in spherical Hecke algebras of $U_{2n}(F)$ in the same spirit of [Ree91] and [Che22]. It shall be pointed out that we need explicitly constructed oldforms to compute the attached Rankin-Selberg integrals, but at the same times, we also need to compute the attached Rankin-Selberg integrals to make sure that our constructions give non-zero oldforms. When $\tau$ is tempered, we verify that these indeed define bases of oldforms (under the Assumption 1.3).

4. The Assumption 1.3 should be verified through standard approaches. Namely, the Rankin-Selberg $\gamma$-factors should be multiplicative. This allows one to connect the Rankin-Selberg $\gamma$-factors to the ones defined by the Langlands-Shahidi method (cf. [Sha81], [Sha90]). Next, by combining results and arguments in [CPSS1], [CKPSS1] and [MT02], it should be able to further relate Shahidi’s $\gamma$-factors with those defined by the associated Weil-Deligne representations. In general, these two $\gamma$-factors might be equal up to an exponential, but we expect that they are actually equal in our settings.

1.3. Organization of the paper. This paper is organized as follows. In §2 we investigate some properties of the open compact subgroups $K_{n,m}$ which will be used later. In §3 we obtain double coset decompositions of $U_{2n+1}(F)$ by maximal parabolic subgroups and $K_{n,m}$. Then we prove Theorem 1.2 in §4. In §5 we introduce conjecture bases for the spaces of oldforms. The Rankin-Selberg integrals are reviewed in §6. We then compute these integrals attached to newforms and oldforms in §7. As a result, Theorem 1.4 is proved. Finally, in the Appendix, we prove a multiplicity one result for certain Hom spaces, which allows one to define Rankin-Selberg $\gamma$-factors attached to reducible representations.

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2We slightly abuse the notation to regard $\phi_{\gamma}$ as the $L$-parameter of the representation $J(\gamma)$ of the $F$-group $Res_{E/F} GL_1(F)$.
1.4. **Notation and conventions.** Let $F$ be a finite field extension of $\mathbb{Q}_p$ with $p > 2$ and $E$ be the unramified quadratic field extension of $F$. Denote by $x \mapsto \bar{x}$ the action of the non-trivial element in the Galois group $\text{Gal}(E/F)$ on $E$. This action and its notation extend naturally to matrices with entries in $E$. Let $E^1 = \{ x \in E \mid x\bar{x} = 1 \}$ be the group of norm one elements in $E$. Let $\mathfrak{o}_F$ (resp. $\mathfrak{o}_E$) be the valuation ring of $F$ (resp. $E$) and $\mathfrak{p}_F$ (resp. $\mathfrak{p}_E$) be its maximal ideal. Fix a generator $\varpi$ of $\mathfrak{p}_F$ which is also a generator of $\mathfrak{p}_E$. Denote by $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ (resp. $k_E = \mathfrak{o}_E/\mathfrak{p}_E$) the residue field of $F$ (resp. $E$) and $q_F = q$ (resp. $q_E$) the cardinality of $k_F$ (resp. $k_E$). Then we have $q_E = q^2$. Let $| \cdot |_E$ be the absolute value on $E$ with $| \varpi |_E = q_E^{-1}$. Fix an element $\delta \in \mathfrak{o}_E^*$ with $\delta = -\delta$. Fix an unramified additive character $\psi_F$ of $F$, namely, $\psi_F$ is an additive character of $F$ that is trivial on $\mathfrak{o}_F$ but not on $\mathfrak{p}_F^{-1}$. Put $\psi_E(x) = \psi_F(\frac{x\varpi}{2q})$ for $x \in E$. Then $\psi_E$ defines an unramified additive character of $E$ which is also trivial on $F$. Let $J_N \in \text{GL}_N(E)$ be the element defined inductively by

$$J_1 = (1) \quad \text{and} \quad J_N = \begin{pmatrix} J_{N-1} \\ 1 \end{pmatrix}$$

and $I_N$ be the identity matrix in $\text{GL}_N(E)$. If $a \in \text{GL}_N(E)$, we denote $a^* = J_N^{-1}aJ_N^{-1}$.

Suppose that $G$ is an $\ell$-group in the sense of [BZ76, Section 1]. We denote by $\delta_G$ the modular function of $G$. If $K \subset G$ is an open compact subgroup, then $\mathcal{H}(G/K)$ denotes the convolution algebra which consists of smooth compact supported bi-$K$-invariant $\mathbb{C}$-valued functions on $G$. In this work, by a representation of $G$ we mean a smooth complex representation with finite length. If $\pi$ is a representation of $G$, then its underlying (abstract) space is usually denoted by $V_\pi$. Finally, if $S_0$ is a subset of a set $S$, we denote by $\bar{S}_0$ the characteristic function of $S_0$.

2. Unitary groups and their open compact subgroups

In this section, we set up the unitary groups and their parametrization. We then introduce the open compact subgroups used to define newforms and investigate some of their properties. The goal is to obtain decompositions of these open compact subgroups (cf. Lemma 2.2), which are important in the sequel.

2.1. **Unitary groups.** Let $V_N$ be an $N$-dimensional vector space over $E$ equipped with a non-degenerated Hermitian form

$$\langle \cdot, \cdot \rangle : V_N \times V_N \to E,$$

which is $E$-linear in the first variable and satisfying $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V_N$. We assume that $V_N$ admits an order basis

$$e_1, \ldots, e_n, v_0, f_0, \ldots, f_1$$

whose associated Gram matrix is $J_N$. Here $n = \lceil \frac{N}{2} \rceil$ and we do not have $v_0$ when $N$ is even. Let $U(V_N)$ be the unitary group of $V_N$, i.e. the connective reductive linear algebra group over $F$ defined by

$$U(V_N) = \{ g \in \text{GL}(V_N) \mid \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V_N \}.$$ 

Notice that $U(V_N)$ is unramified over $F$ by our assumption on the ordered basis. This ordered basis also allows us to realize $U(V_N)$ as a subgroup $U_N(F)$ of $\text{GL}_N(E)$ given by

$$U_N(F) = \{ g \in \text{GL}_N(E) \mid 1^t \bar{g} J_N g = J_N \}.$$ 

In particular, we have $U_1(F) = E^1$ and it can be identified as the center of $U_N(F)$ through the embedding

$$x \mapsto xJ_N.$$ 

To distinguish the parity of $N$, we also denote $G_n = U_{2n+1}(F)$ and $H_r = U_{2r}(F)$ with $n \geq 0$ and $r > 0$ in the followings.
2.2. Subgroups and embeddings. We identify various subgroups of $U(V_{2n+1})$ with lower rank unitary groups and general linear groups via embeddings. More specifically, if $0 \leq n_0 \leq n$ is an integer, we embed $U(V_{2n_0+1})$ into $U(V_{2n_1+1})$ as the subgroup fixing $e_1, \ldots, e_{n-n_0}, f_{n-n_0}, \ldots, f_1$. In coordinates, we have

$$G_{n_0} \ni g_0 \mapsto \begin{pmatrix} I_{n-n_0} & g_0 & I_{n-n_0} \\ g_0 & I_{n-n_0} \\ I_{n-n_0} \end{pmatrix} \in G_n.$$ 

On the other hand, if $1 \leq r \leq n$, then we identify $U(V_{2r})$ as the subgroup fixing $e_{r+1}, \ldots, e_n, v_0, f_n, \ldots, f_{r+1}$. In coordinates, we have

$$H_r \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n$$

for some $a, b, c, d \in \text{Mat}_{r \times r}(E)$. The general linear group $GL(V_r)$ embeds as the subgroup of $U(V_{2n+1})$ stabilizing the subspace spanned by $e_1, \ldots, e_r$ which in addition fixing the vectors $e_{r+1}, \ldots, e_n, v_0, f_n, \ldots, f_{r+1}$. In coordinates, we have

$$GL_r(E) \ni a \mapsto \hat{a} := \begin{pmatrix} a & I_{2(n-r)+1} \\ 0 & a^* \end{pmatrix} \in G_n$$

where $a^* := J_r a^t J_r$. More generally, for a subset $S$ of $GL_r(E)$, we denote $\hat{S} = \{ \hat{a} \mid a \in S \}$. In this paper, we do not distinguish the these groups with their images in $G_n$. These shall not cause serious confusions. In particular, we see that $GL_r(E)$ is indeed contained in $H_r$. Let $A_r \subset GL_r(E)$ be the diagonal tours and $T_r = \hat{A}_r$. Then $T_r$ is the maximal diagonal tours of $H_r$ and the maximal diagonal tours of $G_n$ is $E^1 \cdot T_n$.

Finally, let $B_r \subset G_n$ be the upper triangular Borel subgroup and $N_r \subset B_r$ be its unipotent radical.

2.2.1. Root subgroups. Let $n \geq 1$ and put $N = 2n + 1$. We introduce certain root subgroups of $G_n$ that will be used in the later computations. Let $i$ be an integer between 1 and $N$, we define $i^* = N + 1 - i$. Then $1 \leq i^* \leq N$ and $(i^*)^* = i$. For integers $1 \leq i, j \leq N$, let $E_{ij}$ be the $N$-by-$N$ matrix whose $(i,j)$-entry is 1 while all other entries are 0. Let $1 \leq i < j \leq n$, $1 \leq k \leq n$ and $y \in E$. We define the following elements in $G_n$:

$$\chi_{i, i^* j}(y) = I_N + y E_{ij} - y E_{j^* i^*},$$

$$\chi_{i, j^* i}(y) = I_N + y E_{ij^*} - y E_{j^* i},$$

$$\chi_{k, j}(y) = I_N + y E_{k, n+1} - y E_{n+1, k} - 2^{-1} y E_{kk}$$

and also

$$\chi_{i, i^* j}(y) = \chi_{i, i^* j}^T(y), \quad \chi_{i, j^* i}(y) = \chi_{i, j^* i}^T(y), \quad \chi_{k, j}(y) = \chi_{k, j}^T(y).$$

More generally, if $S \subseteq E$ is a subset and $\alpha \in \{ \pm e_i, \pm e_j \mid 1 \leq i \leq j \leq n \}$, we denote

$$\chi_\alpha(S) = \{ \chi_\alpha(s) \mid s \in S \}.$$ 

Note that if $1 \leq r \leq n$, then $\chi_\alpha(E) \subset H_r$ for every $\alpha \in \{ \pm e_i, \pm e_j \mid 1 \leq i < j \leq r \}$. On the other hand, we have

$$\chi_{e_i e_j}(E) \cap H_r = \{ I_N \}$$

for all $1 \leq k \leq n$.

2.2.2. Weyl elements. Let $W_{G_n}$ be the spherical Weyl group of the diagonal tours $E^1 \cdot T_n$ in $G_n$. Then $W_{G_n}$ is isomorphic to $\mathfrak{S}_n \times \mathbb{Z}_2^n$, where $\mathfrak{S}_n$ is the permutation group of the set $I_n := \{ 1, 2, \ldots, n \}$. To obtain a set of representatives of $W_{G_n}$ in $G_n$, let $W_{GL_n} \subset GL_n(E)$ be the subgroup consisting of permutation matrices. For a (possibly empty) subset $S \subseteq I_n$ and an element $y \in E^*$, we put

$$w_S(y) = \sum_{i \leq j \leq n, i \notin S} (E_{ii} + E_{i^* i^*}) + E_{n+1, n+1} + \sum_{j \in S} (y^{-1} E_{jj^*} + y E_{j^* j}).$$

Now if we attach an element $y_S \in E^*$ for each subset $S \subseteq I_n$, then

$$\{ w \cdot w_S(y_S) \mid w \in W_{GL_n}, S \subseteq I_n \}$$

gives a complete set of representatives of $W_{G_n}$ in $G_n$. Observe that this set of representatives is actually contained in $H_n$. In fact, the spherical Weyl group $W_{H_n}$ of $T_n$ in $H_n$ is also isomorphic to $\mathfrak{S}_n \times \mathbb{Z}_2^n$; hence (2.1) also gives a complete set of representatives of $W_{H_n}$ in $H_n$. In general, if we replace $n$ in (2.1) with any $1 \leq r \leq n$, then we get a complete set of representatives of the spherical Weyl group $W_{H_r}$ of $T_r$ in $H_r$. 


2.3. Open compact subgroups. For each integer \( m \geq 0 \), let \( K_{n,m} \) be the open compact subgroup of \( G_n \) defined by

\[
K_{n,m} = \left( \begin{array}{ccc}
\mathfrak{o}_E & \mathfrak{o}_E & p_E^m \\
\mathfrak{p}_E^m & 1 + p_E^m & \mathfrak{o}_E \\
\mathfrak{p}_E & p_E & \mathfrak{o}_E
\end{array} \right) \cap G_n.
\]

These are the open compact subgroups used to define newforms and oldforms of \( G_n \) introduced by Miyazaki \([\text{Miy13a}]\) when \( n = 1 \) and by Atobe-Oi-Yasuda \([\text{AOY22}]\) in general. Note that \( K_{n,0} \) is the hyperspecial maximal compact subgroup of \( G_n \) and \( K_{n,m} \cap G_{n_0} = K_{n_0,m} \) where \( 0 \leq n_0 \leq n \). Put

\[ R_{r,m} = K_{n,m} \cap H_r. \]

for \( 1 \leq r \leq n \). Then \( R_{r,0} \) and \( R_{r,1} \) are non-conjugate hyperspecial maximal compact subgroups of \( H_r \). In general, if \( m = e + 2\ell > 1 \) for some \( e = 0, 1 \) and \( \ell \geq 1 \), then \( R_{r,m} \) is conjugate to \( R_{r,e} \) by the tours element

\[ \left( \begin{array}{cc}
\omega^e I_r & 0 \\
0 & \omega^{-e} I_r
\end{array} \right) \in T_r. \]

Let

\[ \mathcal{W}_m = \{ \hat{w} \cdot w_S(\omega^m) \mid w \in \mathcal{W}_{G_n}, S \in T_n \}. \]

Then \( \mathcal{W}_{G_n} \) gives a complete set of representatives of \( \mathcal{W}_{G_n} \) (cf. \([\text{Miy13a}]\)) which is in addition contained in \( R_{n,m} \).

**Lemma 2.1.** Let \( e = 0, 1 \). We have \( G_n = B_n K_{n,e} \).

**Proof.** When \( e = 0 \), the assertion is the usual Iwasawa decomposition for \( G_n \). So it suffices to prove the assertion when \( e = 1 \). For this, we follow the proof of \([\text{Miy13a}]\) Lemma 2.1. The reduction of \( K_{n,0} \) modulo \( p_E \) is isomorphic to \( U_{2n+1}(k_E/k_F) \). It then follows from the Bruhat decomposition of \( U_{2n+1}(k_E/k_F) \) that

\[ G_n = B_n K_{n,0} = B_n \mathcal{W}_0(K_{n,0} \cap K_{n,1}). \]

Now since \( \mathcal{W}_0 \subset B_n \mathcal{W}_1 \), we conclude that

\[ G_n = B_n \mathcal{W}_0(K_{n,0} \cap K_{n,1}) = B_n \mathcal{W}_1(K_{n,0} \cap K_{n,1}) = B_n K_{n,1} \]

as desired. \( \square \)

**Lemma 2.2.** Let \( s_1, s_2 \in \mathfrak{S}_n \) and \( m \geq 1 \). Put \( E_m^{s_1} = E^1 \cap (1 + p_E^m) \), viewed as a subgroup of the center of \( G_n \). Then we have

\[
K_{n,m} = E_m^{s_1} \prod_{j=1}^{n} \chi_{\varepsilon_{s_2(j)}}(p_E^m) \prod_{i=1}^{n} \chi_{\varepsilon_{s_1(i)}}(\mathfrak{o}_E) R_{n,m}
\]

(2.4)

**Proof.** The proof of this lemma is actually similar to that of \([\text{Isa13}]\) Proposition 7.1.3. Here we only prove the second identity as the proof of the first one is similar. Since the subgroups \( \chi_{\varepsilon_i}(\mathfrak{o}_E) \) and \( \chi_{-\varepsilon_i}(p_E^m) \) are contained in \( K_{n,m} \) (cf. \([\text{Miy13a}]\)), it suffices to show that the LHS contains \( K_{n,m} \). To do so, we use the observation that the stabilizer of \( v_0 \) in \( K_{n,m} \) is \( R_{n,m} \). More precisely, for a given \( k \in K_{n,m} \), we are going to show that there exist \( y_1, \ldots , y_n, z_1, \ldots , z_n \) in \( \mathfrak{o}_E \), and \( b_n \in E_m^{s_1} \) such that

\[ b_n^{-1} \prod_{i=1}^{n} \chi_{\varepsilon_{s_2(n+1-i)}}(y_{n+1-i}) \prod_{j=1}^{n} \chi_{\varepsilon_{s_1(n+1-j)}}(\omega^m z_{n+1-j}) k v_0 = v_0. \]

Then since

\[ b_n^{-1} \prod_{i=1}^{n} \chi_{\varepsilon_{s_2(n+1-i)}}(y_{n+1-i}) \prod_{j=1}^{n} \chi_{\varepsilon_{s_2(n+1-j)}}(\omega^m z_{n+1-j}) k \in K_{n,m} \]

the assertion follows from the above observation.
Form the shape of $K_{n,m}$, we can write

$$kv_0 = \sum_{i=1}^{n} a_i e_i + b v_0 + \sum_{j=1}^{n} \varpi^m c_j f_j$$

for some $a_i, c_j$ in $\mathfrak{o}_E$ and $b \in 1 + \mathfrak{p}_E^m$. Let $z_1 \in \mathfrak{o}_E$ to be chosen and put $s_2(1) = \ell$. We have

$$\chi_{-\ell}(\varpi^m z_1)kv_0 = \sum_{i=1}^{n} a_i e_i + b v_0 + \varpi^m (c_1 - b z_1 - 2^{-1} a_\ell \varpi^m z_1) f_\ell + \sum_{j=1}^{n} \varpi^m c_j f_j$$

where $b_1 := b + \varpi^m a_\ell z_1 \in 1 + \mathfrak{p}_E^m$. Since 2 and $\varpi$ are units in $\mathfrak{o}_E$ and $\varpi$ is a prime element of both $E$ and $F$, we can apply the proof of Hensel’s lemma to conclude that there exists $z_1 \in \mathfrak{o}_E$ such that

$$c_\ell - b z_1 - 2^{-1} a_\ell \varpi^m z_1 = 0.$$ 

Continue this process, we get $z_1, \ldots, z_n$ in $\mathfrak{o}_E$ such that

$$\prod_{j=1}^{n} \chi_{-\epsilon_2(n+1-j)}(\varpi^m z_{n+1-j})kv_0 = b_n v_0 + \sum_{i=1}^{n} a_i e_i$$

for some $b_n \in 1 + \mathfrak{p}_E^m$. To proceed, let $y_1 \in \mathfrak{o}_E$ to be chosen and put $s_1(1) = r$. We have

$$\chi_{-\ell}(y_1) \prod_{j=1}^{n} \chi_{-\epsilon_2(n+1-j)}(\varpi^m z_{n+1-j})kv_0 = b_n v_0 + (a_r - b_n y_1) e_r + \sum_{1 \leq i \leq n} a_i e_i.$$ 

Certainly, we can find $y_1 \in \mathfrak{o}_E$ so that $a_r - b_n y_1 = 0$. Continue this process, we can obtain $y_1, \ldots, y_n \in \mathfrak{o}_E$ such that

$$k' v_0 = b_n v_0 \quad \text{where} \quad k' := \prod_{i=1}^{n} \chi_{-\epsilon_2(n+1-i)}(y_{n+1-i}) \prod_{j=1}^{n} \chi_{-\epsilon_2(n+1-j)}(\varpi^m z_{n+1-j})k \in K_{n,m}.$$ 

It remains to show that $b_n \in E^1_m$. But this follows immediately from the following identity:

$$b_n \bar{b}_n = (b_n v_0, b_n v_0) = (k' v_0, k' v_0) = (v_0, v_0) = 1.$$ 

This completes the proof. □

Lemma 2.2 has the following consequence. Let $\ell \geq 0$ be an integer and put

$$t_\ell = \begin{pmatrix} \varpi^\ell I_n & 1 \\ -\varpi^{-\ell} I_n \end{pmatrix} \in G_n.$$ 

Let $m \geq 0$ be an integer and write $m = e + 2\ell$ for some $e \in \{0, 1\}$ and $\ell \geq 0$. Define

$$K_{m,n}^0 = t_\ell K_{n,m} t_\ell^{-1}.$$ 

Then $K_{m,n}^0 = K_{n,m}$ when $m = 0, 1$ and Lemma 2.2 implies the following decompositions for $m \geq 1$:

$$K_{n,m}^0 = E^1_m \prod_{j=1}^{n} \chi_{s_2(j)}(\mathfrak{p}_E^{\ell e}) \prod_{i=1}^{n} \chi_{s_1(i)}(\mathfrak{p}_E^{\ell e}) R_{n,e}$$

(2.5)

$$= E^1_m \prod_{i=1}^{n} \chi_{s_1(i)}(\mathfrak{p}_E^{\ell e}) \prod_{j=1}^{n} \chi_{s_2(j)}(\mathfrak{p}_E^{\ell e}) R_{n,e}.$$ 

Here $s_1, s_2$ are elements in $\mathfrak{S}_n$. In particular, we have the following filtrations:

$$K_{n,e}^0 \supset K_{n,e+2}^0 \supset \cdots \supset K_{n,e+2\ell}^0 \supset \cdots \supset R_{n,e} = \bigcap_{m \equiv e \pmod{2}} K_{n,m}^0$$

(2.6)

for $e = 0, 1$. To investigate the double coset decompositions in the next section, we use $K_{n,m}^0$ instead of $K_{n,m}$.
3. Double coset decompositions

Let $r$ be an integer with $1 \leq r \leq n$. Let $P_{n,r}$ be the parabolic subgroup of $G_n$ containing $B_n$ with the Levi decomposition $P_{n,r} = M_{n,r} \times N_{n,r}$, where $M_{n,r} = \mathbb{GL}_r(E) \times G_{n-r}$ (cf. 2.2), and $N_{n,r} \subset N_n$. Let $\tilde{P}_{n,r} \supset M_{n,r}$ be the opposite of $P_{n,r}$. We first investigate the double coset decomposition:

$$\tilde{P}_{n,r} \backslash G_n / K_{n,m}^0.$$

**Lemma 3.1.** Let $m \geq 0$ be an integer and write $m = e + 2\ell$ for some $e, 0, 1$ and $\ell \geq 0$. Then the set

$$\left\{ \chi_{e,i}(\varpi^d) \mid 0 \leq d \leq \ell \right\}$$

gives a complete set of representatives of $\tilde{P}_{n,r} \backslash G_n / K_{n,m}^0$. Here $i$ is any integer between 1 and $r$.

**Proof.** It suffices to prove the lemma for $i = 1$ since $\chi_{e,i}(y)$ and $\chi_{e,i}(y)$ are conjugate by an element in $\mathcal{W}_{GL_r}$ for $1 \leq i \leq r$, and $\mathcal{W}_{GL_r}$ (cf. 2.2) is contained in both $\tilde{P}_{n,r}$ and $K_{n,m}^0$. We begin with the case when $m = 0, 1$. Recall that $K_{n,m}^0 = K_{n,m}$ and note that $e = m$ and $\ell = 0$ in this case. Since $w_{\mathcal{L}}(\varpi^e) \in K_{n,e}$ and $\tilde{B}_n = w_{\mathcal{L}}(\varpi^e)B_n w_{\mathcal{L}}(\varpi^e)^{-1}$ we get from Lemma 2.1 that

$$G_n = w_{\mathcal{L}}(\varpi^e)G_n w_{\mathcal{L}}(\varpi^e)^{-1} = w_{\mathcal{L}}(\varpi^e)B_n K_{n,e} w_{\mathcal{L}}(\varpi^e)^{-1} = \tilde{B}_n K_{n,e} = \tilde{P}_{n,r} K_{n,e}.$$

As $\chi_{e,1}(1) \in K_{n,e}$, the case $m = 0, 1$ is proved.

Suppose from now on that $m \geq 2$ so that $\ell \geq 1$. We split the proof into three steps. The first step is to show that the set

$$\left\{ \prod_{i=1}^{r} \chi_{e,i}(\varpi^{d_i}) \mid 0 \leq d_1 \leq \cdots \leq d_r \leq \ell \right\}$$

contains a complete set of representatives of $\tilde{P}_{n,r} \backslash G_n / K_{n,m}^0$. Observe that by 3.1,

$$\tilde{P}_{n,r} \backslash G_n / K_{n,m}^0 = (\tilde{P}_{n,r} \cap K_{n,e}) \backslash K_{n,e} / K_{n,m}^0.$$

We therefore separate the case $e = 0$ from $e = 1$. Assume first that $e = 1$. Then we have the decomposition

$$K_{n,1} = E_{1} \prod_{j=1}^{n} \chi_{-\epsilon_j}(p_{E}) \prod_{i=1}^{r} \chi_{e,i}(\varphi_{E}) \prod_{i=1}^{r} \chi_{e,i}(\varphi_{E}) R_{n,1}$$

by Lemma 2.2. Since the subgroups $\chi_{e,i}(p_{E})$ for $1 \leq i \leq r$ and $R_{n,1}$ are contained in $K_{n,m}^0$ (cf. 2.3), while the subgroups $E_{1} \chi_{e,j}(p_{E})$ for $1 \leq j \leq n$ and $\chi_{e,j}(p_{E})$ for $r+1 \leq j \leq n$ are contained in $\tilde{P}_{n,r} \cap K_{n,1}$, we see that

$$\left\{ \prod_{i=1}^{r} \chi_{e,i}(\varpi^{d_i} y_i) \mid d_i \geq 0, y_i \in \mathfrak{o}_{E}^{\times} \text{ for } 1 \leq i \leq r \text{ and one of } d_i \leq \ell \right\}$$

contains a complete set of representatives of the double coset. Because we are allowing to conjugate elements in the subgroup

$$T_n \cap R_{n,1} = \left\{ \text{diag}(y_1, \ldots, y_n, 1, \bar{y}_n^{-1}, \ldots, \bar{y}_1^{-1}) \mid y_1, \ldots, y_n \in \mathfrak{o}_{E}^{\times} \right\}$$

a complete set of representatives can be chosen in the set

$$\left\{ \chi(d_1, \ldots, d_r) := \prod_{i=1}^{r} \chi_{e,i}(\varpi^{d_i}) \mid d_i \geq 0 \text{ for } 1 \leq i \leq r \text{ and one of } d_i \leq \ell \right\}.$$

To reduce further, we conjugate elements in $\mathcal{W}_{GL_r}$. If $w \in \mathcal{W}_{GL_r}$ is the element corresponding to $s \in \mathfrak{S}_r$, then

$$\hat{w} \cdot \chi(d_1, \ldots, d_r) \cdot \hat{w}^{-1} = \chi(d_{s(1)}, \ldots, d_{s(r)}).$$

In particular, we may in addition assume that $d_1 \leq \cdots \leq d_r$ in (3.3). Since

$$\tilde{P}_{n,r} \chi(d_1, \ldots, d_r) K_{n,m}^0 = \tilde{P}_{n,r} \prod_{i=1}^{j} \chi_{e,i}(\varpi^{d_i}) K_{n,m}^0 = \tilde{P}_{n,r} \chi(d_1, \ldots, d_j, \ell, \ldots, \ell) K_{n,m}^0,$$

if there exists $0 \leq j < r$ such that $d_j < \ell \leq d_{j+1}$ (with $d_0 := 0$), our assertion for the case $e = 1$ follows.
Suppose that \( e = 0 \). In this case, we don’t have an analogue decomposition for \( K_{n,0} \), so we use the following decomposition
\[
K_{n,0} = (\bar{N}_n \cap K_{n,0})(N_n \cap K_{n,0})(E^1 \cdot T_n \cap K_{n,0})\mathcal{U}_0
\]
instead, where \( \bar{N}_n \) is the unipotent radical of \( B_n \). Recall that \( \mathcal{U}_0 \subset R_{n,0} \) is given by (2.3). Clearly, we have
\[
\bar{N}_n \cap K_{n,0} \subset \bar{P}_{n,r} \cap K_{n,0} \quad \text{and} \quad (E^1 \cdot T_n \cap K_{n,0})\mathcal{U}_0 \subset K_{n,0}.
\]
On the other hand, since
\[
(N_n \cap K_{n,0}) = \left( \prod_{i=1}^n \chi_{\epsilon_i}(\mathfrak{o}_E) \right) \left( \prod_{i=1}^r \chi_{\epsilon_i}(\mathfrak{o}_E) \right) (N_n \cap R_{n,0})
\]
and the subgroups \( \chi_{\epsilon_i}(\mathfrak{p}_E) \) for \( 1 \leq i \leq r \) and \( R_{n,0} \) are contained in \( K_{n,m}^0 \) by (2.5), and moreover, the subgroups \( \chi_{\epsilon_i}(\mathfrak{o}_E) \) for \( r + 1 \leq i \leq n \) are contained in \( \bar{P}_{n,r} \cap K_{n,0} \), we are reducing to the similar situation as in the case \( e = 1 \). Now the same argument proves the assertion for the case \( e = 0 \). This completes the first step.

The next step is to show that
\[
\{ \chi_{\epsilon_i}(\omega^d) \mid 0 \leq d \leq \ell \}
\]
contains a complete set of representatives of \( \bar{P}_{n,r} \setminus \bar{G}_n / K_{n,m}^0 \). Since the assertion for \( r = 1 \) is already proved in the first step, we may assume \( r > 1 \). To proceed, the following identity is needed
\[
(3.4) \quad \chi_{\epsilon_{k-1}}(\omega^{d'}) \chi_{\epsilon_k}(\omega^d) = \chi_{\epsilon_{k-1}+\epsilon_k}(\omega^{d-d'}) \chi_{\epsilon_{k-1}+\epsilon_k}(-\omega^{d-d'}) \chi_{\epsilon_{k-1}+\epsilon_k}(2^{-1}\omega^{d+d'})
\]
for \( 1 < k \leq n \) and \( d,d' \in \mathbb{Z} \). Observe that if \( k \leq r \) and \( d' \leq d \), then \( \chi_{\epsilon_{k-1}+\epsilon_k}(\omega^{d-d'}) \in \bar{P}_{n,r} \) and both of \( \chi_{\epsilon_{k-1}+\epsilon_k}(-\omega^{d-d'}) \) and \( \chi_{\epsilon_{k-1}+\epsilon_k}(2^{-1}\omega^{d+d'}) \) are contained in \( R_{n,m} \). Now let
\[
g = \prod_{i=1}^r \chi_{\epsilon_i}(\omega^d) = g_0 \chi_{\epsilon_{r-1}+\epsilon_r}(\omega^{d_{r-1}}) \chi_{\epsilon_{r-1}}(\omega^{d_r})
\]
be an element in the set given by (3.2), where \( g_0 = \prod \chi_{\epsilon_i}(\omega^d) \) and we define \( g_0 = I_{2n+1} \) when \( r = 2 \). Applying (3.4) and noting that \( \chi_{-\epsilon_{r-1}+\epsilon_r}(y) \) commutes with \( \chi_{\epsilon_i}(z) \) for every \( 1 \leq i \leq r-2 \), we get that
\[
\bar{P}_{n,r} \bar{G}_n^0 = \bar{P}_{n,r} g_0 \chi_{-\epsilon_{r-1}+\epsilon_r}(\omega^{d_{r-1}}) \chi_{\epsilon_{r-1}}(\omega^{d_{r-1}}) \chi_{-\epsilon_{r-1}+\epsilon_r}(-\omega^{d_{r-1}}) \chi_{\epsilon_{r-1}}(2^{-1}\omega^{d_{r-1}}) \chi_{\epsilon_{r-1}}(\omega^{d_{r-1}}) K_{n,m}^0
\]
\[
= \bar{P}_{n,r} \chi_{\epsilon_{r-1}+\epsilon_r}(\omega^{d_{r-1}}) g_0 \chi_{\epsilon_{r-1}}(\omega^{d_{r-1}}) K_{n,m}^0
\]
\[
= \bar{P}_{n,r} \prod_{i=1}^{r-1} \chi_{\epsilon_i}(\omega^d) K_{n,m}^0.
\]
The assertion then follows from continuing this process. This finishes the second step.

The final step is to show that the double cosets
\[
\bar{P}_{n,r} \chi_{\epsilon_i}(\omega^d) K_{n,m}^0
\]
for \( 0 \leq d \leq \ell \) are all distinct. Suppose in contrast that there exist \( 0 \leq d' < d \leq \ell \) and \( h \in \bar{P}_{n,r} \), \( k \in K_{n,m}^0 \) such that
\[
(3.5) \quad \chi_{\epsilon_i}(\omega^d) = h \chi_{\epsilon_i}(\omega^{d'}) k.
\]
To obtain a contradiction, note that
\[
h = \chi_{\epsilon_i}(\omega^d) k^{-1} \chi_{\epsilon_i}(-\omega^{d'}) \in K_{n,m}^0 \cap \bar{P}_{n,r}
\]
implies
\[
(3.6) \quad h^{-1} f_i = \sum_{j=1}^r a_{ij} f_j,
\]
for some \( a_{ij} \in \mathfrak{o}_E \) for \( 1 \leq i,j \leq r \). Also, from the shape of \( K_{n,m}^0 \), we can write
\[
k v_0 = \sum_{i=1}^n \omega^i a_i e_i + b v_0 + \sum_{j=1}^n \omega^{\ell} c_j f_j.
\]
for some \(a_i, c_j\) in \(\mathfrak{o}_E\) and \(b \in 1 + p_E^{m}\). Then we have
\[
\chi_\epsilon_1(\omega^d)k v_0 = \omega^d b' c_1 + \sum_{i=2}^n \omega^f a_i c_1 + (b - \omega^d c_1)v_0 + \sum_{j=1}^n \omega^f c_j f_j
\]
where \(b' := b + \omega^f c_1 - 2^{-1} \omega^d + \omega^f c_1 \in \mathfrak{o}_E^\times\). Now let \(1 \leq i \leq r\). We are going to compute \(\langle \chi_\epsilon_1(\omega^d)v_0, f_i \rangle\) in two ways. On one hand,
\[
\langle \chi_\epsilon_1(\omega^d)v_0, f_i \rangle = \langle v_0 + \omega^d c_1, f_i \rangle = \omega^d \delta_{i1}.
\]
On the other hand, from \((3.6)\) and \((3.7)\),
\[
\langle \chi_\epsilon_1(\omega^d)v_0, f_i \rangle = \langle \chi_\epsilon_1(\omega^d)k v_0, h^{-1} f_i \rangle = \omega^d b' a_{11} + \sum_{j=2}^n \omega^f a_j a_{1j}.
\]
We thus obtain
\[
\omega^d b' a_{11} + \sum_{j=2}^n \omega^f a_j a_{1j} = \omega^d \delta_{i1}.
\]
From this identity, we see that \(a_{11} \in p_E\) for \(2 \leq i \leq n\) since \(d' < \ell\). It then follows that \(a_{11} \in \mathfrak{o}_E^\times\) as \(h\) (and hence \(h^{-1}\)) is contained in \(K_{n,m}^0 \cap \tilde{Q}_{n,r}\) (cf. \((3.0)\)). But this would imply
\[
\omega^d = \omega^d b' a_{11} + \sum_{j=2}^n \omega^f a_j a_{1j} \in \omega^d \mathfrak{o}_E^\times
\]
which contradicts to the assumption \(d' < d\). This concludes the final step and also the proof of Lemma 3.1. \(\square\)

To state the next lemma, we need some notation. For a given \(h \in \mathcal{P}_{n,r}\), we denote by \(a_h \in M_{n,r}\) the “Levi part” of \(h\) under the Levi decomposition \(\mathcal{P}_{n,r} = M_{n,r} \times \tilde{N}_{n,r}\). On the other hand, we let \(\Gamma_{r,m}^0 \subseteq \text{GL}_r(\mathfrak{o}_E)\) be the open compact subgroup defined by
\[
\Gamma_{r,m}^0 = (r-1) \begin{pmatrix} \mathfrak{o}_E & \mathfrak{p}_E^m \\ \mathfrak{o}_E & 1 + \mathfrak{p}_E^m \end{pmatrix} \cap \text{GL}_r(\mathfrak{o}_E).
\]

**Lemma 3.2.** Let \(m = c + 2\ell \geq 0\) be an integer with \(e = 0, 1\) and \(\ell \geq 0\). Let \(r, d\) be integers with \(1 \leq r \leq n\) and \(0 \leq d \leq \ell\). Let \(\mathcal{M}_{n,r}^d \subseteq M_{n,r}\) be the subgroup defined by
\[
\mathcal{M}_{n,r}^d = \{a_h | h \in \chi_\epsilon_1(\omega^d)K_{n,m}^0, \chi_\epsilon_1(\omega^d)^{-1} \cap \mathcal{P}_{n,r}\}.
\]
Then we have \(\mathcal{M}_{n,r}^d = \Gamma_{r,d}^0 \times K_{n-r, e+2d}^0\).

**Proof.** For convenient, we also denote \(s_{r,d} = \chi_\epsilon_1(\omega^d), U_E^0 = \mathfrak{o}_E^\times\) and \(U_E^j = 1 + \mathfrak{p}_E^j \subseteq \mathfrak{o}_E^\times\) for \(j > 0\). Note that the lemma holds trivially when \(m = 0, 1\) since in this case, \(\ell = d = 0\) and \(s_{r,0} \in K_{n,m}^0 = K_{n,m}\). So we assume in the rest of the proof that \(m \geq 2\) (so that \(\ell \geq 1\)). We start with the case \(r = 1\). The first step is to show
\[
\mathcal{U}_E^{d - \ell} \times K_{n-1, c+2d}^0 \subseteq \mathcal{M}_{n,1}^d.
\]
To do so, for a given \(a\) in the LHS of \((3.8)\), our strategy is to find \(h \in \mathcal{P}_{n,1}\) with \(a_h = a\) and \(s_{1,d}^{-1} h s_{1,d} \in K_{n,m}^0\). This would imply \(a \in \mathcal{M}_{n,1}^d\). Now let us write
\[
s_{1,d} = \begin{pmatrix} 1 & \alpha & \beta \\ & 1 \end{pmatrix}
\]
with \(\alpha = (0, \ldots, 0, \omega^d, 0, \ldots, 0)\) \(\alpha' = -^t \alpha\) and \(\beta = -2^{-1} \omega^{2d}\).
Then
\[
s_{1,d}^{-1} = \begin{pmatrix} 1 & -\alpha & \beta \\ & 1 \end{pmatrix}.
\]
Let
\[ h' = \begin{pmatrix} a & h \\ \tilde{a}^{-1} \end{pmatrix} \in \tilde{P}_{n,1} \]
with \( a \in U_{\tilde{E}}^d \) and \( h \in R_{n-1,c} \). Then a direct computation shows
\[
s_{1,d}^{-1} h' s_{1,d} = \begin{pmatrix} a & a\alpha - \alpha h & a\beta - \alpha h a' + \beta \tilde{a}^{-1} \\ h & h a' & \tilde{a}^{-1} \\ \tilde{a}^{-1} \end{pmatrix} = \begin{pmatrix} a & (a-1)\alpha & a\beta + \beta \tilde{a}^{-1} + 2\delta \\ h & h a' & a'(1 - \tilde{a}^{-1}) \end{pmatrix} \in K_{n,m}^0.
\]
It follows that
\[(3.9) \quad \tilde{U}_{\tilde{E}}^{d} \times R_{n-1,c} \subseteq \tilde{M}_{n,1}^d.\]
To establish (3.8), it remains to show that \( \hat{i} \times E_{m}^1 \), \( \hat{i} \times \chi_{\epsilon_j}(p_E^d) \) and \( \hat{i} \times \chi_{-\epsilon_j}(p^{e+d}) \) are contained in \( \tilde{M}_{n,1}^d \) for \( 2 \leq j \leq n \). Here we regard \( E_{m}^1 \) as a subgroup in the center of \( G_{n-1} \) and we define \( E_{0}^1 = E^1 \). We check this case by case. First let \( y \in E_{m}^1 \). Then
\[
\begin{pmatrix} 1 \\ yJ_{n-1} \end{pmatrix} \in \tilde{P}_{n,1}
\]
and we have
\[
s_{1,d}^{-1} \begin{pmatrix} 1 \\ yJ_{n-1} \end{pmatrix} s_{1,d} = \begin{pmatrix} 1 & \alpha(1-y) & 2\beta - \alpha y a' \\ yJ_{n-1} & yJ_{n-1} & a'(y-1) \end{pmatrix} \in K_{n,m}^0.
\]
This gives \( \hat{i} \times E_{m}^1 \subseteq \tilde{M}_{n,1}^d \). Next let \( y \in \mathfrak{o}_{E} \) and \( 2 \leq j \leq n \). Then
\[
\chi_{-\epsilon_1+\epsilon_j}(-y) = \begin{pmatrix} 1 \\ xJ_{n-1} \end{pmatrix} \quad \text{and} \quad \chi_{\epsilon_j}(\varpi^d y) = \begin{pmatrix} 1 \\ h \end{pmatrix}
\]
and we write
\[
\chi_{-\epsilon_1+\epsilon_j}(-y) = \chi_{\epsilon_j}(\varpi^d y) \quad \text{and} \quad \chi_{\epsilon_j}(\varpi^d y) = \begin{pmatrix} 1 \\ h \end{pmatrix}
\]
with
\[
x = (0, \ldots, 0, -y, 0, \ldots, 0), \quad x' = -\tilde{x}J_{n-1} \quad \text{and} \quad h \in G_{n-1}.
\]
then
\[
s_{1,d}^{-1} \chi_{-\epsilon_1+\epsilon_j}(-y) \chi_{\epsilon_j}(\varpi^d y) s_{1,d} = \begin{pmatrix} 1 & \alpha - \alpha h + \beta x' h & 2\beta - \alpha h a' + \beta x' h a' \\ x & x \alpha + h - \alpha x' h & x \beta + h a' - \alpha x' h a' - a' \\ 0 & x' h & 1 + x' h a' \end{pmatrix} \in K_{n,m}^0.
\]
It follows that \( \hat{i} \times \chi_{\epsilon_j}(p_E^d) \subseteq \tilde{M}_{n,1}^d \). Similarly, if \( y \in \mathfrak{o}_{E} \) and \( 2 \leq j \leq n \), then
\[
\chi_{-\epsilon_1-\epsilon_j}(\varpi^d y) \chi_{-\epsilon_j}(\varpi^{e+d} y) \in \tilde{P}_{n,1}
\]
and we have
\[
s_{1,d}^{-1} \chi_{-\epsilon_1-\epsilon_j}(\varpi^d y) \chi_{-\epsilon_j}(\varpi^{e+d} y) s_{1,d} = K_{n,m}^0.
\]
Thus we also have \( \hat{i} \times \chi_{-\epsilon_j}(p_E^d) \subseteq \tilde{M}_{n,1}^d \). Now (3.9) follows from these and (3.8). Indeed, if \( e + 2d \geq 1 \), then this is a consequence of (2.5). On the other hand, if \( e = d = 0 \), then we use the facts that \( K_{n-1,0} \) is generated by the subgroups \( E^1, \tilde{R}_{n-1,0} \) and \( \chi_{\epsilon_j}(\mathfrak{o}_E) \) for \( 2 \leq j \leq n \).

To prove the reverse inclusion, first note that \( s_{1,d} \in K_{n,c+2d}^0 \) and \( K_{n,m}^0 \subseteq K_{n,c+2d}^0 \). It follows that
\[
s_{1,d} K_{n,m}^0 s_{1,d}^{-1} \subseteq \tilde{P}_{n,1} \subseteq K_{n,c+2d}^0 \cap \tilde{P}_{n,1},
\]
and hence
\[
\tilde{M}_{n,1}^d \subseteq \tilde{E}_E \times K_{n-1,c+2d}^0.
\]
Thus it remains to show that if \( s_{1,d}^{-1}hs_{1,d} \in K_{n,m}^0 \), where
\[
h = \begin{pmatrix} a \\ h' \\ \bar{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1_{2n-1} \\ x & x' \\ y \end{pmatrix} \in \tilde{P}_{n,1}
\]
then \( a \in U_{E}^{f=d} \). For this, let us compute
\[
\{ s_{1,d}^{-1}hs_{1,d}v_0, f_1 \} = \{ hs_{1,d}v_0, s_{1,d}f_1 \}
\]
\[
= \{ hs_{1,d}v_0, f_1 - \omega^d v_0 - 2^{-1} \omega^{2d} e_1 \}
\]
\[
= \{ s_{1,d}v_0, h^{-1}f_1 \} - \{ hs_{1,d}v_0, \omega^d v_0 + 2^{-1} \omega^{2d} e_1 \}
\]
\[
= \{ v_0 + \omega^d e_1, \bar{a} f_1 \} - \omega^d \{ hs_{1,d}v_0, v_0 + \omega^d e_1 \} + 2^{-1} \omega^{2d} \{ hs_{1,d}v_0, e_1 \}
\]
\[
= \omega^d a - \omega^d \{ hs_{1,d}v_0, s_{1,d}v_0 \} + 2^{-1} \omega^{2d} \{ hs_{1,d}v_0, s_{1,d}e_1 \}
\]
\[
= \omega^d a - \omega^d \{ s_{1,d}^{-1}hs_{1,d}v_0, v_0 \} + 2^{-1} \omega^{2d} \{ s_{1,d}^{-1}hs_{1,d}v_0, e_1 \}.
\]
Here we use the facts that \( h^{-1}f_1 = \bar{a} \) and \( s_{1,d}e_1 = e_1 \). Since \( s_{1,d}^{-1}hs_{1,d} \in K_{n,m}^0 \), the shape of \( K_{n,m}^0 \) implies that
\[
\{ s_{1,d}^{-1}hs_{1,d}v_0, f_1 \} \in p_E^f, \quad \{ s_{1,d}^{-1}hs_{1,d}v_0, v_0 \} \in U_E^m \quad \text{and} \quad \{ s_{1,d}^{-1}hs_{1,d}v_0, e_1 \} \in p_E^{e+f}.
\]
From these we conclude \( a \in U_{E}^{f=d} \). This proves the lemma when \( r = 1 \).

Suppose now that \( r > 1 \). The proof of this case is similar to that of \( r = 1 \). In fact, we will apply the result for \( r = 1 \). Again, we first establish the following inclusion
\[
(3.10) \quad \tilde{\Pi}_{r,\ell,d}^r \times K_{n-r,c+2d}^0 \subseteq \tilde{M}_{n,r}^d
\]
which is clearly a consequence of
\[
(3.11) \quad \tilde{I}_r \times K_{n-r,c+2d}^0 \subseteq \tilde{M}_{n,r}^d
\]
and
\[
(3.12) \quad \tilde{\Pi}_{r,\ell,d}^r \times I_{2(n-r)+1} \subseteq \tilde{M}_{n,r}^d.
\]
To prove (3.11), note that \( s_{r,d} \in G_{n-r+1} \) (recall the embedding \( G_{n-r+1} \hookrightarrow G_n \) (cf. 2.24)) and hence
\[
s_{r,d}K_{n-r+1,m}^{-1} s_{r,d}^{-1} \subseteq G_{n-r+1}.
\]
Since \( G_{n-r+1} \cap \tilde{P}_{n,r} = \tilde{P}_{n-r+1} \), we find that
\[
s_{r,d}K_{n-r+1,m}^{-1} s_{r,d}^{-1} \cap \tilde{P}_{n,r} = s_{r,d}K_{n-r+1,m}^{-1} s_{r,d}^{-1} \cap G_{n-r+1} \cap \tilde{P}_{n,r} = s_{r,d}K_{n-r+1,m}^{-1} s_{r,d}^{-1} \cap \tilde{P}_{n-r+1}.
\]
Now we can apply the result for \( r = 1 \) (with \( n \) replaced by \( n - r + 1 \)) to obtain (3.11). Next, we show (3.12). Let \( a \in \Gamma_{r,\ell,d}^r \) and write
\[
s_{r,d} = \begin{pmatrix} 1_n & \alpha & \beta \\ \alpha' & 1 \end{pmatrix} \quad \text{and} \quad \tilde{a} = \begin{pmatrix} A \\ 1 \end{pmatrix}
\]
with
\[
t \alpha = (0, \ldots, 0, \omega^d, 0, \ldots, 0), \quad \alpha' = -t \alpha J_n, \quad \beta = -2^{-1} \omega^{2d} E_{rr} \in \text{Mat}_{n \times n}(E) \quad \text{and} \quad A = \begin{pmatrix} a \\ I_{n-r} \end{pmatrix}.
\]
Then we have
\[
s_{r,d}^{-1} \tilde{a} s_{r,d} = \begin{pmatrix} A & A \alpha - \alpha \beta - \alpha' A^* \\ 1 & \alpha' - \alpha A^* \end{pmatrix} \in K_{n,m}^0.
\]
This shows (3.12) and hence (3.10).

Now we prove the reverse inclusion. First note that since \( s_{r,d} \in K_{n,c+2d}^0 \) and \( K_{n,m}^0 \subseteq K_{n,c+2d}^0 \), we have
\[
s_{r,d}K_{n,m}^{-1} s_{r,d}^{-1} \cap \tilde{P}_{n,r} \subseteq K_{n,c+2d}^0 \cap \tilde{P}_{n,r}.
\]
This implies
\[(3.13) \quad M_{n,r}^d \subseteq \text{GL}_r(\mathfrak{o}_E) \times K_{n-r,\varepsilon+2d}^0.
\]
In particular, the case \(d = \ell\) is proved since \(\Gamma_{r,0}^0 = \text{GL}_r(\mathfrak{o}_E) \) and we have \((3.10)\). So assume \(d < \ell\) and let \(h \in \hat{P}_{n,r}\) such that \(s_{r,d}^{-1}hs_{r,d} \in K_{n,m}^0\). Then \((3.13)\) implies
\[
h^{-1}f_j = \sum_{k=1}^ra_{jk}f_k
\]
for some \(a_{jk} \in \mathfrak{o}_E\) with \(1 \leq j, k \leq r\). We need to show that \(a_{r, r} \in 1 + p_{E}^{\ell-d}\) and \(a_{r, j} \in p_{E}^{\ell-d}\) for \(1 \leq j \leq r-1\). The idea of which is similar to the case \(r = 1\). Form the shape of \(K_{n,m}^0\), we see that
\[
(s_{r,d}^{-1}hs_{r,d}, v_0, f_j) = (s_{r,d}v_0, f_j) = \sum_{k=1}^r (v_0 + \varpi^d) a_{jk} f_k = \varpi^d a_{r, r}.
\]
These imply \(a_{r, r} \in p_{E}^{\ell-d}\) for \(1 \leq j \leq r-1\). If \(j = r\), then
\[
(s_{r,d}^{-1}hs_{r,d}, v_0, f_r) = (s_{r,d}v_0) = \sum_{k=1}^r (v_0 + \varpi^d)e_r + 2^{-1}\varpi^d e_r
\]
and hence \(a_{r, r} \in 1 + p_{E}^{\ell-d}\) as desired. This finishes the proof. \(\square\)

Let \(\Gamma_{r,m} \subseteq \text{GL}_r(\mathfrak{o}_E)\) be the usual "congruence subgroup", namely,
\[
\Gamma_{r,m} = \left\{\begin{array}{c}(r-1) \quad 1 \\
1 \quad 0_E \quad p_{m}^E \quad 1 + p_{E}^m \end{array}\right\} \cap \text{GL}_r(\mathfrak{o}_E).
\]

Then we have the following corollary:

**Corollary 3.3.** Let \(r, m\) be integers with \(1 \leq r \leq n\) and \(m \geq 0\). Write \(m = e + 2\ell\) for some \(e \geq 0\) and \(\ell \geq 0\). Then for any \(1 \leq j \leq r\), the set
\[
\{\chi_{-e}((\varpi^{e+d})^d) \mid 0 \leq d \leq \ell\}
\]
forms a complete set of representatives of \(P_{n,r}\backslash G_{n,m}K_{n,m}^0\). Moreover, we have
\[
M_{r,m}^d := \{a_h \mid h \in \chi_{-e}((\varpi^{e+d})^d)K_{n,m}^0, (\varpi^{e+d})^{-1} \cap P_{n,r}\} = \tilde{\Gamma}_{r,\ell-d} \times K_{n-r,\varepsilon+2d}^0
\]
where \(a_h \in M_{n,r}\) denotes the "Levi part" of \(h \in P_{n,r}\) under the Levi decomposition \(P_{n,r} = M_{n,r} \times N_{n,r}\).

**Proof.** Let \(S = \{1, 2, \ldots, r\} \subseteq \mathbb{I}_n\) and \(w = w_S(\varpi^e)\) (cf. \((2.2)\)). Then \(w \in R_{n,e} \subseteq K_{n,m}^0\) and we have \(wP_{n, rw^{-1}} = P_{n, r}\) with
\[
w \text{diag}(a, g_0, a^*) w^{-1} = \text{diag}(t_a^{-1}, g_0, J_0 a J_0^{-1})
\]
for \(\text{diag}(a, g_0, a^*) \in M_{n,r}\), where \(a \in \text{GL}_r(E)\) and \(g_0 \in G_{n-r}\). Since the map \(a \mapsto t_a^{-1}\) gives an isomorphism from \(\Gamma_{r,m}^0\) onto \(\Gamma_{r,m}\) and we have \(w_{e_j}(\varpi^{e+d}) w^{-1} = \chi_{-e}(\varpi^{e+d})^{-1}\) for \(1 \leq j \leq r\), the corollary follows immediately from Lemma \((3.1)\) and Lemma \((3.2)\).

## 4. Proof of Theorem 1.2

Let us proof Theorem 1.2 in this section. To begin with, we need some preparations.
4.1. Preliminaries. The following lemma describe a property of the local Langlands correspondence for $GL_\tau$, which should be well-known to the experts. However, since we can’t locate a proper reference, we shall provide a proof here. To state the lemma, let $\sigma$ be an irreducible representation of $GL_N(E)$ with the associated $L$-parameter $\phi_\sigma: WD_E \to GL_N(\mathbb{C})$ (cf. [HT01], [Hen00], [Sch13]). Define $\pi^\tau$ to be the representation of $GL_N(E)$ by $\pi^\tau(\alpha) = \pi(\bar{\alpha})$ for $\alpha \in GL_N(E)$. On the other hand, fix $w_0 \in WD_E \setminus WD_F$ and define a representation $\phi^\tau_{w_0}$ of $WD_E$ by $\phi^\tau_{w_0}(w)w_0^{-1}$ for $w \in WD_E$. The representation $\phi^\tau_{w_0}$ is independent of the choice of $w_0$. Now the lemma can be stated as follow:

Lemma 4.1. Under the local Langlands correspondence for $GL_N(E)$, we have $\phi^\tau_{w_0} \cong \phi_{\pi^\tau}$.

PROOF. When $N = 1$, the assertion follows from the local class field theory (cf. [Tat79]). For general $N$, the proof consists of two steps. The first step is to reduce the proof to the case when $\sigma$ is supercuspidal. This part is quite straightforward and it follows from the Bernstein-Zelevinsky classification (cf. [Zel80]) and the explicit local Langlands correspondence modulo the supercuspidal ones (cf. [Wed00] Section 4.2)). The second step is obviously to prove the lemma when $\sigma$ is supercuspidal. In particular, $\sigma$ is generic. The proof is by induction on $N$ together with the local converse theorem for $GL_N$ (see [JL18] and the references therein).

Let $\sigma$ be an irreducible supercuspidal representation of $GL_N(E)$ with $N \geq 2$. Let $\sigma'$ be the irreducible representation of $GL_N(E)$ whose $L$-parameter $\phi_{\sigma'}$ is isomorphic to $\phi^\tau_{w_0}$. Note that since $\sigma$ is supercuspidal, $\phi_{\sigma'}$ is irreducible. It follows that $\phi^\tau_{w_0}$ is also irreducible and hence $\sigma'$ is again supercuspidal. Moreover, $\sigma'$ and $\sigma^\tau$ have the same central character. Indeed, under the local Langlands correspondence, the central character $\omega_\sigma$ corresponds to $\det(\phi_\sigma)$, and hence character of $\sigma^\tau$, which is clearly equal to $\omega_\sigma^\tau$, corresponds to $\det(\phi_{\sigma'})^\tau$. As the central character of $\sigma'$ corresponds to $\det(\phi_{\sigma'}) = \det(\phi_\sigma)^\tau$, the assertion follows. Recall that our goal is to show $\sigma' \cong \sigma^\tau$. So suppose inductively that the assertion holds for every irreducible supercuspidal representations $\tau$ of $GL_r(E)$ with $1 \leq r \leq N - 1$. Thus we have

\begin{equation}
\phi^\tau_{\tau} \cong \phi_{\tau^\tau}
\end{equation}

for every such $\tau$. To apply the local converse theorem, we have to consider the $\gamma$-factor $\gamma(s, \sigma \times \tau, \psi)$ attached to $\sigma, \tau$ and a non-trivial additive character $\psi$ of $E$ defined by the Rankin-Selberg integrals for $GL_N \times GL_r$ (cf. [JPSSS83]). Using these local integrals, one can check directly that

\begin{equation}
\gamma(s, \sigma' \times \tau^\tau, \psi^\tau) = \gamma(s, \sigma \times \tau, \psi)
\end{equation}

where $\psi^\tau$ is the character of $E$ defined by $\psi^\tau(x) = \psi(x)$ for $x \in E$. On the other hand, we also have

\begin{equation}
\gamma(s, \phi^\tau_{\sigma} \otimes \phi^\tau_{\tau}, \psi^\tau) = \gamma(s, \phi_{\sigma} \otimes \phi_{\tau}, \psi).
\end{equation}

To verify this, first notice that

\begin{equation}
\gamma(s, \phi^\tau_{\sigma} \otimes \phi^\tau_{\tau}, \psi^\tau) = \gamma(s, \sigma' \times \tau^\tau, \psi^\tau) = \epsilon(s, \sigma' \times \tau^\tau, \psi^\tau) = \epsilon(s, \phi^\tau_{\sigma} \otimes \phi^\tau_{\tau}, \psi^\tau)
\end{equation}

as both of the $L$-factors appeared in the $\gamma(s, \sigma' \times \tau^\tau, \psi^\tau)$ are equal to 1 (cf. [JPSSS83 Proposition 8.1]). Here we also use the fact that these local factors are preserved under the local Langlands correspondence. Since same reasoning shows $\gamma(s, \phi_{\sigma} \otimes \phi_{\tau}, \psi) = \epsilon(s, \phi_{\sigma} \otimes \phi_{\tau}, \psi)$, we are reducing to show

\begin{equation}
\epsilon(s, \phi^\tau_{\sigma} \otimes \phi^\tau_{\tau}, \psi^\tau) = \epsilon(s, \phi_{\sigma} \otimes \phi_{\tau}, \psi).
\end{equation}

But this is a consequence of the following identity:

\begin{equation}
\epsilon(s, \phi^\tau_{\sigma}, \psi^\tau) = \epsilon(s, \phi_{\sigma}, \psi)
\end{equation}

for every admissible representations $\phi$ of $WD_E$, whose validity can be verified by Tate’s integral formula (cf. [Tat79]) and the inductivity of the $\epsilon$-factors.

Now we can complete the proof. In fact, by (4.1), (4.2), and (4.3), we have

\begin{equation}
\gamma(s, \sigma' \times \tau, \psi) = \gamma(s, \sigma' \times \tau^\tau, \psi^\tau) = \gamma(s, \phi_{\sigma} \times \phi_{\tau}, \psi^\tau) = \gamma(s, \phi^\tau_{\sigma} \times \phi_{\tau}, \psi) = \gamma(s, \phi^\tau_{\sigma} \times \phi_{\tau}, \psi^\tau) = \gamma(s, \sigma' \times \tau, \psi)
\end{equation}

for every supercuspidal representations $\tau$ of $GL_r(E)$ with $1 \leq r \leq N - 1$. Since $\sigma^\tau$ and $\sigma'$ have the same central, we conclude that $\sigma^\tau \cong \sigma'$ by the local converse theorem for $GL_N$ (cf. [JL18]).

\footnote{Actually, $\lfloor \frac{N}{2} \rfloor$ is enough.}
Lemma 4.1 has two corollaries. To state the first one, let $\sigma$ be an irreducible representation of $GL_N(E)$ with the associated $L$-parameter $\phi_\sigma$ as before. Since $\psi_E$ unramified, the $\epsilon$-factor $\epsilon(s, \phi_\sigma, \psi_E)$ can be written as
$$
\epsilon(1/2, \phi_\sigma, \psi_E)q^{-a_\sigma(2s-1)}
$$
for some integer $a_\sigma \geq 0$ and non-zero complex number $\epsilon(1/2, \phi_\sigma, \psi_E)$. Denote by $\tilde{\sigma}$ the contragredient of $\sigma$. Then we have:

**Corollary 4.2.** Let $\sigma$ be an irreducible representation of $GL_N(E)$. Then $a_{\tilde{\sigma}} = a_\sigma$.

**Proof.** Since $\psi_E$ is trivial on $F$, one has $\psi_E^c = \psi_E^{-1}$. In particular, $\psi_E^c$ is again unramified, and hence $a_{\tilde{\sigma}}$ also appears in the exponent of $\epsilon(s, \phi_\sigma, \psi_E^c)$. Since
$$
\epsilon(s, \phi_\sigma, \psi_E^c) = \epsilon(s, \phi_\sigma^c, \psi_E) = \epsilon(s, \phi_\sigma, \psi_E)
$$
by Lemma 4.1 and 4.3, we see that $a_{\tilde{\sigma}} = a_\sigma$. Next, since $\phi_\sigma \cong \tilde{\phi}_\sigma$, where $\tilde{\phi}_\sigma$ is the contragredient representation of $\phi_\sigma$ and
$$
\epsilon(s, \phi_\sigma, \psi_E)\epsilon(1-s, \phi_\sigma, \psi_E^c) = 1
$$
by [Tat79] Equation (3.4.7), we conclude that $a_\sigma = a_{\tilde{\sigma}}$. This completes the proof. \(\square\)

**Corollary 4.3.** Let $\phi : WD_E \to GL_N(\mathbb{C})$ be an $L$-parameter. If $\tilde{\phi} \cong \phi^c$, then $L(s, \tilde{\phi}) = L(s, \phi)$.

**Proof.** Let $\sigma$ be an irreducible representation of $GL_N(E)$ corresponds to $\phi$ under the local Langlands correspondence. Then the assumption on $\phi$ and Lemma 4.1 imply
$$
L(s, \tilde{\sigma}) = L(s, \phi^c) = L(s, \phi) = L(s, \sigma^c)
$$
where the first and the last $L$-factors are defined by the Rankin-Selberg integrals in [JPSS89]. Thus it suffices to show that $L(s, \sigma^c) = L(s, \sigma)$. But this follows immediately from the definition of the integrals. \(\square\)

To state the next lemma, let $\tilde{J}_N \in GL_N(E)$ be the element defined inductively by
$$
\tilde{J}_1 = (1) \quad \text{and} \quad \tilde{J}_N = \begin{pmatrix} (N-1)(1/L_{N-1}) \end{pmatrix}
$$
and $\theta : GL_{2n+1}(E) \to GL_{2n+1}(E)$ be the involution given by
$$
(4.5) \quad a^\theta = \tilde{J}_{2n+1} a^{-1} \tilde{J}_{2n+1}^{-1}.
$$
Let $\tilde{K}_{2n+1,m} \subset GL_{2n+1}(E)$ to be the open compact subgroup defined by
$$
(4.6) \quad \tilde{K}_{2n+1,m} = \omega_m \Gamma_{2n+1,m} \omega_m^{-1} \quad \text{where} \quad \omega_m := \begin{pmatrix} I_n & 1 \\ \omega^m I_n & 1 \end{pmatrix} \in GL_{2n+1}(E).
$$
Then $\tilde{K}_{2n+1,m}$ consists of matrices $k$ of the form
$$
\begin{pmatrix} n & 1 & n \\ n & 0_E & 0_E \\ 1 & p_m^E & 1 + p_m^E \end{pmatrix}
$$
with $\det(k) \in \phi^c_{\tilde{E}}$, and is invariant under $\theta$. Let $\Pi$ be an irreducible generic representation of $GL_{2n+1}(E)$ and define $\Pi^\theta$ to be an irreducible generic representation of $GL_{2n+1}(E)$ acting on $V_\sigma$ with the action $\Pi^\theta(a) = \Pi(a^\theta)$. We assume that $\Pi \cong \Pi^\theta$. Since $(\Pi^\theta)^\theta = \Pi$, there exists an intertwining map $I : \Pi \longrightarrow \Pi^\theta = \Pi$ with $I^2 = \text{id}$. This $I$ is unique up to $\pm 1$ and can be normalized in the following way. On one hand, since $\tilde{K}_{2n+1,m}$ is $\theta$-invariant, $I$ preserves the space $V_{\Pi}^{\tilde{K}_{2n+1,m}}$ for every $m \geq 0$. On the other, [4.0] and the theory of local newforms for generic representations of $GL_N$ ([JPSS89], [Jac12]) imply
$$
dim V_{\Pi}^{\tilde{K}_{2n+1,m}} = 1.
$$
Thus we can require that $I$ is the identity map on the space $V_{\Pi}^{\tilde{K}_{2n+1,m}}$. In particular, the trace of $I$ on the space $V_{\Pi}^{\tilde{K}_{2n+1,m}}$ is 1. The next lemma computes the trace of $I$ on the space $V_{\Pi}^{\tilde{K}_{2n+1,m}}$ for every $m \geq 0$. 
Lemma 4.4. Let notation be as above. We have
\[ \text{tr}(I; \mathcal{V}_H^{K_{2n+1}, m}) = \left( \frac{m-a_H}{2} + n \right) \]
for every \( m \geq 0 \).

Proof. Let \( I_m : \mathcal{V}_H \to \mathcal{V}_H \) be the \( \mathbb{C} \)-linear isomorphism defined by \( I_m = \Pi(\omega_m^{-1}) \circ I \circ \Pi(\omega_m) \). Then from (4.6) and the fact that and \( I \) preserves \( \mathcal{V}_H^{K_{2n+1}, m} \), we see that \( I_m \) preserves \( \mathcal{V}_H^{K_{2n+1}, m} \) and
\[ \text{tr}(I; \mathcal{V}_H^{K_{2n+1}, m}) = \text{tr}(I_m; \mathcal{V}_H^{K_{2n+1}, m}) \]
So it suffices to compute the trace of \( I_m \) on \( \mathcal{V}_H^{K_{2n+1}, m} \). To do so, we first recall the results of Jacquet–Piatetski-Shapiro–Shalika ([JPSS81], see also [Jac12]) and Reeder ([Ree91]). Let \( \mathcal{H} = \mathcal{H}(\text{GL}_{2n}(E)/\text{GL}_{2n}(\mathfrak{o}_E)) \) be the spherical Hecke algebra of \( \text{GL}_{2n}(F) \). We embed \( \text{GL}_{2n}(E) \) into \( \text{GL}_{2n+1}(E) \) via \( a \mapsto \left( \begin{array}{c} a \\ 1 \end{array} \right) \) and define the action of \( \mathcal{H} \) on \( \mathcal{V}_H^{\text{GL}_{2n}(E)} \) by
\[ f \ast v = \int_{\text{GL}_{2n}(E)} f(a) \Pi \left( \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & 1 \end{array} \right) \right) v \text{det}(a) |_E^{1/2} da \]
for \( f \in \mathcal{H} \) and \( v \in \mathcal{V}_H^{\text{GL}_{2n}(E)} \). Here the Haar measure \( da \) on \( \text{GL}_{2n}(E) \) is chosen so that \( \text{vol}(\text{GL}_{2n}(\mathfrak{o}_E), da) = 1 \).

Note that \( \mathcal{V}_H^{K_{2n+1}, m} \subseteq \mathcal{V}_H^{\text{GL}_{2n}(E)} \) for every \( m \geq 0 \) and we have the identity
\[ (f \ast f') \ast v = f \ast (f' \ast v) \]
for every \( f, f' \in \mathcal{H} \) and \( v \in \mathcal{V}_H^{\text{GL}_{2n}(E)} \). Indeed, a direct computation shows \( (f \ast f') \ast v = f' \ast (f \ast v) \) for every \( f, f' \in \mathcal{H} \) and \( v \in \mathcal{V}_H^{\text{GL}_{2n}(E)} \). But since \( \mathcal{H} \) is abelian, the identity (4.7) follows.

From a result of [JPSS81], we know that \( \mathcal{V}_H^{K_{2n+1}, m} = 0 \) for \( 0 \leq m < a_H \) and \( \mathcal{V}_H^{K_{2n+1}, m} \neq 0 \) for all \( m \geq a_H \) with \( \dim_{\mathbb{C}} \mathcal{V}_H^{K_{2n+1}, m} = 1 \). Fix a non-zero element \( v_0 \in \mathcal{V}_H^{K_{2n+1}, a_H} \). Then our normalization on \( I \) implies
\[ I_{a_H}(v_0) = v_0. \]

To describe a basis of \( \mathcal{V}_H^{K_{2n+1}, m} \) for \( m > a_H \), we recall a result of Reeder ([Ree91]). For \( a \in \text{GL}_{2n}(E) \), we denote
\[ [a] = \text{GL}_{2n}(\mathfrak{o}_E) a \text{GL}_{2n}(\mathfrak{o}_E). \]

For \( 0 \leq i \leq 2n \), we write
\[ \omega_i = \text{diag}(\omega, \ldots, \omega, 1, \ldots, 1) \in \text{GL}_{2n}(E) \]
and put
\[ f_i = q_E^{i(2n-i)/2} I_{1/2}(\omega_i). \]

Then
\[ \beta_\ell = \left\{ f_0 f_1 \cdots f_{2n} v_0 \mid \ell_i \geq 0 \text{ for } 0 \leq i \leq 2n \text{ and } \ell_0 + \ell_1 + \cdots + \ell_{2n} = \ell \right\} \]
is a basis of \( \mathcal{V}_H^{K_{2n+1}, a_H+i} \) for every \( \ell \geq 0 \).

To proceed, we prove the following identity, which is the key to the proof of this lemma. Let \( f \in \mathcal{H}, v \in \mathcal{V}_H^{K_{2n+1}, m} \) and suppose that \( f \ast v \in \mathcal{V}_H^{K_{2n+1}, m+1} \). Then we have
\[ I_{m+1}(f \ast v) = f' \ast I_m(v) \]
with \( f' \in \mathcal{H} \) defined by
\[ f'(a) = q_E^n f(a^{-1}) \text{det}(a)^{-1} \]
for \(a \in \text{GL}_2(E)\), where we view \(\varpi\) as an element in the center of \(\text{GL}_2(E)\). The proof of this identity is by a straightforward computation:

\[
I_{m+1}(f \ast v) = \int_{\text{GL}_2(E)} f(a)(I_{m+1} \circ \Pi)(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix})v| \det(a) | \frac{1}{2} da
\]

\[
= \int_{\text{GL}_2(E)} f(a)(\Pi(\omega_{m+1}^{-1} \circ I \circ \Pi)(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix})\omega_{m+1}^{-1} \omega_m^{-1})I(\Pi(\omega_m)v)| \det(a) | \frac{1}{2} da
\]

\[
= \int_{\text{GL}_2(E)} f(a)\Pi(\omega_{m+1} \omega_m^{-1} \theta (a^{-1} 1)\omega_m^{-1} \theta)I_m(v)| \det(a) | \frac{1}{2} da
\]

\[
= \int_{\text{GL}_2(E)} f(a)\Pi(\varpi^{-1} J^{1} \omega J_{2n}^{-1} 1)I_m(v)| \det(a) | \frac{1}{2} da
\]

\[
= f^i \ast I_m(v)
\]

where \(J := (I_n - I_n) \vartheta_{2n} \in \text{GL}_2(o_E)\). This proves (7.2).

Recall that \(f_i \in \mathcal{H}\) is given by (4.9) for \(0 \leq i \leq 2n\). One can check directly that

\[
(4.12) 
\]

\[
f_i = q^{-i} f_{2n-i}.
\]

Now we are ready to compute \(\text{tr}(\Pi_{\mathcal{H}} \vartheta_{2n+1,m})\). We may assume \(m \geq a_H\) since otherwise the space \(\vartheta_{2n+1,m}\) is zero and so is the trace. Let \(m = a_H + \ell\) for some \(\ell \geq 0\). To obtain the trace, we compute the matrix of \(I_m\) with respect to the basis (4.10). Let \(v = f_{f_0} f_{f_1} \cdots f_{f_{2n}} \ast v_0 \in \beta_{\ell}\) so that \(\ell_0 + \ell_1 + \cdots + \ell_{2n} = \ell\). Then we have

\[
I_m(v) = I_{a_H + \ell}(v) = q_{E}^{\ell_0} f_{f_0} f_{f_1} \cdots f_{f_{2n}} \ast v_0
\]

by (4.7), (4.8), (4.10) and (4.12). From this we see that \(I_{a_H + \ell}(v)\) is constant multiple of an element in \(\beta_{\ell}\), and moreover,

\[
I_{a_H + \ell}(v) \in Cv
\]

if and only if \(\ell_i = \ell_{2n-i}\) for \(0 \leq i \leq 2n\), in which case we have \(I_{a_H + \ell}(v) = v\). It follows that

\[
\text{tr}(I_{a_H + \ell} \vartheta_{2n+1,m}) = |\{(\ell_0, \ldots, \ell_{2n}) \in \mathbb{Z}_{>0}^{2n+1} | \ell_i = \ell_{2n-i}\}|
\]

\[
= |\{(\ell_0, \ldots, \ell_n) \in \mathbb{Z}_{>0}^{n+1} | 2\ell_0 + \cdots + 2\ell_{n-1} + \ell_n = \ell\}|
\]

\[
= |\{(\ell_0, \ldots, \ell_{n-1}) \in \mathbb{Z}_{>0}^{n+1} | \ell_0 + \cdots + \ell_{n-1} \leq [\ell/2]\}|
\]

\[
= \left\lfloor \frac{\ell}{2} \right\rfloor + n.
\]

This completes the proof. \(\square\)

4.2. Proof of Theorem 1.2 In the following proof, we will retain the notation in the previous subsection. Let \(\pi\) be an irreducible generic representation of \(G_n\) with the associated \(L\)-parameter \(\phi_{\pi} : WD_E \rightarrow \text{GL}_{2n+1}(\mathbb{C})\) which is conjugate orthogonal (cf. [GGPT2 Section 8], [Mok15]). Then by the local Langlands correspondence for \(\text{GL}_{2n+1}\), \(\phi_{\pi}\) corresponds to an irreducible representation \(\Pi\) of \(\text{GL}_{2n+1}(E)\). This \(\Pi\) is conjugate self-dual and therefore \(\Pi \cong \Pi^\vee\). Assume first that \(\pi\) is tempered. Then \(\Pi\) is also tempered and hence generic. Let \(I : \Pi \rightarrow \Pi^\vee\) be the normalized intertwining map as in the paragraph before Lemma 4.4. Then the proofs of [AYO22] Theorems 4.3, 4.4] imply

\[
\dim_C \vartheta_{2n+1,m} = \text{tr}(I; \Pi_{\mathcal{H}}\vartheta_{2n+1,m})
\]

for every \(m \geq 0\). Now the desired identity for tempered \(\pi\) follows from Lemma 4.4.
Suppose that $\pi$ is non-tempered from now on. Then by the Langlands’ classification ([Sil78]) and standard module conjecture ([CS98], [Mü01]),

$$\pi \cong \tau_1 \times \cdots \times \tau_k \times \pi_0 := \text{Ind}_{\mathbb{Z}^k}^G(\tau_1 \otimes \cdots \otimes \tau_k \otimes \pi_0)$$

(normalized induced)

(4.13)

where $1 \leq r \leq n$ is an integer, $P = (\tau_1, \ldots, \tau_k)$ is a partition of $r$, $P_{\mathbb{Z}^k} \subset G_{\mathbb{Z}}$ is the parabolic subgroup containing $B_{\mathbb{Z}}$ whose Levi subgroup isomorphic to $\text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k} \times G_{n-r}$, $\tau_j$ is an irreducible essentially square integrable representations of $\text{GL}_{r_j}$ for $1 \leq j \leq k$, and $\pi_0$ is an irreducible tempered generic representation of $G_{n-r}$. The associated $L$-parameter then decompose accordingly

$$\phi_{\pi} = \phi_{\tau_1} \otimes \cdots \otimes \phi_{\tau_k} \otimes \phi_{\pi_0} \otimes \phi_{\pi_1} \otimes \phi_{\pi_2} \otimes \cdots \otimes \phi_{\pi_l}$$

where $\phi_{\tau_j}$ (resp. $\phi_{\pi_0}$) is the $L$-parameter attached to $\tau_j$ (resp. $\pi_0$) for $1 \leq j \leq k$. Since

$$\epsilon(s, \phi_{\pi}, \psi_E) = \epsilon(s, \phi_{\pi_0}, \psi_E) \prod_{j=1}^k \epsilon(s, \phi_{\tau_j}, \psi_E) \epsilon(s, \phi_{\tau_j}, \psi_E)$$

we find that

$$a_\pi = a_{\pi_0} + 2 \sum_{j=1}^k a_{\tau_j}$$

by Corollary 4.2

Since $K_{n,m}$ is conjugate to $K_{n,m}^0$ for each $m$, it suffices to prove (1.2) with $V_\pi^{K_{n,m}}$ replaced by $V_\pi^{K_{n,m}^0}$. We do this by the induction on $k$. Suppose that $k = 1$ so that $r = \tau_1$, $\pi = \tau_1 \times \pi_0$ and $a_\pi = a_{\pi_0} + 2a_{\tau_1}$. Write $m = e + 2\ell$ for some $e = 0,1$ and $\ell \geq 0$. Then Corollary 3.3 implies

$$\dim_{\mathbb{C}} V_\pi^{K_{n,m}} = \sum_{d=0}^\ell \dim_{\mathbb{C}} (V_{\tau_1} \otimes V_{\pi_0})^{\Gamma_{n-r}} \sum_{d=0}^\ell \left( \dim_{\mathbb{C}} V_\tau^{\Gamma_{r,e-d}} \right) \left( \dim_{\mathbb{C}} V_{\pi_0}^{K_{n-r,a_e+2d}} \right).$$

Observe that if $\ell - d < a_{\tau_1}$ or $e + 2d < a_{\pi_0}$, then $V_{\tau_1}^{\Gamma_r,e-d} = 0$ or $V_{\pi_0}^{K_{n-r,a_e+2d}} = 0$ accordingly by Theorem 1.1 and the theory of local newforms for $\text{GL}_r$ (cf. [JPSS81]). It follows that $V_\pi^{K_{n,m}^0} = 0$ if $m < a_\pi$. Now suppose that $m \geq a_\pi$. If $m$ and $a_\pi$ have the same parity, then we can write $m = a_\pi + 2\ell_1 = a_{\pi_0} + 2(a_{\tau_1} + \ell_1)$ for some $\ell_1 \geq 0$. Note that $m$ and $a_{\pi_0}$ also have the same parity. Then the above observation and the dimension formulae for $\pi_0$ (cf. (1.2)) and $\tau_1$ (cf. [Re91]) give

$$\dim_{\mathbb{C}} V_\pi^{K_{n,m}^0} = \sum_{d=0}^\ell \left( \dim_{\mathbb{C}} V_{\tau_1}^{\Gamma_{r,e-d}} \right) \left( \dim_{\mathbb{C}} V_{\pi_0}^{K_{n-r,a_e+2d}} \right)$$

$$= \sum_{d=0}^\ell \left( \dim_{\mathbb{C}} V_{\tau_1}^{\Gamma_{r+\ell_1-1-e-d}} \right) \left( \dim_{\mathbb{C}} V_{\pi_0}^{K_{n-r,a_{\pi_0}+\ell_1+2d}} \right)$$

$$= \sum_{d=0}^\ell \left( \frac{r - 1 + \ell_1 - d}{\ell - d} \right) \left( \ell_1 + n - r \right)$$

$$= \left( \frac{\ell_1 + n}{n} \right).$$

The last equality follows from the combinatorial identity in ([Gou72], (3.2)]. This proves (1.2) when $m$ and $a_\pi$ have the same parity. The proof when $m$ and $a_\pi$ have the opposite parity is similar. We just need to replaced $a_\pi$ by $a_\pi + 1$. The proof for $k = 1$ is now complete. Suppose inductively that (1.2) holds for $k-1$ and $\pi$ is of the form (1.1). Then by induction in stage, we can write $\pi \cong \pi_1 \times \pi_1$ where $\pi_1 := \tau_2 \times \cdots \tau_k \times \pi_0$ is an irreducible generic representation of $G_{n-\tau_1}$. Since $a_{\pi_1} = a_{\pi_1} + 2a_{\tau_1}$ and (1.2) holds for $\pi_1$, by the induction hypothesis, we can apply the argument for $k = 1$ to obtain (1.2) for $\pi$. This finishes the proof. \qed

5. Conjectural bases for oldforms

Now we come to the second part of this paper, namely, we would like to compute the Rankin-Selberg integrals attached newforms and also oldforms. As we will see, many statements and their proofs in this part of the paper are similar to that of [Che22]. So in these cases, we will simply write down the statements and refer their proofs to those in loc. cit. However, we will indicate the modifications whenever needed.
5.1. **Level raising operators.** As in the literature, the conjectural bases for oldforms are obtained from certain level raising procedures. To define the level raising operators, we begin with the following lemma whose proof is similar to that of [Che22, Lemma 3.1].

**Lemma 5.1.** Let $dh$ be a Haar measure on $H_r$. Then we have $\text{vol}(R_{r,m}, dh) = \text{vol}(R_{r,0}, dh)$ for all $m \geq 0$.

**Proof.** Let $t_{r,m} = \text{diag}(\omega^m I_r, I_r)$. Then $t_{r,m} \in \text{GU}_2(F)$, the similitude unitary group with $2r$ variables and $t_{r,m}R_{r,m}t_{r,m}^{-1} = R_{r,0}$. Now we can apply the proof of [Che22, Lemma 3.1] to obtain the lemma. Note that the intersection of the center of $\text{GU}_2(F)$ and $\text{U}_2(F)$ is $E^1$, which is compact. So the proof of loc. cit. is still applicable. 

Following [Ree91] and [Tsa13, Che22], the level raising operators coming from the elements in the Hecke algebras $\mathcal{H}(H_n/|R_{n,m})$ for $m \geq 0$. To describe a $\mathbb{C}$-linear basis of $\mathcal{H}(H_n/|R_{n,m})$, let

$$\mathcal{P} = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \}.$$ 

Given $\lambda \in \mathcal{P}$, we denote

$$\omega^\lambda = \text{diag}(\omega^{\lambda_1}, \ldots, \omega^{\lambda_n}, \omega^{-\lambda_n}, \ldots, \omega^{-\lambda_1}) \in T_n$$

and put

$$\varphi_{\lambda,m} = 1_{R_{n,m}} \omega^\lambda R_{n,m} \in \mathcal{H}(H_n/|R_{n,m}).$$

Then we have

$$\mathcal{H}(H_n(F)/|R_{n,m}) = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C} \cdot \varphi_{\lambda,m}.$$ 

as $\mathbb{C}$-linear spaces.

Now let $\pi$ be an irreducible generic representation of $G_n$ and $\nu \in \mathcal{V}^{R_{n,m}}_\pi$. We define

$$\varphi \ast \nu = \int_{H_n} \varphi(h) \nu(h^{-1}) vdh \quad (\text{vol}(R_{n,0}, dh) = 1).$$

where $\varphi \in \mathcal{H}(H_n/|R_{n,m})$. Clearly, we have $\varphi \ast \nu \in \mathcal{V}^{R_{n,m}}_\pi$. In particular, if $\nu \in \mathcal{V}^{K^0_{n,m}}_\pi$ and $\varphi \in \mathcal{H}(H_n/|R_{n,e})$, where $e = 0, 1$ is such that $m \equiv e (\text{mod } 2)$, then $\varphi \ast \nu \in \mathcal{V}^{K^0_{n,m'}}_\pi$ for some $m' \equiv e (\text{mod } 2)$ by the filtration (2.5).

The following lemma tells us what $m'$ is.

**Lemma 5.2.** Let $\nu \in \mathcal{V}^{K^0_{n,m}}_\pi$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}$. Then we have $\varphi_{\lambda,e} \ast \nu \in \mathcal{V}^{K^0_{n,m+2\lambda_1}}_\pi$.

**Proof.** The proof of this lemma is similar to that of [Tsa13, Proposition 8.1.1]. However, since Tsai’s proof is very brief, we shall fill in some details here. Let us write $m = e + 2 \ell$ for some $\ell \geq 0$. Let $v_1 = \nu(\omega^{-\lambda}) \nu$ and

$$v_2 = \int_{R_{n,e}} \pi(h) v_1 dh.$$

Then $v_1$ is $\omega^{-\lambda} K^0_{n,m} \omega^\lambda$-invariant and $v_2$ is a non-zero multiple of $\varphi_{\lambda,e} \ast \nu$. Thus it suffices to show that $v_2$ is fixed by $K^0_{n+2\lambda_1}$. For this, we use the decomposition (2.6) of $K^0_{n,m}$ when $m \geq 1$ and the fact that $K^0_{n,0} = K_{n,0}$ is generated by $E^1$, $R_{n,0}$ and the root groups $\chi_{+\epsilon}(o_E)$ for $1 \leq i \leq n$.

Certainly, $v_2$ is fixed by $R_{n,e}$ and $E^1_{n+2\lambda_1}$. Therefore, we only need to check that $v_2$ is invariant under $\chi_{+\epsilon}(p^e_{\ell+1})$ and $\chi_{-\epsilon}(p^e_{\ell+1})$ for $1 \leq i, j \leq n$ by the facts just mentioned. By conjugating these root groups by Weyl elements in $W$ (cf. (2.6)), we actually only need to check that $v_2$ is $\chi_{-\epsilon}(p^e_{\ell+1})$-invariant. To do so, we claim that $h \cdot u \cdot h^{-1} \in \omega^{-\lambda} K^0_{n,m} \omega^\lambda$ for all $h \in R_{n,e}$ and $u \in \chi_{-\epsilon}(p^e_{\ell+1})$. Assuming the claim, we find that

$$\pi(u) v_2 = \int_{R_{n,e}} \pi(uh) v_1 dh = \int_{R_{n,e}} \pi(h \cdot u \cdot h^{-1}) v_1 dh = \int_{R_{n,e}} \pi(h) v_1 dh = v_2$$

and the proof follows.

To verify the claim, note that $\omega^{-\lambda} K^0_{n,m} \omega^\lambda$ contains $\chi_{+\epsilon}(p^e_{\ell+1})$, $\chi_{-\epsilon}(p^e_{\ell+1})$ for $1 \leq i, j \leq n$ and $\omega^{-\lambda} R_{n,e} \omega^\lambda$.

Moreover, $R_{n,e}$ is generated by $T_n \cap R_{n,e}, \chi_{+\epsilon}(o_E)$, $\chi_{-\epsilon}(o_E), \chi_{+\epsilon+\epsilon}(o_E)$, $\chi_{-\epsilon+\epsilon}(o_E)$ for $1 \leq i < j \leq n$ and also $\chi_{2\epsilon_k}(p^e_{\ell+1})$ for $1 \leq k \leq n$. Here for $y \in F$ and $1 \leq k \leq n$, we put

$$\chi_{2\epsilon_k}(x) = I_{2n+1} + x \delta E_{2n+2-k,k} \quad \text{and} \quad \chi_{-2\epsilon_k}(x) = I_{2n+1} + x \delta E_{2n+2-k,k}.$$
where $E_{ij}$ denotes the $(2n+1)$-by-$(2n+1)$ matrix whose $(i,j)$-entry is 1 and all other entries are 0, and 
\[ \chi_{\pm 2s}(S) = \{ \chi_{\pm 2s}(x) \mid x \in S \} \]
for any subset $S$ of $F$. Now let $y \in p_{E}^{(v + \lambda)}$ and put $u = \chi_{e_{n}}(y)$. To show that $h^{-1}uh \in \omega^{-\lambda}K_{n,m}^{0}\omega^{\lambda}$ for every $h \in R_{n,e}$, we may assume that $h$ is an element in one of the subgroups generating $R_{n,e}$. Then the claim is proved via case by case verification. For example, one checks that

\[ \chi_{e_{i}}(y_{n}) = \chi_{e_{i}}(y_{n}2^{-1}y\tilde{y}^2) \chi_{e_{i}}(-yz) \]

for $1 \leq i \leq n-1$. In particular, if $z \in \mathcal{O}_{E}$, then the RHS of the above identity is contained in $\omega^{-\lambda}K_{n,m}^{0}\omega^{\lambda}$. Indeed, since $y \in p_{E}^{v + \lambda} \subseteq p_{E}^{v - \lambda}$ and $-yz \in p_{E}^{v + \lambda} \subseteq p_{E}^{v - \lambda}$, we see that both $\chi_{e_{n}}(y)$ and $\chi_{e_{i}}(-yz)$ are contained in $\omega^{-\lambda}K_{n,m}^{0}\omega^{\lambda}$. On the other hand, since $2^{-1}y\tilde{y}^2 \in p_{E}^{v + 2\lambda}$, we find that $\chi_{e_{i}}(2^{-1}y\tilde{y}^2)\omega^{-\lambda}R_{n,e}\omega^{\lambda}$.

This completes the proof.

Now we can define the level raising operators used in the constructions of the conjectural bases. Put

\[ \mu_{\ell} = (\ell, \ldots, \ell) \in \mathcal{P} \]

for $\ell \geq 0$. Let $v \in V_{\pi}^{K_{n,m}}$ and write $m = e + 2\ell$ for some $e = 0, 1$ and $\ell \geq 0$. Then $\pi(\omega^{\mu_{\ell}})v \in V_{\pi}^{K_{n,m}^{0}}$; hence

\[ \varphi_{\lambda,e} \ast \pi(\omega^{\mu_{\ell}})v \in V_{\pi}^{K_{n,m+2\lambda}} \]

by Lemma 5.2 where $\lambda = (\lambda_{1}, \ldots, \lambda_{n}) \in \mathcal{P}$. Since $V_{\pi}^{K_{n,m+2\lambda}} \subseteq V_{\pi}^{K_{n,m}}$ if $m'$ and $m$ have the same parity and $m' \geq m + 2\lambda$, we thus obtain a level raising operator $\eta_{\lambda,m,m'}$ from $V_{\pi}^{K_{n,m}}$ to $V_{\pi}^{K_{n,m'}}$ given by

\[ \eta_{\lambda,m,m'}(v) = \pi(\omega^{-\mu_{\ell'}})\varphi_{\lambda,e} \ast \pi(\omega^{\mu_{\ell}})v \]

where $m' = e + 2\ell'$ for some $\ell' \geq 0$ and we understand that $\omega^{-\mu_{\ell'}} = (\omega^{\mu_{\ell'}})^{-1}$. Note that $\eta_{\mu_{0},m,m}$ is the identity map by Lemma 5.1 and $\eta_{\lambda,m,m}$ raises by an even integer. To raise levels to those with opposite parity, it natural to consider the operators from $V_{\pi}^{K_{n,m}}$ to $V_{\pi}^{K_{n,m+1}}$ defined by

\[ v \mapsto \text{vol}(K_{n,m} \cap K_{n,m+1}, dg)^{-1} \int_{K_{n,m+1}} \pi(g)vdg. \]

However, we don’t consider these operators here. The reason will be clear in the next subsection.

5.2. Conjectural bases. Let $\pi$ be an irreducible generic representation of $G_{n}$ and $v_{\pi} \in V_{\pi}^{K_{n,a_{\pi}}}$ be a newform. By Theorem 1.2 the space $V_{\pi}^{K_{n,a_{\pi}+1}}$ is also one-dimensional. So we simply let $v'_{\pi}$ be its basis. Now let $m \geq a_{\pi}$ be an integer. If $m$ and $a_{\pi}$ have same parity, then we put

\[ B_{\pi,m} = \left\{ \eta_{\lambda,a_{\pi},m}(v_{\pi}) \mid \lambda = (\lambda_{1}, \ldots, \lambda_{n}) \in \mathcal{P} \text{ with } 2\lambda_{1} \leq m - a_{\pi} \right\}. \]

On the other hand, if $m$ and $a_{\pi}$ have opposite parity, then we set

\[ B_{\pi,m} = \left\{ \eta_{\lambda,a_{\pi}+1,m}(v'_{\pi}) \mid \lambda = (\lambda_{1}, \ldots, \lambda_{n}) \in \mathcal{P} \text{ with } 2\lambda_{1} \leq m - a_{\pi} - 1 \right\}. \]

We conjecture that $B_{\pi,m}$ is a basis of $V_{\pi}^{K_{n,m}}$ for each $m \geq a_{\pi}$. In fact, it’s not hard to see that the set $B_{\pi,m}$ has right cardinality, namely, the cardinality of $B_{\pi,m}$ is equal to the dimension of $V_{\pi}^{K_{n,m}}$ given by [1.2]. We introduce these conjectural bases in order to compute the Rankin-Selberg integrals attached to oldforms. Logically, one has to first verify that these $B_{\pi,m}$ are linearly independent and then compute the integrals attached to the elements in $B_{\pi,m}$. Here we directly compute the integrals (under the Assumption 1.3) without knowing the linear independence. As a consequence of our computations, we show that $B_{\pi,m}$ are linearly independent when $\pi$ is tempered. This also gives another reason why we don’t write $v'_{\pi}$ as a level raising of $v_{\pi}$, since we can compute the Rankin-Selberg integral attached to $v_{\pi}$ without appealing to level raising operators, but only use the fact that $V_{\pi}^{K_{n,a_{\pi}+1}}$ is one-dimensional. Of course, one can still ask whether or not

\[ \int_{K_{n,a_{\pi}+1}} \pi(g)v_{\pi}dg \neq 0. \]

But we don’t investigate this problem due to our purpose.
6. Rankin-Selberg integrals

In this section, we introduce the local Rankin-Selberg integrals attached to generic representations of $U_{2n+1}$ for $GL_r$ with $1 \leq r \leq n$ constructed in \cite{Tam91} (for $r = n$) and \cite{BAS09} (for $r < n$). In this section only, we take $\psi_0$ to be an arbitrary non-trivial additive character of $F$, $\theta \in E^*$ with $\theta = -\theta$ and let $\psi$ be the additive character of $E$ defined by $\psi(x) = \psi(x/29)$, so that $\psi$ is trivial on $F$. To describe these integrals, let $Z_r \subset GL_r(F)$ be the upper triangular maximal unipotent subgroup and $\psi_{Z_r}$ be a non-degenerate character of $Z_r$ defined by

$$\tilde{\psi}_{Z_r}(z) = \psi(z_{12} + z_{23} + \cdots + z_{r-1,r})$$

for $z = (z_{ij}) \in Z_r$. Let $\tau$ be a representation of $GL_r(F)$, which is of Whittaker type in the sense that

$$\dim_{C} \text{Hom}_{Z_r}(\tau, \psi_{Z_r}) = 1.$$

In particular, irreducible generic representations of $GL_r(E)$ are of Whittaker type. We fix a non-zero element $\Lambda_{\tau, \psi}$ in $\text{Hom}_{Z_r}(\tau, \psi_{Z_r})$. Define $\tau^\ast$ to be the representation of $GL_r(F)$ on $V_\tau$ with the action $\tau^\ast(a) = \tau(a^\ast)$. Notice that $\tau^\ast$ is also of Whittaker type and we fix $\Lambda_{\tau^\ast, \psi} \in \text{Hom}_{Z_r}(\tau^\ast, \psi_{Z_r})$ with $\Lambda_{\tau^\ast, \psi} = \Lambda_{\tau, \psi} \circ \tau(d_r)$ where

$$d_r = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (-1)^{r-1} & \cdots & -1 & 1 \end{pmatrix} \in GL_r(E).$$

Let $s$ be a complex number and $\tau_s$ be a representation of $GL_r(E)$ on $V_\tau$ with the action $\tau_s(a) = \tau(a)|\det(a)|^{s+\frac{1}{2}}$.  

6.1. Induced representations. Let $V_r \subset H_r$ be the upper triangular maximal unipotent subgroup and $Q_r \subset H_r$ be the parabolic subgroup containing $V_r$ with the Levi decomposition $Q_r = M_r \ltimes Y_r$ where

$$M_r = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^\ast \end{pmatrix} \mid a \in GL_r(E) \right\} \cong GL_r(E)$$

and $Y_r \subset V_r$. By pulling back the homomorphism $Q_r \rightarrow Q_r/Y_r \cong GL_r(E)$, $\tau_s$ becomes a representation of $Q_r$ on $V_r$. We then form a normalized induced representation

$$\rho_{\tau,s} = \text{Ind}_{Q_r}^{H_r} \tau_s$$

of $H_r$. The representation space $I_r(\tau, s)$ of $\rho_{\tau,s}$ consists of smooth functions $\xi_s : H_r \rightarrow V_r$ satisfying

$$\xi_s(mnh) = \delta_{Q_r}^\frac{1}{2}(m)\tau_s(m)\xi_s(h)$$

for $m \in M_r$, $n \in Y_r$ and $h \in H_r$, and the action of $H_r$ on $I_r(\tau, s)$ is by the right translation.

6.2. Intertwining maps. There is an intertwining map $M(\tau, s)$ from $I_r(\tau, s)$ to $I_r(\tau^\ast, 1 - s)$ given by the integral

$$M(\tau, s)\xi_s(h) = \int_{Y_r} \xi_s(w_r^{-1}uh)du$$

for $\Re(s) >> 0$ and by meromorphic continuation in general. Here

$$w_r = \begin{pmatrix} I_r \\ I_r \end{pmatrix} \in H_r$$

and the Haar measure $du$ is defined as a product measure, where each root group is isomorphic to $E$ or $F$, and we take the self-dual measure on them with respect to $\psi$ or $\psi_0$. We normalize $M(\tau, s)$ by Shahidi’s local coefficients $c_\theta(2s - 1, \tau, As, \psi_0)$. To define them, let us put

$$f_{\xi_s}(h) = \Lambda_{\tau, \psi}(\xi_s)(h).$$

Similarly, for a given $\xi^\ast_s \in I_r(\tau^\ast, s)$, let $f_{\xi^\ast_s}(h) = \Lambda_{\tau^\ast, \psi}(\xi^\ast_s)(h)$. Then the local coefficients are given by

$$\int_{V_r} f_{\xi_s}(w_ruh)\psi(u_{r,r+1})du = c_\theta(2s - 1, \tau, As, \psi_0) \int_{V_r} f_{M(\tau, s)\xi_s}(w_ruh)\psi(u_{r,r+1})du.$$

where $u = (u_{ij}) \in V_r$ and the Haar measure $du$ on $V_r$ is arbitrary. The normalized intertwining maps $M_{\psi_0, \theta}^{\dagger}(\tau, s)$ are then defined by

$$M_{\psi_0, \theta}^{\dagger}(\tau, s) = c_\theta(2s - 1, \tau, As, \psi_0)M(\tau, s).$$
6.3. **Rankin-Selberg integrals.** Recall that $N_n \subset G_n$ is the upper triangular maximal unipotent subgroup. Define a non-degenerated character $\psi_{N_n}$ of $N_n$ by

$$
\psi_{N_n}(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1,n} + u_{n,n+1})
$$

for $u = (u_{ij}) \in N_n$. Let $\pi$ be a representation of $G_n$ that is of Whittaker type, i.e.

$$
dim_{\mathbb{C}} \text{Hom}_{N_n}(\pi, \psi_{N_n}) = 1.
$$

In particular, irreducible generic representations of $G_n$ are of Whittaker type. We fix a non-zero element $\Lambda_{\pi, \psi}$ in $\text{Hom}_{N_n}(\pi, \psi_{N_n})$. For $v \in \mathcal{V}_\pi$, let $W_v$ be the associated Whittaker function with respect to $\psi$, that is, $W_v$ is a $\mathbb{C}$-valued function on $G_n$ given by $W_v(y) = \Lambda_{\pi, \psi}(\pi(y)r)$.

For a given $x \in \text{Mat}_{(n-r)\times r}(E)$, we denote

$$(6.1) \quad \tilde{x} = \begin{pmatrix} I_r & x & I_{n-r} \\ x & -J_r \tilde{x} J_{n-r} & I_r \end{pmatrix} \in G_n.$$ 

Then the Rankin-Selberg integral $\Psi_{n,r}(v \otimes \xi_s)$ attached to $v \in \mathcal{V}_\pi$ and $\xi_s \in I_r(\tau, s)$ is given by

$$(6.2) \quad \Psi_{n,r}(v \otimes \xi_s) = \int_{\mathcal{V}_\pi \setminus H_r} \int_{\text{Mat}_{(n-r)\times r}(E)} W_v(\tilde{x}h) f_{\xi_s}(h) dx dh. $$

The Haar measures $dx$ and $dh$ can be any at this moment and we embed $H_r$ into $G_n$ via the embedding given in (2.2). As usual, these integrals, which are originally absolute convergence in some half planes, have meromorphic continuations to whole complex plane, and give rise to rational functions in $q_E^{-s}$. Moreover, the following functional equations

$$(6.3) \quad \Psi_{n,r}(v \otimes M^\dagger_{\psi_0, \theta}(\tau, s) \xi_s) = \gamma_0(s, \pi \times \tau, \psi_0) \Psi_{n,r}(v \otimes \xi_s)$$

are satisfied for every $v \in \mathcal{V}_\pi$ and $\xi_s \in I_r(\tau, s)$, where $\gamma_0(s, \pi \times \tau, \psi_0)$ is a nonzero rational function in $q_E^{-s}$ depending only on $\psi_0$, $\theta$ and (the classes of) $\pi$, $\tau$.

**Remark.** We have some comments about these integrals. First, our definition of $\Psi_{n,r}(v \otimes \xi_s)$ is slightly different from (but equivalent to) that of [BAS09] because we want to emphasize the resemblance between these integrals and the ones attached to generic representations of special orthogonal groups (cf. [Som99]). Second, in [BAS09], the integrals are attached to irreducible generic representations and the results about them, namely, admit meromorphic continuations, are rational functions in $q_E^{-s}$ and can be made into a non-zero constant, are proved for irreducible ones. However, their proofs are easily apply to our more general settings. The only less obvious statement is the existence of the functional equations. In fact, in loc. cit., the authors did not investigate the functional equations of these integrals. On the other hand, Q. Zhang proved the existence of the functional equations for irreducible representations in [Zhi19] by applying the uniqueness of Bessel model for $U_{2n+1}$ established in [AGRS10] and [GGPT12]. In the Appendix, we will extend Zhang’s result to representations of Whittaker type.

The following lemma explains why we only consider unramified representations of $GL_r(E)$ when computing the Rankin-Selberg integrals.

**Lemma 6.1.** Suppose that $v \in \mathcal{V}_\pi^{K_{r,m}}$ is a nonzero element. Then the integrals $\Psi_{n,r}(v \otimes \xi_s)$ vanish for all $\xi_s \in I_r(\tau, s)$ if $\tau$ is not unramified. Furthermore, if $r = n$, $\tau$ is unramified and $W_v$ is identically zero on $T_n$, then the integrals $\Psi_{n,n}(v \otimes \xi_s)$ also vanish for all $\xi_s \in I_n(\tau, s)$.

**Proof.** The proof of this lemma is similar to that of [Che22, Lemma 4.3]. Indeed, the decomposition $H_r = Q_r R_{r,m}$ and the fact $M_r \cap R_{r,m}$ give the isomorphism $I_r(\tau, s)^{R_{r,m}} \cong \mathcal{V}_\tau^{GL_n(\sigma_E)}$ between $\mathbb{C}$-linear spaces. Now if $\xi_s \in I_r(\tau, s)$, then the proof in loc. cit. implies that

$$
\Psi_{n,r}(v \otimes \xi_s) = \Psi_{n,r}(v \otimes \xi'_s)
$$

for some $\xi'_s \in I_r(\tau, s)^{R_{r,m}}$. In particular, the integral $\Psi_{n,r}(v \otimes \xi_s) = 0$ if $\tau$ is not unramified.
Next suppose that $r = n$, $\tau$ is unramified and $W_v$ is identically zero on $T_n$. Then it follows from the Iwasawa decomposition $H_r = B_{H_n} R_n, m$ that
\[
\Psi_{n,n}(v \otimes \xi_s) = \Psi_{n,n}(v \otimes \xi'_s) = \int_{T_n} W_v(t) f_{E'}(t) \delta_{B_{H_n}}^{-1}(t) dt = 0.
\]
Here $B_{H_n} = T_n \times V_n$ stands for the upper triangular Borel subgroup of $H_n$. This concludes the proof. \[\square\]

We also need the following lemma.

**Lemma 6.2.** Let $\pi$ (resp. $\tau$) be an irreducible tempered generic representation of $G_n$ (resp. $\text{GL}_r(E)$). Then the Rankin-Selberg integrals attached to $\pi$ and $\tau$ converge absolutely for $\Re(s) > 0$.

**Proof.** By the Iwasawa decomposition $H_r = B_{H_r} R_{r,0}$, the identification $\hat{A}_r = T_r$ and the right $R_{r,0}$-finiteness, we see that each Rankin-Selberg integral can be written as a finite sum of the integrals of the form
\[
\int_{A_r} \int_{\text{Mat}((n-r), r)(E)} W(\hat{x}\hat{y}) W'(y) \delta_{Q_r}(\hat{y}) \delta_{B_{H_r}}^{-1}(\hat{y}) |\det(y)|_E^{s-r-n} \, dx \, dy
\]
where $W$ (resp. $W'$) is the Whittaker function of an element in $\mathcal{V}_\pi$ (resp. $\mathcal{V}_\tau$) with respect to $\psi$ (resp. $\hat{\psi}$), and $B_{H_r} = T_r \times V_r$ is the upper triangular Borel subgroup of $H_r$. Since $\hat{y}^{-1} \hat{x}\hat{y} = \hat{x}'\hat{y}$, we get from changing the variable that the integral becomes
\[
\int_{A_r} \int_{\text{Mat}((n-r), r)(E)} W(\hat{y}\hat{x}) W'(\hat{a}) \delta_{Q_r}(\hat{y}) \delta_{B_{H_r}}^{-1}(\hat{y}) |\det(y)|_E^{s-r-n} \, dx \, dy.
\]
At this point we invoke [BAS09, Lemma 4.1], which asserts that the function
\[
x \mapsto W(\hat{y} x)
\]
on $\text{Mat}((n-r), r)(E)$ has compact support, which is independent of $y$. It follows that the integral can be further written as a finite sum of the integrals of the form (with possibly different $W$)
\[
\int_{A_r} W(\hat{y}) W'(y) \delta_{Q_r}(\hat{y}) \delta_{B_{H_r}}^{-1}(\hat{y}) |\det(y)|_E^{s-r-n} \, dy.
\]
To continue, we parametrize
\[
y = \text{diag}(y_1, \ldots, y_r, y_{r+1}, \ldots, y_{r+1} y_{r+1} y_{r+1} \ldots y_r)
\]
for $y_1, \ldots, y_r$ in $E^\times$ and use asymptotic expansions of Whittaker functions (cf. [CSS80, Section 6], [LM09]) to express $W(\hat{y})$ (resp. $W'(\hat{y})$) as a finite sum of the functions of the form
\[
\delta_{B_{H_n}}(\hat{y}) \prod_{j=1}^r \varphi_j(y_j) \chi_j(y_j) \quad \text{(resp.} \quad \delta_{B_{GL_r}}(\hat{y}) \prod_{j=1}^r \varphi'_j(y_j) \chi'_j(y_j))
\]
for some locally constant, compact support functions $\varphi_j, \varphi'_j$ on $E$ and finite functions $\chi_j, \chi'_j$ on $E^\times$. Here $B_{GL_r} = A_r \times Z_r$ is the upper triangular Borel subgroup of $\text{GL}_r(E)$. Since
\[
\delta_{B_{H_n}}(\hat{y}) \delta_{B_{GL_r}}(\hat{y}) \delta_{Q_r}(\hat{y}) \delta_{B_{H_r}}^{-1}(\hat{y}) |\det(y)|_E^{s-r-n} = |\det(y)|_E^s = \prod_{j=1}^r |y_j|^s
\]
we are now reducing to estimate the following integral
\[
\prod_{j=1}^r \int_{E^\times} \varphi_j(y_j) \varphi'_j(y_j) \chi_j(y_j) \chi'_j(y_j) |y_j|^s \, d^s y_j
\]
which converges absolutely for $\Re(s) > 0$ by the temperedness assumption and a result of Waldspurger (cf. [Wal03 Proposition III.2.2]). This finishes the proof. \[\square\]
6.4. Unramified representations. Because of Lemma 6.3, we shall only consider unramified $\tau$. Then as in [Che22, we assume that $\tau$ is an induced representation of Langlands' type. This includes all unramified generic representations, but also reducible ones. To describe this type of representations, let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ be an $r$-tuple of nonzero complex numbers. Then there is a unique unramified character $\chi_{\underline{\alpha}}$ of $A_{r}$ given by

$$\chi_{\underline{\alpha}}(\text{diag}(y_1, y_2, \ldots, y_r)) = \prod_{i=1}^{r} e^{-\log y_i \frac{[|\alpha_i|]}{\alpha_i}}.$$

By extending $\chi_{\underline{\alpha}}$ to a character of the upper triangular Borel subgroup $B_{GL_r} = A_r \times Z_r$ of $GL_r(E)$, we can form a normalized induced representation $\tau_{\underline{\alpha}}$ of $GL_r(E)$. This is an unramified representation of $GL_r(E)$ whose $GL_r(\mathfrak{o}_E)$-fixed subspace is one-dimensional. Moreover, every unramified irreducible representation of $GL_r(E)$ can be realized as a constituent of $\tau_{\underline{\alpha}}$ for some $\underline{\alpha}$. Note that $\tau_{\underline{\alpha}} = \tau_{\underline{\alpha}^*}$ with $\underline{\alpha}^* := (\alpha_r^{-1}, \alpha_{r-1}^{-1}, \ldots, \alpha_1^{-1})$.

An unramified representation $\tau$ of $GL_r(E)$ is called an induced representation of Langlands’ type if $\tau \cong \tau_{\underline{\alpha}}$ for some $\underline{\alpha}$ with

$$|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_r|.$$  

These unramified representations may be reducible, but they have the following nice properties that allow us to work with: (i) The $C$-linear space $\text{Hom}_{\mathbb{C}}(\tau, \psi_F)$ is one-dimensional. (ii) The intertwining map $u \mapsto W_u$ from $u \in V_r$ to its associated Whittaker function is an isomorphism (cf. [JS98, Jac12, Lemma 2]). (iii) $\tau^r$ is again induced of Langlands’ type. We denote by $J(\tau)$ the unique irreducible quotient of $\tau$, which is unramified (cf. [Mat13, Corollary 1.2]) with the Satake parameters $\alpha_1, \alpha_2, \ldots, \alpha_r$.

Now let $\tau = \tau_{\underline{\alpha}}$ be an unramified representation of $GL_r(E)$ that is an induced representation of Langlands’ type. We denote by $\phi_{J(\tau)}$ the $L$-parameter of $J(\tau)$. Then the following lemma compares Langlands-Shahidi’s local coefficients with arithmetic $\gamma$-factors.

Lemma 6.3. We have $c_{\delta}(s, \tau, A_\delta, \psi_F) = \gamma(s, \phi_{J(\tau)}, A_\delta, \psi_F)$.

Proof. This follows from the multiplicativity of $c_3(s, \tau, A_\delta, \psi_F)$ and $\gamma(s, \phi_{J(\tau)}, A_\delta, \psi_F)$, as well as the choice of $\delta$ and $\psi_F$ (cf. [Sha18 Proposition 2.3.1]).

To state the next lemma, we fix a non-zero element $v_\tau \in \mathcal{Y}_{\tau, GL_r(\mathfrak{o}_E)}$. Then the decomposition $H_r = Q_r R_{r,m}$ implies that the space $I_r(\tau, s)^{R_{r,m}}$ is one-dimensional, so we can fix a basis $\xi_{\tau,s}$ with $\xi_{\tau,s}(I_r) = v_\tau$.

Lemma 6.4. We have

$$M(\tau, s) \xi_{\tau,s}^m = \omega_{\tau,s}(\varpi)^m \cdot \frac{L(2s - 1, \phi_{J(\tau)}, A_s)}{L(2s, \phi_{J(\tau)}, A_s)} \cdot \xi_{\tau,s}^{m, 1-s},$$

where $\omega_{\tau,s}$ is the central character of $\tau_s$.

Proof. The proof of this lemma is similar to that of [Che22, Lemma 6.4]. Indeed, it is slightly simpler than the one in loc. cit. as we do have additional Weyl elements here.

7. Rankin-Selberg integrals attached newforms and oldforms

We shall prove Theorem 1.4 in this section. Let us begin with the setups.

7.1. Setups. As before, let $\pi$ be an irreducible generic representation of $G_n$ with the associated $L$-parameter $\phi_{\pi} : WD_E \to GL_{2n+1}(\mathbb{C})$. Fix a non-zero $\Lambda_{\underline{\alpha}, \mathfrak{o}_E} \in \text{Hom}_{\mathbb{C}}(\pi, \psi_{N_{Z_r}})$ and a newform $v_\tau \in \mathcal{Y}_{\tau, GL_r(\mathfrak{o}_E)}$. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ be a $r$-tuple of non-zero complex numbers and $\underline{\bar{\alpha}}$ be its rearrangement so that (6.4) holds. Let $\tau = \tau_{\underline{\alpha}}$ which is an unramified representation of $GL_r(E)$ that is an induced representation of Langlands’ type (cf. [6.4]). Let $J(\tau)$ be the unique unramified quotient of $\tau$ and $\phi_{J(\tau)}$ be the $L$-parameter of $J(\tau)$. Fix non-zero elements $v_\tau \in \mathcal{Y}_{\tau, GL_r(\mathfrak{o}_E)}$ and $\Lambda_{\tau, \mathfrak{o}_E} \in \text{Hom}_{\mathbb{C}}(\pi, \psi_{N_{Z_r}})$. Since $\psi_E$ is unramified, we may assume $\Lambda_{\tau, \mathfrak{o}_E}(v_\tau) = 1$ (cf. [Shi76, Jac12]). Let $\xi_{\tau,s}$ be the basis of the one-dimensional space $I_r(\tau, s)^{R_{r,m}}$ with $\xi_{\tau,s}(I_r) = v_\tau$.

The Haar measures appeared in the Rankin-Selberg integrals are chosen as follows. First, we take $dx$ to be the Haar measure on $\text{Mat}_{(n-r) \times r}(E)$ with $\text{vol}(\text{Mat}_{(n-r) \times r}(\mathfrak{o}_E), dx) = 1$. On the other hand, the Haar measures
To describe the proposition, let $\delta_r$ denote the modulus function of the upper triangular Borel subgroup $T_r \subseteq V_r$ of $H_r$. We indicate that Lemma 5.1 implies that $dt$ does not depend on $m$. Moreover, as argued in the proof of [AOY22, Theorem 4.4], this $\delta_r$-invariant, its not hard to see that the support of $W_v$ when restricting to $T_1$ is contained in
\[ \{ \text{diag}(y_1, \ldots, y_n, 1, \tilde{y}_1^{-1}, \ldots, \tilde{y}_1^{-1}) | 0 < |y_1|_E \leq |y_2|_E \leq \cdots \leq |y_n|_E \leq 1 \}. \]

In the following two lemmas, we will reveal more properties of $W_v$.

**Lemma 7.1.** Let $a \in GL_r(F)$ and $x \in \text{Mat}_{(n-r) \times r}(E) \cdot \text{Mat}_{(n-r) \times r}(\phi_E)$ with $r < n$. Then we have $W_v(\tilde{a}x) = 0$.

**Proof.** We refer to [Che22] and [6.1] for the notation of $\tilde{a}$ and $\tilde{x}$ respectively. The proof of this lemma the same as [Che22] Lemma 6.5]. Observe that the matrices used in loc. cit. correspond to the root matrices $\chi_{\epsilon_{k} \cdot \epsilon_{s-r}}(y)$ and $\chi_{\epsilon_{k}}(y)$ here. \qed

The proof of the next lemma is inspired by the proof of [AOY22, Theorem 4.4] which is one of the keys for showing that $\Lambda_{\pi, \psi_E}(v_n) \neq 0$ when $\pi$ is tempered.

**Lemma 7.2.** Suppose that $\pi$ is tempered and $v$ is non-zero. Then $W_v$ can not be identically zero on $T_n$.

**Proof.** The proof is by contradiction. So let us suppose that $W_v$ is identically zero on $T_n$. Let $(\cdot, \cdot)_\pi$ be an $G_n$-equivalent Hermitian pairing on $\mathcal{V}_\pi$ and $f_v(g) = (\pi(g)v, v)_\pi$ be a matrix coefficient of $\pi$. Since $v \neq 0$, we have $f_v(I_{2n+1}) \neq 0$. Then from the proof of [GS12, Lemma 12.5]), there exists an irreducible tempered representation $\pi'$ of $H_n$ and a matrix coefficient $f'$ of $\pi'$ such that
\[ \int_{H_n} f_v(h) f'(h) dh \neq 0. \]

Moreover, as argued in the proof of [AOY22, Theorem 4.4], this $\pi'$ must be unramified and $f'$ can be chosen to be bi-$R_{n,m}$-invariant. It follows that $\pi' = \rho_{\tau, 1/2}$ (cf. [6.1] for some irreducible unramified tempered representation $\tau$ of $GL_n(E)$). Note that $\tau$ is necessarily generic and we may assume $f'(h) = (\pi'(h)\xi_{\tau, 1/2}^{\epsilon}, \xi_{\tau, 1/2}^{\epsilon})_{\pi'}$, where $(\cdot, \cdot)_{\pi'}$ stands for an $H_n$-equivalent Hermitian pairing on $I_n(\tau, 1/2)$.

Now the proof follows immediately from Lemma 6.1 and Lemma 6.2. Indeed, (7.1) implies
\[ v' \otimes \xi_{1/2} \mapsto \int_{H_n} (\pi(h)v', v)(\pi(h)\xi_{1/2}, \xi_{\tau, 1/2})_{\pi'} dh \]

defines a non-zero element in $\text{Hom}_{H_n}(\pi \otimes \pi', \mathbb{C})$. On the other hand, by Lemma 6.2 and [BAS09, Proposition 4.9],
\[ v' \otimes \xi_{1/2} \mapsto \Psi_{n,n}(v' \otimes \xi_{1/2}) \]

also defines a non-zero element in the same Hom space. It then follows from the multiplicity one result (cf. AGRS10) and Lemma 6.1 that
\[ \int_{H_n} f_v(h) f'(h) dh = c \Psi_{n,n}(v \otimes \xi_{1/2}) = 0 \]

where $c$ is a non-zero constant. This contradicts to (7.1); hence the result. \qed

The following proposition, whose proof is similar to that of [Che22, Proposition 6.7], is the key for computing the Rankin-Selberg integrals and is where we need the Assumption 4.3; namely, we assume that
\[ \gamma_{\phi}(s, \pi \times \tau, \psi_E) = \gamma(s, \phi_{\pi} \otimes \phi_{\tau}, \psi_E). \]

To describe the proposition, let $S_r$ be the $\mathbb{C}$-algebra of symmetric polynomials in
\[ (X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_r, X_r^{-1}) \]
and $\mathcal{S}^0_r$ be its subalgebra consisting of elements $f$ satisfying

$$f(X_1, \ldots, X_r^{-1}, \ldots, X_r) = f(X_1, \ldots, X_j, \ldots, X_r)$$

for $1 \leq j \leq r$. For each $m$, we have the algebra isomorphism (Satake isomorphism)

$$\mathcal{S}^0_{r,m} : \mathcal{H}(H/[R_{r,m}]) \cong \mathcal{H}(T_r/[T_r \cap R_{r,m}]) = \mathbb{C}[A_{r,c}]^{\mathfrak{g}_{r,c}(\mathbb{Z}/22)}$$

where $A_{r,c}$ is the diagonal torus of $GL_r(\mathbb{C})$ and

$$\mathcal{S}^0_{r,m}(f)(t) = \frac{\mathfrak{m}^{s-1/2} \alpha_1, \ldots, \mathfrak{m}^{s-1/2} \alpha_r}{L(s, \phi_{J(r)}, \psi_E) \Psi_{n,r}(v \otimes \mathcal{S}^0_{r,m})} L(s, \phi_{\mathfrak{m}^{s-1/2} \alpha_1}, \psi_E)$$

for every $s \in \mathbb{C}$. The functional equation

$$\Xi_{n,r}(v; X_1^{-1}, \ldots, X_r^{-1}) = (X_1 \cdots X_r)^{a_{s,m}} \Xi_{n,r}(v; X_1, \ldots, X_r)$$

holds. The kernel of $\Xi_{n,r}$ is given by

$$\ker(\Xi_{n,r}) = \{ v \in \mathcal{V}_E^K | W_v(t) = 0 \text{ for every } t \in T_r \}. $$

The relation

$$\Xi_{n,r}(v; X_1, \ldots, X_{r-1}, 0) = \Xi_{n,r-1}(v; X_1, \ldots, X_{r-1})$$

holds for $v \in \mathcal{V}_E^{K_{n-m}}$ and $2 \leq r \leq n$. We have

$$\Xi_{n,m}(v; X_1, \ldots, X_n) = \mathcal{S}_{n,m}(v; X_1, \ldots, X_n)$$

for $\varphi \in \mathcal{H}(H/[R_{n,m}])$. Proof. As mentioned, the proof is similar to that of [Che22, Proposition 6.7]. So here we merely give a sketch of the proof. Let $v \in \mathcal{V}_E^K$ and $W(a; \alpha_1, \ldots, \alpha_r; \psi_E) \vdash A_{r,\psi_E}(\tau(a) v_r)$ for $a \in GL_r(E)$. Note that the scalar $W(a; \alpha_1, \ldots, \alpha_r; \psi_E)$ is the value of the polynomial $W(a; X_1, \ldots, X_r; \psi_E) \in \mathcal{S}_r$ at $(\alpha_1, \ldots, \alpha_r)$ (cf. [Jac12, Section 2], see also [Che22, Section 5.3]). Then by Lemma 7.1 and the choice of the measures, we have

$$\Psi_{n,r}(v \otimes \mathcal{S}^0_{r,m}) = \sum_{E} \int_{a \in \mathbb{Z}_r \setminus GL_r(E)} |\det(a)|^{-\frac{1}{2}} W_v(\tilde{a}) W(a; \alpha_1, \ldots, \alpha_r; \psi_E) |\det(a)|^{s-\frac{1}{2} + n} da$$

for $\Theta(s) > 0$. This is derived in a similar way as in the proof of [Che22, Lemma 6.6]. Observe that the power of $|\det(a)|$ is different from that of loc. cit. due the the difference between the modulus functions of $Q_r$. Now consider the following integral

$$\Psi_{n,r}(v; X_1, \ldots, X_r) = \int_{a \in \mathbb{Z}_r \setminus GL_r(E)} |\det(a)|^{-\frac{1}{2}} W_v(\tilde{a}) W(a; X_1, \ldots, X_r; \psi_E) |\det(a)|^{-n} da$$

which is indeed a finite sum; hence gives rise to a homogeneous polynomial of degree $\ell$ in $S_r$. Furthermore, there exists an integer $N_v$ depending only on $v$ such that $\Psi_{n,r,\ell}(v; X_1, \ldots, X_r) = 0$ for all $\ell < N_v$. We thus define the following formal Laurent series

$$\Psi_{n,r}(v; X_1, \ldots, X_r) = \sum_{\ell \geq N_v} \Psi_{n,r,\ell}(v; X_1, \ldots, X_r) Y^\ell = \sum_{\ell \geq N_v} \Psi_{n,r,\ell}(v; X_1, \ldots, X_r) Y^\ell$$
with coefficient in $S_r$. It then follows from the construction that
\[ \Psi^m_{n,r}(v; \alpha_1, \ldots, \alpha_r; q_E^{s+1}) = \Psi_{n,r}(v \otimes \xi_{r,s}) \]
for $\Re(s) \gg 0$.

Next, let $P_{\phi_\tau}(Y) \in \mathbb{C}[Y]$ be such that $L(s, \phi_\tau) = P_{\phi_\tau}(q_E^s)^{-1}$ and put
\[ P_{\phi_\tau}(X_1, \ldots, X_r; Y) = \prod_{j=1}^r P_{\phi_\tau}(q_E^{-\frac{j}{2}} X_j Y). \]
Since $\phi_\tau$ is conjugate self-dual, Corollary 4.3 gives
\[ L(s, \phi_\tau \otimes \phi_{J(\tau)}) = P_{\phi_\tau}(\alpha_1, \ldots, \alpha_r; q_E^{s+1})^{-1} \quad \text{and} \quad L(1-s, \phi_\tau \otimes \phi_{J(\tau)}) = P_{\phi_\tau}(\alpha_1^{-1}, \ldots, \alpha_r^{-1}; q_E^{-s+1})^{-1}. \]

On the other hand, let
\[ P_{As}(X_1, \ldots, X_r; Y) = \prod_{1 \leq k \leq r} (1 - q_E^{-1} X_k Y^2) \prod_{1 \leq k \leq r} (1 - q_E^{-\frac{k}{2}} X_k Y). \]
Then by definition
\[ L(2s, \phi_{J(\tau)}; As) = L(2-s, \phi_{J(\tau)}; As) = P_{As}(\alpha_1^2, \ldots, \alpha_r^2; q_E^{-s+1}). \]

Now we define
\[ \Xi^m_{n,r}(v; X_1, \ldots, X_r; Y) = \frac{P_{\phi_\tau}(X_1, \ldots, X_r; Y) \Psi^m_{n,r}(v; X_1, \ldots, X_r; Y)}{P_{As}(X_1, \ldots, X_r; Y)} \]
which is again a formal Laurent series with coefficients in $S_r$, and we have
\[ \Xi^m_{n,r}(v; \alpha_1, \ldots, \alpha_r; q_E^{s+1}) = \frac{L(2s, \phi_{J(\tau)}; As) \Psi_{n,r}(v \otimes \xi_{r,s}^m)}{L(s, \phi_\tau \otimes \phi_{J(\tau)})} \]
for $\Re(s) \gg 0$ from the constructions. Moreover, by replacing $\alpha_1, \ldots, \alpha_r$ and $s$ with $\alpha_1^{-1}, \ldots, \alpha_r^{-1}$ and $1-s$ respectively, we see that
\[ \Xi^m_{n,r}(v; \alpha_1^{-1}, \ldots, \alpha_r^{-1}; q_E^{-s+1}) = \frac{L(2-2s, \phi_{J(\tau)}; As) \Psi_{n,r}(v \otimes \xi_{r,s}^m)}{L(1-s, \phi_\tau \otimes \phi_{J(\tau)})}. \]
for $\Re(s) \ll 0$.

To connect two formal Laurent series $\Xi^m_{n,r}(v; X_1, \ldots, X_r; Y)$ and $\Xi^m_{n,r}(v; X_1^{-1}, \ldots, X_r^{-1}; Y^{-1})$, we have to apply the functional equation \textbf{(6.3)} and this is the place that we need the Assumption 1.3. More precisely, Lemma 6.3 and Lemma 6.4 imply
\[ M^\dagger_{\phi, \beta}(\tau, s) \xi_{r,s}^m = \omega_{r,s}(\varpi)^m \cdot \frac{L(2-2s, \phi_{J(\tau)}; As)}{L(2s, \phi_{J(\tau)}; As)} \cdot \xi_{r,s}^{m-1}. \]
Then the functional equation \textbf{(6.3)} and the Assumption 1.3 give
\[ \frac{L(2-2s, \phi_{J(\tau)}; As) \Psi_{n,r}(v \otimes \xi_{r,s}^{m-1})}{L(1-s, \phi_\tau \otimes \phi_{J(\tau))}} = \omega_{r,s}(\varpi)^{-m} \epsilon(s, \phi_\tau \otimes \phi_{J(\tau)}, \psi) \cdot \frac{L(2s, \phi_{J(\tau)}; As) \Psi_{n,r}(v \otimes \xi_{r,s}^m)}{L(s, \phi_\tau \otimes \phi_{J(\tau))}}. \]
Now if we put
\[ \epsilon_{\phi_\tau, m}(X_1, \ldots, X_r; Y) = (X_1 \cdots X_r)^{a_\tau^{-m}} Y^{(a_\tau^{-m})} \]
then
\[ \epsilon_{\phi_\tau, m}(\alpha_1, \ldots, \alpha_r; q^{s+1}) = \omega_{r,s}(\varpi)^{-m} \epsilon(s, \phi_\tau \otimes \phi_{J(\tau)}, \psi) \]
for $s \in \mathbb{C}$ (cf. \textbf{1.1}) and the relations above imply
\[ \Xi^m_{n,r}(v; X_1^{-1}, \ldots, X_r^{-1}; Y^{-1}) = \epsilon_{\phi_\tau, m}(X_1, \ldots, X_r; Y) \Xi^m_{n,r}(v; X_1, \ldots, X_r; Y). \]
In particular, $\Xi^m_{n,r}(v; X_1, \ldots, X_r; Y)$ is a finite sum; hence it makes sense to define
\[ \Xi^m_{n,r}(v; X_1, \ldots, X_r) = \Xi^m_{n,r}(v; X_1, \ldots, X_r; 1) \in S_r. \]
Furthermore, the same reasoning in the proof of [Che22, Proposition 6.7] gives

\[ \Xi^{n,r}_{a}(v; YX_1, \ldots, YX_r) = \Xi^{n,r}_{a}(v; X_1, \ldots, X_r; Y). \]

Together, the existence of the linear maps \( \Xi^{m}_{a,n,r} \) and the first assertion are proved. The proofs of the rest of the assertions are similar to that of loc. cit. We point out that when \( \pi \) is unramified and \( v \in V_{\pi}^{K,n,0} \), the assertion (4) is essentially obtained in the proof of [BAS09, Theorem 8.1]. However, same argument actually applies even if \( \pi \) is not unramified and \( v \in V_{\pi}^{K,n,m} \) for \( m > a_{\pi} \).

7.3. Proof of Theorem 1.4. Now we are ready to verify Theorem 1.4. We claim that

\[ (7.2) \quad \Xi^{n}_{a,n}(v_{\pi}; X_1, \ldots, X_n) = \Lambda_{\pi,\psi_E}(v_{\pi}). \]

Then the proof follows. In fact, (7.3) will be implied by Proposition (1) and (4). On the other hand, if \( \pi \) is tempered, then Lemma 7.2 and Proposition 7.3 will tell us that \( \Lambda_{\pi,\psi_E}(v_{\pi}) \neq 0 \).

To prove (7.2), let \( Y_j \) be the \( j \)-th elementary symmetric polynomial in \( X_1, \ldots, X_n \) for \( 1 \leq j \leq n \). Then we have

\[ S_n = \mathbb{C}[Y_1, \ldots, Y_{n-1}, Y^*_n]. \]

Note that \( Y_n = X_1X_2\cdots X_n \) gives a \( \mathbb{Z} \)-grading \( S_n \) by the degree of \( Y_n \):

\[ S_n = \bigoplus_{\ell \in \mathbb{Z}} \mathbb{C}[Y_1, \ldots, Y_{n-1}] Y_n^\ell. \]

Now the key observation is that

\[ (7.3) \quad \Xi^{m}_{a,n}(v; X_1, \ldots, X_n) \in \bigoplus_{\ell \geq 0} S_{n,\ell} \]

for every \( v \in V_{\pi}^{K,n,m} \) and \( m \geq 0 \). This follows from the property of \( W_\psi \) remarked in the beginning of §7.2 and the fact that

\[ \deg_{Y_n} W(\text{diag}(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_n}); X_1, \ldots, X_n; \psi_E) = \lambda_n \]

where \( \lambda_1 \geq \cdots \geq \lambda_n \) are integers. Then Proposition 7.3 (2) gives

\[ \Xi^{a_n}_{m,n}(v_{\pi}; X_1^{-1}, \ldots, X_n^{-1}) = \Xi^{a_n}_{m,n}(v_{\pi}; X_1^{-1}, \ldots, X_n^{-1}) = \Xi^{a_n}_{m,n}(v_{\pi}; X_1, \ldots, X_n); \]

hence by (7.3), we must have \( \Xi^{a_n}_{m,n}(v_{\pi}; X_1, \ldots, X_n) = c \) for some constant \( c \).

To complete the proof, it remains to show that \( c = \Lambda_{\pi,\psi_E}(v_{\pi}) \). By Proposition (1), we find that

\[ (7.4) \quad cL(s, \phi_{\pi} \otimes \phi_{\psi_E}) = L(2s, \phi_{\psi_E}^{\text{abs}}); \text{As}) \Psi_{a,n}(v_{\pi} \otimes \xi^{a_n}_{m,n}) \]

Then since

\[ \Psi_{a,n}(v \otimes \xi^{a_n}_{m,n}) = \sum_{\ell \geq 0} \int_{\alpha | \det(a)^{1/2}} q^{\ell \frac{1}{2}} W(\alpha; A_1, \ldots, A_n; \psi_E) \delta_{B_m}(\alpha) \delta_{Q_n}(\alpha) |\det(a)|^{-\frac{1}{2}} da \]

by the Iwasawa decomposition \( H_n = B_nR_{n, a_{\pi}} \), the identification \( T_n = \hat{A}_n \) and again the property of \( W_{v_{\pi}} \) mentioned in the beginning of §7.2 we see that the constant term in the LHS of (7.4) is \( \Lambda_{\pi,\psi_E}(v_{\pi}) \). Note that here we also use the fact that \( W(I_n; A_1, \ldots, A_n; \psi_E) = 1 \). This finishes the proof.

7.4. Rankin-Selberg integrals attached to oldforms. In this paper, we also compute the Rankin-Selberg integrals attached to oldforms. We in fact compute the integrals attached to elements in the conjectural basis \( \mathcal{B}_{\pi,m} \) of \( V_{\pi}^{K,n,m} \) proposed in §5.2. The key for the computations is again Proposition 7.3 but additional works are required. Recall the notation

\[ \mu_{\ell} = (\ell, \ldots, \ell) \in \mathcal{S}_n \quad \text{and} \quad \varpi^{\mu_{\ell}} = \text{diag}(\varpi^{\ell}, \ldots, \varpi^{\ell}) \in T_n \]

and the convention \( \varpi^{-\mu_{\ell}} = (\varpi^{\mu_{\ell}})^{-1} \). Now let \( v \in V_{\pi}^{R_{n,m}} \). Then \( \pi(\varpi^{\mu_{\ell}})v \in V_{\pi}^{R_{n,m+2}} \) and we have the following lemma.
Lemma 7.4. Let $v \in \mathcal{V}_\pi^{R_{n,m}}$. We have
\[
\Xi_{n,n}^{m+2}(v;X_1,\ldots,X_n) = q_{E}^{\mathfrak{a}_{n-m}}(X_1\cdots X_n)\Xi_{n,n}^{m}(v;X_1,\ldots,X_n).
\]

Proof. This follows from the identical arguments in the proof of [Che22, Lemma 7.1].

Recall that $\mathfrak{v}_v^{n,\pi} \in \mathcal{V}_\pi^{K_{n,a_\pi}}$ stands for a basis. The next lemma compute the integrals attached to $\mathfrak{v}_v^{n,\pi}$. To state and to prove it, we denote by $\mathcal{Y}_j(X_1,X_2,\ldots,X_r)$ the $j$-th elementary symmetric polynomial in $X_1,X_2,\ldots,X_r$ for $1 \leq j \leq r$ and put $\mathcal{Y}_0(X_1,X_2,\ldots,X_r) = 1$. Then the relations
\[
\mathcal{Y}_j(X_1,X_2,\ldots,X_{r-1},0) = \mathcal{Y}_j(X_1,X_2,\ldots,X_{r-1}) \quad \text{and} \quad \mathcal{Y}_r(X_1,X_2,\ldots,X_{r-1},0) = 0
\]
hold for $0 \leq j \leq r - 1$.

Lemma 7.5. We have
\[
\Xi_{n,n}^{a_\pi+1}(v;X_1,\ldots,X_n) = \Lambda_{\pi,\psi}(v;X_1,\ldots,X_n).
\]

Moreover, if $\pi$ is tempered, then $\Lambda_{\pi,\psi}(v) \neq 0$.

Proof. By (7.3), we can write
\[
\Xi_{n,n}^{a_\pi+1}(v;X_1,\ldots,X_n) = \sum_{\ell} b_\ell \mathcal{Y}_1(X_1,\ldots,X_n)\ell_1 \mathcal{Y}_2(X_1,\ldots,X_n)\ell_2 \cdots \mathcal{Y}_n(X_1,\ldots,X_n)\ell_n
\]
for some $\ell = (\ell_1,\ldots,\ell_n) \in \mathbb{Z}_{\geq 0}^n$ and $b_\ell \in \mathbb{C}$. Then since
\[
\mathcal{Y}_j(X_1^{-1},\ldots,X_r^{-1}) = \mathcal{Y}_{r-j}(X_1,\ldots,X_r)\mathcal{Y}_r(X_1,\ldots,X_r)^{-1}
\]
for $0 \leq j \leq r$, the functional equation in Proposition (4.3) (2) gives
\[
\Xi_{n,n}^{a_\pi+1}(v;X_1,\ldots,X_n) = \sum_{\ell} b_\ell \mathcal{Y}_1(X_1,\ldots,X_n)\ell_1 \mathcal{Y}_2(X_1,\ldots,X_n)\ell_2 \cdots \mathcal{Y}_n(X_1,\ldots,X_n)^{1-\ell_1-\ell_2-\cdots-\ell_n}.
\]
As $1 - \ell_1 - \ell_2 - \cdots - \ell_n \geq 0$, we find that $\ell_j \leq 1$ for $1 \leq j \leq n$; hence (7.3) becomes
\[
\Xi_{n,n}^{a_\pi+1}(v;X_1,\ldots,X_n) = \sum_{j=0}^n b_j \mathcal{Y}_j(X_1,\ldots,X_n). \tag{7.5}
\]

To proceed, we apply Proposition (7.3) (4) to get
\[
\Xi_{n,r}^{a_\pi+1}(v;X_1,\ldots,X_r) = \sum_{j=0}^r b_j \mathcal{Y}_j(X_1,\ldots,X_r)
\]
for $1 \leq r \leq n$. Then the functional equation for $\Xi_{n,r}^{a_\pi+1}(v;X_1,\ldots,X_r)$ implies
\[
\Xi_{n,r}^{a_\pi+1}(v;X_1,\ldots,X_r) = \mathcal{Y}_r(X_1,\ldots,X_r)\Xi_{n,r}^{a_\pi+1}(v;X_1^{-1},\ldots,X_r^{-1}) = \sum_{j=0}^r b_j \mathcal{Y}_{r-j}(X_1,\ldots,X_r). \tag{7.6}
\]

By comparing the constant terms, we find that $b_j = b_0$ for $1 \leq j \leq n$; thus
\[
\Xi_{n,n}^{a_\pi+1}(v;X_1,\ldots,X_n) = b_0 \sum_{j=0}^n \mathcal{Y}_j(X_1,\ldots,X_n).
\]

Finally, exactly the same arguments as in the proof of Theorem (1.4) above show that $b_0 = \Lambda_{\pi,\psi}(v)$ and $\Lambda_{\pi,\psi}(v) \neq 0$ if $\pi$ is tempered. This completes the proof.

Now we can describe how to compute Rankin-Selberg integrals attached to elements in $\mathcal{B}_\pi$.

Theorem 7.6. Let $\mathcal{B}_\pi$ be the set defined in (2.2) for each $m \geq a_\pi$. Then the Rankin-Selberg integrals $\mathcal{Y}_n(v \otimes \mathcal{L}_{\pi,m}^{\pi})$ attached to $v \in \mathcal{B}_\pi$ and $\mathcal{L}_{\pi,m}^{\pi}$ can be computed by using Proposition (7.3) (1), (4) and (5) together with Lemma (7.4), Lemma (7.5) and (1.4).
Proof. The proof is almost clear from the statement. However, there is one point that we need to clarify, namely, in the definition of \( \mathcal{B}_{\pi,m} \), the Hecke algebra elements involved are in \( \mathcal{H}(H_n/R_{n,e}) \) with \( e = 0, 1 \); while in Proposition 7.3 (5), the Satake isomorphisms \( \mathcal{S}_{n,m} \) are for any \( m \geq 0 \). To explain this, let \( a = a_\pi \) or \( a_\pi + 1 \) and \( v_0 \in V_{\pi}^{K_{n,m}} \) be a basis. Write \( a = e + 2\ell \) for some \( e = 0, 1 \) and \( \ell \geq 0 \). Let \( m \geq a \) be an integer whose parity is the same as \( a \), so that \( m = e + 2\ell' \) for some \( \ell' \geq \ell \). Then \( \mathcal{B}_{\pi,m} \) consists of the elements of the form

\[
v = \pi(\omega^{-\ell'}) \varphi_{\lambda,e} * \pi(\omega^{\ell'})(v_0)
\]

for some \( \lambda \in \mathcal{P} \). Now a straightforward computation shows that

\[
v = \pi(\omega^{-\ell'}) \varphi_{\lambda,a} * v_0;
\]

hence Proposition 7.3 (5) and Lemma 7.4 give

\[
\Xi_{n,n}(v; X_1, \ldots, X_n) = q_E^{(a_{n-1}v(m-a))} \left( X_1 \cdots X_n \right)^{a_{n-1}a} \mathcal{S}_{n,a}(\varphi_{\lambda,a}) \cdot \Xi_{n,n}(v_0; X_1, \ldots, X_n).
\]

This concludes the proof. \( \square \)

As a corollary, we see that \( \mathcal{B}_{\pi,m} \) defines a basis of \( V_{\pi}^{K_{n,m}} \) for each \( m > a_\pi + 1 \) when \( \pi \) is tempered.

Corollary 7.7. Suppose that \( \pi \) is tempered. Then under the Assumption 7.3, the set \( \mathcal{B}_{\pi,m} \) defines a basis of \( V_{\pi}^{K_{n,m}} \) for each \( m > a_\pi + 1 \).

Proof. This follows immediately from 7.3, Lemma 7.4 and 7.6. \( \square \)

Appendix

The aim of this appendix is to make sense the functional equation (8.3) even when the representations involved are reducible. So let \( \psi \) be a non-trivial additive character of \( E \) and \( \pi \) (resp. \( \tau \)) be a representation of \( G_n \) (resp. \( \text{GL}_r(E) \)) that is of Whittaker type. Let \( Z' \) and \( Y' \) be subgroups of \( G_n \) given by

\[
Z' = \left\{ \begin{pmatrix} I_r & 0 & 0 & b \\ 0 & I_{n-r} & x & c \\ 0 & 0 & I_{n-r} & \alpha' \\ z^* & 1 & 0 & I_r \end{pmatrix} \in G_n \mid z \in \mathbb{Z}_{n-r} \right\} \quad \text{and} \quad Y' = \left\{ \begin{pmatrix} I_r & 0 & 0 & b \\ a & I_{n-r} & x' & 0 \\ 0 & 1 & \alpha' & I_r \end{pmatrix} \in G_n \mid z \in \mathbb{Z}_{n-r} \right\}.
\]

Put \( Y = Z'Y' \) which is a subgroup of \( G_n \); we define a character \( \psi_Y \) of \( Y \) by \( \psi_Y(z'y') = \psi_z(x_{n-r}) \psi(x_{n-r}) \), where \( tx = (x_1, \ldots, x_{n-r}) \in E^{n-r} \). Note that \( \psi_Y(hy) = \psi_Y(y) \) for \( h \in H_r \) and \( y \in Y \) (recall the embedding \( H_r \to G_n \in \text{2.3} \)). Thus \( \psi_Y \) extends to a character of \( H_r \cdot Y \subset G_n \) which we again denoted by \( \psi_Y \). Now one checks that the Rankin-Selberg integral (8.2) gives rise to an element in

\[
\text{Hom}_{H_r \cdot Y} \left( \tau |_{H_r \cdot Y} \otimes \rho_{\tau,s}, \psi_Y \right)
\]

for \( \Re(s) > 0 \) by the integral and by meromorphic continuation in general. Here we let \( Y \) acts trivially on \( \rho_{\tau,s} \). Then we prove the following:

Theorem A. The dimension of the Hom space (8.4) is equal to one, except for finitely many values of \( q_E^s \).

Proof. The proof is obtained by adopting the method of Soudry, which paves a similar result for the Rankin-Selberg integrals attached to \( \text{SO}_{2n+1} \times \text{GL}_r \) (cf. 7.3 Theorem 8.2). The proof uses the \( P_{n+1} \)-theory developed by Gel'fand-Kazhdan (cf. 7.2) and Bernstein-Zelevinsky (7.7), where \( P_{n+1} \) is the unipotent subgroup of \( \text{GL}_{n+1}(E) \). More precisely, let \( P_{n+1} \subset G_{n+1} \) be the subgroup given by

\[
P_{n+1} \subset \text{GL}_{n+1}(E) \rtimes N_{n,n}.
\]

Then the exact sequence

\[
1 \to N_{n,n} \to P_{n+1} \to P_{n+1} \to 1
\]

implies that the normalized Jacquet module \( (\pi_{N_{n,n}}, V_{\pi,N_{n,n}}) \) of \( \pi \) with respect to \( N_{n,n} \) can be regarded as a representation of \( P_{n+1} \). Now the assumption that \( \pi \) is of Whittaker type and the proof of (7.7) give the following finite sequence of \( P_{n+1} \)-modules

\[
0 = 0 \cap V_0 \subset V_1 \subset \cdots \subset V_M = V_{\pi,N_{n,n}}
\]
such that \( \mathcal{V}_i \setminus \mathcal{V}_i \) is an irreducible \( P_{n+1} \)-module for \( 1 \leq i \leq M \), \( \mathcal{V}_1 \cong \text{ind}_{Z_{n+1}}^{P_{n+1}} \psi_{Z_{n+1}} \) and \( \mathcal{V}_1 \setminus \mathcal{V}_M \) does not contain \( \text{ind}_{Z_{n+1}}^{P_{n+1}} \psi_{Z_{n+1}} \) as a subquotient. Here "ind" stands for the compact induction. Already at this stage, we can follow the proof of Soudry word for word; however, we shall provide more details in the followings.

First note that

\[
\text{Hom}_{H,\psi_Y}(\pi|_{H,\psi_Y} \otimes \rho_{\tau,s}, \psi_Y) = \text{Hom}_{H,\psi_Y}(\pi_{\psi_Y} \otimes \rho_{\tau,s}, \mathbb{C})
\]

where \( \pi_{\psi_Y} \) is the twisted Jacquet module of \( \pi \) with respect to \( (Y, \psi_Y) \). Then by the Frobenius reciprocity,

\[
\text{Hom}_{H,\psi_Y}(\pi_{\psi_Y} \otimes \rho_{\tau,s}, \mathbb{C}) \cong \text{Hom}_{GL_r}\big((\pi_{\psi_Y})_Y, \otimes \tau_s, \mathbb{C}\big).
\]

Observe that the normalized Jacquet module \((\pi_{\psi_Y})_Y\) of \( \pi_{\psi_Y} \) with respect to \( Y_r \) is isomorphic to \( \pi_{Y_r,\psi_Y} \), where \( Y := Y \cdot Y_r \) is a subgroup of \( P_{n,n} \) (again, the embedding in \( 2,3 \) is used) and we extend \( \psi_Y \) trivially across \( Y_r \) to obtain a character \( \psi_Y \) of \( Y \). Moreover, \( N_{n,n} \) is normal in \( Y \) and \( N_{n,n}Y \) is isomorphic to the following subgroup of \( P_{n+1} \)

\[
\mathcal{R} = \left\{ e(x,z,y) = \begin{pmatrix} I_r & 0 & 0 \\ x & z & y \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \text{Mat}_{(n-r) \times r}(E), z \in Z_{n-r}, y \in E^{n-r} \right\}
\]

and \( \psi_Y \) on \( N_{n,n}Y \) is mapped to \( \psi_{\mathcal{R}}(e(x,z,y)) = \psi_{Z_{n-r}}(z)\psi(y_{n-r}) \), where \( y = (y_1, \ldots, y_{n-r}) \in E^{n-r} \). The group \( GL_r(E) \) is embedded in \( P_{n+1} \) via the standard way, i.e.

\[
a \mapsto \begin{pmatrix} a \\ I_{n-r+1} \end{pmatrix} \in P_{n+1}
\]

for \( a \in GL_r(E) \). It follows that \( \pi_{Y_r,\psi_Y} \cong (\pi_{N_{n,n}})_{\mathcal{R},\psi_Y} \) as \( GL_r(E) \)-modules. By applying Frobenius reciprocity once more, we find that

\[
\text{Hom}_{GL_r}\big((\pi_{Y_r,\psi_Y})_Y, \otimes \tau_s, \mathbb{C}\big) \cong \text{Hom}_{GL_r}\big((\pi_{N_{n,n}})_{\mathcal{R},\psi_Y} \otimes \tau_s, \mathbb{C}\big)
\]

(7.9)

\[
\cong \text{Hom}_{P_{n+1}}\left(\pi_{N_{n,n}} \otimes \left(\delta_{P_{n+1}} \otimes \text{ind}_{GL_r(E) \times \mathcal{R}}^{P_{n+1}} (\tau_s \otimes \psi_{\mathcal{R}}^{-1})\right) \mathbb{C}\right)
\]

\[
\cong \text{Hom}_{P_{n+1}}\left(\pi_{N_{n,n}} \otimes \left(\text{ind}_{GL_r(E) \times \mathcal{R}}^{P_{n+1}} (\tau_s \otimes \psi_{\mathcal{R}}^{-1})\right) \mathbb{C}\right)
\]

where in the last line above, we use the fact that the map \( f \mapsto \delta_{P_{n+1}} f \) defines an isomorphism

\[
\delta_{P_{n+1}} \otimes \text{ind}_{GL_r(E) \times \mathcal{R}}^{P_{n+1}} (\tau_s \otimes \psi_{\mathcal{R}}^{-1}) \cong \text{ind}_{GL_r(E) \times \mathcal{R}}^{P_{n+1}} (\tau_s+1 \otimes \psi_{\mathcal{R}}^{-1})
\]

between \( P_{n+1} \)-modules.

To proceed, recall that irreducible representations of \( P_{n+1} \) are of the form

\[
\text{ind}_{\mathcal{R}_m}^{P_{n+1}} \sigma \otimes \psi_{Z_{n+1-m}}
\]

where \( \mathcal{R}_m \subset P_{n+1} \) is the subgroup given by

\[
\mathcal{R}_m = \left\{ \begin{pmatrix} a & b \\ z & \end{pmatrix} \mid a \in GL_m(E), b \in \text{Mat}_{m \times (n+1-m)}(E), z \in Z_{n+1-m} \right\}
\]

for \( 0 \leq m \leq n \), \( \sigma \) is an irreducible representation of \( GL_r(E) \) and we extend the representation \( \sigma \otimes \psi_{Z_{n+1-m}} \) of \( GL_m(E) \times Z_{n+1-m} \) to \( \mathcal{R}_m \) trivially across \( b \). Then by combining (7.9) with (7.8), it suffices to show that

\[
\text{Hom}_{P_{n+1}}\left(\text{ind}_{\mathcal{R}_m}^{P_{n+1}} \sigma \otimes \psi_{Z_{n+1-m}} \right) \otimes \left(\text{ind}_{GL_r(E) \times \mathcal{R}}^{P_{n+1}} (\tau_s \otimes \psi_{\mathcal{R}}^{-1})\right) \mathbb{C} = 0
\]

for all but finitely many values of \( q^{-\xi} \), unless \( m = 0 \), in which case we are dealing with

\[
\text{Hom}_{P_{n+1}}\left(\text{ind}_{Z_{n+1}}^{P_{n+1}} \psi_{Z_{n+1}} \right) \otimes \left(\text{ind}_{GL_r(E) \times \mathcal{R}}^{P_{n+1}} (\tau_s \otimes \psi_{\mathcal{R}}^{-1})\right) \mathbb{C}
\]

which we show to be one-dimensional. Now we are in exactly the same situation as in [Sou93 Pages 55-57]. This completes the proof.

\[\square\]
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