Transverse and longitudinal coupled nonlinear vibration of an axially moving Euler beam under variable tension

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Abstract. When describing the vibration characteristics of the transverse parameters of an axially moving beam, the effects of nonlinear factors need to be introduced. Under normal circumstances, it is difficult to solve the nonlinear vibration equations of horizontal and vertical coupling with mixed partial derivative terms of time and space using approximate analytical methods. The longitudinal displacement in the vibration process is very small and much smaller than the lateral displacement, so the equation is simplified to a nonlinear vibration governing equation only about the lateral displacement. However, in these studies, no specific numerical basis for the reason for the simplification is given. Therefore, this paper considers the coupling model of the Euler beam's lateral and longitudinal vibrations, chooses the Galerkin truncation method to decouple and numerically solve it, and obtains the numerical solution of the lateral and longitudinal vibration displacements. Based on the calculation results of the direct multi-scale method and the differential quadrature method of the simplified model, and the solution results of the eight-order Galerkin truncation method of the coupled model, the steady-state response amplitudes of the coupled model and the simplified model are compared to verify the accuracy of the simplified model. Therefore, it provides a strong theoretical basis for the simplified model that ignores the longitudinal vibration.

1. Introduction

In the current literature on the continuum of axially variable-speed motion, most of them assume that the axial tension of the continuum is uniform. However, according to Newton's second law, the assumption of uniformity of axial tension is obviously inaccurate because of the acceleration of the axially moving body. Chen and Tang studied the parametric resonance of axially moving structures with longitudinally varying tension and velocity pulsation¹³, and soon applied the model to the parametric vibration of Euler beams². Ding et al. studied the parametric resonance and internal resonance of the axially accelerated viscoelastic Euler beam⁴, and revealed the bifurcation and chaos of the supercritical axially accelerated viscoelastic beam⁴. Tang et al. analyzed the parametric resonance of an axially accelerated beam with longitudinally varying tension⁵, and the steady-state response of an axially accelerated viscoelastic beam under time-varying tension⁶. Tang and Ma analyzed for the first time the
3:1 internal resonance and instability boundary of an axially moving beam under the correlation between perturbation speed and perturbation tension\[7,8\].

2. The transverse and longitudinal coupling mathematical model of the axially moving Euler beam under variable tension

Considering that the density is $\rho$, the cross-sectional area is $A$, the modulus of elasticity is $E$, the coefficient of viscoelasticity is $\alpha$, the length between the ends of the support is $L$, the initial axial tension is $P_0$ (that is, the tension when the beam is at rest), The time-varying speed $\Gamma(t)$ moves along the axial direction. Applying Euler-Bernoulli beam theory, the coupled plane vibration of the beam consists of longitudinal displacement $u(x,t)$ and lateral displacement $v(x,t)$. Figure 1 shows the physical model of a viscoelastic Euler beam with axially variable motion.

![Fig 1 The physical model of an axially accelerating viscoelastic Euler-Bernoulli beam](image)

The governing equation in dimensionless form is

$$
\begin{align*}
    u_{xx} + 2\gamma \nu_{xx} + c_d(u_t + \gamma u_x) + (\gamma^2 - k_N^2) u_{xx} + \gamma \nu_x - k_N^2 \nu_{xx} = 0 \\
    v_{xx} + 2\gamma \nu_{xx} + \left[-1 + (1-\eta)\gamma^2 - (x-1)\gamma^2\right] v_{xx} + k_N^2 v_{xxxx} + k v + c_d(v_t + \gamma v_x) + \alpha(v_{xxxx} + \gamma v_{xxxx}) - k_N^2 u_{xx} v_x - k_N^2 u_{xx} v_{xx} - \frac{3}{2} k_N^2 v_x^2 v_{xx} = 0
\end{align*}
$$

In the above formula, $k_r$ represents the stiffness coefficient of the beam, and $k_N$ represents the nonlinear coefficient of the beam. Since the simply supported boundary is often used as the boundary condition in engineering applications, the simply supported boundary of the smooth sleeve is selected, and the boundary condition form is

$$v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0$$

3. Galerkin truncation

For the coupled governing equations (1) and (2) of viscoelastic Euler beams in axial motion under the action of axial variable tension, it is generally difficult to solve by approximate analytical method, so the Galerkin truncation method will be used to separate the variables, combined with the Runge-Kutta algorithm Solve. In the numerical method of solving nonlinear structures, Galerkin truncation method is often used. The principle is based on the variational thought of equations corresponding to fictions, discretizing the separation variables of partial differential equations into ordinary differential equations for numerical solution. Select a finite number of trial functions for superposition, and make the weighted integral of the superposition result in the solution domain and on the boundary satisfy the original equation. The weight function is selected as the trial function itself, which can be simplified into a set of ordinary differential equations that satisfy the boundary conditions and are easy to numerically solve.

Assume that the longitudinal and transverse vibration displacement variables of the beam's motion equation meet the following equations:

$$u(x,t) = \sum_{m=1}^{M} q_m(t) \phi_m(x)$$

(4)
\( v(x,t) = \sum_{m=1}^{N} p_m(t) \phi_m(x) \) \hspace{1cm} (5)

Where \( \phi_m(x) \) represents the \( m \) order characteristic function of the lateral vibration of the simply supported beam at both ends, \( p_m(t) (m=1,2,\ldots,M) \) and \( \rho_m(t) (m=1,2,\ldots,M) \) respectively represent the \( m \) generalized coordinates of longitudinal and transverse vibration. The characteristic function \( \phi_m(x) \) that satisfies the boundary condition equation (3) can be the modal function of the static linear elastic beam, expressed as

\[ \phi_m(x) = \sin(m\pi x) \] \hspace{1cm} (6)

Substituting equation (6) into equations (4) and (5), it can be seen that the boundary condition equation (3) is satisfied.

Substituting equations (4), (5), and (6) into the relevant variables in equations (1) and (2), we can get

\[ u_{x_1} = \sum_{m=1}^{M} \hat{u}_m \sin(m\pi x), \quad u_{x_2} = \sum_{m=1}^{M} \hat{u}_m m\pi \cos(m\pi x), \quad v_{x_1} = \sum_{m=1}^{N} p_m m\pi \cos(m\pi x), \]

\[ v_{x_1} = \sum_{m=1}^{M} p_m \left(-m^2\pi^2\right) \sin(m\pi x), \quad u_{x_2} = \sum_{m=1}^{M} q_m \sin(m\pi x), \quad u_{x_2} = \sum_{m=1}^{M} q_m \cos(m\pi x), \]

\[ v_{x_1} = \sum_{m=1}^{M} q_m \left(-m^2\pi^2\right) \sin(m\pi x), \quad v_{x_1} = \sum_{m=1}^{N} \hat{p}_m \sin(m\pi x), \quad v_{x_2} = \sum_{m=1}^{N} \hat{p}_m \sin(m\pi x), \]

\[ v_{x_{1111}} = \sum_{m=1}^{N} p_m \left(m^4\pi^4\right) \sin(m\pi x), \quad v_{x_{1111}} = \sum_{m=1}^{N} p_m \left(m^4\pi^4\right) \cos(m\pi x) \]

Second-order Ordinary Differential Equations

\[ \ddot{\dot{q}}_m - 4\gamma \sum_{m=1, n=0}^{M} \frac{mn \left[1 - (-1)^{m+n}\right]}{m^2 - n^2} \dot{q}_m + k_n \pi \left[ \sum_{m=1, n=0}^{M} \frac{1}{2} m^2 p_m p_n \right] \]

\[ + \sum_{m=1, n=0}^{M} \frac{1}{2} m^2 p_m p_n + \sum_{m=1, n=0}^{M} \frac{1}{2} m^2 p_m p_n \]

\[ + k_n \pi^2 \left[ \sum_{m=1, n=0}^{M} \left(1 - (-1)^{m+n}\right) q_m - \left(\gamma^2 - k_n \pi^2\right) n^2 \pi^2 q_m = 0 \right] \]

\[ \ddot{p}_m + k_n \pi^2 p_m - \left(1 - \eta\right) \gamma^2 + \frac{1}{2} \gamma - 1 \pi^2 n^2 \pi^2 p_m - \]

\[ \sum_{m=1, n=0}^{M} \frac{4\gamma mn \left[1 - (-1)^{m+n}\right]}{m^2 - n^2} \dot{p}_m - m^2 \left[ \frac{(-1)^{m+n} - 1 - (-1)^{m+n}}{(m+n)^2} \right] \gamma p_m \]

\[ + c_d \left[ \dot{p}_n - 2\gamma \sum_{m=1, n=0}^{M} \frac{mn \left[1 - (-1)^{m+n}\right]}{m^2 - n^2} p_m \right] + k_n \pi^2 p_m + \]

\[ \alpha \left[ n^4 \pi^4 p_m - 2\pi^4 \gamma \sum_{m=1, n=0}^{M} \frac{m^4 n \left[1 - (-1)^{m+n}\right]}{m^2 - n^2} p_m \right] + \]
4. Numerical simulation
In the numerical simulation calculation of this paper, the original parameters of a given beam are L=1.0m, E=210.0×10^{10} Pa, A=0.04×0.03 m^2, ρ=7850 kg/m and P_0=4.725×10^4 N and the corresponding infinite quantity is obtained. Dimension parameters k_N=23.0940, k_f=0.2. Consider other parameters as γ_0=0.86, γ_1=0.1, α=0.0001 and c_d=0.001. Considering the frequency of velocity perturbation, select the frequency of resonance of the first-order harmonic parameters, that is, ω=2ω_1+σ. Among them, σ represents the detuning parameter, which indicates the degree of change of the velocity perturbation frequency ω around 2ω_1. ω_1 is the first-order natural frequency of the linear free vibration of the beam system, that is, ω_1=2.9171^9.

\[
\begin{align*}
 k_N^2 \pi^2 \left[ \sum_{m-k-n=0}^{N-M} \left( -\frac{1}{2} mk^2 p_m q_k \right) + \sum_{k-m-n=0}^{N-M} \frac{1}{2} mk^2 p_m q_k + \sum_{m-n-k=0}^{N-M} \frac{1}{2} mk^2 p_m q_k \right] + \\
 k_f^2 \pi^2 \left[ \sum_{m-k-n=0}^{N-M} \left( -\frac{1}{2} mk^2 q_m p_k \right) + \sum_{k-m-n=0}^{N-M} \frac{1}{2} mk^2 q_m p_k + \sum_{m-n-k=0}^{N-M} \frac{1}{2} mk^2 q_m p_k \right] + \\
 k_N^2 \pi^2 \left[ \frac{3}{8} n^4 p^3 + \sum_{m-k-n=0}^{N} \frac{3}{4} m^2 n^2 p_m p_k + \sum_{j=m+k-n}^{N} \left( -\frac{3}{8} k^2 p_m p_k \right) \right] + \\
 \sum_{m-n-k=0}^{N} \frac{3}{8} k^2 p_m p_k + \sum_{k-n-m=0}^{N} \left( -\frac{3}{8} k^2 p_m p_k \right) + \\
 \sum_{j=m+k+n}^{N} \left( \frac{3}{8} k^2 p_m p_k \right) + \sum_{k=m+n}^{N} \left( -\frac{3}{8} k^2 p_m p_k \right) + \\
 \sum_{j=m+n+k}^{N} \left( \frac{3}{8} k^2 p_m p_k \right) + \sum_{k=m+n}^{N} \left( -\frac{3}{8} k^2 p_m p_k \right) + \\
 \sum_{j=m+k+n}^{N} \left( \frac{3}{8} k^2 p_m p_k \right) + \sum_{k=m+n}^{N} \left( -\frac{3}{8} k^2 p_m p_k \right)
\end{align*}
\] (9)

Fig 2 The Comparison of the multiple method and differential quadrature method and the eighth-order Galerkin truncation of the beam on stable steady-state response
Given the parameters $\sigma=0 $, $\gamma_0=0.86 $, $\gamma_1=0.1 $, $\alpha=0.0001 $, $c_\text{d}=0.001$ and $\varepsilon=0$, the period of the dimensionless velocity perturbation amplitude is $T=2\pi/\omega$. As shown in Figure 3, the time history diagram, time history magnification diagram, phase diagram, spectrum diagram and Poincaré cross section of the first-order harmonic parameter resonance under the horizontal and vertical coupled vibration model are given. Observing Figures 3(a) and Figures 3(b), it can be seen that after the vibration of the axially moving beam enters a steady state, the longitudinal vibration amplitude and the transverse vibration amplitude differ by $10^{-3}$ in dimension, that is, the longitudinal amplitude is much smaller than the transverse amplitude. It can be seen from Figures 3(c) and Figures 3(d) that the phase diagram of longitudinal vibration exhibits irregular attenuation motion, and the phase diagram of lateral vibration is periodic motion.

5. Conclusion
This paper describes the steady-state vibration response of a viscoelastic Euler beam under axially variable tension. First, the radially varying tension is introduced, the viscous damping coefficient and support stiffness are introduced, the transverse and longitudinal coupling mathematical model of the viscoelastic Euler beam with axially variable motion is established, and the corresponding homogeneous boundary conditions are established. Based on the eighth-order Galerkin truncation method and the fourth-order Runge-Kutta method, the nonlinear partial differential governing equations are numerically solved. The vibration amplitude results of the coupled model and the simplified model under the approximate analytical method and different numerical methods are compared and analyzed. The specific numerical basis for ignoring the influence of longitudinal vibration displacement is given. Finally, given parameter values, the horizontal and longitudinal vibrations of the beam system are shown.
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