Hadronic Matrix Elements and Radiative $B \rightarrow K^*\gamma$ Decay

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Abstract

Within the standard model, we calculate the radiative $B \rightarrow K^*\gamma$ decay rate based on a Bethe-Salpeter description for the meson wave functions and the hadronic matrix elements. With a reasonable choice of parameters the branching ratio $\text{BR}(B \rightarrow K^*\gamma)$ is found to be $(3.8 - 4.6) \times 10^{-5}$, which is in agreement with the CLEO experimental data. We also find with $m_b = 5.12$ GeV the ratio $R \equiv \Gamma(B \rightarrow K^*\gamma)/\Gamma(b \rightarrow s\gamma) = (10 - 12)\%$, which can be slightly larger if a smaller $m_b$ is chosen. In this approach, the light degrees of freedom in mesons are treated as light constituent quarks with relativistic kinematics, and the form factors in the decay amplitude are essentially determined by the relativistic kinematics and the overlap of wave functions of the initial and final state mesons. Due to the large recoil momentum of the $K^*$ meson at the $B$ meson rest frame, the form factors are sensitive to the overlap integral of the meson wave functions, which are determined dynamically by a QCD-motivated inter-quark potential. Relativistic effects on the meson wave functions mainly due to the Breit-Fermi interactions are found to be significant in determining the decay rate.

I. INTRODUCTION

The interest in studying rare $B$ decays lies with the fact that these decays, induced by the flavor changing $b \rightarrow s\gamma$ neutral currents, are controlled by the one-loop electromagnetic penguin...
diagrams. They play important roles in testing loop effects in the standard model $SU(3)_C \times SU(2)_L \times U(1)$ and in searching for the so-called “New Physics” because they involve many important standard model parameters such as the top quark mass $m_t$ and the Cabibbo-Kobayashi-Maskawa matrix elements $|V_{tb}|$ and $|V_{ts}|$.

It has been shown that at the quark level the inclusive decay $b \rightarrow s\gamma$ has a sizable enhancement due to the QCD corrections\cite{1}. However, it is unfortunate that we can only observe the exclusive channels, such as $B \rightarrow K^*\gamma$, which are hadron transitions and therefore plagued by uncertainties in determining the weak hadronic form factors. Recently the CLEO Collaboration reported the result\cite{2}

\begin{equation}
BR(B \rightarrow K^*\gamma) = (4.5 \pm 1.5 \pm 0.9) \times 10^{-5},
\end{equation}

which makes it possible to test various models including the standard model and models involving new physics, provided that the estimate of hadronic matrix elements is under control.

By now, there have been many methods to calculate the hadronic matrix elements, such as the nonrelativistic quark model\cite{3} \cite{4} \cite{5}, the HQET (Heavy Quark Effective Theory) where the strange quark $s$ is assumed to be heavy\cite{6}, the HQET combined with chiral symmetries where the $s$ is considered light\cite{7}, and the QCD sum rules\cite{8}. The predicted value of $R \equiv \Gamma(B \rightarrow K^*\gamma)/\Gamma(b \rightarrow s\gamma)$ is very different among various models, ranging from 4.5\%\cite{5} to 40\%\cite{8}. Theoretically, the difficulty is mainly due to the large recoil momentum of the $K^*$ meson in this $B$ decay process.

As an attempt to tackle this problem and, in particular, to incorporate the relativistic effects of the underlying dynamics, in this paper we will use the Bethe-Salpeter description for hadronic form factors and meson wave functions\cite{9} \cite{10} to calculate the hadronic matrix elements involved in this rare $B$ decay. In this approach, the light degrees of freedom in mesons are treated as light constituent quarks with relativistic kinematics, and the form factors in the decay amplitude are essentially determined by the relativistic kinematics and the overlap of wave functions of the initial and final state mesons. The form factors may then be related to the inter-quark dynamics through the meson wave functions. This method might allow us to calculate the form factors in a wide range of values of the squared momentum transfer, $q^2$, provided that a good understanding about the covariant decay amplitude and BS wave functions is achieved. The remainder of this paper is organized as follows. In Sec.II we review the method we use. In Sec.III we apply our method to calculate the rate of the radiative decay $B \rightarrow K^*\gamma$ and compare it with those given by the nonrelativistic Schrödinger equation and the naïve scaling law. Conclusions are given in the last section.

\section{II. FORMALISM}

The standard model $SU(3)_C \times SU(2)_L \times U(1)$ has gained great successes and it has now been
thought as the fundamental theory to describe the weak-electromagnetic and strong interactions. However, there is one problem which frequently obscures theoretical predictions in the standard model. That is the so-called nonperturbative (long distance) effects that cannot be dealt with effectively at present.

Nonperturbative effects play significant roles in the weak decays of the heavy flavor mesons. Dealing with the hadronic matrix elements in these decays, one assumes that perturbative short distance ($\sim \frac{1}{M_W}$) effects and nonperturbative long distance ($\sim \frac{1}{\Lambda_{QCD}}$) effects can be treated separately. Perturbative effects can be calculated using the well known short distance techniques of QCD as modifications to the weak Hamiltonian. Nonperturbative effects (including the exchange of soft gluons, the creation of quark-antiquark pairs from vacuum and the final state interactions) are absorbed into the initial and final hadronic states. When dealing with the hadronic matrix elements one must have a good mastery of the hadron wave functions connected with the QCD nonperturbative dynamics. Unfortunately, at present nonperturbative effects are still difficult to be calculated from first principles. Therefore one often relies on various phenomenological models such as the Bauer-Stech-Wirbel (BSW) model, the Isgur-Scora-Grinstein-Wise nonrelativistic quark model, etc.

In quantum field theory, a basic description for the bound states is the Bethe-Salpeter equation. Define the Bethe-Salpeter wave function of the bound state $|P\rangle$ with an overall momentum $P$ of a quark $\psi(x_1)$ and an antiquark $\overline{\psi}(x_2)$

$$\chi(x_1, x_2) = \langle 0 | T\psi(x_1)\overline{\psi}(x_2) | P\rangle,$$

(2)

where $T$ represents time-order product, and transform it into the momentum space

$$\chi_P(q) = e^{-iP\cdot X} \int d^4xe^{-iq\cdot x}\chi(x_1, x_2).$$

(3)

Here we use the standard center of mass and relative variables

$$X = \eta_1 x_1 + \eta_2 x_2, \quad x = x_1 - x_2,$$

(4)

where $\eta_i = \frac{m_i}{(m_1 + m_2)} (i = 1, 2)$. Then in momentum space the bound state BS equation reads

$$(\not{p}_1 - m_1)\chi_P(q)(\not{p}_2 + m_2) = \frac{i}{2\pi} \int d^4kG(P, q - k)\chi_P(k),$$

(5)

where $p_1$ and $p_2$ represent the momenta of quark and antiquark respectively, $G(P, q - k)$ is the interaction kernel which dominates the inter-quark dynamics. According to Eq. (5) we have

$$p_1 = \eta_1 P + q, \quad p_2 = \eta_2 P - q.$$
Note that in Eq. (5) \( m_1 \) and \( m_2 \) represent the effective constituent quark masses so that we could use the effective free propagators of quarks instead of the full propagators. This is an important approximation and simplification for light quarks. Furthermore, because of the lack of a fundamental description for the nonperturbative QCD dynamics, we have to make some approximations for the interaction kernel of quarks.

i) To solve Eq. (5) one must have a good command of the potential between two quarks. However, the reliable information about the potential only comes from the lattice QCD result, which shows that the potential for a heavy quark-antiquark pair \( Q \bar Q \) in the static limit is well described by a long-ranged linear confining potential (Lorentz scalar \( V_S \)) and a short-ranged one gluon exchange potential (Lorentz vector \( V_V \)), i.e.,

\[
V_S(\vec{r}) = \lambda r, \quad V_V(\vec{r}) = -\frac{4}{3} \frac{\alpha_s(r)}{r},
\]

The lattice QCD result for the \( Q \bar Q \) potential is strongly supported by the heavy quarkonium spectroscopy including both spin-independent and spin-dependent effects. In the next section we will employ the potential below regardless of whether the quarks are heavy or not

\[
V(r) = V_S(r) + \gamma_\mu \otimes \gamma^\mu V_V(r),
\]

\[
V_S(r) = \lambda r \left( 1 - e^{-\alpha r} \right),
\]

\[
V_V(r) = -\frac{4}{3} \frac{\alpha_s(r)}{r} e^{-\alpha r},
\]

where the introduction of the factor \( e^{-\alpha r} \) is to avoid the infrared (IR) divergence and also to incorporate the color screening effects of the dynamical light quark pairs on the “quenched” \( Q \bar Q \) potential [13]. It is clear that when \( \alpha r \ll 1 \) the potentials given in (8) become identical with that given in (7). In momentum space

\[
G(\vec{p}) = G_S(\vec{p}) + \gamma_\mu \otimes \gamma^\mu G_V(\vec{p}),
\]

\[
G_S(\vec{p}) = -\frac{\lambda}{\alpha} \delta^3(\vec{p}) + \frac{\lambda}{\pi^2} \frac{1}{(\vec{p}^2 + \alpha^2)^2},
\]

\[
G_V(\vec{p}) = -\frac{2}{3\pi^2} \frac{\alpha_s(\vec{p})}{\vec{p}^2 + \alpha^2},
\]

where \( \alpha_s(\vec{p}) \) is the well known running coupling constant and is assumed to become a constant of
\( O(1) \) as \( \vec{p}^2 \to 0 \)

\[
\alpha_s(\vec{p}) = \frac{12\pi}{27} \frac{1}{\ln(a + \frac{\vec{p}^2}{\Lambda_{QCD}})}.
\]  

(10)

The constants \( \lambda, \alpha, a, \) and \( \Lambda_{QCD} \) are the parameters that characterize the potential. In the computation in the next section we will use

\[
\lambda = 0.183 GeV^2, \quad \alpha = 0.06 GeV, \quad a = 2.7183, \quad \Lambda_{QCD} = 0.15 GeV,
\]

(11)

and will discuss the sensitivity of the results to the values of parameters later.

ii) In solving Eq. (5), in order to avoid the notorious problem due to the excitation of the relative time variable we have to employ the “instantaneous approximation”. Meanwhile, we will neglect the negative energy projectors in the quark propagators because in general the negative energy projector only contributes to quantities of higher orders in \( \frac{1}{m_Q} \), where \( m_Q \) represents the mass of the heavy quark.

We write therefore in the “instantaneous approximation”

\[
(\not{p}_1 - m_1)\chi_P(q)(\not{p}_2 + m_2) = \frac{i}{2\pi} \int d^4k \tilde{G}(\vec{P}, \vec{q} - \vec{k})\chi_P(k),
\]

(12)

where \( \tilde{G}(\vec{P}, \vec{q} - \vec{k}) \) represents the “instantaneous” part of the potential \( G(P, q - k) \). This suggests the derivation of an equation for the three dimensional BS wave function

\[
\Phi_{\vec{p}}(\vec{q}) = \int dq^0 \chi_P(q^0, \vec{q})
\]

(13)

by dividing both sides of Eq. (12) by the propagators of two quarks, then integrating over \( k^0 \) and neglecting the negative energy projectors

\[
\Phi_{\vec{p}}(\vec{q}) = \frac{1}{P^0 - E_1 - E_2} \Lambda_+^1 \gamma^0 \int d^3k \tilde{G}(\vec{P}, \vec{q} - \vec{k})\Phi_{\vec{p}}(\vec{k}) \gamma^0 \Lambda_-^2,
\]

(14)

where

\[
\Lambda_+^1 = \frac{1}{2E_1}(E_1 + \gamma^0 \vec{\gamma} \cdot \vec{p}_1 + m_1 \gamma^0),
\]

\[
\Lambda_-^2 = \frac{1}{2E_2}(E_2 - \gamma^0 \vec{\gamma} \cdot \vec{p}_2 - m_2 \gamma^0),
\]

(15)
are the remaining positive energy projectors of the quark and antiquark respectively. Here, \( E_1 = \sqrt{m_1^2 + p_1^2} \), \( E_2 = \sqrt{m_2^2 + p_2^2} \). From Eq. (14) it is easy to see that
\[
\Lambda_+^{\dagger} \Phi_{-}^{\dagger}(\vec{q}) = \Phi_{-}^{\dagger}(\vec{q}), \\
\Phi_{-}^{\dagger}(\vec{q}) \Lambda_-^2 = \Phi_{-}^{\dagger}(\vec{q}). \tag{16}
\]
Considering the constraint of Eqs. (16), and the requirement of space reflection of \( \gamma^0 \Phi_{-}^{\dagger}(\vec{q}) \gamma^0 = -\Phi_{-}^{\dagger}(-\vec{q}) \) for a negative parity meson, and the constraint of the general form of the meson wave function in the rest frame (see Eq. (19) below), then the wave function \( \Phi_{-}^{\dagger}(\vec{q}) \) in a moving frame can be written as follows:
\[
\Phi_{-}^{\dagger}(\vec{q}) = \Lambda_+^{\dagger} \gamma^0 (1 + \frac{P}{M}) \gamma_5 \gamma^0 \Lambda_-^2 \varphi_{P}^{\dagger}(\vec{q}) = \frac{\gamma_1 + m_1}{2E_1} \gamma_5 \frac{\gamma_2 - m_2}{2E_2} \varphi_{P}^{\dagger}(\vec{q}), \\
\Phi_{-}^{\dagger}(\vec{q}) = \Lambda_+^{\dagger} \gamma^0 (1 + \frac{P}{M}) \varphi_{P}^{\dagger}(\vec{q}) = \frac{\gamma_1 + m_1}{2E_1} \varphi_{f}^{\dagger}(\vec{q}), \tag{17}
\]
where \( \Phi_{-}^{\dagger}(\vec{q}) \) and \( \Phi_{-}^{\dagger}(\vec{q}) \) are the three dimensional BS wave functions of the \( 0^- \) meson and \( 1^- \) (S-wave) meson respectively. \( P^\mu \) and \( M \) are the 4-momentum and mass of the meson. \( \varphi = \gamma_\mu e^\mu \), \( e^\mu \) is the polarization vector of \( 1^- \) meson. \( \varphi_{P}^{\dagger}(\vec{q}) \) and \( \varphi_{f}^{\dagger}(\vec{q}) \) are scalar functions of \( \vec{P} \) and \( \vec{q} \) in general.

Note that in Eq.(17) the appearance of the positive energy projectors for quark and antiquark in the meson wave functions at the moving frame is an immediate consequence of Eq.(16). It is easy to see that if taking the heavy quark limit \( m_1 \rightarrow \infty \), then \( p_1^\mu \rightarrow P^\mu \), Eq. (17) becomes
\[
\Phi_{-}^{\dagger}(\vec{q}) = \frac{1}{\gamma^0} (1 + \gamma_\mu) \gamma_5 \gamma^0 \Lambda_-^2 \varphi_{v}(\vec{q}), \\
\Phi_{-}^{\dagger}(\vec{q}) = \frac{1}{\gamma^0} (1 + \gamma_\mu) \gamma_5 \gamma^0 \Lambda_-^2 \varphi_{v}(\vec{q}), \tag{18}
\]
where \( \gamma^\mu = \frac{P^\mu}{M} \), and \( \varphi_{v} = f_{v} \), which is due to vanishing color-magnetic force in the heavy quark limit. This indicates that in the heavy quark limit the BS wave functions respect the flavor-spin symmetry, and the light degrees of freedom are described by \( \varphi_{v} \), the wave function of the light constituent quark, which is to be determined dynamically by the BS equation. In the rest frame of the meson (\( P = 0 \))
\[
\Phi_{-}^{\dagger}(\vec{q}) = \Lambda_+^{\dagger} \gamma^0 (1 + \gamma^0) \gamma_5 \gamma^0 \Lambda_-^2 \varphi(\vec{q}), \\
\Phi_{-}^{\dagger}(\vec{q}) = \Lambda_+^{\dagger} \gamma^0 (1 + \gamma^0) \varphi(\vec{q}). \tag{19}
\]
It is easy to show\(^4\) that Eq. (19) is the most general form for the \(0^−\) and \(1^−\) (S-wave) \(q_1q_2\) meson wave functions at the rest frame (e.g. for the \(0^−\) meson wave function there are four independent scalar functions but with the constraint of Eq.(16) those scalar functions can be reduced to one and expressed exactly as Eq.(19)), and Eq. (17) may be obtained by boosting the spinors from the rest frame to the moving frame.\(^4\)

Substituting Eq. (19) into Eq. (14), one derives the equations for \(\varphi(\vec{q})\) and \(f(\vec{q})\) in the meson rest frame\(^3\)

\[
M\varphi_1(\vec{q}) = (E_1 + E_2)(E_1m_1 + E_2m_2)\ \varphi(\vec{q})
\]

\[
-\frac{E_1E_2 + m_1m_2 + \vec{q}^2}{4E_1E_2} \int d^3k (G_S(\vec{q} - \vec{k}) - 4G_V(\vec{q} - \vec{k}))\varphi_1(\vec{k})
\]

\[
-\frac{(E_1m_2 + E_2m_1)}{4E_1E_2} \int d^3k (G_S(\vec{q} - \vec{k}) + 2G_V(\vec{q} - \vec{k}))\frac{m_1 + m_2}{E_1 + E_2}\varphi_1(\vec{k})
\]

\[
+ \frac{E_1 + E_2}{4E_1E_2} \int d^3k G_S(\vec{q} - \vec{k}) (\vec{q} \cdot \vec{k}) \frac{m_1 + m_2}{E_1m_2 + E_2m_1}\varphi_1(\vec{k})
\]

\[
+ \frac{m_1 - m_2}{4E_1E_2} \int d^3k (G_S(\vec{q} - \vec{k}) + 2G_V(\vec{q} - \vec{k})) (\vec{q} \cdot \vec{k}) \frac{E_1 - E_2}{E_1m_2 + E_2m_1}\varphi_1(\vec{k}),
\] (20)

where

\[
\varphi_1(\vec{q}) = \frac{(m_1 + m_2 + E_1 + E_2)(E_1m_1 + E_2m_1)}{4E_1E_2(m_1 + m_2)}\varphi(\vec{q}),
\] (21)

\[
Mf_1(\vec{q}) = (E_1 + E_2)f_1(\vec{q})
\]

\[
-\frac{1}{4E_1E_2} \int d^3k G_S(\vec{q} - \vec{k}) - 2G_V(\vec{q} - \vec{k})) (E_1m_2 + E_2m_1)f_1(\vec{k})
\]

\[
-\frac{E_1 + E_2}{4E_1E_2} \int d^3k G_S(\vec{q} - \vec{k}) \frac{m_1 + m_2}{E_1m_2 + E_2m_1} f_1(\vec{k})
\]

\[
+ \frac{E_1E_2 - m_1m_2 + \vec{q}^2}{4E_1E_2} \int d^3k (G_S(\vec{q} - \vec{k}) + 4G_V(\vec{q} - \vec{k})) (\vec{q} \cdot \vec{k}) f_1(\vec{k})
\]

\[
-\frac{E_1m_2 - E_2m_1}{4E_1E_2} \int d^3k G_S(\vec{q} - \vec{k}) - 2G_V(\vec{q} - \vec{k})) (\vec{q} \cdot \vec{k}) \frac{E_1 - E_2}{m_2 + m_1} f_1(\vec{k})
\]

\[
-\frac{E_1 + E_2 - m_2 - m_1}{2E_1E_2} \int d^3k G_S(\vec{q} - \vec{k}) (\vec{q} \cdot \vec{k})^2 \frac{1}{E_1 + E_2 + m_1 + m_2} f_1(\vec{k})
\]

\[
-\frac{m_2 + m_1}{E_1E_2} \int d^3k G_V(\vec{q} - \vec{k}) (\vec{q} \cdot \vec{k})^2 \frac{1}{E_1 + E_2 + m_1 + m_2} f_1(\vec{k}),
\] (22)
where
\[ f_1(\vec{q}) = -\frac{m_1 + m_2 + E_1 + E_2}{4E_1E_2} f(\vec{q}). \] (23)

In the nonrelativistic limit for both quark and antiquark, Eq. (20) and (22) can be expanded in terms of $\vec{q}^2/m_1^2$ and $\vec{q}^2/m_2^2$, and they are identical with the Schrödinger equation to the zeroth order, and with the Breit equation to the first order. If the antiquark is light and becomes relativistic then Eq. (20) and (22) will include the higher order relativistic corrections. These equations will be solved numerically.

In the BS description the transition matrix element for $|P,M⟩ \rightarrow |P',M'⟩$ is given by
\[ \langle P',M' |\overrightarrow{Q}\Gamma Q | P, M⟩ = (2\pi)^i i \int d^4p_2 Tr\left\{\chi^\dagger_{P'}(\vec{q}')\Gamma\chi_{P}(\vec{p}_2 + m_2)\right\}, \] (24)

where $\chi_{P'}(\vec{q}') = -\gamma^0\chi^\dagger_{P'}(\vec{q}')\gamma^0$. Eq. (24) can be reduced into a simpler form expressed in terms of the three dimensional BS wave functions, on condition that the negative energy projectors in quark propagators are neglected and the kernel is independent of the relative time variable
\[ \langle P',M' |\overrightarrow{Q}\Gamma Q | P, M⟩ = (2\pi)^3 \int d^3p_2 Tr\left\{\Phi^\dagger_{P'}(\vec{q}')\gamma^0\Gamma\Phi_{P}(\vec{q})\right\}, \] (25)

where the quark may change its flavor while the antiquark remains a spectator (see Fig. 1). Then the normalization of the wave function $\Phi_{P'}(\vec{q})$ reads
\[ (2\pi)^3 \int d^3q Tr\left\{\Phi^\dagger_{P'}(\vec{q})\Phi_{P}(\vec{q})\right\} = 2E = 2\sqrt{M^2 + \vec{P}^2} \] (26)

Based on the formalism above with the solutions for the BS wave functions, we can calculate the form factors in various processes. We can also calculate the decay constants and the mass differences between $0^−$ and $1^−$ mesons, etc. Along these lines a rather extensive investigation has been made and the results are found in agreement with the experiments\[9\]. In the next section we’ll apply this formalism to compute the decay rate of the radiative $B \rightarrow K^*\gamma$ decay.

III. The Decay $B \rightarrow K^*\gamma$ Within the Standard Model

Within the standard model, the inclusive $b \rightarrow s\gamma$ decay is governed by the electromagnetic penguin operator, for $m_s \ll m_b$\[15\]
\[ \mathcal{H}_{eff} = C m_b e^\alpha \sigma_{\mu\nu} q''(1 + \gamma_5)b, \] (27)
where $\epsilon^\mu$ and $q^\mu$ are the polarization vector and momentum of the photon. The constant C includes the QCD corrections and the dependence upon the CKM matrix elements and the heavy quark masses

$$C = \frac{G_F e}{2\sqrt{2} \pi^2} V_{tb}V_{tb} F_2(\frac{m_t^2}{m_W^2}).$$  

(28)

where

$$F_2(x) = r^{-\frac{16}{27}} \left\{ \bar{F}_2(x) + \frac{116}{27} \left[ \frac{1}{5} (r^{10/23} - 1) + \frac{1}{14} (r^{28/23} - 1) \right] \right\}$$  

(29)

with $r = \frac{\alpha_s(m_b)}{\alpha_s(m_W)}$ and $\bar{F}_2(x)$ given by

$$\bar{F}_2(x) = \frac{x}{(x - 1)^3} \left[ \frac{2}{3} x^2 + \frac{5}{12} x - \frac{7}{12} - \frac{3 x^2 - 2x}{2(x - 1)} \ln x \right].$$  

(30)

The function $F_2(x)$ depends weakly upon the top quark mass with a value increasing from 0.55 to 0.68 in the range $90 GeV \leq m_t \leq 210 GeV$.

The amplitude for $B(P) \rightarrow K^*(k, \eta) \gamma(q, \epsilon)$ is

$$A(B \rightarrow K^*\gamma) = C m_b \langle K^*(k, \eta) | \bar{s} \sigma_{\mu\nu} (1 + \gamma_5) b | B(P) \rangle \epsilon^\mu q^\nu,$$  

(31)

where $\eta$ and $k$ are the polarization vector and momentum of the $K^*$ meson and $P$ is the momentum of the $B$ meson. The hadronic matrix element involved in this process can be expressed in terms of its Lorentz structures as follows (or equivalently with form factors $f_1, f_2$ and $f_3$)

$$\langle K^*(k, \eta) | \bar{s} \sigma_{\mu\nu} b | B(P) \rangle = \epsilon^{\mu\nu\alpha\beta} (A \eta^\alpha_P P_\beta + B \eta^\alpha k_\beta + C \eta^\ast \cdot P P_\alpha k_\beta),$$  

(32)

where the form factors A, B and C are functions of $q^2 = (P - k)^2$. Using the Dirac matrix identity

$$\sigma^{\mu\nu}\gamma_5 = -\frac{1}{2} i\epsilon^{\mu\nu\alpha\beta} \sigma_{\alpha\beta},$$  

(33)

the matrix element of the current $\bar{s} \sigma_{\mu\nu} \gamma_5 b$ is given by the same form factors

$$\langle K^*(k, \eta) | \bar{s} \sigma_{\mu\nu} \gamma_5 b | B(P) \rangle = i [ A(\eta^{*\mu} P^\nu - \eta^* P^\mu) + B(\eta^{*\mu} k^\nu - \eta^* k^\mu) + C \eta^* \cdot P (P^\mu k^\nu - P^\nu k^\mu) ] .$$  

(34)

For the real photon these form factors are evaluated at $q^2 = 0$ and the form factor C does not contribute to this transition. The decay width calculation is straightforward,

$$\Gamma(B \rightarrow K^*\gamma) = \frac{|C|^2 (m_B^2 - m_{K^*}^2)^2 m_b^2}{4\pi m_B^4} |A(0) + B(0)|^2$$  

(35)
We will calculate the form factors $A$ and $B$ at $q^2 = 0$ in our BS formalism. It is convenient to do the computation in the rest frame of $B$ meson ($\vec{P}=0$) in which the wave functions of $B^-(b\bar{u})$ and $K^*(s\bar{u})$ can be expressed as follows

$$
\Phi_B(\vec{q}) = \frac{\not{p}_b + m_b}{2E_b}(1 + \gamma^0)\gamma_5 \frac{\not{p}_u - m_u}{2E_u} \varphi(\vec{q}),
$$

$$
\Phi_{K^*}(\vec{q}') = \frac{\not{p}_s + m_s}{2E_s}(1 + \frac{k}{M_{K^*}}) \gamma^0 \frac{\not{p}_u - m_u}{2E_u} f(k, q'),
$$

(36)

where $\vec{q}$ and $\vec{q}'$ are the internal relative momentum of $B$ meson and $K^*$ meson respectively: $\vec{q} = -\vec{p}_u$, $\vec{q}' = \frac{m_s}{m_s + m_u} \vec{k} - \vec{p}_u$. Substituting Eqs. (36) into Eq. (24) we get

$$
\langle K^*(k, \eta) | \bar{s} \gamma_\mu \gamma_5 b | B(P) \rangle = (2\pi)^3 \int d^3p_u Tr \{ \gamma^\mu (1 + \frac{k}{M_{K^*}}) \gamma_5 \gamma^0 \gamma_5 \} \varphi(\vec{p}_u) f(\frac{m_u}{m_s + m_u} \vec{k} - \vec{p}_u),
$$

(37)

where the $\bar{u}$ is a spectator in the transition, and

$$
E_s = \sqrt{m_s^2 + p_s^2} = \sqrt{m_s^2 + (\vec{k} - \vec{p}_u)^2},
$$

$$
E_b = \sqrt{m_b^2 + p_b^2} = \sqrt{m_b^2 + p_u^2}, \quad E_u = \sqrt{m_u^2 + p_u^2}.
$$

(38)

Here we have employed the identity

$$
\frac{\not{p}_u - m_u}{2E_u} \gamma^0 \frac{\not{p}_u - m_u}{2E_u} = \frac{\not{p}_u - m_u}{2E_u}.
$$

(39)

Considering the form factors are independent of the polarization of $K^*$ and Lorentz indices, we may choose some specific polarization (e.g. a transverse one) and Lorentz indices to do calculation. From Eq. (37) we obtain

$$
B = -\frac{(2\pi)^3}{\eta^{ij}k^i - \eta^{ij}k^j} \int d^3p_u \quad Tr \quad \{ \gamma^\mu (1 + \frac{k}{M_{K^*}}) \gamma_5 \gamma^0 \gamma_5 \} \varphi(\vec{p}_u) f(\frac{m_u}{m_s + m_u} \vec{k} - \vec{p}_u),
$$

(40)

$$
A = \frac{1}{M_B} \left[ (2\pi)^3 \int d^3p_u \quad Tr \quad \{ \gamma^\mu (1 + \frac{k}{M_{K^*}}) \gamma_5 \gamma^0 \gamma_5 \} \varphi(\vec{p}_u) f(\frac{m_u}{m_s + m_u} \vec{k} - \vec{p}_u) - B \sqrt{M_{K^*}^2 + k^2} \right].
$$

(41)
Apart from two approximations i.e. neglecting the dependence of the kernel on the relative time and neglecting the contribution of higher order negative energy projectors, the expressions (40) and (41) are rather general and model independent. To calculate the values of A and B we need to know the scalar wave functions $\varphi$ and $f$, which depend on the dynamical model to be used. Here we will use the potential model described in (7)—(11) to solve for $\varphi$ and $f$. Substituting $\varphi$ and $f$ obtained by solving Eqs. (20),(21) and Eqs. (22),(23) into Eq. (40) and Eq. (41) we get

$$|A(0) + B(0)| = 0.62 \tag{42}$$

and

$$BR(B \rightarrow K^*\gamma) = 3.8 \times 10^{-5}, \tag{43}$$

where we have used $|V_{ts}| = 0.042$, $m_t = 150GeV$, $|V_{tb}| \simeq 1$, $m_b = 5.12GeV$, $m_s = 0.55GeV$, $m_u = 0.33GeV$ and $\tau_B \simeq 1.3ps$. Our result is close to those of Ref.[7] ($|A(0) + B(0)| = 0.53$) and Ref.[16] ($|A(0) + B(0)| = 0.46$ in our notation). Taking into account the experimental uncertainties of $|V_{ts}|$(ranging from 0.030 to 0.054[17]), we find the branching ratio $BR(B \rightarrow K^*\gamma)$ ranges from $2 \times 10^{-5}$ to $6 \times 10^{-5}$ which agrees with the CLEO data given in (1) within errors. The ratio of the exclusive rate to inclusive rate, $R$, is expressed in term of $A$ and $B$ as follows

$$R \equiv \frac{\Gamma(B \rightarrow K^*\gamma)}{\Gamma(b \rightarrow s\gamma)} \approx \frac{m_B^3(m_B^2 - m_{K^*}^2)^3}{m_B^3(m_b^2 - m_s^2)^3} \frac{1}{4} |A(0) + B(0)|^2 \tag{44}$$

Substituting Eq. (42) into Eq. (44) we get

$$R = 10\%, \tag{45}$$

In comparison we have solved the nonrelativistic Schrödinger equations for the scalar wave functions $\varphi$ and $f$ in the nonrelativistic limit using the same potential as that in the BS equations i.e. Eqs. (20) — (23) which are automatically reduced to the nonrelativistic Schrödinger equations to the lowest order in $\vec{q}^2/m_i^2 (i = 1, 2)$. We find the value of $|A(0) + B(0)|$ is about 0.32, significantly smaller than that given by Eq. (42).

Meanwhile, we have also assumed the scalar wave functions are the Gaussian as in some nonrelativistic quark model[3][4][5]

$$\varphi(\vec{q}) \propto e^{-\vec{q}^2/a^2}, \quad f(\vec{q}) \propto e^{-\vec{q}^2/b^2}. \tag{46}$$

There the parameters $a$ and $b$ were obtained by the variational method in the nonrelativistic Schrödinger equation[6]. Here we will use a simpler method to estimate their values. The parameters $a$ and $b$ are connected with the mean value of the internal momentum squared of the $B$ and $K^*$ mesons

$$a^2 = 4/3 \langle \vec{q}^2 \rangle_B, \quad b^2 = 4/3 \langle \vec{q}^2 \rangle_{K^*}. \tag{47}$$
Using the “virial theorem”

$$\langle T \rangle = \frac{1}{2} \left\langle r \frac{\partial V}{\partial r} \right\rangle$$

(48)

and extrapolating the naïve scaling law obtained from the study of the heavy quarkonium: $V(r) \sim C \ln r$ where $C = 0.73 GeV$ \[18\] to the $B$ and $K^*$ mesons, we get

$$\langle \vec{q}^2 \rangle = \mu C,$$

(49)

where $\mu$ represents the reduced mass of the meson system. Therefore, for $B$ meson and $K^*$ meson

$$a^2 = 0.30 GeV^2, \quad b^2 = 0.20 GeV^2,$$

(50)

which are consistent with those obtained in the nonrelativistic quark models\[3\] \[4\] \[5\]. Using Eqs. (46), (50) and Eqs. (41), we find the value of $|A(0) + B(0)|$ is about 0.31, almost in coincidence with our result based on the nonrelativistivic Schrödinger equation solutions.

From the results we obtained above we see that the decay rates given by the relativistic BS wave functions distinguish them significantly from those given by the nonrelativistic Schrödinger wave functions. This reflects the dynamical differences between these two descriptions and suggests that the relativistic effects must be considered when we deal with the systems containing light quarks. The scalar functions of different descriptions are compared in Fig. 2 and Fig. 3. From Fig. 2 and Fig. 3 we see that the wave function of $B$ meson in the BS description is “fatter” than that in the nonrelativistic Schrödinger description and that the $0^-$ meson wave function has a longer “tail” than the $1^-$ meson. These are relativistic effects which are mainly due to the well known Breit-Fermi interactions, including both spin-dependent and spin-independent terms. In fact, in the $0^-$ channel there is a very short-ranged ($\delta$ function like) spin-spin force between quarks, which is attractive and lowers the energy level of the $0^-$ state and pulls the quarks towards the origin. Consequently, in momentum space the $0^-$ meson wave function becomes “fatter” and has a longer “tail”. Whereas for the $1^-$ mesons the spin-spin force is repulsive and weak (three times weaker than that for the corresponding $0^-$ mesons), and is compensated and even overwhelmed by other attractive spin-independent relativistic corrections. These dynamical ingredients are contained in Eq. (20) for the $0^-$ meson with Eq. (22) for the $1^-$ meson, and can be explicitly seen by the nonrelativistic reduction of these equations in terms of $\vec{q}^2/m_1^2$ and $\vec{q}^2/m_2^2$.

In line with the observation made in Ref\[4\], we find that our results depend rather strongly upon the wave functions. When we change the “characteristic momentum” parameters $a$ and $b$ in Eq. (46), we find that the value of $|A(0) + B(0)|$ changes rapidly. This is because the momentum transferred to $K^*$ is very large and $K^*$ is so far away from the zero recoil limit that the decay rate essentially depends upon the overlap of the wave functions of the initial and final meson states,
and in this kinematic region the overlap becomes particularly sensitive to the broadness of the wave functions in the momentum space. The more broad the wave functions are, the larger the overlap and hence the decay rate.

The numerical values of branching ratio (43) and ratio $R$ (45) are obtained by using the potential parameters (11) and quark mass parameters given after (43). It is significant to examine the sensitivity of the results to those parameters. In doing so, we first use the same quark parameters as before but change the potential parameters. We find that the value of $|A(0) + B(0)|$ is insensitive to $\alpha$ and $a$ (see (11)) (note that the screening effect of $\alpha$ is mainly on higher excited states rather than the ground state mesons, and that the strength of the running Coulomb force at large distances, which is associated with $a$, also has little effects on the ground state heavy mesons). On the other hand, $|A(0) + B(0)|$ is increased as the string tension $\lambda$ and $\Lambda_{QCD}$ increase. This is because, larger $\lambda$ and $\Lambda_{QCD}$ result in stronger attractive inter-quark forces, and therefore make the meson wave functions more compact in coordinate space and more broad in momentum space, and hence increase the overlap integral of meson wave functions and the value of $|A(0) + B(0)|$.

With a popular choice of $\lambda = (0.18 - 0.20)$ GeV$^2$ and $\Lambda_{QCD} = (0.15 - 0.20)$ GeV and other parameters unchanged, we find instead of (43) and (45)

$$BR(B \rightarrow K^{*}\gamma) = (3.8 - 4.6) \times 10^{-5},$$

and

$$R \equiv \frac{\Gamma(B \rightarrow K^{*}\gamma)}{\Gamma(b \rightarrow s\gamma)} = (10 - 12)\%.$$ (52)

Next, we use the same potential parameters as (11), but change the quark mass parameters. We find that $|A(0) + B(0)|$ is insensitive to the $b$ quark mass. This is because, as naively shown in (49), the internal momenta of quarks in a meson are mainly determined by the reduced mass. With $m_u = 0.33$ GeV, $m_b = (4.70 - 5.12)$ GeV, the reduced mass $\mu = (0.308 - 0.310)$ GeV is almost unchanged, and hence $|A(0) + B(0)|$ remains rather stable. However, from (44) it is clear that the ratio $R$ is sensitive to $m_b$. With $m_b$ going down to 4.70 GeV from 5.12 GeV, $R$ will be increased by about 30% entirely due to a smaller $m_b$ in the mass ratios in $R$, while $R$ is decreased by less than 10% due to a slightly smaller value of $|A(0) + B(0)|$. It is obvious that a larger value of $R$ will favour a smaller value of $m_b$.

Very recently the CLEO Collaboration has reported the first result of inclusive $b \rightarrow s\gamma$ decay branching ratio:

$$BR(b \rightarrow s\gamma) = (2.32 \pm 0.51 \pm 0.29 \pm 0.22) \times 10^{-4},$$ (53)

and

$$R \equiv \frac{\Gamma(B \rightarrow K^{*}\gamma)}{\Gamma(b \rightarrow s\gamma)} = 0.19 \pm 0.08.$$ (54)
Our predicted value (52) for $R$ is smaller than the experimental value (54). However, as discussed above, if we use a smaller value for $m_b$, the calculated value for $R$ can be increased by, say, about $(10-20)\%$. Nevertheless, it seems to be difficult for our model to reach the large central value of $R = 0.19$ measured by CLEO. Further investigations are still needed.

IV. SUMMARY and CONCLUSION

In this paper we solve the BS equations for mesons and employ the covariant form for the wave functions and the transition matrix elements to calculate the form factors involved in the radiative $B \rightarrow K^\ast \gamma$ decay. In principle, we can calculate the form factors at any values of the squared momentum transfer. In practice, we have made two approximations, i.e., neglecting the dependence of the kernel on the relative time and neglecting the contribution of the higher order negative energy projectors. We obtain a rather general form for the form factors (40) and (41). These expressions are rather model independent (apart from the two approximations mentioned above). We then use a QCD-motivated one gluon exchange plus linear confinement potential as the kernel to solve the BS equation. With a rather popular choice for the potential parameters and quark masses, we get the scalar wave functions $\varphi$ and $f$, and then calculate the form factors $A$ and $B$. For $|V_{ts}| = 0.042$, $|V_{tb}| = 1$, $m_t = 150 GeV$ we find that the branching ratio $BR(B \rightarrow K^\ast \gamma) = (3.8 - 4.6) \times 10^{-5}$, and $R = (10 - 12)\%$. $R$ may take a slightly larger value of $(12 - 14)\%$ if, instead of choosing $m_b = 5.12 GeV$, a smaller $b$ quark mass say $m_b = (4.7 - 4.9) GeV$ is used.

Because $B \rightarrow K^\ast \gamma$ is a large recoil process, the relativistic effects are important. There are two sources of relativistic effects. One is from relativistic kinematics, which may be seen in (40) and (41) where the form factors are expressed in terms of Dirac spinors, and where $E_u \gg m_u, E_s \gg m_s$. In (40) and (41), due to the large recoil momentum of the $K^\ast$ meson $|\vec{k}| = \left(\frac{M_B^2 - M_{K^\ast}^2}{2M_B}\right) \approx \frac{M_B}{2}$, the overlap integral of the wave functions of $B$ and $K^\ast$ becomes much smaller than that in the zero recoil limit. In connection with this, there is another source of relativistic effects, i.e., the dynamical effect on the meson wave functions. With the large recoil momentum the overlap is small and therefore is particularly sensitive to the meson wave functions, which are determined by inter-quark forces. To see this we have used three forms for $\varphi$ and $f$ in (40) and (41): (a) the solutions of BS equation; (b) the solutions of zeroth order Schrödinger equation with the same inter-quark potential; (c) the Gaussian wave function. We find that the three results are quite different and the relativistic effects on the wave functions $\varphi$ and $f$ are indeed important. With (b) and (c) the obtained decay rates are similar but smaller by more than a factor of 2 than with (a). This is because, with BS equation the Breit-Fermi interactions induced by relativistic motion will broaden the wave functions in momentum space. The important effect is that the $B$
meson wave function is broadened by the color magnetic i.e. the hyperfine spin-spin force and other spin-independent terms induced by one gluon exchange, which also gives the $B^{*-} - B$ mass splitting with a right size. With the broadened wave functions the overlap integral is increased and a larger decay rate is achieved. In this connection, the hadronic matrix element involved in the decay $B \rightarrow K^*\gamma$ may indeed provide a test of some important ingredients in the inter-quark dynamics. The effects of Breit-Fermi Hamiltonian on the wave functions have also been shown in charmonium decays.\(^{(20)}\) The observed suppressions for the electric dipole transitions $\psi' \rightarrow \gamma \chi_J$ and $\chi_J \rightarrow \gamma \psi(J=0,1,2)$, as well as $^{1}P_1 \rightarrow \gamma \eta_c$ are probably due to these effects, because the broadened wave functions in momentum space (hence narrowed in coordinate space) by the Breit-Fermi interactions will reduce the dipole transition overlap integrals, and therefore the rates.

Of course, there are theoretical uncertainties in our approach, such as the neglect of the retardation effects and the contribution of the negative energy projectors, the lack of knowledge for the correction to the static inter-quark potential due to the light quark motion, and the gluon (hard and soft) exchange nonspectator effects that are difficult to compute in the quark models at present. Hopefully, our results can serve as an useful estimate of this decay. Definite conclusions strongly depend upon the reduction of these uncertainties in the theoretical computations which, we hope, will be improved in the future.

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Note added. After this work was completed and submitted for publication the CLEO Collaboration reported a result of the inclusive $b \rightarrow s\gamma$ decay rate (Ref.[19]). We have therefore included a brief discussion for it in our calculations.

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**Figure Captions**

**Figure 1:** Diagram for the meson transition $| P, M \rangle \rightarrow | P', M' \rangle$. Here the quark changes its
flavor and momentum via the vertex $\Gamma$, while the antiquark remains a spectator.

**Figure 2:** The $B$ meson wave function $\varphi(\vec{p})$, defined in Eq. (19) and normalized by Eq. (26).
The solid line represents the BS equation solution by solving Eqs. (20) and (21) while the dashed
line represents the Schrödinger equation solution by solving Eqs. (20) and (21) but only keeping
the lowest order terms in the nonrelativistic expression.

**Figure 3:** The $K^*$ meson wave function $f(\vec{p})$, defined in Eq. (18) and normalized by Eq. (20).
The solid line represents the BS equation solution by solving Eqs. (22) and (23) while the dashed
line represents the Schrödinger equation solution by solving Eqs. (22) and (23) but only keeping
the lowest order terms in the nonrelativistic expression.