The enriched Thomason model structure on 2-categories

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Abstract. We prove that categories enriched in the Thomason model structure admit a model structure that is Quillen equivalent to the Bergner model structure on simplicial categories, providing a new model for (∞, 1)-categories. Along the way, we construct model structures on modules and monoids in the Thomason model structure and prove that any model structure on the category of small categories that has the same weak equivalences as the Thomason model structure is not a cartesian model structure.

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1 Introduction

The Thomason model structure on the category Cat of small categories was established by Thomason [1980] and Cisinski [1999.c]. It is Quillen equivalent to the Kan–Quillen model structure on simplicial sets, which means that small categories can be used as models for spaces in homotopy theory. These models had already been proved to be very productive by the time Thomason’s paper came out, e.g., in Quillen’s Theorems A and B in higher algebraic K-theory (Quillen [1973]) or in computing homotopy colimits of diagrams of spaces (Thomason [1979]).

It is natural to ask whether the above Quillen equivalence has an analogue for (∞, 1)-categories. That is to say, does the category of small categories enriched in the Thomason model structure (i.e., strict 2-categories) admit a model structure that is Quillen equivalent to the Bergner model structure on simplicial categories? This paper answers this question in the affirmative.

Theorem 1.1. (Theorem 5.11 and Theorem 5.14.) The category CatCat of small 2-categories admits a left proper combinatorial model structure (the enriched Thomason model structure) whose weak equivalences are Dwyer–Kan equivalences and acyclic fibrations are Dwyer–Kan acyclic fibrations. The Quillen equivalence $cSd^2 \dashv \text{Ex}^2 N$ (Proposition 3.9) induces a Quillen equivalence

$$L \dashv R, \quad L: \text{Cat}_{sSet} \to \text{Cat}_{Cat}, \quad R: \text{Cat}_{Cat} \to \text{Cat}_{sSet},$$

where the right adjoint functor $R$ applies the functor $\text{Ex}^2 N$ to every hom-object. That is to say, the model categories of small simplicial categories (Bergner [2004.c]) and small 2-categories with the enriched Thomason model structure (Theorem 5.11) are Quillen equivalent.

The Quillen equivalence $L \dashv R$ (Theorem 5.14) immediately connects (via zigzags of Quillen equivalences) the enriched Thomason model structure on small 2-categories to all the other models for small (∞, 1)-categories such as quasicategories (Joyal [2008.a, Theorem 6.12]), relative categories (Barwick–Kan [2010.1, Theorem 6.1]), Segal categories (Hirschowitz–Simpson [1998.c, Théorème 2.3]; see also Bergner [2003, Theorems 5.1 and 7.1]), complete Segal spaces (Rezk [1998.a, Theorem 7.2]), simplicial categories (Bergner [2004.d, Theorem 1.1]), marked simplicial sets (Lurie [2017, Proposition 3.1.3.7]), etc., with Quillen equivalences between them established by Bergner [2003, Theorem 6.3, 7.5, 8.6], Joyal–Tierney [2006.a, Joyal [2007.a], Barwick–Kan [2010.1, Theorem 6.1, 6.2], Lurie [2017, Propositions 3.1.5.3 and 3.1.5.6]. See
Bergner [2006.b, 2018.a] for a review of these Quillen equivalences. The adjective “enriched Thomason” refers to \((\infty, 1)\)-categories and distinguishes this model structure from the Thomason model structure on 2-categories (§1.3), which models \((\infty, 0)\)-categories. It would be interesting to see how the existing results for other definitions of \((\infty, 1)\)-categories (see, for example, Lurie [2017], Cisinski [2019], Riehl–Verity [2022]) can be (re)formulated in the setting of 2-categories.

In support of the main theorem cited above, we prove that the Thomason model structure enjoys a collection of properties that makes it similar to the Kan–Quillen model structure on simplicial sets, with a prominent exception of the pushout product axiom. The latter property immediately precludes the possibility of using existing model-categorical tools to construct the model structure of Theorem 5.11 such as the theorems of Lurie [2017, Proposition A.3.2.4] and Muro [2012.a, Theorem 1.1], which construct a model structure on the category of small categories enriched in a combinatorial monoidal model category satisfying some additional conditions (Lurie: every object is cofibrant and weak equivalences are closed under filtered colimits; Muro: the monoid axiom holds).

**Theorem 1.2.** (Proposition 3.8, Proposition 3.9, Proposition 4.3) The Thomason model structure (Definition 2.3), on the category \(\mathbf{Cat}\) of small categories is a proper combinatorial model category that is tractable (Definition 2.10), pretty small (Definition 2.11), h-monoidal (Definition 3.6), flat (Definition 3.3), and satisfies the properties of the monoid axiom (Definition 3.7) other than the nonacyclic part of the pushout product axiom (Definition 3.1). Every model structure on the same category with the same weak equivalences is not a cartesian model structure. Furthermore, the Quillen equivalence

\[
c_{\mathbf{Sd}}^2 \dashv \mathbf{Ex}_2^{\mathbf{N}}, \quad c_{\mathbf{Sd}}^2: \mathbf{sSet} \to \mathbf{Cat}, \quad \mathbf{Ex}_2^{\mathbf{N}}: \mathbf{Cat} \to \mathbf{sSet}
\]

satisfies the conditions of a weak monoidal Quillen equivalence in the sense of Schwede–Shipley [2002, Definition 3.6], except for the nonacyclic part of the pushout product axiom.

1.3. **Previous work** Used in §1.3

Lack [2004.a, Theorem 4] constructs a model structure on the category of small 2-categories whose weak equivalences are equivalences of 2-categories. Worytkiewicz–Hess–Parent–Tonks [2004.a, 2015.b], Ara–Maltsiniotis [2013.c, Chiche 2012.b, Théorème 7.9], Ara [2016.b] construct a Thomason model structure (modeling \((\infty, 0)\)-categories) on the category of small 2-categories and show it is Quillen equivalent to the Kan–Quillen model structure on simplicial sets.

Fiore–Paoli [2008.b] introduce a Thomason model structure on small strict \(n\)-fold categories (obtained by iterating the internal category construction \(n\) times starting from the category of sets) and prove it is Quillen equivalent to simplicial sets.

Raptis [2010.a] constructs a Thomason model structure on the category of small posets, shows it to be Quillen equivalent to the Thomason model structure on small categories, and proves that the Thomason model structure is not cartesian. Bruckner [2015.c] constructs a Thomason model structure on acyclic categories (categories without inverses and nonidentity endomorphisms). Bruckner–Pegel [2016.a] give examples of cofibrant posets in the Thomason model structure.

Meier–Ozornova [2014.a] show that the category of weak equivalences of a Barwick–Kan partial model category (Barwick–Kan [2011.a]) is Thomason-fibrant. Meier [2015.a] shows that the category of weak equivalences of a fibration category is Thomason-fibrant.

1.4. **Prerequisites**

We assume familiarity with basics of the following topics from homotopy theory. Appropriate references will be given throughout the text.

- Simplicial homotopy theory, including simplicial sets, simplicial maps, simplicial weak equivalences, Kan’s subdivision–extension adjunction \(\mathbf{Sd} \dashv \mathbf{Ex}\), the fundamental category–nerve adjunction \(\mathbf{c} \dashv \mathbf{N}\). See Gabriel–Zisman [1967.b], Goerss–Jardine [1999.a].
- Model categories, including model structures, Quillen adjunctions, transferred model structures, monoidal model categories. See Quillen [1967.a], Hovey [1999.b], Hirschhorn [2003], Barwick [2007.b], Schwede–Shipley [1998.a].
- Enriched categories, enriched operads, and algebras over enriched operads. See Kelly [1982, 1972].
1.5. **Acknowledgments**

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2 The Thomason model structure and its properties

In this section we establish the properties of the Thomason model structure that we need later.

Recall the Kan–Quillen model structure on the category \( \text{sSet} \) of simplicial sets (Quillen [1967, Theorem II.3.3]). Recall the adjunction between Kan’s subdivision and extension functors

\[
\text{Sd} \dashv \text{Ex}, \quad \text{Sd}: \text{sSet} \to \text{sSet}, \quad \text{Ex}: \text{sSet} \to \text{sSet} \quad \text{(Kan [1956, \S3, \S6])}
\]

and the adjunction between the fundamental category and nerve functors

\[
\text{c} \dashv \text{N}, \quad \text{c}: \text{sSet} \to \text{Cat}, \quad \text{N}: \text{Cat} \to \text{sSet} \quad \text{(Gabriel–Zisman [1967, \II.4])},
\]

where \( \text{Cat} \) denotes the category of small categories and functors.

**Remark 2.1.** A simplicial set is a compact object in \( \text{sSet} \) if and only if it has finitely many nondegenerate simplices. A small category is a compact object in \( \text{Cat} \) if and only if it admits a presentation with finitely many generators and relations. Thus, the functors \( \text{Sd} \) and \( \text{c} \) send representable simplicial sets \( \Delta^n \) to compact objects. Therefore, \( \text{Sd} \) and \( \text{c} \) preserve compact objects and \( \text{Ex} \) and \( \text{N} \) preserve filtered colimits. Used in Proposition 1.1.

Recall the definition of a transferred model structure.

**Definition 2.2.** (Crans [1993, Theorem 3.3], Hirschhorn [2003, Theorem 11.3.2].) Suppose \( C \) is a model category and \( U: D \to C \) is a right adjoint functor. The transferred model structure on \( D \) (if it exists) is the unique model structure whose weak equivalences and fibrations are created by the functor \( U \), meaning a morphism \( f \) in \( D \) is a weak equivalence if and only if \( U(f) \) is a weak equivalence in \( C \) and likewise for fibrations. Used in Propositions 2.14, 2.13, 5.6.

**Definition 2.3.** The Thomason model structure (Thomason [1980, Theorem 4.9], Cisinski [1999c]) on the category \( \text{Cat} \) of small categories is transferred (Definition 2.2) along the right adjoint functor

\[
\text{Ex}^2 \text{N}: \text{Cat} \to \text{sSet},
\]

meaning its weak equivalences (Thomason weak equivalences), fibrations, and acyclic fibrations are created by the functor \( \text{Ex}^2 \text{N} \), whereas cofibrations and acyclic cofibrations are defined using the left lifting property. Used in Propositions 1.2, 2.14, 2.13, 5.5, 5.6.

**Remark 2.4.** There are other model structures on \( \text{Cat} \), such as the model structure of Joyal–Tierney [1991, Theorem 4], in which weak equivalences are precisely equivalences of categories. In this paper, \( \text{Cat} \) is always considered with the Thomason model structure.

**Proposition 2.5.** Weak equivalences in \( \text{Cat} \) are closed under compositions, transfinite compositions, filtered colimits, finite products, and satisfy the 2-out-of-3 property. Used in Propositions 1.2, 2.14, 5.5, 5.6.

**Proof.** This follows from the analogous properties for \( \text{sSet} \) combined with the fact that the functor \( \text{Ex}^2 \text{N} \) preserves filtered colimits because its left adjoint functor \( \text{c} \text{Sd}^2 \) sends every simplex \( \Delta^n \) to a compact object in \( \text{Cat} \).

Recall the notion of an \( h \)-cofibration in a model category. (The letter “\( h \)” stands for Hurewicz, meaning \( h \)-cofibrations abstract away some of the properties of Hurewicz cofibrations.)
Definition 2.6. (Grothendieck; Batanin–Berger [2013,b, Definition 1.1].) A morphism \( f: X \to Y \) in a model category \( C \) (more generally, a relative category, i.e., a category equipped with a subcategory of weak equivalences) is an \( h \)-cofibration if the cobase change functor along \( f \)

\[
f_! : X/C \to Y/C
\]
preserves weak equivalences, where \( X/C \) denotes the undercategory (alias coslice category) of \( X \in C \). An \textit{acyclic }\( h \)-cofibration is an \( h \)-cofibration that is also a weak equivalence.

Recall the following definition due to Cisinski, following Thomason.

Definition 2.7. (Cisinski [1999,c, Définition 1].) A \textit{Cisinski–Dwyer map} is an inclusion of a full subcategory \( \iota: A \to B \) such that

- The functor \( \iota \) is a sieve: if \( Y \in A \) and \( f: X \to Y \) is a morphism in \( B \), then \( X \in A \) and \( f \) is a morphism in \( A \).
- The inclusion \( j: A \to Z \) admits a retraction \( r: Z \to A \). Here \( Z \subset B \) is the cosieve generated by \( A \) in \( B \), i.e., the full subcategory of \( B \) comprising objects \( Y \in B \) such that there is a morphism \( X \to Y \) in \( B \) for which \( X \in A \).
- The retraction \( r \) admits a natural transformation \( \varepsilon: ir \to \text{id}_Z \).
- The natural transformation \( \varepsilon \circ \iota \) is the identity natural transformation of functors \( A \to Z \).

An \textit{acyclic Cisinski–Dwyer map} is a Cisinski–Dwyer map that is also a weak equivalence in \( \text{Cat} \).

Remark 2.8. The original definition of Dwyer maps (Thomason [1980, Definition 4.1]) further requires that \( \iota \) is left adjoint to \( r \), in which case \( \varepsilon \) can be taken to be the counit. This condition is too strong for Thomason [1980, Lemma 5.3.3] (whose proof is omitted there), which claims that Dwyer maps are closed under retracts. Cisinski [1999,c, Lemme 1 and the two following paragraphs] constructs an example of a retract of a Dwyer map that is not a Dwyer map. Cisinski [1999,c, Définition 1] defined Cisinski–Dwyer maps and showed that Thomason's necessary criterion for cofibrations in \( \text{Cat} \) [1980, Proposition 5.4] and the proof of left properness of \( \text{Cat} \) [1980, Corollary 5.5] are valid when Cisinski–Dwyer maps are substituted for Dwyer maps.

Proposition 2.9. Cisinski–Dwyer maps and acyclic Cisinski–Dwyer maps are closed under cobase changes, composition, transfinite composition, retracts, and products with a fixed object in \( \text{Cat} \). All cofibrations in \( \text{Cat} \) are Cisinski–Dwyer maps [Definition 2.7] and all Cisinski–Dwyer maps are \( h \)-cofibrations [Definition 2.6] in \( \text{Cat} \).

Proof. Cisinski–Dwyer maps are closed under retracts by Cisinski [1999,c, Lemme 4]. Cisinski–Dwyer maps are closed under compositions and transfinite compositions by Cisinski [1999,c, Remarque] and Thomason [1980, Lemma 5.3]. Cisinski–Dwyer maps are closed under cobase changes by Cisinski [1999,c, Remarque] and Thomason [1980, Proposition 4.3]. To show that Cisinski–Dwyer maps are closed under products with a fixed object \( W \in \text{Cat} \), observe that if

\[
(i: A \to B, r: Z \to A, \varepsilon: ir \to \text{id}_Z)
\]
is a tuple exhibiting \( i \) as a \textit{Cisinski–Dwyer map}, then

\[
(W \times i: W \times A \to W \times B, r: W \times Z \to W \times A, W \times \varepsilon: W \times ir \to W \times \text{id}_Z)
\]
is a tuple exhibiting \( W \times i \) as a \textit{Cisinski–Dwyer map}.

All cofibrations in \( \text{Cat} \) are Cisinski–Dwyer maps by Cisinski [1999,c, Proposition 2]. To show that every Cisinski–Dwyer map \( i: A \to B \) is an \( h \)-cofibration, pick any weak equivalence \( f: X \to Y \), together with a morphism \( A \to X \). The induced map

\[
X \sqcup_A B \to Y \sqcup_A B
\]
is a weak equivalence in \( \text{Cat} \), meaning

\[
N(X \sqcup_A B) \to N(Y \sqcup_A B)
\]

is a weak equivalence in \( \text{sSet} \). By Cisinski [1999.d, Remarque] and Thomason [1980, Proposition 4.3], the latter morphism is weakly equivalent to

\[
NX \sqcup_{NA} NB \to NY \sqcup_{NA} NB.
\]

This morphism is a weak equivalence because \( N: NA \to NB \) is a cofibration of simplicial sets and the model category \( \text{sSet} \) is left proper.

Finally, acyclic Cisinski–Dwyer maps satisfy the same set of properties by Proposition 2.5 and the previously established properties of Cisinski–Dwyer maps.

**Definition 2.10.** (Barwick [2007.b, Definition 1.3 (arXiv); 1.21 (journal)].) A model category is tractable if it is combinatorial (as defined by Jeffrey H. Smith, see Dugger [1998.B, Definition 2.4]) and it admits a set of generating (acyclic) cofibrations with cofibrant domains.

**Remark 2.11.** By Barwick [2007.b, Corollary 1.12 (arXiv); 2.7 (journal)], a combinatorial model category that admits a set of generating cofibrations with cofibrant domains is Definition 2.10, i.e., admits a set of generating acyclic cofibrations with cofibrant domains.

The following definition is one of many ways to formalize the idea of a compactly generated model category. The specific definition chosen here is motivated by the fact that it is well behaved with respect to transfers of model structures, as shown in Pavlov–Scholbach [2015.d, Proposition 5.3(ii)].

**Definition 2.12.** (Pavlov–Scholbach [2015.d, Definition 2.1].) A model category \( C \) is pretty small if there is a cofibrantly generated model category structure \( D \) on \( C \) with the same weak equivalences, possibly smaller class of cofibrations, and such that the domains and codomains of some set of generating cofibrations of \( D \) are compact objects.

Although the cited work of Thomason and Cisinski does prove the existence of the Thomason model structure, we find it beneficial to give a short modern proof, while reusing only a fraction of their results.

Recall the transfer theorem for model structures, which we state in the special case of locally presentable categories.

**Proposition 2.13.** (Crans [1993, Theorem 3.3], Hirschhorn [2003, Theorem 11.3.2]; see also Garner–Kedziorek–Riehl [2018.B] for a generalization.) Suppose \( U: D \to C \) is a right adjoint functor between locally presentable categories and \( D \) is equipped with a combinatorial model structure with \( I \) and \( J \) as sets of generating (acyclic) cofibrations. Then the transferred model structure (Definition 2.3) on \( D \) exists if and only if the functor \( U \) sends to weak equivalences all transfinite compositions of cobase changes of elements of \( F(J) \), where \( F \) is the left adjoint of \( U \). In this case, the model structure on \( D \) is combinatorial with \( F(I) \) and \( F(J) \) are sets of generating (acyclic) cofibrations.

**Theorem 2.14.** (Thomason [1980, Theorem 4.9], Cisinski [1999.d].) The Thomason model structure exists and is a proper tractable (Definition 2.10) pretty small (Definition 2.12) combinatorial model structure.

**Proof.** The functor \( \text{Ex}^2 \)\( N \) is a right adjoint functor between locally presentable categories that preserves filtered colimits and its left adjoint preserves compact objects. Thus, by the transfer theorem (Proposition 2.13), it suffices to show that transfinite compositions of cobase changes of generating acyclic cofibrations are weak equivalences in \( \text{Cat} \). By Proposition 2.9, generating acyclic cofibrations in \( \text{Cat} \) are acyclic Cisinski–Dwyer maps, which in their turn are closed under cobase changes and transfinite compositions. Thus, the transferred model structure on \( \text{Cat} \) exists and is combinatorial. By Proposition 2.9, every cofibration in \( \text{sSet} \) is an \( h \)-cofibration, hence the model structure on \( \text{Cat} \) is left proper. Since the model structure on \( \text{sSet} \) is right proper, so is the model structure on \( \text{Cat} \). The model structure on \( \text{Cat} \) is tractable because the domains \( \text{sD}^2 \partial \Delta^n \) and \( \text{sD}^2 \Lambda^n \) of generating (acyclic) cofibrations are cofibrant since \( \text{sD}^2 \) is a left Quillen functor and all simplicial sets are cofibrant. The model structure on \( \text{Cat} \) is pretty small because the domains and codomains \( \text{sD}^2 \Delta^n \) of generating (acyclic) cofibrations are compact since \( \text{sD}^2 \) preserves compact objects by Remark 2.1 and the simplicial sets \( \partial \Delta^n, \Lambda^n \), and \( \Delta^n \) have finitely many nondegenerate simplices, hence are compact.
3 Monoidal properties of the Thomason model structure

In this section, we investigate properties of the Thomason model structure connected to its cartesian monoidal structure. An example due to Raptis [2010,a, §3] shows that the Thomason model structure is not cartesian. Nevertheless, we can establish some other properties, which suffice to establish a model structure on modules over monoids in the Thomason model structure (Proposition 3.10).

Recall the notion of a monoidal model category from Schwede–Shipley [1998,a].

Definition 3.1. (Schwede–Shipley [1998,a, Definition 2.1 (arXiv); Definition 3.1 (journal)].) Suppose $M$ is a closed monoidal category equipped with a model structure. We say that $M$ is a monoidal model category if the monoidal product functor

$$M \times M \to M$$

is a left Quillen bifunctor. The latter condition is also known as the pushout product axiom. If the monoidal structure is cartesian, we talk about a cartesian model structure.

Often, an additional condition on the monoidal unit is included in the definition of a monoidal model category, such as the unit axiom (Hovey [1999,b, Lemma 4.2.7]), the strong unit axiom (Muro [2013,a, Definition A.9]), or the very strong unit axiom (Batanin–Berger [2013,b, §1.20]). In our case, the monoidal unit will always be cofibrant, which is the strongest of the unit axioms.

The following proposition is due to Raptis [2010,a, §3].

Proposition 3.2. (Raptis [2010,a, §3].) The Thomason model structure is not cartesian. More precisely, the pushout product of the acyclic cofibration $\{0\} \to \{0 \to 1\}$ with itself is not a Cisinski–Dwyer map and therefore not a cofibration.

Below, Definition 3.3 (flat model structure), Definition 3.6 (h-monoidal model structure), Definition 3.7 (the monoid axiom) are stated for model categories equipped with a monoidal structure that need not satisfy the pushout product axiom.

Definition 3.3. (Pavlov–Scholbach [2015,d, Definition 3.2.4].) A model category equipped with a monoidal structure is flat if the pushout product of a cofibration and a weak equivalence is a weak equivalence.

Proposition 3.4. The Thomason model structure on $\mathbf{Cat}$ is flat (Definition 3.3).

Proof. Suppose $A \to B$ is a cofibration and $X \to Y$ is a weak equivalence in $\mathbf{Cat}$. Denote by $P \to B \times Y$ their pushout product:

$$A \times X \to A \times Y$$

$$B \times X \to B \times Y$$

$$\downarrow \quad \downarrow$$

$$\quad P$$

The map $A \times X \to B \times X$ is a Cisinski–Dwyer map and an h-cofibration by Proposition 2.9. The map $A \times X \to A \times Y$ is a weak equivalence by Proposition 2.5. Thus, the canonical map $B \times X \to P$ is a weak equivalence. Since $B \times X \to B \times Y$ is a weak equivalence by Proposition 2.5, by the 2-out-of-3 property the map $P \to B \times Y$ is also a weak equivalence.

Corollary 3.5. The Thomason model structure on $\mathbf{Cat}$ satisfies the acyclic part of the pushout product axiom: the pushout product of a cofibration and an acyclic cofibration is a weak equivalence.

Definition 3.6. (Batanin–Berger [2013,b, Definition 1.11].) A model category $C$ equipped with a monoidal structure is h-monoidal if the monoidal product of any object $A \in C$ with an (acyclic) cofibration in $C$ is an (acyclic) h-cofibration.

Recall the Schwede–Shipley monoid axiom [1998,a, Definition 2.2 (arXiv); Definition 3.2 (journal)].
Definition 3.7. A model category $C$ with a monoidal structure satisfies the monoid axiom if every transfinite composition of cobase changes of tensor products of an object in $C$ and an acyclic cofibration is a weak equivalence in $C$. Used in 1.2, 3.2*, 3.8, 3.8*, 3.10*.

Proposition 3.8. The Thomason model structure on $\text{Cat}$ (Definition 2.3) is a proper combinatorial model category that is tractable (Definition 2.10), pretty small (Definition 2.12), h-monoidal (Definition 3.6), flat (Definition 3.3), and satisfies the monoid axiom (Definition 3.7). Used in 1.2, 3.10*.

Proof. By Theorem 2.14, the Thomason model structure is a proper combinatorial tractable pretty small model category.

By Proposition 2.9, cofibrations in $\text{Cat}$ are Cisinski–Dwyer maps, the product of an object in $\text{Cat}$ and a Cisinski–Dwyer map is again a Cisinski–Dwyer map, and every Cisinski–Dwyer map is an h-cofibration, hence the Thomason model structure is h-monoidal.

The monoid axiom (Definition 3.7) is established in Pavlov–Scholbach [2015.d, Lemma 3.2.3] (which does not require the pushout product axioms), using the fact that the Thomason model structure is h-monoidal and pretty small. We reproduce the proof of the cited result here: by Proposition 2.9, maps of the form $M \times j$ (where $M \in \text{Cat}$ and $j$ is an acyclic cofibration in $\text{Cat}$) are acyclic Cisinski–Dwyer maps, and again by Proposition 2.9, acyclic Cisinski–Dwyer maps are closed under cobase changes and transfinite compositions.

Proposition 3.9. The Quillen equivalence

$$c\text{Sd}^2 \dashv \text{Ex}^2 N : \text{sSet} \to \text{Cat}, \quad \text{Ex}^2 N : \text{Cat} \to \text{sSet}$$

satisfies the conditions for a weak monoidal Quillen equivalence in the sense of Schwede–Shipley [2003, Definition 3.6], except for the nonacyclic part of the pushout product axiom for $\text{Cat}$. That is, $\text{sSet}$ is a cartesian model category, $\text{Cat}$ satisfies the acyclic pushout product axiom (Corollary 3.5), the right adjoint $\text{Ex}^2 N$ is a lax (in fact, strong) monoidal functor (i.e., preserves finite products), the left adjoint $c\text{Sd}^2$ preserves the monoidal unit (i.e., the terminal object), and the canonical map

$$c\text{Sd}^2(A \times B) \to c\text{Sd}^2 A \times c\text{Sd}^2 B$$

is a weak equivalence for any simplicial sets $A$ and $B$. Both functors in the adjunction preserve and reflect weak equivalences. In particular, the unit and counit maps are weak equivalences. Used in 1.2, 3.14, 3.15.

Proof. The only statement that remains to be proved is that the comonoidal map

$$c\text{Sd}^2(A \times B) \to c\text{Sd}^2 A \times c\text{Sd}^2 B$$

is a weak equivalence for any simplicial sets $A$ and $B$. This follows from the fact that the functor $c$ preserves finite products and there is a natural weak equivalence $\text{Sd}^2 \to \text{id}$ induced by the last vertex map.

We conclude this section by constructing a model structure on the category of modules over a strict monoidal category, i.e., a monoid in the Thomason model structure. Recall that any monoidal category can be strictified to a strict monoidal category.

Proposition 3.10. Suppose $M$ is a monoid in the Thomason model structure, i.e., a strict monoidal category. The category of (strict) modules over $M$ admits a model structure transferred along the forgetful functor to the Thomason model structure (Definition 2.3). Used in 3.0*.

Proof. (Compare Schwede–Shipley [1998.a, Remark 3.2 (arXiv); Remark 4.2 (journal)].) By the transfer theorem (Proposition 2.13) it suffices to show that the forgetful functor sends transfinite compositions of cobase changes of maps of the form $M \times j$, where $j \in J$ is a generating acyclic cofibration for the Thomason model structure, to weak equivalences in the Thomason model structure. This is the content of the monoid axiom established in Proposition 3.8.
4 Nonexistence of cartesian model structures

In this section we show that any choice of a model structure on the category of small categories equipped with \([\text{Thomason weak equivalences}]\) must be a noncartesian model structure. This means that the existing approaches to constructing model structures on monoids (Schwede–Shipley [1998.2]), enriched categories (Muro [2012.a]), or algebras over colored symmetric operads (Pavlov–Scholbach [2014.b]) do not work, since these make heavy use of the pushout product axiom. Nevertheless, in the next section we do show the existence of model structures on monoids and enriched categories by replicating the original argument of Thomason in this setting.

Proposition 4.1. Suppose the category of small categories is equipped with a model structure whose weak equivalences are \([\text{Thomason weak equivalences}]\). Then the terminal category is cofibrant and at least one of the inclusions \(\{0\} \to \{0 \to 1\}\), \(\{1\} \to \{0 \to 1\}\) is an acyclic cofibration. Used in [2.3]

Proof. Factor the map \(q: \emptyset \to \{0\}\) as a cofibration \(p: \emptyset \to A\) followed by an acyclic fibration \(q: A \to \{0\}\). Since \(q\) is a \([\text{Thomason weak equivalence}]\) there is an object \(a \in A\), which induces a morphism \(\iota: \{0\} \to A\). The maps \(\iota\) and \(q\) exhibit \(\{0\}\) as a retract of the cofibrant object \(A\), hence the category \(\{0\}\) is cofibrant. In particular, acyclic fibrations must be surjective on objects.

Factor the weak equivalence \(\{0\} \to \{0 \to 1\}\) as an acyclic cofibration \(p: \{0\} \to F\) followed by an acyclic fibration \(q: F \to \{0 \to 1\}\). Any acyclic fibration has the right lifting property with respect to the cofibration \(\emptyset \to \{0\}\), hence \(q\) is surjective on objects. Since \(q\) is a \([\text{Thomason weak equivalence}]\), the arrow \(0 \to 1\) must be in the image of \(q\).

Introduce a preorder relation \(P\) on the set \(S\) of objects of \(F\): we have \(x \leq y\) if there is an arrow \(x \to y\).

Consider also an equivalence relation \(R\) on the same set \(S\): we have \(xRy\) if \(x \leq y\) and \(y \leq x\). The preorder \(P\) becomes an order on the quotient \(S/R\). By Szpilrajn’s extension theorem [1930], we can extend the partial order on \(S/R\) to a total order on \(S/R\), which induces a total preorder \(Q\) on \(S\).

The equivalence class \(E\) (with respect to the equivalence relation \(R\) on \(S\)) of \(p(0) \in F\) must necessarily have at least incoming or outgoing arrow connecting it (and hence connecting \(p(0)\)) to a different equivalence class \(E'\), since otherwise \(E\) is a connected component of \(F\), which is impossible because such a connected component cannot contain an object \(z \in F\) such that \(q(z) = 1\). Since \(Q\) is a total order, we have \(E < E'\) or \(E' < E\), so we analyze each of those cases separately.

Suppose \(z \to p(0)\) is an arrow in \(F\) such that \(z < p(0)\) with respect to \(Q\). Consider the inclusion \(\{0 \to 1\} \to F\) that sends \(0 \to 1\) to \(z \to p(0)\) and the retraction \(F \to \{0 \to 1\}\) that sends \(z \mapsto 0\) if \(z < p(0)\) and \(z \mapsto 1\) if \(z \geq p(0)\). The definition of \(Q\) guarantees that the latter construction yields a functor. We exhibited the injective functor \(\{0\} \to \{0 \to 1\}\) that sends \(0 \mapsto 1\) as a codomain retract of the inclusion \(\{0\} \to F\). Thus, the inclusion \(\{1\} \to \{0 \to 1\}\) is a cofibration.

Suppose \(p(0) \to z\) is an arrow in \(F\) such that \(p(0) < z\) with respect to \(Q\). Consider the inclusion \(\{0 \to 1\} \to F\) that sends \(0 \to 1\) to \(p(0) \to z\) and the retraction \(F \to \{0 \to 1\}\) that sends \(z \mapsto 0\) if \(z \leq p(0)\) and \(z \mapsto 1\) if \(z > p(0)\). The definition of \(Q\) guarantees that this is a functor. We exhibited the inclusion \(\{0\} \to \{0 \to 1\}\) as a codomain retract of the inclusion \(\{0\} \to F\).

Thus, at least one of the inclusions \(\{0\} \to \{0 \to 1\}\), \(\{1\} \to \{0 \to 1\}\) is a cofibration. □

I thank Georges Maltsiniotis for suggesting the choice of the map \(A \to D\) in the following proposition.

Proposition 4.2. (Maltsiniotis, 2022.) Consider the posets \(B = \{0 \to 1\}^3\), \(A = B \setminus \{111\}\), \(C = A \setminus \{000\}\). Turn these posets into categories, and consider also the categories given by pushouts in the category of small categories \(D = A \sqcup_C A\) and \(E = D \sqcup_A B\). The canonical map \(D \to E\) is not a \([\text{Thomason weak equivalence}]\).

\[
\begin{array}{ccc}
C & \longrightarrow & A \\
\downarrow & & \downarrow \\
A & \longrightarrow & D \\
\downarrow & & \downarrow \\
B & \longrightarrow & E.
\end{array}
\]
Proof. The inclusion \( \iota \) of posets into categories preserves the pushout \( A \sqcup_C A \), i.e., the canonical map

\[
\iota A \sqcup_C \iota A \to \iota (A \sqcup_C A)
\]

is an isomorphism because \( C \subset A \) is an upward closed subset. Henceforth, we omit \( \iota \) from the notation.

Thus, the category \( D \) is the nerve of the poset whose underlying set is \( A \sqcup \{000'\} \), where \( 000' \) is ordered in the same way as \( 000 \), and the elements \( 000 \) and \( 000' \) are incomparable. Furthermore, the nerve functor preserves the pushout \( A \sqcup_C A \), i.e., the canonical map

\[
N A \sqcup N C \to N (A \sqcup_C A)
\]

is an isomorphism. Thus, the homotopy type of \( D \) can be computed as the homotopy pushout of \( N A \leftarrow N C \to N A \). The nerve of \( A \) is contractible. The nerve of \( C \) is six 1-simplices connected together, i.e., a circle. Thus, \( ND \) is weakly equivalent to the 2-sphere.

We remark that working in the pushout of categories \( D \sqcup_A B \) is crucial for the above argument. In particular, if we worked with the pushout of nerves \( ND \sqcup_{NA} NB \), then the above argument would construct simplicial homotopies between the corresponding triples of edges, which glue together into a noncollapsible 2-sphere. In the pushout of categories \( D \sqcup_A B \) this 2-sphere is collapsed to a single morphism \( 000' \to 111 \).

Proposition 4.3. Suppose the category of small categories is equipped with a model structure whose weak equivalences are Thomason weak equivalences. Then this model structure is not cartesian.

Proof. By Proposition 4.1, at least one of the inclusions \( \{0\} \to \{0 \to 1\} \), \( \{1\} \to \{0 \to 1\} \) is a cofibration in this model structure. The two cases are symmetric, so without loss of generality we assume \( \{0\} \to \{0 \to 1\} \) is a cofibration. The pushout product of three copies of this cofibration is the nerve of inclusion of posets

\[
\nu: \{0 \to 1\}^3 \setminus \{111\} \subset \{0 \to 1\}^3,
\]

i.e., the map \( \nu: A \to B \) from Proposition 4.2. If the pushout product axiom is satisfied, the map \( \nu \) must be an acyclic cofibration, in particular, its cobase changes must be Thomason weak equivalences. However, Proposition 4.2 constructs a cobase change of \( \nu \) that is not a Thomason weak equivalence.
5 Main theorems

In this section we construct the enriched Thomason model structure on small 2-categories (Theorem 5.11) and prove it is Quillen equivalent to the Bergner model structure on small simplicial categories (Theorem 5.14). Thus, strict 2-categories equipped with the enriched Thomason model structure provide another model for \((\infty, 1)\)-categories. We start with the simpler case of a model structure on monoids in the Thomason model structure (Theorem 5.6). The established machinery for constructing model structures on monoids (Schwede–Shipley [1998.a]) is not applicable because Proposition 4.3 shows that any approach based on the pushout product axiom cannot work. However, we do make use of the Schwede–Shipley filtration (Proposition 5.4) on cobase changes of free morphisms of monoids.

Definition 5.1. Suppose \(C\) and \(D\) are locally presentable closed symmetric monoidal categories. Denote by

\[ \text{Mon}_C \]

the category of monoids in \(C\). Denote by

\[ F \dashv U: C \rightleftarrows \text{Mon}_C \]

the free-forgetful adjunction between \(C\) and monoids in \(C\). We also write \(F_C \dashv U_C: C \rightleftarrows \text{Mon}_C\) if there is more than one possibility for \(C\). Given an adjunction \(L \dashv R: C \rightleftarrows D\), where the right adjoint functor \(R\) is strong monoidal (in our case: \(c \text{Sd}^2 \dashv \text{Ex}^2: \text{sSet} \rightleftarrows \text{Cat}\)), denote by

\[ L \dashv R: \text{Mon}_C \rightleftarrows \text{Mon}_D \]

the adjunction between monoids in \(C\) and monoids in \(D\), where the right adjoint \(R\) applies the functor \(R\) to the underlying object of a monoid in \(D\). Used in 5.4, 5.5, 5.15.

Definition 5.2. Given a morphism \(f: K \to L\) in a cocomplete symmetric monoidal category \(C\) and \(i \geq 0\), denote by

\[ f^{\boxtimes i}: f^{\boxtimes i} \to L^{\boxtimes i} \]

the \(i\)-fold pushout product of \(f\) with itself.

Remark 5.3. The notation \(f^{\boxtimes i}\) refers to the domain of \(f^{\boxtimes i}\). If \(C\) is the category of sets with its cartesian monoidal structure and \(f\) is an inclusion of sets, then \(f^{\boxtimes i}\) is the set \(L^{\boxtimes i} \setminus (L \setminus K)^{\boxtimes i}\).

Recall the following presentation of cobase changes of free morphisms of monoids due to Schwede–Shipley [1998.a] proof of Lemma 5.2 (arXiv); 6.2 (journal)].

Proposition 5.4. (Schwede–Shipley [1998.a] proof of Lemma 5.2 (arXiv); 6.2 (journal)].) Assuming the notation of Definition 5.1, given a morphism \(f: K \to L\) in \(C\) and a morphism \(F(K) \to X\) in \(\text{Mon}_C\), in the pushout square

\[
\begin{array}{ccc}
F(K) & \longrightarrow & X \\
\downarrow & & \downarrow \\
F(L) & \longrightarrow & P
\end{array}
\]

the map \(U(X) \to U(P)\) is the transfinite composition of the chain

\[ U(X) = P_0 \to P_1 \to P_2 \to \cdots \to P_\infty = U(P), \]

where for every \(i > 0\) the morphism \(P_{i-1} \to P_i\) fits into the pushout square

\[
\begin{array}{ccc}
X^{\boxtimes (i+1)} & \longrightarrow & P_{i-1} \\
\downarrow & & \downarrow \\
L^{\boxtimes i} \otimes X^{\boxtimes (i+1)} & \longrightarrow & P_i
\end{array}
\]
where the attaching map on top is defined inductively by using the previously defined maps

\[ L^{\otimes j} \otimes X^{\otimes (j+1)} \to P_j \to P_{i-1} \]

for \( j < i \). Used in 5.0*, 5.5*, 5.11*, 5.13.

**Proposition 5.5.** Assuming the notation of [Definition 5.1], given \( n \geq 0 \) and a commuting diagram of pushout squares in the category \( \text{Mon}_{\text{CAT}} \)

\[
\begin{array}{ccc}
F(c \text{Sd}^2 \partial \Delta^n) & \to & X \\
\downarrow & & \downarrow \\
F(c \text{Sd}^2 \Delta^n) & \to & P \\
\end{array}
\]

if the map \( X \to Y \) is a [Thomason weak equivalence], then so is the map \( P \to Q \). That is to say, applying \( F \circ c \circ \text{Sd}^2 \) to \( \partial \Delta^n \to \Delta^n \) yields an \( h \)-cofibration (Definition 2.6) in \( \text{Mon}_{\text{CAT}} \). Used in 5.6*, 5.11*.

**Proof.** Our proof follows the analogous proof of Thomason [1980, Proposition 4.3], combining it with the filtration of Proposition 5.4. Recall that the inclusion \( \text{Sd}^2 \partial \Delta^n \to \Delta^n \) is the nerve of a map of posets \( \iota: A \to B \), which factors as

\[
c \text{Sd}^2 \partial \Delta^n = A \xrightarrow{\kappa} W \xrightarrow{\mu} B = c \text{Sd}^2 \Delta^n,
\]

where \( W \) is the cosieve generated by \( A \) in \( B \). Recall also that \( W \) exhibits \( A \) as a categorical analogue of a neighborhood deformation retract; indeed, the inclusion \( A \to W \) is isomorphic to the inclusion \( A \times \{0\} \to A \times \{0 \to 1\} \), so in particular, it admits a retraction \( W \to A \) and the composition \( W \to A \to W \) admits a natural transformation to the identity functor.

Consider the induced diagram of pushout squares:

\[
\begin{array}{ccc}
FA & \to & X \\
\downarrow & & \downarrow \\
FW & \to & R \\
\downarrow & & \downarrow \\
FB & \to & P \\
\end{array}
\]

The rest of the proof proceeds in two steps: first, we show that the map \( R \to S \) is a [Thomason weak equivalence], and then we prove the same claim for the map \( P \to Q \).

To show that \( R \to S \) is a [Thomason weak equivalence], we exhibit the map \( FA \to FW \) as a monoidal analogue of a [Cisinski–Dwyer map]. First, the retraction \( W \to A \) induces a retraction \( FW \to FA \) and therefore a retraction \( R \to X \). In particular, the composition \( X \to R \to X \) is the identity map. To show that the composition \( R \to X \to R \) is a [Thomason weak equivalence], we apply the functor \( U \) and construct a natural transformation

\[
\varepsilon: U(R \to X \to R) \to \text{id}_{UR}
\]

using the filtration of Proposition 5.4. Thus, present the functor \( UX \to UR \) as the transfinite composition

\[
UX = R_0 \to R_1 \to R_2 \to \cdots \to R_{\infty} = UR,
\]

where for every \( i > 0 \) the map \( R_{i-1} \to R_i \) is a cobase change of the map

\[
\begin{array}{ccc}
X \times (i+1) \times (\kappa \square i \to W \times i),
\end{array}
\]

where \( \kappa: A \to W \) is the inclusion map. The natural transformation

\[
\varepsilon: I \times R_{\infty} \to R_{\infty}, \quad I = \{0 \to 1\}
\]
weak equivalence.

Denote by \( W \) and \( I \) an object of \( As \).

As observed by Thomason [1980, Proof of Proposition 4.3], the commutative square

\[
\begin{array}{ccc}
\varepsilon: I \times R_i & \to & R_i,
\end{array}
\]

which are constructed by induction on \( i \geq 0 \). We take \( \varepsilon_0: I \times X \to X \) to be the projection functor. This also ensures that \( \varepsilon \circ \chi \) is the identity natural transformation of functors \( X \to R \), where \( \chi: X \to R \) is the inclusion map, in analogy to the definition of a Cisinski–Dwyer map. By inductive assumption, we have already constructed the natural transformation \( \varepsilon_{i-1}: I \times R_{i-1} \to R_{i-1} \).

The natural transformation \( \varepsilon_i \) is constructed as follows:

\[
\begin{align*}
\varepsilon_i: I \times R_i &= I \times (R_{i-1} \sqcup (I \times X_{(i+1)} \times \kappa \sqcup i) \times W^{\times i}) \\
&\cong (I \times R_{i-1}) \sqcup (I \times X_{(i+1)} \times \kappa \sqcup i) \times W^{\times i} \\
&\to R_i \sqcup X_{(i+1)} \times \kappa \sqcup i \times W^{\times i} = R_i,
\end{align*}
\]

where the last map has as its first component \( \varepsilon_{i-1} \) and the other two components are induced by the identity on \( X^{\times (i+1)} \) and the natural transformation of functors

\[
\alpha_i: (W^{\times i} \to A^{\times i} \to W^{\times i}) \Rightarrow \text{id}_{W^{\times i}}
\]

given by taking the product of the identity functor on \( A^{\times i} \) with the natural transformation

\[
(I^{\times i} \to \{1\}^{\times i} \to I^{\times i}) \Rightarrow \text{id}_{I^{\times i}}, \quad x \mapsto (0^{\times i} \to x).
\]

By inspection, the map \( \alpha_i \) is compatible with the map \( \varepsilon_{i-1} \), so we indeed get a morphism of pushouts \( \varepsilon_i: I \times R_i \to R_i \). This proves that the map \( X \to R \) is a Thomason weak equivalence. The same argument also proves that \( Y \to S \) is a weak equivalence. By the 2-out-of-3 property, the morphism \( R \to S \) is also a weak equivalence.

We now proceed to show that the map \( P \to Q \) is a Thomason weak equivalence. Following Thomason, denote by \( V \subset B \) the full subcategory given by the objects of \( B \) that are not in \( A \). The inclusions \( V \to B \) and \( W \cap V \to W \) are cosieves. Here \( V \setminus W \) has a single vertex, the barycenter of \( S_{d-1} \Delta^n \), which is the initial object of \( V \). The inclusion \( W \cap V \to W \) is isomorphic to the inclusion

\[
A \times \{1\} \to A \times \{0 \to 1\} \cong W.
\]

As observed by Thomason [1980, Proof of Proposition 4.3], the commutative square

\[
\begin{array}{ccc}
W \cap V & \to & W \\
\downarrow & & \downarrow \\
V & \to & B
\end{array}
\]

is cocartesian because the left and top maps are inclusions of cosieves. Thus, we have a commutative diagram of cocartesian squares

\[
\begin{array}{ccc}
F(W \cap V) & \to & FW \to R \to S \\
F\lambda & & \downarrow \\
FV & \to & FB \to P \to Q.
\end{array}
\]

From now on, we ignore the second column with the map \( F\mu \) and analyze the maps \( R \to P \) and \( S \to Q \) using the filtration of Proposition 5.4 with respect to the map \( F\lambda \). The \( i \)th step in the filtration of the map \( UR \to UP \) has the form

\[
\begin{array}{ccc}
R^{\times (i+1)} \times \lambda \sqcup i & \to & P_{i-1} \\
\downarrow & & \downarrow \\
R^{\times (i+1)} \times V^{\times i} & \to & P_i,
\end{array}
\]

\[12\]
where $\lambda: W \cap V \to V$ is the inclusion map. The left map is a cosieve because $V$ adds a single vertex to $W \cap V$ given by the barycenter, so in the pushout product, $V^{\times i}$ adds a single vertex to $\lambda^{\square i}$ given by the barycenter in every factor, and such a vertex does not admit any morphisms from $\lambda^{\square i}$. Next, we show that the inclusion

$$Y = R^{\times (i+1)} \times \lambda^{\square i} \to P_{i-1}$$

is a cosieve. Given an object

$$z = r_0 v_1 v_2 \cdots r_{i-1} v_i r_i \in Y$$

(where $r_k \in R$, $v_k \in V$, and juxtaposition denotes the multiplication operation), morphisms of the form $z \to z'$ in $P_{i-1}$ are by construction monoidal products of morphisms

$$r_0 \to r'_0, v_1 \to v'_1, r_1 \to r'_1, \ldots$$

because $V \cap W$ is a cosieve in $W$. Thus, for such a morphism $z \to z'$ the object $z'$ is in the image of $Y$ because $V \cap W$ is a cosieve in $V$. By the same argument, the map

$$Y = R^{\times (i+1)} \times \lambda^{\square i} \to Q_{i-1}$$

is also a cosieve inclusion.

As observed by Thomason [1980, Proof of Proposition 4.3], a pushout square of cosieves in $\mathbf{Cat}$ is a homotopy pushout square in $\mathbf{Cat}$. Thus, the cobase change square yielding $P_{i-1} \to P_i$ is homotopy cocartesian. The same argument shows that the square

$$\begin{array}{ccc}
S^{\times (i+1)} \times \lambda^{\square i} & \longrightarrow & Q_{i-1} \\
S^{\times (i+1)} \times \lambda^{\square i} & \downarrow & \\
S^{\times (i+1)} \times V^{\times i} & \longrightarrow & Q_i,
\end{array}$$

is homotopy cocartesian. The morphism $R \to S$ induces a natural transformation from the former square to the latter square. In this natural transformation, the component $P_{i-1} \to Q_{i-1}$ is a weak equivalence by induction, with the base case $i = 1$ yielding the map $R \to S$, which is a Thomason weak equivalence by the first part of the proof. The other two components are given by taking the product of the map $R^{\times (i+1)} \to S^{\times (i+1)}$ with the category $\lambda^{\square i}$ respectively $V^{\times i}$. By Proposition 2.3, these products are weak equivalences because $R \to S$ is a weak equivalence. Therefore, all three components are weak equivalences. The homotopy pushout of weak equivalences is again a weak equivalence. Hence the map $P_i \to Q_i$ is a weak equivalence in $\mathbf{Cat}$. Therefore, the map $P \to Q$ is also a weak equivalence in $\mathbf{Cat}$ by Proposition 2.3.

**Theorem 5.6.** The category $\text{Mon}_{\mathbf{Cat}}$ of monoid objects in small categories (i.e., strict monoidal categories) admits a proper model structure transferred (Definition 2.2) along the right adjoint forgetful functor $\text{Mon}_{\mathbf{Cat}} \to \mathbf{Cat}$ from the Thomason model structure (Definition 2.3) on $\mathbf{Cat}$. Used in 5.0*, 5.11*, 5.13, 5.14*.

**Proof.** The set of generating (acyclic) cofibrations in $\text{Mon}_{\mathbf{Cat}}$ is given by $F(c \text{Sd}^2(I))$ respectively $F(c \text{Sd}^2(J))$. Weak equivalences in $\mathbf{Cat}$ are closed under transfinite compositions by Proposition 2.3. Thus, by Pavlov–Scholbach [2015a, Lemma 2.5(iv)] the class of h-cofibrations in $\text{Mon}_{\mathbf{Cat}}$ is a weakly saturated class, i.e., is closed under cobase changes, transfinite compositions, and retracts. Since $J$ is a subset of the weak saturation of $I$, we deduce that $F(c \text{Sd}^2(J))$ is a subset of the weak saturation of $F(c \text{Sd}^2(I))$; in particular, it consists of h-cofibrations by Proposition 5.3. This means that cobase changes of elements of $F(c \text{Sd}^2(J))$ are weak equivalences. Hence so are their transfinite compositions (by Proposition 2.3) and retracts. By Hirschhorn [2003, Theorem 11.3.2] this proves the existence of the transferred model structure. Since cofibrations are h-cofibrations, the transferred model structure is left proper, whereas right properness is inherited from $\mathbf{Cat}$.

An alternative proof can be given by citing Lurie [2017, Proposition A.2.6.15], which requires the class of weak equivalences in $\text{Mon}_{\mathbf{Cat}}$ to be perfect (follows from Proposition 2.3), the generating cofibrations to be h-cofibrations (holds by Proposition 5.3), and morphisms with the right lifting property with respect to all generating cofibrations to be weak equivalences (holds by adjunction between $\mathbf{Cat}$ and $\text{Mon}_{\mathbf{Cat}}$).
Definition 5.7. If $V$ is a monoidal category, then $\text{Cat}_V$ denotes the category of small $V$-enriched categories and $V$-enriched functors. In the special case of $V = \text{Cat}$ with the cartesian monoidal structure, we talk about small 2-categories. \(\text{ Used in 4.13, 4.14, 4.15} \)

Remark 5.8. The category $\text{Cat}_V$ is locally presentable whenever $V$ is a locally presentable closed monoidal category. For a detailed proof, see Kelly–Lack [2001, Theorem 4.5].

Definition 5.9. (Muro [2012,a]) Suppose $V$ is a model category with a monoidal structure and $F:C \to D$ is a $V$-enriched functor between $V$-enriched categories.

- The functor $F$ is essentially surjective if it becomes an essentially surjective functor in the usual sense after applying the functor $A \mapsto \{1, A\}$ to every hom-object, where $[-,-]$ denotes the hom-set in the homotopy category of $V$ and $1$ denotes the monoidal unit in $V$.
- The functor $F$ is a Dwyer–Kan equivalence if the functor $F$ is essentially surjective, and for every pair of objects $x, y \in C$ the induced morphism $C(x,y) \to D(Fx, Fy)$ is a weak equivalence in $V$.
- The functor $F$ is a Dwyer–Kan acyclic fibration if the functor $F$ is surjective on objects, and for every pair of objects $x, y \in C$ the induced morphism $C(x,y) \to D(Fx, Fy)$ is an acyclic fibration in $V$.

\(\text{ Used in 1.1, 5.10, 5.11} \)

Definition 5.10. (Muro [2012,a]) Suppose $V$ is a model category with a monoidal structure. The Dwyer–Kan model structure on $\text{Cat}_V$ (if it exists) is a model structure with weak equivalences and acyclic fibrations as in Definition 5.9.

Theorem 5.11. The category $\text{Cat}_{\text{Cat}_V}$ of small 2-categories admits a left proper combinatorial model structure (the enriched Thomason model structure) whose weak equivalences are Dwyer–Kan equivalences and acyclic fibrations are Dwyer–Kan acyclic fibrations as in Definition 5.10 with $V = \text{Cat}$ being equipped with the Thomason model structure. \(\text{ Used in 4.13, 4.14, 4.15} \)

Proof. Like in the proof of Theorem 5.6, we invoke Lurie [2017, Proposition A.2.6.15]. As before, the class of Dwyer–Kan weak equivalences is a perfect class by Proposition 2.5 and Muro [2012,a, Proposition 9.2]. Take

\[ I' = \{ 0 \to \{0\} \} \cup T_{0,1}(I) \]

as a set of generating cofibrations for $\text{Cat}_{\text{Cat}_V}$. Here $T_{0,1}:V \to \text{Cat}_V$ is a functor that sends an object $A \in V$ to the category $T_{0,1}(A)$ defined as follows. The set of objects of $T_{0,1}(A)$ is $\{0, 1\}$. The only nontrivial hom-object of $T_{0,1}(A)$ is $T_{0,1}(A)(0, 1) = A$. By Muro [2012,a, Corollary 4.8], morphisms with the right lifting property with respect to $I'$ are Dwyer–Kan equivalences. Thus, it remains to show that elements of $I'$ are $I$-cofibrations. The morphism $0 \to \{0\}$ is an $I$-cofibration because Dwyer–Kan equivalences are closed under disjoint unions.

To show that the elements of $T_{0,1}(I)$ are $I$-cofibrations, we use the same argument as in Proposition 5.5. Given $n \geq 0$ and a commuting diagram of pushout squares in the category $\text{Cat}_{\text{Cat}_V}$

\[
\begin{array}{cccc}
T_{0,1}(cSd^2 \partial \Delta^n) & \longrightarrow & X & \longrightarrow & Y \\
T_{0,1}(i) & \downarrow & \downarrow & & \\
T_{0,1}(cSd^2 \Delta^n) & \longrightarrow & P & \overset{s}{\longrightarrow} & Q,
\end{array}
\]

we claim that if the map $X \to Y$ is a Thomason weak equivalence, then so is the map $s:P \to Q$. That is to say, applying $T_{0,1} \circ c \circ Sd^2$ to $\partial \Delta^n \to \Delta^n$ yields an $I$-cofibration (Definition 2.6) in $\text{Cat}_{\text{Cat}_V}$. By Muro [2012,a, Proposition 9.1], homotopically essentially surjective functors are closed under cobase changes. Thus, it suffices to show that $P \to Q$ is homotopically fully faithful, i.e., for any objects $x, y \in P$, the induced map

\[ P(x,y) \to Q(s(x), s(y)) \]

is a Thomason weak equivalence. (Recall that $X \to P$ induces an identity map on the sets of objects.)

Given a morphism $f:K \to L$ in $\text{Cat}$, consider the following cocartesian square in $\text{Cat}_{\text{Cat}_V}$:

\[
\begin{array}{ccc}
T_{0,1}(K) & \longrightarrow & X \\
T_{0,1}(f) & \downarrow & \downarrow \\
T_{0,1}(L) & \longrightarrow & P.
\end{array}
\]
Here the top attaching map sends \(0 \mapsto a, 1 \mapsto b\) for some objects \(a, b \in X\). As shown by Muro [2012, §5], unfolding Proposition 5.4 yields a presentation of the morphism \(X(x, y) \to P(x, y)\) (for some \(x, y \in X\)) as the transfinite composition of cobase changes of morphisms (Muro [2012,a, (5.5)])

\[
X(b, y) \times X(b, a)^{\times(i-1)} \times X(x, a) \times f^\otimes_i.
\]

As in Proposition 5.5, we factor the map \(\iota = c \text{Sd}^2 \delta_n\) as

\[
A = c \text{Sd}^2 \partial \Delta^n \xrightarrow{\kappa} W \xrightarrow{\mu} c \text{Sd}^2 \Delta^n = B
\]

and apply the above filtration first to \(f = \kappa\) and then to \(f = \lambda\), where \(\lambda; W \cap V \to V\) is defined in Proposition 5.5. If we now compare the above map to the map \(X^{\times(i+1)} \times \mathbb{N}^i\) (respectively \(R^{\times(i+1)} \times \mathbb{N}^i\)) used in the single-object case in Proposition 5.5, we see that the object \(X^{\times(i+1)}\) (respectively \(R^{\times(i+1)}\)) is manipulated formally in the entire proof. The remainder of the proof now proceeds identically to Proposition 5.5, replacing \(X^{\times(i+1)}\) with

\[
X(b, y) \times X(b, a)^{\times(i-1)} \times X(x, a)
\]

everywhere in the first part of the proof and replacing \(R^{\times(i+1)}\) with

\[
R(b, y) \times R(b, a)^{\times(i-1)} \times R(x, a)
\]

everywhere in the second part of the proof.

**Remark 5.12.** Since Theorem 5.11 shows that \(\text{Cat}_{\text{cat}}\) is left proper, one can ask a similar question about the Bergner model structure on \(\text{Cat}_{\text{cat}}\). Indeed, Lurie [2017, Proposition A.3.2.4] (available in arXiv v1 from 2006) and Cisinski–Moerdijk [2011,b, Corollary 8.10] show that the Bergner model structure is left proper.

**Remark 5.13.** One might wonder whether it may be possible to extend Theorem 5.6 to construct a model structure on algebras over operads. In this case, the Schwede–Shipley filtration of Proposition 5.4 is replaced by the Elmendorf–Mandell filtration [2004.b, §12], which replaces the map

\[
f^\otimes_i \otimes X^{\otimes(i+1)}; f^\otimes_i \otimes X^{\otimes(i+1)} \to L^{\otimes_i} \otimes X^{\otimes(i+1)}
\]

by the map

\[
f^\otimes_i \otimes \text{Env}(O, X); f^\otimes_i \otimes \text{Env}(O, X)_i \to L^{\otimes_i} \otimes \text{Env}(O, X)_i,
\]

where \(\text{Env}(O, X)\) is the enveloping operad of the \(O\)-algebra \(X\), defined using a universal property, and \(\text{Env}(O, X)_i\) denotes its object of operations of arity \(i\). For symmetric operads, we must also mod out by the action of the symmetric group \(\Sigma_i\). In the case of monoids (i.e., \(O\) is the associative nonsymmetric operad), we have \(\text{Env}(O, X) = X^{\otimes(i+1)}\), which recovers the Schwede–Shipley filtration. The proof of Theorem 5.6 used the fact that if \(X \to Y\) is a weak equivalence, then so is \(\text{Env}(O, X) \to \text{Env}(O, Y)\). This is trivial if \(\text{Env}(O, X) = X^{\otimes(i+1)}\), but for an arbitrary operad \(O\) we have no control over \(\text{Env}(O, X)\) unless we impose very strong conditions on \(O\) such as cofibrancy. It seems plausible that arguments of this type could be used to construct an enriched Thomason model structure on nonsymmetric colored operads enriched in \(\text{Cat}\) along the lines of the model structure of Cisinski–Moerdijk [2011,b, §1] for symmetric simplicial operads. The symmetric case creates further problems, for example, by Hackney–Robertson–Yau [2014,a, §4], the model category of single-colored simplicial operads is not left proper, whereas our strategy relies on showing that cofibrations are h-cofibrations, which implies left properness. The counterexample uses operations of arity 0 in an essential way, which leaves open the question whether reduced colored symmetric operads enriched in \(\text{Cat}\) admit an enriched Thomason model structure.

**Theorem 5.14.** The Quillen equivalence of Proposition 3.9 induces a Quillen equivalence

\[
L \dashv R, \quad L: \text{Cat}_{\text{Set}} \to \text{Cat}_{\text{cat}}, \quad R: \text{Cat}_{\text{cat}} \to \text{Cat}_{\text{Set}};
\]

where the right adjoint functor \(R = \text{Cat}_{\text{Ex}}^2\) applies the functor \(\text{Ex}^2\) to every hom-category in a given 2-category. That is to say, the model category of small simplicial categories (Bergner 2004.a) and small
2-categories with the enriched Thomason model structure (Theorem 5.11) are Quillen equivalent. Restricting to categories with a single object yields a Quillen equivalence

\[ L \dashv R, \quad L: \text{Mon}_{\text{Set}} \to \text{Mon}_{\text{Cat}}, \quad R: \text{Mon}_{\text{Cat}} \to \text{Mon}_{\text{Set}}. \]

**Proof.** The functor \( R \) is well defined because the functor \( \text{Ex}^2 \mathcal{N} \) preserves finite products up to a natural isomorphism. The functor \( R \) preserves small limits and filtered colimits since these limits and colimits are computed on the level of underlying graphs by [Muro 2012: Proposition 2.6]. Thus, \( R \) is an accessible continuous functor between locally presentable categories, hence a right adjoint functor by the adjoint functor theorem.

By Proposition 3.1, the functor \( R \) preserves and reflects weak equivalences. The explicit description of generating cofibrations in \( \text{Cat}_{\text{Set}} \) (Bergner [2004d, Proposition 3.2]) as maps \( \emptyset \to \{0\} \) and \( T_{0,1}(\partial \Delta^n \to \Delta^n) \) immediately implies that \( L \) preserves cofibrations. The explicit description of generating acyclic cofibrations \( j: K \to K' \) in \( \text{Cat}_{\text{Set}} \) (Bergner [2004d, Propositions 2.3 and 2.5]) shows that their domains and codomains are cofibrant. In the commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{j} & K' \\
\downarrow & & \downarrow \\
RLK & \xrightarrow{RLj} & RLK'
\end{array}
\]

the vertical maps are unit maps for cofibrant objects \( K \) and \( K' \), which are weak equivalences by Proposition 5.15. Since the map \( j \) is a weak equivalence, by the 2-out-of-3 property so is \( R(L(j)) \). Since the functor \( R \) reflects weak equivalences, the map \( L(j) \) is a weak equivalence. Thus, the functor \( L \) preserves cofibrations and sends generating acyclic cofibrations to weak equivalences. Therefore, the adjunction \( L \dashv R \) is a Quillen adjunction.

To show that the adjunction \( L \dashv R \) is a Quillen equivalence, consider the derived unit map of a cofibrant object \( A \in \text{Cat}_{\text{Set}} \). Since the functor \( R \) preserves weak equivalences, the derived unit map of \( A \) can be computed as the unit map of \( A \), which is a weak equivalence by Proposition 5.15.

Consider the derived counit map

\[ \varepsilon_B: L(Q(R(B))) \to B \]

of a fibrant object \( B \in \text{Cat}_{\text{Cat}} \), where \( Q: \text{Cat}_{\text{Set}} \to \text{Cat}_{\text{Set}} \) is a cofibrant replacement functor. Since \( R \) reflects weak equivalences, to show that \( \varepsilon_B \) is a weak equivalence, it suffices to show that \( R(\varepsilon_B) \) is a weak equivalence. The composition of the unit map

\[ Q(R(B)) \to R(L(Q(R(B)))) \]

with the map

\[ R(\varepsilon_B): R(L(Q(R(B)))) \to R(B) \]

is the cofibrant replacement map \( Q(R(B)) \to R(B) \) of \( R(B) \), which is a weak equivalence. Since the unit map of the cofibrant object \( Q(R(B)) \) is a weak equivalence by Proposition 5.15, by the 2-out-of-3 property the map \( R(\varepsilon_B) \) is a weak equivalence, hence so is the map \( \varepsilon_B \).

Thus, the derived unit and derived counit of the Quillen adjunction \( L \dashv R \) are natural weak equivalences, hence \( L \dashv R \) is a Quillen equivalence.

The case of monoids is treated in the same way (an enriched category with one object can be identified with a monoid), using Proposition 5.15 and the set of generating cofibrations described in Theorem 5.6.

**Proposition 5.15.** The unit map \( X \to RLX \) of a cofibrant object \( X \in \text{Cat}_{\text{Set}} \) for the adjunction

\[ L \dashv R: \text{Cat}_{\text{Set}} \rightleftarrows \text{Cat}_{\text{Cat}} \]

(Theorem 5.14) is a weak equivalence in \( \text{Cat}_{\text{Set}} \). The same is true for unit maps in the adjunction \( L \dashv R: \text{Mon}_{\text{Set}} \rightleftarrows \text{Mon}_{\text{Cat}} \) (Definition 5.1).

**Proof.** We prove by induction on a cofibrant object \( X \in \text{Cat}_{\text{Set}} \) that the unit map \( X \to RLX \) of \( X \) is a weak equivalence. If \( X = \emptyset \), the unit map of \( X \) is the identity map \( \emptyset \to \emptyset \), which is a weak equivalence. Suppose...
the unit map $X \to RLX$ of $X$ is a weak equivalence and the map $X \to P$ is a base change of a generating cofibration $i$ in $\text{Cat}_{\text{Set}}$. If $i$ is the map $\emptyset \to \{0\}$, then $P = X \sqcup \{0\}$ and the unit map of $P$ is simply the coproduct of the unit map of $X$ and the identity map on $\{0\}$, hence is a weak equivalence.

If $i$ is the map $T_{0,1}(\delta_n)$, where $\delta_n : \partial \Delta^n \to \Delta^n$ is a generating cofibration of simplicial sets, then the base change square for $T_{0,1}(\delta_n)$ in the category $\text{Cat}_{\text{Set}}$ is mapped by the functor $L$ to the base change square for $T_{0,1}(cSd^2(\delta_n))$ in the category $\text{Cat}_{\text{Set}}$, since $L(T_{0,1}(\delta_n))$ can be computed as $T_{0,1}(cSd^2(\delta_n))$.

Consider the unit natural transformation of commutative squares

$$
\begin{array}{ccc}
T_{0,1}(\partial \Delta^n) & \longrightarrow & X \\
T_{0,1}(\delta_n) & \downarrow & \\
T_{0,1}(\Delta^n) & \longrightarrow & P
\end{array}
\quad
\begin{array}{ccc}
RT_{0,1}(cSd^2 \partial \Delta^n) & \longrightarrow & RLX \\
RT_{0,1}(cSd^2 \delta_n) & \downarrow & \\
RT_{0,1}(cSd^2 \Delta^n) & \longrightarrow & RLP
\end{array}
$$

The left square is homotopy cocartesian because $T_{0,1}(\delta_n)$ is a cofibration in $\text{Cat}_{\text{Set}}$, and the latter category is left proper. The components

$$
T_{0,1}(\partial \Delta^n) \to RT_{0,1}(cSd^2 \partial \Delta^n), \quad T_{0,1}(\Delta^n) \to RT_{0,1}(cSd^2 \Delta^n)
$$

are weak equivalences by Proposition 3.9 since the functors $R$ and $L$ commute past $T_{0,1}$ in the obvious way. The component $X \to RLX$ is a weak equivalence by assumption. Thus, the component $P \to RLP$ is a weak equivalence if and only if the right square is homotopy cocartesian.

Since the map $T_{0,1}(cSd^2 \delta_n)$ is an $\text{h}$-cofibration in $\text{Cat}_{\text{Set}}$ by Theorem 5.11, the base change square

$$
\begin{array}{ccc}
T_{0,1}(cSd^2 \partial \Delta^n) & \longrightarrow & LX \\
T_{0,1}(cSd^2 \delta_n) & \downarrow & \\
T_{0,1}(cSd^2 \Delta^n) & \longrightarrow & LP
\end{array}
$$

is homotopy cocartesian. Thus, we have to show that the functor $R$ sends this square to a homotopy cocartesian square. Since the map $RT_{0,1}(cSd^2 \delta_n)$ is a cofibration (hence an $\text{h}$-cofibration) in $\text{Cat}_{\text{Set}}$, we have to show that the canonical map

$$
\hat{P} = RT_{0,1}(cSd^2 \Delta^n) \sqcup RT_{0,1}(cSd^2 \partial \Delta^n) RLX \to R(T_{0,1}(cSd^2 \Delta^n) \sqcup T_{0,1}(cSd^2 \partial \Delta^n) LX) \cong RLP
$$

is a weak equivalence. The domain and codomain of this map, when considered as objects in the undercategory of $RLX$, admit filtrations on their hom-objects as in Theorem 5.11 using the base change squares for the maps $RT_{0,1}(cSd^2 \delta_n)$ and $T_{0,1}(cSd^2 \delta_n)$ given above, and applying the functor $\text{Ex}^2 N$ to the latter filtration.

Recall that the construction of the filtration in Theorem 5.11 starts by observing that the inclusion $\text{sd}^2 \partial \Delta^n \to \text{sd}^2 \Delta^n$ is the nerve of a map of posets $\kappa : A \to B$, which factors as

$$
cSd^2 \partial \Delta^n = A \xrightarrow{\kappa} W \longrightarrow^\mu B = cSd^2 \Delta^n,
$$

where $W$ is the cosieve generated by $A$ in $B$. We also have the induced commutative diagrams

$$
\begin{array}{ccc}
RT_{0,1}(A) & \longrightarrow & RLX \\
RT_{0,1}(\kappa) & \downarrow & \\
RT_{0,1}(W) & \longrightarrow & \hat{P}
\end{array}
\quad
\begin{array}{ccc}
T_{0,1}(A) & \longrightarrow & LX \\
T_{0,1}(\kappa) & \downarrow & \\
T_{0,1}(W) & \longrightarrow & \hat{R}
\end{array}
\quad
\begin{array}{ccc}
RT_{0,1}(A) & \longrightarrow & RLX \\
RT_{0,1}(\kappa) & \downarrow & \\
RT_{0,1}(W) & \longrightarrow & R\hat{R}
\end{array}
$$

where the two pairs of squares on the left are base change squares and the right diagram is given by applying the functor $R$ to the middle diagram.
By Bergner [2004, Proposition 3.2], the map \( R(T_{0,1}(\kappa)) \) is an acyclic fibration in \( \text{Cat}_{sSet} \), hence the map \( RLX \to \tilde{R} \) is a weak equivalence. By [Theorem 5.14], the map \( T_{0,1}(\kappa) \) is an acyclic fibration in \( \text{Cat}_{\text{Cat}} \), so the map \( LX \to \tilde{R} \) is a weak equivalence. Since the functor \( R \) preserves weak equivalences, the map \( RLX \to R\tilde{R} \) is a weak equivalence in \( \text{Cat}_{sSet} \). Thus, the top squares in all three diagrams are coface changes and homotopy coface change squares. The natural transformation from the first diagram to the third diagram has identities as its components for the left column, as well as the top right corner. Thus, the component \( \tilde{R} \to R\tilde{R} \) is also a weak equivalence.

Next, consider the bottom coface change squares in the above three diagrams. As explained in the proof of [Theorem 5.14], after taking hom-objects for a fixed pair of objects \( x \) and \( y \) of \( X \), we get filtrations on the bottom right maps \( \hat{R}(x, y) \to \hat{P}(x, y), \hat{R}(x, y) \to \hat{P}(x, y) \), and (after applying the functor \( \text{Ex}^2 N \)) the map \( \text{Ex}^2 N(\hat{R}(x, y)) \to \text{Ex}^2 N(\hat{P}(x, y)) \).

For the map \( \hat{R}(x, y) \to \hat{P}(x, y) = \hat{L}(x, y) \) the \( i \)th map \( \hat{P}_{i-1} \to \hat{P}_i \) in the filtration is a coface change in \( \text{Cat} \) of the map

\[
\beta = \hat{R}(b, y) \times \hat{R}(b, a) \times (\text{Ex}^2 N) \mu^{[i]}
\]

This coface change was shown to yield a homotopy coface square in [Theorem 5.14]. Since the functor \( \text{Ex}^2 N \) is a right Quillen equivalence that preserves weak equivalences, the image of the latter homotopy coface square under \( \text{Ex}^2 N \) is again homotopy cocartesian.

Likewise, for the map \( \hat{R}(x, y) \to \hat{P}(x, y) \) the \( i \)th map \( \bar{P}_{i-1} \to \bar{P}_i \) in the filtration is a coface change in \( sSet \) of the map

\[
\alpha = \hat{R}(b, y) \times \hat{R}(b, a) \times (\text{Ex}^2 N) \mu^{[i]},
\]

which is a homomorphism of simplicial sets, hence the corresponding commutative square is homotopy cocartesian.

Thus, we have a natural transformation of homotopy cocartesian squares

\[
\begin{array}{ccc}
\text{dom} \alpha & \longrightarrow & \hat{P}_{i-1} \\
\alpha \downarrow & & \downarrow \\
\text{codom} \alpha & \longrightarrow & \hat{P}_i \\
\end{array}
\quad
\begin{array}{ccc}
\text{Ex}^2 N \text{ dom} \beta & \longrightarrow & \text{Ex}^2 N \bar{P}_{i-1} \\
\text{Ex}^2 \text{ N} \beta \downarrow & & \downarrow \\
\text{Ex}^2 \text{ N} \text{ codom} \beta & \longrightarrow & \text{Ex}^2 N \bar{P}_i. \\
\end{array}
\]

The components \( \text{dom} \alpha \to \text{Ex}^2 N \text{ dom} \beta \) and \( \text{codom} \alpha \to \text{Ex}^2 N \text{ codom} \beta \) are weak equivalences in \( sSet \) because the functor \( \text{Ex}^2 N \) preserves finite products, the map \( \tilde{R} \to R\tilde{R} \) was shown to be a weak equivalence above, and the morphism of arrows

\[
(\text{Ex}^2 N) \mu^{[i]} \to \text{Ex}^2 N(\mu^{[i]})
\]

is a weak equivalence. For the latter, observe that the punctured \( i \)-dimensional cube whose colimit yields the domain of \( \mu^{[i]} \) is a homotopy cocartesian cube, and the functor \( \text{Ex}^2 N \) is a right Quillen equivalence that preserves weak equivalences, hence it preserves homotopy colimit diagrams. Thus, if the map \( \bar{P}_{i-1} \to \text{Ex}^2 N \bar{P}_{i-1} \) is a weak equivalence, then so is the map \( \bar{P}_i \to \text{Ex}^2 N \bar{P}_i \).

The map \( \bar{P}_0 \to \text{Ex}^2 N \bar{P}_0 \) coincides with the map \( \hat{R}(x, y) \to \text{Ex}^2 N(\hat{R}(x, y)) \), which was shown to be a weak equivalence above. By induction, the map \( \bar{P}_i \to \text{Ex}^2 N \bar{P}_i \) is a weak equivalence for all \( i \geq 0 \). Since simplicial weak equivalences are closed under filtered colimits, the map \( \bar{P}(x, y) \to \text{Ex}^2 N(\bar{P}(x, y)) \) is also a weak equivalence, which completes the proof. \( \blacksquare \)

**Remark 5.16.** The Quillen equivalence of [Theorem 5.14] immediately connects (via zigzags of Quillen equivalences) the **enriched Thomason model structure** on small 2-categories to all the other models for \((\infty, 1)\)-categories such as quasicategories, relative categories (Barwick–Kan [2010,b]), Segal categories, complete Segal spaces, marked simplicial sets, etc. See Bergner [2006, 2018.a] for a review of these Quillen equivalences.
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