Quantum Hall plateau transition in the lowest Landau level of disordered graphene

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We investigate, analytically and numerically, the effects of disorder on the density of states and on the localization properties of the relativistic two dimensional fermions in the lowest Landau level. Employing a supersymmetric technique, we calculate the exact density of states for the Cauchy (Lorentzian) distribution for various types of disorders. We use a numerical technique to establish the localization-delocalization (LD) transition in the lowest Landau level. For some types of disorder the LD transition is shown to belong to a different universality class, as compared to the corresponding nonrelativistic problem. The results are relevant to the integer quantum Hall plateau transitions observed in graphene.

I. INTRODUCTION

Recent experiments have unravelled a fascinating set of phenomena in atomically thin layer of hexagonally arranged carbon atoms known as graphene. The quasiparticles of graphene are $(2+1)$-dimensional massless Weyl fermions. In the context of condensed matter physics their properties are strikingly different from non-relativistic fermions. And phenomena that are hard to realize for the relativistic case, such as the Klein paradox or the Zitterbewegung are accessible in graphene. It is perhaps not an exaggeration to remark that many subtleties and a rich set of phenomenology are waiting to be discovered.

A. Quantum Hall Effect

A highlight has been the observation of an unconventional quantum Hall effect and the corresponding theoretical development. In graphene the filling fractions are $\nu_f = \pm (n + \frac{1}{2})$ for magnetic field $B < 9 T$, where $n$ is an integer. The factor of 4 comes from the two fold spin degeneracy and the two fold nodal degeneracy of the Landau levels. The Zeeman splitting is negligible compared to the cyclotron frequency and the disorder broadening of the Landau levels. The factor of half is due to a zero mode in the Landau level spectrum of Dirac fermions.

For stronger magnetic fields, $20 T < B < 45 T$, plateaus appear at $\nu_f = 0, \pm 1, \pm 2q$, where $q$ is an integer. The plateaus at $\nu_f = 0, \pm 1$ can be explained by the lifting of both the spin and the nodal degeneracies in the lowest Landau level (LLL), but those at $\nu_f = \pm 4, \pm 6, \ldots$ reflect only the removal of spin degeneracy in higher Landau levels. The removal of nodal degeneracy requires electron-electron interaction. Mechanisms suggested include $SU(4)$ ferromagnetism, sublattice symmetry breaking due to short range interactions, and the generation of a mass gap by magnetic catalysis. $SU(4)$ quantum hall ferromagnetism predicts plateaus at all odd integer filling fractions. However, apart from $\nu_f = \pm 1$, the plateaus at $\nu_f = \pm 3, \pm 5, \ldots$ have not yet been observed.

B. Localization-delocalization transition

The special quantization rules in graphene are explained by the relativistic Landau levels, modified perhaps by interactions, but for the existence of Hall plateaus the Laughlin argument is necessary. According to this argument the extended states at the center of a Landau band are separated by the localized states elsewhere. If the Fermi energy falls in the mobility gap, the plateaus are explained by a gauge invariance argument that is remarkably robust. The underlying phenomenon, therefore, is a localization-delocalization (LD) transition at the band center. The conventional integer quantum Hall (IQH) plateau transition has been widely studied, and it is known that the localization length exponent $\nu \approx 7$. Can we prove that the same argument applies to graphene, and, if so, does the LD transition belong to the same universality class?

C. Disorder and Dirac fermions

In the absence of a magnetic field, Dirac fermions in the presence of disorder have been widely studied in systems as varied as gapless semiconductors, gapless superconductors and IQH plateau transitions. As compared to nonrelativistic fermions, the localization problem of Dirac fermions is richer because of a number of discrete symmetries. More specifically, if the disorder is particle-hole symmetric, for example a random gauge field, the LD transition takes place at zero energy and is reflected in the single particle density of states (DOS), in contrast to the conventional metal-insulator transition where the DOS is smooth through the LD transition. Surprisingly, there is a line of fixed points with continuously varying exponents depending on the disorder coupling constant. Some of the
unusual behavior of disordered Dirac fermions may be expected to realize in graphene. One such effect that has received considerable attention is the weak (anti)localization phenomenon.\cite{36,37,38,39,40} However, relativistic Landau levels in the presence of disorder have not yet received much attention.\cite{11,41,42,43} Here we provide a reasonably complete study of the possible effects.

There is another important reason why LD transitions in the relativistic Landau level should be carefully analyzed. In the conventional IQH effect, the spin-degenerate plateau transition corresponds to $\nu \sim 4.6$ when it is assumed that the LD transition takes place at a single energy at the band center.\cite{44,45,46,47} This has led to extensive theoretical investigation of the LD transition in the spin-degenerate Landau band.\cite{48,49,50,51,52} When spin-orbit scattering is included, the LD transition is found to occur at two distinct energies, away from the band center. Scaling analysis about these distinct energies provide, once again, that $\nu \approx 7/3$, as in the spin-polarized system. The scaling about a single energy at the band center leads to the effective exponent $\nu \sim 4.6$. One should anticipate a similar discrepancy between the spin and the nodal polarized IQH effect and the fourfold degenerate IQH effect in graphene.

D. Graphene in the lowest Landau level

For simplicity we shall concentrate on the spin polarized lowest Landau level (LLL) of graphene and analyze the LD transition both in the presence and in the absence of nodal degeneracy. An interesting example of a controlled analytic calculation in the disordered Landau level problem is the DOS in the LLL. This was first computed exactly by Wegner\cite{53} by examining the Euler trails of the impurity diagrams for the white noise disorder and was subsequently extended by Brezin et al.\cite{54} by using a supersymmetric (SUSY) technique. Here we also obtain some exact results for the DOS in the disordered relativistic LLL using SUSY techniques.

The most general model of disorder consists of a random potential, a random mass, a random gauge field and a random internode scattering; however, the random gauge field leaves the LLL unperturbed. After projection to the spin-polarized LLL, we study the following Hamiltonian:

$$\hat{H}_{\text{LLL}} = m\eta_3 + \sum_{j=0}^{3} V_j(\vec{r})\eta_j, \quad (1)$$

where $V_0(\vec{r})$, $V_3(\vec{r})$ represent potential and mass disorders respectively and $V_1(\vec{r})$ and $V_2(\vec{r})$ describe internode scattering effects. A mass $m$ of the fermions have been included to study the effect of the removal of the nodal degeneracy. For simplicity we have omitted the constant Zeeman energy. The $2 \times 2$ matrix $\eta_0$ is the identity matrix and $\eta_1$, $\eta_2$ and $\eta_3$ are the three Pauli matrices.

E. Summary of results

Because of a large number of cases involved, it is useful to summarize the results for the LD transition. Let $g_0$, $g_3$, $g_1$ and $g_2$ denote the widths of the Gaussian random distributions corresponding to the random potential, random mass, and random internode scatterings, respectively.

1. $m = 0$

The list of possible cases are:

1. $g_0 \neq 0$ and $g_3 = g_1 = g_2 = 0$.
2. $g_3 \neq 0$, $g_0 = g_1 = g_2 = 0$.
3. $g_2 \neq 0$, $g_0 = g_3 = g_1 = 0$.
4. $g_1 \neq 0$, $g_0 = g_3 = g_2 = 0$.
5. $g_0 \neq 0$, $g_3 \neq 0$ and $g_1 = g_2 = 0$.
6. $g_0 \neq 0$, $g_2 \neq 0$ and $g_3 = g_1 = 0$.
7. $g_0 \neq 0$, $g_1 \neq 0$ and $g_3 = g_2 = 0$.
8. $g_3 \neq 0$, $g_2 \neq 0$, $g_0 = g_1 = 0$.
9. $g_3 \neq 0$, $g_1 \neq 0$, $g_0 = g_2 = 0$.
10. $g_2 \neq 0$, $g_1 \neq 0$ and $g_0 = g_3 = 0$.
11. $g_0 \neq 0$, $g_3 \neq 0$, $g_2 \neq 0$ and $g_1 = 0$.
12. $g_0 \neq 0$, $g_3 \neq 0$, $g_1 \neq 0$ and $g_2 = 0$.
13. $g_0 \neq 0$, $g_2 \neq 0$, $g_1 \neq 0$ and $g_3 = 0$.
14. $g_3 \neq 0$, $g_2 \neq 0$, $g_1 \neq 0$ and $g_0 = 0$.
15. $g_0 \neq 0$, $g_3 \neq 0$, $g_2 \neq 0$ and $g_1 \neq 0$.

In the cases (1) and (2), when disorder does not mix the two nodes, the LD transitions belong to the conventional IQH universality class with $\nu \approx 7/3$. It is interesting to note that mass disorder produces LD transition in the LLL, whereas for zero magnetic field random mass is known to be an irrelevant perturbation for the $(2+1)$-dimensional Dirac fermions.\cite{55} The Hamiltonians for (2), (3) and (4) involve only a single Pauli matrix at a time, related to each other by unitary transformations. Thus, (2), (3) and (4) are equivalent to each other and have $\nu \approx 7/3$. Because unitary transformations leave the identity matrix invariant, the same argument implies that (5), (6) and (7) are equivalent to each other and once again $\nu \approx 7/3$.

The cases (8), (9) and (10) involve a pair of Pauli matrices and are equivalent to each other. In (8) the Hamiltonian has a discrete symmetry $\eta_1 \hat{H} \eta_1 = -\hat{H}$, often called a particle-hole symmetry. The cases (9) and (10) have the same discrete symmetry with respect to
The case (10) has been analyzed by Hikami et al. for a spin degenerate nonrelativistic LLL. When \( g_1 = g_2 \), DOS diverges at the band center and has two symmetrically located peaks away from it. The LD transition takes place at these three distinct energies. Away from the band center the LD transition has the exponent \( \nu \sim 2.98 \) and the transition at the band center corresponds to a different exponent. If \( g_1 \neq g_2 \), the divergence of the DOS at the band center disappears, but the two symmetrically placed peaks away from the band center still exist. We find that the LD transition at these two energies have continuously varying exponents depending on the ratio \( g_2/g_1 \).

The cases (11), (12) and (13) are equivalent. The Hamiltonians in these cases are respectively the Hamiltonians for the cases (8), (9) and (10), augmented by the identity matrix corresponding to the potential disorder. Potential disorder breaks the discrete symmetry mentioned above, and there is no divergence of the DOS at the band center. The DOS is still peaked at two symmetrically placed energies away from the band center. The LD transitions occur at energies away from the band center. If \( g_0 \) is much smaller than the two remaining coupling constants, \( \nu \) follows trends similar to (8), (9), and (10). If \( g_0 \) is comparable or larger, we find \( \nu \sim 7/3 \).

In (14) all three Pauli matrices are present. The discrete symmetry of (8), (9) and (10) are absent, and the LD transitions take place at two symmetrically placed energies away from the band center. When all the coupling constants are equal, the exponent \( \nu \sim 3.6 \). Depending on the relative strengths of the coupling constants the exponents vary continuously. If any particular coupling constant is significantly larger than the rest, \( \nu \sim 7/3 \). By adding \( g_0 \) we obtain (15). If \( g_0 \) is smaller than the rest, the situation is similar to (14). If \( g_0 \) is larger than the rest, \( \nu \sim 7/3 \).

2. \( m \neq 0 \)

When \( m \neq 0 \), the LD transitions occur at two symmetrically placed energies about the band center, and these energies are greater than or equal to \( m \). In the absence of internode scattering, the transitions occur at \( E = \pm m \) and the exponent \( \nu \sim 7/3 \). If the strength of the intranode scattering is larger than \( m \), the bands at \( \pm m \) overlap and effectively correspond to the nodally degenerate case.

If \( g_0 = g_3 = 0 \) and only one of the internode couplings is present, the DOS diverges at \( E = \pm m \) with an exponent of 0.5 and is identically zero for \( |E| < m \). The LD transitions occur at \( E = \pm m \) and have a continuously varying exponent. When the disorder is strong compared to \( m \), \( \nu \sim 7/3 \), and, in the opposite limit, \( \nu \) approaches unity. If we include small intranode scattering the situation is similar. If the intranode scattering strength is greater than the internode scattering, \( \nu \sim 7/3 \).

When \( g_0 = g_3 = 0 \) and both internode couplings are present, the DOS diverges at \( E = \pm m \) with an exponent \( \nu \sim 0.47 \). However, the LD transitions occur at energies larger than \(|m| \). We have analyzed a case where \( g_1 = g_2 \). The exponent varies continuously. If the internode scattering strength is larger than \( m \), \( \nu \sim 3.8 \), and in the opposite limit \( \nu \) approaches unity. This behavior is stable against intranode scattering if its strength is smaller than both \( m \) and the internode scattering. If intranode scattering strength is larger than the internode scattering, \( \nu \sim 7/3 \).

**F. Roadmap**

Our paper is organized as follows: In Sec. II we describe the Dirac fermion model. In Sec. III we describe various possible disorders and their forms when projected to the LLL. In Sec IV we calculate the averaged density of states using supersymmetry. In the sections V, VI, and VII we describe the numerical studies of the LD transition projected to the lowest Landau level. Section VIII is a brief concluding section. In the Appendix A we provide some mathematical details of the density of states calculation. In Appendix B we describe the recursive Green function technique used for numerical calculations and finally in Appendix C we outline the procedure of data collapse involved in the finite size scaling of the localization length.

**II. DIRAC FERMIONS AND LANDAU LEVELS OF GRAPHENE**

The low energy quasiparticles in graphene are well described by the Lorentz invariant form as the sum over two inequivalent nodes (the Fermi velocity \( v_F \approx 10^6 \text{m/s} \))

\[
H_0 = -i\hbar v_F \int d^2r \bar{\Psi}_\sigma (\gamma^1 D_x + \gamma^2 D_y) \Psi_\sigma, \tag{2}
\]

where \( \Psi_\sigma = \Psi_\uparrow^\dagger \gamma^0 \) and the summation over spin \( \sigma = \pm 1 \) is understood. The four component Dirac spinor \( \Psi_\sigma = (\psi_{K,A}, i\psi_{K,B}, i\psi_{K',B}, -i\psi_{K',A}) \), where the component \( \psi_{K,A} \) is constructed by superposing Bloch functions close to one of the two inequivalent nodes \((K, K')\) of the Brillouin zone, corresponding to one of the two sublattices \((A, B)\) of the hexagonal graphene lattice. The notation \( D = (\partial - i\xi A) \) stands for the covariant derivative, \( A \) being the vector potential. The \( \gamma \)-matrices are defined by \( \gamma^0 = (\tau_3, i\tau_1, i\tau_2) \otimes \eta_3 \); the Pauli matrix \( \tau \) operates on the two components corresponding to the sublattice indices, and the Pauli matrix \( \eta \) operates on the components corresponding to the nodal indices. To be explicit:

\[
\gamma^0 = \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\tau_1 & 0 \\ 0 & -i\tau_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\tau_2 & 0 \\ 0 & -i\tau_2 \end{pmatrix}. \tag{3}
\]

To include a Zeeman term, we add

\[
H_z = E_z \int d^2r \bar{\Psi}_\sigma \gamma^0 \sigma_3^{\sigma\sigma'} \Psi_{\sigma'}, \tag{4}
\]
where $E_z = g\mu_B B$ is the Zeeman energy and $\sigma_j$ is a Pauli matrix operating on the spin indices. The Zeeman term breaks the $SU(2)$ symmetry of the spin space down to $U(1)$. The energy eigenvalues of the Hamiltonian operator

$$\hat{H}_0 = -i\hbar v_F \gamma^0 (\gamma^1 D_x + \gamma^2 D_y) + E_z \sigma_3,$$

are well known:

$$E_{n\sigma} = s \sqrt{2n|eB|\hbar^2 /c - \sigma E_z}, n = 0, 1, 2, ..., $$

where $s = \pm 1$ refer to the particle and the hole branches. In the presence of disorder Landau levels get broadened into a band, and the amount of broadening depends on the strength of the disorder. When the disorder is very strong, the half-width of the broadened band can be larger than $E_z$, and experimentally this corresponds to the spin degeneracy of the Landau bands. In the spin degenerate situation, the observed filling factors is given by $\nu = 4(n + \frac{1}{2})^{10}$.

The LLL wave function in the absence of the disorder in the symmetric gauge $A = (-By/2, Bz/2, 0)$ can be written as

$$U(z, \bar{z}) = e^{-z\bar{z}/4\hbar^2} \begin{pmatrix} f_1(z) & 0 \\ 0 & f_2(z) \end{pmatrix},$$

where $eB > 0$. The functions $f_1(z)$ and $f_2(z)$ are holomorphic functions of the complex coordinates $z = x + iy$; $\bar{z} = x - iy$, and $l_B = \sqrt{c/eB}$ is the magnetic length. Hence, in the zero mode, the first and the second node have nonzero amplitudes coming only from the sublattices A and B respectively.

Two distinct onsite energies on the two sublattices correspond to a charge density modulation at the lattice scale. As a result, the particle and the hole branches acquire an energy gap. When linearized about the inequivalent nodes, this energy gap appears as a parity preserving mass of the Dirac fermions. To be explicit, the linearized hamiltonian will have two new terms: the chemical potential term $[(V_A + V_B)/2] \Psi \gamma^0 \Psi$ and the mass term $[(V_A - V_B)/2] \Psi \gamma^0 \Psi$, where $V_A$ and $V_B$ are the site energies at the sublattices A and B.

Although the non-interacting quasiparticles are massless in the absence of site modulation, they can acquire a parity conserving mass due to interaction effects. This spontaneous symmetry breaking is facilitated by the presence of the magnetic field, a phenomenon known as “magnetic catalysis” of chiral symmetry breaking.\textsuperscript{26,27} The effect has been argued to be the reason behind the quantum hall plateaus at $\nu = 0, \pm 1$ observed in strong magnetic fields.\textsuperscript{17,18} Though it is beyond the scope of the present paper to consider electronic interactions, we will pay some attention to the noninteracting problem with a finite mass. Our philosophy is to analyze the consequences of having a mass (possible in an interacting theory) on the LD transition. So, we shall include the term $m \Psi \gamma^0 \Psi$ in the effective Hamiltonian to examine the effect of mass. In the presence of such a mass term, the nodal degeneracy of $E_{0,s,\sigma} = -\sigma E_z$ is removed and it splits into four levels $E_{0,1,\sigma} = m - \sigma E_z$ and $E_{0,-1,\sigma} = -m - \sigma E_z$. Each of these levels has the degeneracy $|eB|/2\pi$. If the applied chemical potential is smaller than $|E_z - m|$, there will be a plateau at $\nu = 0$. If $|E_z - m| < |\nu| < E_z + m$, $\nu = \pm 1$ plateaus will appear depending on the sign of $\mu$. Next possible values of quantized plateaus are $\nu = \pm 2$. The introduction of the mass term does not, however, lift the nodal degeneracy of the higher Landau levels, and the energy levels $E_{n\geq 1,\sigma} = s \sqrt{m^2 + 2n|eB|\hbar^2 / c - \sigma E_z}$ has the degeneracy $|eB|/2\pi$. Therefore, when a mass is included quantized plateaus appear at $\nu = 0, \pm 1, \pm 2q$, where $q$ is an integer.

### III. RANDOMNESS

There are many sources of disorder in graphene: vacancies, interstitials, substrate disorder and lattice distortions due to dislocations. In principle there could also be random spin-orbit coupling. However, due to the small atomic mass of carbon, spin-orbit coupling is very weak compared to other energy scales. For simplicity, we shall primarily be interested in the spin polarized limit and ignore the random spin-orbit coupling.

Point defects and substrate disorder can be described by introducing random site energies in the tight binding model. In the presence of substrate disorder there can also be a random modulation of the charge densities between the two sublattices. These effects can be described by a random chemical potential $V_0(r)\Psi \gamma^0 \Psi$ and a random mass $V_3(r)\Psi \gamma^5 \Psi$ in the continuum limit.

Because true long range crystalline order is not possible in two dimensions at any finite temperature, topological defects, dislocations and disclinations will be present. Effects of these topological defects will result in random hopping amplitudes $\delta_{AB}$ and hence intranode as well as internode scattering. However, these scattering processes will take place between states on different sublattices. The following two bilinears, $V_2(r)\Psi \gamma^3 \Psi$ and $V_1(r)\Psi \gamma^5 \Psi$, describe the internode scattering terms arising from random hopping. The two mutually anticommuting matrices,

$$\gamma^3 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

also anticommute with $\gamma^\mu$; $I$ is the identity matrix.

In the continuum limit, the most general impurity Hamiltonian is a $4 \times 4$ matrix:

$$H^{imp} = \int d^2 r \Psi^\dagger \sigma_i \begin{pmatrix} D_{11}(r) & D_{12}(r) \\ D_{21}(r) & D_{22}(r) \end{pmatrix} \Psi, \quad (9)$$

where $D_{ij}(r)$ are $2 \times 2$ matrices. Here $D_{11} = D_{11}^\dagger$ and...
\[ \hat{H}_{\text{LLL}}^\text{imp} = \sum_{j=0}^{3} V_j(r) \eta_j, \quad (10) \]

where we have denoted the \( I \) matrix by \( \eta_0 \).

### IV. AVERAGE DENSITY OF STATES

Using a four component bosonic spinor \( \phi \) and a four component Grassmann spinor \( \chi \) the average retarded Green function for a noninteracting problem can be written as

\[ G_R^R(E; r, r') = -i \prod_{j=0}^{3} \mathcal{D}[\phi^*] \mathcal{D}[\phi] \mathcal{D}[\chi^*] \mathcal{D}[\chi] \mathcal{D}[V_j] P[V_j] \phi^*(r) \phi(r') e^{S_R}, \quad (11) \]

where \( P[V_j] \) is the probability distribution of \( V_j \) and

\[ S_R = i \int d^2r \left[ \phi^+(E - \hat{H}_0 - \hat{H}_\text{imp} + i\delta) \phi + \chi^+(E - \hat{H}_0 - \hat{H}_\text{imp} + i\delta) \chi \right]. \quad (12) \]

The average density of states is given by

\[ \bar{\rho}(E) = -\frac{1}{\pi} \text{Im} \ G_R^R(E; r, r). \quad (13) \]

After performing the disorder averages we can write

\[ G_R^R(E; r, r') = -i \int \mathcal{D}[\phi^*] \mathcal{D}[\phi] \mathcal{D}[\chi^*] \mathcal{D}[\chi] \phi^*(r) \phi(r') e^{A^R}, \quad (14) \]

where the action \( A^R \) involves interactions among the fields generated by the disorder averaging procedure. After projection to the LLL, the action \( A^R \) can be expressed in terms of a two-component holomorphic bosonic spinor

\[ \phi(z) = \begin{pmatrix} v_1(z) \\ v_2(z) \end{pmatrix}, \quad (15) \]

and a two component holomorphic Grassmann spinor

\[ \chi(z) = \begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix}. \quad (16) \]

In terms of these fields the action is given by

\[ A^R = A_j^R + \sum_{j=0}^{3} A_j^D, \]

\[ A_j^R = \epsilon \int d^2ze^{-z/z_{\eta j}}(\phi^+ \phi + \chi^+ \chi), \]

\[ A_j^D = \int d^2z h_j \left[ e^{-z/z_{\eta j}}(\phi^+ \eta_j \phi + \chi^+ \eta_j \chi) \right]. \quad (17) \]

where \( \epsilon = E + i\delta \) and

\[ h_j(\kappa) = \ln \left( \int e^{-i\kappa V_j} P[V_j] D\phi \right), \quad (18) \]

is the effective interaction of the fields generated by the averaging over the random variable \( V_j \). For the Cauchy distribution, defined by

\[ P[V_j(\vec{r})] = \frac{g_j}{\pi} \frac{1}{g_j^2 + V_j^2(\vec{r})}, \quad (19) \]

we have

\[ h_j(\kappa) = -g_j |\kappa|. \quad (20) \]

If the disorder distribution is Gaussian white noise, defined by

\[ P[V_j(\vec{r})] = N \exp \left[ -\frac{1}{2g_j^2} \int d^2r V_j^2(\vec{r}) \right], \quad (21) \]

we get

\[ h_j(\kappa) = -\frac{1}{2} g_j \kappa^2. \quad (22) \]

The above action is invariant under the translation followed by a gauge transformation. Due to this invariance, the spatial dependence of the average retarded Green function is same as the spatial dependence of the pure system’s Green function

\[ G_{\text{pure}}^R(E, z_1, z_2) = \frac{\exp[-(|z_1|^2 + |z_2|^2 - 2z_1 \bar{z}_2)]}{2\pi_l^2 \epsilon(E + i\delta)}. \quad (23) \]

The disorder averaged Green function can be written as

\[ \mathcal{G}_R^R(E, z_1, z_2) = C(E + i\delta, g_j) \exp[-(|z_1|^2 + |z_2|^2 - 2z_1 \bar{z}_2)], \quad (24) \]

where \( g_j \)’s are coupling constants of various types of disorder and \( C(E + i\delta, g_j) \) is a gauge invariant proportionality constant which depends on the energy and disorder strengths. This gauge invariant proportionality constant is what we need to calculate to find the average density of states.

For the calculation of the average Green function’s dependence on the energy and disorder coupling constants we introduce two new Grassmann variables \( \theta \) and \( \bar{\theta} \) and enlarge the Euclidean coordinate space into a superspace of coordinates \((x, y, \theta, \bar{\theta})\). Integrals over the Grassmann
coordinates are normalized as $\pi \int d\phi d\bar{\phi} = 1$. The norm of a coordinate vector is defined as $x^2 + y^2 + \theta \bar{\theta}$. This norm is invariant under the superspace rotations. In addition to the ordinary rotations in the Euclidean subspace and the symplectic transformations in the Grassmann subspace, the superspace rotations involve transformations which mix $(x, y)$ and $(\theta, \bar{\theta})$ in the following manner:

\[
\begin{align*}
\vec{r} &\rightarrow \vec{r} + 2\vec{t}_1 \Omega \theta + 2\vec{t}_2 \Omega \bar{\theta} \\
\theta &\rightarrow \theta + 4(\vec{t}_2 \cdot \vec{r}) \Omega \\
\bar{\theta} &\rightarrow \bar{\theta} - 4(\vec{t}_1 \cdot \vec{r}) \Omega.
\end{align*}
\] (25)

In the above set of transformations $\vec{t}_{1,2}$ are two arbitrary Euclidean vectors and $\Omega$ is a Grassmann number. We also define two holomorphic superfields and their conjugates as

\[
\begin{align*}
\Phi(z, \theta) &= \phi(z) + \frac{\theta}{\sqrt{2} B} \chi(z), \\
\bar{\Phi}(z, \theta) &= \phi^\dagger(z) + \frac{\chi^\dagger(z)}{\sqrt{2} B} \theta.
\end{align*}
\] (26)

In terms of these superfields the pure part of the action can be expressed as

\[
A^R_j = 2\pi i \Gamma^2 R \int d^2 z d\theta d\bar{\theta} e^{-(z\bar{z} + \theta\bar{\theta})/2\Gamma^2} \Phi \bar{\Phi},
\] (27)

which is manifestly invariant under superspace rotations. After the disorder contributions to the action are expressed in terms of these new superfields, we have to demonstrate these to be invariant under superspace rotations. In order to be supersymmetric $A^R_j$’s have to be local in the supercoordinate space and this is only possible if they do not involve any quartic fermionic interactions. We note that

\[
\begin{align*}
\hbar_j \left[ e^{-z\bar{z} + \theta\bar{\theta}} (\phi^\dagger \eta_j \phi + \chi^\dagger \eta_j \chi) \right] \\
= \hbar_j \left[ e^{-z\bar{z} + \theta\bar{\theta}} \phi^\dagger \eta_j \phi \right] \\
+ \hbar_j' \left[ e^{-z\bar{z} + \theta\bar{\theta}} \phi^\dagger \eta_j \phi \right] e^{-z\bar{z} + \theta\bar{\theta}} \chi^\dagger \eta_j \chi \\
+ \frac{1}{2} \hbar'' j \left[ e^{-z\bar{z} + \theta\bar{\theta}} \phi^\dagger \eta_j \phi \right] e^{-z\bar{z} + \theta\bar{\theta}} (\chi^\dagger \eta_j \chi)^2,
\end{align*}
\] (28)

where $\hbar_j'$ and $\hbar'' j$ correspond to the first and second derivatives of $h_j$ with respect to its argument. The Taylor series truncates at the quadratic order as the higher powers of $\chi^\dagger \eta_j \chi$ are identically zero according to the anticommutation rules. We also note that $(\chi^\dagger \eta_j \chi)^2 = -2w_1 w_2 \eta_j \chi$ for $j = 1, 2, 3$ and $(\chi^\dagger \eta_0 \chi)^2 = 2w_1 w_2 w_3 \eta_j \chi$. If $\hbar_j''$ does not vanish we get four-fermion interactions.

If there were one bosonic and one Grassmann fields instead of spinors, as in the problem solved by Brézin et al.\textsuperscript{54} no four fermionic terms would be generated, and the action for an arbitrary disorder distribution would be local in the superspace coordinates. For the case under consideration, such a simplification is not possible in general. However, for Cauchy distribution the disorder averaged action is quadratic and can be made manifestly supersymmetric. Thus, the calculation of the DOS reduces to a calculation of a zero dimensional field theory over two complex bosonic fields.

\[\textbf{A. Cauchy Distribution}\]

\[1. m = 0\]

The action is given by

\[
A^R = \int d^2 x d\theta d\bar{\theta} e^{-(z\bar{z} + \theta\bar{\theta})/2\Gamma^2} \left[ i\epsilon (\phi^\dagger \phi + \chi^\dagger \chi) - \sum_{j=0}^{3} g_j |\phi^\dagger \eta_j \phi + \chi^\dagger \eta_j \chi| \right].
\] (29)

Using the superfields $\Phi$ and $\bar{\Phi}$ the action can be written as,

\[
A^R = 2\pi l_B^2 \int d^2 z d\theta d\bar{\theta} e^{-(z\bar{z} + \theta\bar{\theta})/2\Gamma^2} \left[ i\epsilon \Phi \overline{\Phi} - \sum_{j=0}^{3} g_j |\Phi \eta_j \Phi| \right],
\] (30)

which is manifestly invariant under rotation and magnetic translation in superspace. Because of this symmetry, the DOS can be reduced to a simple expression involving integrals over two ordinary complex variables. Expressed in terms of two radial and two angular variables, it is

\[
\overline{p}(E) = \frac{1}{2\pi l_B^2} \frac{1}{\sqrt{2\pi}} \ln \left\{ \int_0^\infty d(r_1^2/2) \int_0^\infty d(r_2^2/2) \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \exp \left[ iE(r_1^2 + r_2^2) - g_0 |r_1^2 + r_2^2| - g_3 |r_1^2 - r_2^2| - 2g_1 r_1 r_2 |\cos(\alpha_1 - \alpha_2)| - 2g_2 r_1 r_2 |\sin(\alpha_1 - \alpha_2)| \right] \right\}.
\] (31)
For simplicity, consider the cases where we keep only one of the internode scattering, or the random mass term, along with the potential disorder. We get:

(i) \( g_1 = g_2 = 0 \)

\[
\bar{\rho}(E) = \frac{1}{2\pi^2 I_B^2} \left[ \frac{g_0}{g_0^2 + E^2} + \frac{g_0 + g_3}{(g_0 + g_3)^2 + E^2} \right],
\]

(ii) \( g_3 = g_2 = 0 \),

\[
\bar{\rho}(E) = \frac{1}{2\pi^2 I_B^2} \left[ \frac{g_0}{g_0^2 + E^2} + \frac{g_0 + g_1}{(g_0 + g_1)^2 + E^2} \right].
\]

The answer for the case (iii) \( g_3 = g_1 = 0 \) is identical to the case (ii). The DOS obtained for these three cases are identical, as the Hamiltonian involves only one Pauli matrix at a time, and these matrices are related by unitary transformations.

Consider now \( g_1 = g_2 = g_{IN} \) and \( g_3 = 0 \). We obtain, defining by \( I \) the expression within the curly parenthesis in Eq. \( 31 \)

\[
I = \frac{\pi}{(a_1^2 - 2g_{IN}^2)} \left[ \pi - 4 \frac{g_{IN}}{\sqrt{a_1^2 - g_{IN}^2}} \tan^{-1} \left( \frac{g_{IN}}{\sqrt{a_1^2 - g_{IN}^2}} \right) - 2\sqrt{2} \frac{g_{IN}}{a_1} + 2\pi \frac{g_{IN}}{\sqrt{a_1^2 - g_{IN}^2}} \right],
\]

where \( a = g_0 - i\epsilon \). The details of the evaluation of the multiple integrals are provided in the Appendix A. The expression for the DOS obtained from this expression is lengthy and not very illuminating, but it is important to note that because of the presence of the term \( \tan^{-1} \left( g_{IN}/\sqrt{a_1^2 - g_{IN}^2} \right) \), we obtain a \( \ln E \) divergence at the band center when \( g_0 = 0 \). Based on symmetry, similar behavior will be obtained when a combination of two Pauli matrices are considered. This should be contrasted with the \((\ln E)^2\) divergence obtained by Hikami et al.\(^{55}\).

2. \( m \neq 0 \)

When the fermion is massive, we will ignore the mass disorder part. The density of states is given by

\[
\bar{\rho}(E) = \frac{\partial}{\partial \epsilon_1} \ln \left\{ \int_0^\infty d(r_1^2/2) \int_0^\infty d(r_2^2/2) \right\},
\]

where \( \epsilon_{1,2} = E \pm m + i\delta \). Again if we take only one of the internode scattering terms (\( g_2 = 0 \)) for simplicity, the expression within the curly parenthesis in Eq. \( 31 \) \( I \), becomes

\[
I = -\frac{\pi^2}{ab + g_1 \sqrt{ab}}
\]

The density of states is then given by

\[
\bar{\rho}(E) = \frac{1}{4\pi^2 I_B^2} \sum_{\sigma = \pm 1} \frac{g_0}{g_0^2 + (E + \sigma m)^2} + \frac{g_0 (R \cos \beta + g_1 \sqrt{R} \cos \frac{\beta}{2})}{R^2 + g_1^2 R + 2 g_1 R \cos \frac{\beta}{2}} + \frac{E (R \sin \beta + g_1 \sqrt{R} \sin \frac{\beta}{2})}{R^2 + g_1^2 R + 2 g_1 R \cos \frac{\beta}{2}},
\]

where \( R = \sqrt{(g_0^2 + m^2 - E^2)^2 + 4g_1^2 E^2} \) and \( \tan \beta = (2g_0 E/(g_0^2 + m^2 - E^2)) \). The above expression takes particularly simple form when \( g_0 = 0 \). It becomes

\[
\bar{\rho}(E^2 > m^2) = \frac{1}{4\pi^2 I_B^2} \left[ \delta(E + m) + \delta(E - m) + \frac{2Eg_1}{(E^2 - m^2 + g_1^2)\sqrt{E^2 - m^2}} \right],
\]

\[
\bar{\rho}(E^2 < m^2) = 0.
\]

If both internode scatterings are present and \( g_1 = g_2 = g_{IN} \), the integral is given by

\[
I = -\frac{\pi}{(ab - 2g_{IN}^2)} \left[ \pi - 4 \frac{g_{IN}}{\sqrt{ab - g_{IN}^2}} \tan^{-1} \left( \frac{g_{IN}}{\sqrt{ab - g_{IN}^2}} \right) - 2\sqrt{2} \frac{g_{IN}}{\sqrt{ab - g_{IN}^2}} + 2\pi \frac{g_{IN}}{\sqrt{ab - g_{IN}^2}} \right]
\]

The expression for the DOS is tedious. However for \( g_0 = 0 \), the feature that the DOS is zero for \( E^2 < m^2 \) is still valid. In this case for energies close to \( \pm m \), \( \bar{\rho}(E) \sim \ln |E - m|/\sqrt{|E - m|} \).
V. HALL PLATEAU IN THE LOWEST LANDAU LEVEL

Similar to the method described in Ref. 58, we generate the matrix elements of the Dirac Hamiltonian after projecting to the lowest Landau level. In our problem, the element \( \langle k|H|k' \rangle \) itself is a \( 2 \times 2 \) matrix:

\[
\langle k|H|k' \rangle = \int dx dy \psi_k^\dagger(x, y) H^\text{imp}_{\text{LLL}}(x, y) + m \eta_3 \psi_k(x, y) = m \eta_3 \delta_{k, k'} + V(k, k'),
\]

where \( \psi_k(x, y) \) is the lowest Landau level wave function in the Landau gauge. We choose all the \( V_j \)'s to follow independent Gaussian white noise distributions such that \( V_j(x, y)V_{j'}(x', y') = g_j^2 \delta_{j, j'} \delta(x - x') \delta(y - y') \). Then the elements of the \( 2 \times 2 \) matrix \( V(k, k') \) can be computed explicitly—for example,

\[
V(k, k')_{11} = \frac{1}{\sqrt{\pi L_y}} e^{-\frac{i l_B^2 (k-k')^2}{2}} \int d\xi \ e^{-\xi^2} \left[ g_0 u_0(l_B \xi + \frac{k + k'}{2} l_B, k' - k) + g_3 u_3(l_B \xi + \frac{k + k'}{2} l_B, k' - k) \right],
\]

where \( u_j(x, k) \) is a complex random variable defined to be the Fourier transforms of \( V_j(x, y) \) along the \( y \) direction normalized by the width \( g_j \), namely:

\[
u_j(x, k) = \frac{1}{g_j \sqrt{L_y}} \int dy V_j(x, y) e^{iky}.
\]

Because each of the disorder fields has zero correlation length, and there are no correlations between them,

\[
\overline{u_i(x, k)u_j(x', k')} = \delta_{i, j} \delta(x - x') \delta(k + k')
\]

It is straightforward to compute the statistical properties of the matrix elements. The averages are:

\[
\overline{V(k, k')_{i,j}} = 0 ; \quad i, j = 1, 2
\]

As to correlations, the only non-vanishing pairs are:

\[
\begin{align*}
&\overline{V(k_1, k_2)_{11}V(k_3, k_4)_{11}} = \overline{V(k_1, k_2)_{22}V(k_3, k_4)_{22}} = \frac{\eta_0^2 + \eta_3^2}{\sqrt{\pi L_y}} \exp\left[-\frac{l_B^2}{2}((k_1 - k_2)^2 + (k_4 - k_3)^2)\right] \delta_{k_1, k_2, k_4, k_3} \\
&\overline{V(k_1, k_2)_{11}V(k_3, k_4)_{22}} = \overline{V(k_1, k_2)_{22}V(k_3, k_4)_{11}} = \frac{\eta_0^2 - \eta_3^2}{\sqrt{\pi L_y}} \exp\left[-\frac{l_B^2}{2}((k_1 - k_2)^2 + (k_4 - k_3)^2)\right] \delta_{k_1, k_2, k_4, k_3} \\
&\overline{V(k_1, k_2)_{12}V(k_3, k_4)_{12}} = \overline{V(k_1, k_2)_{21}V(k_3, k_4)_{21}} = \frac{\eta_0^2 - \eta_3^2}{\sqrt{\pi L_y}} \exp\left[-\frac{l_B^2}{2}((k_1 - k_2)^2 + (k_4 - k_3)^2)\right] \delta_{k_1, k_2, k_4, k_3} \\
&\overline{V(k_1, k_2)_{12}V(k_3, k_4)_{21}} = \overline{V(k_1, k_2)_{21}V(k_3, k_4)_{12}} = \frac{\eta_0^2 + \eta_3^2}{\sqrt{\pi L_y}} \exp\left[-\frac{l_B^2}{2}((k_1 - k_2)^2 + (k_4 - k_3)^2)\right] \delta_{k_1, k_2, k_4, k_3}
\end{align*}
\]

For numerical implementation, we discretize and use the integer \( l \) to label the \( x \) coordinate. We then generate a set of complex random variables \( u_j(l, k) \), that are \( \delta \)-correlated as in Ref. 13. Finally, we approximate the integrals by sums. Explicitly, the matrix elements are...
with \( A = \sum_j e^{-2a^2 j^2} \) and \( a^2 = \pi/2M^2 \). Here \( M \) is the length of the system in the \( y \) direction, the unit being \( \sqrt{2\pi l_B} \), that is, \( M = L_y/\sqrt{2\pi l_B} \), chosen to be an integer. The integers \( k \) and \( k' \) label the wave vectors. Since the matrix elements decay exponentially, we can neglect them for \( k' > 2M \). A cutoff is also necessary for the recursive Green’s function technique that we use.

We compute the density of states \( \rho(E) \) by directly diagonalizing the Hamiltonian. We have checked that \( \rho(E) \) is independent of \( M \), for sufficiently large \( M \); \( M = 32 \) seems to be sufficient; the total number of momentum states \( N_k \) is chosen to be 1000, which is half the dimension of the Hamiltonian matrix to be diagonalized, as there are two fermions for each \( k \). Typically, an average over 100 disorder realizations is used.

The recursive Green’s function technique, similar to that in Ref. \( 58 \), is used to explore the localization properties. The details are described in Appendix \( \text{[C]} \). We first compute the localization lengths for a finite system, \( \lambda \nu(E_i) \), at a set of energies, \( \{E_i\}_{i=1}^{N_E} \), in systems with transverse dimensions \( \{M_y\}_{j=1}^{N_j} \). Since there are two types of fermions, in general there can be two distinct localization lengths; however, in most cases discussed below, they are identical within our numerical accuracy, and we will not generally distinguish them. Assuming finite-size scaling, \( \lambda \nu(M / E) = f(M^{1/\nu} (E - E_c)) \), where \( f(x) \) is a universal function, the data is collapsed to obtain the localization length exponent \( \nu \), and the critical energy \( E_c \). Strictly, scaling holds only for large enough systems in the vicinity of critical energy. Here the energies \( \{E_i\} \) are chosen close to the critical energy, \( E_c \), and the validity of the scaling law is verified by the success of data collapse. For the details of the procedure involving data collapse, see Appendix \( \text{[C]} \).

The numerical calculations about the localization properties were mostly performed for a quasi-one dimensional system with the transverse dimensions \( M = 8, 16, 32, 64 \). The total number of momentum states is \( N_k = 5 \times 10^4 \). Because of the \( 2 \times 2 \) character of the Hamiltonian matrix elements, the numerical calculations are more demanding than those in Ref. \( 58 \). The data are typically averaged over 100 disorder configurations to reduce fluctuations. Energies \( \{E_i\} \) were chosen close to the critical energy and measured in units of \( 2(\sum_j g_j^2)^{1/2} \) similar to Ref. \( 58 \).

Our program is also validated by the case \( g_0 = 0.5, g_1 = g_2 = g_3 = 0, \) and \( m = 0 \). In this case, the two types of fermions are independent. Because the LLL wave function is identical to the non-relativistic one, the properties should be the same as in Ref. \( 58 \). Numerical computations show a single peak in the density of states and a localization length exponent of \( \nu = 2.41 \pm 0.08 \); both agree well with the previous results.

VI. LD TRANSITION FOR THE MASSLESS CASE

A. One disorder field

Consider first the cases where only one type of disorder has nonzero strength. Numerically, we considered (1) \( g_1 = 0.5, g_0 = g_1 = g_2 = 0, \) (2) \( g_1 = 0.5, g_0 = g_2 = g_3 = 0, \) and (3) \( g_2 = 0.5, g_0 = g_1 = g_3 = 0 \). In all of these cases, the delta function density of states in the pure system is broadened into a simple bell shape function due to disorder. The results of successful data collapse, not shown here, yield critical exponents \( \nu = 2.46 \pm 0.09, \nu = 2.48 \pm 0.11 \) and \( \nu = 2.45 \pm 0.08 \) for cases (1), (2), and (3), respectively, that is, they are the same within the error bars.

The critical exponents are all equal to that of single type of fermions subject to potential disorder. This can be understood as follows. The original Hamiltonian matrix is in the basis \( \{|k_1, 1\}, |k_1, 2\rangle, |k_2, 1\rangle, |k_2, 2\rangle, \ldots \rangle \), where 1 and 2 label the type of fermions. If we reorder the basis as \( \{|k_1, 1\}, |k_1, 2\rangle, \ldots, |k_1, 2\rangle, |k_2, 2\rangle, \ldots \rangle \), the Hamiltonian becomes a \( 2 \times 2 \) block matrix with diagonal blocks representing intranode parts, and the off-diagonal blocks representing internode scatterings. Explicitly, it is in the form:

\[
H = \begin{pmatrix}
g_0 U_0 + g_3 U_3 + mI & g_1 U_1 - ig_2 U_2 \\
g_1 U_1 + ig_2 U_2 & g_0 U_0 - g_3 U_3 - mI
\end{pmatrix}
\]
where \( U_i, i = 0, \ldots, 3 \) are statistically independent random Landau matrices, and \( I \) is the identity matrix. The case with only \( g_3 \) nonzero, has the same structure, and hence same statistical properties, as the case when only \( g_0 \) is nonzero. When only \( g_1 \) is nonzero, we can, by a unitary transformation given by,

\[
T = B \begin{pmatrix} I & I \\ I & -I \end{pmatrix},
\]

where \( B \) is a normalization constant, bring the Hamiltonian back to the block diagonal form, resulting in a structure corresponding to two types of independent fermions in the presence of mass disorder. Thus, the critical exponent is the same as the case when only \( g_0 \) is non-zero. The same argument also applies to the case when only \( g_2 \) is non-zero.

**B. Two disorder fields**

If one of the two disorder fields is \( V_0(x, y) \), and another is \( V_1(x, y) \), or \( V_2(x, y) \), or \( V_3(x, y) \), an appropriate unitary transformation about an axis by \( \pi/2 \) will map one possible case to another. For instance, the transformation in Eq. (43) will transform the case with \( V_0(x, y) \) and \( V_2(x, y) \) to \( V_3(x, y) \). It is therefore sufficient to consider only the case with just \( V_0(x, y) \) and \( V_3(x, y) \). However, from Eq. (47), the Hamiltonian is block diagonal, and the blocks \( g_0 U_0 + g_2 U_2 \) and \( g_0 U_0 - g_2 U_2 \) are statistically equivalent to a new block \( \sqrt{g_0^2 + g_2^2} U \), with \( U \) a new random matrix satisfying the same statistical properties as \( U_i \)'s; see Eq. (45). That is, the Hamiltonian for \( g_0 \neq 0 \) and \( g_2 \neq 0 \) is statistically the same as that corresponding to a potential disorder \( g_0 = \sqrt{g_0^2 + g_2^2} \).

Our numerical computations confirm this argument. The data collapse was found to be successful, assuming \( \nu_c = 0 \), and the critical exponents are \( \nu = 2.45 \pm 0.06 \) for \( g_0 = g_3 = 0.5, g_1 = g_2 = 0, \) and \( \nu = 2.45 \pm 0.11 \) for \( g_0 = g_1 = 0.5, g_2 = g_3 = 0. \)

Next, we choose two disorder fields from \( V_1(x, y) \), \( V_2(x, y) \), and \( V_3(x, y) \). There are three possible combinations. In fact, these three cases are not independent; we can map one case to another by an appropriate unitary transformation corresponding to a rotation by \( \pi/2 \) about a certain axis. Therefore, it is sufficient to consider only one of the three cases; for example, let us choose \( V_3(x, y) \) (mass disorder), and \( V_1(x, y) \) (internode coupling).

We set \( g_1 = g_3 = 0.5, g_0 = g_2 = 0. \) The density of states and the localization lengths are plotted in Fig. 1 for \( E > 0 \); there is symmetry under \( E \rightarrow -E \). The extended states are no longer at \( E = 0 \) but shifted to \( E = E_c \sim \pm 0.42 \). At \( E_c \sim \pm 0.42 \), we study the localization properties using the data in the range of \(|E| > E_c \), since data in the range \(|E| < E_c \) are close to both critical points and are likely to result in inaccurate results. The maximum system size used is \( M = 64 \). Because we do not have a priori knowledge of \( E_c \), the statistical procedure discussed in Appendix \([C]\) is employed to determine \( E_c \), hence the critical exponent \( \nu \). The data collapse is shown in Fig. 2. The critical exponent for this parameter set is found to be \( \nu = 3.23 \pm 0.26 \), distinct from the nonrelativistic case of \( \nu \sim 7/3 \).

The present problem can be exactly mapped onto the spin-orbit scattering involving the two-state Landau level problem discussed in Ref. \([61]\); our results are in full agreement. From Fig. 1 there appears to be a divergence in the DOS at the band center, corresponding to a possible LD transition at \( E = 0 \). As shown above, the
the extent of the band increases, as $V_{\text{int}}$ decreases disorder in the limits $g_{1} \lessgtr g_{3}$ although the data collapse becomes insensitive to the deviation from this value is the largest when $g_{1} \approx g_{3}$. Nonetheless, the results are suggestive of a continuously exponent is suggested in Fig. 4. It is interesting to study the behavior as the ratio $g_{1}/g_{3}$ is varied. The result for the DOS is shown in Fig. 3. Note that the energy $E$ is in the unit of $2\sqrt{g_{1}^{2} + g_{3}^{2}}$. So the extent of the band increases, as $g_{1}$ increases. The divergence of the DOS at the band center is a unique feature when $g_{1}$ and $g_{3}$ are equal, while in the extreme limits there may be a slight dip at $E = 0$.

As to $\nu$, a continuously exponent is suggested in Fig. 4. In the limits $g_{3} \gg g_{1}$ or $g_{3} \ll g_{1}$, only one type of disorder dominates, hence the value $\nu \sim 7/3$ is plausible. The deviation from this value is the largest when $g_{1} \sim g_{3}$, although the data collapse becomes insensitive to the value of $\nu$ in the same regime, resulting in larger error. Nonetheless, the results are suggestive of a continuously varying critical exponent.

In the Hamiltonian, the mass disorder $V_{3}(x, y)$ and the internode scattering disorder $V_{1}(x, y)$ are accompanied by the Pauli matrices $\eta_{3}$ and $\eta_{1}$. If we apply a unitary transformation corresponding to a rotation of $\pi/2$ about $y$ axis, where $C$ is a normalization factor, the disorder Hamiltonian \( H \) will be transformed such that $g_{3} \rightarrow g_{1}$, $g_{1} \rightarrow -g_{3}$. Because we are studying statistical properties of the system, and all distribution functions are symmetric about zero, the negative sign in front of the $g_{3}$ is of no importance. This means that this unitary transformation effectively interchanges $g_{1}$ and $g_{3}$, hence map the regime $g_{3} > g_{1}$ to the regime $g_{3} < g_{1}$. Note the symmetry between the two regimes in Fig. 3 and Fig. 4.

C. Three disorder fields

The important case in this category is when $g_{1} \sim g_{2} \sim g_{3}$; other cases can be roughly understood in terms of the cases discussed above. For numerical computation, we take $g_{1} = g_{2} = g_{3} = 0.5$. The DOS is shown in Fig. 4. Compared to the case when only $g_{1} = g_{3} = 0.5$, discussed above, the divergence of the DOS at $E = 0$ is missing, but the two peaks at $E = \pm 0.46$ survive. This is suggestive of nonexistence of extended states at the band center, but a LD transition at $E \sim \pm 0.46$, which is confirmed by the scaling curve shown in Fig. 3 and a critical exponent of $\nu = 3.6 \pm 0.3$ is obtained. The error bar is large due to a substantial degree of disorder, but the exponent is distinctly different from the value $\nu = 7/3$, indicating a new universality class.
The constant mass term results in new physics when combined with the internode coupling, and the resulting phenomena are different from the case when the mass disorder and the internode coupling are combined, as in the previous section.

VII. LD TRANSITION FOR THE MASSIVE CASE

The constant mass term results in new physics when combined with the internode coupling, and the resulting phenomena are different from the case when the mass disorder and the internode coupling are combined, as in the previous section.
the analytical result for the Cauchy distribution, namely, 

\[ \lambda_M/M \text{ on energy } E \text{ for different system sizes } M. \]  

The shaded area is used for scaling. Critical exponent is found to be \( \nu = 1.82 \pm 0.06 \), with the choice of \( E_c = m \).

**FIG. 8:** (Color online) The scaling curve for the case \( g_0 = g_2 = g_3 = 0, g_1 = 0.4, \) and \( m = 0.15 \). Insert: dependence of \( \lambda_M/M \) on energy \( E \) for different system sizes \( M \). The shaded area is used for scaling. Critical exponent is found to be \( \nu = 1.82 \pm 0.06 \), with the choice of \( E_c = m \).

The dependence of the critical exponent \( \nu \) on \( g_1 \) (normalized by the mass \( m \)). The parameters are \( g_0 = g_2 = g_3 = 0 \), and \( m = 0.15 \). The dashed line indicates the level of \( \nu = 2.33 \).

**FIG. 9:** The dependence of the critical exponent \( \nu \) on \( g_1 \) (normalized by the mass \( m \)). The parameters are \( g_0 = g_2 = g_3 = 0 \), and \( m = 0.15 \). The dashed line indicates the level of \( \nu = 2.33 \).

**B. Two disorder fields**

The most relevant case corresponding to graphene is the one with two types of internode scattering of comparable magnitude. Therefore, we choose \( g_1 = g_2 = m = 0.5 \). The DOS is shown in Fig. 10. Note that the energy is now measured in units of \( 2\sqrt{g_1^2 + g_2^2} = \sqrt{2} \), so that the divergence is located at \( E = m \), which is \( m = 0.5/\sqrt{2} = 0.354 \). From the insert in Fig. 10 we find a slope of \( -0.47 \pm 0.01 \), which is consistent with the analytical result for the Cauchy distribution, namely, \( \rho(E) \propto \ln |E - m|/\sqrt{|E - m|} \). Also note the gap in the DOS.

The crossing point in Fig. 10 indicates \( E_c \sim 0.55 \) instead of \( E_c = m \). The data collapse is shown in Fig. 11 with a critical exponent of \( \nu = 3.8 \pm 0.2 \). This critical exponent is not close to any of the values found for a finite mass with a single internode coupling, as in Fig. 8. However, it is reasonably close to the exponent for \( m = g_0 = g_2 = 0, g_1 \sim g_3 \) (see Fig. 9), which is also equivalent to the case \( m = g_0 = g_3 = 0, g_1 \sim g_2 \), as discussed above. Note that \( m = 0.5 \) is much smaller than the bandwidth \( 2\sqrt{g_1^2 + g_2^2} = 1.414 \), and this critical exponent indicates that the presence of small finite mass will have little effect on the critical exponent as long as two internode couplings are finite.

At the other limit, when \( m \) is large enough compared to the bandwidth, the critical exponent \( \nu \to 1 \). Thus, it is reasonable to believe that the exponent also varies continuously, as a function of \( g_1/m \), provided that \( g_1 = g_2 \), and the behavior is similar to in Fig. 9 except that \( \nu \to 3.8 \) in the limit \( m \to 0 \).

**C. Three disorder fields**

Because potential disorder is always present in experiments on graphene, we would like to discuss the case when \( g_0 \neq 0, m \neq 0 \), and \( g_1 = g_2 \).

First, when \( m \) is the smallest parameter, it can be neglected, and hence the massless case discussed above is recovered. Our numerical computations gave a critical exponent of \( \nu \sim 3.8 \) when \( m \ll g_0 \ll g_1 = g_2 \), and \( \nu \sim 2.3 \) in the limit \( m \ll g_1 = g_2 \ll g_0 \), because now the
potential disorder is more important than the rest.

When $g_0$ is the smallest, it will have little effect. Therefore, as discussed in the previous subsection, there will be a continuously varying exponent from $\nu \sim 1.0$ for $g_0 \ll g_1 = g_2 \ll m$ to $\nu \sim 3.8$ for $g_0 \ll m \ll g_1 = g_2$.

Finally, if $g_1 = g_2$ are smaller than the rest, the internode scattering is no longer important, and hence the two nodal fermions will be decoupled. The exponent will therefore be always $\nu \sim 2.3$ regardless of the relationship between $m$ and $g_0$.

FIG. 11: (Color online) The scaling curve for the case $g_1 = g_2 = m = 0.5$, $g_0 = g_3 = 0$. Insert: the dependence of $\lambda_M/M$ on the energy $E$ for different system sizes $M$. The shaded area is used for scaling.

VIII. CONCLUSIONS

We have analyzed the effects of disorder on the LD transition in the LLL of graphene. Because both types of internode scattering, present in the LLL, arise from the random hopping, they will have roughly the same strength. Because the sources of the mass disorder and the internode scattering are different, their strengths will be generically different. In some special cases of disorder combinations we have found new universality classes of LD transition in contrast to the conventional IQH.

Our results for the LD transitions in the LLL have direct experimental relevance for the plateau transitions in graphene. Consider first the cases where both the spin and the nodal degeneracies are completely removed. A number of authors have shown that the inclusion of a finite mass and Zeeman energy can explain the appearance of plateaus at $\nu_f = 0, \pm 1, \pm 2q$.

Because experiments resolve the spin and nodal splitting, intranode scattering which always broadens the Landau levels is weak compared to $E_z$ and $m$. If the internode scattering strength is larger than the intranode scattering strength, we expect that in the lowest Landau level $0 \rightarrow \pm 1, 1 \rightarrow 2$ and $-1 \rightarrow -2$ plateau transitions can have different universality classes, in contrast to the conventional IQH effect. In the opposite limit, when the intranode scattering is considerably stronger, the plateau transitions will fall into the conventional IQH universality class with $\nu \sim 2.3$.

For the spin and nodally degenerate plateaus, the potential scattering is strong compared to the Zeeman energy and the mass gap. Theoretically, from our analysis of the massless cases we can infer the plateau transitions to be of the conventional IQH type. However, in experiments, if scaling with respect to the band center is invoked, an effective exponent for these plateau transitions will be observed.

Plateaus at $\nu = \pm 4, \pm 6, \ldots$ involve higher Landau levels. In the higher Landau levels both the random potential at the lattice scale and the random hopping will have nonzero intranode as well as internode scattering contributions. This will complicate the analysis of LD transitions in these levels. Because inclusion of a finite mass does not lift the nodal degeneracies of higher Landau levels, the effect of internode scattering can be strong. In view of the degeneracy factor, there is the possibility of observing an effective exponent for these higher plateau transitions.

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APPENDIX A: INTEGRAL

For the Cauchy distribution, in the absence of mass disorder, the calculation of the DOS in Eq. (35) involves the integral
\[ I = \frac{1}{4} \int_0^\infty dx \int_0^\infty dy \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \exp \left[ -ax - by - 2g_2 \sqrt{xy} \sin(\alpha_1 - \alpha_2) - 2g_1 \sqrt{xy} \cos(\alpha_1 - \alpha_2) \right], \] (A1)

where \( a = g_0 - i\epsilon_1 \) and \( b = g_0 - i\epsilon_2 \). In the massless case \( a = b = g_0 - i\epsilon \). After performing the integrals over one of the angles, the double integral over the angles is reduced to

\[ I_{\text{ang}} = 8\pi \int_0^{\pi/2} d\alpha_1 \exp[-2\Delta \sqrt{xy} \cos(\alpha_1 - \beta)], \] (A2)

where \( \Delta = \sqrt{g_1^2 + g_2^2} \) and \( \tan \beta = g_2/g_1 \). Now, expanding the exponential in a power series, integrals over \( x \) and \( y \) can be easily performed. For the angular integral we use the relation

\[ \int_0^{\pi/2} d\alpha \cos^j(\alpha - \beta) = -\frac{1}{l + 1} \left[ \sin^{l+1} \beta F_2 \left( \frac{l + 1}{2}, 1; \frac{3 + l}{2}, \sin^2 \beta \right) + \cos^{l+1} \beta F_2 \left( \frac{l + 1}{2}, 1; \frac{3 + l}{2}, \cos^2 \beta \right) \right], \] (A3)

where \( F_2 \) is Gauss’s hypergeometric function and obtain

\[ I = -\frac{2\pi}{\Delta ab} \sum_{j=0}^\infty \frac{\Gamma^2((l/2) + 1)}{\Gamma(l + 2)} g_j \left(-\frac{2g_j}{\sqrt{ab}}\right)^l \]
\[ \times F_2 \left( \frac{l + 1}{2}, 1; \frac{3 + l}{2}, 1 \right). \] (A4)

The summation over \( l \) can be performed by the following trick: (1) use the integral representation of \( F_2 \), (2) perform a power series summation which is simple in these cases, and (3) complete the integration over the auxiliary variable introduced for the integral representation. For simplicity we will specialize to the cases: (i) \( g_1 \neq 0, g_2 = 0 \) and (ii) \( g_1 = g_2 = g_{1N} \).

(i) \( g_1 \neq 0, g_2 = 0 \): In this case we have

\[ I = -\frac{2\pi}{\sqrt{ab}} \sum_{l=0}^\infty \frac{\Gamma^2((l/2) + 1)}{\Gamma(l + 2)} (-g_{1N})^l \]
\[ \times F_2 \left( \frac{l + 1}{2}, 1; \frac{3 + l}{2}, 1 \right). \] (A5)

We now use the following two relations

\[ F_2 \left( \frac{l + 1}{2}, 1; \frac{3 + l}{2}, 1 \right) = \sqrt{\pi} \Gamma((3 + l)/2) / \Gamma((l + 2)/2), \] (A6)

\[ \sum_{l=0}^\infty (-x)^l \frac{\Gamma((l/2) + 1)\Gamma((3 + l)/2)}{\Gamma(l + 2)} = \frac{\sqrt{\pi}}{2 + x}, \] (A7)

to obtain Eq. (6.5).

(ii) \( g_1 = g_2 = g_{1N} \): In this case we have

\[ I = -\frac{2\sqrt{\pi}}{ab} \sum_{l=0}^\infty \frac{\Gamma^2((l/2) + 1)}{\Gamma(l + 2)} (-g_{1N})^l \]
\[ \times F_2 \left( \frac{l + 1}{2}, 1; \frac{3 + l}{2}, 1/2 \right). \] (A8)

Using the integral representation

\[ F_2 \left( \frac{l + 1}{2}, 1; \frac{3 + l}{2}, 1 \right) = \frac{\Gamma((3 + l)/2)}{\Gamma(1/2)\Gamma((l/2) + 1)} \int_0^1 dt \frac{dt}{\sqrt{t(1-t)^2 + g_{1N}\sqrt{t(1-t)}}} \]
\[ \times \left( \frac{1-t}{1-t/2} \right)^{1/2} \left( 1 - \frac{t}{2} \right)^{-1/2} \] (A9)

and Eq. (A7) we get

\[ I = -\pi \sqrt{2ab} \int_0^1 \frac{dt}{\sqrt{ab\sqrt{t(1-t)^2} + g_{1N}\sqrt{t(1-t)}}} \] (A10)

After performing the integral over \( t \) we get Eq. (39). After setting \( m = 0 \) one recovers Eq. (41).

**APPENDIX B: RECURSIVE GREEN’S FUNCTION**

Because there are two types of fermions in our problem, the recursive Green’s function technique is a little more complicated than that introduced by Huckenstein. All the matrix elements, such as those in Eqns. (6.9) in Ref. 58, become \( 2 \times 2 \) matrices; hence, all the operations, such as the multiplication and inversion are matrix operations.

Denote the \( 2 \times 2 \) matrix \( \langle i|G|j \rangle \) simply by \( G(i, j) \). Suppose \( G^{(K)}(i, j) \), \( i, j = 1, \ldots, K \), the Green’s function containing \( K \) momentum states is available, then as we add another momentum state, the recursion relations for \( G^{(K+1)}(i, j) \) are:
\[ G^{(K+1)}(K+1, K+1) = \left( E - V(K+1, K+1) - \sum_{i,j} V(i, K+1) G^{(K)}(i,j) V(j, K+1) \right)^{-1} \]  

\[ G^{(K+1)}(i, K+1) = \left[ \sum_{j} G^{(K)}(i,j) V(j, K+1) \right] G^{(K+1)}(K+1, K+1) : i \leq K \]

\[ G^{(K+1)}(i,j) = G^{(K)}(i,j) + G^{(K+1)}(i,K+1) G^{(K+1)}(K+1, K+1)^{-1} G^{(K+1)}(K+1,j) : i,j \leq K \]

These matrix inversions here can be accurately computed because the sizes are small. Corresponding to the two types of fermions, we are interested in \( G^{(K)}(1,K) \) and \( G^{(K)}(1,K-1) \). To overcome this difficulty, suppose we have obtained \( \{G^{(K)}(i,j)\} \) and \( \{G^{(K+1)}(i,j)\} \) in \( K = 1, \ldots, K \), such that their elements \( G^{(K)}(j,m,n) = G^{(K)}(1,j,m,n)/G^{(K)}(1,K)m,m \), from the recursion relations \( \text{(B2)} \), we obtain:

\[ q_{n+1}^{(K)} = \left[ \sum_{j} g^{(K)}(j) V(j, K+1) \right] G^{(K+1)}(K+1, K+1) \]

\[ g^{(K+1)}(i,m,n) = \frac{1}{q_{m+1}^{(K)}} \left[ g^{(K)}(i,m,n) + \sum_{j} g^{(K)}(j) V(j, K+1) G^{(K+1)}(K+1,i) \right] , \quad i \leq K \]

\[ g^{(K+1)}(i,m,n) = \frac{1}{q_{m+1}^{(K)}} \left( \sum_{j} g^{(K)}(j) V(j, K+1) G^{(K+1)}(K+1,i) \right) , \quad i = K+1 \]

APPENDIX C: DATA COLLAPSE

In this appendix, we describe how we can extract the exponent \( \nu \) and the critical energy \( E_c \), if necessary, based on the computed localization lengths in finite systems \( \lambda_{M}(E) \) assuming a single parameter scaling assumption.

Suppose we have obtained \( \{\lambda_{M}(E)/M\} \) in systems with \( \{M_i\}_{i=1}^{N_M} \) for \( \{E_j\}_{j=1}^{N_E} \), each with a standard deviation \( \{\sigma_{M_{i,j}}\} \). Our goal is to find out the proper values of \( \nu \) and \( E_c \) such that all the \( N_E \times N_M \) data points collapse on to a single curve:

\[ \frac{\lambda_{M}(E)}{M} = f \left( M^{1/\nu}(E - E_c) \right) \]  

Since \( f(x) \) is unknown, it is difficult to characterize the quality of the data collapse. To overcome this difficulty, we proceed as follows: suppose that we are given a pair of values \((\nu, E_c)\), we can attempt to represent the unknown function \( f(x; \nu, E_c) \) by a polynomial of degree \( N \) by simply performing a general fit to Eq. \( \text{(C2)} \) given below, based on a total of \( N_E \times N_M \) data points \( \{x_i, y_i, \sigma_y\} : (\log[M_j^{1/\nu}(E_i - E_c)], \log[\lambda_{M}(E_j)/M_j], \sigma_{M_{i,j}}) \} \):

\[ \log \frac{\lambda_{M}(E)}{M} = \sum_{k=0}^{N} a_k \left[ \log(M^{1/\nu}(E - E_c)) \right]^k , \quad \text{(C2)} \]

where \( \{a_i\}_{i=0}^{N} \) are the coefficients to be fitted. In the computer implementation, the order of polynomials was chosen to be \( N = 5 \), since no significant changes were noted by increasing \( N \) to 9. The quality of this fit is represented by the variable \( S \) defined as:

\[ S(\nu, E_c) = \sum_{i=1}^{N_E \times N_M} \left( \frac{y_i - f(x_i; \nu, E_c)}{\sigma_i} \right)^2 \]  

If the preset values \((\nu, E_c)\) are not correct, the data points
will be scattered, resulting in a large value of $S$, which in turn indicates a poor data collapse. However, when $(\nu, E_c)$ attain the correct localization length exponent and the correct critical energy, respectively, $S$ will be minimized. Following this procedure, by minimizing $S$ with the standard gradient descent method, we are able to determine correctly both the critical energy $E_c$ and the localization length exponent $\nu$. Because the scaling law is only valid in the close vicinity of the critical energy, once $E_c$ is obtained from above procedure, we have to check, for the purpose of self consistency, that all energies used in the data collapse are indeed close to $E_c$.

As for the statistical error of $\nu$, the usual procedure is to assume that the minimized $S_{\text{min}}$ follows a $\chi^2$ distribution, and hence the error bar can be drawn corresponding to a certain confidence probability. However, this is not the case in this problem, since $S_{\text{min}}$ does not follow the $\chi^2$ distribution due to the nonlinear form of the estimated parameters $\nu$ and $E_c$ in \(\text{(C2)}\). To draw an error bar for $\nu$ statistically correctly, recall that we have the original data \(\{ (x_i, y_i, \sigma_{y_i}) \}\). We generate a large number of data sets synthetically \(\{ (x_i^{(k)}, y_i^{(k)}, \sigma_{y_i}^{(k)}) \}\), for $k = 1, 2, \ldots, N_s$, such that $x_i^{(k)} = x_i$, $\sigma_{y_i}^{(k)} = \sigma_{y_i}$, and $y_i^{(k)}$ a random variable distributed in the Gaussian form with a mean of $y_i$ and a standard deviation of $\sigma_{y_i}$. Next, we perform exactly the same procedure to get $\nu^{(k)}$ for each synthetic data set \(\{ (x_i^{(k)}, y_i^{(k)}, \sigma_{y_i}^{(k)}) \}\), as was performed in actual data set \(\{ (x_i, y_i, \sigma_{y_i}) \}\) for estimating the $\nu$ and $E_c$. Finally, the error bar for $\nu$ is drawn as the estimated standard deviation:

\[
\sigma_{\nu} = \left[ \frac{1}{N_s - 1} \sum_{k=1}^{N_s} (\nu^{(k)} - \bar{\nu})^2 \right]^{1/2} \quad \text{(C4)}
\]

where $N_s$ is the number of synthetic data sets, and $\bar{\nu}$ is the average of $\nu^{(k)}$. $N_s = 10^4$ in computer implementation.

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