SOLVING RANDOM EQUATIONS IN GARSIDE GROUPS USING LENGTH FUNCTIONS

MARTIN HOCK AND BOAZ TSABAN

Abstract. We give a systematic exposition of memory-length algorithms for solving equations in noncommutative groups. This exposition clarifies some points untouched in earlier expositions. We then focus on the main ingredient in these attacks: Length functions. After a self-contained introduction to Garside groups, we describe length functions induced by the greedy normal form and by the rational normal form in these groups, and compare their worst-case performances. In the case of Artin’s Braid group, we show that a better approach for estimating the minimal length in Artin generators is measuring the length in Birman-Ko-Lee (BKL) generators of the rational BKL form. This is proved theoretically for the worst case, and experimentally for the generic case.

1. Solving random equations

All groups considered in this paper are multiplicative noncommutative groups, with an efficiently solvable word problem, that is, there is an efficient algorithm for deciding whether two given (finite products of) elements in the group are equal as elements of the group. Throughout this paper, $G$ denotes such a group.

Problems involving solutions of equations in groups have a long history, and are nowadays also explored towards applications in public-key cryptography [13]. We mention some of the more elegant problems of this type.

Problem 1 (Conjugacy Search). Given conjugate $a, b \in G$, find $x \in G$ such that $b = xax^{-1}$.

Problem 2 (Root Search). Given $a \in G$, find $x \in G$ such that $a = x^2$, provided that such $x$ exists.

Problem 3 (Decomposition Search). Let $H$ be a proper subgroup of $G$. Given $a, b \in G$, find $x, y \in H$ such that $b = xay$, provided that there exist such $x, y$.

The second author was partially supported by the Koshland Center for Basic Research.
We will discuss the meaning of the terms “given” and “find”, appearing in Problems 1–3 later.

Problems 1–3, as well as many additional ones, can be stated generally as follows. By free-group word $w(t_1, \ldots, t_k)$ we mean a product of variables $t_{i_1}^\epsilon_1 \cdot t_{i_2}^\epsilon_2 \cdot \ldots \cdot t_{i_n}^\epsilon_n$ for any choice of a positive integer $n$ and elements $i_1, \ldots, i_n \in \{1, \ldots, k\}$ and $\epsilon_1, \ldots, \epsilon_n \in \{1, -1\}$, such that no cancellation is possible, that is, for each $j = 1, \ldots, n$, if $i_j = i_{j+1}$, then $\epsilon_j \neq -\epsilon_{j+1}$.

**Problem 4** (Solution Search). Fix $H_1, \ldots, H_k \leq G$ and a free-group word $w(t_1, \ldots, t_{k+n})$. Given parameters $p_1, \ldots, p_n \in G$ and an element $c \in G$, find $x_1 \in H_1, \ldots, x_k \in H_k$ such that $c = w(x_1, \ldots, x_k, p_1, \ldots, p_n)$, provided that there exist such $x_1, \ldots, x_k$.

Problem 4 deals with the solution of a single solvable equation (with parameters). It can also be stated for systems of several equations. The algorithms proposed here easily generalize to cover this case, cf. [9].

1.1. **Making the problems meaningful.** It suffices to discuss Problem 4.

First, all given information must be coded in some compact form. For example, the subgroups $H_1, \ldots, H_k$ of $G$ may be described by lists of generators and relations, all (the list, the generators, and the relations) of manageable length.

Second, the problem may require that it be possible to find a solution for each possible instance of the problem, or for a certain portion of the instances. Alternatively, the instances of the problem may be chosen according to a certain distribution $D$, and we may require that a solution can be found with a high-enough probability (a probabilistic model).

Finally, by “find” we mean “find efficiently”, i.e., use an algorithm with a feasible running time. Otherwise, in most cases of interest the problems are solvable. E.g., if $G$ is a finitely generated group with solvable word problem, then we can solve Problem 4 by enumerating $G^k$ recursively, and trying all possible solutions until one is found. This algorithm always succeeds in a finite running time, but usually this running time is infeasible.

In this discussion, all quantitative terms (compact, efficient, significant, etc.) have two natural interpretations: Concrete (e.g., of size less than 1GB) or asymptotic (e.g., polynomial in the size of the input).

1.2. **The probabilistic model.** With an eye towards applications, we will always use the probabilistic version of the problems, where we wish to find (efficiently) a solution with a significant probability, provided
that the instances of the problem are chosen according to a certain known distribution $D$.

More precisely, in Problem 4 we fix a distribution $D$ on $G^{k+n}$ such that for each $(x_1, \ldots, x_k, p_1, \ldots, p_n)$ in the support of $D$, we have that $x_1 \in H_1, \ldots, x_k \in H_k$. An instance of the problem is generated as follows: A secret tuple $(x_1, \ldots, x_k, p_1, \ldots, p_n)$ in $G^{k+n}$ is chosen according to the distribution $D$, and we are given $p_1, \ldots, p_n$ and an element $c \in G$ equal to $w(x_1, \ldots, x_k, p_1, \ldots, p_n)$ in $G$. We must then search for elements $\tilde{x}_1 \in H_1, \ldots, \tilde{x}_k \in H_k$ such that with a significant probability, $c = w(\tilde{x}_1, \ldots, \tilde{x}_k, p_1, \ldots, p_n)$ in $G$.

By peeling off known parameters on the left of the given word $w(x_1, \ldots, x_k, p_1, \ldots, p_n)$, we may assume that it begins with a variable $x_i$ (possibly inverted). If we are able to find $x_i$ (with a significant probability), we can treat it as a parameter henceforth, and proceed to the next leading variable after peeling off all parameters on the left. Continuing in this manner, we find suggestions for all variables, and can check whether we obtained a solution.

This reduces the general Problem 4 to the following (more difficult) problem.

**Problem 5** (Leading-Variable Search). Fix $H_1, \ldots, H_k \leq G$ and a free-group word $t_1 \cdot w(t_1, \ldots, t_{k+n})$. Given parameters $p_1, \ldots, p_n \in G$ and an element $c = x_1 \cdot w(x_1, \ldots, x_k, p_1, \ldots, p_n)$ in $G$ such that $x_1 \in H_1, \ldots, x_k \in H_k$, find $x_1$.

Problem 5 makes sense only in the probabilistic model, because in general there could be more than one solution to a given equation. In certain settings, it may be much more difficult than the original Problem 4.

Problem 5 can be reduced to the following neatly stated problem.

**Problem 6** (Factorization Search). Fix $H \leq G$. Given an element $c = xy \in G$ with $x \in H$, find $x$.

Problems 6 and 5 are equivalent: Given an instance of Problem 5, we can take $x = x_1$ and $y = w(x_1, \ldots, x_k, p_1, \ldots, p_n)$ to get an instance of Problem 6, which if solved successfully, would give us $x_1$. On the other hand, given an instance of Problem 6, we can take $w(t_1) = t_1$ and the instance is $x \cdot w(y)$, so Problem 5 applies.

In summary, any algorithm solving Problem 6 with a significant probability, may be used to solve arbitrary equations (Problem 4), though with smaller success probability.

1.3. **Decision problems.** All mentioned problems also have a decision version. For example, the Congugacy Problem is: Given $a, b \in G$,
are they conjugate? From the *probabilistic* point of view, a solution to
the search version also implies a solution to the decision version, in the
following sense.

Assume that $A$ is an algorithm searching for solutions of equations of
a certain type (e.g., $b = xax^{-1}$), and that its running time is bounded,
say by a polynomial function of the length of its input. We define a *decision*
algorithm $A'$ with running time bounded by the same polynomial:
Given an instance of the equation to be checked, run $A$ on this
instance for the expected polynomial time, and then terminate it if it
did not terminate already. If a solution was found, the decision of $A'$
is *Yes*. Otherwise, it is *No*.

Assume that the instances of the equation are distributed according
to some distribution $E$. This induces a distribution $D$ on the *solvable*
equations, by conditioning that the chosen equation be solvable. Let
$p$ be the probability that $A$ finds a solution to (necessarily, solvable)
equations distributed according to $D$.

For each specific instance of the equation, $A'$ is correct in probability
at least $p$: If this instance has a solution, it will be found by $A$ in
probability $p$, in which case $A'$ decides “Yes”. And if this instance has
no solution, then in probability 1, $A$ will not find a solution (simply
because there is none), and $A'$ decides “No”.

This can also be viewed as follows: Let $q = 1 - p$. The probability
that $A'$ comes up with a wrong answer is:

$$P(\text{Wrong decision}) =$$

$$= P(\text{Decision} = \text{Yes} \mid \nexists \text{Solution}) \cdot P(\nexists \text{Solution}) +$$

$$+ P(\text{Decision} = \text{No} \mid \exists \text{Solution}) \cdot P(\exists \text{Solution}) =$$

$$= 0 \cdot P(\nexists \text{Solution}) + q \cdot P(\exists \text{Solution}) =$$

$$= q \cdot P(\exists \text{Solution}).$$

In particular, this probability is at most $q$, and the worst case is
when $P(\exists \text{Solution})$ is 1, in which the oracle always produces solvable
instances, and we are actually in the *search* version of the problem.

This justifies, to some extent, restricting attention to search problems
when working in the probabilistic model.

## 2. The memory-length algorithm

The potential usefulness of length functions for solving Problem 6
was identified in [10]. This was extended in [9] to the following algo-


2.1. The memory-length algorithm. Let $H \leq G$ be generated by elements $a_1, \ldots, a_m$ of $G$. Assume that an efficiently computable function $\ell : G \to \mathbb{R}_{\geq 0}$ is given, such that $\ell(abw)$ tends to be greater than $\ell(w)$ for $w \in G, a, b \in \{a_1, \ldots, a_m\}^{\pm 1}$.

An instance of Problem 6 is chosen according to a certain distribution $D$, and we are given $c$ which is equal in $G$ to $xy$.

Let $x = a_{j_1}^{\epsilon_1} a_{j_2}^{\epsilon_2} \cdots a_{j_n}^{\epsilon_n}$ be a (shortest) expression of $x$ in the generators $a_1, \ldots, a_m$. By standard arguments, we may assume that $n$ is known.

The algorithm generates an ordered list of $M$ sequences of length $n$, with the aim that with a significant probability, the sequence $((j_1, \epsilon_1), (j_2, \epsilon_2), \ldots, (j_n, \epsilon_n))$ (which codes $X$) appears in the list, and tends to be among its first few members. It consists of the following steps:

**Step 1.** For each $j = 1, \ldots, m$ and each $\epsilon \in \{1, -1\}$, compute $a_j^{\epsilon} c = a_j^{\epsilon} xy$, and give $(j, \epsilon)$ the score $\ell(a_j^{\epsilon} c)$. Keep in memory the $M$ elements $(j, \epsilon)$ with the best (=lowest) scores.

**Steps $s = 2, 3, \ldots, n$.** For each sequence $((j_1, \epsilon_1), \ldots, (j_{s-1}, \epsilon_{s-1}))$ out of the $M$ sequences stored in the memory, each $j_s = 1, \ldots, m$, and each $\epsilon_s \in \{1, -1\}$, compute

$$\ell(a_{j_s}^{-\epsilon_s}(a_{j_{s-1}}^{-\epsilon_{s-1}} \cdots a_{j_1}^{-\epsilon_1} c)) = \ell(a_{j_s}^{-\epsilon_s} a_{j_{s-1}}^{-\epsilon_{s-1}} \cdots a_{j_1}^{-\epsilon_1} xy),$$

and assign this score to the sequence $((j_1, \epsilon_1), \ldots, (j_s, \epsilon_s))$. Keep in memory only the $M$ sequences with the best scores.

The algorithm terminates after $n$ steps, with $M$ proposals for $((j_1, \epsilon_1), (j_2, \epsilon_2), \ldots, (j_n, \epsilon_n))$.

It is not difficult to see that the complexity of this algorithm is $n(n+4m+1)M/2$ group operations and evaluations of $\ell$. It is interesting to note that this algorithm may also be useful for solving the following.

**Problem 7** (Shortest) Subgroup Membership Search. Given $a_1, \ldots, a_m \in G$ and $x \in \langle a_1, \ldots, a_m \rangle$, find a (shortest possible) expression of $x$ as a product of elements from the set $\{a_1, \ldots, a_m\}^{\pm 1}$.

2.2. Sufficiency for the general problem. Assume that the algorithm succeeds, with a significant probability, to have the leading element $x$ in the final list. Then we have the following.

If there is only one unknown variable in the equation (e.g., Problems [1][3]), then we can check (in running time $M$) all elements in the list and find one which is a solution to the problem.
In the general case (Problem 4) there are several unknown variables, and we can iterate the algorithm by checking each suggestion in the list. The overall complexity is in principle $M^k$. However, the suggestions for each variable are ordered more or less according to their likelihood, and it suffices to check, for some $N \ll M$, the $N$ most likely solutions. This reduces the complexity to $N^k$, or more precisely to $N_1 \cdot N_2 \cdots N_k$, where $N_k$ is the number of elements required at the $k$th step, and it is likely that $N_{i+1} \ll N_i$ for each $i$.

2.3. Improvements. Certain simple modifications in the memory-length algorithm increase its success rates. We refer the reader to [15] for details.

2.4. The length function. For this algorithm to be meaningful and useful, one must have a good and efficiently computable length function on the group $G$. Our introduction of the memory-length algorithm suggests a natural model for comparing length functions for appropriateness to this method. We explore this below, after introducing a new proposal for a length function on the braid group. The braid group is, thus far, the most popular in applications related to cryptography [13]. Most of these cryptographic applications give rise to an equation, whose solution would imply the insecurity of the application. Thus, it is natural to look for good length functions on this group. See [13] for more details.

3. Excursion: Garside groups

We are going to consider two Garside structures on the braid group (to be defined). This section is an essentially self-contained introduction to Garside groups, and may be skipped by readers who are familiar with this concept, and by readers who do not insist on understanding all details of this paper.

Garside groups were introduced by Dehornoy and Paris [5], and later in a more general form by Dehornoy [4]. We treat the latter, more general case. All unproved assertions, as well as most of the proved ones, are from [5].

3.1. Garside Monoids and Groups. Let $M$ be a monoid with cancellation. $x \in M$ is an atom if $x \neq 1$, and $x = ab$ for $a, b \in M$ implies $a = 1$ or $b = 1$. $M$ is atomic if $M$ is generated by its atoms, and for each $a \in M$, the maximum number of atoms in an expression of $a$ as a product of atoms, denoted $\|a\|$, exists. It follows that $\|ab\| \geq \|a\| + \|b\|$ for all $a, b \in M$. In particular, as $1 = 1 \cdot 1$, we have that $\|1\| \geq \|1\| + \|1\|$, and thus $\|1\| = 0$. For $a \neq 1$, $\|a\| > 0$. 
Let $M$ be an atomic monoid. For $a, b \in M$, $a$ is a left divisor of $b$ if there is $c \in M$ such that $ac = b$. Similarly, $a$ is a right divisor of $b$ if there is $c \in M$ such that $ca = b$. $a \in M$ is a Garside element of $M$ if its left divisors and right divisors coincide, and include all atoms of $M$.

$M$ is a Garside monoid if it is atomic, has a Garside element, and for all $a, b \in M$, a greatest common divisor $a \land b$ and a least common multiple $a \lor b$ of $a$ and $b$ exist in $M$, both with respect to left divisibility.

For $a, b \in M$, the complement $a \setminus b$ is the unique $c \in M$ such that $ac = b$. The closure of the set of atoms under the operations of complement and least common multiple is the set $S$ of simple elements of $M$. The least common multiple of all elements of $S$, if it exists (e.g., if $M$ is finitely generated), is called the fundamental element of $M$ and denoted $\delta$. $\delta$, if it exists, is the least Garside element of $M$.

$G$ is a Garside group if it is the group of fractions of a Garside monoid $M$. In this case, the elements of $M$ are called the positive elements of $G$. In the remainder of this section, $M$ is a Garside group with a fundamental element $\delta$, and $G$ is the Garside group of fractions of $M$.

### 3.2. Greedy Normal Form. For $x \in M$ with $x \neq 1$, the simple element $\delta \land x \neq 1$. Define $\partial(x) = (\delta \land x)^{-1}x$. Then $\partial(x) \in M$, and as $x = (\delta \land x)\partial(x)$, $\|x\| \geq \|\delta \land x\| + \|\partial(x)\| > \|\partial(x)\|$. Define simple elements $s_1, s_2, \ldots$, as follows. Set $x_1 = x$, and for each $i = 1, \ldots, r$, let $s_i = \delta \land x_i$, and $x_{i+1} = \partial(x_i)$. $\|x\| = \|x_1\| > \|x_2\| > \cdots \geq 0$, and thus there is a minimal $n$ such that $x_{n+1} = 1$. $x = s_1 \cdots s_n$. Let $k \geq 0$ be maximal with $s_i = \delta$, and define $p_i = s_{k+i}$, $i = 1, \ldots, r$, $r = n - k$. The expression

$$x = \delta^kp_1 \cdots p_r$$

is called the greedy normal form of $x$.

Consider now $x \in G \setminus M$. If $x = \delta^ks$ and $s \in M$, then $k < 0$. Take the maximal integer $k$ such that $x = \delta^ks$ for some $s \in M$. Fix such $s$, and let $\delta^0p_1 \cdots p_r = p_1 \cdots p_r$ be the greedy normal form of $s$. The greedy normal form of $x$ is then again defined to be $\delta^kp_1 \cdots p_r$.

By the construction, we have that $p_{i+1} \land p_i^{-1}\delta = (p_{i+1} \cdots p_r \land \delta) \land p_i^{-1}\delta = p_{i+1} \cdots p_r \land (\delta \land p_r^{-1}\delta) = x_{i+1} \land p_r^{-1}\delta = 1$ for all $i = 1, \ldots, r - 1$, and that $p_r \neq 1$. We say in such cases that the sequence $p_1, \ldots, p_r$ is left-weighted.

### 3.3. Rational Normal Form. Following Thurston [6, Chapter 9], Dehornoy and Paris define the rational normal form of an element $x \in G$. To this end, we need the following.

\[^1\text{Also called mixed or symmetric normal form.}\]
Theorem 8 (Dehornoy-Paris [5]). For each \( x \in G \), there is a unique pair \((u,v)\) in \( M \times M \) such that \( x = u^{-1}v \) and \( u \wedge v = 1 \).

Let \( x \in G \), and let \( u, v \in M \) be as in Theorem 8. Let \( s_1 \cdots s_k, p_1 \cdots p_l \) be the greedy normal form of \( u, v \), respectively. The rational normal form of \( x \) is the expression

\[
x = (s_1 \cdots s_k)^{-1}(p_1 \cdots p_l).
\]

All \( s_i, p_j \) are simple, \( s_1 \wedge p_1 = 1 \), and the sequences \( s_1, \ldots, s_k \) and \( p_1, \ldots, p_l \) are both left-weighted. (The special cases where \( k = 0 \) or \( l = 0 \) are also allowed.)

For each \( a \in G \), define \( \tau(a) = a^\delta = \delta^{-1} a \delta \). \( \tau \) is an inner automorphism of \( G \), and its \( n \)th iterate at \( a \) is \( \tau^n(a) = a^{\delta^n} \). \( \tau \) maps simple elements to simple elements: For each simple \( s \), let \( p \) be such that \( sp = \delta \). Then \( p \) is simple, and thus there is a simple \( q \) with \( pq = \delta \). Then

\[
s \delta = spq = \delta q,
\]

and thus \( s^\delta = q \) is simple. In particular, \( M \) is invariant under \( \tau \). Any automorphism of \( G \) mapping positive elements to positive elements, maps atoms to atoms. It follows that \( \tau \) is a permutation of the atoms of \( M \).

One can obtain the rational normal form from the greedy normal form. To see this, we use the following.

Lemma 9. If \( s, p \) are simple and \( sp \) is left-weighted, then so are \( s^\delta p^\delta \) and \( s^{-1}p^{-1} \).

**Proof.** If \( ac = b \) are all positive, then \( a^{\delta \pm 1} c^{\delta \pm 1} = (ac)^{\delta \pm 1} = b^{\delta \pm 1} \), and \( c^{\delta \pm 1} \in M \). Thus, \( \tau^{\pm 1} \) both map left divisors to left divisors, and therefore

\[
(a \wedge b)^{\delta \pm 1} = a^{\delta \pm 1} \wedge b^{\delta \pm 1}
\]

for all \( a, b \in M \). Now, assume that \( sp \) is left-weighted. Then

\[
(s^{\delta^{\pm 1}})^{-1} \delta \wedge p^{\delta^{\pm 1}} = (s^{-1} \delta)^{\delta^{\pm 1}} \wedge p^{\delta^{\pm 1}} = (s^{-1} \delta \wedge p)^{\delta^{\pm 1}} = 1^{\delta^{\pm 1}} = 1,
\]

showing that \( s^\delta p^\delta \) is left-weighted. \( \square \)

Proposition 10. If \( s, p \) are simple and \( sp \) is left-weighted, then so are \((p^{\delta^k})^{-1} \delta)((s^{\delta^{k+1}})^{-1} \delta)\), for all integer \( k \).

**Proof.** Assume that \( sp \) is left-weighted. Then so is \((p^{-1} \delta)((s^{\delta})^{-1} \delta)\):

\[
(p^{-1} \delta)^{-1} \delta \wedge ((s^{\delta})^{-1} \delta) = p^\delta \wedge (s^{-1} \delta)^{\delta} = (p \wedge (s^{-1} \delta))^{\delta} = 1^{\delta} = 1.
\]

By Lemma 9 \((p^{\delta^k})^{-1} \delta)((s^{\delta^{k+1}})^{-1} \delta) = ((p^{-1} \delta)((s^{\delta})^{-1} \delta))^{\delta^k} \) is also left-weighted. \( \square \)
Let \( \delta^k p_1 \cdots p_r \) be the greedy normal form of \( x \). Consider three possible cases.

Case 1: \( k \geq 0 \). Then \( \delta^k p_1 \cdots p_r \) is already a rational normal form (with a trivial negative part).

Case 2: \( k = -m < 0 \) and \( m \geq r \). By definition, \( \delta^{-n} a = a \delta^{n} \delta^{-n} \) for all \( a \) and all \( n \). Using this, we have that

\[
\delta^{-m} p_1 \cdots p_r = \delta^{-1} p_1^{\delta^{m-1} \cdot \delta^{-1} p_2^{\delta^{m-2} \cdot \cdots \delta^{-1} p_r^{\delta^{m-r} \cdot \delta^{-1} (m-r)}} = \left( \delta^{m-r} \cdot (p_r^{\delta^{m-r}})^{-1} \delta \cdot \cdots \cdot (p_2^{\delta^{m-2}})^{-1} \delta \cdot (p_1^{\delta^{m-1}})^{-1} \delta \right)^{-1}.
\]

By Proposition 10, the last inverted expression is left-weighted, and thus we have a rational form, with a trivial positive part.

Case 3: \( k = -m < 0 \) and \( m < r \). In the same manner, we have that

\[
\delta^{-m} p_1 \cdots p_r = \delta^{-1} p_1^{\delta^{m-1} \cdot \delta^{-1} p_2^{\delta^{m-2} \cdot \cdots \delta^{-1} p_r^{\delta^{m-r} \cdot \delta^{-1} p_{m+1} \cdot \cdots \cdot p_r}} = \left( p_m^{-1} \delta \cdot \cdots \cdot (p_2^{\delta^{m-2}})^{-1} \delta \cdot (p_1^{\delta^{m-1}})^{-1} \delta \right)^{-1} (p_{m+1} \cdot \cdots \cdot p_r),
\]

By Proposition 10, each of the bracketed expressions is left-weighted. Thus, this expression is in rational normal form.

4. Several length functions on Garside groups

Let \( M \) be a Garside monoid with fundamental element \( \delta \), and \( G \) be its group of quotients.

Assumption 11. We assume that for each simple \( s \in M \), the minimal length \( \ell(s) \) of an expression of \( s \) as a product of atoms can be efficiently computed.

There is always an algorithm for computing \( \ell(s) \): Enumerate all words of length 1, 2, 3, \ldots, until one equal to \( s \) is found. The running time is bounded by \( k^{\ell(a)} \leq k^{\|a\|} \), where \( k \) is the number of atoms. But this is in general infeasible. When Assumption 11 fails, one may use below any estimation of \( \ell \) instead of the true function.

Fortunately, in the specific monoids in which we are interested, all relations are length-preserving, and thus \( \ell(s) \) is just the length of any expression of \( s \) as a product of atoms. Thus, Assumption 11 is true in our applications.

Example 12 (Artin’s presentation of \( B_N \)). Consider the monoid \( B_N^+ \) generated by \( \sigma_1, \ldots, \sigma_{N-1} \), subject to the relations

\[
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}; \\
\sigma_i \sigma_j &= \sigma_j \sigma_i \text{ when } |i - j| > 1.
\end{align*}
\]
The quotient group of this monoid is the braid group $B_N$ on $N$ strings. $B_N^+$ is a Garside monoid with atoms $\sigma_1, \ldots, \sigma_{N-1}$, and fundamental element

$$\delta = (\sigma_1 \cdots \sigma_{N-1})(\sigma_1 \cdots \sigma_{N-2}) \cdots (\sigma_1 \sigma_2)\sigma_1.$$ 

The positive elements of $B_N$ are the words in $\sigma_1, \ldots, \sigma_{N-1}$ not involving inverses of generators. As the relations are length preserving, all expressions of a positive element as a product of atoms have the same length. Thus, for $a \in M$, $\|a\|$ is the length of a (any) presentation of $a$.

Elements of $B_N$ can be identified with braids having $N$ strings, where each generator $\sigma_i$ performs a half-twist on the $i$th and $i + 1$st strings. This way, $\delta$ is a half-twist of the full set of strings. The simple elements correspond to positive braids in which any two strings cross at most once. A simple element is described uniquely by the permutation it induces on the strings, and every permutation of the $N$ strings corresponds to a simple element.

**Example 13 (BKL presentation of $B_N$).** Generalizing the geometric interpretation in Example 12 to allow half-twists of the $i$th and the $j$th string for arbitrary $i, j$, Birman, Ko, and Lee [2] introduced the following presentation of the braid group $B_N$. The monoid $BKL_N^+$ is generated by $a_{t,s}$, $1 \leq s < t \leq N$, subject to the relations

$$a_{t,s}a_{r,q} = a_{r,q}a_{t,s} \quad \text{if} \quad (t-r)(t-q)(s-r)(s-q) > 0;$$
$$a_{t,s}a_{s,r} = a_{t,r}a_{t,s} = a_{s,r}a_{t,r} \quad \text{if} \quad t > s > r.$$

Also here, the relations are length preserving, and thus the norm is equal to the number of atoms in any expression of the element.

This monoid also has the braid group $B_N$ as its quotient group. In terms of Artin’s presentation (Example 12), the Birman-Ko-Lee (BKL) generators can be expressed by

$$a_{t,s} = (\sigma_{t-1} \cdots \sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1} \cdots \sigma_{t-1}^{-1}).$$

$BKL_n^+$ is a Garside monoid with fundamental element

$$\delta = a_{n,n-1}a_{n-1,n-2} \cdots a_{2,1}.$$ 

Here too, a simple element is described uniquely by the permutation it induces on the strings. However, not every permutation of the $n$ strings corresponds to a simple element.

**Definition 14.** Let $M$ be a Garside monoid with Garside group $G$. The **greedy** (respectively, **rational** length of an element $x \in G$ is the sum of the minimal lengths of all simple elements (including the inverted ones) in the greedy (respectively, rational) normal form of $x$. 

Specifically, if the greedy normal form of $x$ is $\delta^k s_1 \cdots s_r$, then the greedy length of $x$ is $k \cdot \ell(\delta) + \ell(s_1) + \cdots + \ell(s_r)$, and if the rational length of $x$ is $(s_1 \cdots s_k)^{-1} p_1 \cdots p_l$, then the rational length of $x$ is $\ell(s_1) + \cdots + \ell(s_k) + \ell(p_1) + \cdots + \ell(p_l)$.

**Proposition 15.** For each $a \in M$, $\ell(a^r) = \ell(a)$.

**Proof.** Let $n = \ell(a)$, and $a = a_1 \cdots a_n$ with $a_1, \ldots, a_n$ atoms. Then $a^r = a_1^r \cdots a_n^r$. As conjugation by $\delta$ moves atoms to atoms, $\ell(a^r) \leq n = \ell(a)$. Similarly, if $m = \ell(a^s)$ and $a^s = b_1 \cdots b_m$ with $b_1, \ldots, b_m$ atoms, then $a = a^s \delta^{-1} = b_1^{\delta^{-1}} \cdots b_m^{\delta^{-1}}$, and as conjugation by $\delta$ moves atoms to atoms, $\ell(a) \leq m = \ell(a^s)$. \hfill \square

The presentation in the previous section of the rational normal form in terms of the greedy normal form gives the following.

**Corollary 16.** The rational length of an element with greedy normal form $\delta^{-m} s_1 \cdots s_r$, where $0 < m \leq r$, is

$$
\ell(s_1^{-1}\delta) + \cdots + \ell(s_m^{-1}\delta) + \ell(s_{m+1}) + \cdots + \ell(s_r),
$$

and similarly for the cases where $m \leq 0$ or $0 < r < m$.

**Corollary 17.** If the relations of $M$ are length-preserving, then the rational length of an element with greedy normal form $\delta^k s_1 \cdots s_r$ can be obtained by removing $2 \sum_{i=1}^{\min(r,k)} \ell(s_i)$ from its greedy normal length.

**Proof.** If the relations of $M$ are length-preserving, we have that $\ell(ab) = \ell(a) + \ell(b)$ for all $a, b \in M$, and thus for simple $s$, $\ell(\delta) = \ell(s) + \ell(s^{-1}\delta)$, that is, $\ell(s^{-1}\delta) = \ell(\delta) - \ell(s)$. \hfill \square

This shows, in particular, that the length function considered in [8, 9] in the case of the Artin presentation of $B_N$ is in fact the rational length for the Artin presentation of $B_N$. This was first pointed out to us by Dehornoy.

### 4.1. Quasi-geodesics in Garside groups.

Even when the relations are length-preserving, it is generally not the case that an efficient algorithm for computing the minimal length $\ell(x)$ is available. Even if the monoid relations are length-preserving, finding $\ell(x)$ for $x$ not in the monoid (nor in its inverse) may be a difficult task. Indeed, assuming $P \neq NP$, there is no polynomial-time algorithm computing $\ell(x)$ with respect to the Artin presentation of $B_N$, for arbitrary $N$ and $x \in B_N$ [14]. Fortunately, in Garside groups $\ell(x)$ can be approximated. For simplicity, we treat the case of length-preserving relations, so that $\ell$ is easy to compute on positive elements.
Theorem 18. Let $M$ be a Garside monoid with length preserving relations and fundamental element $\delta$, and let $G$ be its fractions group. For each $x \in G$:

1. If $x \in M$, then $\ell_G(x) = \ell_R(x) = \ell(x)$.
2. If $x \in M^{-1}$, then $\ell_R(x) = \ell(x)$.
3. $\ell(x) \leq \ell_R(x) \leq \ell_G(x) \leq 2(\ell(\delta) - 1)\ell(x)$.
4. $\ell_R(x) \leq (\ell(\delta) - 1)\ell(x)$.

Moreover, these bounds cannot be improved.

Proof. (1) For $x \in M$, each normal form gives some positive presentation of $x$, and thus the corresponding length is the same as the minimal length.

(2) Fix $x \in M^{-1}$. Then $\ell_R(x) = \ell_R(x^{-1})$, and by (1), $\ell_R(x^{-1}) = \ell(x^{-1}) = \ell(x)$.

(3) The first inequality is clear. The second follows from Corollary 17. We prove the third. Let

\[ x = a_1^{\epsilon_1} \cdots a_m^{\epsilon_m} \]

with $m = \ell(x)$, $a_1, \ldots, a_m$ atoms, and $\epsilon_1, \ldots, \epsilon_m \in \{1, -1\}$. For each atom $a$, let $\bar{a}$ be the simple element such that $\bar{a}a = \delta$. Then $a^{-1} = \delta^{-1}\bar{a}$.

Rewrite each negative atom in the equation in this form, and move all occurrences of $\delta^{-1}$ to the left, using the relation $a\delta^{-1} = \delta^{-1}a\delta^{-1}$. Let $n = |\{i : \epsilon_i = -1\}|$. We obtain a presentation

\[ x = \delta^{-n}b_1 \cdots b_m, \]

with each $b_i$ being (up to an application of $\tau$) an integer number of times, which preserves length by Proposition 15) $a_i$ if $\epsilon_i = 1$, and $\bar{a}_i$ otherwise. In particular, $\ell(b_i) = 1$ if $\epsilon_i = 1$, and $\ell(\bar{a}_i) = \ell(\delta) - 1$ otherwise.

Let $\delta^ks_1 \cdots s_j$ be the left-weighted form of $b_1 \cdots b_m$. Then the greedy normal form of $x$ is $\delta^{-n+k}s_1 \cdots s_j$, which cannot be longer than $\delta^{-n}\delta^k s_1 \cdots s_j$. As expressions of positive elements all have the same length, the length of $\delta^ks_1 \cdots s_j$ is exactly that of $b_1 \cdots b_m$. Thus,

\[
\ell_G(x) \leq n\ell(\delta) + \ell(b_1 \cdots b_m) = n\ell(\delta) + \ell(b_1 \cdots b_m) = n\ell(\delta) + n(\ell(\delta) - 1) + (m - n) = n(2\ell(\delta) - 2) + m \leq (2\ell(\delta) - 1)m,
\]

as $m \leq n$.\footnote{The step before last is added to emphasize that for random words, the upper bound is far from being optimal. Indeed, in this case we have $n \approx m/2$, which gives roughly half of the mentioned bound. There is an elbow room for improvements in the random case.}
This can be proved as in the proof of (3). Alternatively, one can use Charney’s Theorem [3], extended to general Garside groups by Dehornoy and Paris [5], that the number of simple elements in the rational normal form is minimal amongst presentations of \( x \) as a product of simple elements (possibly inverted): If \( x \in M^{\pm 1} \), we can use (1) or (2) and there is nothing to prove. Otherwise, let \( x = a_{1}^{y_{1}} \cdots a_{m}^{y_{m}} \) be a minimal presentation of \( x \). In particular each \( a_{i}^{y_{i}} \) is a (possibly inverted) simple element. Thus, the number \( n \) of simple elements in the rational form of \( x \) is at most \( m \). As \( x \notin M^{\pm 1} \), no simple element in the rational form of \( x \) is \( \delta \). It follows that \( \ell_{R}(x) \leq (\ell(\delta) - 1)m \).

(1) shows that the lower bounds cannot be improved. To see that the mentioned upper bounds cannot be improved, consider \( \ell_{G}(a^{-m}) \) and \( \ell_{R}(a^{m}b^{-m}) \) for \( m \) positive and distinct non-commuting atoms \( a, b \).

Theorem 18 shows that \( \ell_{R} \) gives a better approximation than \( \ell_{G} \), and gives a theoretical motivation for the results described in [8]. Having both experimental [8] and theoretical evidence for the superiority of \( \ell_{R} \) over \( \ell_{G} \), we concentrate henceforth on the former.

### 4.2. Quasi-geodesics in embedded Garside groups

We need not stop here, and may consider, as in the case of \( B_{N} \), two distinct Garside structures of the same group, such that one of them embeds in the other. Let \( M_{1}, M_{2} \) be Garside monoids with fundamental elements \( \Delta, \delta \), respectively, such that each atom of \( M_{1} \) is also an atom of \( M_{2} \), and the group of fractions of \( M_{1} \) coincides with that of \( M_{2} \). Then we may take a length in one Garside structure as an estimation for the length in the other. We will denote the used structure by a superscripted index. By Theorem 18

\[
\ell_{R}^{2}(x) \leq (\ell^{2}(\delta) - 1)\ell^{2}(x) \leq (\ell^{2}(\delta) - 1)\ell^{1}(x);
\]

\[
\ell_{R}^{1}(x) \leq (\ell^{1}(\Delta) - 1)\ell^{1}(x).
\]

Thus, if \( \ell^{2}(\delta) < \ell^{1}(\Delta) \), \( \ell_{R}^{2}(x) \) has a smaller approximation factor at its upper bound.

For the lower bound, let \( A_{2} \) be the set of atoms of \( M_{2} \), and set

\[
\alpha = \max\{\ell^{1}(a) : a \in A_{2}\}.
\]

Then \( \ell^{1}(x) \leq \alpha \ell^{2}(x) \), and thus

\[
\ell^{1}(x) \leq \alpha \ell^{2}(x) \leq \alpha \ell_{R}^{2}(x).
\]

This gives the following.
Theorem 19. In the above notation,

\[
\frac{1}{\alpha} \ell^1(x) \leq \ell^2_R(x) \leq (\ell^2(\delta) - 1)\ell^1(x). \quad \Box
\]

The advantage of Theorem 19 is that the distortion factors are symmetrized around the used length function \(\ell^2_R(x)\). Our main application is the following.

4.3. The case of the braid group. Consider the braid group as generated by the Artin monoid \(B_N^+\) as well as by the BKL monoid \(BKL_N^+\) (Examples 12–13), and let \(\Delta\) and \(\delta\) be their respective fundamental elements. Consider the minimal lengths \(\ell^1\) for the Artin structure, and \(\ell^2\) for the BKL structure of \(B_N\), respectively.

\[
\ell^1(\Delta) = N(N - 1)/2, \quad \text{whereas} \quad \ell^2(\delta) = N - 1.
\]

For each atom \(a_{t,s}\) of \(BKL_N^+\), \(\ell^1(a_{t,s}) \leq 2(t - s - 1) + 1 = 2(t - s) - 1\). In particular, the maximum \(\alpha\) of all these lengths satisfies

\[
\alpha \leq 2N - 3.
\]

By Theorem 19, we have that \(\ell^2_R\), the length in BKL generators of the rational normal form in the BKL structure of \(B_N\), is quite symmetrical close to the minimal Artin length:

**Corollary 20.** For each \(x \in B_N\):

\[
\frac{1}{2N - 3} \ell^1(x) \leq \ell^2_R(x) \leq (N - 2)\ell^1(x). \quad \Box
\]

For comparison, measuring the minimal Artin length by working solely with the Artin structure of \(B_N\), we only have (by Theorem 18):

\[
\ell^1(x) \leq \ell^1_R(x) \leq (\ell^1(\Delta) - 1)\ell^1(x) = \frac{N^2 - N - 2}{2} \ell^1(x).
\]

The gain may be viewed as follows: In the latter case, we have a constant (in \(N\)) error factor from below, and quadratic error from above. In Corollary 20, both errors are linear, that is, the errors are symmetrized by dividing by \(O(N)\) terms.

Another matter, which we cannot prove at present, is that the lower bound in Corollary 20 seems to be a big underestimate in the generic case. It seems to us that in the generic case, the lower bound factor should not be much smaller than 1 (indeed, it may be greater than 1).

In summary, we have theoretical evidence suggesting that estimating the minimal length in Artin generators by using rational BKL normal form should be better than the same estimation using rational Artin normal form. We now verify this with experimental results.
5. Experimental results

5.1. Initial experiments. For the Artin presentation, it is shown in [8] that the rational Artin length is much better than greedy Artin length, at least with regards to solving random equations with difficult parameters. Our initial experiments showed that this is also the case for the BKL presentation: The rational BKL length is better than greedy BKL length.

In the initial phase of this project, we have compared various length functions induced by various alternative ways of measuring lengths of elements, and found out that only the rational BKL length outperforms the rational Artin length when the problem’s parameters are getting difficult. The remainder of this report is therefore dedicated to the comparison of these two leading candidates.

5.2. A detailed comparison. We adopt the basic framework of [1, 9, 8]: The equations are in a finitely generated group $G = \langle a_1, \ldots, a_{\text{ng}} \rangle \leq B_{\text{ns}}$, where $\text{ns}$ denotes the number of strings and $\text{ng}$ denotes the number of generators of $G$. Each generator $a_i$ is a word in $B_{\text{ns}}$ obtained by multiplying $\text{wl}$ (word length) independent uniformly random elements of $\{\sigma_1, \ldots, \sigma_{\text{ns}-1}\}^{\pm 1}$. In $G$, we build a sentence $X$ of length $\text{sl}$ (sentence length):

$$X = a_1 a_2 \cdots a_{\text{sl}}$$

(For the while, we restrict $\text{sl} \leq \text{ng}$).

We begin with a description of a test suitable for groups $G$ which are close to being free. For each $i \in \{1, \ldots, \text{ng}\}$ and each $\epsilon \in \{1, -1\}$, we give the generator $a_i^\epsilon$ the score $\ell(a_i^\epsilon X)$, sort the generators according to their scores (position 1 is for the shortest length), and reorder each block of identical scores by applying a random permutation. We then keep in a histogram the position of $a_1$.

We do one such computation for each sample of $G$ and $X$.

While $a_1 a_2 \cdots a_{\text{sl}}$ is not the way a random $\text{sl}$ sentence in $G$ was defined, this does not make the problem easier: We use each group $G$ to produce only one such sentence.

To partially compensate for the fact that $G$ need not be free, we do the following. There could be several $i \in \{1, \ldots, \text{ng}\}$ such that $X = a_i a_1 \cdots a_{i-1} a_{i+1} \cdots a_{\text{sl}}$. Let $\text{COR}$ denote the set of these $a_i$, the correct first generators. After sorting all generators as above, instead of looking for the position of $a_1$, we look at the lowest position an element of $\text{COR}$ attained.
Remark 21. A more precise, but infeasible, way to construct COR would be to find all shortest presentations of $X$ as a product of elements from \{a_1, \ldots, a_m\}^\pm, and let COR be the set of the first generators in these presentations. For the parameters we have checked, we believe that this should not make a big difference. The results in Section 5.6 support this hypothesis.

We have also checked one case where $\text{SL} > \text{NG}$. In this case we defined

$$X = a_{i_1} a_{i_2} \cdots a_{i_{\text{SL}}},$$

where $i_j = (j - 1 \mod \text{NG}) + 1$ for $j = 1, \ldots, \text{SL}$, and made the obvious adjustments.

In summary, for each set of parameters $(\text{NS}, \text{WL}, \text{NG}, \text{SL})$ mentioned below, and for $\ell$ being either the rational Artin or the rational BKL length, we have repeated the following at least 1,000 times: Choose $a_1, \ldots, a_{\text{NG}}$, compute $X$, compute COR, sort all generators $a_i^\epsilon$ according to the lengths $\ell(a_i^\epsilon X)$, find the lowest position attained by an element of COR, and store this position number in the histogram.

After dividing the numbers in the histogram by the numbers of samples made, we obtain the distribution of the best position of a correct generator. In light of the intended application described in the first two sections, a natural measure to the effectiveness of $\ell$ is the graph of the accumulated probability, showing for each $x = 1, \ldots, 2 \text{NG}$ the probability that some correct generator attained a position $\leq x$.

The results of our experiments are divided into 4 sets such that in each set of experiments, only one parameter varies. This shows the effect of that parameter on the difficulty of the problem. The varying parameter takes 3 possible values, so we have 3 pairs (since there are two length functions) of graphs. Each pair of graphs has its own line style, so to allow plotting all 6 graphs on the same figure.

For all pairs, one of the graphs is always above or almost the same as the other. Fortunately, in all cases, it is the rational BKL length which is above the rational Artin length, so there is no need to supply this information in the figure.

Finally, since the accumulated distributions all reach 1 for $x = 2\text{NG}$, the graphs are more interesting for the smaller values of $x$. We therefore plot only the first 35 values of $x$.

5.3. **When the sentence length varies.** Fix $\text{NS} = 64, \text{WL} = 8, \text{NG} = 128$. Figure 1 shows the accumulated probabilities for $\text{SL} \in \{32, 64, 128\}$. 
5.4. **When the word length varies.** For \( \text{NS} = \text{SL} = 64, \text{NG} = 128, \) and \( \text{WL} \in \{8, 16, 32\} \), we obtain the graphs in Figure 2. The problem gets easier when \( \text{WL} \) increases, since this way \( G \) gets closer to a free group (where the length approach is optimal). The remarkable observation is that the harder the problem becomes (by making \( \text{WL} \) smaller), the greater the improvement of the rational BKL length over the rational Artin length becomes.

5.5. **When the number of generators varies.** Now set \( \text{NS} = \text{SL} = 64, \text{WL} = 8, \) and let \( \text{NG} \in \{32, 64, 128\} \). The graphs appear in Figure 3. Here too, the more difficult the problem becomes (by increasing the number of generators), the greater the advantage of BKL over Artin is. Moreover, the graphs show that doubling \( \text{NG} \) has little influence on the performance of the rational BKL length, whereas it seriously degrades the performance of the rational Artin length.

5.6. **When the number of strings varies.** Finally, set \( \text{WL} = 8, \text{SL} = 64, \text{NG} = 128, \) and let \( \text{NS} \in \{16, 32, 64\} \). Here, the problem becomes easier when we increase \( \text{NS} \) (Figure 4). This is *not* in accordance with earlier results in [8, 9], and is perhaps due to the fact that we allow any correct generator, whereas in the earlier works we only counted \( \alpha_1 \) a success. Indeed, the more strings there are, the greater the chances are that words of length 8 commute. On the other hand, the graphs

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**Figure 1.** When SL varies
show that while the BKL approach benefits a great deal when the number of strings is doubled, this is not quite so for the Artin approach.
This means that the improvement in success rates due to commuting generators is not substantial.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{When \textit{ns} varies}
\end{figure}

6. Concluding remarks and proposed future research

Memory-length algorithms give a powerful heuristic method to solve arbitrary equations in noncommutative groups, and consequently a variety of otherwise intractable problems. These algorithms rely on a good length function on the group in question. In the past, greedy Artin length was used as a length function on the braid group, and it was realized that rational Artin length gives better results.

In this paper, we suggested to use rational BKL length to measure the minimal Artin length, and gave theoretical as well as experimental evidence for the advantage of the new function over rational Artin length, at least when randomization is modelled as in \cite{1}.

The main drawback in our estimations is that they give much larger lengths than the minimal length. Some interesting directions for possible improvements are:

(1) As we have seen, the rational form can be computed from the greedy normal form by “removing” $\delta$-s from the leading simple elements. We may be more greedy, and remove the available $\delta$-s from the (leftmost) longest simple elements in the greedy
normal form. This gives a new normal form in $B_N$, which has shorter length in terms of atoms. The resulting length function may be yet better than the one proposed here.

(2) For each $x$ and each proposal for a length function of $x$, we can take the minimum of the lengths of several elements whose minimal length is not smaller than that of $x$, including: $x$, $x^{-1}$, $x^{d_k}$ for each $k = 1, \ldots, m - 1$, where $m$ is the minimal with $d_m$ central.

(3) Since we use left-oriented normal forms in our estimations, we can also try the corresponding right-oriented normal forms, and take the minimum.

(4) We can iterate conjugation by $d$ and inverses (and other operations which are not increasing the minimal length) with shortening heuristics like Dehornoy handle-reduction. In [12] this was done only to a very limited extent.

(5) In [12], Dehornoy handle-reduction was applied to the greedy normal form to obtain an estimation of the minimal length. We conjecture that applying Dehornoy handle-reduction to the rational normal form would give better estimations.

Acknowledgements. We thank Joan Birman and Dima Ruinskiy for their comments on earlier versions of the paper. We also thank Patrick Dehornoy and Sang Jin Lee for informative discussions concerning our notation.

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**DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF WISCONSIN, MADISON, WI 53706, USA**

*E-mail address: mdhock@gmail.com*

**DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT-GAN 52900, ISRAEL; AND DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL**

*E-mail address: tsaban@math.biu.ac.il*

*URL: http://www.cs.biu.ac.il/~tsaban*