EXTERIOR CONTROLLABILITY PROPERTIES OF A NONLOCAL MOORE–GIBSON–THOMPSON EQUATION

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Abstract. The three concepts of exact, null and approximate controllabilities are analyzed from the exterior of the Moore–Gibson–Thompson equation associated with the fractional Laplace operator subject to the nonhomogeneous Dirichlet type exterior condition. Assuming that $b > 0$ and $\alpha - \frac{2}{N} > 0$, we show that if $0 < s < 1$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded Lipschitz domain, then there is no control function $g$ such that the following system

$$
\begin{aligned}
\begin{cases}
  u_{ttt} + \alpha uu_{tt} + c^2(-\Delta)^s u + b(-\Delta)^s u_t = 0 & \text{in } \Omega \times (0,T), \\
  u = g(x) \chi_{\partial \Omega \times (0,T)} & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
  u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, u_{ttt}(\cdot,0) = u_2 & \text{in } \Omega,
\end{cases}
\end{aligned}
$$

is exact or null controllable at time $T > 0$. However, we prove that for every $0 < s < 1$, the system is indeed approximately controllable for any $T > 0$ and $g \in \mathcal{D}(\partial \Omega \times (0,T))$, where $\partial \Omega \subset \mathbb{R}^N \setminus \Omega$ is an arbitrary non-empty open set.

1. Introduction

In the present work we investigate the following third order nonlocal partial differential equation

$$
\begin{aligned}
\begin{cases}
  u_{ttt} + \alpha uu_{tt} + c^2(-\Delta)^s u + b(-\Delta)^s u_t = 0 & \text{in } \Omega \times (0,T), \\
  u = g(x) \chi_{\partial \Omega \times (0,T)} & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
  u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, u_{ttt}(\cdot,0) = u_2 & \text{in } \Omega,
\end{cases}
\end{aligned}
$$

that we call a linearized nonlocal version of the so called Moore-Gibson-Thompson (MGT) equation [24, 25, 26]. In (1.1), $\Omega \subset \mathbb{R}^N$ is a bounded open set with a Lipschitz continuous boundary $\partial \Omega$, $\alpha$, $b$, $c$ are real numbers, $(-\Delta)^s$ ($0 < s < 1$) is the fractional Laplace operator (see (3.2)), $u = u(x,t)$ is the state to be control and $g = g(x,t)$ is the control function which is localized in an open set $\partial \Omega \subset \mathbb{R}^N \setminus \Omega$.

Despite the wide range of applications of the local MGT equation such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultra-sound cleaning, etc., there have been quite a few works about their controllability properties [27].

For the notion controllability of PDEs from the exterior of the domain where the PDE is solved, the nonlocal case seems to be more suitable to handle because, on the one hand, the associated stationary (time independent) system is ill-posed (see e.g. [40]) if the control function $g$ is prescribed at the boundary $\partial \Omega$ and, on the other hand, it has been very recently shown by Warma [40] that for nonlocal PDEs associated with the fractional Laplacian, the exterior control is the right notion that replaces the classical boundary control problems (that is, when the control function is localized on a subset $\omega$ of the boundary $\partial \Omega$) associated with local operators such as the Laplace operator or general second order elliptic operators.

On the other hand, the MGT equation (which is originally a nonlinear equation) arises from modeling high amplitude sound waves. The classical nonlinear acoustics models include Kuznetsov’s equation, the

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Westervelt equation and the Kokhlov-Zabolotskaya-Kuznetsov equation. A thorough study of the linearized models is a good starting point for better understanding the well-posedness and asymptotic behaviors of the nonlinear models. We refer to \cite{6, 7, 20, 21, 22, 24, 25, 30} and the references therein for the derivation of the local version of the MGT equation and the physical meaning of the parameters $\alpha$, $c$ and $b$. A complete analysis concerning well-posedness, regularity, stability and asymptotic behavior of solutions has been established in the above mentioned references. However, due to the nature of the applications, it is desirable to know how the dynamics of the model changes by means of external controls or forces.

We shall show that if $b > 0$ and $(u_0, u_1, u_2)$ belongs to a suitable Banach space, then for every function $g \in L^2((0, T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$, the system \eqref{1.1} has a unique weak solution $(u, u_t, u_{tt})$ satisfying the regularity $u \in C([0, T]; L^2(\Omega)) \cap C^2((0, T]; W^{-s,2}(\Omega))$. In such case, the set of reachable states can be defined as follows:

$$\mathcal{R}((u_0, u_1, u_2), T) := \left\{ (u(\cdot, T), u_t(\cdot, T), u_{tt}(\cdot, T)) : g \in L^2((0, T); W^{s,2}(\mathbb{R}^N \setminus \Omega)) \right\}.$$  

The classical three notions of controllability for this system can then be defined as follows.

- We shall say that the system \eqref{1.1} is null controllable at time $T > 0$ if
  $$(0, 0, 0) \in \mathcal{R}((u_0, u_1, u_2), T).$$
- The system will be said to be exact controllable at $T > 0$ if
  $$\mathcal{R}((u_0, u_1, u_2), T) = L^2(\Omega) \times W^{-s,2}(\Omega) \times W^{-s,2}(\Omega).$$
- Finally we will say that the system is approximately controllable at $T > 0$ if
  $$\mathcal{R}((u_0, u_1, u_2), T)$$
  is dense in $L^2(\Omega) \times W^{-s,2}(\Omega) \times W^{-s,2}(\Omega)$.

From the above definitions, it is easy to see that null or exact controllability implies approximate controllability. We refer to Section 3 for the definition of the spaces involved.

Our first main result states that if $b > 0$ and $\alpha - \frac{d}{2} > 0$, then the system \eqref{1.1} is not exact or null controllable at time $T > 0$. As a substitute, we obtain that the system is indeed approximately controllable at any $T > 0$ and for every $g \in \mathcal{D}(\mathcal{O} \times (0, T))$ where $\mathcal{O}$ is an arbitrary non-empty open subset of $\mathbb{R}^N \setminus \Omega$. This is the best possible result that can be obtained regarding the controllability of the system \eqref{1.1}.

We remark that in our study of controllability, we shall assume that $b > 0$ because if $b = 0$, then the system \eqref{1.2} is ill-posed \cite{21} and, if $b < 0$, then we lost the good behavior of the eigenvalues of the fractional Laplacian operator.

As far as we know, the present work is the first one that analyses the controllability properties for the nonlocal MGT equation using an exterior control function $g$. We can mention that our recently paper \cite{27} is the first work dealing with the interior controllability issues for the local MGT equation using the concept of moving control, but associated with the Laplace operator.

When $g = 0$, letting $(-\Delta)^s_D$ be the realization of $(-\Delta)^s$ in $L^2(\Omega)$ with the zero exterior condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$, then the associated system can be written as the following evolution system:

$$\begin{aligned}
\begin{cases}
u_{ttt} + \alpha u_{tt} + c^2 (-\Delta)^s_D u + b (-\Delta)^s_D u_t = 0 & \text{in } \Omega \times (0, T), \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, u_{tt}(\cdot, 0) = u_2 & \text{in } \Omega.
\end{cases}
\end{aligned}$$

The well-posedness of an abstract version of \eqref{1.2} with $(-\Delta)^s_D$ replaced with a generic self-adjoint operator $A$ with domain $\mathcal{D}(A)$ dense in a Hilbert space $H$ has been completely examined in \cite{6, 7, 20, 21, 22, 24, 25, 26, 30} and their references by using semigroup methods. Some nonlinear models and some versions including memory terms have been also intensively studied by Lasiecka and Wang \cite{24, 25, 26} where they have obtained some fundamental and beautiful results.

We notice that $(-\Delta)^s_D$ is a self-adjoint operator in $L^2(\Omega)$ with dense domain and has a compact resolvent (see Section 3), hence it enters in the framework of the above mentioned references. However, in \eqref{1.1} we have a non zero exterior condition which did not satisfy the conditions contained in the above references. For this reason, in the present article we shall also include new results of existence and regularity of solutions to our nonhomogeneous system.
In contrast, we observe that with the non zero exterior condition \( u = g \in \mathbb{R}^N \setminus \Omega \), the associated operator \((-\Delta)^s\) is no longer a generator of a \(C_0\)-semigroup and hence semigroup methods cannot be used directly to prove the well-posedness of the system (1.1). This makes the study of (1.1) harder than the zero exterior condition. To overcome this difficulty, we shall exploit a new technique which has been developed by Warma in [28, 40, 41] to solve fractional diffusion equations, fractional super diffusive equation and strong damping wave equations. This original method shall allow us not only to prove well-posedness but also to have an explicit representation of solutions in terms of series which is crucial for the analysis of the controllability of the system.

To summarize, the main novelties of the present paper can be formulated as follows.

1. For the first time, a nonlocal version of the MGT equation associated with the fractional Laplacian operator with non-zero exterior condition has been studied. Some well-posedness results and an explicit representation of solutions in terms of series of the nonhomogeneous exterior value nonlocal evolution system (1.1) have been established.
2. We have shown that the system is not null or exact controllable at time \( T > 0 \).
3. The unique continuation property of solutions to the adjoint system associated with (1.1) has been established. This result is obtained by carefully exploiting the unique continuation property for the eigenvalues problem of \((-\Delta)^s\) recently obtained in [40] and by using some powerful tools from PDEs and complex analysis.
4. The final important result is the approximate controllability of the system which is a direct consequence of the unique continuation property of the dual system.

Fractional order operators have emerged as a modeling alternative in various branches of science. For instance, a number of stochastic models for explaining anomalous diffusion have been introduced in the literature; among them we quote the fractional Brownian motion; the continuous time random walk; the Lévy flights; the Schneider grey Brownian motion; and more generally, random walk models based on evolution equations of single and distributed fractional order in space (see e.g. [10, 16, 29, 35, 42]). In general, a fractional diffusion operator corresponds to a diverging jump length variance in the random walk. We refer to [8, 36] and the references therein for a complete analysis, the derivation and the applications of the fractional Laplace operator. For further details we also refer to [12, 13] and their references.

The remaining of the paper is structured as follows. In Section 2 we state the main results of the article. The first one (Theorem 2.3) says that if \( b > 0 \) and \( \alpha - \frac{s^2}{4} > 0 \), then the system (1.1) is not exact or null controllable at time \( T > 0 \). Our second main result (Theorem 2.6) is the unique continuation property for the adjoint system associated with (1.1). The third main result (Theorem 2.7) states that the system is approximately controllable at any \( T > 0 \) and for every \( g \in \mathcal{D}(0 \times (0, T)) \). This last result will be obtained as a direct consequence of the above mentioned unique continuation property. In Section 3 we introduce the function spaces needed throughout the paper, give a rigorous definition of the fractional Laplacian and some known results that will be used in the proofs of our main results. The proofs of the well-posedness and an explicit representation of solutions to the system (1.1) and the associated dual system are contained in Section 4. Finally, in Section 5 we give the proofs of our main results on controllability.

2. Main results

In this section we state the main results of the article. Throughout the remainder of the paper, without any mention, \( \alpha, b, c \) and \( 0 < s < 1 \) are real numbers and \( \Omega \subset \mathbb{R}^N \) denotes a bounded open set with a Lipschitz continuous boundary. Given a measurable set \( E \subset \mathbb{R}^N \), we shall denote by \( \langle \cdot, \cdot \rangle_{L^2(E)} \) the scalar product in \( L^2(E) \). We refer to Section 3 for a rigorous definition of the function spaces and operators involved. Let \( W_0^{-s,2}(\Omega) \) be the energy space and denote by \( W^{-s,2}(\Omega) \) its dual. We shall denote by \( \langle \cdot, \cdot \rangle_{-\frac{s}{2}, \frac{s}{2}} \) the duality pair between \( W^{-s,2}(\Omega) \) and \( W_0^{s,2}(\Omega) \) (see Section 3).
Next, we introduce our notion of solution. Let \((u_0, u_1, u_2) \in L^2(\Omega) \times W^{-s,2}(\Omega) \times W^{-s,2}(\Omega)\) and consider the following two systems:

\[
\begin{align*}
  v_{ttt} + \alpha v_{tt} + c^2(-\Delta)^s v + b(-\Delta)^s v_t &= 0 & \text{in } \Omega \times (0,T), \\
  v &= g & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
  v(\cdot,0) &= 0, \quad v_t(\cdot,0) = 0, \quad v_{tt}(\cdot,0) = 0 & \text{in } \Omega,
\end{align*}
\]  

(2.1)

and

\[
\begin{align*}
  w_{ttt} + \alpha w_{tt} + c^2(-\Delta)^s w + b(-\Delta)^s w_t &= 0 & \text{in } \Omega \times (0,T), \\
  w &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
  w(\cdot,0) &= u_0, \quad w_t(\cdot,0) = u_1, \quad w_{tt}(\cdot,0) = u_2 & \text{in } \Omega.
\end{align*}
\]  

(2.2)

Then it is clear that \(u = v + w\) solves the system (1.1). We next introduce our notion of weak solution to the system (2.1).

**Definition 2.1.** Let \(g\) be a given function. A function \((v, v_t, v_{tt})\) is said to be a weak solution of (2.1), if the following properties hold.

- **Regularity:**
  \[
  \begin{align*}
  v &\in C([0,T];L^2(\Omega)) \cap C^2([0,T];W^{-s,2}(\Omega)), \\
  v_{ttt} &\in C((0,T);W^{-s,2}(\Omega)).
  \end{align*}
  \]  

(2.3)

- **Variational identity:** For every \(w \in W^{s,2}_0(\Omega)\) and a.e. \(t \in (0,T)\),
  \[
  \langle v_{ttt} + \alpha v_{tt}, w \rangle + \langle (-\Delta)^s(c^2 v + bv), w \rangle = 0.
  \]

- **Initial and exterior conditions:**
  \[
  v(\cdot,0) = 0, \quad v_t(\cdot,0) = 0, \quad v_{tt}(\cdot,0) = 0 \quad \text{in } \Omega \quad \text{and} \quad v = g \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T).
  \]

By Definition 2.1, for a weak solution \((v, v_t, v_{tt})\) of (2.1), we have that

\[
(v(\cdot,T), v_t(\cdot,T), v_{tt}(\cdot,T)) \in L^2(\Omega) \times W^{-s,2}(\Omega) \times W^{-s,2}(\Omega).
\]

Concerning existence and uniqueness of weak solutions to the system (1.1) we have the following result.

**Theorem 2.2.** For every \((u_0, u_1, u_2) \in W^{s,2}_0(\Omega) \times W^{s,2}_0(\Omega) \times L^2(\Omega)\) and \(g \in \mathcal{D}(\mathbb{R}^N \setminus \Omega) \times (0,T)\), the system (1.1) has a unique weak solution \((u, u_t, u_{tt})\) given by

\[
u(x,t) = \sum_{n=1}^{\infty} \left( A_n(t)u_{0,n} + B_n(t)u_{1,n} + C_n(t)u_{2,n} \right) \varphi_n(x)
\]

\[
+ \sum_{n=1}^{\infty} \left( \int_0^t \left( g(\cdot,\tau), N_n \varphi_n \right)_{L^2(\mathbb{R}^N \setminus \Omega)} \frac{1}{\lambda_n} \left( C_n''(t - \tau) + \alpha C_n''(t - \tau) \right) d\tau \right) \varphi_n(x),\n\]

where \((\varphi_n)_{n \in \mathbb{N}}\) is the orthonormal basis of eigenfunctions of the operator \((-\Delta)^s_D\) associated with the eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) and \(A_n(t), B_n(t), C_n(t)\) are defined in Proposition 4.3 below.

Our first main result concerning controllability says that if \(b > 0\) and \(\alpha - \frac{c^2}{b} > 0\), then the system (1.1) is not exact or null controllable.

**Theorem 2.3.** Let \(b > 0\) and \(\alpha - \frac{c^2}{b} > 0\). Then the system (1.1) is not exact or null controllable at time \(T > 0\).

Since (1.1) is not exact or null controllable if \(b > 0\) and \(\alpha - \frac{c^2}{b} > 0\), then we shall study if it can be approximately controllable. It is straightforward to verify that the study of the approximate controllability of (1.1) can be reduced to the case \(u_0 = u_1 = u_2 = 0\). We refer to [23, 28, 34, 39, 40, 43] for more details.

From now on, without any mention, we shall assume that
Definition 2.4. Let \( \psi \) be viewed as the dual system associated with (2.1). Our notion of weak solution to (2.4) is as follows.

\[
\begin{aligned}
\text{Definition 2.4. Let } \psi \in W_0^{s,2}(\Omega) \times W_0^{s,2}(\Omega) \times L^2(\Omega). \text{ A function } (\psi, \psi_t, \psi_{tt}) \text{ is said to be a weak solution of (2.4), if for a.e. } t \in (0, T), \text{ the following properties hold.}
\end{aligned}
\]

- Regularity and final data:
  \[
  \psi \in C^1([0, T]; W_0^{s,2}(\Omega)) \cap C^2([0, T]; L^2(\Omega)),
  \]
  \[
  \psi_{tt} \in C((0, T); W^{-s,2}(\Omega))
  \]
  \[
  \psi(\cdot, T) = \psi_0, \ \psi_t(\cdot, T) = \psi_1 \text{ and } \psi_{tt}(\cdot, T) = \psi_2 \text{ in } \Omega.
  \]
- Variational identity: For every \( w \in W_0^{s,2}(\Omega) \) and a.e. \( t \in (0, T) \),
  \[
  \langle \psi_{tt} + \alpha \psi_t, w \rangle_{\frac{1}{2} +} + \langle (-\Delta)^s(c^2 \psi - b \psi), w \rangle_{\frac{1}{2} +} = 0.
  \]

We have the following existence result.

Theorem 2.5. For every \( (\psi_0, \psi_1, \psi_2) \in W_0^{s,2}(\Omega) \times W_0^{s,2}(\Omega) \times L^2(\Omega) \), the dual system (2.4) has a unique weak solution \( (\psi, \psi_t, \psi_{tt}) \) given by
\[
\psi(x, t) = \sum_{n=1}^{\infty} \left( \psi_{0,n} A_n(T - t) - \psi_{1,n} B_n(T - t) + \psi_{2,n} C_n(T - t) \right) \phi_n(x),
\]

where \( A_n(t), B_n(t) \) and \( C_n(t) \) are given in (4.7), (4.8) and (4.9), respectively. In addition the following assertions hold.

(a) There is a constant \( C > 0 \) such that for all \( t \in [0, T] \),
\[
\|\psi(\cdot, t)\|_{W_0^{s,2}(\Omega)}^2 + \|\psi_t(\cdot, t)\|_{W_0^{s,2}(\Omega)}^2 + \|\psi_{tt}(\cdot, t)\|_{L^2(\Omega)}^2 \leq C \left( \|\psi_0\|_{W_0^{s,2}(\Omega)}^2 + \|\psi_1\|_{W_0^{s,2}(\Omega)}^2 + \|\psi_2\|_{L^2(\Omega)}^2 \right),
\]
and
\[
\|\psi_{tt}(\cdot, t)\|_{W^{-s,2}(\Omega)}^2 \leq \left( \|\psi_0\|_{W_0^{s,2}(\Omega)}^2 + \|\psi_1\|_{W_0^{s,2}(\Omega)}^2 + \|\psi_2\|_{L^2(\Omega)}^2 \right).
\]

(b) We have that \( \psi \in L^\infty((0, T); D((-\Delta)_D^s)) \).

(c) The mapping \( N_\varepsilon : [0, T] \ni t \mapsto N_\varepsilon \psi(\cdot, t) \in L^2(\mathbb{R}^N \setminus \Omega) \),

\[
\text{can be analytically extended to the half-plane } \Sigma_T := \{ z \in \mathbb{C} : \text{Re}(z) < T \}.
\]
Here, \( N_\varepsilon \psi \) is the nonlocal normal derivative of \( \psi \) defined in (3.7) below.

The next result says that the adjoint system (2.4) satisfies the unique continuation property for evolution equations which is our second main result.

Theorem 2.6. Let \( (\psi_0, \psi_1, \psi_2) \in W_0^{s,2}(\Omega) \times W_0^{s,2}(\Omega) \times L^2(\Omega) \) and let \( (\psi, \psi_t, \psi_{tt}) \) be the unique weak solution of (2.4). Let \( \Omega \subset \mathbb{R}^N \setminus \Omega \) be an arbitrary non-empty open set. If \( N_\varepsilon \psi = 0 \) in \( \Omega \times (0, T) \), then \( \psi = 0 \) in \( \Omega \times (0, T) \).

The last main result concerns the approximate controllability of (1.1).
**Theorem 2.7.** The system (1.1) is approximately controllable for any $T > 0$ and $g \in D(\mathbb{O} \times (0, T))$, where $\mathbb{O} \subset \mathbb{R}^N \setminus \Omega$ is an arbitrary non-empty open set. That is,

$$
\mathcal{R}((0, 0, T)^* L^2(\mathbb{O}) \times W^{-s, 2}(\mathbb{O})) = L^2(\mathbb{O}) \times W^{-s, 2}(\mathbb{O}),
$$

where $(u, u_t, u_{tt})$ is the unique weak solution of (1.1) with $u_0 = u_1 = u_2 = 0$.

**3. Preliminaries**

In this section we give some notations and recall some known results as they are needed in the proof of our main results.

We start with the fractional order Sobolev spaces. Given $0 < s < 1$, we let

$$
W^{s, 2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\},
$$

and we endow it with the norm

$$
\|u\|_{W^{s, 2}(\Omega)} := \left( \int_\Omega |u(x)|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
$$

We set

$$
W^{s, 2}_0(\Omega) := \{ u \in W^{s, 2}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}.
$$

For more information on fractional order Sobolev spaces, we refer to [8, 17, 19, 37].

Next, we give a rigorous definition of the fractional Laplace operator. Let

$$
\mathcal{L}^1_s(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable, } \int_{\mathbb{R}^N} \frac{|u(x)|}{(1 + |x|)^{N+2s}} \, dx < \infty \right\}.
$$

For $u \in \mathcal{L}_s^1(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$
(-\Delta)_s^\varepsilon u(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x - y| > \varepsilon\}} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N,
$$

where $C_{N,s}$ is a normalization constant given by

$$
C_{N,s} := \frac{s2^{2s} \Gamma \left( \frac{2s + N}{2} \right)}{\pi^{\frac{N}{2}} \Gamma(1 - s)}.
$$

The **fractional Laplacian** $(-\Delta)^s$ is defined by the following singular integral:

$$
(-\Delta)^s u(x) := C_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy = \lim_{\varepsilon \to 0} (-\Delta)^\varepsilon_s u(x), \quad x \in \mathbb{R}^N,
$$

provided that the limit exists. We notice that $\mathcal{L}^1_s(\mathbb{R}^N)$ is the right space for which $v := (-\Delta)_s^\varepsilon u$ exists for every $\varepsilon > 0$, $v$ being also continuous at the continuity points of $u$. The fractional Laplacian can be also defined as the pseudo-differential operator with symbol $|\xi|^{2s}$. For more details on the fractional Laplace operator we refer to [4, 5, 8, 11, 13, 14, 37, 38] and their references.

Next, we consider the following Dirichlet problem:

$$
\begin{cases}
(-\Delta)^s u = 0 & \text{in } \Omega, \\
u = g & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
$$

**Definition 3.1.** Let $g \in W^{s, 2}(\mathbb{R}^N \setminus \Omega)$ and $\tilde{g} \in W^{s, 2}(\mathbb{R}^N)$ be such that $\tilde{g}|_{\mathbb{R}^N \setminus \Omega} = g$. A $u \in W^{s, 2}(\mathbb{R}^N)$ is said to be a weak solution of (3.3) if $u - \tilde{g} \in W^{s, 2}_0(\mathbb{R}^N)$ and

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = 0, \quad \forall \ v \in W^{s, 2}_0(\mathbb{R}^N).
$$

The following existence result is taken from [18] (see also [15]).
Proposition 3.2. For every $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, there is a unique $u \in W^{s,2}(\mathbb{R}^N)$ satisfying (3.3) in the sense of Definition 3.1. In addition, there is a constant $C > 0$ such that

$$
\|u\|_{W^{s,2}(\mathbb{R}^N)} \leq C\|g\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)}.
$$

Now, we consider the realization of $(-\Delta)^s$ in $L^2(\Omega)$ with the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$. More precisely, we consider the closed and bilinear form

$$
\mathcal{F}(u, v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy, \quad u, v \in W^{s,2}_0(\Omega).
$$

Let $(-\Delta)^s_D$ be the selfadjoint operator in $L^2(\Omega)$ associated with $\mathcal{F}$ in the sense that

$$
\begin{cases}
D((-\Delta)^s_D) = \{ u \in W^{s,2}_0(\Omega), \ f \in L^2(\Omega), \ \mathcal{F}(u, v) = (f, v)_{L^2(\Omega)} \ \forall \ v \in W^{s,2}_0(\Omega) \}, \\
(-\Delta)^s_D u = f.
\end{cases}
$$

More precisely, we have that

$$
D((-\Delta)^s_D) := \{ u \in W^{s,2}_0(\Omega), \ (-\Delta)^s u \in L^2(\Omega) \}, \quad (-\Delta)^s_D u := (-\Delta)^s u.
$$

Then $(-\Delta)^s_D$ is the realization of $(-\Delta)^s$ in $L^2(\Omega)$ with the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$. It is well-known (see e.g. [36, 40]) that $(-\Delta)^s_D$ has a compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ satisfying $\lim_{n \to \infty} \lambda_n = \infty$. In addition, the eigenvalues are of finite multiplicity. Let $(\varphi_n)_{n \in \mathbb{N}}$ be the orthonormal basis of eigenfunctions associated with $(\lambda_n)_{n \in \mathbb{N}}$. Then $\varphi_n \in D((-\Delta)^s_D)$ for every $n \in \mathbb{N}$, $(\varphi_n)_{n \in \mathbb{N}}$ is total in $L^2(\Omega)$ and satisfies

$$
\left\{ \begin{array}{l}
(-\Delta)^s \varphi_n = \lambda_n \varphi_n \quad \text{in} \ \Omega, \\
\varphi_n = 0 \quad \text{in} \ \mathbb{R}^N \setminus \Omega.
\end{array} \right.
$$

With this setting, we have that for $\gamma \geq 0$, we can define the $\gamma$-powers of $(-\Delta)^s_D$ as follows:

$$
\begin{cases}
D((-\Delta)^s_D)^\gamma := \{ u \in L^2(\Omega) : \sum_{n=1}^{\infty} |\lambda_n^\gamma (u, \varphi_n)_{L^2(\Omega)}|^2 < \infty \}, \\
((-\Delta)^s_D)^\gamma u := \sum_{n=1}^{\infty} \lambda_n^\gamma (u, \varphi_n)_{L^2(\Omega)}.
\end{cases}
$$

Using (3.5), we can easily show that $D((-\Delta)^s_D)^{\frac{1}{2}} = W^{s,2}_0(\Omega)$ and for $u \in W^{s,2}_0(\Omega)$ we have that

$$
\|u\|_{W^{s,2}_0(\Omega)}^2 = \sum_{n=1}^{\infty} |\lambda_n^{\frac{1}{2}} (u, \varphi_n)_{L^2(\Omega)}|^2,
$$

defines an equivalent norm on $W^{s,2}_0(\Omega)$. If $u \in D((-\Delta)^s_D)$, then

$$
\|u\|_{D((-\Delta)^s_D)}^2 = \|(-\Delta)^s_D u\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |\lambda_n (u, \varphi_n)_{L^2(\Omega)}|^2.
$$

In addition, for $u \in W^{-s,2}(\Omega)$, we have that

$$
\|u\|_{W^{-s,2}(\Omega)}^2 = \sum_{n=1}^{\infty} |\lambda_n^{-\frac{1}{2}} (u, \varphi_n)_{L^2(\Omega)}|^2.
$$

In that case, using the so called Gelfand triple (see e.g. [1]) we have the following continuous embeddings

$$
W^{s,2}_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-s,2}(\Omega).
$$

Next, for $u \in W^{s,2}(\mathbb{R}^N)$ we introduce the nonlocal normal derivative $\mathcal{N}_s$ given by

$$
\mathcal{N}_s u(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N \setminus \Omega,
$$

where $C_{N,s}$ is the constant given in (3.1). By [15, Lemma 3.2], for every $u \in W^{s,2}(\mathbb{R}^N)$, we have that $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega)$.

The following unique continuation property which shall play an important role in the proof of our main results has been recently obtained in [40, Theorem 3.10].
Lemma 3.3. Let \( \lambda > 0 \) be a real number and \( \emptyset \subset \mathbb{R}^N \setminus \overline{\Omega} \) a non-empty open set. If \( \varphi \in D((-\Delta)^s_D) \) satisfies
\[ (-\Delta)^s_D \varphi = \lambda \varphi \quad \text{in} \quad \Omega \quad \text{and} \quad N_s \varphi = 0 \quad \text{in} \quad \emptyset, \]
then \( \varphi = 0 \) in \( \mathbb{R}^N \).

Remark 3.4. The following important identity has been recently proved in [40, Remark 3.11]. Let \( g \in W^{s,2}((\mathbb{R}^N \setminus \Omega)) \) and \( U_g \in W^{s,2}(\mathbb{R}^N) \) the associated unique weak solution of the Dirichlet problem (3.3). Then
\[ \int_{\mathbb{R}^N \setminus \Omega} g \lambda_n \varphi_n \, dx = -\lambda_n \int_{\Omega} \varphi_n U_g \, dx. \]

For more details on the Dirichlet problem associated with the fractional Laplace operator we refer the interested reader to [2, 3, 4, 5, 18, 32, 33, 37, 40] and their references.

The version given here can be obtained by using a simple density argument (see e.g. [40]).

Proposition 3.5. Let \( u \in W^{s,2}(\mathbb{R}^N) \) be such that \( (-\Delta)^s u \in L^2(\Omega) \). Then for every \( v \in W^{s,2}(\mathbb{R}^N) \), we have
\[ \frac{C_{N,s}}{2} \int_{\mathbb{R}^2N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy, \]
so that for such functions, the identity (3.8) becomes
\[ \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\Omega} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy. \]

We conclude this section with the following observation.

Remark 3.6. We mention the following facts.

(a) Firstly, we notice that
\[ \mathbb{R}^2N \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega). \]

(b) Secondly, if \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \) or \( v = 0 \) in \( \mathbb{R}^N \setminus \Omega \), then
\[ \int_{\mathbb{R}^2N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy, \]
so that for such functions, the identity (3.8) becomes
\[ \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\Omega} v(-\Delta)^s u \, dx + \int_{\mathbb{R}^N \setminus \Omega} v N_s u \, dx. \]

4. Series representation of solutions

In this section we prove a representation in terms of series of weak solutions to the system (2.1) and the dual system (2.4). Evolution equations with non-homogeneous boundary or exterior conditions are in general not so easy to solve since one cannot apply directly semigroup methods due the fact that the associated operator is in general not a generator of a semigroup. For this reason, we shall give more details in the proofs. The representation of solutions in term of series shall play an important role in the proofs of our main results.

Throughout the remainder of the article, without any mention, we shall denote by \( (\varphi_n)_{n \in \mathbb{N}} \) the orthonormal basis of eigenfunctions of the operator \((-\Delta)^s_D\) associated with the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}} \).

We also recall that \( \beta > 0 \) and \( \alpha - \frac{\beta^2}{\alpha^2} > 0 \).

4.1. Series solutions of the system (2.1). We have shown in Section 2 that a solution \((u, u_t, u_{tt})\) of (1.1) can be written as \( u = v + w \) where \((v, v_t, v_{tt})\) solves (2.1) and \((w, w_t, w_{tt})\) is a solution of (2.2).

Consider the system (2.2) which is equivalent to the following system:
\[
\begin{align*}
    w_{ttt} + \alpha w_{tt} + c^2 (-\Delta)^s_D w + b (-\Delta)^s_D w_t &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
    w(\cdot, 0) &= u_0, \quad w_t(\cdot, 0) = u_1, \quad w_{tt}(\cdot, 0) = u_2 \quad \text{in} \quad \Omega.
\end{align*}
\]
Let
\[
W = \begin{pmatrix} w \\ w_t \\ w_{tt} \end{pmatrix} \quad \text{and} \quad W_0 = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}.
\]

Then (4.1) can be rewritten as the following first order Cauchy problem:
\[
\begin{cases}
W_t + AW = 0 & \text{in } \Omega \times (0, T), \\
W(\cdot, 0) = W_0 & \text{in } \Omega,
\end{cases}
\]
where the operator matrix \(A\) with domain \(D(A) = D((-\Delta)_D^s) \times D((-\Delta)_D^s) \times L^2(\Omega)\) is given by
\[
A := \begin{pmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ c^2(-\Delta)_D^s & b(-\Delta)_D^s & \alpha I \end{pmatrix}.
\]

Let \(D([-\Delta]^\gamma_D)\) be the space defined in (3.5) and let
\[
\mathcal{H} := D([-\Delta]^\gamma_D) \times D([-\Delta]^\gamma_D) \times L^2(\Omega) = W_0^{s,2}(\Omega) \times W_0^{s,2}(\Omega) \times L^2(\Omega)
\]
be endowed with the graph norm
\[
\|U_0\|_{\mathcal{H}}^2 = ||(-\Delta)_D^s\frac{1}{2}u_0\|_{L^2(\Omega)}^2 + ||(-\Delta)_D^s\frac{1}{2}u_1\|_{L^2(\Omega)}^2 + ||u_2\|_{L^2(\Omega)}^2.
\]

Noticing that the operator \((-\Delta)_D^s\) enters in the framework of [21], we have the following result taken from [21, Theorem 1.1] which has been proved with a generic operator \(A\).

**Theorem 4.1.** The operator \(-\mathcal{A}\) generates a strongly continuous group in \(\mathcal{H}\). As a consequence, for every \(W_0 := (u_0, u_1, u_2) \in \mathcal{H}\), the system (4.2), hence, the system (4.1), has a unique strong solution \(W\) given by \(W(t) = e^{-t\mathcal{A}}W_0\), where \((e^{-t\mathcal{A}})_{t \geq 0}\) is the strongly continuous semigroup generated by the operator \(-\mathcal{A}\).

Knowing that the system is well-posed, we are interested to have an explicit representation of solutions which is crucial for the study of the controllability of the system.

From the work of Marchand, McDevitt and Triggiani [30], we have the following result.

**Lemma 4.2.** Each pair \(\{\lambda_n, \varphi_n\}\) of eigenvalues and eigenfunctions of \((-\Delta)_D^s\) generates eigenvalues \(\{\lambda_{n,j}\}_{n \in \mathbb{N}}, j = 1, 2, 3\), of \(A\) given as the roots of the following cubic equation:
\[
\lambda_n^3 + \alpha \lambda_n^2 + (\lambda_n b) \lambda_n + \lambda_n c^2 = 0.
\]

Besides, under the condition that \(\gamma := \alpha - \frac{c^2}{b} > 0\), one root \(\lambda_{n,1}\) is real and the other two \(\lambda_{n,2}\) and \(\lambda_{n,3}\) are complex conjugates, all with negative real parts. Moreover, the eigenvalues satisfy the following asymptotic behavior:
\[
\begin{align*}
\lim_{n \to \infty} \lambda_{n,1} &= -\frac{c^2}{b}, \\
\lim_{n \to \infty} \frac{\lambda_{n,j}}{(\text{Re } \lambda_{n,2})} &= \frac{1}{\beta}, \\
\text{Re } \lambda_{n,2} &\sim -\frac{c^2}{b}, \\
|\text{Im } \lambda_{n,2}| &\sim \sqrt{b} \lambda_n \to \infty, \text{ as } n \to \infty.
\end{align*}
\]

Throughout the rest of the article we assume that
\[
\gamma = \alpha - \frac{c^2}{b} > 0.
\]

Next we give the representation of solutions in terms of series.

**Proposition 4.3.** Let \((u_0, u_1, u_2) \in W_0^{s,2}(\Omega) \times W_0^{s,2}(\Omega) \times L^2(\Omega)\). Then the unique solution \((w, w_t, w_{tt})\) of the system (4.1) is given by
\[
w(x, t) = \sum_{n=1}^{\infty} \left( A_n(t)(u_0, \varphi_n)_{L^2(\Omega)} + B_n(t)(u_1, \varphi_n)_{L^2(\Omega)} + C_n(t)(u_2, \varphi_n)_{L^2(\Omega)} \right) \varphi_n(x),
\]
where
\[
A_n(t) = \frac{\lambda_{n,2} \lambda_{n,3}}{\xi_{n,1}} e^{\lambda_{n,1} t} - \frac{\lambda_{n,1} \lambda_{n,3}}{\xi_{n,2}} e^{\lambda_{n,2} t} + \frac{\lambda_{n,1} \lambda_{n,2}}{\xi_{n,3}} e^{\lambda_{n,3} t},
\]
where

\[ B_n(t) = -\frac{\lambda_{n,2} + \lambda_{n,3}}{\xi_{n,1}} e^{\lambda_{n,1} t} + \frac{\lambda_{n,1} + \lambda_{n,3}}{\xi_{n,2}} e^{\lambda_{n,2} t} - \frac{\lambda_{n,1} + \lambda_{n,2}}{\xi_{n,3}} e^{\lambda_{n,3} t}, \]  

(4.8)

and

\[ C_n(t) = \frac{1}{\xi_{n,1}} e^{\lambda_{n,1} t} - \frac{1}{\xi_{n,2}} e^{\lambda_{n,2} t} + \frac{1}{\xi_{n,3}} e^{\lambda_{n,3} t}. \]  

(4.9)

Here, \( \lambda_{n,j} \) are the solutions of (4.4) and

\[ \begin{align*}
\xi_{n,1} &= (\lambda_{n,1} - \lambda_{n,2})(\lambda_{n,1} - \lambda_{n,3}), \\
\xi_{n,2} &= (\lambda_{n,1} - \lambda_{n,2})(\lambda_{n,2} - \lambda_{n,3}), \\
\xi_{n,3} &= (\lambda_{n,1} - \lambda_{n,3})(\lambda_{n,2} - \lambda_{n,3}).
\end{align*} \]  

(4.10)

Proof. Using the spectral theorem of selfadjoint operators, we can proceed with the method of separation of variables. That is, we look for a solution \( (w, w_t, w_{tt}) \) of (4.1) in the form

\[ w(x, t) = \sum_{n=1}^{\infty} (w(\cdot, t), \varphi_n)_{L^2(\Omega)} \varphi_n(x). \]  

(4.11)

For the sake of simplicity we let \( w_n(t) = (w(\cdot, t), \varphi_n)_{L^2(\Omega)} \). Replacing (4.11) in the first equation of (4.1), then multiplying both sides with \( \varphi_k \) and integrating over \( \Omega \), we get that \( w_n(t) \) satisfies the following ordinary differential equation (ODE)

\[ w_n''(t) + \alpha w_n'''(t) + c^2 \lambda_n w_n(t) + b \lambda_n w'_n(t) = 0. \]

From Lemma 4.2 and letting \( u_{0,n} = (u_0, \varphi_n)_{L^2(\Omega)}, u_{1,n} = (u_1, \varphi_n)_{L^2(\Omega)} \) and \( u_{2,n} = (u_2, \varphi_n)_{L^2(\Omega)} \) so that

\[ u_0 = \sum_{n=1}^{\infty} u_{0,n} \varphi_n, \quad u_1 = \sum_{n=1}^{\infty} u_{1,n} \varphi_n, \quad u_2 = \sum_{n=1}^{\infty} u_{2,n} \varphi_n, \]

we get

\[ w(x, t) = \sum_{n=1}^{\infty} \sum_{j=1}^{3} a_{n,j} e^{\lambda_{n,j} t} \varphi_n(x), \]

where

\[ \begin{align*}
a_{n,1} &= \lambda_{n,2} \lambda_{n,3} u_{0,n} - \frac{(\lambda_{n,2} + \lambda_{n,3})}{\xi_{n,1}} u_{1,n} + \frac{1}{\xi_{n,1}} u_{2,n}, \\
a_{n,2} &= -\frac{\lambda_{n,1} \lambda_{n,3}}{\xi_{n,2}} u_{0,n} + \frac{(\lambda_{n,1} + \lambda_{n,3})}{\xi_{n,2}} u_{1,n} - \frac{1}{\xi_{n,2}} u_{2,n}, \\
a_{n,3} &= \frac{\lambda_{n,1} \lambda_{n,2}}{\xi_{n,3}} u_{0,n} - \frac{(\lambda_{n,1} + \lambda_{n,2})}{\xi_{n,3}} u_{1,n} + \frac{1}{\xi_{n,3}} u_{2,n}.
\end{align*} \]

Therefore, we obtain the following expression of \( w \):

\[ \begin{align*}
w(x, t) &= \sum_{n=1}^{\infty} \left[ u_{0,n} \left( \frac{\lambda_{n,2} \lambda_{n,3}}{\xi_{n,1}} e^{\lambda_{n,1} t} - \frac{\lambda_{n,1} \lambda_{n,3}}{\xi_{n,2}} e^{\lambda_{n,2} t} + \frac{\lambda_{n,1} \lambda_{n,2}}{\xi_{n,3}} e^{\lambda_{n,3} t} \right) \right] \varphi_n(x) \\
&\quad + \sum_{n=1}^{\infty} \left[ u_{1,n} \left( -\frac{(\lambda_{n,2} + \lambda_{n,3})}{\xi_{n,1}} e^{\lambda_{n,1} t} + \frac{(\lambda_{n,1} + \lambda_{n,3})}{\xi_{n,2}} e^{\lambda_{n,2} t} - \frac{(\lambda_{n,1} + \lambda_{n,2})}{\xi_{n,3}} e^{\lambda_{n,3} t} \right) \right] \varphi_n(x) \\
&\quad + \sum_{n=1}^{\infty} \left[ u_{2,n} \left( \frac{1}{\xi_{n,1}} e^{\lambda_{n,1} t} - \frac{1}{\xi_{n,2}} e^{\lambda_{n,2} t} + \frac{1}{\xi_{n,3}} e^{\lambda_{n,3} t} \right) \right] \varphi_n(x).
\end{align*} \]  

(4.12)

Letting \( A_n(t), B_n(t) \) and \( C_n(t) \) be given as in (4.7), (4.8) and (4.9), respectively, we obtain that (4.6) follows from (4.12).
Moreover, there is a constant of (4.15), and there is a constant $C > 0$. We proof the theorem in several steps. We have shown in Proposition 3.2 that for every $g$ and
\begin{equation}
\begin{aligned}
A_n(0)u_{0,n} + B_n(0)u_{1,n} + C_n(0)u_{2,n}
\end{aligned}
\end{equation}
and
\begin{equation}
\begin{aligned}
A_n'(0)u_{0,n} + B_n'(0)u_{1,n} + C_n'(0)u_{2,n}
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\varphi_n(x) = u_{0,n} \varphi_n(x) = u_0(x),
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
w(t, 0) = \sum_{n=1}^{\infty} \left( A_n'(0)u_{0,n} + B_n'(0)u_{1,n} + C_n'(0)u_{2,n} \right) \varphi_n(x) = \sum_{n=1}^{\infty} u_{1,n} \varphi_n(x) = u_1(x),
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
w_{tt}(t, 0) = \sum_{n=1}^{\infty} \left( A_n''(0)u_{0,n} + B_n''(0)u_{1,n} + C_n''(0)u_{2,n} \right) \varphi_n(x) = \sum_{n=1}^{\infty} u_{2,n} \varphi_n(x) = u_2(x).
\end{aligned}
\end{equation}

It is straightforward to show that $w$ given in (4.6) has the regularity (2.3). Since we are not interested with solutions of (4.1), we leave the verification to the interested reader. The proof is finished.

Next, we consider the non-homogeneous system (2.1).

**Theorem 4.4.** For every $g \in D((\mathbb{R}^N \setminus \Omega) \times (0, T))$, the system (2.1) has a unique weak (classical solution) $(v, v_t, v_{tt})$ such that $v \in C^\infty([0, T]; W^{s,2}(\mathbb{R}^N))$ and is given by
\begin{equation}
\begin{aligned}
v(x, t) = \sum_{n=1}^{\infty} \left( \int_0^t \left( g(\cdot, \tau), N_n \varphi_n \right)_{L^2(\mathbb{R}^N)} \frac{1}{\lambda_n} \left[ C_n''(t-\tau) + \alpha C_n''(t-\tau) \right] d\tau \right) \varphi_n(x).
\end{aligned}
\end{equation}

Moreover, there is a constant $C > 0$ such that for all $t \in [0, T]$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,
\begin{equation}
\begin{aligned}
\|\partial_t^m v(t)\|_{W^{s,2}(\mathbb{R}^N)} \leq C \left( \|\partial_t^m g\|_{L^\infty([0, T], W^{s,2}(\mathbb{R}^N \setminus \Omega))} + \|\partial_t^m g(t)\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)} \right).
\end{aligned}
\end{equation}

**Proof.** We proof the theorem in several steps.

**Step 1:** Consider the following elliptic Dirichlet exterior problem:
\begin{equation}
\begin{aligned}
\begin{cases}
(-\Delta)^s \phi = 0 & \Omega, \\
\phi = g & \mathbb{R}^N \setminus \Omega.
\end{cases}
\end{aligned}
\end{equation}

We have shown in Proposition 3.2 that for every $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, there is a unique $\phi \in W^{s,2}(\mathbb{R}^N)$ solution of (4.15), and there is a constant $C > 0$ such that
\begin{equation}
\|\phi\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)} \leq C \|g\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)}.
\end{equation}

Since $g$ depends on $(x, t)$, then $\phi$ also depends on $(x, t)$. If in (4.15) one replaces $g$ by $\partial_t^m g$, $m \in \mathbb{N}$, then the associated unique solution is given by $\partial_t^m \phi$ for every $m \in \mathbb{N}_0$. From this, we can deduce that $\phi \in C^\infty([0, T], W^{s,2}(\mathbb{R}^N))$.

Now let $v$ be a solution of (2.1) and set $w := v - \phi$. Then a simple calculation gives
\begin{equation}
\begin{aligned}
w_{ttt} + \alpha w_{tt} + c^2(-\Delta)^s w + b(-\Delta)^s w_t \\
= v_{ttt} - \phi_{ttt} + \alpha v_{tt} - \alpha \phi_t + c^2(-\Delta)^s v - c^2(-\Delta)^s \phi + b(-\Delta)^s v_t - b(-\Delta)^s \phi_t \\
= v_{ttt} + \alpha v_{tt} + c^2(-\Delta)^s v + b(-\Delta)^s v_t - \phi_{ttt} - \alpha \phi_t \\
= -\phi_{ttt} - \alpha \phi_t \quad \text{in } \Omega \times (0, T).
\end{aligned}
\end{equation}

In addition
\begin{equation}
\begin{aligned}
w = v - \phi = g - g = 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0, T),
\end{aligned}
\end{equation}
and
\begin{equation}
\begin{aligned}
w_t(\cdot, 0) = v_t(\cdot, 0) - \phi_t(\cdot, 0) = -\phi_t(\cdot, 0) \quad \text{in } \Omega, \\
w_{tt}(\cdot, 0) = v_{tt}(\cdot, 0) - \phi_{tt}(\cdot, 0) = -\phi_{tt}(\cdot, 0) \quad \text{in } \Omega, \\
w_{ttt}(\cdot, 0) = v_{ttt}(\cdot, 0) - \phi_{ttt}(\cdot, 0) = -\phi_{ttt}(\cdot, 0) \quad \text{in } \Omega.
\end{aligned}
\end{equation}
Since $g \in \mathcal{D}((\mathbb{R}^N \setminus \Omega) \times (0, T))$, we have that $\phi(\cdot, 0) = \partial_t \phi(\cdot, 0) = \partial_{tt} \phi(\cdot, 0) = 0$ in $\Omega$. We have shown that a solution $(v, v_t, v_{tt})$ of (2.1) can be decomposed as $v = \phi + w$, where $(w, w_t, w_{tt})$ solves the system

$$
\begin{align*}
\left\{
\begin{array}{ll}
w_{tt} + \alpha w_{tt} + c^2 (-\Delta)^s w + b (-\Delta)^s w_t = -\phi_{ttt} - \alpha \phi_{tt} & \text{in } \Omega \times (0, T), \\
w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
w(\cdot, 0) = 0, w_t(\cdot, 0) = 0, w_{tt}(\cdot, 0) = 0 & \text{in } \Omega.
\end{array}
\right.
\end{align*}
$$

(4.17)

We notice that $\phi \in C^\infty([0, T]; W^{s,2}(\mathbb{R}^N))$.

**Step 2:** We observe that letting

$$
W = \begin{pmatrix} w \\ w_t \\ w_{tt} \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 \\ 0 \\ -\phi_{ttt} - \alpha \phi_{tt} \end{pmatrix},
$$

then the system (4.17) can be rewritten as the following first order Cauchy problem

$$
\begin{align*}
\begin{cases}
W_t + AW = \Phi & \text{in } \Omega \times (0, T), \\
W(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \text{in } \Omega,
\end{cases}
\end{align*}
$$

(4.18)

where $A$ is the matrix operator defined in (4.3). Using [21, Corollary 1.2], we get that the first order Cauchy problem (4.18) has a unique classical solution $W$ and hence, (4.17) has a unique weak (classical) solution $(w, w_t, w_{tt})$ such that $w \in C^\infty([0, T]; W^{s,2}(\mathbb{R}^N))$ and is given by

$$
w(x, t) = -\sum_{n=1}^{\infty} \left( \int_0^t \left( \phi_{\tau \tau} \tau, \tau \right) + \alpha \phi_{\tau \tau} \tau, \tau \right)_{L^2(\Omega)} C_n(t - \tau) d\tau \right) \varphi_n(x),
$$

(4.19)

where we recall that $C_n$ is given in (4.9). Integrating (4.19) by parts we get that

$$
w(x, t) = -\sum_{n=1}^{\infty} \left( \int_0^t \left( \phi_{\tau \tau} \tau, \tau \right)_{L^2(\Omega)} C_n(t - \tau) + \alpha C_n(t - \tau) \right) d\tau \right) \varphi_n(x)
- \sum_{n=1}^{\infty} \left( \phi_{\tau \tau} \tau, \tau \right)_{L^2(\Omega)} C_n(t - \tau) \right) \varphi_n(x)
- \sum_{n=1}^{\infty} \left( \phi_{\tau \tau} \tau, \tau \right)_{L^2(\Omega)} C_n(t - \tau) \right) \varphi_n(x)
- \alpha \sum_{n=1}^{\infty} \left( \phi_{\tau \tau} \tau, \tau \right)_{L^2(\Omega)} C_n(t - \tau) \right) \varphi_n(x)
- \alpha \sum_{n=1}^{\infty} \left( \phi_{\tau \tau} \tau, \tau \right)_{L^2(\Omega)} C_n(t - \tau) \right) \varphi_n(x).
$$

We observe that $C_n(0) = C_n'(0) = 0$ and $C_n''(0) = 1$ for all $n \in \mathbb{N}$. Since $\phi(\cdot, 0) = \phi_t(\cdot, 0) = \phi_{tt}(\cdot, 0) = 0$, we get that

$$
w(x, t) = -\phi(x, t) - \sum_{n=1}^{\infty} \left( \int_0^t \left( \phi_{\tau \tau} \tau, \tau \right)_{L^2(\Omega)} C_n(t - \tau) + \alpha C_n(t - \tau) \right) d\tau \right) \varphi_n(x).
$$

(4.20)

Using the fact that $\varphi_n$ satisfies (3.4) and the integration by parts formulas (3.8)-(3.9), we obtain
From (4.20) and (4.21) we can deduce that

\[
\left(\phi(\cdot, \tau), \lambda_n \varphi_n\right)_{L^2(\Omega)} = \left(\phi(\cdot, \tau), (-\Delta)\varphi_n\right)_{L^2(\Omega)}
\]

\[
= \left((-\Delta)\varphi, \tau), \varphi_n\right)_{L^2(\Omega)} - \int_{\mathbb{R}^N \setminus \Omega} \left(\phi N_s \varphi_n - \varphi_n N_s \phi\right) dx
\]

\[
= - \int_{\mathbb{R}^N \setminus \Omega} gN_s \varphi_n dx.
\]  \quad (4.21)

From (4.20) and (4.21) we can deduce that

\[
w(x, t) = -\phi(x, t) + \sum_{n=1}^{\infty} \left(\int_{0}^{t} \left(g(\cdot, \tau), N_s \varphi_n\right)_{L^2(\mathbb{R}^N \setminus \Omega)} \frac{1}{\lambda_n} \left(C''_n(t - \tau) + \alpha C''_n(t - \tau)\right) d\tau\right) \varphi_n(x).
\]

We have shown (4.13). Since \(\phi, w \in C^\infty(\mathbb{R}^N; W^{s,2}(\mathbb{R}^N))\), it follows that \(v \in C^\infty([0, T]; W^{s,2}(\mathbb{R}^N))\).

**Step 3:** Using (4.19) and calculating, we get that for every \(t \in [0, T]\),

\[
\left\|(-\Delta)_D^s w(\cdot, t)\right\|_{L^2(\Omega)} = \left\|\sum_{n=1}^{\infty} \lambda_n \left(\int_{0}^{t} \left(\phi_{ttt}(\cdot, \tau) + \alpha \phi_{ttt}(\cdot, \tau), \varphi_n\right)_{L^2(\Omega)} C_n(t - \tau) d\tau\right) \varphi_n\right\|_{L^2(\Omega)}
\]

\[
\leq \int_{0}^{t} \left\|\sum_{n=1}^{\infty} \lambda_n \left(\phi_{ttt}(\cdot, \tau) + \alpha \phi_{ttt}(\cdot, \tau), \varphi_n\right)_{L^2(\Omega)} C_n(t - \tau) d\tau \varphi_n\right\|_{L^2(\Omega)}
\]

\[
\leq \int_{0}^{t} \left(\sum_{n=1}^{\infty} \left(\phi_{ttt}(\cdot, \tau) + \alpha \phi_{ttt}(\cdot, \tau), \varphi_n\right)_{L^2(\Omega)} \right)^2 \left|\lambda_n C_n(t - \tau)\right|^2 \frac{1}{2} d\tau.
\]  \quad (4.22)

Using the asymptotic behavior (4.5), we obtain (see Lemma 4.5 below) that there is a constant \(C > 0\), independent of \(n\), such that

\[
|\lambda_n C_n(t)| \leq C, \quad \forall \ n \in \mathbb{N} \quad \text{and} \quad t \in [0, T].
\]  \quad (4.23)

It follows from (4.22), (4.23) and (4.16) that for every \(t \in [0, T]\),

\[
\left\|(-\Delta)_D^s w(\cdot, t)\right\|_{L^2(\Omega)} \leq C \int_{0}^{t} \left\|\phi_{ttt}(\cdot, \tau) + \alpha \phi_{ttt}(\cdot, \tau)\right\|_{L^2(\Omega)} d\tau
\]

\[
\leq C \int_{0}^{t} \left\|\phi_{ttt}(\cdot, \tau) + \alpha \phi_{ttt}(\cdot, \tau)\right\|_{W^{s,2}(\mathbb{R}^N)} d\tau
\]

\[
\leq C \int_{0}^{t} \left\|g_{ttt}(\cdot, \tau) + \alpha g_{ttt}(\cdot, \tau)\right\|_{W^{s,2}(\mathbb{R}^N)} d\tau
\]

\[
\leq CT \left\|g_{ttt} + \alpha g_{ttt}\right\|_{L^\infty([0, T]; W^{s,2}(\mathbb{R}^N) \setminus \Omega)}.
\]  \quad (4.24)

Using (4.24), we get that for every \(t \in [0, T]\),

\[
\left\|v(\cdot, t)\right\|_{W^{s,2}(\mathbb{R}^N)} \leq C \left(\left\|(-\Delta)_D^s w(\cdot, t)\right\|_{L^2(\Omega)} + \left\|\phi(\cdot, t)\right\|_{W^{s,2}(\mathbb{R}^N)}\right)
\]

\[
\leq C \left(\left\|g_{ttt} + \alpha g_{ttt}\right\|_{L^\infty([0, T]; W^{s,2}(\mathbb{R}^N) \setminus \Omega)} + \left\|g(\cdot, t)\right\|_{W^{s,2}(\mathbb{R}^N) \setminus \Omega}\right).
\]

We have shown (4.14) for \(m = 0\). Proceeding by induction on \(m\) we can easily deduce (4.14) for every \(m \in \mathbb{N}_0\). The proof is finished.

We conclude this subsection with the proof of our main result on existence and uniqueness of weak solutions for the system (1.1).

**Proof of Theorem 2.2.** We have shown in Section 2 that a solution \((u, u_t, u_{tt})\) of (1.1) can be decomposed into \(u = v + w\) where \((v, v_t, v_{tt})\) solves (2.1) and \((w, w_t, w_{tt})\) is a solution of (2.2). Now the result follows from Proposition 4.3 and Theorem 4.4. \qed
4.2. Series solutions of the dual system. Now we consider the dual system (2.4). Let
\[ \psi_0, n := (\psi_0, \varphi_n)_{L^2(\Omega)}, \quad \psi_1, n := (\psi_1, \varphi_n)_{L^2(\Omega)}, \quad \text{and} \quad \psi_2, n := (\psi_2, \varphi_n)_{L^2(\Omega)}. \]
Throughout this subsection we will denote \( D_n(t) = A_n(t), E_n(t) = -B_n(t) \) and \( F_n(t) = C_n(t) \), where \( A_n(t), B_n(t) \) and \( C_n(t) \) are given in (4.7), (4.8) and (4.9), respectively. We begin with the following technical Lemma that is crucial in the proof of Theorem 2.5.

**Lemma 4.5.** There is a constant \( C > 0 \) (independent of \( n \)) such that for every \( t \in [0, T] \),
\[
\max \left\{ |D_n(t)|^2, |D_n'(t)|^2, \left| \frac{D''_n(t)}{\lambda_n^2} \right|^2, \left| \frac{D''_n(t)}{\lambda_n^7} \right|^2 \right\} \leq C, \quad (4.25)
\]
\[
\max \left\{ |E_n(t)|^2, |E_n'(t)|^2, \left| \frac{E''_n(t)}{\lambda_n^2} \right|^2, \left| \frac{E''_n(t)}{\lambda_n^7} \right|^2 \right\} \leq C, \quad (4.26)
\]
and
\[
\max \left\{ \left| \frac{\lambda_n^2}{\xi_n^2} F_n(t) \right|^2, \left| \lambda_n F_n(t) \right|^2, \left| \frac{\lambda_n^2}{\xi_n^2} F'_n(t) \right|^2, \left| \frac{F''_n(t)}{\lambda_n^7} \right|^2 \right\} \leq C. \quad (4.27)
\]

**Proof.** We rewrite the functions \( D_n(t), E_n(t) \) and \( F_n(t) \) as follows:
\[
D_n(t) = \sum_{j=1}^3 D_{n,j} e^{\lambda_{n,j} t}, \quad E_n(t) = \sum_{j=1}^3 E_{n,j} e^{\lambda_{n,j} t}, \quad F_n(t) = \sum_{j=1}^3 F_{n,j} e^{\lambda_{n,j} t},
\]
where
\[
D_{n,1} = \frac{\lambda_{n,2} \lambda_{n,3}}{\xi_{n,1}}, \quad D_{n,2} = -\frac{\lambda_{n,1} \lambda_{n,3}}{\xi_{n,2}}, \quad D_{n,3} = \frac{\lambda_{n,1} \lambda_{n,2}}{\xi_{n,3}},
\]
\[
E_{n,1} = \frac{\lambda_{n,2} + \lambda_{n,3}}{\xi_{n,1}}, \quad E_{n,2} = -\frac{\lambda_{n,1} + \lambda_{n,3}}{\xi_{n,2}}, \quad E_{n,3} = \frac{\lambda_{n,1} + \lambda_{n,2}}{\xi_{n,3}},
\]
\[
F_{n,1} = \frac{1}{\xi_{n,1}}, \quad F_{n,2} = -\frac{1}{\xi_{n,2}}, \quad F_{n,3} = \frac{1}{\xi_{n,3}}.
\]

We proof the Lemma in several steps. First of all, we notice that sine \( \lambda_{n,1} < 0 \) and \( \text{Re}(\lambda_{n,j}) < 0 \), for \( j = 2, 3 \), it follows that \( |e^{\lambda_{n,j} t}| \leq 1 \) for \( j = 1, 2, 3 \).

**Step 1:** Observe that
\[
\left| \frac{\lambda_n^2}{\lambda_{n,2}} \right| = \frac{1}{\left| \frac{\text{Re}(\lambda_{n,2})}{\lambda_n^2} - \frac{|\text{Im}(\lambda_{n,2})|}{\lambda_n^7} \right|}.
\]
Since \( \text{Re}(\lambda_{n,2}) \sim -\frac{1}{n}, |\text{Im}(\lambda_{n,2})| \sim \sqrt{b_n} \lambda_n \) and \( \lambda_n \to +\infty \) as \( n \to \infty \), it follows that the sequence \( \left\{ \left| \frac{\lambda_n^2}{\lambda_{n,2}} \right| \right\}_{n \in \mathbb{N}} \) is convergent. Using that \( \lambda_{n,3} = \overline{\lambda_{n,2}} \) we also obtain that the sequence \( \left\{ \left| \frac{\lambda_n^2}{\lambda_{n,3}} \right| \right\}_{n \in \mathbb{N}} \) is convergent. Thus these two sequences are bounded.

**Step 2:** We claim that the sequence \( \left\{ |\lambda_{n,j} D_{n,j}| \right\}_{n \in \mathbb{N}} \) is convergent. Indeed, since \( \lambda_{n,3} = \overline{\lambda_{n,2}} \), it suffices prove the cases \( j = 1, 2 \). We observe that
\[
|\lambda_{n,1}D_{n,1}| = |\lambda_{n,1} \frac{\lambda_{n,2}\lambda_{n,2}}{(\lambda_{n,1} - \lambda_{n,2})(\lambda_{n,1} - \lambda_{n,2})}| = |\lambda_{n,1} - \lambda_{n,2}|^2
\]

\[
\leq |\lambda_{n,1}| \left| \frac{\text{Re}(\lambda_{n,2})}{\text{Im}(\lambda_{n,2})} + 1 \right| \left( \frac{|\text{Re}(\lambda_{n,1})|}{|\text{Im}(\lambda_{n,1})|} - \frac{|\text{Im}(\lambda_{n,1})|}{|\text{Im}(\lambda_{n,1})|} \right) \right|^2.
\]

and

\[
|\lambda_{n,2}D_{n,2}| = |\lambda_{n,1} \frac{\lambda_{n,2}\lambda_{n,2}}{(\lambda_{n,1} - \lambda_{n,2})(\lambda_{n,2} - \lambda_{n,2})}| = |\lambda_{n,1} - \lambda_{n,2}|^2
\]

\[
\leq |\lambda_{n,1}| \left| \frac{\text{Re}(\lambda_{n,2})}{\text{Im}(\lambda_{n,2})} + 1 \right| \left( \frac{|\text{Re}(\lambda_{n,1})|}{|\text{Im}(\lambda_{n,1})|} - \frac{|\text{Im}(\lambda_{n,1})|}{|\text{Im}(\lambda_{n,1})|} \right) \right|^2.
\]

From (4.5), we can deduce the convergence of the sequence \( \{|\lambda_{n,j}D_{n,j}|\}_{n \in \mathbb{N}} \).

**Step 3:** Now we prove (4.25). Notice that

\[
|D_{n,j}| = \left| \frac{1}{\lambda_{n,j}} \right| |\lambda_{n,j}D_{n,j}|,
\]

and since \( \{|\lambda_{n,j}D_{n,j}|\}_{n \in \mathbb{N}} \) and \( \{ |\lambda_{n,j}| \}_{n \in \mathbb{N}} \) are convergent, we obtain that \( \{ |D_{n,j}| \}_{n \in \mathbb{N}} \), for \( j = 1, 2 \), is convergent. Thus, the sequence \( \{ |D_{n}(t)| \}_{n \in \mathbb{N}} \) is bounded.

The case of \( \{ |D'_{n}(t)| \}_{n \in \mathbb{N}} \) is a simple consequence of Step 2 due to the fact that

\[
D'_{n}(t) = \sum_{j=1}^{3} \lambda_{n,j}D_{n,j}e^{\lambda_{n,j}t}.
\]

For the convergence of the sequence \( \{ |D''_{n}(t)| \}_{n \in \mathbb{N}} \), we observe that

\[
\frac{D''_{n}(t)}{\lambda_{n}^2} = \sum_{j=1}^{3} \frac{\lambda_{n}^2}{\lambda_{n}^2} D_{n,j}e^{\lambda_{n,j}t} = \sum_{j=1}^{3} \frac{\lambda_{n,j}}{\lambda_{n}^2} \lambda_{n,j}D_{n,j}e^{\lambda_{n,j}t}.
\]

Since \( \{|\lambda_{n,j}D_{n,j}|\}_{n \in \mathbb{N}} \) and \( \{ |\lambda_{n,j}| \}_{n \in \mathbb{N}} \) are convergent, we obtain that \( \{ \frac{|D''_{n}(t)|}{\lambda_{n}^2} \}_{n \in \mathbb{N}} \) is bounded. From the above computation we also deduce that the sequence \( \{ \frac{|D''_{n}(t)|}{\lambda_{n}^2} \}_{n \in \mathbb{N}} \) is bounded. Therefore, we can deduce that (4.25) holds.

**Step 4:** To prove (4.26), we observe the following:

\[
E_{n,1} = -\left( \frac{1}{\lambda_{n,3}} + \frac{1}{\lambda_{n,2}} \right) D_{n,1}, \quad E_{n,2} = \left( \frac{1}{\lambda_{n,3}} + \frac{1}{\lambda_{n,1}} \right) D_{n,2},
\]

\[
E_{n,3} = -\left( \frac{1}{\lambda_{n,2}} + \frac{1}{\lambda_{n,1}} \right) D_{n,3},
\]

\[
\lambda_{n,1}E_{n,1} = -\left( \frac{1}{\lambda_{n,3}} + \frac{1}{\lambda_{n,2}} \right) \lambda_{n,1}D_{n,1}, \quad \lambda_{n,2}E_{n,2} = \left( \frac{1}{\lambda_{n,3}} + \frac{1}{\lambda_{n,1}} \right) \lambda_{n,2}D_{n,2},
\]

\[
\lambda_{n,3}E_{n,3} = -\left( \frac{1}{\lambda_{n,2}} + \frac{1}{\lambda_{n,1}} \right) \lambda_{n,3}D_{n,3},
\]
We proof the theorem in several steps. Here we include more details.

Therefore, it is easy to see that

\[
\frac{\lambda_{n}^{2}}{\lambda_{n}^{2}}E_{n,1} = - \left( \frac{1}{\lambda_{n,3}} + \frac{1}{\lambda_{n,2}} \right) \frac{\lambda_{n,1}}{\lambda_{n}^{2}} \lambda_{n,1}D_{n,1}, \quad \frac{\lambda_{n}^{2}}{\lambda_{n}^{2}}E_{n,2} = \left( \frac{1}{\lambda_{n,3}} + \frac{1}{\lambda_{n,1}} \right) \frac{\lambda_{n,2}}{\lambda_{n}^{2}} \lambda_{n,2}D_{n,2},
\]

\[
\frac{\lambda_{n}^{2}}{\lambda_{n}^{2}}E_{n,3} = - \left( \frac{1}{\lambda_{n,2}} + \frac{1}{\lambda_{n,1}} \right) \lambda_{n,3} \lambda_{n,3}D_{n,3}.
\]

Then, using the previous steps we obtain (4.26).

Step 5: We claim that \( \{\lambda_{n}F_{n,j}\}_{n \in \mathbb{N}} \) is convergent. Indeed,

\[
|\lambda_{n}F_{n,j}| \leq \left| \frac{\lambda_{n}}{\xi_{n,1}} \right| + \left| \frac{\lambda_{n}}{\xi_{n,2}} \right| + \left| \frac{\lambda_{n}}{\xi_{n,3}} \right|
\]

\[
= \left| \frac{\lambda_{n}}{\lambda_{n,2}\lambda_{n,3}} \right| \left| D_{n,1} \right| + \left| \frac{\lambda_{n}}{\lambda_{n,1}\lambda_{n,2}\lambda_{n,3}} \right| \left| D_{n,2} \right| + \left| \frac{\lambda_{n}}{\lambda_{n,1}\lambda_{n,2}\lambda_{n,3}} \right| \left| D_{n,3} \right|
\]

We observe that

\[
\left| \frac{\lambda_{n}}{\lambda_{n,1}\lambda_{n,2}\lambda_{n,3}} \right| = \left| \frac{1}{\lambda_{n,1}} \right| \left| \frac{1}{\lambda_{n,2}} \left| \frac{1}{\lambda_{n,3}} \right| \right| = \left| \frac{1}{\lambda_{n,1}} \left| \frac{1}{\lambda_{n,2}} \left| \frac{1}{\lambda_{n,3}} \right| \right| \right|
\]

Using the convergence property (4.5), we obtain the desired result.

Step 6: Finally, we prove (4.27). Observe that

\[
\frac{\lambda_{n}^{2}}{\lambda_{n}^{2}}F_{n}(t) = \frac{\lambda_{n}^{2}}{\lambda_{n,2}\lambda_{n,3}}D_{n,1}e^{\lambda_{n,1}t} - \frac{\lambda_{n}^{2}}{\lambda_{n,1}\lambda_{n,2}\lambda_{n,3}}D_{n,2}e^{\lambda_{n,2}t} + \frac{\lambda_{n}^{2}}{\lambda_{n,1}\lambda_{n,2}\lambda_{n,3}}D_{n,3}e^{\lambda_{n,3}t}
\]

Therefore, it is easy to see that \( \{\frac{\lambda_{n}^{2}}{\lambda_{n}^{2}}F_{n}(t)\}_{n} \) is a bounded sequence.

With a similar argument, we can deduce that the following sequences \( \left\{ \frac{\lambda_{n}^{2}}{\lambda_{n}^{2}}F_{n}(t) \right\}_{n} \) and \( \left\{ \frac{\lambda_{n}^{2}}{\lambda_{n}^{2}}F_{n}(t) \right\}_{n} \) are bounded. The proof is finished.

Proof of Theorem 2.5. Let

\[
\psi_{0} = \sum_{n=1}^{\infty} \psi_{0,n} \varphi_{n}, \quad \psi_{1} = \sum_{n=1}^{\infty} \psi_{1,n} \varphi_{n}, \quad \psi_{2} = \sum_{n=1}^{\infty} \psi_{2,n} \varphi_{n}.
\]

We proof the theorem in several steps. Here we include more details.

Step 1: Proceeding in the same way as the proof of Proposition 4.3, we easily get that

\[
\psi(x,t) = \sum_{n=1}^{\infty} \left[ D_{n}(T-t)\psi_{0,n} + E_{n}(T-t)\psi_{1,n} + F_{n}(T-t)\psi_{2,n} \right] \varphi_{n}(x),
\]

where \( D_{n}(t) = A_{n}(t), \quad E_{n}(t) = -B_{n}(t) \) and \( F_{n}(t) = C_{n}(t) \). In addition, a simple calculation gives \( \psi(x) = \psi_{0}(x), \quad \psi_{1}(x) = -\psi_{1}(x) \) and \( \psi_{2}(x) = \psi_{2}(x) \) for a.e. \( x \in \Omega \).

Let us show that \( \psi \) satisfies the regularity and variational identity requirements. Let \( 1 \leq n \leq m \) and set

\[
\psi_{m}(x,t) = \sum_{n=1}^{m} \left[ D_{n}(T-t)\psi_{0,n} + E_{n}(T-t)\psi_{1,n} + F_{n}(T-t)\psi_{2,n} \right] \varphi_{n}(x).
\]
For every \( m, \tilde{m} \in \mathbb{N} \) with \( m > \tilde{m} \) and \( t \in [0, T] \), we have that
\[
\| \psi_m(x, t) - \psi_{\tilde{m}}(x, t) \|^2_{W_0^{s,2}(\Omega)} = \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} D_n(T - t) \psi_{0,n} + \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} E_n(T - t) \psi_{1,n} + \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} F_n(T - t) \psi_{2,n} \right|^2
\leq 2 \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} D_n(T - t) \psi_{0,n} \right|^2 + 2 \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} E_n(T - t) \psi_{1,n} \right|^2 + 2 \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} F_n(T - t) \psi_{2,n} \right|^2.
\] (4.28)

Using (4.25), (4.26) and (4.27) we get from (4.28) that for every \( m, \tilde{m} \in \mathbb{N} \) with \( m > \tilde{m} \) and \( t \in [0, T] \),
\[
\| \psi_m(x, t) - \psi_{\tilde{m}}(x, t) \|^2_{W_0^{s,2}(\Omega)} \leq C \left( \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} \psi_{0,n} \right|^2 + \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} \psi_{1,n} \right|^2 + \sum_{n=m+1}^{\infty} \left| \psi_{2,n} \right|^2 \right) \rightarrow 0,
\]
as \( m, \tilde{m} \to \infty \). We have shown that the series
\[
\sum_{n=1}^{\infty} \left[ D_n(T - t) \psi_{0,n} + E_n(T - t) \psi_{1,n} + F_n(T - t) \psi_{2,n} \right] \varphi_n \rightarrow \psi(\cdot, t) \text{ in } W_0^{s,2}(\Omega),
\]
and that the convergence is uniform in \( t \in [0, T] \). Hence, \( \psi \in C([0, T]; W_0^{s,2}(\Omega)) \). Using (4.25), (4.26) and (4.27) again we get that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),
\[
\| \psi(\cdot, t) \|^2_{W_0^{s,2}(\Omega)} \leq C \left( \| \psi_0 \|^2_{W_0^{s,2}(\Omega)} + \| \psi_1 \|^2_{W_0^{s,2}(\Omega)} + \| \psi_2 \|^2_{L^2(\Omega)} \right).
\] (4.29)

**Step 2:** Next, we show that \( \psi_1 \in C([0, T]; W_0^{s,2}(\Omega)) \). Indeed, we have
\[
(\psi_m)_t(x, t) = -\sum_{n=1}^{m} \left[ D_n'(T - t) \psi_{0,n} + E_n'(T - t) \psi_{1,n} + F_n'(T - t) \psi_{2,n} \right] \varphi_n(x).
\]
Proceeding as above, we obtain that the series
\[
\sum_{n=1}^{\infty} \left[ D_n'(T - t) \psi_{0,n} + E_n'(T - t) \psi_{1,n} + F_n'(T - t) \psi_{2,n} \right] \varphi_n \rightarrow \psi_1(\cdot, t) \text{ in } W_0^{s,2}(\Omega),
\]
and the convergence is uniform in \( t \in [0, T] \). As in the previous case, using (4.25), (4.26) and (4.27), we get that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),
\[
\| \psi_1(\cdot, t) \|^2_{L^2(\Omega)} \leq C \left( \| \psi_0 \|^2_{W_0^{s,2}(\Omega)} + \| \psi_1 \|^2_{W_0^{s,2}(\Omega)} + \| \psi_2 \|^2_{L^2(\Omega)} \right).
\] (4.30)

**Step 3:** Next, we claim that \( \psi_{tt} \in C([0, T]; L^2(\Omega)) \). Calculating, we get that
\[
(\psi_m)_{tt}(x, t) = \sum_{n=1}^{m} \left[ D_n''(T - t) \psi_{0,n} + E_n''(T - t) \psi_{1,n} + F_n''(T - t) \psi_{2,n} \right] \varphi_n(x).
\]
As in Step 1, we obtain that for every \( m, \tilde{m} \in \mathbb{N} \) with \( m > \tilde{m} \) and \( t \in [0, T] \)
\[
\| \partial_{tt} \psi_m(x, t) - \partial_{tt} \psi_{\tilde{m}}(x, t) \|^2_{L^2(\Omega)} \leq 2 \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} D_n''(T - t) \psi_{0,n} \right|^2 + 2 \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} E_n''(T - t) \psi_{1,n} \right|^2 + 2 \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} F_n''(T - t) \psi_{2,n} \right|^2 \leq C \left( \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} \psi_{0,n} \right|^2 + \sum_{n=m+1}^{\infty} \left| \frac{\lambda_n^s}{\lambda_{\tilde{m}}^s} \psi_{1,n} \right|^2 + \sum_{n=m+1}^{\infty} \left| \psi_{2,n} \right|^2 \right) \rightarrow 0 \text{ as } m, \tilde{m} \to \infty.
\]
Again, we can easily deduce that the series
\[ \sum_{n=1}^{\infty} \left[ D_n''(T-t)\psi_{0,n} + E_n''(T-t)\psi_{1,n} + F_n''(T-t)\psi_{2,n} \right] \varphi_n \rightarrow \psi_{tt}(\cdot, t) \text{ in } L^2(\Omega), \]
and the convergence is uniform in \( t \in [0, T] \). In addition using (4.25), (4.26) and (4.27), we get that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),
\[
\|\psi_n(\cdot, t)\|_{L^2(\Omega)}^2 \leq C \left( \|\varphi_n\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_1\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_2\|_{L^2(\Omega)}^2 \right). \tag{4.31}
\]
The estimate (2.5) follows from (4.29), (4.30) and (4.31).

**Step 4:** We show that \( \psi_{tt} \in C([0, T]; W^{-s,2}(\Omega)) \). Using (3.6), (4.25), (4.26) and (4.27), we get that for every \( t \in [0, T] \),
\[
\|(-\Delta)^{\frac{s}{2}}\psi(\cdot, t)\|_{W^{-s,2}_0(\Omega)}^2 \\
\leq \sum_{n=1}^{\infty} \left( \left| \lambda_n^{\frac{s}{2}} \lambda_n D_n(T-s)\psi_{0,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} \lambda_n E_n(T-s)\psi_{1,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} \lambda_n F_n(T-s)\psi_{2,n} \right|^2 \right) \\
\leq \sum_{n=1}^{\infty} \left( \left| \lambda_n^{\frac{s}{2}} D_n(T-s)\psi_{0,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} E_n(T-s)\psi_{1,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} F_n(T-s)\psi_{2,n} \right|^2 \right) \\
\leq C \left( \|\psi_0\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_1\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_2\|_{L^2(\Omega)}^2 \right). \tag{4.32}
\]
Using (3.6), (4.25), (4.26) and (4.27) again we get that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),
\[
\|(-\Delta)^{\frac{s}{2}}\psi(\cdot, t)\|_{W^{-s,2}_0(\Omega)}^2 \\
\leq \sum_{n=1}^{\infty} \left( \left| \lambda_n^{\frac{s}{2}} D_n'(T-s)\psi_{0,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} E_n'(T-s)\psi_{1,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} F_n'(T-s)\psi_{2,n} \right|^2 \right) \\
\leq \sum_{n=1}^{\infty} \left( \left| \lambda_n^{\frac{s}{2}} D_n'(T-s)\psi_{0,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} E_n'(T-s)\psi_{1,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} F_n'(T-s)\psi_{2,n} \right|^2 \right) \\
\leq C \left( \|\psi_0\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_1\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_2\|_{L^2(\Omega)}^2 \right). \tag{4.33}
\]
Finally, using (3.6), (4.25), (4.26) and (4.27) again we get
\[
\|\psi_{tt}(\cdot, t)\|_{W^{-s,2}_0(\Omega)}^2 \leq \sum_{n=1}^{\infty} \left( \left| \lambda_n^{\frac{s}{2}} D_n''(T-s)\psi_{0,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} E_n''(T-s)\psi_{1,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} F_n''(T-s)\psi_{2,n} \right|^2 \right) \\
\leq \sum_{n=1}^{\infty} \left( \left| \lambda_n^{\frac{s}{2}} D_n''(T-s)\psi_{0,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} E_n''(T-s)\psi_{1,n} \right|^2 + \left| \lambda_n^{\frac{s}{2}} F_n''(T-s)\psi_{2,n} \right|^2 \right) \\
\leq C \left( \|\psi_0\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_1\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_2\|_{L^2(\Omega)}^2 \right). \tag{4.34}
\]
Since \( \psi_{tt}(\cdot, t) = -\alpha\psi_t(t, t) - c^2(-\Delta)^{\frac{s}{2}}\psi(\cdot, t) + b(-\Delta)^{\frac{s}{2}}\psi(\cdot, t) \), it follows from (4.32), (4.33) and (4.34) that
\[
\|\psi_{tt}(\cdot, t)\|_{W^{-s,2}_0(\Omega)}^2 \leq C \left( \|\psi_0\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_1\|_{W^{0,2}_0(\Omega)}^2 + \|\psi_2\|_{L^2(\Omega)}^2 \right),
\]
and we have also shown (2.6). We can also easily deduce that \( \psi_{tt} \in C([0, T); W^{-s,2}(\Omega)) \).

**Step 5:** We claim that \( \psi \in L^{\infty}((0, T); D((-\Delta)^{\frac{s}{2}})) \). It follows from the estimate (2.5) that \( \psi \in L^{\infty}((0, T); L^2(\Omega)) \). Since \( D((-\Delta)^{\frac{s}{2}}) \times D((-\Delta)^{\frac{s}{2}}) \times L^2(\Omega) \) is dense in the Banach space \( W^{s,2}_0(\Omega) \times W^{s,2}_0(\Omega) \times
\(L^2(\Omega)\), it suffices to consider \((\psi_0, \psi_1, \psi_2) \in D((-\Delta)_D^s) \times D((-\Delta)_D^s) \times L^2(\Omega)\). Proceeding as above we get that
\[
\|\psi(\cdot, t)\|^2_{D((-\Delta)_D^s)} = \|(-\Delta)_D^s \psi(\cdot, t)\|^2_{L^2(\Omega)}
\leq 2 \sum_{n=1}^{\infty} \left( |D_n(T-t)\lambda_n \psi_{0,n}|^2 + |E_n(T-t)\lambda_n \psi_{1,n}|^2 + |\lambda_n F_n(T-t)\psi_{2,n}|^2 \right). \tag{4.35}
\]

It follows from (4.35), (4.25), (4.26) and (4.27) that
\[
\|\psi(\cdot, t)\|^2_{D((-\Delta)_D^s)} \leq C \left( \|\psi_0\|^2_{D((-\Delta)_D^s)} + \|\psi_1\|^2_{D((-\Delta)_D^s)} + \|\psi_2\|^2_{L^2(\Omega)} \right).
\]

Thus \(\psi \in L^\infty((0,T); D((-\Delta)_D^s))\) and we have shown the claim.

**Step 6:** It is easy to see that the mapping \([0, T] \ni t \to \psi(\cdot, t) \in L^2(\mathbb{R}^N \setminus \Omega)\) can be analytically extended to \(\Sigma_T\). We also recall that for every \(t \in [0, T]\) fixed, we have that \(\psi(\cdot, t) \in D((-\Delta)_D^s) \subset W^{s,2}(\mathbb{R}^N)\). Therefore, \(N_s v(\cdot, t)\) exists and belongs to \(L^2(\mathbb{R}^N \setminus \Omega)\).

We claim that
\[
N_s \psi(x, t) = \sum_{n=1}^{\infty} \left( D_n(T-t)\psi_{0,n} + E_n(T-t)\psi_{1,n} + F_n(T-t)\psi_{2,n} \right) N_s \varphi_n(x), \tag{4.36}
\]
and the series is convergent in \(L^2(\mathbb{R}^N \setminus \Omega)\) and that the convergence is uniform in \(t \in [0, T]\). Indeed, let \(\eta > 0\) be fixed but arbitrary and let \(t \in [0, T-\eta]\). Let \(n, m \in \mathbb{N}\) with \(n > m\). Since \(N_s : W^{s,2}(\mathbb{R}^N) \to L^2(\mathbb{R}^N \setminus \Omega)\) is bounded, then using (4.25), (4.26) and (4.27), we get that there is a constant \(C > 0\) such that
\[
\left\| \sum_{n=m+1}^{\infty} \left( D_n(T-t)\psi_{0,n} + E_n(T-t)\psi_{1,n} + F_n(T-t)\psi_{2,n} \right) N_s \varphi_n \right\|^2_{L^2(\mathbb{R}^N \setminus \Omega)}
\leq C \left( \sum_{n=m+1}^{\infty} |\psi_{0,n}|^2 + \sum_{n=m+1}^{\infty} |\psi_{1,n}|^2 + \sum_{n=m+1}^{\infty} |\psi_{2,n}|^2 \right) \to 0 \text{ as } m \to \infty.
\]

Thus, \(N_s\) is given by (4.36) and the series is convergent in \(L^2(\mathbb{R}^N \setminus \Omega)\) uniformly in any compact subset of \([0, T]\).

Besides, we obtain the following continuous dependence on the data for the nonlocal normal derivative. Let \(m \in \mathbb{N}\) and consider
\[
\psi_m(x, t) = \sum_{n=1}^{m} \left( D_n(T-t)\psi_{0,n} + E_n(T-t)\psi_{1,n} + F_n(T-t)\psi_{2,n} \right) N_s \varphi_n(x).
\]

Using the fact that the operator \(N_s : W^{s,2}_0(\Omega) \to L^2(\mathbb{R}^N \setminus \Omega)\) is bounded, the continuous embedding \(W^{s,2}_0(\Omega) \hookrightarrow L^2(\Omega)\), (4.25) and (4.26) and (4.25), we get that there is a constant \(C > 0\) such that for every \(t \in [0, T]\),
The system

\[ L^2(\mathbb{R}^N \setminus \Omega) \]

is analytic in \( \Sigma_T \).

Next, since the functions \( D_n(z) \), \( E_n(z) \) and \( F_n(z) \) are entire functions, it follows that the function

\[ \sum_{n=1}^{m} \left[ D_n(T - z)\psi_{0,n} + E_n(T - z)\psi_{1,n} + F_n(T - z)\psi_{2,n} \right] N_s \varphi_n \]

is analytic in \( \Sigma_T \).

Let \( \tau > 0 \) be fixed but otherwise arbitrary. Let \( z \in \mathbb{C} \) satisfy \( \text{Re}(z) \leq T - \tau \). Then proceeding as above by using (4.25), (4.26) and (4.27), we get that

\[
\begin{align*}
\left\| \sum_{n=m+1}^{\infty} \psi_{0,n} D_n(T - z) N_s \varphi_n \right\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 &+ \left\| \sum_{n=m+1}^{\infty} \psi_{1,n} E_n(T - z) N_s \varphi_n \right\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 \\
+ \left\| \sum_{n=m+1}^{\infty} \psi_{2,n} F_n(T - z) N_s \varphi_n \right\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 &\leq C \sum_{n=m+1}^{\infty} \lambda_n^{1/n} |\psi_{0,n}|^2 + \sum_{n=m+1}^{\infty} \lambda_n^{1/n} |\psi_{1,n}|^2 + C \sum_{n=m+1}^{\infty} |\psi_{2,n}|^2 \to 0 \quad \text{as } m \to \infty.
\end{align*}
\]

We have shown that

\[ N_s \psi(\cdot, z) = \sum_{n=1}^{\infty} \psi_{0,n} D_n(T - z) N_s \varphi_n + \sum_{n=1}^{\infty} \psi_{1,n} E_n(T - z) N_s \varphi_n + \sum_{n=1}^{\infty} \psi_{2,n} F_n(T - z) N_s \varphi_n, \quad (4.39) \]

and the series is convergent in \( L^2(\mathbb{R}^N \setminus \Omega) \) uniformly in any compact subset of \( \Sigma_T \). Thus, \( N_s \psi \) given in (4.39) is also analytic in \( \Sigma_T \). The proof is finished. \( \square \)

5. **Proof of the main controllability results**

In this section we prove the main results stated in Section 2.

5.1. **The lack of exact or null controllability result.** We start with the proof of the lack of null/exact controllability of the system (1.1). For this purpose, we will use the following concept of controllability.

**Definition 5.1.** The system (1.1) is said to be spectrally controllable if any finite linear combination of eigenvectors

\[ u_0 = \sum_{n=1}^{M} u_{0,n} \varphi_n, \quad u_1 = \sum_{n=1}^{M} u_{1,n} \varphi_n, \quad u_2 = \sum_{n=1}^{M} u_{2,n} \varphi_n, \]

can be steered to zero by a control function \( g \).
Next, let \((u, u_t, u_{tt})\) and \((\psi, \psi_t, \psi_{tt})\) be the weak solutions of (1.1) and (2.4), respectively. Multiplying the first equation in (1.1) by \(\psi\), then integrating by parts over \((0, T)\) and over \(\Omega\) and using the integration by parts formulas (3.8)-(3.9), we get

\[
\int_{\Omega} \left( u_{tt} \psi - u_t \psi_t + u \psi_{tt} + \alpha (u_t \psi - uu_t) + bu (-\Delta)^s \psi \right) \bigg|_{t=0}^{t=T} \, dx = \int_0^T \int_{\mathbb{R}^N \setminus \Omega} \left( c^2 g(x,t) + bg_t(x,t) \right) N_x \psi(x,t) \, dx dt.
\]  

(5.1)

Using the identity (5.1) and a density argument to pass to the limit, we obtain the following criterion of null and exact controllabilities.

**Lemma 5.2.** The following assertions hold.

(a) The system (1.1) is null controllable if and only if for each initial condition \((u_0, u_1, u_2) \in W_0^{s,2}(\Omega)^3 \times W_0^{s,2}(\Omega)^3 \times W_0^{s,2}(\Omega)\), there exists a control function \(g \in L^2((0,T); W_0^{s,2}(\mathbb{R}^N \setminus \Omega))\) such that the weak solution \((\psi, \psi_t, \psi_{tt})\) of the dual system (2.4) satisfies

\[
\begin{align*}
&- \langle u_2, \psi(0) \rangle_{L^2(\Omega)} + \langle u_1, \psi_t(0) \rangle \frac{1}{2}, -\frac{1}{2} - \langle u_0, \psi_{tt}(0) \rangle \frac{1}{2}, -\frac{1}{2} \\
&- \alpha \langle u_1, \psi(0) \rangle_{L^2(\Omega)} + \alpha \langle u_0, \psi_t(0) \rangle \frac{1}{2}, -\frac{1}{2} - b \langle u_0, (-\Delta)^s \psi(0) \rangle \frac{1}{2}, -\frac{1}{2}
\end{align*}
\]

\[
= \int_0^T \int_{\mathbb{R}^N \setminus \Omega} \left( c^2 g(x,t) + bg_t(x,t) \right) N_x \psi(x,t) \, dx dt,
\]

for each \((\psi_0, \psi_1, \psi_2) \in L^2(\Omega) \times W^{-s,2}(\Omega) \times W^{-s,2}(\Omega)\).

(b) The system (1.1) is exact controllable at time \(T > 0\), if and only if there exists a control function \(g \in L^2((0,T); W_0^{s,2}(\mathbb{R}^N \setminus \Omega))\) such that the solution \((\psi, \psi_t, \psi_{tt})\) of (2.4) satisfies

\[
\begin{align*}
&\langle u_{tt}(\cdot, T), \psi_0 \rangle_{L^2(\Omega)} - \langle u_t(\cdot, T), \psi_1 \rangle \frac{1}{2}, -\frac{1}{2} + \langle u(\cdot, T), \psi_2 \rangle \frac{1}{2}, -\frac{1}{2} \\
&+ \alpha \langle u_{tt}(\cdot, T), \psi_0 \rangle_{L^2(\Omega)} - \langle u_t(\cdot, T), \psi_1 \rangle \frac{1}{2}, -\frac{1}{2} + b \langle u(\cdot, T), (-\Delta)^s \psi_0 \rangle \frac{1}{2}, -\frac{1}{2}
\end{align*}
\]

\[
= \int_0^T \int_{\mathbb{R}^N \setminus \Omega} \left( c^2 g(x,t) + bg_t(x,t) \right) N_x \psi(x,t) \, dx dt,
\]

for each \((\psi_0, \psi_1, \psi_2) \in L^2(\Omega) \times W^{-s,2}(\Omega) \times W^{-s,2}(\Omega)\).

Now, we are able to give the proof of the first main result of this article.

**Proof of Theorem 2.3.** Using Definition 5.1, we prove that no non-trivial finite linear combination of eigenvectors can be driven to zero in finite time.

Write the initial data in Fourier series

\[
u_0 = \sum_{n=1}^{\infty} u_{0,n} \varphi_n, \quad u_1 = \sum_{n=1}^{\infty} u_{1,n} \varphi_n, \quad u_2 = \sum_{n=1}^{\infty} u_{2,n} \varphi_n,
\]

and suppose that there exists \(M \in \mathbb{N}\) such that

\[
u_{0,n} = u_{1,n} = u_{2,n} = 0, \quad \forall \ n \geq M.
\]

Assume that the system (1.1) is spectrally controllable. Then, there exists a control function \(g\) such that the solution \((u, u_t, u_{tt})\) of (1.1) with \(u_0, u_1, u_2\) given by (5.2)-(5.3) satisfy \(u(\cdot, T) = u_t(\cdot, T) = u_{tt}(\cdot, T) = 0\) in \(\Omega\). From Lemma 5.2 we have

\[
\begin{align*}
&- \langle u_2, \psi(0) \rangle_{L^2(\Omega)} + \langle u_1, \psi_t(0) \rangle \frac{1}{2}, -\frac{1}{2} - \langle u_0, \psi_{tt}(0) \rangle \frac{1}{2}, -\frac{1}{2} \\
&- \alpha \langle u_1, \psi(0) \rangle_{L^2(\Omega)} + \alpha \langle u_0, \psi_t(0) \rangle \frac{1}{2}, -\frac{1}{2} - b \langle u_0, (-\Delta)^s \psi(0) \rangle \frac{1}{2}, -\frac{1}{2}
\end{align*}
\]

\[
= \int_0^T \int_{\mathbb{R}^N \setminus \Omega} \left( c^2 g(x,t) + bg_t(x,t) \right) N_x \psi(x,t) \, dx dt,
\]

(5.4)
for any solution \((\psi, \psi_t, \psi_{tt})\) of the dual system (2.4).

We consider the following trajectories:
\[
\psi(x, t) = e^{\lambda_n j(T-t)} \varphi_n(x), \quad j = 1, 2, 3.
\] (5.5)

Replacing (5.5) in (5.4) we obtain, for any \(n \in [1, M - 1]\), the following system:
\[
- u_{2,n} e^{\lambda_n j T} + u_{1,n} \lambda_n e^{\lambda_n j T} - u_{0,n} \lambda_n^2 e^{\lambda_n j T} - \alpha e^{\lambda_n j T} (u_{1,n} - u_{0,n} \lambda_n) - bu_{0,n} \lambda_n e^{\lambda_n j T}
\]
\[
= \int_0^T \int_{\mathbb{R}^N \setminus \Omega} (c^2 g(x, t) + b g(t, x)) e^{\lambda_n j (T-t)} N_s \varphi_n(x) dx dt.
\] (5.6)

Multiplying (5.6) by \(e^{-\lambda_n j T}\), for each \(j = 1, 2, 3\), we obtain that the moment problem is to find some \(g\) that satisfies
\[
- u_{2,n} + u_{1,n} \lambda_n - u_{0,n} \lambda_n^2 - \alpha (u_{1,n} - u_{0,n} \lambda_n) - bu_{0,n} \lambda_n
\]
\[
= \int_0^T \int_{\mathbb{R}^N \setminus \Omega} (c^2 g(x, t) + b g(t, x)) e^{-\lambda_n j T} N_s \varphi_n(x) dx dt.
\] (5.7)

Next, following the works [31, 41], we define the complex function
\[
F(z) = \int_0^T \left( \int_{\mathbb{R}^N \setminus \Omega} (c^2 g(x, t) + b g(t, x)) N_s \varphi_n(x) dx \right) e^{iz t} dt.
\] (5.8)

According to the Paley–Wiener theorem, \(F\) is an entire function. Due to (5.3), from (5.7) we obtain that \(F\) satisfies \(F(i \lambda_{n,j}) = 0\), for all \(n \geq M\). Besides, we know that \(\lambda_{n,1} \to -\frac{c'}{T} \) as \(n \to \infty\) (see Lemma 4.2). Then, \(F\) is zero in a set with finite accumulation point. This implies that \(F \equiv 0\). It follows from (5.7) and (5.8) that
\[
\begin{pmatrix}
\alpha \lambda_{n,1} - \lambda_{n,1}^2 - b \lambda_n & \lambda_{n,1} - \alpha & -1 \\
\alpha \lambda_{n,2} - \lambda_{n,2}^2 - b \lambda_n & \lambda_{n,2} - \alpha & -1 \\
\alpha \lambda_{n,3} - \lambda_{n,3}^2 - b \lambda_n & \lambda_{n,3} - \alpha & -1
\end{pmatrix}
\begin{pmatrix}
u_{0,n} \\
u_{1,n} \\
u_{2,n}
\end{pmatrix}
= \begin{pmatrix}0 \\
0 \\
0\end{pmatrix}.
\]

Calculating we get that
\[
\det(B) = (\lambda_{n,1} - \lambda_{n,2}) (\lambda_{n,1} - \lambda_{n,3}) (\lambda_{n,2} - \lambda_{n,3}) \neq 0.
\]

Hence, the matrix \(B\) is invertible and we can then conclude that \(u_{0,n} = u_{1,n} = u_{2,n} = 0\), for \(n < M\). Thus the trivial state is the only one which can be steered to zero. We have shown that the system is not spectrally controllable. It is clear from the proof that this implies that the system is not exact or null controllable. The proof is finished.

\[\Box\]

### 5.2. The unique continuation property.

**Proof of Theorem 2.6.** Let \(\Omega \subset \mathbb{R}^N \setminus \Omega\) be an arbitrary non-empty open set. Suppose that \(N_s \psi = 0\) in \(\Omega \times (0, T)\). Then, for all \((x, t) \in \Omega \times (0, T)\) we have that
\[
N_s \psi(x, t) = \sum_{n=1}^{\infty} \left(D_n(T-t) \psi_{0,n} + E_n(T-t) \psi_{1,n} + F_n(T-t) \psi_{2,n}\right) N_s \varphi_n(x) = 0.
\]

Since \(N_s \psi\) can be analytically extended to \(\Sigma_T\), it follows that for all \((x, t) \in \Omega \times (-\infty, T)\),
\[
N_s \psi(x, t) = \sum_{n=1}^{\infty} \left(D_n(T-t) \psi_{0,n} + E_n(T-t) \psi_{1,n} + F_n(T-t) \psi_{2,n}\right) N_s \varphi_n(x) = 0.
\] (5.9)

Let \(\{\lambda_k\}_{k \in \mathbb{N}}\) be the set of all eigenvalues of the operator \((-\Delta)^\frac{T}{2}\) and let \(\{\varphi_{k,j}\}_{1 \leq j \leq k}\) be an orthonormal basis for \(\ker(\lambda_k - (-\Delta)^\frac{T}{2})\). Then, (5.9) can be rewritten as
Using the analytic continuation in \( z \) that the operator \( \mathcal{N} \), we recall that we obtain that there is a constant \( N \) on the data of \( \mathcal{N} \) (see (4.38)), and letting

\[
\psi_m(\cdot, t) := \sum_{k=1}^{m_k} \left( \sum_{j=1}^{m_k} \psi_{0,k_j} \mathcal{N} \varphi_{k_j}(x) \right) e^{z(t-T)} D_k(T-t) + \sum_{k=1}^{m_k} \left( \sum_{j=1}^{m_k} \psi_{1,k_j} \mathcal{N} \varphi_{k_j}(x) \right) e^{z(t-T)} E_k(T-t)
+ \sum_{k=1}^{m_k} \left( \sum_{j=1}^{m_k} \psi_{2,k_j} \mathcal{N} \varphi_{k_j}(x) \right) e^{z(t-T)} F_k(T-t),
\]

we obtain that there is a constant \( C > 0 \) such that for every \( t \in [0, T] \),

\[
\| \psi_m(\cdot, t) \|_{L^2(\mathbb{R}^N \setminus \Omega)} \leq Ce^{\eta(t-T)} \left( \| \psi_0 \|_{W^{s,2}(\mathbb{M})} + \| \psi_1 \|_{W^{s,2}(\mathbb{M})} + \| \psi_2 \|_{L^2(\Omega)} \right).
\]

The right hand side of (5.11) is integrable over \( t \in (-\infty, T) \) and

\[
\int_{-\infty}^{T} e^{\eta(t-T)} \left( \| \psi_0 \|_{W^{s,2}(\mathbb{M})} + \| \psi_1 \|_{W^{s,2}(\mathbb{M})} + \| \psi_2 \|_{L^2(\Omega)} \right) dt = \frac{1}{\eta} \left( \| \psi_0 \|_{W^{s,2}(\mathbb{M})} + \| \psi_1 \|_{W^{s,2}(\mathbb{M})} + \| \psi_2 \|_{L^2(\Omega)} \right).
\]

By the Lebesgue dominated convergence theorem, we can deduce that

\[
\int_{-\infty}^{T} e^{z(t-T)} \left[ \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} \psi_{0,k_j} \mathcal{N} \varphi_{k_j}(x) \right) D_k(T-t) + \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} \psi_{1,k_j} \mathcal{N} \varphi_{k_j}(x) \right) E_k(T-t)
+ \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} \psi_{2,k_j} \mathcal{N} \varphi_{k_j}(x) \right) F_k(T-t) \right] dt
= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \left( G_k(z) \psi_{0,k_j} + H_k(z) \psi_{1,k_j} + I_k(z) \psi_{2,k_j} \right) \mathcal{N} \varphi_{k_j}(x), \quad x \in \mathbb{R}^N \setminus \Omega, \ \text{Re}(z) > 0,
\]

where

\[
\begin{align*}
G_k(z) &= \frac{\lambda_k}{\xi_k(1-z-\lambda_k)} - \frac{\lambda_{k,1}}{\lambda_{k,2} + \lambda_{k,3}} + \frac{\lambda_{k,1}}{\lambda_{k,2}}, \\
H_k(z) &= -\frac{\xi_{k,1}(z-\lambda_{k,1})}{\xi_{k,2}(1-z-\lambda_{k,2})} + \frac{\xi_{k,2}}{\xi_{k,3}(1-z-\lambda_{k,3})}, \\
I_k(z) &= \frac{1}{\xi_{k,1}(z-\lambda_{k,1})} - \frac{\xi_{k,2}}{\xi_{k,3}(z-\lambda_{k,2})}.
\end{align*}
\]

We recall that \( \xi_{k,1}, \xi_{k,2} \) and \( \xi_{k,3} \) are given in (4.10).

From (5.10) we get that

\[
\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \left( G_k(z) \psi_{0,k_j} + H_k(z) \psi_{1,k_j} + I_k(z) \psi_{2,k_j} \right) \mathcal{N} \varphi_{k_j}(x) = 0, \quad x \in \mathcal{O}, \ \text{Re}(z) > 0.
\]

Using the analytic continuation in \( z \), we obtain that (5.12) holds for every \( z \in \mathbb{C} \setminus \{ \lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3} \}_{k \in \mathbb{N}}. \)
We take a small circle about \( \lambda_k, h \), for some \( h \in \{1, 2, 3\} \), but not including \( \{\lambda_{l,j}\}_{l \neq k, j \neq h} \), with \( j \in \{1, 2, 3\} \).

Then, integrating over that circle we get the following system:

\[
\begin{align*}
\sum_{j=1}^{m_k} \left[ \frac{\lambda_k,2 \lambda_k,3}{\xi_k,1} \psi_{0,k_j} - \frac{\lambda_k,2 + \lambda_k,3}{\xi_k,1} \psi_{1,k_j} + \frac{1}{\xi_k,1} \psi_{2,k_j} \right] N_s \varphi_{k_j}(x) &= 0, \quad x \in \Omega, \\
\sum_{j=1}^{m_k} \left[ -\frac{\lambda_k,1 \lambda_k,3}{\xi_k,2} \psi_{0,k_j} + \frac{\lambda_k,1 + \lambda_k,3}{\xi_k,2} \psi_{1,k_j} - \frac{1}{\xi_k,2} \psi_{2,k_j} \right] N_s \varphi_{k_j}(x) &= 0, \quad x \in \Omega, \\
\sum_{j=1}^{m_k} \left[ \frac{\lambda_k,1 \lambda_k,2}{\xi_k,3} \psi_{0,k_j} - \frac{\lambda_k,1 + \lambda_k,2}{\xi_k,3} \psi_{1,k_j} + \frac{1}{\xi_k,3} \psi_{2,k_j} \right] N_s \varphi_{k_j}(x) &= 0, \quad x \in \Omega.
\end{align*}
\]

Let

\[
\begin{align*}
\psi^1_k &= \sum_{j=1}^{m_k} \left[ \frac{\lambda_k,2 \lambda_k,3}{\xi_k,1} \psi_{0,k_j} - \frac{\lambda_k,2 + \lambda_k,3}{\xi_k,1} \psi_{1,k_j} + \frac{1}{\xi_k,1} \psi_{2,k_j} \right] \varphi_{k_j}, \\
\psi^2_k &= \sum_{j=1}^{m_k} \left[ -\frac{\lambda_k,1 \lambda_k,3}{\xi_k,2} \psi_{0,k_j} + \frac{\lambda_k,1 + \lambda_k,3}{\xi_k,2} \psi_{1,k_j} - \frac{1}{\xi_k,2} \psi_{2,k_j} \right] \varphi_{k_j}, \\
\psi^3_k &= \sum_{j=1}^{m_k} \left[ \frac{\lambda_k,1 \lambda_k,2}{\xi_k,3} \psi_{0,k_j} - \frac{\lambda_k,1 + \lambda_k,2}{\xi_k,3} \psi_{1,k_j} + \frac{1}{\xi_k,3} \psi_{2,k_j} \right] \varphi_{k_j}.
\end{align*}
\]

It follows from (5.13), (5.14) and (5.15) that \( N_s \psi^1_k = N_s \psi^2_k = N_s \psi^3_k = 0 \) in \( \Omega \). We have shown that

\[
(-\Delta)^s \psi^l_k = \lambda_k \psi^l_k \quad \text{in} \quad \Omega \quad \text{and} \quad N_s \psi^l_k = 0, \quad l = 1, 2, 3.
\]

From Lemma 3.3, we deduce that \( \psi^l_k = 0 \), for every \( k \in \mathbb{N} \) and \( l = 1, 2, 3 \). Since the system \( \{\varphi_{k_j}\}_{1 \leq j \leq m_k} \) is linearly independent in \( L^2(\Omega) \), we get that

\[
\begin{pmatrix}
\lambda_k,2 \\
-\lambda_k,1 \\
\lambda_k,1 \\
\lambda_k,3
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\xi_k,1} \\
\frac{1}{\xi_k,2} \\
\frac{1}{\xi_k,3}
\end{pmatrix}
\begin{pmatrix}
\psi_{0,k} \\
\psi_{1,k} \\
\psi_{2,k}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

A simple calculation shows that the determinant of the matrix \( A \) is given by

\[
\det(A) = \frac{-i}{2 \text{Im}(\lambda_k,2) [\text{Re}(\lambda_k,2) - \lambda_k,1]^2 + (\text{Im}(\lambda_k,2))^2} \neq 0.
\]

Since the matrix \( A \) is invertible, we can deduce that

\[
\psi_{0,k} = \psi_{1,k} = \psi_{2,k} = 0, \quad k \in \mathbb{N}.
\]

Since the solution \( (\psi, \psi_t, \psi_{tt}) \) of the adjoint system is unique, we can conclude that \( \psi = 0 \) in \( \Omega \times (0, T) \). The proof is finished.

5.3. The approximate controllability. We obtain the result as a direct consequence of the unique continuation property for the adjoint system (Theorem 2.6).

**Proof of Theorem 2.7.** Let \( g \in \mathcal{D}(\mathbb{R} \times (0, T)) \), \( (u, u_t, u_{tt}) \) the unique weak solution of (1.1) with \( u_0 = u_1 = u_2 = 0 \) and let \( (\psi, \psi_t, \psi_{tt}) \) be the unique weak solution of (2.4) with \( (\psi_0, \psi_1, \psi_2) \in W^{s,2}(\overline{\Omega}) \times W^{s,2}(\overline{\Omega}) \times L^2(\Omega) \). Firstly, it follows from Theorems 4.4 that \( u \in C^\infty([0, T]; W^{s,2}(\mathbb{R}^N)) \). Thus \( u(\cdot, T) \in L^2(\Omega), u_t(\cdot, T) \in W^{s,2}(\overline{\Omega}) \) and \( u_{tt}(\cdot, T) \in W^{s,2}(\overline{\Omega}) \). Secondly, it follows from Theorem 2.5 that \( \psi \in L^2((0, T); L^2(\Omega)) \). Therefore, using the identity (5.1) we can deduce that
\[ \langle u_{tt}(\cdot, T), \psi_0 \rangle_{L^2} - \langle u_t(\cdot, T), \psi_1 \rangle_{L^2} + \langle u(\cdot, T), \psi_2 \rangle_{L^2} \\
+ \alpha \left( \langle u_t(\cdot, T), \psi_0 \rangle_{L^2} - \langle u(\cdot, T), \psi_1 \rangle_{L^2} \right) + b(u(\cdot, T), (-\Delta)^s\psi_0)_{L^2} \]
\[ = \int_0^T \int_{\mathbb{R}^N \setminus \Omega} \left( c^2 g(x, t) + b g_1(x, t) \right) N_\alpha \psi(x, t) dx dt, \quad (5.16) \]

If \((\psi_0, \psi_1, \psi_2) \in D((-\Delta)^s_D) \times \cdots \times L^2(\Omega) \hookrightarrow W^{s, 2}_0(\Omega) \times W^{s, 2}_0(\Omega) \times L^2(\Omega),\) then (5.16) becomes
\[ \langle u_{tt}(\cdot, T), \psi_0 \rangle_{L^2} - \langle u_t(\cdot, T), \psi_1 \rangle_{L^2} + \langle u(\cdot, T), \psi_2 \rangle_{L^2} + \alpha \left( \langle u_t(\cdot, T), \psi_0 \rangle_{L^2} - \langle u(\cdot, T), \psi_1 \rangle_{L^2} \right) + b(u(\cdot, T), (-\Delta)^s\psi_0)_{L^2} = 0, \quad (5.17) \]

Since \(D((-\Delta)^s_D) \times \cdots \times L^2(\Omega)\) is dense in \(W^{s, 2}_0(\Omega) \times W^{s, 2}_0(\Omega) \times L^2(\Omega),\) to prove that the set \(\{(u(\cdot, T), u_t(\cdot, T), u_{tt}(\cdot, T)) : g \in \mathcal{D}(\mathcal{O} \times (0, T))\}\) is dense in \(L^2(\Omega) \times W^{-s, 2}(\Omega) \times W^{-s, 2}(\Omega),\) it suffices to show that if \((\psi_0, \psi_1, \psi_2) \in D((-\Delta)^s_D) \times \cdots \times L^2(\Omega)\) is such that
\[ \langle u_{tt}(\cdot, T), \psi_0 \rangle_{L^2} - \langle u_t(\cdot, T), \psi_1 \rangle_{L^2} + \langle u(\cdot, T), \psi_2 \rangle_{L^2} + \alpha \left( \langle u_t(\cdot, T), \psi_0 \rangle_{L^2} - \langle u(\cdot, T), \psi_1 \rangle_{L^2} \right) + b(u(\cdot, T), (-\Delta)^s\psi_0)_{L^2} = 0, \quad (5.18) \]

for any \(g \in \mathcal{D}(\mathcal{O} \times (0, T)),\) then \(\psi_0 = \psi_1 = \psi_2 = 0.\)

Indeed, let \((\psi_0, \psi_1, \psi_2) \in D((-\Delta)^s_D) \times \cdots \times L^2(\Omega)\) satisfy (5.18). It follows from (5.17) and (5.18) that
\[ \int_0^T \int_{\mathbb{R}^N \setminus \Omega} \left( c^2 g(x, t) + b g_1(x, t) \right) N_\alpha \psi(x, t) dx dt = 0, \]

for any \(g \in \mathcal{D}(\mathcal{O} \times (0, T)).\) Recall that \(b > 0.\) By the fundamental lemma of the calculus of variations, we can deduce that
\[ N_\alpha \psi = 0 \quad \text{in} \quad \mathcal{O} \times (0, T). \]

It follows from Theorem 2.6 that \(\psi = 0 \quad \text{in} \quad \Omega \times (0, T).\) Since the solution \((\psi, \psi_t, \psi_{tt})\) of (2.4) is unique, we can conclude that \(\psi_0 = \psi_1 = \psi_2 = 0 \quad \text{in} \quad \Omega.\) The proof is finished. \(\square\)

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