Further study on Horozov-Iliev’s method of estimating the number of limit cycles

Xiaoyan Chen¹ & Maoan Han²,*

¹Department of Computer Science and Mathematics, Changsha University, Changsha 410022, China; ²Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

Email: xychen@hnu.edu.cn, mahan@shnu.edu.cn

Received June 22, 2021; accepted November 29, 2021; published online May 6, 2022

Abstract In the study of the number of limit cycles of near-Hamiltonian systems, the first order Melnikov function plays an important role. This paper aims to generalize Horozov-Iliev’s method to estimate the upper bound of the number of zeros of the function.

Keywords near-Hamiltonian system, piecewise smooth system, Melnikov function, limit cycle

MSC(2020) 34C05, 34C07, 37G15

Citation: Chen X Y, Han M A. Further study on Horozov-Iliev’s method of estimating the number of limit cycles. Sci China Math, 2022, 65: 2255–2270, https://doi.org/10.1007/s11425-021-1933-7

1 Introduction

Consider the planar differential system

\[
\begin{align*}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y),
\end{align*}
\] (1.1)

where \(P(x, y)\) and \(Q(x, y)\) are real polynomials in the variables \(x\) and \(y\) of degree less than or equal to \(n\). As we know, the determination of the number and positions of limit cycles for the system (1.1) is the second part of Hilbert’s 16th problem [9]. This problem is difficult, and is still open even for quadratic systems. In 1983, Arnold posed a weak Hilbert’s 16th problem which asks the maximum number of zeros of the Abelian integral

\[
I(h) = \oint_{H=h} (gdx - fdy),
\]

where \(H, f\) and \(g\) are all the real polynomial functions of \(x\) and \(y\). This problem has been studied by many researchers (see [5, 6, 12] and the references therein).

In this paper, we consider a system of the form

\[
\begin{align*}
\dot{x} &= \frac{\partial H(x, y)}{\partial y} + \epsilon p(x, y), \\
\dot{y} &= -\frac{\partial H(x, y)}{\partial x} + \epsilon q(x, y),
\end{align*}
\] (1.2)

*Corresponding author

© Science China Press and Springer-Verlag GmbH Germany, part of Springer Nature 2022
math.scichina.com link.springer.com
where $\epsilon \in \mathbb{R}$ is a small parameter, and $H(x, y)$, $p(x, y)$ and $q(x, y)$ are smooth functions of $x$ and $y$. Suppose that the system (1.2) $|_{\epsilon = 0}$ has a family of periodic orbits $L_h$ defined by $H(x, y) = h$ for $h \in (\alpha, \beta)$.

As we know, the first order Melnikov function of the system (1.2) has the following form:

$$M(h) = \int_{L_h} (q(x, y)dx - p(x, y)dy), \quad h \in (\alpha, \beta)$$

(see [5]). In 2010, Liu and Han [15] introduced the first order Melnikov function and established its formula when (1.2) is a piecewise smooth system. Recently, the formula of the first order Melnikov function $M(h)$ was extended to piecewise smooth near-Hamiltonian or near-integrable systems with multiple switching lines or curves (see [22, 23]). As we know, the total number of zeros of $M(h)$ can control the number of limit cycles bifurcating from a period annulus. More precisely, Han and Yang [8] showed that if $M(h)$ has at most $k$ zeros in $h \in (\alpha, \beta)$, multiplicities taken into account, then for small $|\epsilon| > 0$, the system (1.2) has at most $k$ limit cycles bifurcating from the period annulus defined by $L_h$, multiplicities taken into account. One can find in [5, 7, 8, 15] more information on the relationship between the number of zeros of $M(h)$ and the number of limit cycles.

The main aim of this paper is to generalize a known method to estimate the upper bound of the maximum number of zeros of $M(h)$ in $h$ on $(\alpha, \beta)$. Up to now, there have been certain classical and fundamental approaches in this aspect. Let us briefly introduce some of them as follows.

(i) Horozov-Iliev’s method. Suppose that there exist functions $G$ and $F$ defined on $(\alpha, \beta)$ such that

$$\left(\frac{M(h)}{G(h)}\right)' = \frac{F(h)}{G^2(h)}, \quad h \in (\alpha, \beta) \\setminus \mathbb{S},$$

(1.3)

where $\mathbb{S}$ is the set of the zeros of $G(h)$ on $(\alpha, \beta)$. Then we have

$$\lambda \leq \mu + p + 1,$$

(1.4)

where $\lambda$, $\mu$ and $p$ are the numbers of zeros of $M(h)$, $F(h)$ and $G(h)$ on $(\alpha, \beta)$, respectively (see [10, Lemma 4.2]). Obviously, (1.3) provides a method to estimate the upper bound of the number of zeros of $M(h)$ on $(\alpha, \beta)$. Since it was firstly obtained by Horozov and Iliev in 1998, we call it Horozov-Iliev’s method. In fact, this method has been widely applied by researchers in recent years (see [14, 24, 28, 30] and the references therein).

(ii) Chebyshev criterion. Let $(f_1, f_2, \ldots, f_n)$ be an ordered set of $C^\infty$ functions on the interval $(\alpha, \beta)$. The ordered set $(f_1, f_2, \ldots, f_n)$ is called an extended complete Chebyshev system (in short ECT-system) on $(\alpha, \beta)$ if for all $1 \leq k \leq n$, any nontrivial linear combination $\sum_{i=1}^{k} a_i f_i(h) = 0$ has at most $k-1$ isolated zeros on $(\alpha, \beta)$ counted with multiplicities. Chebyshev criterion says that the ordered set $(f_1, f_2, \ldots, f_n)$ is an ECT-system if and only if for each $k = 1, 2, \ldots, n$,

$$W_k(h) = \begin{vmatrix}
 f_1(h) & f_2(h) & \cdots & f_k(h) \\
 f_1'(h) & f_2'(h) & \cdots & f_k'(h) \\
 \vdots & \vdots & \ddots & \vdots \\
 f_1^{(k-1)}(h) & f_2^{(k-1)}(h) & \cdots & f_k^{(k-1)}(h)
\end{vmatrix} \neq 0, \quad h \in (\alpha, \beta)$$

(see [11]). Chebyshev theory was also widely used and extended by many researchers (see [1, 3, 16, 17, 26, 29] and the references therein).

(iii) Petrov’s method or the method of the argument principle. This method was firstly used by Petrov to study the perturbation of elliptic Hamiltonians of degree 3 and degree 4 in the 1980s (see [18–21]). The main idea of this method is to extend $h$ to a complex number and estimate the number of zeros of $M(h)$ by the argument principle in a suitable domain (see [4, 13, 27] and the references therein).

In [23], Wang and Han considered a piecewise smooth near-Hamiltonian system of the following form:

$$\begin{pmatrix}
 \dot{x} \\
 \dot{y}
\end{pmatrix} = \begin{pmatrix}
 \frac{\partial H_k(x, y)}{\partial y} + \epsilon p_k(x, y) \\
 \frac{\partial H_k(x, y)}{\partial x} + \epsilon q_k(x, y)
\end{pmatrix}, \quad (x, y) \in D_k, \quad k = 1, 2, 3, 4,$$

(1.5)
where
\[ p_k(x, y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j, \quad q_k(x, y) = \sum_{i+j=0}^{n} b_{ij} x^i y^j, \]  
(1.6)

and
\[ D_1 = \{(x, y) \mid x > 0, y > 0\}, \quad D_2 = \{(x, y) \mid x < 0, y > 0\}, \]
\[ D_3 = \{(x, y) \mid x < 0, y < 0\}, \quad D_4 = \{(x, y) \mid x > 0, y < 0\}, \]

\( \epsilon \) is small enough, and \( H_k \in C^\infty \). Specially, Wang and Han [23] studied the system (1.5) for the following four cases:

**Case (1)** \( H_1(x, y) \) is given by
\[ H_1(x, y) = \lambda_1 (x - 1) (y - 1), \quad (x, y) \in D_1, \]  
(1.7)

and \( H_k(x, y), k = 2, 3, 4 \) are given by
\[ H_k(x, y) = -\frac{w_k}{2} (x^2 + y^2), \quad (x, y) \in D_k, \]  
(1.8)

where \( \lambda_1 > 0 \) and \( w_k > 0 \).

**Case (2)** \( H_1(x, y) \) is given by (1.7), \( H_k(x, y), k = 3, 4 \) are given by (1.8), and
\[ H_2(x, y) = -\lambda_2 (x + 1) (y - 1), \quad (x, y) \in D_2, \]  
(1.9)

where \( \lambda_2 > 0 \).

**Case (3)** \( H_i(x, y), i = 1, 2, 4 \) are given by (1.7)–(1.9), respectively, and
\[ H_3(x, y) = \lambda_3 (x + 1) (y + 1), \quad (x, y) \in D_3, \]  
(1.10)

where \( \lambda_3 > 0 \).

**Case (4)** \( H_i(x, y), i = 1, 2, 3 \) are given by (1.7), (1.9) and (1.10), respectively, and
\[ H_4(x, y) = -\lambda_4 (x - 1) (y + 1), \quad (x, y) \in D_4, \]  
(1.11)

where \( \lambda_4 > 0 \).

Wang and Han [23] obtained that for each one of the above cases, the system (1.5) has a family of periodic orbits given by \( L_h = \bigcup_{k=1}^{4} L^k_h \) for \( h \in (0, 1) \), where
\[ L^k_h = \left\{ (x, y) \mid H_k(x, y) = -\frac{w_k}{2} (1 - h)^2, \quad (x, y) \in D_k \right\} \]

for \( H_k(x, y) \) defined by (1.8) and
\[ L^k_h = \left\{ (x, y) \mid H_k(x, y) = \lambda_k h, \quad (x, y) \in D_k \right\} \]

for \( H_k(x, y) \) defined by (1.7) and (1.9)–(1.11). Clearly, if \( h \to 1 \), \( L_h \) approaches the origin which is a center for the cases (1)–(3) and a generalized singular point for the case (4). If \( h \to 0 \), \( L_h \) approaches a compound homoclinic loop \( L_0 \) with a saddle (1, 1) for the case (1) and a compound \( k \)-polycycle \( L_0 \) for the case (k), \( k = 2, 3, 4 \). For the definition of a compound homoclinic loop or \( k \)-polycycle, see [23].

Denote by \( Z_k(n) \) the maximum number of zeros of the first order Melnikov function of the system (1.5) for the case (k) on the open interval (0, 1) for all possible \( p_i \) and \( q_i \) satisfying (1.6) and \( i, k = 1, 2, 3, 4 \). For the case (k), denote by \( N^k_{\text{Hopf}}(n) \) the maximum number of limit cycles produced in Hopf bifurcation near the origin for all possible \( p_i \) and \( q_i \) satisfying (1.6) and \( i = 1, 2, 3, 4, k = 1, 2, 3 \). Denote by \( N^k_{\text{Homoc}}(n) \) and by \( N^k_{\text{Hetec}}(n) \), respectively, the maximum numbers of limit cycles produced in the compound homoclinic bifurcation and the compound \( k \)-polycycle bifurcation for all possible \( p_i \) and \( q_i \) satisfying (1.6) for the case (k), \( i = 1, 2, 3, 4, k = 2, 3, 4 \). Then the main results in [23] can be stated as follows.
Theorem 1.1 (See [23]). Consider the system (1.5) with \( n \geq 2 \). We have

(i) \( N^\text{Homoc}(n) \geq n + 2 + \frac{[n+1]}{2} \);

(ii) \( N^k_\text{Hopf}(n) \geq n + 2 + \frac{[n+1]}{2} \) for \( k = 1, 2, 3 \);

(iii) \( N^k_\text{Hetec} \geq n + 2 + \frac{[n+1]}{2} \) for \( k = 2, 3, \) and \( N^4_\text{Hetec} \geq n + 1 + \frac{[n+1]}{2} \);

(iv) \( n + 2 + \frac{[n+1]}{2} \leq Z_k(n) \leq n + 1 + 2\frac{[n+1]}{2} \) for \( k = 1, 2, 3, \) and \( n + 1 + \frac{[n+1]}{2} \leq Z_4(n) \leq n + 1 + 2\frac{[n+1]}{2} \).

In [25], Yang considered the following two piecewise smooth near-integrable systems:

\[
\begin{align*}
\frac{\dot{x}}{\dot{y}} &= \left( \sqrt{2}xy + \epsilon f_k(x,y), \frac{\sqrt{2}}{x^2}(1 - x^2 + 2y^2) + \epsilon g_k(x,y) \right), \quad (x, y) \in D_k, \quad k = 1, 2, 3, 4 \tag{1.12} \\
\frac{\dot{x}}{\dot{y}} &= \left( \frac{\sqrt{2}}{x^2}xy + \epsilon f_k(x,y), \frac{\sqrt{2}}{x^2}(2 - 2x + y^2) + \epsilon g_k(x,y) \right), \quad (x, y) \in D_k, \quad k = 1, 2, 3, 4, \tag{1.13}
\end{align*}
\]

where

\[
f_k(x,y) = \sum_{i+j=0}^n c_{kij}x^iy^j, \quad g_k(x,y) = \sum_{i+j=0}^n d_{kij}x^iy^j,
\]

\[
D_1 = \{(x, y) \mid x > 1, y > 0\}, \quad D_2 = \{(x, y) \mid x > 1, y < 0\},
\]

\[
D_3 = \{(x, y) \mid x < 1, y < 0\}, \quad D_4 = \{(x, y) \mid x < 1, y > 0\}
\]

and \( \epsilon \) is small enough. From [25], the system (1.12) \( |\epsilon| = 0 \) has two isochronous centers \((\pm 1, 0)\) and the families of periodic orbits around the centers \((\pm 1, 0)\) are given by \( \mathcal{T}_h = \{(x, y) \mid H(x,y) = h\} \) for \( h \in (-\infty, -1) \cup (0, +\infty) \), where

\[
H(x,y) = \frac{1}{x} \left( \frac{1}{2}y^2 + \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} \right).
\]

One also knows from [25] that the system (1.13) \( |\epsilon| = 0 \) has one isochronous center \((1, 0)\) and the family of periodic orbits around the center \((1, 0)\) is given by \( \mathcal{L}_h = \{(x, y) \mid H(x,y) = h\} \) for \( h \in (0, 1) \), where

\[
H(x,y) = \frac{1}{x^2} \left( \frac{1}{2}y^2 + x^2 - 2x + 1 \right).
\]

The main results obtained in [25] are as follows.

Theorem 1.2 (See [25]). (i) The number of limit cycles of the system (1.12) bifurcating from the period annuli around the isochronous centers \((\pm 1, 0)\) is not more than \( 11n + 34 \) for \( n \geq 1 \).

(ii) The number of limit cycles of the system (1.13) bifurcating from the period annulus around the isochronous center \((1, 0)\) is not more than \( 21n - 25 \) for \( n \geq 3 \) and \( 39 \) for \( n = 1, 2 \).

This paper aims to generalize Horozov-Iliev’s method and apply it to improve the two theorems above. The rest of this paper is organized as follows. In Section 2, we establish and prove our main theorem which is a development of Horozov-Iliev’s method. In Sections 3 and 4, we apply our main theorem to estimate the numbers of limit cycles of the systems (1.5), (1.12) and (1.13), respectively. By using our main theorem, we obtain sharper and more precise estimation of the numbers of limit cycles for these three systems. In Section 5, we present a conclusion.

2 Development of Horozov-Iliev’s method

In this section, we establish the following theorem.

Theorem 2.1. Let \( M : (\alpha, \beta) \to \mathbb{R} \) be a \( C^k \) function and \( k \gg 1 \). Suppose that there exist two \( C^k \) functions \( G \) and \( \tilde{F} \) defined on \((\alpha, \beta)\) such that

\[
\left( \frac{M(h)}{G(h)} \right)^{(m)} = \frac{\tilde{F}(h)}{G^{m+1}(h)}, \quad h \in (\alpha, \beta) \setminus \mathcal{S}, \tag{2.1}
\]

where
where \(1 \leq m \leq k\) and \(S\) is the set of zeros of \(G(h)\) in \(h\) on \((\alpha, \beta)\). Then we have

\[
\lambda \leq \mu + mp + m,
\]

where \(\lambda\), \(\mu\) and \(p\) are the numbers of zeros of \(M(h)\), \(\bar{F}(h)\) and \(G(h)\) in \(h\) on \((\alpha, \beta)\) including the multiplicities, respectively.

**Proof.** We prove (2.2) for two cases of \(p = 0\) and \(p \neq 0\), separately.

If \(p = 0\), then by (2.1), \((M(h))/G(h))^{(m)}\) has \(\mu\) zeros on \((\alpha, \beta)\), multiplicities taken into account. According to [8, Lemma 2.3], it follows that \((M(h))/G(h))^{(m-1)}\) has at most \(\mu + 1\) zeros on \((\alpha, \beta)\), multiplicities taken into account. In a similar way, we can obtain that \((M(h))/G(h))^{(m)}\) has at most \(\mu + m\) zeros on \((\alpha, \beta)\), multiplicities taken into account. Therefore, (2.2) holds for \(p = 0\).

If \(p \neq 0\), there are exactly \(s\) different zeros of \(G(h)\) on \((\alpha, \beta)\), denoted by \(h_1, \ldots, h_s\), for some integer \(s\) satisfying \(1 \leq s \leq p\). We can assume that \(h_1 < h_2 < \cdots < h_s\) and they have multiplicities \(a_1, \ldots, a_s\), respectively. Then

\[
a_1 + \cdots + a_s = p, \quad a_1 \geq 1, \ldots, a_s \geq 1.
\]

For \(i = 1, 2, \ldots, s\), we assume that \(h_i\) is a zero of \(M(h)\) and \(\bar{F}(h)\) with multiplicities \(l_i\) and \(t_i\), respectively. If \(M(h_i) \neq 0\) \((h_i) \neq 0\) for some \(i\), then we take \(l_i = 0\) \((t_i = 0)\). Thus, \(l_i \geq 0\) and \(t_i \geq 0\) for \(i = 1, 2, \ldots, s\). Next, we prove \(l_i \leq t_i\), \(i = 1, 2, \ldots, s\). By [8, Lemma 2.2], there exist functions \(m_i \in C^{k-l_i}, g_i \in C^{k-a_i}\), and \(f_i \in C^{k-t_i}\) satisfying \(m_i(h_i) \neq 0, g_i(h_i) \neq 0\) and \(f_i(h_i) \neq 0\) such that for \(i = 1, 2, \ldots, s\),

\[
M(h) = (h - h_i)^{l_i} m_i(h),
\]

\[
G(h) = (h - h_i)^{a_i} g_i(h),
\]

\[
\bar{F}(h) = (h - h_i)^{t_i} f_i(h).
\]

Then for \(i = 1, 2, \ldots, s\),

\[
\frac{M(h)}{G(h)} = \frac{(h - h_i)^{l_i-a_i} m_i(h)}{g_i(h)} \quad \text{(2.3)}
\]

and

\[
\frac{\bar{F}(h)}{G^{a_i+1}(h)} = \frac{(h - h_i)^{l_i-(a_i+1)} f_i(h)}{[g_i(h)]^{a_i+1}} \quad \text{(2.4)}
\]

Thus, by (2.3), for \(i = 1, 2, \ldots, s\),

\[
(M(h))/G(h))^{(m)} = (h - h_i)^{l_i-a_i-m} \phi_i(h), \quad 0 < |h - h_i| \ll 1,
\]

where the function \(\phi_i\) is \(C^{k-m}\) for \(0 < |h - h_i| \ll 1\). Here, \(\phi_i(h_i)\) may be zero or not zero for \(i = 1, 2, \ldots, s\). Then from (2.1) and (2.4), we have

\[
l_i - a_i - m \leq t_i - (m + 1) a_i, \quad i = 1, 2, \ldots, s,
\]

i.e.,

\[
l_i \leq t_i - m a_i + m, \quad i = 1, 2, \ldots, s.
\]

Hence, by \(a_i \geq 1\), we have \(l_i \leq t_i, \ i = 1, 2, \ldots, s\).

Let \(h_0 = \alpha\) and \(h_{s+1} = \beta\), and we assume that \(M(h)\) and \(\bar{F}(h)\), respectively, have \(\lambda_i\) and \(\mu_i\) zeros on \((h_i, h_{i+1})\) for \(i = 0, 1, \ldots, s\), multiplicities taken into account. Then by the proof of the case \(p = 0\), we
have $\lambda_i \leq \mu_i + m$, $i = 0, 1, \ldots, s$. Therefore, the total number of zeros of $M(h)$ on $(\alpha, \beta)$ is estimated by

$$
\lambda = \sum_{j=1}^{s} t_j + \sum_{j=0}^{s} \lambda_j \\
\leq \sum_{j=1}^{s} t_j + \sum_{j=0}^{s} (\mu_j + m) \\
= \sum_{j=1}^{s} t_j + \sum_{j=0}^{s} \mu_j + (s + 1)m \\
= \mu + (s + 1)m \\
\leq \mu + mp + m,
$$

since $s \leq p$. The proof is ended.

\begin{proof}
\end{proof}

**Remark 2.2.** From the above proof, it is easy to see that Theorem 2.1 also holds for $C^\infty$ functions $M(h)$, $G(h)$ and $\tilde{F}(h)$ defined in closed intervals or half-closed and half-open intervals or unbounded intervals. Obviously, the method to find an upper bound of the number of zeros of $M(h)$ introduced in Theorem 2.1 is a generalization of Horozov-Iliev's method. Moreover, it improves Horozov-Iliev's method, due to multiplicities taken into account in Theorem 2.1.

3 The number of limit cycles of the system (1.5)

In this section, we estimate the number of limit cycles bifurcating from the period annulus $\bigcup_{h \in (0,1)} L_h$ of the system (1.5) for the cases (1)–(4). For this purpose, we first introduce some conclusions obtained in [23].

For $n \geq 2$, denote by $M_k(h)$ the first order Melnikov function of the system (1.5) for the case $(k)$, $k = 1, 2, 3, 4$. Then from [23, Lemmas 4.1, 5.1, 6.1 and 7.1], we have that for $k = 1, 2, 3$,

$$
M_k(h) = \sum_{i=0}^{n+1} u_{ki}(1 - h)^{i+1} + \sum_{i=1}^{n} v_{ki}h^i(1 - h) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} r_{ki}h^i(1 - h)\ln h, \quad h \in (0,1) \tag{3.1}
$$

and

$$
M_4(h) = \sum_{i=0}^{n} u_{4i}(1 - h)^{i+1} + \sum_{i=1}^{n} v_{4i}h^i + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} r_{4i}h^i\ln h, \quad h \in (0,1), \tag{3.2}
$$

where $u_{ki}, v_{kj}, r_{km}, i = 0, 1, \ldots, n, j = 1, \ldots, s, m = 1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor$, $k = 1, 2, 3, 4$, and $u_{s,n+1}, s = 1, 2, 3$ are independent coefficients and can be taken as free parameters.

Now, we are in a position to show one of our main results.

**Theorem 3.1.** Consider the system (1.5) with $n \geq 2$. We have

1. If $Z_k(n) = n + 2 + \lfloor \frac{n+1}{2} \rfloor$ for $k = 1, 2, 3$, and $Z_4(n) = n + 1 + \lfloor \frac{n+1}{2} \rfloor$;

2. the number of limit cycles bifurcating from the period annulus $\bigcup_{h \in (0,1)} L_h$ is $n + 2 + \lfloor \frac{n+1}{2} \rfloor$ for the case $(k)$, $k = 1, 2, 3$ and $n + 1 + \lfloor \frac{n+1}{2} \rfloor$ for the case $(4)$, multiplicities taken into account, by the first order Melnikov function.

**Proof.** Obviously, for $k = 1, 2, 3$, $M_k(h)$ given in (3.1) can be analytically extended to the interval $(0,1)$, and then it can be rewritten as

$$
M_k(h) = P_{k,n+2}(h) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} A_{ki}h^i\ln h, \quad 0 < h \leq 1,
$$

where $P_{k,n+2}(h)$ is a polynomial of degree $n + 2$, and

$$A_{ki} = \begin{cases} r_{ki}, & i = 1, \\ r_{ki} - r_{k,i-1}, & 1 < i < \left\lceil \frac{n + 1}{2} \right\rceil + 1, \\ -r_{k\left\lceil \frac{n + 1}{2} \right\rceil}, & i = \left\lceil \frac{n + 1}{2} \right\rceil + 1. \end{cases}$$

Consider the $(\left\lceil \frac{n + 1}{2} \right\rceil + 2)$-th order derivative of $M_k(h)$, $k = 1, 2, 3$. We have

$$(M_k(h))^{(\left\lceil \frac{n + 1}{2} \right\rceil + 2)} = P_{k,n-\left\lceil \frac{n + 1}{2} \right\rceil}(h) + \sum_{i=1}^{\left\lceil \frac{n + 1}{2} \right\rceil + 1} A_{ki}(h^i \ln h)^{(\left\lceil \frac{n + 1}{2} \right\rceil + 2)}, \quad k = 1, 2, 3.$$ 

Notice that

$$(\ln h)^{(m)} = \frac{(-1)^{m-1}(m-1)!}{h^m}, \quad m \geq 1.$$ 

Then for $m \geq i + 1$, we have

$$(h^i \ln h)^{(m)} = \sum_{j=0}^{i} C_m^j (i - 1) \cdots (i - j + 1) h^{i-j} \frac{(-1)^{m-j-1}(m-j-1)!}{h^{m-j}} = B_{mi} \frac{1}{h^{m-i}},$$

where

$$B_{mi} = \sum_{j=0}^{i} C_m^j (i - 1) \cdots (i - j + 1)(-1)^{m-j-1}(m-j-1)!.$$ 

Hence, we have

$$(M_k(h))^{(\left\lceil \frac{n + 1}{2} \right\rceil + 2)} = P_{k,n-\left\lceil \frac{n + 1}{2} \right\rceil}(h) + \sum_{i=1}^{\left\lceil \frac{n + 1}{2} \right\rceil + 1} \frac{A_{ki} B_{i+1} h^{i+1}}{h^{\left\lceil \frac{n + 1}{2} \right\rceil + 2}},$$

Then by Theorem 2.1, we have that for each $k = 1, 2, 3$, $M_k(h)$ has at most $n + \left\lceil \frac{n + 2}{2} \right\rceil + 3$ zeros in $(0, 1]$, multiplicities taken into account. From (3.1), we have $M_k(1) = 0$ for $k = 1, 2, 3$. Thus, for each $k = 1, 2, 3$, $M_k(h)$ has at most $n + \left\lceil \frac{n + 1}{2} \right\rceil + 2$ zeros in $(0, 1)$, multiplicities taken into account. Combining Theorem 1.1, we have

$$Z_k(n) = n + \left\lceil \frac{n + 1}{2} \right\rceil + 2$$

for each $k = 1, 2, 3$ and $n \geq 2$.

Similarly, $M_4(h)$ can be rewritten as

$$M_4(h) = P_{n+1}(h) + \sum_{i=1}^{\left\lceil \frac{n + 1}{2} \right\rceil} r_i h^i \ln h, \quad 0 < h < 1.$$
Consider the $([\frac{n+1}{2}] + 1)$-th order derivative of $M_4(h)$ as follows:

\[(M_4(h))^{([\frac{n+1}{2}] + 1)} = P_{n-[\frac{n+1}{2}]}(h) + \sum_{i=1}^{[\frac{n+1}{2}]} r_{4i}B_{[\frac{n+1}{2}]+1,i}h^i\]

\[= P_{n-[\frac{n+1}{2}]}(h) + \sum_{i=0}^{[\frac{n+1}{2}]-1} r_{4i+1}B_{[\frac{n+1}{2}]+1,i+1}h^i\]

\[= P_{n-[\frac{n+1}{2}]}(h) + \frac{P_n(h)}{h^{[\frac{n+1}{2}]+1}}.\]

Then by Theorem 2.1, $M_4(h)$ has at most $n+1+[\frac{n+1}{2}]$ zeros in $h$ on $(0, 1)$, multiplicities taken into account. Combining Theorem 1.1, we have $Z_4(n) = n+1+[\frac{n+1}{2}]$ for $n \geq 2$.

Notice that for $n \geq 2$ and $k = 1, 2, 3$, $Z_4(n) = n+2+[\frac{n+1}{2}]$. Then by [8, Theorem 4.4], we have that the number of limit cycles of the system (1.5) for the cases (1)–(3) bifurcating from the period annulus $\bigcup_{h \in (0, 1)} L_h$ is at most $n+2+[\frac{n+1}{2}]$, multiplicities taken into account. Recall that Theorem 1.1 states that the number of limit cycles produced in Hopf bifurcation near the center (i.e., $0 < 1 - h \ll 1$) is at least $n+2+[\frac{n+1}{2}]$. Hence, the number of limit cycles bifurcating from the period annulus $\bigcup_{h \in (0, 1)} L_h$ is $n+1+[\frac{n+1}{2}]$ for the cases (1)–(3), multiplicities taken into account.

Similarly, we can obtain that the number of limit cycles bifurcating from the period annulus $\bigcup_{h \in (0, 1)} L_h$ is $n+1+[\frac{n+1}{2}]$ for the case (4), multiplicities taken into account. The proof is finished.

**Remark 3.2.** Theorem 3.1 is an improvement of Theorem 1.1 on the upper bound of $Z_k(n)$ for $k = 1, 2, 3, 4$. Moreover, Theorem 3.1 provides the maximum number of limit cycles of the system (1.5) bifurcating from the period annulus $\bigcup_{h \in (0, 1)} L_h$ for the cases (1)–(4), multiplicities taken into account.

### 4 The numbers of limit cycles of the systems (1.12) and (1.13)

In this section, we estimate the numbers of limit cycles bifurcating from the periodic orbits around the isochronous centers for the systems (1.12) and (1.13), respectively. For this purpose, we introduce some results obtained in [25]. In fact, Yang [25] derived the first order Melnikov functions of the systems (1.12) and (1.13) by establishing some Picard-Fuchs equations. Before presenting these first order Melnikov functions, we give some notations firstly.

For $h \in (0, +\infty)$, let

\[f_{i,1}(h) = a_i(h^2 + h) - \sqrt{2b_i}|h|^\frac{3}{2} - \sqrt{2b_i}(h^2 + h) \arctan \sqrt{|h|},\]

\[f_{i,2}(h) = c_i \sqrt{h^2 + h} - 2b_i h,\]

\[f_{i,3}(h) = a_i h - \sqrt{2b_i}[(h + 1) \arctan \sqrt{|h|} - \sqrt{|h|}],\]

\[f_{i,4}(h) = \frac{c_i}{2} \ln |2\sqrt{h^2 + h} + 2h + 1|,\]

where $b_1 = 1$, $b_2 = -1$, and $a_i$ and $c_i$, $i = 1, 2$ are constants.

For $h \in (-\infty, -1)$, let

\[u_1(h) = -4 \sqrt{h^2 + h},\]

\[u_2(h) = c_3 h,\]

\[u_3(h) = c_3 (h^2 + h),\]

\[u_4(h) = 4\sqrt{h^2 + h} - 2(2h + 1) \ln |2\sqrt{h^2 + h} + 2h + 1|,\]

where $c_3$ is a constant.
For \( h \in (0, 1) \), let

\[
\begin{align*}
g_{i,1}(h) & = \tilde{a}_ih, \\
g_{i,2}(h) & = \tilde{b}_i\sqrt{h}, \\
g_{i,3}(h) & = 2\tilde{a}_i - 2\tilde{a}_i\sqrt{1-h} + \tilde{c}_i\sqrt{2h} - \sqrt{2}\tilde{c}_i\sqrt{1-h} \arcsin\sqrt{n}, \\
g_{i,4}(h) & = \tilde{c}_i\sqrt{h} - \tilde{b}_i(1-h)\ln\frac{1+\sqrt{n}}{1-\sqrt{n}} + \tilde{c}_i(1-h)\ln(1-h) + \tilde{c}_i, 
\end{align*}
\] (4.3)

where \( \tilde{c}_1 = 1, \tilde{c}_2 = -1, \) and \( \tilde{a}_i, \tilde{b}_i, i = 1, 2 \) are constants.

The following lemma gives the first order Melnikov functions of the systems (1.12) and (1.13).

**Lemma 4.1** (See [25]).

(i) Consider the system (1.12). For \( h \in (0, +\infty) \),

\[
M(h) = \sum_{i=1}^{2}[\alpha_i(h)f_{i,1}(h) + \beta_i(h)f_{i,2}(h) + \gamma_i(h)f_{i,3} + \delta_i(h)f_{i,4}(h)] + P_{2n-1}(\sqrt{h}),
\] (4.4)

where \( P_{2n-1}(u) \) is a polynomial in \( u \) of degree at most \( 2n-1 \), and \( \alpha_k(h), \beta_k(h), \gamma_k(h) \) and \( \delta_k(h) \) are polynomials of \( h \) with

\[
\begin{align*}
deg\alpha_k(h), \deg\beta_k(h) & \leq n - 2, \quad \deg\gamma_k(h) \leq n - 1, \quad \deg\delta_k(h) \leq 2, \quad n \geq 3, \\
\alpha_k(h) & \equiv \text{constant}, \quad \deg\beta_k(h), \deg\gamma_k(h), \deg\delta_k(h) \leq 1, \quad n = 1, 2, \quad k = 1, 2.
\end{align*}
\]

For \( h \in (-\infty, -1) \),

\[
M_0(h) = \frac{1}{2h+1}[\alpha_3(h)u_1(h) + \beta_3(h)u_2(h) + \gamma_3(h)u_3(h) + \delta_3(h)u_4(h)],
\] (4.5)

where \( \alpha_3(h), \beta_3(h), \gamma_3(h) \) and \( \delta_3(h) \) are polynomials of \( h \) with

\[
\begin{align*}
deg\alpha_3(h), \deg\beta_3(h) & \leq n - 1, \quad \deg\gamma_3(h) \leq n - 3, \quad \deg\delta_3(h) \leq 2, \quad n \geq 3, \\
deg\alpha_3(h) & \leq 2, \quad \deg\beta_3(h), \deg\gamma_3(h), \deg\delta_3(h) \leq 1, \quad n = 1, 2.
\end{align*}
\]

(ii) Consider the system (1.13). For \( h \in (0, 1) \), if \( n \geq 3 \), then

\[
M(h) = \frac{1}{(h-1)^{n-2}} \left\{ \sum_{i=1}^{2}[\alpha_i(h)g_{i,1}(h) + \beta_i(h)g_{i,2}(h) + \gamma_i(h)g_{i,3}(h) + \delta_i(h)g_{i,4}(h)] + P_{3n-3}(\sqrt{h}) \right\},
\] (4.6)

where \( P_{3n-3}(u) \) is a polynomial in \( u \) of degree at most \( 3n-3 \), and \( \alpha_k(h), \beta_k(h), \gamma_k(h) \) and \( \delta_k(h) \) are polynomials of \( h \) with

\[
\begin{align*}
deg\alpha_k(h), \deg\delta_k(h) & \leq n - 2, \quad \deg\beta_k(h), \deg\gamma_k(h) \leq n - 1, \quad k = 1, 2.
\end{align*}
\]

If \( n = 1, 2 \), then

\[
M(h) = \frac{1}{h-1} \left\{ \sum_{i=1}^{2}[\alpha_i(h)g_{i,1}(h) + \beta_i(h)g_{i,2}(h) + \gamma_i(h)g_{i,3}(h) + \delta_i(h)g_{i,4}(h)] + P_3(\sqrt{h}) \right\},
\] (4.7)

where \( P_3(u) \) is a polynomial in \( u \) of degree at most \( 5 \), and \( \alpha_k(h), \beta_k(h), \gamma_k(h) \) and \( \delta_k(h) \) are polynomials of \( h \) with

\[
\begin{align*}
deg\alpha_k(h), \deg\delta_k(h) & \leq 1, \quad \deg\beta_k(h), \deg\gamma_k(h) \leq 2, \quad k = 1, 2.
\end{align*}
\]

Now, we estimate the upper bounds of the numbers of limit cycles for the systems (1.12) and (1.13) by Theorem 2.1. Our results are shown in the following two theorems.
Theorem 4.2. For the system (1.12), the number of limit cycles bifurcating from the period annulus

\[ \bigcup_{h \in (-\infty, -1) \cup (0, +\infty)} T_h \]

around the isochronous centers \((\pm 1, 0)\) is not more than \(8n + 2\) for \(n \geq 3\), counting multiplicities and 21 for \(n = 1, 2\), counting multiplicities, by the first order Melnikov function.

Proof. By (4.1), (4.2), (4.4) and (4.5), we have that for

\[ M(h) = \begin{cases} P_2(h) + P_1(h)\sqrt{h} + P_1(h)\sqrt{h^2 + h} + P_2(h) \arctan \sqrt{h} + \tilde{P}_1(h) g(h), & n = 1, 2, \\ P_n(h) + P_{n-1}(h)\sqrt{h} + P_{n-1}(h)\sqrt{h^2 + h} + P_n(h) \arctan \sqrt{h} + P_2(h) g(h), & n \geq 3, \end{cases} \]

and for \(h \in (-\infty, -1)\),

\[ M_0(h) = \frac{1}{2h + 1} \left\{ P_2(h) + P_2(h)\sqrt{h^2 + h} + P_2(h) g(h), \quad n = 1, 2, \\ P_n(h) + P_{n-1}(h)\sqrt{h^2 + h} + P(n) g(h), \quad n \geq 3, \right. \]

where

\[ g(h) = \ln |2\sqrt{h^2 + h} + 2h + 1|, \]

and \(P_i(h), \tilde{P}_i(h)\) and \(\tilde{P}_i(h)\) are polynomials of degree not more than \(i\). For \(n \geq 3\), consider the \((n+1)\)-th order derivative of \(M(h)\). For this purpose, we first notice that

\[ (\arctan \sqrt{h})' = \frac{1}{2(1 + h)\sqrt{h}} \quad \text{and} \quad (\ln(2\sqrt{h^2 + h} + 2h + 1))' = \frac{1}{\sqrt{h} \sqrt{1 + h}}. \]

Then according to

\[ \left( \frac{1}{\sqrt{h}} \right)^{(n)} = \frac{(-1)^n(2n - 1)!!}{2^n h^{n+\frac{1}{2}}}, \]

\[ \left( \frac{1}{\sqrt{1 + h}} \right)^{(n)} = \frac{(-1)^n(2n - 1)!!}{2^n (1 + h)^{n+\frac{1}{2}}} \]

and

\[ \left( \frac{1}{1 + h} \right)^{(n)} = \frac{(-1)^n n!}{(1 + h)^{n+1}}, \]

we have

\[ (\arctan \sqrt{h})^{(n+1)} = \frac{\tilde{P}_n(h)}{(1 + h)^{n+1}h^{n+\frac{1}{2}}} \]

and

\[ (\ln(2\sqrt{h^2 + h} + 2h + 1))^{(n+1)} = \frac{\mathcal{P}_n(h)}{h^{n+\frac{1}{2}}(1 + h)^{n+\frac{1}{2}}}, \]

where \(\tilde{P}_n(h)\) and \(\mathcal{P}_n(h)\) are polynomials of degree not more than \(n\). Similarly, according to

\[ (\sqrt{h})^{(n)} = \frac{(-1)^{n-1}(2n - 3)!!}{2^n h^{n-\frac{3}{2}}} \quad \text{and} \quad (\sqrt{1 + h})^{(n)} = \frac{(-1)^{n-1}(2n - 3)!!}{2^n (1 + h)^{n-\frac{3}{2}}}, \]

we obtain

\[ (\sqrt{h}(1 + h))^{(n)} = \frac{\tilde{P}_n(h)}{h^{n-\frac{1}{2}}(1 + h)^{n-\frac{3}{2}}}. \]
where $\tilde{P}_n(h)$ is a polynomial of degree not more than $n$. Based on the above results, we have for $n \geq 3$,

\[
(M(h))^{(n+1)} = \sum_{i=0}^{n-1} C^i_{n+1} P_{n-1-i}(h)(\sqrt{h})^{(n+1-i)} + \sum_{i=0}^{n-2} C^i_{n+1} P_{n-2-i}(h)(\sqrt{h^2 + h})^{(n+1-i)}
\]

\[
+ \sum_{i=0}^{n} C^i_{n+1} P_{n-i}(h)(\arctan \sqrt{h})^{(n+1-i)} + \sum_{i=0}^{2} C^i_{n+1} P_{2-i}(h)(g(h))^{(n+1-i)}
\]

\[
= \sum_{i=0}^{n-1} C^i_{n+1} P_{n-1-i}(h)(-1)^{n-i}(2n-2i-1)!! + \sum_{i=0}^{n-2} C^i_{n+1} P_{n-2-i}(h)\tilde{P}_{n+1-i}(h)
\]

\[
+ \sum_{i=0}^{n} C^i_{n+1} P_{n-i}(h)\tilde{P}_{n-1}(h) + \sum_{i=0}^{2} C^i_{n+1} P_{2-i}(h)\tilde{P}_{n-1}(h)
\]

\[
= \frac{P_n(h)}{h^{n+\frac{1}{2}}} + \frac{P_{2n-1}(h)}{(h^2 + h)^{n+\frac{1}{2}}} + \frac{P_{2n}(h)}{(1 + h)^{n+1}h^{n+\frac{1}{2}}} + \frac{P_{2n+2}(h)}{(h^2 + h)^{n+\frac{3}{2}}}
\]

\[
= \frac{P_{2n}(h) + \sqrt{1 + h}P_{2n-1}(h) + \sqrt{1 + h}P_{2n}(h) + \sqrt{1 + h}P_{2n+2}(h)}{(1 + h)^{n+1}h^{n+\frac{1}{2}}}. 
\]

Then $(M(h))^{(n+1)} = 0$ is equivalent to

\[
\frac{P_{2n}(h) + \sqrt{1 + h}P_{2n-1}(h)}{(1 + h)^{n+1}h^{n+\frac{1}{2}}} = 0.
\]

It yields that $(M(h))^{(n+1)}$ has at most $4n$ zeros. Then, according to Theorem 2.1, $M(h)$ has at most $5n + 1$ zeros on the interval $(0, +\infty)$ for $n \geq 3$, multiplicities taken into account. Similarly, we have that for $n = 1, 2$,

\[
(M(h))^{(3)} = \frac{P_4(h) + P_6(h)\sqrt{1 + h}}{h^2(1 + h)^3}.
\]

Thus, by Theorem 2.1, $M(h)$ has at most 11 zeros on the interval $(0, +\infty)$ for $n = 1, 2$, multiplicities taken into account.

For $h \in (-\infty, -1)$, we have

\[
\ln(-2\sqrt{h^2 + h - 2h - 1})^{(n+1)} = \frac{P_n(h)}{(h^2 + h)^{n+\frac{1}{2}}}
\]

So if $n \geq 3$, then the $(n+1)$-th order derivative of $M_0(h)(2h + 1)$ is

\[
(M_0(h)(2h + 1))^{(n+1)} = \frac{P_{2n}(h)}{(h^2 + h)^{n+\frac{1}{2}}}
\]

Then by Theorem 2.1, $M_0(h)$ has at most $3n + 1$ zeros on $(-\infty, -1)$ for $n \geq 3$, multiplicities taken into account. Similarly, we have that for $n = 1, 2$,

\[
(M_0(h)(2h + 1))^{(4)} = \frac{P_0(h)}{(h^2 + h)^{\frac{1}{2}}}
\]

Then by Theorem 2.1, $M_0(h)$ has at most 10 zeros on $(-\infty, -1)$ for $n = 1, 2$, multiplicities taken into account.

From the above analysis and [8, Theorem 4.4], we can conclude that the number of limit cycles of the system (1.12) bifurcating from the period annulus $\bigcup_{h \in (-\infty, -1) \cup (0, +\infty)} \mathcal{T}_h$ around the isochronous centers $(\pm 1, 0)$ is not more than $8n + 2$ (counting multiplicities) for $n \geq 3$ and 21 (counting multiplicities) for $n = 1, 2$. The proof is ended.

□
For the system
\[ 2266 \text{ Chen X Y} \]
\[ \text{where} \quad P \geq 0, \]
\[ \text{for this purpose, we first state some facts, i.e.,} \]
\[ (4.3) \quad \text{and} \quad (4.6), \]
\[ \text{we have that for} \quad n = 1, 2, \]
\[ \text{by the first order Melnikov function.} \]

**Proof.** According to (4.3) and (4.6), we have that for \( n \geq 3, \)
\[ M(h) = \frac{1}{(h - 1)^{n-2}} \left[ P_{n-1}(h) + \tilde{P}_{n-1}(h) \sqrt{h} + \tilde{P}_{n-1}(h) \sqrt{1-h} \right. \]
\[ + P_{n-1}(h) \sqrt{1-h} \arcsin \sqrt{h} + \tilde{P}_{n-1}(h) \ln \frac{1 + \sqrt{h}}{1 - \sqrt{h}} \]
\[ + \tilde{P}_{n-1}(h) \ln(1-h) + P_{3n-3}(\sqrt{h}) \right], \quad h \in (0, 1), \]

where \( P_i(u), \tilde{P}_i(u), \tilde{P}_i(u), \tilde{P}_i(u) \) and \( \tilde{P}_i(u) \) are polynomials of \( u \) with degree \( i \). Notice that
\[ P_{3n-3}(\sqrt{h}) = \begin{cases} P_{n-2+\lfloor \frac{n}{2} \rfloor}(h) \sqrt{h} + P_{n-2+\lfloor \frac{n}{2} \rfloor}(h), & n = 2k, \\ P_{n-2+\lfloor \frac{n}{2} \rfloor}(h) \sqrt{h} + P_{n-1+\lfloor \frac{n}{2} \rfloor}(h), & n = 2k + 1. \end{cases} \]

Then we have that for \( n \geq 3, \)
\[ M(h) = \frac{1}{(h - 1)^{n-2}} \left[ P_{m-1}(h) + P_{n-2+\lfloor \frac{n}{2} \rfloor}(h) \sqrt{h} + \tilde{P}_{n-1}(h) \sqrt{1-h} \right. \]
\[ + P_{n-1}(h) \sqrt{1-h} \arcsin \sqrt{h} + \tilde{P}_{n-1}(h) \ln \frac{1 + \sqrt{h}}{1 - \sqrt{h}} + \tilde{P}_{n-1}(h) \ln(1-h) \right], \]

where
\[ m = n + \left[ \frac{n-1}{2} \right]. \quad (4.8) \]

Next, we consider the \( m \)-th order derivative of \( M_1(h) \), where
\[ M_1(h) = M(h)(h - 1)^{n-2}. \]

For this purpose, we first state some facts, i.e.,
\[ (\arcsin \sqrt{h})' = \frac{1}{2\sqrt{h} \sqrt{1-h}}, \]
\[ \left( \ln \frac{1 + \sqrt{h}}{1 - \sqrt{h}} \right)' = \frac{1}{1 - h \sqrt{h}}, \]
\[ (\ln(1-h))^{(n+1)} = -\frac{n!}{(1-h)^{n+1}}, \]
\[ (\arcsin \sqrt{h})^{(n+1)} = \frac{P_n(h)}{[h(1-h)]^{n+\frac{3}{2}}}, \]
\[ (\sqrt{1-h} \arcsin \sqrt{h})^{(n)} = \frac{P_{n-1}(h)}{(1-h)^{n-1}h^{n-1}} + \frac{-2(n-3)!!}{2^n(1-h)^{n-1}} \arcsin \sqrt{h} \]
and
\[ \left( \ln \frac{1 + \sqrt{h}}{1 - \sqrt{h}} \right)^{(n+1)} = \frac{P_n(h)}{(1-h)^{n+1}h^{n+\frac{3}{2}}}. \]

Then the \( m \)-th order derivative of \( M_1(h) \) is as follows:
\[ (M_1(h))^{(m)} = \sum_{i=0}^{n-2+\lfloor \frac{n}{2} \rfloor} C_m P_{n-2+\lfloor \frac{n}{2} \rfloor-i}(h)(\sqrt{h})^{(m-i)} + \sum_{i=0}^{n-1} C_m \tilde{P}_{n-1-i}(h)(\sqrt{1-h})^{(m-i)} \]
\[\begin{align*}
&= \sum_{i=0}^{n-1} C_i^n \tilde{P}_{n-1-i}(h)(\sqrt{1 - h \arcsin \frac{\sqrt{h}}{h}})^{(m-i)} + \sum_{i=0}^{n-1} C_i^n \tilde{P}_{n-1-i}(h)\left( \ln \frac{1 + \sqrt{h}}{1 - \sqrt{h}} \right)^{(m-i)} \\
&+ \sum_{i=0}^{n-2} \left( -1 \right)^{m-i} C_i^n (2m - 2i - 3)!! P_{n-2+i}\tilde{P}_{n-1-i}(h) + \sum_{i=0}^{n-1} C_i^n (2m - 2i - 3)!! \tilde{P}_{n-1-i}(h) \\
&+ \sum_{i=0}^{n-1} C_i^n \tilde{P}_{n-1-i}(h) P_{m-1-i}(h) + \sum_{i=0}^{n-1} (2m - 2i - 3)!! \tilde{P}_{n-1-i}(h) \arcsin \sqrt{h}
&+ \sum_{i=0}^{n-1} C_i^n (m - i - 1) \tilde{P}_{n-1-i}(h) + \sum_{i=0}^{n-1} C_i^n (m - i - 1) \arcsin \sqrt{h}
&+ \sum_{i=0}^{n-1} C_i^n (m - i - 1) \tilde{P}_{n-1-i}(h)
&= \frac{P_{n-2+i}(h)}{(1-h)^{m-i-\frac{1}{2}}} + \frac{\tilde{P}_{n-1}(h)}{(1-h)^{m-i-\frac{1}{2}}} + \frac{\tilde{P}_{n-1}(h) P_{m-1-i}(h)}{(1-h)^{m-i-\frac{1}{2}}} + \frac{\tilde{P}_{n-1}(h) P_{m-1-i}(h)}{(1-h)^{m-i-\frac{1}{2}}} \arcsin \sqrt{h}
&= \frac{P_{n-m-2}(h) + \frac{P_{n-2+i}(h)}{\sqrt{h} \sqrt{1 - h}} + \frac{P_{n-2+m}(h)}{\sqrt{h} \sqrt{1 - h}} + \frac{P_{n-m-2}(h) \arcsin \sqrt{h}}{(1-h)^{m-i-\frac{1}{2}}} + \frac{P_{n+m-2}(h) \arcsin \sqrt{h}}{(1-h)^{m-i-\frac{1}{2}}} }{(1-h)^{m-i-\frac{1}{2}} h^{m-1}}.
\end{align*}\]

Now, we consider the \((n + m - 1)\)-th order derivative of \(M_2(h)\), where

\[M_2(h) = (M_1(h))^{(n)} (1-h)^{m-\frac{1}{2}} h^{m-1}.\]

Notice that

\[\left( \frac{1}{\sqrt{h} \sqrt{1 - h}} \right)^{(n)} = \frac{P_n(h)}{[h(1-h)]^{n+\frac{1}{2}}} \]

and

\[\left( \frac{\sqrt{1 - h}}{\sqrt{h}} \right)^{(n)} = \frac{\tilde{P}_n(h)}{[h(1-h)]^{n+\frac{1}{2}}} \]

Set \(t = n + m - 1\). Then we have

\[\begin{align*}
(M_2(h))^{(t)} &= \sum_{i=0}^{t} C_i^t P_{n+m-2+i}(h)\left( \frac{1}{\sqrt{1 - h \arcsin \frac{\sqrt{h}}{h}}} \right)^{(t-i)} + \sum_{i=0}^{n+m-2} C_i^t \tilde{P}_{n+m-2-i}(h)\left( \frac{1}{\sqrt{1 - h}} \right)^{(t-i)} \\
&+ \sum_{i=0}^{n+m-2} C_i^t \tilde{P}_{n+m-2-i}(h)\left( \frac{\sqrt{1 - h \arcsin \frac{\sqrt{h}}{h}}}{h} \right)^{(t-i)} + \sum_{i=0}^{n+m-2} C_i^t \tilde{P}_{n+m-2-i}(h)\arcsin \sqrt{h} \left( \frac{\sqrt{1 - h \arcsin \frac{\sqrt{h}}{h}}}{h} \right)^{(t-i)} \\
&= \sum_{i=0}^{t} C_i^t P_{n+m-2+i}(h) P_{t-i}(h) + \sum_{i=0}^{n+m-2} C_i^t \tilde{P}_{n+m-2-i}(h) P_{t-i}(h) + \sum_{i=0}^{n+m-2} C_i^t \tilde{P}_{n+m-2-i}(h) P_{t-i}(h) \\
&+ \sum_{i=0}^{n+m-2} C_i^t \tilde{P}_{n+m-2-i}(h) P_{t-i}(h) + \sum_{i=0}^{n+m-2} C_i^t \tilde{P}_{n+m-2-i}(h) P_{t-i}(h) \\
&= \frac{P_{n+m-t+2+i}(h) P_{t-i}(h)}{[h(1-h)]^{t+i-\frac{1}{2}}} + \frac{\tilde{P}_{n+m-2}(h) P_{t-i}(h)}{[h(1-h)]^{t+i-\frac{1}{2}}} + \tilde{P}_{n+m-2}(h) P_{t-i}(h) \\
&+ \frac{P_{n+m-t+2+i}(h) P_{t-i}(h)}{[h(1-h)]^{t+i-\frac{1}{2}}} + \frac{P_{n+m-t+2+i}(h) P_{t-i}(h)}{[h(1-h)]^{t+i-\frac{1}{2}}} \\
&= \frac{P_{n+m+t-2+i}(h)}{[h(1-h)]^{t+i-\frac{1}{2}}} + \frac{P_{n+m+t-2+i}(h)}{[h(1-h)]^{t+i-\frac{1}{2}}} \arcsin \sqrt{h} = 0.
\end{align*}\]
Thus, the number of zeros of \((M_2(h))^{(t)}\) is at most
\[
2n + 2m + 2t - 4 + 2\left\lfloor \frac{n}{2} \right\rfloor.
\]
By Theorem 2.1, the number of zeros of \(M_2(h)\) is at most
\[
2n + 2m + 3t - 4 + 2\left\lfloor \frac{n}{2} \right\rfloor.
\]
multiplicities taken into account. By (4.9) and Theorem 2.1, the number of zeros of \(M_1(h)\) is at most
\[
2n + 3m + 3t - 4 + 2\left\lfloor \frac{n}{2} \right\rfloor.
\]
multiplicities taken into account. Thus, by (4.8) and \(t = n + m - 1\), the number of zeros of \(M(h)\) is at most \(15n - 13\) (\(15n - 11\) if \(n \geq 3\) is even (odd), multiplicities taken into account. The first conclusion follows.

By (4.3) and (4.7), we have \(M(h) = \frac{M_3(h)}{h^{3-t}}\) for \(n = 1, 2\), where
\[
M_3(h) = P_3(h) + \tilde{P}_2(h)\sqrt{h} + \tilde{P}_2(h)\sqrt{1 - h} + \tilde{P}_2(h)\sqrt{1 - h} \arcsin \sqrt{h}
\]
\[
\quad + \tilde{P}_2(h) \ln \frac{1 + \sqrt{h}}{1 - \sqrt{h}} + \tilde{P}_2(h) \ln (1 - h).
\]

Then the 3rd order derivative of \(M_3(h)\) is as follows:
\[
(M_3(h))^{(3)} = \sum_{i=0}^{2} C_i^3 \tilde{P}_{2-i}(h) (\sqrt{h})^{(3-i)} + \sum_{i=0}^{2} C_i^3 \tilde{P}_{2-i}(h) (\sqrt{1 - h})^{(3-i)}
\]
\[
+ \sum_{i=0}^{2} C_i^3 \tilde{P}_{2-i}(h) \left( \ln \frac{1 + \sqrt{h}}{1 - \sqrt{h}} \right)^{(3-i)} + \sum_{i=0}^{2} C_i^3 \tilde{P}_{2-i}(h) (\sqrt{1 - h} \arcsin \sqrt{h})^{(3-i)}
\]
\[
+ \sum_{i=0}^{2} C_i^3 \tilde{P}_{2-i}(h) (\ln (1 - h))^{(3-i)}
\]
\[
= \sum_{i=0}^{2} (-1)^{2-i} C_i^3 (3 - 2i)! \tilde{P}_{2-i}(h) + \sum_{i=0}^{2} (-1)^{2-i} C_i^3 (3 - 2i)! \tilde{P}_{2-i}(h)
\]
\[
\frac{2^{3-i} h^{\frac{1}{2}}}{(1 - h)^{3-i} h^{\frac{1}{2}}} + \sum_{i=0}^{2} C_i^3 \tilde{P}_{2-i}(h) (P_{2-i}(h) \ln (1 - h))^{(3-i)}
\]
\[
= \frac{\tilde{P}_3(h)}{h^{\frac{1}{2}}} + \frac{\tilde{P}_2(h)}{(1 - h)^{\frac{1}{2}}} + \frac{P_2(h) \ln (1 - h)}{(1 - h)^{\frac{1}{2}}} \arcsin \sqrt{h} + \frac{\tilde{P}_2(h) \ln (1 - h)}{(1 - h)^{\frac{1}{2}}} \arcsin \sqrt{h}
\]
\[
= \frac{P_3(h) + P_2(h) \sqrt{1 - h}}{\sqrt{h} \sqrt{1 - h}} + \frac{P_2(h) \ln (h)}{\sqrt{h} \sqrt{1 - h}} + \frac{\tilde{P}_2(h) \ln (h)}{\sqrt{h} \sqrt{1 - h}} + \frac{\tilde{P}_2(h) \ln (1 - h)}{(1 - h)^{\frac{1}{2}}}
\]
\[
= \frac{P_4(h) + \tilde{P}_4(h) \sqrt{1 - h}}{\sqrt{h} \sqrt{1 - h}} + \frac{\tilde{P}_4(h) \ln (h)}{\sqrt{h} \sqrt{1 - h}} + \frac{\tilde{P}_4(h) \ln (1 - h)}{(1 - h)^{\frac{1}{2}}}
\]
\[
\quad + \tilde{P}_4(h) \arcsin \sqrt{h}.
\]
Let
\[
M_4(h) = P_4(h) + \tilde{P}_4(h) \frac{\sqrt{1 - h}}{\sqrt{h}} + \frac{\tilde{P}_4(h) \ln (h)}{\sqrt{h} \sqrt{1 - h}} + \frac{\tilde{P}_4(h) \ln (1 - h)}{(1 - h)^{\frac{1}{2}}} + \tilde{P}_4(h) \arcsin \sqrt{h}.
\]

Notice that
\[
\left( \frac{\sqrt{1 - h}}{\sqrt{h}} \right)^{(n)} = \frac{P_n(h)}{h^{n+\frac{1}{2}} (1 - h)^{n-\frac{1}{2}}}.
\]
Then we have

\[ (M_4(h))^{(5)} = \sum_{i=0}^{4} C_i^4 \hat{P}_{4-i}(h) \left( \frac{\sqrt{1 - h}}{\sqrt{h}} \right)^{(5-i)} + \sum_{i=0}^{4} C_i^5 \overline{P}_{4-i}(h) \left( \frac{1}{\sqrt{1 - h}} \right)^{(5-i)} + \sum_{i=0}^{4} C_i^6 \hat{\overline{P}}_{4-i}(h) (\arcsin \sqrt{h})^{(5-i)} \]

\[ = \sum_{i=0}^{4} C_i^4 \hat{P}_{4-i}(h) P_{4-i}(h) \sum_{i=0}^{4} C_i^5 \overline{P}_{4-i}(h) P_{4-i}(h) \sum_{i=0}^{4} C_i^6 \hat{\overline{P}}_{4-i}(h) \overline{P}_{4-i}(h) \]

\[ = \frac{\hat{P}_3(h) P_4(h)}{h^{3/2} (1 - h)^{3/2}} + \frac{\overline{P}_3(h) P_4(h)}{h^{3/2} (1 - h)^{3/2}} + \frac{\overline{\overline{P}}_3(h) P_4(h)}{h^{3/2} (1 - h)^{3/2}} \]

\[ = \frac{P_{10}(h) + P_9(h) \sqrt{h}}{h^{3/2} (1 - h)^{3/2}}. \]

Obviously, \((M_4(h))^{(5)} = 0\) is equivalent to

\[ P_{10}(h) + P_9(h) \sqrt{h} = 0. \]

Then \((M_4(h))^{(5)}\) has at most 20 zeros. By Theorem 2.1, \(M_4(h)\) has at most 25 zeros, multiplicities taken into account. Thus, by (4.10) and Theorem 2.1, \(M_3(h)\) has at most 28 zeros, multiplicities taken into account. It implies that \(M(h)\) has at most 28 zeros for \(n = 1, 2\), multiplicities taken into account.

From the above analysis and [8, Theorem 4.4], we can conclude that the number of limit cycles of the system (1.13) bifurcating from the period annulus \(\bigcup_{h \in (0,1]} \tilde{L}_h\) around the isochronous center \((1, 0)\) is not more than \(15n - 11\) (counting multiplicities) for \(n \geq 3\) and 28 (counting multiplicities) for \(n = 1, 2\). The proof is ended.

**Remark 4.4.** Theorems 4.2 and 4.3 improve the conclusions (i) and (ii) of Theorem 1.2, respectively.

5 Conclusion

According to the proofs of Theorems 3.1, 4.2 and 4.3, it is easy to see that we can get rid of the logarithm function, the arc sine function and the arc tangent function by Theorem 2.1. If the first order Melnikov function \(M(h)\) is a linear combination of power functions and some other elementary functions, such as the logarithm function or inverses of trigonometric functions, one can try to use Theorem 2.1 to get rid of these elementary functions to find an upper bound of the number of zeros of \(M(h)\). In fact, Cen et al. [1], Chen and Han [2] and Xiong and Han [24] used this idea to get rid of some logarithm functions to estimate the upper bound of the number of zeros of the first order Melnikov function.

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant Nos. 11931016 and 11771296) and Hunan Provincial Education Department (Grant No. 19C1898).

**References**

1. Cen X, Liu C, Yang L, et al. Limit cycles by perturbing quadratic isochronous centers inside piecewise polynomial differential systems. J Differential Equations, 2018, 265: 6083–6126
2. Chen X, Han M. A linear estimate of the number of limit cycles for a piecewise smooth near-Hamiltonian system. Qual Theory Dyn Syst, 2020, 19: 61
3. Gasull A, Li C, Torregrosa J. A new Chebyshev family with applications to Abel equations. J Differential Equations, 2012, 252: 1635–1641
4. Gavrilov L, Iliev I D. Two-dimensional Fuchsian systems and the Chebyshev property. J Differential Equations, 2003, 191: 105–120
5 Han M. Bifurcation Theory of Limit Cycles. Mathematics Monograph Series 25. Beijing: Science Press, 2013
6 Han M, Li J. Lower bounds for the Hilbert number of polynomial systems. J Differential Equations, 2012, 252: 3278–3304
7 Han M, Sheng L. Bifurcation of limit cycles in piecewise smooth systems via Melnikov function. J Appl Anal Comput, 2015, 5: 809–815
8 Han M, Yang J. The maximum number of zeros of functions with parameters and application to differential equations. J Nonlinear Model Anal, 2021, 3: 13–34
9 Hilbert D. Mathematical problems. Bull Amer Math Soc (NS), 1902, 8: 437–479
10 Horozov E, Iliev I D. Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians. Nonlinearity, 1998, 11: 1521–1537
11 Karlin S J, Studden W J. Tchebycheff Systems: With Applications in Analysis and Statistics. Pure and Applied Mathematics, vol. 15. New York-London-Sydney: Interscience Publishers/John Wiley & Sons, 1966
12 Li C, Li W, Llibre J, et al. Linear estimate for the number of zeros of Abelian integrals for quadratic isochronous centres. Nonlinearity, 2000, 13: 1775–1800
13 Li S, Liu C. A linear estimate of the number of limit cycles for some planar piecewise smooth quadratic differential system. J Math Anal Appl, 2015, 428: 1354–1367
14 Liang F, Han M, Romanovski V G. Bifurcation of limit cycles by perturbing a piecewise linear Hamiltonian system with a homoclinic loop. Nonlinear Anal, 2012, 75: 4355–4374
15 Liu X, Han M. Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems. Internat J Bifur Chaos, 2010, 20: 1379–1390
16 Mañosas F, Villadelprat J. Bounding the number of zeros of certain Abelian integrals. J Differential Equations, 2011, 251: 1656–1669
17 Novaes D D, Torregrosa J. On extended Chebyshev systems with positive accuracy. J Math Anal Appl, 2017, 448: 171–186
18 Petrov G S. Number of zeros of complete elliptic integrals. Funct Anal Appl, 1984, 18: 148–150
19 Petrov G S. Elliptic integrals and their nonoscillation. Funct Anal Appl, 1986, 20: 37–40
20 Petrov G S. Complex zeros of an elliptic integral. Funct Anal Appl, 1987, 21: 247–248
21 Petrov G S. The Chebyshev property of elliptic integrals. Funct Anal Appl, 1988, 22: 72–73
22 Tian H, Han M. Limit cycle bifurcations of piecewise smooth near-Hamiltonian systems with a switching curve. Discrete Contin Dyn Syst Ser B, 2021, 26: 5581–5599
23 Wang Y, Han M, Constantinescu D. On the limit cycles of perturbed discontinuous planar systems with 4 switching lines. Chaos Solitons Fractals, 2016, 83: 158–177
24 Xiong Y, Han M. Bifurcation of limit cycles by perturbing a piecewise linear Hamiltonian system. Abstr Appl Anal, 2013, 2013: 575390
25 Yang J. Picard-Fuchs equation applied to quadratic isochronous systems with two switching lines. Internat J Bifur Chaos, 2020, 30: 2050042
26 Yang J. Complete hyper-elliptic integrals of the first kind and the Chebyshev property. J Nonlinear Model Anal, 2020, 2: 431–446
27 Yang J, Zhao L. Limit cycle bifurcations for piecewise smooth integrable differential systems. Discrete Contin Dyn Syst Ser B, 2017, 22: 2417–2425
28 Yang J, Zhao L. Bounding the number of limit cycles of discontinuous differential systems by using Picard-Fuchs equations. J Differential Equations, 2018, 264: 5734–5757
29 Zalik R A. Some properties of Chebyshev systems. J Comput Appl Anal, 2011, 13: 20–26
30 Zhao Y, Zhang Z. Linear estimate of the number of zeros of Abelian integrals for a kind of quartic Hamiltonians. J Differential Equations, 1999, 155: 73–88