An Alternative Presentation of the Symmetric-Simplicial Category

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Abstract

The category $\text{Fin}$ of symmetric-simplicial operators is obtained by enlarging the category $\text{Ord}$ of monotonic functions between the sets $\{0, 1, \ldots, n\}$ to include all functions between the same sets. Marco Grandis [Gra01a] has given a presentation of $\text{Fin}$ using the standard generators $d_i$ and $s_i$ of $\text{Ord}$ as well as the adjacent transpositions $t_i$ which generate the permutations in $\text{Fin}$. The purpose of this note is to establish an alternative presentation of $\text{Fin}$ in which the codegeneracies $s_i$ are replaced by quasi-codegeneracies $u_i$. We also prove a unique factorization theorem for products of $d_i$ and $u_j$ analogous to the standard unique factorizations in $\text{Ord}$. This presentation has been used by the author to construct symmetric hypercrossed complexes (to be published elsewhere) which are algebraic models for homotopy types of spaces based on the hypercrossed complexes of [CC91].

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1 Introduction

In order to motivate the subject of this note, we bring together two distinct lines of historical development. First, recall that \textbf{Ord} is the category whose objects are the standard finite ordered sets

\[ [n] := \{0, 1, \ldots, n\} \text{ for } n \geq 0 \]

and whose morphisms are all monotonic functions

\[ f : [n] \rightarrow [m] \]

\[ i < j \implies f(i) \leq f(j). \]

Simplicial sets, which are by definition contravariant functors from \textbf{Ord} to the category \textbf{Set} of all sets and mappings, are used in Homotopy Theory and related fields as combinatorial models for topological spaces, among other things. For this reason, \textbf{Ord} is often referred to as the \textit{simplicial category} and denoted \( \Delta \), and the category of simplicial sets is then denoted \( \text{Set}_\Delta := \text{Fun}(\Delta^{op}, \text{Set}) \).

In Pursuing Stacks ([Gro83]), Alexander Grothendieck proposed replacing \textbf{Ord} in the definition of simplicial set with an arbitrary small category \( \Gamma \) and looking for Quillen model category structures on the category \( \text{Set}_\Gamma \) of \( \Gamma \)-sets (defined as \( \text{Set}_\Gamma := \text{Fun}(\Gamma^{op}, \text{Set}) \)) in order to investigate the possibilities for doing homotopy theory there. He laid special emphasis on certain geometrically motivated examples of \( \Gamma \), including the category which is the subject of this paper, namely the category denoted \( \text{Fin} \) (denoted by him \( \hat{\Delta} \)) whose objects are the same as those of \textbf{Ord} but whose morphisms consist of all functions \( f : [n] \rightarrow [m] \) for each \( m, n \geq 0 \).

A short while later, W.G. Dwyer, Michael Hopkins and Daniel Kan proved a result showing that for a certain class of categories \( \Gamma \), including \( \Gamma = \text{Fin} \), one may define a model structure on \( \text{Set}_\Gamma \) such that the resulting homotopy theory is equivalent to the usual one on \( \text{Set}_\Delta \) (see [DHK85]). A later observation of F. William Lawvere in [Law88] also suggested studying \( \text{Set}_\Gamma \) for \( \Gamma = \text{Fin} \), inspiring Marco Grandis to take up the subject ([Gra88], [Gra01a], [Gra01b], [Gra02], [Gra03]). Note that \( \text{Fin} \) contains the group \( \text{Sym}[n] \) of all permutations of the set \([n]\) for each \( n \geq 0 \), as well as the category \textbf{Ord}. For this reason, \( \text{Fin} \) is referred to by Grandis as the \textit{symmetric-simplicial category}, and we shall do so as well.
We turn briefly to the other line of historical development relevant for us here. Motivated by the fact that the category $\text{SGrp} := \text{Fun}(\Delta^\text{op}, \text{Grp})$ of simplicial groups possesses a homotopy theory equivalent to that of pointed connected spaces (they play the role of loop spaces, see [May67] or [GJ99]), P. Carrasco and A. M. Cegarra discovered a nonabelian Dold-Kan theorem for simplicial groups and used it to describe homological-algebraic models for classical homotopy types ([CC91]), which they dubbed hypercrossed complexes.

Since the author has shown (to appear elsewhere) that symmetric-simplicial groups also have a homotopy theory equivalent to that of pointed connected spaces, it is of interest to ask what sort of homological-algebraic objects can arise from nonabelian Dold-Kan decompositions (in the sense of [CC91]) of symmetric-simplicial groups. The author has shown ([Ant10]) that, in addition to the decompositions obtained via a direct application of [CC91] to the underlying simplicial group of a symmetric-simplicial group, there also exist new Dold-Kan decompositions which can be obtained by making judicious use of the algebra of the category $\text{Fin}$. These decompositions give rise in turn to new homological-algebraic models for homotopy types, which we call symmetric hypercrossed complexes, that are simpler than the original hypercrossed complexes in the sense that a great deal of the algebraic data constituting them vanishes (to appear elsewhere).

The new Dold-Kan decompositions are obtained using an alternative presentation of the category $\text{Fin}$, whose verification is the main purpose of the present note. Grandis gave a presentation in [Gra01a] (reviewed in section 2 below) of $\text{Fin}$ that uses the standard presentation of $\text{Ord}$ as well as the Moore presentations of the symmetric groups $\text{Sym}[n]$ via adjacent transpositions. In the alternative presentation of $\text{Fin}$, the monotonic elementary codegeneracies $s_i \in \text{Ord}$ in Grandis’s presentation are replaced by certain nonmonotonic surjections $u_i \in \text{Fin}$ which we call the elementary quasi-codegeneracies (see Definition 2.7).

In section 3 we prove this alternative presentation by relating it directly to Grandis’s presentation. In section 4 we also show that the morphisms of the subcategory of $\text{Fin}$ generated by the $d_i$ and $u_i$ are characterized by the following two conditions.

- They take 0 to 0.
- They are strictly monotonic outside of the preimage of 0.
We call such morphisms *quasi-monotonic* and denote the subcategory of \textbf{Fin} consisting of quasi-monotonic functions by \textbf{qOrd}. Finally we show that \textbf{qOrd} admits unique factorizations analogous to those of \textbf{Ord}. These results are relied upon in [Ant10] to derive the alternative Dold-Kan decompositions for symmetric-simplicial groups mentioned above.

2 Grandis’s Presentation of the Symmetric-Simplicial Category

We begin by recalling the well-known presentation of category \textbf{Ord} via generators and relations (see [May67], [Lam68], [ML70], [GJ99] et. al.). The generators are given in the following definition.

**Definition 2.1** The elementary coface maps are defined by

\[
d_i = d_i^{(n)} : [n-1] \rightarrow [n] \quad \text{for } n \geq 1 \text{ and } 0 \leq i \leq n
\]

\[
k \mapsto \begin{cases} k & \text{for } k \leq i \\ k + 1 & \text{for } k > i \end{cases}
\]

and the elementary codegeneracy maps are defined as follows.

\[
s_i = s_i^{(n)} : [n+1] \rightarrow [n] \quad \text{for } n \geq 0 \text{ and } 0 \leq i \leq n
\]

\[
k \mapsto \begin{cases} k & \text{for } k \leq i \\ k - 1 & \text{for } k > i \end{cases}
\]

**Remark 2.2** One may put this definition into words by saying that \(d_i\) is the unique monotonic injection \([n-1] \rightarrow [n]\) whose image contains everything except the element \(i\), and \(s_i\) is the unique monotonic surjection \([n+1] \rightarrow [n]\) for which each range element has a single pre-image except for the element \(i\), which has two pre-images.

These generators satisfy the following *cosimplicial identities*. It is proved in [ML70] that \textbf{Ord} is isomorphic to the abstract category obtained by imposing the cosimplicial identities on the free category having objects \([n]\) and generators \(s_i^{(n)}, d_i^{(n)}\).
The Cosimplicial Identities.

\[
\begin{align*}
\text{d}_i \text{d}_j &= \begin{cases} 
\text{d}_{i+1} \text{d}_i & \text{if } i \leq j \\
\text{d}_j \text{d}_{i-1} & \text{if } i > j 
\end{cases} \\
\text{d}_i \text{s}_j &= \begin{cases} 
\text{s}_{i+1} \text{d}_i & \text{if } i \leq j \\
\text{s}_i \text{d}_{i+1} & \text{if } i > j 
\end{cases} \\
\text{s}_i \text{d}_j &= \begin{cases} 
\text{d}_{j-1} \text{s}_i & \text{if } i \leq j - 2 \\
\text{id} & \text{if } i = j - 1 \text{ or } j \\
\text{d}_j \text{s}_{i-1} & \text{if } i \geq j + 1 
\end{cases} \\
\text{s}_i \text{s}_j &= \begin{cases} 
\text{s}_{j-1} \text{s}_i & \text{if } i < j \\
\text{s}_j \text{s}_{i+1} & \text{if } i \geq j 
\end{cases}
\end{align*}
\]

Remark 2.3 These identities are usually written in a nonredundant form. Here, and in all other presentations below, we have included all possible situations that arise when interchanging two generators, thus incurring a certain amount of redundancy.

Traditionally, the action of Ord on simplicial objects is written on the left. This necessitates reversing the cosimplicial identities given above (and then reorganizing indices). The reversed identities are called the simplicial identities and are included here for reference.

The Simplicial Identities.

\[
\begin{align*}
\text{d}_i \text{d}_j &= \begin{cases} 
\text{d}_{j-1} \text{d}_i & \text{if } i < j \\
\text{d}_j \text{d}_{i+1} & \text{if } i \geq j 
\end{cases} \\
\text{s}_i \text{s}_j &= \begin{cases} 
\text{s}_{i+1} \text{s}_i & \text{if } i \leq j \\
\text{s}_i \text{s}_{i-1} & \text{if } i > j 
\end{cases} \\
\text{d}_i \text{s}_j &= \begin{cases} 
\text{s}_{j-1} \text{d}_i & \text{if } i < j \\
\text{id} & \text{if } i = j \text{ or } j + 1 \\
\text{s}_j \text{d}_{i-1} & \text{if } i \geq j + 2 
\end{cases} \\
\text{s}_i \text{d}_j &= \begin{cases} 
\text{d}_{j+1} \text{s}_i & \text{if } i < j \\
\text{d}_j \text{s}_{i+1} & \text{if } i \geq j 
\end{cases}
\end{align*}
\]

Remark 2.4 The table above (as well as the others to come) is arranged so that all identities in the right column follow from the identities to their left. Some redundancies also remain within the left column.

A presentation of Fin via generators and relations has been given by Marco Grandis in [Gra01a]. In addition to the generators \(\text{d}_i\) and \(\text{s}_i\) of Ord, his presentation also makes use of the following generating permutations.
Definition 2.5  The adjacent transpositions are defined as follows.

\[ t_i = t_i^{(n)} : [n] \rightarrow [n] \text{ for } n \geq 1 \text{ and } 0 \leq i \leq n - 1 \]

\[
k \mapsto \begin{cases} 
  k & \text{for } k \neq i, i + 1 \\
  i + 1 & \text{for } k = i \\
  i & \text{for } k = i + 1
\end{cases}
\]

These transpositions satisfy certain relations constituting a well-known presentation of the symmetric group on \(n+1\) elements, ascribed to the American mathematician E.H. Moore (1862-1932).

\[
\begin{align*}
t_i^2 &= \text{id} \\
t_i t_j &= t_j t_i \text{ if } |i - j| \geq 2 \\
t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}
\end{align*}
\]

In addition to these as well as the simplicial identities, Grandis’s presentation also includes relations allowing one to interchange a transposition with a face or degeneracy operator. His relations are given below in contravariant form, that is, as the relations defining \(\text{Fin}^{op}\), so that they are suitable for writing the action on a symmetric-simplicial object on the left.
The Symmetric-Simplicial Identities (Grandis).

\[
d_{i}d_{j} = \begin{cases} 
  d_{j-1}d_{i} & \text{if } i < j \\
  d_{j}d_{i+1} & \text{if } i \geq j 
\end{cases}
\]

\[
d_{i}s_{j} = \begin{cases} 
  s_{j-1}d_{i} & \text{if } i < j \\
  \text{id} & \text{if } i = j \text{ or } j + 1 \\
  s_{j}d_{i-1} & \text{if } i \geq j + 2 
\end{cases}
\]

\[
s_{i}s_{j} = \begin{cases} 
  s_{j+1}s_{i} & \text{if } i \leq j \\
  s_{j}s_{i-1} & \text{if } i > j 
\end{cases}
\]

\[
t_{i}t_{j} = \begin{cases} 
  t_{j-1}d_{i} & \text{if } i < j \\
  d_{i+1} & \text{if } i = j \\
  d_{i-1} & \text{if } i = j + 1 \\
  t_{i}d_{i} & \text{if } i \geq j + 2 
\end{cases}
\]

\[
t_{i}s_{j} = \begin{cases} 
  s_{j}t_{i} & \text{if } i \leq j - 2 \\
  t_{i+1}s_{j-1}t_{i} & \text{if } i = j - 1 \\
  s_{j} & \text{if } i = j \\
  t_{i-1}s_{j+1}t_{i-1} & \text{if } i = j + 1 \\
  s_{j}t_{i-1} & \text{if } i \geq j + 2 
\end{cases}
\]

\[
s_{i}d_{j} = \begin{cases} 
  d_{j+1}s_{i} & \text{if } i < j \\
  d_{j}s_{i+1} & \text{if } i \geq j 
\end{cases}
\]

In order to give an alternate presentation of the category $\text{Fin}^{op}$, it is convenient to introduce the following operators first.
**Definition 2.6** The following maps in $\text{Fin}$ will be called the *standard cyclic permutations*.

$$z_i = z_i^{(n)} : [n] \rightarrow [n] \text{ for } n \geq 1 \text{ and } 0 \leq i \leq n$$

$$k \mapsto \begin{cases} 
  k + 1 & \text{for } 0 \leq k \leq i - 1 \\
  0 & \text{for } k = i \\
  k & \text{for } k > i 
\end{cases}$$

Note that $z_i$ is an $(i + 1)$-cycle on the elements $0, 1, \ldots, i$. In particular, $z_0$ is the identity. One may equivalently take the following formula in $\text{Fin}^{op}$ as a definition of the corresponding symmetric-simplicial operator $z_i$ for $n \geq 0$ and $0 \leq i \leq n$.

$$z_i = z_i^{(n)} := t_{i-1} \cdots t_1 t_0$$

The alternative presentation of $\text{Fin}^{op}$ given below keeps the elementary face operators and transpositions as generators but substitutes for the elementary degeneracies the following.

**Definition 2.7** The following maps in $\text{Fin}$ will be referred to as the *elementary quasi-codegeneracy maps*.

$$u_i = u_i^{(n)} : [n+1] \rightarrow [n] \text{ for } n \geq 0 \text{ and } 1 \leq i \leq n + 1$$

$$k \mapsto \begin{cases} 
  0 & \text{for } k = 0 \text{ or } i \\
  k & \text{for } 1 \leq k \leq i - 1 \\
  k - 1 & \text{for } k > i 
\end{cases}$$

In particular, $u_1$ coincides with $s_0$. Note $u_0$ is not defined. One may equivalently define the *elementary quasi-degeneracy operators* $u_i$ in $\text{Fin}^{op}$ in terms of the $s_i$ and $z_i$ by means of the following formula holding in $\text{Fin}^{op}$ for $i \geq 1$.

$$u_i := z_{i-1}^{-1}s_{i-1}z_{i-1}$$

The following theorem gives a presentation of $\text{Fin}^{op}$ in terms of the generators $d_i, u_i, t_i$. 

8
Theorem 2.8 The generators $d_i$, $u_i$, and $t_i$ together with the following relations constitute a presentation of $\text{Fin}^\alpha$.

$$d_i d_j = \begin{cases} d_{j-1} d_i & \text{if } i < j \\ d_j d_{i+1} & \text{if } i \geq j \end{cases}$$

$$d_i u_j = \begin{cases} z_{j-1} & \text{if } i = 0 \\ u_{j-1} d_i & \text{if } 0 \neq i < j \\ \text{id} & \text{if } i = j \\ u_j d_{i-1} & \text{if } i > j \end{cases}$$

$$u_i d_j = \begin{cases} d_{j+1} u_i & \text{if } i \leq j \\ d_j u_{i+1} & \text{if } i > j \neq 0 \\ d_1 u_{i+1} t_0 & \text{if } j = 0 \end{cases}$$

$$u_i u_j = \begin{cases} u_{j+1} u_i & \text{if } i \leq j \\ u_j u_{i-1} & \text{if } i > j \end{cases}$$

$$t_i t_j = \begin{cases} t_{j-1} d_i & \text{if } i < j \\ d_{i+1} & \text{if } i = j \\ d_{i-1} & \text{if } i = j + 1 \\ t_j d_i & \text{if } i \geq j + 2 \end{cases}$$

$$t_i d_j = \begin{cases} t_{j-1} d_i & \text{if } i < j \\ d_{j+1} t_i t_{i+1} & \text{if } i = j - 1 \\ d_j t_{i+1} & \text{if } i \geq j \end{cases}$$

$$t_i u_j = \begin{cases} u_{j+1} t_i & \text{if } 0 \neq i \leq j - 2 \\ u_j & \text{if } 0 \neq i = j - 1 \\ u_{j+1} & \text{if } i = j \\ u_j t_{i-1} & \text{if } i > j \end{cases}$$

$$t_j u_i = \begin{cases} t_{j+1} u_i & \text{if } i \leq j \\ t_j t_{j+1} u_{i-1} & \text{if } i = j + 1 \\ t_j u_i & \text{if } i \geq j + 2 \end{cases}$$

$$t_0 u_1 = u_1$$

$$t_0 u_i t_0 u_j = \begin{cases} u_{j+1} t_0 u_i t_0 & \text{if } 2 \leq i \leq j \\ u_j t_0 u_i t_0 & \text{if } 2 \leq j < i \end{cases}$$

The next section is devoted to proving this theorem.

Remark 2.9 For our purposes, this presentation has some advantages over that of Grandis. For instance, Corollary 4.13 is a consequence of the rule for $t_i u_j$ (contrast with the rule for $t_i s_j$). The rule for $d_i u_j$ for $i > 0$ in particular is responsible for the vanishing of a great many brackets universally in symmetric hypercrossed complexes (this will be demonstrated in a forthcoming article).
The above presentation also has some notable disadvantages, particularly in the inability to move \( t_0 \) past any \( u_i \) for \( i \geq 2 \), as well as in the identity 
\[
d_0 u_i = z_{i-1},
\]
which makes the full definition of \( d_0 \) in symmetric hypercrossed complexes dependent on \( t_0 \).

**Remark 2.10** It is readily verified that all relations in the right column follow from the relations in the left column. All references to the statement of Theorem 2.8 will be understood as referring to relations of the left column only.

Here are some other useful operators in \( \text{Fin}^{op} \).

**Definition 2.11** In the statement of Theorem 2.8, note the overlapping conditions in the identities for \( u_i d_j \). Indeed the equations
\[
u_i d_i = d_{i+1} u_i = d_i u_{i+1} =: r_i
\]
hold for all \( 1 \leq i \leq n \). We refer to the \( r_i \) as replacement operators.

**Proposition 2.12** For each \( n \geq 1 \), the replacement operators
\[
r_i : [n] \rightarrow [n] \text{ for } 1 \leq i \leq n
\]
constitute a family of mutually commuting idempotents in \( \text{Fin}^{op} \).

\[
r_i^2 = r_i,
\]
\[
r_i r_j = r_j r_i
\]

**Proof.** This is most easily verified using the following formula for \( r_i \) as a function in \( \text{Fin} \).
\[
r_i(k) = \begin{cases} 
0 & \text{if } k = 0 \text{ or } i \\
1 & \text{otherwise}
\end{cases}
\]
Alternatively, it is a fun exercise to prove the assertion using the identities of Theorem 2.8 and Definition 2.11.

\[\diamondsuit\]
3 Proof of the Alternative Presentation of the Symmetric-Simplicial Category

In this section, we give the proof of Theorem 2.8.

**Proof of Theorem 2.8.** The framework for the proof is as follows. Form the free category with objects $[n]$ for $n \geq 0$ and generators $d_i = d^{(n)}_i : [n] \to [n + 1]$ for $0 \leq i \leq n + 1$, $t_i = t^{(n)}_i : [n] \to [n]$ for $0 \leq i \leq n - 1$, and $s_i = s^{(n)}_i : [n] \to [n - 1]$ for $0 \leq i \leq n - 1$.

and let $\mathcal{D}$ denote the quotient of this free category by those relations involving only the $d_i$ and $t_i$ (note these relations are common to both Grandis’s presentation and the one proposed by the theorem). According to [Gra01a], the imposition on $\mathcal{D}$ of the remaining relations of Grandis involving the $s_i$ produces the category $\text{Fin}^{op}$. Letting $u_i$ stand for $z_i^{-1} s_{i-1} z_i^{-1} \in \mathcal{D}$, we wish to show that the imposition of the relations in the statement of the theorem involving the $u_i$ also produces $\text{Fin}^{op}$. For this it suffices to show that each relation involving the $u_i$ is a consequence of those involving the $s_i$ and those of $\mathcal{D}$, and that each of Grandis’s relations involving the $s_i$ is a consequence of those involving the $u_i$ and those of $\mathcal{D}$. All the various statements constituting these assertions are proved in several propositions below, and the proof of the theorem is completed after that.

**Lemma 3.1** The following relations hold in the group $\text{Sym}[n]^{op}$.

$$z_i^{-1} z_j = \begin{cases} z_j t_0 z_i^{-1} & \text{if } i < j \\ \text{id} & \text{if } i = j \\ z_j^{-1} t_0 z_i^{-1} & \text{if } i > j \end{cases}$$

**Proof.** Each of the above identities can be deduced from

$$z_j^{-1} z_i z_j = z_{i+1} t_0$$

so it suffices to prove the latter. To see this, note that since $i$ is less than $j$, the effect of the conjugation action of $z_j$ on

$$z_i = t_{i-1} \ldots t_0$$

is
Lemma 3.2 The following identities hold in $\mathcal{Q}$ (hence also in $\text{Fin}^{\text{op}}$).

\[
t_i z_j = \begin{cases} 
  z_j t_{i+1} & \text{if } i \leq j - 2 \\
  z_{j-1} & \text{if } i = j - 1 \\
  z_{j+1} & \text{if } i = j \\
  z_j t_i & \text{if } i \geq j + 1 
\end{cases} \quad \quad z_i^{-1} t_j = \begin{cases} 
  t_j z_i^{-1} & \text{if } i < j \\
  z_{i+1} & \text{if } i = j \\
  z_i^{-1} & \text{if } i = j + 1 \\
  t_{j+1} z_i^{-1} & \text{if } i \geq j + 2 
\end{cases} \\
\]

\[
d_i z_j = \begin{cases} 
  z_{j-1} d_{i+1} & \text{if } i < j \\
  d_0 & \text{if } i = j \\
  z_j d_i & \text{if } i > j 
\end{cases} \quad \quad z_i^{-1} d_j = \begin{cases} 
  d_j z_i^{-1} & \text{if } i < j \\
  d_{j+1} z_i^{-1} & \text{if } i \geq j 
\end{cases} 
\]

Proof. For the upper batches of identities, use the relations of the Moore presentation in a straightforward manner. The proofs of the lower batches are similar to but easier than those of Lemma 3.4 below. All are left to the reader.

Proposition 3.3 In each of the following equivalences, imposing upon $\mathcal{Q}$ the relation on the left hand side produces the same result as imposing its correspondent on the right hand side.

\[
d_is_j = s_{j-1}d_i \quad \text{for } i < j \quad \iff \quad d_is_i = u_{j-1}d_i \quad \text{for } 0 \neq i \leq j \\
d_is_{i-1} = \text{id} \quad \iff \quad d_is_i = \text{id} \\
d_is_i = \text{id} \quad \iff \quad d_0u_{i+1} = z_i \\
d_is_j = s_jd_{i-1} \quad \text{for } i \geq j + 2 \quad \iff \quad d_is_j = u_jd_{i-1} \quad \text{for } i > j \\
\]

\[
t_is_j = s_j t_i \quad \text{for } i \leq j - 2 \quad \iff \quad t_is_i = u_j t_i \quad \text{for } 0 \neq i \leq j - 2 \\
t_{j-1}s_j = t_j s_{j-1} t_{j-1} \quad \iff \quad t_is_j = u_{i+1} \\
t_{i+1}s_i = t_is_{i+1} t_i \quad \iff \quad t_is_i = u_{i+1} \\
t_is_j = s_j t_{i-1} \quad \text{for } i \geq j + 2 \quad \iff \quad t_is_j = u_j t_{i-1} \quad \text{for } i > j \\
\]

Proof. The verifications all follow the same pattern, so we prove the first one for illustration and leave the rest to the reader. For $i < j$, one has

\[
d_is_j = d_is_ju_{j+1}z_j^{-1} \quad \text{(By Def. of } u_{j+1}) \\
= z_{j-1}d_{i+1}u_{j+1}z_j^{-1} \quad \text{(By Lemma 3.2)} 
\]

\[\text{(12)}\]
and also

\[ s_{j-1}d_i = z_{j-1}u_jz_{j-1}^{-1}d_i \quad \text{(By Def. of } u_j) \]
\[ = z_{j-1}u_jd_{i+1}z_j^{-1} \quad \text{(By Lemma 3.2)} \]

and since the outermost terms of each of these results coincide and are invertible, one obtains (after a reparametrization) the following two-way implication as desired.

\[ d_is_j = s_{j-1}d_i \text{ for } i < j \iff d_iu_j = u_{j-1}d_i \text{ for } 0 \neq i \leq j \]

\[ \text{Lemma 3.4} \]

Let \( \mathcal{D} \) denote the category obtained by imposing on \( \mathcal{D} \) the following relations from the statement of Theorem 2.8.

\[ t_iu_j = u_jt_i \text{ for } 0 \neq i \leq j - 2 \quad (1) \]
\[ t_iu_{i+1} = u_i \quad (2) \]
\[ t_iu_i = u_{i+1} \text{ for } i \geq 1 \quad (3) \]
\[ t_iu_j = u_jt_{i-1} \text{ for } i > j \quad (4) \]
\[ t_0u_1 = u_1 \quad (5) \]

Then the following identities hold in \( \mathcal{D} \).

\[
z_j^{-1}u_i = \begin{cases} 
   t_0u_{i+1}t_0z_j^{-1} & \text{if } i < j \\
u_1 & \text{if } i = j \\
t_0u_0z_j^{-1} & \text{if } i > j 
\end{cases}
\]

\[
u_is_j = \begin{cases} 
   z_{j+1}t_0u_{i+1}t_0 & \text{if } i \leq j \\
z_{j}t_0u_it_0 & \text{if } i > j 
\end{cases}
\]

\textbf{Proof.} The identities on the right can be directly deduced from those on the left, so we prove only the latter. The case \( i < j \) is demonstrated as follows.

\[
z_j^{-1}u_i = t_0 \cdots t_{j-1}u_i \quad \text{(Def. of } z_j) \]
\[ = t_0(t_1 \cdots t_{i-1})t_i(t_{i+1} \cdots t_{j-1})u_i \]
\[ = t_0(t_1 \cdots t_{i-1})u_i(t_i \cdots t_{j-2}) \quad \text{(By (1))} \]
\[ = t_0u_{i+1}(t_1 \cdots t_{i-1})(t_i \cdots t_{j-2}) \quad \text{(By (3))} \]
\[ = t_0u_{i+1}t_0t_0(t_1 \cdots t_{i-1})(t_i \cdots t_{j-2}) \quad (t_0^2 = \text{id}) \]
\[ = t_0u_{i+1}t_0z_j^{-1} \quad \text{(Def. of } z_{j-1}) \]
The proof of the case $i > j$ is similar but easier. The case $i = j$ follows from (4) and repeated application of (2).

Lemma 3.5 Let $Q$ be as in Lemma 3.4. Then the following holds in $Q$.

\[ u_i u_j = \begin{cases} u_{j+1} u_i & \text{if } i \leq j \\ u_j u_{i-1} & \text{if } i > j \end{cases} \]

Proof. Calculate as follows for $i \leq j$.

\[
\begin{align*}
u_i u_j &= z_i u_1 z_j u_1 & \text{(By Lemma 3.4)} \\
&= z_i z_{j+1} t_0 u_1 t_0 u_1 & \text{(By Lemma 3.4)} \\
&= z_{j+1} z_i + 1 t_0 u_1 t_0 u_1 & \text{(By Lemma 3.1)} \\
&= z_{j+1} u_1 z_i u_1 & \text{(By Lemma 3.4)} \\
&= u_{j+1} u_i & \text{(By Lemma 3.4)}
\end{align*}
\]

The case $i > j$ follows immediately from the case $i \leq j$.

Proposition 3.6 Let $Q$ be as in Lemma 3.4. Then the following holds in $Q$.

\[ t_i s_i = s_i \text{ for all } i \]

Proof. One computes as follows.

\[
\begin{align*}
t_i s_i &= t_i z_i u_{i+1} z_i^{-1} & \text{(Def. of } u_{i+1}) \\
&= z_{i+1} u_{i+1} z_i^{-1} & \text{(By Lemma 3.1)} \\
&= z_i z_{i+1} u_1 z_i^{-1} & \text{(By Lemma 3.3)} \\
&= z_i z_{i+1} t_0 u_1 z_i^{-1} & \text{(By Lemma 3.1 with } j = i + 1) \\
&= z_i z_{i+1} u_1 z_i^{-1} & \text{(By Lemma 3.1 with } j = i + 1) \\
&= z_i u_{i+1} z_i^{-1} & \text{(By Lemma 3.4)} \\
&= s_i & \text{(Def. of } u_{i+1})
\end{align*}
\]
Proposition 3.7 Let $\mathcal{D}$ be as in Lemma 3.4. Then the following equivalences of algebraic relations hold (in the same sense as in Lemma 3.5).

\[ s_is_j = s_{j+1}s_i \text{ for } i < j \quad \iff \quad t_0u_it_0u_j = u_{j+1}t_0u_it_0 \text{ if } 2 \leq i \leq j \]
\[ s_is_i = s_{i+1}s_i \text{ for } all \ i \geq 0 \quad \iff \quad u_1u_j = u_{j+1}u_1 \text{ for } all \ j \geq 1 \]

Proof. Calculate as follows for $i < j$.

\[ s_is_j = z_iu_{i+1}z_i^{-1}z_ju_{j+1}z_j^{-1} \quad \text{(By Def. 2.7)} \]
\[ = z_iu_{i+1}z_jt_0z_{i+1}^{-1}u_{j+1}z_j^{-1} \quad \text{(By Lemma 3.1)} \]
\[ = z_iu_{j+1}z_jt_0u_{i+2}t_0t_0u_ju_{i+1}t_0z_{i+1}^{-1}z_j^{-1} \quad \text{(By Lemma 3.4 twice)} \]
\[ = (z_iu_{j+1}t_0)(u_{i+2}t_0u_{j+1}t_0)(z_{i+1}^{-1}z_j^{-1}) \]

\[ s_{j+1}s_i = z_{j+1}u_{j+1}u_{j+1}z_{i+2}^{-1}z_iu_{i+1}z_i^{-1} \quad \text{(By Def. 2.7)} \]
\[ = z_{j+1}u_{j+1}u_{j+1}u_{j+1}z_i^{-1}z_i \quad \text{(By Lemma 3.1)} \]
\[ = z_{j+1}z_{j+1}u_{j+1}u_{j+1}u_{j+1}z_i^{-1} \quad \text{(By Lemma 3.4 twice)} \]
\[ = (z_{j+1}z_{j+1})(t_0u_{j+1}t_0u_{j+1})(t_0z_j^{-1}z_i^{-1}) \]

Now by Lemma 3.1, the respective outer terms of the two expressions coincide, and since these terms are invertible, one obtains the equality of the inner terms, that is

\[ s_is_j = s_{j+1}s_i \text{ for } i < j \quad \iff \quad t_0u_it_0u_j = u_{j+1}t_0u_it_0 \text{ for } 2 \leq i \leq j \]

Similarly, for the case $i = j$ one computes

\[ s_is_i = z_iu_{i+1}z_i^{-1}z_iu_{i+1}z_i^{-1} \quad \text{(By Def. 2.7)} \]
\[ = z_iu_{i+1}u_{i+1}z_i^{-1} \quad \text{(By Lemma 3.3)} \]
\[ s_{i+1}s_i = z_{i+1}u_{i+1}u_{i+1}z_{i+1}^{-1}z_iu_{i+1}z_i^{-1} \quad \text{(By Def. 2.7)} \]
\[ = z_{i+1}u_{i+1}z_iu_{i+1}z_i^{-1} \quad \text{(By Lemma 3.1)} \]
\[ = z_{i+1}z_{i+1}t_0u_{i+2}t_0t_0u_{i+1}t_0z_{i+1}^{-1}u_{i+1}z_i^{-1} \quad \text{(By Lemma 3.4)} \]
\[ = z_{i+1}z_{i+1}u_{i+2}t_0t_0u_{i+1}z_i^{-1} \quad \text{(By Lemma 3.4)} \]
\[ = z_{i+1}z_{i+1}u_{i+2}u_{i+1}z_i^{-1} \quad \text{(By Lemma 3.4)} \]
and similarly as before one obtains the following.

\[ s_i s_i = s_{i+1} s_i \text{ for all } i \geq 0 \iff u_1 u_j = u_{j+1} u_1 \text{ for all } j \geq 1 \]

In order to finish the proof of Theorem 2.8, it is convenient to introduce the following notation.

**Definition 3.8** Let the symbol \( R[d, u] \) denote the set of relations stated in Theorem 2.8 involving only the operators \( d_i \) and \( u_j \). Similarly use the symbols \( R[u, u] \) and \( R[t, u] \). It is also convenient to write \( R[t+, u] \) for those relations of \( R[t, u] \) involving only \( t_i \) with \( i > 0 \) and \( R[t_0, u] \) for those relations of \( R[t, u] \) involving \( t_0 \) but not \( t_i \) with \( i > 0 \).

Additionally \( R[d, s], R[s, s] \) and \( R[t, s] \) are used to refer to the analogous sets of relations from Grandis’s presentation.

**Proof of Theorem 2.8 (continued).** For one direction, assume that all of Grandis’s relations hold, so that we are working in the category \( \text{Fin}^{op} \).

From Proposition 3.3 all identities \( R[d, u] \) and \( R[t+, u] \) are obtained. These identities fulfill the hypotheses of Lemmas 3.4 and 3.5, and thus \( R[u, u] \) is obtained. Finally, these same lemmas enable us to apply Proposition 3.7 and so \( R[t_0, u] \) is also obtained (with the exception of \( t_0 u_1 = u_1 \), but this is just the same as \( t_0 s_0 = s_0 \)).

For the reverse direction, assume all identities \( R[d, u], R[t, u] \) and \( R[u, u] \) are imposed on \( \mathcal{D} \). In the resulting category, all identities \( R[d, s] \) and \( R[t, s] \) obtain by Propositions 3.3 and 3.6. By Proposition 3.7, the identities \( R[s, s] \) also obtain.

**Remark 3.9** It is a corollary of Propositions 3.3 and 3.6 that, in Grandis’s presentation \[Gra01a\], the relations \( t_i s_i = s_i \) for \( i > 0 \) are redundant. Tracing this, one finds that they are a consequence of the relation \( t_0 s_0 = s_0 \) as well as the other relations for exchanging \( t_i \) and \( s_j \).

Similarly and perhaps surprisingly, Lemma 3.5 says that the relations for \( u_i u_j \) are redundant in the alternate presentation of Theorem 2.8. They are a consequence of the relations for exchanging \( t_i \) and \( u_j \) as well as of the Moore relations for the \( t_i \).
4 The Algebra of the Symmetric-Simplicial Category

In the subsections of this final section we collect together a number of facts about the algebraic structure of the category $\text{Fin}$, viewed from the point of view of the alternative presentation given in Theorem 2.8. For this purpose we introduce the following notational device.

**Definition 4.1** A multi-index $\alpha$ of length $k$ and dimension $\leq n$ is a strictly increasing sequence of indices

$$\alpha = \{i_1 < \ldots < i_k\}$$

satisfying $1 \leq i_p \leq n$ for all $p$. The length $|\alpha|$ of $\alpha$ is the number $k$ of indices in $\alpha$.

**Definition 4.2** The quasi-codegeneracy $u_{\alpha} \in \text{Fin}$ corresponding to $\alpha$ is the composition of elementary quasi-codegeneracies

$$u_{\alpha} : [n] \to [n - |\alpha|]$$

$$u_{\alpha} := u_{i_1}u_{i_2}\ldots u_{i_k}$$

(note the indices increase from left to right) and the coface $d_{\alpha} \in \text{Fin}$ corresponding to $\alpha$ is

$$d_{\alpha} : [n - |\alpha|] \to [n]$$

$$d_{\alpha} := d_{i_k}d_{i_{k-1}}\ldots d_{i_1}$$

(note the indices decrease from left to right).

4.1 The Quasi-Monotonic Functions

The goal of this subsection is to characterize the functions in $\text{Fin}$ obtained as compositions of the form $d_{\alpha}u_{\beta}$ as the quasi-monotonic functions (defined below), to show that they constitute a subcategory of $\text{Fin}$ and finally to prove that the expressions $d_{\alpha}u_{\beta}$ themselves constitute a family of unique factorizations for that subcategory. This amounts to an analog of the usual unique factorization theorem for $\text{Ord}$ with the $u_i$ taking the place of the $s_i$, we which recall here (see [May67], [ML70], [Lam68] for a proof).
Proposition 4.3  Each simplicial operator $f : [n] \to [m] \in \text{Ord}$ has a unique factorization

$$f = d_{i_1} \ldots d_{i_k} s_{j_1} \ldots s_{j_l}$$

with strictly increasing sequences of indices

$$0 \leq i_1 < \ldots < i_k \leq m$$

$$0 \leq j_1 < \ldots < j_l \leq n,$$

that is, reading from left to right, the indices of the degeneracies decrease and the indices of the faces increase.

The following property will ultimately be shown to characterize functions of the form $d_\alpha u_\beta$.

Definition 4.4  Let a function $f \in \text{Fin}$ be called quasi-monotonic if it satisfies the following two conditions.

QM1. $f(0) = 0$

QM2. $f(p) \neq 0 \neq f(q)$ and $p < q \implies f(p) < f(q)$

The second condition says that $f$ is strictly increasing outside of $f^{-1}(0)$.

Proposition 4.5  The quasi-monotonic functions are closed under composition.

Proof. Let $f$ and $g$ be quasi-monotonic. It suffices to check for $p < q$ that

$$(g \circ f)(p) \neq 0 \neq (g \circ f)(q) \implies (g \circ f)(p) < (g \circ f)(q).$$

By QM1 for $g$ it must be the case that $f(p) \neq 0 \neq f(q)$, and it follows that $f(p) < f(q)$ since $f$ is quasi-monotonic. Then condition QM2 for $g$ applies to give exactly $g(f(p)) < g(f(q))$.  \hfill \blacktriangleleft

Lemma 4.6  Let $\alpha$ be a multi-index and let the nonnegative integer $p$ belong to the domain of $u_\alpha$. Then $u_\alpha(p) = 0$ if and only if $p = 0$ or $p$ belongs to $\alpha$. Moreover, if $p$ is neither 0 nor in $\alpha$ then

$$u_\alpha(p) = p - \# \{ i \in \alpha \mid i < p \}.$$
Proof. We start with the “if” direction. First, if $p$ is 0, then $u_\alpha(p) = 0$ since by Definition 2.7 all quasi-codegeneracies send 0 to 0. Now assume $p$ belongs to $\alpha$. Then one may factor $u_\alpha$ as $u_{\alpha^<p}u_pu_{\alpha^>p}$ where $\alpha^<p$ consists of those indices in $\alpha$ that are less than $p$ and $\alpha^>p$ consists of those indices in $\alpha$ that are greater than $p$. By Definition 2.7 all $u_i$ with $i > p$ send $p$ to $p$, so that $u_{\alpha^>p}(p) = p$. Then we have

$$u_\alpha(p) = u_{\alpha^<p}u_pu_{\alpha^>p}(p)$$
$$= u_{\alpha^<p}u_p(p)$$
$$= u_{\alpha^<p}(0)$$
$$= 0$$

as claimed.

We turn to prove the “only if” direction, so we assume $p$ is neither 0 nor in $\alpha$. Then one may factor $u_\alpha$ as $u_{\alpha^<p}u_{\alpha^>p}$ with $\alpha^<p$ and $\alpha^>p$ having the same meanings as above. As before $u_{\alpha^>p}(p) = p$, so that $u_\alpha(p) = u_{\alpha^<p}(p)$. If $\alpha^<p$ is empty then $u_{\alpha^<p}$ is the identity and we conclude $u_\alpha(p) = p \neq 0$ as required. If $\alpha^<p$ is not empty, say $\alpha^<p = \{i_1, i_2, \ldots, i_j\}$. Then by Definition 2.7 we can evaluate

$$u_{\alpha^<p}(p) = u_{i_1}u_{i_2} \ldots u_{i_j}(p)$$
$$= u_{i_1}u_{i_2} \ldots u_{i_{j-1}}(p - 1)$$
$$:$$
$$= u_{i_1}u_{i_2} \ldots u_{i_{j-1}}(p - l)$$
$$:$$
$$= p - j.$$ 

where one notes that at each step, the argument $p - l$ decreases by 1 while the rightmost index $i_{j-l}$ decreases by at least 1, so that $i_{j-l} < p - l$ holds for all $l$ and therefore the calculation may always proceed to the next step. Since $j$ is the number of indices in $\alpha^<p$ that are less than $p$, and since all indices in $\alpha^<p$ are between 1 and $p - 1$ inclusive, we deduce that $j$ is at most $p - 1$. Then $u_\alpha(p) = p - j \geq 1$ and we conclude that $u_\alpha(p)$ is not 0 as claimed.

The final assertion of the Lemma can be read out of the proof of the “only if” direction just given.

♦
Lemma 4.7 Quasi-monotonic surjections are uniquely determined by their zeros.

Proof. Let \( h : [n] \rightarrow [m] \) be a quasi-monotonic surjection. Note \( h \) restricts to a surjection \([n] \setminus h^{-1}(0) \rightarrow [m] \setminus \{0\}\) and by QM2 the restriction must be strictly monotonic, therefore also a bijection. Hence \( h \) is uniquely determined on \([n] \setminus h^{-1}(0)\) and therefore on all of \([n]\). ♦

Proposition 4.8 The quasi-monotonic injections are precisely the functions \( d_\alpha \). The quasi-monotonic surjections are precisely the functions \( u_\alpha \).

Proof. That all functions of the form \( d_\alpha \) or \( u_\beta \) are quasi-monotonic follows from Lemma 4.5 and the fact that, with the exception of \( d_0 \), all \( d_i \) and \( u_i \) are quasi-monotonic. Moreover \( d_0 \) is not a factor of \( d_\alpha \) since multi-indices \( \alpha \) by definition do not contain 0.

Let \( g \) be a quasi-monotonic injection. Then \( g \) must be monotonic because by QM2, \( g \) is increasing off of \( g^{-1}(0) \) and by QM1 \( g^{-1}(0) = \{0\} \), i.e., \( g \) is increasing on the rest of the domain of \( g \). Then \( g \) has a unique factorization \( g = d_{i_k} \ldots d_{i_1} \) in \( \text{Ord} \) with \( i_1 < \ldots < i_k \), and moreover \( i_1 \) cannot be 0 because then 0 would not be in the image of \( g \). So \( \alpha = \{i_1 < \ldots < i_k\} \) satisfies the definition of multi-index and \( g \) is equal to \( d_\alpha \).

Finally let \( h \) be a quasi-monotonic surjection. Let \( \alpha \) be the multi-index consisting of the zeros of \( h \), excluding 0 itself. By Lemma 4.6, the function \( u_\alpha \) has precisely the same zeros as \( h \). Then by Lemma 4.7, \( h \) must coincide with \( u_\alpha \). ♦

Proposition 4.9 The quasi-monotonic functions are precisely the functions of the form \( d_\alpha u_\beta \).

Proof. That all functions of the form \( d_\alpha u_\beta \) are quasi-monotonic follows from Lemma 4.5 just as in the preceding proposition.

To see that every quasi-monotonic \( f \) has the form \( d_\alpha u_\beta \), first factor \( f \) as \( f = g \circ h \) where \( g \) is a monotonic injection and \( h \) is a surjection (this is possible for any \( f \)). Then \( g \) has a factorization \( d_{i_k} \ldots d_{i_1} \) by Proposition 4.3. Since 0 is in the image of \( f \), \( d_0 \) does not occur, that is, none of the \( i_j \) is 0,
and hence we may write down the quasi-monotonic surjection $u_{i_1} \ldots u_{i_k}$. It is a left-inverse for $g$, so by composing with it one discovers that
\[ h = (u_{i_1} \ldots u_{i_k}) \circ g \circ h = (u_{i_1} \ldots u_{i_k}) \circ f \]
is quasi-monotonic by Lemma 4.5. Then $h$ is a quasi-monotonic surjection, so by Proposition 4.8, $h$ has the form $u_{j_1} \ldots u_{j_l}$.

**Proposition 4.10** Quasi-monotonic functions $f \in \text{Fin}$ have unique factorizations of the form $f = d_\alpha u_\beta$.

**Proof.** Existence of factorizations was demonstrated in the previous proposition. To prove uniqueness, let a quasi-monotonic function $f$ have two factorizations
\[ f = d_\alpha u_\beta = d_{\alpha'} u_{\beta'} \]
where $\alpha, \alpha', \beta, \beta'$ are multi-indices. It follows that
\[ \text{Im}(d_\alpha) = \text{Im}(d_\alpha u_\beta) = \text{Im}(d_{\alpha'} u_{\beta'}) = \text{Im}(d_{\alpha'}) \]
and since monotonic injections are uniquely determined by their images, one concludes $d_\alpha = d_{\alpha'}$ and then $\alpha = \alpha'$ by Proposition 4.3. Then cancelling these by composing both sides with a left-inverse, one obtains $u_\beta = u_{\beta'}$ and finally $\beta = \beta'$ by Lemmas 4.6 and 4.7.

### 4.2 Exchanging Transpositions and Quasidegeneracies

In this subsection we consider interchanging permutations with a quasidegeneracy $u_\gamma$. The reader is warned that we state results for $\text{Fin}^{\text{op}}$ instead of for $\text{Fin}$.

The following definition is motivated by the effect of repeatedly using the identity for $t_i u_j$ from Theorem 2.8 to push $t_i$ to the right across the factors of $u_\gamma$ one at a time.

**Definition 4.11** For any multi-index $\gamma$ and index $i \neq 0$, let $t_{i \sim \gamma}$ stand for the following permutation in $\text{Fin}^{\text{op}}$.

\[
t_{i \sim \gamma} := \begin{cases} 
\text{id} & \text{if } i \text{ or } i+1 \in \gamma \\
t_{\gamma'} & \text{for } \gamma' := i - \# \{ j \in \gamma \mid j < i \} & \text{if } i \notin \gamma \text{ and } i+1 \notin \gamma
\end{cases}
\]
In the next lemma, \( t_i(\gamma) \) denotes the multi-index obtained by applying \( t_i \) as a function to the elements of \( \gamma \). Note that the indices in \( t_i(\gamma) \) are to be rearranged in sequential order even if \( t_i \) puts the indices of \( \gamma \) out of order.

**Lemma 4.12** The following identity holds in \( \text{Fin}^{op} \) for any multi-index \( \gamma \) and index \( i \neq 0 \).

\[
t_i u_\gamma = u_{t_i(\gamma)} t_{i^\sim\gamma}
\]

**Proof.** Write

\[
u_\gamma = u_\zeta u_\theta u_\alpha
\]

where \( \zeta \) consists of the indices of \( \gamma \) greater than or equal to \( i + 2 \), \( \theta \) consists of those of \( i \) and \( i + 1 \) belonging to \( \gamma \), and \( \alpha \) consists of those indices of \( \gamma \) less than \( i \).

From the following identities from Theorem 2.8

\[
\begin{align*}
t_i u_i &= u_{i+1} \\
t_i u_{i+1} &= u_i \\
t_i u_{i+1} u_i &= u_i u_i = u_{i+1} u_i
\end{align*}
\]

one concludes

\[
t_i u_\theta = u_{t_i(\theta)}
\]

whenever \( \theta \) is not empty. In this case, one calculates

\[
\begin{align*}
t_i u_\gamma &= t_i u_\zeta u_\theta u_\alpha \\
&= u_\zeta t_i u_\theta u_\alpha & \text{(By Theorem 2.8)} \\
&= u_\zeta u_{t_i(\theta)} u_\alpha \\
&= u_{t_i(\gamma)}
\end{align*}
\]

as desired.

If \( \theta \) is empty, that is, neither \( i \) nor \( i + 1 \) belongs to \( \gamma \), then the calculation becomes

\[
\begin{align*}
t_i u_\gamma &= t_i u_\zeta u_\alpha \\
&= u_\zeta t_i u_\alpha & \text{(By Theorem 2.8)} \\
&= u_\zeta u_{t_i-|\alpha|} \\
&= u_\gamma t_{i-|\alpha|} & \text{(By Theorem 2.8)} \\
&= u_{t_i(\gamma)} t_{i^\sim\gamma}
\end{align*}
\]
also as desired.

For the final corollary below, we let $\text{Sym}_n'$ denote the group of permutations of $[n]$ leaving $0 \in [n]$ fixed.

**Corollary 4.13** For any permutation $\pi \in (\text{Sym}_n')^\text{op}$ and multi-index $\gamma$ of length $k$, there exists a permutation $\pi' \in (\text{Sym}_{n-k}')^\text{op}$ such that

$$\pi u_\gamma = u_{n-1(\gamma)} \pi'$$

**Proof.** Factor $\pi$ as a product of operators $t_i$ and use the previous lemma to push each one past $u_\gamma$. In this process, all transpositions in the factorization of $\pi$ pile up in order in the subscript of $u_\gamma$. Since $\pi$ belongs to $\text{Fin}^\text{op}$, its factorization occurs in the order opposite to that of its factorization in the permutation group $\text{Sym}_n'$. Applying the transpositions as functions in this reversed order to $\gamma$ is therefore the same as applying the permutation $\pi^{-1}$ to $\gamma$.

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