DYNAMICAL HARTREE-FOCK-BOGOLIUBOV APPROXIMATION OF INTERACTING BOSONS

JACKY JIA WEI CHONG

ABSTRACT. We consider a many-body Boson system with pairwise particle interaction given by $N^{3\beta - 1}v(N^{\beta}x)$ where $0 < \beta < \frac{1}{3}$ and $v$ a non-negative spherically symmetric function. Our main result is the extension of the local-in-time Fock space approximation of the exact dynamics of the quasi-free state proved in [GM17] to a global-in-time approximation. The key ingredient in establishing the Fock space approximation is our quantitative result on the uniform in $N$ global wellposedness of the 3D time-dependent Hartree-Fock-Bogoliubov (TDHFB) equations.

1. INTRODUCTION

We consider a system of $N$ interacting spinless Bosons in three dimensional space whose evolution is governed by the $N$-body linear Schrödinger equation

$$
\left( \frac{1}{i} \frac{\partial}{\partial t} - \sum_{j=1}^{N} \Delta x_{j} + \frac{1}{N} \sum_{i>j} v_{N}(x_{i} - x_{j}) \right) \Psi_{N}(t, x_{1}, \ldots, x_{N}) = 0
$$

where $x_{i} \in \mathbb{R}^{3}$ and $v_{N}(x) = N^{3\beta}v(N^{\beta}x)$. In particular, we work exclusively in the setting of repulsive interaction.

One of the main interests of this paper is to study the effective dynamics describing the evolution of the above many-body system and provide a quantitative method for tracking the evolution of the many-body quantum system in state space. Unfortunately, the problem of tracking the exact dynamics of Bosonic systems in state space with arbitrary initial condition, at least to the author’s knowledge, is still not tractable with the current available tools. Nevertheless, if we restrict ourselves to a class of initial conditions, which we will make precise later, we are able to obtain some positive results in the direction of understanding the exact evolution in state space via studying some effective dynamics of the system.

Another interesting question that one could study is the range of $\beta$ associated to the interaction potential $v_{N}$. In three dimensional space, the analysis of the effective dynamics becomes more difficult as $\beta$ approaches 1. Physically, for $0 < \beta < \frac{1}{3}$, we are in the regime of weakly interacting dense gas. But once $\frac{1}{3} \leq \beta \leq 1$, we enter the self-interacting regime or sometimes called the strongly-interacting diluted gas regime. Let us remark that in three dimensional space the case $\beta = 1$ is the endpoint case in the heuristic
energy analysis. More precisely, when $\beta = 1$ both kinetic and interaction potential energy scale in a similar manner in $N$ which suggests a more refined analysis is needed since heuristically we cannot treat the interactions as a mere perturbation of the non-interaction case, or, at the very least, not a small perturbation.

In recent years, many have contributed to the studies of effective dynamics for many particle systems. In the case of $\beta = 0$ with repulsive Coulomb interactions, Erdős and Yau in [EY01] prove the qualitative result, via the method of BBGKY hierarchy, that the one-particle marginal density $\gamma_{N,t}^{(1)}$ associated to the wave function $\Psi_{N,t}$ with asymptotically factorized initial state, i.e. $\Psi_{N,0} \rightarrow \phi_{\otimes N}$ as $N \rightarrow \infty$, converges to $|\phi_t\rangle\langle \phi_t|$ in trace norm in the mean-field limit of $N \rightarrow \infty$ where $\phi_t$ satisfies the Hartree equation
\[ \frac{1}{i} \frac{\partial}{\partial t} \phi(t, x) - \Delta_x \phi(t, x) + \left( \frac{1}{|\cdot|} * |\phi_t|^2 \right) \phi(t, x) = 0. \tag{1} \]

Using the Fock space method introduced by Hepp in [Hep74] and subsequently extended by Ginibre and Velo in [GV79a, GV79b], Rodnianski and Schlein in [RS09] provide a rate of convergence of the one-particle marginal associated to the many-body quantum system towards the Hartree dynamics in trace norm, that is, \[ \text{Tr} \left| \gamma_{N,t}^{(1)} - |\phi_t\rangle\langle \phi_t| \right| \lesssim \frac{e^{Kt}}{\sqrt{N}}. \]

The estimate was later improved to $e^{Kt}N^{-1}$ in [ES09, CLST1]. Using a second-order correction Fock space method introduced by Grillakis, Machedon and Margetis in [GMM10, GMM11], Kuz in [Kuz15b] provides a rate of convergence of the many-body quantum system to the Hartree dynamics in the sense of Fock space marginal density\[ 2 \] which in turn establishes the validity of the approximation for time $t$ of the order $\sqrt{N}$. Similar results are derived in [FKS09, KP10] but the approaches are completely different from the above methods.

For the case $0 < \beta \leq 1$, Erdős, Schlein and Yau in a series of papers [ESY06, ESY07, ESY10, ESY09] show qualitatively that the many-body dynamics with factorized initial data converges to the cubic nonlinear Schrödinger dynamics when $0 < \beta < 1$ or the Gross-Pitaevskii dynamics.

---

1. We adopt the standard notation $A \lesssim B$ to mean there exists a constant, depending on some parameters, such that $A \leq CB$.

2. One should note the main result in Rodnianski and Schlein’s paper is their result on the rate of convergence of the one-particle Fock marginal towards the Hartree dynamics. Whereas, the significance of Kuz’s paper is that she was able to show that the mean-field estimate is actually valid for a much longer period of time than most proceeding results had indicated.
when $\beta = 1$. More precisely, they prove that $\gamma_{N,t}^{(1)} \to |\phi_t\rangle\langle \phi_t|$ in trace norm where $\phi_t$ satisfies

$$\frac{1}{i} \frac{\partial}{\partial t} \phi(t, x) - \Delta_x \phi(t, x) = \begin{cases} 
- (\int v) |\phi_t|^2 \phi_t & \text{if } 0 < \beta < 1 \\
-8\pi a |\phi_t|^2 \phi_t & \text{if } \beta = 1
\end{cases}$$

where $a$ is the scattering length corresponding to the potential $v$. Results on the rate of convergence of Fock space marginals can be found in [KP10, BdOS15, Kuz15b].

After identifying the mean-field dynamics, it is natural to study the quantum fluctuation around it. A natural setting to account for the fluctuation is in the Bosonic Fock space

$$\mathcal{F}_s(h) = \mathbb{C} \oplus \bigoplus_{n \geq 1} \text{Sym}(h^\otimes n)$$

where $h = L^2(\mathbb{R}^3)$. Introducing $\mathcal{F}_s$ allows us to deal with states with varying number of particles, which in our model are the excitation and condensate elements. Recent works on evolution of coherent state in Fock space with quantum fluctuation can be found in [RS09, GMM10, GMM11, Che12, GM13a, GM13b, Kuz15b, Kuz15a, RCS17, NN17, Cho16]. Hence, by accounting for some quantum fluctuation, one is able to estimate the evolution of the coherent state in Fock space norm, which in effect allows one to obtain $L^2$-norm approximation of the evolution of many-body quantum system with factorized initial data. We refer the reader to [LSSY05, Gol16, GMM17] for a complete survey of the subject.

2. Background and Main Result

2.1. Background and Earlier Results. In this section, we provide a brief summary of the results obtained in [GM13a, GM17] along with relevant notations and background materials necessary to capture the quantum fluctuation dynamics about quasi-free states.

Let us denote the one-particle base space by $h := L^2(\mathbb{R}^3, dx)$ endowed with the inner product $\langle \cdot, \cdot \rangle_h$ which is linear in the second variable and conjugate linear in the first variable. We define the Bosonic Fock space over $h$ to be the closure of

$$\mathcal{F}_s(h) = \mathcal{F}_s := \mathbb{C} \oplus \bigoplus_{n = 1}^{\infty} \text{Sym}(h^\otimes n)$$

with respect to the norm induced by the Fock inner product

$$\langle \varphi, \psi \rangle_{\mathcal{F}} = \varphi_0 \psi_0 + \sum_{n=1}^{\infty} \langle \varphi_n, \psi_n \rangle_{h^\otimes n}$$

where $\varphi = (\varphi_0, \varphi_1, \ldots), \psi = (\psi_0, \psi_1, \ldots) \in \mathcal{F}_s(h)$. The vacuum, denoted by $\Omega$, is defined to be the Fock vector $(1, 0, 0, \ldots) \in \mathcal{F}_s$.

---

3 c.f. Ch 10 in [Sol13].
For every field \( \phi \in \mathfrak{h} \) we can define the associated creation and annihilation operators on \( F_s \), denote respectively by \( a^\dagger(\phi) \) and \( a(\bar{\phi}) \), as follow:

\[
(a^\dagger(\phi)\psi)_n(x_1, \ldots, x_n) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi(x_j)\psi_{n-1}(x_1, \ldots, \hat{x}_j, \ldots, x_n)
\]

\[
(a(\bar{\phi})\psi)_n(x_1, \ldots, x_n) := \sqrt{n+1} \int dx \bar{\phi}(x)\psi_{n+1}(x, x_1, \ldots, x_n)
\]

on sector with the property that \( a(\phi)\Omega = 0 \). In particular, we could define the corresponding creation and annihilation distribution-valued operators denote by \( a^\dagger_x \) and \( a_x \) as follow:

\[
(a^\dagger_x\psi)_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(x - x_j)\psi_{n-1}(x_1, \ldots, \hat{x}_j, \ldots, x_n)
\]

\[
(a_x\psi)_n := \sqrt{n+1} \psi_{n+1}(x, x_1, \ldots, x_n)
\]

Hence, we have the relations

\[
a^\dagger(\phi) = \int dx \{\phi(x)a^\dagger_x\} \quad \text{and} \quad a(\bar{\phi}) = \int dx \{\bar{\phi}(x)a_x\}.
\]

Let us note that creation and annihilation operators \( a(\bar{\phi}) \) and \( a^\dagger(\phi) \) associated to the field \( \phi \) are unbounded, densely defined, closed operators. Moreover, one could formally verify that pair \( (a^\dagger_x, a_x) \) satisfies the canonical commutation relation (CCR): \([a_x, a^\dagger_y] = \delta(x - y), [a_x, a_y] = [a^\dagger_x, a^\dagger_y] = 0\), and the number operator defined by

\[
N := \int dx \ a^\dagger_x a_x
\]

is a diagonal operator on \( F_s \) that counts the number of particles in each sector; for instance, \((N\psi)_n = n\psi_n\).

For each \( \phi \in \mathfrak{h} \), we associate a skew-Hermitian operator operator

\[
\mathcal{A}(\phi_t) = \mathcal{A}(t) := a(\bar{\phi}) - a^\dagger(\phi)
\]

and define the Weyl operator to be the corresponding unitary \( e^{-\sqrt{N}\mathcal{A}(\phi)} \).

Then the coherent state associated to \( \phi \) is given by

\[
\psi(\phi) := e^{-\sqrt{N}\mathcal{A}(\phi)}\Omega.
\]

Using the Baker-Campbell-Hausdorff formula, one can show

\[
\psi(\phi) = (\ldots, c_n\phi^\otimes n, \ldots) \quad \text{where} \quad c_n = \left(e^{-N\|\phi\|_2^2}N^n/n!\right).
\]

In this context, \( \phi \) is called the condensate wave-function and the Fock space marginal associated to the scheme \( e^{-\sqrt{N}\mathcal{A}(\phi_t)}\Omega \) where \( \phi_t \) satisfies the cubic NLS offers a first-order (mean-field) approximation to exact evolution of the coherent state in trace norm; see [RS09]. However, to track the exact
dynamics in Fock space, we need to introduce the pair excitation function 
\( k(x, y) = k(y, x) \) and its corresponding quadratic operator \( B(k) \) with kernel

\[
B(k_t)(x, y) = \int dx dy \left\{ \bar{k}(t, x, y)a_x a_y - k(t, x, y)a_x^\dagger a_y^\dagger \right\}.
\]

From the pair excitation we concoct a new approximation scheme, which is a second-order correction \(^4\) to the mean-field approximation, given by

\[
\psi_{\text{approx}} = e^{iNt}\chi(t)e^{-\sqrt{NA}(t)}e^{-B(t)}\Omega
\]

where \( \chi(t) \) is some phase factor to be determined. With some appropriate choice of evolution equations for \( \phi \) and \( k \) we will later see that \( 2 \) will indeed allow use to track the exact dynamics of the evolution of quasi-free state in Fock space.

For a fixed \( N \in \mathbb{N} \), we are interested in the time-evolution generated by Fock Hamiltonian operator associated to \( N \), which is a diagonal operator, denoted by \( H_N \), on the Fock space defined by

\[
(H_N\psi)_n = \left( \sum_{j=1}^n \Delta_{x_j} - \frac{1}{N} \sum_{i<j}^n v_N(x_i - x_j) \right) \psi_n =: H_{N,n}\psi_n
\]

where \( v_N(x) = N^{3\beta}v(N^\beta x) \). Rewrite \( H_N \) using creation and annihilation operators we get

\[
H_N := \int dx dy \left\{ \Delta_{x}\delta(x - y)a_x^\dagger a_y - \frac{1}{2N} \int dx dy \left\{ v_N(x - y)a_x^\dagger a_y^\dagger a_x a_y \right\} \right\}.
\]

In light of the Fock Hamiltonian, we are interested in the solution to the following Cauchy problem in Fock space

\[
\frac{1}{i} \frac{\partial}{\partial t} \psi = H_N\psi \quad \text{with initial datum} \quad \psi_0 = e^{-\sqrt{NA}(\phi_0)}e^{-B(k_0)}\Omega
\]

which we shall formally write as

\[
\psi_{\text{exact}} = e^{itH_N}e^{-\sqrt{NA}(\phi_0)}e^{-B(k_0)}\Omega.
\]

Let \( \mathcal{M} = e^{-\sqrt{NA}}e^{-B} \). Following \([\text{GM13a}, \text{GM17}]\), we work with the reduced dynamic. More specifically, since \( \mathcal{M} \) is unitary then it follows

\[
\left\| \psi_{\text{exact}}(t) - e^{iN\int_0^t \chi_0(s) \, ds}\psi_{\text{approx}}(t) \right\|_F = \left\| e^{-iN\int_0^t \chi_0(s) \, ds}\psi_{\text{red}}(t) - \Omega \right\|_F
\]

where

\[
\psi_{\text{red}} = e^{B(t)}e^{\sqrt{NA}(t)}e^{i(H_N)e^{-\sqrt{NA}(A_0)}e^{-B_0}\Omega}.
\]

\(^4\)In the mathematical physics literature, \( e^{\mathcal{A}} \) is called the infinite dimensional Segal-Shale-Weil representation of the double cover of the group of symplectic matrices of integral operators. The elements of the corresponding \( C^* \)-algebra are called Bogoliubov transformations (cf. chapter 4 of \([\text{Fol89}]\) and chapter 11 of \([\text{DG13}]\) ).
Thus, to estimate the Fock space error, we need to be able to control
\[ \frac{1}{i} \frac{\partial}{\partial t} \psi_{\text{red}} = H_{\text{red}} \psi_{\text{red}} \] where \( H_{\text{red}} = \frac{1}{i}(\partial_t M^*) M + M^* H M \)
we see that
\[ \left( \frac{1}{i} \frac{\partial}{\partial t} - H_{\text{red}} + X_0 \right) \left( e^{-iN \int_0^t \gamma_0(s) \, ds} \psi_{\text{red}} - \Omega \right) = H_{\text{red}} \Omega - X_0 \Omega \]
with
\[ H_{\text{red}} \Omega = (X_0, X_1, X_2, X_3, X_4, 0, 0, \ldots). \]

Thus, to estimate the Fock space error, we need to be able to control \( H_{\text{red}} \Omega \). A direct calculation reveals that \( X_3 \) and \( X_4 \) are heuristically small since they are proportional to \( N^{-1/2} \) and \( N^{-1} \), respectively. On the other hand, \( X_1 \) and \( X_2 \) are proportional to \( N^{1/2} \) and constant respectively. Hence, \( X_1 = X_2 = 0 \) are natural conditions to impose on \( \phi_t \) and \( k_t \).

Following [GM17], we define the monomial \( P_{n,m} := a_{x_1} \cdots a_{x_n} y_1 a_{y_m} \) and consider the \( L \)-matrices whose kernels are defined by
\[ L_{n,m}(t, x_1, \ldots, x_n; y_1, \ldots, y_m) = \frac{1}{N(a+m)2} \langle M \Omega, P_{n,m} M \Omega \rangle. \]

In particular, let us focus on the matrices \( L_{0,1}, L_{1,1} \) and \( L_{0,2} \), which we will denote by \( \phi, \Gamma \) and \( \Lambda \) respectively. It is shown in [GM17] that the conditions \( X_1 = X_2 = 0 \) is equivalent to the fact that \( (\phi, \Gamma, \Lambda) \) forms a closed system of coupled nonlinear equations
\[
\begin{align*}
\left\{ \frac{1}{i} \frac{\partial}{\partial t} - \Delta_{x_1} \right\} \phi(x_1) &= -\int dy \left\{ v_N(x_1 - y) \rho_T(t, y) \right\} \phi(x_1) \quad (4a) \\
&\quad - \int dy \left\{ v_N(x_1 - y) \Gamma(y, x_1) - \phi(y) \phi(x_1) \phi(y) \right\} \\
&\quad - \int dy \left\{ v_N(x_1 - y) \Lambda(x_1, y) - \varphi(y) \varphi(x_1) \phi(y) \right\} \\
&\quad \left\{ \frac{1}{i} \frac{\partial}{\partial t} - \Delta_{x_1} + \Delta_{x_2} \right\} \Gamma(x_1, x_2) = -\int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) \Lambda(x_1, y) \Lambda(x_1, y) \right\} \\
&\quad - \int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) \Gamma(x_1, y) \Gamma(y, x_2) \right\} \\
&\quad - \int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) \rho_T(t, y) \Gamma(x_1, x_2) \right\} \\
&\quad + 2 \int dy \left\{ (v_N(x_1 - y) - v_N(x_2 - y)) |\phi(y)|^2 \phi(x_1) \phi(x_2) \right\}
\end{align*}
\]
Theorem 2.1. Let $\frac{1}{3} \leq \beta < \frac{2}{3}$ and $v \in S$ a nonnegative interaction potential satisfying the condition that $|\tilde{v}| \leq \tilde{w}$ for some $w \in S$. Suppose $(\phi_t, \Gamma_t, \Lambda_t)$ are solutions to the TDHFB equations with some smooth initial conditions $(\phi_0, \Gamma_0, \Lambda_0)$ satisfying the following regularity condition uniformly in $N$: for some $\varepsilon > 0$ and $0 \leq i \leq 1, 0 \leq j \leq 2$

\[
\left\| \frac{1}{N} \left( v_N(x_1 - y) + v_N(x_2 - y) \right) \rho_t(t, y) \Lambda(x_1, x_2) \right\|_{L^2(dx)} \lesssim 1
\]

\[
\left\| \left( v_N(x_1 - y) + v_N(x_2 - y) \right) \rho_t(t, y) \Lambda(x_1, x_2) \right\|_{L^2(dx)} \lesssim 1
\]

\[
\left\| \left( v_N(x_1 - y) + v_N(x_2 - y) \right) \rho_t(t, y) \Lambda(x_1, x_2) \right\|_{L^2(dx)} \lesssim 1
\]

\[
\left\| \partial_t \left( v_N(x_1 - y) + v_N(x_2 - y) \right) \rho_t(t, y) \Lambda(x_1, x_2) \right\|_{L^2(dx)} \lesssim 1
\]

\[
\left\| \left( v_N(x_1 - y) + v_N(x_2 - y) \right) \rho_t(t, y) \Lambda(x_1, x_2) \right\|_{L^2(dx)} \lesssim 1
\]

where $\rho_t(t, x) = \Gamma(t, x, x)$. Note, we have suppressed the time dependence to compactify the notation. We will refer to these equations as the time-dependent Hartree-Fock-Bogoliubov (TDHFB) equations.

Independently and in a different frame work, Bach, Breteaux, Chen, Fröhlich, and Sigal derived equations closely related to the above equations in [BBCFS]. In particular, the two sets of equations are equivalent in the case of pure states.

By direct calculation, it is shown in [GM17] that

\[
\Gamma(t, x, y) = \tilde{\phi}(t, x)\phi(t, y) + \frac{1}{N} \left( \text{sh}(k) \circ \text{sh}(k) \right) (t, x, y)
\]

\[
\Lambda(t, x, y) = \phi(t, x)\phi(t, y) + \frac{1}{2N} \text{sh}(2k)(t, x, y)
\]

where we have

\[
\text{sh}(k) := k + \frac{1}{3!} k \circ \bar{k} \circ k + \ldots \quad \text{and} \quad \text{ch}(k) := \delta + \frac{1}{2!} \bar{k} \circ k + \ldots
\]

The local wellposedness of (4) were established in [GM17] using techniques from dispersive PDEs. Consequently, the authors were able to obtain Fock space estimate for small time. The following theorem summarizes the results of [GM17].
Then there exists constants $\delta = \delta(\varepsilon), \kappa = \kappa(\varepsilon), C = C(\varepsilon, \beta)$, a phase function $\chi(t)$, depending on $N$, and $T_0$ ($T_0 \sim 1$) independent of $N$ such that we have the Fock space estimate

$$
\left\| e^{it\mathcal{H}} e^{-\sqrt{N}A(\phi_0)} e^{-B(k_0)} \Omega - e^{it\chi(t)} e^{-\sqrt{N}(\phi_t)} e^{-B(k_t)} \Omega \right\|_\mathcal{F} \leq \frac{C}{N^{1/6}}
$$

for all $0 \leq t \leq T_0$.

The main goal of this paper is to extend Theorem 2.1 to obtain a global in time result. Let us state the main result of this paper

**Theorem 2.2.** Let $\frac{1}{3} \leq \beta < \frac{2}{3}$ and $v \in \mathcal{S}$ a nonnegative interaction potential satisfying the condition that $|\vec{v}| \leq \hat{w}$ for some $w \in \mathcal{S}$. Suppose $(\phi_t, \Gamma_t, \Lambda_t)$ are solutions to the TDHFB equations with some smooth initial conditions $(\phi_0, \Gamma_0, \Lambda_0)$ satisfying the following regularity condition uniformly in $N$: for some $\varepsilon > 0$ and $0 \leq i \leq 1, 0 \leq j \leq 2$

$$
\left\| (\nabla_x)^{1/2+\varepsilon} \partial_t^i \nabla_x^j \phi(t, \cdot) \right\|_{L^2(dx)} \lesssim 1,
\left\| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} \partial_t^i \nabla_x^j \phi(t, \cdot) \right\|_{L^2(dx, dy)} \lesssim 1,
\left\| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} \partial_t^i \nabla_x^j \Lambda(t, \cdot) \right\|_{L^2(dx, dy)} \lesssim 1.
\left\| \nabla_t^j \partial_t \phi(0, x, y) \right\|_{L^2(dx, dy)} \lesssim 1.
$$

Then there exists constants $\delta = \delta(\varepsilon), \kappa = \kappa(\varepsilon), C = C(\varepsilon, \beta)$ and a phase function $\chi(t)$, depending on $N$, such that we have the Fock space estimate

$$
\left\| e^{it\mathcal{H}} e^{-\sqrt{N}A(\phi_0)} e^{-B(k_0)} \Omega - e^{it\chi(t)} e^{-\sqrt{N}(\phi_t)} e^{-B(k_t)} \Omega \right\|_\mathcal{F} \leq \frac{C \exp\left(\kappa T^{1+\frac{1}{\beta}}\right)}{N^{1/6}}
$$

for all $0 \leq t \leq T$.

As remarked in §2 of [GM17], we need to first prove the following apriori estimates

$$
\left\| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} \Lambda(t, \cdot) \right\|_{L^2(dx, dy)} \lesssim C(t),
\left\| \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \Gamma(t, \cdot) \right\|_{L^2(dx, dy)} \lesssim 1,
\left\| \nabla_x^{1/2+\varepsilon} \phi(t, \cdot) \right\|_{L^2(dx)} \lesssim 1,
$$

and use them to obtain appropriate norm bounds on the solutions of the TDHFB equations; see Propositions 4.3, 4.6, 4.9, 5.1. Afterward, by replicating the proof of Theorem 2.1 in §9 and 10 of [GM17], one can obtain the desired Fock space estimate.
3. Global Estimates for the TDHFB Equations

In this section we prove, for a sufficiently small \( \varepsilon > 0 \), the following estimates:

\[
\| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \Lambda(t, \cdot) \|_{L^2(dx) L^2(dy)} \leq C(t) \tag{5}
\]

\[
\| \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \Gamma(t, \cdot) \|_{L^2(dx) L^2(dy)} \leq C \tag{6}
\]

\[
\| \nabla_x^{1/2+\varepsilon} \phi(t, \cdot) \|_{L^2(dx)} \leq C \tag{7}
\]

hold uniformly in \( N \) for any fixed time \( t \). The proof of estimates (5)-(7) relies on the conservation laws established in [GM13a]. Therefore, for the convenience of the reader, we shall restate the conservation laws for the TDHFB equations in the following proposition. Before stating the proposition, let us recall the total particle number and energy, denoted by \( N \) and \( E \) respectively, can be evaluated explicitly:

\[
N = N \left\{ \int dx \; |\phi(x)|^2 + \frac{1}{N} \int dxdy \; |sh(k)(x,y)|^2 \right\} \tag{8}
\]

and

\[
E = N \left\{ \int dx \; |\nabla \phi(x)|^2 + \frac{1}{2N} \int dxdy \; |\nabla_{x,y} sh(k)(x,y)|^2 \right. \\
\left. + \frac{1}{2N} \int dxdydz \; v_N(x-y)|\phi(x) sh(k)(y,z) + \phi(y) sh(k)(x,z)|^2 \\
+ \frac{1}{4} \int dxdy \; v_N(x-y) \left\{ 2|\Lambda(x,y)|^2 + |\Gamma(x,y)|^2 + \Gamma(x,x)\Gamma(y,y) \right\} \right. \tag{9}
\]

**Proposition 3.1** (Conservation Laws). Suppose \((\phi_t, \Gamma_t, \Lambda_t)\) solves the time-dependent Hartree-Fock-Bogoliubov equations with nonnegative \( v \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R}) \). Then the total particle number and energy is conserved.

The reader should be aware of the fact that we are assuming, for \( N \) fixed, \( N \) and \( E \) are proportional to \( N \), i.e. \( N = N \) and \( E \sim N \) or, equivalently, \( N^{-1}E \sim 1 \). In other words, we are assuming the energy per particle is constant and independent of \( N \).

Now, as an immediate corollary of the conservation laws, we prove estimate (6) and (7).

**Corollary 3.2.** Let \( \phi_t \) and \( \Gamma_t \) be solutions to the TDHFB equations. Then, for any \( |\varepsilon| \leq 1/2 \), we have the estimates

\[
\| \nabla_x^{1/2+\varepsilon} \nabla_x^{1/2+\varepsilon} \Gamma(t, \cdot) \|_{L^2(dx) L^2(dy)} \lesssim 1
\]

\[
\| \nabla_x^{1/2+\varepsilon} \phi(t, \cdot) \|_{L^2(dx)} \lesssim 1
\]

which hold uniformly in \( N \) and independent of \( t \).
Next, taking derivatives of the operator identity

\[ N \] 

\[ \text{Hence it remains to show} \]

\[ t \] 

Using Plancherel identity, we establish the estimate

\[ \text{Proof.} \]

\[ \text{Lemma 3.3.} \]

and

\[ \text{(13) and (14) we obtain the estimate} \]

\[ \text{Hence interpolating (10) and (11) yields the desired result.} \]

For the remainder of this section, we shall prove estimate (5) holds and show that \( C(t) \) is a sublinear function. To this end, let us begin by making the observation that to prove estimate (5) is equivalent to showing

\[ N^{-1} \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \sh(2k_t) \|_{L^2(dx,dy)} \lesssim C(t) \]

holds for some sufficiently small \( \varepsilon > 0 \). Furthermore, to aid us in proving estimate (12), we use the operator identity

\[ \sh(2k) = 2 \sh(k) \circ \ch(k) = 2 \sh(k) + 2 \sh(k) \circ p \]

and the triangle inequality to obtain a preliminary estimate

\[ \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \sh(2k_t) \|_{L^2(dx,dy)} \]

\[ \lesssim \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \sh(k_t) \|_{L^2} + \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \sh(k_t) \circ p_t \|_{L^2} \]

\[ =: I_1(t) + I_2(t). \]

Hence it remains to show \( N^{-1} I_i(t) \lesssim C(t) \) for \( i = 1, 2 \).

To estimate \( I_2(t) \) we use the following lemma

**Lemma 3.3.** We have the following estimates

\[ N^{-1} \| \nabla_x^{1/2} \sh(k_t) \circ \nabla_y^{1/2} p_t \|_{L^2(dx,dy)} \lesssim 1 \]

and

\[ N^{-1} \| \nabla_x \sh(k_t) \circ \nabla_y p_t \|_{L^2(dx,dy)} \lesssim 1 \]

which are independent of time \( t \). In particular, by interpolating estimates (13) and (14) we obtain the estimate

\[ N^{-1} \| \nabla_x^{1/2+\varepsilon} \sh(k_t) \circ \nabla_y^{1/2+\varepsilon} p_t \|_{L^2(dx,dy)} \lesssim 1 \]

for all \( 0 \leq \varepsilon \leq 1/2 \).

**Proof.** Using Plancherel identity, we establish the estimate

\[ N^{-1} \| \nabla_x^{1/2} \sh(k_t) \circ \nabla_y^{1/2} p_t \|_{L^2(dx,dy)} \]

\[ \lesssim N^{-1} \| \nabla_x \sh(k_t) \circ p_t \|_{L^2(dx,dy)} + N^{-1} \| \sh(k) \circ \nabla_y p_t \|_{L^2(dx,dy)} \]

\[ \lesssim N^{-1} \| \nabla_x \sh(k_t) \|_{L^2} \| p_t \|_{L^2} + N^{-1} \| \sh(k_t) \|_{L^2} \| \nabla_y p_t \|_{L^2}. \]

Next, taking derivatives of the operator identity

\[ \sh(k) \circ \sh(k) = p \circ p + 2p \]
DYNAMICAL HARTREE-FOCK-BOGOLIUBOV APPROXIMATION

gives us the identity
\[ \nabla_x \text{sh}(k) \circ \nabla_y \text{sh}(k) = \nabla_x p \circ \nabla_y p + 2 \nabla_x \nabla_y p. \]

In particular, we have that
\[ \| \nabla_x \text{sh}(k) \|_{L^2(dxdy)}^2 = \| \nabla_x p \circ \nabla_y p + 2 \nabla_x \nabla_y p \|_{\text{tr}} \geq \| \nabla_x p \|_{L^2(dxdy)}^2 \]

since both \( \nabla_x \nabla_y (p \circ p + 2p) \) and \( 2 \nabla_x \nabla_y p \) are positive trace class operators. Hence combining estimate (16) with the conservation laws, we obtain the estimate
\[ N^{-1} \| \nabla_x^{1/2} \text{sh}(k_t) \circ \nabla_y^{1/2} p_t \|_{L^2} \lesssim N^{-1} \| \nabla_x \text{sh}(k_t) \|_{L^2} \| \text{sh}(k_t) \|_{L^2} \lesssim 1. \]

Likewise, we have shown
\[ N^{-1} \| \nabla_x \text{sh}(k_t) \circ \nabla_y p_t \|_{L^2} \lesssim N^{-1} \| \nabla_x \text{sh}(k_t) \|_{L^2} \lesssim 1. \]

To estimate \( I_1(t) \) we will need to prove a couple preliminary lemmas.

**Lemma 3.4.** We have the following estimates
\[ \| \nabla_{x,y} \Lambda(t, \cdot) \|_{L^2(dxdy)} \lesssim 1 \]
(17)
and
\[ \| \nabla_{x,y} \Gamma(t, \cdot) \|_{L^2(dxdy)} \lesssim 1 \]
(18)
which holds uniformly in \( N \) and independent of time \( t \).

**Proof.** This is an immediate corollary of Lemma 3.3. \( \square \)

**Lemma 3.5.** Let \( \Lambda_t \) be a solution to the TDHFB equations. Then we have the following energy estimate
\[ \| \nabla_x \nabla_y \Lambda(t, \cdot) \|_{L^2(dxdy)} \lesssim \| \nabla_x \nabla_y \Lambda_0 \|_{L^2(dxdy)} + N^{\text{some power}} t. \]
(19)

**Proof.** For convenience, let us restate the equation for \( \Lambda_t \) which is
\[ (S + V) \Lambda = -(v_N \Lambda) \circ \Gamma - \widetilde{\Gamma} \circ (v_N \Lambda) \]
\[ - (v_N \widetilde{\Gamma}) \circ \Lambda - \Lambda \circ (v_N \Gamma) \]
\[ + 2(v_N \ast |\phi|^2)(x)\phi(x)\phi(y) + 2(v_N \ast |\phi|^2)(y)\phi(y)\phi(x) =: F \]
(20)
where \( v_N \Lambda = v_N(x - y)\Lambda(x, y) \) and
\[ V = \frac{1}{N} v_N + (v_N \ast \text{Tr} \Gamma)(x) + (v_N \ast \text{Tr} \Gamma)(y). \]

Differentiating equation (20) by \( \nabla_x \nabla_y \) gives us the following equation
\[ (S + V)(\nabla_x \nabla_y \Lambda) = [S + V, \nabla_x \nabla_y] \Lambda + \nabla_x \nabla_y F \]
and
\[ [S + V, \nabla_x \nabla_y] = N^{-1}(\nabla_x \nabla_y v_N)\Lambda + N^{-1}\nabla_y v_N \nabla_x \Lambda + N^{-1}\nabla_x v_N \nabla_y \Lambda \]
\[ + [(\nabla_y v_N) \ast \text{Tr} \Gamma(y)] \nabla_x \Lambda + [(\nabla_x v_N) \ast \text{Tr} \Gamma(x)] \nabla_y \Lambda. \]
Using the energy method, we obtain the estimate
\[
\frac{d}{dt} \left\| \nabla_x \nabla_y \Lambda_t \right\|_{L^2}^2 = 2 \, \text{Re} \left( \partial_t \nabla_x \nabla_y \Lambda_t, \nabla_x \nabla_y \Lambda_t \right)
\]
\[
= 2 \, \text{Re} \left( (S + V)(\nabla_x \nabla_y \Lambda_t), \nabla_x \nabla_y \Lambda_t \right)
\]
\[
\leq 2 \left\| [S + V, \nabla_x \nabla y] \Lambda_t + \nabla_x \nabla_y F \right\|_{L^2} \left\| \nabla_x \nabla_y \Lambda_t \right\|_{L^2}
\]
which then leads to the energy estimate
\[
\left\| \nabla_x \nabla_y \Lambda(t, \cdot) \right\|_{L^2(dx dy)} \leq \left\| \nabla_x \nabla_y \Lambda_0 \right\|_{L^2(dx dy)} + \int_0^t ds \left\| [S + V, \nabla_x \nabla y] \Lambda(s, \cdot) + \nabla_x \nabla_y F(s) \right\|_{L^2(dx dy)}.
\]
We are now ready to control the forcing terms. First, for the commutator term we have the estimate
\[
\left\| [S + V, \nabla_x \nabla y] \Lambda(t, \cdot) \right\|_{L^2}
\]
\[
\leq N^{-1} \left\| (\nabla_x \nabla_y v_N) \Lambda(t, \cdot) \right\|_{L^2} + 2N^{-1} \left\| \nabla_y v_N \nabla_x \Lambda(t, \cdot) \right\|_{L^2}
\]
\[
+ 2 \left\| \left[ (\nabla_x v_N) + \text{Tr} \Gamma(y) \right] \nabla_x \Lambda(t, \cdot) \right\|_{L^2}
\]
\[
\lesssim N^{5\beta-1} \left\| \Lambda(t, \cdot) \right\|_{L^2} + N^{4\beta-1} \left\| \nabla_x \Lambda(t, \cdot) \right\|_{L^2}
\]
\[
+ N^{4\beta-1} \left\| \Gamma(t, x, x) \right\|_{L^1(dx)} \left\| \nabla_x \Lambda(t, \cdot) \right\|_{L^2} \lesssim N^{5\beta-1}
\]
The other forcing term in estimate (21) can be handled similarly. We shall estimate only one of the terms since the proof is exactly the same for the other terms. Observe, for the \((v_N \Lambda \circ \Gamma)\) we have
\[
\left\| \nabla_x \nabla_y (v_N \Lambda \circ \Gamma) \right\|_{L^2(dx dy)}
\]
\[
\leq \left\| \nabla_x (v_N \Lambda(t, \cdot)) \right\|_{L^2(dx dy)} \left\| \nabla_y \Gamma(t, \cdot) \right\|_{L^2(dx dy)} \lesssim N^{4\beta}.
\]
Hence combining all the estimates yields the desired estimate. \(\square\)

Lemma 3.6. There exists \(\varepsilon > 0\) such that
\[
N^{-1} \left\| \nabla_x^{1/2 + \varepsilon} \nabla_y^{1/2 + \varepsilon} \text{sh}(k_t) \right\|_{L^2(dx dy)} \lesssim C(t)
\]
where \(C(t)\) is a sublinear function.

Proof. Applying Lemma 3.5 gives us the estimate
\[
N^{-1} \left\| \nabla_x \nabla_y \text{sh}(k_t) \right\|_{L^2(dx dy)}
\]
\[
\lesssim N^{-1} \left\| \nabla_x \nabla_y \text{sh}(2k_t) \right\|_{L^2(dx dy)} + N^{-1} \left\| \nabla_x \text{sh}(k_t) \circ \nabla_y p_t \right\|_{L^2(dx dy)}
\]
\[
\lesssim \left\| \nabla \phi_t \right\|_{L^2} + \left\| \nabla_x \nabla_y \Lambda(t, \cdot) \right\|_{L^2} + N^{-1} \left\| \nabla_x \text{sh}(k_t) \circ \nabla_y p_t \right\|_{L^2}
\]
\[
\lesssim 1 + N^{\text{some power} \, t}.
\]
Interpolating the above estimate with the estimate
\[
\frac{1}{N} \left\| \nabla_x^{1/2} \nabla_y^{1/2} \text{sh}(k_t) \right\|_{L^2(dx dy)} \lesssim \frac{1}{N} \left\| \nabla_x \text{sh}(k_t) \right\|_{L^2(dx dy)} \lesssim \frac{1}{\sqrt{N}}
\]
we have shown that there exists \(\varepsilon > 0\) such that
\[
N^{-1} \left\| \nabla_x^{1/2 + \varepsilon} \nabla_y^{1/2 + \varepsilon} \text{sh}(k_t) \right\|_{L^2} \lesssim C(t)
\]
where \( C(t) \) is a sub-linear function.

\[ \text{Remark 3.7.} \] The indefinite usage of exponent in the statement of Lemma 3.5 is to de-emphasize the role that the exponent plays in the paper. Nevertheless, for the interested reader, it is not hard to see that the power should be \( 4\beta \) and \( C(t) \sim t^{\frac{1}{1+8\beta}} \) for \( t \gg 1 \), where \( 0 < \beta < \frac{2}{3} \). Moreover, we also have
\[
\varepsilon = \frac{1}{2(1+8\beta)}.
\]

4. Global Wellposedness of the TDHFB Equations

Let us define the norms which we shall use to prove the uniform in \( N \) global wellposedness of the TDHFB equations as in [GM17] for \( 0 < \beta < \frac{2}{3} \). Fix \( \varepsilon \) as in Remark 3.7 and define
\[
N_{[T_0,T_1]}(\Lambda) := \sup_{z} \| (\nabla_x)^{1/2+\varepsilon} \Lambda(t, x + z, x) \|_{L^2([T_0,T_1] \times \mathbb{R}^3)}
\]
\[
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} \Lambda(t, x, y) \|_{L^\infty([T_0,T_1] \times L^2(\mathbb{R}^6))}
\]
\[
N_{[T_0,T_1]}(\Gamma) := \sup_{z} \| |(\nabla_x)^{1/2+\varepsilon} \Gamma(t, x + z, x) \|_{L^2([T_0,T_1] \times \mathbb{R}^3)}
\]
\[
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} \Gamma(t, x, y) \|_{L^\infty([T_0,T_1] \times L^2(\mathbb{R}^6))}
\]
\[
N_{[T_0,T_1]}(\phi) := \| (\nabla_x)^{1/2+\varepsilon} \phi \|_{L^\infty([T_0,T_1] \times L^2_2)} + \| (\nabla_x)^{1/2+\varepsilon} \phi \|_{L^2([T_0,T_1] \times L^2_2)}
\]

For convenience we shall denote the sum of the three norms by
\[
N_{[T_0,T_1]}(X) := N_{[T_0,T_1]}(\phi) + N_{[T_0,T_1]}(\Gamma) + N_{[T_0,T_1]}(\Lambda).
\]

and if \([T_0, T_1] = [0, T]\) then we denote \( N_{[0,T]}(X) := N_T(X) \) (similarly for the other norms). Moreover, we shall adopt the notation
\[
N_{[T_0,T_1]}(DX) := N_{[T_0,T_1]}(D\phi) + N_{[T_0,T_1]}(D\Gamma) + N_{[T_0,T_1]}(D\Lambda)
\]
where \( D \) is some differential operator.

The goal of this section is to prove the global wellposedness of solutions for the TDHFB equations. However, it suffices to prove the an apriori estimate of the form
\[
N_T(X) \lesssim F(T)
\]
for some positive real-valued function \( F \) defined on all of \([0, \infty)\). We shall begin by proving a couple lemmas to aid us in establishing the above apriori estimate.

**Lemma 4.1.** Let \((\phi_t, \Gamma_t, \Lambda_t)\) be solutions to the TDHFB equations. Then there exists \( \delta > 0 \) such that we have the following estimate
\[
N_{[T_0,T_1]}(X) \lesssim C(T_0) + (T_1 - T_0)^\delta \ C(T_1) \ N_{[T_0,T_1]}(X)
\]
\[ C(T) := \| \langle \nabla_x \rangle^{1/2+\varepsilon} \phi(T, \cdot) \|_{L^2(dx)} + \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \Gamma(T, \cdot) \|_{L^2(dx dy)} + \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \Lambda(T, \cdot) \|_{L^2(dx dy)}. \]

**Proof.** Recall the equation for \( \phi \) is given by

\[ S \phi = - (v_N \Lambda) \circ \phi - (v_N \Gamma) \circ \phi - (v_N * \text{Tr} \Gamma) \cdot \phi + 2(v_N * |\phi|^2) \phi =: F \]

Then by Proposition 5.8 in [GM17] we have the estimate

\[ N_{[T_0, T_1]}(\phi) \]

\[ \lesssim \| \langle \nabla_x \rangle^{1/2+\varepsilon} \phi(T_0, \cdot) \|_{L^2} + \| \langle \nabla_x \rangle^{1/2+\varepsilon} F \|_{L^2([T_0, T_1])L^{6/5+}} \]

\[ \lesssim \| \langle \nabla_x \rangle^{1/2+\varepsilon} \phi(T_0, \cdot) \|_{L^2} + (T_1 - T_0)^{\delta} \| \langle \nabla_x \rangle^{1/2+\varepsilon} F \|_{L^2([T_0, T_1])L^{5/5+}}. \]

The symbol 6/5+ denotes a fixed number slightly bigger than 6/5 with dependence on \( \varepsilon \). Observe for the forcing term \( (v_N * \text{Tr} \Gamma) \cdot \phi \) we have the estimate

\[ \| \nabla_x^{1/2+\varepsilon} (v_N * \text{Tr} \Gamma) \cdot \phi \|_{L^2([T_0, T_1])L^{6/5+}} \]

\[ \lesssim \| \nabla_x^{1/2+\varepsilon} \text{Tr} \Gamma \|_{L^2([T_0, T_1])L^2} \| \phi \|_{L^\infty([T_0, T_1])L^{3+}} \]

\[ + \| \text{Tr} \Gamma \|_{L^2([T_0, T_1])L^{3+}} \| \nabla_x^{1/2+\varepsilon} \phi \|_{L^\infty([T_0, T_1])L^2} \]

\[ \lesssim \hat{N}_{[T_0, T_1]}(\Gamma). \]

Likewise, for the forcing term \( (v_N \Lambda) \circ \phi \) we have the estimate

\[ \| \nabla_x^{1/2+\varepsilon} (v_N \Lambda) \circ \phi \|_{L^2([T_0, T_1])L^{6/5+}} \]

\[ \lesssim \int dy \ v_N(y) \| \nabla_x^{1/2+\varepsilon} \Lambda(x, x - y) \|_{L^2([T_0, T_1])L^2} \| \phi \|_{L^\infty([T_0, T_1])L^{3+}} \]

\[ + \int dy \ v_N(y) \| \Lambda(x, x - y) \|_{L^2([T_0, T_1])L^{3+}} \| \nabla_x^{1/2+\varepsilon} \phi \|_{L^\infty([T_0, T_1])L^2} \]

\[ \lesssim N_{[T_0, T_1]}(\Lambda). \]

The remaining nonlinear terms \( (v_N \Gamma) \circ \phi \) and \( (v_N * |\phi|^2) \cdot \phi \) can be handled in similar manners as above. Thus, we have shown

\[ N_{[T_0, T_1]}(\phi) \lesssim C(T_0) + (T_1 - T_0)^{\delta} N_{[T_0, T_1]}(X). \]

Next, recall the equation for \( \Gamma \) is given by

\[ S_{\pm} \Gamma = - (v_N \Lambda) \circ \hat{\Lambda} + \Lambda \circ (v_N \Lambda) - (v_N \hat{\Gamma}) \circ \hat{\Gamma} + \hat{\Gamma} \circ (v_N \hat{\Gamma}) \]

\[ - (v_N * \text{Tr} \Gamma) \cdot \hat{\Gamma} + \hat{\Gamma} \cdot (v_N * \text{Tr} \Gamma) \]

\[ + 2(v_N * |\phi|^2) \cdot \hat{\Gamma} - 2\hat{\Gamma} \cdot (v_N * |\phi|^2) =: G \]
Again, by Proposition 5.8 in [GM17], we have
\[
\tilde{N}_{[T_0, T_1]}(\Gamma) \lesssim \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} \Gamma(T_0, \cdot) \|_{L^2(dx dy)} \\
+ (T_1 - T_0)^\delta \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} G \|_{L^2([T_0, T_1]) L^{5/\delta} L^5_y}.
\]

For the forcing term \((\nu_N \ast \text{Tr} \Gamma) \cdot \Gamma\), we use Young's convolution inequality to obtain the following estimate
\[
\| \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \tilde{\Gamma} \cdot (\nu_N \ast \text{Tr} \Gamma) \|_{L^2([T_0, T_1]) L^{5/\delta} L^5_y} \lesssim \| \nabla_x^{1/2+\varepsilon} \text{Tr} \Gamma \|_{L^2([T_0, T_1]) L^2(dx)} \\
+ \| \nabla_y^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \text{Tr} \Gamma \|_{L^2([T_0, T_1]) L^2(dx dy)} \| \text{Tr} \Gamma \|_{L^2([T_0, T_1]) L^{5/\delta} (dx)}.
\]

Next, for the forcing term \(\Lambda \circ (\nu_N \tilde{\Lambda})\) we apply the \(\Lambda\)-estimate from the previous section to obtain the estimate
\[
\| \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \Lambda \circ (\nu_N \tilde{\Lambda}) \|_{L^2([T_0, T_1]) L^{5/\delta} L^5_y} \lesssim \int dz \| \nabla_x^{1/2+\varepsilon} \tilde{\Lambda}(x, x-z) \nabla_y^{1/2+\varepsilon} \Lambda(x-z, y) \|_{L^2([T_0, T_1]) L^{6/5+\varepsilon} L^5_y} \\
\lesssim \int dz \| \nabla_x^{1/2+\varepsilon} \tilde{\Lambda}(x, x-z) \|_{L^2 L^2} \| \nabla_y^{1/2+\varepsilon} \Lambda \|_{L^\infty([T_0, T_1]) L^{4+\varepsilon} L^5_y} \lesssim C(T_1) N_{[T_0, T_1]}(\Lambda)
\]

Hence it follows
\[
\tilde{N}_{[T_0, T_1]}(\Gamma) \lesssim C(T_0) + (T_1 - T_0)^\delta C(T_1) N_{[T_0, T_1]}(X).
\]

Similarly, we could show
\[
N_{[T_0, T_1]}(\Lambda) \lesssim C(T_0) + (T_1 - T_0)^\delta C(T_1) N_{[T_0, T_1]}(X).
\]

Combining all the above results yields the desired result. \(\square\)

To obtain an apriori estimate of the form (23) we need to employ the following elementary lemma.

**Lemma 4.2.** Let \(\delta > 0\) and \(C > 0\). Then there exists a monotone sequence of positive real number \(T_k\) such that
\[
\lim_{k \to \infty} T_k = \infty \quad \text{and} \quad (T_{k+1} - T_k)^\delta T_{k+1} \leq \frac{1}{4C} \quad \forall k \in \mathbb{N}.
\]

**Proof.** Consider the sequence \(T_k\) defined by
\[
T_k := \frac{1}{(4C)^{1/(1+\delta)}} \frac{\delta}{1+\delta} \left( 1 + \frac{1}{2^{1/(1+\delta)}} + \ldots + \frac{1}{k^{1/(1+\delta)}} \right) \\
\leq \frac{1}{(4C)^{1/(1+\delta)}} k^{\delta/(1+\delta)}.
\]
Proof. Taking the time derivative of (20) yields
\[\delta T_{k+1} - T_k \leq \frac{1}{4C} \left( \frac{\delta}{1+\delta} \right) \delta \leq \frac{1}{4C}.\]
\[\square\]

**Proposition 4.3.** Let \(T > 0\) and \(\delta > 0\). Suppose \((\phi, \Gamma, \Lambda)\) are solutions to the TDHFB equations, then we have the following apriori estimate
\[N_T(X) \lesssim 1 + T^{2+\frac{1}{2}}. \tag{27}\]

**Proof.** Define \(T_k\) as in the previous lemma with a sufficiently large \(C > 0\). Using estimate (24) and sub-linearity of \(C(T)\), we obtain the estimate
\[N_{[T_k, T_{k+1}]}(X) \lesssim C(T_0) + T_k \tag{28}\]
which means
\[N_{T_{k+1}}(X) \leq N_{T_k}(X) + N_{[T_k, T_{k+1}]}(X) \lesssim N_{T_k}(X) + 1 + T_k \]
\[\lesssim (1 + k) + T_1 + \ldots + T_k. \]

Switching to continuous \(T\)-variable yields the desired estimate
\[N_T(X) \lesssim (1 + T^{1+\frac{1}{2}}) + \int_0^T x^{\frac{\delta}{1+\delta}} \, dx \lesssim 1 + T^{2+\frac{1}{2}}. \quad \square\]

**Lemma 4.4.** Let \((\phi_t, \Gamma_t, \Lambda_t)\) be solutions to the TDHFB equations. Then we have the following estimates
\[N_{[T_0, T_1]}(\partial_t X) \lesssim C_1(T_0) + (\Delta T)^\delta N_{[T_0, T_1]}(X)N_{[T_0, T_1]}(\partial_t X), \tag{29}\]
\[N_{[T_0, T_1]}(\nabla_{x+y} X) \lesssim C_2(T_0) + (\Delta T)^\delta N_{[T_0, T_1]}(X)N_{[T_0, T_1]}(\nabla_{x+y} X) \tag{30}\]
where
\[C_1(T) := \| \langle \nabla_x \rangle^{1/2+\varepsilon} \partial_t \phi(T, \cdot) \|_{L^2(dx)} \]
\[+ \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \partial_t \Gamma(T, \cdot) \|_{L^2(dx dy)} \]
\[+ \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \partial_t \Lambda(T, \cdot) \|_{L^2(dx dy)} \]
and
\[C_2(T) := \| \langle \nabla_x \rangle^{1/2+\varepsilon} \nabla_x \phi(T, \cdot) \|_{L^2(dx)} \]
\[+ \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \nabla_{x+y} \Gamma(T, \cdot) \|_{L^2(dx dy)} \]
\[+ \| \langle \nabla_x \rangle^{1/2+\varepsilon} \langle \nabla_y \rangle^{1/2+\varepsilon} \nabla_{x+y} \Lambda(T, \cdot) \|_{L^2(dx dy)}. \]

**Proof.** Taking the time derivative of (29) yields
\[(S + N^{-1} v_N) \partial_t \Lambda = -(v_N \Lambda) \circ \partial_t \Gamma - (v_N \ast \text{Tr} \Gamma)(x) \partial_t \Lambda \]
+ similar terms =: F

By proposition 5.9 in [GM17] and Cauchy-Schwarz, we have
\[ N_{[\tau_0,\tau_1]}(\partial_\Gamma) \lesssim C_1(\tau_0) + (\Delta \tau)^\delta \| \langle \nabla_x \rangle \langle \nabla_y \rangle \partial_\Gamma \|_{L^2([\tau_0,\tau_1])L_x^{6/5}L_y^2}. \]

We shall look at two generic cases, as stated above, to deduce the desired estimate. The first case is to estimate the term \( (v_N \Lambda) \circ \partial_\Gamma \), which goes as follow
\[
\| \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} (v_N \Lambda) \circ \partial_\Gamma \|_{L^2([\tau_0,\tau_1])L_x^{6/5}L_y^2} \\
\lesssim \int dz \, v_N(z) \| \nabla_x^{1/2+\varepsilon} \Lambda(x,z) \nabla_y^{1/2+\varepsilon} \partial_\Gamma(x,z,y) \|_{L^2([\tau_0,\tau_1])L_x^{6/5}L_y^2} \\
+ \int dz \, v_N(z) \| \Lambda(x,z) \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \partial_\Gamma(x,z,y) \|_{L^2([\tau_0,\tau_1])L_x^{6/5}L_y^2} \\
\lesssim \int dz \, v_N(z) \| \nabla_x^{1/2+\varepsilon} \Lambda(x,z) \|_{L^2_x L_x^{3+\varepsilon}} \| \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \partial_\Gamma(x,y) \|_{L^\infty_x L_y^2} \\
+ \int dz \, v_N(z) \| \Lambda(x,z) \|_{L^2_x L_x^{3+\varepsilon}} \| \nabla_x^{1/2+\varepsilon} \nabla_y^{1/2+\varepsilon} \partial_\Gamma(x,y) \|_{L^\infty_x L_y^2} \\
\lesssim N_{[\tau_0,\tau_1]}(\Lambda) \tilde{N}_{[\tau_0,\tau_1]}(\partial_\Gamma)
\]

The second case we estimate is for the term \( (v_N \ast \text{Tr} \Gamma) \cdot \partial_\Lambda \) which goes as follows
\[
\| \langle \nabla_x \rangle \langle \nabla_y \rangle \partial_\Gamma(x,y) \|_{L^2([\tau_0,\tau_1])L_x^{6/5}L_y^2} \\
\lesssim \| (v_N \ast (\langle \nabla_x \rangle \langle \nabla_y \rangle \partial_\Gamma)(x) \partial_\Lambda(x,y) \|_{L^2([\tau_0,\tau_1])L_x^{6/5}L_y^2} \\
+ \| (v_N \ast (\langle \nabla_x \rangle \langle \nabla_y \rangle \partial_\Gamma)(x) \partial_\Lambda(x,y) \|_{L^2([\tau_0,\tau_1])L_x^{6/5}L_y^2} \\
\lesssim \tilde{N}_{[\tau_0,\tau_1]}(\Gamma) N_{[\tau_0,\tau_1]}(\partial_\Lambda)
\]

Hence combining the above estimates we get that
\[
N_T(\partial_\Lambda) \lesssim C_1(\tau_0) + (\tau_1 - \tau_0)^\delta \{ N_T(\Lambda) \tilde{N}_T(\partial_\Gamma) + N_T(\partial_\Lambda) \tilde{N}_T(\Gamma) + N_T(\partial_\phi) N_T(\phi) \} \\
\lesssim C_1(\tau_0) + (\tau_1 - \tau_0)^\delta N_{[\tau_0,\tau_1]}(X) N_{[\tau_0,\tau_1]}(\partial_\Gamma)
\]

Similarly, we can show
\[
N_T(\partial_\Gamma) \lesssim C_1(\tau_0) + (\tau_1 - \tau_0)^\delta N_{[\tau_0,\tau_1]}(X) N_{[\tau_0,\tau_1]}(\partial_\Gamma)
\]
and
\[
N_T(\partial_\phi) \lesssim C_1(\tau_0) + (\tau_1 - \tau_0)^\delta N_{[\tau_0,\tau_1]}(X) N_{[\tau_0,\tau_1]}(\partial_\Gamma)
\]

Therefore, summing up the three inequalities yields (29). Moreover, the proof of (30) is exactly the same since \( \nabla_{x+y} \) commutes with \( \frac{1}{N} v_N(x-y) \), i.e. \( [\nabla_{x+y}, N^{-1} v_N(x-y)] = 0 \). \( \square \)

Remark 4.5. The proof of Lemma 4.4 is essentially the same as Lemma 4.1. However, in the proof of Lemma 4.4 we were unable to use the conservation
laws of the TDHFB equations which leads to the above quadratic term bound.

Using the above lemma we could again prove apriori estimates for both the $\partial_t X$ and $\nabla_{x+y} X$ as in the following proposition

**Proposition 4.6.** Let $T > 0$ and $\delta > 0$. Suppose $(\phi_t, \Gamma_t, \Lambda_t)$ are solutions to the TDHFB equations, then we have the following apriori estimates

$$N_T(\partial_t X) \lesssim \exp\left(\alpha T^{1+\frac{1}{\delta}}\right),$$

$$N_T(\nabla_{x+y} X) \lesssim \exp\left(\alpha' T^{1+\frac{1}{\delta}}\right)$$

for some $\alpha, \alpha' > 0$, which are independent of $T$.

**Proof.** We shall again choose the sequence $T_k$ defined by (26) for some sufficiently large $C > 0$. Applying estimate (28) and Lemma 4.4 yield the estimate

$$N_{[T_k, T_{k+1}]}(\partial_t X) \lesssim C_1(T_k) + (T_{k+1} - T_k)^\delta N_{[T_k, T_{k+1}]}(X) N_{[T_k, T_{k+1}]}(\partial_t X) \lesssim C_1(T_k) + (T_{k+1} - T_k)^\delta (1 + T_{k+1}) N_{[T_k, T_{k+1}]}(\partial_t X).$$

Then for $k$ sufficiently large it follows

$$N_{[T_k, T_{k+1}]}(\partial_t X) \lesssim C_1(T_k).$$

In particular, we get the estimate

$$N_T(\partial_t X) \leq N_{T_k}(\partial_t X) + \sum_{j=0}^{k} C_1(T_j)$$

By switching to the continuous $T$-variable we obtain the estimate

$$N_T(\partial_t X) \lesssim C_1(T_0) + \int_0^T d\tau \ C_1(\tau) \tau^{1/\delta} \lesssim C_1(T_0) + \int_0^T d\tau \ N_\tau(\partial_t X) \tau^{1/\delta}$$

Apply Gronwall’s inequality yield

$$N_T(\partial_t X) \lesssim \exp\left(\alpha T^{1+\frac{1}{\delta}}\right).$$

The proof for (32) is exactly the same. $\square$

**Remark 4.7.** Moreover, from the apriori estimate (31), we could deduce

$$\|\partial_t \phi\|_{L^1([0,T] \times L^2(\mathbb{R}^3))} \leq T \|\partial_t \phi\|_{L^\infty([0,T] \times L^2(\mathbb{R}^3))} \lesssim T \exp\left(\alpha T^{1+\frac{1}{\delta}}\right)$$

$$\|\partial_t \Gamma\|_{L^1([0,T] \times L^2(\mathbb{R}^6))} \leq T \|\partial_t \Gamma\|_{L^\infty([0,T] \times L^2(\mathbb{R}^6))} \lesssim T \exp\left(\alpha T^{1+\frac{1}{\delta}}\right)$$

$$\|\partial_t \Lambda\|_{L^1([0,T] \times L^2(\mathbb{R}^6))} \leq T \|\partial_t \Lambda\|_{L^\infty([0,T] \times L^2(\mathbb{R}^6))} \lesssim T \exp\left(\alpha T^{1+\frac{1}{\delta}}\right).$$
Then by 1D Gagliardo-Nirenberg inequality we have that $\phi \in C([0, T] \times L^2(\mathbb{R}^3))$ and $\Gamma, \Lambda \in C([0, T] \times L^2(\mathbb{R}^6))$, i.e. $\phi, \Gamma$ and $\Lambda$ are strong solutions to the nonlinear equations.

Let us conclude this section with some a priori estimates for the higher order derivatives of $\phi, \Gamma$ and $\Lambda$ which we will use later on to estimate $sh(2k)$.

**Lemma 4.8.** Let $\delta > 0$. Suppose $\phi, \Gamma$ and $\Lambda$ are solutions to the nonlinear equations, then we have the following estimates

\[
N_{[T_0, T_1]}(\partial_t \nabla_{x+y} X) \lesssim C_3(T_0) + (T_1 - T_0)^{\delta} N_{[T_0, T_1]}(X) N_{[T_0, T_1]}(\partial_t \nabla_{x+y} X) \\
+ (T_1 - T_0)^{\delta} N_{[T_0, T_1]}(\partial_t X) N_{[T_0, T_1]}(\nabla_{x+y} X)
\]

(33)

\[
N_{[T_0, T_1]}(\partial_t \nabla_{x+y}^2 X) \lesssim C_4(T_0) + (T_1 - T_0)^{\delta} N_{[T_0, T_1]}(X) N_{[T_0, T_1]}(\nabla_{x+y}^2 X) \\
+ (T_1 - T_0)^{\delta} N_{[T_0, T_1]}^2(\partial_t \nabla_{x+y} X) N_{[T_0, T_1]}(\nabla_{x+y} X)
\]

(34)

\[
N_{[T_0, T_1]}(\partial_t \nabla_{x+y}^2 X) \lesssim C_5(T_0) + (T_1 - T_0)^{\delta} N_{[T_0, T_1]}(X) N_{[T_0, T_1]}(\partial_t \nabla_{x+y}^2 X) \\
+ (T_1 - T_0)^{\delta} N_{[T_0, T_1]}^2(\partial_t \nabla_{x+y} X) N_{[T_0, T_1]}(\partial_t X)
\]

(35)

where

\[
C_3(T) = \| (\nabla_x)^{1/2+\varepsilon} \partial_t \nabla_x \phi(T, \cdot) \|_{L^2(dx)} \\
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} (\partial_t \nabla_{x+y} \Gamma)(T, \cdot) \|_{L^2(dxdy)} \\
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} (\partial_t \nabla_{x+y} \Lambda)(T, \cdot) \|_{L^2(dxdy)}
\]

(33) \[
C_4(T) = \| (\nabla_x)^{1/2+\varepsilon} \nabla_x \phi(T, \cdot) \|_{L^2(dx)} \\
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} (\nabla_{x+y}^2 \Gamma)(T, \cdot) \|_{L^2(dxdy)} \\
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} (\nabla_{x+y}^2 \Lambda)(T, \cdot) \|_{L^2(dxdy)}
\]

(34) \[
C_5(T) = \| (\nabla_x)^{1/2+\varepsilon} \partial_t \nabla_x \phi(T, \cdot) \|_{L^2(dx)} \\
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} (\partial_t \nabla_{x+y}^2 \Gamma)(T, \cdot) \|_{L^2(dxdy)} \\
+ \| (\nabla_x)^{1/2+\varepsilon} (\nabla_y)^{1/2+\varepsilon} (\partial_t \nabla_{x+y}^2 \Lambda)(T, \cdot) \|_{L^2(dxdy)}
\]

(35)

**Proof.** The proof is exactly the same lemma 4.4 \qed
Proposition 4.9. Let $T > 0$ and $\delta > 0$. Suppose $(\phi, \Gamma, \Lambda)$ be solutions to the TDHFB equations, then we have the following apriori estimates

\[ N_T(\partial_t \nabla_{x+y} X) \lesssim \exp \left( \kappa T^{1+\frac{1}{\delta}} \right), \]
\[ N_T(\nabla_{x+y}^2 X) \lesssim \exp \left( \kappa' T^{1+\frac{1}{\delta}} \right), \]
\[ N_T(\partial_t \nabla_{x+y}^2 X) \lesssim \exp \left( \kappa'' T^{1+\frac{1}{\delta}} \right) \]

for some $\kappa, \kappa', \kappa'' > 0$, which are independent of $T$ and uniform in $N$.

Proof. Let us begin by choosing the same sequence $T_k$ defined by (26) for some sufficiently large $C > 0$. By the previous lemma we obtain the estimate

\[ N_{[T_k, T_{k+1}]}(\partial_t \nabla_{x+y} X) \lesssim C_3(T_k) + (T_{k+1} - T_k) \delta N_{[T_k, T_{k+1}]}(\partial_t X) N_{[T_k, T_{k+1}]}(\nabla_{x+y} X). \]

Then for $k$ sufficiently large we obtain the estimate

\[ N_{[T_k, T_{k+1}]}(\partial_t \nabla_{x+y} X) \lesssim C_3(T_k) + \frac{C_1(T_k)C_2(T_k)}{T_{k+1}}. \]

Hence it follows

\[ N_{T_{k+1}}(\partial_t \nabla_{x+y} X) \leq N_{T_k}(\partial_t \nabla_{x+y} X) + N_{[T_k, T_{k+1}]}(\partial_t \nabla_{x+y} X) \]
\[ \lesssim N_{T_k}(\partial_t \nabla_{x+y} X) + C_3(T_k) + \frac{C_1(T_k)C_2(T_k)}{T_{k+1}} \]
\[ \lesssim C_3(T_0) + \sum_{j=1}^{k} \left[ C_3(T_j) + \frac{C_1(T_j)C_2(T_j)}{T_j} \right]. \]

Switching to continuous $T$-variable yield the estimate

\[ N_T(\partial_t \nabla_{x+y} X) \lesssim C_3(T_0) + \int_0^T d\tau N_\tau(\nabla_{x+y} X) N_\tau(\partial_t X)^{1/\delta - 1} \]
\[ + \int_0^T d\tau N_\tau(\partial_t \nabla_{x+y} X)^{1/\delta} \]
\[ \lesssim C_3(T_0) + T^{1/\delta} \exp \left( kT^{1+\frac{1}{\delta}} \right) \]
\[ + \int_0^T d\tau N_\tau(\partial_t \nabla_{x+y} X)^{1/\delta}. \]

Using Gronwall’s inequality, we obtain the estimate

\[ N_T(\partial_t \nabla_{x+y} X) \lesssim (1 + T^{1/\delta} \exp \left( kT^{1+\frac{1}{\delta}} \right) \exp \left( cT^{1+\frac{1}{\delta}} \right) \]
\[ \lesssim \exp \left( \kappa T^{1+\frac{1}{\delta}} \right) \]

for some $\kappa > 0$. The proofs of the other two estimates are similar. \qed
5. Estimates of $\text{sh}(2k)$

The purpose of this section is to obtain estimates for $\text{sh}(2k)$, which will be used to obtain Fock space estimates. Recall the equation for $\text{sh}(2k)$ is given by

$$S(\text{sh}(2k)) = -2v_N \Lambda - (v_N \Lambda) \circ p_2 - \bar{p}_2 \circ (v_N \Lambda)$$

$$- (\langle v_N * \text{Tr} \Gamma \rangle(x) + (v_N * \text{Tr} \Gamma)(y)) \text{sh}(2k)$$

$$- (v_N \Gamma) \circ \text{sh}(2k) - \text{sh}(2k) \circ (v_N \Gamma)$$

(39)

where $S = \frac{1}{t} \partial_t - \Delta_R^6$.

**Proposition 5.1.** Let $\text{sh}(2k)$ satisfy (39) with some initial conditions. Then for any fixed $T > 0$ and $0 \leq j \leq 2$ we have that

$$\| \nabla_{x+y}^j \text{sh}(2k)(t, \cdot) \|_{L^2(dx \, dy)} \lesssim \exp \left( \alpha_j T^{1+\frac{j}{2}} \right)$$

(40)

$$\sup_x \| \text{sh}(2k)(t, x, \cdot) \|_{L^2(dy)} \lesssim \exp \left( \alpha' T^{1+\frac{1}{2}} \right).$$

(41)

for some $\alpha_j, \alpha > 0$.

To prove the above proposition we will need a couple lemmas

**Lemma 5.2.** Let $s_0^0$ be the solution to

$$S_s^0 = -2v_N(x-y)\Lambda$$

$$s_0^0(0, x, y) = \text{sh}(2k)(0, x, y)$$

on the interval $[0, T]$. Then there exists $\kappa_j > 0$ for $0 \leq j \leq 2$ such that

$$\| \nabla^j_{x+y} s_0^0(t, \cdot, \cdot) \|_{L^2(dx \, dy)} \lesssim \exp \left( \kappa_j T^{1+\frac{j}{2}} \right)$$

for all $t \in [0, T]$.

**Proof.** Observe we could write the solution as

$$s_0^0(t, x, y) = e^{it\Delta_{x,y}} \text{sh}(2k)(0, x, y) + i \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(x-y) \Lambda(s) \, ds$$

then it follows

$$\| s_0^0(t, \cdot) \|_{L^2_{x,y}} \leq \| \text{sh}(2k_0) \|_{L^2_{x,y}} + \left\| \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(x-y) \Lambda(s) \, ds \right\|_{L^2_{x,y}}.$$ 

Let us focus on the nonlinear term. By a change of variables, we get

$$\left\| \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(x-y) \Lambda(s) \, ds \right\|_{L^2(dx \, dy)}$$

$$\lesssim \left\| P_{|\xi| > 1} \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(y) \Lambda \left( s, \frac{x+y}{2}, -\frac{x-y}{2} \right) \, ds \right\|_{L^2(dx \, dy)}$$

$$+ \left\| P_{|\xi| \leq 1} \int_0^t e^{i(t-s)\Delta_{x,y}} v_N(y) \Lambda \left( s, \frac{x+y}{2}, -\frac{x-y}{2} \right) \, ds \right\|_{L^2(dx \, dy)}.$$
Let us denote $\Lambda(s, \frac{x+y}{2}, \frac{x+y}{2})$ by $\tilde{\Lambda}(s, x, y)$. For the first term we shall rewrite the integral using integration by parts, i.e.

\[
\int_0^t e^{i(t-s)\Delta_{x,y}v_N(y)}\tilde{\Lambda}(s) \, ds \sim -\int_0^t ds \frac{\partial}{\partial s} e^{i(t-s)\Delta_{x,y}v_N(y)}\tilde{\Lambda}(s)
\]

\[
= \int_0^t e^{i(t-s)\Delta_{x,y}v_N(y)}\frac{\partial}{\partial s} \tilde{\Lambda}(s)
\]

\[
+ e^{it\Delta}\Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(0)) - \Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(t))
\]

then it follows

\[
\left\| P_{|\xi|\geq1} \int_0^t e^{i(t-s)\Delta_{x,y}v_N(y)}\tilde{\Lambda}(s, \cdot) \, ds \right\|_{L^2(dx,dy)}
\]

\[
\lesssim \int_0^t ds \left\| P_{|\xi|\geq1} \Delta_{x,y}^{-1}(v_N(y)\frac{\partial}{\partial s} \tilde{\Lambda}(s, \cdot)) \right\|_{L^2(dx,dy)}
\]

\[
+ \left\| P_{|\xi|\geq1} \Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(0, \cdot)) \right\|_{L^2} + \left\| P_{|\xi|\geq1} \Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(t, \cdot)) \right\|_{L^2}.
\]

Next, by Plancherel, we obtain the estimate

\[
\int_0^t ds \left\| P_{|\xi|\geq1} \Delta_{x,y}^{-1}(v_N(y)\frac{\partial}{\partial s} \tilde{\Lambda}(s, \cdot)) \right\|_{L^2(dx,dy)}
\]

\[
\lesssim \int_0^t ds \left\| \frac{v_N \partial_s \tilde{\Lambda}(s, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_{L^2(|\xi|\geq1,d\xi)L^2(dy)}
\]

\[
\lesssim \int_0^t ds \left\| v_N(y) \partial_s \tilde{\Lambda}(s, x, y) \right\|_{L^1(dy)L^2(dx)}
\]

\[
\sim \int_0^t ds \left\| \partial_s \Lambda(s, x + y, x) \right\|_{L^\infty(dy)L^2(dx)}
\]

\[
\lesssim \sqrt{t} \left\| \partial_s \Lambda(s, x + y, x) \right\|_{L^2([0,t])L^\infty(dy)L^2(dx)} \lesssim \exp \left( \kappa_0 t^{1 + \frac{1}{6}} \right).
\]

For the third and second term, we have the estimate

\[
\left\| P_{|\xi|\geq1} \Delta_{x,y}^{-1}(v_N(y)\tilde{\Lambda}(t, x, y)) \right\|_{L^2(dx,dy)}
\]

\[
\lesssim \left\| \Lambda(t, x, x - y) \right\|_{L^\infty(dy)L^2(dx)}
\]

\[
\lesssim \left\| \Lambda(s, x, x - y) \right\|_{L^\infty([0,t])L^\infty(dy)L^2(dx)}
\]

\[
\lesssim \left\| \partial_s \Lambda(s, x, x - y) \right\|_{L^2([0,t])L^\infty(dy)L^2(dx)} \lesssim \exp \left( \kappa_0 t^{1 + \frac{1}{6}} \right)
\]

Thus, we have shown

\[
\left\| s_0(t, x, y) \right\|_{L^\infty([0,T])L^2_{x,y}} \lesssim \exp \left( \kappa_0 t^{1 + \frac{1}{6}} \right).
\]
In particular, it’s easy to check
\[
\| \nabla^j_x y^s_a(t, x, y) \|_{L^\infty([0, T])L^2_x}^2 \lesssim \| (\partial_t \nabla^j_x y^A)(s, x, x - y) \|_{L^2([0, T])L^\infty_x L^2_y}^2 \\
\lesssim N_T(\partial_t \nabla^j_x y^A X) \lesssim \exp \left( t \kappa^{1+\frac{1}{2}} \right).
\]

Lemma 5.3. Let \( s_a \) be the solution to
\[
\tilde{S}s_a = -2v_N \Lambda
\]
\( s_a(0, x, y) = \text{sh}(2k)(0, x, y) \)
on the interval \([0, T]\). Then there exists \( \kappa_j > 0 \) for \( 0 \leq j \leq 2 \) such that
\[
\| \nabla^j_x y^s_a(t, x, y) \|_{L^2} \leq \exp \left( \kappa_j t^{1+\frac{1}{2}} \right)
\]
for all \( t \in [0, T] \).

Proof. Recall \( \tilde{S} = S + V \) where
\[
V(u) = ((v_N \ast \text{Tr} \Gamma)(x) + (v_N \ast \text{Tr} \Gamma)(y))u + (v_N \Gamma) \circ u + u \circ (v_N \Gamma).
\]
Using the previous result, we see that
\[
\tilde{S}s_a = -V(s_a^0) s_a^1(0, x, y) = 0
\]
where \( s_a = s_a^1 + s_a^0 \). It’s not hard to see
\[
(\tilde{S} - V)(\nabla^j_x y^s_a) = [S, \nabla^j_x y^s_a] s_a^1 - \nabla^j_x y^V(s_a^0)
\]
\[
= [V, \nabla^j_x y^s_a] s_a^1 - \nabla^j_x y^V(s_a^0).
\]
Using energy estimate, we have
\[
\| \nabla^j_x y^s_a \|_{L^2} \leq \int_0^t ds \| [V, \nabla^j_x y^s_a] s_a^1 \|_{L^2} + \| \nabla^j_x y^V(s_a^0) \|_{L^2}.
\]
Let us consider the case when \( j = 0 \). Observe we have
\[
\| s_a^1 \|_{L^2} \leq \int_0^t ds \| V(s_a^0) \|_{L^2}.
\]
Observe
\[
\int_0^t ds \| (v_N \ast \text{Tr} \Gamma)(x) \cdot s_a^0 \|_{L^2(dx)} \lesssim \| s_a^0 \|_{L^\infty(dx)L^2(dy)}
\]
\[
\leq \int_0^t ds \| v_N \ast \text{Tr} \Gamma(x) \|_{L^2(dx)} \| s_a^0 \|_{L^\infty(dx)L^2(dy)}
\]
\[
\leq \int_0^t ds \| v_N \ast \text{Tr} \Gamma(x) \|_{L^2(dx)} \| s_a^0(x, x + y) \|_{L^\infty(dx)} \| L^2(dy)
\]
\[
\lesssim t^{1/2} \sup_{0 \leq s \leq t} \| (\nabla^j_x y^s_a)(s, x, y) \|_{L^2(dy)} \| L^2(dx)
\]
\[
\lesssim \exp \left( \kappa_0 t^{1+\frac{1}{2}} \right)
\]
\[
\int_0^t ds \| (v_N \Gamma) \circ s_a^0 \|_{L^2(dx dy)} \\
\leq \int_0^t ds \int dz v_N(z) \| \Gamma(x, x - z) s_a^0(x - z, y) \|_{L^2(dx dy)} \\
\lesssim \int_0^t ds \int dz v_N(z) \| \Gamma(x, x - z) \|_{L^\infty(dx)} \| s_a^0 \|_{L^2(dx dy)} \\
\lesssim \exp \left( \kappa_0 T^{1 + \frac{1}{\delta}} \right).
\]

The proof for \( j = 1, 2 \) is the same.

\[\square\]

**Proof of Proposition 5.1.** The proof is exactly the same as the one given for Theorem 7.1 in [GM17].

\[\square\]

**References**

[127x690] [BBCFS] V. Bach, S. Breteaux, T. Chen, J. Fröhlich, and I. M. Sigal, *The time-dependent hartree-fock-bogoliubov equations for bosons*, arXiv preprint arXiv:1602.05171 (2016), 1–36.

[127x487] [BCS17] C. Boccato, S. Cenatiempo, and B. Schlein, *Quantum many-body fluctuations around nonlinear schrödinger dynamics*, Annales Henri Poincaré 18 (2017), no. 1, 113–191.

[127x368] [BdOS15] N. Benedikter, G. de Oliveira, and B. Schlein, *Quantitative derivation of the Gross-Pitaevskii equation*, Communications on Pure and Applied Mathematics 68 (2015), no. 8, 1399–1482.

[127x302] [Che12] X. Chen, *Second order corrections to mean field evolution for weakly interacting bosons in the case of three-body interactions*, Archive for Rational Mechanics and Analysis 203 (2012), no. 2, 455–497.

[127x270] [Cho16] J.J.W. Chong, *Dynamics of large boson systems with attractive interaction and a derivation of the cubic focusing NLS in \( \mathbb{R}^3 \)*, arXiv preprint arXiv:1608.01615 (2016), 1–23.

[127x248] [CLS11] L. Chen, J.O. Lee, and B. Schlein, *Rate of convergence towards hartree dynamics*, Journal of Statistical Physics 144 (2011), no. 4, 872–903.

[127x226] [DG13] J. Dereziński and C. Gérard, *Mathematics of quantization and quantum fields*, Cambridge University Press, 2013.

[127x204] [ES09] L. Erdös and B. Schlein, *Quantum dynamics with mean field interactions: a new approach*, Journal of Statistical Physics 134 (2009), no. 5, 859–870.

[127x171] [ESY06] L. Erdös, B. Schlein, and H.-T. Yau, *Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate*, Communications on Pure and Applied Mathematics 59 (2006), no. 12, 1659–1741.

[127x171] [ESY07] ———, *Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems*, Inventiones Mathematicae 167 (2007), no. 3, 515–614.

[127x171] [ESY09] ———, *Rigorous derivation of the gross-pitaevskii equation with a large interaction potential*, Journal of the American Mathematical Society 22 (2009), no. 4, 1099–1156.

[127x171] [ESY10] ———, *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate*, Annals of Mathematics 172 (2010), no. 1, 291–370.
[EY01] L. Erdős and H.-T. Yau, *Derivation of the nonlinear Schrödinger equation from a many-body Coulomb system*, Advances in Theoretical and Mathematical Physics 5 (2001), no. 6, 1169–1205.

[FKS09] J. Fröhlich, A. Knowles, and S. Schwarz, *On the mean-field limit of bosons with coulomb two-body interaction*, Communications in Mathematical Physics 288 (2009), no. 3, 1023–1059.

[Fol89] G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, no. 122, Princeton University Press, 1989.

[GM13a] M. Grillakis and M. Machedon, *Beyond mean field: On the role of pair excitations in the evolution of condensates*, Journal of Fixed Point Theory and Applications 14 (2013), no. 1, 91–111.

[GM13b] ———, *Pair excitations and the mean field approximation of interacting Bosons, I*, Communications in Mathematical Physics 324 (2013), no. 2, 601–636.

[GM17] ———, *Pair excitations and the mean field approximation of interacting Bosons, II*, Communications in Partial Differential Equations 42 (2017), no. 1, 24–67.

[GMM10] M. Grillakis, M. Machedon, and D. Marletas, *Second-order corrections to mean field evolution of weakly interacting Bosons. I.*, Communications in Mathematical Physics 294 (2010), no. 1, 273–301.

[GMM11] ———, *Second-order corrections to mean field evolution of weakly interacting Bosons. II*, Advances in Mathematics 228 (2011), no. 3, 1788–1815.

[GMM17] ———, *Evolution of the boson gas at zero temperature: Mean-field limit and second-order correction*, Quarterly of Applied Mathematics 75 (2017), no. 1, 69–104.

[Gol16] François Golse, *On the dynamics of large particle systems in the mean field limit*, Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity, Springer, 2016, pp. 1–144.

[GV79a] J. Ginibre and G. Velo, *The classical field limit of scattering theory for non-relativistic many-boson systems. I.*, Communications in Mathematical Physics 66 (1979), no. 4, 37–76.

[GV79b] ———, *The classical field limit of scattering theory for non-relativistic many-boson systems. II*, Communications in Mathematical Physics 68 (1979), no. 1, 45–68.

[Hep74] K. Hepp, *The classical limit for quantum mechanical correlation functions*, Communications in Mathematical Physics 35 (1974), no. 4, 265–277.

[KP10] Antti Knowles and Peter Pickl, *Mean-field dynamics: singular potentials and rate of convergence*, Communications in Mathematical Physics 298 (2010), no. 1, 101–138.

[Kuz15a] E. Kuz, *Exact evolution versus mean field with second-order correction for Bosons interacting via short-range two-body potential*, arXiv preprint arXiv:1511.00487 (2015), 1–38.

[Kuz15b] ———, *Rate of convergence to mean field for interacting Bosons*, Communications in Partial Differential Equations 40 (2015), no. 10, 1831–1854.

[LSSY05] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason, *The mathematics of the bose gas and its condensation*, vol. 34, Springer Science & Business Media, 2005.

[NN17] P. T. Nam and M. Napoliowski, *Bogoliubov correction to the mean-field dynamics of interacting bosons*, Advances in Theoretical and Mathematical Physics 21 (2017), no. 3, 683–738.

[RS09] I. Rodnianski and B. Schlein, *Quantum fluctuations and rate of convergence towards mean field dynamics*, Communications in Mathematical Physics 291 (2009), no. 1, 31–61.
[Sol14] Jan Philip Solovej, *Many body quantum mechanics*, Lecture Notes. Summer (2014), no. 1–102.

Department of Mathematics, University of Maryland, College Park, MD 20742

E-mail address: jwchong@math.umd.edu