Quantum Information metric for time-dependent quantum systems and higher-order corrections

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It is well established that quantum criticality is one of the most intriguing phenomena which signals the presence of new states of matter. Without prior knowledge of the local order parameter, the quantum information metric (or fidelity susceptibility) can indicate the presence of a phase transition as well as it measures distance between quantum states. In this work, we calculate the distance between quantum states which is equal to the fidelity susceptibility in quantum model for a time-dependent system describing a two-level atom coupled to a time-driven external field. As inspired by the Landau-Lifshitz quantum model, we find in the present work information metric induced by fidelity susceptibility. We, for the first time, derive a higher-order rank-3 tensor as a third-order fidelity susceptibility. Having computed quantum noise function in this simple time-dependent model we show that the noise function eternally lasts long in our model.

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I. INTRODUCTION

Quantum criticality is one of the most intriguing phenomena which is crucial for interpreting a wide variety of experiments. As well known, it signals the presence of new states of matter [1]. In order to observe exotic features at quantum critical point, one has to study systems in the thermodynamic regime involving large numbers of interacting particles, which encounter experimental and theoretical limitations [2]. Despite consisting only of a single-mode cavity field and a two-level atom, the authors of Ref.[3] show that the Rabi system exhibits a quantum phase transition (QPT). They demonstrate that the superradiant QPT primarily studied for systems of many atoms can be achieved with systems of a single one.

In recent years, there was a great deal of interest in studying QPTs from different perspectives of quantum information science [5], e.g., quantum entanglement [6, 7] and quantum fidelity [8, 9]. At the phase transition point, physical observables exhibit sigular behavior governing the most dramatic manifestations of the laws of statistical and quantum mechanics. In order to probe the phase transition, the fidelity susceptibility draws one of the most promising machines in which no prior knowledge of the order parameter and the symmetry of the system are required [14, 30]. Regarding these works, the connection between the quantum information theory and condensed matter physics can be in principle achieved which might allow us to deepen our understanding in the various condensed matter phenomena. Notice that the concept of the fidelity susceptibility was originally introduced in Ref.[8].

An obvious physical example of QPTs using the quantum fidelity approach recently is given in [11]. It was illustrated that at two sides of the critical point \( g_c \) of a quantum many body system the ground state
wavefunctions have different structures Ref.\[12\]. Then, consequently, this may lead to the overlap of the
two ground states which are separated by a small distance $\delta g$ in the parameter space and then might
emerge. In general, at the critical point $g_c$, the distance can be parameterized via $|\langle \Psi_0(g) | \Psi_0(g + \delta g) \rangle|$ which is minimum. Therefore, the structure of the ground state of a quantum many-body system experiences a significant change because the system is driven across the transition point adiabatically. As a consequence, we expect that the fidelity susceptibility should be maximum (or even diverse) at the transition point, \[8\]. We notice that in various systems many authors have investigated the QPTs from the fidelity point of view \[8\] \[11\] \[13\] \[36\]. They have shown that the fidelity susceptibility can be considered to be a simple approach in determining the universality of quantum phase transitions \[13\] \[36\]. Interestingly, in \[40\], the quantum information metric gravity dual in conformal field theories has just been examined.

In this work, we study the fidelity susceptibility in quantum model for a time-dependent system decribing a two-level atom coupled to a time-driven external field. We analytically investigate the behavior of fidelity susceptibility in the time driven quantum model when the potential $V$ is time-dependent. The organization of the paper is as follows. In Sec.II we compute fidelity susceptibility for a simple two level exact model. In Sec.III we explore the mathematical foundations for fidelity susceptibility in time-dependent systems. In Sec.IV the two level Landau-Lifshitz quantum model is analyzed. In Sec.V an experimental method based on noise function is proposed. In Sec.VI, the higher-order correction to fidelity susceptibility is calculated. Finally, we conclude our findings in the last section.

II. FIDELITY SUSCEPTIBILITY FOR AN EXACT TWO-LEVEL SYSTEM

In this section we give an example of two level quantum system driven by a time-dependent potential. Let us consider a Hamiltonian of the whole system with a driving parameter $\lambda$ as

$$H = H_0 + \lambda V(t).$$

(1)

Notice that as $\lambda$ changes, one or several phase transition(s) may take place because of the competition between $H_0$ and $V(t)$. Here the quantum fidelity measures the distance on the manifold of $\lambda$, which is defined as the overlap between the ground-state wavefunctions at two different values of the driving parameter $^1$:

$$| \langle \psi(\lambda) | \psi(\lambda + \delta \lambda) \rangle | \approx 1 - \frac{(\delta \lambda)^2}{2} \chi_F + \mathcal{O}(\delta \lambda)^3.$$  

(2)

To examine the wave function $|\psi(\delta \lambda)\rangle$, we expand it in terms of non-perturbation stationary states

$$|\psi(\delta \lambda)\rangle = \sum c_n(t) |\phi_n^0(\lambda)\rangle,$$

(3)

Here, $H_0 |\phi_n^0(\lambda)\rangle = \lambda \delta(t) |\phi_n^0(\lambda)\rangle$. We suppose that $|\phi_n^0(\lambda)\rangle$ satisfies the orthogonality property such that $\langle \phi_m^0(\lambda) | \phi_n^0(\lambda) \rangle = \delta_{mn}$. With the expansion of Eq.(1) and using the orthogonality condition, we obtain a pair of first-order differential equations:

$$\frac{dc_n(t)}{dt} = \frac{\lambda}{i\hbar} V(t) c_n(t).$$

(4)

Now we consider the Dirac delta function as a time driving system and suppose that $V(t) = \delta(t)$. Performing the integration gives us the following exact solutions:

$$c_n(t) = c_n(0)e^{\frac{\lambda}{i\hbar} \theta(t)},$$

(5)

where $\theta(t)$ is the Heaviside step function. Consequently, the total wavefunction of the two-level quantum system can be simply written by

$$|\psi(\lambda)\rangle = c_1(t) |\phi_1^0\rangle + c_2(t) |\phi_2^0\rangle = a(\lambda)e^{-(\frac{\lambda}{E_2}) \theta(t)\text{-}i\frac{E_2 t}{\hbar}} |\phi_1^0\rangle + b(\lambda)e^{-(\frac{\lambda}{E_2}) \theta(t)\text{-}i\frac{E_1 t}{\hbar}} |\phi_2^0\rangle.$$  

(6)

If the wavefunction is influenced under adiabatic change $\lambda \rightarrow \lambda + \delta \lambda$, we have

$$|\psi(\lambda)\rangle = a(\lambda)e^{-i\frac{\lambda+\delta\lambda}{E_2} \theta(t)\text{+}i\frac{E_2 t}{\hbar}} |\phi_1^0\rangle + b(\lambda)e^{-i\frac{\lambda+\delta\lambda}{E_2} \theta(t)\text{+}i\frac{E_1 t}{\hbar}} |\phi_2^0\rangle.$$  

(7)

$^1$ It is worth noting that the term with $\mathcal{O}(\delta \lambda)^3$ is novel and its derivation takes place in Sec.VI.
By defining the following notations:

\[ A(\lambda) = a(\lambda)e^{-i[\lambda e^{-i\theta} + \frac{E_m}{\hbar}]} \quad \text{and} \quad B(\lambda) = b(\lambda)e^{-i[\lambda e^{-i\theta} + \frac{E_m}{\hbar}]}, \]

we can simply find \( \chi_F \) as follows:

\[ \langle A(\lambda)\phi_1^{(0)} + B(\lambda)\phi_2^{(0)} | A(\lambda + \delta\lambda)\phi_1^{(0)} + B(\lambda + \delta\lambda)\phi_2^{(0)} \rangle = A^*(\lambda)A(\lambda + \delta\lambda)\langle \phi_1^{(0)} | \phi_1^{(0)} \rangle + B^*(\lambda)B(\lambda + \delta\lambda)\langle \phi_2^{(0)} | \phi_2^{(0)} \rangle \]

The modulus of the inner product yields

\[ |\langle \psi(\lambda)|\psi(\lambda + \delta\lambda)\rangle|^2 = |A(\lambda)|^2 + |B(\lambda)|^2 = |a(\lambda)|^2 + |b(\lambda)|^2 = 1. \]

Consequently we find from Eq. (2) that \( \chi_F = 0 \). In next section we formulate \( \chi_F \) for a general potential using time dependent perturbation theory.

**III. MATHEMATICAL FORMULATION OF FIDELITY SUSCEPTIBILITY IN TIME-DEPENDENT DRIVING SYSTEMS**

In this section we will formulate fidelity susceptibility for a general time-deriving system with two levels. Let us consider a physical system with non-perturbative time dependent Hamiltonian \( H_0 \) in operator form:

\[ i\hbar \frac{\partial}{\partial t} \psi_k^{(0)} = H_0 \psi_k^{(0)}. \]

Our aim is to find perturbed wavefunctions with Hamiltonian \( H = H_0 + V(t) \) when \( |V(t)| \ll |H_0| \). Note that here \( V \) is considered to have off diagonal components, i.e, \( V_{m\neq n} = \langle \psi_m^{(0)} | V | \psi_n^{(0)} \rangle \neq 0 \). Suppose that the perturbative solution for \( H \) can be technically written in the following form:

\[ \Psi = \sum_k a_k \psi_k^{(0)}. \]

where \( a_k = a_k(t) \). Substituting \[12\] into Schrödinger equation and multiplying by \( \psi_k^{(0)} \), we obtain:

\[ i\hbar \frac{d a_m}{d t} = \sum_k V_{mk}(t)a_k, \]

where

\[ V_{mk}(t) = \int \psi_k^{(0)} \hat{V} \psi_m^{(0)} dt = V_{mk} e^{i\frac{E_m - E_k}{\hbar} t}. \]

Using iteration method up to the first order, i.e. \( a_k^{(0)} + a_k^{(1)} \) where \( a_k^{(0)} = a_k(t = 0) \), we can find the ordinary differential equation for the first-order perturbation,

\[ i\hbar \frac{d a_k^{(1)}}{d t} = V_{kn}(t). \]

Finally, up to the first order perturbation theory, the total wave function is written as

\[ \Psi_n = \sum_k a_{kn}(t) \psi_k^{(0)}. \]

Performing an integration, we obtain

\[ a_k^{(1)} = -\frac{i}{\hbar} \int V_{kn}(t) dt = -\frac{i}{\hbar} \int V_{kn} e^{i\omega_{kn} t} dt. \]

In this case, to figure out how \( \chi_F \) looks like, we need the ground state wavefunction to be,

\[ \psi_n = \sum_k (\delta_{kn} + a_k^{(1)}) \psi_k^{(0)} = \psi_n^{(0)} + \sum_k a_k^{(1)} \psi_k^{(0)}. \]
We find out that the normalization of the perturbed wavefunction is clearly greater than unity:

\[ \langle \psi_n | \psi_n \rangle = \langle \psi_n^{(0)} + \sum_k a_{kn}^{(1)} \psi_k^{(0)} | \psi_n^{(0)} + \sum_m a_{mn}^{(1)} \psi_m^{(0)} \rangle = 1 + 0 + \sum_k |a_{kn}^{(1)}|^2 > 1. \]  

(19)

Let us further analyze our result for a two level system. The perturbed wavefunction for the ground state \( E_1 \) is given by,

\[ \psi_1 = \psi_1^{(0)} + a_{11}^{(1)} \psi_1^{(0)} + a_{12}^{(1)} \psi_2^{(0)} = \psi_1^{(0)} + \lambda_1 U_{11} \psi_1^{(0)} + \lambda_2 W_{12} \psi_2^{(0)} \]

(20)

\[ \psi_1^{(1)} = (1 + \lambda_1 U_{11}) \psi_1^{(0)} + \lambda_2 W_{12} \psi_2^{(0)}. \]  

(21)

Here we suppose that \( a_{11}^{(1)} = \lambda_1 U_{11}, a_{12}^{(1)} = \lambda_2 W_{12} \). Let us calculate the inner product which is satisfied to yield the fidelity susceptibility,

\[ \langle \psi_1(\lambda) | \psi_1(\lambda + \delta \lambda) \rangle = \langle \psi_1(\lambda) | \psi_1(\lambda) \rangle + \delta \lambda \langle \psi_1(\lambda) | \partial_\lambda \psi_1(\lambda) \rangle = 1 + \lambda_1 U_{11}|^2 + |\lambda_2 W_{12}|^2 + \delta \lambda_1 \langle \psi(\lambda) | U_{11^*} \psi_1^{(0)} \rangle \]

\[ = 1 + \lambda_1 U_{11}|^2 + |\lambda_2 W_{12}|^2 + U_{11}(1 + \lambda_1 U_{11^*}) \delta \lambda_1 + W_{12}(\lambda_2 W_{12^*}) \delta \lambda_2 \]

\[ = 1 + (\lambda_1)^2 |U_{11}|^2 + 2 \lambda_1 \text{Re}(U_{11}) + (\lambda_2)^2 |W_{12}|^2 + U_{11}(1 + \lambda_1 U_{11^*}) \delta \lambda_1 + W_{12}(\lambda_2 W_{12^*}) \delta \lambda_2. \]  

(22)

We can rewrite it as follows:

\[ \langle \psi_1(\lambda) | \psi_1(\lambda + \delta \lambda) \rangle = \langle \psi_1(\lambda) | \psi_1(\lambda) \rangle + U_{11}(1 + \lambda_1 U_{11^*}) \delta \lambda_1 + \lambda_2 |W_{12}|^2 \delta \lambda_1. \]  

(23)

In order to obtain the final results, the following expressions are needed,

\[ \frac{\langle \psi_1(\lambda) | \psi_1(\lambda + \delta \lambda) \rangle}{\langle \psi_1(\lambda) | \psi_1(\lambda) \rangle} = 1 + \frac{U_{11}(1 + \lambda_1 U_{11^*}) \delta \lambda_1 + \lambda_2 |W_{12}|^2 \delta \lambda_2}{|1 + \lambda_1 U_{11}|^2 + |\lambda_2 W_{12}|^2} \]  

(24)

\[ \left| \frac{\langle \psi_1(\lambda) | \psi_1(\lambda + \delta \lambda) \rangle}{\langle \psi_1(\lambda) | \psi_1(\lambda) \rangle} \right|^2 = 1 + \frac{|U_{11}(1 + \lambda_1 U_{11^*}) \delta \lambda_1 + (W_{12})^2 \lambda_2 \delta \lambda_2|^2}{(|1 + \lambda_1 U_{11}|^2 + |\lambda_2 W_{12}|^2)^2} + 2 \text{Re} \frac{U_{11}(1 + \lambda_1 U_{11^*}) \delta \lambda_1 + \lambda_2 W_{12}|^2 \delta \lambda_2}{|1 + \lambda_1 U_{11}|^2 + |\lambda_2 W_{12}|^2}. \]  

(25)

Finally we suggest the following expression for the fidelity susceptibility \( \chi_F \) for a time-driving system

\[ \chi_{ij} = \left[ \frac{\langle \psi_1(\lambda) | \partial_\lambda_i \psi_1(\lambda) \rangle}{\langle \psi_1(\lambda) | \psi_1(\lambda) \rangle} \right] \left[ \frac{\langle \psi_1(\lambda) | \partial_\lambda_j \psi_1(\lambda) \rangle}{\langle \psi_1(\lambda) | \psi_1(\lambda) \rangle} \right] + 2 \frac{\langle \psi_1(\lambda) | \partial_\lambda_i \partial_\lambda_j \psi_1(\lambda) \rangle}{\langle \psi_1(\lambda) | \psi_1(\lambda) \rangle} \delta_{ij}. \]  

(26)

Note that \( ds^2 = \chi_{ij} \delta \lambda_i \delta \lambda_j \) defines a Riemannian metric on a manifold \( M \) which is a family of (positive definite) inner products – for all differentiable vector fields \( \lambda_1, \lambda_2 \) on \( M \), that defines a smooth function \( M \to \mathbb{R}^2 \) on coordinate space \( (\lambda_i)^2 \). An explicit form for the metric can be written as follows:

\[ ds^2 = \chi_{11} \delta \lambda_1^2 + 2 \text{Re}(\chi_{12}) \delta \lambda_1 \delta \lambda_2 + \chi_{22} \delta \lambda_2^2, \]

(27)

or its equivalent form,

\[ ds^2 = \chi_{ij}(t) d\lambda_i d\lambda_j, \]  

(28)

It is adequate to consider it as co-dimension one spacelike slicing of the bulk spacetime:

\[ ds^2_{2+1} = \chi_{ij}(t) d\lambda_i d\lambda_j - dt^2. \]  

(29)

### IV. FIDELITY SUSCEPTIBILITY IN THE LANDAU-LIFSHITZ TWO LEVEL MODEL

In the previous section we introduced a general formulation for fidelity susceptibility for time deriving potential. In this section, we will investigate a concrete example, inspired from Landau-Lifshitz cookbooks.
The first-order perturbation, \( F \), reads,

\[
\langle H_0 \rangle_0 \leq \langle H_0 \rangle_{\text{Excited state}},
\]

(30)

The energy levels for the unperturbed Hamiltonian \( H_0 \) is defined as \( E_a = E_1, E_2 \) and it is convenient to define a frequency basis for the system,

\[
\omega_{12} = \frac{E_2 - E_1}{\hbar} > 0.
\]

(31)

As a two-level system, \( E_1 = E_0 = E_{\text{min}} \), consequently we have:

\[
E_2 > E_1.
\]

(32)

The following two total wavefunctions of a two-level system define a frequency basis for the system,

\[
\chi_4.
\]

The system under consideration is a two-level quantum system initially prepared in ground state. The aim is to calculate \( \chi_F \) matrix using (26). The ground state is defined by \( n = 0 \) and it satisfies:

\[
\langle H_0 \rangle_0 \leq \langle H_0 \rangle_{\text{Excited state}}.
\]

(30)

The following two total wavefunctions of a two-level system \( E_2 > E_1 \) are defined using the orthogonality realization:

\[
\Psi_1 = \sum_k a_{k1}(t)\psi^{(0)}_k, \quad \Psi_2 = \sum_k a_{k2}(t)\psi^{(0)}_k,
\]

(33)

where

\[
a_{kn} = a_{kn}^{(0)} + a_{kn}^{(1)} = \delta_{kn} - \frac{i}{\hbar} \int V_{kn} e^{i\omega_{kn}t} dt.
\]

(34)

Next we propose a specific form of the potential as

\[
V = Fe^{-i\omega t} + Ge^{i\omega t},
\]

(35)

where \( F \) and \( G \) are time-independent operators. If \( V_{nm} = V^{*}_{mn} \), then we obtain \( G_{nm} = F^{*}_{mn} \). In this situation, the matrix element takes the form,

\[
V_{kn}(t) = V_{kn} e^{i\omega_{kn}t} = F_{kn} e^{i(w_{kn} - \omega)t} + G_{kn} e^{i(w_{kn} + \omega)t} = F_{kn} e^{i(w_{kn} - \omega)t} + F^{*}_{kn} e^{i(w_{kn} + \omega)t}.
\]

(36)

Substituting (36) into (34) and performing an integration, we obtain

\[
a_{kn}^{(1)} = -\frac{F_{kn} e^{i(w_{kn} - \omega)t}}{h(\omega_{kn} + \omega)} - \frac{F^{*}_{kn} e^{i(w_{kn} + \omega)t}}{h(\omega_{kn} - \omega)},
\]

(37)

where we have assumed that \( \omega_{kn} \neq \pm \omega \). Note that the matrix element for an arbitrary operator \( O \) is given by:

\[
O_{mn}(t) = O_{mn}^{(0)}(t) + O_{mn}^{(1)}(t) = O_{mn}^{(0)} e^{i\omega_{mn}t} + O_{mn}^{(1)}(t),
\]

(38)

where

\[
O_{mn}^{(1)}(t) = e^{i\omega_{mn}t} \left( \sum_k \frac{O_{nk}^{(0)} F_{km}}{h(\omega_{km} - \omega)} + \frac{O_{kn}^{(0)} F_{nk}}{h(\omega_{kn} + \omega)} \right) e^{-i\omega t} + \left[ \frac{O_{nk}^{(0)} F^{*}_{mk}}{h(\omega_{mk} + \omega)} + \frac{O_{km}^{(0)} F^{*}_{kn}}{h(\omega_{kn} - \omega)} \right] e^{i\omega t}.
\]

(39)

To be more concrete when choosing \( O = H \) and \( H_{nk}^{(0)} = E_k \delta_{nk} \), the matrix form for \( H \) in zeroth order reads,

\[
H_{nm} = F_{nm} \delta_{nm} e^{i\omega_{nm}t} - e^{i\omega_{nm}t} \left( \sum_k \frac{E_k \delta_{nk} F_{km}}{h(\omega_{km} - \omega)} + \frac{E_k \delta_{mk} F_{nk}}{h(\omega_{kn} + \omega)} \right) e^{-i\omega t} + \left[ \frac{E_k \delta_{nk} F^{*}_{mk}}{h(\omega_{mk} + \omega)} + \frac{E_k \delta_{mk} F^{*}_{kn}}{h(\omega_{kn} - \omega)} \right] e^{i\omega t}.
\]

(40)

If \( F \) is real, i.e., \( F_{mn} = F^{*}_{nm} \), we obtain the following expression for a matrix representation of \( H \) up to the first-order perturbation,

\[
H_{nm} = F_{nm} \delta_{nm} e^{i\omega_{nm}t} - e^{i\omega_{nm}t} F_{nm} \omega_{nm} \left( \frac{e^{-i\omega t}}{\omega_{nm} - \omega} + \frac{e^{i\omega t}}{\omega_{nm} + \omega} \right).
\]

(41)
Note that the diagonal elements are commonly parametrized by $H_{nn} = E_n$ and the off diagonal ones are

$$H_{n\neq m} = -e^{i\omega_{nm}t}F_{nm}\omega_{nm} \left( \frac{e^{-i\omega t}}{\omega_{nm} - \omega} + \frac{e^{i\omega t}}{\omega_{nm} + \omega} \right).$$

(42)

For the two-level system, it is still plausible to obtain

$$H_{12} = (H_{21})^* = \omega_0 F_{12} \left( \frac{e^{i(\omega - \omega_0)t}}{\omega - \omega_0} - \frac{e^{-i(\omega + \omega_0)t}}{\omega + \omega_0} \right).$$

(43)

The wavefunction coefficients read as follows:

$$a_{11}^{(1)} = i \frac{F_{11}}{\hbar \omega} \sin(\omega t),$$

(44)

and

$$a_{21}^{(1)} = -\frac{F_{12}^*}{\hbar} \left( \frac{e^{-i\omega t}}{\omega_0 - \omega} + \frac{e^{i\omega t}}{\omega_0 + \omega} \right).$$

(45)

Therefore, the total perturbed wavefunction for the ground state is given by,

$$\Psi_1 = \left( \frac{F_{11}}{\hbar \omega} \sin(\omega t) \right) \psi_1^{(0)} - \frac{F_{12}^*}{\hbar} \left( \frac{e^{-i\omega t}}{\omega_0 - \omega} + \frac{e^{i\omega t}}{\omega_0 + \omega} \right) \psi_2^{(0)}.$$  

(46)

It is reasonable to parametrize perturbed matrix elements as follows:

$$F_{11} = \lambda_1 V_{11}$$

(47)

$$F_{12} = \lambda_2 W_{12}.$$  

(48)

In terms of these parameters, we obtain

$$\Psi_1 = \left( i \frac{V_{11}}{\hbar \omega} \sin \omega t \right) \lambda_1 \psi_1^{(0)} - i \frac{W_{12}^*}{\hbar} e^{i\omega t} \left( \frac{e^{-i\omega t}}{\omega_0 - \omega} + \frac{e^{i\omega t}}{\omega_0 + \omega} \right) \lambda_2 \psi_2^{(0)}.$$  

(49)

By defining two auxiliary functions,

$$\alpha(t) = i \frac{V_{11}}{\hbar \omega} \sin \omega t,$$

(50)

$$\beta(t) = -i \frac{W_{12}^*}{\hbar} \left( \frac{e^{i(\omega_0 - \omega)t}}{\omega_0 - \omega} + \frac{e^{i(\omega_0 + \omega)t}}{\omega_0 + \omega} \right),$$

(51)

and using (26), we end up with the matrix elements for $\chi_F$ as follows:

$$\chi_{11} = \frac{1}{2} \left( \frac{1}{\lambda_1} + \left| \frac{\beta}{\alpha} \right|^2 \left( \frac{\lambda_2}{\lambda_1} \right)^2 \right),$$

(52)

$$\chi_{12} = \frac{1}{2} \left( \frac{1}{\lambda_1} + \left| \frac{\beta}{\alpha} \right|^2 \left( \frac{\lambda_2}{\lambda_1} \right)^2 \right),$$

(53)

$$\chi_{22} = \frac{1}{2} \left( \frac{1}{\lambda_1} + \left| \frac{\beta}{\alpha} \right|^2 \left( \frac{\lambda_2}{\lambda_1} \right)^2 \right).$$

(54)

Note that here $\lambda_2 \neq \lambda_1$ to have the non singular metric $\chi_{ij}$. In our model,

$$\left| \frac{\beta}{\alpha} \right|^2 = \frac{2\omega W_{12}^*}{V_{11}} \frac{\omega^2 \cos^2 (\omega_0 t) + \omega_0^2 \sin^2 (\omega_0 t) \cot^2 (\omega t)}{\left( \omega^2 - \omega_0^2 \right)^2}.$$  

(55)

We are interested in high frequencies where $\omega \gg \omega_0$. In this case we have

$$\left| \frac{\beta}{\alpha} \right|^2 \approx \left( \frac{4W_{12}^2}{V_{11}} \right)^2 \cos^2 (\omega_0 t).$$

(56)
Finally, by defining \( \gamma \equiv \frac{|W(t)|^2}{V(t)} > 0 \), we have the following approximated form for fidelity susceptibility at high frequencies and ultraviolet (UV) regime as follows:

\[
\chi_{11} = \frac{1}{2\lambda_1} \frac{1}{1 + \gamma \cos^2(\omega_0 t) \left( \frac{\lambda_2}{\lambda_1} \right)^2}, \tag{57}
\]

\[
\chi_{12} = \frac{1}{2\lambda_1} \frac{1}{1 + \gamma \cos^2(\omega_0 t) \left( \frac{\lambda_2}{\lambda_1} \right)^2}, \tag{58}
\]

\[
\chi_{22} = \frac{1}{2\lambda_1} \frac{\gamma \cos^2(\omega_0 t)}{1 + \gamma \cos^2(\omega_0 t) \left( \frac{\lambda_2}{\lambda_1} \right)^2}. \tag{59}
\]

The information metric, measures the distance between two quantum states close to each other in UV regime and is given as follows:

\[
ds^2 = \frac{1}{2\lambda_1(1 + \gamma \cos^2(\omega_0 t) \left( \frac{\lambda_2}{\lambda_1} \right)^2)} \left[ d\lambda_1^2 + 2 \left( 1 + \gamma \cos^2(\omega_0 t) \left( \frac{\lambda_2}{\lambda_1} \right)^2 \right) d\lambda_2 + \gamma \cos^2(\omega_0 t) d\lambda_2^2 \right]. \tag{60}
\]

We believe [54] that this metric could be dual to a non relativistic time dependent bulk theory via Maldacena’s AdS/CFT correspondence [55] in a same methodology as presented in [56].

V. MEASUREMENT \( \chi_F \) USING QUANTUM NOISE SETUPS

In recent years, the time-dependent systems phase transitions have been investigated in references [47] - [53]. In Ref. [47], the universal scaling behavior in a one-dimensional quantum Ising model subject to time-dependent sinusoidal modulation in time of its transverse magnetic field has been illustrated. This scaling behavior existed in various quantities, e.g. concurrence, entanglement entropy, magnetic and fidelity susceptibility. Based on an Ising spin chain and with periodically varying external magnetic field along the transverse direction the authors, in Ref. [48], investigated the microscopic quantum correlations dynamics of the bipartite entanglement and quantum discord.

In this section, we mainly focus on frequency spectrum of the quantum system resulting from the quantum noise function. Let us assume a generalized Hamiltonian \( H = H_0 + \lambda V \) with \( \lambda \) denoting the control parameter. The quantum noise spectrum of the driven Hamiltonian \( V \) can be defined as

\[
S_Q(\omega) = \sum_{n \neq 0} |\langle \phi_n | V | \phi_0 \rangle|^2 \delta(\omega - E_n + E_0), \tag{61}
\]

where \( |\phi_n\rangle \) is the eigenstate of the Hamiltonian \( H(\lambda) \) and we assumed \( E_n \) as non-degenerate energy levels of the whole system. Note that the quantum noise function \( S_Q(\omega) \) can be constructed from the excited states \( E_n > E_0 \). In our model, the ground state wavefunction is given in Eq. [49]. Here we can rewrite the noise function [61] using matrix element given in Eq. [36] as follows:

\[
S_Q(\omega) = |\lambda_2|^2|W_{12}^*|^2 + |W_{12}|^2 e^{2i\omega t} = 2|\lambda_2 W_{12}|^2 \cos^2 \omega t. \tag{62}
\]

We plot the noise function \( S_Q(\omega) \) versus time \( (t) \) and frequency \( (\omega) \) illustrated in Fig. [1].

As well known, the fidelity susceptibility plays an important role in QPTs stemming from the fact that it is always possible to describe the universality classes of QPTs without specifying the type of the symmetry of the system. However, it is adequate to ask whether we can measure \( \chi_F \) using experimental setups. It has been shown that recently the \( \xi_F \) is related to the quantum noise spectrum of the time-driven Hamiltonian [44]. It is remarkable to relate \( \xi_F \) to \( S_Q(\omega) \) using Kronig-Penney transformation:

\[
\chi_F = \int_{-\infty}^{\infty} d\omega \frac{S_Q(\omega)}{\omega^2}. \tag{63}
\]

Bearing in mind that the following definition of derivative of any analytic function \( f(z) \) provides a useful tool:

\[
f(z) : \mathbb{C} \to \mathbb{C}, \quad f^{(n)}(z) = \frac{\Gamma(n + 1)}{2\pi i} \int \frac{f(w)}{(w-z)^{n+1}} dw, \tag{64}
\]
where \( \Gamma(n + 1) = \int_0^{\infty} e^{-t^n} dt \) is a Gamma function. Using (64) we clearly observe that (65):

\[
\chi_F = \pi i \frac{d S_Q(\omega)}{d\omega^2} \mid_{\omega=0}.
\]

It is clearly stated that \( S_Q(\omega) \) can be measured in laboratory, see Ref.\[45\]. Consequently we verify that the \( \chi_F \) could be measured in the laboratory, as well. Particularly the Landau-Lifshitz model with \( \chi_F \) presented in Eqs.(54)-(56) provides a useful machinery to study the universal scaling behavior of \( \chi_F \).

### VI. \( \mathcal{O}(\delta \lambda)^3 \) MISSING TERM

In this section, we highlight higher order corrections up to the \( \mathcal{O}(\delta \lambda)^3 \) of \( \chi_F \). Let us consider a two-level quantum system where the system of equations is given by the following form:

\[
(H_0 + \lambda V(t)) \sum_k a_k(t) \psi_k^{(0)} = i \hbar \frac{\partial}{\partial t} \sum_k a_{k1}(t) \psi_k^{(0)}.
\]

From the above expression, we derive the basic equation:

\[
i \hbar \frac{da_m}{dt} = \sum_k V_{mk}(t) a_k.
\]

In order to proceed our analysis, here, we suppose that the system has only two non-degenerate levels namely \( k = 1, 2 \). In this case, the potential matrix is supposed to be

\[
V_{mk} = \lambda \begin{bmatrix} V(t) & W(t) \\ W^*(t) & V(t) \end{bmatrix}.
\]

From the potential matrix derived above, the set of differential equations are written as follows:

\[
\frac{da_1(t)}{dt} = -i \frac{\lambda}{\hbar} (V(t)a_1(t) + W(t)a_2(t)),
\]

\[
\frac{da_2(t)}{dt} = -i \frac{\lambda}{\hbar} (W^*(t)a_1(t) + V(t)a_2(t)).
\]

It is possible to reduce the system to an uncoupled second-order differential equations for \( a_1(t) \):

\[
\ddot{a}_1(t) - \dot{a}_1(t)p(t) - \frac{\lambda q(t)}{\hbar^2 W(t)} a_1(t) = 0,
\]
where
\[ p(t) = \frac{\hbar \dot{W}(t) - 2i\lambda V(t)W(t)}{\hbar W(t)}, \] (71)
and
\[ q(t) = W(t)(-\lambda |W(t)|^2 - i\hbar \dot{V}(t) + AV(t)^2) + i\hbar V(t)\dot{W}(t). \] (72)

Using the same formalism as previous sections, with \( \lambda \ll 1 \) being a smallness parameter, we expand the ground state wavefunction coefficient as \( a_1(t) = a(t) + \lambda b(t) + \lambda^2 c(t) \). After substituting it in the evolution equation, we obtain:

\[
O(\lambda^0) : \dot{a} + \partial_t \ln W(t) \dot{a} \approx 0, \tag{73}
\]
\[
O(\lambda^1) : \dot{b} - \partial_t \ln W(t) \dot{b} + \frac{ia}{\hbar} \left(2V(t) - V(t)\partial_t \ln W(t) + \dot{V}\right) \approx 0, \tag{74}
\]
\[
O(\lambda^2) : \dot{c} - \partial_t \ln W(t) \dot{c} + \frac{i}{\hbar} \left(b(\dot{V}(t) - \partial_t \ln W(t)V(t)) + 2V(t)\dot{b}\right) + \frac{a}{\hbar^2} \left(V(t)^2 + |W(t)|^2\right) \approx 0. \tag{75}
\]

Integrating the above equations yields the following solutions:

\[
a(t) = C_1 + C_2 \int W(t) \, dt, \tag{76}
\]
\[
b(t) = \int \left(-\int \frac{f(t)}{W(t)} \, dt + C_3\right) W(t) \, dt + C_4, \tag{77}
\]
\[
c(t) = \int \left(-\int \frac{g(t)}{W(t)} \, dt + C_5\right) W(t) \, dt + C_6, \tag{78}
\]

where
\[
f(t) = \frac{i\dot{V}(t) \left(C_1 + \int W(t) \, dt C_2\right)}{\hbar} - \frac{i\dot{W}(t)V(t) \left(C_1 + C_2 \int W(t) \, dt\right)}{\hbar W(t)} + \frac{2iW(t)V(t)C_2}{\hbar}, \tag{79}
\]
and
\[
g(t) = \frac{-i\dot{V}(t)}{\hbar} \int W(t) \left(\int \frac{f(t)}{W(t)} \, dt + C_3\right) \frac{\dot{W}(t)}{W(t)} \, dt \, dt + C_5 \frac{i\dot{V}(t)}{\hbar} \int \frac{f(t)}{W(t)} \, dt \, dt - \frac{2iW(t)V(t)}{\hbar} \int \frac{f(t)}{W(t)} \, dt \, dt + C_5 \frac{i\dot{V}(t)}{\hbar} \frac{W(t)}{W(t)} - \frac{iC_4 V(t)\dot{W}}{\hbar W(t)} + \frac{2iW(t)V(t)}{\hbar} \left(\int W(t) \, dt + \frac{C_1}{\hbar^2}\right) (W(t)^2 - V(t)^2). \tag{80}
\]

with \( C_i \)s being constants of integration. Using Eq. (26), we can compute the fidelity susceptibility
\[
\chi_F = \left[\frac{(\psi(\lambda)|\partial_\lambda \psi(\lambda))}{\langle \psi|\psi \rangle}\right]^2 + 2 \frac{\langle \psi(\lambda)|\partial_\lambda^2 \psi(\lambda) \rangle}{\langle \psi|\psi \rangle}, \tag{81}
\]
where
\[
|\psi\rangle = (a + \lambda b + \lambda^2 c)|\psi_1^{(0)}\rangle + W^{-1}(t) \left[\frac{i\hbar}{\lambda}(\dot{a} + \lambda \dot{b} + \lambda^2 \dot{c}) - V(t)(a + \lambda b + \lambda^2 c)\right]|\psi_2^{(0)}\rangle. \tag{82}
\]

Note that
\[
\langle \psi|\psi \rangle = |a + \lambda b + \lambda^2 c|^2 + |W(t)|^{-2} \left[\frac{i\hbar}{\lambda}(\dot{a} + \lambda \dot{b} + \lambda^2 \dot{c}) - V(t)(a + \lambda b + \lambda^2 c)^2\right], \tag{83}
\]
\[
\langle \psi|\partial_\lambda \psi \rangle = (a^* + \lambda b^* + \lambda^2 c^*)(b + 2\lambda c) + |W(t)|^{-2} \left[\frac{i\hbar}{\lambda}(\dot{a} + \lambda \dot{b} + \lambda^2 \dot{c}) - V(t)(a + \lambda b + \lambda^2 c)\right] \times \left[i\hbar(-\lambda^{-2} \dot{a} + \dot{c}) - V(t)(b + 2\lambda c)\right]. \tag{84}
\]
The expression given in Eq. (81) is the usual fidelity susceptibility, where the coefficient of $\delta \lambda^2$ is hidden. It is noteworthy to figure out higher order terms, i.e., the coefficient of $\delta \lambda^2$ using the expressions given above. Remember that

\[ |\psi(\lambda + \delta \lambda)| = |\psi(\lambda)| + \delta \lambda \partial_\lambda |\psi(\lambda)| + \frac{\delta \lambda^2}{2} \partial_\lambda^2 |\psi(\lambda)|. \]  

Let us compute the following inner product:

\[ \langle \psi(\lambda)|\psi(\lambda + \delta \lambda) \rangle \approx \langle \psi(\lambda)|\psi(\lambda) \rangle + \delta \lambda \langle \psi(\lambda)|\partial_\lambda \psi(\lambda) \rangle + \delta \lambda^2 \langle \psi(\lambda)|\partial_\lambda^2 \psi(\lambda) \rangle + ... , \]

where the ellipses denote higher order (correction) terms. Consequently, we obtain the following expression:

\[ \left| \frac{\langle \psi(\lambda)|\psi(\lambda + \delta \lambda) \rangle}{\langle \psi(\lambda)|\psi(\lambda) \rangle} \right| \approx 1 + \delta \lambda \left( \frac{\langle \psi(\lambda)|\partial_\lambda \psi(\lambda) \rangle}{\langle \psi|\psi \rangle} \right) + \delta \lambda^2 \left( \frac{\langle \psi(\lambda)|\partial_\lambda^2 \psi(\lambda) \rangle}{\langle \psi|\psi \rangle} \right) \]

\[ \approx 1 + \delta \lambda \left( \frac{\langle \psi(\lambda)|\partial_\lambda \psi(\lambda) \rangle}{\langle \psi|\psi \rangle} \right) + \frac{\delta \lambda^2}{2} \left( \left[ \frac{\langle \psi(\lambda)|\partial_\lambda \psi(\lambda) \rangle}{\langle \psi|\psi \rangle} \right]^2 + 2 \frac{\langle \psi(\lambda)|\partial_\lambda^2 \psi(\lambda) \rangle}{\langle \psi|\psi \rangle} \right) \]

Next we can define the third-order fidelity susceptibility as follows:

\[ \zeta_F = \frac{\langle \psi(\lambda)|\partial_\lambda \psi(\lambda) \rangle \langle \psi(\lambda)|\partial_\lambda^2 \psi(\lambda) \rangle}{\langle \psi|\psi \rangle^2}. \]  

Now using Eqs. (82-84) we obtain:

\[ \langle \psi|\partial_\lambda^2 \psi \rangle = 2\lambda c(\alpha(t) + \lambda b + \lambda^2 c) + |W(t)|^{-2} \left[ \frac{i\hbar}{\lambda} \left( \hat{a} + \lambda \hat{b} + \lambda^2 \hat{c} \right) - V(t)(\alpha + \lambda b + \lambda^2 c) \right] \times \]

\[ \times \left[ 2\hbar \lambda^{-3} \hat{a} - 2V(t) \hat{c} \right]. \]  

The above equation defines a higher order correction to the usual fidelity susceptibility. The corresponding metric is a Finsler manifold in which the general information metric is characterized by the following form:

\[ ds^2 = \chi_{ij} d\lambda_i d\lambda_j + (\zeta_{ijk} d\lambda_i d\lambda_j d\lambda_k)^{\frac{3}{2}} + ... . \]  

It is worth noting that the distant between two quantum states in any quantum theory can be quantified not only by fidelity but also with higher order cubic quantity defined by $\zeta_F$. We note here that the corresponding tensor form for $\zeta_F$ is given by:

\[ (\zeta_F)_{ijk} = \frac{\langle \psi(\lambda)|\partial_\lambda \psi(\lambda) \rangle \langle \psi(\lambda)|\partial_\lambda \partial_\lambda \psi(\lambda) \rangle}{\langle \psi|\psi \rangle^2}. \]  

It will be very interesting to find bulk dual for this new tensor in a similar way recently suggested for fidelity susceptibility as a maximal volume in the AdS spacetime \[56\].

VII. SUMMARY

In this work, we have presented a simple and straightforward approach to compute distance between quantum states responsible for the fidelity susceptibility in quantum model for a time-dependent system describing a two-level atom coupled to a time-driven external field. Analytically we have investigated the behavior of fidelity susceptibility in the time-driven quantum model in which the potential $V$ is time-dependent. Interestingly, the information metric induced by fidelity susceptibility can be nicely achieved. We also plotted the obtained noise function and found that the noise function eternally lasts long in our model. We have also derived for the first time a higher-order rank-3 tensor as third-order fidelity susceptibility for having a model beyond fidelity susceptibility.

It will be very interesting to find bulk dual for this new tensor in a similar way recently suggested for fidelity susceptibility as a maximal volume in the AdS spacetime \[56\]. Moreover, as mentioned in Refs. \[57\] \[58\], our understanding of quantum gravity may be satisfied using quantum information theory along with holography. This may allow us to further examine a possible connection between the
fidelity susceptibility and holographic complexity and may shed new light on the deeper understanding of quantum gravity.

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