Is the Heisenberg uncertainty relation really violated?

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I. INTRODUCTION

The uncertainty relation for quantum states is that the fluctuations for the canonically conjugate observables $\hat{q}$ and $\hat{p}$ must satisfy the relation $\sigma(\hat{q})\sigma(\hat{p}) \geq \hbar/2$. This can easily be proven using the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$. It is closely connected with the complementarity or the wave-particle duality of quantum states. Practically it places limitations in the preparation of wave packets.

On the other hand, the uncertainty relation that was discussed by Heisenberg earlier in terms of the hypothetical gamma-ray microscope is for the quantum measurement. Namely, if one measures an observable $\hat{q}$ with a given accuracy (or with error) $\epsilon$, then the conjugate observable $\hat{p}$ necessarily suffers the disturbance $\eta$, which satisfies the inequality

$$\epsilon \eta \geq \frac{\hbar}{2}$$

The Heisenberg inequality can be derived by utilizing a quantum mechanical model of measurement processes. As the first step of measurement the unitary interaction between the object and a probe of measuring apparatus is considered. The probe is a part of measuring apparatus and works as a front end that interacts with the object quantum-mechanically. After the interaction, the probe variable $Q$, which contains some information on the object variable $\hat{q}$, is read out by the other part of the measuring apparatus and is fixed as a classical value. This indirect measurement model is helpful to discuss the error and disturbance associated with measurement.

It has been pointed out that for some types of interactions the Heisenberg inequality seems to be violated. For example, in the contractive measurement, $\epsilon = 0$ and $\eta < \infty$ are satisfied simultaneously and the inequality is violated; $\epsilon \eta = 0$. The other extreme case of $\eta = 0$ and $\epsilon < \infty$ is also possible.

In order to reconcile with the above cases, a new uncertainty relation was proposed by Ozawa:

$$\epsilon \eta + \epsilon \sigma(\hat{p}) + \sigma(\hat{q}) \eta \geq \frac{\hbar}{2}$$

The Ozawa inequality is attracting considerable interests because it admits the cases that violate the limit posed by the Heisenberg inequality. It might be possible to devise a measurement scheme which is free from the Heisenberg uncertainty principle. Unfortunately, however, no systematic experiments that demonstrate the violation of the Heisenberg limit or that make some use of the new bound have been performed so far.

From experimentalists’ view point, the definition of the noise and disturbance operators, whose expectation values give the error and disturbance, is the first obstacle. Each of them contains two observables, one for before and the other for after the interaction. Moreover because they are non-commutable, the separate measurements are of no use and a joint measurement seems required essentially. However, this difficulty is only an artifact associated with the use of the Heisenberg picture in the theory. It will be shown that the use of Schrödinger picture gives more simple perspective. With the probability distributions of corresponding observable, separately measured before and after the interaction, we can examine the error and disturbance.

Yet there seems another problem in the definition of the noise and disturbance operators. In the derivation of the Ozawa inequality, a broad class of measurement interaction is assumed. On the other hand, the definition of the noise and disturbance operators are just borrowed from the case of ideal measurement. In the case of general interactions, we have to consider the amplification (or deamplification) of variables due to the unitary transformation. The gain, which is unity for the ideal measurement, must be taken into account. We will show that redefining the error and disturbance operators properly, the violation of the Heisenberg inequality is canceled for a broad class of interaction.

A related problem in the discussion of violation of the
The Heisenberg inequality is that the following is admitted unconditionally: the finiteness of disturbance \( \eta \) is implied from the finite standard deviation \( \sigma(\hat{p}') \) of the object momentum after the interaction. Even for the case of finite standard deviation, if the distribution of \( \hat{p}' \) is completely uncorrelated with the original distribution of \( \hat{p} \), then the disturbance should be considered infinitely large so as to destroy the information completely. Similarly the finiteness of the variance \( \sigma(Q') \) of the probe position after the interaction does not imply the finite error \( \epsilon \). The error should be considered infinite if the information on \( \hat{q} \) is not transferred to \( Q' \) at all. These claims will be confirmed by examining the change of probability distributions with the Schrödinger picture (Sec. VII).

II. THE UNCERTAINTY RELATION FOR THE STANDARD MODEL

The measurement process can be described with the object system to be measured and the probe. The probe is the front end of the measuring device and is assumed to be treated quantum-mechanically. The measured observable (position) is denoted by \( \hat{q} \) and its conjugate observable (momentum) by \( \hat{p} \). For the probe, the conjugate observables \( \hat{Q} \) and \( \hat{P} \) are introduced. We assume, \( [\hat{q}, \hat{p}] = [\hat{Q}, \hat{P}] = i\hbar \) and the eigenkets for \( \hat{q} \) and \( \hat{Q} \) are defined with \( \hat{q}|q\rangle = q|q\rangle \) and \( \hat{Q}|Q\rangle = Q|Q\rangle \), respectively.

The probe is prepared in a fixed known state \( |\psi\rangle \). This part can be modeled with von Neumann type (projection) measurement.

For the moment, we assume that the unitary operator \( \hat{U} \) satisfies the relation \( \hat{Q}' = \hat{U}^\dagger \hat{Q} \hat{U} = \hat{Q} + \hat{q}, \hat{p}' = \hat{U}^\dagger \hat{p} \hat{U} = \hat{p} - \hat{P} \) (3) where \( \hat{Q}' \) and \( \hat{p}' \) are the quantities after the interaction (in the Heisenberg picture). This interaction corresponds to the case of ideal measurement.

Rewriting (3) as

\[ \hat{Q}' = \hat{q} + \hat{E}, \quad \hat{p}' = \hat{p} + \hat{D}, \]

we find the definition of two operators

\[ \hat{E} := \hat{Q}' - \hat{q} (= \hat{Q}), \quad \hat{D} := \hat{p}' - \hat{p} (= - \hat{P}). \]

The former operator corresponds to the accuracy or the error added to \( \hat{q} \) and the latter operator corresponds to the disturbance against \( \hat{p} \).

The second-order moments of \( \hat{E} \) and \( \hat{D} \) for an initial state \( |\psi_{\text{tot}}\rangle = |\psi\rangle |\Psi\rangle \) give the error \( \epsilon \) and \( \eta \) as

\[ \epsilon := (\hat{E}^2)^{1/2} = \sigma(\hat{Q}), \quad \eta := (\hat{D}^2)^{1/2} = \sigma(\hat{P}), \]

respectively. For an operator \( \hat{A} \), \( \langle \hat{A} \rangle \) represents the expectation value and \( \sigma(\hat{A}) = (\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2)^{1/2} \) is the standard deviation with respect to a given state. Hereafter, for simplicity, we assume that the initial state \( |\Psi\rangle \) of probe satisfies the conditions \( \langle \hat{Q} \rangle = \langle \hat{P} \rangle = 0 \). Then the equalities hold in Eq. (6), from which Eq. (6) we have the Heisenberg uncertainty relation (HUR)

\[ \epsilon \eta \geq \sigma(\hat{Q}) \sigma(\hat{P}) \geq \frac{\hbar}{2} \]

for the indirect measurements with ideal interaction (3).

III. INTERACTION FOR MEASUREMENT OF CONTINUOUS VARIABLES

The interaction (3) is for the ideal measurement of a continuous variable. Using the eigenstates for the positions of object and probe, the action of \( \hat{U} \) can be written as

\[ \hat{U} : |q\rangle|Q\rangle \rightarrow |q\rangle|Q + q\rangle. \]

The probe position \( \hat{Q} \) is deflected by the variable \( q \) to be measured, while \( \hat{q} \) itself is not affected by the interaction.

Here we introduce a generalized form of interaction \( \hat{U} \): for \( a, b, c, d \in \mathbb{R} \), it is defined as

\[ \hat{U} : |q\rangle|Q\rangle \rightarrow \sqrt{\Delta} |dq + cQ\rangle |aQ + bq\rangle. \]

The positions \( \hat{q}, \hat{Q} \) are linearly transformed through the interaction (20). The factor \( \sqrt{\Delta} \) with \( \Delta = ad - bc > 0 \) is determined from the unitary condition of \( \hat{U} \) (Appendix).

With this unitary transformation, the variables for the object and probe are transformed

\[ \begin{bmatrix} \hat{Q}' \\ \hat{q}' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \hat{Q} \\ \hat{q} \end{bmatrix}, \quad \begin{bmatrix} \hat{p}' \\ \hat{P} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} \hat{p} \\ \hat{P} \end{bmatrix}, \]

Corresponding to Eq. (3), the portions related to the uncertainty relation are

\[ \hat{Q}' = a\hat{Q} + b\hat{q}, \quad \hat{p}' = a'\hat{p} - b'\hat{P}, \]

where \( a' = a/\Delta, \quad b' = b/\Delta \). The parameters \( a, b, a', b' \) are relevant to the uncertainty relation. We can assume \( a \geq 0 \) and \( b \geq 0 \) as discussed in Appendix. The parameter \( b \) corresponds to the gain from \( \hat{q} \) to \( \hat{Q}' \) and \( a' \) that from \( \hat{p} \) to \( \hat{p}' \).

IV. STANDARD FORMS OF INTERACTION

We can classify the unitary transformation (10) into three types and associate a standard form to each class.

Before and after the interaction, we apply the following scale transformations

\[ \hat{Q} \rightarrow A \hat{Q}, \quad \hat{P} \rightarrow A^{-1} \hat{P}, \quad \hat{q} \rightarrow \lambda \hat{q}, \quad \hat{p} \rightarrow \lambda^{-1} \hat{p}, \]

\[ \hat{q}' \rightarrow \mu^{-1} \hat{q}, \quad \hat{p}' \rightarrow \mu \hat{p}, \]  

where \( A, \lambda, \) and \( \mu \) are non-zero, real constants. Then the coefficient matrix transforms as

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda a & \lambda b \\ \mu c & \mu d \end{bmatrix}. \] 

(13)

In the case of \( ab \neq 0 \), by setting \( A = 1/a, \lambda = 1/b, \mu = ab/\Delta \), we can simplify the matrices as

\[ \begin{bmatrix} \hat{Q}' \\ \hat{q}' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ b/c & a'd \end{bmatrix} \begin{bmatrix} \hat{Q} \\ \hat{q} \end{bmatrix}, \quad \begin{bmatrix} \hat{P}' \\ \hat{p}' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -b/c & a'd \end{bmatrix} \begin{bmatrix} \hat{P} \\ \hat{p} \end{bmatrix}. \] 

(14)

In the case \( a = 0, b \neq 0 \), we can set \( \lambda = 1/b, A = -1/c, \mu = 1 \), to obtain

\[ \begin{bmatrix} \hat{Q}' \\ \hat{q}' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & d/b \end{bmatrix} \begin{bmatrix} \hat{Q} \\ \hat{q} \end{bmatrix}, \quad \begin{bmatrix} \hat{P}' \\ \hat{p}' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & d/b \end{bmatrix} \begin{bmatrix} \hat{P} \\ \hat{p} \end{bmatrix}. \] 

(15)

Similarly in the case \( a \neq 0, b = 0 \), by setting \( \lambda = 1/d, A = 1/a, \mu = 1 \), we have

\[ \begin{bmatrix} \hat{Q}' \\ \hat{q}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} \hat{Q} \\ \hat{q} \end{bmatrix}, \quad \begin{bmatrix} \hat{P}' \\ \hat{p}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -c/a & 1 \end{bmatrix} \begin{bmatrix} \hat{P} \\ \hat{p} \end{bmatrix}. \] 

(16)

Now we have the three standard forms for measurement interaction;

\[ \begin{align*}
&\text{(O)} \quad \hat{Q}' = \hat{Q} + \hat{q}, \quad \hat{p}' = \hat{p} - \hat{P}, \\
&\text{(A)} \quad \hat{Q}' = \hat{q}, \quad \hat{p}' = -\hat{P}, \\
&\text{(B)} \quad \hat{Q}' = \hat{Q}, \quad \hat{p}' = \hat{p}
\end{align*} \] 

(17)

For Type (B), we have

\[ \hat{E} = \hat{Q} - \hat{q}, \quad \hat{D} = 0, \]

\[ \epsilon \geq \sqrt{\sigma^2(\hat{Q}) + \sigma^2(\hat{q})}, \quad \eta = 0. \] 

(20)

The conjugate variable is conserved: \( \hat{p}' = \hat{p} \), therefore, no disturbances occur, while the error can be finite. This type of interaction is singular in the sense that the probe variable \( \hat{Q}' \) does not depend on the variable \( \hat{q} \) to be measured. The case of no-interaction (\( c = 0 \)) is contained in this type. The case of ideal measurement for the object’s momentum \( (c/a = -1) \) is also contained.

V. OZAWA’S INEQUALITY

We have seen that the Heisenberg uncertainty relation holds for Type (O) interaction but seems broken for Types (A) and (B), actually, \( \epsilon \eta = 0 \) for these cases.

A new inequality which is valid for all types has been proposed by Ozawa [10]:

\[ \epsilon \eta + \epsilon \sigma(\hat{p}) + \sigma(\hat{q}) \eta \geq \frac{\hbar}{2} \] 

(21)

The Ozawa uncertainty relation (OUR) includes the standard deviations \( \sigma(\hat{q}) \) and \( \sigma(\hat{p}) \) of the initial object state in addition to \( \epsilon \) and \( \eta \).

For Type (A), OUR sets a finite lower bound of disturbance as \( \eta \geq (\hbar/2)\sigma(\hat{q})^{-1} \geq \sigma(\hat{p}) \) despite of \( \epsilon = 0 \). For type (B), the finite error \( \epsilon \geq \sigma(\hat{q}) \) for \( \eta = 0 \).

We explore the relation between HUR and OUR. Assuming the minimum uncertainty for the initial state of object; \( \sigma(\hat{q}) \sigma(\hat{p}) = \hbar/2 \), we can introduce normalized variables \( \bar{\epsilon} = \epsilon/\sigma(\hat{q}) \) and \( \bar{\eta} = \eta/\sigma(\hat{p}) \). The normalization gives

\[ \bar{\epsilon} \bar{\eta} \geq 1 \quad \text{(HUR)}, \]

\[ \bar{\epsilon} \bar{\eta} + \bar{\epsilon} + \bar{\eta} \geq 1 \quad \text{(OUR)}. \] 

(22)

(23)

As shown in Fig. 1 there is an appreciable gap between the two bounds and the bound of OUR always violates HUR.

Using Eq. (5), the error and disturbance operators \( \hat{E} \) and \( \hat{D} \) for the general interaction \( (d/b) \) are

\[ \hat{E} = a\hat{Q} + (b - 1)\hat{q}, \quad \hat{D} = (a' - 1)\hat{p} - b'\hat{P}. \] 

(24)

The expectation values of \( \hat{E}^2 \) and \( \hat{D}^2 \) for an initial state are

\[ \epsilon^2 = \langle \hat{E}^2 \rangle = a^2\sigma^2(\hat{Q}) + (b - 1)^2\sigma^2(\hat{q}) \]

\[ \eta^2 = \langle \hat{D}^2 \rangle = (a' - 1)^2\sigma^2(\hat{p}) + b'^2\sigma^2(\hat{P}). \] 

(25)

For simplicity we have assumed \( \langle \hat{q} \rangle = \langle \hat{p} \rangle = 0 \). (By introducing new variables: \( \hat{q}_1 = \hat{q} - \langle \hat{q} \rangle, \hat{Q}' = \hat{Q} - a'\langle \hat{q} \rangle, \hat{p}_1 = \hat{p} - \langle \hat{p} \rangle, \hat{p}'_1 = \hat{p}' - d'\langle \hat{p} \rangle \), we can remove the mean values.)
The Ozawa inequality has been introduced as a remedy against the violation of the Heisenberg inequality in the cases of (A) and (B), which represent somewhat singular interactions.

We propose another method of remedy for the violation of the Heisenberg uncertainty relation. We replace the definition of noise and disturbance operators considering the gain of the interaction.

We use the general (unscaled) interaction (11) with the parameters \(a, b,\) and \(\Delta\). For the moment, \(ab \neq 0\) is assumed. Equation (11) can be rewritten as

\[
\hat{Q}' = b(\hat{q} + \hat{E}_s), \quad \hat{p}' = a'(\hat{p} + \hat{D}_s),
\]

where \(b\) can be considered as the gain with which the input \(\hat{q}\) is amplified to generate the output \(\hat{Q}'\). Similarly \(a' = a/\Delta\) is the gain from \(\hat{p}\) to \(\hat{p}'\). Here, we have replaced the definition of \(\hat{E}\) and \(\hat{D}\) by

\[
\hat{E}_s := \frac{1}{b} \hat{Q}' - \hat{q} = \frac{a}{b} \hat{Q},
\]

\[
\hat{D}_s := \frac{1}{a'} \hat{p}' - \hat{p} = -\frac{b'}{a} \hat{P} = -\frac{b}{a} \hat{P},
\]

each of which corresponds to the input-referred noise (Figs. 2 and 3).

In the reference [9], the gain of phase-sensitive amplification in backaction-evading interaction is properly incorporated for the definition of error [Eq. (21a)] and disturbance [Eq. (21c)], but for other cases the gain seems ignored [for example Eq. (23c)].
that for of a in Eq. (31) is questionable because they diverge despite variable we can safely use the Heisenberg inequality.  

Now we have found that the redefined error and disturbance satisfy the Heisenberg inequality for \( ab \neq 0 \).

The case \( a = 0 \) can be considered as the limiting case of \( a \to 0 \) with \( \Delta \) kept constant. From Eq. (31), we see that for \( \epsilon_s \to 0 \), the uncertainty product \( \epsilon_s \eta_s \) is conserved and Eq. (32) is always satisfied. The case of \( b = 0 \) can be treated similarly as the limit of \( b \to 0 \).

Thus, revising the definition of error and disturbance appropriately, we can defend the Heisenberg uncertainty relation for general class of measurement interaction. At least, for the linear type interaction of continuous variable we can safely use the Heisenberg inequality.

**VII. UNCERTAINTY IN PROBABILITY DISTRIBUTION**

One may argue that the new definition of \( \epsilon_s \) and \( \eta_s \) in Eq. (31) is questionable because they diverge despite of the finiteness of \( \sigma(\hat{Q}') \) and \( \sigma(\hat{P}') \). For example, in the limit of \( b \to 0 \), the standard deviation of \( \hat{Q} \) is finite but \( \epsilon_s \) diverges. Therefore, \( \epsilon_s \) which stays finite as seen in Eq. (24) seems more appropriate. We will show that this intuitive argument is not correct.

In the uncertainty relations, the second-order moments are used as the quantitative measures of uncertainties. In stead here we will use the probability distribution of each variable in order to accurately examine the effect of interaction.

**A. Ideal measurement case**

First we study the case of ideal measurement, i.e., Type (O). The initial state for the total system is

\[
|\Psi_{\text{tot}}\rangle = |\psi\rangle \otimes |\Phi\rangle = |\psi\rangle|\Phi\rangle,
\]

where \( |\psi\rangle \) and \( |\Phi\rangle \) are the initial states for the object and probe, respectively. In the Schrödinger picture, the measurement operator \( \hat{U} \) brings the state into

\[
|\Psi'_{\text{tot}}\rangle = \hat{U}(|\psi\rangle|\Phi\rangle).
\]

We denote the initial probability distributions for \( \hat{q} \), \( \hat{Q} \), \( \hat{p} \), and \( \hat{P} \) respectively as

\[
f(q) := |\langle q|\psi\rangle|^2 = |\psi(q)|^2, \\
F(Q) := |\langle Q|\Psi\rangle|^2 = |\Psi(Q)|^2, \\
g(p) := |\langle p|\Phi\rangle|^2 = |\Phi(p)|^2, \\
G(-P) := |\langle P|\Psi\rangle|^2 = |\Psi(P)|^2.
\]

The probability distribution for \( \hat{Q} \) (after the interaction) is

\[
F'(Q) = \int_{-\infty}^{\infty} |\Psi'_{\text{tot}}(q, Q)|^2 dq,
\]

where

\[
|\Psi'_{\text{tot}}(q, Q)\rangle = \langle q|\langle Q|\hat{U}|\psi\rangle|\Phi\rangle\rangle = \langle q|\langle Q|\hat{U}|\Psi\rangle\rangle = \langle q|\langle Q|\Psi_{\text{tot}}\rangle\rangle
\]

\[
= \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dQ' \delta(q-Q') \langle q'|\langle Q'|\Psi_{\text{tot}}\rangle\rangle
\]

\[
= \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dQ' \delta(q-q') \delta(Q-Q') \psi(q') \Psi(Q') = \psi(q) \Psi(Q-q)
\]

is the wavefunction of the entire system. Now we have the change of the distribution as

\[
F'(Q) = \int_{-\infty}^{\infty} dq f(q) F(Q-q) = (f * F)(Q),
\]

where \( (f * g)(x) = \int_{-\infty}^{\infty} dy f(y) g(x-y) \) represents the convolution integral.

The distribution of probe position \( \hat{Q}' \) after the interaction, \( F'(Q) \), is a convolution of the initial distribution of
the object position \( f(q) = |\psi(q)|^2 \) with that of the probe position \( F(Q) = |\Psi(Q)|^2 \).

If \( F(Q) \) is a sharp function (close to the delta function), then the initial distribution \( f(q) \) can be reproduced. On the other hand, if \( F(Q) \) is a broad function, the distribution is blurred. Namely, the width of \( F(Q) \) determines the accuracy of the measurement.

The distribution of object momentum \( \hat{p}' \) after the interaction is

\[
g'(p) = \int_{-\infty}^{\infty} |\Psi_{\text{tot}}(p, P)|^2 dP, \tag{39}\]

where

\[
\Phi_{\text{tot}}(p, P) = \langle p|(P)|\Phi_{\text{tot}}' \rangle = \langle p|(P)\tilde{U}|\Psi_{\text{tot}} \rangle
= \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dp'' \langle p|\tilde{U}|p'' \rangle \langle p''|(P')|\Psi_{\text{tot}} \rangle
= \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dp'' \delta(p - p' + P')\delta(P - P')\phi(p')\Phi(P')
= \phi(p + P)\Phi(P) \tag{40}\]

is the wavefunction represented by the momentum basis. Thus, we have

\[
g'(p) = \int_{-\infty}^{\infty} dP g(p + P)G(-P) = (g * G)(p), \tag{41}\]

which is essentially the convolution of the initial distribution \( g(p) \) with the distribution \( G(P) \) of the probe momentum \( \tilde{P} \). For a narrow \( G(P) \), the distribution \( g(p) \) is conserved. On the other hand, for a wide \( G(P) \), \( g(p) \) is destructed. The width of \( G(P) \) corresponds to the strength of disturbance.

Because \( \Psi(Q) \) and \( \Phi(P) \) are the Fourier-transform pair:

\[
\Phi(P) = \langle P|\Psi \rangle = \int_{-\infty}^{\infty} dQ \langle Q|\Psi \rangle \langle Q|P \rangle
= \frac{1}{\sqrt{2\pi}h} \int_{-\infty}^{\infty} dQ \Psi(Q) e^{-iQ/P}, \tag{42}\]

it is impossible to reduce the widths of \( F(Q) = |\Psi(Q)|^2 \) and \( G(-P) = |\Phi(P)|^2 \), simultaneously.

We represent the first order moment and the variance of a distribution \( f(\cdot) \) as

\[
m(f) = \int_{-\infty}^{\infty} qf(q) dq,
\]

\[
\sigma^2(f) = \int_{-\infty}^{\infty} (q - m(f))^2 f(q) dq, \tag{43}\]

respectively. Using \( \sigma^2(f * g) = \sigma^2(f) + \sigma^2(g) \), from Eqs. (43) and (41), we obtain

\[
\sigma^2(F') = \sigma^2(f) + \sigma^2(F),
\]

\[
\sigma^2(g') = \sigma^2(g) + \sigma^2(G). \tag{44}\]

These increments in the variances correspond to \( \epsilon^2 \) and \( \eta^2 \) and we have again the uncertainty relation

\[
\epsilon^2 \eta^2 = [\sigma^2(F') - \sigma^2(f)] [\sigma^2(g') - \sigma^2(g)]
= \sigma^2(F)\sigma^2(G) \geq \frac{\hbar^2}{4}. \tag{45}\]

for the ideal measurement.

**B. General cases**

For the general transformation \( F' \), the probability distributions of \( Q' \) and \( \hat{p}' \) are

\[
F'(Q) = \frac{1}{\Delta} \int_{-\infty}^{\infty} f(-\epsilon'Q + a'q)F(d'Q - b'q)dq,
\]

\[
g'(p) = \Delta \int_{-\infty}^{\infty} g(dp + bP)G(-cp - aP)dP. \tag{46}\]

In the case of \( ab \neq 0 \), these can be written as

\[
F'(Q) = (f_{1/\alpha} * f_{1/\omega})_{1/\Delta}(Q),
\]

\[
g'(p) = (g_{1/\alpha} * G_{1/\Delta})(p), \tag{47}\]

where we define

\[
f_{k}(x) := kf(kx), \tag{48}\]

for a function \( f(x) \) and a real constant \( k > 0 \). The graph of the function \( f_{k}(\cdot) \) can be obtained from that of \( f(\cdot) \) by stretching horizontally by factor \( 1/k \) and vertically by factor \( k \). We note that the area is conserved;

\[
\int_{-\infty}^{\infty} f_{k}(x) dx = \int_{-\infty}^{\infty} f(x) dx, \tag{49}\]

and also note \( m(f_{k}) = (1/k)m(f) \) and \( \sigma^{2}(f_{k}) = (1/k^{2})\sigma^{2}(f) \). The variances of Eq. (44) are

\[
\sigma^{2}(F') = b^{2} \left( a^{2}(f) + \frac{a^{2}}{b^{2}}\sigma^{2}(F) \right),
\]

\[
\sigma^{2}(g') = a^{2} \left( \sigma^{2}(g) + \frac{b^{2}}{a^{2}}\sigma^{2}(G) \right). \tag{50}\]

from which we can regard the error and disturbance as

\[
\epsilon^{2}_{*} \eta^{2}_{*} = \left( \frac{a}{b} \right)^{2} \sigma^{2}(F), \quad \eta^{2}_{*} = \left( \frac{b}{a} \right)^{2} \sigma^{2}(G). \tag{51}\]

These are consistent with Eq. (11). Again we have the uncertainty relation for the case of \( ab \neq 0 \) as

\[
\epsilon_{*} \eta_{*} = \sigma(F)\sigma(G) = \sigma(\hat{Q})\sigma(\hat{P}) \geq \frac{\hbar}{2}. \tag{52}\]

For the case of \( a = 0 \) or \( b = 0 \), Eq. (10) simply becomes

\[
(a = 0) \quad F'(Q) = f(Q), \quad g'(p) = G(p),
(b = 0) \quad F'(Q) = F(Q), \quad g'(p) = g(p). \tag{53}\]
For the case \( a = 0 \), the distribution of \( \hat{q} \) is faithfully transferred to that for \( \hat{Q}' \) and therefore no errors creep in; \( \epsilon_* = 0 \). On the other hand, the distribution of \( \hat{p}' \) is replaced with that of \( \hat{P} \), which contains no information on \( \hat{p} \). This situation can be considered that the information is completely destroyed with infinitely large disturbance, \( \eta_* = \infty \).

For the case \( b = 0 \), we have infinitely large error: \( \epsilon_* = \infty \), because no information is transferred from \( \hat{q} \) to \( \hat{Q}' \). On the other hand, no disturbances is applied, \( \eta_* = 0 \), because \( \hat{p} \) is conserved.

### C. Infinitely large error and disturbance

In order to clarify further the meaning of infinitely large errors and disturbances, let us consider the case \( a = 0 \) as a limit of \( a \to 0 \) for Eq. \( (47) \).

First we remember that the delta function can be defined as a limit of parameterized functions

\[
f_{1/a}(x) = \frac{1}{a} f(x/a) \to \delta(x) \quad (a \to 0)
\]

for an arbitrary function \( f(x) \) with unit area.

For \( b = 1 \), \( a \to 0 \), the first equation of \( (47) \) becomes

\[
F'(Q) = (f_\Delta * F_{\Delta/a})_{1/a}(Q) = (f * F_{1/a})(Q)
\]

\[
\to (f * \delta)(Q) = f(Q) \quad (a \to 0).
\]

We confirm that \( F'(Q) \) is an exact copy of \( f(q) \) and the error-free \( (\epsilon_* = 0) \) measurement is achieved.

The second equation of \( (47) \) with \( b = 1 \) approaches

\[
g'(p) = (g_{1/a} * G)_\Delta(p) \to (\delta * G)_\Delta(p) = G_\Delta(p) \quad (a \to 0),
\]

where the original distribution \( g(p) \) approaches the delta function and its shape is lost completely. This fact justifies the infinitely large disturbance \( (\eta_* = \infty) \) despite of the finite width of \( g'(p) \).

For another way of understanding, Eq. \( (56) \) can be rewritten as

\[
g'(p) = (g * G_a)_{1/a}(p) \to G_\Delta(p) \quad (a \to 0).
\]

This equation can be understood as follows: the original distribution \( g(\cdot) \) is convoluted with \( G_a(\cdot) \), whose width is infinitely large or scales as \( 1/a \). The resultant distribution \( (g * G_a)(\cdot) \) has infinitely large width. However, the rescaling with \( 1/a' \) results in the finite-width distribution \( G_\Delta(\cdot) \). This equation helps to remove the common misunderstanding that the finite width implies the finite disturbance or equivalently that the infinite disturbance implies the infinite width.

Thus, by considering the cases \( a = 0 \) and \( b = 0 \) as limits of cases \( ab \neq 0 \), we have confirmed that the Heisenberg uncertainty relation \( (32) \) holds also in these cases.

### VIII. CONCLUSION

If we borrow the definitions of the error and disturbance from the ideal measurement case, the Heisenberg inequality is apparently violated for generalized measurements. A loosened bound posed by Ozawa is one of the methods for remedy. Here we have revised the definition of the error and disturbance considering the gain of generalized measurement interaction. With these new measures, the validity of the Heisenberg inequality is recovered.

Examining the changes of distribution functions caused by the general measurement interaction, the physical meanings of infinitely large errors and disturbances are clarified.

We note that infinitely small, or no errors or disturbances associated with finite widths, which are just the dual cases, are normally admitted.

In conclusion, with the proper definition for the error and disturbance, we may not need to discard the Heisenberg inequality for the measurement at least within the framework of the continuous linear interactions.

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### APPENDIX: INTERACTION FOR MEASUREMENT

We consider the interaction \( \hat{U} \) which provides the linear relation between the object and probe positions. It transforms the basis ket \( \lvert \psi \rangle \langle Q \rangle \) as

\[
\hat{U} : \lvert \psi \rangle \langle Q \rangle \mapsto \lvert \psi' \rangle \langle Q' \rangle = \omega \lvert dq + cQ \rangle \langle aQ + bq \rangle,
\]

where \( a, b, c, d \in \mathbb{R} \) are constant and \( \omega \) will be determined from the unitary condition.

In the position basis, \( \hat{U} \) is represented as

\[
\hat{U} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dq' dQ dq' dQ' dQ dQ \\
\int U(q', Q', q, Q) \lvert q' \rangle \langle Q' \rangle |q \rangle \langle Q |.
\]

(A.2)

In order to satisfy Eq. \( (A.1) \), the matrix element should be of the following form:

\[
U(q', Q', q, Q) = \omega \lvert dq' - cQ \rangle \delta(Q' - aQ - bq),
\]

(A.3)
where $\delta(\cdot)$ is Dirac’s delta function. Substitution of (A.3) into the unitary condition:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq' dQ' U^*(q', Q', q'') U(q', Q, q, Q) = \delta(q'' - q)\delta(Q'' - Q),
$$

(A.4)

yields

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq' dQ' |\omega|^2 \delta(q' - dq'' - cQ'')\delta(Q' - aQ'' - bq'')
\times \delta(q'' - dq - cQ)\delta(Q' - aQ - bq)
= |\omega|^2 \delta(d(q'' - q) + c(Q'' - Q))\delta(a(Q'' - Q) + b(q'' - q))
= \frac{|\omega|^2}{|ad - bc|} \delta(q'' - q)\delta(Q'' - Q),
$$

(A.5)

where we have used the formula:

$$
\delta(ax + by)\delta(cx + dy) = \delta(x)\delta(y)/|ad - bc|.
$$

(A.6)

Thus the unitary condition is given as

$$
\omega = \sqrt{|\Delta|}, \quad \Delta = ad - bc (\neq 0).
$$

(A.7)

For the unitary operator which is generated continuously with a Hamiltonian, $\Delta > 0$ can be assumed because the identity operator, $a = d = 1$, $b = c = 0$, satisfies $\Delta = 1 > 0$. The sign of $a$ can be inverted by changing the sign of $\hat{Q}$ and $\hat{q}$. The sign of $b$ can also be inverted with $\hat{q}$ and $\hat{q}'$. Therefore, without loss of generality we can assume $a \geq 0$ and $b \geq 0$.

The momentum presentation of $\hat{U}$ is

$$
V(p', P', p, P) = \langle p' | (P' | \hat{U} | p) | P \rangle
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq' dq'dQ'dQ
U(q', Q', q, Q)\langle p' | q' \rangle \langle P' | Q' \rangle \langle q | p \rangle \langle Q | P \rangle
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq' dq'dQ'dQ
U(q', Q', q, Q) e^{i(\hat{p} q' + P')/\hbar} e^{i(\hat{P} Q' + p')/\hbar}
= \frac{\omega}{(2\pi \hbar)^2} \int_{-\infty}^{\infty} dq'dQ e^{i[(p' - dp' - bP')q + (P' - aP' - cp')]Q}/\hbar
= \omega \delta(p - dp' - bP')\delta(P - aP' - cp')
= \omega^{-1} \delta(p' - a'p + b'P)\delta(P' - d'P + c'p),
$$

(A.8)

where $a' = a/\Delta$, $b' = b/\Delta$, $c' = c/\Delta$ and $d' = d/\Delta$. We have utilized Eq. (A.6). The eigenket of momenta, $|p\rangle|P\rangle$, is transformed as

$$
\hat{U} : |p\rangle|P\rangle \mapsto \frac{1}{\sqrt{\Delta}} |a'p - b'P\rangle | -c'p + d'P \rangle.
$$

(A.9)

From Eqs. (A.1) and (A.3), we can confirm the commutation relation:

$$
[\hat{Q}', \hat{P}'] = [a\hat{Q} + bq, -c'\hat{p} + d'\hat{P}]
= ad' [\hat{Q}, \hat{P}] - bc'[\hat{q}, \hat{p}] = i\hbar. \tag{A.10}
$$

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