When the universe was about 10 μseconds old, a first order cosmological quark-hadron phase transition occurred at a critical temperature of around 200 MeV. In this work, we study the quark-hadron phase transition in the context of brane-world cosmologies, in which our Universe is a three-brane embedded in a five-dimensional bulk, and within an effective model of QCD. We analyze the evolution of the physical quantities, relevant for the physical description of the early universe, namely, the energy density, temperature and scale factor, before, during, and after the phase transition. To study the cosmological dynamics and evolution we use both analytical and numerical methods. In particular, due to the high energy density in the early Universe, we consider in detail the specific brane world model case of neglecting the terms linearly proportional to the energy density with respect to the quadratic terms. A small brane tension and a high value of the dark radiation term tend to decrease the effective temperature of the quark-gluon plasma and of the hadronic fluid, respectively, and to significantly accelerate the transition to a pure hadronic phase. By assuming that the phase transition may be described by an effective nucleation theory, we also consider the case where the Universe evolved through a mixed phase with a small initial supercooling and monotonically growing hadronic bubbles.

PACS numbers: 04.50.-h, 98.80.Cq, 98.80.Bp, 98.80.Jk

I. INTRODUCTION

The possibility that our 4D universe may be viewed as a brane hypersurface embedded in a higher dimensional bulk space has attracted considerable interest lately. A scenario with an infinite fifth dimension in the presence of a brane can generate a theory of gravity which mimics purely four-dimensional gravity, both with respect to the classical gravitational potential and with respect to gravitational radiation [1]. The gravitational self-couplings are not significantly modified in this model. This result has been obtained from the study of a single 3-brane embedded in five dimensions, with the 5D metric given by \( ds^2 = e^{-2f(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \), which can produce a large hierarchy between the scale of particle physics and gravity due to the appearance of the warp factor [1]. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane. In contrast to the compactified case, this follows because the near-brane geometry traps the massless graviton. Hence this model allows the presence of large or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a 5-dimensional anti-de Sitter space-time.

The effective gravitational field equations on the brane-world, in which all the matter forces except gravity are confined on the 3-brane in a 5-dimensional space-time with \( Z_2 \)-symmetry have been obtained, by using an elegant geometric approach, in [2, 3]. The correct signature for gravity is provided by the brane with positive tension. If the bulk space-time is exactly anti-de Sitter, then generically the matter on the brane is required to be spatially homogeneous. The electric part of the 5-dimensional Weyl tensor \( E_{I\bar{J}} \) gives the leading order corrections to the conventional Einstein equations on the brane. This implies a modification of the basic equations describing the cosmological and astrophysical dynamics, which has been extensively considered [4]. For reviews of the dynamics and geometry of the brane Universes, as well as for the discussions of the cosmological implications see [5].

According to standard cosmology, as it expanded and cooled, the early Universe is expected to have undergone
a series of symmetry-breaking phase transitions, at which topological defects may have formed. Phase transitions
are labelled first or second order, according to whether the position of the vacuum state in field space changes
discontinuously or continuously, as the critical temperature is crossed. A first order phase transition proceeds by
bubble nucleation and expansion. When at least \((4 - n)\) of these bubble collide, for \(n = 0, 1, 2\), an \(n\)-dimensional
topological defect may form in the region between them [6]. Recent lattice QCD calculations for two quark flavors
suggest that QCD makes a transition at a temperature of \(T_c \approx 150\) MeV [7]. This phase transition, which may have
occurred in the early Universe, could lead to the formation of relic quark-gluon plasma objects, which still survive
today.

A first order quark-hadron phase transition in the expanding Universe can be described generically as follows [6]. As
the color deconfined quark-gluon plasma cools below the critical temperature \(T_c \approx 150\) MeV, it becomes energetically
favorable to form color confined hadrons (primarily pions and a tiny amount of neutrons and protons, due to the
conserved net baryon number). However, the new phase does not show up immediately. As is characteristic for a
first order phase transition, some supercooling is needed to overcome the energy expense of forming the surface of
the bubble and the new hadron phase. When a hadron bubble is nucleated, latent heat is released, and a spherical
shock wave expands into the surrounding supercooled quark-gluon plasma. This reheats the plasma to the critical
temperature, preventing further nucleation in a region passed by one or more shock fronts. Generally, bubble growth
is described by deflagrations, with a shock front preceding the actual transition front. The nucleation stops when
the whole Universe has reheated to \(T_c\). This part of the phase transition passes very fast, in about 0.05 \(\mu s\), during
which the cosmic expansion is totally negligible. After that, the hadron bubbles grow at the expense of the quark
phase and eventually percolate or coalesce. The transition ends when all quark-gluon plasma has been converted to
hadrons, neglecting possible quark nugget production. The physics of the quark-hadron phase transition, as well as
the cosmological implications of this process have been extensively discussed in the framework of general relativistic
cosmology in [8] - [21].

In the context of brane-world scenarios, the Friedmann equation contains deviations to the 4D case, which results
in an increased expansion in early times. This in general has important effects on the cosmological evolution, and in
particular on cosmological phase transitions. First order phase transitions have also been considered, in the framework
of the brane-world model, in [22]. Due to the effects coming from the extra-dimensions, phase transitions require a
higher nucleation rate to complete, and baryogenesis and particle abundances could be suppressed. The evolution of
topological defects is also affected, but the increased expansion cannot solve the monopole and domain wall problems.

In this work, we consider the quark-hadron phase transition in the brane-world scenario. By assuming that the
phase transition is of the first order, we study in detail the evolution of the relevant cosmological parameters (energy
density, temperature, scale factor, etc) of the quark-gluon and hadron phases, and the phase transition itself. It is
important to emphasize that in the early universe the energy density is high, so that one may neglect the terms
linearly proportional to the energy density with respect to the quadratic terms. This is carried out in detail in this
work. An important parameter to describe the phase transition is the hadron fraction, whose time evolution describes
the conversion process. We also consider the effect of the dark radiation term on the phase transition. The brane
world effects (the quadratic corrections to the matter energy-momentum tensor, described by the numerical value
of the brane tension, and the dark radiation term) lead to an overall decrease of the temperature of the very early
universe, and accelerate the transition to the pure hadronic phase.

This paper is organized in the following manner. In Section II, we briefly outline, for self-completeness and self-
consistency, the field equations in brane-world models and the basics of the quark-hadron phase transition. In the
latter, we lay down the equations of state and the relevant physical quantities that are analyzed in the remaining
Sections. In Section III we analyze in detail the quark-hadron phase transition. In Section IV, we consider bubble
nucleation in the brane-world scenario, by assuming that the phase transition may be described by an effective
nucleation theory. We discuss and summarize our results in Section V. In the present paper we use a system of units
so that the speed of light is \(c = 1\).

II. GEOMETRY, BRANE-WORLD FIELD EQUATIONS, EQUATIONS OF STATE AND
CONSEQUENCES

A. The field equations in the brane-world models

In the brane-world models the effective four-dimensional gravitational field equations on the brane take the form
[2, 3]:

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^4 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu},
\]

(1)
where $\Lambda = k^2_4 (\Lambda_5 + k^2_3 \lambda^2/6)/2$ and $k^4_3 = k^4_3 \lambda/6$. $T_{\mu\nu}$ is the matter energy-momentum tensor on the brane and $T = T^\mu_\mu$ is the trace of the energy-momentum tensor. The Einstein equation in the bulk also implies the conservation of the energy momentum tensor of the matter on the brane [3]. The first correction term relative to Einstein’s general relativity is the inclusion of a quadratic term $S_{\mu\nu}$ in the energy-momentum tensor, arising from the extrinsic curvature terms in the projected Einstein tensor, and is given by

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu}^\alpha T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} (3 T_{\alpha\beta} T_{\alpha\beta} - T^2).$$

The second correction term, $E_{\mu\nu}$, is the projection of the 5-dimensional Weyl tensor, $C_{ABCD}$, onto the brane, and is defined as $E_{\mu\nu} = \delta^A_\mu \delta^B_\nu C_{ABCD} n^D n^C$, and encompasses nonlocal bulk effects. The only general known property of this nonlocal term is that it is traceless, i.e., $E_{\mu\nu} = 0$. The symmetry properties of $E_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field $u^\mu$ as

$$E_{\mu\nu} = - \frac{6}{\lambda k^2_4} \left[ U \left( u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + P_{\mu\nu} + 2 Q_{(\mu} u_{\nu)} \right],$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to $u^\mu$, $U$ is a scalar, $Q_\mu$ a spatial vector and $P_{\mu\nu}$ a spatial, symmetric and trace-free tensor, respectively. For homogeneous and isotropic Friedmann-Robertson-Walker (FRW) type cosmological models $Q_\mu = P_{\mu\mu} = 0$ [5], and hence the only non-zero contribution from the 5-dimensional Weyl tensor from the bulk is given by the scalar term $U$.

There are several constraints which have been obtained for the brane tension $\lambda$. Thus from big bang nucleosynthesis constraints it follows $\lambda \geq 1$ MeV$^4$ [23]. A much stronger bound for $\lambda$ arises from null results of submillimeter tests of Newton’s law, giving $\lambda \geq 10^8$ GeV$^4$ [24]. An astrophysical lower limit on $\lambda$, which is independent of the Newton law and the cosmological limits has been derived in [23], leading to $\lambda > 5 \times 10^8$ MeV$^4$.

In this work, we assume that the space-time geometry is the flat FRW metric, given by

$$ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right).$$

For the matter energy-momentum tensor on the brane we restrict our analysis to the case of the perfect fluid energy-momentum tensor,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu},$$

where $\rho$ and $p$ are the energy density and isotropic pressure of the cosmological fluid on the brane, respectively.

Thus, the gravitational field equations, corresponding to the line element (4) become

$$3 \frac{\dot{a}^2}{a^2} = \Lambda + k^2_4 \rho + \frac{k^2_3}{2 \lambda} \dot{a}^2 + \frac{6}{\lambda k^2_4} U,$$

$$\frac{2 \dot{a}}{a} + \frac{\ddot{a}}{a^2} = \Lambda - k^2_4 p - \frac{k^2_3}{2 \lambda} \dot{a} + \frac{k^3_3}{2 \lambda} \dot{p}^2 - \frac{2}{\lambda k^2_4} U,$$

$$\dot{\rho} + 3(\rho + p) \frac{\dot{a}}{a} = 0,$$

$$\ddot{U} + 4 \frac{\dot{a}}{a} \dot{U} = 0.$$  

Eq. (9) can be immediately integrated to give the following general expression for the “dark energy” $U$:

$$U = \frac{U_0}{a^4},$$

with $U_0 > 0$ a constant of integration.

B. Quark-hadron phase transition

In this Section, we outline the relevant physical quantities of the quark-hadron phase transition, which will be applied in the following sections in the context of the brane-world scenario. Note that the scale of the cosmological QCD transition is given by the Hubble radius $R_H$ at the transition, which is $R_H \sim m_{Pl}/T_c^2 \sim 10 \text{ km}$, where $T_c$ is the critical temperature. The mass inside the Hubble volume is $\sim 1 M_\odot$. The expansion time scale is $10^{-5}$ s, which
should be compared with the timescale of QCD, 1 fm/c ≈ 10^{-23} s. Even the rate of the weak interactions exceeds the Hubble rate by a factor of 10^7. Therefore, in this phase photons, leptons, quarks and gluons (or pions) are lightly coupled and may be described as a single, adiabatically expanding fluid [12].

In order to study the quark-hadron phase transition it is necessary to specify the equation of state of the matter, in both quark and hadron state. Giving an equation of state is equivalent to give the pressure as a function of the temperature $T$ and chemical potential $\mu$. At high temperatures the quark chemical potentials are equal, because weak interactions keep them in chemical equilibrium, and the chemical potentials for leptons are assumed to vanish. Thus the chemical potential for a baryon is defined by $\mu_B = 3\mu_q$. The baryon number density of an ideal Fermi gas of three quark flavors is given by $n_B \approx T^2 \mu_B/3$, leading to $\mu_B/T \sim 10^{-9}$ at $T > T_c$. At low temperatures $\mu_B/T \sim 10^{-2}$. Therefore the assumption of a vanishing chemical potential at the phase transition temperature in both quark and hadron phase represents an excellent approximation for the study of the equation of state of the cosmological matter in the early Universe. In addition to the strongly interacting matter, we assume that in each phase there are present leptons and relativistic photons, satisfying equations of state similar to that of hadronic matter [6].

The equation of state in the quark phase takes the form of the MIT bag model equation of state, interpolating with a chiral field formed with the $\pi$ meson field and a scalar field. If the temperature effects can be ignored, the equation of state in the quark phase can generally be given in the form

$$\rho_q = 3a_q T^4 + V(T), \quad p_q = a_q T^4 - V(T), \quad \text{(11)}$$

where $a_q = (\pi^2/90) g_q$, with $g_q = 16 + (21/2) N_F + 14.25 = 51.25$ and $N_F = 2$. $V(T)$ is the self-interaction potential. For $V$ we adopt the expression [19]

$$V(T) = B + \gamma_T T^2 - \alpha_T T^4, \quad \text{(12)}$$

where $B$ is the bag constant, $\alpha_T = 7\pi^2/20$, and $\gamma_T = m_s^2/4$, with $m_s$ the mass of the strange quark in the range $m_s \in (60-200)$ MeV. The form of the potential $V$ corresponds to a physical model in which the quark fields are interacting with a chiral field formed with the $\pi$ meson field and a scalar field. If the temperature effects can be ignored, the equation of state in the quark phase takes the form of the MIT bag model equation of state, $p_q = (\rho_q - 4B)/3$. Results obtained in low energy hadron spectroscopy, heavy ion collisions and phenomenological fits of light hadron properties give $B^{1/4}$ between 100 and 200 MeV [25].

In the hadron phase we take the cosmological fluid to consist of an ideal gas of massless pions and of nucleons described by the Maxwell-Boltzmann statistics, with energy density $\rho_h$ and pressure $p_h$, respectively. The equation of state can be approximated by

$$p_h(T) = \frac{1}{3} a_\pi T^4, \quad \text{(13)}$$

where $a_\pi = (\pi^2/90) g_h$ and $g_h = 17.25$.

The critical temperature $T_c$ is defined by the condition $p_q(T_c) = p_h(T_c)$ [6], and is given, in the present model, by

$$T_c = \sqrt{\frac{\gamma_T + \sqrt{\gamma_T^2 + 4 (a_q + \alpha_T - a_\pi) B}}{2 (a_q + \alpha_T - a_\pi)}}. \quad \text{(14)}$$

For $m_s = 200$ MeV and $B^{1/4} = 200$ MeV the transition temperature is of the order $T_c \approx 125$ MeV. Since the phase transition is of first order, all the physical quantities, like the energy density, pressure and entropy exhibit discontinuities across the critical curve. The ratio of the quark and hadron energy densities at the critical temperature, $\rho_q(T_c)/\rho_h(T_c)$, is of the order of 3.62 for $m_s = 200$ MeV and $B^{1/4} = 200$ MeV. If the temperature effects in the self-interaction potential $V$ are neglected, $\alpha_T = \gamma_T \approx 0$, then we obtain the well-known relation between the critical temperature and the bag constant, $B = (g_q - g_h) \pi^2 T_c^4/90$ [6].

### III. DYNAMICS OF THE BRANE UNIVERSE DURING THE QUARK-HADRON PHASE TRANSITION

The quantities to be traced through the quark-hadron phase transition in the brane-world cosmological scenario are the energy density $\rho$, the temperature $T$ and the scale factor $a$. These quantities are determined by the gravitational field equations (6) and (8) and by the equations of state (11), (12) and (13). We shall consider now the evolution of the brane-world before, during and after the phase transition.

Before the phase transition, $T > T_c$ the brane-world is in the quark phase. With the use of the equations of state of the quark matter, and the Bianchi identity on the brane, Eq. (8) can be written in the form

$$\frac{\dot{a}}{a} = \frac{3a_q - \alpha_T \dot{T}}{3a_q T} - \frac{1}{6} \frac{\gamma_T}{a_q} \frac{\dot{T}}{T^3}. \quad \text{(15)}$$
and can be integrated to give the following scale factor-temperature relation:

$$a(T) = a_0 T^{\alpha_T - 3a_B} \exp \left( \frac{1}{12} \frac{\gamma_T}{a_B} \frac{1}{T^2} \right),$$

(16)

where $a_0$ is a constant of integration.

With the use of Eq. (15), from the gravitational field equations we obtain an expression, describing the evolution of the temperature of the brane Universe in the quark phase, given by

$$\frac{dT}{dt} = -\frac{T^3 \sqrt{A_0 T^8 + A_1 T^6 + A_2 T^4 + A_3 T^2 + A_4 T^4 A_5} \exp \left( -2A_6 \frac{1}{B_T} \right) + A_7}{A_5 T^2 + A_6},$$

(17)

where we have denoted

$$A_0 = k_B^2 (3a_B - \alpha_T)^2 / 2\lambda, \quad A_1 = k_B^2 (3a_B - \alpha_T) / \lambda, \quad A_2 = k_B^2 \left[ \gamma_T^2 / 2\lambda + (3a_B - \alpha_T)(1 + B / \lambda) \right],$$

(18)

$$A_3 = k_B^2 (1 + B / \lambda) \gamma_T, \quad A_4 = 6U_0 / \lambda k_B^2 a_B^4, \quad A_5 = (3a_B - \alpha_T) / 3a_B, \quad A_6 = \gamma_T / 3a_B, \quad A_7 = \lambda + k_B^2 B (1 + B / 2\lambda).$$

(19)

The variation of the temperature as a function of the parameter $\tau = k_B t$ in the quark matter filled brane-world is represented, in the case of a vanishing cosmological constant $\Lambda = 0$, for different values of the brane tension $\lambda$ and for a fixed numerical value of the constant $A_4 = 6U_0 / \lambda k_B^2 a_B^4$, in Fig. 1.

![Graph](image)

**FIG. 1:** Variation of the temperature of the quark fluid on the brane as a function of $\tau = k_B t$ for different values of the brane tension $\lambda$: $\lambda = 5 \times 10^6$ MeV$^4$ (solid curve), $\lambda = 5 \times 10^9$ MeV$^4$ (dotted curve), $\lambda = 5 \times 10^{10}$ MeV$^4$ (short dashed curve) and $\lambda = 5 \times 10^{16}$ MeV$^4$ (long dashed curve). We have assumed that in this phase the cosmological constant is vanishingly small. For the bag constant we have chosen the value $B^{1/4} = 200$ MeV, while $A_4 = 0.01$.

The variation of the temperature of the brane in the quark phases for different values of the constant $A_4$, corresponding to different numerical values of the integration constant $U_0$, describing the effect of the dark radiation term on the cosmological evolution, and considering a fixed value for the brane tension $\lambda$, is represented in Fig. 2.

In order to have an analytical insight into the evolution of the cosmological quark matter in the brane-world and on the effects of the extra-dimensions on the phase transition, we consider the simple case in which the temperature corrections can be neglected in the self-interaction potential $V$. In this case $V = B = constant$, and the equation of state of the quark matter is given by the bag model equation of state, $p_q = (\rho_q - 4B) / 3$. Thus, Eq. (8) can immediately be integrated to give the following scale factor-temperature relation:

$$a(T) = a_0 T, \quad (20)$$

where $a_0$ is a constant of integration. Since in the initial stages of the evolution of the Universe the density of the matter is very high, we suppose that in Eq. (6) we can neglect the term proportional to the energy density with
FIG. 2: Variation of the temperature of the quark fluid on the brane as a function of $\tau = k_4 t$ for different values of the dark radiation term $U$: $A_4 = 10^2$ (solid curve), $A_4 = 10^3$ (dotted curve), $A_4 = 10^5$ (short dashed curve) and $A_4 = 10^6$ (long dashed curve). We have assumed that in this phase the cosmological constant is vanishingly small. For the bag constant and for the brane tension we have chosen the values $B^{1/4} = 200 \text{ MeV}$ and $\lambda = 5 \times 10^8 \text{ MeV}^4$, respectively.

respect to the term containing $\rho^2$. The contributions of the cosmological constant and of the dark energy are also negligible small. Therefore the evolution of the quark phase of the brane-world is described by the equation

$$\frac{\dot{a}}{a} \approx \frac{k_4}{\sqrt{6\lambda}} \rho.$$  \hspace{1cm} (21)

Hence the time dependence of the temperature can be obtained from the equation

$$\frac{dT}{dt} \approx -\frac{k_4}{\sqrt{6\lambda}} (3a_q T^5 + BT),$$  \hspace{1cm} (22)

with the general solution given by

$$T(t) \approx \frac{B^{1/4} C^{1/4} \exp \left( -\frac{k_4 B t}{\sqrt{6\lambda}} \right)}{\left[ 1 - 3a_q C \exp \left( -\frac{4k_4 B}{\sqrt{6\lambda}} t \right) \right]^{1/4}}.$$  \hspace{1cm} (23)

The integration constant $C$ is related to the temperature $T_0$ of the quark matter at the time $t_0$ by means of the relation

$$C = \frac{T_0^4 \exp \left( \frac{4k_4 B}{\sqrt{6\lambda}} t_0 \right)}{(3a_q T_0^4 + B)^{1/4}},$$  \hspace{1cm} (24)

leading to

$$T(t) \approx \frac{B^{1/4} T_0 \exp \left( -\frac{k_4 B}{\sqrt{6\lambda}} (t - t_0) \right)}{\left\{ B + 3a_q T_0^4 - 3a_q T_0^4 \exp \left[ -\frac{4k_4 B}{\sqrt{6\lambda}} (t - t_0) \right] \right\}^{1/4}}.$$  \hspace{1cm} (25)

During the phase transition, the temperature and the pressure are constants, $T = T_c$ and $p = p_c$, respectively. The entropy $S = sa^3$ and the enthalpy $W = (\rho + p) a^3$ are conserved quantities. During the phase transition $\rho(t)$ decreases from $\rho_q (T_c) \equiv \rho_Q$ to $\rho_h (T_c) \equiv \rho_H$. For phase transition temperature of $T_c = 125 \text{ MeV}$ we have $\rho_Q \approx 5 \times 10^9 \text{ MeV}^4$ and $\rho_H \approx 1.38 \times 10^9 \text{ MeV}^4$, respectively. For the same value of the temperature the value of the pressure of
the cosmological fluid during the phase transition is \( p_c \approx 4.6 \times 10^8 \text{ MeV}^4 \). It is convenient, following [6], to replace \( \rho (t) \) by \( h(t) \), the volume fraction of matter in the hadron phase, by defining

\[
\rho (t) = \rho_H h(t) + \rho_Q [1 - h(t)] = \rho_Q [1 + n h(t)],
\]

where we denoted \( n = (\rho_H - \rho_Q) / \rho_Q \). At the beginning of the phase transition \( h(t_c) = 0 \), where \( t_c \) is the time corresponding to the beginning of the phase transition, and \( \rho (t_c) \equiv \rho_Q \), while at the end of the transition \( h(t_h) = 1 \), where \( t_h \) is the time at which the phase transition ends, corresponding to \( \rho (t_h) \equiv \rho_H \). For \( t > t_h \) the Universe enters in the hadronic phase.

From Eq. (8) we obtain

\[
\frac{\dot{a}}{a} = -\frac{1}{3} \frac{(\rho_H - \rho_Q) \dot{h}}{\rho_Q + (\rho_H - \rho_Q) h} = \frac{1}{3} \frac{\dot{h}}{1 + rh},
\]

where we denoted \( r = (\rho_H - \rho_Q) / (\rho_Q + p_c) \). The above equation immediately leads to

\[
a(t) = a(t_c) [1 + rh(t)]^{-1/3},
\]

where we have used the initial condition \( h(t_c) = 0 \). The evolution of the fraction of the matter in the hadron phase is described by the equation

\[
\frac{dh}{dt} = -\sqrt{3} \left( \frac{1}{r} + h \right) \sqrt{\Lambda + k_4^2 \rho_Q + k_4^2 \rho_Q^2 / 2\lambda + k_4^2 \rho_Q \left( 1 + \frac{\rho_Q}{\lambda} \right) h + k_4^2 \rho_Q^2 n^2 / 2\lambda - 6U_0 / (k_4^2 \lambda a^4(t_c)) (rh + 1)^{1/3}.}
\]

It is also convenient to plot the variation of the hadron fraction as a function of the different parameters. The variation of the hadron fraction \( h \) as a function of the parameter \( \tau = k_4 t \) is represented, for different values of the brane tension and for a fixed value of \( U_0 \) in Fig. 3.

![FIG. 3: Variation of the hadron fraction \( h \) as a function of the parameter \( \tau = k_4 t \) during the quark-hadron phase transition on the brane, for different values of the brane tension: \( \lambda = 5 \times 10^8 \text{ MeV}^4 \) (solid curve), \( \lambda = 5 \times 10^9 \text{ MeV}^4 \) (dotted curve), \( \lambda = 5 \times 10^{10} \text{ MeV}^4 \) (short dashed curve) and \( \lambda = 5 \times 10^{12} \text{ MeV}^4 \) (long dashed curve). We have assumed that the effect of the cosmological constant can be neglected. The temperature during the phase transition is \( T_c = 125 \text{ MeV} \). For the bag constant we have chosen the value \( B^{1/4} = 200 \text{ MeV} \), while \( A_4 = 0.01 \).](image)

The time variation of the hadron fraction during the phase transition for a fixed brane tension \( \lambda \) and for different values of the integration constant \( U_0 \) is represented in Fig. 4.

If the quadratic term in the energy density, describing the effects of the extra-dimensions, dominates the evolution of the Universe, and by neglecting the dark energy contribution, the evolution of the hadron fraction during the phase transition is described by the equation

\[
\frac{dh}{dt} = -\frac{3k_4 \rho_Q}{r \sqrt{6\lambda}} (1 + nh) (1 + rh),
\]

where we have used the initial condition

\[
\rho (t) = \rho_H h(t) + \rho_Q [1 - h(t)] = \rho_Q [1 + n h(t)],
\]
FIG. 4: Variation of the hadron fraction $h$ as a function of the parameter $\tau = k_4 t$ during the quark-hadron phase transition on the brane, for different values of the dark radiation term $u_0 = 6 k_0 / k_4^2 \lambda a^4 (t_c)$: $u_0 = 10^{10}$ MeV$^{-2}$ (solid curve), $u_0 = 10^{11}$ MeV$^{-2}$ (dotted curve), $u_0 = 10^{12}$ MeV$^{-2}$ (short dashed curve) and $u_0 = 10^{13}$ MeV$^{-2}$ (long dashed curve). We have assumed that the effect of the cosmological constant can be neglected. The temperature during the phase transition is $T_c = 125$ MeV. For the bag constant we have chosen the value $B^{1/4} = 200$ MeV, while $\lambda = 5 \times 10^8$. With the general solution given by

$$h(t) = \exp \left[ -\frac{3(n-r) k_4 \rho Q}{r \sqrt{6} \lambda} (t - t_c) \right] - 1 \quad \text{(31)}$$

where we have also used the initial condition $h(t_c) = 0$.

The respective evolution of the scale factor during the phase transition for the extra-dimensions dominated evolution of the cosmological fluid is described by the equation

$$a(t) = a(t_c) \left( \frac{n-r}{n-r+1} \right) \left[ n-r \exp \left( -\frac{3(n-r) k_4 \rho Q}{r \sqrt{6} \lambda} (t - t_c) \right) \right]^{1/3}. \quad \text{(32)}$$

The phase transition ends when $h(t)$ has increased to 1. If the evolution is dominated by the terms coming from the extra-dimensions in the bulk, the time $t_h$ at which the phase transition ends is

$$t_h = t_c + \frac{\sqrt{6} \lambda r \ln \frac{1+n}{1+r}}{3 k_4 (n-r) \rho Q} \quad \text{(33)}$$

At the end of the phase transition the scale factor of the Universe has the value $a(t_h) = a(t_c) (r+1)^{-1/3}$.

Finally, after the phase transition, the energy density of the pure hadronic matter is $\rho_h = 3 p_h = 3 a \pi T^4$. The Bianchi identity Eq. (8) gives $a(T) = a(t_h) T_c / T$. The temperature dependence of the brane Universe in the hadronic phase is governed by the equation

$$\frac{dT}{dt} = -\frac{T}{\sqrt{3}} \Lambda + \left( 3 a \pi k_4^2 + \frac{6 k_0}{\lambda k_4^2 a^4 (t_h) T_c^4} \right) T^4 + \frac{9a^2 k_4^2}{2 \lambda T^8}. \quad \text{(34)}$$

As before, it is convenient to plot the variation of the temperature in terms of the various parameters. The variation of the temperature of the hadron fluid filled brane Universe as a function of the brane tension $\lambda$ is represented in Fig. 5.

The time variation of the temperature for the brane in the hadron phase for different values of the dark energy coefficient $k_0$ is represented in Fig. 6.
FIG. 5: Variation of the temperature of the hadron fluid on the brane, as a function of \( \tau = k_4 t \), for different values of the brane tension \( \lambda \): \( \lambda = 5 \times 10^8 \text{ MeV}^4 \) (solid curve), \( \lambda = 5 \times 10^9 \text{ MeV}^4 \) (dotted curve), \( \lambda = 5 \times 10^{10} \text{ MeV}^4 \) (short dashed curve) and \( \lambda = 5 \times 10^{11} \text{ MeV}^4 \) (long dashed curve). We have assumed that in this phase the cosmological constant is vanishingly small. For the bag constant we have chosen the value \( B^{1/4} = 200 \text{ MeV} \), while the dark radiation term has been fixed so that

\[
6 \frac{U_0}{\lambda k_4 a_4(t_h)} T_c^4 = 0.01.
\]

FIG. 6: Variation of the temperature of the hadronic fluid on the brane, as a function of \( \tau = k_4 t \), for different values of the dark radiation term \( u_0 = 6 \frac{U_0}{\lambda k_4 a_4(t_h)} T_c^4 \): \( u_0 = 10^8 \) (solid curve), \( u_0 = 10^9 \) (dotted curve), \( u_0 = 10^{10} \) (short dashed curve) and \( u_0 = 10^{11} \) (long dashed curve). We have assumed that in this phase the cosmological constant is vanishingly small. For the bag constant and for the brane tension we have chosen the values \( B^{1/4} = 200 \text{ MeV} \) and \( \lambda = 5 \times 10^8 \text{ MeV}^4 \), respectively.

If the extra-dimensional terms dominates the four-dimensional gravitational effects, then the time variation of the temperature of the cosmological hadron fluid is given by

\[
T(t) \approx \left[ \frac{12k_4 a_4}{\sqrt{6\lambda}} (t - t_h) + T_c^{-4} \right]^{-1/4},
\]

where we have used the initial condition \( T(t_h) = T_c \).

In the extra-dimensions dominated hadron phase the scale factor of the Universe evolves according to the law

\[
a(t) \approx a(t_h) T_c \left[ \frac{12k_4 a_4}{\sqrt{6\lambda}} (t - t_h) + T_c^{-4} \right]^{1/4}.
\]
**IV. BUBBLE NUCLEATION IN THE BRANE-WORLD COSMOLOGICAL SCENARIOS**

In order to describe the process of formation and evolution of microscopic quark nuclei in the cosmological fluid on the brane one must use nucleation theory. The goal of nucleation theory is to compute the probability that a bubble or droplet of the $A$ phase appears in a system initially in the $B$ phase near the critical temperature [26]. Homogeneous nucleation theory applies when the system is pure. Nucleation theory is applicable for first-order phase transitions when the matter is not dramatically supercooled or superheated. If substantial supercooling or superheating is present, or if the phase transition is second order, then the relevant dynamics is spinodal decomposition [27].

In this context, note that in the analysis of the phase transition considered in the previous section we do not take into account the possibility that the quark plasma could undergo a supercooling phase, in which it stays in the quark phase below the critical temperature. If the supercooling is not too strong, the phase transition can be described by an effective nucleation theory, as mentioned above. As the temperature decreases, there is a probability that a droplet of hadrons is nucleated from the quark plasma. The nucleation probability density is given by [6]

$$ p(t) = p_0 T_c^4 \exp \left[ -\frac{u_0}{(1 - T^4)^2} \right], \quad (37) $$

where $\bar{T} = T/T_c$. Once a droplet of hadrons is formed, it starts to expand explosively [28], with a velocity $v_f$, smaller than the sound speed. Contemporarily, a much quicker shock wave is generated. More and more bubbles are created while the temperature decreases, until the shock waves collide and re-heat the plasma to the critical temperature.

To calculate the fraction of the volume which at a time $t$ is turned to the hadronic phase in the small supercooling scenario, we should sum over the volumes at the time $t$ of the bubbles which are nucleated at a previous time $t_p$, times the probability to create a bubble at that time $t_p$, which is given by

$$ f(t) = \int_{t_i}^{t} dt' p(t') \frac{4\pi}{3} [v_f(t - t_p)]^3, \quad (38) $$

where $t_i$ is the time at which the critical temperature is reached, and the nucleation process started. Using Eqs. (22)-(23), we can express this integral in terms of normalized temperature. After some algebra we get

$$ f(\bar{T}) = \int_{\bar{T}}^{1} d\bar{T}' \frac{p_0 v_{fr}^3}{B^3} \left( \frac{6\lambda}{\kappa_4^2} \right)^2 \exp \left[ -\frac{u_0}{(1 - T_c^4)^2} \right] \left\{ \log \left[ \frac{3a_q\bar{T}_p^4 + b}{3a_q\bar{T}_p^4 + \bar{T}_p^4} \right] \right\}^3, \quad (39) $$

where $b \equiv B/T_c^4$.

The integrand is extremely peaked around its maximum, so the integral is dominated by the value of the function at the maximum, which can be found by solving the equation:

$$ \log \left[ \frac{3a_q\bar{T}_p^4 + b}{3a_q\bar{T}_p^4 + \bar{T}_p^4} \right] ^4 = \frac{8\bar{T}_0}{(1 - \bar{T}_p^4)^3} \left[ \frac{1 + 12a_q\bar{T}_p^2}{3a_q\bar{T}_p^4 + b} \right] + 12a_q\bar{T}_p^2 = 3 \left( \frac{4}{\bar{T}_p^4} - \frac{12a_q\bar{T}_p^2}{3a_q\bar{T}_p^4 + b} \right). \quad (40) $$

To evaluate this maximum, we note that the integrated function falls very rapidly to zero as $\bar{T}_p \rightarrow 1$ because of the exponential, so the maximum must be very close to the lower end of the interval. Thus we can set $\bar{T}_{p,max} \simeq \bar{T} + x$ with $x/\bar{T} \ll 1$ and approximate the first derivative (40) to first order in $x$. Moreover, since we will assume only a small supercooling, the final temperature $\bar{T}$ is very close to 1, so that the first term in the square bracket of the left hand side in Eq. (40) is much larger than the other two, which are of order 1 and can be discarded. Eventually we obtain

$$ x(\bar{T}) = \frac{(1 - \bar{T}^4)^3 (6a_q\bar{T}^4 + b)}{8\bar{T}_0 b T^3}, \quad (41) $$

so that the integral (39) can be approximated as

$$ f(\bar{T}) \simeq x(\bar{T}) \frac{p_0 v_{fr}^3}{B^3} \left( \frac{6\lambda}{\kappa_4^2} \right)^2 \exp \left[ -\frac{u_0}{(1 - (\bar{T} + x(\bar{T}))^4)^2} \right] \left\{ \log \left[ \frac{3a_q(\bar{T} + x(\bar{T}))^4 + b}{3a_q(\bar{T} + x(\bar{T}))^4 + b} \right] \right\}^3. \quad (42) $$
FIG. 7: Variation of the hadronic fraction $f$ as a function of the normalized temperature in the small supercooling scenario, for different values of the brane tension $\lambda$: $\lambda = 5 \times 10^8$ MeV$^4$ (solid curve), $\lambda = 5 \times 10^9$ MeV$^4$ (dotted curve), $\lambda = 5 \times 10^{10}$ MeV$^4$ (short dashed curve) and $\lambda = 5 \times 10^{16}$ MeV$^4$ (long dashed curve). The bag constant and the critical temperature have the values: $B^{1/4} = 200$ MeV and $T_c = 125$ MeV, respectively.

FIG. 8: Variation of the hadronic fraction $f$ as a function of the parameter $\tau = k_4 t$ in the small supercooling scenario, for different values of the brane tension: $\lambda = 5 \times 10^8$ MeV$^4$ (solid curve), $\lambda = 5 \times 10^9$ MeV$^4$ (dotted curve), $\lambda = 5 \times 10^{10}$ MeV$^4$ (short dashed curve) and $\lambda = 5 \times 10^{16}$ MeV$^4$ (long dashed curve). The bag constant and the critical temperature have the values: $B^{1/4} = 200$ MeV and $T_c = 125$ MeV, respectively.

The plot of $f(\bar{T})$ is presented in Fig. 7 for values of the parameters used previously, and setting the front velocity $v_{fr} = 10^{-3}$ and the other constants $w_0, p_0 = 1$. The fraction of hadronic matter stays very close to zero until the supercooling temperature is between $\bar{T} = 0.98$ and $\bar{T} = 0.97$, then it jumps to 1 very rapidly. The same behavior can be verified in the time dependent plot of Fig. 8. The comparison between Fig. 8 and Fig. 3 shows clearly the difference between the first order phase transition and the supercooling. In the former, the hadronic fraction grows smoothly over a decade from zero to one, while in the latter it changes dramatically and almost instantaneously. Physically, what happens is that, at a certain temperature below the critical value, an enormous amount of hadronic bubbles are nucleated almost everywhere in the plasma, which grow explosively to transform all the plasma into hadrons. The picture is similar to what happen in standard cosmology [6], in which the small supercooling phase transition is also very rapid with respect to the simple first order phase transition, at a temperature ($T \simeq 0.98 T_c$), which is very similar to the one we have obtained. Another remarkable feature that is present in the supercooling scenario is the important dependence of the transition temperature on the brane tension, about 10% difference between the lowest and the highest, which eventually leads to a different time at which the transition occurs.
V. DISCUSSIONS AND FINAL REMARKS

In the context of brane-world scenarios, in the high density cosmological phase the Friedmann equation contains deviations to the $4D$ case, which imposes fundamental phenomenological consequences on the cosmological evolution, and in particular on the cosmological phase transitions. In this work, we have analyze the evolution of the relevant physical quantities, namely, the energy density, temperature and scale factor before, during and after the phase transition. It is important to emphasize that in the early universe the energy density is extremely high, so that one can neglect the terms linearly proportional to the energy density with respect to the quadratic terms. This approximation has been considered in detail throughout this work. In the high density regime the Hubble function is proportional to the energy density of the cosmological matter, which drastically affects the cosmological dynamics of the universe. Moreover, in the early universe the dark radiation term, the projection of the Weyl tensor from the bulk, which appears in form of a radiation-like term in the field equations, may also play an important role. The magnitude of the brane world effects can be characterized by the numerical value of the brane tension $\lambda$. In the limit $\lambda \to \infty$ we recover standard general relativity [5]. On the other hand, the study of the quark-hadron phase transition is also very important from an observational point of view, since the inhomogeneities generated at the QCD phase transition might have a noticeable effect on nucleosynthesis [9].

By fully taking into account the brane world effects we have found that the temperature evolution of the universe in the brane world scenario is different from the idealized standard FRW model. The temperature of the early universe in the quark phase is smaller in the brane world scenario, as one can see from Fig. 1, where the long dashed curve corresponds (approximately) to the general relativistic limit. Hence a small value of the brane tension would significantly reduce the temperature of the quark-gluon plasma, and accelerate the phase transition to the hadronic era. An increase of the dark radiation term for fixed bag constant $B$ and brane tension $\lambda$ gives the same effect, as one can see from Fig. 2. Once the quark-hadron phase transition starts, the hadron fraction $h$ is again strongly dependent on the brane tension. For small values of $\lambda$, $h(t)$ is much higher than in the standard general relativity, as one can see from Fig. 3. The increase of the dark radiation on the brane also strongly accelerates the formation of the hadronic phase (see Fig. 4), and decreases the time interval necessary for the transition. A small brane tension and a high energy density of the dark radiation also tend to reduce the temperature of the hadronic fluid. Of course, the temperature evolution also depends upon the relativistic degrees of freedom in the equation of state and upon the equation of state. In addition to this, by assuming that the phase transition may be described by an effective nucleation theory, we have also considered the case where the Universe evolved through a mixed phase with a small initial supercooling, and monotonically growing hadronic bubbles. It was shown that at a certain temperature below the critical value, an enormous amount of hadronic bubbles are nucleated, which grow explosively to transform all the plasma into hadrons, indicating that the small supercooling phase transition is very rapid with respect to the simple first order phase transition.

Many details of the QCD phase transition are not yet conclusively understood. Even the order of transition is still a matter of debate. An advance in the understanding of the numerical values of the QCD coupling constants would be very helpful in obtaining accurate cosmological conclusions. Such an advance may also provide a powerful method for testing on a cosmological scale the theoretical predictions of the brane world models and the possible existence of the extra-dimensions.

Acknowledgments

We would like to thank the anonymous referee, whose comments and suggestions helped us to significantly improve the manuscript. The work of GDR is supported by I.N.F.N. The work of TH is supported by the RGC grant HKU 702507P of the government of the Hong Kong SAR. FSNL was funded by Fundação para a Ciência e Tecnologia (FCT)–Portugal through the grant SFRH/BPD/26269/2006.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370-3373 (1999); L. Randall and R. Sundrum, Phys. Rev. Lett 83, 4690-4693 (1999).
[2] T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D62, 024012 (2000).
[3] M. Sasaki, T. Shiromizu and K. Maeda, Phys. Rev. D62, 024008 (2000); K. Maeda, S. Mizuno and T. Torii, Phys. Rev. D68, 024033 (2003).
[4] H. B. Kim and H. D. Kim, Phys. Rev. D61, 064003 (2000); K. Maeda and D. Wands, Phys. Rev. D62, 124009 (2000); P. Binétruy, C. Deffayet and D. Langlois, Nucl. Phys. B 565, 269 (2000); R. Maartens, Phys. Rev. D62, 084023 (2000); A. Campos and C. F. Sopuerta, Phys. Rev. D63, 104012 (2001); A. Campos and C. F. Sopuerta, Phys. Rev. D64, 104011
(2001); C.-M. Chen, T. Harko and M. K. Mak, Phys. Rev. D64, 044013 (2001); D. Langlois, Phys. Rev. Lett. 86, 2212 (2001); C.-M. Chen, T. Harko and M. K. Mak, Phys. Rev. D64, 124017 (2001); J. D. Barrow and R. Maartens, Phys. Lett. B532, 153 (2002); C.-M. Chen, T. Harko, W. F. Kao and M. K. Mak, Nucl. Phys. B636, 159 (2002); M. Szydlowski, M. P. Dabrowski and A. Krawiec, Phys. Rev. D66, 064003 (2002); T. Harko and M. K. Mak, Class. Quantum Grav. 20, 407 (2003); C.-M. Chen, T. Harko, W. F. Kao and M. K. Mak, JCAP 0311, 005 (2003); T. Harko and M. K. Mak, Class. Quantum Grav. 21, 1489 (2004); M. K. Mak and T. Harko, Phys. Rev. D 70, 024010 (2004); T. Harko and M. K. Mak, Phys. Rev. D69, 064020 (2004); M. Maziazhvili, Phys. Lett. B627, 197 (2005); S. Mukohyama, Phys. Rev. D72, 061901 (2005); M. Maziashvili, Phys. Lett. B627, 197 (2005); S. Mukohyama, Phys. Rev. D72, 061901 (2005); M. K. Mak and T. Harko, Phys. Rev. D71, 104022 (2005); L. Á. Gergely and Z. Kovacs, Phys. Rev. D72, 064015 (2005); T. Harko and K. S. Cheng, Astrophys. J. 636, 8 (2006); L. A. Gergely, Phys. Rev. D74, 024002, (2006); N. Pires, Zong-Hong Zhu, J. S. Alcaniz, Phys. Rev. D75, 064003 (2007); T. Harko and K. S. Cheng, Phys. Rev. D76, 044013 (2007); A. Viznyuk and Y. Shtanov, Phys. Rev. D76, 064009 (2007); Z. Keresztes, L. Á. Gergely, B. Nagy and G. M. Szabó, PMC Physics A 1, 4 (2007); G. M. Szabó, L. Á. Gergely and Z. Keresztes, PMC Physics A 1, 8 (2007); Z. Kovacs and L. Á. Gergely, Phys. Rev. D77, 024003 (2008); C. G. Boehmer, T. Harko and F. S. N. Lobo, Class. Quant. Grav. 25, 045015 (2008); T. Harko and V. S. Sabau, Phys. Rev. D77, (2008) 104009; T. Harko, W. F. Choi, K. C. Wong and K. S. Cheng, JCAP 06, 002 (2008); L. Á. Gergely, arXiv:0806.3857 (2008); L. Á. Gergely, arXiv:0806.4006 (2008).

[5] P. Brax and C. van de Bruck, Class. Quant. Grav. 20, R201 (2003); R. Maartens, Living Rev. Rel. 7, 7 (2004); P. Brax, C. van de Bruck and A.C. Davis, Rept. Prog. Phys. 67, 2183 (2004); B. A. Bassett, S. Tsujikawa and D. Wands, Rev. Mod. Phys. 78, 537 (2006).

[6] K. Kajantie and H. Kurki-Suonio, Phys. Rev. D 34, 1719 (1986).

[7] Z. G. Tan and A. Bonasera, Nucl. Phys. A784, 368 (2007).

[8] J. Ignatius, K. Kajantie, H. Kurki-Suonio and M. Laine, Phys. Rev. D49, 3854 (1994).

[9] J. Ignatius, K. Kajantie, H. Kurki-Suonio and M. Laine, Phys. Rev. D50, 3738 (1994).

[10] H. Kurki-Suonio and M. Laine, Phys. Rev. D51, 5431 (1995).

[11] H. Kurki-Suonio and M. Laine, Phys. Rev. D54, 7163 (1996).

[12] M. B. Christiansen and J. Madsen, Phys. Rev. D53, 5446 (1996).

[13] L. Rezzolla, J. C. Miller and O. Pantano, Phys. Rev. D52, 3202 (1995).

[14] L. Rezzolla and J. C. Miller, Phys. Rev. D53, 5411 (1996).

[15] L. Rezzolla, Phys. Rev. D54, 1345 (1996).

[16] L. Rezzolla, Phys. Rev. D54, 6072 (1996).

[17] A. Bhattacharyya, J.-e Alam, S. S. P. Roy, B. Sinha, S. Raha and P. Bhattacharjee, Phys. Rev. D61, 083509 (2000).

[18] A.-C. Davis and M. Lilley, Phys. Rev. D64, 043502 (2000).

[19] N. Borghini, W. N. Cottingham and R. Vinh Mau, J. Phys. G26, 771 (2000).

[20] H. I. Kim, B.-H. Lee and C. H. Lee, Phys. Rev. D64, 067301 (2001).

[21] J. Ignatius and D. J. Schwarz, Phys. Rev. Lett. 86, 2216 (2001).

[22] S. C. Davis, W. B. Perkins, A.-C. Davis and I. R. Vernon, Phys. Rev. D63, 083518 (2001).

[23] C. Germani and R. Maartens, Phys. Rev. D64, 124010 (2001).

[24] R. Maartens, D. Wands, B. A. Bassett and I. P. C. Heard, Phys. Rev. D62, 041301(R) (2000).

[25] T. D. Lee and Y. Pang, Phys. Rept. 221, 251 (1992).

[26] L. D. Landau and E. M. Lifshitz, Statistical Physics, Oxford, Pergamon Press (1980).

[27] P. Shukla and A. K. Mohanty, Phys. Rev. C64, 054910 (2001).

[28] M. Gyulassy, K. Kajantie, H. Kurki-Suonio and L. D. McLerran, Nucl. Phys. B 237, 477 (1984).