Multiscalar-Metric Gravity: Cosmological Constant Screening and Emergence of Massive-Graviton Dark Components of the Universe

Yury F. Pirogov*

Theory Division, Logunov Institute for High Energy Physics of NRC "Kurchatov Institute", Protvino, Moscow region, 142281 Russia

Received February 24, 2022; revised February 24, 2022; accepted April 10, 2022

Abstract—In the multiscalar-metric frameworks, the issues of the vacuum energy/cosmological constant (CC) screening due to Weyl-scale enhancement of the Diff gauge symmetry, along with emergence of the massive dark gravity components through the gravitational Higgs mechanism are considered. A generic dark gravity model is developed, with two extreme versions of the model of particular interest based on general relativity (GR) and its classically equivalent Weyl transverse alternative, being compared and argued to be, generally, inequivalent. The so constructed spontaneously broken Weyl Transverse Relativity (WTR) is proposed as a viable beyond-GR effective field theory of gravity, with screening of the Lagrangian CC, superseded by the induced one, and emergence of the massive tensor and scalar gravitons as dark gravity components. A basic concept with the spontaneously broken Diff gauge symmetry/relativity—in particular, WTR vs. GR—as a principle source of the emergent dark gravity components of the Universe is put forward.

DOI: 10.1134/S0202289322030070

1. INTRODUCTION

The issue of vacuum energy/cosmological constant (CC) in General Relativity (GR) due to the so-called zero-point quantum fluctuations is well-known already for a long time.1 But only recently, in the wake of the appearance of convincing observational evidence for (an effective) CC responsible for the late-time accelerated expansion of the Universe, as expressed by the cosmological Standard Model (SM), or, otherwise, the LCDM model, the CC (or, more generally, the vacuum energy) problem seems to become the most crucial one in the realm of contemporary fundamental physics.2 Though not meaning any explicit discrepancy with observations, this problem signifies, nevertheless, at least a tension between the two present-day basic physical theories: the theory of gravity—GR, and the theory of matter and its interactions—quantum field theory (QFT)—expressed more particularly by the particle SM. Namely, in the effective field theory (EFT) framework, the CC problem is, in fact, at least threefold:3

(I) The (un)naturalness problem: What forbids the huge Lagrangian CC, expected in the EFT framework, from explicit manifestations, with a nonzero tiny observational of CC being (technically) "unnatural"?

(II) The coincidence problem: What determines such a tiny value of the observational CC, so that the latter manifests itself in the inflating expansion of the Universe just “lately” on the cosmological time scale?

(III) The quantum stability problem: What prevents the tiny observed CC from being drastically renormalized by radiative corrections, a priori expected in the EFT framework?

Solving these and related problems presents a challenge to modern fundamental physics and could ultimately imply either a revision of QFT, or a GR modification, with a variety of the latter ones proposed up to now.4 More particularly, to solve the CC problem, modified gravity should, conceivably, be treated as an EFT valid at relatively low energy scales (as compared to the fundamental one given by the Planck mass). At that, GR as EFT is well-known, first, to be constructed from a symmetric second-rank tensor field—the metric $g_{\mu\nu}$, with its determinant $g$ defining a

---

1 E-mail: pirogov@ihep.ru
2 For the observational status of CC see, e.g., [2]. For a modern view of the vacuum energy/CC problem, see, e.g., [3, 4].
3 For the importance of the EFT approach to the CC problem, cf., e.g., [5].
4 For modifications of gravity see, e.g., [6–8], and in particular, [9, 10] for massive gravity.
four-volume element $\sqrt{-g}d^4x$ (in the space-time dimension $n = 4$), with the space-time measure $\sqrt{-g}$, and, second, to be based on the gauge symmetry of the (full) diffeomorphisms (Diff’s) incorporating, in particular, the longitudinal ones. The fact that the space-time measure $\sqrt{-g}$ transforms under longitudinal diffeomorphisms proves to be in GR (and its siblings) an ultimate reason of the emergent CC problem. By this token, we take the issue of $g$ as a guide for choosing a proper route towards solving the CC problems, with some principle steps leading to this goal shortly indicated below.

(i) Unimodular relativity. First of all, long before the CC problem became so acute, for solving such a problem, in [11] (with numerous subsequent elaborations) it was proposed to substitute GR with Unimodular Relativity (UR), known also as Unimodular Gravity, based on full Diff symmetry, or on the transverse diffeomorphism (TDiff $\subset$ Diff) symmetry. By construction, instead of the conventional ten-component (at $n = 4$) metric $g_{\mu\nu}$, UR/TDiff uses a reduced nine-component one $g^a_{\mu\nu}$, with the determinant $\hat{g} = \kappa$ given by a fixed absolute/non-dynamical scalar density $\kappa$ (for simplicity, say, $\kappa = -1$). The latter defines simultaneously the space-time volume element $\sqrt{-\kappa}d^4x$. The Lagrangian CC within UR proves to be irrelevant, being superseded by an integration constant having yet no clear-cut physical meaning. Due to this, UR remains, in a sense, dynamically incomplete. Nevertheless, solving the first part of the CC problem, UR definitely presents a step in the right direction, though hardly sufficient to completely solve the problem, and implies further elaboration.

(ii) Weyl transverse relativity. In the latter respect, in [13] proposed was (cf. also [14]) a ten-component modification of UR constructed from a relative tensor $\hat{g}_{\mu\nu} \equiv g_{\mu\nu}/(-g)^{1/4}$ (at $n = 4$), with a fixed determinant (put for definiteness $\hat{g} = -1$), the former being liable to be converted into a true tensor through multiplication by $(-\kappa)^{1/4}$. By construction, the respective EFT of gravity—to be called Weyl Transverse Relativity (WTR)—satisfies a gauge symmetry, WTDiff, consisting of TDiff enhanced by Weyl scale transformations, and provides ultimately an improved, compared to UR, solution of the CC problem. Like UR/TDiff, its modification WTR/WTDiff may naturally give a justification of the disappearance of the (huge) Lagrangian CC. At the classical level (under covariant conservation of the energy-momentum tensor), WTR/WTDiff proves to be equivalent to GR/Diff, with an arbitrary induced CC to be treated as a global degree of freedom. Moreover, at the quantum level, WTR/WTDiff could ensure quantum stability of the residual classical CC against radiative corrections [17, 18]. Besides, on the field-theoretical side, WTR may be obtained as a viable alternative to GR proceeding in a self-consistent manner, in complete analogy to GR, from a theory of the precisely two-d.o.f. self-interacting massless graviton [19]. Altogether, solving (at least partly) the first and third parts of the CC problem and possessing a solid field-theoretic status, WTR/WTDiff could thus constitute a viable alternative to GR/Diff, realizing one more step in the right direction, with next one(s) still needed. More particularly, within WTR per se (like UR), there appears no indication on an infrared mass scale for a tiny residual CC, with the second part of the CC problem remaining to be resolved, too.

(iii) Higgs mechanism for gravity. The looked-for infrared mass scale could naturally be provided by a massive graviton, serving as the putative dark energy (DE) mimicking the observed CC. At that, the original tensor-graviton mass term due to Fierz and Pauli [20] explicitly violates the Diff gauge symmetry, and was later shown to result in the massive-massless perturbative discontinuity [21, 22], to remedy which, a nonperturbative mechanism for recovering the transition was proposed [23]. It was also found [24] that such an explicit Diff gauge symmetry violation results in emergence of a ghost, with violation of unitarity. Thus, with massless GR treated as a Diff gauge theory, a consistent ghost-free theory of massive gravity would, conceivably, imply not an explicit violation but a spontaneous breaking of such a gauge symmetry by a counterpart of the Higgs mechanism for gravity. An attempt to approach such a goal, modifying GR through the Higgs mechanism in terms of a quartet of scalar fields, was undertaken in [25] (see also [26, 27]), in the wake of which a simple solution to the problem elaborated further in [31, 32] was proposed [28–30]. In this respect, GR augmented by the Higgs mechanism for gravity could naturally provide a required scale for the late exponential expansion of the Universe, associating such a scale with a tiny tensor-graviton mass. The latter, on the other hand, should naturally be treated as a parameter of the theory (given still ad hoc) like

---

5 For the relevance of TDiff gauge symmetry as a substitute of the full Diff one for a consistent description of the massless tensor graviton, see [12].
6 The notation in Introduction is in accord with what follows.
7 For a development of WTR, known also as Weyl Transverse Gravity, cf., e.g., [15, 16].
8 This fact alone could, conceivably, justify WTR (and its siblings) to be further explored at least on par with GR (and its siblings).
any other fundamental parameter of Nature. Out of the four (at \( n = 4 \)) scalar fields used in such a Higgs mechanism within GR, three components can, at a proper choice of the (quasi-)Higgs potential, be absorbed as additional components of the massive tensor graviton. At that, one more component may be stated to become a ghost independently of the potential, though in a sufficiently high order of the perturbation theory [32]. Besides, in the massive-modified GR, still remains unclear the reason of absence of the (huge) Lagrangian CC, as well as a radiative stability of the Higgs mechanism (in the absence of a protecting symmetry). So, the Higgs mechanism in GR is, by its own, hardly sufficient to exhaust the CC problem, too.

(iv) **Quartet-metric gravity.** Altogether, merging the above-mentioned directions of gravity modification to reconcile various parts of the CC problem, with two principle ingredients of the properly modified gravity—an effective metric \( g_{\mu\nu}, \, g = \kappa \), and a (quasi-)Higgs field—to stem from the same source—the scalar quartet—may be obligatory.\(^9\) A new concept should, conceivably, be invoked to this end, with a change of the space-time paradigm [35, 36]. According to the latter, among the arbitrary kinematic observer’s coordinates, there exist some distinct dynamic ones, associated ultimately with the vacuum and given by a quartet (at \( n = 4 \)) of peculiar (invertible) scalar fields. Such a scalar quartet is to be considered on par with the metric as a basic field variable to construct an EFT of the so-called quartet-metric gravity. Introducing in such a framework, on the one hand, a proper effective metric, without adding a (quasi-)Higgs potential for a massive tensor graviton, one can eliminate the (huge) Lagrangian CC as in WTR. On the other hand, introducing the proper (quasi-)Higgs potential without modifying the metric, one can produce, as in GR, a massive tensor graviton serving, conceivably, as DE. Superimposing these two routes of gravity modification within the quartet-modified WTR may, conceivably, solve both the problem of screening the (huge) Lagrangian CC and emergence of the (tiny) observable one supplemented by the associated dark gravity components—massive tensor and scalar gravitons. This is a goal of the present paper.

In Section 2, the basics of the quartet-metric (dark) gravity are exposed. In Section 3, a sufficiently general prototype dark gravity model is constructed to realize the concept of spontaneously broken Diff
gauge symmetry/relativity as a source of the emergent dark gravity components. Two extreme versions of such a generic model of particular interest, corresponding, respectively, to the spontaneous breaking of GR vs. its classically equivalent alternative—WTR—are worked out in Sections 4 and 5. These versions are compared in the context of CC screening and emergence of the dark gravity components through the gravitational Higgs mechanism, and are argued to be, generally, non-equivalent. The consistency of the so constructed spontaneously broken WTR as a theory of massive tensor and scalar gravitons is demonstrated in the weak-field limit. In the Conclusion, viability of spontaneously broken WTR as a beyond-GR EFT of gravity, solving the CC problem through screening the Lagrangian CC, superseding it by the induced one and supplementing it by massive tensor and scalar gravitons as the dark gravity components, is argued. The importance of further developing the basic concept of spontaneously broken Diff gauge symmetry/relativity—in particular, WTR vs. GR—as a principle reason for emergence of the dark gravity components of the Universe is stressed.\(^10\)

2. MULTISCALAR-METRIC DARK GRAVITY: BASICS

2.1. **Multiscalar-Extended Field Set**

Under the term “dark gravity” there will be understood a merger of the (conventional) gravity with the emergent dark gravitational components. The EFT of the multiscalar-metric dark gravity may basically be defined through the generally covariant action functional\(^11\)

\[
S[g_{\mu\nu}, X^a] = \int \mathcal{L}(g_{\mu\nu}, X^a) \, d^n x
\]

in terms of a Lagrangian density \( \mathcal{L} \) depending on a tensor field, the basic metric \( g_{\mu\nu} \) (possessing an inverse \( g^{-1\mu\nu} \)) extended through a set of scalar fields \( X^a \), with \( a, b, \ldots = 0, 1, \ldots, n-1 \) being global Lorentz indices, and \( n \) the space-time dimension. The scalar multiplet \( X^a = X^a(x) \)—the multiscalar—is peculiar by the fact of being (in a patchwise manner) invertible, \( x^\mu = x^\mu(X) \). This allows us to use it as some distinct dynamical space-time coordinates

\(^{10}\) Though adhering primarily to the space-time dimension \( n = 4 \), not to be formally confined to this value, the term “scalar quartet,” valid at \( n = 4 \), is replaced in what follows with a more general one—the “multiscalar”—and, accordingly, “quartet-metric” with “multiscalar-metric,” etc.

\(^{11}\) Here we start directly from the EFT level. For a justification of EFT of the multiscalar-metric gravity as an affine-Goldstone nonlinear model, see [37].
$X^\alpha = \delta^\alpha_\mu X^\mu(x)$ among the arbitrary kinematic ones $x^\mu$, as well as to introduce the respective dynamical frame $X^\mu_\mu \equiv \partial_\mu X^\mu$, $\det(X^\mu_\mu) \neq 0$, possessing an inverse $X^{-1\mu}_\alpha \equiv X^{\mu}_\alpha$. We stress that such a multiscalar is not an auxiliary (non-dynamical) variable introduced just to restore general covariance, but a faithful dynamical variable on par with the basic metric. Ultimately, the dynamical coordinates $X^\alpha$ may be treated as those associated with the (dynamical) vacuum. The action $S$ is, moreover, postulated to be invariant under reparametrizations of $X^\alpha$ given by Lorentz transformations, and finite constant shifts, $X^\alpha \to X^\alpha + C^\alpha$. By this token, $X^\alpha$ should, in fact, enter through a derivative term $\kappa_{\mu\nu}$ constructed as

$$\kappa_{\mu\nu} \equiv \partial_\mu X^\alpha \partial_\nu X^\beta \eta_{\alpha\beta},$$

with $\eta_{\alpha\beta}$ the Lorentz-invariant Minkowski symbol. By default, $\kappa_{\mu\nu}$, serving as a (quasi-)metric, possesses an inverse $\kappa^{-1\mu\nu}$ (at least patchwise with the proper matching), with

$$\kappa \equiv \det(\kappa_{\mu\nu}) = (\det(\partial_\mu X^\alpha))^2\det(\eta_{\alpha\beta}) < 0$$

to ensure non-degeneracy and invertibility of such a (quasi-)metric.\(^{12}\) In terms of the independent field variables, the multiscalar-metric action, in fact, looks like

$$S[\mu_\nu, X^\alpha] = \int \mathcal{L}(g_{\mu\nu}, \kappa_{\mu\nu}) d^n x.$$\(^{(4)}\)

Further, in terms of $\kappa$ and $g = \det(g_{\mu\nu}) < 0$, one can define an effective scalar field

$$\sigma = \ln(\sqrt{-g}/\sqrt{-\kappa}),$$

to be called the scalar graviton. By this token, in terms of the fields $g_{\mu\nu}$ and $\kappa_{\mu\nu}$, one can introduce the effective ones as a conformal metric $\bar{g}_{\mu\nu}$ and a tensor field $\bar{\kappa}_{\mu\nu}$ given by a correlator (partial contraction) of the two metrics as follows:

$$\bar{g}_{\mu\nu} \equiv e^\bar{w} g_{\mu\nu}, \quad \bar{\kappa}_{\mu\nu} \equiv \bar{g}^{\mu\rho} \kappa_{\rho\nu},$$

where $\bar{w} = \bar{w}(\sigma)$ is a gravitational scale factor,\(^{13}\) $\bar{g}_{\mu\nu} \equiv \bar{g}^{-1}_{\mu\nu} = e^{-\bar{w}}g^{-1}_{\mu\nu}$ is an inverse of $\bar{g}_{\mu\nu}$, and $\det(\bar{\kappa}_{\mu\nu}) = \kappa/\bar{g} = e^{-2(\bar{w} + \sigma)}$, $\bar{g} \equiv \det(\bar{g}_{\mu\nu}) < 0$.\(^{14}\) The tensor field $\bar{\kappa}_{\mu\nu}$ may be treated as a realization of the (a priori loose) DE. By means of these effective fields, merging gravity and its dark components, the effective action of the purely multiscalar-metric dark gravity may be presented quite generally as

$$S[\mu_\nu, X^\alpha] = \int \bar{L}(\bar{g}_{\mu\nu}, \bar{\kappa}_{\mu\nu}, \sigma)\sqrt{-\bar{g}} d^n x,$$\(^{(7)}\)

where $\bar{L}$ is an effective Lagrangian defining, in line with $\bar{w}$, a particular dark gravity model. Since not all of the variables in such a form of the effective action are independent, the action is ultimately still a function of the independent basic variables $g_{\mu\nu}$ and $X^\alpha$. Lastly, by means of the frame $X^\mu_\mu \equiv \partial_\mu X^\alpha$ and its inverse $X^{\mu}_\alpha$, the DE tensor $\bar{\kappa}_{\mu\nu}$ may equivalently be converted into a scalar form as $\bar{H}^a_b \equiv X^\alpha \bar{a}^a_{\mu\nu} X^\beta_b$, with $\det(\bar{g}) = \det(\bar{H}^a_b)$, where

$$\bar{H}^a_\eta^{\mu\nu} \equiv \bar{H}^{ab} = \bar{g}^{\kappa\lambda} \partial_\kappa X^\alpha \partial_\lambda X^b$$

is a modification of the original (quasi-)Higgs field for gravity [28–30]. Ultimately, the effective action in the form (7), with the dependence on the DE field $\bar{g}_{\mu\nu}$ expressed through the scalar potential $\bar{V}_{\eta}(\bar{g}_{\mu\nu}) = \bar{V}_H(\bar{H}^{ab})$, is aimed at spontaneous breaking of the Diff gauge symmetry/relativity, producing the dark gravity components ultimately due to the multiscalar.

2.2. Weyl-Scale Enhanced Diff Gauge Symmetry

Besides a set of fields, any EFT is basically characterized by a pattern of its symmetries. For the generic multiscalar-metric dark gravity, the effective action (7) is assumed to be generally covariant and gauge-invariant under the (full) Diff symmetry given by a Lie derivative $D_\xi$. The latter depends on an (infinitesimal) vector parameter $\xi^\mu(x)$ and may by default be constructed in terms of a covariant derivative $\nabla_\lambda(g_{\mu\nu})$ defined by the basic metric $g_{\mu\nu}$, subsequently expressed through the effective fields as $g_{\mu\nu} = e^{-\bar{w}(\sigma)}\bar{g}_{\mu\nu}$. Under $D_\xi$, all basic fields transform conventionally in accord with their covariant tensor structure still irrespective of their nature. To account for the different origin of the basic fields, consider the (local) Weyl scale transformations distinguishing such fields as follows:\(^{15}\)

$$\Delta_\xi g_{\mu\nu} = \zeta g_{\mu\nu}, \quad \Delta_\xi X^\alpha = 0,$$\(^{(9)}\)

where $\zeta(x)$ is an arbitrary (infinitesimal) scalar parameter, so that\(^{16}\)

$$\Delta_\xi g = 4\zeta g.$$\(^{(10)}\)

\(^{12}\) To this end, we assume the (patchwise) Lorentz-preservation in the field space of $X^\alpha$, up to the physically equivalent affine redefinitions of the fields $X^\alpha$ and the metric $\eta_{\alpha\beta}$.

\(^{13}\) The dependence $\bar{w}(\sigma)$ presents the simplest case. More generally, one could envisage within EFT the dependence of the effective metric scale $\bar{w}$ on the matter fields, the environment, etc, with the ultraviolet and infrared behavior of the effective metric being a priori quite different.

\(^{14}\) By default, the space-time indices are now lowered (raised) through the effective metric $\bar{g}_{\mu\nu}$ ($\bar{g}^{\mu\nu}$) if not stated explicitly otherwise.

\(^{15}\) A priori, the Weyl scale symmetry is assumed to be specifically a metric one, with the conventional matter fields, like $X^\alpha$, to be inert under Weyl rescalings, unless stated otherwise.

\(^{16}\) For definiteness, we restrict the consideration here and in what follows to the space-time dimension $n = 4$. 

GRAVITATION AND COSMOLOGY Vol. 28 No. 3 2022
\[
\Delta \zeta \left( g_{\mu \nu}/(-g)^{1/4} \right) = 0, \\
\Delta \zeta \kappa_{\mu \nu} = \Delta \zeta \kappa = 0. 
\]  
(10)

For the effective fields, this implies in turn
\[
\Delta \zeta \sigma = 2 \zeta, \\
\Delta \zeta \bar{g}_{\mu \nu} = \zeta (1 + 2 \bar{w}') \bar{g}_{\mu \nu}, \\
\Delta \zeta \bar{g} = 4 \zeta (1 + 2 \bar{w}') \bar{g}, \\
\Delta \zeta \bar{\alpha}_{\mu}^{\nu} = - \zeta (1 + 2 \bar{w}') \bar{\alpha}_{\mu}^{\nu}, 
\]  
(11)

where \( \bar{w}' = d\bar{w}/d\sigma \). Generally, this reflects the explicit Weyl scale violation in the multiscalar-metric dark gravity.\(^\text{17}\) In the exceptional case \( \bar{w}' = -1/2 \), one independent component of \( \bar{g}_{\mu \nu} \), namely, the determinant \( g \), is missing from \( \bar{g}_{\mu \nu} \) and \( \bar{\alpha}_{\mu}^{\nu} \). At that, the residual dependence of the effective action (7) on \( g \) is mediated only by \( \sigma \). With \( g \) being a principle source of the CC problem, such a Weyl scale symmetry, signifying otherwise the \( g \)-independence, is of particular importance for CC screening in an exceptional case of a generic dark gravity model to be exposed in what follows.

3. SPONTANEOUS DIFF GAUGE SYMMETRY/RELATIVITY BREAKING

3.1. Generic Dark Gravity Model

To be more particular with the (multiscalars) spontaneous breaking of the Diff gauge symmetry (or, otherwise, relativity) consider a prototype dark gravity model merging tensor gravity and the associated dark gravity components, with the effective Lagrangian in a restricted but still rather general partitioned form,
\[
\tilde{L} = \tilde{L}_G(\bar{g}_{\mu \nu}, \bar{\alpha}_{\mu}^{\nu}) + \tilde{L}_M(\bar{g}_{\mu \nu}, \sigma, \phi_I) + \Delta \tilde{L},
\]  
(12)

consisting of an extended gravity Lagrangian \( \tilde{L}_G \), an extended matter Lagrangian \( \tilde{L}_M \), and the rest, \( \Delta \tilde{L} \). By default, \( \tilde{L}_G \) may depend, a priori arbitrarily, on the tensor fields \( \bar{g}_{\mu \nu} \) and \( \bar{\alpha}_{\mu}^{\nu} \) (and thus on \( \kappa_{\mu \nu} \)), while \( \tilde{L}_M \) is assumed to be independent from \( \bar{\alpha}_{\mu}^{\nu} \) and to depend only minimally on \( \bar{g}_{\mu \nu} \), but a priori arbitrarily on the scalar graviton \( \sigma \) and the generic matter fields \( \phi_I, I = 1, \ldots \) The mixing part \( \Delta \tilde{L} \) depending, generally, on all variables, is assumed to be negligible in the main approximation.\(^\text{18}\) More particularly, adopt for \( \tilde{L}_G \) the partitioned form
\[
\tilde{L}_G = \tilde{L}_A + \tilde{L}_g(\bar{g}_{\mu \nu}) - \bar{V}_w(\bar{\alpha}_{\mu}^{\nu}),
\]  
(13)

where the conventional gravity part, \( \tilde{L}_g \), is taken for simplicity in the GR-like form for \( \bar{g}_{\mu \nu} \):
\[
\tilde{L}_g = - \frac{1}{2} M_{pl}^2 R(\bar{g}_{\mu \nu}),
\]  
(14)

with \( M_{pl} = 1/(8\pi G_N)^{1/2} \) the Planck mass and \( R = R(\bar{g}_{\mu \nu}) \) the Ricci scalar curvature.\(^\text{19}\) This term is supplemented with a constant one,
\[
\tilde{L}_A \equiv - M_{pl}^2 \Lambda,
\]  
(15)

where \( \Lambda \) is an effective Lagrangian CC. Lastly, \( \bar{V}_w \) is a scalar potential depending on the traces of products of the DE tensor field \( \bar{\alpha}_{\mu}^{\nu} \),\(^\text{20}\)
\[
\bar{V}_w = M_{pl}^2 \sum \bar{v}_{n_1 n_2 \ldots} \left( \text{tr} (\bar{\alpha}_{\mu}^{\nu})^n \right),
\]  
(16)

with a priori arbitrary coefficients \( \bar{v}_{n_1 n_2 \ldots} \). The powers \( n_1 = 0, 1, 2, \ldots \) are to be positive and finite for the potential to be analytic near \( \bar{\alpha}_{\mu}^{\nu} = 0 \), allowing in such a case a smooth limit to GR. The potential may be normalized as \( \bar{V}_w |_{\Lambda = 0} = 0 \) at an a priori arbitrary background point in the unbroken or spontaneously broken phases, say, at \( \bar{\alpha}_{\mu}^{\nu} = 0 \), \( \bar{\alpha}_{\mu}^{\nu} = \delta_{\mu}^{\nu} \), respectively, defining, in turn, the proper normalization of \( \Lambda \). In what follows, we restrict the consideration by the simplest quartic potential which proves to be sufficient in the leading approximation (see Section 5). The consistency of the theory in higher-order approximations may require a further elaboration of the potential, including, in particular, fractional powers of \( \bar{\alpha}_{\mu}^{\nu} \), as well as negative ones, corresponding to \( (\bar{\alpha}_{\mu}^{\nu})^{-1} \equiv \bar{\alpha}_{\mu}^{\nu} = \kappa^{-1} \epsilon_{\mu \nu} g_{\lambda \sigma} \), etc.\(^\text{21}\) Finally, the extended matter Lagrangian \( \tilde{L}_M \) (incorporating, conceivably, some kinds of the particle DM in addition to the scalar-graviton dark component) remains a priori arbitrary, elaborating further the purely tensor dark gravity due to \( \tilde{L}_G \).

3.2. Generic Field Equations

A straightforward way of deriving the dynamical field equations (FEs) for EFT of the multiscalar-metric dark gravity is to extremize its effective action with respect to the independent variations of the

\(^{17}\)Note, though, that the exclusively derivative couplings of \( \sigma \) still admit the residual global Weyl scale symmetry \( g_{\mu \nu} \rightarrow e^{\delta_0} g_{\mu \nu} \), with constant shifts \( \sigma \rightarrow \sigma + 2\delta_0 \).

\(^{18}\)In the spirit of EFT, \( \tilde{L}_G \) is assumed to be of the order of the Planck mass squared, \( M_{pl}^2 \), while \( \tilde{L}_M \) and \( \Delta \tilde{L} \) to be relatively suppressed at the scale of \( M_{pl} \).

\(^{19}\)In principle, any modification of \( \tilde{L}_g \), like \( f(\mathcal{R}) \), etc., is conceivable too.

\(^{20}\)At that, \( \text{det}(\bar{\alpha}_{\mu}^{\nu}) = \kappa/\bar{g} = e^{-2\epsilon_{\mu \nu} \epsilon_{\lambda \sigma} g_{\lambda \sigma}} \) is explicitly missing, being expressed, in principle, through \( \sigma \) accounted in \( \tilde{L}_M \).

\(^{21}\)A priori, one may admit another mode of spontaneous breaking, with the potential \( \bar{V}_w \) depending, instead of \( \bar{\alpha}_{\mu}^{\nu} \), on its inverse \( \bar{\alpha}_{\mu}^{-\nu} \), subject to the analyticity requirement near \( \bar{\alpha}_{\mu}^{-\nu} = 0 \) and possessing by the flat background at the symmetric point \( \bar{\alpha}_{\mu}^{-\nu} = \bar{\alpha}_{\nu}^{\mu} \).
primary metric $g_{\mu\nu}$ (or, rather, its inverse $g^{-1\mu\nu}$) and the multiscalar $X^a$, expressing afterwards the coefficients at the independent variations $\delta g^{-1\mu\nu}$ and $\delta X^a$ through the effective fields $\bar{g}_{\mu\nu}$ (and its inverse $\bar{g}^{\mu\nu}$), as well as $\sigma$ (in particular, $\bar{w}'(\sigma)$). By this token, the effective variations look like

$$\delta \bar{g}^{\mu\nu} = e^{-\bar{w}} \delta g^{-1\mu\nu} - \bar{w}' \bar{g}^{\mu\nu} \delta \sigma,$$

$$\delta \sqrt{-\bar{g}} = -\frac{1}{2} \bar{g}_{\kappa\lambda} \delta \bar{g}^{\kappa\lambda},$$

$$\delta \sqrt{-\kappa} = \frac{1}{2} \kappa^{-1\kappa\lambda} \delta \kappa_{\kappa\lambda},$$

with a prime meaning a derivative with respect to $\sigma$, as well as

$$\delta \bar{w}^{\mu}_{\nu} = \bar{g}^{\mu\kappa} \delta \kappa_{\nu\lambda} + \kappa^{\nu\lambda} \delta \bar{g}^{\mu\kappa}.$$  \hspace{1cm} (18)

In the above, one should also account for the constraints (5) and (2) (implying $\sigma$ and $\kappa_{\mu\nu}$ to be the effective composite fields), resulting to

$$\delta \sqrt{-g} = -\frac{1}{2} e^{-w} \bar{g}_{\kappa\lambda} \delta \bar{g}^{\kappa\lambda},$$

$$\delta \sqrt{-\kappa} = \frac{1}{2} \kappa^{-1\kappa\lambda} \delta \kappa_{\kappa\lambda},$$

in the respective variation of $\sigma$

$$\delta \sigma = \frac{\delta \sqrt{-g}}{\sqrt{-g}} - \frac{\delta \sqrt{-\kappa}}{\sqrt{-\kappa}},$$

$$= -\frac{1}{2} (e^{-w} \bar{g}_{\kappa\lambda} \delta \bar{g}^{\kappa\lambda} + \kappa^{-1\kappa\lambda} \delta \kappa_{\kappa\lambda}),$$  \hspace{1cm} (20)

where

$$\delta \kappa_{\mu\nu} = (X^a_{\mu} \delta X^b_{\nu} + X^a_{\nu} \delta X^b_{\mu}) \eta_{ab},$$

$$\delta X^a_\mu \equiv \partial_\mu \delta X^a.$$  \hspace{1cm} (21)

Extremizing then the effective action (7) given by the effective Lagrangian (12)–(16), with respect to the independent variations $\delta g^{-1\mu\nu}$ and $\delta \kappa_{\mu\nu}$ (the latter expressed finally through $\delta X^a$), we get the generic (under an arbitrary $\bar{w}$) FEs:

$$R_{\mu\nu} - \frac{1}{4} \bar{R} \bar{g}_{\mu\nu} - \frac{1}{M^2_{Pl}} (\bar{T}_{\mu\nu} - \frac{1}{4} \bar{T} \bar{g}_{\mu\nu}) = 0,$$

$$-\frac{1}{4} (1 + 2\bar{w}') \left( \bar{R} + 4\Lambda + \frac{1}{M^2_{Pl}} \bar{T} \right) + \frac{1}{2} \frac{\delta \bar{L}_M}{\delta \sigma} = 0,$$

$$\nabla^\lambda \left[ \frac{1}{M^2_{Pl}} \bar{g}^{\lambda\rho} \frac{\partial \bar{V}_w}{\partial \bar{g}^{\lambda\rho}} X_{\kappa\alpha} \right] + \frac{1}{8} \left( \bar{R} + 4\Lambda + \frac{1}{M^2_{Pl}} \bar{T} \right) X^\lambda_a = 0,$$

$$\frac{\delta \bar{L}_M}{\delta \phi_I} = 0.$$  \hspace{1cm} (22)

for the three sectors of the multiscalar–metric dark gravity—the tensor–traceless, tensor–trace and multiscalar, respectively—supplemented by a FE for matter. Here and in what follows, $\partial/\partial \sigma$ and $\delta/\delta \sigma$ mean, respectively, the partial and total (incorporating the derivatives with respect to the derivatives of the proper fields) variational derivatives.\footnote{At that, the scalar–graviton FE as such is absent, with the composite $\sigma$ being, generally, off–mass–shell, $\delta \bar{L}_M/\delta \sigma \neq 0$.} Besides, $X_{\kappa\alpha} \equiv X^b_\kappa \eta_{ba}$, and

$$\bar{T}_{\mu\nu} \equiv \bar{T}_{\mu\nu} + \bar{T}_{\mu\nu},$$

$$\bar{T} \equiv \bar{g}^{\kappa\lambda} \bar{T}_{\kappa\lambda} = \bar{T}_{\alpha} + \bar{T}_{\mu\nu},$$  \hspace{1cm} (23)

is the total energy–momentum tensor for the DE field $\bar{\phi}^{\nu}_{\mu}$ and the extended matter $M = (\sigma, \phi_I)$, with the following partial contributions:

$$\bar{T}_{\mu\nu} = -\left( \frac{\partial \bar{V}_w}{\partial \bar{g}^{\mu\lambda}} \bar{g}_{\nu\lambda} + \frac{\partial \bar{V}_w}{\partial \bar{g}^{\nu\lambda}} \bar{g}_{\mu\lambda} + \bar{V}_w \bar{g}_{\mu\nu} \right),$$

$$\bar{T}_{\mu\nu} = \frac{2}{\sqrt{-g}} (\sqrt{-g} \bar{L}_M) = 2 \frac{\partial \bar{L}_M}{\partial \bar{g}^{\mu\nu}} - \bar{L}_M \bar{g}_{\mu\nu}.$$  \hspace{1cm} (24)

A priori, the derived generic FEs do not automatically imply covariant conservation of the total energy–momentum tensor, to be discussed below.

### 3.3. Geometry Consistency Condition

To explicitly account for the Riemannian structure of space–time in terms of the effective metric $\bar{g}_{\mu\nu}$, apply the proper covariant derivative $\nabla_{\mu}$ to the sum of the trace–free and trace (times $\bar{g}_{\mu\nu}$) parts of the tensor–gravity FEs. Due to the reduced Bianchi identity, $\nabla^{\lambda}(\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu}) = 0$, this gives the geometric consistency condition

$$\frac{1}{2} \frac{\partial}{\partial \mu} \left( \bar{w}' (\bar{R} + 4\Lambda + \frac{1}{M^2_{Pl}} \bar{T}) \right) = \frac{1}{M^2_{Pl}} \nabla^{\lambda} \Xi_{\lambda\mu}.$$  \hspace{1cm} (25)

to be satisfied for all solutions to the multiscalar–metric FEs. In the above, the tensor

$$\Theta_{\mu\nu} \equiv \bar{T}_{\mu\nu} + \bar{\theta}_{\mu\nu} = \bar{T}_{\mu\nu} + \bar{T}_{\mu\nu} + \bar{\theta}_{\mu\nu},$$  \hspace{1cm} (26)

is an extension of the total energy–momentum tensor accounting also for the off–mass–shell contribution

$$\bar{\theta}_{\mu\nu} \equiv -\frac{\partial \bar{L}_M}{\partial \bar{g}_{\mu\nu}}.$$  \hspace{1cm} (27)

of the (composite) scalar graviton $\sigma$.\footnote{Note that the value $\bar{\theta}_{\mu\nu} \neq 0$ additionally closes or opens a space–time region, i.e., acts in this region as an effective dark matter (DM) or DE, depending on $\delta \bar{L}_M/\delta \sigma$ being positive or negative, respectively.} A priori, $\Theta_{\mu\nu}$ is not bound to covariantly conserve, like any its part (26) separately, in particular, $\bar{\theta}_{\mu\nu}$. Covariant conservation of $\Theta_{\mu\nu}$ can still be imposed at an
arbitrary \( \tilde{w} \) as an additional restriction on the solutions. The two extreme versions of the so constructed generic dark gravity model, with \( \tilde{w}' = 0 \) and \(-1/2\), being of special interest—as the spontaneously broken GR vs. WTR, respectively—are worked out in more detail below.

### 3.4. Linear Approximation

To illustrate the spontaneous breaking of the Diff gauge symmetry/relativity, expand the basic fields \( g_{\mu\nu} \) and \( X^a \) against the background ones \( g_{*\mu\nu} \) and \( X^a_* \), supplemented by some perturbations:

\[
g_{\mu\nu} \equiv g_{*\mu\nu}(x) + h_{\mu\nu}(x), \\
X^a \equiv X^a_*(x) + \chi^a(x).
\]  
(28)

Choose then the proper background coordinates \( x^*\nu \equiv \delta^a_{\nu} X^a_*(x) \), with the flat background looking in such coordinates as follows:

\[
g_{*\alpha\beta} = \delta_{\alpha\beta}, \\
\tilde{a}^{\pm1}_{\alpha\beta} = \delta_{\beta}^{\alpha}, \quad \sigma_* = 0,
\]  
(29)

where \( \eta_{\alpha\beta} \) is the Minkowski symbol. Identifying the global Lorentz indices \( a, b, \ldots \) with the flat background ones \( \alpha, \beta, \ldots \) results then, for the basic fields, in the linear approximation (LA) in the relations

\[
g^{\pm1}_{\alpha\beta} = \eta_{\alpha\beta} \pm h_{\alpha\beta}(x), \\
X^{\pm1}_{\alpha} = \delta^{\alpha}_{\alpha} \pm \partial_{\alpha} \chi^{\alpha}(x_*), \\
\kappa^{\pm1}_{\alpha} = \eta_{\alpha\beta} \pm k_{\alpha\beta}(x_*), \\
k_{\alpha\beta} \equiv \partial_{\alpha} \chi_{\beta}(x_*) + \partial_{\beta} \chi_{\alpha}(x_*),
\]  
(30)

with \( \partial_{\alpha} \equiv \partial/\partial x^a_\alpha \) and \( \chi_{\alpha} \equiv \eta_{\alpha\beta} \chi^{\beta} \), etc. Here and henceforth, the tensors are explicitly marked vs. their inverse values by the superscripts \( \pm1 \). For the effective fields, this in turn implies in the LA:

\[
g^{\pm1}_{\alpha\beta} = \eta_{\alpha\beta} \pm \tilde{h}_{\alpha\beta}, \\
\tilde{a}^{\pm1}_{\alpha\beta} = \delta_{\beta}^{\alpha} \pm \tilde{f}_{\beta},
\]  
(31)

and \( \sigma = 0 \). Designating in the LA \( \tilde{w} = \tilde{w} \tilde{\sigma} \), with \( \tilde{w} \) a constant interpolating between \( \tilde{w} = 0 \) for modified GR and \( \tilde{w} = -1/2 \) for modified WTR, we get in the same approximation:

\[
\tilde{h}_{\alpha\beta} \equiv h_{\alpha\beta} + \tilde{\alpha}/2(h - 2\partial \chi) \eta_{\alpha\beta}, \\
\tilde{f}_{\alpha\beta} \equiv f_{\alpha\beta} + \tilde{f}_{\beta} = \tilde{h}_{\alpha\beta} - k_{\alpha\beta}, \\
\tilde{f} = (1 + 2\tilde{\alpha})(h - 2\partial \chi), \\
s = (1/2)(h - 2\partial \chi),
\]  
(32)

with \( \partial \chi \equiv \partial_{\alpha} \chi^\alpha, \ h = \eta_{\alpha\beta} \eta^{\alpha\beta}, \ f = \tilde{f}_{\alpha\beta} \eta^{\alpha\beta}, \) etc. By construction, under the Diff gauge symmetry enhanced by the Weyl rescalings, for the basic fields it holds:

\[
\chi^\alpha \rightarrow \chi^\alpha + \xi^\alpha,
\]

\[
k_{\alpha\beta} \rightarrow k_{\alpha\beta} + (\partial_{\alpha} \xi_{\beta} + \partial_{\beta} \xi_{\alpha}), \\
\partial_{\alpha} (\chi_{\beta} + \xi_{\beta}) + \partial_{\beta} (\chi_{\alpha} + \xi_{\alpha}),
\]

\[
h_{\alpha\beta} \rightarrow h_{\alpha\beta} + (\partial_{\alpha} \xi_{\beta} + \partial_{\beta} \xi_{\alpha}) + \zeta \eta_{\alpha\beta}, \\
h \rightarrow h + 2\partial \xi + 4\zeta,
\]  
(33)

with arbitrary functions \( \xi^\alpha(x_*) \) and \( \zeta(x_*) \) as gauge parameters. For the effective fields, this results in LA in the transformations

\[
\tilde{h}_{\alpha\beta} \rightarrow \tilde{h}_{\alpha\beta} + (\partial_{\alpha} \xi_{\beta} + \partial_{\beta} \xi_{\alpha}) + (1 + 2\tilde{\alpha})\zeta \eta_{\alpha\beta}, \\
\tilde{f}_{\alpha\beta} \rightarrow \tilde{f}_{\alpha\beta} + (1 + 2\tilde{\alpha})\zeta \eta_{\alpha\beta},
\]  
(34)

Otherwise, \( \tilde{h}_{\alpha\beta} \) is a tensor under Diff, coinciding in the “unitary” gauge \( \chi^\alpha = 0 \) (and thus \( k_{\alpha\beta} = 0 \)), according to (32), with the Diff gauge-invariant \( \tilde{f}_{\alpha\beta} \), whereas \( s \) is a scalar under Diff. Hence, the fields \( \tilde{f}_{\alpha\beta} \) and \( s \) may be used to describe a spontaneous Diff gauge symmetry/relativity breaking in dark gravity in the explicitly Diff gauge-invariant terms. At that, \( \tilde{f}_{\alpha\beta} \) at \( \tilde{\alpha} \neq -1/2 \) and \( s \) explicitly violate the Weyl-scale gauge symmetry, with \( s \) serving, in effect, as a Weyl scale in disguise.\(^{24}\)

### 4. Spontaneously Broken General Relativity

Let first \( \tilde{w}' = 0 \), so that without loss of generality \( \tilde{w} = w = 0 \) and \( g_{\mu\nu} = g_{\mu\nu} \), with the bar sign to be dropped-off everywhere. In the Lagrangian \( L = L_G + L_M \), both \( L_G \) and \( L_M \) are Diff gauge invariant, with the explicitly violated Weyl scale symmetry. The FEs (22) can now be combined as follows:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} - \frac{1}{M^2} \Theta_{\mu\nu} = 0,
\]

\[
\nabla_{\lambda}(g^{\lambda\nu} \frac{\partial V}{\partial \epsilon_\nu} X_{\alpha} + \frac{1}{2} \frac{\delta L_M}{\delta \sigma} X^\lambda_{\alpha}) = 0,
\]

\[
\frac{\delta L_M}{\delta \phi_1} = 0,
\]  
(35)

with the scalar-graviton FE \( \delta L_M/\delta \sigma = 0 \) as such being, generally, missing. Due to the reduced Bianchi identity, the extended total energy-momentum tensor \( \Theta_{\mu\nu} \) should, for the geometrical consistency, be covariantly conserved for all solutions to the FEs, \( \nabla^\lambda \Theta_{\lambda\nu} \equiv 0 \), though separately each of its three contributions to the total sum (26) may generally, not satisfy this requirement. In general, the FEs (35) correspond to the multiscalar-modified/spontaneously

\(^{24}\) To be more concise, the scalar graviton, being a measure of the (Weyl) scale transformations, may be called the systolon, with the tensor graviton being conventionally the graviton.
broken GR, with the Lagrangian CC $\Lambda$ supplemented by the emergent tensor- and scalar-graviton dark components contributing through $\Theta_{\mu\nu}$. Under the conventional FE for the scalar graviton $\sigma$, $\delta L_M/\delta \sigma = 0$, and the absence of tensor DE, $V_{\bar{w}} \equiv 0$, the multiscalar part of the FEs (35) is clearly trivial, with the tensor FE reducing due to $T_{\mu\nu} = 0$ to that of unbroken GR supplemented by a scalar field $\sigma$.

In the LA, at $\vec{a} = \alpha = 0$, one has $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$, with the tensor and scalar dark gravity fields

$$f_{\alpha\beta} = h_{\alpha\beta} - k_{\alpha\beta}, \quad f = h - 2\partial \chi, \quad s = (1/2)(h - 2\partial \chi),$$

(36)

transforming under the enhanced Diff gauge symmetry as follows:

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} + (\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha) + \zeta \eta_{\alpha\beta},$$

$$f_{\alpha\beta} \rightarrow f_{\alpha\beta} + \zeta \eta_{\alpha\beta}, \quad s \rightarrow s + 2\zeta,$$

(37)

resulting in Weyl scale symmetry to be explicitly violated. At that, the trace metric part $h$ enters in both the Diff gauge-invariant tensor field $f_{\alpha\beta}$ and scalar $s$, and, generally, results in a ghost. In the case with missing $\sigma$, $\delta L_M/\delta \sigma \equiv 0$, under a proper choice of the potential $V_{\bar{w}}$, there appears, in the broken phase, a massive tensor graviton with $h$ as a ghost missing. But the ghost inevitably reappears in a sufficiently high order of the perturbation theory [28–32]. This may, conceivably, be traced back to the absence of a protecting symmetry. On the other hand, in the reduced case $V_{\bar{w}} = 0$, the tensor graviton remains massless, with $h$ in the transverse gauge $\partial \chi = 0$ becoming the physical massive scalar graviton.25

The general tensor-scalar mixed case beyond the LA, with both $V_{\bar{w}} \neq 0$ and $\delta L_M/\delta \sigma \neq 0$, is to be investigated.

Altogether, though being able to produce the dark gravity components, spontaneously broken GR at its face value can hardly help in solving the entire CC problem, the latter still to be resolved by some other means. Nevertheless, an alternative to GR classically equivalent to the latter up to CC—WTR—proves to be a priori well suited to this end, as is discussed in more detail below, with the case of GR reserved as the canonical reference one.

5. SPONTANEOUSLY BROKEN WEYL TRANSVERSE RELATIVITY

5.1. Full Nonlinear Theory

5.1.1. General field equations. Let then $\bar{w}' = -1/2$, with the effective gravity scale factor $\bar{w} = \bar{w} \equiv -\sigma/2$. Replacing in this case everywhere the bar-sign with a hat, one has for the effective metric

$$\hat{g}_{\mu\nu} = e^{-\sigma/2}g_{\mu\nu} = (\kappa/g)^{1/4}g_{\mu\nu}, \quad \hat{g} = \kappa,$$

(38)

with $\det(\hat{\delta}_{\mu\nu}) = \kappa/\hat{g} = 1$. In the effective Lagrangian $\hat{L} = \hat{L}_G + \hat{L}_M$, the pure tensor-gravity part $\hat{L}_G$ possesses Diff gauge symmetry enhanced by the “hidden” local Weyl scale symmetry. Due to that, one independent component in $\hat{L}_G$ is, in fact, missing. With account for $\hat{L}_M$, the missing component reappears again as $\sigma$ signifying an explicit violation of the local Weyl scale symmetry. The FEs (22) now read

$$\hat{R}_{\mu\nu} - \frac{1}{4}\hat{R}\hat{g}_{\mu\nu} - \frac{1}{M_{Pl}^2}\left(\hat{T}_{\mu\nu} - \frac{1}{4}\hat{T}\hat{g}_{\mu\nu}\right) = 0,$$

$$\nabla_\lambda \left[ \frac{1}{M_{Pl}^4} \hat{g}^{\lambda\rho} \frac{\partial \hat{T}_{\rho\sigma}}{\partial \hat{g}^{\sigma\alpha}} X^\alpha + \frac{1}{8} (\hat{R} + 4\hat{\Lambda} + \frac{1}{M_{Pl}^2}\hat{T}) X^\lambda \right] = 0,$$

$$\frac{\delta \hat{L}_M}{\delta \sigma} = \frac{\delta \hat{L}_M}{\delta \phi_I} = 0,$$

(39)

with $\hat{T}_{\mu\nu} = \hat{T}_{\mu\nu} + \hat{T}_{M\mu\nu}$ given by Eq. (24). The tensor-gravity FE (39) is trace-free, being in principle in the context of the CC problem.26 In the multiscalar-metric framework, such a FE alone is, in fact, incomplete, to be supplemented by a multiscalar one. At that, the Lagrangian CC $\Lambda$ clearly influences the tensor gravity just indirectly through the multiscalar FE. A crucial point here is that the dynamical measure $\sqrt{-\kappa}$ becoming, in a sense, non-gravitating, depends entirely on the multiscalar $X^\alpha$, but not on the basic metric $g_{\mu\nu}$. The scalar graviton behaves now like a conventional scalar particle, with $\delta_{\mu\nu} \equiv -\delta \hat{L}_M/\delta \hat{g}_{\mu\nu} = 0$. Due to the general covariance and the conventional FEs for $\sigma$ and $\phi_I$, the extended matter energy-momentum tensor $\hat{T}_{M\mu\nu}$ should be covariantly conserved, $\nabla^\lambda \hat{T}_{M\lambda\mu\nu} = 0$, whereas the DE $\hat{T}_{\mu\nu}$ and thus the total $\hat{T}_{\mu\nu}$ may, generally, not satisfy this requirement.27

5.1.2. Covariant conservation condition. Impose additionally on a class of solutions to Eqs. (39) the condition of covariant conservation of the total energy-momentum tensor $\hat{\Theta}_{\mu\nu} = \hat{T}_{\mu\nu}$ in Eq. (25),

25 For the reduced case $V_{\bar{w}} \equiv 0$, with the (ultralight) physical $\sigma$ serving as a scalar DE superseding the Lagrangian CC, see [38]. For the similar case with $\sigma$ as a scalar DM, cf. [39].

26 For viability of the trace-free tensor-gravity FE in cosmology, cf., e.g., [40, 41].

27 In addition to the scalar graviton $\sigma$, the extended matter Lagrangian $\hat{L}_M = \hat{L}_M(\hat{g}_{\mu\nu}, \sigma, \phi_I)$ could, in principle, account for the (more conventional) DM components. In the absence of matter, the purely scalar-graviton Lagrangian $\hat{L}_M = \hat{L}_s(\hat{g}_{\mu\nu}, \sigma)$ could be sought for, e.g., in a most general form for the scalar field $\sigma$ satisfying the second-order FE to avoid the explicit Ostrogradsky instabilities [42, 43].
resulting upon integration in the relation
\[ \dot{R} + 4\dot{\Lambda}_0 + \frac{1}{M^2_{Pl}} \dot{T} = 0, \quad (40) \]
where \( \dot{\Lambda}_0 \) is an integration constant, in particular, \( \dot{\Lambda}_0 = 0 \) or the Lagrangian CC \( \dot{\Lambda} \). Incorporating this relation refines, in turn, the tensor-gravity and multiscalar FE as follows:
\[ \frac{1}{M^2_{Pl}} \dot{g}_{\mu\nu} - \dot{\Lambda}_0 \dot{g}_{\mu\nu} - \frac{1}{M^2_{Pl}} \dot{T}_{\mu\nu} = 0, \]
\[ \ddot{\nabla} \lambda \left( \frac{1}{M^2_{Pl}} \dot{g}_{\mu\nu} \frac{\partial V_{\alpha}}{\partial \dot{\chi}_{\alpha}} \chi_\alpha + \frac{1}{2} (\dot{\Lambda} - \dot{\Lambda}_0) \chi_\alpha \right) = 0, \]
\[ \frac{\delta L_M}{\delta \sigma} = \frac{\delta L_M}{\delta \sigma_I} = 0, \quad (41) \]
with the Lagrangian CC \( \dot{\Lambda} \) clearly screened from the tensor gravity FE, and \( \dot{T}_{\mu\nu} \) explicitly covariantly conserved due to the reduced Bianchi identity. Moreover, due to \( \nabla^\lambda T_{\mu\nu} = 0 \) there should now separately fulfill \( \nabla^\lambda T_{\alpha\lambda\nu} = 0 \). Under an arbitrary \( \dot{V}_{\alpha\nu} \), FEs (41) describe the multiscalar-modified/spontaneously broken WTR, with DE manifesting itself through the covariantly conserved \( \dot{T}_{\alpha\mu\nu\lambda} \), as well as the induced Lagrangian CCs \( \dot{\Lambda}_0 \) and \( \dot{\Lambda} \), respectively. In the case \( \dot{V}_{\alpha\nu} \equiv 0 \), the multiscalar part of the FEs (41) factorizes due to \( \dot{g} = \kappa \) as \( \partial_\chi (\sqrt{-\kappa} X_\chi) \). By this token, \( \kappa \) can be chosen arbitrarily, looking formally non-dynamical (similarly to UR). With account for \( \dot{\Lambda}_{\alpha\mu\nu\lambda} = 0 \), the FEs in this case reduce to those of UR, equivalent classically to GR up to CC, with the matter set (containing, conceivably, DM) extended through the scalar graviton \( \sigma \) as a conventional scalar particle.\(^{28}\)

5.2 Weak-Field Limit

5.2.1. Linear approximation. At \( \dot{\alpha} = \dot{\alpha} = -1/2 \), in flat background one has in the LA
\[ \dot{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{4} (h - 2\partial_\chi) \eta_{\alpha\beta}, \quad \dot{h} = 2\partial_\chi, \]
\[ \dot{f}_{\alpha\beta} \equiv f_{\alpha\beta} - k_{\alpha\beta}, \quad \dot{f} = 0, \]
\[ s = \frac{1}{2} (h - 2\partial_\chi). \quad (42) \]
Under the Weyl-scale enhanced Diff gauge transformations one gets
\[ \dot{h}_{\alpha\beta} \rightarrow \dot{h}_{\alpha\beta} + (\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha), \]
\[ \dot{f}_{\alpha\beta} \rightarrow \dot{f}_{\alpha\beta}, \quad s \rightarrow s + 2\zeta. \quad (43) \]
This means that \( \dot{h}_{\alpha\beta} \), being a tensor under Diff, is Weyl-scale invariant. \( \dot{f}_{\alpha\beta} \) is completely invariant under Weyl-scale enhanced Diff gauge symmetry, while \( s \), being a scalar under Diff, transforms inhomogeneously under Weyl rescalings. Hence, the fields \( \dot{f}_{\alpha\beta} \) and \( s \) may be used to describe the spontaneously broken Diff gauge symmetry in explicitly Diff-invariant terms, realizing the unitary gauge \( (\chi^\alpha = 0, \dot{h}_{\alpha\beta} = \dot{f}_{\alpha\beta}) \) for dark gravity. At that, one independent DE component in the tensor \( \dot{f}_{\alpha\beta} \) is missing due to \( \dot{f} = 0 \) as a manifestation of Weyl scale symmetry. At the end, such a component reappears again through the scalar graviton \( \sigma \) as a conventional particle, reflecting ultimately an explicit local Weyl scale violation.\(^{29}\) The account, beyond the LA, for higher orders in the presence of the protecting Weyl scale symmetry, as well as local Weyl-scale violation due to \( \sigma \), are to be investigated.

5.2.2. Massive tensor graviton. To clarify the viability of multiscalar-modified/spontaneously broken WTR as a theory of gravitons, retain first only the Weyl-scale invariant pure tensor-gravity Lagrangian \( \dot{L}_G \), putting \( \dot{L}_M (\sigma, \phi_I) = 0 \) neglecting matter and the explicit scalar-graviton contribution. Choosing then, as independent variables for the DE tensor field \( \dot{\alpha}_\mu^\nu \), its trace \( \dot{\alpha} \) and the traceless part \( \dot{\alpha}_\mu^\nu \), respectively, as
\[ \dot{\alpha} \equiv \dot{\alpha}_\mu^\nu \delta_\mu^\kappa, \quad \dot{\alpha}_\mu^\nu \equiv \dot{\alpha}_\mu^\nu - \frac{1}{4} \dot{\alpha} \delta_\mu^\nu, \quad (44) \]
consider the even quartic potential \( \dot{V}_\alpha (\dot{\alpha}_\mu^\nu, \dot{\alpha}) \)
\[ \dot{V}_\alpha = \frac{1}{2} M^2_{Pl} \dot{\nu}_2 \left[ \hat{\beta}_2 \dot{\alpha}_\mu^\nu \dot{\alpha}_\mu^\nu + \beta_4 \left( \frac{1}{4} \dot{\alpha}^2 - 1 \right)^2 \right], \quad (45) \]
where \( \dot{\nu}_2 \rightarrow 0 \) and \( \beta_4 \) are some constant parameters.\(^{30}\) For definiteness, \( \dot{V}_\alpha \) is normalized as \( \dot{V}_\alpha |_{\dot{\alpha} = 0} = 0 \) in flat background at \( \dot{\alpha}_\mu^\nu = \delta_\mu^\nu, \dot{\alpha} = 4 \), with an additional constant contribution assigned by default to the Lagrangian CC \( \dot{\Lambda} \). Such a potential is peculiar by possessing two extrema, at \( \dot{\alpha} = 0 \) and \( |\dot{\alpha}| = 4 \), which may be ascribed at \( \beta_4 \rightarrow 0 \) to the unbroken trivial and broken flat backgrounds, respectively. In the trivial background at \( \dot{\alpha}_\mu^\nu = 0, \dot{\alpha} = 0 \), the potential \( \dot{V}_\alpha |_{\dot{\alpha} = 0} \) at

\(^{28}\) Note that imposing ab initio the UR restriction \( g = \kappa \), eliminating, in fact, one independent component would result in the LA in \( h = \dot{h} = 2\partial_\chi \) (still preserving \( \dot{f} = 0 \)), and \( \dot{h}_{\alpha\beta} = \dot{h}_{\alpha\beta} \), as well as \( s = 0 \), with lost Weyl scale symmetry.

\(^{30}\) In particular, a similar potential \( \dot{V}_\alpha \) at a proper \( \beta_4 \) is used for the consistent description of the massive tensor graviton within spontaneously broken GR [28].
\( \beta_4 > 0 \) lies higher, being positive and unstable near \( \hat{\alpha}_* = 0 \). In a vicinity of the flat background at \( \hat{\alpha}_* = \delta_\beta^* \), the potential \( \hat{V}_a \) in the quadratic approximation is\(^{31}\)

\[
\hat{V}_a = \frac{1}{2} M_p^2 \hat{v}_2 f^a_\beta f^\beta_a + \frac{1}{4} (\hat{\beta}_4 - 1) \hat{f}^2,
\]

being in the LA, due to \( \hat{f} = 0 \), in fact, independent of \( \hat{f} \) irrespective of \( \hat{\beta}_4 \). This gives in turn:

\[
\bar{T}_{a\alpha\beta} = 2 M_p^2 \hat{v}_2 \hat{f}_{a\beta}, \quad \bar{T}_a \equiv \bar{T}_{a\alpha\beta} \eta^\alpha\beta = 0.
\]

Due to the Diff gauge symmetry, under the unitary gauge \( \chi^\alpha = 0 \) (and thus \( k_\alpha = 0 \)), accounting for (42), one can substitute in \( \bar{R}_{a\beta} \) the tensor-graviton field \( \hat{h}_{a\beta} \), with \( \hat{h} = 0 \), by the DE one \( \hat{f}_{a\beta} \), with \( \hat{f} = 0 \), getting in the LA

\[
\bar{R}_{a\beta} = \frac{1}{2} \left( \partial_\alpha \partial_\gamma \hat{f}^\gamma_\beta + \partial_\beta \partial_\gamma \hat{f}^\gamma_\alpha - \partial^2 \hat{f}_{a\beta} \right),
\]

\[
\bar{R} = \partial_\alpha \partial_\beta \hat{f}^\gamma_\delta.
\]

This results in the LA in the following tensor and multiscalar FEs (39):

\[
\partial_\alpha \partial_\gamma \hat{f}^\gamma_\beta + \partial_\beta \partial_\gamma \hat{f}^\gamma_\alpha - \partial^2 \hat{f}_{a\beta},
\]

\[
- \frac{1}{2} \partial_\alpha \partial_\beta \hat{f}^\gamma_\delta \eta_{a\beta} = m_g^2 \hat{f}_{a\beta},
\]

\[
\partial_\alpha (m_g^2 \hat{f}^\alpha_\beta - \frac{1}{2} \partial_\gamma \partial_\delta \hat{f}^\gamma_\delta \delta^\beta_\alpha) = 0,
\]

clearly independent from \( \hat{\Lambda} \). The latter property can be explicitly confirmed under an arbitrary \( \chi^\alpha \) due to the Diff gauge symmetry. In the above, we have put \( \hat{v}_2 \equiv m_g^2 / 4 \), with \( m_g \) to be associated with the tensor-graviton mass. It proves that the multiscalar FE follows, in fact, from the tensor one, with all nine dark-gravity components remaining thus independent. Meanwhile, one missing component corresponds to the neglected scalar graviton, whose inclusion would explicitly violate the hidden Weyl scale symmetry. To reduce the number of independent components, impose additionally the covariant conservation condition (25), resulting in the LA in the transversality condition

\[
\partial_\gamma \hat{f}^\gamma_\alpha = 0,
\]

followed by \( \hat{\Lambda}_0 = 0 \). This finally results in the FE

\[
(\partial^2 + m_g^2) \hat{f}_{a\beta} = 0
\]

for the gauge-invariant \( \hat{f}_{a\beta} \), coinciding with the gauge-variant \( \hat{h}_{a\beta} = h_{a\beta} - \frac{1}{2} m_g \hat{h}_{a\beta} \) under the unitary gauge \( \chi^\alpha = 0 \). The FEs (50) and (51) describe precisely the five independent DE degrees of freedom without ghosts, similarly to the Fierz-Pauli case for a massive tensor graviton within spontaneously broken GR [28]. The covariant conservation condition in the spontaneously broken WTR, like GR, proves thus to be of principle to ensure the consistency of the massive tensor graviton description. Still, the conceivable violation of the covariant conservation condition in the context of DE is of particular interest.

5.2.3. Massless limit. Lastly, in the massless limit \( m_g \rightarrow 0 \), in the FEs (49) there emerges a three-parameter residual gauge symmetry

\[
\hat{f}_{a\beta} \rightarrow \hat{f}_{a\beta} - (\partial_\alpha \hat{\phi}_\beta + \partial_\beta \hat{\phi}_\alpha),
\]

\[
\partial_\alpha \hat{\phi} = 0, \quad \partial^2 \hat{\phi}_\alpha = 0,
\]

satisfied also by the constraint (50). This reduces the number of independent tensor-graviton components on-mass-shell \( m_g = 0 \) to two, making thus the massless limit in the LA consistent. In fact, in this limit \( \hat{f}_{a\beta} \) reduces to the conventional massless unimodular \( \hat{h}_{a\beta} \), with \( \hat{h} = 0 \). The presence of the residual gauge symmetry ensuring, in the spontaneously broken WTR, a smooth massless limit to UR is thus crucial for the explicit consistency of the massive tensor-graviton description in the LA.

5.2.4. Scalar graviton. Finally, retaining the tensor-graviton description through the Weyl-scale invariant Lagrangian \( \hat{L}_G \), with the three (at \( n = 4 \)) gauge components converted into additional physical ones for a massive tensor graviton, one could add the explicitly Weyl-scale violating \( \hat{L}_M(\sigma) \) to consistently realize the last component contained in the multiscalar \( X^\alpha \). Under neglection of matter, this would result in the LA in a FE for the scalar graviton \( s \) as a conventional particle:

\[
(\partial^2 + m_s^2) s = 0,
\]

with a scalar-graviton mass \( m_s \). In the above, the gauge-invariant scalar field \( s \) coincides with the gauge-variant metric component \( h/2 \) under the transverse gauge \( \partial \chi = 0 \) (embodifying, in particular, the unitary one \( \chi^\alpha = 0 \)). Going beyond the LA and accounting for mixing of the scalar graviton \( \sigma \) with matter remain to be studied.

6. CONCLUSION

The generic multiscalar-metric framework provides a promising route for modification of gravity,
with spontaneous breaking of the Diff gauge symmetry/relativity as a basic reason for dark gravity—a merger of gravity with emergent dark components. The multiscalar-modified/spontaneously broken WTR, built on this route, incorporating the Weyl scale symmetry and the Higgs mechanism for gravity, may well serve as a viable beyond-GR EFT of gravity. It is able to screen the Lagrangian CC $\hat{\Lambda}$, with emergence of the induced one $\hat{\Lambda}$ supplemented by massive tensor and scalar gravitons as dark gravity components. As an EFT, the so constructed dark gravity model, being applied to the (extensions of the) particle SM, allows, in principle, to account also for spontaneous breaking of the internal symmetries. This should be followed by stepwise modifications of the Lagrangian CC and matching of the emergent descriptions at the phase transition scales. In the leading approximation, the spontaneously broken WTR proves to be well-behaved as a theory of the massive tensor and scalar gravitons without ghosts. A further development of the basic concept of the spontaneously broken Diff gauge symmetry/relativity—in particular, WTR vs. GR—as a principle source of the emergent dark gravity components of the Universe is urgent.

ACKNOWLEDGMENTS

The author is sincerely grateful to S. S. Gershtein for encouraging discussions.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

REFERENCES

1. Ya. B. Zel’dovich, “The cosmological constant and elementary particles,” JETP Lett. 6, 316 (1967).
2. S. M. Carroll, “The cosmological constant,” Living Rev. Rel. 4, 1 (2001); astro-ph/0004075.
3. S. Weinberg, “The cosmological constant problem,” Rev. Mod. Phys. 61, 1 (1989).
4. J. Martin, “Everything you always wanted to know about the cosmological constant problem (but were afraid to ask),” Comptes Rendus Physique 13, 566 (2012); arXiv: 1205.3365.
5. C. P. Burgess, “The cosmological constant problem: Why it’s hard to get dark energy from micro-physics,” The Les Houches Summer School Post-Planck Cosmology; arXiv: 1309.4133.
6. S. Capozziello and M. De Laurentis, “Extended theories of gravity,” Phys. Rep. 509, 167 (2011); arXiv: 1108.6266.
7. T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, “Modified gravity and cosmology,” Phys. Rep. 513, 1 (2012); arXiv: 1106.2476.
8. S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, “Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution,” Phys. Rep. 692, 1 (2017); arXiv: 1705.11098.
9. V. A. Rubakov and P. G. Tinyakov, “Infrared-modified gravities and massive gravitons,” Phys. Usp. 51, 759 (2008); arXiv: 0802.4379.
10. C. de Rham, “Massive gravity,” Living Rev. Rel. 17, 7 (2014); arXiv: 1401.4173.
11. J. L. Anderson and D. Finkelstein, “Cosmological constant and fundamental length,” Am. J. Phys. 39, 901 (1971).
12. J. J. van der Bij, H. van Dam, and Y. J. Ng, “The exchange of massless spin-two particles,” Physica A 116, 307 (1982).
13. K.-I. Izawa, “Derivative expansion in quantum theory of gravitation,” Prog. Theor. Phys. 93, 615 (1995); hep-th/9410111.
14. E. Alvarez, D. Blas, J. Garriga, and E. Verdaguer, “Transverse Fierz-Pauli symmetry,” Nucl. Phys. B 756, 148 (2006); hep-th/0606019.
15. E. Alvarez and R. Vidal, “Weyl transverse gravity (WTDi) and the cosmological constant,” Phys. Rev. D 81, 084057 (2010); arXiv: 1001.4458.
16. I. Oda, “Classical Weyl transverse gravity,” Eur. Phys. J. C 77, 284 (2017); arXiv: 1610.05441.
17. R. Carballo-Rubio, “Longitudinal diffeomorphisms obstruct the protection of vacuum energy,” Phys. Rev. D 91, 124071 (2015); arXiv: 1502.05278.
18. E. Alvarez, S. Gonzalez-Martin, M. Herrero-Valea, and C. P. Martin, “Unimodular gravity redux,” Phys. Rev. D 92, 061502 (2015); arXiv: 1505.00022.
19. C. Barcelo, R. Carballo-Rubio, and L. J. Garay, “Absence of cosmological constant problem in special relativistic field theory of gravity,” Ann. Phys. 398, 9 (2018); arXiv: 1406.7713.
20. M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field,” Proc. Roy. Soc. Lond. A 173, 211 (1939).
21. H. van Dam and M. J. G. Veltman, “Massive and massless Yang-Mills and gravitational fields,” Nucl. Phys. D 22, 397 (1970).
22. V. I. Zakharov, “Linearized gravitation theory and the graviton mass,” JETP Lett. 12, 312 (1970).
23. A. I. Vainshtein, “To the problem of nonvanishing graviton mass,” Phys. Lett. B 39, 393 (1972).
24. D. G. Boulware and S. Deser, “Can graviton have a finite range?,” Phys. Rev. D 6, 3368 (1972).
25. G. ’t Hooft, “Unitarity in the Brout-Englert-Higgs mechanism for gravity,” arXiv: 0708.3184.
26. Z. Kakushadze, “Gravitational Higgs mechanism and massive gravity,” Int. J. Mod. Phys. A 23, 1581 (2008); arXiv: 0709.1673.
27. D. A. Demir and N. K. Pak, “General tensor Lagrangians from gravitational Higgs mechanism,” Class. Quantum Grav. 26, 105018 (2009); arXiv: 0904.0089.
28. A. H. Chamseddine and V. Mukhanov, “Higgs for graviton: simple and elegant solution,” JHEP 1008, 011 (2010); arXiv: 1002.3877.
29. I. Oda, “Higgs mechanism for gravitons,” Mod. Phys. Lett. A 25, 2411 (2010); arXiv: 1003.1437.
30. I. Oda, “Remarks on Higgs mechanism for gravitons,” Phys. Lett. B 690, 322 (2010); arXiv: 1004.3078.
31. L. Alberte, A. H. Chamseddine, and V. Mukhanov, “Massive gravity: resolving the puzzles,” JHEP 1012, 023 (2010); arXiv: 1008.5132.
32. L. Alberte, A. H. Chamseddine, and V. Mukhanov, “Massive gravity: exorcising the ghost,” JHEP 1104, 004 (2011); arXiv: 1011.0183.
33. E. I. Guendelman and A. B. Kaganovich, “The principle of non-gravitating vacuum energy and some of its consequences,” Phys. Rev. D 53, 7020 (1996); gr-qc/9605026.
34. E. I. Guendelman and A. B. Kaganovich, “Gravitational theory without the cosmological constant problem,” Phys. Rev. D 55, 5970 (1997); gr-qc/9611046.
35. Yu. F. Pirogov, “Quartet-metric general relativity: scalar graviton, dark matter and dark energy,” Eur. Phys. J. C 76, 215 (2016); arXiv: 1511.04742.
36. Yu. F. Pirogov, “Quartet-metric gravity and dark components of the Universe,” Int. J. Mod. Phys.: Conf. Series 47, 1860101 (2018); arXiv: 1712.00612.
37. Yu. F. Pirogov, “Affine-Goldstone/quartet-metric gravity and beyond,” Phys. Atom. Nucl. 82, 503 (2019); arXiv: 1807.02160.
38. Yu. F. Pirogov, “Quartet-metric/multi-component gravity: scalar graviton as emergent dark substance,” JCAP 01, 055 (2019); arXiv: 1811.12923.
39. Yu. F. Pirogov, “Unimodular bimode gravity and the coherent scalar-graviton field as galaxy dark matter,” Eur. Phys. J. C 72, 2017 (2012); arXiv: 1111.1437.
40. G. F. R. Ellis, H. van Elst, J. Murugan, and J.-P. Uzan, “On the trace-free Einstein equations as a viable alternative to General Relativity,” Class. Quantum Grav. 28, 225007 (2011); arXiv: 1008.1196.
41. G. F. R. Ellis, “The trace-free Einstein equations and inflation,” Gen. Rel. Grav. 46, 1619 (2014); arXiv: 1306.3021.
42. P. G. Bergmann, “Comments on the scalar-tensor theory,” Int. J. Theor. Phys. 1, 25 (1968).
43. G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” Int. J. Theor. Phys. 10, 363 (1974).