TWO-LOOP SUPERSTRINGS AND S-DUALITY ¹

Eric D’Hoker⁹, Michael Gutperle,⁹ and D.H. Phong⁶

⁹ Department of Physics and Astronomy
University of California, Los Angeles, CA 90095, USA

⁶ Department of Mathematics
Columbia University, New York, NY 10027, USA

Abstract

The two-loop contribution to the Type IIB low energy effective action term $D^4 R^4$, predicted by SL(2,Z) duality, is compared with that of the two-loop 4-point function derived recently in superstring perturbation theory through the method of projection onto super period matrices. For this, the precise overall normalization of the 4-point function is determined through factorization. The resulting contributions to $D^4 R^4$ match exactly, thus providing an indirect check of SL(2,Z) duality. The two-loop Heterotic low energy term $D^2 F^4$ is evaluated in string perturbation theory; its form is closely related to the $D^4 R^4$ term in Type II, although its significance to duality is an open issue.

¹Research supported in part by National Science Foundation grants PHY-01-40151, PHY-02-45096, and DMS-02-45371.
1 Introduction

The dualities [1, 2] between various superstring theories and M theory provide strong constraints on their effective actions. In [3] it was conjectured that the $SL(2, \mathbb{Z})$ duality (or S-duality) of Type IIB string theory completely determines the $R^4$ term in the superstring effective action. The dependence of the $R^4$ term on the axion/dilaton scalar field $\tau = \chi + ie^{-\phi}$ is given by a non-holomorphic Eisenstein series (or Maass waveform [4]) $E_3/2(\tau, \bar{\tau})$. A remarkable consequence of this conjecture is that the $R^4$ term receives contributions only from tree-level and one-loop orders in string perturbation theory (in addition there are non-perturbative D-instanton contributions).

Further evidence for this conjecture was provided in [5, 6], where the $R^4$ term was derived from a one-loop amplitude of eleven dimensional supergravity. In [7] it was shown that the exact form of the $R^4$ and related terms [8, 9] is completely determined by supersymmetry and S-duality. Utilizing U-duality symmetries, these results for the $R^4$ term were extended to toroidally compactified Type II string theory in [10, 11, 12, 13, 14]. Related conjectures for $R^4 H^4 g^{-4}$ terms were discussed in [15].

The $R^4$ term is expected to receive no renormalizations beyond one-loop, since it can be expressed as an integral over half the (linearized and on-shell) Type IIB superspace, and thus is protected by supersymmetry. Dimensional arguments suggest that terms with additional derivatives should enjoy a similar protection. In [16, 17], a two-loop calculation in eleven dimensional supergravity was used to show that the axion/dilaton dependence of $D^4 R^4$ term in the superstring effective action is governed by a non-holomorphic Eisenstein series $E_{5/2}(\tau, \bar{\tau})$. The functional form of the Eisenstein series is such that it generates only tree-level and two-loop contributions in string perturbation theory to this term.

Recently, a systematic method for constructing the superstring $N$-point function to two-loops from first principles has been developed [18, 19, 20, 21] (see also [22] for a review). With this method, we have a completely explicit expression for the superstring 4-point function of massless NS bosons [21, 23]. In particular, this construction provides a first principles proof that the two-loop amplitude does not contribute to the $R^4$ term in the effective action [21]. This result may be interpreted as an indirect check of the conjectured $SL(2, \mathbb{Z})$ duality structure of the $R^4$ terms. The vanishing of the two-loop contribution to the $R^4$ terms had also been argued earlier by many authors [24, 25, 26]. However, the calculations they had to rely on were gauge-slice dependent and did not provide reliable checks.

The purpose of the present paper is to compare the contribution to the $D^4 R^4$ term deduced from the two-loop superstring amplitude [21, 23] to the one predicted by [16] on the basis of $SL(2, \mathbb{Z})$ duality. Since we shall be comparing two non-vanishing contributions, their overall normalizations will have to be determined with particular care. Actually, the
two-loop 4-point function was determined in [21, 23] only up to an overall constant, left undetermined due to the occurrence of an unknown overall normalization constant in the chiral bosonization formulas of [27]. Several attempts have been made at calculating these constants, but even just for the bc ghost spin 1 bosonization formula, no generally agreed upon values seems to have been attained as yet [28].

Precise overall normalization constants are of critical importance for the detailed comparison between results from duality and superstring perturbation theory that we shall carry out in this paper. One of our main tasks here is to determine precisely the overall normalization of the two-loop 4-point function. We accomplish this through factorization constraints to tree-level and one-loop. Many different conventions used in the literature, as well as the occurrence of an occasional typo, have forced us to carefully rederive also the basic tree-level and one-loop amplitudes. A systematic account is provided in Appendices B and C.

With these precise overall normalizations, we find complete agreement between the two-loop perturbative value for the $D^4R^4$ term and the value predicted by $SL(2, \mathbb{Z})$ duality, thus providing another indirect check for S-duality.

Finally, the Heterotic strings [29] inherit half of the supersymmetry and some of the non-renormalization effects of Type II theories in perturbation theory, even though their non-perturbative structure is very different from that of Type II strings. In particular, the $\text{Spin}(32)/\mathbb{Z}_2$ Heterotic string is expected to be dual to the Type I string [30, 31], (for a review, see [32]). In [21], it was shown that in the low energy effective action for the Heterotic strings, terms in $F^4$, $F^2F^2$, $R^2F^2$ and $R^4$ receive no two-loop contributions. While the non-renormalization of the $F^4$ and $F^2F^2$ terms had been argued earlier in [33], that of the $R^4$ terms appears to be at odds with an earlier duality analysis [31]. To clarify these issues, it may be helpful to better understand the renormalization properties of terms that involve also additional derivatives. In this paper, we calculate the two-loop contribution to the $D^2F^2F^2$ term, and find that its magnitude involves a remarkable modular form. The interplay between this result and S-duality is an issue left for later study.

The remainder of the paper is organized as follows. In section 2, the predictions for the Type IIB low energy effective action terms $R^4$ and $D^4R^4$, both from S-duality and from perturbation theory are summarized, compared and found to be in perfect agreement. The precise normalization of the two-loop massless NS-NS 4-point function in Type II is obtained by factorization onto one-loop components in sections 3. From this 4-point function, the perturbative contribution to the $D^4R^4$ term is derived in section 4. The analogous calculation of the two-loop corrections to $D^2F^2F^2$ terms in the Heterotic string is provided in section 5. In Appendix A, basics of genus 1 and 2 moduli spaces are summarized. Careful calculations of the normalization of the 4-point functions are given to tree-level in Appendix B and to one-loop in Appendix C. Regulator dependence is discussed in Appendix D.
2 Summary of S-Duality and Perturbative Results

In this section, we collect the predictions of S-duality for the $R^4$ and $D^4 R^4$ terms in the low energy effective action of Type IIB, and express them in a notation that will facilitate their comparison with the results from perturbation theory. The latter are summarized in this section, but their derivation is postponed until sections 3 and 4. A detailed comparison between both sets of predictions reveals perfect agreement.

2.1 S-duality prediction for $R^4$ and $D^4 R^4$ terms

The massless spectrum of Type IIB string theory contains two scalar fields, the NS-NS dilaton $\phi$ and the R-R axion $\chi$. The theory has a remarkable non-perturbative $SL(2, \mathbb{Z})$ S-duality symmetry under which a complex combination $\tau = \chi + i e^{-\phi}$ transforms as

$$\tau \rightarrow a \tau + b \overline{c \tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \quad (2.1)$$

In the Einstein frame, the action for the graviton is given by

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_E} R_E \quad (2.2)$$

where the superscript $E$ indicated that the metric $G_E$ and the Ricci scalar $R_E$ are expressed in the Einstein frame. S-duality acts naturally in the Einstein frame where the metric is invariant under the transformation (2.1). Therefore, it is expected that all terms in the effective action which involve only the metric and the dilaton/axion fields should have a dependence on $\tau$ and $\bar{\tau}$ which is invariant under (2.1).

The Einstein frame is not the natural frame for expressing the coupling of the string sigma model to the space time metric. Instead, one uses the string frame metric $G$, which is related to the Einstein frame metric by

$$G_{\mu\nu} = e^{\phi/2} G_{E\mu\nu} \quad (2.3)$$

The gravitational part of the action in the string frame becomes

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} R \quad (2.4)$$

where $R$ is now the Ricci scalar for the metric $G$. Henceforth, we shall use the string frame metric throughout. Notice that in the Einstein-Hilbert action, only the combination $\kappa_{10}^2 e^{2\phi}$ enters, a combination that is unaffected by a shift in $\phi$ compensated by a multiplicative...
factor in $\kappa_{10}^2$. The normalization chosen here for $\phi$ is such that the S-duality transformation law for $\tau = \chi + ie^{-\phi}$ is canonical, as in (2.1).

It was conjectured in [3] that the $R^4$ terms in the Type IIB effective action take the following form, (in the string frame),

$$S_{R^4} = C_{R^4} \int d^{10}x \sqrt{-G} R^4 e^{-\frac{1}{2}\phi} 2\zeta(3) E_{3/2}(\tau, \bar{\tau})$$

(2.5)

The expression $R^4$ stands for $R^4 = t_s t_8 R^4$, where $t_8$ is the well-known kinematic tensor which enters both the tree-level and one-loop superstring 4-point functions and is defined here as it was in [34, 35]. The overall constant $C_{R^4}$ will be discussed later. Furthermore, $\zeta(s)$ is the Riemann zeta function, and $E_{3/2}(\tau, \bar{\tau})$ is the non-holomorphic Eisenstein series of weight $s = 3/2$. For general $s$, the non-holomorphic Eisenstein series $E_s$ is defined by

$$2\zeta(2s) E_s(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{|m + n\tau|^{2s}}$$

(2.6)

and satisfies the following differential equation

$$4\tau_2^2 \partial_\tau \partial_{\bar{\tau}} E_s(\tau, \bar{\tau}) = s(s - 1) E_s(\tau, \bar{\tau})$$

(2.7)

In [7] it was shown that (2.7) is a consequence of supersymmetry. It is easy to see that (2.7) has two solutions of the form $c_{-s} \tau_2^s$ and $c_{s-1} \tau_2^{1-s}$ corresponding to two particular orders of perturbation theory. The series expression (2.6) is the unique S-duality invariant solution with these perturbative terms. In [6] the form of the $R^4$ term (2.5) was derived from a one-loop calculation in eleven dimensional supergravity. The expansion of the non-holomorphic Eisenstein series $2\zeta(3) E_{3/2}(\tau, \bar{\tau})$ is given in [4]

$$2\zeta(3) E_{3/2}(\tau, \bar{\tau}) = 2\zeta(3) e^{-\frac{3}{2}\phi} + \frac{2\pi^2}{3} e^{\frac{1}{2}\phi} + \text{non-perturbative}$$

(2.8)

It follows that the $R^4$ term in the effective action (2.5) gets perturbative contributions only from tree-level and one-loop. The vanishing of two-loop contributions was proven in [21].

The $R^4$ term may be expressed as an integral over half the (linearized on-shell) IIB superspace and should therefore be protected by supersymmetry. Dimensional analysis suggests that terms with up to six derivatives acting on $R^4$ should still be protected by supersymmetry. In [16] a two-loop calculation in eleven dimensional supergravity was used to calculate the $D^4 R^4$ terms explicitly in the Type IIB effective action. It was found that it has a S-duality invariant form (still expressed in the string frame),

$$S_{D^4 R^4} = C_{D^4 R^4} \int d^{10}x \sqrt{-G} D^4 R^4 e^{\frac{1}{2}\phi} \zeta(5) E_{5/2}(\tau, \bar{\tau})$$

(2.9)
For $E_{5/2}$ the expansion in large negative $\phi$ results in two perturbative terms, given by

$$2\zeta(5)E_{5/2} = 2\zeta(5)e^{-\frac{2}{3}\phi} + \frac{\pi^4}{90} \times \frac{8}{3} e^{\frac{2}{3}\phi} + \text{non-perturbative} \quad (2.10)$$

From (2.9) it is clear that the two terms come from a tree-level and a two-loop contribution, but that the one-loop contribution is absent. This is in accord with [17], where the one-loop contribution to this term in the action was shown to be zero.

The normalization constants $C_{R^4}$ in (2.5) and $C_{D^4R^4}$ in (2.9) can be determined from the tree-level 4-point function

$$A_0^{(4)} = R^4\kappa_{10}^2 e^{-2\phi} \frac{\Gamma(-s\alpha'/4)\Gamma(-t\alpha'/4)\Gamma(-u\alpha'/4)}{\Gamma(1 + s\alpha'/4)\Gamma(1 + t\alpha'/4)\Gamma(1 + u\alpha'/4)} \quad (2.11)$$

as follows. We use the formula

$$\frac{\Gamma(1 - z)}{\Gamma(1 + z)} = e^{2\gamma z} \exp \left(\sum_{k=1}^{\infty} \frac{2\zeta(2k + 1)z^{2k+1}}{2k + 1}\right) \quad (2.12)$$

to derive the expansion of the 4-point function in powers of $s$, $t$ and $u$,

$$A_0^{(4)} = R^4\kappa_{10}^2 e^{-2\phi} \frac{2^6}{(\alpha')^3stu} \exp \left(\sum_{k=1}^{\infty} \frac{2\zeta(2k + 1)\alpha'2k+1}{2k + 1}(s^{2k+1} + t^{2k+1} + u^{2k+1})\right) \quad (2.13)$$

Expanding to the order needed for the study of $R^4$ and $D^4R^4$ terms yields,

$$A_0^{(4)} = R^4\kappa_{10}^2 e^{-2\phi} \left(\frac{2^6}{(\alpha')^3stu} + 2\zeta(3) + \frac{\zeta(5)}{16}(\alpha')^2(s^2 + t^2 + u^2)\right) \quad (2.14)$$

The first term in (2.14) arises through 1-particle reducible Feynman diagrams from the Einstein-Hilbert action. The second term in (2.14) gives the following tree-level contribution to the $R^4$ terms in the effective action [36]

$$A_0^{(R^4)} = R^4\kappa_{10}^2 2\zeta(3) e^{-2\phi} \quad (2.15)$$

Now the $R^4$ terms in the effective action are given by the formula (2.5) in terms of $2\zeta(3)E_{3/2}(\tau, \bar{\tau})$. The expansion (2.8) for $2\zeta(3)E_{3/2}(\tau, \bar{\tau})$ gives then the relative coefficients between the tree-level and the one-loop contributions to the $R^4$ terms in the effective action.

---

1 Throughout, we shall omit the overall momentum conservation factor $(2\pi)^4\delta^{(10)}(k_1 + k_2 + k_3 + k_4)$ when expressing the scattering amplitudes, and use the Mandelstam variables, $s = -(k_1 + k_2)^2$, $t = -(k_2 + k_3)^2$, and $u = -(k_1 + k_3)^2$. 

6
It follows that the normalization of the one-loop contributions to the \( R^4 \) terms corresponding to the normalization (2.14) for the 4-point function is given by

\[
A^{(R^4)}_1 = R^4 \kappa^2 \frac{2\pi^2}{3}
\]  

(2.16)

The third term in (2.14) gives the following tree-level contribution to the \( D^4 R^4 \) terms,

\[
A^{(D^4 R^4)}_0 = R^4 (s^2 + t^2 + u^2) \kappa^2 \frac{(\alpha')^2}{24}\zeta(5)e^{-2\phi}
\]  

(2.17)

The relative coefficients of the tree-level and the two-loop contribution to the \( D^4 R^4 \) term can be read off from the expansion (2.10) of \( 2\zeta(5)E_{5/2}(\tau, \bar{\tau}) \). It follows that the two-loop contribution must be given by

\[
A^{(D^4 R^4)}_2 = R^4 \kappa^2 \frac{(\alpha')^2}{24} \frac{4\pi^4}{270} e^{2\phi}(s^2 + t^2 + u^2)
\]  

(2.18)

Note that the one-loop contribution to \( D^4 R^4 \) indeed vanishes in view of [17].

### 2.2 Normalizations of Superstring Perturbation Theory

The starting point for superstring perturbation theory is the action \( I_m \) and vertex operators for massless NS-NS states \( V \) on a worldsheet \( \Sigma \), normalized as follows,

\[
I_m = \lambda \chi(\Sigma) + \frac{1}{2\pi\alpha'} \int_{\Sigma} d^{2|2}z E D_\mu X^\mu D_+ X^\mu
\]

\[
V(\epsilon, \bar{\epsilon}, k) = \kappa \epsilon^\mu \bar{\epsilon}^\nu \int_{\Sigma} d^{2|2}z E D_\mu X^\mu D_+ X^\nu e^{ik \cdot X}
\]  

(2.19)

Here, \( \chi(\Sigma) = 2 - 2h \) is the Euler number for a closed surface \( \Sigma \) with \( h \) handles, and \( \lambda \) is the corresponding coupling constant\(^2\) governing the perturbative loop expansion in powers of the string coupling \( g_s \sim e^{-2\lambda} \). Furthermore, \( z = (z, \theta) \) is the super-coordinate on the worldsheet \( \Sigma \), and the superfield \( X^\mu \) is given in terms of component fields by \( X^\mu = x^\mu + \theta \psi^\mu_+ + \bar{\theta} \psi^\mu_- + i\theta\bar{\theta} F^\mu \), where \( F \) is an auxiliary field. Finally, \( \kappa \) is the normalization constant of the massless vertex operators, which is fixed in terms of the other parameters using unitarity. Its precise value will not be needed here.

The general supergeometry expressions for the various ingredients in (2.19) are given in [37, 18]. On a superconformally flat worldsheet, we have \( E = 1 \), and \( D_+ = \partial_\theta + \theta \partial_z \).

\(^2\)While \( \lambda \) and the vacuum expectation value of the dilaton \( \phi \) are clearly related objects, their precise normalizations may differ by an additive constant (see Appendix D), and the form of S-duality transformations, expressed in terms of \( \lambda \), may not be canonical as in (2.1).
\[ D_- = \partial \bar{\theta} + \bar{\theta} \partial \theta. \] Using the convention \( \int d^2 \theta (\partial \bar{\theta}) = 1 \), the above definitions give us back the standard component expressions for the action and the vertex operators,

\[ I_m = \lambda \chi(\Sigma) + \frac{1}{2\pi \alpha'} \int \partial^2 \Sigma \left( \partial x^\mu \partial x^\mu - \psi_+^\mu \partial \psi_+^\mu - \psi_-^\mu \partial \psi_-^\mu \right). \]

\[ V(\epsilon, \bar{\epsilon}, k) = \kappa \varepsilon^\mu \bar{\varepsilon}^\nu \int \partial^2 \Sigma (\partial x^\mu + ik^\rho \psi_+^\rho)(\bar{\partial} x^\nu + ik^\sigma \psi_-^\sigma) e^{ik \cdot x}. \] (2.20)

For tree-level and one-loop orders, these expressions are sufficient, but at higher genus, the supermoduli must be carefully taken into account as well, and (2.20) has additional dependence on Beltrami differentials and worldsheet gravitini [37, 20].

### 2.3 Perturbative Predictions

With the above normalizations and conventions, the Type II superstring 4-point functions for massless NS-NS bosons to tree-level, one-loop and two-loop orders are given as follows. To tree-level, we have

\[ \mathcal{A}_0^{(4)}(\epsilon_i, k_i) = C_0 Q_0 e^{-2\lambda} \kappa^4 K \bar{K} \frac{2\pi \Gamma(-\alpha'/4)\Gamma(-\alpha'/2)\Gamma(-\alpha'/4)}{\Gamma(1 + \alpha'/4)\Gamma(1 + \alpha'/4)\Gamma(1 + \alpha'/4)} \] (2.21)

To one-loop, we have

\[ \mathcal{A}_1^{(4)}(\epsilon_i, k_i) = C_1 \kappa^4 K \bar{K} \int_{\mathcal{M}_1} \frac{|d\tau|^2}{(\text{Im} \tau)^2} \prod_{i=1}^4 \frac{d^2 z_i}{\text{Im} \tau} \exp \left\{ -\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right\} \] (2.22)

and to two-loop order, we have

\[ \mathcal{A}_2^{(4)}(\epsilon_i, k_i) = C_2 e^{2\lambda} \kappa^4 K \bar{K} \int_{\mathcal{M}_2} \frac{|d^3 \Omega|^2}{(\text{det} \text{Im} \Omega)^2} \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp \left( -\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right) \] (2.23)

Here \( G(z, w) \) are the conformally invariant Green’s functions for genus 1 and genus 2,

\[ G(z, w) = -\ln |E(z, w)|^2 + 2\pi (\text{Im} \Omega)_{ij} \left( \text{Im} \int_z^w \omega_i \right) \left( \text{Im} \int_z^w \omega_j \right) \] (2.24)

The quadri-holomorphic 1-form \( \mathcal{Y}_S \) is given by

\[ 3\mathcal{Y}_S = (t - u) \Delta(1, 2) \Delta(3, 4) + (s - t) \Delta(1, 3) \Delta(4, 2) + (u - s) \Delta(1, 4) \Delta(2, 3) \] (2.25)

3The amplitudes predicted from S-duality were denoted by Latin capital letters \( A \), while the amplitudes resulting from superstring perturbation theory will be denoted by calligraphic capitals \( \mathcal{A} \).

4One-loop moduli are customarily denoted by \( \tau \) and \( \bar{\tau} \); clearly they are not to be confused with the complex dilaton/axion field \( \tau = \chi + ie^{-\phi} \)
where the basic bi-holomorphic antisymmetric 1-form $\Delta$ is defined by
\[\Delta(z, w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z).\]  
(2.26)

The kinematic factor $K$ is normalized as follows,
\[K \equiv (f_1 f_2)(f_3 f_4) + (f_1 f_3)(f_2 f_4) + (f_1 f_4)(f_2 f_3) - 4(f_1 f_2 f_3 f_4)\]
using the notations,
\[f^\mu_\nu_i \equiv \varepsilon^\mu_{i k} \varepsilon^\nu_{i l} - \varepsilon^{\nu}_{i k} \varepsilon^{\mu}_{i l},
(f_i f_j) \equiv f^\mu_\nu_i f_{\mu j} f^{\nu}_{i l},
(f_i f_j f_k f_l) \equiv f^\mu_\nu_i f_{\mu j} f^{\nu}_{k l} f_{i l}.\]
(2.28)

The expression $R_4$, used in the preceding subsection, is related to $K$ in the following manner,
\[K K^\ast = 2^{6} R_4.\]

The constants $C_0$ and $Q_0$ for the tree-level amplitude will be calculated in Appendix B, $C_1$ will be calculated in Appendix C, and $C_2$ will be calculated by factorization in section 3.

### 2.4 Matching S-Duality and Perturbative predictions

For tree-level, one- and two-loop, we have from (2.21), (2.22), and we shall establish in (4.7),
\[A_0^{(1)}(\epsilon, k) = 2\pi C_0 Q_0 e^{-2\lambda} \kappa^4 K K \Gamma(-\alpha's/4)\Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)\]
\[A_1^{(R^4)}(\epsilon, k) = \frac{32\pi}{3} C_1 \kappa^4 K K \Gamma(1 + \alpha's/4)\Gamma(1 + \alpha't/4)\Gamma(1 + \alpha'u/4)\]
\[A_2^{(D^4 R^4)}(\epsilon, k) = \frac{2\kappa^4 (\alpha')^2 (s^2 + t^2 + u^2)}{270} C_2 e^{2\lambda} K K \]
\[(2.30)\]

The one-loop term was approximated to order $R^4$, using $\int \Sigma d^2 z_i = 2 \text{Im} \tau$, the formula (2.22) for the one-loop 4-point function, and the fact that $\int_{\Sigma_1} |d\tau|^2 (\text{Im} \tau)^{-2} = 2\pi/3$. 


The predictions of S-duality and superstring perturbation theory require the matching of (2.11), (2.16), and (2.18) with the three expressions in (2.30). Using the conversion relation $KK = 2^6 R^4$, these matching conditions are equivalent to the following relations,

\[
\begin{align*}
(h = 0) & \quad \kappa_{10}^2 e^{-2\phi} = 2\pi C_0 Q_0 \kappa^4 e^{-2\lambda} 2^6 \\
(h = 1) & \quad \kappa_{10}^2 \frac{2\pi^2}{3} = \frac{32\pi}{3} C_1 \kappa^4 2^6 \\
(h = 2) & \quad \kappa_{10}^2 e^{2\phi} \frac{4\pi^4}{270} \frac{(\alpha')^2}{4^4} = \rho \frac{2^6 \pi^3}{270} C_2 \kappa^4 e^{2\lambda} (\alpha')^2 2^6 
\end{align*}
\] (2.31)

These three relations must hold for effectively two unknowns, namely $\kappa_{10}^2 / \kappa^4$ and $\exp\{\phi - \lambda\}$. Matching thus requires that a single relation between the coefficients $C_0 Q_0$, $C_1$ and $C_2$ hold,

\[
C_1^2 = 2\pi^2 C_0 Q_0 C_2
\] (2.32)

As will be shown in section 3, equation (3.33), this relation is precisely the factorization condition on the two-loop 4-point function used to determine $C_2$, and is manifestly satisfied by (2.29).

One also obtains the relation between the couplings $\phi$ and $\lambda$, using the specific values of the constants $C_0, Q_0, C_1$,

\[
e^{2\phi} = 2^6 \sqrt{2} \pi^2 e^{2\lambda}
\] (2.33)

Notice that $\rho$ does not depend upon the details of the constants $C_0, C_1, C_2, Q_0$, as long as they satisfy the factorization equation (2.32).
3 Normalization of superstring two-loop amplitudes

The expression for the two-loop 4-point function, derived in [21, 23], gives the amplitude only up to an overall constant factor \( C_2 \) which is independent of moduli and of the Mandelstam variables. This constant \( C_2 \) is a priori unknown due to the fact that the chiral determinants of the Dirac operators on Riemann surfaces are themselves known only up to multiplicative constants depending only on the genus. In the present section we determine the exact value of \( C_2 \) from physical factorization to lower-loop amplitudes.

For convenience, we recall the two-loop 4-point function of (2.23),

\[
\mathcal{A}_2^{(4)}(\epsilon_i, k_i) = C_2 e^{2\lambda} K^4 \kappa^4 \int_{\mathcal{M}_2} \frac{|d\Omega|^2}{(\text{det Im}\Omega)^{1/2}} \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp \left( -\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right) \tag{3.1}
\]

The antisymmetric biholomorphic 1-form \( \Delta \) is defined in (2.26), and \( \mathcal{Y}_S \) in (2.25). To calculate \( C_2 \), we shall work out the separating degeneration limit of \( \mathcal{A}_2^{(4)} \) to two one-loop amplitudes. Thus, we need the precise asymptotics, including constants depending only on the genus, of the Green’s function \( G(z, w) \), of the quantity \( \mathcal{Y}_S \), and of the volume forms, in the limit where the genus 2 surface \( \Sigma \) degenerates to two separated genus one surfaces \( \Sigma_1 \cup \Sigma_2 \).

3.1 Degeneration formulas for \( E(z, w) \), \( G(z, w) \), and \( \mathcal{Y}_S \)

We begin with the degeneration formulas for the period matrix, abelian differentials, and prime forms. They are all well-known [27, 37, 38]. However, in order to obtain precise values for the constants of importance to us, consistently with our notations, we provide a detailed derivation of the precise asymptotics for the prime form.

We shall right away restrict our considerations to genus 2. Choose a standard basis for the homology 1-cycles, \( A_I, B_I \), with \( I = 1, 2 \) and \( \#(A_I, B_I) = \delta_{IJ} \). The period matrix will be parametrized by

\[
\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \begin{pmatrix} \tau_{11} & 0 \\ 0 & \tau_{22} \end{pmatrix} + \mathcal{O}(t) \tag{3.2}
\]

The degeneration limit corresponds to letting \( t \to 0 \), so that \( \tau_{12} \to 0 \) and \( \tau_{II}' \) tend to finite limits \( \tau_{II}' \to \tau_{II} \). The moduli \( \tau_{11} \) and \( \tau_{22} \) are then the moduli of the separated tori. The resulting asymptotics may be expressed in terms of genus 1 \( \vartheta \)-functions. We recall the definition, mainly in order to fix our conventions

\[
\vartheta[k_I](z_I, \tau_{II}) \equiv \sum_{m \in \mathbb{Z}} \exp \left\{ i\pi (m + \kappa_I')^2 \tau_{II} + 2i\pi (m + \kappa_I') (z_I + \kappa_I''') \right\} \tag{3.3}
\]

The \( \vartheta \)-function satisfies the heat equation \( \vartheta''_{II} \vartheta[k_I](z_I, \tau_{II}) = 4\pi i \vartheta_{II} \vartheta[k_I](z_I, \tau_{II}) \), and the product relations \( \vartheta'(0, \tau_{II}) = \vartheta[\mu_0]'(0, \tau_{II}) = -\pi \vartheta[\mu_2] \vartheta[\mu_3] \vartheta[\mu_4](\tau_{II}), \) where
\(\kappa_1, I = 1, 2,\) stand for any genus 1 spin structures while the spin structures \(\mu_2, \mu_3, \mu_4\) are the three distinct even spin structures, and \(\nu_0\) is the unique odd spin structure. As usual, we set \(\vartheta_1(z, \tau) \equiv \vartheta[\nu_0](z, \tau)\). Formulas for \(\vartheta\)-function degenerations are standard, and we have (see e.g. [19], eq. (5.10)),

\[
\vartheta^\left[\kappa_1 \kappa_2\right](z, \Omega) = \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{\tau_{12}}{2\pi i}\right)^p \partial^p_{z_1} \vartheta[\kappa_1](z_1, \tau_{11}) \partial^p_{z_2} \vartheta[\kappa_2](z_2, \tau_{22})
\]

where \(z = (z_1, z_2) \in \mathbb{C}^2\). In practice, we shall only need the following special cases,

\[
\vartheta^\left[\mu \nu_0\right](z, \Omega) = \vartheta[\mu](z_1, \tau_{11}) \vartheta_1(z_2, \tau_{22}) + \mathcal{O}(t)
\]

\[
\partial_1 \vartheta^\left[\mu \nu_0\right](0, \Omega) = 2\tau_{12} \partial_{\tau_{11}} \vartheta[\mu](0, \tau_{11}) \vartheta_1'(0, \tau_{22}) + \mathcal{O}(t^2)
\]

\[
\partial_2 \vartheta^\left[\mu \nu_0\right](0, \Omega) = \vartheta[\mu](0, \tau_{11}) \vartheta_1'(0, \tau_{22}) + \mathcal{O}(t)
\]

where \(\vartheta_1\) denotes differentiation of the first argument, as usual, and \(\mu\) is any even spin structure.

The degeneration formulas for the holomorphic Abelian differentials are given by [38] and [37], eq. (6.59). Their expressions for general \(\Omega\) are denoted by \(\omega_t^I(z)\), \(I = 1, 2\), while the holomorphic differentials on the separated components \(\Sigma_I\) are \(\omega_I(z)\). Also, we denote by \(\omega_p^{(I)}(z)\) the Abelian differential of the second kind on \(\Sigma_I\) with a double pole at \(z = p\), and unit residue. (To be completely clear, if \(z \in \Sigma_I\), then \(\omega_J(z) = 0\) and \(\omega_p^{(J)}(z) = 0\) when \(J \neq I\).) The asymptotics to first order in \(t\) is then given by

\[
\omega_t^I(z) = \begin{cases} 
\omega_1(z) + \frac{t}{4} \omega_1(p_1) \omega_p^{(1)}(z) & z \in \Sigma_1 \\
\frac{t}{4} \omega_1(p_1) \omega_p^{(2)}(z) & z \in \Sigma_2
\end{cases}
\]

\[
\omega_t^2(z) = \begin{cases} 
\frac{t}{4} \omega_2(p_2) \omega_p^{(1)}(z) & z \in \Sigma_1 \\
\omega_2(z) + \frac{t}{4} \omega_2(p_2) \omega_p^{(2)}(z) & z \in \Sigma_2
\end{cases}
\]

Here, the points \(p_1\) and \(p_2\) are the limiting points on the surfaces \(\Sigma_1\) and \(\Sigma_2\) respectively, left of the degeneration limit. The normalizations of these differentials are as follows,

\[
\int_{A_I} \omega_J = \delta_{IJ} \tau_{11} \int_{B_I} \omega_J = \delta_{IJ} \tau_{II}
\]

\[
\int_{A_I} \omega_p^{(J)} = 0 \quad \int_{B_I} \omega_p^{(J)} = 2\pi i \delta_{IJ} \omega_1(p)
\]

\[
\int_{A_I} \omega_t^J = \delta_{IJ} \quad \int_{B_I} \omega_t^J = \Omega_{IJ}
\]

12
It follows that we have\(^5\) \(\tau_{12} = \frac{\pi i}{2} t \overline{\omega_1(p_1)\overline{\omega_2(p_2)}} + \mathcal{O}(t^2),\) \(\tau'_{11} = \tau_{11} + \frac{\pi i}{2} t \overline{\omega_1(p_1)}^2 + \mathcal{O}(t^2),\) \(\tau_{22}' = \tau_{22} + \frac{\pi i}{2} t \overline{\omega_2(p_2)}^2 + \mathcal{O}(t^2).\) Parametrizing the tori \(\Sigma_I\) by the standard parallelogram, with vertices \(0, 1, \tau_{II}, \tau_{II} + 1,\) gives \(\overline{\omega_1}(z) = 1\) for \(z \in \Sigma_I\) and \(v_I(z) = 0\) for \(z \notin \Sigma_I.\) Hence we have the following simplified expressions,

\[
\tau_{12} = \frac{\pi i}{2} t + \mathcal{O}(t^2), \quad \tau'_{11} = \tau_{11} + \frac{\pi i}{2} t + \mathcal{O}(t^2), \quad \tau_{22}' = \tau_{22} + \frac{\pi i}{2} t + \mathcal{O}(t^2). \tag{3.8}
\]

We can derive now the asymptotics for the genus 2 prime form \(E(z, w).\) In general, \(E(z, w)\) is defined by

\[
E(z, w; \Omega) = \frac{\vartheta(\nu)(\int_z^w \omega_I, \Omega)}{h_{\nu}(z, \Omega)h_{\nu}(w, \Omega)} \tag{3.9}
\]

Here, \(\nu\) is a genus 2 odd spin structure; \(E(z, w; \Omega)\) is actually independent of \(\nu,\) and we shall choose it as below. The spinor \(h_{\nu}\) is defined by,

\[
h_{\nu}(z, \Omega)^2 = \omega_I(z) \partial_I \vartheta[\nu](0, \Omega) \quad \nu = \begin{bmatrix} \mu \\ \nu_0 \end{bmatrix} \tag{3.10}
\]

We shall need the degeneration limit only when \(z \in \Sigma_1\) and \(w \in \Sigma_2,\) which we henceforth assume to be the case. To leading order, we have

\[
\int_z^w \omega_I' = \int_{p_1}^z \overline{\omega_1} = z - p_1 + \mathcal{O}(t) \\
\int_z^w \omega_J' = \int_w^{p_2} \overline{\omega_2} = -w + p_2 + \mathcal{O}(t) \tag{3.11}
\]

To leading order, the limit of the \(\vartheta\)-function factor in the prime form gives

\[
\vartheta[\nu]\left(\int_w^z \omega_I, \Omega\right) = \vartheta[\mu](z - p_1, \tau_{11}) \vartheta(0, p_2 - w, \tau_{22}) + \mathcal{O}(t) \tag{3.12}
\]

The calculation of the spinors \(h_{\nu}\) proceeds analogously, and we have

\[
h_{\nu}(z, \Omega)^2 h_{\nu}(w, \Omega)^2 = \vartheta[\mu](0, \tau_{11})^2 \vartheta'(0, \tau_{22})^2 \left[2\tau_{11} \ln \vartheta[\mu](0, \tau_{11}) + \frac{1}{4} t \overline{\omega_1(1)}(z)\right] \tag{3.13}
\]

Using again the heat equation to recast the \(\tau_{11}\)-derivative in terms of \(z\)-derivatives, and the relation between \(\tau_{12}\) and \(t,\) the term in brackets becomes,

\[
\left[\cdots\right] = \frac{t}{4} \left\{ \partial_z \partial_{p_1} \ln \vartheta_1(0, \tau_{11}) + \partial_z^2 \vartheta[\mu](0, \tau_{11})/\vartheta[\mu](0, \tau_{11}) \right\} = \frac{t}{4} S_\mu(z, p_1)^2 \tag{3.14}
\]

\(^5\)These formulas are naturally interpreted as the period matrix deformations due to a Beltrami differential supported at the points \(p_1\) and \(p_2,\) with magnitude \(t.\)
where $S_{\mu}$ is the genus 1 Szegő kernel for even spin structure $\mu$. The first and second lines above are related by a form of the Fay identity for genus 1. The Szegő kernel is given by

$$S_{\mu}(z, p_1) = \vartheta[\mu](z - p_1, \tau_{11}) \vartheta'[0, \tau_{11}] \vartheta'[0, \tau_{22}] \vartheta[0, \tau_{11}] \vartheta[z - p_1, \tau_{11}] (3.15)$$

Putting all together, we have

$$h_{\nu}(z)h_{\nu}(w) = \frac{1}{2} \sqrt{t} \frac{\vartheta[\mu](z - p_1, \tau_{11}) \vartheta'[0, \tau_{11}] \vartheta'[0, \tau_{22}]}{\vartheta[0, \tau_{11}] \vartheta[z - p_1, \tau_{11}]} (3.16)$$

As a result, we have

$$E(z, w; \Omega) = 2t^{-\frac{1}{2}} E_1(z, p_1; \tau_{11}) E_1(p_2, w; \tau_{22}) + O(t^{\frac{1}{2}}) = \left(2\pi i/\tau_{12}\right)^{\frac{1}{2}} E_1(z, p_1; \tau_{11}) E_1(p_2, w; \tau_{22}) + O(\tau_{12}^{-\frac{1}{2}}) (3.17)$$

where the genus 1 prime form is given by $E_1(z, w; \tau) = \vartheta[z - w, \tau]/\vartheta'[0, \tau]$. 

Next, we derive the asymptotics of the Green’s function $G(z, w)$. The Green’s function $G(z, w)$ was defined in (2.24). Its leading asymptotics arises from the degeneration of the prime form and produces a $\ln |t|$ term. The subleading asymptotics is constant as $t \to 0$, and must also be retained. Asymptotics in lower order terms are not needed. The precise formula is quite simple,

$$G(z, w; \Omega) = - \ln |E(z, p_1; \tau_{11})|^2 + 2\pi (\text{Im} \tau_{11})^{-1} (\text{Im} (z - p_1))^2$$

$$- \ln |E(w, p_2; \tau_{22})|^2 + 2\pi (\text{Im} \tau_{22})^{-1} (\text{Im} (w - p_2))^2$$

$$+ \ln(|t|/4) + o(1) (3.18)$$

or, using the form of the genus 1 Green functions,

$$G(z, w; \Omega) = \ln(|\tau_{12}|/2\pi) + G(z, p_1; \tau_{11}) + G(w, p_2; \tau_{22}) + o(1). (3.19)$$

Finally, making use of the limits of the holomorphic Abelian differentials established earlier, $\lim_{t \to 0} \omega_I(z) = 1$ or 0 depending on whether $z \in \Sigma_I$ or not, as well as the fact that $s + t + u = 0$, we readily find the asymptotics for $\mathcal{Y}_S$,

$$\lim_{t \to 0} \mathcal{Y}_S = -s. (3.20)$$
3.2 Combinatorics of the degeneration

Three important combinatorial considerations need to be taken into account in order to obtain the correct normalization of the 2-loop measure and amplitude from factorization.

1. The formulas for the Siegel fundamental domain require that the genus 1 components of the degeneration be ordered [39, 40], in the sense that

   \[ \text{Im}(\tau_{11}) \leq \text{Im}(\tau_{22}) \]  \hspace{1cm} (3.21)

   For an integrand that is symmetric under \( \tau_{11} \leftrightarrow \tau_{22} \), this ordering is equivalent to taking the product over the two genus 1 moduli spaces, and including a factor of 1/2.

2. The pair \( z_1, z_2 \) must run over both genus 1 components of the degeneration. This produces a factor of 2 when the formula is written with \( z_1, z_2 \) running only over one of the genus 1 components.

3. As pointed out in [40], one must impose the identification \( t \sim -t \), or equivalently \( \tau_{12} \sim -\tau_{12} \). The explanation is as follows. The genus 1 components each have a (conformal) automorphism \( z \rightarrow -z \). Applying this transformation to only one genus 1 component changes \( t \rightarrow -t \), in view of the plumbing fixture definition \( t = zw \), where \( z \) and \( w \) lie on opposite genus 1 components.

The two combinatorial factors due to points 1. and 2. above cancel one another if one takes the prescription that one integrates over the full genus 1 moduli spaces of the two components and assumes that the pair \( z_1, z_2 \) runs only over a single component, while the other pair \( z_3, z_4 \) runs over the other component only.

3.3 Factorization of the two-loop 4-point amplitude

We shall now evaluate the contribution to the residue of the pole in \( s \) at \( s = -q^2 \) where \( q = k_1 + k_2 = -k_3 - k_4 \) of the two-loop amplitude resulting from the separating degeneration alone. There are other singularities at \( s = -q^2 \), which are of no interest to us. The first is when two vertex operators come close together, producing a pole; the second is when a non-separating degeneration occurs, producing a branch cut\(^6\).

By translation invariance on each torus, the points \( p_1 \) and \( p_2 \) are arbitrary, and the three genus 2 moduli \( \tau_{11}, \tau_{22} \) and \( \tau_{12} \) span the moduli coordinates on the two tori components.

\(^{6}\)For a detailed discussion of the poles and branch cuts for the superstring box graph, see [41].
\[ \Sigma_1, \Sigma_2 \text{ as well as the plumbing fixture, without the need for } p_1 \text{ and } p_2. \text{ Thus, to leading asymptotics, the measure has the following limit,} \]
\[
\lim_{\tau_{12} \to 0} \frac{|d^2 \Omega|^2}{(\det \text{Im} \Omega)^5} = \frac{|d\tau_{12}|^2}{|d\tau_{11}|^2} \frac{|d\tau_{22}|^2}{(\text{Im} \tau_{11})^5 (\text{Im} \tau_{22})^5} \]  
(3.22)

Combining all ingredients \( G(z, w) \) and \( \mathcal{Y}_S \) for the limit, as calculated above, we get
\[
A_2^{(4)}(\epsilon_i, k_i) \sim s^2 C_2 K \bar{K} (2\pi)^{\alpha'/2} \int |d\tau_{12}|^2 |\tau_{12}|^{-\alpha'/2} B_1(k_1, k_2, q)B_1(k_3, k_4, -q) \]  
(3.23)

where the 3-point functions are given by (we suppress the dependence of \( B_1^{(3)} \) on the polarization vectors \( \epsilon_i \)),
\[
B_1^{(3)}(k_1, k_2, -q) = \int_{M_1} \frac{|d\tau_{11}|^2}{|d\tau_{11}|^5} \int d^2 z_1 d^2 z_2 \exp \frac{\alpha' s}{4} \{ G(z_1, z_2) - G(z_1, p_1) - G(z_2, p_1) \} \]  
(3.24)

\[
B_1^{(3)}(k_3, k_4, q) = \int_{M_1} \frac{|d\tau_{22}|^2}{|d\tau_{22}|^5} \int d^2 z_3 d^2 z_4 \exp \frac{\alpha' s}{4} \{ G(z_3, z_4) - G(z_3, p_2) - G(z_4, p_2) \} \]

The above kinematical factors are exactly as one would expect for a 3-point function with the external momenta \((k_1, k_2, -q)\) in the first and \((k_3, k_4, q)\) in the second, in view of the kinematical relations
\[
-k_1 \cdot (-q) = -k_2 \cdot (-q) = -k_3 \cdot q = -k_4 \cdot q = \frac{1}{2} s \]  
(3.25)

Notice that, at the value \( s = 4/\alpha' \), the 3-point functions \( B_1^{(3)} \) themselves have a pole in \( s \). This is as expected, and the pole arises from the integration regions \( z_1 \sim z_2 \) and \( z_3 \sim z_4 \). The factorization formula should be understood to hold for \( s \) in the neighborhood of \( 4/\alpha' \).

It remains to evaluate the pole itself. We are interested only in the first massive pole at \( s = 4/\alpha' \), which is also the first singularity (i.e. the smallest value of \( s \) for which there is a singularity) of the \( \tau_{12} \)-integral. Both \( B_1 \) factors are independent of \( \tau_{12} \) in the limit \( \tau_{12} \to 0 \). Recall from the previous subsection that one must identify \( \tau_{12} \) with \(-\tau_{12} \), so that the integration over \( \tau_{12} \) is actually over a disk with opposite identified, \( D_\varepsilon/\mathbb{Z}_2 \), where \( D_\varepsilon = \{ \tau_{12} \in \mathbb{C}, |\tau_{12}| < \varepsilon \} \). Instead of the disk, it is convenient to cutoff the \( \tau_{12} \)-integral by an exponential instead with \( \tau_{12} \in \mathbb{C} \), since the integral is then easily analytically continued in \( s \). Retaining only the pole parts, we have
\[
(2\pi)^{\alpha'/2} \int |d\tau_{12}|^2 |\tau_{12}|^{-\alpha'/2} = (2\pi)^{\alpha'/2} \int_{\mathbb{C}/\mathbb{Z}_2} |d\tau_{12}|^2 |\tau_{12}|^{-\alpha'/2} e^{-|\tau|} = -\frac{16\pi^3/\alpha'}{s - 4/\alpha'} \]  
(3.26)
Here the $Z_2$ factor accounts for the identification $\tau_{12} \rightarrow -\tau_{12}$. Putting all together, we have

$$A_2^{(4)}(\epsilon_i, k_i) = -\delta(k) \frac{2^6 \pi^3 C_2/\alpha'}{s - 4/\alpha'} e^{2\lambda} K \bar{K} B_1^{(3)}(k_1, k_2, -q) B_1^{(3)}(k_3, k_4, q).$$

(3.27)

It should be understood here that we are concerned only with that region of moduli space where the points on each torus are kept separated from one another.

### 3.4 Factorization of the one-loop 4-point amplitude

From the normalized one-loop 4-point amplitude $A_1^{(4)}$ of (2.22), (derived in Appendix C), we obtain the residue at the massive pole $s = 4/\alpha'$, which arises when the two insertion points $z_3$ and $z_4$ come close together. In the region of moduli space where $z_4$ comes close to $z_3$, we eliminate $z_4$ in favor of $z = z_4 - z_3$, but keep all other points. In this way, we get

$$A_1^{(4)}(\epsilon_i, k_i) = C_1 K \bar{K} \kappa^4 \int_{\mathcal{M}_1} \frac{|d\tau|^2}{(\text{Im } \tau)^6} \int |dz|^2 |z|^{-\alpha'/s/2} \prod_{i=1,2,3} \int d^2 z_i \times \exp \left\{ -\frac{\alpha'}{2} \sum_{1 \leq i < j \leq 3} k_i \cdot k_j G(z_i, z_j) \right\}. \quad (3.28)$$

The $z$-integral is familiar and, retaining only the pole parts in $s$, we have

$$\int |dz|^2 |z|^{-\alpha'/s/2} = \int_C |dz|^2 |z|^{-\alpha'/s/2} e^{-|z|} = \frac{8\pi / \alpha'}{s - 4/\alpha'}. \quad (3.29)$$

The remaining integral is proportional to the 3-point function factors $B_1$ introduced in the factorization of the 2-loop amplitude in (3.24). In (3.24), however, $B_1$ was expressed as an integral over 2 of the three vertex points, while the integral in (3.28) has 3 vertex point integrations. Using translation invariance on the torus worldsheet, one of these three integrations may be carried out. This fixes the third point and produces a worldsheet volume factor $2\text{Im } \tau$. The factor of $\text{Im } \tau$ reduces the denominator in (3.28) from $(\text{Im } \tau)^{-6}$ to $(\text{Im } \tau)^{-5}$ in (3.24). The extra factor of 2 needs to be retained, and hence

$$A_1^{(4)}(\epsilon_i, k_i) = C_1 K \bar{K} \kappa^4 \frac{16\pi / \alpha'}{s - 4/\alpha'} B_1(k_3, k_4, q), \quad q = -k_3 - k_4. \quad (3.30)$$

### 3.5 Factorization of the tree-level 4-point amplitude

The normalized tree-level 4-point amplitude $A_0^{(4)}$ of (2.21), (calculated in Appendix B), may also be factored onto the massive pole at $s = 4/\alpha'$, and we find,

$$A_0^{(4)}(\epsilon_i, k_i) = \frac{8\pi}{\alpha'} C_0 Q_0 e^{-2\lambda} K \bar{K} \frac{1}{s - 4/\alpha'}. \quad (3.31)$$
3.6 The factorization constraint

Finally, we can implement the factorization constraint between the two-loop, one-loop, and tree-level superstring amplitudes. Expressing the pole in terms of a common factor,

\[ A_0^{(4)}(k_i) = \frac{8\pi/\alpha'}{s-4/\alpha'} \kappa^4 K\bar{K} \left( C_0 Q_0 e^{-2\lambda} \right) \]

\[ A_1^{(4)}(k_i) = \frac{8\pi/\alpha'}{s-4/\alpha'} \kappa^4 K\bar{K} \left( 2C_1 B_1^{(3)} \right) \]

\[ A_2^{(4)}(k_i) = \frac{8\pi/\alpha'}{s-4/\alpha'} \kappa^4 K\bar{K} \left( 8\pi^2 C_2 e^{+2\lambda} B_1^{(3)} B_1^{(3)} \right), \tag{3.32} \]

The particular factorization of the tree-level, one and two-loop string amplitudes we are considering in (3.32) is illustrated in Figure 1.

Figure 1: Factorization of the tree-level (a) to (a'), one-loop (b) to (b') and two-loop (c) to (c') 4-point functions on a massive intermediate state.
The consistency of the factorization of the two-loop amplitude on the first massive pole with the tree-level and one-loop factorization requires the following relation

\[ C_1^2 = 2\pi^2 C_0 Q_0 C_2 \] (3.33)

Using the values obtained for the tree-level amplitudes in Appendix B and for the one-loop amplitude in Appendix C,

\[ C_0 = \frac{1}{2^6} \quad \quad Q_0 = \frac{\sqrt{2}}{2^6 \pi^6 (\alpha')^5} \quad \quad C_1 = \frac{1}{2^8 \pi^2 (\alpha')^5} \] (3.34)

we obtain

\[ C_2 = \frac{\sqrt{2}}{2^6 (\alpha')^5}. \] (3.35)

The \(\alpha'\) factor agrees on dimensional grounds. For completeness, we list the two-loop 4-point function expressed in terms of the S-duality normalization of the dilaton,

\[ A_2^{(4)}(\epsilon_i, k_i) = \frac{e^{2\phi} \pi^4 K \bar{K}}{2^{12} \pi^2 (\alpha')^5} \int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{(\text{det} \text{Im} \Omega)^5} \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp \left( -\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right) \] (3.36)
4 The $D^4 R^4$ term from the two-loop amplitude

In view of the presence of the factor $|Y_S|^2$ in the two-loop amplitude, and the fact that $Y_S$ itself is linear in the kinematic variables $s, t, u$, the term $D^4 R^4$ will arise from setting $k_i = 0$ in the exponential in (2.23). The integration over $\Sigma^4$ then reduces to the following integrals,

$$\int_{\Sigma^4} |Y_S|^2 = \left| s \Delta(1, 4) \Delta(2, 3) - t \Delta(1, 2) \Delta(3, 4) \right|^2 = \mathcal{L}_1 + \mathcal{L}_2$$

(4.1)

where

$$\mathcal{L}_1 = (s^2 + t^2) \left( \int_{\Sigma^2} |\Delta(1, 2)|^2 \right)^2$$

$$\mathcal{L}_2 = -2st \int_{\Sigma^4} \Delta(1, 4) \Delta(2, 3) \overline{\Delta(1, 2) \Delta(3, 4)}$$

(4.2)

The integrations may be carried out using the Riemann bilinear relation,

$$\int_{\Sigma} \omega_I \omega^*_{J} \equiv -i \int_{\Sigma} \omega^*_I \wedge \omega_J = 2 \text{Im} \Omega_{IJ}$$

(4.3)

The double integral for $\mathcal{L}_1$ and the quadruple integral for $\mathcal{L}_2$ are readily carried out using (4.3), and we find the following results,

$$\mathcal{L}_1 = 64(s^2 + t^2)(\det \text{Im} \Omega)^2$$

$$\mathcal{L}_2 = 64st (\det \text{Im} \Omega)^2$$

(4.4)

Putting all together, and using $4s^2 + 4t^2 + 4st = 2(s^2 + t^2 + u^2)$, we have

$$\int_{\Sigma^4} |Y_S|^2 = 32(s^2 + t^2 + u^2)(\det \text{Im} \Omega)^2$$

(4.5)

In this limit, the amplitude reduces to the following expression,

$$\mathcal{A}_2^{(D^4 R^4)}(\epsilon_i, k_i) = 8V_2 C_2 e^{2\lambda}(\alpha')^2 (s^2 + t^2 + u^2) \kappa^4 K \bar{K}$$

(4.6)

Here, we have restored the dependence on $\alpha'$ for later use, and $V_2$ is the volume of genus 2 moduli space $\mathcal{M}_2$, i.e. the volume of the fundamental domain of $Sp(4, \mathbb{Z})/\mathbb{Z}_2$. Using the explicit formula for the volume given in Appendix 8A, we have

$$\mathcal{A}_2^{(D^4 R^4)}(\epsilon_i, k_i) = \frac{2^6\pi^3}{270} C_2 e^{2\lambda}(\alpha')^2 (s^2 + t^2 + u^2) \kappa^4 K \bar{K}.$$  

(4.7)
5 Heterotic $D^2F^2F^2$ and $D^2F^4$ terms

Extensive results are available on the Heterotic string contribution to the low energy effective action arising from tree-level and one-loop orders [44]. As the two-loop contribution to the 4-point function is now also available, we may add to these results as follows.

Two-loop 4-point amplitudes in the Heterotic string with space-time and gauge structure $F^4$ and $F^2F^2$ were derived in [21] eq. (1.22), and are given by

$$A_{F^4} = C_H K \int_\mathcal{M}_2 \frac{|d^2\Omega|^2}{(\text{det} \text{Im}\Omega)^5 \Psi_{10}(\Omega)} \int_{\Sigma^4} \mathcal{W}_{F^4} \tilde{\mathcal{Y}}_S \exp \left( - \sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right)$$ (5.1)

Here, $\tilde{K}$ is the standard kinematical factor $2^3 t_8 F^4$, $C_H$ is an overall normalization constant and $\tilde{\mathcal{Y}}_S$ is the same quadri-holomorphic form familiar from the Type II amplitudes in (2.25). The leading order term in small momenta is obtained when the exponential factor equals 1 as $k_i \to 0$. The remaining factor $\tilde{\mathcal{Y}}_S$ has two extra factors of momenta. This guarantees that the $F^4$ and $F^2F^2$ terms receive no two-loop renormalization, as shown in [21], but thus also implies non-vanishing corrections of the type $D^2F^4$ and $D^2F^2F^2$.

We shall evaluate this contribution here for the simplest case, namely for the $E_8 \times E_8$ theory, when two external states (say 1,2) are in the first $E_8$, while the remaining states (3,4) are in the second $E_8$. The form of $\mathcal{W}_{F^4}$ is then particularly simple, and given by

$$\mathcal{W}_{F^4} = \frac{1}{4} \text{tr}(T^{a_1}T^{a_2})\text{tr}(T^{a_3}T^{a_4})^2 F^{(2)}_4(z_1, z_2)F^{(2)}_4(z_3, z_4)$$ (5.2)

In [21], the function $F^{(2)}_4$ was computed explicitly in equation (12.7) and is given by

$$F^{(2)}_4(z, w) = \Psi_4 \partial_z \partial_w \ln E(z, w) + \frac{\pi i}{2} \omega_{IJ}(z_1)\omega_{IJ}(z_2)\partial_{IJ}\Psi_4$$ (5.3)

Here, $\Psi_4$ is the unique modular form of weight 4, defined by $\Psi_4 = \sum_\delta \vartheta(\delta)(0, \Omega)^8$. Using the symmetries of $\tilde{\mathcal{Y}}_S$, the integration reduces to

$$\int_{\Sigma^4} \mathcal{W}_{F^4} \tilde{\mathcal{Y}}_S = -se^{IJ} \epsilon^{KL} \int_{\Sigma^4} F^{(2)}_4(1, 2)F^{(2)}_4(3, 4)\omega^*_I(1)\omega^*_J(2)\omega^*_K(3)\omega^*_L(3)$$ (5.4)

The integral over $\Sigma^4$ may be computed using (4.3) as well as the Riemann relation

$$\int_{\Sigma} \omega^* \partial_z \partial_w \ln E(z, w) = 2\pi \omega_I(w)$$ (5.5)

It is convenient to first evaluate the double integral

$$\int_{\Sigma^2} F^{(2)}_4(1, 2)\omega^*_I(1)\omega^*_K(2) = 4\pi (\text{Im}\Omega)_{IK} \Psi_4 + 2\pi i (\text{Im}\Omega)_{IM} \partial_{MN} \Psi_4 (\text{Im}\Omega)_{NK}$$ (5.6)
and hence
\[ \int_{\Sigma^4} \mathcal{W}_{F^4} \tilde{Y}_S = 8\pi^2 s (\det \text{Im} \Omega)^2 \Psi_4^2 \det \left( \partial_{I,J} \ln \Psi_4 - 2i(\text{Im} \Omega)^{-1}_{I,J} \right) \]  (5.7)

Putting all together, we have in this limit,
\[ A_{F^4} = s K 8\pi^2 C_H \text{tr} (T^{a_1} T^{a_2}) \text{tr} (T^{a_3} T^{a_4}) \int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{(\det \text{Im} \Omega)^3} \Psi_{F^4} (\Omega) \]
\[ \Psi_{F^4} (\Omega) = \frac{\Psi_4 (\Omega)^2}{\Psi_{10} (\Omega)} \det \left[ \partial_{I,J} \ln \left\{ \Psi_4 (\det \text{Im} \Omega)^4 \right\} \right] \]  (5.8)

where we have used the relation \( \partial_{I,J} \ln \det \text{Im} \Omega = -i/2 (\text{Im} \Omega)^{-1}_{I,J} \). Notice that \( \Psi_{F^4} \) is a non-analytic modular function, i.e. a modular form of weight 0. Indeed, combining the following transformation laws,
\[ \tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1} \]
\[ \Psi_4 (\tilde{\Omega}) = \det (C\Omega + D)^4 \Psi_4 (\Omega) \]
\[ \det \text{Im} \tilde{\Omega} = |\det (C\Omega + D)|^{-2} \det \text{Im} \Omega \]  (5.9)

we find the following transformation law,
\[ \Psi_4 (\tilde{\Omega}) (\det \text{Im} \tilde{\Omega})^4 = \det (C\Omega^* + D)^{-4} \Psi_4 (\Omega) (\det \text{Im} \Omega)^4 \]  (5.10)

Taking the log and differentiating in \( \Omega_{I,J} \) produces a rank 2 modular tensor
\[ \partial_{I,J} \ln \left\{ \Psi_4 (\det \text{Im} \Omega)^4 \right\} \]  (5.11)
whose determinant is a modular form of weight 2, making \( \Psi_{F^4} \) a modular function.

**Acknowledgments**

We are grateful to Costas Bachas, Michael Green, Sam Grushevsky, Boris Pioline, Jacob Sturm, Tomasz Taylor and Richard Wentworth for useful conversations, correspondences, and references.
A The $Sp(2h, \mathbb{Z})/\mathbb{Z}_2$ fundamental domains

We give a synopsis of the key facts about genus 1 and 2 moduli spaces and their measures needed in the sequel.

A.1 Genus 1

The genus 1 moduli space is given by the fundamental domain of $Sp(2, \mathbb{Z})/\mathbb{Z}_2$, namely

$$M_1 = \left\{ \tau \in \mathbb{C}; \text{Im}(\tau) > 0, \ |\tau| \geq 1, \ |\text{Re}(\tau)| \leq \frac{1}{2} \right\}$$  \hspace{1cm} (A.1)

Its volume for the Poincaré metric is as follows,

$$V_1 = \int_{M_1} \frac{|d\tau|^2}{(\text{Im} \, \tau)^2} = \frac{2\pi}{3} \quad |d\tau|^2 \equiv |d\bar{\tau} \wedge d\tau|$$  \hspace{1cm} (A.2)

A.2 Genus 2

The genus 2 moduli space is considerably more complicated. It is given by the fundamental domain of $Sp(4, \mathbb{Z})/\mathbb{Z}_2$,

$$M_2 = \left\{ \Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \text{ satisfying (1), (2), and (3)} \right\}$$  \hspace{1cm} (A.3)

where the three conditions are given by, (see e.g. [39] and [40]),

(1) \quad |\text{Re}(\tau_{11})| \leq \frac{1}{2}, \ |\text{Re}(\tau_{22})| \leq \frac{1}{2}, \ |\text{Re}(\tau_{12})| \leq \frac{1}{2}

(2) \quad 0 \leq |2\text{Im}(\tau_{12})| \leq \text{Im}(\tau_{11}) \leq \text{Im}(\tau_{22})

(3) \quad |\text{det}(C\Omega + D)| \geq 1 \quad \text{for all} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z})$$  \hspace{1cm} (A.4)

The volume of the genus 2 moduli space was computed by Siegel [39], and is given as follows,

$$\int_{M_2} \frac{d^3\text{Re}(\Omega)d^3\text{Im}(\Omega)}{(\text{det } \text{Im } \Omega)^3} = \frac{\pi^3}{270}$$  \hspace{1cm} (A.5)

In our conventions for the measure, this takes the form,

$$V_2 = \int_{M_2} \frac{|d^3\Omega|^2}{(\text{det } \text{Im } \Omega)^3} = \frac{8\pi^3}{270} \quad |d^3\Omega|^2 = \left| \wedge_{I \leq J} d\tau_{IJ} \right|^2.$$  \hspace{1cm} (A.6)

Notice that in the separating degeneration limit, $M_2$ tends to the symmetrized product of $M_1$ and $M_2$, since their moduli are ordered,

$$\lim_{\tau_{12} \to 0} M_2 = M_1(\tau_{11}) \times M_1(\tau_{22}) \quad \text{Im}(\tau_{11}) \leq \text{Im}(\tau_{22})$$  \hspace{1cm} (A.7)
B Tree-level superstring amplitudes

In [37], an expression for the tree-level superstring amplitudes was derived only up to a constant due to the combined determinants of matter and ghost fields on the sphere. The scalar as well as the \( bc \) ghost determinants have been derived by Weisberger [42]. We give here a rigorous derivation for all spins. We also take the opportunity to correct some typos in [37]. Since numerical constants are crucial for our present purposes, we explain their origin in some detail.

\[ g_{mn} = \frac{2\delta_{mn}}{(1 + |z|^2)^2} \quad (B.1) \]

where \( \delta_{z\bar{z}} = \delta_{\bar{z}z} = 1 \) and \( \delta_{zz} = \delta_{\bar{z}\bar{z}} = 0 \). The Laplacians of interest are,

\[ \Delta^+_{(n)} = -2\nabla^z_{(n+1)} \nabla^{(n)}_z \quad (B.2) \]

for forms of spin \( n \in \mathbb{Z}/2 \). The cases \( n = 0, n = 1, n = -1/2 \) and \( n = +1/2 \) respectively correspond to scalars, the \( bc \) ghosts, the Dirac spinor, and the \( \beta\gamma \) superghosts. Henceforth, we shall assume that \( n \geq -1/2 \); in the contrary case, simply interchange \( n \) and \( -n - 1 \). The eigenvalues of the Laplacians are given by,

\[ (\ell - n)(\ell + n + 1) \quad \{ \ell = n + 1, n + 2, \ldots \} \quad \text{multiplicity } (2\ell + 1) \quad (B.3) \]

Modes with \( \ell = n \) and \( l = -n - 1 \) correspond to zero modes of \( \nabla^{(n)}_z \) and \( \nabla^z_{(n+1)} \), while modes with smaller \( \ell \) correspond to non-normalizable solutions. The determinant of the Laplacian (with zero modes suitably removed), is given by

\[ \ln \det' \Delta_n = -\zeta'_n(0) \quad (B.4) \]

where the corresponding \( \zeta \)-function is defined by

\[ \zeta_n(s) = \sum_{\ell=-n+1}^{\infty} \frac{2\ell + 1}{(\ell - n)^s(\ell + n + 1)^s} = \sum_{\ell=1}^{\infty} \frac{2\ell + 2n + 1}{\ell^s(\ell + 2n + 1)^s} \quad (B.5) \]

Using Feynman parameters to combine both factors in the denominators of the sum, and expressing the result with the help of the Hurwitz \( \zeta \)-function, which is defined by

\[ \zeta(s, u) = \sum_{m=0}^{\infty} \frac{1}{(m + u)^s} \quad (B.6) \]

24
we have
\[
\zeta(n)(s) = \frac{\Gamma(2s - 1)}{\Gamma(s)\Gamma(s - 1)} \int_0^1 d\alpha \alpha^{s-2}(1-\alpha)^{s-2} \zeta(2s - 1, 1 + \alpha(2n + 1)) \quad (B.7)
\]

The prefactor of Gamma functions vanishes to first order in \(s\) at \(s = 0\), but the integral has poles at \(s = 0\). It suffices to isolate the parts of the integrand that produce the poles. They arise from terms constant and/or vanishing linearly at \(\alpha = 0, 1\). We use the notation \(\sigma = 2s - 1\), and expand as follows,
\[
\zeta(\sigma, 1 + \alpha(2n + 1)) = \frac{1}{2} \{ \zeta(\sigma, 1) + \zeta(\sigma, 2n + 2) \} + a_1(\alpha - \frac{1}{2}) + \alpha(1 - \alpha)\varphi_n(\sigma, \alpha)
\]
\[
\varphi_n(\sigma, \alpha) = \frac{1}{2} \{ \varphi_n(\sigma, 0) + \varphi_n(\sigma, 1) \} + a_2(\alpha - \frac{1}{2}) + \alpha(1 - \alpha)\psi_n(\sigma, \alpha) \quad (B.8)
\]
The coefficients \(a_1\) and \(a_2\) are independent of \(\alpha\), so that the contribution of those terms to the integral over \(\alpha\) vanishes by the symmetry \(\alpha \leftrightarrow (1 - \alpha)\) of the integral. The functions \(\varphi_n\) and \(\psi_n\) are analytic in \(s\), just as \(\zeta\) was. The integrals of the terms independent of \(\alpha\) and linear in \(\alpha(1 - \alpha)\) are calculated using the Euler beta-function integral. The terms in \(\varphi_n\) above may be computed by taking the derivative in \(\alpha\) of (B.8) at \(\alpha = 0, 1\), and using the formula
\[
\frac{\partial \zeta(\sigma, \alpha)}{\partial \alpha} = -\sigma \zeta(\sigma + 1, \alpha),
\]
\[
\varphi_n(\sigma, 1) + \varphi_n(\sigma, 0) = \sigma(2n + 1) (\zeta(\sigma + 1, 2n + 2) - \zeta(\sigma + 1, 1)) \quad (B.9)
\]
Combining all results, we obtain,
\[
\zeta(n)(s) = \zeta^0(n)(s) + \frac{\Gamma(2s - 1)}{\Gamma(s)\Gamma(s - 1)} \int_0^1 d\alpha \alpha^s(1-\alpha)^s\psi_n(2s - 1, \alpha) \quad (B.10)
\]
where the reduced function is given by
\[
\zeta^0(n)(s) = \zeta(2s - 1, 1) + \zeta(2s - 1, 2n + 2) + \frac{1}{2}(s-1)(2n+1) (\zeta(2s, 2n + 2) - \zeta(2s, 1))
\]
\[
= 2\zeta_R(2s - 1) - \sum_{\ell=1}^{2n+1} \frac{1}{\ell^{2s-1}} + \frac{1}{2}(1-s)(2n+1) \sum_{\ell=1}^{2n+1} \frac{1}{\ell^{2s}} \quad (B.11)
\]
Here \(\zeta_R\) is the Riemann \(\zeta\)-function. The \(\zeta^0(n)\) part of (B.10) admits an analytic continuation in \(s\) throughout the complex plane \(C\) because \(\zeta_R\) does, while the integral term in (B.10) is analytic in \(s\) as long as \(\text{Re}(s) > -1\). In particular, the entire expression is now well-defined and analytic around \(s = 0\), and its value there may be readily computed.

To evaluate \(\zeta'(n)(0)\), it suffices to differentiate (B.10) is \(s\) and to set \(s = 0\),
\[
\zeta'(n)(0) = \zeta^0'(n)(0) + \frac{1}{2} \int_0^1 d\alpha \psi_n(-1, \alpha) \quad (B.12)
\]
To calculate $\psi_n(-1, \alpha)$, we use its definition in (B.8), combined with the following formula, which may be found in [43],

$$\zeta(-1, u) = \frac{1}{2} u (1 - u) - \frac{1}{12}$$  \hspace{1cm} (B.13)

As a result, $\varphi_n(-1, \alpha) = (2n + 1)^2/2$ and thus $\psi_n(-1, \alpha) = 0$, so that the integral in (B.10) vanishes. The remaining terms are obtained from differentiating (B.11), and we find,

$$\zeta'_n(0) = 4\zeta'_R(-1) - \frac{1}{2} (2n + 1)^2 + \sum_{\ell=1}^{2n+1} (2\ell - 2n - 1) \ln \ell$$  \hspace{1cm} (B.14)

By the same methods, we may read off also the following useful values,

$$\zeta_n(0) = -n - \frac{2}{3}$$  \hspace{1cm} (B.15)

The special cases $n = 0, n = \pm 1/2$, and $n = 1$ give the following $\zeta'_n(0)$,

$$\zeta'_0(0) = 4\zeta'_R(-1) - \frac{1}{2} \zeta'_1(0) = 4\zeta'_R(-1) - 2 + 2 \ln 2$$
$$\zeta'_1(0) = 4\zeta'_R(-1) - \frac{9}{2} + 3 \ln 3 + \ln 2$$  \hspace{1cm} (B.16)

Finally, we use the general formula that $\zeta_{\Delta(n)}(s) = \lambda^{-s} \zeta_{\Delta(n)}(s)$ to rescale the Laplace operators and the determinants,

$$\det' \lambda \Delta(n) = \lambda^{-n-2/3} \det' \Delta(n) = \lambda^{-n-2/3} \det' \Delta(n)$$  \hspace{1cm} (B.17)

Putting together all the determinants that enter the superstring, we have

$$\left( \frac{4\pi^2 \alpha' \det' \Delta(0)}{\int_{\Sigma} \sqrt{g}} \right)^{-5} \det'(2\Delta(1)) \left( \frac{\det \Delta(-1/2)}{\det'(2\Delta(1/2))} \right)^5 = \frac{\sqrt{2}}{27(\pi \alpha')^5}$$  \hspace{1cm} (B.18)

**B.2 The conformal Killing vector volume**

The measure $|\delta \rho_b|^2$ of $SL(2, \mathbb{C})$ (denoted $d\mu = |\delta \rho_b|^2$ in [37], eq.(2.108)), may be defined in a variety of related ways. It is convenient to express these in terms of the analytic measure
\(\delta \rho_b\). First, we have the formula of [37], eq. (2.108),

\[
\delta \rho_b = \frac{\delta z_0 \delta z_1 \delta z_\infty}{(z_0 - z_1)(z_1 - z_\infty)(z_\infty - z_0)}
\]  \hspace{1cm} (B.19)

Parametrizing the points as the images under \(g \in SL(2, \mathbb{C})\) of three fixed points 0, 1, \(\infty\),

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \quad \quad z_n = \frac{an + b}{cn + d}
\]  \hspace{1cm} (B.20)

for \(n = 0, 1, \infty\) and \(ad - bc = 1\), the measure \(\delta \rho_b\) may be expressed in terms of \(a, b, c, d\),

\[
\delta \rho_b = -a \delta b \land \delta c \land \delta d + b \delta a \land \delta c \land \delta d - c \delta a \land \delta b \land \delta d + d \delta a \land \delta b \land \delta c
\]  \hspace{1cm} (B.21)

The calculation of the conformal Killing volume now proceeds as follows. The integration over all vector fields \(v^m\) parametrizing infinitesimal metric deformations is only over those \(v^m\) orthogonal to the conformal Killing vector fields. On the other hand, we divide out by the entire volume of all vector fields. We want to trade in the measure \(|\delta \rho_b|^2\) for the integration over the conformal Killing vector fields \(v^m\). To see the effect, it suffices to determine the ratio of these two measures. Given that we are dealing with the invariant measure on a group \(SL(2, \mathbb{C})\), we need to do this just around the identity in \(SL(2, \mathbb{C})\).

The vector fields \(v^z\) are the variations in \(z\) due to variations in \(a, b, c, d\),

\[
\delta v^z = \sum_{n=0,1,2} \delta \epsilon_n z^n = \delta b + (\delta a - \delta d)z - \delta cz^2
\]  \hspace{1cm} (B.22)

Hence the volume form may be expressed in terms of the \(\delta \epsilon_n\),

\[
\delta \rho_b = -\delta \epsilon_0 \land \delta \epsilon_1 \land \delta \epsilon_2
\]  \hspace{1cm} (B.23)

On the other hand, the measure for the conformal Killing vectors \(|\delta^3 v|^2\) is governed by the metric on the space of Killing vectors,

\[
||\delta v||^2 = \int d^2 z \sqrt{g_{mn}} \delta v^m \delta v^n = \sum_{n=0,1,2} |\delta \epsilon_n|^2 N_n^2
\]  \hspace{1cm} (B.24)

Hence we have \(|\delta^3 v|^2 = N_0^2 N_1^2 N_2^2 |\delta \rho_b|^2\). With the round metric of (B.1), the normalization factors are evaluated as follows,

\[
N_n^2 = 2 \int d^2 z (g_{zz})^2 |z^n|^2 = 16\pi \int_0^\infty dr^2 \frac{r^{2n}}{(1 + r^2)^4}
\]  \hspace{1cm} (B.25)

and we find, \(N_0^2 = N_1^2 / 2 = N_2^2 = 16\pi / 3\), in agreement with [42], and

\[
|\delta \rho_b|^2 = \frac{1}{N_0^2 N_1^2 N_2^2} |\delta^3 v|^2 = 2 \left(\frac{3}{16\pi}\right)^3 |\delta^3 v|^2
\]  \hspace{1cm} (B.26)
B.3 The superconformal Killing spinor volume

The superconformal case is analogous to the conformal case, but the group of interest is now the supergroup $OSp(1,1)$. The derivation given here is parallel to the procedure of [37] §III. L, page 976. We isolate the measure $|\delta \rho_s|^2$ on $OSp(1,1)$ in terms of $z_{ij} = z_i - z_j - \theta_i \theta_j$,

$$\delta \rho_s = \frac{\delta z_0 \delta z_1 \delta z_\infty \delta \theta_0 \delta \theta_1 \delta \theta_\infty}{z_{01} z_{1\infty} z_{\infty 0}}(z_{01} \theta_\infty + z_{1\infty} \theta_0 + z_{\infty 0} \theta_1 + \theta_0 \theta_1 \theta_\infty)$$  \hspace{1cm} (B.27)

The last factor is related to the invariant $\Delta$ of [37, 45]. The $OSp(1,1)$ transformations act

$$T(z, \theta) = \begin{pmatrix} az + b + \alpha \theta \\ cz + d + \beta \theta \\ \gamma z + \epsilon + A \theta \end{pmatrix}$$  \hspace{1cm} (B.28)

where the matrices

$$T = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \epsilon & A \end{pmatrix} \quad K = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$  \hspace{1cm} (B.29)

satisfy $T^*KT = K$. We parametrize the variation of three points

$$\begin{align*}
(z_0, \theta_0) &= T(0, \theta) \\
(z_1, \theta_1) &= T(1, 0) \\
(z_\infty, \theta_\infty) &= T(z, 0)
\end{align*}$$  \hspace{1cm} (B.30)

infinitesimally around the identity transformation $T = I$. The general variation is given by

$$\begin{align*}
\delta z_n &= \delta b + (\delta a - \delta d)z_n - \delta c z_n^2 + (\delta \alpha - z_n \delta \beta) \theta_n \\
\delta \theta_n &= \delta \epsilon + z_n \delta \gamma + \delta A \theta_n - \theta_n (z_n \delta c + \delta d) + \delta \theta \delta \eta_n
\end{align*}$$  \hspace{1cm} (B.31)

The measure may now be computed, and we find,

$$\delta \rho_s = \theta \delta \theta \wedge \delta \gamma \wedge \delta \epsilon \wedge (\delta a - \delta d) \wedge \delta b \wedge \delta c$$  \hspace{1cm} (B.32)

The normalizations for the two conformal Killing spinors are now given by

$$N_n^2 = 2 \int d^2 z (g_{zz})^{3/2} |z^{-\frac{2}{3}}|^2 \quad n = \frac{1}{2}, \frac{3}{2}$$  \hspace{1cm} (B.33)

Explicit evaluation gives

$$N_2^2 = N_3^2 = 2^{5/2} \pi$$  \hspace{1cm} (B.34)

The measures are related by

$$|\delta \rho_s|^2 = N_2^2 N_3^2 |\delta^2 \zeta|^2 |\delta^3 \rho_b|^2 = \frac{27}{64 \pi} |\delta^3 v \delta^2 \zeta|^2$$  \hspace{1cm} (B.35)

The relation of $|\delta^3 \rho_b|^2$ to $|\delta^3 v|^2$ is as in the bosonic case (B.26).
B.4 The coefficient $Q_0$

Following the gauge-fixing procedure of [37], the coefficient $Q_0$ is found to be given by

$$Q_0 = \frac{N_0^2 N_+^2}{N_0^2 N_+^2 N_2^2} \left( \frac{\det \Delta_{(-1/2)}}{\pi \alpha' \det' \Delta_{(0)}} \right)^5 \frac{\det' (2 \Delta_{(1)})}{\det' (2 \Delta_{(1/2)})} = \frac{\sqrt{2}}{64 \pi^6 (\alpha')^5}$$  \hspace{1cm} (B.36)

B.5 The massless tree-level $N$-point function

This calculation was carried out in [37], III.I, but some of the coefficients are inaccurate. The correct expressions are as follows. Let the superfield propagator on the superplane (projected from the supersphere) be given by,

$$G(z, z') = -\ln \left( |z - z' - \theta \theta'|^2 + \epsilon^2 \right)$$  \hspace{1cm} (B.37)

where $\epsilon$ gives the correct short-distance prescription.

To compute the scattering amplitudes, it is actually convenient to view $\epsilon$ and $\bar{\epsilon}$ as Grassmann odd numbers and to work with the vertex generating function

$$V^*(\epsilon, \bar{\epsilon}, k) = \int_{\Sigma} d^2z \exp \{ ik \cdot X + \epsilon^\mu D_+ X^\mu + \bar{\epsilon}^\mu D_- X^\mu \}$$  \hspace{1cm} (B.38)

where it is understood that this quantity must be expanded precisely to first order in $\epsilon$ and to first order in $\bar{\epsilon}$, a prescription that will be indicated by $|\epsilon \bar{\epsilon}|$. We then have,

$$\left\langle \prod_{i=1}^{N} V(\epsilon_i, \bar{\epsilon}_i, k_i) \right\rangle = \left. \prod_{i=1}^{N} V^*(\epsilon_i, \bar{\epsilon}_i, k_i) \right|_{|\epsilon \bar{\epsilon}|}$$  \hspace{1cm} (B.39)

It is straightforward to compute now,

$$\left\langle \prod_{i=1}^{N} V^*(\epsilon_i, \bar{\epsilon}_i, k_i) \right\rangle = \prod_{i=1}^{N} \int_{\Sigma} d^2z_i \exp \left\{ \mathcal{G}_N - \sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right\}$$  \hspace{1cm} (B.40)

where

$$\mathcal{G}_N = \sum_{i \neq j} \left( ik_i \cdot \epsilon_j D_+^i + ik_j \cdot \bar{\epsilon}_i D_-^j - \frac{1}{2} \epsilon_i \cdot \epsilon_j D_+^i D_+^j - \frac{1}{2} \epsilon_i \cdot \bar{\epsilon}_j D_+^i D_-^j - \frac{1}{2} \bar{\epsilon}_i \cdot \epsilon_j D_-^i D_-^j - \frac{1}{2} \bar{\epsilon}_i \cdot \bar{\epsilon}_j D_-^i D_-^j \right) G(z_i, z_j)$$  \hspace{1cm} (B.41)
Neglecting contact terms, as we are instructed to do by the “cancelled propagator argument”, we have the following expressions,

\[\mathcal{D}^+_i G(z_i, z_j) = -\frac{\theta_{ij}}{z_{ij}} \quad \mathcal{D}^-_i G(z_i, z_j) = -\frac{\bar{\theta}_{ij}}{\bar{z}_{ij}}\]

\[\mathcal{D}^+_i \mathcal{D}^+_i G(z_i, z_j) = -\frac{1}{z_{ij}} \quad \mathcal{D}^-_i \mathcal{D}^-_i G(z_i, z_j) = -\frac{1}{\bar{z}_{ij}}\]

\[\mathcal{D}^+_i \mathcal{D}^-_i G(z_i, z_j) = 0 \quad \mathcal{D}^-_i \mathcal{D}^+_i G(z_i, z_j) = 0\] (B.42)

where we use the notation,

\[\theta_{ij} = \theta_i - \theta_j \quad z_{ij} = z_i - z_j - \theta_i \theta_j\]

\[\bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j \quad \bar{z}_{ij} = \bar{z}_i - \bar{z}_j - \bar{\theta}_i \bar{\theta}_j\] (B.43)

Thus, we have the following systematic expression for \(G_N = G_N^{(+)} + G_N^{(-)}\),

\[G_N^{(+)} = \sum_{i \neq j}^N \left( \frac{1}{2} \bar{\epsilon}_i \cdot \epsilon_j \frac{1}{z_{ij}} - i k_i \cdot \epsilon_j \frac{\theta_{ij}}{z_{ij}} \right)\]

\[G_N^{(-)} = \sum_{i \neq j}^N \left( \frac{1}{2} \bar{\epsilon}_i \cdot \bar{\epsilon}_j \frac{1}{\bar{z}_{ij}} - i k_i \cdot \bar{\epsilon}_j \frac{\bar{\theta}_{ij}}{\bar{z}_{ij}} \right)\] (B.44)

The superconformal volume is factored out as follows,

\[\langle \prod_{i=1}^N V^*(\epsilon_i, \bar{\epsilon}_i, k_i) \rangle = \left( \int |\delta \rho_s|^2 \right) \int d^2 \theta_{N-2} \prod_{i=1}^{N-3} d^2 z_i \mathcal{W}_N^{(+)} \mathcal{W}_N^{(-)} \prod_{i<j}^{N-1} |z_{ij}|^{2k_i k_j}\]

\[\mathcal{W}_N^{(+)} = \lim_{z_N \to \infty} \left( z_N \exp\{G_N^{(+)}\} \right)_{\epsilon}\]

\[\mathcal{W}_N^{(-)} = \lim_{z_N \to \infty} \left( z_N \exp\{G_N^{(-)}\} \right)_{\bar{\epsilon}}\] (B.45)

Here, it is understood that

\[z_{N-2} = \bar{z}_{N-2} = 0 \quad \theta_N = \bar{\theta}_N = 0\]

\[z_{N-1} = \bar{z}_{N-1} = 1 \quad \theta_{N-1} = \bar{\theta}_{N-1} = 0\] (B.46)

and the subscript \(|_{\epsilon}\) (respectively \(|_{\bar{\epsilon}}\)) stands for the prescription of retaining only terms that are linear in each \(\epsilon_i\) (respectively in each \(\bar{\epsilon}_i\)).

For \(N = 3\), only the integration over \(\theta_1\) remains, and we have

\[\mathcal{W}_3^{(+)} = i(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot k_1) + i(\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot k_2) + i(\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot k_3)\] (B.47)
B.5.1 The 4-point function

We expand $G_4^{(+)}$ and use the fact that $\theta_3 = \theta_4 = 0$ to simplify the result. We obtain,

$$G_4^{(+)} = \epsilon_1 \cdot \epsilon_2 \frac{1}{z_{12}} + \epsilon_1 \cdot \epsilon_3 \frac{1}{z_{13}} + \epsilon_1 \cdot \epsilon_4 \frac{1}{z_{14}} + \epsilon_2 \cdot \epsilon_3 \frac{1}{z_{23}} + \epsilon_2 \cdot \epsilon_4 \frac{1}{z_{24}} + \epsilon_3 \cdot \epsilon_4 \frac{1}{z_{34}}$$

$$-i(k_1 \cdot \epsilon_2 + k_2 \cdot \epsilon_1) \frac{\theta_1}{z_{12}} - i(k_1 \cdot \epsilon_3 + k_3 \cdot \epsilon_1) \frac{\theta_1}{z_{13}} - i(k_1 \cdot \epsilon_4 + k_4 \cdot \epsilon_1) \frac{\theta_1}{z_{14}}$$

$$-i(k_2 \cdot \epsilon_3 + k_3 \cdot \epsilon_2) \frac{\theta_2}{z_{23}} - i(k_2 \cdot \epsilon_4 + k_4 \cdot \epsilon_2) \frac{\theta_2}{z_{24}}$$

(B.48)

Next, exponentiate while retaining only contributions that are linear in each $\epsilon_i$,

$$W_4^{(+)} = -(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) \frac{1}{z_{12}} - (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) \frac{1}{z_{13}} - (\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3) \frac{1}{z_{23}}$$

$$-(\epsilon_1 \cdot \epsilon_2) \left( (\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_2) \frac{\theta_1 \theta_2}{z_{12}z_{13}} + (\epsilon_4 \cdot k_1)(\epsilon_3 \cdot k_2) \frac{\theta_1 \theta_2}{z_{12}z_{23}} \right)$$

$$+5 \text{ permutations of the last line}$$

(B.49)

The basic integral formula needed for 4-point functions is\(^7\)

$$\int d^2z \, z^A(1 - z)^B \bar{z}^\bar{A}(1 - \bar{z})^\bar{B} = 2\pi \frac{\Gamma(1 + A)\Gamma(1 + B)}{\Gamma(2 + A + B)} \cdot \frac{\Gamma(-1 - \bar{A} - \bar{B})}{\Gamma(-\bar{A})\Gamma(-\bar{B})}$$

(B.50)

for $A - \bar{A}, B - \bar{B} \in \mathbb{Z}$. The integral is invariant under $(A, B) \leftrightarrow (\bar{A}, \bar{B})$, and factors according to the left-moving $z$-dependence of the exponents $A, B$, and the right-moving $\bar{z}$-dependence of the exponents $\bar{A}, \bar{B}$. The superspace integrals we need for $c, \bar{c} = 0, 1$ are as follows,

$$\int d^{2|2}z_1 \int d^2\theta_2(\theta_1 \theta_2)^c(\bar{\theta}_1 \bar{\theta}_2)^\bar{c} z_{12}^A(1 - z_1)^B \bar{z}_{12}^{\bar{A}}(1 - \bar{z}_1)^{\bar{B}}$$

$$= 2\pi(-)^c \frac{\Gamma(1 + A)\Gamma(1 + B)}{\Gamma(1 + A + B + c)} \cdot \frac{\Gamma(-\bar{c} - \bar{A} - \bar{B})}{\Gamma(-\bar{A})\Gamma(-\bar{B})}$$

(B.51)

which is also invariant under the interchange $(A, B, c) \leftrightarrow (\bar{A}, \bar{B}, \bar{c})$. The integrals needed here are given as follows, for $a, \bar{a}, b, \bar{b}, c, \bar{c} \in \mathbb{Z}$,

$$\int d^{2|2}z_1 \int d^2\theta_2(\theta_1 \theta_2)^c(\bar{\theta}_1 \bar{\theta}_2)^{\bar{c}} z_{12}^{\frac{a}{2}}(1 - z_1)^{-\frac{a}{2} - b} \bar{z}_{12}^{\frac{\bar{a}}{2}}(1 - \bar{z}_1)^{-\frac{\bar{a}}{2} - b}$$

$$= 2\pi \mathcal{R} \mathcal{R} \frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(1 + s/2)\Gamma(1 + t/2)\Gamma(1 + u/2)}$$

(B.52)

\(^7\)We use the following conventions (which differ from those used in [37]), $d^2z = -id\bar{z} \wedge dz$, $d^2\theta = d\bar{\theta} d\theta$, so that $\int d^2\theta \theta \bar{\theta} = 1$, and $\int d^2\theta_1 \int d^2\theta_2 (\theta_1 \theta_2)(\bar{\theta}_1 \bar{\theta}_2) = -1$.
where
\[
\mathcal{R} = \frac{(-)^c \Gamma(1 - s/2 - a) \Gamma(1 + t/2) \Gamma(1 - u/2 - b)}{\Gamma(-s/2) \Gamma(1 + t/2 - a - b + c) \Gamma(-u/2)}
\]
\[
\tilde{\mathcal{R}} = \frac{\Gamma(1 + s/2) \Gamma(-t/2 + \tilde{a} + \tilde{b} - \tilde{c}) \Gamma(1 + u/2)}{\Gamma(s/2 + \tilde{a}) \Gamma(-t/2) \Gamma(u/2 + b)}
\]

The various values needed here for the calculation are given in the table below.

| form of prefactor | \(a\) | \(b\) | \(c\) | \(\mathcal{R}\) | sign | net factor |
|-------------------|------|------|------|----------------|------|------------|
| \(-1/(z_{12})\)   | 1    | 0    | 0    | \(-tu/4\)     | -    | \(+tu/4\)  |
| \(-1/(z_{13})\)   | 0    | 1    | 0    | \(-st/4\)     | +    | \(-st/4\)  |
| \(-1/(z_{23})\)   | 0    | 0    | 0    | \(+su/4\)     | +    | \(+su/4\)  |
| \(-\theta_1\theta_2/(z_{12}z_{13})\) | 1    | 1    | 1    | \(-t/2\)      | +    | \(-t/2\)   |
| \(-\theta_1\theta_2/(z_{12}z_{23})\) | 0    | 0    | 1    | \(+u/2\)      | +    | \(+u/2\)   |
| \(-\theta_1\theta_2/(z_{13}z_{23})\) | 1    | 1    | 0    | \(+s/2\)      | +    | \(+s/2\)   |

A similar table holds true for \(\tilde{\mathcal{R}}\).

**B.5.2 Final formula for the 4-point function**

\[
\mathcal{W}_4^{(+)} \rightarrow \frac{1}{4} tu(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) - \frac{1}{4} st(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) + \frac{1}{4} su(\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3)
\]
\[
-\frac{1}{2} t(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_2) - \frac{1}{2} u(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot k_2)(\epsilon_4 \cdot k_1)
\]
\[
+\frac{1}{2} s(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot k_3)(\epsilon_4 \cdot k_1) + \frac{1}{2} t(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot k_1)(\epsilon_4 \cdot k_3)
\]
\[
-\frac{1}{2} s(\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot k_4)(\epsilon_3 \cdot k_1) - \frac{1}{2} u(\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_4)
\]
\[
-\frac{1}{2} s(\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot k_3)(\epsilon_4 \cdot k_2) - \frac{1}{2} u(\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot k_2)(\epsilon_4 \cdot k_3)
\]
\[
+\frac{1}{2} s(\epsilon_2 \cdot \epsilon_4)(\epsilon_1 \cdot k_4)(\epsilon_3 \cdot k_2) + \frac{1}{2} t(\epsilon_2 \cdot \epsilon_4)(\epsilon_1 \cdot k_2)(\epsilon_3 \cdot k_4)
\]
\[
-\frac{1}{2} u(\epsilon_3 \cdot \epsilon_4)(\epsilon_1 \cdot k_4)(\epsilon_2 \cdot k_3) - \frac{1}{2} t(\epsilon_3 \cdot \epsilon_4)(\epsilon_1 \cdot k_3)(\epsilon_2 \cdot k_4)
\]

Factoring out the polarization vectors (taking into account that they anti-commute with one another), one gets

\[
\mathcal{W}_4^{(+)} \rightarrow -\epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \epsilon_4^\sigma L_{\mu \nu \rho \sigma}
\]
with

\[
L_{\mu\nu\rho\sigma} = -\frac{1}{4}tu\eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{4}st\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{4}su\eta_{\mu\sigma}\eta_{\nu\rho}
\]

\[
+ \frac{1}{2}s(\eta_{\mu\rho}k^\nu_3k^\sigma_1 + \eta_{\mu\sigma}k^\nu_4k^\rho_1 + \eta_{\nu\rho}k^\mu_3k^\sigma_2 + \eta_{\nu\sigma}k^\mu_4k^\rho_2)
\]

\[
+ \frac{1}{2}t(\eta_{\mu\nu}k^\rho_1k^\sigma_2 + \eta_{\mu\rho}k^\nu_1k^\sigma_3 + \eta_{\nu\sigma}k^\mu_2k^\rho_4 + \eta_{\rho\sigma}k^\mu_3k^\nu_4)
\]

\[
+ \frac{1}{2}u(\eta_{\mu\sigma}k^\rho_1k^\nu_4 + \eta_{\mu\nu}k^\rho_2k^\sigma_1 + \eta_{\nu\rho}k^\mu_2k^\sigma_3 + \eta_{\rho\sigma}k^\mu_3k^\nu_3)
\]

(B.56)

To identify this expression with the \( K \)-factor normalized earlier, we identify the coefficients in \( \epsilon_1 \cdot \epsilon_2 \), and we find,

\[
K = (\epsilon_1 \cdot \epsilon_2) \{2tu(\epsilon_3 \cdot \epsilon_4) - 4t(\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_2)\} + \text{perm}
\]

(B.57)

Hence, \( K = 8\epsilon_1^\mu \epsilon_2^\rho \epsilon_3^\nu \epsilon_4^\sigma L_{\mu\nu\rho\sigma} \) and

\[
\left\langle \prod_{i=1}^{4} V(\epsilon_i, \bar{\epsilon}_i, k_i) \right\rangle = 2^{-6} \left( \int |\delta \rho_s|^2 \right) K\bar{K} \frac{2\pi\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(1 + s/2)\Gamma(1 + t/2)\Gamma(1 + u/2)}
\]

(B.58)

Thus, the 4-point function for 4 gravitons is given by (2.21) with \( C_0 = 2^{-6} \).
C One-loop superstring amplitudes

We first give a detailed derivation of the determinant formulas for genus 1, paying special attention to their absolute normalization.

C.1 Summary of determinant formulas

Although the full amplitude is independent of the specific parametrizations used, the intermediate formulas are dependent. We use the following notations,

\[ \Sigma = \{ z \in \mathbb{C}, z = \sigma_1 + \tau \sigma_2, 0 \leq \sigma_{1,2} \leq 1 \} \]

\[ \mathcal{M}_1 = \{ \tau \in \mathbb{C}, 0 < \text{Im}(\tau), |\text{Re}(\tau)| \leq \frac{1}{2}, 1 \leq |\tau| \} \] (C.1)

and the metric is given by \( ds^2 = 2g_{zz}dzd\bar{z} = 2dzd\bar{z} \). With this metric, the area of the surface \( \Sigma \) equals \( 2\text{Im}(\tau) \). It is often convenient to set \( z = x + iy, \) with \( x, y \in \mathbb{C} \). The Cauchy-Riemann operators are then,

\[ 2\partial = 2 \frac{\partial}{\partial z} = \partial_x - i \partial_y \quad \quad 2\bar{\partial} = 2 \frac{\partial}{\partial \bar{z}} = \partial_x + i \partial_y \] (C.2)

With these conventions, we have the following values for the functional determinants,

\[ \det'\Delta = 2(\text{Im}\tau)^2|\eta(\tau)|^4 \]

\[ (\det' P_1^\dagger P_1)_{\frac{1}{2}} = \det'2\Delta = (\text{Im}\tau)^2|\eta(\tau)|^4 \]

\[ (\det 2\bar{\partial})_{\delta} = (\text{det}\bar{\partial})_{\delta} = \frac{\partial[\delta(\tau)]}{\eta(\tau)} \]

\[ (\det P_1^\dagger P_1)_{\frac{1}{2}} = \frac{(\text{det}2\partial)_{\delta}}{\eta(\tau)} \] (C.3)

C.2 Determinants of Laplacians on the torus

The differential operators are all proportional to the Cauchy-Riemann operators or their associated Laplacian, but the boundary conditions depend on whether we deal with a tensorial or a spinorial field with spin structure \( \delta \). The general boundary conditions are,

\[ \varphi(x+1,y) = -\exp\{2\pi i \delta'\}\varphi(x,y) \]

\[ \varphi(x+\text{Re}(\tau), y+\text{Im}(\tau)) = -\exp\{2\pi i \delta''\}\varphi(x,y) \] (C.4)

The Laplace operators for various tensor weights are all proportional to the Laplace operator on scalars, \( \Delta_{(0)} = -2\partial\bar{\partial} \). A basis of functions for these boundary conditions is given by

\[ \varphi_{mn}[\delta](x,y) = \exp\left\{ \frac{2\pi i}{\text{Im}(\tau)}((m+a)(\text{Im}(\tau)x - \text{Re}(\tau)y) + (n+b)y) \right\} \quad m, n \in \mathbb{Z} \] (C.5)
with \( a = \delta' + 1/2 \) and \( b = \delta'' + 1/2 \). This basis diagonalizes the Laplace operators,

\[
2\Delta \varphi_{mn}[\delta] = \frac{4\pi^2}{(\text{Im} \tau)^2} |n + b - (m + a)\tau|^2 \varphi_{mn}[\delta] \quad (C.6)
\]

For generic \((a, b) \neq (0, 0)\), the \( \zeta \)-function for the Laplace operator is defined by

\[
\zeta_{\delta}(s) = \sum_{m,n \in \mathbb{Z}} \frac{1}{|n + b - (m + a)\tau|^{2s}} \quad (C.7)
\]

The determinant is obtained by

\[
\ln(\det 2\Delta)_{\delta} = -\zeta'_{\delta}(0) - 2\zeta_{\delta}(0) \ln \left( \frac{\text{Im} \tau}{2\pi} \right) \quad (C.8)
\]

The values of the \( \zeta \)-function are as follows. As long as \((a, b) \neq (0, 0)\), we have \( \zeta_{\delta}(0) = 0 \), and \( -\zeta'_{\delta}(0) = \ln \left| \frac{\psi[\delta](0, \tau)}{\eta(\tau)} \right|^2 \) \( (\det 2\Delta)_{\delta} = \left| \frac{\psi[\delta](0, \tau)}{\eta(\tau)} \right|^2 \) \( (C.9) \)

and by holomorphic factorization, we have (up to an overall constant phase),

\[
(\det 2\bar{\partial})_{\delta} = \frac{\psi[\delta](0, \tau)}{\eta(\tau)} \quad (C.10)
\]

For periodic boundary conditions, \( a = b = 0 \), we first subtract the term \( m = n = 0 \) in the case \((a, b) \neq (0, 0)\) and then take the limit \( a, b \to 0 \). The relevant \( \zeta \)-function is

\[
\zeta(s) = \sum_{(m,n) \neq (0,0)} \frac{1}{|n - m\tau|^{2s}} = \lim_{(a,b) \to (0,0)} \left( \zeta_{\delta}(s) - \frac{1}{|b - a\tau|^{2s}} \right) \quad (C.11)
\]

with

\[
\ln \det'(2\Delta) = -\zeta'(0) - 2\zeta(0) \ln \left( \frac{\text{Im} \tau}{2\pi} \right) \quad (C.12)
\]

Using \( \psi[\delta](0, \tau) \sim (b - a\tau)\psi'_1(0, \tau) \), and \( \psi'_1(0, \tau) = -2\pi\eta(\tau)^3 \), we get,

\[
\det'(2\Delta) = (\text{Im} \tau)^2 |\eta(\tau)|^4 \quad (C.13)
\]

It is also useful to have the scaled up version of determinants,

\[
\det'(2\lambda\Delta) = \frac{1}{\lambda} (\text{Im} \tau)^2 |\eta(\tau)|^4 \quad (C.14)
\]
C.3 The 1-loop superstring measure

Putting together the entire measure, we have

\[ d\mu_1[\delta] = \frac{1}{2} \left( \frac{4\pi^2 \alpha' \text{det} \Delta}{\int \sqrt{g}} \right)^{-5} (\text{det} \delta)^{10} (\text{det} P_1^{\dagger} P_2)_{\delta}^{-\frac{1}{2}} \left( \frac{(\text{det} P_1^{\dagger} P_1)^{\frac{1}{2}}}{2 \text{Im} \tau} \right) |d\tau|^2 \frac{1}{(\text{Im} \tau)^2} \]  

(C.15)

where the last factor \(|d\tau|^2/\tau_2^2\) is the normalized Weil-Peterson measure on \(M_1\). Substituting the values of the various determinants, we have

\[ d\mu_1[\delta] = Q_1 |\vartheta[\delta](0, \tau)|^8 (\text{Im} \tau)^6 |\eta(\tau)|^{24} |d\tau|^2 \]  

(C.16)

where the coefficient \(Q_1\) is given by

\[ Q_1 = \frac{1}{4(4\pi^2 \alpha')^5} \]  

(C.17)

A first factor of 1/2 arises from dividing out by the volume factor of \(SL(2, \mathbb{Z})\) instead of \(PSL(2, \mathbb{Z})\), as has been argued for in [46] and in [47]. The difference between these two groups consists of the (conformal) automorphism of the worldsheet \(\Sigma\) which send \(z \to -z\).

A second factor of 1/2 arises because of our conventions for \(|d\tau|^2 = |d\tau \wedge d\bar{\tau}|\). Compared to [37], this has an extra factor of 2 by its very definition.

As a result of the extra factor of 1/4 in \(d\mu_1\) above, the coefficient \(Q_1\) now also has an extra factor of 1/4. It is straightforward to see that, for the superstring amplitude with even spin structure \(\delta\), this final normalization agrees with the corresponding normalization for the bosonic 1-loop amplitude, as computed in [46] and in [47], using the Polyakov integral, and in [48] using operator methods.

C.4 The 1-loop superstring amplitudes

To complete the calculation of the 4-point 1-loop amplitude, we must carry out the GSO summation of the contractions of all \(\psi\)-fields in the vertex operators. All other contractions vanish by the Riemann identity on the torus. The expansion of the vertex operator contraction, was given for any genus in [21], eq (7.2), with the normalization of the kinematical factor \(K\) given in [21], eq (6.1). The two required summation identities are as follows,

\[ S_1 = \sum_{\delta} \langle \nu_0 | \delta \rangle \vartheta[\delta](0)^4 S_{\delta}(z_1, z_2)^2 S_{\delta}(z_3, z_4)^2 \]

\[ S_2 = \sum_{\delta} \langle \nu_0 | \delta \rangle \vartheta[\delta](0)^4 S_{\delta}(z_1, z_2) S_{\delta}(z_2, z_3) S_{\delta}(z_3, z_4) S_{\delta}(z_4, z_1) \]  

(C.18)

Since each quantity is holomorphic in each \(z_i\), both are \(z_i\)-independent. By setting \(z_3 = z_1\) in both, we obtain the same expressions; hence we have \(S_2 = S_1\). To evaluate \(S_1\), we use
Fay’s formula for the torus,

\[ S_\delta(x, y)^2 = \partial_x \partial_y \ln E(x, y) + \vartheta[\delta]''(0, \tau) / \vartheta[\delta](0, \tau) \]  \hspace{1cm} (C.19)

The first term does not contribute and the second may be recast in the following form using the heat equation,

\[ \vartheta[\delta]''(0, \tau) / \vartheta[\delta](0, \tau) = 4\pi i \partial_\tau \ln \vartheta[\delta](0, \tau) \]  \hspace{1cm} (C.20)

Thus, we have

\[ S_1 = (4\pi i)^2 \sum_\delta \langle \nu_0 | \delta \rangle \vartheta[\delta](0)^4 \left( \partial_\tau \ln \vartheta[\delta](0, \tau) \right)^2 \]  \hspace{1cm} (C.21)

This sum may be worked out using the following identities, based on [19], eq (5.36-37),

\[ \partial_\tau \ln \left( \vartheta[\mu_2](0, \tau) / \eta(\tau) \right) = \frac{i\pi}{12} \left( \vartheta[\mu_3](0, \tau)^4 + \vartheta[\mu_4](0, \tau)^4 \right) \]  \hspace{1cm} (C.22a)

\[ \partial_\tau \ln \left( \vartheta[\mu_3](0, \tau) / \eta(\tau) \right) = \frac{i\pi}{12} \left( \vartheta[\mu_2](0, \tau)^4 - \vartheta[\mu_4](0, \tau)^4 \right) \]  \hspace{1cm} (C.22b)

\[ \partial_\tau \ln \left( \vartheta[\mu_4](0, \tau) / \eta(\tau) \right) = \frac{i\pi}{12} \left( -\vartheta[\mu_2](0, \tau)^4 - \vartheta[\mu_3](0, \tau)^4 \right) \]  \hspace{1cm} (C.22c)

where, as usual, we have

\[ \mu_2 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \quad \mu_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mu_4 = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \]  \hspace{1cm} (C.23)

Hence

\[ S_1 = \frac{\pi^4}{9} \left[ -\vartheta[\mu_2]^4 (\vartheta[\mu_3]^4 + \vartheta[\mu_4]^4)^2 + \vartheta[\mu_3]^4 (\vartheta[\mu_2]^4 - \vartheta[\mu_4]^4)^2 \right. \\
\left. \quad -\vartheta[\mu_4]^4 (\vartheta[\mu_2]^4 + \vartheta[\mu_3]^4)^2 \right] \]  \hspace{1cm} (C.24)

With the help of the Jacobi identity, this simplifies to \( S_1 = -(2\pi)^4 \eta(\tau)^{12} \). Thus, using the notations of [21], eq (7.2), we have

\[ \sum_\mu \langle \nu_0 | \mu \rangle \vartheta[\mu](0)^4 W_0[\mu] = -4\pi^4 \eta(\tau)^{12} K \]  \hspace{1cm} (C.25)

And the full 1-loop amplitude is given by (2.22), with the value of \( C_1 \),

\[ C_1 = (4\pi^4)^2 Q_1 = \frac{1}{2^8 \pi^2 (\alpha')^5}. \]  \hspace{1cm} (C.26)
D Regularization Dependence of Determinants

The Polyakov integral can only give superstring amplitudes up to a proportionality factor of the form $e^{c(2h-2)}$, which should be absorbed in a shift of the dilaton expectation value. This is due to the need for regularization and renormalization of the path integrals, and different schemes differ by such a proportionality factor.

D.1 Difference between regularization schemes

Here we illustrate this phenomenon by comparing explicitly the zeta function regularization with the short time cut-off regularization.

In the zeta function regularization for the determinant of a Laplacian $\Delta$, the quantity $\ln \det' \Delta$ can be defined as $-\zeta'(0)$, where $\zeta(s) = \sum' \lambda^{-s}$, and the prime denotes summation over the non-zero eigenvalues $\lambda$ of $\Delta$. For our purposes, it is convenient to write $\zeta(s)$ as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} (\text{Tr} \ e^{-t\Delta} - N)$$

where $N$ is the number of zero modes of $\Delta$. Now the integral in $t$ between 1 and $\infty$ is convergent, and produces an entire function of $s$. Thus, to evaluate $\zeta'(0)$, it suffices to determine explicitly the analytic continuation of the integral over $t \in (0,1)$. For this, write

$$\int_0^1 (\text{Tr} \ e^{-t\Delta} - N) t^{s-1} dt = \int_0^1 (\frac{A_{-1}}{t} + A_0) t^{s-1} dt + \int_0^1 (\text{Tr} \ e^{-t\Delta} - \frac{A_{-1}}{t} - (A_0 + N)) t^{s-1} dt$$

where $\text{Tr} \ e^{-t\Delta} = \sum_{i=-1}^N A_i t^i = \mathcal{O}(t^{N+1})$ is the asymptotic expansion of the trace of the heat kernel for small time. The second integral on the right is holomorphic at $s = 0$. The first integral can be evaluated explicitly to be $A_{-1}(s-1)^{-1} + A_0 s^{-1}$ for $\text{Re} \ s > 1$, and hence it is given by this same formula by analytic continuation. Altogether, we find

$$\ln \det' \Delta = A_{-1} - A_0 (\frac{1}{s\Gamma(s)})'(0) - \int_0^1 \frac{dt}{t} (\text{Tr} \ e^{-t\Delta} - \frac{A_{-1}}{t} - (A_0 + N))$$

$$- \int_1^\infty \frac{dt}{t} (\text{Tr} \ e^{-t\Delta} - N).$$

We compare this value with the result of small time cut-off regularization and renormalization, $\det^* \Delta$, which is defined by

$$\ln \det^* \Delta = -P.V. \int_\epsilon^\infty \frac{dt}{t} (\text{Tr} \ e^{-t\Delta} - N),$$

where $P.V.$ indicates taking the finite part in the expansion in $\epsilon$ for small $\epsilon$. Once again, the integral over $1 \leq t < \infty$ converges, and we need only consider the integral over $\epsilon < t \leq 1$.  

38
Carrying out the rearrangement as in (D.2), but with lower limit $\epsilon$ and $s = 0$, we find
\[
\int_{\epsilon}^{1} \frac{dt}{t} (\text{Tr} e^{-t\Delta} - N) = -\frac{A_{-1}}{\epsilon} + (A_{0} + N) \ln \epsilon \\
+ A_{-1} - \int_{0}^{1} \frac{dt}{t} (\text{Tr} e^{-t\Delta} - \frac{A_{-1}}{t} - (A_{0} + N)), \tag{D.5}
\]
and hence
\[
\ln \det' \Delta = \ln \det^* \Delta - A_{0} \left( \frac{1}{s \Gamma(s)} \right)'(0) \tag{D.6}
\]
Thus the two regularization schemes differ by an additive factor proportional to $A_{0} = \zeta(0)$.

However, when $\Delta$ is the Laplacian on a field of $U(1)$ weight $n$, dimensional considerations show readily that the $t^{0}$ term of the heat kernel must be proportional to the curvature of $\Sigma$. The coefficient $A_{0}$ is the integral of this term, and is proportional to the Euler characteristic $\chi(\Sigma) = 2 - 2h$. This establishes the desired form for the difference between regularization and renormalization schemes.

**D.2 Failure of ultra-locality**

The appearance of the factor $e^{\epsilon(2 - 2h)}$ is also related to the failure of ultra-locality. Ultra-locality states that \cite{46, 42},
\[
\int \mathcal{D}x^{\mu} e^{-\lambda \pi ||x||_{g}^{2}} = e^{-\mu(\lambda)} \int_{\Sigma} \sqrt{g}
\]
where $\lambda$ is a real positive constant, $\mu(\lambda)$ is a function of $\lambda$, $||x||_{g}$ is the $L^{2}$-norm for the worldsheet metric $g$ and $\mathcal{D}x^{\mu}$ the associated functional measure. Explicit calculation of the above functional integral, however, shows that a term proportional to the Euler number also arises, in full accord with standard renormalization theory, so that ultra-locality does not hold when $h \neq 1$. This was already pointed out by Weisberger \cite{42}.

To see this explicitly, we calculate the integral in two different ways, first by bringing out the Laplacian $\Delta_{g}$, and then by bringing out the operator $\lambda \Delta_{g}$. Their equality gives
\[
\left( \frac{\text{det}' \Delta_{g}}{\int \sqrt{g}} \right)^{-d/2} \int dx^{\mu}_{0} \int \mathcal{D}x^{\mu} e^{-\pi \lambda ||x||_{g}^{2}} = \left( \frac{\text{det}' \lambda \Delta_{g}}{\int \sqrt{g}} \right)^{-d/2} \int dx^{\mu}_{0} \int \mathcal{D}x^{\mu} e^{-\pi ||x||_{g}^{2}} \tag{D.8}
\]
Using now specifically zeta regularization for the determinants, we have
\[
(\text{det}' \lambda \Delta_{g})^{-d/2} = \lambda^{-\zeta_{\Delta_{g}}(0)d/2} (\text{det}' \Delta_{g})^{-d/2}, \tag{D.9}
\]
and hence
\[
\int \mathcal{D}x^{\mu} e^{-\lambda \pi ||x||_{g}^{2}} = \lambda^{-\zeta_{\Delta_{g}}(0)d/2} \int \mathcal{D}x^{\mu} e^{-\pi ||x||_{g}^{2}} \tag{D.10}
\]
Since the expression $\zeta_{\Delta_{g}}(0)$ is proportional to the Euler number, our claim follows.
References

[1] C. M. Hull and P. K. Townsend, “Unity of superstring dualities,” Nucl. Phys. B 438, 109 (1995) [arXiv:hep-th/9410167].

[2] E. Witten, “Some comments on string dynamics,” arXiv:hep-th/9507121.

[3] M. B. Green and M. Gutperle, “Effects of D-instantons,” Nucl. Phys. B 498, 195 (1997) [arXiv:hep-th/9701093].

[4] A. Terras, Harmonic analysis on symmetric spaces and applications I, Springer, New York-Berlin 1985.

[5] M. B. Green and P. Vanhove, “D-instantons, strings and M-theory,” Phys. Lett. B 408, 122 (1997) [arXiv:hep-th/9704145].

[6] M. B. Green, M. Gutperle and P. Vanhove, “One loop in eleven dimensions,” Phys. Lett. B 409, 177 (1997) [arXiv:hep-th/9706175].

[7] M. B. Green and S. Sethi, “Supersymmetry constraints on type IIB supergravity,” Phys. Rev. D 59, 046006 (1999) [arXiv:hep-th/9808061].

[8] M. B. Green, M. Gutperle and H. H. Kwon, “Light-cone quantum mechanics of the eleven-dimensional superparticle,” JHEP 9908, 012 (1999) [arXiv:hep-th/9907155]; M. B. Green, M. Gutperle and H. H. Kwon, “λ16 and related terms in M-theory on T2,” Phys. Lett. B 421, 149 (1998) [arXiv:hep-th/9710151].

[9] S. de Haro, A. Sinkovics and K. Skenderis, “A supersymmetric completion of the R**4 term in IIB supergravity,” Phys. Rev. D 67, 084010 (2003) [arXiv:hep-th/0210080]; K. Peeters, P. Vanhove and A. Westerberg, “Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace,” Class. Quant. Grav. 18, 843 (2001) [arXiv:hep-th/0010167].

[10] B. Pioline, “A note on non-perturbative R**4 couplings,” Phys. Lett. B 431, 73 (1998) [arXiv:hep-th/9804023].

[11] E. Kiritsis and B. Pioline, “On R**4 threshold corrections in type IIB string theory and (p,q) string instantons,” Nucl. Phys. B 508, 509 (1997) [arXiv:hep-th/9707018].

[12] N. A. Obers and B. Pioline, “Eisenstein series in string theory,” Class. Quant. Grav. 17, 1215 (2000) [arXiv:hep-th/9910115].

[13] N. A. Obers and B. Pioline, “Eisenstein series and string thresholds,” Commun. Math. Phys. 209, 275 (2000) [arXiv:hep-th/9903113].

[14] N. Berkovits, “Construction of R**4 terms in N = 2 D = 8 superspace,” Nucl. Phys. B 514, 191 (1998) [arXiv:hep-th/9709116].
[15] N. Berkovits and C. Vafa, “Type IIB R**4 H**(4g-4) conjectures,” Nucl. Phys. B 533, 181 (1998) [arXiv:hep-th/9803145].

[16] M. B. Green, H. h. Kwon and P. Vanhove, “Two loops in eleven dimensions,” Phys. Rev. D 61, 104010 (2000) [arXiv:hep-th/9910055].

[17] M. B. Green and P. Vanhove, “The low energy expansion of the one-loop type II superstring amplitude,” Phys. Rev. D 61, 104011 (2000) [arXiv:hep-th/9910056].

[18] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings I, Main Formulas”, Phys. Lett. B529 (2002) 241-255; hep-th/0110247;
“Two-Loop Superstrings II, The chiral Measure on Moduli Space”, Nucl. Phys. B636 (2002) 3-60; hep-th/0110283;
“Two-Loop Superstrings III, Slice Independence and Absence of Ambiguities”, Nucl. Phys. B636 (2002) 61-79; hep-th/0111016.

[19] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings IV, The Cosmological Constant and Modular Forms”, Nucl. Phys. B639 (2002) 129-181; hep-th/0111040.

[20] E. D’Hoker and D. H. Phong, “Two-loop superstrings. V: Gauge slice independence of the N-point function,” arXiv:hep-th/0501196.

[21] E. D’Hoker and D. H. Phong, “Two-loop superstrings. VI: Non-renormalization theorems and the 4-point function,” arXiv:hep-th/0501197.

[22] E. D’Hoker and D.H. Phong, “Lectures on two-loop superstrings”, in String Theory, Proceedings of the 2002 International String Theory Conference in Huangzhou, China, K. Liu and S.T. Yau, Eds, (International Press, 2004), hep-th/0211111.

[23] Z. J. Zheng, J. B. Wu and C. J. Zhu, “Two-loop superstrings in hyperelliptic language. I: The main results,” Phys. Lett. B 559, 89 (2003) [arXiv:hep-th/0212191];
C. J. Zhu, “Two-loop computation in superstring theory,” arXiv:hep-th/0301018;
Z. J. Zheng, J. B. Wu and C. J. Zhu, “Two-loop superstrings in hyperelliptic language. II: The vanishing of the cosmological constant and the non-renormalization theorem,” Nucl. Phys. B 663, 79 (2003) [arXiv:hep-th/0212198];
Z. J. Zheng, J. B. Wu and C. J. Zhu, “Two-loop superstrings in hyperelliptic language. III: The four-particle amplitude,” Nucl. Phys. B 663, 95 (2003) [arXiv:hep-th/0212219];
J. B. Wu and C. J. Zhu, “Comments on two-loop four-particle amplitude in superstring theory,” JHEP 0305, 056 (2003) [arXiv:hep-th/0303152].

[24] A. Morozov, “On the two-loop contribution to the superstring four-point function”, Phys. Lett. B 209 (1988) 473-476;
O. Yasuda, “Factorization of a two loop Four Point Superstring Amplitude”, Phys. Rev. Lett. 60 (1988) 1688; erratum-ibid 61 (1988) 1678;
R. Iengo and C.J. Zhu, “Two-loop computation of the four-particle in heterotic string theory”, Phys. Lett. B 212 (1988) 313;
[25] O. Lechtenfeld, “Factorization and modular invariance of multiloop superstring amplitudes in the unitary gauge”, Nucl. Phys. B 338 (1990) 403-414.

[26] R. Iengo and C.J. Zhu, “Explicit modular invariant two-loop superstring amplitude relevant to $R^4$”, JHEP 06 (1999) 011.

[27] E. Verlinde and H. Verlinde, “Chiral Bosonization, determinants and the string partition function”, Nucl. Phys. B288 (1987) 357.

[28] J. Jorgenson, “Degenerating hyperbolic Riemann surfaces and an evaluation of the constant in Deligne's arithmetic Riemann-Roch theorem”, 1991 preprint; H. Gillet, C. Soulé, “Analytic torsion and the arithmetic Todd genus”, Topology 30 (1991) 21; C. Soulé, “Géometrie d’Arakelov des surfaces arithmetiques”, Astérisque 177-178 (1989) 327; R. Wentworth, “Asymptotics of determinants from functional integration”, J. Math. Phys. 32 (1991) 1767-1773.

[29] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, “Heterotic String Theory. 1. The Free Heterotic String,” Nucl. Phys. B 256, 253 (1985); “Heterotic String Theory. 2. The Interacting Heterotic String,” Nucl. Phys. B 267 (1986) 75.

[30] E. Witten, “String theory dynamics in various dimensions,” Nucl. Phys. B 443, 85 (1995) [arXiv:hep-th/9503124]; A. Dabholkar, “Ten-dimensional heterotic string as a soliton,” Phys. Lett. B 357, 307 (1995) [arXiv:hep-th/9506160]; C. M. Hull, “String-string duality in ten-dimensions,” Phys. Lett. B 357, 545 (1995) [arXiv:hep-th/9506194]; J. Polchinski and E. Witten, “Evidence for Heterotic - Type I String Duality,” Nucl. Phys. B 460, 525 (1996) [arXiv:hep-th/9510169].

[31] A. A. Tseytlin, “On SO(32) heterotic - type I superstring duality in ten dimensions,” Phys. Lett. B 367, 84 (1996) [arXiv:hep-th/9510173]; A. A. Tseytlin, “Heterotic - type I superstring duality and low-energy effective actions,” Nucl. Phys. B 467, 383 (1996) [arXiv:hep-th/9512081].

[32] I. Antoniadis, H. Partouche and T. R. Taylor, Nucl. Phys. Proc. Suppl. 61A, 58 (1998) [Nucl. Phys. Proc. Suppl. 67, 3 (1998)] [arXiv:hep-th/9706211]; C. P. Bachas, “Lectures on D-branes,” arXiv:hep-th/9806199.

[33] S. Stieberger and T. R. Taylor, “Non-Abelian Born-Infeld action and type I - heterotic duality. I: Heterotic F**6 terms at two loops,” Nucl. Phys. B 647, 49 (2002) [arXiv:hep-th/0207026]; ”II: Nonrenormalization theorems,” Nucl. Phys. B 648, 3 (2003) [arXiv:hep-th/0209064].

[34] M. B. Green and J. H. Schwarz, “Supersymmetrical String Theories,” Phys. Lett. B 109, 444 (1982).
[35] M.B. Green, J.H. Schwarz, and E. Witten, “Superstring Theory”, Cambridge University Press (1987), Vols I and II.

[36] D. J. Gross and E. Witten, “Superstring Modifications Of Einstein’s Equations,” Nucl. Phys. B 277, 1 (1986).

[37] E. D’Hoker and D.H. Phong, “The geometry of string perturbation theory”, Rev. Modern Physics 60 (1988) 917-1065.

[38] J. Fay, Theta Functions on Riemann Surfaces, Springer Lecture Notes in Mathematics No 352 (Springer, Berlin) 1973.

[39] C.L. Siegel, “Symplectic Geometry”, Am. J. Math. 65 (1943) 1-86;
    H. Klingen, Introductory Lectures to Siegel Modular Forms, Cambridge University Press, 1990.

[40] G. W. Moore, “Modular Forms And Two Loop String Physics,” Phys. Lett. B 176, 369 (1986).

[41] E. D’Hoker and D.H. Phong, “Momentum analyticity, and finiteness of the one-loop superstring amplitude”, Phys. Rev. Lett. 70 (1993), 3692, hep-th/9302003;
    E. D’Hoker and D.H. Phong, “The box graph in superstring theory”, Nucl. Phys. B440 (1995) 24, hep-th/9410152;
    E. D’Hoker and D.H. Phong, “Dispersion relations in string theory”, Theor.Math.Phys.98:306-316,1994, (Teor.Mat.Fiz.98:442-455,1994) e-Print Archive: hep-th/9404128.

[42] W. I. Weisberger, “Normalization Of The Path Integral Measure And The Coupling Constants For Bosonic Strings,” Nucl. Phys. B 284, 171 (1987).

[43] A. Erdélyi, Higher Transcendental Functions’, Vol. I, Malabar, Florida, Krieger, 1981.

[44] Y. Cai and C. A. Nunez, “Heterotic String Covariant Amplitudes And Low-Energy Effective Action,” Nucl. Phys. B 287, 279 (1987);
    Y. Kikuchi and C. Marzban, “Low-Energy Effective Lagrangian Of Heterotic String Theory,”
    Phys. Rev. D 35, 1400 (1987);
    D. J. Gross and J. H. Sloan, “The Quartic Effective Action For The Heterotic String,”
    Nucl. Phys. B 291, 41 (1987);
    J. R. Ellis, P. Jetzer and L. Mizrachi, “One Loop String Corrections To The Effective Field Theory,”
    Nucl. Phys. B 303, 1 (1988).

[45] K. Aoki, Commun. Math. Phys. 117, 405 (1988).

[46] J. Polchinski, “Evaluation Of The One Loop String Path Integral,” Commun. Math. Phys. 104, 37 (1986).

[47] E. D’Hoker and D.H. Phong, “Multiloop amplitudes for the bosonic Polyakov string”, Nucl. Phys. B 269 (1986) 205-234.
[48] J. Polchinski, “String Theory”, Cambridge University Press, (1998), Vol I and II.