A note on nontrivial intersection for selfmaps of complex Grassmann manifolds

Thaís F. M. Monis
tfmonis@rc.unesp.br

Northon C. L. Penteado
northoncanevari@gmail.com

Sérgio T. Ura
sergioura@gmail.com

Peter Wong
pwong@bates.edu

Abstract

Let $G(k, n)$ be the complex Grassmann manifold of $k$-planes in $\mathbb{C}^{k+n}$. In this note, we show that for $1 < k < n$ and for any selfmap $f : G(k, n) \to G(k, n)$, there exists a $k$-plane $V^k \in G(k, n)$ such that $f(V^k) \cap V^k \neq \{0\}$.

1 Introduction

The problem of determining the fixed point property (f.p.p.) for Grassmann manifolds has been studied by many authors (for example [7], [5], [6]).

Let

$$\mathbb{F}M(n_1, \ldots, n_k) = \frac{U_\mathbb{F}(n)}{U_\mathbb{F}(n_1) \times \cdots \times U_\mathbb{F}(n_k)},$$

where $n_1 + \cdots + n_k = n$. Here, $\mathbb{F}$ stands for one of the fields $\mathbb{R}$, $\mathbb{C}$ or the skew field $\mathbb{H}$, and

$$U_\mathbb{F}(n) = \begin{cases} O(n) & \text{the orthogonal group of order } n \text{ if } \mathbb{F} = \mathbb{R}, \\ U(n) & \text{the unitary group of order } n \text{ if } \mathbb{F} = \mathbb{C}, \\ Sp(n) & \text{the symplectic group of order } n \text{ if } \mathbb{F} = \mathbb{H}. \end{cases}$$

In [4], Glover and Homer have given the following necessary condition for $\mathbb{F}M(n_1, \ldots, n_k)$ to have the f.p.p..

Theorem 1 ([4], Theorem 1). If $\mathbb{F}M(n_1, \ldots, n_k)$ has the f.p.p., then $n_1, \ldots, n_k$ are distinct integers and, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, at most one is odd.

The above theorem gives rise to the following conjectures:

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**Conjecture 1.** If \( n_1, \ldots, n_k \) are all distinct then \( HM(n_1, \ldots, n_k) \) has the f.p.p..

**Conjecture 2.** If \( n_1, \ldots, n_k \) are all distinct and at most one is odd then \( FM(n_1, \ldots, n_k) \) has the f.p.p., for \( F = \mathbb{R} \) and \( F = \mathbb{C} \).

The above conjectures were already proved to be true in the following cases:

- Projective spaces (\( FM(1, n-1) \));
- If \( n_2 \) and \( n_3 \) are distinct positive even integers and \( n_3 \geq 2n_2^2 - 1 \) then \( CM(1, n_2, n_3) \) has the f.p.p. (\[H\]).
- If \( 1, n_2 \) and \( n_3 \) are distinct positive integers and \( n_3 \geq 2n_2^2 - 1 \), then \( HM(1, n_2, n_3) \) has the f.p.p. (\[H\]).
- If \( n_2 < n_3 \) are even integers greater than 1 and either \( n_2 \leq 6 \) or \( n_3 \geq n_2^2 - 2n_2 - 2 \), then \( RM(1, n_2, n_3) \) has the f.p.p. (\[H\]).
- If \( n_1, n_2, n_3 \) are positive integers such that at most one is odd, \( n_1 \leq 3, n_3 \geq n_2^2 - 1 \), and \( \lfloor n_1/2 \rfloor < \lfloor n_2/2 \rfloor < \lfloor n_3/2 \rfloor \), then \( RM(n_1, n_2, n_3) \) has the f.p.p. (\[H\]).
- If \( F = \mathbb{C} \) or \( H, FM(2, q) \) has the f.p.p. for all \( q > 2 \) (\[7\]).
- \( \mathbb{R}(2, q) \) has the f.p.p. for all \( q = 4k \) or \( q = 4k + 1, k = 1, 2, 3, \ldots \) (\[7\]).
- For \( p \leq 3 \) and \( q > p \) or \( p > 3 \) and \( q \geq 2p^2 - p - 1 \), \( CM(p, q) \) has the f.p.p. iff \( pq \) is even (\[5\]).
- For \( p \leq 3 \) and \( q > p \) or \( p > 3 \) and \( q \geq 2p^2 - p - 1 \), \( HM(p, q) \) always has the f.p.p. (\[5\]).

The main tool used to prove the above results is the calculation of the Lefschetz number of a self-map of such a space. Let’s focus on the case of complex Grassmann manifolds \( \mathbb{C}(k, n) = G(k, n) \), the space of \( k \)-planes in \( \mathbb{C}^{k+n} \). Let \( \gamma^k \) be the canonical \( k \)-plane bundle over \( G(k, n) \). If

\[
ch(\gamma^k) = 1 + c_1 + \cdots + c_k, \quad c_i \in H^{2i}(G(k, n); \mathbb{Q}),
\]

is the total Chern class of \( \gamma^k \), then the cohomology ring \( H^*(G(k, n); \mathbb{Q}) \) is given by:

\[
H^*(G(k, n); \mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c_k]/I_{k,n},
\]
where \( I_{k,n} \) is the ideal generated by the elements \((c^{-1})_{n+1}, \ldots, (c^{-1})_{n+k}\). Here, \((c^{-1})_{q}\) is the part of the formal inverse of \( c \) in dimension \( 2q \) (see [6], Theorem 2.1). Then, \( c_1 \) is the only generator in dimension 2. Therefore, given a self-map \( f : G(k, n) \to G(k, n) \), \( f^*(c_1) = mc_1 \) for some coefficient \( m \).

**Theorem 2** ([6], Theorem 1). Let \( k \leq 3 \) and \( n > k \) or \( k > 3 \) and \( n \geq 2k^2 - k - 1 \). Then every graded ring endomorphism of \( H^*(G(k, n); \mathbb{Q}) \) is an Adams endomorphism\(^1\). Consequently, if \( f : G(k, n) \to G(k, n) \) is a self-map with \( f^*(c_1) = mc_1 \) then \( f^*(c_i) = m^i c_i, \ i = 1, \ldots, k \).

The classification of the graded ring endomorphisms of \( H^*(G(k, n); \mathbb{Q}) \) is fundamental in the study of f.p.p. for \( G(k, n) \) because of the following.

**Proposition 1.** An Adams endomorphism of \( H^*(G(k, n); \mathbb{Q}) \) has Lefschetz number zero if and only if its degree is \(-1\) and \( kn \) is odd.

*Proof.* See [4], Proposition 4.

In [6], M. Hoffman was able to prove the following.

**Theorem 3** ([6], Theorem 1.1). Let \( k < n \) and \( h \) be a graded ring endomorphism of \( H^*(G(k, n); \mathbb{Q}) \) with \( h(c_1) = mc_1, m \neq 0 \). Then \( h(c_i) = m^i c_i, 1 \leq i \leq k \).

If \( k < n \) and \( h \) is a graded ring endomorphism of \( H^*(G(k, n); \mathbb{Q}) \) with \( h(c_1) = 0 \), it is still unclear about what \( h \) looks like in general. The conjecture is that, in this case, \( h \) must be the null homomorphism. If one can prove such conjecture then the problem of determining the f.p.p. for \( G(k, n) \) will be completely solved.

In this note, we prove a much more modest result for complex Grassmann manifolds than a fixed point theorem. Our main theorem is the following.

**Theorem 4** (Main Result). Let \( k > 1 \) and \( k < n \). Then for every continuous map \( f : G(k, n) \to G(k, n) \) there exists a \( k \)-plane \( V^k \in G(k, n) \) such that \( V^k \cap f(V^k) \neq \{0\} \).

\(^1\)An Adams endomorphism of \( H^*(G(k, n); \mathbb{Q}) \) is a endomorphism \( \varphi \) of the form \( \varphi(x) = \lambda^i x \) for \( x \in H^{2i}(G(k, n); \mathbb{Q}) \). The coefficient \( \lambda \) is called the degree of \( \varphi \).
The motivation for this work is the paper [8] where the author gave an alternative proof for the f.p.p. of $CP^{2n}$ using characteristic classes. In fact, a closer look at the proof of the main result in [8] indicates that the same argument would also yield an alternative proof of the f.p.p. for $RP^{2n}$ by replacing Chern classes with Stiefel-Whitney classes. We should also point out that a non-trivial intersection result similar to Theorem 4 has been obtained in [1] for maps between two different Grassmann manifolds.

2 Proof of the Main Theorem

Throughout this paper, $G(k, n)$ denotes the complex Grassmann manifold of $k$-planes in $C^{k+n}$.

Note that, since $G(k, n)$ and $G(n, k)$ are homeomorphic, $\gamma^k$ and $\gamma^n$ can be seen as subbundles of the trivial bundle $G(k, n) \times C^{k+n}$, which is denoted by $\epsilon^{k+n}$, and, under such identification,

$$\gamma^k \oplus \gamma^n = \epsilon^{k+n}.$$

**Lemma 1.** Let $\text{ch}(\gamma^n) = 1 + \bar{c}_1 + \cdots + \bar{c}_n$ be the total Chern class of the bundle $\gamma^n$. Then, a general formula for the class $\bar{c}_i$ in terms of the Chern classes of $\gamma^k$ is given by

$$\bar{c}_i = \sum_{\|\alpha\|=i} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!} \text{ch}(\gamma^k)^{\alpha},$$

where $\alpha$ represents the $k$-uple $\alpha = (a_1, \ldots, a_k)$, $\|\alpha\| = a_1 + 2a_2 + \cdots + ka_k$, $|\alpha| = a_1 + a_2 + \cdots + a_k$, $\alpha! = a_1!a_2!\cdots a_k!$ and $\text{ch}(\gamma^k)^{\alpha} = c_1^{a_1} \cdot \cdots \cdot c_k^{a_k}$.

**Proof.** The proof is given recursively in the index $i$.

As $\gamma^k \oplus \gamma^n = \epsilon^{k+n}$, we have

$$\text{ch}(\gamma^k) \sim \text{ch}(\gamma^n) = \text{ch}(\epsilon^{k+n}) = 1$$

in $H^*(G(k, n); \mathbb{Z})$. So

$$(1 + c_1 + \cdots + c_k) \sim (1 + \bar{c}_1 + \cdots + \bar{c}_n) = 1$$

and then

$$1 = 1$$

$$0 = c_1 + \bar{c}_1$$

$$0 = c_2 + c_1 \sim \bar{c}_1 + \bar{c}_2$$

$$\cdots$$
Then
\[ \bar{c}_j = - \sum_{i=1}^{j} c_i \sim \bar{c}_{j-i} \]
for all \( j = 1, \ldots, n \), with the convention \( c_i = 0 \) when \( i > k \). Thus,

(i) \( \bar{c}_1 = -c_1 \);  
(ii) \( \bar{c}_2 = -(c_1 \sim -c_1) - c_2 = c_1^2 - c_2 \);  
(iii) Suppose
\[ \bar{c}_j = \sum_{|\alpha| = j} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^\alpha, \]
for \( j = 1, \ldots, m - 1 < n \).

Then
\[
\bar{c}_m = - \sum_{i=1}^{m} c_i \sim \bar{c}_{m-i} \\
= - \sum_{i=1}^{m} \left( c_i \sim \sum_{|\alpha| = m-i} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^\alpha \right) \\
= \sum_{i=1}^{m} \left( c_i \sim \sum_{|\alpha| = m-i} (-1)^{|\alpha|+1} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^\alpha \right) \\
= \sum_{i=1}^{m} \left( \sum_{|\alpha| = m-i} (-1)^{|\alpha|+1} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^\alpha \sim c_i \right) \\
= \sum_{i=1}^{m} \sum_{|\alpha| = m-i} (-1)^{|\alpha+e_i|} \frac{|\alpha|!}{\alpha!} ch(\gamma^k)^{\alpha+e_i} \quad (e_i = (0, \ldots, 0, 1, 0, \ldots 0)) \\
= \sum_{|\beta|=m} (-1)^{|\beta|} X(\beta)ch(\gamma^k)^\beta \quad (\beta = \alpha + e_i)
where

\[
X(\beta) = \sum_{b_i \neq 0} \frac{|\beta - e_i|!}{(\beta - e_i)!} \\
= \sum_{b_i \neq 0} \frac{(|\beta| - 1)!b_i}{\beta!} \\
= \frac{m}{\beta!} \sum_{b_i \neq 0} (|\beta| - 1)!b_i \\
= \frac{(|\beta| - 1)! \sum_{i=1}^{m} b_i}{\beta!} \\
= \frac{|\beta|!}{\beta!} \\
= \frac{|\beta|!}{\beta!}.
\]

\[\square\]

2.1 Proof of Theorem 4

Suppose, to the contrary, there exists a continuous map \(f : G(k, n) \to G(k, n)\) such that \(V^k \cap f(V^k) = \{0\}\) for every \(k\)-plane \(V^k \in G(k, n)\). Then the direct sum \(\gamma^k \oplus f^* \gamma^k\) can be seen as a subbundle of the trivial bundle \(\mathcal{E}^{k+n}\). Let \(\eta^{n-k}\) be the normal bundle of \(\gamma^k \oplus f^* \gamma^k\) in \(\mathcal{E}^{k+n}\). Then

\[
\text{ch}(\gamma^k) \sim \text{ch}(f^* \gamma^k) \sim \text{ch}(\eta^{n-k}) = 1.
\]

(2.1)

It follows that

\[
\text{ch}(f^* \gamma^k) \sim \text{ch}(\eta^{n-k}) = 1 + \bar{c}_1 + \cdots + \bar{c}_n.
\]

(2.2)

Let

\[
\text{ch}(f^* \gamma^k) = 1 + \bar{c}_1 + \cdots + \bar{c}_k, \quad \bar{c}_i \in H^{2i}(G(k, n); \mathbb{Q}),
\]

(2.3)

and

\[
\text{ch}(\eta^{n-k}) = 1 + t_1 + \cdots + t_{n-k}, \quad t_j \in H^{2j}(G(k, n); \mathbb{Q}).
\]

(2.4)

We will show that it is impossible for

\[
\bar{c}_n = \bar{c}_k \sim t_{n-k}.
\]

(2.5)

The proof of the impossibility of the above equality will be split into several cases.
Case 1: $1 < k \leq 3$. Since $c_1 \in H^2(G(k,n); \mathbb{Q})$ is the only generator in dimension 2, $f^*(c_1)$ is a multiple of $c_1$, let’s say $f^*(c_1) = mc_1$. Following [7] and [5], for $k \leq 3$ and $k < n$, every endomorphism of the ring $H^*(G(k,n); \mathbb{Q})$ that preserves dimension is an Adams endomorphism. Therefore, if $f^*(c_1) = mc_1$ then $f^*(c_2) = m^2c_2, \ldots, f^*(c_k) = m^kc_k$. Thus

$$ch(f^\gamma) = f^*(ch(\gamma)) = 1 + mc_1 + m^2c_2 + \cdots + m^kc_k.$$ 

It follows that
\[
\bar{c}_n = m^kc_k \sim t_{n-k},
\]
in contradiction with Lemma [1].

Case 2: $k > 3$. This case will be split in four cases.

Case 2(i): $n = l(k-1) + r$ with remainder $r \neq 1$, that is, $1 < r < k-1$ or $r = 0$. In this case, $r$ is of the form $r = 2i$ or $r = 2i + 3$, for some integer $i \geq 0$. In case of $r = 2i$, the class $c_{k-1}^l \sim c_2^j$ does not appear in $\bar{c}_k \sim t_{n-k}$ but, by Lemma [1] it appears in $\bar{c}_n$, contradicting $\bar{c}_n = \bar{c}_k \sim t_{n-k}$. In case of $r = 2i + 3$, the class $c_{k-1}^l c_2^j c_3$ does not appear in $\bar{c}_k \sim t_{n-k}$ but, by Lemma [1] it appears in $\bar{c}_n$, contradicting $\bar{c}_n = \bar{c}_k \sim t_{n-k}$.

Case 2(ii): $k > 4$ and $n = (l+1)(k-1) + 1$. In this case, we have

$$n = (l+1)(k-1) + 1 = l(k-1) + k$$

and, since $n > k, l \geq 1$. We can write $n = (l+1)(k-1) + 1$ in the form

$$n = (l-1)(k-1) + 2(k-2) + 3$$

and, since we are supposing $k > 4, k-2 > 2$. With these information, one can check that the class $c_{k-1}^{m-1} \sim c_{k-2}^2 \sim c_3$ cannot appear in $\bar{c}_k \sim t_{n-k}$. On the other hand, by Lemma [1] the class $c_{k-1}^{m-1} \sim c_{k-2}^2 c_3$ appears in $\bar{c}_n$. Therefore, $\bar{c}_n = \bar{c}_k \sim t_{n-k}$ is impossible.

Case 2(iii): $k = 4, n = (l+1)(k-1) + 1$ and $l$ even, say $l = 2j$. In this case, $n-k = 3l$ and, since $n > 1, l \geq 1$. Let

$$\bar{c}_4 = c_4^1 + \alpha c_2^2 + \theta c_4 + \text{other terms}$$

$$t_{3l} = c_4^{3l} + \alpha' c_2^{3j} + \beta c_3 + \text{other terms}.$$
Thus, in the product $\tilde{c}_4 \sim t_{3l}$, $\alpha\alpha'$ is the coefficient of $c_2^{3j+2}$, $\alpha\beta$ is the coefficient of $c_2^2 \sim c_3^l$ and $\theta\beta$ is the coefficient of $c_4 \sim c_3^l$. From Lemma 1 together with the fact that $\tilde{c}_4 \sim t_{3l} = \tilde{c}_n$, it follows that

$$\alpha\alpha' = \frac{(3j+2)!}{(3j+2)!1!},$$
$$\alpha\beta = \frac{(l+2)!}{l!2!},$$
$$\theta\beta = \frac{(l+1)!}{l!1!}.$$  

Thus

$$\alpha\alpha' = 1,$$
$$\alpha\beta = \frac{(l+2)(l+1)}{2},$$
$$\theta\beta = l + 1.$$

Then, we conclude that $\alpha = \pm 1$, $\beta = \pm \frac{(l+2)(l+1)}{2}$ and $|\beta| = \frac{(l+2)(l+1)}{2}$ divides $\theta\beta = l + 1$. It follows that $l = 0$, but $l \geq 1$, a contradiction!

Case 2(iv): $k = 4$, $n = (l+1)(k-1) + 1$ and $l$ odd, say $l = 2j + 1$. Again, $n - k = 3l$ and, since $n > 1$, $l \geq 1$. Let

$$\tilde{c}_4 = c_4 + \alpha c_2 + \gamma c_1 c_3 + \text{other terms}$$
$$t_{3l} = c_1^{3l} + \alpha' c_1 c_2^{3j+1} + \beta c_3^l + \text{other terms}.$$  

It follows that, in the product $\tilde{c}_4 \sim t_{3l}$, $\alpha\alpha'$ is the coefficient of $c_1 \sim c_2^{3j+3}$, $\alpha\beta$ is the coefficient of $c_2^2 \sim c_3^l$, $\theta\beta$ is the coefficient of $c_4 \sim c_3^l$ and $\gamma\beta$ is the coefficient of $c_1 \sim c_3^{l+1}$. Since $\tilde{c}_n = \tilde{c}_4 \sim t_{3l}$, together with Lemma 1

$$\alpha\alpha' = \frac{(3j+4)!}{1!(3j+3)!},$$
$$\alpha\beta = \frac{(l+2)!}{l!2!},$$
$$\theta\beta = \frac{(l+1)!}{l!1!},$$
$$\gamma\beta = \frac{(l+2)!}{1!(l+1)!}.$$  

Thus
\[ \alpha \alpha' = 3j + 4 \]
\[ \alpha \beta = \frac{(l + 2)(l + 1)}{2} \]
\[ \theta \beta = l + 1 \]
\[ \gamma \beta = l + 2. \]

From the two last equalities above, it follows that \( \beta \) divides \( l + 1 \) and \( l + 2 \). Therefore, \( \beta = 1 \).

It follows that \( \alpha = \frac{(l+2)(l+1)}{2} \) and, since \( \alpha \) divides \( 3j + 4 \),
\[ \frac{(l + 2)(l + 1)}{2} \leq 3j + 4 = \frac{3l + 5}{2}. \]
Therefore, \( l^2 \leq 3 \). Since \( l \) is an integer not smaller than 1, it follows that \( l = 1 \). Then,
\[ 3j + 4 = \frac{3l + 5}{2} = 4 \]
is divisible by \( \frac{(l+2)(l+1)}{2} = 3 \), a contradiction! \( \square \)

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Thaís F. M. Monis  
Departamento de Matemática, IGCE, Univ Estadual Paulista.  
email: tfmonis@rc.unesp.br

Northon C. L. Penteado  
Departamento de Matemática, IGCE, Univ Estadual Paulista.  
email: northoncanevari@gmail.com

Sérgio T. Ura  
Departamento de Matemática, IGCE, Univ Estadual Paulista.  
email: sergioura@gmail.com

Peter Wong  
Department of Mathematics, Bates College  
e-mail: pwong@bates.edu