Almost triangular Markov chains on $\mathbb{N}$

Luis Fredes
(Work with J.F. Marckert)

IMB
**Transition matrix:** \( M = [M_{i,j}]_{i,j \in S} \) with non-negative real entries that sum up to one on each row. A Markov chain \( Y \) with transition matrix \( M \) satisfies

\[
P(X_{n+1} = b | X_n = a) = M_{a,b}.
\]

---

**Irreducible:** every pair \( a, b \in S \) has a finite sequence \( \ell = \ell(a, b) \) of steps with positive probability (\( M_\ell a, b > 0 \)) such that \( b \) can be reached from \( a \).

**From now on we assume irreducibility**

**Recurrence / Transience:** A chain with transition matrix \( M \) is called recurrent if for all/one state \( a \in S \) the probability to return to \( a \) is one, otherwise the chain is called transient.

**Positive recurrence:** The expected return time of all/one state \( a \in S \) is finite, i.e. \( E_a(\tau_a + a) < +\infty \).

**Invariant measure:** A measure \( \pi \) on \( S \) is said to be invariant by \( M \) if, \( X_a \in S \) \( \pi_a M_a, b = \pi_b \) for all \( b \in S \).
**Transition matrix:** $M = [M_{i,j}]_{i,j \in S}$ with non-negative real entries that sum up to one on each row. A Markov chain $Y$ with transition matrix $M$ satisfies

$$\mathbb{P}(X_{n+1} = b | X_n = a) = M_{a,b}.$$ 

**Irreducible:** every pair $a, b \in S$ has a finite sequence $\ell = \ell(a, b)$ of steps with positive probability ($M_{a,b}^\ell > 0$) such that $b$ can be reached from $a$.

**From now on we assume irreducibility**

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**Invariant measure:** A measure $\pi$ on $S$ is said to be invariant by $M$ if,

$$\sum_{a \in S} \pi_a M_{a,b} = \pi_b \quad \text{for all } b \in S.$$
Almost-upper triangular □ and almost-lower triangular △

\[ U := \begin{bmatrix}
U_{0,0} & U_{0,1} & U_{0,2} & U_{0,3} & U_{0,4} & \cdots \\
U_{1,0} & U_{1,1} & U_{1,2} & U_{1,3} & U_{1,4} & \cdots \\
0 & U_{2,1} & U_{2,2} & U_{2,3} & U_{2,4} & \cdots \\
0 & 0 & U_{3,2} & U_{3,3} & U_{3,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad L := \begin{bmatrix}
L_{0,0} & L_{0,1} & 0 & 0 & 0 & \cdots \\
L_{1,0} & L_{1,1} & L_{1,2} & 0 & 0 & \cdots \\
L_{2,0} & L_{2,1} & L_{2,2} & L_{2,3} & 0 & \cdots \\
L_{3,0} & L_{3,1} & L_{3,2} & L_{3,3} & L_{3,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \]

Birth and death processes (BDP)

\[ T := \begin{bmatrix}
T_{0,0} & T_{0,1} & 0 & 0 & 0 & \cdots \\
T_{1,0} & T_{1,1} & T_{1,2} & 0 & 0 & \cdots \\
0 & T_{2,1} & T_{2,2} & T_{2,3} & 0 & \cdots \\
0 & 0 & T_{3,2} & T_{3,3} & T_{3,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \]
Motivation

Almost-upper triangular $\square$ and almost-lower triangular $\square$

Birth and death processes (BDP)
Theorem: Tridiagonal case (Karlin & McGregor ’57)

The following measure \( \pi \) with \( \pi_0 = 1 \)

\[
\pi_a = \prod_{j=1}^{a} \frac{T_{j-1,j}}{T_{j,j-1}} \quad \text{for all } a \geq 1,
\]

is the unique invariant by \( T \) up to a constant factor and the chain is

- **positive recurrent** if and only if

\[
\sum_{k \geq 1} \prod_{j=1}^{k} \frac{T_{j-1,j}}{T_{j,j-1}} < +\infty
\]

- **recurrent** if and only if

\[
\sum_{k \geq 0} \prod_{j=1}^{k} \frac{T_{j,j-1}}{T_{j,j+1}} = +\infty.
\]
Theorem: Almost-upper triangular case (F. -Marckert ’21)

The following measure $\pi$ with $\pi_0 = 1$

$$\pi_a := \frac{\det(Id - U_{[0,a-1]})}{\prod_{j=1}^{a} U_{j,j-1}}$$

for all $a \geq 1$.

is the unique invariant by $U$ up to a constant factor and the chain is

• **positive recurrent** if and only if

$$\sum_{a=1}^{\infty} \frac{\det(Id - U_{[0,a-1]})}{\prod_{j=1}^{a} U_{j,j-1}} < \infty.$$ 

• **recurrent** if and only if

$$\lim_{b \to +\infty} U_{1,0} \frac{\det(Id - U_{[2,b-1]})}{\det(Id - U_{[1,b-1]})} = 1.$$
Plan of the talk

- Proof of the theorem for Almost-upper triangular.
- Almost-lower triangular.
- Link between almost-upper and almost-lower.
- Recovering the results of BDP with our results.
- Another model.
Combinatorial warm up I: Matrix tree theorem

ST(G) = set of spanning trees of G.

Matrix-tree theorem [Kirchhoff]

$$|\text{ST}(G)| = \det \left( \text{Laplacian}^{(r)}_G \right),$$

where $\text{Laplacian}^{(r)}_G$ is the Laplacian matrix of $G$ deprived of the line and column associated to $r$.

$$\text{Laplacian}_G(i, j) = \left[ \deg(u_i) \mathbb{1}_{i=j} - |\{u_i, u_j\} \in E| \right]$$
Combinatorial warm up I: Matrix tree theorem

\( \text{ST}(G) = \text{set of spanning trees of } G. \)

**Matrix-tree theorem [Kirchhoff]**

\[
|\text{ST}(G)| = \det \left( \text{Laplacian}_{G}^{(r)} \right),
\]

where \( \text{Laplacian}_{G}^{(r)} \) is the Laplacian matrix of \( G \) deprived of the line and column associated to \( r \).

\[
\text{Laplacian}_{G}(i, j) = \left[ \deg(u_i) \mathbb{1}_{i=j} - |\{u_i, u_j\} \in E| \right]
\]

\[
\text{Laplacian}_G = \begin{pmatrix}
3 & -1 & 0 & -1 & 0 & -1 \\
-1 & 5 & -2 & -1 & -1 & 0 \\
0 & -2 & 3 & 0 & -1 & 0 \\
-1 & -1 & 0 & 3 & -1 & 0 \\
0 & -1 & -1 & -1 & 4 & -1 \\
-1 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

\[
|\text{ST}(G)| = \det \left( \text{Laplacian}_{G}^{(A)} \right) = 98.
\]
$\text{ST}(G, r):= \text{set of spanning trees of } G \text{ (finite graph) rooted at } r.$

$W_M(T, r) := \prod_{\bar{e} \in T} M_{\bar{e}}$ with edges pointing towards the root $r$.

**Weighted Matrix-tree theorem [Kirchhoff]**

$$\sum_{T \in \text{ST}(G, r)} W_M(T, r) = \det \left( \text{Id} - M^{(r)} \right),$$

where $M^{(r)}$ is the matrix $M$ deprived of the line and column $r$. 
\( \text{ST}(G, r) := \text{set of spanning trees of } G \) \textbf{(finite graph)} rooted at \( r \).

\[
W_M(T, r) := \prod_{\bar{e} \in T} M_{\bar{e}} \text{ with edges pointing towards the root } r.
\]

\textbf{Weighted Matrix-tree theorem [Kirchhoff]}

\[
\sum_{T \in \text{ST}(G, r)} W_M(T, r) = \det \left( \text{Id} - M^{(r)} \right),
\]

where \( M^{(r)} \) is the matrix \( M \) deprived of the line and column \( r \).
Forest($N, R$) = set of spanning forests of $N$ containing the roots in $R \subset N$. We consider each tree oriented towards its root.

**Proposition (F.-Marckert’21)**

Consider the graph $G = (\mathbb{N}, \{(i, j) : U_{i,j} > 0\})$, weighted by the transition matrix $U$ almost-upper triangular. We have

$$
\sum_{F \in \text{Forest}(\mathbb{N}, [x, +\infty))} W_{U}(F) = \det((\text{Id} - U)_{[0,x-1]}).
$$

**Markov chain tree theorem**

The invariant probability measure $\rho$ of $M$ satisfies

$$
\rho_{\nu} \stackrel{(\text{Alg})}{=} \frac{\det(I - M^{(\nu)})}{Z} \stackrel{(\text{MTT})}{=} \frac{\sum_{T \in \text{ST}(G, \nu)} W_{M}(T, \nu)}{Z}.
$$
Proof: uniqueness and expression of the invariant measure

Uniqueness of the invariant measure: Triangular system

\[ \pi_b = \sum_{a \leq b+1} \pi_a U_{a,b} \iff \pi_{b+1} = (\pi_b - \sum_{a \leq b} \pi_a U_{a,b}) / U_{b+1,b} \]

Explicit expression and positive recurrence criteria:
Set \( U(n) \) as the matrix in \( \mathbb{N} \cap [0, n] \) (chain truncated at \( n \))

\[
\begin{cases}
U(n)_{i,j} = U_{i,j}, & \text{for } i \in [0, n], j \in [0, n-1] \\
U(n)_{i,n} = \sum_{j \geq n} U_{i,j}.
\end{cases}
\]

From the Markov chain tree theorem (\( U(n) \) lives on a finite state space)

\[ \rho_a(n) = \alpha_n \det(\text{Id} - U(n)^{(a)}) \]
Claim: there exist $\alpha_n'$ such that

$$
\text{det}(\text{Id} - U(n)^{(a)}) = \text{det}(\text{Id} - U_{[0,a-1]}) \prod_{j=a+1}^{n} U_{a,a-1}
$$

$$
= \alpha_n' \text{det}(\text{Id} - U_{[0,a-1]}) / \prod_{j=1}^{a} U_{a,a-1}.
$$
Claim: there exist $\alpha'_n$ such that

$$\det(\text{Id} - U(n)^{(a)}) = \det(\text{Id} - U_{[0,a-1]}) \prod_{j=a+1}^{n} U_{a,a-1}$$

$$= \alpha'_n \det(\text{Id} - U_{[0,a-1]})/ \prod_{j=1}^{a} U_{a,a-1}.$$
Claim: there exist $\alpha'_n$ such that

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Claim: there exist $\alpha'_{n}$ such that

$$\det(\text{Id} - U(n)^{(a)}) = \det(\text{Id} - U_{[0,a-1]}) \prod_{j=a+1}^{n} U_{a,a-1}$$

$$= \alpha'_{n} \det(\text{Id} - U_{[0,a-1]}) \prod_{j=1}^{a} U_{a,a-1}.$$

All transitions on the RHS of first line come from points in $[0, n]$ and go to points in $[0, n - 1]$, in particular here $U(n)$ and $U$ coincide.

$$\prod_{j=1}^{n} U_{a,a-1}^{(n)} = \prod_{j=1}^{n} U_{a,a-1}.$$
Gathering everything

\[
\frac{\rho(n) \alpha}{\alpha' n} = \frac{\det(\text{Id} - U_{[0,a-1]})}{\prod_{j=1}^{a} U_{a,a-1}} = \pi_a
\]

the result follows from the following lemma.

**Lemma (F.-Marckert’21)**

*The transition matrix $U$ admits $\pi$ as an invariant positive measure, if and only if there exists a sequence $(c_n, n \geq 0)$ such that $c_n \rho(n) \to \pi$ weakly.*
Informally: some “elements” that are stacked.

Formally: a set of letters $\mathcal{P}$ is given and a binary relation $R$:
- $x R y$ means that $x$ commutes with $y$ (that is $xy = yx$),
- $x R y$ means that $x$ does not commute with $y$.

Heap of dominos: $\mathcal{P} = \{a, b, c, d, e\}$ $a R b$, $b R c$, $c R d$, $d R e$.

Heaps: Equivalence classes of words

$w \sim w'$ if they are equal up to a finite number of allowed commutations of consecutive letters.

General heap: (left) Equivalence class of words describing the history of the stack $= baeddecb = baeddceb = ebaddbce = ....$

Trivial heap: (right) All the pieces on the ground $ae = ea$
**Figure:** Heaps of squares. They do not commute if they share a side.

**Figure:** Heaps of oriented cycles. They do not commute if they share a vertex.

**Figure:** Heaps of outgoing edges. They do not commute if they start at the same point.

**Figure:** Heaps of dominoes. They do not commute if they share one extremity.
Heap of pieces

For a heap $H$

$$Weight(H) = \prod_{e \in H} w(e)$$

where $w : \mathcal{P} \rightarrow \mathbb{R}$ (or any formal commutative set)

Inversion lemma

$$\sum_{H \in \text{Heaps}} Weight(H) = \frac{1}{\sum_{H \in \text{TrivialHeaps}} (-1)^{|H|} Weight(H)}$$

Example: Weight $x$ for each piece, $Weight(H) = x^{|H|}$,

$$\sum_{H \in \text{Heaps}} Weight(H) = \frac{1}{1 - 5x + 6x^2 - x^3}$$
In particular for the heaps of oriented cycles (HC) avoiding 0 with weights given by $M = (M_{i,j})_{i,j \in [0,n]}$

$$\sum_{HC \in \text{Heaps avoiding 0}} W(HC) = \left( \sum_{HC \in \text{TrivialHeaps avoiding 0}} (-1)^{|HC|} W_M(HC) \right)^{-1}$$

$$= \left( \sum_{C \in \mathcal{C}} (-1)^{N(C)} \prod_{c \text{ cycles of } C} \prod_{e \in c} M_e \right)^{-1}$$

$$= \det \left( I - M^{(0)} \right)^{-1}$$

where the last sum ranges over $\mathcal{C} = \text{set of collection of disjoint oriented cycles of length } \geq 1 \text{ avoiding 0.}$
Generalised inversion lemma

For \( \mathcal{P} \) a set of pieces and \( \mathcal{M} \subset \mathcal{P} \) one has

\[
\sum_{H \in \text{Heaps}(\mathcal{P}), \text{maximal piece } \in \mathcal{M}} \text{Weight}(H) = \frac{\sum_{H \in \text{TrivialHeaps}(\mathcal{P} \setminus \mathcal{M})} (-1)^{|H|} \text{Weight}(H)}{\sum_{H \in \text{TrivialHeaps}(\mathcal{P})} (-1)^{|H|} \text{Weight}(H)}
\]
For $Y$ a Markov chain with transition matrix $U$ consider the hitting time of the set $A$ as $\tau_A = \inf\{n \geq 0 : Y_n \in A\}$

$$u_b = \mathbb{P}(\tau_0(Y) \leq \tau_{[b, +\infty)}(Y) | Y_0 = 1)$$

**Recurrence** iff $u_b \to 1$ as $b \to \infty$. 

Luis Fredes (Université de Bordeaux)

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**Recurrence** iff $u_b \to 1$ as $b \to \infty$.

**Proof**: Let $w$ be any path starting from 1, ending at 0 and staying completely in $[0, b - 1]$. 
For $Y$ a Markov chain with transition matrix $U$ consider the hitting time of the set $A$ as $\tau_A = \inf\{n \geq 0 : Y_n \in A\}$

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**Proof**: Let $w$ be any path starting from 1, ending at 0 and staying completely in $[0, b - 1]$.

- Notice that $w = w_1(1, 0)$ where $w_1$ is a path starting and ending at 1 and staying completely in $[1, b - 1]$, therefore it can be seen as a heap of cycles in $[1, b - 1]$ with maximal piece adjacent to 1.
For a Markov chain with transition matrix $U$ consider the hitting time of the set $A$ as

$$\tau_A = \inf\{n \geq 0 : Y_n \in A\}$$

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- By the generalized inversion lemma we have that

$$u_b = \frac{\det(Id - U_{[2, b-1]})}{\det(Id - U_{[1, b-1]})} U_{1,0}$$
For $Y$ a Markov chain with transition matrix $U$ consider the hitting time of the set $A$ as $\tau_A = \inf\{n \geq 0 : Y_n \in A\}$

$$u_b(x) = \mathbb{P}(\tau_0(Y) \leq \tau_{[b, +\infty)}(Y) | Y_0 = x)$$

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- Notice that $w = w_1(1, 0)$ where $w_1$ is a path starting and ending at 1 and staying completely in $[1, b - 1]$, therefore it can be seen as a heap of cycles in $[1, b - 1]$ with maximal piece adjacent to 1.
- By the generalized inversion lemma we have that

$$u_b(x) = \frac{\det(I_d - U_{[x+1, b-1]})}{\det(I_d - U_{[1, b-1]})} \prod_{j=1}^{x} U_{j, j-1}$$
Theorem: Almost-lower triangular criteria (F.-Marckert’21)

Invariant measure

- **Finite state space** $\mathbb{N} \cap [0, s]$ the following measure $\pi$ with $\pi_0 = 1$

$$\pi_a = \pi_0 \det(\text{Id} - L_{[a+1, s]}) \prod_{i=1}^{a} L_{i-1,i} \quad \text{for all } a \in [1, s].$$

is the unique invariant by $L$ up to a constant factor.

- **Infinite state space** $\mathbb{N}$ the Markov chain may have none, one or multiple invariant measures (we give examples of each case).

- The Markov chain is **recurrent** if and only if

$$\lim_{b \to +\infty} \frac{\prod_{j=1}^{b-1} L_{j,j+1}}{\det(\text{Id} - L_{[1,b-1]})} = 0.$$
Consider an irreducible transition matrix $U$, with invariant measure $\pi$. Set $L$ as

$$L_{i,j} = \frac{\pi_j}{\pi_i} U_{j,i}.$$

Then,

- $L$ is an irreducible transition matrix on $\mathbb{N}$, with invariant measure $\pi$ too.
- $L$ is recurrent if and only if $U$ is recurrent.
- $L$ is positive recurrent if and only if $U$ is positive recurrent.
Invariant measure and positive recurrence criteria: tridiagonal case
The uniqueness of the invariant measure with $\pi_0 = 1$ gives the equality of two formulas for the invariant measure (Karlin-McGregor and F.-Marckert)

$$\frac{\det(\text{Id} - T_{[0,a-1]})}{\prod_{j=1}^{a} T_{j,j-1}} = \prod_{j=1}^{a} \frac{T_{j-1,j}}{T_{j,j-1}} \quad \forall a \geq 1$$

then, the positive recurrence criteria is recovered.
Recovering the results of Karlin & McGregor with ours

Recurrence criteria
Set $D_{i,j} = \det(\text{Id} - T_{[i,j]})$, then

$$D_{i,j} = (1 - T_{i,i})D_{i+1,j} - T_{i,i+1}T_{i+1,i}D_{i+2,j}, \quad (1)$$

and set

$$Z_{i,j} := \frac{D_{i,j}}{D_{i+1,j}T_{i,i-1}}$$

and notice that our result translates into: the MC with transition matrix $T$ is recurrent iff

$$\lim_{b \to +\infty} T_{1,0} \frac{\det(\text{Id} - T_{[2,b-1]})}{\det(\text{Id} - T_{[1,b-1]})} = \lim_{b \to +\infty} \frac{1}{Z_{1,b-1}} = 1.$$ 

(1) rewrites

$$Z_{i,j} = \frac{(1 - T_{i,i})}{T_{i,i-1}} - \frac{T_{i,i+1}}{Z_{i+1,j}},$$

which gives the convergents of a continued fraction.
These can be solved and give that

\[
Z_{1,b-1} = c_1 + \frac{a_2}{c_2} + \frac{a_3}{c_2 + \cdots + \cdots + \cdots + \frac{a_{b-1}}{c_{b-2}}}
\]

So that \( \lim_{b \to \infty} Z_{1,b-1} = 1 \) is equivalent to Karlin & McGregor’s recurrence criteria.

\[
\sum_{k=0}^{k} \prod_{j=1}^{k} \frac{T_{j,j-1}}{T_{j,j+1}} = +\infty.
\]
The Repair shop Markov chain is defined by \( X_n = (X_{n-1} - 1)_+ + Z_n \), with \((Z_i)_{i \in \mathbb{N}}\) i.i.d. family having \( P(Z_k = i) = a_i \), i.e. with kernel

\[
A := \begin{bmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
0 & a_0 & a_1 & a_2 & a_3 & \cdots \\
0 & 0 & a_0 & a_1 & a_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \cdots \cdots \\
\end{bmatrix}
\]

and denote by \( m \) the expectation of \( Z_1 \) (i.e. \( m = \sum_{i \in \mathbb{N}} ia_i \)).

Criteria for the repair shop (Brémaud ’13)

The Repair shop Markov chain is

- **positive recurrent** iff \( m < 1 \),
- **recurrent** iff \( m \leq 1 \).