Can an infrared-vanishing gluon propagator confine quarks?

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ABSTRACT

It is shown that the solution of the quark Dyson-Schwinger equation in QCD obtained with a gluon propagator of the form $D(q) \sim q^2/[q^4 + b^4]$ and a quark-gluon vertex that is free of kinematic singularities does not describe a confined particle and that there is always a value of $b^2 = b_c^2$ such that chiral symmetry is not dynamically broken for $b^2 > b_c^2$.

1. Introduction

Dynamical Chiral Symmetry Breaking (D\chiSB) and confinement are two crucial features of quantum chromodynamics (QCD). They are responsible for the nature of the hadronic spectrum; D\chiSB ensuring the absence of low mass scalar partners of the pion and confinement ensuring the absence of free quarks, for example. A natural method for studying both D\chiSB and confinement in QCD is the complex of Dyson-Schwinger Equations (DSEs).

One goal of DSE studies is to develop this nonperturbative approach to the point where it is as firmly founded as lattice QCD and calculationally competitive. Although more needs to be done in order to achieve this goal there has been a good deal of progress, especially in the study of Abelian gauge theories where direct and meaningful comparisons can be made, and agreement obtained, between the results of lattice and DSE studies.

Herein a recent study of the fermion DSE in which the gluon propagator vanishes at $q^2 = 0$, aimed at determining whether such a gluon propagator can support D\chiSB and/or generate a confining quark propagator, is described. There has been renewed interest in such a form of the gluon propagator, which was argued in Ref. [4] to be associated with the elimination of Gribov copies, because of the recent work of Häbel et. al. and Zwanziger.

In Sec. 2 the DSE for the quark propagator is described in detail. There is also a discussion of what is known about the dressed gluon propagator and quark-gluon

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vertex in QCD. The analysis of $D\chi_{SB}$ is reported in Sec. 3 and quark confinement is discussed in Sec. 4. The results are summarised in Sec. 5.

2. Dyson-Schwinger Equation for the Fermion Self Energy

In Minkowski space, with metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and in a general covariant gauge, the inverse of the dressed quark propagator can be written as

$$S^{-1}(p) = \not{p} - m - \Sigma(p) = Z^{-1}(p^2) \left( \not{p} - M(p^2) \right) = A(p^2) \not{p} - B(p^2), \quad (1)$$

with: $m$ the renormalised, explicit chiral symmetry breaking mass (if present); $\Sigma(p)$ the self-energy; $M(p^2) = B(p^2)/A(p^2)$ the dynamical quark mass function; and $Z(p^2) = A^{-1}(p^2)$ the momentum-dependent renormalisation of the quark wavefunction. The unrenormalised DSE for the inverse propagator is

$$S^{-1}(p) = \not{p} - m^{\text{bare}} - \frac{4}{3} g^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu S(k) \Gamma^\nu(k, p) D_{\mu\nu}(\not{p} - \not{k}), \quad (2)$$

where $D_{\mu\nu}(q^2)$ is the dressed gluon propagator and $\Gamma^\nu$ is the proper quark-gluon vertex, which is illustrated in Fig. 1.

The renormalised, massless ($m = 0$) DSE can be written as

$$\Sigma_R(p) = (1 - Z_S) \not{p} + i Z_T \frac{4}{3} g^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu S_R(k) \Gamma^\nu_R(k, p) D_{\mu\nu}^R((p - k)^2), \quad (3)$$

where $Z_S$ and $Z_T$ are quark-propagator and quark-gluon-vertex renormalisation constants, respectively, which depend on the renormalisation scale, $\mu$, and ultraviolet cutoff, $\Lambda$. Hereafter we suppress the label $R$.

The solution of this equation provides information about $D\chi_{SB}$. The quark condensate, $\langle \overline{q}q \rangle \propto \text{tr}[S(x = 0)]$, is a chiral symmetry order parameter. If there is a solution of Eq. (3) with $B \neq 0$ then the quark has generated a mass via interaction with its own gluon field and the chiral symmetry is therefore dynamically broken. The

\[
\text{Fig. 1. The Dyson-Schwinger equation for the quark self-energy.}
\]

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where $Z_S$ and $Z_T$ are quark-propagator and quark-gluon-vertex renormalisation constants, respectively, which depend on the renormalisation scale, $\mu$, and ultraviolet cutoff, $\Lambda$. Hereafter we suppress the label $R$.
solution also provides information about quark confinement, as discussed in Sec. 4.

2.1a Gluon Propagator

In a general covariant gauge the dressed gluon propagator, which is diagonal in colour space, can be written:

\[ D_{\mu\nu}(q^2) = \left( g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2} \right) \frac{1}{1 - \Pi(q^2)} + \xi \frac{q_{\mu}q_{\nu}}{q^2} \frac{1}{q^2}, \]

(4)

where \( \Pi(q^2) \) is the gluon vacuum polarisation and \( \xi \) is the gauge parameter. In covariant gauges the longitudinal piece of this propagator is not modified by interactions, which follows from the Slavnov-Taylor identities in QCD.

The Dyson-Schwinger equation for the gluon propagator is given diagrammatically in Fig. 2. The symmetrisation factors of 1/2 and 1/6 arise from the usual Feynman rules, which also require a negative sign [unshown] to be included for every fermion and ghost loop. This equation has been studied extensively\[5,7,8,9\]. There have also been attempts to determine the gluon propagator from numerical simulations of lattice-QCD\[10,11\].

The results of the DSE and lattice studies are summarised in Sec. 5.1 of Ref. [1] and are represented in Fig. 3. This figure illustrates that for spacelike-\( q^2 > 1 \text{ GeV}^2 \) the gluon propagator is given by the two-loop, QCD renormalisation group result, with the next order correction being \(<10\%\). For spacelike-\( q^2 < 1 \text{ GeV}^2 \), however, the form of the propagator is not known. The DSE studies of Refs. [7,8,9] suggest a regularised infrared singularity, represented by \( 1/q^4 \) in the figure. That of Ref. [5], which differs
mainly in that the Ansatz used for the triple-gluon vertex has kinematic singularities, suggests an infrared vanishing form, characterised by $q^2/(q^4 + b^4)$, which has also been argued to be the form necessary to completely eliminate Gribov copies.

The lattice Landau-gauge studies of Ref. [10] favour the massive vector boson form, $1/(q^2 + m^2_g)$, which is broadly consistent with the improved studies of Ref. [11]. On $16^3 \times 40$ and $24^3 \times 40$ lattices at $\beta = 6.0$ these numerical results allowed a fit of the form $q^2/(q^4 + b^4)$, with $b \sim 340$ MeV, but a fit using a standard massive particle propagator could not be ruled out. On a lattice of dimension $16^3 \times 24$ at $\beta = 5.7$ it was found that the gluon propagator was best fit by a standard massive vector boson propagator with mass $\sim 600$ MeV. There is a problem with these studies, however, which is indicated by the dashed vertical line at the right of Fig. 3. With present technology, the domain of spacelike-$q^2 < 1$ GeV is actually inaccessible in lattice studies and, since all forms of the propagator are very nearly the same outside this domain, it is clear that these results are both qualitatively and quantitatively unreliable.

This brief discussion indicates that at present one can only say that

$$ D(q^2) \equiv \frac{1}{q^2 [1 - \Pi(q^2)]} = \frac{q^2}{q^4 + b^4} \quad (5) $$

is not implausible in QCD, at least at small $q^2$. For this reason it is of interest to determine whether such a form of $D(q^2)$ can lead to $D_{\chi SB}$ and a confining quark propagator, which will provide further insight into the validity of this form of gluon propagator. In keeping with the ultraviolet behaviour illustrated in Fig. 3, the results obtained with the “ultraviolet-improved” form:

$$ \frac{g^2}{4\pi} D(q^2) = \alpha(\tau; q^2) \frac{q^2}{q^4 + b^4} \quad (6) $$

with $\alpha(\tau; q^2) = (d\pi)/\left(\ln \left[\tau + q^2/\Lambda^2_{QCD}\right]\right)$, $d = 12 /[33 - 2N_f]$, $N_f = 4$, where $\tau \geq 1$ is an infrared regularising parameter, are described herein.

2.1b Gluon Condensate

The gluon condensate can be calculated from the nonperturbative part of the gluon propagator. Using Eq. (5) one obtains

$$ \langle \alpha_S GG \rangle_{\mu} = -\frac{3b^4}{\pi^2} \ln \left(\frac{\mu^2}{\Lambda^2_{QCD}}\right) \quad (7) $$

For $b = 340$ MeV, $\Lambda_{QCD} = 200$ MeV and $\mu = 1$ GeV this yields $\langle \alpha_S GG \rangle_{\mu} = -0.0047$ GeV$^4$, which should be compared with the value inferred from QCD sum rules: $\langle \alpha_S GG \rangle_{\mu} \sim 0.04$ GeV$^4$. The “wrong sign” is due to the fact that Eq. (5) is
weaker than perturbative gluon exchange for all spacelike-\(q^2\) and is an harbinger of the results to follow.

2.2 Quark-Gluon Vertex

The quark-gluon vertex satisfies its own DSE but hitherto there have been no studies of this equation in QCD. Making use of the “Abelian approximation” the Slavnov-Taylor identity for this vertex reduces to the Ward-Takahashi identity familiar from QED:

\[ k_\mu \Gamma^\mu(p, q) = S^{-1}(p) - S^{-1}(q) , \]

\[ k = (p - q) , \]

and the quark-gluon vertex can be written in the general form:

\[ \Gamma^\mu(p, q) = \Gamma^{BC}_\mu(p, q) + \sum_{i=1}^{8} T^i_\mu(p, q) g^i(p^2, p \cdot q, q^2) , \]

where the eight tensors, \(T^i_\mu\), are transverse, \(k^\mu T^i_\mu(p, q) = 0\) and

\[ \Gamma^{BC}_\mu(p, q) = \Sigma_A(p, q) \gamma_\mu + (p + q)_\mu \left\{ \Delta_A(p, q) \frac{1}{2} [\gamma \cdot p + \gamma \cdot q] - i \Delta_B(p, q) \right\} \]

\[ \delta^2 \left( k, p \right) = \left( k^2 - p^2 \right)^2 + \left[ B^2(k^2) + A^2(p^2) \right] \left( k^2 - p^2 \right) ^2 \]

This Ansatz, Eq. (9) with Eq. (11) and \(T^i_\mu = 0\) for \(i \neq 6\), satisfies the Ward-Takahashi Identity, is free of kinematic singularities (i.e., has a well defined limit as \((p \to q)\), reduces to the bare vertex in the free field limit in the manner prescribed by perturbation theory, transforms correctly under charge conjugation and Lorentz transformations and preserves multiplicative renormalisability in the quark DSE. (Of these properties the vertex \(\Gamma^{BC}_\mu\) satisfies all but the last and, for the most part, the calculations described herein were performed with the Ansatz \(\Gamma^{BC}_\mu = \Gamma^{BC}_\mu\) )

It should be noted that the absence of kinematic singularities is an important and physically reasonable constraint. To understand this one can simply consider an analogy with PCAC. In the case of the axial-current vertex, \(\Gamma^5\), there is a kinematic singularity in the chiral limit, which is identified with the massless pion excitation. A kinematic singularity in the quark-gluon vertex would therefore entail the existence...
of an hitherto unknown, massless excitation in QCD.

2.3 Quark DSE in Euclidean Space

In the Abelian approximation one has \( Z_s = 1 = Z_f \) at one-loop in Landau gauge. Using this result here considerably simplifies Eq. (13), which can be written in Euclidean space, with metric \( \delta_{\mu\nu} = \text{diag}(1, 1, 1, 1) \) and \( \gamma_\mu \) hermitian,

\[
\Sigma(p) = \frac{4}{3} g^2 \int^\Lambda \frac{d^4 k}{(2\pi)^4} \gamma_\mu S(k) \Gamma_\nu(k, p) D_{\mu\nu}((p - k)^2)
\]

(13)

where

\[
S^{-1}(p) = i \gamma \cdot p + \Sigma(p) = i \gamma \cdot p A(p^2) + B(p^2)
\]

(14)

and all the other elements in this equation are taken to be specified by the expressions given above evaluated at Euclidean (spacelike) values of their arguments. The important, subtle considerations associated with the transformation between Minkowski and Euclidean space are discussed in Sec. 2.3 of Ref. [1].

Equation (13) is actually a pair of coupled, nonlinear integral equations for \( A(p^2) \) and \( B(p^2) \). With \( \Gamma_\mu = \Gamma_\mu^{BC} \) one obtains

\[
B(p^2) = \frac{16\pi}{3} \int^\Lambda \frac{d^4 k}{(2\pi)^4} \alpha(\tau; (p - k)^2) \frac{(p - k)^2}{(p - k)^4 + b^4} \frac{1}{A^2(k^2)k^2 + B^2(k^2)} \times \left\{ 3B(k^2) \frac{A(k^2) + A(p^2)}{2} + \left[ B(k^2) \Delta A(k^2, p^2) - A(k^2) \Delta B(k^2, p^2) \right] h(p, k) \right\}
\]

(15)

with \( h(p, k) = 2 \frac{[k^2p^2 - (k \cdot p)^2]}{(p - k)^2} \) and a similar but more complicated equation for \( A(p^2) \), while with \( \Gamma_\mu = (\Gamma_\mu^{BC} + T_\mu^a g^a) \), from Eq. (11), one has

\[
B(p^2) = \text{RHS of (15)} + \frac{16\pi}{3} \int^\Lambda \frac{d^4 k}{(2\pi)^4} \alpha(\tau; (p - k)^2) \frac{(p - k)^2}{(p - k)^4 + b^4} \frac{B(k^2) \Delta A(k^2, p^2)}{A^2(k^2)k^2 + B^2(k^2)} \frac{(k^2 - p^2)}{2d(k, p)} \times 3(k^2 - p^2)
\]

(16)

with, again, a similar but more complicated equation for \( A(p^2) \).

The results discussed below were obtained from an iterative, numerical solution of these equations on a logarithmic grid of \( x = p^2/\Lambda_{\text{QCD}}^2 \) and \( y = k^2/\Lambda_{\text{QCD}}^2 \) points. The solutions were independent of the seed-solution and grid choice and also of the UV cutoff, which was \( \Lambda^2 = 5 \times 10^8 \Lambda_{\text{QCD}}^2 \).

3. Dynamical Chiral Symmetry Breaking?

The gauge-invariant quark condensate is an order parameter for D\( \chi \)SB and is obtained from the trace of the quark propagator:

\[
\langle \bar{q} q \rangle_\mu = -\frac{3}{4\pi^2} \ln \left( \frac{\mu^2}{\Lambda_{\text{QCD}}^2} \right) \lim_{\Lambda^2 \to \infty} \left( \ln \left( \frac{\Lambda^2}{\Lambda_{\text{QCD}}^2} \right) \right)^{-d} \int_0^\Lambda d^4 s \frac{B(s)}{sA(s)^2 + B(s)^2}
\]

(17)
where $\mu$ is the renormalisation point for the condensate, which is usually fixed at 1 GeV. This is the parameter that is used to study $D\chi_{SB}$ in lattice QCD. A nonzero value signals $D\chi_{SB}$.

In Fig. 4 the condensate obtained from the numerical solution of Eq. (15) and the associated equation for $A(p^2)$ for values of $\ln \tau$ in the domain $[0.0, 0.7]$ and $b^2$ in $[0.1, 1.0]$ is plotted. This figure shows regions of unbroken and dynamically broken chiral symmetry. Notably there is no $D\chi_{SB}$ for the value of $b^2 \sim 3 \Lambda_{QCD}^2$ inferred from lattice simulations.

The numerical results suggest that $\langle \bar{q}q \rangle_\mu$ rises continuously from the transition boundary and hence that the transition is second order. Assuming therefore that, for a given value of $\ln \tau$, the order parameter behaves as $\langle \bar{q}q \rangle_\mu(b^2) \approx C (1 - b^2/b_c^2)^\beta$ one obtains $\beta = 0.572$ with $\sigma_\beta = 0.020$. Including the $T_6^\mu$ term in the vertex only leads to a small quantitative change in the results. For example, with $\ln \tau = 0.6$ one finds $\beta_{T_6} = 0.579$, $\sigma_{\beta_{T_6}} = 0.015$. This suggests that for any vertex that is free of kinematic singularities there is a critical value of $b^2 = b_c^2$ such that there is no $D\chi_{SB}$ for $b^2 > b_c^2$.

4. Quark Confinement?

In order to determine whether the quark propagator obtained as a solution to Eq. (13) with an infrared-vanishing gluon propagator can represent a confined particle we follow Ref. [16] and adapt a method commonly used in lattice QCD to estimate bound state masses. Writing $\sigma_S(s) = B(s)/[sA(s)^2 + B(s)^2]$, $s = p^2$, defining

$$\Delta_S(T) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \sigma_S(y^2) e^{iyT}$$

and, for notational convenience, $E(T) = -\ln [\Delta_S(T)]$, it follows from the axioms of field theory that if there is a stable asymptotic state with the quantum number of the quark then

$$\lim_{T \to \infty} \frac{dE(T)}{dT} = m ;$$

where $m \geq 0$ is the mass of this excitation; i.e., this limit yields the dynamically generated quark mass.

As a simple example one can consider the Nambu-Jona-Lasinio model in which the dressed quark propagator is $S(p) = 1/[i\gamma \cdot p + M]$ and hence $\sigma_S(s) = M/[s + M^2]$. In this case $\Delta_S(T) = \exp(-MT)/2$, which, from Eq. (19), yields $m = M$, as one would expect.

This confinement test was applied to the numerical solutions in the following cases: 1) The propagators obtained with $\ln \tau = 0.1$ and $b^2$ in the range $[0.1, 1.0]$; 2) The propagator obtained with $\ln \tau = 0$ and $b^2 = 0.35$, which yields the largest value of $(-\langle \bar{q}q \rangle_\mu)$ on the $(b^2, \ln \tau)$ domain considered; 3) Two propagators obtained with $(b^2, \ln \tau) = (0.1, 0.6)$ - one using $\Gamma_\mu = \Gamma_\mu^{BC}$ and the other using $\Gamma_\mu = (\Gamma_\mu^{BC} + T_6^\mu g^6)$. Plots of $E'(T)$ for the family of propagators obtained with $\ln \tau = 0.1$ are presented in Fig. 5. (It should be noted that a quark propagator with complex conjugate poles)
leads to a form of $\Delta_S(T)$ with zeros and hence to $E'(T)$ with zeros and poles; a strong signal of which there is no sign in Fig. 5. Since the behaviour of all the other solutions was qualitatively the same as that described by the results presented in Fig. 5 one concludes that, independent of $b$, $D(q) \sim q^2/[q^4 + b^4]$ does not yield a confining quark propagator.

5. Summary

A study of the quark Dyson-Schwinger equation in QCD using a gluon propagator that vanishes as $q^2 \to 0$, Eq. (6), and a dressed quark-gluon vertex, $\Gamma_{\mu}(p, q)$, that has no kinematic singularities; i.e., has a well defined limit as $p \to q$, is described. The results indicate that such a gluon propagator can only support $D\chi_{SB}$ for values of $\ln \tau$ and $b^2$ less than certain critical values and does not confine quarks. The results are qualitatively independent of the model dependent elements in this study. One is therefore lead to conclude that the dressed gluon propagator in QCD does not vanish in the infrared.

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Fig. 3. Results of studies of the gluon propagator in QCD. Typical values of $m_g = 0.5$ GeV and $b = 0.7$ GeV have been used.
Fig. 4. Criticality plot for \((-\langle \bar{q}q \rangle)_{\mu}^{\frac{1}{3}}\) as a function of ln \(\tau\) and \(b^2\). The condensate, \((-\langle \bar{q}q \rangle_{\mu})^{\frac{1}{4}}\), is in units of MeV, scaled to \(\mu^2 = 1\) GeV, and \(b^2\) is in units \(\Lambda_{QCD}^2\) [\(b^2 = 0.49 \Rightarrow b \sim 140\) MeV]; the gluon regulator \(\tau\) is dimensionless.
Fig. 5. Dressed-quark-mass curves for the family of propagators with the minimal Ball-Chiu vertex and $\ln \tau = 0.1$. The masses are in units of $\Lambda_{QCD}$. 
