POINCARE POLYNOMIALS AND LEVEL RANK DUALITIES IN THE $N = 2$ COSET CONSTRUCTION

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Abstract

We review the coset construction of conformal field theories; the emphasis is on the construction of the Hilbert spaces for these models, especially if fixed points occur. This is applied to the $N = 2$ superconformal cosets constructed by Kazama and Suzuki. To calculate heterotic string spectra we reformulate the Gepner construction in terms of simple currents and introduce the so-called extended Poincaré polynomial. We finally comment on the various equivalences arising between models of this class, which can be expressed as level rank dualities.

1 Introduction

Conformal field theory in general and $N = 2$ superconformal field theories in particular have provided a deeper understanding of many issues in both mathematics and physics.

In physics, string theory is still one of the most exiting paths towards a quantization of gravity and also implications in the realm of statistical physics, e.g. for universality classes of two-dimensional critical behaviour, have been established. Moreover, deep and beautiful

1Invited talk given at the III. International Conference on Mathematical Physics, String Theory and Quantum Gravity, Alushta, Ukraine, June 1993. To appear in Theor. Math. Phys.
2supported by Studienstiftung des deutschen Volkes
relations to other types of quantum field theories, e.g. Toda theories, Chern-Simons theories and integrable systems have been unraveled. Conformal field theories provide also an important testing ground for the algebraic theory of superselection sectors.

Mathematics has largely benefitted from conformal field theory as well: it gave rise to a new and rich interplay between various branches: algebraic and differential geometry, number theory, infinite dimensional Lie algebras, the theory of $C^*$ algebras, commutative algebra, just to enumerate some of them.

The interest in $N = 2$ superconformal theories was initially motivated in string theory by the fact that – together with charge quantization – $N = 2$ superconformal symmetry on the world sheet yields a space time supersymmetric string spectrum, but nowadays an independent motivation to study these models comes from their beautiful intrinsic structure and their deep connection to other objects in mathematical physics, e.g. to two-dimensional topological quantum field theories and possibly even to general conformal field theories.

Among the various ways to construct these models – non-linear sigma models with the target space a Calabi-Yau manifold, (infra-red) fixed points of the renormalization group flow on Landau-Ginzburg potentials etc. – exactly solvable models are distinguished by two important properties. Not only can one calculate in these models – at least in principle – exactly, i.e. non-perturbatively correlation functions, but, what is even more fundamental, the full field content of these models is known. This allows for an explicit calculation of the behaviour under modular transformations, giving complete information on quantum dimensions and fusion rules. So it is in these models that one hopes to really identify the fundamental symmetries of quantum physics. Our main interest in this talk will be in coset theories, where the mathematical framework is the theory of affine Lie algebras, but, of course, there are also other completely solvable formulations, e.g. the Coulomb gas approach.

2 The coset construction

Starting from a complex, affine (untwisted) Kac-Moody algebra at level $k$:

$$[J^a_n, J^b_m] = f^{ac}_{eb} J^c_{n+m} + k m \delta_{m+n,0} \kappa^{ab}$$  \hspace{1cm} (2.1)

we obtain by the Sugawara construction

$$L_n := \frac{1}{2k + 2g^\vee} \sum_{m,a,b} \kappa_{ab} : J^a_{m+n} J^b_{-m} :$$  \hspace{1cm} (2.2)

the Virasoro algebra:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}$$  \hspace{1cm} (2.3)

with central charge $c = \frac{k \dim g}{k + g^\vee}$. Many quantities of interest of a WZW theory can be entirely described in terms of the simple Lie algebra $g$ that is generated by the zero mode currents. So in the equations above $\kappa$ denotes the Killing form of $g$ and $g^\vee$ the dual Coxeter number; colons indicate normal ordering. Imposing unitarity, which is a natural requirement in string theory,
but not necessarily in statistical mechanics, we end up with a rational WZW theory at integer level $k$.

The Virasoro central charges obtained this way always obey $c \geq \text{rank} \, g$, with equality exactly for simply-laced Lie algebras at level $k = 1$; so the various minimal series cannot be obtained this way. This was one motivation for the coset construction [1], which turned out to be a powerful tool to construct new conformal field theories.

Consider a subalgebra $h$ embedded in $g$. It is easy to show that

$$L_{g/h}^n := L_g^n - L_h^n, \quad (2.4)$$

generates a Virasoro algebra with central charge $c_{g/h} = c_g - c_h$.

While on the level of the symmetry algebra the construction is straightforward, we have to emphasize the fact that we are still far away from having constructed a conformal field theory. The question we have to worry about is: what is the Hilbert space of the theory? As it turns out, this is a highly non-trivial question and at present no completely satisfactory answer to it is known.

A first guess is to decompose any irreducible unitary representation $\mathcal{H}_\Lambda^g$ of $g$ in irreducible representations $\mathcal{H}_\lambda^h$ of the subalgebra $h$:

$$\mathcal{H}_\Lambda^g = \bigoplus_{\lambda} \mathcal{H}_\Lambda^A \otimes \mathcal{H}_\lambda^h, \quad (2.5)$$

what leads to the following relation between the characters:

$$\chi_\Lambda^g(\tau) = \sum_{\lambda} b_{\Lambda}^A(\tau) \chi_\lambda^h(\tau). \quad (2.6)$$

The Hilbert space of the coset is then supposed to be all $\mathcal{H}_\Lambda^A$, its characters are the branching functions $b_{\Lambda}^A$.

But right here we start running into problems. To see what happens let us have a look at the critical Ising model, i.e. the minimal model with $c = \frac{1}{2}$, which has three primary fields: the vacuum with conformal weight $\Delta = 0$, the twist field $\sigma$ with $\Delta = 1/16$ and the energy operator $\epsilon$ with $\Delta = \frac{1}{2}$. It can be described by the coset:

$$\frac{(A_1)_1 \oplus (A_1)_1}{(A_1)_2}. \quad (2.7)$$

Here subscripts are used to indicate the levels of the various algebras. A field is labeled by three quantum numbers: $\Phi_{pqr}^g$, where unitarity restricts $p$ and $q$ to be 0 or 1, and $0 \leq r \leq 2$.

We immediately see that group theoretical selection rules for the coupling of the two $A_1$ representations force all fields with $p + q - r \neq 0 \text{ mod } 2$ to vanish. A second glance at the branching functions of the coset (2.7) reveals that, apparently, each field occurs twice in the spectrum. The vacuum, e.g. is represented by $\Phi_{000}^0$ or $\Phi_{111}^1$, $\sigma$ by $\Phi_{011}^0$ or $\Phi_{101}^1$, and $\epsilon$ by $\Phi_{111}^0$ or $\Phi_{000}^2$. A more careful analysis reveals that moreover modular invariance is spoiled.

Here a new requirement we impose on a conformal field theory enters: modular invariance. In string theory, in a string loop expansion à la Polyakov, modular invariance is used to factor
out the ‘large’ diffeomorphisms in the diffeomorphism group. In fact, we require even more
than just modular invariance of the partition function; we also require that the characters form
an unitary representation of the modular group.

The situation encountered in the coset representation of the critical Ising model generalizes:
some branching functions vanish while other non-vanishing branching functions turn out to be
identical. There are three possible reasons for a branching function to vanish: group theoretical
selection rules like in the case of the Ising model, the occurrence of null states and a slightly
more complicated combination argument, but for which no example is known. On the other
hand, in general, $S$-matrix elements between vanishing and non-vanishing branching functions
do not vanish, so it is impossible to simply delete the corresponding rows and lines in the
$S$-matrix without spoiling unitarity of the $S$-matrix.

Before explaining the way out of this situation, it is appropriate to recall some results from
the theory of simple currents (for a review see [3]). A simple current $J$ is a primary field in the
theory for which the fusion product with any field of the theory yields just one other field:

$$J \star \Phi = \Phi'.$$  \hfill (2.8)

We define the monodromy charge $Q$ of $\Phi$ relative to $J$ as $Q = \Delta(J) + \Delta(\Phi) - \Delta(\Phi').$ In the
case of WZW theories $Q$ can be shown to be simply the conjugacy class.

Simple currents give rise to modular invariants which can be different from the diagonal one.
Important for our purposes is the case of integer spin simple currents. In this case a non-trivial
modular invariant exists in which only fields with vanishing monodromy charge occur. It is
given by:

$$Z = \sum_{a, Q(a)=0} \frac{N_0}{N_a} \left| \sum_{i=0}^{N_a-1} \chi_{J^a} \right|^2$$  \hfill (2.9)

Here $N_a$ is the length of the orbit of the simple current $J$, $N_0$ is the length on the orbit which
contains the vacuum, i.e. $N_0$ is the order of the simple current.

These invariants are suited to take care of group theoretical selection rules: non-vanishing
fields can be characterized by the fact that they have vanishing monodromy charge relative to
a subgroup of the group of simple currents of $g$ tensored with (the complement of) $h$.\footnote{This is the case for conformal embeddings (which yield trivial cosets with $c = 0$) and also for the ‘maverick’
cosets described in [2].}

We find thus that the true fields of the theory are the orbits of the identification group; in
the literature this is referred to as ‘field identification’ [4,5]. In fact, this seems to be a generic
feature of reduction procedures; e.g. in a system with first class constraints it is well known
that we have to factor out the action of a ‘gauge’ group, too, to obtain a consistent physical
system.

In our case problems arise if orbits of different lengths occur. Namely, we want to keep just
one representative of every orbit to have a unique vacuum; in other words, we have to divide
$Z$ by $N_0^2$. But this would lead to non-integer coefficients in the partition function of shorter
orbits, what is incompatible with the interpretation of $Z$ as a partition function. The shorter

\footnote{The modular invariants of the maverick cosets are in general not simple current invariants.}
orbits which are termed ‘fixed points’ require thus a special treatment, and here we clearly need some additional input.

The idea suggested by the prefactors of the complete squares in (2.9) is that every fixed point of length \( N_f < N_0 \) has to be resolved in \( N_0/N_f \) distinct physical fields. This procedure possibly introduces some arbitrariness, as \( S \)-matrix and characters for the individual physical fields are a priori unknown.

For the \( S \)-matrix elements involving fixed points we make the following ansatz [3]:

\[
\tilde{S}_{f,i} = \frac{N_f N_g}{N_0} S_{fg} + \Gamma_{ij}^{fg},
\]

(2.10)

where the indices \( i, j \) count resolved fields and \( S \) is the naive \( S \)-matrix. Modular invariance can be shown to imply the following sum rules for the characters \( \mathcal{X}_f \) and the \( S \)-matrix elements of fixed points:

\[
\sum_i \mathcal{X}_{f_i} = \mathcal{X}_f \quad (2.11)
\]

\[
\sum_i \Gamma_{ij}^{fg} = 0 = \sum_j \Gamma_{ij}^{fg} \quad (2.12)
\]

In most cases the \( \Gamma \) matrices and also the character modifications needed to fulfill (2.11) can be described in terms of a different WZW theory, which is usually called ‘fixed point theory’. A list of these fixed point theories can be found in [3].

We emphasize that only after having found a consistent resolution of the fixed points we have really constructed a conformal field theory. Unfortunately, no general results concerning existence or uniqueness of the resolution are known.

### 3 \( N = 2 \) coset theories

In the sequel we will focus our attention on a special subclass of cosets models, namely the \( N = 2 \) coset models constructed by Kazama and Suzuki [6, 7]. As is well known, the \( N = 2 \) algebra is generated by the stress energy tensor \( T \), two spin 3/2 supercurrents \( G^\pm \) and one spin 1 \( u(1) \)-current \( J \). With respect to \( J \) the supercurrents \( G^\pm \) have charge \( \pm 1 \).

For the following considerations it is convenient to define

\[
G^{(1)} := \frac{1}{\sqrt{2}}(G^+ + G^-), \quad G^{(2)} := \frac{1}{i\sqrt{2}}(G^+ - G^-).
\]

(3.1)

Then \( T \) and \( G^{(1)} \) or \( G^{(2)} \) generate an \( N = 1 \) superconformal algebra.

We are now going to describe the construction proposed by Kazama and Suzuki [7]. Given a reductive subalgebra \( h \) of a simple Lie algebra \( g \), the embedding \( h \hookrightarrow g \) induces also an embedding (‘tangent space embedding’) of \( h \hookrightarrow \mathfrak{so}(2d) \), where \( 2d = \dim g - \dim h \). We can thus consider the diagonal embedding

\[
h \hookrightarrow g_k \oplus \mathfrak{so}(2d),
\]

(3.2)
where subscripts again denote levels. The choice of this embedding can be motivated by a supersymmetric extension of the coset construction using super Kac-Moody algebras. In particular, the so(2d) part corresponds to bosonized free fermions, and it is clear that the corresponding modular invariant should be chosen.

These models can be shown to have always $N = 1$ supersymmetry, where the supercurrent is given by

$$G^{(1)} = G^{g/h} = G^g - G^h = \frac{2}{k}(\kappa_{ab} : j^a \tilde{J}^b : - \frac{i}{3k} f_{abc} : j^a j^b j^c :).$$ (3.3)

Here $\kappa$ is the Killing form, $f$ are the structure constants of $g$. The bar over the indices indicates that the sum is only over elements in the orthogonal complement of $h$ in $g$ relative to the Killing form. $\tilde{J}$ are the purely bosonic currents and $j^a$ the fermionic currents transforming in the vector representation of so(2d).

One may now ask in which cases the symmetry algebra can be enlarged to an $N = 2$ algebra. To investigate this question we make the most general ansatz in terms of normal ordered products of fields for a spin $3/2$ current and plug it into the $N = 2$ algebra:

$$G^{(2)} = \frac{2}{k}(h_{ab} : j^a \tilde{J}^b : - \frac{i}{3k} S_{abc} : j^a j^b j^c :).$$ (3.4)

This leads to a system of algebraic equations for $h$ and $S$. This system was analyzed in [6] in the case of regular embeddings $h \rightarrow g$, and for special embeddings in [8]. The resulting classification can be summarized as follows: a subalgebra $h$ of $g$ yields an $N = 2$ supersymmetric coset of the form (3.2) if and only if the Dynkin diagram of $h$ can be obtained from the (non-extended) Dynkin diagram of $g$ by removing at least one node. The situation turns out to be particularly simple if the node corresponds to a cominimal weight, i.e. a weight with Coxeter label equal to one. One can easily show [9] that then the corresponding homogeneous space is hermitian symmetric. In fact most of the examples considered in the literature belong to this special subclass, in particular to the so-called projective cosets

$$\frac{\text{SU}(n+1)_k \oplus \text{SO}(2n)_1}{\text{SU}(n)_{k+1} \oplus U(1)}.$$ (3.5)

The $N = 2$ minimal models can be recovered from the projective cosets by setting $n = 1$.

At this point one may ask whether all rational $N = 2$ conformal field theories can be represented in the form (3.2). We cannot answer this question. Actually one representation of an $N = 2$ coset is known that is not of this form, namely at $c = 1$:

$$\frac{(A_1)_2 \oplus (A_1)_2}{(A_1)_2^P}.$$ (3.6)

But, as this model is a minimal model, it can alternatively also be described in the form (3.2).

In [9] all tensor products of $N = 2$ cosets that have central charge $c = 9$ and are thus suitable for the inner sector of a heterotic string compactification have been classified. To state the result of our classification we count cosets; as different modular invariants of the same coset yield different conformal field theories (and thus superstring vacua) the number of the latter...
is certainly higher. Unfortunately, it is unknown, as a classification of modular invariants of general WZW theories is still missing.

There are 168 tensor products of minimal models, 190 tensor products of projective cosets and minimal models, 123 tensor products involving other hermitian symmetric cosets [4] and 198 tensor products involving at least one coset that is not a hermitian symmetric coset [9]. Here some trivial group theoretical identifications have already been taken into account, such as $C_2 \equiv B_2$. Using non-trivial relations, such as the level rank dualities discussed below (compare section 4 and [10]) the number of distinct coset conformal field theories is reduced even further, e.g. for those involving non-hermitian symmetric cosets from 198 to 112.

4 Explicit Calculations in $N = 2$ coset theories

As should be clear from the preceding discussion, an $N = 2$ coset has to be given a sense as a conformal field theory, just like any other coset theory. Therefore first the $u(1)$ part has to be specified and the levels of the simple ideals $h_i$ of $h$. The latter can be shown to be $k_i = I_i(k + g^0) - h_i^\vee$, where $I_i$ is the index of the embedding $h_i \hookrightarrow g$.

Then group theoretical selection rules or, equivalently, the identification rules have to be determined. In [9] a simple formula for the number of identification currents was derived which is useful to check the completeness of a set of selection rules:

$$|G_{id}| = Q_{i_0} \prod I_C(h_i), \quad (4.1)$$

where $Q_{i_0}$ is the $u(1)$ charge of the missing simple root (whose node in the Dynkin diagram has been deleted) and $I_C$ is the index of connection, which is equal to the number of conjugacy classes.

Interesting quantities are the number $N_{27}$ of massless generations and $N_{27}$ of anti-generations of the heterotic string compactification which has a $c = 9$ tensor product of coset theories in its inner sector. For simplicity, we restrict ourselves to the diagonal modular invariant.

Several problems have to be overcome: firstly, in coset models, only the fractional part of the conformal weight can be easily obtained. The integer part can be obtained in principle from a character decomposition using the Weyl-Kac character formula, but this is extremely tedious and in practice hardly feasible. A way out is to work with Ramond ground states which, due to spectral flow, provide equivalent information. An index-like argument [11] shows that a complete set of representatives $\Phi_{\tilde{\Lambda}^x}^{A_{\lambda}}$ for all Ramond ground states is given by the formula

$$\tilde{\lambda} = w(\Lambda + \rho_g) - \rho_h, \quad (4.2)$$

where $w$ runs over all elements of the Weyl group of $g$ such that $\tilde{\lambda}$ is a highest weight of $h$, including an $u(1)$-weight. As this formula has been derived from the Weyl-Kac character formula, it automatically takes care of null states. For arbitrary states null states are a severe problem; this makes e.g. the determination of $E_6$ singlets very cumbersome in practice. In fact, their number has only been determined for tensor products of minimal models, where the
representation theory of the $N = 2$ algebra provides an independent powerful handle on null states.

Also, in general, the superconformal charge $q$ can be read off easily only mod 2; only for Ramond ground states the following formula \cite{12, 13, 9} holds:

$$q = \frac{d}{2} - l(w) - \frac{\xi_0 Q}{k + g^\vee}, \quad (4.3)$$

where $l(w)$ is the length of the Weyl group element given by (4) and $\xi_0$ some factor of proportionality. Using equation (4.3) it was shown \cite{9} that for any known $N = 2$ coset theory the set of Ramond ground states is symmetric under charge conjugation.

Once we know all Ramond ground states or, equivalently, due to chiral flow, the ring of chiral primary fields, we can in principle implement the method of $\beta$ vectors \cite{14}. In practice, this turns out to be rather inconvenient, but there is a different approach, which has the additional benefit to provide also the very important insight that the spectra do not depend on the details of the resolution procedure for the fixed points, contrary to what one might expect.

In \cite{15} modular invariance was used to express the Euler number $\chi = 2(N_{27} - N_{27})$ in terms of the Poincaré polynomial $P$:

$$\chi = \frac{1}{M} \sum_{r,s=0}^{M-1} P(e^{2\pi d(r,s)/M}). \quad (4.4)$$

Here $M$ is the smallest common denominator of all $u(1)$-charges in the chiral ring and $d(\cdot, \cdot)$ stands for the largest common divisor of two integers.

Fixed points of the identification group have to be carefully taken into account, of course. However, one can show that any orbit containing at least one representative of (4) yields after resolution exactly one Ramond ground state, no matter how long the orbit is. It is important to realize that this result holds irrespective of the details of the resolution procedure and that, as a consequence of (4.4), $\chi$ does not depend on these details either.

5 The extended Poincaré polynomial

The method of the extended Poincaré polynomial finally allows for a separate calculation of $N_{27}$ and $N_{27}$.

If we want to build out of a tensor product $C_9$ of $N = 2$ superconformal field theories with $c = 9$ a heterotic string theory, we have to perform several projections. We will sketch below how this can be described in terms of simple currents and explain the resulting prescription encoded in the ‘extended Poincaré polynomial’. In a second step we shall comment on the case $c = 3 + 6n$. This case is of much practical interest as we have to resort to it in some cases to remove ambiguities in the resolution of fixed points.

After splitting off the contribution of the bosonic space time coordinates and applying the bosonic string map \cite{16}, we can describe the heterotic string in a conformal field theory language as the tensor product

$$(D_5)_1 \oplus (E_8)_1 \oplus C_9. \quad (5.1)$$
The first two factors will provide for the right movers the gauge multiplet, for the left movers, they describe the contribution of the fermionic coordinate fields. As the only purpose of the $E_8$ factor is to provide a phase in the $S$-matrix such that the fermions are correctly reproduced, we will drop it in our discussion from now on.

First, to obtain supersymmetry on the world sheet, we have to align the boundary conditions in the various theories such that all fields are either Ramond or Neveu–Schwarz. Therefore, it is important to note that $T_F := \Phi_0^0$ is a spin $1/2$ simple current of order 2 that is present in any $N = 2$ coset theory. Its monodromy charge is 0 for fields in the Neveu–Schwarz sector and 1/2 in the Ramond sector.

Alignment is thus equivalent to enlarging the chiral algebra by all bilinears $T_F(i)T_F(j)$ (which have conformal dimension 3). In the $D_5$ part we set $T_F(0) := v$, which is, just like any other field of $(D_5)_1$, a simple current.

Space time supersymmetry requires the projection on even $\mathbb{Z}$ values of the $u(1)$ charges [14].

To implement this projection we note that the Ramond ground state $R_0$ with highest $u(1)$ charge is always a simple current with conformal dimension $h = \frac{c}{24}$. One can show [9] that there is always one representative of $R_0$ of the form $\Phi_0^0$; from this explicit form its monodromy charge can be easily seen to be half of the superconformal charge. The desired projection is thus equivalent to including the integer spin simple current $S_{tot} := (s, R_0)$ in the chiral algebra. Here $s$ is the spinor simple current of $(D_5)_1$. $S_{tot}$ has been termed spinor current in [4]. We will see below that its presence in the chiral algebra assures the existence of a space time gravitino in the corresponding heterotic string spectrum.

In a conformal field theory language a heterotic string theory thus amounts to a conformal field theory (5.1) with the modular invariant generated by the integer spin simple current $S_{tot}$ and all bilinear combinations $T_F(i)T_F(j)$.

It is now easy to recover the massless spectrum of the heterotic string. To obtain the proper interpretation we recall that in one chiral sector of the theory, e.g. for left movers, we have to apply the bosonic string map: the $D_5 \oplus E_8$ part is mapped on a $so(2)_1$ theory by interchanging vector and scalar and changing the sign of the spinor and conjugate spinor representation in the partition function. This map preserves the modular transformation properties and allows for a description of the fermionic coordinates of the string.

As we work in a purely bosonic description, fields are massless if $\Delta = \bar{\Delta} = 1$. Let us first explain how in this formulation the generic part of the string spectrum arises which provides the supergauge- and supergravity-multiplets. Two fields that occur in any $N = 2$ theory in the inner sector are the vacuum and the two Ramond ground states with highest and lowest $u(1)$-charge. The massless right moving fields that are tensored with the vacuum of the inner sector have $\Delta = 1$ and, due to the charge selection rule, $q = 0, \pm 2$. These conditions are fulfilled for the currents of $E_8 \oplus D_5$ and the transverse bosons. In the modular invariant described above these fields are paired with the following left movers: $(v, 0)$ what yields for the transverse bosons the graviton (as well as an antisymmetric tensor and the dilaton as the trace) and for the currents the gauge multiplets. Applying $S_{tot}$ in the left moving sector yields the superpartners of the gauge bosons and the graviton.

\footnote{Here we formulate the condition after applying the bosonic string map, what explains the difference to what the reader might expect, namely projection on odd values [14].}
In the right moving sector, we also find in the complete square of the identity the fields \( S_\text{tot} \) and \( S_\text{tot}^\dagger \), as well as the 0 of \( D_5 \) tensored with the \( u(1) \)-current of the \( N = 2 \) algebra. According to the well known branching of the adjoint representation of \( E_6 \) to the adjoint representation, the spinor, conjugate spinor and scalar of \( D_5 \), these fields extend the gauge symmetry from \( E_8 \oplus D_5 \) to \( E_8 \oplus E_6 \). In particular cases, if more fields are present, one can even further extend both the gauge symmetry for right movers and the supersymmetry for the left movers.

To explain how massless (anti-)generations transforming in the 27 resp. \( \overline{27} \) representations of \( E_6 \) arise, we remark that massless states that are vectors of \( D_5 \) have \( \Delta = \frac{1}{2} \) and \( q = \pm 1 \) in \( C_9 \), i.e. they are (anti-)chiral fields. Acting twice with \( S_\text{tot}^\dagger \) on the vector primary field with \( q = 1 \) yields a spinor tensored with a Ramond ground state and in a second step 0 tensored with an anti-chiral state with \( q = -2 \); these states combine in a 27 of \( E_6 \).

Starting with an anti-chiral field and applying \( S_\text{tot} \) instead we obtain states transforming in a \( \overline{27} \) of \( E_6 \). These states can be paired with spinors or conjugate spinors in the left moving sector; together they give rise to the generations and anti-generations and their CPT conjugates.

To extract information on the spectra we introduce the following notation: denote by \( h^{p,q} \) the number of fields which are in both the left and the right moving part of \( C_9 \) chiral primaries and have superconformal charge \( p \) resp. \( q \); \( p, q \) are integers smaller than \( d := c/3 \). These numbers can be seen as analogues to the Hodge numbers of a Calabi-Yau threefold. In fact, we find the usual symmetries: \( h^{p,q} = h^{q,p} \), as we started from a left right symmetric invariant, and \( h^{p,q} = h^{d-q, d-p} \), due to the conjugation symmetry on the chiral ring. Note that if the vacuum is not paired with any chiral primary field other than the unique chiral primary field with \( q = \frac{4}{3} \), we have \( h^{0,1} = h^{0,2} = 0 \); in the corresponding heterotic string compactification neither gauge symmetry nor space time supersymmetry is extended. As this is the most interesting case we will restrict ourselves to it from now on. The Euler number is given as usual \( \chi := \sum (-1)^{p+q} h^{p,q} \).

The discussion above shows that the number \( N_{27} \) of massless generations transforming in the 27 representation of \( E_6 \) is equal \( \sum \) to \( h^{1,1} \), or equivalently to the number of fields in the theory, which are in both sectors spinors of \( D_5 \) tensored with a Ramond ground state with superconformal charge \( -\frac{1}{2} \). The massless anti-generations \( N_{\overline{27}} \) transforming in the \( \overline{27} \) of \( E_6 \) can be correspondingly characterized by the fields which are spinors and Ramond ground state with charge \( -\frac{1}{2} \) in one sector and conjugate spinors and Ramond ground state with charge \( +\frac{1}{2} \) in the other sector.

We are thus interested in the structure of the relevant simple current orbits. Let us first look at the orbits of \( S_\text{tot} \) : as we are only interested in the massless spectrum we start with an arbitrary Ramond ground state \( (s, R^{(1)}, \ldots) \). Suppose now that, on the orbit, we encounter \( (J_{\epsilon_1^0 v}, T_{F(i)}^\epsilon R_i^\epsilon) \), where \( \epsilon_i \) is 0 or 1 . This state – which is massive unless all \( \epsilon_i \) vanish – is paired in the simple current invariant with the original state in the other sector of the theory. But the chiral algebra contains also all bilinears of the form \( (J_v, T_{F(i)}) \): we thus find within the same complete square of the partition function the corresponding massless state, for which all \( \epsilon_i \) vanish, too. If the \( D_5 \) part is a spinor this yields a generation; conjugate spinors correspond to anti-generations.

\(^6\)Our notation is different from the one used for Calabi-Yau manifolds: there the superconformal charge in both sectors is defined with a relative minus sign, so the number of generations corresponds to the Hodge number \( h^{1,-1} = h^{1,3} \) of the manifold.
The information on the orbit of $S_{\text{tot}}$ is very conveniently encoded in the extended Poincaré polynomial [4]. To start with, we define it on each factor of the tensor product separately. As any simple current has finite order, the orbit has some periodicity which we first factor out for convenience: for any Ramond ground state $R$ we define $N_R$ to be the smallest power of the spinor current such that $(S^{N_R} R)$ is equal to $R$ or $T_F R$. We define $\epsilon(R)$ to be $+1$ in the first and $-1$ in the second case. The extended Poincaré polynomial is now defined as:

$$P(t, x) = \sum_R \frac{t^q}{1 - \epsilon(R)x^{N_R}} \left[ \sum_{m \in F_+} x^m - \sum_{n \in F_-} x^n \right].$$

(5.2)

The sum is over all Ramond ground states $R$, $q$ is the superconformal charge of the chiral primary field connected via spectral flow. The sets $F_{\pm}$ are defined by the prescription: $m \in F_+$ iff $(S_{\text{tot}})^m R$ is a Ramond ground state and $n \in F_-$ iff $(S_{\text{tot}})^n R$ is $T_F$ applied to a Ramond ground state; in particular all $m, n$ are even. The extended Poincaré polynomial is not a polynomial in the new variable $x$, but rather a series with periodic coefficients. We remark that we recover the ordinary Poincaré polynomial as $P(t, 0)$.

We obtain the extended Poincaré polynomial for a tensor product by the following multiplication: given the extended Poincaré polynomials $P_i(t, x_i)$ of the factors, first perform the ordinary product of polynomials and then delete all terms in which the powers of the $x_i$ do not coincide. This procedure implements the simple observation that, in order to have a Ramond ground state of the tensor product, we need Ramond ground states in each factor of the theory.

The statements about the corresponding string compactification can be rephrased in terms of the extended Poincaré polynomial. First, note that our assumption that the symmetry is not enlarged translates into the requirement that the polynomial in $x$ multiplying $t^0$ is equal to $1 + x^2$. $N_{27}$ and $N_{\overline{27}}$ can be read off from the polynomial in $x$ multiplying $t^1$: let $p(x) = \sum a_m x^m$ denote one period of this series. As the action of any of the bilinears $(v, T_F(i))$ and of $(S_{\text{tot}})^2$ changes the conjugacy class in the $D_5$ theory we find generations if $a_m > 0$ and $m = 0 \bmod 4$ or $a_m < 0$ and $m = 2 \bmod 4$; the other cases correspond to anti-generations. Put differently we find $N_{27} + N_{\overline{27}} = \sum |a_m|$ and $N_{27} - N_{\overline{27}} = p(i)$.

The formalism of the extended Poincaré polynomial allows for an easy calculation [4, 9] of $N_{27}$ and $N_{\overline{27}}$ even if fixed points are present. Namely, after writing down those parts of the extended Poincaré polynomial which are not affected by fixed points and taking into account some evident structure of the fixed points we are left with only a few candidates for the extended Poincaré polynomials. As pointed out above, the ordinary Poincaré polynomial and hence the Euler number do not depend on the resolution procedure. For all possible extended Poincaré polynomials of a given theory we calculate $\chi$ from $N_{27}$ and $N_{\overline{27}}$. If we do not get the correct Euler number (4.4), we can exclude the candidate. This works surprisingly well. But, as the Hodge numbers turn out to be relatively robust against changes of the parameters left in the extended Poincaré polynomial, it is important to have many tensor products in which a given model appears in order to have enough consistency conditions to single out the true extended Poincaré polynomial.

This is our main motivation to generalize this formalism to tensor products of cosets with conformal charge $c = 3 + 6n$. Here in general, we have no string interpretation at hand, so we can replace the $D_5$ factor by some other $D_d$ factor. However, we have to require that the current
(s, R₀), with conformal weight d/8 + c/24 has integer spin. This fixes d to d = −2n − 1 mod 8. (We recover the previous situation for d = 5, n = 1.) It is important to note that, as the S matrices of D_d and D_{d+4} coincide, the choice of d does not affect the fusion rules. (Note however, that the T matrices coincide only for D_d and D_{d+24}.)

We now implement analogous projections, i.e. take the simple current invariant induced by all bilinears in the T_F(i) and (s, R₀), and obtain the extended Poincaré polynomial by exactly the same prescription as in the c = 9 case. Again massless states that are spinors or conjugate spinors in D_d are Ramond ground states of C_{3+6n}. The charge selection rule implies that states paired with spinors have superconformal charge q ≡ −1/2 mod 2 and for conjugate spinors +1/2 mod 2. The chiral primary fields connected via spectral flow have thus charge q ≡ n mod 2 for spinors resp. n + 1 for conjugate spinors. This shows that we can recover the Euler number from the polynomials multiplying all odd powers of t in the extended Poincaré polynomial and summing up all contributions. Comparing this result with the result of (4.4), which was derived in [15] for all c = 3 + 6n, we obtain new consistency conditions on the coefficients in the extended Poincaré polynomial that can arise in the resolution procedure. We remark that in general we can only read off \( \sum q^h_{p,q}(\pm 1) \) from the extended Poincaré polynomial; this is sufficient to determine all Hodge numbers separately only for n ≤ 1.

6 Old and New Level Rank Dualities

We have already seen (3.6) that distinct Lie algebraic cosets can describe the same conformal field theory. In fact, this phenomenon occurs rather frequently: in [10] the equivalence of four series of N = 2 coset models was established. As these identities arise by exchanging (a simple function of) the rank and the level of the algebras involved, they are commonly referred to as ‘level rank dualities’.

Based on well-known level rank dualities between WZW theories [17], which are however not isomorphisms between these theories, we were able to set up an isomorphism between the following N = 2 coset models (the notation is taken from [4] and [9]):

\[
\begin{align*}
(B, 2n + 1, 2k + 1) &\equiv (B, 2k + 1, 2n + 1) \\
(B, 2n, 2k + 1) &\equiv (B, 2k + 1, 2n),_D \\
(BB, m + 2, 1) &\equiv (CC, 2, 2m + 1) \\
(CC, n, k) &\equiv (CC, k + 1, n - 1).
\end{align*}
\]

The subscript D in the last line indicates that one must take the D-type modular invariant for these models rather than the diagonal one. The last two identities have been suggested in [7] because of the corresponding symmetry of the central charges. They have also been observed in the string spectra and in the extended Poincaré polynomials in [4].

In [10] the identity of these models was established by the construction of one-to-one correspondence \( \mathcal{T} \) between all primary fields (including a careful treatment of the fixed points). As an example, in the case of the last duality (6.4) which reads in the full notation

\[
\frac{(C_n)_k \oplus (D_{2n-1})_1}{(C_{n-1})_{k+1} \oplus u(1)}_{2(2k+n+1)} \equiv \frac{(C_{k+1})_{n-1} \oplus (D_{2k+1})_1}{(C_k)_n \oplus u(1)}_{2(2k+n+1)},
\]

12
\( \mathcal{T} \) is constructed by relating \((C_n)_k\) and \((C_k)_n\) resp. \((C_{n-1})_k+1\) and \((C_{k+1})_{n-1}\) by the level rank duality of WZW theories [17] and by a prescription for the map on the \(u(1)\) and \(D_d\) parts of the theory.

The mapping \( \mathcal{T} \) can be shown to preserve both ring structures present in these models: on the one hand, as \( \mathcal{T} \) preserves the \(S\)-matrix – and thus the fusion ring – and respects the conformal dimensions, it is an intertwiner for the whole modular group. On the other hand, the chiral ring structure and the superconformal charges are respected, too. Moreover, reasoning along the lines given in [18] it should not be too hard to show that also the branching functions coincide.

The arguments given in [10] show that the two respective theories are exactly identical as conformal field theories, and do not merely represent different points in the moduli space of one theory. In fact, changing the moduli generically also changes the conformal dimensions and even the fusion structure of the theory, as can be easily seen e.g. when looking at the situation at \(c = 1\). (An arbitrary marginal deformation of a rational conformal field theory does not even lead to rational conformal field theory.)

We expect that the techniques used in the cases above also allow for an explanation of the level rank duality in the \(A\)-series on the level of Hilbert spaces which was proven in [7] for the symmetry algebras.

## 7 Conclusions

One may, of course, extend the analysis presented above and also include non diagonal modular invariants. For many purposes a complete survey of the spectra occurring in this class of \(N = 2\) models would be helpful. One might get a better feeling of whether the old suspicion is true that all rational conformal field theories are related in some way to coset models, possibly including additional orbifoldizations.

It is also important to obtain more information about the massless spectrum of these theories, e.g. the number of \(E_6\) singlets. One may also ask whether the models presented in this talk admit a description as a Calabi-Yau manifold or via a Landau-Ginzburg potential. Finally it would be interesting to get some insight in whether all coset models with \(N = 2\) superconformal symmetry admit a description in the form (3.2).

**Acknowledgments:** The results presented in this talk are mostly the outcome of a very pleasant collaboration with Jürgen Fuchs. I would also like to thank M. Kreuzer, W. Lerche, A.N. Schellekens and M.G. Schmidt for stimulating and helpful discussions. Finally I would like to thank the organizers of the III Conference on Mathematical Physics for their efforts and for having given me the possibility to present these results. Financial support from the Studienstiftung des deutschen Volkes is gratefully acknowledged.
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