Meta Distribution of SIR in Ultra-Dense Networks with Bipartite Euclidean Matchings

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Abstract—Ultra-dense networks maximise spatial spectral efficiency through spatial reuse. The way in which their dense limit is approached is difficult to understand mathematically, because the Euclidean combinatorial theory, despite being simple to state, is often difficult to work with, and often involves a stochastic model. Small changes in the way these models are defined can make significant differences to the qualitative properties of their predictions, such as concerning the meta distribution of the SIR, the data capacity, or their efficiency. In this paper we focus on predictions, such as concerning the meta distribution of the SIR, which is apparently independent of the communication strategy chosen [2]. See the highly cited works of Gupta and Kumar, and the following groundbreaking work of Francescetti, Di Renzo, Andrews, Haenggi, Baccelli, and others [3]–[9]. In an abstract setting, various ideas from “Euclidean combinatorial optimisation” [10], a field where combinatorial structures in d-dimensional Euclidean space are optimised somehow, are therefore useful when theorising about this limit, and in other scenarios where spatially stochastic network models are used in wireless communications [11]–[18].

In this paper, and following on from previous work [19], [20], we consider this problem in two-dimensional (2d) Euclidean space without boundary, from the perspective of optimal transport, which is an important topic surrounding transportation theory, measure theory, analysis and operations research, providing procedures for efficiently moving a set of ‘resources’ to a set of ‘destinations’, i.e. from mines to silos. With n mines and m silos, one pairs mines with silos so that each mine is assigned a unique silo, and each silo a unique mine [21]. Also, Mézard and Parisi, observed early links between Euclidean matching theory, and the physics of spin glass, a magnetic alloy important in condensed matter physics and complexity [22].

Consider user equipments (UEs) and base stations (BSs). Two binomial point processes in the Euclidean plane are paired-off in an edge-independent way known as a Euclidean matching, in particular a bipartite one, shown in Fig. 1 as studied by G. Sicuro [23], and we ask what optimal transport looks like in this setting. What is the maximum data rate in bits per second which the UE can pass to the BS [19]? This scenario is also relevant for machine-to-machine communication and sensor networks, where a massive number of devices form communication pairs, changing over time, aiming at distributing relevant data and measurement info over the full network. An upper bound on the maximum data rate concerns a functional defined on the space of bipartite matchings known as “transport plans”, which is minimised given the constraints each interferer places on its neighbouring transmitters. Certain transport plans will lead to some transmitter-receiver (Tx-Rx) pairs failing to meet an Signal-to-Interference Ratio (SIR) threshold with given reliability, so the introduction of the meta-distribution of the SIR is important in this setting. This then leads to the studies of spatial capacity [24]–[26]. We discuss the meta-distribution of the SIR in Section II.

Mathematically, the reliability is defined as the conditional probability $P_s(\theta) = \mathbb{P}(\text{SIR} > \theta | \Phi)$, where $\theta$ is the operation threshold, and $\Phi$, in our case, corresponds to the point process of transmitters and receivers in a bipartite matching. Assuming unit-mean exponential random variables for the power fading and Bernoulli variates with parameter $\xi$ for the activity of each interferer, the reliability can be read as [5] Appendix

$$P_s(\theta) = \prod_k \left( 1 - \xi + \frac{\xi}{1 + \theta r^\eta x_k^{-\eta}} \right),$$

where $r$ is the distance of the transmitter-receiver link whose reliability is computed, and the product is over the locations $x_k$ of interferers conditioned on the realization of $\Phi$.

Various Euclidean models have been studied from this perspective. For example, the bipolar model, where $N$ transmitter per unit volumes are distributed randomly in the infinite plane,
and then each is assigned a receiver at a fixed distance in a uniformly random direction, has a certain fraction of Tx-Rx pairs who can achieve an SIR greater than \( \theta \) with reliability greater than \( u \in [0, 1] \), while others fall below. The bipartite matching of Euclidean points is distinct since it incorporates Euclidean correlations, and variable link distances which shorten with \( N \to \infty \). This has subtle but, as we show, very important effects on the statistics of the meta-distribution. The details of this follow in Sections II and III.

In Section II, we discuss the meta-distribution of the SIR. In Section III, we introduce the bipartite Euclidean matching, which is the unique setting of our results, and we derive the moments of the meta-distribution of the SIR for it. While doing so, we compare the performance accuracy of various link distance models, including the popular bipolar model with nearest-neighbour communication. Finally, in Section IV, we discuss these results in light of spin glass and communication theory, and conclude.

II. META DISTRIBUTION

The classical analysis of wireless networks using stochastic geometry assumes suitable distributions for different parts of the system and averages the performance indicator (outage probability, data rate, etc.) over the ensemble of possible network states [27]. In equation (1), we have already averaged over the distributions of the fading channel and the activity of interferers. If we average over the point process of interferers too, the result might not represent accurately the reliability of individual links in the network, especially when the standard deviation of the Random Variable (RV) in (1) is large. Because of that, we may calculate the distribution (or at least the moments) of the RV in (1) instead [28].

The complementary Cumulative Distribution Function (CDF) of the RV \( P_s(\theta) \), i.e., \( P(P_s(\theta) > u), u \in (0, 1) \) is usually referred to as the meta-distribution of the SIR. Given the operation threshold \( \theta \), this metric gives the percentage of random spatial realizations where the typical receiver meets the SIR target \( \theta \) with reliability (probability calculated over fading and activity, see equation (1)) at least \( u \in (0, 1) \). For ergodic point processes, this percentage is equal to the fraction of links achieving a reliability higher than \( u \) for each spatial realization (fixed but unknown locations).

Thus far, the moments of the meta-distribution of the SIR, \( M_b(\theta) = \mathbb{E}[P_s(\theta)^b] \), have been investigated for Poisson bipolar, cellular and heterogeneous wireless networks using the Probability Generating Functional (PGFL) of Poisson Point Process (PPP) [5], [6]. The meta-distribution has also allowed to go beyond classical mean-field models [29], and maximize the actual density of active links in Poisson bipolar networks meeting a reliability constraint [24].

Assuming that the interferers follow a PPP of intensity \( \lambda \), the integer moments of the meta-distribution can be calculated by raising (1) to the \( b \)-th power and expressing the spatial average using the PGFL.

\[
M_b(\theta) = \exp \left( -2\pi \xi \int_0^\infty \left( 1 - \left( 1 - \frac{\xi \theta^\eta}{x^\eta + \theta^\eta} \right)^b \right) x \, dx \right),
\]

Next, we concentrate on the distribution of the link distance \( r \) in Euclidean bipartite matchings.

III. SYSTEM MODEL AND META DISTRIBUTION

With \( \lambda > 0 \) and \( N \sim \text{Poisson}(\lambda) \), consider the Poisson point process (PPP) \( X_N \subset [0,1]^d \) of \( N \) points, and also the PPP \( Y_N \subset [0,1]^d \), also of \( N \) points, drawn from the hypercube with periodic boundary conditions, which is equivalent to e.g. a torus when \( d = 2 \), with \( 2N \) points distributed uniformly at random over its (flat) surface. Form a perfect bipartite matching \( M_N \) of these points by assigning \( N \) of the pairs with one end in each of \( X \) and \( Y \) ‘active’ in such a way that every point is incident to exactly one active pair.

Call the Euclidean lengths of these edges \( d_1, d_2, \ldots, d_N \). We reserve \( d \) for the dimension of the hypercube. For \( C, \eta > 0 \), in [28], one assigns each edge its own data capacity \( C_i = \log_2 (1 + d_i^{-\eta}) \) based on the Shannon-Hartley theorem, since if the received signal power is well modelled by \( P_i = C d_i^{-\eta} \), taking \( \eta \) for the path loss exponent, then such a function of the edge length corresponds to the theoretical upper bound on the data capacity of a link over distance \( d_i \), given the bandwidth and noise are set to unity. For each matching, we therefore have a length \( L_M = \sum_i d_i \) and a capacity \( C_M = \sum_{i=1}^{N} \log_2 (1 + d_i^{-\eta}) \). The perfect matching which minimises \( L_M \) is the shortest perfect matching, or just shortest matching. In this paper, we consider a perfect bipartite matching \( M_N \) between the two sets minimizing the sum of Euclidean distances \( d_1, \ldots, d_N \). We seek the proportion of links in the matching, which are able to achieve an SIR greater than \( \theta \) with probability at least \( u \). Instead of
analyzing the spatial performance of wireless ad hoc networks using a simple bipolar model as in [5], [28], we use the theory of Euclidean matchings for the geometry of the links in the network with the meta-distribution of SIR for its performance.

We briefly note that the bipolar matchings produced here are via the algorithm of Duff and Koster used in the MatLab (R2019b) ‘matchpairs’ toolbox [30]. They are often only close to optimal, due to the tractability of obtaining the unique shortest assignment.

Let us assume a realization of both $X$ and $Y$ of intensity $\lambda > 0$, modeling the set of transmitters in a wireless ad hoc network deployed in the unit square $[0, 1]^2$. Given the number $N$ of transmitters, we deploy uniformly at random $(N-1)$ receivers, and in addition to that we place a receiver at the center of the square. Under periodic boundary conditions, all receivers in the network experience the same interference field. In addition, we assume that the point process of transmitters and receivers $\Phi$ is ergodic. This is not as straightforward to prove as in Poisson bipolar and cellular networks in [5], [28], because nearby links experience correlated distances in a Euclidean matching. Under the ergodicity assumption (whose proof is deferred to a future study), the receiver at the origin is hereafter referred to as the typical receiver, and the meta-distribution of the SIR at the typical receiver will correspond to the fraction of links in the matching experiencing certain reliability.

In the performance evaluation of wireless ad hoc networks, the bipolar model assumes a fixed and known link distance $r = R$ [5], [28]. Under the assumption that the nearest interferer can come arbitrarily close to the receiver, the moments of the meta-distribution of the SIR follow from equation (2).

$$M_b(\theta) = \exp \left( -2\lambda \pi \sum_{k=1}^{b} (-1)^{k+1} \left( \frac{b}{k} \zeta \int_0^{\infty} x (\theta R^2) \frac{dx}{(\theta R^2+x)^{\frac{k}{2}}} \right) \right)$$

$$= \exp \left( -2\lambda \pi \sum_{k=1}^{b} (-1)^{k+1} \left( \frac{b}{k} \frac{R^2 \Gamma(\delta) \Gamma(k-\delta)}{\theta^{-\delta} \zeta^{-k} \eta \Gamma(k)} \right) \right) \tag{3}$$

where $\delta = \frac{1}{2}$. Note that the above equation is equivalent to Eq. (5) using the diversity polynomials.

In the literature, the link distance $R$ is assigned a value either independently of the intensity of interferers $\lambda$ or proportional to the nearest-neighbor distance $R \propto \lambda^{-1/2}$ [31, Eq. (3.29)]. The latter agrees with the intuition, i.e., it takes into account the fact that for increasing density of users, a higher intensity of interferers should come along with shorter distances for the transmitter-receiver link. It is easy to see from (3) that the average success probability for nearest-neighbor communication becomes independent of the intensity $\lambda$. However, the scaling $\lambda^{-1/2}$ ignores the correlations of link distances inherent in a Euclidean matching.

We note that for $\lambda \to \infty$, which describes the case of ultra-dense networks, the sum of link distances in bipartite Euclidean matchings in 2d space scales as $\sqrt{\lambda \log \lambda}$ [32]. Recently, the pre-constant of the leading-order term has been also derived, showing that the scaling is proportional to $\sqrt{\frac{\lambda \log \lambda}{2\pi}}$ [33, 34].

Equ. (6)]. Therefore we may assume that the mean link distance in a bipartite Euclidean matching scales as $\sqrt{\log \lambda / 2\pi \lambda}$. One may now substitute this value instead of $R$ in (3) and proceed with the calculation of the moments. For instance, the first moment takes the following simple form:

$$M_1 = \frac{1}{\lambda^2}, \quad \lambda \to \infty, \quad c = \frac{\pi \xi \beta}{\eta} \sqrt{\frac{2\pi}{\eta}}. \tag{4}$$

Equation (4) can be seen as a complementary result to [31, Eq. (3.29)]. It indicates that the average success probability decreases for denser bipolar networks, while all other parameters remain fixed. Note that the new bipolar model assumes a fixed and known link distance, which is equal to the leading-order term of the mean link distance in a bipartite Euclidean matchings. As such it takes into account some of the correlation of link distances across the network through their mean. On the other hand, the classical bipolar model in [5], [28] specifies the link distance assuming that the link distances of Tx-Rx pairs are independent.

The new bipolar model in (4) may offer an improved estimate for the link distance in a bipartite network, but it still neglects the fact that the link distances in the matching would naturally follow a distribution. Another popular distance model in wireless communications research is the Rayleigh distribution, inspired by the void probability of PPP. In our system setup, we will consider the distribution $f(r) = \frac{r^2}{\lambda \pi} \exp(-\frac{r^2}{\lambda \pi})$, which has a mean equal to $\sqrt{\frac{\log \lambda}{2\pi \lambda}}$. The moments of the meta-distribution are calculated after substituting $r$ instead of $R$ in (3) and averaging over the $f(r)$ above yielding

$$M_b(\theta) = \frac{\pi^2 \lambda}{\pi^2 \lambda + 2 \log(\lambda) \cdot C_b(\theta)}, \tag{5}$$

where $C_b(\theta) = 2\pi \lambda \sum_{k=1}^{b} (-1)^{k+1} \left( \frac{b}{k} \frac{\Gamma(\delta) \Gamma(k-\delta)}{\pi \nu \zeta^{-k} \eta \Gamma(k)} \right) \Gamma(\delta) \Gamma(k-\delta)$.

Note that the Rayleigh distribution for the link distances is commonly associated with the performance evaluation of downlink cellular systems with nearest base station association. For a density $\lambda$ of base stations and the typical receiver at the origin, the link distance follows the Rayleigh distribution, inspired by the void probability of PPP. In our system setup, we will consider the distribution $f(r) = \frac{r^2}{\lambda \pi} \exp(-\frac{r^2}{\lambda \pi})$, which has a mean equal to $\sqrt{\frac{\log \lambda}{2\pi \lambda}}$. In addition, the nearest interfering base station must be located further than the serving base station. This is taken into account by setting the lower integration limit in the first line of (3) equal to $r$ instead of zero. The expression in (5) corresponds to a different system, i.e., the Rayleigh distribution has a different mean to reflect bipolar matchings in ultra-dense ad hoc networks instead of cellular systems, and the nearest interferer can also come closer to the receiver than its associated transmitter.

In bipartite Euclidean matchings the distance distribution for short links behaves linearly [22, Eq. (7)]. As a result, a good candidate model might also be the Gamma distribution $f(r) = \frac{r^{\nu-1} \exp(-r/\nu)}{\Gamma(\nu)}$ with scale $\nu = 2$. The shape $\beta$ can be set to match the mean of the link distances, i.e., $\beta = \sqrt{\frac{\log \lambda}{2\pi \lambda}}$. In Fig. 2 we see that the Gamma distribution provides better fit to the
CCDF

CDF

the complementary error function.

\[ \tau \]

with rate \( f \) and averaging over an exponential distribution with link distance \( \lambda \) with Poisson interferers of intensity \( \lambda \), is calculated by substituting \( r \) in bipartite matchings. Clearly, both bipolar models are unable to reflect accurately the distribution of link gains in the network.

For the exponential distance model, the moments of the meta-distribution are calculated by substituting \( r \) instead of \( R \) in (3) and averaging over an exponential distribution \( f(r) = \tau e^{-\tau r} \) with rate \( \tau = \frac{1}{2 \pi \lambda} \).

\[
M_b(\theta) = e^{\frac{\theta^2}{2C_b}} \text{erfc} \left( \frac{\tau}{2\sqrt{C_b}} \right) \sqrt{\frac{\pi \tau^2}{4C_b}},
\]

where \( C_b(\theta) \equiv C_b \) for brevity and \( \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} \, dt \) is the complementary error function.

For the Gamma model, we average (3) over the distribution \( f(r) = \frac{1}{\beta^\lambda} e^{-\frac{r}{\beta}} \), where \( \beta = \frac{\log \lambda}{2\pi \lambda} \), yielding

\[
M_b(\theta) = \frac{1}{2\beta^2 C_b} - \frac{\sqrt{\pi \beta^2 C_b}}{4\beta^3 C_b^{3/2}} e^{\frac{1}{2\beta^2 C_b}} \text{erfc} \left( \frac{1}{2\beta \sqrt{C_b}} \right).
\]

In Fig. 3 we approximate the simulated meta-distribution using the various link distance models. In order to do that we calculate the first two moments for each model, see (3)–(7), and we map them to Beta distributions. We have checked that the Beta approximations provide a very good fit to the simulated meta-distributions for all the models. Note that due to the simplicity of (3)–(7) higher moments can be easily computed and used to numerically invert the characteristic function of the meta-distributions [5, Eq: (12)]. We see in Fig. 3 that the performance of the bipolar model is poor over the full range of the distribution and is clearly unsuitable to model the SIR in bipartite Euclidean matchings. The other models can capture the trend of the meta-distribution, with the Gamma model giving almost a perfect fit.

Let us assume that the network density \( \lambda \) increases while the product \( \lambda \xi \) is kept fixed. For a bipolar model with link distance \( R \) independent of the intensity as in [5], the average success probability should not change. By visual inspection of Fig. 5 and Fig. 4 we see that for bipartite Euclidean matchings the average success probability increases for denser networks with fixed \( \lambda \xi \). This is because the useful link distance (on average) decreases, while the mean interference level remains the same. Another remark is that denser networks augment the difference between the traditional and the new bipolar model. Finally, in Fig. 5 we plot the outage probability for various thresholds \( \theta \). The Gamma model still outperforms all others. The bipolar model might be accurate at low thresholds, however, its prediction for the variance of the meta-distribution remains poor, see for instance Fig. 4.
IV. DISCUSSION AND CONCLUSIONS

The incorporation of a more sophisticated Euclidean model with variable link distance has been studied, and show to be significantly different from a simple model effectively trying to capture the same Tx-Rx pairing scenario. The careful introduction of Euclidean combinatorial theory into ultra-dense networks is crucial, as we see with such a large deviation in the qualitative properties of the meta-distribution statistics between the bipolar model and the bipartite Euclidean matchings.

A central development is to move beyond a random link assumption, incorporating the recent developments of Sicuro et al. in the theory of Euclidean matchings. In this paper, we have used just the leading-order term for the mean link distance in bipartite Euclidean matchings. Incorporating higher-order moments in our analysis would allow us to devise more accurate models for the performance of the network. Also, working on more general curved spaces may help understand the effect of geometry better. Nevertheless, the meeting of spin glass physics, and dense communication network theory, has significant potential for both fields.

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