Time-period flow of a viscous liquid past a body

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Time-periodic solutions to the Navier-Stokes equations that govern the flow of a viscous liquid past a three-dimensional body moving with a time-periodic velocity are investigated. The net motion of the body over a full time-period is assumed to be non-zero. In this case the appropriate linearization of the system is constituted by the time-periodic Oseen equations in a three-dimensional exterior domain. A priori $L^q$ estimates are established for this linearization. Based on these estimates, existence of a solution to the fully non-linear Navier-Stokes problem is obtained by the contraction mapping principle.

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1 Introduction

Consider a three-dimensional body moving with a prescribed time-periodic velocity $v_b(t)$ in a viscous liquid governed by the Navier-Stokes equations. If the body occupies a

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bounded, simply connected domain $B \subset \mathbb{R}^3$, the liquid flowing past it occupies the corresponding exterior domain $\Omega := \mathbb{R}^3 \setminus B$. The motion of the liquid can then be described, in a coordinate system attached to the body, by the following system of equations:

\[
\begin{cases}
\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f \quad \text{in } \mathbb{R} \times \Omega, \\
\text{div } u = 0 \quad \text{in } \mathbb{R} \times \Omega, \\
uu u = u^* \quad \text{on } \mathbb{R} \times \partial \Omega, \\
\lim_{|x| \to \infty} u(t, x) = -v_b(t).
\end{cases}
\]  

(1.1)

Here, $u : \mathbb{R} \times \Omega \to \mathbb{R}^3$ denotes the Eulerian velocity field and $p : \mathbb{R} \times \Omega \to \mathbb{R}$ the pressure field of the liquid. As is natural for time-periodic problems, the time axis is taken to be the whole of $\mathbb{R}$, and so $(t, x) \in \mathbb{R} \times \Omega$ denotes the time variable $t$ and spatial variable $x$ of the system, respectively. A body force $f : \mathbb{R} \times \Omega \to \mathbb{R}^3$ and velocity distribution $u^* : \mathbb{R} \times \partial \Omega \to \mathbb{R}^3$ of the liquid on the surface of the body have been included. The constant coefficient of kinematic viscosity of the liquid is denoted by $\nu$. An investigation of time-periodic solutions, that is, solutions $(u, p)$ satisfying for some fixed $T > 0$

\[
uu u(t + T, x) = u(t, x), \quad p(t + T, x) = p(t, x),
\]  

(1.2)

corresponding to time-periodic data of the same period,

\[
uu v_b(t + T) = v_b(t), \quad f(t + T, x) = f(t, x), \quad u^*(t + T, x) = u^*(t, x),
\]  

(1.3)

will be carried out.

We assume the net motion of the body over a full time-period is non-zero, that is,

\[
uu \int_0^T v_b(t) \, dt \neq 0.
\]  

(1.4)

We shall not treat the case of a vanishing net motion. The distinction between the two cases is justified by the physics of the problem. In the former case, the body performs a nonzero translatory motion, which induces a wake in the region behind it. In the latter case, the motion of the body would be purely oscillatory without a wake. The different properties of the solutions in the two cases also influence the mathematical analysis of the problem. If the net motion of the body has a nonzero translatory component, the appropriate linearization of (1.1) is a time-periodic Oseen system. If the net motion over a period is zero, the linearization is a time-periodic Stokes system. The investigation in this paper is based on suitable $L^q$ estimates for solutions to the time-periodic Oseen system. Similar estimates do not hold for the corresponding Stokes system, in which case a different approach is needed. It will further be assumed that the motion of the body is directed along a single axis, say

\[
uu v_b(t) = u_\infty(t) e_1, \quad u_\infty(t) \in \mathbb{R}.
\]  

(1.5)
This assumption is made for technical reasons only.

We shall in Theorem 6.1 establish existence of a solution to (1.1) for data \( f, v_b \) and \( u_s \) sufficiently restricted in “size”. The solution is strong both in the sense of local regularity and global summability. The proof is based on the contraction mapping principle and suitable \( L^q \) estimates of solutions to a linearization of (1.1). More specifically, we linearize (1.1) around \( v_b \) and obtain, due to (1.4), a time-periodic Oseen system. In Theorem 5.1 and Corollary 5.5 we identify a time-periodic Sobolev-type space that is mapped homeomorphically onto a time-periodic \( L^p \) space by the Oseen operator. We then employ embedding properties of the Sobolev-type space to show existence of a solution to the fully nonlinear problem (1.1) by a fixed-point argument. A similar result was obtained for a two-dimensional exterior domain in [5].

The study of time-periodic solutions to the Navier-Stokes equations was originally suggested by Serrin [27]. The first rigorous investigations of the classical time-periodic Navier-Stokes problem in bounded domains are due to Prodi [25], Yudovich [35] and Prouse [26]. Further properties and extensions to other types of domains and problems have been studied by a number of authors over the years: [13], [30], [23], [22], [32], [19], [20], [21], [14], [34], [2], [6], [7], [31], [33], [28], [4], [16], [17], [18] [24], [8]. Of these articles, [6, 7, 17, 16, 4, 18] treat the same type of flow past a body (1.1) that is investigated in the following. While weak solutions to an even more general problem are established in [6, 7], the corresponding whole-space problem in dimension two and three is studied in [4, 16, 17, 18].

2 Notation

Constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear. The notation \( C(\xi) \) is used to emphasize the dependence of a constant on a parameter \( \xi \).

The notation \( B_R \) is used to denote balls in \( \mathbb{R}^n \) centered at 0 with radius \( R > 0 \).

The symbol \( \Omega \) denotes an exterior domain of \( \mathbb{R}^n \), that is, an open connected set that is the complement of the closure of a simply connected bounded domain \( B \subset \mathbb{R}^n \). Without loss of generality, it is assumed that \( 0 \in B \). Two constants \( R_0 > R_* > 0 \) with \( B \subset \subset B_{R_*} \) remain fixed. Moreover, the domains \( \Omega_R := \Omega \cap B_R, \Omega_{R_1,R_2} := \Omega \cap B_{R_2} \setminus B_{R_1} \) and \( \Omega^R := \Omega \setminus B_R \) are introduced.

Points in \( \mathbb{R} \times \mathbb{R}^n \) are denoted by \((t,x)\). Throughout, \( t \) is referred to as the time and \( x \) as the spatial variable. For a sufficiently regular function \( u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \), \( \partial_t u := \partial_x u \) denotes spatial derivatives.

3 Preliminaries

A framework shall be employed based on the function space of smooth and compactly supported \( T \)-time-periodic functions:

\[
C_{0,\text{per}}^\infty (\mathbb{R} \times \overline{\Omega}) := \{ f \in C^\infty (\mathbb{R} \times \Omega) \mid f(t+T, x) = f(t, x) \ \& \ f \in C_0^\infty ([0, T] \times \overline{\Omega}) \}.
\]
For simplicity, the interval \((0, T)\) of one time period is sometimes denoted by \(T\). An \(L^q\) norm on the time-space domain \(\mathbb{T} \times \Omega\) is defined by

\[
\|f\|_q := \|f\|_{q, \mathbb{T} \times \Omega} := \left( \frac{1}{T} \int_0^T \int_\Omega |f(t, x)|^q \, dx \, dt \right)^{\frac{1}{q}}, \quad q \in [1, \infty).
\]

Lebesgue spaces of time-periodic functions are defined by

\[
L^q_{\text{per}}(\mathbb{R} \times \Omega) := C^\infty_{0,\text{per}}(\mathbb{R} \times \overline{\Omega}) \| \|_q.
\]

It is easy to see that the elements of \(L^q_{\text{per}}(\mathbb{R} \times \Omega)\) coincide with the \(T\)-time-periodic extension of functions in \(L^q((0, T) \times \Omega)\). The Lebesgue space \(L^q(\Omega)\) is treated as the subspace of functions in \(L^q_{\text{per}}(\mathbb{R} \times \Omega)\) that are time-independent. For such functions the \(L^q(\Omega)\) norm coincides with the norm \(\|f\|_q\) introduced above.

Sobolev spaces of \(T\)-time-periodic functions are also introduced as completions of \(C^\infty_{0,\text{per}}(\mathbb{R} \times \Omega)\) in appropriate norms:

\[
W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) := C^\infty_{0,\text{per}}(\mathbb{R} \times \overline{\Omega}) \| \|_{1,2,q}, \quad \|u\|_{1,2,q} := \left( \sum_{|\beta| \leq 1} \|\partial^\beta_t u\|_q^q + \sum_{|\alpha| \leq 2} \|\partial^\alpha_x u\|_q^q \right)^{\frac{1}{q}}.
\]

These Sobolev spaces are clearly subspaces of the classical anisotropic Sobolev spaces \(W^{1,2,q}((0, T) \times \Omega)\). Analogously, homogeneous Sobolev spaces of time-periodic functions

\[
D^{1,q}_{\text{per}}(\mathbb{R} \times \Omega) := C^\infty_{0,\text{per}}(\mathbb{R} \times \overline{\Omega}) \langle \langle 1, q \rangle \rangle, \quad (p)_{1,q} := \|\nabla p\|_q + \frac{1}{T} \int_0^T \int_{\Omega \cap \{0\}} |p(t,x)| \, dx \, dt
\]

are defined. It is easy to see that \(D^{1,q}_{\text{per}}(\mathbb{R} \times \Omega)\) can be identified with \(L^q((0, T); D^{1,q}(\Omega))\), where \(D^{1,q}(\Omega)\) is the classical homogeneous Sobolev space.

In a similar manner, Lebesgue and Sobolev spaces of time-periodic vector-valued functions are defined for any Banach space \(X\) respectively as

\[
L^q_{\text{per}}(\mathbb{R}; X) := C^\infty_{\text{per}}(\mathbb{R}; X) \| \|_{L^q((0,T), X)}, \quad W^{m,q}_{\text{per}}(\mathbb{R}; X) := C^\infty_{\text{per}}(\mathbb{R}; X) \| \|_{W^{m,q}((0,T), X)}.
\]

One readily verifies that \(L^q_{\text{per}}(\mathbb{R}; X)\) coincides with the \(T\)-periodic extensions of functions in the Lebesgue space \(L^q_{\text{per}}((0, T); X)\). One may further verify for sufficiently regular domains, say \(\Omega\) of class \(C^1\), that

\[
W^{1,2,q}_{\text{per}}(\mathbb{R} \times \Omega) = W^{1,2,q}_{\text{per}}(\mathbb{R}; L^q(\Omega)) \cap L^q_{\text{per}}(\mathbb{R}; W^{2,q}(\Omega)) = \{ u \in L^q_{\text{per}}(\mathbb{R} \times \Omega) \ | \ \|u\|_{1,2,q} < \infty \}.
\]

Sufficiently regular \(T\)-time-periodic functions \(u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}\) can be decomposed into what will be referred to as a \textit{steady-state} part \(Pu : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}\) and \textit{oscillatory} part...
\[ \mathcal{P}_\perp u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \text{ by} \]
\[ \mathcal{P} u(t, x) := \frac{1}{T} \int_0^T u(x, s) \, ds \quad \text{and} \quad \mathcal{P}_\perp u(t, x) := u(t, x) - \mathcal{P} u(t, x) \]  
whenever these expressions are well-defined. Note that the steady-state part of a \( T \)-time-periodic function \( u \) is time-independent, and the oscillatory part \( \mathcal{P}_\perp u \) has vanishing time-average over the period. Also note that \( \mathcal{P} \) and \( \mathcal{P}_\perp \) are complementary projections, that is, \( \mathcal{P}^2 = \mathcal{P} \) and \( \mathcal{P}_\perp = \text{Id} - \mathcal{P} \). Based on these projections, the following sub-spaces are defined:

\[ C^\infty_{0, \perp, \perp}(\mathbb{R} \times \Omega) := \{ f \in C^{\infty}_{0, \perp, \perp}(\mathbb{R} \times \Omega) \mid \mathcal{P} f = 0 \}, \]
\[ L^q_{\perp, \perp}(\mathbb{R} \times \Omega) := \{ f \in L^q_{\perp, \perp}(\mathbb{R} \times \Omega) \mid \mathcal{P} f = 0 \}, \]
\[ W^{1,2,q}_{\perp, \perp}(\mathbb{R} \times \Omega) := \{ u \in W^{1,2,q}_{\perp, \perp}(\mathbb{R} \times \Omega) \mid \mathcal{P} u = 0 \}. \]

Intersections of these spaces are denoted by

\[ L^{q,r}_{\perp, \perp}(\mathbb{R} \times \Omega) := L^q_{\perp, \perp}(\mathbb{R} \times \Omega) \cap L^r_{\perp, \perp}(\mathbb{R} \times \Omega), \]
\[ W^{1,2,q,r}_{\perp, \perp}(\mathbb{R} \times \Omega) := W^{1,2,q}_{\perp, \perp}(\mathbb{R} \times \Omega) \cap W^{1,2,r}_{\perp, \perp}(\mathbb{R} \times \Omega) \]
and equipped with the canonical norms. Similar subspaces of homogeneous Sobolev spaces are defined by

\[ D^{1,q}_{\perp, \perp}(\mathbb{R} \times \Omega) := \left\{ p \in D^1_{\perp, \perp}(\mathbb{R} \times \Omega) \mid \mathcal{P} p = 0 \right\}, \]
\[ D^{1,q,r}_{\perp, \perp}(\mathbb{R} \times \Omega) := D^{1,q}_{\perp, \perp}(\mathbb{R} \times \Omega) \cap D^{1,r}_{\perp, \perp}(\mathbb{R} \times \Omega), \]
\[ D^{1,q}_{\perp, \perp, R_0}(\mathbb{R} \times \Omega) := \left\{ p \in D^1_{\perp, \perp}(\mathbb{R} \times \Omega) \mid \mathcal{P} u = 0 \land \int_{\Omega_{R_0}} p(t, x) \, dx = 0 \right\}, \]
\[ D^{1,q,r}_{\perp, \perp, R_0}(\mathbb{R} \times \Omega) := D^{1,q}_{\perp, \perp, R_0}(\mathbb{R} \times \Omega) \cap D^{1,r}_{\perp, \perp, R_0}(\mathbb{R} \times \Omega). \]

All spaces above are clearly Banach spaces.

Finally, we introduce for \( \lambda > 0 \) and \( q \in (1, 2) \) the Sobolev-type space

\[ X^q_\lambda(\Omega) := \{ v \in L^q_{\text{loc}}(\Omega) \mid \| v \|_{X^q_\lambda} < \infty \}, \]
\[ \| v \|_{X^q_\lambda} := \lambda^{\frac{1}{2}} \| v \|_{L^q} + \lambda^{\frac{1}{2}} \| \nabla v \|_{L^q} + \lambda \| \partial_\nu v \|_q + \| \nabla^2 v \|_q , \]
which is used to characterize the velocity field of a steady-state Oseen system in three-dimensional exterior domains. An appropriate function space for the corresponding pressure term is the homogeneous Sobolev space

\[ D_{R_0}^{1,q}(\Omega) := \left\{ v \in D^{1,q}(\Omega) \mid \int_{B_{R_0}} p \, dx = 0 \right\} \]
equipped with the norm \( \langle \cdot \rangle_{1,q} := \| \nabla v \|_q \). As mentioned above, \( D^{1,q}(\Omega) \) denotes the classical homogeneous Sobolev space.
4 An Embedding Theorem

Embedding properties of the Sobolev spaces of time-periodic functions defined in the previous section shall be established. As such properties may be of use in other applications as well, we consider in this section an exterior domain $\Omega \subset \mathbb{R}^n$ of arbitrary dimension $n \geq 2$.

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be an exterior domain of class $C^1$ and $q \in (1, \infty)$. Assume that $\alpha \in [0, 2]$ and $p_0, r_0 \in [q, \infty]$ satisfy

\[
\begin{align*}
\begin{cases}
r_0 \leq \frac{2q}{2 - \alpha q} & \text{if } \alpha q < 2, \\
r_0 < \infty & \text{if } \alpha q = 2, \\
r_0 \leq \infty & \text{if } \alpha q > 2,
\end{cases}
\quad
\begin{cases}
p_0 \leq \frac{nq}{n - (2 - \alpha)q} & \text{if } (2 - \alpha)q < n, \\
p_0 < \infty & \text{if } (2 - \alpha)q = n, \\
p_0 \leq \infty & \text{if } (2 - \alpha)q > n,
\end{cases}
\end{align*}
\]

(4.1)

and that $\beta \in [0, 1]$ and $p_1, r_1 \in [q, \infty]$ satisfy

\[
\begin{align*}
\begin{cases}
r_1 \leq \frac{2q}{2 - \beta q} & \text{if } \beta q < 2, \\
r_1 < \infty & \text{if } \beta q = 2, \\
r_1 \leq \infty & \text{if } \beta q > 2,
\end{cases}
\quad
\begin{cases}
p_1 \leq \frac{nq}{n - (1 - \beta)q} & \text{if } (1 - \beta)q < n, \\
p_1 < \infty & \text{if } (1 - \beta)q = n, \\
p_1 \leq \infty & \text{if } (1 - \beta)q > n.
\end{cases}
\end{align*}
\]

(4.2)

Then

\[
\forall u \in W_{\text{per}}^{1,2,q}(\mathbb{R} \times \Omega) : \quad \|u\|_{L_{\text{per}}^{r_0,q}(\mathbb{R}; L^{p_0,1}(\Omega))} + \|\nabla u\|_{L_{\text{per}}^{r_1,1}(\mathbb{R}; L^{p_1,1}(\Omega))} \leq C_1\|u\|_{1,2,q},
\]

(4.3)

with $C_1 = C_1(T, n, \Omega, r_0, p_0, r_1, p_1)$.

**Proof.** The regularity of $\Omega$ is sufficient to ensure existence of a continuous extension operator $E : W_{\text{per}}^{1,2,q}(\mathbb{R} \times \Omega) \to W_{\text{per}}^{1,2,q}(\mathbb{R} \times \mathbb{R}^n)$ as in the case of classical Sobolev spaces. Consequently, it suffices to show (4.3) for functions $u \in W_{\text{per}}^{1,2,q}(\mathbb{R} \times \mathbb{R}^n)$. For this purpose, we identify $W_{\text{per}}^{1,2,q}(\mathbb{R} \times \mathbb{R}^n)$ with a Sobolev space of functions defined on the group $G := (\mathbb{R}/2\pi \mathbb{Z}) \times \mathbb{R}^n$. Endowed with the quotient topology induced by the quotient map $\pi : \mathbb{R} \times \mathbb{R}^n \to (\mathbb{R}/2\pi \mathbb{Z}) \times \mathbb{R}^n$, $G$ becomes a locally compact abelian group. A Haar measure is given by the product of the Lebesgue measure on $\mathbb{R}^n$ and the normalized Lebesgue measure on the interval $[0, T) \simeq \mathbb{R}/T\mathbb{Z}$. Also via the quotient map, the space of smooth functions on $G$ is defined as

\[
C^\infty(G) := \{ f : G \to \mathbb{R} \mid f \circ \pi \in C^\infty(\mathbb{R} \times \mathbb{R}^n) \}.
\]

(4.4)

Sobolev spaces $W^{1,2,q}(G)$ can then be defined as the closure of $C^\infty(G)$, the subspace of compactly supported functions in $C^\infty(G)$, in the norms $\|\cdot\|_{1,2,q}$. It is easy to verify that $W^{1,2,q}(G)$ and $W_{\text{per}}^{1,2,q}(\mathbb{R} \times \mathbb{R}^n)$ are homeomorphic. Further details can be found in [17] and [15]. The availability of the Fourier transform $\mathcal{F}_G$ is an immediate advantage in the
group setting. Denoting points in the dual group \( \hat{G} := \mathbb{Z} \times \mathbb{R}^n \) by \((k, \xi)\), we obtain the representation

\[
\partial_j \mathcal{P} \perp u = \mathcal{F}^{-1}_G \left[ \frac{i\xi_j (1 - \delta_Z(k))}{|\xi|^2 + i \frac{2\pi}{k}} \mathcal{F}_G [\partial_t u - \Delta u] \right],
\]

where \( \delta_Z \) denotes the delta distribution on \( Z \), that is, \( \delta_Z(0) = 1 \) and \( \delta_Z(k) = 0 \) for \( k \neq 0 \). Since \( \mathcal{F}_G = \mathcal{F}_{\mathbb{R}/2\pi\mathbb{Z}} \circ \mathcal{F}_{\mathbb{R}^n} \), it follows that

\[
\partial_j \mathcal{P} \perp u = \mathcal{F}^{-1}_{\mathbb{R}/2\pi\mathbb{Z}} \left[ (1 - \delta_Z)|k|^{-\frac{1}{2}\beta} \right] *_{\mathbb{R}/2\pi\mathbb{Z}} \mathcal{F}^{-1}_{\mathbb{R}^n} \left[ |\xi|^{\beta-1} \right] *_{\mathbb{R}^n} F, \quad (4.5)
\]

with

\[
F := \mathcal{F}^{-1}_G \left[ \frac{|k|^{\frac{1}{2}\beta} |\xi|^{1-\beta} i\xi_j (1 - \delta_Z(k))}{|\xi|^2 + i \frac{2\pi}{k}} \right].
\]

Owing to the fact that \( M \) has no singularities, one can utilize a so-called transference principle and verify that \( M \) is an \( L^q(G) \) Fourier multiplier for all \( q \in (1, \infty) \); see [17] for the details on such an approach. It follows that \( F \in L^q(G) \) with \( \|F\|_q \leq C \|u\|_{1,2,q} \).

We now consider \( r_1, p_1 \in (1, \infty) \) that satisfy (4.6) and (4.7). We then recall (4.5) to...
\begin{align*}
\| \partial_j P u \|_{L^r(\mathbb{R}/2\pi\mathbb{Z}; L^p(\Omega))} &= \left( \int_{\mathbb{R}/2\pi\mathbb{Z}} \| \mathcal{F}^{-1} [ [\xi]^{1/2} \ast_{\mathbb{R}/2\pi\mathbb{Z}} F(t, \cdot)] \|_{L^p_{\Omega}}^r \, dt \right)^{\frac{1}{r}} \\
&\leq c_0 \left( \int_{\mathbb{R}/2\pi\mathbb{Z}} \| \gamma \ast_{\mathbb{R}/2\pi\mathbb{Z}} F(t, \cdot) \|_{L^q}^r \, dt \right)^{\frac{1}{r}} \\
&\leq c_1 \left( \int_{\mathbb{R}^n} \| \gamma \ast_{\mathbb{R}/2\pi\mathbb{Z}} F(x, \cdot) \|_{L^q}^r \, dx \right)^{\frac{1}{r}} \leq c_2 \| F \|_q \leq c_3 \| u \|_{1,2,q},
\end{align*}

where Minkowski’s integral inequality is employed to conclude the second inequality above. Classical Sobolev embedding yields \( \nabla P u \in L^p_{\Omega} \) with \( \| \nabla P u \|_{L^p_{\Omega}} \leq c_4 \| u \|_{1,2,q}. \) By the above, it thus follows that \( \| \nabla u \|_{L^r(\mathbb{R}/2\pi\mathbb{Z}; L^p_{\Omega})} \leq c_5 \| u \|_{1,2,q}. \) By interpolation, the same estimate follows for all \( r_1, p_1 \in [q, \infty) \) satisfying (4.2). In a similar manner, the estimate \( \| u \|_{L^r(\mathbb{R}/2\pi\mathbb{Z}; L^p_{\Omega})} \leq c_6 \| u \|_{1,2,q} \) can be shown for parameters \( r_0, p_0 \in [q, \infty) \) satisfying (4.1). This concludes the theorem.

5 Linearized problem

A suitable linearization of (1.1) is given by the time-periodic Oseen system

\begin{equation}
\begin{cases}
\partial_t u - \nu \Delta u + \lambda \partial_1 u + \nabla p = F & \text{in } \mathbb{R} \times \Omega, \\
\text{div } u = 0 & \text{in } \mathbb{R} \times \Omega, \\
u = 0 & \text{on } \mathbb{R} \times \partial \Omega, \\
\lim_{|x| \to \infty} u(t, x) = 0, \quad u(t + T, x) = u(t), \quad p(t + T, x) = p(t),
\end{cases}
\end{equation}

where \( \lambda > 0. \) The goal in this section is to identify a Banach space \( X \) of functions \( (u, p) \) satisfying (5.1) such that the differential operator on the left-hand side in (5.1)

\[ \mathcal{L} (u, p) := \partial_t u + \lambda \partial_1 u - \Delta u + \nabla p \]

becomes a homeomorphism \( \mathcal{L} : X \to L^q_{\text{per}}(\mathbb{R} \times \Omega)^3. \) In other words, what can be referred to as “maximal \( L^q \) regularity” of the time-periodic system (5.1) shall be established.

The projections \( P \) and \( P_\perp \) shall be used to decompose (5.1) into two problems. More specifically, for data \( F \in L^q_{\text{per}}(\mathbb{R} \times \Omega)^3 \) a solution to (5.1) is investigated as the sum of a solution corresponding to the steady-state part of the data \( P F, \) and a solution corresponding to the oscillatory part \( P_\perp F. \) We start with the latter and consider data in the space \( L^q_{\text{per}, \perp}(\mathbb{R} \times \Omega)^3. \) In this case, appropriate \( L^q \) estimates can be established irrespectively of whether \( \lambda \) vanishes or not. The case \( \lambda = 0 \) is therefore included in the theorem below.
Theorem 5.1. Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^2$, $q \in (1, \infty)$ and $\lambda \in [0, \lambda_0]$. For any vector field $F \in L^2_{\text{per}, \perp}(\mathbb{R} \times \Omega)^3$ there is a solution
\[(u, p) \in W^{1,2,q}_{\text{per}, \perp}(\mathbb{R} \times \Omega)^3 \times D^{1,q}_{\text{per}, \perp}(\mathbb{R} \times \Omega)\] (5.2)
to (5.1) which satisfies
\[\|u\|_{1,2,q} + \|\nabla p\|_q \leq C_2\|F\|_q,\] (5.3)
with $C_2 = C_2(q, \Omega, \nu, \lambda_0)$. If $r \in (1, \infty)$ and $(\tilde{u}, \tilde{p}) \in W^{1,2,r}_{\text{per}, \perp}(\mathbb{R} \times \Omega)^3 \times D^{1,r}_{\text{per}, \perp}(\mathbb{R} \times \Omega)$ is another solution, then $\tilde{u} = u$ and $\tilde{p} = p + d(t)$ for some $T$-periodic function $d : \mathbb{R} \to \mathbb{R}$.

The proof of Theorem 5.1 will be based on three lemmas. The first lemma states that the theorem holds in the case $q = 2$.

Lemma 5.2. Let $\Omega$ and $\lambda$ be as in Theorem 3. For any $F \in L^2_{\text{per}, \perp}(\mathbb{R} \times \Omega)^3$ there is a solution $(u, p) \in W^{1,2,2}_{\text{per}, \perp}(\mathbb{R} \times \Omega)^3 \times D^{1,2}_{\text{per}, \perp}(\mathbb{R} \times \Omega)$ to (5.1). Moreover, the solution obeys the estimate
\[\|u\|_{1,2,2} + \|\nabla p\|_2 \leq C_3 \|F\|_2,\] (5.4)
with $C_3 = C_3(\lambda_0, T, \Omega)$.

Proof. The proof in [5, Lemma 5] can easily be adapted to establish the desired statement. For the sake of completeness, a sketch of the proof is given here. For a Hilbert space $H$, we introduce the function space $L^2_{\text{per}}(\mathbb{R}; H)$ whose elements are the $T$-periodic extensions of the functions in $L^2((0, T); H)$. Classical theory on Fourier series, in particular the theorem of Parseval, is available for such spaces. Since clearly $L^2_{\text{per}}(\mathbb{R} \times \Omega) = L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))$, we may express the data $F$ as a Fourier series $F = \sum_{k \in \mathbb{Z}} F_k e^{\frac{2\pi}{T} kt}$ with Fourier coefficients $F_k \in L^2(\Omega)^3$. Since $\mathcal{P}F = 0$, it follows that $F_0 = 0$. Consider for each $k \in \mathbb{Z} \setminus \{0\}$ the system
\[
\begin{cases}
  ik^2 \frac{2\pi}{T} u_k - \nu \Delta u_k + \lambda \partial_1 u_k + \nabla p_k = F_k & \text{in } \Omega, \\
  \text{div } u_k = 0 & \text{in } \Omega, \\
  u_k = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Standard methods from the theory on elliptic systems can be employed to investigate this problem. As a result, existence of a solution $(u_k, p_k) \in W^{2,2}(\Omega) \times (D^{1,2}(\Omega) \cap L^6(\Omega))$ which satisfies
\[
\frac{2\pi}{T} |k| \|u_k\|_2 + \|\nabla^2 u_k\|_2 + \|\nabla p_k\|_2 \leq c_0 \|F_k\|_2,
\] (5.5)
with $c_0$ independent on $k$, can be established. Observe that $D^{1,2}(\Omega) \cap L^6(\Omega)$ is a Hilbert space in the norm $\|\nabla \cdot\|_2$. Thus, by (5.5) and Parseval’s theorem, the vector fields
\[
u := \sum_{k \in \mathbb{Z} \setminus \{0\}} u_k e^{\frac{2\pi}{T} kt}, \quad p := \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k e^{\frac{2\pi}{T} kt},
\]
are well-defined as elements of $L^2_{\text{per}}(\mathbb{R}^2; W^{2,2}(\Omega))$ and $L^2_{\text{per}}(\mathbb{R}^2; D^{1,2}(\Omega))$, respectively, with $\partial_t u \in L^2_{\text{per}}(\mathbb{R}; L^2(\Omega))$. Parseval’s theorem yields

$$\|\partial_t u\|_{L^2_{\text{per}}(\mathbb{R};L^2(\Omega))} + \|u\|_{L^2_{\text{per}}(\mathbb{R};W^{2,2}(\Omega))} + \|\nabla p\|_{L^2_{\text{per}}(\mathbb{R};L^2(\Omega))} \leq c_1 \|F\|_{L^2_{\text{per}}(\mathbb{R};L^2(\Omega))}.$$  

Finally observe that $\mathcal{P}u = \mathcal{P}p = 0$ as both Fourier coefficients $u_0$ and $p_0$ vanish by definition of $u$ and $p$. We thus conclude that $(u, p) \in W^{1,2,2}_{\text{per}, \perp}(\mathbb{R} \times \Omega)^3 \times D^{1,2}_{\text{per}, \perp}(\mathbb{R} \times \Omega)$ and satisfies (5.4). By construction, $(u, p)$ is a solution to (5.1).

The next lemma states that the assertions in Theorem 5.1 are valid if $\Omega$ is replaced with a bounded domain.

**Lemma 5.3.** Let $D \subset \mathbb{R}^3$ be a bounded domain of class $C^2$, $q \in (1, \infty)$ and $\lambda \in [0, \lambda_0]$. For any vector field $F \in L^3_{\text{per}, \perp}(\mathbb{R} \times D)^3$ there is a solution

$$(u, p) \in W^{1,2,2}_{\text{per}, \perp}(\mathbb{R} \times D)^3 \times D^{1,2}_{\text{per}, \perp}(\mathbb{R} \times D)$$

(5.6)

to

$$\begin{cases}
\partial_t u - \nu \Delta u + \lambda \partial_t u + \nabla p = F & \text{in } \mathbb{R} \times D, \\
\operatorname{div} u = 0 & \text{in } \mathbb{R} \times D, \\
u = 0 & \text{on } \mathbb{R} \times \partial D,
\end{cases}$$

(5.7)

which satisfies

$$\|u\|_{1,2,q} + \|\nabla p\|_q \leq C_4 \|F\|_q,$$

(5.8)

with $C_4 = C_4(q, \nu, \lambda, \lambda_0)$. If $r \in (1, \infty)$ and $(\tilde{u}, \tilde{p}) \in W^{1,2,r}_{\text{per}, \perp}(\mathbb{R} \times D)^3 \times D^{1,r}_{\text{per}, \perp}(\mathbb{R} \times D)$ is another solution, then $\tilde{u} = u$ and $\tilde{p} = p + d(t)$ for some $\mathcal{T}$-periodic function $d : \mathbb{R} \rightarrow \mathbb{R}$.

**Proof.** One may verify that the proof in [5, Lemma 9] for a two-dimensional domain also holds for a three-dimensional domain. For the sake of completeness, a sketch is given here. By density of $C^\infty_{0,\text{per}, \perp}(D)$ in $L^3_{\text{per}, \perp}(\mathbb{R} \times D)$, it suffices to consider only $F \in C^\infty_{0,\text{per}, \perp}(D)^3$. Starting point is a solution $(u, p) \in W^{1,2,2}_{\text{per}, \perp}(\mathbb{R} \times D)^3 \times D^{1,2}_{\text{per}, \perp}(\mathbb{R} \times D)$, the existence of which can be shown by the same methods as in the proof of Lemma 5.2. By Sobolev’s embedding theorem, it may be assumed that this solution is continuous in the sense that $u \in C^\infty_{\text{per}}(\mathbb{R}; L^2_\sigma(D))$. Clearly, for this solution a $t_0 \in [0, \mathcal{T}]$ can be chosen such that $u(t_0) \in W^{2,2}(D)$ and $\operatorname{div} u(t_0) = 0$. By Sobolev’s embedding theorem, it follows that $u(t_0) \in L^3_\sigma(D)$. Consider now the initial-value problem

$$\begin{cases}
\partial_t v = \nu \Delta v - \nabla p & \text{in } (t_0, \infty) \times D, \\
\operatorname{div} v = 0 & \text{in } (t_0, \infty) \times D, \\
v = 0 & \text{on } (t_0, \infty) \times \partial D, \\
v(t_0, \cdot) = u(t_0, \cdot) & \text{in } D.
\end{cases}$$

(5.9)
It is well known that the Stokes operator $A := \mathcal{P}_H \Delta$ generates a bounded analytic semigroup on $L^2(D)$; see [9]. Consequently, the solution to the initial-value problem (5.9) given by $v := \exp (A(t - t_0)) u(t_0)$ satisfies
\[ \forall t > t_0 : \quad \| \partial_t v(t) \|_q + \| Av(t) \|_q \leq c_0 (t - t_0)^{-1}; \tag{5.10} \]
see for example [1, Theorem II.4.6]. Also by classical results, see for example [10, Theorem 2.8], there exists a solution $(w, \pi) \in W^{1,2,q}((t_0, \infty) \times D) \times W^{0,1,q}((t_0, \infty) \times D)$ to
\[
\begin{aligned}
\partial_t w &= \nu \Delta w - \nabla \pi + F & \text{in } (t_0, \infty) \times D, \\
\text{div } w &= 0 & \text{in } (t_0, \infty) \times D, \\
w &= 0 & \text{on } (t_0, \infty) \times \partial D, \\
w(t_0, \cdot) &= 0 & \text{in } D
\end{aligned}
\tag{5.11}
\]
that is continuous in the sense $w \in C_{\text{per}}((t_0, \infty); L^2(D))$ and satisfies
\[ \forall \tau \in (t_0, \infty) : \quad \| v \|_{W^{1,2,q}((t_0, \tau) \times D)} + \| \pi \|_{W^{0,1,q}((t_0, \tau) \times D)} \leq c_1 \| F \|_{L^q((t_0, \tau) \times D)} \tag{5.12} \]
with $c_1$ independent on $\tau$. Since $u$ and $v + w$ solve the same initial-value problem, a standard uniqueness argument implies $u = v + w$. Due to the $T$-periodicity of $u$ and $F$, it follows for all $m \in \mathbb{N}$ that
\[
\int_0^T \| \partial_t u(t) \|_q^q + \| Au(t) \|_q^q \, dt = \frac{1}{m} \int_{2T}^{(m+2)T} \| \partial_t u(t) \|_q^q + \| Au(t) \|_q^q \, dt \\
\leq c_2 \frac{1}{m} \int_0^\infty t^{-q} \, dt + c_3 \frac{1}{m} \| F \|_{L^q((0,m+1)T \times D)}^q \\
\leq c_4 \frac{1}{m} \frac{1}{m} T^{1-q} + c_5 \frac{m + 1}{m} \| F \|_{L^q((0,T) \times D)}^q.
\tag{5.13}
\]
Now let $m \to \infty$ to conclude
\[
\| \partial_t u \|_{L^q_{\text{per}}(\mathbb{R} \times D)} + \| Au \|_{L^q_{\text{per}}(\mathbb{R} \times D)} \leq c_6 \| F \|_{L^q_{\text{per}}(\mathbb{R} \times D)}.
\]
The estimate $\| \nabla u \|_{L^q_{\text{per}}(\mathbb{R} \times D)} \leq c_6 \| Au \|_{L^q_{\text{per}}(\mathbb{R} \times D)}$ is a consequence of well-known $L^q$ theory for the Stokes problem in bounded domains; see for example [3, Theorem IV.6.1]. Consequently, the estimate $\| u \|_{1,2,q} \leq c_7 \| F \|_q$ follows by employing Poincaré’s inequality $\| u \|_q \leq c_8 \| \partial_t u \|_q$. Now modify the pressure $p$ by adding a function depending only on $t$ such that $\int_D p(t, x) \, dx = 0$, which ensures the validity of Poincaré’s inequality for $p$. A similar estimate is then obtained for $p$ by isolating $\nabla p$ in (5.7). This establishes (5.8).

To show the statement of uniqueness, a duality argument can be employed. For this purpose, let $\varphi \in C_{0,\text{per}}^\infty(\mathbb{R} \times D)$ and let $(\psi, \eta)$ be a solution to the problem
\[
\begin{aligned}
\partial_t \psi &= -\nu \Delta \psi - \nabla p + \varphi & \text{in } \mathbb{R} \times D, \\
\text{div } \psi &= 0 & \text{in } \mathbb{R} \times D, \\
\psi &= 0 & \text{on } \mathbb{R} \times \partial D, \\
\psi(t + \tau, x) &= \psi(t, x)
\end{aligned}
\tag{5.14}
\]

adjoint to \((5.7)\). The existence of a solution \((\psi, \eta)\) follows by the same arguments from above that yield a solution to \((5.7)\). Since \(\varphi \in C^{\infty}_{0,\text{per}}(\mathbb{R} \times D)\), the solution satisfies \((\psi, \eta) \in W^{1,2,s}_{\text{per}}(\mathbb{R} \times D) \times W^{1,2}_{\text{per}}(\mathbb{R} \times D)\) for all \(s \in (1, \infty)\). The regularity of \((\psi, \eta)\) ensures validity of the following computation:
\[
\int_0^T \int_D (w - \tilde{w}) \cdot \varphi \, dx \, dt = \int_0^T \int_D (w - \tilde{w}) \cdot (\partial_t \psi + \nu \Delta \psi + \nabla p) \, dx \, dt
\]
\[
= \int_0^T \int_D (\partial_t [w - \tilde{w}] - \nu \Delta [w - \tilde{w}] + \nabla [\pi - \tilde{\pi}]) \cdot \psi \, dx \, dt = 0.
\]
Since \(\varphi \in C^{\infty}_{0,\text{per}}(\mathbb{R} \times D)\) was arbitrary, \(\tilde{w} - w = 0\) follows. In turn, \(\nabla \pi = \nabla \tilde{\pi}\) and thus \(\tilde{\pi} = \pi + d(t)\) follows. \(\square\)

The final lemma concerns estimates of the pressure term in \((5.1)\). The following lemma was originally proved for a two-dimensional exterior domain in \([5]\). We employ the ideas from \([5, \text{Proof of Lemma 6}]\) and establish the lemma in a slightly modified form for a three-dimensional exterior domain.

**Lemma 5.4.** Let \(\Omega\) and \(\lambda\) be as in Theorem 5.3 and \(s \in (1, \infty)\). There is a constant \(C_5 = C_5(R_0, \Omega, s)\) such that a solution \((u, p) \in W^{1,2,r}_{\text{per}}(\mathbb{R} \times \Omega)^3 \times D^{1,2,r}_{\text{per},\perp}(\mathbb{R} \times \Omega)\) to \((5.1)\) corresponding to data \(F \in L^r_{\text{per},\perp}(\mathbb{R} \times \Omega)^3\) for some \(r \in (1, \infty)\) satisfies for a.e. \(t \in \mathbb{R}\):
\[
\|p(t, \cdot)\|_{2, s, \Omega_{R_0}} \leq C_5 \left( \|F(t, \cdot)\|_{s} + \|\nabla u(t, \cdot)\|_{s, \Omega_{R_0}} + \|\nabla u(t, \cdot)\|_{s, \Omega_{R_0}}^{\frac{1}{2}} \right)
\]
\[
\|\nabla p(t, \cdot)\|_{s, \Omega^c} \leq C_6 \left( \|F(t, \cdot)\|_{s} + \|p(t, \cdot)\|_{s, \Omega^c} \right).
\]

**Proof.** For the sake of simplicity, the \(t\)-dependency of functions is not indicated. All norms are taken with respect to the spatial variables only. Consider an arbitrary \(\varphi \in C^{\infty}_{0,\text{per}}(\overline{\Omega})\). Observe that for any \(\psi \in C^{\infty}_{0,\text{per}}(\mathbb{R})\) holds
\[
\int_0^T \int_\Omega \partial_t u \cdot \nabla \varphi \, dx \, dt = \int_0^T \int_\Omega \text{div} u \, \partial_t \varphi \, dx \, dt = 0,
\]
which implies \(\int_\Omega \partial_t u \cdot \nabla \varphi \, dx = 0\) for a.e. \(t\). Moreover,
\[
\int_\Omega \partial_t u \cdot \nabla \varphi \, dx = - \int_\Omega \text{div} u \cdot \partial_t \varphi \, dx = 0.
\]

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Hence it follows from (5.1) that \( p \) is a solution to the weak Neumann problem for the Laplacian:

\[
\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega \nabla p \cdot \nabla \varphi \, dx = \int_\Omega F \cdot \nabla \varphi + \Delta u \cdot \nabla \varphi \, dx.
\]

Recall that

\[
p \in L^1_{\text{loc}}(\Omega) \quad \nabla p \in L^r(\Omega)^3 \quad \int_{\Omega_{R_0}} p \, dx = 0.
\]  

(5.17)

It is well-known that the weak Neumann problem for the Laplacian in an exterior domain is uniquely solvable in the class (5.17); see for example [3, Section III.1] or [29]. We can thus write \( p \) as a sum \( p = p_1 + p_2 \) of two solutions (in the class above) to the weak Neumann problem

\[
\forall \varphi \in C_\infty^0(\Omega) : \int_\Omega \nabla p_1 \cdot \nabla \varphi \, dx = \int_\Omega F \cdot \nabla \varphi \, dx
\]

and

\[
\forall \varphi \in C_\infty^0(\Omega) : \int_\Omega \nabla p_2 \cdot \nabla \varphi \, dx = \int_\Omega \Delta u \cdot \nabla \varphi \, dx,
\]

respectively. The \textit{a priori} estimate

\[
\forall q \in (1, \infty) : \| \nabla p_1 \|_q \leq c_0 \| F \|_q \tag{5.18}
\]

is well-known. An estimate of \( p_2 \) shall now be established. Consider for this purpose an arbitrary function \( g \in C_0^\infty(\Omega_{R_0}) \) with \( \int_{\Omega_{R_0}} g \, dx = 0 \). Existence of a vector field \( h \in C_0^\infty(\Omega_{R_0}) \) with \( \text{div} \, h = g \) and

\[
\forall q \in (1, \infty) : \| h \|_{1,q} \leq c_1 \| g \|_q
\]

is well-known; see for example [3, Theorem III.3.3]. Let \( \Phi \) be a solution to the following weak Neumann problem for the Laplacian:

\[
\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega \nabla \Phi \cdot \nabla \varphi \, dx = \int_\Omega h \cdot \nabla \varphi \, dx.
\]

By classical theory, such a solution exists with

\[
\forall q \in (1, \infty) : \Phi \in C^\infty(\Omega) \quad \| \nabla \Phi \|_{1,q} \leq c_2 \| h \|_{1,q} \leq c_3 \| g \|_q.
\]

Since \( \Phi \) is harmonic in \( \mathbb{R}^3 \setminus \overline{B_{R_0}} \), the following asymptotic expansion as \( |x| \to \infty \) is valid:

\[
\partial^\alpha \Phi(x) = \partial^\alpha c_4 + \partial^\alpha \Gamma_\nu(x) \cdot \int_{\partial B_\rho} \frac{\partial \Phi}{\partial n} \, dS + O(|x|^{-2-|\alpha|}),
\]

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where $c_4$ is a constant and $\Gamma_L : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$, $\Gamma_L(x) := (4\pi|x|)^{-1}$ the fundamental solution of the Laplacian in $\mathbb{R}^3$. Observing that

\[
\int_{\partial B} \frac{\partial \Phi}{\partial n} dS = \int_{\Omega} \frac{\partial \Phi}{\partial n} dS + \int_{\Omega} \Delta \Phi dx = 0 + \int_{\Omega} \text{div } h dx = \int_{\Omega} g dx = 0,
\]

we thus deduce $\nabla \Phi(x) = O(|x|^{-3})$ as $|x| \to \infty$. Similarly, we see that $p_2 = c_5 + O(|x|^{-1})$.

We therefore conclude that

\[
\lim_{\rho \to \infty} \int_{\partial B} \rho p_2 \frac{\partial \Phi}{\partial n} dS = 0.
\]

We can thus compute

\[
\int_{\Omega} p_2 g dx = \lim_{\rho \to \infty} \int_{\Omega_{\rho}} p_2 \Delta \Phi dx = -\lim_{\rho \to \infty} \left( \int_{\partial \Omega_{\rho}} \rho \frac{\partial \Phi}{\partial n} dS - \int_{\Omega_{\rho}} \nabla p_2 \cdot \nabla \Phi dx \right),
\]

where $\nabla \Phi$ further implies $\partial_i u_j \partial_k \Phi \in L^1(B^{2R_0})$. We can thus find a sequence $\{\rho_k\}_{k=1}^{\infty}$ of positive numbers with $\lim_{k \to \infty} \rho_k = \infty$ such that

\[
\lim_{\rho_k \to \infty} \int_{\partial B_{\rho_k}} \partial_j u_i \partial_i \Phi n_j - \partial_j u_i \partial_j \Phi n_i dS = 0.
\]

Returning to (5.19), we continue the computation and find that

\[
\int_{\Omega} p_2 g dx = -\lim_{k \to \infty} \int_{\Omega_{\rho_k}} \partial_j u_i \partial_i \Phi n_j - \partial_j u_i \partial_j \Phi n_i dS = -\int_{\partial \Omega} \partial_j u_i \partial_i \Phi n_j - \partial_j u_i \partial_j \Phi n_i dS = -\int_{\partial \Omega} \nabla u : (\nabla \Phi \otimes n - n \otimes \nabla \Phi) dS.
\]

Applying first the H"older and then a trace inequality (for example [3, Theorem II.4.1]), we deduce

\[
\left| \int_{\Omega} p_2 g dx \right| \leq c_6 \|\nabla u\|_{s,\partial \Omega} \|\nabla \Phi\|_{\frac{s}{n-2},\partial \Omega} \\
\leq c_7 \|\nabla u\|_{s,\partial \Omega} \|\nabla \Phi\|_{1,\frac{3s}{n-2},\Omega_{R_0}} \leq c_8 \|\nabla u\|_{s,\partial \Omega} \|g\|_{\frac{3s}{n-2},\Omega_{R_0}}.
\]

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Recalling that $\int_{\Omega R_0} p_2 \, dx = 0$, we thus obtain

$$\|p_2\|_{\frac{3}{2},\Omega R_0} = \sup_{g \in C_0^\infty(\Omega R_0)} \frac{1}{\int_{\Omega R_0} g \, dx = 0} \left| \int_{\Omega} p_2 \, g \, dx \right| \leq c_9 \|\nabla u\|_{s,\partial \Omega}.$$  

Another application of a trace inequality (see again [3, Theorem II.4.1]) now yields

$$\|p_2\|_{\frac{3}{2},\Omega R_0} \leq c_{10} \left( \|\nabla u\|_{s,\Omega R_0} + \|\nabla u\|_{s,\Omega R_0}^{\frac{2}{3}} \right).$$

Recalling (5.18), we employ Sobolev’s embedding theorem to finally conclude

$$\|p\|_{\frac{3}{2},\Omega R_0} \leq c_{11} \left( \|p_1\|_{s,\Omega R_0} + \|p_2\|_{\frac{3}{2},\Omega R_0} \right) \leq c_{12} \left( \|F\|_s + \|\nabla u\|_{s,\Omega R_0} + \|\nabla u\|_{s,\Omega R_0}^{\frac{2}{3}} \right)$$

and thus (5.15). To show (5.16), we introduce $R \in (R_*, \rho)$ and a “cut-off” function $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R})$ with $\chi = 1$ on $\Omega R$ and $\chi = 0$ on $\Omega R_*$. We then put $\pi := \chi p$ and observe from (5.1) that $\pi$ is a solution to the weak Neumann problem for the Laplacian

$$\forall \varphi \in C_0^\infty(\overline{\Omega}) : \int_{\Omega} \nabla \pi \cdot \nabla \varphi \, dx = \langle F_1, \varphi \rangle + \langle F_2, \varphi \rangle$$

with

$$\langle F_1, \varphi \rangle := \int_{\Omega} (2p \nabla \chi + \chi F) \cdot \nabla \varphi \, dx, \quad \langle F_2, \varphi \rangle := \int_{\Omega} (\nabla \chi \cdot F + \Delta \chi p) \varphi \, dx.$$

We clearly have

$$\sup_{\|\nabla \varphi\|_s = 1} |\langle F_1, \varphi \rangle| \leq c_{13} \left( \|\pi\|_{s,\Omega \rho} + \|F\|_s \right).$$

Since $\chi = 1$ on $\Omega R$, we further observe that

$$\sup_{\|\nabla \varphi\|_s = 1} |\langle F_2, \varphi \rangle| = \sup_{\|\nabla \varphi\|_s = 1} \sup_{\varphi \in \Omega_R} |\langle F_2, \varphi \rangle| \leq c_{14} \left( \|F\|_s + \|\pi\|_{s,\Omega \rho} \right),$$

where Poincaré’s inequality is used to obtain the last estimate. A standard a priori estimate for the weak Neumann problem (5.20) now implies (5.16). \hfill \Box

**Proof of Theorem 5.1.** By density of $C_0^\infty(\mathbb{R} \times \Omega)$ in $L^q_{\text{per.}, \perp}(\mathbb{R} \times \Omega)$, it suffices to consider only $F \in C_0^\infty(\mathbb{R} \times \Omega)$ in $L^q_{\text{per.}, \perp}(\mathbb{R} \times \Omega)$, then it suffices to consider only $F \in C_0^\infty(\mathbb{R} \times \Omega)$. The starting point will be the solution $(u, p) \in W^{1,2,1}_{\text{per.}, \perp}(\mathbb{R} \times \Omega)^3 \times D^{1,2,1}_{\text{per.}, \perp}(\mathbb{R} \times \Omega)$ from Lemma 5.2. By adding to $p$ a function that only
depends on time, we may assume without loss of generality that \( \int_{\Omega_{t_0}} p \, dx = 0 \). For the scope of the proof, we fix a constant \( \rho \) with \( R_s < \rho < R_0 \).

We shall establish two fundamental estimates. To show the first one, we introduce a cut-off function \( \psi_1 \in C^\infty(\mathbb{R}^3; \mathbb{R}) \) with \( \psi_1(x) = 1 \) for \( |x| \geq \rho \) and \( \psi_1(x) = 0 \) for \( |x| \leq R_s \).

We let \( I_L : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \), \( I_L(x) := (4\pi|x|)^{-1} \) denote the fundamental solution to the Laplace operator and put

\[
V : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \quad V = \nabla I_L \ast \nabla \psi_1 \cdot u,
\]

\[
P : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \quad P = I_L \ast \left( (\partial_t - \Delta + \lambda \partial_t)(\nabla \psi_1 \cdot u) \right),
\]

\[
w : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \quad w(t, x) := \psi_1(x) u(t, x) - V(t, x),
\]

\[
\pi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \quad \pi(t, x) := \psi_1(x) p(t, x) - P(t, x).
\]

Then \((w, \pi)\) is a solution to the whole-space problem

\[
\begin{align*}
\partial_t w - \Delta w + \lambda \partial_t w + \nabla \pi &= \psi_1 F - 2\nabla \psi_1 \cdot \nabla u - \Delta \psi_1 u + \lambda \partial_t \psi_1 u + \nabla \psi_1 p \quad \text{in } \mathbb{R} \times \mathbb{R}^3, \\
\text{div } w &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^3.
\end{align*}
\]

The precise regularity of \((w, \pi)\) is not important at this point. It is enough to observe that \( w \) and \( \pi \) belong to the space of tempered time-periodic distributions \( \mathcal{S}'_{\text{per}}(\mathbb{R} \times \mathbb{R}^3) \), which is easy to verify from the definition \((5.21)\) and the regularity of \( u \) and \( p \). It is not difficult to show, see \[16, Lemma 5.3\], that a solution \( w \) to \((5.22)\) is unique in the class of distributions in \( \mathcal{S}'_{\text{per}}(\mathbb{R} \times \mathbb{R}^3) \) satisfying \( \mathcal{P} w = 0 \). Consequently, \( w \) coincides with the solution from \[17, Theorem 2.1\] and therefore satisfies

\[
||w||_{1,2,s} \leq c_0 \left( ||\psi_1 F - 2\nabla \psi_1 \cdot \nabla u - \Delta \psi_1 u + \lambda \partial_t \psi_1 u + \nabla \psi_1 p||_s + ||\nabla \psi_1 u||_{s,T \times \Omega_\rho} + ||\nabla u||_{s,T \times \Omega_\rho} + ||p||_{s,T \times \Omega_\rho} \right)
\]

for all \( s \in (1, \infty) \). Clearly,

\[
||\nabla V||_s + ||\nabla^2 V||_s \leq c_2 \left( ||u||_{s,\Omega_\rho} + ||\nabla u||_{s,\Omega_\rho} \right).
\]

Since \( u = w + V \) for \( x \in \Omega^\rho \), we conclude

\[
||\nabla u||_{s,T \times \Omega_\rho} + ||\nabla^2 u||_{s,T \times \Omega_\rho} \leq c_3 \left( ||F||_s + ||u||_{s,T \times \Omega_\rho} + ||\nabla u||_{s,T \times \Omega_\rho} + ||p||_{s,T \times \Omega_\rho} \right)
\]

for all \( s \in (1, \infty) \). For a similar estimate of \( u \) and \( \partial_t u \), we turn first to \((5.1)\) and then apply \((5.16)\) to deduce

\[
||\partial_t u||_{s,T \times \Omega_\rho} \leq c_4 \left( ||F||_s + ||u||_{s,T \times \Omega_\rho} + ||\nabla u||_{s,T \times \Omega_\rho} + ||p||_{s,T \times \Omega_\rho} \right)
\]

\[
\leq c_5 \left( ||F||_s + ||u||_{s,T \times \Omega_\rho} + ||\nabla u||_{s,T \times \Omega_\rho} + ||p||_{s,T \times \Omega_\rho} \right).
\]

Since \( \mathcal{P} u = 0 \), Poincaré’s inequality yields \( ||u||_{s,T \times \Omega_\rho} \leq c_6 ||\partial_t u||_{s,T \times \Omega_\rho} \). We have thus shown

\[
||u||_{1,2,s,T \times \Omega_\rho} \leq c_7 \left( ||F||_s + ||u||_{s,T \times \Omega_\rho} + ||\nabla u||_{s,T \times \Omega_\rho} + ||p||_{s,T \times \Omega_\rho} \right)
\]

(5.23)
for all $s \in (1, \infty)$.

Next, we seek to establish a similar estimate for $u$ over the bounded domain $\Omega = \mathbb{T} \times \Omega_{\rho}$. For this purpose, we introduce a “cut-off” function $\psi_2 \in C^\infty(\mathbb{R}^3; \mathbb{R})$ with $\psi_2(x) = 1$ for $|x| \leq \rho$ and $\psi_2(x) = 0$ for $|x| > R_0$. We then introduce a vector field $V$ with

$$V \in W^{1,2,2}_{\text{per,\perp}}(\mathbb{R} \times \mathbb{R}^3), \quad \text{supp} V \subset \mathbb{R} \times \Omega_{\rho,R_0}, \quad \text{div} V = \nabla \psi_2 \cdot u,$$

$$\forall s \in (1, \infty), \quad |V|_{1,2,s} \leq c_8 \left( \|u\|_{s,T \times \Omega_{\rho,R_0}} + \|\nabla u\|_{s,T \times \Omega_{\rho,R_0}} + \|\partial_t u\|_{s,T \times \Omega_{\rho,R_0}} \right). \tag{5.24}$$

Since

$$\int_{\Omega_{\rho,R_0}} \nabla \psi_2 \cdot u \, dx = \int_{\Omega_{\rho,R_0}} \text{div} (\psi_2 u) \, dx = \int_{\partial\Omega_{R_0}} u \cdot n \, dS = 0,$$

the existence of a vector field $V$ with the properties above can be established by the same construction as the one used in [3, Theorem III.3.3]; see also [15, Proof of Lemma 3.2.1]. We now let

$$w : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \quad w(t,x) := \psi_2(x) u(t,x) - V(t,x),$$

$$\pi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \quad \pi(t,x) := \psi_2(x) p(t,x). \tag{5.25}$$

Then $(w, \pi) \in W^{1,2,2}_{\text{per,\perp}}(\mathbb{R} \times \Omega_{R_0}) \times D^{1,2}_{\text{per,\perp}}(\mathbb{R} \times \Omega_{R_0})$ is a solution to the problem

$$\begin{cases}
\partial_t w = \Delta w + \lambda \partial_1 w + \nabla \pi \\
\psi_2 F - 2\nabla \psi_2 \cdot \nabla u - \Delta \psi_2 u + \lambda \partial_1 \psi_2 u + \nabla \psi_2 p + |\partial_t - \Delta + \lambda \partial_1| V \\
\text{div} w = 0 \\
w = 0
\end{cases}$$

in $\mathbb{R} \times \Omega_{R_0}$, in $\mathbb{R} \times \Omega_{R_0}$, on $\mathbb{R} \times \partial\Omega_{R_0}$.

By Lemma 5.3, we thus deduce

$$|w|_{1,2,s} \leq c_9 \|\psi_2 F - 2\nabla \psi_2 \cdot \nabla u - \Delta \psi_2 u + \lambda \partial_1 \psi_2 u + \nabla \psi_2 p + |\partial_t - \Delta + \lambda \partial_1| V|_s \leq c_{10} \left( \|F\|_s + \|u\|_{s,T \times \Omega_{\rho,R_0}} + \|\nabla u\|_{s,T \times \Omega_{\rho,R_0}} + \|p\|_{s,T \times \Omega_{\rho,R_0}} + \|V|_{1,2,s} \right)$$

for all $s \in (1, \infty)$. Since $\Omega_{\rho,R_0} \subset \Omega_{\rho}$, we can combine (5.24) and (5.23) to estimate

$$|u|_{1,2,s,\Omega_{\rho}} \leq c_{11} \left( \|F\|_s + \|u\|_{s,T \times \Omega_{\rho}} + \|\nabla u\|_{s,T \times \Omega_{\rho}} + \|p\|_{s,T \times \Omega_{\rho}} \right),$$

which was the intermediate goal at this stage. Combining the estimate above with (5.23), we have

$$|u|_{1,2,s} \leq c_{12} \left( \|F\|_s + \|u\|_{s,T \times \Omega_{\rho,R_0}} + \|\nabla u\|_{s,T \times \Omega_{R_0}} + \|p\|_{s,T \times \Omega_{R_0}} \right) \tag{5.26}$$

for all $s \in (1, \infty)$.

We now move on to the final part of the proof. We emphasize that estimate (5.26) has been established for all $s \in (1, \infty)$, but we do not actually know whether the right-hand side is finite or not. At the outset, we only know the right-hand side is finite for $s \leq 2$.  

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We shall now use a boot-strap argument to show that is also the case for \( s \in (2, \infty) \). For this purpose, we need the embedding properties of \( W_{\text{per}}^{1,2,s} (\mathbb{R} \times \Omega) \) stated in Theorem 4.1. Choosing for example \( \alpha = \beta = \frac{1}{2} \) in Theorem 4.1, we obtain the implication
\[
\forall s \in [2, \infty) : \quad u \in W_{\text{per}}^{1,2,s} (\mathbb{R} \times \Omega) \implies u, \nabla u \in L_{\text{per}}^{2,s} (\mathbb{R} \times \Omega).
\]
(5.27)

We now turn to estimate (5.15) of the pressure term. By Hölder’s inequality,
\[
\int_0^T \left( \| \nabla u(t, \cdot) \|_{s,\Omega_R^0} \| \nabla u(t, \cdot) \|_{1,s,\Omega_R^0} \right)^{\frac{2-s}{s}} \, dt \leq \left( \int_0^T \| \nabla u(t, \cdot) \|_{s,\Omega_R^0} \, dt \right)^{\frac{s-2}{s}} \| u \|_{1,2,s}.
\]

Utilizing again Theorem 4.1, this time with \( \beta = 1 \), we see that the right-hand side above is finite for all \( s \in [2, \infty) \), provided \( u \in W_{\text{per}}^{1,2,s} (\mathbb{R} \times \Omega) \). Due to the normalization of the pressure \( p \) carried out in the beginning of the proof, Lemma 5.4 can be applied to infer from (5.15) that
\[
\forall s \in [2, \infty) : \quad u \in W_{\text{per}}^{1,2,s} (\mathbb{R} \times \Omega) \implies p \in L_{\text{per}}^{2,s} (\mathbb{R} \times \Omega_R^0).
\]
(5.28)

Combining (5.26) with the implications (5.27) and (5.28), we find that
\[
\forall s \in [2, \infty) : \quad u \in W_{\text{per}}^{1,2,s} (\mathbb{R} \times \Omega) \implies u \in W_{\text{per}}^{1,2,\frac{2-s}{s}} (\mathbb{R} \times \Omega).
\]
(5.29)

Starting with \( s = 2 \), we can now boot-strap (5.29) a sufficient number of times to deduce that \( u \in W_{\text{per}}^{1,2,s} (\mathbb{R} \times \Omega) \) for any \( s \in (2, \infty) \). Knowing now that the right-hand side of (5.26) is finite for all \( s \in (1, \infty) \), we can use interpolation and (5.15) in combination with Young’s inequality to deduce
\[
\| u \|_{1,2,s} \leq c_{13} \left( \| F \|_s + \| u \|_{s,T \times \Omega_R^0} \right)
\]
for all \( s \in (1, \infty) \). It then follows directly from (5.1) that
\[
\| u \|_{1,2,s} + \| \nabla p \|_s \leq c_{13} \left( \| F \|_s + \| u \|_{s,T \times \Omega_R^0} \right)
\]
(5.30)
for all \( s \in (1, \infty) \).

If \( (U, \mathfrak{p}) \in W_{\text{per}, \perp}^{1,2,r} (\mathbb{R} \times \Omega) \times D_{\text{per}, \perp}^{1,r} (\mathbb{R} \times \Omega) \) with \( r \in (1, \infty) \) is a solution to (5.1) with homogeneous right-hand side, then \( U = \nabla \mathfrak{p} = 0 \) follows by a duality argument. More specifically, since for arbitrary \( \varphi \in C_{0,\text{per}, \perp}^{\infty} (\mathbb{R} \times \Omega)^3 \) existence of a solution \( (W, \Pi) \in W_{\text{per}, \perp}^{1,2,r'} (\mathbb{R} \times \Omega) \times D_{\text{per}, \perp}^{1,r'} (\mathbb{R} \times \Omega) \) to (5.1) with \( \varphi \) as the right-hand side has just been established, the computation
\[
0 = \int_0^T \int_\Omega \left( \partial_t U - \Delta U + \lambda \partial_t U + \nabla \mathfrak{p} \right) \cdot W \, dx dt
\]
(5.31)

\[
= - \int_0^T \int_\Omega U \cdot (\partial_t W - \Delta W + \lambda \partial_t W + \nabla \Pi) \, dx dt = \int_0^T \int_\Omega U \cdot \varphi \, dx dt
\]
is valid. It follows that $U = 0$ and in turn, directly from (5.1), that also $\nabla \Psi = 0$.

We now return to the estimate (5.30). Owing to the fact a solution to (5.1) with
homogeneous right-hand is necessarily zero, which we have just shown above, a standard
contradiction argument, see for example [5, Proof of Proposition 2], can be used to
eliminate the lower order term on the right-hand side in (5.30) to conclude

$$
\|u\|_{1,2,q} + \|\nabla p\|_q \leq c_{14} \|F\|_q. 
\tag{5.32}
$$

It is easy to verify that $C_{0,\perp}(T \times \Omega)$ is dense in $L^q_{\perp}(T \times \Omega)$. By a density argument,
the existence of a solution $(u, p) \in W_{\perp}^{1,2,q}(\mathbb{R} \times \Omega)^3 \times D_{\perp}^{1,q}(\mathbb{T} \times \Omega)$ to (5.1) that satisfies
(5.32) follows for any $F \in L^q_{\perp}(T \times \Omega)^3$.

Finally, assume $(\tilde{u}, \tilde{p}) \in W_{\perp}^{1,2,r}(\mathbb{R} \times \Omega) \times D_{\perp}^{1,r}(\mathbb{R} \times \Omega)$ is another solution to (5.1) with
$r \in (1, \infty)$. The duality argument used in (5.31) applied to the difference $(u - \tilde{u}, p - \tilde{p})$
yields $u = \tilde{u}$ and $\nabla p = \nabla \tilde{p}$. The proof of theorem is thereby complete.

By combining Theorem 5.1 with well-known $L^q$ estimates for the steady-state Oseen
system, we can formulate what can be referred to as “maximal $L^q$ regularity” of the
time-periodic Oseen system (5.1).

**Corollary 5.5.** Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^2$, $\lambda \in (0, \lambda_0]$ and $q \in (1, 2)$.

Define

$$
\mathcal{A}^q_{\lambda}(\Omega) := \{v \in X^q(\Omega) \mid \text{div} \, v = 0, \ v = 0 \text{ on } \partial \Omega\}.
$$

Moreover, let $r \in (1, \infty)$ and

$$
W_{\perp}^{1,2,q,r}(\mathbb{R} \times \Omega) := \{w \in W_{\perp}^{1,2,q,r}(\mathbb{R} \times \Omega) \mid \text{div} \, w = 0, \ w = 0 \text{ on } \partial \Omega\}.
$$

Then the $T$-time-periodic Oseen operator

$$
A_0 : \left(\mathcal{A}^q_{\lambda}(\Omega) \oplus W_{\perp}^{1,2,q,r}(\mathbb{R} \times \Omega)\right) \times \left(D_{R_0}^{1,q}(\Omega) \oplus D_{\perp}^{1,q,r}(\mathbb{R} \times \Omega)\right) \to L^q(\Omega) \oplus L^{q,r}(\mathbb{R} \times \Omega),
$$

$$
A_0(v + w, p + \pi) := \partial_t w - \nu \Delta(v + w) + \lambda \partial_t (v + w) + \nabla(p + \pi)
\tag{5.33}
$$

is a homeomorphism with $\|A_0^{-1}\|$ depending only on $q, r, \Omega, \nu$ and $\lambda$. If $q \in (1, \frac{3}{2})$, then
$\|A_0^{-1}\|$ depends only on the upper bound $\lambda_0$ and not on $\lambda$ itself.

**Proof.** It is well-known that the steady-state Oseen operator, that is, the Oseen operator
from (5.33) restricted to time-independent functions, is a homeomorphism as a mapping
$A_0 : \mathcal{A}^q_{\lambda}(\Omega) \times D_{R_0}^{1,q}(\Omega) \to L^q(\Omega)$; see for example [3, Theorem VII.7.1]. By Theorem 5.1,
its follows that also the time-periodic Oseen operator is a homeomorphism as a mapping
$A_0 : W_{\perp}^{1,2,q,r}(\mathbb{R} \times \Omega) \times D_{\perp}^{1,q,r}(\mathbb{R} \times \Omega) \to L^{q,r}_{\perp}(\mathbb{R} \times \Omega)$. Since clearly $\mathcal{P}$ and $\mathcal{P}_\perp$
commute with $A_0$, it further follows that $A_0$ is a homeomorphism as an operator in
the setting (5.33). The dependency of $\|A_0^{-1}\|$ on the various parameters follows from [3,
Theorem VII.7.1] and Theorem 5.1.
6 Fully Nonlinear Problem

Existence of a solution to the fully nonlinear problem (1.1) shall now be established. We employ a fixed point argument based on the estimates established for the linearized system (5.1) in the previous section. For this approach to work, we need to assume (1.4) to ensure that (5.1) is indeed a suitable linearization of (1.1). Moreover, we need to assume (1.5) to ensure that all terms appearing on the right-hand side after linearizing (1.1) are subordinate to the linear operator on the left-hand side.

For convenience, we rewrite (1.1) by replacing $u$ with $u + v_b$ and obtain the following equivalent problem:

\[
\begin{aligned}
\begin{cases}
    \partial_t u + (u - v_b) \cdot \nabla u = \nu \Delta u - \nabla p + f & \text{in } \mathbb{R} \times \Omega, \\
    \text{div } u = 0 & \text{in } \mathbb{R} \times \Omega, \\
    u = u_* & \text{in } \mathbb{R} \times \partial \Omega, \\
    \lim_{|x| \to \infty} u(t, x) = 0.
\end{cases}
\end{aligned}
\]  

(6.1)

**Theorem 6.1.** Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $C^2$. Assume that $v_b(t) = -u_\infty(t) e_1$ for a $T$-periodic function $u_\infty : \mathbb{R} \to \mathbb{R}$ with $\lambda := \mathcal{P} u_\infty > 0$. Moreover, let $q \in \left[ \frac{3}{2}, \frac{4}{3} \right]$. There is an $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ there is an $\varepsilon_0 > 0$ such that for all $f \in L^q_{\text{per}}(\mathbb{R} \times \Omega)^3$, $u_\infty \in L^\infty_{\text{per}}(\mathbb{R})$ and

\[
u \Delta u + \nabla \Pi \perp = \mathcal{W}
\]

satisfying

\[
\begin{aligned}
    &\|f\|_q + \|P_{\perp} f\|_{\frac{3q}{3q-2q}} + \|P_{\perp} u_\infty\|_{\infty} + \|u_*\|_{W^{1,q}_{\text{per}}(\mathbb{R}^2, \mathbb{R}^2(e_1))} \\
    \leq \varepsilon_0
\end{aligned}
\]  

(6.2)

there is a solution

\[
(u, \Pi) \in \left( X^q_{\lambda}(\Omega) \oplus W^{1,2,q}_{\text{per}, \perp}(\mathbb{R}^2, \mathbb{R}^3(\partial \Omega)) \right)^3 \times \left( D^{1,q}(\Omega) \oplus D^{1,q}_{\text{per}, \perp}(\mathbb{R}^2, \mathbb{R}^2(\partial \Omega)) \right)
\]  

(6.3)

to (6.1).

**Proof.** In order to “lift” the boundary values in (6.1), that is, rewrite the system as one of homogeneous boundary values, a solution $(\mathcal{W}, \Pi \perp) \in W^{1,2,q, \frac{3q}{3q-2q}}_{\text{per}, \perp}(\mathbb{R} \times \Omega) \times D^{1,q, \frac{3q}{3q-2q}}_{\text{per}, \perp}(\mathbb{R} \times \Omega)$ to

\[
\begin{aligned}
    -\nu \Delta \mathcal{W} + \nabla \Pi \perp = \mathcal{W} & \quad \text{in } \mathbb{R} \times \Omega, \\
    \text{div } \mathcal{W} = 0 & \quad \text{in } \mathbb{R} \times \Omega, \\
    \mathcal{W} = \mathcal{P}_{\perp} u_* & \quad \text{on } \mathbb{R} \times \partial \Omega,
\end{aligned}
\]  

(6.5)

is introduced. Observe that (6.5) is a Stokes resolvent-type problem. One can therefore use standard methods to solve (6.5) in $T$-time periodic function spaces and obtain a solution that satisfies

\[
\forall r \in \left( 1, \frac{3q}{3q-2q} \right) : \|\mathcal{W}\|_{1,2,r} + \|\nabla \Pi \perp\|_r \leq c_0 \|u_*\|_{W^{1,q}_{\text{per}}(\mathbb{R}^2, \mathbb{R}^2(e_1))},
\]  

(6.6)
where $c_0 = c_0(r, q, \Omega, \nu)$. Furthermore, classical results for the steady-state Oseen problem [3, Theorem VII.7.1] ensure existence of a solution $(\mathcal{V}, \Pi_s) \in X_q^1(\Omega) \times D^{1,q}(\Omega)$ to

$$
\begin{cases}
-\nu \Delta \mathcal{V} + \lambda \partial_1 \mathcal{V} + \nabla \Pi_s = 0 & \text{in } \Omega, \\
\text{div } \mathcal{V} = 0 & \text{in } \Omega, \\
\mathcal{V} = P u_s & \text{on } \partial \Omega,
\end{cases}
$$

which satisfies, since $q \geq \frac{6}{5}$ implies $\frac{3q}{3q-2} \geq 2$,

$$
\forall r \in (1, 2) : \|\mathcal{V}\|_{X_q^1(\Omega)} + \|\nabla \Pi_s\|_r \leq c_1 \|P u_s\|_{W^{\frac{3q}{3q-2}}(\Omega)},
$$

where $c_1 = c_1(r, \Omega, \nu)$. We shall now establish existence of a solution $(u, p)$ to (6.1) on the form

$$u = v + \mathcal{V} + w + \mathcal{W}, \quad p = p + \Pi_s + \pi + \Pi_{\perp},$$

where $(v, p) \in X_q^1(\Omega) \times D^{1,q}(\Omega)$ is a solution to the steady-state problem

$$
\begin{cases}
-\nu \Delta v + \lambda \partial_1 v + \nabla \pi = \mathcal{R}_1(v, w, \mathcal{V}, \mathcal{W}) & \text{in } \Omega, \\
\text{div } v = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
$$

with

$$
\mathcal{R}_1(v, w, \mathcal{V}, \mathcal{W}) := -v \cdot \nabla v - v \cdot \nabla \mathcal{V} - \mathcal{V} \cdot \nabla v - \mathcal{V} \cdot \nabla \mathcal{V}
- P [w \cdot \nabla w] - P [w \cdot \nabla \mathcal{W}] - P [\mathcal{W} \cdot \nabla w] - P [\mathcal{W} \cdot \nabla \mathcal{W}]
- P [P_{\perp} u_{\infty} \partial_1 w] - P [P_{\perp} u_{\infty} \partial_1 \mathcal{W}] + P f,
$$

and $(w, \pi) \in W^{1,2,q,\frac{3q}{3q-2}}_{\text{per, } \perp}(\mathbb{R} \times \Omega) \times D^{1,q,\frac{3q}{3q-2}}_{\text{per, } \perp}(\mathbb{R} \times \Omega)$ a solution to

$$
\begin{cases}
\partial_t w - \nu \Delta w + \lambda \partial_1 w + \nabla \pi = \mathcal{R}_2(v, w, \mathcal{V}, \mathcal{W}) & \text{in } \mathbb{R} \times \Omega, \\
\text{div } w = 0 & \text{in } \mathbb{R} \times \Omega, \\
w = 0 & \text{on } \mathbb{R} \times \partial \Omega,
\end{cases}
$$

with

$$
\mathcal{R}_2(v, w, \mathcal{V}, \mathcal{W}) := -P_{\perp} [w \cdot \nabla w] - P_{\perp} [w \cdot \nabla \mathcal{W}] - P_{\perp} [\mathcal{W} \cdot \nabla w] - P_{\perp} [\mathcal{W} \cdot \nabla \mathcal{W}]
- v \cdot \nabla w - v \cdot \nabla \mathcal{W} - w \cdot \nabla v - w \cdot \nabla \mathcal{V}
- \mathcal{V} \cdot \nabla w - \mathcal{V} \cdot \nabla \mathcal{W} - \mathcal{W} \cdot \nabla v - \mathcal{W} \cdot \nabla \mathcal{V}
- P_{\perp} u_{\infty} \partial_1 v - P_{\perp} u_{\infty} \partial_1 \mathcal{V} - P_{\perp} \partial_1 [P_{\perp} u_{\infty} \partial_1 w] - P_{\perp} \partial_1 [P_{\perp} u_{\infty} \partial_1 \mathcal{W}]
- \partial_t \mathcal{W} - \mathcal{W} + \lambda \partial_1 \mathcal{W} + P_{\perp} f.
$$
The systems (6.10) and (6.11) emerge as the result of inserting (6.9) into (6.1) and subsequently applying first $\mathcal{P}$ then $\mathcal{P}_\perp$ to the equations. Recalling the function spaces introduced in Corollary 5.5, we define the Banach space

$$K^q_\Lambda(\mathbb{R} \times \Omega) := \Lambda^{\frac{3q}{4}}(\mathbb{R} \times \Omega) \cap D_{R_0}^{1,q}(\mathbb{R} \times \Omega) \cap D^{1,q}_{\perp}(\mathbb{R} \times \Omega).$$

We can obtain solutions $(v, p)$ and $(w, \pi)$ to (6.10) and (6.11), respectively, as a fixed point of the mapping

$$\mathcal{N} : K^q_\Lambda(\mathbb{R} \times \Omega) \to K^q_\Lambda(\mathbb{R} \times \Omega),$$

$$\mathcal{N}(v + w, p + \pi) := A^{-1}_0(\mathcal{R}_1(v, w, V, W) + \mathcal{R}_2(v, w, V, W)).$$

We shall show that $\mathcal{N}$ is a contracting self-mapping on a ball of sufficiently small radius. For this purpose, let $\rho > 0$ and consider some $(v + w, p + \pi) \in K^q_\Lambda \cap B_{\rho}$. Suitable estimates of $\mathcal{R}_1$ and $\mathcal{R}_2$ in combination with a smallness assumption on $\varepsilon_0$ from (6.3) are needed to guarantee that $\mathcal{N}$ has the desired properties. We first estimate the terms of $\mathcal{R}_1$. Since $q \in \left[\frac{2}{3}, \frac{3}{4}\right]$ implies $\frac{4q}{3q - 2} \leq 2 \leq \frac{3q}{3q - 2}$, we can employ first Hölder’s inequality and then interpolation to estimate

$$\|v \cdot \nabla v\|_q \leq \|v\|_{\frac{2q}{3q - 2}} \|\nabla v\|_2 \leq \lambda^{-\frac{1}{2}} \|v\|_{X^q_\Lambda} \|\nabla v\|^{\frac{3q - 2}{3q}} \|\nabla v\|^{\frac{1}{2} - \theta}$$

with $\theta = \frac{10q - 12}{q}$. It thus follows by the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^{\frac{3q}{3q - 2}}(\Omega)$ that

$$\|v \cdot \nabla v\|_q \leq c_2 \lambda^{-\frac{1}{2} - \frac{\theta}{2}} \|v\|_{X^q_\Lambda} \|\nabla^2 v\|^{1 - \theta} \leq c_2 \lambda^{-\frac{3q - 3}{2}} \|v\|_{X^q_\Lambda} \leq c_2 \lambda^{-\frac{3q - 3}{2}} \rho^2.$$  \hspace{1cm} (6.13)

The other terms in the definition of $\mathcal{R}_1$ can be estimated in a similar fashion to conclude in combination with (6.3) that

$$\|\mathcal{R}_1(v, w, V, W)\|_{L^q(\Omega)} \leq c_3(\lambda^{-\frac{3q - 3}{2}} \rho^2 + \lambda^{-\frac{1}{2}} \rho \varepsilon_0 + \rho \varepsilon_0 + \varepsilon_0^2 + \varepsilon_0).$$

An estimate of $\mathcal{R}_2$ is required both in the $L^q_{\perp}(\mathbb{R} \times \Omega)$ and $L^{\frac{3q}{3q - 2}}_{\perp}(\mathbb{R} \times \Omega)$ norm. Observe that

$$\|\mathcal{P}_\perp u_\infty \partial_1 v\|_{L^q_{\perp}(\mathbb{R} \times \Omega)} \leq c_4 \|\mathcal{P}_\perp u_\infty\|_{\infty} \|\partial_1 v\|_q \leq c_4 \lambda^{-\frac{1}{2}} \varepsilon_0 \rho.$$  \hspace{1cm} (6.14)

With the help of the embedding properties in Theorem 4.1, the other terms in $\mathcal{R}_2$ can be estimated to obtain

$$\|\mathcal{R}_2(v, w, V, W)\|_{L^q_{\perp}(\mathbb{R} \times \Omega)} \leq c_3(\lambda^{-\frac{1}{2}} \varepsilon_0 \rho + \rho^2 + \rho \varepsilon_0 + \varepsilon_0^2 + \lambda \varepsilon_0 + \varepsilon_0).$$

The embedding properties can also be used to establish an $L^{\frac{3q}{3q - 2}}_{\perp}(\mathbb{R} \times \Omega)$ estimate of $\mathcal{R}_2$. For example,

$$\|w \cdot \nabla w\|_{L^q_{\perp}(\mathbb{R} \times \Omega)} \leq c_6 \|w\|_{L^{\frac{3q}{3q - 2}}_{\perp}(\mathbb{R}; L^\infty(\Omega))} \|\nabla w\|_{L^\infty_{\perp}(\mathbb{R}; L^{\frac{3q}{3q - 2}}(\Omega))} \leq c_7 \rho^2,$$

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where Theorem 4.1 is utilized with $\alpha = 0$ and $\beta = 1$ in the last inequality. For this particular utilization of Theorem 4.1, it is required that $q \geq \frac{6}{5}$. Further note that

$$\|P_\perp u_\infty \partial_1 v\|_{\frac{3q}{3-q}} \leq \varepsilon_0 \|v\|_{X^1} \leq \varepsilon_0 \rho,$$

which explains the choice of the exponent $\frac{3q}{3-q}$ in the setting of the mapping $\mathcal{N}$. The rest of the terms in $\mathcal{R}_2$ can be estimated to conclude

$$\|\mathcal{R}_2(v, w, \mathcal{V}, \mathcal{W})\|_{\frac{3q}{L_{\per}}(\mathbb{R} \times \Omega)} \leq c_8 \left( \rho^2 + \rho \varepsilon_0 + \varepsilon_0 + \lambda \varepsilon_0 + \varepsilon_0 \right).$$

We can now conclude from Corollary 5.5, recall that $\|A^{-1}_O\|$ is independent on $\lambda$, the estimate

$$\mathcal{N}(v + w, \pi)_{\mathcal{X}_3^1} \leq \|A^{-1}_O\| \cdot \left( \|\mathcal{R}_1\|_{L^r(\Omega)} + \|\mathcal{R}_2\|_{\frac{3q}{q} L_{\per}^{\frac{3q}{3-q}}(\mathbb{R} \times \Omega)} \right)$$

$$\leq c_9 \left( \lambda^{-\frac{3q-3}{q}} \rho^2 + \lambda^{-1} \varepsilon_0 \rho + \lambda^{-\frac{1}{2}} \rho \varepsilon_0 + \rho \varepsilon_0 + \varepsilon_0 + \lambda \varepsilon_0 + \varepsilon_0 \right).$$

In particular, $\mathcal{N}$ becomes a self-mapping on $B_\rho$ if

$$c_9 \left( \lambda^{-\frac{3q-3}{q}} \rho^2 + \lambda^{-1} \varepsilon_0 \rho + \lambda^{-\frac{1}{2}} \rho \varepsilon_0 + \rho \varepsilon_0 + \varepsilon_0 + \lambda \varepsilon_0 + \varepsilon_0 \right) \leq \rho.$$ 

One may choose $\varepsilon_0 := \lambda^2$ and $\rho := \lambda$ to find the above inequality satisfied for sufficiently small $\lambda$. For such choice of parameters, one may further verify that $\mathcal{N}$ is also a contraction. By the contraction mapping principle, existence of a fixed point for $\mathcal{N}$ follows. This concludes the proof. \hfill \square

**Remark 6.2.** As for classical anisotropic Sobolev spaces, the trace operator for time-periodic Sobolev spaces is continuous and surjective for any $r \in (1, \infty)$ in the setting

$$\text{Tr} : W^{1,2\cdot r}_{\per} (\mathbb{R} \times \Omega) \to W^{1-\frac{1}{2} \cdot r}_{\per} (\mathbb{R} \times \partial \Omega) := W^{1-\frac{1}{2} \cdot r}_{\per} (\mathbb{R} ; L^r(\Omega)) \cap L^r(\mathbb{R} ; W^{2-\frac{1}{2} \cdot r}(\Omega)).$$

Consequently, for the Sobolev space in (6.4), in which a solution $u$ is established in Theorem 6.1, we find that

$$\text{Tr} : W^{1,2\cdot q}_{\per, \perp} (\mathbb{R} \times \Omega) \to W^{1-\frac{3\cdot q-3}{3\cdot q}}_{\per, \perp} (\mathbb{R} \times \partial \Omega)$$

is continuous and surjective. It therefore seems natural that Theorem 6.1 would hold for arbitrary boundary values $u_*$ in the space on the right-hand side in (6.15). In other words, it seems we are requiring too much regularity on $u_*$ in (6.2). We leave as an open question as to whether or not the regularity assumptions on the boundary values can be weakened.
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