Center-Outward Sign- and Rank-Based Quadrant, Spearman, and Kendall Tests for Multivariate Independence

Hongjian Shi, Mathias Drton, Marc Hallin and Fang Han

Abstract

Defining multivariate generalizations of the classical univariate ranks has been a long-standing open problem in statistics. Optimal transport has been shown to offer a solution by transporting data points to grid approximating a reference measure (Chernozhukov et al., 2017; Hallin, 2017; Hallin et al., 2021a). We take up this new perspective to develop and study multivariate analogues of popular correlations measures including the sign covariance, Kendall’s tau and Spearman’s rho. Our tests are genuinely distribution-free, hence valid irrespective of the actual (absolutely continuous) distributions of the observations. We present asymptotic distribution theory for these new statistics, providing asymptotic approximations to critical values to be used for testing independence as well as an analysis of power of the resulting tests. Interestingly, we are able to establish a multivariate elliptical Chernoff–Savage property, which guarantees that, under ellipticity, our nonparametric tests of independence when compared to Gaussian procedures enjoy an asymptotic relative efficiency of one or larger. Hence, the nonparametric tests constitute a safe replacement for procedures based on multivariate Gaussianity.
1 Introduction

The problem of testing for independence between two random variables with unspecified densities has been among the very first applications of rank-based methods in statistical inference. Spearman’s correlation coefficient was proposed in the early 1900s (Spearman, 1904), and Kendall’s rank correlation goes back to Kendall (1938), long before Wilcoxon (1945) gave his rank sum and signed rank tests for location.

The multivariate version of the same problem—testing independence between two random vectors with unspecified densities—is significantly harder, crucially due to the difficulty of defining a multivariate counterpart to univariate ranks. Indeed, for \( d > 1 \) the real space \( \mathbb{R}^d \) lacks a canonical ordering. As a result, the problem of defining, in dimension \( d > 1 \), concepts of signs and ranks enjoying the properties that make the traditional ranks so successful in univariate statistical inference has been an open problem for more than half a century. One of the most important properties is the exact distribution-freeness (for i.i.d. samples from absolutely continuous distributions). In an important new development involving optimal transport, the concept of center-outward ranks and signs was proposed recently by Chernozhukov et al. (2017), Hallin (2017), and Hallin et al. (2021a) and enjoys a property of “maximal distribution-freeness”, contrary to earlier concepts put forth in work such as Puri and Sen (1971); Oja (2010); Liu and Singh (1993); Zuo and He (2006); Hallin and Paindaveine (2002b,a).

For testing independence between two random vectors, the first attempt to provide a rank-based alternative to the Gaussian likelihood ratio method of Wilks (1935) was developed in Chapter 8 of Puri and Sen (1971) and, for almost thirty years, has remained the only rank-based approach to the problem. The proposed tests, however, are based on componentwise rankings and are not distribution-free—unless, of course, both vectors have dimension one, in which case we are back to the traditional context of bivariate independence (see, e.g., Chapter III.6 of Hájek and Šidák (1967)). This issue persists in more recent work, e.g., that of Puri and Sen (1971), Randles (1989), Gieser (1993), Gieser and Randles (1997), Taskinen et al. (2003, 2004), and Taskinen et al. (2005).

We note here that the above work does provide test statistics that are asymptotically distribution-free in subclasses such as elliptical distributions. From the perspective we take here, such subclasses are too restrictive. Moreover, there is a crucial difference between finite-sample and asymptotic distribution-freeness. Indeed, one should be wary that a sequence of tests \( \psi_n^{(n)} \) with asymptotic size \( \lim_{n \to \infty} \mathbb{E}_P[\psi_n^{(n)}] = \alpha \) under any element \( P \) in a
class \( \mathcal{P} \) of distributions does not necessarily have asymptotic size \( \alpha \) under unspecified \( P \in \mathcal{P} \): the convergence of \( E_P[\psi(n)] \) to \( \alpha \), indeed, typically is not uniform over \( \mathcal{P} \), so that, in general, \( \lim_{n \to \infty} \sup_{P \in \mathcal{P}} E_P[\psi(n)] \neq \alpha \). Genuinely distribution-free tests \( \phi(n) \), where \( E_P[\psi(n)] \) does not depend on \( P \), do not suffer that problem, and this is why finite-sample distribution-freeness is a fundamental property.

Palliating these limitations of the existing procedures by defining genuinely distribution-free—now over the class of all absolutely continuous distributions—multivariate extensions of the quadrant, Spearman, and Kendall tests, based on the concept of center-outward ranks and signs, is thus highly desirable. It is the objective this paper.

While this paper is focusing on quadrant, Spearman, and Kendall tests of independence, other tests have been considered in the literature. Center-outward ranks and signs have been used recently by Shi et al. (2021a) in the construction of distribution-free versions of distance covariance tests for multivariate independence, and a general framework for designing distribution-free tests of multivariate independence that are consistent and statistically efficient based on center-outward ranks and signs has been developed in Shi et al. (2021b). Multivariate ranks (based on measure transportation to the unit cube rather than the unit ball) have been used similarly in Ghosal and Sen (2021), Deb and Sen (2021).

Center-outward ranks and signs also have been used successfully in other statistical problems: construction of R-estimators (Hallin et al., 2021b, 2020b) in VARMA models, rank tests for multiple-output regression and MANOVA (Hallin et al., 2020a), and two-sample goodness-of-fit tests (Deb and Sen, 2021; Deb et al., 2021; Hallin and Mordant, 2021). We show here how center-outward ranks and signs naturally allow us to define distribution-free multivariate versions of the popular quadrant, Spearman, and Kendall tests.

The paper is organized as follows. Section 2 briefly reviews the notion of center-outward ranks and signs, and Section 3 introduces our tests of multivariate independence based on center-outward ranks and signs. In Section 4, we establish an elliptical Chernoff–Savage property for our center-outward test based on van der Waerden scores, which uniformly dominates, against Konijn alternatives, Wilks’ test for multivariate independence, and we also derive an analog of Hodges and Lehmann (1956)’s result for the problem under study. This paper ends with a short conclusion in Section 5. All the proofs are relegated to appendix.
2 Center-outward distribution functions, ranks, and signs

2.1 Definitions

Denoting by $S_d$ and $S_{d-1}$, respectively, the open unit ball and the unit hypersphere in $\mathbb{R}^d$, let $U_d$ stand for the spherical\(^1\) uniform distribution over $S_d$. Let $P$ belong to the class $\mathcal{P}_d$ of Lebesgue-absolutely continuous distributions over $\mathbb{R}^d$. The main result in McCann (1995) then implies the existence of an a.e. unique convex (and lower semi-continuous) function $\phi : \mathbb{R}^d \to \mathbb{R}$ with gradient $\nabla \phi$ such that $\nabla \phi \# P = U_d$. Call center-outward distribution function of $P$ any version $F_\pm$ of this a.e. unique gradient.

Further properties of $F_\pm$ require further regularity assumptions. Assume that $P$ is in the so-called class $\mathcal{P}_d^+ \subset \mathcal{P}_d$ of distributions with nonvanishing densities—namely, the class of distributions with density $f := dP/d\mu_d$ ($\mu_d$ the $d$-dimensional Lebesgue measure) such that, for all $D \in \mathcal{P}_d^+$, there exist constants $\lambda_{D,P}^-$ and $\lambda_{D,P}^+$ satisfying

$$0 < \lambda_{D,P}^- \leq f(z) \leq \lambda_{D,P}^+ < \infty$$

for all $z$ with $\|z\| \leq D$.

Then, it follows from Figalli (2018) that there exists a version of $F_\pm$ defining a homeomorphism between the punctured unit ball $S_d \setminus \{0\}$ and $\mathbb{R}^d \setminus F_\pm^{-1}(\{0\})$; that version has a continuous inverse $Q_\pm$ (with domain $S_d \setminus \{0\}$), which naturally qualifies as $P$’s center-outward quantile function. Figalli’s result is extended, in del Barrio et al. (2020), to a more general\(^3\) class $\mathcal{P}_d^\pm$ of absolutely continuous distributions, while the definition of $F_\pm$ given in Hallin et al. (2021a) aims at selecting, for each $P \in \mathcal{P}_d$, a version of $\nabla \phi$ which, whenever $P \in \mathcal{P}_d^+$, is yielding that homeomorphism. For the sake of simplicity, since we are not interested in quantiles, we stick here to the a.e. unique definition given above for $P \in \mathcal{P}_d$, and, whenever asymptotic statements are made, to $P \in \mathcal{P}_d^+$.

Turning to sample quantities, denote by $Z^{(n)} := (Z_1^{(n)}, \ldots, Z_n^{(n)})$, $n \in \mathbb{N}$ a triangular array of i.i.d. $d$-dimensional random vectors with distribution $P$. Associated with $Z^{(n)}$ is the empirical center-outward distribution function $F_\pm^{(n)}$\(^2\).

---

\(^1\)Namely, the spherical distribution with uniform (over $[0,1]$) radial density—equivalently, the product of a uniform over the distances to the origin and a uniform over the unit sphere $S_{d-1}$. For $d = 1$, $U_1$ coincides with the Lebesgue uniform over $(-1,1)$.

\(^2\)We borrow from measure transportation the convenient notation $T \# P$ ($T : \mathbb{R}^d \to \mathbb{R}^d$ pushes $P$ forward to $T \# P$) for the distribution of $T(Z)$ under $Z \sim P$.

\(^3\)Namely, $\mathcal{P}_d^+ \subseteq \mathcal{P}_d^\pm \subseteq \mathcal{P}_d$.
mapping the $n$-tuple $Z_1^{(n)}, \ldots, Z_n^{(n)}$ to a “regular” grid $\mathcal{G}_n$ of the unit ball $S_d$. That regular grid $\mathcal{G}_n$ is obtained as follows:

(a) first factorize $n$ into $n = n_R n_S + n_0$, with $0 \leq n_0 < \min(n_R, n_S)$;

(b) next consider a “regular array” $\mathcal{G}_{n_S} := \{s_1^{n_S}, \ldots, s_{n_S}^{n_S}\}$ of $n_S$ points on the sphere $S_{d-1}$ (see the comment below);

(c) construct the grid consisting in the collection $\mathcal{G}_n$ of the $n_Rn_S$ points of the form

$$(r/(n_R + 1))s_s^{n_S}, \quad r = 1, \ldots, n_R, \quad s = 1, \ldots, n_S,$$

along with $(n_0$ copies of) the origin in case $n_0 \neq 0$: in total $n - (n_R - 1)$ or $n$ distinct points, thus, according as $n_0 > 0$ or $n_0 = 0$.

By “regular” we mean “as regular as possible”, in the sense, for example, of the low-discrepancy sequences of the type considered in numerical integration, Monte-Carlo methods, and experimental design. The only mathematical requirement needed for the asymptotic results below is the weak convergence, as $n_S \to \infty$, of the uniform discrete distribution over $\mathcal{G}_{n_S}$ to the uniform distribution over $S_{d-1}$. A uniform i.i.d. sample of points over $S_{d-1}$ (almost surely) satisfies such a requirement. However, one easily can construct arrays that are “more regular” than an i.i.d. one. For instance, one could see that $n_S$ or $n_S - 1$ of the points in $\mathcal{G}_n$ are such that $-s_s^{n_S}$ also belongs to $\mathcal{G}_{n_S}$, so that $\|\sum_{s=1}^{n_S}s_s^{n_S}\| = 0$ or 1 according as $n_S$ is even or odd. One also could consider factorizations of the form $n = n_R n_S + n_0$ with $n_S$ even, then require $\mathcal{G}_{n_S}$ to be symmetric with respect to the origin, yielding $\sum_{s=1}^{n_S}s_s^{n_S} = 0$.

The empirical counterpart $F_{\pm}^{(n)}$ of $F_{\pm}$ is defined as the (bijective, once the origin is given multiplicity $n_0$) mapping from $Z_1^{(n)}, \ldots, Z_n^{(n)}$ to the grid $\mathcal{G}_n$ that minimizes $\sum_{i=1}^{n}\|F_{\pm}^{(n)}(Z_i^{(n)}) - Z_i^{(n)}\|^2$. That mapping is unique with probability one; in practice, it is obtained via a simple optimal assignment (pairing) algorithm (a linear program; see Hallin et al. (2021a) for details).

Call center-outward rank of $Z_i^{(n)}$ the integer (in $\{1, \ldots, n_R\}$ or $\{0, \ldots, n_R\}$) according as $n_0 = 0$ or not

$$R_{i,\pm} := (n_R + 1)\|F_{\pm}^{(n)}(Z_i^{(n)})\| \quad i = 1, \ldots, n$$

and center-outward sign of $Z_i^{(n)}$ the unit vector

$$S_{i,\pm}^{(n)} := F_{\pm}^{(n)}(Z_i^{(n)})/\|F_{\pm}^{(n)}(Z_i^{(n)})\| \quad \text{for } F_{\pm}^{(n)}(Z_i^{(n)}) \neq 0;$$

\footnote{Note that this implies that $n_0/n = o(1)$ as $n \to \infty$. See Mordant (2021, Chapter 7.4) for a suggestion of selecting $n_R$ and $n_S$.}

\footnote{See also Hallin and Mordant (2021) for a spherical version of the so-called Halton sequences.}
put $S_{i;=}^{(n)} = 0$ for $F_{i;=}^{(n)}(Z_{i;=}^{(n)}) = 0$.

Some desirable finite-sample properties, such as strict independence between the ranks and the signs, only hold for $n_0 = 0$ or $1$, due to the fact that the mapping from the sample to the grid is no longer injective for $n_0 \geq 2$. This, which has no asymptotic consequences (since the number $n_0$ of tied values involved is $o(n)$ as $n \to \infty$), is easily taken care of by the following tie-breaking device:

(i) randomly select $n_0$ directions $s_1^0, \ldots, s_{n_0}^0$ in $S_{nS}$, then

(ii) replace the $n_0$ copies of the origin with the new gridpoints

$$[1/2(n_R + 1)]s_1^0, \ldots, [1/2(n_R + 1)]s_{n_0}^0.$$ (2.2)

The resulting grid (for simplicity, the same notation $G_n$ is used) no longer has multiple points, and the optimal pairing between the sample and this grid is bijective; the $n_0$ smallest ranks, however, take the non-integer value $1/2$.

2.2 Main properties

This section summarizes some of the main properties of the concepts defined in Sections 2.1; further properties and the proofs can be found in Hallin et al. (2021a), Hallin et al. (2020a) and Hallin (2022).

Proposition 2.1. Let $F_{i;=}^{(n)}$ denote the center-outward distribution function of $P \in \mathcal{P}_d$. Then,

(i) $F_{i;=}^{(n)}$ is a probability integral transformation of $\mathbb{R}^d$: namely, $Z \sim P$ iff $F_{i;=}^{(n)}(Z) \sim U_d$; by construction, $\|F_{i;=}^{(n)}(Z)\|$ is uniform over $[0,1)$, $F_{i;=}^{(n)}(Z)/\|F_{i;=}^{(n)}(Z)\|$ is uniform over the sphere $S_{d-1}$, and they are mutually independent.

Let $Z_1^{(n)}, \ldots, Z_n^{(n)}$ be i.i.d. with distribution $P \in \mathcal{P}_d$ and center-outward distribution function $F_{i;=}^{(n)}$. Then,

(ii) $(F_{i;=}^{(n)}(Z_1^{(n)}), \ldots, F_{i;=}^{(n)}(Z_n^{(n)}))$ is uniformly distributed over the $n!/n_0!$ permutations with repetitions of the gridpoints in $\mathcal{S}_n$ with the origin counted as $n_0$ indistinguishable points (resp. the $n!$ permutations of $\mathcal{S}_n$ if either $n_0 \leq 1$ or the tie-breaking device described in Section 2.1 is adopted);

(iii) if either $n_0 = 0$ or the tie-breaking device described in Section 2.1 is adopted, the $n$-tuple of center-outward ranks $(R_1^{(n)}, \ldots, R_{n_0}^{(n)})$ and the $n$-tuple of center-outward signs $(S_1^{(n)}, \ldots, S_{n_0}^{(n)})$ are mutually independent;

(iv) if either $n_0 \leq 1$ or the tie-breaking device described in Section 2.1 is adopted.
adopted, \((F_{\pm}^{(n)}(Z_1^{(n)}), \ldots, F_{\pm}^{(n)}(Z_n^{(n)}))\) is strongly essentially maximal ancillary.\(^6\)

Assuming, moreover, that \(P \in \mathcal{P}_d^+\),
(v) (Glivenko–Cantelli)
\[
\max_{1 \leq i \leq n} \left\| F_{\pm}^{(n)}(Z_i^{(n)}) - F_{\pm}(Z_i) \right\| \to 0 \text{ a.s. as } n \to \infty.
\]

Center-outward distribution functions, ranks, and signs also inherit, from the invariance of squared Euclidean distances, elementary but quite remarkable invariance and equivariance properties under orthogonal transformations and global rescaling. Denote by \(F_{\pm}^Z\) the center-outward distribution function of \(Z\) and by \(F_{\pm}^{Z,(n)}\) the empirical distribution function of an i.i.d. sample \(Z_1, \ldots, Z_n\) associated with a grid \(\varnothing_n\).

**Proposition 2.2.** Let \(\mu \in \mathbb{R}^d\), \(k \in \mathbb{R}^+\), and denote by \(O\) a \(d \times d\) orthogonal matrix. Then,
(i) \(F_{\pm}^{\mu + kOZ}(\mu + Oz) = OF_{\pm}^{Z}(z), \ z \in \mathbb{R}^d;\)
(ii) denoting by \(F_{\pm}^{\mu + kOZ,(n)}\) the empirical distribution function of the sample \(\mu + kOZ_1, \ldots, \mu + kOZ_n\) associated with the grid \(O\varnothing_n\) (hence by \(F_{\pm}^{Z,(n)}\) the empirical distribution function of the sample \(Z_1, \ldots, Z_n\) associated with the grid \(\varnothing_n\)),
\[
F_{\pm}^{\mu + kOZ,(n)}(\mu + kOZ_i) = OF_{\pm}^{Z,(n)}(Z_i), \quad i = 1, \ldots, n. \quad (2.3)
\]

3 Rank-based tests for multivariate independence

3.1 Center-outward test statistics for multivariate independence

In this section, we describe the test statistics we are proposing for testing independence between two random vectors. Consider a sample
\[(X_1', X_2'), (X_1', X_2'), \ldots, (X_1', X_2');\]
of \(n\) i.i.d. copies of some \((d_1 + d_2) = d\)-dimensional random vector \((X_1', X_2')'\) with Lebesgue-absolutely continuous distribution \(P \in \mathcal{P}_d\) and density \(f\). We are interested in the null hypothesis under which \(X_1\) and \(X_2\), with unspecified marginal distributions \(P_1\) (density \(f_1\)) and \(P_2\) (density \(f_2\)), respectively, are mutually independent: \(f\) then factorizes into \(f = f_1f_2\).

\(^6\)See Section 2.4 and Appendices D.1 and D.2 of Hallin et al. (2021a) for a precise definition and a proof of this essential property.
Denote by $R^{(n)}_{ki;\pm}$ and $S^{(n)}_{ki;\pm}$, $i = 1, 2, \ldots, n$ the center-outward rank and the sign of $X_{ki}$ computed from $X_{k1}, X_{k2}, \ldots, X_{kn}$, $k = 1, 2$, respectively. For the simplicity of notation, assume, without loss of generality as $n \to \infty$, that the grid used for computing those ranks and signs is such that $\sum_{s=1}^{n_s} s_s^{n_s} = 0$, for $d = d_1, d_2$. Also assume that $n_0 = 0$ or 1 (if necessary, after implementing the tie-breaking device described in Section 2.1). This implies that $\sum_{i=1}^{n} S^{(n)}_{ki;\pm} = 0$ for $k = 1, 2$, and moreover, that
\[
\sum_{i=1}^{n} J_k \left( R^{(n)}_{ki;\pm} / (n_R + 1) \right) S^{(n)}_{ki;\pm} = 0
\]
for any score functions $J_k : [0, 1) \to \mathbb{R}$, $k = 1, 2$.

Consider the $d_1 \times d_2$ matrices
\[
W^{(n)}_{\text{sign}} := \frac{1}{n} \sum_{i=1}^{n} S^{(n)}_{1i;\pm} S^{(n)'}_{2i;\pm}, \tag{3.1}
\]
\[
W^{(n)}_{S} := \frac{1}{n(n_R + 1)^2} \sum_{i=1}^{n} R^{(n)}_{1i;\pm} R^{(n)}_{2i;\pm} S^{(n)}_{1i;\pm} S^{(n)'}_{2i;\pm}, \tag{3.2}
\]
\[
W^{(n)}_{k} := \binom{n}{2}^{-1} \sum_{i<i'} \text{sign} \left[ \left( R^{(n)}_{1i;\pm} S^{(n)}_{1i;\pm} - R^{(n)}_{1i';\pm} S^{(n)}_{1i';\pm} \right) \right.
\times \left. \left( R^{(n)}_{2i;\pm} S^{(n)}_{2i;\pm} - R^{(n)}_{2i';\pm} S^{(n)}_{2i';\pm} \right) \right], \tag{3.3}
\]
where $\text{sign}[M]$ stands for the matrix collecting the signs of the entries of a real matrix $M$, and
\[
W^{(n)}_{J} := \frac{1}{n} \sum_{i=1}^{n} J_1 \left( \frac{R^{(n)}_{1i;\pm}}{n_R + 1} \right) J_2 \left( \frac{R^{(n)}_{2i;\pm}}{n_R + 1} \right) S^{(n)}_{1i;\pm} S^{(n)'}_{2i;\pm}, \tag{3.4}
\]
where the score functions $J_k : [0, 1) \to \mathbb{R}$, $k = 1, 2$ are the square-integrable differences of two monotone increasing functions, with
\[
0 < \sigma^2_{J_k} := \int_{0}^{1} J_k^2(u) du < \infty. \tag{3.5}
\]

Those matrices defined in (3.1)–(3.4) clearly constitute matrices of cross-covariance measurements based on center-outward ranks and signs (for $W^{(n)}_{\text{sign}}$, signs only). For $d_1 = 1 = d_2$, it is easily seen that $W^{(n)}_{\text{sign}}$, $W^{(n)}_{S}$, and $W^{(n)}_{k}$, up to scaling constants, reduce to the quadrant, Spearman, and Kendall test statistics, while $W^{(n)}_{J}$ yields a score-based extension of Spearman’s correlation coefficient.
3.2 Asymptotic representation and asymptotic normality

Each of the rank-based matrices defined in (3.1)–(3.4) has an asymptotic representation in terms of i.i.d. variables. More precisely, defining $S_{ki;\pm}$ as $F_{k;\pm}(X_{ki})/\|F_{k;\pm}(X_{ki})\|$ if $F_{k;\pm}(X_{ki}) \neq 0$ and 0 otherwise for $k = 1, 2$, let

$$W_{\text{sign}}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} S_{1i;\pm} S_{2i;\pm}',$$

(3.6)

$$W_{S}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} F_{1;\pm}(X_{1i}) F_{2;\pm}(X_{2i}),$$

(3.7)

$$W_{K}^{(n)} := \left(\frac{n}{2}\right)^{-1} \sum_{i< i'} \text{sign} \left[ \left( F_{1;\pm}(X_{1i}) - F_{1;\pm}(X_{1i'}) \right) \right.$$

$$\times \left( F_{2;\pm}(X_{2i}) - F_{2;\pm}(X_{2i'}) \right) \left. \right]^{'},$$

(3.8)

and

$$W_{J}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} J_{1} \left( \|F_{1;\pm}(X_{1i})\| \right) J_{2} \left( \|F_{2;\pm}(X_{2i})\| \right) S_{1i;\pm} S_{2i;\pm}'.$$

(3.9)

The following asymptotic representation results then hold under the null hypothesis of independence (hence, also under contiguous alternatives).

**Proposition 3.1.** Under the null hypothesis of independence, as $n_{R}$ and $n_{S}$ tend to infinity, $\text{vec}(W_{\text{sign}}^{(n)} - W_{\text{sign}}^{(n)})$, $\text{vec}(W_{S}^{(n)} - W_{S}^{(n)})$, $\text{vec}(W_{K}^{(n)} - W_{K}^{(n)})$, and, provided that $J_{1}$ and $J_{2}$ are the square-integrable differences of two monotone increasing functions, $\text{vec}(W_{J}^{(n)} - W_{J}^{(n)})$ is $o_{q.m.}(n^{-1/2})$.

The asymptotic normality for $\text{vec}W_{\text{sign}}^{(n)}$, $\text{vec}W_{S}^{(n)}$, $\text{vec}W_{K}^{(n)}$, and $\text{vec}W_{J}^{(n)}$ follows immediately from the asymptotic representation results and the standard central-limit behavior of $\text{vec}W_{\text{sign}}^{(n)}$, $\text{vec}W_{S}^{(n)}$, $\text{vec}W_{K}^{(n)}$, and $\text{vec}W_{J}^{(n)}$.

**Proposition 3.2.** Under the null (independence) hypothesis, as $n_{R}$ and $n_{S}$ tend to infinity, $n^{1/2}\text{vec}W_{\text{sign}}^{(n)}$, $n^{1/2}\text{vec}W_{S}^{(n)}$, $n^{1/2}\text{vec}W_{K}^{(n)}$, and $n^{1/2}\text{vec}W_{J}^{(n)}$ are asymptotically normal with mean vectors $0_{d_{1}d_{2}}$ and covariance matrices

$$\frac{1}{d_{1}d_{2}}I_{d_{1}d_{2}}, \quad \frac{1}{9d_{1}d_{2}}I_{d_{1}d_{2}}, \quad \frac{4}{9}I_{d_{1}d_{2}}, \quad \text{and} \quad \frac{\sigma_{0}^{2} \sigma_{1}^{2}}{d_{1}d_{2}}I_{d_{1}d_{2}},$$

respectively.
3.3 Center-outward sign, Spearman, Kendall, and score tests

Associated with $W_{\text{sign}}^{(n)}$, $W_{S}^{(n)}$, $W_{K}^{(n)}$, and $W_{J}^{(n)}$ are the sign, Spearman, Kendall, and score test statistics

\[ T_{\text{sign}}^{(n)} := nd_1d_2\|W_{\text{sign}}^{(n)}\|_F^2, \quad T_{S}^{(n)} := 9nd_1d_2\|W_{S}^{(n)}\|_F^2, \]
\[ T_{K}^{(n)} := \frac{9n}{4}\|W_{K}^{(n)}\|_F^2, \quad \text{and} \quad T_{J}^{(n)} := \frac{nd_1d_2}{\sigma_{J_1}^2\sigma_{J_2}^2}\|W_{J}^{(n)}\|_F^2, \]

respectively, where $\|M\|_F$ stands for the Frobenius norm of a matrix $M$, and $\sigma_{J_k}^2$, $k = 1, 2$ are defined as in (3.5).

In view of the asymptotic normality results in Proposition 3.2, the tests (denoted respectively by $\psi_{\text{sign}}^{(n)}$, $\psi_{S}^{(n)}$, $\psi_{K}^{(n)}$, and $\psi_{J}^{(n)}$) rejecting the null hypothesis whenever $T_{\text{sign}}^{(n)}$, $T_{S}^{(n)}$, $T_{K}^{(n)}$, or $T_{J}^{(n)}$ exceed the $(1 - \alpha)$-quantile $\chi_{d_1,d_2;1-\alpha}^2$ of a chi-square distribution with $d_1d_2$ degrees of freedom has asymptotic level $\alpha$. These tests are strictly distribution-free, however, and exact critical values can be computed or simulated as well. The tests based on $T_{\text{sign}}^{(n)}$, $T_{S}^{(n)}$, and $T_{K}^{(n)}$ are multivariate extensions of the traditional quadrant, Spearman, and Kendall tests, respectively, to which they reduce for $d_1 = 1 = d_2$.

4 Local asymptotic power

While there is only one way for two random vectors $X_1$ and $X_2$ to be independent, their mutual dependence can take many forms. The classical benchmark, in testing for bivariate independence, is a “local” form of an independent component analysis model that goes back to Konijn (1956). A multivariate extension of such alternatives has been considered also by Gieser and Randles (1997) and Taskinen et al. (2003) in the elliptical context. We extend it further here to more general, non-elliptical situations.

4.1 Generalized Konijn alternatives

Let $X^* = (X_1^*, X_2^*)'$, where $X_1^*$ and $X_2^*$ be mutually independent random vectors, with absolutely continuous distributions $P_1$ over $\mathbb{R}^{d_1}$ and $P_2$ over $\mathbb{R}^{d_2}$ and densities $f_1$ and $f_2$, respectively; then $X^*$ has density $f = f_1f_2$ over $\mathbb{R}^d$. Consider

\[ X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = M_\delta \begin{pmatrix} X_1^* \\ X_2^* \end{pmatrix} := \begin{pmatrix} (1 - \delta)I_{d_1} + \delta M_1 \\ \delta M_2 \end{pmatrix} \begin{pmatrix} X_1^* \\ X_2^* \end{pmatrix} \]

(4.1)
where $\delta \in \mathbb{R}$ and $M_1 \in \mathbb{R}^{d_1 \times d_2}$, $M_2 \in \mathbb{R}^{d_2 \times d_1}$ are nonzero. For given $P_1$, $P_2$, $M_1$, and $M_2$, the distribution $P^X$ of $X$ belongs to a one-parameter family $\mathcal{P}^X := \{P^X_\delta \mid \delta \in \mathbb{R}\}$.

On $f_1$ and $f_2$, we make the following assumption.

**Assumption 4.1.**

(K1) The densities $f_1$ and $f_2$ are such that
\[
\int_{\mathbb{R}^{d_k}} x f_k(x) dx = 0 \quad \text{and} \quad 0 < \int_{\mathbb{R}^{d_k}} xx' f_k(x) dx =: \Sigma_k < \infty, \quad k = 1, 2.
\]

(K2) The functions $x_k \mapsto (f_k(x_k))^{1/2}$, $k = 1, 2$ admit quadratic mean partial derivatives\(^7\)
\[
D_\ell[(f_k)^{1/2}], \quad \ell = 1, \ldots, d_k, \quad k = 1, 2.
\]

(K3) Letting
\[
\varphi := (\varphi_1', \varphi_2')' := (\varphi_1;1, \ldots, \varphi_1;d_1, \varphi_2;1, \ldots, \varphi_2;d_1)',
\]
with
\[
\varphi_{k;\ell} := -2D_\ell[(f_k)^{1/2}]/(f_k)^{1/2} \overset{a.e.}{=} -\partial_\ell f_k/f_k, \quad \ell = 1, \ldots, d_k, \quad k = 1, 2,
\]
it holds that, for $k = 1, 2$ and $\ell = 1, \ldots, d_k$, $0 < \int_{\mathbb{R}^{d_k}} (\varphi_{k;\ell}(x))^2 < \infty$, and\(^8\)
\[
J_k := \text{Var}(X_k' \varphi_k(X_k)) = \int_{\mathbb{R}^{d_k}} (x' \varphi_k(x) - d_k)^2 f_k(x) dx < \infty.
\]

It should be stressed, however, that these assumptions are not to be imposed on the observations in order for our tests to be valid but only intend to provide an analytically convenient benchmark for the comparison of local power. Let
\[
I_k := \int_{\mathbb{R}^{d_k}} \varphi(x) \varphi'(x) f_k(x) dx < \infty.
\]

Under $P_0^X$, $X_1 = X_1^1$ and $X_2 = X_2^2$ are mutually independent; for $\delta \neq 0$, call $P^X_\delta$ a (generalized) Konijn alternative to $P_0^X$. Sequences of the form $P^X_{\delta^{-1/2} \tau}$ with $\tau \neq 0$, as we shall see, constitute local alternatives to the null hypothesis of independence in a sample of size $n$. More precisely, the following LAN property holds in the vicinity of $\delta = 0$.

\(^7\)Existence of quadratic mean partial derivatives is equivalent to quadratic mean differentiability; this was shown in Lind and Roussas (1972) and independently rediscovered by Garel and Hallin (1995, Lemma 2.1).

\(^8\)Integration by parts yields $\int_{\mathbb{R}^{d_k}} \varphi_k(x) f_k(x) dx = 0$, $\int_{\mathbb{R}^{d_k}} x' \varphi_k(x) f_k(x) dx = d_k$, and $\int_{\mathbb{R}^{d_k}} x \varphi_k(x)' f_k(x) dx = I_k$, $k = 1, 2$; see also Garel and Hallin (1995, page 555).
Proposition 4.1. Let $P_1$ and $P_2$ satisfy Assumption 4.1. Then, denoting by $X^{(n)} := (X_1, \ldots, X_n)$, $n \in \mathbb{N}$ a triangular array of $n$ independent copies of $X \sim P_0^X$, for given nonzero $M_1$ and $M_2$, the family $P_X$ of Konijn alternatives is LAN at $\delta = 0$ with root-$n$ contiguity rate, central sequence

$$
\Delta^{(n)}(X^{(n)}) := \sum_{i=1}^{n} \left[ X_{1i}'M_2'\varphi_2(X_{2i}) + X_{2i}'M_1'\varphi_1(X_{1i}) - \left( X_{1i}'\varphi_1(X_{1i}) - d_1 \right) - \left( X_{2i}'\varphi_2(X_{2i}) - d_2 \right) \right]
$$

and Fisher information

$$
\gamma^2 := J_1 + J_2 + \text{vec}'(\Sigma_1) \text{vec}(M_2'\mathcal{I}_2M_2) + \text{vec}'(\Sigma_2) \text{vec}(M_1'\mathcal{I}_1M_1) + \text{tr}(M_1M_2) + \text{tr}(M_2M_1).
$$

Namely, under $P_0^X$,

$$
\Lambda^{(n)}(X^{(n)}) := \log \frac{dP_X}{dP_0^{X}}(X^{(n)}) = \tau \Delta^{(n)}(X^{(n)}) - \frac{1}{2} \gamma^2 + o_P(1)
$$

and $\Delta^{(n)}(X^{(n)})$ is asymptotically normal, with mean zero and variance $\gamma^2$ as $n \to \infty$.

4.2 Limiting distributions and Pitman efficiencies

In this section, we aim to establish elliptical Chernoff–Savage and Hodges–Lehmann results for our center-outward test based on van der Waerden and Wilcoxon scores comparing to Wilks’ test, respectively; compare Chernoff and Savage (1958) and Hodges and Lehmann (1956). To this end, we first derive the limiting distributions of $T^{(n)}_J$ and $T^{(n)}_K$ under the sequence of alternatives $P_1^{X^{(n-1/2)}_J}$.

Proposition 4.2. Let $P_1$ and $P_2$ satisfy Assumption 4.1. Then, if observations are $n$ independent copies with distribution $P_1^{X^{(n-1/2)}_J}$, for given nonzero $M_1$ and $M_2$,

(i) the limiting distribution of the test statistic $T^{(n)}_J$ is noncentral chi-square with $d_1d_2$ degrees of freedom and noncentrality parameter

$$
\frac{\tau^2 d_1d_2}{\sigma_1^2 \sigma_2^2} \left\| \mathbb{E}_{H_0} \left[ J_1(F_{1;\pm}(X_1))RJ_2(F_{2;\pm}(X_2))' \right] \right\|_F^2,
$$

where $R := X_1'M_2'\varphi_2(X_2) + X_2'M_1'\varphi_1(X_1)$ and

$$
J_k(u) := J_k(\|u\|) \frac{u}{\|u\|} 1_{\{\|u\| \neq 0\}}, \quad u \in \mathbb{S}_d;
$$
(ii) the limiting distribution of the test statistic $T^{(n)}_{K}$ is noncentral chi-square with $d_1d_2$ degrees of freedom and noncentrality parameter

$$9\tau^2\|E_{H_0}\left[F_{\Pi_{1:1}}(X_1)RF_{2:1}(X_2)\right]\|_F^2,$$

where

$$(F_{k:1}(X_k))_j := 2F_{kj}\left((F_{k:1}(X_k))_j\right)-1$$

(recall $F_{kj}$ denotes the cumulative distribution function of $(F_{k:1}(X_k))_j$).

Suppose that all the conditions in Proposition 4.2 hold. Then the limiting alternative distribution of Wilks’ (log) likelihood ratio test statistic is also noncentral chi-square, with $d_1d_2$ degrees of freedom and noncentrality parameter

$$\tau^2\|\Sigma_1^{1/2}M_2^\prime\Sigma_2^{-1/2} + \Sigma_1^{-1/2}M_1\Sigma_2^{1/2}\|_F^2;$$

see, e.g., page 919 of Taskinen et al. (2005).

Now we are ready to compute the asymptotic relative efficiencies of our center-outward rank tests with respect to Wilks’ likelihood ratio test.

**Proposition 4.3.** Let $P_1$ and $P_2$ be elliptically symmetric distributions, namely, admit densities of the form

$$f_k(x_k) \propto (\det(\Sigma_k))^{-1/2} \phi_k \left(\sqrt{x_k^\prime \Sigma_k^{-1} x_k}\right), \quad k = 1, 2,$$

satisfying Assumption 4.1. Then, the Pitman asymptotic relative efficiency (ARE) of the center-outward test based on score functions $J_k$, $k = 1, 2$ with respect to Wilks’ test (denoted by $\psi_{\Pi}^{(n)}$) is

$$\text{ARE}(\psi_{J}^{(n)}, \psi_{\Pi}^{(n)}) = \frac{\|D_1C_2\Sigma_1^{1/2}M_2^\prime\Sigma_2^{-1/2}D_2C_1\Sigma_1^{-1/2}M_1\Sigma_2^{1/2}\|_F^2}{d_1d_2\sigma_1^2\sigma_2^2\|\Sigma_1^{1/2}M_2^\prime\Sigma_2^{-1/2}D_2\Sigma_1^{-1/2}M_1\Sigma_2^{1/2}\|_F^2},$$

where

$$C_k \equiv C_k(J_k, \phi_k) := E[J_k^{-1}(U)\rho_k(\tilde{F}_k^{-1}(U))],$$

$$D_k \equiv D_k(J_k, \phi_k) := E[J_k^{-1}(U)\tilde{F}_k^{-1}(U)],$$

$\rho_k := -\phi_k'/\phi_k$, $\tilde{F}_k$ denotes the cumulative distribution function of $\|Y_k\|$ with $Y_k := \Sigma_k^{-1/2}X_k$, and $U$ stands for a random variable uniformly distributed over $(0, 1)$. In particular, if $\Sigma_1M_2^\prime = M_1\Sigma_2$, we have

(i) $\text{ARE}(\psi_{J_1}^{(n)}, \psi_{\Pi}^{(n)}) \geq 1$, where $J_1^{aw}$, $k = 1, 2$ are the van der Waerden score functions $J_k^{aw}(u) := (F_{\chi^2_k}^{-1}(u))^{1/2}$ with $F_{\chi^2_k}$ the $\chi^2_k$ cumulative
distribution function;

(ii) $\text{ARE}(\psi_{JW}^{(n)}, \psi_{N}^{(n)}) \geq \Omega(d_1, d_2) \geq 9/16$, where the Wilcoxon score functions are defined as $J_k^\psi(u) := u$ for $k = 1, 2$, and

$$
\Omega(d_1, d_2) := \frac{9(2c_{d_1}^2 + d_1 - 1)^2(2c_{d_2}^2 + d_2 - 1)^2}{1024d_1d_2c_{d_1}^2c_{d_2}^2},
$$

$$
c_d := \inf \left\{ x > 0 \mid \left( \sqrt{x}B_{\sqrt{2d-1}/2}(x) \right)' = 0 \right\},
$$

$$
B_a(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + a + 1)} \left( \frac{x}{2} \right)^{2m+a}.
$$

Gieser (1993) notices that the Pitman ARE depends on the underlying covariance structure ($\Sigma_1$ and $\Sigma_2$) for $X_1$ and $X_2$ with elliptically symmetric distributions, while most the existing literature (e.g. Gieser (1993), Gieser and Randles (1997), Taskinen et al. (2003, 2004), Taskinen et al. (2005), Hallin and Paindaveine (2008) and Deb et al. (2021)) focuses on the spherically symmetric case. The proposition above fills this gap by providing the explicit formula of ARE with general $\Sigma_k$’s. The claim (i) shows Pitman non-admissibility under ellipticity of Wilks’ test, which is uniformly dominated by our center-outward test with van der Waerden scores, for elliptically symmetric distributions. This is comparable with Theorem 4.1 in Deb et al. (2021). Claim (ii) is a multivariate extension of Hodges and Lehmann (1956)’s result; the minimum of $\Omega(d_1, d_2)$, 9/16, is achieved when $d_1, d_2 \to \infty$. One can find more numerical values of $\Omega(d_1, d_2)$ for fixed $d_1, d_2$ in Hallin and Paindaveine (2008, Table 3).

5 Conclusion

Optimal transport provides an entirely new approach to rank-based statistical inference in dimension $d \geq 2$. The new multivariate ranks retain many of the favorable properties one is used to with the classical univariate ranks. Here, we demonstrate how the new multivariate ranks can be used for a definition of multivariate versions of popular rank correlations such as Kendall’s tau or Spearman’s rho. We show how the new multivariate rank correlations yield fully distribution-free, yet powerful and computationally efficient tests of independence. A highlight of our results is the fact that the use of van der Waerden scores allows one to design a nonparametric test whose asymptotic efficiency under arbitrary elliptical densities never drops below that of Wilks’ test—not even under a Gaussian model.
References

[1] Barbour, A. D. and Eagleson, G. K. (1986). Random association of symmetric arrays. *Stochastic Anal. Appl.*, 4(3):239–281.

[2] Bhattacharya, R. N. and Ranga Rao, R. (1986). *Normal Approximation and Asymptotic Expansions* (Rpt. ed.). Robert E. Krieger Publishing Co., Inc., Melbourne, FL.

[3] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.*, 29(4):972–994.

[4] Chernozhukov, V., Galichon, A., Hallin, M., and Henry, M. (2017). Monge-Kantorovich depth, quantiles, ranks and signs. *Ann. Statist.*, 45(1):223–256.

[5] Deb, N., Bhattacharya, B. B., and Sen, B. (2021). Efficiency lower bounds for distribution-free Hotelling-type two-sample tests based on optimal transport. Available at arXiv:2104.01986v2.

[6] Deb, N. and Sen, B. (2021+). Multivariate rank-based distribution-free nonparametric testing using measure transportation. *J. Amer. Statist. Assoc.* (in press).

[7] del Barrio, E., González-Sanz, A., and Hallin, M. (2020). A note on the regularity of optimal-transport-based center-outward distribution and quantile functions. *J. Multivariate Anal.*, 180:104671, 13.

[8] Figalli, A. (2018). On the continuity of center-outward distribution and quantile functions. *Nonlinear Anal.*, 177(part B):413–421.

[9] Garel, B. and Hallin, M. (1995). Local asymptotic normality of multivariate ARMA processes with a linear trend. *Ann. Inst. Statist. Math.*, 47(3):551–579.

[10] Ghosal, P. and Sen, B. (2021+). Multivariate ranks and quantiles using optimal transport: consistency, rates, and nonparametric testing. *Ann. Statist.* (in press).

[11] Gieser, P. W. (1993). A *new nonparametric test for independence between two sets of variates*. PhD thesis, University of Florida. Available at https://ufdc.ufl.edu/AA00003658/00001 and https://www.proquest.com/docview/304041219.
[12] Gieser, P. W. and Randles, R. H. (1997). A nonparametric test of independence between two vectors. *J. Amer. Statist. Assoc.*, 92(438):561–567.

[13] Hájek, J. and Šidák, Z. (1967). *Theory of Rank Tests*. Academic Press, New York-London; Academia Publishing House of the Czechoslovak Academy of Sciences, Prague.

[14] Hallin, M. (2017). On distribution and quantile functions, ranks and signs in $\mathbb{R}^d$: a measure transportation approach. Available at https://ideas.repec.org/p/eca/wpaper/2013-258262.html.

[15] Hallin, M. (2022). Measure transportation and statistical decision theory. *Annu. Rev. Stat. Appl.*, 9(1). (in press).

[16] Hallin, M., del Barrio, E., Cuesta-Albertos, J., and Matrán, C. (2021a). Distribution and quantile functions, ranks and signs in dimension $d$: A measure transportation approach. *Ann. Statist.*, 49(2):1139–1165.

[17] Hallin, M., Hlubinka, D., and Hudecová, Š. (2020a). Fully distribution-free center-outward rank tests for multiple-output regression and MANOVA. Available at arXiv:2007.15496v1.

[18] Hallin, M., La Vecchia, D., and Liu, H. (2020b). Rank-based testing for semiparametric VAR models: a measure transportation approach. Available at arXiv:2011.06062v1.

[19] Hallin, M., La Vecchia, D., and Liu, H. (2021+b). Center-outward R-estimation for semiparametric VARMA models. *J. Amer. Statist. Assoc.* (in press).

[20] Hallin, M. and Mordant, G. (2021). On the finite-sample performance of measure transportation-based multivariate rank tests. Available at arXiv:2111.04705v2.

[21] Hallin, M. and Paindaveine, D. (2002a). Optimal procedures based on interdirections and pseudo-Mahalanobis ranks for testing multivariate elliptic white noise against ARMA dependence. *Bernoulli*, 8(6):787–815.

[22] Hallin, M. and Paindaveine, D. (2002b). Optimal tests for multivariate location based on interdirections and pseudo-Mahalanobis ranks. *Ann. Statist.*, 30(4):1103–1133.
[23] Hallin, M. and Paindaveine, D. (2008). Chernoff-Savage and Hodges-Lehmann results for Wilks’ test of multivariate independence. In Beyond parametrics in interdisciplinary research: Festschrift in honor of Professor Pranab K. Sen, volume 1 of Inst. Math. Stat. (IMS) Collect., pages 184–196. Inst. Math. Statist., Beachwood, OH.

[24] Hannan, E. J. (1956). The asymptotic powers of certain tests based on multiple correlations. J. Roy. Statist. Soc. Ser. B, 18(2):227–233.

[25] Hodges, Jr., J. L. and Lehmann, E. L. (1956). The efficiency of some nonparametric competitors of the $t$-test. Ann. Math. Statist., 27(2):324–335.

[26] Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution. Ann. Math. Statist., 19(3):293–325.

[27] Hoeffding, W. (1951). A combinatorial central limit theorem. Ann. Math. Statist., 22(4):558–566.

[28] Kendall, M. G. (1938). A new measure of rank correlation. Biometrika, 30(1/2):81–93.

[29] Konijn, H. S. (1956). On the power of certain tests for independence in bivariate populations. Ann. Math. Statist., 27(2):300–323.

[30] Lehmann, E. L. and Romano, J. P. (2005). Testing Statistical Hypotheses (3rd ed.). Springer Texts in Statistics. Springer, New York.

[31] Lind, B. and Roussas, G. (1972). A remark on quadratic mean differentiability. Ann. Math. Statist., 43(3):1030–1034.

[32] Liu, R. Y. and Singh, K. (1993). A quality index based on data depth and multivariate rank tests. J. Amer. Statist. Assoc., 88(421):252–260.

[33] McCann, R. J. (1995). Existence and uniqueness of monotone measure-preserving maps. Duke Math. J., 80(2):309–323.

[34] Mordant, G. (2021). Transporting probability measures: some contributions to statistical inference. PhD thesis, UCL-Université Catholique de Louvain. Available at http://hdl.handle.net/2078.1/250201.

[35] Oja, H. (2010). Multivariate Nonparametric Methods with R: An Approach Based on Spatial Signs and Ranks, volume 199 of Lecture Notes in Statistics. Springer, New York.
[36] Paindaveine, D. (2004). A unified and elementary proof of serial and nonserial, univariate and multivariate, Chernoff-Savage results. Stat. Methodol., 1(1-2):81–91.

[37] Parzen, E. (2004). Quantile probability and statistical data modeling. Statist. Sci., 19(4):652–662.

[38] Puri, M. L. and Sen, P. K. (1971). Nonparametric Methods in Multivariate Analysis. John Wiley & Sons, Inc., New York-London-Sydney.

[39] Randles, R. H. (1989). A distribution-free multivariate sign test based on interdirections. J. Amer. Statist. Assoc., 84(408):1045–1050.

[40] Shi, H., Drton, M., and Han, F. (2021+a). Distribution-free consistent independence tests via center-outward ranks and signs. J. Amer. Statist. Assoc. (in press).

[41] Shi, H., Hallin, M., Drton, M., and Han, F. (2021+b). On universally consistent and fully distribution-free rank tests of vector independence. Ann. Statist. (in press).

[42] Shorack, G. R. (2017). Probability for Statisticians (2nd ed.). Springer Texts in Statistics. Springer, Cham, Switzerland.

[43] Spearman, C. (1904). The proof and measurement of association between two things. Amer. J. Psychol., 15(1):72–101.

[44] Taskinen, S., Kankainen, A., and Oja, H. (2003). Sign test of independence between two random vectors. Statist. Probab. Lett., 62(1):9–21.

[45] Taskinen, S., Kankainen, A., and Oja, H. (2004). Rank scores tests of multivariate independence. In Theory and applications of recent robust methods, Stat. Ind. Technol., pages 329–341. Birkhäuser, Basel.

[46] Taskinen, S., Oja, H., and Randles, R. H. (2005). Multivariate nonparametric tests of independence. J. Amer. Statist. Assoc., 100(471):916–925.

[47] van der Vaart, A. W. (1998). Asymptotic Statistics, volume 3 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, United Kingdom.

[48] Wilcoxon, F. (1945). Individual comparisons by ranking methods. Biometrics Bulletin, 1(6):80–83.
A Appendix

Proof of Proposition 3.1. We only need to prove the result for vec\(\mathbf{W}_J^{(n)}\) (of which vec\(\mathbf{W}_{\text{sign}}^{(n)}\) and vec\(\mathbf{W}_S^{(n)}\) are particular cases) and vec\(\mathbf{W}_K^{(n)}\).

(a) Starting with vec\(\mathbf{W}_J^{(n)}\), letting

\[ Y_{ki} := J_k \left( \frac{R_{ki,\pm}^{(n)}}{n_R + 1} \right) S_{ki,\pm}^{(n)} \quad \text{and} \quad Y_{ki} := J_k \left( \| \mathbf{F}_{ki,\pm}(\mathbf{X}_{ki}) \| \right) S_{ki,\pm}, \quad k = 1, 2, \]

rewrite \(n^{-1/2}(\mathbf{W}_J^{(n)} - \mathbf{W}_J^{(n)})_{j\ell}, j = 1, \ldots, d_1, \ell = 1, \ldots, d_2\) as

\[ n^{-1/2} \sum_{i=1}^{n} \left( \left( Y_{1i}^{(n)} \right)_{j} \left( Y_{2i}^{(n)} \right)_{\ell} - \left( Y_{1i} \right)_{j} \left( Y_{2i} \right)_{\ell} \right). \]

Let us show that

\[ E \left[ \left\{ n^{-1/2} \sum_{i=1}^{n} \left( Y_{1i} \right)_{j} \left( Y_{2i} \right)_{\ell} \right\}^2 \right], \quad (A.1) \]

\[ E \left[ \left\{ n^{-1/2} \sum_{i=1}^{n} \left( Y_{1i}^{(n)} \right)_{j} \left( Y_{2i}^{(n)} \right)_{\ell} \right\}^2 \right], \quad (A.2) \]

and

\[ E \left[ \left\{ n^{-1/2} \sum_{i=1}^{n} \left( Y_{1i}^{(n)} \right)_{j} \left( Y_{2i}^{(n)} \right)_{\ell} \right\} \left\{ n^{-1/2} \sum_{i=1}^{n} \left( Y_{1i} \right)_{j} \left( Y_{2i} \right)_{\ell} \right\} \right], \quad (A.3) \]

tend to the same limit as \(n_R\) and \(n_S\) tend to infinity.

First consider (A.1):

\[ E \left[ \left\{ n^{-1/2} \sum_{i=1}^{n} \left( Y_{1i} \right)_{j} \left( Y_{2i} \right)_{\ell} \right\}^2 \right] = E \left[ \left( Y_{11} \right)_{j}^2 \right] \cdot E \left[ \left( Y_{21} \right)_{\ell}^2 \right], \]

due to the independence between \(Y_{1i} \) and \(Y_{2i} \), and the independence between \(Y_{ki} \) and \(Y_{ki'} \), \(k = 1, 2, i \neq i' \), \(j = 1, \ldots, d_1, \ell = 1, \ldots, d_2 \).
Turning to (A.2), since \(((Y_{11}^{(n)})_j, \ldots, (Y_{1n}^{(n)})_j)\) and \(((Y_{21}^{(n)})_{\ell}, \ldots, (Y_{2n}^{(n)})_{\ell})\) are independent for \(j = 1, \ldots, d_1, \ell = 1, \ldots, d_2,\) and in view of Proposition 2.1(ii), we have (see, e.g., Theorem 2 in Hoeffding (1951)),

\[
E \left[ n^{-1/2} \sum_{i=1}^{n} (Y_{1i}^{(n)})_j (Y_{2i}^{(n)})_{\ell} \right]^2 = \frac{1}{n(n-1)} \sum_{u_1 \in \mathcal{G}_{d_1}} \left( (J_1(u_1))_j \right)^2 \sum_{u_2 \in \mathcal{G}_{d_2}} \left( (J_2(u_2))_{\ell} \right)^2,
\]

where the right-hand side, by properties of the grid \(\mathcal{G}\) and the fact that score functions \(J_k, k = 1, 2\) have bounded variation, tends to

\[E[(J_1(V_1))^2] \cdot E[(J_2(V_2))^2] = E[(Y_{11})_j^2] \cdot E[(Y_{21})_{\ell}^2] \]

with \(V_k \sim U_{d_k}, k = 1, 2.\)

Next we evaluate the term in (A.3):

\[
E \left[ n^{-1/2} \sum_{i=1}^{n} (Y_{1i}^{(n)})_j (Y_{2i}^{(n)})_{\ell} \times n^{-1/2} \sum_{i=1}^{n} (Y_{1i}^{(n)})_j (Y_{2i}^{(n)})_{\ell} \right]
\]

\[
= n^{-1} \left\{ \sum_{i=1}^{n} \left( E \left[ (Y_{1i}^{(n)})_j (Y_{1i}^{(n)})_j \right] \right) \left( E \left[ (Y_{2i}^{(n)})_{\ell} (Y_{2i}^{(n)})_{\ell} \right] \right) \right. \\
+ \left. \sum_{i \neq i'} \left( E \left[ (Y_{1i'}^{(n)})_j (Y_{1i'}^{(n)})_j \right] \right) \left( E \left[ (Y_{2i'}^{(n)})_{\ell} (Y_{2i'}^{(n)})_{\ell} \right] \right) \right\}. \quad (A.4)
\]

Since, for \(k = 1, 2,\)

\[E \left[ (Y_{ki}^{(n)})_j (Y_{ki})_j \right] + \sum_{i', i' \neq i} E \left[ (Y_{k'i'}^{(n)})_j (Y_{ki})_j \right] = E \left[ (Y_{ki})_j \sum_{i' = 1}^{n} (Y_{k'i'}^{(n)})_j \right] = 0,
\]

and by symmetry, for all \(i \neq i',\)

\[E \left[ (Y_{ki}^{(n)})_j (Y_{ki})_j \right] = E \left[ (Y_{k1}^{(n)})_j (Y_{k1})_j \right],
\]

\[E \left[ (Y_{k'i'}^{(n)})_j (Y_{ki})_j \right] = E \left[ (Y_{k2}^{(n)})_j (Y_{k1})_j \right],
\]

we deduce that the right-hand side of (A.4) equals

\[
n^{-1} \left\{ n \left( E \left[ (Y_{11}^{(n)})_j (Y_{11})_j \right] \right) \left( E \left[ (Y_{21}^{(n)})_{\ell} (Y_{21})_{\ell} \right] \right) \right. \\
+ n(n-1) \left( (n-1)^{-1} E \left[ (Y_{11}^{(n)})_j (Y_{11})_j \right] \right) \left( (n-1)^{-1} E \left[ (Y_{21}^{(n)})_{\ell} (Y_{21})_{\ell} \right] \right) \left. \right\}
\]

\[= \frac{n}{n-1} E \left[ (Y_{11}^{(n)})_j (Y_{11})_j \right] \cdot E \left[ (Y_{21}^{(n)})_{\ell} (Y_{21})_{\ell} \right]. \quad (A.5)
\]
Continuity and the Glivenko-Cantelli result in Hallin et al. (2021a) imply that
\[ Y_{k1}^{(n)} - Y_{k1} \rightarrow 0 \quad \text{a.s.} \]
while
\[
\lim_{n \to \infty} E[\|Y_{k1}^{(n)}\|^2] = \lim_{n_R \to \infty} n_R^{-1} \sum_{r=1}^{n_R} J_k^2 \left( \frac{r}{n_R + 1} \right) = \int_0^1 J_k^2(u) du = E[\|Y_{k1}\|^2] < \infty.
\]
It then follows (see, e.g., part (iv) of Theorem 5.7 in Shorack (2017, Chap. 3)) that
\[
E[\|Y_{k1}^{(n)} - Y_{k1}\|^2] \rightarrow 0, \quad k = 1, 2 \quad \text{as } n_R, n_S \rightarrow \infty;
\]
in particular,
\[
E[\|Y_{k1}^{(n)} - Y_{k1}\|_{2j}^2] \rightarrow 0
\]
and thus
\[
E\left(\left(\frac{Y_{k1}^{(n)}}{Y_{k1}}\right)_{2j}\right) \rightarrow 0.
\]

(b) The case of the Kendall matrix vec\(W_k^{(n)}\) is slightly different, although the arguments in the proof are quite similar. We consider the Hájek projection of U-statistics (see, e.g., Proof of Theorem 7.1 in Hoeffding (1948)) for
\[
\left( W_k^{(n)} \right)_{j\ell} = \left( \frac{n}{2} \right)^{-1} \sum_{i < i'} \text{sign}\left( (F_{1;\pm}(X_{1i}) - F_{1;\pm}(X_{1i'}))_{j} \right) \times \text{sign}\left( (F_{2;\pm}(X_{2i}) - F_{2;\pm}(X_{2i'}))_{\ell} \right).
\]
It holds by Application 9(d) in Hoeffding (1948) that
\[
\left( W_k^{(n)} \right)_{j\ell} = \frac{2}{n} \sum_{i=1}^{n} \left\{ 2F_{1j} \left( (F_{1;\pm}(X_{1i}))_{j} \right) - 1 \right\} \times \left\{ 2F_{2\ell} \left( (F_{2;\pm}(X_{2i}))_{\ell} \right) - 1 \right\} + o_{\text{q.m.}}(n^{-1/2}), \quad (A.6)
\]
where \(F_{1j}, F_{2\ell}\) denote the cumulative distribution functions of \((F_{1;\pm}(X_1))_{j}\) and \((F_{2;\pm}(X_2))_{\ell}\), respectively.
We also have the Hájek projection of combinatorial statistics (see, e.g., page 242 of Barbour and Eagleson (1986); also refer to Chapter II.3.1 of Hájek and Šidák (1967)) for

\[
\left( W_k^{(n)} \right)_{j\ell} = \left( \frac{n}{2} \right)^{-1} \sum_{i < i'} \text{sign}\left( (F_{1;\pm}^{(n)}(X_{1i}) - F_{1;\pm}^{(n)}(X_{1i'}))_j \right) \times \text{sign}\left( (F_{2;\pm}^{(n)}(X_{2i}) - F_{2;\pm}^{(n)}(X_{2i'}))_{\ell} \right),
\]

which implies

\[
\left( W_k^{(n)} \right)_{j\ell} = \frac{2}{n} \sum_{i=1}^{n} \left\{ 2F_{1;\mid\text{mid}}^{(n)}(F_{1;\pm}^{(n)}(X_{1i})) - 1 \right\} \times \left\{ 2F_{2;\mid\text{mid}}^{(n)}(F_{2;\pm}^{(n)}(X_{2i})) - 1 \right\} + o_{\text{q.m.}}(n^{-1/2}), \tag{A.7}
\]

where \( F_{1;\mid\text{mid}}, F_{2;\mid\text{mid}} \) denote the mid-cumulative distribution functions (Parzen, 2004) of \( (F_{1;\pm}^{(n)}(X_1))_j \) and \( (F_{2;\pm}^{(n)}(X_2))_{\ell} \), respectively. Here the mid-cumulative distribution function of a random variable \( X \) is defined as

\[
F_{\mid\text{mid}}(x) := \frac{P(X \leq x) + P(X < x)}{2}.
\]

Finally, we can show that the difference between the right-hand sides of (A.7) and (A.6) is \( o_{\text{q.m.}}(n^{-1/2}) \) along the same lines as in the proof for part (a), which completes the proof. \( \square \)

**Proof of Proposition 4.1.** It follows from the quadratic mean differentiability of \( f_{1/2}^{1/2} \) and \( f_{2/2}^{1/2} \) and the differentiability with respect to \( \delta \) of \( M_{\delta} \) that, denoting by \( [V]_1 \) and \( [V]_2 \), respectively, the first \( d_1 \) and last \( d_2 \) components of a \( d \)-dimensional vector \( V \),

\[
\delta \mapsto \left( \frac{dP^X_{\delta}}{d\mu_d}(x) \right)^{1/2} = \left( |\det(M_{\delta})|^{-1} \right)^{1/2} f_1([M_{\delta}^{-1}x]_1)f_2([M_{\delta}^{-1}x]_2)
\]

also is differentiable in quadratic mean. The quadratic expansion of the log-likelihood ratio \( \log \left( \frac{dP^{X}_{\delta+n^{-1/2}}}{dP^X_{\delta}}(X^{(n)}) \right) \) follows (see, e.g., Theorem 12.2.3 (i) in Lehmann and Romano (2005)), yielding, at \( \delta = 0 \), the second-order asymptotic representation (4.4). The explicit forms (4.2) and (4.3) of the central sequence \( \Delta^{(n)}(X^{(n)}) \) and the Fisher information \( \gamma^2 \) for \( \delta = 0 \) are obtained via elementary differentiation. The asymptotic normality result for \( \Delta^{(n)}(X^{(n)}) \) follows from part (ii) of the same Theorem 12.2.3 in Lehmann and Romano (2005). \( \square \)
Proof of Proposition 4.2. We only give the proof for $T_{k}(n)$; the proof for $T_{k}(n)$ is similar and hence omitted. Applying the multivariate central limit theorem (Bhattacharya and Ranga Rao, 1986, Equation (18.24)) to the asymptotic form of $\Lambda_{(n)}(X(n))$ (see Proposition 3.1), we deduce that, under the null hypothesis,

$$\left(n^{1/2}\text{vec}W_{J}^{(n)}, \Lambda_{(n)}(X(n))\right) \rightsquigarrow N_{d_{1}d_{2}+1}\left(\begin{pmatrix} 0_{d_{1}d_{2}} \\ -\frac{1}{2} \tau \gamma^{2} \end{pmatrix}, \begin{pmatrix} \sigma_{J}^{2}I_{d_{1}d_{2}} & \tau \nu \\ \tau \nu' & \frac{1}{2} \tau^{2} \gamma^{2} \end{pmatrix}\right)$$

where $\sigma_{J}^{2} := \sigma_{I}^{2}(d_{1}d_{2})$,

$$\nu := \text{Cov} \left[\text{vec}\left(J_{1}(F_{1};+)(X_{1})\right) J_{2}(F_{2};+(X_{2}))'\right],$$

$$X_{1}'M_{2}^{'}\varphi_{2}(X_{2}) + X_{2}'M_{1}\varphi_{1}(X_{1}) - \left(X_{1}'\varphi_{1}(X_{1}) - d_{1}\right) - \left(X_{2}'\varphi_{2}(X_{2}) - d_{2}\right),$$

and thus, by Proposition 4.1,

$$\left(n^{1/2}\text{vec}W_{J}^{(n)}, \Lambda_{(n)}(X(n))\right) \rightsquigarrow N_{d_{1}d_{2}+1}\left(\begin{pmatrix} 0_{d_{1}d_{2}} \\ -\frac{1}{2} \tau \gamma^{2} \end{pmatrix}, \begin{pmatrix} \sigma_{J}^{2}I_{d_{1}d_{2}} & \tau \nu \\ \tau \nu' & \frac{1}{2} \tau^{2} \gamma^{2} \end{pmatrix}\right).$$

Then we employ a corollary to Le Cam’s third lemma (van der Vaart, 1998, Example 6.7) to obtain that, under the alternative sequences,

$$n^{1/2}\text{vec}W_{J}^{(n)} \rightsquigarrow N_{d_{1}d_{2}}(\tau \nu, \sigma_{J}^{2}I_{d_{1}d_{2}}).$$

The result follows. \(\square\)

Proof of Proposition 4.3. Direct computation yields

$$E_{H_{0}}\left[J_{1}(F_{1};+(X_{1}))(X_{1}'M_{2}^{'}\varphi_{2}(X_{2})) J_{2}(F_{2};+(X_{2}))'\right]$$

$$= E_{H_{0}}\left[J_{1}\left(Y_{1}\|\|Y_{1}\|\|\right)(Y_{1}'\Sigma_{1}^{1/2}M_{2}^{'}\varphi_{2}(\Sigma_{2}^{1/2}Y_{2})) J_{2}\left(Y_{2}\|\|Y_{2}\|\|\right)\right]$$

$$= E_{H_{0}}\left[\frac{Y_{1}}{\|\|Y_{1}\|\|} J_{1}\left(\tilde{F}_{1}(\|\|Y_{1}\|\|)\right) \left(Y_{1}'\Sigma_{1}^{1/2}M_{2}^{'}\varphi_{2}(\Sigma_{2}^{1/2}Y_{2})\right) J_{2}\left(\tilde{F}_{2}(\|Y_{2}\|)\right)\frac{Y_{2}'}{\|\|Y_{2}\|\|}\right]$$

$$= D_{1}C_{2}\Sigma_{1}^{1/2}M_{2}^{'}\Sigma_{2}^{-1/2}.$$

The first result then follows from Hannan (1956, Equation (5)). The next part follows from the proof of Propositions 1 and 2 in Hallin and Paindaveine (2008); see also Theorem 1 in Paindaveine (2004) and Proposition 7 in Hallin and Paindaveine (2002a). \(\square\)