EXPOSITIONAL POLYNOMIALS, STIRLING NUMBERS, AND EVALUATION OF SOME GAMMA INTEGRALS

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Abstract. This article is a short elementary review of the exponential polynomials (also called single-variable Bell polynomials) from the point of view of Analysis. Some new properties are included and several Analysis-related applications are mentioned. At the end of the paper one application is described in details - certain Fourier integrals involving \( \Gamma(a + it) \) and \( \Gamma(a + it) \Gamma(b - it) \) are evaluated in terms of Stirling numbers.

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1. Introduction.

We review the exponential polynomials \( \phi_a(x) \) and present a list of properties for easy reference. Exponential polynomials in Analysis appear, for instance, in the rule for computing derivatives like \( \left( \frac{d}{dt} \right)^n e^{ax} \) and the related Mellin derivatives

\[
(x \frac{d}{dx})^n f(x), \quad (\frac{d}{dx} - x)^n f(x).
\]

Namely, we have
or, after the substitution $x = e^t$,

$$
\left( x \frac{d}{dx}\right)^n e^{ax} = \phi_n(ax) e^{ax}.
$$

(1.2)

We also include in this review two properties relating exponential polynomials to Bernoulli numbers, $B_k$. One is the semi-orthogonality

$$
\int_{-\infty}^{0} \phi_n(x) \phi_m(x) e^{2x} \frac{dx}{x} = (-1)^n \frac{2^{n+m}-1}{n+m} B_{n+m},
$$

(1.3)

where the right hand side is zero if $n + m$ is odd. The other property is (2.10).

At the end we give one application. Using exponential polynomials we evaluate the integrals

$$
\int_{\mathbb{R}} e^{-it\lambda} t^n \Gamma(a + it) dt,
$$

(1.4)

and

$$
\int_{\mathbb{R}} e^{-it\lambda} t^n \Gamma(a + it) \Gamma(b - it) dt,
$$

(1.5)

for $n = 0, 1, \ldots$ in terms of Stirling numbers.

2. Exponential polynomials

The evaluation of the series

$$
S_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad n = 0, 1, 2, \ldots
$$

(2.1)

has a long and interesting history. Clearly, $S_0 = 1$, with the agreement that $0^0 = 1$. Several reference books (for instance, [31]) provide the following numbers.

$$
S_1 = e, \quad S_2 = 2e, \quad S_3 = 5e, \quad S_4 = 15e, \quad S_5 = 52e, \quad S_6 = 203e, \quad S_7 = 877e, \quad S_8 = 4140e.
$$
As noted by H. Gould in [19, p. 93], the problem of evaluating $S_n$ appeared in the Russian journal *Matematicheskii Sbornik*, 3 (1868), p.62, with solution ibid , 4 (1868-9), p. 39.) Evaluations are presented also in two papers by Dobiński and Ligowski. In 1877 G. Dobiński [15] evaluated the first eight series $S_1, \ldots, S_8$ by regrouping:

\[
S_1 = \sum_{k=1}^{\infty} \frac{k}{k!} = 1 + \frac{2}{2!} + \frac{3}{3!} + \ldots = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots = e
\]

\[
S_2 = \sum_{k=1}^{\infty} \frac{k^2}{k!} = 1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \ldots = 1 + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \ldots = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + \left(\frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \ldots\right) = e + S_1 = 2e,
\]

and continuing like that to $S_8$. For large $n$ this method is not convenient. However, later that year Ligowski [27] suggested a better method, providing a generating function for the numbers $S_n$.

\[
e^{xe^x} = \sum_{k=0}^{\infty} \frac{e^{kx}}{k!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{k^n}{k!} \frac{x^n}{n!} = \sum_{n=0}^{\infty} S_n \frac{x^n}{n!}.
\]

Further, an effective iteration formula was found

\[
S_n = \sum_{j=0}^{n-1} \binom{n-1}{j} S_j
\]

by which every $S_n$ can be evaluated starting from $S_1$.

These results were preceded, however, by the work [23] of Johann August Grunert (1797-1872), professor at Greifswalde. Among other things, Grunert obtained formula (2.2) below from which the evaluation of (2.1) follows immediately.

The structure of the series $S_n$ hints at the exponential function. Differentiating the expansion
\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

and multiplying both sides by \( x \) we get

\[ xe^x = \sum_{k=0}^{\infty} \frac{kx^k}{k!} \]

which, for \( x = 1 \), gives \( S_1 = e \). Repeating the procedure, we find \( S_2 = 2e \) from

\[ x(xe^x)' = (x + x^2)e^x = \sum_{k=0}^{\infty} \frac{k^2 x^k}{k!} \]

and continuing like that, for every \( n = 0, 1, 2, \ldots \), we find the relation

\[ (x \frac{d}{dx})^n e^x = \phi_n(x) e^x = \sum_{k=0}^{\infty} \frac{k^n x^k}{k!} \] (2.2)

where \( \phi_n \) are polynomials of degree \( n \). Thus,

\[ S_n = \phi_n(1)e, \quad \forall \ n \geq 0. \]

The polynomials \( \phi_n \) deserve a closer look. From the defining relation (2.2) we obtain

\[ x(\phi_n e^x)' = x(\phi_n' + \phi_n) e^x = \phi_{n+1} e^x \]

i.e.

\[ \phi_{n+1} = x(\phi_n' + \phi_n) \] (2.3)

which helps to find \( \phi_n \) explicitly starting from \( \phi_0 \).

\[ \phi_0(x) = 1 \]
\[ \phi_1(x) = x \]
\[ \phi_2(x) = x^2 + x \]
\[ \phi_3(x) = x^3 + 3x^2 + x \]
\[ \phi_4(x) = x^4 + 6x^3 + 7x^2 + x \]
and so on. Another interesting relation that easily follows from (2.2) is

\[ \phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{x}{k} \phi_k(x). \]  

(2.4)

Here is a short proof. Starting from

\[ \phi_k(x) e^x = \sum_{j=0}^{\infty} \frac{j^k x^j}{j!} \]

we compute

\[ \sum_{k=0}^{\infty} \binom{x}{k} \phi_k(x) e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=0}^{\infty} \binom{x}{k} j^k = \sum_{j=0}^{\infty} \frac{x^j}{j!} (j+1)^n \]

\[ = \frac{1}{x} \sum_{j=0}^{\infty} \frac{(j+1)^n x^{j+1}}{(j+1)!} = \frac{1}{x} \phi_{n+1}(x) e^x \]

and (2.4) is ready.

From (2.3) and (2.4) one finds immediately

\[ \phi'_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} \phi_k(x). \]

(2.5)

Obviously, \( x = 0 \) is a zero for all \( \phi_n \), \( n > 0 \). It can be seen that all the zeros of \( \phi_n \) are real, simple, and nonpositive. The nice and short induction argument belongs to Harper [24].

The assertion is true for \( n = 1 \). Suppose that for some \( n \) the polynomial \( \phi_n \) has \( n \) distinct real non-positive zeros (including \( x = 0 \)). Then the same is true for the function

\[ f_n(x) = \phi_n(x) e^x. \]

Moreover, \( f_n \) is zero at \( -\infty \) and by Rolle’s theorem its derivative

\[ \lim_{x \to -\infty} f_n(x) = 0. \]
\[ \frac{d}{dx} \phi_n(x) = \frac{d}{dx} \left( \phi_n(x)e^x \right) \]

has \( n \) distinct real negative zeros. It follows that the function

\[ \phi_{n+1}(x)e^x = x \frac{d}{dx} \left( \phi_n(x)e^x \right) \]

has \( n+1 \) distinct real non-positive zeros (adding here \( x = 0 \)).

The polynomials \( \phi_n \) can be defined also by the exponential generating function (extending Ligowski’s formula)

\[ \exp[^{(e^x-1)}] = \sum_{n=0}^{\infty} \phi_n(x) \frac{x^n}{n!} . \tag{2.6} \]

It is not obvious, however, that the polynomials defined by (2.2) and (2.6) are the same, so we need the following simple statement.

**Proposition 1.** The polynomials \( \phi_n(x) \) defined by (2.2) are exactly the partial derivatives \( \frac{\partial}{\partial z} z^n \exp[^{(e^z-1)}] \) evaluated at \( z = 0 \).

(2.6) follows from (2.2) after expanding the exponential \( \exp[^{(e^z-1)}] \) in double series and changing the order of summation. A different proof will be given later.

Setting \( z = 2k\pi i \), \( k = \pm 1, \pm 2, \ldots \), in the generating function (2.6) one finds

\[ \exp[^{(2ki\pi)}] = 1, \exp[^{(e^{ki\pi-1})}] = e^{ki\pi} = 1, \]

which shows that the exponential polynomials are linearly dependent

\[ 1 = \sum_{n=0}^{\infty} \phi_n(x) \frac{(2k\pi i)^n}{n!} \text{ or } 0 = \sum_{n=1}^{\infty} \phi_n(x) \frac{(2k\pi i)^n}{n!}, k = \pm 1, \pm 2, \ldots . \tag{2.7} \]

In particular, \( \phi_n \) are not orthogonal for any scalar product on polynomials. (However, they have
the semi-orthogonality property mentioned in the introduction and proved in Section 4.)

Comparing coefficient for $z$ in the equation

$$e^{(x+y)z^2} = e^{xz} e^{yz}$$

yields the binomial identity

$$\phi_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \phi_k(x) \phi_{n-k}(y).$$  \hfill (2.8)

With $y = -x$ this implies the interesting “orthogonality” relation for $n \geq 1$

$$\sum_{k=0}^{n} \binom{n}{k} \phi_k(x) \phi_{n-k}(-x) = 0.$$  \hfill (2.9)

Next, let $B_n, n \geq 0, 1, \ldots$, be the Bernoulli numbers. Then for $p = 0, 1, \ldots$, we have

$$\int_{0}^{x} \phi_p(t) dt = \frac{1}{p+1} \sum_{k=1}^{p+1} \binom{p+1}{k} B_{p+1-k} \phi_k(x).$$  \hfill (2.10)

For proof see Example 4 in [3, p.51], or [6].

**Some historical notes**

As already mentioned, formula (2.2) appears in the work of Grunert [23], on p. 260, where he gives also the representation (3.3) below and computes explicitly the first six exponential polynomials. The polynomials $\phi_n$ were studied more systematically (and independently) by S. Ramanujan in his unpublished notebooks. Ramanujan’s work is presented and discussed by Bruce Berndt in [3, Part 1, Chapter 3]. Ramanujan, for example, obtained (2.6) from (2.2) and also proved (2.4), (2.5) and (2.10). Later, these polynomials were studied by E.T. Bell [1] and Jacques Touchard [39], [40]. Both Bell and Touchard called them “exponential” polynomials, because of their relation to the exponential function, e.g. (1.1), (1.2), (2.2) and (2.6). This name was used also by Gian-Carlo Rota [34]. As a matter of fact, Bell introduced in [1] a more general class of polynomials of many variables, $\Psi_{n,k}$, including $\phi_n$ as a particular case. For this reason $\phi_n$ are known also as the single-variable Bell polynomials [13], [20], [21], [41]. These polynomials are also a special case of the actuarial polynomials introduced by Toscano [38] which, on their part, belong to the more general class of Sheffer polynomials [7].
The exponential polynomials appear in a number of papers and in different applications - see [4], [5], [6], [29], [32], [33], [34] and the references therein. In [35] they appear on p. 524 as the horizontal generating functions of the Stirling numbers of the second kind (see below (3.3)).

The numbers

\[ b_n = \phi_n(1) = \frac{1}{e} S_n \]  

(2.11)

are sometimes called exponential numbers, but a more established name is Bell numbers. They have interesting combinatorial and analytical applications [2], [8], [14], [16], [20], [21], [25], [30], [37], [38]. An extensive list of 202 references for Bell numbers is given in [18].

We note that equation (2.2) can be used to extend \( \phi_n \) to \( \phi_z \) for any complex number \( z \) by the formula

\[ \phi_z(x) = e^{-z} \sum_{k=0}^{\infty} \frac{k^z x^k}{k!} \]  

(2.12)

(Butzer et al. [9], [10]). The function appearing here is an interesting entire function in both variables, \( x \) and \( z \). Another possibility is to study the polyexponential function

\[ e_z(x, \lambda) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\lambda)^z}, \]

(2.13)

where \( \Re \lambda > 0 \). When \( s \) is a negative integer, the polyexponential can be written as a finite linear combination of exponential polynomials (see [6]).

3. Stirling numbers and Mellin derivatives

The iteration formula (2.3) shows that all polynomials \( \phi_n \) have positive integer coefficients. These coefficients are the Stirling numbers of the second kind \( \{ n \atop k \} \) (or \( S(n,k) \)) - see [12], [14], [17], [22], [26], [35]. Given a set of \( n \) elements, \( \{ n \atop k \} \) represents the number of ways this set can be partitioned into \( k \) nonempty subsets \( (0 \leq k \leq n) \). Obviously,
\{\frac{n}{1}\} = 1, \{\frac{n}{n}\} = 1 \text{ and a short computation gives } \{\frac{n}{2}\} = 2^{n-1} - 1. \text{ For symmetry one sets } \\
\{\frac{0}{0}\} = 1, \{\frac{0}{n}\} = 0. \text{ The definition of } \{\frac{n}{k}\} \text{ implies the property } \\
\{\frac{n+1}{k}\} = k \{\frac{n}{k}\} + \{\frac{n}{k-1}\} \\
(3.1) \\
(see \ p.259 \ in \ [22]) \text{ which helps to compute all } \{\frac{n}{k}\} \text{ by iteration. For instance, } \\
\{\frac{n}{3}\} = \frac{3^{n-1} - 2^n + 1}{2}. \\
A \text{ general formula for the Stirling numbers of the second kind is } \\
\{\frac{n}{k}\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n. \\
(3.2) \\
Proposition 2. \text{ For every } n = 0, 1, 2, \ldots \text{ } \\
\phi_n(x) = \{\frac{n}{0}\} + \{\frac{n}{1}\} x + \{\frac{n}{2}\} x^2 + \ldots + \{\frac{n}{n}\} x^n = \sum_{k=0}^{n} \{\frac{n}{k}\} x^k. \\
(3.3) \\
The proof is by induction and is left to the reader. Setting here } x = 1 \text{ we come to the well-known representation for the numbers } S_n \\
S_n = e \{\frac{n}{0}\} + \{\frac{n}{1}\} + \{\frac{n}{2}\} + \ldots + \{\frac{n}{n}\}. \\
It is interesting that formula (3.3) is very old - it was obtained by Grunert \[23, \ p \ 260\] together with the representation (3.2) for the coefficients which are called now Stirling numbers of the second kind. In fact, coefficients of the form \\
\{\frac{n}{k}\} k! \text{ appear in the computations of Euler - see } [17]. \\
It is good to note that the representation (3.2) quickly follows from (3.3) and (2.2). First \text{ we write } \\
\sum_{k=0}^{n} \{\frac{n}{k}\} x^k = e^{-x} \sum_{k=0}^{\infty} \frac{k^n x^k}{k!} = (\sum_{j=0}^{\infty} \frac{(-1)^{j} x^j}{j!}) \{\sum_{k=0}^{\infty} \frac{k^n x^k}{k!}\}. \\
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then we multiply the two series by Cauchy’s rule and compare coefficients. Thus we come to (3.2). This proof shows very well that the right-hand side in (3.2) is zero when \( k > n \).

Next we turn to some special differentiation formulas. Let \( D = \frac{d}{dx} \).

**Mellin derivatives.** It is easy to see that the first equality in (2.2) extends to equation (1.2), where \( a \) is an arbitrary complex number i.e.,

\[
(xD)^n e^{ax} = \phi_n(ax) e^{as}
\]

by the substitution \( x \to ax \). Even further, this extends to

\[
(xD)^n e^{ax^p} = \phi_n(ax^p) e^{as^p}
\]

for any \( a, p \) and \( n = 0, 1, \ldots \) (simple induction and (2.3)). Again by induction, it is easy to prove that

\[
(xD)^n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k D^k f(x)
\]

for any \( n \)-times differentiable function \( f \). This formula was obtained by Grunert [23, pp 257-258] (see also p. 89 in [19], where a proof by induction is given).

As we know the action of \( xD \) on exponentials, formula (3.5) can be “discovered” by using Fourier transform. Let

\[
F[f](t) = \int_{\mathbb{R}} e^{-i\xi t} f(x) \, dx
\]

be the Fourier transform of some function \( f \). Then

\[
f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} F[f](t) \, dx
\]

\[
(xD)^n f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} \phi_n(ixt) F[f](t) \, dx
\]

\[
\sum_{k=0}^{n} \binom{n}{k} x^k F^{-1}[(it)^k F[f]](x) = \sum_{k=0}^{n} \binom{n}{k} x^k D^k f(x)
\]

Next we turn to formula (1.1) and explain its relation to (1.2). If we set \( x = e^t \), then for
any differentiable function $f$

$$\frac{d}{dt} f = \left( \frac{d}{dx} f \right) \frac{dx}{dt} = \left( \frac{d}{dx} f \right) e^t = (xD)f$$

and we see that (1.1) and (1.2) are equivalent.

$$(\frac{d}{dt})^n e^{az^t} = (xD)^n e^{ax} = \phi_n(ax) e^{ax} = \phi_n(ae^t) e^{ax} \quad (3.8)$$

**Proof** of Proposition 1. We apply (1.1) to the function $f_x(z) = e^{x(e^z-1)} = e^{xe^z} e^{-z}$

$$(\frac{d}{dz})^n f_x(z) = \phi_n(xe^z) f_x(z)$$

From here, with $z = 0$

$$(\frac{d}{dz})^n f_x(z) \bigg|_{z=0} = \phi_n(x)$$

as needed.

Now we list some simple operational formulas. Starting from the obvious relation

$$(xD)^n x^k = k^n x^k, \quad n = 0, 1, ..., k \in \mathbb{R}. \quad (3.9)$$

for any function of the form

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \quad (3.10)$$

we define the differential operator

$$f(xD) = \sum_{n=0}^{\infty} a_n (xD)^n$$
with action on functions $g(x)$,

$$f(xD)g(x) = \sum_{n=0}^{\infty} a_n (xD)^n g(x). \quad (3.11)$$

When $g(x) = x^k$, (3.9) and (3.11) show that

$$f(xD)x^k = \sum_{n=0}^{\infty} a_n k^nx^k = f(k)x^k.$$ 

If now

$$g(x) = \sum_{k=0}^{\infty} c_k x^k$$

is a function analytical in a neighborhood of zero, the action of $f(xD)$ on this function is given by

$$f(xD)g(x) = \sum_{k=0}^{\infty} c_k f(k)x^k, \quad (3.12)$$

provided the series on the right side converges. When $f$ is a polynomial, formula (3.12) helps to evaluate series like

$$\sum_{k=0}^{\infty} c_k f(k)x^k$$

in a closed form. This idea was exploited by Schwatt [36] and more recently by the present author in [4]. For instance, when $g(x) = e^x$ equation (3.12) becomes

$$\sum_{k=0}^{\infty} f(k) \frac{x^k}{k!} = e^x \sum_{n=0}^{\infty} a_n \phi_n(x). \quad (3.13)$$
As shown in [4] this series transformation can be used for asymptotic series expansions of certain functions.

**Leibniz Rule.** The higher order Mellin derivative \((x D)^n\) satisfies the Leibniz rule

\[
(x D)^n (fg) = \sum_{k=0}^{n} \binom{n}{k} [(x D)^{n-k} f] [(x D)^k g]
\]  

(3.14)

The proof is easy, by induction, and is left to the reader. We shall use this rule to prove the following proposition.

**Proposition 3.** For all \(n, m = 0, 1, 2, \ldots\)

\[
\phi_{n+m}(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{j} j^{n-k} x^j \phi_k(x)
\]  

(3.15)

**Proof.**

\[
\phi_{n+m}(x) = (x D)^n e^x = (x D)^n (x D)^m e^x = (x D)^n (\phi_m(x) e^x),
\]

which by the Leibniz rule (3.14) equals

\[
\sum_{k=0}^{n} \binom{n}{k} [(x D)^{n-k} \phi_m(x)] [(x D)^k e^x].
\]

Using (3.2) and (3.9) we write

\[
(x D)^{n-k} \phi_m(x) = \sum_{j=0}^{n} \binom{n}{j} j^{n-k} x^j,
\]

and since also

\[
(x D)^k e^x = \phi_k(x) e^x,
\]
we obtain (3.15) from (3.14). The proof is completed.

Setting $x = 1$ in (3.14) yields an identity for the Bell numbers.

$$b_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{j} \binom{m}{k} j^{n-k} b_k.$$  \hspace{1cm} (3.16)

This identity was recently published by Spivey [37], who gave a combinatorial proof. After that Gould and Quaintance [21] obtained the generalization (3.15) together with two equivalent versions. The proof in [21] is different from the one above.

Using the Leibniz rule for $xD$ we can prove also the following extension of property (2.9)

**Proposition 4.** For any two integers $n, m \geq 0$

$$(xD)^n \Phi_m(x) = \sum_{k=0}^{n} \binom{n}{k} k^n x^k = \sum_{k=0}^{n} \binom{n}{k} \Phi_{m+k}(x) \Phi_{n-k}(-x).$$  \hspace{1cm} (3.17)

The proof is simple. Just compute

$$(xD)^n \Phi_m(x) = (xD)^n \left[ (e^{-x}) (\Phi_m(x) e^x) \right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[ (xD)^{n-k} e^{-x} \right] \left[ (xD)^k (\Phi_m(x) e^x) \right]$$

and (3.17) follows from (1.2).

For completeness we mention also the following three properties involving the operator $Dx$. Proofs and details are left to the reader.

$$(Dx)^n e^{ax} = \frac{\Phi_{n+1}(ax)}{ax} e^{ax},$$  \hspace{1cm} (3.18)
\[ (Dx)^n f(x) = \sum_{k=0}^{n} \binom{n+1}{k+1} x^k D^k f(x), \quad \text{(3.19)} \]

and
\[ f(Dx) g(x) = \sum_{k=0}^{\infty} c_k f(k+1) x^k, \quad \text{(3.20)} \]

analogous to (1.2), (3.5), and (3.12) correspondingly.

For a comprehensive study of the Mellin derivative we refer to [11].

**More Stirling numbers.** The polynomials \( \phi_n, n = 0, 1, \ldots \) form a basis in the linear space of all polynomials. Formula (3.3) shows how this basis is expressed in terms of the standard basis \( 1, x, x^2, \ldots \). We can solve for \( x^k \) in the equations (3.3) and express the standard basis in terms of the exponential polynomials

\[
\begin{align*}
1 &= \phi_0 \\
x &= \phi_1 \\
x^2 &= -\phi_1 + \phi_2 \\
x^3 &= 2\phi_1 - 3\phi_2 + \phi_3 \\
x^4 &= -6\phi_1 + 11\phi_2 - 6\phi_3 + \phi_4,
\end{align*}
\]

etc. The coefficients here are also special numbers. If we write
\[ x^n = \sum_{k=0}^{n} (-1)^{n-k} \left[ \begin{array}{c} n \\ k \end{array} \right] \phi_k \quad \text{(3.21)} \]

then \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) are the (absolute) Stirling numbers of first kind, as defined in [22]. (The numbers \( \left[ \begin{array}{c} n \\ k \end{array} \right] \)
are non-negative. The symbol \( s(n, k) = (-1)^{n-k}\binom{n}{k} \) is used for Stirling numbers of the first kind with changing sign - see [14], [18] and [26] for more details. \( \binom{n}{k} \) is the number of ways to arrange \( n \) objects into \( k \) cycles. According to this interpretation,

\[
\binom{n}{k} = (n - 1)\binom{n-1}{k} + \binom{n-1}{k-1}, \quad n \geq 1.
\]

4. Semi-orthogonality of \( \phi_n \)

Proposition 5. For every \( n, m = 1, 2, \ldots \), we have

\[
\int_0^\infty \phi_n(-x) \phi_m(-x) e^{-2x} \frac{dx}{x} = (-1)^{n-1} \frac{2^{n+m} - 1}{n + m} B_{n+m}.
\]  

(4.1)

Here \( B_k \) are the Bernoulli numbers. Note that the right hand side is zero when \( k + m \) is odd, as all Bernoulli numbers with odd indices > 1 are zeros.

Using the representation (3.3) in (4.1) and integrating termwise we obtain an equivalent form of (4.1)

\[
\sum_{k=0}^n \sum_{j=0}^m (-1)^{k+j} \binom{n}{k} \binom{m}{j} \frac{(k+j-1)!}{2^{k+j}} = (-1)^{n-1} \frac{2^{n+m} - 1}{n + m} B_{n+m}.
\]  

(4.2)

This (double sum) identity extends the known identity [22, p.317, Problem 6.76]

\[
\sum_{j=0}^m (-1)^j \binom{m}{j} \frac{j!}{2^{j+1}} = \frac{2^{m+1} - 1}{m+1} B_{m+1}.
\]  

(4.3)

Namely, (4.3) results from (4.2) for \( n = 1 \). The presence of \( (-1)^{n-1} \) at the right hand side in (4.1) is not a “break of symmetry”, because when \( n + m \) is even, then \( n \) and \( m \) are both even or
both odd.

Proof of the proposition. Starting from

\[ \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx \]  \hspace{1cm} (4.4)

we set \( x = e^\lambda, \quad z = a + it \), to obtain the representation

\[ \Gamma(a + it) = \int_{-\infty}^\infty e^{i \lambda t} e^{a \lambda} e^{-\lambda^2} \, d\lambda, \]  \hspace{1cm} (4.5)

which is a Fourier transform integral. The inverse transform is

\[ e^{a \lambda} e^{-\lambda^2} = \frac{1}{2\pi} \int_\mathbb{R} e^{-i \lambda t} \Gamma(a + it) \, dt. \]  \hspace{1cm} (4.6)

When \( a = 1 \) this is

\[ -e^\lambda e^{-e^\lambda} = \frac{d}{d\lambda} e^{-\lambda^2} = \frac{-1}{2\pi} \int_\mathbb{R} e^{-i \lambda t} \Gamma(1 + it) \, dt. \]  \hspace{1cm} (4.7)

Differentiating (4.7) \( n - 1 \) times for \( \lambda \) we find

\[ (\frac{d}{d\lambda})^n e^{-\lambda^2} = \Phi_n (-e^\lambda) e^{-\lambda^2} = \frac{-1}{2\pi} \int_\mathbb{R} e^{-i \lambda t} (-it)^{n-1} \Gamma(1 + it) \, dt, \]  \hspace{1cm} (4.8)

and Parceval’s formula yields the equation

\[ \int_\mathbb{R} \Phi_n (-e^\lambda) \Phi_m (-e^\lambda) e^{-2x^2} \, d\lambda = \frac{1}{2\pi} \int_\mathbb{R} (-it)^{n-1} (it)^{m-1} |\Gamma(1 + it)|^2 \, dt \]

or, with \( x = e^\lambda \)

\[ \int_0^\infty \Phi_n (-x) \Phi_m (-x) e^{-\frac{2x^2}{x}} \, dx = \frac{(-1)^n i^{n+m}}{2\pi} \int_\mathbb{R} \frac{x^{n+m-2} \pi i t}{\sinh(\pi t)} \, dt. \]  \hspace{1cm} (4.9)
The right hand side is 0 when \( n + m \) is odd. When \( n + m \) is even, we use the integral [31, p.351]

\[
\int_0^\infty \frac{t^{2p-1}}{\sinh(\pi t)} \, dt = \frac{2^{2p-1}}{2p} (-1)^p B_{2p} \tag{4.10}
\]

to finish the proof.

Property (4.1) resembles the semi-orthogonal property of the Bernoulli polynomials

\[
\int_0^1 B_n(x) B_m(x) \, dx = (-1)^{n-1} \frac{n! \, m!}{(n+m)!} B_{n+m} \tag{4.11}
\]

- see, for instance, [35, p.530].

5. Gamma integrals.

We use the technique in the previous section to compute certain Fourier integrals and evaluate the moments of \( \Gamma(a + it) \) and \( \Gamma(a + it) \Gamma(b - it) \).

**Proposition 6.** For every \( n = 0, 1, \ldots \), and \( a, b > 0 \) we have

\[
\int_{\mathbb{R}} e^{-\lambda t} t^n \Gamma(a+it) \Gamma(b-it) \, dt \tag{5.1}
\]

\[
eq i^n 2\pi e^{-\mu} \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \binom{k}{m} (-1)^m a^{n-k} \frac{\Gamma(a+b+m)}{(1+e^{-\mu})^{a+b+m}} ;
\]

\[
\int_{\mathbb{R}} e^{-\lambda^2 t} t^n \Gamma(a+it) \, dt \tag{5.2}
\]
In particular, when $\lambda = \mu = 0$, we obtain the moments

$$\Gamma_n(a, b) = \int_{\mathbb{R}} t^n \Gamma(a + it) \Gamma(b - it) \, dt$$

(5.3)

$$= i^n \pi \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} (-1)^m a^{n-k} \frac{\Gamma(a+b+m)}{2^{a+b+m-1}}.$$  

(5.4)

When $n = 0$ in (5.1) we have the known integral

$$\int_{\mathbb{R}} e^{-it} \Gamma(a+it) \Gamma(b-it) \, dt = 2\pi \Gamma(a+b) e^{-b} (1 + e^{-t})^{a-b},$$

(5.5)

which can be found in the form of an inverse Mellin transform in [28].

**Proof.** Using again equation (4.6)

$$e^{a\lambda} e^{-b} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it} \Gamma(a+it) \, dt$$

(5.6)

we differentiate both side $n$ times

$$\left( \frac{d}{d\lambda} \right)^n [e^{a\lambda} e^{-b}] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it} (-it)^n \Gamma(a+it) \, dt,$$

and then, according to the Leibniz rule and (1.1) the left hand side becomes.
\[
\left( \frac{d}{d\lambda} \right)^n [e^{\lambda x} e^{-\lambda y}] = e^{\lambda x} e^{-\lambda y} \sum_{k=0}^{n} \binom{n}{k} \phi_k (-\lambda)^{n-k} a^{n-k}.
\]

Therefore,
\[
e^{\lambda x} e^{-\lambda y} \sum_{k=0}^{n} \binom{n}{k} \phi_k (-\lambda)^{n-k} a^{n-k} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda t} e^{i\mu t} \Gamma(a+it) \Gamma(b-it) dt.
\]

and (5.2) follows from here.

Replacing \( \lambda \) by \( \lambda - \mu \) we write (5.6) in the form
\[
e^{\lambda x} e^{-\lambda y} \sum_{k=0}^{n} \binom{n}{k} \phi_k (-\lambda)^{n-k} a^{n-k} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda t} e^{i\mu t} \Gamma(b+it) \Gamma(b-it) dt.
\]

and then Parceval’s formula for Fourier integrals applied to (5.7) and (5.8) yields
\[
e^{-\mu x} \sum_{k=0}^{n} \binom{n}{k} \phi_k (-\lambda)^{n-k} \int_{\mathbb{R}} e^{(a+b)\lambda} e^{-\lambda(t+x)} \phi_k (-\lambda)^{n-k} d\lambda
\]
\[
= \frac{(-i)^n}{2\pi} \int_{\mathbb{R}} e^{-i\mu t} t^n \Gamma(a+it) \Gamma(b-it) dt.
\]

Returning to the variable \( x = e^{\lambda} \) we write this in the form
\[
\frac{1}{2\pi} \int_{\mathbb{R}} e^{-\mu x} t^n \Gamma(a+it) \Gamma(b-it) dt
\]
\[
= i^n e^{-\mu x} \sum_{k=0}^{n} \binom{n}{k} \phi_k(-x) x^{a+b-1} e^{-x(1+\mu x)} dx
\]
\[
= i^n e^{-\mu x} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} \phi_{k-j}(-1)^j \int_{0}^{\infty} x^{a+b+j-1} e^{-x(1+\mu x)} dx
\]
\[ i^n e^{-b^2} \sum_{k=0}^{n} \sum_{j=0}^{k} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} k \\ j \end{array} \right) a^{n-k} (-1)^j \frac{\Gamma(a+b+j)}{(1+e^{-\nu})^{a+b+j}} \]

which is (5.1). The proof is complete.

Next, we observe that for any polynomial

\[ p(t) = \sum_{n=0}^{m} a_n t^n \quad (5.11) \]

one can use (5.4) to write the following evaluation

\[ \int_{\mathbb{R}} p(t) \Gamma(a+it) \, dt = \sum_{n=0}^{m} a_n G_n(a). \quad (5.12) \]

In particular, when \( a = 1 \) we have

\[ G_n(1) = 2\pi i^n e^{-1} \phi_{n+1}(-1), \quad (5.13) \]

and therefore,

\[ \int_{\mathbb{R}} p(t) \Gamma(1+it) \, dt = \frac{2\pi}{e} \sum_{n=0}^{m} a_n t^n \phi_{n+1}(-1). \quad (5.14) \]

More applications can be found in the recent papers [4], [5] and [6].

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