Complete Semiclassical Treatment
of the
Quantum Black Hole Problem

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Two types of semiclassical calculations have been used to study quantum effects in black hole backgrounds, the WKB and the mean field approaches. In this work we systematically reconstruct the logical implications of both methods on quantum black hole physics and provide the link between these two approaches. Our conclusions completely support our previous findings based solely on the WKB method: quantum black holes are effectively p-brane excitations and, consequently, no information loss paradox exists in this problem.

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I. INTRODUCTION

In four or higher dimensions two types of semiclassical calculations, both applied during the 1970’s, have been used in the study of quantum effects in black hole backgrounds. The first one is the familiar Euclidean path integral formulation of general relativity (GR) in the semiclassical WKB approximation. The standard interpretation of this quantity by Hawking and others [1–4] was taken to be the canonical partition function of a gas of black holes of equal mass in thermal equilibrium at inverse Hawking temperature $\beta_H$, the latter quantity being related to the mass of the black hole. A second type of calculation aimed at studying the quantum field theory of particles immersed in the classical black hole background [5]. It was found that the effect of the non-trivial topology of the black hole spacetime, due to its horizon, was to modify the quantization procedure by doubling the number of degrees of freedom of the given quantum fields as compared to the expected number in a topologically trivial, horizonless, spacetime. This quantum field theory then acquired a mathematical structure analogous to flat space quantum field theories at finite temperature (e.g. the thermofield dynamics formalism [6]). Furthermore, the temperature of the gas of particles in the black hole background agreed with that found in the WKB calculation. It seemed as though thermodynamics was naturally coming out of quantum black hole theory. This led Hawking to propose a new set of physical laws – black hole thermodynamics.

Over time however, an increasing number of theorists raised concerns over the potential violations of quantum mechanics implied by this new black hole thermodynamics. In particular, unitarity of time evolution would be broken as particles in pure states originally absorbed by the black hole would tunnel out of it and end up in the above described thermal state. Inconsistency of the thermal picture for black holes could also be detected in the form of a negative canonical specific heat, a necessarily positive definite quantity.

In this work no toy model (whether in 2 or other dimensions) is used. We address the real problem. The purpose of this paper is to systematically re-assemble our knowledge of quantum black hole physics in 4 or higher dimensions solely on the basis of the two
calculational methods described above for dealing with quantum effects in gravity problems. Conjectures are eliminated as much as possible. A clear picture of the nature of quantum black holes finally emerges, a picture in agreement with the currently known laws of physics. Such a picture is the one described by the present authors in previous publications [7–10]. What is new in this paper is the consistent incorporation of the second computational semiclassical method and the elucidation of its role in the understanding of the overall picture. This was not included in our previous work.

II. THE WKB METHOD

In the 1970’s one of the very few methods available to deal with the problem of extracting useful and finite information from quantum gravity theory was the WKB semiclassical approximation [1–4]. It was pioneered by Hawking, among others, in the context of black holes. This technique provides an approximate evaluation of the path integral of Euclidean field theories by finding saddle points. The result is generically given by the exponential of minus the Euclidean action (over $\hbar$) evaluated at classical solutions (instantons) of the Euclidean field equations, multiplied by an overall factor which is determined by the quantum corrections. It is a clearly nonperturbative calculation.

Because of its connection with instantons, the WKB approximation to the path integral provides a useful formula for the tunneling probability per unit volume of quantum particles in barrier penetration problems.

In the case of a $D$-dimensional Schwarzschild black hole an analytical continuation from real to imaginary time must be performed to evaluate the above tunneling probability across the horizon. In this process a conical singularity in the Euclidean spacetime will develop, however. The situation is remedied by further demanding that the imaginary time dimension be constrained to form a compact circle with circumference $\beta_{H}$, the well-known Hawking inverse “temperature”. The gravitational instantons that are the Euclidean black holes are therefore in fact periodic instantons [12].
Since the tunneling probability is an effective measure of the ratio of a single particle state having escaped the black hole to the number of quantum states inside the black hole, we therefore arrive at the following WKB formula for the black hole quantum degeneracy of states \( \rho(m) \) at mass level \( m \),

\[
\rho(m) \sim c e^{S_E(m)/\hbar},
\]

(2.1)

where \( c \) represents the quantum field theoretical corrections and \( S_E \) is the Euclidean action of the Euclidean black hole (evaluated from the horizon outward). The integral over Euclidean time in Eq.(2.1) is to be performed for a single period. The picture thus presented is completely quantum mechanical.

On the other hand, ever since the mid-fifties when Matsubara [13] proposed his imaginary time formalism for equilibrium quantum field theory at finite temperature, the Euclidean path integral formulation has also been interpreted as the canonical partition function \( Z(\beta) \) of a gas in thermal equilibrium. The inverse temperature \( \beta_H \) is again the period of the Euclidean time. Hawking and others [4-4], in the mid-seventies, chose this second interpretation of the path integral in the black hole context, a pioneering effort to understand the statistical mechanics of these unusual objects. From there the statistical density of states \( \Omega(E) \) of a gas of black holes with average energy \( E = M \) can be found. The canonical partition function is written as

\[
Z(\beta_H) \sim e^{-S_E(\beta_H)/\hbar} = e^{-\beta_H F(\beta_H)},
\]

(2.2)

where \( F(\beta_H) \) is the Helmholtz free energy. The corresponding entropy is

\[
S_H = \beta_H M - \beta_H F(\beta_H) = \beta_H M - S_E,
\]

(2.3)

and so the statistical mechanical density of states is now given as,

\[
\Omega_H(M) = e^{S_H(M)}.
\]

(2.4)

Let us consider explicitly the simple problem of the \( D \)-dimensional Schwarzschild black hole. The Euclidean metric is
\[ ds^2 = e^{2\Phi} d\tau^2 + e^{-2\Phi} dr^2 + r^2 d\Omega_{D-2}^2, \]  

where

\[ e^{2\Phi} = 1 - \left( \frac{r_+}{r} \right)^{D-3}, \]

and \( r_+ \) is the horizon radius.

The condition of the vanishing of the conical singularity of the spacetime (2.5) yields the Hawking inverse temperature

\[ \beta_H = \frac{2\pi}{\left[ e^\Phi \partial_r e^\Phi \right]_{r=r_+}} = \frac{4\pi r_+}{D - 3}. \]

The Euclidean action for such a \( D \)-dimensional black hole has been repeatedly derived. The result is \( (\hbar = 1) \),

\[ S_E = \frac{A_{D-2}}{16\pi} \beta_H r_+^{D-3}, \]

where \( A_D \) is the area of a unit \( D \)-sphere. The relation between the horizon \( r_+ \) and the black hole mass \( M \) is given by

\[ M = \frac{(D-2)}{16\pi} A_{D-2} r_+^{D-3}. \]

Therefore,

\[ S_E(M) = \frac{\beta_H M}{D - 2} = \sigma(D) M^{\frac{D-2}{D-3}}, \]

where,

\[ \sigma(D) = \frac{4^{\frac{D-1}{2}} \pi^{\frac{D-2}{2}}}{(D-3)(D-2)^{\frac{D-2}{2}} A_{D-2}^{\frac{D-1}{2}}}. \]

The Hawking entropy is now given as

\[ S_H(M) = (D - 3)S_E(M) = (D - 3)\sigma(D) M^{\frac{D-2}{D-3}}, \]

Therefore,
\[ \rho(M) \sim e^{\sigma(D)M \frac{D-3}{D-2}}, \quad (2.13) \]

and

\[ \Omega_H(M) \sim \rho^{D-3}(M). \quad (2.14) \]

According to the last equation, although both the quantum and Hawking density of states agree in 4 dimensions the thermodynamical interpretation predicts a vastly enhanced number of states in higher dimensions as compared to the quantum interpretation.

Let us now analyze very closely the implications of the thermodynamical interpretation. First as is clear from Eq.(2.2), the canonical partition function is finite in the WKB approximation. On the other hand, the exact partition function is the Laplace transform of the density of states (2.14). We get

\[ Z(\beta) = \int_0^{\infty} dE e^{-\beta E} e^{(D-3)\sigma(D)E^{\frac{D-2}{D-3}}}. \quad (2.15) \]

For \( D \geq 4 \), the above expression diverges badly for all \( \beta \), implying the non-existence of the canonical partition function for all temperatures. The non-existence of the canonical ensemble can also be seen from the calculation on the canonical specific heat. It is found to be negative. The saddle point approximation therefore must fail and the canonical and microcanonical ensembles are not equivalent. The study of the statistical mechanics of black holes must proceed in the microcanonical ensemble. The result of Eqs.(2.2) and (2.14) must therefore be wrong. Black hole thermodynamics is simply not a viable option. Eq.(2.15) gives the proof.

This leaves us with only one alternative, the fully quantum interpretation of the WKB formula as the tunneling probability, yielding directly the quantum black hole degeneracy of states Eqs.(2.1) and (2.13).

Comparison of the degeneracy of states (2.13) with those of known non-local quantum theories yields immediately the classification of \( D \)-dimensional Schwarzschild black holes as the quantum excitation modes of a \( \left( \frac{D-2}{D-4} \right) \)-brane. Black holes are therefore fully elementary.
particles. Massless particles such as photons may then be regarded as extreme quantum black holes with a horizon of zero radius.

The study of the statistical mechanics of a gas of such objects in the microcanonical ensemble (the unique approach) reveals their conformal nature through two characteristics, the validity of the statistical bootstrap property and the duality (crossing symmetry) of the $S$-matrix. The equilibrium state of a gas of $N$ black holes is found \textsuperscript{[7–10]} to be the one for which there is a single very massive black hole in the gas and $(N - 1)$ massless others, a state very far from thermal equilibrium. The microcanonical specific heat, of course, is negative. The above fully quantum picture therefore resolves completely the so-called information loss paradox.

III. MEAN-FIELD THEORY

The results of the preceding section were arrived at within the semiclassical WKB approximation. Besides the WKB method there is another semiclassical technique (mean field theory) which aims at quantizing field theories in the classical black hole background \textsuperscript{[3]}.

The problem of quantization can be approached from the viewpoint of scattering theory, in which quantum fields are scattered off the black hole horizon. Because of the horizon, two causally disconnected spacelike regions coexist and field quantization in each sector makes use of a different Hilbert (Fock) space. When quantum fields scatter off the horizon, mixing occurs between the corresponding two sets of modes. This is expressed mathematically as a Bogoliubov transformation. This doubling of the number of degrees of freedom of the theory due to the non-trivial topology of the black hole spacetime, turns out to bear considerable resemblance to the mathematical structure of modern field theories at finite temperature (e.g. the thermofield dynamics formalism \textsuperscript{[4]}). In particular, the vacuum state for the outgoing particle can be formally written as follows,

$$
|\text{out}, 0 > = Z^{-1/2}(\beta) \sum_{n=0}^{\infty} e^{-\beta n \omega/2} |n > \otimes |\bar{n} > ,
$$

(3.1)
in which $|n>$ and $|\tilde{n}>$ are the Fock spaces of the two causally disconnected regions (the observer sees only $|n>$ directly).

It follows that the physically observable correlation functions are those obtained by making use of the vacuum (3.1). It is easy to show that any expectation value with respect to the above vacuum is equivalent to a statistical average in the canonical ensemble. For any observable $A$ we have

$$<\text{out}, 0 | A | \text{out}, 0 > = \sum_{n=0}^{\infty} e^{-\beta n \omega} <n| A |n> / Z(\beta),$$

(3.2)

where the partition function $Z(\beta)$ is given by

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-n \beta \omega}.$$  

(3.3)

The inverse temperature $\beta$ is

$$\beta = \frac{2 \pi}{\kappa},$$

(3.4)

where $\kappa$ is the surface gravity of the black hole. Therefore $\beta = \beta_H$, in agreement with the WKB result.

A direct consequence of Eq.(3.2) is the fact that the mass $m$ particle number density has the following Planckian (thermal) distribution for an observer outside the event horizon

$$n_k(m; \beta_H) = \frac{1}{e^{\beta_H \omega_k(m)} - 1},$$

(3.5)

a fact interpreted as a loss of information during the scattering process with the black hole. Since the scattering process started with particles in a pure state (the “in” vacuum given by $|\text{in}, 0 > = |0 > \otimes |\tilde{0}>$), Eq.(3.5) implies a loss of unitarity during the scattering process, a violation of quantum mechanics (in the field theory limit),

$$|\text{out}, 0 > = S^{-1}(\beta)|\text{in}, 0 > \ ; \ S^{-1} \neq S^\dagger.$$  

(3.6)

Clearly the results of the previous section show that a thermodynamical description of quantum black hole physics is inappropriate. Thus the legitimate question: what about the result of Eq.(3.5)? In the following section we explain how one reconciles both semiclassical considerations.
IV. SEMICLASSICAL BLACK HOLES

In this section, we explain how the seemingly contradictory semiclassical results of the preceding two sections actually do reconcile.

The inescapable conclusion of section II is the non-local nature of the semiclassical black holes. On the other hand, the result of Eq.(3.5) for the particle number density is that of a local field theory. This result therefore cannot be accepted at face value and cannot determine alone the true vacuum of our black hole problem.

Clearly, and it is here that back reaction effects start entering the picture, one needs to consider the total particle number density of all the particle modes of the full non-local quantum gravity theory. For a complete treatment, Eq.(3.5) must then be replaced by the following general expression,

\[ n_k(\beta_H) = \int_0^\infty dm \rho(m) n_k(m; \beta_H) . \]  \hspace{1cm} (4.1)

Eq.(4.1) leads to the following canonical partition function,

\[ Z(\beta_H) = \exp\left(-\frac{V}{(2\pi)^{D-1}} \int_0^\infty \int_{-\infty}^{\infty} dm \rho(m) \ln \left[1 - e^{-\beta_H \omega_k(m)}\right]\right) , \] \hspace{1cm} (4.2)

where \( \omega_k(m) = \sqrt{\vec{k}^2 + m^2} \).

The “thermal vacuum” Eq.(3.1) is now generalized as follows,

\[ |_{out,0} > = Z^{-1/2}(\beta) \prod_{m,k} \sum_{n_k,m=0}^{\infty} \prod_{m,k} e^{-\frac{\beta}{2} n_k,m \omega_k,m} |n_k,m > \otimes |\tilde{n}_{k,m} > . \] \hspace{1cm} (4.3)

Recalling that

\[ Z(\beta_H) = \int_0^\infty dE e^{-\beta_H E} \Omega(E) , \] \hspace{1cm} (4.4)

and comparing the above equation with Eq.(4.2), one finally arrives at Hagedorn’s old self-consistency condition \cite{14},

\[ \exp\left(-\frac{V}{(2\pi)^{D-1}} \int_0^\infty \int_{-\infty}^{\infty} dm \rho(m) \ln \left[1 - e^{-\beta_H \omega_k(m)}\right]\right) = \int_0^\infty dE e^{-\beta_H E} \Omega(E) . \] \hspace{1cm} (4.5)
We are seeking solutions obeying the statistical bootstrap requirement,

\[
\frac{\rho(E)}{\Omega(E)} \to 1 \ ; \ (E \to \infty) .
\] (4.6)

As is well known, string theories are the only possible solutions of the above self-consistency condition,

\[
\rho(m) \sim e^{bm} \ ; \ (m \to \infty) ,
\] (4.7)

provided \(\beta_H > b\), where \(b^{-1}\) is the so-called Hagedorn temperature.

The WKB results of section II exclude however string theories as quantum black hole theories. Black holes are p-brane excitations with \(p = \frac{D-2}{D-4}\) and so \(p > 1\). Black hole solutions are therefore excluded as solutions of the conditions (4.5) and (4.6) as they yield an infinite canonical partition function. Therefore, the thermal vacuum Eq.(4.3) is the false vacuum. Again one finds that thermal equilibrium is alien to quantum black hole physics. Semiclassical quantization as discussed in the previous section is the wrong starting point for field quantization in a black hole spacetime. Note that to arrive at this conclusion, one needs consider the full non-locality of the quantum gravity theory by including the effects of all the excitation modes (back reactions) of the theory.

The obvious next question is how to quantize fields in black hole backgrounds. This is not an easy question to answer. However, it may be possible that, recalling the nature of the non-thermal equilibrium state of a gas of black holes, one might need some kind of generalization of usual (“canonical”) quantum field theory to the so-called microcanonical quantum field theory.

V. CONCLUSION

In this work, we presented a complete treatment of the semiclassical approaches to quantum black hole physics.

In section II, we reviewed the resolution of the so-called information loss paradox, as provided in our earlier works \([4,11]\). In section IV, we provided the solution to the thermal
spectrum problem by taking full account of the non-locality of the quantum theory of gravity. Only then could the results of both semiclassical calculations be brought to agreement.

Again all our considerations are consistent with the view that quantum $D$-dimensional black holes are excitation modes of $\frac{D-2}{D-4}$ branes.

Of course a deeper understanding of these results remains the subject of future endeavors.

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