Computing projective equivalences of special algebraic varieties

Michal Bizzarri\textsuperscript{b}, Miroslav Lávička\textsuperscript{a,b}, Jan Vršek\textsuperscript{a,b}

\textsuperscript{a}Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic
\textsuperscript{b}NTIS – New Technologies for the Information Society, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic

Abstract

This paper is devoted to the investigation of selected situations when the computation of projective (and other) equivalences of algebraic varieties can be efficiently solved with the help of finding projective equivalences of finite sets on the projective line. In particular, we design a unifying approach that finds for two algebraic varieties $X, Y$ from special classes an associated set of automorphisms of the projective line (the so called good candidate set) consisting of candidates for the construction of possible mappings $X \rightarrow Y$. The functionality of the designed method is presented on computing projective equivalences of rational curves, on determining projective equivalences of rational ruled surfaces, on the detection of affine transformations between planar curves, and on computing similarities between two implicitly given algebraic surfaces. When possible, symmetries of given shapes are also discussed as special cases.

Key words: Projective transformation, symmetry, rational curve, rational ruled surface, algebraic surface

1. Introduction

Projective (or affine, or similar, or isometric) equivalencies and symmetries of geometric shapes is a fundamental concept in nature, science, engineering, architecture, etc. For instance, symmetries in the natural world has inspired people to integrate symmetry when designing tools, buildings, or artwork. Many biochemical processes are governed by symmetries. Hence, detecting a suitable class of equivalencies in given geometric data is a problem in geometry processing that has attracted attention of researchers from different scientific areas for many years. Numerous algorithms have been formulated to identify, extract, encode, and apply geometric equivalencies or symmetries and numerous applications immediately benefit from gained equivalency or symmetry information. Nowadays, geometric equivalencies and symmetries play a significant role also in computer graphics, computer vision, or in pattern recognition.

In short, the main goal is to decide whether two given geometric shapes are related by a suitable transformation and in the affirmative case to detect all such equivalences. In many applications, it is sufficiently enough to find approximate equivalencies and symmetries of the given shapes, only. For this, one can identify mainly the following two practical reasons – first, the input shape is approximate (it is an simplified model of some real object), or second, computations cannot be provided exactly (solving of complicated systems of non-linear equations). However, one question still remains, i.e., how to solve the problem exactly at least for some special algebraic varieties.
This problem has become an active research area especially in recent years and one can find several papers focused on the detection and computation of symmetries and some equivalences of curves, see e.g. Huang and Cohen (1994); Brass and Knauer (2004): Lebmeir and Richter-Gebert (2008); Lebmeir (2009), or series of papers Alcázar (2014); Alcázar et al. (2014a,b, 2015, 2018). The first paper devoted to the broadest group, i.e., to the general projective equivalences, has appeared quite recently, see Hauer and Jüttler (2018). In this paper, the authors study equivalences of curves with respect to the projective group in an arbitrary space dimension. The formulated symbolic-numerical algorithm (based on Gröbner bases computation) is universal and provides good computational results for all presented examples with coefficients from $\mathbb{Q}$. Later, the problem of deterministically computing the symmetries of a planar implicitly given curve and the problem of deterministically checking whether or not two implicitly given, planar algebraic curves are similar, i.e., equal up to a similarity transformation, was investigated in Alcázar et al. (2018). Nonetheless, solving this problem for surfaces in 3-space, at least for special classes, remains still an open question which deserves further research.

The paper is organized as follows. Section 2 recalls some basic facts concerning projective (and other) equivalences, finite rotation groups and Grassmannians. As the paper is focused on studying suitable situations that can be reduced to the computation of equivalences of finite sets on the projective line, we present in Section 3 two algorithms devoted to the detection of equivalences of finite point sets using cross-ratios and to the detection of equivalences of finite sets given by a polynomial relation. The formulated computational method is then used in Section 4 on computing projective (and other) equivalences of several types of algebraic varieties, in particular on projective equivalences of rational curves, on projective equivalences of rational ruled surfaces, on the detection of affine transformations mapping a planar curve to another planar curve and on computing similarities between two implicitly given algebraic surfaces. The functionality of the designed unifying approach is documented on several examples. Finally, we conclude the paper in Section 5.

2. Preliminaries

First we recall some fundamental facts and notions whose knowledge is anticipated in the following sections.

2.1. Projective transformations

Recall that the projective space $\mathbb{P}^n_\mathbb{K}$ of dimension $n$ over the field $\mathbb{K}$ is the set of all lines through the origin in $\mathbb{K}^{n+1}$. It can be interpreted as the quotient $\mathbb{K}^{n+1} \setminus \{0\}/\sim$, where $\sim$ denotes the equivalence relation of points lying on the same line going through the origin:

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \iff (x_0, \ldots, x_n) = \lambda(y_0, \ldots, y_n) \text{ for some } \lambda \in \mathbb{K}^*, \quad (1)$$

where $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Hence a point in the projective space can be considered as an equivalence class in $\mathbb{K}^{n+1}$ split by $\sim$. To distinguish the class from its representative we will write colons between the coordinates and the square brackets instead of the round brackets, i.e., $x = [x_0 : x_1 : \cdots : x_n]$.

Throughout the paper we will work mainly over the field of complex numbers, in which case we will write $\mathbb{P}^n$ instead of $\mathbb{P}^n_\mathbb{C}$.

Fixing a hyperplane $\omega : x_0 = 0$ as a hyperplane at infinity, or an ideal hyperplane, we obtain the affine space $\mathbb{A}^n_\mathbb{K}$ embedded into $\mathbb{P}^n_\mathbb{K}$ via $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$. Conversely a point $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n \setminus \omega = \mathbb{A}^n_\mathbb{K}$ has the affine coordinates $x = (x_1/x_0, \ldots, x_n/x_0)$. The oval quadric $\Omega : x_0 = x_1^2 + \cdots + x_n^2 = 0$ lying in $\omega \subset \mathbb{P}^n_\mathbb{K}$ is called the absolute quadric. It may induce a metric in the affine space which then becomes the Euclidean space $\mathbb{E}^n_\mathbb{K}$. Note that $\Omega$ consists of imaginary points only.

Let us denote $\text{Aut}(\mathbb{P}^n_\mathbb{K})$ the set of all projective transformations of $\mathbb{P}^n_\mathbb{K}$, i.e., $\text{Aut}(\mathbb{P}^n_\mathbb{K}) \cong \mathbb{P} \text{GL}_{n+1}(\mathbb{K})$. A projective transformation mapping the hyperplane at infinity $\omega$ onto itself and thus also $\mathbb{A}^n_\mathbb{K}$
Proposition 1. Two ordered quadruples $A$ known, any projective transformation $A \subset \mathbb{R}^n$ denoted by $\text{Aff}(A)$ onto itself is called an affine transformation. The set of all affine transformations forms a group denoted by $\text{Aff}(\mathbb{R})$. Since we will be interested in $\text{Aff}(\mathbb{C})$ only we will write simply $\text{Aff}$ for this group without any danger of confusion. A similarity is an affine transformation in the Euclidean space $\mathbb{R}^n$ which preserves the absolute quadric. The group of direct similarities is generally denoted $\text{Sim}(\mathbb{R})$. Let us remark that analogously to the affine case we will deal with $\text{Sim} := \text{Sim}(\mathbb{R})$ only. Finally, isometries are similarities which preserve distances.

The matrix representation of any transformation of $\mathbb{P}^n$ can be written in the form

$$\overline{A} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{00} & \tilde{a} \\ \ast & A \end{pmatrix} . \tag{2}$$

The affine transformations correspond to the case $a_{00} \neq 0$ and $\tilde{a} = (0, \ldots, 0)$. The additional assumption on the transformation to be a similarity is fulfilled by the condition $A^\top A = A I$, where $\lambda \in \mathbb{R}^{>0}$, and especially for $\lambda = 1$ we obtain an isometry.

Let $\mathcal{G}$ be a subgroup of $\text{Aut}(\mathbb{P}^n)$ and let $A, B \subset \mathbb{P}^n$. We will write

$$\mathcal{G}_{A,B} := \{ \phi \in \mathcal{G} : \phi(A) = B \} \tag{3}$$

for the set of equivalences between $A$ and $B$. In the case $B = A$ the set $\mathcal{G}_{A,A}$ forms a group and we will denote it $\mathcal{G}_A$.

2.2. $\text{Aut}(\mathbb{P}^1)$ and its finite subgroups

As known, any projective transformation $\mathbb{P}^n \to \mathbb{P}^n$ is uniquely determined specifying $n + 2$ pairs of points in a general position. In particular, a projective automorphism of $\mathbb{P}^1$ is specified by three points and their images. Thus for an ordered quadruple $\{a_1, \ldots, a_4\}$ there exists the unique $\phi \in \text{Aut}(\mathbb{P}^1)$ such that $\phi(a_1) = [1 : 1]$, $\phi(a_2) = [0 : 1]$ and $\phi(a_3) = [1 : 0]$. If we write $\phi(a_4) = [s : t]$ then the cross-ratio of the quadruple is defined to be a number

$$[a_1, a_2; a_3, a_4] = \frac{t}{s} . \tag{4}$$

Proposition 1. Two ordered quadruples $\{a_1, \ldots, a_4\}$ and $\{b_1, \ldots, b_4\}$ in $\mathbb{P}^1$ are projectively equivalent if and only if $[a_1, a_2; a_3, a_4] = [b_1, b_2; b_3, b_4]$.

The Riemann sphere $\hat{\mathbb{C}}$ is the set $\mathbb{C} \cup \{\infty\}$, where $\infty$ is a formal point not in $\mathbb{C}$. The homeomorphism $\varphi : \hat{\mathbb{C}} \to \mathbb{P}^1$ given by $\varphi(z) = [1 : z]$ for $z \in \mathbb{C}$ and $\varphi(\infty) = [0 : 1]$ identifies the Riemann sphere $\hat{\mathbb{C}}$ with the complex projective line $\mathbb{P}^1$. Moreover since the stereographic projection naturally identifies the Riemann sphere with the unit sphere $S^2 \subset \mathbb{R}^3$, we obtain three homeomorphic copies of the sphere: the sphere $S^2$ itself, the Riemann sphere $\hat{\mathbb{C}}$ and the projective line $\mathbb{P}^1_{\mathbb{C}}$. In this way the group $\text{Aut}(\mathbb{P}^1)$ can be identified with conformal homeomorphisms of the sphere, see [Shurman and Levenberg 1997] pg. 26 for detailed explanation and for the proof of the following proposition.

Proposition 2. Any finite automorphism group of the sphere is conjugate to the rotation group.

Moreover all the types of finite rotation groups are classified as follows

Proposition 3. Each finite rotation group of the sphere is isomorphic to one of the following groups:

1. cyclic groups $C_n$,
2. dihedral groups $D_n$,
3. the symmetry groups of tetrahedron $T$, octahedron $O$ or icosahedron $I$. 


2.3. Grassmannians

The set of all projective subspaces of dimension \(k\) in \(\mathbb{P}^n_k\) forms a projective variety; the so called Grassmannian \(G(k, n)\). We are mainly interested in two cases. First, the Grassmannian \(G(n - 1, n)\) of all subspaces of dimension \(n - 1\) in \(\mathbb{P}^n\), which is again a projective space, called the dual space and denoted \((\mathbb{P}^n_k)^\vee\). Second, the variety of lines in \(\mathbb{P}^3_k\), which is a quadratic hypersurface in \(\mathbb{P}^5\) and denoted \(G = G(1, 3)\). For an introduction to the theory of Grassmannians see e.g. Pottmann and Walther [2001] or Harris [1992].

Let us focus on the group of automorphisms of \(G\). Any projective transformation \(\phi : \mathbb{P}^3 \to \mathbb{P}^3\) maps lines to lines. It turns out that it induces a transformation of \(\mathbb{P}^3\) preserving the Grassmannian \(G\). In fact it induces an injective group homomorphism \(\mathrm{Aut}(\mathbb{P}^3_k) \to \mathrm{Aut}(\mathbb{P}^3_k)_G\), we will write \(\hat{\phi}\) for the transformation induced by \(\phi\). Let \(\mathrm{Aut}(\mathbb{P}^3_k)_G\) denotes the image of \(\mathrm{Aut}(\mathbb{P}^3_k)\) under this homomorphism. Then it is a subgroup of index 2 in \(\mathrm{Aut}(\mathbb{P}^5_k)_G\). Its complement \(\mathrm{Aut}(\mathbb{P}^3_k)^\hat{\chi}\) is formed by transformations induced by regular projective mappings \(\mathbb{P}^3 \to (\mathbb{P}^1)^3\), cf. Harris [1992] Theorem 10.19.

3. Projective equivalences of finite subsets of \(\mathbb{P}^1\)

The paper is devoted to studying selected situations when the computation of projective (or other) equivalences of certain algebraic curves and surfaces can be simply solved with a unifying approach for determining projective equivalences of associated finite sets on the projective line. Hence, in this section we formulate two algorithms devoted to the detection of equivalences of finite point sets on the projective line given either directly or as the roots of a polynomial.

3.1. Finite subsets as the collections of points

Consider two finite subsets \(A\) and \(B\) of \(\mathbb{P}^1\), obviously they can be projectively equivalent only if they have the same cardinality. Since any transformation of the projective line is determined by three points we see that \(\#\mathrm{Aut}(\mathbb{P}^1)_{A,B}\) is non-empty whenever \(\#A = \#B = 3\). In fact in this case it is isomorphic to the permutation group on three elements. We have already seen (Proposition \([\square]\)) that four point sets are not projectively equivalent in general – surprisingly if there exists a projectivity mapping \(A\) to \(B\) then it is not unique.

**Lemma 4.** Let \(A\) and \(B\) be two subsets of \(\mathbb{P}^n\) such that \(\mathrm{Aut}(\mathbb{P}^n)_{A,B}\) is non-empty and finite. Then \(\#\mathrm{Aut}(\mathbb{P}^n)_{A,B} = \#\mathrm{Aut}(\mathbb{P}^n)_{A} = \#\mathrm{Aut}(\mathbb{P}^n)_{B}\)

**Proof.** To prove \(\#\mathrm{Aut}(\mathbb{P}^n)_{A,B} = \#\mathrm{Aut}(\mathbb{P}^n)_{A}\) we construct the mapping \(\mathrm{Aut}(\mathbb{P}^n)_{A} \to \mathrm{Aut}(\mathbb{P}^n)_{A,B}\) as follows. Fix \(\phi \in \mathrm{Aut}(\mathbb{P}^n)_{A,B}\) and define \(\psi \mapsto \phi \circ \psi\). It is easily seen that it is bijective. The second part is analogous. \(\square\)

**Lemma 5.** Let \(A = \{a_1, \ldots, a_4\} \subset \mathbb{P}^1\) be a set consisting of four distinct points. Then \(\mathrm{Aut}(\mathbb{P}^1)_A\) is a group of order at least four. More precisely \(\mathrm{Aut}(\mathbb{P}^1)_A = \mathbb{Z}_2 \times \mathbb{Z}_2\) unless \([a_1, a_2; a_3, a_4] \in \{-1, 2, -1, e^{2\pi i}\}\)

**Proof.** Altogether there are 24 permutations on four elements, whereas there exist at most six different values of their cross-ratios. Thus by Proposition \([\square]\) the group \(\mathrm{Aut}(\mathbb{P}^1)_A\) has the order at least four. Moreover when the cross-ratio is different from \(-1, 2, 1, e^{2\pi i}\) then there are exactly six values and it is known that the subgroup of permutations on four points preserving their cross ratio is the Klein group \(\mathbb{Z}_2 \times \mathbb{Z}_2\). \(\square\)

On contrary if two projectively equivalent sets have more than four points then the transformation is generically unique.
However, the forms carry more information because of possible higher multiplicities of its roots.

3.2. Finite sets as the roots of a polynomial

A collection of $n$ points in $\mathbb{P}^1$ can be given as a set of roots of homogeneous form of degree $n$. However, the forms carry more information because of possible higher multiplicities of its roots. Let $F(x_0, x_1)$ and $G(x_0, x_1)$ be the forms. Any $\phi \in \text{Aut}(\mathbb{P}^1)$ acts naturally on the set of forms of degree $n$ by $F \mapsto F \circ \phi$. We define

$$\text{Aut}(\mathbb{P}^1)_{F,G} = \{ \phi \in \text{Aut}(\mathbb{P}^1) \mid \exists \lambda \in \mathbb{C}^* : G \circ \phi = \lambda F \}.$$  (12)
If $A$ and $B$ are the sets of roots of $F(x_0, x_1)$ and $G(x_0, x_1)$ respectively, then obviously any $\phi \in \text{Aut}(\mathbb{P}^1)_{F,G}$ induces a transformation mapping $A$ to $B$. Thus $\text{Aut}(\mathbb{P}^1)_{F,G} \subset \text{Aut}(\mathbb{P}^1)_{A,B}$. If all the roots of $F$ and $G$ are simple then we have the equality. Nonetheless the inclusion may be proper, in general.

**Algorithm 2** Detection of equivalences of finite sets given by polynomial relation.

**Input:** $F(x_0, x_1)$ and $G(x_0, x_1)$
1: Consider the ideal $I$ generated by the coefficients of the polynomial $G(a_{00}x_0 + a_{10}x_1, a_{01}x_0 + a_{11}x_1) - F(x_0, x_1)$ w.r.t. $x_0, x_1$.
2: Compute the reduced Gröbner basis $GB = \{g_1, \ldots, g_\ell\}$ of $I$ w.r.t. a suitable ordering of the variables $a_{00}, a_{10}, a_{01}, a_{11}$.
3: if $GB = \{1\}$ then
4: $\text{Aut}(\mathbb{P}^1)_{F,G} = \emptyset$
5: else
6: Find a solution of the system of equations $g_1 = 0, \ldots, g_\ell = 0$.
7: $\text{Aut}(\mathbb{P}^1)_{F,G}$ consists of automorphisms given by all the solutions $a_{00}, a_{10}, a_{01}, a_{11}$
8: end if

**Output:** $\text{Aut}(\mathbb{P}^1)_{F,G}$.

**Remark 7.** In Algorithm 2 we compute the Gröbner basis of the ideal generated by the coefficients of $G(a_{00}x_0 + a_{10}x_1, a_{01}x_0 + a_{11}x_1) - F(x_0, x_1)$, whereas two forms are projectively equivalent if one can be mapped to the other up to a complex multiple $\lambda$, cf. [12]. However we do not need to consider the additional parameter $\lambda$ (which would cost some computational time) since the matrix $A$ of the transformation is also determined uniquely up to a complex multiplication and hence it can ensure $\lambda = 1$.

Although the Algorithm 2 requires to solve a large system of non-linear equations, recall that two general forms of degree at least four are not projectively equivalent and thus we have the ideal $I = \{1\}$. For example, in this case CAS Mathematica or CAS Maple give a decision even for polynomials of degree 20 within a few seconds. Next, the general polynomial of degree at least five possesses no projective automorphism and thus by Lemma 3 the transformation is unique for two equivalent generic forms of high degree. Again in this case the answer is obtained within a few seconds as the Gröbner basis possesses a special structure containing linear forms in $a_{00}, a_{10}, a_{01}, a_{11}$ which uniquely determine the automorphism. In addition, the basis also contains one nonlinear term responsible for a particular choice of $a_{00}, a_{10}, a_{01}, a_{11}$ (of course, describing the same automorphism for all choices), cf. Example 8 and (16).

**Example 8.** Consider two forms of degree six

$$F = 571x_0^6 - 426x_1x_0^5 - 1827x_1^2x_0^4 + 8532x_1^3x_0^3 - 11259x_1^4x_0^2 + 12150x_1^5x_0 - 3645x_1^6$$

and

$$G = -569x_0^6 + 430x_1x_0^5 + 1758x_1^2x_0^4 + 3891x_1^3x_0^3 + 6054x_1^4x_0^2 + 2105x_1^5x_0 + 2055x_1^6$$

The Gröbner basis of the ideal generated by the coefficients of

$$G(a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1) - F(x_0, x_1)$$

has the form

$$\{1771561a_{11}^6 - 46656, a_{10}7 - a_{11}, 2a_{01} + 5a_{11}, 6a_{00} - 7a_{11}\}.$$  

(16)

Since the transformation is unique up to a scalar multiplication, we can omit the first polynomial and solve the linear system only, i.e., we obtain

$$a_{01} \rightarrow -\frac{15a_{00}}{7}, \quad a_{10} \rightarrow \frac{6a_{00}}{7}, \quad a_{11} \rightarrow \frac{6a_{00}}{7}$$

(17)
yielding the transformation of \( \mathbb{P}^1 \) represented by the matrix

\[
\mathbf{A} = \begin{pmatrix} 7 & -15 \\ 6 & 6 \end{pmatrix}
\]

mapping the roots of \( F \) to the roots of \( G \).

4. Projective and other equivalences of selected algebraic varieties

In this section we will discuss several problems which can be reduced to the computation of equivalences of finite sets in \( \mathbb{P}^1 \). The general setting of our problem is following. Let be given \( G \) a subgroup of \( \text{Aut}(\mathbb{P}^n) \) and two algebraic varieties \( X, Y \subset \mathbb{P}^n \). Our goal is to compute \( G_{X,Y} \) using the approach introduced in the previous section. Hence we find suitable forms \( F(x_0, x_1) \) and \( G(x_0, x_1) \) associated to the varieties \( X \) and \( Y \) together with the inclusion

\[
\iota : G_{X,Y} \hookrightarrow \text{Aut}(\mathbb{P}^1)_{F,G}.
\]

The idea is that for \( \phi \in \text{Aut}(\mathbb{P}^1)_{F,G} \) it is simple to decide whether there exists \( \psi \in G_{X,Y} \) such that \( \phi = \iota(\psi) \). In this sense \( \text{Aut}(\mathbb{P}^1)_{F,G} \) consists of candidates for possible mappings \( X \rightarrow Y \). If \( \text{Aut}(\mathbb{P}^1)_{F,G} \) is not too large compared to \( G_{X,Y} \) — in particular if they have the same dimension, then we will call it a good candidate set of \( G_{X,Y} \).

4.1. Projective equivalences of rational curves

When studying projective equivalences of algebraic varieties then it is natural to start with the further simplest case after the collections of points, i.e., with rational curves. Recently, Hauer and Jüttler (2018) published a paper devoted to the detection of equivalences and symmetries of rational curves with respect to the group of projective transformations including the subgroup of affine transformations. We continue in this investigation and present an algorithm based on computing projective equivalences of finite point sets.

By a rational curve of degree \( d \) in \( \mathbb{P}^n \) we mean the image of a morphism \( \mathbb{P}^1 \rightarrow \mathbb{P}^n \) given by

\[
p(s, t) = [p_0(s, t) : p_1(s, t) : \cdots : p_n(s, t)],
\]

where \( p_i(s, t) \) are homogeneous polynomials of degree \( d \) without a common factor. Moreover the mapping is assumed to be a birational morphism, i.e., it is almost everywhere injective. In what follows, we assume the curve to be non-degenerate, i.e., it is not contained in any hyperplane, or equivalently all the polynomials \( p_i \) are linearly independent over \( \mathbb{C} \). Obviously a curve can be non-degenerate only if \( d \geq n \).

Since the degree of a curve is a projective invariant, the equivalent curves must have the same degree. Recall that any parameterization \( p(s, t) \) of a rational curve of degree \( d \) in \( \mathbb{P}^n \) is an image of the rational normal curve

\[
C_d : \quad c_d(s, t) = [s^d : s^{d-1} t : \cdots : t^d]
\]

under some projection \( \mathbb{P}^{d!}\setminus M \rightarrow \mathbb{P}^n \), where \( M \) is a linear subspace of dimension \( d - n - 1 \). In particular, if \( p_l(s, t) = \sum_{j=0}^d p_{lj} s^{d-j} t^j \), then the projection is given by the matrix \( (p_{lj})_{i,j=0}^n \) and the subspace \( M \) is generated by the kernel of the matrix. Clearly there exists a projective transformation taking one parameterization to the other one if and only if the projection matrices have the same kernels.

**Proposition 9.** Two parameterizations of non-degenerate rational curves of degree \( n \) in \( \mathbb{P}^n \) are always projectively equivalent.
Hence in what follows we focus on transformations between curves of degree \( d > n \). Let \( C \subset \mathbb{P}^n \) be parameterized by \( p(s, t) \) then the osculating \( k \)-planes, \( k = 1, \ldots, n - 1 \), having the contact of order at least \( k + 1 \) are spanned by \( \frac{\partial^k p(s, t)}{\partial s^k}, \frac{\partial^k p(s, t)}{\partial s^{k-1}\partial t}, \ldots, \frac{\partial^k p(s, t)}{\partial t^k} \). For \( k = n - 1 \) we obtain osculating hyperplanes. Stall points are the points where the osculating hyperplane has the contact higher than expected. They are given by the condition

\[
\Delta_p(s, t) = \det \left[ \frac{\partial^k p(s, t)}{\partial s^k}, \frac{\partial^k p(s, t)}{\partial s^{k-1}\partial t}, \ldots, \frac{\partial^k p(s, t)}{\partial t^k} \right] = 0. \tag{22}
\]

In particular, the homogeneous form \( \Delta_p(s, t) \) has degree \((d - n)(n + 1)\) and thus on any non-degenerate curve with degree \( d > n \), there exist only finitely many stalls. A projective transformation takes osculating \( k \)-planes of the curve to osculating \( k \)-planes of its image curve, in particular stalls are mapped to stalls.

**Theorem 10.** Let \( C, D \subset \mathbb{P}^n \) be non-degenerate rational curves of degrees \( d > n \) and let \( p : \mathbb{P}^1 \to C \) and \( q : \mathbb{P}^1 \to D \) be the birational morphisms parameterizing them. Then \( \text{Aut}(\mathbb{P}^1)_{\Delta_p, \Delta_q} \) is a candidate group for \( \text{Aut}(\mathbb{P}^n)_{C, D} \) and the inclusion \( \iota : \text{Aut}(\mathbb{P}^n)_{C, D} \hookrightarrow \text{Aut}(\mathbb{P}^1)_{\Delta_p, \Delta_q} \) is given by \( \iota : \phi \mapsto q^{-1} \circ \phi \circ p \).

**Proof.** Assume that there exists a projective transformation \( \phi \) taking the curve \( C : p(s, t) \) to the curve \( D : q(u, v) \) and thus there exists the reparameterization \( \psi \) making the following diagram commutative

\[
\begin{array}{ccc}
C & \xrightarrow{\phi} & D \\
p & \uparrow & q \\
\mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1
\end{array}
\tag{23}
\]

Now, given \( \psi \in \text{Aut}(\mathbb{P}^1)_{\Delta_p, \Delta_q} \) we would like to decide whether it is the image of some \( \phi \in \text{Aut}(\mathbb{P}^n)_{C, D} \). By \( \psi \in \text{Aut}(\mathbb{P}^1)_{\Delta_p, \Delta_q} \) this happens if and only if the parameterizations \( p \) and \( q \circ \psi \) are projectively equivalent. Nonetheless this is equivalent to the condition that the matrices of the coefficients of these parameterizations have the same kernels. The method is summarized in Algorithm 3.

**Algorithm 3** Projective equivalences of rational curves.

**Input:** Curves \( C : p(s, t) \) and \( D : q(s, t) \)

1. Compute the forms \( \Delta_p(s, t) \) and \( \Delta_q(s, t) \), cf. (22).
2. Find the candidate group \( \text{Aut}(\mathbb{P}^1)_{\Delta_p, \Delta_q} \) composed of automorphisms \( \psi \) described by matrices \( \overline{B}_\psi \), see Section 3.2.
3. For all \( \psi \in \text{Aut}(\mathbb{P}^1)_{\Delta_p, \Delta_q} \) compute the kernels \( K_p \) and \( K_{q\cdot\psi} \) of the matrices of the coefficients of \( p(s, t) \) and \( q(\overline{B}_\psi(s, t)^\top) \).
4. If \( K_p = K_{q\cdot\psi} \) then
5. By solving linear equations corresponding to \( \overline{A}p(s, t) - q(\overline{B}_\psi(s, t)^\top) \) compute the projective transformation \( \phi \) given by \( \overline{A} \) and include it into \( \text{Aut}(\mathbb{P}^n)_{C, D} \).
6. end if

**Output:** \( \text{Aut}(\mathbb{P}^n)_{C, D} \).

The case of projective equivalences between rational quartics in \( \mathbb{P}^4 \) was already studied in \( \text{Telling 1936} \) pg. 42, with the result that two quartics with same osculating polynomial (up to reparameterization) are projectively equivalent. Let us briefly recall the arguments. For a quartic parameterization \( p(s) = [p_0(s) : \cdots : p_4(s)] \) of \( C \) the condition that \( p(s_i), i = 1, \ldots, 4 \) are coplanar is symmetric algebraic relation in \( s_i \) which is moreover linear in each parameter – because when
given three points on \( C \) then the fourth point is determined uniquely. Hence this relation has the form
\[
\Phi(s_1, s_2, s_3, s_4) = \varphi_4 s_1 s_2 s_3 s_4 + \varphi_3 \sum_{i<j<k} s_i s_j s_k + \varphi_2 \sum_{i<j} s_i s_j + \varphi_1 \sum_i s_i + \varphi_0. \tag{24}
\]
This polynomial is unique up to a scalar and it is a polarized form of a quartic polynomial
\[
\Phi(s, s, s, s) = \varphi_4 s^4 + 4 \varphi_3 s^3 + 6 \varphi_2 s^2 + 4 \varphi_1 s + \varphi_0. \tag{25}
\]
The roots of this polynomial correspond to points on the curve \( C \) with their osculating plane of contact order four. Thus this is precisely (up to a scalar multiple) our osculating polynomial \( \Delta_p(s) \).

From the construction of (24) it is clear that it is invariant under projective transformations. Conversely let be given two curves with the same osculating polynomial and thus with the same \( \Delta_p(s) \) as well. Then there exists a correspondence between quadruples of coplanar points and thus between planes. This provides a projective transformation taking one curve to the other. This proof, in fact, does not work only for quartics in \( \mathbb{P}^3 \) between planes. This provides a projective transformation taking one curve to the other. This proposition states that it is invariant under projective transformations.

**Proposition 11.** [Telling] Let \( C, D \subseteq \mathbb{P}^n \) be rational curves of degree \( n+1 \) and let \( \text{Aut}(\mathbb{P}^3) \Delta_p, \Delta_q \) be a candidate group. Then the mapping \( \iota: \text{Aut}(\mathbb{P}^n)_{C,D} \rightarrow \text{Aut}(\mathbb{P}^3)_{\Delta_p, \Delta_q} \) is a bijection.

**Example 12.** Consider two rational quartics in \( \mathbb{P}^3 \)
\[
C : p(s, t) = [75s^4 - 296s^3 t + 424s^2 t^2 - 272st^3 + 64t^4 : 9s^4 - 16s^3 t - 8s^2 t^2 + 32st^3 - 16t^4 : 13s^4 - 20s^3 t - 8s^2 t^2 + 32st^3 - 16t^4 : -53s^4 + 104s^3 t - 40s^2 t^2 - 48st^3 + 32t^4] \tag{26}
\]
and
\[
D : q(s, t) = [32s^4 + 96s^3 t + 64s^2 t^2 + 36st^3 + 9t^4 : -80s^4 - 128s^3 t - 48s^2 t^2 - 4st^3 + 7t^4 : -32s^4 - 32s^3 t + 16s^2 t^2 + 16st^3 + 6t^4 : 64s^4 + 160s^3 t + 144s^2 t^2 + 64st^3 + 16t^4]. \tag{27}
\]
First, we compute the osculating polynomials
\[
\Delta_p = 3s^4 + 20s^3 t - 72s^2 t^2 + 64st^3 - 16t^4 \tag{28}
\]
and
\[
\Delta_q = 8s^4 + 24s^3 t + 12s^2 t^2 - 2st^3 - t^4. \tag{29}
\]
Now, employing Algorithm 3 we obtain four different reparameterizations of \( \Delta_q \) yielding \( \Delta_p \).
Using Proposition 11 we know that to each reparameterization there will exist a corresponding projective transformation mapping \( C \) to \( D \). For the sake of brevity we present only one case, e.g., the reparameterization
\[
s \mapsto s - 2t, \quad t \mapsto 4t - 4s \tag{30}
\]
leads to the following projective transformation given by the matrix
\[
\mathcal{A} = \begin{bmatrix}
1 & 13 & 16 & 2 \\
9 & -16 & -24 & -10 \\
4 & -18 & -6 & -6 \\
-2 & -12 & 10 & -4
\end{bmatrix}. \tag{31}
\]

4.2. Projective equivalences of rational ruled surfaces

Let \( S \) be a rational ruled surface in \( \mathbb{P}^3 \), i.e., a surface generated by a rational one-dimensional family of lines. Such a family is parameterized by a rational curve on the Grassmannian \( \mathbb{G} \subseteq \mathbb{P}^5 \). Hence it is tempting to use the methods from the previous section to study the ruled surfaces as
Proof. It is clear that the map indeed takes the set $\text{Aut}(\hat{\Sigma}_{S,R})$ onto $\text{Aut}^+(G)_{\hat{S},\hat{R}}$. The bijectivity then follows from the fact that it is a restriction of the isomorphism.

We saw that two non-degenerate rational cubics in $\mathbb{P}^3$ were always projectively equivalent, whereas this was not true for quartics any more. Let us investigate the same question for rational ruled surfaces in $\mathbb{P}^3$, too. There are two possibilities: $S$ is either a projection of $\Sigma_{0,3}$ or $\Sigma_{1,2}$, where the first one is a cone over a rational cubic. The theory of projective equivalences between cones is clearly equivalent to the theory of projective equivalences of planar curves. Two rational planar cubics are projectively equivalent whenever they have equivalent osculating polynomials, by Proposition $\text{[9]}$. Since a general rational cubic has its osculating polynomial with three distinct roots (the case of a nodal cubic) we conclude that two generic projections of $\Sigma_{0,3}$ are projectively equivalent.

The generic projection of $\Sigma_{1,2}$ contains a chain of four special lines, see Piene (2005) for numeric formulas for the degree of singular locus, number of torsal lines, etc. The singular locus of $S$ is a line $\Gamma$. Through each point of $\Gamma$ there pass two rulings, except of two pinch points $g_1, g_2$. The generic projection of $\Sigma$ is then a rational curve of degree $d$, and we will denote it by $\hat{S}$. Let $\hat{g}_1, \hat{g}_2$ be the images of $g_1, g_2$ under the projection $\pi$. The double curve is traced twice and thus it admits a parameterization (written non-homogeneously) $\hat{g}_1 = [1 : 0 : 0 : 0], \hat{g}_2 = [0 : 1 : 0 : 0], \hat{m}_1 = [0 : 0 : 1 : 0]$ and $\hat{m}_2 = [0 : 0 : 0 : 1]$. The surface is then obtained by joining corresponding points on lines $M$ and $\Gamma$. W.l.o.g. parameterize $M$ homogeneously as $s_1\hat{m}_1 + s_2\hat{m}_2$, i.e., the points $\hat{m}_i$ correspond to parameter values $s_j = 0$ for $i \neq j$. The double curve is traced twice and thus it admits a parameterization $g(s_1, s_2) = g_1(s_1, s_2)\hat{g}_1 + g_2(s_1, s_2)\hat{g}_2$ for some quadratic forms $g_i(s_1, s_2)$. The fact that the points $\hat{g}_i$ lie on torsal rulings means that these points are pinch points on the double line $\Gamma$ and the parameterization $g(s_1, s_2)$ fails to be regular at these points. Together with the conditions on compatibility with parameterization of $M$ we arrive at the possible parameterizations $\alpha s_1^2\hat{g}_1 + \beta s_2^2\hat{g}_2$, where $\alpha : \beta \in \mathbb{P}^1$.

To sum up we just constructed a family of ruled surfaces admitting a parameterization (written non-homogeneously)

$$[s : 1 : \alpha s^2 t : \beta t];$$

Nevertheless it is easy to see that all such surfaces are projectively equivalent. Let us just mention that this analysis shows that the group of automorphisms of a generic projection of $\Sigma_{1,2}$ is two-dimensional and has two disconnected components. The first one is formed by transformations preserving all the lines $M$, $\Gamma$ and $L_i$, whereas the second component swaps two torsal rulings $L_i$. 

10
Closer look at the group of automorphisms of ruled cubic reveals that there exist non-trivial transformations which preserve each ruling. To imagine this, consider at the moment a cylinder in the affine space. It is obviously invariant under translations with the direction of its axis. These transformations do not interchange the rulings of the surface. In other words if $\hat{\phi} \in \text{Aut}^+(G)$ is the associated transformation and $\hat{S}$ the curve on the Grassmannian then $\hat{\phi}$ is the identity when restricted to $\hat{S}$. For a rational ruled surface we define

$$N_S = \left\{ \phi \in \text{Aut}(\mathbb{P}^3)_S : \hat{\phi}|_S = \text{id} \right\}. \quad (35)$$

The reason for the existence of non-trivial $N_S$ is the fact that the curve $\hat{S}$ can be contained in the subspace of dimension less than 5. Let us write $\text{span}(\hat{S})$ for the smallest subspace containing $\hat{S}$. If $\hat{\phi} \in N_S$ then the projective transformation $\hat{\phi}$ must be the identity on the whole subspace $\text{span}(\hat{S})$. We will briefly discuss properties of the group $N_S$ in dependence on the dimension of this subspace.

$\dim \text{span}(\hat{S}) = 5$. In this case any $\hat{\phi}$ which is identity on $\hat{S}$ must be the identity on the whole space $\mathbb{P}^5$. Therefore $N_S = \{\text{id}\}$.

$\dim \text{span}(\hat{S}) = 4$. Now, $\text{span}(\hat{S})$ is a hyperplane in $\mathbb{P}^5$. The section of the $G$ by a hyperplane is called a linear complex. For an introduction to the theory of linear complexes see e.g. (Pottmann and Wallner 2001, Chapter 3). Here we will recall only some necessary notions. There exist two different kinds of complexes. Since $G$ is a hyperquadric in $\mathbb{P}^5$ it induces a correspondence between points and hyperplanes. It associates to each point its polar hyperplane and vice versa. If the point $p$ is contained in $G$ then the section by the polar hyperplane is called singular complex. Otherwise the complex is said to be regular.

Let $H \subset \mathbb{P}^5$ be a hyperplane. A transformations $\mu \in \text{Aut}(\mathbb{P}^5)$ leaving all points of $H$ invariant is called perspective collineation and there exists a point $p \in \mathbb{P}^5$ such that for each $x \in \mathbb{P}^5$ the triple $p, x$ and $\mu(x)$ is collinear – see (Pottmann and Wallner 2001, Theorem 1.1.9). The point $p$ is called a center. The additional requirement that the quadric $G$ must be invariant under $\mu$ as well, forces $p$ to be the pole of the hyperplane w.r.t. $G$.

Thus for a singular complex the only transformation preserving $G$ and leaving each point of the hyperplane invariant is the identity. There exists additional transformation except of identity in regular case, namely the reflection induced by $G$ and $H$. However it is known that such a transformation is induced by a mapping from $\mathbb{P}^3$ to the dual space $(\mathbb{P}^3)^\perp$. In other words $\mu \notin \text{Aut}^+(G)$, see (Pottmann and Wallner 2001, Section 3.1) for the detailed discussion. Hence we conclude that $N_S = \{\text{id}\}$ in the case $\dim \text{span}(\hat{S}) = 4$, too.

$\dim \text{span}(\hat{S}) = 3$. In this case $\text{span}(\hat{S}) \cap G$ has dimension two and thus it the so called congruence. The space polar to $\text{span}(\hat{S})$ is a line. If the line is contained in $G$, then it corresponds to a pencil of lines passing through a point in $\mathbb{P}^3$. The polar space then intersects $G$ in the set of lines intersecting each line in this pencil. These are the lines which lie in the plane of the pencil or lines passing through the vertex. Thus $\hat{S} \cap G$ is irreducible and consists of union of two 2-planes; each of them corresponding to one type of lines. However $\hat{S}$ is irreducible and thus it must be contained in exactly one of these planes, which is a contradiction with the assumption $\dim \text{span}(\hat{S}) = 3$.

Hence assume that the line polar to $\text{span}(\hat{S})$ is not contained in the Grassmannian. Then it intersects it in two (not necessarily) distinct points. These two points (if distinct) correspond to lines $M, N \subset \mathbb{P}^3$ and the congruence in this case consists of their transversals. Thus a ruled surface is then formed by a one-dimensional family of lines intersecting both $M$ and $N$. A detailed description of the linear of congruences can be found again in (Pottmann and Wallner 2001, Section 3.2).

In this case the group $N_S$ is not trivial any more. To see this let $H \subset \mathbb{P}^5$ be a space of dimension 3 and $H^\perp$ its polar line w.r.t. $G$. Choose two points in $H^\perp \setminus G$ then a composition of two reflections
induced by these points is a transformation preserving $G$ and fixing every point of $H$. From above we know that the original transformation is a composition $\mathbb{P}^3 \to (\mathbb{P}^3)^\vee \to \mathbb{P}^3$ and thus it is an element of $\text{Aut}(\mathbb{P}^3)$. Moreover since $H$ and $H^\perp$ span the whole space $\mathbb{P}^5$ in this case we can conclude that all these transformations can be naturally identified with projective automorphisms $H^\perp \to H^\perp$ leaving the intersection $H^\perp \cap G$ invariant. Therefore we conclude that $\dim N_S = 1$ in this case.

$\dim \text{span}(\hat{S}) = 2$. There are basically two options in this case. First, curve $\hat{S}$ is the section of $G$ by the plane span ($\hat{S}$), i.e., it is a conic section and thus the surface $S$ is a quadric. Second span ($\hat{S}$) is contained in $G$. In this case the surface $S$ is either a plane or a cone. In the conical case the group of all projective transformations of $\mathbb{P}^3$ leaving the vertex invariant can be identified with a group of projective transformations of $\mathbb{P}^2$. Thus $N_S$ is again non-trivial.

$\dim \text{span}(\hat{S}) = 1$. Since the curve $\hat{S}$ is a line, the surface $S$ must be a plane in $\mathbb{P}^3$.

The cases $\deg S = 1, 2$ were excluded, and thus the only case with non-trivial group occurs for $\dim \text{span}(\hat{S}) = 3$ or for the cones. Since a generic rational curve of degree at least four is not contained in three-dimensional space, we see that most of surfaces of degree at least four possess the trivial subgroup $N_S$.

### 4.3. Transformations of affine curves

Another problem which can be reduced to the computation of projective equivalences between finite sets of points in $\mathbb{P}^1$ is the detection of affine transformations mapping a planar curve $C$ to a planar curve $D$, i.e., the computation of $\text{Aff}_{C,D}$. Note that for rational curves $C$ and $D$ we could use methods from Subsection 4.1 to detect $\text{Aut}(\mathbb{P}^2)_{C,D}$. The affine transformations then form its subset preserving $\omega$. Hence the curves are not assumed to be necessarily rational in this part. We only require that they are irreducible. Thus they are given by their irreducible defining polynomials $F(x_0, x_1, x_2)$ and $G(x_0, x_1, x_2)$ of degrees $d$. Write $F(x_0, x_1, x_2) = F_d(x_1, x_2) + F_{d-1}(x_1, x_2)x_0 + \cdots F_0(x_1, x_2)x_0^d$ and similarly for the form $G$. As the affine transformations between lines or conics are easy to find we will omit these cases and assume $d > 2$.

The affine transformations are exactly the projective transformations $\mathbb{P}^2 \to \mathbb{P}^2$ preserving the ideal line $\omega : x_0 = 0$ and thus the matrix representation of any affine transformation can be written as

$$
\begin{pmatrix}
  a_{00} & 0 & 0 \\
  a_{10} & a_{11} & a_{12} \\
  a_{20} & a_{21} & a_{22}
\end{pmatrix}, \quad a_{00} \neq 0 \text{ and } a_{11}a_{22} - a_{12}a_{21} \neq 0. \quad (36)
$$

Write $A$ for the matrix $(a_{ij})_{ij=1}^2$. The affine transformation acts on $\omega$ via $[x_1 : x_2]^\top \mapsto A[x_1 : x_2]^\top$. This defines a group homomorphism $\mu : \text{Aff} \to \text{Aut}(\mathbb{P}^1)$.

**Theorem 14.** Let $C$ and $D$ be curves as above. Then $\text{Aut}(\mathbb{P}^1)_{F_0, G_0}$ is a good candidate set for $\text{Aff}$ where the inclusion map is given by the restriction $\iota = \mu|_{\text{Aff}_{C,D}}$.

**Proof.** Any affine transformation between $C$ and $D$ must map the ideal points of $C$ to the ideal points of $D$. Hence its restriction to $\omega$ can be naturally viewed as a transformation from $\text{Aut}(\mathbb{P}^1)_{F_0, G_0}$. Since $C$ and $D$ are irreducible there exist exactly $d$ intersections of each curve with $\omega$ and thus $\text{Aut}(\mathbb{P}^1)_{F_0, G_0}$ is finite whenever $d > 2$. Thus in order to show that it is a good candidate set we remain to prove that $\iota$ is injective. Assume a contradiction. Let there exists two different transformations $\phi_1, \phi_2 \in \text{Aff}_{C,D}$ which are mapped to the same transformation in $\text{Aut}(\mathbb{P}^1)$. It is easy to see that in this case $\phi_2 \circ \phi_1^{-1}$ is a translation or scaling in $\text{Aff}$. However the only irreducible algebraic curves invariant under these transformations are lines. \qed
Now, let be given $\phi \in \text{Aut}(\mathbb{P}^1)_{F_d,G_d}$, i.e., there exists a regular matrix $A$ and $\lambda \in \mathbb{C}^*$ such that $G_d(A(x_1,x_2)^\top) = \lambda F_d(x_1,x_2)$. In order to find its preimage given by $A$, cf. [36], in $\text{Aff}_{C,D}$ under $\iota$ it is enough to compute $a_{00}$, $a_{10}$ and $a_{20}$ such that $G(A(x_0,x_1,x_2)^\top) = \lambda F(x_0,x_1,x_2)$. This leads to a system of polynomial equations in $a_{00}$, $a_{10}$ and $a_{20}$. Although it might seem to be complicated, we know that the system has no or exactly one solution depending on the existence of the preimage. In addition, writing $G(A(x_0,x_1,x_2)^\top) = G'_d(x_1,x_2) + G'_{d-1}(x_1,x_2)x_0 + \cdots G'_0(x_1,x_2)x_0^5$ one can show that the subsystem of equations corresponding to $G'_{d-1}(x_1,x_2) = \lambda F_{d-1}(x_1,x_2)$ is linear. Moreover it can be easily seen that for curves in general position this system has a unique solution and thus we can avoid solving non-linear systems.

\textbf{Algorithm 4} Affine equivalences of planar curves.

\textbf{Input}: Curves $C: F(x_0,x_1,x_2) = 0$ and $D: G(x_0,x_1,x_2) = 0$, both of degree $d$.

1: Compute $\text{Aut}(\mathbb{P}^1)_{F_d,G_d}$, cf. Algorithm 2.
2: For each $A \in \text{Aut}(\mathbb{P}^1)_{F_d,G_d}$ construct a matrix $A$, cf. [36], with $a_{11}, a_{12}, a_{21}, a_{22}$ given by $A$ and $a_{00}, a_{10}, a_{20}$ as free parameters.
3: Set $G'(x_0,x_1,x_2) = G(A(x_0,x_1,x_2)^\top)$.
4: if the linear system corresponding to $G'_{d-1}(x_1,x_2) = F_{d-1}(x_1,x_2)$ has a solution then
5: include map corresponding to the matrix $A$, i.e., the solution $a_{00}, a_{10}, a_{20}$ together with $A$, into $\text{Aff}_{C,D}$.
6: end if

\textbf{Output}: $\text{Aff}_{C,D}$.

\textbf{Example 15}. Consider two algebraic curves $C$ and $D$ of degree 5 given by the forms

\begin{align*}
F &= -23x_0^5 - 109x_1x_0^4 - 7x_2x_0^4 - 179x_1^2x_0^3 + 5x_2^2x_0^3 - 54x_1x_2x_0^2 - 22x_1^3x_0^2 \\
&\quad \quad - 4x_2^3x_0 - 6x_1x_2^2x_0^2 - 40x_1^2x_2x_0^2 + 70x_1^3x_0 - 2x_2^4x_0 - 12x_1x_2^3x_0 - 28x_2^2x_0
\end{align*}

and

\begin{align*}
G &= x_0^5 - 2x_1x_0^4 + x_2x_0^4 + x_2^3x_0^3 + 9x_2^2x_0^3 + 4x_1x_2x_0^3 + 10x_1^3x_0^2 + 42x_2^3x_0^2 \\
&\quad + 65x_1x_2^2x_0 + 41x_1^2x_2x_0 + 10x_1^4x_0 + 63x_2^2x_0 + 139x_1x_2^3x_0 + 128x_1^2x_0
\end{align*}

By computing the automorphisms $\text{Aut}(\mathbb{P}^1)_{F_d,G_d}$ of the forms

\begin{align*}
F_d &= 49x_1^4 + 13x_2x_1^3 - 6x_2^2x_1^2 + 2x_2^3x_1 + 5x_1^2x_1 + x_2^3, \\
G_d &= 2x_1^5 + 18x_2x_1^4 + 61x_2^2x_1^3 + 102x_2^3x_1^2 + 87x_2^4x_1 + 31x_2^5
\end{align*}

we arrive at

\begin{equation}
A = \begin{pmatrix}
-4 & -3 \\
3 & 1
\end{pmatrix}
\end{equation}

When setting $G'(x_0,x_1,x_2) = G(A(x_0,x_1,x_2)^\top)$, the condition $G'_{d-1}(x_1,x_2) = F_{d-1}(x_1,x_2)$ leads to the system of linear equations

\begin{align*}
a_{00} &= 5a_{10} = 5a_{20} &= -2, \\
-6a_{00} &= -28a_{10} = -36a_{20} &= -12, \\
-4a_{00} &= -18a_{10} = -30a_{20} &= -28, \\
6a_{00} &= 46a_{10} = 76a_{20} &= -28, \\
139a_{00} &= 103a_{10} = 219a_{20} &= 70,
\end{align*}

which has the following solution

\begin{align*}
a_{00} &= -2, \quad a_{10} = -3, \quad a_{20} = 3.
\end{align*}
Altogether we obtain the resulting transformation described by the matrix

\[
\begin{pmatrix}
-2 & 0 & 0 \\
-3 & -4 & -2 \\
3 & 3 & 1
\end{pmatrix}
\]  

(44)

which maps C to D.

4.4. Similarities and symmetries of surfaces

In this section we solve the problem of the detection of direct similarities of algebraic surfaces \(R\) and \(S\), i.e. finding \(\text{Sim}_{R,S}\), by the computation of projective equivalences between finite sets of points in \(\mathbb{P}^1\). Again we will employ Algorithm 2.

The real algebraic surfaces \(R, S\) are given as real solutions of polynomial equations \(F(x_0, x_1, x_2, x_3) = 0\) and \(G(x_0, x_1, x_2, x_3) = 0\), where \(F, G\) are typically defined over \(\mathbb{Q}\) or its finite extension. We make a natural assumption that the polynomials \(F, G\) are irreducible over \(\mathbb{C}\) and that \(\text{dim}_{\mathbb{R}} R = \text{dim}_{\mathbb{R}} S = 2\). Since the degree of a surface is a projective invariant, we assume that both surfaces have the same degree. In addition we have (see e.g. Alcázar and Hermoso [2016] for a more detailed analysis):

**Proposition 16.** If \(R, S\) are not both cylinders, cones or surfaces of revolution then \(\text{Sim}_{R,S}\) is finite.

There exist efficient algorithms for recognizing surfaces invariant under translations (cylinders), scalings (cones) and a one parameter set of rotations (surfaces of revolution) and we assume that \(R\) and \(S\) are not surfaces of these types.

Write \(R_C\) and \(S_C\) for the zero locus of \(F\) and \(G\) in \(\mathbb{P}^3_{\mathbb{C}}\). The group \(\text{Sim}\) acts naturally on \(\mathbb{P}^3_{\mathbb{C}}\) and it allows to consider also \(\text{Sim}_{R_C, S_C}\). Since any \(\phi \in \text{Sim}_{R,S}\) maps the real points of \(R\) to the real points of \(S\) we obtain \(\text{Sim}_{R_C, S_C} \subset \text{Sim}_{R,S}\). Nevertheless the inclusion may be proper. The following lemma legitimizes our assumptions on the surfaces.

**Lemma 17.** Let \(R\) and \(S\) be irreducible surfaces with \(\text{dim}_{\mathbb{R}} R = \text{dim}_{\mathbb{R}} S = 2\) then \(\text{Sim}_{R_C, S_C} = \text{Sim}_{R,S}\).

**Proof.** This follows directly from the fact that there is (up to a constant factor) a unique irreducible polynomial vanishing exactly on \(R\) whenever \(\text{dim}_{\mathbb{R}} R = 2\).

Lemma 17 enables us to replace the real surface \(R\) by the complex one. Let us write

\[
F(0, x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^k \tilde{F}(x_1, x_2, x_3),
\]

(45)

where \(\sum x_i^2\) does not divide \(\tilde{F}\). Then we set \(R_{\Omega}\) to be the intersection of the absolute conic \(\Omega\) and the curve \(\tilde{F} = 0\) in the plane at infinity, together with the intersection multiplicities. Hence set theoretically \(R_{\Omega} = (\mathbb{R}_C \cap \omega) \setminus \Omega \cap \Omega\) and it is a finite subset of \(\Omega\). Since every similarity \(\phi\) preserves the absolute conic we have \(\phi(R_{\Omega}) = R_{\Omega}\). The conic \(\Omega\) is a smooth rational curve and thus there exists an isomorphism \(\mu : \mathbb{P}^1_{\mathbb{C}} \to \Omega\), for example it can be given by

\[
[s : t] \mapsto [0 : 2st : s^2 - t^2 : i(s^2 + t^2)].
\]

(46)

The pull-back \(\mu^* \tilde{F}\) is a form of degree \(2\deg R\) on \(\mathbb{P}^1\) such that its zero-set is exactly the pre-image of the set \(R_{\Omega}\) together with the multiplicities. Analogously, we obtain \(\mu^* \tilde{G}\) for the surface \(S\).

**Theorem 18.** Let \(R\) and \(S\) be surfaces as above. If \(\tilde{F}\) and \(\tilde{G}\) are not constants, then \(\text{Aut}(\mathbb{P}^1, \mu^* \tilde{F}, \mu^* \tilde{G})\) is a good candidate set for \(\text{Sim}_{R,S}\) where the inclusion map is given by the restriction \(\iota = \mu|_{\text{Sim}_{R,S}}\).
The proof is analogous to the proof of Theorem 14 with the specification that \( \iota \) is not injective only for cylinders and cones, which were excluded from our considerations. \( \square \)

Let \( \phi \in \text{Aut}(\mathbb{P}^1_{\mu^*}) \), then \( \psi = \mu \circ \phi \circ \mu^{-1} \) is an automorphism of \( \Omega \) mapping the set \( R_\Omega \) to \( S_\Omega \). Since \( \psi \) is an automorphism of a conic section in \( \mathbb{P}^2 \) there exists a projective transformation \( \Psi \) of \( \mathbb{P}^2 \) preserving \( \Omega \) such that \( \Psi|_\Omega = \psi \). Write \( A \) for a matrix representing \( \Psi \). We can determine the remaining coefficients \( a_{00}, \ldots, a_{03} \) of the matrix \( \mathbf{X} \) of the possible similarity. In particular we solve the system of linear equations analogously as in Section 4.3. The method is summarized in the Algorithm 5.

**Algorithm 5** Similarities of algebraic surfaces.

**Input:** Surfaces \( R : F(x_0, x_1, x_2, x_3) = 0 \) and \( S : G(x_0, x_1, x_2, x_3) = 0 \), both of degree \( d \).

1. Compute \( \text{Aut}(\mathbb{P}^1_{\mu^* \tilde{F}_{\mu^*}}) \), cf. Algorithm 2.
2. For each \( \varphi \in \text{Aut}(\mathbb{P}^1_{\mu^* \tilde{F}_{\mu^*}}) \) construct a projective transformation \( \Psi \) (a matrix \( A \)) of \( \mathbb{P}^2 \) preserving \( \Omega \) such that \( \Psi|_\Omega = \mu \circ \phi \circ \mu^{-1} \).
3. For each such \( A \) construct a matrix \( \mathbf{X} \), cf. (2), given by \( A, \tilde{a} = (0, \ldots, 0) \) and \( a_{00}, \ldots, a_{30} \) as free parameters.
4. Set \( G'(x_0, x_1, x_2, x_3) = G(\mathbf{X}(x_0, x_1, x_2, x_3)^\top) \).
5. If the linear system corresponding to \( G'_{d-1}(x_1, x_2, x_3) = F_{d-1}(x_1, x_2, x_3) \) has a solution then include map corresponding to the matrix \( \mathbf{X} \) (i.e., the solution \( a_{00}, \ldots, a_{30} \) together with \( A \)) into \( \text{Sim}_{S,R} \).
6. end if

**Output:** \( \text{Sim}_{S,R} \).

Let us remark, that a self-similarity of an algebraic surface is immediately an isometry, hence \( \text{Sim}_S \) determines a group of symmetries. Since we assume that the surface is neither a cylinder, a cone or a surface of revolution, we know that \( \text{Sim}_S \) is trivial, cyclic, dihedral or a group of symmetries of a platonic solid, cf. Proposition 3.

We conclude this section by visualising the candidate group of the surface. This is achieved by the following construction of the map between \( \Omega \) and the unit sphere \( S^2 \subset \mathbb{R}^3 \). The point from \( \mathbb{P}^3 \) is contained in \( \Omega \) if and only if it has coordinates \( [0 : \mathbf{p}] \) such that \( \text{Re}(\mathbf{p}) \cdot \text{Im}(\mathbf{p}) = 0 \) and \( |\text{Re}(\mathbf{p})| = |\text{Im}(\mathbf{p})| \). Consider a mapping \( \gamma : \Omega \to S^2 \) defined by

\[
\gamma : \mathbf{p} \mapsto \mathbf{i} \frac{\text{Re}(\mathbf{p}) \times \text{Im}(\mathbf{p})}{|\text{Re}(\mathbf{p})||\text{Im}(\mathbf{p})|},
\]

A rotation \( \phi \) can be viewed simultaneously as a mapping \( \phi : \Omega \to \Omega \) or \( \phi : S^2 \to S^2 \). Since for a regular matrix \( \mathbf{M} \) and two vectors \( \mathbf{a}, \mathbf{b} \) it holds \( (\mathbf{M}\mathbf{a}) \times (\mathbf{M}\mathbf{b}) = \det \mathbf{M} \cdot \mathbf{M}^{-\top}(\mathbf{a} \times \mathbf{b}) \), the mapping \( \gamma \) leads to a commutative diagram

\[
\begin{array}{ccc}
\Omega \quad \phi & \longrightarrow & \Omega \\
\downarrow \quad \gamma & & \downarrow \quad \gamma \\
S^2 \quad \phi & \longrightarrow & S^2
\end{array}
\]

For the sake of simplicity we assume that the surface \( S \) intersects \( \Omega \) with multiplicities one (otherwise we should consider also the multiplicities of the intersections and modify the approach accordingly). Denote \( \Lambda := \gamma(S_\Omega) \) the image of the intersections of the surface \( S \) with the absolute conic on the unit sphere in \( \mathbb{R}^3 \). Let \( 0 < d_1 < \cdots < d_k < 2 \) be all the possible distances between points of \( \Lambda \) (except of antipodal points) and write

\[
\Lambda_i := \{ \mathbf{p} \in \Lambda : \exists \mathbf{q} \in \Lambda \text{ such that } |\mathbf{p} \cdot \mathbf{q}| = d_i \},
\]

15
for $i \in \{1, \ldots, k\}$. Since isometries preserve the distances we see that any isometry preserving $\Lambda$ must preserve each $\Lambda_i$. Conversely if every $\Lambda_i$ is preserved by some isometry, then the same is true for their union $\Lambda$. And thus we have the following formula for the candidate group

$$\text{Aut}(\mathbb{P}^1)_{\mu, \tilde{F}} \cong \bigcap_{i=1}^{k} \text{Sim}_{\Lambda_i}$$

(50)

The procedure is illustrated in Fig. 1 when we arrive at the case $\text{Sim}_S = \bigcap_{i=1}^{k} \text{Sim}_{\Lambda_i} \cong D_4$. Furthermore, in Fig. 2 we see a situation where $\text{Sim}_S \cong T$ is a proper subgroup of the candidate group $\bigcap_{i=1}^{k} \text{Sim}_{\Lambda_i} \cong O$.

![Figure 1](image1.png)

Figure 1: The surface $(x^2 + y^2)^3 - 4x^2y^2(x^2 + 1)$ with $\text{Sim}_S \cong D_4$.

![Figure 2](image2.png)

Figure 2: Chair surface with the tetrahedral symmetry $\text{Sim}_S \cong T$.

5. Conclusion

An identification of a suitable class of equivalencies in given geometric data is a topic interesting not only from the theoretical but also from the practical point of view. From this reason, computing projective equivalences of distinguished algebraic varieties has become an active research area and various situations are incessantly investigated. And as direct computations (although they can be formulated easily) are getting quite complicated even for trivial inputs, alternative efficient approaches are still required and investigated.
In this paper, we studied several situations that can be transformed to determining equivalences of finite subsets of the projective line. This makes the designed method computationally suitable e.g. for finding projective equivalences of rational curves, determining projective equivalences of rational ruled surfaces, detecting affine transformations between planar algebraic curves, and computing similarities between two implicitly given algebraic surfaces. The designed algorithms were implemented in the CAS Mathematica and their functionality was documented on several examples.

Acknowledgments

The authors were supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

References

Alcázar, J. G., 2014. Efficient detection of symmetries of polynomially parametrized curves. Journal of Computational and Applied Mathematics 255, 715 – 724.
Alcázar, J. G., Hermoso, C., 2016. Involutions of polynomially parametrized surfaces. Journal of Computational and Applied Mathematics 294, 23 – 38.
Alcázar, J. G., Hermoso, C., Muntingh, G., 2014a. Detecting similarity of rational plane curves. Journal of Computational and Applied Mathematics 269, 1 – 13.
Alcázar, J. G., Hermoso, C., Muntingh, G., 2014b. Detecting symmetries of rational plane and space curves. Computer Aided Geometric Design 31 (3), 199 – 209.
Alcázar, J. G., Hermoso, C., Muntingh, G., 2015. Symmetry detection of rational space curves from their curvature and torsion. Computer Aided Geometric Design 33, 51 – 65.
Alcázar, J. G., Hermoso, C., Muntingh, G., 2018. Similarity detection of rational space curves. Journal of Symbolic Computation 85, 4 – 24, 41th International Symposium on Symbolic and Algebraic Computation (ISSAC’16).
Alcázar, J. G., Lávička, M., Vršek, J., Jan. 2018. Symmetries and similarities of planar algebraic curves using harmonic polynomials. ArXiv e-prints (arXiv:1801.09962).
Brass, P., Knauer, C., 2004. Testing congruence and symmetry for general 3-dimensional objects. Computational Geometry 27 (1), 3 – 11, computational Geometry - EWCG’02.
Harris, J., 1992. Algebraic Geometry: A First Course. Springer-Verlag.
Hauer, M., Jüttler, B., 2018. Projective and affine symmetries and equivalences of rational curves in arbitrary dimension. Journal of Symbolic Computation 87, 68 – 86.
Huang, Z., Cohen, F. S., Jun 1994. Affine-invariant b-spline moments for curve matching. In: 1994 Proceedings of IEEE Conference on Computer Vision and Pattern Recognition. pp. 490–495.
Lebmeir, P., 2009. Feature detection for real plane algebraic curves. Ph.D. thesis, Technische Universität München.
Lebmeir, P., Richter-Gebert, J., 2008. Rotations, translations and symmetry detection for complexified curves. Computer Aided Geometric Design 25 (9), 707 – 719, classical Techniques for Applied Geometry.
Piene, R., 2005. Singularities of some projective rational surfaces. In: Computational Methods for Algebraic Spline Surfaces. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 171–182.
Pottmann, H., Wallner, J., 2001. Computational Line Geometry. Springer.
Shurman, J., Levenberg, J., 1997. Geometry of the Quintic. Wiley-Interscience publication. Wiley.
Telling, H., 1936. The Rational Quartic Curve in Space of Three and Four Dimensions: Being an Introduction to Rational Curves. Cambridge tracts in mathematics and mathematical physics. Cambridge University Press.