Combinatorial properties of non-archimedean convex sets

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Convexity in valued fields

- Introduced by Monna in 1940’s, extensively studied in non-archimedean functional analysis.

- **Notation.** $K$ a valued field (e.g. $\mathbb{Q}_p$), with value group $\Gamma = \Gamma_K$, valuation $\nu = \nu_K : K \to \Gamma_\infty := \Gamma \sqcup \{\infty\}$, valuation ring $\mathcal{O} = \mathcal{O}_K = \nu^{-1}([0, \infty])$, maximal ideal $\mathfrak{m} = \mathfrak{m}_K = \nu^{-1}((0, \infty])$, and residue field $k = \mathcal{O}/\mathfrak{m}$. The residue map $\mathcal{O} \to k$ will be denoted $\alpha \mapsto \bar{\alpha}$.

- For $d \in \mathbb{N}_{\geq 1}$, a set $X \subseteq K^d$ is convex if, for any $n \in \mathbb{N}_{\geq 1}$, $x_1, \ldots, x_n \in X$, and $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \ldots + \alpha_n = 1$ we have $\alpha_1 x_1 + \ldots + \alpha_n x_n \in X$ (in the vector space $K^d$).

- The family of convex subsets of $K^d$ will be denoted $\text{Conv}_{K^d}$. 
Convex combinations

Given an arbitrary set $X \subseteq K^d$, its **convex hull** $\text{conv}(X)$ is the convex set given by the intersection of all convex sets containing $X$, equivalently the set of all convex combinations from $X$:

$$\text{conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}, \alpha_i \in \mathcal{O}, x_i \in X, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$ 

**Prop.** Let $K$ be a valued field and $X \subseteq K^d$. If $X$ is closed under 3-element convex combinations (in the sense that if $x, y, z \in X$ and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$, then $\alpha x + \beta y + \gamma z \in X$), then $X$ is convex.

**Prop.** 2-element convex combinations suffice iff $k \nsubseteq \mathbb{F}_2$. 
Convex subsets of $\mathbb{R}^n$ vs convex subsets of $K^n$

- Parallel: combinatorics of convex subsets of $\mathbb{R}^n$ vs definable subsets of $\mathbb{R}^n$ vs. definable subsets of $\mathbb{Q}_p$.

- **Example** (Marker). Naming a single (bounded) convex subset of $\mathbb{R}^2$ in the field of reals allows to define the set of integers. Indeed, we can define a continuous and piecewise linear function $f : [0, 1] \rightarrow [0, 1]$ such that

$$C := \{(x, y) : x \in [0, 1], 0 \leq y \leq f(x)\}$$

is convex but the set of points where $f$ is not differentiable is exactly $\left\{\frac{1}{n} : n \in \mathbb{N}_{\geq 2}\right\}$. Now in the field of reals with a predicate for $C$ we can define $f$ and the set of points where it is not differentiable, hence $\mathbb{N}$ is also definable.

- In contrast, turns out that convex sets in $K^n$ are tame both model theoretically and combinatorially, so we get the best of both worlds.
Convex subsets and $\mathcal{O}$-submodules of $K^d$

- **Prop.** Nonempty convex subsets of $K^d$ are precisely the translates of $\mathcal{O}$-submodules of $K^d$.

- **Proof.** First, $\mathcal{O}$-submodules of $K^d$ are clearly convex and contain 0. Conversely, suppose $C \subseteq K^d$ is convex and $0 \in C$. Then for any $\alpha \in \mathcal{O}$ and $x \in C$, $\alpha x = \alpha x + (1 - \alpha)0 \in C$. And for any $x, y \in C$, $x + y = 1 \cdot x + 1 \cdot y - 1 \cdot 0 \in C$. Therefore $C$ is an $\mathcal{O}$-submodule. And set can be translated to contain 0 (affine maps preserve convexity).

- From this, easy to see that the convex subsets of $K = K^1$ are exactly $\emptyset$ and the quasi-balls (i.e. sets $B = \{x \in K^d : \nu(x - c) \in \Delta\}$ for some $c \in K$ and an upwards closed subset $\Delta$ of $\Gamma_\infty$).
Def. A valued field $K$ is *spherically complete* if every nested family of (closed or open) valuational balls has non-empty intersection.

Thm. Suppose $K$ is a spherically complete valued field, $d \in \mathbb{N}_{\geq 1}$, and let $C \subseteq K^d$ be an $\mathcal{O}$-submodule. Then there exists a complete flag of vector subspaces
\[
\{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = K^d
\]
and a decreasing sequence of nonempty, upwards-closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_d$ of $\Gamma_\infty$ such that
\[
C = \{v_1 + \ldots + v_d \mid v_i \in F_i, \nu(v_i) \in \Delta_i\}.
\]
Further properties of this presentation

- $\Delta_d = \{ \gamma \in \Gamma_\infty \mid \forall v \in K^d, \nu(v) = \gamma \implies v \in C \}$. That is, $\Delta_d$ is the quasi-radius of the largest quasi-ball around 0 contained in $C$.

- $F_{d-1}$ can be chosen to be any linear hyperplane $H$ in $K^d$ such that every element of $C$ differs from an element of $H$ by a vector in $K^d$ with valuation in $\Delta_d$.

- **Cor.** If $K$ is a spherically complete valued field and $d \in \mathbb{N}_{\geq 1}$, then the non-empty convex subsets of $K^d$ are precisely the affine images of $\nu^{-1}(\Delta_1) \times \ldots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$.

- By contrast to Marker’s example: if $K$ is a spherically complete, then every convex subset of $K^d$ is definable in the expansion of the field $K$ by a predicate for each Dedekind cut of the value group (definable in Shelah expansion of $K$ by externally definable sets, so e.g. NIP if $K$ was). In particular, if $K$ has value group $\mathbb{Z}$, then all convex subsets of $K^d$ form a definable family.
Combinatorial consequences

- Using this (combinatorial properties below pass to spherical completions), we can get:

- **Thm.** Let $K$ be a valued field and $d \geq 1$. Then the family $\text{Conv}_{K^d}$ has *breadth* $d$. That is, any nonempty intersection of finitely many convex subsets of $K^d$ is the intersection of at most $d$ of them. (Not true for convex subsets of $\mathbb{R}^2$!)

- **Cor.** The *Helly number* of $\text{Conv}_{K^d}$ is $d + 1$. I.e., given any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$, if every $(d + 1)$-subset of $\{S_1, \ldots, S_n\}$ has nonempty intersection, then $\bigcap_{i \in [n]} S_i \neq \emptyset$.

- **Cor.** $\text{Conv}_{K^d}$ has VC-dimension $d + 1$ and dual VC-dimension $d$. 
Combining this with Matoušek’s theorem, we obtain:

Cor. The fractional Helly number of the family Conv$_K$ is at most $d + 1$ (exactly $d + 1$ if $K$ is infinite). I.e. for every $\alpha \in \mathbb{R}_{>0}$ there exists $\beta \in \mathbb{R}_{>0}$ so that: for any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in$ Conv$_K$ (possibly with repetitions), if there are $\geq \alpha \binom{n}{d+1} (d + 1)$-element subsets of the multiset $\{S_1, \ldots, S_n\}$ with a non-empty intersection, then there are $\geq \beta n$ sets from $\{S_1, \ldots, S_n\}$ with a non-empty intersection.

Moreover, $\beta$ can be chosen depending only on $d$ and $\alpha$ (and not on the field $K$).
Finally, combining these, we obtain an analog of the Boros-Füredi/Bárány selection lemma over valued fields (answering a question of Peterzil and Kaplan):

**Thm.** For each $d \geq 1$ there is a constant $c = c(d) > 0$ such that: for any valued field $K$ and any finite $X \subseteq K^d$ (say $n := |X|$), there is some $a \in X$ contained in the convex hulls of at least $c\binom{n}{d+1}$ of the $\binom{n}{d+1}$ subsets of $X$ of size $d + 1$. 