The Algebraic Curve of 1-loop Planar $\mathcal{N} = 4$ SYM

Sakura Schäfer-Nameki

II. Institut für Theoretische Physik der Universität Hamburg
Luruper Chaussee 149, 22761 Hamburg, Germany
sakura.schafer-nameki@desy.de

Abstract

The algebraic curve for the $\mathfrak{psu}(2,2|4)$ quantum spin chain is determined from the thermodynamic limit of the algebraic Bethe ansatz. The Hamiltonian of this spin chain has been identified with the planar 1-loop dilatation operator of $\mathcal{N} = 4$ SYM. In the dual $AdS_5 \times S^5$ string theory, various properties of the data defining the curve for the gauge theory are compared to the ones obtained from semiclassical spinning-string configurations, in particular for the case of strings on $AdS_5 \times S^1$ and the $\mathfrak{su}(2,2)$ spin chain agreement of the curves is shown.
1. Introduction

Integrable structures in $d = 4 \mathcal{N} = 4$ $SU(N)$ Super-Yang Mills theory (SYM) have recently been utilized to put to test gauge/string holography [1] realized in terms of the AdS/CFT correspondence [2,3]. Extending the seminal work of Minahan and Zarembo [4], the key observation of Beisert and Staudacher [5,6] is the identification of the 1-loop dilatation operator of planar $\mathcal{N} = 4$ SYM with the Hamiltonian of a quantum spin chain for the Lie super-algebra $\mathfrak{psu}(2,2|4)$. Put into the context of the AdS/CFT correspondence, one would expect to find a corresponding integrable structure in the $AdS_5 \times S^5$ string theory. Evidence to this effect was obtained by following the proposals in [7,8] by Frolov and Tseytlin [9,10] and subsequently in [11,12,13,14] by considering semi-classical string configurations, with large spins on the $AdS_5$ and/or $S^5$. Another line of research successfully matched the local charges of the integrable systems [20,21], and comparisons of the non-local charges have appeared in [22,23]. Progress towards a quantum Bethe-ansatz for the (notoriously difficult to quantize) $AdS_5 \times S^5$ string was made in [24,25,26]. Integrability seems to persist beyond 1-loop [27,20,28,29], however a mismatch has emerged at 3-loops, the origin of which has been conjectured to be an order of limits problem [28,30].

The main objective of this paper is to expand on the methods of [32,33,34], the idea of which is the following: An integrable system can be characterized by an algebraic curve, which is constructed out of the transfer matrix, and in particular contains information about the local charges. In the case of quantum spin chains the curve is extracted from the transfer matrix in the thermodynamic limit, which in the context of AdS/CFT would need to be compared to the string sigma-model curve for large spins in the limit $\lambda/J^2 \to 0$, where $\sqrt{\lambda}$ is the string tension and $J$ the angular momentum on the $S^5$. One would expect that a necessary condition for the integrity of the AdS/CFT correspondence is the agreement of the algebraic curves of the respective quantum/classical integrable systems. As a first step, in this note the curve for the full $\mathcal{N} = 4$ SYM theory will be constructed.

The agreement of the gauge and string theory curves in the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ subsectors was proven in [32,33] and was then shown to hold in the $\mathfrak{so}(6)$ subsector of AdS/CFT by Beisert, Kazakov and Sakai [34]. In this note these methods are applied in a straight
forward manner to the $\mathfrak{su}(2, 2)$ subsector, which is closed at 1-loop order and the resulting curve is identified with the one for the sigma-model on $AdS_5 \times S^1$.

The present note is structured as follows: in section 2 the super-spin chain for $\mathfrak{psu}(2, 2|4)$ is discussed, including a recap of the Bethe ansatz of and a derivation of the transfer matrix eigenvalues. Following a succinct discussion of the thermodynamic limit in section 3, the algebraic curve is derived from the Bethe equations in section 4. A comparison to some known properties of the full $AdS_5 \times S^5$ sigma-model curve is given in section 5 and the agreement of the curves in the $\mathfrak{su}(2, 2)$ subsector and the $AdS_5 \times S^1$ sigma-model, respectively, is shown in section 6. We conclude in section 7. Appendix A collects formulae for transfer-matrices of super-spin chains and in appendix B the algebraic curve for the $\mathfrak{su}(2, 2)$ spin chain with anti-ferromagnetic ground state, which is of relevance for the QCD spin chain.

2. The $\mathfrak{psu}(2, 2|4)$ Spin Chain

In this section some known facts about spin chains with Lie super-algebra symmetries are reviewed, flushing out various points that are different from the more common case of Lie algebras. Apart from fixing notation and spelling out the Bethe equations, we give an explicit expression for the eigenvalue of the transfer matrix for the $\mathfrak{psu}(2, 2|4)$ spin chain, which will be essential for constructing the algebraic curve.

2.1. R-matrix

The Lie super-algebra of interest for the spin chain associated to the dilatation operator of $\mathcal{N} = 4$ SYM is $\mathfrak{psu}(2, 2|4)$. More precisely, the planar 1-loop dilatation operator of this theory has been shown to be identical to the Hamiltonian of a $\mathfrak{psu}(2, 2|4)$ integrable spin chain. The integrability was inferred by an explicit construction of an R-matrix. Denote by $\mathcal{R}_{ij}(u)$ the R-matrix acting on the tensor product $\mathcal{V}_i \otimes \mathcal{V}_j$, where $\mathcal{V}_i$ denotes the $\mathfrak{psu}(2, 2|4)$-module located at site $i$ of the spin chain and $\mathcal{V}_0 = \mathcal{V}_{aux}$ be the auxiliary space. The R-matrix satisfies the Yang-Baxter equations

$$\mathcal{R}_{ij}(u_i - u_j)\mathcal{R}_{ik}(u_i - u_k)\mathcal{R}_{jk}(u_j - u_k) = \mathcal{R}_{jk}(u_j - u_k)\mathcal{R}_{ik}(u_i - u_k)\mathcal{R}_{ij}(u_i - u_j). \quad (2.1)$$

Further define the transfer matrix along the entire spin chain of length $L$ by

$$\mathcal{T}(u) = \mathcal{R}_{01}(u)\mathcal{R}_{02}(u) \cdots \mathcal{R}_{0L}(u). \quad (2.2)$$
It acts on the auxiliary space, where it will be formally written in block-form

\[
\mathcal{T}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.
\] (2.3)

The simplest choice for \( V_{aux} \) is the fundamental representation, i.e., the \( 4|4 \) for \( \text{psu}(2,2|4) \), for the trace of which we shall provide an expression below.

2.2. Bethe equations

For a Lie algebra or Lie super-algebra \( g \), define a set of Bethe roots \( u^{(k)}_i, k = 1, \ldots, r = \text{rank}(g) \) and \( i = 1, \ldots, J_k \), where \( J_k \) denotes the excitation number for the \( k \)th root. Further, define \( J = \sum J_k \) as the total excitation number and let \( L \) be the length of the spin chain. Then the corresponding Bethe equations were determined in [36] to be

\[
\left( \frac{u^{(k)}_i - \frac{i}{2} V_k}{u^{(k)}_i + \frac{i}{2} V_k} \right)^L = \prod_{l=1}^r \prod_{j=1}^{J_l} \frac{u^{(k)}_i - u^{(l)}_j - \frac{i}{2} M_{kl}}{u^{(k)}_i - u^{(l)}_j + \frac{i}{2} M_{kl}}. \] (2.4)

Translational invariance along the spin chain implies further that

\[
1 = \prod_{k=1}^r \prod_{i=1}^{J_k} \frac{u^{(k)}_i + \frac{i}{2} V_k}{u^{(k)}_i - \frac{i}{2} V_k} = e^{iP}. \] (2.5)

The \( g \)-dependent data entering these equations are the Cartan matrix \( M_{kl} \) and the vector of Dynkin labels \( V_k \) of the representation that is located at the respective spin chain sites. Eq. (2.3) yields furthermore a quantization condition for the total momentum \( P \).

The validity of this Bethe ansatz (most of the Bethe ansatz methodology goes thought for super-algebras, up to some sign changes, which are detailed in appendix A) for Lie super-algebras has been established in [37]. The Cartan matrix for \( \text{psu}(2,2|4) \) which is most suitable for the present analysis has been discussed in [6,30] and is based on the Dynkin diagram in Figure 1.

![Figure 1](image-url)
The nodes $\otimes$ denote as usual the fermionic roots. The choice of Dynkin diagram for Lie super-algebras is not unique, and an alternative choice is discussed in the appendix, which will be useful for the reduction to $\mathfrak{su}(2, 2)$ in section 6. The Cartan matrix for the Dynkin diagram in figure 1 can be put into the form

$$
M = \begin{pmatrix}
-2 & 1 & & & \\
1 & 0 & -1 & & \\
-1 & 2 & -1 & & \\
-1 & 2 & -1 & & \\
-1 & 0 & 1 & & \\
1 & -2 & & & \\
\end{pmatrix}.
$$

(2.6)

In fact, this is the Cartan matrix only up to rearrangements of rows, which however leave the Bethe equations invariant, cf. [30].

In the present setup, single-trace operators in the SYM theory are identified with states of a spin chain with periodic boundary conditions, where the spins at each site transform in a representation of $\mathfrak{psu}(2, 2|4)$. The Dynkin labels of the highest weights will be denoted by

$$
V = (s_1, r_1; q_1, q, q_2; r_2, s_2),
$$

(2.7)

where $[q_1, q, q_2]$ are the Dynkin labels for $\mathfrak{so}(6)$ and $[s_1, s_2]$ of $\mathfrak{so}(3, 1)$. Each state is characterized further by the bare dimension $\Delta_0$, the length $L$ of the spin-chain as well as the hyper-charge $B$. The multiplet, in which the elementary component fields of $\mathcal{N} = 4$ SYM transform, and which will thus be the representation at the lattice sites of the spin chain, is the so-called field-strength multiplet and has highest weight vector

$$
V_\mathbf{F} = (0, 0; 0, 1; 0, 0).
$$

(2.8)

2.3. Transfer-matrix Eigenvalues

From now on we fix the representation at site $i$ to be $\mathcal{V}_i = \mathbf{F}$ and consider various choices for $\mathcal{V}_{aux}$. The super-trace of the transfer matrix in the auxiliary space

$$
T_\mathbf{R}(u) = \text{STr}_\mathbf{R} \mathcal{T}(u), \quad \mathcal{V}_{aux} = \mathbf{R},
$$

(2.9)

contains the integrals of motion, or local/global charges, when expanded in the spectral parameter $u$. Let us choose $\mathcal{V}_{aux} = \mathbb{C}^N = \mathbb{C}^{4+4}$ to be the fundamental representation and
consider the decomposition of $\mathcal{T}(u)$ in (2.3) into blocks acting on $\mathbb{C}^{N-1} \oplus \mathbb{C}$, so that $C_k(u)$ is an $(N-1)$-column. The state space is then generated by these oscillators $C_k(u^{(k)})$

$$\left| \{u^{(k)}_i\}, L \right> = \prod_{k=1}^r \prod_{i=1}^{J_k} C_k \left( u^{(k)}_i \right) \left| 0, L \right>.$$  (2.10)

The action of $\mathcal{T}(u)$ upon it is determined from the exchange relations between $A$ and $D$ with $C$ resulting from the Yang-Baxter equations (2.1). Define

$$R_k(u) = \prod_{i=1}^{J_k} (u - u^{(k)}_i).$$  (2.11)

The eigenvalue of the trace of the transfer matrix for the $u(4|4)$ spin chain is computed for $\mathcal{V}_{aux} = 4|4$ in appendix A, and for the SYM spin-chain it is

$$\frac{(u+i/2)^L}{u^L} \mathcal{T}_{4|4}(u) = \frac{(u+i/2)^L}{u^L} \frac{R_1(u+i)}{R_1(u)}$$

$$+ \frac{(u+i/2)^L}{u^L} \frac{R_1(u-i)}{R_1(u)} \frac{R_2(u+i)}{R_2(u)}$$

$$- \frac{(u+i/2)^L}{u^L} \frac{R_2(u+i)}{R_2(u)} \frac{R_3(u-i)}{R_3(u)}$$

$$+ \frac{(u-i/2)^L}{u^L} \frac{R_4(u+i)}{R_4(u)} \frac{R_5(u-i)}{R_5(u)}$$

$$- \frac{(u-i/2)^L}{u^L} \frac{R_5(u+i)}{R_5(u)} \frac{R_6(u-i)}{R_6(u)}$$

$$+ \frac{(u-i/2)^L}{u^L} \frac{R_6(u-i)}{R_6(u)} \frac{R_7(u+i)}{R_7(u)}$$

$$+ \frac{(u-i/2)^L}{u^L} \frac{R_7(u-i)}{R_7(u)}.$$  (2.12)

The local charges of the spin chain are extracted from the trace of the transfer matrix with $\mathcal{V}_{aux} = \mathcal{V}_F$. The direct computation of this is involved, as the representation in question is infinite-dimensional. However, the relation between $\mathcal{T}_F$ and the local charges $Q_r$ is

$$\mathcal{T}_F(u + u_0) = \exp \left( i \sum_{r=2}^\infty u^{r-1}Q_r \right),$$  (2.13)

where the expansion of the transfer matrix trace is around the point $u_0$, where the R-matrix reduces to a projector. The coefficients $Q_r$ in this expansion give rise to the local
charges. This expansion is an important input for the construction of the algebraic curve and has been quoted in [38,30]

\[ T_R(u + u_0) = \prod_{i=1}^{r} \prod_{j=1}^{J_i} \frac{u - u_j^{(l)} - i V_i}{u - u_j^{(l)} + i V_i} + O(u^L). \] (2.14)

Note that this is precisely the term that appears in the cyclicity constraint (2.3). For \( R = F \) the asymptotics are

\[ T_F(u + u_0) = \prod_{j=1}^{J_4} \frac{u - u_j^{(4)} - i/2}{u - u_j^{(4)} + i/2} + O(u^L) = \frac{R_4(u - i/2)}{R_4(u + i/2)} + O(u^L). \] (2.15)

Further, the asymptotics of the higher local charges are

\[ Q_r = \frac{
}{r - 1} \sum_{i=1}^{r} \sum_{j=1}^{J_i} \left( \frac{1}{(u_j^{(l)} + i V_i)^{r-1}} - \frac{1}{(u_j^{(l)} - i V_i)^{r-1}} \right). \] (2.16)

3. Thermodynamic Limit

The first step in determining the algebraic curve of the above system is to consider the thermodynamic limit \( L \to \infty \). This procedure has been explained for the \( su(2) \) spin chain in [32] and will now be discussed for general \( g \). Taking the logarithm of (2.4) yields

\[ L \log \left( \frac{u_i^{(k)} - i V_k}{u_i^{(k)} + i V_k} \right) = \sum_{l=1}^{r} \sum_{j=1, j \neq i}^{J_l} \log \left( \frac{u_i^{(k)} - u_j^{(l)} - i M_{kl}}{u_i^{(k)} - u_j^{(l)} + i M_{kl}} \right) - 2\pi i n_i^{(k)}, \] (3.1)

where \( n_i^{(k)} \in \mathbb{Z} \) are the mode numbers, arising due to taking the logarithm. Define \( x_i^{(k)} = u_i^{(k)}/L \) and perform the large \( L \) and \( J \) limit of this equation

\[ \frac{V_k}{x_i^{(k)}} = \sum_{l=1}^{r} \frac{1}{J_l} \sum_{j=1, j \neq i}^{J_l} \frac{M_{kl}}{x_i^{(k)} - x_j^{(l)}} - 2\pi i n_i^{(k)}. \] (3.2)

For fixed \( n_k \) define the densities

\[ \rho_k(x) = \sum_{j=1}^{J_k} \delta(x - x_j^{(k)}), \] (3.3)

and the corresponding resolvents

\[ G_k(x) = \frac{1}{J_k} \sum_{j=1}^{J_k} \frac{1}{x - x_j^{(k)}}. \] (3.4)
In the limit, the following replacement is made

$$\sum_{j=1}^{J_k} \to L \int_{C_k} dv \rho_k(v), \quad (3.5)$$

where $C_k$ denotes the curve along which the Bethe roots condense. The densities are normalized as

$$\int_{C_k} \rho_k(u) = J_k. \quad (3.6)$$

The resolvent become

$$G_k(u) = \frac{1}{J_k} \int_{C_k} dv \frac{\rho_k(v)}{v - u}, \quad (3.7)$$

and the Bethe equations limit to

$$\int_{C} dv \frac{\rho_k(v)M_{kf}(v)}{v - u} = -\frac{V_k}{u} + 2\pi n_i^{(k)}, \quad u \in C_i^{(k)}, \quad (3.8)$$

where $C = \cup_k C_k$ and each of the curves $C_k$ associated to simple roots is on the other hand $C_k = \cup_j C_j^{(k)}$. Further, $f(u) = l$ for $u \in C_l \subset C$. The principal value $f$ results from the restriction of the sum in (3.2) to $j \neq i$. Equivalently we have

$$M_{kk} G_k(u) + \sum_{l \neq k} M_{kl} G_l(u) = -\frac{V_k}{u} + 2\pi n_i^{(k)}, \quad u \in C_i^{(k)}. \quad (3.9)$$

Slashes denote principal values, that is

$$\mathcal{G}(u) = \frac{1}{2}(G(u+) + G(u-)). \quad (3.10)$$

For the discussion of the algebraic curve and its properties we shall require the asymptotics of the resolvents around $u = \infty$. The analysis is identical to the one in [34]. At infinity their expansion is

$$G_k(u) = -\frac{1}{u} \int_{C_k} dv \rho_k(v) + O\left(\frac{1}{u^2}\right) = -\frac{J_k}{u} + O\left(\frac{1}{u^2}\right). \quad (3.11)$$

The relation between the excitations numbers and the Dynkin labels (2.7) was obtained in [6]

$$J_k = \begin{pmatrix}
\frac{1}{2} \Delta_0 - \frac{1}{2} (L - B) - \frac{1}{2} s_1 \\
\Delta_0 - (L - B) \\
\Delta_0 - q - \frac{3}{2} q_1 - \frac{3}{4} q_2 \\
\Delta_0 - \frac{1}{2} q_1 - \frac{1}{2} q_2 \\
\Delta_0 - \frac{1}{2} q - \frac{3}{2} q_1 - \frac{3}{4} q_2 \\
\Delta_0 - (L + B) \\
\frac{1}{2} \Delta_0 - \frac{1}{2} (L + B) - \frac{1}{2} s_2
\end{pmatrix}. \quad (3.12)$$
4. The Algebraic Curve

The algebraic curve is implicitly defined in the Bethe equations (3.9). In the standard procedure to relate (3.9) to an algebraic curve a set of quasi-momenta $p^R = (p_1(u), \cdots, p_8(u))$ is defined. One way to determine them directly is to compute the transfer matrices for $\text{psu}(2,2\mid4)$ in the representation $R$ and extract the $p_i$ from their large $L$ limit. For the $\text{so}(6)$ sector this was done in [34]. Here the same procedure is applied to the transfer matrix eigenvalue (2.12).

4.1. Fermionic roots

In the Lie super-algebra case there is an additional subtlety due to the fermionic Lie algebra roots: as these square to zero, the corresponding diagonal entry in the Cartan matrix vanishes, which reflects itself in the absence of the diagonal, self-interacting term for the Bethe roots associated to the fermionic Lie algebra root. The above argument for deriving the thermodynamic limit needs to be treated with some care, as the corresponding continuum equation ceases to be a singular integral equation.

We shall discuss the case of the fermionic root $\alpha_2$, the case of $\alpha_6$ works in an identical fashion. The first four Bethe equations, which are relevant for this discussion with $V = V_F$ are

$$1 = \prod_{j \neq i} \frac{u_i^{(1)} - u_j^{(1)} + i}{u_i^{(1)} - u_j^{(1)} - i} \prod_{j=1} J_2 \frac{u_i^{(1)} - u_j^{(2)} - \frac{i}{2}}{u_i^{(1)} - u_j^{(2)} + \frac{i}{2}}$$

$$1 = \prod_{j=1} J_1 \frac{u_i^{(2)} - u_j^{(1)} - \frac{i}{2}}{u_i^{(2)} - u_j^{(1)} + \frac{i}{2}}$$

$$1 = \prod_{j=1} J_3 \frac{u_i^{(3)} - u_j^{(1)} + \frac{i}{2}}{u_i^{(3)} - u_j^{(1)} - \frac{i}{2}}$$

$$V(u_i^{(4)})^L = \prod_{j=1} J_2 \frac{u_i^{(3)} - u_j^{(2)} + \frac{i}{2}}{u_i^{(3)} - u_j^{(2)} - \frac{i}{2}} \prod_{j \neq i} J_3 \frac{u_i^{(3)} - u_j^{(3)} - i}{u_i^{(3)} - u_j^{(3)} + i} \prod_{j=1} J_4 \frac{u_i^{(3)} - u_j^{(4)} - \frac{i}{2}}{u_i^{(3)} - u_j^{(4)} + \frac{i}{2}}$$

where $V(u_i^{(4)})$ is the RHS of (2.4). The roots $u_i^{(k)} = 0$ for $k = 5, 6, 7$, which is admissible. First note that the excitation numbers satisfy the bounds $0 \leq J_1 \leq J_2 \leq J_3 \leq J_4$. Secondly, the bosonic Bethe roots come in complex conjugate pairs, unless they are real.
Consider the simpler case when in addition \( u_i^{(1)} = 0 \). The second equation in (4.1) turns into

\[
1 = \prod_{j=1}^{J_3} \frac{u_i^{(2)} - u_j^{(3)} + i}{u_i^{(2)} - u_j^{(3)} - i},
\]

which is an algebraic equation of degree \( J_3 - 1 \) in \( u_i^{(2)} \). For each \( u_k^{(3)} \), \( \bar{u}_k^{(3)} \) appears as well. From the second equation follows then by taking the absolute value that \( u_i^{(2)} \in \mathbb{R} \) (a related argument appears in [39]). So in this case, the solutions to the Bethe equations are of the type that all bosonic Bethe roots come in complex conjugate pairs (or are real) and the fermionic Bethe roots are real and solutions to the algebraic equation (1.2).

Including the \( u_i^{(1)} \) roots, the second equation yields a polynomial equation of degree \( J_1 + J_3 - 1 \) for the Bethe roots \( u_i^{(2)} \). Reality does not follow in this case. The trick to solve this was introduced in the condensed matter literature in [40] and used further in [11,12,13]. The idea is to introduce a string of Bethe roots for each fermionic Bethe root. In order to spell out the procedure for the present purposes, make the ansatz for the fermionic roots \( u_k^{(2)} \) and \( u_k^{(6)} \) as follows*

\[
\begin{aligned}
\quad u_k^{(2)} &= \left\{ \begin{array}{ll}
    v_k^{(2)} & k = 1, \ldots, J_2 \\
    v_k^{(1)} + i/2 & k = J_2 + 1, \ldots, J_2 + J'_1 \\
    v_k^{(1)} - i/2 & k = J'_1 + J_2 + 1, \ldots, 2J'_1 + J_2 \\
    v_k^{(6)} & k = 1, \ldots, J_6 \\
    v_k^{(7)} + i/2 & k = J_6 + 1, \ldots, J_6 + J'_7 \\
    v_k^{(7)} - i/2 & k = J_6 + J'_7 + 1, \ldots, J_6 + 2J'_7,
\end{array} \right.
\end{aligned}
\]

and values \( v_k^{(1)} \) and \( v_k^{(7)} \) for \( u_k^{(1)} \) and \( u_k^{(7)} \), respectively, where all \( v_j^{(k)} \in \mathbb{R} \). This has to hold strictly only in the \( L = \infty \) limit. The total number of Bethe roots for \( \alpha_2 \) is then \( J_2 + 2J'_1 \) etc..

We shall discuss the case of the fermionic root \( \alpha_2 \), the case of \( \alpha_6 \) works in an identical fashion. Plugging (4.3) into the second set of Bethe equations gives

\[
\begin{aligned}
1 &= \prod_{j=1}^{J'_1} \frac{v_i^{(1)} - u_j^{(1)} - i}{v_i^{(1)} - u_j^{(1)} + i} \prod_{j \neq i} \frac{v_i^{(1)} - v_j^{(1)} - i}{v_i^{(1)} - v_j^{(1)} + i} \prod_{j=1}^{J_3} \frac{v_i^{(1)} - u_j^{(3)} + i}{v_i^{(1)} - u_j^{(3)} - i} \\
1 &= \prod_{j=1}^{J'_1} \frac{v_i^{(2)} - u_j^{(1)} - i/2}{v_i^{(2)} - u_j^{(1)} + i/2} \prod_{j=1}^{J_2} \frac{v_i^{(2)} - v_j^{(1)} - i/2}{v_i^{(2)} - v_j^{(1)} + i/2} \prod_{j=1}^{J_3} \frac{v_i^{(2)} - u_j^{(3)} + i/2}{v_i^{(2)} - u_j^{(3)} - i/2}.
\end{aligned}
\]

* A proof, that this string-ansatz yields the complete set of solutions, does not seem to exist.
In particular, the Bethe roots $v_i^{(1)}$ now have a self-interacting term, as opposed to the real Bethe roots $v_i^{(2)}$ associated to $\alpha_2$. Switching the roots $v_i^{(1)}$ off reduces the Bethe equation to one without self-interactions, for the real roots $v_i^{(2)}$.

So in summary the distribution of Bethe roots comprises of complex Bethe roots assigned to the bosonic Lie algebra roots, which condense in complex contours (denoted $C_i^{(k)}$), as well as real Bethe roots associated to the fermionic Lie algebra roots, and triplets of roots (e.g., for $\alpha_2$ these are $v_k^{(2)}, v_k^{(2)} \pm i/2$), assigned to $\alpha_1/2$ and $\alpha_6/7$. The real roots associated to the fermionic roots are algebraically determined in terms of the other Bethe roots.

4.2. Construction of the Algebraic Curve

For the present case of interest let us first spell out the Bethe equations (3.9) for the bosonic Lie algebra roots

\begin{align}
-2G_1(u) + G_2(u) &= 2\pi n_j^{(1)} - \frac{V_{j1}}{u}, \quad u \in C_j^{(1)}, \\
-G_2(u) + 2G_3(u) - G_4(u) &= 2\pi n_j^{(3)} - \frac{V_{j3}}{u}, \quad u \in C_j^{(3)}, \\
-G_3(u) + 2G_4(u) - G_5(u) &= 2\pi n_j^{(4)} - \frac{V_{j4}}{u}, \quad u \in C_j^{(4)}, \\
-G_4(u) + 2G_5(u) - G_6(u) &= 2\pi n_j^{(5)} - \frac{V_{j5}}{u}, \quad u \in C_j^{(5)}, \\
G_6(u) - 2G_7(u) &= 2\pi n_j^{(7)} - \frac{V_{j7}}{u}, \quad u \in C_j^{(7)}.
\end{align}

Each of the above lines with mode number $n_j^{(k)}$ corresponds to the $k$th root and it is assumed that $u \in C_j^{(k)}$. I.e., for each root the densities $\rho_k(u)$ have support on the union of curves $C_k = C_1^{(k)} \cup \cdots \cup C_{A_k}^{(k)}$. The Bethe equations associated to the fermionic roots give rise to the algebraic constraints

\begin{align}
\frac{1}{J_1} \sum_{j=1}^{J_1} \frac{1}{u_i^{(2)} - u_j^{(1)}} - \frac{1}{J_3} \sum_{j=1}^{J_3} \frac{1}{u_i^{(2)} - u_j^{(3)}} &= 2\pi n_i^{(2)}, \\
\frac{1}{J_6} \sum_{j=1}^{J_6} \frac{1}{u_i^{(6)} - u_j^{(5)}} - \frac{1}{J_7} \sum_{j=1}^{J_7} \frac{1}{u_i^{(6)} - u_j^{(7)}} &= 2\pi n_i^{(6)}.
\end{align}

In the case of only real values for the fermionic Bethe roots, $u_j^{(k)} = v_j^{(k)}$ for $k = 2, 6$, these can be rewritten as

\begin{align}
G_1(u) - G_3(u) &= 2\pi n_j^{(2)} - \frac{V_{j2}}{u}, \quad u \in R^{(2)}, \\
-G_5(u) + G_7(u) &= 2\pi n_j^{(2)} - \frac{V_{j2}}{u}, \quad u \in R^{(6)},
\end{align}
where $\mathcal{R}^{(k)} \subset \mathbb{R}$, so (4.7) are algebraic equations, which have to be satisfied for a collection of real points.

The situation changes, if we allow complex values for the fermionic roots. Then, the thermodynamic limit needs to be taken for the equations (4.4). Let $H_k(u)$ be the resolvent for the additional real centers $v_j^{(k)}$, $k = 1, 7$, thus

$$G_1(u) + H_1(u) - G_3(u) = \pi m_j^{(2)}, \quad u \in \mathcal{R}^{(2)}$$
$$G_1(u) + H_1(u) - G_3(u) = 2\pi l_j^{(2)}, \quad u \in \mathcal{S}^{(2)}$$
$$G_7(u) + H_7(u) - G_5(u) = \pi m_j^{(6)}, \quad u \in \mathcal{R}^{(6)}$$
$$G_7(u) + H_7(u) - G_5(u) = 2\pi l_j^{(6)}, \quad u \in \mathcal{S}^{(6)}.$$

The principal value now arises from the self-interacting term for the Bethe root $v_1^{(1)}$ in (4.4). The second equation is again algebraic and both equations are for real values of $u$.

Now we turn to the construction of the algebraic curve. Assume that the fermionic Bethe roots are real. The aim is to rewrite the equations (4.5) in terms of the quasi-momenta $p_k(u)$, which will be defined shortly, such that they take the form

$$M_{kk} \tilde{G}_k + \sum_{j \neq k} M_{kj} \tilde{G}_j(u) = \mathcal{P}_k(u) - \mathcal{P}_{k+1}(u) = 2\pi n_j^{(k)}, \quad u \in \mathcal{C}_j^{(k)}$$

for $k \neq 2, 6$, where the singular terms in (4.3) have been absorbed into the resolvents, $\tilde{G}_k(u)$. For the fermionic roots, the Bethe equations in terms of quasi-momenta are

$$\sum_{j \neq k} M_{kj} \tilde{G}_j(u) = p_k(u) - p_{k+1}(u) = 2\pi n_j^{(k)}, \quad k = 2, 6,$$

for $u \in \mathcal{R}_j^{(k)}$ real.

![Figure 2](image-url)  
**Figure 2** First four sheets of the algebraic curve.
In summary: the various sheets of the algebraic curve are labeled by \( p_k(u) \), and for bosonic roots are glued together along the cuts \( C_k \), whereas for the fermionic roots the Bethe roots lie on the real axis, where the quasi-momenta of the sheets satisfy an algebraic relation. In figure 2 the first four sheets assigned to \( p_k(u) \) are depicted schematically. The curves within each sheet are Bethe roots distributed along the complex cuts \( C_j^{(k)} \) for the bosonic roots, and the real Bethe roots on sheets 2 and 3 lie on the real axis. The vertical dotted lines indicate the identifications of the sheet functions, encoded in (4.9) (blue) and (4.10) (red stripy).

The quasi-momenta \( p_k \) are obtained from the eigenvalue \( T_{4|4} \) of the transfer matrix (2.12) in the large \( L \) limit

\[
T_{4|4}(u) \to \sum_{k=1}^{8} \epsilon_k \exp(ip_k), \tag{4.11}
\]

with (cf.(2.12))

\[
\epsilon_k = (+++---++). \tag{4.12}
\]

The terms in (2.12) limit to

\[
\frac{R_k(u+s)}{R_k(u+t)} \to \exp((t-s)G_k(u)), \quad \frac{(u+s)L}{(u+t)L} \to \exp\left(\frac{(s-t)}{u}\right). \tag{4.13}
\]

The representation relevant for the SYM spin chain has Dynkin labels \( V_F \), see (2.8). Define the singular resolvents by

\[
\tilde{G}_1(u) = G_1(u) - \frac{1}{2u}, \quad \tilde{G}_2(u) = G_2(u) - \frac{1}{u}, \quad \tilde{G}_3(u) = G_3(u) - \frac{1}{2u},
\]

\[
\tilde{G}_4(u) = G_4(u), \quad \tilde{G}_5(u) = G_5(u) - \frac{1}{2u}, \quad \tilde{G}_6(u) = G_6(u) - \frac{1}{u}, \quad \tilde{G}_7(u) = G_7(u) - \frac{1}{2u}. \tag{4.14}
\]

The quasi-momenta for this representation then turn out to be

\[
p_1(u) = -\tilde{G}_1(u)
p_2(u) = -\tilde{G}_2(u) + \tilde{G}_1(u)
p_3(u) = +\tilde{G}_3(u) - \tilde{G}_2(u)
p_4(u) = +\tilde{G}_4(u) - \tilde{G}_3(u)
p_5(u) = +\tilde{G}_5(u) - \tilde{G}_4(u)
p_6(u) = +\tilde{G}_6(u) - \tilde{G}_5(u)
p_7(u) = -\tilde{G}_7(u) + \tilde{G}_6(u)
p_8(u) = +\tilde{G}_7(u), \tag{4.15}
\]
and need to satisfy the Bethe equations (4.5) in their incarnation (4.9). Note that

\[ \sum_{k=1}^{8} \epsilon_k p_k = 0. \]  

(4.16)

With these redefinitions, the Bethe equations take the compact form of a Riemann-Hilbert problem

\[ \varphi_k(u) - \varphi_{k+1}(u) = 2\pi n_j^{(k)}, \quad u \in C_j^{(k)}, \]  

(4.17)

for \( k = 1, 3, 4, 5, 7 \). In addition, the algebraic equations (4.6) have to be satisfied. The functions \( p_k(u) \) with \( k = 1, \cdots, 8 \) in (4.17) determine a function \( p_F(u) \) defined on a Riemann surface, where each \( p_k \) is \( p_F(u) \) restricted to the \( k \)th sheet.

The asymptotics of the quasi-momenta at \( u = \infty \) are obtained from (3.11) and (3.12)

\[
\begin{align*}
    p_1(u) &= \frac{1}{u} \left( +\frac{1}{2}\Delta_0 - \frac{1}{2}s_1 \right) + O\left( \frac{1}{u^2} \right) \\
    p_2(u) &= \frac{1}{u} \left( +\frac{1}{2}\Delta_0 + \frac{1}{2}s_1 \right) + O\left( \frac{1}{u^2} \right) \\
    p_3(u) &= \frac{1}{u} \left( +\frac{3}{4}q_1 + \frac{1}{2}q + \frac{1}{4}q_2 \right) + O\left( \frac{1}{u^2} \right) \\
    p_4(u) &= \frac{1}{u} \left( -\frac{1}{4}q_1 + \frac{1}{2}q + \frac{1}{4}q_2 \right) + O\left( \frac{1}{u^2} \right) \\
    p_5(u) &= \frac{1}{u} \left( -\frac{1}{4}q_1 - \frac{1}{2}q + \frac{1}{4}q_2 \right) + O\left( \frac{1}{u^2} \right) \\
    p_6(u) &= \frac{1}{u} \left( -\frac{1}{4}q_1 - \frac{1}{2}q - \frac{3}{4}q_2 \right) + O\left( \frac{1}{u^2} \right) \\
    p_7(u) &= \frac{1}{u} \left( -\frac{1}{2}\Delta_0 - \frac{1}{2}s_2 \right) + O\left( \frac{1}{u^2} \right) \\
    p_8(u) &= \frac{1}{u} \left( -\frac{1}{2}\Delta_0 + \frac{1}{2}s_2 \right) + O\left( \frac{1}{u^2} \right). 
\end{align*}
\]

(4.18)

Thus, all quasi-momenta have the asymptotics \( p_i(u) \sim 1/u + O(1/u^2) \).

From the relation between the quasi-momenta and resolvents the asymptotics at \( u = 0 \) are

\[
    p_k(u) = \begin{cases} 
        +\frac{1}{2u} + O(1) & k = 1, 2, 3, 4 \\
        -\frac{1}{2u} + O(1) & k = 5, 6, 7, 8. 
    \end{cases} 
\]

(4.19)
The asymptotics of the local charge expansion (2.16) is obtained as follows, taking into account that the expansion around \( u_0 \) becomes an expansion around \( u = 0 \) in the thermodynamic limit. Eq. (2.15) in the thermodynamic limit yields by (4.13)

\[
\tilde{G}_4(u) = G_4(u) = p_1(u) + p_2(u) + p_3(u) + p_4(u) = \frac{2}{u} + \sum_{r=1}^{\infty} u^r Q_r .
\] (4.20)

In summary, by (4.18) and (4.19), the function \( p(u) \) defined by the quasi-momenta on each of the eight sheets is therefore not regular. An algebraic curve can be engineered out of it by removing the singularities. Consider

\[
y(u) = \epsilon u^2 \frac{dp(u)}{du} .
\] (4.21)

or written on each of the sheets

\[
y_k(u) = \epsilon_k u^2 \frac{dp_k(u)}{du} .
\] (4.22)

\( y(u) \) then has no poles and satisfies an octic equation

\[
\sum_{d=0, d \neq 7}^{8} P_d(u) y^d = P_8(u) \prod_{d=1}^{8} (y - y_k(u)) = 0 ,
\] (4.23)

where \( y_k(u) = u^2 p'_k(u) \) and the coefficient of \( y^7 \) vanishes by (4.16). The argument runs essentially the same way as in [34]: \( p(u) \) has poles at 0 and \( \infty \). Assume that \( y(u) \) takes constant values at these points. Then (4.23) is satisfied only if the \( P_k(u) \) have the same degree \( 2d \) and the constant term does not vanish – in this case (4.23) gives rise to a polynomial equation in \( y \) with constant coefficients, which can be solved for finite \( y \). The curve thus has \( 8(2d+1) - 1 \) parameters left. As explained in [34] it is necessary to ensure the right number of square-root branch cuts, which are determined from the square-root poles in \( y \). These are \( d \) in number. The absence of other unwanted cuts is obtained by requiring the discriminant for the polynomial on the LHS of (4.23)

\[
D(y) = P_8(u)^{14} \prod_{i<j}^{8} (y_i - y_j)^2 ,
\] (4.24)

\[\diamond\] There is a slightly subtle point here: in (2.15) only the terms up to order \( u^L \) are determined. However, as explained in sec. 4.5 of [30], these terms may well contribute in the thermodynamic limit, yielding terms of \( O(1/u) \). However assuming that the term will be a combination of quasi-momenta \( p_k \), the \( O(1/u) \) term can be determined indirectly from the asymptotics of \( p_k(u) \) at \( u = 0 \).
to be a perfect square. The discriminant condition reduces the number of parameters by 13\(d\). Further, the pole-structure at \(u = 0\) fixes further 7 coefficients, leaving 3\(d\) parameters, which are fixed by the period-integrals: \(d\) parameters are fixed by requiring that the A-cycle period-integrals (i.e., integrals circumnavigating one cut within a sheet) vanish. Further, the Bethe equations in the Riemann-Hilbert form are equivalent to the integrality of the B-periods (i.e., integrals along cycles, which connect two sheets) with the integers given by the \(n_j^{(k)}\) in the Bethe equations. This fixes another \(d\) parameters. Finally, choosing the filling fractions gives rise to the remaining \(d\) parameters, of which one is fixed by the momentum condition resulting from (2.5), cf. [32].

5. Comments on the Algebraic Curve for the Sigma-model on \(AdS_5 \times S^5\)

Some very crude comparisons between the \(\mathcal{N} = 4\) SYM algebraic curve and the (yet-to-be-determined) algebraic curve for the string theory on \(AdS_5 \times S^5\) can be made. In the spirit of the discussions in [32,33,34] the idea is that the properties of the quasi-momenta determine various properties of the curve, as e.g., seen in the last section. Thus, a necessary requirement for the matching of the two curves is that the asymptotics of the \(p_k\) agree.

On the string theory side, some important properties of the quasi-momenta were determined recently by Arutyunov and Frolov [44] by constructing a Lax representation of the full string sigma-model. We shall compare the there-obtained asymptotics of the quasi-momenta with the ones derived in section 4.

5.1. Asymptotics of the Sigma-Model Quasi-Momenta

Denote the spectral parameter of the string sigma-model by \(x\). Then the asymptotics for the quasi-momenta were obtained in [44] for \(x\), which will be compared to the asymptotics in the spectral parameter \(u\) used so far in this paper.

The expressions for \(p_k(u)\) for \(u = \infty\) obtained in (4.18) are mapped to the ones in [44] by identifying the representation labels in (3.12) with the various spins of \(AdS_5\) \((S_1, S_2)\) and \(S^5\) \((J_1, J_2, J_3)\) by means of

\[
\begin{align*}
q_1 &= J_2 - J_3, & q_2 &= J_1 - J_2, & q &= J_2 + J_3, \\
q_s &= S_1 - S_2, & q_s &= S_1 + S_2.
\end{align*}
\]

\footnote{Thanks to the authors of [34] for making v3 of their paper available in advance, where this is discussed in the \(SO(6)\) case.}
By [44], the asymptotics of the quasi-momenta \( q_i(x) \) of the sigma-model at \( x = 0 \) and \( x = \infty \) are:

\[
\begin{align*}
(x = 0) & \quad (x = \infty) \\
q_1(x) &= x \frac{2\pi}{\sqrt{\lambda}} (-\Delta_0 + S_1 - S_2) & q_1(x) &= \frac{2\pi}{x \sqrt{\lambda}} (+\Delta_0 - S_1 + S_2) \\
q_2(x) &= x \frac{2\pi}{\sqrt{\lambda}} (-\Delta_0 - S_1 + S_2) & q_2(x) &= \frac{2\pi}{x \sqrt{\lambda}} (+\Delta_0 + S_1 - S_2) \\
q_3(x) &= x \frac{2\pi}{\sqrt{\lambda}} (+J_3 - J_1 - J_2) & q_3(x) &= \frac{2\pi}{x \sqrt{\lambda}} (-J_3 + J_1 + J_2) \\
q_4(x) &= x \frac{2\pi}{\sqrt{\lambda}} (-J_3 - J_1 + J_2) & q_4(x) &= \frac{2\pi}{x \sqrt{\lambda}} (+J_3 + J_1 - J_2) \\
q_5(x) &= x \frac{2\pi}{\sqrt{\lambda}} (-J_3 + J_1 - J_2) & q_5(x) &= \frac{2\pi}{x \sqrt{\lambda}} (+J_3 - J_1 + J_2) \\
q_6(x) &= x \frac{2\pi}{\sqrt{\lambda}} (+J_3 + J_1 + J_2) & q_6(x) &= \frac{2\pi}{x \sqrt{\lambda}} (-J_3 - J_1 - J_2) \\
q_7(x) &= x \frac{2\pi}{\sqrt{\lambda}} (+\Delta_0 + S_1 + S_2) & q_7(x) &= \frac{2\pi}{x \sqrt{\lambda}} (-\Delta_0 - S_1 - S_2) \\
q_8(x) &= x \frac{2\pi}{\sqrt{\lambda}} (+\Delta_0 - S_1 - S_2) & q_8(x) &= \frac{2\pi}{x \sqrt{\lambda}} (-\Delta_0 + S_1 + S_2).
\end{align*}
\]

The quasi-momenta \( q_k(x) \) in (5.2) for \( k = 3, 4, 5, 6 \) reproduce the ones for the five-sphere, and for \( k = 1, 2, 7, 8 \) for AdS. The asymptotics at \( x \to \pm 1 \) are vastly more complicated ((5.5) and (5.8) in [44]).

5.2. Comparison to the \( \text{psu}(2, 2|4) \) Spin Chain

The comparison between gauge theory and string theory is made by taking the limit \( L \to \infty \) in the SYM theory (which has been accounted for by considering the spin chain in the thermodynamic limit) and likewise the large angular momentum limit of the string theory. More precisely, define \( J = \sum J_i \) and \( S = \sum S_i \), and consider the limit

\[
\frac{\lambda}{J^2} \to 0, \quad J \to \infty, \quad \frac{J_i}{J}, \frac{S_i}{J} = \text{fixed} < \infty.
\]

First, the relation between the spectral parameters \( u \) and \( x \) needs to be sorted out. From the asymptotics of the quasi-momenta at \( \infty \) one naively reads off \( \frac{1}{u} \sim \frac{4\pi}{x \sqrt{\lambda}} \). In order to match the \( x, u \to 0 \) asymptotics, one infers that

\[
u = \frac{\sqrt{\lambda}}{4\pi J^x}.
\]

* An overall minus sign is introduced compared to [44], which does not spoil the symmetry of \( p_k(x) \) under \( x \to 1/x \) and simplifies the comparison to the gauge theory.
Rescaling the various spins \((S_i, \Delta_0, J_i)\) by \(J\), the asymptotics at \(u, x = \infty\), (4.18) and (5.2), agree.

At \(u = 0\) the comparison is a bit more subtle. From the gauge theory result (4.19) we expect simple poles with strengths \(\pm 1/2\) at \(u = 0\). Setting \(u = \pm \sqrt{\lambda/4\pi J}\), which amounts to \(u \rightarrow 0\) in the limit (5.3), implies by (5.4) that this is to be compared with the limit \(x \rightarrow \pm 1\) of the string theory quasi-momenta, which are rather involved expressions, whose \(\lambda/J^2 \rightarrow 0\) limit is not easily extracted. We shall discuss this elsewhere in more detail.

Clearly it would be desirable to construct the complete curve for the sigma-model on \(AdS_5 \times S^5\), and perform a comparison of the curves, as is done in the case of \(\mathbb{R} \times S^5\) in [34] and for \(AdS_5 \times S^1\) in section 6.

6. The Algebraic Curve for the \(su(2, 2)\) ‘Subsector’

In this section the reduction of the algebraic curve to the subsector which is the dual to spinning strings on \(AdS_5 \times S^1\) is considered. The curve and Bethe ansatz are obtained for the sigma-model and are shown to agree with the ones of the \(su(2, 2)\) spin chain. A word of caution is in place here: we should point out that the truncation to \(su(2, 2)\) does not yield a subsector of \(\mathcal{N} = 4\) in general, however it is a closed subsector at 1-loop [5]. One can truncate the spin chain to this symmetry algebra and show that the quasi-momenta have the same asymptotics with the ones obtained for the \(AdS_5 \times S^1\) sigma-model. The corresponding spinning string solutions with multiple \(AdS\)-spins and one spin on the \(S^5\) were first obtained in [10].

6.1. The \(su(2, 2)\) Spin Chain

The reduction from \(psu(2, 2|4)\) to \(su(2, 2)\) is most transparent by picking the distinguished Dynkin diagram of \(psu(2, 2|4)\), as done in appendix A\(^5\). The quantities computed with this choice will be distinguished with a prime. The first question to address is which \(su(2, 2)\)-representation is to sit at each spin chain site. The sub-module of \(F\), which is obtained by acting with \(su(2, 2)\) on the highest weight \(V_F\) (which for the distinguished Dynkin diagram is \(V_{F'} = (0, -3, 2)\)) would be one choice. However the highest weight state here corresponds in the SYM theory to a highly excited state, and thus discussing a

---

\(^5\) Recently the \(su(2, 2)\) spin chain has also been discussed in the context of integrability in large \(N\) QCD [35]. There, the representation at each site is \(F\).
termodynamic limit in this case does not make sense. Nevertheless the curve in this case is computed in appendix B, as it is of relevance for the QCD spin chain \[35\].

The correct choice for the representation for the \(\mathfrak{su}(2,2)\) subsector of \(\mathcal{N} = 4\) SYM has highest weight \(V_2^{\mathcal{Z}} = (0,-1,0)\). It is built on the physical vacuum and thus the Bethe ansatz equations describe excitations around this BPS-state, whereas \(\mathbf{F}'\) would have a vacuum energy of \(3L\) \[3\] and the excitations would be built on a non-BPS state and thus unstable\(^\#\). This yields quasi-momenta, with asymptotics matching the ones of the string sigma-model. The non-compact \(\mathfrak{su}(2,2)\)-module with highest weight \(\mathcal{Z}' = (0,-1,0)\) corresponds in \(\mathcal{N} = 4\) SYM to the sector with vacuum \(|0\rangle = \text{Tr}Z^L\), with the scalar \(Z = \Phi_{34}\), which in the oscillator representation of \[3\] is \(|Z\rangle = c_3^\dagger c_4^\dagger|0\rangle\). The states are obtained by acting on \(|0\rangle\) with the oscillators \(a_i^\dagger b_j^\dagger, i,j \in \{1,2\}\), which correspond to space-time derivatives \(D_{ij}\).

In order to determine the asymptotics of the quasi-momenta, one requires to first find the relation between the representation labels and excitation numbers. In \[3\] the excitation numbers \(n_a\) etc. for the oscillators were obtained for a state of the type \((a_1^\dagger)^{k_1}(a_2^\dagger)^{k_2}(b_1^\dagger)^{l_1}(b_2^\dagger)^{k_1+k_2+l_1}|0\rangle\). Furthermore, the combination of oscillators corresponding to the roots \(\alpha_k\) of \(\mathfrak{su}(2,2)\) are

\[
\alpha_1 : \ b_2^\dagger b_1^\dagger, \quad \alpha_2 : \ b_1^\dagger a_1^\dagger, \quad \alpha_3 : \ a_2^\dagger a_1^\dagger,
\]

so that the excitation numbers \(J_k'\) for the roots are \(J_1 = n_{b_2}\), \(J_2 = n_{b_2} + n_{b_1}\) and \(J_3 = n_{b_2} + n_{b_1} - n_{a_1} = n_{a_2}\). In summary, the relation between excitation numbers for the roots and the representation labels are thus

\[
J_k' = \begin{pmatrix}
-1/2 + \frac{1}{2}(\Delta_0 - s_2) \\
-1 + \Delta_0 \\
-1/2 + \frac{1}{2}(\Delta_0 - s_1)
\end{pmatrix}.
\]

The singular resolvents and quasi-momenta are extracted from the transfer matrix in appendix A, where all the \(G_k' = 0\) for \(k = 4 \cdots 7\)

\[
\tilde{G}_1'(u) = G_1'(u) - \frac{1}{2u}, \\
\tilde{G}_2'(u) = G_2'(u) - \frac{1}{u}, \\
\tilde{G}_3'(u) = G_3'(u) - \frac{1}{2u},
\]

\(^\#\) Thanks to N. Beisert for explaining this point to me.
and
\[ p'_1(u) = \tilde{G}'_1(u) \]
\[ p'_2(u) = \tilde{G}'_2(u) - \tilde{G}'_1(u) \]
\[ p'_3(u) = \tilde{G}'_3(u) - \tilde{G}'_2(u) \]
\[ p'_4(u) = -\tilde{G}'_3(u). \]

The corresponding Bethe equations in the large \( L \) limit take the form as expected from the general considerations in (3.9)

\[ 2 \mathcal{G}'_1(u) - \mathcal{G}'_2(u) = p'_1(u) - p'_2(u) - \frac{V_{j_1}'}{u} = 2 \pi n_j^{(1)} - \frac{V_{j_1}'}{u}, \quad u \in \mathcal{C}_j^{(1)} \]
\[ -\mathcal{G}'_1(u) + 2 \mathcal{G}'_2(u) - \mathcal{G}'_3(u) = p'_3(u) - p'_4(u) - \frac{V_{j_3}'}{u} = 2 \pi n_j^{(2)} - \frac{V_{j_3}'}{u}, \quad u \in \mathcal{C}_j^{(2)} \]
\[ 2 \mathcal{G}'_3(u) - \mathcal{G}'_2(u) = p'_3(u) - p'_4(u) - \frac{V_{j_3}'}{u} = 2 \pi n_j^{(1)} - \frac{V_{j_3}'}{u}, \quad u \in \mathcal{C}_j^{(3)}, \quad (6.5) \]

where with the definition of the singular resolvents (6.3), the RHS is precisely given by \( V'_Z = (0, -1, 0) \).

The asymptotics of the quasi-momenta are extracted in the same way as before from (3.11) and (6.2) at \( u = \infty \)

\[ p'_1(u) = \frac{1}{u} \left( -\frac{1}{2} \Delta_0 + \frac{1}{2} s_2 \right) + O \left( \frac{1}{u^2} \right) \]
\[ p'_2(u) = \frac{1}{u} \left( -\frac{1}{2} \Delta_0 - \frac{1}{2} s_2 \right) + O \left( \frac{1}{u^2} \right) \]
\[ p'_3(u) = \frac{1}{u} \left( \frac{1}{2} \Delta_0 + \frac{1}{2} s_1 \right) + O \left( \frac{1}{u^2} \right) \]
\[ p'_4(u) = \frac{1}{u} \left( \frac{1}{2} \Delta_0 - \frac{1}{2} s_1 \right) + O \left( \frac{1}{u^2} \right) \].

The asymptotics at \( u = 0 \) are obtained from (2.14). At finite \( L \) the expansion is

\[ T_Z(u + i) = \frac{R_2(u + i/2)}{R_2(u - i/2)} + O(u^L), \quad (6.7) \]

which for \( L \to \infty \) limits to

\[ T_Z(u + i) \to \exp \left( -i \tilde{G}'_2(u) \right). \]

Thus at \( u = 0 \) the combination of resolvents

\[ -\tilde{G}'_2(u) = -p'_1(u) - p'_2(u) = \frac{1}{u} + \sum_{r=0}^{\infty} u^r Q_r. \]

19
gives rise to the local charges $Q_r$. The asymptotics of all quasi-momenta at zero are

$$p'_1(u) = -\frac{1}{2u} + O(1), \quad p'_2(u) = -\frac{1}{2u} + O(1), \quad p'_3(u) = +\frac{1}{2u} + O(1), \quad p'_4(u) = +\frac{1}{2u} + O(1).$$

The curve for this subsector is written in terms of $y_k = u^2 \frac{d^2 p_k}{du^2}$ as

$$\sum_{i=0, i \neq 3}^4 P_i(u) y_i = P_4(u) \prod_{k=1}^4 (y - y_k(u)) = 0. \quad (6.11)$$

Again the moduli-count is as before and pretty much identical to the $\mathfrak{so}(6)$ case discussed in [34]: $\sum_k p'_k = 0$, thus the $P_d$ together have $4(2d + 1) - 1$ parameters, of which $3$ are fixed by requiring the asymptotics at $u = 0$. The discriminant condition fixes $5d$ and the vanishing of the A-periods removes another $d$, leaving $2d$, which are fixed by the B-period integrals ($d$), the filling fractions ($d - 1$) and momentum constraint (1). The latter follows from integrating up the Bethe equations using that $G_2(u)$ at order $u^0$ is $2\pi m$ and reduces the number of $d$ filling fractions to $d - 1$.

### 6.2. Classical Sigma-model on $AdS_5 \times S^1$

Classically the string moving on $AdS_5 \times S^1$ is described by a coset model on

$$AdS_5 \times S^1 = \frac{SO(4,2)}{SO(4,1)} \times U(1). \quad (6.12)$$

The construction that will be most suited for a discussion of the integrable structure is based on the Lax-pair representation obtained recently by Arutyunov and Frolov in [44].

For the comparison we quote the relevant properties of the sigma-model integrable system in [44]. Denote the spectral parameter on the sigma-model side by $x$. Then the asymptotics of the quasi-momenta at $x = \infty$ are

$$q_1(x) = \frac{1}{x} \frac{2\pi}{\sqrt{\lambda}} (-\Delta_0 + S_1 + S_2)$$
$$q_2(x) = \frac{1}{x} \frac{2\pi}{\sqrt{\lambda}} (-\Delta_0 - S_1 - S_2)$$
$$q_3(x) = \frac{1}{x} \frac{2\pi}{\sqrt{\lambda}} (+\Delta_0 + S_1 - S_2)$$
$$q_4(x) = \frac{1}{x} \frac{2\pi}{\sqrt{\lambda}} (+\Delta_0 - S_1 + S_2),$$

(6.13)
and at $x = 0$ (there are no non-trivial cycles and thus no windings in the $AdS$ part)

$$
q_1(x) = x \frac{2\pi}{\sqrt{\lambda}} (+\Delta_0 - S_1 - S_2)
$$

$$
q_2(x) = x \frac{2\pi}{\sqrt{\lambda}} (+\Delta_0 + S_1 + S_2)
$$

$$
q_3(x) = x \frac{2\pi}{\sqrt{\lambda}} (-\Delta_0 - S_1 + S_2)
$$

$$
q_4(x) = x \frac{2\pi}{\sqrt{\lambda}} (-\Delta_0 + S_1 - S_2),
$$

(6.14)

as well as $x = \pm 1$, with $m$ the winding number of the string on the $S^1$,

$$
q_1(x) = q_2(x) = -q_3(x) = -q_4(x) = -\frac{\pi}{x \mp 1} \left( \frac{J}{\sqrt{\lambda}} \pm m \right).
$$

(6.15)

The spins are related to the ones appearing in the gauge theory expressions by (5.1).

The Lax connections formed out of left and right invariant currents for the sigma-model are related by a gauge transformation and by replacing $x \rightarrow 1/x$ [4]. The associated transfer matrices are therefore related by the same transformation, and the eigenvalues must agree up to the replacement $x \rightarrow 1/x$

$$
q_i(x) = -q_i(1/x).
$$

(6.16)

With these asymptotics of the quasi-momenta, the curve is constructed as follows: First remove again the singularities, which now occur at $x = 0$ as well as $x = \pm 1$, and define the variables $z_k$ which are regular on each of the four sheets defined by the quasi-momenta $q_k$: $z_k = x \left( x - \frac{1}{x} \right)^2 \frac{dq_k(x)}{dx}$, so that $z_k(1/x) = z_k(x)$. $z$ satisfies a quartic equation of the type (6.11). The moduli count runs in an identical fashion to the one for the $so(6)$ curve in [34] and we refrain from repeating it here and point out some differences: the symmetry (6.16) gives rise to the distribution of cuts such that the cut $C_i$ connects the sheets $i$ and $i + 1$. However as one readily verifies, the moduli count is unchanged by this. Further, the main difference is the asymptotics of $q_k$ at $x = \pm 1$, which will in particular enter the Bethe equations. A point worth elaborating upon is the identification of the additional condition on the $d$ filling fractions. In the gauge theory, the constraint arose from the momentum quantization condition, which reduced the choice of filling fractions to $d - 1$. It was derived by integrating the Bethe equations in the Riemann-Hilbert form. In the next subsection we shall derive the sigma-model analog of these.
6.3. Bethe Equations for the Sigma-Model

Using the isomorphism between $\mathfrak{su}(2,2)$ and $\mathfrak{so}(4,2)$, the construction of the Bethe-type equations for the sigma-models proceeds along the lines of $\mathfrak{so}(6)$. Applying the methods put forward in [34] and the asymptotics at $x = \pm 1$ in (6.15) the resolvents satisfy

$$
2 \mathcal{G}_1(x) - G_2(x) = 2\pi n_j^{(1)} - V_1 \mathcal{L}, \quad x \in C_j^{(1)}
$$

$$
-G_1(x) + 2 \mathcal{G}_2(x) - G_3(x) = 2\pi n_j^{(2)} - V_2 \mathcal{L}, \quad x \in C_j^{(2)}
$$

$$
2 \mathcal{G}_3(x) - G_2(x) = 2\pi n_j^{(3)} - V_3 \mathcal{L}, \quad x \in C_j^{(3)},
$$

where $V = (0, -1, 0)$ is again the vector of Dynkin labels for the representation and

$$
\mathcal{L} = 2\pi \left( \frac{\sqrt{\lambda} + m}{x - 1} + \frac{\sqrt{\lambda} - m}{x + 1} \right).
$$

The equations (6.17) are manufactured such that they are equivalent to $q_i - q_{i+1} = 2\pi n_j^{(i)}$. The analog of the momentum constraint is then obtained by integrating the equations with $\int_{C_k} \rho_k(x)$ (where $\rho_k(x)$ are the densities for the singular resolvents $\tilde{G}_k(x)$) and using the normalization, $\int_{C_k} \rho_k(x)/x = -\tilde{G}_k(0)$, which can be computed from the asymptotics of the quasi-momenta at $x = 0$ and (6.18).

6.4. Comparison Gauge/String Theory

To show that the algebraic curves agree, the defining equations need to be matched, as well as the asymptotics of the quasi-momenta need to agree. The comparison is made at small $\lambda$ as before. The spectral parameters $u$ and $x$ are related by

$$
x = \frac{4\pi J}{\sqrt{\lambda}} u.
$$

Firstly, the number of sheets defining the curves, which is determined by the number of quasi-momenta, agrees. Secondly, the asymptotics at $u = \infty$, (5.6) and (6.13), match after rescaling $\Delta_0, S_1, S_2$ by $J$.

\[\text{\dag}\quad\text{These equations are the analogs of (5.21) in [34], where the non-singular part of the resolvent has been merged into $G_k$ and the part containing the singular term at $x = \pm 1$ is explicitly written out on the RHS.}\]
The asymptotics at \( u = 0 \) are compared as follows. On the gauge theory side, the asymptotics were determined in (6.10). The comparison to the gauge theory requires to take the sigma-model quantities in the limit
\[
\frac{\lambda}{J^2} \to 0, \quad \Delta_0 = S + J, \quad J \to \infty, \quad \frac{S_i}{J} = \alpha_i < \infty.
\] (6.20)

The asymptotics for the quasi-momenta at \( u = 0 \) are obtained by setting \( u = \pm \frac{\sqrt{\lambda}}{4\pi J} \) (which in the limit (6.20) becomes \( u \to 0 \)) i.e., by considering the asymptotics (6.15) at \( x = \pm 1 \) in the sigma-model. Then
\[
\pm q_k(u \to 0) = -\frac{\pi}{x - 1} \left( \frac{J}{\sqrt{\lambda} + m} \right) - \frac{\pi}{x + 1} \left( \frac{J}{\sqrt{\lambda} - m} \right)
= -\frac{1}{4} \left( \frac{1}{u - \frac{\sqrt{\lambda}}{4\pi J}} \left( 1 + \frac{\sqrt{\lambda}}{J m} \right) + \frac{1}{u - \frac{\sqrt{\lambda}}{4\pi J}} \left( 1 + \frac{\sqrt{\lambda}}{J m} \right) \right).
\] (6.21)

So that in the limit \( \lambda/J^2 \to 0 \) we have
\[
q_1(u) = -\frac{1}{2u}, \quad q_2(u) = -\frac{1}{2u}, \quad q_3(u) = \frac{1}{2u}, \quad q_4(u) = \frac{1}{2u}.
\] (6.22)

The combination entering the generating function of local charges in (6.9) has asymptotics
\[
-q_1(u) - q_2(u) = \frac{1}{u} + O(u^0).
\] (6.23)

again setting \( u = \pm \frac{\sqrt{\lambda}}{4\pi J} \to 0 \), which is in agreement with the gauge theory asymptotics in (6.9). Together with the fact that the defining equations for \( y_k \) and \( z_k \) are mapped into each other, this implies the agreements of the curves in the \( \mathfrak{su}(2, 2) \) subsector.

As an aside, note, that the comparison at \( u = 0 \) breaks down for the \( \mathfrak{su}(2, 2) \) spin chain with representation \( F' \). The latter asymptotics were computed in appendix B and do not yield the correct pole strengths compared to (6.22).

Finally, the Bethe equations (6.5) and (6.17), are shown to agree. This follows by rewriting (6.17) in terms of \( u \) and then expanding the RHS, i.e.
\[
\mathcal{L} = \frac{J}{x - 1} + \frac{J}{x + 1} = \frac{1}{2\pi} \left( \frac{u}{u^2} - \frac{m\lambda}{16\pi^2 J^2} \right)
= \frac{1}{2\pi} \left( \frac{1}{u} + \left[ \frac{1}{16\pi^2 u^3} + \frac{m}{4\pi u^2} \right] \frac{\lambda}{J^2} + O\left( \frac{\lambda^2}{J^4} \right) \right).
\] (6.24)
So to lowest order $\lambda/J^2$ this reproduces the Bethe equations obtained in the gauge theory. At order $\lambda/J^2$ an $m$-dependent term appears, which via the Virasoro constraint can be seen to be a non-local term. The corresponding two-loop result in the gauge theory is not known and it may in fact not be consistent to discuss the $\mathfrak{su}(2,2)$ sector at more than 1-loop. Nevertheless, truncating to $\mathfrak{sl}(2)$, we get agreement with the result of [33], where it was suggested that the $m$-dependent term may be foreboding a breakdown of the correspondence at two-loops. Very recently however it was shown by Staudacher in [45], that in the $(2)$ sector the agreement goes through up to two-loops, the main point being that as proposed in [28], the map between the spectral parameters $u$ and $x$ is modified when including higher loop effects.

7. Conclusions

The main two objectives of this note were to adapt the methods in [34] in order to construct the algebraic curve for the $\mathfrak{psu}(2,2|4)$ super-spin chain as well as to show the matching of the curve of the $\mathfrak{su}(2,2)$ subsector and the sigma-model for spinning strings on $AdS_5 \times S^1$, respectively. The latter analysis was a rather straight-forward application of the results on the $\mathfrak{su}(4)$ subsector, similar to the resemblance in the discussions for the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ cases. The main difference being the distinct asymptotics at $x = \pm 1$, which resulted in a prediction for higher loops. A comparison to the gauge theory should be preceded by an analysis along the lines of [29], checking, whether in the thermodynamic limit the $SU(2,2)$ sector is closed at higher loops.

In appendix B an alternative algebraic curve was presented, which resulted from the $\mathfrak{su}(2,2)$ with representation $\mathbf{F}'$. The asymptotics of the quasi-momenta for these are distinct from the ones of the string sigma-model. It may be of interest to understand the relevance of this spin-chain, which is obtained from the “Beast”ly Bethe ansatz in [8] and thus seems to be most suitable for describing states with large hyper-charge $B$. Identifying the corresponding string configurations may elucidate the role of $B$ on the string side.

Finally, needless to mention, to complete the discussion on the $\mathfrak{psu}(2,2|4)$ spin chain, it would be formidable to construct the curve for the full $AdS_5 \times S^5$ string, presumably relying on the constructions in [44]. Very recently an extremely interesting proposal has appeared [15], emphasizing the S-matrix aspect of integrability. It may well be interesting to study the properties of the full $\mathfrak{psu}(2,2|4)$ curve in this light.
Acknowledgements

I thank Gleb Arutjunov, Andrei Onishchenko and Henning Samtleben, and in particular Niklas Beisert for useful discussions and comments. This work is partially supported by the Deutsche Forschungsgesellschaft, the DAAD and the European RTN Program MRTN-CT-2004-503369.

Appendix A. Transfer-matrices for \( \mathfrak{gl}(m|n) \)

The properties of the transfer matrices for Lie (super-)algebras have been discussed in [46, 37], in particular the eigenvalues of the transfer matrix for Lie super-algebras based on classical Lie algebras are determined therein. Consider a \( \mathfrak{gl}(m|n) \) spin chain with representation of highest weight \((\mu_1, \ldots, \mu_m | \nu_1, \ldots, \nu_n)\). Further define

\[
a(u) = \frac{u + 1}{u}, \quad S_i(u) = \frac{u + l/2}{u - l/2}.
\]  

Then the eigenvalue \( T(u) \) of the super-trace \( t(u) \) in the fundamental representation, i.e., the auxiliary space is chosen to be \( \mathcal{V}_{aux} = m|n \), of the transfer matrix on a state specified by the Bethe roots \( v_i^{(k)} \), \( k = 1 \cdots m + n - 1 \), and is (for the distinguished Dynkin diagram)

\[
T(u) = (u + \mu_1)^L \prod_{i=1}^{J_1} a(v_i^{(1)} - u) \\
+ \sum_{k=2}^{m} (u + \mu_k)^L \prod_{i=1}^{J_{k-1}} a(u - v_i^{(k-1)}) \prod_{i=1}^{J_k} a(v_i^{(k)} - u) \\
- (u - \nu_1)^L \prod_{i=1}^{J_m} a(v_i^{(m)} - u) \prod_{i=1}^{J_{m+1}} a(u - v_i^{(m+1)}) \\
- \sum_{k=2}^{n-1} (u - \nu_k)^L \prod_{i=1}^{J_{m+k-1}} a(v_i^{(m+k-1)} - u) \prod_{i=1}^{J_{m+k}} a(u - v_i^{(m+k)}) \\
- (u - \nu_n)^L \prod_{i=1}^{J_{m+n-1}} a(v_i^{(m+n-1)} - u).
\]  

25
The Bethe equations for this algebra read

\[
(S_1(v_j^{(1)}))^L = \prod_{i=1,i\neq j}^{J_1} S_2(v_j^{(1)} - v_i^{(1)}) \prod_{i=1}^{J_2} S_1(v_j^{(1)} - v_i^{(2)}) \\
1 = \prod_{i=1}^{J_1} S_1(v_j^{(2)} - v_i^{(1)}) \prod_{i=1,i\neq j}^{N_2} S_2(v_j^{(2)} - v_i^{(2)}) \prod_{i=1}^{J_3} S_1(v_j^{(2)} - v_i^{(3)}) \\
\ldots
\]

\[
1 = \prod_{i=1}^{J_{n-1}} S_1(v_j^{(n-1)} - v_i^{(n-1)}) \prod_{i=1}^{J_{n+1}} S_1(v_j^{(n)} - v_i^{(n+1)})
\]

\[
1 = \prod_{i=1}^{J_{n+1}} S_1(v_j^{(n+1)} - v_i^{(n)}) \prod_{i=1,i\neq j}^{N_{n+1}} S_2(v_j^{(n+1)} - v_i^{(n+1)}) \prod_{i=1}^{J_{n+2}} S_1(v_j^{(n+1)} - v_i^{(n+2)}) \\
\ldots
\]

\[
1 = \prod_{i=1}^{J_{n+m-2}} S_1(v_j^{(n+m-1)} - v_i^{(n+m-2)}) \prod_{i=1,i\neq j}^{J_{n+m-1}} S_2(v_j^{(n+m-1)} - v_i^{(n+m-1)})
\]

(A.3)

The conventions are mapped to the ones used standard-wise in the applications to
gauge theory by replacing

\[
u \rightarrow -iu, \quad v_i^{(1)} \rightarrow u_i = -i(v_i^{(1)} + \frac{1}{2}).
\]

(A.4)

E.g. then the eigenvalue of the transfer matrix (A.2) reduces for \(su(2)\) to the standard form

\[
T(u)/(u + 1)^L \rightarrow \prod_k^{J_1} \frac{u - u_k - \frac{i}{2}}{u - u_k + \frac{i}{2}} + \left(\frac{u}{u + i}\right)^L \prod_k^{J_1} \frac{u - u_k + \frac{3i}{2}}{u - u_k + \frac{1}{2}}.
\]

(A.5)

Similarly, for \(su(4)\) it yields the expressions

\[
T(u)/(u + 1)^L \rightarrow \prod_k^{J_1} \frac{u - u_k^{(1)} - \frac{3i}{2}}{u - u_k^{(1)} - \frac{i}{2}} + \left(\frac{u}{u + i}\right)^L \prod_k^{J_2} \frac{u - u_k^{(1)} + \frac{i}{2}}{u - u_k^{(1)} - \frac{i}{2}} + \left(\frac{u}{u + i}\right)^L \prod_k^{J_3} \frac{u - u_k^{(2)} - \\frac{i}{2}}{u - u_k^{(2)} + \frac{i}{2}} + \left(\frac{u}{u + i}\right)^L \prod_k^{J_3} \frac{u - u_k^{(3)} + \frac{3i}{2}}{u - u_k^{(3)} + \frac{1}{2}}.
\]

(A.6)
for the maps \( u \rightarrow -iu \) as well as

\[
v_i^{(1)} \rightarrow u_i^{(1)} = -i(v_i^{(1)} - \frac{1}{2}), \quad v_i^{(2)} \rightarrow u_i^{(2)} = -i(v_i^{(2)} + 0), \quad v_i^{(3)} \rightarrow u_i^{(3)} = -i(v_i^{(3)} + \frac{1}{2}).
\]

(A.7)

Finally, for \( u(2,2|4) = \mathfrak{gl}(4|4) \) the transfer matrix eigenvalue in (A.2) result for the slightly different choice of Cartan matrix (corresponding to the distinguished ‘Beastly’ Dynkin diagram) for the \( \mathfrak{sl}(4|4) \) sub-super-algebra

\[
M' = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& -1 & 0 & 1 & \\
1 & -2 & 1 & & \\
1 & & & & -2
\end{pmatrix}
\]

(A.8)

as

\[
T'(u) = (u + \mu_1) L \frac{R_1(u - i)}{R_1(u)} + (u + \mu_2) L \frac{R_1(u + i)}{R_1(u)} \frac{R_2(u - i)}{R_2(u)} + (u + \mu_3) L \frac{R_2(u + i)}{R_2(u)} \frac{R_3(u - i)}{R_3(u)}
\]

\[
+ (u + \mu_4) L \frac{R_3(u + i)}{R_3(u)} \frac{R_4(u - i)}{R_4(u)} - (u - \mu_5) L \frac{R_4(u - i)}{R_4(u)} \frac{R_5(u + i)}{R_5(u)}
\]

\[
- (u - \mu_6) L \frac{R_5(u - i)}{R_5(u)} \frac{R_6(u + i)}{R_6(u)} - (u - \mu_7) L \frac{R_6(u - i)}{R_6(u)} \frac{R_7(u + i)}{R_7(u)}
\]

\[
- (u - \mu_8) L \frac{R_7(u - i)}{R_7(u)}.
\]

(A.9)

Finally, the quasi-momenta for this case are computed. In the thermodynamic limit \( L \rightarrow \infty \) the transfer matrix takes the from \( T = \sum_k \epsilon_k e^{ip_k}, \) with signs \( \epsilon_k \) depending on the chosen Dynkin diagram. From (4.13) the quasi-momenta can be read off as

\[
p'_1 = +\tilde{G}'_1(u),
\]

\[
p'_2 = +\tilde{G}'_2(u) - \tilde{G}'_1(u), \quad p'_3 = +\tilde{G}'_3(u) - \tilde{G}'_2(u), \quad p'_4 = +\tilde{G}'_4(u) - \tilde{G}'_3(u),
\]

\[
p'_5 = -\tilde{G}'_5(u) + \tilde{G}'_4(u), \quad p'_6 = -\tilde{G}'_6(u) + \tilde{G}'_5(u), \quad p'_7 = -\tilde{G}'_7(u) + \tilde{G}'_6(u),
\]

\[
p'_8 = -\tilde{G}'_7(u).
\]

(A.10)

The distinguished Dynkin diagram is \( e.g. \), suitable for the discussion of the \( \mathfrak{su}(2,2) \) sub-sector, as done in section 6, where also the singular resolvents \( \tilde{G}'_k \) for this sector are detailed.
Appendix B. Algebraic Curve for the QCD Spin Chain

In the main body of the paper we discussed the su(2, 2) spin chain and its associated algebraic curve, when the representation at each spin chain site has highest weight \( V_Z = (0, -1, 0) \). An alternative choice, would be to truncate the \( \mathbf{F} \) representation of \( \mathfrak{psu}(2, 2|4) \) to \( \mathfrak{su}(2, 2) \). From the SYM point of view, this sector is built on the ground state corresponding to \( \text{Tr}\mathcal{F}^L \), which has vacuum energy \( 3L \) and is distinct from the one in section 6 (which on the contrary is half-BPS and has vanishing vacuum energy). In fact it being a highly excited state in the SYM it does not make sense to consider the thermodynamic limit around it. However the discussion here may be of interest for the QCD spin chain, where precisely this representation for \( \mathfrak{su}(2, 2) \) is chosen [35], \( \text{i.e.} \), the vacuum is the anti-ferromagnetic ground state. So this appendix should be understood as constructing the algebraic curve for the QCD spin chain (rather than that for a subsector of \( \mathcal{N} = 4 \) SYM).

The reduction to this Lie sub-algebra is most transparent with the distinguished (“Beast”) choice for the Dynkin diagram of \( \mathfrak{psu}(2, 2|4) \) [3]. The highest weight of \( \mathbf{F} \) in this case is \( V_\mathbf{F} = (0, -3, 2; 0, 0, 0) \). This appendix contains a succinct discussion of the algebraic curve when the representation at each site is the sub-module of \( \mathbf{F} \), obtained by acting with \( \mathfrak{su}(2, 2) \) on \( V_\mathbf{F} \).

The relation between the excitation numbers and weights for the distinguished Dynkin diagram were obtained in [3] and the relevant parts for the present \( \mathfrak{su}(2, 2) \) chain are

\[
J_k' = \begin{pmatrix}
-1 + \frac{1}{2}(\Delta_0 - s_2) \\
-2 + \frac{1}{2}(\Delta_0 - s_1) \\
0 + \frac{1}{2}(\Delta_0 - s_1)
\end{pmatrix},
\]

where this is for states with \( B = L = 1 \), \( \text{i.e.} \), \( B \) is chosen large before taking the thermodynamic limit.

The singular resolvents and quasi-momenta are extracted from the transfer matrix in appendix A, where all the \( G_k' = 0 \) for \( k = 4 \cdots 7 \)

\[
\tilde{G}_1'(u) = G_1'(u) - \frac{1}{u},
\]

\[
\tilde{G}_2'(u) = G_2'(u) - \frac{2}{u},
\]

\[
\tilde{G}_3'(u) = G_3'(u)
\]

(\( B.2 \))
and
\[ p'_1(u) = \tilde{G}'_1(u) \]
\[ p'_2(u) = \tilde{G}'_2(u) - \tilde{G}'_1(u) \]
\[ p'_3(u) = \tilde{G}'_3(u) - \tilde{G}'_2(u) \]
\[ p'_4(u) = -\tilde{G}'_3(u) . \]  \hspace{1cm} (B.3)

The Bethe equations in the thermodynamic limit take the form as expected from the general considerations in (3.9) with the weight vector \( V_n = (0, -3, 2) \)

\[ 2 \mathcal{G}'_1(u) - \mathcal{G}'_2(u) = \mathcal{P}'_1(u) - \mathcal{P}'_2(u) - \frac{V'_j}{u} = 2\pi n_j^{(1)} - \frac{V'_j}{u} , \quad u \in C_j^{(1)} \]
\[ -\mathcal{G}'_1(u) + 2 \mathcal{G}'_2(u) - \mathcal{G}'_3(u) = \mathcal{P}'_2(u) - \mathcal{P}'_3(u) - \frac{V'_j}{u} = 2\pi n_j^{(2)} - \frac{V'_j}{u} , \quad u \in C_j^{(2)} \]  \hspace{1cm} (B.4)

\[ 2 \mathcal{G}'_3(u) - \mathcal{G}'_2(u) = \mathcal{P}'_3(u) - \mathcal{P}'_4(u) - \frac{V'_j}{u} = 2\pi n_j^{(1)} - \frac{V'_j}{u} , \quad u \in C_j^{(3)} , \]

Invoking (B.1) and (B.1) the asymptotics at \( u = \infty \) are found to be

\[ p'_1(u) = \frac{1}{u} \left( + \frac{1}{2} \Delta_0 - \frac{1}{2} s_2 \right) + O \left( \frac{1}{u^2} \right) \]
\[ p'_2(u) = \frac{1}{u} \left( + \frac{1}{2} \Delta_0 + \frac{1}{2} s_2 \right) + O \left( \frac{1}{u^2} \right) \]
\[ p'_3(u) = \frac{1}{u} \left( - \frac{1}{2} \Delta_0 - \frac{1}{2} s_1 \right) + O \left( \frac{1}{u^2} \right) \]
\[ p'_4(u) = \frac{1}{u} \left( - \frac{1}{2} \Delta_0 + \frac{1}{2} s_1 \right) + O \left( \frac{1}{u^2} \right) . \]  \hspace{1cm} (B.5)

The asymptotics at \( u = 0 \) are obtained from (2.14). At finite \( L \) the expansion is

\[ T_F(u + i) = \frac{R_2(u + \frac{3}{2} i) R_3(u - i)}{R_2(u - \frac{3}{2} i) R_3(u + i)} + O(u^L) , \]  \hspace{1cm} (B.6)

which for \( L \to \infty \) limits to

\[ T_F(u + i) \to \exp \left( i(-3\tilde{G}'_2(u) + 2\tilde{G}'_3(u)) \right) . \]  \hspace{1cm} (B.7)

Thus at \( u = 0 \) the combination of resolvents

\[ -3\tilde{G}'_2(u) + 2\tilde{G}'_3(u) = -p'_1(u) - p'_2(u) + 2p'_3(u) = \frac{6}{u} + \sum_{r=0}^{\infty} u^r Q_r . \]  \hspace{1cm} (B.8)
gives rise to the local charges $Q_r$. The asymptotics of the quasi-momenta at zero are

\[ p_1'(u) = -\frac{1}{u} + O(1), \quad p_2'(u) = -\frac{1}{u} + O(1), \quad p_3'(u) = +\frac{2}{u} + O(1), \quad p_4'(u) = 0 + O(1). \quad (B.9) \]

Note that the asymptotics at $u = \infty$ agree with the ones for the representation $Z$ (up to an overall sign), however the ones at $u = 0$ (6.10) and (B.9) are distinct.

The curve for this subsector is written in terms of $y_k = u^2 \frac{d^2 p_k}{du^2}$ as

\[ \sum_{i=0, i \neq 3}^4 P_i(u) y^i = P_4(u) \prod_{k=1}^4 (y - y_k(u)) = 0. \quad (B.10) \]

The moduli-count is identical to the one for the curve in section 6. However, due to the different asymptotics of the quasi-momenta, the curves are distinct. It may be interesting to determine the dual string configuration to the spin-chain discussed in this appendix.
References

[1] G. 't Hooft, *A Planar Diagram Theory For Strong Interactions*, Nucl. Phys. B 72, 461 (1974).

[2] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)]; hep-th/9711200.

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2, 253 (1998); hep-th/9802150.

[4] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills*, JHEP 0303, 013 (2003); hep-th/0212208.

[5] N. Beisert, *The complete one-loop dilatation operator of N = 4 super Yang-Mills theory*, Nucl. Phys. B 676, 3 (2004); hep-th/0307015.

[6] N. Beisert and M. Staudacher, *The N = 4 SYM integrable super spin chain*, Nucl. Phys. B 670, 439 (2003); hep-th/0307042.

[7] D. Berenstein, J. M. Maldacena, H. Nastase, *Strings in flat space and pp-waves from N = 4 super Yang Mills*, JHEP 0204, 013 (2002); hep-th/0202021.

[8] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, Nucl. Phys. B 636, 99 (2002); hep-th/0204051.

[9] S. Frolov and A. A. Tseytlin, *Semiclassical quantization of rotating superstring in AdS$_5 \times S^5$, JHEP 0206, 007 (2002); hep-th/0204226.

[10] S. Frolov and A. A. Tseytlin, *Multi-spin string solutions in AdS$_5 \times S^5$, Nucl. Phys. B 668 (2003) 77; hep-th/0304255.

[11] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, JHEP 0309, 010 (2003); hep-th/0306139.

[12] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, *Spinning strings in AdS$_5 \times S^5$ and integrable systems*, Nucl. Phys. B 671, 3 (2003); hep-th/0307191.

[13] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, *Precision spectroscopy of AdS/CFT*, JHEP 0310, 037 (2003); hep-th/0308117.

[14] G. Arutyunov, J. Russo and A. A. Tseytlin, *Spinning strings in AdS$_5 \times S^5$: New integrable system relations*, Phys. Rev. D 69, 086009 (2004); hep-th/0311004.

[15] A. A. Tseytlin, *Semiclassical strings and AdS/CFT*, hep-th/0409296.

[16] A. V. Kotikov and L. N. Lipatov, *DGLAP and BFKL equations in the N = 4 supersymmetric gauge theory*, Nucl. Phys. B 661, 19 (2003), [Erratum-ibid. B 685, 405 (2004)]; hep-ph/0208220.

[17] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, *Three-loop universal anomalous dimension of the Wilson operators in N = 4 SUSY Yang-Mills model*, Phys. Lett. B 595, 521 (2004); hep-th/0404092.
[18] V. M. Braun, S. E. Derkachov and A. N. Manashov, *Integrability of three-particle evolution equations in QCD*, Phys. Rev. Lett. 81, 2020 (1998); hep-ph/9805225.

[19] A. V. Belitsky, V. M. Braun, A. S. Gorsky and G. P. Korchemsky, *Integrability in QCD and beyond*, hep-th/0407232.

[20] G. Arutyunov and M. Staudacher, *Two-loop commuting charges and the string / gauge duality*, hep-th/0403077.

[21] J. Engquist, *Higher conserved charges and integrability for spinning strings in AdS$_5 \times S^5$*, JHEP 0404, 002 (2004) hep-th/0402092.

[22] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the AdS$_5 \times S^5$ superstring*, Phys. Rev. D 69, 046002 (2004); hep-th/0305116.

[23] L. Dolan, C. R. Nappi and E. Witten, *A relation between approaches to integrability in superconformal Yang-Mills theory*, JHEP 0310, 017 (2003) hep-th/0308089.

[24] G. Arutyunov and M. Staudacher, *Matching higher conserved charges for strings and spins*, JHEP 0403, 004 (2004); hep-th/0310182.

[25] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, JHEP 0410, 016 (2004) hep-th/0406256.

[26] N. Beisert, *Spin chain for quantum strings*, hep-th/0409054.

[27] D. Serban and M. Staudacher, *Planar N = 4 gauge theory and the Inozemtsev long range spin chain*, JHEP 0406, 001 (2004); hep-th/0401057.

[28] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar N = 4 super Yang-Mills*, JHEP 0407, 075 (2004); hep-th/0405001.

[29] J. A. Minahan, *Higher loops beyond the SU(2) sector*, JHEP 0410, 053 (2004); hep-th/0405243.

[30] N. Beisert, *The dilatation operator of N = 4 super Yang-Mills theory and integrability*, hep-th/0407277.

[31] J. Lucietti, S. Schafer-Nameki and A. Sinha, *On the exact open-closed vertex in plane-wave light-cone string field theory*, Phys. Rev. D 69, 086005 (2004); hep-th/0311231.

[32] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, JHEP 0405, 024 (2004); hep-th/0402207.

[33] V. A. Kazakov and K. Zarembo, *Classical/quantum integrability in non-compact sector of AdS/CFT*, hep-th/0410105.

[34] N. Beisert, V. A. Kazakov and K. Sakai, *Algebraic curve for the SO(6) sector of AdS/CFT*, hep-th/0410253.

[35] N. Beisert, G. Ferretti, R. Heise and K. Zarembo, *One-Loop QCD Spin Chain and its Spectrum*, hep-th/0412029.

[36] E. Ogievetsky, P. Wiegmann and N. Reshetikhin, *The Principal Chiral Field In Two-Dimensions On Classical Lie Algebras: The Bethe Ansatz Solution And Factorized Theory Of Scattering*, Nucl. Phys. B 280, 45 (1987).
[37] P. P. Kulish, *Integrable Graded Magnets*, J. Sov. Math. **35**, 2648 (1986) [Zap. Nauchn. Semin. **145**, 140 (1985)].

[38] N. Y. Reshetikhin and P. B. Wiegmann, *Towards The Classification Of Completely Integrable Quantum Field Theories*, Phys. Lett. B **189**, 125 (1987).

[39] M. Takahashi, *One-Dimensional Electron Gas with Delta-Function Interaction at Finite Temperature*, Prog. Theo. Phys. **46**, 1388 (1971).

[40] M. Takahashi, *Many-body Problem of Attractive Fermions with Arbitrary Spin in One Dimension*, Prog. Theo. Phys. **44**, 899 (1970).

[41] F. H. L. Essler and V. E. Korepin, *Spectrum of Low-Lying Excitations in a Supersymmetric Extended Hubbard Model*; [cond-mat/9307019](http://arxiv.org/abs/cond-mat/9307019).

[42] K. Schoutens, *Complete solution of a supersymmetric extended Hubbard model*, Nucl. Phys. B **413**, 675 (1994).

[43] H. Saleur, *The continuum limit of sl(N/K) integrable super spin chains*, Nucl. Phys. B **578**, 552 (2000); [solv-int/9905007](http://arxiv.org/abs/solv-int/9905007).

[44] G. Arutyunov and S. Frolov, *Integrable Hamiltonian for classical strings on AdS$_5 \times S^5$*; [hep-th/0411089](http://arxiv.org/abs/hep-th/0411089).

[45] M. Staudacher, *The Factorized S-Matrix of CFT/AdS*; [hep-th/0412188](http://arxiv.org/abs/hep-th/0412188).

[46] P. P. Kulish and N. Y. Reshetikhin, *Diagonalization Of Gl(N) Invariant Transfer Matrices And Quantum N Wave System (Lee Model)*, J. Phys. A **16**, L591 (1983).