VIRTUAL PULLBACKS IN DONALDSON-THOMAS
THEORY OF CALABI-YAU 4-FOLDS

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ABSTRACT. Recently, Oh and Thomas constructed algebraic virtual cycles for moduli spaces of sheaves on Calabi-Yau 4-folds. The purpose of this paper is to provide a virtual pullback formula between these Oh-Thomas virtual cycles. We find a natural compatibility condition between 3-term symmetric obstruction theories that induces a virtual pullback formula. There are two types of applications.

Firstly, we introduce a Lefschetz principle in Donaldson-Thomas theory, which relates the tautological DT4 invariants of a Calabi-Yau 4-fold with the DT3 invariants of its divisor. As corollaries, we prove the Cao-Kool conjecture on the tautological Hilbert scheme invariants for very ample line bundles and the Cao-Kool-Monavari conjecture on the tautological DT/PT correspondence for line bundles with Calabi-Yau divisors when the tautological complexes are vector bundles.

Secondly, we present a correspondence between the Oh-Thomas virtual cycles on the moduli spaces of pairs and the moduli spaces of sheaves by combining the virtual pullback formula and a pushforward formula for virtual projective bundles. As corollaries, we prove the Cao-Maulik-Toda conjecture on the primary PT/GV correspondence for irreducible curve classes and the Cao-Toda conjecture on the primary JS/GV correspondence under the coprime condition, assuming the Cao-Maulik-Toda conjecture on the primary Katz/GV correspondence. Moreover, we also prove tautological versions of these two correspondences.

CONTENTS

Introduction 2
1. Square root virtual pullback 12
2. Functoriality 23
3. Lefschetz principle 38
4. Pairs/Sheaves correspondence 58
Appendix A. Torus localization without quasi-projectivity 71
Appendix B. Virtual pullback in K-theory 75
Appendix C. Reduction of symmetric complexes 77
References 80

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Introduction

Background. Donaldson-Thomas invariants are virtual numbers of counting sheaves, defined as the integrals of cohomology classes over the virtual cycles of moduli spaces of sheaves. For Calabi-Yau 3-folds and Fano 3-folds, such virtual cycles were constructed by Thomas [58] through 2-term perfect obstruction theories [4, 43]. The associated Donaldson-Thomas invariants have been studied intensively during the last two decades and it turned out that they have rich structures and properties (e.g., correspondence to Gromov-Witten invariants [46, 47] and rationality [51, 52], motivic property [3, 33], modularity [25, 57], etc.).

Generalizing Donaldson-Thomas theory to higher-dimensional algebraic varieties is not obvious. The standard method of constructing virtual cycles in [4, 43] does not work for higher-dimensional varieties since the natural obstruction theories on the moduli spaces of sheaves are no longer 2-term. In particular, for Calabi-Yau 4-folds, the obstruction theories are 3-term symmetric, which are never 2-term.

In the groundbreaking work [8], Borisov and Joyce constructed real virtual cycles for schemes with 3-term symmetric obstruction theories based on the Darboux theorem [9, 5] and derived differential geometry. Thus Donaldson-Thomas invariants for Calabi-Yau 4-folds can be defined via the Borisov-Joyce virtual cycles. However, computation of these DT4 invariants through the Borisov-Joyce virtual cycles is believed to be very difficult.

Recently, Oh and Thomas [49] lifted Borisov-Joyce virtual cycles to Chow groups by generalizing Cao-Leung’s algebraic approach [15]. The key idea is to localize Edidin-Graham’s square root Euler class [23] by an isotropic section via Kiem-Li’s cosection-localized Gysin map [34]. This algebraic method enables us to compute DT4 invariants in some cases.

Currently, there are three known computational tools in DT4 theory:

1. Reduction to Edidin-Graham/Behrend-Fantechi classes [15];
2. Torus localization [49];
3. Cosection localization [36].

These tools are shown to be effective when they can be applied, i.e., the moduli space is smooth/virtually smooth or has a torus action/cosection (cf. [15, 12, 16, 17, 13, 20, 19, 14]). However, since there are many examples that are not in the above cases, it is desired to develop additional tools.

Main Result: Virtual pullback formula. The purpose of this paper is to provide a new computational tool for DT4 invariants: a virtual pullback formula between Oh-Thomas virtual cycles.

Recall [45] that Manolache introduced the notion of virtual pullbacks as a relative version of Behrend-Fantechi virtual cycles [4]. More specifically, given a morphism $f: \mathcal{X} \to \mathcal{Y}$ of schemes equipped with a relative 2-term perfect obstruction theory $\phi_f: \mathbb{K}_f \to \mathbb{L}_f$, Manolache constructed a map

$$f^!: A_*(\mathcal{Y}) \to A_*(\mathcal{X}),$$
called virtual pullback, satisfying the functorial property. In particular, if
the schemes $\mathcal{X}$ and $\mathcal{Y}$ are also equipped with 2-term perfect obstruction
theories $\phi_\mathcal{X}$ and $\phi_\mathcal{Y}$ that fit into a compatibility diagram

\[
\begin{array}{ccc}
K_Y & \xrightarrow{f^*\phi_Y} & K_f \\
\downarrow & & \downarrow \\
L_Y & \xrightarrow{\phi_f} & L_f \\
\end{array}
\]

then there is a virtual pullback formula

\[
f^!|\mathcal{Y}\rangle^\text{vir}_{BF} = [\mathcal{X}\rangle^\text{vir}_{BF} \in A_*(\mathcal{X})
\]
between the associated Behrend-Fantechi virtual cycles.

The main result of this paper is a generalization of the virtual pullback
formula (0.2) to the Oh-Thomas virtual cycles:

**Theorem 0.1** (Virtual pullback formula). Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism
of quasi-projective schemes equipped with the following obstruction theories:

D1) symmetric obstruction theories $\phi_\mathcal{X} : E_\mathcal{X} \to L_\mathcal{X}$ and $\phi_\mathcal{Y} : E_\mathcal{Y} \to L_\mathcal{Y}$ of
tor-amplitude $[-2, 0]$, oriented, and isotropic (see Definition 1.10);

D2) a perfect obstruction theory $\phi_f : E_f \to L_f$ of tor-amplitude $[-1, 0]$.

Assume that there exist morphisms of distinguished triangles

\[
\begin{array}{ccc}
\mathbb{D}^\vee[2] & \xrightarrow{\alpha^\vee} & E_\mathcal{X} \\
\downarrow & & \downarrow \\
f^*E_\mathcal{Y} & \xrightarrow{\beta} & \mathbb{D} \\
\downarrow & & \downarrow \\
f^*\phi_\mathcal{Y} & \xrightarrow{\phi_f} & E_f \\
\end{array}
\]

for some perfect complex $\mathbb{D}$ and maps $\alpha, \beta, \phi_f$ such that
$\phi_\mathcal{X} = \phi_f^\vee \circ \alpha$. We further assume that the orientations of $E_\mathcal{X}$ and $E_\mathcal{Y}$ are compatible with
the isomorphism $\det(E_\mathcal{X}) \cong f^*\det(E_\mathcal{Y})$ induced by (0.3). Then we have a
virtual pullback formula

\[
f^!|\mathcal{Y}\rangle^\text{vir}_{OT} = [\mathcal{X}\rangle^\text{vir}_{OT} \in A_*(\mathcal{X})
\]
between the associated Oh-Thomas virtual cycles.

In DT4 theory, there is a general philosophy that we should use operations
that are symmetric. The compatibility diagram (0.3) is a typical example.
If we use the compatibility diagram (0.1) in the situation of Theorem 0.1,
then we will have

\[
f^*E_\mathcal{Y} \cong \text{cone}(E_\mathcal{X} \to E_f)[-1],
\]
which violates the fact that both $E_\mathcal{X}$ and $f^*E_\mathcal{Y}$ are symmetric complexes.
Thus we need to consider a symmetric operation such as

\[
f^*E_\mathcal{Y} \cong \text{cone}(\text{cone}(E_\mathcal{Y}^\vee[2] \to E_\mathcal{X}) \to E_f)[-1],
\]
which is exactly what the compatibility diagram \(0.3\) does. The above operation \(0.5\) will be called the reduction of the symmetric complex \(E_X\) by the isotropic subcomplex \(E_f\), which generalizes the reduction \(K^\perp/K\) of an orthogonal bundle \(E\) by an isotropic subbundle \(K\).

We briefly sketch how we prove Theorem \(0.1\). Overall, the proof has a similar structure to the standard arguments of functoriality of various pullbacks \(24, 37, 45, 55, 21, 35\). Specifically, in our setting, we first introduce a relative version of Oh-Thomas virtual cycles: square root virtual pullbacks for relative 3-term symmetric obstruction theories. Then the crucial step is to construct a relative 3-term symmetric obstruction theory \(\phi_h\) for the composition

\[
h : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y} \times \mathbb{A}^1 \to M_{\mathcal{Y}/\text{Spec}(\mathbb{C})}^O,
\]

where \(M_{\mathcal{Y}/\text{Spec}(\mathbb{C})}^O\) denotes the deformation space \(24, 40\), such that the relative intrinsic normal cone \(\mathcal{E}_{\mathcal{X} \times \mathbb{A}^1/M_{\mathcal{Y}/\text{Spec}(\mathbb{C})}}\) is isotropic. After constructing \(\phi_h\), a deformation argument reduces Theorem \(0.1\) to the functoriality for \(X \to Y \to C\). Further replacing the cone stack \(\mathcal{E}_Y\) by some cone \(C\) that covers \(\mathcal{E}_Y\), the blowup method in \(36\) will complete the proof.

The virtual pullback formula in Theorem \(0.1\) was motivated by the Cao-Kool conjecture \(12\) on the tautological Hilbert scheme invariants and the Cao-Maulik-Toda conjectures \(16, 17\) on the genus zero Gopakumar-Vafa type invariants. Now Theorem \(0.1\) turns out that these conjectures (under some assumptions) are corollaries of more general phenomena: Lefschetz principle and Pairs/Sheaves correspondence.

**Application I: Lefschetz principle.** Recall \(37\) that the quantum Lefschetz principle relates the Gromov-Witten invariants of an algebraic variety with the Gromov-Witten invariants of its divisor. The virtual pullback formula in Theorem \(0.1\) provides an analogous formula in Donaldson-Thomas theory.

Let \(X\) be a Calabi-Yau 4-fold, i.e., a smooth projective variety \(X\) such that \(K_X \cong \mathcal{O}_X\). The Hilbert scheme \(I_{n,\beta}(X)\) of curves \(C\) with \(\text{ch}(\mathcal{O}_C) = (0, 0, 0, \beta, n)\) carries an Oh-Thomas virtual cycle

\[
[I_{n,\beta}(X)]_{\text{vir}}^{\text{OT}} \in A_n(I_{n,\beta}(X))
\]

by \(49\), which depends on the choice of an orientation \(11\). For a line bundle \(L\) on \(X\), define the tautological complex as

\[
\mathcal{L}_{n,\beta} := R\pi_*(\mathcal{O}_Z \otimes L)
\]

where \(Z\) denotes the universal family and \(\pi : I_{n,\beta}(X) \times X \to I_{n,\beta}(X)\) denotes the projection map. Suppose that there is a smooth divisor \(D\) with \(\mathcal{O}_X(D) = L\) and let \(i : D \hookrightarrow X\) be the inclusion map. Then the Hilbert scheme \(I_{n,\beta}(D)\) of curves \(C\) on \(D\) with \(\text{ch}(i_*\mathcal{O}_C) = (0, 0, 0, \beta, n)\) carries a Behrend-Fantechi virtual cycle

\[
[I_{n,\beta}(D)]_{\text{vir}}^{\text{BF}} \in A_{-\beta, D}(I_{n,\beta}(D))
\]

by \(46, 47, 51\).
**Theorem 0.2 (Lefschetz principle).** Let $X$ be a Calabi-Yau 4-fold and $L$ be a line bundle. Let $D$ be a smooth divisor with $\mathcal{O}_X(D) = L$. Fix a curve class $\beta \in H_2(X, \mathbb{Q})$ and an integer $n \in \mathbb{Z}$. If $H^1(C, L) = 0$ for all $[C] \in I_{n, \beta}(X)$, then for any orientation on $I_{n, \beta}(X)$, there exist canonical signs $\sigma(e)$ for connected components $I_{n, \beta}(D)^e$ of $I_{n, \beta}(D)$ such that

\[
\sum_e (-1)^{\sigma(e)} (j_e)_* [I_{n, \beta}(D)^e]_{\text{vir}}^{\text{BF}} = e(\mathcal{L}_{n, \beta}) \cap [I_{n, \beta}(X)]_{\text{vir}}^{\text{OT}}
\]

where $j_e : I_{n, \beta}(D)^e \hookrightarrow I_{n, \beta}(X)$ denotes the inclusion map.

The defining equation of the divisor $D$ induces the tautological section $\tau$ of the vector bundle $\mathcal{L}_{n, \beta}$ over $I_{n, \beta}(X)$ whose zero locus is $I_{n, \beta}(D)$.

\[
\begin{array}{ccc}
\mathcal{L}_{n, \beta} & \xrightarrow{\tau} & I_{n, \beta}(D) \\
\downarrow & & \downarrow j \\
I_{n, \beta}(X) & \xrightarrow{\tau} & I_{n, \beta}(X).
\end{array}
\]

The key idea of proving Theorem 0.2 is to construct a natural 3-term symmetric obstruction theory on $I_{n, \beta}(D)$ as a non-trivial extension of the standard 2-term perfect obstruction theory on $I_{n, \beta}(D)$. Then by comparing this 3-term symmetric obstruction theory with the standard 3-term symmetric obstruction theory on $I_{n, \beta}(X)$, we can apply the virtual pullback formula in Theorem 0.1,

\[
j^! [I_{n, \beta}(X)]_{\text{vir}}^{\text{OT}} = [I_{n, \beta}(D)]_{\text{vir}}^{\text{OT}} = \sum_e (-1)^{\sigma(e)} [I_{n, \beta}(D)^e]_{\text{vir}}^{\text{BF}},
\]

which implies the Lefschetz formula (0.6).

Theorem 0.2 also holds for moduli spaces of pairs. For instance, let $P_{n, \beta}(X)$ be the moduli space of Pandharipande-Thomas stable pairs $(F, s)$ with $\text{ch}(F) = (0, 0, 0, \beta, n)$ [21, 54]. Then we have two virtual cycles

\[
[P_{n, \beta}(X)]_{\text{vir}}^{\text{OT}} \in A_n(P_{n, \beta}(X)) \quad \text{and} \quad [P_{n, \beta}(D)]_{\text{vir}}^{\text{BF}} \in A_{-\beta}D(P_{n, \beta}(X))
\]

where $P_{n, \beta}(D)$ is the moduli space of stable pairs on $D$. If $H^1(X, F) = 0$ for all $(F, s) \in P_{n, \beta}(X)$, then we have a Lefschetz formula

\[
\sum_e (-1)^{\sigma(e)} (j_e)_* [P_{n, \beta}(D)^e]_{\text{vir}}^{\text{BF}} = e(\mathcal{L}_{n, \beta}) \cap [P_{n, \beta}(X)]_{\text{vir}}^{\text{OT}}
\]

where $\mathcal{L}_{n, \beta} := R\pi_* (\mathbb{F} \otimes L)$ denotes the tautological complex, $\mathbb{F}$ denotes the universal family, and all the other notations are given analogously.

There are two immediate corollaries of Theorem 0.2. Firstly, when $\beta = 0$, the tautological complex $L^{[n]} = L_{n, 0}$ on the Hilbert scheme of points $X^{[n]} = I_{n, 0}(X)$ is always a vector bundle. Thus the Lefschetz principle computes the tautological invariants in terms of the MacMahon function.
Corollary 0.3 (Tautological Hilbert scheme invariants). Let $X$ be a Calabi-Yau 4-fold and $L$ be a line bundle. If there is a smooth connected divisor $D$ such that $\mathcal{O}_X(D) = L$, then there exists a choice of orientations such that
\[
\sum_{n \geq 0} \int_{[X^{[n]}]} e(L^{[n]}) \cdot q^n = M(-q)^{c_3(X) c_1(L)} \]
where $M(q) = \prod_{n \geq 1} (1 - q^n)^{-n}$ denotes the MacMahon function.

The above formula (0.7) was conjectured by Cao-Kool in [12, Conjecture 1.2] for any line bundle $L$. Corollary 0.3 proves that the Cao-Kool conjecture holds when $L$ has a smooth connected divisor (e.g., when $L$ is very ample).

Secondly, when $\beta \neq 0$, the Lefschetz principle for a Calabi-Yau divisor gives us a tautological DT/PT correspondence.

Corollary 0.4 (Tautological DT/PT correspondence). Let $X$ be a Calabi-Yau 4-fold and $L$ be a line bundle which has a smooth connected Calabi-Yau divisor. Let $\beta \in H_2(X, \mathbb{Q})$ be a curve class satisfying the followings:

A1) For all pure 1-dimensional closed subschemes $C$ of $X$ with $[C] = \beta$, we have $H^1(C, L) = 0$.

A2) For all $n$, the inclusion maps $I_{n, \beta}(D) \hookrightarrow I_{n, \beta}(X)$ and $P_{n, \beta}(D) \hookrightarrow P_{n, \beta}(X)$ induce injective maps between the sets of connected components.

Then there exists a choice of orientations such that
\[
\sum_{n \geq 0} \int_{[I_{n, \beta}(X)]^{vir}} e(L_{n, \beta}) \cdot q^n = \sum_{n \geq 0} \int_{[P_{n, \beta}(X)]^{vir}} e(L_{n, \beta}) \cdot q^n.
\]

The 4-fold DT/PT correspondence was first conjectured by Cao-Kool in [13, Conjecture 0.3] for primary insertions. Later, the tautological DT/PT correspondence (0.8) was conjectured by Cao-Kool-Monavari in [14, Conjecture 0.13] for any $X$, $L$, and $\beta$. Corollary 0.4 shows that the Cao-Kool-Monavari conjecture holds when $L$ has a smooth Calabi-Yau divisor and the tautological complexes are vector bundles.

It was considered to be difficult to check the tautological DT/PT correspondence (0.8) for compact Calabi-Yau 4-folds. The main reason is that the reduction to Behrend-Fantechi method does not work for the DT moduli spaces $I_{n, \beta}(X)$, even in the special geometries with irreducible curve classes due to free-roaming points. In Example 3.5, we will present some simple examples that satisfy the assumptions in Corollary 0.4. This provides an evidence for the Cao-Kool-Monavari conjecture.

Application II: Pairs/Sheaves correspondence. In many cases, maps between moduli spaces of sheaves or complexes can be realized as virtual projective bundles. Since there is a general pushforward formula for virtual projective bundles, a virtual pullback formula for these cases is practically effective for computing invariants. We provide a correspondence between the moduli of stable pairs and the moduli of stable sheaves as an example.
Let us first briefly explain what we call a virtual projective bundle in this paper. Let \( \mathcal{X} \) be any scheme and \( \mathbb{K} \) be a 2-term perfect complex of amplitude \([0, 1]\). The virtual projective bundle associated to \( \mathbb{K} \) is a pair of a projective cone
\[
p : \mathbb{P}(\mathbb{K}) := \text{Proj} \text{Sym}^* (h^0(\mathbb{K}^\vee)) \to \mathcal{X}
\]
and a natural 2-term perfect obstruction theory
\[
\mathbb{L}^{\text{vir}}_{\mathbb{P}(\mathbb{K})/\mathcal{X}} := \text{cone}(\mathcal{O}_{\mathbb{P}(\mathbb{K})} \to p^* \mathbb{K}(1))^\vee \to \mathbb{L}_{\mathbb{P}(\mathbb{K})/\mathcal{X}}.
\]
For any cycle class \( \alpha \in A_*(\mathcal{X}) \) and a K-theory class \( \xi \in K^0(\mathcal{X}) \), we have a pushforward formula
\[
p_* (c_m(p^* \xi(1)) \cap p^! \alpha) = \sum_{0 \leq i \leq m} \binom{s - i}{m - i} \cdot c_i(\xi) \cap c_{m-i+1-r}(-\mathbb{K}) \cap \alpha
\]
where \( r \) is the rank of \( \mathbb{K} \) and \( s \) is the rank of \( \xi \).

Let \( X \) be a Calabi-Yau 4-fold. The moduli space \( P_{n,\beta}(X) \) of PT stable pairs \((F, s)\) with \( \text{ch}(F) = (0, 0, 0, \beta, n) \) and the moduli space \( M_{n,\beta}(X) \) of stable sheaves \( G \) with \( \text{ch}(G) = (0, 0, 0, \beta, n) \) have Oh-Thomas virtual cycles
\[
[P_{n,\beta}(X)]^{\text{vir}}_{\text{OT}} \in A_n (P_{n,\beta}(X)) \quad \text{and} \quad [M_{n,\beta}(X)]^{\text{vir}}_{\text{OT}} \in A_1 (M_{n,\beta}(X))
\]
by [49]. When the curve class \( \beta \) is irreducible, then all pure 1-dimensional sheaves \( G \) with \( \text{ch}_3(G) = \beta \) are stable so that \( M_{n,\beta}(X) \) is proper, and the forgetful map
\[
p : P_{n,\beta}(X) \to M_{n,\beta}(X) : (F, s) \mapsto F
\]
is well-defined.

**Theorem 0.5** (Pairs/Sheaves correspondence). Let \( X \) be a Calabi-Yau 4-fold, \( \beta \in H_2(X, \mathbb{Q}) \) be an irreducible curve class, and \( n \) be an integer. Assume that there exists a universal family \( \mathbb{G} \) on \( M_{n,\beta}(X) \). Then the forgetful map \( p : P_{n,\beta}(X) \to M_{n,\beta}(X) \) is the virtual projective bundle of the tautological complex \( R\pi_* \mathbb{G} \). Moreover, there exists a choice of orientations such that the following pullback/pushforward formulas hold:

1. (Pullback formula) We have
\[
[p^!(M_{n,\beta}(X))]^{\text{vir}}_{\text{OT}} = p^! (M_{n,\beta}(X))^{\text{vir}}_{\text{OT}}
\]
where \( p^! \) denotes the virtual pullback of the virtual projective bundle.

2. (Pushforward formula) For any vector bundle \( E \) on \( X \), we have
\[
p_* (c_{n-1}(\mathcal{E}_{n,\beta}) \cap [P_{n,\beta}(X)]^{\text{vir}}_{\text{OT}}) = \binom{N}{n-1} \cdot [M_{n,\beta}(X)]^{\text{vir}}_{\text{OT}}
\]
where \( \mathcal{E}_{n,\beta} := R\pi_* (\mathbb{F} \otimes E) \) denotes the tautological complex on \( P_{n,\beta}(X) \), \( [\mathcal{O} P_{n,\beta}(X) \times X \to \mathbb{F}] \) denotes the universal pair of \( P_{n,\beta}(X) \), and \( N := n \cdot \text{rank}(E) + \int_\beta c_1(E) \) denotes the rank of \( \mathcal{E}_{n,\beta} \).

Here both the projection maps \( P_{n,\beta}(X) \times X \to P_{n,\beta}(X) \) and \( M_{n,\beta}(X) \times X \to M_{n,\beta}(X) \) are denoted by \( \pi \).
Theorem 0.5 is a 4-fold analog of Pandharipande-Thomas’ formula [52, Theorem 4] on Calabi-Yau 3-folds. The main difference is that they used motivic techniques, while we use a virtual cycle approach. Our approach is a generalization of Cao-Maulik-Toda’s approach in [17, Proposition 2.10], where they considered the case when Oh-Thomas virtual cycles reduce to Behrend-Fantechi virtual cycles.

The correspondence of virtual cycles in Theorem 0.5 induces correspondences of the primary invariants and the tautological invariants. Recall [16, 17] that the primary invariants for a cohomology class \( \gamma \in H^4(X, \mathbb{Q}) \) are defined as
\[
P_{n,\beta}(\gamma) := \int_{[P]^{\text{vir}}} (\pi_* (\text{ch}_3(F) \cup \gamma))^n, \quad M_{n,\beta}(\gamma) := \int_{[M]^{\text{vir}}} \pi_* (\text{ch}_3(G) \cup \gamma).
\]
On the other hand, the tautological invariants for a line bundle \( L \) (cf. [12, 19]) can be defined as
\[
P_{n,\beta}(L) := \int_{[P]^{\text{vir}}} c_n(R\pi_*(\mathbb{F} \otimes L)), \quad M_{n,\beta}(L) := \int_{[M]^{\text{vir}}} c_1(R\pi_*(\mathbb{G} \otimes L)).
\]
Here we abbreviated the virtual cycles \([P]^{\text{vir}}\) and \([M]^{\text{vir}}\) by \([P]\) and \([M]\), respectively.

**Corollary 0.6** (Primary PT/GV correspondence). Let \( X \) be a Calabi-Yau 4-fold and \( \beta \in H_2(X, \mathbb{Q}) \) be an irreducible curve class. Then there exists a choice of orientations such that
\[
P_{1,\beta}(\gamma) = M_{1,\beta}(\gamma)
\]
for any \( \gamma \in H^4(X, \mathbb{Q}) \).

In [17, Conjecture 0.1], Cao-Maulik-Toda conjectured a primary PT/GV correspondence
\[
P_{1,\beta}(\gamma) = \sum_{\beta_1 + \beta_2 = \beta} M_{1,\beta_1}(\gamma) \cdot P_{0,\beta_2}
\]
(0.9)
for any \( \beta \in H_2(X, \mathbb{Q}) \) and \( \gamma \in H^4(X, \mathbb{Q}) \). Corollary 0.6 proves that the Cao-Maulik-Toda conjecture (0.9) holds for irreducible curve classes.

**Corollary 0.7** (Tautological PT/GV correspondence). Let \( X \) be a Calabi-Yau 4-fold, \( \beta \) be an irreducible curve class, and \( n \) be an integer. Assume that there is a universal family \( G \) of \( M_{n,\beta}(X) \). Then there exists a choice of orientations such that
\[
P_{n,\beta}(L) = \begin{cases} 
-M_{n,\beta}(\mathcal{O}_X) & \text{if } n = 0 \\
M_{n,\beta}(L) - M_{n,\beta}(\mathcal{O}_X) & \text{if } n \geq 1
\end{cases}
\]
for any line bundle \( L \) with \( \int_\beta c_1(L) = 0 \).

\[^1\]Here we replaced the genus zero Gopakumar-Vafa type invariants \( n_{0,\beta}(\gamma) \) in [39] by the primary stable sheaf invariants \( M_{1,\beta}(\gamma) \) based on Cao-Maulik-Toda’s Katz/GV conjecture in [16, Conjecture 0.2].
Theorem 0.5 can be generalized to reducible curve classes as follows: Assume that $M_{n,\beta}(X)$ has no strictly semi-stable sheaves. The moduli space of Joyce-Song type stable pairs in $\mathcal{P}_{JS}^{n,\beta}(X) = \{(F, s) : F \in M_{n,\beta}(X) \text{ and } s : \mathcal{O}_X \to F \text{ non-zero}\}$ have an Oh-Thomas virtual cycle $[P_{JS}^{n,\beta}(X)]_{vir} \in A_n(P_{JS}^{n,\beta}(X))$ by [20, 49]. Now we have a well-defined forgetful map $p : P_{JS}^{n,\beta}(X) \to M_{n,\beta}(X) : (F, s) \mapsto F$ for a reducible curve class $\beta$.

**Corollary 0.8 (JS/GV correspondence).** Let $X$ be a Calabi-Yau 4-fold, $\beta \in H_2(X, \mathbb{Q})$ be a curve class, and $n$ be an integer. Assume that $\int_{\beta} c_1(\mathcal{O}_X(1))$ and $n$ are coprime. Then the forgetful map $p : P_{JS}^{n,\beta}(X) \to M_{n,\beta}(X)$ is the virtual projective bundle of the tautological complex $R\pi_*\mathcal{G}$. Furthermore, there exists a choice of orientations such that the formula

$$[P_{JS}^{n,\beta}(X)]_{OT}^{vir} = p_! [M_{n,\beta}(X)]_{OT}^{vir}$$

holds. Consequently, we have a primary JS/GV correspondence

$$P_{JS}^{n,\beta}(\gamma) = M_{1,\beta}(\gamma)$$

for any $\gamma \in H^4(X, \mathbb{Q})$, and a tautological JS/GV correspondence

$$P_{n,\beta}^{JS}(L) = \begin{cases} -M_{n,\beta}(\mathcal{O}_X) & \text{if } n = 0 \\ M_{n,\beta}(L) - M_{n,\beta}(\mathcal{O}_X) & \text{if } n \geq 1 \end{cases}$$

for any line bundle $L$ with $\int_{\beta} c_1(L) = 0$. Here the primary invariants $P_{n,\beta}^{JS}(\gamma)$ and the tautological invariants $P_{n,\beta}^{JS}(L)$ are defined analogously.

The primary JS/GV correspondence (0.10) in Corollary 0.8 was conjectured by Cao-Toda in [20, Conjecture 0.3].

**Generalizations.** We can generalize the virtual pullback formula in Theorem 0.1 to Deligne-Mumford stacks without assuming the quasi-projectivity. The essential ingredient for this generalization is the Kimura sequence for Artin stacks, which will appear in a forthcoming paper [2] with Younghan Bae.

Chang-Kiem-Li [21] discovered that Graber-Pandharipande’s torus localization formula [26] for Behrend-Fantechi virtual cycles can be deduced by Manolache’s virtual pullback formula. Analogously, Oh-Thomas’s torus localization formula [49, Theorem 7.1] can be deduced by the virtual pullback formula in Theorem 0.1. This virtual pullback approach allows us to remove the quasi-projectivity hypothesis in the torus localization formula.

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In [20, Conjecture 0.3], Cao-Toda conjectured a primary JS/GV correspondence for any $n$ and $\beta$. Corollary 0.8 proves that the Cao-Toda conjecture holds when $\int_{\beta} c_1(\mathcal{O}_X(1))$ and $n$ are coprime.
Proposition 0.9 (Torus localization). Consider a separated DM stack $\mathcal{X}$ equipped with a $T := \mathbb{G}_m$-action and a $T$-equivariant $3$-term symmetric obstruction theory. Assume that the fixed locus $\mathcal{X}^T$ has the $T$-equivariant resolution property. Then we have

$$i_* \left( \frac{\lbrack \mathcal{X}^T \rbrack \text{vir}}{\sqrt{\epsilon(N^{\text{vir}})}} \right) = [\mathcal{X}] \text{vir} \in A^*_t(\mathcal{X}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t^{\pm 1}]$$

where $i : \mathcal{X}^T \hookrightarrow \mathcal{X}$ denotes the inclusion map (see Appendix A).

We can also generalize the virtual pullback formula in Theorem 0.1 to K-theory by replacing Manolache’s virtual pullback with the K-theoretic twisted virtual pullback (cf. [48, 55]). Consequently, we also have K-theoretic versions of the Lefschetz principle and Pairs/Sheaves correspondence.

**Future works and open problems.** We will apply the virtual pullback formula in Theorem 0.1 to surface counting problems in a forthcoming paper [1] with Younghan Bae and Martijn Kool. We will provide a Lefschetz principle for surface counting moduli spaces and introduce various virtual projective bundles that the virtual pullback formula can be applied.

In the Lefschetz principle (Theorem 0.2), it is desirable to compute the signs $\sigma(e)$ in (0.6). If the signs $\sigma(e)$ all coincide, then we will have a simpler Lefschetz formula

$$j_* [I_{n,\beta}(D)]^{\text{vir}}_{BF} = \epsilon(L_{n,\beta}) \cap [I_{n,\beta}(\mathcal{X})]^{\text{vir}}_{OT}$$

without the signs. This will allow us to remove the assumption A2 in the tautological DT/PT correspondence (Corollary 0.4).

It would be interesting to know whether there is a derived algebraic geometry interpretation of the virtual pullback formula in Theorem 0.1. A naive approach is to consider a morphism $f : X \to Y$ of $(-2)$-shifted symplectic derived schemes, but this is almost never quasi-smooth. The compatibility condition (0.3) suggests that we should consider a third derived scheme $X' \subseteq X$ with a quasi-smooth map $f : X' \to Y$ such that $(X')^{\text{cl}} = (X)^{\text{cl}}$ and $f^*(\omega_Y) = \omega_X |_{X'}$. However, it is not obvious to the author what the geometric meaning of these conditions is and why the virtual pullback formula (0.4) should hold in this setting from the derived algebraic geometry perspective.

In [27], Gross-Joyce-Tanaka conjectured a powerful wall-crossing formula for DT4 invariants, as an analog of the motivic wall-crossing formula for DT3 invariants [33]. We hope that the virtual pullback formula in Theorem 0.1 can be used to prove the wall-crossing conjecture.

We expect that the virtual pullback formula in Theorem 0.1 can be generalized to the reduced virtual cycles and the cosection-localized virtual cycles introduced in [36] under some natural compatibility conditions.

**Related works.** In [18], Cao and Qu also presented Corollary 0.3 by an independent method. They developed another version of a virtual pullback formula that applies to a different setting. Basically, they considered a morphism $f : \mathcal{X} \to \mathcal{Y}$ of schemes such that $\mathcal{X}$ and $f$ are equipped with 2-term
perfect obstruction theories, and \( Y \) is equipped with a 3-term symmetric obstruction theory whose pullback to \( X \) splits by a 2-term perfect obstruction theory. Roughly speaking, their formula is an intermediate version of Manolache’s virtual pullback formula and the virtual pullback formula (Theorem 0.1) in this paper. However, Cao-Qu’s formula and the virtual pullback formula in Theorem 0.1 are not directly related.

In [6], Bojko proved the Cao-Kool conjecture [12, Conjecture 1.2] for all line bundles based on the results for very ample line bundles (Corollary 0.3) and Gross-Joyce-Tanaka’s conjectural wall-crossing formula [27].

Outline. In §1 we introduce the notion of square root virtual pullbacks as a relative version of Oh-Thomas virtual cycles. We also explore some basic properties of the isotropic condition of 3-term symmetric obstruction theories. In §2 we prove functoriality of square root virtual pullbacks, which is the relative version of the virtual pullback formula in Theorem 0.1. In §3 we prove the Lefschetz principle and its corollaries. In §4 we introduce virtual projective bundles and prove Pairs/Sheaves correspondence. In Appendix A we generalize square root virtual pullbacks to algebraic stacks and prove the torus localization formula. In Appendix B we generalize square root virtual pullbacks to K-theory. In Appendix C we prove some elementary properties of the reduction operation of symmetric complexes.

Notations and conventions.
- All schemes and algebraic stacks are assumed to be of finite type over the field of complex numbers \( \mathbb{C} \).
- For a morphism \( f : \mathcal{X} \to \mathcal{Y} \) of algebraic stacks, we denote \( \mathbb{L}_f \) the full cotangent complex [31] (cf. [51]) and denote \( \mathbb{L}_f = \tau_{\geq -1} \mathbb{L}_f \) the truncated cotangent complex (cf. [30]).
- For any algebraic stack \( \mathcal{X} \), we denote \( A_* (\mathcal{X}) \) the Chow group of Kresch [40] with \( \mathbb{Q} \) coefficients.
- For any object \( E \in D^\leq_{\text{coh}} (\mathcal{X}) \) on an algebraic stack \( \mathcal{X} \), we denote \( C(\mathcal{Y}) := \text{Spec} \mathcal{S}^* \mathcal{X} \) the associated abelian cone stack. For any coherent sheaf \( Q \), we denote \( C(Q) := \text{Spec} S^* Q \) the associated abelian cone.
- In this paper, all perfect obstruction theories are assumed to be of tor-amplitude \([-1, 0]\), and all symmetric obstruction theories are assumed to be of tor-amplitude \([-2, 0]\) and oriented.

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1. Square root virtual pullback

In this section, we construct a square root virtual pullback for a three-term symmetric obstruction theory and prove its basic properties.

1.1. Square root Euler class. In this preliminary subsection, we briefly review the square root Euler class $\sqrt{e}(E)$ of a special orthogonal bundle $E$ and its localization $\sqrt{e}(E, s)$ by an isotropic section $s$ from \cite{23, 49, 36}. Roughly speaking, the localized square root Euler classes $\sqrt{e}(E, s)$ are local models of the square root virtual pullbacks.

Let $E$ be a special orthogonal bundle of even rank $2n$ over a scheme $X$. In \cite{23}, Edidin-Graham constructed an algebraic characteristic class $\sqrt{e}(E) \in A^n(X)$, called the square root Euler class of $E$. An important property of the square root Euler class is the reduction formula

\begin{equation}
\sqrt{e}(E) = e(K) \circ \sqrt{e}(K^\perp/K)
\end{equation}

for an isotropic subbundle $K$ of $E$.

Suppose the special orthogonal bundle $E$ has given an isotropic section $s \in \Gamma(X, E)$. In \cite{49}, Oh-Thomas proved that the square root Euler class $\sqrt{e}(E)$ can be localized to the zero locus $X(s)$ of the isotropic section $s$. More precisely, they constructed a bivariant class $\sqrt{e}(E, s) \in A^n_{X(s)}(X)$ satisfying $\iota_* \circ \sqrt{e}(E, s) = \sqrt{e}(E)$ for the inclusion map $\iota : X(s) \hookrightarrow X$. This was achieved by combining Edidin-Graham’s flag variety \cite{23}, Fulton-MacPherson’s deformation to the normal cone \cite{24}, and Kiem-Li’s cosection localization \cite{34}.

An alternative construction of the localized square root Euler class $\sqrt{e}(E, s)$ was introduced in \cite{36} via a blowup method. This blowup method provided some additional functorial properties of $\sqrt{e}(E, s)$. In particular, this gives us a refined version of the reduction formula in (1.1).

Proposition 1.1 (\cite{36} Lemma 4.5). Let $K$ be an isotropic subbundle of $E$ such that $s \cdot K = 0$. Let $s_1 \in \Gamma(X, K^\perp/K)$ be the induced isotropic section and let $s_2 = s|_{X(s_1)} \in \Gamma(X(s_1), K|_{X(s_1)})$ be the restriction. Then we have

\begin{equation}
\sqrt{e}(E, s) = e(K|_{X(s_1)}, s_2) \circ \sqrt{e}(K^\perp/K, s_1) \in A^n_{X(s)}(X).
\end{equation}

Roughly speaking, the formula (1.2) is a local model of the functoriality of square root virtual pullbacks.

1.2. Symmetric complex. In DT4 theory \cite{49, 8, 15}, three-term symmetric complexes play the role of two-term perfect complexes in virtual intersection theory \cite{1, 15}. In this subsection, we study basic properties of these three-term symmetric complexes and the abelian cone stacks associated to them.

Let us first fix the notion of symmetric complexes.
Definition 1.2 (Symmetric complex). A symmetric complex $E$ on a scheme $X$ consists of the following data:

1. A perfect complex $E$ of tor-amplitude $[-2, 0]$ on $X$.
2. A non-degenerate symmetric form $\theta$ on $E$, i.e., a morphism $\theta : O_X \to (E \otimes E)[-2]$ in the derived category of $X$ such that
   (a) $\theta = \sigma \circ \theta$, where $\sigma : E \otimes E \to E \otimes E$ is the transition map, and
   (b) the induced map $\iota_{\theta} : E^\vee \to E[-2]$ is an isomorphism.
3. An orientation $o$ of $E$, i.e., an isomorphism $o : O_X \to \det(E)$ of line bundles such that $\det(\iota_{\theta}) = o \circ o^\vee$.

In this paper, all symmetric complexes are assumed to be of amplitude $[-2, 0]$ and oriented, unless stated otherwise.

Proposition 1.3 (Symmetric resolution). Any symmetric complex $E$ on a quasi-projective scheme $X$ has a symmetric resolution, i.e., an isomorphism

$$[B \to E^\vee \to B^\vee] \xrightarrow{\cong} E$$

in the derived category of $X$ satisfying the following properties:

1. $E$ is a special orthogonal bundle and $B$ is a vector bundle.
2. Under the isomorphism (1.3), the symmetric form is represented by the chain map

$$\begin{array}{cccc}
E^\vee & \xrightarrow{i_{\theta}} & B & \xrightarrow{d} & E & \xrightarrow{d^\vee o q} & B^\vee \\
E[-2] & \xrightarrow{i_{\theta}} & B & \xrightarrow{q \circ d} & E^\vee & \xrightarrow{d^\vee} & B^\vee
\end{array}$$

where $q : E \to E^\vee$ is the quadratic form of $E$.
3. The orientation of $E$ is given by the canonical isomorphism between the determinant line bundles $\det(E) \cong \det(E)$ induced by (1.3).

Moreover, given a map $\delta : E \to K$ from a symmetric complex $E$ to a perfect complex $K$ of tor-amplitude $[-1, 0]$, if $h^0(\delta)$ is surjective, then there exist a symmetric resolution (1.3) of $E$ and a resolution of $K$ by a complex of vector bundles such that the map $\delta$ can be represented by a degreewise surjective chain map

$$\begin{array}{cccc}
E & \xrightarrow{\delta} & B & \xrightarrow{} & E^\vee & \xrightarrow{} & B^\vee \\
K & \xrightarrow{} & 0 & \xrightarrow{} & K^\vee & \xrightarrow{} & D^\vee
\end{array}$$
of chain complexes.

Proof. We omit the proofs since the statements follow directly from the proof of [49 Proposition 4.1].
Recall that an important operation on a special orthogonal bundle $E$ is the reduction $K^\perp/K$ by an isotropic subbundle $K$. A similar operation exists for symmetric complexes. We first introduce a notion of an isotropic subcomplex of a symmetric complex.

**Definition 1.4 (Isotropic subcomplex).** Let $E$ be a symmetric complex on a scheme $X$. An isotropic subcomplex of $E$ is a perfect complex $K$ equipped with a map $\delta : E \to K$ satisfying the following properties:

1. $K$ is perfect of tor-amplitude $[-1, 0]$;
2. $h^0(\delta) : h^0(E) \to h^0(K)$ is surjective;
3. $K$ is isotropic, i.e., the induced symmetric form $\delta^* : \mathcal{O}_X \to (E \otimes E)[-2] \to (K \otimes K)[-2]$ on $K$ is zero.

We can form a reduction of a symmetric complex by an isotropic subcomplex.

**Proposition 1.5 (Reduction).** Let $X$ be a quasi-projective scheme. For any symmetric complex $E$ on $X$ and an isotropic subcomplex $K$ with $\delta : E \to K$, there exists a reduction
diagram

of $E$ by $K$, i.e., a symmetric complex $G$ on $X$, denoted by (1.4), satisfying the following properties:

1. There exists a morphism of distinguished triangles

for some $D, \alpha, \beta$, where $\alpha^\vee, \beta^\vee$ are the duals of $\alpha, \beta$ with respect to the identifications $E^\vee[2] \cong E$ and $G^\vee[2] \cong G$.

2. The orientation of $G$ is induced from the orientation of $E$ under the canonical isomorphism $\det(G) \cong \det(E)$ given by (1.5).

**Proof.** By Proposition 1.3, we can choose a symmetric resolution $E \cong [B \to E^\vee \to B^\vee]$ and a resolution $K \cong [K^\vee \to D^\vee]$ such that $\delta : E \to K$ can be represented by a surjective chain map. Then $D$ is a subbundle of $B$ and $K$ is an isotropic subbundle of $E$. Thus we can define the reduction as

$G = \left[ (B/D) \to (K^\perp/K) \to (B/D)^\vee \right]$, which fits into the diagram (1.5). The distinguished triangles in the diagram (1.5) give us isomorphisms

$\det(E) \cong \det(D) \otimes \det(K^\vee) \cong \det(G) \otimes \det(K) \otimes \det(K^\vee) \cong \det(G)$

between the determinant line bundles. It completes the proof. \qed
Remark 1.6. In Appendix [C] we will prove that the reduction in Proposition [1.5] is unique as a symmetric complex.

Recall that any object $E$ in the derived category of a scheme $\mathcal{X}$, concentrated in non-positive degrees, defines an abelian cone stack by [4, Proposition 2.4]. In other words, there is a contravariant functor

$$\mathcal{C} : D_{\text{coh}}^{\leq 0}(\mathcal{X}) \to \{\text{abelian cone stacks over } \mathcal{X}\} : E \mapsto h^1/h^0((\tau_{\geq -1}E)'_\mathcal{X}).$$

We will also denote $\mathcal{C}(E)$ by $\mathcal{C}_E$.

For a symmetric complex $E$, the associated abelian cone stack $\mathcal{C}_E$ has a quadratic function $q_E : \mathcal{C}_E \to \mathbb{A}^1_{\mathcal{X}}$ induced from the symmetric form $\theta$ of $E$. This quadratic function $q_E$ will be used to define the isotropic condition of symmetric obstruction theories in §1.3.

**Proposition 1.7 (Quadratic function).** For each symmetric complex $E$ on a scheme $\mathcal{X}$, there exists a function $q_E : \mathcal{C}_E \to \mathbb{A}^1_{\mathcal{X}}$ on the associated abelian cone stack $\mathcal{C}_E$ satisfying the following properties:

1. If $E = E[1]$ for a special orthogonal bundle $E$, then $q_E = q_E : \mathcal{C}_E \to \mathbb{A}^1_{\mathcal{X}}$, is the quadratic function on $E = \text{Spec } S^*E'$ defined by the quadratic form $q_E \in \Gamma(\mathcal{X}, S^2E')$ of $E$.

2. For any morphism $f : \mathcal{Y} \to \mathcal{X}$ of schemes, we have $f^*q_E = q_{f^*E} : f^*\mathcal{C}_E \to \mathbb{A}^1_{\mathcal{Y}}$.

3. If $\mathcal{G}$ is the reduction of $E$ by an isotropic subcomplex $\mathcal{K}$, then the diagram

$$
\begin{array}{ccc}
\mathcal{C}_E & \xrightarrow{q_E} & \mathbb{A}^1_{\mathcal{X}} \\
\mathcal{C}_D & \xrightarrow{c_\alpha} & \mathcal{C}_E \\
\mathcal{C}_G & \xrightarrow{q_\mathcal{G}} & \mathbb{A}^1_{\mathcal{X}} \\
\end{array}
$$

commutes, where $\alpha, \beta, \mathcal{D}$ are given as in Proposition [1.3].

Moreover, the function $q_E$ is uniquely determined by the above properties. We call $q_E : \mathcal{C}_E \to \mathbb{A}^1_{\mathcal{X}}$ the quadratic function.

**Proof.** We first construct the quadratic function $q_E$ when $\mathcal{X}$ is a quasi-projective scheme. Choose a symmetric resolution $[B \to E' \to B'] \cong E$ of the symmetric complex $E$ as in Proposition [1.3]. Let $Q = \text{coker}(B \to E')$ be the cokernel and $C_Q = \text{Spec } S^*Q$ be the abelian cone associated to $Q$. Then we have $\mathcal{C}_E = [C_Q/B]$ by definition. Consider the function

$$q_E|_{C_Q} : C_Q \hookrightarrow E \to \mathbb{A}^1_{\mathcal{X}},$$

where $q_E$ is the quadratic function of the special orthogonal bundle $E$. 
We claim that the function \( q_E : C_E \rightarrow A^1_X \) in (1.7) is invariant under the action of \( B \) on \( C_Q \) so that it descends to a function

\[
q_E : C_E = [C_Q/B] \rightarrow A^1_X
\]
on the abelian cone stack \( C_E \). Indeed, it suffices to show that the two maps

\[
B \times C_Q \xrightarrow{\sigma} C_Q \xrightarrow{q_E} E \xrightarrow{q_E} A^1_X
\]
coincide, where \( \sigma \) is the action and \( p_2 \) is the projection. Equivalently, it suffices to show that the two maps

\[
\emptyset_X \xrightarrow{q_E} S^2(E^\vee) \xrightarrow{S^2(c)} S^2(Q) \xrightarrow{S^2(1,0)} S^2(Q \oplus B^\vee)
\]
coincide, where \( c : Q \rightarrow B^\vee \) is the canonical map induced by \( d^\vee : E^\vee \rightarrow B^\vee \). Note that there is a canonical decomposition

\[
S^2(Q \oplus B^\vee) = (S^2Q) \oplus (Q \otimes B^\vee) \oplus (S^2B^\vee).
\]

Then we can deduce that the two maps in (1.9) coincide since the two maps

\[
B \xrightarrow{d} E \xrightarrow{q_E} E^\vee \rightarrow Q \quad \text{and} \quad B \xrightarrow{d} E \xrightarrow{q_E} E^\vee \xrightarrow{d^\vee} B^\vee
\]
are zero. It proves the claim.

We now claim that the map \( q_E : C_E \rightarrow A^1_X \) in (1.8) is independent of the choice of the symmetric resolution. Indeed, consider two symmetric resolutions

\[
[B_1 \rightarrow E_1^\vee \rightarrow B_1^\vee] \cong E \cong [B_2 \rightarrow E_2^\vee \rightarrow B_2^\vee]
\]
of \( E \). Let \( Q_1 = \text{coker}(B_1 \rightarrow E_1^\vee) \) and \( Q_2 = \text{coker}(B_2 \rightarrow E_2^\vee) \) be the cokernels. Let

\[
q_1, q_2 : C_E \Rightarrow A^1_X
\]
be the two functions induced by the functions \( q_{E_1}|_{C_{Q_1}} : C_{Q_1} \rightarrow A^1 \) and \( q_{E_2}|_{C_{Q_2}} : C_{Q_2} \rightarrow A^1 \), respectively. We will show that \( q_1 = q_2 \). Since the statement is local on \( X \), we may assume that \( X \) is an affine scheme. Then the identity map \( \text{id} : \text{E}^\vee \rightarrow \text{E}^\vee \) can be represented by a chain map

\[
E^\vee \xrightarrow{B_2} E_2 \xrightarrow{q_2} B_2^\vee
\]

for some \( u, v, w \), where \( q_1 : E_1 \cong E_1^\vee \) and \( q_2 : E_2 \cong E_2^\vee \) are the symmetric forms. Hence, we obtain a commutative diagram

\[
C_{Q_2} \xrightarrow{r} E_2 \\
\downarrow v \\
C_{Q_1} \xrightarrow{u} E_1
\]
for a unique dotted arrow \( r : C_{Q_2} \to C_{Q_1} \). Consider a diagram

\[
\begin{array}{cccc}
E & \xrightarrow{\theta} & B_2 & \xrightarrow{d_2} E_2 & \xrightarrow{d_2^* q_2} B_2^\vee \\
\downarrow{\iota_\theta} & & \downarrow{wu} & \downarrow{q_2} & \downarrow{u^* w^\vee} \\
E[-2] & & B_2 & \xrightarrow{q_2 d_2} E_2^\vee & \xrightarrow{d_2^*} B_2^\vee
\end{array}
\]

of two chain maps representing the same map \( \iota_\theta : E^\vee \to E[-2] \) in the derived category. Since \( \mathcal{X} \) is affine, there exist maps \( h : E_2 \to B_2 \) and \( k^\vee : B_2^\vee \to E_2^\vee \) such that

\[
q_2 - v^\vee q_1 v = q_2 d_2 h + k^\vee d_2^* q_2 : E_2 \to E_2^\vee.
\]

Equivalently, if we let \( q_1 \in \Gamma(\mathcal{X}, E_1^\vee \otimes E_1^\vee) \), \( q_2 \in \Gamma(\mathcal{X}, E_2^\vee \otimes E_2^\vee) \) by abuse of notation, then we have

\[
(1.10) \quad q_2 = (v^\vee \otimes v^\vee)(q_1) + (h^\vee d_2^* \otimes 1)(q_2) + (1 \otimes k^\vee d_2^*)(q_2)
\]

as global sections in \( \Gamma(\mathcal{X}, E_2^\vee \otimes E_2^\vee) \). If we compose (1.10) with the canonical map \( E_2^\vee \otimes E_2^\vee \to Q_2 \otimes Q_2 \), then we obtain a commutative diagram

\[
\begin{array}{cccc}
C_{Q_2} & \xrightarrow{q_{E_2}} & A_1^1 \mathcal{X} \\
\downarrow{r} & & \downarrow{} \\
C_{Q_1} & \xrightarrow{q_{E_1}} & A_1^1 \mathcal{X}
\end{array}
\]

since the composition

\[
\mathcal{O}_\mathcal{X} \xrightarrow{q_2} E_2^\vee \otimes E_2^\vee \to B_2^\vee \otimes Q_2
\]

is zero. Since the projection map \( C_{Q_2} \to \mathcal{C}_E = [C_{Q_2}/B_2] \) is smooth and surjective, we have \( q_1 = q_2 \). It proves the claim.

We have shown that there exists a well-defined quadratic function \( q_{E_2} : \mathcal{C}_E \to A_1^1 \mathcal{X} \) for a symmetric complex \( E \) on a quasi-projective scheme \( \mathcal{X} \). This can be generalized to an arbitrary scheme \( \mathcal{X} \) since we can always construct the quadratic function locally, and then glue them globally. The result in the previous paragraph assures that this is possible. Moreover, (1) and (2) follow directly from the construction.

We will now prove (1.6) for an isotropic subcomplex \( \mathbb{K} \) of \( E \). By Proposition 1.3, we can choose a symmetric resolution \( [B \to E^\vee \to B^\vee] \cong E \) and a resolution \( [0 \to K^\vee \to D^\vee] \cong \mathbb{K} \) such that \( \delta : E \to \mathbb{K} \) can be represented by a surjective chain map. Then we have induced resolutions

\[
\mathbb{D} \cong [(B/D) \to (K^\perp)^\vee \to B^\vee] \quad \text{and} \quad \mathbb{G} \cong [(B/D) \to (K^\perp/K)^\vee \to (B/D)^\vee]
\]

where the second one is a symmetric resolution. Let \( Q = \text{coker}(B \to E^\vee) \) as before, and let

\[
R = \text{coker}((B/D) \to (K^\perp)^\vee) \quad \text{and} \quad S = \text{coker}((B/D) \to (K^\perp/K)^\vee)
\]
be the cokernels. We claim that the two squares

\[
\begin{array}{ccc}
K^\perp & \longrightarrow & E \\
\downarrow & & \downarrow q_E \\
K^\perp/K & \longrightarrow & A_1^1 \\
C_R & \longrightarrow & C_Q \\
\downarrow q_E & & \downarrow q_E \\
C_S & \longrightarrow & A_1^1
\end{array}
\]

commute. Indeed, the left square in (1.11) commutes since the quadratic form on the reduction \(K^\perp/K\) is induced from the quadratic form of \(E\). The right square in (1.11) commutes since it is a restriction of the left square in (1.11). Since the projection map \(C_R \to \mathcal{C}_R = [C_R/B]\) is smooth and surjective, the right square in (1.11) implies (1.6).

Note that for any symmetric resolution \([B \to E \to B^\vee]\) \(\cong \mathbb{E}\), the symmetric complex \(\mathbb{E}\) is the reduction of \(E[1]\) by \(B[1]\). The uniqueness of the quadratic function \(q_{\mathbb{E}}\) follows from this observation.

□

Using the language of derived algebraic geometry, the quadratic function \(q_{\mathbb{E}} : \mathcal{C}_E \to A_1^1\) on the abelian cone stack \(\mathcal{C}_E\) of a symmetric complex \(\mathbb{E}\) has the following simple description:

**Remark 1.8.** Consider the total space \(\text{Tot}(E^\vee[1])\) of the perfect complex \(E^\vee[1]\) of tor-amplitude \([-1, 1]\), which is a derived Artin stack. Then we have a weak homotopy equivalence

\[
\text{Map}_{\text{dSt}_X}(\text{Tot}(E^\vee[1]), A_1^1) \cong \text{Map}_{\text{Lqc}(X)}(O_X, S^0(E[-1]))
\]

between the mapping spaces. Thus the symmetric form \(\theta \in \Gamma(X, S^2(E[-1]))\) defines a function

\[
(1.12) \quad \text{Tot}(E^\vee[1]) \to A_1^1.
\]

The quadratic function \(q_{\mathbb{E}} : \mathcal{C}_E \to A_1^1\) in Proposition 1.7 is the restriction of the function (1.12) to its classical truncation \(\text{Tot}(E^\vee[1])_{\text{cl}} = \mathcal{C}_E\).

1.3. **Symmetric obstruction theory.** To construct a square root virtual pullback for a symmetric obstruction theory, we need an additional assumption called the isotropic condition. In this subsection, we define the isotropic condition and explore its basic properties.

We first fix the definition of **symmetric obstruction theories**.

**Definition 1.9 (Symmetric obstruction theory).** A **symmetric obstruction theory** for a morphism \(f : X \to Y\) of schemes is a morphism \(\phi : E \to L_f\) in the derived category of \(X\) such that

1. \(E\) is a symmetric complex in the sense of Definition 1.2,
2. \(\phi\) is an obstruction theory in the sense of Behrend-Fantechi [4], i.e.,
   \(h^0(\phi)\) is bijective and \(h^{-1}(\phi)\) is surjective,

where \(L_f = \tau^{\geq -1}L_f\) is the truncated cotangent complex.
Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism of schemes equipped with a symmetric obstruction theory \( \phi : \mathcal{E} \to \mathcal{L}_f \). The obstruction theory \( \phi \) induces a closed embedding

\[
\mathcal{C}_f \hookrightarrow \mathcal{C}_E
\]

of the intrinsic normal cone \( \mathcal{C}_f \) to the virtual normal cone \( \mathcal{C}_E \) by [4, Proposition 2.6]. Since \( \mathcal{E} \) is a symmetric complex, we have a quadratic function

\[
q_E : \mathcal{C}_E \to \mathbb{A}^1_X
\]
on the virtual normal cone \( \mathcal{C}_E \) by Proposition 1.7.

**Definition 1.10 (Isotropic condition).** We say that a symmetric obstruction theory \( \phi : \mathcal{E} \to \mathcal{L}_f \) satisfies the isotropic condition if the intrinsic normal cone \( \mathcal{C}_f \) is isotropic in the virtual normal cone \( \mathcal{C}_E \), i.e., the restriction

\[
q_E|_{\mathcal{C}_f} : \mathcal{C}_f \hookrightarrow \mathcal{C}_E \to \mathbb{A}^1_X
\]
of the quadratic function \( q_E \) on \( \mathcal{C}_E \) to \( \mathcal{C}_f \) vanishes.

We now compare the definition of the isotropic condition in Definition 1.10 and the isotropic condition of Oh-Thomas in [49, Proposition 4.3].

**Lemma 1.11 (Comparison of isotropic conditions).** Assume that there exists a symmetric resolution \( \mathcal{E} \cong [B \to E^\vee \to B^\vee] \). Let \( \mathcal{F} = [E^\vee \to B^\vee] \) be the stupid truncation. Consider a fiber diagram

\[
\begin{array}{ccc}
C & \longrightarrow & C_Q \\
\downarrow^p & & \downarrow \\
\mathcal{C}_f & \longrightarrow & \mathcal{C}_E \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\mathcal{E} & \longrightarrow & \mathcal{E} \rightarrow \mathcal{E}/B \\
\end{array}
\]

where \( Q = \text{coker}(B \to E^\vee) \), and the bottom horizontal arrows are the closed embeddings induced by the maps

\[
\mathcal{F} = [0 \to E^\vee \to B^\vee] \hookrightarrow \mathcal{E} = [B \to E^\vee \to B^\vee] \hookrightarrow \mathcal{L}_f.
\]

Then \( \phi \) satisfies the isotropic condition if and only if the subcone \( C \) of \( E \) is isotropic.

**Proof.** From the commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & E^\vee[1] \longrightarrow B^\vee[1] \\
\downarrow & & \downarrow \\
\mathcal{E} & \longrightarrow & \mathcal{F}^\vee[2] \longrightarrow B[1]
\end{array}
\]

we deduce that \( \mathcal{E} \) is the reduction of \( E[1] \) by \( B[1] \). Hence by Proposition 1.7(3), we obtain a commutative diagram

\[
\begin{array}{ccc}
C_Q & \longrightarrow & C \\
\downarrow & & \downarrow \\
\mathcal{C}_E & \longrightarrow & \mathcal{E} \rightarrow \mathbb{A}^1_X
\end{array}
\]
where $C_{F^\vee[2]} = C_Q$. From the diagram (1.13), we obtain

$$p^*(q_E|_{C_f}) = q_E|C.$$  

Since the projection map $p : C \rightarrow C_f$ is smooth and surjective, $q_E|_{C_f}$ vanishes if and only if $q_E|C$ vanishes. It completes the proof. \hfill $\square$

A more refined version of the above lemma is the following criterion:

**Proposition 1.12** (Criterion for isotropic condition). Consider a factorization

$$
\begin{array}{ccc}
\tilde{f} & \rightarrow & Y \\
\downarrow & & \downarrow \\
X & \rightarrow & \tilde{Y}
\end{array}
$$

of $f$ by a closed immersion $\tilde{f}$ and a smooth morphism $\tilde{f}$. Assume that there is a symmetric resolution (1.3) of $E$ such that $\phi : E \rightarrow \mathcal{L}_f$ is represented by a chain map

$$
\begin{array}{ccc}
E & \rightarrow & B^{qod} \\
\downarrow & & \downarrow \\
\mathcal{L}_f & \rightarrow & E^\vee \rightarrow B^\vee
\end{array}
$$

where $\mathcal{I} = I_{\tilde{f}}$ is the ideal sheaf. Then $\phi$ satisfies the isotropic condition if and only if the composition

$$C_{X/\tilde{Y}} \hookrightarrow N_{X/\tilde{Y}} \xrightarrow{C(\phi)} E \xrightarrow{q_E} \mathbb{A}^1$$

vanishes.

**Proof.** As in Lemma 1.11, we have a commutative diagram

$$
\begin{array}{ccc}
C_f & \rightarrow & C \\
\downarrow & & \downarrow \\
\mathcal{C}_f & \rightarrow & \mathcal{C}_E \rightarrow [E/B].
\end{array}
$$

Since $E$ is the reduction of $E[1]$ by $B[1]$, we have

$$r^*(q_E|_{C_f}) = q_E|C_f$$

by Proposition 1.7(3). Since the projection map $r : C_f \rightarrow \mathcal{C}_f$ is smooth and surjective, the two vanishing conditions are equivalent. \hfill $\square$
1.4. **Square root virtual pullback.** Let \( f : X \to Y \) be a morphism of quasi-projective schemes equipped with a symmetric obstruction theory \( \phi : E \to L_f \) satisfying the isotropic condition. By Proposition 1.3, we can choose a symmetric resolution \([B \to E^\vee \to B^\vee] \cong E\). The stupid truncation \([E^\vee \to B^\vee] = F\) gives us a fiber diagram

\[
\begin{array}{ccc}
C & \longrightarrow & C_Q \\
\downarrow & & \downarrow \\
\mathcal{E}_f & \longrightarrow & [E/B]
\end{array}
\]

where \( C \) is an isotropic subcone of \( E \) by Lemma 1.11. Let \( \tau \in \Gamma(C, E|_C) \) be the tautological section. Then the zero section \( 0 : X \hookrightarrow C \) is the zero locus of the tautological section \( \tau \).

**Definition 1.13** (Square root virtual pullback). The **square root virtual pullback** is defined as the composition:

\[
\sqrt{f^!} : A_*(Y) \xrightarrow{\text{sp}_f} A_*(\mathcal{E}_f) \xrightarrow{\text{p}^*} A_{*+\text{rank}(B)}(C) \xrightarrow{\sqrt{e}(E|_C, \tau)} A_{*+\frac{1}{2}\text{rank} E}(X),
\]

where \( \text{sp}_f : A_*(Y) \to A_*(\mathcal{E}_f) \) denotes the specialization map \([44, 45]\), and \( \sqrt{e}(E|_C, \tau) \) denotes the localized square root Euler class \([49]\).

**Lemma 1.14** (cf. \[49\]). **The square root virtual pullback is independent of the choice of the symmetric resolution.**

**Proof.** The proof is analogous to that in \([49\] pp. 28-31). We just need to replace the fundamental classes of cones in \([49\] with the specialization maps. The functorial properties in \([36\] Lemma 4.4\] and \([36\] Corollary 4.7\] assures that this is possible. We omit the details. □

Consider a fiber diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow & \text{g'} & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

of quasi-projective schemes. Let \( a_g : \mathcal{E}_{f'} \hookrightarrow g'^*\mathcal{E}_f \) denote the canonical closed embedding \([44, \text{Proposition 2.26}]\). The composition

\[
(1.14) \quad \phi' : g'^*E \xrightarrow{g'^*\phi} g'^*L_f \to L_{f'}
\]

is also a symmetric obstruction theory for \( f' : X' \to Y' \), and satisfies the isotropic condition by Proposition 1.7(2). Thus we have an induced square root virtual pullback

\[
\sqrt{(f')^!} : A_*(Y') \to A_*(X').
\]

By abuse of notation, we also denote \( \sqrt{(f')^!} \) by \( \sqrt{f^!} \).
Proposition 1.15 (Bivariance). The square root virtual pullback is a bivariant class, i.e.,

1. If \( g : Y' \to Y \) is projective, \( \sqrt{f^*} \circ g_* = g'_* \circ \sqrt{f'} \).
2. If \( g : Y' \to Y \) is flat and equi-dimensional, \( \sqrt{f^*} \circ g^* = (g')^* \circ \sqrt{f'} \).
3. If \( g : Y' \to Y \) is a regular immersion, \( \sqrt{f^*} \circ g! = g! \circ \sqrt{f'} \).

Proof. By the proof of [45, Theorem 4.1] and [45, Theorem 4.3] (cf. [35, Lemma 4.10]), the map

\[ A_*(Y') \xrightarrow{sp_{Y'}} A_*(E_{Y'}) \xrightarrow{(a_g)_*} A_*(E_f|_{X'}) \]

commutes with projective pushforwards, flat pullbacks, and Gysin pullbacks of regular immersions. Since the localized square root Euler class \( \sqrt{e(E|_{C'}, \tau)} \) is a bivariant class by [36, Lemma 4.4], the square root virtual pullback \( \sqrt{f^*} = \sqrt{e(E|_{C'}, \tau)} \circ p^* \circ sp_f \) is also a bivariant class.

Definition 1.16 (Oh-Thomas virtual cycle). Let \( X \) be a quasi-projective scheme equipped with a symmetric obstruction theory satisfying the isotropic condition. The Oh-Thomas virtual cycle is defined as

\[ [X]^{vir} := \sqrt{p}[\text{Spec}(\mathbb{C})] \in A_*(X) \]

where \( p : X \to \text{Spec}(\mathbb{C}) \) denotes the projection map.

Example 1.17 (-2-shifted symplectic derived scheme). Let \( X \) be a derived scheme equipped with a -2-shifted symplectic form and an orientation. Then the classical truncation \( X = X_{cl} \) carries a symmetric obstruction theory \( \mathbb{L}_X|_X \to \mathbb{L}_X \to L_X \). The Darboux theorem [9] assures that this obstruction theory satisfies the isotropic condition. Indeed, this can be shown directly from MacPherson’s graph construction [24, Remark 5.1.1] and the criterion in Proposition 1.12. Thus \( X \) carries an Oh-Thomas virtual cycle \( [X]^{vir} \in A_*(X) \) when \( X \) is quasi-projective.

1.5. Reduction formula. When a symmetric obstruction theory can be factored by a perfect obstruction theory, we can decompose the square root virtual pullback into Manolache’s virtual pullback [45] and Edidin-Graham’s square root Euler class [23].

Proposition 1.18 (Reduction formula). Let \( f : X \to Y \) be a morphism of quasi-projective schemes equipped with a symmetric obstruction theory \( \phi : E \to L_f \). Let \( K \) be an isotropic subcomplex of \( E \) with respect to \( \delta : E \to K \) such that \( h^0(\delta) \) is bijective and \( h^{-1}(\delta) \) is surjective. Assume that there is a map \( \psi : K \to L_f \) such that \( \phi = \psi \circ \delta \). Then the reduction of \( E \) by \( K \) is \( G[1] \) for some special orthogonal bundle \( G \), \( \phi \) satisfies the isotropic condition, \( \psi \) is a perfect obstruction theory, and we have a reduction formula

\[ \sqrt{f^*_\phi} = \sqrt{e(G)} \circ f^!_{\psi} \]

where \( \sqrt{f^*_\phi} \) denotes the square root virtual pullback for \( \phi : E \to L_f \) and \( f^!_{\psi} \) denotes the virtual pullback for \( \psi : K \to L_f \).
Proof. By Proposition 1.3, we can choose a symmetric resolution $[B \to E^\vee \to B^\vee] \cong \mathbb{E}$ and a resolution $[0 \to K^\vee \to D^\vee] \cong \mathbb{K}$ such that the map $\delta : \mathbb{E} \to \mathbb{K}$ is represented by a surjective chain map. Since $h^0(\delta)$ is bijective and $h^{-1}(\delta)$ is surjective, the map $E^\vee \to B^\vee \times D^\vee K^\vee$ is surjective. Replacing $[K^\vee \to D^\vee] \cong [B^\vee \to B^\vee \times D^\vee K^\vee \to B^\vee]$, we may assume that $D = B$.

Let $G = K^\perp / K$ be the reduction of $E$ by $K$. Then the reduction of $E$ by $K$ is $G[1]$. Form a fiber diagram

\[
\begin{array}{ccc}
C & \to & K \\
\downarrow p & & \downarrow C_K \\
\mathcal{C}_f & \to & \mathcal{C}_E
\end{array}
\]

where $\mathbb{F} = [E^\vee \to B^\vee]$ is the stupid truncation of $[B \to E^\vee \to B^\vee]$. Since $K$ is an isotropic subbundle of $E$, the subcone $C$ is also isotropic. By Lemma 1.11, the symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_f$ satisfies the isotropic condition. Recall that

\[
f^1_\psi = e(K|_C, \tau) \circ p^* \circ \text{sp}_f \quad \text{and} \quad \sqrt{f^1_\phi} = \sqrt{e}(E|_C, \tau) \circ p^* \circ \text{sp}_f
\]

where $\text{sp}_f : A_\ast(Y) \to A_\ast(\mathcal{C}_f)$ denotes the specialization map and $\tau \in \Gamma(C, K|_C)$ denotes the tautological section. The formula (1.2) in Proposition 1.1 completes the proof. \qed

Corollary 1.19 (Local complete intersection). Assume that $f : \mathcal{X} \to \mathcal{Y}$ is a local complete intersection morphism equipped with a symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_f$. Then the reduction of $\mathbb{E}_f$ by $\mathbb{L}_f$ is $G[1]$ for some special orthogonal bundle $G$ over $\mathcal{X}$, the symmetric obstruction theory $\phi_f$ is isotropic, and we have $\sqrt{f^1} = \sqrt{e}(G) \circ f^1$.

2. Functoriality

In this section, we prove functoriality of square root virtual pullbacks.

2.1. Main result. Consider a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\downarrow g & & \downarrow \text{go}_f \\
\mathcal{Z} & & 
\end{array}
\]

of quasi-projective schemes. The canonical distinguished triangle of cotangent complexes

\[
f^\ast \mathbb{L}_g \to \mathbb{L}_{g \circ f} \to \mathbb{L}_f \to
\]

induces a distinguished triangle

\[
\tau^\geq -1 f^\ast \mathbb{L}_g \xrightarrow{a} \mathbb{L}_{g \circ f} \xrightarrow{b} \mathbb{L}^\prime_f 
\]

where $\mathbb{L}_g := \tau^\geq -1 \mathbb{L}_g$ and $\mathbb{L}_{g \circ f} := \tau^\geq -1 \mathbb{L}_{g \circ f}$ are the truncated cotangent complexes, and $\mathbb{L}^\prime_f$ is the cone of $a$. Let $r : \mathbb{L}^\prime_f \to \tau^\geq -1 \mathbb{L}^\prime_f \cong \mathbb{L}_f$ denote the canonical map.
Definition 2.1 (Compatibility condition). Given a commutative diagram (2.1), we say that a triple \((\phi_f, \phi_g, \phi_{gof})\) of symmetric obstruction theories \(\phi_g : E_g \to L_g\), \(\phi_{gof} : E_{gof} \to L_{gof}\), and a perfect obstruction theory \(\phi_f : E_f \to L_f\) is compatible if

1. there exist two morphisms of distinguished triangles

\[
\begin{array}{ccc}
D & \xrightarrow{\alpha} & E_{gof} \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\tau^{-1}f^*L_g & \xrightarrow{a} & L_{gof} \\
\end{array}
\]

such that \(\phi_{gof} = \phi'_{gof} \circ \alpha\) and \(\phi_f = r \circ \phi'\); and

2. the orientation of \(E_{gof}\) is given by the orientation of \(E_g\) via the canonical isomorphism \(\text{det}(E_{gof}) \cong \text{det}(f^*E_g)\) induced by (2.3).

The compatibility condition in Definition 2.1 is slightly more general than that in Theorem 0.1 since the truncated cotangent complexes are used instead of the full cotangent complexes. This generality will be needed in the applications in §3 and §4.

Theorem 2.2 (Functoriality). Consider a commutative diagram (2.1) of quasi-projective schemes equipped with a compatible triple \((\phi_f, \phi_g, \phi_{gof})\) of obstruction theories. Assume that \(\phi_g\) and \(\phi_{gof}\) satisfy the isotropic condition in Definition 1.10. Then we have

\[
\sqrt{(g \circ f)^!} = f^! \circ \sqrt{g^!}.
\]

In particular, if \(\mathcal{Z} = \text{Spec}(\mathbb{C})\), then we have a virtual pullback formula

\[
[\mathcal{X}]_{\text{vir}} = f^![\mathcal{Y}]_{\text{vir}}
\]

between the Oh-Thomas virtual cycles.

The isotropic condition for \(\phi_{gof}\) is redundant in Theorem 2.2.

Lemma 2.3. Given a compatible triple \((\phi_f, \phi_g, \phi_{gof})\) of obstruction theories for (2.1), if \(\phi_g\) satisfies the isotropic condition, then so does \(\phi_{gof}\).

Proof. By Proposition 1.7(3), the diagram

\[
\begin{array}{ccc}
\mathcal{C}_{gof} & \xrightarrow{} & \mathcal{C}(D) \\
\downarrow{f^*E_g} & & \downarrow{q(E_{gof})} \\
\mathcal{A}_\mathcal{X} & \xrightarrow{} & \mathcal{A}_\mathcal{Y}
\end{array}
\]

commutes. By Proposition 1.7(2), we have \(q(f^*E_g) = f^*q(E_g)\). Hence the isotropic condition for \(\phi_g\) implies the isotropic condition for \(\phi_{gof}\). \(\square\)
The rest of this section is devoted to the proof of Theorem 2.2. Basically, the structure of the proof is similar to the standard arguments of functoriality in [24, 37, 45, 21, 55, 35]. The additional ingredients for Theorem 2.2 are the followings:

1. Blowup method introduced in [36] (cf. [34]);
2. Reduction operation of symmetric complexes in Proposition 1.5;
3. Criterion for isotropic condition in Proposition 1.12.

2.2. Special case via blowup. We first prove Theorem 2.2 when \( g \) is a closed immersion using the blowup method as in [36, Lemma 4.5].

**Lemma 2.4.** Theorem 2.2 holds if \( g : Y \to Z \) is a closed immersion.

**Proof.** Let \( \tilde{Z} = \text{Bl}_Y Z \) be the blowup of \( Z \) along \( Y \). Form fiber diagrams

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{g}} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Z
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{g}} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Z
\end{array}
\end{array}
\]

By the blowup sequence [21, Example 1.8.1], we have a surjective map

\[(p_*, g_*) : A_*(\tilde{Z}) \oplus A_*(Y) \to A_*(Z).\]

Since the both sides of (2.4) commute with projective pushforwards by Proposition 1.15, and the compatibility condition in Definition 2.1 is stable under the base change of \( Z \), it suffice to prove (2.4) for

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{g}} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Z
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{g}} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Z
\end{array}
\]

Since the reduction formula in Proposition 1.18 proves the latter case, it remains to prove the former case. After replacing \( \tilde{X}, \tilde{Y}, \tilde{Z} \) by \( X, Y, Z \), we may assume that the closed immersion \( g : Y \to Z \) is a local complete intersection morphism.

Since \( g : Y \to Z \) is a regular immersion, the symmetric obstruction theory \( \phi_g \) can be expressed as

\[\phi_g : E_g = G^{\vee}[1] \to L_g = L_g = N^{\vee}[1].\]

for some special orthogonal bundle \( G \) over \( Y \), where \( N = N_{Y/Z} \) is the normal bundle of \( Y \) in \( Z \). Since \( \phi_g \) satisfies the isotropic condition, \( N \) is an isotropic subbundle of \( G \).

Form a morphism of distinguished triangles

\[
\begin{array}{ccc}
f^*L_g & \xrightarrow{\eta} & E_f \\
\downarrow \psi & & \downarrow \phi'_f \\
f^*L_g & \xrightarrow{\phi'_f} & L'_f
\end{array}
\]

(2.5)
for some perfect complex $K$ and maps $\psi$ and $\eta$. Then $\psi : K \to L_{qof}$ is a perfect obstruction theory in the sense of Behrend-Fantechi. By Manolache’s virtual pullback formula \cite[Theorem 4.8]{manolache2008virtual}, the diagram (2.5) gives us

\[(g \circ f)^!_\psi = f^! \circ g^!,\]

where $(g \circ f)^!_\psi$ denotes the virtual pullback for $\psi$ and $g^!$ denotes the ordinary Gysin pullback.

Form a morphism of distinguished triangles

\[
\begin{array}{ccc}
\beta & \beta & \beta \\
\downarrow f^\ast \phi_g & \downarrow f^\ast \phi_g & \downarrow f^\ast \phi_g \\
\gamma & \gamma & \gamma \\
\end{array}
\]

for some map $\zeta$. Then $K$ is an isotropic subcomplex of $E_{qof} \to K$ since $\phi_g$ satisfies the isotropic condition. Note that the reduction of $E_{qof}$ by $E_f$ is $f^\ast E_g$, and the reduction of $f^\ast E_g = f^\ast G[1]$ by $f^\ast L_g = f^\ast N^\vee[1]$ is $f^\ast (N^\perp/N)[1]$. We claim that

\[(2.7) \quad [K^\vee [2] \to E_{qof} \to K] = f^\ast (N^\perp/N)[1].\]

Indeed, choose a symmetric resolution of $E_{qof}$ and resolutions of $K$, $E_f$ such that the two maps $E_{qof} \xrightarrow{\zeta \circ \alpha} K \xrightarrow{\eta} E_f$ are represented by surjective chain maps. As in the proof of Proposition \ref{prop:isotropic_subcomplex}, we may assume that the degree 0 terms of the resolutions of $E_{qof}$, $K$, $E_f$ are isomorphic since $h^0(\zeta \circ \alpha)$, $h^0(\eta)$ are bijective and $h^{-1}(\zeta \circ \alpha)$, $h^{-1}(\eta)$ are surjective. Then (2.7) follows immediately.

By the reduction formula in Proposition \ref{prop:isotropic_subcomplex}, we have

\[(2.8) \quad \sqrt{(g \circ f)^!_\psi \circ \zeta \circ \alpha} = \sqrt{e(N^\perp/N) \circ (g \circ f)^!_\psi}\]

where the left-hand side of (2.8) is the square root virtual pullback for the induced symmetric obstruction theory $\psi \circ \zeta \circ \alpha : E_{qof} \to L_{qof}$. Combining the two equations (2.6) and (2.8), we obtain

\[
\sqrt{(g \circ f)^!_\psi \circ \zeta \circ \alpha} = f^! \circ \sqrt{g^!}.
\]

since $\sqrt{g^!} = \sqrt{e(N^\perp/N) \circ g^!}$ by Corollary \ref{cor:symmetric_virtual_pullback}.

Finally, consider the two morphisms

\[
\begin{array}{ccc}
f^\ast E_g & \beta & \beta \\
\downarrow f^\ast \phi_g & \downarrow \phi^\ast \phi_g & \downarrow \phi^\ast \phi_g \\
\gamma & \gamma & \gamma \\
\end{array}
\]

Here the compatibility condition (2.5) is slightly general than in Manolache \cite{manolache2008virtual} since we are considering the truncated cotangent complexes. However, exactly the same proof works for this generalized compatibility condition.
of distinguished triangles. By Lemma 2.3, the maps
\[(1 - t) \cdot (\phi'_{gof}) + (t) \cdot (\psi \circ \zeta) \circ \alpha : E_{gof} \to L_{gof}\]
for all \(t \in \mathbb{A}^1\) give us a family of symmetric obstruction theories satisfying the isotropic condition. Since square root virtual pullbacks are bivariant classes by Proposition 1.15, we can deduce
\[\sqrt{(g \circ f)^!} = \sqrt{(g \circ f)^!_{\psi \circ \zeta \circ \alpha}}\]
by a deformation argument. It completes the proof. \(\Box\)

2.3. Cone stack case. We then consider the case when \(Z = C\) is a cone stack over \(Y\). After choosing a presentation \(C \to C\) of \(C\) by a cone \(C\), we will reduce the situation as
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\xrightarrow{k \circ f} & & \xrightarrow{k} \mathcal{C} \\
\end{array}
\]
by lifting the obstruction theories. Since the zero section \(\tilde{k} : \mathcal{Y} \to C\) is a closed embedding, we can apply Lemma 2.4 to the latter case in (2.9) and deduce Theorem 2.2 for the former case in (2.9).

Remark 2.5. Note that everything in §1 can be generalized to a morphism \(f : \mathcal{X} \to \mathcal{Y}\) from a quasi-projective scheme \(\mathcal{X}\) to an algebraic stack \(\mathcal{Y}\) in a straightforward manner. Indeed, we can define a symmetric obstruction theory, the isotropic condition, and the square root virtual pullback as in Definition 1.9, Definition 1.10, and Definition 1.13. The criterion in Proposition 1.12 holds if we further assume that \(f\) is quasi-projective. Proposition 1.15 also holds if we further assume that \(g\) is quasi-projective.

Lemma 2.6. Consider a commutative diagram
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\xrightarrow{k \circ f} & & \xrightarrow{k} \mathcal{C} \\
\end{array}
\]
of quasi-projective schemes \(\mathcal{X}\), \(\mathcal{Y}\) and a cone stack \(\mathcal{C}\) over \(\mathcal{Y}\), where \(k : \mathcal{Y} \to \mathcal{C}\) is the zero section. Consider a symmetric obstruction theory \(\phi_k : E_k \to L_k\) satisfying the isotropic condition and a perfect obstruction theory \(\phi_f : E_f \to L_f\). Note that \(L_k \circ f \cong \tau^{-1} f^* L_k \oplus L_f\) by [55, Lemma 2.10]. Then
\[\phi_{k \circ f} = \left(\begin{array}{cc}
f^* \phi_k & \xi \\
0 & \phi_f
\end{array}\right) : f^* E_k \oplus (E_f \oplus E_f^\vee[2]) \to \tau^{-1} f^* L_k \oplus L_f = L_{k \circ f}\]
is a symmetric obstruction theory satisfying the isotropic condition for any map \(\xi\) and gives us
\[\sqrt{(k \circ f)^!} = f^! \circ \sqrt{k^!}.\]
Proof. It is easy to show that \((ϕ_f, ϕ_k, ϕ_{kof})\) form a compatible triple of obstruction theories with obvious distinguished triangles. Hence \(ϕ_{kof}\) is a symmetric obstruction theory satisfying the isotropic condition by Lemma 2.3. It remains to prove the formula (2.10).

Note that replacing \(ξ\) by \(t⋅ξ\) for any \(t \in \mathbb{C}\) gives us a symmetric obstruction theory. Hence, by a deformation argument, we may assume that \(ξ = 0\).

By Proposition 1.3, we can choose a symmetric resolution \(E_k \cong [B \to E] \to \mathcal{Y}\). Let \(C = \mathcal{C} \times [E/B] E\) and consider a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\downarrow k & & \downarrow k \\
\mathcal{C} & \xrightarrow{\tilde{k}} & C
\end{array}
\]

where \(\tilde{k} : \mathcal{Y} \to C\) denotes the zero section. The inclusion \(C \subseteq E\) defines a symmetric obstruction theory \(ϕ_{\tilde{k}} : E\tilde{k} = E[1] \to L\tilde{k}\), which is isotropic by Lemma 1.11.

\[
ϕ_{kof} = (f^*ϕ_k \oplus ϕ_f \oplus 0) : f^*E\tilde{k} \oplus (E_f \oplus E_f[2]) \to \tau_{\geq -1}f^*L\tilde{k} \oplus L_f = L_{kof}.
\]

Then \((ϕ_f, ϕ_{\tilde{k}}, ϕ_{kof})\) is also a compatible triple of obstruction theories and thus \(ϕ_{kof}\) is isotropic by Lemma 2.3. Since \(\tilde{k} : \mathcal{Y} \to C\) is a closed immersion, Lemma 2.4 gives us

\[(2.11) \quad \sqrt{(k \circ f)^!} = f^! \circ \sqrt{(\tilde{k})^!}.
\]

Choose a resolution \(E_f \cong [K^\vee \to D^\vee]\) and form a fiber diagram

\[
\begin{array}{ccc}
\mathcal{C}_{kof} & \xrightarrow{f^*N_k \times N_f} & E \times [K/D] \times K^\vee \\
\downarrow r & & \downarrow r \\
\mathcal{C}_{kof} & \xrightarrow{f^*N_k \times N_f} & [E/B] \times [K/D] \times K^\vee
\end{array}
\]

where the horizontal arrows are closed embedding and the vertical arrows are smooth morphisms. Since \(M_{kof}^2 \to M_{kof}^c\) is smooth, we can easily show

\[
r^* \circ sp_{kof} = sp_{kof} \circ p^*.
\]

Therefore, we have

\[(2.12) \quad \sqrt{(k \circ f)^!} = \sqrt{e(E \oplus (K \oplus K^\vee), \tau) \circ s^* \circ r^* \circ sp_{kof}}
\]

\[= \sqrt{e(E \oplus (K \oplus K^\vee), \tau) \circ s^* \circ sp_{kof} \circ p^*} = \sqrt{(k \circ f)^!} \circ p^*\]
where $\tau \in \Gamma(C', E \oplus (K \oplus K^\vee))$ denotes the tautological section. On the other hand, we have

\begin{equation}
\sqrt{k'} = \sqrt{\mathcal{E}|_C, \tau} \circ p^* = \sqrt{(k')^\vee} \circ p^*.
\end{equation}

since $sp_k = \text{id}$ (cf. [55, Lemma 2.8]). The three equations (2.11), (2.12) and (2.13) completes the proof. □

2.4. Deformation to the normal cone. As in [24, 37, 45, 55, 35], we will use a deformation argument to replace the commutative diagram (2.1) as

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
g \circ f & \downarrow & \downarrow g \\
\mathcal{Z} & \xrightarrow{k \circ f} & \mathcal{C}_g
\end{array}
\]

where $\mathcal{C}_g$ denotes the intrinsic normal cone of $g$. Then Lemma 2.6 will complete the proof of Theorem 2.2.

Given a commutative diagram (2.1) of quasi-projective schemes, let $M_g^0$ be the deformation space of $g : \mathcal{Y} \to \mathcal{Z}$. Consider the canonical cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{g \circ f} & \mathcal{Z} \\
\downarrow i_0 & & \downarrow i_1 \\
\mathcal{X} \times \mathbb{A}^1 & \xrightarrow{h} & (M_g^0)^{\vee} \\
\downarrow k \circ f & & \downarrow \text{id} \\
\mathcal{X} & \xrightarrow{k \circ f} & \mathcal{C}_g
\end{array}
\]

where $(M_g^0)^{\vee}$ denotes the open substack of $M_g^0$ consists of the fibers of $M_g^0 \to \mathbb{P}^1$ over $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{0\}$, and $k : \mathcal{Y} \to \mathcal{C}_g$ denotes the zero section. Note that we have a canonical isomorphism $L_k = L_g$ between the truncated cotangent complexes.

**Proposition 2.7.** There exists a symmetric obstruction theory

\[\phi_h : E_h \to L_h\]

satisfying the isotropic condition such that the fibers of $\phi_h$ over $\lambda \in \mathbb{A}^1$ (see (1.14) are given as follows:

1. If $\lambda \neq 0$, then

\begin{equation}
(\phi_h)_{\lambda} = \phi_{g \circ f} : E_{g \circ f} \to L_{g \circ f}.
\end{equation}

2. If $\lambda = 0$, then

\begin{equation}
(\phi_h)_0 = \left(\begin{array}{cccc}
f^* \phi_g & \xi & 0 \\
0 & r \circ \eta & 0
\end{array}\right) : f^*E_g \oplus E_f \oplus E_f^\vee[2] \to \tau_{\lambda}^{-1} f^*L_g \oplus L_f = L_{k \circ f}
\end{equation}
for some maps $\xi$ and $\eta : E_f \to L'_f$ such that $\eta$ fits into a morphism (2.16)

$$f^*E_g \xrightarrow{\beta} D \xrightarrow{\gamma} E_f \xrightarrow{\phi_g} E_f$$

$\gamma \cdot f^*L_g \xrightarrow{a} L_{gof} \xrightarrow{\phi_{gof}} L'_f$ 

of distinguished triangles over $X$.

Let us first assume that Proposition 2.7 holds. Then Theorem 2.2 follows immediately from the following simple argument.

Proof of Theorem 2.2. By Proposition 2.7 and the bivariance of square root virtual pullbacks in Proposition 1.15, we have (2.17)

$$\sqrt{(g \circ f)^!} = \sqrt{(k \circ f)^!} \circ sp_g$$

where $sp_g : A_*(Z) \to A_*(C_g)$ denotes the specialization map. By Lemma 2.6, we have (2.18)

$$\sqrt{(k \circ f)^!} = f^! \eta$$

where $f^! \eta$ denotes the virtual pullback associated to the perfect obstruction theory $E_f \xrightarrow{\eta} L'_f \xrightarrow{\eta} L_f$. Since the maps

$$(1 - t) \cdot f_f + t \cdot \eta : E_f \to L'_f$$

for all $t \in \mathbb{C}$ fit into the diagram (2.16) as the dotted arrow, a deformation argument proves (2.19)

$$f^! = f^! \eta.$$ 

Since $\sqrt{g} = \sqrt{k^!} \circ sp_g$ by Definition 1.13, the three equations (2.17), (2.18), and (2.19) proves the functoriality (2.4).

The rest of this subsection is devoted to the proof of Proposition 2.7.

2.4.1. Review on $M_g^\alpha$ and $L_h$. We first review the basic facts on the deformation space $M_g^\alpha$ and the cotangent complex $L_h$ from [24, 37].

Consider a factorization of the diagram (2.11) as

(2.20)

$$\begin{array}{c}
\chi' \xrightarrow{\tilde{f}} \tilde{Y} \\
\downarrow f \\
\chi \\
\end{array} \xrightarrow{\tilde{g}} \tilde{Z} \xrightarrow{\tilde{f}} \tilde{Z}' \xrightarrow{\tilde{g}} Z$$

where the square is cartesian, the horizontal arrows are closed embeddings, and the vertical arrows are smooth. Let $I = I_{\tilde{Y}/\tilde{Z}}$ and $J = I_{\chi/\tilde{Z}}$ be the
ideal sheaves. Then the canonical maps between the truncated cotangent complexes induced by the distinguished triangle \((2.2)\) can be represented by a canonical right exact sequence

\[
\begin{array}{ccc}
\tau_{\geq -1} f^* L_g & \rightarrow & L_{g \circ f} \\
\| & \| & \| \\
I/IJ & \rightarrow & J/J^2 \\
\| & \| & \|
\end{array}
\rightarrow J/(J^2 + I) \rightarrow 0
\]

of chain complexes.

The factorization \((2.20)\) induces a factorization of the map \(h\) as

\[
\begin{array}{ccc}
\mathcal{X} \times \mathbb{A}^1 & \xrightarrow{\bar{h}} & (M_{\bar{g}}^0)' \\
\downarrow h & & \downarrow \bar{h} \\
(\mathcal{X} \times \mathbb{A}^1) & \rightarrow & (M_{\bar{g}}^0)'
\end{array}
\]

where \((M_{\bar{g}}^0)'\) is the open substack of \(M_{\bar{g}}^0\) consists of the fibers of \(M_{\bar{g}}^0 \rightarrow \mathbb{P}^1\) over \(\mathbb{A}^1 = \mathbb{P}^1 \setminus \{0\}\). The horizontal arrow \(\bar{h}\) in \((2.2)\) is a closed embedding and the vertical arrow \(\bar{h}\) in \((2.22)\) is a smooth morphism. By \([24, \S 5]\), the deformation space \((M_{\bar{g}}^0)'\) is

\[
(M_{\bar{g}}^0)' = \text{Spec} (\cdots \oplus I^2 T^{-2} \oplus IT^{-1} \oplus \mathcal{O}_{\overline{Z}} T^0 \oplus \mathcal{O}_{\overline{Z}} T^1 \oplus \mathcal{O}_{\overline{Z}} T^2 \oplus \cdots)
\]

over \(\overline{Z} \times \mathbb{A}^1\), and the ideal sheaf of the closed subscheme \(\mathcal{X} \times \mathbb{A}^1\) in \((M_{\bar{g}}^0)'\) is

\[
\mathcal{I} = \cdots \oplus I^2 T^{-2} \oplus IT^{-1} \oplus JT^0 \oplus JT^1 \oplus JT^2 \oplus \cdots
\]

**Lemma 2.8** (cf. \([37]\) Proposition 1)). *There is a distinguished triangle*

\[
\begin{array}{ccc}
\tau_{\geq -1} f^* L_g & \xrightarrow{(T,a)} & \tau_{\geq -1} f^* L_g \oplus L_{g \circ f} \\
\| & \| & \|
\end{array}
\rightarrow L'_h \rightarrow 0
\]

for some \(L'_h\) such that \(\tau_{\geq -1} L'_h \cong L_h\).
Moreover, the induced maps between the truncated cotangent complexes can be represented by a canonical right exact sequence

\[(2.24)\]

\[
\tau_{\geq -1} f^* L_g \rightarrow \tau_{\geq -1} f^* L_g \oplus L_{g\circ f} \rightarrow L_h
\]

\[
\frac{I}{IJ}[T] \rightarrow (\frac{I}{IJ} \oplus \frac{J}{J^2})[T] \rightarrow (\frac{T^{-1}}{T^2})[T] \rightarrow 0
\]

\[
0 \rightarrow (\Omega_{\mathcal{G}|\mathcal{X}})[T] \rightarrow (\Omega_{\mathcal{G}|\mathcal{X}} \oplus \Omega_{\mathcal{G}|\mathcal{X}})[T] \rightarrow \Omega_{\mathcal{H}|\mathcal{X} \times \mathbb{A}^1}[T] \rightarrow 0
\]

of chain complexes, where \(\frac{T}{T^2} = \frac{I}{J}T^{-1} \oplus \frac{J}{J^2}T^0 \oplus \cdots\).

**Proof.** The statements follow from the proof of [37, Proposition 1]. Indeed, the upper right exact sequence in (2.24) is given in Case 1 in the proof of [37, Proposition 1]. On the other hand, in Case 3 of the proof of [37, Proposition 1], it is explained that the lower exact sequence in (2.24) can be deduced by the conormal sequences and the result in Case 1. Finally, if we form the distinguished triangle (2.23), then we have \(\tau_{\geq -1} L_h' \cong L_h\) by the right exact sequence (2.24). \(\square\)

We will need the following technical lemma for proving the isotropic condition in Lemma 2.12.

**Lemma 2.9.** There exists a dotted arrow that fits into the diagram

\[(2.25)\]

\[
\mathcal{C}_h \rightarrow \mathcal{C}_g|\mathcal{X} \times \mathbb{A}^1
\]

\[
\mathfrak{N}_h \rightarrow \mathfrak{N}_g|\mathcal{X} \times \mathbb{A}^1
\]

where the map \(\mathfrak{N}_h \rightarrow \mathfrak{N}_g|\mathcal{X} \times \mathbb{A}^1\) of the intrinsic normal sheaves is induced by the map \(\tau_{\geq -1} f^* L_g \rightarrow L_h\) of the truncated cotangent complexes in (2.24).

**Proof.** Note that the normal cone \(C_h\) of \(\mathcal{X} \times \mathbb{A}^1\) in \((M_G^2)'\) is

\[
C_h = \text{Spec} \left( \bigoplus_{n \geq 0} \frac{I^n}{T^{n+1}} \right) \quad \text{where} \quad \frac{I^n}{T^{n+1}} = \bigoplus_{1 \leq i \leq n} \frac{I^iJ^{n-i-1}}{I^iJ^{n-1}} \cdot T^{-i} \bigoplus_{j \geq 0} \frac{J^n}{J^{n+1}} \cdot T^j.
\]

Hence the map of graded \(\mathcal{O}_\mathcal{X}[T]\)-algebras

\[(2.26)\]

\[
\bigoplus_{n \geq 0} T^{-n} : \bigoplus_{n \geq 0} \left( \frac{I^n}{I^nJ}[T] \right) \rightarrow \bigoplus_{n \geq 0} \left( \bigoplus_{1 \leq i \leq n} \frac{I^iJ^{n-i-1}}{I^iJ^{n-1}} \cdot T^{-i} \bigoplus_{j \geq 0} \frac{J^n}{J^{n+1}} \cdot T^j \right)
\]
gives us a commutative diagram
\[(2.27)\]
\[
\begin{array}{ccc}
C \sim h & \rightarrow & C \sim g \\
\downarrow & & \downarrow \\
N \sim h & \rightarrow & N \sim g
\end{array}
\]
where the (-1)-term of the map \((2.26)\) is equal to the map \((I/IJ)[T] \rightarrow I/I^2\) in \((2.24)\). The commutative diagram \((2.25)\) follows from \((2.27)\). □

2.4.2. Construction of \(\phi_h\). We now construct a symmetric obstruction theory \(\phi_h\) for \(h\) using the reduction operation in Proposition 1.5.

By Proposition 1.5, we can form a reduction
\[E_h := \left[ E^\vee f[2] \overset{(\delta^\vee,0,T)}{\rightarrow} E_{gof} \oplus (E_f \oplus E_f^\vee[2]) \overset{(\delta,T,0)}{\rightarrow} E_f \right] \]
where \(T \in \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})\) is the coordinate function. Then the fibers of the symmetric complex \(E_h\) over \(\lambda \in \mathbb{A}^1\) can be expressed as follows:
\[(E_h)_\lambda = \begin{cases} 
E_{gof} & \text{if } \lambda \neq 0 \\
 f^*E_g \oplus (E_f \oplus E_f^\vee[2]) & \text{if } \lambda = 0 
\end{cases} .
\]

Lemma 2.10. There exists a symmetric obstruction theory \(\phi_h : E_h \rightarrow L_h\) such that the fibers are given as in \((2.14)\) and \((2.15)\).

Proof. Form a morphism of distinguished triangles
\[(2.28)\]
\[
\begin{array}{ccc}
E^\vee f[2] & \overset{(T,\delta^\vee)}{\rightarrow} & E^\vee f[2] \oplus D^\vee[2] \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{(T,\delta^\vee)}{\rightarrow} & E^\vee f[2] \oplus E_{gof} \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{s_1}{\rightarrow} & B_1 \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{u}{\rightarrow} & E_f \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{s_2}{\rightarrow} & B_2 \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{\tau}{\rightarrow} & B_1[1]
\end{array}
\]
for some perfect complexes \(B_1, B_2\) and maps \(s_1, s_2, u\). Applying the octahedral axiom to \((2.28)\), we obtain a distinguished triangle
\[(2.29)\]
\[
\begin{array}{ccc}
B_1 & \overset{u}{\rightarrow} & B_2 \\
\downarrow & & \downarrow \\
E_f & \overset{u}{\rightarrow} & B_1[1]
\end{array}
\]
such that \(v \circ s_2 = (0,\delta)\).

We claim that there is a commutative diagram
\[
\begin{array}{ccc}
E^\vee f[2] \oplus D^\vee[2] & \overset{1 \oplus \alpha^\vee}{\rightarrow} & E^\vee f[2] \oplus E_{gof} \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{s_1}{\rightarrow} & E_f \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{s_2}{\rightarrow} & E_f \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{(0,\phi_g\beta^\vee)}{\rightarrow} & E_f \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{(0,\phi_g\beta^\vee)}{\rightarrow} & E_f \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{(0,\phi_g\beta^\vee)}{\rightarrow} & E_f \\
\downarrow & & \downarrow \\
E^\vee f[2] & \overset{(0,\phi_g\beta^\vee)}{\rightarrow} & E_f
\end{array}
\]
induced by (2.28). Indeed, the distinguished triangles in (2.28) show that there exist the dotted arrows $\psi_1$ and $\psi_2$ which factor the curved arrows $(0, \phi_g \circ \beta^\vee)$ and $(0, \phi_{go}f)$. Then the commutativity of the total square implies the commutativity of the bottom square since $\text{Hom}(E_f[3], L_{go}f) = 0$.

Form a morphism of distinguished triangles

\[
\begin{array}{ccccccc}
B_1 & \xrightarrow{(T,u)} & B_1 \oplus B_2 & \xrightarrow{\psi_1} & B_1[1] \\
\downarrow \psi_1 & & \downarrow \psi_1 \oplus \psi_2 & & \downarrow \phi_h' \\
\tau_{\geq -1}f^*L_g(T,a) & \xrightarrow{(T,a)} & \tau_{\geq -1}f^*L_g \oplus L_{go}f & \xrightarrow{\phi_h'} & E_0 
\end{array}
\]

for some map $\phi_h'$, where the upper distinguished triangle is given by the distinguished triangle (2.29) (see Lemma 2.11 below) and the lower distinguished triangle is (2.23). Let $\phi_h : E_h \xrightarrow{\phi_h'} L_h' \to \tau_{\geq -1}L_h' \cong L_h$ be the composition. The diagram (2.31) shows that $h^0(\psi_1)$, $h^0(\psi_2)$ are bijective and $h^{-1}(\psi_1)$, $h^{-1}(\psi_2)$ are surjective. The long exact sequence associated to (2.31) shows that $\phi_h$ is an obstruction theory.

Clearly, the restriction of the diagram (2.31) over $\lambda \neq 0 \in A^1$ gives us the obstruction theory for $g \circ f$,

\[(\phi_h)_{\lambda} = \phi_{go}f : E_{go}f \to L_{go}f.\]

On the other hand, the restriction of the diagram (2.31) over $\lambda = 0 \in A^1$ gives us a morphism

\[
\begin{array}{ccccccc}
(E^\vee_f[2] \oplus f^*E_g) \oplus (E^\vee_f[2] \oplus B) & \xrightarrow{1 \oplus (0,\gamma)} & (E^\vee_f[2] \oplus f^*E_g) \oplus E_f & \\
\downarrow (0,\phi_g) \oplus (0,\phi_{go}f) & & \downarrow \phi'_h & & \downarrow (\phi'_h)_0 \\
\tau_{\geq -1}f^*L_g \oplus L_{go}f & \xrightarrow{1 \oplus b} & \tau_{\geq -1}f^*L_g \oplus L'_{f} & 
\end{array}
\]

of distinguished triangles. Therefore we have (2.15) for some maps $\eta : E_f \to L'_{f}$ and $\xi : E_f \to \tau_{\geq -1}f^*L_g$ such that $\eta$ fits into (2.16).

We need the following lemma to complete the proof of Lemma 2.10

**Lemma 2.11.** There is a distinguished triangle

\[
(2.32)\quad B_1 \xrightarrow{(T,u)} B_1 \oplus B_2 \to E_h \to B_1[1]
\]

associated to the short exact sequence (2.37) below.

**Proof.** We can simply deduce an isomorphism

\[
\text{cone}(B_1 \xrightarrow{(T,u)} B_1 \oplus B_2) \cong \text{cone}(B_2 \oplus E_f \xrightarrow{(v,T)} E_f)[-1] = E_h
\]

from the distinguished triangle (2.29) using the octahedral axiom twice. However, we will construct an explicit short exact sequence using resolutions
of complexes to prove the isotropic condition of \( \phi_h \) in Lemma 2.12 below. We will find a short exact sequence representing (2.29), then we will have an isomorphism
\[
\text{coker}(B_1 \xrightarrow{(T,u)} B_1 \oplus B_2) \cong \ker(B_2 \oplus E_f \xrightarrow{(v,T)} E_f) = E_h
\]
of chain complexes.

Choose a symmetric resolution and a resolution
\[
(B \rightarrow E \rightarrow B^\vee) \cong E_{g,f} \quad \text{and} \quad (K^\vee \rightarrow D^\vee) \cong E_f,
\]
respectively, such that \( \delta : E_{g,f} \rightarrow E_f \) can be represented by a surjective chain map. The resolutions (2.33) induce a symmetric resolution and a resolution
\[
(B \oplus D \rightarrow K \rightarrow (B \oplus D)^\vee) \cong f^*E_g \quad \text{and} \quad (B \oplus D \rightarrow K \rightarrow B^\vee) \cong D
\]
of \( f^*E_g \) and \( D_1 \), respectively. Consider the two subbundles
\[
D_1 = (1, T) \cdot D \subseteq B \oplus D \quad \text{and} \quad K_1 = (1, T) \cdot K \subseteq E \oplus K
\]
on \( X \times \mathbb{A}^1 \), given by the images of the embeddings \((1, T) : D \rightarrow B \oplus D\) and \((1, T) : K \rightarrow E \oplus K\). Let
\[
K_1^\perp \subseteq E \oplus (K \oplus K^\vee)
\]
be the orthogonal complement of \( K_1 = K_1 \oplus 0 \) in the special orthogonal bundle \( E \oplus (K \oplus K^\vee) \). The two perfect complexes \( B_1 \) and \( B_2 \) in (2.28) have resolutions
\[
(B_D \rightarrow K_{1,1} \rightarrow (B_D)^\vee) \cong B_1 \quad \text{and} \quad (B_D \rightarrow E_K \rightarrow B^\vee) \cong B_2,
\]
respectively, and the symmetric complex \( E_h \) has a symmetric resolution
\[
(B_D \rightarrow K_{1,1} \rightarrow (B_D)^\vee) \cong E_h.
\]
The canonical short exact sequence of chain complexes
\[
(2.37)
\]

The diagram represents the short exact sequence with the following structure:

- The horizontal arrows represent the chain maps.
- The vertical arrows are the maps induced by the resolution and symmetric resolutions.
- The sequences are exact, meaning the kernel of each arrow is the image of the previous one.

The notation \((T,u)\) and \((v,T)\) indicate the transformation maps associated with the pullback operations.
gives us the desired distinguished triangle \( \text{(2.32)} \), where the map \( \kappa \) is
\[
\left( \frac{K^\perp \oplus K}{K_1} \right) \oplus \left( \frac{E \oplus K}{K_1} \right) \xrightarrow{\kappa} \frac{K^\perp}{K_1} \subseteq \frac{E \oplus K \oplus K^\vee}{K_1}
\]
for \( x \in K^\perp, y \in K, z \in E \), and \( w \in K \).

2.4.3. Isotropic condition for \( \phi_h \). We finally prove the isotropic condition of \( \phi_h \) in Lemma 2.10 using the criterion in Proposition 1.12.

Lemma 2.12. \( \phi_h \) satisfies the isotropic condition.

Proof. We use the notations as above. More precisely, we will use the factorizations \( \text{(2.20)}, \text{(2.22)} \), the resolutions of cotangent complexes in \( \text{(2.21)}, \text{(2.24)} \), and the resolutions of perfect complexes and symmetric complexes in \( \text{(2.33)}, \text{(2.34)}, \text{(2.35)}, \text{(2.36)} \), and the short exact sequence \( \text{(2.37)} \). Since the statement is local, we may assume that all schemes in \( \text{(2.20)} \) are affine.

Since \( \mathcal{X} \) is affine, there is a chain map
\[
\begin{array}{c}
\text{(2.38)} \\
f^*E_g \\
\tau^{\geq -1}f^*L_g \\
\end{array}
\]
representing \( f^*\phi_g \). By Proposition 1.12 we have a commutative diagram
\[
\begin{array}{c}
\xymatrix{
C_g|\mathcal{X} \ar[r] & N_g|\mathcal{X} \ar[r] & C(Q) \ar[r] & K^\perp/K \\
\mathcal{E}_g|\mathcal{X} \ar[r] & \mathcal{N}_g|\mathcal{X} \ar[r]^{\mathcal{E}(\phi_g)} & \mathcal{C}(f^*E_g) \ar[r]^{\mathcal{E}(f^*E_g)} & \mathcal{E}(f^*E_g) \ar[r] & \mathcal{E}(f^*E_g) \\
\mathcal{C}(\Phi_g) \ar[r] & \mathcal{C}(\Phi_g) \ar[r] & \mathcal{C}(\Phi_g) \ar[r] & \mathcal{C}(\Phi_g) \ar[r] & \mathcal{C}(\Phi_g) \\
\end{array}
\]
where \( Q = \text{coker}(B/D \to K^\perp/K) \). Hence the composition
\[
\begin{array}{c}
\text{(2.39)} \\
C_g|\mathcal{X} \ar[r] & N_g|\mathcal{X} \ar[r] & C(Q) \ar[r] & K^\perp/K \ar[r] & K^\perp/K \\
\end{array}
\]
vanishes by the isotropic condition of \( \phi_g \).

The chain map \( \text{(2.38)} \) induces a chain map representing \( \psi_1 \)
\[
\begin{array}{c}
\text{(2.40)} \\
\mathbb{E}_1 \oplus D_1 \ar[r]^{\psi_1} & K^\perp \oplus K_1 \ar[r]^{(B/D)^\vee} & (B/D)^\vee \\
\tau^{\geq -1}f^*L_g \\
\end{array}
\]
since \( \text{Hom}(\mathbb{E}_f[3], \tau^{\geq -1}f^*L_g) = 0 \). Then the \((-1)\)-term \( \Psi_1 \) is the composition
\[
\Psi_1 : (K^\perp \oplus K)/K_1 \xrightarrow{(1,0)} K^\perp/K \xrightarrow{\Phi_g} I/IJ.
\]
Choose any chain map

\[
\begin{array}{cccccc}
\mathbb{E}_h & B \oplus D & K_1^\perp & (B \oplus D)^\vee \\
\downarrow \phi_h & \downarrow & \downarrow \Phi_h & \downarrow \\
\mathbb{L}_h & 0 & K_1 & \Omega_{\eta_h} X
\end{array}
\]

representing \( \phi_h \). By the criterion of isotropic condition in Proposition 1.12, it suffices to show that the composition

\[
C_h \hookrightarrow N_h \xrightarrow{C(\phi_h)} K_1^\perp/K_1 \xrightarrow{q_{K_1^\perp/K_1}} \mathbb{A}^1
\]

vanishes.

Consider the commutative diagram

\[
\begin{array}{ccc}
\mathbb{E}_h & \mathbb{E}_1 & \mathbb{E}_h \\
\downarrow \phi_h & \downarrow \psi_1 & \downarrow \\
\mathbb{L}_h & \mathbb{L}_h \end{array}
\]

in the derived category of \( \mathcal{X} \times \mathbb{A}^1 \), induced by \( (2.31) \). The chain maps in \( (2.37), (2.24), (2.40) \), and \( (2.41) \) represent the maps in the diagram \( (2.42) \). Since \( \mathcal{X} \times \mathbb{A}^1 \) is affine, these chain maps representing the diagram \( (2.42) \) commute up to a homotopy. This homotopy should be given by a map \( x : (B/D)^\vee \to \mathcal{I}/\mathcal{I}^2 \). Considering the \((-1)-terms\), there is a diagram

\[
\begin{array}{ccc}
K_1^\perp & K & (B/D)^\vee \\
\downarrow \Psi_1 & \downarrow 1 \oplus 1 \oplus 0 & \downarrow x \\
(I/IJ)[T] & (I/IJ)[T] & \mathcal{I}/\mathcal{I}^2
\end{array}
\]

where the diagonal composition in \( (2.43) \) is the sum of the other two compositions in \( (2.43) \).

Let \( K_2 \) be the isotropic subbundle of \( K_1^\perp/K_1 \) given by the embedding

\[
(0, 1, 0) : K \hookrightarrow K_1^\perp/K_1 \subset \frac{E \oplus K \oplus K^\vee}{K_1}.
\]

Then \( K_1^\perp \oplus K_1^\perp/K_1 \to K_1^\perp/K_1 \oplus \Phi_h, \mathcal{I}/\mathcal{I}^2 \) as subbundles of \( K_1^\perp/K_1 \) and \( K_2^\perp/K_2 \cong K_1^\perp/K_1 \) as special orthogonal bundles. The composition

\[
K_2 \hookrightarrow K_1^\perp/K_1 \xrightarrow{\Phi_h} \mathcal{I}/\mathcal{I}^2
\]
vanishes since the two maps

\[ K_2 \to K_2^\perp \xrightarrow{\Psi_1} (I/IJ)[T] \quad \text{and} \quad K_2 \to K_2^\perp \to (B/D)^\vee \]

vanish. Let

\[ \Phi'_h : K_2^\perp/K_2 \to (K_1^\perp/K_1)/K_2 \to \mathcal{I}/\mathcal{I}^2. \]

be the induced map. From the commutative diagram

\[
\begin{array}{cccccc}
C(\Phi_h) & \xrightarrow{\Phi_h} & N_\tilde{h} & \xrightarrow{\Psi_1} & K_2^\perp & \xrightarrow{q_{K_2^\perp/K_1}} \\
& & & & & \mathbb{A}^1 \\
C(\Phi'_h) & \xrightarrow{\Phi'_h} & K_1^\perp/K_1 & \xrightarrow{q_{K_1^\perp/K_1}} & \mathbb{A}^1
\end{array}
\]

it suffices to show that the composition

\[ C_{\tilde{h}} \xhookleftarrow{N_\tilde{h}} \xrightarrow{C(\Phi'_h)} K_1^\perp/K_1 \xrightarrow{q_{K_1^\perp/K_1}} \mathbb{A}^1 \]

vanishes. The diagram (2.38) induces a diagram

\[
\begin{array}{cccccc}
K_1^\perp/K_1 & \xrightarrow{\Phi_g} & (B/D)^\vee \\
\xrightarrow{\Phi_h} & & \xrightarrow{x} & \mathcal{I}/\mathcal{I}^2 \\
(I/IJ)[T] & \xrightarrow{T^{-1}} & \mathcal{I}/\mathcal{I}^2
\end{array}
\]

where the diagonal arrow is the sum of the two other compositions. Since the two compositions

\[ B/D \to K_1^\perp/K_1 \xrightarrow{\Phi_g} I/IJ \quad \text{and} \quad B/D \to K_1^\perp/K_1 \to (B/D)^\vee \]

vanishes by (2.38), it suffices to show that the composition

\[ C_{\tilde{h}} \xhookleftarrow{N_\tilde{h}} \xrightarrow{C(\Phi_g)} K_1^\perp/K_1 \xrightarrow{q_{K_1^\perp/K_1}} \mathbb{A}^1 \]

vanishes. The commutative square (2.27) in Lemma 2.9 and the vanishing of the composition (2.39) completes the proof. \[ \square \]

3. Lefschetz principle

In this section, we prove the Lefschetz principle in Donaldson-Thomas theory.

3.1. Main result. Let \( X \) be a smooth projective Calabi-Yau 4-fold. Fix a curve class \( \beta \in H_2(X, \mathbb{Q}) \) and an integer \( n \in \mathbb{Z} \). Let

\[
I_{n,\beta}(X) = \{ \text{closed subschemes } Z \text{ of } X \text{ with } \text{ch}(\Theta_Z) = (0,0,0,\beta,n) \},
\]

\[
P_{n,\beta}(X) = \{ \text{stable pairs } (F,s) \text{ on } X \text{ with } \text{ch}(F) = (0,0,0,\beta,n) \}
\]

be the Hilbert scheme of curves and the moduli space of stable pairs. In particular, when \( \beta = 0 \), we let \( I_{n,0}(X) = X^{[n]} \) be the Hilbert scheme of points and \( P_{n,0}(X) = \emptyset \).
Let $P(X)$ denote one of the above two moduli spaces. Then there is a universal family

$$\mathbb{I} = \left[ \mathcal{O}_{P(X) \times X} \rightarrow \mathbb{F} \right]$$

of pairs on $X$. When $P(X) = I_{n,\beta}(X)$ is the Hilbert scheme, $\mathbb{F}$ is the structure sheaf of the universal family. The perfect complex $\mathbb{I}$ defines an open embedding

$$P(X) \hookrightarrow \text{Perf}(X)^{\text{spl}}_{\mathcal{O}_X}$$

to the moduli space of simple perfect complexes on $X$ with fixed trivial determinant $[32, 44, 60, 56]$. Hence $P(X)$ carries a (-2)-shifted symplectic derived enhancement $[53]$ and an orientation $[11]$, which induces an Oh-Thomas virtual cycle

$$[P(X)]^{\text{vir}}_{\text{OT}} \in A_n(P(X))$$

by $[49]$. Let $L$ be an line bundle on $X$. We will study the tautological invariants

$$\int_{[P(X)]^{\text{vir}}} c_n(\mathbb{R}\pi_*(\mathbb{F} \otimes L))$$

in terms of the divisors of $L$. Here $\pi : P(X) \times X \rightarrow P(X)$ denotes the projection map.

Let $D$ be a smooth divisor of $X$ such that $\mathcal{O}_X(D) = L$. Define the moduli space $P(D)$ on $D$ as

$$P(D) = \left\{ \bigsqcup_{\beta'} I_{n,\beta'}(D) \text{ if } P(X) = I_{n,\beta}(X) \right\} \left\{ \bigsqcup_{\beta'} P_{n,\beta'}(D) \text{ if } P(X) = P_{n,\beta}(X) \right\}$$

where the disjoint union is taken over all $\beta' \in H_2(D, \mathbb{Q})$ such that $i_*\beta' = \beta$, and $i : D \hookrightarrow X$ denotes the inclusion map. Then $P(D)$ carries a Behrend-Fantechi virtual cycle

$$[P(D)]^{\text{vir}}_{\text{BF}} \in A_{-\beta \cdot D}(P(D))$$

associated to the standard perfect obstruction theory.

**Theorem 3.1 (Lefschetz principle).** Let $X$ be a Calabi-Yau 4-fold and $D$ be a smooth divisor with $\mathcal{O}_X(D) = L$. Fix a curve class $\beta' \in H_2(X, \mathbb{Q})$ and an integer $n$. Let $P(X)$ be one of the two moduli spaces $I_{n,\beta}(X)$ and $P_{n,\beta}(X)$. Assume that the tautological complex $\mathbb{R}\pi_*(\mathbb{F} \otimes L)$ is a vector bundle concentrated in degree 0. Then for any orientation on $P(X)$, there exist canonical signs $\sigma(e)$ for connected components $P(D)^e$ of $P(D)$ such that

$$\sum_e (-1)^{\sigma(e)} (j_e)_* [P(D)^e]^{\text{vir}}_{\text{BF}} = e(\mathbb{R}\pi_*(\mathbb{F} \otimes L)) \cap [P(X)]^{\text{vir}}_{\text{OT}},$$

where $j_e : P(D)^e \hookrightarrow P(X)$ denotes the inclusion map.

In principle, the signs $\sigma(e)$ are uniquely determined by natural triangles, but the author does not know how to compute them. If we have an affirmative answer to Question $[62]$ below, then we can remove the signs $\sigma(e)$ in
and have the following simpler formula
\[ j_*(P(D))_{\text{vir}}^{BF} = e(R\pi_* (\mathcal{F} \otimes L)) \cap [P(X)]_{\text{vir}}^{OT} \]
where \( j : P(D) \hookrightarrow P(X) \) denotes the inclusion map.

**Question 3.2.** Given \( X, D, P(X) \) as in Theorem 3.1, is there a choice of orientations on connected components of \( P(X) \) such that the signs \( \sigma(e) \) all coincide?

Before we prove our main theorem (Theorem 3.1) in this section, we present two immediate corollaries.

**Corollary 3.3** (Tautological Hilbert scheme invariants). Let \( X \) be a Calabi-Yau 4-fold and \( L \) be a line bundle. If there is a smooth connected divisor \( D \) such that \( \mathcal{O}_X(D) = L \), then there exists a choice of orientations such that
\[
\sum_{n \geq 0} \int_{[X^n]_{\text{vir}}} e(L^n) \cdot q^n = M(-q) \int_X c_3(X)c_1(L)
\]
where \( M(q) = \prod_{n \geq 1} (1 - q^n)^{-n} \) denotes the MacMahon function.

**Proof.** Since \( D^n \) is connected, the Lefschetz principle gives us
\[
\int_{[X^n]_{\text{vir}}} e(L^n) = \int_{[D^n]_{\text{vir}}} 1.
\]
By [41][42], the generating series of the degree zero MNOP invariants [46][47] of a smooth projective 3-fold \( D \) can be expressed as
\[
\sum_{n \geq 0} \int_{[D^n]_{\text{vir}}} 1 \cdot q^n = M(-q) \int_D c_3(T_D \otimes K_D).
\]
By an elementary argument, we can deduce
\[
\int_D c_3(T_D \otimes K_D) = \int_X c_3(T_X)c_1(L)
\]
(cf. [12] (2.5)). It completes the proof. \( \square \)

**Corollary 3.4** (Tautological DT/PT correspondence). Let \( X \) be a Calabi-Yau 4-fold and \( L \) be a line bundle. Assume that there is a smooth connected divisor \( D \) of \( X \) which is a Calabi-Yau 3-fold and \( \mathcal{O}_X(D) = L \). Let \( \beta \in H_2(X,\mathbb{Q}) \) be a curve class satisfying the following two conditions for both \( I_{n,\beta}(X) \) and \( P_{n,\beta}(X) \) and for all \( n \):
- **A1)** The tautological complex \( R\pi_* (\mathcal{F} \otimes L) \) is a vector bundle.
- **A2)** The inclusion map \( j : P(D) \hookrightarrow P(X) \) induces an injective function between the sets of connected components of \( P(D) \) and \( P(X) \).

Then there exists a choice of orientations such that
\[
\frac{\sum_{n \geq 0} \int_{I_{n,\beta}(X)^{\text{vir}}} c_n(R\pi_* (\mathcal{F} \otimes L)) \cdot q^n}{\sum_{n \geq 0} \int_{X^n} c_n(L^n) \cdot q^n} = \sum_{n \geq 0} \int_{P_{n,\beta}(X)^{\text{vir}}} c_n(R\pi_* (\mathcal{F} \otimes L)) \cdot q^n.
\]
Proof. Applying the Lefschetz principle to the three moduli space \( I_{n,\beta}(X) \), \( P_{n,\beta}(X) \), and \( I_{n,0}(X) = X[n] \), the 3-fold DT/PT correspondence \[10, 59\]
\[
\sum_{n \geq 0} \int_{[I_{n,\beta}(D)]_{\text{vir}}} 1 \cdot q^n = \sum_{n \geq 0} \int_{[P_{n,\beta}(D)]_{\text{vir}}} 1 \cdot q^n
\]
completes the proof. \( \square \)

We present an example that satisfies the assumptions in Corollary 3.4.

**Example 3.5.** Let \( X = Y \times E \) be the product of a Calabi-Yau 3-fold \( Y \) and an elliptic curve \( E \). Let \( L = \mathcal{O}_X(Y \times \{pt\}) \). Let \( \beta \in H_2(Y) \subseteq H_2(X) \) be a curve class. Assume that for any pure 1-dimensional closed subscheme \( C \) of \( Y \) with \([C] = \beta\), there is a collection of rigid smooth rational curves \( C_1, C_2, \ldots, C_r \) on \( Y \) such that
\[
C = \bigcup_{1 \leq i \leq r} C_i \quad \text{and} \quad \# \left( C_i \cap \left( \bigcup_{j < i} C_j \right) \right) \leq 1 \text{ for all } i.
\]
Then the assumptions in Corollary 3.4 are satisfied and the tautological DT/PT correspondence holds.

The rest of this section is devoted to the proof of Theorem 3.1. We briefly sketch the structure of the proof.

1. In §3.2, we prove that \( P(D) \) is the zero locus of the tautological section \( \tau \) of the tautological bundle \( R\pi_* (\mathbb{F} \otimes L) \) in \( P(X) \).
2. In §3.3, we construct a natural 3-term symmetric obstruction theory on \( P(D) \) which gives us the same virtual cycle \([P(D)]_{\text{vir}} = [P(D)]_{\text{vir}}^{BF}\) (up to signs).
3. In §3.4, we show the compatibility of the 3-term symmetric obstruction theories for \( P(X) \) and \( P(D) \).
4. In §3.5, we apply the virtual pullback formula in §2 and finish the proof of Theorem 3.1.

**3.2. Comparison of moduli spaces.** Under the notations in Theorem 3.1, define the **tautological section** as the composition
\[
\tau : \mathcal{O}_{P(X)} \xrightarrow{s} R\pi_* (\mathbb{F}) \xrightarrow{f_D} R\pi_* (\mathbb{F} \otimes L)
\]
where \( s : \mathcal{O}_{P(X) \times X} \to \mathbb{F} \) is the universal section and \( f_D \in \Gamma(X, L) \) is the defining equation of the divisor \( D \). Consider a diagram
\[
\begin{array}{ccc}
R\pi_* (\mathbb{F} \otimes L) & \xrightarrow{\tau} & P(D) \\
\downarrow & & \downarrow j \\
P(X) & \xrightarrow{P(D)} & P(X).
\end{array}
\]

**Proposition 3.6.** \( P(D) \) is the zero locus of \( \tau \) in \( P(X) \).
Proposition [3.6] is a geometric version of the Lefschetz principle in Theorem [3.1]. If we prove Proposition [3.6], then we will have a Gysin pullback

\[(3.2)\quad j^!: A_*(P(X)) \to A_*(P(D))\]
such that \(j_! \circ j^! = e(\mathbb{R}\pi_* (\mathbb{F} \otimes L))\).

**Proof of Proposition 3.6.** Let \(P(X) = I_{n,\beta}(X)\) be the Hilbert scheme. Then it is easy to show that \(P(D)\) is the zero locus of \(\tau\) (cf. [12, Proposition 2.4]). Indeed, consider a morphism \(T \to P(X)\) from a scheme \(T\) that corresponds to a closed subscheme \(Z \subseteq T \times X\). Then the pullback \(\tau|_T\) of the tautological section corresponds to the composition

\[(3.3)\quad \mathcal{O}_T \otimes L^\vee \xrightarrow{f_D} \mathcal{O}_{T \times X} \to \mathcal{O}_Z\]

under the adjunction \(\pi^* \dashv \pi_*\). Since (3.3) vanishes if and only if \(Z\) is contained in \(T \times D\), the zero locus of \(\tau\) is exactly \(P(D)\).

Now assume that \(P(X) = P_{n,\beta}(X)\) is the moduli space of stable pairs. Let \(V\) be the zero locus of \(\tau\) in \(P(X)\). Obviously, we have \(P(D) \subseteq V\) since the pullback \(\tau|_P(D)\) is zero. Conversely, we will show that \(V \subseteq P(D)\). Choose any morphism \(T \to V\) from a scheme \(T\). Then the composition \(T \to V \to P(X)\) corresponds to a family \(s : \mathcal{O}_{T \times X} \to F\) of stable pairs. Since the pullback \(\tau|_T\) of the tautological section vanishes, the composition

\[(3.4)\quad \mathcal{O}_T \otimes L^\vee \xrightarrow{f_D} \mathcal{O}_{T \times X} \xrightarrow{s} F\]

also vanishes by the adjunction. To show that the map \(T \to V\) factors through \(P(D)\), we need to show that \(F\) is scheme-theoretically supported in \(T \times D\). Equivalently, it suffices to show that the map

\[(3.4)\quad f_D \otimes 1_F : L^\vee \otimes F \to F\]

vanishes. Let \(Q = \text{coker}(s : \mathcal{O}_{T \times X} \to F)\) be the cokernel of the section \(s\) and \(G = L^\vee \cdot F \subseteq F\) be the image of the map (3.4). Since the map (3.4) vanishes away from the support of \(Q\), the reduced support of \(G\) is contained in the support of \(Q\). Let \(s_t : \mathcal{O}_X \to F_t\) be the fiber of \(s : \mathcal{O}_{T \times X} \to F\) over \(t \in T\). Then \(Q_t = \text{coker}(s_t : \mathcal{O}_X \to F_t)\) is a zero-dimensional sheaf, and \(F_t\) is a pure 1-dimensional sheaf, by the stability of the pair \((F_t, s_t)\). Hence the fiber \(G_t\) of \(G\) over \(t \in T\) is also a 0-dimensional sheaf, and thus \(\text{Hom}_X(G_t, F_t) = 0\).

By Lemma 3.7 below, we have

\[\text{Hom}_{T \times X}(G, F) = 0.\]

Hence \(G = 0\) and the map (3.4) is also zero. \(\square\)

We need the following lemma to complete the proof of Proposition 3.6.

**Lemma 3.7.** Let \(\pi : \mathcal{X} \to T\) be a smooth projective morphism of schemes. Let \(\mathcal{X}_t = \mathcal{X} \times_T \{t\}\) be the fiber over \(t \in T\). Consider coherent sheaves \(\mathcal{F}\) and \(\mathcal{G}\) on \(\mathcal{X}\). Assume that \(\mathcal{F}\) is flat over \(T\). Then

\[(3.5)\quad \text{Hom}_{\mathcal{X}_t}(\mathcal{G}_t, \mathcal{F}_t) = 0 \text{ for all } t \in T \implies \text{Hom}_X(\mathcal{G}, \mathcal{F}) = 0,\]

where \(\mathcal{G}_t\) and \(\mathcal{F}_t\) are the pullbacks of the sheaves \(\mathcal{G}\) and \(\mathcal{F}\) to \(\mathcal{X}_t\).
Proof. Since the statement is local, we may assume that $T$ is a quasi-projective scheme. Let $i_t : \mathcal{X}_t \hookrightarrow \mathcal{X}$ be the inclusion map. We claim that

$$\text{Hom}_{\mathcal{X}_t}(Li^*_t \mathcal{G}, \mathcal{F}_t) = \text{Hom}_{\mathcal{X}_t}(\mathcal{G}, \mathcal{F}_t).$$

Indeed, we have a distinguished triangle

$$\begin{align*}
R_1 & \longrightarrow Li^*_t \mathcal{G} \longrightarrow \mathcal{G}_t \longrightarrow R_1[1]
\end{align*}$$
on $\mathcal{X}_t$ such that $R_1$ is concentrated in degree $\leq -1$. Hence

$$\text{Hom}_{\mathcal{X}_t}(R_1, \mathcal{F}_t) = \text{Hom}_{\mathcal{X}_t}(R_1[1], \mathcal{F}_t) = 0,$$

which proves the claim.

Since $\mathcal{X}$ is quasi-projective, we can find an exact sequence

$$0 \to R_2 \to E^{-1} \to E^0 \to \mathcal{G} \to 0$$

for some vector bundles $E^0$ and $E^{-1}$ on $\mathcal{X}$. Letting $E = [E^{-1} \to E^0]$, we can form a distinguished triangle

$$\begin{align*}
R_2[1] & \longrightarrow E \longrightarrow \mathcal{G} \longrightarrow R_2[2]
\end{align*}$$
on $\mathcal{X}$. As above, we have

$$\text{Hom}_{\mathcal{X}_t}(Li^*_t \mathcal{G}, \mathcal{F}_t) = \text{Hom}_{\mathcal{X}_t}(\mathcal{E}_t, \mathcal{F}_t) \quad \text{and} \quad \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}) = \text{Hom}_{\mathcal{X}}(\mathcal{G}, \mathcal{F})$$

for all $t \in T$, where $\mathcal{E}_t = Li^*_t E$. Hence it suffices to show (3.5) for $E$.

Consider the perfect complex $R \text{Hom}_\pi(\mathcal{E}, \mathcal{F})$ on $T$. By the base change theorem, we have

$$R \text{Hom}_\pi(\mathcal{E}, \mathcal{F})|_{t \in T} = R \text{Hom}_{\mathcal{X}_t}(\mathcal{E}_t, \mathcal{F}_t).$$

Hence if $\text{Hom}_{\mathcal{X}_t}(\mathcal{E}_t, \mathcal{F}_t) = 0$ for all $t \in T$, then $R \text{Hom}_\pi(\mathcal{E}, \mathcal{F})$ has tor-amplitude $\geq 1$ so that

$$\text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}) = H^0(\mathcal{R} \Gamma \circ R \text{Hom}_\pi(\mathcal{E}, \mathcal{F})) = 0.$$ It completes the proof. \qed

Remark 3.8. There is an alternative proof of Proposition 3.6 by comparing the pair obstruction theories\footnote{These obstruction theories are different with the obstruction theories used to define the virtual cycles in \[\text{§3.1}].} of $P(D)$ and $P(X)$ using derived algebraic geometry. This approach allows us to generalize Proposition 3.6 and Theorem 3.1 to other moduli spaces of pairs (e.g., moduli space $P^d_{n, \beta}(X)$ of $Z$-stable pairs \[\text{[20]}\]). We sketch the proof here.

Let $P(X)$ be the derived enhancement of $P(X)$ as a derived moduli space of pairs. Then the tangent complex at a $\mathbb{C}$-point $(F, s) \in P(X)$ can be expressed as

$$T_{P(X), (F, s)} = R \text{Hom}_X(I, F)$$

where $I = \{0_X \to F\}$. The tautological complex $R \pi_* (\mathcal{F} \otimes L)$ and the tautological section $\tau$ on $P(X)$ extend to $P(X)$ naturally. Let $\mathbf{V}$ be the derived
zero locus of the extended tautological section. Clearly, the classical truncation of $V$ is $V$. The canonical distinguished triangle of tangent complexes for the map $V \to P(X)$ at a $\mathbb{C}$-point $(F, s) \in V$ is

\[
\begin{align*}
R \text{Hom}_X(J, F) & \to R \text{Hom}_X(I, F) \to R \text{Hom}_X(L^\vee, F) \\
\text{T}_{V, (F, s)} & \to \text{T}_{P(X), (F, s)} \to \text{T}_{V/P(X), (F, s)}[1]
\end{align*}
\]

where $I = [\mathcal{O}_X \to F]$ and $J = [\mathcal{O}_D \to F]$.

Let $P(D)$ be the derived enhancement of $P(D)$ as a derived moduli space of pairs. Then there exists a canonical map $P(D) \to V$. The distinguished triangle of tangent complexes for $P(D) \to V$ at a $\mathbb{C}$-point $(F, s) \in P(D)$ is

\[
\begin{align*}
R \text{Hom}_{D}(J, F) & \to R \text{Hom}_X(J, F) \to R \text{Hom}_{D}(J, F \otimes L)[-1] \\
\text{T}_{P(D), (F, s)} & \to \text{T}_{V, (F, s)} \to \text{T}_{P(D)/V, (F, s)}[1]
\end{align*}
\]

where $J = [\mathcal{O}_D \to F]$. Hence the relative cotangent complex $L_{P(D)/V}$ is concentrated in degrees $\leq -2$. Therefore the induced map $P(D) \to V$ between the classical truncations is an étale morphism. Using the stability conditions, it is easy to show that $P(D) \to V$ is bijective. Hence $P(D) \to V$ is an isomorphism.

**Remark 3.9.** Proposition 3.6 holds even when the tautological complex $R\pi_*(F \otimes L)$ is not a vector bundle. In this case, we have a cartesian diagram

\[
\begin{array}{ccc}
P(D) & \to & P(X) \\
\downarrow & & \downarrow \tau \\
P(X) & \to & C(h^0(R\pi_*(F \otimes L)^\vee))
\end{array}
\]

where $C(h^0(R\pi_*(F \otimes L)^\vee)$ is the abelian cone associated to the coherent sheaf $h^0(R\pi_*(F \otimes L)^\vee)$.

However, the derived enhancement of the cartesian diagram (3.6)

\[
\begin{array}{ccc}
P(D) & \to & P(X) \\
\downarrow & & \downarrow \\
P(X) & \to & R\pi_*(F \otimes L)
\end{array}
\]

is not homotopy-cartesian, where $F$ denotes the extended universal family on $P(X)$. 
3.3. Symmetric obstruction theory on $P(D)$. Given $X, L, D, P(X)$ as in Theorem 3.1, consider the commutative diagram

\[
P(D) \times D \xrightarrow{d} P(D) \times X \xrightarrow{\pi_X} P(D)
\]

of schemes. Recall that the Behrend-Fantechi virtual cycle $[P(D)]^\text{vir}_{BF}$ is constructed from the 2-term perfect obstruction theory

\[
\psi_D : \mathcal{RHom}_{\pi_D}(\mathcal{J}, \mathcal{J} \otimes L)_0[2] \xrightarrow{At(\mathcal{J})} \mathbb{L}_{P(D)}
\]

induced by the Atiyah class $At(\mathcal{J}) : \mathcal{J} \to \mathcal{J} \otimes \mathbb{L}_{P(D) \times D}[1]$ of the universal complex $\mathcal{J} = [0_{P(D) \times D} \to \mathcal{F}]$.

**Proposition 3.10.** There exists a canonical 3-term symmetric complex $\mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})_\#[3]$ on $P(D)$ with a distinguished triangle

\[
\mathcal{RHom}_{\pi_D}(\mathcal{J}, \mathcal{J})_0[3] \xrightarrow{\psi_D} \mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})_\#[3] \xrightarrow{\epsilon} \mathcal{RHom}_{\pi_D}(\mathcal{J}, \mathcal{J} \otimes L)_0[2]
\]

for some map $\epsilon$. Here we identified

\[
(\mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})_\#[3])^\vee[2] = \mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})_\#[3]
\]

\[
(\mathcal{RHom}_{\pi_D}(\mathcal{J}, \mathcal{J} \otimes L)_0[2])^\vee[2] = \mathcal{RHom}_{\pi_D}(\mathcal{J}, \mathcal{J})_0[3]
\]

via the symmetric form of $\mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})_\#[3]$ and the relative Serre duality. Moreover, the symmetric form of $\mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})_\#[3]$ is induced from the standard symmetric form of $\mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})[3]$ in the sense of Lemma 3.12 below.

Let us first assume that Proposition 3.10 holds. Then we can define a 3-term symmetric obstruction theory on $P(D)$ as the composition

\[
\phi_D : \mathcal{RHom}_{\pi_X}(d_\ast \mathcal{J}, d_\ast \mathcal{J})_\#[3] \xrightarrow{\psi_D} \mathcal{RHom}_{\pi_D}(\mathcal{J}, \mathcal{J} \otimes L)_0[2] \xrightarrow{\psi_D} \mathbb{L}_{P(D)}.
\]

By the reduction formula in Proposition 1.18, the associated Oh-Thomas virtual cycle is identical to the Behrend-Fantechi virtual cycle

\[
[P(D)]^\text{vir}_{OT} = \sum \epsilon (-1)^{\sigma(\epsilon)} [P(D)\epsilon]^\text{vir}_{BF}
\]

up to canonical signs $(-1)^{\sigma(\epsilon)}$ on connected components $P(D)\epsilon$ of $P(D)$.

The rest of this subsection is devoted to the proof of Proposition 3.10. For simplicity, we will use the following abbreviations:

\[
\mathcal{D} = P(D) \times D, \quad \mathcal{X} = P(D) \times X, \quad \mathcal{P} = P(D).
\]

We start with the following well-known lemma.
Lemma 3.11. There is a canonical distinguished triangle
\[
(\mathbf{3.10}) \quad \mathbf{R} \mathbf{Hom}(\mathcal{J}, \mathcal{J}) \xrightarrow{\xi^\vee} \mathbf{R} \mathbf{Hom}(d_* \mathcal{J}, d_* \mathcal{J}) \xrightarrow{\xi} \mathbf{R} \mathbf{Hom}(\mathcal{J}, \mathcal{J} \otimes L)[-1]
\]
on $\mathcal{X}$, where $\mathbf{R} \mathbf{Hom}_d := \mathbf{R} d_* \mathbf{R} \mathbf{Hom}_D$. Here we identified
\[
\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J}) = (\mathbf{R} \mathbf{Hom}(\mathcal{J}, \mathcal{J} \otimes L)[-1])^\vee
\]
via the trace map and the Grothendieck-Serre duality.

Proof. Since the self-dual distinguished triangle (3.10) is well-known, we will only sketch how (3.10) is constructed. Note that there is a canonical distinguished triangle
\[
(\mathbf{3.11}) \quad \mathcal{J} \otimes L^\vee[1] \longrightarrow d^* d_* \mathcal{J} \longrightarrow \mathcal{J} \longrightarrow \mathcal{J} \otimes L^\vee[2]
\]
on $\mathcal{D}$ by [7, Lemma 3.3] (see also [28, Corollary 11.4]). Here the first two arrows are given by the adjunctions
\[
d_* \dashv d^* \otimes L[-1] \quad \text{and} \quad d^* \dashv d_*.
\]
Applying $\mathbf{R} \mathbf{Hom}_X(-, \mathcal{J})$ to (3.11), we obtain the desired distinguished triangle (3.10). The self-dualness of the distinguished triangle (3.10) follows from the naturality of the duality maps and the adjunction maps. □

We will construct a symmetric complex $\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J})$ as the "reduction" of the "symmetric complex" $\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J})$ by the "isotropic subcomplex"
\[
\theta^\vee : \mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J}) \xrightarrow{\mathbf{tr}_\xi} d_*(\mathcal{O}_\mathcal{P} \boxtimes L|_D)[-1].
\]
Strictly speaking, $\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J})$ is not a symmetric complex in the sense of Definition 1.2. However, Lemma C.2 in Appendix C shows that we can still form a reduction.

Lemma 3.12. There is a perfect complex $\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J})$ on $\mathcal{X}$ and a self-dual isomorphism
\[
(\mathbf{3.12}) \quad \theta : \mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J}) \cong (\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J})^\#)^\vee,
\]
i.e., $\theta^\vee = \theta$, satisfying the following two properties:
(1) There is a morphism of distinguished triangles
\[
(\mathbf{3.13}) \quad \begin{array}{ccc}
\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J})_R & \xrightarrow{\mu^\vee} & \mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J}) \xrightarrow{\mathbf{tr}_\xi} d_*(\mathcal{O}_\mathcal{P} \boxtimes L|_D)[-1] \\
\end{array}
\]
\[
\begin{array}{ccc}
\mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J})_L & \xrightarrow{\nu^\vee} & \mathbf{R} \mathbf{Hom}_X(d_* \mathcal{J}, d_* \mathcal{J}) \xrightarrow{\mathbf{tr}_\xi} d_*(\mathcal{O}_\mathcal{P} \boxtimes L|_D)[-1] \\
\end{array}
\]
\]

5In [7,28], they only considered smooth schemes but their proofs also work for arbitrary schemes.
for some maps $\mu$ and $\nu$, and perfect complexes $R\text{Hom}_\mathcal{X}(d_*,\mathcal{J})_L$ and $R\text{Hom}_\mathcal{X}(d_*,\mathcal{J})_R := (R\text{Hom}_\mathcal{X}(d_*,\mathcal{J})_L)^\vee$.

(2) There exists a distinguished triangle

\[
\text{RHom}_d(\mathcal{J}, \mathcal{J})_0 \xrightarrow{\varepsilon^\vee} \text{RHom}_\mathcal{X}(d_*,\mathcal{J})_\# \xrightarrow{\eta} \text{RHom}_d(\mathcal{J}, \mathcal{J} \otimes L)_0[-1]
\]

for some map $\varepsilon$ such that the diagram

\[
\begin{array}{ccc}
\text{RHom}_\mathcal{X}(d_*,\mathcal{J})_R & \xrightarrow{\nu^\vee} & \text{RHom}_\mathcal{X}(d_*,\mathcal{J})_R \\
\downarrow{\xi} & & \downarrow{\xi} \\
\text{RHom}_\mathcal{X}(d_*,\mathcal{J})_\# & \xrightarrow{(\varepsilon,0)} & \text{RHom}_d(\mathcal{J}, \mathcal{J} \otimes L)[-1]
\end{array}
\]

commutes. Here we identified $\text{RHom}_d(\mathcal{J}, \mathcal{J} \otimes L) = \text{RHom}_d(\mathcal{J}, \mathcal{J} \otimes L)[0] \oplus d_*(\Omega_{\mathcal{Y}} \boxtimes L|_D)$.

Moreover, the pair of the perfect complex $\text{RHom}_\mathcal{X}(d_*,\mathcal{J})_\#$ and the self-dual isomorphism $\theta$ is uniquely determined by the diagram (3.13).

**Proof.** We first simplify the notations as follows\footnote{The notations $A, B, C, ..., b, c, d, ..., p, q, r, ...$ introduced in the proof of Lemma 3.12 will not be used in the rest of this paper.}. Let

\[
A := \text{RHom}_\mathcal{X}(d_*,\mathcal{J})_R \\
B := \text{RHom}_d(\mathcal{J}, \mathcal{J} \otimes L)_0[-1], \quad C := d_*(\Omega_{\mathcal{Y}} \boxtimes L|_D)[-1]
\]

be the abbreviations of the perfect complexes on $\mathcal{X}$. Note that

\[
\text{Hom}_\mathcal{X}(C^\vee, (B \oplus C)[-1]) = \text{Hom}_\mathcal{X}(d_*\mathcal{O}_{\mathcal{Y}}, \text{RHom}_d(\mathcal{J}, \mathcal{J} \otimes L)[-2]) = 0.
\]

Hence we have

\[
\text{Hom}_\mathcal{X}(C^\vee, B[-1]) = \text{Hom}_\mathcal{X}(C^\vee, C[-1]) = 0.
\]

By Lemma 3.12 we can form a perfect complex $\text{RHom}_\mathcal{X}(d_*,\mathcal{J})_\#$ and a self-dual isomorphism (3.12) with the diagram (3.13). The uniqueness also follows from Lemma C.2. Let

\[
D := \text{RHom}_\mathcal{X}(d_*,\mathcal{J})_L \quad \text{and} \quad E := \text{RHom}_\mathcal{X}(d_*,\mathcal{J})_\#
\]

be the abbreviations of the perfect complexes on $\mathcal{X}$.

Now we will form the distinguished triangle (3.14). By the octahedral axiom, we can form a commutative diagram of four distinguished triangles

\[
\begin{array}{ccc}
B^\vee \oplus C^\vee & \xrightarrow{(p,q)} & B^\vee \oplus C^\vee \\
\downarrow{f} & & \downarrow{(b^\vee,c^\vee)} \\
C & \xrightarrow{(0,1)} & B \oplus C \xrightarrow{(1,0)} B
\end{array}
\]
for some perfect complex $F$ and maps $p$, $q$, $f$, where the middle vertical distinguished triangle is (3.10) and $(b, c) := \xi$. Again by the octahedral axiom, we have a commutative diagram of four distinguished triangles

(3.18)

\[
\begin{array}{ccccccccc}
C^\vee & \rightarrow & C^\vee \\
\downarrow^{(0,1)} & & \downarrow^{q} \\
B^\vee \oplus C^\vee & \rightarrow & F^\vee & \rightarrow & C \\
\downarrow^{(1,0)} & & \downarrow^{g} & & \downarrow^{c \circ f} \\
B^\vee & \rightarrow & G & \rightarrow & C
\end{array}
\]

for some perfect complex $G$ and maps $g$, $r$, where the middle horizontal distinguished triangle is the left vertical distinguished triangle in (3.17). Note that we have $f \circ q = c^\vee$ by the left upper square in (3.17). By the octahedral axiom, we have a commutative diagram of four distinguished triangles

(3.19)

\[
\begin{array}{ccccccc}
C^\vee & \rightarrow & C^\vee \\
\downarrow^{q} & & \downarrow^{c^\vee} \\
F & \rightarrow & A & \rightarrow & B \\
\downarrow^{g} & & \downarrow^{\mu} & & \downarrow^{t} \\
G & \rightarrow & D & \rightarrow & C \\
\end{array}
\]

for some map $s$ and $t$, where the middle vertical distinguished triangle is the dual of the top horizontal distinguished triangle in (3.13).

We claim that we can form a commutative diagram of four distinguished triangles

(3.20)

\[
\begin{array}{ccccccc}
B^\vee & \rightarrow & G & \rightarrow & C \\
\downarrow^{g \circ p} & & \downarrow^{r} & & \downarrow^{d} \\
E & \rightarrow & D & \rightarrow & C \\
\downarrow^{\epsilon} & & \downarrow^{d} & & \downarrow^{t} \\
B & \rightarrow & B
\end{array}
\]

for some maps $\epsilon$ and $e$, where the map $d$ and the middle horizontal distinguished triangle in (3.20) are given by (3.13). To deduce the claim from the octahedral axiom, we need to show that the right upper square in (3.20) commutes, $r = d \circ s$. It suffices to show that

\[r \circ g = d \circ s \circ g : F \rightarrow C\]

since the middle vertical sequence in (3.18) is a distinguished triangle, and \(\text{Hom}_{\mathcal{X}}(C^\vee[1], C) = 0\) by (3.16). We have $r \circ g = c \circ f$ by (3.18) and $d \circ s \circ g = d \circ \mu \circ f = c \circ f$ by (3.19) and (3.13). It proves the claim.
We claim that
\[(3.21)\quad e = \epsilon^\vee : B^\vee \to E.\]
By the distinguished triangle in (3.13), it suffices to show that
\[\nu \circ e = \nu \circ \epsilon^\vee : B^\vee \to E \to D\]
since Hom_\chi(B^\vee, C[-1]) = 0 by (3.10). Consider the commutative diagram
\[
\begin{array}{ccc}
D^\vee & \xrightarrow{\mu^\vee} & A \\
\downarrow{\nu^\vee} & & \downarrow{b} \\
E & \xrightarrow{\nu} & D \\
& \xleftarrow{\epsilon} & B
\end{array}
\]
induced by the commutative diagrams (3.13), (3.19), and (3.20). By taking dual, we obtain
\[(3.22)\quad \nu \circ \epsilon^\vee = \mu \circ b^\vee.\]
On the other hand, consider the commutative diagram
\[
\begin{array}{ccc}
B^\vee \oplus C^\vee & \xrightarrow{(b^\vee, c^\vee)} & F \\
\downarrow{(p, q)} & & \downarrow{f} \\
B^\vee & \xrightarrow{g \circ p} & G \\
& \xleftarrow{\nu \circ e} & D
\end{array}
\]
induced by the commutative diagrams (3.17), (3.18), (3.19), and (3.20). Hence we have
\[(3.23)\quad \nu \circ e = \mu \circ b^\vee.\]
The two equations (3.22) and (3.23) proves the claim (3.21).
By the equation (3.21), the distinguished triangle
\[
\begin{array}{ccc}
B^\vee & \xrightarrow{\epsilon^\vee} & E \\
& \xleftarrow{e} & B
\end{array}
\]
in (3.20) gives us the desired distinguished triangle (3.14). The dual of the equation (3.22)
\[\epsilon \circ \nu^\vee = b \circ \mu^\vee\]
induces the desired commutative diagram (3.15) since \(c \circ \mu^\vee = 0\) and \(\xi = (b, c)\). It completes the proof. \(\square\)

Now Proposition 3.10 follows directly from Lemma 3.12.

**Proof of Proposition 3.10.** Let
\[\mathbf{RHom}_{\pi_X}(d_*J, d_*J)\# := \mathbf{R}(\pi_X)_! \mathbf{RHom}_X(d_*J, d_*J)\# .\]
Then the self-dual isomorphism (3.12) induces a self-dual isomorphism
\[(\mathbf{RHom}_{\pi_X}(d_*J, d_*J)\#) ^\vee \cong \mathbf{RHom}_{\pi_X}(d_*J, d_*J)\#[4] \]
by the naturality of the relative Serre duality map. Similarly, the distinguished triangle \([3.14]\) in Lemma \([3.12(2)]\) gives us the desired distinguished triangle \([5.7]\).

\[\square\]

3.4. **Comparison of obstruction theories.** By Proposition \([3.6]\) and Proposition \([3.3]\) it suffices to show that

\begin{equation}
(3.24) \quad j^! [P(X)]_{OT}^{\text{vir}} = [P(D)]_{OT}^{\text{vir}}
\end{equation}

holds to prove the Lefschetz principle in Theorem \([3.1]\). We would like to deduce \([3.24]\) from the virtual pullback formula in Theorem \([2.2]\). To do this, we will compare the obstruction theories of \(P(X)\) and \(P(D)\).

Given \(X, L, D, P(X)\) as in Theorem \([3.1]\) let

\[
\begin{array}{ccc}
P(D) \times D & \xrightarrow{d} & P(D) \times X \\
\pi_D & \downarrow & \pi_X \\
& & \pi \\
P(D) & \xrightarrow{j} & P(X)
\end{array}
\]

be the canonical commutative diagram. Recall that the Oh-Thomas virtual cycle \([P(X)]_{OT}^{\text{vir}}\) is constructed from the 3-term symmetric obstruction theory

\begin{equation}
(3.25) \quad \phi_X : \mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi}(\mathbb{I}, \mathbb{I})_0[3] \xrightarrow{\text{At}(\mathbb{I})} \mathbb{I} \mathcal{P}(X)
\end{equation}

induced by the Atiyah class \(\text{At}(\mathbb{I}) : \mathbb{I} \to \mathbb{I} \otimes \mathbb{I} \mathcal{P}(X) \times X [1]\) of the universal complex \(\mathbb{I} = [\mathcal{O}_{P(X) \times X} \to \mathcal{F}].\) Let

\[
\mathcal{I} = \mathbb{I}_{P(D) \times X} = [\mathcal{O}_{P(D) \times X} \to d_s \mathcal{F}]
\]

be the restriction, where \(\mathcal{J} = [\mathcal{O}_{P(D) \times D} \to \mathcal{F}]\) is the universal complex of \(P(D)\).

**Proposition 3.13.** The two symmetric obstruction theory \(\phi_X\) in \([3.25]\) and \(\phi_D\) in \([3.8]\) are compatible as follows:

1. There is a reduction diagram (see Proposition \([1.3]\)), i.e. a morphism of distinguished triangles

\begin{equation}
(3.26) \quad \begin{array}{ccc}
\mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi_X}(d_s \mathcal{J}, \mathcal{I})_0[3] & \xrightarrow{\beta} & \mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi_X}(d_s \mathcal{J}, d_s \mathcal{J})_0[3] \\
\beta & \downarrow & \delta \\
\mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi_X}(\mathcal{I}, \mathcal{I})_0[3] & \xrightarrow{\alpha} & \mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi_X}(d_s \mathcal{J}, \mathcal{I})_0[3]
\end{array}
\end{equation}

for some maps \(\alpha, \beta, \gamma, \delta\) and perfect complexes \(\mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi_X}(\mathcal{I}, d_s \mathcal{J})_0[3]\) and \(\mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi_X}(d_s \mathcal{J}, \mathcal{I})_0[3] := (\mathbb{R} \mathcal{H} \mathcal{O} \mathcal{M}_{\pi_X}(\mathcal{I}, d_s \mathcal{J})_0[3])^\vee[4]\) on \(P(D)\).
(2) The Atiyah classes of \( I \) and \( J \) are compatible, i.e., the diagram

\[
\begin{array}{ccc}
\mathcal{R}Hom_{\pi_X}(d_*J, I) \#_0[3] & \xrightarrow{\alpha^\vee} & \mathcal{R}Hom_{\pi_X}(d_*J, d_*J) \#_0[3] \\
\downarrow{g^\vee} & & \downarrow{f} \\
\mathcal{R}Hom_{\pi_X}(I, I)_0[3] & \xrightarrow{\text{At}(I)} & \mathcal{R}Hom_{\pi_D}(J, J \otimes L)_0[2] \\
\downarrow{\text{At}(I)} & & \downarrow{\text{At}(J)} \\
\mathbb{L}P(X)|_P(D) & \xrightarrow{\mathbb{L}P(D)} & \mathbb{L}P(D)
\end{array}
\]

commutes.

In the rest of this subsection, we will prove Proposition 3.13 through several steps. We first fix the notations. For simplicity, we will use the following abbreviations

\( \mathcal{D} = P(D) \times D, \quad \mathcal{X} = P(D) \times X, \quad \text{and} \quad \mathcal{P} = P(D) \)

of schemes as in \( \S \) 3.3. Let

\[
\begin{align*}
\mathcal{J} & \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}[1] \\
\mathcal{I} & \longrightarrow \mathcal{O}_X \longrightarrow d_*\mathcal{F} \longrightarrow \mathcal{I}[1]
\end{align*}
\]

be the two canonical distinguished triangles on \( \mathcal{D} \) and \( \mathcal{X} \), respectively. By applying the octahedral axiom to the two distinguished triangles \( \text{(3.28)} \) and \( \text{(3.29)} \), we obtain a third distinguished triangle

\[
\mathcal{L}^\vee \longrightarrow \mathcal{I} \longrightarrow d_*\mathcal{J} \longrightarrow \mathcal{L}^\vee[1]
\]

on \( \mathcal{X} \) such that \( d_*e = v \circ f \). Here we denoted \( \mathcal{L} = \mathcal{O}_{P(D)} \boxtimes L \).

3.4.1. Prototype. We start with a prototype of the reduction diagram \( \text{(3.26)} \) in Proposition 3.13 where we use

\[
\begin{align*}
\mathcal{R}Hom_{\pi_X}(d_*\mathcal{J}, d_*\mathcal{J}) & \quad \text{and} \quad \mathcal{R}Hom_{\pi_X}(I, d_*\mathcal{J})
\end{align*}
\]

instead of \( \mathcal{R}Hom_{\pi_X}(d_*\mathcal{J}, d_*\mathcal{J})_\# \) and \( \mathcal{R}Hom_{\pi_X}(I, d_*\mathcal{J})_\# \).

**Lemma 3.14.** There is a (not necessarily commutative) diagram of two distinguished triangles

\[
\begin{array}{ccc}
\mathcal{R}Hom_{\pi_X}(d_*\mathcal{J}, I) & \xrightarrow{l_v} & \mathcal{R}Hom_{\pi_X}(d_*\mathcal{J}, d_*\mathcal{J}) \\
\downarrow{\text{pr}_v} & & \downarrow{r_v} \\
\mathcal{R}Hom_{\pi_X}(\mathcal{I}, I)_0 & \xrightarrow{l_v \circ v} & \mathcal{R}Hom_{\pi_X}(I, d_*\mathcal{J}) \\
\downarrow{\eta} & & \downarrow{\eta} \\
\mathcal{R}Hom_{\pi_X}(\mathcal{I}, d_*\mathcal{J}) & \xrightarrow{\eta} & \mathcal{R}Hom_{\pi_X}(d_*\mathcal{J}, \mathcal{L}^\vee)[1]
\end{array}
\]
for some η such that \( r_v \circ η = l_w \). Here \( l_v \) denotes the left composition with \( v \) and \( r_v \) denotes the right composition with \( v \). Also \( ι \) and \( ρ \) are the canonical maps in the direct sum diagram

\[
\begin{array}{ccc}
\text{RHom}_X(I, I) & \xrightarrow{ι} & \text{RHom}_X(I, I) \\
\downarrow r_v & & \downarrow tr \\
\text{RHom}_X(L^\vee, L^\vee) & \xrightarrow{\lambda} & \text{RHom}_X(I, d_* J) \\
\end{array}
\]

where \( tr \) denotes the trace map.

Proof. Applying \( \text{RHom}_X(d_* J, -) \) to (3.30), we can form the upper distinguished triangle in (3.31). The lower distinguished triangle can be obtained by applying the octahedral axiom to the commutative diagram

\[
\begin{array}{ccc}
\text{RHom}_X(I, L^\vee) & \xrightarrow{lu} & \text{RHom}_X(I, I) \\
\downarrow r_w & & \downarrow tr \\
\text{RHom}_X(L^\vee, L^\vee) & \xrightarrow{\lambda} & \text{RHom}_X(I, d_* J) \\
\end{array}
\]

where the three distinguished triangles are the obvious ones given by (3.30) and (3.32). The formula \( r_v \circ η = l_w \) follows from the octahedral axiom. □

3.4.2. We will replace the complexes in (3.31) as

\[
\begin{array}{c}
\text{RHom}_X(d_* J, d_* J) \xrightarrow{r_v \circ ξ^\vee \circ id_J} \text{RHom}_X(I, d_* J) \\
\end{array}
\]

for some perfect complex \( \text{RHom}_X(I, d_* J) \). Then the (non-commutative) diagram (3.31) will become commutative and induce the desired commutative diagram (5.26).

Define the perfect complex \( \text{RHom}_X(I, d_* J) \) as the cone of the composition \( r_v \circ ξ^\vee \circ id_J \) (see (3.10)). Then it fits into a distinguished triangle

\[
\begin{array}{ccc}
d_* O_D & \xrightarrow{r_v \circ ξ^\vee \circ id_J} & \text{RHom}_X(I, d_* J) \\
\downarrow & & \downarrow \lambda \\
d_* O_D[1] & \xrightarrow{\mu \circ r_w} & \text{RHom}_X(I, d_* J) \\
\end{array}
\]

for some map \( \lambda \). Let \( \text{RHom}_X(d_* J, I) := (\text{RHom}_X(I, d_* J))^\vee \) be the dual.

By the octahedral axiom, we obtain a commutative diagram of four distinguished triangles

\[
\begin{array}{ccc}
\text{RHom}_X(L^\vee, d_* J)[-1] & \xrightarrow{r_w} & \text{RHom}_X(L^\vee, d_* J)[-1] \\
\downarrow & & \downarrow \mu \circ r_w \\
d_* O_D & \xrightarrow{ξ^\vee \circ id_J} & \text{RHom}_X(d_* J, d_* J) \\
\end{array}
\]

\[
\begin{array}{ccc}
d_* O_D & \xrightarrow{r_v \circ ξ^\vee \circ id_J} & \text{RHom}_X(I, d_* J) \\
\downarrow & & \downarrow \lambda \\
d_* O_D & \xrightarrow{d_* O_D} & \text{RHom}_X(I, d_* J) \\
\end{array}
\]
for some map $\zeta$, where the middle vertical distinguished triangle is induced by (3.30), the middle horizontal distinguished triangle is the dual of the top horizontal distinguished triangle in (3.13), and the bottom horizontal distinguished triangle is (3.33).

3.4.3. Maps $\alpha$, $\beta$, $\gamma$, $\delta$. Define the maps $\alpha$ and $\beta$ as the compositions:

$$\alpha: \mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{J},d_{*}\mathcal{J})_{\#}^{\zeta} \rightarrow \mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{J},d_{*}\mathcal{J})_{L}^{\zeta} \rightarrow \mathbb{R}\text{Hom}_{X}(I,d_{*}\mathcal{J})_{\#}^{\delta},$$

$$\beta: \mathbb{R}\text{Hom}_{X}(I,I)_{0}^{\iota_{0}} \rightarrow \mathbb{R}\text{Hom}_{X}(I,d_{*}\mathcal{J})^{\lambda} \rightarrow \mathbb{R}\text{Hom}_{X}(I,d_{*}\mathcal{J})_{\#}^{\gamma}.$$  

We define the map $\gamma$ as follows:

**Lemma 3.15.** There exists a commutative diagram of four distinguished triangles

(3.35)

$$
\begin{array}{ccc}
d_{*}\mathcal{O}_{D} & \rightarrow & \mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{O}_{D},\mathcal{L}^{\gamma})[1] \\
\downarrow r_{i} & & \downarrow r_{i} \\
\mathbb{R}\text{Hom}_{X}(I,I)_{0}^{\iota_{0}} & \rightarrow & \mathbb{R}\text{Hom}_{X}(I,d_{*}\mathcal{J})^{\eta} \rightarrow \mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{J},\mathcal{L}^{\gamma})[1] \\
\downarrow & & \downarrow \lambda \\
\mathbb{R}\text{Hom}_{X}(I,I)_{0}^{\beta} & \rightarrow & \mathbb{R}\text{Hom}_{X}(I,d_{*}\mathcal{J})_{\#}^{\gamma} \rightarrow \mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{F},\mathcal{L}^{\gamma})[2]
\end{array}
$$

for some map $\gamma$, where the middle vertical distinguished triangle is (3.33), the right vertical distinguished triangle is induced by (3.28), and the middle horizontal distinguished triangle is the bottom horizontal distinguished triangle in (3.31).

**Proof.** We will use the octahedral axiom to obtain the diagram (3.35). Thus it suffices to show that the upper right square in (3.35) commutes.

We claim that the squares in the diagram

(3.36)

$$
\begin{array}{ccc}
d_{*}\mathcal{O}_{D} & \rightarrow & \mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{O}_{D},\mathcal{L}^{\gamma})[1] \\
\downarrow \xi^{\gamma_{0}d_{*}\mathcal{J}} & & \downarrow r_{i} \\
\mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{J},d_{*}\mathcal{J})_{w} & \rightarrow & \mathbb{R}\text{Hom}_{X}(d_{*}\mathcal{J},\mathcal{L}^{\gamma})[1] \\
\downarrow r_{w} & & \downarrow r_{w} \\
\mathbb{R}\text{Hom}_{X}(I,d_{*}\mathcal{J})_{w} & \rightarrow & \mathbb{R}\text{Hom}_{X}(I,\mathcal{L}^{\gamma})[1]
\end{array}
$$

commute. Indeed, under the adjunction $d_{*} \dashv d^{*} \otimes \mathcal{L}[-1]$, we have a correspondence

$$(\mathcal{J} \xrightarrow{i} \mathcal{O}_{D}) \leftrightarrow (d_{*}\mathcal{J} \xrightarrow{w} \mathcal{L}^{\gamma}[1])$$
between the two canonical maps in (3.28) and (3.30). Hence we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_D & \xrightarrow{\mathbf{R}\text{Hom}_D(\mathcal{O}_D, \mathcal{O}_D)} & \mathbf{R}\text{Hom}_D(\mathcal{J}, d^*d_*\mathcal{J} \otimes \mathcal{L}[-1]) \\
\downarrow & & \downarrow \mathbf{r}_i \\
\mathbf{R}\text{Hom}_D(\mathcal{J}, d^*d_*\mathcal{J} \otimes \mathcal{L}[\mathcal{L}]) & \xrightarrow{\mathbf{lr}_w} & \mathbf{R}\text{Hom}_D(\mathcal{J}, \mathcal{O}_D)
\end{array}
\]

on \( \mathcal{D} \), which proves the commutativity of the upper square in (3.36). The commutativity of the lower square in (3.36) is just the naturality of the composition functors.

If we compose the two maps

\[(3.37) \quad d_*\mathcal{O}_D \Rightarrow \mathbf{R}\text{Hom}_X(d_*\mathcal{J}, \mathcal{X}^\vee[1])\]

in the upper right square in (3.35) with the map

\[\mathbf{r}_v : \mathbf{R}\text{Hom}_X(d_*\mathcal{J}, \mathcal{X}^\vee[1]) \rightarrow \mathbf{R}\text{Hom}_X(\mathcal{I}, \mathcal{X}^\vee[1]),\]

then we obtain the total square in (3.36) since \( \mathbf{r}_v \circ \eta = \mathbf{I}_w \) by Lemma 3.14. Hence the difference of the two maps (3.37) comes from a map

\[(3.38) \quad d_*\mathcal{O}_D \Rightarrow \mathbf{R}\text{Hom}_X(\mathcal{X}^\vee[1], \mathcal{X}^\vee[1]) = \mathcal{O}_X.\]

Since \( \text{Hom}_X(d_*\mathcal{O}_D, \mathcal{O}_X) = 0 \), the map (3.38) is zero and the two maps (3.37) coincide.

We next define the map \( \delta \) as follows:

**Lemma 3.16.** There exists a commutative diagram of four distinguished triangles

\[(3.39) \quad \begin{array}{ccc}
\mathcal{O}_D & \xrightarrow{\mathbf{R}\text{Hom}_X(d_*\mathcal{O}_D, \mathcal{X}^\vee)[1]} & \mathbf{R}\text{Hom}_X(d_*\mathcal{O}_D, \mathcal{X}^\vee)[1] \\
\downarrow & & \downarrow \mathbf{r}_i \\
\mathbf{R}\text{Hom}_X(d_*\mathcal{J}, \mathcal{I}) & \xrightarrow{\mathbf{R}\text{Hom}_X(d_*\mathcal{J}, d_*\mathcal{J})_{\#}} & \mathbf{R}\text{Hom}_X(d_*\mathcal{J}, \mathcal{L}^\vee[1]) \\
\downarrow & & \downarrow \mathbf{r}_e \\
\mathbf{R}\text{Hom}_X(d_*\mathcal{J}, \mathcal{I}) & \xrightarrow{\mathbf{R}\text{Hom}_X(d_*\mathcal{J}, d_*\mathcal{J})_{\#}} & \mathbf{R}\text{Hom}_X(d_*\mathcal{J}, \mathcal{X}^\vee)[2]
\end{array}\]

for some map \( \delta \), where the middle vertical distinguished triangle is the dual of the bottom horizontal distinguished triangle in (3.13), the right vertical distinguished triangle is given by (3.28), and the middle horizontal distinguished triangle is the dual of the right vertical distinguished triangle in (3.34).

**Proof.** As in Lemma 3.15 it suffices to show the commutativity of the upper right square in (3.39). This follows from the commutativity of the upper square in (3.36). \(\square\)
3.4.4. Reduction diagram. Combining the four maps $\alpha$, $\beta$, $\gamma$, $\delta$, we can form the reduction diagram \((3.26)\) in Proposition \(3.13(1)\).

Proof of Proposition \(3.13(1)\). We claim that the diagram (3.40)

\[
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{J}, \mathcal{I})_# \\
\downarrow \beta^\vee
\end{array}
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{J}, d_*\mathcal{J})_# \\
\downarrow \alpha
\end{array}
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{F}, \mathcal{L}^\vee)[2] \\
\end{array}
\]

commutes. We will first show that the left square in (3.40) commutes. Indeed, we have the commutative diagram (3.41)

\[
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{J}, \mathcal{I})_# \\
\downarrow \lambda^\vee
\end{array}
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{J}, d_*\mathcal{J})_R \\
\downarrow \mu^\vee
\end{array}
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{J}, d_*\mathcal{J})_# \\
\downarrow \nu
\end{array}
\]

and an equation $\lambda \circ (r_v \circ \xi^\vee \circ \text{id}_\mathcal{J}) = 0$ by \((3.33)\). Since $\beta = \lambda \circ l_v \circ \iota$ by definition, and \((3.42)\) is zero, the diagram (3.43)

\[
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{J}, \mathcal{I})_0 \\
\downarrow \rho
\end{array}
\begin{array}{c}
\text{RHom}_X(d_*\mathcal{J}, \mathcal{J}) \\
\downarrow \lambda
\end{array}
\]

commutes. The two commutative diagrams \((3.41)\) and \((3.43)\) give us the commutativity of the left square in (3.40).

Now we will prove that the right square in (3.40) commutes. Since

\[
\text{Hom}_X(d_*\mathcal{O}_\mathcal{D}[1], \text{RHom}_X(d_*\mathcal{F}, \mathcal{L}^\vee[2])) = \text{Hom}_\mathcal{D}(\mathcal{F}, \mathcal{O}_\mathcal{D}) \oplus \text{Ext}^{-1}_\mathcal{D}(\mathcal{F}, \mathcal{L}|_\mathcal{D}) = 0,
\]

we have a commutative diagram \((3.44)\).
it suffices to show that  
\[ \gamma \circ \alpha \circ \nu^\vee = \delta \circ \nu^\vee. \]

Consider a diagram

\[
\begin{array}{c}
\text{RHom}_X(d_*, J, d_*, J) R \overset{\nu^\vee}{\longrightarrow} \text{RHom}_X(d_*, J, d_*, J)_R \\
\downarrow \mu^\vee \\
\text{RHom}_X(d_*, J, d_*, J) \overset{\mu}{\longrightarrow} \text{RHom}_X(d_*, J, d_*, J)_L \\
\downarrow r_v \\
\text{RHom}_X(I, d_*, J) \overset{\lambda}{\longrightarrow} \text{RHom}_X(I, d_*, J)_R \\
\downarrow \eta \\
\text{RHom}_X(d_*, J, L')[1] \overset{r_e}{\longrightarrow} \text{RHom}_X(d_*, J, L')[2]
\end{array}
\]

of commutative squares induced by (3.13), (3.34), and (3.35). The two maps \( \eta \circ r_v \) and \( l_w \) are not necessarily equal, but their compositions with \( r_v \),

\[ \text{RHom}_X(d_*, J, d_*, J) \overset{\eta \circ r_v}{\longrightarrow} \text{RHom}_X(I, d_*, J, L')[1] \overset{r_v}{\longrightarrow} \text{RHom}_X(I, L')[1] \]

are equal since \( r_v \circ \eta = l_w \) by Lemma 3.14. Hence the compositions of the two maps \( \eta \circ r_v \) and \( l_w \) with \( r_e \) are also equal since \( d_* e = v \circ f \). By (3.39), we have

\[ \gamma \circ \alpha \circ \nu^\vee = \gamma \circ \zeta \circ \nu \circ \nu^\vee = r_e \circ \eta \circ r_v \circ \mu^\vee = r_e \circ l_w \circ \mu^\vee = \delta \circ \nu^\vee, \]

which proves the claim.

If we apply \( R(\pi_X)_* \) to (3.40), then we obtain the desired diagram (3.26) (see Lemma C.4). It completes the proof.

3.4.5. Atiyah classes. Finally, we compare the Atiyah classes of \( J \) and \( I \) using derived algebraic geometry.

**Proof of Proposition 3.13(2).** We first claim that the diagram (3.44)

\[
\begin{array}{c}
\text{RHom}_{\pi_X}(d_*, J, I)[3] \overset{l_v}{\longrightarrow} \text{RHom}_{\pi_X}(d_*, J, d_*, J)[3] \overset{\zeta}{\longrightarrow} \text{RHom}_{\pi_D}(J, J \otimes L)[2] \\
\downarrow r_v \\
\text{RHom}_{\pi_X}(I, I)[3] \\
\downarrow \text{At}_{P(X) \times X}(I) \\
\text{At}_{P(D) \times X}(I) \\
\downarrow \text{At}_{P(D) \times D}(J) \\
\text{At}_{P(X) \times P(D)} \downarrow \text{At}_{P(D)}
\end{array}
\]

...
commutes. Indeed, the commutativity of the middle square in (3.44) follows from the functoriality of the Atiyah classes

\[
\begin{array}{ccc}
I 
\xrightarrow{v} & d_*J \\
\downarrow & \downarrow \\
At_{P(D) \times X}(I) & At_{P(D) \times X}(d_*J)
\end{array}
\]

The lower triangle in (3.44) is the commutative diagram of tangent maps for

\[
P(D) \xrightarrow{j} P(X) \xrightarrow{1} \text{Perf}(X)
\]

by [58, Appendix A], where \(\text{Perf}(X)\) denotes the derived moduli stack of perfect complexes on \(X\). The right triangle in (3.44) is the commutative diagram of tangent maps for

\[
P(D) \xrightarrow{\mathcal{J}} \text{Perf}(D) \xrightarrow{1} \text{Perf}(X)
\]

where the map \(\text{Perf}(D) \rightarrow \text{Perf}(X)\) between the derived moduli stacks is given by the derived pushforward.

By the square (3.15), the dual of the right bottom square in (3.34), and the dual of the square (3.43), the commutative diagram (3.44) implies the desired commutative diagram (3.27). □

3.5. Comparison of virtual cycles. Finally, we prove the Lefschetz principle in Theorem 3.1 using the results in the previous subsections.

Proof of Theorem 3.1. By Proposition 3.13 we can form two vertical morphisms

(3.45)

\[
\begin{array}{ccc}
\mathcal{R} \text{Hom}_{\pi_X}(d_*J, I) & \rightarrow & \mathcal{R} \text{Hom}_{\pi_X}(d_*J, d_*J) \\
\phi_X & \rightarrow & \phi_D \\
\tau^{-1}L_{P(X)}|_{P(D)} & \rightarrow & \tau^{-1}L_{P(D)}/P(X)
\end{array}
\]

of the three horizontal distinguished triangles for some maps \(\phi_D\) and \(\phi'\). Indeed, there exists a map

\[
\phi'_D : \mathcal{R} \text{Hom}_{\pi_X}(I, d_*J) \rightarrow \tau^{-1}L_{P(D)}
\]

The commutativity of the right triangle in (3.44) for the truncated cotangent complex \(\tau^{-1}L_{P(D)}\) can be deduced by [22, Theorem 2.5] without using derived algebraic geometry.
such that \( \phi_D' \circ \alpha = \phi_D \) since \( \phi_D \circ \delta' = \alpha \circ \phi_X \circ \beta' \circ \gamma' = 0 \). Then we have
\[
a \circ \phi_X = \phi_D' \circ \beta
\]
since \( \text{Hom}_{\mathcal{P}}(\mathcal{D})(\mathcal{R}(\pi_D)(\mathcal{F} \otimes L)[2], \tau^{-1}_{L_{\mathcal{P}(\mathcal{D})}}) = 0 \). Hence we can also find a map \( \phi' \) that fits into the diagram (3.45).

The long exact sequence associated to (3.45) assures that the composition
\[
\phi : (\mathcal{R}(\pi_D)(\mathcal{F} \otimes L)[-1])' \rightarrow L'_{\mathcal{P}(\mathcal{D})/\mathcal{P}(X)} \rightarrow L_{\mathcal{P}(\mathcal{D})}/\mathcal{P}(X)
\]
is a perfect obstruction theory. Note that the associated virtual pullback
\[
j_\phi^! : A_*(P(X)) \rightarrow A_*(P(D))
\]
only depends on the vector bundle \( \mathcal{R}(\pi_D)(\mathcal{F} \otimes L) \) and is independent of the map \( \phi \) by [24, Example 4.1.8] (see Lemma [4.11]). Therefore, the virtual pullback formula (Theorem 2.2) and the reduction formula (3.9) gives us
\[
j_\phi^!(P(X))^{\text{vir}}_{\mathcal{O}T} = j_\phi^!(P(X))^{\text{vir}}_{\mathcal{O}T} = [P(D)]^{\text{vir}}_{\mathcal{O}T} = \sum_e (-1)^{\sigma(e)}[P(D)]^{\text{vir}}_{\mathcal{O}T} \}
\]
where \( j^! \) is the Gysin pullback \([3,2] \) given by the tautological section. \[\square\]

We end this section with a remark on orientations:

**Remark 3.17.** For any given orientation on \( \mathcal{R}Hom_{\pi}(\mathcal{I}, \mathcal{I})_0[3] \), there exists a unique orientation on \( \mathcal{R}Hom_{\pi}(\mathcal{d}_*\mathcal{J}, \mathcal{d}_*\mathcal{J})_\# \) induced from the reduction diagram (3.26). Then the signs \( \sigma(e) \) in (3.9) is uniquely determined by this orientation and the self-dual distinguished triangle (3.7).

## 4. Pairs/Sheaves correspondence

In this section, we compare the Oh-Thomas virtual cycles for moduli spaces of pairs and moduli spaces of sheaves by combining the virtual pull-back formula in \( \S 2 \) and the pushforward formula for virtual projective bundles in \( \S 4.1 \).

### 4.1. Virtual projective bundles.

In this preliminary subsection, we provide a pushforward formula for virtual projective bundles. This pushforward formula is just a reformulation of a classical result in intersection theory \([24]\) from the point of view of virtual intersection theory \([4, 45]\).

**Definition 4.1** (Virtual projective bundle). Let \( \mathbb{K} \) be a perfect complex of tor-amplitude \([0, 1] \) on a quasi-projective scheme \( \mathcal{X} \). The virtual projective bundle \( \mathbb{P}(\mathbb{K}) \) associated to \( \mathbb{K} \) is a pair of

1. a projective cone
\[
p : \mathbb{P}(\mathbb{K}) := \text{Proj Sym}^* h^0(\mathbb{K}^\vee) \rightarrow \mathcal{X},
\]
2. and a relative perfect obstruction theory
\[
\mathbb{L}^{\text{vir}}_{\mathbb{P}(\mathbb{K})/\mathcal{X}} := \text{cone}(\mathcal{O}_{\mathbb{P}(\mathbb{K})} \rightarrow p^*\mathbb{K}(1))' \rightarrow \mathbb{L}_{\mathbb{P}(\mathbb{K})/\mathcal{X}}
\]
given as follows: Choose a resolution $K \cong [K_0 \rightarrow K_1]$ by vector bundles. Then we have a commutative diagram

\[
\begin{array}{ccc}
K_1(1)|_{\mathbb{P}(K_0)} & \xrightarrow{t} & \mathbb{P}(K_0) \\
\mathbb{P}(K) & \xrightarrow{i} & \mathbb{P}(K_0) \\
\downarrow p & & \downarrow q \\
\mathcal{X} & \xrightarrow{i} & \mathbb{P}(K_0)
\end{array}
\]

where $\mathbb{P}(K)$ is the zero locus of the tautological section $t$. Thus

\[
\mathbb{L}^\text{vir}_{\mathbb{P}(K)/\mathcal{X}} = (K_1(1))|_{\mathbb{P}(K)} \xrightarrow{dt} \Omega_{\mathbb{P}(K_0)/\mathcal{X}}|_{\mathbb{P}(K)}
\]

\[
\mathbb{L}_{\mathbb{P}(K)/\mathcal{X}} = I/I^2 \xrightarrow{d} \Omega_{\mathbb{P}(K_0)/\mathcal{X}}|_{\mathbb{P}(K)}
\]

gives us a perfect obstruction theory, where $I$ denotes the ideal sheaf of $\mathbb{P}(K)$ in $\mathbb{P}(K_0)$.

The perfect obstruction theory is independent of the choice of the global resolution and exists without assuming the global resolution, but we will not need these facts in this paper.

**Proposition 4.2** (Pushforward formula for virtual projective bundles). Let $p : \mathbb{P}(K) \rightarrow \mathcal{X}$ be a virtual projective bundle over a quasi-projective scheme $\mathcal{X}$. For any cycle class $\alpha \in A_*(\mathcal{X})$ and a $K$-theory class $\xi \in K^0(\mathcal{X})$, we have

\[
p_* (c_m(p^*\xi(1)) \cap p^! \alpha) = \sum_{0 \leq i \leq m} \binom{s - i}{m - i} \cdot c_i(\xi) \cap c_{m-i+1-r}(\mathcal{O}(1)) \cap \alpha
\]

where $r$ is the rank of $K$ and $s$ is the rank of $\xi$.

**Proof.** Fix a global resolution $K \cong [K_0 \rightarrow K_1]$ and consider the factorization. Manolache’s virtual pullback formula $p' = i' \circ q^*$ implies

\[
p_* (c_m(\xi(1)) \cap p^! \alpha) = q_* (c_m(\xi(1)) \cap c_{r_1}(K_1(1)) \cap q^* \alpha)
\]

where $r_0$ and $r_1$ are the ranks of $K_0$ and $K_1$, respectively. Note that

\[
c_m(\xi(1)) = \sum_{0 \leq i \leq m} \binom{s - i}{m - i} c_i(\xi) c_1(\mathcal{O}(1))^{m-i}
\]

---

8 The virtual projective bundle is the classical truncation of the derived projective bundle.
by [24, Example 3.2.2]. Therefore, we have

\[
p_*(c_m(\xi(1)) \cap p^!\alpha) = \sum_{0 \leq i \leq m} \sum_{0 \leq j \leq r_1} \binom{s-i}{m-i} \cdot c_i(\xi) \cap c_j(K_1) \cap q_*(c_1(\mathcal{O}(1))^{m+r_1-i-j} \cap q^*\alpha) \cap \alpha
\]

\[
= \sum_{0 \leq i \leq m} \sum_{0 \leq j \leq r_1} \binom{s-i}{m-i} \cdot c_i(\xi) \cap c_j(K_1) \cap s_{m+r_1-i-j-r_0+1}(K_0) \cap \alpha
\]

\[
= \sum_{0 \leq i \leq m} \binom{s-i}{m-i} \cdot c_i(\xi) \cap c_{m-i+1-r}(\mathcal{O}) \cap \alpha
\]

where \(s_*(K_0)\) denotes the Segre class of \(K_0\). \(\square\)

4.2. Main result. Let \(X\) be a smooth projective Calabi-Yau 4-fold. Let \(\beta \in H_2(X, \mathbb{Q})\) be a curve class and \(n \in \mathbb{Z}\) be an integer. Let

\[
P_{n,\beta}(X) = \{\text{stable pairs } (F, s) \text{ on } X \text{ with } \text{ch}(F) = (0, 0, 0, \beta, n)\}
\]

\[
M_{n,\beta}(X) = \{\text{stable sheaves } G \text{ on } X \text{ with } \text{ch}(G) = (0, 0, \beta, n)\}
\]

be the moduli spaces of stable pairs and stable sheaves. Then they both have Oh-Thomas virtual cycles

\[
[P_{n,\beta}(X)]^\text{vir} \in A_n(P_{n,\beta}(X)) \quad \text{and} \quad [M_{n,\beta}(X)]^\text{vir} \in A_1(M_{n,\beta}(X)).
\]

Assume that the curve class \(\beta \in H_2(X, \mathbb{Q})\) is irreducible. Then all pure sheaves \(G\) with \(\text{ch}_3(G) = \beta\) are stable so that the moduli space \(M_{n,\beta}(X)\) is proper and does not depend on the polarization \(\mathcal{O}_X(1)\). Moreover, there is a well-defined forgetful map

\[
p : P_{n,\beta}(X) \to M_{n,\beta}(X) : (F, s) \mapsto F.
\]

When there is a universal sheaf \(\mathcal{G}\) on \(M_{n,\beta}(X) \times X\), then the moduli space \(P_{n,\beta}(X)\) of stable pairs can be identified to the virtual projective bundle

\[
P_{n,\beta}(X) = \mathbb{P}(\mathbb{R}\pi_*^M \mathcal{G}) \to M_{n,\beta}(X)
\]

of the perfect complex \(\mathbb{R}\pi_*^M \mathcal{G}\), where \(\pi^M : M_{n,\beta}(X) \times X \to M_{n,\beta}(X)\) denotes the projection map.

**Theorem 4.3** (Pairs/Sheaves correspondence). Let \(X\) be a Calabi-Yau 4-fold, \(\beta \in H_2(X, \mathbb{Q})\) be an irreducible curve class, and \(n \in \mathbb{Z}\) be an integer. Assume that there exists a universal sheaf \(\mathcal{G}\) on \(M_{n,\beta}(X)\). For any orientation on \(M_{n,\beta}(X)\), there exists an induced orientation on \(P_{n,\beta}(X)\) such that the following formulas hold:

(1) **(Pullback formula)** We have

\[
[P_{n,\beta}(X)]^\text{vir} = p'! [M_{n,\beta}(X)]^\text{vir}
\]

where \(p'\) is the virtual pullback of the virtual projective bundle.

---

*In [23], it is stated for vector bundles, but it is easy to show that the result also holds for any K-theory classes.*
(2) (Pushforward formula) For any vector bundle $E$ on $X$, we have

$$p_*(c_m(R\pi^P_* (F \otimes E)) \cap \{P_{n,\beta}(X)^{\text{vir}}\})$$

$$= \begin{cases} 
\binom{N}{n-1} \cdot [M_{n,\beta}]^{\text{vir}} & \text{if } m = n - 1 \\
\left(\binom{N-1}{n-1} \cdot c_1(R\pi^M_* (G \otimes E)) \right) \cap [M_{n,\beta}(X)]^{\text{vir}} & \text{if } m = n \\
-\binom{N}{n} \cdot c_1(R\pi^P_* G) & \text{otherwise}
\end{cases}$$

where $\pi^P : P_{n,\beta}(X) \times X \to P_{n,\beta}(X)$ is the projection map, $r$ is the rank of $E$, and $N = rn + \int_\beta c_1(E)$ is the rank of the tautological complex $R\pi^P_* (F \otimes E)$.

We will prove the pullback formula (4.3) through the virtual pullback formula in Theorem 2.2 by comparing the symmetric obstruction theories of $P_{n,\beta}(X)$ and $M_{n,\beta}(X)$. Then the pushforward formula (4.4) will follow directly from the general pushforward formula for virtual projective bundles in Proposition 4.2.

Before we prove our main theorem in this section (Theorem 4.3), we first provide immediate corollaries. Recall [16, 17] that the primary invariants for a cohomology class $\gamma \in H^4(X, \mathbb{Q})$ are defined as follows:

$$P_{n,\beta}(\gamma) := \int_{[P_{n,\beta}(X)]^{\text{vir}}} (\pi^P_* (\text{ch}_3(F) \cup \gamma))^n$$

$$M_{n,\beta}(\gamma) := \int_{[M_{n,\beta}(X)]^{\text{vir}}} \pi^M_* (\text{ch}_3(G) \cup \gamma)$$

where the primary invariant $M_{n,\beta}(\gamma)$ does not depend on the choice of the universal family $G$.

**Corollary 4.4** (Primary PT/GV correspondence). Let $X$ be a Calabi-Yau 4-fold and $\beta \in H_2(X, \mathbb{Q})$ be an irreducible curve class. Assume that $M_{n,\beta}(X)$ has a universal family. Then there exists a choice of orientations such that

$$P_{n,\beta}(\gamma) = \begin{cases} 
M_{1,\beta}(\gamma) & \text{if } n = 1 \\
0 & \text{if } n \geq 2
\end{cases}$$

for any $\gamma \in H^4(X, \mathbb{Q})$.

**Proof.** By the pushforward formula (4.4), we have

$$p_*[P_{n,\beta}(X)]^{\text{vir}} = \begin{cases} 
[M_{n,\beta}(X)]^{\text{vir}} & \text{if } n = 1 \\
0 & \text{if } n \geq 2.
\end{cases}$$

Since we have

$$\text{ch}_3(F) = \text{ch}_3((p \times 1)^*G \otimes \mathcal{O}(1)) = \text{ch}_3((p \times 1)^*G) = (p \times 1)^*\text{ch}_3(G),$$

the projection formula proves (4.5). \qed
We can define the *tautological invariants* (cf. [12, 19]) for a vector bundle $E$ on $X$ as follows:

$$P_{n,\beta}(E) := \int_{[P_{n,\beta}(X)]^{\vir}} c_n(R\pi_*(F \otimes E))$$

$$M_{n,\beta}(E) := \int_{[M_{n,\beta}(X)]^{\vir}} c_1(R\pi_*(G \otimes E))$$

where the tautological invariant $M_{n,\beta}(E)$ depends on the choice of the universal family $G$.

**Corollary 4.5** (Tautological PT/GV correspondence). *Let $X$ be a Calabi-Yau 4-fold and $\beta$ be an irreducible curve class. Assume that there is a universal family $G$ of $M_{n,\beta}(X)$. For any vector bundle $E$, there exists a choice of orientations such that*

$$P_{n,\beta}(E) = \begin{cases} \left(-\binom{N}{n}\right) \cdot M_{n,\beta}(O_X) & \text{if } n = 0 \\ \left(\frac{N-1}{n-1}\right) \cdot M_{n,\beta}(E) - \binom{N}{n} \cdot M_{n,\beta}(O_X) & \text{if } n \geq 1 \end{cases}$$

*where $N = n \cdot \text{rank}(E) + \int_{\beta} c_1(E)$ is the rank of $R\pi_* F \otimes E$.*

We can generalize Theorem 4.3 to reducible curve classes as follows: When $M_{n,\beta}(X)$ has no strictly semi-stable sheaves, the moduli space of Joyce-Song type stable pairs in [20] (cf. [54, 33]) can be expressed as $P_{n,\beta}^JS(X) = \{(F, s) : F \in M_{n,\beta}(X) \text{ and } s : O_X \to F \text{ is non-zero}\}$.

By [20, Theorem 0.1], $P_{n,\beta}^JS(X)$ is an open subscheme of the moduli space $\text{Perf}(X)^{\text{spl}}_{O_X}$ of simple perfect complexes with fixed trivial determinant. Hence $P_{n,\beta}^JS(X)$ carries an Oh-Thomas virtual cycle $[P_{n,\beta}^JS(X)]^{\vir} \in A_n(P_{n,\beta}^JS(X))$ by [49]. For any curve class $\beta \in H_2(X, \mathbb{Q})$, there is a forgetful map

$$p : P_{n,\beta}^JS(X) \to M_{n,\beta}(X) : (F, s) \mapsto F$$

which identifies $P_{n,\beta}^JS(X)$ with the virtual projective bundle of $R\pi_* M G$. The proof of Theorem 4.3 also works for $P_{n,\beta}^JS(X)$ and gives us analogous pullback formula (4.3) and the pushforward formula (4.4). In particular, we have the following corollary:

**Corollary 4.6** (JS/GV correspondence). *Let $X$ be a Calabi-Yau 4-fold, $\beta \in H_2(X, \mathbb{Q})$ be a curve class, and $n$ be an integer. Assume that $\int_{\beta} c_1(O_X(1))$ and $n$ are coprime. Then there exists a choice of orientations such that*

$$P_{n,\beta}^JS(\gamma) = \begin{cases} M_{1,\beta}(\gamma) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

$$P_{n,\beta}^JS(E) = \begin{cases} \left(-\binom{N}{n}\right) \cdot M_{0,\beta}(O_X) & \text{if } n = 0 \\ \left(\frac{N-1}{n-1}\right) \cdot M_{n,\beta}(E) - \binom{N}{n} \cdot M_{n,\beta}(O_X) & \text{if } n \geq 1 \end{cases}$$

*for any cohomology class $\gamma$ and a vector bundle $E$, where the primary invariant $P_{n,\beta}^JS(\gamma)$ and the tautological invariant $P_{n,\beta}^JS(E)$ are defined analogously.*
The rest of this section is devoted to the proof of Theorem 4.3 (1).

4.3. Comparison of obstruction theories. Given \(X, \beta, n\) as in Theorem 4.3, consider the cartesian diagram

\[
P_{n,\beta}(X) \times X \rightarrow M_{n,\beta}(X) \times X
\]

\[
P_{n,\beta}(X) \rightarrow M_{n,\beta}(X)
\]

of schemes. Recall that the Oh-Thomas virtual cycles on \(P_{n,\beta}(X)\) and \(M_{n,\beta}(X)\) are constructed from the symmetric obstruction theories \(\phi_P: RHom_\pi(\mathbb{I}, \mathbb{I})_0[3] \xrightarrow{At} \mathbb{L}_{P_{n,\beta}(X)}\)

\[
(4.6)
\]

and \(\phi_M: (\tau_{[1,3]} RHom_\pi M(G, G))[3] \xrightarrow{At(G)} \tau^{\geq -1} \mathbb{L}_{M_{n,\beta}(X)}\)

\[
(4.7)
\]

induced by the Atiyah classes of the universal complex \(\mathbb{I} = [\mathcal{O}_{P_{n,\beta}(X) \times X} \rightarrow \mathcal{F}]\) on \(P_{n,\beta}(X) \times X\) and the universal sheaf \(G\) on \(M_{n,\beta}(X) \times X\). Note that the universal sheaf \(\mathcal{F}\) on \(P_{n,\beta}(X) \times X\) can be written as

\[
\mathcal{F} = (p \times 1)^* \mathcal{G} \otimes (\pi^P)^* \mathcal{O}(1)
\]

where \(\mathcal{O}(1)\) is the dual of the tautological line bundle of the virtual projective bundle \(P_{n,\beta}(X) = P(\mathcal{R}_\pi M \mathcal{G})\).

**Proposition 4.7.** The two symmetric obstruction theories \(\phi_P\) and \(\phi_M\) in (4.6) and (4.7) are compatible as follows:

1. We have a reduction diagram (see Proposition 1.5), i.e., a morphism of distinguished triangles

\[
RHom_\pi(\mathbb{I}, \mathcal{F})_\# [2] \xrightarrow{\alpha^\vee} RHom_\pi(\mathbb{I}, \mathbb{I})_0[3] \xrightarrow{\delta} RHom_\pi(\mathcal{F}, \mathcal{O})_\# [4]
\]

\[
(\tau_{[1,3]} RHom_\pi M(\mathcal{F}, \mathcal{F}))[3] \xrightarrow{\beta} RHom_\pi M(\mathcal{F}, \mathbb{I})_\# [4] \xrightarrow{\gamma} RHom_\pi M(\mathcal{F}, \mathcal{O})_\# [4]
\]

for some maps \(\alpha, \beta, \gamma, \delta\) and perfect complexes \(RHom_\pi M(\mathcal{F}, \mathcal{O})_\#\), \(RHom_\pi M(\mathcal{F}, \mathcal{I})_\#\), and \(RHom_\pi M(\mathbb{I}, \mathbb{I})_\# := (RHom_\pi M(\mathcal{F}, \mathbb{I})_\#)^{-\vee}[-4].\)

\[^{10}\text{Here the Atiyah class } At(G) \text{ depends on the choice of the universal family } G, \text{ but the symmetric obstruction theory } \phi_M \text{ does not depend of the choice of } G. \text{ See Remark 4.10.}\]
The Atiyah classes of \(I\) and \(G\) are compatible, i.e., the diagram

\[
\begin{array}{ccc}
\mathbf{R} \mathbf{H} \text{om}_\pi (I, F) \# [2] & \xrightarrow{\alpha^\vee} & \mathbf{R} \mathbf{H} \text{om}_\pi (I, I)_0 [3] \\
\downarrow \beta^\vee & & \downarrow \text{At}(I) \\
(\tau^{[1,3]} \mathbf{R} \mathbf{H} \text{om}_\pi (F, F)) [3] & \xrightarrow{\text{At}(G)} & \mathbf{R} \mathbf{H} \text{om}_\pi (I, I) _0 [3] \\
\tau \geq -1 \mathbb{L} M_{n, \beta}(X) |_{P_{n, \beta}(X)} & \xrightarrow{\text{At}(F)} & \tau \geq -1 \mathbb{L} P_{n, \beta}(X)
\end{array}
\]

commutes.

We will prove Proposition 4.7 through several steps. The proof is similar to that of Proposition 3.13 in §3.4. To simplify the notations, let

\[ P = P_{n, \beta}(X), \quad \mathcal{X} = P_{n, \beta}(X) \times X, \quad \pi = \pi^P, \quad \text{and} \quad \mathcal{M} = M_{n, \beta}(X) \]

be the abbreviations of the schemes. Let

\[
\begin{array}{c}
I \xrightarrow{i} \mathcal{O}_\mathcal{X} \xrightarrow{s} F \xrightarrow{e} I[1]
\end{array}
\]

be the canonical distinguished triangle on \(\mathcal{X}\).

4.3.1. Prototype. We first consider a prototype of the reduction diagram (4.8) where we use

\[ \mathbf{R} \mathbf{H} \text{om}_\pi (F, F), \quad \mathbf{R} \mathbf{H} \text{om}_\pi (F, \mathcal{O}_\mathcal{X}) \quad \text{and} \quad \mathbf{R} \mathbf{H} \text{om}_\pi (F, I) \]

instead of \(\tau^{[1,3]} \mathbf{R} \mathbf{H} \text{om}_\pi (F, F), \mathbf{R} \mathbf{H} \text{om}_\pi (F, \mathcal{O}_\mathcal{X}) \#, \) and \( \mathbf{R} \mathbf{H} \text{om}_\pi (F, I) \# \).

Lemma 4.8. We have a (not necessarily commutative) diagram of two distinguished triangles

\[
\begin{array}{ccc}
\mathbf{R} \mathbf{H} \text{om}_\pi (I, F) [2] & \xrightarrow{\rho \circ l} & \mathbf{R} \mathbf{H} \text{om}_\pi (I, I)_0 [3] \\
\downarrow r_c & & \downarrow r_c \circ l \\
\mathbf{R} \mathbf{H} \text{om}_\pi (F, F) [3] & \xrightarrow{l_c} & \mathbf{R} \mathbf{H} \text{om}_\pi (F, F) [4] \\
\mathbf{R} \mathbf{H} \text{om}_\pi (I, I)_0 & \xrightarrow{\iota \circ \rho} & \mathbf{R} \mathbf{H} \text{om}_\pi (I, I)_0 \\
\uparrow tr & & \uparrow tr \\
\mathbf{R} \mathbf{H} \text{om}_\pi (F, F) [4] & \xrightarrow{l_i} & \mathbf{R} \mathbf{H} \text{om}_\pi (F, F) [4] \\
\end{array}
\]

for some map \(\eta\) such that \(l_c \circ \eta = \text{id}_1 \circ r_s\). Here \(r_i, r_s, r_c\) (resp. \(l_i, l_s, l_c\)) are the right (resp. left) composition maps with the maps \(i, s, e\) in (4.10), and \(\iota, \rho\) are the canonical maps in the direct sum diagram

\[
\begin{array}{ccc}
\mathbf{R} \mathbf{H} \text{om}_\pi (I, I)_0 & \xrightarrow{\iota} & \mathbf{R} \mathbf{H} \text{om}_\pi (I, I)_0 \\
\xrightarrow{\rho \circ \text{id}_1} & & \xrightarrow{\text{id}_1} \mathbf{R} \mathbf{H} \text{om}_\pi (I, I)_0 \\
\end{array}
\]

where \(tr\) denotes the trace map.
Proof. Applying $\mathbb{R}\text{Hom}_{\chi}(\mathcal{F}, -)$ to (4.10), we obtain the lower distinguished triangle in (4.11). Applying the octahedral axiom to the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{\chi} & \xrightarrow{id} & \mathbb{R}\text{Hom}_{\chi}(\mathcal{O}_{\chi}, \mathcal{O}_{\chi}) \\
& \downarrow{r_i} & \\
\mathbb{R}\text{Hom}_{\chi}(\mathcal{I}, \mathcal{I}) & \xrightarrow{l_i} & \mathbb{R}\text{Hom}_{\chi}(\mathcal{I}, \mathcal{O}_{\chi})
\end{array}
$$

with the obvious three distinguished triangles given by (4.10) and (4.12), we also obtain the upper distinguished triangle in (4.11).

Clearly, the middle square in (4.11) commutes. However, the left square in (4.11) may not commute.

4.3.2. We will replace the complexes in (4.11) as

\begin{align*}
\mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{F}) & \xrightarrow{\tau^{[1, 3]\text{tr}_4}} \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{F}) \\
& = \text{cone}(\text{cone}(\mathcal{O}_p \rightarrow \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{F})) \rightarrow \mathcal{O}_p[-4])[1] \\
\mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{O}_{\chi}) & \xrightarrow{\tau^{[1, 3]\text{tr}_4}} \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{O}_{\chi}) \\
& = \text{cone}(\mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{O}_{\chi}) \rightarrow \mathcal{O}_p[-4])[1] \\
\mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{I}) & \xrightarrow{\tau^{[1, 3]\text{tr}_4}} \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{I}) \\
& = \text{cone}(\mathcal{O}_p[-1] \rightarrow \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{I})[4])
\end{align*}

to obtain the commutative diagram (4.8). Here we fix some notations among them.

(1) We can form a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_p & \xrightarrow{id_p} & \mathcal{O}_p \\
& \downarrow{\tau^{\leq 3}\text{tr}_4} & \\
\tau^{[1, 3]}\mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{F}) & \xrightarrow{\mu} & \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{F}) \\
& \downarrow{\nu} & \\
& & \mathcal{O}_p[-4]
\end{array}
$$

of four distinguished triangles for unique maps $\mu$, $\nu$, and $t$ (cf. Lemma [C.2]). Here $\text{tr}_4$ denotes the top trace map.

(2) We can also form the following two distinguished triangles

$$
\begin{align*}
\mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{O}_{\chi}) & \xrightarrow{\kappa} \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{O}_{\chi}) \\
& \rightarrow \mathcal{O}_p[-4] \\
\mathcal{O}_p[-1] & \xrightarrow{l_{\text{tr}_4}} \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{I}) & \xrightarrow{\lambda} \mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{I}) \\
& \rightarrow \mathcal{O}_p[-4]
\end{align*}
$$

over $\mathcal{P}$. Let $\mathbb{R}\text{Hom}_{\pi}(\mathcal{I}, \mathcal{F}) := (\mathbb{R}\text{Hom}_{\pi}(\mathcal{F}, \mathcal{I}))^{\vee}_{\#}[-4]$.
4.3.3. Maps $\alpha, \beta, \gamma, \delta$. We now construct the maps $\alpha, \beta, \gamma, \delta$ and the distinguished triangles in (4.8) as follows:

**Lemma 4.9.** We have the following diagrams:

1. There is a commutative diagram of four distinguished triangles (4.16)

   \[
   \begin{array}{ccc}
   \mathcal{O}_P[3] & \longrightarrow & \mathcal{O}_P[3] \\
   \downarrow \text{id}_F & & \downarrow l_e \circ \text{id}_F \\
   \mathbf{R}\text{Hom}_\pi(F,F)[3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(F,F)[3] \\
   \downarrow \mu & & \downarrow \lambda \\
   (\tau^{\geq 1}\mathbf{R}\text{Hom}_\pi(F,F))[3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(F,F)$#][4] \longrightarrow \mathbf{R}\text{Hom}_\pi(F,F)[4] \\
   \end{array}
   \]

   for some maps $\zeta$ and $\omega$. Here the three given distinguished triangles are those in (4.13), (4.15), and (4.10).

2. There is a commutative diagram of four distinguished triangles (4.17)

   \[
   \begin{array}{ccc}
   \mathbf{R}\text{Hom}_\pi(F,O_X)[3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(F,O_X)[3] \\
   \downarrow \mu_{\text{ol}} & & \downarrow \text{tr}^4_{\text{ol}} \\
   (\tau^{[1,3]}\mathbf{R}\text{Hom}_\pi(F,F))[3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(F,F)[3] \\
   \downarrow \zeta & & \downarrow \beta \\
   (\tau^{[1,3]}\mathbf{R}\text{Hom}_\pi(F,F))[3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(F,F)$#][4] \longrightarrow \mathbf{R}\text{Hom}_\pi(F,F)[4] \\
   \end{array}
   \]

   for some maps $\beta$ and $\gamma$ such that $\kappa \circ \gamma = \omega$. Here the three given distinguished triangles are those in (4.16), (4.14), and (4.13).

3. There is a commutative diagram of four distinguished triangles (4.18)

   \[
   \begin{array}{ccc}
   \mathbf{R}\text{Hom}_\pi(F,O_X)[3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(F,O_X)[3] \\
   \downarrow \eta & & \downarrow \text{tr}^4_{\text{ol}} \\
   \mathbf{R}\text{Hom}_\pi(I,F)[3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(I,F)[3] \\
   \downarrow \rho_{\text{ol}} & & \downarrow \delta \\
   \mathbf{R}\text{Hom}_\pi(I,F)$#][3] & \longrightarrow & \mathbf{R}\text{Hom}_\pi(F,O_X)$#][4] \\
   \end{array}
   \]

   for some maps $\alpha$ and $\delta$ such that $\kappa \circ \delta = r_e \circ l_i \circ \iota$. Here the three given distinguished triangles are those in (4.11), (4.14), and (4.15).

**Proof.** (1) and (2) follow directly from the octahedral axiom. We will only prove (3). It suffices to show that the top right square in (4.18) commutes.
Consider a diagram

\[
\begin{array}{ccc}
\mathbf{R}\text{Hom}_X(F, \mathcal{O}_X) & \xrightarrow{\eta} & \mathbf{R}\text{Hom}_X(I, F) \\
\downarrow r_s & & \downarrow r_e \\
\mathcal{O}_X & \xrightarrow{id_i} & \mathbf{R}\text{Hom}_X(I, I) \\
\end{array}
\]

of commutative squares where the left square in (4.19) commutes by Lemma 4.8. Then we have

\[
tr \circ r_e \circ \eta = tr \circ l_e \circ \eta = tr \circ id_i \circ r_s = r_s = tr \circ l_s.
\]

Applying \( R\pi_* \) to (4.19), we obtain the desired commutative diagram. \( \square \)

### 4.3.4. Reduction diagram

Now we can form the reduction diagram (4.8).

**Proof of Proposition 4.7(1).**  We first prove the commutativity of the left square in (4.8). Indeed, consider the commutative diagram

\[
\begin{array}{ccc}
\mathbf{R}\text{Hom}_\pi(I, F)_{\#}[2] & \xrightarrow{\lambda^\vee} & \mathbf{R}\text{Hom}_\pi(I, F)[2] \\
\downarrow \zeta^\vee & & \downarrow r_e \\
(\tau^{\leq 3}\mathbf{R}\text{Hom}_\pi(F, F))[3] & \xrightarrow{\mu^\vee} & \mathbf{R}\text{Hom}_\pi(F, F)[3] \\
\downarrow \nu^\vee & & \downarrow l_e \\
(\tau^{[1,3]}\mathbf{R}\text{Hom}_\pi(F, F))[3] & \xrightarrow{\nu} & \tau^{\geq 1}\mathbf{R}\text{Hom}_\pi(F, F)[3] \\
\end{array}
\]

given by (4.13) and (4.16). Note that \( \alpha \) and \( \beta \) are defined as the compositions

\[
\alpha = \lambda \circ r_e \circ \tau : \mathbf{R}\text{Hom}_\pi(I, I)_0[3] \to \mathbf{R}\text{Hom}_\pi(F, I)_{\#}[4]
\]

\[
\beta = \zeta \circ \nu : (\tau^{[1,3]}\mathbf{R}\text{Hom}_\pi(F, F))[3] \to \mathbf{R}\text{Hom}_\pi(F, I)_{\#}[4]
\]

in (4.18) and (4.17). Thus it suffices to show

\[
(4.20) \quad \lambda \circ r_e \circ \tau \circ l_e \circ \lambda^\vee = \lambda \circ r_e \circ l_e \circ \lambda^\vee
\]

to prove \( \alpha \circ \alpha^\vee = \beta \circ \beta^\vee \). By the dual diagram of (4.16), we have

\[
(4.21) \quad tr^4 \circ l_e \circ \lambda^\vee = tr^4 \circ r_e \circ \lambda^\vee = 0.
\]

Since the pairing on \( \mathbf{R}\text{Hom}_\pi(I, I)[3] \) is given by the composition

\[
\mathbf{R}\text{Hom}_\pi(I, I)[3] \otimes \mathbf{R}\text{Hom}_\pi(I, I)[3] \xrightarrow{\cup} \mathbf{R}\text{Hom}_\pi(I, I)[6] \xrightarrow{tr^4} \mathcal{O}_\mathcal{P}[2],
\]

the equation (4.21) proves (4.20).

We then prove the commutativity of the right square in (4.8). Since

\[
\text{Hom}_\mathcal{P}(\mathbf{R}\text{Hom}_\pi(I, I)_0[3], \mathcal{O}_\mathcal{P}[-1]) = 0,
\]

we have
it suffices to show that the square in the diagram

\[
\begin{array}{ccc}
\mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{I}, \mathbb{I})_0 [3] & \xrightarrow{\delta} & \mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{F}, \mathcal{O}_X) \# [4] \\
\downarrow_{\alpha = \lambda \circ \rho \circ \iota} & & \downarrow_{\kappa} \\
\mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{F}, \mathbb{I}) \# [4] & \xrightarrow{\gamma} & \mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{F}, \mathcal{O}_X) \# [4]
\end{array}
\]

\[
\mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{F}, \mathcal{O}_X) [4] \xrightarrow{\omega}
\]

commutes. By Lemma (4.9)(2), Lemma (4.9)(3), and equation $\omega \circ \lambda = l_i$ in (4.16), we have

\[\kappa \circ \gamma \circ \alpha = \omega \circ \lambda \circ \rho \circ \iota = l_i \circ \rho \circ \iota = \rho \circ l_i \circ \iota = \kappa \circ \delta,\]

which proves the claim.

By Lemma (4.4) the commutativity of the two squares in (4.8) suffices to form the desired reduction diagram. It completes the proof. \qed

4.3.5. Atiyah classes. Finally, we compare the Atiyah classes.

Proof of Proposition (4.7)(2). We first compare the Atiyah classes of $\mathbb{I}$ and $\mathbb{F}$. The functoriality of Atiyah classes

\[
\begin{array}{ccc}
\mathbb{F} & \xrightarrow{e} & \mathbb{I}[1] \\
\downarrow \mathbf{At}(\mathbb{F}) & & \downarrow \mathbf{At}(1) \\
\mathbb{F} \otimes \mathbb{L}_X[1] & \xrightarrow{e \otimes 1} & \mathbb{I} \otimes \mathbb{L}_X[2]
\end{array}
\]

gives us a commutative diagram

\[
\begin{array}{ccc}
\mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{I}, \mathbb{F}) [2] & \xrightarrow{l_e} & \mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{I}, \mathbb{I}) [3] \\
\downarrow r_e & & \downarrow \mathbf{At}(1) \\
\mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{F}, \mathbb{F}) [3] & \xrightarrow{\mathbf{At}(\mathbb{F})} & \mathbb{L}_P.
\end{array}
\]

Since $\lambda \circ l_e = \zeta \circ \mu$ by (4.16), $\alpha^\vee = \rho \circ l_e \circ \lambda^\vee$ by (4.18), and the composition

\[
\mathbf{R} \pi_\ast \mathcal{O}_X [3] \xrightarrow{\text{id}} \mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{I}, \mathbb{I}) [3] \xrightarrow{\mathbf{At}(1)} \mathbb{L}_P
\]

is zero by [56, Proposition 3.2], the commutative diagram (4.22) induces a commutative diagram

\[
\begin{array}{ccc}
\mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{I}, \mathbb{F}) \# [2] & \xrightarrow{\alpha^\vee} & \mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{I}, \mathbb{I})_0 [3] \\
\downarrow \zeta^\vee & & \downarrow \mathbf{At}(1) \\
(\tau \leq 3 \mathbf{R} \mathbf{H} \text{om}_\pi (\mathbb{F}, \mathbb{F}) ) [3] & \xrightarrow{\mathbf{At}(\mathbb{F})} & \mathbb{L}_P.
\end{array}
\]
Since $\text{Hom}_P(\mathcal{O}_P[4], \tau \geq -1 \mathcal{L}_P) = 0$, the commutative diagram \cite{4.23} gives us the commutativity of the upper square in \cite{11}.

We then compare the Atiyah classes of

$$(p \times 1)^* \mathcal{G} \quad \text{and} \quad F = (p \times 1)^* \mathcal{G} \otimes \pi^* \mathcal{O}_P(1).$$

By \cite{29}, p. 260, we have

\begin{equation}
\text{At}(F) = \text{At}((p \times 1)^* \mathcal{G}) \otimes \text{id}_{\pi^* \mathcal{O}_P(1)} + \text{id}_{(p \times 1)^* \mathcal{G}} \otimes \text{At}(\pi^* \mathcal{O}_P(1)) : F \to F \otimes \mathcal{L}_X[1]. \tag{4.24}
\end{equation}

Also, we can deduce

$$\text{At}(\pi^* \mathcal{O}_P(1)) = (\pi^* \text{At}(\mathcal{O}_P(1)), 0) : \mathcal{O}_X \to \mathcal{L}_X[1] = \pi^* \mathcal{L}_P[1] \oplus (\pi^X)^* \mathcal{L}_X[1]$$

from the commutative diagram

$$\xymatrix{ X \ar[r]^-{\pi} \ar[rrd]_-{\pi^* \mathcal{O}_P(1)} & \mathcal{P} \ar[r]^-{\mathcal{O}_P(1)} & \mathcal{P} \ar[d] \ar[r]^-{\text{At}(\mathcal{F})} & \mathcal{L}_P \ar[l]_-{\text{At}((p \times 1)^* \mathcal{G})} }$$

by \cite{56}, Appendix A, where $\mathcal{P}$ denotes the derived moduli stack of perfect complexes and $\pi^X : \mathcal{P} \times X \to X$ denotes the projection map. Hence the difference of the two induced maps

\begin{equation}
\mathbf{R} \text{Hom}_\pi(F, F)[3] \xrightarrow{\text{At}(F)} \mathcal{L}_P \tag{4.25}
\end{equation}

is the composition

$$\mathbf{R} \text{Hom}_\pi(F, F)[3] \xrightarrow{\text{tr}^4} \mathcal{O}_P[-1] \xrightarrow{\text{At}(\mathcal{O}_P(1))} \mathcal{L}_P.$$ 

By \cite{11}, the compositions of the two maps in \cite{11} with the map

$$\mu^\vee : (\tau \leq 3) \mathbf{R} \text{Hom}_\pi(F, F)[3] \to \mathbf{R} \text{Hom}_\pi(F, F)[3]$$

coincide. Since $\text{Hom}_P(\mathcal{O}_P[4], \tau \geq -1 \mathcal{L}_P) = 0$, the two maps

\begin{equation}
(\tau[1, 3]) \mathbf{R} \text{Hom}_\pi(F, F)[3] \xrightarrow{\text{At}(F)} \tau \geq -1 \mathcal{L}_P \tag{4.26}
\end{equation}

given by the Atiyah classes of $F$ and $(p \times 1)^* \mathcal{G}$ are equal.

Finally, we compare the Atiyah classes of $\mathcal{G}$ and $(p \times 1)^* \mathcal{G}$. From the commutative diagram

$$\xymatrix{ \mathcal{P} \ar[r]^-{p} \ar[rrd]_-{(p \times 1)^* \mathcal{G}} & \mathcal{M} \ar[r]^-{\mathcal{G}} & \mathcal{P} \ar[d] \ar[r]^-{\text{Perf} (X)} & \mathcal{P} \ar[l]_-{\text{At}(F)} }$$

\[\text{In} \] \cite{29}, the formula \cite{4.24} is proved for Atiyah classes on smooth schemes, but \cite{31} Chapitre IV, 2.3.7] shows that it also holds for Atiyah classes on arbitrary schemes.
we can deduce that the triangle
\[(4.27)\]
\[
\begin{array}{cccc}
p^* \mathcal{R} \text{Hom}_\pi(G, G)[3] & \\
\text{At}(G) & \text{At}(p \times 1)^* G & \\
p^* \mathcal{L}_M & \mathcal{L}_P & \\
\end{array}
\]
commutes by \[56\] Appendix A.

Combining the commutative triangle \[(4.27)\] with the equality \[(4.26)\], we deduce the commutativity of the lower triangle in \[(4.9)\]. □

**Remark 4.10.** By the arguments in the second paragraph of the proof of Proposition \[4.7(2)\], we can deduce that the symmetric obstruction theory \(\phi_M\) in \[(4.7)\] is independent of the choice of the universal family \(G\).

### 4.4. Comparison of virtual cycles

Finally, we can prove the pullback formula \[(4.3)\] in Theorem \[4.3\] from the compatibility of the obstruction theories in Proposition \[4.7\] and the virtual pullback formula in Theorem \[2.2\].

**Proof of Theorem \[4.3\].** By Proposition \[4.7\] we can form morphisms of distinguished triangles
\[(4.28)\]
\[
\begin{array}{cccc}
\mathcal{R} \text{Hom}_\pi(I, F)_\# [2] & \xrightarrow{\alpha^\vee} & \mathcal{R} \text{Hom}_\pi(I, I)_\# [3] & \xrightarrow{\delta} & \mathcal{R} \text{Hom}_\pi(F, O_X)_\# [4] \\
\downarrow & & \downarrow & & \downarrow \\
(\tau^{[1,3]} \mathcal{R} \text{Hom}_\pi(F, F))[3] & \xrightarrow{\beta^\vee} & \mathcal{R} \text{Hom}_\pi(F, I)_\# [4] & \xrightarrow{\gamma} & \mathcal{R} \text{Hom}_\pi(F, O_X)_\# [4] \\
\downarrow & & \downarrow & & \downarrow \\
\tau^\geq -1 p^* \mathcal{L}_M & \xrightarrow{\phi_M^\vee} & \tau^\geq -1 \mathcal{L}_P & \xrightarrow{\phi_P^\vee} & \tau^\geq -1 \mathcal{L}_{P/M} \\
\end{array}
\]
for some maps \(\phi_P^\vee\) and \(\phi\) such that \(\phi_P^\vee \circ \alpha = \phi^\vee\), as in the proof of Theorem \[3.1\] in \[3.3\]. By the long exact sequence associated to \[(4.28)\], we deduce that
\[(4.29)\]
\[
\phi : \mathcal{R} \text{Hom}_\pi(F, O_X)_\# [4] \xrightarrow{\phi_P^\vee} \mathcal{L}_P^{M}/M \rightarrow \tau^\geq -1 \mathcal{L}_{P/M} \cong \tau^\geq -1 \mathcal{L}_{P/M}
\]
is a perfect obstruction theory. Since the virtual pullback
\[
p^! : A_*(M) \rightarrow A_*(P)
\]
depends only on the virtual cotangent complex \(\mathcal{R} \text{Hom}_\pi(F, O_X)_\# [4]\), but not on the map \(\phi\) by Lemma \[4.11\] below, the virtual pullback given by the perfect obstruction theory \[(4.29)\] is equal to the virtual pullback given by the perfect obstruction theory of the virtual projective bundle \(P = \mathbb{P}(\mathcal{R} \pi^* M G)\) in Definition \[4.4\].

Since we have a compatible triple of obstruction theories \[(4.28)\], we have a virtual pullback formula
\[
p^! [M]^{\text{vir}} = [P]^{\text{vir}}
\]
by Theorem \[2.2\]. It completes the proof. □
We need Lemma 4.11 below to complete the proof of Theorem 4.3 above.

**Lemma 4.11.** Let $\psi_1, \psi_2 : K \to L_{X/Y}$ be two perfect obstruction theories for a morphism $f : X \to Y$ of quasi-projective schemes. Then the two associated virtual pullbacks coincide.

**Proof.** Since the virtual pullbacks commute with projective pushforwards, it suffices to show $f^!_{\psi_1}(\mathcal{Y}) = f^!_{\psi_2}(\mathcal{Y})$ for an integral scheme $Y$. Choose a global resolution $[K_1 \to K_0] \cong K$ of vector bundles. Consider the fiber diagrams

\[
\begin{array}{ccc}
C_1 & \longrightarrow & K_1 \\
\downarrow \scriptstyle{p_1} & & \downarrow \scriptstyle{p_2} \\
\mathcal{C}_X \cong [K_1/K_0] & \longrightarrow & \mathcal{C}_X \cong [K_1/K_0]
\end{array}
\]

where the two closed embeddings $\mathcal{C}_X \cong [K_1/K_0]$ are given by the two obstruction theories $\psi_1$ and $\psi_2$. Hence we have two short exact sequence

\[
K_0 \longrightarrow C_3 \longrightarrow C_1 \quad \text{and} \quad K_0 \longrightarrow C_3 \longrightarrow C_2
\]

of cones over $X$. Therefore, we have

\[
f^!_{\psi_1}(\mathcal{Y}) = 0^!_{K_1}[C_1] = (s(C_1) \cdot c(K_1))_{vd} = (s(C_3) \cdot c(K_0) \cdot c(K_1))_{vd} = f^!_{\psi_2}(\mathcal{Y})
\]

by [24, Example 4.1.8], where $vd = \dim(Y) + \text{rank}(K)$. \qed

**Appendix A. Torus localization without quasi-projectivity**

Here we generalize square root virtual pullbacks in §1 and its functoriality in §2 to DM morphisms between algebraic stacks. The Kimura sequence in Lemma A.1 is the essential ingredient. As a corollary, we prove the torus localization formula without assuming the quasi-projectivity hypothesis.

**A.1. Kimura sequence.**

**Lemma A.1 ([2] 38).** Let $p : \mathcal{Y} \to \mathcal{X}$ be a projective surjective morphism between algebraic stacks with affine stabilizers. Then we have a right exact sequence

\[
A_*(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}) \xrightarrow{(p_1)_* - (p_2)_*} A_*(\mathcal{Y}) \xrightarrow{p_*} A_*(\mathcal{X}) \longrightarrow 0,
\]

where $p_1, p_2 : \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ denote the projection maps.

For schemes, Lemma A.1 was proved in [38]. The proof of Lemma A.1 for Artin stacks will appear in a forthcoming paper [2].
A.2. Square root virtual pullback and functoriality. Note that the definitions of symmetric complexes (Definition 1.2), quadratic functions (Definition 1.7), symmetric obstruction theories (Definition 1.9), and the isotropic condition (Definition 1.10) can be generalized to algebraic stacks and DM morphisms between algebraic stacks in a straightforward manner.

**Definition A.2.** Let $E$ be a symmetric complex on an algebraic stack $X$. Let $Q(E)$ be the zero locus of the quadratic function $q_E : C(E) \to \mathbb{A}^1_X$ on the associated abelian cone stack $C(E)$. We define the square root Gysin pullback

$$\sqrt{0^!_{\Omega(E)}} : A_*(Q(E)) \to A_*(X)$$

in the following cases:

1. Assume that $X$ is a separated DM stack. By the Chow lemma, there exists a projective surjective map $p : \tilde{X} \to X$ from a quasi-projective scheme $\tilde{X}$. Define the square root Gysin pullback as the unique map that fits into the commutative diagram

$$
\begin{array}{ccc}
A_*(Q(E|_{\tilde{X} \times X})) & \to & A_*(Q(E|_{\tilde{X}})) \\
\sqrt{0^!_{\Omega(E|_{\tilde{X} \times X})}} & \to & \sqrt{0^!_{\Omega(E|_{\tilde{X}})}} \\
A_*(\tilde{X} \times_X \tilde{X}) & \to & A_*(\tilde{X}) \\
\end{array}
$$

where the other two square root Gysin pullbacks are well-defined as square root virtual pullbacks since $\tilde{X}$ and $\tilde{X} \times_X \tilde{X}$ are quasi-projective schemes, and the two horizontal sequences are exact by the Kimura sequence (Lemma A.1). It is easy to show that $\sqrt{0^!_{\Omega(E)}}$ is independent of the choice of the projective cover $p : \tilde{X} \to X$ using the Kimura sequence.

2. More generally, assume that $X = [P/G]$ is the quotient stack of a separated DM stack $P$ by an action of a linear algebraic group $G$. By [61], there exist $G$-representations $V_i$ and a $G$-invariant open subschemes $U_i \subseteq V_i$ such that $U_i/G$ are quasi-projective schemes and $\text{codim}_{V_i \setminus U_i} V_i \geq i$. We may regard $r_i : \mathcal{X}_i := [(P \times U_i)/G] \to X$ as approximations of $X$ by separated DM stacks $\mathcal{X}_i$. By the homotopy property of Chow groups [10, Corollary 2.5.7], we can define the square root Gysin pullback as

$$
\begin{array}{ccc}
A_d(\Omega(E)) & \to & A_d(X) \\
\sqrt{0^!_{\Omega(E)}} & \to & \sqrt{0^!_{\Omega(\mathcal{X}_i)}} \\
A_{d+d_i}(\Omega(E|_{\mathcal{X}_i})) & \to & A_{d+d_i}(\mathcal{X}_i) \\
\end{array}
$$

for big enough $i$ for each $d$, where $d_i$ denotes the relative dimension of $r_i$. It is easy to show that $\sqrt{0^!_{\Omega(E)}}$ is independent of the choice
of the approximation $r_i : \mathcal{X}_i \to \mathcal{X}$ using the homotopy property of Chow groups.

**Definition A.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a DM morphism between algebraic stacks equipped with a symmetric obstruction theory $\phi : E \to L_f$ satisfying the isotropic condition. Then we have a closed embedding $a : \mathcal{C}_f \hookrightarrow \Omega(E)$. Assume that $\mathcal{X}$ is the quotient stack of a separated DM stack by an action of a linear algebraic group. We define the square root virtual pullback as the composition

$$\sqrt{f^!} : A_*(\mathcal{Y}) \xrightarrow{\text{sp}_f} A_*(\mathcal{C}_f) \xrightarrow{a_*} A_*(\Omega(E)) \xrightarrow{\sqrt[0]{\Omega(E)}} A_*(\mathcal{X})$$

where $\text{sp}_f$ denotes the specialization map.

It is easy to show that $\sqrt{f^!}$ commutes with projective pushforwards, smooth pullbacks, and Gysin pullbacks for regular immersions.

**Theorem A.4.** Consider a commutative diagram (2.1) of DM morphisms between algebraic stacks equipped a compatible triple $(\phi_f, \phi_g, \phi_{g\circ f})$ of obstruction theories in the sense of Definition 2.1. Assume that $\phi_g$ and $\phi_{g\circ f}$ satisfy the isotropic condition. We further assume the followings:

(1) $\mathcal{Y}$ is the quotient stack of a separated DM stack by an action of a linear algebraic group,

(2) $f : \mathcal{X} \to \mathcal{Y}$ is quasi-projective, and

(3) $\mathcal{X}$ has the resolution property.

Then we have $\sqrt{(g \circ f)^!} = f^! \circ \sqrt{g^!}$.

**Proof.** Since $\mathcal{X}$ has the resolution property, Proposition 2.7 also holds in this setting. Indeed, the resolution property for $\mathcal{X}$ guarantees Lemma 2.11. The first paragraph of Lemma 2.8 also holds by 37. Hence we can construct a symmetric obstruction theory $\phi_h$ as in Lemma 2.10. The isotropic condition follows from Lemma 2.12 since the isotropic condition can be shown locally.

Now it suffices to show the functoriality for

$$\mathcal{X} \to \mathcal{Y} \to \mathcal{C}_g.$$

Let $\mathcal{Y} = [P/G]$ be the quotient stack of a separated DM stack $P$ by an action of a linear algebraic group $G$. Let $U_i/G \to BG$ be approximations of the classifying stack $BG$ by quasi-projective schemes $U_i/G$. Let $\mathcal{Y}_i = \mathcal{Y} \times_{BG} (U_i/G)$. By the homotopy property of Chow groups, it suffices to show the functoriality for

$$\mathcal{X} \times_\mathcal{Y} \mathcal{Y}_i \to \mathcal{Y}_i \to \mathcal{C}_g \times \mathcal{Y}_i.$$

Since $\mathcal{Y}_i$ is a separated DM stack, we can choose a projective surjective map $\tilde{\mathcal{Y}}_i \to \mathcal{Y}_i$ from a quasi-projective scheme $\tilde{\mathcal{Y}}_i$ by the Chow lemma. By the Kimura sequence (Lemma A.1), it suffices to show the functoriality for

$$\mathcal{X} \times_\mathcal{Y} \tilde{\mathcal{Y}}_i \to \tilde{\mathcal{Y}}_i \to \mathcal{C}_g \times \mathcal{Y}_i.$$
Since \( f : \mathcal{X} \to \mathcal{Y} \) is quasi-projective, \( \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i \) is a quasi-projective scheme. Lemma 2.6 completes the proof.

We expect that Theorem A.4 holds in a much greater generality. However, here we used assumptions that suffices to prove the torus localization formula below for the simplicity of the arguments.

A.3. Torus localization.

**Proposition A.5.** Let \( \mathcal{X} \) be a separated DM stack with a \( T = \mathbb{G}_m \)-action. Let \( \mathcal{X}^T \) be the fixed locus \([40]\). Let \( \phi : \mathbb{E} \to L_{\mathcal{X}} \) be a \( T \)-equivariant symmetric obstruction theory. Then \( \psi : \mathbb{E}|_{\mathcal{X}^T} \to L_{\mathcal{X}^T} \to L_{\mathcal{X}^T} \) is a symmetric obstruction theory for \( \mathcal{X}^T \) by \([26]\). Assume that \([\mathcal{X}^T/T]\) has the resolution property. Then we have

\[
i_* \left( \frac{[\mathcal{X}^T]_{\text{vir}}}{\sqrt{e(N_{\text{vir}})}} \right) = [\mathcal{X}]_{\text{vir}} \in A^*_s(\mathcal{X}) \otimes \mathbb{Q}[t][t^{\pm 1}]
\]

where \( i : \mathcal{X}^T \hookrightarrow \mathcal{X} \) denotes the inclusion map, \([B \to E \to B^\vee] \cong \mathbb{E}|_{\mathcal{X}^T} \) is a \( T \)-equivariant symmetric resolution, \( \sqrt{e(N_{\text{vir}})} := e(B^m)/\sqrt{e(E^m)} \), and \( t \) is the first Chern class of the one-dimensional weight one representation of \( T \).

**Proof.** By \([40]\) Theorem 5.3.5, we may write \([\mathcal{X}]_{\text{vir}} = i_* (\alpha) \) for some \( \alpha \in A^*_s(\mathcal{X}^T) \otimes \mathbb{Q}[t][t^{\pm 1}] \). Consider a modified symmetric obstruction theory

\[
\psi' = (\psi, 0) : \mathbb{E}|_{\mathcal{X}^T} \oplus E^m[1] = [B^f \to E \to (B^f)^\vee] \to L_{\mathcal{X}^T}
\]

for \( \mathcal{X}^T \). By the Whitney sum formula \([36]\) Lemma 4.8, the homotopy property of Chow groups \([40]\) Corollary 2.5.7, and the Kimura sequence (Lemma A.1), we can easily show that

\[
[\mathcal{X}^T]_{\psi'} = \sqrt{e(E^m)} [\mathcal{X}^T]_{\psi'}
\]

holds. Consider a reduction diagram given by the morphism

\[
\begin{array}{ccc}
0 \to 0 \to (B^m)^\vee & \longrightarrow & [B^f \to E \to B^\vee] \longrightarrow [B^f \to E \to (B^f)^\vee] \\
0 \to 0 \to (B^m)^\vee & \longrightarrow & [B \to E \to B^\vee] \longrightarrow [B \to E \to (B^f)^\vee]
\end{array}
\]

of short exact sequences. Applying the functoriality in Theorem A.4 to \([\mathcal{X}^T/T] \to [\mathcal{X}/T] \to BT \), we deduce

\[
\sqrt{e(E^m)} [\mathcal{X}^T]_{\psi'} = [\mathcal{X}^T]_{\psi'} = i'_* [\mathcal{X}]_{\psi'} = i'_* (\alpha) = e(B^m)(\alpha).
\]

Therefore, we have \( i_* \left( \frac{[\mathcal{X}^T]_{\psi'}}{\sqrt{e(E^m)}} \right) = [\mathcal{X}]_{\text{vir}} \).
Appendix B. Virtual pullback in K-theory

B.1. Twisted virtual pullback. For any algebraic stack $\mathcal{X}$, let $K_0(\mathcal{X})$ (resp. $K^0(\mathcal{X})$) denote the Grothendieck group of coherent sheaves (resp. vector bundles) with $\mathbb{Q}$-coefficients. For any line bundle $L$ on a scheme $X$, there exists a unique square root $\sqrt{L} \in K_0(X)$ such that $(\sqrt{L})^2 = L$ and $\sqrt{L} - 1$ is nilpotent (cf. \cite[Lemma 5.1]{??}).

**Definition B.1.** Let $f : X \to Y$ be a morphism of quasi-projective schemes equipped with a perfect obstruction theory $\psi : K \to L_f$. Choose a global resolution $K \to K_0 \times K_1 \to K$ by vector bundles, and let $C = \mathcal{C}_f \times [K_1/K_0] K_1$. The twisted virtual pullback is defined as the composition

$$\hat{f}^! : K_0(Y) \xrightarrow{sp_f} K_0(\mathcal{C}_f) \xrightarrow{p^*} K_0(C) \xrightarrow{\varepsilon(K_1|C, \tau)} K_0(\mathcal{X}) \xrightarrow{\sqrt{\det(K)}} K_0(X)$$

where $p : C \to \mathcal{C}_f$ denotes the projection map, $\tau \in \Gamma(C, K_1|C)$ denotes the tautological section, and $\varepsilon(K_1|C, \tau)$ denotes the localized K-theoretic Euler class.

When $Y = \text{Spec}(\mathbb{C})$ is a point, then $\hat{O}_{\text{vir}}^X = \hat{f}^! (\mathcal{O}_{\text{Spec}(\mathbb{C})})$ is Nekrasov-Okounkov’s twisted virtual structure sheaf \cite{??}.

B.2. Square root virtual pullback.

**Definition B.2.** Let $f : X \to Y$ be a morphism of quasi-projective schemes equipped with a symmetric obstruction theory $\phi : E \to L_f$ satisfying the isotropic condition. Let $[B \to E \to B^\vee] \cong E$ be a symmetric resolution (Proposition \ref{symmetric_resolution}) and let $C = \mathcal{C}_f \times [E/B] E$. We define the twisted square root virtual pullback as the composition

$$\sqrt{\hat{f}^!} : K_0(Y) \xrightarrow{sp_f} K_0(\mathcal{C}_f) \xrightarrow{p^*} K_0(C) \xrightarrow{\sqrt{\varepsilon(E|C, \tau)}} K_0(\mathcal{X}) \xrightarrow{\sqrt{\det(B^\vee)}} K_0(X)$$

where $p : C \to \mathcal{C}_f$ denotes the projection map, $\tau \in \Gamma(C, E|C)$ denotes the tautological section, which is isotropic by Lemma \ref{isotropic_tautological_section} and $\sqrt{\varepsilon(E|C, \tau)}$ denotes the localized square root Euler class \cite{??}.

In particular, if $Y = \text{Spec}(\mathbb{C})$ is a point, then the twisted virtual structure sheaf is defined as $\hat{O}_{\text{vir}}^X := \sqrt{\hat{f}^! (\mathcal{O}_{\text{Spec}(\mathbb{C})})} \in K_0(X)$.

The square root virtual pullback $\sqrt{\hat{f}^!}$ and the twisted virtual structure sheaf $\hat{O}_{\text{vir}}^X$ are independent of the choice of the symmetric resolution (cf. \cite[Proposition 5.10]{??}). Also, the square root virtual pullback $\sqrt{\hat{f}^!}$ commutes with projective pushforwards and lci pullback. Moreover, in the situation of Proposition \ref{reduction_proposition} we have a reduction formula $\sqrt{f^!} = \sqrt{\varepsilon(G)} \cdot f^! \psi$. 
Theorem B.3 (Functoriality). Given a commutative diagram (2.1) of quasi-projective schemes equipped with a compatible triple \((\phi_f, \phi_g, \phi_{gof})\) of obstruction theories, if \(\phi_g\) and \(\phi_{gof}\) satisfy the isotropic condition, then we have

\[ \sqrt{(g \circ f)^!} = \tilde{f}^! \circ \sqrt{g^!}. \]

Proof. The proof is identical to that for Chow theory in §2. □

As corollaries, we have a K-theoretic Lefschetz principle and K-theoretic Pairs/Sheaves correspondence.

Corollary B.4 (Lefschetz principle). In the situation of Theorem 3.1, we have

\[ \sum_e (-1)^{\sigma(e)} (j_e)_* \left( \hat{\mathcal{O}}_{P(D)}^{\text{vir}} \right) = \hat{\mathcal{O}}(R\pi_* (\mathcal{F} \otimes L)) : \hat{\mathcal{O}}_{P(X)}^{\text{vir}} \]

where \(\hat{\mathcal{O}}(E) = \sqrt{\det(E)} \cdot \mathfrak{c}(E)\).

Corollary B.5 (Pairs/Sheaves correspondence). In the situation of Theorem 4.3, we have

\[ \hat{\mathcal{O}}_{P_{n,\beta}}^{\text{vir}}(X) = \hat{p}^! \left( \hat{\mathcal{O}}_{M_{n,\beta}}^{\text{vir}}(X) \right). \]

All the arguments in §3 and §4 immediately work for K-theory except Lemma 4.11. The Lemma B.6 below shows that Lemma 4.11 also holds in K-theory, which proves Corollary B.4 and Corollary B.5.

Lemma B.6 (Virtual Grothendieck-Riemann-Roch). Let \(f : X \to Y\) be a morphism of quasi-projective schemes equipped with a perfect obstruction theory \(\psi : \mathbb{K} \to \mathcal{L}_f\). Then we have a commutative diagram

\[
\begin{array}{ccc}
K_0(Y) & \xrightarrow{\tau_Y} & A_*(Y) \\
\downarrow f^! & & \downarrow \text{td}(\mathcal{K}^\vee) \\
K_0(X) & \xrightarrow{\tau_X} & A_*(X)
\end{array}
\]  

where \(\tau_X\) and \(\tau_Y\) denote the Grothendieck-Riemann-Roch maps in [24].

Proof. Choose a global resolution \([K_0 \to K_1] \cong \mathcal{K}^\vee\) and a factorization of \(f\) by a closed embedding \(\tilde{f}\) and a smooth morphism \(\hat{f}\). Form a fiber digram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \tilde{f} & \swarrow \hat{f} & \downarrow \hat{f} \\
\tilde{X} & \rightarrow & \tilde{Y}
\end{array}
\]

where \(\tilde{X}\) and \(\tilde{Y}\) denote the virtual schemes equipped with a perfect obstruction theory \(\tilde{\psi} : \tilde{\mathbb{K}} \to \tilde{\mathcal{L}}_f\). Then we have a commutative diagram

\[
\begin{array}{ccc}
K_0(\tilde{Y}) & \xrightarrow{\tau_{\tilde{Y}}} & \hat{A}_*(\tilde{Y}) \\
\downarrow \tilde{f}^! & & \downarrow \hat{f}^! \\
K_0(\tilde{X}) & \xrightarrow{\tau_{\tilde{X}}} & \hat{A}_*(\tilde{X})
\end{array}
\]

where \(\tau_{\tilde{X}}\) and \(\tau_{\tilde{Y}}\) denote the virtual Grothendieck-Riemann-Roch maps in [24].
where $p$ is a $K_0$-torsor, and $q$ is a $\mathbb{T}_{\tilde{Y}/Y}$-torsor. Then we have

$$f^! = 0^!_{[K_1/K_0]} \circ sp_f = 0^!_{K_1} \circ p^* \circ (q^*)^{-1} \circ sp_{X/\tilde{Y}} \circ f$$

(B.1)

both in $K$-theory and Chow theory. Since all the operations in (B.1) are for schemes, we have

$$\tau_X \circ f^! = td(-K_1) \cdot td(\mathbb{T}_{\tilde{Y}/Y})^{-1} \cdot td(K_0) \cdot 1 \cdot td(\mathbb{T}_{\tilde{Y}/Y}) \cdot f^! \circ \tau_Y$$

$$= td(\mathbb{K}^Y) \cdot f^! \circ \tau_Y$$

by [24, Theorem 18.2]. It completes the proof. □

Appendix C. Reduction of symmetric complexes

Here we will slightly generalize the notions of symmetric complexes and isotropic subcomplexes in §1.

Definition C.1. Let $X$ be an algebraic stack.

(1) A $d$-shifted symmetric complex is a pair $(E, \theta)$ of a perfect complex $E$ on $X$ and an isomorphism $\theta : E^\vee \cong E[d]$ such that $\theta^\vee[d] = \theta$.

(2) An isotropic subcomplex of $E$ is a pair $(K, \delta)$ of a perfect complex $K$ on $X$ and a map $\delta : E \to K$ such that $\delta^*(\theta) = \delta[d] \circ \theta \circ \delta^\vee = 0$ and $\text{Hom}(K^\vee[-d], \mathbb{K}[-1]) = 0$.

Lemma C.2. Let $E$ be a $d$-shifted symmetric complex on an algebraic stack $X$ and $K$ be an isotropic subcomplex. Then there exists a unique $d$-shifted symmetric complex $G$ that fits into a morphism of distinguished triangles

(C.1) $$\begin{array}{ccc}
\mathbb{D}^\vee[-d] & \xrightarrow{\alpha^\vee} & E \\
\downarrow{\beta^\vee} & & \downarrow{\alpha} \\
G & \xrightarrow{\beta} & \mathbb{D}
\end{array}$$

for some maps $\alpha$, $\beta$ and a perfect complex $\mathbb{D}$. We call $G$ the reduction.

Remark C.3. The above definitions are compatible with the definitions in §1 as follows:

(1) The symmetric complexes in Definition 1.2 are the $(-2)$-shifted symmetric complexes (in the sense of Definition C.1(1)) of tor-amplitude $[-2,0]$ with orientations.

(2) The isotropic subcomplexes in Definition 1.4 are the isotropic subcomplexes (in the sense of Definition C.1(2)) of tor-amplitude $[-1,0]$ whose reductions are of tor-amplitude $[-2,0]$.

Proof of Lemma C.2. Form a distinguished triangle

$$\begin{array}{ccc}
\mathbb{K}^\vee[-d] & \xrightarrow{\delta^\vee} & E \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
\mathbb{D} & \xrightarrow{\epsilon} & \mathbb{K}^\vee[-d+1]
\end{array}$$
for some perfect complex \( \mathcal{D} \) and maps \( \alpha \) and \( \epsilon \). Since \( \delta \circ \delta^\vee = 0 \), there exists a map \( \gamma : \mathcal{D} \to \mathcal{K} \) that fits into the commutative square

\[
E \xrightarrow{\delta} \mathcal{K} \\
\downarrow \alpha \\
\mathcal{D} \xrightarrow{\gamma} \mathcal{K}
\]

as the dotted arrow. Moreover, the map \( \gamma : E \to \mathcal{K} \) is uniquely determined by the commutative square \((C.2)\) since \( \text{Hom}(\mathcal{K}^\vee[-d+1], \mathcal{K}) = 0 \). Now form a distinguished triangle

\[
\mathcal{G} \xrightarrow{\beta} \mathcal{D} \xrightarrow{\gamma} \mathcal{K} \xrightarrow{1} \mathcal{G}[1]
\]

for some perfect complex \( \mathcal{G} \) and a map \( \beta \). Applying the octahedral axiom to the diagram \((C.2)\), we obtain a commutative diagram

\[
K^\vee[-d] \xrightarrow{\delta^\vee} K^\vee[-d] \\
\downarrow \gamma^\vee \\
K[-1] \xrightarrow{\epsilon^\vee} D^\vee[-d] \xrightarrow{\alpha^\vee} E \xrightarrow{\delta} K \\
\downarrow \beta^\vee \\
K[-1] \xrightarrow{\gamma^\vee} G \xrightarrow{\beta} D \xrightarrow{\gamma} K \\
\downarrow \epsilon \\
K^\vee[-d+1] \xrightarrow{\beta^\vee} K^\vee[-d+1]
\]

consists of four distinguished triangles for some map \( b^\vee : D^\vee[-d] \to \mathcal{G} \), where the map \( K^\vee[-d] \to D^\vee[-d] \) is \( \gamma^\vee \) since it factors \( \delta^\vee \). Note that if we dualize the diagram \((C.3)\), then we obtain the same diagram where \( \mathcal{G} \) is replaced by \( \mathcal{G}^\vee[-d] \), and \( \beta \) and \( b \) are replaced by each other. Form two morphisms of distinguished triangles

\[
D^\vee[-d] \xrightarrow{\beta^\vee} G^\vee[-d] \xrightarrow{eb} K^\vee[-d+1] \quad K^\vee[-1] \xrightarrow{\beta^\vee \epsilon^\vee} \mathcal{G}^\vee[-d] \xrightarrow{b} D \\
\downarrow \eta_1 \\
D^\vee[-d] \xrightarrow{b^\vee} G \xrightarrow{\epsilon \beta} K^\vee[-d+1] \quad K^\vee[-1] \xrightarrow{b^\vee \epsilon^\vee} \mathcal{G} \xrightarrow{\beta} D \xrightarrow{\eta_2}
\]

for some maps \( \eta_1 \) and \( \eta_2 \).

We claim that we can choose a map \( \eta_1 = \eta_2 \) that fits into the two commutative diagrams in \((C.4)\) simultaneously. Indeed, first let \( \eta = \eta_1 \) be the map that fits into the left commutative diagram in \((C.4)\). Then we have

\[
\eta \circ \beta^\vee = b^\vee \quad \text{and} \quad \epsilon \circ \beta \circ \eta = \epsilon \circ b.
\]

Since \( \epsilon \circ (\beta \circ \eta - b) = 0 \), there exists a map \( x : G^\vee[-d] \to E \) such that \( \beta \circ \eta - b = \alpha \circ x \). Since \( \gamma \circ (\beta \circ \eta - b) = 0 \), we have \( \gamma \circ \alpha \circ x = \delta \circ x = 0 \).
Hence there exists a map $y : G^\vee[-d] \to D^\vee[-d]$ such that $x = \alpha^\vee \circ y$. Also, $(\beta \circ \eta - b) \circ \beta^\vee = \beta \circ b^\vee - b \circ \beta^\vee = \alpha \circ \alpha^\vee - \alpha \circ \alpha^\vee = 0$ implies $\alpha \circ \alpha^\vee \circ y \circ \beta^\vee = 0$. Hence there exists $z : D^\vee[-d] \to K^\vee[-d]$ such that $\alpha^\vee \circ y \circ \beta^\vee = \delta^\vee \circ z$. Then we have $\alpha^\vee \circ (y \circ \beta^\vee - \gamma^\vee \circ z) = 0$ so that there exists $w : D^\vee[-d] \to K[-1]$ such that $y \circ \beta^\vee - \gamma^\vee \circ z = \epsilon^\vee \circ w$. Note that $w \circ \gamma^\vee = 0$ since $\text{Hom}(K^\vee[-d], K[-1]) = 0$. Hence there exists a map $u : G^\vee[-d] \to K[-1]$ such that $w = u \circ \beta^\vee$. Now if we replace $\eta$ as

$$\eta \mapsto \eta - b^\vee \circ (y - \epsilon^\vee \circ u),$$

then we have

$$\eta \circ \beta^\vee = b^\vee \quad \text{and} \quad \beta \circ \eta = b.$$

Hence $\eta$ fits into the two commutative diagrams in (C.4) simultaneously.

Note that $\eta^\vee$ also fits into the two commutative diagrams in (C.4). Replace $\eta$ by $\frac{\eta + \eta^\vee}{2}$. Then $\eta$ still fits into the two commutative diagrams in (C.4) and $\eta = \eta^\vee$. This proves the existence.

Now we will prove the uniqueness. Let $G'$ be another $d$-shifted symmetric complex that fits into the diagram (C.4) with $\beta$ replaced by $\beta'$. Here we can assume that $D$, $\gamma$, and $\epsilon$ are fixed. By repeating the arguments in the second paragraph, we can form morphisms

\[
\begin{array}{ccccccc}
D^\vee[-d] & \xrightarrow{\beta^\vee} & G & \xrightarrow{\epsilon^\vee} & K^\vee[-d + 1] \\
\downarrow f & & \downarrow f & & \downarrow f \\
D^\vee[-d] & \xrightarrow{\beta'^\vee} & G' & \xrightarrow{\epsilon'^\vee} & K^\vee[-d + 1]
\end{array}
\]

of distinguished triangles for some map $f : G \to G'$ which fits into the both diagram simultaneously. Here we identified $G^\vee[-d] = G$ and $(G')^\vee[-d] = G'$ by their symmetric forms. Then $(f^\vee)^{-1} : G \to G'$ also fits into the both commutative diagrams in (C.5). Hence after replacing $f$ by $\frac{f + (f^\vee)^{-1}}{2}$, we may assume that $f = (f^\vee)^{-1}$. Equivalently, we have a commutative diagram

\[
\begin{array}{ccccccc}
G & \xrightarrow{f} & G' \\
\downarrow & & \downarrow \\
G^\vee[-d] & \xleftarrow{f^\vee} & (G')^\vee[-d]
\end{array}
\]

which means that $G$ and $G'$ are isomorphic as symmetric complexes. \(\square\)

**Lemma C.4.** Let $E$ and $G$ be symmetric complexes (in the sense of Definition [1.2]) and $K$ be an isotropic subcomplex of $E$ with respect to $\delta : E \to K$ (in the sense of Definition [1.4]). Assume that we have a (not necessarily commutative) diagram (C.1) where the two horizontal sequences are distinguished triangles and only the first two squares commute. Then we can form the full reduction diagram (C.1) with the same maps $\alpha$, $\beta$, $\delta$. 


Proof. Form a morphism of distinguished triangles
\[
\begin{array}{ccc}
\mathbb{D}^\vee [2] & \xrightarrow{\alpha^\vee} & \mathbb{E} \\
\beta^\vee & \searrow & \mathbb{K} \\
G & \xrightarrow{\beta} & \mathbb{D}^\vee [1]
\end{array}
\]
for some map \(\phi : \mathbb{K} \to \mathbb{K}\). Since \(\mathbb{K}^\vee[2]\) is of tor-amplitude \([-2, -1]\), \(h^0(\beta^\vee)\), \(h^0(\alpha)\) are bijective and \(h^{-1}(\beta^\vee), h^{-1}(\alpha)\) are surjective. From the morphism of the long exact sequences associated to (C.6), we deduce that \(h^0(\phi)\) is bijective and \(h^{-1}(\phi)\) is surjective. We now claim that \(h^{-1}(\phi)\) is injective.

Since \(\phi \circ \delta = \gamma \circ \alpha = \delta\), there exists \(\psi : \mathbb{D}^\vee [3] \to \mathbb{K}\) such that \(\phi - 1 = \psi \circ x\).

If \(h^{-1}(\phi)(a) = 0\), then \(h^{-1}(x)(a) = 0\) since \(h^0(\beta^\vee)\) is bijective. Hence
\[
h^{-1}(\phi)(a) = a - h^{-1}(\psi) \circ h^{-1}(x)(a) = a = 0.
\]
Hence \(\phi : \mathbb{K} \to \mathbb{K}\) is an isomorphism. Replace \(y\) by \(y \circ \phi = \beta^\vee \circ x\), then we obtain the desired reduction diagram. \(\square\)

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