Quasi-optimal convergence rate for an adaptive hybridizable $C^0$ discontinuous Galerkin method for Kirchhoff plates

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Abstract In this paper, we present an adaptive hybridizable $C^0$ discontinuous Galerkin (HCDG) method for Kirchhoff plates. A reliable and efficient a posteriori error estimator is produced for this HCDG method. Quasi-orthogonality and discrete reliability are established with the help of a postprocessed bending moment and the discrete Helmholtz decomposition. Based on these, the contraction property between two consecutive loops and complexity of the adaptive HCDG method are studied thoroughly. The key points in our analysis are a postprocessed normal–normal continuous bending moment from the HCDG method solution and a lifting of jump residuals from inter-element boundaries to element interiors.

Keywords A posteriori error estimates · Adaptive hybridizable $C^0$ discontinuous Galerkin method · Convergence · Computational complexity · Kirchhoff plate bending problems

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1 Introduction

Hybridization as an implementational technique can be traced back to [51], which eliminates the continuity constraints of the finite element space and enforces it by introducing a Lagrange multiplier. As a result, the original indefinite stiffness matrix can be transformed to a symmetric and positive-definite one, and the globally coupled degrees of freedom will be much fewer. It was observed by Arnold and Brezzi [3] that the Lagrange multiplier can be used to construct a new superconvergent approximation of the original variable by postprocessing. In the last decade, Cockburn and his collaborators studied the hybridization of finite element methods systematically and thoroughly (cf. [17–19]), especially for the second order problems. They presented a characterization of the Lagrange multiplier in a unifying framework in [19], and designed the hybridizable discontinuous Galerkin (HDG) method for the second order problems (cf. [15,20,21]) which overcomes the drawbacks of the traditional discontinuous Galerkin methods. Applying the HDG method for the second order problems in [15], a hybridizable and superconvergent discontinuous Galerkin method for the biharmonic equation based on the Ciarlet–Raviart formulation was given in [16]. And we devised a hybridizable $C^0$ discontinuous Galerkin (HCDG) method for Kirchhoff plate bending problems based on the Hellan–Herrmann–Johnson formulation in [37], which is also superconvergent and will be the focus of this paper.

There have been lots of works with regard to the a posteriori error analysis of the numerical methods for the fourth-order elliptic problems (cf. [44,50]). Reliable and efficient residual-based a posteriori error estimators were given in [1,38,50] for the fourth-order problems discretized by the $H^2$-conforming finite element methods. Nonconforming finite element methods are preferred to discretize the fourth-order problems due to their simplicity. Employing the Helmholtz decomposition of the second order tensors created in [48], a posteriori error analysis for the Morley element method was shown in [32,48], which was then extended to the Kirchhoff plate bending problems with general boundary conditions in [49]. The a posteriori error estimates for the nonconforming rectangular finite element methods were developed in [10], in whose analysis the Helmholtz decomposition was replaced by an abstract error decomposition. As for the Ciarlet–Raviart mixed finite element method, we refer to [13,28] for the residual-based a posteriori error estimator and [40] for the gradient recovery-based a posteriori error estimator. In the existing works on the discontinuous Galerkin methods for the fourth-order problems, the recovery technique and Helmholtz decomposition were the mainly two types of techniques for deriving the residual-based a posteriori error estimates. The recovery technique was used to obtain the a posteriori error estimates for the $C^0$ interior penalty method in [7], the weakly over-penalized symmetric interior penalty method in [8], the interior penalty discontinuous Galerkin (IPDG) method in [27] and the reduced local $C^0$ discontinuous Galerkin method in [36]. By the ideas in [48], the Helmholtz decomposition was also applied for the a posteriori error analysis for the local $C^0$ discontinuous Galerkin method in [53] and the $C^0$ interior penalty method in [30].
However there are very few results involving the convergence analysis of adaptive algorithms for the fourth order problems. Following the paradigm in [12], the convergence and optimality of the adaptive Morley element method were analyzed in [11,33]. The key points in [33] were the local conservative property of the Morley element method observed to prove the quasi-orthogonality and the intergrid transfer operators between two nonconforming spaces used to build the discrete reliability. Applying the Helmholtz decomposition in [48] again, a reliable and efficient a posteriori error estimator was constructed for the Hellan–Herrmann–Johnson (HHJ) method, based on which a adaptive mixed finite element method with any polynomial degree was studied systematically in [35,52]. The discrete Helmholtz decomposition and discrete inf-sup condition were the crucial tools established in [35] for deriving the quasi-orthogonality of the moment field and the discrete reliability of the estimator.

To the best of our knowledge, there are no results on the convergence of the adaptive hybridizable discontinuous Galerkin method for the fourth order problems in the literature.

On the other hand, the convergence of the adaptive IPDG method for the second order problems was first analyzed in [39] under the interior node property. Then in [31], the requirement on the interior node property in the refinement was removed. Under the same assumption in [31,39] that the penalty parameter should be sufficiently large, the quasi-optimal asymptotic rate of convergence for the adaptive IPDG method on nonconforming meshing was obtained in [6]. With the aid of a postprocessed solution, the contraction property for the weakly penalized adaptive discontinuous Galerkin methods was derived in [29] only assuming that the penalty parameter was large enough to guarantee the stability of the methods. Recently, the contraction property was established in [22] for the adaptive HDG method of the Poisson problem when the product of the stabilization parameter and the meshsize of the initial triangulation was sufficiently small. The original technique in their analysis was the lifting of trace residuals from inter-element boundaries to element interiors, which was used to compare the inter-element flux jump residuals between two nested meshes.

In this paper, we present the convergence and optimality of an adaptive HCDG method for Kirchhoff plates. The adaptive HCDG method is based on the standard successive loop

\[
\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.
\]

The HCDG method in [37], the Dörfler marking strategy in [24] and the newest vertex bisection in [5,41,46,47] are employed in SOLVE, MARK and REFINE respectively. The analysis in this paper mainly follows the ideas in [35] and [22,54]. It’s worth mentioning that the exact solution \(\sigma\) was required to be piecewise \(H^1\) in [35,52]. Here we only assume the minimal regularity \(\sigma \in L^2(\Omega, S)\).

Since the bending moment of the HCDG method solution is entirely discontinuous, we first construct a normal–normal continuous bending moment by postprocessing the HCDG method solution, which is the pivot in the analysis. The difference between the postprocessed bending moment error and the original one is characterized by the stability of the postprocessing error with respect to the mesh, which was also
used for adaptive HDG method in [22]. As demonstrated in [22], another crucial feature to analyze the adaptive HCDG method is a lifting of trace residuals from inter-element boundaries to element interiors, which makes the comparison of jump residuals of the inter-element bending moment between two successive meshes possible. A reliable and efficient a posteriori error estimator is the start point for the adaptive algorithm. Taking advantage of the Helmholtz decomposition in [48], the lifting estimates and a well-tailored interpolation operator used in the a priori analysis of the HHJ method (cf. [4, 23, 25, 45]), we construct a reliable residual-based a posteriori error estimator for the HCDG method. The efficiency of the estimator is proved by the standard technique of bubble functions, which has been shown in [35]. It is worth mentioning that our new estimator differs from the one proposed in [35] even if the stabilization parameter vanishes, since the well-tailored interpolation operator used in the proof. The next important ingredient is to create the quasi-orthogonality of the bending moment. To this end, we prove the quasi-orthogonality for the postprocessed bending moment by using the discrete Helmholtz decomposition in [35], which together with the stability of the postprocessing error with respect to the mesh gives the required quasi-orthogonality. Then we show that the adaptive HCDG method is a contraction for the sum of the bending moment error in an energy norm and the scaled error estimator between two consecutive meshes when the product of the stabilization parameter and the meshsize of the initial triangulation was sufficiently small.

Another key ingredient for the complexity of the adaptive HCDG method is the discrete reliability of the error estimator, which is obtained by using the discrete Helmholtz decomposition and the stability of the postprocessing error with respect to the mesh when the product of the stabilization parameter and the meshsize of the initial triangulation was sufficiently small. With two connection operators corresponding to the deflection and the bending moment respectively, we proved the quasi-optimality of the total error under the minimal regularity. Here the total error is defined as the sum of the bending moment error in an energy norm, the data oscillation and the jump residuals of the inter-element bending moment. Then we define a nonlinear approximation class based on the total error. With previous preparations, we exhibit that the adaptive HCDG method generates a decay rate of the total error in terms of the number of degrees of freedom.

The rest of this paper is organized as follows. We review a HCDG method for Kirchhoff plates in Sect. 2. In Sect. 3, a reliable and efficient a posteriori error estimator is constructed for the adaptive HCDG method. We achieve the quasi-orthogonality of the bending moment in Sect. 4. We consider the convergence of the adaptive HCDG method in Sect. 5. And the complexity of the adaptive HCDG method is analyzed in Sect. 6. Numerical results are provided in the last section.

2 The HCDG method for Kirchhoff plates

Given a thin plate occupying a bounded polygonal domain \( \Omega \subset \mathbb{R}^2 \), assume it is clamped on the boundary and acted under a vertical load \( f \in L^2(\Omega) \). Then the mathematical model describing the deflection \( u \) of the plate is governed by (cf. [26, 42])
\[
\begin{align*}
\mathcal{C} \sigma &= \mathcal{K}(u) \text{ in } \Omega, \\
\nabla \cdot (\nabla \cdot \sigma) &= -f \text{ in } \Omega, \\
u &= \partial_n u = 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \( n \) is the unit outward normal to \( \partial \Omega \), \( \nabla \) is the usual gradient operator, \( \nabla \cdot \) stands for the divergence operator acting on tensor-valued or vector-valued functions (cf. [42]), and

\[
\mathcal{K}(u) := (K_{ij}(u))_{2 \times 2}, \quad K_{ij}(u) := -\partial_{ij} u, \quad 1 \leq i, j \leq 2.
\]

Here, \( \mathcal{C} \) is a symmetric and positive definite operator defined as follows: for any second-order tensor \( \tau \),

\[
\mathcal{C} \tau := \frac{1}{1-\nu} \tau - \frac{\nu}{1-\nu^2} (\text{tr} \tau) \mathcal{I}
\]

with \( \mathcal{I} \) a second order identity tensor, \( \text{tr} \) the trace operator acting on second order tensors, and \( \nu \in L^\infty(\Omega) \) the Poisson ratio satisfying \( \inf_{x \in \Omega} \nu > 0 \) and \( \sup_{x \in \Omega} \nu < 0.5 \).

We assume in this paper that \( \nu \) is piecewise constant corresponding to the initial triangulation. It is easy to see that

\[
\mathcal{C}^{-1} \tau = (1-\nu) \tau + \nu \text{tr} (\tau) \mathcal{I}.
\]

### 2.1 Notation

Denote by \( S \) the space of all symmetric \( 2 \times 2 \) tensors (matrices). Given a bounded domain \( G \subset \mathbb{R}^2 \) and a non-negative integer \( r \), let \( H^r(G) \) be the usual Sobolev space of functions on \( G \), and \( H^r(G, \mathbb{X}) \) be the usual Sobolev space of functions taking values in the finite-dimensional vector space \( \mathbb{X} \) for \( \mathbb{X} \) being \( S \) or \( \mathbb{R}^2 \). The corresponding norm and semi-norm are denoted respectively by \( \| \cdot \|_{r, G} \) and \( | \cdot |_{r, G} \). If \( G \) is \( \Omega \), we abbreviate them by \( \| \cdot \|_r \) and \( | \cdot |_r \), respectively. Let \( H^0_0(G) \) be the closure of \( C_0^\infty(G) \) with respect to the norm \( \| \cdot \|_r \). \( P_r(G) \) stands for the set of all polynomials in \( G \) with the total degree no more than \( r \), and \( P_r(G, \mathbb{X}) \) denotes the tensor or vector version of \( P_r(G) \) for \( \mathbb{X} \) being \( S \) or \( \mathbb{R}^2 \), respectively.

Let \( T_0 \) be an initial shape-regular and conforming triangulation of \( \Omega \). Denote by \( T \) any refinement of \( T_0 \) which is also shape-regular and conforming. For each \( K \subset T \), define \( h_K := \sqrt{|K|} \) where \( |K| \) means the area of \( K \). Denote by \( n_K = (n_1, n_2)^T \) the unit outward normal to \( \partial K \) and write \( t_K := (t_1, t_2)^T = (-n_2, n_1)^T \), a unit vector tangent to \( \partial K \). Without causing any confusion, we will abbreviate \( n_K \) and \( t_K \) as \( n \) and \( t \) respectively for simplicity. Let \( E(T) \) be the union of all edges of the triangulation \( T \) and \( E^i(T) \) the union of all interior edges of the triangulation \( T \). For any \( e \in E(T) \), denote by \( h_e \) its length and fix a unit normal vector \( n_e := (n_1, n_2)^T \) and a unit tangent vector \( t_e := (-n_2, n_1)^T \). For a second order tensor-valued function \( \tau \), set

\[
\begin{align*}
M_n(\tau) := n_e^T \tau n_e, & \quad M_{nt}(\tau) := t_e^T \tau n_e, \\
M_{nn_e}(\tau) := n_e^T \tau n, & \quad M_{nt_e}(\tau) := t_e^T \tau n
\end{align*}
\]
on each edge $e \in \mathcal{E}(T)$. In the context of solid mechanics, $M_n(\tau)$ and $M_{nt}(\tau)$ are called normal bending moment and twisting moment respectively when $\tau$ is a moment. For any $G \subset \Omega$, let $\mathcal{O}(G) := \{ K \in T : \overline{K} \cap \overline{G} \neq \emptyset \}$ and $\mathcal{T}(G)$ be the restriction of $\mathcal{T}$ on $G$. Throughout this paper, we also use “$\lesssim \cdots$” to mean that “$\leq C \cdots$”, where $C$ is a generic positive constant independent of the mesh size, which may take different values at different appearances. And $\equiv A$ means $\lesssim A$ and $\lesssim A$.

For later uses, we introduce averages and jumps on edges as in [34]. Consider two adjacent triangles $K^+$ and $K^-$ sharing an interior edge $e$. Denote by $n^+$ and $n^-$ the unit outward normals to the common edge $e$ of the triangles $K^+$ and $K^-$, respectively. For a scalar-valued function $v$, write $v^+ := v|_{K^+}$ and $v^- := v|_{K^-}$. Then define averages and jumps on $e$ as follows:

$$\{ v \} := \frac{1}{2} (v^+ + v^-), \quad [v] := v^+ n_e \cdot n^+ + v^- n_e \cdot n^-.$$

On an edge $e$ lying on the boundary $\partial \Omega$, the above terms are defined by

$$\{ v \} := v, \quad [v] := v n_e \cdot n.$$

For any second order tensor field $\tau$ and vector field $\phi$, define differential operators

$$\text{rot} \, \tau := \left( \begin{array}{cc} \partial_1 \tau_{12} - \partial_2 \tau_{11} \\ \partial_1 \tau_{22} - \partial_2 \tau_{21} \end{array} \right), \quad \text{Curl} \, \phi := \left( \begin{array}{cc} -\partial_2 \phi_1 & \partial_1 \phi_1 \\ -\partial_2 \phi_2 & \partial_1 \phi_2 \end{array} \right),$$

$$\varepsilon^\perp(\phi) := \frac{1}{2} \left( \text{Curl} \phi + (\text{Curl} \phi)^T \right).$$

### 2.2 The HCDG method

In this subsection, we will present a hybridizable $C^0$ discontinuous Galerkin method for problem (2.1). To this end, define three finite element spaces based on the triangulation $T$ as

$$\Sigma_T := \left\{ \tau \in L^2(\Omega, S) : \tau|_K \in P_{k-1}(K, S) \; \forall \; K \in T \right\},$$

$$V_T := \left\{ v \in H^1_0(\Omega) : v|_K \in P_k(K) \; \forall \; K \in T \right\},$$

$$M_T := \left\{ \mu \in L^2(\mathcal{E}(T)) : \mu|_e \in P_{k-1}(e) \; \forall \; e \in \mathcal{E}(T) \; \text{and} \; \mu|_{\partial \Omega} = 0 \right\},$$

with integer $k \geq 1$. We also need the following two more finite element spaces which will be used in the analysis

$$W_T := \left\{ v \in H^1(\Omega, \mathbb{R}^2) : v|_K \in P_k(K, \mathbb{R}^2) \; \forall \; K \in T \right\},$$

$$\Sigma_{T}^{HHJ} := \left\{ \tau \in \Sigma_T : [M_n(\tau)]_e = 0 \; \forall \; e \in \mathcal{E}(T) \right\}.$$
Then the hybridizable $C^0$ discontinuous Galerkin (HCDG) method for problem (2.1) designed in [37] is defined as follows: Find $(\sigma_T, u_T, \lambda_T) \in \Sigma_T \times V_T \times M_T$ such that

\begin{align*}
    a(\sigma_T, \tau) + b_T(\tau, u_T) &= \sum_{K \in T} \int_{\partial K} M_{nn}(\tau) \lambda_T ds,
    \tag{2.2a}

    - b_T(\sigma_T, v) &= \sum_{K \in T} \int_K (M_n(\sigma_T) - M_n(\sigma_T)) \partial_n v ds - \int_\Omega f v dx,
    \tag{2.2b}

    \sum_{e \in E^i(T)} \int_e \left[ \hat{M}_n(\sigma_T) \right] \mu ds = 0,
    \tag{2.2c}

    M_n(\sigma_T)|_{\partial K} = M_n(\sigma_T) + \xi(\partial_n u_T - \lambda_T) n_e \cdot n \quad \forall K \in T
    \tag{2.2d}
\end{align*}

for all $(\tau, v, \mu) \in \Sigma_T \times V_T \times M_T$, where

\begin{align*}
    a(\sigma, \tau) &:= \int_\Omega C \sigma : \tau dx, \\
    b_T(\tau, v) &:= -\sum_{K \in T} \int_K (\nabla \cdot \tau) \cdot \nabla v dx + \sum_{K \in T} \int_{\partial K} M_{nt}(\tau) \partial_t v ds, \\
    \xi|_K &:= \xi_K = C_0 h_K^{1+\gamma} \quad \forall K \in T
\end{align*}

with constants $C_0 \geq 0$ and $\gamma > -1$. Here $\xi$ is called the stabilization parameter. It has been shown in [37] that the HCDG method (2.2a)–(2.2d) possesses superconvergence when $\gamma \geq 1$ or $C_0 = 0$.

Let $C_\xi := \max_{K \in T_0} h_K \xi_K = \max_{K \in T_0} C_0 h_K^{1+\gamma}$. For any $S \subset T$ and $v \in V_T$, define a mesh-dependent norm by

\begin{equation*}
    \|v\|_{2,S}^2 := \sum_{K \in S} |v|_{2,K}^2 + \sum_{K \in S} h_K^{-1} \|\partial_n v\|_{0, \partial K}^2.
\end{equation*}

The bilinear form $b_T(\cdot, \cdot)$ possesses the following inf-sup condition (cf. [35, Lemma 4.2])

\begin{equation*}
    \|v\|_{2,T} \lesssim \sup_{\tau \in \Sigma_T^{HHJ}} \frac{b_T(\tau, v)}{\|\tau\|_{0,T}} \quad \forall v \in V_T,
    \tag{2.3}
\end{equation*}

where

\begin{equation*}
    \|\tau\|_{0,T}^2 := \|\tau\|_0^2 + \sum_{K \in T} h_K \|M_n(\tau)\|_{0, \partial K}^2 \quad \forall \tau \in \Sigma_T.
\end{equation*}

Thanks to the exact sequence of the HHJ method (cf. [14]), we have

\begin{equation*}
    \mathcal{E}^\perp(W_T) \subset \Sigma_T^{HHJ} \quad \text{and} \quad b_T(\mathcal{E}^\perp(\varphi), v) = 0 \quad \forall \varphi \in W_T, v \in V_T.
    \tag{2.4}
\end{equation*}
3 A posteriori error estimates

In this section, reliable and efficient error estimators of the bending moment will be constructed for designing the adaptive algorithm. With the help of an interpolation operator associated with the HHJ method and a postprocessed discrete bending moment, we establish the reliability of the error estimators adopting the techniques used in [35, Lemma 3.1] and [22], i.e. the Helmholtz decomposition for second order tensors and Lemma 1. The efficiency of the error estimators will be proved by the technique of bubble functions (cf. [50]).

3.1 Preliminaries

Hereafter, let $\mathcal{T}^*$ be a shape-regular and conforming refinement of $\mathcal{T}$. Define $I_T : H^2_0(\Omega) \cup V_{T^*} \to V_T$ in the following way: given $w \in H^2_0(\Omega) \cup V_{T^*}$,

$$I_T w(a) = w(a) \text{ for each vertex } a \text{ of } \mathcal{T},$$

$$\int_e (w - I_T w) \mu ds = 0 \quad \forall \mu \in P_{k-2}(e) \text{ for each edge } e \in \mathcal{E}(T),$$

$$\int_K (w - I_T w) v dx = 0 \quad \forall v \in P_{k-3}(K) \text{ for each triangle } K \in \mathcal{T}.$$  (3.1)

According to the definition of $I_T$ and integration by parts, it holds for any $\tau \in \Sigma_T$ and $v \in H^2_0(\Omega) \cup V_{T^*}$ (cf. [4,23])

$$b_T(\tau, v - I_T v) = 0. \quad \text{(3.1)}$$

The following error estimate for the interpolation operator $I_T$ can be found in [4,23, 25,45]. For any $K \in \mathcal{T}$, it holds

$$\|v - I_T v\|_{0,K} + h_K^{3/2} \|\nabla (v - I_T v)\|_{0,\partial K} \lesssim h_K^2 \|v\|_{2,K} \quad \forall v \in H^2_0(\Omega). \quad \text{(3.2)}$$

Adopting the similar argument as in Lemma 4.3 of [35], we also have for any $v \in V_{T^*}$

$$\|v - I_T v\|_{0,K} \lesssim h_K^2 \|v\|_{2,\mathcal{O}_{T^*}(K)} \quad \forall K \in \mathcal{T} \setminus \mathcal{T}^*. \quad \text{(3.3)}$$

With the solution $\sigma_T$ of the HCDG method (2.2a)–(2.2d), we define a postprocessed bending moment $\tilde{\sigma}_T \in \Sigma_T$ as follows: for any $K \in \mathcal{T}$,

$$\int_K \tilde{\sigma}_T : \tau dx = \int_K \sigma_T : \tau dx \quad \forall \tau \in P_{k-2}(K, \mathbb{S}),$$

$$\int_e M_n(\tilde{\sigma}_T) \mu ds = \int_e M_n(\sigma_T) \mu ds \quad \forall \mu \in P_{k-1}(e) \text{ for each edge } e \text{ of } K.$$
Due to (2.2c), we have \( \hat{\sigma}_T \in \Sigma_{HHJ}^T \). It holds by using the scaling argument

\[
\| \hat{\sigma}_T - \sigma_T \|_{0,K}^2 \approx h_K \| M_n(\hat{\sigma}_T) - M_n(\sigma_T) \|_{0,\partial K}^2 \quad \forall \ K \in T. \tag{3.4}
\]

By the definition of \( \hat{\sigma}_T \) and integration by parts, we obtain

\[
b_T(\hat{\sigma}_T - \sigma_T, v) = \sum_{K \in T} \int_{\partial K} M_n(\sigma_T - \hat{\sigma}_T) \partial_n v \, ds \quad \forall \ v \in V_T. \tag{3.5}
\]

Thus (2.2b) can be rewritten as

\[
- b_T(\hat{\sigma}_T, v) = \int_{\Omega} f v \, dx \quad \forall \ v \in V_T. \tag{3.6}
\]

Employing integration by parts, we get from (2.1)

\[
a(\sigma, \tau) + b_T(\tau, u) = - \sum_{K \in T} \int_{\partial K} M_n(\tau) \partial_n u \, ds \quad \forall \ \tau \in \Sigma_T, \tag{3.7}
\]

\[
\int_{\Omega} \sigma : K(v) \, dx = \int_{\Omega} f v \, dx \quad \forall \ v \in H_0^2(\Omega). \tag{3.8}
\]

Subtracting (2.2a) from (3.7) and using (3.1), we obtain the following error equation that for any \( \tau \in \Sigma_T \),

\[
a(\sigma - \sigma_T, \tau) + b_T(\tau, I_T u - u_T) = \sum_{K \in T} \int_{\partial K} M_{nn\tau}(\tau)(\lambda_T - \partial_n u) \, ds. \tag{3.9}
\]

### 3.2 Error estimators

For any \( K \in T \) and integer \( r \geq 0 \), denote by \( Q^r_K \) the \( L^2 \)-orthogonal projection from \( L^2(K) \) onto \( P_r(K) \), and \( Q^r_K \) means the tensor version of \( Q^r_K \). Let \( Q^{-2}_K = Q^{-1}_K = 0 \) and \( Q^{-1}_K = 0 \). For any \( S \subset T \) and \( \tau \in \Sigma_T \), define

\[
\text{osc}^2(f, S) := \sum_{K \in S} h_K^4 \| f - Q^{-3}_K f \|_{0,K}^2,
\]

\[
\tilde{\eta}^2_1(\tau, S) := \sum_{K \in S} h_K \xi_K \| C\tau - Q^{-2}_K(C\tau) \|_{0,K}^2,
\]

\[
\eta^2_1(\tau, f, S) := \text{osc}^2(f, S) + \tilde{\eta}^2_1(\tau, S),
\]

\[
\eta^2_2(\tau, S) := \sum_{K \in S} \left( h_K^2 \| \text{rot}(C\tau) \|_{0,K}^2 + h_K \| [C\tau]_e \|_{0,\partial K}^2 \right),
\]

\[
\eta^2(\tau, f, S) := \eta^2_1(\tau, f, S) + \eta^2_2(\tau, S). \tag{3.10}
\]
To derive the reliability of the error estimator, we need the following lifting of the trace residuals from inter-element boundaries to element interiors, which will be also used in the proofs of the stability of the postprocessing error with respect to the mesh and the quasi-optimality of the total error.

**Lemma 1** For any $K \in T$, we have

\[
\| \partial_{n_e} u_T - \lambda_T \|_{0, \partial K}^2 \approx h_K \| C\sigma_T - Q_K^{k-2}(C\sigma_T) \|_{0,K}^2, \tag{3.11}
\]
\[
\| M_n(\tilde{\sigma}_T) - M_n(\sigma_T) \|_{0, \partial K}^2 \approx h_K \xi_K^2 \| C\sigma_T - Q_K^{k-2}(C\sigma_T) \|_{0,K}^2, \tag{3.12}
\]
\[
\| \tilde{\sigma}_T - \sigma_T \|_{0,K} \approx h_K \xi_K \| C\sigma_T - Q_K^{k-2}(C\sigma_T) \|_{0,K}. \tag{3.13}
\]

**Proof** Applying integration by parts to (2.2a), it holds

\[
\int_{\partial K} M_{nn_e}(\tau)(\partial_{n_e} u_T - \lambda_T)ds = \int_K (C\sigma_T + \nabla^2 u_T) : \tau dx \quad \forall \tau \in P_{k-1}(K, S). \tag{3.14}
\]

First construct $\tau \in P_{k-1}(K, S)$ such that

\[
M_{nn_e}(\tau)|_{\partial K} = \partial_{n_e} u_T - \lambda_T \quad \text{and} \quad \int_K \tau : \xi dx = 0 \quad \forall \xi \in P_{k-2}(K, S).
\]

Using the scaling argument, it follows

\[
\| \tau \|_{0,K} \approx h_K^{1/2} \| \partial_{n_e} u_T - \lambda_T \|_{0, \partial K}.
\]

Hence we obtain from (3.14) that

\[
\| \partial_{n_e} u_T - \lambda_T \|_{0, \partial K}^2 = \int_K (C\sigma_T + \nabla^2 u_T) : \tau dx = \int_K (C\sigma_T - Q_K^{k-2}(C\sigma_T)) : \tau dx \leq \| C\sigma_T - Q_K^{k-2}(C\sigma_T) \|_{0,K} \| \tau \|_{0,K}.
\]

Combining the last two inequalities gives

\[
\| \partial_{n_e} u_T - \lambda_T \|_{0, \partial K}^2 \lesssim h_K \| C\sigma_T - Q_K^{k-2}(C\sigma_T) \|_{0,K}.
\]

Next choosing $\tau = C\sigma_T - Q_K^{k-2}(C\sigma_T)$ in (3.14), we get from the inverse inequality

\[
\| C\sigma_T - Q_K^{k-2}(C\sigma_T) \|_{0,K}^2 = \int_K (C\sigma_T + \nabla^2 u_T) : \tau dx
\]
\[= \int_{\partial K} M_{nn_e}(\tau)(\partial_{n_e} u_T - \lambda_T)ds \]
\[\lesssim h_K^{1/2} \| C\sigma_T - Q_K^{k-2}(C\sigma_T) \|_{0,K} \| \partial_{n_e} u_T - \lambda_T \|_{0, \partial K},
\]

which ends the proof of (3.11). At last, (3.12) can be derived from the definition of $\tilde{\sigma}_T$ and (3.11), and (3.13) can be derived from (3.4) and (3.12).  

\[\square\]
Now we have the following reliability and efficiency of the error estimators for the bending moment.

**Lemma 2** (The reliability and efficiency of the error estimators) There exist positive constants $C_1$ and $C_2$ depending only on the shape-regularity of the triangulations, the polynomial degree $k$ and the tensor $C$ such that

$$
\|\sigma - \sigma_T\|^2_C \leq C_1 \eta^2(\sigma_T, f, T),
$$

(3.15)

$$
\eta^2_2(\sigma_T, T) \leq C_2 \|\sigma - \sigma_T\|^2_C,
$$

(3.16)

$$
\eta^2(\sigma_T, f, T) \leq C_2 \|\sigma - \sigma_T\|^2_C + \eta^2_1(\sigma_T, f, T).
$$

(3.17)

**Proof** The efficiency of the error estimator (3.16) is easily derived by using the technique of bubble functions as in [35, Theorem 3.2]. And the error estimator (3.17) is the direct result of the error estimator (3.16). Then we only focus on the reliability of the error estimator (3.15). Due to the Helmholtz decomposition (cf. [35, Lemma 3.1]) of $\sigma - \sigma_T$, there exist $\psi \in H^2_0(\Omega)$ and $\phi \in H^1(\Omega)$ such that

$$
\sigma - \sigma_T = C^{-1}K(\psi) + e^\perp(\phi),
$$

$$
\|\psi\|^2_2 + \|\phi\|^1_1 \lesssim \|\sigma - \sigma_T\|_0.
$$

(3.18)

Hence we have

$$
\|\sigma - \sigma_T\|^2_C = \int_{\Omega} (\sigma - \sigma_T) : K(\psi) dx + a(\sigma - \sigma_T, e^\perp(\phi)).
$$

(3.19)

For the first term of (3.19), it follows from (3.8), integration by parts and (3.1)

$$
\int_{\Omega} (\sigma - \sigma_T) : K(\psi) dx = \int_{\Omega} f \psi dx - \int_{\Omega} \sigma_T : K(\psi) dx
$$

$$
= \int_{\Omega} f \psi dx + b_T(\sigma_T, \psi) + \sum_{K \in T} \int_{\partial K} M_n(\sigma_T) \partial_n \psi ds
$$

$$
= \int_{\Omega} f \psi dx + b_T(\sigma_T, I_T \psi)
$$

$$
- \sum_{K \in T} \int_{\partial K} M_n(\tilde{\sigma}_T - \sigma_T) \partial_n \psi ds.
$$

Using (3.6) with $v = I_T \psi$ and (3.5), we acquire

$$
\int_{\Omega} f \psi dx + b_T(\sigma_T, I_T \psi) = \int_{\Omega} f(\psi - I_T \psi) dx - b_T(\tilde{\sigma}_T - \sigma_T, I_T \psi)
$$

$$
= \int_{\Omega} f(\psi - I_T \psi) dx
$$

$$
+ \sum_{K \in T} \int_{\partial K} M_n(\tilde{\sigma}_T - \sigma_T) \partial_n(I_T \psi) ds.
$$
Then we get from the last two equalities
\[
\int_\Omega (\sigma - \sigma_T) : K(\psi) dx = \sum_{K \in \mathcal{T}} \int_K (f - Q^k K f)(\psi - I_T \psi) dx
\]
\[+ \sum_{K \in \mathcal{T}} \int_{\partial K} M_n(\hat{\sigma}_T - \sigma_T) \partial_n(I_T \psi - \psi) ds.
\]
Together with the Cauchy–Schwarz inequality, (3.2) and (3.12), it holds
\[
\int_\Omega (\sigma - \sigma_T) : K(\psi) dx \lesssim \eta_1(\sigma_T, f, \mathcal{T}) \|\psi\|_2.
\] (3.20)

Next consider the bound of the second term in (3.19) which can be achieved by using the similar argument of Theorem 3.1 in [35]. Here we will rewrite the proof in a more compact manner. It readily follows from integration by parts
\[
a(\sigma, \varepsilon(\phi)) = \int_\Omega K(u) : \varepsilon(\phi) dx = 0.
\]
Let \(I_{SZ}^T \phi \in W_T\) be the vectorial Scott-Zhang interpolation of \(\phi\) designed in [43]. Then taking \(\tau = \varepsilon(\varepsilon I_{SZ}^T \phi)\) in (2.2a), we obtain from (2.4)
\[
a(\sigma_T, \varepsilon(\varepsilon I_{SZ}^T \phi)) = 0.
\] (3.21)

Hence we get from the last two equalities
\[
a(\sigma - \sigma_T, \varepsilon(\phi)) = -a(\sigma_T, \varepsilon(\phi)) = a(\sigma_T, \varepsilon(\varepsilon I_{SZ}^T \phi - \phi)),
\]
which is nothing but (3.18) in [35]. Thus it follows from integration by parts and the error estimates of \(I_{SZ}^T\)
\[
a(\sigma - \sigma_T, \varepsilon(\phi)) \lesssim \eta_2(\sigma_T, \mathcal{T}) \|\phi\|_1.
\] (3.22)
Finally we can achieve (3.15) by using (3.18)–(3.20) and (3.22).

**Remark 1** Due to (2.2c)–(2.2d), it holds for any \(e \in \mathcal{E}^i(\mathcal{T})\)
\[
[M_n(\sigma_T)] = -2 \left\{ \xi(\partial_{n_e} u_T - \lambda_T) \right\} \quad \text{on edge } e.
\]
By (3.11), it is observed that \(\tilde{\eta}_1^2(\sigma_T, \mathcal{T})\) can be used to measure and control the normal–normal discontinuity of the discrete space \(\Sigma_T\) for the bending moment. The term \(\tilde{\eta}_1^3(\sigma_T, \mathcal{T})\) disappears for the HHJ method, since \(\xi = 0\) and the normal–normal part of the tensor in \(\Sigma_T\) across the interior edges is continuous. However for the general HCDG method with \(\xi \neq 0\), it requires \(\tilde{\eta}_1^3(\sigma_T, \mathcal{T})\) to control the normal–normal discontinuity of the discrete bending moment.
To derive an error estimator involving $\sigma_T$, $u_T$ and $\lambda_T$ simultaneously, we will construct a new approximation to the deflection $u$. Denote

$$V_T^* := \left\{ v \in L^2(\Omega) : v|_K \in P_{k+1}(K) \quad \forall K \in T \right\}.$$ 

With this space, a new approximation $u^*_T \in V_T^*$ to $u$ is defined piecewisely as a solution of the following problem: for any $K \in T$,

$$u_T^*(a_i) = u_T(a_i) \quad \text{for } i = 1, 2, 3, \quad (3.23)$$

$$\int_K C^{-1} \nabla^2 u_T^* : \nabla^2 v dx = - \int_K \sigma_T : \nabla^2 v dx \quad (3.24)$$

for any $v \in P_{k+1}(K)$ with $v(a_i) = 0$ ($i = 1, 2, 3$), where $\{a_i\}_{i=1}^3$ are the three vertices of $K$.

**Lemma 3** With $u_T^* \in V_T^*$ defined by (3.23)–(3.24), we have

$$\| \sigma - \sigma_T \|_C^2 + \| \nabla_T^2 (u - u_T^*) \|_0^2 + \| \partial_n u_T - \lambda_T \|^2 \lesssim \eta^2(\sigma_T, f, T) + \| C\sigma_T + \nabla_T^2 u_T^* \|_0^2, \quad (3.25)$$

$$\eta^2(\sigma_T, f, T) + \| C\sigma_T + \nabla_T^2 u_T^* \|_0^2 \lesssim \| \sigma - \sigma_T \|_C^2 + \| \nabla_T^2 (u - u_T^*) \|_0^2 + \eta^2(\sigma_T, f, T), \quad (3.26)$$

where $\nabla_T^2$ is the elementwise counterpart of $\nabla^2$ with respect to the triangulation $T$, and $\| \partial_n u_T - \lambda_T \|^2 := \sum_{K \in T} \xi \| \partial_n u_T - \lambda_T \|^2_{0, \partial K}$.

**Proof** By (3.11), it holds

$$\| \partial_n u_T - \lambda_T \| \lesssim \tilde{\eta}_1(\sigma_T, T).$$

Then (3.25) follows from (3.15), the last inequality and

$$\| \nabla_T^2 (u - u_T^*) \|_0 \lesssim \| \sigma - \sigma_T \|_C + \| C\sigma_T + \nabla_T^2 u_T^* \|_0.$$ 

Since

$$\| C\sigma_T + \nabla_T^2 u_T^* \|_0 \lesssim \| \sigma - \sigma_T \|_C + \| \nabla_T^2 (u - u_T^*) \|_0,$$ 

we acquire (3.26) from (3.17).

**Remark 2** In the a posteriori error estimates (3.25)–(3.26), we use $u_T^*$ instead of $u_T$ for the consideration that $\| \nabla_T^2 (u - u_T^*) \|_0$ shares the same a priori convergence rate as $\| \sigma - \sigma_T \|_C$ when the exact solution is smooth enough, which is one order higher than that of $\| \nabla_T^2 (u - u_T) \|_0$ (cf. [37, Section 5]).
4 Quasi-orthogonality

Quasi-orthogonality of the bending moment will be derived in this section, which is indispensable in the analysis of the convergence and complexity of the adaptive algorithm. By means of the discrete Helmholtz decomposition in [35], we first create the quasi-orthogonality for the postprocessed bending moment. Moreover, the stability of the postprocessing error with respect to the mesh is derived, which will be used in the proofs of the quasi-orthogonality and the discrete reliability of the error estimator. With these, the quasi-orthogonality will be obtained from the following inequality

\[
|a(\sigma - \sigma_T^*, \sigma_T - \sigma_T^*)| \\
= |a(\sigma - \sigma_T^*, \sigma_T - \sigma_T^*) + a(\sigma - \sigma_T^*, (\sigma_T^* - \sigma_T) - (\sigma_T - \sigma_T))| \\
\leq |a(\sigma - \sigma_T^*, \sigma_T - \sigma_T^*)| + \|\sigma - \sigma_T^*\|_C\|\sigma_T^* - \sigma_T - (\sigma_T - \sigma_T)\|_C. \\
\]

(4.1)

To derive the quasi-orthogonality, we need a discrete operator \( K_{T^*} : V_{T^*} \rightarrow \Sigma_{HJ}\) defined as follows (cf. [35, (4.1)]): given \( v \in V_{T^*} \),

\[
\int_\Omega K_{T^*}(v) : \tau dx = -b_{T^*}(\tau, v) \quad \forall \ \tau \in \Sigma_{T^*}.
\]

According to the definition of \( K_{T^*} \) and the inf-sup condition (2.3), we get (cf. [35, (4.54)]),

\[
\|v\|_{2,T^*} \lesssim \|K_{T^*}(v)\|_0 \quad \forall \ v \in V_{T^*}. \quad (4.2)
\]

We have the following quasi-orthogonality for the postprocessed bending moment.

**Lemma 4** It follows

\[
|a(\sigma - \sigma_T^*, \sigma_T - \sigma_T^*)| \lesssim \text{osc}(f, T^*)\|\sigma - \sigma_T^*\|_C. \quad (4.3)
\]

**Proof** Since \( \sigma_T - \sigma_T^* \in \Sigma_{HJ} \), making use of the discrete Helmholtz decomposition in Lemma 4.1 of [35], there exist \( \psi \in V_{T^*} \) and \( \phi \in W_{T^*} \) such that

\[
\sigma_T - \sigma_T^* = K_{T^*}(\psi) + \varepsilon^\perp(\phi), \quad \|K_{T^*}(\psi)\|_0 + \|\phi\|_1 \lesssim \|\sigma_T - \sigma_T^*\|_0. \quad (4.4)
\]

Picking \( \tau = \varepsilon^\perp(\phi) \) in (3.9) on \( T^* \), it holds from (2.4)

\[
a(\sigma - \sigma_T^*, \varepsilon^\perp(\phi)) = 0.
\]

Hence

\[
a(\sigma - \sigma_T^*, \sigma_T - \sigma_T^*) = a(\sigma - \sigma_T^*, K_{T^*}(\psi)) + a(\sigma - \sigma_T^*, \varepsilon^\perp(\phi)) \\
= a(\sigma - \sigma_T^*, K_{T^*}(\psi)). \quad (4.6)
\]
By the definition of $\mathcal{K}_{T^*}(\psi)$ and (2.4),

$$\|\mathcal{K}_{T^*}(\psi)\|_0^2 = -b_{T^*}(\mathcal{K}_{T^*}(\psi), \psi) = -b_{T^*}(\hat{\sigma}_{T^*} - \check{\sigma}_{T^*}, \psi).$$

It is easy to see that

$$b_{T^*}(\tau, v) = b_T(\tau, v) \quad \forall \tau \in \Sigma_T, \ v \in V_{T^*}.$$ Together with (3.6) on $T^*$, we have

$$\|\mathcal{K}_{T^*}(\psi)\|_0^2 = -\int_\Omega f \psi dx - b_{T^*}(\hat{\sigma}_{T^*}, \psi) = -\int_\Omega f \psi dx - b_T(\hat{\sigma}_{T^*}, \psi).$$

Using (3.6) again with $v = I_T \psi$ and noting the fact that $I_T \psi = \psi$ on $T \cap T^*$, it holds from (3.1)

$$\|\mathcal{K}_{T^*}(\psi)\|_0 = \int_\Omega f(I_T \psi - \psi)dx + b_T(\hat{\sigma}_{T^*}, I_T \psi - \psi) = \int_\Omega f(I_T \psi - \psi)dx.

$$

Then we obtain from the Cauchy–Schwarz inequality, (3.3) and (4.2)

$$\|\mathcal{K}_{T^*}(\psi)\|_0 \lesssim \text{osc}(f, T \setminus T^*). \quad (4.7)$$

Therefore we finish the proof from (4.6)–(4.7) and the Cauchy–Schwarz inequality.

**Lemma 5** For any $\delta > 0$, it follows

$$\sqrt{2^{1+\gamma}\hat{\eta}^2_1(\sigma_{T^*}, T^* \setminus T)} \leq (1 + \delta)\hat{\eta}^2_1(\sigma_T, T \setminus T^*) + (1 + \delta^{-1})\sum_{K \in T \setminus T^*} C_\xi \|C(\sigma_{T^*} - \sigma_T)\|_{0,K}^2. \quad (4.8)$$

**Proof** It is sufficient to prove that for any $K \in T \setminus T^*$,

$$\sqrt{2^{1+\gamma}\hat{\eta}^2_1(\sigma_{T^*}, T^*(K))} \leq (1 + \delta)\hat{\eta}^2_1(\sigma_T, K) + (1 + \delta^{-1})C_\xi \|C(\sigma_{T^*} - \sigma_T)\|_{0,K}^2. \quad (4.9)$$

For each $K' \in T^*(K)$, by the definition of $L^2$-orthogonal projection $Q_{K'}^{k-2}$ and the fact $h_{K'} \leq \frac{1}{\sqrt{2}} h_K$,

$$\sqrt{2^{1+\gamma} h_{K'}\xi_{K'}} \|C\sigma_{T^*} - Q_{K'}^{k-2}(C\sigma_{T^*})\|_{0,K'}^2 \leq h_K \xi_K \|C\sigma_{T^*} - Q_K^{k-2}(C\sigma_{T^*})\|_{0,K'}^2.$$
Summing the last inequality over all $K' \in T^*(K)$, it holds
\[
\sqrt{2}^{1+\gamma} \eta_1^2(\sigma_{T^*}, T^*(K)) \leq h_K \xi_K \|C\sigma_{T^*} - Q_k^{-2}(C\sigma_{T^*})\|_{0,K}^2.
\]

Due to the triangle inequality and the definition of $L^2$-orthogonal projection $Q_k^{-2}$,
\[
\|C\sigma_{T^*} - Q_k^{-2}(C\sigma_{T^*})\|_{0,K} \leq \|C(\sigma_{T^*} - \sigma_T) - Q_k^{-2}(C(\sigma_{T^*} - \sigma_T))\|_{0,K} + \|C\sigma_{T^*} - Q_k^{-2}(C\sigma_T)\|_{0,K} \\
\leq \|C(\sigma_{T^*} - \sigma_T)\|_{0,K} + \|C\sigma_{T^*} - Q_k^{-2}(C\sigma_T)\|_{0,K}.
\]

Thus (4.9) can be obtained by using the last two inequalities, the Young’s inequality and the definition of $C\xi$. \hfill \Box

Next we show the stability of the postprocessing error with respect to the mesh.

**Lemma 6** It follows
\[
\|\tilde{\sigma}_{T^*} - \sigma_{T^*}\|_{0,k} \lesssim C_\xi^2 \|\sigma_{T^*} - \sigma_T\|_{0,k}^2 + C_\xi \eta_1^2(\sigma_T, T\setminus T^*). \tag{4.10}
\]

**Proof** For any $K \in T \cap T^*$, we get from (3.14) on $T$ and $T^*$ that for any $\tau \in P_{k-1}(K, S)$,
\[
\int_{\partial K} M_{n\tau}(\tau)((\partial_n u_{T^*} - \lambda_{T^*}) - (\partial_n u_T - \lambda_T))ds \\
= \int_K (C(\sigma_{T^*} - \sigma_T) + \nabla^2(u_{T^*} - u_T)) : \tau dx.
\]

Then using the similar argument as in the proof of (3.11), it holds
\[
\|\tilde{\sigma}_{T^*} - \sigma_{T^*}\|_{0,k} \lesssim h_K \|C(\sigma_{T^*} - \sigma_T) - Q_k^{-2}(C(\sigma_{T^*} - \sigma_T))\|_{0,K}^2 \leq h_K \|C(\sigma_{T^*} - \sigma_T)\|_{0,K}^2. \tag{4.11}
\]

It is easy to see from the definitions of $\tilde{\sigma}_T$ and $\tilde{\sigma}_{T^*}$ that
\[
\int_K (\tilde{\sigma}_{T^*} - \sigma_{T^*}) - (\tilde{\sigma}_T - \sigma_T) : \tau dx = 0 \quad \forall \tau \in P_{k-2}(K, S).
\]

Thus the scaling argument implies
\[
\|\tilde{\sigma}_{T^*} - \sigma_{T^*}\|_{0,K} \leq h_K \|M_n((\tilde{\sigma}_{T^*} - \sigma_{T^*}) - (\tilde{\sigma}_T - \sigma_T))\|_{0,\partial K}^2 \\
\lesssim h_K \underline{\xi}_K^2 \|((\partial_n u_{T^*} - \lambda_{T^*}) - (\partial_n u_T - \lambda_T))\|_{0,\partial K}^2.
\]
which together with (4.11) indicates
\[
\sum_{K \in T \cap T^*} \| (\hat{\sigma}^* - \sigma^*) - (\hat{\sigma} - \sigma) \|_{0,K}^2 \lesssim C_3^2 \| \sigma^* - \sigma \|_C^2.
\]

On the other side, we get from (3.13)
\[
\sum_{K \in T \setminus T^*} \| (\hat{\sigma}^* - \sigma^*) - (\hat{\sigma} - \sigma) \|_{0,K}^2 \\
\leq 2 \sum_{K \in T^* \setminus T} \| \hat{\sigma}^* - \sigma^* \|_{0,K}^2 + 2 \sum_{K \in T \setminus T^*} \| \hat{\sigma} - \sigma \|_{0,K}^2 \\
\lesssim C_3 \eta_1^2 (\sigma^*, T^* \setminus T) + C_3 \eta_1^2 (\sigma, T \setminus T^*).
\]

The proof is finished by (4.8) with $\delta = 1$ and the last two inequalities.

Hence the quasi-orthogonality is achieved from (4.1), (4.3) and (4.10).

Lemma 7 (Quasi-orthogonality) There exists a positive constant $C_3$ depending only on the shape-regularity of the triangulations, the polynomial degree $k$ and the tensor $C$ such that
\[
|a(\sigma - \sigma^*, \sigma^* - \sigma^*)| \\
\leq C_3 (\max \{ C_3, 1 \} \eta_1^2 (\sigma, T), T \setminus T^*) + C_3^2 \| \sigma^* - \sigma \|_C^2 \| \sigma - \sigma^* \|_C.
\]
\[(4.12)\]

5 Convergence of the AHCDGM

The target of this section is to design an adaptive hybridizable $C^0$ discontinuous Galerkin method and show its convergence.

Based on the error estimator in (3.10), an adaptive hybridizable $C^0$ discontinuous Galerkin method (AHCDGM) using Dörfler marking strategy (cf. [24]) for problem (2.1) is presented in Algorithm 1.

Algorithm 1 Adaptive hybridizable $C^0$ discontinuous Galerkin method.

1: Given a parameter $0 < \vartheta < 1$ and an initial mesh $T_0$. Set $m := 0$.
2: (SOLVE) Solve the HCDG method (2.2a)-(2.2d) on $T_m$ for the discrete solution $(\sigma_m, u_m, \lambda_m) \in \Sigma_{T_m} \times V_{T_m} \times M_{T_m}$.
3: (ESTIMATE) Compute the error indicator $\eta^2 (\sigma_m, f, T_m)$ defined in (3.10).
4: (MARK) Mark a set $\mathcal{M}_m \subset T_m$ with minimal cardinality such that
\[
\eta^2 (\sigma_m, f, \mathcal{M}_m) \geq \vartheta \eta^2 (\sigma_m, f, T_m).
\]
\[(5.1)\]
5: (REFINE) Refine each triangle $K$ in $\mathcal{M}_m$ by the newest vertex bisection to get $T_{m+1}$.
6: Set $m := m + 1$ and go to Step 2.
Owing to the newest vertex bisection, the shape regularity of \([\mathcal{T}_m]\) generated by Algorithm 1 only depends on the initial mesh \(\mathcal{T}_0\) (cf. [5, 41, 46, 47]). If the initial mesh \(\mathcal{T}_0\) satisfies the condition (b) in section 4 of [46], then

\[
\#\mathcal{T}_m - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{m-1} \#\mathcal{M}_j. \tag{5.2}
\]

The relations of the error estimators over two consecutive meshes are exhibited in the next lemma.

**Lemma 8** (Estimator reduction) For any \(\delta > 0\), there exists a positive constant \(C_4\) depending only on the shape-regularity of the triangulations, the polynomial degree \(k\) and the tensor \(\mathcal{C}\) such that

\[
\eta_1^2(\sigma, f, T^*) \leq (1 + \delta)(\eta_1^2(\sigma, f, T) - \Delta \eta_1^2(\sigma, f, T \setminus T^*)) + (1 + \delta^{-1})C_a C_\xi \|\sigma_{T^*} - \sigma_T\|_C^2, \tag{5.3}
\]

\[
\eta_2^2(\sigma_{T^*}, T^*) \leq (1 + \delta)(\eta_2^2(\sigma, T) - (1 - 2^{-1/2})\eta_2^2(\sigma, T \setminus T^*)) + (1 + \delta^{-1})C_4 \|\sigma_{T^*} - \sigma_T\|_C^2, \tag{5.4}
\]

\[
\eta^2(\sigma_{T^*}, f, T^*) \leq (1 + \delta)(\eta^2(\sigma, f, T) - \Delta \eta^2(\sigma, f, T \setminus T^*)) + (1 + \delta^{-1})(C_a C_\xi + C_4) \|\sigma_{T^*} - \sigma_T\|_C^2 \tag{5.5}
\]

with \(A := 1 - \max\{2^{-1/2}, 2^{-(1+\gamma)/2}\}\) and \(C_a := \sup_{\tau \in L^2(\Omega, S)} \frac{\|\mathcal{C}\tau\|^2}{\alpha(\tau, \tau)}\).

**Proof** The inequality (5.4) is just (5.4) in [35]. (5.5) follows immediately from (5.3)–(5.4). Next we show the proof of (5.3). Using the similar argument as in the proof of (4.8), it follows

\[
\text{osc}^2(f, T^* \setminus T) \leq \frac{1}{4} \text{osc}^2(f, T \setminus T^*),
\]

which implies

\[
\text{osc}^2(f, T^*) = \text{osc}^2(f, T \cap T^*) + \text{osc}^2(f, T^* \setminus T)
\leq \text{osc}^2(f, T \cap T^*) + \frac{1}{4} \text{osc}^2(f, T \setminus T^*)
\]

\[
= \text{osc}^2(f, T) - \frac{3}{4} \text{osc}^2(f, T \setminus T^*). \tag{5.6}
\]

From the triangle inequality, the definition of \(L^2\)-orthogonal projection \(Q_{k-2}^k\) and the Young’s inequality,

\[
\bar{\eta}_1^2(\sigma_{T^*}, T \cap T^*)
\leq \sum_{K \in T \cap T^*} h_K \xi_K \left(\|\mathcal{C}\sigma_T - Q_{k-2}^k(\mathcal{C}\sigma_T)\|_{0,K} + \|\mathcal{C}(\sigma_{T^*} - \sigma_T)\|_{0,K}\right)^2
\leq (1 + \delta)\bar{\eta}_1^2(\sigma, T \cap T^*) + (1 + \delta^{-1})C_\xi \sum_{K \in T \cap T^*} \|\mathcal{C}(\sigma_{T^*} - \sigma_T)\|_{0,K}^2.
\]
Then we get from (4.8)

\[ \tilde{\eta}_1^2(\sigma_{T^*}, T^*) \leq (1 + \delta) \left( \tilde{\eta}_1^2(\sigma_T, T) - (1 - 2^{-(1+\gamma)/2})\tilde{\eta}_1^2(\sigma_T, T \setminus T^*) \right) + (1 + \delta^{-1})C_\xi \Vert C(\sigma_{T^*} - \sigma_T) \Vert_0^2. \] (5.7)

Therefore (5.3) is the result of (5.6)–(5.7) and the definition of \( \eta_1^2(\sigma_{T^*}, f, T^*) \).

Now we show the main result of this section, i.e. the contraction of the quasi-error for the AHCDGM.

**Theorem 1** There exist positive constants \( \alpha < 1, \beta_1, \beta_2 \) and \( C_{\xi}^* \) depending only on the shape-regularity of the triangulations, the polynomial degree \( k \) and the tensor \( C \) such that if \( C_\xi \leq C_{\xi}^* \), then

\[ \| \sigma - \sigma_{m+1} \|^2_C + \beta_1 \eta_1^2(\sigma_{m+1}, f, T_{m+1}) + \beta_2 \eta_1^2(\sigma_{m+1}, f, T_{m+1}) \leq \alpha \left( \| \sigma - \sigma_m \|^2_C + \beta_1 \eta_1^2(\sigma_m, f, T_{m}) + \beta_2 \eta_1^2(\sigma_m, f, T_{m}) \right). \] (5.8)

**Proof** Let \( \varepsilon \leq \frac{1}{2}, \delta_1 \leq 1 \) and \( \delta_2 \leq 1 \) be three yet-to-be-determined positive constants.

Set \( C_{\xi}^* = \min \left\{ \frac{\Lambda \delta_1 \varepsilon}{2C_3(\Lambda \delta_1 + C_a)}, 1 \right\} \). Due to quasi-orthogonality (4.12) with \( T = T_m \) and \( T^* = T_{m+1} \) and the Young’s inequality, it holds

\[ \| \sigma - \sigma_{m+1} \|^2_C + \| \sigma_{m} - \sigma_{m+1} \|^2_C = \| \sigma - \sigma_{m} \|^2_C + 2\alpha(\sigma - \sigma_{m+1}, \sigma_{m} - \sigma_{m+1}) \leq \| \sigma - \sigma_{m} \|^2_C + 2C_3(\eta_1^2(\sigma_{m}, f, T_m \setminus T_{m+1}) + C_\xi^2 \| \sigma_{m+1} - \sigma_m \|^2_C) + \frac{C_3^2}{\varepsilon}(\eta_1^2(\sigma_{m}, f, T_m \setminus T_{m+1}) + C_\xi^2 \| \sigma_{m+1} - \sigma_m \|^2_C).
\]

Hence by a direct manipulation,

\[ \| \sigma - \sigma_{m+1} \|^2_C \leq \frac{1}{1 - \varepsilon} \| \sigma - \sigma_{m} \|^2_C + \beta_1 \Lambda (1 + \delta_1) \eta_1^2(\sigma_{m}, f, T_m \setminus T_{m+1}) \frac{1 - \frac{C_3^2 C_\xi^2}{\varepsilon}}{1 - \delta_1 \varepsilon} \| \sigma_{m+1} - \sigma_m \|^2_C \]

with \( \beta_1 = \frac{C_3^2}{\Lambda (1 + \delta_1) \varepsilon (1 - \varepsilon)} \). From (5.3) with \( \delta = \delta_1 \), we have

\[ \beta_1 \eta_1^2(\sigma_{m+1}, f, T_{m+1}) \leq \beta_1 (1 + \delta_1)(\eta_1^2(\sigma_{m}, f, T_{m}) - \Lambda \eta_1^2(\sigma_{m}, f, T_m \setminus T_{m+1})) + \frac{C_3^2 C_a C_\xi}{\Lambda \delta_1 \varepsilon (1 - \varepsilon) \| \sigma_{m+1} - \sigma_m \|^2_C}. \]
Then we obtain from the last two inequalities
\[
\|\sigma - \sigma_{m+1}\|_C^2 + \beta_1 \eta_1^2(\sigma_{m+1}, f, T_{m+1}) \\
\leq \frac{1}{1 - \varepsilon} \|\sigma - \sigma_m\|_C^2 + \beta_1 (1 + \delta_1) \eta_1^2(\sigma_m, f, T_m) \\
- \frac{1}{1 - \varepsilon} \left( 1 - \frac{C^2 C^* (C_a \Lambda \delta_1 + C_a)}{\Lambda \delta_1 \varepsilon} \right) \|\sigma_{m+1} - \sigma_m\|_C^2.
\]

By the definition of $C^*_1$, it follows
\[
\|\sigma - \sigma_{m+1}\|_C^2 + \beta_1 \eta_1^2(\sigma_{m+1}, f, T_{m+1}) \\
\leq \frac{1}{1 - \varepsilon} \|\sigma - \sigma_m\|_C^2 + \beta_1 (1 + \delta_1) \eta_1^2(\sigma_m, f, T_m) - \frac{1}{2(1 - \varepsilon)} \|\sigma_{m+1} - \sigma_m\|_C^2.
\]

Let $\beta_2 = \frac{\delta_2}{2(1 - \varepsilon)(1 + \delta_2)(C_a + C_4)}$. We get from (5.5) with $\delta = \delta_2$
\[
\beta_2 \eta_1^2(\sigma_{m+1}, f, T_{m+1}) \leq \beta_2 (1 + \delta_2) (\eta_2^2(\sigma_m, f, T_m) - \Lambda \eta_2^2(\sigma_m, f, T_m \setminus T_{m+1})) \\
+ \frac{1}{2(1 - \varepsilon)} \|\sigma_{m+1} - \sigma_m\|_C^2.
\]

Adding the last two inequalities, it holds from marking strategy (5.1)
\[
\|\sigma - \sigma_{m+1}\|_C^2 + \beta_1 \eta_1^2(\sigma_{m+1}, f, T_{m+1}) + \beta_2 \eta_2^2(\sigma_{m+1}, f, T_{m+1}) \\
\leq \frac{1}{1 - \varepsilon} \|\sigma - \sigma_m\|_C^2 + \beta_1 (1 + \delta_1) \eta_1^2(\sigma_m, f, T_m) \\
+ \beta_2 (1 + \delta_2) (1 - \Lambda \delta \varepsilon) \eta_2^2(\sigma_m, f, T_m).
\]

Now set
\[
\delta_2 = \frac{\Lambda \delta}{3}, \quad \varepsilon = \min \left\{ \frac{\delta_2 \Lambda \delta}{12C_1(C_a + C_4)}, \frac{1}{2} \right\},
\]
\[
\delta_1 = \min \left\{ \frac{\beta_2 (1 + \delta_2) \Lambda^2 \delta \varepsilon (1 - \varepsilon)}{6C_3^2}, 1 \right\}.
\]

Then it follows from the reliability of the error estimator (3.15) on $T_m$ and the definitions of $\varepsilon$ and $\beta_2$
\[
\frac{2\varepsilon}{1 - \varepsilon} \|\sigma - \sigma_m\|_C^2 \leq \frac{2\varepsilon C_1}{1 - \varepsilon} \eta^2(\sigma_m, f, T_m) \leq \frac{1}{3} \beta_2 (1 + \delta_2) \Lambda \delta \eta_2^2(\sigma_m, f, T_m).
\]

It is also easy to see from the definitions of $\delta_1$, $\beta_1$ and $\delta_2$
\[
2\beta_1 \delta_1 \eta_1^2(\sigma_m, f, T_m) \leq \frac{1}{3} \beta_2 (1 + \delta_2) \Lambda \delta \eta_2^2(\sigma_m, f, T_m),
\]
\[
\beta_2 (\delta_2 + \delta_2^2) \eta^2(\sigma_m, f, T_m) \leq \frac{1}{3} \beta_2 (1 + \delta_2) \Lambda \delta \eta_2^2(\sigma_m, f, T_m).
\]
Adding the last four inequalities, we have

\[
\| \sigma - \sigma_{m+1} \|^2_C + \beta_2 \eta^2(\sigma_{m+1}, f, T_{m+1}) + \beta_2 \eta^2(\sigma, f, T_m) \\
\leq \frac{1 - 2\varepsilon}{1 - \varepsilon} \| \sigma - \sigma_m \|^2_C + \beta_1 (1 - \delta_1) \eta^2(\sigma_m, f, T_m) + \beta_2 (1 - \delta_2) \eta^2(\sigma_m, f, T_m).
\]

Therefore (5.8) is acquired by choosing \( \alpha = \max\{ \frac{1 - 2\varepsilon}{1 - \varepsilon}, 1 - \delta_1, 1 - \delta_2 \} < 1. \)

6 Complexity of the AHCDGM

We discuss the complexity of the AHCDGM in this section. Through introducing two connection operators corresponding to the deflection and the bending moment respectively, we acquire the quasi-optimalty of the total error, which leads to a nonlinear approximation class. In order to achieve the asymptotic estimate for the total error, we also develop the discrete reliability of the error estimator.

6.1 Connection operators

To derive the quasi-optimalty of the total error, two connection operators are provided in this subsection. For any \( K \in \mathcal{T} \), we define a modified Argyris element \( \{ K, V_M^A, N_K \} \) as follows:

- The local shape function space \( V_M^A \) is \( P_{k+4}(K) \);
- A unisolvent set of degrees of freedom \( N_K \) is given for any shape function \( w \in V_M^A \) by (cf. [9])
  - (i) the pointwise evaluations of \( w \) at the three vertices of the triangle,
  - (ii) \( \int_{\partial K} w ud\nu = \frac{\partial w}{\partial n} \nu d\nu \) on each edge \( e \) of the triangle,
  - (iii) \( \int_K w ud\nu = \frac{\partial w}{\partial n} d\nu \) on each edge \( e \),
  - (iv) \( \int_{\partial K} \frac{\partial w}{\partial n} d\nu \) at the three vertices of the triangle,
  - (v) the evaluations of the normal derivatives of \( w \) at \( k \) interior points on each edge,
  - (vi) \( k/(k-2) \) additional interior nodal variables that uniquely determine polynomials in \( P_{k-2}(K) \).

Let \( V_T^A \subset H_0^2(\Omega) \) be the corresponding modified Argyris finite element space with respect to \( \mathcal{T} \). Then define a connection operator \( I_T^A : V_T \to V_T^A \) associated with the deflection as follows: given \( v \in V_T \),

- for any degree of freedom \( D \) corresponding to (i)–(iii) and (vi) of \( N_K \),
  \[
  D(I_T^A v) = D(v),
  \]
- for any degree of freedom \( D \in \partial \Omega \) corresponding to (iv)–(v) of \( N_K \),
  \[
  D(I_T^A v) = 0.
  \]

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– for any degree of freedom $D \in \Omega$ corresponding to (iv)–(v) of $N_K$,

$$
D \left( I_T^{MA} v \right) = \frac{1}{|T_p|} \sum_{K \in T_p} (D(v|_K)),
$$

where $p$ is the nodal point corresponding to $D$, and $T_p$ is the set of triangles in $T$ sharing the common nodal point $p$.

It is obvious that the degrees of freedom of the modified Argyris element can be obtained from the degrees of freedom of the Argyris element by replacing the nodal variables of the $k$-th Lagrange element with the nodal variables of $I_T$. Thus as (3.1), it follows

$$
b_T \left( \tau, v - I_T^{MA} v \right) = 0 \quad \forall \, \tau \in \Sigma_T, v \in V_T.
$$

Due to the similar argument in [35, Lemma 4.3], we have for any $v \in V_T$

$$
\| v - I_T^{MA} v \|_{0,K} + h_K^2 |I_T^{MA} v|_{2,K} \lesssim h_K^2 \| v \|_{2,\mathcal{O}(K)} \quad \forall \, K \in T.
$$

Next we define another connection operator $\Pi_T : \Sigma_T \rightarrow \Sigma^{HHJ}_T$ associated with the bending moment in the following way: given $\tau \in \Sigma_T$, for any element $K \in T$ and any edge $e$ of $K$,

$$
\int_e M_n \left( \Pi_T \tau \right) \mu ds = \int_e \{M_n(\tau)\} \mu ds \quad \forall \, \mu \in P_{k-1}(e),
$$

$$
\int_K (\tau - \Pi_T \tau) : \varsigma dx = 0 \quad \forall \, \varsigma \in P_{k-2}(K, \mathbb{S}).
$$

From the scaling argument and the definition of $\Pi_T$, it readily holds

$$
\| \tau - \Pi_T \tau \|_0^2 \approx \sum_{e \in E^i(T)} h_e \| [M_n(\tau)] \|_{0,e}^2 \quad \forall \, \tau \in \Sigma_T.
$$

Let $Q_T$ be the $L^2$-orthogonal projection from $L^2(\Omega, \mathbb{S})$ onto $\Sigma_T$. We have the following error estimate for the connection operator $\Pi_T$ under the minimal regularity $\sigma \in L^2(\Omega, \mathbb{S})$.

**Lemma 9** For any $\tau \in \Sigma_T$, it follows

$$
\| \sigma - \Pi_T (Q_T \sigma) \|_0^2 \lesssim \| \sigma - \tau \|_0^2 + \text{osc}^2(f, T).
$$

**Proof** It follows from (6.3) with $\tau = Q_T \sigma$

$$
\| \sigma - \Pi_T (Q_T \sigma) \|_0^2 \lesssim \| \sigma - Q_T \sigma \|_0^2 + \| Q_T \sigma - \Pi_T (Q_T \sigma) \|_0^2
$$

$$
\lesssim \| \sigma - Q_T \sigma \|_0^2 + \sum_{e \in E^i(T)} h_e \| [M_n(Q_T \sigma)] \|_{0,e}^2.
$$
Since Lemma 3.3 in [2], we have
\[ \sum_{e \in E^i(T)} h_e \| [M_n(Q_T \sigma)] \|^2_{0,e} \lesssim \| \sigma - Q_T \sigma \|^2_0 + \text{osc}^2(f, T). \]

Therefore by the definition of $Q_T$, it holds for any $\tau \in \Sigma_T$
\[ \| \sigma - \Pi_T(Q_T \sigma) \|^2_0 \lesssim \| \sigma - Q_T \sigma \|^2_0 + \text{osc}^2(f, T) \leq \| \sigma - \tau \|^2_0 + \text{osc}^2(f, T), \]
as required. \qed

### 6.2 Discrete reliability of the error estimator

In this subsection, we prove the discrete reliability of the error estimator by employing the discrete Helmholtz decomposition and the stability of the postprocessing error with respect to the mesh.

**Lemma 10** There exist positive constants $C_5$ and $C^*_\xi$, depending only on the shape-regularity of the triangulations, the polynomial degree $k$ and the tensor $C$ such that if $C_\xi \leq C^*_\xi$, then
\[ \| \sigma_T - \sigma_{T^*} \|^2_C \leq C_5 \eta^2(\sigma_T, f, T \setminus T^*). \] (6.5)

**Proof** Here we use the discrete Helmholtz decomposition (4.4)–(4.5) of $\hat{\sigma}_T - \hat{\sigma}_{T^*}$ again. Since (2.2a) and (2.4) on $T^*$, it holds
\[ a(\sigma_{T^*}, e^\perp(\phi)) = 0. \]

Thus we have from (3.21)
\[ a(\sigma_T - \sigma_{T^*}, e^\perp(\phi)) = a(\sigma_T, e^\perp(\phi)) = a(\sigma_T, e^\perp(\phi - I_{T^*}^{\text{SZ}} \phi)). \]

This is just (4.59) in [35]. Thus using integration by parts, the fact that $I_{T^*}^{\text{SZ}} \phi = \phi$ on any $K \in T \cap T^*$ and the error estimates of $I_{T^*}^{\text{SZ}}$, we get
\[ a(\sigma_T - \sigma_{T^*}, e^\perp(\phi)) \lesssim \eta_2(\sigma_T, T \setminus T^*) \| \phi \|_1. \]

Together with (4.5) and (4.10), it follows
\[ a(\sigma_T - \sigma_{T^*}, e^\perp(\phi)) \lesssim \eta_2(\sigma_T, T \setminus T^*) \| \hat{\sigma}_T - \hat{\sigma}_{T^*} \|_C \lesssim (1 + C_\xi) \eta_2(\sigma_T, T \setminus T^*) \times \left( \| \sigma_T - \sigma_{T^*} \|_C + \tilde{\eta}_1(\sigma_T, T \setminus T^*) \right). \]

According to the Cauchy–Schwarz inequality and (4.7),
\[ a(\sigma_T - \sigma_{T^*}, K_{T^*}(\psi)) \leq \| \sigma_T - \sigma_{T^*} \|_C \| K_{T^*}(\psi) \|_C \lesssim \| \sigma_T - \sigma_{T^*} \|_C \text{osc}(f, T \setminus T^*). \]
Then we get from (4.4)
\[
a(\sigma_{T} - \sigma_{T^*}, \Tilde{\sigma}_{T} - \Tilde{\sigma}_{T^*})
= a(\sigma_{T} - \sigma_{T^*}, e^\perp(\phi)) + a(\sigma_{T} - \sigma_{T^*}, K_{T^*}(\psi))
\lesssim (1 + C_{\xi}) \left( \|\sigma_{T} - \sigma_{T^*}\|_C(\eta_{2}(\sigma_{T}, T \setminus T^*) + \text{osc}(f, T \setminus T^*))
\right.
+ \eta^2(\sigma_{T}, f, T \setminus T^*) \bigg). 
\]

On the other hand, it holds from (4.10)
\[
a(\sigma_{T} - \sigma_{T^*}, (\Tilde{\sigma}_{T} - \Tilde{\sigma}_{T^*}) - (\Tilde{\sigma}_{T} - \sigma_{T}))
\leq \|\sigma_{T} - \sigma_{T^*}\|_C(\Tilde{\sigma}_{T} - \Tilde{\sigma}_{T^*}) - (\Tilde{\sigma}_{T} - \sigma_{T})\|_C
\lesssim \|\sigma_{T} - \sigma_{T^*}\|_C(\eta_{1}(\sigma_{T}, T \setminus T^*) + C_{\xi}\eta(\sigma_{T}, f, T \setminus T^*)). 
\]

Adding the last two inequalities, there exists a positive constant $C_{\xi}^* \Delta^2$ depending only on the shape-regularity of the triangulations, the polynomial degree $k$ and the tensor $C$ such that
\[
2C_{\xi}^*\|\sigma_{T} - \sigma_{T^*}\|_C^2 \leq C_{\xi}^*\|\sigma_{T} - \sigma_{T^*}\|_C^2 + (1 + C_{\xi}) \left( \sqrt{3}\|\sigma_{T} - \sigma_{T^*}\|_C \eta(\sigma_{T}, f, T \setminus T^*) \right)
\]
\[
+ \eta^2(\sigma_{T}, f, T \setminus T^*) \bigg). 
\]

Hence if $C_{\xi} \leq C_{\xi}^*\Delta^2$, the last inequality can be rewritten as
\[
C_{\xi}^*\|\sigma_{T} - \sigma_{T^*}\|_C^2
\leq (1 + C_{\xi}^*\Delta^2) \left( \sqrt{3}\|\sigma_{T} - \sigma_{T^*}\|_C \eta(\sigma_{T}, f, T \setminus T^*) \right)
\]
\[
+ \eta^2(\sigma_{T}, f, T \setminus T^*) \bigg). 
\]

Applying the Young’s inequality, we obtain
\[
C_{\xi}^*\|\sigma_{T} - \sigma_{T^*}\|_C^2
\leq \frac{1}{2}C_{\xi}^*\|\sigma_{T} - \sigma_{T^*}\|_C^2 + (1 + C_{\xi}^*\Delta^2) \left( \frac{3(1 + C_{\xi}^*\Delta^2)}{2C_{\xi}^*\Delta^2} + 1 \right) \eta^2(\sigma_{T}, f, T \setminus T^*),
\]
which is exactly (6.5) when we set $C_{5} = \frac{(1+C_{\xi}^*\Delta^2)(3+5C_{\xi}^*\Delta^2)}{(C_{\xi}^*\Delta^2)^2}$. \hfill \Box

### 6.3 The total error

For any $\tau \in \Sigma_{T}$, define total error as
\[
E_{T}(\tau) := \left( \|\sigma - \tau\|_C^2 + \eta^2(\tau, f, T) \right)^{1/2}.
\]

It is easy to know from (3.16) that
\[
E_{T}^2(\sigma_{T}) \approx \|\sigma - \sigma_{T}\|_C^2 + \beta_{1}\eta^2(\sigma_{T}, f, T) + \beta_{2}\eta^2(\sigma_{T}, f, T). \tag{6.6}
\]
Let $Q_T$ be the $L^2$-orthogonal projection from $L^2(\mathcal{E}(T))$ onto $M_T$.

**Lemma 11** For any $K \in T$, it holds

$$\|\partial_n(I_T u) - Q_T^\alpha(\partial_n u)\|_{0,K}^2 \lesssim h_K \|C\sigma - Q_k^{-2}(C\sigma)\|_{0,K}^2. \quad (6.7)$$

**Proof** Using (3.1) and integration by parts, we have for any $\tau \in P_{k-1}(K, S)$

$$\int_{\partial K} M_{nn}(\tau)(\partial_n(I_T u) - Q_T^\alpha(\partial_n u))ds$$

$$= \int_{\partial K} M_n(\tau)\partial_n(I_T u - u)ds$$

$$= \int_{\partial K} (\tau n) \cdot \nabla(I_T u - u)ds - \int_K (\nabla \cdot \tau) \cdot \nabla(I_T u - u)dx$$

$$= \int_K \nabla^2(I_T u - u) : \tau dx = \int_K (C\sigma + \nabla^2(I_T u)) : \tau dx.$$

Then we can obtain (6.7) by adopting the argument in the proof of (3.11). □

**Lemma 12** For any $\tau \in \Sigma_{HHJ_T}^T$ and $v \in V_T$, we have

$$b_T(\tau, v) + \int_\Omega f v dx \lesssim (\|\sigma - \tau\|_{\mathcal{L}} + \text{osc}(f, T)) \|v\|_{2,T}. \quad (6.8)$$

**Proof** From (6.1), integration by parts, (3.8) and the definition of $I_{T}^{MA}$,

$$b_T(\tau, v) + \int_\Omega f v dx = b_T(\tau, I_{T}^{MA} v) + \int_\Omega f I_{T}^{MA} v dx + \int_\Omega f(v - I_{T}^{MA} v)dx$$

$$= a(\sigma - \tau, K(I_{T}^{MA} v)) + \sum_{K \in T} \int_K (f - Q_k^{-3} f)(v - I_{T}^{MA} v)dx,$$

which combined with the Cauchy–Schwarz inequality and (6.2) ends the proof. □

**Lemma 13** We have the following quasi-optimality of the total error

$$E_T(\sigma_T) \lesssim \inf_{\tau \in \Sigma_T} E_T(\tau).$$

**Proof** It follows from (3.9) that for any $\bar{\tau} \in \Sigma_{HHJ_T}^T$,

$$a(\sigma - \sigma_T, \bar{\tau} - \sigma_T) + b_T(\bar{\tau} - \sigma_T, I_T u - u_T)$$

$$= \sum_{K \in T} \int_{\partial K} M_{nn}(\sigma_T)(\partial_n u - \lambda_T)ds.$$
Combining (2.2b) with $v = I_T u - u_T$ and (2.2c) with $\mu = \lambda_T - Q^\beta_T (\partial_n u)$,

$$\begin{align*}
-b_T(\sigma, I_T u - u_T) - \sum_{K \in T} \int_{\partial K} M_{nn}(\sigma)(\partial_n u - \lambda_T) ds \\
+ \sum_{K \in T} \int_{\partial K} \xi(\partial_n u - \lambda_T)(\partial_n(I_T u) - Q^\beta_T (\partial_n u) - \partial_n u_T + \lambda_T) ds \\
= \int_{\Omega} f(I_T u - u_T) dx.
\end{align*}$$

Thus we get from the last two equalities

$$\begin{align*}
\| \tilde{\tau} - \sigma_T \|_C^2 + \sum_{K \in T} \int_{\partial K} \xi(\partial_n u - \lambda_T)^2 ds \\
= a(\tilde{\tau} - \sigma, \tilde{\tau} - \sigma_T) - b_T(\tilde{\tau}, I_T u - u_T) - \int_{\Omega} f(I_T u - u_T) dx \\
+ \sum_{K \in T} \int_{\partial K} \xi(\partial_n u - \lambda_T)(\partial_n(I_T u) - Q^\beta_T (\partial_n u)) ds.
\end{align*}$$

According to the Cauchy–Schwarz inequality and (6.8) with $\tau = \tilde{\tau}$ and $v = u_T - I_T u$,

$$\begin{align*}
\| \tilde{\tau} - \sigma_T \|_C^2 &+ \sum_{K \in T} \int_{\partial K} \xi(\partial_n u - \lambda_T)^2 ds \\
&\lesssim \| \sigma - \tilde{\tau} \|_C^2 + \sum_{K \in T} \int_{\partial K} \xi(\partial_n(I_T u) - Q^\beta_T (\partial_n u))^2 ds \\
&+ (\| \sigma - \tilde{\tau} \|_C + \text{osc}(f, T)) \| u_T - I_T u \|_{L^2, T}.
\end{align*}$$

By the triangle inequality, (3.11) and (6.7), we have

$$\begin{align*}
\| \sigma - \sigma_T \|_C^2 &+ \tilde{\eta}_1^2(\sigma_T, T) \\
&\lesssim \| \sigma - \tilde{\tau} \|_C^2 + \| \tilde{\tau} - \sigma_T \|_C^2 + \sum_{K \in T} \int_{\partial K} \xi(\partial_n u - \lambda_T)^2 ds \\
&\lesssim \| \sigma - \tilde{\tau} \|_C^2 + \tilde{\eta}_1^2(\sigma, T) + (\| \sigma - \tilde{\tau} \|_C + \text{osc}(f, T)) \| u_T - I_T u \|_{L^2, T}.
\end{align*}$$

On the other hand, using the inf-sup condition (2.3) with $v = u_T - I_T u$ and (3.9), it holds

$$\begin{align*}
\| u_T - I_T u \|_{L^2, T} &\lesssim \sup_{\tau \in \Sigma_T} \frac{b_T(\tau, u_T - I_T u)}{\| \tau \|_{0, T}} = \sup_{\tau \in \Sigma_T} \frac{a(\tau - \sigma_T, \tau)}{\| \tau \|_{0, T}} \\
&\lesssim \| \sigma - \sigma_T \|_C.
\end{align*}$$
Hence we get from the last two inequalities and the Young’s inequality
\[ \|\sigma - \sigma_T\|_C^2 + \tilde{\eta}_1^2(\sigma_T, T) \lesssim \|\sigma - \bar{T}\|_C^2 + \tilde{\eta}_1^2(\sigma, T) + \text{osc}^2(f, T). \]

Now choose \( \bar{T} = \Pi_T(Q_T\sigma) \). We obtain from (6.4) and the triangle inequality that for any \( \tau \in \Sigma_T \),
\[ \|\sigma - \sigma_T\|_C^2 + \tilde{\eta}_1^2(\sigma_T, T) \lesssim \|\sigma - \tau\|_C^2 + \tilde{\eta}_1^2(\tau, T) + \text{osc}^2(f, T). \]

Finally we finish the proof by the arbitrariness of \( \tau \). \( \square \)

**Lemma 14** When \( C_\xi \leq C_{\xi^2}^* \), there exists a positive constant \( C_6 \) depending only on the shape-regularity of the triangulations, the polynomial degree \( k \) and the tensor \( C \) such that for any refinement \( T^* \) of \( T \),
\[ E_{T^*}(\sigma_{T^*}) \leq C_6 E_T(\sigma_T). \quad (6.9) \]

**Proof** For any \( \delta > 0 \), it holds from the Young’s inequality
\[ \|\sigma - \sigma_{T^*}\|_C^2 \leq (1 + \delta)\|\sigma - \sigma_T\|_C^2 + (1 + \delta^{-1})\|\sigma_{T^*} - \sigma_T\|_C^2. \]

Using (5.5) and (6.5), we have
\[ \eta^2(\sigma_{T^*}, f, T^*) \leq (1 + \delta)\eta^2(\sigma_T, f, T) + (1 + \delta^{-1})\left(C_a C_{\xi^2}^* + C_4 - \frac{\delta A}{C_S^*}\right)\|\sigma_{T^*} - \sigma_T\|_C^2. \]

Then adding the last two inequality and choosing \( \delta = \frac{C_5(C_a C_{\xi^2}^* + C_4 + 1)}{A} \), it follows
\[ \|\sigma - \sigma_{T^*}\|_C^2 + \eta^2(\sigma_{T^*}, f, T^*) \leq \frac{\Lambda + C_5(C_a C_{\xi^2}^* + C_4 + 1)}{A} \left(\|\sigma - \sigma_T\|_C^2 + \eta^2(\sigma_T, f, T)\right). \]

On the other hand, it follows from (3.16)
\[ E_T^2(\sigma_T) \approx \|\sigma - \sigma_T\|_C^2 + \eta^2(\sigma_T, f, T). \]

Thus we can complete the proof from the last two inequalities. \( \square \)
6.4 Approximation class and the complexity

For any integer \( N \geq \#T_0 \), let \( \mathbb{T}_N \) be the set of all possible conforming triangulations \( T \) refined from the initial mesh \( T_0 \) satisfying \( \#T \leq N \). Define

\[
A_{s} : = \left\{ (\sigma, f) : |\sigma, f|_{s} := \sup_{N \geq \#T_0} N^{s} \inf_{T \in \mathbb{T}_N} \inf_{\tau \in \Sigma_{T}} E_{T}(\tau) < +\infty \right\}.
\]

Lemma 15 Assume \( C_{\xi} \leq C_{\xi 2}^{*} \). Then for a given \( \chi \in (0, 1) \), we can choose some refinement \( T^{*} \) of \( T \) such that

1. \( E_{T^{*}}(\sigma_{T^{*}}) \leq \chi E_{T}(\sigma_{T}) \),
2. \( \#T^{*} - \#T \lesssim \chi^{-1/s} E_{T}^{-1/s}(\sigma_{T})|\sigma, f|_{s}^{1/s} \).

Proof By the definition of \( A_{s} \) and Lemma 13, there exists a triangulation \( T_{\chi} \) which is some refinement of \( T_0 \) such that

\[
E_{T_{\chi}}(\sigma_{T_{\chi}}) \leq \frac{\chi}{C_{6}} E_{T}(\sigma_{T}),
\]

and

\[
\#T_{\chi} - \#T_0 \lesssim \chi^{-1/s} E_{T}^{-1/s}(\sigma_{T})|\sigma, f|_{s}^{1/s}.
\]

Let \( T^{*} = T \cup T_{\chi} \). Then using (6.9), it holds

\[
E_{T^{*}}(\sigma_{T^{*}}) \leq C_{6} E_{T_{\chi}}(\sigma_{T_{\chi}}) \leq \chi E_{T}(\sigma_{T}).
\]

Finally according to Lemma 3.7 in [12], we have

\[
\#T^{*} - \#T \leq \#T_{\chi} - \#T_0 \lesssim \chi^{-1/s} E_{T}^{-1/s}(\sigma_{T})|\sigma, f|_{s}^{1/s}.
\]

This ends the proof. \( \square \)

Lemma 16 In the Döfler marking, we choose the positive parameter \( \vartheta \) small enough such that

\[
\vartheta < \frac{1}{(C_{2} + 1)(C_{3}^{2}(C_{\xi 2}^{*} + 1) + 1 + C_{5}(1 + (C_{3} C_{\xi 2}^{*})^{2} + 2C_{\xi 2}^{*} C_{a}))). \tag{6.10}
\]

Set

\[
\chi = \sqrt{\frac{1}{2} \left( 1 - (C_{2} + 1)(C_{3}^{2}(C_{\xi 2}^{*} + 1) + 1 + C_{5}(1 + (C_{3} C_{\xi 2}^{*})^{2} + 2C_{\xi 2}^{*} C_{a}))) \vartheta \right)}.
\]

Let \( T^{*} \) be a refinement of \( T_{m} \) such that \( E_{T^{*}}(\sigma_{T^{*}}) \leq \chi E_{T_{m}}(\sigma_{m}) \). When \( C_{\xi} \leq C_{\xi 2}^{*} \), then

\[
\#M_{m} \leq \#T^{*} - \#T_{m}.
\]
Proof According to the definition of $\tilde{\eta}_1$ and the Young’s inequality,

$$\tilde{\eta}_1^2(\sigma_m, T_m \cap T^*) \leq 2\tilde{\eta}_1^2(\sigma_{T^*}, T_m \cap T^*) + 2C_{\xi_2}^* \sum_{K \in T_m \cap T^*} \|C(\sigma_{T^*} - \sigma_m)\|_{0,K}^2.$$ 

Hence

$$\eta_1^2(\sigma_m, f, T_m \cap T^*) = \tilde{\eta}_1^2(\sigma_m, T_m \cap T^*) + \text{osc}^2 (f, T_m \cap T^*) \leq 2\eta_1^2(\sigma_{T^*}, f, T_m \cap T^*) + 2C_{\xi_2}^* C_a \|\sigma_{T^*} - \sigma_m\|_{C}^2,$$

which immediately implies

$$\eta_1^2(\sigma_m, f, T_m) \leq \eta_1^2(\sigma_m, f, T_m \setminus T^*) + 2\eta_1^2(\sigma_{T^*}, f, T^*) + 2C_{\xi_2}^* C_a \|\sigma_{T^*} - \sigma_m\|_{C}^2.$$ 

Due to (4.12) with $T = T_m$ and the Young’s inequality, it holds

$$- 2a(\sigma - \sigma_{T^*}, \sigma_m - \sigma_{T^*}) \leq \|\sigma - \sigma_{T^*}\|_{C}^2 + C_3^2 (\max \{C_{\xi_2}^*, 1\}) \eta_1^2(\sigma_m, f, T_m \setminus T^*) + \left( C_{\xi_2}^* \right)^2 \|\sigma_{T^*} - \sigma_m\|_{C}^2.$$ 

which together with

$$\|\sigma_{T^*} - \sigma_m\|_{C}^2 = \|\sigma - \sigma_m\|_{C}^2 + 2a(\sigma - \sigma_{T^*}, \sigma_m - \sigma_{T^*}) - \|\sigma - \sigma_{T^*}\|_{C}^2$$

means

$$(1 + (C_3 C_{\xi_2}^*)^2) \|\sigma_{T^*} - \sigma_m\|_{C}^2 \geq \|\sigma - \sigma_m\|_{C}^2 - 2\|\sigma - \sigma_{T^*}\|_{C}^2 - C_3^2 (C_{\xi_2}^* + 1) \eta_1^2(\sigma_m, f, T_m \setminus T^*)$$

$$= E_{T_m}^2(\sigma_m) - 2E_{T_m}^2(\sigma_{T^*}) - \eta_1^2(\sigma_m, f, T_m) + 2\eta_1^2(\sigma_{T^*}, f, T^*) - C_3^2 (C_{\xi_2}^* + 1) \eta_1^2(\sigma_m, f, T_m \setminus T^*).$$

Then it follows from (6.11) and (6.5) with $T = T_m$

$$(1 - 2\chi^2) E_{T_m}^2(\sigma_m) \leq E_{T_m}^2(\sigma_m) - 2E_{T_m}^2(\sigma_{T^*}) \leq (1 + (C_3 C_{\xi_2}^*)^2 + 2C_{\xi_2}^* C_a) \|\sigma_{T^*} - \sigma_m\|_{C}^2 + (C_3^2 (C_{\xi_2}^* + 1) + 1) \eta_1^2(\sigma_m, f, T_m \setminus T^*) \leq (C_3^2 (C_{\xi_2}^* + 1) + 1 + C_5 (1 + (C_3 C_{\xi_2}^*)^2 + 2C_{\xi_2}^* C_a)) \eta_2^2(\sigma_m, f, T_m \setminus T^*).$$

Due to (3.16) with $T = T_m$,

$$\frac{1}{C_2 + 1} \eta_2^2(\sigma_m, f, T_m) \leq \frac{1}{C_2} \eta_2^2(\sigma_m, T_m) + \eta_1^2(\sigma_m, f, T_m) \leq E_{T_m}^2(\sigma_m).$$
Combining the last two inequalities, it holds
\[
\frac{1 - 2\chi^2}{C_2 + 1} \eta^2(\sigma_m, f, T_m) \leq (C_3 (C^*_5 + 1) + C_5 (1 + (C_3 C^*_5)^2 + 2C_2^* C_0)) \eta^2(\sigma_m, f, T_m \setminus T^*).
\]

By the choice of \( \chi \), we obtain
\[
\eta^2(\sigma_m, f, T_m \setminus T^*) \geq \vartheta \eta^2(\sigma_m, f, T_m).
\]

Since in the marking strategy we choose the minimal set \( M_m \) such that
\[
\eta^2(\sigma_m, f, M_m) \geq \vartheta \eta^2(\sigma_m, f, T_m).
\]

We conclude that
\[
\#M_m \leq \#(T_m \setminus T^*) \leq \#T^* - \#T_m,
\]
as required.

The following important lemma concerns on the number of elements marked in the marking procedure. It immediately follows Lemmas 15 and 16.

**Lemma 17** Assume that the marking parameter \( \vartheta \) verifies (6.10). Let \( M_m \subset T_m \) be the set with the minimal number of simplices such that
\[
\eta^2(\sigma_m, f, M_m) \geq \vartheta \eta^2(\sigma_m, f, T_m).
\]
When \( C_\xi \leq C^*_\xi \), then
\[
\#M_m \lessapprox E^{-1/s}_{T_m}(\sigma_m) |\sigma, f|^{1/s}_{s_m}.
\]

We are now in a position to derive the asymptotic estimate for the total error.

**Theorem 2** Assume that the marking parameter \( \vartheta \) verifies (6.10) and the initial mesh \( T_0 \) satisfies condition (b) of section 4 in [46]. Let \( (\sigma, u) \) be the solution of problem (2.1) and \( \{T_m, (\sigma_m, u_m, \lambda_m)\} \) be the sequence of meshes and discrete solutions produced by Algorithm 1. If \( (\sigma, f) \in A_s \) and \( C_\xi \leq \min\{C^*_\xi, C^*_\xi \} \), then there holds
\[
\|\sigma - \sigma_m \|_C^2 + \eta_1^2(\sigma_m, f, T_m) \lessapprox \#T_m - \#T_0 - 2s |\sigma, f|^{2s}_{s_m}.
\]

**Proof** Denote \( e_m^2 := \|\sigma - \sigma_m \|_C^2 + \beta_1 \eta_1^2(\sigma_m, f, T_m) + \beta_2 \eta^2(\sigma_m, f, T_m) \), then (6.6) can be rewritten as \( e_m \approx E^{-1/s}_{T_m}(\sigma_m) \). By (5.2) and (6.12), it holds
\[
\#T_m - \#T_0 \lessapprox \sum_{j=0}^{m-1} \#M_j \lessapprox |\sigma, f|^{1/s}_{s_m} \sum_{j=0}^{m-1} E^{-1/s}_{T_j}(\sigma_j). \]
Using (6.6) and (5.8), we have
\[ E_{T_j}^{-1/s}(\sigma_j) \lesssim e_j^{-1/s} \lesssim \alpha^{(m-j)/(2s)} e_m^{-1/s} \lesssim \alpha^{(m-j)/(2s)} E_{T_m}^{-1/s}(\sigma_m). \]

Therefore
\[ \#T_m - \#T_0 \lesssim |\sigma, f|_s^{1/s} E_{T_m}^{-1/s}(\sigma_m) \sum_{j=0}^{m-1} \alpha^{(m-j)/(2s)} \lesssim |\sigma, f|_s^{1/s} E_{T_m}^{-1/s}(\sigma_m). \]

The desired result then follows. \(\square\)

### 7 Numerical experiments

In this section we will numerically test the AHCDGM, i.e. Algorithm 1. Let \( \Omega \) be the L-shaped domain \((-1, 1) \times (-1, 1) \setminus [0, 1) \times (-1, 0] \). Take \( k = 1 \), the Poisson ratio \( \nu = 0.3 \), and the exact singular solution is chosen as
\[ u(r, \theta) = (r^2 \cos^2 \theta - 1)^2 (r^2 \sin^2 \theta - 1)^2 r^{(1+z)} g(\theta), \]
where \( z = 0.544483736782464 \) is a noncharacteristic root of \( \sin^2(z \omega) = z^2 \sin^2 \omega \), \( \omega = \frac{3\pi}{2} \) and
\[
\begin{align*}
g(\theta) &= \left( \frac{1}{z - 1} \sin((z - 1)\theta) - \frac{1}{z + 1} \sin((z + 1)\theta) \right) \\
& \quad \times (\cos((z - 1)\theta) - \cos((z + 1)\theta)) \\
& \quad - \left( \frac{1}{z - 1} \sin((z - 1)\theta) - \frac{1}{z + 1} \sin((z + 1)\theta) \right) \\
& \quad \times (\cos((z - 1)\theta) - \cos((z + 1)\theta)).
\end{align*}
\]
It is easy to check that $u \in H^s(\Omega)$ for any $s < 2 + z$. The initial mesh is given by Fig. 1.

First set $\xi_K = h_K$ for all $K \in T$. The histories of the error $\|\sigma - \sigma_T\|_C$ and the estimator $\eta(\sigma_T, f, T)$ versus the number of degrees of freedom (#dofs) in a ln–ln scale for the AHCDGM with $\vartheta = 0.3, 0.5$ are shown in Fig. 2. We can see from Fig. 2 that $\|\sigma - \sigma_T\|_C \approx \eta(\sigma_T, f, T) \approx (#\text{dofs})^{-0.5}$, which is optimal. Next set $\xi = 0$. For this case, the HCDG method (2.2a)–(2.2d) is just the hybridized HHJ method. The histories
of the error $\|\sigma - \sigma^T\|_C$ and the estimator $\eta(\sigma^T, f, T)$ for $\vartheta = 0.3, 0.5$ are presented in Fig. 3. Again it is observed from Fig. 3 that $\|\sigma - \sigma^T\|_C \approx \eta(\sigma^T, f, T) \approx (\text{#dofs})^{-0.5}$. The performances of the AHCDGM for $\xi_K = h_K$ and $\xi_K = 0$ are very similar. All these numerical results coincide with Theorem 2.

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