Electrostatic patch effect in cylindrical geometry: III. Torques

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Abstract
We continue to study the effect of uneven voltage distribution on two close cylindrical conductors with parallel axes started in our papers (Ferroni and Silbergleit 2011 Class. Quantum Grav. 28 145001; Ferroni and Silbergleit 2011 Class. Quantum Grav. 28 145002) now to find the electrostatic torques. We calculate the electrostatic potential and energy to lowest order in the gap to cylinder radius ratio for an arbitrary relative rotation of the cylinders about their symmetry axis. By energy conservation, the axial torque, independent of the uniform voltage difference, is found as a derivative of the energy in the rotation angle. We also derive both the axial and slanting torques by the surface integration method: the torque vector is the integral over the cylinder surface of the cross product of the electrostatic force on a surface element and its position vector. The slanting torque consists of two parts: one coming from the interaction between the patch and the uniform voltages, and the other due to the patch interaction. General properties of the torques are described. A convenient model of a localized patch suggested in Ferroni and Silbergleit (2011 Class. Quantum Grav. 28 145002) is used to calculate the torques explicitly in terms of elementary functions. Based on this, we analyze in detail patch interaction for one pair of patches, namely the torque dependence on the patch parameters (width and strength) and their mutual positions. Finally, we study the rotational motion of cylinders caused by the patch effect torque.

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1. Introduction
The actual distribution of charges on a metal does not guarantee its surface to be an equipotential because of the impurities and microcrystal structure of the material. This phenomenon, known

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as the patch effect (PE; see the pioneering paper [3] and [4–6] for its experimental study), is responsible for the mutual force and torque between two metallic surfaces at finite distances. The larger the effect, the closer the surfaces, as first confirmed by the calculation of the patch effect force for two parallel conducting planes [7].

PE in a cylindrical geometry was studied in the first two parts of our work [1] and [2] (henceforth referred to as CPEI and CPEII, for ‘cylindrical patch effect’). PE energy and force have been examined there; here, we calculate the torque due to PE between two coaxial cylinders. This calculation completes the study of the electrostatic interaction for a cylindrical capacitor; our analysis largely benefits from the results found in CPEI and II.

PE is important for any precision measurement if its setup includes conducting surfaces close to each other. One is the STEP experiment [8–11], where the differential axial motion of cylindrical test masses will be used to test the universality of free fall to an unprecedented accuracy of about one part in 10^{18}. The axial torque is of a particular interest in this case, since it leads to a rotational motion (studied in section 6), which makes the patch distribution on, say, the outer cylinder time dependent in the rest frame of the inner one; therefore, the PE forces, including the axial one, would also vary with time.

We determine PE torques between the two boundary surfaces of an infinitely long cylindrical capacitor. By the energy conservation, a rotation by some angle \( \gamma \) about the direction \( \hat{\gamma} \) of one of the conductors relative to the other causes an electrostatic torque in the same direction \( \hat{\gamma} \) given by the formula

\[
T_\gamma = -\frac{\partial W(\gamma)}{\partial \gamma};
\]  

(1)

here, \( W(\gamma) \) is the electrostatic energy as a function of the rotation angle. However, due to the specifics of cylindrical geometry, we can properly imagine a rotation only about the symmetry axis of the two infinite cylinders. Tilting, say, the inner one about any other direction leads to the intersection of the cylinders at some finite distance, i.e. to the breaking of the problem geometry. For this reason, we also employ a different method of the torque calculation. The force, \( d\vec{F} \), due to the electrical field \( \vec{E} \) acting on a small area \( dA \) of a conductor with the charge density \( \sigma \) is given by

\[
d\vec{F} = \sigma \vec{E} dA.
\]  

(2)

The resulting element of the torque about the origin at distance \( \vec{r} \) from \( dA \) is then

\[
d\vec{T} = \vec{r} \times d\vec{F},
\]  

(3)

whose expression, integrated all over the body surface, gives the general expression of the torque acting on the conductor.

The energy, the field and the surface charge density which are needed in formulas (1) and (3) are expressed through the electrostatic potential in the gap. For typical experimental conditions, such as in the STEP configuration [9], the gap, \( d = b - a \), is much smaller than either of the cylinder radii, \( a < b \). This, first, justifies the model of infinite cylinders, especially if the patches are predominantly far from the real cylinder edges and, second, it allows for a significant simplification of results to lowest order (l.o.) in \( d/a \).

In the next section, we solve the boundary value problem (BVP) for the potential in the gap with general voltage distributions on the cylinder surfaces. Based on this, we find the energy in section 3, and then the longitudinal PE torque by formula (1). In section 4, we derive all the three components of the PE torque by the surface integration method using formula (3)

\[4\] This formula offers a method of calculation of the patch effect force alternative to the energy method used in CPEI. We have checked that results of both in the case of unshifted cylinders do coincide.
Figure 1. Geometry of the problem and coordinate systems.

(the two expressions for the axial torque agree precisely). In section 5, a model of the localized patch potential introduced in CPEII is described, ending with closed-form expressions for the torques. The latter are then calculated and analyzed in the case when a single patch is present at each of the cylinders. In section 6, we study rotational motion of the cylinders under the action of the axial PE torque. The details of calculations are given in two appendices.

2. Electrostatic potential

We employ both Cartesian and cylindrical coordinates in two frames related to the inner and outer cylinders as shown in figure 1. In the outer frame, an arbitrary point is labeled by the vector radius $\vec{r}$, Cartesian coordinates $\{x, y, z\}$ or cylindrical coordinates $\{\rho, \varphi, z\}$; in the inner frame, the corresponding quantities are $\vec{r}$, $\{x, y, z\}, \{\rho, \varphi, z\}$. The origins of the frames coincide so that the primed and unprimed coordinates are related simply by a rotation about the $z$-axis ($\gamma$ is the rotation angle):

$$x' = x \cos \gamma + y \sin \gamma, \quad y' = -x \sin \gamma + y \cos \gamma, \quad z' = z;$$

the relations in cylindrical coordinates are $\rho' = \rho, \varphi' = \varphi - \gamma, z' = z$. The surfaces of the inner and outer cylinders are thus described by the equations $\rho = a$ and $\rho = b$, respectively, and are assumed to carry arbitrary distributions of electrostatic voltage. Hence, the electrostatic potential, $\Phi$, satisfying the Laplace equation in the gap between the cylinders,

$$\Delta \Phi = 0, \quad \rho > a, \quad \rho < b, \quad 0 \leq \varphi < 2\pi, \quad |z| < \infty,$$

also satisfies the boundary conditions of the first kind at the cylinder surfaces:

$$\Phi|_{\rho=a} = G(\varphi, z); \quad \Phi|_{\rho=b} = V^- + H(\varphi', z) = V^- + H(\varphi - \gamma, z).$$
Here $V^- = \text{const}$ is the uniform potential difference, so all voltages are counted from the uniform voltage of the inner cylinder taken as zero. The non-uniform potential distributions, i.e. the patch voltages, are described by arbitrary smooth enough functions $G(\varphi, z)$ and $H(\varphi, z)$. Same as in CPEII we assume these functions squarely integrable; wherever proper, we will also assume them more smooth, as done in conditions (10), (A11), (A12) and (C5) of CPEI.

For any squarely integrable function $u(\varphi, z)$ we have its Fourier expansion with the Fourier coefficient $u_n(k)$:

$$u(\varphi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} u_n(k) e^{i(kz+n\gamma)},$$

$$u_n(k) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} d\varphi dz u(\varphi, z) e^{-i(kz+n\gamma)}.$$  

For any two such functions $u(\varphi, z)$ and $v(\varphi, z)$, the useful Parseval identity holds:

$$(u, v) \equiv \int_0^{2\pi} \int_{-\infty}^{\infty} d\varphi dz u(\varphi, z)v^\ast(\varphi, z) = \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} u_n(k)v_n^\ast(k);$$

here and elsewhere the star denotes complex conjugation.

According to the boundary condition (6), we split the potential into two parts due to, respectively, the uniform boundary voltages and patches:

$$\Phi_1(\vec{r}) = \Phi_1^u(\vec{r}) + \Phi_1^p(\vec{r}),$$

$$\Phi_1^u|_{\rho=a} = 0, \quad \Phi_1^u|_{\rho=b} = V^-;$$

$$\Phi_1^p|_{\rho=a} = G(\varphi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} G_n(k) e^{i(kz+n\gamma)},$$

$$\Phi_1^p|_{\rho=b} = H(\varphi - \gamma, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} H_n(k) e^{-i\gamma n} e^{i(kz+n\gamma)}.$$  

The function $\Phi_1^u$ is the classical solution for a cylindrical capacitor; to l.o. in $d/a$ it is

$$\Phi_1^u(\vec{r}) = (a/d) V^- \ln (\rho/a).$$

The function $\Phi_1^p$ is obtained by the standard separation of variables in cylindrical coordinates [see cf [12], chapters 5 and 6]. Its representation satisfying formally the Laplace equation is

$$\Phi_1^p(\vec{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} [A_n(k)L_n(k\rho) + B_n(k)K_n(k\rho)] e^{i(kz+n\gamma)},$$

where $L_n(\xi)$, $K_n(\xi)$ are the modified Bessel functions of the first and second kind, respectively (the Macdonald’s function $K_n(\xi)$ definition for the negative values of its argument is taken by the parity of $L_n(\xi)$; so, $K_n(k\rho)$ stands for (sign $k)^n K_n(|k|\rho)$). The unknown $A_n(k)$, $B_n(k)$ are found from the linear system implied by the boundary conditions (11):

$$A_n(k) L_n(ka) + B_n(k) K_n(ka) = G_n(k);$$

$$A_n(k) L_n(kb) + B_n(k) K_n(kb) = H_n(k) e^{-i\gamma n}, \quad n = 0, \pm 1, \pm 2, \ldots.$$  

It is the same system that has been effectively solved, to l.o. in $d/a$, in CPEI (appendix A, coefficients $A_0^0(k)$ and $B_0^0(k)$), with the exception of $e^{-i\psi}$ on the rhs instead of $e^{ikz}$. So, by replacing $H_n(k)e^{-i(kz+\gamma)}$ with $H_n(k)e^{-i\gamma}$ in the answer (A13), CPEI, we obtain
\begin{align}
A_n(k) &= -\frac{a}{d}\{K_n(kb)[G_n(k) - H_n(k)e^{-i\gamma}]\}, \\
B_n(k) &= \frac{a}{d}\{I_n(ka)[G_n(k) - H_n(k)e^{-i\gamma}]\}.
\end{align}
Thus, the electrostatic potential (13), to l.o. in $d/a$, is
\begin{equation}
\Phi(\vec{r}) = -\frac{a}{d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} [G_n(k) - H_n(k)e^{-i\gamma}] \Omega_n(k\rho) e^{i(kz+n\phi)},
\end{equation}
\begin{equation}
\Omega_n(k\rho) = K_n(kb)I_n(k\rho) - I_n(ka)K_n(k\rho).
\end{equation}
Formulas (12), (16) and (17) allow us to calculate both the electric energy and field.

3. Axial torque by the energy method

The uniform potential (12) does not depend on $\gamma$, so the variation of the electrostatic energy due to the rotation comes only from the patch potential (16), same as it happens with the axial PE force (CPEII, section 3). The axial torque, $T_z$, is thus given by formula (1) where $W(\gamma)$ is replaced with $W^p(\gamma)$ that we calculate below.

3.1. Electrostatic energy

Denote $D_\infty$ as the infinite domain between the two cylinders of our capacitor. The patch energy stored there, finite due to the locality of patch distributions, is
\begin{align}
W^p &= \varepsilon_0 \int_{D_\infty} (\nabla \Phi^p)^2 dV \\
&= \varepsilon_0 \left\{ \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi H(\phi - \gamma, z) \left. \frac{\partial \Phi^p}{\partial \rho} \right|_{\rho=b} - \int_{-\infty}^{\infty} dz \int_0^{2\pi} a \, d\phi G(\phi, z) \left. \frac{\partial \Phi^p}{\partial \rho} \right|_{\rho=a} \right\}.
\end{align}
Here we used boundary conditions (11) and the fact that the potential is harmonic in the domain $D_\infty$, see section 3 in CPEI for details. The double integrals above are calculated via Fourier coefficients of the potential and its derivative in $\rho$ by the Parseval identity (8). The Fourier coefficients of the derivatives are found from formula (16); the calculation goes the same way as in CPEI, appendix C, and results in
\begin{equation}
\left. \frac{\partial \Phi^p}{\partial \rho} \right|_{\rho=a,b} = -\frac{1}{d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} [G_n(k) - H_n(k)e^{-i\gamma}] e^{i(kz+n\phi)}.
\end{equation}
To l.o. in $d/a$, this expression holds at both the inner and the outer boundary. Using the Fourier coefficients, $G_n(k)$ and $H_n(k)$, of the boundary functions, we write formula (18) as
\begin{equation}
W^p = \frac{\varepsilon_0 a}{2d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left| G_n(k) - H_n(k)e^{-i\gamma} \right|^2.
\end{equation}
The only part of this that depends on $\gamma$, and thus contributes to the axial torque, is
\begin{equation}
W^p(\gamma) = -\frac{\varepsilon_0 a}{d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \Re\{G_n(k)H_n^*(k)e^{i\gamma}\}.
\end{equation}
3.2. Axial torque

Using (21), we calculate the axial torque by formula (1):

\[ T_z = -\frac{\partial W_p(\gamma)}{\partial \gamma} = -\frac{\varepsilon_0 a}{d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \Im[G_n(k)H_n^*(k) e^{i\gamma n}]. \]  

(22)

This representation is valid, to l.o. in \( d/a \), for an arbitrary rotation \( \gamma \). The torque does not vanish only if patches are at both boundaries \( [G_n, H_n \neq 0] \). A non-zero torque is generally found for \( \gamma = 0 \), unless \( G_n(k) = \lambda H_n(k), \lambda \) real, i.e. the patch distributions at both cylinders are the same up to scaling. The expression of the axial torque perfectly matches that of the axial force in the symmetric configuration (CPEII, formula (21)); the only difference is \( \gamma \) instead of \( z_0 \), and the factor \( n \) in place of \( k \).

4. All torques by the surface integration method

4.1. General formulas for electrostatic torques

According to formula (3), the patch effect torque on the outer cylinder is

\[ \mathbf{T} = \epsilon_0 a \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \sigma(\mathbf{r} \times \mathbf{E}) \Bigg|_{\rho=b} \]

\[ = \epsilon_0 a \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left[ -z \frac{\partial \Phi}{\partial \rho} \cos \phi + \left( \rho \frac{\partial \Phi}{\partial z} - z \frac{\partial \Phi}{\partial \rho} \right) \sin \phi \right] \mathbf{\hat{x}} 
\]

\[ - \left[ z \frac{\partial \Phi}{\partial \phi} \sin \phi + \left( \rho \frac{\partial \Phi}{\partial z} - z \frac{\partial \Phi}{\partial \rho} \right) \cos \phi \right] \mathbf{\hat{y}} + \left( \frac{\partial \Phi}{\partial \rho} \right) \mathbf{\hat{z}} \right|_{\rho=b}. \]  

(23)

Here we expressed the electrical field through the potential, and the charge density as the product of \( \varepsilon_0 \) and the normal component of the field, by the Gauss law. We also set \( b = a + d \approx a \), so the above holds to l.o. in \( d/a \). The Cartesian components of the torque are found using the well-known expressions of cylindrical unit vectors through the Cartesian ones, resulting in

\[ \mathbf{T} = \epsilon_0 a \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left[ \left( -2z \frac{\partial \Phi}{\partial \rho} - z \frac{\partial \Phi}{\partial \rho} \cos \phi + \rho \frac{\partial \Phi}{\partial \phi} \cos \phi + \rho \frac{\partial \Phi}{\partial z} \sin \phi \right) \mathbf{\hat{x}} 
\]

\[ - \left( 2z \frac{\partial \Phi}{\partial \rho} \cos \phi - \rho \frac{\partial \Phi}{\partial \phi} \sin \phi - \rho \frac{\partial \Phi}{\partial z} \cos \phi \right) \mathbf{\hat{y}} \right|_{\rho=b}. \]  

(24)

Because of the bilinear structure of this expression, the splitting (9) of the potential in the sum of \( \Phi^u \) and \( \Phi^p \) implies formally three contributions to the torque: one from the uniform voltages only, the other due to the interaction of patches and uniform voltages, and the third one from the patches only, just like we had it for the force in CPEII. However, uniform voltages do not give any torque, so the first contribution vanishes. Likewise, the interaction between the patches and uniform potential difference gives zero axial torque: the interaction torque is perpendicular to the symmetry axis (slanting torque):

\[ T^\text{Int}_x = \epsilon_0 a \left. \frac{\partial \Phi^U}{\partial \rho} \right|_{\rho=b} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left( -2z \frac{\partial \Phi^p}{\partial \rho} - z \frac{\partial \Phi^p}{\partial \rho} \cos \phi + \rho \frac{\partial \Phi^p}{\partial \phi} \sin \phi \right) \right|_{\rho=b}; \]  

(25)

\[ T^\text{Int}_y = \epsilon_0 a \left. \frac{\partial \Phi^U}{\partial \rho} \right|_{\rho=b} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left( 2z \frac{\partial \Phi^p}{\partial \rho} \cos \phi - z \frac{\partial \Phi^p}{\partial \phi} \sin \phi - \rho \frac{\partial \Phi^p}{\partial z} \cos \phi \right) \right|_{\rho=b}. \]  

(26)
In contrast with that, the torque due to the patch interaction generally has all the components:

\[
T_p^x = \epsilon \alpha \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left\{ \frac{\partial \Phi_p}{\partial \rho} \left[ -z \frac{\partial \Phi_p}{\partial \rho} \cos \phi + \left( \frac{\rho}{\partial z} - z \frac{\partial \Phi_p}{\partial \rho} \right) \sin \phi \right] \right\} \bigg|_{\rho=b}; \quad (27)
\]

\[
T_p^y = -\epsilon \alpha \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left\{ \frac{\partial \Phi_p}{\partial \rho} \left[ z \frac{\partial \Phi_p}{\partial \rho} \sin \phi + \left( \frac{\rho}{\partial z} - z \frac{\partial \Phi_p}{\partial \rho} \right) \cos \phi \right] \right\} \bigg|_{\rho=b}; \quad (28)
\]

\[
T_p^z = \epsilon \alpha \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left\{ \frac{\partial \Phi_p}{\partial \rho} \frac{\partial \Phi_p}{\partial \rho} \right\} \bigg|_{\rho=b}. \quad (29)
\]

Formulas (25)–(29) provide the general representation for the torque based on the surface integration method. Below we use them along with expressions (12) and (16) of the potential in the gap, to find the PE torque on the cylinder.

4.2. Axial torque by the surface integration method

We compute the axial component (29) of the torque employing the Parseval identity (8):

\[
T_p^z = \epsilon \alpha d V^{-d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left[ -\frac{1}{2d} (G_n^*(k) - H_n^*(k)) e^{i\gamma y} \right. \int H_n(k) e^{-i\gamma y} \\
+ \frac{1}{2d} (G_n(k) - H_n(k)) e^{-i\gamma y} \int H_n^*(k) e^{i\gamma y} \right] \\
= -\epsilon \alpha d \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \Im \left[ n G_n(k) H_n^*(k) e^{i\gamma y} \right]. \quad (30)
\]

We used Fourier coefficients of the two derivatives of the potential, the radial one (19), and the angular one obtained by differentiating the second of boundary condition (11):

\[
\left. \frac{\partial \Phi_p}{\partial \phi} \right|_{\rho=b} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \Im \left[ n H_n(k) e^{-i\gamma y} \right]. \quad (31)
\]

The axial torques (30) and (22) by the surface integration and energy method, respectively, are exactly the same; this is an important cross-check of our calculations.

4.3. Slanting torques by the surface integration method

4.3.1. Uniform and patch potential interaction. We start with calculating the torque due to the interaction of uniform and patch potentials. We first substitute the uniform field, \( V^{-d}/d \), in formula (25):

\[
T^{Int}_x = \frac{\epsilon \alpha d}{d} V^{-d} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left\{ -2z \frac{\partial \Phi_p}{\partial z} \sin \phi - z \frac{\partial \Phi_p}{\partial \rho} \sin \phi + \frac{\rho}{\partial z} \frac{\partial \Phi_p}{\partial \rho} \sin \phi \right\} \bigg|_{\rho=b};
\]

to get the \( x \) component. The last term vanishes after integrating in \( z \), so

\[
T^{Int}_x = \frac{\epsilon \alpha d}{d} V^{-d} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \left\{ -2z \frac{\partial \Phi_p}{\partial \rho} \sin \phi - z \frac{\partial \Phi_p}{\partial \rho} \cos \phi \right\} \bigg|_{\rho=b}.
\]
By definition (7) of the Fourier transform, this double integral is equal to $2\pi$ times the Fourier coefficient of the integrand at $n = k = 0$. The needed Fourier coefficients are determined in appendix A, formulas (A.7) and (A.8); using them, we arrive at

$$T_{x}^{int} = -4\pi \frac{\epsilon_{0d}}{d^2} V^{-3} \left[ \frac{\partial}{\partial k} (G_1(k) - H_1(k) e^{-i\gamma}) \right] \bigg|_{k=0} \left[ 1 + O \left( \frac{d}{a} \right) \right]. \quad (32)$$

This final compact formula for the torque has been obtained by employing the property $G_n(-k) = G_n^*(k)$, $H_n(-k) = H_n^*(k)$ of Fourier coefficients of real functions. The estimate of the remainder in expression (32) holds for patch distributions $G(\varphi, z)$ and $H(\varphi, z)$ satisfying conditions (A 11), (A 12) and (C 5), CPEI, and also such that the products $zG(\varphi, z)$ and $zH(\varphi, z)$ are squarely integrable, see formula (A.10) in appendix A in this paper. In addition to this, the combination of derivatives in $k$ at $k = 0$ involved in formula (32) must be finite. The validity of these conditions is assumed everywhere below, including the final expressions of all the torques.

A similar calculation for the $y$ component starts from formula (26) and uses the Fourier coefficients (A.6) and (A.9). It results, to $1.0$ in $d/a$, in

$$T_{y}^{int} = 4\pi \frac{\epsilon_{0d}}{d^2} V^{-3} \left[ \frac{\partial}{\partial k} (G_1(k) - H_1(k) e^{-i\gamma}) \right] \bigg|_{k=0}, \quad (33)$$

with the same remainder as in formula (32).

### 4.3.2. Patch potentials interaction

Now we go for the expressions of the torque caused by the interaction between patches. The $x$ component of this torque, $T_{p}^{x}$, given by formula (27), with the help of the Parseval identity becomes (the proper Fourier coefficients are found in (19), (A.7), (A.8), and (A.12)):

$$T_{p}^{x} = -\frac{\epsilon_{0d}}{2d^2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left[ G_n^*(k) - H_n^*(k) e^{i\gamma} \right]$$

$$\times \frac{\partial}{\partial k} \left[ (G_{n-1}(k) - H_{n-1}(k) e^{-i(n-1)\gamma}) - (G_{n+1}(k) - H_{n+1}(k) e^{-i(n+1)\gamma}) \right] \left[ 1 + O \left( \frac{d}{a} \right) \right].$$

This formula can be simplified further: integrating by part in the first of the two products and shifting there the index $n$ by 1, $n' = n - 1$, leads to the final more compact representation:

$$T_{p}^{x} = \frac{\epsilon_{0d}}{d^2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \Im \left\{ G_n^*(k) - H_n^*(k) e^{i\gamma} \right\} \frac{\partial}{\partial k} \left[ G_{n+1}(k) - H_{n+1}(k) e^{-i(n+1)\gamma} \right]. \quad (34)$$

The other component, $T_{p}^{y}$, can be determined in a similar way starting with expression (28) and combining it with formulas (19), (A.6), (A.9), and (A.11). The result is

$$T_{p}^{y} = -\frac{\epsilon_{0d}}{d^2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \Im \left\{ G_n^*(k) - H_n^*(k) e^{-i\gamma} \right\} \frac{\partial}{\partial k} \left[ G_{n+1}(k) - H_{n+1}(k) e^{-i(n+1)\gamma} \right]. \quad (35)$$

entirely similar to (34). The general analysis of PE torques is completed.
4.4. General properties of patch effect torques

Looking at the results of the calculation of the PE torques, one can come up with a few general conclusions regarding their properties, such as:

(1) uniformly charged cylinders do not give rise to any torque;
(2) the axial torque is inversely proportional to the gap width, and the transverse components are inversely proportional to its square;
(3) patches need to be present on both the cylinders to generate an axial torque;
(4) a non-zero axial torque is generally found when the cylinders are not rotated against each other ($\gamma = 0$), unless the patch voltage distributions on both cylinders are the same up to a factor;
(5) just one patch is enough to generate a slanting torque;
(6) a non-zero slanting torque is generally found when $\gamma = 0$ unless the patch voltage distributions on both cylinders are the same;
(7) the interaction between patches and uniform potentials involves only the first harmonics of the azimuthal angle of the patch distribution.

Some of these conclusions are intuitively clear, like no 1 and no 3 due to symmetry, etc. Nevertheless, all of them are accurately derived here.

5. Single patch at each of the electrodes: a picture of patch interaction

To study the features of PE torques, we analyze them in the case when only one localized patch is found at each of the cylinders, as it was done with the PE force in CPEII. For meaningful and transparent results one naturally needs the torques in a simple enough closed form, which requires some special choice of the generic patch model, a rather delicate task. The localized potential distribution that satisfies this very well has been suggested and developed in CPEII, section 4. We repeat the basic facts about this model and use it to calculate the torques and examine the patch interaction.

5.1. The patch model

The suggested model of the patch potential is

$$V(\varphi - \varphi_*, z - z_*) \equiv V(\varphi - \varphi_*, \lambda, z - z_*, \Delta z) = V_* f(z - z_*) u(\varphi - \varphi_*),$$  \hspace{1cm} (36)

where

$$f(z) = \exp \left[ -\left( \frac{z}{\sqrt{2} \Delta z} \right)^2 \right], \quad u(\varphi) = u(\varphi, \lambda) = \frac{(1 - \lambda)^2}{2 - 2\lambda \cos \varphi + \lambda^2};$$  \hspace{1cm} (37)

(note $|f(z)| \leq 1$, $|u(\varphi)| \leq 1$, and $f(0) = u(0) = 1$). Here $V_* = V(0, 0)$ is the maximum magnitude of the potential (positive or negative); the center of the patch is at $\varphi = \varphi_*$, $z = z_*$, $\Delta z$ denotes the axial size of the patch, and $\lambda$ controls its angular size. Indeed, $\lambda$ is related to $\Delta \varphi$, the angular half-width of the patch (defined as the angle for which $u$ is equal to its mean value, $u(\Delta \varphi) = u_{av}$), by the equalities

$$\cos \Delta \varphi = \lambda, \quad \Delta \varphi = \arccos \lambda.$$  \hspace{1cm} (38)

Fourier coefficients of the patch model (36), (37) are

$$V_n(k) \equiv V_n(k, \lambda, \Delta z) = V_* \tilde{f}(k) e^{-ikz_*} u_n e^{-i\varphi_*}, \quad \tilde{f}(k) = \Delta z \exp \left[ -\left( \frac{k \Delta z}{\sqrt{2}} \right)^2 \right];$$  \hspace{1cm} (39)
Figure 2. Axial torque versus the angular distance between the patches for $\Delta \phi = \pi/8, \pi/4, \pi/2$.

For a single patch at each of the two cylinders, the boundary functions $G(\phi, z)$ and $H(\phi - \gamma, z)$ are given by formula (36) as

$$G(\phi, z) = V(\phi - \phi_1, \lambda_1, z - z_1, \Delta z_1); \quad H(\phi - \gamma, z) = V(\phi - \phi_2 - \gamma, \lambda_2, z - z_2, \Delta z_2).$$

The torques corresponding to these distributions are calculated in appendix B in a closed form. Here we study the patch interaction for a particular case when the sizes of the patches are identical, $\Delta z_1 = \Delta z_2 = \Delta z, \Delta \phi_1 = \Delta \phi_2 = \Delta \phi$, and $V_1 = \pm V_2 = V_0$.

5.2. Axial torque

As shown in our general analysis in sections 3 and 4, the axial torque occurs only when both cylinders carry non-uniform voltages. For the case of two patches of equal sizes and magnitude, its expression is found by formula (B.15) (recall that we have set $\gamma = 0$):

$$T_p^z = \mp \frac{\sqrt{\pi}^3}{4} \frac{a}{d} \varepsilon_0 V_0^2 \Delta z \sin^6 (\Delta \phi) \mu \sin (\phi_1 - \phi_2) \exp \left[ - \left( \frac{z_1 - z_2}{2 \Delta z} \right)^2 \right] \left( 1 + \frac{\lambda^2}{(1 - 2\lambda^2 \cos (\phi_1 - \phi_2) + \lambda^4)^2} \right).$$

The signs $\mp$ in formula (42) stand for patches of equal or opposite potential, respectively. This torque is proportional to the inverse of the relative gap, $d/a$. Its dependence on the axial patch distance is driven by the Gaussian exponent: the axial torque monotonically and rapidly decreases toward zero with increasing values of $|z_1 - z_2|$. The dependence on the angular patch distance, $0 \leq \phi_1 - \phi_2 \leq \pi$, is shown in figure 2 for various values of $\Delta \phi$: the torque goes down when the patch angular width gets smaller.
The axial and azimuthal sizes of the patch play a very distinctive role in expression \((42)\). As \(\Delta z\) grows, the torque magnitude increases and goes linearly to infinity when \(\Delta z \to \infty\). However, this (unphysical) limit is forbidden by the conditions imposed in section 4.3.1: the boundary voltages \((36), (37)\) become independent of \(z\), so their Fourier coefficients \((39)\) behave like \(\delta(k)\) in this limit, forming a strong singularity at \(k = 0\). The same is true for the slanting torques studied below.

In the opposite limit \(\Delta z \to 0\), the torque drops to zero very quickly, due to the negative exponent of \(1/(\Delta z)^2\), unless the patches are in the same cross-section, \(z_1 = z_2\). On the other hand, the influence of the azimuthal patch width is represented predominantly by the sixth power of the sine of \(\Delta \phi\): the torque vanishes for belt-like patches, and goes to zero as \((\Delta \phi)^6\) for \(\Delta \phi \to 0\).

5.3. Slanting torques

5.3.1. Uniform and patch potential interaction. In our case of identical patches expressions \((B.3)\) for the interaction torque simplify to

\[
\begin{align*}
T_x^{\text{Int}} &= -\sqrt{2\pi^3} \frac{a \Delta z}{d^2} e_0 V_0 V^- \sin^2(\Delta \phi)(z_1 \sin \varphi_1 \mp z_2 \sin \varphi_2); \\
T_y^{\text{Int}} &= \sqrt{2\pi^3} \frac{a \Delta z}{d^2} e_0 V_0 V^- \sin^2(\Delta \phi)(z_1 \cos \varphi_1 \mp z_2 \cos \varphi_2).
\end{align*}
\]

Recall that \(V^-\) is the difference between the uniform potentials at the boundaries, see formula \((10)\). The minus or plus sign above is taken when the patches have the same or the opposite voltages, respectively. The torque is inversely proportional to the square of the gap, but the proper dimensionless parameter is not \((d/a)^2\), it is rather \((d^2/a \Delta z)\): the gap relates one time to the cylinder radius, and the other to the axial patch width.

Expressions \((43)\) show that this torque is a superposition of two contributions each coming from a single patch interacting with the uniform voltage \(V^-\). Each of these torques can be expressed as the product of a force acting at the center of the patch and the respective arm, \(z_1\)
or $z_2$. It is interesting that these forces are equal to the zero order interaction forces obtained in CPEII, section 5. For $0 \leq \Delta \varphi \leq \pi/2$ the torque grows with the azimuthal width of the patch; then, for bigger angular sizes, it decreases and goes to zero as the patch becomes annular. Moreover, for $\Delta \varphi \to 0$ it goes to zero as $(\Delta \varphi)^2$.

5.3.2. Patch–patch interaction. The slanting patch torque expressions (B.9) and (B.11) become, in our case of identical patches [see also (B.7)]:

\[ T_{p x} = -2 \sqrt{\pi} a/2 \Delta z d \epsilon_0 V_0^2 \sin^2 (\Delta \varphi) \times [N(z_1 \sin \varphi_1 + z_2 \sin \varphi_2) \mp (z_1 + z_2) M \sin (\varphi_1 + \varphi_2) \exp(-\tilde{z}^2)]; \]
\[ T_{p y} = 2 \sqrt{\pi} a/2 \Delta z d \epsilon_0 V_0^2 \sin^2 (\Delta \varphi) \times [N(z_1 \cos \varphi_1 + z_2 \cos \varphi_2) \mp (z_1 + z_2) M \cos (\varphi_1 + \varphi_2) \exp(-\tilde{z}^2)]; \]
\[ \tilde{z} = \frac{z_1 - z_2}{2 \Delta z}; \quad N = \frac{2 - \lambda}{8}; \quad M = \frac{1 - \lambda}{2} \left[ 1 - \frac{\lambda (1 + \lambda)}{2} \left( \frac{1}{1 - 2 \lambda^2 \cos (\varphi_1 - \varphi_2) + \lambda^2} \right) \right]. \]

The dependence of the torque on the gap is the same as for the uniform voltage–patch interaction.

The first term in the square brackets carries the contributions of each single patch independent of their signs. Its structure is similar to the previous case (43): the torque is the sum of the products of a force acting on each patch and the arm $z_1$ or $z_2$. The second term includes interaction between the patches. It decreases as the distance between them grows, due to the presence of the coefficient $M$ and the Gaussian exponent (see figure 3). The faster the decay, the smaller the widths $\Delta z$ and $\Delta \varphi$. The plus or minus sign of this term is taken depending on whether the patch potentials have the same or opposite signs. The slanting patch torque is essentially proportional to the square sine of $\Delta \varphi$, and to $\Delta z$.

6. Rotational motion caused by the patch effect torque

In this section we investigate the rotation of cylinders about their symmetry axis under the action of the axial PE torque found above. Small oscillations due to slanting torques are not very interesting because they are suppressed in a typical experimental setup, as it is particularly done in the STEP experiment [8, 9]. Also, the spin, i.e. relative rotation of the cylinders about their common axis, makes the patch distributions, and hence the patch forces found in CPEII depending on time, which affects the translational motion as well. We start with the simplest case of one patch on each of the cylinders studied in detail in the previous section and, basing on this, describe then the features of rotational PE motion for a general patch distribution.

6.1. Spin motion in the case of two patches

We use the torque expression (42) to write the motion equation as

\[ I \dot{\gamma} = T_z = \mp T_\ast (\tilde{z}) \frac{\Delta z}{d} \sin^6 (\Delta \varphi) \mu(\gamma, \lambda) \sin \gamma \]

for the rotation of the outer cylinder about the symmetry axis. Here $I$ is the proper moment of inertia, the characteristic value of the torque is

\[ T_\ast (\tilde{z}) = \frac{\pi^{3/2}}{4} \epsilon_0 V_0^2 a e^{-\tilde{z}^2}, \quad \tilde{z} = \frac{z_1 - z_2}{2 \Delta z}. \]
and the dimensionless function $\mu$ is
\[
\mu = \mu(\gamma, \lambda) = \frac{1 + \lambda^2}{(1 - 2\lambda^2 \cos \gamma + \lambda^4)^2} > 0.
\] (47)

Without any loss of generality, we count the rotation angle $\gamma$ from the position where two patches are right one against the other, $\varphi_1 - \varphi_2 = 0$. The torque has the minus (plus) sign when the patch voltages have the same (opposite) sign.

6.1.1. General nonlinear picture of motion. Equation (45) strongly resembles the classical motion equation of the pendulum, with just one additional coefficient $\mu(\gamma, \lambda)$ being a strictly positive non-singular periodic function of $\gamma$. Extending this similarity, we note that, by expression (22) for the axial torque through the patch energy $W_p(\gamma)$, equation (45) has the potential $W_p(\gamma)/I$: it can be equivalently written as
\[
\mathcal{T} \ddot{\gamma} = -\frac{\partial W_p(\gamma)}{\partial \gamma}.
\] (48)

Since the potential is periodic in $\gamma$, the complete qualitative picture of motion is well known (see, for instance, [13], chapter 1). It follows from the energy integral of equation (48) that
\[
\frac{\mathcal{T} \dot{\gamma}^2}{2} + W_p(\gamma) = E_0,
\] (49)
where $E_0$ is the total energy determined by the initial conditions:
\[
E_0 = \frac{\mathcal{T} \dot{\gamma}_0^2}{2} + W_p(\gamma_0), \quad \gamma_0 = \gamma(t_0), \quad \dot{\gamma}_0 = \dot{\gamma}(t_0).
\]

Potential energy $W_p(\gamma)$ is bounded, with the bounds denoted as
\[
-\infty < W_0 = \min_{0 \leq \gamma < 2\pi} W_p(\gamma) < W_\ast = \max_{0 \leq \gamma < 2\pi} W_p(\gamma) < \infty.
\] (50)

The minimum potential energy $W_0 = W_p(\gamma_0)$ corresponds, of course, to a stable equilibrium $\gamma(t) \equiv \gamma_0 = \text{const}$, while the maximum one, $W_\ast = W_p(\gamma_\ast)$, is achieved at an unstable equilibrium point, $\gamma(t) \equiv \gamma_\ast = \text{const}$. This is enough to qualitatively describe the motion.

Indeed, from the energy conservation (49) it is clear that $E_0 \geq W_0$. If $E_0 = W_0$, then the system stays at the stable equilibrium, $\gamma(t) \equiv \gamma_0 = \text{const}$, $\dot{\gamma}(t) \equiv 0$, $t \geq t_0$. If $W_\ast < E_0 < W_\ast$, then the system can never reach the peak of the potential, the rotation angle is bounded at all times, $\gamma_\ast \leq \gamma(t) < \gamma_{\max}$, and the motion is finite, which means that the cylinder oscillates about the stable equilibrium. If, next, $E_0 > W_\ast$, then the system always remains above all the potential wells, the motion is infinite, and the cylinder rotates indefinitely and non-uniformly in one direction depending on the sign of the initial velocity. What remains is the exceptional case $E_0 = W_\ast$, when, in purely mathematical view, the system stays at the equilibrium $\gamma_\ast$; however, this rest point is unstable, so in reality any small perturbation in this or that direction leads, again, either to the oscillational or to the rotational motion.

To make all this even more particular in our case of just two patches, we give the patch energy explicitly, as easily found either by the general expression (21) or, up to an insignificant constant, by the direct integration of the torque on the rhs of equation (45):
\[
W_p(\gamma) = \mp \mathcal{T}_z(\zeta) \frac{\Delta z}{d} \sin^2 \left(\frac{\Delta \varphi}{2}\right) w(\gamma) + \text{const}, \quad w(\gamma) = \frac{(1 + \lambda)^2}{2\lambda^2} \frac{1}{1 - 2\lambda^2 \cos \gamma + \lambda^4}.
\] (51)

There is no essential difference between the two cases with the opposite signs, so we discuss only the case of the minus sign below.
As seen yet from equation (45), in this case we have just two equilibria at each period of the potential, the stable one, $\gamma_- = 0 \pmod{2\pi}$, and the unstable $\gamma_+ = \pi \pmod{2\pi}$. This is also clear from the plot of the patch energy given in figure 4; the horizontal line through the peaks shows the critical energy, $W_\ast$, that separates the finite motions (oscillations) from the infinite ones (rotations). The larger the period of oscillations, the higher the energy $E_0$, i.e. the larger the oscillation amplitude (the limit of small oscillations, when the period is independent of the amplitude, is described below).

The energy integral (49) allows also for the representation of motion in the phase plane $\gamma$, $\dot{\gamma}$. The corresponding plot is given in figure 5; closed orbits in it correspond to oscillatory (finite) motions (the size of these ovals grows with $E_0$). They are separated from the infinite trajectories (rotations) by the so-called heteroclinic curves, which go from one unstable equilibrium to another nearest to it. The total time of motion along these separatrices from one rest point to the other is infinite.

Note that under the STEP experimental conditions, one expects only oscillatory spinning of the cylinders rather than their rotation. The reason is that the cylinders, the outer test mass and its inner magnetic bearing, will be caged (fixed) during the satellite launch; then the test mass will be rather accurately released, practically with no initial rotational velocity, which leads to pure oscillations. This should be true for other experiments as well.

**6.1.2. Small motions near equilibria.** Here we describe small motions of the cylinder near its rest points. Accordingly, we linearize equation (45) by setting

$$\gamma(t) = \gamma_\pm + \delta \gamma(t), \quad |\delta \gamma(t)| \ll 1,$$

and working to l.o. in $\delta \gamma(t)$, which results in

$$\ddot{\delta \gamma} = \mp \omega_\pm^2 \delta \gamma, \quad \omega_\pm^2 = \frac{T_e(z)}{I} \frac{\Delta z}{d} \sin \phi \left( \frac{1 + \lambda z^2}{1 + \lambda z^2} \right).$$

(52)
The cylinder thus oscillates about the stable equilibrium position $\gamma = 0$ with the frequency
\[ f_{pe} = \frac{2\pi}{\omega} = f_0 e^{-\xi^2/2} \left( \sqrt{\frac{\Delta z}{d}} \right) \left( \frac{\sqrt{1 + \cos^2 \Delta \varphi}}{\sin \Delta \varphi} \right) \leq f_0 \left( \sqrt{\frac{\Delta z}{d}} \right) \left( \frac{\sqrt{1 + \cos^2 \Delta \varphi}}{\sin \Delta \varphi} \right); \]
\[ f_0 \equiv \frac{V_0}{4\pi^{1/4} \sqrt{\frac{\epsilon_0 a}{T}}}. \]

The basic frequency $f_0$ is proportional to the patch voltage and goes down with the cylinder mass and radius squared as the inverse square root of their product. The oscillation frequency $f_{pe}$ drops sharply as a Gaussian exponent with the axial distance between patches if their axial width is fixed. It and its upper bound achieved when both patches are in the same cylinder cross-section ($z_1 = z_2$) contain additional dependence on the patch axial width, $\sim (\sqrt{\Delta z}/d)$; $f_{pe}$ diverges for small angular width, $\sim (1/\Delta \varphi)$ when $\Delta \varphi \to +0$. Both the frequency and its upper limit decrease with $\Delta \varphi$ growing from zero to ninety degrees, when the patches become semi-annular, and then increase until the patches circle the full cylinder ($\Delta \varphi = \pi$).

By the same token, the cylinder rotates exponentially away from the unstable position $\gamma_+ = \pi$ with the characteristic time
\[ \tau_{pe} = \frac{1}{\omega_s} = \frac{e^{\xi^2/2}}{2\pi f_0} \left( \frac{\sqrt{d}}{\Delta z} \right) \frac{(1 + \cos^2 \Delta \varphi)^{3/2}}{\sin^3 \Delta \varphi} \geq \frac{1}{2\pi f_0} \left( \frac{\sqrt{d}}{\Delta z} \right) \frac{(1 + \cos^2 \Delta \varphi)^{3/2}}{\sin^3 \Delta \varphi}. \]

The dependences of $\tau_{pe}$ and its lower bound on all the parameters are, evidently, inverse of those in the case of $f_{pe}$, except the divergence in the small angular width, which is stronger here: $\tau_{pe} \sim 1/(\Delta \varphi)^3$ for $\Delta \varphi \to +0$.

6.2. Spin motion for a general patch distribution

The above picture of the spin motion due to the patch effect torque in the case of two patches is valid, in fact, for any boundary voltage distributions as well. Indeed, the electrostatic
patch energy is always bounded and a $2\pi$-periodic function of the rotation angle $\gamma$, hence equations (48)–(50), always hold, along with the argument about the motion based on them. Thus, in the most general case the cylinder either rotates all the way in the same direction, or oscillates about a stable rest point, depending on whether the total energy is above or below its critical value, i.e. the global maximum of potential energy. The only significant difference is that for a general patch distribution the number of equilibria can be larger than 2 (but always even, with the equal number of stable and unstable rest points).

The increase of the number of equilibrium positions leads to a general trend of decreasing the amplitude of a typical oscillatory motion. The remark at the end of section 6.1.1 is also valid in the general case: rotational regimes are not anticipated in the STEP and similar set-ups at all. The STEP conditions are such that any friction or other dissipation should be extremely low, so the typical spin oscillations of a test mass should be damped but very weakly. (One should note, however, that if some resistive energy losses are present, as it appears to have happened with GP-B [14], then this picture might change significantly). These oscillations will make the axial force found in CPEII change periodically with the time. If the basic frequency or some of its harmonics is close to the frequency of the STEP science signal, then it might introduce a systematic error in the experiment, provided that the PE force and the acceleration due to it is large enough. For this and other reasons, careful pre- and post-mission calibrations on orbit are recommended, along with the extensive PE simulations before the STEP flight as described in CPEII, section 6.

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Appendix A. Calculation of the slanting torques

We provide here intermediate results needed for finding the slanting torques. Particular terms that appear under the integrals in formulas (25)–(28) are

\[
\begin{align*}
\frac{z\partial\Phi^p}{\partial\rho} \cos \varphi, & \quad \frac{z\partial\Phi^p}{\partial\rho} \sin \varphi, & \quad \frac{z\partial\Phi^p}{\partial\varphi} \cos \varphi, & \quad \frac{z\partial\Phi^p}{\partial\varphi} \sin \varphi, \\
\frac{z\partial\Phi^p}{\partial\varphi} \sin \varphi, & \quad \frac{\partial\Phi^p}{\partial z} \cos \varphi, & \quad \frac{\partial\Phi^p}{\partial z} \sin \varphi;
\end{align*}
\]

we need to determine the Fourier coefficients of these functions evaluated at the boundary $\rho = b$. The radial and the angular derivatives of the potential involved here are

\[
\begin{align*}
\frac{\partial\Phi^p}{\partial\rho} \bigg|_{\rho = b} &= -\frac{1}{a} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz + n\varphi)} [G_n(k) - H_n(k) e^{-i\gamma}]; \quad (A.2) \\
\frac{\partial\Phi^p}{\partial\varphi} \bigg|_{\rho = b} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz + n\varphi)} [i n H_n(k) e^{-i\gamma}]; \quad (A.3)
\end{align*}
\]

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they come from formulas (19) and (31), respectively. The \( z \) derivative is computed from the second of the boundary conditions (11):

\[
\frac{\partial \Phi^p}{\partial z} \bigg|_{\rho=b} = \frac{i}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz+\gamma \rho)} [k H_n(k) e^{-i\gamma}] .
\] (A.4)

For any function \( u(\varphi, z) \) such that \( z u(\varphi, z) \) is squarely integrable, by the definition of the Fourier transform (7), the following equalities hold:

\[
u(\varphi, z) \cos \varphi = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz+\gamma \rho)} [u_{n-1}(k) + u_{n+1}(k)];
\] (A.5)

\[
u(\varphi, z) \sin \varphi = -\frac{i}{4\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz+\gamma \rho)} [u_{n-1}(k) - u_{n+1}(k)].
\]

Using formulas (A.5) and (A.2) with the radial derivative playing the role of \( u(\varphi, z) \), the first two functions (A.1) are represented in the following way:

\[
\frac{z}{\partial \rho} \cos \varphi = \frac{i}{4\pi d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz+\gamma \rho)} \frac{\partial}{\partial k} [(G_{n-1}(k) - H_{n-1}(k) e^{-i(n-1)\gamma}) + (G_{n+1}(k) - H_{n+1}(k) e^{-i(n+1)\gamma})];
\] (A.6)

\[
\frac{z}{\partial \rho} \sin \varphi = -\frac{1}{4\pi d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz+\gamma \rho)} \frac{\partial}{\partial k} [(G_{n-1}(k) - H_{n-1}(k) e^{-i(n-1)\gamma}) - (G_{n+1}(k) - H_{n+1}(k) e^{-i(n+1)\gamma})].
\] (A.7)

By the same token, using formula (A.3) instead of (A.2), for the two terms proportional to the potential derivative in \( \varphi \), we obtain

\[
\frac{z}{\partial \varphi} \cos \varphi = \frac{i}{4\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz+\gamma \rho)} \frac{\partial}{\partial k} [(n-1)H_{n-1}(k) e^{-i(n-1)\gamma}) + (n+1)H_{n+1}(k) e^{-i(n+1)\gamma})];
\] (A.8)

\[
\frac{z}{\partial \varphi} \sin \varphi = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} e^{i(kz+\gamma \rho)} \frac{\partial}{\partial k} [(n-1)H_{n-1}(k) e^{-i(n-1)\gamma}) - (n+1)H_{n+1}(k) e^{-i(n+1)\gamma})].
\] (A.9)

Note that expressions (A.6)–(A.9) here are valid under the additional conditions

\[
\int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left| \frac{\partial}{\partial k} G_n(k) \right|^2 < \infty, \quad \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left| \frac{\partial}{\partial k} H_n(k) \right|^2 < \infty,
\]

which are equivalent to

\[
\int_{-\infty}^{\infty} dz \int_{0}^{2\pi} d\varphi |zG(\varphi, z)|^2 < \infty; \quad \int_{-\infty}^{\infty} dz \int_{0}^{2\pi} d\varphi |zH(\varphi, z)|^2 < \infty.
\] (A.10)
To determine the remaining two terms in (A.1) containing the derivative of the potential with respect to \( z \), we just need the last two formulas from (A.5) and the expression (A.4) in place of \( u(\varphi, z) \). This results in

\[
\frac{\partial \Phi^\mu}{\partial z} \cos \varphi = -\frac{i}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}k \sum_{n=-\infty}^{\infty} e^{ikz_j + in\varphi_j} k [H_{n-1}(k) e^{-i(n-1)\gamma} + H_{n+1}(k) e^{-i(n+1)\gamma}];
\]

(A.11)

\[
\frac{\partial \Phi^\mu}{\partial z} \sin \varphi = \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}k \sum_{n=-\infty}^{\infty} e^{ikz_j + in\varphi_j} k [H_{n-1}(k) e^{-i(n-1)\gamma} - H_{n+1}(k) e^{-i(n+1)\gamma}].
\]

(A.12)

Formulas (A.6)–(A.12), valid under conditions (A.10), provide all the expressions needed for calculating the integrals in formulas (25)–(28) for the slanting torque.

**Appendix B. Calculation of the torque for a single patch at each of the cylinders**

To get the torque we first need to find the Fourier coefficients of the boundary distributions \( G(\varphi, z) \) and \( H(\varphi - \gamma, z) \) from formulas (41) and their derivative with respect to \( k \). The former can be represented by the Fourier coefficients (39), (40) as

\[
G_n(k) = V_n(k, \lambda_1, \Delta z_1); \quad H_n(k) = V_n(k, \lambda_2, \Delta z_2).
\]

The latter are the derivatives in \( k \) of these expressions, and they are given by the general formula:

\[
\frac{\partial V_i^j(k)}{\partial k} = -V_j \Delta z_j (k(\Delta z_j)^2 + iz_j) \exp \left[ -\left( \frac{k \Delta z_j}{\sqrt{2}} \right)^2 \frac{2}{\lambda} \right] u_n(\lambda_j) e^{-i(kz_j + n\varphi_j)}, \quad j = 1, 2,
\]

(B.2)

with \( V_i^j(k) = G_n(k) \) and \( V_i^j(k) = H_n(k) \).

The torque due to the interaction between the patches and uniform potential difference is a linear function of the derivatives in \( k \) of \( G_n(k) \) and \( H_n(k) \) calculated at \( k = 0 \) and \( n = 1 \). These are determined by formulas (B.2) as

\[
\frac{\partial G_1(k)}{\partial k} \bigg|_{k=0} = -\sqrt{2\pi} V_1 \Delta z_1 (iz_1) \frac{1 - \lambda_1^2}{4} e^{-i\varphi_1};
\]

\[
\frac{\partial H_1(k)}{\partial k} \bigg|_{k=0} = -\sqrt{2\pi} V_2 \Delta z_2 (iz_2) \frac{1 - \lambda_2^2}{4} e^{-i\varphi_2},
\]

where we used equality (40) for the coefficient \( u_n \). In order to calculate the torque we need just to substitute the above expressions in formulas (32) and (33). This leads to the following result:

\[
T_{1}^{\text{Int}} = -\sqrt{2\pi} \frac{\epsilon_{104}^d}{d^2} V^{-1} [V_1 \Delta z_1 (\sin \Delta \varphi_1)^2 z_1 \sin \varphi_1 - V_2 \Delta z_2 (\sin \Delta \varphi_2)^2 z_2 \sin (\varphi_2 + \gamma)];
\]

\[
T_{2}^{\text{Int}} = \sqrt{2\pi} \frac{\epsilon_{104}^d}{d^2} V^{-1} [V_1 \Delta z_1 (\sin \Delta \varphi_1)^2 z_1 \cos \varphi_1 - V_2 \Delta z_2 (\sin \Delta \varphi_2)^2 z_2 \cos (\varphi_2 + \gamma)].
\]

(B.3)

Note that we replaced \( \lambda \) with the more meaningful parameter \( \Delta \varphi \) according to the first of relations (38).

The expressions for the slanting torque due to the interaction between the patches are more cumbersome to find, since one needs to calculate the sum over \( n \) and the integral over \( k \).
of the product of the Fourier coefficients of the boundary distributions and their derivatives in \( k \). For the \( x \) component, we combine formula (34) with (B.2) for the derivatives and (B.1) for the boundary functions, to obtain the expression

\[
T^p_x = 2\pi \frac{\partial^2}{\partial^2 z^2} \int_{-\infty}^{\infty} dk \text{Re} \left\{ -V^2_1 \Delta z_1^2 N(\lambda_1) \left[ k \Delta z_1^2 + i z_1 \right] \exp \left[ -k^2 \Delta z_1^2 \right] \right\}
\]

\[
\quad + \sum_{n=-\infty}^{\infty} \frac{\partial}{\partial \psi} \left[ 2 \sqrt{2\pi} \right] \exp \left[ -\frac{(z_1 - z_2)^2}{2 \left( \Delta z_1^2 + \Delta z_2^2 \right)} \right] \exp \left[ -k \left( \Delta z_1^2 + \Delta z_2^2 \right) \right] k e^{i k (z_1 - z_2)}
\]

\[
\text{for the matter of space we set, without any loss of generality, } \gamma = 0 \text{ and denoted (compare with CPEH, appendix A)}
\]

\[
M_1 = M_1(\lambda_1, \psi_1, \lambda_2, \psi_2) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{\lambda_1, \lambda_2} u_n(\lambda_1) e^{i \psi_1} u_{n+1}(\lambda_2) e^{-i \psi_2}; \quad (B.5)
\]

\[
N(\lambda) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} u_n(\lambda) u_{n+1}(\lambda) = M_1(0, \lambda; 0, \lambda); \quad (B.6)
\]

The values of these coefficients are obtained by summing up geometrical progressions (see coefficient \( u_n \) in formula (40)):}

\[
M_1 = \frac{1 - \lambda_1}{8} \left[ e^{-i(\psi_1 + \psi_2) \lambda_1} \left[ 1 + \frac{\lambda_1}{2} (1 + \lambda_2) \frac{\exp \left[ -i(\psi_1 - \psi_2) \lambda_2 \right]}{D} \right] \right. \]

\[
\left. + e^{-i(\psi_1 - \psi_2) \lambda_2} \left[ 1 + \frac{\lambda_2}{2} (1 + \lambda_1) \frac{\exp \left[ -i(\psi_1 - \psi_2) \lambda_1 \right]}{D} \right] \right)
\]

\[
D = 1 - 2\lambda_1 \lambda_2 \cos(\psi_1 - \psi_2) + (\lambda_1 \lambda_2)^2; \quad (B.7)
\]

\[
N(\lambda) = \frac{1 - \lambda_1^2}{8} (2 - \lambda). \quad (B.8)
\]

All we need now to get \( T^p_x \) is the two integrals in formula (B.4), which are well known:

\[
\int_{-\infty}^{\infty} dk \exp \left[ -\frac{k^2}{2 \left( \Delta z_1^2 + \Delta z_2^2 \right)} \right] = \sqrt{\frac{2\pi}{\Delta z_1^2 + \Delta z_2^2}} \exp \left[ -\frac{2 \left( \Delta z_1^2 + \Delta z_2^2 \right)}{(z_1 - z_2)^2} \right];
\]

\[
\int_{-\infty}^{\infty} dk \exp \left[ -\frac{k^2}{2 \left( \Delta z_1^2 + \Delta z_2^2 \right)} \right] k e^{i k (z_1 - z_2)} = \pm i \sqrt{\frac{2\pi}{\Delta z_1^2 + \Delta z_2^2}} \exp \left[ -\frac{(z_1 - z_2)^2}{2 \left( \Delta z_1^2 + \Delta z_2^2 \right)} \right].
\]

In the case \( z_1 = z_2 \) and \( \Delta z_1 = \Delta z_2 = \Delta z \), they become

\[
\int_{-\infty}^{\infty} dk \exp \left[ -k^2 \Delta z^2 \right] = \frac{1}{\sqrt{\Delta z}}; \quad \int_{-\infty}^{\infty} dk \exp \left[ -k^2 \Delta z^2 \right] k = 0.
\]

Based on these results we are now able to rewrite formula (B.4) for the torque as

\[
T^p_x = -2\pi^3 \frac{\partial}{\partial z^2} \left[ V^2_1 \Delta z_1 N(\lambda_1) \sin \psi_1 + V^2_2 \Delta z_2 N(\lambda_2) \sin \psi_2 \right.
\]

\[
\left. - V_1 V_2 \sqrt{3} (M_1) \left[ \frac{\sqrt{2z_1}}{\sqrt{\Delta z_1^2 + \Delta z_2^2}} \right] \exp \left[ -z^2 \right] \right) \quad \left[ \frac{\sqrt{2z_2}}{\sqrt{\Delta z_1^2 + \Delta z_2^2}} + (z_1 + z_2) \right] \exp \left[ -z^2 \right]. \quad (B.9)
\]
with the new notations

$$I \equiv \sqrt{2} \frac{\Delta z_1 \Delta z_2}{\sqrt{\Delta z_1^2 + \Delta z_2^2}}; \quad \bar{z} \equiv \frac{z_1 - z_2}{\sqrt{2(\Delta z_1^2 + \Delta z_2^2)}}. \quad (B.10)$$

The calculation of the $y$ component of the patch torque does not present any additional difficulties. In the same way as we derived formula (B.9), for $T_p^y$ we find

$$T_p^y = 2\pi \frac{3}{2} \frac{\epsilon_0 d}{d^2} \left\{ V_1^2 z_1 \Delta z_1 N(\lambda_1) \cos \varphi_1 + V_2^2 z_2 \Delta z_2 N(\lambda_2) \cos \varphi_2 \right\}$$

$$- V_1 V_2 \bar{z} (M_1) \left[ \sqrt{2\pi} \frac{(\Delta z_1^2 - \Delta z_2^2)}{\sqrt{\Delta z_1^2 + \Delta z_2^2}} + (z_1 + z_2) \right] \exp \left\{ \bar{z}^2 \right\} \quad (B.11)$$

(again the notations (B.7) and (B.10) are used).

To obtain the closed form representation of the axial torque we combine formula (30) with expressions (B.1). The integral in $k$ there is found above, so the result is

$$T_p^z = 4\pi \frac{3}{2} \frac{\epsilon_0 d}{d^2} V_1 V_2 \bar{z} (M_3) \exp \left\{ \bar{z}^2 \right\}. \quad (B.12)$$

with the coefficient $M_3$ given by

$$M_3 = M_3(\lambda_1, \varphi_1, \lambda_2, \varphi_2) \equiv \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} n u_n(\lambda_1) e^{-i n \varphi_1} u_n(\lambda_2) e^{i n \varphi_2}. \quad (B.13)$$

Formula (40) for $u_n$ leads to an explicit sum of this series reduced to the derivative of a geometric progression:

$$M_3 = -i \left[ \frac{(1 - \lambda_1^2) (1 - \lambda_2^2)}{8} \right] \frac{(1 - \lambda_1^2 \lambda_2^2)}{D^2} \sin (\varphi_1 - \varphi_2). \quad (B.14)$$

Using this we write formula (B.12) in the final explicit form:

$$T_p^z = -\frac{\pi}{4} \frac{\epsilon_0 d}{d^2} V_1 V_2 \bar{z} \left(1 - \lambda_1^2 \right) \left(1 - \lambda_2^2 \right) \sin (\varphi_1 - \varphi_2) \left( \frac{1 - \lambda_1^2 \lambda_2^2}{D^2} \right) \exp \left\{ \bar{z}^2 \right\}. \quad (B.15)$$

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