A DYNAMICAL SYSTEM APPROACH TO THE INVERSE SPECTRAL PROBLEM FOR HANKEL OPERATORS: THE CASE OF COMPACT OPERATORS WITH SIMPLE SINGULAR VALUES

SERGEI TREIL AND ZHEHUI LIANG

Abstract. Given complex sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\mu_n\}_{n=1}^{\infty} \) satisfying the following intertwining property:

\[
|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > ... > |\lambda_n| > |\mu_n| \to 0,
\]

we use a dynamical approach to prove that there exists a unique Hankel operator \( \Gamma \) on \( \ell^2 \), with simple eigenvalues \( \{\lambda_n\}_{n=1}^{\infty} \), and \( \Gamma S \) with simple eigenvalues \( \{\mu_n\}_{n=1}^{\infty} \) respectively, here \( S \) is the shift operator. Specifically, we introduce abstract Borg’s theorem, saying that two sets of real eigenvalues will uniquely define the spectral measure of compact self-adjoint operators, under rank-one perturbation. Then we use this Borg’s theorem to solve the inverse spectral problem of Hankel operators, namely reconstruct Hankel operators from two sets of spectrum. In addition, we give another complex analysis method to describe the trivial kernel condition \( \text{Ker} \Gamma = \{0\} \) in terms of \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\mu_n\}_{n=1}^{\infty} \).

1. Introduction and main results

In [1], it was shown that under certain assumptions we can find a Hankel operator \( \Gamma \), which is unitary equivalent to a given self-adjoint operator \( R \). For this paper, we want to show another proof for the existence and uniqueness of such Hankel operator, under the assumption that \( R \) is a compact operator with distinct eigenvalues.

Definition 1.1. We say an operator \( \Gamma \) on \( \ell^2 = \ell^2(\mathbb{Z}_+) \) is a Hankel operator, if there exists a sequence \( \{\alpha_n\} \in \ell^2 \) satisfies: \( (\Gamma e_j, e_k) = \alpha_{j+k} \), where \( \{e_j\}_{j=0}^{\infty} \) is the standard basis in \( \ell^2 \). And we call \( \{\alpha_n\} \) to be the Hankel coefficients of \( \Gamma \).

Remark 1.2. A bounded operator \( \Gamma \) on \( \ell^2 \) is Hankel iff \( \Gamma S = S^*\Gamma \), where \( S \) is the shift operator on \( \ell^2 \), \( S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots) \).

We want to discuss whether a Hankel operator can be determined by its spectrum characteristics. In [2, P 6], Treil has proved the following result:

Proposition 1.3. \( R \) is a positive self-adjoint operator satisfying the following assumptions

(i) \( R \) is non-invertible;
(ii) \( \dim \text{Ker} R = 0 \) or \( \infty \).

Then there exists a Hankel operator \( \Gamma \), such that \( |\Gamma| \) is unitarily equivalent to \( R \).

In addition, A.V. Megreskii, V.V. Peller, S.R. Treil proved the following result in [1]:

Proposition 1.4. \( R \) is a bounded self-adjoint operator on a Hilbert space \( \mathcal{H} \), let \( \mu \) be a scalar spectral measure of \( R \), and \( \nu \) be its spectral multiplicity function. Then \( R \) is unitarily equivalent to a Hankel operator \( \Gamma \) if and only if the following assumptions hold:

(i) either \( \dim \text{Ker} R = 0 \) or \( \infty \);
(ii) $R$ is non-invertible;
(iii) $|\nu(t) - \nu(-t)| \leq 2$, $\mu_a$-a.e. (absolute continuous part of $\mu$), and $|\nu(t) - \nu(-t)| \leq 1$, $\mu_s$-a.e. (singular part of $\mu$).

Without further into detail about the definition of $\nu(t)$ and the proof of Proposition 1.3 and Proposition 1.4, we have the $\Gamma$ constructed in the two propositions above is not necessarily unique. In this paper, we will show that a compact Hankel operator with simple singular values can be uniquely determined by its spectrum characteristics.

We first present a simple proof for the following result stated in [8].

**Theorem 1.5.** Given sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ of non-zero real numbers, satisfying intertwining relations

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > \ldots > |\lambda_n| > |\mu_n| > \ldots$$

and such that $\lim_{n \to \infty} \lambda_n = 0$, $\lim_{n \to \infty} \mu_n = 0$, there exists a unique self-adjoint compact Hankel operator $\Gamma$ such that non-zero eigenvalues of $\Gamma$ and $\Gamma S$ are simple, and coincide with $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ respectively.

Moreover, $\ker \Gamma = \{0\}$ if and only if both of the following identities hold:

$$\sum_{j=1}^{\infty} \left( 1 - \frac{\mu_j^2}{\lambda_j^2} \right) = \infty,$$

$$\sum_{j=1}^{\infty} \left( \frac{\mu_j^2}{\lambda_j^2} - 1 \right) = \infty.$$  

The proof consists of three essentially separate parts. The first one is a simple operator-theoretic statement (which was essentially proved in [1]), which is presented in section 2 below. The second part, which is the hardest part in [1], is proving the asymptotic stability of some contraction. However in our case of compact operator can be obtained essentially for free. This part is discussed in Section 4.

The third part is an abstract version of the Borg’s two spectra theorem in [8], which is essentially an exercise in complex analysis.

For the last part of this paper, we generalize Theorem 1.5 to the cases of general Hankel operators with simple singular values, but not necessarily self-adjoint. The result is stated as below:

**Theorem 1.6.** Given complex sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$, where $\lambda_k = s_k e^{i\theta_k}$, $\mu_k = t_k e^{i\theta'_k}$, and the module part satisfies the following intertwining relation

$$s_1 > t_1 > s_2 > t_2 > \ldots \to 0,$$

then there exists a unique compact Hankel operator $\Gamma$ such that

(i) $\Gamma, \Gamma_1 = \Gamma S$ have simple singular values $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ respectively;

(ii) For the two antilinear operators $\mathcal{C}\Gamma, \mathcal{C}\Gamma_1$ (here $\mathcal{C}$ is the canonical conjugation defined by (3.1) we have:

$$\mathcal{C}\Gamma u_k = \lambda_k u_k, \quad \mathcal{C}\Gamma_1 v_k = \mu_k v_k;$$

where

$$u = \Gamma^* e_0, \quad u_k = P_{\ker(|\Gamma| - s_k I)} u, \quad v_k = P_{\ker(|\Gamma_1| - t_k I)} u.$$

Moreover, \( \text{Ker} \Gamma = \{0\} \) if and only if both of the following identities hold:

\[
\sum_{j=1}^{\infty} \left( 1 - \frac{t_j^2}{s_j^2} \right) = \infty,
\]

\[
\sum_{j=1}^{\infty} \left( \frac{s_j^2}{t_j^2 + 1} - 1 \right) = \infty.
\]

**Remark 1.7.** For antilinear operators, we can’t use the ”eigenvalues” and ”eigenspaces” notations, because for another vector \((iu_k)\) which belongs to the space spanned by \(u_k\), we have:

\[
\mathcal{C}_{\Gamma\mid_{(\text{Ker} \Gamma)^\perp}} (iu_k) = -i\lambda_k u_k = (-\lambda_k)(iu_k)
\]

Thus the one dimensional space \(\text{Span}\{u_k\}\) corresponds to two values \(\lambda_k, -\lambda_k\), which is controversial.

### 2. An abstract inverse problem theorem

#### 2.1. Plan of the Game.

Let \( \Gamma \) be a self-adjoint Hankel operator. Define \( \Gamma_1 = \Gamma S = S^* \Gamma \), which is also a self-adjoint Hankel operator. We can write

\[
\Gamma_1^2 = \Gamma SS^* \Gamma = \Gamma (I - c_0 e_0^* \Gamma) \Gamma = \Gamma^2 - uu^*,
\]

where \( u := \Gamma e_0 \), and \( u^* \) is the operator symbolizing \( u^* = \langle \cdot, u \rangle \) (we will use the similar terminology \( p^* \) later on without further notation).

Now let us try to go from the opposite direction. Suppose that we are given two self-adjoint operators \( R, R_1 \) on a Hilbert space \( \mathcal{H} \), and \( R, R_1 \) are their modulus operator respectively

\[
R = R J, \quad R_1 = R_1 J_1.
\]

We want to find a Hankel operator \( \Gamma \), such that the Hankel operators \( \Gamma \) and \( \Gamma_1 \) are simultaneously unitary equivalent to \( R \) and \( R_1 \) respectively, meaning that we have \( \Gamma = URU^* \), \( \Gamma_1 = U R_1 U^* \) for some unitary \( U \). We can see from (2.1) that \( R \) and \( R_1 \) should satisfy the relation

\[
R^2 - R_1^2 = pp^*
\]

for some \( p \in \mathcal{H} \).

To solve the inverse problem we first need to find a contraction \( T \), (hopefully unitary equivalent to the backward shift \( S^* \)) such that \( R_1 = T R \). The tool to find \( T \) is the following simple lemma.

**Lemma 2.1 (Douglas Lemma).** Let \( A \) and \( B \) be two bounded operators on a Hilbert space \( \mathcal{H} \) such that

\[
\| Bh \| \leq \| Ah \| \quad \forall h \in \mathcal{H},
\]

or, equivalently, \( B^* B \leq A^* A \).

Then there exists a contraction \( T \) (i.e. \( \| T \| \leq 1 \)) such that \( B = TA \).

Moreover, if \( A \) has dense range, the operator \( T \) is unique.

To avoid non-uniqueness when defining \( T \), we usually assume \( \text{Ker} R = \{0\} \), meaning that we want \( R \) to be unitary equivalent to the essential part of \( \Gamma \), i.e., to the operator \( \tilde{\Gamma} := \Gamma\mid_{(\text{Ker} \Gamma)^\perp} \). In this case the operator \( T \) should be unitarily equivalent to \( S^* \) restricted on \((\text{Ker} \Gamma)^\perp\), meaning that

\[
T = \tilde{V}^* S^* \mid_{(\text{Ker} \Gamma)^\perp} \tilde{V}
\]
for an unitary $\tilde{V}$.

In this situation the vector $p$ from (2.2) should be $p = Rq$ where $\|q\| \leq 1$, we can think of $q$ being

$$q = \tilde{V}^* P_{(\text{Ker } \Gamma)^\perp} e_0.$$

Using the fact that $R_1 = T R = R T^*$ we can rewrite (2.2) as

$$\mathcal{R}(I - T^* T)\mathcal{R} = \mathcal{R} qq^* \mathcal{R},$$

which implies that

$$I - T^* T = qq^* = (\cdot, q)q. \quad (2.3)$$

So we arrive at the following setup: $\mathcal{R}$ and $\mathcal{R}_1$ are self-adjoint operators, $\text{Ker } R = \{0\},$

$$\mathcal{R}^2 - \mathcal{R}_1^2 = pp^*, \quad p = \mathcal{R}q, \quad \|q\| \leq 1, \quad (2.4)$$

and $T$ is the unique contraction satisfying $\mathcal{R}_1 = T \mathcal{R}$; note that in this case $T$ satisfies (2.3).

**Definition 2.2.** We say an operator $T$ is asymptotically stable iff $T^n \to 0$ in the strong operator topology as $n \to \infty$, i.e. iff for $\forall x \in \mathcal{H}$

$$\lim_{n \to \infty} \|T^n x\| \to 0.$$

Recall that in this paper for a Hankel operator $\Gamma$ we denote $\Gamma_1 := \Gamma S = S^* \Gamma$

**Proposition 2.3.** If $T$ is asymptotically stable, then there exist a unique self-adjoint Hankel operator $\Gamma$ and an unitary operator $\tilde{V}: \mathcal{H} \to (\text{Ker } \Gamma)^\perp$ such that

$$\tilde{\Gamma} := \Gamma|_{(\text{Ker } \Gamma)^\perp} = \tilde{V} \mathcal{R} \tilde{V}^*, \quad (2.5)$$

$$\tilde{\Gamma}_1 := \Gamma_1|_{(\text{Ker } \Gamma)^\perp} = \tilde{V} \mathcal{R}_1 \tilde{V}^*, \quad (2.6)$$

$$\Gamma e_0 = \tilde{V} p; \quad (2.7)$$

note that (2.7) implies that

$$P_{(\text{Ker } \Gamma)^\perp} e_0 = \tilde{V} q, \quad (2.8)$$

And the Hankel coefficients $\{\alpha_k\}_{k=0}^\infty$ can be expressed by

$$\alpha_k = \langle T^k p, q \rangle.$$

Moreover, $\text{Ker } \Gamma = \{0\}$ if and only if $\|q\| = 1$ and $q \notin \text{Ran} \{R\}$.

**2.2. Proof of Proposition 2.3.**

**2.2.1. Existence of Hankel operator $\Gamma$.**

**Proof.** Treating (2.3) as an identity for quadratic forms and substituting $x \in \mathcal{H}$ into it we get

$$\|x\|^2 - \|Tx\|^2 = |\langle x, q \rangle|^2.$$

Then we substitute $x$ by $T^k x$ for an arbitrary $k \in \mathbb{N}$, we get

$$\|T^k x\|^2 - \|T^{k+1} x\|^2 = |\langle T^k x, q \rangle|^2,$$

Now taking the sum from $k = 0$ to $k = n$ we get

$$\|x\|^2 - \|T^{n+1} x\|^2 = \sum_{k=0}^n |\langle T^k x, q \rangle|^2.$$
Let \( n \to \infty \) and using the asymptotic stability of \( \mathcal{T} \) we see that

\[
\|x\|^2 = \sum_{k=0}^{\infty} |\langle \mathcal{T}^k x, q \rangle|^2,
\]

which means that the operator \( V : \mathcal{H} \to \ell^2 \)

\[
\mathcal{V}x = (\langle x, q \rangle, \langle \mathcal{T}x, q \rangle, \langle \mathcal{T}^2 x, q \rangle, \ldots) = \left( \langle \mathcal{T}^k x, q \rangle \right)_{k=0}^{\infty}
\]

is an isometry.

We can see that \( \mathcal{VT}x = (\langle \mathcal{T}x, q \rangle, \langle \mathcal{T}^2 x, q \rangle, \langle \mathcal{T}^3 x, q \rangle, \ldots) = S^* \mathcal{V}x \), so \( T \) is unitarily equivalent to either \( S^* \) (if Ran \( V = \ell^2 \)) or to the restriction of \( S^* \) to Ran \( V \), which is a \( S^* \)-invariant subspace (if Ran \( V \neq \ell^2 \)). (Here we use a simple fact that an onto isometry is unitary.)

Let \( \mathcal{V} \) be the operator \( V \) with the target space restricted to Ran \( V \), so \( \mathcal{V} : \mathcal{H} \to \text{Ran} \mathcal{V} \) is an unitary operator. Denoting by \( \mathcal{S}^* := S^* |_{\text{Ran} \mathcal{V}} \) (so \( \mathcal{S}^* = S^* \) if Ran \( V = \ell^2 \)), we see that

\[
\mathcal{VT}^* \mathcal{V}^* = \mathcal{S}^*, \quad \mathcal{VT} \mathcal{V}^* = (\mathcal{S}^*)^* = P_{\text{Ran} \mathcal{V}} S |_{\text{Ran} \mathcal{V}} =: \mathcal{S}^*.
\]

Define

\[
\mathcal{\Gamma} := \mathcal{V} \mathcal{R} \mathcal{V}^*, \quad \mathcal{\Gamma}_1 := \mathcal{V} \mathcal{R}_1 \mathcal{V}^*.
\]

Then the relation \( \mathcal{T} \mathcal{R} = \mathcal{R}_1 = \mathcal{R}_1^* = \mathcal{R} \mathcal{T}^* \) (remember that \( \mathcal{R} \) and \( \mathcal{R}_1 \) are self-adjoint) translates to

\[
\mathcal{\Gamma} \mathcal{S} = \mathcal{\Gamma}_1 = \mathcal{S}^* \mathcal{\Gamma}.
\]

Extending \( \mathcal{\Gamma} \) and \( \mathcal{\Gamma}_1 \) to operators \( \Gamma \) and \( \Gamma_1 \) on the whole space \( \ell^2 \) by setting them to be 0 on \( (\text{Ran} \mathcal{V})^\perp \), we can see that \( \text{Ker} \Gamma = (\text{Ran} \mathcal{V})^\perp \) and that

\[
\Gamma = \mathcal{V} \mathcal{R} \mathcal{V}^*, \quad \Gamma_1 = \mathcal{V} \mathcal{R}_1 \mathcal{V}^*.
\]

Let us now show that \( \Gamma \) satisfies the identity \( \Gamma S = S^* \Gamma \), i.e. that \( \Gamma \) is indeed a Hankel operator. For \( \forall f \in \ell^2 \) we decompose

\[
f = P_{\text{Ran} \mathcal{V}} f + P_{(\text{Ran} \mathcal{V})^\perp} f =: f_1 + f_2.
\]

We know that \( S^* \text{Ran} \mathcal{V} \subset \text{Ran} \mathcal{V} \), so \( S((\text{Ran} \mathcal{V})^\perp) \subset (\text{Ran} \mathcal{V})^\perp \). Therefore \( Sf_2 \perp \text{Ran} \mathcal{V} \), so \( \Gamma Sf_2 = 0 = S^* \Gamma f_2 \). As for \( f_1 \), since \( \text{Ker} \Gamma = (\text{Ran} \mathcal{V})^\perp \), we have

\[
\Gamma Sf_1 = \Gamma P_{\text{Ran} \mathcal{V}} Sf_1 = \Gamma \mathcal{S} f_1 = \mathcal{\Gamma S} f_1 = \mathcal{S}^* \mathcal{\Gamma} f_1 = S^* \Gamma f_1,
\]

so \( \Gamma \) is indeed a Hankel operator.

Now it remains to show the identities (2.5), (2.6) and (2.7). The first two identities are exactly from (2.10). To show (2.7), we first derive a representation for \( \mathcal{V}^* \).

For \( \forall x, y \in \mathcal{H} \), we have

\[
\langle \mathcal{V}x, y \rangle = \sum_k \bar{y}_k \langle \mathcal{T}^k x, q \rangle = \langle x, \sum_k y_k (\mathcal{T}^*)^k q \rangle,
\]
thus $\mathcal{V}^* y = \sum_k y_k (\mathcal{T}^*)^k q$. In specific, we have $\mathcal{V}^* e_j = q$. In conclusion, we have

$$\Gamma e_0 = \mathcal{V} \mathcal{R} \mathcal{V}^* q = \mathcal{V} \mathcal{R} q = \mathcal{V} \tilde{\mathcal{V}} p,$$

and we finish the proof of the existence part.

2.2.2. **Uniqueness of Hankel operator** $\Gamma$.

*Proof.* Suppose that the identities (2.5), (2.6), (2.7) hold for some unitary operator $\tilde{\mathcal{V}} : \mathcal{H} \to \text{clos} \text{Ran} \Gamma = (\text{Ker} \Gamma)^\perp \subset \mathcal{h}^2$.

Since for a Hankel operator $\text{Ker} \Gamma$ is always $S$-invariant, the subspace $(\text{Ker} \Gamma)^\perp$ is $S^*$-invariant, so the restriction $S^*|_{(\text{Ker} \Gamma)^\perp}$ is well defined. The identities (2.5), (2.6) and the definition of $\mathcal{T}$ imply that

$$S^*|_{(\text{Ker} \Gamma)^\perp} = \tilde{\mathcal{V}} \mathcal{T} \tilde{\mathcal{V}}^*.$$ (2.11)

Now denote $\tilde{\Gamma} := \Gamma|_{(\text{Ker} \Gamma)^\perp}$ and $\tilde{\Gamma}_1 = \Gamma_1|_{(\text{Ker} \Gamma)^\perp}$, we restrict both sides of identity (2.1) on $(\text{Ker} \Gamma)^\perp$ and write

$$\tilde{\Gamma}_1^2 = \tilde{\Gamma}^2 - uu^*,$$

where

$$u := \Gamma e_0 = \tilde{\Gamma} P_{(\text{Ker} \Gamma)^\perp} e_0,$$

Then identities (2.5), (2.6) will be unitarily translated to

$$\mathcal{R}^2 = \mathcal{R}^2 - \tilde{p} \tilde{p}^*, \quad \text{where} \quad \tilde{p} = \mathcal{R} \tilde{q}, \; \tilde{q} = \tilde{\mathcal{V}}^* P_{(\text{Ker} \Gamma)^\perp} e_0.$$ Comparing this with $\mathcal{R}_1^2 = \mathcal{R}^2 - pp^*$ we conclude that $p = \alpha \tilde{p}, q = \alpha \tilde{q}$, for an $|\alpha| = 1$.

Then to compute the Hankel coefficients $\{\gamma_k\}_{k=1}^\infty$ of $\Gamma$, we have

$$\gamma_k = \langle \Gamma e_0, S^k e_0 \rangle = \langle (S^*)^k \Gamma e_0, e_0 \rangle = \langle (S^*)^k P_{(\text{Ker} \Gamma)^\perp} e_0, P_{(\text{Ker} \Gamma)^\perp} e_0 \rangle$$

$$= \langle \mathcal{T}^k \mathcal{R} \tilde{q}, \tilde{q} \rangle = \langle \mathcal{T}^k \tilde{p}, \tilde{q} \rangle = \langle \mathcal{T}^k p, q \rangle,$$

meaning that the coefficients $\{\gamma_k\}_{k=0}^\infty$ does not depend on $\tilde{\mathcal{V}}$. So the uniqueness is proved. \hfill $\Box$

2.2.3. **Trivial Kernel Condition of Hankel Operator** $\Gamma$.

*Proof.* As we discussed in the proof of existence part, $\text{Ker} \Gamma$ is trivial if and only if the operator $\mathcal{V}$ defined by (2.9) satisfies $\text{Ran} \mathcal{V} = \mathcal{h}^2$. The latter condition is equivalent to $\tilde{S}^* = S^*$, which happens if and only if $\mathcal{T}$ is unitarily equivalent to $S^*$, or, equivalently, $\mathcal{T}^*$ is unitarily equivalent to $S$.

For the first direction, suppose $\text{Ker} \Gamma = \{0\}$, so $\mathcal{T}$ is unitarily equivalent to $S^*$. We know that

$$I - S S^* = e_0 e_0^*, \quad I - \mathcal{T}^* \mathcal{T} = q q^*,$$

so by the unitary equivalence $\|q\| = \|e_0\| = 1$. If $q \in \text{Ran} \mathcal{R}$, i.e. $q = \mathcal{R} f$, then

$$\mathcal{R} \mathcal{T}^* f = \mathcal{R} \mathcal{T} f = \mathcal{T} q = 0,$$

and since $\text{Ker} \mathcal{R} = \{0\}$, we conclude that $\mathcal{T}^* f = 0$. But if $\text{Ker} \Gamma = \{0\}$, then $\mathcal{T}$ is an isometry, which contradicts to $\mathcal{T}^* f = 0$. So $q \notin \text{Ran} \mathcal{R}$.

For the sufficiency part, suppose $\|q\| = 1$ and $q \notin \text{Ran} \mathcal{R}$. We know that

$$\mathcal{T} \mathcal{R}^2 \mathcal{T}^* = \mathcal{R}_1^2 = \mathcal{R} \mathcal{T}^* \mathcal{T} \mathcal{R} = \mathcal{R}(I - q q^*) \mathcal{R},$$
and that \( \ker(I - qq^*) = \text{Span}\{q\} \). Since \( q \notin \text{Ran}\, R \), we see that \( \ker R(I - qq^*)R = \{0\} \), so \( \ker T^* = \{0\} \).

Now applying (2.3) to vector \( q \) we get that
\[
q - T^*Tq = q,
\]
and since \( \ker T^* = \{0\} \) we see that \( Tq = 0 \).

Then left and right multiplying (2.3) by \( T \) and \( T^* \) respectively, we get
\[
TT^* - T^*TT^* = Tqq^*T^*,
\]
and since \( Tq = 0 \), we have \( TT^* = (T^*)^2 \), which implies that \( TT^* \) is an orthogonal projection. Since \( \ker T^* = \{0\} \), we conclude that \( TT^* = I \), i.e. \( T \) is an isometry.

In addition, we have that \( T \) is asymptotically stable, thus there is no reduced subspace on which \( T \) (and so \( T^* \)) is unitary. The identity (2.3) implies that \( \text{rank}(I - T^*T) = 1 \), i.e. the defect indices of \( D_T = 1 \) so \( T^* \) is unitarily equivalent to the shift operator \( S \), i.e. that
\[
S = V T^* V^*
\]
for some unitary operator \( V : H \to \ell^2 \). Defining \( \Gamma = \overline{V} R \overline{V}^* \), \( \Gamma_1 = \overline{V} R_1 \overline{V}^* \), we can see that
\[
\Gamma_1 = \Gamma S = S^* T,
\]
thus \( \Gamma \) is indeed the Hankel operator with trivial kernel, which satisfies (2.5), (2.6) and (2.7). Since the constructed Hankel operator \( \Gamma \) that satisfies (2.5), (2.6) and (2.7) is unique, we have finished the proof of trivial kernel condition. \( \square \)

3. Generalization for Hankel Operators not Self-Adjoint

In previous section, we have shown that a self-adjoint Hankel operator \( \Gamma \) can be characterized by a rank-one perturbation triple \((R, R_1, p)\) with the following conditions
\begin{enumerate}[(a)]
\item \( R, R_1 \) are self-adjoint operators, \( \ker R = \{0\} \);
\item \( R^2 - R_1^2 = pp^* \), where \( p \) is cyclic with respect to \( R \), \( ||R^{-1}p|| \leq 1 \);
\item The contraction \( T \) defined by \( R_1 = T R \) (implied by Douglas Lemma 2.1) is asymptotically stable.
\end{enumerate}

However, things are slightly different when \( \Gamma \) is not self-adjoint. In this section, we derive a similar result (proposition 3.11) when \( \Gamma \) is not self-adjoint.

3.1. Preparations and Some Terminology. We first introduce the definition of \( C \)-symmetric operators and some terminologies from [9].

Definition 3.1. A function \( C : H \to H \) is called a conjugation on the complex Hilbert space \( H \), if and only if the following conditions holds:
\begin{enumerate}[(i)]
\item conjugate-linear: \( C(\alpha x + \beta y) = \bar{\alpha}Cx + \bar{\beta}Cy \) for all \( x, y \in H \);
\item involutive: \( C^2 = I \);
\item isometric: \( ||Cx|| = ||x|| \) for all \( x \in H \).
\end{enumerate}

Lemma 3.2. If \( C \) is a conjugation on \( H \), then there exists an orthonormal basis \( \{e_n\} \) of \( H \) such that: \( Ce_n = e_n \) for all \( n \). In particular, \( C(\sum \alpha_n e_n) = \sum \overline{\alpha_n} e_n \) for all \( \{\alpha_n\}_{n=1}^\infty \in \ell^2 \). And we call such basis \( \{e_n\}_{n=1}^\infty \) \( C \)-real orthogonal basis.

Definition 3.3. Let \( C \) be a conjugation on Hilbert space \( H \). A bounded linear operator \( T \) on \( H \) is called \( C \)-symmetric if and only if \( T = CT^* C \).
Remark 3.4. For a Hankel operator $\Gamma$ on $\ell^2$, we have: $\Gamma = \mathcal{C}\Gamma^*\mathcal{C}$, where $\mathcal{C}$ is the canonical conjugation on $\ell^2$

$$\mathcal{C}(z_1, z_2, z_3...) = \mathcal{C}(\bar{z}_1, \bar{z}_2, \bar{z}_3...), \quad (3.1)$$

Hence Hankel operator is always $\mathcal{C}$-symmetric.

Furthermore, polar decomposition can also be generalized for complex symmetric operators. The following theorem is also cited from [9].

**Theorem 3.5.** If $T : \mathcal{H} \to \mathcal{H}$ is a bounded $\mathcal{C}$-symmetric operator, then we can write $T = \mathcal{C}\mathcal{J}|T|$, where $\mathcal{J}$ is a conjugation that commutes with $|T| = \sqrt{T^*T}$.

3.2. A Result Related to Rank-one perturbation. We still need more preparation before introducing our Hankel operator setting. In this subsection, we will prove the following result

**Proposition 3.6.** Let $(R, R_1, p)$ be a triple satisfies

(i) $R, R_1$ are two positive self-adjoint operators defined on a Hilbert space $\mathcal{H}$, $p$ is a vector in $\mathcal{H}$;

(ii) $R^2 - R_1^2 = pp^*$.

Then there exists a conjugation $\mathcal{J}_p$ such that

$$\mathcal{J}_p R = R \mathcal{J}_p, \quad \mathcal{J}_p R_1 = R_1 \mathcal{J}_p, \quad \mathcal{J}_p p = p$$

Moreover, $\mathcal{J}_p$ is unique on space $\mathcal{H}_0 := \text{span}\{R^n p : n \geq 0\}$. Especially, if $p$ is a cyclic vector with respect to $R$, i.e. that $\mathcal{H} = \mathcal{H}_0$, then $\mathcal{J}_p$ is unique.

We first prove several lemmas before proving Proposition 3.6.

**Lemma 3.7.** Let $W$ be a self-adjoint operator on Hilbert space $\mathcal{H}$ and vector $p \in \mathcal{H}$. Then there exists a conjugation $\mathcal{J}_p$ commuting with $W$ and preserving $p$, i.e. $\mathcal{J}_p p = p$. Moreover, if $p$ is a cyclic vector with respect to $W$, then the conjugation is unique.

**Proof.** Since everything is defined up to unitary equivalence, we can assume that $W$ is the multiplication by independent variable $M_x$ in space $L^2(\rho)$. Here $\rho$ is the scalar spectral measure of $W$ with respect to $p$.

We want to find out all conjugations which commutes with $M_x$ on $L^2(\rho)$.

First we introduce the canonical conjugation $\mathcal{J}_1$ defined on $L^2(\rho)$ to be: $\mathcal{J}_1 : f(x) \to \overline{f(x)}$.

The following lemma is a modified version from [10, Theorem 2.2]:

**Lemma 3.8.** Let $\mathcal{J}$ be a conjugation on $L^2(\rho)$, where $\rho$ is a compact Borel measure on $\mathbb{R}$. Then the following two conditions are equivalent:

(i) $M_x \mathcal{J} = \mathcal{J} M_x$, where $M_x$ is the operator of multiplication by independent variable $\mathcal{J}_1$ defined on $L^2(\rho)$, $|\phi(x)| = 1$ $\rho$-a.e., such that: $\mathcal{J} f = M_{\phi(x)} \mathcal{J}_1 f$ holds $\rho$-a.e.

**Proof of Lemma 3.8.** We only need to show $(i) \implies (ii)$, while the other direction is trivial. We know that $M_x \mathcal{J}_1 = \mathcal{J} M_x \mathcal{J}_1 = \mathcal{J} \mathcal{J}_1 M_x$, hence $M_x$ commutes with an unitary operator $\mathcal{J}_1$ on $L^2(\rho)$. By [Theorem 3.2, [11]], $\mathcal{J}_1 = M_{\phi}$ for a $\phi(x) \in L^2(\rho)$, $J = M_{\phi(x)} \mathcal{J}_1$. Since $\mathcal{J}_1$ is unitary, we conclude that $|\phi| = 1$ $\rho$-a.e. \hfill \Box

Now back to the proof of Lemma 3.7. By unitary equivalence, we can assume that $W$ is the multiplication by independent variable $M_x$ on $L^2(\rho)$, and $p$ is a function $f(x) \in L^2(\rho)$. By
Lemma 2.2, since $M_x J = J M_x$, we have $J = M_{\phi(x)} J_1$ for a certain function $\phi(x) \in L^\infty(p)$, $|\phi(x)| = 1$ $p-$ a.e. Then from $J f = f$, we have

$$\phi(x) \overline{f(x)} = f(x)$$

Thus we can set $\phi(x) = \frac{f(x)}{\overline{f(x)}}$ where $f(x) \neq 0$, and set $\phi(x)$ to be any unit value where $f(x) = 0$. This gives the existence of such conjugation $J_p$.

Now we move on to the uniqueness part. Denoting $H_0 = \text{span}\{W^n p | n \geq 0\}$, we will show that $J_p$ is unique on $H_0$. In fact, we can show by induction that $J_p W^n p = W^n p$ holds for all $n \in \mathbb{N}$

The equation holds naturally when $n = 0$. Assume that the equation holds for $n = k$, then for $n = k + 1$,

$$J_p W^{k+1} p = W J_p W^k p = W W^k p = W^{k+1} p$$

Thus $J_p$ is uniquely defined on $H_0$.

Lemma 3.9. Let $R$ be a self-adjoint positive, bounded operator on Hilbert space $H$, and $J$ is a conjugation on $H$. Then the following two statements are equivalent:

(i) $R$ commutes with $J$;
(ii) $R^2$ commutes with $J$.

Proof. (i) $\implies$ (ii) is trivial, we only need to prove the other.

Since $R^2$ commutes with $J$, for any polynomial $p$ with real coefficients, $p(R^2)$ also commutes with $J$: $p(R^2) J = J p(R^2)$. Hence we can take a polynomial sequence $p_n(x)$ which converges uniformly to $\phi(x) = \sqrt{x}$ on $\sigma(R^2)$, which is the spectrum of $R^2$. Then by $p(R^2) J = J p(R^2)$ and let $n \to \infty$ we have

$$R J = J R$$

Now back to our setting of triple $(R, R_1, p)$ in Proposition 3.6.

Proof of Proposition 3.6. By Lemma 3.7, there exists a conjugation $J_p$ satisfying: $J_p p = p$ and $J_p R^2 = R^2 J_p$. Hence from $R^2 - pp^* = R_1^2$ we get $J_p R_1^2 = R_1 J_p$. Now applying Lemma 3.9, we have $J_p$ commutes with $R$ and $R_1$.

For the uniqueness part, we know $J_p R^2 = R^2 J_p$ from $J_p R = R J_p$, then from lemma 3.7 such $J_p$ is unique.

3.3. Setting of Hankel operators as C-symmetric operators. Now back to the setting of game when our given Hankel $\Gamma$ is compact but not self-adjoint. We want to describe what spectral characteristics completely determines $\Gamma$. Unlike the self-adjoint case, we give the description only under assumption that the vector $u = \Gamma^* e_0$ is the cyclic vector for the operator $|\Gamma| : = |\Gamma|_{(Ker \Gamma)^\perp}$

Defining $\Gamma_1 := \Gamma S = S^* \Gamma$, we can write

$$|\Gamma_1|^2 = \Gamma_1 \Gamma_1 = \Gamma^* SS^* \Gamma = \Gamma^* (I - e_0 e_0^*) \Gamma = |\Gamma|^2 - uu^*,$$

where $u := \Gamma^* e_0$. The above identity holds without any assumption about the cyclicity of $u$.

Now from Proposition 3.2, there exists a conjugation $\tilde{J}_\Gamma$ such that

$$|\Gamma| \tilde{J}_\Gamma = \tilde{J}_\Gamma |\Gamma|, \quad |\Gamma_1| \tilde{J}_\Gamma = \tilde{J}_\Gamma |\Gamma_1|, \quad \tilde{J}_\Gamma u = u. \quad (3.2)$$
Then we introduce the polar decomposition form of $\Gamma$ and $\Gamma_1$ given by Theorem 3.5 and write
\[ \Gamma = \mathcal{C}_3|\Gamma| = \mathcal{C}\Phi_3|\Gamma|, \quad \mathcal{C}\Gamma_3 = \Phi_3|\Gamma|3_\Gamma = \Phi|\Gamma|; \]
and also similar notations for $\Gamma_1$
\[ \Gamma_1 = \mathcal{C}_3|\Gamma_1| = \mathcal{C}\Phi_3|\Gamma_1|, \quad \mathcal{C}\Gamma_3 = \Phi_3|\Gamma_1|3_\Gamma = \Phi_1|\Gamma_1|, \]
where $\Phi = \mathcal{J}_3$ and $\Phi_1 = \mathcal{J}_1$ are unitary operators commutes with $|\Gamma|, |\Gamma_1|$ respectively. Here $\mathcal{C}\Gamma_3$ and $\mathcal{C}_1\Gamma_3$ are normal operators (Notice that $\Gamma, \Gamma_1$ are not necessarily normal).

In addition, we have
\[ |\Gamma_1|\Phi_13_\Gamma \mathcal{C} = \Gamma_1^* = S^*\Gamma^* = S^*|\Gamma|\Phi_3\mathcal{C}, \]
hence
\[ |\Gamma_1|\Phi_1 = S^*|\Gamma|\Phi. \]
Restricted on $(\text{Ker } \Gamma)^\perp$ and define $\tilde{\Gamma} := \Gamma|_{(\text{Ker } \Gamma)^\perp}$ (here we similarly define $\tilde{\Phi}, \tilde{\Gamma}_1, \tilde{\Phi}_1$, and also $\tilde{S}^* := S^*|_{(\text{Ker } \Gamma)^\perp}$, since $(\text{Ker } \Gamma)^\perp$ is always $S^*$-invariant), thus $\tilde{S} = (\tilde{S}^*)^* = P_{(\text{Ker } \Gamma)^\perp}S|_{(\text{Ker } \Gamma)^\perp}$ and (3.5) translates to $|\tilde{\Gamma}_1|\Phi_1 = \tilde{S}^*|\tilde{\Gamma}|\Phi$. Hence $\tilde{S}^* = |\tilde{\Gamma}_1|\Phi_1^*|\tilde{\Gamma}|^{-1}$ and we have
\[ 3_\Gamma \tilde{S}^* = |\tilde{\Gamma}_1|\Phi_1^*|\tilde{\Gamma}|^{-1}3_\Gamma \]
Denote $\tilde{S}^* := |\tilde{\Gamma}_1|\Phi_1^*|\tilde{\Gamma}|^{-1}$, we have $|\tilde{\Gamma}_1|\Phi_1^* = \tilde{S}^*|\tilde{\Gamma}|\Phi = |\tilde{\Gamma}|\Phi\tilde{S}$. Now define $v := 3_\Gamma e_0$, then
\[ u = \Gamma^* e_0 = |\Gamma|3_\Gamma e_0 = |\Gamma|3_\Gamma e_0 = |\Gamma|\Phi v, \]
and the symbol of Hankel operator $\Gamma$ can be calculated as
\[ \gamma_k = \langle \Gamma e_k, e_0 \rangle = \langle e_0, (S^*)^k u \rangle = \langle 3_\Gamma (S^*)^k u, v \rangle \]
\[ = \langle 3_\Gamma (\tilde{S}^*)^k u, v \rangle = \langle (\tilde{S}^*)^k 3_\Gamma u, v \rangle = \langle (\tilde{S}^*)^k u, v \rangle. \]

Now let's discuss it from the opposite direction. Recall that in the self-adjoint setting we are given a triple $(R, R_1, p)$ satisfying: $R^2 - R_1^2 = pp^*$, here $R, R_1$ are self-adjoint positive operators, with $\text{Ker } R = \{0\}$. In addition, $p$ is a cyclic vector with respect to $R$, with property $\|R^{-1}p\| \leq 1$.

However, for the case when Hankel operator is not self-adjoint, things are more complicated. Corresponding to the unitary $\Phi, \Phi_1$ given in (3.3), (3.4), suppose that we are also given two scalar functions $\phi, \phi_1$ which are unimodular and Borel measurable, then $\phi(R), \phi_1(R_1)$ are unitary operators commutes with $R, R_1$ respectively. Now we want to find a Hankel operator $\Gamma$ such that there is an unitary equivalence between $(R\phi(R), R\phi_1(R_1), p)$ and $(|\Gamma|\Phi, |\tilde{\Gamma}_1|\Phi_1, u)$. In other words, we want the two anti-linear operators $(\mathcal{C}_1)^\perp, (\mathcal{C}_1)^\perp$ are simultaneously unitary equivalent to $R\phi(R)3_p, R_1\phi_1(R_1)3_p$, where $3_p$ is a conjugation satisfying
\[ 3_pR = R3_p, \quad 3_pR_1 = R_13_p, \quad 3_pp = p \]
indicated by Proposition 3.6.

To begin our proof, we need to define a contraction between $R_1\phi_1(R_1)$ and $R\phi(R)$. By Douglas Lemma 2.1, there exists an unique contraction $T$ such that: $R_1\phi_1(R_1) = TR\phi(R)$, i.e. $T = R_1\phi_1(R_1)R^{-1}(\phi(R))^*$. 

**Remark 3.10.** For the self-adjoint case which we have discussed in section 2, we have
\[ R = R\phi(R), \quad R_1 = R_1\phi_1(R_1), \quad R_1 = T R. \]
So it’s a special case when $\phi, \phi_1$ can only take values $\pm 1$. 

Now we are ready to state the main result under the non-self-adjoint setting, we first restate all the settings below. We have two Borel measurable unimodular scalar functions \( \phi, \phi_1 \), and a triple \((R, R_1, p)\) satisfying:

(i) \( R, R_1 \) are self-adjoint positive operators, with \( \text{Ker} R = \{0\} \);
(ii) \( R^2 - R_1^2 = pp^* \), while \( p \in \text{Ran} R \) with property \( \|R^{-1}p\| \leq 1 \);
(iii) \( p \) is a cyclic vector for \( R \).

And we also define \( J_p \) as the unique conjugation satisfying

\[
J_p R = R J_p, \quad J_p R_1 = R_1 J_p, \quad J_p p = p,
\]

followed by Proposition 3.6. Besides, we denote \( T \) as the unique contraction satisfying

\[
R_1 \phi_1(R_1) = \mathcal{T} R \phi(R).
\]  

**Proposition 3.11.** If the contraction \( T \) defined by (3.9) is asymptotically stable, then there exists an unique Hankel operator \( \Gamma \) and an unitary operator \( \overline{V} : \mathcal{H} \to (\text{Ker} \Gamma)^\perp \) such that

\[
\mathcal{C}|_{(\text{Ker} \Gamma)^\perp} = \overline{V} R \phi(R) J_p V^* ,
\]

\[
\mathcal{C}_{11}|_{(\text{Ker} \Gamma)^\perp} = \overline{V} R \phi_1(R_1) J_p V^*,
\]

\[
\Gamma^* e_0 = \overline{V} p
\]

where recall \( \mathcal{C} \) is the canonical conjugation on \( \ell^2 \) defined by (3.1).

Moreover, the coefficients \( \gamma_k \) of the Hankel operator \( \Gamma \) can be calculated as

\[
\gamma_k = (\Sigma^k p, q)
\]

where \( \Sigma = J_p T J_p \) and \( q := R^{-1} (\phi(R))^* p = (\phi(R))^* R^{-1} p \) (note that by (ii) we have \( \|q\| \leq 1 \)).

In addition, \( \text{Ker} \Gamma = \{0\} \) iff \( \|q\| = 1 \) and \( q \notin \text{Ran} R \)

### 3.4. Proof of Proposition 3.1.

The whole proof consists of three different parts: existence, uniqueness and the trivial kernel condition.

**3.4.1. Existence of Hankel operator \( \Gamma \).**

**Proof.** We firstly prove some identities for later use. We show

\[
R_1 \phi_1^*(R_1) = \Sigma R \phi^*(R), \quad R_1 \phi_1(R_1) = R \phi(R) \Sigma^* ;
\]

\[
R(\phi(R))^* J_p q = p.
\]

For (3.14), since \( J_p \) commutes with \( R, R_1 \), together with the definition (3.9), we have

\[
\Sigma R \phi^*(R) = J_p T J_p R \phi^*(R) = J_p T R \phi(R) J_p = J_p R_1 \phi_1(R_1) J_p = J_p R_1 \phi_1^*(R_1),
\]

and for (3.15), since \( p = R \phi(R) q \), we have

\[
R(\phi(R))^* J_p q = R J_p \phi(R) q = J_p R \phi(R) q = J_p p = p.
\]

Since \( \phi(R) \) is unitary and commutes with \( R \), we can write

\[
R^2 = \phi(R)^* R^2 \phi(R).
\]

Similarly we rewrite \( R_1^2 \) as

\[
R_1^2 = (R_1 \phi_1(R_1))^* (R_1 \phi_1(R_1)) = \phi^*(R) R T^* T R \phi(R),
\]

where the last equality follows from the definition of \( T \) in (3.9).
Therefore identity $R^2 - R_1^2 = pp^*$ can be translated to
\[ \phi^*(R) R (I - T^* T) R \phi(R) = pp^* \]
Since $R \phi(R)$ has trivial kernel, using (3.15) we arrive at
\[ I - T^* T = (3p_\ell q) (3p_\ell q)^* . \]  
(3.18)
Apply both sides of (3.18) on $x$ and take inner product with $x$, we get
\[ \| x \|^2 - \| Tx \|^2 = \langle x, 3p_\ell q \rangle^2 . \]
Replace $x$ by $T(x, (T^2 x, \ldots , (T)^{n-1} x)$, and summing up all $n$ equations, we get
\[ \| x \|^2 - \| T^n x \|^2 = \sum_{i=0}^{n-1} \langle 3p_\ell q, T^i x \rangle^2 . \]
Now take $n \to \infty$ and using the asymptotic stability of $T$, we have
\[ \| x \|^2 = \sum_{n=0}^{\infty} \langle T^n x, 3p_\ell q \rangle^2 , \]  
(3.19)
which implies that the operator $\mathcal{V} : H \to \ell^2$ defined by
\[ \mathcal{V} x := (\langle x, 3p_\ell q \rangle, \langle T x, 3p_\ell q \rangle, \langle T^2 x, 3p_\ell q \rangle, \ldots ) = \left( \langle T^k x, 3p_\ell q \rangle \right)_{k=0}^{\infty} \]  
(3.20)
is an isometry. The above identity (3.20) implies that
\[ S^* \mathcal{V} = \mathcal{V} T . \]  
(3.21)
Denote $\tilde{\mathcal{V}}$ as the operator $\mathcal{V}$ with the restricted target space on $\text{Ran} \mathcal{V}$; then the operator $\tilde{\mathcal{V}} : \mathcal{H} \to \text{Ran} \mathcal{V}$ is unitary. Trivially (3.20) implies that $\text{Ran} \mathcal{V}$ is $S^*$-invariant, so we can define $\tilde{S}^* : \text{Ran} \mathcal{V} \to \text{Ran} \mathcal{V}$ as $\tilde{S}^* := S^* |_{\text{Ran} \mathcal{V}}$. Denote by $\tilde{S}$ the adjoint of $\tilde{S}^*$, $\tilde{S} := (\tilde{S}^*)^* = P_{\text{Ran} \mathcal{V}} S |_{\text{Ran} \mathcal{V}}$. Then the identity (3.21) implies that
\[ \tilde{S}^* = \tilde{\mathcal{V}} T \tilde{\mathcal{V}}^* \quad \text{and} \quad \tilde{S} = \tilde{\mathcal{V}} T^* \tilde{\mathcal{V}}^* . \]  
(3.22)
Now we define operator $A = \tilde{\mathcal{V}} R \tilde{\mathcal{V}}^*, B = \tilde{\mathcal{V}} R_1 \tilde{\mathcal{V}}^*, \Phi = \tilde{\mathcal{V}} \phi(R) \tilde{\mathcal{V}}^*, \Phi_1 = \tilde{\mathcal{V}} \phi(R_1) \tilde{\mathcal{V}}^*$. We will construct the Hankel operator in terms of $A, B, \Phi$ and $\Phi_1$. By the unitary relation (3.22) between $S^*$ and $T$, (3.9) can be unitarily translated to
\[ B \phi_1(B) = \tilde{S}^* A \phi(A) , \]
and we also have
\[ A^2 - B^2 = \tilde{\mathcal{V}} (R^2 - R_1^2) \tilde{\mathcal{V}}^* = (\tilde{\mathcal{V}} p)^* (\tilde{\mathcal{V}} p) . \]
Here $A, B$ are positive, self-adjoint operators, hence by Proposition 3.6, there exists a conjugation $3_\Gamma$ satisfying $A 3_\Gamma = 3_\Gamma A, B 3_\Gamma = 3_\Gamma B$ and $3_\Gamma (\tilde{\mathcal{V}} p) = \tilde{\mathcal{V}} p$. In fact, we can just take $3_\Gamma := \tilde{\mathcal{V}} 3_{p_\ell} \tilde{\mathcal{V}}^*$ as
\[ A 3_\Gamma = A \tilde{\mathcal{V}} 3_{p_\ell} \tilde{\mathcal{V}}^* = \tilde{\mathcal{V}} R 3_{p_\ell} \tilde{\mathcal{V}}^* = \tilde{\mathcal{V}} 3_{p_\ell} \tilde{\mathcal{V}}^* A = 3_\Gamma A ; \]
\[ B 3_\Gamma = B \tilde{\mathcal{V}} 3_{p_\ell} \tilde{\mathcal{V}}^* = \tilde{\mathcal{V}} R_1 3_{p_\ell} \tilde{\mathcal{V}}^* = \tilde{\mathcal{V}} 3_{p_\ell} \tilde{\mathcal{V}}^* B = 3_\Gamma B ; \]
\[ 3_\Gamma (\tilde{\mathcal{V}} p) = \tilde{\mathcal{V}} 3_{p_\ell} \tilde{\mathcal{V}}^* (\tilde{\mathcal{V}} p) = \tilde{\mathcal{V}} 3_{p_\ell} p = \tilde{\mathcal{V}} p . \]
Now we define operator $\tilde{\mathcal{G}} : \text{Ran} V \to \ell^2$ as
\[ \tilde{\mathcal{G}} := \mathcal{C}(A \phi(A)) 3_\Gamma = \mathcal{C} \tilde{\mathcal{V}} R \phi(R) 3_{p_\ell} \tilde{\mathcal{V}}^* , \]  
(3.23)
and also operator $\tilde{\Gamma}_1 : \text{Ran} \, V \to \ell^2$ as

$$\tilde{\Gamma}_1 := \mathcal{C} \tilde{\nu} R_1 \phi_1 (R_1) \mathfrak{J}_p \tilde{V}^*,$$

(3.24)

We can show that $|\tilde{\Gamma}| = A$. In fact, since $\phi (R), \tilde{V}$ are all isometries,

$$|\tilde{\Gamma}| = |\mathcal{C} \tilde{\nu} \mathfrak{J}_p \phi^* (R) \tilde{V}^*| = \tilde{V} R \tilde{V}^* = A$$

Similarly, $|\tilde{\Gamma}_1| = \tilde{V} R_1 \tilde{V}^* = B$.

Now we extend $\tilde{\Gamma}$ and $\tilde{\Gamma}_1$ to $\Gamma$ and $\Gamma_1$ by taking them to be 0 on $(\text{Ran} \, V^\perp)$. This is the same as defining the operators $\Gamma, \Gamma_1$ on $\ell^2$ as

$$\Gamma := \mathcal{C} \nu \phi (R) \mathfrak{J}_p R \nu^*; \quad \Gamma_1 := \mathcal{C} \nu \phi_1 (R_1) \mathfrak{J}_p R_1 \nu^*.$$  

(3.25) \hspace{1cm} (3.26)

Let us show that

$$\Gamma_1 = \Gamma S,$$  

(3.27)  

$$S^* \Gamma = \mathcal{C} \Gamma_1,$$  

(3.28)

which implies that $\Gamma$ is a Hankel operator. Indeed, since $S^*$ commutes with the canonical conjugation $\mathcal{C}$, the identity (3.28) can be rewritten as $\mathcal{C} S^* \Gamma = \mathcal{C} \Gamma_1$, which implies that $S^* \Gamma = \Gamma_1$. Together with (3.27) this means that $\Gamma S = S^* \Gamma$, so $\Gamma$ is Hankel.

To prove (3.27), we write

$$\Gamma_1 = \mathcal{C} \nu R_1 \phi_1 (R_1) \mathfrak{J}_p \nu^*$$

by the definition (3.26) of $\Gamma_1$

$$= \mathcal{C} \nu R \phi (R) \mathfrak{J}_p \nu^*$$

by (3.14)

$$= \mathcal{C} \nu R \phi (R) (\mathfrak{J}_p \mathfrak{T}^*) \nu^*$$

by the definition of $\mathfrak{S}$ in the proposition

$$= \mathcal{C} \nu R \phi (R) \mathfrak{J}_p \nu^* S$$

by (3.22)

$$= \Gamma S$$

by the definition of $\Gamma$.

For identity (3.28), we have

$$\mathcal{C} \Gamma_1 = \nu \phi_1 (R_1) R_1 \mathfrak{J}_p \nu^*$$

by the definition of $\Gamma_1$

$$= \nu \mathfrak{T} \phi (R) R_3 \mathfrak{J}_p \nu^*$$

by (3.9)

$$= S^* \nu \phi (R) R_3 \mathfrak{J}_p \nu^*$$

by (3.22)

$$= S^* \mathcal{C} \Gamma$$

by the definition of $\Gamma$.

With (3.27) and (3.28), together with $\mathcal{C} S^* = S^* \mathcal{C}$, we have:

$$\mathcal{C} S^* \Gamma = S^* \mathcal{C} \Gamma = \mathcal{C} \Gamma_1 = \mathcal{C} \Gamma S.$$  

Hence $S^* \Gamma = \Gamma S$, and $\Gamma$ is Hankel. In addition, $\Gamma$ satisfies

$$\mathcal{C} \Gamma |_{(\text{Ker} \, \Gamma)^\perp} = \mathcal{C} \tilde{\Gamma} = \tilde{V} R \phi (R) \mathfrak{J}_p \tilde{V}^*,$$

$$\mathcal{C} \Gamma_1 |_{(\text{Ker} \, \Gamma)^\perp} = \mathcal{C} \tilde{\Gamma}_1 = \tilde{V} R_1 \phi_1 (R_1) \mathfrak{J}_p \tilde{V}^*.$$  

Since $\Gamma$ is Hankel, we know $|\tilde{\Gamma}|^2 - |\tilde{\Gamma}_1|^2 = (\Gamma^* e_0) (\Gamma^* e_0)^*$. But on the other hand, the identity $pp^* = R^2 - R_1^2$ can be unitarily translates to

$$(\tilde{V} p)^* (\tilde{V} p) = A^2 - B^2 = |\tilde{\Gamma}|^2 - |\tilde{\Gamma}_1|^2,$$

Hence $u = \Gamma^* e_0 = \tilde{V} p$. \qed
3.4.2. Uniqueness of Hankel operator $\Gamma$.

Proof. Now let’s discuss the uniqueness by showing that the Hankel symbol $\gamma_k = (\Gamma e_k, e_0)$ must have representation (3.13), which is independent of $\mathcal{V}$.

Suppose we are given Hankel operator $\Gamma$ and $\Gamma_1 = \Gamma S$ satisfying (3.10), (3.11) and (3.12). Denote

$$\tilde{\Gamma} := \Gamma|_{(\text{Ker}\Gamma)\perp}, \quad \tilde{\Gamma}_1 := \Gamma_1|_{(\text{Ker}\Gamma)\perp},$$

then by (3.10) we can write $\tilde{\Gamma}$ as

$$\tilde{\Gamma} = \mathcal{C}(\tilde{V}\phi(R)\tilde{J}_p\tilde{V}^*)\tilde{V}R\tilde{V}^*.$$ (3.29)

where $\tilde{V}R\tilde{V}^*$ is self-adjoint and $\tilde{V}\phi(R)\tilde{J}_p\tilde{V}^*$ is a conjugation. In addition two operators commute. Comparing with the generalized polar decomposition form given in Theorem 3.5, we have $|\tilde{\Gamma}| = \tilde{V}R\tilde{V}^*$. Similarly, $|\tilde{\Gamma}_1| = \tilde{V}R_1\tilde{V}^*$.

Now we want to find a conjugation $\tilde{J}_R$ that commutes with both $|\tilde{\Gamma}|, |\tilde{\Gamma}_1|$ and preserves $u$ (Here $u := \Gamma^*e_0$ given in (3.12)). Here we define the natural extension of $\tilde{V} : \mathcal{H} \to (\text{Ker}\Gamma)\perp$ to $\mathcal{V} : \mathcal{H} \to \ell^2$ (so here $\mathcal{V}$ is an isometry). Then a canonical choice for $\tilde{J}_R$ is $\tilde{J}_R := \mathcal{V}\tilde{J}_p\mathcal{V}^*$, this is because (we denote the restriction of $\tilde{J}_R$ on $(\text{Ker}\Gamma)\perp$ as $\tilde{J}_R := \mathcal{V}\tilde{J}_p\mathcal{V}^*$)

$$\tilde{J}_Ru = \tilde{J}_R(G \tilde{P}_{(\text{Ker}\Gamma)\perp}e_0) = \tilde{J}_R\tilde{V}J_p\tilde{V}^* = \tilde{V}R_1\tilde{J}_p\tilde{V}^* = u,$$

$$\tilde{J}_R|\tilde{\Gamma}| = \tilde{V}\tilde{J}_pR\tilde{V}^* = \tilde{V}R_1\tilde{J}_p\tilde{V}^* = |\tilde{\Gamma}|\tilde{J}_R,$$

$$\tilde{J}_R|\tilde{\Gamma}_1| = \tilde{V}\tilde{J}_pR_1\tilde{V}^* = \tilde{V}R_1\tilde{J}_p\tilde{V}^* = |\tilde{\Gamma}_1|\tilde{J}_R.$$

And here the second and third identity above can be trivially extended to $\tilde{J}_R|\Gamma| = |\Gamma|\tilde{J}_R, \quad \tilde{J}_R|\Gamma_1| = |\Gamma_1|\tilde{J}_R$.

Now recall how we express $\mathcal{C}\tilde{J}_R$ in equation (3.3): $\mathcal{C}\tilde{J}_R = \Phi|\Gamma|$. Thus for the restriction on $(\text{Ker}\Gamma)\perp$, we can write $\mathcal{C}\tilde{J}_R = |\Gamma|\Phi$ as

$$\mathcal{C}\tilde{J}_R = (\tilde{V}\phi(R)\tilde{J}_p\tilde{V}^*)(\tilde{V}R\tilde{V}^*)(\tilde{V}\tilde{J}_p\tilde{V}^*) = (\tilde{V}\phi(R)\tilde{V}^*)(\tilde{V}R\tilde{V}^*)$$

since $|\tilde{\Gamma}| = \tilde{V}R\tilde{V}^*$, we have: $\tilde{\Phi} = \tilde{V}\phi(R)\tilde{V}^*$. Similarly, if we write $\mathcal{C}\tilde{J}_R = \tilde{\Phi}_1|\tilde{\Gamma}_1|$, then $\tilde{\Phi}_1 = \tilde{V}\phi_1(R_1)\tilde{V}^*$.

Now denote $\tilde{S}^* = S^*|_{(\text{Ker}\Gamma)\perp}$, recall that in (3.5) we have already derived $\tilde{S}^*|\tilde{\Gamma}|\Phi = |\tilde{\Gamma}_1|\Phi_1$, thus we have

$$\tilde{S}^*\tilde{V}R\phi(R)\tilde{V}^* = \tilde{V}R_1\phi_1(R_1)\tilde{V}^*$$ (3.30)

Comparing (3.30) with $R_1\phi_1(R_1) = \mathcal{T}R\phi(R)$, we have $\tilde{S}^* = \tilde{V}\mathcal{T}\tilde{V}^*$.

If we define a new operator $\tilde{\mathcal{G}}^* : \ell^2 \to \ell^2$ as

$$\tilde{\mathcal{G}} = |\tilde{\Gamma}|^{-1}\Phi^*\Phi_1|\tilde{\Gamma}_1|,$$

then by the definition of $\mathcal{T}, \Sigma$ in (3.9), we have:

$$\tilde{J}_R\tilde{S}^* = \tilde{\mathcal{G}}^*\tilde{J}_R, \quad \tilde{\mathcal{G}}^* = \tilde{V}\Sigma\tilde{V}^*$$ (3.31)

Denote $v := \tilde{J}_Re_0$, since $q = R^{-1}(\phi(R))^*p$, we get

$$u = \Gamma^*e_0 = |\Gamma|\tilde{J}e_0 = |\Gamma|\Phi\tilde{J}_Re_0 = |\Gamma|\Phi v,$$
And also
\[
q = R^{-1}(\phi(R))^*p = R^{-1}(\phi(R))^*\tilde{\mathcal{V}}^*u = R^{-1}(\phi(R))^*\tilde{\mathcal{V}}^*|\Gamma|\Phi v \\
= R^{-1}(\phi(R))^*\tilde{\mathcal{V}}^*(\tilde{\mathcal{V}}R\tilde{\mathcal{V}}^*)(\tilde{\mathcal{V}}\phi(R)\tilde{\mathcal{V}}^*v) = \tilde{\mathcal{V}}^*v. \tag{3.32}
\]

We intend to show that the Hankel symbol defined by: \(\gamma_k = \langle (\mathcal{S}^*)^ku, e_0 \rangle\) can be expressed by \(p\) and \(q\), and doesn’t depend on \(\mathcal{V}\). Indeed, by the Hankel symbol representation (3.7) and identity (3.31) we get

\[
\gamma_k = \langle \mathcal{J}_1(\mathcal{S}^*)^ku, v \rangle = \langle (\mathcal{G}^*)^k\mathcal{J}_1u, v \rangle \\
= \langle (\mathcal{G}^*)^ku, v \rangle = \langle \mathcal{V}\Sigma^k\mathcal{V}^*\mathcal{V}p, \mathcal{V}q \rangle \\
= (\Sigma^k p, q). \tag{3.33}
\]

Hence from (3.33), the Hankel symbol only relies on the triple \(\langle R, R_1, p \rangle\) and the two unimodular function \(\phi, \phi_1\), which is independent of \(\tilde{\mathcal{V}}\). The uniqueness is proved. \(\square\)

3.4.3. Trivial kernel condition of Hankel operator \(\Gamma\).

**Proof.** As discussed above, Ker \(\Gamma = \{0\}\) if and only if \(\mathcal{V}: \mathcal{H} \to \ell^2\) defined by (3.20) satisfies \(\text{Ran } \mathcal{V} = \ell^2\), and this is equivalent to \(\mathcal{T}\) is unitary equivalent to \(S^*\).

If Ker \(\Gamma = \{0\}\), by
\[
I - SS^* = e_0 e_0^*, \quad I - T^*T = (\mathcal{J}_p q)(\mathcal{J}_p q)^*
\]
we have \(\|q\| = \|\mathcal{J}_p q\| = 1\). Now assume \(q \in \text{Ran } R\) and take \(q = Rx\), we will lead to a contradiction.

Take a vector \(f = \phi_1(R_1)(\phi(R))^*\mathcal{J}_p x\), thus \(x = \mathcal{J}_p \phi(R)(\phi(1(R_1))^*f\) and \(\mathcal{J}_p q = R\phi(R)(\phi(1(R_1))^*f\),

Hence
\[
\mathcal{T}R\phi(R)(\phi_1(R_1))^*f = \mathcal{T}(\mathcal{J}_p q) = S^*e_0 = 0.
\]

But on the other hand we have
\[
\mathcal{T}R\phi(R)(\phi_1(R_1))^* = R_1 = \phi_1(R_1)(\phi(R))^*RT^*,
\]
where the last equality follows from \(R_1\) is a self-adjoint operator. Hence we have
\[
\phi_1(R_1)(\phi(R))^*RT^*f = 0; \quad T^*f = 0,
\]
which contradicts to the fact that \(\mathcal{T}\) is an isometry. Hence \(q \notin \text{Ran } R\).

For the sufficiency part, suppose that \(\|q\| = 1\) and \(q \notin \text{Ran } R\). We show that Ker \(R_1 = \{0\}\).

If \(R_1x = 0\) for a \(x \in \mathcal{H}\), then apply \(x\) on both sides of the following equation
\[
R_1^2 = R\left(I - \langle \cdot, \phi(R)q \rangle \phi(R)q \right)R,
\]
together with Ker \(R = \{0\}\), we have
\[
Rx = \langle Rx, \phi(R)q \rangle \phi(R)q,
\]
This implies \(R(\phi(R))^*x = q\), which contradicts to the assumption \(q \notin \text{Ran } R\).

Now from the definition of \(\mathcal{T}\): \(\langle \phi_1(R_1)^*\rangle R_1 = (\phi(R))^*RT^*\), we know Ker \(\mathcal{T}^* = \{0\}\) since Ker \(R_1 = \{0\}\).

In addition, we apply \(\mathcal{J}_p q\) on both sides of (3.18) we get \(\mathcal{T}^*\mathcal{J}_p q = 0\), together with Ker \(\mathcal{T}^* = \{0\}\), we have \(\mathcal{T}\mathcal{J}_p q = 0\), hence by the definition of \(\Sigma\) in (3.31) we also get \(\Sigma q = 0\).
Now left and right multiplying (3.18) by $T$ and $T^*$ respectively, we get

$$TT^* - TTT^* = T\left(\langle \cdot, J_pq \rangle J_q \right)T^* = 0,$$

hence $TT^*$ is a projection. Furthermore, since $\text{Ker } T^* = \{0\}$, we have $TT^* = I$ and $T^*$ is an isometry.

By Wold Decomposition Theorem [3, Theorem 1.1, p. 3], there exists an orthogonal sum for the whole Hilbert space: $H = H_0 \oplus H_1$, such that $T^*|_{H_0}$ is unitary and $T^*|_{H_1}$ is a unilateral shift. Since $T$ is asymptotically stable proved in subsection 5.4 which has no unitary part, we have $H_0 = \{0\}$ and thus there exists a wandering subspace $L$, such that $H = \bigoplus_{n=0}^{\infty} (T^*)^n L$.

To show that $T^*x$ for an arbitrary $x$ into (3.18), we get

$$\langle T^*x, J_pq \rangle J_q = T^*x - T^*(TT^*)x = T^*x - T^*x = 0,$$

hence $J_pq \perp \text{Ran } T^*$, and $L$ is the space spanned by $J_pq$, which is of dimension 1. This implies $T^*$ is unitarily equivalent to $S$, and $\text{Ker } \Gamma = \{0\}$. \hfill $\Box$

4. Asymptotic stability

Recall that in Proposition 2.3 and Proposition 3.11, we both require the asymptotic stability of a defined contraction $T$. Proving the asymptotic stability of $T$ is usually the hardest part, however, under the assumption that $R, R_1$ is compact, we can get asymptotic stability for free.

Let us recall the setup. We had positive self-adjoint operators $R$ and $R_1$, $\text{Ker } R = \{0\}$, satisfying $R_1^2 = R^2 - pp^*$, where $\|R^{-1}p\| \leq 1$, and we also have vector $p$ is cyclic for $R$; and in this section we also assume that $R, R_1$ are compact. Besides, we are given two unimodular scalar functions $\phi, \phi_1$ which are Borel measurable, and the contraction $T$ is defined by the following identity:

$$R_1 \phi_1(R_1) = TR\phi(R)$$

(4.1)

In this section, we will prove the following proposition.

**Proposition 4.1.** Under the assumptions (i), (ii), (iii) in Proposition 3.11 for $R, R_1, p, \phi, \phi_1$, the contraction $T$ defined by $R_1 \phi_1(R_1) = TR\phi(R)$ (see Douglas Lemma 2.1) is asymptotically stable.

**Remark 4.2.** Notice that the contraction $T$ defined for the self-adjoint case: $R_1 = TR$, where we can write $R = R\mathcal{J} = R\phi(R), R_1 = R_1\mathcal{J}_1 = R_1\phi_1(R_1)$ is just the special case when $\phi_1, \phi$ are unimodular functions that only take values $\pm 1$.

4.1. Preparation. To prove the asymptotic stability of the contraction $T$ we will use the following simple lemma, which is a slight modification of [1, lemma 3.2].

**Lemma 4.3.** Let $\|T\| \leq 1$, and let $K$ be a compact operator with dense range. Assume that an operator $A$ satisfies

$$TK = KA.$$

(4.2)

If $A$ is weakly asymptotically stable, meaning that $A^n \to 0$ in the weak operator topology (W.O.T) as $n \to \infty$, then $T$ is asymptotically stable.
Proof. Iterating (4.2) we get that \( T^n K = KA^n, n \geq 1 \). Take \( x \in \mathcal{H} \). Since \( A^n \to 0 \) in W.O.T. and \( K \) is compact, we have that \( \|KA^n x\| \to 0 \).

So \( \lim_{n \to \infty} \|T^n y\| = 0 \) for all \( y \in \text{Ran } K \). Thus, we have strong convergence on a dense set, and since \( \|T^n\| \leq 1 \), we conclude (by \( \varepsilon/3\)-Theorem) that \( T^n \to 0 \) in the strong operator topology. \( \square \)

**Lemma 4.4.** For the operators \( R \) and \( T \) from (4.1), there exists a unique contraction \( A \), such that
\[
TR^{1/2} = R^{1/2}A
\]

*Proof.* Since \( R_1 \phi_1(R_1) = TR\phi(R) \), we have
\[
R_1^2 = (R_1 \phi_1(R_1))(R_1 \phi_1(R_1))^* = TR\phi(R)(\phi(R))^*RT^* = TR^2T^*.
\]

Hence by \( T^*T \leq I \), we have
\[
R^2 \geq R_1^2 = TR^2T^* \geq TTR^*TR^* = (TR^*)^2
\]

This tells us \( R \geq TR^* \) and
\[
\|R^{1/2}x\| \geq \|R^{1/2}TR^*x\| \quad \text{holds for all} \quad x \in H
\]

Thus by Douglas Lemma 2.1, we can find a contraction denoted as \( A^* \), satisfying
\[
A^*R^{1/2} = R^{1/2}T^*.
\]

Taking the adjoint for the equation above, and we finish the proof of this lemma. \( \square \)

The operator \( A \) constructed in the above Lemma 4.4 satisfies the identity (4.2) with \( K = R^{1/2} \). Since \( R^{1/2} \) is compact, Lemma 4.3 says that the weak asymptotic stability of \( A \) implies the asymptotic stability of \( T \).

### 4.2. Weak Asymptotic Stability of \( A \)

We will show below in Section 4.3 that under our assumptions the operator \( A \) is a strict contraction, meaning that \( \|Ax\| < \|x\| \) for all \( x \neq 0 \).

**Lemma 4.5.** Let \( A : \mathcal{H} \to \mathcal{H} \) be a strict contraction. Then \( A^n \to 0 \) in the weak operator topology (WOT) as \( n \to \infty \).

*Proof.* First we notice that the assumption that \( A \) is a strict contraction implies that \( A \) is completely non unitary, meaning that there is no reducing subspace of \( A \) on which \( A \) acts unitarily. But every completely non-unitary contraction admits the functional model, i.e. it is unitarily equivalent to the model operator \( M_\theta \) on the model space \( K_\theta \), where \( \theta \) is the so-called characteristic function of \( T \). Without going into details, which are not important for our purposes, we just mention that the model space \( K_\theta \) is a subspace of a vector-valued space \( L^2(E) = L^2(T, m; E) \) of square integrable (with respect to the normalized Lebesgue measure \( m \) on \( T \)) functions with values in an auxiliary Hilbert space \( E \). The model operator \( M_\theta \), to which \( A \) is unitary equivalent, is just the compression of the multiplication operator \( M_z \) by the independent variable \( z \)

\[
M_\theta f = P_{K_\theta} M_z f;
\]

recall that the multiplication operator \( M_z \) is defined by \( M_z f(z) = zf(z), z \in \mathbb{T} \).

Since trivially \( M_z \to 0 \) in the weak operator topology of \( B(L^2(T, m; E)) \) as \( n \to +\infty \), we conclude that \( M_\theta^n \to 0 \) as \( n \to +\infty \) in the weak operator topology of \( B(K_\theta) \), and so \( A^n \to 0 \) in the weak operator topology as well. \( \square \)
A is a Strict Contraction. To prove the weak asymptotic stability of A we need to investigate its structure in more detail.

We know that $R_2^2 = R^2 - pp^*$ $\leq R^2$. By the Löwner–Heinz inequality with $\alpha = 1/2$ we have that $R_1 \leq R$, so by Lemma 2.1 there exists an unique contraction $Q$ such that

$$R_1^{1/2} = QR^{1/2}$$  \hspace{1cm} (4.3)

The following simple Proposition (see [1, Lemma 3.5]) gives an expression for operator $A$.

**Proposition 4.6.** The operator $A$ from Lemma 4.4 is given by

$$A = Q^*\phi_1(R_1)Q\phi^*(R).$$

**Proof.** We calculate the representation of $R_1\phi_1(R_1)$ in two different ways. Firstly we have

$$R_1\phi_1(R_1) = TR_1^{1/2}R_1^{1/2}\phi(R) = R_1^{1/2}AR_1^{1/2}\phi(R).$$

On the other hand,

$$R_1\phi_1(R_1) = R_1^{1/2}Q^*QR_1^{1/2}\phi_1(R_1).$$

Since $\text{Ker } R = \{0\}$, combining the two equations above, we get

$$A = Q^*Q^{1/2}\phi_1(R_1)(\phi(R))^*R^{-1/2} = Q^*(R_1^{1/2}\phi_1(R_1)(\phi(R))^*R^{-1/2}) = Q^*\phi_1(R_1)Q(\phi(R))^*,$$

which finishes our proof. \hfill $\Box$

In addition, the following proposition gives the structure of $Q$.

**Proposition 4.7.** Let $H_0$ be the smallest invariant subspace of $R$ that contains $p$. Then the operator $Q$ with respect to the decomposition $H = H_0 \oplus H_0^\perp$ has the block structure

$$Q = \begin{pmatrix} Q_0 & 0 \\ 0 & I \end{pmatrix},$$

where $Q_0$ defined on $H_0$ is a strict contraction (i.e. $\|Q_0x\| < \|x\|$ for all $x \neq 0$).

**Proof.** We know that

$$R_1^2 = R^2 - pp^*,$$  \hspace{1cm} (4.5)

so $R^2$ coincides with $R_1^2$ on $p^\perp$.

One can easily see that $H_0$ is an invariant subspace for $R^2$ and for $R_1^2$, and therefore so is $H_0^\perp$. That means $H_0$ and $H_0^\perp$ are reducing subspaces for both $R^2$ and $R_1^2$, i.e. that these operators in the decomposition $H = H_0 \oplus H_0^\perp$ are block diagonal. Therefore, the same is true for $R_1^{1/2}$.

Easy to see that $R^{1/2}$ and $R_1^{1/2}$ coincide on $H_0^\perp$, which is a reducing space for both operators, so only need to show that $Q_0$ is a strict contraction.

Using (4.5), and the identity $R_1^{1/2} = QR_1^{1/2} = R_1^{1/2}Q^*$, we can write

$$R^2 - pp^* = R_1^2 = R_1^{1/2}Q^*QR_1^{1/2}Q^*Q_0^2QR_1^{1/2}.$$

Recalling that $p = Rq = R\phi(R)q$, we can rewrite the above identity as

$$R_1^{1/2} \left( R - (R_1^{1/2}\phi(R)q)(R_1^{1/2}\phi(R)q)^* \right) R_1^{1/2} = R_1^{1/2}Q^*Q_0^2QR_1^{1/2}Q^*Q_0^2QR_1^{1/2},$$

which, because $\text{Ker } R = \{0\}$, implies that

$$Q^*Q_0^2QR_1^{1/2}Q = R - (R_1^{1/2}\phi(R)q)(R_1^{1/2}\phi(R)q)^*. $$  \hspace{1cm} (4.6)
Applying both sides to \( x \), and taking the inner product with \( x \), we get
\[
(RQ^*Qx, Q^*Qx) = (Rx, x) - |(x, R^{1/2}ϕ(R)q)|^2.
\] (4.7)

Now, take \( x \) such that \( \|Qx\| = \|x\| \). Since \( \|Q\| < 1 \), this happens if and only if \( x = Q^*Qx \). The equation (4.7) can be rewritten in this case as
\[
(Rx, x) = (Rx, x) - |(x, R^{1/2}ϕ(R)q)|^2,
\]
which implies that \( x \perp R^{1/2}ϕ(R)q \). Applying equation (4.6) to such \( x \), and using again the fact that \( Q^*Qx = x \), we get that
\[
Q^*QRx = Rx.
\]
Hence set \( \mathcal{H}_1 := \{h \in \mathcal{H} : h \in \mathcal{H}, \|Qh\| = \|h\|\} = \text{Ker}(I - Q^*Q) \) is an invariant subspace for \( R \), which is orthogonal to \( ϕ(R)R^{1/2}q \). Therefore
\[
\mathcal{H}_1 \perp \overline{\text{span}}\{R^nR^{1/2}ϕ(R)q : n \geq 0\} = \overline{\text{span}}\{R^n p : n \geq 0\} = \mathcal{H}_0,
\]
and so \( Q_0 = Q|_{\mathcal{H}_0} \) is a strict contraction.

Now from Proposition 4.7, we have \( Q \) is a pure contraction since \( p \) is a cyclic vector for \( R \), thus by Proposition 4.6 we know that \( A \) is also a pure contraction.

4.4. Proof of Proposition 4.1.

Proof. To show that \( \mathcal{T} \) is asymptotically stable, it suffices to show that the operator \( A \) defined by
\[
\mathcal{T}R^{1/2} = R^{1/2}A
\]
is weakly asymptotically stable according to Lemma 4.3 and Lemma 4.4. And furthermore by Lemma 4.5, we only need to show \( A \) is a strict contraction.

On the other hand, by Proposition 4.7 we know that \( Q \) is a pure contraction since \( p \) is a cyclic vector with respect to \( R \), thus by Proposition 4.6 \( A \) is also a strict contraction. Thus we have proved the asymptotic stability of \( \mathcal{T} \).

\[\square\]

5. Abstract Borg’s theorem

Let us introduce some terminology from previous sections. Consider a triple \( R, R_1, p \), where \( R, R_1 \) are operators on a Hilbert space \( \mathcal{H} \) and \( p \in \mathcal{H} \) (we call this a triple on \( \mathcal{H} \)). We say that two triples \( R, R_1, p \) and \( \tilde{R}, \tilde{R}_1, \tilde{p} \) on \( \mathcal{H} \) respectively are unitary equivalent if there exists a unitary operator \( U : \mathcal{H} \to \mathcal{H} \) such that
\[
\tilde{R} = URU^*, \quad \tilde{R}_1 = UR_1U^*, \quad \tilde{p} = Up.
\]

Propositions 2.3 and Proposition 4.1 imply the following statement.

Proposition 5.1. Let \( \mathcal{R}, \mathcal{R}_1 \) be self-adjoint compact operators, \( \text{Ker} \mathcal{R} = \{0\} \), satisfying (2.4), and \( p \) is cyclic with respect to \( \mathcal{R} \). Further assuming that both \( \mathcal{R}, \mathcal{R}_1 \) have simple nonzero eigenvalues, and denote them as \( \{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty} \) respectively, which satisfy the intertwining relation (1.1), then there exists a unique self-adjoint Hankel operator \( Γ \) such that the triple \( Γ|_{(\ker Γ)^\perp}, Γ_1|_{(\ker Γ)^\perp}, u := Γe_0 = ΓP_{(\ker Γ)^\perp} e_0 \) is unitarily equivalent to the triple \( \mathcal{R}, \mathcal{R}_1, p \).

Moreover, \( \text{Ker} Γ = \{0\} \) if and only if
\[
\|\mathcal{R}^{-1}p\| = 1, \quad \|\mathcal{R}^{-2}p\| = \infty
\]
For the non self-adjoint setting, Proposition 3.11 and Proposition 4.1 implies the following Proposition

**Proposition 5.2.** Let $R, R_1, p$ be a triple satisfying the following assumptions:

1. $R, R_1$ are self-adjoint positive operators, with Ker $R = \{0\}$;
2. $R^2 - R_1^2 = pp^*$, while $p \in \text{Ran} R$ with property $\|R^{-1}p\| \leq 1$;
3. $p$ is a cyclic vector for $R$;
4. $R, R_1$ are compact operators, with simple eigenvalues $\{s_k\}_{k=1}^{\infty}, \{t_k\}_{k=1}^{\infty}$ which satisfy the intertwining relationship

\[ s_1 > t_1 > s_2 > t_2 > \ldots \rightarrow 0 \quad (5.1) \]

and let $\phi, \phi_1$ be two unimodular Borel measurable functions, such that $\phi(R), \phi_1(R_1)$ has simple eigenvalues $\{e^{i\theta_k}\}_{k=1}^{\infty}, \{e^{i\theta_k}\}_{k=1}^{\infty}$ respectively.

Denote by $J_p$ the unique conjugation satisfying

\[ J_p R = R J_p, \quad J_p R_1 = R_1 J_p, \quad J_p p = p; \]

( the existence and uniqueness of such $J_p$ follow from Proposition 3.6)

Then there exists a unique Hankel operator $\Gamma$ such that, the triple of anti-linear operator $\mathcal{C} \Gamma|_{(\text{Ker} \, \Gamma)^{\perp}}, \mathcal{C} \Gamma_1|_{(\text{Ker} \, \Gamma)^{\perp}}$, $u := \Gamma^* e_0$ is unitary equivalent to the triple $R \phi(R) J_p, R_1 \phi_1(R_1) J_p, p$.

Here $\mathcal{C}$ is the standard conjugation on $\ell^2$ defined by (3.1).

In addition, Ker $\Gamma = \{0\}$ iff

\[ \|R^{-1}p\| = 1, \quad \|R^{-2}p\| = \infty \]

For this section, we want to show that the two sequences of spectral characteristics will uniquely determine a Hankel operator. That is, i.e., in the self-adjoint setting regarding proposition 5.1, we need to show that the intertwining two real sequences $\{\lambda_k\}_{k=1}^{\infty}, \{\mu_k\}_{k=1}^{\infty}$ will uniquely determine such triple $(R, R_1, p)$ up to unitary equivalence. While for the non self-adjoint setting regarding Proposition 5.2, we will show that the two complex sequences $\{\lambda_k\}_{k=1}^{\infty} = \{s_k e^{i\theta_k}\}_{k=1}^{\infty}, \{\mu_k\}_{k=1}^{\infty} = \{t_k e^{i\theta_k}\}_{k=1}^{\infty}$ where the module $\{s_k\}_{k=1}^{\infty}, \{t_k\}_{k=1}^{\infty}$ satisfy the intertwining relation (5.1), can uniquely determine $R, R_1, p, \phi, \phi_1$ up to unitary equivalence.

We need the following Abstract Borg’s Theorem, which will be proved at the end of this section.

**Theorem 5.3 (Abstract Borg’s Theorem).** Given two sequences $\{\lambda_k^2\}_{k \geq 1}$ and $\{\mu_k^2\}_{k \geq 1}$ satisfying intertwining relations (1.1) and such that $\lambda_k^2 \rightarrow 0$ as $k \rightarrow \infty$, there exists an unique (up to unitary equivalence) triple $W, W_1, p$, such that $W = W^* \geq 0$, Ker $W = \{0\}$ is a compact operator with simple eigenvalues $\{\lambda_k^2\}_{k=1}^{\infty}$, and the operator $W_1 = W - pp^*$ has $\{\mu_k^2\}_{k=1}^{\infty}$ as its (simple) non-zero eigenvalues ($W_1$ can also have a simple eigenvalue at 0).

Moreover, $\|W^{-1/2}p\| = 1$ if and only if

\[ \sum_{j=1}^{\infty} \left( 1 - \frac{\mu_j^2}{\lambda_j^2} \right) = \infty, \quad (5.2) \]

and $\|W^{-1}p\| = \infty$ if and only if

\[ \sum_{j=1}^{\infty} \left( \frac{\mu_j^2}{\lambda_j^2} - 1 \right) = \infty. \quad (5.3) \]
Remark 5.4. The original Borg’s theorem [5] states that the potential $q$ of a Schrödinger operator $L$, $Ly = y'' + q(x)y$ on an interval is uniquely defined by the two sets of eigenvalues, corresponding to two specific boundary conditions. Later Levinson [6] extended this result by showing that essentially any non-degenerate pair of self-adjoint boundary conditions would work.

Changing boundary conditions for a Schrödinger operator is essentially a rank one perturbation (by an unbounded operator). Namely, if $L_1$ and $L_2$ are Schrödinger operators on an interval with the same potential, but with two different self-adjoint boundary conditions, then for any $\lambda \notin \sigma(L_1) \cup \sigma(L_2)$ the difference $(L_1 - \lambda I)^{-1} - (L_2 - \lambda I)^{-1}$ is a rank one operator (and the operators $(L_1 - \lambda I)^{-1}$, $(L_2 - \lambda I)^{-1}$ are compact). Thus, by picking a real $\lambda$ the problem can be reduced to rank one perturbations of compact self-adjoint operators.

Our Theorem 5.3 deals with rank one perturbations of (abstract) compact self-adjoint operators, hence the name. We do not assume that our operators came from Schrödinger operators, so we only reconstructing the spectral measure, and are not concerned with the reconstruction of the potential. However, it is well known how to reconstruct the potential from the spectral measure, or, more precisely, from the Titchmarsh–Weyl $m$-function, so it should be possible to get the Borg’s result from our abstract theorem.

Note also, that Theorem 5.3 gives not only uniqueness, but the existence as well.

Remark 5.5. Notice that the triple $(W, W_1, p)$ we constructed in Theorem 5.3 satisfies that $\text{Proj}_{\text{Ker}(W - \lambda_k^2)} p \neq 0$ for all $k$, hence the cyclicity of $p$ with respect to $W$ is automatically satisfied. In fact, if there is a $k$ such that

$$p_k := \text{Proj}_{\text{Ker}(W - \lambda_k^2)} p = 0$$

Then $\lambda_k^2$ is a common eigenvalue for $W, W_1$, which gives us contradiction.

5.1. Abstract Borg’s Theorem Implies Theorem 1.4. In this part, we show how to get triple $(\mathcal{R}, \mathcal{R}_1, p)$ from $(W, W_1, p)$.

Proof. By Theorem 5.3, We can find a triple $(W, W_1, p)$ with the relation: $W_1 = W - pp^*$, here $W$ and $W_1$ are positive, compact operators, and their non-zero eigenvalues are simple which coincide with $\{\lambda_k^2\}_{k=1}^\infty$, $\{\mu_k^2\}_{k=1}^\infty$ respectively. In addition, the triple is unique up to unitary equivalence.

Denote that the eigenvector of $W$ corresponding to $\lambda_k^2$ is $u_k$, and the eigenvector of $W_1$ corresponding to $\mu_k^2$ is $v_k$:

$$W u_k = \lambda_k^2 u_k, \quad W_1 v_k = \mu_k^2 v_k$$

Then we define $\mathcal{R}$ and $\mathcal{R}_1$ by

$$\mathcal{R} u_k := \lambda_k u_k; \quad \mathcal{R}_1 v_k := \mu_k v_k,$$

$$\mathcal{R}_1 |_{\text{Ker} W_1} = 0$$

Here we have $\text{Ker} \, \mathcal{R} = \{0\}$ and $\text{Ker} \, \mathcal{R}_1 = \text{Ker} W_1$, and we can express $\mathcal{R}$ and $\mathcal{R}_1$ by

$$\mathcal{R} x = \sum_k \lambda_k \text{Proj}_{u_k} x, \quad \mathcal{R}_1 x = \sum_k \mu_k \text{Proj}_{v_k} x$$

Hence we get the unique square roots of $W$ and $W_1$ according to the given signs of $\{\lambda_k\}_{k=1}^\infty$, $\{\mu_k\}_{k=1}^\infty$.

Now we have the triple $(\mathcal{R}, \mathcal{R}_1, p)$ satisfies the relation (2.4), and the contraction $\mathcal{T}$ defined by $\mathcal{R}_1 = \mathcal{T} \mathcal{R}$ is asymptotically stable. Hence by Proposition 5.1, there exists an unique
Hankel operator $\Gamma$ such that the triple $\Gamma|_{(\ker \Gamma)\perp}$, $\Gamma_1|_{(\ker \Gamma)\perp}$, $u := \Gamma e_0 = \Gamma P_{(\ker \Gamma)\perp} e_0$ is unitarily equivalent to the triple $R$, $R_1$, $p$, this means that the non-zero eigenvalues of $\Gamma$ and $\Gamma_1$ are simple and coincide with $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$ respectively.

As for the uniqueness, assume that $(\Gamma, \Gamma_1, u := \Gamma e_0)$ and $(\Gamma', \Gamma'_1, u' := \Gamma' e_0)$ are two different triples $(\mathcal{R}, \mathcal{R}_1, p)$, $(\mathcal{R}', \mathcal{R}_1', p)$ on the same space such that $(\Gamma, \Gamma_1, u)$ unitary equivalent to $(\mathcal{R}, \mathcal{R}_1, p)$, and $(\Gamma', \Gamma'_1, u')$ unitary equivalent to $(\mathcal{R}', \mathcal{R}_1', p)$, where $(\mathcal{R}, \mathcal{R}_1)$ and $(\mathcal{R}', \mathcal{R}_1')$ share the same spectral characteristics.

From the equation

$$\mathcal{R}^2 - \mathcal{R}_1^2 = \mathcal{R}'^2 - \mathcal{R}_1'^2 = pp^*,$$

and the uniqueness in the abstract Borg’s theorem 5.3, we have $\mathcal{R}^2 = \mathcal{R}'^2 = W, \mathcal{R}_1^2 = \mathcal{R}_1'^2 = W_1$. If $\mathcal{R} \neq \mathcal{R}'$, then there exists a $k$, such that $\ker (R - \lambda I) \neq \ker (R' - \lambda I)$, thus $\dim \ker (W - \lambda^2 I) \geq 2$ which gives a contradiction. Similarly we have $R_1 = R'_1$, which indicates the uniqueness of such Hankel operator.

5.2. Abstract Borg’s Theorem Implies Theorem 1.5.

**Proof.** By Theorem 5.3, we can find a triple $(W, W_1, p)$ (unique up to equivalence) satisfying

(i) $W_1 = W - pp^*$, where $W, W_1$ are positive, self-adjoint operators;

(ii) $W$ has trivial kernels, $p$ is a cyclic vector for $W$;

(iii) $W, W_1$ have simple nonzero eigenvalues $\{s_k^2\}_{k=1}^{\infty}$, $\{t_k^2\}_{k=1}^{\infty}$ respectively.

Now denote the eigenvectors of $W$ corresponding to $s_k^2$ is $x_k$, and eigenvectors of $W_1$ corresponding to $t_k^2$ is $y_k$.

We define positive self-adjoint operators $R, R_1$ by

$$Rx_k := s_kx_k, \quad R_1y_k := t_ky_k.$$ 

And we define the two unimodular scalar functions $\phi, \phi_1$ on the real line $\mathbb{R}$

$$\phi(x) = \sum_{k=1}^{\infty} 1_{\{x=s_k\}} e^{i\theta_k} + 1_{\mathbb{R}/\{s_k\}} \sum_{k=1}^{\infty} 1_{\{x=s_k\}} e^{i\theta_k} , \quad (5.4)$$

$$\phi_1(x) = \sum_{k=1}^{\infty} 1_{\{x=t_k\}} e^{i\theta_k} + 1_{\mathbb{R}/\{t_k\}} \sum_{k=1}^{\infty} 1_{\{x=t_k\}} e^{i\theta_k} , \quad (5.5)$$

Then it’s easy to see that $\phi(x), \phi_1(x) = 1$ a.e, and $\phi(R), \phi_1(R_1)$ are unitary operators satisfying

$$\phi(R)x_k = e^{i\theta_k} x_k, \quad \phi_1(R_1)y_k = e^{i\theta_k} y_k.$$ 

Thus $R\phi(R), R_1\phi_1(R_1)$ are compact operators with simple eigenvalues $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ respectively.

Now we define a conjugation $J_p$ on $\mathcal{H}$. Since $\mathcal{H} = \text{span}\{R^n p | n \geq 0\}$, we define $J_p$ by

$$J_p R^n p = R^n p \quad \text{for all } n.$$ 

This is the same as defining $J_p$ by

$$J_p p_k = p_k \quad \text{for all } k,$$

where

$$p_k = P|_{\ker (W - \lambda_k^2 I)} p.$$
For all $k$ we have $s_k \mathcal{J}_p p_k = \mathcal{J}_p s_k p_k = \mathcal{J}_p R p_k = R \mathcal{J}_p p_k$, hence $\mathcal{J}_p p_k$ is also an eigenvector for $R$. Since $s_k$ is a simple eigenvalue, we have $\mathcal{J}_p p_k = l_k p_k$ holds for a constant $l_k \in \mathbb{C}$. On the other hand we write

$$p = \mathcal{J}_p p = \sum_{k=1}^{\infty} \mathcal{J}_p p_k = \sum_{k=0}^{\infty} l_k p_k$$

This implies $l_k = 1$ and $\mathcal{J}_p p_k = p_k$ holds for all $k$.

Thus we apply the tuple $(R, R_1, p, \phi, \phi_1, \mathcal{J}_p)$ to Proposition 3.11, there exists a unique Hankel $\Gamma$ and a unitary $\tilde{V}$ satisfying

$$\mathcal{C}_\Gamma|_{(\text{ker } \Gamma)^\perp} = \tilde{V} R \phi(R) \mathcal{J}_p \tilde{V}^*,
$$

$$\mathcal{C}_{\Gamma_1}|_{(\text{ker } \Gamma)^\perp} = \tilde{V} R_1 \phi_1(R_1) \mathcal{J}_p \tilde{V}^*,$$

$$\Gamma^* e_0 = \tilde{V} p.$$

Using the same notation $\tilde{\Gamma}, \tilde{\Gamma}_1, \tilde{\mathcal{J}}_p, \tilde{\mathcal{J}}_1$ as in section 3.4.2, from the proof in section 3.4.2, we know

$$|\tilde{\Gamma}| = \tilde{V} R \tilde{V}^*, \quad |\tilde{\Gamma}_1| = \tilde{V} R_1 \tilde{V}^*;
$$

$$\tilde{\mathcal{J}}_p = \tilde{V} \mathcal{J}_p \tilde{V}^*, \quad \mathcal{J}_1 = \mathcal{V} \mathcal{J}_p \mathcal{V}^*$$

where $\mathcal{V} : \mathcal{H} \to \ell^2$ is the natural extension of $\tilde{V}$.

Hence $\mathcal{C}_\tilde{\Gamma} \mathcal{J}_p = \mathcal{V} R \phi(R) \mathcal{V}^*$ has simple eigenvalues $\{\lambda_k\}_{k=1}^\infty$. Equivalently saying, if we take vector $p_k := \text{Proj}_{\text{ker}(R - s_k I)} p$, and $u_k := \mathcal{V} p_k = \text{Proj}_{\text{ker}(\Gamma - s_k I)} u$, then we have

$$R \phi(R) p_k = \lambda_k p_k, \quad \mathcal{C}_\tilde{\Gamma} \mathcal{J}_p u_k = \lambda_k u_k. \tag{5.6}$$

And a similar result holds for $R_1 \phi_1(R_1)$ and $\mathcal{C}_\tilde{\Gamma} |_{\tilde{\mathcal{J}}_1}$.

Then it’s easy to see that the two conjugate linear operators $R \phi(R) \mathcal{J}_p$, $\mathcal{C}_\tilde{\Gamma}$ satisfy

$$R \phi(R) \mathcal{J}_p p_k = \lambda_k p_k, \quad \mathcal{C}_\tilde{\Gamma} u_k = \lambda_k u_k$$

since $\mathcal{J}_p p_k = p_k$ holds for all $k$ and thus, $\mathcal{J}_1 u_k = \mathcal{V} \mathcal{J}_p p_k = V p_k = u_k$.

And a similar result holds for $R_1 \phi_1(R_1) \mathcal{J}_p$ and $\mathcal{C}_\tilde{\Gamma}$.

Now consider the uniqueness. We only need to show that the set $R, R_1, p, \phi(R), \phi_1(R_1)$ is uniquely determined (all the uniqueness of tuple is defined up to unitary equivalence, which will be omitted without further notice). According to abstract Borgs theorem, the triple $(W, W_1, p)$ is uniquely determined by the intertwining module sequence $\{s_k\}, \{t_k\}$, thus $R = W^{1/2}, R_1 = W_1^{1/2}$ are uniquely determined. We only need to show that $\phi(R)$ and $\phi_1(R_1)$ are unique.

Now according to (3.10), for all $k$, the identity $\mathcal{C}_\Gamma|_{(\text{ker } \Gamma)^\perp} u_k = \lambda_k u_k$ can be unitary translates to $R \phi(R) \mathcal{J}_p p_k = \lambda_k p_k$ where $p_k, u_k$ are defined in (5.6). Recalling that we have already showed in the existence part that $\mathcal{J}_p p_k = p_k$, then together with the requirement that $R$ commutes with $\phi(R)$, we know that for $\forall k$,

$$\lambda_k p_k = \phi(R) R p_k = s_k \phi(R)p_k, \quad \phi(R)p_k = e^{i\varphi_k} p_k$$

Thus $\phi(R)$ is uniquely determined on $\mathcal{H}$. Similarly we also have $\phi_1(R_1)$ is also unique, which finishes the uniqueness of such Hankel $\Gamma$. \qed
5.3. **Proof of the Abstract Borg’s Theorem: Existence and Uniqueness Part.** First of all notice that the intertwining condition (1.1) implies that the vector $p$ must be cyclic for $W$. Since everything is defined up to unitary equivalence, we can then assume without loss of generality that $W$ is the multiplication $M_s$ by the independent variable $s$ in the weighted space $L^2(\rho)$, where $\rho$ is the spectral measure, corresponding to the vector $p$.

Recall, that this spectral measure $\rho(s)$ can be defined as

$$\rho(s) = \int d\rho(s) s - z.$$  (5.7)

Since $W$ is a compact operator with eigenvalues $\{\lambda_k^2\}_{k \geq 1}$, the measure $\rho$ is purely atomic,

$$\rho = \sum_{k \geq 1} a_k \delta_{\lambda_k^2}, \quad a_k > 0.$$  

Note also that in this representation the vector $p$ is represented by the function 1 in $L^2(\rho)$.

Since the choice of $a_k$ can uniquely determine the triple $W, W_1, p$ up to unitary equivalence. We want to find out values of $\{a_k\}_{k=1}^\infty$ according to the given intertwining sequence.

5.3.1. **Cauchy Transform of Spectral Measure.** Let us recall some definitions here. Given a self-adjoint operator $W$, and a vector $p$, then the scalar spectral measure of $W$ with respect to $p$ is given by

$$F(z) = ((W - zI)^{-1}p, p) = \int \frac{d\rho(s)}{s - z}.$$  (5.8)

The following proposition gives the Cauchy transform of an operator under a rank one perturbation.

We will need the following well known result in perturbation theory, cf. [7, Chapter 9], [4, Theorem 5.8.1, p. 335]

**Proposition 5.6** (Aronszajn-Krein formula). Assume that $W^\alpha$ is a rank-one perturbation of operator $W$: $W^\alpha = W + \alpha pp^*$, then the Borel transform of $W$ and $W^\alpha$ defined as

$$F(z) = ((W - zI)^{-1}p, p), \quad F^\alpha(z) = ((W^\alpha - zI)^{-1}p, p)$$

have the following relation

$$F^\alpha(z) = \frac{F(z)}{1 + \alpha F(z)}.$$  

Back to our main problem. Denote the Cauchy transform of $W$ and $W_1$ with respect to $p$ to be

$$F(z) = ((W - zI)^{-1}p, p) = \int \frac{d\rho(s)}{s - z}, \quad (5.9)$$

$$F_1(z) = ((W_1 - zI)^{-1}p, p) = \int \frac{d\rho_1(s)}{s - z}.$$  (5.10)

Applying Proposition 5.6 and take $\alpha = -1$, recognizing $F_1(z)$ the same as $F^{-1}(z)$, and $W^{-1}$ as $W_1$ we have

$$1 - F = \frac{F}{F_1}.$$  (5.11)
5.3.2. Guess and Proof for Function F. In order to find out values for \( \{a_k\}^\infty_{k=1} \), we need to give a guess for the function F.

Denote \( \sigma = \{\lambda_k\}^\infty_{k=1} \cup \{0\} \) and \( \sigma_1 = \{\mu_k\}^\infty_{k=1} \cup \{0\} \), then we know \( F(z) \) has simple poles at \( \{\lambda_k\}^\infty_{k=1} \) and analytic at \( C \setminus \sigma \), \( F_1(z) \) has simple poles at \( \{\mu_k\}^\infty_{k=1} \) and analytic at \( C \setminus \sigma_1 \). Thus \( 1 - F = \frac{F}{F_1} \) should be a function which is analytic function on \( C \setminus \sigma \), and has simple zeros at \( \{\mu_k\}^\infty_{k=1} \) and simple poles at \( \{\lambda_k\}^\infty_{k=1} \).

In the following part of this section, we will prove our guess for the function F, which is:

**Proposition 5.7.** We have the following equation for function F hold

\[
1 - F(z) = \frac{F}{F_1} = \prod_{k=1}^\infty \left( \frac{z - \mu^2_k}{z - \lambda^2_k} \right). \tag{5.12}
\]

We firstly prove that the right hand side of (5.12) converges uniformly on compact subset of \( C \setminus \sigma \).

**Lemma 5.8.** \( \Phi_N(z) = \prod_{k=1}^N \left( \frac{z - \mu^2_k}{z - \lambda^2_k} \right) \) converges uniformly on compact subset of \( C \setminus \sigma \).

**Proof.** We use the trivial fact that the convergence of \( \sum_{k=0}^\infty |f_k(z) - 1| \) implies the convergence of \( \prod_{k=0}^\infty f_k(z) \) (convergence here always means the uniform convergence on compact subset).

Since \( |1 - \frac{z - \mu^2_k}{z - \lambda^2_k}| \) it is sufficient to show that \( \sum_{k=1}^\infty |\frac{\mu^2_k - \lambda^2_k}{z - \lambda^2_k}| \) converges.

For \( z \) in a compact \( K \subset C \setminus \sigma \), and sufficiently large \( k \) (i.e. for all \( k > N \))

\[
|\frac{\mu^2_k - \lambda^2_k}{z - \lambda^2_k}| \leq C(K,N)|\lambda^2_k - \mu^2_k| \leq C(K,N)(\lambda^2_k - \lambda^2_{k-1}),
\]

Thus:

\[
\sum_{k=N}^\infty |\frac{\mu^2_k - \lambda^2_k}{z - \lambda^2_k}| \leq C(K,N)\lambda_N^2 < \infty,
\]

and the convergence follows trivially. \( \square \)

Now denote \( \lim_{N \to \infty} \Phi_N(z) \to \Phi(z) \) on \( C \setminus \sigma \). Before we introduce several properties for this function \( \Phi(z) \), let us recall the definition of Nevanlinna function and its integral representation.

**Definition 5.9.** Nevanlinna function is an analytic function on the open half plane and image has non-negative imaginary part.

**Theorem 5.10** (Integral representation). Every Nevanlinna function \( f \) admits a following integral representation

\[
f(z) = C + Dz + \int_R \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda) \tag{5.13}
\]

where, \( C \) is a real constant, \( D \) is non-negative, \( \mu \) is a Borel measure on \( R \) satisfying \( \int_R \frac{d\mu(\lambda)}{1 + \lambda^2} < \infty \).

Conversely, if a function has this type of form, then it’s a Nevanlinna function, and the representation is unique.
Now let us get back to discuss function \( \Phi(z) \).

**Lemma 5.11.** Function \( \Phi(z) \) satisfies the following properties:

(i) \( \lim_{z \to \infty} \Phi(z) = 1 \);

(ii) function \( -\Phi(z) \) is a Nevanlinna function, restricted on \( \mathbb{C}^+ \);

(iii) \( \Phi(z) \) is symmetric, i.e. \( \Phi(z) = \Phi(\overline{z}) \), in particular, \( \Phi(z) \) is real for all \( x \in \mathbb{R} \setminus \sigma \);

(iv) \( \Phi(z) \) has simple poles at \( \{ \lambda_k^2 \}_{k=1}^\infty \), simple zeros at \( \{ \mu_k^2 \}_{k=1}^\infty \).

**Proof.** (i) This is because

\[
\sum_{k=1}^\infty \log \left| \frac{z - \mu_k^2}{\lambda_k^2 - \lambda_k^2} \right| \leq \sum_{k=1}^\infty \left| \frac{\lambda_k^2 - \mu_k^2}{z - \lambda_k^2} \right|
\]

\[
\leq \sum_{k=1}^\infty \left| \frac{\lambda_k^2 - \lambda_{k+1}^2}{|z| - |\lambda_k|^2} \right|
\]

\[
\leq \frac{|\lambda_1^2|}{|z| - |\lambda_1|^2}
\]

goes to 0 when \( z \to \infty \), hence \( \lim_{z \to \infty} \Phi(z) = 1 \).

(ii) It’s equivalent to show that \( \frac{1}{\Phi(z)} = \prod_{i=1}^\infty \left( \frac{z - \lambda_i^2}{z - \mu_i^2} \right) \) is a Nevanlinna function on \( \mathbb{C}^+ \). Only need to show that \( 0 < \arg \frac{1}{\Phi(z)} < \pi \).

Denote \( Z \) to be the point \( z \), and \( A_1, A_2, \ldots \) to be the point of sequence \( \{ \lambda_n \} \) on the real axis, and \( B_1, \ldots, B_n, \ldots \) to be \( \{ \mu_n \} \). Then \( \arg \frac{1}{\Phi(z)} \) is given by

\[
\angle B_1 Z A_1 + \ldots + \angle B_n Z A_n < \pi,
\]

While this is trivially true because \( \{ \lambda_n \} \) and \( \{ \mu_n \} \) are two interwining sequences, and all those angles don’t intersect with each other.

While (iii) and (iv) are trivially true. \( \square \)

Now we prove the following property.

**Lemma 5.12.** The function \( \Phi(z) \) given above is the only function that satisfies (i),(ii),(iii), and (iv) in Lemma 5.11.

**Proof.** Assume that \( \Phi_1(z) \) is another function that satisfies those properties. Denote their ratio to be: \( \Psi := \Phi_1/\Phi \). Additionally, we have both functions have simple poles at \( \mu_n \) and simple zeros at \( \lambda_n \), hence \( \Psi(z) \) is analytic and zero-free in \( \mathbb{C} \setminus \{0\} \).

In addition, we have \( \lim_{z \to \infty} \Psi(z) = 1 \) since both \( \Phi(z), \Phi_1(z) \) satisfy (i) in Lemma 5.11.

Moreover, for \( x \in \mathbb{R} \setminus \{0\} \) we have \( \Psi(x) > 0 \). Indeed, on \( \mathbb{R} \setminus \sigma \setminus \sigma_1 \) functions \( \Phi_1 \) and \( \Phi \) are real and have the same sign (If \( \Phi(x), \Phi_1(x) \) have different signs for a certain \( x \in \mathbb{R} \setminus \sigma \setminus \sigma_1 \), then \( \lim_{x \to \pm \infty} \Phi(x) \) and \( \lim_{x \to \pm \infty} \Phi_1(x) \) have different signs, which gives a contradiction to (i) in Lemma 5.11), so \( \Psi(x) > 0 \) on \( \mathbb{R} \setminus \sigma \setminus \sigma_1 \). Since \( \Psi \) is continuous and zero-free on \( \mathbb{R} \setminus \{0\} \), this tells us that \( \Psi \) is positive on \( \mathbb{R} \setminus \{0\} \).

Next, let us notice that \( \Psi(z) \) does not take real negative values. If \( \text{Im} \, z > 0 \), then according to (ii) in Lemma 5.11, \( \text{Im} \, \Phi_1(z) < 0, \text{Im} \, \Phi(z) < 0 \), so \( \Psi(z) = \Phi_1(z)/\Phi(z) \) cannot be negative real. If \( \text{Im} \, z < 0 \), the symmetry \( \Psi(\overline{z}) = \overline{\Psi(z)} \) implies the same conclusion. And, as we just discussed above, on the real line \( \Psi \) takes positive real values.
So $\Psi$ omits infinitely many points, therefore by the Picard’s Theorem, 0 is not the essential singularity for $\Psi$. Trivial analysis shows that 0 cannot be a pole, otherwise $\frac{1}{\Psi}$ is analytic at 0, which also contradicts to the fact that $\Psi$ can’t take negative real values on real axis. Hence we have 0 is a removable singularity for function $\Psi$, and $\Psi$ is an entire function. By Liouville’s Theorem, condition $\Psi(\infty) = 1$ implies that $\Psi \equiv 1$ for all $z \in \mathbb{C}$, hence the lemma is proved.

$\square$

**Remark 5.13.** For the last paragraph of the proof for Lemma 5.12, we can also consider the square root $\Psi^{1/2}$, where we take the principal branch of the square root (cut along the negative half-axis). Then $\text{Re} \Psi(z)^{1/2} \geq 0$, so by the Casorati–Weierstrass Theorem 0 cannot be the essential singularity for $\Psi^{1/2}$. Again, trivial reasoning shows that 0 cannot be a pole, so again, $\Psi^{1/2}$ is an entire function. The condition $\Psi^{1/2}(\infty) = 1$ then implies that $\Psi^{1/2}(z) \equiv 1$.

Now back to the proof of Proposition 5.7.

**Proof of Proposition 5.7.** At the beginning of this section, we have mentioned that $\frac{F}{F_1}$ is a function which is analytic on $\mathbb{C} \setminus \sigma$, with simple poles at $\{\lambda_k\}_{k=1}^{\infty}$ and simple zeros at $\{\mu_k\}_{k=1}^{\infty}$, and equals 1 at $\infty$, which corresponds to property (i), (iv) in Lemma 5.11.

Now we can show that function $\frac{F}{F_1}$ also satisfies the property (ii), (iii) in Lemma 5.11.

In fact, for property (ii), to show that $\frac{F}{F_1}$ maps $\mathbb{C}^+$ to $\mathbb{C}^-$, according to (5.11) it's equivalent to show that $F(z)$ maps $\mathbb{C}^+$ to $\mathbb{C}^-$. This is trivially true, because we have shown that

$$F(z) = \sum_{k \geq 1} \frac{a_k}{\lambda_k^2 - z}$$

is analytic on $\mathbb{C}/\sigma$, and each single term $\frac{a_k}{\lambda_k^2 - z}$ has positive imaginary part if $\text{Im}(z) > 0$.

As for property (iii), to show $\frac{F}{F_1}$ is symmetric, it is equivalent to show that $F(z)$ is symmetric, which is also trivially true from (5.14).

So far we have shown that $\frac{F}{F_1}$ and $\Phi(z)$ are two functions that satisfy all four properties (i), (ii), (iii), (iv) given in Lemma 5.11. Hence those two functions coincide, and we proved Proposition 5.7. $\square$

5.3.3. **Coefficients for the scalar spectral measure.** In this section, we calculate the coefficients for the scalar spectral measure $\rho$. We will show the following lemma.

**Lemma 5.14.** The function $\Phi$ defined by (5.12) can be decomposed as

$$\Phi(z) = 1 - \sum_{n \geq 1} \frac{a_n}{\lambda_n^2 - z},$$

where

$$a_n = (\lambda_n^2 - \mu_n^2) \prod_{k \neq n} \left(\frac{\lambda_n^2 - \mu_k^2}{\lambda_n^2 - \lambda_k^2}\right).$$

**Proof.** Consider functions $\Phi_N$ defined as

$$\Phi_N(z) = \prod_{k=1}^{N} \left(\frac{z - \mu_k^2}{z - \lambda_k^2}\right).$$

Trivially

$$\Phi_N(z) = 1 - \sum_{n \geq 1} \frac{a_n^N}{\lambda_n^2 - z}$$

(5.16)
where
\[ a_N^N = (\lambda_n^2 - \mu_n^2) \prod_{k=1 \atop k \neq n}^N \left( \frac{\lambda_n^2 - \mu_k^2}{\lambda_n^2 - \lambda_k^2} \right) > 0 \quad \text{if } n \leq N, \] (5.17)

and \( a_n^N = 0 \) if \( n > N \). This is because both functions have the same poles and the same residues, so their difference is a polynomial. Then let \( z \to \infty \), we know that polynomial equals 0 at \( \infty \). Hence those two functions are equal.

Next, let \( N \to \infty \). We know, see Lemma 5.8 that \( \Phi_N(z) \) converges uniformly to \( \Phi(z) \) in any compact subset \( K \subset \mathbb{C} \setminus \sigma \). Hence, to prove the lemma it remains to show that \( 1 - \sum_{n \geq 1} a_n^N/\lambda_n^2 - z \) converges to \( 1 - \sum_{n \geq 1} a_n/\lambda_n^2 - z \) uniformly.

Take \( z = 0 \) in (5.16). Then we have
\[ 1 - \sum_{n \geq 1} a_n^N/\lambda_n^2 \geq N \prod_{k=1}^n \left( \frac{\mu_k^2}{\lambda_k^2} \right) > 0, \]
so \( \sum_{n \geq 1} a_n^N/\lambda_n^2 \leq 1. \)

Notice that for any fixed \( n \) the sequence \( a_n^N \to a_n \) as \( N \to \infty \), so \( \sum_{n \geq 1} a_n^N/\lambda_n^2 \leq 1. \)

Take an arbitrary compact \( K \subset \mathbb{C} \setminus \sigma \). Clearly for any \( z \in K \),
\[ \left| \frac{a_n^N}{\lambda_n^2 - z} \right| \leq \frac{a_n^N}{\text{dist}(K, \sigma)} \leq \frac{a_n}{\text{dist}(K, \sigma)} \leq \frac{\lambda_n^2}{\text{dist}(K, \sigma)} \cdot \frac{a_n}{\lambda_n^2}, \]
so the condition \( \sum_{n \geq 1} a_n/\lambda_n^2 \leq 1 \) implies that the series \( \sum_{n \geq 1} a_n^N/\lambda_n^2 - z \) converges uniformly on \( K \).

**Remark 5.15.** We also need to calculate the coefficients of \( \rho_1(s) \) (which is the scalar spectral measure of \( W_1 \) with respect to \( p \) given in (5.10)) for a future use. However, this is slightly different from calculating \( \rho(s) \) since \( \text{Ker } W = \{0\} \) but \( \text{Ker } W_1 \) can be non-trivial. We will go back to this calculation in Proposition 5.24.

**5.3.4. Existence and uniqueness of the triple \( W, W_1, p \).** Define a measure \( \rho(s) \) as
\[ \rho = \sum_{k=1}^\infty a_k \delta_{\lambda_k^2}, \]
where \( \{a_k\}_k=1^\infty \) is defined by (5.15). Let \( W \) be the multiplication by independent variable in \( L^2(\rho) \), and let \( p \equiv 1 \). Clearly \( W \) is a positive compact operator with simple eigenvalues \( \{\lambda_k^2\}_k=1^\infty \), and its Borel transform
\[ F(z) = \left( (W - zI)^{-1}p, p \right) \]
is given by
\[ F(z) = \sum_{k=1}^\infty \frac{a_k}{\lambda_k^2 - z}. \]
Take \( W_1 = W - pp^* \), and let \( F_1(z) = \left( (W_1 - zI)^{-1}p, p \right) \). Recall that by Proposition 5.6 we have
\[ 1 - F = \frac{F}{F_1}, \] (5.18)
By Lemma 5.14,
\[ 1 - F(z) = \prod_{k \geq 1} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right). \]
Together with (5.18) this implies that \( F_1 \) has simple poles exactly at points \( \mu_k^2 \), so \( \mu_k^2 \) are the non-zero eigenvalues of \( W_1 \). So the existence of the triple \( W, W_1, p \) is proved.

The uniqueness follows immediately from Lemma 5.12.

5.4. **Proof of the abstract Borg’s theorem: the trivial kernel condition.** In Proposition 2.3, we have shown that the trivial kernel condition is equivalent to:
\[ \| W^{-\frac{1}{2}} p \| = 1, \quad \| W^{-1} p \| = \infty \]  \hfill (5.19)
In this subsection, we will translate this condition in terms of \( \{ \lambda_n \}_{n=1}^\infty \) and \( \{ \mu_n \}_{n=1}^\infty \). Then we derive the equivalent condition for \( \text{Ker} \ W_1 = \{ 0 \} \) (we will see that this condition is also closely related to the two identities in (5.19)). Finally, we give an expression for the coefficients of \( \rho_1(s) = \sum_{k \geq 1} b_k \delta_{\mu_k^2}(s) + b_0 \delta_0(s) \), which is defined as the scalar spectral measure of \( W_1 \) with respect to \( p \):
\[ \langle (W_1 - zI)^{-1} p, p \rangle = \int_{\mathbb{R}} \frac{d\rho_1(s)}{s-z} \]

5.4.1. **Trivial kernel condition of Hankel operator \( \Gamma \).**

**Proposition 5.16.** The trivial kernel condition \( \| W^{-\frac{1}{2}} p \| = 1 \) and \( \| W^{-1} p \| = \infty \) is equivalent to the following

(i) \[ \sum_{j=1}^{\infty} \left( 1 - \frac{\mu_j^2}{\lambda_j^2} \right) = \infty ; \]

(ii) \[ \sum_{j=1}^{\infty} \left( \frac{\mu_j^2}{\lambda_{j+1}^2} - 1 \right) = \infty. \]

**Proof.** Since \( \| \text{Proj}_{\text{Ker}(W - \lambda_i^2 I)} p \| = \sqrt{a_k} \), condition \( \| W^{-\frac{1}{2}} p \| = 1 \) can be written as
\[ \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} = 1 \]  \hfill (5.20)

Now recall equation (5.12)
\[ \prod_{k=1}^{\infty} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right) = 1 - \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2 - z} \]  \hfill (5.21)
Take real \( z < 0 \), \( z \to 0^- \), then by (5.21) and monotone convergence theorem, we have
\[ 1 - \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} = \prod_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^2}, \]  \hfill (5.22)
Hence \( \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} = 1 \) is equivalent to
\[ \prod_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^2} = 0, \]  \hfill (5.23)
which is also equivalent to:
\[
\sum_{k=1}^{\infty} \left( \frac{\mu_k^2}{\lambda_k^2} - 1 \right) = -\infty, \tag{5.24}
\]
gives us the condition for (1.2).

Now for the second condition \(\|W^{-1}p\| = \infty\), it’s equivalent to
\[
\sum_{k=1}^{\infty} a_k \lambda_k^4 = \infty. \tag{5.25}
\]
We rewrite (5.12) as
\[
\prod_{k=1}^{\infty} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right) = 1 - \sum_{k=1}^{\infty} a_k \frac{(\lambda_k^2 - z) + z}{\lambda_k^2(\lambda_k^2 - z)}
= 1 - \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} - \sum_{k=1}^{\infty} \frac{a_k z}{\lambda_k^2(\lambda_k^2 - z)},
\]
Hence we have
\[
-\frac{1}{z} \prod_{k=1}^{\infty} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right) = \frac{\sum_{k=1}^{\infty} a_k}{\lambda_k^2(\lambda_k^2 - z)}. \tag{5.26}
\]
Denote the function of \(z\) in (5.26) as \(G(z)\). Taking \(z = -\lambda_k^2\), and let \(N \to \infty\), then RHS of (5.26) increases monotonically to \(\sum_{k=1}^{\infty} a_k / \lambda_k^4\).
Therefore condition (5.25) is equivalent to
\[
\lim_{N \to \infty} G(-\lambda_N^2) = \lim_{N \to \infty} \frac{1}{\lambda_N^2} \prod_{k=1}^{\infty} \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \right) = \infty. \tag{5.27}
\]
Denote \(G_N(z) = \frac{1}{\lambda_N^2} \prod_{k=1}^{N-1} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right)\), we want to prove the following lemma.

**Lemma 5.17.** The following statements are equivalent:

(i) \(\lim_{N \to \infty} G(-\lambda_N^2) = \infty\);

(ii) \(\lim_{N \to \infty} G_N(-\lambda_N^2) = \infty\);

(iii) \(\lim_{N \to \infty} G_N(0) = \infty\).

**Proof of Lemma 5.17.** (i) \(\iff\) (ii): To prove this equivalence, it is sufficient to show that
\[
0 < C_1 \leq \prod_{k=N}^{\infty} \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \right) \leq C_2 < \infty \tag{5.28}
\]
with constants \(C_1, C_2\) independent of \(N\). Note that for \(k \geq N\), we trivially have
\[
\frac{1}{2} \leq \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \leq 1. \tag{5.29}
\]
The upper bound trivially implies the upper bound in (5.28) with \(C_2 = 1\). To get the lower bound in (5.28), we use the estimate (5.29) and the following inequality
\[
\ln x \geq (\ln 2)(x - 1), \quad \forall x \in [1/2, 1].
\]
Thus
\[ \sum_{k=N}^{\infty} \ln 2 \left( \frac{\mu_k^2 + \mu_{k+1}^2}{\lambda_k^2 + \lambda_{k+1}^2} - 1 \right) = \sum_{k=N}^{\infty} -\ln 2 \frac{\lambda_k^2 - \mu_k^2}{\lambda_k^2 + \lambda_{k+1}^2} \geq -\ln 2 \sum_{k=N}^{\infty} \frac{\lambda_k^2 - \mu_k^2}{\lambda_k^2} \geq -\ln 2, \]
we see that the lower bound in (5.28) holds with \( C_1 = \frac{1}{2}. \)

(ii) \( \iff \) (iii): The function \( G_N \) is analytic and zero free in the disc \( D_N \) of radius \( 2\lambda_N^2 \) centered at \( -\lambda_N^2 \), since we have \( \lambda_N^2 \) is smaller than \( \mu_{n-1}^2 \) and so the function \( \ln |G_N| \) is harmonic in this disc.

In addition, we can show that the function \( \ln |G_N| \) is non-negative for sufficiently large \( N \), with one of the two assumptions (ii), (iii) satisfied. In fact, if we goes from (ii) to (iii), then we find a sufficiently large \( N \), such that \( |G_n(-\lambda_n^2)| > 1 \) when \( n \geq N \). Then applying the maximum principle to function \( \ln |G_n(z)| \), then we have \( |G_n(z)| > 1 \) will be hold on the whole disk closed \( D_n \). Similarly, we can also assume \( \ln |G_n(z)| \) is non-negative when going from (iii) to (ii).

Therefore by Harnack inequality
\[ \frac{1}{3} \ln |G_N(-\lambda_N^2)| \leq \ln |G_N(0)| \leq 3 \ln |G_N(-\lambda_N^2)|, \tag{5.30} \]
which proves the equivalence (ii) \( \iff \) (iii).

\[ \square \]

Remark 5.18. We use the Harnack inequality where showing (5.30). Here the Harnack inequality is stated as below:

**Theorem 5.19 (Harnack inequality).** Let \( f \) be a non-negative function defined on a closed ball \( B(x_0, R) \). If \( f \) is continuous on the closed ball and harmonic on its interior, then for every point \( x \) with \( |x-x_0| = r < R \), we have
\[ \frac{1-(r/R)}{1+(r/R)} f(x_0) \leq f(x) \leq \frac{1+(r/R)}{1-(r/R)} f(x_0) \]

Now by lemma 5.17, condition (5.25) is equivalent to \( \lim_{N \to \infty} G_N(0) = \infty \), hence
\[ \lim_{N \to \infty} \frac{1}{\lambda_1^2} \prod_{k=1}^{N-1} \frac{\mu_k^2}{\lambda_k^2} = \lim_{N \to \infty} \frac{1}{\lambda_N^2} \prod_{k=1}^{N-1} \frac{\mu_k^2}{\lambda_k^2} = \infty, \]
And condition \( \prod_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^2} = \infty \) is equivalent to
\[ \sum_{k=1}^{\infty} \left( \frac{\mu_k^2}{\lambda_k^2} - 1 \right) = \infty \tag{5.31} \]
Thus (5.24) together with (5.31) finish our proof for Proposition 5.16.

\[ \square \]

5.4.2. **Trivial Kernel Condition for \( W_1 \) and the Spectral Measure \( \rho_1(s) \).** With the constructed triple \( (W, W_1, p) \) in Theorem 5.3, we discuss the trivial kernel condition of \( W_1 \) in this subsection, and in addition, we give an expression for the coefficients of \( \rho_1(s) = \sum_{k \geq 1} b_k \delta_{\mu_k^2}(s) + b_0 \delta_0(s) \), which is defined as the scalar spectral measure of \( W_1 \) with respect to \( p \):
\[ \langle (W_1 -zI)^{-1}, p \rangle = \int_{\mathbb{R}} \frac{dp_1(s)}{s-z}. \]
Proposition 5.20. If the coefficients \( \{a_n\}_{n \geq 1} \) given in Lemma 5.14 satisfies \( \sum_{n \geq 1} \frac{a_n}{\lambda_n^2} < 1 \), then \( b_0 = 0 \), and the coefficients \( \{b_n\}_{n \geq 1} \) can be written as:

\[
  b_n = (\lambda_n^2 - \mu_n^2) \prod_{k \neq n} \left( \frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2} \right) \quad (5.32)
\]

Proof. From (5.11) we have

\[
  1 + F_1(z) = \frac{F_1(z)}{F} = \prod_{i=1}^{\infty} \left( \frac{z - \mu_i^2}{z - \lambda_i^2} \right).
\]

If \( \prod_{i=1}^{\infty} \left( \frac{\lambda_i^2}{\mu_i^2} \right) < \infty \) (thus by (5.22) we have \( \sum_{n \geq 1} \frac{a_n}{\lambda_n^2} < 1 \)), then follow a similar procedure as the proof in Lemma 5.14, we write

\[
  1 + \sum_{n} \frac{b_n^N}{\mu_n^2 - z} = \prod_{k=1}^{N} \left( \frac{z - \lambda_k^2}{z - \mu_k^2} \right),
\]

then we have

\[
  b_n^N = (\lambda_n^2 - \mu_n^2) \prod_{k=1, k \neq n}^{N} \left( \frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2} \right) \text{ if } n \geq N,
\]

and \( b_n^N = 0 \) if \( n > N \).

Now take a upper bound \( B \) for \( \prod_{i=1}^{\infty} \left( \frac{\lambda_i^2}{\mu_i^2} \right) \), then we have

\[
  \sum_n \frac{b_n^N}{\mu_n^2} < B - 1.
\]

Since \( b_n^N > 0 \) and \( b_n^N \downarrow b_n \) as \( N \to \infty \), where \( b_n \) has the formula as

\[
  b_n = (\lambda_n^2 - \mu_n^2) \prod_{k \neq n} \left( \frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2} \right);
\]

we have \( \sum_{n \geq 1} \frac{b_n}{\mu_n^2} < B - 1 \) and \( \sum_{n \geq 1} \frac{b_n^N}{\mu_n^2 - z} \) converges uniformly to \( \sum_{n \geq 1} \frac{b_n}{\mu_n^2 - z} \) on any compact \( K \subset \sigma \).

Thus in this case we can write \( \rho_1(s) = \sum_n b_n \delta_{k_n}^2(z) \) where \( b_n \) has the form in (5.32). \( \square \)

Proposition 5.21. For the triple \((W, W_1, p)\) constructed in Theorem 5.3, we have \( \ker W_1 \neq \{0\} \) iff the following two equations hold:

\[
  \sum_{n \geq 1} \frac{a_n}{\lambda_n^2} = 1, \quad \sum_{n \geq 1} \frac{a_n}{\lambda_n^4} < \infty. \quad (5.33)
\]

Hence by Proposition 5.16, (5.33) is also equivalent to:

\[
  \prod_{k=1}^{\infty} \left( \frac{\lambda_k^2}{\mu_k^2} \right) = \infty, \quad \prod_{k=1}^{\infty} \left( \frac{\mu_k^2}{\lambda_{k+1}^2} \right) = \infty. \quad (5.34)
\]
Proof. The proof is very much similar to the proof in Proposition 2.3.

We define two positive self-adjoint operators \( R, R_1 \) as: \( R := W^{1/2}, R_1 = W_1^{1/2} \), and also a contraction \( T := R_1 R^{-1} \) implied by Douglas Lemma 2.1, and also a vector \( q := R^{-1} p \). Then the two equations in (5.33) are equivalent to:

\[
\|q\| = 1, \quad q \in \text{Ran } R
\]

(i) Under the condition that \( \|q\| < 1 \), we have shown in Proposition 5.20, that \( \text{Ker } W_1 = \{0\} \).

(ii) Suppose that \( \|q\| = 1 \) and \( q \notin \text{Ran } R \). Since we have

\[
W_1 = R_1^2 = RT^*TR = R(I - qq^*)R
\]

(5.35)

Since \( \text{Ker}(I - qq^*) = \text{Span}\{q\} \) and \( q \notin \text{Ran } R \), we can see that \( \text{Ker } R(I - qq^*)R = \{0\} \), thus \( \text{Ker } W_1 = \{0\} \).

(iii) Suppose that \( \|q\| = 1 \) and \( q \in \text{Ran } R \). Using (5.35) again, and apply \( u := R^{-1} q \) to (5.35), we have \( (I - qq^*)Ru = 0 \), thus \( W_1 u = 0 \).

Now in contrast with the case of Proposition 5.20 when \( \sum_{k \geq 1} \frac{q_k}{n_k} < 1 \), we consider the case when \( \sum_{k \geq 1} \frac{q_k}{n_k} = 1 \). We first prove the following lemma:

**Lemma 5.22.** For all \( z \notin \{\frac{1}{\mu_j}\}_{j \geq 1} \), we have the following equation

\[
\prod_{j=1}^{\infty} \frac{1 - z \lambda_j^2}{1 - z \mu_j^2} = 1 - b_0 z + \sum_{k \geq 1} \frac{b_k z}{\mu_k^2 z - 1}
\]

is well-defined and holds for some constant \( b_0 \in \mathbb{C} \). Here coefficients \( \{b_k\}_{k \geq 1} \) has the expression in (5.32).

**Proof of Lemma 5.22.** The first step is similar to the proof of Proposition 5.14. We consider rewriting a finite product

\[
\prod_{j=1}^{N} \frac{1 - z \lambda_j^2}{1 - z \mu_j^2} = 1 + \sum_{k=1}^{N} \frac{b_k^N}{\mu_k^2 z} = 1 + \sum_{k=1}^{N} \frac{b_k^N z}{\mu_k^2 z - 1},
\]

(5.36)

then we have the coefficients \( b_k^N \) has the expression

\[
b_k^N = (\lambda_k^2 - \mu_k^2) \prod_{n=1, n \neq k}^{N} \frac{\mu_k^2 - \lambda_n^2}{\mu_k^2 - \mu_n^2},
\]

here \( b_k^N > 0 \) and \( b_n^N \searrow b_n \) as \( N \to \infty \).

Now we rewrite (5.36) and define function \( H_N(z) \) as:

\[
H_N(z) := \frac{1}{z} \left( \prod_{j=1}^{N} \frac{1 - z \lambda_j^2}{1 - z \mu_j^2} - 1 \right) = \sum_{k=1}^{N} \frac{b_k^N}{\mu_k^2 z - 1}.
\]

(5.37)

Let \( z = 0 \), and we take the expansion of infinite product at \( z = 0 \) on LHS, we have the sum

\[
\sum_{k=1}^{N} b_k^N = \sum_{j=1}^{N} (\lambda_j^2 - \mu_j^2)
\]

is bounded from above, thus let \( N \to \infty \), we have \( \sum_{k=1}^{\infty} b_k < \infty \).
Now again for the function $H_N(z)$ defined in (5.37), we write

$$H_N(z) = H_N(0) + \int_0^z H'_N(t) dt.$$  \hspace{1cm} (5.38)

Here $H'_N(t)$ has the expression

$$H'_N(t) = \sum_{j=1}^N \frac{-b_j^N \mu_j^2}{(\mu_j^2 t - 1)^2}.$$  

Thus for a compact set $K$ which doesn’t intersect with any of $\{\frac{1}{\mu_j^2}\}_{j\geq 1}$ and any $t \in K$, we have

$$\left| \frac{-b_j^N \mu_j^2}{(\mu_j^2 t - 1)^2} \right| \leq C(K; \sigma) b_j^N.$$  

Since $\sum_{j\geq 1} b_j < \infty$, we have

$$\sum_{j=1}^N \frac{-b_j^N \mu_j^2}{(\mu_j^2 t - 1)^2} \to \sum_{j=1}^\infty \frac{-b_j \mu_j^2}{(\mu_j^2 t - 1)^2}$$  

uniformly on $K$. Hence we take $N \to \infty$ in (5.38), we get

$$\frac{1}{z} \left( \prod_{j=1}^\infty \frac{1 - z \lambda_j^2}{1 - z \mu_j^2} \right) = \lim_{N \to \infty} H_N(0) + \int_0^z \sum_{j=1}^\infty \frac{-b_j \mu_j^2}{(\mu_j^2 t - 1)^2}.$$  

Hence

$$\frac{1}{z} \left( \prod_{j=1}^\infty \frac{1 - z \lambda_j^2}{1 - z \mu_j^2} - 1 \right) = \sum_{j=1}^\infty \frac{b_j}{\mu_j^2 z - 1} + C$$  

for some constant $C \in \mathbb{C}$. And we finish the proof of lemma 5.22. \hfill $\Box$

Now we replace $z$ by $\frac{1}{z}$ in Lemma 5.22, and we get that for $z \notin \{\mu_j^2\}_{j\geq 1}$, we have the following equation

$$\prod_{j=1}^\infty \frac{\lambda_j^2 - z}{\mu_j^2 - z} = 1 - b_0 \frac{1}{z} + \sum_{k\geq 1} \frac{b_k}{\mu_k^2 z - 1},$$  

where $\{b_k\}_{j\geq 1}$ has the form as (5.32). The only remaining thing to do is to find the value of $b_0$.

**Lemma 5.23.** With the assumptions that $\sum_{k \geq 1} \frac{a_k}{\lambda_k^2} = 1$ and $\sum_{k \geq 1} \frac{a_k}{\mu_k^2} < \infty$, we have

$$b_0 = \left( \sum_{k \geq 1} \frac{a_k}{\lambda_k^2} \right)^{-1}.$$  

**Proof.** If we denote the scalar spectral measure $\rho_1(s) = \sum_{k \geq 1} \delta_{\mu_k^2}(s) + b_0 \delta_0(s)$, and substitute it into

$$1 + F_1(z) = \frac{F_1(z)}{F(z)} = \prod_{k=1}^\infty \left( \frac{z - \lambda_k^2}{z - \mu_k^2} \right) = 1 + \sum_{k \geq 1} \frac{b_k}{\mu_k^2 z - 1} - \frac{b_0}{z}.$$  

Hence we have

$$b_0 = \lim_{z \to 0} \left[ -z \prod_{k=1}^{\infty} \frac{z - \lambda_k^2}{z - \mu_k^2} \right]$$

On the other hand, from (5.26) we have

$$-\frac{1}{z} \prod_{k=1}^{\infty} \frac{z - \mu_k^2}{z - \lambda_k^2} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2 (\lambda_k^2 - z)}.$$  \hspace{1cm} (5.39)

Let $z \to 0$, then we get

$$\frac{1}{b_0} = \sum_{k \geq 1} \frac{a_k}{\lambda_k^4},$$

hence we finish the proof of Lemma 5.23. \hfill \Box

Thus combining the results we get in Proposition 5.20, Lemma 5.22 and Lemma 5.23, we can give an expression for all coefficients of $\rho_1(s)$.

**Proposition 5.24.** Denote the scalar spectral measure of $W_1$ with respect to $p$ defined by (5.10) to be $\rho_1(s)$. Then

$$\rho_1(s) = \sum_{k \geq 1} b_k \delta_{\mu_k^2}(s) + b_0 \delta_0(s)$$

where:

1. $b_n = \left( \lambda_n^2 - \mu_n^2 \right) \prod_{k \neq n} \left( \frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2} \right)$, when $n \geq 1$
2. $b_0 = \begin{cases} 0 & \text{if } \sum_{n} \frac{a_n}{\lambda_n^4} < 1, \text{ or } \sum_{n} \frac{a_n}{\mu_n^4} = 1 \text{ and } \sum_{n} \frac{a_n}{\mu_n^4} = \infty. \\ \left( \sum_{n} \frac{a_n}{\lambda_n^4} \right)^{-1} & \text{if } \sum_{n} \frac{a_n}{\mu_n^4} = 1 \text{ and } \sum_{n} \frac{a_n}{\mu_n^4} < \infty. \end{cases}$ \hspace{1cm} (5.40)

### 6. Summarization

Now we summarize the results we get from this paper.

For the first part of this paper, we show that a Hankel operator can be uniquely determined by the spectral data of two operators satisfying a rank-one perturbation relation under the assumption of asymptotic stability. In section 2, we proved in Proposition 2.3 that a self-adjoint Hankel operator can be uniquely characterized by a rank-one perturbation triple $(R, R_1, p)$. And in section 3, we proved in Proposition 3.11 that a general Hankel operator (which is $\mathcal{C}$-symmetric) can be uniquely determined by a tuple $(R, R_1, p, \phi, \phi_1)$. In other words, the Hankel operator will be uniquely determined by the spectral data of $R\phi, R_1\phi_1$.

For the second part of this paper, we further translate the spectral data of the two operators which we get in the first part. In this part, we consider the case of compact Hankel operators with simple singular values. Under this case, we will get the asymptotic stability of contraction in Proposition 2.3 and Proposition 3.11 for free.

(i) If the Hankel operator $\Gamma$ is self-adjoint, Theorem 1.5 indicates that $\Gamma$ will be uniquely determined by two sequences of real numbers $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$, satisfying an intertwining relations

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > \ldots > |\lambda_n| > |\mu_n| > \ldots \to 0.$$  \hspace{1cm} (6.1)

If we write the Borel transform of $\Gamma^2$ with respect to $u$:

$$\left( (\Gamma^2 - zI)^{-1} u, u \right) = \int \frac{d\rho(s)}{s - z},$$
then the coefficients of the scalar spectral measure $\rho(s) = \sum_{k \geq 1} a_k \delta_{\lambda_k^2}(s)$ is uniquely determined with the expression in (5.15).

(ii) If the Hankel operator $\Gamma$ is not self-adjoint, Theorem 1.6 indicates that $\Gamma$ will be uniquely determined by two sequences of complex numbers $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$, whose modulus part satisfies an intertwining relation

$$|\lambda_1| > |\mu_1| > |\lambda_2| > \ldots \to 0.$$ 

In other words, with the unique conjugation $\tilde{\Gamma}$ which commutes with $|\Gamma|, |\Gamma_1|$ and preserves $u := \Gamma e_0$, and the induced unitary $\Phi, \Phi_1$ given in (3.3), (3.4). We have the non-zero eigenvalues of the two operators $|\Gamma|, |\Gamma_1|$ are simple, and coincide with $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ respectively.

For this second case, we can also say that the Hankel operator $\Gamma$ can be uniquely determined by three factors:

(a) A discrete measure $\rho(s)$, which is the scalar spectral measure of $|\Gamma|^2$ with respect to $u = \Gamma^* e_0$:

$$\left( |\Gamma|^2 - zI \right)^{-1} u, u \right) = \int \frac{d\rho(s)}{s - z}.$$ 

The coefficients of this scalar measure is uniquely determined by the modulus part of sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$.

(b) A unimodular measurable function $\phi : \mathbb{R} \to \mathbb{C}$ (with the expression in (5.4)), which is determined by the phase part of complex sequences $\{\lambda_n\}_{n=1}^{\infty}$. This function induces an operator $\phi(|\Gamma|)$ commutes with $|\Gamma|$ and share the same eigenvectors with $|\Gamma|$.

(c) A unimodular measurable function $\phi_1 : \mathbb{R} \to \mathbb{C}$ (with the expression in (5.5)), which is determined by the phase part of complex sequences $\{\mu_n\}_{n=1}^{\infty}$. This function induces an operator $\phi_1(|\Gamma_1|)$ commutes with $|\Gamma_1|$ and share the same eigenvectors with $|\Gamma_1|$.

References

[1] A. V. Megretskii, V. V. Peller, and S. R. Treil’, The inverse spectral problem for self-adjoint Hankel operators, Acta Math. 174 (1995), no. 2, 241–309.

[2] S. R. Treil’, An inverse spectral problem for the modulus of the Hankel operators, and balanced realizations, Algebra i Analiz, 1990, Volume 2, Issue 2, 158-182

[3] B. S. Nagy, C. Foias, H. Berciﬁci, and L. Kérchy, Harmonic Analysis of Operators on Hilbert Space Second Edition, Springer-Verlag, New York-Dordrecht-Heidelberg-London, 2010

[4] B. Simon, Operator Theory, A Comprehensive course in Analysis, Part 4, AMS Providence, United States, 2015

[5] G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe : Bestimmung der Differentialgleichung durch die Eigenwerte, Acta Math. 78(1946) 1-96

[6] N. Levinson, The inverse Sturm-Liouville problem, Matematisk Tidsskrift, B(1949), pp.25-30

[7] M. S. Birman, M. Z. Solomjak, Spectral Theory of Self-Adjoint Operators in Hilbert space, translated from Russian by S. Khrushchev and V. Peller, Springer-Verlag, Dordrecht Netherlands, 1987

[8] P. Gérard and S. Grellier, Inverse spectral problems for compact Hankel Operators, J. Inst. Math. Jessieu 13(2014), no. 2, 273–301.

[9] S. R. Garcia, E. Prodan, M. Putinar, Mathematical and Physical Aspects Of Complex Symmetric Operators, J. Phys.A 47(2014), no. 35, 353001

[10] M. C. Câmara, K. K. Garlicka, B. Lanucha, M. Ptak, Conjugations in $L^2$ and their invariants, Analysis and Mathematical Physics (2020)

[11] H. Radjavi, P. Rosenthal, Invariant Subspaces, Springer, New York (1973)