ON MAXIMIZING THE SPEED OF A RANDOM WALK IN FIXED ENVIRONMENTS

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Abstract. We consider a random walk in a fixed \( \mathbb{Z} \) environment composed of two point types: \((q, 1-q)\) and \((p, 1-p)\) for \( \frac{1}{2} < q < p \). We study the expected hitting time at \( N \) for a given number \( k \) of \( p \)-drifts in the interval \([1, N-1]\), and find that this time is minimized asymptotically by equally spaced \( p \)-drifts.

1. Introduction

Procaccia and Rosenthal \( \text{[1]} \) studied how to optimally place given number of vertices with a positive drift on top of a simple random walk to minimize the expected crossing time of an interval. They ask about extending their work to the situation where the environment on \( \mathbb{Z} \) is composed of two point types: \((q, 1-q)\) and \((p, 1-p)\) for \( \frac{1}{2} < q < p \). This is the goal of this note. See \( \text{[1]} \) for background and further related work.

Consider nearest neighbor random walks on \( 0, 1, ..., N \) with reflection at the origin. We denote the random walk by \( \{X_n\}_{n=0}^{\infty} \), and by \( \omega(i) \) the transition probability at vertex \( i \):

\[
P(X_{n+1} = i + 1 | X_n = i) = \omega(i)
\]
\[
P(X_{n+1} = i - 1 | X_n = i) = 1 - \omega(i).
\]

First, we prove the following proposition concerning the expected hitting time at vertex \( N \):

**Proposition 1.** For a walk \( \omega \) starting at \( x \), the hitting time \( T_N = \min \{n \geq 0 | X_n = N\} \) satisfies:

\[
E_\omega^x (T_N) = N - x + 2 \sum_{i=x}^{N-1} \sum_{j=1}^{i} \prod_{k=j}^{i} \rho_k,
\]

where \( \rho_i = \frac{1 - \omega(i)}{\omega(i)} \), and \( E_\omega^x (T_N) \) stands for the expected hitting time. In particular:

\[
E_\omega^0 (T_N) = N + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{i} \prod_{k=j}^{i} \rho_k.
\]

**Corollary 2.** The expected hitting time from 0 to \( N \) is symmetric under reflection of the environment, i.e. taking the environment \( \omega'(i) = \omega(N-i) \) gives \( E_{\omega'}^0 (T_N) = E_\omega^0 (T_N) \).
Next we turn to the case of an environment consisting of two types of drifts, 
\((q, 1 - q)\) (i.e. probability \(q\) to go to the right and \(1 - q\) to the left) and \((p, 1 - p)\), 
for some \(\frac{1}{2} < q < p \leq 1\). Assume that \(k\) of the vertices are \(p\)-drifts, and the rest 
are \(q\)-drifts. In [1] it was proven that for \(q = \frac{1}{2}\) equally spaced \(p\)-drifts minimize 
\(E^0_\omega(T_N)\) (for large \(N\)). In this paper we extend this result for \(q > \frac{1}{2}\). We define an 
environment in which the \(p\)-drifts are equally spaced (up to integer effects):
\[
\omega_{N,k}(x) = \begin{cases} 
p & x = \lfloor i \cdot \frac{N-1}{k} \rfloor \text{ for some } 1 \leq i \leq k, 
q & \text{otherwise}
\end{cases}
\]
and prove the following theorem:

**Theorem 3.** For every \(\varepsilon > 0\) there exists \(n_0\) such that for every \(N > n_0\) and 
environment \(\omega\):
\[
\frac{E^0_\omega(T_N)}{N} > \frac{E^0_{\omega_{N,k}}(T_N)}{N} - \varepsilon,
\]
where \(k\) is the number of \(p\)-drifts in \(\omega\).

Finally, we consider the set of environments \(\omega_{ak,k}\) for some \(a \in \mathbb{N}\), and calculate
\[
\lim_{k \to \infty} \frac{E^0_{\omega_{ak,k}}(T_N)}{ak}.
\]

**Proposition 4.** Let \(a \in \mathbb{N}\). Then:
\[
\lim_{k \to \infty} \frac{E^0_{\omega_{ak,k}}(T_N)}{ak} = 1 + 2 \alpha x^2 - 2 ax + (a - 1) \alpha^2 + \left(\frac{a \alpha^2 - (a + 1) \alpha}{(a^2 - 2 \alpha + 1) \alpha} \right) \beta.
\]

2. **Proof of the main theorem**

**Proof of Proposition [1].** Define \(v_x = E^x_\omega(T_N)\) for \(0 \leq x \leq N\). By conditioning on 
the first step:

1. \(v_N = 0\)
2. \(v_0 = v_1 + 1\)
3. \(v_x = p_x v_{x+1} + (1 - p_x) v_{x-1} + 1 \quad 1 \leq x \leq N - 1\).

To solve these equations, define \(a_x = v_x - v_{x-1}\) (for \(1 \leq x \leq N\)) and \(b_x = v_{x+1} - v_{x-1}\) (for \(1 \leq x \leq N - 1\)). Then:

\[
\begin{align*}
    b_x &= a_x + a_{x+1} \\
    a_x &= p_x b_x + 1 \\
    a_1 &= -1
\end{align*}
\]

We get for \(a_x\) the relation \(a_{x+1} = \rho_x a_x - \rho_x - 1\), whose solution is \(a_x = \sum_{j=1}^{x-1} \prod_{k=j}^x \rho_k - 1\), and then:
\[ \begin{align*}
v_x &= \sum_{i=x+1}^{N} (v_{i-1} - v_i) + v_N \\
 &= \sum_{i=x+1}^{N} (-a_i) + v_N \\
 &= N - x + 2 \sum_{i=x}^{N-1} \sum_{j=1}^{i} \prod_{k=j}^{i} \rho_k
\end{align*} \]

**Definition 5.** To evaluate \( E_0^0 (T_N) \) we define:

\[ S_N = \sum_{d=1}^{N-1} \sum_{j=1}^{N-d-1} \prod_{k=j}^{N-1} \rho_k. \]

Next define \( \tilde{\rho}_k \) for \( k \) in the circle \( \mathbb{Z}_{N-1} \), such that for \( 1 \leq k \leq N-1 \) we will have \( \tilde{\rho}_k = \rho_k \) (gluing the point 0 to the point \( N - 1 \)), and then look at:

\[ \tilde{S}_N = \sum_{d=1}^{N-1} \sum_{j=1}^{N-d-1} \prod_{k=j}^{N-1} \tilde{\rho}_k. \]

This way, rather than summing \( \prod_{k=i}^{j} \rho_k \) over subintervals \([i, j]\) of \([1, N-1]\), we sum \( \prod_{k=i}^{j} \tilde{\rho}_k \) over subintervals of the circle \( \mathbb{Z}_{N-1} \).

**Proposition 6.** Define \( \alpha = \frac{1-q}{q}, \beta = \frac{1-p}{p} \). Since \( \beta < \alpha < 1 \):

\[ |\tilde{S}_N - S_N| = \sum_{d=1}^{N-1} \sum_{j=N-d-1}^{N-1} \prod_{k=j}^{N-1} \rho_k \]

\[ \leq \sum_{d=1}^{N-1} d\alpha^d \]

\[ \leq \sum_{d=1}^{\infty} d\alpha^d < C(\alpha) \]

for some constant \( C(\alpha) \) which doesn’t depend on \( N \).

**Definition 7.** Let \( n_i^{(d)} \) be the number of \( p \)-drifts in the interval \([i, i+d-1]\).

Since every drift appears in \( d \) intervals of length \( d \), \( \sum_{i=1}^{N-1} n_i^{(d)} = dk \). Also,
\[ \overline{S}_N = \sum_{d=1}^{N-1} \sum_{i=1}^{N-1} \left( \frac{\beta}{\alpha} \right)^{i(d)} \cdot \alpha^d \]

where \( \sigma_d = \sum_{i=1}^{N-1} \left( \frac{\beta}{\alpha} \right)^{n_i(d)} \cdot \alpha^d \).

Claim 8. For \( n_i^{(d)} \in \mathbb{N} \) the expression \( \sigma_d \) is minimized under the restriction \( \sum_{i=1}^{N-1} n_i^{(d)} = dk \) if \( n_i^{(d)} - n_j^{(d)} \leq 1 \) for all \( i, j \).

Proof. For convenience, we omit \( d \) from the notation, and set \( n = (n_1, ..., n_{N-1}) \). If a vector \( n \) satisfies \( n_i - n_j \leq 1 \forall i, j \), we say \( n \) is almost constant. We will show that \( \sigma \) is minimal for some almost constant vector. Then we show that \( \sigma \) takes on the same value for all almost constant vectors under the restriction, and this completes the proof.

Suppose \( \sigma \) is minimized (under the restriction) by some vector \( n_0 \). If \( n_0 \) is almost constant, we are done. Else, for some \( i, j \) we have that \( n_i^0 - n_j^0 \geq 2 \). We choose \( i, j \) such that \( n_i^0 - n_j^0 \) is maximal. Define:

\[
    n^1_l = \begin{cases} 
        n_i^0 & l \neq i, j \\
        n_i^0 - 1 & l = i \\
        n_j^0 + 1 & l = j 
    \end{cases}
\]

\( n^1 \) satisfies the restriction, and \( \sigma \left( n^0 \right) \geq \sigma \left( n^1 \right) \):

\[
    \sigma \left( n^0 \right) - \sigma \left( n^1 \right) = \sum_{t=1}^{N-1} \left( \frac{\beta}{\alpha} \right)^{n_t^0} \cdot \alpha^d - \sum_{t=1}^{N-1} \left( \frac{\beta}{\alpha} \right)^{n_t^1} \cdot \alpha^d 
\]

\[
    = \alpha^d \left( \left( \frac{\beta}{\alpha} \right)^{n_i^0} + \left( \frac{\beta}{\alpha} \right)^{n_j^0} - \left( \frac{\beta}{\alpha} \right)^{n_i^0-1} - \left( \frac{\beta}{\alpha} \right)^{n_j^0+1} \right) 
\]

\[
    = \alpha^d \left( 1 - \frac{\beta}{\alpha} \right) \left( \left( \frac{\beta}{\alpha} \right)^{n_j^0} - \left( \frac{\beta}{\alpha} \right)^{n_i^0-1} \right) 
\]

\[
    \geq 0,
\]

where the inequality follows from the fact that \( 0 \leq \frac{\beta}{\alpha} < 1 \) and \( n_j^0 < n_i^0 - 1 \). From minimality of \( \sigma \left( n^0 \right) \), we get that \( \sigma \left( n^1 \right) \) is also minimal. This process must end after a finite number of steps \( f \), yielding an almost constant \( n^f \) which minimizes \( \sigma \).

Now for a general almost constant vector \( n \), set \( a = \min \{ n_l : 1 \leq l \leq N - 1 \} \). We have \( n_l \in \{ a, a+1 \} \), so defining \( m_0 \) to be the number of \( a \)'s and \( m_1 \) to be the number of \( a+1 \)'s, we get:
and since \( m_1 < N - 1 \), there is a unique solution for natural \( a, m_1 \). So all almost constant \( n \) (satisfying the restriction) are the same up to ordering, and since \( \sigma \) doesn’t depend on the order, they all give the same value. \( \square \)

**Claim 9.** For every choice of \( M, k \), the placement of \( k \) drifts on the circle \( \mathbb{Z}_M \) in which the \( i \)th drift is at the point \( \lfloor i \cdot \frac{M}{k} \rfloor \) satisfies:

\[
\forall d, i, j \quad n_i^{(d)} - n_j^{(d)} \leq 1.
\]

**Proof.** Place the \( i \)th drift at the point \( \lfloor i \cdot \frac{M}{k} \rfloor \). We calculate the number of drifts in the interval \([x, x + d - 1]\). The first drift inside this interval is:

\[
\left\lfloor i_0 \cdot \frac{M}{k} \right\rfloor \geq x,
\]
\[
i_0 \cdot \frac{M}{k} \geq x,
\]
\[
i_0 \geq \frac{x}{\frac{k}{M}},
\]
\[
i_0 = \left\lfloor x \cdot \frac{k}{M} \right\rfloor.
\]

The last drift inside this interval is:

\[
\left\lfloor i_1 \cdot \frac{M}{k} \right\rfloor \leq x + d - 1
\]
\[
i_1 \cdot \frac{M}{k} < x + d
\]
\[
i_1 < (x + d) \cdot \frac{k}{M}
\]
\[
i_1 = \left\lfloor (x + d) \cdot \frac{k}{M} \right\rfloor - 1.
\]

The number of drifts inside this interval is therefore:
\[ i_1 - i_0 + 1 = \left\lceil (x + d) \cdot \frac{k}{M} \right\rceil - \left\lfloor x \cdot \frac{k}{M} \right\rfloor \]
\[ \geq (x + d) \cdot \frac{k}{M} - x \cdot \frac{k}{M} - 1 \]
\[ = \frac{dk}{M} - 1 \]
\[ i_1 - i_0 + 1 \leq (x + d) \cdot \frac{k}{M} + 1 - x \cdot \frac{k}{M} \]
\[ = \frac{dk}{M} + 1. \]

So for non-integer \( \frac{dk}{M} \) the number of drifts takes on only the two values \( \lfloor \frac{dk}{M} \rfloor, \lceil \frac{dk}{M} \rceil \).

For integer \( \frac{dk}{M} \) we simply have:
\[ i_1 - i_0 + 1 = \left\lceil (x + d) \cdot \frac{k}{M} \right\rceil - \left\lfloor x \cdot \frac{k}{M} \right\rfloor = \frac{dk}{M}. \]

\[ \Box \]

**Claim 10.** \( \tilde{S}_N \) is minimal for the configuration of drifts described by \( \omega_{N,k} \) (where the \( i \)th drift is at vertex \( \lfloor i \cdot \frac{N-1}{k} \rfloor \)).

**Proof.** \( \tilde{S}_N = \sum_{d=1}^{N-1} \sigma_d \), and by claims 8 and 9 each \( \sigma_d \) is minimized by this configuration, therefore the sum is also minimized. \( \Box \)

**Proof of Theorem 3.** From Proposition 6, \( 0 < \tilde{S}_N - S_N < C \). Let \( n_0 = \frac{2C}{\varepsilon} \). Then for \( N > n_0 \):
\[ \frac{E_0^0(T_N)}{N} = \frac{N + 2S_N}{N} \]
\[ = 1 + \frac{2S_N}{N} \]
\[ > 1 + \frac{2\tilde{S}_N}{N} - \varepsilon \]
\[ \geq 1 + 2\frac{S_N}{N} - \varepsilon \]
\[ \geq 1 + 2\frac{S_N}{N} - \varepsilon \]
\[ = \frac{E_0^{\omega_{N,k}}(T_N)}{N} - \varepsilon \]

where we denote by \( S_N \) and \( \tilde{S}_N \) the values calculated for \( \omega_{N,k} \). \( \Box \)

**Proof of Proposition 4.** We evaluate \( \lim_{k \to \infty} \frac{\tilde{S}_N}{ak} \). First, we consider the \( k \) intervals that do not contain any \( \beta \), each of which contributes:
\[ s_0 = \sum_{i=1}^{a-1} (a - i) \alpha^i. \]

Next we consider the \( k \) intervals that contain \( n \geq 1 \) \( \beta \)'s:

\[ s_n = \beta^n \cdot \alpha^{(a-1)(n-1)} \cdot \sum_{r=0}^{a-1} \sum_{s=0}^{a-1} \alpha^{r+s}. \]

Then we get:

\[
\lim_{k \to \infty} \frac{\bar{S}_{ak}}{ak} = \frac{1}{a} \lim_{k \to \infty} \frac{ks_0 + \sum_{n=1}^{k} ks_n}{k} = \frac{1}{a} \cdot \frac{\alpha^{a+2} - a\alpha^3 + (a - 1) \alpha^2 + ((a\alpha^2 - (a + 1) \alpha) \alpha^a + \alpha) \beta}{(\alpha^2 - 2a + 1) \alpha^a \beta - \alpha^4 + 2a^2 - \alpha},
\]

and since \( \lim_{k \to \infty} \frac{\bar{S}_{ak}}{ak} - \bar{S}_{ak} = 0 \) from Proposition 6, the proof is complete. \( \square \)

3. **Further Questions**

1. Show that the optimal environment also minimizes the variance of the hitting time.

2. Can this result be extended to a random walk on \( \mathbb{Z} \) with a given density of drifts (as in \([1]\))? 

3. Can similar results be found for other graphs? For example, \( \mathbb{Z}_2 \times \mathbb{Z}_N \).

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**References**

1. E.B. Procaccia and R. Rosenthal, *The need for speed: maximizing the speed of random walk in fixed environments*, Electronic Journal of Probability **17** (2012), 1–19.