Effects of Vacuum Polarization in Strong Magnetic Fields with an Allowance Made for the Anomalous Magnetic Moments of Particles

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Abstract

Given the anomalous magnetic moments of electrons and positrons in the one-loop approximation, we calculate the exact Lagrangian of an intense constant magnetic field that replaces the Heisenberg-Euler Lagrangian in traditional quantum electrodynamics (QED). We have established that the derived generalization of the Lagrangian is real for arbitrary magnetic fields. In a weak field, the calculated Lagrangian matches the standard Heisenberg-Euler formula. In extremely strong fields, the field dependence of the Lagrangian completely disappears, and the Lagrangian tends to a constant determined by the anomalous magnetic moments of the particles.

1. INTRODUCTION

The quantum corrections to the Maxwellian Lagrangian of a constant electromagnetic field were first calculated by Heisenberg and Euler [1] in 1936. The radiative corrections that correspond to the polarization of an electron-positron vacuum by external electromagnetic fields with diagrams containing different numbers of electron loops are still the focus of attention [2-4]. Estimates suggest that the quantum (radiative) corrections could
reach the Maxwellian energy density of the electromagnetic field only in exponentially strong electromagnetic fields \( F_c \sim \exp(\pi/\alpha)H_c \) \(^1\) [5]. The calculations by Heisenberg and Euler are known to contain no approximations in the intensity of external electromagnetic fields, and their results have been repeatedly confirmed by calculations performed in terms of different approaches. On this basis, several authors have identified the field intensity \( F_c \) with the validity boundary of universally accepted QED. However, it is clear that, although such quantities were greatly overestimated, because the corresponding scale lengths are many orders of magnitude smaller than not only the scale on which weak interactions manifest themselves, but also the Planck length, determining the validity range of traditional QED is currently of fundamental importance. While on the subject of the physics of extremely small distances, we cannot but say that there is a deep analogy between the phenomena that arise for large momentum transfers and the processes in intense electromagnetic fields [2-17]. In fact, the overlapping of the seemingly distinctly different areas of physics is not accidental and is suggested by simple dimension considerations.

An allowance for the electromagnetic field intensity based on the exact integrability of the equations of motion is known to play an important role in studying the quantum effects of the interaction between charged particles and the electromagnetic field. In particular, the standard Schwinger correction to the Bohr magneton

\[
\mu_0 = \frac{e}{2m}
\]

which is called the anomalous magnetic moment of a particle,

\[
\Delta \mu = \mu_0 \frac{\alpha}{2\pi},
\]

manifests itself only in the nonrelativistic limit for weak quasi-static fields [7]. Indeed, when the influence of an intense external field is accurately taken into account, the anomalous magnetic moment of a particle calculated in QED as a one-loop radiative correction decreases with increasing field intensity and increasing energy of the moving particles from the Schwinger value to zero.

\(^1\)Here, we use a system of units with \( \hbar = c = 1, H_c = 4.41 \cdot 10^{13} \text{ G} \) is the characteristic scale of the electromagnetic field intensity in QED, \( e \) and \( m \) are the electron charge and mass, and \( \alpha = 1/137 \) is the fine-structure constant.
In particular, for magnetic fields $H \sim H_c$, the anomalous magnetic moment of an electron is described by the asymptotic formula [7,10]

$$\Delta \mu(H) = \mu_0 \frac{\alpha}{2\pi} \ln \frac{2H}{H_c}$$

(1)

It follows from (1) that $\Delta \mu(H)$ becomes zero only at one point while decreasing with increasing field. A similar expression for the anomalous magnetic moment of an electron in an intense constant crossed field $\mathbf{E} \perp \mathbf{H}$; ($E = H$) at $H p_\perp \gg mH_c$, where $p_\perp$ is the electron momentum component perpendicular to $[\mathbf{E} \times \mathbf{H}]$, can be represented as [11]

$$\Delta \mu(E) = \mu_0 \frac{\alpha \Gamma(1/3)}{9\sqrt{3}} \left(\frac{3p_\perp H}{mH_c}\right)^{-2/3}$$

(2)

Note that $\Delta \mu(E) \neq 0$ in the entire range of parameters.

Numerous calculations of the Lagrangian for an electromagnetic field (see, e.g., [1-3, 5,7, 11]) have been performed by assuming that the magnetic moment of electrons is exactly equal to the Bohr magneton, i.e., at $\Delta \mu = 0$. However, the following question is of considerable importance in elucidating the internal closeness of QED: What effects will an allowance for the anomalous magnetic moments of electrons and positrons produce when calculating the polarization of an electron-positron vacuum by intense electromagnetic fields?

Thus, it is of interest to compare the radiative corrections to the Maxwellian Lagrangian of a constant field calculated by the traditional method with the results that can be obtained from similar calculations by taking into account the nonzero anomalous magnetic moments of particles. The fact that the Lagrangian replacing the Heisenberg-Euler Lagrangian with nonzero anomalous magnetic moments can be calculated by retaining the method of exact solutions of the modified Dirac-Pauli equation [8] in arbitrarily intense

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2The author of [6] took into account the anomalous magnetic moments when analyzing vacuum polarization of arbitrary spin particles. To this end the description particles with anomalous magnetic moments in external electromagnetic fields the second order wave equation was considered. However, the Dirac relativistic wave equations provide a basis for relativistic quantum mechanics and quantum electrodynamics of spinor particles. It is well known that the existence of a anomalous magnetic moment of an electron can be taken into account in the modified Dirac equation by means of Pauli interaction term $\sim \Delta \mu \sigma^{\mu \nu} F_{\mu \nu}$, were $F_{\mu \nu}$—the electromagnetic field strength tensor [8] (see also [7, 18]).
electromagnetic fields also deserves serious attention. In the approach being developed, the suggested theoretical generalization initially contains no constraints on the electromagnetic field intensity.

It should be noted that nonzero anomalous magnetic moments also appear in some of the modified quantum field theories (QFTs) that also describe the electromagnetic interactions. In particular, this is true for a generalization of the traditional QFT known as the theory with "fundamental mass" (see, [12, 13] and references therein). The starting point of this theory is the condition that the mass spectrum of elementary particles is limited. This condition can be represented as

\[ m \leq M, \quad (3) \]

where the new universal parameter \( M \) is called the fundamental mass. Relation (3) is used as an additional fundamental physical principle that underlies the new QFT. A significant deviation from traditional calculations is the fact that the charged leptons in QED with fundamental mass have magnetic moments that are not equal to the Bohr magneton. This is because, apart from the traditional "minimal" term, the new Lagrangian of the electromagnetic interaction includes "nonminimal" terms. Thus, an electron in modified QED has an anomalous magnetic moment from the outset:

\[ \Delta \mu = \mu - \mu_0 = \mu_0 \left( \sqrt{1 + \frac{m^2}{M^2}} - 1 \right) \]

An important aspect of the problem under consideration is that the current state of the art in the development of laser physics [14] allows one to carry out a number of optical experiments to directly measure the contributions from the nonlinear vacuum effects predicted by various generalizations of Maxwellian electrodynamics [15]. Therefore, it should be emphasized that experimental verification of the nonlinear vacuum effects with a high accuracy in the presence of relatively weak electromagnetic fields can also provide valuable information about the validity of QED predictions at small distances [16, 17]. Note in passing that precision measurements of various quantities (e.g., the anomalous magnetic moments of an electron and a muon) at non-relativistic energies, together with studies of the particle interaction at high energies, are of current interest in the same sense.

2. THE CORRECTION TO THE LAGRANGIAN OF AN
Let us consider the correction to the Lagrangian of an electromagnetic field attributable to the polarization of an electron-positron vacuum in the presence of an arbitrarily strong constant magnetic field by taking into account the nonzero anomalous magnetic moments of the particles. To solve this problem, it is convenient, as in the standard approach [5], to represent the electron-positron vacuum as a system of electrons that fill "negative" energy levels. For a constant uniform magnetic field, the Dirac-Pauli equation containing the interaction of a charged lepton with the field (including the anomalous magnetic moment of the particle) has an exact solution [18]. In this case, the energy eigenvalues explicitly depend on the spin orientation with respect to the axis of symmetry specified by the magnetic field direction. Thus, the energy spectrum of an electron that moves in an arbitrarily intense constant uniform magnetic field is

\[
E_n(p, H, \zeta) = m \left[ \frac{p^2}{m^2} + \sqrt{\frac{H}{H_c}}(1 + 2n + \zeta) + \zeta \frac{\Delta \mu}{2\mu_0} \frac{H}{H_c} \right]^{21/2} \tag{5}
\]

where \(p\) is the electron momentum component along the external field \(H\), \(n = 0, 1, 2, \ldots\) is the quantum number of the Landau levels, and \(\zeta = \pm 1\) characterizes the electron spin component along the magnetic field.

Noting that the radiative correction to the classical density of the Lagrangian is equal, to within the sign, to the total energy density of the electron-positron vacuum in the presence of an external field [5]

\[
\mathcal{L}' = -W^H.
\]

Let us calculate \(W^H\) in a constant magnetic field by taking into account the anomalous magnetic moment of the electron. Without dwelling on the details of standard calculations, we represent \(W^H\) as

\[
W^H = -\frac{|eH|}{(2\pi)^2} \int_{-\infty}^{\infty} dp \left[ -\varepsilon_0^+ (p) + \sum_{n=0}^{\infty} \left[ \varepsilon_n^- (p) + \varepsilon_n^+ (p) \right] \right], \tag{6}
\]

where

\[
\varepsilon_n^\pm = \sqrt{p^2 + m^2 \left( \sqrt{1 + 2 \frac{H}{H_c} n \pm \frac{H}{4H_c^2}} \right)^2}. \tag{7}
\]
Using the Laplace and Fourier integral transforms for the functions that define (6) and performing summation over Landau levels, we can obtain the following formula for $\mathcal{L'}$:

$$\mathcal{L'} = -\frac{m^4 \gamma b_1}{8\pi^2} \int_0^\infty \frac{d\eta}{\eta^2} e^{-\eta} \left[ \sinh(b) \right.$$ 
$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-ix} \frac{dx}{x} \cot(-i\gamma \eta/b_1 + x\gamma) \left( \frac{1}{2} \right)_1 F_2 \left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{4}, \frac{3}{4} \right\}, -\frac{b_1^4}{64x^2} \right) \right],$$

(8)

where we use the notation $a_1 = \eta H/2H^*_c$, $b_1 = 1 + (H/4H^*_c)^2$, $b = a_1/b_1$, $\gamma = H/H_c$, $\left( \frac{1}{2} \right)_1 F_2(z)$ is the generalized hypergeometric function. Formula (8) is an exact expression for the Lagrangian with an allowance made for the anomalous magnetic moment calculated in the one-loop approximation in an arbitrarily intense magnetic field. An important deviation from the Heisenberg-Euler Lagrangian is that expression (8) contains the additional field scale

$$H^*_c = \frac{m}{4\Delta \mu}.$$  

(9)

In the theory with fundamental mass, it would be natural to call the quantity

$$H^*_c = \frac{M^2}{e} = \frac{M^2}{m^2} H_c,$$

(10)

a fundamental field.

Passing to the limits of integration over $x$ from zero to $\infty$ in (8) and using the evenness of the function $\left( \frac{1}{2} \right)_1 F_2(z)$, we obtain

$$\mathcal{L'} = -\frac{m^4 \gamma b_1}{8\pi^2} \int_0^\infty \frac{d\eta}{\eta^2} e^{-\eta}$$

$$\left[ \sinh(b) + \frac{2}{\pi} \int_0^\infty \frac{dx}{x} \left( \frac{\sin(2\gamma x)}{\cosh(2y)} - \frac{\sin(x)}{\cos(2\gamma x)} \right) \left( \frac{1}{2} \right)_1 F_2(z) \right],$$

(11)

where $y = \eta \gamma/b_1$, and $z = -b_1^4/64x^2$. 6
In particular, it immediately follows from (11) that

$$\text{Im} \mathcal{L}' = 0$$

The fact that the Lagrangian $\mathcal{L}'$ is real for all possible field intensities suggests the absence of unstable modes; i.e., the vacuum in a constant uniform magnetic field in the case under consideration, as in traditional QED, is stable against the spontaneous production of electron-positron pairs.

Next, let us separate out the integral over $x$ in expression (11). After several obvious substitutions, it reduces to

$$I = \int_0^\infty \frac{du}{u} \left[ \frac{2a_2 \sin(u) \cos(b_2 u) + (1 - a_2^2) \sin(b_2 u)}{[1 + a_2^2 - 2a_2 \cos(u)]} \right] {}_1F_2(z_1),$$  \hspace{1cm} (12)

where $a_2 = e^{-2y}$, $b_2 = (2\gamma)^{-1}$, $z_1 = -\frac{b_4(2\gamma)^2}{64u}$. Since the expansions

$$\frac{\sin(u)}{1 + a_2^2 - 2a_2 \cos(u)} = \sin(u) + a_2 \sin(2u) + a_2^2 \sin(3u) + \cdots;$$

$$\frac{1 - a_2^2}{1 + a_2^2 - 2a_2 \cos(u)} = 1 + 2a_2 \cos(u) + 2a_2^2 \cos(2u) + \cdots,$$

are valid, we obtain for (12)

$$I = \int_0^\infty \frac{du}{u} \left[ -\sin(b_2 u) + 2 \sum_{k=0}^{\infty} a_2^k \sin[u(k + b_2)] \right] {}_1F_2(z_1).$$  \hspace{1cm} (13)

It is easy to see that the following expansion of the function ${}_1F_2(z)$ at zero may be used in a field that is weak compared to the fundamental field $H^*_c$.

$${}_1F_2(z_1) = 1 + \frac{16}{3} z_1 + \frac{256}{105} z_1^2 + \cdots$$  \hspace{1cm} (14)

Hence, we obtain for (13)

$$I = \frac{\pi}{2} \cdot \frac{1 + a_2}{1 - a_2} = \frac{\pi}{2} \cdot \text{cth}(y),$$  \hspace{1cm} (15)

where $y = \eta \gamma$. 
Substituting (15) into (11) and performing standard regularization of the derived integral [5] yields

\[ \mathcal{L}' = -\frac{m^4}{8\pi^2} \int_0^\infty \frac{e^{-\eta}}{\eta^3} \left[ \eta \gamma \text{cth} (\eta \gamma) - 1 - \frac{\eta^2 \gamma^2}{3} \right] d\eta. \]  

(16)

Thus, it follows from (16) that in the limit of a weak field, formula (11) matches the Heisenberg-Euler Lagrangian [1] for an arbitrarily intense constant uniform magnetic field.

Next, let us consider \( H > 4H_c^* \). It is easy to verify that in the limit of extremely strong fields, \( H \gg 16H_c^{*2}/H_c \), we may again use expansion (14) and can obtain the following expression for integral (13):

\[ I = \frac{\pi}{2} \text{cthy}, \]  

(17)

where \( y = 16\eta H_c^{*2}/HH_c \). Results (15) and (17) have a simple meaning: for a sufficiently wide energy gap that separates the electron and positron states, the terms with large numbers \( k \) make the largest contribution to integral (13). However, for magnetic fields close to the fundamental field, \( H \sim 4H^* \), i.e., when the gap width is close to zero, the term with \( k = 0 \) makes the largest contribution to the sum in the integrand of (13). In this case, integral (13) can be calculated exactly. Our calculations yield

\[ I = \frac{\pi}{2} \cdot \text{ch} \left[ \eta \frac{H}{2H_c^*} \frac{1}{1 + (H/4H_c^*)^2} \right]. \]  

(18)

The estimates of integral (13) in the three ranges of magnetic fields (\( H \ll H_c^*, H \sim 4H_c^*, \) and \( H \gg H_c^* \)) can be represented as a single formula:

\[ I = \frac{\pi}{2} \cdot \text{ch} \left[ \eta \frac{H}{2H_c^*} \frac{1}{1 + (H/4H_c^*)^2} \right] \text{ch} \left[ \frac{\eta \gamma}{1 + (H/4H_c^*)^2} \right]. \]  

(19)

Substituting (19) into (11) and regularizing the remaining diverging in-
tegral \(^3\), we obtain
\[
L' = -\frac{m^4}{8\pi^2} \int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta} \left[ \eta b_1 \gamma \frac{\cosh[(1 + a)\eta \gamma/b_1]}{\sinh(\eta \gamma/b_1)} - b_1^2 - \frac{\eta^2}{6} \gamma^2 \left(2 + 6a + 3a^2\right) \right],
\]
were \(a = H_c/2H_c^*\).

3. ASYMPTOTIC RESULTS

In weak fields \((H \ll H_c)\) and in the limit of very strong magnetic fields \((H \gg 16H_c^{*2}/H_c)\) the integrand in (20) admits an expansion into a series. In the first approximation
\[
L' = \frac{m^4\gamma^4}{2880\pi^2b_1^2} \left[8 - 15a^2(2 + a)^2\right] \int_0^\infty d\eta \eta e^{-\eta},
\]
whence it follows that
\[
L' = \frac{m^4\gamma^4}{2880\pi^2b_1^2} \left[8 - 15a^2(2 + a)^2\right]. \tag{21}
\]

Thus, the quantum correction to the Maxwellian Lagrangian in the limit of a weak field \((H \ll H_c)\) can be represented as
\[
L' = \frac{m^4H^4}{360\pi^2H_c^4} \left[1 - \frac{15}{2}a^2 - \frac{15}{2}a^3 - \frac{15}{8}a^4 + O(\gamma^2; \gamma^2a^2)\right], \tag{22}
\]
where the first term matches the standard Heisenberg-Euler formula. The first correction to it attributable to the anomalous magnetic moments of the particles is negative and quadratic in \(a\).

In an extremely strong field \((H \gg 16H_c^{*2}/H_c)\) we can also obtain from (21)
\[
L' = \frac{m^4}{180\pi^2a^4} \left(8 - 60a^2 - 60a^3 - 15a^4\right) \left(1 - \frac{8}{\gamma^2a^2}\right). \tag{23}
\]

\(^3\)First, as usual \([5]\), the part of the integral that does not contain the magnetic field intensity and that represents the energy of the free vacuum elections should be discarded. Second, it is necessary to subtract the contribution proportional to \(H^2\) that has already been included in the unperturbed field energy. Discarding this term is related to renormalizing the field intensity and, hence, the charge. Finally, subtracting a contribution on the order of \(H^4/H_c^{*4}\) basically corresponds to renormalizing an additional parameter of the theory—the anomalous magnetic moment of the particle.
According to (23), in the limit of extremely strong fields, the Lagrangian $\mathcal{L}'$ ceases to depend on the field; i.e., as the field grows, the quantum correction to the density of the Lagrangian in the case under consideration asymptotically approaches the constant

$$\mathcal{L}'_\infty = \frac{\alpha^2}{360\pi^2(\Delta\mu)^4}. \quad (24)$$

In a sense, the result obtained may be compared with the situation observed in the standard model, where the cross sections for several processes cease to increase with energy if, apart from a photon, vector $W^\pm$- and $Z^0$-bosons, an additional diagrams with a Higgs $H$-boson is included in the analysis. An allowance made for this diagram reduces the increasing terms in amplitude and leads to a behavior of the cross sections consistent with the unitary limit. Since the standard model does not predict the mass of the H-boson, it may well be that this particle is much heavier than the t-quark, the heaviest known elementary particle. Thus, $M_H \sim 1$ TeV may prove to be the critical mass that limits the mass spectrum of elementary particles, i.e., acts as the fundamental mass (see (3))

By comparing the correction $\mathcal{L}'$ with the Lagrangian of the Maxwellian field, we can determine the field intensity

$$F_c^* = \sqrt{\frac{256\alpha}{45\pi}} \frac{H_c^*}{H_c}, \quad (25)$$

at which $\mathcal{L}_0$ becomes equal to (24). For $H = F_c^*$, the quantum correction $\mathcal{L}'$ does not yet reach its asymptotic value of $\mathcal{L}'_\infty$. A comparison of $\mathcal{L}_0$ and $\mathcal{L}'$ in other field ranges clearly shows that the corrections $\mathcal{L}'$ are always small compared to the Lagrangian $\mathcal{L}_0$. The relative corrections $\mathcal{L}'/m^4$ derived from (20) for the anomalous magnetic moments of particles $\Delta\mu_1/\mu_0 = 10^{-3}$, $\Delta\mu_2/\mu_0 = 10^{-3.05}$ and $\Delta\mu_3/\mu_0 = 10^{-3.1}$ are plotted against magnetic field intensity $\gamma = H/H_c$ in Fig. 1.

We estimate the Lagrangian for strong magnetic fields by using formula

\[\text{Note in this connection that the central point in the program of research on the Large Hadron Collider (LHC) at CERN is the search for Higgs bosons in a mass range up to 1 TeV.}\]
Figure 1: Upper curve corresponds to $\Delta \mu_3/\mu_0 = 10^{-3.1}$.

(20), which we will represent after the substitution $\gamma \eta/b_1 \to x$ as

$$L' = -\frac{m^4 \gamma^2}{8\pi^2} \int_0^\infty \frac{e^{-b_1 x/\gamma}}{x^3} \left[ x \frac{\text{ch}[x(1 + a)]}{\text{sh}(x)} - 1 - \frac{x^2}{3} \left( 1 + 3a + \frac{3}{2}a^2 \right) \right] dx. \quad (26)$$

For $H_c << H << 16H_c^{*2}/H_c$ the range $1 << x << 16H_c^{*2}/H_c^2$ is important in integral (26). In this case, the hyperbolic functions may be substituted with exponentials, and the integrand in (26) becomes

$$\frac{e^{f_1(a, \gamma, x)}}{x^2} - \frac{e^{f_2(a, \gamma, x)}}{x^3} - \frac{e^{f_2(a, \gamma, x)}}{3x} \left( 1 + 3a + \frac{3}{2}a^2 \right), \quad (27)$$

where

$$f_1(a, \gamma, x) = -\frac{1}{4\gamma}(2 - a\gamma)^2 x$$

$$f_2(a, \gamma, x) = -\frac{1 + \frac{a^2}{4}\gamma^2}{\gamma} x.$$
For $H_c << H < 4H_c^*$ (1 << $\gamma < 2/a$) $f_1 \sim f_2 = -x/\gamma$ and we find from (26) with a logarithmic accuracy that

$$\mathcal{L}' = \frac{m^4 \gamma^2}{24\pi^2} \left(1 + 3a + \frac{3}{2}a^2\right) \ln\gamma,$$

(28)

For $a \to 0$ this formula matches the Heisenberg-Euler Lagrangian in the limit of strong magnetic fields, $H >> H_c$ [5].

If $4H_c^* < H << 16H_c^*^2/H_c$ or $2/a < \gamma << 4/a^2$, then $f_1 \sim f_2 = -\gamma a^2 x/4$, and the range $1 << x << 4/(a^2 \gamma)$ gives the largest contribution to the integral. In this case, we find from (26) that

$$\mathcal{L}' = \frac{m^4 \gamma^2}{24\pi^2} \left(1 + 3a + \frac{3}{2}a^2\right) \left(2\ln\frac{2}{a} - \ln\gamma\right).$$

(29)

For $H >> 16H_c^*^2/H_c$, we return to the cases considered above (see (23)) where the range $x << 1$ gives the largest contribution to integral (26).

If $a\gamma = 2$, i.e., for $H = 4H_c$ the exponent $f_1$, in one of the exponentials in (27) becomes zero. It is easy to verify that this term is attributable to the contribution from the ground energy state $\varepsilon_0$ (see formula (7)) in which the dependence on particle mass completely drops out at this field intensity. There is no such state with a "dropping" mass in the structure of the Heisenberg-Euler Lagrangian for a fixed field. However, if we consider the passage to the limit $m^2 \to 0$, then the Heisenberg-Euler Lagrangian can simulate such an effect. It is easy to see that the total contribution of the ground state is small compared to the contribution of the last term in (27), which owes its origin to the field renormalization in expression (11). A similar conclusion can also be reached by considering integral (16).

The following should be emphasized when commenting the analogy between the limits $m^2 \to 0$ in the Heisenberg-Euler Lagrangian (see formula (16)) and $H \to 4H_c^*$ in (26). As we showed above, for $H = 4H_c^*$ in the modified Lagrangian, just as in the Heisenberg-Euler Lagrangian for $m^2 = 0$, the exponent in the terms whose contribution is vanishingly small against the background of the contributions from the renormalization procedure becomes zero. In other words, in both cases, the ground states in the structure of the integrand are equally preferential, but their contribution to the integral is not dominant.
Neglecting the first and the second terms in (27), we find from (26) that

\[ L' = \frac{m^4}{6\pi^2a^2} \left( 1 + 3a + 3a^2/2 \right) \ln \left( \frac{2}{a} \right). \]

This result agrees with formulas (28) and (29) from which it can be obtained by the substitution \( \gamma = 2/a \). Thus, these functions are continuously joined at \( H = 4H_c^* \).

Finally, let us estimate the Lagrangian \( L' \) in terms of traditional QED, i.e., by taking into account the nonzero anomalous magnetic moments of particles in intense electromagnetic fields attributable to radiative effects. Substituting \( \Delta \mu \) from (1) into the expression \( b_1 = 1 + (\Delta \mu H/m)^2 \) yields an estimate of \( L' \) (26) in the limit of extremely strong fields. For a constant magnetic field, \( \gamma \gg \mu_0/\Delta \mu \)

\[ b_1 \sim \alpha_2 \ln^2(2\gamma), \]

where

\[ \alpha_2 = \alpha^2/16\pi^2. \]

In this case, the exponent in (26) is

\[ f_3 = -b_1 \frac{x}{\gamma} = -\frac{\alpha_2 \ln^2(2\gamma)}{\gamma} \frac{x}{x}. \]

If \( \gamma \gg \alpha_2 \ln^2(2\gamma) \), then the range \( 1 \ll x \ll \gamma/\alpha_2 \ln^2(2\gamma) \) is important in (26). Thus, we can find from (26) that

\[ L' = \frac{m^4 \gamma^2}{24\pi^2} \left[ \ln \gamma - \ln \alpha_2 - 2\ln(\ln 2\gamma) \right]. \]  

The first term in (30) is identical to its estimate in the Heisenberg-Euler theory [5]. The relative effective Lagrangian \( L'/m^4 \) derived from the integral representation (26) (with \( \alpha_2 \sim 3.4 \cdot 10^7 \)) is plotted against magnetic field intensity in Fig. 2 (curve 1). For comparison, the same figure also shows a plot for \( \Delta \mu = 0 \) (curve 2).

By similar arguments in the case of an intense constant crossed field

\[ \gamma^{2/3} \alpha_1 \gg 1, \]
Figure 2: Upper curve corresponds to $\Delta \mu = 0$.

were

$$\alpha_1 = \frac{\alpha^2 \Gamma^2 (1/3)}{972} \left( \frac{3\rho \perp}{m} \right)^{-4/3},$$

we have

$$b_1 \sim \gamma^{2/3} \alpha_1.$$

For exponent (26) in this case, we fined

$$f_3 = -\frac{b_1}{\gamma} = -\frac{\alpha_1}{\gamma^{1/3}} x.$$

If $\gamma^{1/3} \gg \alpha_1$, then the range $1 \ll x \ll \gamma^{1/3}/\alpha_1$ is important in (26).

Thus, we can find from (26)

$$L' = \frac{m^4 \gamma^2}{72 \pi^2} (\ln \gamma - 3 \ln \alpha_1).$$

(31)

It follows from (31), that in intense crossed field the presence of $\Delta \mu(E) \neq 0$ decreases to three times the radiative correction $L'$ then analogous result under the condition $\Delta \mu(E) = 0$

$$\frac{L'}{L_0} = \frac{\alpha}{9\pi} \ln \gamma.$$
The relative effective Lagrangian $\mathcal{L}'/m^4$ at $m/p_\perp = 0.2$ (with $\alpha_1 \sim 10^{-8}$) is plotted against crossed field intensity in Fig. 3 (curve I). For comparison, the same figure shows a plot for $\Delta \mu = 0$ (curve 2).

Note that an allowance made for the anomalous magnetic moments of vacuum particles in terms of universally accepted QED leads to a decrease in the radiative correction to the field energy density. Recall that we reached a similar conclusion by considering the static anomalous magnetic moment that arises, in particular, in the modified field theory. Thus, irrespective of the nature of the anomalous magnetic moment attributable to the dynamic or static types of interaction, we obtain consistent results. Our conclusions are also important in studying the anomalous magnetic moment as the most accurate calculable and measurable (in numerous precision experiments) characteristic of particles.

4. CONCLUSIONS

Our results can be of considerable importance in constructing astrophysical models, in particular, in studying extremely magnetized neutron stars—magnetars; interest in the existence of the latter objects has increased appreciably in recent years (see, e.g., [4] and references therein). According
to models for the macroscopic magnetization of bodies composed mostly of neutrons, the intensity of the magnetic fields frozen into them increases from the surface to the central regions and can reach $10^{15} - 10^{17}$ G [19].

Note also that the radiative effects can be enhanced by external intense electromagnetic fields not only in Abelian, but also in non-Abelian quantum field theories. For example, an allowance made for the influence of an external field on such parameters as the lepton mass and magnetic moment in terms of the standard model leads to nontrivial results. In this case, apart from the electrodynamic contribution, the one-loop mass operator of a charged lepton also contains the contributions from the interaction of $W^\pm$, $Z^0$, and H-bosons with a vacuum. It is easy to see that, in the absence of an external field, the contribution from weak interactions to the radiative shift of the lepton mass $m$ is suppressed by a factor of $(m/M_i)^2$ $(i = W, Z, H)$ compared to the electrodynamic contribution. However, the contributions of weak currents in the ultrarelativistic limit can dominate in intense external fields, as was first noted in [20] (see also [21]).

In close analogy with the quantum corrections to the particle masses, the anomalous magnetic moments of charged leptons in the standard model are attributable to the vacuum radiative effects of electromagnetic and weak interactions and contain the contribution from the hadron polarization of the vacuum. For example, for the anomalous magnetic moment of a muon,

$$a_{\mu}^{SM} = a_{\mu}(QED) + a_{\mu}(weak) + a_{\mu}^{had}.$$  

According to recent theoretical estimates made in the standard model [22], the contributions from electromagnetic and weak interactions can be written as

$$a_{\mu}(QED) = 11658470.57(0.29) \cdot 10^{-10};$$
$$a_{\mu}(weak) = 15.1(0.4) \cdot 10^{-10}.$$  

Although the calculations of the contributions from the hadron polarization of a vacuum to $a_{\mu}^{SM}$ have a history that spans almost forty years, $a_{\mu}^{had}$ is currently known with the largest uncertainty (see, e.g., [23-28]). One of the most reliable estimates for the contributions of the lowest-order hadron polarization of a vacuum that generalizes the data on hadron $t$-decay and $e^+e^-$ annihilation appears as follows [23,24]:  

$$a_{\mu}^{had} = 692(6) \cdot 10^{-10}$$

\[5\] See, however, [25], where the contribution of the highest orders of hadron polarization
The theoretical anomalous magnetic moment of a muon in the standard model takes the form [28]

\[ a_{\mu}^{SM} = 11659177(7) \cdot 10^{-10} \]

The results of one of the most resent \((g - 2)\) experiments aimed at measuring the anomalous magnetic moments of positive polarized muons carried out on a storage ring with superconducting magnets at Brookhaven National Laboratory (BNL) can be represented as

\[ a_{\mu}^{exp} = 11659204(7)(5) \cdot 10^{-10}. \quad (32) \]

The data obtained yield the difference

\[ \Delta_{\mu} = a_{\mu}^{exp} - a_{\mu}^{SM} = 27 \cdot 10^{-10} \quad (33) \]

which exceeds the total measurement errors and the uncertainties of the theoretical estimates. According to the most recent reports from the BNL muon \((g - 2)\) collaboration [29], the relative value of this excess is 2.6. A twofold increase in this accuracy is expected in the immediate future. Clearly, the solution of the muon \((g - 2)\) problem may lead to the appearance of a new theory outside the scope of the standard theory.

Recall in this connection that the anomalous magnetic moment of a muon in the modified theory contains the contribution attributable to the new universal parameter \(M\) from the outset. According to (4),

\[ a_{\mu}(M) = \frac{m_{\mu}^2}{2M^2}, \quad (34) \]

where \(m_{\mu}\) – is the muon mass. It is easy to see that \(a_{\mu}(M)\) is equal in order of magnitude to (32) at \(M \sim 1\) TeV.

The principal conclusion drawn from a comparison of the above estimates is that we cannot rule out the possibility that the observed difference between the theoretical and experimental values for \(\Delta_{\mu}\) is equal to \(a_{\mu}(M)\). As was pointed out above, the parameter \(M\) in the new theory may be related to the Higgs boson mass \((M_H)\). In this case, the difference between \(a_{\mu}^{exp}\) and of a vacuum were calculated, and the recent papers \([26, 27]\), in which the contribution from the third-had order diagrams to \(a_{\mu}^{had}\) attributable to photon-photon scattering was taken into account.
$a_\mu^{SM}$ can provide valuable information about the particle whose mass has not been determined in the standard model. Substituting $m_\mu$ and the anomalous magnetic moment of a muon into (33), we can easily impose the following constants on the $H$-boson mass:

$$1.2 \text{ TeV} \leq M_H \leq 1.8 \text{ TeV}.$$  

The standard model with the Higgs boson mass $M_H \geq 1 \text{ TeV}$ entails several additional features, in particular, the impossibility to describe the weak interactions in the sector of $H, W,$ and $Z$-particles in terms of the perturbation theory [30]. Naturally, the need for constructing a new nonperturbative theory arises in this case. Apart from the condition for the mass spectrum being limited, $m \leq M$ (see (2)), the Higgs mechanisms of mass formation and compensation for the discrepancies can become integral elements of one of the most promising versions of the modified theory—the standard model with fundamental mass.

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