Multiplicities and dimensions in enveloping tensor categories

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1Abstract. In a previous paper, semisimple tensor categories were constructed from certain regular Mal’cev categories. In this paper, we calculate the tensor product multiplicities and the categorical dimensions of the simple objects. This yields also the Grothendieck ring. The main tool is the subquotient decomposition of the generating objects.

1. Introduction

Fix an algebraically closed field $K$ of characteristic 0. By a tensor category we mean a $K$-linear, locally finite, abelian category which is equipped with a symmetric tensor product such that each object has a dual object (rigidity) and such that $\text{End}\, \mathbb{1} = K$. They are also referred to as symmetric tensor categories or pre-Tannakian categories. A typical example is the category of finite dimensional $K$-vector spaces or, more general, representations of a linear algebraic group. Categories of this type are called Tannakian.

First examples of non-Tannakian categories were found by Deligne, see e.g., [Del07], using an interpolation procedure. In [Kno07] a different method for the construction of non-Tannakian categories was devised. It is based on the observation that the category of relations of a base category $\mathcal{A}$ is already rigid, symmetric, and monoidal. After making it $K$-linear and twisting the product of relations by a $K$-valued degree function $\delta$ one obtains a category $\mathcal{T} = \mathcal{T}(\mathcal{A}, \delta)$ which is in many cases semisimple, hence in particular abelian.

The necessary assumptions on $\mathcal{A}$ and $\delta$ recalled in Section 2. They hold, for example, for any category of finite algebraic objects containing a group structure, like finite groups, finite rings, finite modules, Boolean algebras and many more. Deligne’s category from [Del07] is recovered by taking for $\mathcal{A}$ the opposite category of the category of finite sets.

In the present paper we start analyzing the internal structure of $\mathcal{T}$ in case it is semisimple. Besides $\mathcal{T}$ being an interesting object in its own right there is another motivation: Despite
\( \mathcal{T} \) not being an interpolation category \textit{a priori}, in many cases it can be realized as one \textit{a posteriori}. Therefore, it may be possible to transfer properties from \( \mathcal{T} \) to the interpolated categories. This paper contains two examples for this transfer.

As for any semisimple category the most important objects of \( \mathcal{T} \) are the simple ones. In fact, these were already determined in \cite{Kno07} as part of the semisimplicity proof: simple objects of \( \mathcal{T} \) are classified by pairs \((x, \chi)\) where \( x \) is an object of \( \mathcal{A} \) (up to isomorphism) and \( \chi \) is an irreducible character of its automorphism group. The corresponding simple object is denoted by \([x]_{\chi}^0\).

It is now a natural problem how the simple objects relate to the tensor product. There are two classical questions: \textit{a}) how does the tensor product \([x]_{\chi}^0 \otimes [y]_{\psi}^0\) decompose in simple objects and \textit{b}) what is the (internal) dimension \( \dim_{\mathcal{T}} [x]_{\chi}^0 \) of a simple object? Both of these questions are being answered in the present paper.

For \( i) \) define for any three objects \( x_1, x_2, \) and \( x_3 \) the set
\[
(1.1) \quad T(x_1, x_2, x_3) := \{ r \subseteq x_1 \times x_2 \times x_3 \mid r \rightarrow x_1, x_2, x_3; r \rightarrow x_1 \times x_2, x_1 \times x_3, x_2 \times x_3 \}.
\]
Let \( \chi_{T(x_1, x_2, x_3)} \) be the induced permutation character of \( \text{Aut}_{\mathcal{T}}(x_1) \times \text{Aut}_{\mathcal{T}}(x_2) \times \text{Aut}_{\mathcal{T}}(x_3) \).

\textbf{1.1. Theorem} (Corollary 5.5 below). Let \([x]_{\chi_1}^0\) and \([x]_{\chi_2}^0\) be two simple objects of \( \mathcal{T} \). Then
\[
(1.2) \quad [x]_{\chi_1}^0 \otimes [x]_{\chi_2}^0 \cong \bigoplus_{x, \chi} \langle \chi_T(x_1, x_2, x) \mid \chi_1 \otimes \chi_2 \otimes \chi \rangle [x]_{\chi}^0.
\]

For the dimension formula we give two variants. For the simpler one we need the following notation. For any object \( x \) let \( s(x) \) and \( q(x) \) the lattice of subobjects and quotient objects, respectively, where \( q(x) \) is ordered in such a way that \( z = x \) is the minimum. For any automorphism \( g \) of \( x \) let \( q(x)^g = \{ z \in q(x) \mid gz = z \} \). Let \( x_g \in q(x)^g \) be the minimal element \( z \) such that \( g|_z = \text{id}_z \). Let moreover \( \hat{x}_g \in q(x)^g \) be the join of all atoms. Finally, let \( \mu_{\Lambda} \) denote the Möbius function of a lattice \( \Lambda \). Define, in particular,
\[
(1.3) \quad \omega_x := \sum_{y \in s(x)} \mu_{s(x)}(y, z) \delta(y \rightarrow 1).
\]

\textbf{1.2. Theorem} (Corollary 6.10 below). Let \( x \) be an object of \( \mathcal{A} \) and \( \chi \) an irreducible character of \( \mathcal{A} = \text{Aut}_{\mathcal{T}}(x) \). Then \( \dim_{\mathcal{T}} [x]_{\chi}^0 = \langle \chi_{[x]^0} \mid \chi \rangle_\mathcal{A} \) where the class function \( \chi_{[x]^0} \) on \( \mathcal{A} \) is defined as
\[
(1.4) \quad \chi_{[x]^0}(g) := \sum_{x_g \leq z \leq x_g} \mu_{q(x)^g}(x, z) \omega_z.
\]

These simple formulas become quite complicated when interpreted in a concrete category \( \mathcal{A} \). We succeeded in doing so for the case \( \mathcal{A} = \text{Set}^{op} \). This way, we were able to rederive a
formula of Littlewood [Lit58] for stable Kronecker coefficients (the equivalence is proved in Appendix B) and to find an apparently new identity for symmetric functions (for which we found a symmetric function proof a posteriori, see also Appendix B).

The category $\mathcal{T}$ is built from basic objects $[x]$ where $x$ is an object of $\mathcal{A}$. So the main technical result of this paper is the subquotient decomposition which also yields a decomposition of $[x]$ into simple objects. This decomposition builds up from the coarser subobject decomposition of our previous paper [Kno22].

The methods of the paper are mostly category theoretical but we also use a fair amount of lattice theory and symmetric function theory. To not interrupt the flow of the paper we collected the necessary facts in two appendices.

2. Regular categories and their tensor envelopes

In this section we briefly recall terminology, notation, and the construction of $\mathcal{T}(\mathcal{A}, \delta)$. For details see [Kno07, Kno22].

We start with a category $\mathcal{A}$ and use the following terminology: Monomorphisms will be called injective and denoted by $y \hookrightarrow x$. Every injective morphism defines a subobject of its target. The image of a morphism $y \to x$ is the smallest subobject of $x$, the morphism factors through. Morphisms whose image equals the target (i.e., extremal epimorphisms) will be called surjective and denoted by $x \twoheadrightarrow z$.

We assume throughout that $\mathcal{A}$ is finitely complete. The terminal object will be denoted by $1$. We also assume that $\mathcal{A}$ is regular. This means that all morphisms have images and that surjectivity is preserved under pullback.

A relation between objects $x$ and $y$ of $\mathcal{A}$ is a subobject $r$ of $x \times y$. If $s \subseteq y \times z$ is another relation then their product $r \circ s$ is the relation $r \circ s = \text{image}(r \times y s \to r \times z)$. In regular categories, the product of relations is associative. This way, we obtain the category $\text{Rel}(\mathcal{A})$ of relations of $\mathcal{A}$ with the same objects but relations as morphisms.

The category $\mathcal{T}(\mathcal{A}, \delta)$ is a twisted $K$-linear version of $\text{Rel}(\mathcal{A})$. The twist function $\delta$ is a map which assigns to any epimorphism $e$ an element $\delta(e)$ of some fixed base field $K$. It is subject to the requirements $D1$–$D3$ from [Kno07].

The category $\mathcal{T}^0 = \mathcal{T}^0(\mathcal{A}, \delta)$ will have the same objects as $\mathcal{A}$. More precisely, each object $x$ of $\mathcal{A}$ gives rise to an object $[x]$ of $\mathcal{T}$. The morphisms $[x] \to [y]$ are the formal $K$-linear combinations of relations $r \subseteq x \times y$. The morphism induced by the relation $r$ will be denoted by $\langle r \rangle$. The product of two such morphism is the $\delta$-twisted product of relations:

$$\langle r \rangle \langle s \rangle := \delta(r \times_y s \to r \circ s) \langle r \circ s \rangle.$$  

\footnote{This condition can be slightly relaxed to accommodate, e.g., the category of affine spaces [Kno07].}
The category $\mathcal{T}(\mathcal{A}, \delta)$ is the pseudo-abelian completion of the additive completion of $\mathcal{T}^0$, i.e., the category obtained by formally adjoining direct sums and direct summands.

With the tensor product induced by the direct product, $[x] \otimes [y] = [x \times y]$, $\mathcal{T}(\mathcal{A}, \delta)$ becomes a pseudo-abelian, symmetric, monoidal category. Its unital object is $1 = [1]$. It is also rigid with $[x]^\vee = [x]$ for all $x$. The adjoint of $f = \langle r \rangle$ is $f^\vee = \langle r^\vee \rangle$ where $r^\vee$ is the relation obtained by swapping the factors. Every morphism $f : x \to y$ gives rise to a morphism $[f] = \langle \Gamma_f \rangle : [x] \to [y]$ where $\Gamma_f = \text{im}(id \times f : x \to x \times y)$ is the graph of $f$. This way, one obtains a faithful functor $\mathcal{A} \to \mathcal{T}$.

For $\mathcal{T}$ to be semisimple more assumptions on $\mathcal{A}$ are needed. First, $\mathcal{A}$ has to be subobject finite, i.e., every object has only finitely many subobjects. This condition ensures that all morphism spaces $\text{Hom}_{\mathcal{T}}(x, y)$ are finitely generated $\mathbb{K}$-modules.

Another assumption we are going to make is that $\mathcal{A}$ is exact. Here exactness (in the sense of Barr [Bar71]) means that all equivalence relations $r \subseteq x \times x$ are effective, i.e., there always exists a quotient object $x/r$.

The most restrictive condition is the requirement of $\mathcal{A}$ being Mal’cev. This means, e.g., that every reflexive relation $r \subseteq x \times x$ is an equivalence relation. The Mal’cev property enters mostly through the validity of via Goursat’s lemma in $\mathcal{A}$ (see [Kno07, (5.1)]. It says:

In an exact Mal’cev category every relation $r \subseteq x \times y$ is of the form $\overline{x} \times_c \overline{y}$ where $\overline{x} \subseteq x$ and $\overline{y} \subseteq y$ are subobjects and $c$ is a quotient object of both $\overline{x}$ and $\overline{y}$:

$$\begin{array}{ccc}
\overline{x} & \xrightarrow{r} & \overline{y} \\
x & \xleftarrow{c} & y
\end{array}$$

In short, this means that subobjects of $x \times y$ can be described in terms of $x$ and $y$ alone.

Despite the restrictiveness of the Mal’cev condition there is an abundance of important examples. In particular, all algebraic theories containing a group operation are exact Mal’cev. This includes the categories of groups, vector spaces, rings, modules, etc. All abelian categories are exact Mal’cev, as well. Combined with our condition of subobject finiteness, we are dealing with the finite models of such theories, like finite groups, and so on.

There is also one example which is slightly less obvious, namely the category $\text{Set}^{\text{op}}$ which is opposite to the category of finite sets. This is the category which gives rise to Deligne’s category $\text{Rep} S_1$ of [Del07]. It is easily recognized as exact Mal’cev, though, by observing that it is equivalent to the category of finite Boolean algebras.
There are two more conditions. First, we will assume throughout $K$ is algebraically closed of characteristic zero. Secondly, it is convenient to assume that the terminal object $1$ has no proper subobjects. This is equivalent to $\text{End}_\mathcal{T}(1) = K$. In this case we will extend the definition of $\delta$ to objects by

$$\delta(x) := \delta(x \to 1).$$

The condition is quite innocuous since the general case can be reduced to it (see [Kno07, Thm. 3.6]).

Now we recall a numerical criterion for $\mathcal{T}(\mathcal{A}, \delta)$ to be a semisimple tensor category. Since $\mathcal{A}$ is subobject finite, its collection $s(x)$ of subobjects forms a finite set. It is partially ordered by inclusion with $x$ being the maximum. In particular, the M"obius number $\mu_s(y, x)$ is defined for every $y \in s(x)$. To every surjective morphism $e : x \to z$ one assigns the following element of $K$:

$$\omega_e := \sum_{y \leq x \in s(x)} \mu_s(y, x) \delta(e|y).$$

When $s(1) = \{1\}$ we extend $\omega_e$ to objects by setting

$$\omega_x := \omega_{x \to 1}.$$

It is a nontrivial fact, [Kno07, Lemma 8.4], that $\omega_e$ is multiplicative in $e$, i.e., for all $x \to y \to z$ one has

$$\omega_{fe} = \omega_f \omega_e.$$

This reduces the computation of $\omega_e$ to the case when $e$ is indecomposable.

We call $\delta$ non-degenerate if $\omega_e \neq 0$ for all (indecomposable) surjective morphisms $e$. Then the criterion is:

2.1. Theorem ([Kno07, Thm. 6.1 and Thm. 8.3]). Let $\mathcal{A}$ be a regular, subobject finite, exact, Mal’cev category and let $\delta$ be a degree function on $\mathcal{A}$ with values in a field $K$ of characteristic 0. Then $\mathcal{T}(\mathcal{A}, \delta)$ is a semisimple tensor category if and only if $\delta$ is non-degenerate.

2.2. Examples. There are three examples which one might want to keep in mind when reading the paper.

i) The category $\mathcal{A} = \text{Set}^{\text{op}} \cong \text{Bool}$. Here the surjective or injective morphisms are the injective or surjective maps between finite sets, respectively. The general degree function applied to an injective map $e : A \hookrightarrow B$ is $\delta(e) = t^{|B| - |A|}$ where $t \in K$ is any element. The
morphism $e$ is indecomposable if $|B| = |A| + 1$. In that case $\omega_e = t - |A|$. Hence $\mathcal{T}(A, \delta)$ is semisimple if and only if $t \not\in \mathbb{N}$.

ii) The category $A = \text{Vect}(\mathbb{F}_q)$. In this case, the degree function is $\delta(e) = t \dim V - \dim U$ where $e : U \to V$ is surjective. The morphism $e$ is indecomposable if $\dim V = \dim U + 1$ in which case $\omega_e = t - q^{\dim U}$. It follows that $\mathcal{T}(A, \delta)$ is semisimple if and only if $t$ is not of the form $q^n$ with $n \in \mathbb{N}$. Observe that $t = 0$ is permitted.

iii) The category $A = \text{Group}$ of finite groups. In this case, the degree function has infinitely many free parameters $t_S$, one for each simple group $S$. The degree function then takes up the form $\delta(G) = t_{C_1} \ldots t_{C_n}$ where $C_1, \ldots, C_n$ are the composition factors of $G$ counted with multiplicity. A surjective homomorphism $e : G \to \overline{G}$ is indecomposable if its kernel $N$ is a minimal, non-trivial, normal subgroup. The computation of $\omega_e$ involves all subgroups $H \subseteq G$ with $HN = G$. This is doable in every given case but impossible in general even if $G = N$ is simple and $\overline{G} = 1$. Since $H = G$ is always present one can say at least that $\omega_e = \delta(G) + \text{lower order terms}$. In any case, $\mathcal{T}(A, \delta)$ will be semisimple if all variables $t_S$ are algebraically independent over $\mathbb{Q}$.

Let $S$ be an object of $\mathcal{T}$ whose endomorphism ring is a division algebra. Then the non-degeneracy of $\delta$ ensures that every nonzero $\mathcal{T}$-morphism $X \to S$ onto $S$ has a section. This and induction on the size of $x$ is used to prove the following lemma:

2.3. Lemma. Under the assumptions of Theorem 2.1 let $x$ be an object of $A$. Then

i) $[x]$ has a unique direct summand $[x]^1$ with:
   a) $[x]^1$ is isomorphic to a direct summand of an object of the form $[z_1] \oplus \ldots \oplus [z_n]$ where each $z_i$ is proper subquotient of $x$.
   b) Conversely, every morphism $[z] \to [x]$ factors through $[x]^1$ whenever $z$ is a proper subquotient of $x$.

ii) $[x]^1$ has a unique complement $[x]^0$ in $[x]$.

iii) $\mathbb{K}[\text{Aut}_A(x)] \to \text{End}_\mathcal{T}([x]^0)$ is an isomorphism.

The proof of part iii) makes crucial use is made of Goursat’s lemma. It implies that any relation $r \subseteq x \times x$ is either the graph of an automorphism or the induced morphism $\langle r \rangle : [x] \to [x]$ factors through $[c]$ where $c$ is a proper subquotient of $x$ (see diagram (2.2) with $y = x$).

Part iii) also yields a classification of the simple objects of $\mathcal{T}$. Let $A := \text{Aut}_A(x)$. Then $[x]^0$ has an $A$-isotypical decomposition

\begin{equation}
[x]^0 = \bigoplus_x V_\chi \otimes_\mathbb{K} [x]^0_\chi
\end{equation}
where $V_\chi$ is the irreducible $A$-module with character $\chi$ and $[x]^0_\chi$ is a simple object of $\mathcal{T}$. Conversely, every simple object of $\mathcal{T}$ is uniquely of that form.

3. The subobject decomposition

In this section $A$ may be any regular, subobject finite category with arbitrary degree function $\delta$. Our goal is to decompose any object $[x]$ as far as possible as a direct sum. We briefly recall the subobject decomposition which was studied in detail in [Kno22].

Any decomposition requires the construction of idempotents. As a starter, each subobject $y$ of $x$ gives rise to the relation which is just the diagonal embedding $y \rightarrow x \times x$. This defines an endomorphism $p_y = \langle y \rangle$ of $[x]$ which is easily seen to be idempotent.

Now, the $p_y$ for different $y$ commute with each other. Hence all decompositions $[x] = p_y[x] + (1 - p_y)[x]$ have a common refinement

\[(3.1) \quad [x] = \bigoplus_{y \subseteq x} [y]^*,\]

called the subobject decomposition of $[x]$. Here $[y]^* := p_y^*[x]$ where $p_y^*$ is a primitive idempotent in the algebra generated by the $p_y$. These are computed by Möbius inversion:

\[(3.2) \quad p_y^* := \sum_{y' \subseteq y} \mu_s(y', y)p_{y'} \quad \text{and} \quad p_y = \sum_{y' \subseteq y} p_{y'}^*.\]

To be very precise, the subobjects $[y]^* \subseteq [x]$ also depend on $x$ but it is easy to show that the relation $y \rightarrow y \times x$ defines an isomorphism between $[y]^* \subseteq [y]$ and $[y]^* \subseteq [x]$. So we are not distinguishing between these two.

It follows from (3.1) that one can take the objects $[x]^*$ also as generators of the pseudo-abelian tensor category $\mathcal{T}$. In fact, Deligne’s construction of $\text{Rep}_S = \mathcal{T}(\text{Set}^{\text{op}}, \delta)$ does exactly that. The advantage of this route is that the space $\text{Hom}_{\mathcal{T}}([x]^*, [y]^*)$ is much smaller than $\text{Hom}_{\mathcal{T}}([x], [y])$. On the other hand, the definitions of compositions, tensor products, and even the associativity constraint become much more involved. This approach has been carried out in the paper [Kno22].

For the remainder of this paper we make the following assumptions on $A, \delta$:

1. The assumptions of Theorem 2.1 hold, i.e., $A$ is a regular, subobject finite, exact, Mal’cev category and $\delta$ is a non-degenerate degree function on $A$ with values in a field $\mathbb{K}$ of characteristic 0. This implies that $\mathcal{T} = \mathcal{T}(A, \delta)$ is a semisimple tensor category.

2. Additionally, we assume for simplicity that $1$ has no proper subobject or, equivalently, that $\text{End}_{\mathcal{T}}(1) = \mathbb{K}$ and that $\mathbb{K}$ is algebraically closed.
4. The subquotient decomposition

We have seen that every subobject \( y \in s(x) \) gives rise to an idempotent \( p_y \). What about quotients \( e : x \rightarrow z \)? The obvious candidate would be the relation \( r = x \times_z x \subseteq x \times x \). It defines an endomorphism \( q_z = \langle r \rangle = [e]^\vee [e] \) of \( [x] \) which is almost idempotent in that \( q_z^2 = \delta(e) q_z \). Hence \( \delta(e)^{-1} q_z \) is an idempotent on \( [x] \) provided \( \delta(e) \neq 0 \). Unfortunately, \( \delta(e) \) may vanish even if \( \delta \) is non-degenerate. A more serious disadvantage of the idempotents \( q_z \) is the fact that they do not commute with the idempotents \( p_y \).

But still, one can embed \([z]\) into \([x]\) in a canonical way. It is just not defined by a canonical idempotent.

4.1. Lemma. Let \( e : x \rightarrow z \) be surjective. Then \([e]^\vee : [z] \rightarrow [x]\) is injective.

Proof. It suffices to show that the map \([e]^\vee \circ : \operatorname{End}_T([z]) \rightarrow \operatorname{Hom}_T([z],[x])\) is injective. Indeed, since \([z]\) is semisimple, any projection onto the kernel of \([e]^\vee \) would be in the kernel of \([e]^\vee \circ \).

Let \( r \rightarrow z \times z \) be a relation. Then \( \langle r \rangle \in \operatorname{End}_T([z]) \) is a basis element and we have

\[
[e]^\vee \circ \langle r \rangle = \left\langle \begin{array}{c}
a \\
b \\
e \\
x \end{array} \right\rangle \xrightarrow{e^\prime} \left\langle \begin{array}{c}
\ast \\
\ast \\
\ast \\
x \end{array} \right\rangle = \left\langle \begin{array}{c}
\ast \\
\ast \\
\ast \\
x \end{array} \right\rangle.
\]

Indeed, the right equality holds because the morphism \( r \times_z x \rightarrow z \times x \) is injective which can be seen with the commutative diagram

\[
\begin{array}{ccc}
r \times_z x & \xrightarrow{(ae', b')} & z \times x \\
\downarrow{(e', b')} & & \downarrow{(\text{id}_z, \text{id}_x)} \\
(r \times x) & \xrightarrow{(a \times b, \text{id}_x)} & z \times z \times x
\end{array}
\]

Since \( r \) can be recovered from \( r \times_z x \) as image of \( (ae', eb') \) in \( z \times z \) the homomorphism \( [e]^\vee \circ \) induces an injective map from a basis into a basis. Hence it is injective itself. \( \square \)

Next we combine subobjects and quotient objects. Recall that \( z \) is a subquotient of \( x \) if there exists a subobject \( y \rightarrow x \) and a surjection \( y \rightarrow z \). When we remember \( y \) we talking about a subquotient object. More precisely, a subquotient object of \( x \) is an equivalence class of diagrams \( z \leftarrow y \rightarrow x \) where two diagrams are equivalent if there are (necessarily
unique) isomorphisms rendering the following diagram commutative:

\[
\begin{array}{ccc}
  z & \xrightarrow{\sim} & y \\
 \downarrow & \downarrow \sim & \downarrow \\
 z' & \xrightarrow{\sim} & y' \\
 & & \xrightarrow{\sim} \xrightarrow{\sim} x
\end{array}
\]

(4.3)

We denote this subquotient object also by \( z_y \). The set of subquotient objects of \( x \) will be denoted by \( \text{sq}(x) \). It is easy to see that any subquotient \( u \) of a subquotient \( z \) of \( x \) is also a subquotient of \( x \) (see, e.g., (4.11) below).

A quotient object is a subquotient object \( z_y \) with \( y = x \). That is the notion dual to subobject. Let \( q(x) \) be the set of quotient objects of \( x \). Like \( s(x) \) also \( q(x) \) is partially ordered with \( z \leq z' \) if \( x \twoheadrightarrow z \twoheadrightarrow z' \). Thus \( z = x \) is the minimum of \( q(x) \).

Any subquotient object \( z \xleftarrow{e} y \rightleftharpoons x \) induces the morphism

\[
\iota : [z] \rightarrow [x] \quad \text{(4.4)}
\]

which is injective by [Kno22, (4.1)] and Lemma 4.1. Denote its image by \( [z]_y \). It depends only on the subquotient object represented by \( e, i \). Moreover, \( \iota : [z] \rightarrow [z]_y \) is an isomorphism.

4.2. Lemma. For an object \([x]\) of \( A \) let \([x]_1\) as in Lemma 2.3. Then

\[
[x]_1 = \sum_{z \in \text{sq}(x), z \neq x} [z]_y. \quad \text{(4.5)}
\]

Proof. We show that the right hand side \([x]'\) satisfies the characterization of \([x]_1\) from Lemma 2.3(i). There is an surjective morphism \( \bigoplus_{z \in \text{sq}(x)} [z] \twoheadrightarrow [x]' \). Since \( T \) is semisimple it has a section. Hence part \( a \) is satisfied. For part \( b \) let \( f : [z] \rightarrow [x] \) be a morphism with \( z \) being a proper subquotient of \( x \). We may assume that \( f = \langle r \rangle \) for some relation \( r \subseteq z \times x \). By Goursat’s lemma we get a diagram

\[
\begin{array}{ccc}
  r & \xrightarrow{\sim} & x \\
 \downarrow & \downarrow & \downarrow \\
  z & \xrightarrow{\sim} & y \\
 \downarrow & \downarrow & \downarrow \\
 c & \xrightarrow{\sim} & x
\end{array}
\]

(4.6)

Since \( c \) is a subquotient of \( z \) it is a proper subquotient of \( x \). Hence \( f([z]) \subseteq [c]_x \subseteq [x]' \). \( \square \)

Recall that \([x]_1\) has a unique complement \([x]_0\) in \([x]\). Then each \( z_y \in \text{sq}(x) \) induces the subobject

\[
[z]_y^0 := \iota[z]^0 \subseteq [x]. \quad \text{(4.7)}
\]
4.3. Theorem (Subquotient decomposition). Let \([x]\) be an object of \(\mathcal{A}\). Then

\[(4.8) \quad [x] = \bigoplus_{z_y \in \text{sq}(x)} [z]_y^0,\]

This decomposition is compatible with the subobject filtration, i.e.,

\[(4.9) \quad p_y[x] = \bigoplus_{z_u \in \text{sq}(x) \atop u \subseteq y} [z]_u^0.\]

Moreover, the projection \(p^*_y : [x] \to [x]^*\) induces an isomorphism

\[(4.10) \quad \bigoplus_{z \in \text{sq}(x)} [z]_x^0 \cong [x]^*.\]

Proof. Let \(X := \bigoplus_{z_y \in \text{sq}(x)} [z]_y^0\) be the right hand side of (4.8). We claim that the canonical morphism \(\Phi_x : X \to [x]\) is surjective. To this end we use induction on \(|\text{sq}(x)|\). Thus we assume that \(\Phi_z\) is surjective for all proper subquotients \(z\) of \(x\). Let \(z_y\) be a subquotient object of \(x\) and \(z_y'\) a subquotient object of \([z]\). Then we obtain the following diagram with Cartesian square:

\[(4.11) \quad \begin{array}{ccc}
    y' & \to & y \\
    \downarrow & & \downarrow \\
    e' & \to & z \\
    \downarrow & & \downarrow \\
    \tau & \to & \tau'
\end{array} \quad \begin{array}{ccc}
    x \\
    \downarrow \\
    \tau
\end{array}\]

Thus \(\tau' \in \text{sq}(x)\). Let \(\iota_{\tau} : [\tau] \to [z]\), \(\iota_y : [z] \to [x]\), and \(\iota_{y'} : [z] \to [x]\) the induced morphisms. Because of \([e][\tau] = [y'][e'][\tau]\) (easy, see [Kno22, Rel2]) we have \(\iota_y \circ \iota_{\tau} = \iota_{y'}\).

Hence Lemma 4.2 implies

\[(4.12) \quad [x] = [x]^0 + [x]^1 = [x]^0 + \sum_{z_y \in \text{sq}(x) \atop z_y \neq x} t_y[z] = [x]^0 + \sum_{z_y \in \text{sq}(x) \atop z_y \neq x} t_y[z] = \sum_{\tau \in \text{sq}(z)} t_{\tau}[\tau]^0 \sum_{\tau' \in \text{sq}(x)} t_{\tau'}[\tau']^0 = \Phi_x(X),\]

proving our claim.

Now we claim that \(\dim_k \text{End}_\tau([x]) = \dim_k \text{End}_\tau(X)\). Because \(X\) is semisimple this will prove that \(\Phi_x\) is an isomorphism. The dimension of \(\text{End}_\tau([x])\) is, by definition, the number of subobjects \(r\) of \(x \times x\). By Goursat’s lemma, each \(r\) can be represented as a diagram

\[(4.13) \quad \begin{array}{ccc}
    \tau & \to & \tau' \\
    \downarrow & & \downarrow \\
    x & \to & x
\end{array} \quad \begin{array}{ccc}
    \tau \\
    \downarrow \\
    z
\end{array} \quad \begin{array}{ccc}
    \tau' \\
    \downarrow \\
    x
\end{array}\]
which is unique up to isomorphism. Thus subobjects of \( x \times x \) are classified by triples \((s_1, s_2, \varphi)\) where \( s_1 = z_\tau, s_2 = z_\varphi \in \text{sq}(x)\) are subquotient objects of \( x \) and \( \varphi : z \to z \) is an isomorphism between the corresponding subquotients. Let \( \widehat{A} \) denote a set of representatives of the isomorphism classes of objects of \( A \) and, for \( z_0 \in \widehat{A} \), let \( \text{sq}_{z_0}(x) \) be the set of subquotient objects \( z_y \) with \( z \cong z_0 \). Then

\[
(4.14) \quad \dim_K \text{End}_T[x] = |s(x \times x)| = \sum_{z \in \widehat{A}} |\text{sq}_z(x)|^2 |\text{Aut}_A(z)|.
\]

On the other side we have \( \text{Hom}([z]^0, [z_2]^0) = 0 \) unless \( z_1 \cong z_2 \). Thus,

\[
(4.15) \quad \dim_K \text{End}_T(X) = \sum_{z \in \widehat{A}} |\text{sq}_z(x)|^2 \quad \dim_K \text{End}_T([z]^0) = \sum_{z \in \widehat{A}} |\text{sq}_z(x)|^2 |\text{Aut}_A(z)|.
\]

This proves the claim and therefore (4.8). Then (4.9) follows from (4.8) with \( x \) replaced by \( y \) and the fact that \( \{z_u \in \text{sq}(x) \mid u \subseteq y\} = \text{sq}(y) \). The isomorphism (4.10) follows immediately by observing that \( [x]^* \cong [x]/\sum_{y \subseteq x}[y] \).

4.4. Remark. The decomposition (4.10) combined with the subobject decomposition (3.1) yields

\[
(4.16) \quad [x] = \bigoplus_{y \subseteq x} [y]^* \cong \bigoplus_{z_y \in \text{sq}(x)} [z]^0.
\]

This decomposition is not the same as the subquotient decomposition (4.8). It is rather its associated graded. This is because in general \([z]^0_x \nsubseteq [x]^*\).

5. Tensor product multiplicities

In this section we derive a formula for the tensor product of any two simple objects of \( T \) and thereby compute the Grothendieck ring. In this section we use the following notation:

Given \( n \geq 0 \) objects \( x_1, \ldots, x_n \) of \( A \) we put

\[
(5.1) \quad \underline{x} := \prod_j x_j \text{ and } x^i := \prod_{j \neq i} x_j.
\]

Moreover, let

\[
(5.2) \quad T(x_1, \ldots, x_n) := \{r \subseteq \underline{x} \mid r \to x_i \text{ and } r \nrightarrow x^i \text{ for all } i\}.
\]

5.1. Remark. For \( n = 0 \) we have \( \underline{x} = 1 \) and therefore \( T(\emptyset) = \{1\} \). For \( n = 1 \) we have \( x^1 = 1 \) and therefore \( T(1) = \{1\} \) and \( T(x_1) = \emptyset \) when \( x_1 \neq 1 \). For \( n = 2 \) we have \( x^1 = x_3 \). Hence for \( r \) to lie in \( T(x_1, x_2) \) all projections \( r \to x_i \) must be isomorphisms. Thus \( T(x_1, x_2) \) consists of graphs of isomorphisms \( x_1 \to x_2 \).

5.2. Theorem. Let \( x_1, \ldots, x_n \) be objects of \( A \). Then there is a canonical isomorphism

\[
(5.3) \quad \varphi : \langle T(x_1, \ldots, x_n) \rangle_K \cong \text{Hom}_T(1, [x_1]^0 \otimes \ldots \otimes [x_n]^0).
\]
Proof. By definition there is an isomorphism

\[ \varphi : \langle s(x) \rangle_K \cong \text{Hom}_T(\mathbb{1}, [x]). \]

Using \([x_i] = [x]^0 \oplus \sum_{z \not\equiv (x_i)} z_y\) (see Lemma 2.3 and Lemma 4.2) one gets

\[ [x] = \bigotimes_i [x_i]^0 \oplus X \]

where \(X\) is the sum of all \([z]_y \subseteq [x]\) of the form \(z = \prod_i z_i, y = \prod_i y_i\) such that at least one subquotient object \((z_i)_y\) of \(x_i\) proper.

Let \(T'\) be the complement of \(T(x_1, \ldots, x_n)\) in \(s(x)\). We claim that \(\varphi(T') \subseteq \text{Hom}_T(\mathbb{1}, X)\).

Fix \(r \in T'\) and let \(y_i \subseteq x_i\) and \(r_i \subseteq x^i\) be the images of \(r\). Goursat’s lemma applied to \(r \subseteq \overline{x} = x^i \times x_i\) yields a diagram

\[ \begin{array}{ccc} y_i & \xrightarrow{a'} & y_i \\ \downarrow & & \downarrow \\ x^i & \xrightarrow{a} & x^i \\ \downarrow & & \downarrow \\ x_i & \xleftarrow{b} & x_i \end{array} \]

where the square is Cartesian. By assumption, there is an \(i\) such that \(r \rightarrow x_i\) is not surjective or \(r \rightarrow x^i\) is not injective. Since therefore \(b\) or \(a'\) is not an isomorphism, \((z_i)_y\) is a proper subquotient object of \(x_i\). Since \(\langle r \rangle\) factors through \([z]_y\) with \(z = x^i \times z_i\) and \(y = x^i \times y_i\) its image lies in \(X\).

Next we claim that

\[ \varphi : \langle T' \rangle_K \rightarrow \text{Hom}_T(\mathbb{1}, X) \]

is bijective. This will prove the theorem since then \(\varphi\) will map \(\langle T(x_1, \ldots, x_n) \rangle\) isomorphically onto \(\bigoplus_i [x_i]^0\).

Since injectivity follows from (5.4) it suffices to show surjectivity. Let \((z_i)_y\) be a subquotient object of \(x_i\) for each \(i\) and assume that at least one is proper. Put \(z = \prod_i z_i\). Then we have a commutative diagram

\[ \begin{array}{ccc} \langle s(z) \rangle_K & \overset{\varphi_z}{\longrightarrow} & \text{Hom}_T(\mathbb{1}, [z]) \\ \downarrow \iota & & \downarrow \iota' \\ \langle s(x) \rangle_K & \overset{\varphi}{\longrightarrow} & \text{Hom}_T(\mathbb{1}, [x]) \end{array} \]

where both \(\iota\) and \(\iota'\) are componentwise induced by the diagrams

\[ \begin{array}{ccc} y_i & \xrightarrow{a} & y_i \\ \downarrow & & \downarrow \\ x^i & \xrightarrow{a'} & x^i \\ \downarrow & & \downarrow \\ x_i & \xleftarrow{b} & x_i \end{array} \]
Observe that \( \iota \) has its image in \( \langle T' \rangle_K \). In fact, let \((z_i)_{yi} \) be the proper subquotient object of \( x_i \). Then diagram (5.6) with \( r \) replaced by \( \iota(r) \) shows that \( \iota(r) \to x_i \) is not surjective or \( \iota(r) \to x^i \) is not injective, hence \( \iota(r) \in T' \). Since \( \varphi_z \) is an isomorphism and \( X \) is generated by the images of \( \iota' \) the claim follows. \( \square \)

5.3. Remark. For \( n \leq 1 \) the theorem says \( \text{Hom}_{T}(1, [1]^0) = K \) (which is trivial since \([1]^0 = [1] = 1\) and \( \text{Hom}_{T}(1, [x_1]^0) = 0 \) for \( x_1 \neq 1 \). For \( n = 2 \) we get \( \text{Hom}_{T}(1, [x_1]^0 \otimes [x_2]^0) = 0 \) if \( x_1 \neq x_2 \) and \( \text{Hom}_{T}(1, [x]^0 \otimes [x]^0) = K[\text{Aut}(x)] \) in accordance with Lemma 2.3iii).

Now put \( A(x) := \text{Aut}(T(x)) \) and \( A := \prod_i A(x_i) \). By abuse of notation let \( \chi_T = \chi_{T(x_1, \ldots, x_n)} \) be the character of the \( A \)-module \( \langle T(x_1, \ldots, x_n) \rangle_K \). Since it is a permutation character we get the explicit formula

\[
(5.10) \quad \chi_T(g) = |T(x_1, \ldots, x_n)| = |\{ r \in T(x_1, \ldots, x_n) \mid gr = r \}|
\]

5.4. Corollary. For \( n \geq 0 \) let \( x_1, \ldots, x_n \) be objects of \( A \) and \( \chi_1 \otimes \ldots \otimes \chi_n \) an irreducible character of \( A \). Then

\[
\dim_K \text{Hom}(1, [x_1]^0 \otimes \ldots \otimes [x_n]^0) = \langle \chi_T \mid \chi_1 \otimes \ldots \otimes \chi_n \rangle_A = \sum_{r \in T/A} \langle \chi_{\text{triv}} \mid \chi_1 \otimes \ldots \otimes \chi_n \rangle_{A_r} =
\]

\[
(5.11) \quad = \frac{1}{|A|} \sum_{g \in A} |T^g| \chi_1(g_1) \cdots \chi_n(g_n) = \frac{1}{|A|} \sum_{r \in T/A, g \in A} \chi_1(g_1) \cdots \chi_n(g_n)
\]

where \( g = (g_1, \ldots, g_n) \in A, T = T(x_1, \ldots, x_n), T/A \) is a set of orbit representatives.

Proof. By (2.7) we have

\[
(5.12) \quad \text{Hom}_{T}(1, \bigotimes_i [x_i]^0) = \bigoplus_{\chi_1, \ldots, \chi_n} \text{Hom}_{T}(1, \bigotimes_i [x_i]_{\chi_i}) \otimes \bigotimes_i V_{\chi_i}.
\]

Hence (5.3) implies that \( \dim_K \text{Hom}_{T}(1, \bigotimes_i [x_i]_{\chi_i}) \) is the multiplicity of \( \bigotimes_i V_{\chi_i} \) in \( \langle T \rangle_K \). The decomposition of \( T \) into orbits gives \( \chi_T = \sum_{r \in T/A} \text{ind}_{A_r}^{\chi} \chi_{\text{triv}} \) where \( A_r \) is the stabilizer of \( r \) in \( A \). The second formula follows. The third formula follows from (5.10) and the fourth is just a reformulation of the third. \( \square \)

The case \( n = 3 \) corresponds to a multiplicity formula for the tensor product of simple objects. In the next statement we let \( nX \) stand for \( X^{\otimes n} \).

5.5. Corollary. Let \( x_1 \) and \( x_2 \) be objects of \( A \). Then

\[
(5.13) \quad [x_1]^0 \otimes [x_2]^0 \cong \bigoplus_{\chi} \langle \chi_{T(x_1, x_2, x)} \mid \chi_1 \otimes \chi_2 \otimes \chi^\vee \rangle [x]^0
\]

where \( x \) runs through all subquotients of \( x_1 \times x_2 \) and \( \chi \) through all characters of \( A(x) \).
Proof. Let \([x_1]_1 \otimes [x_2]_2 \cong \sum_{x, \chi} m^x_{\chi_1, \chi_2} [x]_x\). Observe that \(([x]_x)^\vee = [x]_x^\vee\). Now application of \(X \mapsto \dim \text{Hom}_T(\mathbb{1}, X \otimes [x]_{x_1})\) and (5.11) yield \(\langle \chi_T \mid \chi_1 \otimes \chi_2 \otimes \chi_\vee \rangle = m^x_{\chi_1, \chi_2}\). Finally, in order for \(T \neq \emptyset\) there must exist \(r \subseteq x_1 \times x_2 \times x\) with \(r \mapsto x_1 \times x_2\) and \(r \mapsto x\) which implies that \(x\) is a subquotient of \(x_1 \times x_2\). \(\square\)

5.6. Example. We verify formula (5.13) in the case of \(\mathcal{A} = \text{Set}^{\text{op}}\). Let \(x = (n) := \{1, \ldots, n\}\). The irreducible representations \(V_\lambda\) of \(A(x) = S_n\) are in \(1-1\)-correspondence with partitions \(\lambda\) of \(n\). In particular, for every \(\lambda\) there is a simple object \(\{\lambda\} := [\lambda]_0^n\).

To the irreducible character \(\chi_\lambda\) is assigned the Schur function \(s_\lambda\). Then the tensor product corresponds to the Kronecker product of symmetric functions:

\[(5.14) \quad \text{When } V_\lambda \otimes V_\mu = \bigoplus_\nu m^\nu_{\lambda\mu} V_\nu \quad \text{then } s_\lambda \ast s_\mu := \sum_\nu m^\nu_{\lambda\mu} s_\nu.\]

Likewise, the tensor product of simple objects of \(\mathcal{T}\) gives rise to a product, as well:

\[(5.15) \quad \text{When } \{\lambda\} \otimes \{\mu\} = \bigoplus_\nu M^\nu_{\lambda\mu} \{\nu\} \quad \text{then } s_\lambda \star s_\mu := \sum_\nu M^\nu_{\lambda\mu} s_\nu.\]

This product was introduced by Murnaghan [Mur38, Mur55] as stable Kronecker product in another form: For a partition \(\lambda\) and \(n > 0\) let \(\lambda^\tau := (n - |\lambda|, \lambda)\). It is a partition of \(n\) as soon as \(n \geq |\lambda| + \lambda_1\). Then we have the stability formula

\[(5.16) \quad m_{\lambda^\tau}^{\lambda^\mu} = M_{\lambda^\mu} \quad \text{for } n \gg 0.\]

That these two definitions coincide follows from [Del07, Prop. 6.4] (see also [EA16]).

Using the scalar product on symmetric functions for which the Schur functions form an orthonormal basis one defines the (double) skew Schur functions by

\[(5.17) \quad s_{\lambda \setminus \mu\nu} := \sum_\tau \langle s_\lambda \mid s_\mu s_\nu s_\tau \rangle s_\tau.\]

We will show in Appendix B that Corollary 5.5 is equivalent to the following result of Littlewood:

5.7. Theorem (Littlewood [Lit58, Thm. IX]). Let \(\lambda\) and \(\mu\) be partitions. Then

\[(5.18) \quad s_\lambda \ast s_\mu = \sum_{\alpha, \beta, \gamma} \langle s_\alpha \ast s_\beta \rangle s_{\lambda \setminus \alpha \beta} s_{\mu \setminus \beta \gamma}.\]

Using formula (5.13), we are able to determine the Grothendieck ring \(K(\mathcal{T})\). Recall that \(\hat{\mathcal{A}}\) denotes the set of isomorphism classes of objects of \(\mathcal{A}\). The representation ring of a finite group \(G\) is denoted by \(\mathcal{R}(G)\). It will be considered as a subring of the ring \(\mathbb{K}[G]^G\) of \(\mathbb{K}\)-valued class functions.
5.8. Corollary. As an additive group

\[ K(\mathcal{T}) = \bigoplus_{x \in \hat{A}} \mathcal{R}(\text{Aut}_T(x)). \]

Moreover, the product of \( f_1 \in \mathcal{R}(A(x_1)) \) and \( f_2 \in \mathcal{R}(A(x_2)) \) is given by \( f_1 \star f_2 = \sum_x f_1 \star_x f_2 \) with \( x \in \hat{A} \) running over all subquotients of \( x_1 \times x_2 \) and

\[ f_1 \star f_2 = \langle \chi_{T(x_1,x_2,x)} \mid f_1 \otimes f_2 \rangle_{A(x_1) \times A(x_2)} \in \mathcal{R}(A(x)). \]

\[ f_1 \star f_2 = \langle \chi_{T(x_1,x_2,x)} \mid f_1 \otimes f_2 \rangle_{A(x_1) \times A(x_2)} \in \mathcal{R}(A(x)). \]

Proof. Let \( a_i := |A(x_i)| \). The simple object \([x]_\chi^0\) corresponds to the function \( \chi \in \mathcal{R}(A(x)) \). Hence, (5.13) implies

\[ (f_1 \star f_2)(h) = \frac{1}{a_1 a_2 |A(x)|} \sum_{\chi} \langle \chi_{T(x_1,x_2,x)} \mid \chi_1 \otimes \chi_2 \otimes \chi^\vee \rangle \chi(h) = \]

\[ = \frac{1}{a_1 a_2} \sum_{g_1,g_2} \chi_{T(g_1,g_2)} f_1(g_1) f_2(g_2) \frac{1}{|A(x)|} \sum_{\chi} \chi(g^{-1}) \chi(h) = \]

\[ = \frac{1}{a_1 a_2} \sum_{g_1,g_2} \chi_{T(g_1,g_2,h)} f_1(g_1) f_2(g_2) = \]

\[ = \langle \chi_{T(\ast, \ast, h)} \mid f_1 \otimes f_2 \rangle_{A(x_1) \times A(x_2)}. \] \( \square \)

In general, the components \( f_1 \star f_2 \) are difficult to compute, except for the highest one corresponding to \( x = x_1 \times x_2 \). Observe that in that case \( A(x_1) \times A(x_2) \) is a subgroup of \( A(x) \).

5.9. Proposition. Let \( x = x_1 \times x_2 \) Then

\[ f_1 \star f_2 = \text{ind}^{A(x_1) \times A(x_2)}_{A(x_1) \times A(x_2)} f_1 \otimes f_2. \]

Proof. Let \( r \in T(x_1,x_2,x) \). Since \( r \to x^3 = x_1 \times x_2 = x \) is injective and \( r \to x_3 = x \) is surjective, \( r \) realizes \( x \) as a subquotient of itself. By [Kno07, Lemma 2.6], both morphisms are isomorphisms. So \( r \) is the graph = \( \Gamma_h \) of an isomorphism \( h : x_1 \times x_1 \simeq x \). Let \( g_1 \in A(x_1), g_2 \in A(x_2) \) and \( g \in A(x) \). Then \( \Gamma_h \) is fixed by \( (g_1,g_2,g) \) if and only if \( h^{-1} gh = g_1 \times g_2 \). This implies

\[ \chi_{T(g_1,g_2,g)} = |T(g_1,g_2,g)| = |\{ h \in A(x) \mid h^{-1} gh = g_1 \times g_2 \}| \]

and therefore

\[ f_1 \star f_2(g) = \left( \frac{1}{|A(x_1) \times A(x_2)|} \right) \sum_{h \in A(x)} (f_1 \otimes f_2)(h^{-1} gh) = \text{ind}^{A(x_1) \times A(x_2)}_{A(x_1) \times A(x_2)} f_1 \otimes f_2. \] \( \square \)

5.10. Remark. In the case \( \mathcal{A} = \text{Set}^{\text{op}} \) this means that the top degree component of the stable Kronecker product is the usual product of symmetric functions.
Let \( \overline{K}(\mathcal{T}) = K(\mathcal{T}) \) but with multiplication \( f_1 f_2 = f_1 *_{x_1 \times x_2} f_2 \) for \( f_i \in \mathcal{R}(A(x_i)) \). Then \( \overline{K}(\mathcal{T}) \) becomes a commutative ring.

5.11. Definition. An **valuation** of \( A \) is a function \( v : \hat{A} \to \mathbb{R}_{\geq 0} \) with the following properties:

i) \( v(x_1 \times x_2) = v(x_1) + v(x_2) \)

ii) If \( z \prec x \) then \( v(z) < v(x) \).

Every \( A \) which has a forgetful functor \( F : A \to \text{Sets} \) preserving products, injectivity, and surjectivity has a valuation namely \( v(x) = \log |F(x)| \).

A valuation induces a filtration on the group \( K(\mathcal{T}) \) by

\[
(5.25) \quad K_{\leq a}(\mathcal{T}) := \bigoplus_{x \in \hat{A}, v(x) \leq a} \mathcal{R}(A(x))
\]

Since \( f_1 * f_2 \neq 0 \) implies \( x \preceq x_1 \times x_2 \) and therefore \( v(x) \leq v(x_1) + v(x_2) \) this is also a filtration of rings. Let \( \text{gr}_v K(\mathcal{T}) := \bigoplus_{a \geq 0} K_{\leq a}(\mathcal{T})/K_{<a}(\mathcal{T}) \) be the associated graded ring. Then

5.12. Corollary. Let \( v \) be a valuation of \( A \). Then \( \text{gr}_v K(\mathcal{T}) = \overline{K}(\mathcal{T}) \).

6. Dimensions

Since \( \text{End}_\mathcal{T} \mathbb{1} = K \) one can define an **internal trace** for any endomorphism \( \varphi \) of an object \( X \) of \( \mathcal{T} \) by

\[
(6.1) \quad \mathbb{1} \xrightarrow{\varphi} X \otimes X^\vee \xrightarrow{\varphi \otimes 1} X \otimes X^\vee \xrightarrow{\sim} X^\vee \otimes X \xrightarrow{\text{tr}_X \varphi} \mathbb{1}
\]

The **internal dimension** of an object \( X \) is then defined as the trace of the identity:

\[
\dim X := \text{tr}_X \text{id}_X.
\]

More generally, for an idempotent \( p \) on \( X \) we have \( \dim p(X) = \text{tr}_X p \).

6.1. Lemma. Let \( x \) be an object of \( A \). Then

\[
(6.2) \quad \dim_{\mathcal{T}}[x] = \delta(x), \quad \dim_{\mathcal{T}}[x]^* = \omega_x, \quad \dim[x]^0 = \sum_{z \in q(x)} \mu_q(x, z) \omega_z.
\]

Proof. For \( X = [x] \) the diagram (6.1) leads to \( \dim_{\mathcal{T}}[x] = \langle \Delta x \Rightarrow \mathbb{1} \rangle = \delta(x) \). The second formula is derived as follows:

\[
(6.3) \quad \dim[x]^* \overset{(3.2)}{=} \sum_{y \subseteq x} \mu_s(y, x) \text{tr}_y p_y = \sum_{y \subseteq x} \mu_s(y, x) \dim_{\mathcal{T}}[y] = \sum_{y \subseteq x} \mu_s(y, x) \delta(y) \overset{(2.4)}{=} \omega_x.
\]

The last formula is obtained by Möbius inversion of (4.10).
The simple objects \([\mathcal{C}]_0\) appear in the isotypical decomposition (2.7) of \([\mathcal{C}]_0\). To compute their dimension we need to calculate its \(A(x)\)-character. But first we need to compute the characters of \([\mathcal{C}]_0\) and \([\mathcal{C}]^*_0\).

Let \(r \in s(x \times x)\) be a relation. Then we define its fixed point object as the subobject

\[
x^r := r \cap \Delta_x = r \times x \subseteq x.
\]

One checks easily that

\[
\text{tr}_x(r) = \dim_{T}[x^r] = \delta(x^r).
\]

If \(r\) is the graph of an automorphism \(g\) of \(x\) then we write \(x^r = x^g\). It is easily seen that \(x^g\) is also the maximal subobject on which \(g\) acts as identity.

When a finite group \(A\) acts on an object \(X\) of \(\mathcal{T}\) then one defines the character of this action as the \(K\)-valued class function \(\chi_X(g) := \text{tr}_X(g)\).

**6.2. Proposition.** Let \(g \in A(x)\). Then

\[
\chi_{[x]}(g) = \dim_{T}[x^g] \quad \text{and} \quad \chi_{[x]^*}(g) = \frac{\omega_x}{|A(x)|}\chi_{\text{reg}}(g) = \begin{cases} \omega_x, & \text{if } g = \text{id}_x; \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** The first assertion follows from (6.5). For the second we use induction on \(|s(x)|\). So we assume that the claim is correct for all proper subobjects of \(x\). If \(g = \text{id}_x\) the assertion is just Lemma 6.1. Otherwise there is, according to the subobject decomposition (3.1), a \(A\)-equivariant decomposition

\[
[x] = \bigoplus_{y \subseteq x^g} [y]^* \oplus \bigoplus_{y \nsubseteq x^g} [y]^* = [x^g] \oplus [x]^* \oplus X \quad \text{with} \quad X = \bigoplus_{y \nsubseteq x^g} [y]^*.
\]

Only the summands \([y]^*\) of \(X\) with \(gy = y\) enter the computation of \(\text{tr}_X(g)\). For these, we have \(y^g = y \cap x^g \neq y\) and therefore \(g|_y \neq \text{id}_y\). The induction hypothesis implies \(\text{tr}_{[y]^*}(g) = 0\). Hence

\[
\text{tr}_{[x]^*}(g) = \text{tr}_{[x]}(g) - \text{tr}_{[x^g]}(g) - \text{tr}_X(g) = \dim_{T}[x^g] - \dim_{T}[x^g] - 0 = 0. \quad \square
\]

Next, we compute the character of \([\mathcal{C}]_0\) using equivariant Möbius inversion on (4.10). The procedure is basically the same as that of Assaf-Speyer [AS20, §5] except that we avoid the explicit use of simplicial complexes.

For \(z \in q(x)\) and \(n \geq 0\) let \(\text{Ch}^n(x, z)\) be the \(K\)-vector spaces spanned by all chains in \(q(x)\)

\[
x = z_0 > \ldots > z_n = z.
\]
These combine to a graded vector space \( \text{Ch}^*(x, z) := \bigoplus_n \text{Ch}^n(x, z) \). Moreover, let
\[
(6.10) \quad \text{Ch}^*(x) := \bigoplus_{z \in q(x)} \text{Ch}^*(x, z) \otimes_K [z]^*.
\]
This is a \( \mathbb{Z} \)-graded object of \( \mathcal{T} \). Assume a group \( A \) acts on \( x \). Then \( \text{Ch}^*(x) \) is an object of the \( A \)-equivariant category \( \mathcal{T}^A \). Let \( K_A(\mathcal{T}) \) be its Grothendieck group. The class of an object \( X \) of \( \mathcal{T}^A \) will be denoted by \( [X]_A \). Observe, that for any subgroup \( A' \subseteq A \) there is a natural induction homomorphism \( \text{ind}^{A'}_A : K^A(\mathcal{T}) \rightarrow K^A(\mathcal{T}) \). The following is well-known when \( \mathcal{T} \) is the category of vector spaces. We repeat the arguments for \( \mathcal{T} \).

6.3. Lemma. Let \( A \) act on the object \( x \) of \( A \). Then the following identity holds in \( K_A(\mathcal{T}) \):
\[
(6.11) \quad [x^0]^A = \chi(\text{Ch}^*(x)) := \sum_{n} (-1)^n \{\text{Ch}^n(x)\}_A.
\]

Proof. We argue by induction on \( |q(x)| \) and assume the formula to be correct for all proper quotients \( z \) of \( x \). It follows from the subquotient decomposition (4.10) of \([x]^*\) that
\[
(6.12) \quad [x^0]^A = [x]^*_A - \bigoplus_{x > z} [z^0]^A.
\]
Let \( q'/A \subseteq q(x) \setminus \{x\} \) be a set of \( A \)-orbit representatives. Using the \( A \)-isomorphism
\[
(6.13) \quad \bigoplus_{x > z} [z^0] = \bigoplus_{z \in q'/A} \text{ind}^A_{A_z} [z^0].
\]
the induction hypothesis yields
\[
(6.14) \quad \{ \bigoplus_{x > z} [z^0] \}_A = \sum_{z \in q'/A} \chi(\text{Ch}^*(z)) = \sum_{z \in q'/A} \chi(\text{ind}^A_{A_z} \text{Ch}^*(z)) = \chi\left( \bigoplus_{x > z} \text{Ch}^*(z) \right).
\]
By prepending \( x \) to a chain we see that
\[
(6.15) \quad \text{Ch}^*(x) = \text{Ch}^0(x) \oplus \bigoplus_{x > z} \text{Ch}^{*+1}(z)
\]
Because of \( \text{Ch}^0(x) = [x]^* \) this implies
\[
(6.16) \quad \{ \bigoplus_{x > z} [z^0] \}_A = \{[x]^*_A - \chi(\text{Ch}^*(x))\}
\]
and the assertion follows. \( \square \)

We now derive two formulas for the \( A(x) \)-character of \([x]^0\). Recall that \( \text{Ch}^*(x, z) \) can be turned into a complex with the differential
\[
(6.17)
\]
The lattice \( x/z := q(x)_{\leq z} \) is modular (see Theorem A.1). Therefore (see Theorem A.6) the homology of \( Ch^*(x/z) \) is concentrated in degree \( r := rk x/z \) and the group \( A(x)_z \) acts on \( H_r(Ch^*(x/z)) \) with a character \( h_{x/z} \).

Define moreover the “inertia group” of \( z \in q(x) \) as
\[
(6.18) \quad A(x/z) := \{ g \in A(x)_z \mid g|_z = id_z \},
\]
let \( \hat{x} \) be the socle (i.e., the meet of all atoms) of \( q(x) \), and let \( (-1)^{x/z} := (-1)^{rk x/z} \). Then we have:

6.4. Theorem. The character of \( [x]^0 \) is
\[
(6.19) \quad \chi_{[x]^0} = \sum_{z \in q(x/\hat{x})} \frac{(-1)^{x/z} \omega_z}{|A(x)/A(x/z)|} \operatorname{ind}_{A(x/z)}^{A(x)} h_{x/z}.
\]

Proof. Write, as in (6.13),
\[
(6.20) \quad \chi(Ch^*(x)) = \sum_{z \in q(A(x))} \operatorname{ind}_{A(x)_z}^{A(x)} \chi(Ch^*(x, z)) \{[z]^*\}_{A(x)_z}.
\]

Thus
\[
(6.21) \quad \chi_{[x]^0} = \sum_{z \in q(A(x))} \operatorname{ind}_{A(x)_z}^{A(x)} (-1)^{x/z} h_{x/z} \chi_{[z]^*}.
\]

Now (6.6) implies that
\[
(6.22) \quad h_{x/z} \chi_{[z]^*} = \frac{|A(x/z)|}{|A(x)_z|} \omega_z \operatorname{ind}_{A(x/z)}^{A(x)} h_{x/z}.
\]

The assertion follows with \( q(x/\hat{x}) \) replaced by \( q(x) \). Now recall that \( H_*(x/z) = 0 \) unless \( \mu_q(x, z) \neq 0 \) (see Theorem A.6) which, according to Lemma A.2, is equivalent to \( z \leq \operatorname{soc} q(x) = \hat{x} \).

6.5. Corollary. Let \( \chi \) be an irreducible character of \( A(x) \). Then
\[
(6.23) \quad \dim_T [x]^0_\chi = \sum_{z \in q(x/\hat{x})} \frac{(-1)^{x/z} \omega_z}{|A(x)/A(x/z)|} \langle h_{x/z} \mid \chi \rangle_{A(x/z)}
\]

6.6. Example. Let \( x \) be a simple object, i.e., \( |q(x)| = 2 \). Since \( A(x/x) = 1, A(x/1) = A(x) \), and \( h_{x/z} \) is the trivial character we get
\[
(6.24) \quad \dim [x]^0_\chi = \frac{\deg \chi}{|A(x)|} \begin{cases} 
\omega_x - |A(x)| & \text{if } \chi = \chi_{\text{triv}} \\
\omega_x & \text{otherwise}
\end{cases}
\]
For example if \( p \) is a prime then \( \dim[Z_p]^0_{\chi_{\text{triv}}} = \frac{t_p - 1}{p-1} - 1 = \frac{t_p - p}{p-1} \) where \( t_p = \delta(Z_p) \).

6.7. Example. Assume, more generally, that \( x \) has a unique minimal quotient \( z \) such that \( \widehat{x} = z \). Then

\[
(6.25) \quad \dim_T[x]^0 = \frac{\deg \chi}{|A(x)|} \cdot \begin{cases} 
\omega_x - |A(x/z)| \omega_z, & \text{if } \chi|_{A(x/z)} \text{ is trivial}, \\
\omega_x, & \text{otherwise}
\end{cases}
\]

Let for example \( x = S_3 \), the symmetric group in the category of finite groups then \( z = S_3/A_3 = \mathbb{Z}_2 \). We have \( A(x) = S_3 \) and \( A(x/z) = A(x) \). Since \( \omega_z = t_2 - 1 \) and \( \omega_{x-z} = t_3 - 3 \) (where \( t_2 = \delta(\mathbb{Z}_2) \) and \( t_3 = \delta(\mathbb{Z}_3) \)) we get

\[
(6.26) \quad \dim_T[S_3]^0 = \begin{cases} 
\frac{1}{6}(t_2 - 1)(t_3 - 9) \quad \chi = \chi_{\text{triv}} \\
\frac{1}{6}(t_2 - 1)(t_3 - 3) \quad \chi = \chi_{\text{sign}} \\
\frac{1}{3}(t_2 - 1)(t_3 - 3) \quad \deg \chi = 2
\end{cases}
\]

Our second approach is to compute \( \chi_{[x]^0}(g) \) using the Hopf fixed point formula (see [Sun94]).

To state the result, let \( q^g := q(x)^g := \{ z \in q(x) \mid g(z) = z \} \) be the fixed point sublattice of \( q(x) \). It is modular, as well. Let \( \mu_{q^g} \) be its Möbius function and let \( \widehat{x}_g \) be its socle.

If \( z \in q^g \) then \( g \) will act on \( z \). Suppose \( g \) acts on two objects \( z_1, z_2 \in q^g \) trivially. Since \( z_1 \wedge z_2 \) is a subobject of \( z_1 \times z_2 \) (namely the image of \( x \) under the diagonal morphism), the element \( g \) will also act trivially on \( z_1 \wedge z_2 \). This shows that there is an object \( x_g \in q^g \) such that \( g|_z = \text{id}_z \) if and only if \( z \geq x_g \).

6.8. Theorem. The character of \( [x]^0 \) at \( g \in A(x) \) is

\[
(6.27) \quad \chi_{[x]^0}(g) = \sum_{z \in q^g \atop x_g \leq z \leq x_g} \mu_{q(x)^g}(x, z) \omega_z
\]

Proof. The action of \( g \) on \( \text{Ch}^n(x) \) is a permutation representation inside \( T \). Since the trace of \( g \) on any orbit which is not a fixed points is zero we see that the trace of \( g \) on \( \text{Ch}^n(x) \) is equal to that on \( \text{Ch}^n(x, z; g) \) where the latter is the subspace of \( \text{Ch}^n(x, z) \) spanned by the \( g \)-invariant chains. Thus we get

\[
\text{tr}(g : \text{Ch}^n(x)) = \text{tr} \left( g : \bigoplus_{z \in q^g} \text{Ch}^n(x, z; g) \otimes_K [z]^* \right) =
\]

\[
= \sum_{z \in q^g} \text{tr}(g : \text{Ch}^n(x, z; g)) \quad \text{tr}_{[z]^*}(g) =
\]

\[
(6.28) \quad \overset{(6.6)}{=} \sum_{z \in q^g \atop x_g \leq z} \text{tr}(g : \text{Ch}^n(x, z; g)) \omega_z.
\]
Since $\text{Ch}^n(x, z; g)$ is just the chain complex of $q^g$ and $g$ acts trivially on it, its trace equals the dimension which is $\mu_{q^g}(x, z)$ (see (A.8)). Since this Möbius number is 0 unless $z \leq \hat{x}_g$, the assertion follows. □

6.9. Remarks. i) Equation (6.27) is more economical than (6.19) since $\hat{x}_g \leq \hat{x}_{\text{id}} = \hat{x}$ for all $g$. In fact, for any atom $z \in q^g$ let $z_0 \in q$ be the join of all atoms $z' \in q$ with $z' \leq z$. Since $z_0 \in q^g$ and $z_0 \leq z$ it follows $z = z_0$. This proves that $\hat{x}_g$ is the join of atoms of $q$ and therefore $\hat{x}_g \leq \hat{x}$.

ii) It may happen that $x_g \nleq \hat{x}_g$, see, e.g., the third conjugacy class of Example 6.11 below. In that case $\chi_{[x]^{0}}(g) = 0$.

The second dimension formula reads:

6.10. Corollary.

$$(6.29) \quad \dim_T[x]^{0}_\chi = \left\langle \chi_{[x]^{0}} \mid \chi \right\rangle = \frac{1}{|A(x)|} \sum_{g \in A(x)} \chi(g) \sum_{x_g \leq z \leq \hat{x}_g} \mu_{q^g}(x, z) \omega_z.$$  

6.11. Example. Let $x = \mathbb{F}_2^3$ in the category of $\mathbb{F}_2$-vector spaces. Then $A(x) = GL(3, \mathbb{F}_2)$ is the simple group of order 168. Its six conjugacy classes are represented by the following matrices. Below we indicated the lattice $q(x)^g$ (or rather the kernels with $x_g$ in bold and $\hat{x}_g$ underlined) and $\chi_{[x]^{0}}(g) = \sum_{z \leq x_g} \mu_{q(x)^g}(x, z) \omega_z$.

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\]

$q(\mathbb{F}_2^3)$

\[
t^3 - 14t^2 + 49t - 44 = -(t - 1)(t - 4)
\]

$$0 \quad -1 \quad -1 \quad -(t - 2)$$
Here $e_{ij} := e_i + e_j$ and $e_i e_j = \langle e_i, e_j \rangle$. This implies the following dimension formulas where $\chi_d$ signifies an irreducible character of degree $d$.

\begin{align*}
\dim [F^3]_{2/\chi_1}^0 &= \frac{1}{108} (t - 1)(t - 2)(t - 2^5) \\
\dim [F^3]_{2/\chi_3}^0 &= \frac{3}{108} (t - 1)(t - 2)(t - 2^2) \\
\dim [F^3]_{2/\chi_5}^0 &= \frac{5}{108} (t - 1)(t - 2)(t - 2^2) \\
\dim [F^3]_{2/\chi_6}^0 &= \frac{6}{108} (t - 1)(t - 2^2)(t - 2^4) \\
\dim [F^3]_{2/\chi_7}^0 &= \frac{7}{108} (t - 1)(t - 2)(t - 2^3) \\
\dim [F^3]_{2/\chi_8}^0 &= \frac{2}{108} (t - 2)(t - 2^2)(t - 2^3)
\end{align*}

(6.30)

6.12. Example. Let $A = \text{Set}^{\text{op}}$. We adopt the notation of [Mac95], i.e. let $\chi^\lambda$ be the irreducible character of $S_n$ corresponding to a partition $\lambda$ of $n$. For another partition $\mu$ of $n$ let $\chi^\mu$ be the value of $\chi^\lambda$ in a permutation $g_\mu$ of cycle type $\mu$. Let $m_i(\mu) = \mu_i' - \mu_{i+1}'$ be the number of parts equal to $i$, so $m_i(\mu)$ is also the number of fixed points of $g_\mu$. The order of the centralizer of $g_\mu$ is $z_\mu := \prod_{i}(i^{m_i(\mu)} m_i(\mu)!)$.

Let $x = (n) := \{1, \ldots, n\}$ be an object of $\text{Set}^{\text{op}}$. Then $x_{g_\mu} = (n)^{g_\mu} \cong (m)$ and $q(x)^{g_\mu} = \Psi((n))^{g_\mu} \cong \Psi((b))$ where $m = m_1(\mu)$ and $b = \ell(\mu) = \mu_1'$. Then

\begin{equation}
\chi_{[x]^\circ}(g_\mu) = \sum_{Z \subseteq [n]^{g_\mu}} \mu \omega^{Z, [n]} \omega_Z = \sum_{k=0}^{m} (-1)^{b-k} \binom{m}{k} (t)_k = (-1)^{\ell(\mu)} C_{m_1(\mu)}
\end{equation}

where $(t)_k := t(t - 1) \ldots (t - k + 1)$ and

\begin{equation}
C_m(t) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} (t)_k
\end{equation}

is, up to a sign, the $m$-th Charlier polynomial for the parameter value $a = 1$. Combining this with a calculation of Deligne one obtains the following identity:

6.13. Corollary. Let $f(x) := (t - x_1) \ldots (t - x_n)$ and $\delta := (n - 1, \ldots, 0)$, Then

\begin{equation}
\dim [n]_{\chi^\lambda}^0 = \sum_{\mu} (-1)^{\ell(\mu)} x_\mu^{-1} \chi^\mu_{m_1(\mu)} = \frac{\deg \chi^\lambda}{n!} f(\lambda + \delta).
\end{equation}

Proof. The first equality follows from $\dim [n]_{\chi^\lambda}^0 = \langle \chi_{[n]^0} | \chi^\lambda \rangle$. The right hand term is the dimension of $[x]_{\chi^\lambda}^0$ as computed by Deligne [Del07, 7.4].

6.14. Remark. In Appendix B we give a non-category theoretical proof of the right hand equality thereby obtaining an independent proof of Deligne’s formula.

We conclude this paper with an interesting observation namely that in each example the dimension of a simple object has nice factorization. We show now that this is no coincidence. Let $K_0 \subseteq K$ be the $\mathbb{Q}$-algebra generated by all values $\delta(e)$ with $e$ surjective and indecomposable. Then we have:
6.15. Theorem. Let \( S \) be a simple object of \( \mathcal{A} \). Then there are indecomposable surjective morphisms \( e_1, \ldots, e_n \) (not necessarily distinct) such that \( \dim T S \) divides \( \omega e_1 \ldots \omega e_n \) within \( K_0 \).

Proof. Let \( K_1 \) be \( K_0 \) with all inverses \( \omega e^{-1}, e \) surjective, adjoined. Let \( K' \) be another field and let \( \varphi : K_1 \to K' \) be a ring homomorphism. Then \( \delta' = \varphi \circ \delta \) is by construction a non-degenerate degree function. Hence \( T' := T(A, \delta') \) is a semisimple tensor category. Let now \( S = [x]_0 \) be the simple object of \( T \). Then \( S' = [x]_0' \) the corresponding simple object of \( T' \). It follows from any of the explicit formulas above that first of all \( d := \dim T S \) is an element of \( K_0 \) and that \( \dim T' S' = \varphi(d) \). According to [Del07, Prop. 5.7(iii)], the dimension of any simple object of \( T' \) is nonzero. This implies \( \varphi(d) \neq 0 \) for all possible \( \varphi \) and therefore that \( d \) is invertible in \( K_1 \). Thus there is \( a \in K_0 \) and \( \omega = \omega e_1 \ldots \omega e_n \in K_0 \) with \( \frac{a}{\omega} \cdot d = 1 \) which shows \( ad = \omega \) as asserted. \( \square \)

6.16. Examples. i) For \( \mathcal{A} = \text{Set}^{\text{op}} \) and \( t = \delta(\{0\}) \) transcendental over \( \mathbb{Q} \) we have \( K_0 = \mathbb{Q}[t] \) and \( \omega = t - a, a \in \mathbb{N} \) for \( e \) indecomposable. Thus \( \dim T S \) is a product of linear factors \( t - a, a \in \mathbb{N} \). This follows of course also from Deligne’s formula for \( S \).

ii) For \( \mathcal{A} = \text{Vect}(\mathbb{F}_q) \) and \( t = \delta(\mathbb{F}_q) \) transcendental over \( \mathbb{Q} \) we have \( K_0 = \mathbb{Q}[t] \) and \( \omega = t - q^a, a \in \mathbb{N} \) for every indecomposable \( \omega e \). Thus \( \dim T S \) is a product of linear factors \( t - q^a, a \in \mathbb{N} \).

Appendix A. The Möbius function of the lattice of quotient objects

In this section, we review some facts for modular lattices and their relation to Mal’cev categories. The reason for our interest is the following theorem of Pedicchio:

A.1. Theorem ([Ped91, Cor. 2.3]). Let \( x \) be an object of a regular exact Mal’cev category. Then its set of quotients \( q(x) \) is a modular lattice.

Recall that a lattice \( \Lambda \) is modular if is obeys the modular identity

(A.1) \( x \land (y \lor z) = (x \land y) \lor z \) whenever \( x \leq z \).

It is clear that with \( \Lambda \) also every interval

(A.2) \( x/z := \{y \in \Lambda \mid x \leq y \leq z\} \)

is modular, as well. The same holds for every sublattice.

In a modular lattice, all maximal chains

(A.3) \( \widehat{0} = z_0 < z_1 < \ldots < z_n = \widehat{1} \)

have the same length. This number \( n \) is the rank \( \text{rk} \Lambda \) of \( \Lambda \).
We are mostly interested in the Möbius function of a modular lattice. Assume therefore that $\Lambda$ is finite with minimum $\hat{0}$ and maximum $\hat{1}$. Recall that an atom is an element $z \in \Lambda$ with $|\hat{0}/z| = 2$. The socle $\text{soc}(\Lambda)$ of $\Lambda$ is the join of its atoms. Recall also that a lattice is complemented if for every $x \in \Lambda$ there is $x' \in \Lambda$ with $x \wedge x' = \hat{0}$ and $x \vee x' = \hat{1}$. It is known that if a modular lattice is complemented then all of its intervals are complemented, as well.

**A.2. Lemma.** Let $\Lambda$ be a finite modular lattice. Then for $z \in \Lambda$ the following are equivalent:

i) $\mu(\hat{0}, z) \neq 0$.

ii) $z$ is the join of atoms.

iii) $[\hat{0}, z]$ is complemented.

iv) $z \leq \text{soc}(\Lambda)$.

**Proof.** i) $\Rightarrow$ ii) See [Sta12, 3.9.5]. ii) $\Rightarrow$ iii) See [Bir79, IV §5, Thm. 6]. iii) $\Rightarrow$ i) This is proved in the same way as [Sta12, Prop. 3.10.1] together with the remark that in a complemented lattice the sum in loc.cit. (3.33) is not empty. ii) $\Rightarrow$ iv) Obvious. iv) $\Rightarrow$ iii) The interval $[\hat{0}, \text{soc}(\Lambda)]$ is complemented by the implication ii) $\Rightarrow$ iii). Thus also $[\hat{0}, z]$ is complemented. □

Even though it is not going to be used in this paper it may of interest to know that finite complemented modular lattice have been classified by Birkhoff [Bir79]:

**A.3. Theorem.** Every finite complemented modular lattice is product of indecomposable finite complemented modular lattices with factors which are unique up to order. Every indecomposables is isomorphic to one of the following lattices:

rk = 1 The two element Boolean lattice $B_1 := \{\hat{0}, \hat{1}\}$.

rk = 2 The lattices $M_{q+1} := \{\hat{0} < a_0, \ldots, a_q < \hat{1}\}$, $q \geq 2$.

rk = 3 The subspace lattice of a finite projective plane.

rk $\geq$ 4 The subspace lattice $L_n(q)$ of $\mathbb{F}_q^n$.

Using [Sta12, (3.34) and Prop. 3.8.2.] one obtains:

**A.4. Corollary.** Let $\Lambda$ be a finite complemented modular lattice. If $\Lambda$ is indecomposable then

\[ \mu_\Lambda(\hat{0}, \hat{1}) = (-1)^r q^{\binom{r}{2}} \]

where $r = \text{rk} \Lambda$ and $q$ is as above. When $\Lambda = \prod_i \Lambda_i$ then

\[ \mu_\Lambda(\hat{0}, \hat{1}) = \prod_i \mu_{\Lambda_i}(\hat{0}, \hat{1}) \]
A.5. Remark. The subspace lattice of $\mathbb{F}_q^3$ is a special case of the $\text{rk} = 3$-case but there exists a plethora of others. Therefore it is a nice surprise (by Jónsson [Jó53]) that the latter do not occur as quotient lattice $q(x)$, at least when $x$ is a model for a Mal’cev algebraic theory. Likewise, in the same context, the lattices $M_m$ appear only when $q$ is a prime power (use [MMT87, Lemma 4.153 and Thm. 4.155]). Thus for algebraic theories only subspace lattices of vector spaces $\mathbb{F}_q^n$, $n \geq 1$, occur. Presumably this holds for all of our categories $\mathcal{A}$.

It is well known that Möbius numbers are categorified by the homology of the complex of chains. Let $\Lambda$ be a finite lattice. For $n \geq 0$ let $\text{Ch}^n(\Lambda)$ be the $\mathbb{K}$-vector space spanned by all chains

$$\text{(A.6)} \quad \emptyset = a_0 < \ldots < a_n = \hat{1}.$$ 

These spaces are connected by the differential

$$\text{(A.7)} \quad \delta : \text{Ch}^n(\Lambda) \to \text{Ch}^{n-1}(\Lambda) : (a_0 < \ldots < a_n) \mapsto \sum_{i=1}^{n-1} (-1)^i (a_0 < \ldots < \hat{a_i} < \ldots < z_n).$$

A.6. Theorem. Let $\Lambda$ be a finite lattice. Then $(\text{Ch}^*(\Lambda), \delta)$ is a chain complex. If $\Lambda$ is modular of rank $r$ then its homology is concentrated in degree $r$ and the homology group $H(\Lambda)$ in degree $r$ has dimension $(-1)^r \mu_{\Lambda}(\emptyset, \hat{1})$.

Proof. See e.g. [Bac80, Prop. 4.2 and Prop. 3.5].

If a group $A$ acts on $\Lambda$ then the character of $H(\Lambda)$ will be denoted by $h_{\Lambda}$. According to the Hopf trace formula we have

$$\text{(A.8)} \quad h_{\Lambda}(g) = (-1)^{\text{rk}_A} \mu_{A^g}(\emptyset, \hat{1}).$$

See e.g. [Sun94, (1.3)]. Here $\Lambda^g \subseteq \Lambda$ is the fixed point sublattice of $g \in A$. Since $\Lambda^g$ is also modular, this implies $h_{\Lambda}(g) = 0$ unless the action of $g$ is “semisimple”, i.e., every $g$-stable $a \in \Lambda$ has a $g$-stable complement. In that case, $\Lambda^g$ is complemented and the Möbius number can be computed using Corollary A.4. Note that when one does these computations for $\Lambda = L_n(q)$ and $g \in \text{GL}(n, \mathbb{F}_q)$ one obtains simply the Steinberg character at $g$ (see e.g. [CLT80]).

Appendix B. Symmetric function identities

In this appendix we provide proofs for two identities in the main text.

B.1. Proof of Theorem 5.7.

Let $X_1, X_2, X_3$ be finite sets with $a_i := |X_i|$ elements. Thus $A = S_{a_1} \times S_{a_2} \times S_{a_3}$. An element of $T(X_1, X_2, X_3)$ is a finite set $R$ together with injective maps $X_i \to R$ such
that $R$ is the union of any two images. Thus, the set $T/A$ is parameterized by the four numbers

\[(B.1) \quad n_0 := |X_1 \cap X_2 \cap X_3| \text{ and } n_i := |(X_j \cap X_k) \setminus X_i|\]

for all permutations $(i, j, k)$ of $(1, 2, 3)$. These numbers are subject to the constraints

\[(B.2) \quad a_i = n_0 + n_j + n_k.\]

The stabilizer of $R$ in $A$ is $A_R = S_{n_0} \times S_{n_1} \times S_{n_2} \times S_{n_3}$. Its embedding into $A$ is the composition of two homomorphisms, namely

\[(B.3) \quad S_{n_0} \times S_{n_1} \times S_{n_2} \times S_{n_3} \xrightarrow{\alpha_1} S_{a_1} \times S_{a_2} \times S_{a_3} \times S_{n_3} \xrightarrow{\beta_1} S_{n_1} \times S_{n_2} \times S_{n_3},\]

where the left arrow is four times a diagonal embedding and the right embedding is columnwise.

We calculate the character of $\langle A/A_R \rangle K = \text{ind}_{A_R}^A \chi_{\text{triv}}$ in terms of Schur functions in two steps. First recall the formulas

\[(B.4) \quad \text{ind}_{S_n}^{S_n \times S_n} \chi_{\text{triv}} = \sum_{\lambda} s_{\lambda} \otimes s_{\lambda}\]

and

\[(B.5) \quad \text{ind}_{S_n}^{S_n \times S_n \times S_n} \chi_{\text{triv}} = \sum_{|\alpha| = |\beta| = n} s_{\alpha} \otimes s_{\beta} \otimes (s_{\alpha} * s_{\beta}).\]

Recall also that

\[(B.6) \quad \text{ind}_{S_m \times S_n}^{S_m + n} s_{\lambda} \otimes s_{\mu} = s_{\lambda}s_{\mu}.\]

Then

\[(B.7) \quad \text{ind}_{A_R}^A \chi_{\text{triv}} = \sum_{\alpha, \beta, \gamma_1, \gamma_2, \gamma_3} s_{\alpha}s_{\gamma_3}s_{\gamma_2} \otimes s_{\beta}s_{\gamma_1}s_{\gamma_3} \otimes (s_{\alpha} * s_{\beta})s_{\gamma_2}s_{\gamma_1}\]

where the summation variables are partitions subject to the constraints

\[(B.8) \quad |\alpha| = |\beta| = n_0 \text{ and } |\gamma_i| = n_i.\]

For three partitions $\lambda, \mu, \nu$ with $|\lambda| = a_1$, $|\mu| = a_2$, and $|\nu| = a_3$ we deduce from equations (5.15), (5.3), and (B.7)

\[\langle s_{\lambda} * s_{\mu} \mid s_{\nu} \rangle = M'_{\lambda \mu} = \sum_{R \in T/A} \langle \text{ind}_{A_R}^A \chi_{\text{triv}} \mid s_{\lambda} \otimes s_{\mu} \otimes s_{\nu} \rangle =\]

\[= \sum_{\alpha, \beta, \gamma_1, \gamma_2, \gamma_3} \langle s_{\alpha}s_{\gamma_3}s_{\gamma_2} \mid s_{\lambda} \rangle \langle s_{\beta}s_{\gamma_1}s_{\gamma_3} \mid s_{\mu} \rangle \langle (s_{\alpha} * s_{\beta})s_{\gamma_2}s_{\gamma_1} \mid s_{\nu} \rangle\]

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Only the constraint $|\alpha| = |\beta|$ needs to be kept since all other possibilities contribute summands equal to 0. After rearranging the terms of (B.9) we get the assertion:

\[
\langle s_\lambda \ast s_\mu \mid s_\nu \rangle = \sum_{|\alpha| = |\beta|} \left( \langle s_\alpha \ast s_\beta \rangle \sum_{\gamma_3} \langle s_\alpha s_\gamma s_\gamma \mid s_\lambda \rangle \langle s_\gamma \mid s_\gamma \rangle \sum_{\gamma_1} \langle s_\beta s_\gamma s_\gamma \mid s_\mu \rangle \langle s_\gamma \mid s_\nu \rangle \right)
\]

(B.10)

\[
= \sum_{|\alpha| = |\beta|} \langle s_\alpha \ast s_\beta \rangle \langle s_\lambda \mid \alpha \gamma_3 s_\mu \mid \beta \gamma_3 \rangle \langle s_\nu \rangle
\]

\hfill \square

### B.2. Proof of Corollary 6.13.

Multiplying all terms of (6.33) by $\chi_\lambda^\mu$ and summing over $\lambda$ one obtains the equivalent formula

\[
\chi_{[n]}(g_\mu) = (-1)^{\ell(\mu)}C_{m_1(\mu)} = \sum_{\lambda} \frac{\deg n!^\lambda f(\lambda + \delta)\chi_\lambda^\mu.}
\]

Consider the differential operators $D := f(x, \partial_x)$ and $D := a_\delta^{-1}\tilde{D}a_\delta$ (for unexplained notation see [Mac95, pp. 467–468]). Then $\tilde{D}(a_{\lambda+\delta}) = f(\lambda + \delta)a_{\lambda+\delta}$ and therefore $D(s_\lambda) = f(\lambda + \delta)s_\lambda$. From $p_{1^n} = \sum_\lambda \deg \chi_\lambda^s_\lambda$ and $\langle s_\lambda \mid p_\mu \rangle = \chi_\lambda^\mu$ one gets that the right hand side of (B.11) equals $\frac{1}{n!}\langle D(p_{1^n}) \mid p_\mu \rangle$.

Now observe that, by Harish-Chandra’s isomorphism, $D$ is the radial part of a biinvariant differential operator $C$ on $M(n, \mathbb{C})$. Other such operators are the Capelli operators

\[
C_k := \sum_{I, J \subseteq [n]} \det A_{IJ} \det \partial A_{IJ}, \quad k = 0, \ldots, n.
\]

The radial part of $C_k$ is the polynomial $P_{1^k}(x) := s_1^*(x - \delta)$ where $s_\lambda^*$ is a shifted Schur function (see [OO97, Cor. 6.6]). From the relation $f(x) = \sum_{k=0}^n (-1)^k(t)_{n-k}P_{1^k}$ (follows from [OO97, (12.4)]) by multiplying by $(u \downarrow v)$ and setting $u = -t + n - 1$ we get $C = \sum_{k=0}^n (-1)^k(t)_{n-k}C_k$.

The polynomial $p_{1^n} = e_1^n = (x_1 + \ldots + x_n)^n$ is the radial part of $h(A) = (\text{tr} A)^n$ and we have $\det \partial A_{IJ}(h) = 0$ for $I \neq J$ and $\det \partial A_{I1}(h) = (n)_{|I|}(\text{tr} A)^n - |I|$. It follows that the radial part of $C_k((\text{tr} A)^n)$ is $(n)_k e_k e_1^{n-k}$ and therefore

\[
C(p_{1^n}) = \sum_{k=0}^n (-1)^{n-k} \frac{n!}{k!} \langle t \rangle_k e_{n-k} e_1^k.
\]

From $e_{n-k} = \sum_{|\tau|=n-k}(-1)^{n-k-\ell(\tau)}z_\tau^{-1}p_\tau$ ([Mac95, Ch. I (2.14')]), $p_\tau e_1^k = p_{\tau 1^k}$, and $\langle p_\lambda \mid p_\mu \rangle = z_\mu \delta_{\lambda\mu}$ ([Mac95, Ch. I (4.7)]) we get

\[
\langle \frac{1}{n!}D(p_{1^n}) \mid p_\mu \rangle = \sum_{k: \tau 1^k = \mu} (-1)^{\ell(\mu)-k} \frac{1}{z_\tau} \frac{1}{k!} \langle t \rangle_k = (-1)^{\ell(\mu)} \sum_{k=0}^m \frac{m_1(\mu)}{k!} (-1)^k \langle t \rangle_k
\]

as claimed. \hfill \square

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