Topological Matrix Models, Liouville Matrix Model and $c = 1$ String Theory

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Abstract: This is a review of some beautiful matrix models related to the moduli space of Riemann surfaces as well as to noncritical $c = 1$ string theory at self-dual radius. These include the Penner model and the $W_{\infty}$ model, which have different origins but are equivalent to each other. In the final section, which is new material, it is shown that these models are also equivalent to a Liouville matrix model. We speculate that this might be interpreted in terms of $N$ D-instantons of the $c = 1$ string.

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1. Introduction

In recent months there has been a revival of interest in the noncritical $c = 1$ bosonic string\textsuperscript{[1, 2, 3]} (and its worldsheet supersymmetric counterparts\textsuperscript{[4, 5, 6]}). New insight has been gained into this relatively simple string theory, using several developments that came after the previous matrix revolution: D-branes\textsuperscript{[7]}, M(atrix) Theory\textsuperscript{[8, 9]}, AdS/CFT\textsuperscript{[10]} and the understanding of boundary states in Liouville theory\textsuperscript{[11, 12, 13]}.
Most of the recent work has centred on matrix quantum mechanics, whose double-scaled limit is believed to represent the $c = 1$ string (at least perturbatively). The basic idea is that the matrix of this model is the matrix-valued tachyon on the worldline of a collection of $N$ D0-branes.

There are also claims that a nonperturbatively consistent version of matrix quantum mechanics can be formulated. This is supposed to be equivalent not to the bosonic $c = 1$ string, but to a worldsheet supersymmetric version, the $\hat{c} = 1$ type 0B string\[4, 5\]. We will not discuss this latter theory here. But many of the observations in the present review presumably can, and should, be generalised to the noncritical type 0B (and 0A) string background.

In its simplest (un-orbifolded) form, the $c = 1$ string has a translationally invariant space/time direction $X$. If it is Euclidean, it can be compactified on a circle of radius $R$. In that case, all physical quantities (partition function and amplitudes) depend on $R$. The Euclidean theory can be interpreted as a finite-temperature theory with $R$ labelling the inverse temperature.

The value $R = 1$ (in units where $\alpha' = 1$) is special because the $c = 1$ CFT is then self-dual under

$$R \rightarrow \frac{1}{R}$$

In this article we will focus on this background: the Euclidean $c = 1$ string with the $X$ direction compactified at the self-dual radius. For short, we will refer to the theory as “$c = 1, R = 1$”.

There are many indications that string theory is topological in this background. The term “topological string theory” is usually taken to mean a string theory where the matter sector has a twisted $\mathcal{N} = 2$ superconformal algebra with central charge zero. To make it a string theory, this matter then has to be coupled to topological worldsheet gravity\[14, 15\], which can be described by a set of bosonic and fermionic ghosts with total central charge zero. The fermionic charge of the twisted $\mathcal{N} = 2$ algebra is the BRST charge defining physical states.

A classic example of such a theory is a superstring on a Calabi-Yau background, whose superconformal worldsheet theory has $\mathcal{N} = 2$ supersymmetry with central charge $c = 9$. On making the standard topological twist of the superconformal algebra:

$$T(z) \rightarrow T(z) + \frac{1}{2} \partial J(z)$$

where $J(z)$ is the $U(1)$ current, the algebra acquires central charge 0 and we have a suitable matter system for a topological string\[16\].

It is known that ordinary string theory, including the ghost sector, can be considered as topological matter\[17, 18, 19\], which makes the distinction between topological and non-topological theories less clear-cut. Even critical bosonic and superstrings are known to be “topological” in this sense. The best distinction one can make is
that the latter theories are “already twisted” while the conventional topological theories are formulated as $\mathcal{N} = 2$ superconformal theories and then given a topological twist.

Topological string theories are generally related to the topology of the moduli space of Riemann surfaces. For example, “pure” topological gravity describes intersection theory on cohomology classes associated to moduli space, or to vector bundles on moduli space\[13\].

Some of the indications that $c = 1, R = 1$ is topological arise from its relation to other theories. The partition function of $c = 1, R = 1$ is closely related to that of the Penner matrix model\[20, 21\], a model constructed to count the Euler characteristic of the moduli space of punctured Riemann surfaces. Amplitudes in $c = 1, R = 1$ are summarised in the form of $W_\infty$ constraints\[22\] and the partition function of the perturbed theory is a $\tau$-function of an integrable hierarchy\[1\]. This $\tau$-function in turn can be written as a matrix model, the $W_\infty$ matrix model\[24\] (this has been previously referred to as the “Kontsevich-Penner model”\[22\] and a “Kontsevich-type model for $c = 1$”\[24\]).

Another indication of the topological nature of $c = 1, R = 1$ is that it is dual to the topological 2d black hole\[19, 25, 26\]. The latter theory indeed starts life as an $\mathcal{N} = 2$ superconformal field theory in two dimensions, where the second supersymmetry is a consequence of the Kazama-Suzuki construction. This theory is also dual to topological Landau-Ginsburg theory\[27, 28, 29\], which is a convenient formulation for explicitly computing amplitudes and comparing them to those of the original $c = 1, R = 1$ theory as a test of the duality.

Finally, $c = 1, R = 1$ has been shown to be dual to topological strings on a conifold singularity\[30\]. Recently it has also been argued\[31\] that $c = 1, R = \infty$, which might be more “physical”, is dual to an infinite-order orbifold of the conifold. Such infinite-order orbifolds can be understood in the language of deconstruction\[32, 33\]. In particular, the above orbifold is believed to be dual to the $(2,0)\, CFT$ in 6 dimensions\[34\] (an alternative limit instead gives rise to the nonrelativistic type IIA string\[35\]). Thus it may be that the noncompact $c = 1$ string can also be usefully formulated as a topological theory. In this case one could even try to continue the $X$ direction back to Minkowski signature and see what happens in the dual theory. Progress on these issues is, however, quite limited to date.

The various developments described above are important because the “topological” context is usually simpler than the “dynamical” one. Moreover, the former is often embedded in the latter, an example being Gopakumar-Vafa duality\[36\], in which an open-closed topological string duality can be lifted\[37\] to an open-closed string duality in the full superstring theory. The computations in the topological theory are related to specific types of amplitudes in the dynamical theory.

\footnote{A different approach to $W_\infty$ constraints and associated matrix models is described for example in Ref.\[23\].}
In this article we will focus on the Penner and $W_\infty$ matrix models. These have been dubbed “topological matrix models” because they originate in the description of the moduli space of Riemann surfaces and also because they are dual to $c = 1$, $R = 1$. These are models of constant matrices and are quite different, in spirit as well as in details, from the familiar $c = 1$ matrix quantum mechanics. In particular, the nature and role of the large-$N$ limit is rather different.

The plan of this article is as follows. In Section 2 we start by briefly reviewing the Liouville and the matrix quantum mechanics approaches to $c = 1$ string theory. After that we collect some explicit results from matrix quantum mechanics that will be useful in the subsequent discussions.

In Section 3 we will explain how the Penner model is related to the triangulation of the moduli space of Riemann surfaces, rather than the triangulation of random surfaces. This requires some discussion of meromorphic quadratic differentials on Riemann surfaces. Section 4 deals with the $W_\infty$ model that is built from correlators of $c = 1$, $R = 1$. The partition function is a $\tau$-function of the Toda hierarchy. Relationships between this model and the Penner model as well as the Kontsevich model are discussed. Essentially the $W_\infty$ model is a perturbation of the Penner model by tachyon couplings, but we will see that the relation between them involves a somewhat subtle change of variables that may have a physical significance.

To date, both these models have been constructed a posteriori by computing appropriate amplitudes of the $c = 1$, $R = 1$ string from matrix quantum mechanics, and then summing them up into an appropriate generating function. This is a somewhat ad hoc state of affairs and does not provide a fundamental explanation of how they arise, nor does it indicate how these models can be generalised to other backgrounds.

In Section 5, we will present some observations which might help to remedy this situation. It will be shown that the $W_\infty$ model can be rewritten as a “Liouville matrix model” – a model of constant matrices with an exponential plus linear potential. This way of rewriting the model is possible only because of its specific form, including the dependence on coupling constants.

The presence of a matrix with an exponential plus linear potential is strongly suggestive of D-instantons moving in the background of a linear dilaton as well a cosmological (exponential) term arising from the closed string tachyon. We will try to develop the analogy, but will not give a conclusive argument that the $W_\infty$ model is really the world-point action on $N$ D-instantons in the $c = 1$, $R = 1$ string theory. If this connection can really be demonstrated, it would indicate a new type of holographic relationship, analogous to the IKKT matrix model\cite{IKKT} of critical string theory. It might also point to some new relationships between the perturbation series of string theory and the topology of moduli space.

An important recent development is the construction of a new matrix model called the “normal matrix model” (NMM)\cite{NMM}. Like the $W_\infty$ model, this one too is...
built directly out of the $W_\infty$ solution for the partition function of $c = 1, R = 1$ (and also other integer $R$). However, it does not seem to be known at present what is the precise relation of NMM to the $W_\infty$ and Penner models. We will briefly discuss the NMM at the end in order to exhibit some similarities to the other models that are the main subject of this article. Interestingly, in Ref. [38] a duality has been proposed between matrix quantum mechanics and NMM, where the two models are associated respectively to the non-compact and compact cycles of a certain complex curve. This suggests an elegant answer to the question of how $c = 1$ matrix quantum mechanics and topological matrix models are related. The discussion of the normal matrix model presented here will unfortunately be very brief.

2. $c = 1$ at $R = 1$

In this section we will summarise some results from the matrix quantum mechanics approach to $c = 1$ string theory. An excellent review is Ref. [39], where derivations of the formulae to be presented below, as well as extensive references, can be found. First we will briefly discuss the definition of $c = 1$ string theory, both from the continuum perspective and through matrices.

2.1 Continuum formulation of $c = 1$

The continuum description of noncritical $c = 1$ strings starts from a matter conformal field theory of a free scalar field $X$, with energy-momentum tensor:

$$T^X_{zz} = -\partial X \partial X$$

This system has unit central charge, and its basic conformal fields are the vertex operators

$$V = : e^{ikX} :$$

as well as polynomials in $\partial^n X$ for various $n$, and products of these with the vertex operators. The coordinate $X$ is taken to be the time direction, though one also frequently studies a Euclideanised theory where $X$ is taken to be spacelike.

A string theory with the critical central charge arises by coupling the above CFT to a Liouville field with energy-momentum tensor:

$$T^\phi = -\partial \phi \partial \phi + 2 \partial^2 \phi$$

This Liouville field has central charge $c = 25$, so that the string is in this sense critical. However, it is supposed that this field arises as the scale factor of the worldsheet metric, which adjusts itself self-consistently to carry this central charge. The Liouville field is spacelike.
Observables of this theory are obtained by computing the BRST cohomology, just as is done for critical string theory. An important class of observables are the “tachyons”, which in Euclidean signature are given by:

\[ T_k = e^{(2-|k|)\phi} e^{ikX} \]  

The name “tachyon” is a misnomer, as these are actually massless states in two dimensions.

There are also other physical modes in the BRST cohomology, the so-called “discrete states” \([40, 41, 42]\), which arise at special values of the momentum. These can be thought of as the two dimensional analogues of the graviton and other tensor modes of a closed string theory. In two dimensions, tensor fields have no propagating field-theoretic degrees of freedom, but in this theory they do have residual discrete modes that survive in the BRST cohomology. The tachyons described above can scatter into these discrete states, a fact which will become important when we discuss amplitudes.

2.2 Matrix formulation of \( c = 1 \)

A very powerful description of \( c = 1 \) strings can be obtained starting from a random matrix integral:

\[ \int [dM] e^{-\beta \int_0^{2\pi R} dx \ tr \left( \frac{1}{2} \dot{M}^2 + \frac{1}{2} M^2 - \frac{1}{3!} M^3 \right)} \]  

where the matrix measure is given by:

\[ [dM] \equiv \prod_i dM_{ii} \prod_{i<j} dM_{ij} dM_{ij}^* \]  

A perturbative expansion of this integral leads to 't Hooft-type Feynman diagrams with cubic vertices. Each such diagram specifies a unique Riemann surface topology on which it can be drawn as a random lattice with three lines meeting at every point. The faces are arbitrary \( n \)-gons. The dual lattice has \( n \) lines meeting at a point but the faces are triangles. The result is a triangulated Riemann surface.

Thus the matrix quantum mechanics can be taken to represent the discretisation of a Riemann surface. Summing over all triangulations of the Riemann surface amounts to summing over all inequivalent conformal classes, so the result should be equivalent to a string theory. Indeed, this is the \( c = 1 \) string. The time direction that we called \( X \) in the continuum case is associated to the time coordinate \( t \) on which the matrix coordinate depends, while the Liouville direction in the continuum language is related to the infinite number of eigenvalues of the matrix at large \( N \) \([43]\).

To solve this string theory, the matrix is first diagonalised. Then the eigenvalue-dependent part of the measure becomes:

\[ [dM] = \prod_i d\lambda_i \Delta^2(\lambda) \]  

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where $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the Vandermonde determinant. In terms of eigenvalues, the kinetic term of the Hamiltonian is:

$$\frac{1}{\Delta(\lambda)} \sum_i \frac{d^2}{d\lambda_i^2} \Delta(\lambda_i)$$

acting on wave functions $\chi(\lambda_i)$. By redefining the wave function as follows:

$$\chi(\lambda_i) \rightarrow \Psi(\lambda_i) = \Delta(\lambda_i)\chi(\lambda_i)$$

we have a simpler Hamiltonian but the new wave function $\Psi$ is fermionic, since interchange of any two eigenvalues gives a minus sign. So the matrix eigenvalues behave as $N$ fermions moving in the given potential.

The ground state of the theory is found by filling up the first $N$ levels (after regulating the potential) to the Fermi level $-\mu_F$. Next we take a double-scaling limit

$$\mu_F \rightarrow 0, \quad \beta \rightarrow \infty$$

with $\mu = \beta \mu_F$ fixed. In this limit, for purposes of perturbation theory the eigenvalues behave like fermions in an inverted harmonic oscillator potential.

The physical quantities (partition function, correlation functions) of this model will be given perturbatively as an expansion in powers of

$$g_s^2 \equiv \frac{1}{\mu^2}$$

Nonperturbative effects will typically arise as terms like

$$e^{-1/g_s} \sim e^{-\mu}$$

in the amplitudes.

Computing the density of eigenvalues, one is able to evaluate the partition function and free energy in a genus expansion. Correlators of observables can also be computed, as we will indicate in a subsequent section.

This much of the discussion is valid for any compactification radius $R$, including the limit $R \rightarrow \infty$. But at $R = 1$ we will see that the formulae acquire special properties.
2.3 Free Energy

Because computations in matrix quantum mechanics are highly technical, we will only quote some relevant results and refer the reader to Ref. [39] and references therein for the derivations.

The free energy $F(\mu) \equiv \log Z(\mu)$ of the $c = 1$ string was first obtained for arbitrary $R$ by Gross and Klebanov [44]:

$$\frac{\partial^2 F(\mu)}{\partial \mu^2} = \text{Re} \int_0^\infty \frac{dt}{t} e^{-i\mu t} \frac{t/2}{\sinh t/2} \frac{t/2R}{\sinh t/2R}$$

(2.13)

This is to be understood as an asymptotic expansion in powers of $1/\mu^2 = g_s^2$. The dependence on the compactification radius $R$ arises from the last factor in the integrand. In the limit $R \to \infty$, this factor goes to 1.

The above formula can be integrated twice in $\mu$ to find the free energy. The integration constants are non-universal terms that will be unimportant.

Performing the expansion in $1/\mu^2$, we find:

$$\frac{\partial^2 F(\mu)}{\partial \mu^2} = -\log \mu + \sum_{g=1}^\infty \frac{f_g(R)}{(4R)^g} \mu^{-2g}$$

(2.14)

where

$$f_g(R) = (2g - 1)! \sum_{k=0}^g |2^{2k} - 2|2^{2(g-k)} - 2||B_{2k}| \frac{|B_{2(g-k)}|}{(2k)! (2(g-k))!} R^{g-2k}$$

(2.15)

and $B_{2k}$ are the Bernoulli numbers.

Something special happens at $R = 1$. One can use a bilinear identity on Bernoulli numbers, due to Gosper [2]:

$$\sum_{i=0}^n \frac{(1-2^{1-i})(1-2^{i-n+1})B_{n-i}B_i}{(n-i)! i!} = \frac{(1-n)B_n}{n!}$$

(2.16)

Using this identity, we get:

$$\left. \frac{\partial^2 F(\mu)}{\partial \mu^2} \right|_{R=1} = -\log \mu + \sum_{g=1}^\infty \frac{2g-1}{2g} |B_{2g}| \mu^{-2g}$$

(2.17)

Integrating, the free energy is found to be:

$$F(\mu)_{R=1} = \sum_{g=0}^\infty \frac{|B_{2g}|}{2g(2g-2)} \mu^{2-2g}$$

(2.18)

\footnote{For just about everything you need to know about Bernoulli numbers, including this identity, see Ref. [4].}
For genus \( g = 0, 1 \) the coefficients are formally divergent (because of the dropped integration constants).

It is remarkable that the above expression arises from doing matrix quantum mechanics in the double scaling limit. As we will see, it is closely related to the (virtual) Euler characteristic of the moduli space of Riemann surfaces. This is the first of many special properties of \( c = 1, R = 1 \) that we will encounter.

### 2.4 Tachyon correlators

Let us first consider the zero-momentum tachyon. The corresponding operator \( T_0 = e^{2\phi} \) is called the cosmological operator, because it is conjugate to the cosmological constant \( \mu \):

\[
\langle T_0 T_0 \cdots T_0 \rangle_g \sim \frac{1}{s!} \frac{\partial^s}{\partial \mu^s} \mathcal{F}(\mu) \tag{2.19}
\]

Next consider \( T_k \) for general nonzero \( k \). At \( R = \infty \), \( k \) is a continuous real variable, while at \( R = 1 \), it is quantised to be a (positive or negative) integer. Because \( \mathcal{X} \) is a free field, the total \( k \) is conserved:

\[
\langle T_{k_1} T_{k_2} \cdots T_{k_n} \rangle_g = 0 \quad \text{unless} \quad \sum_{i=1}^n k_i = 0 \tag{2.20}
\]

In matrix quantum mechanics, one must identify the operators that correspond to the massless tachyon modes. This procedure is less straightforward than computing the BRST cohomology in the continuum case. It turns out that the correct local operators are defined as moments of loop operators\,[46, 47], up to normalisation – about which we will say more below. Define:

\[
\mathcal{O}(l, k) = \int dx e^{ikx} \text{tr} e^{-l\mathcal{M}(x)} \sim l^k \mathcal{P}_k + \cdots \tag{2.21}
\]

where we keep the leading term for small \( l \).

Correlators of the \( \mathcal{P}_k \) the follow by computing correlation functions of the eigenvalue density. For general \( R \), these computations were performed in Refs.\,[18, 24]. For example, the three point function is given by:

\[
\langle P_{k_1} P_{k_2} P_{k_3} \rangle = R \delta(k_1 + k_2 + k_3) \mu^{2(|k_1|+|k_2|+|k_3|)} \Gamma(1 - |k_1|) \Gamma(1 - |k_2|) \Gamma(1 - |k_3|) \times
\]

\[
\left[ 1 - \frac{1}{24R}(|k_3| - 1)(|k_3| - 2) \right] \left\{ R(k_1^2 + k_2^2 - |k_3| - 1) - \frac{1}{R} \right\} \mu^{-2} + \mathcal{O}(\mu^{-4}) \tag{2.22}
\]

It is a remarkable achievement of matrix quantum mechanics that this result is actually known to all orders in \( 1/\mu^2 \). Despite appearances, it is totally symmetric under interchange of \( k_1, k_2, k_3 \), and also invariant under \( k_i \to -k_i \) as required by translational symmetry in the \( \mathcal{X} \) coordinate.
By contrast, amplitude computations in the continuum Liouville theory have only been done for genus $g = 0$\cite{Witten1985}. The three-point function in this formulation was found to be:

$$
\langle T_{k_1} T_{k_2} T_{k_3}\rangle_{g=0} = R \delta(k_1 + k_2 + k_3) \mu^{\frac{1}{2}(|k_1| + |k_2| + |k_3| - 2)} \frac{\Gamma(1 - |k_1|) \Gamma(1 - |k_2|) \Gamma(1 - |k_3|)}{\Gamma(|k_1|) \Gamma(|k_2|) \Gamma(|k_3|)}
$$

It follows that the continuum tachyons $T_k$ and the matrix model puncture operators $P_k$ are related by:

$$
T_k = \frac{1}{\Gamma(|k|)} P_k
$$

At zero momentum this represents an infinite change of scale between the two descriptions\textsuperscript{3}. For other values of $k$ it is just a finite change.

We see that whenever $k$ is a nonzero integer, the amplitudes for both $T_k$ and $P_k$ develop poles. Integer momenta are precisely the ones for which additional discrete states (to which we briefly referred in the Introduction) exist in the theory. The divergences in tachyon amplitudes are believed to be due to the production of these states in intermediate channels.

At $R = 1$, since $k$ is always an integer, every correlator of $T_k$ or $P_k$ (except for $P_0$) is divergent. Hence it is necessary to define “amputated tachyon operators” $T_k$:

$$
T_k = \frac{\Gamma(1 + |k|)}{\Gamma(1 - |k|)} T_k = \frac{1}{\Gamma(1 - |k|)} P_k
$$

The correlators of these operators have no poles\textsuperscript{4}. Indeed, the three-point function of amputated tachyons is:

$$
\langle T_{k_1} T_{k_2} T_{k_3}\rangle = \delta(k_1 + k_2 + k_3) \mu^{\frac{1}{2}(|k_1| + |k_2| + |k_3| - 2)} |k_1 k_2 k_3| \times
$$

$$
\left[ 1 - \frac{1}{24}(|k_3| - 1)(|k_3| - 2)(k_1^2 + k_2^2 - |k_3| - 2) \mu^{-2} + O(\mu^{-4}) \right]
$$

The absence of poles in the amplitudes for amputated tachyons suggests that we are really working with a somewhat different theory, in which the other discrete states are no longer present. The discrete tachyons appear to be the only surviving states. In particular, this means that the radius perturbation, which should be turned on to continuously deform $R$ away from its self-dual value, is absent. In this sense, the

\textsuperscript{3}This is responsible for the fact that the cosmological operator is not really $e^{2\phi}$ but $\phi e^{2\phi}$. Indeed, we see that

$$
P_0 = \lim_{k \to 0} \frac{T_k}{k} \sim -\phi e^{2\phi} + \text{infinite term}
$$

\textsuperscript{4}In the Minkowski theory these leg pole factors are phases. Their physical role has been discussed by Polchinski\cite{Polchinski1986}. 

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theory is “stuck” at $R = 1^5$. The above properties characterise the topological $c = 1$ background$^6$.

It is intriguing that the coefficient of the $\mu^{-2}$ term is an integer for every $k_1, k_2, k_3$ satisfying $k_1 + k_2 + k_3 = 0$. Explicit computation of higher-order terms in this and other correlators suggests that this is always the case, though there does not seem to be an explicit statement or proof in the literature. If every term is an integer, this would certainly be special to $R = 1$, as this property can be seen not to hold for any other $R$.

An exact generating function for all discrete-tachyon correlators, in every genus, is known$^{22}$. This will be described in a subsequent section.

3. Riemann Surfaces and the Penner Matrix Model

As has been indicated, the above results were derived starting from matrix quantum mechanics, using the free fermion description. The first indication that these are related to the topology of Riemann surfaces arose in the study of the Penner matrix model$^{21}$, via the work of Distler and Vafa$^{21}$. To explain this model, we must first discuss some mathematical issues relating to the moduli space of Riemann surfaces of genus $g$.

3.1 Moduli space of Riemann surfaces and its topology

The moduli space of a compact Riemann surface of genus $g$ without punctures, $\mathcal{M}_g$, is a manifold of complex dimension $3g - 3$ (for $g > 1$). It arises as the quotient of the Teichmüller space by the action of the mapping class group. Since this action has fixed points, the moduli space $\mathcal{M}_g$ has orbifold-like singularities.

What is the simplest topological invariant of $\mathcal{M}_g$? For a $D$-dimensional smooth manifold, we can define the the Euler characteristic $\chi$ by making a simplicial decomposition $S$ of the manifold and then evaluating:

$$\chi = \sum_{i \in S} (-1)^{d_i}$$

where $d_i$ is the dimension of the $i$th simplex, and the sum is over all the simplices in the complex $S$.

In the presence of orbifold singularities, the natural quantity to define is the virtual Euler characteristic $\chi_V$. Here each term in the sum is divided by the order of a discrete group $\Gamma_i$ that fixes the $i$th simplex. Thus:

$$\chi_V = \sum_{i \in S} \frac{(-1)^{d_i}}{\#(\Gamma_i)}$$

$^5$One can take orbifolds to get other integer values of $R$.

$^6$Note that “winding tachyons”, which are winding modes around the $X$ direction, should still exist, though they have not yet been satisfactorily understood in topological matrix models.
It was found by Harer and Zagier\cite{51} that for the moduli space of unpunctured Riemann surfaces, the virtual Euler characteristic is:

\[ \chi_g = \frac{B_{2g}}{2g(2g-2)} \]  

(3.3)

where \( B_{2g} \) are the Bernoulli numbers.

One can also consider the moduli space \( \mathcal{M}_{g,s} \) of Riemann surfaces of genus \( g \) with \( s \) punctures. This has complex dimension \( 3g - 3 + s \) (for \( g > 1 \), or \( g = 1, s \geq 1 \), or \( g = 0, s \geq 3 \)). Its virtual Euler characteristic is given by:

\[ \chi_{g,s} = \frac{(-1)^s(2g - 3 + s)(2g - 1)}{(2g)!s!} B_{2g} = (-1)^s \binom{2g - 3 + s}{s} \chi_g \]  

(3.4)

The above results for \( \chi_g \) and \( \chi_{g,s} \) are related in a suggestive way. In string theory, one expects that \( s \) punctures can be created by differentiating the vacuum amplitude \( s \) times with respect to the cosmological constant. And indeed, as noticed by Distler and Vafa\cite{21}, one has the identity:

\[ \frac{1}{s!} \frac{\partial^s}{\partial \mu^s} (\chi_g \mu^{2-2g}) = \chi_{g,s} \mu^{2-2g-s} \]  

(3.5)

Thus, if we could invent a model whose genus \( g \) partition function is \( \chi_g \), we might expect it to bear some relation to string theory. Below we will see that this is indeed the case.

### 3.2 Quadratic differentials and fatgraphs

The above results were obtained by triangulating the moduli space of punctured Riemann surfaces. Such a triangulation was constructed by Harer\cite{52} in terms of quadratic differentials, using a theorem due to Strebel\cite{53}. It is instructive to sketch how this was done.

On any Riemann surface with a finite number of marked points, one can define a meromorphic quadratic differential

\[ \eta = \eta_{zz}(z) \, dz^2 \]  

(3.6)

which has poles at the locations of the marked points. This can equivalently be considered as a holomorphic differential on the punctured Riemann surface, which has these marked points removed.

Under a change of coordinates:

\[ z \rightarrow z'(z) \]  

(3.7)

a quadratic differential transforms as:

\[ \eta'_{zz'}(z') = \left( \frac{\partial z}{\partial z'} \right)^2 \eta_{zz}(z) \]  

(3.8)
This differential can be used to invariantly define the length of a curve $\gamma$ on the Riemann surface. The length is simply defined to be:

$$|\gamma|_\eta = \int_\gamma \sqrt{|\eta(z)||dz|}$$

(3.9)

Indeed, defining a new coordinate via

$$dw = \sqrt{\eta(z)}dz$$

(3.10)

we see that this length is the ordinary length of the curve in the Euclidean sense, in the $w$ coordinate.

Now a quadratic differential is not in general unique, but it was shown by Strebel [53] that with some extra properties, such a differential is unique up to multiplication by a positive real number. For this, let us consider a geodesic curve under the metric defined above. At any point, such a curve will be called horizontal if $\eta$ is real and positive along it, and vertical if $\eta$ is real and negative. The horizontal curves define flows along the Riemann surface.

The flow pattern is regular except at zeroes and poles of $\eta$. Here the flows exhibit interesting properties. At an $n$th-order zero of the quadratic differential, precisely $n + 2$ horizontal curves meet at a point. To see this, let us write down what the differential looks like near this zero and along the radial direction:

$$\eta \sim z^n (dz)^2 \sim e^{i(n+2)\theta} dr^2$$

(3.11)

From this we see that as we encircle the zero, there are precisely $n + 2$ values of the angle $\theta$ at which this differential is positive, or horizontal.

At a double pole, if the coefficient is real and negative, the flows form concentric circles around the point. This follows from the fact that near the pole, and along the angular direction, the differential looks like:

$$\eta \sim -c\frac{dz^2}{z^2} \sim c \, d\theta^2$$

(3.12)

Thus, in the $\theta$ direction, the differential is positive, or horizontal, at all points surrounding the double pole.

Other behaviours are possible at poles of order $n = 1$ or $n \geq 3$, or if the coefficient of $\eta$ at a double pole is complex (for a comprehensive description, see Figs.5-13 of Ref.[53]). But we do not need to consider these other behaviours because we will restrict our quadratic differentials not to have such poles.

In fact, we will specialise to a class of quadratic differentials which have a double pole at a point $P$, at which the coefficient $c$ is required to be real and negative. Next, we require that all smooth horizontal trajectories (i.e., those that do not pass through zeroes of $\eta$) form closed curves. Quadratic differentials satisfying all these conditions exist, and are called horocyclic.
An illustration of the flow pattern associated to a horocyclic quadratic differential is given in Fig.2. Note that the vertex has five lines meeting at a point, indicating a third-order zero (the differential also has additional zeroes not shown in the diagram).

**Figure 2:** Riemann surface with the flow pattern of a horocyclic quadratic differential.

Strebel’s theorem states that on every Riemann surface of genus $g$ with 1 puncture, for fixed values of its moduli, there exists a unique horocyclic quadratic differential with a double pole at the puncture. (The uniqueness is up to multiplication by a real positive number).

Thus, by studying how these quadratic differentials vary as we vary the moduli, we get information about the moduli space $\mathcal{M}_{g,1}$ of a once-punctured Riemann surface. In view of some mathematical results relating the moduli spaces of Riemann surfaces of a fixed genus $g$ but different numbers of punctures $s$\cite{52}, this construction actually enables us to study the moduli space $\mathcal{M}_{g,s}$ of $s$-punctured Riemann surfaces as well.

The main motivation for having gone through this mathematical description here is that we will now see the emergence of “fatgraphs”, otherwise known as ’t Hooft diagrams or large-$N$ diagrams. Most of the flows are closed and smooth, but there are singular ones that branch into $n+2$-point vertices at $n$th order zeroes of $\eta$. We can think of these singular flows as defining a Feynman diagram, whose vertices are the branch points. Each double pole of $\eta$ is a point around which the flows form a loop, hence the number of loops of the diagram is the number of double poles, which is the number of punctures of the original Riemann surface. Finally, because the flows that do not pass through a zero are smooth, each singular flow can be “thickened” into a smooth ribbon in a unique way, and we arrive at a fatgraph.

The fatgraphs with a single face triangulate the moduli space $\mathcal{M}_{g,1}$ in the following way. Consider the lengths of each edge of a fatgraph, as computed in the metric defined in Eq. (3.4) above. We have an overall freedom of scaling the whole Riemann surface, which clearly does not change the moduli. So to vary the moduli, we must
change the lengths of the different edges keeping the total length fixed. By Strebel’s correspondence between quadratic differentials and moduli, this sweeps out a region of the moduli space of the Riemann surface. The dimensionality of this region will be $E - 1$ where $E$ is the number of edges of the graph.

As Harer argues, this region swept out will not be the whole moduli space, but a simplex of it. In a simplicial decomposition, at the boundary of a simplex we find a lower-dimensional simplex. In terms of fat graphs, a boundary occurs whenever a length goes to zero and two vertices meet. As an example, when two three-point vertices meet, we obtain a four-point vertex. The new graph thus obtained sweeps out a lower-dimensional simplex in the moduli space as its remaining lengths are varied, and this process continues.

Now the virtual Euler characteristic can be defined directly in terms of fatgraphs. We consider the set of all fatgraphs of a given genus $g$ and a single puncture. Let us call the set $S$ and label each distinct graph by an integer $i \in S$. Let $\Gamma_i$ be the automorphism group of a fatgraph. Then, defining $d_i = E - 1$, we write

$$\chi_V = \sum_{i \in S} \frac{(-1)^{d_i}}{\#(\Gamma_i)}$$

which is analogous to Eq.(3.2) above, except that now the sum is over fatgraphs rather than over simplices. In particular, the automorphism group of the fatgraph is the same as the group that fixes the corresponding simplex. Very low-dimensional simplices do not contribute to $\chi_V$ since at some point the corresponding fatgraph has an automorphism group of infinite order, in which case the denominator of Eq.(3.13) becomes infinite and the corresponding term vanishes.

Let us check how the correspondence between fatgraphs and quadratic differentials works out in practice. The fatgraphs we have been considering have $V$ vertices, $E$ edges and 1 face. These integers satisfy:

$$V - E + 1 = 2 - 2g$$

where $g$ is the genus of the Riemann surface on which the graph is drawn.

We also have the relations

$$V = \sum_k v_k, \quad E = \frac{1}{2} \sum_k kv_k$$

where $v_k$ is the number of $k$-point vertices. From these relations, we get:

$$\sum_k (k - 2)v_k = 4g - 2$$

All integer solutions of this equation, i.e. all choices of the set $\{v_k\}$ for fixed $g$, are valid graphs that correspond to simplices in the triangulation of $\mathcal{M}_{g,1}$. 
Let us recast the above equation as

$$\sum_k (k - 2)v_k - 2 = 4g - 4 \quad (3.17)$$

Then, noticing that $k - 2$ is the order of the zero represented by a $k$-point vertex, we see that the first term on the left is the total number of zeroes (weighted with their multiplicity) of the quadratic differential corresponding to the given fatgraph. Moreover, the differential has precisely one double pole, so the second term is minus the (weighted) number of poles. Thus this result is in accord with a theorem which states that for meromorphic quadratic differentials on a Riemann surface of genus $g$,

$$\#(\text{zeroes}) - \#(\text{poles}) = 4g - 4 \quad (3.18)$$

A particular solution that is always available is $v_3 = V$, $v_k = 0$, $k \geq 4$. Clearly this gives the maximum possible number of vertices and therefore also edges. In this case,

$$V = 4g - 2 \quad (3.19)$$

and we see from Eq. (3.15) that the number of edges is

$$E = \frac{3}{2}V = 6g - 3 \quad (3.20)$$

Thus the dimension of the space spanned by varying the lengths of the graph keeping $s$ lengths fixed, is:

$$E - 1 = 6g - 4 \quad (3.21)$$

which is the real dimension of $\mathcal{M}_{g,1}$. Thus the graphs with only cubic vertices span a top-dimensional simplex in moduli space. All other graphs arise by collapse of one or more lines, merging two or more 3-point vertices to create higher $n$-point vertices. These correspond to simplices of lower dimension in the moduli space.

As an example, in genus 3 with one puncture, the moduli space has real dimension 14. Hence the graph that describes the “bulk” of moduli space should have 15 edges and 10 3-point vertices. Also, because there is a single puncture, the graph should have just one face. In summary, we have

$$V = 10, \quad E = 15, \quad s = 1 \quad \Rightarrow \quad V - E + s = -4 = 2 - 2g \quad (3.22)$$

as expected.

### 3.3 The Penner model

In 1986, Penner\[20\] constructed a matrix model that provides a generating functional for $\chi_{g,s}$. The Penner model is defined in terms of $N \times N$ random matrices whose
“fatgraphs” are precisely the ones described in the previous subsection. The free energy $\mathcal{F} = \log \mathcal{Z}$ of this model then has the expansion:

$$\mathcal{F} = \sum_g \mathcal{F}_g = \sum_{g,s} \chi_{g,s} N^{2-2g} t^{2-2g-s}$$

where $t$ is a parameter of the model. The term $s = 0$ is not present in the sum.

Let us first present the model and then show why it is correct. The model is given by an integral over Hermitian random matrices:

$$\mathcal{Z} = \mathcal{N}_P \int [dQ] e^{-Nt \text{tr} \sum_{k=2}^\infty \frac{1}{k} Q^k}$$

$$= \mathcal{N}_P \int [dQ] e^{Nt \text{tr} (\log(1-Q) + Q)}$$

where $\mathcal{N}_P$ is a normalisation factor given by:

$$\mathcal{N}_P^{-1} = \int [dQ] e^{-Nt \text{tr} \frac{1}{2} Q^2}$$

and the matrix measure $[dQ]$ is given by:

$$[dQ] \equiv \prod_i dQ_{ii} \prod_{i<j} dQ_{ij} dQ^*_{ij}$$

This action has all powers $k \geq 2$ of the random matrix appearing in it. The model is to be considered as a perturbation series around $Q \sim 0$.

To show that this model is correct, we must show that its fatgraphs are in one to one correspondence with those arising from quadratic differentials as discussed in the previous section. Thus we must show that the partition function involves a sum over all (connected and disconnected) fatgraphs for a fixed genus $g$ and number of faces $s$, multiplied by the weighting factor

$$\frac{(-1)^{E-s}}{\#(\Gamma_i)} N^{2-2g} t^{2-2g-s}$$

By taking the logarithm of the partition function we end up with the sum over connected graphs in the familiar way.

In the above expression $\Gamma_i$, the automorphism group, is the collection of maps of a given fatgraph to itself such that (i) the set of vertices is mapped onto itself, (ii) the set of edges is mapped to itself, and (iii) the cyclic ordering of each vertex is preserved. A key result due to Penner[20] is that the order of this group is equal to:

$$\#(\Gamma_i) = \frac{1}{V!} \prod_{\text{vertices}} \frac{1}{k^i} \times C$$

where $C$ is the combinatoric factor labelling how many distinct contractions lead to the same graph. Now this is exactly the factor that arises if we obtain our diagrams
as the expansion of the matrix integral Eq. (3.24), with the \( V! \) coming from the order of expansion of the exponent, the \( \frac{1}{k} \) factors being built into the action, and the combinatoric factor being the usual one.

The remaining factors are straightforward. Each vertex has a factor \(-Nt\), each edge is a propagator given by \((Nt)^{-1}\), and each face gives the usual index sum \(N\). This completes our informal derivation of the Penner model\(^7\).

### 3.4 Double-scaled Penner model

In principle, the Penner model has nothing to say about the moduli spaces \( \mathcal{M}_{g,0} \) of unpunctured Riemann surfaces. As we have noted, the expansion of the partition function does not contain terms with \( s = 0 \).

However, it was noticed by Distler and Vafa\(^{[21]}\) that in a suitable double-scaling limit, the model seems to describe unpunctured Riemann surfaces. Their observation goes as follows. Start with the equations:

\[
F = \sum_g F_g = \sum_{g,s} \chi_{g,s} N^{2-2g} t^{2-2g-s} \tag{3.30}
\]

\[
\chi_{g,s} = \frac{(-1)^s (2g - 3 + s)! (2g - 1)}{(2g)! s!} \quad B_{2g} = (-1)^s \binom{2g - 3 + s}{s} \chi_g \tag{3.31}
\]

where we have inserted the known value of \( \chi_{g,s} \).

Let us work with genus \( g > 1 \). It is straightforward to explicitly perform the sum over \( s \), and one gets:

\[
F_g = \frac{B_{2g}}{2g(2g - 2)} (Nt)^{2-2g} \left((1 + t)^{2-2g} - 1\right) \tag{3.32}
\]

Now Distler and Vafa took the limit \( N \to \infty \) and \( t \to t_c = -1 \), keeping fixed the product \( N(1 + t) = -\nu \). This leads immediately to the result:

\[
F_g = \frac{B_{2g}}{2g(2g - 2)} \nu^{2-2g} = \sum_g \chi_g \nu^{2-2g} \tag{3.33}
\]

and we have recovered the virtual Euler characteristic of \( \mathcal{M}_{g,0}! \)

Thus the Penner model, originally designed to study the moduli space of punctured Riemann surfaces, describes unpunctured ones too – but only in the special double-scaling limit above. We can remove some of the mystery of the double-scaling limit by defining, for any finite \( N \) and \( t \), the parameter

\[
\nu = -N - Nt \tag{3.34}
\]

in terms of which the matrix model can be written:

\[
Z = \mathcal{N}_P \int [dQ] e^{-(\nu + N) \text{tr} \left( \log(1 - Q) + Q \right)} \tag{3.35}
\]

\(^7\)Another derivation, as well as proposed generalisations, can be found in Ref.\(^{[54]}\).
Now the double-scaling limit is simply the ordinary large-$N$ limit, $N \to \infty$, with $\nu$ held fixed.

From the discussions of the previous sections, it should be evident that the double-scaled Penner model is closely related to $c = 1$, $R = 1$. Its free energy:

$$F_P(\nu) = \sum_g \frac{B_{2g}}{2g(2g-2)} \nu^{2-2g}$$  \hspace{1cm} (3.36)$$

is almost identical to the free energy of the $c = 1$, $R = 1$ string theory:

$$F_{c=1}(\mu) = \sum_g \frac{|B_{2g}|}{2g(2g-2)} \mu^{2-2g}$$  \hspace{1cm} (3.37)$$

However, there is an important issue of signs. It is well-known that the Bernoulli numbers alternate in sign:

$$|B_{2g}| = (-1)^{g-1} B_{2g}$$  \hspace{1cm} (3.38)$$

Therefore if we define $\nu = -i\mu$, we can write:

$$F_P(-i\mu) = \sum_{g=0}^{\infty} \frac{|B_{2g}|}{2g(2g-2)} \mu^{2-2g} = F_{c=1}(\mu)$$  \hspace{1cm} (3.39)$$

Thus the double-scaled Penner model (at imaginary cosmological constant) has the same free energy as the $c = 1$, $R = 1$ string. This intriguing correspondence calls for an explanation.

In the last section we will try to argue that the double-scaled Penner model could be related to the Euclidean $c = 1$, $R = 1$ theory, and could perhaps be derived from it, as the matrix integral for $N$ D-instantons of the latter theory. If true, this might both explain the above coincidence and also provide a role for D-instantons of the noncritical $c = 1$ string.

4. The $W_\infty$ Matrix Model

Let us now return to the solution of the $c = 1$, $R = 1$ matrix quantum mechanics. We will now focus on its tachyon amplitudes.

4.1 $c = 1$ amplitudes and $W_\infty$

As was already mentioned, for this theory a generating functional for all tachyon correlators to all genus has been obtained\cite{22}. This functional $F(t, \bar{t})$ depends on an infinite set of couplings $t_k, \bar{t}_k$ such that:

$$\langle T_{k_1} \ldots T_{k_n} T_{-l_1} \ldots T_{-l_m} \rangle = \frac{\partial}{\partial t_{k_1}} \ldots \frac{\partial}{\partial t_{k_n}} \frac{\partial}{\partial \bar{t}_{l_1}} \ldots \frac{\partial}{\partial \bar{t}_{l_m}} F(t, \bar{t}) \bigg|_{t=\bar{t}=0}$$  \hspace{1cm} (4.1)$$
where on the LHS we have connected amplitudes. We see that the couplings \(t_k\) are sources for tachyons of positive momentum \(k\), while the couplings \(\bar{t}_k\) are sources for tachyons of negative momentum \(-k\). These tachyons are “amputated” in the sense of Eq.(2.20).

Instead of constructing the generating functional directly in terms of \(t_k, \bar{t}_k\), it turns out necessary to encode the parameters \(t_k\) into a constant \(N \times N\) matrix \(A\). The \(t_k\) are defined in terms of \(A\) by the relation:

\[
t_k = \frac{1}{\nu k} \text{tr} A^{-k}
\]

which is sometimes called the Kontsevich-Miwa transform. As \(N \to \infty\), this matrix can encode infinitely many independent parameters \(t_k\).

Now using matrix quantum mechanics at \(R = 1\), it was shown\[22\] that \(Z(t, \bar{t}) = e^{F(t, \bar{t})}\) is a \(\tau\)-function of the Toda hierarchy, satisfying an infinite set of constraints that form a \(W_\infty\) algebra. These constraints were subsequently rewritten in the following form\[24\], which is the version that we will need for the subsequent discussion:

\[
\frac{1}{(-\nu)} \frac{\partial Z}{\partial \bar{t}_n} = \frac{1}{(-\nu)^n} (\det A)^\nu \text{tr} \left( \frac{\partial}{\partial A} \right)^n (\det A)^{-\nu} Z(t, \bar{t})
\]

Here \(\nu = -i\mu\), and \(\mu\) is the cosmological constant. The correlators determined by the above expression are actually invariant under \(k_i \to -k_i\), though this is far from manifest, since \(t_n\) and \(\bar{t}_n\) do not appear symmetrically.

The \(W_\infty\) constraints can be integrated to give a matrix model\[24\] as follows:\footnote{The same constraints were integrated in Ref.\[22\] but owing to some technical errors, as explained in Ref.\[24\], the resulting matrix model obtained in Ref.\[22\] was not correct.}

Let us start by assuming that \(Z(t, \bar{t})\) is an integral over constant matrices \(M\):

\[
Z(t, \bar{t}) = (\det A)^\nu \int [dM] e^{\text{tr} V(M, A, \bar{t})}
\]

where \([dM] = \prod_i dM_{ii} \prod_{i<j} dM_{ij} dM_{ij}^\ast\). The potential \(V\) is determined by imposing the above differential equation:

\[
\left[ \frac{1}{(-\nu)} \frac{\partial}{\partial \bar{t}_n} - \frac{1}{(-\nu)^n} \text{tr} \left( \frac{\partial}{\partial A} \right)^n \right] \int [dM] e^{\text{tr} V(M, A, \bar{t})} = 0
\]

This determines:

\[
V(M, t, \bar{t}) = -\nu \left( MA + \sum_{k=1}^{\infty} \bar{t}_k M^k \right) + f(M)
\]

where \(f(M)\) is a function independent of \(A, \bar{t}\).

The function \(f(M)\) can be determined using a boundary condition, arising from conservation of the tachyon momentum. Momentum conservation tells us that if we
set all the \( t_k \) equal to zero, then \( Z(A, 0) \) has to be independent of \( A \). From Eqns. (4.4) and (4.9), we get:

\[
Z(t, 0) = (\det A)^\nu \int [dM] e^{-\nu \text{tr} MA + \text{tr} f(M)}
\]

Upon changing variables \( M \to MA^{-1} \), we see that:

\[
[dM] \to (\det A)^{-N}[dM]
\]

It follows that:

\[
Z(t, 0) = (\det A)^{\nu - N} \int [dM] e^{-\nu \text{tr} M + \text{tr} f(MA^{-1})}
\]

\[
= \int [dM] e^{-\nu \text{tr} M + \text{tr} f(MA^{-1}) + (\nu - N) \text{tr} \log A}
\]

This uniquely determines the function \( f(M) \) to be:

\[
f(M) = (\nu - N) \log M
\]

Putting everything together, we see that the generating function of all tachyon amplitudes in the \( c = 1, R = 1 \) string theory is:

\[
Z(t, \bar{t}) = (\det A)^\nu \int [dM] e^{\text{tr} (-\nu MA + (\nu - N) \log M - \nu \sum_{k=1}^{\infty} \bar{t}_k M^k)}
\]

\[
= \int [dM] e^{\text{tr} (-\nu M + (\nu - N) \log M - \nu \sum_{k=1}^{\infty} \bar{t}_k (MA^{-1})^k)}
\]

This is the \( W_\infty \) matrix model[24]. The second form was obtained by redefining \( M \to MA^{-1} \). We also recall that

\[
t_k = \frac{1}{\nu k} \text{tr} A^{-k}
\]

Note that for the matrix integral to be well-defined, the matrix \( M \) (more precisely, its eigenvalues) must be positive semi-definite.

The \( W_\infty \) model has a number of interesting properties. One of these is the puncture equation, which says that the shift

\[
A \to A - \epsilon I, \quad \bar{t}_1 \to \bar{t}_1 + \epsilon
\]

changes the free energy \( F \) by a simple additive factor:

\[
F(A - \epsilon I, \bar{t}_1 + \delta_{k,1} \epsilon) = F(A, \bar{t}) - \nu^2 \sum_{k=1}^{\infty} \epsilon^k t_k
\]

This is easiest to see when the model is written in the form Eq.(4.12). We see that there are two linear terms in \( M \), one multiplying \( A \) and the other multiplying \( \bar{t}_1 \).
Their variations under the above transformation cancel out, leaving only the effect of varying the determinant factor in front of the integral. Taking logarithms gives the above behaviour of the free energy.

Another property of the $W_\infty$ model is that it is invariant under the simultaneous rescaling $t_k \rightarrow \lambda^{-k} t_k$ and $\bar{t}_k \rightarrow \lambda^k \bar{t}_k$, for any (in principle, complex) parameter $\lambda$. This is just momentum conservation. To see this invariance, note first that by Eq.(4.2) the rescaling $t_k \rightarrow \lambda^{-k} t_k$ is the same as $A \rightarrow \lambda A$. Now the expression Eq.(4.13) is manifestly invariant under $A \rightarrow \lambda A, \bar{t}_k \rightarrow \lambda^k \bar{t}_k$.

There have been other matrix models in the literature which have a form similar to that of the $W_\infty$ model\textsuperscript{9}. Some examples are the Chekhov-Makeenko model\textsuperscript{50}, the NBI matrix model\textsuperscript{57, 58} and the normal matrix model\textsuperscript{38}. The first of these is similar in structure to our $W_\infty$ matrix model but the coefficients are different. The second one is proposed as a modification of the IKKT matrix model by coupling it to a new matrix with a log plus linear interaction. The third example, the normal matrix model, will be discussed in Sec. 5.3.

4.2 Relation to the Penner model

The matrix integral of the $W_\infty$ model is convergent for real positive $\nu$. Indeed, it is the matrix analogue of the $\Gamma$-function. But we are presently interested only in its perturbative expansion in inverse powers of $\nu^2$. If we send $\nu \rightarrow -\nu$, we get the same perturbation series, even though the integral is formally no longer convergent and has to be defined by analytic continuation (like the $\Gamma$-function itself).

It turns out that this transformation is necessary in order to recover the Penner model from the $W_\infty$ model. Carrying out the change $\nu \rightarrow -\nu$, the resulting matrix integral is:

$$Z(t,0) = \int [dM] e^{\text{tr} (\nu M - (\nu + N) \log M)}$$

(4.17)

Now make the change of variables:

$$M = \alpha (1 - Q)$$

(4.18)

where $\alpha$ is a parameter to be determined.

Thus the $W_\infty$ matrix model is transformed into the following matrix integral:

$$Z = \mathcal{N} \int [dQ] e^{-\left(\nu + N\right) \text{tr} \log(1 - Q) - \nu \alpha Q}$$

(4.19)

where $\mathcal{N}$ is a normalisation constant.

Choosing $\alpha = 1 + N/\nu$, we get the standard form of the Penner model:

$$Z = \mathcal{N} \int [dQ] e^{Nt} \text{tr} \left( \log(1 - Q) + Q \right)$$

(4.20)

\textsuperscript{9}An early attempt to incorporate correlators in the Penner model is Ref.\textsuperscript{53}. 

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\textsuperscript{9}An early attempt to incorporate correlators in the Penner model is Ref.\textsuperscript{53}. 

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where \( t = -1 - \nu/N \).

Notice that in the “double-scaled” limit of the Penner model, we have \( N \to \infty \) with \( \nu \) finite, so the parameter \( \alpha \) goes to infinity. As a result, the Penner model perturbation series, around \( Q \sim 0 \), actually corresponds to the \( W_\infty \) model in the region of large \( M \).

A word about normalisation is called for. The \( W_\infty \) model determines correlators of tachyons of nonzero integer momentum (positive or negative). If insertions of \( T_0 \) are also required, they can be obtained by differentiating in \( \nu \). However, in the derivation leading up to Eq.(4.13), the overall \( \nu \)-dependent normalisation was not fixed, as this does not affect correlators for which at least one tachyon has nonzero momentum (by momentum conservation, this means that at least two tachyons have nonzero momentum).

This is why Eq.(4.20) has a normalisation constant in front of it. We could of course have at the outset chosen the normalisation in Eq.(4.13) to reproduce the correct normalisation factor of the Penner model given in Eq.(3.26). But amusingly, this is not really necessary, for the following reason. We can extract correlators of zero-momentum tachyons from the \( W_\infty \) model by taking a formal limit \( k \to 0 \) in the corresponding combinatorial formulae. That this gives the right answer is demonstrated in Ref.[29].

### 4.3 Relation to the Kontsevich model

We have not so far discussed the original Kontsevich matrix model[59], but it is appropriate to make a brief mention of it here. Like Penner’s model, this model too was formulated to solve a combinatoric problem, this time concerning stable cohomology classes on moduli space. Via Witten’s results relating these cohomology classes to pure topological gravity and \( c < 1 \) strings[15], this matrix model actually describes \( c < 1 \) (rather than \( c = 1 \)) noncritical strings.

The Kontsevich model is described by random matrices with the following matrix integral:

\[
Z_K(t) = \int [dX] e^{-\text{tr} \frac{1}{2} \Lambda X^2 + i \text{tr} X^3/6} \tag{4.21}
\]

The \( t_i \) on which the partition function depends are an infinite set of parameters defined by:

\[
t_k = -(2k - 1)!! \text{ tr } \Lambda^{-2k-1} \tag{4.22}
\]

Differentiating the free energy \( F_K = \log Z_K \) with respect to the \( t_i \) gives rise to an infinite set of correlation functions of operators that are analogous to the “amputated discrete tachyons” of previous sections. In turn, these correlators are the amplitudes of a class of \( c < 1 \) string theories based on \((2, q)\) minimal models, the ones originally studied as double-scaled matrix models[60, 61, 62].
It was noticed by Kontsevich that his model is related to the matrix Airy function:

$$A(Y) = \int [d\hat{X}] e^{i \text{tr} \left( \hat{X}^3/3 - \hat{X}Y \right)}$$  \hspace{1cm} (4.23)$$

Indeed, the change of variables:

$$\hat{X} = 2^{-1/3} X + Y^{1/2}, \quad Y = -2^{-2/3} \Lambda^2$$  \hspace{1cm} (4.24)$$

brings Eq.(4.23) into the form of Eq.(4.21), with, however, a significant prefactor depending on $\Lambda$. Thus the perturbation series of one can be expressed in terms of the perturbation series of the other.

It was also noted in Ref.[59] that the matrix Airy function Eq.(4.23) is to be viewed as an asymptotic expansion at large $Y$, for which the saddle point at $\hat{X} \sim Y^{1/2}$ dominates and therefore the matrix $\hat{X}$ is also large. On the other hand, the corresponding perturbation series for the model of Eq.(4.21) is around $X \sim 0$. This is strikingly similar to the relation between the $W_\infty$ model of Sec. 4 and the Penner model of Sec. 3.3. As noted in Sec. 4.2, the former model expanded about large values of the matrix $M$ gets mapped to the latter model expanded about $Q \sim 0$.

We can actually obtain the matrix Airy function directly as a special case of the $W_\infty$ model of Eq.(4.13). Starting with Eq.(4.13) at finite $N$, set $\nu = N$ and $\bar{t}_k = \delta_{k,3}$. The result is:

$$Z(t) = (\det A)^N \int dM e^{-N \text{tr} MA - N \text{tr} M^3}$$  \hspace{1cm} (4.25)$$

After a further rescaling of both $M$ and $A$, this is just proportional to the integral Eq.(4.23). Thus the matrix Airy function arises from the $W_\infty$ model by “condensing” a specific negative-momentum tachyon, $t_3$.

Generalisations of the Kontsevich model to describe noncritical strings based on $(p, q)$ minimal models do exist[63]. They are related to generalisations of the matrix Airy integral where the cubic term is replaced by a higher power. Clearly these too can be obtained from the $W_\infty$ model by condensing, or turning on, $\bar{t}_k$ for $k > 3$.

For string theory amplitudes, we are interested not just in the matrix integral but also the $A$-dependent or $\Lambda$-dependent normalisations. So these observations do not show that $c < 1$ strings are obtained from the $c = 1$ background by “condensing” discrete tachyons, but they do seem to suggest this. Specifically, turning on $\bar{t}_p$ or condensing $T_{-p}$ is related to the family of $(p - 1, q)$ minimal models coupled to gravity, for fixed $p$ and all coprime $q$. More work is needed to precisely establish this relationship.

5. Liouville Matrix Model and D-Instantons

In this section we will rewrite the $W_\infty$ model in a form that involves a Liouville-like potential. This form will be suggestive of a D-instanton interpretation.
5.1 Unperturbed $W_{\infty}$ and the Liouville Matrix Model

Let us start with the model as written in Eq. (4.17), where all the $\bar{t}_k$ have been set to zero. Notice that the coefficients in the action have a very specific dependence. The linear term is multiplied by a constant $\nu$ that remains fixed in the large-$N$ limit, while the log term has a coefficient $\nu - N$. Also, as we have seen, the matrix $M$ has to be positive semi-definite. Both these features look slightly unnatural in a matrix model, and suggest a redefinition of variables.

Define a new $N \times N$ matrix $\Phi$ by:

$$M = e^\Phi$$  \hspace{1cm} (5.1)

Under this transformation, the matrix measure transforms as:

$$[dM] (\det M)^{-N} = [d\Phi]$$  \hspace{1cm} (5.2)

and the integral in Eq.(4.17) becomes:

$$Z = \int [d\Phi] e^{-S} = \int [d\Phi] e^{-\nu \tr(e^\Phi - \Phi)}$$  \hspace{1cm} (5.3)

where $[d\Phi]$ is an (analytically continued) matrix measure appropriate for unitary matrices:

$$[d\Phi] \equiv \prod_i d\phi_i \prod_{i<j} \left( \sinh \frac{1}{2}(\phi_i - \phi_j) \right)^2$$  \hspace{1cm} (5.4)

Since $M$ is Hermitian and positive semi-definite, it follows that $\Phi$ is Hermitian with no further restrictions. Thus we no longer have to deal with a positivity constraint. More remarkably, we have also got rid of the explicit $N$ in the exponent. The matrix model in the $\Phi$ variable is simply a matrix integral with a potential that has a Liouville type exponential term, as well as a linear term. We will refer to it as the Liouville matrix model.

5.2 Interpretation of the Model

How do we interpret this model? The potential looks very much like that on the worldsheet of a fundamental string in the $c = 1$ background. If we think of $\Phi$ as the Liouville field, the two terms are reminiscent of the Liouville potential (cosmological operator) and the linear dilaton potential respectively.

However, there is a different and more plausible interpretation. The coefficient of the entire matrix action is $\nu$, the cosmological constant. In noncritical string theories this corresponds to the inverse string coupling: $\nu = 1/g_s$. Thus the action of the Liouville matrix model has a factor $1/g_s$ in front of it. This is highly suggestive of a D-brane action. Since the matrices are constant, the D-brane in question should be a D-instanton.
In critical superstring theory, a single D-instanton has essentially no action corresponding to it. But $N$ D-instantons do have an action, consisting of all the commutator terms that arise in the nonabelian Yang-Mills action:

$$S = \frac{1}{g_s} \text{tr} [A_\mu, A_\nu]^2 + \cdots$$  \hspace{1cm} (5.5)

Fermionic terms and higher-order commutators are also present. All these commutator terms arise because the scalars transverse to the instanton, and their fermionic partners, are promoted to matrices when there are $N$ instantons. Systems of D-instantons for large $N$ are associated to the IKKT matrix model of type IIB string theory.

Suppose we had $N$ D-instantons in $c = 1$ string theory. What action should we propose for them? One could try to address this by constructing a boundary state and computing open-string amplitudes. This approach would work only at weak coupling, that is far out near the region of large negative Liouville field $\phi$. In this region, the D-instanton will feel two forces, one due to the cosmological term, i.e. the Liouville potential, which is an excitation of the closed-string tachyon, and the other due to the linear dilaton. The former would drive the brane towards weak coupling, while the latter drives it to strong coupling. It is at least plausible that the two competing forces have an equilibrium position somewhere in the region of $\phi \sim 0$. However, it would be hard to gather reliable information in that region, since the theory is strongly coupled there.

Hence we leave this question for future work. What is quite striking, though, is that without making any assumptions at all, we have obtained a Liouville matrix model which precisely consists of a competing exponential and linear potential. This might be considered as evidence for a D-instanton interpretation of the $W_\infty$ matrix model.

This interpretation also adds something to the discussion in the previous section on the relationship of the $W_\infty$ to the Penner model. Recall that the transformation from the $W_\infty$ to the Penner matrix, Eq.(4.18) involves a large parameter $\alpha = 1 + N/\nu$, such that at large $N$, small values of the Penner matrix $Q \sim 0$ are mapped to large values of $M$. Now we see that in the Liouville variable this is the region of large $\Phi$, or the strong coupling region. This could be related to the fact that in the D0-brane description of $c = 1$ strings via the ZZ boundary state, the branes are thought of as being localised at large $\phi$.

Some ingredients are clearly missing from the story. We did not find a reason why the cosmological constant in this theory should be imaginary. Also, a D-instanton action should depend on two additional variables: the transverse scalar $X$ describing the $c = 1$ direction, and the open-string tachyon $T_{\text{open}}$. We will see that something

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10I am grateful to Ashoke Sen for discussions on this point.
like $X$ does appear when we consider tachyon perturbations, but we do not seem to find a role for $T_{\text{open}}$, unless it is somehow “mixed up” with $\Phi$.

### 5.3 Tachyon Perturbations

We proposed that the matrix model on $N$ D-instantons is equivalent to the $W_\infty$ model without tachyon perturbations (i.e. at $t, \bar{t} = 0$). What happens when we turn on tachyon perturbations?

If the proposal is correct, the D-instanton action should match the full $W_\infty$ action as a function of $t, \bar{t}$:

$$Z(t, \bar{t}) = \int [dM] e^{\text{tr} \left(-\nu M + (\nu - N) \log M - \nu \sum_{k=1}^\infty \bar{t}_k (MA^{-1})^k\right)}$$

where

$$t_k = \frac{1}{\nu k} \text{tr} A^{-k}$$

Let us focus on the matrix $A$. If it is Hermitian then (for real $\nu$) the $t_k$ are real. But we could equally well choose $A$ to be unitary, in which case the $t_k$ are complex, since we only need to differentiate $F(t, \bar{t})$ near $t, \bar{t} = 0$.

Let us therefore choose $A$ to be unitary, and parametrise it as:

$$A = e^{iX}$$

where $X$ is Hermitian.

Now, in the Liouville-like variable $\Phi$, the $W_\infty$ matrix model with tachyon perturbations can be written:

$$Z = \int [d\Phi] e^{-S} = \int [d\Phi] e^{-\nu \text{tr} \left(e^{\Phi - \Phi + \sum_{k=1}^\infty \bar{t}_k (e^{\Phi - iX})^k}\right)}$$

with

$$t_k = \frac{1}{\nu k} \text{tr} (e^{-ikX})$$

To understand this better, let us consider the simplest case of $N = 1$, i.e. $1 \times 1$ matrices. In this case, the perturbing term becomes:

$$\sum_{k=1}^\infty \bar{t}_k (e^{\Phi - iX})^k \rightarrow \sum_{k=1}^\infty \bar{t}_k e^{k(\phi - ix)}$$

The RHS looks like half of the mode expansion of a 2D Euclidean scalar field. This suggests that we identify $x$ with the (compact) Euclidean time direction. Indeed, the periodicity is correct, since $\exp(ikx)$ is single-valued at $R = 1$ (i.e. $x \rightarrow x + 2\pi$) precisely for integer $k$. It is plausible that this is how a single D-instanton couples to closed-string tachyons of negative momentum.
For $N$ D-instantons we need to go back to the noncommuting matrices, i.e. make the replacement:

$$e^{k(\phi-ix)} \rightarrow \text{tr} \left( e^\Phi e^{-iX} \right)^k$$

(5.12)

This amounts to a specific prediction for the matrix ordering on $N$ D-instantons of $c = 1, R = 1$ string theory. When the exponentials on the RHS are expanded, we find an infinite sequence of commutator terms between $\Phi$ and $X$.

The most puzzling aspect of this framework is that $X$ is not a dynamical variable unlike $\Phi$ (i.e. a random matrix to be integrated over), but rather a fixed background. It seems to be determined self-consistently, like a condensate, in terms of the positive momentum tachyon couplings, via the Kontsevich-Miwa relation Eq.(5.10).

We can get an indication how this might come about, by going back to the simple case of $N = 1$ and making the following plausible ansatz for the tachyon couplings:

$$\Delta S = \sum_{k=1}^{\infty} \left( t_k e^{k(\phi+ix)} + \bar{t}_k e^{k(\phi-ix)} \right)$$

(5.13)

Next, assume that $x$ is determined consistently by making the above term stationary:

$$\frac{\partial}{\partial x} \Delta S = 0$$

(5.14)

which leads to:

$$t_k e^{k\phi} e^{ikx} = \bar{t}_k e^{k\phi} e^{-ikx}$$

(5.15)

or

$$\frac{t_k}{\bar{t}_k} = e^{-2ikx}$$

(5.16)

Now $x \rightarrow -x$ is a symmetry of the theory that interchanges $t_1$ with $\bar{t}_1$. The unique solution of the above equation that respects this symmetry is:

$$t_1 = e^{-ix}, \quad \bar{t}_1 = e^{ix}$$

(5.17)

The first of these equations is a miniature version of the Kontsevich-Miwa transform, which is the best we can expect since we chose $N = 1$.

Suppose we use the above solution for $t_1$, and think of it as determining $x$. If we also leave $\bar{t}_1$ arbitrary (this procedure does not explain why we should do that) then we find that:

$$\Delta S \sim \sum_{k=1}^{\infty} \bar{t}_k e^{k\phi} e^{-ikx}$$

(5.18)

and $t_1$ is determined by

$$t_1 = e^{-ix}$$

(5.19)

which is more or less the right story for the tachyon couplings of the $W_\infty$ model at $N = 1$. A more complete analysis, particularly at general $N$, would hopefully shed some light on the non-dynamical nature of $X$ in this model.
Interestingly there is another matrix model in the literature, the normal matrix model\[38\], which is very similar to the Penner model but treats $t, \bar{t}$ in a manifestly symmetric way. It is defined in terms of a complex matrix $Z$ that is constrained to be normal:

$$[Z, Z^\dagger] = 0$$ \hspace{1cm} (5.20)

and the matrix integral is:

$$Z(t, \bar{t})_{NMM} = \int [dZ \, dZ^\dagger] e^{\text{tr} \left( -\nu ZZ^\dagger + (\nu - M) \log ZZ^\dagger - \nu \sum_{k=1}^{\infty} (t_k Z^k + \bar{t}_k Z^\dagger)^k \right)}$$ \hspace{1cm} (5.21)

(A variant of this model exists for any finite radius $R$). The relationship of this model to the $W_\infty$ model deserves to be examined in detail, but we will not be able to do this here. Let us just observe that if we parametrise:

$$Z = e^\Phi e^{iX}$$ \hspace{1cm} (5.22)

then the unperturbed part of the action depends only on $\phi$ and is similar to the unperturbed $W_\infty$ model.

The normality constraint amounts to:

$$[\Phi, X] = 0$$ \hspace{1cm} (5.23)

and the tachyon perturbation of this action is:

$$\Delta S = -\nu \sum_{k=1}^{\infty} (t_k e^{ik(\Phi+iX)} + \bar{t}_k e^{ik(\Phi-iX)})$$ \hspace{1cm} (5.24)

We see that in this model, both the $\Phi$ and $X$ directions are represented by dynamical matrices, and the tachyon perturbation is similar to the one proposed above in Eq. (5.13).

Indeed, the normal matrix model is supposed to be equivalent to the $W_\infty$ model at large $N$, as both are equivalent to $c = 1, R = 1$ by construction, though a precise equivalence in terms of matrices has not been demonstrated. In view of our conjecture that the action of the $W_\infty$ model is that on $N$ D-instantons with the transverse coordinate $X$ treated as non-dynamical, it seems natural to speculate that the NMM describes the same system but with dynamical $X$. It is plausible that both models become equivalent at large $N$, though superficially they do not appear to be equivalent at finite $N$.

6. Conclusions

We have seen that $c = 1, R = 1$ is a very special theory. It is completely solved to all orders in perturbation theory, and its free energy and correlators are known in a
genus expansion. Its partition function is related to the topology of moduli space, while the correlators are related to integrable hierarchies.

All the above information is elegantly encoded in the $W_\infty$ matrix model. Here we have shown that this model maps to a matrix model with a Liouville-plus-linear interaction. This is proposed to be the matrix theory on $N$ D-instantons of the $c = 1$ string theory. It is important to confirm whether this proposal is correct and/or needs modification.

Extensions of these ideas and relationships from $c = 1$ to $\hat{c} = 1$ (type 0A, 0B) strings will be interesting to pursue. In the supersymmetric case, there is naively no self-dual radius since the T-dual of type 0B theory is type 0A theory. However, there is a subtle “affine quotient” theory discovered in Refs. [64, 65] and studied in Ref. [5], which is T-dual to itself and therefore does have a self-dual radius. It is this theory to which the considerations of this article are most likely to generalise.

Traditionally, topological string theories are tied to the genus expansion by their very definition, in that the partition function and amplitudes are defined genus by genus. But if, as is currently believed, the type 0A and 0B string theories truly have a nonperturbative completion, then the corresponding topological formulations might have one as well. If the latter can be summarised as matrix models, it is reasonable to hope that these super-analogues of the Penner and $W\infty$ matrix models are not just generating functions of a genus expansion, but also contain physically meaningful nonperturbative terms. This could prove extremely important for the understanding of nonperturbative string theory.

The above observations could also have a bearing on the many topological theories related to $c = 1, R = 1$, such as the topological black hole and topological conifold.

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