Pairwise disjoint perfect matchings in \( r \)-edge-connected \( r \)-regular graphs

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Abstract

Thomassen [Problem 1 in Factorizing regular graphs, J. Combin. Theory Ser. B, 141 (2020), 343-351] asked whether every \( r \)-edge-connected \( r \)-regular graph of even order has \( r - 2 \) pairwise disjoint perfect matchings. We show that this is not the case if \( r \equiv 2 \mod 4 \). Together with a recent result of Mattiolo and Steffen [Highly edge-connected regular graphs without large factorizable subgraphs, J. Graph Theory, 99 (2022), 107-116] this solves Thomassen’s problem for all even \( r \).

It turns out that our methods are limited to the even case of Thomassen’s problem. We then prove some equivalences of statements on pairwise disjoint perfect matchings in highly edge-connected regular graphs, where the perfect matchings contain or avoid fixed sets of edges.

Based on these results we relate statements on pairwise disjoint perfect matchings of 5-edge-connected 5-regular graphs to well-known conjectures for cubic graphs, such as the Fan-Raspaud Conjecture, the Berge-Fulkerson Conjecture and the 5-Cycle Double Cover Conjecture.

Keywords: perfect matchings, regular graphs, factors, \( r \)-graphs, edge-colorings, class 2 graphs

1 Introduction and motivation

We consider finite graphs that may have parallel edges but no loops. A graph without parallel edges is called simple. Let \( r \geq 0 \) be an integer. A graph \( G \) is \( r \)-regular if every vertex has
degree \( r \), where the degree \( d_G(v) \) of a vertex \( v \) is the number of edges that are incident with \( v \). An \( r \)-graph is an \( r \)-regular graph \( G \), where every odd set \( X \subseteq V(G) \) is connected by at least \( r \) edges to its complement \( V(G) \setminus X \). For \( k \in \{1, \ldots, r\} \), a \( k \)-factor of \( G \) is a spanning \( k \)-regular subgraph of \( G \). The edge set of a 1-factor is called a perfect matching of \( G \). Moreover, for \( k \geq 2 \), a \( k \)-PDPM of an \( r \)-regular graph \( G \) is a set of \( k \) pairwise disjoint perfect matchings of \( G \).

An \( r \)-regular graph is class 1 if it has an \( r \)-PDPM. Otherwise, it is class 2. On one side, dense simple \( r \)-regular graphs are class 1 [4]. On the other side, for every \( r \geq 3 \) there are \((r - 2)\)-edge-connected \( r \)-regular graphs of even order without a perfect matching, see for example [1] p. 47 ff.

Every \( r \)-graph has a perfect matching [16] and we call an \( r \)-graph poorly matchable if any two perfect matchings intersect. Every class 2 3-graph is poorly matchable. A natural question is what is the maximum number \( t \) such that every \( r \)-graph has a \( t \)-PDPM? For \( r \)-graphs, the answer is given by Rizzi [15], who showed that for every \( r \geq 3 \), there are poorly matchable \( r \)-graphs. However, all graphs he constructed have a 4-edge-cut. It seems that the situation changes for highly edge-connected \( r \)-graphs.

Let \( m(r) \) be the maximum number \( t \) such that every \( r \)-edge-connected \( r \)-graph has \( t \) pairwise disjoint perfect matchings. This gives rise to the following problem.

**Problem 1.1.** Determine \( m(r) \) for all \( r \geq 2 \).

Clearly, \( m(r) \geq 1 \), \( m(2) = 2 \), and \( m(3) = 1 \). Furthermore, \( m(4) = 1 \) by the result of Rizzi [15]. Thomassen [17] conjectured that there exists a natural number \( r_0 \) such that \( m(r) \geq 2 \) for every \( r \geq r_0 \). In other words, he conjectured that for all \( r \geq r_0 \), there are no poorly matchable \( r \)-edge-connected \( r \)-graphs. Thus, \( r_0 \geq 5 \).

Class 1 graphs are of no interest for the study of \( m(r) \). They have \( r \) pairwise disjoint perfect matchings. For \( r \geq 3 \), class 2 \( r \)-graphs have at most an \((r - 2)\)-PDPM. Thus, \( m(r) \leq r - 2 \), if there is an \( r \)-edge-connected class 2 \( r \)-graph. Note that \( r \)-edge-connected class 2 \( r \)-graphs are known for all \( r \geq 3 \) and \( r \neq 5 \). The case \( r = 5 \) is briefly addressed at the end of this section and in Section 5.

Thomassen (Problem 1 of [17]) proposed the following question for the value of \( m(r) \).

**Problem 1.2** (Thomassen [17]). For all \( r \geq 3 \), is it true that \( m(r) = r - 2 \)?

**Problem 1.2** is solved for \( r \equiv 0 \mod 4 \) in [12]. It is shown that \( m(r) \leq r - 3 \) in this case. In Section 3 we prove the following statement.

**Theorem 1.3.** If \( r \equiv 2 \mod 4 \), then \( m(r) \leq r - 3 \).

Together with the results of [12, 15] we obtain the following corollary.

**Corollary 1.4.** If \( r > 2 \) is even, then \( m(r) \leq r - 3 \).
The graphs that prove Corollary 1.4 have a 2-vertex-cut. It is easy to see that for odd \( r \), an \( r \)-edge-connected \( r \)-graph is 3-vertex-connected. This shows that our methods are limited to the case when \( r \) is even.

Thus, the main motivation for Section 4 is the study of Problems 1.1 and 1.2 for odd \( r \). We prove that every \( r \)-edge-connected \( r \)-graph has \( k \in \{2, \ldots, r - 2\} \) pairwise disjoint perfect matchings if and only if every \( r \)-edge-connected \( r \)-graph has \( k \) pairwise disjoint perfect matchings that contain (or that avoid) a fixed edge. For odd \( r \), we prove the stronger statement that every \( r \)-edge-connected \( r \)-graph has an \( (r - 2) \)-PDPM if and only if for every \( r \)-edge-connected \( r \)-graph and every \( \left\lfloor \frac{r}{2} \right\rfloor \) adjacent edges, there is an \( (r - 2) \)-PDPM of \( G \) containing all \( \left\lfloor \frac{r}{2} \right\rfloor \) edges.

In Section 5 these results are used to prove tight connections between (possible) answers to Problem 1.1 for \( r = 5 \) and some well-known conjectures on cubic graphs. In particular, we prove the following statement.

**Theorem 1.5.** If \( m(5) \geq 2 \), then the Fan-Raspaud Conjecture holds. Moreover, if \( m(5) = 5 \), then both the 5-Cycle Double Cover Conjecture and the Berge-Fulkerson Conjecture hold.

The condition \( m(5) = 5 \) of the second statement of Theorem 1.5 seems to be surprising since it is equivalent to saying that every 5-edge-connected 5-regular graph is class 1. So far, we have not succeeded in constructing 5-edge-connected 5-regular class 2 graphs. Also, intensive literature research and computer-assisted searches in graph databases did not lead to the desired success. We conclude this introduction with the following problem, which surprisingly seems to be unsolved.

**Problem 1.6.** Is there any 5-edge-connected 5-regular class 2 graph?

For planar graphs, the answer to the above question is “no”. Guenin [8] proved that all planar 5-graphs are class 1. Indeed, it is conjectured by Seymour [16] that every planar \( r \)-graph is class 1. So far, this conjecture is proved to be true for all \( r \leq 8 \), see [3] also for further references.

## 2 Basic definitions and results

In this section we introduce some definitions and tools. For undefined notation and terminology the readers are referred to [2].

If \( u, v \) are two vertices of a graph \( G \), we denote by \( \mu_G(u, v) \) the number of parallel edges connecting \( u \) and \( v \). In case \( \mu_G(u, v) = 1 \) we also say that \( e = uv \) is a simple edge. The **underlying graph** of \( G \) is the simple graph \( H \) with \( V(H) = V(G) \) and \( \mu_H(u, v) = 1 \) if and only if \( \mu_G(u, v) \geq 1 \). For any two disjoint sets \( U, W \subseteq V(G) \), denote by \( [U, W]_G \) the set of edges with exactly one endpoint in each of \( U \) and \( W \). For convenience, we simply write \( [u, W]_G \) for
\( [U, W]_G \) if \( U = \{u\} \), and just write \( \partial_G(U) \) if \( W = V(G) \setminus U \). Moreover, \( \partial_G(U) \) is called an edge-cut of \( G \). The index \( G \) is sometimes omitted if there is no ambiguity. The graph induced by \( U \) is denoted by \( G[U] \). Similarly, we use \( G - U \) instead of \( G[V(G) \setminus U] \), and use \( G - u \) instead of \( G - \{u\} \) if \( U = \{u\} \). A \( k \)-circuit is a circuit of length \( k \) and a cycle is a union of pairwise edge-disjoint circuits. A set of \( k \) vertices, whose removal increases the number of components of a graph, is called a \( k \)-vertex-cut. A graph \( G \) on at least \( k + 1 \) vertices is called \( k \)-connected if \( G - X \) is connected for every \( X \subseteq V(G) \) with \( |X| \leq k - 1 \). Similarly, a graph \( G \) is called \( k \)-edge-connected if for every non-empty \( X \subset V(G) \), \( |\partial_G(X)| \geq k \). Moreover, a graph \( G \) is called cyclically \( k \)-edge-connected if \( G - S \) has at most one component containing a circuit for any edge-cut \( S \subseteq E(G) \) with \( |S| \leq k - 1 \).

A multiset \( \mathcal{M} \) consists of objects with possible repetitions. We denote by \( |\mathcal{M}| \) the number of objects in \( \mathcal{M} \). For a positive integer \( k \), we define \( k \mathcal{M} \) to be the multiset consisting of \( k \) copies of each element of \( \mathcal{M} \). Let \( G \) be a graph and \( N \) a multiset of edges of the complete graph on \( V(G) \). The graph \( G + N \) is obtained by adding a copy of all edges of \( N \) to \( G \). This operation might generate parallel edges. More precisely, if \( N \) contains exactly \( t \) edges connecting the vertices \( u \) and \( v \) of \( G \), then \( \mu_{G + N}(u, v) = \mu_G(u, v) + t \).

We will frequently use the following simple fact.

**Observation 2.1.** Let \( G \) be a graph with a perfect matching \( M \). For any subset \( X \subseteq V(G) \), if \( |X| \) is odd, then \( |\partial_G(X) \cap M| \) is odd.

Many results in the following sections rely on the properties of the Petersen graph, which will be denoted by \( P \) throughout this paper. We need to fix a drawing of \( P \) and consider its six perfect matchings. Let \( v_1 \ldots v_5 v_1 \) and \( u_1 u_3 u_5 u_2 u_4 u_1 \) be the two disjoint 5-circuits of \( P \) such that \( u_i v_i \in E(P) \) for each \( i \in \{1, \ldots, 5\} \). Set \( M_0 = \{u_i v_i : i \in \{1, \ldots, 5\}\} \). Clearly, \( M_0 \) is a perfect matching of the Petersen graph. For each \( i \in \{1, \ldots, 5\} \), let \( M_i \) be the only other perfect matching containing \( u_i v_i \), see Figure 1.

![Figure 1: The 6 perfect matchings of the Petersen graph.](image-url)
Let $P^M = P + \sum_{j=1}^{k} N_j$ be the graph obtained from $P$ and $\mathcal{M}$, where $\mathcal{M} = \{N_1, \ldots, N_k\}$ is a multiset of perfect matchings of $P$. For a perfect matching $N$ of $P^M$, if it is a copy of $M_j$ of $P$, then we say that $N$ is of type $j$ in $P^M$. Let $\mathcal{M}'$ be a multiset of perfect matchings of $P^M$. We denote by $\mathcal{M}'_P$ the multiset of perfect matchings of $\mathcal{M}'$ interpreted as perfect matchings of $P$. That is, for each $j \in \{0, \ldots, 5\}$, $\mathcal{M}'_P$ contains exactly $k$ copies of $M_j$ if and only if $\mathcal{M}'$ contains exactly $k$ perfect matchings of type $j$.

**Lemma 2.2 ([12]).** Let $\mathcal{M}$ be a multiset of $k$ perfect matchings of $P$. If $\mathcal{M}' = \{M'_1, \ldots, M'_{k+1}\}$ is a $(k+1)$-PDPM of $P^M$, then $\mathcal{M} \subseteq \mathcal{M}'_P$.

We also need the following lemma, whose proof is basically the same as that of Lemma 2.4 in [12] but its statement is more general. To keep the paper self-contained, the proof is presented.

**Lemma 2.3.** Let $\mathcal{M}$ be a multiset of $k$ perfect matchings of $P$. If for every $u, v \in V(P^M)$, $\mu_{P^M}(u, v) \leq \lfloor \frac{k+3}{2} \rfloor$, then $P^M$ is a $(k+3)$-edge-connected $(k+3)$-regular class 2 graph.

**Proof.** By construction, $P^M$ is $(k+3)$-regular. Moreover, it is class 2, see Theorem 3.1 of [7]. Let $X \subset V(P^M)$ be the subset with $|\partial_{P^M}(X)|$ minimum. This implies that $P^M[X]$ is connected. If $|X|$ is odd, then by Observation 2.1, every perfect matching of $\mathcal{M}$ intersects $\partial_{P^M}(X)$. Hence, $|\partial_{P^M}(X)| \geq 3 + k$. If $|X|$ is even, then it suffices to consider the cases $|X| \in \{2, 4\}$. Since $P$ has girth 5, the subgraph $P[X]$ is a path on either 2 or 4 vertices or it is isomorphic to $K_{1,3}$. Since $\mu_{P^M}(u, v) \leq \lfloor \frac{k+3}{2} \rfloor$ for every $u, v \in V(P^M)$ it follows that $\partial_{P^M}(X)$ contains at least $\lfloor \frac{k+3}{2} \rfloor$ edges for each vertex of degree 1 in $P[X]$. Consequently, $|\partial_{P^M}(X)| \geq k + 3$. \qed

Meredith [13] constructed $r$-edge-connected class 2 $r$-graphs for every $r \geq 8$. One can observe that $P + M_0 + M_1 + M_2$ and $P + M_0 + M_1 + M_2 + M_3$ are respectively a 6-edge-connected 6-regular graph and a 7-edge-connected 7-regular graph. Such graphs are class 2 by [7] (Theorem 3.1). However, we cannot easily construct a 5-edge-connected 5-regular graph in the same way. Indeed, adding two perfect matchings to $P$ generates a 4-edge-cut. It is also well known that for each $r \in \{3, 4\}$ there are $r$-edge-connected class 2 $r$-graphs. Surprisingly, it seems that so far no 5-edge-connected 5-regular class 2 graph is known, see Problem 1.6.

### 3 Proof of Theorem 1.3

In this section we construct a $(4k+2)$-edge-connected $(4k+2)$-graph $G_k$ without a 4k-PDPM for each integer $k \geq 1$. As in [12], we first construct a graph $P_k$ by adding perfect matchings to the Petersen graph and a graph $Q_k$ by using two copies of $P_k$. Then, we construct a graph $S_k$ and 'replace’ some edges of $S_k$ by copies of $Q_k$ to obtain the graph $G_k$ with the desired properties.
3.1 The graphs $P_k$ and $Q_k$

For each $k \geq 1$, let

\[ P_k = P + k(M_0 + M_1 + M_2) + (k - 1)M_5, \]

as shown in Figure 2.

![Figure 2: The graph $P_2$.](image)

Let $P^1_k$ and $P^2_k$ be two distinct copies of $P_k$. For each $w \in V(P_k)$, the vertex of $P^1_k$ ($P^2_k$, respectively) that corresponds to $w$ is denoted by $w^1$ ($w^2$, respectively). Now, we obtain the graph $Q_k$ from $P^1_k$ and $P^2_k$ by removing the $2k+1$ parallel edges connecting $u^1_i$ and $v^1_i$ from $P^i_k$, for each $i \in \{1, 2\}$, and identifying $u^1_1$ and $u^2_1$ to a new vertex, denoted by $u_{Q_k}$. Note that the degree of $u_{Q_k}$ in $Q_k$ is $4k+2$. For a graph $G$ containing $Q_k$ as a subgraph, let $E^i_k = [v^1_i, V(G) \setminus V(Q_k)]_G$ for each $i \in \{1, 2\}$. The subgraph $Q_2$ and the edge-sets $E^1_2$ and $E^2_2$ are shown in Figure 3.

![Figure 3: The subgraph $Q_2$ and the edge-sets $E^1_2$ and $E^2_2$.](image)

The following lemma is similar to Lemma 2.5 in [12] ($Q_k$ is different), and it can be proved analogously. In order to keep the paper self-contained, we present the proof here.

**Lemma 3.1.** Let $G$ be a graph that contains $Q_k$ as an induced subgraph. Let $\mathcal{N} = \{N_1, \ldots, N_{4k}\}$ be a set of pairwise disjoint perfect matchings of $G$ and let $N = \bigcup_{i=1}^{4k} N_i$. If $\partial(V(Q_k)) = E^1_k \cup E^2_k$,
\[|E|^1_k \cap N| = |E|^2_k \cap N| = 2k.\]

**Proof.** Every perfect matching of \(G\) intersects \(\partial(V(Q_k))\) precisely once since \(|V(Q_k)|\) is odd and \(\{v^1_1, v^2_1\}\) is a 2-vertex-cut. It remains to show that \(E^1_k\) intersects precisely \(2k\) elements of \(N\). Recall that \(Q_k\) is constructed by using two copies of \(P + \sum_{M \in M} M\), where

\[M = \{M_0, M_1, M_2, \ldots, M_0, M_1, M_2, M_3, \ldots, M_5\}.\]

We argue by contradiction. Without loss of generality, suppose that \(|E^1_k \cap N| < 2k\), which is equivalent to \(|E^2_k \cap N| > 2k\). Every perfect matching of \(N\) that intersects \(E^1_k\) also intersects the set \([u_{Q_k}, V(P^2_k)]\)_G, and vice versa. Consequently, the existence of \(N\) implies that there is a set \(N^\prime\) of \(4k\) pairwise disjoint perfect matchings in \(P^1_k\) such that \(M \not\subseteq N^\prime\), a contradiction to Lemma 2.2. Hence, \(|E^1_k \cap N| = |E^2_k \cap N| = 2k.\]

3.2 The graph \(S_k\)

For every \(k \geq 1\), let \(S_k\) be the graph with vertex set \(\{x_i, y_i, z_i, w : i \in \{1, \ldots, 4k + 2\}\}\) and edge set \(A_k \cup kB_k \cup (k + 1)C_k \cup (2k + 1)(D_k \cup E_k)\) where

\[A_k = \{wz_i : i \in \{1, \ldots, 4k + 2\}\},\]
\[B_k = \{z_ix_i, z_iy_i : i \in \{1, \ldots, 4k + 2\}\},\]
\[C_k = \{x_iy_i : i \in \{1, \ldots, 4k + 2\}\},\]
\[D_k = \{y_ix_{i+1} : i \in \{1, \ldots, 4k + 2\}\},\]
\[E_k = \{z_iz_{i+2k+1} : i \in \{1, \ldots, 2k + 1\}\},\]

and the indices are added modulo \(4k + 2\), see Figure 4.

**Lemma 3.2.** For all \(k \geq 1\), \(S_k\) is \((4k + 2)\)-edge-connected and \((4k + 2)\)-regular.

**Proof.** By definition, \(S_k\) is \((4k + 2)\)-regular. Let \(X \subset V(G)\) be a non-empty set. First, we consider the case that there are two vertices \(u, v \in \{x_i, y_i : i \in \{1, \ldots, 4k + 2\}\}\) such that \(X\) contains exactly one of them. Clearly, there are \(4k + 2\) pairwise edge-disjoint \(uv\)-paths in \(S_k\), which only contain edges of \(kB_k \cup (k + 1)C_k \cup (2k + 1)D_k\). Hence, \(|\partial_{S_k}(X)| \geq 4k + 2\). Therefore, without loss of generality we may assume \(\{x_i, y_i : i \in \{1, \ldots, 4k + 2\}\} \cap X = \emptyset\). Since \(S_k\) is \((4k + 2)\)-regular and \(\mu_{S_k}(u, v) \leq 2k + 1\) for every \(u, v \in V(S_k)\), we have \(|X| \notin \{1, 2\}\). Hence, \(X\) either contains at least three vertices of \(\{z_i : i \in \{1, \ldots, 4k + 2\}\}\) or \(w\) and exactly two vertices of \(\{z_i : i \in \{1, \ldots, 4k + 2\}\}\). In the first case, \(\partial(X)\) contains at least \(6k\) edges of \(kB_k\). In the second case, \(\partial(X)\) contains \(4k\) edges of \(A_k\) and at least \(4k\) edges of \(kB_k\), which completes the proof. \(\square\)
3.3 The graph $G_k$

For every $k \geq 1$, let $G_k$ be the graph obtained from $S_k$ as follows. First, remove all edges of $(2k + 1)(D_k \cup E_k)$. Then, for every edge $e = uv \in D_k \cup E_k$, add a copy $Q^e_k$ of $Q_k$, connect $u$ with the vertex corresponding to $v_1^1$ by $2k + 1$ new parallel edges and connect $v$ with the vertex corresponding to $v_1^2$ by $2k + 1$ new parallel edges, see Figure 4.

In order to prove that $G_k$ has the desired properties, we need the following two observations.

**Observation 3.3.** Let $G$ be a graph and let $u, v \in V(G)$ with $\mu_G(u, v) = t$. Let $H$ be the graph obtained from $G$ by identifying $u$ and $v$ to a (new) vertex $w$ and removing all resulting loops. If $G$ is $2t$-edge-connected and $2t$-regular, then $H$ is $2t$-edge-connected and $2t$-regular.

**Proof.** Assume that $G$ is $2t$-edge-connected and $2t$-regular. Since $\mu_G(u, v) = t$, it follows that $\partial_G(\{u, v\}) = 2t$ and hence, $H$ is $2t$-regular. Let $X \subset V(H)$ be a non-empty set. If $w \in X$, then $|\partial_H(X)| = |\partial_G(X \setminus \{w\} \cup \{u, v\})| \geq 2t$. If $w \notin X$, then $|\partial_H(X)| = |\partial_G(X)| \geq 2t$. \hfill $\square$

**Observation 3.4.** Let $G$ and $G'$ be two graphs and let $u, v \in V(G)$ and $u', v' \in V(G')$ such that $\mu_G(u, v) = \mu_{G'}(u', v') = t$. Let $H$ be the graph obtained from $G$ and $G'$ as follows. Remove the $t$ parallel edges between $u$ and $v$ and the $t$ parallel edges between $u'$ and $v'$. Add $t$ parallel edges between $u$ and $u'$ and $t$ parallel edges between $v$ and $v'$. If $G$ and $G'$ are $2t$-edge-connected and $2t$-regular, then $H$ is $2t$-edge-connected and $2t$-regular.

**Proof.** Clearly, $H$ is $2t$-regular. Note that $G - [u, v]_G$ and $G' - [u', v']_{G'}$ are $t$-edge-connected. Let $X \subset V(H)$ be a non-empty set.
Figure 5: The graph $G_1$, where the boxes are copies of $Q_1$.

**Case 1.** $X \cap V(G) = V(G)$ or $X \cap V(G') = V(G')$.

Say $V(G) \subseteq X$, then $\partial_H(X)$ contains either (i) $[u, u']_H$ and $[v, v']_H$ or (ii) one of $[u, u']_H$ and $[v, v']_H$ and a $t_1$-edge-cut of $G' - [u', v']_{G'}$ with $t_1 \geq t$ or (iii) a $t_2$-edge-cut of $G' - [u', v']_{G'}$ with $t_2 \geq 2t$.

**Case 2.** $X \cap V(G) \neq V(G)$ and $X \cap V(G') \neq V(G')$.

If $X \cap V(G) \neq \emptyset$ and $X \cap V(G') = \emptyset$, then $|\partial_H(X)| = |\partial_G(X \cap V(G))| - t + |\partial_{G'}(X \cap V(G'))| - t \geq 2t$. If $X \cap V(G') = \emptyset$, then $|\partial_H(X)| \geq |\partial_G(X)| \geq 2t$. □

**Theorem 3.5.** For all $k \geq 1$, $G_k$ is a $(4k+2)$-edge-connected $(4k+2)$-graph without $4k$ pairwise disjoint perfect matchings.

**Proof.** By Lemma 2.3, $P_k$ is $(4k+2)$-edge-connected and $(4k+2)$-regular. Hence, by Observations 3.3 and 3.4, the graph $Q_k + (2k+1)\{v_1^1, v_1^2\}$ is $(4k+2)$-edge-connected and $(4k+2)$-regular. Thus, $G_k$ is $(4k+2)$-edge-connected and $(4k+2)$-regular by Observation 3.4 again. Furthermore, the order of $G_k$ is $|V(S_k)| + 19|D_k \cup E_k|$, which is even. Suppose to the contrary that $G_k$ has $4k$ pairwise disjoint perfect matchings. Let $N \subseteq E(G_k)$ be the union of them and let $wz_i \in N$. Lemma 3.1 implies $|N \cap \partial_{G_k}(\{x_i, y_i, z_i\})| = 6k + 1$. On the other hand, every perfect matching contains an odd number of edges of $\partial_{G_k}(\{x_i, y_i, z_i\})$ by Observation 2.1. Therefore, $|N \cap \partial_{G_k}(\{x_i, y_i, z_i\})|$ is even, a contradiction. □

Theorem 3.5 implies that $m(r) \leq r - 3$ if $r \equiv 2 \mod 4$. Thus, Theorem 1.3 and Corollary 1.4 are proved.
4 Equivalences for statements on the existence of a $k$-PDPM

The graph $G_k$ from the previous section has many 2-vertex-cuts. The following observation shows that such a construction will not apply for the odd case of Problem 1.2.

**Observation 4.1.** For odd $r \geq 3$, every $r$-edge-connected $r$-graph is 3-connected.

**Proof.** Let $G$ be an $r$-edge-connected $r$-graph. Clearly, $G$ is of even order and 2-connected. Suppose that there are two vertices $v_1, v_2$ such that $G - \{v_1, v_2\}$ is not connected. Then $G - \{v_1, v_2\}$ has exactly two components $A$ and $B$. Since the order of $G$ is even, $A$ and $B$ are either both of even order or both of odd order. In the first case, $|\partial(V(A))| + |\partial(V(B))| \leq |\partial(v_1)| + |\partial(v_2)| = 2r$. Since $A$ and $B$ are of even order, $|\partial(V(A))|$ and $|\partial(V(B))|$ are both even. Hence, it follows that either $|\partial(V(A))| < r$ or $|\partial(V(B))| < r$ since $r$ is odd. In the second case, $|\partial(V(A) \cup \{v_1\})| + |\partial(V(B) \cup \{v_1\})| = |\partial(v_1)| + |\partial(v_2)| = 2r$. Thus, $|\partial(V(A) \cup \{v_1\})| < r$ or $|\partial(V(B) \cup \{v_1\})| < r$ since $A$ and $B$ are of odd order. Therefore, both cases lead to a contradiction with the assumption that $G$ is $r$-edge-connected. 

We are going to prove some equivalent statements about the existence of a $k$-PDPM in $r$-edge-connected $r$-graphs.

**Definition 4.2.** Let $G$ and $H$ be two disjoint $r$-regular graphs with $u \in V(G)$ and $v \in V(H)$. Let $(G, u)(H, v)$ be the set of all graphs obtained by replacing the vertex $u$ of $G$ by $(H, v)$, that is, deleting $u$ from $G$ and $v$ from $H$, and then adding $r$ edges between $N_G(u)$ and $N_H(v)$ such that the resulting graph is regular.

**Lemma 4.3.** If $G$ and $H$ are two disjoint $r$-edge-connected $r$-regular graphs with $u \in V(G)$ and $v \in V(H)$, then every graph in $(G, u)(H, v)$ is $r$-regular and $r$-edge-connected.

**Proof.** Suppose to the contrary that there exists a graph $G' \in (G, u)(H, v)$ with a set $X \subseteq V(G')$ such that $|\partial_{G'}(X)| \leq r - 1$. If $X \subseteq V(G - u)$ or $X \subseteq V(H - v)$, then $|\partial_G(X)| = |\partial_{G'}(X)| \leq r - 1$ or $|\partial_H(X)| = |\partial_{G'}(X)| \leq r - 1$, a contradiction. Hence, by symmetry, we assume $X \cap V(G - u) = X_1$, $X \cap V(H - v) = X_2$, $X \cap V(G - u) = X_3$ and $X \cap V(H - v) = X_4$, where $X_i = V(G') - X$ and $X_i \neq \emptyset$ for each $i \in \{1, 2, 3, 4\}$. Since $|\partial_{G'}(X)| \leq r - 1$, we have $|[X_1, X_3]_{G'}| \leq \lceil \frac{r - 1}{2} \rceil$ or $|[X_2, X_4]_{G'}| \leq \lceil \frac{r - 1}{2} \rceil$. It implies that $G - u$ or $H - v$ has an edge-cut of cardinality at most $\lceil \frac{r - 1}{2} \rceil$, which contradicts the assumption that both $G$ and $H$ are $r$-edge-connected. 

Let $e$ be an edge of a graph $G$ and $\mathcal{M}$ be a $k$-PDPM of $G$. We say that $\mathcal{M}$ contains $e$ if there is an $N \in \mathcal{M}$ such that $e \in N$. Otherwise, we say that $\mathcal{M}$ avoids $e$. In what follows we show that if every $r$-edge-connected $r$-graph has a $k$-PDPM, then every $r$-edge-connected $r$-graph has a $k$-PDPM containing or avoiding a fixed set of edges.
\textbf{Theorem 4.4.} Let $r \geq 4$ and $2 \leq k \leq r - 2$. The following statements are equivalent.

(i) Every $r$-edge-connected $r$-graph has a $k$-PDPM.

(ii) For every $r$-edge-connected $r$-graph $G$ and every $e \in E(G)$, there exists a $k$-PDPM of $G$ containing $e$.

(iii) For every $r$-edge-connected $r$-graph $G$ and every $e \in E(G)$, there exists a $k$-PDPM of $G$ avoiding $e$.

(iv) For every $r$-edge-connected $r$-graph $G$, every $v \in V(G)$ and $e \in \partial_G(v)$, there are at least $s = r - \left\lceil \frac{r-k}{2} \right\rceil - 1$ edges $e_1, \ldots, e_s$ in $\partial_G(v) \setminus \{e\}$ such that, for each $i \in \{1, \ldots, s\}$, there exists a $k$-PDPM of $G$ containing $e_i$ and $e$.

\textbf{Proof.} Clearly, each of (ii), (iii) and (iv) implies (i). Thus, it suffices to prove that (i) implies (ii); (i) implies (iii); and (ii) implies (iv).

(i) $\Rightarrow$ (ii), (iii). Assume that statement (i) is true and let $G$ be an $r$-edge-connected $r$-graph with an edge $vu_1$. We use the same construction for both implications. Let $C_{2r} = u_1u_2 \ldots u_{2r}u_1$ be a circuit of length $2r$. Denote $U_o = \{u_i : i \text{ is odd}\}$ and $U_e = \{u_i : i \text{ is even}\}$. We construct a new graph $H$ from $C_{2r}$ as follows. Replace each edge of $C_{2r}$ by $\frac{r-1}{2}$ parallel edges, if $r$ is odd, and replace the edge $u_iu_{i+1}$ ($u_{i+1}u_i$ and $u_{2r}u_{1}$, respectively) of $C_{2r}$ by $\frac{r}{2}$ ($\frac{r-2}{2}$, respectively) parallel edges for each $u_i \in U_o$ ($u_{i} \in U_e \setminus \{u_{2r}\}$, respectively), if $r$ is even. Add two new vertices, denoted by $u$ and $u'$, such that $u$ is adjacent to each vertex in $U_o$ and $u'$ is adjacent to each vertex in $U_e$, see Figure [3]. Clearly, $H$ is $r$-regular and $r$-edge-connected.

Let $I = \{i : i \in \{1, \ldots, 2r\}, i \text{ is odd}\}$ and for every $i \in I$ let $G^i$ be a copy of $G$, in which the vertices are labeled accordingly by using an upper index. For example, $v^i$ is the vertex of $G^i$ that corresponds to the vertex $v$ of $G$. Following the procedure described in Definition 4.2, we construct another new graph $H'$ from $H$ by successively replacing each vertex $u_i \in U_o$ of $H$ by $(G^i, v^i)$ such that for each $i \in I$ the vertex $v^i$ is adjacent to $u$ (see Figure [7]). By Lemma 4.3, $H'$ is $r$-regular and $r$-edge-connected. Note that $H'$ is an $r$-graph since it is of even order.

In order to prove statements (ii) and (iii) we observe the following. Let $M$ be an arbitrary perfect matching of $H'$ and for every $i \in I$, let $m_i = |\partial_{H'}(V(G^i - v^i)) \cap M|$. The set $M$ contains exactly one edge incident with $u$ and one edge incident with $u'$. Thus, by the construction of $H'$ we have $\sum_{i \in I} m_i = |M \cap \partial_{H'}(U_o)| = |I|$. Observation 2.1 implies $m_i \geq 1$ and hence, $m_i = 1$ for every $i \in I$. Thus, every perfect matching of $H'$ can be translated into a perfect matching of $G^i$ for each $i \in I$.

Now, by statement (i), $H'$ has a $k$-PDPM $\mathcal{N}$. Furthermore there are two integers $i, j \in I$ such that $\mathcal{N}$ contains $vu^i_1$ and avoids $vu^j_1$. By the above observation, the graph $G^i$ has a $k$-PDPM containing $v^i_1u^i_1$ and $G^j$ has a $k$-PDPM avoiding $v^j_1u^j_1$, which proves statements (ii) and (iii).

(ii) $\Rightarrow$ (iv). Let $G$ be an $r$-edge-connected $r$-graph and let $e_1 = vu_1 \in E(G)$. Suppose $|\{e \in \partial_G(v) \setminus \{e_1\} : \text{there exists a } k\text{-PDPM of } G \text{ containing } e, e_1\}| < s$. As a consequence, $\partial_G(v) \setminus \{e_1\}$ contains at least $t = r - 1 - (s - 1) = \left\lceil \frac{r-k}{2} \right\rceil + 1$ edges $e_2, \ldots, e_{t+1}$, such that for
We show that there is no $k$-PDPM of $G$ containing $e_1$ and $e_j$. For each $j \in \{2, \ldots, t+1\}$ denote $e_j = uv_j$.

Let $K_4$ be the complete graph of order 4 and let $V(K_4) = \{u_1, u_2, u_3, u_4\}$. We construct a new $r$-regular graph $H$ from $K_4$ by replacing each edge of $\{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$ by $\frac{r-1}{2}$ parallel edges if $r$ is odd, and replacing each edge of $\{u_1u_2, u_3u_4\}$ ($\{u_2u_3, u_4u_1\}$, respectively) by $\frac{r}{2}$ ($\frac{r-2}{2}$, respectively) parallel edges if $r$ is even, see Figure 9. Clearly, $H$ is $r$-edge-connected.

For each $i \in \{1, 3\}$, let $G^i$ be a copy of $G$ in which the vertices and edges are labeled accordingly by using an upper index and let $V^i = \{v^i_j : j \in \{2, \ldots, t+1\}\}$. Following the procedure in Definition 4.2, we construct another new graph $H'$ from $H$ by successively replacing each vertex $u_i \in \{u_1, u_3\}$ of $H$ by $(G^i, v^i)$ such that $v^i_1$ is adjacent to $v^i_2$ and $|u_2, V^1 \cup V^3|_{H'}$ contains as many edges as possible, see Figure 9. The graph $H'$ is $r$-regular and $r$-edge-connected by Lemma 4.3. By statement (ii), $H'$ has a $k$-PDPM $\mathcal{N}' = \{N_1, \ldots, N_k\}$ containing $u_2u_4$. Clearly, $v^i_1v^i_2$ and $u_2u_4$ are in the same perfect matching of $\mathcal{N}$ and so each $N_i \in \mathcal{N}$ contains exactly one edge of $\partial_{H'}(V(G^1-v^i))$ and one edge of $\partial_{H'}(V(G^3-v^i))$ by Observation 2.1. Thus, $N_i \cap [u_2, V^1 \cup V^3]_{H'} = \emptyset$ for each $i \in \{1, \ldots, k\}$. Now we consider the following two cases.

Case 1. $r$ is odd.

Since $t = \left\lfloor \frac{r-k}{2} \right\rfloor + 1 \leq \frac{r-1}{2}$, the set $[u_2, V^i]_{H'}$ contains $t$ edges for each $i \in \{1, 3\}$ by the construction of $H'$. Note that $N_i \cap [u_2, V^1 \cup V^3]_{H'} = \emptyset$ for each $i \in \{1, \ldots, k\}$. Hence, the $k$-PDPM $\mathcal{N}$ of $H'$ contains at most $r-2t = r-2\left\lfloor \frac{r-k}{2} \right\rfloor \leq r-2(\frac{r-1}{2} - 1) = k-1$ edges in $\partial_{H'}(u_2)$, a contradiction.

Case 2. $r$ is even.

Case 2.1. $k = 2$.

Since $t = \left\lfloor \frac{r-k}{2} \right\rfloor + 1 = \frac{r}{2}$, the set $[u_2, V^1 \cup V^3]_{H'}$ contains $2t-1 = r-1$ edges. Hence, the $k$-PDPM $\mathcal{N}$ of $H'$ contains at most $r - (2t-1) = 1$ edges in $\partial_{H'}(u_2)$, a contradiction.

Case 2.2. $k > 2$.

Since $t = \left\lfloor \frac{r-k}{2} \right\rfloor + 1 \leq \frac{r-4}{2} + 1 = \frac{r}{2} - 1$, we have that $[u_2, V^i]_{H'}$ contains $t$ edges for each $i \in \{1, 3\}$ by the construction of $H'$. Hence, the $k$-PDPM $\mathcal{N}$ of $H'$ contains at most $r - 2t = r - 2\left\lfloor \frac{r-k}{2} \right\rfloor + 1 \leq r - 2\left(\frac{r-k}{2} - 1\right) = k-1$ edges in $\partial_{H'}(u_2)$, a contradiction again.

For the special case $k = r - 2$, we can obtain a stronger result as follows.

**Theorem 4.5.** Let $k \geq 1$. The following statements are equivalent.

(i) Every $(2k+1)$-edge-connected $(2k+1)$-graph has a $(2k-1)$-PDPM.

(ii) For every $(2k+1)$-edge-connected $(2k+1)$-graph $G$ and every $k$ edges sharing a common vertex, there exists a $(2k-1)$-PDPM of $G$ containing this $k$ edges.

**Proof.** It suffices to prove that statement (i) implies statement (ii). Let $G$ be a $(2k+1)$-edge-connected $(2k+1)$-graph and let $v \in V(G)$ be a vertex with $\partial_G(v) = \{e_i : i \in \{1, \ldots, 2k+1\}\}$. We show that there is a $(2k-1)$-PDPM of $G$ that contains the edges $e_1, \ldots, e_k$. 12
Figure 6: Two examples for the graph $H$ obtained from $C_{2r}$ as in the proof of Theorem 4.4.

Figure 7: Two examples for the graph $H'$ obtained from $G$ and $H$ as in the proof of Theorem 4.4.
Figure 8: Two examples for the graph $H$ obtained from $K_4$ as in the proof of Theorem 4.4.

Figure 9: Two examples for the graph $H'$ obtained from $G$ and $H$ as in the proof of Theorem 4.4.
Denote $e_i = vv_i$ for each $i \in \{1, \ldots, 2k+1\}$. Let $G^1$ be a copy of $G$ in which the vertices and edges are labeled accordingly by using an upper index. As described in Definition 4.2, construct a new graph $H$ from $G$ by replacing $v$ with $(G^1, v^1)$ such that the set of new edges is given by $\{v_{2k+1}v^1_{2k+1}\} \cup E_1 \cup E_2$, where $E_1 = \{v_i v^1_i : i \in \{1, \ldots, k\}\}$ and $E_2 = \{v^1_i v_{i+k} : i \in \{1, \ldots, k\}\}$. By Lemma 4.3, $H$ is $(2k+1)$-edge-connected and $(2k+1)$-regular. Thus, by statement (i) and Theorem 4.4 there is a $(2k+1)$-PDPM $N$ of $H$ avoiding $v_{2k+1}v^1_{2k+1}$. By Observation 2.1 every perfect matching of $N$ contains exactly one edge of $\partial_H(V(G) \setminus \{v\})$ and hence, $N$ contains either every edge of $E_1$ or every edge of $E_2$. In the first case, $G$ has a $(2k-1)$-PDPM that contains $e_1, \ldots, e_k$; in the second case, $G^1$ has a $(2k-1)$-PDPM that contains $e^1_1, \ldots, e^1_k$. This proves statement (ii). \qed

5 5-graphs

We explore some consequences of the non-existence of 5-edge-connected class 2 5-graphs. The edge-connectivity may play a crucial role in this case as, by a result of Rizzi [15], there are poorly matchable 4-edge-connected 5-graphs.

Let $G$ be a cubic graph and let $\mathcal{F} = \{F_1, \ldots, F_t\}$ be a multiset of subsets $F_i$ of $E(G)$. For an edge $e$ of $G$, we denote by $\nu_\mathcal{F}(e)$ the number of elements of $\mathcal{F}$ containing $e$. A Fan-Raspaud triple, or FR-triple, is a multiset $\mathcal{T}$ of three perfect matchings of $G$ such that $\nu_\mathcal{T}(e) \leq 2$ for all $e \in E(G)$. A 5-cycle double cover, or 5-CDC, is a multiset $\mathcal{C}$ of five cycles in $G$ such that, for every edge $e \in E(G)$, $\nu_\mathcal{C}(e) = 2$. A Berge-Fulkerson cover, or BF-cover, is a multiset $\mathcal{T}$ of six perfect matchings of $G$ such that $\nu_\mathcal{T}(e) = 2$ for all $e \in E(G)$. We recall the following three well-known conjectures and a result from [9].

**Conjecture 5.1** (Fan-Raspaud Conjecture [5]). *Every bridgeless cubic graph has an FR-triple.*

**Conjecture 5.2** (5-cycle double cover Conjecture, see [18]). *Every bridgeless cubic graph has a 5-cycle double cover.*

**Conjecture 5.3** (Berge-Fulkerson Conjecture [6]). *Every bridgeless cubic graph has a BF-cover.*

**Theorem 5.4** (Kaiser and Škrekovski [9]). *Every bridgeless cubic graph has a 2-factor that intersects every edge-cut of cardinality 3 and 4. Moreover, any two adjacent edges can be extended to such a 2-factor.*

As shown in [14], the following conjecture is equivalent to the Fan-Raspaud Conjecture.

**Conjecture 5.5** (Mkrtchyan and Vardanyan [14]). *Let $G$ be a bridgeless cubic graph. For every $e \in E(G)$ and $i \in \{0, 1, 2\}$, there is an FR-triple $\mathcal{T}$ with $\nu_\mathcal{T}(e) = i$.***

In the same paper, they also pointed out the following observation but without proof. To keep the paper self-contained, we present a short proof here.
Observation 5.6 (Mkrtchyan and Vardanyan [14]). A minimum possible counterexample $G$ to Conjecture 5.5 with respect to $|V(G)|$ is 3-edge-connected.

Proof. Suppose that $G$ is a minimum counterexample to Conjecture 5.5 with respect to $|V(G)|$. Then, there is $e \in E(G)$ and $i \in \{0, 1, 2\}$ such that no FR-triple $T$ satisfies $\nu_T(e) = i$. Suppose that there is a set $X \subseteq V(G)$ with $u, v \in X$ and $\partial_G(X) = \{ux, vy\}$. Let $H_1 = G[X] + \{uv\}$ and $H_2 = G - X + \{xy\}$. Notice that both $H_1$ and $H_2$ are bridgeless cubic graphs. If $e \in \{ux, vy\}$, since $|V(G)|$ is minimum, there is an FR-triple $T_1$ of $H_1$ and an FR-triple $T_2$ of $H_2$ such that $\nu_{T_1}(uv) = \nu_{T_2}(xy) = i$. Then $T_1$ and $T_2$ can be used to construct an FR-triple $T$ of $G$ with $\nu_T(e) = i$, a contradiction. Hence, without loss of generality we may assume $e \in E(H_1)$. Since $|V(G)|$ is minimum, there is an FR-triple $T_1$ of $H_1$ and an FR-triple $T_2$ of $H_2$ such that $\nu_{T_1}(e) = i$ and $\nu_{T_2}(xy) = \nu_{T_1}(uv)$. Again, $T_1$ and $T_2$ can be used to construct an FR-triple $T$ of $G$ with $\nu_T(e) = i$, a contradiction. \qed

5.1 Relation to the Fan-Raspaud Conjecture

In this subsection we show that the Fan-Raspaud Conjecture is true if there is no poorly matchable 5-edge-connected 5-graph.

Theorem 5.7. If $m(5) \geq 2$, then Conjecture 5.5 is true.

Proof. By contradiction, suppose that $m(5) \geq 2$ and Conjecture 5.5 is false. Let $G$ be a minimum counterexample to Conjecture 5.5 with respect to $|V(G)|$. Then, there is an edge $e = uv$ of $G$ and an $i \in \{0, 1, 2\}$ such that no FR-triple $T$ satisfies $\nu_T(e) = i$. By Observation 5.6 $G$ is 3-edge-connected.

First, we consider the case $i = 0$. By Theorem 5.4 there is a 2-factor $F$ of $G$ such that $e \in E(F)$ and $F$ intersects every edge-cut of cardinality 3 and 4. Let $H = G + E(F)$ and let $e'$ be the new edge parallel to $e$. Since $G$ is 3-edge-connected, the graph $H$ is 5-edge-connected by the choice of $F$. Since $m(5) \geq 2$, it follows with Theorem 4.4 (iv) that for each edge $e_0 \in \partial_H(v) \setminus \{e, e'\}$, there are at least three edges $e_1, e_2, e_3 \in \partial_H(v) \setminus \{e_0\}$ such that for each $j \in \{1, 2, 3\}$ there exists a 2-PDPM containing $e_j$ and $e_0$. This implies that $H$ has two disjoint perfect matchings $N_1$ and $N_2$ such that $e$ and $e'$ are in none of them. In the graph $G$, let $N'_1$ and $N'_2$ be the perfect matchings corresponding to $N_1$ and $N_2$, respectively. Let $N_3 = E(G) \setminus E(F)$. Since $N_1$ and $N_2$ are disjoint, every edge of $N'_1 \cap N'_2$ belongs to $E(F)$, i.e. $T = \{N'_1, N'_2, N_3\}$ is an FR-triple of $G$. Furthermore $\nu_T(e) = 0$, a contradiction.

Next suppose $i \in \{1, 2\}$. By Theorem 5.4 we can choose a 2-factor $F$ of $G$ such that $e \notin E(F)$ and $F$ intersects every edge-cut of cardinality 3 and 4. Again, the graph $H$ defined by $H = G + E(F)$ is 5-edge-connected. Since $m(5) \geq 2$, by statements (ii) and (iii) of Theorem 4.4 $H$ has two disjoint perfect matchings $N_1$ and $N_2$ such that $e$ is in exactly $i - 1$ of them.
Therefore, $T = \{N_1', N_2', N_3\}$ is an FR-triple of $G$ with $\nu_T(v) = i$ where $N_1'$ and $N_2'$ are the perfect matchings of $G$ that correspond to $N_1$ and $N_2$, respectively, and $N_3 = E(G) \setminus E(F)$. This leads to a contradiction again.

If $m(5) \geq 2$, then in particular every 5-edge-connected 5-graph with an underlying cubic graph has two disjoint perfect matchings. By adjusting Theorem 4.4, one can show the following strengthening of Theorem 5.7 (for a sketch of the proof, see Appendix A).

**Theorem 5.8.** If every 5-edge-connected 5-graph whose underlying graph is cubic has two disjoint perfect matchings, then Conjecture 5.5 is true.

### 5.2 Relation to the 5-cycle double cover Conjecture

Now we focus on the consequences of the non-existence of 5-edge-connected class 2 5-graphs. Let $k \geq 3$ be an integer. A $k$-wheel $W_k$ is a $k$-circuit $C_k$ plus one additional vertex $w$ adjacent to all vertices of $C_k$.

**Theorem 5.9.** The following statements are equivalent.

(i) Every 5-edge-connected 5-graph is class 1.

(ii) Every 5-edge-connected 5-graph with an underlying cubic graph is class 1.

**Proof.** The first statement implies trivially the second one. We prove now the other implication. Let $G$ be a 5-edge-connected 5-graph. For every vertex $v$ of $G$, let $W_v^5$ be a copy of the graph $W_5 + E(C_5)$. Moreover, let $w^v$ and $C_v^5$ be the vertex and, respectively, the circuit of $W_v^5$ corresponding to $w$ and $C_5$ in $W_5$. Following the procedure described in Definition 4.2, successively replace every vertex $v$ of $G$ with $(W_v^5, w^v)$ to obtain a new graph $H$, which is 5-regular and 5-edge-connected. Moreover, its underlying graph is cubic and so $H$ is class 1 by statement (ii). Hence, $H$ has a 5-PDPM, denoted by $N = \{N_1, \ldots, N_5\}$. Since $|V(C_v^5)|$ is odd, by Observation 2.1 we have that, for all $i \in \{1, \ldots, 5\}$, $|N_i \cap \partial_H(V(C_v^5))| = 1$. Hence, the restriction $N_i'$ of $N_i$ to the graph $G$ is a perfect matching of $G$. Moreover, $\{N_1', \ldots, N_5'\}$ is a 5-PDPM of $G$. Therefore, $G$ is class 1.

It is well known that a counterexample of minimum order to Conjecture 5.2 is a cyclically 4-edge-connected cubic class 2 graph.

**Theorem 5.10.** If $m(5) = 5$, then Conjecture 5.2 is true.

**Proof.** Let $K$ be the graph obtained from a 4-wheel by doubling the edges of the outer circuit and of one spoke. Note that $K$ has one vertex of degree 6, which we denote by $w$, and four vertices of degree 5.

Let $G$ be a minimum counterexample to Conjecture 5.2 with respect to $|V(G)|$. Then, $G$ is cubic and cyclically 4-edge-connected. Thus, the graph $2G = G + E(G)$ is 6-edge-connected.
For every vertex \( v \) of \( G \), let \( K^v \) be a copy of \( K \) and let \( w^v \) be the vertex of \( K^v \) corresponding to \( w \) in \( K \). Analogously to Definition 4.2, let \( H \) be the graph obtained by replacing each vertex \( v \) of \( 2G \) by \( (K^v, w^v) \), in such a way that parallel edges of \( 2G \) are incident with the same vertex of \( K^v \). Then, \( H \) is a 5-edge-connected 5-graph and therefore, it has a 5-PDPM \( \mathcal{N} = \{N_1, \ldots, N_5\} \). For every \( v \in V(2G) \), there exist exactly three perfect matchings of \( \mathcal{N} \), say \( N_1', N_2', N_3' \), such that \( |N_i' \cap \partial_H(K^v - w^v)| = 2 \) for each \( i \in \{1, 2, 3\} \). Hence, for every \( j \in \{1, \ldots, 5\} \), the restriction of each \( N_j \in \mathcal{N} \) on \( G \) induces an even subgraph \( C'_j \) of \( G \). Moreover, we have \( \nu_C(e) = 2 \) for each \( e \in E(G) \), where \( C = \{C'_1, \ldots, C'_5\} \). So \( C \) is a 5-CDC of \( G \).

\[ \square \]

5.3 Relation to the Berge-Fulkerson Conjecture

Observation 5.11. If \( m(5) = 5 \), then Conjecture 5.3 is true.

Proof. Assume \( m(5) = 5 \) and suppose that \( G \) is a counterexample to the Berge-Fulkerson Conjecture such that the order of \( G \) is minimum. Let \( F \) be a 2-factor of \( G \). As shown in \([10]\), \( G \) is cyclically 5-edge-connected and hence, \( G + E(F) \) is 5-edge-connected. Therefore, \( G + E(F) \) has five pairwise disjoint perfect matchings. The corresponding five perfect matchings of \( G \) and \( E(G) \setminus E(F) \) are a BF-cover of \( G \), a contradiction.

\[ \square \]

5.4 Properties of a minimum possible 5-edge-connected class 2 5-graph

We are going to prove some structural properties of a smallest possible 5-edge-connected class 2 5-graph. Let \( G \) be a graph and \( v \in V(G) \). A lifting of \( G \) at \( v \) is the following operation. Remove two edges \( vx, vy \) where \( x \neq y \) and add a new edge \( xy \). In this case we say \( vx \) and \( vy \) are lifted to \( xy \).

Theorem 5.12 (Mader [11]). Let \( G \) be a finite graph and let \( v \in V(G) \) such that \( d(v) \geq 4 \), \( |N(v)| \geq 2 \) and \( G - v \) is connected. There is a lifting of \( G \) at \( v \) such that, for every pair of distinct vertices \( u, w \in V(G) \setminus \{v\} \), the number of edge-disjoint uw-paths in the resulting graph equals the number of edge-disjoint uw-paths in \( G \).

Statement (ii) of the following theorem is already mentioned in \([3]\) for planar \( r \)-graphs without proof.

Theorem 5.13. Let \( G \) be a 5-edge-connected class 2 5-graph such that the order of \( G \) is as small as possible. The following statements hold.

(i) Every 5-edge-cut of \( G \) is trivial, i.e. if \( X \subset V(G) \) and \( |\partial(X)| = 5 \), then \(|X| = 1 \) or \(|V(G) \setminus X| = 1 \).

(ii) Every 3-vertex-cut is trivial, i.e. if \( X \subset V(G), |X| = 3 \) and \( G - X \) is not connected, then one component of \( G - X \) is a single vertex.
Proof. (i). The proof follows easily and is left to the reader.

(ii). By contradiction, suppose that \( X = \{v_1, v_2, v_3\} \subset V(G) \) is a 3-vertex-cut of \( G \) such that none of the components of \( G - X \) is a single vertex. By Observation 4.1 and the edge-connectivity of \( G \), the graph \( G - X \) has at most three components. First, we consider the case that \( G - X \) has exactly three components. Denote the vertex sets of these three components by \( A, B \) and \( C \). We have that \( |\partial_G(S)| = 5 \), for each \( S \in \{A, B, C\} \), and so \( |A| = |B| = |C| = 1 \) by statement (i), a contradiction.

Next, we assume that \( G - X \) has exactly two components whose vertex sets are denoted by \( A \) and \( B \). Since \( G \) has even order, we may assume \( |A| \) is odd and \( |B| \) is even. For each \( i \in \{1, 2, 3\} \), set \( n_i = |\partial_G(B) \cap \partial_G(v_i)| \) and let

\[
a = \frac{1}{2} (n_1 + n_2 - n_3), \quad b = \frac{1}{2} (-n_1 + n_2 + n_3), \quad c = \frac{1}{2} (n_1 - n_2 + n_3).
\]

We have that \( n_1 + n_2 + n_3 = |\partial_G(B)| \) is even, since \( |B| \) is even. Thus, all of \( a, b, c \) are integers. Furthermore, \( 5 \leq |\partial_G(B \cup \{v_3\})| = n_1 + n_2 + (5 - n_3) \) and hence, \( a \geq 0 \). Analogously, we obtain \( b, c \geq 0 \). Therefore, we can define a new graph \( H_1 \) as follows (see Figure 10).

\[
H_1 = (G - B) + a \{v_1 v_2\} + b \{v_2 v_3\} + c \{v_3 v_1\}.
\]

By the definitions of \( a, b, c \), the graph \( H_1 \) is 5-regular. Moreover \( H_1 \) is also 5-edge-connected. Indeed, let \( Y \subseteq V(H_1) \). We can assume, without loss of generality, that \( |Y \cap \{v_1, v_2, v_3\}| \leq 1 \) (otherwise, we argue by taking its complement). By the choices of \( a, b \) and \( c \), we have \( |\partial_{H_1}(Y)| = |\partial_G(Y)| \geq 5 \) and so \( H_1 \) is 5-edge-connected.

Let \( H' \) be the graph obtained from \( G \) by identifying all vertices in \( A \) to a new vertex \( u \) and removing all resulting loops, see Figure 10. Then, \( H' \) is 5-edge-connected and every vertex is of degree 5 except \( u \). Since \( |A| \) is odd, we have that \( |\partial_G(A)| \) is odd. Hence, the vertex \( u \) has an odd degree of at least 5 in \( H' \). Now, by Theorem 5.12 a new 5-edge-connected 5-graph \( H_2 \) can be obtained from \( H' \) by \( \frac{1}{2}(d_H(u) - 5) \) liftings at \( u \), see Figure 10. We will refer to the edges of \( H_2 \) obtained by a lifting at \( u \) as lifting edges and denote the set of all lifting edges by \( \mathcal{L} \).

By the minimality of \( |V(G)| \), \( H_1 \) has a 5-PDPM \( \{N_1^1, \ldots, N_3^3\} \) and \( H_2 \) has a 5-PDPM \( \{N_1^1, \ldots, N_3^3\} \). Since \( u \) has at most three neighbors in \( H_2 \), every perfect matching of \( H_2 \) contains at most one lifting edge. For each \( i \in \{1, \ldots, 5\} \), let \( N_i \) be the subset of edges of \( H' \) defined as follows.

\[
N_i = \begin{cases} \quad N_i^2 & \text{if } N_i^2 \cap \mathcal{L} = \emptyset; \\ \quad (N_i^2 \setminus \{e\}) \cup \{e_1, e_2\} & \text{if } N_i^2 \cap \mathcal{L} = \{e\} \text{ and } e_1, e_2 \text{ are the two edges lifted to } e. \end{cases}
\]

Every perfect matching of \( H_1 \) contains either one or three edges of \( \partial_{H_1}(A) \) by Observation 2.1. Let \( s_1 \) be the number of integers \( i \in \{1, \ldots, 5\} \) with \( |N_i^1 \cap \partial_{H_1}(A)| = 3 \), let \( s_2 = |\mathcal{L}| \) and let \( s' \) be the number of integers \( j \in \{1, \ldots, 5\} \) with \( |N_j \cap \partial_{H'}(u)| = 3 \). We have that \( s_2 = s' \). Moreover,
we have $\partial_G(A) = 3s_1 + (5 - s_1) = 5 + 2s_2$ and so $s_1 = s_2 = s'$. Note that $\partial_{H_1}(A) = \partial_G(A)$ and recall that $H'$ is obtained from $G$ by identifying all vertices in $A$ to $u$. As a consequence, the sets of edges $N_1, \ldots, N_5$ of $H'$ and the perfect matchings $N^1_1, \ldots, N^1_5$ of $H_1$ can be combined to obtain a 5-PDPM of $G$, a contradiction.

Figure 10: An example for the graphs $H_1$, $H'$ and $H_2$ obtained from $G$ in Theorem 4.4.

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A A sketch of the proof of Theorem 5.8

In order to prove Theorem 5.8 we adjust Theorem 4.4 as follows.

**Theorem A.1.** The following statements are equivalent.

(i) Every 5-edge-connected 5-graph with an underlying cubic graph has a 2-PDPM.

(ii) For every 5-edge-connected 5-graph $G$ with an underlying cubic graph and every simple edge $e \in E(G)$, there is a 2-PDPM containing $e$.

(iii) For every 5-edge-connected 5-graph $G$ with an underlying cubic graph and every simple edge $e \in E(G)$, there is a 2-PDPM avoiding $e$.

(iv) For every 5-edge-connected 5-graph $G$ with an underlying cubic graph and every simple edge $e \in E(G)$ and every two parallel edges $e_1, e_2$ adjacent with $e$, there is a 2-PDPM containing $e$ and avoiding $e_1, e_2$.

**Proof.** Clearly, each of (ii), (iii) and (iv) implies (i). Thus, it suffices to prove that (i) implies (ii); (ii) implies (iii); and (iii) implies (iv).

(i) $\Rightarrow$ (ii), (iii). Let $G$ be a 5-edge-connected 5-graph whose underlying graph is cubic and let $e = vv_1$ be a simple edge of $G$. Let $H$ and $H'$ be the graphs constructed in the part *(i)* $\Rightarrow$ (ii), (iii)* of the proof of Theorem 4.4 by using $C_2r$ and $r$ copies of $G$ in the case $r = 5$ (see Figures 6(a) and 7(a)). Clearly, the graph $H'$ can be constructed from $H$ such that every vertex of $V(H') \setminus \{u, w\}$ has degree 3 in the underlying graph of $H'$. Let $W = W_5 + E(C_5)$. Now, according to Definition 4.2, replace $u$ by $(W^1, w^1)$ and replace $u'$ by $(W^2, w^2)$, where $W^l$ is a copy of $W$ and $w^l$ is the vertex of $W^l$ corresponding to $w$, for $l \in \{1, 2\}$. The resulting graph, denoted by $H''$, is a 5-edge-connected 5-graph by Lemma 4.3. Since its underlying graph is cubic, $H''$ has two disjoint perfect matchings $N_1, N_2$ by statement (i). Let $N = N_1 \cup N_2$ and recall that $I = \{1, 3, 5, 7, 9\}$. For every $i \in I$ and $j \in \{1, 2\}$, Observation 2.1 implies $m_{ij} \in \{1, 3\}$, where $m_{ij} = |\partial_{H''}(V(G^i) \setminus \{v^i\}) \cap N_j|$. Furthermore, we have $|\partial_{H''}(V(W^l) \setminus \{w^l\}) \cap N_j| \in \{1, 3\}$ for every $l, j \in \{1, 2\}$ also by Observation 2.1. Thus, $|\partial_{H''}(V(W^l) \setminus \{w^l\}) \cap N| \in \{2, 4\}$ for every $l \in \{1, 2\}$. As a consequence, there is an integer $i \in I$ such that $N$ does not contain the unique edge in $[v^i_1, V(W^1) \setminus \{w^1\}]_{H''}$. We have $m_{i1} = m_{i2} = 1$. Therefore, $G^i$ has two disjoint perfect matchings such that $v^i_1v^i_2$ is in none of them, which proves statement (iii). For statement (ii), we consider the following cases.

Case 1. $|N \cap \partial_{H''}(V(W^1) \setminus \{w^1\})| = |N \cap \partial_{H''}(V(W^2) \setminus \{w^2\})| = 2$.

In this case, $H'$ has a 2-PDPM, and hence, statement (ii) follows by the same argumentation as in the proof of Theorem 4.4, part *(i)* $\Rightarrow$ (ii), (iii)*.

Case 2. Without loss of generality $|N_1 \cap \partial_{H''}(V(W^1) \setminus \{w^1\})| = 3$.

In this case, there is an integer $i \in I$, say $i = 1$, such that $N_1$ contains the unique edge in $[v^1_1, V(W^1) \setminus \{w^1\}]_{H''}$ and the unique edge in $[v^1_1v^2_1, V(W^1) \setminus \{w^1\}]_{H''}$. The set $N_1$ contains exactly one edge incident with $w_2$ and thus, $m_{11} = 1$ or $m_{31} = 1$ by the construction of $H''$.  

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Therefore, \( G^1 \) has two disjoint perfect matchings such that \( v^1v_1^1 \) is in one of them or \( G^3 \) has two disjoint perfect matchings such that \( v^3v_3^1 \) is in one of them, which proves statement (ii).

**Case 3.** Without loss of generality \( |N_1 \cap \partial_{H''}(V(W^2) \setminus \{w^2\})| = 3 \).

In this case, there is an integer \( i \in I \), say \( i = 3 \), such that \( N_1 \) contains the unique edge in \([u_{i-1}, V(W^2) \setminus \{w^2\}]_{H''}\) and the unique edge in \([u_{i+1}, V(W^2) \setminus \{w^2\}]_{H''}\). As a consequence, \( N_1 \) contains the unique edge in \([v^3_1, V(W^1) \setminus \{w^1\}]_{H''}\) and \( m_{31} = 1 \). Therefore, \( G^3 \) has two disjoint perfect matchings such that \( v^3v_3^1 \) is in one of them, which proves statement (ii).

\((ii) \Rightarrow (iv)\). Let \( G \) be a 5-edge-connected 5-graph whose underlying graph is cubic, let \( v \in V(G) \) and let \( N_G(v) = \{v_1, v_2, v_3\} \). Furthermore, let \( \mu_G(v, v_1) = 1, \mu_G(v, v_2) = \mu_G(v, v_3) = 2 \), let \( e \) be the edge connecting \( v \) and \( v_1 \) and let \( e_1, e_2 \) be the two parallel edges connecting \( v \) and \( v_2 \). We show that there are two disjoint perfect matchings such that their union contains \( e \) but neither \( e_1 \) nor \( e_2 \).

Let \( G^1 \) and \( G^3 \) be two copies of \( G \) in which the vertices and edges are labeled accordingly by using an upper index. Let \( H \) be the graph constructed in the part "(ii) \Rightarrow (iv)" of the proof of Theorem 4.4 by using \( K_4 \) in the case \( r = 5 \), see Figure 8 (a). According to Definition 4.2, construct a new graph \( H' \) from \( H \) by replacing \( u_1 \) with \( (G^1, v^1) \) and replacing \( u_3 \) with \( (G^3, v^3) \) such that \( \mu_{H'}(v^1_1, v^3_1) = 1 \) and \( \mu_{H'}(v^1_2, u_2) = \mu_{H'}(v^3_2, u_2) = \mu_{H'}(v^3_3, u_4) = 2 \). The graph \( H' \) is 5-edge-connected and 5-regular by Lemma 4.3 and its underlying graph is cubic. Therefore, by statement (ii) there are two disjoint perfect matchings \( N_1, N_2 \) of \( H' \) such that \( u_2u_4 \in N_1 \). By Observation 2.1, we have \( v^1_1v^3_1 \in N_1 \) and \( |\partial_{H'}(V(G') \setminus \{v^1\}) \cap N_j| = 1 \) for every \( i \in \{1, 3\} \) and every \( j \in \{1, 2\} \). Furthermore, \( N_1 \cup N_2 \) either does not contain the two edges connecting \( v^1_2 \) and \( u_2 \) or does not contain the two edges connecting \( v^3_2 \) and \( u_2 \). In the first case, \( G^1 \) has two disjoint perfect matchings such that their union contains \( e^1 \) but neither \( e^1_1 \) nor \( e^1_2 \); in the second case \( G^3 \) has two disjoint perfect matchings such that their union contains \( e^3 \) but neither \( e^3_1 \) nor \( e^3_2 \). This proves statement (iv).

\( \square \)

Theorem 5.8 can be proved like Theorem 5.7 by using Theorem A.1 instead of Theorem 4.4.