Long-time behavior for three dimensional compressible viscous and heat-conductive gases

Xiaoping Zhai, Zhi-Min Chen*

School of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China

Abstract

Large-time behavior of solutions to the compressible Navier-Stokes equations for viscous and heat-conductive gases in \( \mathbb{R}^3 \) is examined. Under a suitable condition involving only the low frequencies of the initial data, optimal time decay rates for the non-isentropic compressible Navier-Stokes flows are obtained, by developing some energy arguments given by [Xin and Xu, arXiv:1812.11714v2].

Keywords: Time decay estimates; compressible Navier-Stokes equations; Besov space

Mathematics Subject Classification (2010): 76N15; 35Q30; 35L65; 35K65

1. Introduction and the main result

In this paper, we consider the three-dimensional non-isentropic compressible Navier-Stokes equations:

\[
\begin{aligned}
\frac{\partial \rho}{\partial t} + \text{div} (\rho u) &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \text{div} (\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \text{div} u + \nabla P &= 0, \\
\rho(\partial_t \theta + u \cdot \nabla \theta) + P \text{div} u - \kappa \Delta \theta &= 2\mu |D(u)|^2 + \lambda (\text{div} u)^2, \\
(\rho, u, \theta)|_{t=0} &= (\rho_0, u_0, \theta_0)
\end{aligned}
\]

(1.1)

for the velocity \( u = (u_1, u_2, u_3) \), the density \( \rho \), the absolute temperature \( \theta \), the pressure \( P \) and the deformation tensor \( D(u) = \frac{1}{2} (\nabla u + (\nabla u)^T) \). The viscosity coefficients \( \mu \) and \( \lambda \) are subject to the standard physical restrictions:

\[
\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0.
\]

*Corresponding author

Email addresses: zhaixp@szu.edu.cn (Xiaoping Zhai), zmchen@szu.edu.cn (Zhi-Min Chen)
\( \kappa > 0 \) denotes the heat conductivity coefficient. For a general fluid, \( P \) is a function of \( \rho \) and \( \theta \). Here, we only consider the case of perfect heat conducting and viscous gases, i.e., \( P(\rho, \theta) = A\rho \theta \) for some constant \( A > 0 \). Without loss of generality, we assume that \( A = 1 \) throughout the paper.

If the effect of the temperature is neglected and thus the pressure is a function of \( \rho \), Eq. (1.1) reduces to the isentropic compressible Navier-Stokes equations, which have been widely studied (see [3, 5, 6, 11, 14, 15, 17, 18, 19, 20, 22, 40] and the references therein). However, the system (1.1), involving the heat conducting effect, is much more complicated due to the occurrence of stronger nonlinearity, and thus has attracted increasing attention from mathematicians. Significant progress has been made in the understanding of the existence, regularity and asymptotic behavior of the solutions over the past several decades.

In the absence of vacuum, the initial density \( \rho_0 \) is bounded away from zero. Nash [34] obtained the local existence of classical solutions in Sobolev spaces. Itaya [26] showed the existence of the local classical solutions in Hölder spaces, and this result was further derived by Tani [36] in a domain with boundary. Later on, Matsumura and Nishida [32] proved the existence of global small solutions and stability of a basic steady-state solution (see also [29] for the one-dimensional situation). For the existence and asymptotic behavior of weak solutions, Jiang [27] considered the one-dimensional case, Jiang [28] examined a spherically symmetric structure in two- and three-dimensional spaces, Hoff and Jenssen [21] discussed spherically and cylindrically symmetric structures in the three-dimensional space, and Feireisl [18] investigated the problem in two- and three-dimensional bounded domains.

In the presence of vacuum, long-time solutions may not exist. If \( \kappa = 0 \), it is obtained by Xin [38] that any non-zero smooth solution with initial compactly supported density would blow up in a finite time. Feireisl [17] got the existence of variational solutions in \( \mathbb{R}^d (d \geq 2) \) under the condition that the temperature equation is satisfied only as an inequality in the sense of distributions. For the equality equations in the sense of distributions, Bresch and Desjardins [2] proposed some different assumptions from [17] and obtained the existence of
global weak solutions to the compressible Navier-Stokes equations with large initial data in $T^3$ and $\mathbb{R}^3$. Huang and Li \cite{22} established the global existence and uniqueness of classical solutions to the three-dimensional compressible Navier-Stokes system with smooth initial data, which are of small energy but possibly large oscillations, and with initial density allowed to vanish. Wen and Zhu \cite{37} established the global existence of spherically and cylindrically symmetric classical solutions to the three-dimensional compressible Navier-Stokes equations. We also emphasize some blowup criterions given by Huang and Li \cite{22, 23, 24} and Huang et al. \cite{25} (see also the references therein). The readers may refer to Chen et al. \cite{5} and Wen and Zhu \cite{37} for more recent developments on this subject.

Let us recall the critical regularity framework of (1.1). Rigorously speaking, Eq. (1.1) does not possess any scaling invariance with respect to $(u, \rho, \theta, P)$. However, in the absence of the pressure, we define the scaling transformation:

$$
\rho(t, x) \rightarrow \rho(\lambda^2 t, \lambda x), \quad u(t, x) \rightarrow \lambda u(\lambda^2 t, \lambda x), \quad \theta(t, x) \rightarrow \lambda^2 \theta(\lambda^2 t, \lambda x) \quad \lambda > 0. \quad (1.2)
$$

A function space is said to be critical with respect to (1.1) if the space norm is invariant with respect to the scaling transformation (1.2). For example, the product space $\dot{B}^d_{p,1}(\mathbb{R}^d) \times \dot{B}^{d-1}_{p,1}(\mathbb{R}^d) \times \dot{B}^{d-2}_{p,1}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, is a critical space for the system (1.1).

In the framework of critical spaces, a breakthrough was made by Danchin \cite{11} for the isentropic compressible Navier-Stokes equations, where he proved the local wellposedness with large initial data and global solutions with small initial data. This result was further extended by Charve et al. \cite{3}, Chen et al. \cite{5}, Haspot \cite{20} and the authors \cite{8}. For the non-isentropic compressible Navier-Stokes equations (1.1), Chikami and Danchin \cite{9} and Danchin \cite{12} considered the local wellposedness problem. Global small solutions in a critical $L^p$ Besov framework were obtained respectively by Danchin \cite{13} for $p = 2$ and Danchin and He \cite{14} for more general $p$. It should be mentioned that the critical Besov space used in \cite{9} and \cite{14} seems the largest one in which the system (1.1) is well-posed. Indeed, Chen et al. \cite{7} proved the ill-posedness of (1.1) in $\dot{B}^{\frac{3}{p}-1}_{p,1} \times \dot{B}^{\frac{5}{p}-1}_{p,1} \times \dot{B}^{\frac{5}{p}-2}_{p,1}(\mathbb{R}^3)$ for $p > 3$.

In the present paper, we consider the large time behaviour of three-dimensional global solutions to (1.1), stemmed from the pioneering work of Matsumura and Nishida \cite{32} on
the decay estimate
\[ \sup_{t \geq 0} \langle t \rangle^{\frac{d}{2}} \| (\rho, u, \theta)(t) - (\bar{\rho}, 0, \bar{\theta}) \|_{L^2} < \infty \quad \text{with} \quad \langle t \rangle = \sqrt{1 + t^2}, \] (1.3)
when the initial data are perturbed around the equilibrium state \((\bar{\rho}, 0, \bar{\theta})\). Similar decay estimates have been established in the half-space or exterior domains (see, for example, [30, 31, 33]). Subsequently, the \(L^2\) decay estimate (1.3) was improved by Ponce [35] to the \(L^p\) one at the rate 
\[ O(\langle t \rangle^{\frac{d}{2}(1 - \frac{1}{p})}) \] with \(2 \leq p \leq \infty\) and \(d = 2, 3\). Inspired by a series of work [15, 16, 32, 39, 40], the present study aims at showing optimal time-decay estimates for (1.1) within the critical regularity framework of the solutions constructed in [14].

Let \((\bar{\rho}, \bar{u} = 0, \bar{\theta})\) be an equilibrium state with constants \(\bar{\rho} > 0\) and \(\bar{\theta} > 0\). We look for the solutions of (1.1) such that
\[ (\rho(x, t), u(x, t), \theta(x, t)) \to (\bar{\rho}, 0, \bar{\theta}) \quad \text{for} \quad |x| \to \infty, \; t > 0. \]

For convenience, let \(\bar{\rho} = \bar{\theta} = 1\) and define
\[ A u = \mu \Delta u + (\lambda + \mu) \nabla \text{div} u, \quad \rho = 1 + a, \quad \theta = 1 + \vartheta, \quad I(a) = \frac{a}{1 + a}, \quad J(a) = \ln(1 + a). \]

Then, we can rewrite (1.1) into the following system:
\[ \begin{aligned}
\partial_t a + \text{div} u &= -\text{div} (a u), \\
\partial_t u - A u + \nabla (a + \vartheta) &= -u \cdot \nabla u - I(a) Au + I(a) \nabla a - \vartheta \nabla J(a), \\
\partial_t \vartheta - \kappa \Delta \vartheta + \text{div} u &= -\text{div} (\vartheta u) - \kappa I(a) \Delta \vartheta + \frac{2 \mu |D(u)|^2 + \lambda (\text{div} u)^2}{1 + a}, \\
(a, u, \vartheta)|_{t=0} &= (a_0, u_0, \vartheta_0). 
\end{aligned} \] (1.4)

For \(z \in S'(\mathbb{R}^3)\), the low and high frequency parts are expressed as
\[ z^\ell \overset{\text{def}}{=} \sum_{2^j \leq j_0} \hat{\Delta}_j z \quad \text{and} \quad z^h \overset{\text{def}}{=} \sum_{2^j > j_0} \hat{\Delta}_j z \] (1.5)
for a large integer \(j_0 \geq 1\). The corresponding truncated semi-norms are defined as follows:
\[ ||z||_{\dot{B}^s_{p,1}}^\ell \overset{\text{def}}{=} ||z^\ell||_{\dot{B}^s_{p,1}} \quad \text{and} \quad ||z||_{\dot{B}^s_{p,1}}^h \overset{\text{def}}{=} ||z^h||_{\dot{B}^s_{p,1}}. \]
The definitions of the operator \(\hat{\Delta}_j\) and the space \(\dot{B}^s_{p,1}\) are given in the next section. Without loss of generality, we take \(\mu = \lambda = \kappa = 1\) in (1.4).

The present analysis is closely related to the following results.
Theorem 1.1. (Danchin and He [14]) Let $2 \leq p < 3$, $(a^0, u^0, \theta^0) \in \dot{B}^{\frac{1}{2}, 1}_2(\mathbb{R}^3)$, $u^1_0 \in \dot{B}^{\frac{3}{p} - 1}_p(\mathbb{R}^3)$, $a^h_0 \in \dot{B}^{\frac{3}{p} - 2}_p(\mathbb{R}^3)$, $\theta^h_0 \in \dot{B}^{\frac{3}{p} - 2}_p(\mathbb{R}^3)$. The initial data satisfy the assumption

$$X_0 \overset{\text{def}}{=} \|(a_0, u_0, \theta_0)\|_{\dot{B}^{\frac{1}{2}, 1}_2} + ||a_0||_{\dot{B}^{\frac{3}{p} - 1}_p} + ||u_0||_{\dot{B}^{\frac{3}{p} - 1}_p} + ||\theta_0||_{\dot{B}^{\frac{3}{p} - 2}_p} \leq \varepsilon$$

(1.6)

for a constant $\varepsilon > 0$ sufficiently small. Then the system (1.4) has a unique global solution $(a, u, \theta)$ so that

$$(a^\ell, u^\ell, \theta^\ell) \in C([0, \infty); \dot{B}^{\frac{1}{2}, 1}_2(\mathbb{R}^3)) \cap L^1(0, \infty; \dot{B}^{\frac{5}{2}}_{2, 1}(\mathbb{R}^3)),$$

$u^h \in C([0, \infty); \dot{B}^{\frac{3}{p} - 1}_p(\mathbb{R}^3)) \cap L^1(0, \infty; \dot{B}^{\frac{3}{p} + 1}_p(\mathbb{R}^3))$,

$a^h \in C([0, \infty); \dot{B}^{\frac{3}{p} - 2}_p(\mathbb{R}^3)) \cap L^1(0, \infty; \dot{B}^{\frac{3}{p}}_{2, 1}(\mathbb{R}^3)),$

$\theta^h \in C([0, \infty); \dot{B}^{\frac{3}{p} - 2}_p(\mathbb{R}^3)) \cap L^1(0, \infty; \dot{B}^{\frac{3}{p}}_{2, 1}(\mathbb{R}^3))$.

Moreover, there holds

$$X(t) + \int_0^t Y(s) ds \leq C X_0, \quad \text{for } t > 0,$$

(1.7)

where, $C$ is a generic constant,

$$X(t) \overset{\text{def}}{=} \|(a, u, \theta)\|_{\dot{B}^{\frac{1}{2}, 1}_2} + ||u||_{\dot{B}^{\frac{3}{p} - 1}_p} + ||a||_{\dot{B}^{\frac{3}{p} - 1}_p} + ||\theta||_{\dot{B}^{\frac{3}{p} - 2}_p},$$

(1.8)

$$Y(t) \overset{\text{def}}{=} \|(a, u, \theta)\|_{\dot{B}^{\frac{5}{2}}_{2, 1}} + ||(a, \theta)||_{\dot{B}^{\frac{3}{p} + 1}_p} + ||u||_{\dot{B}^{\frac{3}{p} + 1}_p}.$$  (1.9)

Theorem 1.2. (Danchin and Xu [16]) In addition to the conditions of Theorem 1.1, assume that there exists a small constant $\varepsilon > 0$ such that

$$\|(a_0, u_0, \theta_0)\|_{\dot{B}^{\frac{1}{2}, 1}_2} \leq \varepsilon,$$

(1.10)

for

$$\frac{3}{2} - \frac{6}{p} < \sigma \leq -\frac{1}{2}.$$  (1.11)

Then, for $t \geq 0$ and $\sigma \leq s \leq \frac{5}{2}$,

$$\langle t \rangle^{\frac{\sigma - p}{2}} \|(a, u, \theta)(t)\|_{\dot{B}^{\frac{1}{2}, 1}_2} + \langle t \rangle^{2 - \sigma} ||a(t)||_{\dot{B}^{\frac{3}{p} + 1}_p}$$

$$+ \langle t \rangle^{2 - \sigma} ||u(t)||_{\dot{B}^{\frac{3}{p} - 1}_p} + \langle t \rangle^{2 - \sigma} ||\theta(t)||_{\dot{B}^{\frac{3}{p} - 2}_p} + t^{2 - \sigma} \|\nabla (u, \theta)(t)\|_{\dot{B}^{\frac{3}{p} + 1}_p} \leq C X_0.$$
The present paper can be regarded as a further study on the previous theorem without the additional smallness assumption \((1.10)\). For the compressible Navier-Stokes equations without \((1.10)\) (let \(\theta_0 = 0\)), the optimal time decay has been obtained by Xin and Xu [39]. Now we enlarge the range of \(\sigma\) in \((1.11)\) and remove \((1.10)\) for the compressible Navier-Stokes equations involving heat-conductive gases. Compared to the compressible Navier-Stokes equations discussed in [39], the equations \((1.4)\) contain the stronger nonlinear terms \(I(a)\Delta \theta\) and \(I(a)|D(u)|^2\), which lead to analysis difficulties.

The main result of the present paper reads as follows.

**Theorem 1.3.** In addition to the conditions of Theorem 1.1, assume that

\[
2 \leq p < 3, \quad \frac{3}{2} - \frac{6}{p} < \sigma < \frac{1}{2}, \quad \frac{3}{p} - \frac{3}{2} + \sigma \leq \alpha \leq \frac{3}{p} - 1, \quad \frac{3}{p} - \frac{3}{2} + \sigma \leq \beta \leq \frac{3}{p} - 2,
\]

the initial data \((a_0^l, u_0^l, \theta_0^l) \in \dot{B}^{\sigma}_{2,1}(\mathbb{R}^3)\). Then the decay rate estimates

\[
\|\Lambda^\alpha (a, u)\|_{L^p} \leq C(1+t)^{-\frac{3}{2} \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{\alpha - \sigma}{2}} \quad \text{and} \quad \|\Lambda^\beta \theta\|_{L^p} \leq C(1+t)^{-\frac{3}{2} \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{\beta - \sigma}{2}}
\]

hold true for a positive constant \(C\) and the positive differential operator \(\Lambda^s = \mathcal{F}^{-1} |\xi|^s \mathcal{F}\) with \(\mathcal{F}\) the Fourier transformation.

**Remark 1.4.** Compared with Theorem 1.2 on the decay rates, Theorem 1.3 does not need the smallness assumption \((1.10)\). Moreover, the \(\sigma\) range is larger than that in \((1.11)\) and thus more decay properties can be derived.

2. Preliminaries

For convenience, we use the symbols \(\|(a, b)\|_X = \|a\|_X + \|b\|_X\) and \(F \lesssim G\), which represents the inequality \(F \leq CG\) for a generic constant \(C\).

In this section, we recall some basic facts on Littlewood-Paley theory (see [1] for instance). Let \(\chi \geq 0\) and \(\varphi \geq 0\) be two smooth radial functions so that the support of \(\chi\) is contained in the ball \(\{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\}\), the support of \(\varphi\) is contained in the annulus \(\{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}\) and

\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \neq 0.
\]
Let $\mathcal{F}$ be the Fourier transform. The homogeneous dyadic blocks $\hat{\Delta}_j$ and the low-frequency cutoff operators $\hat{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$
\hat{\Delta}_j u = \mathcal{F}^{-1}(\phi(2^{-j} \cdot) \mathcal{F} u), \quad \hat{S}_j u = \mathcal{F}^{-1}(\chi(2^{-j} \cdot) \mathcal{F} u).
$$

Let us remark that, for any homogeneous function $A$ of order 0 smooth outside 0, we have

$$
\forall p \in [1, \infty], \quad \|\hat{\Delta}_j (A(D)f)\|_{L^p} \leq C \|\hat{\Delta}_j f\|_{L^p}.
$$

Denote by $\mathcal{S}_h'(\mathbb{R}^3)$ the space of tempered distributions subject to the condition

$$
\lim_{j \to -\infty} \hat{S}_j u = 0.
$$

Then we have the decomposition

$$
u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \quad \forall u \in \mathcal{S}_h'(\mathbb{R}^3).
$$

We recall the definition of homogeneous Besov spaces.

**Definition 2.1.** For $1 \leq p, r \leq \infty$ and $-\infty < s < \infty$, we set

$$
\dot{B}^s_{p,r}(\mathbb{R}^3) = \left\{ u \in \mathcal{S}_h'(\mathbb{R}^3) \mid \|u\|_{\dot{B}^s_{p,r}} < \infty \right\},
$$

where

$$
\|u\|_{\dot{B}^s_{p,r}} = \left( \sum_{j \in \mathbb{Z}} (2^{js} \|\hat{\Delta}_j u\|_{L^p})^r \right)^{\frac{1}{r}}.
$$

Let us now state some Besov space properties to be used repeatedly in this paper.

**Lemma 2.2.** (a) For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, then there holds the embedding

$$
\|u\|_{\dot{B}^s_{p,1}} \lesssim \|\nabla u\|_{\dot{B}^{s-1}_{p,1}} \lesssim \|u\|_{\dot{B}^s_{p,1}} \quad \text{for} \quad u \in \dot{B}^s_{p,1}(\mathbb{R}^3).
$$

(b) For $1 \leq p \leq \infty$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 > s_2$, then there holds

$$
\|u^\ell\|_{\dot{B}^{s_1}_{p,1}} \lesssim \|u^\ell\|_{\dot{B}^{s_2}_{p,1}}, \quad \|u^h\|_{\dot{B}^{s_1}_{p,1}} \lesssim \|u^h\|_{\dot{B}^{s_2}_{p,1}} \quad \text{for} \quad u \in \dot{B}^{s_1}_{p,1}(\mathbb{R}^3) \cap \dot{B}^{s_2}_{p,1}(\mathbb{R}^3).
$$

(c) For $s \in \mathbb{R}$ and $1 \leq p < q \leq \infty$, then we have the embedding

$$
\dot{B}^s_{p,1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s_{\frac{3}{p} - \frac{3}{q}}}_{q,1}(\mathbb{R}^3).
$$
The following Bernstein’s lemma will be repeatedly used throughout this paper:

**Lemma 2.3.** (see [1]) Let $B$ be a ball and $C$ an annulus of $\mathbb{R}^3$ centered at the origin. For an integer $0 \leq k \leq 2$ and reals $1 \leq p \leq q \leq \infty$, there hold

$$
\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \lesssim \sigma^k \|u\|_{L^p} \quad \text{for \ supp \ } \mathcal{F}u \subset \sigma B,
$$

$$
\sigma^k \|u\|_{L^p} \lesssim \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \lesssim \sigma^k \|u\|_{L^p} \quad \text{for \ supp \ } \mathcal{F}u \subset \sigma C
$$

with respect to scaling parameter $\sigma > 0$.

**Lemma 2.4.** (\cite{1}) For $0 < s_1 < s_2$, $0 \leq \theta \leq 1$ and $1 \leq p \leq \infty$, there hold the following interpolation inequality

$$
\|u\|_{\dot{B}_{sp,1}^\theta} \lesssim \|u\|_{\dot{B}_{sp,1}^{1-\theta}} \|v\|_{\dot{B}_{sp,1}^{1-\theta}} \quad \text{with \ } s = \theta s_1 + (1-\theta) s_2.
$$

The Bony decomposition is very effective in the estimate of nonlinear terms in fluid motion equations. Here, we recall the decomposition in the homogeneous context:

$$
uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),
$$

where

$$
\dot{T}_u v \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \Delta_j v \quad \text{and} \quad \dot{R}(u,v) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v \quad \text{with} \quad \tilde{\Delta}_j \overset{\text{def}}{=} \sum_{|j-j'| \leq 1} \Delta_j v.
$$

The paraproduct $\dot{T}$ and the remainder $\dot{R}$ operators satisfy the following continuous properties.

**Lemma 2.5.** (see [1]) For $s, t \in \mathbb{R}$, $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, there hold the inequalities

$$
\|\dot{T}_u v\|_{\dot{B}_{sp_1,1}^{s+t}} \lesssim \|u\|_{\dot{B}_{sp_1,1}^{s}} \|v\|_{\dot{B}_{sp_2,1}^{s}}, \quad s \leq 0,
$$

$$
\|\dot{R}(u,v)\|_{\dot{B}_{sp_1,1}^{s+t}} \lesssim \|u\|_{\dot{B}_{sp_1,1}^{s}} \|v\|_{\dot{B}_{sp_2,1}^{s}}, \quad s + t > 0,
$$

$$
\|\dot{R}(u,v)\|_{\dot{B}_{sp_1,\infty}^{s+t}} \lesssim \|u\|_{\dot{B}_{sp_1,1}^{s}} \|v\|_{\dot{B}_{sp_2,\infty}^{s+t}}.
$$

The following product law plays central roles when we estimate the couple terms appeared in the equations:
Lemma 2.6. ([15, Proposition A.1]) Assume that $1 \leq p, q \leq \infty$,

$$s_1 \leq \frac{3}{q}, \quad s_2 \leq 3 \min \left\{ \frac{1}{p}, \frac{1}{q} \right\} \quad \text{and} \quad s_1 + s_2 > 3 \max \left\{ 0, \frac{1}{p} + \frac{1}{q} - 1 \right\}.$$

Then we have, for $(f, g) \in \dot{B}^{s_1}_{q,1}(\mathbb{R}^3) \times \dot{B}^{s_2}_{p,1}(\mathbb{R}^3)$,

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{3}{q}}_{p,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{q,1}} \|g\|_{\dot{B}^{s_2}_{p,1}}.$$

In order to deal with composition functions in the Besov spaces, we also need the following proposition:

Proposition 2.7. (see [1]) For $\alpha < 0 < \beta$, let $G$ be a smooth function defined on the open interval $(\alpha, \beta)$ so that $G(0) = 0$. Assume that $f : \mathbb{R} \mapsto I \subset (\alpha, \beta)$ for an interval $I$. Then we have the estimate

$$\|G(f)\|_{\dot{B}^s_{p,1}} \lesssim \|f\|_{\dot{B}^s_{p,1}} \quad \text{for} \quad 1 \leq p \leq \infty, \quad s > 0. \quad (2.1)$$

3. The proof of Theorem 1.3

The proof is to show the decay estimate of the solution $(a, u, \vartheta)$ given in Theorem [1.1]. This decay estimate is simply derived from a Lyapunov-type differential inequality for the solution. This inequality lies on the viability of the uniform bound

$$\|(a, u, \vartheta)\|_{\dot{B}^\sigma_{2,1}} \leq C \quad \text{for} \quad \sigma < \frac{1}{2} \quad (3.1)$$

This bound can be derived from an energy estimate of (1.4). To show the energy estimate, we start with the following energy estimate the linearized equation system of (1.4).

3.1. Low frequency energy estimate of the linearized system of (1.4)

For $(a, u, \vartheta)$ the smooth solution to the following linearized system of (1.4):

$$\begin{aligned}
\partial_t a + \text{div} u &= f_1, \\
\partial_t u - \Delta u + \nabla(a + \vartheta) &= f_2, \\
\partial_t \vartheta - \Delta \vartheta + \text{div} u &= f_3, \\
(a, u, \vartheta)|_{t=0} &= (a_0, u_0, \vartheta_0),
\end{aligned} \quad (3.2)$$
we show the energy estimate, for $\gamma \in \mathbb{R}$,

$$\|(a, u, \vartheta)\|_{B^{2,1}_2}^\gamma + \int_0^t \|(a, u, \vartheta)\|_{B^{2,2}_2}^\gamma \, ds \lesssim \|(a_0, u_0, \vartheta_0)\|_{B^{2,1}_2}^\gamma + \int_0^t \|(f_1, f_2, f_3)\|_{B^{2,1}_2}^\gamma \, ds. \quad (3.3)$$

Indeed, let $\omega = \Lambda^{-1} \text{curl} \, u$ be the incompressible part of $u$ and $v = \Lambda^{-1} \text{div} \, u$ be the compressible part of $u$. We see that $\omega$ satisfies the heat equation:

$$\partial_t \omega - \Delta \omega = \Lambda^{-1} \text{curl} \, f_2, \quad \omega(0) = \omega_0. \quad (3.4)$$

A standard energy argument applied to (3.4) implies

$$\|(\omega)\|_{B^{2,1}_2}^\gamma + \int_0^t \|(\omega)\|_{B^{2,2}_2}^\gamma \, ds \lesssim \|(\omega_0)\|_{B^{2,1}_2}^\gamma + \int_0^t \|(f_2)\|_{B^{2,1}_2}^\gamma \, ds. \quad (3.5)$$

On the other hand, it is easy to check that $(\Delta_j a, \Delta_j v, \Delta_j \vartheta)$ satisfies the equations

$$\begin{cases}
\partial_t \Delta_j a + \Lambda \Delta_j v = \hat{\Delta}_j f_1, \\
\partial_t \Delta_j v - \Delta \Delta_j v - \Delta_j \Lambda(a + \vartheta) = \Lambda^{-1} \Delta_j \text{div} \, f_2, \\
\partial_t \Delta_j \vartheta - \Delta \Delta_j \vartheta + \Lambda \Delta_j v = \hat{\Delta}_j f_3.
\end{cases} \quad (3.6)$$

Taking the $L^2$ inner product of (3.6) with $(\Delta_j a, \Delta_j v, \Delta_j \vartheta)$, and using the following cancellation

$$\int_{\mathbb{R}^3} \Delta_j \Lambda v \cdot \Delta_j a \, dx + \int_{\mathbb{R}^3} \Delta_j \Lambda a \cdot \Delta_j \vartheta \, dx - \int_{\mathbb{R}^3} \Delta_j \Lambda (a + \vartheta) \cdot \Delta_j v \, dx = 0,$$

we get

$$\frac{1}{2} \frac{d}{dt} \left( \|\Delta_j a\|_{L^2}^2 + \|\Delta_j v\|_{L^2}^2 + \|\Delta_j \vartheta\|_{L^2}^2 \right) + \|\Delta_j \Lambda v\|_{L^2}^2 + \|\Delta_j \Lambda \vartheta\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} \hat{\Delta}_j f_1 \cdot \Delta_j a \, dx + \int_{\mathbb{R}^3} \hat{\Delta}_j \Lambda^{-1} \text{div} \, f_2 \cdot \Delta_j v \, dx + \int_{\mathbb{R}^3} \hat{\Delta}_j f_3 \cdot \Delta_j \vartheta \, dx. \quad (3.7)$$

Applying $\Lambda$ to the first equation of (3.6) gives

$$\partial_t \hat{\Delta}_j \Lambda a - \Delta_j \Lambda v = \hat{\Delta}_j \Lambda f_1. \quad (3.8)$$

Taking the $L^2$ inner product of (3.8) with $\hat{\Delta}_j \Lambda a$ implies

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j \Lambda a\|_{L^2}^2 - \int_{\mathbb{R}^3} \Delta_j \Lambda v \cdot \Delta_j \Lambda a \, dx = \int_{\mathbb{R}^3} \Delta_j \Lambda f_1 \cdot \Delta_j \Lambda a \, dx. \quad (3.9)$$
To find the hidden dissipation of \( a \) in the low frequency, we get by testing the second equation of (3.6) by \( \dot{\Delta_j} a \) and (3.8) by \( \dot{\Delta_j} v \) respectively that

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} \dot{\Delta_j} v \cdot \dot{\Delta_j} a \, dx \right) + \| \dot{\Delta_j} a \|^2_{L^2} - \| \dot{\Delta_j} a \|^2_{L^2} - \int_{\mathbb{R}^3} \dot{\Delta_j} a \cdot \dot{\Delta_j} a \, dx = \int_{\mathbb{R}^3} \dot{\Delta_j} f_1 \cdot \dot{\Delta_j} a \, dx + \int_{\mathbb{R}^3} \dot{\Delta_j} a \cdot \dot{\Delta_j} a \, dx. \tag{3.10}
\]

Multiplying (3.7) by 2 and (3.10) by \(-1\) respectively, and summing up the resultant equations with (3.9), we have the energy equality involving the dissipation for \( a \):

\[
\frac{1}{2} \frac{d}{dt} \mathcal{L}^2_j + 2 \| \dot{\Delta_j} a \|^2_{L^2} + 2 \| \dot{\Delta_j} \theta \|^2_{L^2} + \int_{\mathbb{R}^3} \dot{\Delta_j} a \cdot \dot{\Delta_j} a \, dx
\]

\[
= 2 \int_{\mathbb{R}^3} \dot{\Delta_j} f_1 \cdot \dot{\Delta_j} a \, dx + 2 \int_{\mathbb{R}^3} \dot{\Delta_j} a \cdot \dot{\Delta_j} v \, dx + 2 \int_{\mathbb{R}^3} \dot{\Delta_j} f_3 \cdot \dot{\Delta_j} \theta \, dx
\]

\[
- \int_{\mathbb{R}^3} \dot{\Delta_j} f_1 \cdot \dot{\Delta_j} v \, dx - \int_{\mathbb{R}^3} \dot{\Delta_j} a \cdot \dot{\Delta_j} a \, dx + \int_{\mathbb{R}^3} \dot{\Delta_j} f_1 \cdot \dot{\Delta_j} a \, dx, \tag{3.11}
\]

where

\[
\mathcal{L}_j \overset{\text{def}}{=} \left( 2 \| \dot{\Delta_j} a \|^2_{L^2} + 2 \| \dot{\Delta_j} a \|^2_{L^2} + 2 \| \dot{\Delta_j} \theta \|^2_{L^2} - 2 \int_{\mathbb{R}^3} \dot{\Delta_j} v \cdot \dot{\Delta_j} a \, dx \right)^{\frac{1}{2}}.
\]

It readily seen that the equivalence

\[
\mathcal{L}^2_j \lesssim \| \Delta_j a \|^2_{L^2} + \| \dot{\Delta_j} v \|^2_{L^2} + \| \dot{\Delta_j} \theta \|^2_{L^2} \lesssim \mathcal{L}^2_j \tag{3.12}
\]

holds true for low frequency components \((j \leq j_0)\). Hence, by Young inequality, Hölder inequality and Bernstein inequalities, we have

\[
\frac{1}{2} \frac{d}{dt} \mathcal{L}^2_j + 2^{2j} \mathcal{L}^2_j \lesssim \| (\Delta_j f_1, \Delta_j f_2, \dot{\Delta_j} f_3) \|_{L^2} \mathcal{L}_j, \quad j < j_0, \tag{3.13}
\]

which implies that

\[
\| (\Delta_j a, \dot{\Delta_j} v, \dot{\Delta_j} \theta) \|_{L^2} + 2^{2j} \int_0^t \| (\Delta_j a, \dot{\Delta_j} v, \dot{\Delta_j} \theta) \|_{L^2} \, ds
\]

\[
\lesssim \| (\Delta_j a_0, \Delta_j v_0, \Delta_j \theta_0) \|_{L^2} + \int_0^t \| (\Delta_j f_1, \Delta_j f_2, \dot{\Delta_j} f_3) \|_{L^2} \, ds, \quad j \leq j_0. \tag{3.14}
\]

Multiplying (3.14) by \(2^{\gamma j}\) and summing up the resultant inequalities with respect to \(j \leq j_0\), we have

\[
\| (a, v, \theta) \|_{B_{2,1}^{\gamma}} + \int_0^t \| (a, v, \theta) \|_{B_{2,1}^{\gamma+2}} ds \lesssim \| (a_0, v_0, \theta_0) \|_{B_{2,1}^{\gamma}} + \int_0^t \| (f_1, f_2, f_3) \|_{B_{2,1}^{\gamma}} ds \tag{3.15}
\]

which combined with (3.5) and (3.15) gives the desired the low frequencies estimate (3.3).
3.2. Nonlinear estimates for showing the uniform boundedness

As \((a, u, \theta)\) is the global small solution of \((1.4)\) given in Theorem 1.1 and \((3.2)\) is the linearized equation system of \((1.4)\), the application of \((3.3)\) to \((1.4)\) with \(\gamma = \sigma\) for \(\frac{3}{2} - \frac{6}{p} < \sigma < \frac{1}{2}\) gives

\[
\| (a, u, \theta) \|_{B^{s}_{2,1}} + \int_{0}^{t} \| (a, u, \theta) \|_{B^{s}_{2,1}}^{2} \, ds \\
\lesssim \| (a_{0}, u_{0}, \theta_{0}) \|_{B^{s}_{2,1}} + \int_{0}^{t} \| u \cdot \nabla u \|_{B^{s}_{2,1}} \, ds + \int_{0}^{t} \| \nabla (\partial u) \|_{B^{s}_{2,1}} \, ds + \int_{0}^{t} \| \div (a u) \|_{B^{s}_{2,1}} \, ds + \int_{0}^{t} \| \partial \nabla I(a) \|_{B^{s}_{2,1}} \, ds
\]

\[
+ \int_{0}^{t} \| I(a) A u \|_{B^{s}_{2,1}} \, ds + \int_{0}^{t} \| \nabla (\partial u) \|_{B^{s}_{2,1}} \, ds + \int_{0}^{t} \| (1 + I(a)) (2 |Du|^{2} + (\div u)^{2}) \|_{B^{s}_{2,1}} \, ds. \\
\text{(3.16)}
\]

To deal with the nonlinear terms on the right-hand side of the previous equation, we need the following product laws:

\[
\| fg \|_{B^{s}_{2,1}} \lesssim \| f \|_{B^{s}_{2,1}} \| g \|_{B^{\frac{3}{p}-1}_{p,1}}, \quad \frac{3}{p} < \sigma \leq \frac{3}{p}, \quad 2 \leq p < 3,
\]

\[
\| fg^{h} \|_{B^{s}_{2,1}} \lesssim \| f \|_{B^{\frac{3}{p}-1}_{p,1}} \| g^{h} \|_{B^{\frac{3}{p}-1}_{p,1}}, \quad \frac{3}{2} - \frac{6}{p} < \sigma \leq \frac{3}{p}, \quad 2 \leq p < 3.
\]

The first one is given by Lemma 2.6. To prove the second one, we use Bony’s decomposition:

\[
f g^{h} = T_{f} g^{h} + \tilde{R}(g^{h}, f) + T_{f} g^{h}.
\]

Applying Lemma 2.5 and the condition \(1 - \frac{3}{p} < 0\) implies that

\[
\| T_{f} g^{h} \|_{B^{s}_{2,1}} \lesssim \| T_{f} g^{h} \|_{B^{\frac{3}{p}-1}_{p,1}} \lesssim \| g^{h} \|_{B^{\frac{3}{p}-1}_{p,1}} \| f \|_{B^{\frac{3}{p}-1}_{p,1}} \lesssim \| g^{h} \|_{B^{\frac{3}{p}-1}_{p,1}} \| f \|_{B^{\frac{3}{p}-1}_{p,1}}
\]

\[
\text{(3.19)}
\]

where we have used the high frequency property of \(g^{h}\) and the fact \(1 - \frac{3}{p} < \frac{3}{p} - 1\) in the last inequality. Similarly, by using the low frequency property and the condition \(\frac{3}{2} - \frac{6}{p} < \sigma < \frac{1}{2}\), the rest term can be estimated from Lemma 2.5 that

\[
\| \tilde{R}(g^{h}, f) \|_{B^{s}_{2,1}} \lesssim \| \tilde{R}(g^{h}, f) \|_{B^{\frac{3}{p}-1}_{p,1}} \lesssim \| R(g^{h}, f) \|_{B^{0}_{p/2,\infty}}
\]

\[
\leq \| g^{h} \|_{B^{\frac{3}{p}-1}_{p,1}} \| f \|_{B^{\frac{3}{p}-1}_{p,1}} \lesssim \| g^{h} \|_{B^{\frac{3}{p}-1}_{p,1}} \| f \|_{B^{\frac{3}{p}-1}_{p,1}}.
\]

\[
\text{(3.20)}
\]
Moreover, with the aid of the H"older inequality and Bernstein’s inequality, we have

$$
\| T_f \S^h \|_{B^1_{2,1}} \lesssim \sum_{j \leq j_0, |j-k| \leq 4} 2^{j(\frac{3}{2} - \frac{6}{p})} \| \Delta_j (S_{k-1} f \Delta_k \S^h) \|_{L^2} \\
\lesssim \sum_{j \leq j_0, |j-k| \leq 4} 2^{j(\frac{3}{2} - \frac{6}{p})} \| \Delta_k f \|_{L^p} \| \Delta_k \S^h \|_{L^p} \\
\lesssim \sum_{j \leq j_0, |j-k| \leq 4} 2^{j(\frac{3}{2} - \frac{6}{p})} (\sum_{k' \leq k-2} \| \Delta_{k'} f \|_{L^p} \| \Delta_k \S^h \|_{L^p})
$$

which together with a similar derivation of (3.19) yields

$$
\| T_f \S^h \|_{B^1_{2,1}} \lesssim \sum_{j \leq j_0, |j-k| \leq 4} 2^{j(\frac{3}{2} - \frac{6}{p})} (\sum_{k' \leq k-2} 2^{k(1 - \frac{3}{p})} \| \Delta_k \S^h \|_{L^p})
$$

The combination of (3.19), (3.20) and (3.21) yields (3.18).

In estimating the nonlinear items of (3.16) by using (3.17), (3.18) and the decomposition $u = u^\ell + u^h$, we find that

$$
\| u \cdot \nabla u \|_{B^1_{2,1}} \lesssim \| u^\ell \cdot \nabla u^\ell \|_{B^1_{2,1}} + \| u^h \cdot \nabla u^h \|_{B^1_{2,1}} + \| u \cdot \nabla u^h \|_{B^1_{2,1}} \\
\lesssim (\| \nabla u^\ell \|_{B^3_{p,1}} + \| u^h \|_{B^3_{1,1}}) \| \nabla u^\ell \|_{B^3_{2,1}} + (\| u^\ell \|_{B^3_{1,1}} + \| u^h \|_{B^3_{1,1}}) \| \nabla u^h \|_{B^3_{1,1}}.
$$

By using Bernstein’s estimate and the properties of low and high frequencies, the previous estimate becomes

$$
\| u \cdot \nabla u \|_{B^1_{2,1}} \lesssim \left( \| u \|_{B^\frac{5}{2}_{2,1}} + \| u^h \|_{B^3_{1,1}} \right) \| u^\ell \|_{B^3_{2,1}} + \left( \| u \|_{B^\frac{1}{2}_{2,1}} + \| u^h \|_{B^3_{1,1}} \right) \| u \|_{B^3_{1,1}} \\
\lesssim \mathcal{U}(t) \| u \|_{B^\frac{5}{2}_{2,1}} + \mathcal{X}(t) \mathcal{V}(t),
$$

for $\mathcal{U}(t)$ and $\mathcal{X}(t)$ defined in (1.9) and (1.6).

By (3.17), Lemma 2.3 and the properties of low and high frequencies, we estimate the following nonlinear terms

$$
\| u \cdot \nabla \theta^\ell \|_{B^1_{2,1}} + \| \nabla u^\ell \|_{B^1_{2,1}} \\
\lesssim \left( \| u \|_{B^\frac{5}{2}_{2,1}} + \| u^h \|_{B^3_{1,1}} \right) \| u^\ell \|_{B^3_{2,1}} + \| u \|_{B^\frac{5}{2}_{2,1}} \| \nabla \theta^\ell \|_{B^3_{2,1}} + \| \theta^h \|_{B^3_{1,1}} \| \nabla u^\ell \|_{B^3_{2,1}} \\
\lesssim \left( \| \theta \|_{B^\frac{5}{2}_{2,1}} + \| u^\ell \|_{B^3_{2,1}} + \| u^h \|_{B^3_{1,1}} \right) \left( \| u^\ell \|_{B^3_{2,1}} + \| u \|_{B^\frac{5}{2}_{2,1}} \| \theta^\ell \|_{B^3_{2,1}} + \| \theta^h \|_{B^3_{1,1}} \| u \|_{B^3_{2,1}} \right).
$$

(3.23)
By (3.18) and Lemma 2.4

\[ \|u \cdot \nabla^h \|_{B^\epsilon_{2,1}}^\ell + \| \nabla \operatorname{div} u^h \|_{B^\epsilon_{2,1}}^\ell \leq (\|u\|_{B^\epsilon_{2,1}}^{3/2} + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla^h \|_{B^\epsilon_{2,1}} + (\|u\|_{B^\epsilon_{2,1}}^{3/2} + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla \operatorname{div} u^h \|_{B^\epsilon_{2,1}}^\ell) \| \phi \|_{B^\epsilon_{2,1}}^\ell \] 

\[ \leq (\|u\|_{B^\epsilon_{2,1}}^{1/2} + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla^h \|_{B^\epsilon_{2,1}} + \|u\|_{B^\epsilon_{2,1}}^{1/2} + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla \operatorname{div} u^h \|_{B^\epsilon_{2,1}}^\ell) (\|u\|_{B^\epsilon_{2,1}}^{1/2} + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla \operatorname{div} u^h \|_{B^\epsilon_{2,1}}^\ell) \] 

(3.24)

Combining (3.23) and (3.24) with the identity

\[ \nabla \phi = u \cdot \nabla \phi + \phi \nabla u^\ell + u \cdot \nabla^h \phi + \phi \nabla u^h, \] 

(3.25)

gives

\[ \| \nabla \phi \|_{B^\epsilon_{2,1}}^\ell \leq (\| \phi \|_{B^\epsilon_{2,1}}^\ell + \| \phi \|_{B^\epsilon_{2,1}}^\ell + \|u\|_{B^\epsilon_{2,1}}^{1/2} + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla \operatorname{div} u^h \|_{B^\epsilon_{2,1}}^\ell) \] 

\[ + (\| \phi \|_{B^\epsilon_{2,1}}^\ell + \| \phi \|_{B^\epsilon_{2,1}}^\ell + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla \operatorname{div} u^h \|_{B^\epsilon_{2,1}}^\ell) (\|u\|_{B^\epsilon_{2,1}}^{1/2} + \|u\|_{B^\epsilon_{2,1}}^{-1} \| \nabla \operatorname{div} u^h \|_{B^\epsilon_{2,1}}^\ell) \] 

\[ \leq Y(t) \| (\phi, u) \|_{B^\epsilon_{2,1}}^\ell + Y(t) X(t). \] 

(3.26)

Similarly, we have

\[ \| \nabla (au) \|_{B^\epsilon_{2,1}}^\ell \leq Y(t) \| (a, u) \|_{B^\epsilon_{2,1}}^\ell + Y(t) X(t). \] 

(3.27)

Moreover, to deal with the nonlinear term \( I(a) \nabla a \), we have to use the estimate (2.1) in a Besov space \( B^s_{p,1} \) for \( s > 0 \). However, for the Besov space \( B^\epsilon_{2,1} \) in the present estimation, the condition \( \epsilon > 0 \) cannot be guaranteed. Thus in order to make use of Proposition 2.7 or (2.1), we employ the formulation \( I(a) = a - aI(a) \) and argue in the same way as the derivation of (3.22) to obtain that

\[ \| I(a) \nabla a \|_{B^\epsilon_{2,1}}^\ell \leq \| a \|_{B^\epsilon_{2,1}}^\ell \| \nabla a^\ell \|_{B^\epsilon_{2,1}}^\ell + \| a \|_{B^\epsilon_{2,1}}^\ell \| \nabla a^\ell \|_{B^\epsilon_{2,1}}^\ell \] 

\[ + \| aI(a) \|_{B^\epsilon_{2,1}}^\ell \| a \|_{B^\epsilon_{2,1}}^\ell + \| I(a) \|_{B^\epsilon_{2,1}}^\ell \| \nabla a \|_{B^\epsilon_{2,1}}^\ell \] 

\[ \leq (\| a \|_{B^\epsilon_{2,1}}^\ell + \| a \|_{B^\epsilon_{2,1}}^\ell) \| a \|_{B^\epsilon_{2,1}}^\ell + \| a \|_{B^\epsilon_{2,1}}^\ell \| a \|_{B^\epsilon_{2,1}}^\ell + \| a \|_{B^\epsilon_{2,1}}^\ell \| \nabla a \|_{B^\epsilon_{2,1}}^\ell \] 

\[ \leq (\| a \|_{B^\epsilon_{2,1}}^\ell + \| a \|_{B^\epsilon_{2,1}}^\ell) \| a \|_{B^\epsilon_{2,1}}^\ell + \| a \|_{B^\epsilon_{2,1}}^\ell \| a \|_{B^\epsilon_{2,1}}^\ell + \| a \|_{B^\epsilon_{2,1}}^\ell \] 

\[ \leq Y(t)(1 + X(t)) \| a \|_{B^\epsilon_{2,1}}^\ell + X(t) Y(t). \] 

(3.28)
Similarly, we have

\[ \|I(a)Au\|_{B^2_{p,1}} \lesssim (\|u\|_{B^2_{p,1}} + \|a\|_{B^{3/2}_{p,1}}^{1/2} + \|a\|_{B^{1/2}_{p,1}}^{1/2} + (\|a\|_{B^{3/2}_{p,1}}^{1/2})^2) \|(a, u)\|_{B^2_{2,1}} \]

\[ \lesssim \mathcal{Y}(t)\mathcal{X}(t) + \mathcal{Y}(t)(1 + \mathcal{X}(t)) \|(a, u)\|_{B^2_{2,1}}. \quad (3.29) \]

Furthermore, to estimate the nonlinear term \(\partial \nabla J(a)\), we continue to use the argument for the derivation of (3.22) and Lemma 2.4 to obtain

\[ \|\partial \nabla J(a)\|_{B^2_{p,1}} \lesssim \|\partial^\ell \nabla (J(a))^\ell\|_{B^2_{p,1}} + \|\partial^\ell \nabla (J(a))^h\|_{B^2_{p,1}} + \|\partial^h \nabla J(a)\|_{B^2_{p,1}} \]

\[ \lesssim \|\partial^\ell\|_{B^2_{p,1}} \|\nabla (J(a))^\ell\|_{B^2_{p,1}} + \|\partial^\ell\|_{B^2_{p,1}} \|\nabla (J(a))^h\|_{B^2_{p,1}} + \|\nabla J(a)\|_{B^2_{p,1}} \]

\[ \lesssim \|\partial^\ell\|_{B^2_{p,1}} \|a\|_{B^2_{p,1}} + \|\partial^\ell\|_{B^2_{p,1}} \|a\|_{B^2_{p,1}} + \|\partial^h\|_{B^2_{p,1}} \|a\|_{B^2_{p,1}} \]

\[ \lesssim \mathcal{Y}(t)\mathcal{X}(t) + \mathcal{Y}(t)\|\partial\|_{B^2_{p,1}}. \quad (3.30) \]

Next attempt is to consider the nonlinear term \(I(a)\Delta \partial\) involving composition functions and is more elaborate. Its component involving the low frequency \(\partial^\ell\) is estimated as

\[ \|I(a)\Delta \partial^\ell\|_{B^2_{p,1}} \lesssim \|(a + aI(a))\Delta \partial^\ell\|_{B^2_{p,1}} \]

\[ \lesssim \|a^\ell \Delta \partial^\ell\|_{B^2_{p,1}} + \|a^h \Delta \partial^\ell\|_{B^2_{p,1}} + \|aI(a)\Delta \partial^\ell\|_{B^2_{p,1}} \]

\[ \lesssim \|\partial^\ell\|_{B^2_{p,1}} \|a^\ell\|_{B^2_{p,1}} + \|a^h\|_{B^2_{p,1}} \|\partial\|_{B^2_{p,1}} + \|a\|^{2/3}_{B^{1/2}_{p,1}} \|\partial\|_{B^2_{p,1}} \]

\[ \lesssim \mathcal{Y}(t)(1 + \mathcal{X}(t)) \|(a, \partial)\|_{B^2_{p,1}}. \quad (3.31) \]

The analysis on the part \(I(a)\Delta \partial^h\) involving the high frequency \(\partial^h\) is more complicated. We need a new product law different to (3.17) and (3.18). To do so, we use the Bony decomposition

\[ I(a)\Delta \partial^h = T_{\Delta \partial^h}I(a) + R(\Delta \partial^h, I(a)) + T_{I(a)} \Delta \partial^h \quad (3.32) \]

and Lemmas 2.3 and 2.5 to estimate the first two items of right-hand side of the previous
Hence, it follows from Lemma 2.3 that
\[
\| T_{\Delta \vartheta_h} I(a) + R(\Delta \vartheta^h, I(a)) \|_{B^\gamma_{2,1}} \lesssim \| T_{\Delta \vartheta_h} I(a) + R(\Delta \vartheta^h, I(a)) \|_{B^\gamma_{2,\infty}}^{\frac{3}{2} - \frac{6}{p}} \\
\lesssim \| T_{\Delta \vartheta_h} I(a) + R(\Delta \vartheta^h, I(a)) \|_{B^0_{p/2,\infty}} \\
\lesssim \| I(a) \|_{B^{-3/2}_{p,1}} \| \Delta \vartheta^h \|_{B^{3/2}_{p,1}}^{\frac{3}{2} - \frac{2}{p}} \\
\lesssim \| a \|_{B^{-3/2}_{p,1}} \| \Delta \vartheta^h \|_{B^{3/2}_{p,1}}^{\frac{3}{2} - \frac{2}{p}} \\
\lesssim (\| a^\ell \|_{B^{2}_{2,1}} + \| a^h \|_{B^{3}_{p,1}}) \| \Delta \vartheta^h \|_{B^{3/2}_{p,1}}^{\frac{3}{2} - \frac{2}{p}}. \tag{3.33}
\]

For the last term on the right-hand side of (3.32), we notice that
\[
\dot{\Delta}_k \Delta \vartheta^h \equiv 0 \text{ for } k < j_0 - 1, \\
\dot{\Delta}_j \dot{\Delta}_k \Delta \vartheta^h \equiv 0 \text{ for } |k - j| > 4.
\]

Hence, it follows from Lemma 2.3 that
\[
\| T_{I(a)} \Delta \vartheta^h \|_{B^\gamma_{2,1}}^{\frac{3}{2} - \frac{6}{p}} \lesssim \sum_{j \leq j_0} 2^{j(\frac{3}{2} - \frac{6}{p})} \| \Delta_j \left( \sum_{k \geq j_0 - 1} \dot{\Delta}_k I(a) \Delta_k \Delta \vartheta^h \right) \|_{L^2} \\
\lesssim \sum_{j \leq j_0} 2^{j(\frac{3}{2} - \frac{6}{p})} \sum_{j_0 - 1 \leq k \leq j_0 + 4} \| \dot{\Delta}_j \dot{\Delta}_{k-1} I(a) \Delta_k \Delta \vartheta^h \|_{L^2} \\
\lesssim \sum_{j \leq j_0} 2^{j(\frac{3}{2} - \frac{6}{p})} \sum_{j_0 - 1 \leq k \leq j_0 + 4} \left( \sum_{k' \leq k - 2} \| \Delta_{k'} I(a) \|_{L^2} \right) \| \Delta_k \Delta \vartheta^h \|_{L^p} \\
\lesssim \sum_{j \leq j_0} 2^{j(\frac{3}{2} - \frac{6}{p})} \sum_{j_0 - 1 \leq k \leq j_0 + 4} \left( \sum_{k' \leq k - 2} 2^{3p/2} \| \Delta_{k'} I(a) \|_{L^2} \right) \| \Delta_k \Delta \vartheta^h \|_{L^p} \\
\lesssim \| I(a) \|_{B^\gamma_{2,1}}^{\frac{1}{2}} \sum_{j \leq j_0} 2^{j(1 - \frac{3}{2})} \sum_{j_0 - 1 \leq k \leq j_0 + 4} \| \dot{\Delta}_k \Delta \vartheta^h \|_{L^p} \\
\lesssim \| a \|_{B^\gamma_{2,1}}^{\frac{1}{2}} \| \Delta \vartheta^h \|_{B^{\frac{3}{2}}_{p,1}}^{\frac{3}{2} - \frac{2}{p}}.
\]

Hence, we have the product law
\[
\| I(a) \Delta \vartheta^h \|_{B^\gamma_{2,1}}^{\frac{3}{2} - \frac{6}{p}} \lesssim (\| a \|_{B^\gamma_{2,1}}^{\frac{3}{2} - \frac{6}{p}} + \| a \|_{B^\gamma_{p,1}}^{\frac{3}{2} - \frac{6}{p}}) \| \vartheta \|_{B^{\frac{3}{2}}_{p,1}} \lesssim \mathcal{Y}(t) \mathcal{X}(t). \tag{3.34}
\]

Therefore, by (3.31) and (3.34), we have
\[
\| I(a) \Delta \vartheta \|_{B^\gamma_{2,1}}^{\frac{3}{2} - \frac{6}{p}} \lesssim \mathcal{Y}(t) \mathcal{X}(t) + \mathcal{Y}(t)(1 + \mathcal{X}(t)) \| \vartheta \|_{B^{\frac{3}{2}}_{p,1}}^{\frac{3}{2} - \frac{6}{p}}. \tag{3.35}
\]
Finally, we consider the last nonlinear term in (3.16). As in the derivation of (3.22), we use (3.17) and (3.18) to obtain that

$$
\|(1 + I(a))(2|Du|^2 + (\text{div } u)^2)\|_{B_{\xi,1}^\ell}^{\ell} \leq \|(1 + I(a))\frac{3}{B_{p,1}}\|_{B_{\xi,1}^\ell} + \|\nabla u\|_{B_{\xi,1}^\ell}^{\ell} \leq (1 + \|a\|_{B_{\xi,1}^\ell}^{\ell} + \|u\|_{B_{\xi,1}^\ell}^{\ell}) \|(1 + \|a\|_{B_{\xi,1}^\ell}^{\ell} + \|u\|_{B_{\xi,1}^\ell}^{\ell})\|_{B_{\xi,1}^\ell}^{\ell} \leq (1 + \mathcal{X}(t))\mathcal{Y}(t)\mathcal{X}(t) + \mathcal{Y}(t)(1 + \mathcal{X}(t))\|u\|_{B_{\xi,1}^\ell}^{\ell}.
$$

(3.36)

Inserting (3.22), (3.26), (3.27), (3.29), (3.30), (3.33) and (3.36) into (3.16), we can get

$$
\|(a, u, \vartheta)\|_{B_{\xi,1}^\ell}^{\ell} + \int_0^t \|(a, u, \vartheta)\|_{B_{\xi,1}^{\ell+2}}^{\ell+2} ds \leq \|(a_0, u_0, \vartheta_0)\|_{B_{\xi,1}^\ell}^{\ell} + \int_0^t \mathcal{Y}(s)(1 + \mathcal{X}(s))\mathcal{Y}(s)\mathcal{X}(s) ds + \int_0^t \mathcal{Y}(s)(1 + \mathcal{X}(s))\|a, u, \vartheta\|_{B_{\xi,1}^{\ell+1}}^{\ell+1} ds.
$$

(3.37)

From Theorem 1.1, we deduce that

$$
\int_0^t \mathcal{Y}(s)(1 + \mathcal{X}(s)) ds + \int_0^t (1 + \mathcal{X}(s))\mathcal{Y}(s)\mathcal{X}(s) ds \leq (1 + \mathcal{X}_0)^2 \mathcal{X}_0.
$$

(3.38)

Hence, by the Gronwall inequality and the condition \(\frac{3}{2} - \frac{6}{p} < \sigma \leq \frac{3}{p}\), we obtain from (3.37) the desired uniform bound (3.1) or

$$
\|(a, u, \vartheta)\|_{B_{\xi,1}^\ell}^{\ell} \leq C
$$

(3.39)

for \(t \geq 0\), where constant \(C > 0\) is dependent of \(\mathcal{X}_0\) and \(\|(a_0, u_0, \vartheta_0)\|_{B_{\xi,1}^\ell}^{\ell}\).

### 3.3. Lyapunov-type differential inequality

We show the following Lyapunov-type differential inequality:

$$
\frac{d}{dt}\mathcal{X}(t) + C\mathcal{X}^{\frac{n-2p}{1-2p}}(t) \leq 0, \quad t > 0
$$

(3.40)

for a constant \(C > 0\).
Indeed, a similar proof of [39, Lemmas 4.1, 4.2] implies the following inequality
\[
\frac{d}{dt}(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{1}} + \|a\|^{h}_{B_{p,1}^{\frac{1}{2}}} + \|u\|^{h}_{B_{p,1}^{\frac{3}{2}}}) + C\left(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{1}} + \|(a, \vartheta)\|^{h}_{B_{p,1}^{\frac{3}{2}}} + \|u\|^{h}_{B_{p,1}^{\frac{3}{2}}}ight) \leq 0
\] (3.41)

or
\[
\frac{d}{dt}X(t) + CY(t) \leq 0, \quad t > 0. \tag{3.42}
\]

Now, for any \(\frac{5}{2} - \frac{6}{p} < \sigma < \frac{1}{2}\) and \(\eta_1 = \frac{4}{5 - 2\sigma}\), it follows from interpolation inequality in Lemma 2.4 and (3.1) that
\[
\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{1}} \lesssim \left(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{1}}\right)^{\eta_1} \left(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{5}}\right)^{1 - \eta_1} \\
\lesssim \left(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{5}}\right)^{\eta_1} \left(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{5}}\right)^{1 - \eta_1} \lesssim \left(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{5}}\right)^{1 - \eta_1}
\]
or the low frequency part estimate
\[
\left(\|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{1}}\right)^{1 - \eta_1} \lesssim \|(a, u, \vartheta)\|^{\ell}_{B_{2,1}^{5}}. \tag{3.43}
\]

To obtain the corresponding high frequency part estimate, we use the smallness property of \(X\) shown in (1.7) and the high frequency property to produce
\[
\left(\|u\|^{h}_{B_{p,1}^{\frac{3}{2}}} + \|a\|^{h}_{B_{p,1}^{\frac{1}{2}}} + \|\vartheta\|^{h}_{B_{p,1}^{\frac{1}{2}}}\right)^{\frac{1}{1 - \eta_1}} \lesssim \|u\|^{h}_{B_{p,1}^{\frac{3}{2}}} + \|a\|^{h}_{B_{p,1}^{\frac{3}{2}}} + \|\vartheta\|^{h}_{B_{p,1}^{\frac{3}{2}}}. \tag{3.44}
\]

The combination of (3.43) and (3.44) together with the definition of the symbols \(Y\) and \(X\) in Theorem 1.1 shows that we have the lower bound estimate for the integrand in the previous equation
\[
X^{1 - \frac{s - 2\sigma}{1 - 2\sigma}}(t) = X^{1 - \eta_1}(t) \lesssim Y(t), \tag{3.45}
\]
which yields the desired Lyapunov-type differential inequality (3.40).

3.4. Decay estimate

Dividing (3.40) by \(X^{1 - \frac{s - 2\sigma}{1 - 2\sigma}}(t)\) and integrating the resultant inequality, we have
\[
X^{1 - \frac{s - 2\sigma}{1 - 2\sigma}}(t) \geq X^{1 - \frac{s - 2\sigma}{1 - 2\sigma}}(0) + \left(\frac{5 - 2\sigma}{1 - 2\sigma} - 1\right)Ct,
\]

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that is,

\[ \mathcal{X}(t) \leq (\mathcal{X}_0^{-\frac{4}{1-2\sigma}} + \frac{4}{1-2\sigma} + t) \leq (1 + t)^{-\frac{1-2\sigma}{4}}, \]

which implies

\[ \| (a, u, \vartheta) \|_{B_{p,1}^9} \lesssim \| (a, u, \vartheta) \|_{B_{p,1}^{8.5}} + \| a \|_{B_{p,1}^{8.5}}^h + \| u \|_{B_{p,1}^{8.5}}^h + \| \vartheta \|_{B_{p,1}^{8.5}}^h \lesssim (1 + t)^{-\frac{1-2\sigma}{4}}. \quad (3.46) \]

For \( \frac{3}{p} - \frac{3}{2} + \sigma \leq \alpha \leq \frac{3}{p} - 1 \), by the interpolation inequality we have

\[ \| (a, u, \vartheta) \|_{B_{p,1}^\alpha} \lesssim \| (a, u, \vartheta) \|_{B_{p,1}^{8.5}}^{\eta_2} \left( \| (a, u, \vartheta) \|_{B_{p,1}^{8.5}}^h \right)^{1-\eta_2}, \quad \eta_2 = \frac{\frac{3}{p} - 1 - \alpha}{\frac{1}{2} - \sigma} \in [0, 1], \]

which combines (3.39) with (3.46) gives

\[ \| (a, u, \vartheta) \|_{B_{p,1}^\alpha} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{p}) - \frac{a-\sigma}{2}}. \quad (3.47) \]

In the light of \( \frac{3}{p} - \frac{3}{2} + \sigma \leq \alpha \leq \frac{3}{p} - 1 \), we see that

\[ \| (a, u) \|_{B_{p,1}^\beta} \leq C(\| a \|_{B_{p,1}^{8.5}}^h + \| u \|_{B_{p,1}^{8.5}}^h) \leq C(1 + t)^{-\frac{1-2\sigma}{4}}, \]

from which and (3.47) gives

\[ \| (a, u) \|_{B_{p,1}^\beta} \leq C(\| (a, u) \|_{B_{p,1}^8} + \| (a, u) \|_{B_{p,1}^\beta}) \]

\[ \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{p}) - \frac{a-\sigma}{2}} + C(1 + t)^{-\frac{1-2\sigma}{4}} \]

\[ \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{p}) - \frac{a-\sigma}{2}}. \]

Similarly, for \( \frac{3}{p} - \frac{3}{2} + \sigma \leq \beta \leq \frac{3}{p} - 2 \), we can get

\[ \| \vartheta \|_{B_{p,1}^\beta} \leq C(\| \vartheta \|_{B_{p,1}^\beta}^h + \| \vartheta \|_{B_{p,1}^\beta}^h) \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{p}) - \frac{\beta-\sigma}{2}}. \]

Thanks to the embedding relation \( \hat{B}_{p,1}^0(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3), \) one infer the desired decay estimates

\[ \| \mathcal{A}^\alpha (a, u) \|_{L^p} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{p}) - \frac{a-\sigma}{2}}, \quad \| \mathcal{A}^\beta \vartheta \|_{L^p} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{p}) - \frac{\beta-\sigma}{2}}. \]

Consequently, the proof of Theorem 1.3 is complete.

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