On the stability of graph independence number

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Abstract

Let $G$ be a graph on $n$ vertices of independence number $\alpha(G)$ such that every induced subgraph of $G$ on $n - k$ vertices has an independent set of size at least $\alpha(G) - \ell$. What is the largest possible $\alpha(G)$ in terms of $n$ for fixed $k$ and $\ell$? We show that $\alpha(G) \leq n/2 + C_{k,\ell}$, which is sharp for $k - \ell \leq 2$.

We also use this result to determine new values of the Erdős-Rogers function.

1 Introduction

Background. All graphs considered here are finite, undirected, and simple. For graph $G$ and property $\mathcal{P}$, the resilience of $\mathcal{P}$ measures how much one should change $G$ in order to destroy $\mathcal{P}$. Assume $G$ has $\mathcal{P}$, then the global resilience of $\mathcal{P}$ refers to the minimum number $r$ such that by removing $r$ edges from $G$ one can obtain a graph not having $\mathcal{P}$, and the local resilience refers to the minimum number $r$ such that by removing at each vertex at most $r$ edges one can obtain a graph not having $\mathcal{P}$. For example, Turán’s theorem (see [16]) characterizes the global resilience of having a $k$-clique in complete graphs, and Dirac’s theorem (see [4]) characterizes the local resilience of having a Hamiltonian cycle in complete graphs. Moreover, the local resilience of various properties is extensively studied in [15].

Note that both global resilience and local resilience focus on removing edges. What about removing vertices? As far as we are aware of, this vertex-removal version of resilience is never discussed. To distinguish from removing edges, we shall always use the word stability when discussing removing vertices throughout this paper.

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To be more specific, we are going to study the resilience of graph independence number with respect to removing vertices. For vertices \( v_1, \ldots, v_k \), denote the induced subgraph of \( G \) on \( V(G) \setminus \{v_1, \ldots, v_k\} \) by \( G \setminus \{v_1, \ldots, v_k\} \).

For non-negative integers \( k > \ell \geq 0 \), a graph \( G(V, E) \) is called \((k, \ell)-stable\), if for every \( k \) vertices \( v_1, \ldots, v_k \) of \( G \),

\[
\alpha(G \setminus \{v_1, \ldots, v_k\}) \geq \alpha(G) - \ell.
\]

That is, the independence number \( \alpha(G) \) drops by at most \( \ell \) after removing any \( k \) vertices from \( V(G) \).

This is related to a classical problem of Erdős and Rogers. Extending the problem of studying Ramsey numbers, in [9] they defined the function

\[
f_{s, s+1}(n) \overset{\text{def}}{=} \min \{ \max \{|S| : S \subseteq V(G) \text{ and the induced subgraph } G[S] \text{ contains no } K_s \} \},
\]

where the minimum is taken over all \( K_{s+1}\)-free graphs \( G \) on \( n \) vertices. They established a lower bound on Ramsey number \( R(k, \ell) \) by arguing that \( f_{s, s+1}(n) \leq n^{1-\varepsilon(s)} \) for some positive constant \( \varepsilon(s) \).

Throughout the years, the upper bounds and lower bounds on \( f_{s, s+1}(n) \) for different pairs \((s, t)\) have been extensively studied (see [2, 11, 12, 13, 14, 7, 18, 6, 10]). The case \( t = s + 1 \) has received the most attention. The best known bounds (see [5] and [6]) in this case are

\[
\Omega(n^{1/2}(\log n)^{1/2}(\log \log n)^{-1/2}) \leq f_{s, s+1}(n) \leq O(n^{1/2}(\log n)^{4s^2}).
\]

For a detailed survey on Erdős–Rogers function, see [8]. For a recent study of an alternative local-global perspective on the Erdős–Rogers function, see [3].

The study of Erdős–Rogers function has focused on the case when \( s, t \) are fixed and \( n \) tends to infinity. Our results imply the exact value of \( f_{s, s+1}(n) \) when \( s > n/2 \).

**Results.** Unlike the previous work on the Erdős–Rogers problem, in this paper we study the behavior of independence number on very large induced subgraphs of a graph. That is, given a \((k, \ell)\)-stable graph \( G(V, E) \), what can be said about \( \alpha(G) \) in terms of \( n \overset{\text{def}}{=} |V| \) as \( n \to \infty \)? Our main result is the following upper bound.

**Theorem 1.** Suppose \( G(V, E) \) is a \((k, \ell)\)-stable graph with \(|V(G)| = n\) and \( k > \ell \geq 0 \), then

\[
\alpha(G) \leq \left\lfloor \frac{n - k + 1}{2} \right\rfloor + \ell.
\]

This upper bound on \( \alpha(G) \) determines a new class of Erdős–Rogers function values.
Theorem 2. \( f_{s,t}(n) = n - t \) for every integers \( s, t, n \) such that \( s > \lfloor \frac{n-t+1}{2} \rfloor \) and \( s + t \leq n + 1 \).

To prove Theorem 1, the crucial step is to prove its special case when \((k, \ell) = (1, 0)\). This special case also has an interesting corollary.

Corollary 3. Suppose \( G(V, E) \) is an \( n \)-vertex graph. If \( \alpha(G \setminus \{v\}) = \alpha(G) \) holds for \( m \) choices \( v \in V \), then \( \alpha(G) \leq \lfloor n - \frac{m}{2} \rfloor \).

Next, we explore the tightness of Theorem 1. We call a graph attaining equality in Theorem 1 tight \((k, \ell)\)-stable. For example, every balanced complete bipartite graph is a tight \((1, 0)\)-stable graph.

Obviously, the complete graph \( K_{k+1} \) is tight \((k, 0)\)-stable for every \( k \). However, it is harder to find tight \((k, \ell)\)-stable graphs on larger vertex sets. The next theorems are in that direction.

Theorem 4. Suppose \( n > k > \ell \geq 0 \).

(i) For every \( \ell \), if \( k = \ell + 1 \), then there exists an \( n \)-vertex tight \((k, \ell)\)-stable graph for every \( n \).

(ii) For every \( \ell \), if \( k = \ell + 2 \), then there exists an \( n \)-vertex tight \((k, \ell)\)-stable graph for every \( n \).

Theorem 5. For every \( \ell \geq 0 \), if \( k = \ell + 3 \), then there exists a sequence of \((\ell + 3, \ell)\)-stable graphs \( G_m(V_m, E_m) \) with \( |V_m| \to \infty \) such that

\[
\alpha(G_m) = \frac{|V_m|}{2} - O\left(\sqrt{|V_m|}\right).
\]

Paper organization. This paper is split into two parts.

In the first part, we prove the upper bounds (Theorem 1). We then apply them to prove Theorem 2 and Corollary 3. These occupy Section 2 through Section 4.

In the second part, we study the tightness of the upper bounds. We prove Theorems 4 and 5 in Sections 5 and 6.

Finally, we devote Section 7 to a partial characterization of tight \((1, 0)\)-stable graphs and tight \((2, 0)\)-stable graphs. In particular, we prove tight lower and upper bounds on number of edges in tight \((1, 0)\)-stable graphs.

2 Proof of Theorem 1 assuming \((k, \ell) = (1, 0)\)

Let \( G(V, E) \) be a \((1, 0)\)-stable graph. Take a maximum-sized independent set \( Y \) of \( V \). Since \( |Y| = \alpha(G) \), it suffices to show that Hall’s condition (see, e.g., [17]) holds from \( Y \) to \( V \setminus Y \). To be specific,
for a set $A \subseteq V$, define

$$N(A) \overset{\text{def}}{=} \{u \in V \setminus A : u \text{ is a neighbor of some } v \in A\}.$$  

Our goal is to prove that $|N(Y')| \geq |Y'|$ for every subset $Y' \subseteq Y$.

Assume, for contradiction’s sake, that $|N(Z)| < |Z|$ for some $Z \subseteq Y$. We may assume that $Z$ is minimal. Choose an arbitrary $z_0 \in Z$, which exists since $Z \neq \emptyset$.

Because $\alpha(G) = \alpha(G \setminus \{z_0\})$, we can find another maximum-sized independent set $X$ with $z_0 \notin X$. Define $Z_1 = X \cap Z$ and $Z_2 = Z \setminus Z_1$. Put

$$U \overset{\text{def}}{=} (X \setminus N(Z_2)) \cup Z_2.$$  

We claim that $U$ is independent and that $|U| > \alpha(G)$, which would be a contradiction.

First, we show that $U$ is independent. Note that both $X$ and $Z$ are independent, hence both $X \setminus N(Z_2)$ and $Z_2$ are independent. Since there is no edge between $Z_2$ and $X \setminus N(Z_2)$, we see that $U$ is independent.

Next, we show that $U$ can also be written as $(X \setminus (N(Z) \setminus N(Z_1))) \cup Z_2$. It suffices to show that $X \setminus (N(Z) \setminus N(Z_1)) = X \setminus N(Z_2)$. Suppose $x \in X \setminus (N(Z) \setminus N(Z_1))$ and $z \in Z_2$ are adjacent. Then $x \in N(Z_2)$ and hence $x \in N(Z)$. Since $Z_1 \subseteq X$ and $X$ is independent, there is no edge between $Z_1$ and $X$, hence $x \notin N(Z_1)$. Thus, $x \in X$ while $x \notin N(Z) \setminus N(Z_1)$, which is a contradiction. We conclude that $X \setminus (N(Z) \setminus N(Z_1)) \subseteq X \setminus N(Z_2)$. Since $N(Z) \setminus N(Z_1) \subseteq N(Z_2)$, the opposite inclusion holds as well. Hence $U = (X \setminus (N(Z) \setminus N(Z_1))) \cup Z_2$.

Finally, we show that $|U| > |X| = \alpha(G)$. Note that $Z_2 \cap X = \emptyset$ by definition, so we have

$$|U| = |X \setminus (N(Z) \setminus N(Z_1))| + |Z_2| \geq |X| - |N(Z) \setminus N(Z_1)| + |Z_2|.$$  

It suffices to show that $|N(Z) \setminus N(Z_1)| < |Z_2|$. Since $z_0 \in Z \setminus X$, so $Z_1 \subseteq Z$, hence $|N(Z_1)| \geq |Z_1|$ by the minimality assumption of $Z$. Thus,

$$|N(Z) \setminus N(Z_1)| = |N(Z)| - |N(Z_1)| \leq |N(Z)| - |Z_1| < |Z| - |Z_1| = |Z_2|,$$

hence $|U| > \alpha(G)$.

This is the promised contradiction. Hence the proof is complete. \qed
3 Proofs of Theorem 1 and Corollary 3

In this section, we derive Theorem 1 and Corollary 3 from what we proved in Section 2.

Proof of Theorem 1. The proof is by induction on $\ell$ and $k$.

Suppose $\ell = 0$, and we are going to show that

$$\alpha(G) \leq \left\lfloor \frac{n - k + 1}{2} \right\rfloor. \quad (2)$$

(2) holds when $k = 1$ since we proved it in Section 2. Assume (2) is established for $k - 1$, and suppose $G(V, E)$ is a $(k, 0)$-stable graph with $|V| = n$. For any vertex $v \in V$, by definition we have that $G \setminus \{v\}$ is $(k - 1, 0)$-stable with $\alpha(G \setminus \{v\}) = \alpha(G)$, so the induction hypothesis tells us that

$$\alpha(G) = \alpha(G \setminus \{v\}) \leq \left\lfloor \frac{(n - 1) - (k - 1) + 1}{2} \right\rfloor = \left\lfloor \frac{n - k + 1}{2} \right\rfloor,$$

which concludes the inductive proof of (2).

Assume (1) is established for $\ell - 1$, and suppose $G(V, E)$ is a $(k, \ell)$-stable graph with $|V| = n$. If there exists $v_1, \ldots, v_{k-1} \in V$ such that $\alpha(G') = \alpha(G) - \ell$, here $G' \overset{\text{def}}{=} G \setminus \{v_1, \ldots, v_{k-1}\}$. Then by definition $G'$ is a $(1, 0)$-stable graph, hence what we proved in Section 2 tells us that

$$\alpha(G) \leq \alpha(G') + \ell = \left\lfloor \frac{n - (k - 1)}{2} \right\rfloor + \ell = \left\lfloor \frac{n - k + 1}{2} \right\rfloor + \ell.$$

Otherwise, $\alpha(G)$ drops by at most $\ell - 1$ after removing any $(k - 1)$-element subset of $V$, and by definition $G$ is $(k - 1, \ell - 1)$-stable, hence the induction hypothesis tells us that

$$\alpha(G) \leq \left\lfloor \frac{n - (k - 1) + 1}{2} \right\rfloor + (\ell - 1) = \left\lfloor \frac{n - k}{2} \right\rfloor + \ell.$$

Putting these together gives us that (1) holds for $k + 1$, which concludes the inductive proof. $\square$

Proof of Corollary 3. Let $S$ be the set of vertices $v$ satisfying $\alpha(G \setminus \{v\}) = \alpha(G)$. By the assumption $|S| \geq m$. Let $S = V(G) \setminus S$ be the complement of $S$. A key observation is that every maximum-sized independent set of $G$ contains $S$.

Define

$$S_1 \overset{\text{def}}{=} \{v \in S : N(v) \cap \overline{S} = \varnothing\}, \quad S_2 \overset{\text{def}}{=} \{v \in S : N(v) \cap \overline{S} \neq \varnothing\}.$$

Let $G_1$ be the induced subgraph of $G$ on $S_1$, and we claim that $\alpha(G_1 \setminus \{v\}) = \alpha(G_1)$ for every vertex $v$ of $S_1$. Indeed, for $v \in S_1$, let $X$ be an independent set of size $\alpha(G)$ in $G \setminus \{v\}$. Then $X_1 \overset{\text{def}}{=} X \cap S_1$
has to be an independent set of $G_1$ of size $\alpha(G_1)$, for otherwise there exists some independent set $X'_1$ of $G_1$ such that $|X'_1| > |X_1|$, hence $X' \equiv X'_1 \cup \overline{S}$ is an independent set in $G$ with $|X'| > \alpha(G)$, which is impossible. Thus, the existence of $X_1$ verifies our claim.

Since every maximum-sized independent set of $G$ contains $\overline{S}$, we have $\alpha(G) = |\overline{S}| + \alpha(G_1)$, hence it follows from what we proved in Section 2 that

$$\alpha(G) = |\overline{S}| + \alpha(G_1) \leq n - |S| + |S_1|/2 \leq n - |S| + |S|/2 \leq n - m/2.$$  

We remark that the bound in Corollary 3 is tight, as witnessed by a disjoint union of a balanced complete bipartite graph $K_{\lceil m/2 \rceil, \lceil m/2 \rceil}$ and an independent set of size $n - 2\lceil m/2 \rceil$ (possibly empty) when $m < n$. For the case $m = n$, see Section 5.

### 4 Proof of Theorem 2

By considering the graph complements, there is no difference between cliques and independent sets in the definition of $f_{s,s+t}$. For convenience we replace all cliques by independent sets.

Define $G_n^m \equiv \{ G : G$ is a graph on $n$ vertices with $\alpha(G) \leq m - 1 \}$. Our goal is to verify that

- on one hand, there is $H \in G_n^{s+t}$ such that for every $t - 1$ vertices $u_1, \ldots, u_{t-1}$ of $H$,
  $$\alpha(H \setminus \{u_1, \ldots, u_{t-1}\}) \geq s;$$

- on the other hand, for every $G \in G_n^{s+t}$, there are $t$ vertices $v_1, \ldots, v_t$ of $G$ such that
  $$\alpha(G \setminus \{v_1, \ldots, v_t\}) \leq s - 1.$$

The first part is seen by considering any $n$-vertex graph $H$ with $\alpha(H) = s + t - 1$. Such an $H$ exists because $s + t \leq n + 1$. Note that the second part is trivial when $\alpha(G) \leq s - 1$, so we assume that $\alpha(G) = s - 1 + m$ for some positive integer $m \leq t$. The key is to observe that $G$ is not $(t, m-1)$-stable. Indeed, if $G$ were $(t, m-1)$-stable, then Theorem 1 would imply that

$$s - 1 + m = \alpha(G) \leq \left\lfloor \frac{n - t + 1}{2} \right\rfloor + (m - 1),$$

hence $s \leq \left\lfloor \frac{n - t + 1}{2} \right\rfloor$, which is a contradiction. Now that $G$ is not $(t, m-1)$-stable, there are $t$ vertices $v_1, \ldots, v_t$ of $G$ such that

$$\alpha(G \setminus \{v_1, \ldots, v_t\}) \leq \alpha(G) - m = s - 1,$$

which concludes the proof of the second part. The proof of Theorem 2 is complete.  

\[ \square \]
5 Proof of Theorem 4

With the help of the lemma below, the general \((k, \ell)\)-stability in Theorem 4 can be reduced to the simpler \((k - \ell, 0)\)-stability.

**Lemma 6.** Suppose \(n > k > \ell \geq 0\). If \(G\) is an \(n\)-vertex tight \((k, \ell)\)-stable graph, then \(G \sqcup K_1\), the disjoint union of \(G\) and an isolated vertex, is an \((n + 1)\)-vertex tight \((k + 1, \ell + 1)\)-stable graph.

**Proof.** Note that \(G\) is tight \((k, \ell)\)-stable. Hence
\[
\alpha(G \sqcup K_1) = \alpha(G) + 1 = \left\lfloor \frac{n-k+1}{2} \right\rfloor + \ell + 1 = \left\lfloor \frac{n+1}{2} - (k+1) + 1 \right\rfloor + (\ell + 1),
\]
which attains equality in Theorem 1. So it suffices to prove that \(G \sqcup K_1\) is \((k+1, \ell+1)\)-stable.

Let \(v'\) be the isolated vertex in \(G \sqcup K_1\), and suppose a subset of \(k + 1\) vertices \(\{v_1, \ldots, v_{k+1}\}\) are removed from \(V(G \sqcup K_1)\). If \(v'\) is removed, we may assume \(v_{k+1} = v'\), hence by the stability of \(G\),
\[
\alpha(G \sqcup K_1 \setminus \{v_1, \ldots, v_{k+1}\}) = \alpha(G \setminus \{v_1, \ldots, v_k\}) \geq \alpha(G) - \ell = \alpha(G \sqcup K_1) - (\ell + 1).
\]
Otherwise, \(v_1, \ldots, v_{k+1} \in V(G)\), then by the stability of \(G\),
\[
\alpha(G \sqcup K_1 \setminus \{v_1, \ldots, v_{k+1}\}) = \alpha((G \setminus \{v_1, \ldots, v_k\}) \setminus \{v_{k+1}\}) + 1
\geq \alpha(G \setminus \{v_1, \ldots, v_k\}) \geq \alpha(G) - \ell = \alpha(G \sqcup K_1) - (\ell + 1).
\]
Thus, \(G \sqcup K_1\) is \((k+1, \ell+1)\)-stable, and the proof is complete. \(\square\)

Evidently, Lemma 6 helps reduce the proof of Theorem 4(i), Theorem 4(ii), and Theorem 5 in general to the proof of their special cases when \((k, \ell) = (1, 0), (2, 0), \text{ and } (3, 0)\).

**Proof of Theorem 4.** For \(n \geq 2\), denote by \(K_n\) the complete bipartite graph \(K_{n/2, n/2}\) when \(n\) is even, or the complete tripartite graph \(K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor, 1}\) when \(n\) is odd. Then \(\alpha(K_n) = \lfloor n/2 \rfloor\). After removing any one vertex \(v\), at least one \(\lfloor n/2 \rfloor\)-vertex part is left untouched, so \(\alpha(K_n \setminus \{v\}) = \lfloor n/2 \rfloor = \alpha(K_n)\). Hence \(K_n\) is always \((1, 0)\)-stable. By applying Lemma 6, we see that Theorem 4(i) holds.

As for Theorem 4(ii), it suffices to construct an \(n\)-vertex \((2, 0)\)-stable graph for every \(n \geq 3\).

Consider *cycles* \(C_n\) and *wheels* \(W_n\). Here an *wheel* refers to an \((n - 1)\)-cycle \(C_{n-1}\) with another vertex connected to every vertex of the cycle. For example, \(C_5\) and \(W_6\) are shown in the figure below:

![Figure 1: Graphs C5 and W6.](image)
We claim that \( C_n \) is tight \((2,0)\)-stable when \( n \) is odd, and \( W_n \) is tight \((2,0)\)-stable when \( n \) is even.

Denote the \( n \)-vertex path graph as \( P_n \), evidently \( \alpha(P_n) = \lceil n/2 \rceil = [(n + 1)/2] \). When \( n \) is odd, obviously \( \alpha(C_n) = (n - 1)/2 \). Suppose two disjoint paths \( P_a \) and \( P_b \) are left after vertices \( u, v \) being removed from \( C_n \) (note that \( a, b \) can be zero), then \( a + b = n - 2 \), and
\[
\alpha(C_n \setminus \{u, v\}) = \left\lceil \frac{a + 1}{2} \right\rceil + \left\lceil \frac{b + 1}{2} \right\rceil = \frac{a + b + 1}{2} = \frac{n - 1}{2} = \alpha(C_n),
\]
since \( a \) and \( b \) are of different parity. So \( C_n \) is tight \((2,0)\)-stable. When \( n \) is even, obviously \( \alpha(W_n) = \alpha(C_{n-1}) \). Suppose two vertices \( u, v \) are removed from \( W_n \), then at most two vertices are removed from the induced subgraph \( C_{n-1} \). Since \( C_{n-1} \) is \((2,0)\)-stable, we see that
\[
\alpha(C_{n-1}) = \alpha(C_{n-1} \setminus \{u, v\}) \leq \alpha(W_n \setminus \{u, v\}) \leq \alpha(W_n) = \alpha(C_{n-1}),
\]
so \( \alpha(W_n) = \alpha(C_{n-1}) = n/2 - 1 \), and hence \( W_n \) is tight \((2,0)\)-stable.

Thus, an \( n \)-vertex tight \((2,0)\)-stable graph always exists. By applying Lemma 6, we are done.

It is worth mentioning that no 6-vertex tight \((3,0)\)-stable graph exists. The proof is simple. Assume \( G \) is a 6-vertex tight \((3,0)\)-stable graph, then \( \alpha(G) = \lceil (6 - 2)/2 \rceil = 2 \), so there is no triangle subgraph \( K_3 \) in the complement graph \( \overline{G} \). Also, if three vertices of \( G \) form a triangle, by removing the other 3 vertices the independence number drops to 1, which contradicts with the \((3,0)\)-stability. Thus, there is no triangle subgraph \( K_3 \) in \( G \). Now we reach a contradiction with the fact that Ramsey number \( R(3,3) = 6 \), so no 6-vertex tight \((3,0)\)-stable graph exists.

6 Proof of Theorem 5

We are going to construct a sequence of \((3,0)\)-stable graphs \( G_m(V_m, E_m) \) with \( |V_m| \to \infty \) such that
\[
\alpha(G_m) = |V_m|/2 - O\left(\sqrt{|V_m|}\right).
\]
Note that the existence of such a sequence directly implies Theorem 5 by Lemma 6.

Suppose \( m \in \mathbb{N} \) (\( m \geq 3 \) for technical reasons). Take
\[
V_m \overset{\text{def}}{=} \mathbb{Z}/(2m^2 + 2m)\mathbb{Z}, \quad E_m \overset{\text{def}}{=} \{(i, j) : |i - j| \equiv m \text{ or } m + 1 \pmod{2m^2 + 2m}\},
\]
we claim that \( G_m \overset{\text{def}}{=} (V_m, E_m) \) is \((3,0)\)-stable with \( \alpha(G_m) = m^2 \). Since
\[
m^2 = (2m^2 + 2m)/2 - O\left(\sqrt{2m^2 + 2m}\right),
\]
Theorem 5 is shown once we verify the claim.
First, we prove that $\alpha(G_m) = m^2$. It is easily seen that the vertices

$$0, \quad 2m + 1, \quad 4m + 2, \quad \cdots, \quad (m - 1)(2m + 1),$$

$$1, \quad 2m + 2, \quad 4m + 3, \quad \cdots, \quad (m - 1)(2m + 1) + 1,$$

$$2, \quad 2m + 3, \quad 4m + 4, \quad \cdots, \quad (m - 1)(2m + 1) + 2,$$

$$\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots$$

$$m - 1, \quad 3m, \quad 5m + 1, \quad \cdots, \quad (m - 1)(2m + 1) + m - 1$$

form an independent set of size $m^2$ in $G_m$. Hence, it suffices to show that $\alpha(G_m) \leq m^2$.

We prove this by contradiction. Suppose $X \subset V_m$ is an independent set of size $m^2 + 1$. Partition the vertices into $m$ groups of size $2m + 2$ each as follows:

$$V_0 \overset{\text{def}}{=} \{0, m + 0, 2m + 0, \cdots, (2m + 1)m + 0\},$$

$$V_1 \overset{\text{def}}{=} \{1, m + 1, 2m + 1, \cdots, (2m + 1)m + 1\},$$

$$V_2 \overset{\text{def}}{=} \{2, m + 2, 2m + 2, \cdots, (2m + 1)m + 2\},$$

$$\vdots$$

$$V_{m-1} \overset{\text{def}}{=} \{m - 1, 2m - 1, 3m - 1, \cdots, (2m + 2)m - 1\}.$$

Note that $|X \cap V_j| \leq m + 1$ since every two consecutive points (including the first and the last) are adjacent in $G_m$. Thus, by pigeonhole principle $|X \cap V_j| = m + 1$ for some $j$. Due to the obvious translation symmetry in $G_m$, we may assume without loss of generality that

$$X \cap V_0 = \{0, 2m, 4m, \ldots, 2m^2\}.$$

Since $X$ is an independent set, from the construction of $E_m$ we see that

$$m, \quad 3m, \quad 5m, \quad \ldots, \quad (2m + 1)m \notin X,$$

$$m + 1, \quad 3m + 1, \quad 5m + 1, \quad \ldots, \quad (2m + 1)m + 1 \notin X,$$

$$m - 1, \quad 3m - 1, \quad 5m - 1, \quad \ldots, \quad (2m + 1)m - 1 \notin X.$$

The key is to pair up every other vertex of $G_m$ as follows:

$$\{1, m + 2\}, \quad \{2m + 1, 3m + 2\}, \quad \cdots, \quad \{(2m \cdot m + 1, (2m + 1)m + 2)\},$$

$$\{2, m + 3\}, \quad \{2m + 2, 3m + 3\}, \quad \cdots, \quad \{(2m \cdot m + 2, (2m + 1)m + 3)\},$$

$$\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots$$

$$\{m - 2, 2m - 1\}, \quad \{3m - 2, 4m - 1\}, \quad \cdots, \quad \{(2m + 1)m - 2, (2m + 2)m - 1\}.$$
Note that the vertices in each pair are adjacent in $G_m$. Hence they are not in $X$ simultaneously. So

$$|X| \leq |X \cap V_0| + \#\text{pairs} = (m + 1) + (m - 2)(m + 1) = m^2 - 1,$$

which contradicts with $|X| = m^2 + 1$. Thus, $\alpha(G) = m^2$, as desired.

**Then, we show that $G_m$ is $(3, 0)$-stable.** Think about $V(G_m)$ as $2m^2 + 2m$ evenly distributed points on a circle. Here two points are adjacent in $G_m$ if and only if they form an interval of length $m$ or $m + 1$ on the circle. We are going to prove that no matter which 3 points are deleted, we can still pick $m^2$ points such that no two of them are adjacent to each other in $G_m$.

Suppose 3 points $u, v, w$ (clockwise in this order) are deleted, and the circle is cut into three pieces with $a, b, c$ consecutive points in $u \sim v, v \sim w, w \sim u$, respectively ($a, b, c$ can be zero), then $a + b + c = 2m^2 + 2m - 3$. Here $u \sim v$ refers to the clockwise arc from $u$ to $v$ on the circle. The idea is to chop these pieces into many *normal groups*, $(2m + 1)$-consecutive-point groups, and three small consecutive-point groups, then to pick some $m$ consecutive points out of each normal group, and to deal with the remaining small groups carefully. Set

$$a' \overset{\text{def}}{=} a \mod 2m + 1, \quad b' \overset{\text{def}}{=} b \mod 2m + 1, \quad c' \overset{\text{def}}{=} c \mod 2m + 1.$$

**Case 1.** $a' + b' + c' = m - 3$. Pick the first $m$ points clockwise in each normal group. We have picked $m^2$ points in total, and every two of them are either among some $m$ consecutive points or are at least $m + 2$ apart from each other, so these points form an independent set of size $m^2$.

**Case 2.** $a' + b' + c' = 3m - 2$. By symmetry we may assume without loss of generality that $a' \geq m$.

Clockwise chop $u \sim v$ into $a', 2m + 1, \ldots, 2m + 1$-point groups, chop $v \sim w$ into $2m + 1, \ldots, 2m + 1, b'$-point groups, chop $w \sim u$ into $2m + 1, \ldots, 2m + 1, c'$-point groups. Then we pick points as below:

- Pick the first $m$ points clockwise in the $a'$-point group (recall that $a' \geq m$).
- Starting from the last picked point clockwise in the $a'$-point group and moving clockwise on the circle, we pick the first *available* $m$ points from each normal group (*available* means forming an interval of length at least $m + 2$ on the circle from the last point that has been picked).

So far we have picked $m^2$ points. By the way we picked these points,

- the consecutive $a' - m$ points clockwise before $v$ are unpicked;
• the consecutive $b' + (a' - m) + 1$ points clockwise before $w$ are unpicked;

• the consecutive $c' + \min\{b' + (a' - m) + 2, m + 1\}$ points clockwise before $u$ are unpicked.

According to the way we picked these points, the open interval between consecutive picked points including $v$ and the open interval between consecutive picked points including $w$ (possibly the same) both contain at least $m + 1$ points. As for the open interval between consecutive picked points including $u$, it also contains at least $m + 1$ points, as $a' + b' + c' = 3m - 2$ implies that

$$c' + \min\{b' + (a' - m) + 2, m + 1\} + 1 \geq m + 1.$$ 

Thus, the $m^2$ points we picked form an independent set.

Case 3. $a' + b' + c' = 5m - 1$. The fact $a', b', c' \leq 2m$ implies that $m - 1 \leq a', b', c' \leq 2m$. By symmetry we may assume $a' = \min\{a', b', c'\}$. Clockwise chop $u \leadsto v$ into $2m + 1, \ldots, 2m + 1, a'$-point groups, chop $v \leadsto w$ into $b', 2m + 1, \ldots, 2m + 1$-point groups, chop $w \leadsto u$ into $2m + 1, \ldots, 2m + 1, c'$-point groups. Then we pick points as below:

• Starting from $u$ and moving clockwise on $u \leadsto v$, we pick the $(m + 1)$’th through the $2m$’th point from each normal group.

• Starting from $v$ and moving clockwise, we pick the first $m$ points from the $b'$-point group (recall that $b' \geq m$). Starting from the last picked point and moving clockwise on $v \leadsto w$, we pick the first available $m$ points from each normal group.

• Starting from $u$ and moving counterclockwise, we pick the first $m$ points from the $c'$-point group (recall that $c' \geq m$). Starting from the last picked point and moving counterclockwise on $w \leadsto u$, we pick the first available $m$ points from each normal group.

So far we have picked $m^2$ points. By the way we picked these points,

• the consecutive $m$ points clockwise after $u$ are unpicked (because $a' \geq m - 1$);

• the consecutive $a' + 1$ points clockwise before $v$ are unpicked;

• the consecutive $b' - m$ points clockwise before $w$ are unpicked, and the consecutive $c' - m$ points clockwise after $w$ are unpicked.
Since $a' \geq m - 1$, the open interval between consecutive picked points including $u$ and the open interval between consecutive picked points including $v$ (possibly the same) both contain at least $m + 1$ points. As for the open interval between consecutive picked points including $w$, it also contains at least $m + 1$ points, as $a' = \min\{a', b', c'\}$ implies that $a' < 2m$, hence

$$(b' - m) + (c' - m) + 1 = (5m - 1) - a' - 2m + 1 \geq m + 1.$$ 

Thus, the $m^2$ points we picked form an independent set.

By combining the three cases, we conclude that $G_m$ is $(3, 0)$-stable.

The claim is seen by combining the above, and the proof is complete. 

We remark that Theorem 1 is also asymptotically tight when $k = \ell + 4$. In fact, every graph $G_m$ with $V_m \overset{\text{def}}{=} \mathbb{Z}/(2m^2 + 2m + 1)\mathbb{Z}$ and $E_m \overset{\text{def}}{=} \{(i, j) : |i - j| \equiv m \text{ or } m + 1 \pmod{2m^2 + 2m + 1}\}$ is $(4, 0)$-stable, and the proof is basically the same as the proof of Theorem 5 but slightly subtler.

7 More on tight $(1, 0)$-stable and tight $(2, 0)$-stable graphs

As we have seen in the proof of Theorem 4, there is an $n$-vertex $(1, 0)$-stable graph for every $n \geq 2$ and an $n$-vertex $(2, 0)$-stable graph for every $n \geq 3$. In this section, we care about the uniqueness of such tight stable graphs. That is, for some given $n$, we want to know whether every $n$-vertex tight $(1, 0)$-stable graph is isomorphic to $K_n$ and whether every $n$-vertex $(2, 0)$-stable graph is isomorphic to $C_n$ or $W_n$.

For tight $(1, 0)$-stable graphs, the uniqueness does not hold (assume $n \geq 4$). For example, denote by $\mathcal{M}_n$ the disjoint union of $n/2$ edges when $n$ is even, and the disjoint union of $\lfloor n/2 \rfloor - 1$ edges and one triangle when $n$ is odd. Evidently, the graph $\mathcal{M}_n$ is tight $(1, 0)$-stable for every $n$. Moreover, every $n$-vertex graph that is sandwiched between $\mathcal{M}_n$ and $K_n$ is also tight $(1, 0)$-stable.

Here we would like to mention that $n$-vertex $(n \geq 3)$ path $P_n$ is another class of tight $(2, 1)$-stable graphs, which can be easily checked. This implies that there exist tight $(2, 1)$-stable graphs other than those constructed in Lemma 6.

For $(2, 0)$-stable graphs, the uniqueness does not hold when $n$ is even (assume $n \geq 6$). One can check that when

$$V_k \overset{\text{def}}{=} \mathbb{Z}/2k\mathbb{Z}, \quad E_k \overset{\text{def}}{=} \{(i, j) : |i - j| \equiv 1 \text{ or } k \pmod{2k}\},$$
$G_k \overset{\text{def}}{=} (V_k, E_k)$ is tight $(2,0)$-stable for every $k \geq 3$. However, we are unaware of any $n$-vertex tight $(2,0)$-stable graph other than the odd cycle $C_n$ when $n$ is odd. We doubt that $C_n$ is the only $n$-vertex tight $(2,0)$-stable graph. Note that $C_n$, $W_n$, and the tight $(2,0)$-stable graphs constructed above all have Hamiltonian cycles inside. We also suspect that every tight $(2,0)$-stable graph contains a Hamiltonian cycle.

**Characterization of tight $(1,0)$-stable graphs.** One might wonder whether a tight $(1,0)$-stable graph $G$ (especially when $|V(G)|$ is even) is always sandwiched between a perfect matching and a complete bipartite graph (i.e. $M_n \subseteq G \subseteq K_n$). In fact, $M_n \subseteq G$ is already shown by Hall’s theorem in the proof in Section 2, while $G \subseteq K_n$ does not necessarily happen. An example is the graph below.

![Figure 2: A tight $(1,0)$-stable graph $G$ with $G \not\subseteq K_n$.](image)

Nevertheless, we can still prove tight upper and lower bounds on number of edges:

**Theorem 7.** Suppose $G(V, E)$ is a tight $(1,0)$-stable graph with $|V(G)| = n$, then

$$|E(M_n)| \leq |E(G)| \leq |E(K_n)|.$$  

**Proof.** We consider the cases when $n$ is even and when $n$ is odd separately.

**Case 1.** $n$ is even. Suppose $n = 2k$, then $\alpha(G) = k$ implies that the complement graph $\overline{G}$ contains no $(k + 1)$-clique. Hence Turán’s theorem implies that

$$|E(\overline{G})| \leq |E(K_{2,\ldots,2})| = 2k^2 - 2k.$$  

Here $K_{2,\ldots,2}$ refers to the complete $k$-partite graph in which each part contains 2 vertices. So

$$|E(G)| \geq \binom{2k}{2} - (2k^2 - 2k) = k = |E(M_n)|,$$

which gives the desired lower bound.

For the upper bound, if there is a vertex $v$ with $\deg(v) \geq k + 1$, then $v$ is not contained in any maximum-sized independent set since $\alpha(G) = k$. Hence $G \setminus \{v\}$ is $(1,0)$-stable as well. However, $\alpha(G \setminus \{v\}) = k$, which contradicts Theorem 1. Thus, every vertex of $G$ is of degree at most $k$, hence

$$|E(G)| \leq \frac{1}{2} \cdot 2k \cdot k = k^2 = |E(K_n)|.$$
which gives the desired upper bound.

**Case 2.** *n* is odd. Suppose *n* = 2\(k+1\), then \(\alpha(G) = k\) implies that the complement graph \(\overline{G}\) contains no \((k+1)\)-clique. Hence Turán’s theorem implies that

\[
|E(\overline{G})| \leq |E(K_{2\ldots,2,3})| = 2k^2 - 2.
\]

Here \(K_{2\ldots,2,3}\) refers to the complete \(k\)-partite graph in which \(k-1\) parts contain 2 vertices in each and 1 part contains 3 vertices. So

\[
|E(G)| \geq \binom{2k+1}{2} - (2k^2-2) = k + 2 = |E(M_n)|,
\]

which gives the desired lower bound.

For the upper bound, if there is a vertex \(v\) with \(\deg(v) \geq k + 2\), then \(v\) is not contained in any maximum-sized independent set since \(\alpha(G) = k\). Hence \(G \setminus \{v\}\) is \((1,0)\)-stable as well. So

\[
|E(G \setminus \{v\})| \leq k^2\text{ by what we just proved, hence}
\]

\[
|E(G)| \leq k^2 + \deg(v) \leq k^2 + 2k = |E(K_n)|.
\]

Otherwise, every vertex in \(G\) is of degree at most \(k + 1\), hence

\[
|E(G)| \leq \frac{1}{2}(2k + 1)(k + 1) \leq k^2 + 2k = |E(K_n)|.
\]

The proof is done by combining the two cases.

\[\square\]

8 Open problems

1. Is the odd cycle \(C_n\) the only \(n\)-vertex tight \((2,0)\)-stable graph for odd \(n \geq 3\)? By exhaustive computer search, we verified this for \(n = 3, 5, 7, 9, 11\). For even \(n\), we suspect that every connected \(n\)-vertex tight \((2,0)\)-stable graph contains a Hamiltonian cycle.

2. Theorem 1 is not always tight. For example, as we remarked in Section 5, there exists no 6-vertex tight \((3,0)\)-stable graph. It would be interesting to find the largest independence number of \(n\)-vertex \((k,\ell)\)-stable graphs for every \(n, k,\) and \(\ell\).

3. We do not know if Theorem 1 is asymptotically tight, i.e., if there exist \(n\)-vertex \((k,\ell)\)-stable graphs with independence number \(n/2 - o(n)\), for every \(k\) and \(\ell\).

**Remark.** This problem is answered positively by Theorem 1.4 in [1].
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