Analogues of the Artin factorization formula for the automorphic scattering matrix and Selberg zeta-function associated to a Kleinian group

Joshua S. Friedman

Abstract. For Kleinian groups acting on hyperbolic three-space, we prove factorization formulas for both the Selberg zeta-function and the automorphic scattering matrix. We extend results of Venkov and Zograf from Fuchsian groups, to Kleinian groups, and we give a proof that is simple and extendable to more general groups.

1. Introduction

In [VZ83] Venkov and Zograf gave an analogue of Artin’s well-known factorization formula. More specifically, they gave factorization formulas for both the Selberg zeta-function and automorphic scattering matrix that are associated to a Fuchsian group (in the context of the Selberg spectral theory of automorphic functions). They proved the following:

Theorem (Venkov-Zograf). Let \( \Gamma \) be a Fuchsian group, \( \Gamma_1 \) a finite-index normal subgroup of \( \Gamma \), then
\[
Z(s, \Gamma_1, 1) = Z(s, \Gamma, U^1) = \prod_{\vartheta \in (\Gamma_1 \backslash \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_{\vartheta}}.
\]

Here \( (\Gamma_1 \backslash \Gamma)^* \) is the set of all pairwise inequivalent irreducible unitary representations of the group \( \Gamma_1 \backslash \Gamma \); for \( \vartheta \in (\Gamma_1 \backslash \Gamma)^* \), \( n_{\vartheta} \) is the dimension of \( \vartheta \); \( Z(s, \Gamma, \vartheta) \) is the Selberg zeta-function; \( 1 \) is the trivial representation of \( \Gamma_1 \), and \( U^1 \) is its induced representation to \( \Gamma \).

In fact, they proved that for any finite-dimensional unitary representation \( \chi \) of \( \Gamma_1 \),
\[
Z(s, \Gamma_1, \chi) = Z(s, \Gamma, U^\chi).
\]

Venkov and Zograf also proved:

Theorem (Venkov-Zograf). Let \( \Gamma \) be a cofinite Fuchsian group, \( \Gamma_1 \) a finite-index normal subgroup of \( \Gamma \), then
\[
\phi(s, \Gamma_1, \chi)\Omega(\Gamma_1, \chi)^{1-2s} = \phi(s, \Gamma, U^\chi)\Omega(\Gamma, U^\chi)^{1-2s}.
\]

Here \( \phi(s, \cdot, \cdot) \) is the determinant of the automorphic scattering matrix, and \( \Omega(\cdot, \cdot) \) is a constant depending on the group and unitary representation.

In this paper, we give a simple proof of these two theorems—a proof that extends to higher dimensions—for the case of cofinite Kleinian groups acting on hyperbolic three-space. Our proof also applies to the Fuchsian case, and slightly strengthens Equation 1.3. Our proof implies that
\[
\phi(s, \Gamma_1, \chi) = \phi(s, \Gamma, U^\chi)
\]
and
\[
\Omega(\Gamma_1, \chi) = \Omega(\Gamma, U^\chi).
\]

Next we state our results and switch from Fuchsian to Kleinian groups. Let \( \Gamma < \text{PSL}(2, \mathbb{C}) \) be a cofinite Kleinian group, \( \Gamma_1 \) a finite-index normal subgroup of index \( n \). Let \( \chi \in \text{Rep}(\Gamma_1, V) \) be a
finite-dimensional unitary representation of $\Gamma_1$ in $V$, and let $\psi = U^\chi \in \text{Rep}(\Gamma, V^n)$ be its induced representation to $\Gamma$.

Our main results are as follows:

**Theorem.** Let $\Gamma$ be an arbitrary Kleinian group, $\Gamma_1$ a finite-index normal subgroup of $\Gamma$. Let $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Then for all $s > 1$,

$$Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi).$$

Here $Z(s, \cdot, \cdot)$ is the Selberg zeta-functions (defined in §2.2).

**Theorem.** Let $\Gamma$ be an arbitrary Kleinian group, $\Gamma_1$ a finite-index normal subgroup of $\Gamma$. Then for all $s > 1$,

$$Z(s, \Gamma_1, 1) = Z(s, \Gamma, U^1) = \prod_{\vartheta \in (\Gamma_1 \setminus \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_\vartheta}.$$  Here $(\Gamma_1 \setminus \Gamma)^*$ is the set of all pairwise inequivalent irreducible unitary representations of the group $\Gamma_1 \setminus \Gamma$; for $\vartheta \in (\Gamma_1 \setminus \Gamma)^*$, $n_\vartheta$ is the dimension of $\vartheta$; $Z(s, \Gamma, \vartheta)$ is the Selberg zeta-function; $1$ is the trivial representation of $\Gamma_1$, and $U^1$ is its induced representation to $\Gamma$.

Let $\mathcal{S}(s, \Gamma_1, \chi), \mathcal{S}(s, \Gamma, \psi)$ be automorphic scattering matrices (defined in §5), and set

$$\phi(s, \Gamma_1, \chi) \equiv \det \mathcal{G}(s, \Gamma_1, \chi),$$

$$\phi(s, \Gamma, \psi) \equiv \det \mathcal{G}(s, \Gamma, \psi).$$

**Theorem.** Let $\Gamma$ be a cofinite Kleinian group, $\Gamma_1$ a finite-index normal subgroup of $\Gamma$. Let $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Then for all $s \in \mathbb{C}$,

$$\phi(s, \Gamma_1, \chi) = \phi(s, \Gamma, \psi).$$

I would like to thank Professor Leon Takhtajan for originally suggesting this problem to me, and for reading through this paper. I would also like to thank Peter Zograf for reading this paper and for useful discussions. I would also like to thank the anonymous referee for correcting some errors.

## 2. Preliminaries

In this section we state the preliminary results that will be needed. Our main references are [Fri05a, Fri05b, EGM98], and [Ven82], and [Hej76, Hej83]. Unless stated otherwise, throughout this section $\Gamma < \text{PSL}(2, \mathbb{C})$ is a cofinite Kleinian group and $\chi \in \text{Rep}(\Gamma, V)$ is a finite-dimensional unitary representation of $\Gamma$ in $V$.

### 2.1. Cofinite Kleinian Groups.

Let $\Gamma < \text{PSL}(2, \mathbb{C})$ be a cofinite Kleinian group acting on hyperbolic three-space $\mathbb{H}^3$. Let $V$ be a finite-dimensional complex inner product space with inner-product $\langle \cdot, \cdot \rangle_V$, and let $\text{Rep}(\Gamma, V)$ denote the set of finite-dimensional unitary representations of $\Gamma$ in $V$. Let $\mathcal{F} \subset \Gamma$ be a fundamental domain for the action of $\Gamma$ in $\mathbb{H}^3$.

Let $\chi \in \text{Rep}(\Gamma, V)$. The Hilbert space of $\chi-$automorphic functions is the set of measurable functions

$$\mathcal{H}(\Gamma, \chi) \equiv \{ f : \mathbb{H}^3 \to V \mid f(\gamma P) = \chi(\gamma) f(P) \ \forall \gamma \in \Gamma, P \in \mathbb{H}^3, \ \text{and} \ \langle f, f \rangle \equiv \int_{\mathcal{F}} \langle f(P), f(P) \rangle_V \ d\nu(P) < \infty \}.$$  Finally, let $\Delta = \Delta(\Gamma, \chi)$ be the corresponding positive self-adjoint Laplace-Beltrami operator on $\mathcal{H}(\Gamma, \chi)$.

Next we briefly define the concept of a singular unitary representation. Let $\mathbb{P}$ be the the boundary of $\mathbb{H}^3$, the Riemann sphere. For every $\zeta \in \mathbb{P}$ let $\Gamma_\zeta$ denote the stabilizer subgroup of $\zeta$ in $\Gamma$,

$$\Gamma_\zeta \equiv \{ \gamma \in \Gamma \mid \gamma \zeta = \zeta \}.$$
and let $\Gamma'$ be the maximal torsion-free parabolic subgroup of $\Gamma$ (the maximal parabolic subgroup of $\Gamma$ that does not contain elliptic elements). A point $\zeta \in \mathbb{P}^1$ is called a cusp of $\Gamma$ if $\Gamma'$ is a free abelian group of rank two. Two cusps $\zeta_1, \zeta_2$ are $\Gamma$–equivalent if $\zeta_1 \in \Gamma \zeta_2$, that is their $\Gamma$–orbits coincide.

Every cofinite Kleinian group has finitely many equivalence classes of cusps, so we fix a set $\{\zeta_\alpha\}_{\alpha=1}^{\kappa(\Gamma)}$ of representatives of these equivalence classes. For notational convenience we set $\Gamma_\alpha \equiv \Gamma_{\zeta_\alpha}$ and $\Gamma'_\alpha \equiv \Gamma'_{\zeta_\alpha}$.

For each cusp $\zeta_\alpha$ fix an element $B_\alpha \in \text{PSL}(2, \mathbb{C})$, a lattice $\Lambda_\alpha = \mathbb{Z} \oplus \mathbb{Z}\tau_\alpha$, $\text{Im}(\tau_\alpha) > 0$, and a root of unity $\epsilon_\alpha$ of order 1, 2, 3, 4, or 6 with the following conditions being satisfied:

1. $\zeta_\alpha = B_\alpha^{-1}\infty$,
2. $B_\alpha \Gamma_\alpha' B_\alpha^{-1} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \ b \in \Lambda_\alpha \right\}$,
3. $B_\alpha \Gamma_\alpha B_\alpha^{-1} = \left\{ \begin{pmatrix} \epsilon & \epsilon b \\ 0 & \epsilon^{-1} \end{pmatrix} \middle| \ b \in \Lambda_\alpha, \ \epsilon \text{ is some power of } \epsilon_\alpha \right\} / \{\pm I\}$

The group $B_\alpha \Gamma_\alpha' B_\alpha^{-1}$ acts on $\mathbb{C}$ via the lattice $\Lambda_\alpha$. See [EGM98] Theorem 2.1.8 for more details.

For each cusp $\zeta_\alpha$ of $\Gamma$, define the singular space

$$V_\alpha \equiv \{ v \in V \mid \chi(\gamma)v = v, \ \forall \gamma \in \Gamma_\alpha \},$$

and the almost singular space

$$V'_\alpha \equiv \{ v \in V \mid \chi(\gamma)v = v, \ \forall \gamma \in \Gamma'_\alpha \},$$

where $1 \leq \alpha \leq \kappa(\Gamma)$.

A representation $\chi \in \text{Rep}$ is singular at the cusp $\zeta_\alpha$ of $\Gamma$ iff the subspace $V_\alpha \neq \{0\}$. If a cusp is not singular, it is called regular.

For each cusp $\zeta_\alpha$, set $k_\alpha = \dim_{\mathbb{C}} V_\alpha$, and

$$k(\Gamma, \chi) \equiv \sum_{\alpha=1}^{\kappa(\Gamma)} k_\alpha.$$

### 2.2. Selberg Zeta-Function.

In this section we define the Selberg zeta-function $Z(s, \Gamma, \chi)$ associated to $\Gamma$ and $\chi \in \text{Rep}(\Gamma, \chi)$. We allow $\Gamma$ to be an arbitrary Kleinian group. See [Fri05a] and [EGM98] Sections 5.2, 5.4, for more details.

Suppose $T \in \Gamma$ is loxodromic (we consider hyperbolic elements as loxodromic elements). Then $T$ is conjugate in $\text{PSL}(2, \mathbb{C})$ to a unique element of the form

$$D(T) = \begin{pmatrix} a(T) & 0 \\ 0 & a(T)^{-1} \end{pmatrix}$$

such that $a(T) \in \mathbb{C}$ has $|a(T)| > 1$. Let $N(T)$ denote the norm of $T$, defined by

$$N(T) \equiv |a(T)|^2,$$

and let by $\mathcal{C}(T)$ denote the centralizer of $T$ in $\Gamma$. There exists a (primitive) loxodromic element $T_0$, and a finite cyclic elliptic subgroup $\mathcal{E}$ of order $m(T)$, generated by an element $E_T$ such that

$$\mathcal{C}(T) = \langle T_0 \rangle \times \mathcal{E}.$$ 

Here $\langle T_0 \rangle = \{ T_0^n \mid n \in \mathbb{Z} \}$. Next, let $t_1, \ldots, t_n$, and $t'_1, \ldots, t'_m$ denote the eigenvalues of $\chi(T_0)$ and $\chi(E_T)$ respectively. The elliptic element $E_T$ is conjugate in $\text{PSL}(2, \mathbb{C})$ to an element of the form

$$\begin{pmatrix} \zeta(T_0) & 0 \\ 0 & \zeta(T_0)^{-1} \end{pmatrix},$$

where $\zeta(T_0)$ is some power of $\epsilon_\alpha$. The relationship $\zeta(T_0) \zeta(T_0)^{-1} = 1$ holds for all $\alpha$.
where here \( \zeta(T_0) \) is a primitive \( 2m(T) \)-th root of unity.

For \( \Re(s) > 1 \) the Selberg zeta-function \( Z(s, \Gamma, \chi) \) is defined by

\[
Z(s, \Gamma, \chi) = \prod_{(T_0) \in \mathcal{R}} \prod_{j=1}^{\dim V} \prod_{c(T,j,k) = 1} \left(1 - t_j a(T_0)^{-2k} a(T_0)^{-2l} N(T_0)^{-s-1}\right).
\]

Here the product with respect to \( T_0 \) extends over a maximal reduced system \( \mathcal{R} \) of \( \Gamma \)-conjugacy classes of primitive loxodromic elements of \( \Gamma \). The system \( \mathcal{R} \) is called reduced if no two of its elements have representatives with the same centralizer. See [EGM98] Section 5.4 for more details. The function \( c(T, j, l, k) \) is defined by

\[
c(T, j, l, k) = \tau_j(T_0)^{2l} \zeta(T_0)^{-2k}.
\]

In [Fri05a], we gave the meromorphic continuation of \( Z(s, \Gamma, \chi) \) to the left half plane under certain technical assumptions.

A practical way of understanding the zeta-function is via its logarithmic derivative. For \( \Re(s) > 1 \),

\[
\frac{d}{ds} \log Z(s, \Gamma, \chi) = \sum_{(T) \in \mathcal{R}} \frac{\text{tr}(\chi(T)) \log N(T_0)}{m(T)|a(T) - a(T)^{-1}|^2 N(T)^{-s}}.
\]

Let \( W(s, \Gamma, \chi) = \frac{d}{ds} \log Z(s, \Gamma, \chi) \). Then, for \( \Re s > 1 \),

\[
Z(s, \Gamma, \chi) = e^{\int W(s, \Gamma, \chi) \, ds + C}
\]

where \( C \) is chosen so

\[
\lim_{s \to \infty} Z(s, \Gamma, \chi) = 1
\]

will be satisfied.

### 2.3. Selberg Theory of \( \Delta \).

In this section we assume that \( \Gamma \) is cofinite, and we state some needed results concerning the Selberg trace formula associated to \( \Gamma \) and \( \chi \in \text{Rep}(\Gamma, V) \). We will not need the full trace formula found in [Fri05a, Fri05b]. Rather, only parts of its proof will be needed. More details can be found in [Fri05a, Fri05b, EGM98], and [Ven82].

For \( P = z + rj, \ P' = z' + r'j \in \mathbb{H}^3 \) set

\[
\delta(P, P') = \cosh(d(P, P')) = \frac{|z - z'|^2 + r^2 + r'^2}{2rr'},
\]

where \( d \) is the hyperbolic distance in \( \mathbb{H}^3 \).

For \( k \in \mathcal{S}([1, \infty)) \), a Schwartz-class function set

\[
K(P, P') = k(\delta(P, P')).
\]

Note that for any \( \gamma \in \text{PSL}(2, \mathbb{C}) \);

\[
K(\gamma P, \gamma P') = K(P, P') \quad \text{and} \quad K(P, \gamma P') = K(\gamma^{-1} P, P').
\]

For \( \Theta \subset \Gamma, \chi \in \text{Rep}(\Gamma, V) \), define

\[
K(P, P', \Theta, \psi) = \sum_{\gamma \in \Theta} \chi(\gamma) K(P, \gamma P').
\]

The series above converges absolutely and uniformly on compact subsets of \( \mathbb{H}^3 \times \mathbb{H}^3 \).

---

1. We assumed that \( \Gamma \) was cofinite and had only one class of cusps. We also showed that the presence of cuspidal elliptic fixed points can cause the zeta-function to have branch points.

2. Note that by standard approximation techniques, we can (and will) look at functions that are not in \( \mathcal{S}([1, \infty)) \).

What we really require is that \( \zeta(T_0) \) converges absolutely, and uniformly on compacts subsets of \( \mathbb{H}^3 \times \mathbb{H}^3 \); and that \( h \) (see (2.3)) decays sufficiently fast so that it can be used in the Selberg trace formula (see [EGM98]). In fact, we can start with \( h \), a holomorphic function on \( \{ s \in \mathbb{C} \mid \text{Im}(s) < 2 + \delta \} \) for some \( \delta > 0 \), satisfying \( h(1 + z^2) = O(1 + |z|^2)^{3/2 - \varepsilon} \) as \( |z| \to \infty \); and recover the function \( k \).
For \( \lambda \in \mathbb{C}, \lambda = 1 - s^2 \), the Selberg–Harish-Chandra transform of \( k \) is the function \( h \), defined by

\[
(2.5) \quad h(\lambda) = h(1 - s^2) = \frac{\pi}{s} \int_1^\infty k \left( 1 + \frac{t}{s} \right) \left( t^s - t^{-s} \right) \left( t - \frac{1}{t} \right) \frac{dt}{t}, \quad \lambda = 1 - s^2.
\]

In addition, let

\[
g(x) = \frac{1}{2\pi} \int h(1 + t^2)e^{-itx} \, dt.
\]

We can start with the function \( h \) and work backwards to find \( k \) [EGM98 Chapter 3]. The pair \( h, g \) is said to be admissible if \( h \) is a holomorphic function on \( \{ s \in \mathbb{C} \mid |\operatorname{Im}(s)| < 2 + \delta \} \) for some \( \delta > 0 \), satisfying \( h(1 + z^2) = O(1 + |z|^2)^{3/2-\epsilon} \) as \( |z| \to \infty \).

For \( v, w \in V \) let \( v \otimes w \) be the linear operator in \( V \), defined by, \( v \otimes w(x) = \langle x, w \rangle v \), where \( x \in V \).

**Lemma 2.1 (Fri05b).** Let \( k \in \mathcal{S}([1, \infty)) \) and \( h : \mathbb{C} \to \mathbb{C} \) be the Selberg–Harish-Chandra Transform of \( k \). Then

\[
(2.6) \quad K(P, Q, \Gamma, \chi) = \sum_{m \in D} h(\lambda_m) e_m(P) \otimes e_m(Q)
\]

\[
+ \frac{1}{4\pi} \sum_{\alpha = 1}^\kappa \sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma_\alpha]}{|\Lambda_\alpha|} \int_R h(1 + t^2) E_{\alpha l}(P, it) \otimes \overline{E_{\alpha l}(Q, it)} \, dt.
\]

The sum and integrals converge on compact subsets of \( \mathbb{R}^3 \times \mathbb{R}^3 \). Here \( D \) is an indexing set of the eigenfunctions \( e_m \) of \( \Delta \) with corresponding eigenvalues \( \lambda_m \), \( E_{\alpha l}(P, s) \) are the Eisenstein series associated to the singular cusps of \( \Gamma \), \( k_\alpha = \dim_{\mathbb{C}} V_\alpha \), and \( |\Lambda_\alpha| \) is the Euclidean area of a fundamental domain for the lattice \( \Lambda_\alpha \). If a cusp is regular it is omitted from the sum in (2.6).

Next, we split up \( K_\Gamma \) as a sum of two kernels. The first kernel

\[
H_\Gamma(P, Q) = \frac{1}{4\pi} \sum_{\alpha = 1}^\kappa \sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma_\alpha]}{|\Lambda_\alpha|} \int_R h(1 + t^2) E_{\alpha l}(P, it) \otimes \overline{E_{\alpha l}(Q, it)} \, dt,
\]

is not of Hilbert-Schmidt class, while the second kernel

\[
L_\Gamma(P, Q) = \sum_{m \in D} h(\lambda_m) e_m(P) \otimes e_m(Q)
\]

is of trace class.

Suppose \( Y > 0 \) is sufficiently large. Then for all \( A > Y \), there exists a compact set \( \mathcal{F}_A \subset \mathbb{H}^3 \) such that

\[
(2.7) \quad \mathcal{F} \equiv \mathcal{F}_A \cup \mathcal{F}_1(A) \cup \cdots \cup \mathcal{F}_n(A)
\]

is a fundamental domain for \( \Gamma \). The sets \( \mathcal{F}_\alpha(A) \) are cusp sectors (see [EGM98] Proposition 2.3.9). It follows that

\[
(2.8) \quad \lim_{A \to \infty} \left( \int_{\mathcal{F}_A} \operatorname{tr}_V(K_\Gamma(P, P)) \, dv(P) - \int_{\mathcal{F}_A} \operatorname{tr}_V(H_\Gamma(P, P)) \, dv(P) \right) = \int_\mathcal{F} \operatorname{tr}_V(L_\Gamma(P, P)) \, dv(P) = \sum_{m \in D} h(\lambda_m) < \infty.
\]

The infinite sum is absolutely convergent.

Let \( \mathcal{G}(s) \) denote the automorphic scattering matrix associated to \( \Gamma \) and \( \chi \) (see [Fri05b]), and let

\[
\phi(s) = \det \mathcal{G}(s).
\]
Upon applying the vector form of the Maaß-Selberg relations (see [Roe66], [Ven82], [Fri05b]), we obtain

\[ (2.9) \int_{\mathcal{F}} \text{tr}_{V}(H_{\Gamma}(P, P)) \, dv(P) = \]

\[ g(0) k(\Gamma, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \phi'(it) h(1 + t^2) \, dt + \frac{h(1) \text{tr} \Theta(0)}{4} + o(1). \]

The integral on the right hand side converges absolutely.

### 3. Induced Representations

Let \( \Gamma \) be an arbitrary Kleinian group, and let \( \Gamma_{1} \triangleleft \Gamma \) be a finite-index normal subgroup of index \( n \). Let \( \mathcal{F}, \mathfrak{F} \) be fundamental domains of \( \Gamma, \Gamma_{1} \) respectively, with \( \{\alpha_{i}\}_{i=1}^{n} \) a complete set of representatives for the right-cosets of \( \Gamma_{1} \setminus \Gamma \), satisfying

\[ (3.1) \mathfrak{F} = \bigcup_{i=1}^{n} \alpha_{i}(\mathcal{F}). \]

Let \( V \) be a finite-dimensional hermitian vector space, and let \( \chi \) be a finite-dimensional unitary representation of \( \Gamma_{1} \) in \( V \). Set

\[ \varpi(\gamma) = \begin{cases} \chi(\gamma), & \gamma \in \Gamma_{1} \\ 0, & \gamma \notin \Gamma_{1}. \end{cases} \]

Let \( \psi \equiv U^{\chi} \) be the induced representation of \( \chi \) from \( \Gamma_{1} \) to \( \Gamma \). More explicitly,

\[ \psi : \Gamma \mapsto \text{GL}(V^n) \]

and for \( \gamma \in \Gamma \) and \( v_{i} \in V \),

\[ (3.2) \psi(\gamma) \left( \sum_{i=1}^{n} \otimes v_{i} \right) = \sum_{i=1}^{n} \otimes \sum_{j=1}^{n} \varpi(\alpha_{i} \gamma \alpha_{j}^{-1}) v_{j}. \]

It follows from (3.2) that for \( \gamma \in \Gamma \),

\[ (3.3) \text{tr}_{V^n} \psi(\gamma) = \sum_{i=1}^{n} \text{tr}_{V} \varpi(\alpha_{i} \gamma \alpha_{i}^{-1}). \]

We will need the following result (see [RS71] and [VZ83] Theorem 2.1):

**Lemma 3.1.** There exists an isometry between the Hilbert spaces \( \mathcal{H}(\Gamma_{1}, \chi) \) and \( \mathcal{H}(\Gamma, \psi) \), which takes the operator \( \Delta(\Gamma_{1}, \chi) \) to \( \Delta(\Gamma, \psi) \).

We abuse notation and call a set \( \Theta \subset \Gamma \) *normal* if for all \( \gamma \in \Gamma \), we have \( \gamma \Theta \gamma^{-1} = \Theta \).

**Lemma 3.2.** Let \( \Gamma_{1} \triangleleft \Gamma \) be a finite-index, normal subgroup of \( \Gamma \) of index \( n \), \( \chi \in \text{Rep}(\Gamma_{1}, V) \), \( \psi \equiv U^{\chi} \in \text{Rep}(\Gamma, V^n) \), and let \( \Theta \subset \Gamma \) be a normal subset. Suppose that

\[ \int_{\mathcal{F}} \text{tr}_{V^n} K(P, P, \Theta, \psi) \, dv(P) \]

converges absolutely. Then

\[ \int_{\mathcal{F}} \text{tr}_{V^n} K(P, P, \Theta, \psi) \, dv(P) = \int_{\mathfrak{F}} \text{tr}_{V} K(P, P \cap \Gamma_{1}, \chi) \, dv(P). \]
Proof. By (3.3),
\[
\int_{\mathcal{F}} \text{tr}_{V^\infty} K(P, P, \Theta, \psi) \, dv(P) = \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma \in \Theta} \text{tr}_V \chi(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P)
\]
\[
= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma \in \Theta} \text{tr}_V \chi(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P)
\]
\[
= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma \in \Theta \cap \Gamma_1} \text{tr}_V \chi(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P)
\]
(by normality).

Now, since \(\Theta\) and \(\Gamma_1\) are normal, \(\gamma \in \Theta \cap \Gamma_1\) implies that \(\alpha_i \gamma \alpha_i^{-1} \in \Theta \cap \Gamma_1\). And as \(\gamma\) goes through each element of \(\Theta \cap \Gamma_1\), so does \(\alpha_i \gamma \alpha_i^{-1}\). So, for each \(i\), setting \(\beta = \alpha_i \alpha_i^{-1}\) we obtain
\[
\sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\gamma \in \Theta \cap \Gamma_1} \text{tr}_V \chi(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) \, dv(P) = \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_V \chi(\beta) K(P, \alpha_i^{-1} \beta \alpha_i P) \, dv(P)
\]
(3.4)
\[
= \sum_{i=1}^{n} \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_V \chi(\beta) K(\alpha_i P, \beta \alpha_i P) \, dv(P)
\]
(3.5)
\[
= \sum_{i=1}^{n} \int_{\alpha_i(\mathcal{F}) \cap \Gamma_1} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_V \chi(\beta) K(Q, \beta Q) \, dv(Q)
\]
(3.6)
\[
= \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \text{tr}_V \chi(\beta) K(Q, \beta Q) \, dv(Q)
\]
where in (3.5) we set \(Q = \alpha_i P\); in (3.4) we used Equation (3.3) and we used the fact that \(dv(\alpha_i Q) = dv(Q)\); and in (3.6) we tiled the fundamental domain \(\mathcal{F}\) according to (3.1).

\(\square\)

Lemma 3.2 works well when the point-pair-invariant \(k\) gives rise to an integral kernel of trace-class. However, a careful look at the proof, shows that we could replace \(\mathcal{F}\) by a truncated fundamental domain \(\mathcal{F}_A\).

For the rest of this section, we assume that \(\Gamma\) is cofinite. Let \(Y > 0\) be sufficiently large so that for \(A > Y\), \(\mathcal{F}\) decomposes into \(\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}^A\), where \(\mathcal{F}_A\) is compact and and \(\mathcal{F}^A\) is a union of cusp sectors (2.7). Since
\[
\mathcal{F} = \bigcup_{i=1}^{n} \alpha_i(\mathcal{F})
\]
it follows that \(\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}^A\) with \(\mathcal{F}_A = \bigcup_{i=1}^{n} \alpha_i(\mathcal{F}_A)\) and \(\mathcal{F}^A = \bigcup_{i=1}^{n} \alpha_i(\mathcal{F}^A)\); \(\mathcal{F}^A\) is also a union of cusp sectors.

**Lemma 3.3.** Let \(\Gamma_1 \triangleleft \Gamma\) be a finite-index, normal subgroup of \(\Gamma\) of index \(n\), \(\chi \in \text{Rep}(\Gamma_1, V)\), \(\psi \equiv U^x \in \text{Rep}(\Gamma, V^n)\), and let \(\Theta \subset \Gamma\) be a normal subset. Suppose that
\[
\int_{\mathcal{F}_A} \text{tr}_{V^\infty} K(P, P, \Theta, \psi) \, dv(P)
\]
converges absolutely. Then
\[
\int_{\mathcal{F}_A} \text{tr}_{V^\infty} K(P, P, \Theta, \psi) \, dv(P) = \int_{\mathcal{F}_A} \text{tr}_V K(P, P, \Theta \cap \Gamma_1, \chi) \, dv(P).
\]
4. Factorization Formula of the Selberg zeta-function

In this section we prove the analogue of the Venkov-Zograf factorization formula ([VZ83]) for arbitrary Kleinian groups.

Throughout this section, \( \Gamma \) is an arbitrary Kleinian group and \( \Gamma_1 \vartriangleleft \Gamma \) is a finite-index normal subgroup of index \( n \).

**Theorem 4.1.** Suppose that \( \Gamma_1 \vartriangleleft \Gamma \), \( \chi \in \text{Rep}(\Gamma_1, V) \), and that \( \psi = U^\chi \in \text{Rep}(\Gamma, V^n) \). Then for \( \text{Re}(s) > 1 \),

\[
Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi).
\]

**Proof.** By (2.2), it suffices to show that

\[
W(s, \Gamma_1, \chi) = W(s, \Gamma, \psi).
\]

Let \( \Theta = \Gamma^{\text{lox}} \subset \Gamma \) be the set of all loxodromic elements of \( \Gamma \). Note that \( \Theta \) is normal since the conjugate of a loxodromic element is loxodromic. For \( \text{Re} s > 1 \), \( \delta > 1 \), set

\[
k_s(\delta) = \frac{1}{4\pi} \left( \frac{\delta + \sqrt{\delta^2 - 1}}{\sqrt{\delta^2 - 1}} \right)^{-s}.
\]

For \( P, P' \in \mathbb{H}^3 \), set \( K_s(P, P') = k_s(\delta(P, P')) \), and, using (2.4), define \( K_s(P, P', \Theta, \psi) \) and \( K_s(P, P', \Theta \cap \Gamma_1, \chi) \). By [EGM98] page 185-198 and [Fri05a] Lemma 6.3, it follows that

\[
W(s, \Gamma_1, \chi) = \int_{\mathbb{H}^3} \text{tr}_V K_s(P, P, \Theta \cap \Gamma_1, \chi) \, dv(P),
\]

and

\[
W(s, \Gamma, \psi) = \int_{\mathbb{H}^3} \text{tr}_{V^n} K_s(P, P, \Theta, \psi) \, dv(P).
\]

The result now follows from Lemma 3.2. \( \square \)

Compare our proof with the proof given in [VZ83] Theorem 3.1.

If we let \( \chi = 1 \), the trivial one-dimensional representation, it follows that \( \psi = U^1 \) can be decomposed into irreducible sub-representations, explicitly:

\[
\psi = \bigoplus_{\vartheta \in (\Gamma_1 \setminus \Gamma)^*} n_{\vartheta} \vartheta,
\]

where \((\Gamma_1 \setminus \Gamma)^*\) denotes the set of all pairwise inequivalent irreducible unitary representations of the group \( \Gamma_1 \setminus \Gamma \), \( n_{\vartheta} \) is the dimension of \( \vartheta \). For more details see [Kir76].

Next, by (2.1) and (2.2) it follows that for \( \vartheta_1, \vartheta_2 \in \text{Rep}(\Gamma, V_i) \) \((i = 1, 2)\),

\[
Z(s, \Gamma, \vartheta_1 \oplus \vartheta_2) = Z(s, \Gamma, \vartheta_1)Z(s, \Gamma, \vartheta_2).
\]

We have

**Theorem 4.2.** Let \( \Gamma_1 \vartriangleleft \Gamma \). Then for \( \text{Re} s > 1 \),

\[
Z(s, \Gamma_1, 1) = Z(s, \Gamma, U^1) = \prod_{\vartheta \in (\Gamma_1 \setminus \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_{\vartheta}}.
\]

\(^3\text{Of course, each } \chi \text{ must be extended from } \Gamma_1 \setminus \Gamma \text{ to } \Gamma.\)
5. Factorization Formula for the Determinant of the Scattering Matrix

In this section we prove the analogous result to Theorem 4.1 for the automorphic scattering matrix. Throughout this section, \( \Gamma \) is a cofinite Kleinian group.

Let \( \Gamma_1 \triangleleft \Gamma, \chi \in \text{Rep}(\Gamma_1, V) \), and \( \psi = U^\chi \). Let \( k \in S([1, \infty)) \), and let \( h \) be given by (2.5). It follows from Lemma 3.1 that \( \Delta(\Gamma_1, \chi) \) and \( \Delta(\Gamma, \psi) \) have the same eigenvalues. Hence (from (2.3))

\[
\int_{\mathcal{F}} \text{tr}_{V^n} L(P, P, \Gamma, \psi) \, dv(P) = \int_{\mathcal{F}} \text{tr}_V L(P, P, \Gamma, \chi) \, dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_m).
\]

Now, since

\[
\int_{\mathcal{F}_A} \text{tr}_{V^n} K(P, P, \Gamma, \psi) \, dv(P) = \int_{\mathcal{F}_A} \text{tr}_V K(P, P, \Gamma, \chi) \, dv(P),
\]

it follows from Equation 2.8 that

\[
\int_{\mathcal{F}_A} \text{tr}_{V^n} H(P, P, \Gamma, \psi) \, dv(P) = \int_{\mathcal{F}_A} \text{tr}_V H(P, P, \Gamma, \chi) \, dv(P) + o(1) \quad A \to \infty.
\]

However, from Equation 2.9 we obtain

\[
(5.1) \quad \int_{\mathcal{F}_A} \text{tr}_{V^n} H(P, P, \Gamma, \psi) \, dv(P) = g(0)k(\Gamma, \psi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}(it)} h(1 + t^2) \, dt + \frac{h(1) \text{tr} \mathcal{S}(0)}{4} + o(1),
\]

and

\[
(5.2) \quad \int_{\mathcal{F}_A} \text{tr}_V H(P, P, \Gamma, \chi) \, dv(P) = g(0)k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}(it)}{\phi_{\Gamma_1}(it)} h(1 + t^2) \, dt + \frac{h(1) \text{tr} \mathcal{S}_{\Gamma_1}(0)}{4} + o(1),
\]

Hence, we have

\[
(5.3) \quad g(0)k(\Gamma, \psi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_{\Gamma}(it)}{\phi_{\Gamma}(it)} h(1 + t^2) \, dt + \frac{h(1) \text{tr} \mathcal{S}(0)}{4} = g(0)k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}(it)}{\phi_{\Gamma_1}(it)} h(1 + t^2) \, dt + \frac{h(1) \text{tr} \mathcal{S}_{\Gamma_1}(0)}{4} + o(1).\]

Equation (5.3) is true for all admissible pairs \( h, g \). Hence we must have

\[
k(\Gamma, \psi) = k(\Gamma_1, \chi),
\]

\[
\text{tr} \mathcal{S}(0) = \text{tr} \mathcal{S}_{\Gamma_1}(0),
\]

and

\[
(5.4) \quad \frac{\phi'_{\Gamma}(z)}{\phi_{\Gamma}(z)} = \frac{\phi'_{\Gamma_1}(z)}{\phi_{\Gamma_1}(z)};
\]

We will prove (5.4) shortly.

**Theorem 5.1.** Let \( \Gamma_1 \triangleleft \Gamma, \chi \in \text{Rep}(\Gamma_1, V) \), and \( \psi = U^\chi \in \text{Rep}(\Gamma, V^n) \). Let

\[
\phi_{\Gamma_1}(s) \equiv \det \mathcal{S}_{\Gamma_1}(s) \equiv \det \mathcal{S}(s, \Gamma_1, \chi),
\]

\[
\phi_{\Gamma}(s) \equiv \det \mathcal{S}(s, \Gamma, \psi).
\]

Then for all regular \( s \),

\[
\phi_{\Gamma_1}(s) = \phi_{\Gamma}(s).
\]
PROOF. For $r > 0, z \in \mathbb{C}$, let $h(z) = e^{-rz^2}$, (and it follows)
$$g(x) = \frac{e^{-r}}{\sqrt{4\pi r}} \frac{1}{x^2 + r^2}. $$
By considering the asymptotics of $\phi$, it follows that
$$k(\Gamma, \psi) = k(\Gamma_1, \chi),$$
and
$$\text{tr} \mathcal{S}_\Gamma(0) = \text{tr} \mathcal{S}_{\Gamma_1}(0).$$
Hence
$$\int_{\mathbb{R}} \frac{\phi'}{\phi} (it) h(1 + t^2) \, dt = \int_{\mathbb{R}} \frac{\phi'}{\phi} (it) h(1 + t^2) \, dt,$$
which implies that
$$\int_{\mathbb{R}} \frac{\phi'}{\phi} (it) e^{-rt^2} \, dt = \int_{\mathbb{R}} \frac{\phi'}{\phi} (it) e^{-rt^2} \, dt \quad (r > 0).$$
Now, by the functional equation for $\mathcal{S}_\Gamma(s)$ (see [Fri05a])
$$\mathcal{S}_\Gamma(s) \mathcal{S}_\Gamma(-s) = 1.$$ 
So
$$\phi'(s)\phi(-s) = 1.$$ 
Hence $\phi'(it)$ is an even function of $t$; so is $\frac{\phi'}{\phi} (it)$. Thus
$$\int_{0}^{\infty} \frac{\phi'}{\phi} (it) e^{-rt^2} \, dt = \int_{0}^{\infty} \frac{\phi'}{\phi} (it) e^{-rt^2} \, dt \quad (r > 0).$$
Next, the substitution $u = t^2$ allows us to rewrite the above integral as a Laplace transform, and by uniqueness (the Laplace transform is invertible), it follows that $\frac{\phi'}{\phi} (it) = \frac{\phi'}{\phi} (it)$, and (by analytic continuation)
$$\frac{\phi'}{\phi} (s) = \frac{\phi'}{\phi} (s).$$
Integrating and exponentiating, gives us a constant $C_1$, so that
$$\phi_{\Gamma} = C_1 \cdot \phi_{\Gamma_1}.$$ 
However, the functional equation (for $\mathcal{S}_\Gamma(s)$) implies that
$$\phi_{\Gamma}(0) = (-1)^{(kr - \text{tr} \mathcal{S}_{\Gamma}(0))/2} = (-1)^{(kr_1 - \text{tr} \mathcal{S}_{\Gamma_1}(0))/2} = \phi_{\Gamma_1}(0) \neq 0.$$ 
Hence $C_1 = 1$. \qed

References

[EGM98] J. Elstrodt, F. Grunewald, and J. Mennicke, Groups acting on hyperbolic space, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, Harmonic analysis and number theory.

[Fri05a] Joshua S. Friedman, The Selberg trace formula and Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations, Math. Z. 250 (2005), no. 4, 939–965.

[Fri05b] Joshua S. Friedman, The Selberg trace formula and Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations, Ph.D. thesis, Stony Brook University, 2005, http://arxiv.org/abs/math.NT/0612807.

[Hej76] Dennis A. Hejhal, The Selberg trace formula for PSL(2, $\mathbb{R}$). Vol. I, Springer-Verlag, Berlin, 1976, Lecture Notes in Mathematics, Vol. 548.

[Hej83] ________, The Selberg trace formula for PSL(2, $\mathbb{R}$). Vol. 2, Lecture Notes in Mathematics, vol. 1001, Springer-Verlag, Berlin, 1983.

[Kir76] A. A. Kirillov, Elements of the theory of representations, Springer-Verlag, Berlin, 1976, Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220.

[Roe66] Walter Roelcke, Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II, Math. Ann. 167 (1966), 292–337; ibid. 168 (1966), 261–324.

[RS71] D. B. Ray and I. M. Singer, $\ell$-torsion and the Laplacian on Riemannian manifolds, Advances in Math. 7 (1971), 145–210. MR MR0295381 (45 #4447).

[Ven82] A. B. Venkov, Spectral theory of automorphic functions, Proc. Steklov Inst. Math. (1982), no. 4(153), ix+163 pp. (1983), A translation of Trudy Mat. Inst. Steklov. 153 (1981).
[VZ83] A. B. Venkov and P. G. Zograf, *Analouges of Artin's factorization formulas in the spectral theory of automorphic functions associated with induced representations of Fuchsian groups*, Math. USSR Izvestiya **21** (1983), no. 3, 435–443.

DEPARTMENT OF MATHEMATICS AND SCIENCES, UNITED STATES MERCHANT MARINE ACADEMY, 300 STEAMBOAT ROAD, KINGS POINT, NY 11024

E-mail address: CrownEagle@gmail.com
E-mail address: friedmanj@usmma.edu
E-mail address: joshua@math.sunysb.edu