ON FUNCTION THEORY IN QUANTUM DISC:
COVARIANCE

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1 Modules over a Hopf algebra

We follow the approach of V. G. Drinfeld and M. Jimbo to constructing the quantum groups
theory [4]. Everywhere in the sequel the deformation parameter $q$ will be assumed to be a
number from the interval $(0, 1)$.

The quantum universal enveloping algebra $U_q\mathfrak{sl}_2$ is a Hopf algebra over $\mathbb{C}$
determined by the generators $K, K^{-1}, E, F$ and the relations

$$KK^{-1} = K^{-1}K = 1, \quad K^{\pm 1}E = q^{\pm 2}EK^{\pm 1}, \quad K^{\pm 1}F = q^{\mp 2}FK^{\pm 1},$$

$$EF - FE = (K - K^{-1})/(q - q^{-1}),$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$  

Note that

$$\varepsilon(E) = \varepsilon(F) = \varepsilon(K^{\pm 1} - 1) = 0,$$ 

$$S(K^{\pm 1}) = K^{-\mp 1}, \quad S(E) = -K^{-1}E, \quad S(F) = -FK,$$

with $\varepsilon : U_q\mathfrak{sl}_2 \to \mathbb{C}$ and $S : U_q\mathfrak{sl}_2 \to U_q\mathfrak{sl}_2$ being respectively the counit and the antipode of $U_q\mathfrak{sl}_2$.

It was shown in [3] that $U_q\mathfrak{sl}_2$ can be derived from the (topological) Hopf algebra $U_h\mathfrak{sl}_2$
over the ring of formal series $\mathbb{C}[[h]]$. The latter Hopf algebra is determined by its generators
$X^+, X^-, H$ and the relations

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \text{sh}(hH/2)/\text{sh}(h/2),$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X^\pm) = X^\pm \otimes e^{hH/4} + e^{-hH/4} \otimes X^\pm,$$

$$\varepsilon(H) = \varepsilon(X^\pm) = 0, \quad S(H) = -H, \quad S(X^\pm) = -e^{\pm h/2}X^\pm.$$ 

$U_h\mathfrak{sl}_2$ is a deformation of the ordinary universal enveloping algebra $U\mathfrak{sl}_2$, and the formal
passage to a limit as $h \to 0$ in the determining relations of $U_h\mathfrak{sl}_2$ yields $U\mathfrak{sl}_2 \simeq U_h\mathfrak{sl}_2/h \cdot U_h\mathfrak{sl}_2.$

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All the $U_q\mathfrak{sl}_2$-modules in the sequel will be assumed to be $\mathbb{R}$-graded, with $K^{-1}v = \exp(\deg(v)h/2) \cdot v$ for any homogeneous element $v$. This restriction allows one to replace $U_q\mathfrak{sl}_2$ by $U_q\mathfrak{b}_+$ while passing to the limit as $q \to 1$ as well as in tedious calculations:

$$q = e^{-h/2}, \quad K = e^{-hH/2}, \quad E = X^{+}e^{-hH/4}, \quad F = e^{hH/4}X^{-}. \quad (1.1)$$

Everywhere below $A$ will stand either for the Hopf algebra $U_q\mathfrak{sl}_2$ or the subalgebra $U_q\mathfrak{b}_+ \subset U_q\mathfrak{sl}_2$, generated by $K, K^{-1}, E$.

Tensor product of $A$-modules $V_1, V_2$ is defined as follows:

$$A \xrightarrow{\Delta} A \otimes A \to \text{End}(V_1) \otimes \text{End}(V_2) \cong \text{End}(V_1 \otimes V_2),$$

and the trivial $A$-module $\mathbb{C}$ as

$$A \xrightarrow{\zeta} \text{End}(\mathbb{C}) \cong \mathbb{C}.$$  

The morphisms of $A$-modules $\eta : V \to \mathbb{C}$ are also called invariant integrals.

The dual $A$-module $V^*$ is defined via the antipode $S$ as follows:

$$(al)(v) = l(S(a)v), \quad \forall a \in A, \ v \in V, \ l \in V^*.$$ 

It follows from the definition of the antipode $S$ that the natural pairing $V^* \otimes V \to \mathbb{C}$ is an invariant integral.

**Proposition 1.1** Linear functional $\eta : V_1 \otimes V_2 \to \mathbb{C}$ is an invariant integral iff

$$\eta(\alpha v_1 \otimes v_2) = \eta(v_1 \otimes S(\alpha)v_2) \quad \text{for all} \quad \alpha \in A, \ v_1 \in V_1, \ v_2 \in V_2. \quad (1.2)$$

**Proof.** (1.2) means that the map $V_1 \to V_2^*$ given by the linear functional $\eta$ is a morphism of $A$-modules. Hence it suffices to verify (1.2) for the generators $K^{\pm 1}, E, F$, which is straightforward. On the other hand, in the case (1.2) is satisfied, $\eta$ becomes an invariant integral due to the definition of the antipode $S$. \hfill $\Box$

Given $A$-modules $V_1, V_2$, we consider the vector space of finite dimensional linear operators $\text{Hom}_\mathbb{C}(V_1, V_2)_f = \{ L \in \text{Hom}_\mathbb{C}(V_1, V_2) | \dim(LV_1) < \infty \}$.

**Proposition 1.2** There exists a unique $A$-module structure on $\text{Hom}_\mathbb{C}(V_1, V_2)_f$ such that the linear map $\text{Hom}_\mathbb{C}(V_1, V_2)_f \otimes V_1 \to V_2, \ L \otimes v \mapstoLv, \ L \in \text{Hom}_\mathbb{C}(V_1, V_2)_f, \ v \in V_1, \ is \ an \ A$-module morphism.

**Proof.** The existence is evident in view of the canonical isomorphism of the vector spaces $i : \text{Hom}_\mathbb{C}(V_1, V_2)_f \xrightarrow{\sim} V_2 \otimes V_2^*$. It remains to demonstrate that for any $A$-module structure on $\text{Hom}_\mathbb{C}(V_1, V_2)_f$ with the above properties, $i$ is a morphism of $A$-modules. This can be easily verified for the generators $K, K^{-1}, E, F$. \hfill $\Box$

**Corollary 1.3** The finite dimensional linear operators in $V$ form an $A$-module $\text{End}(V)_f \cong V \otimes V^*$. \hfill $\Box$
Proposition 1.4 The linear functional

$$\text{End}(V_f) \to \mathbb{C}, \quad L \mapsto \text{tr}(LK^{-1})$$ (1.3)

is an invariant integral.

Proof. Let $i_{st} : V \hookrightarrow V^{**}$ be the standard embedding of vector spaces. It follows from the relation $S^2(a) = K^{-1}aK$ that the map $i : V \hookrightarrow V^{**}, \quad i : v \mapsto i_{st} \cdot K^{-1}v$ is an $A$-module morphism. It induces an invariant integral

$$\text{End}(V_f) \simeq V \otimes V^* \overset{\text{id} \otimes \text{id}}{\to} V^{**} \otimes V^* \to \mathbb{C}.$$ 

Evidently, this integral coincides with that in (1.3). \qed

To conclude, note that the tensor product of $A$-module morphisms is again an $A$-module morphism. More exact statement is that $A$-modules and their morphisms constitute a tensor category [3].

2 Covariant algebras and bimodules

Consider an algebra $F$, which is also an $A$-module. $F$ is said to be an $A$-module (covariant) algebra if the multiplication

$$m : F \otimes F \to F, \quad m : f_1 \otimes f_2 \mapsto f_1f_2; \quad f_1, f_2 \in F,$$

is an $A$-module morphism [1].

In the case of a unital algebra $F$ the above definition also includes the assumption

$$\forall a \in A \quad a \cdot 1 = \varepsilon(a)1.$$ (2.1)

An element $v$ of a $A$-module $V$ is called an invariant if the map $\mathbb{C} \to V, \quad z \mapsto z \cdot v, \quad z \in \mathbb{C},$ is an $A$-module morphism. In this context, (2.1) claims that the unit of $F$ is an invariant.

One can view the algebra $\text{End}(V_f) \simeq V \otimes V^*$ and the tensor algebra $T(V) = \mathbb{C} + V + V \otimes V + \ldots,$ associated to the $A$-module $V$, as the examples of covariant algebras.

A bimodule $M$ over a covariant algebra $F$ is said to be $A$-module (or covariant) if $M$ itself is an $A$-module, and the maps

$$F \otimes M \to M, \quad f \otimes m \mapsto fm,$$

$$M \otimes F \to M, \quad m \otimes f \mapsto mf, \quad f \in F, \quad m \in M,$$

are $A$-module morphisms.

Impose the notation $U_q\mathfrak{su}(1,1) = (U_q\mathfrak{sl}_2,*)$, $U_h\mathfrak{su}(1,1) = (U_h\mathfrak{sl}_2,*)$ for the Hopf $*$-algebras whose involutions are given by

$$h^* = h, \quad H^* = H, \quad (X^\pm)^* = -X^\mp,$$

$$E^* = -KF, \quad F^* = -EK^{-1}, \quad K^* = K.$$
A covariant algebra $F$ is called a covariant $*$-algebra if it is equipped with an involution, and
\[ \forall f \in F \quad (\xi f)^* = (S(\xi))^* f^* \tag{2.2} \]
for all elements $\xi$ of the $*$-Hopf algebra.

It is very well known [6, 9] that $\text{Pol}(\mathbb{C})_q$ can be equipped with a structure of a covariant $*$-algebra. This result was extended onto the case of "prehomogeneous vector spaces of parabolic type" in [10], where the detailed calculations were also performed for the case of $\text{Pol}(\mathbb{C})_q$. In particular, one has [10]:

\[ K^{\pm}z = q^{\pm 2}z, \quad Fz = q^{1/2}, \quad Ez = -q^{1/2}z^2, \]
\[ K^{\pm 1}z^* = q^{\mp 2}z^*, \quad Ez^* = q^{-3/2}, \quad Fz^* = -q^{5/2}z^{*2}; \tag{2.3} \]

("in a different notation" we have

\[ Hz = 2z, \quad X^-z = e^{-h/4}, \quad X^+z = -e^{h/4}z^2, \]
\[ H z^* = -2z^*, \quad X^+z^* = e^{h/4}, \quad X^-z^* = -e^{-h/4}z^{*2}. \tag{2.4} \]

Let $F$ be a covariant algebra, $M$ a covariant bimodule over $F$, and $\eta : M \to \mathbb{C}$ — an invariant integral. Proposition 1.1 implies the following "formula of integration by parts":

**Proposition 2.1** \( \forall f \in F, \psi \in M, a \in A \)

\[ \int (af)\psi d\eta = \int f(S(a)\psi)d\eta, \quad \int (a\psi)f d\eta = \int \psi(S(a)f)d\eta. \]

3 The covariant algebra $D(U)_q$

We refer to [12] for definitions of $\text{Pol}(\mathbb{C})_q$-bimodules $D(U)_q$, $D(U)'_q$ of finite functions and distributions respectively in the quantum disc. It follows from the definitions that $D(U)'_q$ may be identified with the space of formal series

\[ f = \sum_{jk} a_{jk} z^jz^{*k}. \tag{3.1} \]

The subspace $\text{Pol}(\mathbb{C})_q$ is constituted by finite sums of the form (3.1), and the structure of algebra in $\text{Pol}(\mathbb{C})_q$ and that of bimodule in $D(U)'_q$ are given by the commutation relation $z^*z = q^2zz^* + 1 - q^2$. It follows from the definition of $D(U)_q$ that $f \in D(U)_q$ iff

\[ z^{*N} \cdot f = f \cdot z^N = 0 \tag{3.2} \]

for some $N \in \mathbb{N}$. The space $D(U)'_q$ is equipped with the topology of coefficientwise convergence, so $\text{Pol}(\mathbb{C})_q$ and $D(U)_q$ become its dense linear subspaces (see [12]).

It follows from the covariance of $\text{Pol}(\mathbb{C})_q$, together with (2.3), (3.1), (3.2), that the action of the Hopf algebra $U_q\mathfrak{sl}(1,1)$ can be extended by continuity onto $D(U)'_q$ and hence transferred onto $D(U)_q$. Of course, $D(U)_q$ is a covariant $*$-algebra, and $D(U)'_q$ a covariant $D(U)_q$-bimodule.
The work [12] contains a construction of an isomorphism $T$ between $D(U)_q$ and the algebra of matrices $(l_{mn})_{m,n \in \mathbb{Z}_+}$ with finitely many nonzero matrix elements. Impose a finite function $f_0$ in the quantum disc such that

$$l_{mn}(f_0) = \begin{cases} 
1 & m = n = 0 \\
0 & m^2 + n^2 \neq 0
\end{cases}. $$

Let $C[z]_q$ and $C[z^*]_q$ stand for covariant subalgebras of $\text{Pol}(\mathbb{C})_q$, generated respectively by $z$ and $z^*$. It follows from the definitions that $D(U)_q$ is a covariant $\text{Pol}(\mathbb{C})_q$-bimodule. As for $f_0$, one evidently has $f_0 \cdot f_0 = f_0$, and

**Proposition 3.1**

1. $\{f \in D(U)_q \mid z^* f = f z = 0\} = \mathbb{C} f_0$
2. $D(U)_q = C[z]_q \cdot f_0 \cdot C[z^*]_q$. 

Let $\bar{H} = \{f \in D(U)_q \mid f z = 0\}$. The relation (2.3) and the covariance of the $\text{Pol}(\mathbb{C})_q$-bimodule $D(U)_q$ imply

**Proposition 3.2** $U_q \mathfrak{b}_+ \cdot \bar{H} \subset \bar{H}$.

It is also easy to deduce

**Proposition 3.3** $\bar{H} = C[z]_q f_0 = \text{Pol}(\mathbb{C})_q f_0 = D(U)_q f_0$.

According to propositions 3.2 and 3.3, the vector space $\bar{H}$ is a $U_q \mathfrak{b}_+$-module and a $D(U)_q$-module. We denote the corresponding representations of $U_q \mathfrak{b}_+$ and $D(U)_q$ in $\bar{H}$ respectively by $\bar{\Gamma}$ and $\bar{T}$.

We also use the notation $y = 1 - zz^*$. The following is a straightforward consequence of the definitions.

**Proposition 3.4** $K^{\pm 1} \psi(y) = \psi(y)$ for any $\psi(y) \in D(U)_q$.

**Theorem 3.5** The linear functional

$$\eta : D(U)_q \to \mathbb{C}, \quad \eta : f \mapsto \text{tr}(\bar{T}(f)\bar{\Gamma}(K^{-1}))$$

is an invariant integral.

**Proof.** It is easy to equip $\bar{H}$ with a structure of pre-Hilbert space in such a way that $\bar{T}(f^*) = \bar{T}(f)^*$ for all $f \in D(U)_q$. By a virtue of proposition 3.4 one has $K^{-1} f_0 = f_0$. This implies $K^{-1}(z^n f_0) = q^{-2n} z^n f_0$, and hence the linear functional $\eta$ is real: $\eta(f^*) = \eta(f)$ for all $f \in D(U)_q$. In view of (2.2), it suffices to show the $U_q \mathfrak{b}_+$-invariance of this linear functional. For that, by proposition 1.4, it suffices to demonstrate that the morphism of algebras $\bar{T} : D(U)_q \to \text{End}(\bar{H})_f$ is also a morphism of $U_q \mathfrak{b}_+$-modules. To see that, one has to
apply the covariance of $D(U)_q$ and to use the same argument as in the proof of proposition 1.2.

**Remark 3.6.** The non-negativity of the invariant integral $\eta$ follows from the existence of a positive scalar product under which $\bar{T}(z^*) = T(z)^*$ (see section 1 of [12]).

Recall [12] that each element $f \in D(U)_q$ admits a unique decomposition

$$f = \sum_{j > 0} z^j \psi_j(y) + \psi_0(y) + \sum_{j > 0} \psi_{-j}(y) z^{-j}. \quad (3.4)$$

Here $\psi_j$ are finite functions defined on $q^{2Z+}$ such that $\text{card}\{(i,j) \mid \psi_j(q^{2i}) \neq 0\} < \infty$.

**Remark 3.7.** The linear functional

$$\int_{U_q} f d\nu = (1 - q^2) \sum_{j=0}^{\infty} \psi_0(q^{2j}) q^{-2j}$$

used in [12] differs only by a scalar multiple from the linear functional $\eta$, and hence is an invariant integral.

**Proposition 3.8**

$$X^+ f_0 = -\frac{e^{-h/4}}{1 - e^{-h}} f_0, \quad X^+ f_0 = -\frac{e^{-3h/4}}{1 - e^{-h}} f_0 z^* \quad (3.5)$$

**Proof.** In virtue of (2.2), it suffices to verify the first relation. It was shown above that $X^+ f_0 \subset H = \mathbb{C}[z] f_0$. Besides, $H(X^+ f_0) = 2X^+ f_0$. This implies $X^+ f_0 = c_+ z f_0$ for some constant $c_+$. The value of $c_+$ can be found via an application of $X^+$ to the relation $z^* \cdot f_0 = 0$:

$$0 = X^+(z^* f_0) = e^{h/4} f_0 + e^{h/2} z^* c_+ z f_0, \quad 0 = e^{h/4} + e^{h/2}(1 - e^{-h}) c_+.$$

The proofs of the basic results announced in [12] require the following statement.

**Theorem 3.9** $f_0$ generates the $U_q\mathfrak{sl}_2$-module $D(U)_q$.

**Proof.** Use the covariance of $D(U)_q$, proposition 3.8 and (2.4) to get

$$X^+(z^j f_0) = a_j z^{j+1} f_0, \quad \left\{ \begin{array}{l}
X^-(f_0 z^{*k}) = b_k f_0 z^{*(k+1)} \\
X^-(z^{j+1} f_0 z^{*k}) = c_j z^j f_0 z^{*k} + d_{jk} z^{j+1} f_0 z^{*(k+1)}
\end{array} \right. \quad (3.7)$$

and the inequalities $a_j < 0$, $b_k < 0$, $c_{jk} > 0$, $d_{jk} < 0$ for all $j, k \in Z_+$.

Consider the subspaces $L_m, m \in Z_+$:

$$L_m = \{ f \in D(U)_q \mid f = \sum_{j=0}^{m} \sum_{i=0}^{\infty} a_{ij} z^j f_0 z^{*j}, a_{ij} \in \mathbb{C} \}.$$

It suffices to prove that for all $m$, $U_q\mathfrak{sl}_2 \cdot f_0 \supset L_m$. We proceed by induction. Firstly, $L_0 = U_q \mathfrak{b}_+ \cdot f_0$ because of (3.6) and $a_{j} \neq 0$ for all $j \in Z_+$. Secondly, $L_{m+1} \subset L_m + X^- L_m$ due to (3.7) and $b_k \neq 0$, $c_{jk} \neq 0$, $d_{jk} \neq 0$ for all $j, k \in Z_+$. □
Corollary 3.10 (Uniqueness of the invariant integral)
\[ \dim \text{Hom}_{U_q\mathfrak{sl}_2}(D(U)_q, \mathbb{C}) \leq 1. \]

Proof. If \( V \) is a free \( U_q\mathfrak{sl}_2 \)-module, \( \dim \text{Hom}_{U_q\mathfrak{sl}_2}(V, \mathbb{C}) = 1 \). On the other hand, theorem 3.9 assures the embedding of the vector spaces \( \text{Hom}_{U_q\mathfrak{sl}_2}(D(U)_q, \mathbb{C}) \hookrightarrow \text{Hom}_{U_q\mathfrak{sl}_2}(V, \mathbb{C}). \) \( \square \)

Finally, note that one can prove the existence and the uniqueness of an invariant integral (\( \dim \text{Hom}_{U_q\mathfrak{sl}_2}(D(U)_q, \mathbb{C}) = 1 \)) in many different ways. These are not those facts themselves that make an interest, but the explicit formula (3.3) for an invariant integral. A similar formula was published in [7].

4 Invariance of the operator \( \square \)

Consider a module \( V \) over a Hopf *-algebra \( A \). By its definition, the antimodule \( V^* \) coincides with \( V \) as an Abelian group, but the multiplication by scalars and the \( A \)-action in \( V \) are given by
\[
(\lambda, v) \mapsto \overline{\lambda} \cdot v; \quad (a, v) \mapsto (S(a))^*v.
\]

A sesquilinear form \( V_1 \times V_2 \rightarrow \mathbb{C} \) is said to be invariant if the associated linear functional \( \eta : V_2 \otimes V_1 \rightarrow \mathbb{C} \) is an invariant integral.

It follows from the invariance of the scalar product in \( V \) that
\[
\forall a \in A, \ v', v'' \in V \quad (av', v'') = (v', a^*v '').
\]
In fact, by a virtue of proposition 1.1,
\[
(av', v'') = \eta(v'' \otimes av') = \eta(S^{-1}(a)v'' \otimes v') = \eta(a^*v'' \otimes v') = (v', a^*v '').
\]

Proposition 4.1 The scalar product \( \langle f_1, f_2 \rangle = \int_{U_q} f_1^* f_2 d\nu \) in \( D(U)_q \) is invariant.

Proof. This is a straightforward consequence of invariance of the integral \( \int_{U_q} f d\nu \) proved in the previous section. \( \square \)

It is easy to show that the structure of a covariant algebra in \( \Omega(\mathbb{C})_q \) introduced in [10] can be transferred by a continuity onto \( \Omega(U)_q \). The differential \( d : \Omega(U)_q \rightarrow \Omega(U)_q \) is certainly a morphism of \( U_q\mathfrak{sl}_2 \)-modules.

Consider the integral \( \Omega(U)^{(1,1)}_q \rightarrow \mathbb{C}, \int_{U_q} f d\nu \) introduced in [12, section 4]. This linear functional is an invariant integral since the linear operator
\[
\Omega(U)^{(1,1)}_q \rightarrow D(U)_q; \quad f d\nu \mapsto f \cdot (1 - zz^*)^2
\]
is a morphism of \( U_q\mathfrak{sl}_2 \)-modules.
The work [12] implements the bimodules $\Omega(\mathbb{C})^{(0,+)}_{\lambda,q}$, $\lambda \in \mathbb{R}$, over the algebra $\Omega(\mathbb{C})^{(0,+)}_q$. It was shown in [10] that these bimodules are covariant, and the operator $\overline{\mathcal{J}} : \Omega(\mathbb{C})^{(0,+)}_{\lambda,q} \to \Omega(\mathbb{C})^{(0,+)}_{\lambda,q}$ is a morphism of $U_q\mathfrak{sl}_2$-modules. Furthermore, for the generator $v_\lambda$ involved into the definition of $\Omega(\mathbb{C})^{(0,+)}_{\lambda,q}$, one has (10): $\overline{\mathcal{J}} v_\lambda = 0$,

$$K^{\pm 1} v_\lambda = q^{\pm \lambda} \cdot v_\lambda, \quad F v_\lambda = 0.$$ 

One can deduce also $Ev_\lambda = -q^{1/2} \frac{1 - q^{2\lambda}}{1 - q^2} z v_\lambda$ via observing that $Ev_\lambda \neq 0$, $Ev_\lambda = \text{const} \cdot z v_\lambda$,

$$F(Ev_\lambda) = -(EF - FE) v_\lambda = -q^{\lambda} q^{-\lambda} \frac{q - q^{-1}}{q - q^{-1}} v_\lambda, \quad F(z v_\lambda) = F(z)(K^{-1} v_\lambda) = q^{1/2} q^{-\lambda} v_\lambda.$$

One can find in [12] the construction of the covariant algebra $\Omega(U)_q^{(0,+)}$ and the bimodules $\Omega(U)^{(0,+)}_{\lambda,q}$ over this algebra as completions in some special topology. It is easy to show that the actions of the Hopf algebra $U_q\mathfrak{sl}_2$ and the operator $\overline{\mathcal{J}}$ can be transferred by a continuity from $\Omega(\mathbb{C})^{(0,+)}_{\lambda,q}$ onto $\Omega(U)^{(0,+)}_{\lambda,q}$. Thus we obtain the covariant algebra, covariant bimodules and morphisms $\overline{\mathcal{J}} : \Omega(U)_q^{(0,+)} \to \Omega(U)^{(0,+)}_{\lambda,q}$ of $U_q\mathfrak{sl}_2$-modules.

Concerning the following proposition, we refer the reader to the relations (2.4), (2.5) of [12] which determine the scalar products in $\Omega(U)_q^{(0,0)}$ and $\Omega(U)^{(0,0)}_{\lambda,q}$.

**Proposition 4.2** The scalar products in $\Omega(U)_q^{(0,0)}$ and $\Omega(U)^{(0,0)}_{\lambda,q}$ are invariant.

**Proof.** One can show that the linear operators

$$j_0 : \Omega(U)_q^{(0,0)} \otimes \Omega(U)^{(0,0)}_{\lambda,q} \to \Omega(U)^{(0,0)}_q,$$

$$j_0 : (f_2 \cdot v_\lambda) \otimes (f_1 \cdot v_\lambda) \mapsto f_2^* \cdot f_1(1 - zz^*)^{-\lambda},$$

$$j_1 : \Omega(U)^{(0,1)}_q \otimes \Omega(U)^{(0,1)}_{\lambda,q} \to \Omega(U)^{(1,1)}_q,$$

$$j_1 : (f_2 v_\lambda dz^*) \otimes (f_1 v_\lambda dz^*) \mapsto f_2^* \cdot f_1(1 - zz^*)^{-\lambda} dz dz^*$$

are the morphisms of $U_q\mathfrak{sl}_2$-modules. So it remains to apply the invariance of the integrals

$$\Omega(U)_q^{(0,0)} \to \mathbb{C}, \quad \Omega(U)^{(1,1)}_q \to \mathbb{C},$$

$$f \mapsto \int f d\nu, \quad f : dz dz^* \mapsto \int f \cdot (1 - zz^*)^2 d\nu,$$

which was proved above. $\square$

An immediate consequence of proposition 4.2 is

**Theorem 4.3** The restrictions of linear operators

$$\overline{\mathcal{J}} : L^2(d\mu)_q \to L^2(d\nu)_q, \quad \square = -\overline{\mathcal{J}} \overline{\mathcal{J}} : L^2(d\nu)_q \to L^2(d\nu)_q$$

onto the dense linear subspaces $\Omega(U)^{(0,1)}_q$ and $D(U)_q$ are morphisms of $U_q\mathfrak{sl}_2$-modules.
If $V$ is a module over the Hopf algebra $U_q\mathfrak{sl}_2$, then the invariance of $v \in V$ means that $Ev = Fv = (K^{\pm 1} - 1)v = 0$.

**Proposition 4.4** The map which takes a distribution $k \in D(U_q)'$ to the linear functional $D(U_q) \to \mathbb{C}$, $f \mapsto \int_U k \cdot fd\nu$, realizes an isomorphism between the $U_q\mathfrak{sl}_2$-module $D(U_q)'$ and the $U_q\mathfrak{sl}_2$-module dual to $D(U_q)$.

**Proof.** It suffices to use proposition 2.1, the invariance of integral, and the definition of a dual $U_q\mathfrak{sl}_2$-module. \hfill \Box

Transfer by a continuity the $U_q\mathfrak{sl}_2$-action from the algebra $(\mathbb{C}[q^\mathbb{Z}]) \otimes \mathbb{C}[q^\mathbb{Z}]$ onto its completion $D(U_q)'$.

**Proposition 4.5**

1. The map that takes a "kernel" $K \in D(U \times U)'_q$ to the linear operator $D(U)_q \to D(U)_q'$, $f \mapsto \int_U K(z,\zeta)f(\zeta)d\nu$ is one-to-one.

2. The integral operator $D(U)_q \to D(U)'_q$ is a morphism of $U_q\mathfrak{sl}_2$-modules iff its kernel is invariant.

**Proof.** The first statement is a direct consequence of the definitions, and the second one follows from proposition 2.1. In fact, the invariance of a kernel $K$ is equivalent to being a solution of the "partial differential equation":

\[(\xi \otimes 1)K = (1 \otimes S^{-1}(\xi))K, \quad \xi \in U_q\mathfrak{sl}_2.\] \hfill \Box

**Remark 4.6.** The invariance of \Box and proposition 4.5 hint that \Box^{-1} is an integral operator with invariant kernel.

## 5 The operator $\Box$ and the Casimir element $\Omega$

Consider the element

\[\Omega = FE + \frac{1}{(q^{-1} - q)^2}(q^{-1}K^{-1} + qK - (q^{-1} + q))\] \hfill (5.1)

of $U_q\mathfrak{su}(1,1)$. In view of (1.1) one also has

\[\Omega = X^{-}X^{+} + \frac{1}{\text{sh}^2(h/2)}\text{sh}(Hh/4)\text{sh}((H + 2)h/4).\] \hfill (5.2)

(At the limit $h \to 0$ we get $\Omega = X^{-}X^{+} + \frac{1}{4}H(H + 2)$. It is well known (see [3]) and is easy to verify that $\Omega$ is in the center of $U_q\mathfrak{su}(1,1)$.)

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It is worthwhile to note that

\[ \Omega^* = \Omega, \quad S(\Omega) = \Omega. \]  \hspace{1cm} (5.3)

In fact,

\[ (FE)^* = FE, \quad (K^{\pm 1})^* = K^{\pm 1}, \quad S(FE) = EF, \]

\[ S(\Omega) - \Omega = \frac{K^{-1} - K}{q^{-1} - q} - \frac{q^{-1}K^{-1} + qK - q^{-1}K - qK^{-1}}{(q^{-1} - q)^2} = 0. \]

Remind that the Laplace-Beltrami operator \( \square \) was defined in [12] as \( \square = -\partial^* \partial \).

**Lemma 5.1** For any function \( \psi(t) \) with a finite carrier inside \( q^{-2Z} \),

\[ \square \psi(x) = -Dx(1 - q^{-1}x)D\psi(x), \]

with \( x = (1 - zz^*)^{-1} \), \( D : f(t) \mapsto (f(q^{-1}t) - f(qt))/(q^{-1}t - qt) \).

**Proof.** We use here the scalar product considered in section 4 with \( \lambda = 0 \). To prove the lemma, it suffices to obtain the relations

\[ \|\psi(x)\|^2 = q^2 \int_1^\infty |\psi(t)|^2 d_{q^{-2}}t, \]  \hspace{1cm} (5.4)

\[ \|\mathcal{D}\psi(x)\|^2 = q^2 \int_1^\infty (Dt(q^{-1}t - 1)D\psi(t))\overline{\psi(t)}d_{q^{-2}}t, \]  \hspace{1cm} (5.5)

with

\[ \int_1^\infty f(t)d_{q^{-2}}t \overset{\text{def}}{=} (q^{-2} - 1) \sum_{m=0}^\infty f(q^{-2m})q^{-2m}, \]

and \( \psi \) is a function with compact support \( \text{supp} \psi \in q^{-2Z} \).

(5.4) is obvious, and (5.5) follows from the relation

\[ \mathcal{D}f(y) = -z \frac{f(y) - f(q^2y)}{y - q^2y} dz^*, \]  \hspace{1cm} (5.6)

with \( y = 1 - zz^* \). (5.6) implies that

\[ \mathcal{D}\psi(x) = -z(1 - q^2) x(\psi(x) - \psi(q^{-2}x))dz^* \]

for any function \( \psi \) with a compact carrier \( \text{supp} \psi \in q^{-2Z} \). Hence

\[ \|\mathcal{D}\psi(x)\|^2 = (1 - q^2)^{-1} \sum_{m=0}^\infty \left| \psi(q^{-2m}) - \psi(q^{-2m+2}) \right|^2 (1 - q^{2m+2})q^{-2m} = \]

\[ q^{-2}(q^2 - 1)^{-2} \int_1^\infty |\psi(t) - \psi(q^{-2}t)|^2 (1 - q^{2t^{-1}})d_{q^{-2}}t = \]
\[
= - \int_1^\infty \left| \frac{\psi(t) - \psi(q^{-2}t)}{t - q^{-2}t} \right|^2 \cdot t(1 - q^{-2}t) dq^{-2}t.
\]

We are to apply the "integration by parts" formula
\[
\int_0^\infty u_1(x) \cdot \frac{u_2(q^{-2}x) - u_2(x)}{q^{-2}x - x} dq^2x = -q^2 \int_0^\infty \frac{u_1(x) - u_1(q^2x)}{x - q^2x} \cdot u_2(x) dq^2x,
\]
with \(\int_0^\infty u(x) dq^2x = (1 - q^2) \sum_{m=-\infty}^{\infty} u(q^{2m})q^{2m}\). In this way we obtain
\[
\square \psi = -B_+ t(1 - q^{-2}t)B_- \psi,
\]
with \(B_+, B_-\) being the linear operators given by \(B_\pm f(t) = \frac{f(t) - f(q^\pm t)}{t - q^\pm t}\). Finally we have
\[
\square \psi = -D(q(t)(1 - q^{-1}t)(B_- \psi)) \bigg|_{qt} = -D t(1 - q^{-1}t) D \psi.
\]

**Proposition 5.2** For all \(f \in D(U)_q\) one has \(q \square f = \Omega f\).

**Proof.** Let \(f_j(x) = \begin{cases} 1, & x = q^{-2j} \\ 0, & x \neq q^{-2j} \end{cases}\),
with \(x = (1 - z z^*)^{-1}\). By a virtue of theorems 3.9, 4.3, it suffices to prove that \(\square f_0 = \Omega f_0\). Now (2.4), (3.5) imply that
\[
\Omega f_0 = X^{-} X^+ f_0 = X^{-} \left( -\frac{e^{-h/4}}{1 - e^{-h} z} f_0 \right) =
\]
\[
= -\frac{e^{-h/2}}{1 - e^{-h} f_0} + \frac{e^{-3h/2}}{(1 - e^{-h})^2} z f_0 z^* - \frac{1}{q^{-1} - q} f_0 + \frac{q^2}{q^{-1} - q} f_1.
\]
It remains to apply lemma 5.1:
\[
\square f_0 = -D x(1 - q^{-1}x) D f_0 = -\frac{f_0}{1 - q^2} + \frac{q^2 f_1}{1 - q^2}.
\]

**Corollary 5.3** The Laplace-Beltrami operator \(\square\) is extendable by a continuity from the linear subspace \(D(U)_q \subset D(U)'_q\) onto the entire distribution space \(D(U)'_q\).

**Corollary 5.4** For any function \(\psi(t)\) on \(q^{-2}\mathbb{Z}_+\) one has
\[
\Omega \psi(x) = q \square \psi(x) = q D x(q^{-1}x - 1) D \psi(x),
\]
with \(x = (1 - z z^*)^{-1}\).
Consider the Hilbert space $L^2(d\nu)_q^{(0)}$ of such functions on $q^{-2Z_+}$ that $\int_1^{\infty} |f(x)|^2 d_{q^{-2}x} < \infty$. Let $\Box^{(0)} : L^2(d\nu)_q^{(0)} \rightarrow L^2(d\nu)_q^{(0)}$, $\Box^{(0)} : f(x) \mapsto -Dx(1-q^{-1}x)Df(x)$, be "the radial part" of $\Box$.

**Lemma 5.5** There exist constants $0 < c_1 \leq c_2$ such that $c_1 \leq -\Box^{(0)} \leq c_2$.

**Proof.** It follows from the definitions that the operator $\Box^{(0)}$ is selfadjoint, bounded, and non-positive: $0 \leq -\Box^{(0)} \leq c_2$. It remains to show that 0 is not in the spectrum of $\Box^{(0)}$.

A similar result for the operator $L^2(d\nu)_q^{(0)}$ is easily available via considering an expansion in eigenfunctions. On the other hand, the continuous spectra of the operators $\Box^{(0)}$ and $q^{-1}Dx^2D$ coincide since their difference is a compact operator (see [3]). Hence 0 does not belong to the continuous spectrum of $\Box^{(0)}$. It remains to prove that 0 is also not an eigenvalue of this operator. Otherwise, there should exist such a function $\psi(y) \in L^2(d\nu)_q^{(0)}$ that $\Box\psi = 0$. That is,

$$-z^q \frac{\psi(y) - \psi(q^2y)}{y - q^2y} dz^* = 0.$$

Hence $\psi(y)$ is a constant, and $\int_{U_q} |\psi|^2 d\nu = \infty$. Thus, we get a contradiction due to our assumption that 0 is an eigenvalue of $\Box^{(0)}$. \qed

To finish our observations, we sketch the proof of [12, lemma 3.1]. Let us prove that $c_1 \leq -\Box \leq c_2$.

Consider a vector space with a basis $\{e_k \}_{k \in \mathbb{Z}}$ and a complex number $l$. Let $V^{(l)}$ stand for a $U_q\mathfrak{sl}_2$-module given by

$$X^\pm e_k = \pm \frac{\text{sh}(k \pm 1)h/2}{\text{sh}(h/2)} e_{k \pm 1}, \quad He_k = 2ke_k.$$

This parametrization of $U_q\mathfrak{sl}_2$-modules is justified by the possibility of realizing them in the spaces of functions with fixed homogeneity degree $l$ on a quantum cone (see [11][13]).

It follows from the definitions that

$$\Omega v = \frac{\text{sh}(lh/2)\text{sh}((l+1)h/2)}{\text{sh}^2(h/2)} v, \quad v \in V^{(l)},$$

and with $l \notin \mathbb{Z}$, $V^{(l)}$ is a simple $U_q\mathfrak{sl}_2$-module. We need only those $U_q\mathfrak{sl}_2$-modules $V^{(l)}$ for which $\text{sh}(lh/2)\text{sh}((l+1)h/2) < 0$. If $l_1 - l_2 < \frac{2\pi i}{h} \mathbb{Z}$ or $l_1 + l_2 + 1 < \frac{2\pi i}{h} \mathbb{Z}$, then obviously $V^{(l_1)} \approx V^{(l_2)}$. Conversely, if $V^{(l_1)} \approx V^{(l_2)}$, then $\text{sh}(l_1h/2)\text{sh}((l_1+1)h/2) = \text{sh}(l_2h/2)\text{sh}((l_2+1)h/2)$ and hence $l_1 - l_2 < \frac{2\pi i}{h} \mathbb{Z}$ or $l_1 + l_2 + 1 < \frac{2\pi i}{h} \mathbb{Z}$.

The above observations allow one to restrict oneself to the $U_q\mathfrak{sl}_2$-modules $V^{(l)}$ with $l \in \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$,

$$\mathcal{L}_1 = \{ l \in \mathbb{C} \mid -\frac{1}{2} < \text{Re} \, l < 0, \text{Im} \, l = 0 \},$$
\[ \mathcal{L}_2 = \{ l \in \mathbb{C} | \text{Re} \ l = -\frac{1}{2}, \ 0 \leq \text{Im} \ l < \frac{\pi i}{\hbar} \}, \]

\[ \mathcal{L}_3 = \{ l \in \mathbb{C} | \text{Re} \ l > -\frac{1}{2}, \ \text{Im} \ l = \frac{\pi i}{\hbar} \}. \]

It is easy to prove the existence and the uniqueness of an invariant scalar product in \( V^{(l)} \) with \((e_0, e_0) = 1\). This scalar product is positive:

\[ (e_i, e_j) = 0, \quad i \neq j; \quad (e_k, e_k) > 0, \quad k \in \mathbb{Z}. \]

The irreducible \( \ast \)-representations of \( U_q \mathfrak{sl}_2 \) in \( V^{(l)} \), \( l \in \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \), are representations respectively from the additional series, the principal unitary series, and the strange series (see [8, 11]).

It follows from theorem 3.9 that the selfadjoint linear operator \( \Box^{(0)} \) has a simple spectrum. Hence (see [14]), there exists a Borel measure \( dm(l) \) on the compact

\[ \mathcal{L}_0 = \{ l \in \mathcal{L} | c_1 \leq -\frac{\text{sh}(lh/2)\text{sh}((l+1)h/2)}{\text{sh}^2(h/2)} e^{h/2} \leq c_2 \} \]

and a unitary operator \( u : D(U)_q \rightarrow L^2(dm) \) such that \( uf_0 = 1 \), and for all \( f \in D(U)_q \)

\[ u\Box^{(0)} f = e^{h/2}\frac{\text{sh}(lh/2)\text{sh}((l+1)h/2)}{\text{sh}^2(h/2)} uf. \]

(Note that \( dm(l) \) and \( u \) are uniquely determined by the properties listed above.)

Consider the completions \( \overline{V^{(l)}} \) of the pre-Hilbert spaces \( V^{(l)} \), \( l \in \mathcal{L}_0 \), and the direct integral

\[ \overline{V} = \bigoplus \int_{\mathcal{L}_0} V^{(l)} \ dm(l) \] (see [3]). By the construction, the Casimir element \( \Omega \) is well defined in the Hilbert space \( \overline{V} \), and \( c_1 \leq -q^{-1}\Omega \leq c_2 \). Now it suffices to present an isometric operator \( i : D(U)_q \rightarrow \overline{V} \) with \( i\Omega f = \Omega tf \) for all \( f \in D(U)_q \).

Let \( l \in \mathcal{L} \), and \( \Phi_l(x) \) a function on \( q^{-2\mathbb{Z}} \) which is a solution of the boundary problem

\[ \Box^{(0)} \Phi_l(x) = e^{h/2}\frac{\text{sh}(lh/2)\text{sh}((l+1)h/2)}{\text{sh}^2(h/2)} \Phi_l(x), \quad \Phi_l(1) = 1. \quad (5.9) \]

In this case the distribution \( \Phi_l(x), \ x = (1 - zz^\ast)^{-1}, \) in the quantum disc is a solution of the equation

\[ \Omega \Phi_l = \frac{\text{sh}(lh/2)\text{sh}((l+1)h/2)}{\text{sh}^2(h/2)} \Phi_l. \]

The following lemma is a direct consequence of the definitions.

Lemma 5.6

1. For each \( l \in \mathcal{L} \) there exists a unique injective morphism of \( U_q \mathfrak{sl}_2 \)-modules \( i^{(l)} : V^{(l)} \rightarrow D(U)_q \) such that \( i^{(l)}e_0 = \Phi_l \).

2. For each \( l \in \mathcal{L} \) there exists a unique linear operator \( j^{(l)} : D(U)_q \rightarrow \overline{V^{(l)}} \) such that for all \( f \in D(U)_q \), \( v \in \overline{V^{(l)}} \) one has \( (j^{(l)}f, v) = \int_{U_q} (i^{(l)}v)^\ast f \ d\nu \).
Proposition 5.7

1. The linear operator \( i : D(U)_q \to \bigoplus_{L_0} \overline{V(l)} dm(l) \), \( i : f \mapsto j(l)f \) is isometric and \( i\Omega f = \Omega if \) for all \( f \in D(U)_q \).

2. The isometry \( i \) is extendable by a continuity up to a unitary operator \( \tilde{i} : L^2(d\nu)_q \to \bigoplus_{L_0} \overline{V(l)} dm(l) \).

**Proof.** The relation \( i\Omega f = \Omega if \), \( f \in D(U)_q \) and the invariance of the scalar product \([f_1, f_2] = (if_1, if_2), f_1, f_2 \in D(U)_q \) follow directly from the fact that \( i(l) \) are morphisms of \( U_q\mathfrak{sl}_2 \)-modules. Now one can deduce from the uniqueness of the unitary operator \( u \) that

\[
uf(x) = q^2 \int_1^\infty \Phi_l(x)f(x)d_q^{-2}x. 
\]

Hence \([f, f] = \int f^*fd\nu \) for all \( f \in D(U)_q \) which are functions of \( x = (1 - zz^*)^{-1} \). The sesquilinear form \([f, f] - \int f^*fd\nu \) is zero on a subspace of finite functions of the form \( f(x) \).

One can readily apply theorem 3.9 that such sesquilinear form is identically zero. Thus the isometricity is proved. It remains to show that the linear subspace \( iD(U)_q \) is dense in the Hilbert space \( \bigoplus_{L_0} \overline{V(l)} dm(l) \). Note that its closure \( L \) is an invariant subspace of the multiplication operator by the function \( \text{sh}(\hbar/2)\text{sh}((l + 1)\hbar/2) \). This function is real on the compact \( L_0 \) and separates its points. Hence \( L \) is a common invariant subspace for multiplication operators by functions \( \varphi \in L^\infty(dm) \). It remains to observe that the subspaces \( j(l)D(U)_q \) are dense in \( \overline{V(l)} \) for all \( l \in L_0 \) due to the injectivity of the "conjugate" operators \( i(l) : \overline{V(l)} \hookrightarrow D(U)_q' \). It is well known (see [2]) that the listed properties of \( L \) imply \( L = \bigoplus_{L_0} \overline{V(l)} dm(l) \). \( \square \)

**Corollary 5.8** \( c_1 \leq -\Box \leq c_2 \).

This finishes the proof of [12, lemma 3.1].

**Appendix. Covariant *-algebra Fun(U)_q**

Consider the vector space

\[
\text{Fun}(U)_q \overset{df}{=} \text{Pol}(\mathbb{C})_q + D(U)_q \subset D(U)_q'.
\]

It is easy to show that the multiplication, the action of the Hopf algebra \( U_q\mathfrak{sl}_2 \) and the involution \(*\) can be transferred by a continuity from \( \text{Pol}(\mathbb{C})_q \) onto \( \text{Fun}(U)_q \). The elements \( z, z^*, f_0 \) generate this covariant *-algebra. The complete list of relations has the form

\[
z^*z = q^2zz^* + 1 - q^2; \quad z^*f_0 = f_0z = 0; \quad f_0f_0 = f_0.
\]

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The action of the Hopf algebra $U_q sl_2$ is determined by (1.1), (2.4), (3.5), and the relation $H \cdot f_0 = 0$. The involution $*$ possesses the following obvious properties: $*: z \mapsto z^*$, $*: f_0 \mapsto f_0$. One has the exact sequence of morphisms of covariant $*$-algebras

$$0 \to D(U)_q \to \text{Fun}(U)_q \to \text{Pol}(\mathbb{C})_q \to 0.$$ 

Every element $f \in D(U)_q'$ admits a decomposition into series

$$f = \sum_{jk} a_{jk}(f) z^j \cdot f_0 \cdot z^{*k}, \quad a_{jk} \in \mathbb{C}.$$ 

This decomposition is unique. $f \in D(U)_q'$ is a finite function iff $\text{card}\{ (j, k) \in \mathbb{Z}_+^2 | a_{jk}(f) \neq 0 \} < \infty$.

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