Intersection local times for interlacements

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Abstract

We define renormalized intersection local times for random interlacements of Lévy processes in $\mathbb{R}^d$ and prove an isomorphism theorem relating renormalized intersection local times with associated Wick polynomials.

1 Introduction

A random interlacement in $\mathbb{R}^d$ is a particular Poisson process $\mathcal{I}_x$ of paths in $\mathbb{R}^d$, [17] [18] [19]. We consider the $n$-fold intersections of random interlacements of Lévy processes in $\mathbb{R}^d$. This entails studying functionals of the form

$$\gamma_{n,\epsilon}(\nu) \overset{df}{=} \left( \sum_{\omega \in \mathcal{I}_x} \int f_\epsilon(Y_t(\omega) - x) \, dt \right)^n \, d\nu(x),$$

(1.1)

where $Y_t(\omega) = \omega(t)$ for a path $\omega \subset \mathbb{R}^d$, $f_\epsilon$ is an approximate $\delta$-function at zero and $\nu$ is a finite measure on $\mathbb{R}^d$. Ideally we would like to take the limit of $\gamma_{n,\epsilon}(\nu)$ as $\epsilon$ goes to 0, but in general the limit is infinite for all $n \geq 2$. To deal with this we use a technique called renormalization, which consists of forming a linear combination of the $\{\gamma_{k,\epsilon}(\nu)\}_{k=1}^n$ which has a finite limit, $L_n(\nu)$, as $\epsilon \to 0$. We study the behavior of $L_n(\nu)$ as a function of $\nu$.

Renormalized intersection local time (rilt) for Markov processes originated with the work of Varadhan [20] who studied planar Brownian rilt for its role in quantum field theory. Renormalized intersection local time turns out to be the right tool for the solution of certain “classical” problems such as the asymptotic expansion of the area of the Wiener sausage in the plane and the

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range of random walks, [3], [7], [8]. For further work on rilt see Dynkin [5], Bass and Khoshnevisan [2], Rosen [16] and Marcus and Rosen [12].

We set $L_{1,\epsilon}(\nu) = \gamma_{1,\epsilon}(\nu)$ and define recursively

$$L_{n,\epsilon}(\nu) = \gamma_{n,\epsilon}(\nu) - \sum_{j=1}^{n-1} c_{n,j,\epsilon} L_{j,\epsilon}(\nu)$$

where the $c_{n,j,\epsilon}$ are constants which diverge as $\epsilon \to 0$; see (4.4) and (4.5).

We show that for a wide class of random interlacements and finite compactly supported measures $\nu$ on $\mathbb{R}^d$

$$L_n(\nu) := \lim_{\epsilon \to 0} L_{n,\epsilon}(\nu) \text{ exists in all } L^p.$$ (1.3)

We refer to $L_n(\nu)$ as the $n$-fold renormalized intersection local time of $I_\alpha$ with respect to $\nu$.

As indicated, a random interlacement $I_\alpha$ is a Poisson process of paths associated with a transient Markov process. A Poisson process is determined by its intensity measure. For random interlacements the intensity measure $\alpha \mu_m$, $\alpha > 0$, is a measure on bilateral paths which measures geometric properties of the paths, rather than their particular parametrization. More precisely, $\mu_m$ is invariant under time shifts. Before giving the precise characterization of $\mu_m$ and providing references for further details, let us give some indication of how $\mu_m$ looks for Brownian motion in $\mathbb{R}^3$. Let $K$ be a compact subset of $\mathbb{R}^3$. Consider the set $A$ of paths $X$ which, up to time shift, hit $K$ for the first time at $t = 0$, lie in some set $A^+ \subseteq C(R^1_+, R^3)$ for $t \geq 0$, and $A^- \subseteq C(R^1_-, R^3)$ for $t \leq 0$. Then we want

$$\mu_m(A) = \int P^x(A^+) P^x_K(A^- \circ r) e_K(dy).$$ (1.4)

Here $e_K(dy)$ is the equilibrium measure of $K$ for Brownian motion in $\mathbb{R}^3$, $P^x$ is the usual probability for (one-sided) Brownian paths starting at $x$, $P^x_K$ is $P^x$ conditioned never to return to $K$ and $r(\omega)(t) = \omega(-t)$. What follows is a more precise characterization of $\mu_m$.

In this paper we deal only with random interlacements of symmetric Lévy processes in $\mathbb{R}^d$. However, random interlacements can be defined quite generally. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient Borel right process with a locally compact state space $S$ and potential densities $u(x, y)$ with respect to a $\sigma$-finite excessive measure $m$. We use $P_t$ to denote the semigroup for $X$. We assume that $m$ is dissipative, that is, that $\int u(x, y) f(y) m(dy) < \infty$ $m$-a.e. for each non-negative $f \in L^1(m)$. 

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Let $W$ denote the set of paths $\omega : R^1 \mapsto S \cup \Delta$ which are $S$ valued and right continuous on some open interval $(\alpha(\omega), \beta(\omega))$ and $\omega(t) = \Delta$ otherwise. Let $Y_t = \omega(t)$, and define the shift operators

$$(\sigma_t \omega)(s) = \omega(t+s), \quad s, t \in R^1.$$ (1.5)

Set $F = \sigma(Y_s, s \in R^1)$ and $F_t = \sigma(Y_s, s \leq t)$. Let $A$ denote the $\sigma$-algebra of shift invariant events in $F$. The quasi-process associated with $X$ is the measure $\mu_m$ on $(W, A)$ which satisfies the following two conditions:

$$(i) : \quad \mu_m \left( \int_{R^1} f(Y_t) \, dt \right) = m(f)$$ (1.6)

and (ii): if $T$ is any intrinsic stopping time, then $Y_{T+t}, t > 0$ is Markovian with semigroup $P_t$ under $\mu_m|_{\{T \in R\}}$. An $F_t+$ stopping time $T$ is called intrinsic if $\alpha \leq T \leq \beta$ on $\{T < \infty\}$ and $T = t + T \circ \sigma_t$ for all $t \in R^1$. Note that since our times run over $R^1$, this definition does not contain the usual condition that $T \geq t$. If $T < t$, then $T \circ \sigma_t$ will be negative. A first hitting time is an example of an intrinsic stopping time. The quasi-process associated with $X$ will exist under the conditions of the previous paragraph, see [4, XIX]. The name ‘quasi-process’ refers to the fact that $\mu_m$ is only defined on the $\sigma$-algebra $A$ of shift invariant sets, hence for example, if $B \subseteq S$, one cannot ask for the measure of the set $\{Y_t \in B\}$.

Random interlacements are the ‘soup’ of a quasi-process. More precisely, for any $\alpha > 0$, the random interlacement $I_\alpha$ associated with $X$ is the Poisson process in $W$ with intensity measure $\alpha \mu_m$. We let $P_\alpha$ denote probabilities for the process $I_\alpha$.

In the rest of this paper we take $X = \{X(t), t \in R^+\}$ to be a symmetric Lévy process in $R^d$ that is either transient to begin with or made transient by killing at the end of an independent exponential time with mean $1/\kappa$, and we take Lebesgue measure $dx$ as our $\sigma$-finite excessive measure $m$. For simplicity, we assume that $X$ is radially symmetric with characteristic exponent of the form $\psi(\cdot), \frac{1}{2}n < \beta \leq d$. (1.7)

where $\psi$ is regularly varying at infinity with index $\beta$ satisfying

$$\left(1 - \frac{1}{2n}\right) d < \beta \leq d.$$ (1.8)

We assume further that $1/\psi(\cdot)$ is locally integrable, (this is automatic if we are dealing with an exponentially killed process), and

$$\int_{R^d} 1/\psi(|\xi|) \, d\xi = \infty,$$ (1.9)

\[3\]
which is automatic if $\beta < d$. We mention that Lebesgue measure $dx$ is dissipative for such processes, hence the random interlacement $\mathcal{I}_\alpha$ associated with such $X$ exist.

The potential densities $u(x, y) = u(x - y)$, which may be infinite on the diagonal, are always weakly positive definite. We use $\mathcal{G}^m$ to denote the set of positive measures $\nu$ which are bounded with bounded potential and for which

$$\int \int (u(x, y))^m \, d\nu(x) \, d\nu(y) < \infty.$$  \hspace{1cm} (1.10)

For compact $K \subseteq \mathbb{R}^d$ we use $\mathcal{G}^m_K$ to denote the set of $\nu \in \mathcal{G}^m$ with support in $K$. It should be understood that when we say $\nu \in \mathcal{G}^m_K$, that this is with respect to the potential of some given Lévy process. We point out that for all $n \geq 2$, $\mathcal{G}^n_K$ will be empty unless $d = 1$ or 2.

**Theorem 1.1** Let $\mathcal{I}_\alpha$ be the random interlacement associated with a Lévy process as above. Let $\nu$ be a finite measure in $\mathcal{G}^n_K$, for some compact $K \subset \mathbb{R}^d$. Then (1.3) holds.

Our main result is an isomorphism theorem relating the renormalized intersection local times $L_n(\nu)$ with associated Wick polynomials. The definition of Wick powers $: G^j : (\nu)$ is recalled in Section 2 and that of the ‘mixed terms’ $\left( : G^j : \times L_{n-j} \right) (\nu)$ is given in (5.20).

**Theorem 1.2 (Isomorphism Theorem)** For any $\alpha > 0$, compact $K \subset S$ and countable set $D \subseteq \mathcal{G}^n_K$,

$$\left\{ \sum_{j=0}^{n} \binom{n}{j} \frac{G^j}{2^j} \times L_{n-j} \right\} (\nu), \nu \in D, \mathcal{P}_{\alpha^2} \times \mathcal{P}_G \right\} \text{ law } \left\{ \sum_{j=0}^{2n} \binom{2n}{j} \alpha^{(2n-j)} \frac{G^j}{2^{j/2}} \right\} \nu \in D, \mathcal{P}_G \}.$$  \hspace{1cm} (1.11)

In particular, when $n = 2$ this says that for any $\alpha > 0$, compact $K \subset S$ and countable set $D \subseteq \mathcal{G}^4_K$,

$$\left\{ L_2(\nu) + 2 \left( \frac{G^2}{2} \times L_1 \right) (\nu) + \frac{G^4}{2^2} \right\} \nu \in D, \mathcal{P}_{\alpha^2} \times \mathcal{P}_G \right\} \text{ law } \left\{ \frac{G^4}{2^2} + 4\alpha \frac{G^3}{2^{3/2}} + 6\alpha^2 \frac{G^2}{2} + 4\alpha^3 \frac{G(\nu)}{2^{1/2}} + \alpha^4 |\nu| \right. \nu \in D, \mathcal{P}_G \}.$$  \hspace{1cm} (1.12)
At first glance these isomorphism theorems may seem too complicated to work with. However, we have found similar isomorphism theorems very useful, see [15] and especially [12, p. 33]. Here is a particularly straightforward example which mirrors our results in [12] for ordinary intersection local times of Lévy processes. We are concerned with the continuity of \( \{L_n(\nu), \nu \in \mathcal{V}\} \), where \( \mathcal{V} \) is some metric space.

Let \( \tau_{2n}(\xi) \) denote the Fourier transform of \((u(x))^2 \) so that
\[
\int \tau_{2n}(\xi) |\hat{\nu}(\xi)|^2 \, d\xi = \int \int (u(x,y))^2 \, d\nu(x) \, d\nu(y). \tag{1.13}
\]
For any finite positive measure \( \nu \) on \( \mathbb{R}^d \), let \( \nu_x(A) = \nu(A - x) \).

**Theorem 1.3** Under the hypotheses of Theorem [1.1], if \( \nu \in \mathcal{G}_2^n \) is such that
\[
\int_1^\infty \left( \int_{|\xi| \geq x} \tau_{2n}(\xi) |\hat{\nu}(\xi)|^2 \, d\xi \right)^{1/2} \frac{(\log x)^{n-1}}{x} \, dx < \infty, \tag{1.14}
\]
then \( \{L_n(\nu_x), x \in \mathbb{R}^m\} \) is continuous almost surely.

In particular, for the random interlacement associated with exponentially killed Brownian motion in \( \mathbb{R}^2 \), this is the case when
\[
|\hat{\nu}(\xi)| = O \left( \frac{1}{(\log |\xi|)^{2n+r}} \right) \quad \text{as} \quad |\xi| \to \infty. \tag{1.15}
\]

Furthermore for the random interlacement associated with a Lévy process \( X \) in \( \mathbb{R}^2 \) with Lévy exponent asymptotic to \( \lambda^2/(\log |\lambda|)^a \), \( a > 0 \), as \( \lambda \to \infty \), (see [12, p. 5]), \( \{L_n(\nu_x), x \in \mathbb{R}^m\} \) is continuous almost surely if (1.15) holds with \( 2n \) replaced by \( 2n(1 + a/2) \).

The \( n = 1 \) case of our theorem,
\[
\begin{align*}
\left\{ \frac{1}{2} : G^2 : (\nu) + L_1(\nu), \nu \in D, P_{\alpha^2} \times P_G \right\}
\end{align*}
\]
\[
\text{law} \\equiv \left\{ \frac{1}{2} : G^2 : (\nu) + \sqrt{2} \alpha G(\nu) + \alpha^2 |\nu|, \nu \in D, P_G \right\},
\]
is essentially due to Sznitman. Formally, it says that there is an equivalence in law between
\[
G^2/2 + L_1 \quad \text{and} \quad \left( G/2^{1/2} + \alpha \right)^2. \tag{1.17}
\]
Taking the $n$'th power of both sides suggest there should be an equivalence in law between
\[
\left(\frac{G^2}{2} + L_1\right)^n = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{G^{2j}}{2^j} \times L_1^{n-j}\right)
\]  
and
\[
\left(\frac{G}{2^{1/2} + \alpha}\right)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} \frac{G^j}{2^{j/2}} \alpha^{(2n-j)},
\]  
This is very suggestive of our Isomorphism Theorem (1.11), except that neither the powers $G^j$ nor $L_1^{n-j}$ make sense without renormalization. It seems remarkable that the subtractions needed for the Wick powers $G^j$ : $(\nu)$ and those needed for the $L_1^{n-j}(\nu)$ match up to preserve the simple form of (1.11).

Our isomorphism theorem, Theorem 1.2 for intersection local times of random interlacements has a strong resemblance to the isomorphism theorem, [12, Theorem 4.2], for intersection local times $\bar{L}_n(\nu)$ of a Lévy process killed at the end of an independent exponential time $\lambda$ which states that
\[
\left\{ \sum_{j=0}^{n} \binom{n}{j} \left(\frac{G^{2j}}{2^j} \times \bar{L}_{n-j}\right)(\nu), \nu \in D, f(X_\lambda)P^\rho \times P_G \right\} \overset{\text{law}}{=} \left\{ \frac{G^{2n}}{2^n} : (\nu), \nu \in D, G(\rho)G(f \cdot dx) P_G \right\}. 
\]  
This holds for any measure $\rho \in G^1_K$ and ‘nice’ function $f$, where $f(X_\lambda)P^\rho(F) = \int P^\rho(F f(X_\lambda)) d\rho(x)$. Once again, the $n = 1$ case of this theorem, essentially due to Dynkin, suggests an equivalence in law between
\[
G^2/2 + \bar{L}_1 \quad \text{and} \quad G^2/2
\]  
under the appropriate measures. As before, taking the $n$'th power of both sides formally leads to (1.20), modulo the renormalizations needed for the terms to make sense. We do not see how to exploit the formal resemblance between (1.20) and our isomorphism theorem, Theorem 1.2 directly, so we must proceed from scratch.

In Section 2 we recall various properties of quasi-processes, random interlacements and Wick powers. The proof of our main Theorems require detailed estimates and involved combinatorics. Before proceeding with this, in Section 3 we show how to obtain the analogue of our isomorphism theorem, Theorem 1.2 in a toy model: interlacements of a random walk in $Z^d$. Since local times exist in this model, there is no problem with the existence of renormalized
intersection local times, and the combinatorics needed for the proof of Theorem 1.2 is greatly simplified. Section 4 defines the renormalized intersection local times for random interlacements and proves Theorem 1.1. We are able to use many of the results and techniques from [9], so we only point out the necessary changes.

The proof of our isomorphism theorem is quite complicated. In Section 5 we give the proof of our isomorphism theorem, subject to Lemma 5.1 which is proven in Section 6. In these sections we are able to use many estimates and combinatorial arguments from [12]. Again, we concentrate on the differences. Our main effort is in Section 6 which requires very different combinatorics.

We mention that [12] is proven under assumptions that are somewhat different than those of the present paper, which is modeled on [9]. However, the reader will have no trouble using the estimates of [9] Section 8 in place of those used in [12], so we will freely use the results of [12].

The heuristic formula (1.1) involves both self-intersections of paths in the Poisson process \( I_\alpha \) and intersections between different paths in \( I_\alpha \). In Section 7 we show how to make this explicit.

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2 Preliminaries on interlacements and Wick powers

If \( L^\nu_t, t \geq 0 \) denotes the CAF on \( \Omega \) with Revuz measure \( \nu \) then there is an extension to \( W \), which we also denote by \( L^\nu_t, t \in \mathbb{R}^1 \) with the property that

\[
\mu_m \left( \int_{\mathbb{R}^1} f(Y_t) \, dL^\nu_t \right) = \nu(f),
\]

for all measurable \( f \), see [4, XIX, (26.5)]. Note in particular that for any bounded compactly supported measurable function \( g \)

\[
L^g_t \, d\mu_t = \int_{-\infty}^{t} g(Y_s) \, ds.
\]

Lemma 2.1 For any \( \nu_1, \cdots, \nu_k \), with support in some compact \( K \subset \mathbb{R}^d \)

\[
\mu_m \left( \prod_{j=1}^{k} L_{\infty}^{\nu_j} \right) = \sum_{\pi \in \text{Perm}([1,k])} \int \prod_{j=1}^{k-1} u(y_j, y_{j+1}) \prod_{j=1}^{k} dv_{\pi(j)}(y),
\]

where \( \text{Perm}([1,k]) \) denotes the set of permutations of \([1,k]\).
Proof:  Let $T_K$ denote the first hitting time of $K$. Then since the $L_{t_j}^\nu_j$ do not begin to grow until time $T_K$

$$\mu_m \left( \int_{\{ -\infty < t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty \}} \prod_{j=1}^{k} dL_{t_j}^\nu_j \right) \tag{2.4}$$

$$= \mu_m \left( \int_{\{ 0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty \}} \prod_{j=1}^{k} dL_{T_K+t_j}^\nu_j \right).$$

Hence by the second property of $\mu_m$ this equals

$$\mu_m \left( \int_{0}^{\infty} h(Y_{T_K+t_1}) \, dL_{T_K+t_1}^\nu_1 \right), \tag{2.5}$$

where

$$h(x) = E^x \left( \int_{\{ 0 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k < \infty \}} \prod_{j=2}^{k} dL_{t_j}^\nu_j \right) \tag{2.6}$$

$$= \int u(x, y_2) \prod_{j=2}^{k-1} u(y_j, y_{j+1}) \prod_{j=2}^{k} d\nu_j(y).$$

Hence, using once again the fact that the $L_{t_j}^\nu_j$ do not begin to grow until time $T_K$ and then (2.1) and

$$\mu_m \left( \int_{\{ -\infty < t_1 \leq \cdots \leq t_{k-1} \leq t_k < \infty \}} \prod_{j=1}^{k} dL_{t_j}^\nu_j \right) \tag{2.7}$$

$$= \mu_m \left( \int_{R^k} h(Y_{t_1}) \, dL_{t_1}^\nu_1 \right)$$

$$= \int \prod_{j=1}^{k-1} u(y_j, y_{j+1}) \prod_{j=1}^{k} d\nu_j(y),$$

and (2.3) follows, since, up to sets of Lebesgue measure zero

$$R^k = \sum_{\pi \in \text{Perm}([1, k])} \{ -\infty < t_{\pi(1)} \leq \cdots \leq t_{\pi(k-1)} \leq t_{\pi(k)} < \infty \}. $$

\[ \blacksquare \]
In particular (2.3) shows that

$$\mu_m \left( (L^\nu_\infty \right)^k = k! \int \prod_{j=1}^{k-1} u(y_j, y_{j+1}) \prod_{j=1}^k \nu(dy_j). \tag{2.8}$$

For bounded compactly supported functions $g_i, i = 1, \ldots, k$, on $R^d$, (2.3) and (2.2) show that

$$\mu_m \left( \prod_{i=1}^k \int_{-\infty}^\infty g_i(Y_t) dt \right) \tag{2.9}$$

$$= \sum_{\pi \in \text{Perm}(1, k)} \int u(y_{\pi(1)}, y_{\pi(2)}) \cdots u(y_{\pi(k-1)}, y_{\pi(k)}) \prod_{i=1}^k g_i(y_i) dm(y_i).$$

Simply take $\nu_i(dx) = g_i(x) dm(x), i = 1, \ldots, k$.

The potential density $u(x)$ for a symmetric Lévy process is always weakly positive definite, so there exists a mean zero Gaussian field $G(\nu)$ for all $\nu \in G^1$ with covariance

$$E(G(\nu)G(\nu')) = \int \int u(x-y) d\nu(x) d\nu'(y). \tag{2.10}$$

Since we are interested in the case when $u(0) = \infty$, there is no mean zero Gaussian process $G_x$ with covariance $u(x-y)$. In order to define a substitute for powers of $G$ we proceed as follows.

Let $f(y)$ be a positive smooth function supported in the unit ball of $R^d$ with $\int f(x) dx = 1$. Set $f_\epsilon(y) = \epsilon^{-d} f(y/\epsilon)$, and $f_{\epsilon,x}(y) = f_\epsilon(y - x)$. It follows from our assumptions on $X$ that

$$u_{\epsilon,\epsilon'}(x, x') := \int u(y, y') f_{\epsilon,x}(y) f_{\epsilon',x'}(y') dy dy' < \infty \tag{2.11}$$

for $\epsilon, \epsilon' > 0$. Hence $f_{\epsilon,x}(y) dy \in G^1$ and if we set $G_{x,\epsilon} = G(f_{\epsilon,x}(y) dy$ it follows that

$$E \left( G_{x,\epsilon} G_{x',\epsilon'} \right) = u_{\epsilon,\epsilon'}(x, x'). \tag{2.12}$$

For $\nu \in G^2$ we let : $G^2 : (\nu)$ denote the Wick square corresponding to $\nu$, a particular second order Gaussian chaos defined as

$$: G^2 : (\nu) = \lim_{\epsilon \to 0} \int (G^2_{x,\epsilon} - E(G^2_{x,\epsilon})) d\nu(x). \tag{2.13}$$
(See [13] for details, as well as [12] Lemma 3.3] for this and the analogue for the higher order Wick powers : \( G^n : (\nu) : \).)

For the proof of the next Theorem only it will be convenient to use another approximation to \( G \). It follows from our assumptions that \( X \) has symmetric and positive definite transition densities \( p_t(x, y) \) with potential densities

\[
u(x, y) \leq \int_0^\infty p_t(x, y) \, dt,
\]

and that

\[
u_{\epsilon}(x, y) := \int_\epsilon^\infty p_t(x, y) \, dt < \infty
\]

for each \( x, y \) and \( \epsilon > 0 \). Hence \( p_{\epsilon}(x, y) \, dy \in G^1 \) and if we set \( \bar{G}_{x, \epsilon} = G(p_{\epsilon}(x, y) \, dy) \) it follows that

\[
E(\bar{G}_{x, \epsilon} \bar{G}_{x', \epsilon'}) = u_{\epsilon} + \epsilon'(x, x').
\]

If \( \nu \in G^2 \) then

\[
\nu : G^2 : (\nu) = \lim_{\epsilon \to 0} \int (\bar{G}_{x, \epsilon}^2 - E(\bar{G}_{x, \epsilon}^2)) \, d\nu(x).
\]

with convergence in all \( L^p \). \( \bar{G}_{x, \epsilon} \) is convenient since \( u_{\epsilon}(x, y) \uparrow u(x, y) \). However, in the sequel, because of interlacements it will be important to work with compactly supported \( f_{\epsilon} \) as in the previous paragraph.

Set

\[
L_1(\nu) := \sum_{\omega \in I_\alpha} L^\nu(\omega).
\]

The next result which will follow from the master formula for Poisson processes, is a generalization of the Isomorphism Theorems of Sznitman, [18, 19, 17]. We let \( |\nu| \) denote the mass of \( \nu \).

**Theorem 2.1** For any \( \alpha > 0 \), compact \( K \subset S \) and countable \( D \subset G^2_K \),

\[
\left\{ \frac{1}{2} : G^2 : (\nu) + L_1(\nu), \nu \in D, \mathcal{P}_{\alpha^2} \times P_G \right\}
\]

\[ \overset{\text{law}}{=} \left\{ \frac{1}{2} : G^2 : (\nu) + \sqrt{2}\alpha G(\nu) + \alpha^2|\nu|, \nu \in D, P_G \right\} \]

**Proof of Theorem 2.1** Because everything is additive in \( \nu \) we can write (2.19) as

\[
\mathcal{P}_{\alpha^2} \times P_G \left( \exp \left( \delta L_1(\nu) + \frac{\delta}{2} : G^2 : (\nu) \right) \right)
\]

\[ = P_G \left( \exp \left( \frac{\delta}{2} : G^2 : (\nu) + \delta \sqrt{2}\alpha G(\nu) + \delta \alpha^2|\nu| \right) \right) \]
for $\delta$ small. Equivalently, we show that

$$P_{\alpha^2} \left( e^{\delta L_1(\nu)} \right) = P_G \left( \exp \left( \frac{\delta}{2} : G^2 : (\nu) + \delta \sqrt{2} \alpha G(\nu) + \delta \alpha^2 |\nu| \right) \right) . \quad (2.21)$$

We first note that using (2.13), the Gaussian moment formula and the monotone convergence theorem we have

$$P_G \left( \exp \left( \frac{\delta}{2} : G^2 : (\nu) \right) \right) = \lim_{\epsilon \to 0} P_G \left( \exp \left( \frac{\delta}{2} \int (\bar{G}_{x,\epsilon} - E (\bar{G}_{x,\epsilon}^2)) \, d\nu(x) \right) \right) . \quad (2.22)$$

and

$$P_G \left( \exp \left( \frac{\delta}{2} : G^2 : (\nu) + \delta \sqrt{2} \alpha G(\nu) + \delta \alpha^2 |\nu| \right) \right) = \lim_{\epsilon \to 0} P_G \left( \exp \left( \frac{\delta}{2} \int ((\bar{G}_{x,\epsilon} + \sqrt{2} \alpha)^2 - E (\bar{G}_{x,\epsilon}^2)) \, d\nu(x) \right) \right) . \quad (2.23)$$

Therefore,

$$\frac{P_G \left( \exp \left( \frac{\delta}{2} : G^2 : (\nu) + \delta \sqrt{2} \alpha G(\nu) + \delta \alpha^2 |\nu| \right) \right)}{P_G \left( \exp \left( \frac{\delta}{2} : G^2 : (\nu) \right) \right)} = \lim_{\epsilon \to 0} \frac{P_G \left( \exp \left( \frac{\delta}{2} \int ((\bar{G}_{x,\epsilon} + \sqrt{2} \alpha)^2 - E (\bar{G}_{x,\epsilon}^2)) \, d\nu(x) \right) \right)}{P_G \left( \exp \left( \frac{\delta}{2} \int (\bar{G}_{x,\epsilon} - E (\bar{G}_{x,\epsilon}^2)) \, d\nu(x) \right) \right)} . \quad (2.24)$$

A simple Gaussian computation, see [14, Lemma 5.2.1], shows that this is

$$= \lim_{\epsilon \to 0} \exp \left( \alpha^2 \left( \sum_{n=1}^{\infty} \delta^n \prod_{j=1}^{n-1} u_2(x_j, x_{j+1}) \prod_{j=1}^{n} \nu(dx_j) \right) \right)$$

$$= \exp \left( \alpha^2 \left( \sum_{n=1}^{\infty} \delta^n \prod_{j=1}^{n-1} u(x_j, x_{j+1}) \prod_{j=1}^{n} \nu(dx_j) \right) \right) .$$

by the monotone convergence theorem.

On the other hand, by the master formula for Poisson processes, [10],

$$P_{\alpha^2} \left( e^{\delta L_1(\nu)} \right) = \exp \left( \alpha^2 \mu_m \left( e^{\delta L_{\infty}^{\nu}} - 1 \right) \right) , \quad (2.25)$$
and it follows from (2.8) that
\[
\mu_m \left( e^{\delta L_{\infty}} - 1 \right) = \sum_{n=1}^{\infty} \delta^n \frac{\mu_m ((L^n}_\nu)^n)}{n!} = \sum_{n=1}^{\infty} \delta^n \int \prod_{j=1}^{n-1} u(x_j, x_{j+1}) \prod_{j=1}^{n} \nu(dx_j).
\]

This completes the proof of (2.21) and hence of (2.19).

The next lemma, which describes the moment structure of random interlacements, follows from (2.9) and the master formula for Poisson processes. It will be used in the next section. For any bounded compactly supported function \(g\) on \(\mathbb{R}^d\) we set
\[
L_1(g) := \int_{\mathbb{R}^d} g(x) \, dx.
\]

Then by (2.18) and (2.2)
\[
L_1(g) = \sum_{\omega \in I_\alpha} \int g(Y_t(\omega)) \, dt.
\]

**Lemma 2.2** Let \(g_j, j = 1, \ldots, k\) be bounded compactly supported functions on \(\mathbb{R}^d\). Then
\[
\mathcal{P}_\alpha \left( \prod_{i=1}^{k} L_1(g_i) \right) = \sum_{B_1 \cup \cdots \cup B_j = [1, k]} \alpha^j \prod_{l=1}^{j} \mu_m \left( \prod_{i \in B_l} \int_{-\infty}^{\infty} g_i(Y_t) \, dt \right)
\]
\[
= \sum_{B_1 \cup \cdots \cup B_j = [1, k]} \alpha^j \int \left( \prod_{l=1}^{j} \sum_{\pi \in \text{Perm}(B_l)} u(y_{\pi(1)}, y_{\pi(2)}) \cdots u(y_{\pi(|B_l| - 1)}, y_{\pi(|B_l|)}) \right) \prod_{i=1}^{k} g_i(y_i) \, dy_i
\]

where the sum in the second and third line is over all partitions of \([1, k]\) and \(\text{Perm}(B_l)\) denotes the set of permutations of \(B_l = \{1_l, 2_l, \ldots, |B_l|\}\).
In this section we take $X$ to be a symmetric continuous time transient random walk in $\mathbb{Z}^d$. We take $m$ to be counting measure, and as before we denote the potential as $u(x, y) = u(x - y)$. Here $u$ is finite. Let

$$L_1(y) := \sum_{\omega \in I^d} L^y_\omega(\omega)$$

(3.1)

where $L^y_\omega(\omega)$ is the total local time of the path $\omega$ at $y \in \mathbb{Z}^d$. We refer to this as a toy model for intersections because when local times exist, intersection local times are straightforward, and in particular there is no real need for renormalization. Nevertheless, it gives us an opportunity to exhibit some of the combinatorics involved in proving our main theorem without the need to deal with approximations and error bounds.

In the following we abbreviate $u = u(0)$. Then if $\{G_x, x \in \mathbb{Z}^d\}$ denotes the Gaussian process with covariance $u(x, y)$, the Wick powers $G^n_x$ are defined as

$$G^n_x := \sum_{j=0}^{[n/2]} (-1)^j \binom{n}{2j} \frac{(2j)!}{j!2^j} u^{n-2j} G_x^{2j}.$$

(3.2)

Thus $G^n_x$ is an $n$th degree polynomial in $G_x$, and it has generating function

$$\sum_{n=0}^{\infty} \frac{s^n \cdot G^n_x}{n!} = e^{s G_x - s^2 u / 2}.$$

(3.3)

See [12, (3.8), (3.16)]. We will use the convention that if $f(x) = \sum_{n=0}^{\infty} a_n z^n$ is analytic, then $f(G_x) := \sum_{n=0}^{\infty} a_n \cdot G^n_x$.

Since $G^2_x := G^2_x - u$, Theorem 2.1 for interlacements of random walks takes the form: for any $\alpha > 0$, compact $K \subset \mathbb{Z}^d$,

$$\left\{ \frac{1}{2} G^2_x + L_1(x), x \in K, \mathcal{P}_\alpha \times \mathcal{P}_G \right\} \overset{\text{law}}{=} \left\{ \frac{1}{2} G^2_x + \sqrt{2} \alpha G_x + \alpha^2, x \in K, \mathcal{P}_G \right\}.$$

(3.4)

It is interesting to compare this with the generalized second Ray-Knight Theorem, [6].

3 A toy model: interlacements of random walks in $\mathbb{Z}^d$
The goal of this section is to prove Theorem 1.2 for random walks:

\[ \left\{ \sum_{j=0}^{n} \binom{n}{j} \left( \frac{G_x^{2j}}{2^j} \right) L_{n-j}(x) \right\}, \ x \in K, \mathcal{P}_{\alpha^2} \times P_G \] (3.5)

Here \( L_0(x) = 1 \) and \( L_n(x) \) for \( n > 1 \) is defined by

\[ \sum_{n=0}^{\infty} \frac{s^n L_n(x)}{n!} = e^{s^2 + s L_1(x)}. \] (3.6)

Note that

\[ \left( G_x/\sqrt{2} + \alpha \right)^2 = \frac{1}{2} G_x^2 + \sqrt{2} \alpha G_x + \alpha^2. \] (3.7)

Let \( \mathcal{M}(\mathcal{J}_K) \) denote the set of functions measurable with respect to \( \mathcal{J}_K =: \sigma \left( \left( G_x/\sqrt{2} + \alpha \right)^2; x \in K \right) \). We define the ring homomorphism

\[ \Phi : \mathcal{M}(\mathcal{J}_K) \rightarrow \mathcal{M}(\mathcal{J}_K \times \mathcal{F}) \] (3.8)
as the measurable extension of the mapping \( \Phi \) such that \( \Phi(1) = 1 \) and

\[ \Phi \left( \prod_{i=1}^{n} \left( G_{x_i}/\sqrt{2} + \alpha \right)^2 \right) = \prod_{i=1}^{n} \left( \frac{G_{x_i}^2}{2} + L_1(x_i) \right), \quad n = 1, \ldots, \] (3.9)

where \( \mathcal{F} \) is the \( \sigma \)-algebra generated by the random interlacement. With this notation (3.1) can be reformulated as follows: Let \( (h_1, h_2, \ldots) \) be a sequence of \( \mathcal{J}_K \) measurable functions. Then for any Borel measurable non-negative function \( F \) on \( R^\infty \)

\[ E_G \mathcal{P}_{\alpha^2} \left( F(\Phi(h_1), \Phi(h_2), \ldots) \right) = E_G \left( F(h_1, h_2, \ldots) \right). \] (3.10)

Hence to prove (3.5) it suffices to show that if we define

\[ J_n(x) := \left( G_x/\sqrt{2} + \alpha \right)^{2n} := \sum_{j=0}^{2n} \binom{2n}{j} \alpha^{(2n-j)} \frac{G_x^j}{2^j}, \] (3.11)

then

\[ J_n(x) \in \sigma \left( \left( G_x/\sqrt{2} + \alpha \right)^2 \right), \] (3.12)
and
\[ \Phi(J_n(x)) = \sum_{j=0}^{n} \binom{n}{j} \left( \frac{G_{2j}^2}{2j} : L_{n-j}(x) \right). \] (3.13)

We abbreviate \( G = G_x, L_n = L_n(x) \). The following Lemma is proven below.

Lemma 3.1
\[ \sum_{n=0}^{\infty} \frac{s^n : (G/\sqrt{2})^{2n}}{n!} = (1 + us)^{-1/2} \exp \left( \frac{s(G/\sqrt{2})^2}{1 + us} \right), \] (3.14)

and
\[ \sum_{n=0}^{\infty} \frac{s^n : (G/\sqrt{2} + \alpha)^{2n}}{n!} = (1 + us)^{-1/2} \exp \left( \frac{s(G/\sqrt{2} + \alpha)^2}{1 + us} \right). \] (3.15)

(3.12) follows from (3.15), and then applying \( \Phi \) to both sides of (3.15) we obtain
\[ \sum_{n=0}^{\infty} \frac{s^n \Phi : (G/\sqrt{2} + \alpha)^{2n}}{n!} = (1 + us)^{-1/2} \exp \left( \frac{s(G/\sqrt{2} + \alpha)^2}{1 + us} \right) \]
\[ = (1 + us)^{-1/2} \exp \left( \frac{sG^2/2 + L_1}{1 + us} \right) \]
\[ = (1 + us)^{-1/2} \exp \left( \frac{sG^2/2}{1 + us} \right) \exp \left( \frac{sL_1}{1 + us} \right) \]
so that by (3.14) and (3.6)
\[ \sum_{n=0}^{\infty} \frac{s^n \Phi : (G/\sqrt{2} + \alpha)^{2n}}{n!} \]
\[ = \left( \sum_{m=0}^{\infty} \frac{s^m : (G/\sqrt{2})^{2m}}{m!} \right) \left( \sum_{j=0}^{\infty} \frac{s^j L_j}{j!} \right) \] (3.16)
which easily proves (3.13).

Proof of Lemma 3.1. By [11 8.957.1], the generating function for the Hermite polynomials \( H_n \) is
\[ \sum_{n=0}^{\infty} \frac{z^n H_n(x)}{n!} = e^{2tx - t^2}. \] (3.17)
Setting $x = G/\sqrt{2u}$, $z = s\sqrt{u/2}$ and comparing with (3.3) we see that

$$G^n := (u/2)^n/2 H_n \left( G/\sqrt{2u} \right). \tag{3.18}$$

If we use the notation $L_n^\kappa(x)$ for the $n$‘th order Laguerre polynomial of index $\kappa$, it follows from [11, 8.972.2] that

$$H_{2n}(x) = (-1)^n n! 2^{2n} L_n^{-1/2}(x^2), \tag{3.19}$$

hence using the previous formula

$$G^{2n} : 2^n = (-u)^n n! L_n^{-1/2}(G^2/2u). \tag{3.20}$$

Therefore

$$\sum_{n=0}^{\infty} \frac{s^n \cdot (G/\sqrt{2})^{2n}}{n!} = \sum_{n=0}^{\infty} (-us)^n L_n^{-1/2}(G^2/2u) \tag{3.21}$$

$$\quad = (1 + us)^{-1/2} \exp \left( \frac{sG^2/2}{1 + us} \right)$$

by [11, 8.975.1]. This gives (3.14).

By (3.2), or (3.20)

$$G^{2n} := P_n(G^2) \tag{3.22}$$

for some $n$‘th degree polynomial $P_n$. In view of (3.14), (3.15) is equivalent to the following: for any $c \in R^1$

$$G + c \rightleftharpoons P_n((G + c)^2) \tag{3.23}$$

for the same polynomial $P_n$.

But by (3.3)

$$\left( \sum_{i=0}^{\infty} \frac{s^i \cdot G^i}{i!} \right) \left( \sum_{j=0}^{\infty} \frac{s^j c^j}{j!} \right) = e^{sG - s^2 u/2} e^{sc}, \tag{3.24}$$

or equivalently

$$\sum_{n=0}^{\infty} \frac{s^n \cdot (G + c)^n}{n!} = e^{s(G+c) - s^2 u/2}, \tag{3.25}$$

which establishes (3.23).
4 Renormalized intersection local times

Let
\[ L_1(x,\epsilon) := L_1(f_{\epsilon,x}) = \sum_{\omega \in \mathcal{I}} \int f_{\epsilon,x}(Y_t(\omega)) \, dt. \]  
(4.1)

\( L_1(x,\epsilon) \) can be thought of as the approximate total local time of the random interlacement at the point \( x \in \mathbb{R}^d \). When \( u(0) = \infty \), local times do not exist and we cannot take the limit of \( L_1(x,\epsilon) \) as \( \epsilon \to 0 \). Nevertheless, it is often the case that renormalized intersection local times exist. We proceed to define renormalized intersection local times.

We begin with the definition of the chain functions
\[ \text{ch}_k(r) = \int u(ry_1,ry_2) \cdots u(ry_k,ry_{k+1}) \prod_{j=1}^{k+1} f(y_j) \, dy_j, \quad k \geq 1. \]  
(4.2)

Note that \( \text{ch}_k(r) \) involves \( k \) factors of the potential density \( u \), but \( k+1 \) variables of integration. For any \( \sigma = (k_1,k_2,\ldots) \) let
\[ |\sigma| = \sum_{i=1}^{\infty} ik_i \quad \text{and} \quad |\sigma|_+ = \sum_{i=1}^{\infty} (i+1)k_i. \]  
(4.3)

We define recursively
\[ L_n(x,r) = L_1^n(x,r) - \sum_{\{\sigma | 1 \leq |\sigma| < |\sigma|_+ \leq n\}} J_n(\sigma,r), \]  
(4.4)

where
\[ J_n(\sigma,r) = \frac{n!}{\prod_{i=1}^{\infty} k_i!(n-|\sigma|_+)} \prod_{i=1}^{\infty} (\text{ch}_i(r))^{k_i} L_{n-|\sigma|}(x,r). \]  
(4.5)

(Note that \( n - |\sigma|_+ \geq 0 \).

To help in understanding (4.4) we note that
\[ L_2(x,r) = L_1^2(x,r) - 2 \text{ch}_1(r)L_1(x,r) \]  
(4.6)

and
\[ L_3(x,r) = L_1^3(x,r) - 6 \text{ch}_1(r)L_2(x,r) - 6\text{ch}_2(r)L_1(x,r) \]  
(4.7)
\[ = L_1^3(x,r) - 6 \text{ch}_1(r)L_1^2(x,r) + (12 \text{ch}_1^2(r) - 6 \text{ch}_2(r))L_1(x,r). \]
It is interesting to note that one can define $L_n(x,r)$ directly, because $L_n(x,r) = B_n(L_1(x,r))$ where the polynomials $B_n(u)$ satisfy

$$
\sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} \frac{\text{ch}_j(r) t^j}{j!} \right)^n t^n B_n(u) = e^{tu}, \quad (4.8)
$$

where we set $\text{ch}_0(r) = 1$, $B_0(u) = 1$. We remark that for our toy model of Section 3, $\text{ch}_j(r) = u^j$ and $(4.8)$ takes the form

$$
\sum_{n=0}^{\infty} \left( \frac{t}{1-tu} \right)^n L_n = e^{t L_1}. \quad (4.9)
$$

Setting $t = s/(1 + su)$ we obtain $(3.9)$. The next theorem gives the joint moments of the $L_n(\nu)$.

**Theorem 4.1** Let $I_\alpha$ be the random interlacement associated with a Lévy process with potential density $u$ as in Theorem 3.1. Let $n = n_1 + \cdots + n_k$, and $\nu_i \in \mathcal{G}_{2n_i}$. Then

$$
\mathcal{P}_\alpha \left( \prod_{i=1}^{k} L_{n_i}(\nu_i) \right) \quad (4.10)
$$

$$
= \prod_{i=1}^{k} n_i! \sum_{B_1 \cup \cdots \cup B_j = [1,n]} \alpha^j \int \sum_{\pi \in \mathcal{M}_a} \prod_{l=1}^{j} \prod_{i=1}^{|B_l|-1} u(x_{\pi(i_l)}, x_{\pi(i_l+1)_l}) \prod_{m=1}^{k} d\nu_m(x_m)
$$

where the first sum is over all partitions of $[1,n]$, we denote the elements of $B_l$ by $\{1_l, 2_l, \ldots, |B_l|_l\}$ and $\mathcal{M}_a$ is the set of maps $\pi : [1,n] \mapsto [1,k]$ with $|\pi^{-1}(m)| = n_m$ for each $m$ and such that for each $l$, if $\pi(l) = m$ then $\pi((i + 1)_l) \neq m$. (The subscript ‘a’ in $\mathcal{M}_a$ stands for alternating).

**Proof of Theorems 1.1 and 4.1** The proof is very similar to the proof of [9] Theorem 1.3. We begin by showing that for the approximate identities
fr,x and bounded functions \( f_i = f_{r_i,x_i}, i = 1, \ldots, m, \)

\[
P_\alpha \left( L_n(x, r) \prod_{i=1}^{m} L_1(f_i) \right) = \sum_{B_1 \cup \cdots \cup B_j \subseteq \{1, m+n\}} \alpha^j \int \left( \prod_{l=1}^{j} \sum_{\pi_l \in \text{Perm}_{m,n}(B_l)} \sum_{i=1}^{m+n} \prod_{i=1}^{m} f_i(y_i) \prod_{i=m+1}^{m+n} f_{r,x}(y_i) dy_i + \int E_r(x, z) \prod_{i=1}^{m} f_i(z_i) dz_i \prod_{i=m+1}^{m+n} f_{r,x}(z_i) dz_i, \right)
\]

where \( \text{Perm}_{m,n}(B_l) \) is the subset of permutations \( \pi_l \) of \( B_l = \{1, 2, \ldots, |B_l|\} \) with the property that for all \( i_l, (i+1)_l \in B_l, \) if \( \pi_l(i_l) \in [m+1, m+n], \) then \( \pi_l((i+1)_l) \in [1, m]. \) That is, under the permutation \( \pi_l, \) no two elements of \( [m+1, m+n] \) are adjacent.

The last term in (4.11) is an error term. It is actually the sum of many terms, some of which may depend on some of the \( z_1^1, \ldots, z_n^m. \) We use \( z \) to designate \( z_1^1, \ldots, z_n^m. \) Since the \( f \)'s are probability density functions we write last term in (4.11) as an expectation,

\[
\mathcal{E}_f(E_r(x, z)) := \int E_r(x, z) \prod_{i=1}^{m} f_i(z_i) dz_i \prod_{i=m+1}^{m+n} f_{r,x}(z_i) dz_i. \tag{4.12}
\]

We show later that for \( \nu \in \mathcal{G}_{K_n}^n, \)

\[
\lim_{r \to 0} \sup_{|z_i| \leq 1} \int E_r(x, z) d\nu(x) = 0, \tag{4.13}
\]

which implies that

\[
\lim_{r \to 0} \int \mathcal{E}_f(E_r(x, z)) d\nu(x) = 0. \tag{4.14}
\]

(Note that since \( f \) is supported on the unit ball in \( R^d \) we can take \( |z_i| \) bounded uniformly for all indices \( i. \)) We deal with all the additional error terms that are introduced similarly.

Assume that (4.11) is proved for \( L_{n'}(x, r), n' < n. \) For any \( \sigma = (k_1, k_2, \ldots) \) let \( \text{Perm}_{m+n}(\sigma) \) denote the set of permutations \( \tilde{\sigma} \) of \( [1, m+n] \) of the form
\(\vec{\pi} = (\pi_1, \ldots, \pi_j),\) (where \(\pi_i\) is a permutation of \(B_i = \{1_i, 2_i, \ldots, |B_i|\}\) with \(B_1 \cup \cdots \cup B_j = [1, m + n]\)) that contain \(k_i\) chains of order \(i = 1, 2, \ldots\) in \([m + 1, m + n]\). (A chain of order \(i \geq 1\) is a sequence \(\pi_l(j_l), \pi_l((j + 1)_l), \ldots, \pi_l((j + i)_l)\) in \([m + 1, m + n]\) for some \(l\) which is maximal in the sense that \(\pi_l((j - 1)_l)\), and \(\pi_l((j + i + 1)_l)\) are not in \([m + 1, m + n]\), possibly because \(j - 1 = 0\) or \(j + i = |B_l|\)).

Let \(\text{Perm}_{m,n}\) denote the set of permutations \(\pi'\) of \([1, m + n]\) of the form \(\pi' = (\pi_1, \ldots, \pi_j),\) where each \(\pi_i\) is a permutation in \(\text{Perm}_{m,n}(B_i)\) with \(B_1 \cup \cdots \cup B_j = [1, m + n]\). As in the proof of [9, Theorem 1.3], we see that the term for any \(\vec{\pi} \in \text{Perm}_{m+n}(\sigma)\) in the evaluation of

\[
P_\alpha \left( L_1^n(x, r) \prod_{i=1}^m L_1(f_i) \right),
\]  

is the same as the term in (4.11) for a particular permutation \(\pi' \in \text{Perm}_{m,n-|\sigma|}\) in the evaluation of

\[
P_\alpha \left( \prod_{i=1}^\infty (\text{ch}_i(r))^{k_i} L_{n-|\sigma|}(x, r) \prod_{i=1}^m L_1(f_i) \right),
\]  

up to error terms \(H_r(x, z)\). (We note that \(\pi'\) will be associated with a partition \(B'_1 \cup \cdots \cup B'_j = [1, m + n - |\sigma|]\), where \(B'_l \subseteq B_l\) for each \(l\).) And as in that proof we can do the combinatorics to show that up to the error terms, the contribution to (4.11) from \(\text{Perm}_{m+n}(\sigma)\) is equal to

\[
\frac{n!}{\prod_{i=1}^\infty k_i!(n-|\sigma|)!} P_\alpha \left( \prod_{i=1}^\infty (\text{ch}_i(r))^{k_i} L_{n-|\sigma|}(x, r) \prod_{i=1}^m L_1(f_i) \right)
\]  

\[
= P_\alpha \left( J_n(\sigma, r) \prod_{i=1}^m L_1(f_i) \right).
\]  

If we let \(\text{Perm}_{m+n}\) denote the set of permutations \(\vec{\pi}\) of \([1, m + n]\) of the form \(\vec{\pi} = (\pi_1, \ldots, \pi_j),\) where each \(\pi_i\) is a permutation of \(B_i\) with \(B_1 \cup \cdots \cup B_j = [1, m + n]\), then considering (4.14) and the fact that \(\text{Perm}_{m+n} - \text{Perm}_{m,n} = \cup_{|\sigma| \geq 1} \text{Perm}_{m+n}(\sigma)\), we see that the induction step in the proof of (4.11) is proved.

We iterate the steps used in the proof of (4.11), and use the fact that each
of the $L_{n_i}(x_i, r)$ are sums of multiples of $L(x_i, r)$ to obtain

$$P_{\alpha} \left( \prod_{i=1}^{k} L_{n_i}(x_i, r_i) \right) = \sum_{j=\frac{1}{B_1 \cup \cdots \cup B_j = [1, m+n]}}^{\alpha} \alpha^j \quad (4.18)$$

$$\int \left( \prod_{l=1}^{j} \sum_{\pi_l \in \text{Perm}_{n_1, \ldots, n_k}(B_i)} u(y_{\pi_l(1)}, y_{\pi_l(2)}) \cdots u(y_{\pi_l(|B_i|-1)}, y_{\pi_l(|B_i|)}) \right) \prod_{j=1}^{n} f_{r_{y(l); x_{y(l)}}}(y_j) \ dy_j + E_{E_{r_1, \ldots, r_k}(x_1, \ldots, x_k, z)},$$

where $\text{Perm}_{n_1, \ldots, n_k}(B_i)$ is the set of permutations $\pi_l$ of $B_i = \{1, 2, \ldots, |B_i|\}$ with the property that for all $m$, when $\pi_l(m_l) \in \left[1 + \sum_{p=1}^{i-1} n_p, \sum_{p=1}^{i} n_p \right] := C_i$ then $\pi_l((m_l + 1)j) \notin C_i$, for all $i \in [1, k]$, and $g(m_l) = i$ when $m_l \in C_i$. (In the last term in (4.18) we use the notation introduced in (4.12).)

The error terms are very similar to those described in the proof of [9, Theorem 1.3], the only difference being that there we worked with the loop measure whereas here we have the quasi-process measure $\mu_m$.

Set

$$L_{n,r}(\nu) = \int L_{n}(x, r) \ d\nu(x). \quad (4.19)$$

Following the proof of [9 Theorem 1.3] we can then show that of Theorem [11] and then of Theorem 4.1.

## 5 Proof of the Isomorphism Theorem

We define the cycle functions

$$c_{y_k}(\epsilon) = \int u(\epsilon y_1, \epsilon y_2) \cdots u(\epsilon y_k-1, \epsilon y_k) u(\epsilon y_k, \epsilon y_1) \prod_{j=1}^{k} f(y_j) \ dy_j. \quad (5.1)$$

For any $\sigma = (k_1, k_2, \ldots; m_2, m_3, \ldots)$ we set

$$|\sigma| = \sum_{i=1}^{\infty} ik_i + \sum_{j=2}^{\infty} jm_j, \quad |\sigma|_+ = \sum_{i=1}^{\infty} (i + 1)k_i + \sum_{j=2}^{\infty} jm_j. \quad (5.2)$$

For example, $\sigma = (2; 0, 1)$ means that $k_1 = 2, m_3 = 1$ and all other $k_i, m_j = 0$, and $|\sigma| = 5, |\sigma|_+ = 7$. 

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Set \( \tilde{H}_0(x, \epsilon) = 1 \) and \( \tilde{H}_1(x, \epsilon) = H_1(x, \epsilon) = \frac{G^2(x, \epsilon)}{2} + 2^{1/2} \alpha G(x, \epsilon) + \alpha^2 \).

We then define inductively
\[
\tilde{H}_n(x, \epsilon) = H_1^0(x, \epsilon) - \sum_{\{\sigma \mid 1 \leq |\sigma| \leq |\sigma_+| \leq n\}} I_{n, \epsilon}(\sigma) \tilde{H}_{n-|\sigma|}(x, \epsilon), \tag{5.3}
\]
where
\[
I_{n, \epsilon}(\sigma) = \frac{n!}{(n - |\sigma_+|)!} \prod_{i=1}^{\infty} k_i! \prod_{j=2}^{\infty} m_j! \prod_{i=1}^{\infty} (\text{ch}_i(\epsilon))^{k_i} \prod_{j=2}^{\infty} \left( \frac{\text{cy}_j(\epsilon)}{2j} \right)^{m_j}. \tag{5.4}
\]

For any \( \alpha > 0 \) let
\[
H_n(\nu) := 2^{n} \sum_{j=0}^{n} \binom{2n}{j} \alpha^{(2n-j)} : G^j : (\nu). \tag{5.5}
\]

The following is the key technical result of this paper.

**Lemma 5.1** For any \( \nu \in G_{2n'}^2 \) and \( \alpha \geq 0 \)
\[
\lim_{\epsilon \to 0} \int \tilde{H}_n(x, \epsilon) \, d\nu(x) = H_n(\nu), \tag{5.6}
\]
for all \( 1 \leq n \leq n' \).

When \( \alpha = 0 \) this is [12, Lemma 4.3].

It is easy to check that
\[
\tilde{H}_2(x, \epsilon) = H_1^0(x, \epsilon) - I_{2, \epsilon}(1; 0) H_1(x, \epsilon) - I_{2, \epsilon}(0; 1) = H_1^0(x, \epsilon) - 2\text{ch}_1(\epsilon) H_1(x, \epsilon) - \frac{\text{cy}_2(\epsilon)}{2}, \tag{5.7}
\]
compare (4.6).

Lemma 5.1 will be proven in the next section. Now we return to \( L^n(x, \epsilon) \)
in order to establish our Isomorphism Theorem.

It follows from (4.4)-(4.5) that
\[
L^n_1(x, \epsilon) = \sum_{\{\sigma \mid 0 \leq |\sigma| \leq |\sigma_+| \leq n\}} \frac{n!}{\prod_{i=1}^{\infty} k_i!} \prod_{i=1}^{\infty} (\text{ch}_i(\epsilon))^{k_i} \frac{L_{n-|\sigma_+|}(x, \epsilon)}{(n - |\sigma_+|)!}. \tag{5.8}
\]

We now rewrite this in a way which is easier to deal with. Set \( \text{ch}_0(\epsilon) = 1 \).
Recall that \( |\sigma_+| - |\sigma| = \sum_{i=1}^{\infty} k_i \). Setting \( k_0 = n - |\sigma_+| \) we then have \( n - |\sigma| = \)
\sum_{i=0}^{\infty} k_i$, the total number of chains when we include \( k_0 \) chains of order 0. We can then rewrite (5.8) as

\[
L_1^n(x, \epsilon) = \sum_{k=0}^{n} \frac{n!}{k!} \prod_{i=0}^{\infty} (\chi_i(\epsilon))^{k_i} L_{\sum_{i=0}^{\infty} k_i}(x, \epsilon).
\]

We claim that

\[
H_1^n(x, \epsilon) = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{i_1, \ldots, i_k} \prod_{a=1}^{r} \left( cy_{i_a}(\epsilon) \right) \prod_{b=1}^{k} \chi_{j_b}(\epsilon) H_k(x, \epsilon).
\]

where the second sum is over sequences of \( k \) integers \( j_i \geq 0 \). To see this, we simply note that if there are a total of \( k \) chains of which \( k_i \) are of length \( i \), there are \( k! / \prod_{i=0}^{\infty} k_i! \) distinct ways to order them. Also, we have used the fact that

\[
\sum_{i=0}^{\infty} k_i = \sum_{i=0}^{\infty} (i+1)k_i + k_0 = \bar{\sigma}_+ + (n - |\bar{\sigma}|_+) = n.
\]

A similar analysis then shows that

\[
H_1^n(x, \epsilon) = \sum_{k=0}^{n} \frac{n!}{k!} \sum_{i_1, \ldots, i_k} \prod_{a=1}^{r} \left( cy_{i_a}(\epsilon) \right) \prod_{b=1}^{k} \chi_{j_b}(\epsilon) \tilde{H}_k(x, \epsilon).
\]

We can write (5.11) and (5.10) as

\[
H_1^n(x, \epsilon) = \sum_{k=0}^{n} A_{n,k} \tilde{H}_k(x, \epsilon), \quad L_1^n(x, \epsilon) = \sum_{k=0}^{n} B_{n,k} L_k(x, \epsilon).
\]

where

\[
A_{n,k} = \frac{n!}{k!} \sum_{r=0}^{\infty} \frac{1}{r!} \prod_{i_1, \ldots, i_r, j_1, \ldots, j_k} \prod_{a=1}^{r} \left( cy_{i_a}(\epsilon) \right) \prod_{b=1}^{k} \chi_{j_b}(\epsilon)
\]

and

\[
B_{n,k} = \frac{n!}{k!} \sum_{j_1, \ldots, j_k} \prod_{b=1}^{k} \chi_{j_b}(\epsilon).
\]

**Proof of Theorem 1.2:** Using the above, this follows as in [12, Section 4]. In somewhat more detail, let \( \mathcal{M}(\mathcal{H}_K) \) denote the set of functions
measurable with respect to $\mathcal{H}_K =: \sigma(H_1(\mu); \mu \in G_2^1)$. We define the ring homomorphism
\[
\Phi : \mathcal{M}(\mathcal{H}_K) \rightarrow \mathcal{M}(\mathcal{H}_K \times \mathcal{F})
\]
as the measurable extension of the mapping $\Phi$ such that $\Phi(1) = 1$ and
\[
\Phi \left( \prod_{i=1}^{n} H_1(\mu_i) \right) = \prod_{i=1}^{n} \left( \frac{G^2 : (\mu_i)}{2} + L_1(\mu_i) \right), \quad n = 1, \ldots,
\]
where $\mathcal{F}$ is the $\sigma$-algebra generated by $X$. With this notation Theorem 2.1 can be reformulated as follows: Let $(h_1, h_2, \ldots)$ be a sequence of $\mathcal{H}_K$ measurable functions. Then for any $C$ measurable non-negative function $F$ on $\mathbb{R}^\infty$
\[
E_G P_{\alpha^2} (F(\Phi(h_1), \Phi(h_2), \ldots)) = E_G (F(h_1, h_2, \ldots)).
\]
It follows by induction from (5.12) that $\widetilde{H}_n(x, \epsilon) \in \mathcal{H}$, and using (5.12) and (5.16) we have
\[
\sum_{k=0}^{n} A_{n,k} \Phi(\widetilde{H}_k(x, \epsilon)) = \Phi(H^1_n(x, \epsilon)) = \left( \frac{G^2 : (x, \epsilon)}{2} + L(x, \epsilon) \right)^n.
\]
The proof of [12, Lemma 4.5] then shows that
\[
\Phi(\widetilde{H}_n(x, \epsilon)) = \sum_{m=0}^{n} \binom{n}{m} \Psi_m(x, \epsilon)L_{n-m}(x, \epsilon)
\]
where $\Psi_m(x, \epsilon)$ is such that
\[
\left( \frac{G^{2m}}{2m} \times L_{n-m} \right) (\nu) \overset{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \int \frac{G^{2m}}{2m} : (x, \epsilon)L_{n-m}(x, \epsilon) d\nu(x)
= \lim_{\epsilon \rightarrow 0} \int \Psi_m(x, \epsilon)L_{n-m}(x, \epsilon) d\nu(x)
\]
for all $0 \leq m \leq n$. Then, using Lemma 5.1 the proof of our Theorem will follow from the fact that $\Phi$ is an isometry in $L^2$.

To better appreciate the nature of $\Psi_m(x, \epsilon)$ and (5.20) we note that when $m = n$, (5.20) is
\[
\frac{G^{2n}(\nu)}{2^n} = \lim_{\epsilon \rightarrow 0} \int \Psi_n(x, \epsilon) d\nu(x).
\]
6 Proof of Lemma 5.1

Proof of Lemma 5.1: Recall (5.12). The key to proving our Lemma is to show that for any \( \nu \in G^2_{K/2} \)

\[
\int H^n_1(x, \epsilon) \, d\nu(x) = \sum_{k=0}^{n} A_{n,k} H_k(\nu) + o(u(\epsilon)^{-(n'-n)}). \tag{6.1}
\]

Our Lemma then follows easily. See the end of the proof of [12, Lemma 4.3], starting from (4.84).

Recall that

\[
H_1(\nu) = \frac{G^2 : (\nu)}{2} + \sqrt{2} \alpha G(\nu) + \alpha^2 |\nu|.
\]

Therefore, see [12, Lemma 3.3]

\[
H_1(x, \epsilon) \overset{df}{=} H_1(f_{x,\epsilon} \cdot dx') = \lim_{\delta \to 0} \int \left( \frac{G^2_{x',\delta} :}{2} + 2^{1/2} \alpha G_{x',\delta} + \alpha^2 \right) f_{x,\epsilon}(x') \, dx'
\]

where the limit is taken in \( L^2 \). Since it follows from [9, Lemma 8.6] that \( f_{x,\epsilon} \cdot dx' \in G^2_{K/2} \), \( H_1(x, \epsilon) \) is one of the basic random variables that generate \( \mathcal{H} \). As explained in the proof of [12, Theorem 4.2], for fixed \( \epsilon > 0 \), \( H \overset{def}{=} \{ H_1(x, \epsilon), x \in \mathbb{R}^n \} \) can be taken to be continuous almost surely. Furthermore, the convergence in (6.2) is almost sure and in \( L^p \), for all \( p \), since Gaussian chaos processes have all moments.

Clearly

\[
H^n_1(x, \epsilon) = \lim_{\delta \to 0} \prod_{i=1}^{n} \int \left( \frac{G^2_{x_i,\delta} :}{2} + 2^{1/2} \alpha G_{x_i,\delta} + \alpha^2 \right) f_{x,\epsilon}(x_i) \, dx_i. \tag{6.3}
\]

We define

\[
H^n_1,\nu = \int H^n_1(x, \epsilon) \, d\nu(x). \tag{6.4}
\]

Since the right-hand side of (6.3) converges in \( L^2 \) uniformly in \( x \) as \( \delta \to 0 \), we see that

\[
H^n_1,\nu = \lim_{\delta \to 0} \int \prod_{i=1}^{n} \left( \frac{G^2_{x_i,\delta} :}{2} + 2^{1/2} G_{x_i,\delta} \alpha + \alpha^2 \right) f_{x,\epsilon}(x_i) \, dx_i \, d\nu(x) \tag{6.5}
\]

in \( L^2 \). (In fact by [11, Lemma 3.3] it also converges almost surely and in \( L^p \) for all \( p \geq 0 \).)
Expand \( \prod_{i=1}^{n} \left( \frac{G_{x_i,\delta}^2}{2} + 2^{1/2} \alpha G_{x_i,\delta} + \alpha^2 \right) \) as a sum of Wick products. Using [12, (2.15)] we can write
\[
\prod_{i=1}^{n} \left( \frac{G_{x_i,\delta}^2}{2} + 2^{1/2} \alpha G_{x_i,\delta} + \alpha^2 \right) = \sum_{A,B,C} 2^{|B|/2} \alpha^{|B|+|C|} \sum_{R,S,T,U,V} \frac{1}{|R|} \sum_{\text{pairings } \mathcal{P}} \prod_{k=1}^{2} u_{\delta,\delta}(x_{\tilde{P}_{k,1}} - x_{\tilde{P}_{k,2}}) : \prod_{i \in T} G_{x_i,\delta}^2 \prod_{j \in S \cup V} G_{x_j,\delta} : \tag{6.6}
\]
where the first sum runs over all partitions \( A \cup B \cup C = \{1, 2, \ldots, n\} \), and the second sum runs over all partitions \( R \cup S \cup T = A, U \cup V = B \) with \( |S| + |U| \) even, and the third sum runs over all pairings \( \mathcal{P} \) of the set \( (R \times \{1, 2\}) \cup S \cup U \) such that \( \tilde{P}_{k,1} \neq \tilde{P}_{k,2} \), where letting \( (P_{k,1}, P_{k,2}) \) denote the \( k \)-th pair of the pairing \( \mathcal{P} \), we set \( \tilde{P}_{k,1} = i \) if either \( P_{k,1} = i \times 1 \) or \( i \times 2 \) for \( i \in R \), or \( P_{k,1} = i \) for \( i \in S \) or \( U \), and similarly for \( \tilde{P}_{k,2} \). Here we use the fact that for \( i \in S \) one of the two \( G_{x_i,\delta} \) terms is allocated to the \( u_{\delta,\delta} \) terms and the other to the Wick product. Since there are two ways to do this the \( 1/2 \) is cancelled. (6.6) can be rewritten as
\[
\prod_{i=1}^{n} \left( \frac{G_{x_i,\delta}^2}{2} + 2^{1/2} \alpha G_{x_i,\delta} + \alpha^2 \right) = \sum_{R,S,T,U,V,C} 2^{|U|+|V|/2} \alpha^{|U|+|V|+2|C|} \mathcal{E}_\delta(x_1, \ldots, x_n; R, S, U) : \prod_{i \in T} G_{x_i,\delta}^2 \prod_{j \in S \cup V} G_{x_j,\delta} : \tag{6.7}
\]
where
\[
\mathcal{E}_\delta(x_1, \ldots, x_n; R, S, U) \overset{\text{def}}{=} \frac{1}{2|R|} \sum_{\text{pairings } \mathcal{P}} \prod_{k=1}^{2} u_{\delta,\delta}(x_{\tilde{P}_{k,1}} - x_{\tilde{P}_{k,2}}). \tag{6.8}
\]
Using this, (6.5), Fubini's theorem and the definition of \( f_{x,\epsilon} \) we have

\[
H^n_{1,\epsilon,\nu} = \sum_{R,S,T,U,V,C} 2^{(|U|+|V|)/2} \alpha^{(|U|+|V|+2|C|)} \lim_{\delta \to 0} \int E_{\delta}(x_1, \ldots, x_n; R, S, U) \left( \int \prod_{i \in T} \frac{G_x^2}{2} G_{x+i,\delta} \prod_{j \in S \cup V} dx \right) \prod_{i=1}^n f_{\epsilon}(x_i) dx_i. \tag{6.9}
\]

Let us now consider

\[
\int E_{\delta}(x_1, \ldots, x_n; R, S, U) \prod_{i=1}^n f_{\epsilon}(x_i) dx_i \tag{6.10}
\]

\[
= \frac{1}{2^{|R|}} \sum_{\text{pairings } P \text{ of } (R \times \{1,2\}) \cup \{S \cup U\} \text{ such that } P_{k,1} \neq P_{k,2}} \int \prod_{k=1}^{|R|+(|S|+|U|)/2} u_{\delta,\epsilon}(x_{P_{k,1}} - x_{P_{k,2}}) \prod_{i=1}^n f_{\epsilon}(x_i) dx_i.
\]

We reorganize this in a form which is more useful. Fix some pairing \( P \) in the sum and pick any factor \( u_{\delta,\epsilon}(x_i - x_j) \) in the product corresponding to \( P \). If both \( i,j \in S \cup U \) we think of \( i,j \) as forming a chain of order one. \( u_{\delta,\epsilon}(x_i - x_j) \) is the factor associated with this chain. If say \( j \in R \), there will be one other factor in the product corresponding to \( P \) which contains \( x_j \), say \( u_{\delta,\epsilon}(x_j - x_k) \). If both \( i,k \in S \cup U \), we think of \( i,j,k \) as forming a chain of order two. \( u_{\delta,\epsilon}(x_i - x_j)u_{\delta,\epsilon}(x_j - x_k) \) is the factor associated with this chain. If either \( i \) or \( k \) or both are in \( R \), we continue to find the other factors containing them, and continue in this manner until we can go no further. Two possibilities arise. Either we end up with a chain of elements \( i_1, i_2, \ldots, i_v \) with end points \( i_1, i_v \) \in S \cup U \) and intermediate points \( i_2, \ldots, i_{v-1} \in R \) and associated factor

\[
\prod_{j=2}^v u_{\delta,\epsilon}(x_{i_j} - x_{i_{j-1}}) \tag{6.11}
\]

(such a chain is said to be of order \( v - 1 \)), or we have, what we call, a cycle \( i_1, i_2, \ldots, i_v \) with all elements in \( R \) and associated factor

\[
u_{\delta,\epsilon}(x_{i_1} - x_{i_v}) \prod_{j=2}^v u_{\delta,\epsilon}(x_{i_j} - x_{i_{j-1}}) \tag{6.12}
\]

(such a cycle is said to be of order \( v \)).
In this way the product
\[
\prod_{k=1}^{|R|+(|S|+|U|)/2} u_{\delta,\delta}(x_{\widetilde{P}_{k,1}} - x_{\widetilde{P}_{k,2}})
\]  
(6.13)
in (6.25) associated with \(P\) breaks up into a product of factors associated with the chains and cycles of \(P\). Note that each \(P\) appearing in (6.25) will necessarily have precisely \(|\delta|\) chains and cycles of \(P\), so we can permute the points of the cycle in \((l-1)!\) distinct ways, and the \(|\delta|\) cycles of \(P\) of length \(l\) have precisely \(|\delta|\) cycles of length \(l\) from, while for each chain of length \(l\) we can permute the elements of the mid-chain in \((l-1)!\) ways, and the \(|\delta|\) end points can be permuted among

\[
\frac{|R|!}{\prod_{l=2}^\infty l^{m_l}m_l!(l-1)!k_l!(l-1)!} \prod_{l=1}^\infty \left(\frac{(l-1)!}{2}\right)^{m_l} \frac{|\delta|!}{2^{\sum_{l=2}^\infty k_l}|\delta|!} \frac{|\delta|!}{2^{\sum_{l=2}^\infty k_l}|\delta|!} |\delta|!.
\]  
(6.15)

Here, the first factor gives the number of ways to partition \(R\) into \(m_l\) cycles of length \(l\), \(l = 2, \ldots\) and \(k_l\) mid-chains (i.e. chains with end points deleted) of length \(l-1\), \(l = 1, \ldots\). To get the remainder of (6.15), we note that in each cycle of length \(l\) we can permute the points of the cycle in \((l-1)!\) distinct ways, except that we must divide by 2 to take into account the mirror image if \(l > 2\), (we will explain shortly where the factor \(1/2\) for cycles of length \(l = 2\) comes from), while for each chain of length \(l\) we can permute the mirror image if \(l > 2\), and the \(|\delta|\) end points can be permuted among
themselves in \((|S| + |U|)!\) ways, except that for any of the \(p = \sum_{l=1}^{\infty} k_l\) given chains we mustn’t count an interchange of the end points of the same chain, since that has already been counted when we considered the permutations of the mid-chain. Finally, recall that the pairings are actually parings of \((R \times \{1, 2\}) \cup S \cup U\), not of \(R \cup S \cup U\), so that for any given pairing we can get analogous but distinct pairings by interchanging \(i \times 1\) with \(i \times 2\) for each \(i \in R\). The only exception is that for any cycle of length \(l = 2\) we get 2 rather than 4 distinct pairings. Altogether this gives rise to \(2^{\sum_{l=1}^{\infty} k_l}\) distinct pairings. (This explains where the factor \(1/2\) for cycles of length \(l = 2\) in (6.15) comes from). Therefore, combining (6.14) and (6.15) we see that

\[
\int \mathcal{E}_\delta(x_1, \ldots, x_n; R, S, U) \prod_{i=1}^{n} f_i(x_i) \, dx_i = \frac{1}{2^{\sum_{l=1}^{\infty} k_l}} \prod_{l=1}^{\infty} \left( \frac{(l-1)!}{2} \right)^{m_l} \left( \frac{(|S| + |U|)!}{(l-1)!} \right)^k \prod_{l=1}^{\infty} \left( \frac{\text{cy}_{l, \delta}(\epsilon)}{2l} \right)^{m_l}.
\]

Let

\[
h(s) := \int_{|\xi| \leq s} 1/\psi(|\xi|) \, d\xi.
\]

By (1.9), \(\lim_{s \to \infty} h(s) = \infty\). We mention some results which are analogues of results used in \[12\]. We will prove these results under the assumptions of this paper at the end of this section.

**Lemma 6.1** For any \(\delta > 0\)

\[
\text{ch}_{k, \delta}(\epsilon) \leq \text{ch}_k(\epsilon) = O \left( (h(1/\epsilon))^k \right) \quad \text{as } \epsilon \to 0 \quad (6.18)
\]

and

\[
\text{cy}_{k, \delta}(\epsilon) = \text{cy}_k(\epsilon) = O \left( (h(1/\epsilon))^k \right) \quad \text{as } \epsilon \to 0. \quad (6.19)
\]
In addition, for \( \epsilon > 0 \) fixed
\[
\lim_{\delta \to 0} c_{h_k, \delta}(\epsilon) = c_{h_k}(\epsilon) \quad \text{and} \quad \lim_{\delta \to 0} c_{y_k, \delta}(\epsilon) = c_{y_k}(\epsilon).
\] (6.20)

**Lemma 6.2** For \( \nu \in \mathcal{G}^{2n'}_K \)
\[
\sup_{|x| \leq \epsilon} \left\| \int : \prod_{i=1}^k G_{x+i, \delta}^2 \prod_{j \in S \cup V} G_{x+j, \delta} : d\nu(x) - : G_{\delta}^2 \nu : \right\|_2 = o \left( \left( \frac{1}{h(1/\epsilon)} \right)^{-\left( n' - k/2 \right)} \right) \quad \text{as} \ \epsilon \to 0.
\] (6.21)

Since \( \sum_{l=1}^{\infty} k_l l + \sum_{l=2}^{\infty} m_l l = |R| + (|S| + |U|)/2 \) it follows from (6.16) and Lemma 6.1 that
\[
\int \mathcal{E}_{\delta}(x_1, \ldots, x_n; R, S, U) \prod_{i=1}^n f_{x_i}(x_i) \, dx_i = O \left( \left( \frac{1}{h(1/\epsilon)} \right)^{|R| + (|S| + |U|)/2} \right)
\] (6.22)
as \( \epsilon \to 0 
\).

Using this and Lemma 6.2 we see that for \( \nu \in \mathcal{G}^{2n'}_K \)
\[
\left\| \int \mathcal{E}_{\delta}(x_1, \ldots, x_n; R, S, U) \right\|
\begin{cases}
\int : \prod_{i \in T} G_{x+i, \delta} \prod_{j \in S \cup V} G_{x+j, \delta} : d\nu(x) - : G_{\delta}^2 \nu : \right\|_2 \\
\sup_{|x| \leq \epsilon} \left\| \int : \prod_{i \in T} G_{x+i, \delta} \prod_{j \in S \cup V} G_{x+j, \delta} : d\nu(x) - : G_{\delta}^2 \nu : \right\|_2
\end{cases}
\leq O \left( \left( \frac{1}{h(1/\epsilon)} \right)^{|R| + (|S| + |U|)/2} \right) o \left( \left( \frac{1}{h(1/\epsilon)} \right)^{-\left( n' - (|T| + (|S| + |V|)/2) \right)} \right)
= o \left( \left( \frac{1}{h(1/\epsilon)} \right)^{-\left( n' - n \right)} \right)
\]
for all \( \delta > 0 \).

Note that it follows from (6.16) and Lemma 6.1 that
\[
\lim_{\delta \to 0} \int \mathcal{E}_{\delta}(x_1, \ldots, x_n; R, S, U) \prod_{i=1}^n f_{x_i}(x_i) \, dx_i
\]
\[
= \sum_{\sigma = (k_1, \ldots, m_2 \ldots) \atop |\sigma|_+ = |R| + |S| + |U|} 2^{-(|S| + |U|)/2} \frac{|R|! |\sigma| + |U|!}{\prod_{l=1}^\infty (m_l!) (k_l!)} \prod_{l=1}^\infty (c_{y_l}(\epsilon))^{k_l} \left( \frac{c y_l(\epsilon)}{2^l} \right)^{m_l}.
\] (6.24)
Using this, (6.23) and (6.9) we see that for $\nu \in \mathcal{G}^{2\alpha}_K$

$$H_{1_\nu}^n = \sum_{R,S,T,U,V,C} 2^{|U|+|V|/2} 2^{\alpha |U|+|V|+|C|} \lim_{\delta \to 0} \int \mathcal{E}_\delta(x_1, \ldots, x_n; R, S, U)$$

$$\prod_{i \in R,S,T,U,V} f(x_i) \frac{dx_i}{2^{|T|}} : G_{\delta}^{|U|/2+|V|} : \nu : +o((u(\epsilon))^{-(n'-n)}) \quad (6.25)$$

$$= \sum_{k=0}^{2n} \sum_{R,S,T,U,V,C} 2^{|U|+|V|/2} 2^{\alpha |U|+|V|+|C|} \lim_{\delta \to 0} \int \mathcal{E}_\delta(x_1, \ldots, x_n; R, S, U)$$

$$\prod_{i \in R,S,T,U,V} f(x_i) \frac{dx_i}{2^{|T|}} : G_{\delta}^k : \nu : +o((u(\epsilon))^{-(n'-n)})$$

in $L^2$, as $\epsilon \to 0$. We remark that if $|R| + |S| + |U| = 0$ we will have $|\sigma| = 0$.

Recalling the definition of $I_{n,\epsilon}(\sigma)$, see (5.4), and then combining (6.16) with (6.25) we see that for $\nu \in \mathcal{G}^{2\alpha}_K$

$$H_{1_\nu}^n = \sum_{k=0}^{2n} \sum_{R,S,T,U,V,C} 2^{|U|+|V|/2-|S|+|V|} 2^{\alpha |U|+|V|+|C|} \frac{|R|!(|S| + |U|)!}{n!}$$

$$\prod_{l=1}^{\infty} (\text{ch}_l(\epsilon))^k(\text{cy}_l(\epsilon))^k \frac{2}{2^{|T|}}$$

$$+o((u(\epsilon))^{-(n'-n)}) \quad (6.26)$$

in $L^2$, as $\epsilon \to 0$.

Let us introduce the abbreviation

$$Z(k, |\sigma_+|, p) = \{2|T| + |S| + |V| = k, |R| + |S| + |U| = |\sigma_+|, |S| + |U| = 2p\}. \quad (6.27)$$
we can reorganize (6.26) as

\[(|U| + |V|)/2 - (|S| + |U|)/2 - |T|\] (6.28)

\[= |V| - (2|T| + |S| + |V|)/2 = |V| - k/2\]

and

\[|U| + |V| + 2|C|\]

(6.29)

\[= 2n - 2|R| - 2|S| - 2|T| - |U| - |V|\]

\[= 2n - 2(|R| + |S| + |U|) + (|S| + |U|) - 2|T| + |S| + |V|\]

\[= 2n - 2|\sigma|_+ + 2p - k = 2(n - |\sigma|) - k,\]

we can reorganize (6.26) as

\[H_{1,\nu}^n = \sum_{\{\sigma | 0 \leq |\sigma| \leq |\sigma|_+ \leq n\}} \sum_{k=0, R, S, T, U, V, C}^{2n} \sum_{Z(k, |\sigma|_+, \nu)}^{2n} \frac{(n - |\sigma|_+)!}{|T|!!|V|!!|C|!!} \cdot G^{k \nu} : \frac{\alpha^k}{2^{k/2}} I_{n, \nu}(\sigma) + o((u(\epsilon))^{-(n' - n)}).\]

Since there are \(n!/(|R|!!|S|!!|T|!!|U|!!|V|!!|C|!!)\) ways to partition \([1, n]\) into sets of size \(|R|, |S|, |T|, |U|, |V|, |C|\) we see that

\[H_{1,\nu}^n = \sum_{\{\sigma | 0 \leq |\sigma| \leq |\sigma|_+ \leq n\}} \sum_{k=0, |R|, |S|, |T|, |U|, |V|, |C|}^{2n} \sum_{Z(k, |\sigma|_+, \nu)}^{2n} \frac{(n - |\sigma|_+)!}{|T|!!|V|!!|C|!!} \cdot G^{k \nu} : \frac{\alpha^k}{2^{k/2}} I_{n, \nu}(\sigma) + o((u(\epsilon))^{-(n' - n)}).\]

Now write (6.31) as

\[H_{1,\nu}^n = \sum_{\{\sigma | 0 \leq |\sigma| \leq |\sigma|_+ \leq n\}} \left( \sum_{k=0}^{2n} \rho(\sigma, k) \alpha^{2(n - |\sigma|) - k} : G^{k \nu} : \frac{\alpha^k}{2^{k/2}} \right) I_{n, \nu}(\sigma) + o((u(\epsilon))^{-(n' - n)}),\]

where

\[\rho(\sigma, k) = \sum_{|R|, |S|, |T|, |U|, |V|, |C|}^{2n} \frac{(n - |\sigma|_+)!}{|T|!!|V|!!|C|!!} \left( \frac{|S| + |U|}{|S|} \right)^2 |V|.\]
Lemma 6.3

\[ \rho(\sigma, k) = \binom{2(n - |\sigma|)}{k}. \]  

(6.34)

Using this in (6.32) and recalling the definition (5.5) of \( H_{n-|\sigma|}(\nu) \) we have

\[ H^n_{1, \nu} = \sum_{\{\sigma|0 \leq |\sigma| \leq |\sigma|_+ \leq n\}} I_{n, \nu}(\sigma) H_{n-|\sigma|}(\nu) + o((u(\epsilon))^{-(n'-n)}). \]  

(6.35)

Then by the argument which led to (5.12) we obtain (6.1). As explained in the beginning of this section, (6.1) will complete the proof of Lemma 5.1.

Proof of Lemma 6.3

Under our constraint \( Z(k, |\sigma|_+, p) \)

\[ 2|T| + |S| + |V| = k, \ |R| + |S| + |U| = |\sigma|_+, \ |S| + |U| = 2p. \]

Once we have specified \( 0 \leq |S| \leq 2p \) and \( 0 \leq |V| \leq k - |S| \), then \( |U|, |R|, |T| \) are determined and consequently so is \( |C| \). Furthermore, we have

\[ |T| = (k - |S| - |V|)/2 \]

and

\[ |C| = n - (|R| + |S| + |T| + |U| + |V|) \]

\[ = n - |\sigma|_+ - (|T| + |V|) = n - |\sigma|_+ - (k - |S| + |V|)/2. \]  

(6.37)

Thus, setting \( m = n - |\sigma|_+ \) we have

\[ \rho(\sigma, k) = \sum_{|S| = 0}^{2p} \binom{2p}{|S|} \sum_{|V| = 0}^{k-|S|} \binom{k-|S|-|V|/2}{m} \binom{m}{m-(k-|S|+|V|)/2} 2^{|V|} \]  

and our Lemma then follows if we can show that for all \( k, m, p \in \mathbb{Z}_+ \)

\[ \sum_{s=0}^{2p} \binom{2p}{s} \sum_{v=0}^{k-s} \binom{k-s-v/2}{v} \binom{m}{m-(k-s+v)/2} 2^v = \binom{2m+2p}{k}, \]  

(6.39)

where the sum over \( v \) is only taken over those values of \( v \) such that \( (k-s-v)/2 \) and \( m-(k-s+v)/2 \) are non-negative integers.
To see (6.39), note first that
\[
\binom{2m + 2p}{k} = \sum_{s=0}^{2p} \binom{2p}{s} \binom{2m}{k-s},
\]
which can be seen by examining the coefficient of \(x^k\) on both sides of
\[
(x + 1)^{2m+2p} = (x + 1)^{2p}(x + 1)^{2m}.
\]
Thus, it will suffice to prove that
\[
\sum_{v=0}^{k-s} \binom{(k-s-v)/2}{v} \binom{m}{m - (k-s+v)/2} 2^v = \binom{2m}{k-s},
\]
which we rewrite with \(q = k-s\) as
\[
\sum_{v=0}^{q} \binom{(q-v)/2}{v} \binom{m}{m - (q+v)/2} 2^v = \binom{2m}{q}.
\]
Now, the coefficient of \(x^q\) in \((x + 1)^{2m}\) is \(\binom{2m}{q}\). But also, as
\[
(x + 1)^{2m} = (x^2 + 2x + 1)^m = \sum_{i,v} \binom{m}{i, v} (2x)^i (1)^{m-i-v} 2^v x^{2i+v},
\]
we see that the only \((v,i)\) that contribute to \(x^q\) are those with \(0 \leq v \leq q\) and \(i = (q-v)/2\), and then only if \(q\) and \(v\) have the same parity. Thus, the coefficient of \(x^q\) is simply
\[
\sum_{v=0}^{q} \binom{(q-v)/2}{v} \binom{m}{m - (q-v)/2 - v} 2^v = \sum_{v=0}^{q} \binom{(q-v)/2}{v} \binom{m}{m - (q+v)/2} 2^v.
\]
(6.40)

This completes the proof of Lemma 6.3 and hence of Lemma 5.1.

Proof of Lemma 6.1: The bounds on \(ch_k(\epsilon), cy_k(\epsilon)\) come from [9, Lemma 8.6]. These bounds are obtained by rewriting \(ch_k(\epsilon), cy_k(\epsilon)\) in terms of the Fourier transform of \(u\). Since \(u_{\delta,\delta} = f_{\delta} * u * f_{\delta}\), we have
\[
\hat{u}_{\delta,\delta}(\lambda) = (\hat{f}(\delta\lambda))^2 \hat{u}(\lambda).
\]
(6.41)
Since by assumption \( \int f(x) \, dx = 1 \), we have \( |\hat{f}(\delta \lambda)| \leq 1 \) for all \( \delta, \lambda \) so that

\[
|\hat{u}_{\delta, \delta}(\lambda)| \leq \hat{u}(\lambda); \quad (6.42)
\]

and \( \lim_{\delta \to 0} \hat{f}(\delta \lambda) = \hat{f}(0) = 1 \) for each \( \lambda \). Using this in the proof of \cite{[9]} Lemma 8.6] the rest of our Lemma follows easily.

**Proof of Lemma 6.2:** Since \( G_k^\delta \nu := \int : G_{x, \delta}^k : \, d\nu(x) \) we can write

\[
\int \prod_{i=1}^k G_{x+z_i, \delta} : \, d\nu(x) - : G_k^\delta \nu : \quad (6.43)
\]

\[
= \sum_{j=1}^k \int \left( \prod_{i=1}^{j-1} G_{x+z_i, \delta} \right) (G_{x+z_j, \delta} - G_{x, \delta}) G_{x, \delta}^{k-j} : \, d\nu(x)
\]

\[
= \sum_{j=1}^k \int \left( \prod_{i=1}^{j-1} G_{x+z_i, \delta} \right) (\Delta_{j,x} G_{x, \delta}) G_{x, \delta}^{k-j} : \, d\nu(x)
\]

where \( \Delta_{j,x} f(x) = f(x + z_j) - f(x) \). We can then compute

\[
E \left( \int \left( \prod_{i=1}^{j-1} G_{x+z_i, \delta} \right) (\Delta_{j,x} G_{x, \delta}) G_{x, \delta}^{k-j} : \, d\nu(x) \right)^2 \right) \quad (6.44)
\]

as a sum of terms of the form

\[
\int \left( \Delta_{j,x} \Delta_{j,y} u_{\delta, \delta}((x-y) + a_1) \prod_{l=2}^k u_{\delta, \delta}((x-y) + a_l) \right) \, d\nu(x) \, d\nu(y) \quad (6.45)
\]

or

\[
\int \left( \Delta_{j,x} u_{\delta, \delta}((x-y) + a_1) \Delta_{j,y} u_{\delta, \delta}((x-y) + a_2) \prod_{l=3}^k u_{\delta, \delta}((x-y) + a_l) \right) \, d\nu(x) \, d\nu(y),
\]

where the \( a_l \) can be either 0 or some combination of the \( z_m, m = 1, \ldots, k \). We consider \( (6.46), (6.45) \) is similar and easier. Write

\[
u_{\delta, \delta}((x-y) + a_l) = \int e^{i\lambda_l((x-y)+a_l)} \hat{u}_{\delta, \delta}(\lambda_l) \, d\lambda_l, \quad (6.47)
\]

\[
\Delta_{j,x} u_{\delta, \delta}((x-y) + a_1) = \int \left( e^{i\lambda_1 x_j} - 1 \right) e^{i\lambda_1((x-y)+a_1)} \hat{u}_{\delta, \delta}(\lambda_1) \, d\lambda_1, \quad (6.48)
\]

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and similarly for $\Delta_{j,y} u_{\delta,\delta}(x - y + a_2)$. Multiplying these out as in (6.46) and then integrating with respect to $d\nu(x)\,d\nu(y)$ we see that (6.46) is bounded by

$$
\int |e^{i\lambda_1 x_j} - 1| |e^{i\lambda_2 x_j} - 1| |\hat{\nu} (\lambda_1 + \cdots + \lambda_k)|^2 \prod_{l=1}^k \hat{u}_{\delta,\delta}(\lambda_l) \,d\lambda_l. \tag{6.49}
$$

Using (6.42) and the Cauchy-Schwarz inequality, it suffices to bound

$$
\int |e^{i\lambda_1 x_j} - 1|^2 |\hat{\nu} (\lambda_1 + \cdots + \lambda_k)|^2 \prod_{l=1}^k \hat{u}(\lambda_l) \,d\lambda_l \tag{6.50}
$$

$$
= \int |\hat{\nu}(\lambda)|^2 \tau_{k,r'z'}(\lambda) \,d\lambda,
$$

where

$$
\tau_{k,r'z'}(\lambda) = \int |e^{i\lambda_1 x_j} - 1|^2 |\hat{u}(\lambda_1)| \tau_{k-1}(\lambda - \lambda_1) \,d\lambda_1, \tag{6.51}
$$

and $\tau_{k-1}$ is the $k - 1$ fold convolution of $\hat{u}$. Our Lemma then follows from [9, Lemma 8.4].

7 Decomposition of intersection local times

The renormalized intersection local times $L_n(\nu)$, a renormalized limit of

$$
\int \left( \sum_{\omega \in I_\alpha} \int f_\epsilon(Y_t(\omega) - x) \,dt \right)^n d\nu(x), \tag{7.1}
$$

involves both self-intersections of paths in the Poisson process $I_\alpha$ and intersections between different paths in $I_\alpha$. In this section we show how to make this explicit.

We first mimic the construction of $L_n(\nu)$, but for a single path under the quasi-process measure $\mu_m$. Let

$$
L_1(x, \epsilon) := \int f_{\epsilon,x}(Y_t) \,dt. \tag{7.2}
$$

Using the notation from Section 4 we define recursively

$$
L_n(x, r) = L_1^0(x, r) - \sum_{\{\sigma| 1 \leq |\sigma| < |\sigma|_+ \leq n\}} I_n(\sigma, r), \tag{7.3}
$$

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where
\[ I_n(\sigma, r) = \frac{n!}{\prod_{i=1}^{\infty} k_i !(n - |\sigma|_+)} \prod_{i=1}^{\infty} (\text{ch}(r))^{k_i} \mathcal{L}_{n-|\sigma|}(x, r). \]  
(7.4)

(Note that \( n - |\sigma|_+ \geq 0 \).)

Set
\[ \mathcal{L}_{n,r}(\nu) = \int \mathcal{L}_{n}(x, r) \, d\nu(x). \]  
(7.5)

Following the proof of Theorems [1, 1] and [4, 1] we can show that if \( \nu \in G_{2n}^2 \), then
\[ \mathcal{L}_{n}(\nu) := \lim_{\epsilon \to 0} \mathcal{L}_{n,\epsilon}(\nu) \]  
exists in all \( L^p(\mu_m) \),
(7.6)

and if \( n = n_1 + \cdots + n_k \) and \( \nu_i \in G_{2n_i}^2 \), then
\[ \mu_m \left( \prod_{i=1}^{k} \mathcal{L}_{n_i}(\nu_i) \right) = \prod_{i=1}^{k} n_i! \int \sum_{\pi \in \mathcal{M}_a} \prod_{i=1}^{n_1} u(x_{\pi(i)}, x_{\pi(i+1)}) \prod_{m=1}^{k} d\nu_m(x_m) \]  
(7.7)

where \( \mathcal{M}_a \) is the set of maps \( \pi : [1, n] \to [1, k] \) with \( |\pi^{-1}(m)| = n_m \) for each \( m \) and such that for each \( l \), if \( \pi(i) = m \) then \( \pi(i + 1) \neq m \). (The subscript ‘a’ in \( \mathcal{M}_a \) stands for alternating).

We refer to the \( \mathcal{L}_{n}(\nu) \) as \( n \)-fold self-intersection local times. If we now set
\[ \mathcal{K}_{n}(\nu) = \sum_{\omega \in \mathcal{T}_a} \mathcal{L}_{n}(\nu)(\omega), \]  
(7.8)

that is we add together the \( n \)-fold self-intersection local times \( \mathcal{L}_{n}(\nu)(\omega) \) for each \( \omega \in \mathcal{T}_a \), then using the moment formula for Poisson processes we find that the \( \mathcal{K}_{n}(\nu) \) satisfy moment formulas similar to (4.10) except that for each partition \( B_1 \cup \cdots \cup B_j = [1, n] \) and each \( m \), there will be some \( l \) with \( \pi^{-1}(m) \in B_l \).

We now construct an intersection local time involving intersections between different paths in \( \mathcal{T}_n \). We follow [9, Section 7], but the situation here is easier since all \( \mathcal{L}_{n}(x, r) \) are \( \mu_m \) integrable. For any set \( A \), we use \( S_n(A) \subset A^n \) to denote the subset of \( A^n \) with distinct entries. That is, if \( (a_i, \ldots, a_i) \in S_n(A) \) then \( a_i \neq a_i \) for \( i \neq i \). Let
\[ \mathcal{K}_{I_1, \ldots, I_n}(x, r) = \sum_{(\omega_1, \ldots, \omega_n) \in S_n(\mathcal{I}_a)} \prod_{j=1}^{n} \mathcal{L}_{I_j}(x, r)(\omega_{i_j}). \]  
(7.9)

It follows as in the proof of [9, Theorem 7.1] that if \( l = l_1 + \cdots + l_n \) and \( \nu \in G_{2l}^2 \),
Then
\[ \mathcal{K}_{I_1, \ldots, I_n}(\nu) := \lim_{\epsilon \to 0} \int \mathcal{K}_{I_1, \ldots, I_n}(x, r) \, d\nu(x) \]  
(7.10)
exists in all $L^p(\mathcal{P}_n)$. Note that $K_{\ell_1,\ldots,\ell_n}(\nu)$ involves $n$ distinct paths in $\mathcal{T}_\alpha$. As in [9, Theorem 7.2] we have

**Theorem 7.1** For $\nu \in \mathcal{G}_K^{2l}$

$$L_n(\nu) = \sum_{D_1 \cup \cdots \cup D_l = [1,n]} K_{|D_1|,\ldots,|D_l|}(\nu),$$

(7.11)

where the sum is over all partitions of $[1,n]$.

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