A roadmap to hadronic supercriticalities: a comprehensive study of the parameter space for high-energy astrophysical sources

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Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT

Hadronic supercriticalities are radiative instabilities that appear when large amounts of energy are stored in relativistic protons. When the proton energy density exceeds some critical value, a runaway process is initiated resulting in the explosive transfer of the proton energy into electron-positron pairs and radiation and the increase of the photon-to-proton efficiency. We perform a comprehensive study of the parameter space by investigating the onset of hadronic supercriticalities for a wide range of source parameters (i.e., magnetic field strengths of $10^{-1} - 10^{-2}$ G and radii of $10^{11} - 10^{16}$ cm) and maximum proton Lorentz factors ($10^{3} - 10^{9}$).

We show that supercriticalities are possible for the whole range of source parameters related to compact astrophysical sources, like gamma-rays bursts, cores and jets of active galactic nuclei. We also provide an in-depth look at the physical mechanisms of hadronic supercriticalities and show that magnetized relativistic plasmas are excellent examples of non-linear dynamical systems in high-energy astrophysics.

Key words: instabilities – radiation mechanisms: non-thermal – galaxies: active – gamma-ray burst: general

1 INTRODUCTION

The non-thermal extended photon spectra of high-energy emitting sources, such as active galactic nuclei (AGN) and gamma-ray bursts (GRBs), suggest the acceleration of radiating particles to ultra-relativistic energies. Although leptonic models are widely favoured in explaining the observed $\gamma$--ray flux through the emission of relativistic electron-positron pairs, hadronic scenarios that invoke relativistic protons (and heavier nuclei) remain a viable alternative, while having direct implications for the sites of cosmic-ray acceleration and neutrino production (e.g., Mannheim 1993; Halzen & Zas 1997; Mücke & Protheroe 2001; Böttcher & Dermer 1998; Kazanas et al. 2002; Asano & Inoue 2007; Murase 2008).

Especially after the discovery of the astrophysical high-energy neutrino flux by IceCube (IceCube Collaboration 2013) and the most recent association of high-energy neutrinos with the $\gamma$--ray emitting blazar TXS 0506+056 (Aartsen et al. 2018a,b), hadronic models became more relevant than ever before (Dermer et al. 2014; Petropoulou et al. 2015a; Cerruti et al. 2015; Keivani et al. 2018; Cerruti et al. 2019; Xue et al. 2019; Petropoulou et al. 2019; Oikonomou et al. 2019; Murase & Shoemaker 2019; Kimura et al. 2019).

The basic premise of a hadronic scenario as applied to the compact high-energy emitting region(s) of an astrophysical source can be summarised as follows. The model assumes the presence of a relativistic proton population that interacts with its environment in two main ways. First, the gyromotion of protons in the magnetic fields of the source produces synchrotron emission, and secondly, the photohadronic interactions with low-energy photons (henceforth, soft photons) lead to the production of many secondary particles. More specifically, photopair (Bethe-Heitler) production creates relativistic electron-positron pairs, while photopion production injects charged and neutral pions ($\pi^{\pm}, \pi^{0}$). The pions decay almost instantaneously to create very energetic $\gamma$--rays from $\pi^{0}$ decay and more electron-positron pairs from $\pi^{\pm}$ decays. $\gamma$--rays can, also, interact with soft photons via $\gamma\gamma$ absorption, thus producing more pairs. Neutrons and neutrinos are another by-product of photopion interactions. While neutrinos will escape from the source, neutrons may interact with ambient photons in a similar way as protons, decay inside the source or decay after they escape from the source.

These interactions, which relate all stable particle species to one another, form a non-linear network of physical processes. Their non-linear character is manifested more strongly when the proton energy losses are not driven by external agents, like magnetic fields...
and/or external radiation fields. In this case, feedback loops based on different combinations of the processes described above might become operative, if certain conditions are fulfilled. This may happen, for example, if the injection of pairs produced from photohadronic interactions significantly increases the number of soft photons in the source (e.g., by synchrotron radiation), which, in turn, serve as targets for further cooling of the protons. This feedback network has been first studied by Stern & Svensson (1991) and Stern et al. (1992) by means of Monte Carlo simulations, and later by Kirk & Mastichiadis (1992); Mastichiadis et al. (2005) and Petropoulou & Mastichiadis (2012) using the kinetic equation approach. Similarly, when the luminosity of the proton-injected γ-rays grows sufficiently, this high-energy radiation is automatically quenched, providing a supplementary soft target photon population for proton cooling, which results in more γ-rays (Stawarz & Kirk 2007; Petropoulou & Mastichiadis 2011; Petropoulou et al. 2013).

As it was shown analytically in Kirk & Mastichiadis (1992) and Petropoulou & Mastichiadis (2012), the above networks are sustained when a feedback condition and a marginal stability criterion are simultaneously satisfied. The former depends on the maximum energy of the proton distribution and the latter on the density of relativistic protons inside the source. Regardless of the details of the networks involved, the result is an exponential growth of protons, which are produced either directly or indirectly from protons, leading inevitably to proton cooling through photohadronic interactions. The conditions in the source just before the onset of the photon number exponentiation indicate a critical point for the system; if the protons inside the source have reached a certain critical density, they become supercritical. Any attempt to further increase their density will make the system undergo a phase transition by efficiently removing energy from the protons due to photohadronic interactions and transferring it to photons, pairs, and neutrinos. This physical situation is in many ways analogous to “sand piles” where just one more grain of sand can cause the sudden collapse of the pile.

The onset of supercriticality depends on various parameters, such as the magnetic field inside the source, the energy content and the maximum cutoff energy of the protons, and others. In an astrophysical environment one could envisage various ways that lead to supercriticalities. The simplest scenario is to consider that protons, after being accelerated to relativistic energies, are injected inside a confining volume. In the absence of strong losses and fast escape, the injected protons accumulate and their density could become supercritical. The onset of supercriticality (i.e., linear phase of photon outgrowth) can be studied by analytical means as long as the network of the driving physical processes can be isolated and the rate equations simplified (Kirk & Mastichiadis 1992; Petropoulou & Mastichiadis 2012). However, the study of the system in the nonlinear phase of the photon outgrowth can be followed only with numerical means. Despite the considerable amount of work that has been done so far on various aspects of hadronic supercriticalities (see the references listed above), a systematic search of the relevant parameter space is still missing.

Here, we numerically explore the phase space of hadronic supercriticalities for a wide range of parameters relevant to nonthermal emitting astrophysical sources. Our goal is to make a comprehensive list of all possible supercriticalities, classify the temporal behaviour of the system, and gain a better understanding of the driving physical processes.

This paper is structured as follows. In Section 2, we outline the physical model and the methodology of our numerical investigation, and continue in Section 3 with a presentation of the parameter space of hadronic supercriticalities. In Section 4, we identify the networks of the physical process that drive the hadronic system to the supercritical regime. In Section 5, we study the effects of various secondary model parameters on our results. We briefly discuss the astrophysical implications of this study in Section 6. We continue in Section 7 with a summary and discussion of our results, and conclude in Section 8.

2 METHODS

In order to investigate numerically the phase space of hadronic supercriticalities, we adopt the standard framework of a one-zone radiation model. We consider a spherical source of radius \( R \), containing a tangled magnetic field of strength \( B \), where pre-accelerated relativistic protons with a power-law energy distribution are uniformly injected at a rate given by

\[
Q_p(\gamma_p, t) = Q_p,0 \gamma_p^{-\delta} H(\gamma_p - \gamma_{p,\text{min}})H(\gamma_{p,\text{max}} - \gamma_p)H(t),
\]

where \( \gamma_{p,\text{min}} \) and \( \gamma_{p,\text{max}} \) correspond to the minimum and maximum proton Lorentz factors, respectively, \( \delta \) is the power-law index of the proton distribution and \( H(\cdots) \) is the Heaviside step function. The proton injection compactness \( \ell_p \) is defined as

\[
\ell_p = \frac{\sigma_T L_p}{4\pi R m_p c^3} = \frac{1}{3} Q_p,0 t_{\text{tr}} \sigma_T R \int_{\gamma_{p,\text{min}}}^{\gamma_{p,\text{max}}} \gamma_p^{-s+1} d\gamma_p,
\]

where \( L_p \) is the proton injection luminosity, \( t_{\text{tr}} = R/c \) is the source light-crossing timescale, \( c \) is the speed of light, and \( \sigma_T \) is the Thomson cross section. Similarly, we define the magnetic field compactness, which is a dimensionless measure of the magnetic energy density \( U_B \)

\[
U_B = \frac{\sigma_T R U_B}{m_e c^2},
\]

where \( m_e \) is the electron rest mass. The photon compactness is accordingly defined as

\[
\ell_\gamma = \frac{L_\gamma \sigma_T}{4\pi R m_e c^3} = 3 \sigma_T R \frac{U_\gamma}{m_e c^2} \left( 1 + \frac{\tau_r}{3} f(\varepsilon) \right)^{-1},
\]

where \( L_\gamma \) and \( U_\gamma \) are the bolometric photon luminosity and energy density, respectively, \( \tau_r \) is the Thomson optical depth, and \( f(\varepsilon) \) is a function of the photon’s energy \( \varepsilon \) in units of \( m_e c^2 \) (for the full expression, see Mastichiadis & Kirk 1995).

Upon entering the source, the relativistic protons interact with the magnetic field and any soft photons present, creating radiation and secondaries. We do not consider the presence of external photon fields, as this would introduce more free parameters, i.e., we make the implicit assumption that the internal photon energy density is always dominant. For the same reason, we do not consider, until Section 5, the injection of an accelerated electron (primary) population in order to study the supercriticalities of a pure hadronic model first.

We assume that pions and muons decay instantaneously into secondary particles, and that neutrinos do not interact with soft photons before they escape the source (i.e., the source is optically thin to neutron-photon interactions). Neutrinos escape the source without any interactions at the light-crossing time. Thus, at any given time, there are three stable particle species in the source, namely, protons, photons, and electron-positron pairs (henceforth, electrons). The temporal evolution of their distributions can be described by three coupled kinetic equations:

\[
\frac{\partial n_i}{\partial t} + \frac{n_i}{t_{i,\text{esc}}} + L_i = Q_i,
\]
where \( n_i \) is the differential number density of particle species \( i \), and \( t_{\text{esc}} \) is the respective escape timescale (for photons, \( t_{\gamma,\text{esc}} = t_{cr} = R/c \)). The operators \( Q_i \) and \( L_i \) denote the injection (source) and loss (sink) terms, respectively. These terms include the following processes: synchrotron radiation for both electrons and protons, photopair (Bethe-Heitler) production, photopion production, photon-photon (\( \gamma\gamma \)) pair production, inverse Compton scattering, synchrotron self absorption, pair annihilation, and photon down-scattering on cold electrons (for details, see Mastichiadis et al. 2005; Dimitrakoudis et al. 2012).

The free parameters of the model are: the radius of the source \( R \), the magnetic field strength \( B \), the proton injection compactness \( \ell_p \) (or, equivalently \( Q_{p,0} \)), the minimum and maximum proton Lorentz factors \( \gamma_{p,\text{min}} \) and \( \gamma_{p,\text{max}} \), respectively), the proton and electron escape timescales \( t_{p,\text{esc}} \) and \( t_{e,\text{esc}} \), and the power-law index \( s \). In order to reduce the number of free parameters we set \( \gamma_{p,\text{min}} = 1 \), and \( t_{e,\text{esc}} = t_{cr} \) and we keep them constant for all runs. Also, for the next sections we set \( t_{p,\text{esc}} = 1000 \) \( t_{cr} \) and \( s = 2 \), and discuss the implications of this choice in Section 5.

For each parameter set (i.e., \( R, B, \gamma_{p,\text{max}}, \) and \( \ell_p \)) we solve numerically equation (5) by utilizing the code of Mastichiadis & Kirk (1995), with improved production rates and emissivities introduced by Mastichiadis & Kirk (1997); Mastichiadis et al. (2005). We use as our initial conditions \( n_i(\gamma, t = 0) = 0 \) for all species and let the system evolve until some time \( t_{\text{end}} \), which is taken to be much larger than all other relevant timescales of the system, except when noted.

3 THE PHASE SPACE OF HADRONIC SUPERCriticalities

In this section, we use the numerical code described above to perform a systematic search of the parameter space of hadronic supercriticalities and study the temporal behaviour of a hadronic system in this regime. In Section 3.1 we show a typical example of the way a hadronic system moves from the subcritical to the supercritical regime by performing consecutive runs of increasing injection proton luminosity. In Section 3.2 we perform a systematic search of the parameter space of hadronic supercriticalities for various combinations of the source radius \( R \) and its magnetic field strength \( B \) which can be of relevance to astrophysical sources. Finally, in Section 3.3 we investigate the role of the maximum proton energy in the development of the supercriticality.

3.1 Transition to supercriticality: the role of the proton injection compactness

We begin our numerical analysis by showing a representative case of the transition to supercriticality. Here, we perform consecutive runs for increasing values of \( \ell_p \), as illustrated in Fig. 1, for \( R = 10^{15} \) cm, \( B = 10^{1.5} \) G, \( \gamma_{p,\text{max}} = 10^{0.5} \), and \( t_{p,\text{esc}} = 10^5 \) \( t_{cr} \). A similar example was also presented in Petropoulou & Mastichiadis (2018), but for the sake of completeness, we outline the procedure here, as it is central for the understanding of this paper.

For low values of \( \ell_p \) (dotted red and dashed blue lines) the system is still in the subcritical regime and it reaches a steady state, typically after a few hundred crossing times. The photon emission, whose light curve is shown in the figure, is mostly the result of proton synchrotron radiation and it increases linearly with \( \ell_p \). As \( \ell_p \) increases, the proton number density exceeds some critical value, which will be defined in Section 3.2 and the system is driven to supercriticality. This transition is manifested with a sudden increase in the radiative efficiency (see Fig. 2 and relevant discussion in next paragraph) and bursty light curves. The bursts may occur (quasi)periodically (limit cycle behaviour; see solid green and dashed orange lines) with a period that depends on \( \ell_p \). More specifically, higher values of \( \ell_p \) lead to more frequent outbursts, while the first burst progressively appears at earlier times. For even higher values of \( \ell_p \), the number of bursts in the light curve reduces to one before the photon compactness reaches a constant value (dash-dotted purple line); henceforth, we refer to such cases as violent relaxations. This is essentially a high-efficiency steady-state regime, where the photon emission is produced via photodisproportion interactions on the proton-synchrotron radiation. Therefore, we can summarize the general behaviour of the system with increasing \( \ell_p \) as: subcritical (low steady state) \( \rightarrow \) supercritical (limit cycles) \( \rightarrow \) supercritical (violent relaxation).
The characteristic supercritical behaviour of the photon light curve (see also Stern & Svensson 1991; Mastichiadis et al. 2005) can be explained from the fact that once the protons enter the supercritical regime, photons grow exponentially with \( \alpha_t \propto e^{4t} (\alpha > 0) \) until they cool abruptly the protons. Therefore photons reach eventually a maximum density before they escape the source, thus producing a characteristic outburst. The index \( \alpha \) is a function of the system’s parameters (e.g., magnetic field strength, source radius, and others). The higher its value, the stronger the supercriticality.

A result of the transition to the supercriticality is the increase of the radiative efficiency of the system. This is illustrated in Fig. 2, which shows the ratio of the total radiated energy in photons, \( E_\gamma \), to the total injected energy in relativistic protons, \( E_p \), as a function of the proton injection compactness \( \ell_p \) for the parameters used in Fig. 1 (same colour coding used) and \( t_{\text{crit}} = 1200 \text{yrs} \). All runs for low values of \( \ell_p \) (e.g., \( \ell_p \sim 10^{-6} - 10^{-5} \)) correspond to low-efficiency steady states dominated by proton synchrotron radiation (subcritical regime). As \( \ell_p \) increases (\( \ell_p \gtrsim 10^{-4} \)), photodisproportionation interactions become progressively more important in both photon production and proton cooling, since these processes depend on the product of the number density of protons and photons. This results in an increase of the system’s radiative efficiency, however the behaviour is still subcritical. The phase transition to supercriticality occurs with the appearance of even one outburst (\( \ell_p \) values around green point). This results in a dramatic increase of the efficiency as a large fraction of the energy stored in protons is burned explosively. As a result, the efficiency curve turns abruptly upwards, reaching values close to 0.1 (entrance to supercriticality). The increase in the efficiency continues during the whole limit cycle phase and only when the system reaches the violent relaxation phase it starts moving asymptotically to efficiencies close to unity. Because of the neutrino emission, the photon efficiency will never reach values close to one, even if the protons radiate all their energy away. In short, Fig. 2 shows a representative curve for a hadronic system’s radiative efficiency as this transits from the subcritical regime to supercriticality. Although the actual values for the efficiency and critical proton compactness depend on the chosen parameters, the overall shape is retained for a wide range of initial parameters.

While the previous example shows a rather typical way to enter the supercritical regime, this is by no means the only one. This is illustrated in Fig. 3, where we show the \( \ell_B - \ell_p \) parameter space of a hadronic system for the same parameters as in Fig. 1, i.e., \( R = 10^{15} \text{cm} \), \( \gamma_{p,\text{max}} = 10^{6.5} \) and \( \ell_{p,\text{esc}} = 10^{12} \text{cm} \). For comparison purposes, we also indicate with filled diamonds the parameters used for the runs in Fig. 1. One can identify four regions of interest in the \( \ell_B - \ell_p \) plane. For both low \( \ell_B \) and \( \ell_p \) values (region i) the system is sub-critical and protons are inefficient in radiating away their energy. For higher \( \ell_B \) values (and low values of \( \ell_p \)) the system is still subcritical, but proton synchrotron cooling and synchrotron radiation become efficient (region ii). The supercritical regime (depicted as grey area) occupies the right-hand side of the \( \ell_B - \ell_p \) plane (regions iii and iv), i.e., it requires high values of \( \ell_B \) to allow the proton number density to build up to the required critical values. The dark grey region is where limit cycles appear while lighter grey areas correspond to supercritical solutions that end up eventually in a steady state through violent relaxations (see top and bottom right panels). Finally, region iii of the parameter space is characterized by both fast proton cooling (synchrotron and photodisproportionation), and by an abrupt transition to a high-efficiency steady state. Typically, this transition occurs very quickly after the onset of proton injection and its exact time behaviour depends on the competition between the two cooling processes.

### 3.2 The \( B - R \) phase space

So far, we have discussed the system’s transition to supercriticality for fixed \( R \) and \( \gamma_{p,\text{max}} \). We can extend the parameter space search even further by looking for supercriticalities in the \( B - R \) plane. To do so, we have to define the minimum proton injection compactness required for the transition to supercriticality. We therefore define as critical proton compactness \( \ell_{p,\text{crit}} \) that value of \( \ell_p \) that produces the first photon flare at \( t_{\text{APP}} = t_{\text{app}} \). According to this definition, the green symbol in Figs. 1-3 indicates \( \ell_{p,\text{crit}} \). However, a transition to supercriticality (grey coloured region) can also occur for slightly lower values of \( \ell_p \). These values can still lead to the production of bursts of radiation but at longer \( t_{\text{APP}} \). If no outburst is produced, which is the case for region ii (and sometimes region iii) in Fig. 3, \( \ell_{p,\text{crit}} \) cannot be defined. Regardless, our definition of \( \ell_{p,\text{crit}} \) is sufficient for the parameter space search that follows.

For each \( \ell_{p,\text{crit}} \) value, we also compute the corresponding proton energy density \( U_{p,\text{crit}} \), which is the maximum density that the protons acquire before the first photon outburst occurs, and the ratio \( U_{p,\text{crit}}/U_{\text{th}} \), which is a measure of the deviation from energy equipartition at the onset of supercriticality. Our results are summarised in Fig. 4, which shows the logarithmic \( B - R \) phase space of hadronic supercriticalities for four values of \( \gamma_{p,\text{max}} \) for the parameter ranges employed, see Table 1. In all panels, one can identify two regions of interest:

- The upper right corner of the phase space (white region) corresponds to the fast proton synchrotron cooling regime, where protons radiate efficiently their energy via synchrotron. The system always reaches a steady state irrespective of the value of \( \ell_p \) (see also regimes ii and iii in Fig. 3). In this region, the critical proton compactness cannot be defined, as no outburst occurs at \( t < t_{\text{app}} \).
- The lower left corner (various shades of grey) corresponds to the slow synchrotron cooling regime, where a transition to supercriticality is always possible. Different shades are used to denote different manifestations of supercriticality: the quasi-periodic outbursts or limit cycles (denoted as LC in the plots) are depicted in dark grey, while the violent relaxations (VR) to a steady state are shown with light grey (see also regime iv in Fig. 3). For \( B, R \) values drawn from the lower left dark-coloured region, the gyroradius of the highest energy protons, i.e., \( r_g = \gamma_{p,\text{max}} m_p c/\epsilon B \), is larger than the core radius. For the upper right white region, the system never attains a steady state. Depending on the chosen parameters, different regimes are encountered:

| Physical quantity | Minimum value | Maximum value |
|-------------------|---------------|---------------|
| \( R \) (cm)      | \( 10^{11} \) | \( 10^{16} \) |
| \( B \) (G)       | \( 10^9 \)   | \( 10^5 \)   |
| \( \gamma_{p,\text{max}} \) | \( 10^3 \) | \( 10^9 \) |

Table 1. Range of values of the radius \( R \), magnetic field strength \( B \), and Lorentz factor \( \gamma_{p,\text{max}} \) used throughout the paper.

1 There is an analogy here with the synchrotron self-Compton process of leptonic plasmas.

2 The system’s temporal behaviour can be understood qualitatively with analytical means – see Appendix A.
than the assumed source radius (Hillas 1984). Thus, this part of the parameter space is physically forbidden.

We generally find that for an increasing $R$ the manifestation of supercriticality changes, as limit cycles give their place to violent relaxations, especially for large values of $\gamma_{p,\max}$. This is due to the fact that the conditions for supercriticality depend on the photon compactness which is related to the size of the source (see equation 4).

Frequent limit cycles with large-amplitude photon outbursts appear both for small and large values of $\gamma_{p,\max}$, while this behaviour becomes weaker, by showing violent relaxations, for intermediate values of this parameter. This rather complicated dependence of supercriticality’s manifestations on the maximum proton energy is explained in Section 4.

The dependence of the supercriticality on $B$ is also complicated. However, there is an overall tendency, as $B$ increases and $R$ is kept fixed, for the system to successively transit from a violent relaxation to a limit cycle behaviour and, finally, to steady state (i.e., from region iv to region iii in Fig. 3). Note also that in all panels we have drawn the line $t_{p,\syn}(\gamma_{p,\max}) = t_{p,\osc}$, where $t_{p,\syn}(\gamma_{p,\max})$ is the synchrotron cooling timescale of the maximum energy protons. This line, however, does not describe very accurately the transition between the sub- and supercritical regimes and it has to be taken more as an indication.

The transition to supercriticality requires generally higher energy densities in protons than in magnetic fields. The ratio $U_{p,\crit}/U_B$ decreases for increasing magnetic field strengths (see contours in all four panels). Although this trend is generic, the exact values of the ratio $U_{p,\crit}/U_B$ depend on the maximum proton energy $\gamma_{p,\max}$ as well as on the power-law index $s$ (see Section 5.2).

Finally, our parameter exploration showed that supercriticalities may appear for a wide range of $B$, $R$ and $\gamma_{p,\max}$ values, thus making them potentially relevant for all types of compact astrophysical objects (for more details, see Section 6).

3.3 The role of the maximum proton energy

We next turn on investigating in more detail the effects of the maximum proton energy $\gamma_{p,\max}$ on the critical proton energy density $U_{p,\crit}$. For this purpose, we search for the onset of supercriticality as a function of $\gamma_{p,\max}$. Our results are presented in Fig. 5 where the ratio $U_{p,\crit}/U_B$ is plotted as a function of $\gamma_{p,\max}$ for a source of radius $R = 10^{15}$ cm and different values of the magnetic field $B$ ranging from $B = 10$ G up to $10^4$ G with an increment of 0.5 in logarithm. Overall, we find that the critical proton energy density $U_{p,\crit}$ decreases for higher values of $\gamma_{p,\max}$. There is a strong dependence of $U_{p,\crit}$ on $\gamma_{p,\max}$, $\lesssim 10^4$, but the dependency weakens for high values (i.e., $\gamma_{p,\max} \gtrsim 10^7$). For $\gamma_{p,\max} < 10^4$, it is becoming increasingly difficult to transit to the supercritical regime, and for all practical purposes one can safely assume that no supercriticality occurs below this value. Generally, as $B$ increases (i.e., moving from curves to a g), we find that the system reaches a steady state without showing any outbursts for a wide range of $\gamma_{p,\max}$ values. The concept of a critical proton energy density becomes irrelevant for the entire lower part of the diagram (i.e., below curve c), which is intentionally left blank.

4 The physical mechanisms of supercriticality

In the previous sections, we showed that, for a wide range of values for the main physical parameters ($B$, $R$, and $\gamma_{p,\max}$), there exists a critical value of the proton compactness which drives the system into supercriticality. In this section, we identify the physical processes that play a dominant role in this transition.
As a guide, we will use the analytical results of Kirk & Mastichiadis (1992) and Petropoulou & Mastichiadis (2012) – henceforth KM92 and PM12 respectively. As it was demonstrated in these studies, hadronic supercriticalities arise as a result of specific networks of physical processes (or feedback loops). In what follows, we briefly present the basic ideas behind these papers before applying them to this study’s findings.

4.1 Photopair or Photopion - Electron Synchrotron Loop

The premise behind the photopair-electron synchrotron ($P_eS$) and the photopion-electron synchrotron ($P\pi S$) loops is fairly simple: electron-positron pairs that are created directly in photopair or indirectly (through the decay of charged mesons) in photopion production processes radiate synchrotron photons. These, in turn, can serve as targets for additional photopair or photopion interactions of protons, thus leading to the production of more pairs, which will radiate via synchrotron and so on. Thus, a loop of physical processes describing the exponentiation of pairs and photons is formed.

KM92 demonstrated that both the $P_eS$ and $P\pi S$ loops can lead the system to supercriticality and derived, using simple analytical formulas, the threshold and the marginal stability criteria that are required to make the above feedback loops operate. In the case of the $P_eS$ loop, the feedback criterion is that the proton Lorentz factor should exceed a critical value given by:

$$\gamma_{p,crit} = \left( \frac{2 m_p}{b \eta_{\pi}} \right)^{1/3}$$

while the corresponding value in the case of the $P\pi S$ loop is

$$\gamma_{p,\pi,crit} = \left( \frac{1}{b \eta_{\pi}} \frac{m_\pi^2}{m_e} \left( 1 + \frac{m_\pi}{2 m_e} \right) \right)^{1/3}$$

where $m_p$, $m_e$, and $m_\pi$ are the proton, electron and pion masses, respectively, $b = B/B_{crit}$, $B_{crit} = 4.4 \times 10^{13}$ G is the Schwinger magnetic field, and $\eta_{\pi} \gtrsim 150$ relates, on average, the Lorentz factor of the secondary pairs produced from charged pion decay to the Lorentz factor of the parent proton (Dimitrakoudis et al. 2012). By comparing the above relations, we find that $\gamma_{p,crit} < \gamma_{p,\pi,crit}$ is always satisfied.

\footnote{This relation has not been derived explicitly in KM92.}
These critical values of the proton Lorentz factor are derived by requiring that the protons are energetic enough to pair-produce or pion-produce on the synchrotron photons emitted by their secondaries. Both expressions are approximate as they were derived by requiring that the protons are energetic enough to pair-produce or pion-produce on the synchrotron photons emitted by their secondaries. However, as we will show later, even under these rough approximations, these relations can qualitatively describe the behaviour of the system for low $\gamma_{p,\text{max}}$.

The marginal stability criterion basically requires that the proton density is high enough so that the synchrotron photons radiated from the secondaries can produce either more pairs ($\text{PeS}$) or more pions ($\text{PrS}$) before they escape from the source. Because the derivation of the marginal proton number density $n_{p,\text{marg}}$ is less straightforward than the one for the critical proton Lorentz factor, we simply repeat the finding of KM92, namely $n_{p,\text{marg}} > n_{p,\text{crit}}$. This means that if the feedback criterion is satisfied for both loops, then the PeS loop occurs more easily as it requires lower proton densities.

### 4.2 Photo-Quenching Loop

PM12 examined another feedback loop relevant in hadronic systems – we will use the abbreviation $\gamma Q$ – that was based on the instability of automatic photon quenching (Stawarz & Kirk 2007; Petropoulou & Mastichiadis 2011). This describes essentially the spontaneous outgrowth of electron-positron pairs and their synchrotron radiation in a source when its $\gamma-$ray compactness exceeds some critical value and, as such, it does not necessarily involve protons. Petropoulou & Mastichiadis (2011) showed that there is a regime in the diagram of the differential radiated luminosity versus photon energy that is “forbidden” in the sense that $\gamma-$rays cannot exist in steady state there. If $\gamma-$rays enter this regime they become self-quenched as they spontaneously produce soft photons which, in turn, absorb them. This process is, in many ways, analogous to the loops discussed in the previous section with $\gamma-$rays replacing protons and $\gamma\gamma$ absorption replacing photopair or photopion production. The feedback criterion for automatic photon quenching translates into a critical value for the photon energy

$$\epsilon_{\gamma,\text{crit}} = 2b^{-1/3}$$

while the marginal stability criterion sets a critical value for the $\gamma-$ray compactness, which can be written approximately as (see equation 34 of Petropoulou & Mastichiadis 2011)

$$\epsilon_{\gamma,\text{marg}}^{\text{inj}} = \frac{3b^2\epsilon_e^2}{8} \left[ \left( \frac{b\epsilon_e}{2} \right)^{3/2} - \frac{8}{b^2} \right]^{-1}.$$  

In this study, we are seeking a proton-related mechanism that is, directly or indirectly, producing $\gamma-$rays that simultaneously satisfy the above criteria. These mechanisms are summarised below.

- Proton synchrotron radiation. It can produce directly the required $\gamma-$rays.
- Photopair production process. It can produce $\gamma-$rays indirectly through the synchrotron/ICS radiation of its secondary electron-positron pairs.
- Photopion production process. It can produce $\gamma-$rays either directly through the channel $n^0 \rightarrow 2\gamma$ or indirectly through the synchrotron/ICS radiation of the secondary electron-positron pairs produced in $\pi^+ \rightarrow \mu^+ \rightarrow \pi^0 + \gamma$.

Out of the aforementioned processes, only proton synchrotron requires rather extreme conditions to meet the threshold condition (i.e., very large values of $B$ and $\gamma_{p,\text{max}}$, Petropoulou et al. 2014) – see Appendix B for estimates on the required thresholds of all processes. In the next subsection we investigate all the aforementioned loops with the help of the numerical code described in Section 2.

### 4.3 The $U_{p,\text{crit}} - \gamma_{p,\text{max}}$ phase space

Using the code and switching off all processes except those processes that are key for the PrS and PeS loops, we compute $U_{p,\text{crit}}$ and plot it against $\gamma_{p,\text{max}}$ in Fig. 6. Panel (a) is using photopair production as its main proton loss mechanism, panel (b) is using photopion production, while both have only electron synchrotron switched on. Here, we have adopted $R = 10^{15}$ cm and $B = 10^{13}$ G, but our results are largely independent of this choice. We find that the combination of only these processes is adequate to produce the phenomenology addressed in the previous sections where the full code was used (e.g., flaring, limit cycle behaviour, and others).

The solid green line in panel (a) shows the critical energy density of the PeS loop, while the solid green line in panel (b) shows the corresponding curve for the PrS loop. On the same plot we also show the analytical results of KM92. The vertical dotted lines in panels (a) and (b) indicate $\gamma_{p,\text{crit}}$ and $\gamma_{p,\text{marg}}$ derived from equations 6 and 7, respectively. These have to be compared with the numerically obtained values of $\approx 3 \times 10^3$ and $\approx 10^3$, respectively – see the ascending green lines in Fig. 6. The use of the full synchrotron emissivity causes the numerically derived values of $\gamma_{p,\text{crit}}$ to fall below the analytical ones that were derived using $\delta-$functions instead. The horizontal dotted lines (in both panels) show the marginal proton energy density (for either feedback loop) derived by KM92 using the marginal stability condition. For the adopted values of $B$ and $R$, these read $U_{p,\text{marg}}^{\text{PrS}} \approx 10^{5.6}$ erg cm$^{-3}$.

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4 This is computed in exactly the same way as when all processes are included in the code.
and $U_{p, \text{crit}}^{P_{\gamma\gamma}+p-syn} \simeq 10^{7.3}$ erg cm$^{-3}$ (see equations 6 and 8 of KM92, respectively). It is worth mentioning that the numerical results are very close to the analytical ones bearing in mind that $U_{p, \text{crit}}$ and $U_{p, \text{marg}}$ are not defining the same quantity; $U_{p, \text{crit}}$ is the maximum energy density that the protons acquire before a photon outburst occurs, while $U_{p, \text{marg}}$ is the minimum energy density required to bring the system to supercriticality. In all cases $U_{p, \text{crit}} > U_{p, \text{marg}}$.

The fact that the numerical values of $U^{P_{\gamma\gamma}+P_{\gamma\gamma}+P_{\pi\pi}}$ and $U^{P_{\gamma\gamma}+P_{\gamma\gamma}+P_{\pi\pi}}$ are very close to the blue line in Fig. 6, which is the critical density when all processes are taken into account, suggests that the $P_{\gamma\gamma}$ and $P_{\pi\pi}$ loops are the main networks of physical processes behind the supercritical behaviour of the system for $\gamma_{p, \text{max}} \lesssim 10^5$. Indeed, we find numerically that the addition of proton synchrotron emission or $\gamma\gamma$ pair production to the above loops can only slightly decrease the critical proton energy densities for $\gamma_{p, \text{max}} \lesssim 10^5$ (compare e.g., dashed orange and solid green lines). However, for high enough values of $\gamma_{p, \text{max}}$ (here $\gtrsim 10^8$), we find that $U_{p, \text{crit}}$ decreases significantly when both proton synchrotron and $\gamma\gamma$ absorption are taken into account (compare dashed orange and solid green lines in both panels). These results imply that there is another network of processes driving the system's supercritical behavior at high $\gamma_{p, \text{max}}$ values. This network ($\gamma Q$) relies mainly on $\gamma\gamma$ absorption and synchrotron radiation.

Inspecting both panels of Fig. 6, one can see the contribution of each of the three basic networks discussed above in the case where all processes are used (blue line). The supercriticality starts operating for values of $\gamma_{p, \text{max}} > \gamma_{p, \text{crit}}$, albeit at high values of $U_{p, \text{crit}}$. As $\gamma_{p, \text{max}}$ increases, there is a sharp drop of the required critical proton density, and at intermediate values of $\gamma_{p, \text{max}}$ it is the $P_{\gamma\gamma}$ loop that is mainly responsible for the onset of the non-linearity. Finally, at high values of $\gamma_{p, \text{max}}$ it is the $\gamma Q$ network that takes over as it requires the lowest possible critical density – in the present example the onset of this loop results in a more than one order of magnitude decrease of $U_{p, \text{crit}}$. Furthermore, it can be deduced that the aforementioned three loops cover all the possible ways of producing supercriticalities, i.e., while it might be possible, in principle, for other networks to operate, they can do so at the cost of a higher critical density.

Finally, inspection of Fig. 3 together with Fig. 6 reveals that the supercriticality is strong (i.e., faster exponential growth of photons) when only one loop operates (e.g., the $\gamma Q$ loop for high enough proton energies). When the parameters are such that two loops operate simultaneously (e.g., the $P_{\gamma\gamma}$ and $\gamma Q$ loops for intermediate values of $\gamma_{p, \text{max}}$), their superposition weakens the supercriticality – compare panels (b) and (d) in Fig. 4.

4.3.1 Synchrotron cooling

With the help of the aforementioned results one can find some explanation for the rather perplexing role of proton synchrotron cooling on the manifestation of supercriticality (see Fig. 4). Indeed, if the key physical idea behind hadronic supercriticalities is the build-up of a critical proton density in the source, then panel (d) in Fig. 4 cannot be explained, as we find supercritical behaviours even for strong synchrotron cooling (i.e., the grey coloured region extends beyond the dashed line). These results can be understood as follows. Synchrotron cooling should not play a critical role in the manifestation of supercriticality, unless a feedback loop that involves protons directly dominates ($P_{\gamma\gamma}$ or $P_{\pi\pi}$). Indeed, for low enough $\gamma_{p, \text{max}}$ values (see e.g., panel (a) in Fig. 4), it is expected that either $P_{\gamma\gamma}$ or $P_{\pi\pi}$ will be the main driver of the supercriticality (see also Fig. 6). This explains why, in this case, supercriticality is found for a wide range of parameters with $t_{\gamma\gamma} < t_{p, \text{osc}}$. On the other hand, the $\gamma Q$ loop depends only indirectly on the proton density. Even cooled protons can produce $\gamma$-rays with a compactness greater than the critical value, which is needed for the onset of the $\gamma Q$ loop. Therefore, in parameters regimes where the $\gamma Q$ loop prevails, proton synchrotron cooling does not play such an important role in determining the development of supercriticalities (e.g., panel (d) in Fig. 4). Even so, when proton synchrotron cooling becomes very fast, then the variability dies out, and the system always reaches a steady-state (see top right corner in all panels of Fig. 4).

5 EFFECTS OF OTHER PARAMETERS

In this section, we investigate the role of various other parameters and factors that were not included in our calculations so far.
3.1 Primary electrons

The calculations presented in the previous sections were based upon the assumption that only relativistic protons were injected in the source. Despite the fact that this assumption helped us define the hadronic supercriticality (Section 3) and identify the driving physical networks (Section 5), it is not a realistic one because most acceleration mechanisms that could operate in astrophysical sources can, in principle, accelerate both protons and electrons to high energies. Therefore, in this section, we add an injection term in the electron kinetic equation (see equation 5), with the objective of studying their effects on hadronic supercriticalities. In order to keep the involved free parameters at a minimum, we assume that the electrons are also accelerated into a power law having the same index as protons. We also assume $\gamma_{e,\text{min}} = \gamma_{p,\text{min}} = 1$ and $\gamma_{e,\text{max}} = \gamma_{p,\text{max}}$ – our results turn out to be rather insensitive to these assumptions. Yet, there is one important parameter, namely the electron injection compactness, $\ell_e$, which is defined analogously to the proton compactness with $m_e$ replacing $m_p$ in equation 2, and is a dimensionless measure of the injected electron luminosity.

Fig. 7 shows the $\ell_e - \ell_p$ parameter space of hadronic supercriticalities for one choice of $R, B$, and $\gamma_{p,\text{max}}$ values, and note that similar results are found for other sets of parameters. Supercriticality occurs for parameter combinations drawn from the grey-coloured region. Particularly, the dark-coloured region corresponds to solutions with limit-cycle behaviour, while the colour gradient indicates the progressive transition from violent relaxations to high compactness steady-state solutions. A few more things about Fig. 7 are worth mentioning.

- Low values of $\ell_e$ have absolutely no effect on the manifestation of the supercriticality, as expected.
- High values of $\ell_e$ bring the system to a steady state by weakening the limit-cycle behaviour. Primary electrons also contribute to the production of photons via synchrotron and Compton processes. If $\ell_e$ is high enough, then the resulting photon number densities will also be high, thus causing proton cooling through photohadronic processes, and leading the system to a steady state. In this particular example, the transition to a steady-state occurs for values of $\ell_e$ corresponding to electron luminosities $\gtrsim 0.01 L_p$. Moreover, the transition from a limit-cycle behaviour to a steady state is also a function of $\gamma_{p,\text{max}}$, with lower values increasing the required value of $\ell_e$ (not explicitly shown in the figure).

It is also noteworthy that the injection of primary relativistic electrons does not decrease the required value of $U_{p,\text{crit}}$. The main reason behind this is that primary electrons do not contribute to the networks leading to supercriticality, as this is an intrinsic property of a purely hadronic system. Finally, similar results are found if one considers the presence of external radiation fields (i.e., not produced by the relativistic particles in the source). In this case, external photons would serve as an extra cooling agent for the relativistic protons, and their presence at high densities would stabilize the hadronic system.

5.2 Power-law index of the proton energy spectrum

Up until this section, all results have been obtained for relativistic protons with a power-law of index $s = 2$. This is a standard value obtained, for example, from first order Fermi acceleration at strong non-relativistic shocks. However, the spectral index, which depends on the acceleration process and is free parameter for the purposes of this study, can affect the onset of hadronic supercriticality.

Figure 7. The parameter space $\log \ell_e - \log \ell_p$ of hadronic supercriticalities for fixed $R = 10^{12} \text{ cm}$, $B = 10^{3.5} \text{ G}$ and $\gamma_{p,\text{max}} = 10^{6.5}$. The dark-coloured region corresponds to limit-cycle solutions. The colour gradient indicates the progressive transition from violent relaxations to a high compactness steady-state solution. No critical proton compactness can be defined in the white-coloured region. The dashed lines correspond to an electron-to-proton luminosity ratio of 0.1% and 1%.

Here, we examine the dependence of supercriticality on the slope $s$ of the power-law proton energy distribution. Our results are shown in Fig. 8 where the ratio of the critical proton energy density to the magnetic field energy density, $U_{p,\text{crit}}/U_B$, is given as a function of the spectral index for $\gamma_{p,\text{max}} = 10^{6.5}$ and three pairs of $R, B$ values (see figure’s inset legend). In all cases, the dependence of $U_{p,\text{crit}}/U_B$ on $s$ is similar: For steep spectra (i.e., $s > 2$) the ratio $U_{p,\text{crit}}/U_B$ is high, but then decreases as the proton energy spectrum flattens, reaching a broad minimum around $s \approx 1.5$. For even flatter spectra ($s \lesssim 1.5$), the critical proton density becomes almost independent to the slope.

These findings can be qualitatively understood, if one recalls that only protons above a certain energy (i.e., $\gamma_p > \gamma_{p,\text{crit}}$) are relevant for the onset of supercriticality through the relevant energy threshold conditions (see Section 4 and Fig. 6). In other words,
\( \gamma_{p,\text{crit}} \) is effectively the minimum Lorentz factor for computing the critical energy density of the proton distribution. As in our calculations we have assumed that \( \gamma_{p,\text{min}} = 1 \), steep spectral indices mean that a large fraction of protons have \( \gamma_p < \gamma_{p,\text{crit}} \), thus contributing to the total energy density but without being relevant for supercriticality. Thus, a higher total proton energy density is needed to compensate for the large number of “inactive” protons. For flat indices (i.e., \( s \lesssim 1.5 \)), we find a very weak dependence of \( U_{p,\text{crit}}/U_B \) on \( s \), as the proton critical energy density is now mainly determined by the maximum Lorentz factor of the distribution, which is kept fixed.

### 5.3 Proton escape timescale

The proton escape timescale \( t_{p,\text{esc}} \) is another parameter of the problem that was kept fixed \( (t_{p,\text{esc}} = 10^3 t_e) \) for the purposes of our previous analysis. It is also a parameter which controls \( t_{p,\text{crit}} \), i.e., the value of the proton injection compactness parameter which is required to bring the hadronic system into supercriticality, through the solution of the equation \( t_{\text{app}}(t_{p,\text{esc}}) = t_{p,\text{esc}} \), where \( t_{\text{app}} \) is the appearance time of the first peak. Note, however, that the required critical proton density is independent of the choice of \( t_{p,\text{esc}} \) as, in the absence of strong losses, one can write \( n_{p,\text{crit}} \propto t_{p,\text{crit}} t_{p,\text{esc}} \).

Clearly, when \( t_{p,\text{esc}} \) is small, implying a fast escape timescale, one would have a high value of \( t_{p,\text{esc}} \) in order to meet the marginal stability criterion and vice versa.

Fig. 9 depicts the behaviour of \( t_{\text{app}} \) as a function of \( t_{p,\text{esc}} \) for three different values of \( t_p \). The top curve corresponds to \( t_{p,\text{crit}} \), and the cross denotes the point \( t_{\text{app}} = t_{p,\text{esc}} = 10^3 t_e \). By increasing \( t_{p,\text{esc}} \) we get an asymptotic behaviour for \( t_{\text{app}} \) which remains close to \( 10^3 t_e \) and justifies the used criterion for entrance to supercriticality. However, for given \( t_p \), there exists a minimum value of \( t_{p,\text{esc}} \) below which the marginal stability criterion cannot be met and the system becomes subcritical (shaded region in the figure).

We find the same overall behaviour if we are to repeat the procedure for different values of \( t_p \). For \( t_p > t_{p,\text{crit}} \), as are the two cases shown in Fig. 9 (dashed orange and solid blue lines), the asymptotic value of \( t_{\text{app}} \) decreases, and the same happens for the minimum value of \( t_{p,\text{esc}} \) required to bring the system into supercriticality.

### 6 ASTROPHYSICAL IMPLICATIONS

In this section, we discuss hadronic supercriticalities in the context of compact astrophysical sources. For sources that invoke relativistic motions of the emitting region (e.g., GRBs and AGN jets), all quantities, such critical energy densities and length scales, refer to their co-moving frame values (unless stated otherwise). The application of our results to compact astrophysical sources is inviting, as hadronic supercriticalities:

- produce spontaneous rapid bursts which last a few crossing times of the source. Therefore, they can in principle be applied to both transient astrophysical systems, like GRBs (Petropoulou & Mastichiadis 2018), and persistent yet flaring sources, like AGN cores and jets.
- appear on length scales relevant to GRBs \((\sim 10^{13} - 10^{14} \text{ cm})\) and jetted AGN \((\sim 10^{15} - 10^{17} \text{ cm})\). As the onset of supercriticalities depends on the source compactness, we find that supercriticality weakens for larger radii (i.e., the growth rate of instability decreases) and eventually disappears; for instance, we do not find any signs of supercriticality at sizes of the radio lobes of AGN \((\gtrsim 10^{20} \text{ cm})\).
- appear if \( \gamma_{p,\text{max}} > 10^3 \). Such proton energies can be accommodated by most of the proposed models for particle acceleration and they could produce neutrinos with energies as low as a few hundreds of GeV.

We provide next some energetic estimates of supercritical bursts as related to the size and magnetic field of the source. For this purpose, we define a quantity of supercriticality which is potentially relevant for astrophysical applications, i.e., the critical proton energy, \( E_{\text{crit}} \). This is the amount of energy in relativistic protons that is required for a source to become supercritical, and it can be
power in the $B - R$ plane for $\gamma_{p,\text{max}} = 10^5$ and two values of the Lorentz factor relevant to GRBs (panel a) and AGN jets (panel b), i.e., $\Gamma = 200$ and 20, respectively. As a rule of thumb, higher values of $P_{\text{jet}}$ correspond to sources of larger radii. Despite the higher $\Gamma$ value of GRBs, the critical jet power is lower than the one obtained for AGN jets due to the smaller co-moving source radii related to GRBs. Although the predicted highly super-Eddington jet powers disfavour the applicability of hadronic supercriticalities to the steady AGN emission, they can still be of relevance for flaring episodes. We plan to explore this possibility in a future publication.

Fig. 12 shows the luminosity $L_{\gamma}$ and the corresponding photon compactness $\ell_{\gamma}$ (inset) at the peak of the outburst from the runs that produced Fig. 10 for two combinations of $B$ and $R$ that bring the system into the supercritical regime: $B^2R = 10^{18} \, \text{G}^2\text{cm}$ (solid blue line) and $B^2R = 10^{19} \, \text{G}^2\text{cm}$ (dashed orange line). For small $R$ the photon compactness at peak is quite high – see inset plot, which means that one expects strong $\gamma - \gamma$ attenuation. However, this condition relaxes as the size of the source increases. This implies that the resulting photon spectrum produced during a supercritical outburst will show stronger $\gamma - \gamma$ emission for sources with larger size. The curves in the main plot of Fig. 12 show the maximum luminosity of a supercritical flare as a function of the source radius $R$. For even higher values of $B$ the system will reach a steady state and all flaring activity will cease. We note, finally, that this luminosity assumes that the source is at rest. If the source is moving relativistically with respect to the observer, one has to transform the luminosities to the observer’s frame, by using the appropriate Doppler boosting.

Finally, Table 2 shows typical values of the isotropic luminosity $L_{\gamma, \text{obs}}^{0}$ calculated at the peak of the first outburst (in the observer’s frame) for a wide range of $R, B$ values that are relevant for GRBs, and AGN (cores and jets). The table also lists the typical variability timescale, $\tau_{\gamma, \text{obs}}$, which is the respective light-crossing time (in the observer’s frame). To compute $L_{\gamma, \text{obs}}^{0}$ and $\tau_{\gamma, \text{obs}}$, the choice of $B^2R = \text{constant}$ implies that the proton cooling timescale remains constant for the runs that obey this relation.
for AGN cores, we predict photon flares with peak luminosities $L_{\gamma,\text{obs}}$ of $5 \times 10^{50} - 10^{53}$ erg s$^{-1}$ for AGN cores.

Even though we did not make any attempt to simulate specific sources, the predicted luminosities and variability timescales are of the same order of magnitude as those observed in GRBs (for a recent review, see Zhang 2018). Similarly, for $R$, $B$ values relevant to AGN cores, we predict proton flares with peak luminosities $\sim 10^{43} - 10^{44}$ erg s$^{-1}$ with hour-long to day-long variability timescales. Interestingly, TeV emission with variability on day-long timescales and luminosities $O(10^{41})$ erg s$^{-1}$ has been detected from the radio galaxy M87 (Aharonian et al. 2006; Aleksić et al. 2012), thus pointing to a compact $\gamma$–ray source. Although we have not discussed the high-energy photon spectrum in this work, supercritical hadronic flares in AGN cores appear a promising topic for future study. Lastly, for parameters related to AGN jets, the predicted peak luminosities are much higher than typically observed for TeV blazars, even if a lower $\Gamma$ value is adopted. However, these estimates match the values reported for the most luminous $\gamma$–ray flares from flat spectrum radio quasars (e.g., Našer 2013). Still, the properties of the resulting $\gamma$–ray spectrum need further investigation.

### Table 2. Typical variability timescale and isotropic luminosity at the peak of the outburst caused by supercriticality (in the observer’s frame) for $B$ and $R$ ranges relevant for GRBs, AGN cores and AGN jets, with $\Gamma$ equal to 200, 1 and 20, respectively. In all cases, $\gamma_{p,\text{max}} = 10^3$.

| Source       | $B$ (G)    | $R$ (cm)  | $t_{\gamma,\text{obs}}$ (s) | $L_{\gamma,\text{obs}}^{\text{iso}}$ (erg s$^{-1}$) |
|-------------|-----------|-----------|-----------------------------|---------------------------------|
| GRBs        | $10^2 - 10^4$ | $10^{11} - 10^{13}$ | $0.018 - 1.8$ | $5 \times 10^{50} - 10^{53}$ |
| AGN cores   | $10^2 - 10^3$ | $3 \times 10^{13} - 3 \times 10^{14}$ | $10^3 - 10^4$ | $2 \times 10^{43} - 3 \times 10^{44}$ |
| AGN jets    | $1 - 10^2$  | $10^{15} - 10^{16}$ | $1.6 \times 10^3 - 1.6 \times 10^4$ | $5 \times 10^{49} - 10^{51}$ |

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### 7 SUMMARY AND DISCUSSION

Superluminalities are an intriguing property of hadronic relativistic plasmas. They are, essentially, radiative instabilities that appear when large amounts of energy are stored in slowly cooling protons. The inability of protons to efficiently radiate their energy away can cause their energy density to exceed some critical value and initiate a runaway process which results in the explosive transfer of the proton-stored energy into electron-positron pairs and radiation. The idea has some analogies to supercritical nuclear piles in the sense that there are some clearly identifiable networks of processes (loops) that operate leading to runaway – however, in our case it is the photons, rather than the neutrons, which are the agent. Despite the fact that the concept of hadronic superluminalities was put forward almost three decades ago (Stern & Svensson 1991, Stern et al. 1992, KM92), there has not been as of yet a comprehensive study neither of the parameter space that leads to such situations, nor of the behaviour of the source during the onset of the non-linear outgrowth. In this work, we have addressed both questions with more emphasis on the former, thus providing a complete roadmap to hadronic superluminalities.

The runaway that ensues when protons enter the superluminal regime produces, in most cases, a flare in photons, and the obtained light curves are distinctively different from the subcritical ones leading to steady-state – see Fig. 1. The subsequent behaviour of the system in superluminality depends on various factors. If the protons continue to be injected in the system at the same rate – an assumption that we made in our analysis – then the system can exhibit either a (quasi-periodic) limit cycle behaviour, or a violent relaxation ending up to a steady state. The limit cycle behaviour can be explained on physical grounds from the fact that superluminality is caused when protons of a certain energy reach a certain critical density – see KM92. The exponentiation of photons causes strong proton losses through photodisproportion interactions. Therefore, protons are pushed back to the subcritical regime and the runaway dies out. If, however, the injection of fresh protons continues, then once again protons will accumulate, and eventually they will enter the superluminal regime causing a new photon runaway and the cycle repeats. This is the description of a classic “prey-predator” system, as it was first shown in PM12 (see also Appendix A).

In order to set the framework that would enable us to examine the superluminality, we have used the premises of the one-zone model that has been widely used both for leptonic and hadronic modelling of high-energy sources. Protons were assumed to be injected with a constant rate inside a spherical source of radius $R$ and magnetic field $B$ while, at the same time, they were allowed to cool by synchrotron radiation and photodisproportion processes and to physically escape from the source. The evolution of the system was followed by means of a numerical code that solves the continuity equations for protons, electrons, and photons. Assuming a generic power-law energy spectrum for the injected protons and using the high-energy cutoff $\gamma_{p,\text{max}}$ and the injection compactness $\ell_p$ (equation 2) as the main parameters, we performed runs for various combinations of $R$, $B$, and $\gamma_{p,\text{max}}$. Having as guidelines previous analytical works (KM92, PM12) and parameters of astrophysical interest, we chose characteristic values of $R$ ranging from $10^{12}$ cm to $10^{16}$ cm, that are relevant to GRBs and AGN cores and jets, and let the magnetic field strength vary from 1 G to 100 kG. We also considered a very wide range of $\gamma_{p,\text{max}}$ values extending from $10^3$ to $10^9$ (see Table 1).

We demonstrated that (for most parameters) there exists a critical value of the injected compactness, $\ell_p,\text{crit}$, that drives the system into a state characterised by rapid photon outgrowth, even if initially the source did not contain any photons – see e.g., Fig. 3. We were able to determine unequivocally the value of $\ell_p,\text{crit}$, because, in the vast majority of the cases, the system enters very abruptly in the supercritical regime, i.e., for $\ell_p = \ell_p,\text{crit} - \delta \ell_p$ the system is still stable and it suffices a small increment of $\delta \ell_p \ll \ell_p$ to push it into the supercritical regime, as illustrated in Fig. 1. The phase space of hadronic superluminalities and the temporal behavior of the system in the non-linear phase of the instability outgrowth are summarized in Fig. 4.

Although hadronic superluminalities emerge for a wide range of source parameter combinations, we showed that all cases require a particle dominated source with $U_{p,\text{crit}} \gg U_B$. The ratio, however, decreases with increasing $B$, and can be as low as $10^2 - 10^3$ for strong magnetic fields (i.e., $B \sim 10^3 - 10^5$ G) – see Fig. 5. Lower values of the energy density ratio can also be achieved by considering harder injection proton spectra, with slopes $s < 2$ – see Fig. 8. We have discussed the properties of hadronic superluminalities under the assumption of continuous injection of relativistic...
tic protons in the source, and showed that whenever their energy density (which is a function of time) exceeds $U_{\text{pr, crit}}$, the system becomes supercritical. Thus, $U_{\text{pr}} / U_B \gtrsim U_{\text{pr, crit}} / U_B \gg 1$ has to be satisfied only for a short duration of time, of the order of a few crossing times. Thus, the necessary departure from equipartition in the source has to be viewed in light of its transient nature.

Supercriticalities are an intrinsic property of hadronic systems. Although purely leptonic systems (i.e., sources with an energetically sub-dominant or even absent population of relativistic protons) can exhibit non-linear behaviour because of synchrotron self-Compton processes, they do not show any features of supercriticality due to the fast electron cooling (e.g., Petropoulou et al. 2015b). Still, the presence of a population of accelerated electrons in a hadronic system does not change the key features of hadronic supercriticalities presented in this work, as long as their luminosity is lower than $0.01L_p$ – see Fig. 7. Above this value, the radiation produced by electrons quenches the relevant hadronic supercriticalities. Generally, we can say that strong photon fields either external or internal to the source, e.g., from primary electrons, can lead the system quickly to a steady state as the supercriticalities under study depend on the internally produced radiation. Any other field, as the ones produced from primary electrons, acts competitively and tends to stabilize the system.

We also compared our results with those presented in the analytical studies of KM92 and PM12, and found good agreement. Particularly, we showed that the network of processes examined by KM92 and PM12 are the most important and can explain, in a very satisfactory manner, the behaviour of the system as a function of $\gamma_{\text{p, max}}$ – see Figs. 5 and 6. In short, photohadronic interactions and electron synchrotron radiation are responsible for the loops at low to mid values of $\gamma_{\text{p, max}}$ (KM92), while at larger values photoquenching and the ensuing electromagnetic cascades that cool explosively the relativistic protons take over (PM12). Other loops of processes (e.g., those with inverse Compton scattering replacing (electron) synchrotron radiation) could also operate, but they do so at the cost of a higher critical proton density.

In this paper, we explored hadronic supercriticalities by injecting pre-accelerated protons in a fixed volume, and then let particles evolve under the presence of energy losses alone. Alternatively, these radiative instabilities can be studied in the case where proton acceleration and radiation take place in the same region. In this scenario, protons should be accelerated to high enough energies and reach high enough densities as to satisfy both the required feedback and critical stability criteria of hadronic supercriticalities – see Section 4. It is possible that the system will exhibit different temporal behaviour than the one studied here, especially when the saturation proton Lorentz factor (i.e., where the energy gains from acceleration are balanced by the energy losses from radiation and inelastic collisions) is close to the critical proton Lorentz factor of $\gamma_\text{S}$ loop.

We have studied the properties of hadronic supercriticalities under the assumption of particle injection in source with a fixed volume. It is, however, possible that the source itself is expanding. In this case, the build-up of a critical proton density in the source may be inhibited by its increasing volume and/or the ensuing adiabatic energy losses. Therefore, the question that naturally arises is whether supercriticalities can even exist in such a scenario, and if yes, in what way they are manifested. Some preliminary results which show that supercriticalities do exist even in expanding sources are presented in Florou & Mastichiadis (2019).

8 CONCLUSIONS

In this work, we provide a roadmap to hadronic supercriticalities, i.e., radiative instabilities that can develop in hadronic relativistic magnetized plasmas. By performing a numerical investigation over a wide range of source parameters and maximum proton energies, we have determined the conditions that drive a hadronic system to supercriticality, and systematized its rich temporal behaviour during the non-linear phase of the instability growth. By isolating certain physical processes, we have identified specific networks (originally predicted by analytical calculations) that are responsible for the onset and sustainability of these instabilities. Even though hadronic supercriticalities can emerge for a wide range of astrophysically relevant parameters, a quantitative application of these ideas to high-energy compact sources, such as GRBs, AGN jets and cores, remains to be tested.

ACKNOWLEDGEMENTS

IF and SB acknowledge that this research is co-financed by Greece and the European Union (European Social Fund-ESF) through the Operational Programme “Human Resources Development, Education and Lifelong Learning” in the context of the project “Strengthening Human Resources Research Potential via Doctorate Research” (MIS-5000432), implemented by the State Scholarships Foundation (IKY). M.P. acknowledges support from the Lyman Jr. Spitzer Postdoctoral Fellowship. Correspondence regarding this work should be sent to A. Mastichiadis (amastich@phys.uoa.gr) or M. Petropoulou (m.petropoulou@astro.princeton.edu).

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APPENDIX A: A TOY MODEL OF HADRONIC SUPERCriticalITIES

In this Appendix, we show that the behaviour of the hadronic system, as described by equation 5 and depicted in Figs. 1 and 3, can be explained in simple terms with the help of a toy model.

Following KM92, we assume that a source contains monoenergetic protons of Lorentz factor $\gamma_p$ which, when unstable, produce through the $pS$ or $pP$ loops (see Section 4) electron-positron pairs of Lorentz factor $\gamma_e$. These pairs (for simplicity, we will refer to them as electrons) radiate monochromatic synchrotron photons with energy $x_e = b\gamma_e^3$ (in units of $m_e c^2$), where $b = B/B_{crit}$. However, at high photon luminosities, electrons can lose energy through inverse Compton scattering, producing photons at energies $x_e = \gamma_e x_e = b\gamma_e^3$, and this acts competitively to the synchrotron losses.

Based on the above, we consider the following four particle distributions inside the source, namely the protons having a number density $n_p$, the hard respectively production of soft photons produced from proton synchrotron radiation. However, their number density is assumed to be negligible, $n_s >> n_p$.

\[
\frac{d\dot{n}_p}{d\tau_{p,esc}} = Q_p + \frac{n_p}{\tau_{p,esc}} + \frac{n_p}{\tau_{p,syn}}
\]

where $Q_p$ is the injection rate and $\tau_{p,esc}$, $\tau_{p,\gamma}$, and $\tau_{p,syn}$ are the proton escape, the photohadronic and the synchrotron loss timescale, respectively (in units of $R/c$). All number densities are normalised to $\sigma_T R$. This relation assumes that only soft photons contribute to proton losses, an assumption partly justified from the fact that $n_s \gg n_h$.

The synchrotron loss timescale can be written under these normalisations as

\[
\tau_{p,syn} = \left(\frac{4}{3} \frac{m_n}{m_p} \ell_B n_p \gamma_p \right)^{-1}
\]

where $\ell_B$ is defined by relation 3.

In analogy to equation A1, we can write a continuity equation for secondary electrons produced in the source

\[
\dot{n}_e + \frac{n_e}{\tau_{e,esc}} = Q_e + \frac{n_e}{\tau_{e,syn}} + \frac{n_e}{\tau_{e,ics}}
\]

where $\tau_{e,esc}$ is the electron timescale, $\tau_{e,syn}$ is the synchrotron loss timescale at (dimensionless) energy $\gamma_e$ given by

\[
\tau_{e,syn} = \frac{1}{3} \ell_B n_e \gamma_e^{-1}
\]

and $\tau_{e,ics}$ is the corresponding inverse Compton loss timescale (in the Thomson regime) given by

\[
\tau_{e,ics} = \frac{2}{3} \frac{\gamma_e^2}{n_e \ell_B n_e \gamma_e^{-1}}
\]

In our toy model, electrons (and positrons) are produced via photodisruptive processes with a typical Lorentz factor $\gamma_e$. Thus, the electron injection term $Q_e$ takes the form

\[
Q_e = \alpha \frac{\gamma_e m_e}{n_p(n_p + n_h)} H(n_p - n_{p,crit})
\]

where $H(x)$ is the Heaviside function. This term simulates the fact that secondary injection starts when the proton density exceeds the critical value $n_{p,crit}$. Note that the electron injection term is written so that it conserves energy, i.e., the energy lost by protons is injected as energy in the secondaries in the electron equation.

Finally, we complete the system by writing two equations for the two photon populations (soft and hard), assuming that they do not interact through $\gamma^2\gamma$ absorption. These equations describe the energy channeled into soft and hard photons by electron synchrotron and inverse Compton scattering, respectively. The soft-photon equation reads

\[
\dot{n}_s + n_s = \frac{4}{3} \ell_B b^{-1} n_e
\]

and the equation for hard photons is

\[
\dot{n}_h + n_h = \frac{4}{3} x_s n_e n_s
\]

Note that, as written, the system of equations A1, A4, A8, and A9, implies that only the soft synchrotron photons $n_s$ produce hard photons through inverse Compton scattering.

Assuming that $n_p = n_e = n_h = 0$ initially, we can distinguish the following cases in the regime of weak magnetic fields and negligible proton synchrotron losses:

- If $Q_p$ is such as $n_p(\tau) < n_{p,crit}$ at all times, the system never becomes supercritical. Instead, it reaches a steady state characterized by $n_p = Q_p / \tau_{p,esc}$. In this case, no secondary particles are produced, i.e., $n_e = n_h = n_h = 0$ at all times. This regime corresponds to the low-efficiency steady states found numerically in our study see e.g., Figs. 1 and 3; the only difference with the toy model’s prediction is that the photon number density is not equal to zero, but are produced by the very inefficient proton synchrotron process (which is neglected in our toy model).
• If $Q_p$ is such as $n_p(\tau_e) > n_{p, crit}$, the system becomes supercritical and secondary particles start being produced. Electrons are injected through photohadronic processes (equation A4) and produce soft photons through synchrotron or Compton (equation A8) which, in turn, cool the protons via photohadronic processes (equation A1). We can further distinguish the following cases:

- If ICS losses are ignored (or, equivalently, if $\tau_{e, syn} \ll \tau_{e, i ca}$, the system of equations is equivalent to the predator-prey equations (or Lotka-Volterra equations), and yields limit-cycle solutions, as shown in Fig. 1.

- If the parameters are such that $\ell_q = x_p n_a \gg \ell_B$ during some phase of the system’s evolution, then Compton cooling becomes faster than synchrotron cooling (i.e., $\tau_{e, i ca} < \tau_{e, syn}$), and electron energy is radiated mostly through Compton channel, producing hard photons. Thus, it does not contribute to the proton feedback loop, but acts instead as a damping factor – see also PM12.

By numerically solving the toy model’s set of equations, one can gain insight to the temporal behaviours depicted in Fig. 3. The low-efficiency steady states found for parameters from the lower left corner of the phase diagram correspond to cases with $n_a < n_{p, crit}$ at all times. When $\ell_B$ is small and the system is in the supercritical regime (i.e., see right-hand side of the plot), then during the first outburst $\ell_q \gg \ell_B$. This drives the system to a high-efficiency steady state through a violent relaxation because of the strong Compton damping. As $\ell_B$ increases, electron synchrotron balances inverse Compton, and the system gets into the limit cycle regime with negligible or very low Compton damping. Finally, when $\ell_B$ increases further, the proton synchrotron term in equation A1 becomes dominant, cooling the protons and never allowing them to reach the supercritical value $n_{p, crit}$. In this case, the feedback loop will not operate.

APPENDIX B: THE $\gamma Q$ LOOP

While the $P e S$ and $P \pi S$ loops (see Section 4.1) are rather straightforward as they involve protons directly, the $\gamma Q$ loop (Section 4.2) is more subtle and requires special attention. Here, we will repeat and expand on some arguments first presented in PM12.

B1 An example of subcritical photon emission

Fig. B1 shows the temporal evolution of a typical multiwavelength (MW) spectrum obtained when a proton power-law is injected in a source of radius $R$ and magnetic field $B$. Snapshots are displayed every $50\ell_{cr}$ (thin blue lines), and are computed when all processes taken into account. The steady-state spectrum (computed when electron and proton synchrotron radiation, photopion production, and $\gamma \gamma$ pair production are included) is also shown (thick red line) for comparison. For the parameters used (i.e., $R = 10^{15}$ cm, $B = 10^{-5} - 10^{-6}$ G, $\gamma_{p, max} = 10^{6.5}$, and $\ell_p = 10^{-4}$, similar to those used in Fig. 1), the system is in the subcritical regime. The MW photon spectrum at all times consists of three distinct peaks. Before discussing the temporal evolution of the photon emission, let us take a closer look at the processes producing the three components of the photon spectrum.

The low-energy peak is produced by proton synchrotron radiation which peaks approximately at an energy (in $m_e c^2$ units)

$$x_{p, syn} \simeq \frac{m_e}{m_p} b \gamma_{p, max}^2 \tag{B1}$$

The high-energy peak is produced from the $\pi^0$ decay and peaks at

$$x_e = \eta_{\pi} \gamma_{p, max} \tag{B2}$$

where $\eta_{\pi} \simeq 350$ (Dimitrakoudis et al. 2012). Finally, the medium-energy peak is produced from the synchrotron radiation of the electron-positron secondaries from (i) $\pi^\pm$ decay and from (ii) $\gamma \gamma$ interactions between the hard photons of the $\pi^0$ peak with the soft ones from the proton synchrotron peak. It turns out that the synchrotron spectra of these two components peak at very similar energies. Taking, for simplicity, the second one we find

$$x_{\pm, syn} \simeq b \left(\frac{x_e}{2}\right)^2 \simeq \frac{1}{4} \eta_{\pi}^2 \gamma_{p, max}^2 \tag{B3}$$

which, interestingly enough, lies in the GeV-TeV band for a wide range of parameters.

A necessary condition for the above picture to occur is that proton synchrotron photons are energetic enough as to pion-produce on the protons. This gives a rather strong lower limit on $\gamma_{p, max}$

$$\gamma_{p, max} \simeq \left(\frac{m_e m_p}{m_e^2}\left(1 + \frac{m_e}{2 m_p}\right)\right)^{1/3} b^{-1/3} \tag{B4}$$

or, numerically,

$$\gamma_{p, max} \simeq 2.8 \times 10^6 B^{-1/3} \tag{B5}$$

where $B$ is in Gauss. Note that when the above condition is met, then the $\gamma$-rays from $\pi^0$ decay will also be above the threshold for pair-production on the proton synchrotron photons; therefore, the middle peak will be supplied by both channels (i) and (ii).

At early times, well before a steady state is established (lower curves), the luminosity of the second peak is low, but grows approximately quadratically with time, as it is supplied by an increasing rate of secondaries. The fact that the steady-state spectrum, computed for a subset of processes (electron and proton synchrotron radiation, photopion production, and $\gamma \gamma$ pair production), is very similar to the steady state solution obtained when all processes are taken into account, suggests that only a few process determine the
photon emission. For example, the position of the peaks in the MW spectrum appear identical and it is only the Bethe-Heitler production that produces extra radiation that fills partly the region between the proton synchrotron and secondary pairs peak.

Finally, the shaded region in Fig. B1 (see equation 34 of Petropoulou & Mastichiadis 2011 or equation 9 in this paper) indicates a “forbidden region” for the photon emission, in the sense that $\gamma$-rays cannot exist there in steady state. Whenever the compactness of $\gamma$-ray photons falls within this region, $\gamma$-rays are automatically quenched (Stawarz & Kirk 2007; Petropoulou & Mastichiadis 2011), i.e., the $\gamma$-rays annihilate on the synchrotron radiation of spontaneously created electron-positron pairs. For the parameters used in this example, the steady state spectrum lies well below this region. In the next section, we discuss the photon emission in the supercritical regime, by increasing $\ell_{\nu}$ and forcing the $\gamma$-ray photons to enter this forbidden area.

### B2 Supercritical photon emission and the $\gamma Q$ loop

Fig. B2 shows a zoom-in to the top light curve depicted in Fig. 1 around the first peak for $\ell_{p} = 10^{-4} \pm 1.25$ (panel a). It also plots the MW spectra before the flare (panel b), at the peak of the flare (panel c), and at the decay phase of the flare (panel d). The spectra shown in panel (a) have not yet reached the quenching regime (shaded region) and are similar to those shown in Fig. B1. However, once the $\gamma$-rays enter this regime, the $\gamma Q$ loop (first described in PM12) starts operating. The excess $\gamma$-rays turn spontaneously into electron-positron pairs which emit synchrotron photons. These synchrotron photons, on the one hand, contribute to the attenuation of $\gamma$-rays but, more importantly, cool the protons via photopion production, and as a consequence more pions are produced, which in turn results in the production of more secondary $\gamma$-rays. Therefore, a strong feedback loop, which starts as soon as the $\gamma$-rays enter the quenching regime, is developed that explosively removes part of the energy stored in protons. In panel (c), the MW spectrum is dominated by the leptonic secondaries and also by strong $\gamma \gamma$ attenuation due to the increasing photon compactness. Therefore, at the peak of the flare, the photon spectrum peaks at rather low $\gamma$-ray energies. The photon outburst continues either until the high-energy protons cool or the $\gamma$-ray emission moves out of the quenching zone. Once one of the above occurs, the feedback stops operating and the non-linearly produced photons escape freely from the system (panel d). However, since we have assumed a constant injection of protons, the cycle will start all over again producing, eventually, a limit cycle behaviour. Note in panel (d) that, at the end of the cycle, the photon compactness has dropped below the quenching regime.

Finally, one can approximately derive the necessary conditions for the loop described above to operate. The first condition requires that the energy of photons in the pair synchrotron peak is above the threshold for automatic $\gamma$-quenching which is (Stawarz & Kirk 2007; Petropoulou & Mastichiadis 2011)

$$x_{\pm, \text{syn}} \geq \left( \frac{8}{b} \right)^{1/3}. \quad (B6)$$

Using equation B3 together with the previous relation one finds a lower limit for the proton maximum Lorentz factor

$$\gamma_{p, \text{max}} \geq \frac{2^{3/2}}{\eta_{e0}} b^{-2/3} \quad (B7)$$

or numerically

$$\gamma_{p, \text{max}} \geq 5.6 \times 10^6 B^{-2/3}. \quad (B8)$$

The second condition is that the non-linear photons which are created by the spontaneous quenching of the $\gamma$-rays (equation B3) are energetic enough as to pion produce on the protons. Given that the non-linear photons are created at energies

$$x_{\pm} = b \gamma_{p} = \frac{b}{4} x_{\pm, \text{syn}}, \quad (B9)$$

this condition yields

$$\gamma_{p, \text{max}} \geq \left( \frac{64}{\eta_{e0}^{1/2} m_{e}} m_{p} \left( 1 + \frac{m_{e}}{2 m_{p}} \right) \right)^{1/5} b^{-3/5} \quad (B10)$$

or numerically

$$\gamma_{p, \text{max}} \geq 9.6 \times 10^6 B^{-3/5}. \quad (B11)$$

Therefore, for $B = 10^{1.5}$ G (see also Fig. 6), we expect that the $\gamma Q$ loop should start operating at $\gamma_{p, \text{max}} \simeq 10^8$, which is indeed the case.
Figure B2. Zoom-in at the first peak of the supercritical light curve with $\log \ell_{p} = -4.125$ shown in Fig. 1. The times marked with vertical dashed lines denote the areas where the multi-wavelength spectrum approaches the quenching regime (panel b), the flaring state (panel c), and the decay phase (panel d). The various curves in panels (b) to (d) show snapshots from the following time intervals: $(0 - 939)t_{cr}$ (panel b), $(939 - 1021)t_{cr}$ (panel c), and $(1021 - 1070)t_{cr}$ (panel d). Time intervals between snapshots are not the same between panels.