Abstract

Let $G$ be countable group and $M$ be a proper cocompact even-dimensional $G$-manifold with orbifold quotient $\tilde{M}$. Let $D$ be a $G$-invariant Dirac operator on $M$. It induces an equivariant $K$-homology class $[D] \in K^G_0(M)$ and an orbifold Dirac operator $\tilde{D}$ on $\tilde{M}$. Composing the assembly map $K^G_0(M) \to K_0(C^*(G))$ with the homomorphism $K_0(C^*(G)) \to \mathbb{Z}$ given by the representation $C^*(G) \to \mathbb{C}$ of the maximal group $C^*$-algebra induced from the trivial representation of $G$ we define $\text{index}([D]) \in \mathbb{Z}$. In the second section of the paper we show that $\text{index}(\tilde{D}) = \text{index}([D])$ and obtain explicit formulas for this integer. In the third section we review the decomposition of $K^G_0(M)$ in terms of the contributions of fixed point sets of finite cyclic subgroups of $G$ obtained by W. Lück. In particular, the class $[D]$ decomposes in this way. In the last section we derive an explicit formula for the contribution to $[D]$ associated to a finite cyclic subgroup of $G$.
1 Introduction

Let $G$ be a countable group and $M$ be a proper cocompact even-dimensional $G$-manifold with orbifold quotient $\overline{M}$. In the literature, orbifolds which can be represented as a global quotient of a smooth manifold by a proper action of a discrete group are often called good orbifolds.

Let $D$ be a $G$-invariant Dirac operator on $M$ acting on sections of a $G$-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded Dirac bundle $F \to M$. It induces an equivariant $K$-homology class $[D] \in K^G_0(M)$ and an orbifold Dirac operator $\overline{D}$ on $\overline{M}$ with index $\text{index}(\overline{D}) \in \mathbb{Z}$. In the following we briefly describe these objects.

We can identify $\overline{D}$ with the restriction of $D$ to the subspace of $G$-invariant sections $C^\infty(M, F)^G$. The operator $\overline{D}$ is an example of an elliptic operator on an orbifold. Index theory for elliptic operators on orbifolds has been started with [Kaw81] (see also [Kaw79], [Kaw78] for special cases, and [Far92b], [Far92c], [Far92a] for alternative approaches). In particular, we have $\dim \ker(\overline{D}) < \infty$, and we can define

$$\text{index}(\overline{D}) := \dim \ker(\overline{D}^+) - \dim \ker(\overline{D}^-).$$

In the present paper we use the analytic definition of equivariant $K$-homology using equivariant $KK$-theory

$$K^G(M) := KK^G(C_0(M), \mathbb{C}).$$

The class $[D] \in KK^G(C_0(M), \mathbb{C})$ is represented by the Kasparov module $(\mathcal{E}, \mathcal{F})$ with $\mathcal{E} := L^2(M, F)$ and $\mathcal{F} := D(D^2 + 1)^{-1/2}$ (see Subsection 2.1 for more details).

Let $C^*(G)$ denote the unreduced group $C^*$-algebra of $G$. In general, the theory of the present paper would not work with the reduced group $C^*$-algebra $C^*_r(G)$. The key point is that finite-dimensional unitary representations of $G$ extend to representations of $C^*(G)$, but not to $C^*_r(G)$ in general.

We now consider the assembly map

$$\text{ass} : K^G_0(M) \to K_0(C^*(G)).$$

We use an analytic description of the assembly map which is part of Definition 2.1, and we refer to [MN06], [DL98] and [BM04] for modern treatements of assembly maps in general.
Composing the assembly map with the homomorphism $I_1 : K_0(C^*(G)) \to K_0(\mathbb{C}) \cong \mathbb{Z}$ given by the representation $1 : C^*(G) \to \mathbb{C}$ induced from the trivial representation of $G$ we define

\[ \text{index}([D]) := I_1 \circ \text{ass}([D]) \in \mathbb{Z}. \]

As a special case of the first main result Theorem 2.2 we get the equality

\[ \text{index}(\bar{D}) = \text{index}([D]). \] (1)

Theorem 2.2 deals with the slightly more general case where the trivial representation $\text{triv}$ of $G$ is replaced by an arbitrary finite-dimensional unitary representation of $G$. We think, that equation (1) was known to specialists, at least as a folklore fact.

The next result of the present paper is a nice local formula for $\text{index}([D])$. The main feature of local index theory is that one can calculate the index of a Dirac operator on a closed smooth manifold in terms of an integral of a local index form. A standard reference for local index theory is the book [BGV92]. Local index theory generalizes to Dirac operators on orbifolds. The index formulas in [Kaw81] and [Far92b] express the index of the Dirac operator on the orbifold as a sum of integrals of local index forms over the various strata. In the case of a good orbifold $G\backslash M$ the strata correspond to the fixed point manifolds $M^g$ of the elements $g \in G$. There are various ways to organize these contributions. For the purpose of the present paper we need a formula which expresses the index as a sum of contributions associated to the conjugacy classes of finite cyclic subgroups of $G$. We will state this formula in Corollary 2.4 (we refrain from giving a detailed statement here since this would require the introduction of too much of notation). In principle one could deduce the formula given in Corollary 2.4 by reorganising the previous results [Kaw81] and [Far92b]. But we found it simpler to prove the formula directly using the heat equation approach to local index theory and the local calculations from equivariant index theory [BGV92].

The proper cocompact $G$-manifold $M$ can be given the structure of a finite $G$-CW-complex. The equivariant $K$-homology of proper $G$-CW-complexes has been studied intensively in connection with the Baum-Connes conjecture. Rationally, $K^G(M)$ decomposes as a sum of contributions of conjugacy classes $(C)$ of finite cyclic subgroups $C \subset G$ (see [Lue02b] for a detailed statement). This decomposition is a consequence of a result of [Lue02b] which is finer since it only requires to invert the primes dividing the orders of the finite subgroups of $G$. We thus can write $[D]$ as a sum of contributions $[D](C)$ where $(C)$ runs over the set of conjugacy classes of finite cyclic subgroups of $G$. Our last result Theorem 4.3 is the calculation of $[D](C)$. In the proof we use the index formula Corollary 2.4 as follows. By a result of [LO01b] the equivariant $K$-theory $K^G_0(M)$ has a description in terms of finite-dimensional $G$-equivariant vector bundles $E \to M$. We first derive a cohomological index formula Theorem 4.1 for the pairing of a $K$-homology class coming
from a finite cyclic subgroup \( C \subset G \) with the class \([E] \in K^0_G(M)\). In the proof we use the relation (1).

We then observe that the pairing of \([D]\) with \([E]\) is the index of the twisted operator \([DE]\) which can be written as a sum of contributions of conjugacy classes of finite subgroups by 2.4. We obtain \([D](C')\) be a comparison of the formulas in Theorem 4.1 and Corollary 2.4 and variation of \(E\).

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2 Assembly and orbifold index

2.1 The equivariant \(K\)-homology class of an invariant Dirac operator

Let \(G\) be a countable discrete group. Let \(M\) be a smooth proper cocompact \(G\)-manifold, i.e. a \(G\)-manifold such that the stabilizer \(G_x\) is finite for all \(x \in M\), and \(G \setminus M\) is compact. We further assume that \(M\) is equipped with a complete \(G\)-invariant Riemannian metric \(g^M\) and a \(G\)-homogeneous Dirac bundle \((F, \nabla^F, \circ, (.,.)_F)\). Here \(\circ : TM \otimes F \to F\) is the Clifford multiplication, \(\nabla^F\) is a Clifford connection, \((.,.)_F\) is the hermitian scalar product, and these structures satisfy the usual compatibility conditions (see [BGV92], Ch.3) and are, in addition, \(G\)-invariant.

For simplicity we assume that \(\dim(M)\) is even and that the Dirac bundle is \(\mathbb{Z}/2\mathbb{Z}\)-graded. In fact, the odd-dimensional case can easily be reduced to the even dimensional case by taking the product with \(S^1\).

We use equivariant \(KK\)-theory in order to define equivariant \(K\)-homology. Thus let \(KK^G\) be the equivariant \(KK\)-theory introduced in [Kas88] (see also [Bla98]). Let \(C_0(M)\) be the \(G\)-\(C^*\)-algebra of continuous functions on \(M\) vanishing at infinity. Then by definition \(K^G_0(M) = KK^G(C_0(M), \mathbb{C})\). The Dirac operator \(D\) associated to the invariant Dirac bundle \(F\) induces a class \([D] \in K^G_0(M)\) as follows. We form the \(\mathbb{Z}/2\mathbb{Z}\)-graded \(G\)-Hilbert space \(\mathcal{E} := L^2(M, F)\). Then \(C_0(M)\) acts on \(\mathcal{E}\) by multiplication. Furthermore, we consider the bounded \(G\)-invariant operator \(\mathcal{F} := D(D^2 + 1)^{-1/2}\) which is defined by applying the function calculus to the unique (see [Che73]) selfadjoint extension of \(D\). Then \([D]\) is represented by the Kasparov module \((\mathcal{E}, \mathcal{F})\).

\(^1\)The factor \(\frac{1}{\text{ord}(g)}\) in (2.3) was missing.
2.2 Descent and index

Let $C^*(G)$ denote the (non-reduced) group $C^*$-algebra of $G$. It has the universal property, that any unitary representation of $G$ extends to representation of $C^*(G)$. In particular, if $\rho : G \to U(V_\rho)$ is an unitary representation of $G$ on a finite-dimensional Hilbert space $V_\rho$, then there is an extension $\rho : C^*(G) \to \text{End}(V_\rho)$. On the level of $K$-theory it induces a homomorphism (using Morita invariance and $K_0(\mathbb{C}) \cong \mathbb{Z}$) $I_\rho : K_0(C^*(G)) \to K_0(\text{End}(V_\rho)) \cong \mathbb{Z}$. In particular, if $\rho = 1$ is the trivial representation, then we also write $I := I_1$. Note that $I_\rho$ can be written as a Kasparov product $\otimes C^*(G)[\rho]$, where $[\rho] \in KK(C^*(G), \text{End}(V(\rho)))$ is represented by the Kasparov module $(V_\rho, 0)$.

Let $C^*(G, C_0(M))$ be the (non-reduced) cross product of $G$ with $C_0(M)$. Then there is the descent homomorphism $j^G : K_0^G(M) \cong KK(C_0(M), \mathbb{C}) \to KK(C^*(G, C_0(M)), C^*(G))$ introduced in [Kas88], 3.11. Following [GHT00] we choose any cut-off function $\chi \in C^\infty_c(M)$ with values in $[0, 1]$ such that $\sum_{g \in G} g^*\chi^2 \equiv 1$. Then we define the projection $P \in C^*(G, C_0(M))$ by $P(g) = (g^{-1})^*\chi$. Let $[P] \in K_0(C^*(G, C_0(M)) \cong KK(\mathbb{C}, C^*(G, C_0(M)))$ be the class induced by $P$, which is independent of the choice of $\chi$.

Definition 2.1 We define $\text{index}_\rho : K_0^G(M) \to \mathbb{Z}$ to be the composition

$$K_0^G(M) \xrightarrow{j^G} KK(C^*(G, C_0(M)), C^*(G)) \xrightarrow{[P] \otimes C^*(G, C_0(M))} KK(\mathbb{C}, C^*(G, C_0(M))) \xrightarrow{I_\rho} \mathbb{Z}.$$

In particular, we set $\text{index} := \text{index}_1$.

2.3 Index and Orbifold index

The quotient $\tilde{M} := G \backslash M$ is a smooth compact orbifold carrying an orbifold Dirac bundle $\tilde{F} := G \backslash F$ with associated orbifold Dirac operator $\tilde{D}$. In our case the space of smooth sections $C^\infty(\tilde{M}, \tilde{F})$ can be identified with the $G$-invariant sections $C^\infty(M, F)^G$. Then $\tilde{D}$ coincides with the restriction of $D$ to this subspace. It is well-known that $\dim(\ker \tilde{D}) < \infty$ so that we can define the index $\text{index}(\tilde{D}) := \dim_s(\ker \tilde{D}) \in \mathbb{Z}$, where the subscript ”$s$” indicates that we take the super dimension.

If $\rho : G \to U(V_\rho)$ is a finite-dimensional unitary representation of $G$, then we define the orbifold bundle $\tilde{V}(\rho) := G \backslash M \times V_\rho$ and let $\tilde{D}_\rho$ be the twisted operator associated to $\tilde{F} \otimes \tilde{V}(\rho)$. The space $C^\infty(\tilde{M}, \tilde{F} \otimes \tilde{V}(\rho))$ can be identified with $(C^\infty(M, F) \otimes V_\rho)^G$ such that $\tilde{D}_\rho$ is the restriction of $D \otimes 1$ to this subspace. Still we can define $\text{index}(\tilde{D}_\rho)$.

Theorem 2.2 $\text{index}(\tilde{D}_\rho) = \text{index}_\rho([D])$

Proof. We first apply $j^G$ to the Kasparov module $(L^2(M, F), \mathcal{F})$ representing $[D]$. According to [Kas88], 3.11., $j^G([D])$ is represented by $(C^*(G, L^2(M, F)), \tilde{F})$, where $C^*(G, L^2(M, F))$...
is a $C^*(G)$-right-module admitting a left action by $C^*(G, C_0(M))$. It is a closure of the space of finitely supported functions $f : G :\rightarrow L^2(M, F)$. The operator $\tilde{F}$ is given by $(\tilde{F}f)(g) = (Ff)(g)$. The $C^*(G)$-valued scalar product is given by $\langle f_1, f_2 \rangle(g) = \sum_{h \in G} \langle f_1(h), f_2(hg) \rangle$. Furthermore, the left action of $C^*(G, C_0(M))$ is given by $(\phi f)(g) = \sum_{h \in G} \phi(h)(hf)(g)$.

Using associativity of the Kasparov product we can compute index$_{\rho}$ by first applying $\otimes_{C^*(G)}[\rho]$ and then $[P] \otimes_{C^*(G, C_0(M))}$. Using that $C^*(G, L^2(M, F)) \otimes_{C^*(G)} V_{\rho} \cong L^2(M, F) \otimes V_{\rho}$ by $f \otimes v \mapsto \sum_{g \in G} f(g) \rho(g)v$ we conclude that $j^G([D]) \otimes_{C^*(G)} [\rho]$ is represented by the Kasparov module $(L^2(M, F) \otimes V_{\rho}, \tilde{F})$, where $\tilde{F} = F \otimes \text{id}_{V_{\rho}}$. The left-action of $C^*(G, C_0(M))$ is given by $(\phi f) = \sum_{h \in G} \phi(h)(h \otimes \rho(h))f$.

Finally we compute $[P] \otimes_{C^*(G, C_0(M))} (j^G([D]) \otimes_{C^*(G)} [\rho])$. We represent $[P]$ by the Kasparov module $(PC^*(G, C_0(M)), 0)$. We must understand $PC^*(G, C_0(M)) \otimes_{C_0(M)} (L^2(M, F) \otimes V_{\rho})$.

There is a natural unitary inclusion $L : L^2(\tilde{M}, F \otimes \tilde{V}(\rho)) \hookrightarrow L^2(M, F) \otimes V_{\rho}$. If $f \in L^2(\tilde{M}, F \otimes \tilde{V}(\rho))$ is considered as an element $\hat{f}$ of $(L^2_{\text{loc}}(M, F) \times V_{\rho})^G$ in the natural way, then $L(f) := \chi \hat{f}$. The projection $LL^*$ onto the range of $L$ is given by

$$LL^*(f) = \sum_{g \in G} (g^{-1})^* \chi g f.$$ 

It now follows from the definition of $P$ that

$$PC^*(G, C_0(M)) \otimes_{C^*(G, C_0(M))} (L^2(M, F) \otimes V_{\rho}) = P(L^2(M, F) \otimes V_{\rho})$$

$$\cong L^2(\tilde{M}, F \otimes \tilde{V}(\rho))$$

The operator $\tilde{D}$ has a natural selfadjoint extension (also denoted by $\tilde{D}$) such that we can form $\overline{\tilde{F}} := \tilde{D}(1 + \tilde{D})^{-1/2}$. We claim that $[P] \otimes_{C^*(G, C_0(M))} (j^G([D]) \otimes_{C^*(G)} [\rho])$ is represented by the Kasparov module $(L^2(\tilde{M}, F \otimes \tilde{V}(\rho)), \overline{\tilde{F}})$. The assertion of the Theorem immediately follows from the claim. In order to show the claim we employ the characterization of the Kasparov product in terms of connections (see [Kas88], 2.10). In our situation we have only to show that $\overline{\tilde{F}}$ is a $\overline{\tilde{F}}$-connection.

For Hilbert-$C^*$-modules $X, Y$ over some $C^*$-algebra $A$ let $L(X, Y)$ and $K(X, Y)$ denote the spaces of bounded and compact adjoinable $A$-linear operators (see [Bla98] for definitions). For $\xi \in PC^*(G, C_0(M))$ we define $\theta_\xi \in L(L^2(M, F) \otimes V_{\rho}, P L^2(M, F) \otimes V_{\rho})$ by $\theta_\xi(f) = \xi f$. Since $\mathcal{F}$ and $\overline{\mathcal{F}}$ are selfadjoint we only must show that $\theta_\xi \circ \overline{\mathcal{F}} - (L\overline{\mathcal{F}})^* \circ \theta_\xi \in K(L^2(M, F) \otimes V_{\rho}, P L^2(M, F) \otimes V_{\rho})$. We have $\xi \overline{\mathcal{F}} - (L\overline{\mathcal{F}})^* \xi = [\xi, \overline{\mathcal{F}}] + (\overline{\mathcal{F}} - L\overline{\mathcal{F}})^* P \xi$. Since $[\xi, \overline{\mathcal{F}}]$ is compact it suffices to show that $(\overline{\mathcal{F}} - L\overline{\mathcal{F}})^* P$ is compact. We consider $\tilde{D} := (1 - P)D(1 - P) + L\tilde{D} \overline{\mathcal{F}}$. Then we have $\tilde{D} = D + Q$, where $Q$ is a zero order non-local operator. Let $\overline{\mathcal{F}} := \tilde{D}(1 + \tilde{D})^{-1/2}$. Then $(\overline{\mathcal{F}} - L\overline{\mathcal{F}})^* P = (\overline{\mathcal{F}} - \overline{\mathcal{F}}) P$. Let $\tilde{\chi} \in C_c^\infty(M)$ be such that $\chi \tilde{\chi} = \chi$. Then we have $(\overline{\mathcal{F}} - \overline{\mathcal{F}}) P = (\overline{\mathcal{F}} - \overline{\mathcal{F}}) \chi P$. Therefore it
suffices to show that $(\hat{F} - \tilde{F})\tilde{\chi}$ is compact. This can be done using the integral representations for $\hat{F}$ and $\tilde{F}$ as in [Bun95].

\[ \square \]

### 2.4 The local index theorem

In this the present subsection we derive a local index theorem which is a formula for $\text{index}_p([D])$ in terms of integrals of characteristic forms over the various singular strata of $\bar{M}$.

Let $W \in C^\infty(M \times M, F \boxtimes F^*)^G$ be an invariant section which satisfies an estimate

\[ |W(x, y)| \leq C \exp(-c \text{dist}(x, y)^2) \]

for some $c > 0$, $C < \infty$. Since $\bar{M}$ is compact the manifold $M$ has bounded geometry, and in particular, it has at most exponential volume growth. Therefore, $W$ defines an integral operator $\bar{W}$ on $L^2(\bar{M}, \bar{F} \otimes \bar{V}_\rho)$ by

\[ \bar{W}f(x) := \int_M (W(x, y) \otimes \text{id}_{V_\rho}) f(y) dy. \]

This operator is in fact of trace class. We claim that

\[ \text{Tr} \bar{W} = \int_{\bar{M}} \sum_{g \in G} \text{tr}(W(x, gx)g_x) dx \text{tr}\rho(g), \]

where $g_x$ denotes the linear map $g_x : F_x \to F_{gx}$. In order to see the claim note that $\text{Tr} \bar{W} = \text{Tr} LW L^*$, and $R := LW L^*$ is the integral operator on $L^2(M, F) \otimes V_\rho$ given by the integral kernel $R(x, y) = \sum_{g \in G} \chi(x)W(x, gy)g_y \chi(y) \otimes \rho(g)$.

Again, since $M$ and $F$ have bounded geometry the heat kernel $W_t$, $t > 0$, i.e. the integral kernel of $\exp(-tD^2)$, satisfies the Gaussian estimate (2). Moreover, $\bar{W}_t$ is precisely $\exp(-t\bar{D}^2)$. By the McKean-Singer formula we have

\[ \text{index}(\bar{D}_\rho) = \text{Tr}_s \bar{W}_t \]

for any $t > 0$, where $\text{Tr}_s$ is the super trace. We obtain the local index formula by evaluating

\[ \lim_{t \to 0} \text{Tr}_s \bar{W}_t \]

If $g \in G$, then let $M^g$ denote the fixed point submanifold of $g$. If $M^g \neq \emptyset$, then $g$ is of finite order. Furthermore, let $Z_G(g)$ denote the centralizer of $g$ in $G$. Then $Z_G(g) \backslash M^g$ is compact. For $g \in G$ let $(g) \in C(G)$ denote the conjugacy class of $g$, where $C(G)$ denotes the set of conjugacy classes. By $\mathcal{F}(G)$ we denote the set of elements of finite order, and by $\mathcal{FC}(G)$ we denote the set of conjugacy classes of $G$ of finite order.
The formula (3) can we rewritten as follows.

\[
\text{tr}_s W = \int_M \sum_{g \in G} \text{tr}_s(W(x, gx)g_x)dx \, \text{tr}\rho(g)
\]
\[
= \sum_{(g) \in C(G)} \int_{G \backslash M} \sum_{h \in Z_G(g) \backslash G} \text{tr}_s(W(x, hgh^{-1}x)(hgh^{-1})_x)dx \, \text{tr}\rho(hgh^{-1})
\]
\[
= \sum_{(g) \in C(G)} \int_{Z_G(g) \backslash M} \text{tr}_s(W(x, gx)g_x)dx \, \text{tr}\rho(g) .
\]

If \( W = W_t \) is the heat kernel, then due to the usual gaussian estimates the integral \( \int_{Z_G(g) \backslash M} \text{tr}_s(W(x, gx)g_x)dx \) localizes at \( Z_G(g) \backslash M^g \) as \( t \to 0 \). There is a \( Z_G(g) \)-invariant density \( U(g) \in C^\infty(M^g, |\Lambda^{\text{max}}|T^*M^g)^{Z_G(g)} \) which is locally determined by the Riemannian structure \( g^M \) and the Dirac bundle \( F \) such that

\[
\lim_{t \to \infty} \int_{Z_G(g) \backslash M} \text{tr}_s(W_t(x, gx)g_x)dx = \frac{1}{\text{ord}(g)} \int_{Z_G(g) \backslash M^g} U(g) \text{tr}\rho(g) .
\]

An explicit formula for \( U(g) \) is given in [BGV92], Ch. 6.4, and it will be recalled below. We conclude that

\[
\text{index}_\rho([D]) = \sum_{(g) \in C_F(G)} \frac{1}{\text{ord}(g)} \int_{Z_G(g) \backslash M^g} U(g) \text{tr}\rho(g) .
\]

The fixed point manifold \( M^g \) is a totally geodesic Riemannian submanifold of \( M \) with induced metric \( g^{M^g} \). Let \( R^{M^g} \) denote its curvature tensor. We define the form \( \hat{A}(M^g) \in \Omega(M^g, \text{Or}(M^g)) \) by

\[
\hat{A}(M^g) = \det^{1/2} \left( \frac{R^{M^g}/4\pi i}{\sinh(R^{M^g}/4\pi i)} \right),
\]

where \( \text{Or}(M^g) \) denote the orientation bundle (the orientation bundle occurs since we must choose an orientation in order to define \( \det^{1/2} \)).

Furthermore, we define the \( G \)-equivariant bundle \( F/S := \text{End}_{\text{Cliff}(T^*M)}(F) \). It comes with a natural connection \( \nabla^{F/S} \). By \( R^{F/S} \) we denote its curvature. Following [BGV92], 6.13, we define the form \( \text{ch}(g, F/S) \in \Omega(M^g, \Lambda^{\text{max}} N \otimes \text{Or}(M)) \) by

\[
\text{ch}(g, F/S) = \frac{2^{\text{codim}_M(M^g)}}{\sqrt{\det(1 - g^N)}} \text{str}(\sigma_{\text{codim}_M(M^g)}(g^F) \exp(-R_0^{F/S}/2\pi i)) .
\]

Here \( g^N \) is the restriction of \( g \) to the normal bundle \( N \) of \( M^g \). Note that \( \det(1 - g^N) > 0 \) so that \( \sqrt{\det(1 - g^N)} \) is well-defined. Furthermore \( g^F \) is the action of \( g \) on the fibre of \( F|_{M^g} \). Since \( g^F \) commutes with \( \text{Cliff}(T^*M^g) \) it corresponds to an element of \( \text{Cliff}(N) \otimes \text{End}_{\text{Cliff}(M)}(F) \). \( \sigma_{\text{codim}_M(M^g)} : \text{Cliff}(N) \to \Lambda^{\text{max}} N \) is the symbol map so
that \( \sigma_{\text{codim}(M^g)} g^F \in \text{End}_{\text{Cliff}(M)}(F) \otimes \Lambda^{\text{max}} N \). Furthermore, the restriction \( R^{F/S}_0 \) of the curvature \( R^{F/S} \) to \( M^g \) is a section of \( \Omega(M^g, \text{End}_{\text{Cliff}(M)}(F)|_{M^g}) \). The super trace \( \text{str} : \text{End}_{\text{Cliff}(M)}(F) \rightarrow \mathbb{C} \otimes \text{Or}(M) \) is defined by \( \text{str}(W) = \text{tr}_g(\Gamma W) \), where \( \Gamma = \text{in}^{n/2} \text{vol}_M \) is the chirality operator defined using the orientation of \( M \).

Let \( T_N : \Lambda^{\text{max}} N \rightarrow \mathbb{C} \otimes \text{Or}(N) \) be the normal Beresin integral, where Or\((N)\) is the bundle of normal orientations. Then we have

\[
U(g) := [T_N(\frac{\hat{A}(M^g) \text{ch}(g, F/S)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi i))})]^{\text{max}}.
\]

Here \( R^N \) is the curvature tensor of \( N \), \( \frac{\det^{1/2}(1 - g^N \exp(-R^N))}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi i))} \in \Omega(M^g, \text{Or}(M^g)) \), and \([\cdot]^{\text{max}}\) takes the part of maximal degree. In order to interpret the right-hand side as a density on \( M^g \) we identify \( \Lambda^{\text{max}} T^* M^g \otimes \text{Or}(M^g)^2 \otimes \text{Or}(N) \otimes \text{Or}(M) \) with \( |\Lambda^{\text{max}}| T^* M^g \) in the canonical way.

**Theorem 2.3**

\[
\text{index}_\rho([D]) = \sum_{(g) \in C \mathcal{F}(G)} \frac{\text{tr}_\rho(g)}{\text{ord}(g)} \int_{Z_G(g) \setminus M^g} [T_N(\frac{\hat{A}(M^g) \text{ch}(g, F/S)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi i))})]^{\text{max}}
\]

### 2.5 Cyclic subgroups

We now reformulate the local index theorem in terms of contributions of conjugacy classes of cyclic subgroups. Let \( \mathcal{F} \text{Cyc}(G) \) denote the set of finite cyclic subgroups. If \( C \in \mathcal{F} \text{Cyc}(G) \), then let \( \text{gen}(C) \) denote the set of its generators. The normalizer \( N_G(C) \) and the Weyl group \( W_G(C) := N_G(C)/Z_G(C) \) acts on \( \text{gen}(C) \). There is a natural map \( p : \mathcal{F}(G) \rightarrow \mathcal{F} \text{Cyc}(G), \ g \mapsto < g > \) which factors over conjugacy classes \( \tilde{p} : \mathcal{F}(G) \rightarrow \mathcal{F} \text{Cyc}(G) \). If \( (C) \in \mathcal{F} \text{Cyc}(G) \), then \( \tilde{p}^{-1}(C) \) can be identified with \( W_G(C)/\text{gen}(C) \).

Note that \( M^g = M^{< g >} \), i.e. it only depends on the cyclic subgroup generated by \( g \). Similarly, \( Z_G(g) = Z_G(< g >) \). So we obtain

**Corollary 2.4**

\[
\text{index}_\rho([D]) = \sum_{(C) \in \mathcal{F} \text{Cyc}(G)} \frac{1}{|C|} \sum_{g \in W_G(C) \setminus \text{gen}(C)} \int_{Z_G(C) \setminus M^C} U(g) \text{tr}_\rho(g)
\]

### 2.6 Cap product and twisting

We define \( K^0_C(M) := KK^G(\mathbb{C}, C_0(M)) \). If \( E \) is a \( G \)-equivariant complex vector bundle, then let \( [E] \in K^0_C(M) \) denote the class represented by the Kasparov module \( (C_0(M, E), 0) \), where we define the \( C_0(M) \)-valued scalar product on \( C_0(M, E) \) after choosing a \( G \)-invariant hermitean metric \((.,.)_E \).
Since $C_0(M)$ is commutative any right $C_0(M)$-module is a left- $C_0(M)$-module in a natural way. If we apply this to Kasparaov modules we obtain a map

$$a : KK^G(C, C_0(M)) \to KK^G(C_0(M), C_0(M)).$$

**Definition 2.5** The cap-product $K_0^G(M) \otimes K_0^G(M) \to K_0^G(M)$ is defined by

$$v \cap x := a(v) \otimes_{C_0(M)} x.$$

If we choose on $(E, (.,.))$ a hermitian connection $\nabla^E$, then we can form the twisted Dirac bundle $E \otimes F$ with associated Dirac operator $D_E$. The following fact is well-known.

An elementary proof (for trivial $G$) can be found e.g. in [Bun95].

**Proposition 2.6** $[D_E] = [E] \cap [D]$

### 2.7 A cohomological index formula for twisted operators

Let $R^E$ denote the curvature of the connection $\nabla^E$. For a finite cyclic subgroup $C \subset G$ let $R^E_0$ denote the restriction of $R^E$ to $M^C$. If $g \in \text{gen}(C)$, then we have

$$\text{ch}(g, E \otimes F/S) = \text{ch}(g, F/S) \cup \text{ch}(g, E),$$

where $\text{ch}(g, E) = \text{tr}g^E \exp(-R^E_0/2\pi i)$. Here $g^E$ denotes the action of $g$ on the fibre of $E$.

Thus we can write

$$U_E(g) := \left[ T_N\left( \frac{\hat{A}(M^g)\text{ch}(g, F/S) \cup \text{ch}(g, E)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi i))} \right) \right]_{\text{max}}.$$

We can write $U_E(g) = [\hat{U}(g) \cap \text{ch}(g, E)]_{\text{max}}$, where

$$\hat{U}(g) = T_N\left( \frac{\hat{A}(M^g)\text{ch}(g, F/S)}{\det^{1/2}(1 - g^N \exp(-R^N/2\pi i))} \right). \quad (4)$$

The cohomology $H^*(Z_G(C) \setminus M^C, \mathbb{C})$ of the orbifold $Z_G(C) \setminus M^C$ can be computed using the complex of invariant differential forms $(\Omega^*(M^C)^{Z_G(C)}, d)$. Furthermore, the homology $H_*(Z_G(C) \setminus M^C, \mathbb{C})$ can be identified with the dual of the cohomology, i.e. $H_*(Z_G(C) \setminus M^C, \mathbb{C}) \cong H^*(Z_G(C) \setminus M^C, \mathbb{C})^*$. The closed form $\hat{U}(g) \in \Omega^*(M^C, \text{Or})$ now defines a homology class $[\hat{U}(g)] \in H_*(Z_G(C) \setminus M^C, \mathbb{C})$ such that $[\hat{U}(g)]([\omega]) = \int_{Z_G(C) \setminus M^C} \hat{U}(g) \cap [\omega]_{\text{max}}$ for any closed form $\omega \in \Omega^*(M^C)^{Z_G(C)}$.

Let $\text{ch}(g, E) \in H^*(Z_G(C) \setminus M^C, \mathbb{C})$ denote the cohomology class represented by the closed form $\text{ch}(g, E)$.

**Theorem 2.7**

$$\text{index}_\rho([E] \cap [D]) = \sum_{(C) \in C \setminus FCycG} \frac{1}{|C|} \sum_{g \in W_G(C) \setminus \text{gen}(C)} \langle [\text{ch}(g, E)], [\hat{U}(g)] \rangle \text{tr} \rho(g).$$
3 Chern characters

3.1 The cohomological Chern character

In this Subsection we review the construction of the Chern character given in \cite{LO01a}. There the equivariant $K$-theory is introduced using a classifying space $K_G \mathbb{C}$. If $X$ is a proper $G$-CW complex, then $K^0_G(X) := [X, K_G \mathbb{C}]_G$, where $[.]_G$ denotes the set of homotopy classes of equivariant maps.

Let $K_G(X)$ be the Grothendieck group of $G$-equivariant complex vector bundles. Then there is a natural homomorphism $b : K_G(X) \to K^0_G(X)$, which is an isomorphism if $X$ is finite (\cite{LO01a}, Prop. 1.5).

If $H$ is a finite group, then let $R_C(H)$ denote the complex representation ring of $H$ with complex coefficients. The character gives a natural identification of $R_C(H)$ with the space of complex-valued class functions on $H$, i.e. $\mathbb{C}(C(H))$.

Since we want to work with differential forms later on we simplify matters by working with complex coefficients (the constructions in \cite{LO01a} are finer since they work over $\mathbb{Q}$). For any finite subgroup $H \subset G$ the construction \cite{LO01a}, (5.4), provides a homomorphism

$$\mathbf{ch}^H_X : K^0_G(X) \to H^*(Z_G(H) \backslash X^H) \otimes \mathbb{C}(C(H)).$$

For our purpose it suffices to understand $\mathbf{ch}^H_X(b(\{E\}))$, where $E$ is a $G$-equivariant complex vector bundle over $X$, and $\{E\}$ denotes its class in $K_G(X)$. First of all note that $E_{|X^H}$ is a $N_G(H)$-equivariant bundle over $X^H$. We can further write $E_{|X^H} = \sum_{\phi \in \hat{H}} \text{Hom}_H(V_\phi, E_{|X^H}) \otimes V_\phi$, where $\text{Hom}_H(V_\phi, E_{|X^H})$ is a $Z_G(H)$-equivariant bundle over $X^H$. We therefore obtain an element of $K^0_{Z_G(H)}(X^H) \otimes R(H)$. We now apply the composition

$$K^0_{Z_G(H)}(X^H) \stackrel{pr^*}{\to} K^0_{Z_G(H)}(EG \times X^H) \stackrel{\cong}{\to} K^0_{Z_G(H)}(EG \times Z_G(H) \times X^H) \stackrel{\mathbf{ch}}{\to} H^*(EG \times Z_G(H) \times X^H, \mathbb{C}) \stackrel{(pr^*)^{-1}}{\to} H^*(Z_G(H) \backslash X^H, \mathbb{C})$$

to the first component, and the character $R(H) \to \mathbb{C}(C(H))$ to the second. The result belongs to $H^*(Z_G(H) \backslash X^H, \mathbb{C}) \otimes \mathbb{C}(C(H))$ and is $\mathbf{ch}^H_X(b(\{E\}))$.

If $C$ is a finite cyclic subgroup, then let $r : \mathbb{C}(C(C)) \to \mathbb{C}(\text{gen}(C))$ be the restriction map. Note that $W_G(C)$ acts on $\mathbb{C}(\text{gen}(C))$ as well as on $H^*(Z_G(C) \backslash X^C, \mathbb{C})$. The result \cite{LO01a}, Lemma 5.6, now asserts that if $X$ is finite, then

$$\prod_{(C) \in CFCyc(G)} (1 \otimes r) \mathbf{ch}^G_X : K^0_G(X)_C \to \prod_{(C) \in CFCyc(G)} (H^{ev}(Z_G(C) \backslash X^C, \mathbb{C}) \otimes \mathbb{C}(\text{gen}(C)))^{W_G(C)}$$

is an isomorphism.
3.2 Differential forms

In the present subsection we give a description of the equivariant Chern character using differential forms. Let $M$ be a smooth proper $G$-manifold and $E$ be a $G$-equivariant complex vector bundle over $M$. Then we can find a $G$-invariant hermitian metric $(.,.)_E$ and a $G$-invariant metric connection $\nabla^E$. Let $R^E$ denote the curvature of $\nabla^E$. We define the closed $G$-invariant form $\text{ch}(E) \in \Omega(M)^G$ by $\text{ch}(E) := \text{tr} \exp(-R^E/2\pi i)$. It represents a cohomology class $[\text{ch}(E)] \in H^*(G\setminus M, \mathbb{C})$. Furthermore, we have the class $\text{ch}_{M}^{(1)}(b\{E\}))$, which is given by the following composition

$$
\mathbb{K}_G^0(M) \xrightarrow{\text{pr}_E} \mathbb{K}_G^0(EG \times M) \xrightarrow{\text{pr}_2} \mathbb{K}_1^0(EG \times_G M) \xrightarrow{\text{ch}} H^*(EG \times_G M, \mathbb{C}) \xrightarrow{(\text{pr}_2)^{-1}} H^*(G\setminus M, \mathbb{C}). \tag{6}
$$

Lemma 3.1 $[\text{ch}(E)] = \text{ch}_{M}^{(1)}(b\{E\}))$

Proof. We show that $\text{ch}_{M}^{(1)}(b\{E\}))$ can be represented by the form $\text{ch}(E)$. To do so we employ an approximation $j : \tilde{E}G \to EG$, where $\tilde{E}G$ is a free $G$-manifold and the $G$-map $j$ is $\dim(M) + 1$-connected. This existence of such approximations will be shown in Subsection 3.3. Then we can define $\text{ch}_{M}^{(1)}(b\{E\}))$ by (3) but with $EG$ replaced by $\tilde{E}G$. It is now clear that $\text{pr}_2^*\text{ch}(E) = \text{ch}(G\setminus \text{pr}_1^*E)$. \hfill \Box

Let $C \subset G$ be a finite cyclic subgroup. Furthermore, let $[\text{ch}(g, E)] \in H^*(Z_G(C)\setminus M^C, \mathbb{C})$ denote the cohomology class represented by $\text{ch}(g, E)$. The function $\text{gen}(C) \ni g \mapsto [\text{ch}(g, E)]$ can naturally be considered as an element $[\text{ch}(.), E] \in H^*(Z_G(C)\setminus M^C, \mathbb{C}) \otimes \mathbb{C}(\text{gen}(C))$ which is in fact $W_G(C)$-equivariant.

Proposition 3.2 $[\text{ch}(.), E]) = (1 \otimes r)\text{ch}_{M}^{C}(b\{E\}))$

Proof. First of all note that $R^E_0$ is the curvature of $E_{|M^C}$. Furthermore, the decomposition $E_{|M^C} = \sum_{\phi \in \mathcal{C}} E(\phi) \otimes V_\phi$ is preserved by $R^E_0$, where $E(\phi) = \text{Hom}_C(V_\phi, E_{|M^C})$. Let $R^{E(\phi)}$ be the restriction of the curvature to the subbundle $E(\phi) \otimes V_\phi$. We get for $g \in \text{gen}(C)$

$$
(1 \otimes r)\text{ch}_{M}^{C}(b\{E\})(g) \overset{\text{def}}{=} \sum_{\phi \in \mathcal{C}} \text{ch}_{M}^{(1)}(b\{E(\phi)\})\text{tr}\phi(g)
$$

$$
\overset{\text{Lemma} \text{3.2}}{=} \sum_{\phi \in \mathcal{C}} [\text{ch}(E(\phi))]\text{tr}\phi(g)
$$

$$
= \sum_{\phi \in \mathcal{C}} [\text{tr} \exp(-R^{E(\phi)}/2\pi i)]\text{tr}\phi(g)
$$

$$
= [\text{tr} g^E \exp(-R^E_0/2\pi i)]
$$

$$
= [\text{ch}(g, E)].
$$
3.3 Smooth approximations of $CW$-complexes

The goal of this subsection is to show that the approximation $j : \tilde{E}G \to E\!G$ used in the proof of Lemma 3.1 exists. We start with the following general result.

**Proposition 3.3** If $X$ is a countable finite-dimensional $CW$-complex, then there exists a smooth manifold $M$ and a homotopy equivalence $M \sim X$.

**Proof.** Let $X$ be a finite-dimensional $CW$-complex. Following [Bro62] we call a manifold with boundary $(\bar{M}, \partial \bar{M})$ a tubular neighbourhood of $X$ if there exists a continuous map $F : \partial \bar{M} \to X$ such that the underlying topological space of $\bar{M}$ is the mapping cylinder $C(F) = \partial \bar{M} \times [0,1] \cup_F X$ of $F$, the inclusion $\partial \bar{M} \times [0,1) \hookrightarrow M$ is smooth, and the inclusion $X \hookrightarrow \bar{M}$ is smooth on each open cell of $X$.

Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n = X$ be the filtration of $X$ by skeletons. We obtain $X_{i+1}$ from $X_i$ by attaching a countable number of $i+1$-cells. We will construct a tubular neighbourhood $(\bar{M}, \partial \bar{M})$ of $X$ by induction over the dimension with the special property that the tangent bundle $T\bar{M} \to \bar{M}$ is trivial. It is clear that the collection of points $X_0$ admits such a tubular neighbourhood of any given dimension. Let us now assume that there exists an $i$-dimensional $CW$-complex $Y$ together with a homotopy equivalence $h : Y \to X_i$ and a $m$-dimensional tubular neighbourhood $(W, \partial W)$, $F : \partial W \to Y$ of $Y$, such that $m \geq 2n + 3$ and $TW \to W$ is trivial.

Via a homotopy inverse of $h$ the attaching data for $X_{i+1}$ from $X_i$ by attaching a countable number of $i + 1$-cells.

We consider $\mathbb{R}$ with the cell-structure given by its decomposition into unit intervals. The product $(W \times \mathbb{R}, \partial W \times \mathbb{R})$ is a tubular neighbourhood of $Y \times \mathbb{R}$ in a natural way with retraction $F \times \text{id}_\mathbb{R} : \partial W \times \mathbb{R} \to Y \times \mathbb{R}$. The projection $Y \times \mathbb{R} \to Y$ is a homotopy equivalence.

We now fix an inclusion $J \subseteq Z$ and define attaching maps $\chi_\alpha := \tilde{\chi}_\alpha \times \{\alpha\} : S^i \to Y \times \mathbb{R}$. Let $\tilde{Z}$ denote the complex obtained by attaching the cells to $Y \times \mathbb{R}$. Our choice of attaching maps is made such that these $i + 1$-cells are attached to the $i$-skeleton of $Y$. We have a homotopy equivalence $\tilde{Z} \sim \tilde{Z}$.

In order to improve the attaching maps we argue as in the proof of [Bro62, Theorem II]. Since $2\dim(Y) + 1 = 2i + 1 \leq 2n + 3 \leq m = \dim(W)$ we can deform the attaching...
map $\tilde{\chi}_\alpha$ slightly so that its image is disjoint from $Y$. To do so we adapt the method of the proof of [Whi55, Theorem 11a], and we use the assumption that the open cells of $Y$ are smoothly embedded. Using the mapping cylinder structure $W \setminus Y \cong \partial W \times [0,1)$ we can further deform the attaching map such that it maps to $\partial W$. Finally, using that $2i + 1 \leq \dim(\partial W) = m - 1$ we can deform it to an embedding (see [Whi30, Theorem II]) into $\partial W$. We still denote this deformed attaching map by $\tilde{\chi}_\alpha$, and we use the assumption that the open cells of $Y$ are smoothly embedded. Using the mapping cylinder structure $\partial W \times \{\alpha\} \cong \partial W \times [0,1)$ we can further deform the attaching map such that it maps to $\partial W$. Finally, using that $2i + 1 \leq \dim(\partial W) = m - 1$ we can deform it to an embedding (see [Whi30, Theorem II]) into $\partial W$. We still denote this deformed attaching map by $\tilde{\chi}_\alpha$, and we obtain a new deformed map $\chi_\alpha := \tilde{\chi}_\alpha \times \{\alpha\} : S^i \to \partial W \times \{\alpha\} \subseteq \partial W \times \mathbb{R}$. Since the tangent bundle $T\partial W \to \partial W$ is trivial, we see that the normal bundle $\nu_\alpha$ of the embedding $\tilde{\chi}_\alpha$ is stably trivial. Since $\dim(\nu_\alpha) \geq i + 1$ the bundle $\nu_\alpha$ is in fact trivial.

For each $\alpha \in J$ we now perform the procedure of attaching a handle to $W \times \mathbb{R}$ described in [Bro62, Sec. 2]. We can arrange the construction such that for $\alpha \in J$ it takes place on $W \times (\alpha - 1/4, \alpha + 1/4)$.

The result of this construction is a manifold with boundary $(N, \partial N)$ of dimension $m + 1$ containing an $i + 1$-dimensional CW-complex $Z$, and a map $N \to Z$ which represents $(N, \partial N)$ as a tubular neighbourhood of $Z$, and we have a homotopy equivalence $Z \cong \tilde{Z}$. The construction depends on the choice of trivializations of the normal bundles $\nu_\alpha$. By Lemma [Bro62, Lemma 2.2] we can choose the trivializations of the normal bundles $\nu_\alpha$ such that $TN \to N$ is again trivial.

After we finite iteration of this construction we obtain a manifold with boundary $(\bar{M}, \partial \bar{M})$ which is a tubular neighbourhood of a CW-complex $\tilde{X}$ which admits a homotopy equivalence $\tilde{h} : \tilde{X} \cong \tilde{X}$. The mapping cylinder structure on $\bar{M}$ gives rise to a projection $p : \bar{M} \to \tilde{X}$. We now consider the smooth manifold $M := \bar{M} \setminus \partial \bar{M}$. The composition $\tilde{h} \circ p|_M : M \to X$ is a homotopy equivalence from a smooth manifold to $X$. □

We can now construct the smooth $n$-connected approximation $j : \tilde{EG} \to EG$, where $n \geq 2$. We start with a countable CW-complex $BG$ of the homotopy type of the classifying space of $G$. For example, we can take the standard simplicial model. It is countable since $G$ is countable.

We consider the $n$-skeleton $BG^n \subseteq BG$. It is a finite-dimensional countable CW-complex. By Proposition 3.3 we can find a smooth manifold $\tilde{BG}$ together with a homotopy equivalence $\tilde{j} : \tilde{BG} \to BG^n$. Let $\tilde{j} : \tilde{BG} \to BG$ denote the composition of $\tilde{j}$ with the inclusion $BG^n \hookrightarrow BG$. By construction $\tilde{j}$ is $n$-connected.

Since $\tilde{j}$ induces an isomorphism of fundamental groups it lifts to a $n$-connected map of universal coverings $j : \tilde{EG} \to EG$. 


4 Explicit decomposition of $K$-homology classes

3.4 The homological Chern character

In this subsection we review the construction of the homological Chern character given in [Lü02a, Lü02b]. Let $X$ be a proper $G$-CW-complex. The main constituent of the Chern character is a homomorphism

$$\text{ch}^X_H : H_{ev}(Z_G(H) \backslash X^H, \mathbb{C}) \otimes R_C(H) \to K^G_0(X)$$

for any finite subgroup $H \subset G$.

$$
\begin{array}{c}
H_{ev}(Z_G(H) \backslash X^H, \mathbb{C}) \otimes R_C(H) & \xrightarrow{(pr_2)^{-1} \otimes \text{id}} & H_{ev}(EG \times_{Z_G(H)} X^H, \mathbb{C}) \otimes R_C(H) \\
\text{ch}^{-1} \otimes \text{id} & \xrightarrow{=} & K_0(EG \times_{Z_G(H)} X^H)_C \otimes R_C(H) \\
\cong & \xrightarrow{m} & K_0^G(EG \times X^H)_C \\
\text{Ind}_{Z_G(H) \times H}^G & \xrightarrow{\text{Ind}_{Z_G(H) \times H}^G} & K_0^G(\text{Ind}_{Z_G(H) \times H}^G(EG \times X^H))_C \\
\text{Ind}_{Z_G(H) \times H}^G(pr_2), & \xrightarrow{m} & K_0^G(\text{Ind}_{Z_G(H) \times H}^G X^H)_C \\
& & K_0^G(X)
\end{array}
$$

Here $\text{ch}$ is the homological Chern character, $\text{Ind}_{Z_G(H) \times H}^G$ denotes the induction functor, and $m : \text{Ind}_{Z_G(H) \times H}^G X^H = G \times_{Z_G(H) \times H} X^H \to X$ is the $G$-map $(g, x) \mapsto gx$.

Let $C \subset G$ be a finite cyclic subgroup. Then we have a natural inclusion $r^* : \mathbb{C}(\text{gen}(C)) \to R_C(C) \cong \mathbb{C}C$ such that the image consists of functions which vanish on $C \setminus \text{gen}(C)$. Note that $\mathbb{C}(\text{gen}(C))$ and $H_*(Z_G(H) \backslash X^H, \mathbb{C})$ are left and right $W_G(C)$-modules in the natural way. It follows from [Lü02b], Thm. 0.7, that

$$
\bigoplus_{(C) \in \mathcal{F}_{\text{Cyc}}(G)} \text{ch}^X_C (1 \otimes r^*) : \bigoplus_{(C) \in \mathcal{F}_{\text{Cyc}}(G)} H_{ev}(Z_G(C) \backslash X^C, \mathbb{C}) \otimes_{\mathbb{C}W_G(C)} \mathbb{C}(\text{gen}(C)) \to K^G_0(X)_C
$$

is an isomorphism.

4 Explicit decomposition of $K$-homology classes

4.1 An index formula

Let $E$ be a $G$-equivariant vector bundle over $X$. If $A \in H_*(Z_G(H) \backslash X^H, \mathbb{C}) \otimes R_C(H)$, then we can ask for a formula for index$_\rho([E] \cap \text{ch}^X_H(A))$ in terms of $\text{ch}^H_X(b([E]))$. Let $\epsilon : R(H) \to \mathbb{Z}$ be the homomorphism which takes the multiplicity of the trivial representation. It
extends to a group homomorphism $\epsilon_C : \mathcal{R}_C(H) \to \mathbb{C}$. Using the ring structure of $\mathcal{R}_C(H)$ and the pairing between homology and cohomology we obtain a natural pairing

$$
\langle . , . \rangle_\rho : \left( H_*(Z_G(H) \setminus X^H, \mathbb{C}) \otimes \mathcal{R}_C(H) \right) \otimes \left( H^*(Z_G(H) \setminus X^H, \mathbb{C}) \otimes \mathcal{R}_C(H) \right) \to \mathcal{R}_C(H) \stackrel{\epsilon_\rho}{\to} \mathbb{C}.
$$

**Theorem 4.1** $\text{index}_\rho([E] \cap \text{ch}_X^\mathfrak{X}(A)) = \langle \text{ch}_X^\mathfrak{X}(b\{E\}), A \rangle_\rho$

**Proof.** Let $M$ be a cocompact free even-dimensional $Z_G(H)$-manifold equipped with a invariant Riemannian metric and a Dirac operator $D$ associated to a $Z_G(H)$-equivariant Dirac bundle $F \to M$. Furthermore, let $f = (f_1, f_2) : M \to EG \times X^H$ be a $Z_G(H)$-equivariant continuous map. We form $[D] \in K_0^{Z_G(H)}(M)$ represented by the Kasparov module $(L^2(M, F), \mathcal{F})$. Then $f_*[D] \in K_0^{Z_G(H)}(EG \times X^H)$.

Note that $K_0^{Z_G(H)}(EG \times X^H) \subset \mathbb{C}$ is spanned by elements arising in this form. This can be seen as follows. First observe that every class in $K_0(EG \times X_G\mathfrak{H})$ can be represented in the form $f_*[D]$, where $\bar{f} : N \to EG \times Z_G(H) X^H$ is a map from a closed $\text{Spin}^c$-manifold, and $\bar{D}$ is the $\text{Spin}^c$-Dirac operator on $\bar{N}$. A proof of this result is given in [BHS]. We now consider the pull-back

$$
\begin{array}{ccc}
N & \xrightarrow{f} & EG \times X^H \\
\downarrow & & \downarrow \\
\bar{N} & \xrightarrow{\bar{f}} & EG \times Z_G(H) X^H
\end{array}
$$

The manifold $N$ carries a $Z_G(H)$-invariant $\text{Spin}^c$-structure with associated Dirac operator $D$. The class $f_*[D]$ corresponds to $\bar{f}_*[\bar{D}]$ under the isomorphism $K_0^{Z_G(H)}(EG \times X^H) \cong K_0^{Z_G(H)}(EG \times X^H)$.

Let $\phi \in \bar{H}$ be a finite-dimensional representation. It gives rise to an element $[\phi] \in K_0^H(\ast)_C$ under the natural identification $\mathcal{R}_C(H) \cong K_0^H(\ast)_C$. Let $T : K_0^{Z_G(H)}(EG \times X^H) \otimes K_0^H(\ast)_C \to K_0^G(X)$ be the composition $m_* \circ \text{Ind}_Z^{G/H} \circ (\text{pr}_2)_* \circ \text{Ind}_Z^{G/H} \circ \text{mult}$, which is part of the definition of $\text{ch}_X^Y$. We first study $\text{index}_\rho([E] \cap T(f_*[D] \otimes [\phi]))$. We have $\text{mult} \circ f_*([D] \otimes [\phi]) = f_* \circ \text{mult}([D] \otimes [\phi])$, and $\text{mult}([D] \otimes [\phi]) \in K_0^{Z_G(H) \times H}(M)$ is represented by the Kasparov module $(L^2(M, F) \otimes V_\phi, \mathcal{F} \otimes \text{id})$. Furthermore, $\text{Ind}_Z^{G/H \times H} \circ f_* \circ \text{mult}([D] \otimes [\phi]) = \text{Ind}_Z^{G/H \times H} (f_* \text{Ind}_Z^{G/H \times H} (\text{mult}([D] \otimes [\phi])))$. Explicitly, $\text{Ind}_Z^{G/H \times H} (\text{mult}([D] \otimes [\phi]))$ is represented by a Kasparov module which is constructed in the following way. Consider the exact sequence

$$
0 \to K \to Z_G(H) \times H \to G,
$$

where $K = Z_G(H) \cap H = Z_H(H)$. We identify $K \setminus Z_G(H) \times H$ with the subgroup $Z_G(H)H \subseteq G$. 

Note that we consider $M$ as a $Z_G(H) \times H$-manifold via the action of the first factor. The $Z_G(H)H$-manifold $\bar{M} := K \backslash M$ carries an induced equivariant Dirac bundle $\bar{F}$. We further consider the flat $Z_G(H)H$-equivariant bundle $\tilde{\bar{V}}_\phi := V_\phi \times_K M$ over $\bar{M}$. The twisted bundle $\tilde{F} \otimes \tilde{\bar{V}}_\phi$ is a $Z_G(H)H$-equivariant Dirac bundle. We consider the cocompact proper $G$-manifold $\bar{M} := G \times_{Z_G(H)H} \tilde{\bar{M}}$. The $Z_G(H)H$-equivariant Dirac bundle $\tilde{F} \otimes \tilde{\bar{V}}_\phi$ induces a $G$-equivariant Dirac bundle $\tilde{F}_\phi \to \bar{M}$ in a natural way with associated operator $\tilde{D}_\phi$. Then $\text{Ind}^G_{Z_G(H)H}(\text{mult}([D] \otimes [\phi]))$ is represented by $[\bar{D}_\phi]$. The map $\text{Ind}^G_{Z_G(H)H}(f_*)$ is induced by the $G$-map $\tilde{f} : \bar{M} \to G \times_{Z_G(H)H} (K \backslash EG \times X^H)$ given by $\tilde{f}([g, Km]) := [g, (Kf_1(m), f_2(m))]$. It is now clear that $T([f_*[D] \otimes [\phi]])$ is represented by $h_*[\tilde{D}_\phi]$, where $h : \bar{M} \to X$ is given by $h([g, Km]) = gf_2(m)$.

It follows from the associativity of the Kasparov product that

$$\text{index}_\rho([E] \cap T(f_*[D] \otimes [\phi])) = \text{index}_\rho([E] \cap h_*[\bar{D}_\phi]) = \text{index}_\rho([h^*E] \cap [\bar{D}_\phi]).$$

By Theorem 2.2 and Proposition 2.6 we obtain $\text{index}_\rho([h^*E] \cap [\bar{D}_\phi]) = \text{index}(\bar{D}_{\phi,h^*E,\rho})$, where $\bar{D}_{\phi,h^*E}$ is the $G$-invariant Dirac operator associated to $\tilde{F} \otimes h^*E$, and $\bar{D}_{\phi,h^*E,\rho}$ is the operator on the orbifold $\tilde{M} := G \backslash \tilde{\bar{M}}$ induced by $\bar{D}_{\phi,h^*E}$ and the twist $\rho$. Restriction from $\tilde{M}$ to the submanifold $\{1\} \times \bar{M}$ provides an isomorphism

$$\left(C^\infty(\tilde{M}, \tilde{F}_\phi \otimes h^*E) \otimes V_\rho\right)^G \cong \left(C^\infty(\tilde{M}, \tilde{F} \otimes \tilde{\bar{V}}_\phi \otimes \tilde{f}_2^*E_{|X^H}) \otimes V_\rho\right)^{Z_G(H)H},$$

where $\tilde{f}_2 : \tilde{M} \to X^H$ is induced by $f_2$. Since the action of $H$ on the latter spaces is implemented by the action on the fibres of $\tilde{V}_\phi \otimes \tilde{f}_2^*E_{|X^H} \otimes V_\rho$ we further obtain

$$\left(C^\infty(\tilde{M}, \tilde{F}_\phi \otimes h^*E) \otimes V_\rho\right)^G = C^\infty(\tilde{M}, \tilde{F} \otimes (\tilde{V}_\phi \otimes \tilde{f}_2^*E_{|X^H} \otimes V_\rho)^H)^{K \backslash Z_G(H)}.$$

In the present situation we have $\tilde{M} = Z_G(H) \backslash M = (K \backslash Z_G(H)) \backslash \tilde{\bar{M}}$, i.e. the orbifold is smooth, and it carries the Dirac bundle $\bar{F}$ with associated Dirac operator $\bar{D}$. We define the $(K \backslash Z_G(H))$-equivariant bundle $E_{\phi,\rho} := (V_\phi \otimes E_{|X^H} \otimes V_\rho)^H$ over $X^H$. Furthermore, we consider the quotient $\tilde{f}_2^*E_{\phi,\rho} := (K \backslash Z_G(H)) \backslash \tilde{f}_2^*E_{\phi,\rho}$ over $\bar{M}$. The identifications above show that $\text{index}_\rho(\bar{D}_{\phi,h^*E}) = \text{index}(\tilde{f}_2^*E_{\phi,\rho})$, i.e. it is the index of a twisted Dirac operator. Writing the index of the twisted Dirac operator in terms of Chern characters we obtain

$$\text{index}_\rho([E] \cap T(f_*[D] \otimes [\phi])) = \langle \text{ch}(\tilde{f}_2^*E_{\phi,\rho}), \text{ch}([D]) \rangle.$$

Note that $\tilde{f}_2^*E_{\phi,\rho} = f^*\text{pr}_2^*E_{\phi,\rho}$, where $\text{pr}_2 : EG \times X^H \to X^H$, $\tilde{f} : \bar{M} \to EG \times_{Z_G(H)} X^H$ is induced by $f$, and $\text{pr}_2^*E_{\phi,\rho} := Z_G(H) \backslash \text{pr}_2^*E_{\phi,\rho}$. We conclude that

$$\langle \text{ch}(\tilde{f}_2^*E_{\phi,\rho}), \text{ch}([D]) \rangle = \langle \text{ch}(\text{pr}_2^*E_{\phi,\rho}), \text{ch}(f_*[D]) \rangle.$$

The right-hand side can now be written as

$$\langle \epsilon_{\mathbb{C}}(\text{ch}_X^H(b([E])) \otimes [\phi] \otimes \rho), (\text{pr}_2)_*\text{ch}(f_*[D]) \rangle = \langle \text{ch}_X^H(E), (\text{pr}_2)_*\text{ch}(f_*[D]) \otimes [\phi])_\rho \rangle.$$
Note that \( \text{ch}_{\hat{H}}^X((\text{pr}_2)_* \text{ch}(\bar{f}_*[\hat{D}]) \otimes [\phi]) = T(f_*[D] \otimes [\phi]) \). Therefore we have shown
\[
\text{index}_\rho([E] \cap \text{ch}_{\hat{H}}^X((\text{pr}_2)_* \text{ch}(\bar{f}_*[\hat{D}]) \otimes [\phi]))) = \langle \text{ch}_{\hat{H}}^X(b([E])), \text{ch}(\bar{f}_*[\hat{D}] \otimes [\phi]) \rangle.
\]
Since the classes \((\text{pr}_2)_* \text{ch}(\bar{f}_*[\hat{D}] \otimes [\phi])\) for varying data \(M, F, f, \phi\) span \(H_{ev}(Z_G(H) \setminus X^H, \mathbb{C}) \otimes R_C(H)\) the theorem follows. \(\square\)

### 4.2 Decomposition

**Lemma 4.2** Let \(X\) be a finite proper \(G\)-CW-complex. If \(x \in K^G_0(X)_\mathbb{C}\) and index([\(E\) \cap \(x\)]) = 0 for all \(G\)-equivariant complex vector bundles \(E\) on \(X\), then \(x = 0\).

**Proof.** Because of the isomorphism (4) it suffices to show that if \(A \in H_{ev}(Z_G(C) \setminus X^C, \mathbb{C}) \otimes \mathbb{C}(\text{gen}(C))\) and index([\(E\) \cap \(\text{ch}_C^X(A)\)]) = 0 for all \(E\), then \(A = 0\). By Theorem 4.1 we have index([\(E\) \cap \(\text{ch}_C^X(A)\)]) = \(\langle \text{ch}_C^X(b(\{E\})), A \rangle\). Using the surjectivity of \(b\) and of the isomorphism (3), and the fact that the pairing
\[
\langle \ldots \rangle : (H_{ev}(Z_G(C) \setminus X^C, \mathbb{C}) \otimes \mathbb{C}(\text{gen}(C)))^{W_G(C)} \otimes (H_{ev}(Z_G(C) \setminus X^C, \mathbb{C}) \otimes \mathbb{C}(\text{gen}(C)))^{W_G(C)} \to \mathbb{C}(\text{gen}(C))
\]
is nondegenerate we see that \(\langle \text{ch}_C^X(b(\{E\})), A \rangle = 0\) for all \(E\) indeed implies \(A = 0\). \(\square\)

Let now \(M\) be an even-dimensional proper cocompact \(G\)-manifold equipped with a \(G\)-invariant Riemannian metric \(g^M\) and a \(G\)-equivariant Dirac bundle \(F\) with associated Dirac operator \(D\). Let \([D]_\mathbb{C} \in K^G_0(M)_\mathbb{C}\) be the equivariant \(K\)-homology class of \(D\).

The \(G\)-space \(M\) has the \(G\)-homotopy type of a finite proper \(G\)-CW-complex. In particular, we have the isomorphism (4)
\[
\bigoplus_{(C) \in \mathcal{C}_\mathcal{F}_{\text{Cyc}}(G)} \text{ch}_C^M(1 \otimes r^*) : \bigoplus_{(C) \in \mathcal{C}_\mathcal{F}_{\text{Cyc}}(G)} H_{ev}(Z_G(C) \setminus M^C, \mathbb{C}) \otimes \mathbb{C}(\text{gen}(C)) \to K^G_0(M)_\mathbb{C}.
\]

Therefore, there exist uniquely determined classes \([D](C) \in H_{ev}(Z_G(C) \setminus M^C, \mathbb{C}) \otimes \mathbb{C}(\text{gen}(C))\) such that
\[
\sum_{(C) \in \mathcal{C}_\mathcal{F}_{\text{Cyc}}(G)} \text{ch}_C^M(1 \otimes r^*)([D](C)) = [D]_\mathbb{C}.
\]

**Theorem 4.3** We have the equality
\[
[D](C) = [\hat{U}],
\]
where \([\hat{U}]\) is given by \(\text{gen}(C) \ni g \to [\hat{U}(g)] \in H_{ev}(Z_G(C) \setminus M^C, \mathbb{C})\), and \(\hat{U}(g)\) was defined in (4).
Proof. Let $E$ be any $G$-equivariant complex vector bundle over $M$. Then we have

$$\text{index}([E] \cap [D]_C) = \sum_{(C) \in CFCyc(G)} \langle ch^C_M(b(\{E\})), [D](C) \rangle.$$ 

Using the definition of $\epsilon_C$ and Proposition 3.2 we can write out the summands of right-hand side as follows

$$\langle ch^C_M(b(\{E\})), [D](C) \rangle = \frac{1}{|C|} \sum_{g \in \text{gen}(C)} \langle [ch(g, E)], [D](C)(g) \rangle.$$ 

On the other hand the index formula Theorem 2.7 gives

$$\text{index}([E] \cap [D]_C) = \sum_{(C) \in CFCyc(G)} \frac{1}{|C|} \sum_{g \in \text{gen}(C)} \langle [ch(g, E)], [\hat{U}](g) \rangle.$$ 

Varying $E$ and using Lemma 4.2 we conclude $[D](C) = [\hat{U}]$. \quad \square

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