1. Introduction

Fractional calculus (FC), including integration and differentiation of arbitrary non-integer order, is the generalization of classical integration and differentiation (Oldham & Spanier, 1974). The beauty of FC is that fractional order derivatives and integrals are non-local. The purpose of using fractional models in differential equations is computationally accurate and may be used to obtain and investigate solutions to time fractional partial differential equations.

The beauty of FC is that fractional differentiation of arbitrary non-integer order, is the generalization of classical integration and differentiation (Oldham & Spanier, 1974). The numerical and graphical solutions achieved by the proposed method show that it is computationally accurate and may be used to obtain and investigate solutions to time fractional partial differential equations.
a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation. Eventually at \( q = 1/n \), the system takes the original form of the equation and the final stage of the deformation gives the desired solution.

The Shehu transform (ST) is a generalization of the Laplace and the Sumudu integral transform (Watugala, 1998). Besides, the proposed integral transform is similar to natural transform (Khan & Khan, 2008). The ST becomes Laplace's transform (Spiegel, 1965), when the variable \( \mu = 1 \), and becomes the Yang’s integral transform (Yang, 2016) when the variable \( s = 1 \).

In this study, we used q-homotopy analysis Shehu transform method (q-HASTM) to gain the analytical solution of system (1). The proposed scheme, namely q-HASTM, is an elegant amalgamation of q-HAM and ST. Its superiority is its ability to adjust two strong computational methodologies for probing FDEs. By choosing proper \( h \), we can control the convergence region of solution series in a large permissible domain. The advantage of q-HASTM in that it does not require linearization or discretization, shows little perturbations, has no restrictive assumptions, lessens mathematical computations significantly, offers non-local effect, promises a big solution series in a large permissible domain. The fractional Caputo derivative of the fractional order function \( g \) takes the original form of the equation and the final stage of the deformation gives the desired solution.

Consider a time fractional partial differential system

\[
\begin{cases}
D^\zeta_\tau g(t) = \phi(t, g(t), g(a_1 g(t), b_1 t)), \\
g(0, t) = h(t).
\end{cases}
\]

(5)

where \( 0 < \zeta \leq 1 \), \( D^\zeta_\tau g(t) \) presents the Caputo derivative of \( g(t) \), and \( \phi \) denotes the differential operator.

Taking the ST to both sides of Equation (5) and on simplifying, we get

\[
\begin{align*}
S[D^\zeta_\tau g(t)] & = \left( \frac{S}{s} \right)^\zeta S[\phi(t, g(t), g(a_1 g(t), b_1 t))], \\
S[g(0, t)] & = \frac{S}{s} S[\phi(t, g(t), g(a_1 g(t), b_1 t))].
\end{align*}
\]

(6)

Now, we define a non-linear operator as

\[
N'[\tilde{\eta}(t) = \tilde{S}[\tilde{\eta}(t) = q]] = \left( 1 - [a_1 t]^q \right) S[\phi(t, \tilde{\eta}(t), \tilde{\eta}(a_1 \tilde{\eta}(t), b_1 t))].
\]

(7)

In Equation (7), \( q \in [0, 1] \) is an embedding parameter and \( \tilde{\eta}(t) = \tilde{S}[\tilde{\eta}(t) = q] \) is the real function of \( \eta, \tau \) and \( q \). Liao (1992, 1995) constructed zeroth-order deformation equation such as

\[
(1 - q)S[\tilde{\eta}(t) = \tilde{\eta}(t)] = \tilde{h}(\tilde{\eta}, \tau) q S[\tilde{\eta}(t) = q],
\]

(8)

where \( S \) represents the ST, \( h \) is nonzero auxiliary parameter, \( H(\eta, \tau \neq 0) \) denoted an auxiliary function, and \( \nu(\eta, 0) \) expresses the initial gauss of \( \nu(\eta, \tau) \), and
\( \Upsilon(\eta, \tau; q) \) is unknown function. Let \( q = 0 \) and \( q = 1 \) in Equation (8), we get
\[
\Upsilon(\eta, \tau; 0) = \nu_0(\eta, \tau), \quad \Upsilon(\eta, \tau; \frac{1}{n}) = \nu(\eta, \tau).
\] (9)

Thus, if \( q \) rises from 0 to \( \frac{1}{2} \), the series solution \( \Upsilon(\eta, \tau; q) \) varies from the initial guess \( \nu_0(\eta, \tau) \) to the solution \( \nu(\eta, \tau) \). Upon expanding \( \Upsilon(\eta, \tau; q) \) with the help of Taylor’s series near to \( q \), we have
\[
\Upsilon(\eta, \tau; q) = \nu_0(\eta, \tau) + \sum_{p=1}^{\infty} \nu_p(\eta, \tau) q^p,
\] (10)

where
\[
\nu_p(\eta, \tau) = \frac{1}{p!} \frac{\partial^p \Upsilon(\eta, \tau; q)}{\partial q^p} \bigg|_{q=0}.
\] (11)

By proper choosing of \( \nu_0(\eta, \tau) \), \( h \), and \( H(\eta, \tau) \) the series in Equation (10) converges at \( q = \frac{1}{n} \), we will get
\[
\nu(\eta, \tau) = \nu_0(\eta, \tau) + \sum_{p=1}^{\infty} \nu_p(\eta, \tau) \left(\frac{1}{n}\right)^p.
\] (12)

We define the vector \( \vec{\nu}_p(\eta, \tau) \) as
\[
\vec{\nu}_p(\eta, \tau) = \{\nu_0(\eta, \tau), \nu_1(\eta, \tau), ..., \nu_p(\eta, \tau)\}.
\] (13)

First, differentiating Equation (8) \( p \)-times with respect to \( q \), then evaluate at \( q = 0 \) and finally dividing by \( \Gamma(p+1) \), we have the so-called \( p \)-th-order deformation equation
\[
S[\nu_p(\eta, \tau) - R_p \nu_{p-1}(\eta, \tau)] = hH(\eta, \tau) \mathcal{R}_p \left[ \vec{\nu}_{p-1}(\eta, \tau) \right].
\] (14)

where
\[
\mathcal{R}_p \left[ \vec{\nu}_{p-1}(\eta, \tau) \right] = \frac{1}{(p-1)!} \frac{\partial^{p-1} \Upsilon(\eta, \tau; q)}{\partial q^{p-1}} \bigg|_{q=0}
\] (15)

and
\[
\chi_p = \begin{cases} 0, & p \leq 1 \\ n, & \text{otherwise}
\end{cases}
\] (16)

Taking the inverse ST to both sides of Equation (14) and with the aid of Equations (8) and (15), we get
\[
\nu_p(\eta, \tau) = \chi_p \nu_{p-1}(\eta, \tau)
\]
\[
+ S^{-1} \left[ h H(\eta, \tau) \mathcal{R}_p \left[ \vec{\nu}_{p-1}(\eta, \tau) \right] \right].
\] (17)

Based on Equation (5), \( \mathcal{R}_p \left[ \vec{\nu}_{p-1}(\eta, \tau) \right] \) is defined as
\[
\mathcal{R}_p \left[ \vec{\nu}_{p-1}(\eta, \tau) \right] = S[\nu_{p-1}(\eta, \tau)] - \frac{\mu}{S} \nu_0(\eta, \tau)(1 - \frac{\chi_p}{n})
\]
\[
- \left(\frac{\mu}{S}\right) S \left[ \phi(\eta, \nu_{p-1}(a_0 \eta, b_0 \tau)), \frac{\partial (\nu_{p-1}(a_0 \eta, b_1 \tau))}{\partial \eta}, \ldots, \frac{\partial^p (\nu_{p-1}(a_p \eta, b_p \tau))}{\partial \eta^p} \right],
\] (18)

Finally, we compute \( \nu_p(\eta, \tau) \) by using Equation (17) for \( p \geq 1 \). Hence the \( M \)-th order approximate solution of Equation (5) can be represented as
\[
\nu(\eta, \tau) = \sum_{p=0}^{M} \nu_p(\eta, \tau) \left(\frac{1}{n}\right)^p.
\] (19)

Moreover, for \( M \to \infty \), we get
\[
\nu(\eta, \tau) = \sum_{p=0}^{\infty} \nu_p(\eta, \tau) \left(\frac{1}{n}\right)^p.
\] (20)

The existence of the factor \( \left(\frac{1}{n}\right)^p \) in the q-HASTM solution (20) allow for faster convergence than the standard HAM. Moreover, in the special case \( n = 1 \), the q-HASTM reduces to the standard homotopy analysis Shehu transform method (HASTM).

4. Convergence analysis

In this section, we investigate the convergence analysis of q-HASTM technique.

Theorem 4.1. Let \( \Re(\nu) \) satisfy the Lipschitz condition with the Lipschitz constant \( \delta \). The solution derived with the aid of q-HASTM of the time fractional partial differential system (5) is unique, wherever \( 0 < \sigma < 1 \), where \( \sigma = (n + h) + \zeta \).

Proof. The solution of the time fractional partial differential system (5) is presented as
\[
\nu(\eta, \tau) = \sum_{p=0}^{\infty} \nu_p(\eta, \tau) \left(\frac{1}{n}\right)^p,
\] (21)

where
\[
\nu_p(\eta, \tau) = (\chi_p + h) \nu_{p-1}(\eta, \tau) - \left(1 - \frac{\chi_p}{n}\right) S^{-1} \left[ \left(\frac{\mu}{S}\right) \nu_0(\eta, \tau) - h S^{-1} \left[ \left(\frac{\mu}{S}\right) S \left[ \phi(\eta, \nu_{p-1}(a_0 \eta, b_0 \tau)), \ldots, \frac{\partial^p (\nu_{p-1}(a_p \eta, b_p \tau))}{\partial \eta^p} \right] \right] \right].
\] (22)

Now, let \( \nu \) and \( \nu^* \) be two different solutions of considered time fractional partial differential system, then we have
\[
|\nu - \nu^*| = |(n + h)(\nu - \nu^*)|
\]
\[
+ h S^{-1} \left[ \left(\frac{\mu}{S}\right) S(\Re(\nu - \nu^*)) \right].
\] (23)

With the aid of the convolution theorem, we can obtain
\[
|\nu - \nu^*| \leq (n + h)|\nu - \nu^*| + h \int_{0}^{\tau} |(\Re(\nu - \nu^*))| dt \leq (n + h)|\nu - \nu^*| + h \int_{0}^{\tau} \left[ \left(\frac{\delta}{\zeta + 1}\right) d\zeta \right].
\] (24)
Next, putting up the integral mean value theorem in use, it yields
\[ |\nu - \nu'| \leq (n + h)|\nu - \nu'| + h\left(\delta(\nu - \nu')\right)T, \]
\[ \leq |\nu - \nu'|/\sigma. \]  
(25)

It gives \((1 - \sigma)|\nu - \nu'| \leq 0\). Because \(0 < \sigma < 1\); therefore, \(\nu - \nu' \geq 0\), which implies that \(\nu = \nu'\). Hence the solution is unique.

**Theorem 4.2** (Convergence theorem). Let us consider that \(X\) be a Banach space and there is a nonlinear mapping \(B: X \to X\) and assume that
\[ \|W(\nu) - W(r)\| \leq \sigma\|\nu - r\|, \quad \forall \nu, r \in X. \]  
(26)

Then in view of Banach’s fixed point theory, \(W\) has a fixed point. Furthermore, the sequence generated by the \(q\)-HASTM with an arbitrary selection of \(\nu_0\), \(f_0 \in X\) converges to the fixed point of \(W\) and
\[ \|\nu_m - \nu_n\| \leq \sigma^{n-1}\|\nu_1 - \nu_0\|, \quad \forall \nu, r \in X. \]  
(27)

**Proof.** Let us take a Banach space \((C[1], \|\cdot\|)\) of all continuous functions on \(I\) with the norm expressed as \(\|g(t)\| = \max_{t \in I}|g(t)|\).

Now, we show that the sequence \(\{\nu_n\}\) is a Cauchy sequence in the Banach space.

\[ \|\nu_m - \nu_n\| = \max_{t \in I}|\nu_m(t) - \nu_n(t)| \]
\[ = \max_{t \in I}|(n + h)(\nu_{m+1} - \nu_{n-1}) + h\left(\delta(\nu_{m+1} - \nu_{n-1})\right)\]
\[ \leq \max_{t \in I}|(n + h)|\nu_{m+1} - \nu_{n-1}| + h\left(\delta(\nu_{m+1} - \nu_{n-1})\right)|. \]

Now, making use of the convolution theorem for the ST, it gives
\[ \|\nu_m - \nu_n\| \leq \max_{t \in I}|(n + h)|\nu_{m+1} - \nu_{n-1}| + h\left(\delta(\nu_{m+1} - \nu_{n-1})\right)| \]
\[ \leq \max_{t \in I}|(n + h)|\nu_{m+1} - \nu_{n-1}| + h\left(\delta(\nu_{m+1} - \nu_{n-1})\right)| \]
\[ \leq \sigma^{n-1}\|\nu_1 - \nu_0\|. \]  
(28)

Next, by the application of the integral mean value theorem (Maitama & Zhao, 2019a), we obtain
\[ \|\nu_m - \nu_n\| \leq \max_{t \in I}|(n + h)|\nu_{m+1} - \nu_{n-1}| + h\left(\delta(\nu_{m+1} - \nu_{n-1})\right)| \]
\[ \leq \sigma^{n}\|\nu_1 - \nu_0\|. \]
Let \(m = n + 1\), then we have
\[ \|\nu_{n+1} - \nu_n\| \leq \sigma^{n}\|\nu_n - \nu_{n-1}\| \leq \sigma^{n}\|\nu_1 - \nu_0\|. \]  
(29)

On using the triangular inequality, it yields
\[ \|\nu_m - \nu_n\| \leq \|\nu_{m+1} - \nu_n\| + \|\nu_{n+1} - \nu_{m+1}\| + \cdots + \|\nu_n - \nu_{n-1}\| \]
\[ \leq \sigma^n + \sigma^{n+1} + \cdots + \sigma^{m-1}\|\nu_1 - \nu_0\| \]
\[ \leq \sigma^n[1 + \sigma + \sigma^2 + \cdots + \sigma^{m-n-1}]\|\nu_1 - \nu_0\| \]
\[ \leq \sigma^n \left(1 - \frac{\sigma^{m-n-1}}{1 - \sigma} \right)\|\nu_1 - \nu_0\|. \]

Because \(0 < \sigma < 1\), so \(1 - \sigma^{m-n-1} < 1\), then we have
\[ \|\nu_m - \nu_n\| \leq \sigma^n \|\nu_1 - \nu_0\|. \]  
(30)

But \(\|\nu_1 - \nu_0\| < \infty\), so as \(m \to \infty\) then \(\|\nu_m - \nu_n\| \to 0\). Therefore, the sequence \(\{\nu_n\}\) is Cauchy sequence in \(C[1]\), and so the sequence is convergent.

### 5. Numerical problem

In this section, we consider two numerical problems to prove the accuracy, and efficiency of our proposed method. All the numerical and graphical results for the following two problems are calculated by utilizing the software scilab-6.0.2.

**Problem 1.** Consider the time-fractional generalized Burger’s equation (Sakar et al., 2016) as
\[ _\zeta D^\zeta \nu(\eta, \tau) = \nu \left(\frac{\eta}{2}\right) \frac{\partial}{\partial \eta} \nu \left(\frac{\eta}{2}\right) + \frac{\partial^2}{\partial \eta^2} \nu(\eta, \tau) \]
\[ \quad + \frac{1}{2} \nu(\eta, \tau), \]  
(31)

where \(0 < \zeta \leq 1\), \(_\zeta D^\zeta \nu(\eta, \tau)\) presents the Caputo derivative of \(\nu(\eta, \tau)\) and subject to initial condition
\[ \nu(\eta, 0) = \eta. \]  
(32)

By performing the ST on both sides of Equation (31) and with the help of Equation (32), we get
\[ S[\nu(\eta, \tau)] - \frac{1}{s} S(\nu(\eta, 0)) - \left(\frac{\mu}{s}\right) \zeta S\left[\nu \left(\frac{\eta}{2}\right) \frac{\partial \nu(\eta, \tau)}{\partial \eta} \right. \]
\[ \quad \left. + \frac{\partial^2 \nu(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu(\eta, \tau) \right] = 0. \]  
(33)

According to proposed scheme, we define the non-linear operator as
\[ N[Y(\eta, \tau; q)] = S[Y(\eta, \tau; q)] - \frac{\mu}{s} Y(\eta, \tau; q) \]
\[ S\left[Y \left(\frac{\eta}{2}\right) \frac{\partial Y(\eta, \tau; q)}{\partial \eta} + \frac{\partial^2 Y(\eta, \tau; q)}{\partial \eta^2} + \frac{1}{2} Y(\eta, \tau; q) \right]. \]  
(34)

Form Equation (14) and choosing \(H(\eta, \tau) = 1\), the \(p\)th order deformation equation is given as
\[ S[\nu_p(\eta, \tau) - \frac{1}{s} \nu_{p-1}(\eta, \tau)] = H(\eta, \tau; q) \]
\[ S[\nu_p(\eta, \tau) - \frac{1}{s} \nu_{p-1}(\eta, \tau)] = hR_{\sigma} \left[\nu_{p-1} \right]. \]  
(35)
and by using of Equations (15) and (34) the value of $\mathcal{R}_p[\nu_{p-1}]$ is given as

$$
\mathcal{R}_p[\nu_{p-1}(\eta, \tau)] = S[\nu_{p-1}(\eta, \tau)] - \left(1 - \frac{\lambda_p}{n}\right) \left(\frac{\mu}{s}\right) \eta^{-\zeta} \left[ \sum_{k=0}^{p-1} \frac{\partial^k \nu_{p-k}(\eta, \tau)}{\partial \eta^k} \nu_{p-1-k}(\frac{\eta}{2}, \frac{\tau}{2}) \right] + \frac{\partial^2 \nu_{p-1}(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu_{p-1}(\eta, \tau) \right].
$$

(36)

Operating the inverse ST to Equation (35) and by using Equation (36), we get

$$
\nu_p(\eta, \tau) = (\lambda_p + h)\nu_{p-1}(\eta, \tau) - \left(1 - \frac{\lambda_p}{n}\right) h \eta^{-\zeta} \left[ \sum_{k=0}^{p-1} \frac{\partial^k \nu_{p-k}(\eta, \tau)}{\partial \eta^k} \nu_{p-1-k}(\frac{\eta}{2}, \frac{\tau}{2}) \right] + \frac{\partial^2 \nu_{p-1}(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu_{p-1}(\eta, \tau) \right].
$$

(37)

By putting $p = 1, 2, 3$ in Equation (37) and with Equation (32), we obtain the following results

$$
\nu_1(\eta, \tau) = -h \zeta^{-1} \left[ \left(\frac{\mu}{s}\right) \eta^{-\zeta} \left[ \sum_{k=0}^{p-1} \frac{\partial^k \nu_{p-k}(\eta, \tau)}{\partial \eta^k} \nu_{p-1-k}(\frac{\eta}{2}, \frac{\tau}{2}) \right] + \frac{\partial^2 \nu_{p-1}(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu_{p-1}(\eta, \tau) \right],
$$

(38)

$$
\nu_2(\eta, \tau) = (n + h)\nu_1(\eta, \tau) - h \zeta^{-1} \left[ \left(\frac{\mu}{s}\right) \eta^{-\zeta} \left[ \sum_{k=0}^{p-1} \frac{\partial^k \nu_{p-k}(\eta, \tau)}{\partial \eta^k} \nu_{p-1-k}(\frac{\eta}{2}, \frac{\tau}{2}) \right] + \frac{\partial^2 \nu_{p-1}(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu_{p-1}(\eta, \tau) \right],
$$

$$
\nu_2(\eta, \tau) = -(n + h)h \zeta^{-1} \left[ \left(\frac{\mu}{s}\right) \eta^{-\zeta} \left[ \sum_{k=0}^{p-1} \frac{\partial^k \nu_{p-k}(\eta, \tau)}{\partial \eta^k} \nu_{p-1-k}(\frac{\eta}{2}, \frac{\tau}{2}) \right] + \frac{\partial^2 \nu_{p-1}(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu_{p-1}(\eta, \tau) \right].
$$

(39)

In the same manner, we can get

$$
\nu_3(\eta, \tau) = -(n + h)^2h \zeta^{-1} \left[ \left(\frac{\mu}{s}\right) \eta^{-\zeta} \left[ \sum_{k=0}^{p-1} \frac{\partial^k \nu_{p-k}(\eta, \tau)}{\partial \eta^k} \nu_{p-1-k}(\frac{\eta}{2}, \frac{\tau}{2}) \right] + \frac{\partial^2 \nu_{p-1}(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu_{p-1}(\eta, \tau) \right],
$$

(40)

$$
\nu_4(\eta, \tau) = -(n + h)^3h \zeta^{-1} \left[ \left(\frac{\mu}{s}\right) \eta^{-\zeta} \left[ \sum_{k=0}^{p-1} \frac{\partial^k \nu_{p-k}(\eta, \tau)}{\partial \eta^k} \nu_{p-1-k}(\frac{\eta}{2}, \frac{\tau}{2}) \right] + \frac{\partial^2 \nu_{p-1}(\eta, \tau)}{\partial \eta^2} + \frac{1}{2} \nu_{p-1}(\eta, \tau) \right],
$$

(41)
Hence, the fourth order approximate solution of (31) is given as

\[
\nu(\eta, \tau) = \eta - \frac{\eta h}{\Gamma(\zeta + 1)} (n + h) \eta + \frac{\eta h^2}{\Gamma(\zeta + 1)} (2^{1-\zeta} + 2^{-1}) h^2 \eta + \frac{\eta h^3}{\Gamma(\zeta + 1)} (2^{3-3\zeta})
\]

\[
- (n + h) \frac{\eta h^2}{\Gamma(\zeta + 1)} (2^{1-\zeta} + 1)(n + h) \eta^2 \frac{\eta h^3}{\Gamma(2\zeta + 1)} - \left(2^{3-3\zeta}\right)
\]

\[
+ 2^{1-2\zeta} + 2^{1-\zeta} + 4^{-1} + \frac{\Gamma(2\zeta + 1)}{(\Gamma(\zeta + 1))^2} h^3 \eta \frac{\eta h^3}{\Gamma(3\zeta + 1)}
\]

In particular, when we take \(\zeta = 1, n = 1, \) and \(h = -1,\) then the solution converge to the exact solution of (31) very fastly

\[
\nu(\eta, \tau) = \eta e^\tau.
\]

**Problem 2.** Consider the TFPDEs as given in Sakar et al. (2016) and Singh and Kumar (2018) with proportional delay

\[
\epsilon D^\zeta \nu(\eta, \tau) = \nu \left(\eta, \frac{\tau}{2}\right) \frac{\partial^2}{\partial \eta^2} \nu \left(\eta, \frac{\tau}{2}\right) - \nu(\eta, \tau),
\] (44)

subject to initial condition

\[
\nu(\eta, 0) = \eta^2.
\] (45)

Applying ST to both sides of Equation (44) and on, we get

\[
S[\nu(\eta, \tau)] - \frac{\mu}{s} (\nu(\eta, 0)) - \left(\frac{\mu}{s}\right)^{\zeta} S \left[\nu \left(\eta, \frac{\tau}{2}\right) \frac{\partial^2}{\partial \eta^2} \nu \left(\eta, \frac{\tau}{2}\right) - \nu(\eta, \tau)\right] = 0.
\] (46)

According to proposed technique, the nonlinear operator decomposed as following

\[
N[Y(\eta, \tau; q)] = S[Y(\eta, \tau; q)] - \frac{\mu}{s} (\eta^2) - \left(\frac{\mu}{s}\right)^{\zeta} \left[Y \left(\eta, \frac{\tau}{2}; q\right) \frac{\partial^2}{\partial \eta^2} Y \left(\eta, \frac{\tau}{2}; q\right) - Y(\eta, \tau; q)\right]
\] (47)

Form Equation (14) and choosing \(H(\eta, \tau) = 1,\) the \(p^{th}\) order deformation equation is given as

\[
S[\nu_p(\eta, \tau) - \chi_p \nu_{p-1}(\eta, \tau)] = h\mathfrak{R}_p[\nu_{p-1}],
\] (48)

and by using of Equations (15) and (47) the value of \(\mathfrak{R}_p[\nu_{p-1}]\) is given as

\[
\mathfrak{R}_p[\nu_{p-1}(\eta, \tau)] = S[\nu_{p-1}(\eta, \tau)] - \left(1 - \frac{\mu}{n \cdot s}\right) \frac{\mu}{s} (\eta^2)
\]

\[
- \left(\frac{\mu}{s}\right)^{\zeta} \sum_{k=0}^{p-1} \nu_k \left(\eta, \frac{\tau}{2}\right) \frac{\partial^2}{\partial \eta^2} \nu_{p-1-k} \left(\eta, \frac{\tau}{2}\right) - \nu_{p-1}(\eta, \tau).
\] (49)
Operating the inverse ST to Equation (48) and by using Equation (49), we get

\[
\nu_p(\eta, \tau) = \left( \frac{\lambda_p}{\tau} + h \right) \nu_{p-1}(\eta, \tau) - \left( 1 - \frac{\lambda_p}{\tau} \right) h \eta^2 - h^5 \frac{\tau}{2} \left[ \frac{\mu s}{S} \sum_{k=0}^{n-1} \frac{\partial \nu_{p-k}(\eta, \tau)}{\partial \eta} \nu_{p-1-k}(\eta, \tau) \right] - \nu_{p-1}(\eta, \tau) \]  

(50)

By putting \( p = 1, 2, 3 \) in Equation (50) and with Equation (45), we obtain the following results

\[
\nu_1(\eta, \tau) = -h \eta^2 \left( \frac{\mu s}{S} \sum_{k=0}^{n-1} \frac{\partial \nu_{p-k}(\eta, \tau)}{\partial \eta} \nu_{p-1-k}(\eta, \tau) \right) - \nu_0(\eta, \tau),
\]

(51)

\[
\nu_2(\eta, \tau) = (n + h) \nu_1(\eta, \tau) - h \eta^2 \left( \frac{\mu s}{S} \sum_{k=0}^{n-1} \frac{\partial \nu_{p-k}(\eta, \tau)}{\partial \eta} \nu_{p-1-k}(\eta, \tau) \right) + \frac{\partial \nu_1(\eta, \tau)}{\partial \eta} \nu_0(\eta, \tau) - \nu_1(\eta, \tau),
\]

(52)

In the same manner, we can obtain

\[
\nu_3(\eta, \tau) = -(n + h)^2 h \eta^2 \frac{\tau^2}{\Gamma(\zeta + 1)} + (2^{3-\zeta} - 2)(n + h)h^2 \eta^2 \frac{\tau^2}{\Gamma(\zeta + 1)} + (2^{2-2\zeta} + 2^{2-\zeta}) - (n + h)^3 h^2 \eta^2 \frac{\tau^2}{\Gamma(\zeta + 1)},
\]

(53)

\[
\nu_4(\eta, \tau) = -(n + h)^3 h \eta^2 \frac{\tau^2}{\Gamma(\zeta + 1)} + (2^{3-\zeta} - 3)(n + h)^2 h^2 \eta^2 \frac{\tau^2}{\Gamma(\zeta + 1)} + (2^{2-2\zeta} + 2^{2-\zeta} + 2^{3-2\zeta} - 2^{4-3\zeta} - 3 - (2^{1-2\zeta} + 2^{2-3\zeta})(2^{2-3\zeta} - 2^{4-3\zeta} - 2^{4-4\zeta} - 2^{4-5\zeta} - 1) + 2^{6-6\zeta} + (2^{3-5\zeta} - 2^{1-2\zeta})(2^{2-3\zeta} - 2^{4-4\zeta}))(2^{2-3\zeta} - 2^{4-4\zeta})) \frac{\tau^2}{\Gamma(\zeta + 1)}.
\]

(54)
Similarly, we obtain next terms in the same manner. Hence, we get the fourth order approximate solution of (44) as follow:

\[
\begin{align*}
C_23 (g, s) &= \frac{g^2}{C_0 C_22} h g^2 s f \left( C (f + 1) \right) / C_0 \left( n + C_22 h \right) / C_22 h g^2 s f \left( C (f + 1) \right) / C_0 \left( n + C_22 h \right) / C_22 h^2 g^2 s f \left( C (2f + 1) \right) + \left( \frac{23}{C_0} n - \frac{22}{C_0} \right) \left( n + C_22 h / C_0 \right) / C_22 h^3 g^2 s f \left( C (3f + 1) \right)
\end{align*}
\]

Figure 1. Comparison of the approximate solution of Problem 1 for distinct values \( \zeta \) w.r.t. the exact solution and absolute error \( |\nu_{\text{exa}} (\eta, \tau) - \nu_{\text{app}} (\eta, \tau)| \) with \( n = 1, \zeta = 1, \) and \( h = -1. \)

Similarly, we obtain next terms in the same manner. Hence, we get the fourth order approximate solution of (44) as follow:

\[
\begin{align*}
\nu (\eta, \tau) &= \eta^2 - h \eta^2 \frac{\tau^\zeta}{\Gamma (\zeta + 1)} - (n + h) h \eta^2 \frac{\tau^\zeta}{\Gamma (\zeta + 1)} - (1 - 2^{2-\zeta}) h^2 \eta^2 \frac{\tau^{2\zeta}}{\Gamma (2\zeta + 1)} \\
&\quad - (n + h)^2 h^2 \frac{\tau^\zeta}{\Gamma (\zeta + 1)} + (2^{2-\zeta} - 2)(n + h) h^2 \eta^2 \frac{\tau^{2\zeta}}{\Gamma (2\zeta + 1)} \\
&\quad + (2^{2-\zeta} + 2^{2-\zeta} - 2^{4-\zeta} - 1 - 2^{1-\zeta}) \frac{\Gamma (2\zeta + 1)}{\Gamma (\zeta + 1)} h^3 \eta^2 \frac{\tau^{3\zeta}}{(3\zeta + 1)}
\end{align*}
\]
In particular, when we take $f = 1$, $n = 1$, and $\eta = 1$, then the solution converge to the exact solution of (44) very fastly

$$\nu(\eta, \tau) = \eta^2 e^\tau.$$  \hspace{1cm} (56)

### 6. Results and discussions

In the present study, we can analyzed from Tables 1 and 2 that the numerical results gained by the proposed technique almost same to the schemes...
Figure 4. The $h$-curves of $v(\eta, \tau)$ for distinct values $n$ with $\eta = 0.1$, and $\tau = 0.5$ for Problem 1.

Figure 5. Comparison of approximate solution of Problem 2 for distinct values $\zeta$ w.r.t. the exact solution and absolute error $|v_{\text{exa.}}(\eta, \tau) - v_{\text{app.}}(\eta, \tau)|$ with $n = 1$, $\zeta = 1$, and $h = -1$. 
presented in the literature (Singh & Kumar, 2018; Wang et al., 2019). Figure 1 depicts the comparative analysis between approximate solution for distinct $\zeta$ and the exact solution. We also present the absolute error for Problem 1 at $n = 1$, $h = -1$, and $\eta = 1$.

Figure 2 represents the nature of obtained solution for (31) with distinct $f$ and compression between approximate solution and exact solution (43). Figure 3 presents the behaviour of approximate solution of (31) with distinct $n$ at $g = 0.1$ and $s = 0.5$, which helps to adjust the convergence region.

Figure 4(a–d) explores the $h$-curves of the q-HASTM solution (31) with distinct values of $n$ at $\eta = 0.1$ and $\tau = 0.5$, which helps to adjust the convergence region.

Moreover, Figure 5 represents the behaviour of approximate solution at distinct $\zeta$ in comparison with the exact solution and the absolute error for Problem 2.

Figure 6. Comparison between the approximate solution at $\zeta \leq 1$ and the exact solution with $h = -1$, $n = 1$, and $\eta = 1$ for Problem 2.

Figure 7. $n$-curves of $\nu(\eta, \tau)$ for distinct values of $\zeta$ with $h = -1$, $\eta = 0.1$, and $\tau = 0.5$ for Problem 2.

Figure 8. The $h$-curves of $\nu(\eta, \tau)$ for distinct values $n$ with $\eta = 0.1$, and $\tau = 0.5$ for Problem 2.
7. Conclusions

In this article, we successfully implemented q-HASTM to find the analytical and numerical solutions of time-fractional partial differential equations (TFPDEs). We obtained the analytical and numerical solutions of two applications of TFPDEs to present the effectiveness and accuracy of proposed scheme. Moreover, the q-homotopy analysis transform method provided the convergent series solution with easily determinable components without using any perturbation, linearization or limiting assumption. If we assume $h = -1$, and $n = 1$ in q-HASTM solution, then the q-HASTM solution presented an excellent agreement with the exact solution of TFPDEs. The numerical and graphical solutions obtained by q-HASTM are presented that the proposed technique is computationally very accurate and attractive technique to obtain and investigate the solutions of time-fractional partial differential equations.

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Authors’ contributions

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