The $q$-AGT–W Relations Via Shuffle Algebras

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Abstract: We construct the action of the $q$-deformed $W$-algebra on its level $r$ representation geometrically, using the moduli space of $U(r)$ instantons on the plane and the double shuffle algebra. We give an explicit LDU decomposition for the action of $W$-algebra currents in the fixed point basis of the level $r$ representation, and prove a relation between the Carlsson–Okounkov Ext operator and intertwiners for the deformed $W$-algebra. We interpret this result as a $q$-deformed version of the AGT–W relations.

1 Introduction

Fix $r \in \mathbb{N}$. The moduli space $\mathcal{M}$ of rank $r$ framed sheaves on $\mathbb{P}^2$ is an algebro-geometric incarnation of the moduli space of $U(r)$ instantons, where the Nekrasov partition function naturally appears. More precisely, it has been known from the work of [10,11,25,34] that the partition function of $5dU(r)^k$-gauge theory with bi-fundamental hypermultiplets $m_1, \ldots, m_k$, in the presence of full $\Omega$-background, is:

$$Z_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \text{Tr} \left( A_{m_1}(x_1) \circ \cdots \circ A_{m_k}(x_k) \bigg| _{u_{k+1}=u^i} \right)$$

where the Ext operator (see (4.2) for the precise geometric definition) is:

$$A_{m_i}(x_i) : K_{u_{i+1}} \longrightarrow K_{u^i}$$

and $K_{u^i}$ denotes the equivariant $K$-theory of the moduli space of rank $r$ sheaves, with equivariant weights encoded in the vector of parameters $u^i = (u^1_i, \ldots, u^r_i)$. The partition function of linear quiver gauge theory can be recovered from the operators $A_m(x)$ and their matrix coefficients, as explained in [7]. This partition function has been studied extensively and from many different points of view, see e.g. [26,35].

The main purpose of the present paper is to mathematically state and prove a connection between the rank $r$ Nekrasov partition function and conformal blocks for the
$q$-$W$-algebra (more commonly called “deformed $W$-algebra”) of type $\mathfrak{gl}_r$. The proof of our main Theorem 1.1 uses two main mathematical tools: expressing $q$-$W$-algebras via shuffle algebras, and performing intersection-theoretic computations with the Ext operator (1.2). We interpret our result as a $q$-deformed version of the well-known AGT–$W$ relations between gauge theory and conformal field theory (these were introduced by Alday, Gaiotto and Tachikawa and extended by Wyllard in the undeformed case, and formulated in the $q$-deformed case by Awata and Yamada, see [4,7,39,40] among other references. The physical literature on the subject is vast, see for example [1,22,36] for other points of view).

The algebra we study is the tensor product of the $q$-$W$-algebra of type $\mathfrak{sl}_r$ [3,14] and a $q$-Heisenberg algebra. By close analogy with loc. cit., we show in Sect. 5 that the defining currents of our $q$-$W$-algebra are “elementary symmetric functions”:

$$W_k(z) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} \exp\left[ b^{i_1}(x) \right] \exp\left[ b^{i_2}\left(\frac{x}{q}\right) \right] \cdots \exp\left[ b^{i_k}\left(\frac{x}{q^{k-1}}\right) \right]$$  \hspace{1cm} (1.3)

in a family of bosonic fields $b^1(z), \ldots, b^r(z)$ which satisfy the commutation relations (5.1). The nice thing about the $\mathfrak{gl}_r$ case is that one can send $r \to \infty$, and the resulting limit can be interpreted as the upper half of the double shuffle algebra (as in Sect. 2).

For fixed $r$, the definition (1.3) implies the following relations:

$$W_0(x) = 1, \quad W_k(x) = 0 \quad \text{for all } k > r$$

and:

$$W_k(x)W_{k'}(y) \cdot f_{kk'}\left(\frac{y}{x}\right) = W_{k'}(y)W_k(x) \cdot f_{kk'}\left(\frac{x}{y}\right)$$

$$= \sum_{i=\max(0,k'-k+1)}^{k'} \delta\left(\frac{y}{xq^i}\right) \left[ W_{k'-i}(x)W_{k+i}(y)f_{k'-i,k+i}\left(\frac{y}{x}\right) \bigg|_{x=y=q^i} \right]$$

$$\theta(\min(i,k-k'+i))$$

$$- \sum_{i=\max(0,k-k'+1)}^{k} \delta\left(\frac{x}{yq^i}\right) \left[ W_{k-i}(y)W_{k'+i}(x)f_{k-i,k'+i}\left(\frac{x}{y}\right) \bigg|_{y=x=q^i} \right]$$

$$\theta(\min(i,k'-k+i))$$  \hspace{1cm} (1.4)

The quantity $\theta(s)$ is defined for all $s \in \mathbb{N}$ in (2.63), while the power series $f_{kk'}(z)$ is defined in (2.64) (note that we always expand it in $|z| \ll 1$). In Proposition 5.7, we will explain how formulas (1.4) differ from those of [3,14].

Our strategy is quite well-known to mathematicians and physicists: to recast the AGT–$W$ relations as a connection between the operator $A_m(x)$ and intertwiners for the $q$-$W$-algebra. This starts with Theorem 3.12 below, which states that for arbitrary $r \in \mathbb{N}$ and generic equivariant parameters $\mathbf{u} = (u_1, \ldots, u_r)$, the $K$-theory group $K_u$ of the moduli space of rank $r$ sheaves is isomorphic to the Verma module of the $q$-$W$-algebra, with highest weight prescribed by the equivariant parameters $\mathbf{u}$. Note that our construction and proof are purely geometric, and do not use the isomorphism between the level $r$ representation and a tensor product of $r$ Fock spaces (which was used e.g. in [2]). This geometric definition is fruitful because it can be extended to moduli of sheaves on other surfaces, see [32,33]: