A note on Nikulin surfaces and their moduli spaces

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Abstract. We study a number of natural linear systems carried by any polarized Nikulin surface of genus $g$. We determine their positivity and establish their Brill-Noether theory. Relying upon recent work of Farkas and Rimányi, we compute the class of some natural effective divisors associated to these linear systems on the moduli space of Nikulin surfaces, in this way obtaining—for any genus $g$—roughly $\sqrt{g}/2$ relations in its tautological ring.

1. Introduction

A polarized Nikulin surface of genus $g \geq 2$ is a smooth polarized K3 surface $(X, L)$, where $L$ is a big and nef line bundle with self-intersection $L^2 = 2g - 2$, such that there exists a set of disjoint smooth rational curves $R_1, \ldots, R_8$ on $X$ satisfying $L \cdot R_i = 0$ for each $i = 1, \ldots, 8$. The divisor class of $R := R_1 + \cdots + R_8$ is divisible by 2 in the Picard group of $X$ and, denoting by $e$ the primitive class satisfying $e \otimes 2 = \mathcal{O}_X(R)$, one can describe the Nikulin lattice $\mathfrak{N}$ as the lattice generated by the classes of $R_1, \ldots, R_8$ and $e$. This yields a primitive embedding

$$\Lambda_g := \mathbb{Z} \cdot [L] \oplus \mathfrak{N} \hookrightarrow \text{Pic}(X),$$

from which one can construct the moduli space $\mathcal{F}_{g}^{\mathfrak{N}}$ of Nikulin surfaces of genus $g$, which is irreducible and 11-dimensional [2, 8].

Nikulin surfaces represent a rather special class of K3 surfaces, which have been studied in relation with various topics, including the theory of automorphisms [14, 7], moduli spaces [13, 8], the study of Prym curves [3] and of the birational geometry of their moduli spaces [5, 6, 11, 17].

Following [7], the general point of $\mathcal{F}_{g}^{\mathfrak{N}}$ corresponds to a Nikulin surface $(X, L)$ with Picard number nine and, if $g \geq 3$, the line bundle $L$ induces a morphism which contracts the 8 curves $R_i$ and maps $X$ to a surface in $\mathbb{P}^g$ with 8 ordinary double points. If $g = 4$ or 5, the line bundle $L \otimes e^{-1}$ is ample, and very ample for $g \geq 6$, mapping $X$ to a surface in $\mathbb{P}^{g-2}$. These linear systems and their interplay provide a source of interesting geometry which has been proven useful for several effective constructions in some of the aforementioned works. A detailed description of these projective models has been carried out for Nikulin surfaces of low genus in [7].

The starting point of this work is a systematic approach for the study of these linear systems on Nikulin surfaces of any genus. Consider the series of line bundles

$$L_m = L \otimes e^{-m}, \quad m = 0, 1, 2, \ldots$$
Assuming $g$ is large enough with respect to $m$, how positive is $L_m$? For the general element of $\mathcal{F}_g^9$, the situation is as good as one could expect.

**Theorem 1.1.** Let $(X, L)$ be a Nikulin surface of genus $g$ with Picard number nine. Write $g = 2k^2 + p$, where $k \geq 1$ and $0 \leq p < 4k + 2$. Denote by $L_m = L \otimes e^{-m}$, $m = 0, 1, \ldots, k$.

Then,

(i) For any $1 \leq m \leq k - 1$ (in particular $g \geq 8$) the general member of the linear system $|L_m|$ is a smooth irreducible curve of genus $g_m = g - 2m^2 \geq 6$.

In fact, $L_m$ is very ample and defines an embedding of $X$ in $\mathbb{P}^{8m}$.

(ii) $|L_k|$ contains a smooth irreducible curve of genus $p$. Moreover, $L_k$ is ample for $p = 2$, and very ample for $p \geq 3$.

An instance of the interesting geometry which can arise by studying these linear systems $|L_m|$ is described by the following elementary fact:

**Proposition 1.2.** Under the assumptions and notations of Theorem 1.1, for any $1 \leq m \leq k - 1$, let $D$ be a smooth irreducible curve in the linear system $|L_{m+1}|$. Then, $D$ is embedded (non-specially) by $L_m$ as a non-degenerate linearly normal curve of degree $2(g_m - 2m - 1)$ in $\mathbb{P}^{8m}$.

A special case in any genus $g = 2k^2 + p$ is given by the curves $D \in |L_k|$, embedded by $L_{k-1}$ as linearly normal curves of genus $p$ in $\mathbb{P}^{4k-2+p}$. For example:

- $p = 0$ ($g = 2k^2$): $D$ is the rational normal curve in $\mathbb{P}^{4k-2}$,
- $p = 1$ ($g = 2k^2 + 1$): $D$ is a normal elliptic curve in $\mathbb{P}^{4k-1}$,
- $p = 1$ ($g = 2k^2 + 2$): $D$ is a normal genus $2$ curve in $\mathbb{P}^{4k}$,

and so on. Let us point out that in genus $g = 8$ (i.e. the case $k = 2$ and $p = 0$), the existence of the smooth rational curve $D \in |L_2|$ embedded by $L_1$ as a rational normal curve of degree 6 in $\mathbb{P}^6$ is the starting point of Verra’s construction [17] for the proof of the rationality of $\mathcal{F}_8^9$.

Returning to the linear systems $|L_m|$, the next natural step is to determine the Brill-Noether theory of the hyperplane sections of $X$ in these projective models.

**Theorem 1.3.** Under the same assumptions and notations of Theorem 1.1, all smooth curves in $|L_m|$, for any $0 \leq m \leq k$, are Brill-Noether general, i.e. have maximal Clifford index $\left\lfloor \frac{g_m - 1}{2} \right\rfloor$.

We note that for $m = 1$ the statement was proved by Farkas and Kemeny in their proof of the Brill-Noether theory of odd genus [3].

In the second part of the paper we use the projective geometry of Nikulin surfaces in order to compute the classes of some natural effective divisors in $\mathcal{F}_g^9$. This was directly inspired by the recent paper of Farkas and Rimányi [4], and relies on some beautiful general formulas therein contained. Specifically, for any index $m = 0, \ldots, k$ (assuming $p \geq 4$ when $m = k$), we consider the virtual divisors
\[ \mathcal{D}_m^{(4)} = \{(X, L) \in \mathcal{F}^g : \gamma_m^{(4)}(2) \neq 0\}, \]

where \( \gamma_m(2) \) is the vector space of quadrics containing the image of the map
\[ \varphi_{L_m} : X \to \mathbb{P}^{8m} \text{ and } \gamma_m^{(4)}(2) = \{q \in \gamma_m(2) : \text{rk}(q) \leq 4\} \] (see §3 below for details). Along the lines of [12, §4], we consider the universal Nikulin surface
\[ \pi : \mathcal{X} \rightarrow \mathcal{F}^g \]

and the class
\[ L \in \text{CH}_1(\mathcal{F}^g) \]

which is independent from this choice, and thus yield well defined classes in \( \text{CH}^1(\mathcal{F}^g) \):
\[ \gamma_0 = \kappa_{3,0,0} - \frac{(g-1)}{4}\kappa_{1,0,1}, \quad \gamma_2 = \kappa_{1,0,1} + 6\kappa_{1,2,0}, \]
\[ \gamma_1 = \kappa_{2,1,0} - \frac{(g-1)}{12}\kappa_{0,1,1}, \quad \gamma_3 = \kappa_{0,1,1} + 2\kappa_{0,3,0}. \]

(cf. §4 below for details). It turns out that the class of \( \mathcal{D}_m^{(4)} \) can be computed in terms of these four classes \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \text{CH}^1(\mathcal{F}^g) \), together with the Hodge class \( \lambda \in \text{CH}^1(\mathcal{F}^g) \). Precisely, we have:

**Theorem 1.4.** Let \( g = 2k^2 + p \), for some integers \( k \geq 1 \) and \( 0 \leq p < 4k + 2 \). If \( m = 0, \ldots, k - 1 \) or \( m = k \) and \( p \geq 4 \), then \( \mathcal{D}_m^{(4)} \) is an effective divisor in \( \mathcal{F}^g \) and
\[ [\mathcal{D}_m^{(4)}] = A_m \left( \frac{2}{g_m + 1} \gamma_0 - \frac{6m}{g_m + 1} \gamma_1 + \frac{m^2}{g_m + 1} \gamma_2 - \frac{m^3}{g_m + 1} \gamma_3 + (2g_m - 1)\lambda \right) \]
in \( \text{CH}^1(\mathcal{F}^g) \), where \( g_m = g - 2m^2 \) and \( A_m \in \mathbb{Q} \).

The proof rests upon a Grothendieck-Riemann-Roch calculation on the universal Nikulin surface of genus \( g \) and the application of general formulas from [4] which allow one to express \([\mathcal{D}_m^{(4)}]\) in terms of Chern classes of vector bundles on \( \mathcal{F}^g \).

Although we do not pursue further divisor class computations in this paper, we remark that, thanks to the validity of Green’s conjecture for smooth curves on K3 surfaces [18, 19, 1] one could use the Brill-Noether properties of the linear systems \( L_m \) established in Theorem 1.3 to obtain further series of relations in the tautological ring of \( \mathcal{F}^g \), for example by means of Koszul divisors and Lazarsfeld-Mukai bundles, as described in [4, §9.2-3].
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Notation and conventions

Nikulin surfaces of genus $g$, as defined in the Introduction, exist in all genera and are sometimes called standard Nikulin surfaces. In odd genus (only) there exist also Nikulin surfaces for which $L \cdot e \neq 0$, i.e. one does not have an orthogonal sum decomposition $\mathbb{Z} \cdot [L] \oplus \mathfrak{N}$. Accordingly, the Néron-Severi lattice of the general such K3 surface is rather an overlattice of index 2 of $\Lambda_g$, cf. [8]. In this paper we are only concerned with standard Nikulin surfaces, whence we will omit this specification throughout. We work over the field of complex numbers.

2. Polarized Nikulin surfaces

We briefly recall some basic facts on Nikulin surfaces and refer to [7] for details. Let $Y$ be a K3 surface carrying a symplectic involution $\iota \in \text{Aut}(Y)$. Then $\iota$ has 8 isolated fixed points and the quotient $Y/\iota$ is a K3 surface with eight ordinary double points. Let $\sigma: \tilde{Y} \to Y$ be the blow-up of $Y$ at the eight fixed points of $\iota$. The involution $\iota$ naturally lifts to an involution on $\tilde{Y}$, fixing the eight exceptional divisors $E_1, \ldots, E_8$. We denote the quotient surface by this involution by $X$ (which turns out to be a K3 surface) and we let $f: \tilde{Y} \to X$ be the quotient map.

By construction, the ramification divisor of $f$ is the sum of the rational curves $R_i := f(E_i)$, and is therefore 2-divisible in $\text{Pic}(X)$. We denote by $e$ the primitive class $e \in \text{Pic}(X)$ satisfying the identity

$$e^{\otimes 2} = \Theta_X(R),$$

where $R = R_1 + \cdots + R_8$. The classes of $R_1, \ldots, R_8$ and $e$ span the Nikulin lattice $\mathfrak{N}$, and one gets in this way a primitive embedding $\mathfrak{N} \hookrightarrow \text{Pic}(X)$.

Definition 2.1. A polarized Nikulin surface of genus $g \geq 2$ consists of a K3 surface $X$ and a primitive embedding $j: \Lambda_g \to \text{Pic}(X)$, where $\Lambda_g := \mathbb{Z} \cdot L \oplus \mathfrak{N},$ such
that $L^2 = 2g - 2$ and $j(L)$ is a big and nef class. With a small abuse of notation, we denote a polarized Nikulin surfaces simply by the pair $(X, L)$.

Polarized Nikulin surfaces of genus $g$ form a moduli space denoted by $\mathcal{F}_g^{31}$, which is known to be irreducible and 11-dimensional [2, 8]. The general point of $\mathcal{F}_g^{31}$ corresponds to a K3 surface $X$ with Picard number nine. More precisely, one has $\text{Pic}(X) \simeq \Lambda_g$, see [7, Prop. 2.1].

For the rest of this section we assume $(X, L)$ to be a polarized Nikulin surface of genus $g$ with Picard number nine and write $g = 2k^2 + p$, where $k \geq 1$ and $0 \leq p < 4k + 2$. We denote by $L_m = L \otimes e^{-m}$.

**Lemma 2.2.** $L_m$ is ample if $1 \leq m \leq k - 1$ or $m = k$ and $p \geq 2$.

**Proof.** Since $L_m$ is effective and $(L_m)^2 > 0$, we have that $L_m$ is ample if and only if $L_m$ intersects positively any smooth rational curve on $X$. Let $D$ be an effective divisor and write $D \sim aL - b_1R_1 - \cdots - b_8R_8$, where $a \in \mathbb{Z}$ and $b_i \in \frac{1}{2}\mathbb{Z}$. Then,

$$D \cdot L_m = 2a(g - 1) - m(\sum_{i=1}^{8} b_i).$$

Assuming $D \cdot L_m \leq 0$, we are going to show that $D^2 < -2$ for $1 \leq m \leq k - 1$. Since $L_m$ intersects positively each $R_i$, we can assume that $D$ does not contain $R_i$ as a component, whence $D \cdot R_i = 2b_i \geq 0$. Hence we must have $a > 0$, else $D$ would not be effective. Thus $D \cdot L_m \leq 0$ yields $2a(g - 1) \leq m(b_1R_1 + \cdots + R_8)$, and we can square both sides of this inequality, obtaining

$$4a^2(g - 1)^2 \leq m^2(\sum_{i=1}^{8} b_i)^2 \leq 8m^2 \sum_{i=1}^{8} b_i^2.$$

where the latter estimate follows by the Cauchy-Schwarz inequality. Let us re-write this condition as

$$(2.1) \quad \frac{a^2(g - 1)^2}{m^2} \leq 2 \sum_{i=1}^{8} b_i^2.$$

Assuming the latter inequality, we can estimate the self-intersection of $D$,

$$D^2 = 2a^2(g - 1) - 2\sum_{i=1}^{8} b_i^2 \leq 2a^2(g - 1) - \frac{a^2(g - 1)^2}{m^2} \quad \text{by (2.1)}$$

$$= a^2\left(\frac{g - 1}{m^2}\right)(1 - g_m).$$
Since \(1 - g_m < 0\), in order for the condition \(D^2 < -2\) to be satisfied, it is enough to ask for the following numerical condition:

\[
a^2 \left( \frac{g - 1}{m^2} \right) (g_m - 1) > 2.
\]

By assumption \(g_m = g - 2m^2 \geq 6\), or equivalently \(m^2 \leq \frac{1}{2}(g - 6)\), and using both these estimates we finally get

\[
a^2 \left( \frac{g - 1}{m^2} \right) (g_m - 1) \geq a^2 \cdot 2 \left( \frac{g - 1}{g - 6} \right) \cdot 5 > 10a^2 > 2.
\]

This proves that \(L_m\) is ample for \(1 \leq m \leq k - 1\). Moreover, we observe that the same argument works for \(m = k\) and \(p \geq 2\), in which case \(g_k = p\) and \(k^2 = \frac{1}{2}(g - p)\), so that we still get

\[
a^2 \left( \frac{g - 1}{k^2} \right) (g_k - 1) = a^2 \cdot 2 \left( \frac{g - 1}{g - p} \right) (p - 1) > 2.
\]

This concludes the proof. \(\square\)

**Lemma 2.3.** \(L_m\) is very ample ample for any \(1 \leq m \leq k - 1\) or \(m = k\) and \(p \geq 3\).

**Proof.** So far we have determined the amplitude of \(L_m\) for any \(1 \leq m \leq k\), with \(p \geq 2\) for \(m = k\). Since each such \(L_m\) is big and nef, by the results of Saint-Donat [16] there can be only two types of obstruction for \(L_m\) to be very ample: either the existence of fixed components, or of hyperelliptic curves in \(|L_m|\). Specifically, we have to exclude the following three possibilities:

(a) There exist an elliptic curve \(E\) and a smooth rational curve \(\Gamma\) on \(X\) such that \(L_m = O_X(rE + \Gamma)\), with \(r \geq 2\) and \(E \cdot \Gamma = 1\).

(b) There exists an elliptic curve \(E\) such that \(L_m \cdot E = 2\).

(c) There exists a curve of genus 2 such that \(L_m = O_X(2B)\).

Notice that (c) could only occur in genus \(g_m = 5\), but in fact it is automatically excluded by the properties of the Néron-Severi lattice of the general Nikulin surface \(X\), since the class \(L_m = L \otimes e^{-m}\) is not \(2\)-divisible in \(\text{NS}(X)\), by [7, Prop. 2.1].

We will proceed by excluding cases (a) and (b) simultaneously, by showing that for any elliptic curve \(E\) we have \(L_m \cdot E > 2\). Let us sketch the argument: given an effective divisor \(D \sim aL - b_1R_1 - \cdots - b_8R_8\), we assume \(D \cdot L_m \leq 2\). This yields the inequality

\[(2.2) \quad \frac{[a(g - 1) - 1]^2}{m^2} \leq 2 \sum_{i=1}^{8} b_i^2.\]

Assuming the latter inequality, we can estimate the self-intersection of \(D\),

\[
D^2 = 2a^2(g - 1) - 2 \sum_{i=1}^{8} b_i^2 \\
\leq \frac{1}{m^2}[a^2(g - 1)(1 - g_m) + 2a(g - 1) - 1].
\]
We use the latter expression to impose the condition $D^2 < 0$. This yields to a quadratic equation in $a$, which will always be satisfied for any $a > 1$.

$$a > \frac{1}{g - 2m^2 - 1} \left( \sqrt{\frac{g - 1 + 2m^2}{2g - 2}} + 1 \right).$$

Since $a \geq 1$, the latter inequality will hold for any $a$ as long as we impose the right hand side to be strictly lower than 1. Hence, we study the following condition:

$$f(T) := \frac{1}{g - T - 1} \left( \sqrt{\frac{g - 1 + T}{2g - 2}} + 1 \right) < 1 \quad (T = 2m^2).$$

By direct computation, we find $f(T) < 1$ for any $g > 2$ and $0 < T < \frac{2g^2 - 8g + 7}{2g - 2}$.

Finally, since $T = 2m^2 \leq 2k^2 = g - p$, we are led to the inequality

$$g - p < \frac{2g^2 - 8g + 7}{2g - 2},$$

which holds for all $g > 3$ and $p \geq 3$ (and fails for $p = 2$, not surprisingly). This shows that $L_m$ is very ample and concludes the proof. □

**Lemma 2.4.** $|L_k|$ contains a smooth irreducible curve of genus $p \geq 0$.

*Proof.* By Riemann-Roch, $h^0(X, L_k) \geq 1 + p \geq 1$, whence $L_k$ is effective. Let

$$|L_k| = |M| + F$$

be the moving and fixed part decomposition of the linear system $|L_k|$. We show that either $F = 0$ (whence $|L_k|$ is basepoint free and the statement follows by Bertini’s Theorem) or $M = 0$ and $F$ is a smooth rational curve.

Note that $L_{k-2}$ is effective and $h^1(X, L_{k-2}) = 0$, since $k \geq 2$ by assumption. Consider the short exact sequence

$$0 \rightarrow L_k \rightarrow L_{k-2} \rightarrow \mathcal{O}_R(L_{k-2}) \rightarrow 0,$$

where $\mathcal{O}_X(R) = \mathcal{O}^{\otimes 2}$. Note that $L_{k-2} \cdot R > 0$ and the induced restriction map

$$H^0(X, L_{k-2}) \rightarrow H^0(R, \mathcal{O}_R(L_{k-2})) = \bigoplus_{i=1}^{8} H^0(R_i, \mathcal{O}_{R_i}(L_{k-2}))$$

is surjective. Then $h^1(X, L_k) = 0$ and $h^0(X, L_k) = 1 + p$, by Riemann-Roch.

If $p \geq 1$, the linear system $|L_k|$ has then a non-trivial moving part, and the established vanishing $h^1(X, L_k) = 0$ implies that the general member of $|L_k|$ is irreducible, i.e. $F = 0$ and $|L_k|$ is basepoint free. If $p = 0$, then $M = 0$ and the linear system $|L_k|$ consists of a single effective divisor $F$ of self-intersection $F^2 = -2$. 
Since $F \cdot R_i = k > 0$, the curve $F$ contains none of the $R_i$'s as a reducible component. Let $F = \Gamma_1 + \cdots + \Gamma_r$ be expressed as a sum of its irreducible components (which are smooth rational curves). By the structure of $\text{Pic}(X)$, we can write

$$\Gamma_\ell = a_\ell \cdot L - \sum_{i=1}^{8} b_{\ell,i} R_i.$$ 

For each $1 \leq \ell \leq r$ and $1 \leq i \leq 8$, we have $b_{\ell,i} \geq 0$, since $\Gamma_\ell$ is different from $R_i$. Therefore $a_\ell > 0$, else $\Gamma_\ell$ would not be effective, and by $F = \Gamma_1 + \cdots + \Gamma_r$ and $F \sim L - ke$, we get $a_1 + \cdots + a_r = 1$, whence $r = 1$. \hfill \Box

**Proof of Theorem 1.1.** It follows at once by the previous lemmas. \hfill \Box

Proposition 1.2 was stated in the Introduction as an example of the sort of projective geometry that can arise by studying the maps induced by the line bundles defined in Theorem 1.1, and is not needed in what follows. Its proof is straightforward and left to the reader. Rather, in the following section we focus our attention on the Brill-Noether aspects of the linear systems $|L_m|$ and show how this naturally leads to divisors in $\mathcal{F}_g^{n_1}$.

### 3. Brill-Noether general curves and the divisor $D_m^{rk(4)}$

Let us briefly recall some well-known facts from Brill-Noether theory. Let $C$ be a smooth algebraic curve of genus $g$. For any $A \in \text{Pic}(C)$, we let

$$\text{Cliff}(A) = \deg A - 2h^0(A) + 2.$$ 

The Clifford index of $C$ is by definition

$$\text{Cliff}(C) = \min\{\text{Cliff}(A) : A \in \text{Pic}(C), h^0(A) \geq 2, h^1(A) \geq 2\}.$$ 

Line bundles $A$ satisfying the conditions as in the definition of $\text{Cliff}(C)$ are said to *contribute to the Clifford index* of $C$. Such line bundles exist for any curve as long as $g \geq 4$. When $g < 4$ we adopt the standard convention

$$\text{Cliff}(C) := \begin{cases} -1 & \text{if } g = 0 \\ 0 & \text{if } g = 1, 2 \text{ or } 3 \text{ and } C \text{ is hyperelliptic} \\ 1 & \text{if } g = 3 \text{ and } C \text{ is non-hyperelliptic} \end{cases}$$

For any curve $C$ of genus $g$, one has the inequality

$$\text{Cliff}(C) \leq \left\lfloor \frac{g - 1}{2} \right\rfloor,$$

which is an equality for the general curve of genus $g$.

**Definition 3.1.** Let $S$ be a surface. We say that a curve $C \subset S$, or equivalently the line bundle $\mathcal{O}_S(C)$, decomposes into a sum of movable classes if $C$ is linearly equivalent to a sum $D_1 + D_2$ of two divisors satisfying $h^0(D_i) \geq 2$, for $i = 1, 2$.

For the rest of this section, we let $(X, L) \in \mathcal{F}_g^{n_1}$ be a polarized Nikulin surface of genus $g$. We write $g = 2k^2 + p$, where $k \geq 1$ and $0 \leq p < 4k + 2$ and denote by $L_m = L \otimes e^{-m}$, keeping the same notations of the previous section.
Proposition 3.2. Assume $X$ has Picard number 9. If $m = 0, \ldots, k - 1$ or $m = k$ and $p \geq 4$, the line bundle $L_m$ does not decompose into a sum of movable classes.

Proof. Let us assume by contradiction that there is some smooth curve $D \in |L_m|$ which decomposes into a sum of movable classes. We can clearly assume $D^2 \geq 2$. Among all such decompositions, we choose one $D \sim D_1 + D_2$ such that the intersection $D_1 \cdot D_2$ is minimal. We can then assume that one of the two classes, say $D_1$, is base point free and the general member of $|D_1|$ is a smooth irreducible curve, cf. [10, Prop 2.7]. In particular, $D_1$ is a nef class, and the restriction $(D_1)|_D$ contributes to the Clifford index of $D$, whence $\deg_D(D_1) > 0$. We want to show that $D_2$ is then not a movable class. Up to linear equivalence, we can write

$$D_\ell = a_\ell \cdot L - \sum_{i=1}^{8} b_{\ell,i} R_i, \quad \ell = 1, 2,$$

where $a_\ell \in \mathbb{Z}$ and $2b_{\ell,i} \in \mathbb{Z}$. By intersecting $D_\ell$ with the nef class $L$, we get $a_\ell \geq 0$ for $\ell = 1, 2$. Since $D_1$ is nef, $D_1 \cdot R_i = 2b_{1,i} \geq 0$ for all $i = 1, \ldots, 8$. Finally,

$$D \cdot D_1 = (L - m) \cdot D_1 = a_1 L^2 - m \sum_{i=1}^{8} b_{1,i} > 0,$$

which implies $a_1 > 0$. On the other hand, by $D \sim D_1 + D_2$ we have $a_1 + a_2 = 1$. Therefore $a_2 = 0$, which clearly contradicts $h^0(D_2) \geq 2$. \qed

We can now prove Theorem 1.3 from the Introduction.

Proof of Thm. 1.3. Let $D \in |L_m|$ be a smooth curve. Assuming by contradiction that $D$ has Clifford index $\text{Cliff}(D) < \left\lfloor \frac{D^2 - 1}{2} \right\rfloor$, there exists a line bundle $M$ on $X$ such that $M \otimes O_D$ contributes to the Clifford index of $D$ and $\text{Cliff}(D) = \text{Cliff}(M \otimes O_D)$, by [9]. By the definition of Clifford index it follows that $h^0(M) \geq 2$ and also $h^0(D - M) \geq 2$. Thus $D \sim D_1 + D_2$, with $D_1 \in |M|$ and $D_2 \in |D - M|$ is a decomposition of $D$ into two movable classes, contradicting Proposition 3.2. \qed

We are now going to define the divisors $\mathcal{Y}_m^{k \times 4}$ from the Introduction. The following discussion parallels [4, §9.1]. Let $m = 0, \ldots, k$ (assuming $p \geq 4$ when $m = k$).

Each one of the following conditions

(a) There exists an elliptic pencil $E$ on $X$ such that $E \cdot L_m = 1$.
(b) There exists an elliptic pencil $E$ on $X$ such that $E \cdot L_m = 2$.
(c) There exists a smooth rational curve $R$ on $X$ such that $R \cdot L_m = 0$.

singles out a virtual Noether-Lefschetz divisor in $\mathcal{F}_g^{m}$, i.e. can be rephrased as the condition of the existence of a primitive embedding $\Lambda_g \hookrightarrow \Lambda'_g$, for some lattices $\Lambda'_g$ of rank 10. “Virtual” here means that these conditions may possibly be empty on $\mathcal{F}_g^{m}$, but Theorem 1.3 implies that for general $(X, L)$ the conditions (a) and (b) are not satisfied, whence they define an actual divisor in $\mathcal{F}_g^{m}$ (obviously though, condition (c) defines an actual divisor only for $m \geq 1$).

By the well-known results of Saint-Donat, Nikulin surfaces $(X, L)$ outside the union of these three divisors are such that each $L_m$ is basepoint free, (pseudo-)


ample and non-hyperelliptic, whence the multiplication map
\[ \psi_m : S^2 H^0(X, L_m) \to H^0(X, L_m^\otimes 2) \]
is surjective, cf. [16, Thm. 6.1]. Thus \( I_m(2) := \ker(\psi_m) \) in \( S^2 H^0(X, L_m) \) has codimension \( h^0(X, L_m^\otimes 2) = 4g_m - 2 \), by Riemann-Roch. The closed subscheme
\[ \Sigma_m^{rk} = \{ q : \text{rk}(q) \leq k \} \subset S^2 H^0(X, L_m) \]
has codimension
\[ \binom{g_m + 2 - k}{2}, \]
see e.g. [15]. Therefore, the expected codimension of scheme-theoretic intersection
\[ I_m^{rk}(2) := I_m(2) \cap \Sigma_m^{rk} = \{ q \in I_m(2) : \text{rk}(q) \leq k \} \]
is
\[ \text{codim} I_m(2) + \text{codim} \Sigma_m^{rk} = 4g_m - 2 + \binom{g_m + 2 - k}{2} \]
in \( S^2 H^0(X, L_m) \). In other words, we expect
\[ \dim I_m^{rk}(2) = (k - 4)g_m + \frac{1}{2}(4 + 3k - k^2), \]
which vanishes precisely when \( k = 4 \). We thus expect that for the general Nikulin surface \( I_m^{rk}(2) = 0 \), so that, interpreting \( \psi_m \) as a morphism between vector bundles over \( \mathcal{F}_g^{\Pi} \), it follows that the condition \( I_m^{rk(4)}(2) \neq 0 \) is a divisorial –or eventually empty– condition on \( \mathcal{F}_g^{\Pi} \), i.e. the locus
\[ \mathcal{D}_m^{rk(4)} = \{(X, L) \in \mathcal{F}_g^{\Pi} : I_m^{rk(4)}(2) \neq 0\}, \]
is a virtual divisor in \( \mathcal{F}_g^{\Pi} \).

**Proposition 3.3.** \( \mathcal{D}_m^{rk(4)} \) is a divisor on \( \mathcal{F}_g^{\Pi} \).

**Proof.** We only have to show that the condition is not empty, i.e. that \( \mathcal{D}_m^{rk(4)} \) is not the whole \( \mathcal{F}_g^{\Pi} \). Let \((X, L)\) be a Nikulin surface of Picard number nine. Assume by contradiction \( I_m^{rk(4)}(2) \neq 0 \), i.e. \( X \subset \mathbb{P}^{d_m} \) is contained in a quadric \( Q \) of rank at most 4. We can assume \( \text{rk} Q = 4 \), whence \( \text{Sing} Q = \mathbb{P}^{d_m - 4} \subset \mathbb{P}^{d_m} \) and \( Q \) is the cone over \( \mathbb{P}^1 \times \mathbb{P}^1 \) under the projection \( \mathbb{P}^{d_m} \to \mathbb{P}^5 \) from \( \text{Sing} Q \). The two rulings of \( \mathbb{P}^1 \times \mathbb{P}^1 \) pull back to \( Q \) and cut out on \( X \) two divisors \( D_1, D_2 \), with \( h^0(X, D_i) \geq 2 \), such that \( L \sim D_1 + D_2 \). This contradicts Proposition 3.2. \( \square \)
4. Tautological classes in $\mathcal{F}_g^\text{NI}$

Let $\mathcal{F}_g^\text{NI}$ be the moduli stack of polarized Nikulin surfaces and

$$\pi : \mathcal{X} \longrightarrow \mathcal{F}_g^\text{NI}$$

the universal polarized Nikulin surface of genus $g$. Denote by

$$\mathcal{L} \in \text{Pic}(\mathcal{X})$$

the universal polarization. The Hodge bundle $E$ on $\mathcal{F}_g^\text{NI}$ is defined by

$$E^\vee = R^2 \pi_* \mathcal{O}_\mathcal{X}$$

and we denote its first Chern class by

$$\lambda = c_1(E) \in \text{CH}^1(\mathcal{F}_g^\text{NI}),$$

which we refer to as the Hodge class on $\mathcal{F}_g^\text{NI}$.

The relative tangent bundle $T_\pi \longrightarrow \mathcal{X}$ is defined by the short exact sequence

$$0 \longrightarrow T_\pi \longrightarrow T_{\mathcal{X}} \longrightarrow \pi^* T_{\mathcal{F}_g^\text{NI}} \longrightarrow 0.$$ 

The relative canonical bundle $\omega_\pi = \det(T_\pi)^{-1}$, which restricts to the canonical (whence trivial) line bundle on each fiber, satisfies

$$\omega_\pi = \pi^* (\pi_* \omega_\mathcal{X}) = \pi^* E.$$ 

In particular,

$$c_1(\omega_\pi) = \pi^* \lambda.$$

Note that the choice of $\mathcal{L}$ is canonical only up to a twist $\mathcal{L} \mapsto \mathcal{L} \otimes \pi^* \alpha$, for some $\alpha \in \text{Pic}(\mathcal{F}_g^\text{NI})$. A polarized Nikulin surface $(X, L)$ comes equipped with a distinguished set of 8 smooth rational curves $R_1, \ldots, R_8$ which form, together with

$$e = \frac{1}{2} (R_1 + \cdots + R_8),$$

a basis of the Nikulin lattice $\mathfrak{N}$. We consider the universal bundle

$$E \in \text{Pic}(\mathcal{X})$$

associated to the distinguished class $e \in \text{Pic}(X)$. Again, the choice of $E$ is canonical only up to a twist $E \mapsto E \otimes \pi^* \beta$, for some $\beta \in \text{Pic}(\mathcal{F}_g^\text{NI})$.

Following [12, §4] we consider in the Chow ring of $\mathcal{F}_g^\text{NI}$ the classes

$$\kappa_{a,b,c} = \pi_* \left( c_1(L^a) \cdot c_1(E)^b \cdot c_2(T_{\mathcal{F}_g^\text{NI}})^c \right) \in \text{CH}^{a+b+2c-2}(\mathcal{F}_g^\text{NI}),$$

for some non-negative integers $a, b, c$. The four classes in codimension zero are readily computed by restriction to a fiber $(X, L)$ as follows:

$$\kappa_{2,0,0} = L^2 = 2g - 2,$$

$$\kappa_{0,2,0} = e^2 = -4,$$

$$\kappa_{1,1,0} = L \cdot e = 0,$$

$$\kappa_{0,0,1} = \chi_{\text{top}}(X) = 24.$$
Then Proposition 4.1. Let $\class c$ and denote by $g$ independent from these twists, and thus are canonically defined in $\CH^1(\mathcal{F}_g^{31})$.

By twisting $(\mathcal{L}, \mathcal{E}) \mapsto (\mathcal{L} \otimes \pi^* \alpha, \mathcal{E} \otimes \pi^* \beta)$, these classes will change in general. However, it is straightforward to verify that the following linear combinations are independent from these twists, and thus are canonically defined in $\CH^1(\mathcal{F}_g^{31})$,

$$
\gamma_0 = \kappa_{3,0,0} - \frac{(g-1)}{4}\kappa_{1,0,1}, \quad \gamma_2 = \kappa_{1,0,1} + 6\kappa_{1,2,0},
\gamma_1 = \kappa_{2,1,0} - \frac{(g-1)}{12}\kappa_{0,1,1}, \quad \gamma_3 = \kappa_{0,1,1} + 2\kappa_{0,3,0}.
$$

As in the previous sections, we write

$$
g = 2k^2 + p, \quad k \geq 1, \quad 0 \leq p < 4k + 2
$$

and denote by $g_m = g - 2m^2$ the genus of the line bundle $L_m = L \otimes e^{-m}$.

**Proposition 4.1.** Let $\mathcal{L}_m = \mathcal{L} \otimes e^{-m}$, with $m = 0, \ldots, k - 1$ or $m = k$ and $p \geq 1$. Then $\mathcal{V}_{n,m} := \pi_*(\mathcal{L}_m^{\otimes n})$ is a locally free sheaf of rank $n^2(g_m - 1) + 2$ and first Chern class $c_1(\mathcal{V}_{n,m})$ equal to

$$
\frac{n^3}{6}[\kappa_{300} - 3m\kappa_{210} + 3m^2\kappa_{120} - m^3\kappa_{030}] + \frac{n^2}{12}[\kappa_{101} - m\kappa_{011}] - [1 + \frac{n^2}{2}(g_m - 1)]\lambda.
$$

**Proof.** For $(X, L) \in \mathcal{F}_g^{31}$, we have $h^0(X, L_m^{\otimes n}) = n^2(g_m - 1) + 2$ and $h^i(X, L_m^{\otimes n}) = 0$ for $i = 1, 2$, whence $\pi_*\mathcal{L}_m^{\otimes n}$ is a locally free sheaf of rank $n^2(g_m - 1) + 2$ and higher direct images vanish $R^i\pi_*\mathcal{L}_m^{\otimes n} = 0$. By the Grothendieck-Riemann-Roch formula,

$$
\ch(\pi_*\mathcal{L}_m^{\otimes n}) = \pi_* \left\{ \ch(\mathcal{L}_m^{\otimes n}) \cdot \td(\mathcal{F}_m) \right\}.
$$

Since $\pi$ is of relative dimension 2, the first Chern class of $\pi_*\mathcal{L}_m^{\otimes n}$ equals the push-forward of the degree 3 part of the product on the right-hand side,

$$
c_1(\pi_*\mathcal{L}_m^{\otimes n}) = \pi_* \left\{ \ch(\mathcal{L}_m^{\otimes n}) \cdot \td(\mathcal{F}_m) \right\}_3.
$$

Recall that the Chern character and Todd class can be expressed in terms of Chern classes as follows:

$$
\ch = \text{rk} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots
$$
$$
\td = 1 + \frac{c_1}{2} + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2) + \cdots
$$

Now, since $c_1(\omega_\pi) = \pi^* \lambda$, we get

$$
\pi_*(c_1(\omega_\pi) \cdot c_2(\mathcal{F}_m)) = \lambda \cdot \pi_* c_2(\mathcal{F}_m) = 24\lambda,
\pi_*(c_1(\mathcal{L}_m) \cdot c_1(\omega_\pi)) = \pi_*(c_1(\mathcal{L}_m)^2) \cdot \lambda = (2g_m - 2)\lambda,
\pi_*(c_1(\mathcal{L}_m) \cdot c_1(\omega_\pi)^2) = \pi_*(c_1(\mathcal{L}_m)) \cdot \lambda^2 = 0.
$$
Using these and \( c_1(\mathcal{F}_\pi) = -c_1(\omega_\pi) \), we get that \( c_1(\pi_*\mathcal{L}_m^{\otimes n}) \) equals

\[
\pi_* \left\{ \left( 1 + nc_1(\mathcal{L}_m) + \frac{n}{2} c_1(\mathcal{L}_m)^2 + \frac{n^3}{6} c_1(\mathcal{L}_m)^3 \right) \cdot \left( 1 + \frac{c_1(\mathcal{F}_\pi)}{2} + \frac{c_1(\mathcal{F}_\pi)^2 + c_2(\mathcal{F}_\pi) + c_1(\mathcal{F}_\pi)c_2(\mathcal{F}_\pi)}{12} + \frac{c_1(\mathcal{F}_\pi)c_2(\mathcal{F}_\pi)}{24} \right) \right\}_3
= \frac{n^3}{6} \pi_* (c_1(\mathcal{L}_m)^3) - \frac{n^2}{4} (2g_m - 2) \lambda + \frac{n}{12} \pi_* (c_1(\mathcal{L}_m)c_2(\mathcal{F}_\pi)) - \lambda.
\]

Substituting \( c_1(\mathcal{L}_m) = c_1(\mathcal{L}) - mc_1(\mathcal{E}) \) and expanding the last expression yields to the claimed formula. \( \square \)

We are now ready to prove Theorem 1.4 from the Introduction.

**Proof of Thm. 1.4.** We consider the morphism of vector bundles over \( \mathbb{P}^g \),

\[
\psi_m : S^2 \mathcal{U}_{1,m} \to \mathcal{U}_{2,m}.
\]

The divisor \( \mathcal{D}_{rk4}^m \) is by definition the locus where \( \ker(\psi_m) \) contains quadrics of rank at most four. Applying the general formula [4, Thm. 1.1], we get

\[
[\mathcal{D}_{rk4}^m] = A_m \left( c_1(\mathcal{U}_{2,m}) - 2 \left( \frac{\text{rk } \mathcal{U}_{2,m}}{\text{rk } \mathcal{U}_{1,m}} \right) c_1(\mathcal{U}_{1,m}) \right),
\]

for some coefficients \( A_m \in \mathbb{Q} \). By Proposition 4.1, the class

\[
c_1(\mathcal{U}_{2,m}) - 2 \left( \frac{\text{rk } \mathcal{U}_{2,m}}{\text{rk } \mathcal{U}_{1,m}} \right) c_1(\mathcal{U}_{1,m})
\]

is equal to

\[
\frac{2}{g_m + 1} [\kappa_{300} - 3m \kappa_{210} + 3m^2 \kappa_{120} - m^3 \kappa_{030}] - \frac{g_m - 1}{2(g_m + 1)} [\kappa_{101} - m \kappa_{011}] + (2g_m - 1) \lambda.
\]

Plugging into the latter expression the following straightforward identities:

\[
2 \gamma_0 + m^2 \gamma_2 = 2 \kappa_{300} - \frac{g_m - 1}{2} \kappa_{101} + 6m^2 \kappa_{120},
\]

\[
-m^3 \gamma_3 - 6m \gamma_1 = -2m^3 \kappa_{030} + \frac{g_m - 1}{2} m \kappa_{011} - 6m \kappa_{210},
\]

one gets

\[
\frac{2}{g_m + 1} \gamma_0 - \frac{6m}{g_m + 1} \gamma_1 + \frac{m^2}{g_m + 1} \gamma_2 - \frac{m^3}{g_m + 1} \gamma_3 + (2g_m - 1) \lambda.
\]

The result follows. \( \square \)
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