On the generalized intelligent states and certain related nonclassical states of a quantum exactly solvable nonlinear oscillator

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Abstract
We construct nonlinear coherent states or $f$-deformed coherent states for a nonpolynomial nonlinear oscillator which can be considered as placed in the middle between the harmonic oscillator and the isotonic oscillator (Cariñena et al 2008 J. Phys. A: Math. Theor. 41 085301). The deformed annihilation and creation operators which are required to construct the nonlinear coherent states in the number basis are obtained from the solution of the Schrödinger equation. Using these operators, we construct generalized intelligent states, nonlinear coherent states, Gazeau–Klauder coherent states and the even and odd nonlinear coherent states for this newly solvable system. We also report certain nonclassical properties exhibited by these nonlinear coherent states. In addition to the above, we consider the position-dependent mass Schrödinger equation associated with this solvable nonlinear oscillator and construct nonlinear coherent states, Gazeau–Klauder coherent states and the even and odd nonlinear coherent states for it. We also give explicit expressions of all these nonlinear coherent states by considering a mass profile which is often used for studying transport properties in semiconductors.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In a recent paper Cariñena et al [1] have considered the potential

$$V_f(x) = \frac{1}{2} \left( \omega^2 x^2 + \frac{2g_a(x^2 - a^2)}{(x^2 + a^2)^2} \right), \quad g_a > 0, \quad (1)$$
which can be considered as placed in the middle between the harmonic oscillator and isotonic oscillator potentials and shown that in a particular case, namely
\[ g_a = 2ω_a^2 (1 + 2ω_a^2), \]
and \[ ω = 1, \]
the associated Schrödinger equation
\[
\frac{d^2 \psi(x)}{dx^2} + \frac{m_0}{\hbar^2} \left( 2E - x^2 - \frac{8(2x^2 - 1)}{(2x^2 + 1)^2} \right) \psi(x) = 0
\]
is exactly solvable. The authors have obtained the eigenfunctions in terms of \( P\)-Hermite functions (defined in equation (A.4)), namely
\[
\psi_n(x) = N_n \frac{P_n(x)}{(1 + 2x^2)} e^{-\frac{x^2}{2}}, \quad n = 0, 3, 4, \ldots,
\]
where the normalization constant is given by
\[
N_n = \left[ \frac{(n-1)(n-2)}{2^n n!} \right]^{1/2}, \quad n = 0, 3, 4, \ldots
\]
It is noted that while solving the Schrödinger equation (2) the constants (\( \hbar, m_0 \)) have been taken as 1 for the sake of simplicity. The energy spectrum is given by
\[
E_n = -\frac{3}{2} + n,
\]
which shows that the ground state (\( \psi_0 \)) has an energy \( E_0 \) which is lower than that of the pure harmonic oscillator case [1]. Consequently the above quantum exactly solvable potential was analysed in different perspectives. In the following, we briefly summarize the activities revolving around this inverse square potential.

To start with Fellows and Smith have shown that the solvable potential considered by Cariñena et al is a supersymmetric partner potential of the harmonic oscillator [2]. In a different study, Kraenkel and Senthilvelan have considered the exactly solvable potential given by Cariñena, and obtained a different class of exactly solvable potentials by transforming the Schrödinger equation (2) into the position-dependent mass Schrödinger equation and solving the underlying equation. They have also given bound state energies and their corresponding wavefunctions for the potentials they have considered [3].

The coherent states [4] for the position-dependent mass Schrödinger equation were then constructed recently by the present authors [5] with an illustration of the above exactly solvable nonlinear oscillator. In a very recent paper, Sesma has considered the Schrödinger equation associated with the potential (1) and transformed it into a confluent Heun equation and solved it numerically [6]. For certain specific values of the parameters the author has also constructed quasi-polynomial solutions.

The aim of this paper is to make some progress in constructing nonlinear coherent states for the Schrödinger equation (2) and bring out their statistical properties [7]. We also construct nonlinear coherent states for the position-dependent mass Schrödinger equation associated with this nonlinear oscillator. Nonlinear coherent states are defined as the eigenstates of an operator \( \hat{a} f(\hat{n}) \), which satisfy the eigenvalue equation \( \hat{a} f(\hat{n})(\alpha, f) = \alpha(\alpha, f) \), where \( f(\hat{n}) \) is an operator-valued function of the number operator, \( \hat{n} = \hat{a}^\dagger \hat{a} \), and they are nonclassical [8, 9]. Nonlinear coherent states were first introduced explicitly by de Matos Filho and Vogel [10] and Man’ko et al [11] but before them they were implicitly defined by Shanta et al [12] in a compact form (see also [13]). In subsequent years these states have been studied in depth and have shown to exhibit several nonclassical properties including squeezing, amplitude-squared squeezing, higher order squeezing, anti-bunching, sub-Poissonian statistics and oscillatory number distribution [14–18]. A veritable explosion of activities occurred in the past two decades in the study of nonlinear coherent states and their underlying properties [19, 20].
It is known that once suitable deformed ladder operators are found then the nonlinear coherent states can be constructed in a straightforward manner for the given quantum system. As we mentioned earlier the authors in [2] have obtained the ladder operators for (2) through the supersymmetric technique and shown that the operators are related to the intertwining between the Hamiltonian (2) and the harmonic oscillator. In this work we build the deformed ladder operators from the solution of the Schrödinger equation [21, 22] and show that these operators satisfy the necessary requirements

\[ \hat{A}|n\rangle \propto |n - 1\rangle, \]

\[ \hat{A}^\dagger |n\rangle \propto |n + 1\rangle \]

(6)
to construct the nonlinear coherent states in the Fock space representation [23–25]. To construct these operators we derive two new recurrence relations solely in terms of the \(P\)-Hermite polynomials (vide equations (A.12) and (A.16)). We then rewrite these two recurrence relations suitably and extract the necessary creation and annihilation operators.

Interestingly, we find that the intensity-dependent function \(f(n)\) has zeros at \(n = 1\) and 3. Hence, the Fock space breaks up into a countable number of irreducible representations [11]. These reduced pieces will not allow a unitary representation [26]. System (2) may be considered as an example for the theory proposed in [11, 26].

From the deformed ladder operators, we construct generalized intelligent states. To do so we introduce two Hermitian operators, say \(\hat{W}\) and \(\hat{P}\), in terms of deformed annihilation and creation operators, related to the uncertainty relation [27, 28] \((\Delta \hat{W})^2(\Delta \hat{P})^2 \geq \frac{1}{2} \langle G \rangle^2\) with \(\hat{G} = i[\hat{W}, \hat{P}]\). The states which satisfy the equality relation in the above uncertainty relation are called the intelligent states [29–31]. They also represent the squeezed states [14]. We then construct nonlinear coherent states by taking \(\lambda = 1\) in the generalized intelligent states. We also study certain Hilbert space properties and nonclassical properties for this quantum system. From the deformed Hamiltonian corresponding to this function, \(e_n = nf^2(n)\), we construct the Gazeau–Klauder coherent states which are defined as coherent states that also saturate the Heisenberg uncertainty relation. We also construct even and odd nonlinear coherent states [32, 33]. Motivated by the recent developments in the study of the position-dependent mass Schrödinger equation, we also construct nonlinear coherent states for the exactly solvable position-dependent mass Schrödinger equation associated with equation (2). Finally, we consider a specific mass profile which is quite often used to study transport properties in semiconductors and give explicit expressions for all the coherent states we have considered [34, 35]. The results given in this paper give further insights into the geometrical properties exhibited by this system.

This paper is organized as follows. In section 2, we discuss the method of finding deformed annihilation and creation operators in the number basis for the nonlinear oscillator. In section 3, we construct the generalized intelligent states for this nonlinear oscillator. In section 4, we construct the nonlinear coherent states for this oscillator and show that the states satisfy the completeness condition. We then deduce certain statistical properties associated with the nonlinear coherent states. In section 5, we construct Gazeau–Klauder coherent states for this nonlinear oscillator and bring out their statistical properties. In section 6, we derive even and odd nonlinear coherent states for this system. In section 7, we consider the position-dependent mass Schrödinger equation associated with this inverse-type potential and construct the nonlinear coherent states for it. We also illustrate the theory with an example. Finally we present our conclusions in section 8. In the appendix, we present the derivation of two recurrence relations which are essential to construct the ladder operator for this system (2).
2. Ladder operators

We consider equation (2) as the number operator equation after subtracting the ground state energy $E_0 = -\frac{3}{2}$:

$$\hat{n}|n\rangle = n|n\rangle,$$  

and address the problem of finding deformed creation ($\hat{A}^\dagger$) and annihilation operators ($\hat{A}$) for the given wavefunction [21].

To begin with we recall the two recurrence relations (A.12) and (A.16). Multiplying each of them by $N_n e^{-\frac{x^2}{2}}/(1 + 2x^2)$ and substituting (3) and its derivative in them and rewriting the resultant expressions suitably one can reexpress the recurrence relations in terms of the wavefunction $\psi_n$. The resultant expressions read

$$\left[n - 1 + \frac{2(2x^2 - 1)}{(1 + 2x^2)^2}\right] \psi'_n + \left[nx - \phi + \frac{2(2x^2 - 1)}{(1 + 2x^2)^2}\phi\right] \psi_n = \sqrt{2n(n-1)(n-3)}\psi_{n-1},$$  

$$\left[n + 2(2x^2 - 1)/(1 + 2x^2)^2\right] \psi'_n + \left[nx - 2(2x^2 - 1)/(1 + 2x^2)^2\phi\right] \psi_n = \sqrt{2(n+1)n(n-2)}\psi_{n+1},$$

where prime denotes differentiation with respect to $x$.

From these expressions one can extract the deformed annihilation $\hat{A}$ and creation operators ($\hat{A}^\dagger$) of the form

$$\sqrt{2} \hat{A} = \left[\frac{2(2x^2 - 1)}{(1 + 2x^2)^2} - 1\right]\left[\frac{d}{dx} + \phi\right] + \left[\frac{d}{dx} + x\right] \hat{n},$$  

$$\sqrt{2} \hat{A}^\dagger = \left[\frac{2(2x^2 - 1)}{(1 + 2x^2)^2}\right]\left[\frac{d}{dx} + \phi\right] + \left[-\frac{d}{dx} + x\right] \hat{n}.$$

These deformed ladder operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the relations

$$\hat{A}|n\rangle = \sqrt{n} f(n)|n-1\rangle,$$  

$$\hat{A}^\dagger|n\rangle = \sqrt{n + 1} f(n + 1)|n+1\rangle,$$

with $f(n) = \sqrt{(n-1)(n-3)}$. Hence, the creation and annihilation operators in the Fock space for the Schrödinger equation (7) can be written as

$$\hat{a} = \frac{\hat{A}}{f(\hat{n})}, \quad \hat{a}^\dagger = \frac{\hat{A}^\dagger}{f(\hat{n} + 1)},$$

which in turn satisfy the following relations:

$$\hat{a}|n\rangle = \sqrt{n} |n - 1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n + 1} |n + 1\rangle.$$

Thus, we obtain the Heisenberg algebra in the number basis $|n\rangle$ through the set of the elements ($\hat{a}$, $\hat{a}^\dagger$ and $I$) satisfying the identities

$$[\hat{a}, \hat{a}^\dagger]|n\rangle = |n\rangle, \quad [\hat{a}, \hat{a}^\dagger]|n\rangle = -\hat{a}|n\rangle, \quad [\hat{a}^\dagger, \hat{n}]|n\rangle = \hat{a}^\dagger|n\rangle,$$

where $\hat{n} = \hat{a}^\dagger \hat{a}$. 

4
3. Generalized intelligent states

We define two Hermitian operators, $\hat{W}$ and $\hat{P}$, in terms of the deformed creation and annihilation operators, $\hat{A}$ and $\hat{A}^\dagger$, [31]:

$$\hat{W} = \frac{1}{\sqrt{2}} (\hat{A} + \hat{A}^\dagger), \quad \hat{P} = \frac{i}{\sqrt{2}} (\hat{A}^\dagger - \hat{A}),$$

(17)

which also satisfy the commutation relation $[\hat{W}, \hat{P}] = i\hat{G}$. It has been shown that the Hermitian operators $\hat{W}$ and $\hat{P}$ satisfying this non-canonical commutation relation, the variances, $(\Delta \hat{W})^2$ and $(\Delta \hat{P})^2$, satisfy the Heisenberg uncertainty relation [27] $(\Delta \hat{W})^2(\Delta \hat{P})^2 \geq \frac{1}{4} (\hat{G})^2$. The generalized intelligent states are obtained when the equality in the Heisenberg uncertainty relation is realized [30]. The generalized intelligent states satisfy the eigenvalue equation

$$(\hat{W} + \i \lambda \hat{P}) |\psi\rangle = \sqrt{2\alpha} |\psi\rangle, \quad \lambda, \alpha \in \mathbb{C}. \quad (18)$$

It is proved that the parameter $\lambda$ in (18) is determined by $|\lambda| = \frac{(\Delta \hat{W})}{(\Delta \hat{P})}$ and can be interpreted as the control parameter for the squeezing effect in the states $|\psi\rangle$.

To solve the eigenvalue equation (18), we invoke definition (17) so that the eigenvalue equation (18) becomes

$$[(1 - \lambda) \hat{A} + (1 + \lambda) \hat{A}^\dagger] |\psi\rangle = 2\alpha |\psi\rangle. \quad (19)$$

We assume $|\psi\rangle$ to be of the form

$$|\psi\rangle = |\alpha, f, \lambda\rangle = \sum_{n=0}^\infty c_n |n\rangle, \quad n \neq 1, 2, \quad (20)$$

where the coefficients, $c_n$, $n = 0, 3, 4, 5, \ldots$, are determined by (19). Substituting equation (20) into (19), we get

$$(1 - \lambda) \sum_{n=0}^\infty c_n \hat{A}^\dagger |n\rangle + (1 + \lambda) \sum_{n=0}^\infty c_n \hat{A} |n\rangle = 2\alpha \sum_{n=0}^\infty c_n |n\rangle. \quad (21)$$

Equating the coefficients of $\psi_n$ in equation (21) yields

$$(1 - \lambda) \sqrt{n(n - 1)(n - 3)} c_{n-1} + (1 + \lambda) \sqrt{(n + 1)n(n - 2)} c_{n+1} = 2\alpha c_n. \quad (22)$$

From (22), we find $c_0 = 0$. To obtain the value of other coefficients $c_i$'s, $i = 3, 4, 5, \ldots, n$, we rewrite equation (22) in the form

$$\frac{(1 - \lambda)}{(1 + \lambda)} \sqrt{n} \frac{f(n)}{c_n} \frac{c_{n-1}}{c_n} + \sqrt{n + 1} \frac{f(n + 1)}{c_n} \frac{c_{n+1}}{c_n} = \frac{2\alpha}{(1 + \lambda)}. \quad (23)$$

Defining $B_n = \frac{c_n}{c_{n+1}}$, from equation (23) can be brought into the form

$$B_n = \frac{1}{\sqrt{n + 1} f(n + 1)} \left[ \frac{2\alpha}{(1 + \lambda)} + \frac{\lambda - 1}{\lambda + 1} \frac{n f^2(n)}{B_{n-1}} \right]. \quad (24)$$

Equation (24) can be reutilized to express $B_{n-1}$ in terms of $B_{n-2}$ and $B_{n-2}$ in terms of $B_{n-3}$ and so on. Substituting all these expressions into (24) one arrives at

$$B_n = \frac{1}{\sqrt{n + 1} f(n + 1)} \left[ \frac{2\alpha}{1 + \lambda} + \frac{2\alpha}{1 + \lambda} \frac{\lambda - 1}{\lambda + 1} \frac{n f^2(n)}{(n - 1)f(n - 1)} \right] \frac{1}{B_{n-1}} + \cdots$$

$$+ \frac{2\alpha}{1 + \lambda} \frac{\lambda - 1}{\lambda + 1} \frac{n f^2(n)}{(n - 1)f(n - 1)} \frac{1}{B_{n-1}} + \cdots$$

$$+ \frac{2\alpha}{1 + \lambda} \frac{\lambda - 1}{\lambda + 1} \frac{n f^2(n)}{(n - 1)f(n - 1)} \frac{1}{B_{n-1}} + \cdots$$
Thus, starting from the definition \( c_n = B_{n-1} c_{n-1} \) one finds that all the coefficients in the series (20), \( c'_n, n = 4, 5, 6, \ldots \), can be expressed in terms of \( c_3 \), that is
\[
c_n = B_{n-1} B_{n-2} B_{n-3} \ldots B_{3c3} = \tilde{B}_{n-1} c_3, \quad n = 3, 4, 5, \ldots, \tag{26}
\]
with \( \tilde{B}_2 = 1 \). As a consequence one can express \(|\psi\rangle\) of the form
\[
|\alpha, \tilde{f}, \lambda\rangle = c_3 \sum_{n=3}^{\infty} \tilde{B}_{n-1} ! |n\rangle. \tag{27}
\]

In order to compare (27) with the latter expressions given in the paper we redefine the constants, \( \tilde{B}_{n-1} !, n = 3, 4, 5, \ldots \), as
\[
\tilde{B}_{n-1} ! = \tilde{A}_n ! \sqrt{\tilde{n} ! \tilde{f}(n) !}, \tag{28}
\]
where \( \tilde{A}_n ! = A_n A_{n-1} A_{n-2} \ldots A_4, \tilde{f}(n) ! = f(n) f(n-1) f(n-2) \ldots f(4), \tilde{n} ! = n(n-1)(n-2) \ldots 4, \) and \( \tilde{A}_3 ! = \tilde{f}(3) ! = 3 ! = 1 \). In the new variables, equation (27) reads
\[
|\alpha, \tilde{f}, \lambda\rangle = c_3 \sum_{n=3}^{\infty} \tilde{A}_n ! \sqrt{\tilde{n} ! \tilde{f}(n) !} |n\rangle. \tag{29}
\]
This is known as generalized intelligent states. These are investigated with the value \(|\lambda| = 1\) in the following section.

4. Nonlinear coherent states

In this section, we deduce nonlinear coherent states from the generalized intelligent states by setting \( \lambda = 1 \) in the latter. The eigenvalue equation for \( \hat{A} \) with the eigenfunction \(|\alpha, f\rangle\) in the Hilbert space reads
\[
\hat{A} |\alpha, f\rangle = \alpha |\alpha, f\rangle, \quad \alpha \in \mathbb{C}. \tag{30}
\]
Since \( \lambda = 1 \), we have \( \Delta \hat{W} = \Delta \hat{P} \). Let
\[
|\alpha, f\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad n \neq 1, 2. \tag{31}
\]
Substituting equation (31) into equation (30), one gets
\[
\sum_{n=0}^{\infty} c_n \sqrt{n(n-1)(n-3)} |n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle. \tag{32}
\]
Equating the coefficients of \( \psi_n \) in the above equation (32) yields
\[
c_{n+1} \sqrt{n+1} f(n+1) = \alpha c_n \tag{33}
\]
from which we find \( c_0 = 0 \) and \( c_3 \) is an arbitrary constant. To express \( c_n \) in terms of \( c_3 \) we rewrite the recurrence relation (33) in the form
\[
c_n = \frac{\alpha}{\sqrt{n} f(n)} c_{n-1}. \tag{34}
\]
From equation (34) one can find
\[
c_n = \frac{\alpha^{n-3}}{\sqrt{n} ! \ f(n) !} c_3, \quad n = 3, 4, 5, \ldots. \tag{35}
\]
The coherent states defined for the oscillator (2) do not contain the states with a photon number less than 3. The coherent states contain the states with the photon number starting from 3 now reads
\[ |\alpha, \tilde{f}\rangle = c_3 \sum_{n=3}^{\infty} \frac{\alpha^{n-3}}{\sqrt{n!} \tilde{f}(n)!} |n\rangle. \tag{36} \]

To determine \( c_3 \), we use the normalization condition \( \langle \alpha, \tilde{f} | \alpha, \tilde{f} \rangle = 1 \) which in turn gives
\[ c_3 = \left( \sum_{n=3}^{\infty} \frac{|\alpha|^{2n-6}}{n! \tilde{f}(n)!^2} \right)^{-1/2}. \tag{37} \]

Since \( c_3 \) depends on \( \alpha, \tilde{f} \), we denote it as \( \tilde{N} \). The nonlinear coherent states for the nonlinear oscillator equation (7) after appropriate normalization reads
\[ |\alpha, \tilde{f}\rangle = \tilde{N} (|\alpha|^2) \sum_{n=3}^{\infty} \frac{\alpha^{n-3}}{\sqrt{n!} \tilde{f}(n)!} |n\rangle. \tag{38} \]

### 4.1. Completeness condition

In this subsection, we investigate whether the nonlinear coherent states form a complete system of states in the Hilbert space or not. To establish this we invoke the completeness relation [36, 37]
\[ \frac{1}{\pi} \int \int_C |\alpha, \tilde{f}\rangle W(|\alpha|^2) \langle \alpha, \tilde{f}| \alpha^2 \rangle \, d^2 \alpha = \hat{1}, \tag{39} \]
where \( W(|\alpha|^2) \) is a positive weight function and \( \hat{1} \) is an identity operator. From (39) we obtain
\[ \frac{1}{\pi} \int \int_C \langle \psi | \alpha, \tilde{f}\rangle W(|\alpha|^2) \langle \alpha, \tilde{f}| \Phi \rangle \, d^2 \alpha = \langle \psi | \hat{1} | \Phi \rangle. \tag{40} \]

Substituting \( |\alpha, \tilde{f}\rangle \) and its conjugate (vide equation (38)) into the left-hand side of equation (40), we get (which we call \( G \))
\[ G = \frac{1}{\pi} \sum_{m,n=3}^{\infty} \frac{\langle \psi | n \rangle \langle m | \Phi \rangle}{\sqrt{n! m!} \tilde{f}(n)! \tilde{f}(m)!} \int \int_C \alpha^{n-3} \alpha^{m-3} \tilde{N}^2 (|\alpha|^2) W(|\alpha|^2) \, d^2 \alpha. \tag{41} \]

Taking \( \alpha = r e^{i\theta} \), one can separate the real and imaginary parts and obtain
\[ G = \frac{1}{\pi} \sum_{m,n=3}^{\infty} \frac{\langle \psi | n \rangle \langle m | \Phi \rangle}{\sqrt{n! m!} \tilde{f}(n)! \tilde{f}(m)!} \int_0^\infty \int_0^{2\pi} \tilde{N}^2(r^2) r^{n+m-6} W(r^2) r \, dr \, d\theta. \tag{42} \]

Since the second integral vanishes except \( n = m \) we can bring equation (42) to the form
\[ G = \sum_{n=3}^{\infty} \frac{\langle \psi | n \rangle \langle n | \Phi \rangle}{n! \tilde{f}(n)!} \int_0^\infty \tilde{N}^2(r^2) r^{2n-6} W(r^2) 2r \, dr. \tag{43} \]

Taking \( r^2 = x \) and using the identities \( \tilde{n}! = \frac{\alpha^n}{n!} \) and \( [\tilde{f}(n)!]^2 = [(n-1)(n-3)]! \), we find
\[ G = \sum_{n=3}^{\infty} \frac{6 \langle \psi | n \rangle \langle n | \Phi \rangle}{n! [(n-1)(n-3)]!} \int_0^\infty x^{n-3} \tilde{N}^2(x) W(x) \, dx. \tag{44} \]
Choosing $\tilde{N}^2(x) W(x) = \frac{2}{\pi} \int_{0}^{\infty} x^{n^2-4n+7} K_{n^2-5n+3}(2x) \, dx$, the integral on the right-hand side in (44) can be brought to the form $\int_{0}^{\infty} x^{n^2-3n+4} K_{n^2-5n+3}(2x) \, dx$ which can be integrated in terms of gamma functions [38], namely $\frac{1}{4} \Gamma(n^2 - 4n + 3) \Gamma(n + 1)$. As a result one gets

$$G = \sum_{n=0}^{\infty} (\langle \psi | n \rangle | n \rangle \Phi). \quad (45)$$

We mention here that the nonlinear coherent states contain the states with photon number greater than or equal to 3. As a consequence equation (45) turns out to be a projector for which the state $n = 0$ is excluded.

4.2. Photon statistical properties of nonlinear coherent states $|\alpha, \tilde{f}\rangle$

Having constructed the nonlinear coherent states and studied its completeness, in this subsection, we study certain statistical properties, namely (1) the photon number distribution, (2) sub-Poissonian function and (3) the correlation function, associated with these nonlinear coherent states.

The photon number distribution $P(n)$ of the nonlinear coherent states is defined by [11]

$$P(n) = |\langle n | \alpha, \tilde{f} \rangle|^2. \quad (46)$$

From (38) we find

$$P(n) = \frac{\tilde{N}^2 |\alpha|^2}{n! [f(n)]^2}. \quad (47)$$

We plot the photon number distribution as a function of $n$ in figure 1(a).

To confirm the sub-Poissonian statistics exhibited by our nonclassical states, we evaluate the Mandel parameter [39]

$$Q = \frac{\langle \Delta \hat{n} \rangle^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle} = \frac{\langle \hat{n}^2 \rangle}{\langle \hat{n} \rangle} - 1, \quad (48)$$

which should be negative ($Q < 0$) [40, 41]. First let us calculate $\langle \hat{n} \rangle$:

$$\langle \hat{n} \rangle = \langle \alpha, \tilde{f} | \hat{n} | \alpha, \tilde{f} \rangle = \tilde{N}^2 \sum_{n=3}^{\infty} \frac{12n |\alpha|^2}{n! (n - 1)! (n - 3)!}. \quad (49)$$

Substituting the value of $\tilde{N}$ (vide equation (37)) into (49) and redefining $p = n - 3$ in the resultant expression, we get

$$\langle \hat{n} \rangle = \sum_{p=3}^{\infty} \frac{12p |\alpha|^2}{p! (p+1)! (p+2)! (p+3)!}, \quad (50)$$

where we have used the identity [38]

$$\sum_{p=0}^{\infty} \frac{12p |\alpha|^2}{p! (p+1)! (p+2)! (p+3)!} = 3 \left( \frac{1}{2} F_3(4; 3, 3, 4; |\alpha|^2) \right).$$

We plot the mean photon number $\langle \hat{n} \rangle$ as a function of $r (= |\alpha|^2)$ for the nonlinear coherent states (38) in figure 1(b).

In a similar fashion we find

$$\langle \hat{n}^2 \rangle = 9 \left( \frac{1}{2} F_3(4; 3, 3, 4; |\alpha|^2) \right). \quad (52)$$
Using (50) and (52) we find
\[ g(\alpha) = \frac{1}{\sqrt{2}} F_3(3; 3, 3, 4; |\alpha|^2) \left( \frac{1}{F_3(4; 3, 3, 4; |\alpha|^2)} - \frac{1}{3} \right). \]

The quantity which determines bunching and anti-bunching of a state of the radiation field is simply decided by the second-order correlation function \( g^{(2)}(0) \) (denoted as \( g \)) of nonlinear coherent states (38).

Substituting \( \langle \hat{n}^2 \rangle \) and \( \langle \hat{n} \rangle \) into (48), we get
\[ Q = 3 \left( \frac{1}{\sqrt{2}} \frac{F_3(4; 3, 3, 3; |\alpha|^2)}{F_3(4; 3, 3, 4; |\alpha|^2)} - \frac{1}{\sqrt{2}} \frac{F_3(4; 3, 3, 4; |\alpha|^2)}{F_3(3; 3, 3, 4; |\alpha|^2)} \right) - 1. \]  

The Mandel parameter \( Q \) is depicted in figure 1(c) which confirms the sub-Poissonian statistics established by the nonlinear coherent states for all values of \( r (=|\alpha|^2) \).

The quantity which determines bunching and anti-bunching of a state of the radiation field is simply decided by the second-order correlation function [16, 17], \( g^{(2)}(0) \), which is defined by
\[ g^{(2)}(0) = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle^2} = \frac{\langle \hat{n}^2 \rangle}{\langle \hat{n} \rangle^2} - 1. \]  

Using (50) and (52) we find
\[ g^{(2)}(0) = \frac{1}{\sqrt{2}} \frac{F_3(3; 3, 3, 4; |\alpha|^2)}{F_3(4; 3, 3, 4; |\alpha|^2)} \left( \frac{1}{F_3(4; 3, 3, 4; |\alpha|^2)} - \frac{1}{3} \right). \]  

We depict the values of \( g^{(2)}(0) \) in figure 1(d). It is clear from figure 1(d) that since \( g^{(2)} < 1 \) for \( r > 0 \), the Fock space states describing the light field is anti-bunched. In other words the nonlinear coherent states form a sub-Poissonian statistics.

Finally, we mention here that one can also express the mean photon number \( \langle n \rangle \), Mandel parameter \( Q \) and the second-order correlation function \( g^{(2)}(0) \) in terms of the normalization
constant $\tilde{N}(|\alpha|^2)$ as well [36, 42]. To implement this, we redefine $|\alpha|^2 = y$ so that $\tilde{N}^2(|\alpha|^2) = N(y)$. In this case equations (50), (48) and (54) can be expressed in terms of $N(y)$ of the form

$$\langle \hat{n} \rangle = \frac{3}{2} \frac{N(y)}{N(y)} + 3,$$

$$Q = \left( \frac{y^2 N''(y) + 4y N'(y)}{y N'(y) + 3N(y)} \right) - y \frac{N(y)}{N(y)} - 1,$$

$$g^{(2)}(0) = \left( \frac{N(y)}{y N'(y) + 3N(y)} \right) \left( \frac{y^2 N''(y) + 4y N'(y)}{y N'(y) + 3N(y)} + 2 \right).$$

respectively.

5. Gazeau–Klauder coherent states

In this section, we construct Gazeau–Klauder coherent states [24] for equation (7). To begin with, we consider the Hamiltonian $\tilde{H}$ satisfying the Schrödinger equation

$$\tilde{H} |n\rangle = e_n |n\rangle, \quad e_n = n \beta^2(n).$$

The Hamiltonian $\tilde{H}$ can be factorized in the number basis as

$$\tilde{H} = \hat{B} \hat{B}^\dagger,$$

where $\hat{B}^\dagger, \hat{B}$ are the creation and annihilation operators, respectively defined by [31]

$$\hat{B} = \hat{B} e^{i\gamma (\hat{e}_n \hat{e}_{n-1} - \hat{e}_n - 1)}, \quad \hat{B}^\dagger = e^{-i\gamma (\hat{e}_n \hat{e}_{n-1} - \hat{e}_n - 1)} \hat{B}^\dagger, \quad \gamma \in \mathbb{R}.$$

They act on the states $|n\rangle$ as

$$\hat{B} |n\rangle = \sqrt{e_n} e^{i\gamma (e_n - e_{n-1})} |n - 1\rangle, \quad \hat{B}^\dagger |n\rangle = \sqrt{e_{n+1}} e^{-i\gamma (e_n - e_{n-1})} |n + 1\rangle.$$

The Gazeau–Klauder coherent states $|z, \gamma\rangle$ are defined as the eigenstates of annihilation operator $\hat{B}$. They satisfy the eigenvalue equation [24]

$$\hat{B} |z, \gamma\rangle = z |z, \gamma\rangle, \quad z \in \mathbb{C}.$$ (63)

Substituting

$$|z, \gamma\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad n \neq 1, 2,$$

into equation (63) and equating the coefficients of $\psi_n$ in the resultant equation we get

$$c_{n+1} \sqrt{e_n} e^{i\gamma (e_n - e_{n-1})} = z c_n.$$ (65)

From (65), we find $c_0 = 0$ and one can express $c_n$ in terms of $c_3$ as

$$c_n = \frac{z^{n-3} e^{-i\gamma \epsilon_n}}{\sqrt{n!} \ f(n)! c_3},$$ (66)

with $\epsilon_n = n(n-1)(n-3)$. We mention here that the coherent states defined for the Hamiltonian (59) do not contain the states with a photon number less than 3. The coherent states containing the states with a photon number starting from 3 now reads

$$|z, \gamma\rangle = \tilde{N}(|\alpha|^2) \sum_{n=3}^{\infty} \frac{z^{n-3} e^{-i\gamma \epsilon_n}}{\sqrt{n!} \ f(n)!} |n\rangle.$$ (67)
To determine $\tilde{N}(|z|^2)$, we use the normalization condition, $\langle z, \gamma | z, \gamma \rangle = 1$, which in turn yields

$$
\tilde{N}(|z|^2) = \left( \sum_{n=0}^{\infty} \frac{|z|^{2n-6}}{\hbar^n [f(n)]^2} \right)^{-1/2}. \tag{68}
$$

It is clear that the nonlinear coherent states are continuous in $z \in \mathbb{C}$ and are temporally stable under the evolution operator. The latter can be confirmed by verifying the condition

$$
e^{-iHt}|z, \gamma \rangle = |z, \gamma + t\rangle. \tag{69}
$$

To confirm that the Gazeau–Klauder coherent states resolve the identity one must find a measure $d\mu(z) = W(|z|^2) d^2z$ such that [36]

$$
\frac{1}{\pi} \int_{\mathbb{C}} \langle \psi | z, \gamma \rangle W(|z|^2) | z, \gamma \rangle d^2z = 1, \tag{70}
$$

where the integration should be carried over the entire complex plane. Now let us evaluate $\langle \psi | \hat{I} | \Phi \rangle$. From (70) we have

$$
\frac{1}{\pi} \int_{\mathbb{C}} \langle \psi | z, \gamma \rangle W(|z|^2) | z, \gamma \rangle d^2z = \langle \psi | \hat{I} | \Phi \rangle. \tag{71}
$$

Substituting $|z, \gamma \rangle$ and its conjugate into (71), we get

$$
\frac{1}{\pi} \sum_{m,n=0}^{\infty} \frac{\langle \psi | n \rangle \langle m | \Phi \rangle}{\sqrt{n!m!} f(n)! f(m)!} \int_{\mathbb{C}} z^{n-3} \tilde{N}^2(|z|^2) W(|z|^2) d^2z = \langle \psi | \hat{I} | \Phi \rangle. \tag{72}
$$

The integral on the left-hand side in equation (72) can be evaluated along the same lines as given in equations (42)–(44). As a result one can bring the left-hand side of equation (72) to the form $\sum_{n=0}^{\infty} \langle \psi | n \rangle \langle n | \Phi \rangle$. In this case also, the final expression turns out to be a projector for which the the ground state $|0\rangle$ is excluded.

Using equation (63), one can obtain the action identity [24]

$$
\langle z, \gamma | \hat{H} | z, \gamma \rangle = |z|^2. \tag{73}
$$

The function $e_n = nf^2(n)$ has zeros at $n = 0, 1$ and 3. Since we have zeros in the operators, the Fock space breaks up into a countable number of irreducible representations. As cited in [25] if the zeros are simple zeros, some of the reduced pieces do not allow a unitary representation and the associated coherent states cannot be expressed as Klauder–Perelomov’s one.

We find that the photon number distribution $P(n)$ for the Gazeau–Klauder coherent states of the nonlinear oscillator (2) is in the form

$$
P(n) = |\langle n | z, - \rangle|^2 = \frac{\tilde{N}^2(|z|^2) |z|^{2n-6}}{n! [f(n)]^2}, \tag{74}
$$

where we have used (67) to calculate $P(n)$. The Mandel parameter, $Q$, and the second-order correlation function, $g^{(2)}(0)$, turn out to be

$$
Q = 3 \left( \frac{F_3(4; 3, 3, 3; |z|^2)}{F_3(4; 3, 3, 4; |z|^2)} - \frac{F_3(3; 3, 3, 4; |z|^2)}{F_3(4; 3, 3, 4; |z|^2)} \right) - 1, \tag{75}
$$

$$
g^{(2)}(0) = \left( \frac{F_3(3; 3, 3, 4; |z|^2)}{F_3(4; 3, 3, 4; |z|^2)} \right) \left( \frac{F_3(4; 3, 3, 3; |z|^2)}{F_3(4; 3, 3, 4; |z|^2)} \right) - \frac{1}{3}, \tag{76}
$$

respectively. For $|z| > 0$ we find $Q < 0$ and these negative values of $Q$ indicate that the Gazeau–Klauder coherent states possess the sub-Poissonian distribution. Again for $|z| > 0$ we find $g^{(2)}(0) < 1$ which in turn confirms the anti-bunching of the light field.
6. Even and odd nonlinear coherent states

In this section, we construct even and odd nonlinear coherent states [32] for the Schrödinger equation (2). The even and odd nonlinear coherent states are the symmetric and antisymmetric combination of the nonlinear coherent states. They are two orthonormalized eigenstates of the square of the annihilation operator and essentially have two kinds of nonclassical effects: the even nonlinear coherent states have squeezing but no anti-bunching while the odd nonlinear coherent states have anti-bunching but no squeezing. The even and odd nonlinear coherent states [33] are defined as the eigenstates of $\hat{A}^2$:

$$\hat{A}^2 |\psi\rangle = a |\psi\rangle.$$  (77)

Substituting $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, $n \neq 1, 2$, into equation (77) and expanding we get

$$\sum_{n=0}^{\infty} c_n \sqrt{n(n-1)} f(n) f(n-1) |n-2\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle.$$  (78)

From (78) we find the constant $c_0 = 0$.

To determine the value of the rest of the coefficients we proceed as follows. Equating the coefficients of $|\psi_n\rangle$ in equation (78) and denoting

$$F(n) = \sqrt{n+1} f(n+1) f(n+2)$$  (79)

in that expression we find a relation between $c_{n+2}$ and $c_n$ of the form

$$\sqrt{n+2} F(n) c_{n+2} = \alpha c_n,$$  (80)

from which we obtain

$$\sqrt{n} F(n) c_n = \alpha c_{n-2}.$$  (81)

In the case when $n$ is an even number, we can redefine $n = 2n$ and fix

$$c_{2n} = \frac{\alpha}{\sqrt{2n} F(2n-2)} c_{2n-2}.$$  (82)

Evaluating $c_{2n-2}$ in terms of $c_{2n-4}$ and so on, one can find the value of the even number coefficients as

$$c_{2n} = \frac{\alpha^{n-2}}{\sqrt{2n} \sqrt{2n-2} \cdots \sqrt{2}} F(2n-2) F(2n-4) \cdots F(4) c_4.$$  (83)

On the other hand if $n$ is an odd number we can redefine $n = 2n+1$ and fix

$$c_{2n+1} = \frac{\alpha}{\sqrt{2n+1} F(2n-1)} c_{2n-1}.$$  (84)

Now evaluating $c_{2n-1}$ in terms of $c_{2n-3}$ and so on we find the value of the odd number coefficients as

$$c_{2n+1} = \frac{\alpha^{n-2}}{\sqrt{(2n+1)(2n-1)(2n-3)} \cdots \sqrt{3} F(2n-1) F(2n-3) \cdots F(3)} c_3.$$  (85)

For simplicity we redefine the constants $c_{2n}$ and $c_{2n+1}$ as

$$c_{2n} = \frac{\alpha^{n-2}}{\sqrt{(2n)!!} F(2n-2)!!} c_4, \quad c_{2n+1} = \frac{\alpha^{n-2}}{\sqrt{(2n+1)!!} F(2n-1)!!} c_3,$$  (86)

where

$$\widetilde{(2n)}!! = 2n.(2n-2).(2n-4) \cdots 6,$$  (87)
\[ \tilde{F}(2n - 2)!! = F(2n - 2)F(2n - 4)F(2n - 6) \ldots F(4), \quad (88) \]
\[ \tilde{F}(2n + 1)!! = (2n + 1).(2n - 1).(2n - 3) \ldots 5, \quad (89) \]
\[ \tilde{F}(2n - 1)!! = F(2n - 1)F(2n - 3) \ldots F(3). \quad (90) \]

As a result, one gets even and odd nonlinear coherent states for equation (7) respectively of the form
\[
|\alpha, \tilde{F}, + \rangle = c_4 \sum_{n=1}^{\infty} \frac{\alpha^{n-2}}{\sqrt{(2n)!! \tilde{F}(2n - 2)!!}} |2n\rangle, \quad (91)
\]
\[
|\alpha, \tilde{F}, - \rangle = c_3 \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(2n + 1)!! \tilde{F}(2n - 1)!!}} |2n + 1\rangle, \quad (92)
\]
where \(\tilde{F}(1)!! = \frac{3}{2}!! = 1\).

The constants \(c_4(=N_\text{e})\) and \(c_3(=N_\text{o})\) can be fixed easily through the normalization procedure \(|\alpha, \tilde{F}, \pm|\alpha, \tilde{F}, \pm \rangle = 1\), which turns out to be
\[
N_\text{e} = \left( \sum_{n=2}^{\infty} \frac{|\alpha|^{2n-4}}{(2n)!! [\tilde{F}(2n - 2)!!]^2} \right)^{-1/2},
\]
\[
N_\text{o} = \left( \sum_{n=1}^{\infty} \frac{|\alpha|^{2n-2}}{(2n + 1)!! [\tilde{F}(2n - 1)!!]^2} \right)^{-1/2}. \quad (93)
\]

The photon number distribution and mean photon number of even nonlinear coherent states turns out to be
\[
\langle \hat{n} \rangle = N = 4 \left( \frac{\up{3}F_5(; 2, 2, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)}{\up{3}F_5(; 2, 2, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)} \right),
\]
\[
P(2n) = |\langle 2n|\alpha, \tilde{F}, + \rangle|^2 = \frac{N_\text{e}^2|\alpha|^{2n-4}}{(2n)!! [\tilde{F}(2n - 2)!!]^2}. \quad (94)
\]

We find the Mandel parameter \(Q\) and the second-order correlation function \(g^{(2)}(0)\) of even nonlinear coherent states of the form
\[
Q = 4 \left( \frac{\up{3}F_5(; 3, 2, 2, 3, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)}{\up{3}F_5(; 2, 2, 3, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)} \right) - 4 \left( \frac{\up{3}F_5(; 2, 2, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)}{\up{3}F_5(; 2, 2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)} \right) - 1,
\]
\[
g^{(2)}(0) = \left( \frac{\up{3}F_5(; 2, 2, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)}{\up{3}F_5(; 2, 2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)} \right) \left( \frac{\up{3}F_5(; 3, 2, 2, 3, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)}{\up{3}F_5(; 2, 2, 3, \frac{5}{2}, \frac{5}{2}; |\alpha|^2)} \right) - 1 \right) / 4, \quad (95)
\]
respectively.

We plot the mean photon number \(\langle \hat{n} \rangle\), Mandel parameter \(Q\) and the second-order correlation function \(g^{(2)}(0)\) as a function of \(r(=|\alpha|^2)\) for the even nonlinear coherent states (91) in figures 2(a)–(d), respectively. The Mandel parameter \(Q\) and the second-order correlation function \(g^{(2)}(0)\) depicted in figures 2(c) and (d) confirm the sub-Poissonian statistics established by the even nonlinear coherent states for all positive values of \(r(=|\alpha|^2)\).

On the other hand, the photon number distribution, mean photon number, Mandel parameter \(Q\) and the second-order correlation function \(g^{(2)}(0)\) of odd nonlinear coherent
The states turn out to be

\( \langle \hat{n} \rangle = N = 3 \left( \frac{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})}{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})} \right) \),

\[ P(2n + 1) = |\langle 2n + 1 | \alpha, \bar{F}, - \rangle|^2 = \frac{N_0^2 \alpha^{2n-2}}{(2n + 1)!! \ [\bar{F} (2n - 1)!!]²} \],

\[ Q = 3 \left( \frac{i F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})}{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})} \right) - 3 \left( \frac{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})}{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})} - 1 \right), \]

\[ g^2(0) = \left( \frac{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})}{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})} \right) \left( \frac{i F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})}{\alpha F_3(\frac{3}{2}; 2, 2, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{5}{4})} - 1 \right), \]

respectively. We plot the mean photon number \( \langle \hat{n} \rangle \), Mandel parameter \( Q \) and the second-order correlation function \( g^2(0) \) as a function of \( r = |\alpha|^2 \) for the odd nonlinear coherent states (96) in figures 3(a)–(d), respectively. The Mandel parameter \( Q \) and the second-order correlation function \( g^2(0) \) depicted in figures 3(c) and (d) confirm the sub-Poissonian statistics established by the odd nonlinear coherent states for the values of \( r > 0 \).
7. \(f\)-deformed coherent states for the position-dependent mass nonlinear oscillator

Using the point canonical transformation method, Kraenkel and Senthilvelan [3] have shown that the position-dependent mass Schrödinger equation

\[-\frac{1}{2} \frac{d}{dy} \left( \frac{1}{m(y)} \frac{d\tilde{\psi}_n(y)}{dy} \right) + \tilde{V}(y)\tilde{\psi}_n(y) = \tilde{E}_n \tilde{\psi}_n(y), \tag{97}\]

where

\[\tilde{V}(y) = \frac{1}{2} \left( \eta^2 + \frac{8(2\eta^2 - 1)}{(1 + 2\eta^2)^2} + 3 \right) + \frac{m''}{8m^2} - \frac{7m^2}{32m^3} \]
\[\frac{d\eta}{dy} \leq m^{1/2}(y) \tag{98}\]

is also exactly solvable. In the above \(m(y)\) is the mass distribution function which depends on the coordinate \(y\). The authors have obtained the eigenfunctions and energy eigenvalues of (97) in the form

\[\tilde{\psi}_n(y) = \tilde{N}_n m^{1/4}(y) \frac{\mathcal{P}_n(\eta(y))}{(1 + 2y^2)^2} e^{-2y^2}, \quad \tilde{E}_n = \tilde{E}_n = -\frac{3}{2} + n, \quad n = 0, 3, 4, \ldots \tag{99}\]

For the sake of illustration the authors have considered three different types of mass distributions which are often being considered in semiconductor physics and constructed bound state energies and wavefunctions for these cases.

In the following, we consider the position-dependent mass Schrödinger equation (97) associated with the nonlinear oscillator and construct its nonlinear coherent states.
To determine the deformed ladder operators $\hat{A}, \hat{A}^\dagger$ we recall the definitions

$$\hat{A} \tilde{\psi}_n = c_1(n) \tilde{\psi}_{n-1},$$  
$$\hat{A}^\dagger \tilde{\psi}_n = c_2(n) \tilde{\psi}_{n+1},$$

with $\tilde{\psi}_n$ given in (99). Assuming, $\hat{A}$ and $\hat{A}^\dagger$ are of the same forms (10) and (11) and demanding that they should satisfy conditions (100) and (101) we find that the deformed ladder operators $\hat{A}$ and $\hat{A}^\dagger$ to be of the form

$$\sqrt{2} \hat{A} = \left[ \frac{2(2\eta^2 - 1)}{(1 + 2\eta^2)^2} - 1 \right] \left[ \frac{1}{\sqrt{m(y)}} \frac{d}{dy} + \frac{1}{\sqrt{m(y)}} \frac{d}{dy} \eta \right] \hat{n},$$

$$\sqrt{2} \hat{A}^\dagger = \left[ \frac{2(1 - 2\eta^2)}{(1 + 2\eta^2)^2} \right] \left[ \frac{1}{\sqrt{m(y)}} \frac{d}{dy} + \frac{1}{\sqrt{m(y)}} \frac{d}{dy} \eta \right] \hat{n}.$$  

We mention here that to determine these two operators one needs to derive two recurrence relations in the $P$-Hermite polynomials of the form

$$\left[ n - 1 + \frac{2(2\eta^2 - 1)}{(1 + 2\eta^2)^2} \right] P_n'(\eta(y)) + n(\eta - \phi)P_n(\eta(y)) = 2n(n - 3)P_{n-1}(\eta(y)),$$

$$\left[ n + \frac{2(2\eta^2 - 1)}{(1 + 2\eta^2)^2} \right] P_n'(\eta(y)) + n(\eta + \phi)P_n(\eta(y)) = nP_{n+1}(\eta(y)),$$

where $\phi = \eta + \frac{4n}{(1 + 2\eta^2)}$ and $\eta(y)$ is given in equation (98).

It is now a simple matter to verify

$$\hat{A} \tilde{\psi}_n = \sqrt{n} f(n) \tilde{\psi}_{n-1},$$

$$\hat{A}^\dagger \tilde{\psi}_n = \sqrt{n+1} f(n + 1) \tilde{\psi}_{n+1},$$

with $f(n) = \sqrt{(n - 1)(n - 3)}$.

Since the deformed ladder operators for the PDMSE (97) are also of the same form as in the case of the constant mass Schrödinger equation, except the presence of position-dependent mass terms, the intelligent states and nonlinear coherent states for equation (97) can be constructed in the same way as in the constant mass case. The resultant expressions read

$$|\alpha, \tilde{f}, \lambda\rangle = c_3 \sum_{n=3}^{\infty} \frac{\lambda_{\hat{A}^\dagger}}{\sqrt{n}! f(n)!} \tilde{\psi}_n,$$

and

$$|\alpha, \tilde{f}\rangle = \hat{N} \sum_{n=3}^{\infty} \frac{\alpha^n}{\sqrt{n}! f(n)!} \tilde{\psi}_n.$$  

In the above, the arbitrary constant $\hat{N}$ can be fixed by the normalization procedure which turns out to be

$$\hat{N} = \left( \sum_{n=3}^{\infty} \frac{|\alpha|^{2n-6}}{\hat{n}! [f(n)!]^2} \right)^{-1/2}.$$
The Gazeau–Klauder coherent states for the position-dependent mass Schrödinger equation (98) read

\[ |z, γ\rangle = \tilde{N}(|z|^2) \sum_{n=3}^{∞} \frac{e^{-\frac{3}{2} |z|^2 n}}{\sqrt{n!} n!} \tilde{f}(n) \tilde{ψ}_n, \tag{111} \]

with

\[ \tilde{N}(|z|^2) = \left( \sum_{n=3}^{∞} \frac{|z|^{2n-6}}{n! [f(n)]^2} \right)^{-1/2}. \tag{112} \]

The even and odd nonlinear coherent states for equation (98) can also be constructed along the same line as given in the constant mass case. The resultant expressions read

\[ |α, \tilde{F}, +\rangle = N_e \sum_{n=2}^{∞} \frac{\alpha^{n-2}}{\sqrt{(2n)!} !! \tilde{F}(2n-2)!!} \tilde{ψ}_{2n}, \tag{113} \]

\[ |α, \tilde{F}, -\rangle = N_o \sum_{n=1}^{∞} \frac{\alpha^{n-1}}{\sqrt{(2n+1)!} [\tilde{F}(2n-1)!!]^{-1}} \tilde{ψ}_{2n+1}, \tag{114} \]

where the normalization constants \( N_e \) and \( N_o \) are given by

\[ N_e = \left( \sum_{n=2}^{∞} \frac{|α|^{2n-4}}{(2n)!! [F(2n-2)!!]^2} \right)^{-1/2}, \]

\[ N_o = \left( \sum_{n=1}^{∞} \frac{|α|^{2n-2}}{(2n+1)!! [F(2n-1)!!]^2} \right)^{-1/2}, \tag{115} \]

respectively.

7.1. Example

In the following, we consider a specific mass profile and give the explicit forms of the nonlinear coherent states for the position-dependent mass Schrödinger equation (98). Let us consider the mass profile [43]

\[ m(y) = \frac{(γ + y^2)^2}{(1 + y^2)^2}, \quad γ = \text{constant}, \tag{116} \]

which is found to be useful for studying transport properties in semiconductors [34, 35]. We also note that the normalization constant is the same as the constant mass Schrödinger equation (2). Substituting (116) into equation (97) we get

\[ \left[ -\frac{1}{2} \frac{d}{dy} \left( \frac{1 + y^2}{(γ + y^2)^2} \frac{d}{dy} \right) + \frac{1}{2} \left( \eta^2 + \frac{8(2η^2 - 1)}{(1 + 2η^2)^2} \right) + \frac{γ - 1}{2(γ + y^2)^2} \right] \tilde{ψ}_n(y) = \tilde{E}_n \tilde{ψ}_n(y), \tag{117} \]

where \( η(y) \) is given by

\[ η(y) = \int m^{\frac{1}{2}}(y)dy = y + (γ - 1) \tan^{-1} y, \quad -∞ < η(y) < ∞. \tag{118} \]

The coherent states given in equations (108), (109), (111), (113) and (114) yield
of the nonlinear oscillator potential (1).

The results obtained in this paper will give deep insights into geometrical properties of the nonlinear oscillator potential (1).

Further, we have studied the Hilbert space properties and certain nonclassical properties of these nonlinear coherent states, even and odd nonlinear coherent states, Gazeau–Klauder coherent states and even and odd nonlinear coherent states for this system. Deviating from the conventional way, the ladder operators for this exactly solvable quantum system have been derived from the solution of the Schrödinger equation. Two recursion relations involving nonlinearity function $f(\hat{\eta})$ in the Fock space becomes zero at two values. As a consequence, the generalized intelligent states, even and odd nonlinear coherent states constructed by us decomposes into two sub-sets in the Fock space. We have also constructed the generalized intelligent states, Gazeau–Klauder coherent states and even and odd nonlinear coherent states for this system. Further, we have studied the Hilbert space properties and certain nonclassical properties of these nonlinear coherent states. In addition to the above, we have considered position-dependent mass Schrödinger equation associated with equation (2) and constructed nonlinear coherent states, even and odd nonlinear coherent states, Gazeau–Klauder coherent states for it. This is motivated by the fact that in a wide variety of physical problems an effective mass depending on the position is of utmost relevance. To cite one such situation, in the context of the effective mass of an electron hole in the thin layered quantum wells varies with the composition rate. In such systems, the mass of the electron may change with the composition rate, which depends on the position. As a consequence, attempts have been made to analyse such PDMSE and their underlying properties for a number of potentials and masses. One such mass profile which is found to be useful for studying transport properties in semiconductors is (116). We have given explicit expressions for these nonlinear coherent states for this mass profile. The results obtained in this paper will give deep insights into geometrical properties of the nonlinear oscillator potential (1).

\begin{equation}
|\alpha, f, \lambda \rangle = c_j \sum_{n=3}^{\infty} \frac{\tilde{A}_n!}{\sqrt{n! f(n)!}} \left( \frac{(\gamma + y^2)^2}{(1 + y^2)^2} \right)^{1/4} \frac{P_n(\eta) e^{-\frac{\alpha^2}{2}}}{(1 + 2\eta^2)}, \quad (119)
\end{equation}

\begin{equation}
|\alpha, \tilde{f} \rangle = \tilde{N} \sum_{n=3}^{\infty} \frac{\alpha^{n-3}}{\sqrt{n! f(n)!}} \left( \frac{(\gamma + y^2)^2}{(1 + y^2)^2} \right)^{1/4} \frac{P_n(\eta) e^{-\frac{\alpha^2}{2}}}{(1 + 2\eta^2)}, \quad (120)
\end{equation}

\begin{equation}
|\tilde{z}, \tilde{\gamma} \rangle = \tilde{N}(|z|^2) \sum_{n=3}^{\infty} \frac{z^{n-3} e^{-iyc}}{\sqrt{n! f(n)!}} \left( \frac{(\gamma + y^2)^2}{(1 + y^2)^2} \right)^{1/4} \frac{P_{2n}(\eta) e^{-\frac{\alpha^2}{2}}}{(1 + 2\eta^2)}, \quad (121)
\end{equation}

\begin{equation}
|\alpha, \tilde{F}, + \rangle = N_e \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{\sqrt{(2n)!! F(2(n+1))!!}} \left( \frac{(\gamma + y^2)^2}{(1 + y^2)^2} \right)^{1/4} \frac{P_{2n}(\eta) e^{-\frac{\alpha^2}{2}}}{(1 + 2\eta^2)}, \quad (122)
\end{equation}

\begin{equation}
|\alpha, \tilde{F}, - \rangle = N_o \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(2n+1)!! F(2(n+1))!!}} \left( \frac{(\gamma + y^2)^2}{(1 + y^2)^2} \right)^{1/4} \frac{P_{2n+1}(\eta) e^{-\frac{\alpha^2}{2}}}{(1 + 2\eta^2)}, \quad (123)
\end{equation}

where $\tilde{N}(|\alpha|^2), \tilde{N}(|z|^2), N_e$ and $N_o$ are given in equations (110), (112) and (115), respectively.

8. Conclusion

In this paper, we have considered the newly solvable nonlinear oscillator that is related to the isotonic oscillator and constructed nonlinear coherent states for it. Deviating from the conventional way, the ladder operators for this exactly solvable quantum system have been derived from the solution of the Schrödinger equation. Two recursion relations involving $P$-Hermite polynomials have been derived to obtain the ladder operators. The same methodology has been adopted to explore the generalized/deformed annihilation and creation operators for the position-dependent mass Schrödinger equation also. We have also shown that these operators satisfy the Heisenberg algebra in the Fock space. We found that the nonlinearity function $f(\hat{\eta})$ in the Fock space becomes zero at two values. As a consequence, the generalized intelligent states and nonlinear coherent states constructed by us decomposes into two sub-sets in the Fock space. We have also constructed the generalized intelligent states, Gazeau–Klauder coherent states and even and odd nonlinear coherent states for this system.
Appendix. Recurrence relations

In this section we derive two basic recurrence relations which are useful in constructing ladder operators for the system (5). To begin with we recall the Hermite differential equation,

\[ H_n'' - 2x H_n' + 2n H_n = 0, \]

and two of its associated recurrence relations, that is

\[ H_n' = 2n H_{n-1}, \]

\[ 2x H_n = H_{n+1} + 2n H_{n-1}, \]

where prime denotes differentiation with respect to \( x \). In [1], the authors have introduced a new family of polynomials \( P_n(x) \) (vide equation (8) in [1]) defined by

\[ P_n(x) = H_n + 4n H_{n-2} + 4n(n - 3) H_{n-4}, \quad n = 3, 4, \ldots \]  
\[ P_n'(x) = 4n(1 + 2x^2) H_{n-3} \]

and

\[ \frac{P_n(x)e^{-x^2}}{(1 + 2x^2)^2} = -2 \frac{d}{dx} \left[ \frac{H_{n-3}}{1 + 2x^2} e^{-x^2} \right], \quad n = 3, 4, \ldots \]  

Substituting equations (A.2) and (A.3) into (A.6) we get

\[ P_n(x) = 2(1 + 2x^2) H_{n-2} + 8x H_{n-3}. \]

As our aim is to obtain a recurrence relation that connects \( P_n(x) \) with \( P_n'(x) \) and \( P_{n-1}(x) \), we revoke equation (A.6) in the form

\[ P_{n-1}(x)e^{-x^2} = -2 \frac{d}{dx} \left[ \frac{H_{n-4}}{(1 + 2x^2)^2} e^{-x^2} \right], \quad n = 4, 5, 6, \ldots \]  

Substituting (A.2) into (A.8), we find

\[ \frac{P_{n-1}(x)e^{-x^2}}{(1 + 2x^2)^2} = -\frac{1}{(n - 3)} \frac{d}{dx} \left[ \frac{H_{n-3}'}{(1 + 2x^2)^2} e^{-x^2} \right] \quad n = 4, 5, 6, \ldots \]  

With the help of (A.2) and (A.3), equation (A.9) can be brought to the form

\[ (n - 3)P_{n-1}(x) = [2n(1 + 2x^2) - (6 + 4x^2)]H_{n-3} - 4x H_{n-2}, \quad n = 4, 5, 6, \ldots \]

Multiplying equation (A.7) by 2n and equation (A.10) by \( 1 + 2x^2 \) and adding the latter two equations and simplifying the resultant expression we arrive at

\[ \frac{2x}{(1 + 2x^2)} P_n(x) + (n - 3)P_{n-1}(x) = 2(1 + 2x^2) \left[ n - \frac{3 + 4x^4}{(1 + 2x^2)^2} \right] H_{n-3}. \]

Now multiplying equation (A.11) by 2n and using (A.5) in it, one can obtain a recurrence relation which connects \( P_n'(x) \) with \( P_n(x) \) and \( P_{n-1}(x) \), namely

\[ \left[ n - 1 + \frac{2(2x^2 - 1)}{(1 + 2x^2)^2} \right] P_n'(x) + n(x - \phi) P_n(x) = 2n(n - 3)P_{n-1}(x), \]

where we have defined \( \phi = x + \frac{4x}{1 + 2x^2} \).

Now we derive the second recurrence relation which connects \( P_n'(x) \) with \( P_n(x) \) and \( P_{n+1}(x) \). To do so we consider equation (A.6) in the form

\[ \frac{P_{n+1}(x)e^{-x^2}}{(1 + 2x^2)^2} = -2 \frac{d}{dx} \left[ \frac{H_{n-2}}{(1 + 2x^2)^2} e^{-x^2} \right], \quad n = 2, 3, 4, \ldots \]
Substituting equations (A.2) and (A.3) into (A.13) and simplifying it we obtain
\[ P_{n+1}(x) = -4(n - 2)(1 + 2x^2)H_{n-1} + 2(x + \phi)(1 + 2x^2)H_n. \]  
(A.14)

Multiplying equation (A.7) by \( (x + \phi) \) and subtracting it from (A.14) and simplifying the resultant expression, one obtains
\[ P_{n+1}(x) - (x + \phi)P_n(x) = -4(1 + 2x^2) \left[ n + \frac{2(2x^2 - 1)}{(1 + 2x^2)^2} \right] H_{n-3}. \]  
(A.15)

Now one can replace the Hermite polynomial \( H_{n-3} \) in (A.15) by \( P'_n(x) \) (vide equation (A.5)). As a result one gets
\[ -\left[ n + \frac{2(2x^2 - 1)}{(1 + 2x^2)^2} \right] P'_n(x) + n(x + \phi)P_n(x) = nP_{n+1}(x). \]  
(A.16)

Recurrence relations (A.12) and (A.16) can be used to construct the ladder operators.

References
[1] Cariñena J F, Perelomov A M, Ranada M F and Santander M 2008 J. Phys. A: Math. Theor. 41 085301
[2] Fellows J M and Smith R A 2009 J. Phys. A: Math. Theor. 42 335303
[3] Kraenkel R A and Senthilvelan M 2009 J. Phys. A: Math. Theor. 42 415303
[4] Gazeau J-P 2009 Mod. Phys. Lett. A 24 131105
[5] Kraenkel R A and Senthilvelan M 2009 J. Phys. A: Math. Theor. 42 415303
[6] Sesma J 2010 J. Phys. A: Math. Theor. 43 185303
[7] Vogel W and Welsch D-G 2006 Quantum Optics (Weinheim: Wiley-VCH)
[8] Nieto M M and Truax D R 1993 Phys. Rev. Lett. 71 2843
[9] López-Peña R, Man’ko V I, Marmo G and Zaccarria F 2000 J. Russ. Laser Res. 21 305
[10] de Matos Filho R L and Vogel W 1996 Phys. Rev. A 54 4560
[11] Man’ko V I, Marmo G, Sudarshan E C G and Zaccarria F 1997 Phys. Scr. 55 528
[12] Shanta P, Chaturvedi S, Srimivasan V and Jagannathan R 1994 J. Phys. A: Math. Gen. 27 6433
[13] Roknizadeh R and Tavassoly M K 2004 J. Phys. A: Math. Gen. 37 8111
[14] Walls D F and Zoller P 1981 Phys. Rev. Lett. 47 709
[15] Walls D F 1983 Nature 306 141
[16] Roy B and Roy P 2000 J. Opt. B: Quantum Semiclass. Opt. 2 65
[17] Choquette J J, Cordes J G and Kiang D 2003 J. Opt. B: Quantum Semiclass. Opt. 5 56
[18] Wang J, Feng J, Liu T K and Zhan M S 2002 J. Phys. B: At. Mol. Opt. Phys. 35 2411
[19] Kwek L C and Kiang D 2003 J. Opt. B: Quantum Semiclass. Opt. 5 383
[20] Paul H 1982 Rev. Mod. Phys. 54 1061
[21] Mahr M H and Venkata Satyanarayana M 1986 Phys. Rev. A 34 640
[22] Dodonov V V 2002 J. Opt. B: Quantum Semiclass. Opt. 4 R1–R33
[23] Kis Z, Vogel W and Davidovich L 2001 Phys. Rev. A 64 033401
[24] Man’ko V I, Marmo G and Zaccarria F 2010 Phys. Scr. 81 045004
[25] Dong S-H and Z-Q Ma 2002 Am. J. Phys. 70 520
[26] Jellal A 2002 Mod. Phys. Lett. A. 17 671
[27] Barut A O and Girardello L 1971 Commun. Math. Phys. 19 41
[28] Gazeau J P and Klauder J R 1999 J. Phys. A: Math. Gen. 32 123
[29] Klauder J R and Skagerstam B S 1985 Coherent States: Applications in Physics and Mathematical Physics (Singapore: World Scientific)
[30] Perelomov A M 1986 Generalized Coherent States and Their Applications (Berlin: Springer)
[31] Glauber R J 1963 Phys. Rev. 131 2766
[32] Schrödinger E 1930 Sitzungsber. Preuss. Akad. Wiss. Phys-Math. Klasse vol 19 (Berlin: Springer) p 296
[33] Robertson H P 1929 Phys. Rev. 34 163
[34] Robertson H P 1934 Phys. Rev. 46 794
[35] Aragone C, Guerri G, Salamo S and Tani J L 1974 J. Phys. A: Math. Nucl. Gen. 7 L149
[36] Dodonov V V, Kurnyshev E V and Man’ko V I 1980 Phys. Lett. A 79 150
[37] El kinani A H and Daoud M 2002 J. Math. Phys. 43 714
[38] Mancini S 1997 Phys. Lett. A 233 291
[33] Sivakumar S 1998 Phys. Lett. A 250 257
[34] Koc R, Koca M and Sahinoglu G 2005 Eur. Phys. J. B 48 583
[35] Gossard A C, Miller R C and Wiegmann W 1986 Surf. Sci. 174 131
[36] Klauder J R, Penson K A and Sixdeniers J M 2001 Phys. Rev. A 64 013817
[37] Chaturvedi S 1996 Mod. Phys. Lett. A 11 2805
[38] Gradshteyn I S and Ryzhik I M 1980 Table of Integrals, Series and Products (New York: Academic)
[39] Mandel L 1979 Opt. Lett. 4 205
[40] Antoine J P, Gazeau J P, Monceau P, Klauder J R and Penson K A 2001 J. Math. Phys. 42 2349
[41] Tavassoly M K 2006 J. Phys. A: Math. Gen. 39 11583
[42] Shreecharan T and Shiv Chaitanya K V S 2010 Aspects of coherent states of nonlinear algebras, arXiv:1005.5607v1
[43] Plastino A R, Rigo A, Casas M, Gracias F and Plastino A 1999 Phys. Rev. A 60 4318