Quasi-Hermitian one-dimensional lattice

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Abstract

We show that a non-Hermitian operator with a tridiagonal matrix representation in a finite-dimensional vector space is similar to a Hermitian operator. The required condition is sufficient and simple examples show that it is not necessary. We derive quite general features of the eigenvalues and eigenvectors for a somewhat particular case.

1 Introduction

Non-Hermitian quantum mechanics has become quite popular in recent years [1,2] because of its intrinsic mathematical interest and as a suitable tool for the interpretation of some physical phenomena. In particular, exactly solvable models given in terms of tridiagonal matrices have proved useful for deriving and illustrating some properties of non-Hermitian systems [3–10]. Some non-Hermitian operators exhibit generalized Hermiticity [11] or quasi-Hermiticity [12] that provides the condition for a linear operator to be similar to a self-adjoint one [13].

The purpose of this paper is a discussion of a non-Hermitian operator with a finite tridiagonal matrix representation, similar to those discussed earlier [3–10], though somewhat more general. In section 2 we derive the main result, in
section 3 we discuss some illustrative examples and in section 4 we summarize the main results of this paper and draw conclusions.

2 Quasi-Hermitian one-dimensional lattice

Throughout this paper we consider the quantum-mechanical model given by the Hamiltonian operator

$$H = \sum_{j=1}^{N-1} (\alpha_j |j\rangle \langle j+1| + \beta_j |j+1\rangle \langle j|) + \sum_{j=1}^{N} \omega_j |j\rangle \langle j|,$$

where \{|j\rangle, j = 1, 2, \ldots, N\} is an orthonormal basis set. This operator is Hermitian if \(\omega^*_j = \omega_j\) and \(\alpha_j = \beta_j^*\).

Any eigenvector \(|\psi\rangle\) of \(H\) is a linear combination of the basis vectors

$$|\psi\rangle = \sum_{j=1}^{N} c_j |j\rangle,$$

and it follows from the eigenvalue equation \(H |\psi\rangle = E |\psi\rangle\) that the coefficients \(c_j\) satisfy the three-term recurrence relation

$$\alpha_j c_{j+1} + (\omega_j - E) c_j + \beta_{j-1} c_{j-1} = 0, \quad j = 1, 2, \ldots, N,$$

$$c_0 = c_{N+1} = 0. \quad (3)$$

In what follows we resort to a mathematical argument used earlier by Child et al.\[15\] and Amore and Fernández\[16\] for the truncation of some particular three-term recurrence relations. If we substitute \(c_j = Q_j d_j\) into equation (3) and divide by \(Q_j\) we obtain a secular equation for the new coefficients \(d_j\)

$$\frac{\alpha_j}{Q_j} Q_{j+1} d_{j+1} + (\omega_j - E) d_j + \frac{\beta_{j-1}}{Q_j} Q_{j-1} d_{j-1} = 0. \quad (4)$$

The tridiagonal matrix that gives rise to this secular equation is symmetric if

$$Q_{j+1}^2 = \frac{\beta_j}{\alpha_j} Q_j^2. \quad (5)$$

Therefore, we conclude that if \(\alpha_j, \beta_j,\) and \(\omega_j\) are real and \(\alpha_j \beta_j > 0\), then the tridiagonal matrix \(Q\) is symmetric and, consequently, all its eigenvalues are
real. In fact, if we choose \( Q_j/Q_j = \sqrt{\beta_j/\alpha_j} \) then the secular equation \( (6) \)
becomes
\[
\sqrt{\alpha_j \beta_j} d_{j+1} + (\omega_j - E) d_j + \sqrt{\alpha_{j-1} \beta_{j-1}} d_{j-1} = 0.
\]
(6)

Note that the boundary conditions have not changed because \( c_0 = c_{N+1} = 0 \Rightarrow d_0 = d_{N+1} = 0. \)

If \( c \) is a column vector with elements \( c_j \) and \( H \) the tridiagonal square matrix with elements \( H_{ij} = \langle i|H|j \rangle \) then \( Hc = Ec \) is the secular equation \( (3) \) in matrix form. The column vector \( d_j \), with elements \( d_j \), is related to \( c \) by means of the diagonal invertible matrix \( Q \) with elements \( Q_{ij} = \delta_{ij} \). Therefore, we have
\[
Hc = Ec \Rightarrow HQd = EQd \Rightarrow Q^{-1}HQd = Ed,
\]
where the transformed matrix \( \tilde{H} = Q^{-1}HQ \) is symmetric. It follows from \( \tilde{H}^t = \tilde{H} \), where \( t \) stands for transpose, that \( HQ^2 = Q^2H^t \), where \( Q^2 \) is symmetric, invertible and positive definite. Clearly, the matrix \( H \) discussed here is an example of the operator \( T \) in the theorem proved by Williams \[13\]. The matrix elements of \( Q^2 \) are given by
\[
Q^2_j = Q_1^2 j \prod_{k=1}^{j-1} \frac{\beta_{j-k}}{\alpha_{j-k}},
\]
(8)

where \( Q_1 \) is any real nonzero number.

If we choose the metric \( Q^{-2} \) \[9\] (and references therein) then the resulting eigenvectors \( c_k = Qd_k \) can be orthonormalized according to \( c^j \cdot Q^{-2} \cdot c_k = d_j^t d_k = \delta_{ij} \). If \( D = (d_1 \ d_2 \ldots \ d_N) \) is the \( N \times N \) unitary matrix with the column vector \( d_j \) as its \( j \)-th column \( (D^tD = I, \text{where} \ I \text{is the} \ N \times N \text{identity matrix}) \), then \( D^t \tilde{H}D = E \), where \( E_{ij} = E_j \delta_{ij} \). Therefore, \( S^{-1}HS = E \), where \( S = QD \).

By means of the invertible Hermitian operator
\[
Q = \sum_{j=1}^{N} Q_j |j \rangle \langle j |,
\]
(9)
we can define the Hermitian operator \( \tilde{H} = Q^{-1}HQ \) that satisfies \( HQ^2 = Q^2H^\dagger \), where \( \dagger \) stands for Hermitian conjugate. Clearly, \( H \) satisfies the hypothesis of Williams’ theorem \[13\].
We can easily derive a general result about the eigenvalues and eigenvectors of the Hamiltonian operator (11) when \( \omega_j = \omega \) for all \( j \). In this case we can rewrite the secular equation (3) as

\[
\alpha_j c_{j+1} - \epsilon c_j + \beta_{j-1} c_{j-1} = 0, \quad (10)
\]

where \( \epsilon = E - \omega \). If we substitute \( c_j = (-1)^j \tilde{c}_j \) then this equation becomes

\[
\alpha_j \tilde{c}_{j+1} + \epsilon \tilde{c}_j + \beta_{j-1} \tilde{c}_{j-1} = 0. \quad (11)
\]

We conclude that if \( c_k \) is an eigenvector of \( H \) with eigenvalue \( \epsilon_k \) then \( \tilde{c}_k \) is an eigenvector with eigenvalue \(-\epsilon_k\) and the eigenvalues \( E_k \) of the Hamiltonian (11) are symmetrically distributed about \( \omega \). Besides, if \( N \) is odd then there is always an eigenvalue \( E_k = \omega \) (or \( \epsilon_k = 0 \)).

### 3 Exactly solvable examples

The simplest example is

\[
H = \begin{pmatrix}
\omega_1 & \alpha \\
\beta & \omega_2
\end{pmatrix}, \quad (12)
\]

with eigenvalues

\[
E_{\pm} = \frac{\omega_1 + \omega_2 \pm \sqrt{(\omega_1 - \omega_2)^2 + 4\alpha\beta}}{2}. \quad (13)
\]

We appreciate that these eigenvalues are real for all \( \alpha\beta > - (\omega_1 - \omega_2)^2 / 4 \) which shows that the condition \( \alpha\beta > 0 \) proposed in section 2 is sufficient but not necessary. When \( \omega_1 = \omega_2 = \omega \) the eigenvalues \( E_{\pm} = \omega \pm \sqrt{\alpha\beta} \) exhibit the symmetric distribution about \( \omega \) derived in section 2 and are real when \( \alpha\beta > 0 \).

The second example is given by \( \omega_j = \omega, \alpha_j = \alpha, \beta_j = \beta \) for all \( j \). In this case, equation (6) can be rewritten as

\[
d_{j-1} - x d_j + d_{j+1} = 0, \quad x = \frac{E - \omega}{\sqrt{\alpha\beta}}. \quad (14)
\]
This simple Hückel-like equation [14] can be easily solved and the result is

\[ E_k = \omega + 2 \sqrt{\alpha \beta} \cos \left( \frac{k\pi}{N+1} \right), \quad k = 1, 2, \ldots, N, \]

\[ d_{jk} = \sqrt{\frac{2}{N+1}} \sin \left( \frac{kj\pi}{N+1} \right), \quad j = 1, 2, \ldots, N. \]  

If we choose \( Q_1 = \sqrt{\beta/\alpha} \) then it follows from equation (8) that

\[ c_{jk} = \left( \frac{\beta}{\alpha} \right)^{j/2} d_{jk}. \]  

Note that the eigenvalues and eigenvectors satisfy the general result derived in section 2 for the case \( \omega_j = \omega \).

In a recent paper, Yuce [17] studied a particular case of the model (1) with \( \alpha_j = 1, \quad \beta_j = \gamma, \quad \omega_j = (-1)^j i V_0 \) and \( N \) even and stated that “obtaining the spectrum analytically is challenging”. The eigenvalue equation for this model is exactly solvable when \( \gamma = 1 \) [18]. Although \( \omega_j \) is complex we can nevertheless apply the approach developed in section 2 and obtain

\[ d_{j-1} + \left[ (-1)^j i V_0 - y \right] d_j + d_{j+1} = 0, \quad v_0 = \frac{V_0}{\sqrt{\gamma}}, \quad y = \frac{E}{\sqrt{\gamma}}, \]  

that is exactly solvable [18]. We thus conclude that the energies are given by

\[ E_k^2 = 4\gamma \cos^2 \left( \frac{k\pi}{N+1} \right) - V_0^2, \quad k = 1, 2, \ldots, \frac{N}{2}, \]  

which are real provided that

\[ |V_0| < 2\sqrt{\gamma} \cos \left( \frac{N\pi}{2(N+1)} \right). \]  

This expression predicts that there are no real energies when \( N \to \infty \).

4 Conclusions

We have proved that if \( \alpha_j, \beta_j \) and \( \omega_j \) are real and \( \alpha_j \beta_j > 0 \), then the Hamiltonian (1) is similar to a Hermitian operator and therefore its eigenvalues are real. The reason for starting from the secular equation [3] is that our proof was motivated by an earlier argument proposed by Child et al [15] and Amore and...
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Fernández [16] for the truncation of tridiagonal recurrence relations coming from the application of the Frobenius method to quasi-solvable quantum-mechanical models. In this paper, that mathematical strategy has proved useful for the construction of a similarity transformation between a non-Hermitian and a Hermitian Hamiltonian and has also enabled us to solve the eigenvalue equation for a supposedly non-solvable model. It is worth noting that in the present case of a finite-dimensional vector space the terms Hermitian operator, self-adjoint operator and symmetric operator are equivalent.

Finally, we want to mention that the results developed in section 2 can be slightly generalized as shown in the appendix A.

A Generalization

In this Appendix we generalize the results developed in section 2.

A linear operator $H$ can be expanded in a complete orthonormal basis set $\{|j\rangle\}$ as

$$H = \sum_i \sum_j H_{i,j} |i\rangle \langle j|, \quad H_{i,j} = \langle i| H |j\rangle.$$  \hfill (A.1)

Assume that $|\psi\rangle$ is an eigenvector of $H$ with eigenvalue $E$, $H |\psi\rangle = E |\psi\rangle$, and carry out the transformation

$$|\psi\rangle = Q |\varphi\rangle, \quad Q = \sum_j Q_j |j\rangle \langle j|, \quad Q_j \neq 0,$$  \hfill (A.2)

so that

$$\tilde{H} |\varphi\rangle = E |\varphi\rangle, \quad \tilde{H} = Q^{-1} HQ.$$  \hfill (A.3)

Therefore, the matrix elements of the transformed operator $\tilde{H}$ are

$$\tilde{H}_{i,j} = Q_i^{-1} Q_j H_{i,j}.$$  \hfill (A.4)

If we restrict the analysis to a tri-diagonal Hamiltonian

$$H = \sum_j (H_{j,j+1} |j\rangle \langle j+1| + H_{j+1,j} |j+1\rangle \langle j| + H_{j,j} |j\rangle \langle j|),$$  \hfill (A.5)
and require that \( \tilde{H}_{j,j+1} = \tilde{H}_{j+1,j}^* \), then we derive the relation

\[
\left| \frac{Q_{j+1}}{Q_j} \right|^2 = \frac{H_{j+1,j}^*}{H_{j,j+1}} = R_j > 0. \tag{A.6}
\]

We conclude that any tridiagonal linear operator \( (A.5) \) that satisfies \( H_{i,i} = H_{i,i}^* \) and \( H_{j+1,j}^* = R_j H_{j,j+1}, R_j > 0 \), is similar to an Hermitian operator of the form

\[
\hat{H} = \sum_j \left( \sqrt{R_j} H_{j,j+1} |j\rangle \langle j+1| + \frac{H_{j+1,j}}{\sqrt{R_j}} |j+1\rangle \langle j| + H_{j,j} |j\rangle \langle j| \right), \tag{A.7}
\]

where, for simplicity, we have chosen \( Q_j = Q_j^* \). Therefore,

\[
Q_j = \sqrt{R_{j-1}R_{j-2} \cdots R_1} Q_1 \tag{A.8}
\]

From the results above we can easily derive an additional condition for a cyclic chain to be quasi-Hermitian. If we take into account that \( H_{N,N+1} = H_{N,1} \) and \( H_{N+1,N} = H_{1,N} \) we conclude that we have to add the condition

\[
H_{N,1}^* = R_{N-1}R_{N-2} \cdots R_1 H_{1,N}. \tag{A.9}
\]

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