On the multilevel internal structure of the asymptotic distribution of resonances

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Abstract

We prove that the asymptotic distribution of resonances has a multilevel internal structure for the following classes of Hamiltonians $H$: Schrödinger operators with point interactions in $\mathbb{R}^3$, quantum graphs, and 1-D photonic crystals. In the case of $N \geq 2$ point interactions, the set of resonances $\Sigma(H)$ essentially consists of a finite number of sequences with logarithmic asymptotics. We show how the leading parameters $\mu$ of these sequences are connected with the geometry of the set $Y = \{y_j\}_{j=1}^N$ of interaction centers. The minimal parameter $\mu_{\text{min}}$ corresponds to the sequences with ‘more narrow’ and so more observable resonances. The asymptotic density of such narrow resonances is described by the multiplicity of $\mu_{\text{min}}$, which occurs to be connected with the symmetries of $Y$ and naturally introduces $N - 1$ different classes of configurations of $Y$. In the case of quantum graphs and 1-D photonic crystals, the decomposition of $\Sigma(H)$ into a finite number of asymptotic sequences is proved under additional commensurability conditions. To address the case of a general quantum graph, we introduce families of special counting and asymptotic density functions for two types of curved complex strips. Moreover, we discuss the applicability of specially shaped counting and density functions to the study of the set of resonances for various types of perturbations of the $m$-D Laplacian with odd $m$. 
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1 Introduction

Let $\Delta = \sum_{j=1}^{N} \partial_{x_j}^2$ be the Laplacian operator in the complex Lebesgue space $L^2_C(\mathbb{R}^m)$ with odd $m \geq 1$. For operators $H$ obtained as various types of perturbations of $(-\Delta)$ on compact subsets of $\mathbb{R}^m$, resonances $k$ are defined as poles of the resolvent $(H - z^2)^{-1}$ extended in a generalized sense through $\mathbb{R}$ into the lower complex half-plane $\mathbb{C}_- := \{ z \in \mathbb{C} : \text{Im} z < 0 \}$ (this and other types of definitions can be found, e.g., in [4, 17, 49, 50]).

Reviews on resonances for obstacle and geometrical scattering can be found in [17, 49, 50]. Due to various engineering applications, wave equations, resonances, and related optimization problems associated with noncompact quantum graphs (see [31, 20, 15, 14, 23, 36, 37, 21, 16] and the monographs [11, 41]) and point interactions
(see [4, 29, 5, 25, 38, 6] and the monographs [2, 7]) attracted a substantial attention during the last decade.

The collection of all resonances \( \Sigma(H) \subset \mathbb{C} \) that are associated with an operator \( H \) (in short, resonances of \( H \)) is a multiset, i.e., a set in which an element \( e \) can be repeated a finite number \( m_e \in \mathbb{N} \) of times (this number \( m_e \) is called the multiplicity of \( e \)). The multiplicity of a resonance \( k \) is defined as the multiplicity of the corresponding generalized pole of \( (H - z^2)^{-1} \) (e.g. [17, 50]) or as the multiplicity of a certain analytic function built from the resolvent of \( H \) and generating resonances as its zeros ([43, 4, 33, 14, 15]).

Presently, there are only few classes of Hamiltonians \( H \), where it is known that \( \Sigma(H) \) is a sequence \( \{k_n\}_{n \in \mathbb{Z}} \) with prescribed asymptotics [43, 32, 4, 46, 13, 44, 40] (all of them are 1-D except for [4]). The more or less explicitly solvable examples of [4, 47, 45, 14, 15] indicate that, generally, the distribution of resonances has a more complex pattern and includes a number of asymptotic sequences (see also the discussion in [49] on radial and obstacle scattering). Therefore, the asymptotic behavior of the multiset of resonances \( \Sigma(H) \) near infinity is usually studied in terms of the counting function \( \mathcal{N}(\cdot) \), which is defined by

\[
\mathcal{N}_H(R) := \# \{ k \in \Sigma(H) : |k| \leq R \}. \quad (1.1)
\]

Here and below \( \#E \) is the number of elements of a multiset \( E \).

The study of the asymptotics of \( \mathcal{N}(R) \) as \( R \to \infty \) for scattering poles of Schrödinger Hamiltonians \(-\Delta + V\) in \( \mathbb{R}^m \) with odd \( m \geq 3 \) was initiated in [39] (for the relation between the notions of scattering poles and resonances, see [17, 50]). This study was continued and extended to obstacle and geometric scattering (see [27, 47, 48, 22, 49, 8, 12, 15, 17, 50]), to quantum graphs (see [14, 15, 37, 21] and Section 6.1), and to point interaction Hamiltonians \( H_{a,Y} \) associated with the formal differential expression

\[
-\Delta u(x) + \sum_{j=1}^{N} \mu(a_j) \delta(x - y_j)u(x), \quad x \in \mathbb{R}^3, \; N \in \mathbb{N}, \quad (1.2)
\]

where the interaction centers \( y_j \in \mathbb{R}^3 \) form the family \( Y = \{y_j\}_{j=1}^{N} \) and \( a = (a_j)_{j=1}^{N} \in \mathbb{C}^N \) is the tuple of ‘strength’ parameters (see [4, 38, 6] and Section 2 for details).

The goal of the present paper is to distinguish internal structures in the asymptotic behavior of resonances, to extract their main parameters, and to connect these parameters to geometric properties of the resonator. We are motivated, in particular, by the study of narrow ‘topological’ resonances [23, 16] and want to consider them from the point of view of high-energy asymptotics (see Section 5). The notion of high-energy asymptotics for resonances have been discussed for quantum graphs in [21] (for somewhat related effects in obstacle scattering, see [26, 49]). The extraction of structural parameters can also provide an approach to optimization problems of the type of [13, Section 8], which involve all the set \( \Sigma(H) \) and are much less studied than problems on optimization of an individual resonance [28, 50].

Our initial step to find asymptotic structures in \( \Sigma(H) \) is to use the observation of [14, 15, 38] that the set of resonances is a set of zeroes of an exponential polynomial
for noncompact quantum graphs and for point interaction Hamiltonians (see also [37, 21, 5, 6]). By the Pólya-Dickson theorem, the zeroes of exponential polynomials are concentrated in a finite number of logarithmic strips (see [10] and in the context of quantum graphs [14, 37]).

We show that, in the case of the point interaction Hamiltonian \(H^a_{a,Y}\) associated with (1.2), the corresponding exponential polynomial \(F\) falls into a special class that is associated only with retarded logarithmic strips in the terminology of [9]. The set of zeroes for such exponential polynomials splits into a finite number of asymptotic sequences. Translating this result to the set \(\Sigma(H^a_{a,Y})\) we show that, roughly speaking, \(\Sigma(H^a_{a,Y})\) consists of sequences that for \(t \in \mathbb{Z}\) with large \(|t|\) have asymptotics

\[
2\pi \mu_n t - i\mu_n \ln t + C + o(1)
\]

and whose leading parameters \(\mu_n\) form a finite set \(\{\mu_n\}^M_{n=1} \subset \mathbb{R}_+\). It is natural to say that \(\mu_n\) are the structural parameters of 1st order (the parameters of 2nd order \(\omega_{n,j}\) are hidden inside the constant \(C\), see Theorem 3.4 for details). Each of parameters \(\mu_n\) participates in \(r_n\) asymptotic sequences. It is natural to call the natural number \(r_n\) the multiplicity of \(\mu_n\).

To show in Theorems 4.3-4.4 the interplay between \(\{\mu_n\}^M_{n=1}\) and the metric geometry of the tuple \(Y\), we introduce a sequence of values \(s_m\), which are called \(m\)-sizes of \(Y\) and are generalizations of the size of \(Y\) introduced in [38]. For integer \(m \in [0, N]\), let \(S_{N,m}\) be the set of permutations \(\sigma \in S_N\) of the symmetric group \(S_N\) such that \(\sigma\) has exactly \(N - m\) fixed points, i.e., \(S_{N,m} := \{\sigma \in S_N : \#\{i : \sigma(i) \neq i\} = m\}\). For integer \(m \in \{0\} \cup [2, N]\), we define the \(m\)-size of \(Y\) by

\[
s_m = s_m(Y) := \max_{\sigma \in S_{N,m}} \sum_{j=1}^N |y_j - y_{\sigma(j)}|;
\]

\[
s_1 = s_1(Y) := \text{diam}(Y), \quad \text{where \text{diam}(Y) := \max_{1 \leq j,j' \leq N} |y_j - y_{j'}| is the diameter of } Y.
\]

Then \(s_0 = 0, s_2 = 2\text{diam}Y, s_3\) is the maximal perimeter of a triangle with vertices in \(Y\); 1-size of \(Y\) is defined a the special way since \(S_{N,1} = \emptyset\). The logic behind this is that the equality \(s_1 := (s_2 + s_0)/2\) simplifies the formulation of Theorem 4.4. The \(N\)-size \(s_N\) was called in [38] the size of \(Y\) and used to define Weyl and non-Weyl types of asymptotics for \(\mathfrak{M}_{H^a_{a,Y}}(\cdot)\) (see the beginning of Section 4).

The connection between the parameters \(\mu_n\) and \(m\)-sizes \(s_m\) is given Theorems 4.3-4.4, but it is not straight as it is shown by the examples of Section 4.3.

The structural theorem for \(\Sigma(H^a_{a,Y})\) of the point interaction Hamiltonians is quite special and cannot be directly brought over even to the case of quantum graphs, which is considered in Section 6. However this theorem give a hint how one can define the structural parameters even if the decomposition of \(\Sigma(H^a_{a,Y})\) into asymptotic sequences is not available. With this aim we introduce ‘shaped’ counting and density functions. The main examples of them for this paper are the logarithmic counting functions

\[
\mathfrak{M}^{\log}(\mu, R) := \#\{k \in \Sigma(H) : -\mu \ln(|\Re k| + 1) \leq \Im k \text{ and } |k| \leq R\},
\]
and its logarithmic density function

\[
\text{Ad}^\log(\mu) := \lim_{R \to \infty} \frac{\mathcal{N}^\log(\mu, R)}{R}, \quad \mu \in \mathbb{R}, \text{ and } \text{Ad}^\log(+\infty) := \lim_{R \to \infty} \mathcal{N}_H(R)/R. \tag{1.4}
\]

The above definition of \(\text{Ad}^\log(+\infty)\) is natural because, for \(H_{a,Y}\) and for quantum graphs, this limit has a finite value that was studied in [15, 14, 38, 6] in connection with Weyl-type asymptotics of \(\mathcal{N}_H(\cdot)\). Note that the study of asymptotics of the counting functions corresponding to sectors \(C < \text{Arg} k < 0\) was initiated in [27] because of a very different reason: it is necessary in the case of perturbations of \((-\Delta)\) in even-dimensional spaces since the cut-off resolvents of such operators have a logarithmic-type branching. It seems that the density functions built by the sectors \(C < \text{Arg} k < 0\) do not give information about the internal structure of \(\Sigma(H_{a,Y})\).

For \(H_{a,Y}\) and for quantum graphs, the function \(\text{Ad}^\log : (-\infty, +\infty] \to [0, +\infty)\) is bounded, nondecreasing and satisfies \(\text{Ad}^\log(\mu) = 0\) for \(\mu < 0\) and \(\text{Ad}^\log(\mu) = \text{Ad}^\log(+\infty)\) for large enough \(\mu\) (this follows from the definition, results of [14, 15, 38], and Theorem 3.3). So it defines a bounded measure \(d\text{Ad}^\log(\cdot)\) on \(\mathbb{R}\) with a compact support. The connection of \(\text{Ad}^\log(\cdot)\) with the structure of \(\Sigma(H)\) is obvious in the case of \(H_{a,Y}\) from Theorem 3.4 and in the case of quantum graphs from Theorem 6.4 (the latter evolves from [14, Theorem 3.1], but describes \(\Sigma(H)\) on a more fine structural level). Namely, the measure \(d\text{Ad}^\log(\cdot)\) consists of a finite number of point masses, the minimum \(\mu_{\text{min}}\) of its support and the height of the corresponding jump \(\text{Ad}^\log(\mu_{\text{min}} + 0) - \text{Ad}^\log(\mu_{\text{min}} - 0)\) are the parameters describing the high-energy asymptotics of ‘most narrow’ (and so most physically relevant) resonances (see Section 5 and the discussion in [43, 18]).

For \(H_{a,Y}\), the parameter \(\mu_{\text{min}}\) is always equal \(1/\text{diam} Y\). So the main parameter of the formula (5.1) for the asymptotic density of narrow resonances \(\text{Ad}^\log(\mu_{\text{min}} + 0) - \text{Ad}^\log(\mu_{\text{min}} - 0)\) is the multiplicity \(r\text{\scriptsize{\text{narrow}}}\) of \(\mu_{\text{min}}\), which is an integer number between 2 and \(N\) and is studied by Theorems 5.2 and 5.4. These theorems naturally lead to a conjecture about the connection of the value of \(r\text{\scriptsize{\text{narrow}}}\) with the group of symmetries of \(Y\).

The structural description \(\Sigma(H_G)\) in the case of a quantum graph \(G\) is given in Section 6.1. It is more complex than that of \(\Sigma(H_{a,Y})\). In the case where \(\mu_{\text{min}} = 0\), a certain neutral strip \(\{ z \in \mathbb{C} : |\text{Im} z| \leq \bar{\gamma} \}\) contains an infinite number of resonances, that are not necessarily decomposable into a finite number of asymptotic sequences. In the important case of the Kirchhoff coupling, the situation is the worst possible. The measure \(d\text{Ad}^\log(\cdot)\) consists of one point mass at \(\mu_{\text{min}} = 0\), i.e., there exists only one 1st order structural parameter, which does not describe any structure. The description of the structure of \(\Sigma(H_G)\) should be encoded in the measure \(d\text{Ad}^\text{hor}\) associated with another density function (6.2) built with the use of horizontal strips. However, the distribution of resonances in the non-decomposable cases is connected with difficult questions arising in the exponential polynomial approach to the study of zeros of Riemann zeta function (see the monograph [10] and the references therein). To obtain a complete decomposition of \(\Sigma(H_G)\) into asymptotic sequences we impose the additional commensurability condition \(\ell_{m_1}/\ell_{m_2} \in \mathbb{Q}\) on the lengths \(\ell_m\) of edges of the graph \(G\) (see Theorem 6.1 and the part (iii) of Theorem 6.4 here and below \(\mathbb{Q}\) is the set of...
rational numbers).

In Section 6.2 we brought over the above results to the set of resonances of 1-D photonic crystals using the fact that they can be considered as generalized ‘weighted’ quantum graphs with the Kirchhoff coupling (for another type of coupling for weighted graphs, see [14]).

In the last Section 7 we discuss a suitable choice of shaped asymptotic density functions for the resonances of m-D obstacle scattering and Schrödinger Hamiltonians $-\Delta + V$ comparing the asymptotic results of [26, 47, 48, 8, 12, 45].

**Notation.** We use the convention that if $n_2 < n_1$, then $\{z_n\}_{n=n_1}^{n_2} = \emptyset$. The following standard sets are used: the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ of natural, integer, rational, and real numbers, resp., the lower and upper complex half-planes $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \text{Im}z > 0\}$, open half-lines $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$, open discs $D_\varepsilon(\zeta) := \{z \in \mathbb{C} : |z - \zeta| < \varepsilon\}$, compact $\mathbb{C}$-intervals $[z_1, z_2] := \{sz_1 + (1-s)z_2 : s \in [0,1]\}$ with $z_1, z_2 \in \mathbb{C}$, $\mathbb{C}$-intervals $(z_1, z_2) := [z_1, z_2] \setminus \{z_1, z_2\}$ without endpoints. The above $\mathbb{C}$-intervals are called degenerate if $z_1 = z_2$. In a metric space $U$ with the distance function $\rho_U(\cdot, \cdot)$ (or in a normed space), we use open balls $B_\varepsilon(u_0) := \{u \in U : \rho_U(u, u_0) < \varepsilon\}$ always assuming that $\varepsilon > 0$. For a normed space $U$, $u_0 \in U$, $S \subset U$, and $z \in \mathbb{C}$, we write $zS + u_0 := \{zu + u_0 : u \in S\}$. For a function $g$ defined on $S$, $g[S]$ is the image of $S$. The function $\ln(\cdot) \, (\text{Arg}_0(\cdot))$ is the branch of the natural logarithm multi-function $\ln(\cdot)$ (resp., the complex argument $\text{Arg}(\cdot)$) in $\mathbb{C} \setminus (-\infty, 0]$ fixed by $\ln 1 = i \text{Arg}_0 1 = 0$. For $z \in \mathbb{R}_-$, we put $\ln z = \ln |z| + i \text{Arg}_0 z = \ln |z| + i\pi$. By $\partial_x f$, $\partial_y f$, etc., we denote (ordinary or partial) derivatives with respect to (w.r.t.) $x$, $x_j$, etc.; $\deg p$ stands for the degree of a polynomial $p$, $\deg X_j$ for the degree of a vertex $X_j$ of a graph; $e = [1][2] \ldots [N]$ for the identity permutation. Here and below we use the square brackets notation of the textbook [34] for permutation cycles, omitting sometimes, when it is convenient, the degenerate cycles consisting of one element.

## 2 Resonances of point interaction Hamiltonians

Throughout this paper, the set $Y = \{y_j\}_{j=1}^N$ consist of $N \geq 2$ distinct points $y_1$, $\ldots$, $y_N$ in $\mathbb{R}^3$. Let $a = (a_j)_{j=1}^N \in \mathbb{C}^N$ be the $N$-tuple of the ‘strength’ parameters.

The operator $H_{a,Y}$ associated with (1.2), where $\delta(\cdot - y_j)$ is the Dirac measure placed at the center $y_j \in \mathbb{R}^3$ of a point interaction, is defined in [11][2] for the case of real $a_j$, and in [3][5] for $a_j \in \mathbb{C}$. It is a closed operator in the complex Hilbert space $L^2_C(\mathbb{R}^3)$ and it has a nonempty resolvent set. The spectrum of $H_{a,Y}$ consists of the essential spectrum $[0, +\infty)$ and an at most finite set of points outside of $[0, +\infty)$ [2][5] (all of those points are eigenvalues).

The resolvent $(H_{a,Y} - z^2)^{-1}$ of $H_{a,Y}$ is defined in the classical sense on the set of $z \in \mathbb{C}_+$ such that $z^2$ is not in the spectrum, and has the integral kernel

\[
(H_{a,Y} - z^2)^{-1}(x, x') = G_z(x - x') + \sum_{j,j'=1}^N G_z(x - y_j) [\Gamma_{a,Y}]_{jj'}^{-1} G_z(x' - y_{j'}),
\]

(2.1)
where \( x, x' \in \mathbb{R}^3 \setminus Y \) and \( x \neq x' \), see e.g. \([2, 5]\). Here \( G_z(x - x') := \frac{e^{i|x-x'|}}{4\pi|x-x'|} \) is the integral kernel associated with the resolvent \((-\Delta - z^2)^{-1}\) of the kinetic energy Hamiltonian \(-\Delta\); \([\Gamma_{a,Y}]_{j,j'}^{-1}\) denotes the \( j, j' \)-element of the inverse to the matrix

\[
\Gamma_{a,Y}(z) = \left[(a_j - \frac{i}{4\pi}) \delta_{j,j'} - \tilde{G}_z(y_j - y_{j'})\right]_{j,j'=1}^N, \quad \text{where} \quad \tilde{G}_z(x) := \begin{cases} G_z(x), & x \neq 0 \\ 0, & x = 0 \end{cases} \quad (2.2)
\]

The Krein-type formula \((2.1)\) for the difference of the perturbed and unperturbed resolvents of operators \(H_{a,Y}\) and \(-\Delta\) can be used as a definition of \(H_{a,Y}\) (see \([2]\)). For other equivalent definitions of \(H\) and for the meaning of \(\mu(a_j)\) and \(a_j\) in \((1.2)\), we refer to \([1, 2, 7]\) in the case \(a_j \in \mathbb{R}\), and to \([3, 5]\) in the case \(a_j \not\in \mathbb{R}\). Note that, in the case \(a \in \mathbb{R}^N\), the operator \(H_{a,Y}\) is self-adjoint in \(L^2(\mathbb{R}^3)\); and in the case \(a \in (\mathbb{C} \cup \mathbb{R})^N\), \(H_{a,Y}\) is closed and maximal dissipative (in the sense of \([19]\), or in the sense that \(iH_{a,Y}\) is maximal accretive).

The set of (continuation) resonances \(\Sigma(H_{a,Y})\) associated with the operator \(H_{a,Y}\) (in short, resonances of \(H_{a,Y}\)) is by definition the set of zeroes of the determinant \(\det \Gamma_{a,Y}(\cdot)\), which we will call the characteristic determinant. This is in agreement with the cut-off resolvent pole definition of \([17]\) and slightly differs from the one used in \([1, 2]\) because isolated eigenvalues are now also included into \(\Sigma(H_{a,Y})\). For the origin of this and related approaches to the understanding of resonances, we refer to \([1, 17, 42, 43]\) and the literature therein. The multiplicity of a resonance \(k\) will be understood as the multiplicity of a corresponding zero of the determinant \(\det \Gamma_{a,Y}\), which is an analytic function in \(z\) (see \([2]\)). Equipped with the multiplicity, the set \(\Sigma(H_{a,Y})\) becomes a multiset (for the discussions on multiplicities of resonances see e.g. \([13, 17, 29, 33, 50]\)).

The function \(\det \Gamma_{a,Y}(\cdot)\) is an exponential polynomial, which after a simple transformation becomes of a special type considered in \([9]\) (for the general theory see \([10]\)). Namely, introducing a new variable \(\zeta = -iz\) and denoting \(A_j := 4\pi a_j\), one can see that the modified characteristic determinant \(D(\zeta) := (-4\pi)^N \det \Gamma_{a,Y}(i\zeta)\) can be expanded by the Leibniz formula into the sum of terms

\[
e^{\zeta \alpha(\sigma)} p^{[\sigma]}(\zeta) \quad (2.3)
\]
taken over all permutations \(\sigma\) in the symmetric group \(S_N\). Here the constants \(\alpha(\sigma) \leq 0\) and the polynomials \(p^{[\sigma]}(\cdot)\) have the form

\[
\alpha(\sigma) := - \sum_{j: \sigma(j) \neq j} |y_j - y_{\sigma(j)}|, \quad p^{[\sigma]}(\zeta) := \epsilon_\sigma K_1(\sigma) \prod_{j: \sigma(j) = j} (-\zeta - A_j), \quad (2.4)
\]

where \(K_1(\sigma) := \prod_{j: \sigma(j) \neq j} |y_j - y_{\sigma(j)}|^{-1} > 0\) (in the case \(\sigma = \varepsilon\), \(K_1(\varepsilon) := 1\)), \((2.5)\)

\(\epsilon_\sigma\) is the permutation sign (the Levi-Civita symbol), and \(\varepsilon\) is the identity permutation.

We will say that an exponential polynomial \(g(\cdot)\) is an exp-monomial if it has the form \(e^{\beta p}(\zeta)\), where \(\beta \in \mathbb{C}\) and \(p\) is a polynomial that is nontrivial in the sense that \(p(\cdot) \neq 0\).
3 Logarithmic asymptotic chains of resonances

3.1 The distribution diagram and logarithmic strips

Summing (2.3) and writing the exponential polynomial $D$ in the canonical form $[10, 9]$, one obtains

$$D(\zeta) = \sum_{j=0}^{\nu} P_{\beta_j}(\zeta)e^{\beta_j \zeta}, \quad (3.1)$$

where $\nu \in \mathbb{N} \cup \{0\}$, $\beta_j \leq 0$, and the nontrivial polynomials $P_{\beta_j}$ (with coefficient depending on $a$ and $Y$) are such that $\beta_0 < \beta_1 < \cdots < \beta_\nu = 0$. Clearly, $P_0(z) = P_{\beta_\nu} = \prod_{j=1}^{N}(-\zeta - A_j)$. The coefficients $\beta_j$ in (3.1) (and $\alpha(\sigma)$ in (2.3)) are called frequencies of the corresponding exponential polynomials (resp., exp-monomials). We will use the convention that

$$\text{if } b \notin \{\beta_j\}_{j=0}^{\nu}, \text{ then } P_b(\cdot) \equiv 0. \quad (3.2)$$

Since we rely on the terminology and the results of the theory of zeros of exponential polynomials given through [9, Sections 12.4-8], we try to keep our notation as close as possible to that of [9]. It is difficult to achieve this aim completely, in particular, our frequencies $\beta_j$ are nonpositive, while those of [9] are nonnegative.

It is obvious that

for each $m \in (\mathbb{Z} \cap [0, N]) \setminus \{1\}$, there exists $\sigma \in S_N$ such that $\alpha(\sigma) = (-s_m)$ and $\deg p^{[\sigma]} = N - m. \quad (3.3)$

In the process of summation of exp-monomials of (2.3) some of the terms may cancel so that, for a certain permutation $\sigma \in S_N$, $\alpha(\sigma)$ is not a frequency of $D$. If this is the case, we say that there is frequency cancellation for the pair $\{a, Y\}$ (for two different examples of frequency cancellation see [9] and Section 4.3.2).

Since $N \geq 2$, there exists a nonzero frequency $\alpha(\sigma)$ that does not cancel for $D$, i.e., $\nu \geq 1$ and $\alpha(\sigma)$ is a frequency of $D$ for certain $\sigma \neq e$. This fact was observed in [5, the proof of Lemma 2.1] and, by a different argument, can also be seen from the next lemma.

**Lemma 3.1.** (i) The number $(-s_2)$ is a frequency of $D(\cdot)$. Besides, $\deg P_{-s_2} = N - 2$.

(ii) Let $N \geq 3$. Then $(-s_3)$ is a frequency of $D(\cdot)$. Moreover,

$$s_3 = s_2 \text{ if all the points of the set } Y \text{ lie on one line; otherwise, } s_3 > s_2. \quad (3.4)$$

**Proof.** (i) Consider the class of all transpositions $\sigma = [j_1j_2]$, $j_1 \neq j_2$, such that

$$|y_{j_1} - y_{j_2}| = \text{diam } Y = s_2/2.$$

The corresponding terms of (2.3) are $(-1)e^{-s_2\zeta}(\text{diam } Y)^{-2} \prod_{j \neq j_1,j_2}(-\zeta - A_j)$. The highest order coefficients of their polynomial factors $p^{[\sigma]}$ coincide and cannot cancel each
other. The exp-monomial produced by summation of these terms has the polynomial part of degree $N-2$ and cannot be canceled by other terms of (2.3) because the other terms either have a lower deg $p^{[\sigma]}$, or a different frequency $\alpha(\sigma)$. Thus, $(-s_2)$ is a frequency of $D$.

(ii) The statement (3.4) is obvious from the triangle inequality. In the case $s_3 = s_2$, $(-s_2)$ is a frequency of $D$ due to (i). Assume $s_3 > s_2$. Then it is easy to modify the proof of (i) to show that the frequency $(-s_3)$ does not cancel. \(\Box\)

Consider the points $T_{\beta_j} = \beta_j + i \deg P_{\beta_j} \in \mathbb{C}$, $j = 0, \ldots, \nu$, associated with the canonical form (3.1) of $D$. Note that $T_{\beta_\nu} = T_0 = iN$.

The distribution diagram of $D$ (see [9]) is the polygonal line $\mathcal{L}$ in $\mathbb{C}$ determined by the following properties:

(L1) $\mathcal{L}$ joins $T_{\beta_0}$ with $T_{\beta_\nu}$.

(L2) $\mathcal{L}$ has vertices only at the points of the set $\{T_{\beta_j}\}_{j=1}^\nu$.

(L3) $\mathcal{L}$ is convex upward. (In particular, it is allowed to be of the form of one $\mathbb{C}$-interval $[z_0, z_1]$. The latter is the case whenever $N = 2$, but not only, see Section 4.3.1.)

(L4) There are no points of the set $\{T_{\beta_j}\}_{j=1}^\nu$ above $\mathcal{L}$ in the sense that the $\mathbb{C}$-intervals $(T_{\beta_j}, \beta_j)$ (with excluded endpoints) do not intersect $\mathcal{L}$.

Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_M$ be the successive segments of $\mathcal{L}$ numbered from left to right and such that $\mathcal{L} = \bigcup_{n=1}^M \mathcal{L}_n$ (it is assumed that each segment $\mathcal{L}_n$ is a closed nondegenerate $\mathbb{C}$-interval and the segments are maximal in the sense that two consecutive segments do not belong to the same line). Note that $\mathcal{L}$ has no vertical segments (lying on the lines $\Re z = c$) due to (L4) (see also [9] Section 12.8).

Let $\{T_{n,h}\}_{h=1}^{\nu_n} := \mathcal{L}_n \cap \{T_{\beta_j}\}_{j=0}^\nu$, where $\mathcal{L}_n = [T_{n,1}, T_{n,\nu_n}]$ and the points $T_{n,h}$ on the segment $\mathcal{L}_n$ are numbered from left to right, i.e., $\Re T_{n,1} < \cdots < \Re T_{n,\nu_n}$,

$$T_{1,1} = T_{\beta_0}, \quad T_{M,\nu_M} = T_0 = iN, \quad \text{and, if } 2 \leq n \leq M, \quad T_{n-1,\nu_{n-1}} = T_{n,1}. \quad (3.5)$$

Put $\beta_{n,h} := \Re T_{n,h}$ and $m_{n,h} := \Im T_{n,h} = \deg P_{\beta_{n,h}}$. Then the slopes $\mu_n$ of the segments $\mathcal{L}_n$ are defined by

$$\mu_n := \tan(\arg(T_{n,\nu_n} - T_{n,1})) = \frac{m_{n,\nu_n} - m_{n,1}}{\beta_{n,\nu_n} - \beta_{n,1}}, \quad n = 1, \ldots, M. \quad (3.6)$$

Note that

the sequence $\{\mu_j\}_{j=1}^M$ is strictly decreasing. \(3.7\)

This follows from the convexity (L3) and the definition of the segments $\mathcal{L}_n$.

The Pólya-Dickson theorem on the zeroes of exponential polynomials [9] [10] states that there exists constants $w_n \geq 0$, $n = 1, \ldots, M$, and the log-strips

$$\mathcal{V}(\mu_n, w_n) := \{\zeta \in \mathbb{C} : |\Re(\zeta + \mu_n \ln \zeta)| \leq w_n\}, \quad n = 1, \ldots, M,$$
such that $\mathbb{C} \setminus \bigcup_{j=1}^{M} \mathcal{V}(\mu_n, w_n)$ can be decomposed into a finite number of subsets with the property that in each of them one of the sums $\sum_{T_j \in \mathcal{L}_n} P_{\beta_j}(\zeta) e^{\beta_j \zeta}$ is of predominant order of magnitude over the other terms in (3.1) as $\zeta \to \infty$ [9]. In particular, this implies the following statement.

**Lemma 3.2** ([9]). There is only a finite number of zeroes of $D(\cdot)$ in $\mathbb{C} \setminus \bigcup_{j=1}^{n} \mathcal{V}(\mu_n, w_n)$.

### 3.2 Bunches of asymptotics chains of resonances

The main point of this subsection is that the modified characteristic determinant $D$ belongs to the more special subclass of exponential polynomials that generate only positive parameters $\mu_n$, i.e., in the terminology of [9], all the log-strips $\mathcal{V}(\mu_n, w_n)$ for $D$ are retarded. This allows us to apply the results of [9] about this subclass and in this way to split $\Sigma(H_{a,Y})$ into a finite number of sequences with a prescribed form of asymptotics at $\infty$.

The fact that $\{\mu_n\}_{n=1}^{M} \subset \mathbb{R}_+$ follows from (3.7) and the following statement.

**Proposition 3.3.** $\mu_M = 1/ \text{diam } Y = 2/\mathcal{S}_2$ and $1 \leq M \leq N - 1$.

**Proof.** Let us show that $\mu_M = 2/\mathcal{S}_2$. It follows from Lemma 3.1 that $(-\mathcal{S}_2)$ is a frequency of $D$. Further, the arguments of the proof of Lemma 3.1 show that

$$\deg P_{-\mathcal{S}_2} = N - 2, \quad T_{-\mathcal{S}_2} = -\mathcal{S}_2 + i(N - 2).$$

(3.8)

The convexity (L3), (3.5), and the definition of $\mu_n$ imply that $\mu_M \leq \frac{\text{Im}(T_{M,\nu_M} - T_{-\mathcal{S}_2})}{\text{Re}(T_{M,\nu_M} - T_{-\mathcal{S}_2})} = \frac{2}{\mathcal{S}_2}$.

Let us prove that $\mu_M \geq 1/ \text{diam } Y = 2/\mathcal{S}_2$ by *reductio ad absurdum*. Assume $\mu_M < 1/ \text{diam } Y$. Then there exist $j < \nu$ such that $\frac{N - \deg P_{-\beta_j}}{-\beta_j} < \frac{1}{\text{diam } Y}$. So there exists $\sigma \in S_N$ such that $\alpha(\sigma) = \beta_j$ and $(N - \deg p^{[\sigma]}) \text{diam } Y < -\alpha(\sigma)$. However, it follows from (2.1) that $(-1)\alpha(\sigma) \leq \sum_{j: \sigma(j) \neq j} \text{diam } Y = (N - \deg p^{[\sigma]}) \text{diam } Y$, a contradiction.

By Lemma 3.1 $M \geq 1$. Let us show that $M \leq N - 1$. We see that

$$m_{n,h} = \text{Im} T_{n,h} = \deg P_{\beta_{n,h}} \in (\mathbb{Z} \cap [0, N]) \setminus \{N - 1\}$$

(3.9)

and, from the definitions of $T_{n,h}$, that the sequence $\text{Im} T_{1,1}, \text{Im} T_{2,1}, \ldots, \text{Im} T_{M,1}$, $\text{Im} T_{M,\nu_M}$ is strictly increasing. This implies that $\mathcal{L}$ has at most $N - 1$ segments. $lacksquare$

The following theorem is the main result of the section.

**Theorem 3.4.** There exist numbers $M \in \mathbb{N}$, $M_0 \in \mathbb{N} \cup \{0\}$, finite sequences $\{\mu_n\}_{n=1}^{M} \subset \mathbb{R}_+$, $\{r_n\}_{n=1}^{M} \subset \mathbb{N}$, $\{\omega_n\}_{j=1}^{r_n} \subset \mathbb{C} \setminus \{0\}$ and $\{t_{n,j}\}_{j=1}^{r_n} \subset \mathbb{N}$ for $n = 1, \ldots, M$, $\mathcal{K}_0 = \{k^{[0]}_t\}_{t=1}^{M_0} \subset \mathbb{C}$, and infinite sequences $\mathcal{K}_{n,j}^{\pm} = \{k_{n,j,l}^{\pm}\}_{l=1}^{\infty} \subset \mathbb{C}$ such that:

(i) $\Sigma(H_{a,Y}) = \bigcup_{n=0}^{M} \mathcal{K}_n$ (taking into account multiplicities), where

$$\mathcal{K}_n := \left(\bigcup_{j=1}^{r_n} \mathcal{K}_{n,j}^{-}\right) \cup \left(\bigcup_{j=1}^{r_n} \mathcal{K}_{n,j}^{+}\right), \quad n = 1, \ldots, M.$$
(ii) Each of $K_{n,j}^\pm$ has the following asymptotics as $t \in \mathbb{N}$ goes to $+\infty$:

$$\frac{k_{n,j,t}^\pm}{\mu_n} = \pm 2\pi t - i \ln(t) \mp \pi/2 - i \ln(2\pi \mu_n) + i \ln(\omega_{n,j}) + o(1).$$  \hspace{1cm} (3.10)

(iii) The sequence $\{\mu_n\}_{n=1}^M$ is strictly decreasing and $\mu_M = \frac{1}{\text{diam} Y}$.

(iv) $1 \leq M \leq N - 1$ and $2 \leq r_M \leq \sum_{n=1}^M r_n \leq N$.

The proof given below also explains how the parameters of the asymptotics (3.10) can be found from the distribution diagram $\mathcal{L}$.

**Proof.** We use the notation of Section 3.1 and the numbers defined therein. In particular, $M \geq 1$ is the number of segments of $L$, $\{\mu_n\}_{n=1}^M$ is the sequence defined by (3.6).

Let $c_{n,h}$ be the coefficient corresponding to the monomial of the highest degree for the polynomial $P_{\beta_n}^{\alpha_n}$, i.e., $c_{n,h} = \lim_{\zeta \to \infty} P_{\beta_n}^{\alpha_n}(\zeta) / \zeta^{m_{n,h}} \neq 0$, $h = 1, \ldots, \nu_n$. Consider the polynomial $q_n(\omega) := \sum_{h=1}^{\nu_n} c_{n,h} \omega^{m_{n,h}} - m_{n,1}$ of the degree $r_n := m_{n,r_n} - m_{n,1}$ and the finite sequence of its zeroes $\{\omega_{n,j}\}_{j=1}^{r_n}$ (repeated according to their multiplicities). Note that $c_{n,1} \neq 0$ imply $\omega_{n,j} \neq 0$.

The statement (iii) of the theorem is already proved. Combining (iii) with [9, Theorem 12.8 and 12.10 (d)] (see also [9, (12.8.11-12)], [9, Theorems 12.6-8], and [9, Lemma 12.4] for details), we see that for each $n = 1, \ldots, M$, there exists $R_n \geq 0$ and sequences $\{\zeta_{n,j,t}\}_{t=t_{n,j}^\pm}^{+\infty} \subset \mathbb{C}$, $j = 1, \ldots, r_n$ with the starting numbers $t_{n,j}^\pm \subset \mathbb{N}$ satisfying the following properties:

(a) The sequence $Z_n^\pm$ of all the zeroes of $D$ in $(V(\mu_n, w_n) \cap \mathbb{C}) \setminus D_{R_n}(0)$ is the union of $r_n$ sequences $\{\zeta_{n,j,t}\}_{t=t_{n,j}^\pm}^{+\infty}$ (taking into account multiplicities).

(b) Each of the sequences $\{\zeta_{n,j,t}\}_{t=t_{n,j}^\pm}^{+\infty}$ has the asymptotics

$$\frac{\zeta_{n,j,t}^\pm}{\mu_n} = \ln \omega_{n,j} - \ln |\pm 2\pi \mu_n t + \mu_n \arg(\omega_{n,j}) \mp \mu_n \pi/2| + i(\pm 2\pi t \mp \pi/2) + o(1),$$

cf. [9, Theorem 12.8].

Due to Lemma 3.2, one sees that only a finite sequence $Z_0$ of zeroes of $D$ does not get into the multiset $\bigcup_{j=1}^M Z_j^\pm$. Passing from $\zeta$ to $z = i\zeta$ and from zeroes of $D(\zeta)$ to that of $\det \Gamma_{a,Y}(z)$, we obtain statements (i)-(ii) of Theorem 3.4. Note that $i\zeta_{n,j,t}^\pm = k_{n,j,t}^\pm$ for large enough $t$.

The part $1 \leq M \leq N - 1$ of statement (iv) is proved by Proposition 3.3. To prove $\sum_{n=1}^M r_n \leq N$, note that (3.9) and (3.5) imply $\sum_{n=1}^M r_n = N - m_{1,1} \leq N$. The fact that $r_M \geq 2$ follows from Lemma 3.1 (i) and Proposition 3.3. This completes the proof of Theorem 3.4. \hfill \Box
4 Geometry of $Y$ and parameters of asymptotics

The parameters $\mu_n$ play the role of leading parameters of the asymptotic sequences $\{k_{n,j,t}^{\pm}\}_{t=1}^{+\infty}$ of (3.10). Their role in the distribution of resonances is emphasized by the next statement, which is immediate corollary of Theorem 3.4 (see also [9, formula (12.18.16)]) and the correcting remark to it in the Russian edition of [9]).

**Corollary 4.1.** For the operator $H_{a,Y}$, the function $Ad\log(\cdot)$ defined by (1.4) is a nondecreasing piecewise constant function with a finite number of jumps. These jumps are exactly at the points of the set $\{\mu_n\}_{n=1}^{M} \subset \mathbb{R}_+$, and the height of each jump is equal to

$$Ad\log(\mu_n + 0) - Ad\log(\mu_n - 0) = \frac{\beta_{n,0} - \beta_{n,1}}{\pi} = \frac{r_n}{\pi \mu_n}. \quad (4.1)$$

For the following considerations let us recall that the $m$-sizes $s_m$ of $Y$ were defined in Section 1 and that the value $s_N$ was introduced in [38] and called the size of $Y$. Let $\mathfrak{N}_{H_{a,Y}}(\cdot)$ be the resonance counting function defined by (1.1) with $H = H_{a,Y}$.

It was shown in [38, Theorem 4.1] (see also comments in [6] about the choice of the ‘strength’ tuple $a$) that $\mathfrak{N}_{H_{a,Y}}(R) = \frac{W(a,Y)}{\pi} R + O(1)$, where $0 \leq W(a,Y) \leq s_N$, and it was said that $\mathfrak{N}_{H_{a,Y}}(\cdot)$ has the Weyl-type asymptotics if $W(a,Y) = s_N$ (i.e., roughly speaking, when $W(a,Y)$ attains the maximal possible value). Slightly rephrasing [38], it is natural to say that $W(a,Y)$ is the effective size of $H_{a,Y}$ (see the comments in [6]).

Taking into account Theorem 3.4 we see that

$$\lim_{\mu \to +\infty} Ad\log(\mu) = Ad\log(+\infty) = W(a,Y)/\pi \leq s_N/\pi.$$

With the use of Theorem 3.4 (iii)-(iv) one can strengthen [38, Theorem 4.1] by the following inequality

$$W(a,Y)/\pi = Ad\log(+\infty) \geq Ad\log(\mu_M + 0) = \frac{r_M}{\pi \mu_M} \geq \frac{2 \text{diam } Y}{\pi}. $$

Thus, $W(a,Y) \geq 2 \text{diam } Y = s_2$. The following statement concerns restrict configurations of $Y$ whose the effective size $W(a,Y)$ takes the minimal possible value $s_2$.

**Proposition 4.2.** If $W(a,Y) = 2 \text{diam } Y$, then all the points of $Y$ lie on one line.

**Proof.** If $Y$ does not belong to one line, then Lemma 3.1 (ii) implies that $s_3 > s_2$ and that $\beta_0$ in (3.1) is less or equal than $(-s_3)$. Thus, [38, Theorem 2.1] (or, alternatively, Corollary 4.1 and (3.6)) implies $W(a,Y) > s_2$.

The implication inverse to Proposition 4.2 does not hold true, as one can see from Section 4.3.2 where also a nontrivial example of $H_{a,Y}$ with $W(a,Y) = 2 \text{diam } Y < s_4$ can be found for the case $N = 4$.

The goal of this section is to show that a stronger connection exists between the family of the parameters $\mu_n$ and the metric geometry of the family $Y$ of the interaction centers. This connection is given below by Theorems 4.3 and 4.4 in terms of the $m$-sizes $s_m$. 

4.1 Generic and maximally structured cases

Let $N \geq 2$ be fixed. To parametrize rigorously the family of Hamiltonians $H_{a,Y}$, let us consider in this section $Y$ as a vector in the space $(\mathbb{R}^3)^N$ of ordered $N$-tuples $y = (y_j)_{j=1}^N$ with the entries $y_j \in \mathbb{R}^3$. We consider $(\mathbb{R}^3)^N$ as a linear normed space with the $\ell^2$-norm $|y|_2 = \left(\sum |y_j|^2\right)^{1/2}$. Then the ordered collection $Y$ of centers is identified with an element of the subset $\mathcal{F} \subset (\mathbb{R}^3)^N$ defined by $\mathcal{F} := \{y \in (\mathbb{R}^3)^N : y_j \neq y'_j \text{ for } j \neq j'\}$.

We consider $\mathcal{F}$ as a metric space with the distance function induced by the norm $| \cdot |_2$.

We denote by $M(a,Y)$ and $\mu_n(a,Y)$, $n = 1, \ldots, M(a,Y)$, the parameters $M$ and $\mu_n$, resp., that are associated with the operator $H_{a,Y}$ in the way described in Section 3.

Let us consider the set $\mathcal{F}_{\text{gen}}$ that consists of $Y \subset \mathcal{F}$ that satisfy the following two properties:

(A1) $M(a,Y)$ and $\{\mu_n(a,Y)\}_{n=1}^{M(a,Y)}$ do not depend on the ‘strength’ tuple $a \in \mathbb{C}^N$,

(A2) $\{\mu_n(a,Y)\}_{n=1}^{M(a,Y)}$ is a subset of

\[ \left\{ \frac{m-j}{s_m(Y) - s_j(Y)} : 2 \leq m \leq N, 0 \leq j \leq N - 1, j \neq 1, \text{ and } s_j < s_m \right\}. \]

The following theorem states that, generically, all $\mu_n$ have the form $\frac{m-j}{s_m - s_j}$.

**Theorem 4.3.** There exists a subset of $\mathcal{F}_{\text{gen}}$ that is open and dense in $\mathcal{F}$.

The proof is given in Section 4.2.

The next theorem describe one of the cases when the connection between the sequences of $\mu_n$ and $s_m$ is especially simple and, simultaneously, the asymptotics of resonances is maximally structured in the sense that the number $M$ of parameters $\mu_n$ takes its maximal possible value $N - 1$.

**Theorem 4.4.** Let $a \in \mathbb{C}^N$ and $Y \in \mathcal{F}$ be such that the numbers $s_n = s_n(Y)$, $n = 1, \ldots, N$, and the polynomials $P_b(\cdot)$, $b \in \mathbb{R}$, which are associated with the representation (3.1) for $D(\zeta) = (-4\pi)^N \det \Gamma_{a,Y}(i\zeta)$ and the convention (3.2), have the following properties:

(A3) For $3 \leq n \leq N$, $P_{-s_n}(\cdot) \neq 0$ and $\deg P_{-s_n} = N - n$.

(A4) The sequence $\{s_m\}_{m=1}^N$ is increasing and strictly convex upward in the sense that $s_m - s_{m-1} > s_{m+1} - s_m > 0$ for $2 \leq m \leq N - 1$ (if $N = 2$, this condition is assumed to be fulfilled automatically; recall also that $s_1 := (s_2 + s_0)/2 = \text{diam } Y$).

Then:

(i) $M(a,Y) = N - 1$ and $\mu_{N-m}(a,Y) = \frac{1}{s_{m+1} - s_m}$ for $m = 1, \ldots, N - 1$;

(ii) $r_{N-1} = 2$, $r_n = 1$ for $1 \leq n \leq N - 2$, and after possible exclusion of a finite number of resonances the multiset $\Sigma(H_{a,Y})$ can be decomposed into the $2N$ asymptotic sequences (3.10).
(iii) there exists \( R \geq 0 \) such that every resonance \( k \) with \( |k| \geq R \) is simple (i.e., of multiplicity 1).

Examples in Section 4.3 illustrate how large can be the family of sets \( Y \) having the property that \( H_{a,Y} \) satisfies (A3)-(A4) for every \( a \in \mathbb{C}^N \).

### 4.2 Proofs of Theorems 4.3 and 4.4

Let us introduce the points \( \tilde{T}_{-s_m} := (-s_m) + i(N - m) \) for \( m = 0 \) and for \( 2 \leq m \leq N \). For \( 2 \leq m \leq N \), the point \( \tilde{T}_{-s_m} \) is not an endpoint of any of the segments \( \mathcal{L}_n \), \( n = 1, \ldots, M \) (and so disappears from the computation of the parameters \( \mu_n \)) if and only if exactly one of the following conditions hold:

(C1) \( \tilde{T}_{-s_m} \notin \{ T_{\beta_j} \}_{j=1}^{\nu-1} \) (i.e., after summation of all terms (2.3) with \( \sigma \in S_{N,m} \) and \( \alpha(\sigma) = -s_m \), the leading coefficients of the polynomials \( p^{[\sigma]} \) cancel each other so that \( \text{deg} P_{-s_m} < N - m \) for \( m \leq N - 1 \), or \( P_{-s_N} \equiv 0 \) for \( m = N \)).

(C2) \( \tilde{T}_{-s_m} \in \{ T_{\beta_j} \}_{j=1}^{\nu-1}, \tilde{T}_{-s_m} \in \mathcal{L}_n \) for a certain \( n \), \( 1 \leq n \leq M \), and \( \tilde{T}_{-s_m} \) is not an endpoint of \( \mathcal{L}_n \).

(C3) \( \tilde{T}_{-s_m} \in \{ T_{\beta_j} \}_{j=1}^{\nu-1} \) and \( \tilde{T}_{-s_m} \) lies strictly below a certain segment \( \mathcal{L}_n \), \( 1 \leq n \leq M \), of the distribution diagram (i.e., \((-s_m) \in \text{Re}\mathcal{L}_n \) and \((\tilde{T}_{-s_m} + i\mathbb{R}_+) \cap \mathcal{L}_n \neq \emptyset \)).

The main idea of the proofs of Theorems 4.3 and 4.4 is that some type of control over the cases (C1)-(C3) is needed. While for the proof of Theorem 4.3 we show that generically there are no cancellations of type (C1), the assumptions of Theorem 4.4 exclude the possibility of each of (C1)-(C3).

Let us prove Theorem 4.3. Consider the family \( \mathcal{A}_{\text{gen}} \) consisting of all \( Y \in \mathbb{F} \) such that, for a certain \( a \in \mathbb{C}^N \),

(A5) each of the points \( \tilde{T}_{-s_m} \) (where \( 0 \leq m \leq N \) and \( m \neq 1 \)) belongs to the set \( \{ T_{\beta_j} \}_{j=1}^{\nu} \) associated with \( H_{a,Y} \).

It is easy to note from the form (2.2) of \( \text{det} \Gamma_{a,Y}(\cdot) \) that if \( Y \in \mathcal{A}_{\text{gen}} \), then (A5) is valid for every \( a \in \mathbb{C}^N \). Indeed, (A5) means that there is no cancellation of the leading coefficients of the polynomials \( p^{[\sigma]}(\zeta) \) in (2.3) with the degree \( N - m \) and \( \sigma \) such that \( \alpha(\sigma) = -s_m \). On the other hand, these leading coefficients do not depend on \( a \).

**Proposition 4.5.** There exists a subset of \( \mathcal{A}_{\text{gen}} \) that is open and dense in \( \mathbb{F} \).

**Proof.** It is easy to see that the family \( \mathcal{A}_1 \) introduced in [6, Section 3.2] is a subset of \( \mathcal{A}_{\text{gen}} \). So the proposition is an immediate corollary of [6, Lemma 3.4].

**Lemma 4.6.** \( \mathcal{A}_{\text{gen}} \subset \mathbb{F}_{\text{gen}} \)
Proof. Let \( Y \in A_{\text{gen}} \). Then the validity of conditions (A1)-(A2) of Section 4.1 follows from the definitions of \( M \) and \( \mu_n \) (see Section 3.1) and the following fact: if \( Y \in A_{\text{gen}} \), then the set of points \( T_{n,h}, n = 1, \ldots, M, h = 1, \ldots, \nu_n \), is a subset of the family of points \( \tilde{T}_{-s_m} \), where \( 0 \leq m \leq N \) and \( m \neq 1 \). Indeed, among all the points \( T_{\beta_j} \) with \( \text{Im} \ T_{\beta_j} = N - m \), the point \( \tilde{T}_{0} = T_0 = T_{M,\nu_N} \) (4.2) is the only point of the distribution diagram \( L \) on the line \( \text{Im} \ z = N \), the convexity (L3) of the distribution diagram \( L \) (see Section 3.1) implies the desired statement.

The combination of Proposition 4.5 and Lemma 4.6 concludes the proof of Theorem 4.3.

Let us prove Theorem 4.4. It follows from the assumption (A3) that the cancellation (C1) does not happen for \( m \geq 3 \). For \( m = 2 \), (C1) does not happen because of Lemma 3.1. For \( m = 0 \), (C1) does not happen because (4.2) holds for all \( H_{a,Y} \). Thus, (A3) implies (A5).

It follows from the proof of Lemma 4.6 that only the points \( \tilde{T}_{-s_m} \) participate in the computation of \( M \) and of the parameters \( \mu_n, n = 1, \ldots, M \). The assumption (A4) insures that all the parameters \( \tilde{T}_{-s_m}, \) where \( 0 \leq m \leq N \) and \( m \neq 1 \), do participate this computation. Thus, \( M = N - 1 \). Now, \( \mu_{N-m}(a,Y) = \frac{1}{s_{m+1} - s_m} \) and the other conclusions of statement (ii) follows directly from (3.6) and Theorem 3.4.

To prove statement (iii) of Theorem 4.4 it is enough to apply Theorem 3.4 and to notice that \( \omega_{M,1} \neq \omega_{M,2} \) (the last fact follows from Lemma 3.1 (i) and the procedure for the computation of \( \omega_{n,j} \) described in Section 3.2). This completes the proof of Theorem 4.4.

4.3 Examples and the thickness of the case (A3)-(A4).

4.3.1 Classification in the case of 3 points

Let \( N = 3 \). In this case it is easy to give a complete classification of all possible structures of asymptotics of \( \Sigma(H_{a,Y}) \) due to the fact that \( F_{\text{gen}} = F \). That is, the distribution diagram \( L \) as well as the parameters \( \mu_n \) and \( \omega_{n,j} \) of the sequences (3.10) do not depend on the ‘strength’ tuple \( a \), and satisfy (A2). The set of tuples \( Y \) satisfying (A3)-(A4) is generic in the sense that it is open and dense in \( (\mathbb{R}^3)^N \).

In more details, there are 3 sub-cases of geometry of 3 points \( \{y_j\}_{j=1}^3 \) composing \( Y \) and the lengths \( \ell_{j,m} = |y_j - y_m| \):

Case 1. \( Y \) consists of the vertices of an equilateral triangle, i.e., \( \ell_{1,2} = \ell_{2,3} = \ell_{3,1} = \text{diam} \ Y \). The asymptotic sequences in this case were found in [4] (formally, under the restriction \( a_1 = a_2 = a_3 \in \mathbb{R} \) on the tuple \( a \)). Condition (A3) is satisfied, but the strict convexity part of (A4) is not; \( M = 1, \mu_1 = 1/\text{diam} \ Y, \) and \( r_1 = 3 \).

Case 2. The points \( y_1, y_2, \) and \( y_3 \) of \( Y \) are on one line. Let us number them such that \( \ell_{1,2} + \ell_{2,3} = \ell_{1,3} \). Then \( s_3 = s_2 \) and condition (A3) fails; \( M = 1, \mu_1 = 1/\ell_{1,3}, \) and \( r_1 = 2 \).
Case 3. The generic case, where $y_1, y_2$, and $y_3$ do not belong to one line. Let us number them such that $\ell_{1,2} + \ell_{2,3} > \ell_{1,3} = \text{diam} Y$. Conditions (A3)-(A4) are satisfied: $M = 2, \mu_1 = 1/(\ell_{1,2} + \ell_{2,3} - \ell_{1,3})$ with the multiplicity $r_1 = 1, \mu_2 = 1/\ell_{1,3}$ with the multiplicity $r_2 = 2$.

4.3.2 Perturbations of an example with non-Weyl asymptotics of $\Omega_{H_{a,Y}}$

The case of Theorem 4.4 is not generic when $N = 4$. To show this in Proposition 4.8, we start from an example of $H_{a,Y}$ with non-Weyl type asymptotics (essentially different from the example constructed in [38]).

Example 4.7. Let us fix an arbitrary $a \in \mathbb{C}^4$. Consider $Y = \{y_j\}_{j=1}^4$ with the properties that it is symmetric w.r.t. the origin, lies on one line passing through the origin, and satisfy $|y_1| = |y_4| = c_1$ and $|y_2| = |y_3| = c_2 < c_1$. The exp-monomials of (2.3) that have the lowest possible frequency ($-\mathfrak{s}_4 = 4c_1 + 4c_2$ are $e^{-\mathfrak{s}_4}/4c_2(c_1+c_2)^2$, $e^{\mathfrak{s}_4}/4c_2(c_1+c_2)^2$, and $e^{-\mathfrak{s}_4}/4c_2(c_1+c_2)^2$.

Thus, the following statements are equivalent for this example:

(i) $\Sigma(H_{a,Y})$ has an asymptotics of non-Weyl type (i.e., $W(a,Y) < \mathfrak{s}_N$);
(ii) $W(a,Y) = \mathfrak{s}_2$ (i.e., the effective size attains the minimal possible value $\mathfrak{s}_2 = 4c_1$);
(iii) the frequency ($-\mathfrak{s}_4$) cancels in the sense that $P_{-\mathfrak{s}_4}(\cdot) \equiv 0$
(iv) $c_1/c_2 = 3 + 2\sqrt{2}$.

Note that, for any $c_1 > c_2 > 0, Y \in \mathbb{F}_{\text{gen}}$, but (A3) does not hold since $\mathfrak{s}_3 = \mathfrak{s}_2$.

Proposition 4.8. Let $a \in \mathbb{C}^4$. Let $Y$ be as in Example 4.7 and satisfy $c_1/c_2 = 3 + 2\sqrt{2}$.
Then there exists a neighborhood $\mathbb{B}_\varepsilon(Y)$ of $Y$ such that for all $\tilde{Y} \in \mathbb{B}_\varepsilon(Y)$ at least one of the conditions (A3)-(A4) fails for $H_{a,Y}$.

Proof. If the frequency ($-\mathfrak{s}_4$) cancels, then (A3) fails. Assume now that $\tilde{Y} \in \mathbb{B}_\varepsilon(Y)$ and (A3) holds for $H_{a,Y}$. Then it is easy to see that for small enough $\varepsilon$, (C3) is valid and (A4) fails. □

5 High-energy asymptotics of ‘physical resonances’

The existence of asymptotic structures means that the main quantities that characterize the high-energy asymptotics of $\Sigma(H_{a,Y})$ from the point of view of Physics are: the minimum $\mu^{\text{min}}$ of the support of the measure $d\text{Ad}^{\log}(\cdot)$ associated with the asymptotic density function $\text{Ad}^{\log}(\cdot)$ and the height of the jump at this point $\text{Ad}^{\log}(\mu^{\text{min}} + 0) - \text{Ad}^{\log}(\mu^{\text{min}} - 0)$. Indeed, in scattering experiments usually only ‘sufficiently narrow’ and ‘threshold’ resonances $k$ are detected, and for this reason they are mostly considered to be the ‘physically relevant’ resonances [18, 19]. Such resonances have small (or, for some models, even zero) value of $\text{resonance width} = |\text{Im} k^2|$ (see [12]).

In the case of the point interaction Hamiltonian $H_{a,Y}$, with the growth of the energy $\text{Re} k^2$ the resonance width growth faster for sequences that have larger structural
parameter \( \mu_n \). Thus, only the asymptotic sequences corresponding to \( \mu^\text{min} = \mu_M = 1/\text{diam } Y \) have chances to be detected with reasonable confidence. It is natural to call the density

\[
\text{Ad}^{\log}(\mu_M + 0) - \text{Ad}^{\log}(\mu_M - 0) = \frac{r_M}{\pi \text{diam } Y}
\]

(5.1)

of the resonances in the associated logarithmic strip the asymptotic density of narrow resonances.

Summarizing, we see that for \( H_{a,Y} \) the multiplicity parameter \( r_M \in \mathbb{N} \), which take values between 2 and \( N \), is the most important characteristics of high-energy asymptotics of ‘physical resonances’ for \( H_{a,Y} \).

**Lemma 5.1.** The parameter \( r_M \) is equal to the maximal integer \( m \in [2, N] \) such that

\[
s_m = m \text{diam } Y \quad \text{and} \quad \tilde{T}_{-s_m} \in \mathcal{L}.
\]

**Proof.** The lemma follows immediately from Sections 3.1 and Sections 3.2 and the definition of the points \( \tilde{T}_{-s_m} \) in Section 4.2. \( \square \)

This lemma, in particular, implies that \( r_M \) does not depend on the ‘strength’ tuple \( a \in \mathbb{C}^N \). Let us denote by \( r^{\text{narrow}}(Y) \) the value of \( r_M \) for \( H_{a,Y} \), and by \( \mathcal{F}_2 \) the set of \( Y \in \mathcal{F} \) such that \( r^{\text{narrow}}(Y) = 2 \) (we use here the notation of Section 4.1).

The following theorem states that, generically, \( r_M = 2 \).

**Theorem 5.2.** There exists a subset of \( \mathcal{F}_2 \) that is open and dense in \( \mathcal{F} \).

**Proof.** Let us denote by \( \mathcal{A}_2 \) the set of \( Y \in \mathcal{F} \) such that \( \alpha(\sigma_1) \neq \alpha(\sigma_2) \) holds for any two different transpositions \( \sigma_1, \sigma_2 \in S_N \) (recall that \( \alpha(\sigma) \) was defined by 2.4). Then it follows from [6, Lemma 3.4] that \( \mathcal{A}_2 \) is open and dense in \( \mathcal{F} \). On the other hand, it is obvious from Lemma 5.1 that \( \mathcal{A}_2 \subset \mathcal{F}_2 \). \( \square \)

So, \( r_M > 2 \) means that the asymptotic density of narrow resonances is atypically large. Let us consider in more details how this can happen. First, note that for \( N \geq 3 \),

\[
s_3 = 3 \text{diam } Y \quad \text{implies} \quad r_M \geq 3.
\]

(5.2)

Indeed, \( s_3 = 3 \text{diam } Y \) means exactly that three of the centers \( y_j \) form an equilateral triangle with the side of the length \( \text{diam } Y \). It follows from Lemma 3.1 (ii), the properties (L3)-(L4) of the distribution diagram \( \mathcal{L} \), and Proposition 3.3 that \( \tilde{T}_{-s_3} \in \mathcal{L} \), and so there is no need to check this condition in Lemma 5.1 (for \( m = 3 \)).

By the following example we show that for \( N \geq 4 \) the condition \( s_4 = 4 \text{diam } Y \) (i.e., \( Y \) contains four distinct centers \( y_{j_n}, n = 1, \ldots, 4 \), such that \( \text{diam } Y = \ell_{j_1,j_2} = \ell_{j_3,j_4} \)) does not imply \( r_M \geq 4 \), and, moreover, it does not imply \( r_M \geq 3 \). Here and below, we shall set \( \ell_{i,j} = |y_i - y_j| \).

**Example 5.3.** Let \( Y_4 = \{y_j\}_{j=1}^4 \) be the tuple of distinct \( \mathbb{R}^3 \) points such that

\[
\ell_{1,2} = \ell_{2,3} = \ell_{3,4} = \ell_{4,1} = \text{diam } Y_4 > \ell_{2,4} \geq \ell_{1,3}
\]
and let $a \in \mathbb{C}^4$ be arbitrary. Then $s_4(Y_4) = 4 \text{diam} Y_4$, but the frequency $s_4(Y_4)$ cancels in the process of summation of the exp-monomials (2.3). This can be seen by a simple modification of arguments of [38, Section 5.2] or by direct calculations. Since $s_3 < 3 \text{diam} Y_4$, we see that $M(a, Y_4) = 2$ and $r^{\text{narrow}}(Y_4) = 2$.

Let us introduce one more condition on $Y$ with $N \geq 4$:

(A6) there exist four distinct centers $y_{j_n} \in Y$, $n = 1, \ldots, 4$, such that $\text{diam} Y = \ell_{j_k,j_4} = \ell_{j_3,j_4}$ and $\{y_{j_1}, y_{j_2}\}^{i=1}_4$ cannot be reordered to satisfy the condition of Example 5.3.

**Theorem 5.4.** Let $N \geq 4$. Assume that condition (A6) holds true or $s_3(Y) = 3 \text{diam} Y$. Then $r^{\text{narrow}}(Y) > 2$.

**Proof.** The case $s_3 = 3 \text{diam} Y$ follows from (5.2).

Assume that $s_3 \neq 3 \text{diam} Y$ and that (A6) holds. Note that (A6) implies $s_4 = 4 \text{diam} Y$. Therefore, due to Lemma 5.1, it is enough to prove that $s_4$ does not cancel in the process of summation of the exp-monomials (2.3) corresponding to permutations $\sigma \in S_{n,4}$. Consider the family $B_1$ of unordered sets of 4 points of $Y$ such that, for a certain ordering $\tilde{Y}_4 := \{y_{m_i}\}^{i=1}_4$ of $Y$, this ordering $\tilde{Y}_4$ satisfies the conditions of Example 5.3. To each $\tilde{Y}_4 \in B_1$, which is assumed to be ordered as in Example 5.3, we put into correspondence the two families $E_1(\tilde{Y}_4)$ and $E_2(\tilde{Y}_4)$ which consist of two unordered pairs of unordered centers: $E_1(\tilde{Y}_4) = \{(y_{m_1}, y_{m_2}), \{y_{m_3}, y_{m_4}\}\}$ and $E_2(\tilde{Y}_4) = \{(y_{m_2}, y_{m_3}), \{y_{m_1}, y_{m_4}\}\}$. The assumption $s_3 \neq 3 \text{diam} Y$ implies that there exists a unique unordered pair $\{E_1(\tilde{Y}_4), E_2(\tilde{Y}_4)\}$ of the above-described families corresponding to a tuple $\tilde{Y}_4 \in B_1$ i.e., $s_3 \neq 3 \text{diam} Y$ implies that various reorderings of $\tilde{Y}_4$ that preserve the conditions of Example 5.3 produce the same unordered pair $\{E_1(\tilde{Y}_4), E_2(\tilde{Y}_4)\}$ (for the case where $s_3 = 3 \text{diam} Y$ and $\{y_{m_i}\}^{i=1}_4$ are vertices of a regular tetrahedron the above statement obviously fails).

The class of all pairs of pairs of centers produced by the maps $E_1(\cdot)$ and $E_2(\cdot)$ shall be denoted by $B_2$, i.e., $B_2 = E_1[B_1] \cup E_2[B_2]$.

The class $\mathcal{S}_1$ of all permutations $\sigma \in S_{n,4}$ that have the sign $\epsilon_\sigma = -1$ and produce exp-monomials (2.3) with the frequency $(-s_4)$ consists of permutations $[m_1m_2m_3m_4]$ and $[m_4m_3m_2m_1]$ associated with $Y_4 \in B_1$. These permutations produce the exp-monomials $\frac{e^{2\pi i \zeta j}}{\text{diam} Y_4} \prod_{j=0}^{(n-4)} (-\zeta - A_j)$. To each pair $[m_1m_2m_3m_4]$ and $[m_4m_3m_2m_1]$ of such permutations we put into correspondence the two permutations $[m_1m_2][m_3m_4]$ and $[m_2m_3][m_4m_1]$ associated with $E_1(\tilde{Y}_4)$ and $E_2(\tilde{Y}_4)$, and we unite all permutations generated in this way by $E_1(\tilde{Y}_4)$ and $E_2(\tilde{Y}_4)$ into the class $\mathcal{S}_2$. The permutations $[m_1m_2][m_3m_4]$ and $[m_2m_3][m_4m_1]$ produce the exp-monomials (2.3) of the form $\frac{e^{2\pi i \zeta j}}{\text{diam} Y_4} \prod_{j=0}^{(n-4)} (-\zeta - A_j)$. So after summation of all exp-monomials generated by permutations of the classes $\mathcal{S}_1$ and $\mathcal{S}_2$ the coefficients of the highest order cancel and the associated polynomial has the degree $< N - 4$.

Assumption (A6) means that $\{(y_{j_1}, y_{j_2}), \{y_{j_3}, y_{j_4}\}\}$ does not belong to $B_2$. However, the exp-monomial produced by the associated permutation $\sigma = [j_1j_2][j_3j_4]$ has the sign $\epsilon_\sigma = 1$ and the frequency $(-s_4)$. Hence the frequency $(-s_4)$ does not cancel and, moreover, $\deg P_{-s_4} = N - 4$. This proves $\tilde{T}_{-s_4} \in \mathcal{L}$, and due to Lemma 5.1 completes the proof of the theorem.  
\qed
Under symmetries of $Y$ we understand isometries $\text{Is}$ of $\mathbb{R}^3$ such that $\text{Is}[Y] = Y$. Moreover, two symmetries $\text{Is}_1$ and $\text{Is}_2$ will be considered identical (equivalent) if $\text{Is}_1(y_j) = \text{Is}_2(y_j)$ for all $y_j \in Y$. So the group $\mathcal{G}$ of symmetries of $Y$ consists of the defined above classes of equivalence of isometries of $\mathbb{R}^3$ that map $Y$ onto $Y$.

**Corollary 5.5.** Let $\mathcal{G}$ be the group of symmetries of $Y = \{y_j\}_{j=1}^N$. Assume that there exist $\text{Is} \in \mathcal{G}$ and $y_{j_1}, y_{j_2} \in Y$ with the following properties:

(i) $|y_{j_1} - y_{j_2}| = \text{diam } Y$,

(ii) the sets $\{y_{j_1}, y_{j_2}\}$ and $\{\text{Is}(y_{j_1}), \text{Is}(y_{j_2})\}$ are disjoint,

(iii) the 4-tuple $\{y_{j_1}, y_{j_2}, \text{Is}(y_{j_1}), \text{Is}(y_{j_2})\}$ cannot be reordered such that it satisfies the conditions of Example 5.3.

Then $r^\text{narrow}(Y) > 2$.

**Proof.** The statement follow easily from Theorem 5.4.

Theorem 5.4 and Corollary 5.5 suggest that $r^\text{narrow}(Y) > 2$ whenever $Y$ possesses a rich enough group of symmetries $\mathcal{G}$. This claim can be supported by the following examples.

**Example 5.6.** Let the tuple $a$ be arbitrary. Then $r^\text{narrow}(Y) > 2$ in each of the following cases:

(i) $Y$ is the set of all vertices of a Platonic solid;

(ii) $Y$ is the set of all vertices of a right prism.

These statements can be shown by using Theorem 5.4 or by direct calculation of the exp-monomials with the frequencies $(-m) \text{diam } Y$, $m \in \mathbb{N} \cap [3, N]$.

The $N - 1$ possible values $2, 3, \ldots, N$ of $r^\text{narrow}$ naturally split the family $\mathcal{F}$ set of all possible tuples $Y$ of interaction centers into $N - 1$ classes. In our opinion, the interplay between these classes and the groups of symmetries of $Y$ deserves an additional study.

### 6 Remarks on resonances of quantum graphs

To study the asymptotic structure of $\Sigma(H)$ for other types of Hamiltonians $H$, the above approach requires some adaptation. The next two subsections consider resonances of quantum graphs. We believe that the study of their asymptotic structures can help to understand the phenomenon of narrow ‘topological’ resonances [23, 16].

Subsection 6.2 briefly addresses the resonances of 1-D photonic crystals considering them as as a simple case of ‘weighted’ quantum graphs.

#### 6.1 Structure of resonance asymptotics for quantum graphs

The asymptotics $\mathcal{N}(R) = \frac{C}{R^2} + O(1)$ of the resonance counting function $\mathcal{N}(R)$ for a Hamiltonian $H$ associated with a noncompact quantum graph $\mathcal{G}$ consisting of a finite number of edges have been derived in [14, 15], where, in particular, Weyl and
non-Weyl types of asymptotics for $G$ were introduced depending on the value of the constant $C \geq 0$. The characterizations of these two cases of asymptotics were obtained in terms of types of couplings at the vertices. A number of examples were elaborated (see [37, 21] and the references therein).

We show that, at least under additional conditions, the asymptotic structures which are somewhat similar, but more complex, than those of $\Sigma(H_{a,Y})$, exist in the multiset of resonances $\Sigma(H)$ of a quantum graph. This leads to another versions of the semi-logarithmic strip counting

$$\mathfrak{N}(\mu, \gamma, R) := \#\{k \in \Sigma(H) : -\mu \ln(|\text{Re} k| + 1) - \gamma \leq \text{Im} k, \ |k| \leq R\}, \quad \mu, \gamma \in \mathbb{R},$$

and the asymptotic density function

$$\text{Ad}(\mu, \gamma) := \lim_{R \to \infty} \frac{\mathfrak{N}(\mu, \gamma, R)}{R}$$

depending now on the two parameters $\mu \geq 0$ and $\gamma \in (-\infty, +\infty]$. For $\mu < 0$ and $\gamma \neq +\infty$, $\text{Ad}(\mu, \gamma) = 0$ as before.

With such a definition the effective size of $H$ introduced in [14, 15] is equal to $\frac{\pi \text{Ad}(0, +\infty)}{2}$ and [15, Theorem 3.2] and [14, Theorem 3.3] can be rewritten as the inequality

$$\text{Ad}(0, +\infty) = \text{Ad}(\mu, +\infty) \leq \frac{2}{\pi} \sum_{j \in J} \rho_j,$$

where $\rho_j$ are the lengths of the internal (i.e., bounded) edges of the graph, $J$ is the finite set that indexes all such edges, and $\mu \geq 0$ is arbitrary.

6.1.1 The case of Kirchhoff's boundary condition at vertices

In this subsection we follow the settings of [15] and consider on a non-compact quantum graph $G$ the self-adjoint Hamiltonian $H_G$ associated with the differential expression $-d^2/dx^2$ with the continuity condition and the Kirchhoff (boundary) condition at each of the vertices.

The set of resonances $\Sigma(H_G)$ is the set of $k \neq 0$ such that there exists a resonant mode $f(\cdot)$ (continuous and $L^2_{\text{loc}}$ on $G$) satisfying $f'' = -k^2 f$ on $G$, the Kirchhoff conditions at each vertex, and the classical radiation condition on each lead (i.e., semi-infinite edge). The Kirchhoff condition means that the sum of outgoing derivatives of $f$ at a vertex is equal to 0. The radiation condition after the identification of a lead with $[0, +\infty)$ takes the form $f(x) = f(0)e^{ikx}$.

To make $\Sigma(H_G)$ a multiset, each resonance $k$ is equipped with a multiplicity, which is multiplicity of $k$ as a zero of a specially constructed analytic function $F(z) = \det A(z)$, where $A(z)$ is a matrix produced by a plugging of the exponential fundamental system of solutions $e^{\pm ikx}$ on each edge in the continuity, resp., the Kirchhoff, resp., the radiation conditions (see [15, Theorem 3.1] for details).
Remark 6.1. In the process of construction in [15] of the matrix-valued function $A(\cdot)$, the Kirchhoff conditions are divided by $ik$, which is permissible because $k = 0$ is excluded from the consideration. Let us observe that, on one hand, $k = 0$ always satisfies the resonance conditions, and on the other hand after the transition to the settings with the energy-type spectral parameter $\lambda = k^2$, the point $\lambda = 0$ is not a pole, but a branching point of the generalized extension of the resolvent $(H_G - \lambda)^{-1}$. The exclusion of $k = 0$ is usual for some types of 1-D resonance problems (see e.g. [33, 13, 29, 30]) and makes the above definition of resonances slightly different from that of [17, 50], where 0 is permitted to be a resonance of a finite multiplicity. Obviously, this difference does not influence the results of [15, 14] and of the present paper on the asymptotics at $\infty$; however it influences the formulation of Theorem 6.1 below.

It is shown in [15, Theorem 3.2] that $\Sigma(H_G)$ lies in a certain horizontal strip $\{z \in \mathbb{C} : -\gamma \leq \text{Im} z \leq 0\}$ (formally, the theorem states this for a certain strip $\{|\text{Im} z| \leq \gamma\}$, however $H_G = H_G^* \geq 0$, and so $\Sigma(H_G) \cap \mathbb{C}_+ = \emptyset$). Hence, the measure $d\text{Ad}^{\log}$ does not carry information on the internal structure of $\Sigma(H_G)$. Indeed, it consists of one mass $\text{Ad}^{\log}(+\infty)$ at the point $\mu = 0$ and so gives only the information about the total asymptotic density $\text{Ad}^{\log}(+\infty)$.

To capture the structure of the asymptotics of $\Sigma(H_G)$, let us introduce another asymptotic density function

$$\text{Ad}^{\text{hor}}(\gamma) := \lim_{R \to \infty} \frac{\mathfrak{N}(0, \gamma, R)}{R}, \quad \gamma \in (-\infty, +\infty].$$

(6.2)

To show that $\text{Ad}^{\text{hor}}$ is an adequate tool, we consider the case where the lengths of all internal edges $\rho_j$ are commensurable and make the observation that for this case the measure $d\text{Ad}^{\text{hor}}$ generated by the monotone function $\text{Ad}^{\text{hor}}$ consists of a finite number of point masses.

**Theorem 6.1.** Assume that $\Sigma(H_G) \neq \emptyset$ and the lengths $\rho_{j_1}$ and $\rho_{j_2}$ of two arbitrary internal edges of $G$ satisfy $\rho_{j_1}/\rho_{j_2} \in \mathbb{Q}$. (6.3)

Then there exist a number $\bar{\tau} \in \mathbb{N}$ and infinite sequences $\{k_{n,t}^{\pm}\}_{t \in \mathbb{Z}} \subset \mathbb{C}$, $n = 1, \ldots, \bar{\tau}$, that satisfy

$$\Sigma(H_G) = \bigcup_{n=1}^{\bar{\tau}} \{k_{n,t}^{\pm}\}_{t \in \mathbb{Z}} \setminus \{0\} \quad \text{and} \quad \beta k_{n,t} = 2\pi t - i \text{Ln}|\xi_n| + \text{Arg}_0 \xi_n,$$

(6.4)

where $\xi_n$, $n = 1, \ldots, \bar{\tau}$, are certain complex algebraic numbers satisfying $|\xi_n| \geq 1$.

**Proof.** We can assume that the index set $J$ is of the form $\{1, \ldots, N\}$ and that $\{\rho_j\}_{j=1}^N$ is nondecreasing. The function $F(z) = \det A(z)$ that produces the set of resonances by the rule of [15] Theorem 3.1 is an exponential polynomial obtained by the summation of exp-monomials of the form $(-1)^{\tau_m} K_m \exp \left(iz \sum_{j \in J_m} (-1)^{\bar{\tau}_{m,j}} \rho_j \right)$, where the index $m$ passes through a certain finite index set $\mathcal{M}$, the sequences $\{\tau_m\}_{m \in \mathcal{M}}$, $\{\bar{\tau}_{m,j}\}_{j \in J_m}$, and
\{K_m\}_{m \in \mathcal{M}} \text{ are subsets of } \mathbb{Z}, \text{ and, for each } m \in \mathcal{M}, J_m \text{ is a subset of } J. \text{ Writing } F \text{ in the canonical form we get }
\begin{equation}
F(z) = \sum_{l=0}^{\nu} C_l e^{ib_l z},
\end{equation}
where \{C_l\}_{l=0}^{\nu} \subset \mathbb{Z} \setminus \{0\} and all \(b_l, l = 0, \ldots, \nu\), have the form \(\sum_{j \in J_l} (-1)^{\tau_{l,j}} \rho_j\) with \(\tilde{J}_l \subset J\) and \(\tau_{l,j} \in \mathbb{Z}\). We can also assume that the sequence of \(b_l\) is strictly increasing.

Supposing now that (6.3) holds, one can see that the arguments similar to that of [9, Section 12.4] are applicable to the function \(\tilde{F}(z) := e^{-ib_0 z} F(z)\). Indeed, let \(\beta > 0\) be such that \(\rho_j = \beta d_j\) with \(d_j \in \mathbb{N}\) for all \(j \in J\). Then the numbers \(\tilde{b}_l := b_l - b_0\) are also commensurable and \(\tilde{b}_l = \beta \tilde{d}_l, l = 0, \ldots, \nu,\) with \(\tilde{d}_l \in \mathbb{N}\). Now one can easily obtain the statement of the theorem.

Thus, in the commensurable case of Theorem 6.1, the asymptotic density \(\text{Ad}^{\text{hor}}(\cdot)\) is a piecewise constant function with a finite number of jumps at the points \(L_n|\xi_n|\). Depending on the level of noise, the asymptotic sequences with smaller \(L_n|\xi_n|\) have more chances to be detected in scattering experiments because they produce narrower resonances.

It is natural to call the resonances \(k\) lying on \(\mathbb{R}\) embedded resonances because the corresponding points \(\lambda = k^2\) in the ‘energy’ plane are embedded into the continuous spectrum \(\sigma_{\text{cont}}(H_G) = [0, +\infty)\) of \(H_G\). By [15, Theorem 2.3], every embedded resonance \(k\) corresponds to an eigenvalue \(k^2\) of \(H_G\) embedded into \(\sigma_{\text{cont}}(H_G)\). The case \(c \in \mathbb{Q} \cup [0, 1]\) of the example of [15, Section 6] gives \(H_G\) with infinite number of embedded resonances (in more general settings, equispaced sequences of embedded resonances have been considered in [20]).

The following general statement describes in the commensurable case the situation when \(\text{Ad}^{\text{hor}}(\cdot)\) has a jump at 0.

**Corollary 6.2.** Suppose that (6.3) holds. Then the following statements are equivalent:
1. \(H_G\) has an infinite number of embedded eigenvalues (and so, of embedded resonances);
2. \(H_G\) has at least one nonzero eigenvalue or at least one embedded resonance;
3. \(\min_{1 \leq n \leq \tilde{r}} \left| \xi_n \right| = 1\) in the settings of Theorem 6.1.

**Proof.** The statement follows immediately from Theorem 6.1.

This corollary is applicable to a number of situations, in particular, it implies the existence of an infinite number of embedded eigenvalues for [36, Example III.1].

In the non-commensurable case, the structure of \(\Sigma(H_G)\) and the structure of the measure \(d\text{Ad}^{\text{hor}}\) generated by the monotone function \(\text{Ad}^{\text{hor}}(\cdot)\) deserve an additional study, which is connected with the theory of distribution of zeroes of exponential polynomials (see [10] and references therein).

Thus, the three main parameters of high-energy asymptotics that are available now in the general case are: (i) the total asymptotic density \(\text{Ad}^{\text{hor}}(\gamma_{\max} + 0) = \text{Ad}^{\text{hor}}(+\infty),\)
which was considered in [15, Theorem 1.2]. (ii) the minimum \( \gamma^\text{min} \) and (iii) the maximum \( \gamma^\text{max} \) of \( \text{supp}(d\text{Ad}^\text{hor}) \).

It is possible to strengthen slightly [15, Theorem 1.2] by the following observation.

**Corollary 6.3.** Assume that \( \Sigma(H_G) \neq \emptyset \). Then \( \Sigma(H_G) \) consists of an infinite number of resonances and their asymptotic density \( \text{Ad}^\text{hor}(+\infty) = \lim_{R \to +\infty} \frac{\text{sup} \Sigma(H_G)(R)}{R} \) is positive.

**Proof.** It follows from \( \Sigma(H_G) \neq \emptyset \) that \( \nu \geq 1 \) in (6.5). Then the desired statement can be easily obtained from [9, Theorem 12.5] (see also [15, Theorem 3.2] and references in [14, 15]). \( \square \)

### 6.1.2 The case of general self-adjoint local coupling

In this subsection the local case of a more general Kostrykin-Schrader-Harmer coupling [31, 24] (in short, local KSH-coupling) is considered with the use of the notation and settings of [14] (see also [37] for a detailed exposition and the literature review).

Let \( |G| \) be the number of vertices of the quantum graph \( G \) and \( \{X_n\}_{n=1}^{|G|} \) the set of the vertices. Let \( \deg X_n \) be the degree of the vertex \( X_n \), i.e., the number of edges connected to \( X_n \).

The quantum graph Hamiltonian \( H = H_{G,U} \) and the multiset of its resonances \( \Sigma(H_{G,U}) \) are defined similarly to Section 6.1.1 but with the Kirchhoff and continuity conditions replaced at each \( X_i \) by the local KSH-coupling. The latter means that, at each vertex \( X_i \), the condition (\( U_n - I \)\( \Psi_n + i(U_n + I)\Psi_n = 0 \) is satisfied with a certain unitary \( \deg X_n \times \deg X_n \) matrix \( U_n \), where the vector \( \Psi_n \) consists of the limits of \( f(\cdot) \) at \( X_n \) along every edge \( E_j \) connected to \( X_n \), and the vector \( \Psi'_n \) consists of the corresponding limits of outwards derivatives. The unitary matrix \( U \) in the notation \( H_{G,U} \) is composed of the diagonal blocks \( U_n \), see [14] [37] for details.

The next theorem states, roughly speaking, that generally the structure of \( \Sigma(H_{G,U}) \) is the combination of the types considered in Section 6.1.1 (with the Kirchhoff and continuity coupling) and Section 3 (for point-interactions).

Let \( \tilde{K}(\gamma) := \{k \in \Sigma(H_{G,U}) : \text{Im} \, k \geq -\gamma\} \) and let \( \mu^\text{max} \) be the supremum of the support of the measure \( d\text{Ad}^\text{log} \) for the Hamiltonian \( H_{G,U} \) (\( \mu^\text{max} \) is taken to be equal to \( -\infty \) if \( \text{Ad}^\text{log}(\cdot) \equiv 0 \) on \( \mathbb{R} \)).

**Theorem 6.4.** (i) If \( \mu^\text{max} \leq 0 \), then there exists \( \tilde{\gamma} \in \mathbb{R} \) such that \( \Sigma(H_{G,U}) = \tilde{\Sigma}(\tilde{\gamma}) \).

(ii) Assume that \( \mu^\text{max} > 0 \). Then there exists \( \tilde{\gamma} \in \mathbb{R} \), \( M, N \in \mathbb{N} \), a strictly decreasing finite sequence \( \{\mu^\text{max}_n\}_{n=1}^M \subset \mathbb{R}^+ \), finite sequences \( \{r_n\}_{n=1}^M \subset \mathbb{N} \), \( \{\omega^\text{max}_{n,j}\}_{n=1}^r \subset \mathbb{C} \setminus \{0\} \), \( \{t^\text{max}_{n,j}\}_{n=1}^r \subset \mathbb{N} \) for \( n = 1, \ldots, M \), and infinite sequences \( K^\pm_{n,j} \) (taking into account multiplicities), where \( K_n := (\bigcup_{j=1}^r K^-_{n,j}) \cup (\bigcup_{j=1}^r K^+_{n,j}) \) for \( n = 1, \ldots, M \).

(iii) Each of \( K^\pm_{n,j} \) has the asymptotics (3.10) as \( t \in \mathbb{N} \) goes to \( +\infty \).

(iv) If the set \( \tilde{\Sigma}(\tilde{\gamma}) \) is infinite, then

\[
\lim_{R \to +\infty} \frac{\#\{k \in \tilde{\Sigma}(\tilde{\gamma}) : |k| \leq R\}}{R} = \text{Ad}^\text{hor}(\tilde{\gamma} + 0) = \text{Ad}^\text{hor}(+\infty) > 0.
\]
(iii) Let additionally the commensurability condition \([6.3]\) hold. Then in each of the cases (i) and (ii) the set \(\tilde{K}(\gamma)\) is either finite, or satisfies the following property: there exist numbers \(\beta > 0\), \(\bar{r} \in \mathbb{N}\), \(M_0 \in \mathbb{N}\), finite sequences \(\{\xi_n\}_{n=1}^{\bar{r}} \subset \mathbb{C} \setminus \mathbb{D}_{1}(0)\), \(\{\tilde{r}_n\}_{n=1}^{\bar{r}} \subset \mathbb{N}\), \(\tilde{K}_0 := \{k_t\}_{t=1}^{M_0}\), and infinite sequences \(\{\tilde{k}_{n,t}\}_{t=1}^{+\infty} \subset \mathbb{C}\) such that

\[
\tilde{K}(\gamma) = \tilde{K}_0 \cup \bigcup_{n=1}^{\bar{r}} \left( \{\tilde{k}_{n,t}^-\}_{t=\tilde{r}_n,j}^{+\infty} \cup \{\tilde{k}_{n,t}^+\}_{t=\tilde{r}_n,j}^{+\infty} \right) \quad \text{and}
\]

\[
\beta \tilde{k}_{n,t}^\pm = \pm 2\pi t - i \ln |\xi_n| + \text{Arg} \xi_n + o(1) \quad \text{as } t \to +\infty. \tag{6.6}
\]

**Proof.** A function \(F(z)\) such that \(k \in \Sigma(H_{G,U})\) if and only if \(k \neq 0\) and \(F(k) = 0\) is constructed in \([14]\). Moreover, the multiplicities of the resonances are the multiplicities of the corresponding zeros of \(F\). Let us consider a modified version of \(F\) given by \(\tilde{F}(\zeta) = F(-i\zeta)\). Then \([14, \text{Theorem 3.1}]\) implies that \(\tilde{F}(\zeta)\) is an exponential polynomial of the form \((3.1)\) with a strictly increasing sequence of frequencies \(\beta_j\) that are not necessarily nonpositive. Now the function \(D(\zeta) = e^{-\beta_0} \tilde{F}(\zeta)\) has the same zeros as \(\tilde{F}(\zeta)\) and has the form \((3.1)\) with a strictly increasing sequence of frequencies \(\beta_j\) so that \(\beta_0 = 0\). The construction of the distribution diagram given in Section \(3.1\) and in more details in \([9]\) applies to \(D(\cdot)\). In particular, in the terminology of \([9]\), the zeros of \(D(\cdot)\) lie in a finite number of logarithmic curvilinear strips and possibly one neutral strip \(|\Im z| \leq \gamma_0\) (this is the statement of \([14, \text{Theorem 3.1}]\)). The logarithmic strips contain a infinite number of resonances and are necessarily retarded because the self-adjoint and lower semi-bounded from below operator \(H_{G,U}\) has at most a finite number of resonances in \(\mathbb{C}_+\). Thus, the arguments of Section \(3.2\) applied to the zeroes of \(D\) in logarithmic strips and the arguments of \([9\text{, Sections 12.4-6}]\) applied to the zeros of \(D\) in the neutral strip easily complete the proof of the theorem. \(\square\)

We see now that the two level asymptotic structure of \(\Sigma(H_{G,U})\) is captured by the asymptotic density function \(\text{Ad}(\mu, \gamma)\) of \((6.1)\). The measure \(d\text{Ad}^{\log}(\cdot) = d\text{Ad}(\cdot, 0)\) has point masses at the numbers \(\mu_n\) corresponding to the logarithmic asymptotic sequences and, in the case where the strip \(|\Im z| \leq \tilde{\gamma}\) contains an infinite number of resonances, it has also a point mass at \(0\). If the infimum \(\mu^{\min}\) of the support of the measure \(d\text{Ad}^{\log}(\cdot)\) is equal to \(0\), then the measure \(d\text{Ad}(0, \cdot) = d\text{Ad}^{\text{hor}}(\cdot)\) is responsible for the internal structure of \(\Sigma(H_{G,U})\) in the strip \(|\Im z| \leq \tilde{\gamma}\) and for high-energy asymptotics of narrow ‘physical resonances’.

If \(\mu^{\min} > 0\), then \(\mu^{\min} = \mu_M\). The high-energy asymptotics of ‘physical resonances’ is described in this case by the sequences \(K^{\pm}_{M,j}\), \(j = 1, \ldots, r_M\), and, on a more rough level, by the asymptotic density \(\text{Ad}^{\log}(\mu_M + 0) - \text{Ad}^{\log}(\mu_M - 0)\) in the corresponding logarithmic strip.

### 6.2 Resonances of 1-D photonic crystals

A typical 1-D photonic crystal (multi-layer optical cavity) is described by the variable dielectric permittivity \(\varepsilon(x) > 0\), \(x \in \mathbb{R}\), which is a piecewise constant function on \(\mathbb{R}\)
with a finite number of steps. That is, there exists a finite partition $-\infty = x_{-1} < x_0 < \cdots < x_N < x_{N+1} = +\infty$ with $N \in \mathbb{N}$ such that $\varepsilon(x) = \varepsilon_j \in \mathbb{R}_+$ for all $x \in (x_{j-1}, x_j)$ of $\mathbb{R}$, $n = 0, \ldots, N$. For $1 \leq j \leq N$, the intervals $(x_{j-1}, x_j)$ represents idealized infinite plane layers of a material with the permittivity $\varepsilon_j$. The semi-infinite intervals $(-\infty, x_0)$ and $(x_N, +\infty)$ represents the homogeneous outer medium (it is assumed often that the corresponding permittivities $\varepsilon_0$ and $\varepsilon_{N+1}$ are equal, however this is not important for this section).

For electromagnetic waves that pass normally to the interfaces of the layers, the Maxwell system can be reduced to the wave equation for a nonhomogeneous string $\varepsilon(x)\partial_t^2 v(x, t) = \partial_x^2 v(x, t)$ (see [30] and references therein). The corresponding operator $H_\varepsilon := -\frac{1}{\varepsilon(x)}\partial_x^2$ is self-adjoint and nonnegative in the weighted Hilbert space $L^2_\varepsilon(\mathbb{R}; \varepsilon(x)dx)$.

The resonances of $H_\varepsilon$ are the complex numbers $k \neq 0$ such that there exists a nontrivial solution $f$ to the equation $f'' = -k^2 \varepsilon(x)f(x)$ satisfying radiation conditions on the outer intervals $(-\infty, x_0)$ and $(x_N, +\infty)$. The latter means that $f(x) = f(x_0)e^{-ik(x-x_0)\varepsilon_0^{1/2}}$ for $x < x_0$ and $f(x) = f(x_N)e^{ik(x-x_N)\varepsilon_{N+1}^{1/2}}$ for $x > x_N$. The multiplicity of a resonance $k$ can be defined as the multiplicity of the zero at $k$ of the corresponding Keldysh characteristic determinant (see [33, 28, 30]) or, equivalently, via the algebraic multiplicity of the eigenvalue $k$ of a special operator [33, 13]. The multiplicity of each resonance is finite.

The approach through the corresponding Keldysh characteristic determinant is essentially equivalent to the approach of [15] with the determinant of a matrix-valued function $A(z)$ constructed by the coupling conditions. Indeed, we can consider the 1-D photonic crystal as a \textit{weighted quantum graph} with the simple linear connectivity. It consists of edges $[x_{j-1}, x_j]$, which are internal for $j = 1, \ldots, N$ and external for $j = 0$ and $j = N+1$, equipped with the differential expressions $\frac{1}{\varepsilon_j}\partial_x^2$ with the constant coefficient $1/\varepsilon_j$. The coupling of the graph is given by the conditions of continuity of $f$ and $f'$, $f(x_j - 0) = f(x_j + 0)$ and $f'(x_j - 0) = f'(x_j + 0)$, $j = 0, \ldots, N$. Note that the latter condition is a simple version of Kirchhoff’s condition for the case when only two edges go out of a vertex $x_j$. (This weighted quantum graph slightly does not fit into the class of the weighted graphs of [14] Section 8 because its coupling conditions for the derivatives are different.)

\textbf{Theorem 6.5.} Assume that the multiset of resonances $\Sigma(H_\varepsilon)$ of $H_\varepsilon$ is nonempty. Then:

(i) $\Sigma(H_\varepsilon)$ consists of an infinite number of resonances.

(ii) There exists $\gamma_0 > 0$ such that $\Sigma(H_\varepsilon) \subset \{z \in \mathbb{C} : -\gamma_0 \leq \text{Im} z < 0\}$ and $0 < \text{Ad}_\text{hor}(\gamma_0) < +\infty$, where $\text{Ad}_\text{hor}(\cdot)$ is the asymptotic density function defined by (6.2) for $H = H_\varepsilon$.

(iii) Assume, additionally, that $\frac{(x_j-x_{j-1})\varepsilon_j^{1/2}}{(x_{j+1}-x_j)\varepsilon_{j+1}^{1/2}} \in \mathbb{Q}$ for $1 \leq j \leq N - 1$. Then there exist
numbers \( \beta > 0, r \in \mathbb{N} \), and infinite sequences \( \{k_{n,t}\}_{t \in \mathbb{Z}} \subset \mathbb{C}, n = 1, \ldots, r \), such that

\[
\Sigma(H_\varepsilon) = \bigcup_{n=1}^{r} \{k_{n,t}\}_{t \in \mathbb{Z}} \quad \text{and} \quad \beta k_{n,t} = 2\pi t - i \ln |\xi_n| + \text{Arg}_0 \xi_n,
\]

where \( \xi_n \), \( n = 1, \ldots, r \), are certain complex numbers satisfying \( |\xi_n| > 1 \).

Proof. To obtain the Keldysh characteristic determinant \( F(z) \), the construction of the matrix-valued function \( A(z) \) from [15] can be applied resulting in \( F(z) = \det A(z) \). Since in this process the derivatives \( f'(x_j - 0) \) are coupled only with the derivatives \( f'(x_j - 0) \), the multiplicative factors \( iz \) are eliminated from \( A(z) \) (similarly to [15] and the definition of ‘dissipation frequencies’ in [33]). Hence, \( F(z) \) takes the form (6.5) with certain \( C_i \in \mathbb{R} \setminus \{0\} \). Thus, the arguments of [15], Theorem 6.1, and Corollary 6.3 can be applied to obtain all the statements of the theorem except the statement that \( \Sigma(H_\varepsilon) \cap \mathbb{R} = \emptyset \), which is well-known and is published e.g. in [30]. \( \square \)

7 Discussion on wider classes of m-D Hamiltonians

We have described the inner structure of the multiset of resonances \( \Sigma(H) \) for 3-D Schrödinger Hamiltonians with point interactions and have shown how these results can be adapted to quantum graphs and 1-D photonic crystals. The structural description of \( \Sigma(H) \) allows us to describe the asymptotics of its ‘physically relevant’ part.

The classes of Hamiltonians considered above are special because for them resonances are zeros of exponential polynomials, and so, the structural theorems for the sets of zeros of exponential polynomials are inherited by \( \Sigma(H) \). However the measures generated by monotone asymptotic density functions in curvilinear strips in \( \mathbb{C} \) with suitable shapes may be helpful to capture the first order structural characteristics for more general classes of Hamiltonians \( H \) for which the multiset of resonances \( \Sigma(H) \) can be defined in \( \mathbb{C} \), but not necessarily given in terms of exponential polynomials. We have seen that strips bounded by logarithmic curves and the related function \( \text{Ad}^{\log}(\mu) \) are suitable for the point interaction Hamiltonians, but the case of quantum graphs with the Kirchhoff condition require horizontal strips and the function \( \text{Ad}^{\text{hor}}(\gamma) \).

To make the definition of asymptotic density function more flexible, let us consider a continuous function \( \varphi(\xi, \mu) \) from \([0, +\infty) \times \mathbb{R} \) to \( \mathbb{R} \) that is strictly increasing in \( \mu \) and such that \( \varphi(\xi, 0) = 0 \) and \( \lim_{\mu \to +\infty} \varphi(\xi, \mu) = +\infty \). Then the related counting function can be defined as

\[
\mathcal{N}_\varphi(\mu, R) = \#\{k \in \Sigma(H) : -\varphi(|\text{Re} k|, \mu) \leq \text{Im} k, |k| \leq R\},
\]

and a family of asymptotic density functions \( \text{Ad}_{\varphi, \alpha} \) with an additional parameter \( \alpha > 0 \) and values in \([0, +\infty) \) as

\[
\text{Ad}_{\varphi, \alpha}(\mu) := \limsup_{R \to \infty} \frac{\mathcal{N}_\varphi(\mu, R)}{R^\alpha}, \quad \mu \in \mathbb{R}, \quad \text{Ad}_{\varphi, +\infty}(\mu) := \limsup_{R \to \infty} \frac{\mathcal{N}(R)}{R^\alpha}. \quad (7.1)
\]
Then $\text{Ad}_{\varphi, \alpha}$ is nondecreasing in $\mu \in (-\infty, +\infty]$. To generate a nontrivial bounded measure $d\text{Ad}_{\varphi, \alpha}(\cdot)$ on $\mathbb{R}$ one have to choose appropriate $\varphi(\cdot)$ and $\alpha$.

The choice of the parameter $\alpha$ is a difficult task even for the Hamiltonians $H_V = -\Delta + V$ in $L^2_{\mathbb{C}}(\mathbb{R}^m)$, where we assume that $m \geq 1$ is odd and the potential $V$ is from the space $L_{\text{comp}}^\infty(\mathbb{R}^m)$ of real valued $L^\infty$-functions with a compact support. On one hand, it follows from [48] that $\text{Ad}_{\varphi, m}(+\infty) < +\infty$ for all $V \in L_{\text{comp}}^\infty(\mathbb{R}^m)$; on the other hand, even when $n \geq 3$, $\alpha = 1$ is the largest known value that ensure $\text{Ad}_{\varphi, \alpha}(+\infty) > 0$ for general $V$ [8]. However, $\text{Ad}_{\varphi, m}(+\infty) > 0$ for generic $V \in L_{\text{comp}}^\infty(\mathbb{R}^n)$ [12].

The choice of the shape function $\varphi$ can be motivated by the following principles. The minimum $\mu_{\text{min}}$ of the support of the measure $d\text{Ad}_{\varphi, \alpha}(\cdot)$ is a good candidate to be the main parameter responsible for high-energy asymptotics of ‘physical resonances’, and so the choice of $\varphi(\cdot)$ have to take into account results of the type of those of [35] on the resonance free regions. On other side, it would be desirable that, at least for the well-studied examples like the Laplacian with Dirichlet boundary condition on a sphere [17, 48, 45], the measure $d\text{Ad}_{\varphi}$ would provide some reasonable set of first order structural parameters $\mu$ that correspond to the decomposition of $\Sigma(H)$ into an infinite number of asymptotic sequences. This should lead to the functions $\varphi(\xi, \mu)$ increasing to $+\infty$ as $\xi \to +\infty$ and connected with the asymptotics of the Hankel functions (see [45] and also discussions in [17, 50]). However, the results of [26] on the scattering for two convex obstacles suggest for the study of narrow resonances the same choice $\varphi(x, \mu) = \mu$ like it was taken for (6.2). Thus, similarly to the case of quantum graphs with KSH-couplings, it is natural to expect that the proper choice of the shape function $\varphi$ might be different for different Hamiltonians even if the corresponding scattering problems are in the same class.

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