On the Structure of Correlation Functions in the Normal Matrix Model

Ling-Lie Chau\textsuperscript{a} and Oleg Zaboronsky\textsuperscript{b}

\textsuperscript{a}Department of Physics, University of California at Davis, Davis, CA 95616, chau@physics.ucdavis.edu
\textsuperscript{b}Department of Mathematics, University of California at Davis, Davis, CA 95616, zaboron@math.ucdavis.edu

February 4, 2022

Abstract

We study the structure of the normal matrix model (NMM). We show that all correlation functions of the model with axially symmetric potentials can be expressed in terms of holomorphic functions of one variable. This observation is used to demonstrate the exact solvability of the model. The two-point correlation function is calculated in the scaling limit by solving the BBGKY\textsuperscript{1} chain of equations. The answer is shown to be universal (i.e. potential independent up to a change of the scale). We then develop a two-dimensional free fermion formalism and construct a family of completely integrable hierarchies (which we call the extended-KP(N) hierarchies) of non-linear differential equations. The well-known KP hierarchy is a lower-dimensional reduction of this family. The extended-KP(1) hierarchy contains the (2+1)-dimensional Burgers equations. The partition function of the (N × N) NMM is the \(\tau\) function of the extended-KP(N) hierarchy invariant with respect to a subalgebra of an algebra of all infinitesimal diffeomorphisms of the plane.

\textsuperscript{1} Born, Bogoliubov, Green, Kirkwood, and Yvone, \cite{32}.
1 Introduction

Normal Matrix Model (NMM) was introduced in [5] and its connection to the Quantum Hall Effect was indicated. In [6] this connection was given a precise form by observing that the partition function of NMM coincides with zero-temperature partition function of two-dimensional electrons in the strong magnetic field.

In the present paper we study the mathematical structure of the NMM. The distinct feature of the NMM, in contrast to other matrix models (Hermitian, Unitary, etc.), is its relation to two-plus-one-dimensional \((2+1)\)-d physical systems, rather then the one-plus-one -dimensional \((1+1)\)-d ones. For example, we will exploit the equivalence of NMM and the system of \((2+1)\)-d Coulomb particles to compute correlation functions of the NMM in the scaling limit. We will also show that the partition function of NMM is a \(\tau\)-function of an integrable hierarchy containing \((2+1)\)-d Burgers equations.

The paper is organized as follows. To make the present exposition self-contained we reproduce in Section 2 the results of [6], devoted to the definition of the model, the derivation of the eigenvalue formula for its partition function and the integrability of the model. To emphasize the \((2+1)\)-d nature of the NMM we notice following [6] that the partition function of the model can be interpreted as a classical partition function of Coulomb particles in a plane. In contrast, the partition function of the Hermitian matrix model (HMM) coincides with the classical partition function of Coulomb particles constrained to a string in a plane. We show that the NMM partition function can be written in a determinant form. This enables us to conclude that the partition function of the model is a \(\tau\)-function of Toda lattice with respect to holomorphic-antiholomorphic variations of the potential.

In Section 3 we study the NMM with axially-symmetric potential. First we derive a determinant representation for the correlation functions of the model. We notice that the orthogonal polynomials associated with the integration measure are just powers of the complex variable \(z\). This simplifies the analysis of the correlation functions considerably. We compute the correlation functions for the case when the matrix model potential is a monomial, \(V(M, M^\dagger) = (M \cdot M^\dagger)^k\). The answer is expressed through degenerate hypergeometric functions. Thus NMM provides us with an example of a matrix model which is exactly (and explicitly) solvable beyond the Gaussian case even before taking the continuum limit.

The two-point function of Gaussian NMM decreases exponentially at infinity, which should be compared to the power decay of the two-point function of HMM. While the latter can be obtained from quantum-mechanical computation in the system of free \((1 + 1)\)-d identical fermions in the external potential (see [30] for details), the former follows from the analogous computation in the system of free \((2+1)\)-d identical fermions placed in the external magnetic field.

In general we find that all correlation functions can be expressed in terms of a holomorphic function in one variable. This observation permits us to close a BBGKY chain of equations for the correlation functions and obtain a closed integro-differential equation for the two-point function. The existence of a single equation which determines the two-point function and thus, through determinant representation, all correlation functions, makes the NMM with axially symmetric potential exactly solvable.

In Section 4 we solve the obtained equation in the continuum \(N \to \infty\) limit. The answer
for the two-point function is universal, i.e. potential-independent up to a change of scale.

In Section 5, we develop a $(2+1)$-d free fermion formalism and construct the free fermion representation of the partition function of NMM. This fact has its analogy in the theory of HMM, in which the partition function admits $(1+1)$-d free fermion representation.

In Section 6, we use the formalism developed in Section 5 to construct a family of completely integrable hierarchies (labeled by an integer $N$) of non-linear differential equations. We show that the $N = 1$ hierarchy contains the $(2+1)$-d Burgers equations (multidimensional generalizations of the original Burgers equation [4], were considered first in [27], see [17] for a review). We thus call it the Burgers hierarchy. We give explicit solutions to it. These solutions generalize the Hopf-Cole solutions to the $(2+1)$-d Burgers equations, [18] and [7].

The family constructed provides a multidimensional extension of KP hierarchy which can be explained as follows. It is well known that the KP hierarchy can be formulated in terms of the pseudodifferential operator $W(t, \partial) = 1 + w_1(t)\partial^{-1} + w_2(t)\partial^{-2} + \cdots$ ([12] and [13], see [10] for a review). It is also known that KP hierarchy admits a reduction specified by the condition that $W(t, \partial)\partial^{N}$ is a differential operator, see [34], [35]. We call it the $KP(N)$ hierarchy. The set of equations of KP hierarchy coincides with that of $KP(N)$ hierarchy with $N \to \infty$. We show that the $N$-th representative of our family of hierarchies of non-linear differential equations is a multidimensional extension of $KP(N)$ hierarchy. Thus we call it the extended-$KP(N)$ hierarchy. In the simplest $N = 1$ case the extended-$KP(1)$, or equivalently, the Burgers hierarchy contains the $(2+1)$-d Burgers equations, while the $KP(1)$ hierarchy contains the $(1+1)$-d Burgers equation.

We then classify all formal solutions to the extended-$KP(N)$ hierarchy. We find the one-to-one correspondence between the set of formal solutions and an open subset of an infinite-dimensional Grassmann manifold $Gr(\infty, N)$. This subset consists of all $N$-dimensional subspaces of an infinite-dimensional complex linear space having a non-degenerate projection onto a fixed $N$-dimensional subspace.

Next we show that the partition function of the $N \times N$ NMM is a $\tau$-function of the extended-$KP(N)$ hierarchy. This is achieved by using the bosonization formula which can regarded as a $(2+1)$-d generalization of $(1+1)$-d bosonization formulae.

Finally in Section 7 we discuss the Ward identities for the NMM. It is known that matrix model solutions to the KP hierarchy (like HMM) satisfy Virasoro constraints, i.e. they are annihilated by an infinite set of differential operators spanning a subalgebra of Virasoro algebra. This subalgebra is isomorphic to an algebra of infinitesimal holomorphic polynomial diffeomorphisms of the complex plane. We show that the partition function of NMM is a special solution to the extended-$KP(N)$ hierarchy annihilated by the set of differential operators generating an algebra of all infinitesimal polynomial diffeomorphisms of the plane. The results of this Section can be used in the further analysis of the continuum limit of the NMM.

The proofs of lemmas and theorems presented in the paper are placed between “♦” signs.
2 The Eigenvalue and the Determinant Forms of the Normal Matrix Model

In this Section we will give the definition of the normal matrix model and derive the eigenvalue and determinant formulae for the partition function. We will be utilizing the standard methods of the theory of matrix models (see [33] for review). In the end of this Section, we will derive a connection between NMM and the Toda lattice hierarchy.

The partition function of NMM has the following form:

\[ Z_N = \int_{\{\Gamma: [M,M^\dagger] = 0\}} d\mu(\Gamma) e^{-\text{tr}V(M,M^\dagger)}, \]  

where \( \Gamma \) denotes the set of \( N \times N \) normal matrices, \( d\mu(\Gamma) \) is a measure on \( \Gamma \) induced by flat metric on the space of all \( N \times N \) complex matrices and \( V(z,\bar{z}) \) is a function on \( \mathbb{C} \) such that (1) exists.

As usual, integral (1) can be reduced to the integral over eigenvalues \( \{z_i\}_{i=1}^N \) and \( \{|\bar{z}_i|\}_{i=1}^N \) of matrices \( M \) and \( M^\dagger \) respectively. A corresponding calculation can be easily performed given the explicit expression for \( d\mu(\Gamma) \) in the appropriate local coordinates on \( \Gamma \). So let us sketch the calculation of \( d\mu(\Gamma) \).

First we notice, that any normal matrix \( M \) can be presented in the form \( M = UDU^\dagger \), where the \( D \) is diagonal matrix, \( U \in U(N) \) is a diagonalizing matrix. This decomposition is unique up to multiplying \( U \) by a diagonal unitary matrix from the right. The corresponding equivalence class of \( U \) together with \( D \) defines a convenient coordinate system on \( \Gamma \). Using these coordinates one can calculate the Riemannian metric induced on \( \Gamma \) by flat metric on the space of all matrices:

\[ \| \delta M \|^2 = \text{tr}(\delta D \cdot \delta D^\dagger) + 2\text{tr}(\delta u \cdot D \cdot \delta u \cdot D^\dagger - \delta u \cdot \delta u \cdot D \cdot D^\dagger), \]  

where \( \delta u = U^\dagger \delta U \) is an invariant (Haar) length element on \( U(N) \). Due to the \( U(N) \)-invariance \( \delta u \) is well-defined in terms of our coordinates on \( \Gamma \). Expressing \( \| \delta M \|^2 \) through eigenvalues and matrix elements \( (\delta u)_{ij} \equiv (U^\dagger \delta U)_{ij} \) we get

\[ \| \delta M \|^2 = \sum_{i=1}^N \delta z_i \delta \bar{z}_i - \sum_{i,j=1}^N \delta u_{ij} \delta u_{ji} |z_i - z_j|^2. \]  

On the other hand, \( \| M \|^2 = G_{ab} l^a l^b \), where \( \{l^a\} \) is a cumulative notation for the local coordinates on \( \Gamma \), \( G_{ab} \) is an induced metric on \( \Gamma \). Then \( \mu(\Gamma) = \text{det}(G)^{1/2} \prod_a dl^a \) (see e.g. [11]). Combining these two formulae with (3) we obtain

\[ d\mu(\Gamma) = dU \prod_{i=1}^N dz_i d\bar{z}_i |\Delta(z)|^2, \]  

where \( dU = \prod_{i \neq j} dz_idz_i \) is Haar measure on \( U(N) \) and \( \Delta(z) \equiv \text{det}[z_i^{j-1}]_{1 \leq i,j \leq N} = \prod_{i>j}(z_i - z_j) \) is the Van der Monde determinant.
Finally, substituting (4) into (1) and integrating over the unitary group we arrive at the eigenvalue expression for $Z_N$:

$$Z_N = c(N) \int_{\mathbb{C}^N} (\prod_{i=1}^N dz_i d\bar{z}_i e^{-V(z_i, \bar{z}_i)}) |\Delta(z)|^2,$$

where $c(N)$ is the volume of the unitary group, a constant factor independent from $V$. From now on we replace it by unity.

Next we will deduce a determinant formula for the integral (5) and relate the result to the solutions to Toda lattice hierarchy of differential equations.

The reformulation of (5) in determinant form is standard and is based on the following formula:

$$\det[M_{ik}] \cdot \det[M_{jk}] = \sum_{\sigma \in S^N} \det[M_{i\sigma(j)}M_{j\sigma(j)}],$$

where $M$ is any $N \times N$ matrix, $\sigma$ is an element of symmetric group $S^N$. Applying (6) to the product of Van der Monde determinants $\Delta(z)\Delta(\bar{z})$ in (5) we arrive at the desired form of the partition function NMM:

$$Z_N = N! \cdot \det[Z_{ij}]_{1 \leq i,j \leq N}, \text{ where}$$

$$Z_{ij} \equiv \int_{\mathbb{C}} dz d\bar{z} e^{-V(z, \bar{z})} z_i^{i-1} \bar{z}_j^{j-1}. \quad (8)$$

Let us consider the potential $V(z, \bar{z})$ in the form

$$V_t(z, \bar{z}) = U(z, \bar{z}) - \sum_{k>0} (t_k z^k + \bar{t}_k \bar{z}^k)$$

(9)

with $t_k = 0 = \bar{t}_k$ for $k \gg 1$. We are interested in the behavior of $Z_N[V_t]$ with respect to variations of $t$'s and $\bar{t}$'s. Note that $Z_N[V_t]$ can be considered as a generating function of correlators in NMM with the potential $U(z, \bar{z})$:

$$\langle (\text{tr} M^i)^{j_1} \cdot (\text{tr} M^{\dagger i_2})^{j_2} \cdots \rangle_{NMM} = \frac{1}{Z_N} \int_{\mathbb{C}^N} (\prod_{i=1}^N dz_i d\bar{z}_i e^{V(z_i, \bar{z}_i)}) |\Delta(z)|^2 \left( (\text{tr} M^{i_1})^{j_1} \cdot (\text{tr} M^{\dagger i_2})^{j_2} \cdots \right)$$

$$= \frac{1}{Z_N} \left( \frac{\partial}{\partial t_{i_1}} \right)^{j_1} \left( \frac{\partial}{\partial \bar{t}_{i_2}} \right)^{j_2} \cdots Z_N[V_t] \mid_{t, \bar{t}=0}. \quad (10)$$

From (8) with potential given by (9) we can easily deduce that

$$\frac{\partial Z_{ij}}{\partial t_k} = Z_{(i+k),j}, \quad \frac{\partial Z_{ij}}{\partial \bar{t}_k} = Z_{i,(j+k)} \mid_{i,j \geq 1}. \quad (10)$$

From (7) together with (10) mean that $Z_N[V_t]$ is an $N^\text{th}$ $\tau$-function of Toda lattice (with respect to complex variables $t$'s and $\bar{t}$'s, see [29] and [47] for details). Note that the real version of the integral (5) together with potential (9) was studied in [33], where it was referred to as a “scalar product model”.

In Section 6 we will discuss a relation between NMM with an arbitrary potential and integrable systems which includes the one described above as a particular case.
3 Correlation Functions of the Normal Matrix Model

In this Section we will analyze the structure of correlation functions of NMM with arbitrary axially symmetric polynomial potentials. We will see in this case that the two-point correlation function can be expressed in terms of the (analytical continuation of) one-point function, thus leading to the exact solvability of the model.

It follows from (5) that the distribution function of the eigenvalues in NMM with an axially symmetric potential is

\[ P_N(z, \bar{z}) = \frac{1}{Z_N}|\Delta(z)|^2e^{-\sum_{i=1}^{N}V(|z_i|^2)}, \]

where \( Z_N \) is the partition function of NMM. The \( n \)-point correlation functions (often called reduced distribution functions in statistical physics) are defined as follows:

\[ R_N^{(n)}(z_1, \cdots, z_n) \equiv \frac{N!}{(N-n)!} \int_{C^{N-n}} \prod_{i=n+1}^{N} dz_id\bar{z}_i P_N(z, \bar{z}), \]

where the combinatorial prefactor accounts for the symmetry of the integrand. Note that

\[ R_N^{(1)}(z) = \langle \delta(z - z_1)\delta(\bar{z} - \bar{z}_1) \rangle_{NMM} \]

and coincides therefore with the level density of the NMM.

Correlation functions (12) can be presented in the determinant form. To prove it we note that the expression (11) for the distribution function can be rewritten as

\[ P_N(z, \bar{z}) = \frac{1}{N!}det[K_N(z_i, z_j)]_{1 \leq i, j \leq N}, \]

where

\[ K_N(z_i, z_j) \equiv \sum_{k=0}^{N-1} \phi_k(z_i)\phi_k(z_j), \]

and

\[ \phi_m(z) \equiv \sqrt{c_m}z^m e^{-\frac{1}{2}V(|z|^2)}, \]

where \( c_m > 0 \). The functions \( \phi_m(z) \)'s introduced above are the orthogonal functions of the problem normalized by the following condition:

\[ \int_C dzd\bar{z} \phi_m(\bar{z})\phi_{m'}(z) = \delta_{m, m'}, \]

which yields

\[ \pi c_m \int_0^{\infty} dx \left[ x^m e^{-V(x)} \right] = 1. \]
Apparently, the orthogonal polynomials associated to the NMM with axially symmetric potential are just monomials in \( z \). This explains the certain simplicity of the model.

It is easy to check that the matrix \( K_N(\bar{z}_i, z_j) \) is hermitian and it can be considered as the kernel of projection operator, i.e. the relation

\[
\int_C d\bar{v} d\bar{w} K_N(\bar{u}, v) K_N(\bar{v}, w) = K_N(\bar{u}, w)
\]

is satisfied. Therefore one can make use of the well-known result from the theory of matrix models (see [30], page 89, theorem 5.2.1) to obtain the following representation of the \( n \)-point functions defined in (12):

\[
R_n^{(n)}(z_1, \cdots, z_n) = \det[K_N(\bar{z}_i, z_j)]_{1 \leq i,j \leq n}.
\]

(20)

It is convenient to rewrite the expression for \( K_N(\bar{z}_i, z_j) \) by substituting (16) into (15):

\[
K_N(\bar{z}_i, z_j) = \sum_{m=0}^{N-1} c_m \cdot (\bar{z}_i \cdot z_j)^m e^{\frac{1}{2}(-V(|z_i|^2) - V(|z_j|^2))}
\]

(21)

Therefore, up to a factor which explicitly depends on the potential, \( K_N(\bar{z}, w) \) is completely determined by the function of one complex variable \( k_N(\bar{z} \cdot w) \).

To illustrate the formulae obtained above let us consider the NMM with the Gaussian potential, \( V(|z|^2) = |z|^2 \), in the limit \( N \to \infty \). A simple calculation gives the following answer for the kernel:

\[
K(\bar{z}, w) = \frac{1}{\pi} e^{\left(\bar{z}w - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2\right)}.
\]

(22)

Substituting (22) into (20) we conclude that in the limit \( N \to \infty \) the one-point correlation function is constant and equals to \( \frac{1}{\pi} \), whereas the 2-point correlation function in the same limit is

\[
R^{(2)}(z, w) = \frac{1}{\pi^2} [1 - e^{(-|z-w|^2)}].
\]

(23)

It follows from the last two results that the connected part of the 2-point function is equal to \( \frac{1}{\pi} e^{x p[-|z - w|^2]} \) and decays exponentially at infinity. Let us note that the Gaussian NMM is equivalent to the Gaussian Complex matrix model (see [30], Chapter 15). However this is no longer true for a non-Gaussian matrix model potential.

Computability of NMM goes beyond the Gaussian case. For example it is possible to obtain a closed expression for \( K(\bar{z}, w) \) when the NMM potential is a monomial in \( |z|^2 \): \( V(z, \bar{z}) = |z|^{2k} \), where \( k \) is a positive integer. Corresponding integral in (18) is easily calculated to give the following expression for \( c_m \):

\[
c_m = \frac{k}{\pi} \frac{1}{\Gamma\left(\frac{m+1}{k}\right)}.
\]

(24)
Substituting (24) into (21) one obtains the following answer for the kernel at $N = \infty$:

$$K(\bar{z}, w) = \sum_{p=0}^{k-1} \frac{(\bar{z}w)^p}{\Gamma(p+1)} F_{1,1} \left( 1; \frac{p+1}{k}; (\bar{z}w)^k \right) e^{-\frac{1}{k}|z|^2k - \frac{1}{k}|w|^2k},$$

(25)

where $F_{1,1}(a; b; z)$ is the degenerate hypergeometric function, the solution to the Kummer equation which is analytic at $z = 0$:

$$z \frac{d^2 \omega}{dz^2} + (b - z) \frac{d \omega}{dz} - a \omega = 0.$$ 

(26)

For $k = 1$, (23) reduces to (22). It is well known (see e.g. [16]) that the $F_{1,1}(a, b, z)$ has a Taylor expansion in $z$ with an infinite radius of convergence and defines therefore an entire function on the complex plane. We use this remark below to analyze the correlation functions of NMM in the case of an arbitrary polynomial potential $V$.

Suppose $V(x) = \sum_{d=1}^{d} t_d x^d$ with $t_d > 0$. Then there are positive constants $a_1$ and $a_2$ such that

$$V(x) \leq a_1 + a_2 x^d, \quad x \geq 0.$$ 

(27)

This estimate can be used to prove that for any complex number $u$ and a positive number $R$ such that $|u| \leq R$,

$$\left| k_N(u) \right|_{V(x) = \sum_{n=1}^{d} t_n x^n} \leq e^{-a_1} \left| k_N(R) \right|_{V(x) = a_2 x^d}.$$ 

(28)

But the r.h.s. of (28) is nothing but a partial sum of a series converging by the remark above for any $0 \leq R < \infty$. Therefore $k(u) \equiv \lim_{N \to \infty} k_N$ exists for any $u$ such that $|u| < \infty$ and defines an entire function on the complex plane. But an entire function is uniquely determined by its values on say positive part of the real line, which provides us with the connection between 2- and 1-point correlation functions. To make this connection explicit we notice that by definition

$$R^{(1)}(|z|^2) = k \left( |z|^2 \right) e^{-V(|z|^2)}.$$ 

(29)

Denote by $R^{(1)}(u)$ an entire function of a complex variable $u$ which coincides with the 1-point function above for $u = |z|^2$. Such function exists as $V$ is a polynomial and $k(u)$ has been proved to be an entire function. Moreover it is unique, therefore is determined completely by the 1-point function. We finally notice that the connected part of the 2-point function can be expressed in turn in terms of $R^{(1)}(u)$:

$$R^{(2)}_c(z, w) = \left| R^{(1)}(\bar{z}w) \right|^2 e^{-[V(|z|^2) + V(|w|^2) - V(\bar{z}w) - V(zw)]},$$

(30)

which states the claimed property of the NMM. It’s worth mentioning that there is the counterpart of (30) for any finite $N$ as well.

Relation (30) leads to an exact solvability of NMM. Let us discuss this point in some more details. It follows from (20) and (21) that all correlation functions of the model can
be expressed in terms of $R^{(1)}(u)$. On the other side they can be considered as a solution to the certain chain of integro-differential equations which is discussed below. These two remarks permit one to derive a closed equation for the function $R^{(1)}(u)$ which is what we mean claiming the exact solvability of NMM. To obtain such an equation we rewrite an expression for the distribution function (11) in the following way:

$$P_N(z, \bar{z}) = \frac{1}{Z_N} e^{-\sum_{i=1}^{N} V(|z_i|^2) + 2 \sum_{i<j} \ln|z_i - z_j|}.$$  (31)

(31) can be identified with the distribution function of classical two-dimensional Coulomb gas in equilibrium. The analogous equivalence has been widely exploited in the study of conventional matrix ensembles (see [30] for review).

Such an identification allows one to use the Liouville’s Theorem of classical mechanics to derive a BBGKY chain of equations for the reduced distribution functions (12) (see e.g. [32]). In particular we have the following equations, connecting one- and two-point functions:

$$\frac{\partial R^{(1)}_N(|z|^2)}{\partial z} + R^{(1)}_N(|z|^2) \frac{\partial V(z)}{\partial z} = \int dwd\bar{w} R^{(2)}_N(z, w) \frac{\partial}{\partial z} \ln|z - w|,$$  (32)

and another one obtained from (32) by means of complex conjugation. Substituting (30) into (32) we get a closed equation for $R^{(1)}(u)$:

$$\frac{\partial R^{(1)}_N(|z|^2)}{\partial z} + R^{(1)}_N(|z|^2) \frac{\partial V(z)}{\partial z} = R^{(1)}_N(|z|^2) \int dwd\bar{w} R^{(1)}_N(|w|^2) \frac{\partial}{\partial z} \ln|z - w| - \int dwd\bar{w} \left[R^{(1)}(\bar{z}w)\right]^2 e^{-[V(|z|^2) + V(|w|^2) - V(\bar{z}w) - V(z\bar{w})]} \frac{\partial}{\partial z} \ln|z - w|. $$  (33)

Equation (33), together with a reality condition $R^{(1)}(u) = R^{(1)}(\bar{u})$ and normalization condition $\int dzd\bar{z} R^{(1)}_N(|z|^2) = N$, can be used to determine the entire function $R^{(1)}_N(u)$ and thus all correlation functions of NMM. In what follows we will exploit (33) to study the $N \to \infty$ scaling limit of NMM.

4 NMM in the Continuum Limit

The present Section is devoted to the NMM in $N \to \infty$ scaling (or continuum) limit. We will find the corresponding limits of level density and the two-point correlation function. We will see that in the regions where the limiting level density is non-zero and smooth (no points of phase transition) the form of the two-point function is universal, i.e. independent of the NMM potential up to a rescaling. The scale itself is determined by the level density which happens to be inversely proportional to the square of correlation length.

It is appropriate to mention that our search for universal answers in the NMM has been motivated by the study of universality in the hermitian matrix models, see e.g. [2].

Consider the $N \times N$ normal matrix model with the potential $N \cdot V(|z|^2)$, where $V(x)$ is a real polynomial with positive coefficient in front of the monomial of the highest degree.
We assume $V$ to be “convex,” by which we mean that the following inequality holds for any $z, w \in \mathbb{C}$:

$$V(|z|^2) + V(|w|^2) - V(\bar{z}w) - V(z\bar{w}) \geq 0,$$  \hspace{1cm} (34)

and the equality in (34) is reached iff $z = w$. Setting $w = z + \epsilon$ and expanding the l.h.s. of (34) around $\epsilon = 0$ we get the infinitesimal version of the convexity condition:

$$\Delta V(|z|^2) \geq 0,$$  \hspace{1cm} (35)

where $\Delta \equiv \partial \bar{\partial}$ is a two-dimensional Laplace operator. It is easy to check that any monomial potential of positive degree satisfies (34), therefore any polynomial of positive degree with positive coefficients does.

We will be interested in the asymptotic properties of the model as $N \to \infty$. We will consider two cases. First, we consider the limit $N \to \infty$ when the range of variation of $|z|^2$ is of order 1, which corresponds to the limit of strong potential and large separation of levels. Second, we will investigate the case of strong potential and small separation of levels, i.e. the limit $N \to \infty$ when the range of variation of $|z|^2$ is of order $\frac{1}{N}$.

For the first case we will seek the solution to (33) in the form of the asymptotic expansion in the inverse powers of $N$,

$$R^{(1)}_N(u) = N \sum_{n=0}^{\infty} R^{(1)}_n(u) N^{-n}.$$  \hspace{1cm} (36)

The limiting one-point function is defined to be

$$R^{(1)}(u) = R^{(1)}_0(u) \equiv \lim_{N \to \infty} \frac{R^{(1)}_N(u)}{N}.$$  \hspace{1cm} (37)

Substituting (30) into (33) and equating the terms of the first order in $N$ we get the following equation for the limiting one-point function:

$$R^{(1)}(|z|^2) \left( \frac{\partial V(|z|^2)}{\partial z} - \int dwd\bar{w} R^{(1)}(|w|^2) \frac{\partial}{\partial z} \ln |z - w| \right) = 0,$$

which implies that

$$\frac{\partial V(|z|^2)}{\partial z} = \int dwd\bar{w} R^{(1)}(|w|^2) \frac{\partial}{\partial z} \ln |z - w|,$$  \hspace{1cm} (38)

if we assume that $R^{(1)}(|z|^2) \neq 0$. To derive (38) from (33) we have used the convexity condition and relation (30). Relation (30) was used to show that in the limit considered the connected part of the two-point function is zero almost everywhere. Note that we derived (33) from the Liouville theorem, so its direct consequence, equation (38), provides an exact description of the continuum limit of NMM.

It is interesting to note that equation (38) can be interpreted as an extremum condition for the potential energy $U[\rho]$ of the gas of charged particles of density $\rho(|z|^2) = R^{(1)}(|z|^2)$ placed in the external potential:

$$U[\rho] = \int dzd\bar{z} \rho V(|z|^2) - \frac{1}{2} \int \int dzd\bar{z}dwd\bar{w} \rho(|z|^2) \rho(|w|^2) \ln |z - w|^2.$$  \hspace{1cm} (39)
Such an interpretation was a starting point in the derivation of the level density in the hermitian matrix model presented in \[30\], however it required an assumption that the two-point function factorizes in $N \to \infty$ limit. In contrast, in the NMM we can demonstrate the factorization of the two-point function almost everywhere by analyzing (30).

Equation (38) can be easily solved by differentiating both sides with respect to $\bar{z}$. Noticing that $\ln |z-w|$ is a Green function of two-dimensional Laplacian, i.e. $\Delta \ln |z-w| = \pi \delta^{(2)}(z-w)$, we get

$$R^{(1)}(|z|^2) = \frac{1}{\pi} \Delta V(|z|^2). \quad (40)$$

So we conclude that the limiting level density (one-point function) of NMM at the point $z$ is equal to $\frac{1}{\pi} \Delta V(|z|^2)$, given that the limiting level density is not zero. This answer has a transparent physical meaning: the gas of charged particles which describes the continuous limit of NMM tends to screen completely the external potential.

Having described the spectrum locally one can derive a complete picture assuming that the potential energy of the “gas” of NMM eigenvalues is minimal. In our case $V$ is convex and the minimum energy assumption yields the following answer for the level density:

$$R^{(1)}(|z|^2) = \frac{1}{\pi} \Delta V(|z|^2) \quad \text{if } |z|^2 < r \quad (41)$$
$$R^{(1)}(|z|^2) = 0 \quad \text{if } |z|^2 > r, \quad (42)$$

where the end point $r$ of the spectrum is determined from the normalization condition

$$\int_{|z|^2 < r} dz d\bar{z} R^{(1)}(|z|^2) = 1. \quad (43)$$

Consider now the case of small distances between eigenvalues, for which the two-point correlations are essential. Let $x$ be a real positive number. We assume that the $R^{(1)}(x) > 0$ and there is a closed neighborhood of $x$ in the complex plane where the convergence of the r.h.s. of (37) is uniform (in physical terms, $x$ is not a point of a phase transition in the $N \to \infty$ limit). We will study the limiting behavior of the two-point function of NMM in the vicinity of such a point. Note that the existence of points of uniform convergence does not contradict the information about the structure of correlation functions we have collected so far. For example, the uniform convergence of the sequence of entire functions $R^{(1)}_N$ in the closed neighborhood $U$ of $x$ implies that the limit is a holomorphic function in this neighborhood. But according to (40) $R^{(1)}(x)$ admits an analytic continuation in some neighborhood of $x$ given that $R^{(1)}(x) \neq 0$ (recall that $V(x)$ is a polynomial function).

Moreover, a direct verification shows that in the case of monomial potentials $x$ is a point of uniform convergence if $x$ is not too large and $R^{(1)}(x) \neq 0$. We conclude therefore that the domain of our consideration is not empty. However, it does not consist of the whole plane: it is apparent that the endpoints of the spectrum defined by (43) are excluded from our considerations, as the sequence of continuous functions cannot converge uniformly to a discontinuous one. Thus the results concerning the two-point functions presented below are valid only in the “bulk” of the spectrum.
Consider now the following holomorphic change of coordinates:

\[ z_N = \sqrt{x} \cdot e^{\zeta/\sqrt{N}}, \quad w_N = \sqrt{x} \cdot e^{\eta/\sqrt{N}}, \quad u_N \equiv \bar{z}_N \cdot w_N. \tag{44} \]

The corresponding limit of the one-point function is

\[ R^{(1)}_{(1)}(x, \bar{\zeta} + \eta) \equiv \lim_{N \to \infty} \frac{R^{(1)}(u_N)}{N}. \]

It is a matter of simple considerations to show that if \( \lim_{N \to \infty} g_N(u) = g(u) \) uniformly, \( u \in U \), then \( \lim_{N \to \infty} g_N(u \cdot f_N) = g(u) \) for any sequence \( \{f_N\} \) converging to 1 and such that \( u \cdot f_N \in U \) for any \( N \). Using this remark we see that \( R^{(1)}(x, \bar{\zeta} + \eta) = R^{(1)}(x) \). [Let us also remark that the expression \( R^{(1)}(x, \bar{\zeta} + \eta) = R^{(1)}(x) \) solves the limit of the equation (44) corresponding to small separations between levels. One can consider this remark as yet another consistency check of the assumption about the existence of the point of uniform convergence.] Now we can substitute (44) into (30) and take the limit \( N \to \infty \) of (30) to obtain the following answer for the scaling limit of the two-point function:

\[ R^{(2)}_{(1)}(x, \bar{\zeta}, \eta) = [R^{(1)}(x)]^2 e^{-\pi \cdot R^{(1)}(x) |\bar{\zeta} - \eta|^2}, \tag{45} \]

where we used the result (40) and assumed that \( R^{(1)}(x) \neq 0 \). We conclude from (45) that the scaling limit of the two-point function of NMM exhibits universal behavior since the dependence on potential (which enters the answer through \( R^{(1)}(x) \) only) can be eliminated by changing the scale and normalization. Notice that the square of correlation length is given by inverse level density. Equation (45) can also be interpreted as a completely universal relation between one- and two-point functions of the NMM. A similar phenomenon was previously observed in [3] for the hermitian matrix model and is referred to as “universality of the second type”.

5 Free Fermion Representation of NMM

The aim of the present Section is to rewrite the partition function (1) of NMM in the form of the free fermion correlator. This will permit us to describe an arbitrary variation of NMM potential by means of integrable system of differential equations and generalize therefore the results of Section 2. The technique we use is a straightforward generalization of free fermion methods developed in [40] and reviewed in part in [10] and their application to the theory of matrix models due to the ITEP group (see [33] for review).

Consider an abelian group \( \mathbb{Z} \times \mathbb{Z} \) consisting of pairs of integers. The corresponding group multiplication will be denoted by “+”. To simplify notations we refer to this group as \( G \). An abelian group \( \mathbb{Z} \) acts on \( G \) as follows:

\[ G \times \mathbb{Z} \to G, \]

\[ (g = (i, j), m) \mapsto g \cdot m = (i \cdot m, j \cdot m). \tag{46} \]

Alternatively, \( G \) can be presented in the following form:

\[ G = \bigoplus_{m \in \mathbb{Z}} G_m = G_- \oplus G_+, \tag{47} \]
where
\[ \mathcal{G}_- = \oplus_{m<0} \mathcal{G}_m ; \quad \mathcal{G}_+ = \oplus_{m \geq 0} \mathcal{G}_m, \] (48)
and
\[ \mathcal{G}_m = \left\{ g \in \mathcal{G} \mid g = (l + m, -l) ; \ l \in \mathbb{Z} \right\}, \ m \in \mathbb{Z}. \] (49)

Note that \( \mathcal{G}_0 \) can be visualized as an antidiagonal of \( \mathcal{G} = \mathbb{Z} \times \mathbb{Z} \). It is also worth mentioning that if \( g \in \mathcal{G}_m \) and \( h \in \mathcal{G}_n \), then \( g + h \in \mathcal{G}_{m+n} \). Therefore, \( \mathcal{G} \) is \( \mathbb{Z} \)-graded and (47) is a decomposition of \( \mathcal{G} \) into a sum of components of a fixed degree.

Let us consider an infinite-dimensional Clifford algebra \( \mathcal{A}(\mathcal{G}) \) over complex numbers which corresponds to \( \mathcal{G} \). We define it by means of the following set of generators and relations:
\[ \mathcal{A}(\mathcal{G}) = \left\langle 1, \bar{\psi}_g, \psi_h ; \ g, h \in \mathcal{G} \right| \psi_g, \bar{\psi}_h = \delta_{g,h}, \{ \psi_g, \psi_h \} = 0 = \{ \bar{\psi}_g, \bar{\psi}_h \} \right\rangle. \] (50)

Let \( \mathcal{F}_R \) be the right Fock space, an irreducible left \( \mathcal{A}(\mathcal{G}) \)-module obtained by applying \( \mathcal{A}(\mathcal{G}) \) to the right vacuum vector \( |\text{vac}\rangle \), which is defined as follows:
\[ \psi_g |\text{vac}\rangle = 0, \ g \in \mathcal{G}_- ; \quad \bar{\psi}_h |\text{vac}\rangle = 0, \ h \in \mathcal{G}_+. \] (51)

In the same fashion we introduce the left Fock space \( \mathcal{F}_L \) which is generated by the right action of \( \mathcal{A}(\mathcal{G}) \) on the left vacuum vector \( \langle \text{vac}| \) which is defined below:
\[ \langle \text{vac}| \bar{\psi}_g = 0, \ g \in \mathcal{G}_- ; \quad \langle \text{vac}| \psi_h = 0, \ h \in \mathcal{G}_+. \] (52)

There is a natural non-degenerate pairing between left and right Fock spaces,
\[ \langle \text{vac}|a, b|\text{vac}\rangle \mapsto \langle \text{vac}|a \cdot b|\text{vac}\rangle, \]
where \( a \) and \( b \) are elements of \( \mathcal{A}(\mathcal{G}) \). This pairing is normalized as follows:
\[ \langle \text{vac}|1 \cdot 1|\text{vac}\rangle = 1. \] (53)

From now on we will think of \( \mathcal{A}(\mathcal{G}) \) as an algebra of linear transformations of \( \mathcal{F}_R \) or \( \mathcal{F}_L \) and refer to elements of \( \mathcal{A}(\mathcal{G}) \) as operators.

To each element \( a \in \mathcal{A}(\mathcal{G}) \) one can assign its normal reordering with respect to the vacuum \( |\text{vac}\rangle \), an element of \( \mathcal{A}(\mathcal{G}) \), denoted as : \( a : \). It is obtained from an element \( a \) by permuting the generators in each term of the sum constituting the element \( a \) in such a way that the fermion generators annihilating the right vacuum stand to the right from the rest of generators.

A variety of vectors in the Fock space is provided by means of the following construction. To each finite set \( S \subset \mathcal{G} \) we assign the following vector in \( \mathcal{F}_R \):
\[ |S\rangle = \prod_{g \in S} o_g |\text{vac}\rangle, \] (54)
where $o_g = \psi_g$ if $g \in G_+ \cap S$ and $o_g = \bar{\psi}_g$ if $g \in G_- \cap S$. Note that without fixing a linear order in the set $\mathbb{Z} \times \mathbb{Z}$, the element (54) is defined up to a sign only. The same construction can be applied to obtain a vector in $F_L$ which we will denote as $\langle S \rangle$,

$$\langle S \rangle = \langle \text{vac} \rangle \prod_{g \in S} \bar{o}_g,$$

(55)

where $\bar{o}_g = \bar{\psi}_g$ if $g \in G_+ \cap S$ and $\bar{o}_g = \psi_g$ if $g \in G_- \cap S$. It is easy to see that any element of $F_R$ or $F_L$ can be presented as a linear combination of vectors (54) or (55), respectively.

Consider a semigroup $Q \subset G$ consisting of all pairs $g = (m,n)$, where $m$ and $n$ are non-negative integers. We introduce the following element of the algebra $\mathcal{A}(G)$:

$$H(t) = \sum_{g \in Q} t_g J_g \equiv \sum_{g \in Q} t_g (\sum_{h \in G} \psi_h \bar{\psi}_{h+g}),$$

(56)

where

$$t_g = 0 \text{ if } \text{deg}(g) >> 0,$$

(57)

and we always suppose that $t_0 = 0$. We will call $H(t)$ a Hamiltonian operator or simply Hamiltonian. It is easy to check that the operators $J_g$’s introduced in (56) commute, i.e. $[J_g, J_h] \equiv J_g \cdot J_h - J_h \cdot J_g = 0$, for $g, h \in Q \setminus \{0\}$. It is also a matter of simple computation to verify that $[J_g, \psi_h] = \psi_{g-h}$ and $[J_g, \bar{\psi}_h] = \bar{\psi}_{g+h}$.

Before we introduced the Hamiltonian (56) all our considerations had not been different from the standard fermion construction as $\mathcal{A}(G)$ is isomorphic to a standard Clifford algebra the generators of which are labeled by elements of $\mathbb{Z}$, the isomorphism being established by ordering the set $\mathbb{Z} \times \mathbb{Z}$. The new development starts with the introduction of Hamiltonian (56) since the presented commutation relations between $J_g$’s and fermion generators depend on the group structure in $G = \mathbb{Z} \times \mathbb{Z}$, but $G$ is not isomorphic to $\mathbb{Z}$ as a group.

It follows from the definition (54) that the Hamiltonian operator annihilates vacuum,

$$H(t)|\text{vac}\rangle = 0.$$

(58)

Define a $t$-evolution of an element $a \in \mathcal{A}(G)$ as the following element of (the formal completion of) $\mathcal{A}(G)$:

$$a(t) = e^{H(t)}ae^{-H(t)}.$$

(59)

Due to (58) the $t$-evolution preserves the normal ordering, $:a(t): = :a : (t)$. It is not difficult to derive the following expressions for the $t$-evolution of fermionic generators of $\mathcal{A}(G)$:

$$\psi_g(t) = \sum_{h \in Q} p_h(t)\psi_{g-h},$$

(60)

$$\bar{\psi}_g(t) = \sum_{h \in Q} p_h(-t)\bar{\psi}_{g+h},$$

(61)

where $\{p_g\}_{g \in Q}$ are generalized Schur polynomials,

$$p_g(t) = \delta_{g,0} + t_g + \frac{1}{2!} \sum_{h', h'' \in Q : h' + h'' = g} t_{h'}t_{h''} + \frac{1}{3!} \sum_{h', h'', h''' \in Q : h' + h'' + h''' = g} t_{h'}t_{h''}t_{h'''} + \cdots.$$
The set $\{p_g\}_{g \in Q}$ defined above indeed consists of polynomials: the sum in the r.h.s. of (62) is finite as there is a finite number of ways to present an element of $Q$ as a sum of elements of $Q$ of positive degree. It is also worth mentioning that one can grade the polynomial ring $C[t_h, h \in Q]$ by setting $\text{deg}(t_h) = \text{deg}(h)$. Under such assignment $p_g(t)$ becomes a homogeneous polynomial of degree equal to $\text{deg}(g)$.

We will also need a generating function for the Schur polynomials. Let $z, \bar{z}$ be complex coordinates in the plane. We set $\vec{z}^g \equiv z^a \cdot \bar{z}^b$, where $g = (a, b)$. Let also

$$V(t, \vec{z}) = \sum_{g \in Q} t_g \vec{z}^g. \quad (63)$$

Then

$$e^{V(t, \vec{z})} = \sum_{g \in Q} p_g(t) \vec{z}^g. \quad (64)$$

The generating function (63) can be used to rewrite relations (60) and (61) in a compact form. Let

$$\psi(\vec{z}) = \sum_{g \in G} \psi_g \vec{z}^g \quad (65)$$

be a free fermion field operator. Then a direct computation shows that

$$e^{H(t)} \psi(\vec{z}) e^{-H(t)} = e^{V(t, \vec{z})} \psi(\vec{z}). \quad (66)$$

To derive a free fermion representation of the partition function of NMM we will need a pair of operators called fermionic projectors:

$$P_+ = :e^{\sum_{g \in G_-} \psi_g \bar{\psi}_g}:; \quad (67)$$

$$P_- = :e^{-\sum_{g \in G_+} \psi_g \bar{\psi}_g}:; \quad (68)$$

One can verify the following properties of fermionic projectors:

$$P_+ \bar{\psi}_g = \psi_g P_+ = 0, \text{ if } g \in G_-; \quad (69)$$

$$P_- \psi_g = \bar{\psi}_g P_- = 0, \text{ if } g \in G_+; \quad (70)$$

$$P_+^2 = P_+; \quad P_-^2 = P_-; \quad (71)$$

Eq. (71) is a direct consequence of (69) and (70), which in turn result from the following equivalent representation of fermionic projectors:

$$P_+ = \prod_{g \in G_-} (1 - \bar{\psi}_g \psi_g),$$

$$P_- = \prod_{g \in G_+} (1 - \psi_g \bar{\psi}_g).$$

Finally we are ready to introduce the main object of our interest. Consider the following fermionic correlator:

$$\tau(U, A(\Phi)|t) = \langle U|e^{H(t)} A(\Phi)|U \rangle, \quad (72)$$
where $U \subset Q$ is a finite subset of $Q \subset G$,

$$A(\Phi) = \left\{ e \left\{ \int_{C^2} dz d\bar{z} dw d\bar{w} \ \Phi(\bar{z}, \bar{w}) \psi_+ (\bar{z}) \bar{\psi}_+ (\bar{w}) - \sum_{g \in G_+} \psi_g \bar{\psi}_g \right\} : \right\},$$  \hspace{1cm} (73)

$$\psi_+(\bar{z}) = \sum_{g \in G_+} \psi_g \bar{z}^g, \quad \bar{\psi}_+(\bar{z}) = \sum_{g \in G_+} \bar{\psi}_g \bar{z}^{-g},$$  \hspace{1cm} (74)

and $\Phi$ is a real function (or a distribution) on $C^2$. Our aim is to compute the correlation function (72). Consider the following matrix of an infinite size:

$$A(\Phi)_{g, h} = \int_{C^2} dz d\bar{z} dw d\bar{w} \ \Phi(\bar{z}, \bar{w}) \bar{w} \cdot \cdot \cdot \Phi(\bar{z}_m, \bar{w}_m) \times \psi_+ (\bar{z}_1) \cdot \cdot \cdot \psi_+ (\bar{z}_m) P_- \bar{\psi}_+ (\bar{w}_m^{(-1,-1)}) \cdot \cdot \cdot \bar{\psi}_+ (\bar{w}_1^{(-1,-1)}).$$  \hspace{1cm} (77)

In what follows we will always require the non-degeneracy of the matrix $A_{g, h}(\Phi)$. We also wish to remark that operator $A(\Phi)$ from (73) is a counterpart of the operator used in [22] to obtain a free fermion representation of conventional one-matrix models.

Using relations (76) and the fact that $A(\Phi)|\text{vac}\rangle = |\text{vac}\rangle$ and applying Wick’s theorem one can express $\tau(U, A(\Phi)|t)$ in terms of two-point correlators:

$$\tau(U, A(\Phi)|t) = \det[Z_{g, h}(t)]_{g, h \in U},$$  \hspace{1cm} (79)
where

\[ Z_{g,h}(t) = \int_{\mathbb{C}^2} dzd\bar{z}dwd\bar{w} \bar{z}^g \bar{w}^h \Phi(\bar{z}, \bar{w}) e^{V(t, \bar{z})} \]  

(80)

Suppose now that the set \( U \subset G \) and the function \( \Phi \) which parametrize the \( \tau \)-function are chosen to be the following:

\[ U_N = \left\{ g \in G \mid g = (n, 0), n = 0, 1, \cdots N - 1 \right\} , \]

(81)

\[ \Phi(\bar{z}, \bar{w}) = \Phi_{NMM}(\bar{z}, \bar{w}) = \delta(z - \bar{w}) \cdot \delta(\bar{z} - w) \cdot e^{-|z|^2} . \]

(82)

Substituting (81) and (82) into (79) and (80), we see that

\[ \tau(U_N, A(\Phi_{NMM}) | t) = \det[Z_{i,j}(t)]_{0 \leq i,j \leq N-1} , \]

(83)

which coincides (up to nonessential factor) with the determinant formula (7) and (8) for the partition function of NMM with a polynomial potential equal to \(-V(t, z) + |z|^2\). Thus we proved that

\[ Z_N = \frac{1}{N!} \tau\left(U_N, A(\Phi_{NMM}) \mid t\right) . \]

(84)

To conclude the discussion of the present Section we would like to mention that the partition functions of conventional matrix models (such as Hermitian, Unitary, Complex one-matrix models and Hermitian two-matrix models) can be presented in the form (72) under appropriate choices of \( U \) and \( \Phi \). To obtain the partition function of the hermitian matrix model, for example, set

\[ \Phi(\bar{z}, \bar{w}) = \Phi_{HMM}(\bar{z}, \bar{w}) = \delta(z - \bar{w}) \cdot \delta(\bar{z} - w) \cdot \delta(z - \bar{z}) \cdot e^{-|z|^2} . \]

(85)

Then

\[ \tau\left(U_N, A(\Phi_{HMM}) \mid t\right) = \det[Z_{i,j}]_{0 \leq i,j \leq N-1} \]

(86)

with

\[ Z_{i,j} = \int_{-\infty}^{\infty} dx x^{i+j} e^{\sum_{m>0} T_m x^m} \text{ and } T_m = \sum_{(k,l) \in \mathbb{Q}} t_{(k,l)} - \delta_{m,1} . \]

(87)

Eq. (86) is the determinant form of the partition function of hermitian one-matrix model with the potential \(-\sum_{p>0} T_p x^p\).
6 The Partition Function of NMM as a \( \tau \)-Function of the Extended-KP(\( N \)) Hierarchy

In this Section we are going to derive a system of differential equations associated with correlation function (72). In virtue of (84) all results of the present Section apply as well to the partition function of NMM with an arbitrary polynomial potential.

The part of our construction dealing with free fermions relies heavily on methods developed in [20], [8] and [9], see [10] for a review. Our consequent analysis of the emerging hierarchy of differential equations shows that the original approach of [42] and [43] to the theory of KP equations can be extended to the multidimensional case as well.

Let \( U_N \) and \( A \) be a subset of \( Q \) and an element of the Clifford algebra given by (81) and (73) correspondingly (to simplify notations, from now on we denote \( A(\Phi) \) and \( A(\Phi)_{g,h} \) as \( A \) and \( A_{g,h} \) respectively). The following functions depending on \( z \) and \( t \)'s are called wave functions:

\[
\begin{align*}
    w_p(z, t) &= \frac{\langle U_N \bar{\psi}_{N,p} e^{H(t)} \psi(z) A U_N \rangle}{\langle U_N e^{H(t)} A U_N \rangle}, \\
    w_q(z, t) &= \frac{\langle U_N \bar{\psi}_{q} e^{H(t)} \psi(z) A U_N \rangle}{\langle U_N e^{H(t)} A U_N \rangle},
\end{align*}
\]

(87) (88)

where \( \psi(z) = \sum_{g \in G} \psi_g z^g \) is a free field operator; \( p = (1, 0) \in G, \ q = (0, 1) \in G \). In our notations for the wave functions we suppress the dependence on \( N \) and \( A \) which are supposed to be fixed. We assume that the common denominator in (87) and (88) is not equal to zero when all \( t \)'s are equal to 0. Then the wave functions make sense as formal power series in \( t \)'s.

There is a linear relation imposed on the wave functions which follows from the identity (76). Let us explore it. Consider the Fourier decomposition of wave functions with respect to \( z \):

\[
\begin{align*}
    w_p(z, t) &= \sum_{g \in G} w_p^g(t) z^g, \quad w_p(z, t) = \sum_{g \in G} w_q^g(t) z^g
\end{align*}
\]

(89)

It is easy to prove the following set of relations between coefficients of such decomposition:

\[
\begin{align*}
\sum_{h \in G} A_{g,h} w_p^h(t) &= 0, \quad \sum_{h \in G} A_{g,h} w_q^h(t) = 0 \text{ if } g \in G_0 \cup U_N.
\end{align*}
\]

(90)

Here \( A_{g,h} \) is a matrix defined by (75). To prove (90) we observe that \( w_p^h(t) \) is proportional to

\[
\langle U_N \bar{\psi}_{N,p} e^{H(t)} \psi_h A U_N \rangle.
\]

Multiplying it by \( A_{g,h} \), summing over \( h \), and using (76) we get \( \langle U_N \bar{\psi}_{p} e^{H(t)} A \psi_g U_N \rangle \), which is zero if \( g \in G_0 \cup U_N \). The similar arguments apply if we replace \( w_p^h(t) \) with \( w_q^h(t) \). Therefore, (90) is proven.
Relations (90) have their counterpart in the theory of KP hierarchy (see e.g. [10]) and are of prime importance for our further considerations. Before one can make use of them however, it is desirable to rewrite equations (90) in the form of linear relations between the finite number of unknowns. This is possible because of the following representation of wave functions (87) and (88):

\[ w_p(\vec{z}, t) = \left( \vec{z}^N \cdot \mathbf{p} + \sum_{\mathbf{g} \in U_N} a_{\mathbf{g}}(t) \vec{z}^g \right) e^{V(t, \vec{z})}, \]  
\[ w_q(\vec{z}, t) = \left( \vec{z}^q + \sum_{\mathbf{g} \in U_N} b_{\mathbf{g}}(t) \vec{z}^g \right) e^{V(t, \vec{z})}, \]  
(91)  
(92)

where coefficients \( \{a_{\mathbf{g}}(t), b_{\mathbf{g}}(t)\}_{\mathbf{g} \in U_N} \) depend on \( t \)'s and the choice of \( A \). It is easy to express them in terms of free fermion correlators, but we will not need the explicit expressions. To verify (91) one can perform the following computation: commute the field operator \( \psi(\vec{z}) \) in the numerator of (87) with \( e^{H(t)} \) using (74). Then notice that

\[ \langle U_N | \psi_{N \cdot \mathbf{p}} \psi_{\mathbf{g}} = 0, \mathbf{g} \in G_+ \setminus \left( U_N \cup \{N \cdot \mathbf{p}\} \right) \rangle \quad \text{and} \quad \psi_{\mathbf{h}} e^{H(t)} A | U_N \rangle = 0, \mathbf{h} \in G_-, \]

in which the first equality follows from the definition (55) of the generating vectors of \( \mathcal{F}_L \) while the other one is a consequence of our choice of the operator \( A \) defined in (73) and the law (60) of evolution of fermionic generators:

\[ \psi_{\mathbf{h}} e^{H(t)} A | U_N \rangle = e^{H(t)} \sum_{\mathbf{g} \in Q} A \psi_{\mathbf{h} - \mathbf{g}} p_{\mathbf{g}} (-t) | U_N \rangle = 0, \mathbf{h} \in G_. \]

It only remains to show that the coefficient in front of \( \vec{z}^N \cdot \mathbf{p} \) in (91) is indeed 1. But according to (87) this coefficient is equal to \( \langle U_N | \psi_{N \cdot \mathbf{p}} \psi_{N \cdot \mathbf{p}} | U_N \rangle \), which is 1. Representation (91) has been derived. The derivation of (92) can be performed along the same lines.

Using the fact that \( e^{V(t, \vec{z})} = \sum_{\mathbf{g} \in G} p_{\mathbf{g}}(t) \vec{z}^g \), where we assumed that \( p_{\mathbf{g}} = 0 \) if \( \mathbf{g} \in G \setminus Q \), and comparing representations of wave functions (84) and ((91), (92)) one can rewrite the relations (90) in terms of coefficients \( \{a_{\mathbf{g}}, b_{\mathbf{g}}\}_{\mathbf{g} \in U_N} \):

\[ \sum_{\mathbf{h} \in G} A_{\mathbf{g}, \mathbf{h}} \left( p_{\mathbf{h} - N \cdot \mathbf{p}}(t) + \sum_{k \in U_N} a_k(t) p_{\mathbf{h} - k}(t) \right) = 0 \quad \text{if} \quad \mathbf{g} \in G_+ \cup U_N, \]  
\[ \sum_{\mathbf{h} \in G} A_{\mathbf{g}, \mathbf{h}} \left( p_{\mathbf{h} - q}(t) + \sum_{k \in U_N} b_k(t) p_{\mathbf{h} - k}(t) \right) = 0 \quad \text{if} \quad \mathbf{g} \in G_- \cup U_N. \]  
(93)  
(94)

Looking closer at relations (93) and (94) we see that the only non-trivial ones are those corresponding to \( \mathbf{g} \in U_N \). Indeed, if \( \mathbf{g} \in G_- \) then \( A_{\mathbf{g}, \mathbf{h}} = 0 \) for \( \mathbf{h} \in G_+ \) in accordance with definition (73). Thus left hand sides of relations (93) and (94) are just identical zeros if \( \mathbf{g} \in G_- \).

Let us consider now the following pair of differential operators:

\[ W_p(t, \vec{\partial}) = \vec{\partial}^N \cdot \mathbf{p} + \sum_{\mathbf{g} \in U_N} a_{\mathbf{g}}(t) \vec{\partial}^g, \]  
\[ W_q(t, \vec{\partial}) = \vec{\partial}^q + \sum_{\mathbf{g} \in U_N} b_{\mathbf{g}}(t) \vec{\partial}^g, \]  
(95)  
(96)
which we call wave operators. Here \( \partial \mathbf{g} \equiv \left( \frac{\partial}{\partial p} \right)^{g_p} \left( \frac{\partial}{\partial q} \right)^{g_q} \) and \( \mathbf{g} = g_p \cdot \mathbf{p} + g_q \cdot \mathbf{q} \) is an arbitrary element of \( \mathbf{G} \) decomposed in terms of \( \mathbf{p} \) and \( \mathbf{q} \). In what follows we will denote the complex variables \( t_p \) and \( t_q \) as \( x \) and \( y \), respectively. Thus \( W_p \) and \( W_q \) are polynomials in \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) with coefficients depending on “times” \( t \)’s. Evidently,

\[
w_p(t,z) = W_p(t,\partial)e^{V(t,z)}, \quad w_q(t,z) = W_q(t,\partial)e^{V(t,z)}. \tag{97}\]

Noting also that \( \partial \mathbf{g} p_h(t) = p_{n-h}(t) \) for \( \mathbf{g} \in \mathbf{Q} \), we can rewrite relations \( (93) \) and \( (94) \) in terms of wave operators, which will provide us with an alternative form of equations \( (90) \) suitable for our needs:

\[
W_p h_{\mathbf{g}}(t) = 0 = W_q h_{\mathbf{g}}(t), \quad \mathbf{g} \in U_N, \tag{98}
\]

where \( h_{\mathbf{g}}(t) = \sum_{\mathbf{h} \in \mathbf{G}} A_{\mathbf{g},\mathbf{h}} p_{\mathbf{h}}(t) \). Relations \( (98) \) can be viewed as two \( N \times N \) systems of linear equations with respect to coefficients of wave operators. Here is an explicit solution:

\[
a_{n-p}(t) = \frac{\text{det}(\partial^n p_h, \partial^{n-1} p_h, \ldots, \partial p_h, \partial p_h, \partial^n p_h, \partial^{n-1} p_h, \ldots, \partial p_h, \partial p_h)}{\text{det}(\partial^n p_h, \ldots, \partial p_h)}, \tag{99}
\]

\[
b_{n-p}(t) = \frac{\text{det}(\partial^n p_h, \partial^{n-1} p_h, \ldots, \partial p_h, \partial p_h, \partial^n p_h, \partial^{n-1} p_h, \ldots, \partial p_h, \partial p_h)}{\text{det}(\partial^n p_h, \ldots, \partial p_h)} \tag{100}
\]

where \( n = 0, 1, \ldots, N - 1 \) and \( \mathbf{h} \) is a column vector with elements \( h_{\mathbf{g}}, \mathbf{g} \in U_N \). It follows from the results of the previous Section that the common denominator of \( (99) \) and \( (100) \) is equal exactly to \( \langle U_N | e^{H(t)} A | U_N \rangle \), and is therefore an invertible power series in \( t \)’s. Thus the wave operators are uniquely determined by conditions \( (98) \) and their coefficients are formal power series in “times”.

Let \( \mathbf{B} \) be a ring of differential operators in \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) with coefficients being formal series in variables \( t \)’s. The statement bellow is a natural consequence of the formalism developed:

**Lemma 1** Let \( \mathbf{O} \in \mathbf{B} \) be a differential operator such that

\[
\mathbf{O} h_{\mathbf{g}} = 0, \quad \mathbf{g} \in U_N. \tag{101}
\]

Then there are differential operators \( b_p \in \mathbf{B} \) and \( b_q \in \mathbf{B} \) such that

\[
\mathbf{O} = b_p W_p + b_q W_q. \tag{102}
\]

in other words \( \mathbf{O} = 0 \) mod \( W_p, W_q \). Moreover one can choose \( b_p \) to be of zeroth order in \( \frac{\partial}{\partial y} \).

The proof of Lemma 1 is given in the Appendix. The system of non-linear equations satisfied by coefficients of wave operators is a direct consequence of the formulated statement. To see this we will differentiate each of the relations \( (88) \) with respect to \( t_h \) and use the fact that \( \frac{\partial}{\partial h} h_{\mathbf{g}}(t) = h h_{\mathbf{g}}(t) \), a property of Schur polynomials. As a result we obtain the following set of identities:

\[
\left( \frac{\partial W_p(q)(t,\partial)}{\partial t_h} + W_p(q)(t,\partial) \partial h \right) h_{\mathbf{g}} = 0, \quad \mathbf{g} \in U_N. \tag{103}
\]
It also follows from (98) that
\[ [W_p, W_q] h_g = 0, \quad g \in U_N. \quad (104) \]

Relations (103) and (104) state that operators
\[ \left( \frac{\partial W_p(q, t, \vec{\partial})}{\partial t_h} + W_p(q, t, \vec{\partial}) \vec{\partial}^h \right) \in B \] and
\[ [W_p, W_q] \in B \text{ annihilate the set of functions } h_g, \quad g \in U_N. \] Thus it follows from Lemma 1 that
\[ \left. \frac{\partial W_p(t, \vec{\partial})}{\partial t_h} + W_p(t, \vec{\partial}) \vec{\partial}^h = 0 \right|_{\mod W_p, W_q}, \quad h \in Q, \quad (105) \]
\[ \left. \frac{\partial W_q(t, \vec{\partial})}{\partial t_h} + W_q(t, \vec{\partial}) \vec{\partial}^h = 0 \right|_{\mod W_p, W_q}, \quad h \in Q, \quad (106) \]
\[ [W_p, W_q] = Y \cdot W_p = 0 \quad (\mod W_p), \quad (107) \]
where \( Y \in B \) is completely determined by \( W_p \) and \( W_q \). An explicit expression for the operator \( Y \) in terms of wave operators will be derived later and the answer is given in (129).

Note that the r.h.s. of the (107) is proportional to \( W_p \) only, which follows from the fact that \([W_p, W_q]\) is an operator in \( \frac{\partial}{\partial x} \) only.

For a fixed \( N \), relations (103) and (104) constitute the system of non-linear differential equations for the unknown functions \( \{a_n(t), b_n(t)\}_{n=0}^{N-1} \) subject to the constraint (107). More explicit form of this system will be presented later.

At this point we must comment on the correctness of the definition (105) and (106) of equations of our hierarchy. Depending on how we mode out the parts proportional to \( W_p \) and \( W_q \) we can obtain seemingly different answers for the remainder. The reason for such ambiguity is purely algebraic: the left \( B \)-ideal \( I = \{O \in B | Oh_g = 0, g \in U_N\} \) consisting of all differential operators annihilating \( h_g \) with \( g \in U_N \) is not freely generated by \( W_p, W_q \), there is the relation (107) between generators. Moreover, this relation is the only one. In other words, we have

**Lemma 2** The left ideal \( I \subset B \) can be described as follows in terms of generators and relations:
\[ I = \left( W_p, W_q \left| [W_p, W_q] = 0 \right|_{\mod W_p} \right), \quad (108) \]
which means that for any \( O \in I \) there are \( b_p \in B \) and \( b_q \in B \) such that \( O = b_p W_p + b_q W_q \). Moreover, expressions \( b_p W_p + b_q W_q \) and \( b_p' W_p + b_q' W_q \) are presentations of the same element of \( I \) if and only if there is an element \( c \in B \) such that
\[ b_p = b_p' - c \cdot (W_p + Y) \quad \text{and} \quad b_q = b_q' + c \cdot W_p. \]

Here \( Y \in B \) is an operator defined in (107). A proof of Lemma 2 is given in Appendix.
Therefore we conclude that all possible ways to write the remainders in (105) and (106) lead to the same answer if the relation (107) is taken into account; thus the hierarchy of equations we care about is completely determined by (105), (106) and (107).

The practical way of deducing differential equations from identities (105), (106) and (107) can be extracted from the proof of Lemma 1 in the Appendix. To illustrate the result we present explicitly the simplest equations among (105), (106) and (107) in the case when the set $U_N$ consists of one point, i.e. $N = 1$. The expressions for the wave operators in this case are

\[ W_p = \frac{\partial}{\partial x} + a(t), \]  
\[ W_q = \frac{\partial}{\partial y} + b(t). \] (109) (110)

The corresponding equations for $h = (2,0), (1,1)$ and $(0,2)$ can be written as follows:

\[ \frac{\partial a}{\partial t_{(2,0)}} + 2(\partial_x a)a = \partial_x^2 a, \] (111)  
\[ \frac{\partial b}{\partial t_{(2,0)}} + 2(\partial_x b)a = \partial_x^2 b; \] (112)

\[ \frac{\partial a}{\partial t_{(1,1)}} + (\partial_x a)b + (\partial_y a)a = \partial_x \partial_y a, \] (113)  
\[ \frac{\partial b}{\partial t_{(1,1)}} + (\partial_x b)b + (\partial_y b)b = \partial_x \partial_y b; \] (114)

\[ \frac{\partial a}{\partial t_{(0,2)}} + 2(\partial_y a)b = \partial_y^2 a, \] (115)  
\[ \frac{\partial b}{\partial t_{(0,2)}} + 2(\partial_y b)b = \partial_y^2 b. \] (116)

The additional condition (107) takes the form

\[ \partial_y a - \partial_x b = 0. \] (117)

Let us rewrite equations (111) - (116) and (117) in terms of real variables and real-valued unknown functions. It follows from the reality condition imposed on the potential $V(t, \vec{z})$ that $t_{(2,0)} = \overline{t_{(0,2)}}, t_{(1,1)} = \overline{t_{(1,1)}}$ and $x = t_{(1,0)} = \overline{t_{(0,1)}} = \overline{\gamma}$. Thus $a(t) = \overline{b(t)}$ and we introduce the following new real variables:

\[ v_1(t) \equiv a(t) + b(t), \quad iv_2(t) \equiv b(t) - a(t), \]
\[ r^1 \equiv x + y, \quad ir^2 \equiv x - y \]
\[ \tau_1 \equiv t_{(1,1)}, \quad t_{(2,0)} \equiv 2(\tau_2 + i\tau_3), t_{(2,0)} \equiv 2(\tau_2 - i\tau_3). \]

Note that as far as transformation properties are concerned, $v_i(t)$ with $i = 1, 2$, is a covector field (one-form). Relation (117) states that this one-form is closed:

\[ d\left(v_i(t)dr^i\right) = 0. \] (118)
Equations (111) - (116) written in terms of $v_i(r, \tau)$ acquire the form

$$\frac{\partial \vec{v}_\alpha}{\partial t} + (\vec{v}_\alpha \cdot \nabla)(\vec{v}_\alpha) = \Delta g_\alpha \vec{v}_\alpha, \quad \alpha = 1, 2, 3,$$

(119)

where $\nabla = (\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2})$ is a gradient operator, $g_\alpha$’s are metric tensors, $(g_1)_{ij} = \delta_{ij}, (g_2)_{ij} = \frac{1}{2}(\sigma_3)_{ij}, (g_3)_{ij} = (\sigma_1)_{ij}$ and $\sigma_3 = \text{diag}(1, -1), \sigma_1 = \text{antidiag}(1, 1)$. Matrices $\sigma$’s are the Pauli matrices. An operator $\Delta g_\alpha = g_{ij}^\alpha \frac{\partial^2}{\partial r_i \partial r_j}$ is a Laplace operator of two-dimensional space equipped with metric $g_\alpha$. Finally, $\vec{v}_\alpha(t) = \left( v^1_\alpha(t), v^2_\alpha(t) \right)$ is a vector field corresponding to a covector field $(v_1(t), v_2(t))$ in the presence of the metric $g_\alpha$, $v^i_\alpha = g^{ij}_\alpha v_j$. Note that $g_1$ is a Euclidean metric, while $g_2$ and $g_3$ are equivalent Minkowski metrics.

Relations (119) for $\alpha = 1$ (the Euclidean case) are called two-dimensional Burgers equations, [27]. As a result we see that two-dimensional Burgers equations are included in an infinite hierarchy of non-linear differential equations, equations (105)-(106) and (107) for $N = 1$. This hierarchy is completely integrable in the sense that we can find all solutions in the class of formal power series, see Theorem 1 below.

Burgers equations in one, two and three spatial dimensions together with continuity equation and potentiality condition (118) can be used to model potential turbulence. We refer the reader to [17] for a review and original references. This book also describes an application of three-dimensional analogue of equations (119) to the study of large scale structure of the Universe. The integrable structure we have discussed can prove useful in the systematic analysis of the development of the turbulence in the models based on Burgers equation: the integrable structure of the Burgers hierarchy implies that the dynamics of the system is constrained to the invariant subspaces of the phase space (or “state space” in the terminology of [28]). The addition of the small perturbation in the form of the random force (see e.g. [46]) destroys integrability and the system moves towards chaos through the deterioration of invariant subspaces in accordance with Kolmogorov-Arnold-Moser theory. The presented picture is very close to the existing scenarios of the development of the turbulence (Hopf-Landau scenario for example, see [28] for details) but has a chance to admit a complete qualitative treatment. Moreover, the integrability (in the sense of the presence of integrals of motion) plays an important role in the description of fully developed turbulence of [38] and [39]. For example, Polyakov’s anomaly introduced in [39] is an anomaly of the conservation law. We hope to investigate the consequences of complete integrability of Burgers hierarchy for the theory of Burgers turbulence in the near future.

In the meantime let us discuss the solutions to (113) subject to the condition (118). Solutions to (118) which are defined at every point of the $(r^1, r^2)$-plane are of the following form:

$$v_i(t) = \partial_j \Psi(t),$$

(120)

where $\Psi(t)$ is an arbitrary function of $t$’s. It is called a potential of the covector field $v_i(t)$. Comparing (120) with (99) and (100) we see that the whole class of solutions to equations...
\( \Psi(t) = \ln \langle U_1 | e^{H(t)} A | U_1 \rangle = \ln \left\{ \int dz d\bar{z} \left[ \int d\bar{w} \Phi(z, \bar{w}) e^{V(t, z, \bar{w})} \right] \right\} . \) (121)

We see that \( \Psi(t) \) is a generalization of the Hopf-Cole (18), (7) solution to the Burgers equations (119). The quantity inside the square brackets in the r.h.s. of the above equation is determined by initial conditions. The choice of \( \Phi \) from (73) in accordance with (82) makes it clear that the \((1 \times 1)\) NMM solves the \((2+1)\)-dimensional Burgers hierarchy. We also see that the Hopf-Cole substitution

\[ v_i(t) = \partial_i \Psi(t) = \partial_i \ln(\tau(t)) \] (122)

linearizes the \((2+1)\)-d Burgers hierarchy, (105) and (106) at \( N = 1 \), subject to the constraint (118). Note that the function \( \tau \) appearing in (122) can be considered as a generating function for the solutions of the \((2+1)\)-d Burgers hierarchy and plays in this sense a role of the so called \( \tau \)-function of this hierarchy, see (40) where the notion of the \( \tau \)-function was introduced.

Now we wish to analyze in some details the structure of equations (105), (106) and (107) for an arbitrary \( N \). Their equivalent form is the following:

\[ \frac{\partial W_p(t, \vec{\partial})}{\partial t_h} + W_p(t, \vec{\partial}) \partial^h = O_{p,p}^h W_p + O_{p,q}^h W_q, \quad h \in Q, \] (123)

\[ \frac{\partial W_q(t, \vec{\partial})}{\partial t_h} + W_q(t, \vec{\partial}) \partial^h = O_{q,p}^h W_p + O_{q,q}^h W_q, \quad h \in Q, \] (124)

\[ [W_p, W_q] = Y W_p, \] (125)

where \( Y \) and \( O_{ij} \) with \( i, j = p, q \) are elements of \( B \) and according to Lemma 1 we can choose these operators in such a way that \( O_{i,p} \) with \( i = p \) and \( q \) are of zeroth order in \( \frac{\partial}{\partial y} \). Such a choice permits us to express the r.h.s. of (123)-(125) in terms of wave operators alone. To obtain such an expression we have to define first the right inverses of the wave operators. A way to do this is the following. We set

\[ W_p^{-1} = \partial_x^{-N} \sum_{n \geq 0} d_n \partial_x^{-n}, \]

\[ W_q^{-1} = \partial_y^{-1} \sum_{n \geq 0} \tilde{e}_n \partial_y^{-n}, \]

where \( \{d_n\}_{n=0}^\infty \) are formal power series in \( t \)'s, while \( \{\tilde{e}_n\}_{n=0}^\infty \) are differential operators in \( \partial_x \) with coefficients being formal power series in \( t \)'s. Operators \( \{\tilde{e}_n\}_{n=0}^\infty \) and series \( \{d_n\}_{n=0}^\infty \) are uniquely determined by equations

\[ W_p \cdot W_p^{-1} = 1 = W_q \cdot W_q^{-1}. \] (126)

The multiplication operation “\( \cdot \)” in the above equations is defined by the Leibnitz rule. Relations (123) considered as equations with respect to unknown quantities \( \{d_n\}_{n=0}^\infty \) and \( \{\tilde{e}_n\}_{n=0}^\infty \) possess a unique solution.
Multiplying equations (123) and (124) by $W_q^{-1}$ from the right, extracting the differential parts and using the fact that the $O^h$'s are already differential operators and the $O^h_{i,p}$'s, $i = p$ and $q$, are of zeroth order with respect to $\partial_y$ we find

$$O^h_{i,q} = \left(W_i \partial^h W_q^{-1}\right)_+, \ i = p \text{ and } q,$$

(127)

where the subscript “plus” denotes the operation of extracting the differential part of an operator, i.e. if $O(t, \vec{\partial}) = \sum_{g \in G} c_g \vec{\partial}^g$ then $(O(t, \vec{\partial}))_+ = \sum_{g \in Q} c_g \vec{\partial}^g$.

To compute operators $O^h_{p,p}$ and $O^h_{q,p}$ we substitute (127) back into (123) and (124). Multiplying the resulting equations with $W_p^{-1}$ and projecting them onto differential operators in $\partial_x$ only we get:

$$O^h_{i,p} = \left((W_{p(q)} \partial^h W_q^{-1})_p - W_q W_p^{-1}\right)_+(+,0), \ i = p, q$$

(128)

where the subscripts denote the following operations: for any pseudodifferential operator $O$, $O_- = O - O_+$ and $O_+(+,0)$ is a projection of $O$ onto differential operators in $\partial_x$.

It remains to compute the operator $Y$ entering the r.h.s. of (125). This differential operator is of zeroth order in $\partial_y$ and is therefore equal to

$$Y = \left([W_p, W_q] W_p^{-1}\right)_+(+,0).$$

(129)

Substituting (127) - (129) into (123) - (125) we obtain a system of equations governing the evolution of the wave operators:

$$\frac{\partial W_p}{\partial t_g} + W_p \vec{\partial}^g = \left((W_p \partial^g W_q^{-1})_p - W_q W_p^{-1}\right)_+(+,0) W_p + (W_p \partial^g W_q^{-1}_p)_+ W_q, \ g \in Q$$

(130)

$$\frac{\partial W_q}{\partial t_g} + W_q \vec{\partial}^g = \left((W_q \partial^g W_p^{-1})_p - W_q W_p^{-1}\right)_+(+,0) W_p + (W_q \partial^g W_p^{-1})_+ W_q, \ g \in Q$$

(131)

$$[W_p, W_q] = \left([W_p, W_q] W_p^{-1}\right)_+(+,0) W_p.$$

(132)

Equations (130) and (131) together with the constraint (132) constitute a hierarchy of nonlinear differential equations which can be viewed as a generalization of the Sato equations in the theory of KP equations (see [35] for review). Burgers equations (119) with the condition (118) give the simplest examples of equations (130) - (132) for $N = 1$.

For large enough $N$ equations (130) and (131) contain first $n$ equations of KP hierarchy, $n << N$. To see this, consider the equation (130) for $W_p$ when $g = n \cdot p$ with $n > 1$. In order to present the answer in the standard form we set $W_p \equiv W$ and $t_n \equiv t_{n,p}$ and obtain

$$\frac{\partial W}{\partial t_n} + W \partial^n_x = (W \partial^n_x W^{-1})_+ W, \ n > 1.$$

(133)
The set of equations (133) constitutes a certain reduction of KP hierarchy, which we describe as follows. Take the solution to KP hierarchy (10),

\[ \tilde{W} = 1 + w_1(t) \partial_x^{-1} + w_2(t) \partial_x^{-2} + \cdots, \]  

(134)

which satisfies an additional condition of \( \tilde{W} \cdot \partial^N_x \) being a differential operator. Then the operator \( W = \tilde{W} \cdot \partial^N_x \) solves equations (133). We call the hierarchy of equations (133) the KP\((N)\) hierarchy. This hierarchy has been described in details in [35].

The KP hierarchy itself can be viewed as a limit of KP\((N)\) as \( N \) tends to infinity. Such a limit makes sense due to the stabilization of equations in KP\((N)\) hierarchy: its \( n \)-th equation is independent from \( N \) if \( n << N \). However the hierarchy (133) at finite \( N \) can be of independent interest as well. For instance, the exact version of the statement of the footnote 2 is that KP\((1)\) is an integrable system containing the (1+1)-d Burgers equation.

We can interpret equations (130) and (131) with (132) as an integrable extension of KP\((N)\) hierarchy (133) to higher dimensions. Therefore, we call such system the extended-KP\((N)\) hierarchy. The \( N = 1 \) example considered above supports such an interpretation: the extended-KP\((1)\), or equivalently the (2+1)-d Burgers hierarchy, is a natural extension of the KP\((1)\), or equivalently the (1+1)-d Burgers hierarchy.

Our main result about the extended-KP\((N)\) hierarchy is that it is a completely integrable extension of the KP\((N)\) hierarchy, i.e. all solutions to the extended-KP\((N)\) hierarchy are of form (99) and (100). To be precise we have the following theorem.

**Theorem 1** Let \( C[[t]] \) be a ring of formal power series in \( t, g \in Q \), with complex coefficients. Let \( B \) be a ring of differential operators in \( \partial_x \) and \( \partial_y \), with coefficients belonging to \( C[[t]] \). Operators \( W_p \) and \( W_q \in B \) of the form

\[ W_p(t, \partial_x, \partial_y) = \partial^N_x + \sum_{n=1}^{N} a_n(t) \partial_x^{n-1}, \]  

(135)

\[ W_q(t, \partial_x, \partial_y) = \partial_y + \sum_{n=1}^{N} b_n(t) \partial_x^{n-1}, \]  

(136)

where \( a_n(t) \) and \( b_n(t) \in C[[t]] \) for \( n = 0, \cdots, N - 1 \), solve the system (130) - (132) if and only if there exists a set \( \{ h_n(t) \}_{n=1}^{N} \) of \( N \) elements of \( C[[t]] \) such that

The Wronskian \( W_x(h_1, \cdots, h_N)(t) \) of \( h_1, \cdots, h_N \) with respect to the variable \( x \) is an invertible element of \( C[[t]] \),

\[ \tilde{\partial}^g h_n(t) = \frac{\partial}{\partial t^g} h_n, \; n = 1, \cdots, N; \; g \in Q, \]  

(138)

and

\[ W_p h_n(t) = 0 = W_q h_n(t), \; n = 1, \cdots, N, \]  

(139)
Note that (i) operators $W_p$ and $W_q$ defined in (135) and (136) are the explicit form of the wave operators defined in (95) and (96); (ii) the condition (137) implies in particular the linear independence of $h_1(t), \ldots, h_N(t) \in \mathbb{C}[[t]]$; (iii) relations (133) constitute a system of linear algebraic equations for the coefficients $a_1(t), \ldots, a_N(t)$ and $b_1(t), \ldots, b_N(t)$. This system has a unique solution due to the linear independence of functions $h_1(t), \ldots, h_N(t)$. Thus Theorem 1 indeed describes all solutions to (130) - (132) and the explicit form of these solutions is given by (12) and (100).

The proof of Theorem 1 is based on the following lemma.

**Lemma 3** Let $W_p$ and $W_q$ be any two elements of $\mathcal{B}$ of the form (135) and (136) satisfying the relation (132). Then $\dim(KerW_p \cap KerW_q) = N$ over the ring $\mathbb{C}[[t_p, t_q, t]]$ of formal power series depending on $\{t_g \mid g \in Q \setminus \{p, q\}\}$. Moreover the basis in $KerW_p \cap KerW_q$ can be chosen to satisfy condition (132).

The proof of Lemma 3 is presented in Appendix. However the proof of Theorem 1 based on Lemma 3 is so short and transparent that we present it here.

Suppose that operators $W_p$ and $W_q \in \mathcal{B}$ solve the hierarchy (130) - (132). Let us fix a basis $\tilde{h}_1(t), \ldots, \tilde{h}_N(t)$ of $KerW_p \cap KerW_q$, which satisfies condition (137). Lemma 3 states that any element of $\mathbb{C}[[t]]$ annihilated by $W_p$ and $W_q$ can be decomposed into linear combination of $\tilde{h}_1(t), \ldots, \tilde{h}_N(t)$ with coefficients depending on all $t_g$'s, $g \in Q \setminus \{p, q\}$. Let us apply operator equalities (130) and (131) to $\tilde{h}_1(t), \ldots, \tilde{h}_N(t)$. The r.h.s.'s of the results are identically 0; substracting 0 from the l.h.s.'s, we get:

$$W_p D_g \tilde{h}_i(t) = 0 = W_q D_g \tilde{h}_i(t), \; g \in Q, \; i = 1, \ldots, N, \tag{140}$$

where $D_g \equiv \frac{\partial}{\partial t_g} - \tilde{\partial}^g$. Therefore $D_g \tilde{h}_i(t) \in KerW_p \cap KerW_q$ for each $i$ and $g$. Then there exist $N \times N$ matrices $A_g(t)$ with $g \in Q$ independent of $x$ and $y$ such that

$$D_g \tilde{h}_i(t) = \sum_j [A_g(t)]_j^i \tilde{h}_j(t). \tag{141}$$

These matrices are not unrelated. The fact that $[D_g, D_h] = 0$ together with the linear independence of basic elements $\tilde{h}_1(t), \ldots, \tilde{h}_N(t)$ yields

$$\frac{\partial A_g}{\partial t_h} - \frac{\partial A_h}{\partial t_g} + [A_g, A_h] = 0, \; g, \; h \in Q \setminus \{p, q\}. \tag{142}$$

These zero-curvature-like conditions imply that there is an $(x,y)$-independent non-degenerate matrix $B(t)$, such that

$$A_g = \frac{\partial B}{\partial t_g} B^{-1}. \tag{143}$$

Non-degeneracy of $B(t)$ means that $detB(t)$ is an invertible element of $\mathbb{C}[[t]]$. Consider a new basis of $KerW_p \cap KerW_q$ defined by

$$h_i(t) = \sum_j B_j^i(t) \tilde{h}_j(t), \; i = 1, \ldots, N. \tag{144}$$
Substituting (143) and (144) into (141) we see that
\[ D_g h_i(t) = 0, \ i = 1, \ldots, N. \]  

Therefore the condition (138) of the theorem is satisfied by elements \( h_1(t), \ldots, h_N(t) \). The condition (137) is satisfied as well, since \( W_x(h_1, \ldots, h_N)(t) = det B(t) \cdot W_x(h_1, \ldots, h_N)(t) \) and the r.h.s. of this relation is invertible in \( C[[t]] \). Thus we have proven the "if" part of the theorem (for each pair \( W_p \) and \( W_q \) solving (130) - (132), there exists a set of linearly independent functions \( h_1(t), \ldots, h_N(t) \) satisfying conditions (137) - (138)).

From our considerations which led to the hierarchy (130) - (132) we know that the inverse statement is also true: any pair of operators \( W_p \) and \( W_q \) annihilating \( N \) functions that satisfy condition (137) and (138) solves the extended-\( KP(N) \) hierarchy. This concludes the proof of Theorem 1. \( \diamond \)

Theorem 1 implies a geometric description of the space of solutions to the extended-\( KP(N) \) hierarchy. Before we can give such a description some additional notations are to be introduced. Let \( L \subseteq C[[t]] \) be a complex linear space consisting of elements of \( C[[t]] \) which are annihilated by operators \( D_g \equiv \frac{\partial}{\partial g} - \bar{\partial} g, \ g \in Q \). Consider a set of all \( N \)-dimensional linear subspaces of \( L \) which we identify with an infinite-dimensional Grassmann manifold \( Gr(\infty, N) \). Let us remind that \( Gr(\infty, N) \) is defined as a set of all \( N \)-dimensional linear subspaces of \( C^\infty \) (understood as a Tychonoff product), see [13] for details. We equip \( Gr(\infty, N) \) with the structure of topological space by declaring that the set \( B_\gamma \) consisting of \( N \)-dimensional linear subspaces of \( L \) having non-degenerate projections on the finite-dimensional linear subspace \( \gamma \subseteq C[[t]] \) is open. It follows from standard theorems of analysis (see e. g. [25]) that the set \( B = \{ B_\gamma \}_{\gamma \in C[[t]]} \) together with an \( \emptyset \) constitutes a base for the topology on \( Gr(\infty, N) \). Let \( \left( Gr(\infty, N) \right)_0 \) be an open subset of \( Gr(\infty, N) \) consisting of all \( N \)-dimensional linear subspaces of \( L \) having a non-degenerate projection on the subspace \( \Pi \) of \( C[[t]] \) spanned by \( \{ x^n \}_{n=0}^{N-1} \). The space of solutions to the extended-\( KP(N) \) hierarchy can now be described as follows.

**Corollary 1** There is a one-to-one correspondence between the set of solutions to (130) - (132) and points of \( \left( Gr(\infty, N) \right)_0 \).

\( \diamond \) First of all let us prove that two sets of functions \( h_1(t), \ldots, h_N(t) \) and \( \tilde{h}_1(t), \ldots, \tilde{h}_N(t) \) both satisfying (137) and (138) determine the same pair of operators \( W_p, W_q \) from (133) and (136) which annihilate them if and only if these two sets of functions are related by a constant non-degenerate linear transformation.

The "if" statement is a direct consequence of the Kramer’s formula (see (99) and (100)). Conversely, suppose that \( W_p, W_q \) annihilate two sets of functions \( h_1(t), \ldots, h_N(t) \) and \( \tilde{h}_1(t), \ldots, \tilde{h}_N(t) \) satisfying (137) and (138). Each of these sets span \( Ker W_p \cap Ker W_q \). Therefore, there is a non-degenerate \( N \times N \) matrix \( M(t) \) independent of \( x, y \) such that \( h_i(t) = \sum_j M_{ij}(t) \tilde{h}_j(t) \). Applying the operator \( D_g \) to both sides of the last equality and using the fact that \( D_{\bar{\partial} g}^i(t) = D_{\bar{\partial} g} \tilde{h}_i(t) = 0, i = 1, \ldots, N \), and the linear independence of elements \( \tilde{h}_1, \ldots, \tilde{h}_N \), we get \( \frac{\partial M(t)}{\partial g} = 0, g \in Q \). Therefore, \( M \) is a constant non-degenerate matrix.
Thus any two sets of $N$ elements of $C[[t]]$ obeying (137) and (138) satisfy (139) iff they are related by a constant non-degenerate linear transformation.

Theorem 1 implies that we have just constructed a one-to-one map from the space of solutions to the extended-$KP(N)$ hierarchy to the set of $N$-dimensional linear subspaces of $L$ or, equivalently, $Gr(\infty, N)$. This map is not onto: a linear subspace of $L$ belongs to the image iff one can find a basis $\{h_n\}_{n=1}^N$ of this subspace such that (137) is satisfied. But this condition is equivalent to the non-degeneracy of the projection of our subspace onto the subspace $\Pi$ described above. Thus the image of the map in question is exactly $(Gr(\infty, N))_0$ and the Corollary is proved. ♦

Finally, let us discuss a relation between the solutions (99) and (100) to the extended-$KP(N)$ hierarchy and the partition function of NMM or, more generally, the fermionic correlators (72) with $U = U_N$. Consider the so called vertex operator:

$$X(t, \vec{z}) = e^{V(t, \vec{z})} e^{-V(\frac{1}{n} \frac{\partial}{\partial n}, \vec{z} - \vec{p})},$$ (146)

where $V(\frac{1}{n} \frac{\partial}{\partial n}, \vec{z} - \vec{p}) = \sum_{n>0} \frac{1}{n} \frac{\partial}{\partial n} \vec{z}^{-n} \vec{p}$. It is proved in the Appendix that the wavefunction $w_p(t, \vec{z})$ which encodes half of the solution to the extended-$KP(N)$ hierarchy can be expressed through the fermionic correlator:

$$w_p(t, \vec{z}) = \frac{X(t, \vec{z}) \tau(U_N, A, t)}{\tau(U_N, A, t)}.$$ (147)

This equation can be viewed as a generalization of previously known one-dimensional bosonization formulae [10]. Knowing the fermionic correlator (72) we can compute $w_p(\vec{z}, t)$ or, equivalently the set of functions $\{a_n(t)\}_{n=0}^{N-1}$. The other half of the solution is given by the set $\{b_n(t)\}_{n=0}^{N-1}$.

Even though we do not have at the moment explicit formulae for the functions $b(t)$’s in terms of the fermionic correlators (72) we believe that functions $b(t)$’s are also determined by $\tau(U_N, A, t)$. The reason for such a conjecture is very simple: all solutions at hand are determined completely by the matrix $A_{g, h}$ with $g \in U_N$ and $h \in Q$ which in turn can be restored from $\tau(U_N, A, t)$. This conjecture has been verified in case $N = 1$, see (120). So we conclude that fermionic correlators $\tau(U_N, A, t)$ with operators $A$ from (73) play a role of $\tau$-functions for the solutions (23) to the extended-$KP(N)$ hierarchy.

We hope to continue the investigation of the structure of hierarchies (130) - (132). One of the most important questions to be answered here is following: is there a universal integrable system from which one can obtain all hierarchies (130) and (131) for $N = 1, 2, \cdots$ by means of reductions? The answer to this question is not clear. The reason is that equations (130) and (131) of the extended-$KP(N)$ hierarchy do not stabilize as $N$ becomes larger. This is readily seen from the fact that the number of elements of $Q$ of a given degree $d$ grows with $d$. Thus the relation between extended-$KP(N)$ hierarchies and this hypothetical universal structure should be different from the known relation between $KP(N)$ hierarchies and $KP$ hierarchy.
7 Ward identities in the NMM

It is well-known (see [31], [15], [19], see [33] for review) that partition functions of hermitian, unitary, complex, etc., matrix models exhibit invariance with respect to a subalgebra of an algebra of holomorphic diffeomorphisms of a complex plane, or Virasoro algebra with zero central charge. This invariance can be presented in the form of the so called Virasoro constraints imposed on the partition function of a matrix model. It is also known that Virasoro constraints can be rewritten in the form of loop equations, [36], [21], see [23] for a review. These equations happen to be exactly solvable in certain scaling limits thus providing a powerful tool for a study of matrix models.

The aim of the present Section is to demonstrate that the partition function of NMM is also subject to an infinite set of constraints. These constraints generate a subalgebra of $\text{Diff}(\mathbb{C})$ algebra of all infinitesimal diffeomorphisms of the complex plane.

We start with some heuristic considerations intended to unveil the reasons for the appearance of a subalgebra of the $\text{Diff}(\mathbb{C})$ algebra in the normal matrix model.

Consider a family of 0-dimensional field theories with action $V$ parametrized by the set of coupling constants $\{t_{kl}\}_{k,l \geq 0}$, where $t_{k,l} = t_{l,k}$:

$$V = \sum_{i=1}^{N} \sum_{k,l \geq 0} t_{kl} z_{i}^{k} \bar{z}_{i}^{l}.$$  \hspace{1cm} (148)

This family is equivalent in the field-theoretical framework to the NMM itself as for each fixed set of coupling constants (148) gives a NMM potential entering (5). The algebra of the following reparametrizations acts in the space of field theories (148):

$$z_{i} \rightarrow z_{i} + \epsilon z_{i}^{m+1} \bar{z}_{i}^{n},$$  \hspace{1cm} (149)

$$\bar{z}_{i} \rightarrow \bar{z}_{i} + \bar{\epsilon} z_{i}^{n} \bar{z}_{i}^{m+1},$$  \hspace{1cm} (150)

where $m, n \geq 0$. These reparametrizations can be presented as vector fields on the parameter space of NMM with the potential (148):

$$\delta_{m,n} V = \epsilon w_{m,n} V + \bar{\epsilon} \bar{w}_{m,n} V,$$  \hspace{1cm} (151)

where

$$w_{m,n} = \sum_{k,l \geq 0} k \frac{\partial}{\partial t_{kl}} \frac{1}{l} \frac{\partial}{\partial t_{k+m,l+n}}, \quad \bar{w}_{m,n} = \sum_{k,l \geq 0} l \frac{\partial}{\partial t_{k+l+1,n+1}} \frac{1}{k} \frac{\partial}{\partial t_{kl}}.$$  \hspace{1cm} (152)

These vector fields obey the following commutation relations:

$$[w_{m,n}, w_{p,q}] = (p - m) w_{m+p,n+q},$$

$$[\bar{w}_{m,n}, \bar{w}_{p,q}] = (p - m) \bar{w}_{m+p,n+q},$$

$$[w_{m,n}, \bar{w}_{p,q}] = q \cdot \bar{w}_{n+p,n+q} - n \cdot w_{m+q,n+p},$$  \hspace{1cm} (153)

in which we recognize the relations for the $\text{Diff}(\mathbb{C})$ algebra generated by operators $z^{n+1} \bar{z}^{m+1} \frac{\partial}{\partial z}$ and $\bar{z}^{n+1} z^{m+1} \frac{\partial}{\partial \bar{z}}$. Operators (152) span a subalgebra of $\text{Diff}(\mathbb{C})$ consisting of all polynomial
reparametrization of the plane. So we conclude that the subalgebra of \( \text{Diff}(\mathbb{C}) \) describes classical symmetries of NMM.\(^3\)

Having discovered the \( \text{Diff}(\mathbb{C}) \) symmetry on the “classical” level we gain the hope that it translates to the “quantum” level in the form of corresponding constraints (Ward identities) on the partition function \(^\ast\). To see this directly let us perform the change of variables given by \((149), (150)\) in the integral \((15)\). As a consequence of the fact that the integral doesn’t depend on the choice of the integration variables we obtain the following set of identities:

\[
\int N \prod_{i=1}^{N} dz_i d\bar{z}_i |\Delta(z)|^2 e^{-tr t_{k,l} M^k M^l} \left( -\sum_{p,q \geq 0} pt_{pq} tr (M^{m+p} \bar{M})^{n+q} + \frac{1}{2} (m+1) tr (M^{m} \bar{M}) + \frac{1}{2} \sum_{p=0}^{m} tr (M^{m-p} \bar{M}^p) tr M^p \right. \\
\left. + \frac{1}{2} \sum_{p=0}^{n-1} \sum_{k \neq l} z_k^{n-1-p} z_l^{m+1} \frac{\bar{z}_k - \bar{z}_l}{z_k - z_l} \right) = 0, \tag{154}
\]

and its complex conjugate. Here \( M \) and \( \bar{M} \) are diagonal matrices with \( M_{ii} = z_i, \bar{M}_{ii} = \bar{z}_i \).

Our goal is to rewrite \((154)\) in the form of differential constraints applied to the partition function itself. The obvious obstacle for doing so is the term with double summation in the left hand side of \((154)\). To overcome it we present a fraction \( \frac{\bar{z}_k - \bar{z}_l}{z_k - z_l} \) in the following form:

\[
\frac{\bar{z}_k - \bar{z}_l}{z_k - z_l} = (\bar{z}_k - \bar{z}_l)^2 \int_0^\infty d\omega e^{-\omega(|z_k - z_l|^2 + \eta)}, \tag{155}
\]

where \( \eta > 0 \) is an infinitesimal parameter. Note that if \( z_k = z_l \) the right hand side of \((155)\) is equal to \( \frac{\eta}{\eta} = 0 \), therefore \((155)\) can be considered as a continuation of the fraction at hands to all values of \( z_k, z_l \). The replacement of \( \frac{\bar{z}_k - \bar{z}_l}{z_k - z_l} \) with the integral \((155)\) in \((154)\) alters the integrand on the set of measure 0 and doesn’t change the integral itself. Using this remark we can substitute \((155)\) into \((154)\) to arrive at the desired result:

\[
W_{mn} Z_N = 0; \quad W_{mn} Z_N = 0, \tag{156}
\]

where \( m, n \geq 0 \) and

\[
W_{m,n} = \sum_{k,l \geq 0} k t_{kl} \frac{\partial}{\partial t_{m+k,n+l}} + \left\{ -\frac{1}{2} (m+1) \frac{\partial}{\partial t_{m,n}} \\
+ \frac{1}{2} \sum_{p=0}^{m} \frac{\partial}{\partial t_{m-p,n}} + \frac{1}{2} \sum_{p=0}^{n-1} \int_0^\infty d\omega e^{-\omega} \left( \frac{\partial}{\partial t_{0,n+1-p}} e^{-\omega D} \frac{\partial}{\partial t_{m+1,p}} + \frac{\partial}{\partial t_{0,n-1-p}} e^{-\omega D} \frac{\partial}{\partial t_{m+1,p+2}} - 2 \frac{\partial}{\partial t_{0,n-p}} e^{-\omega D} \frac{\partial}{\partial t_{m+1,p+1}} \right) \right\}, \tag{157}
\]

\[
W_{m,n} = \sum_{k,l \geq 0} l t_{kl} \frac{\partial}{\partial t_{m+k,n+l}} + \left\{ -\frac{1}{2} (m+1) \frac{\partial}{\partial t_{n,m}} \right. \]

\(^3\)The reasonings above are inspired by \[31\].
\[ + \frac{1}{2} \sum_{p=0}^{m} \frac{\partial}{\partial t_{m,p}} \frac{\partial}{\partial t_{p,0}} + \frac{1}{2} \sum_{p=0}^{n-1} \int_0^\infty d\omega e^{-\omega} \times \left( \frac{\partial}{\partial t_{n+1-p,0}} e^{\omega D} \frac{\partial}{\partial t_{p,m}} - 2 \frac{\partial}{\partial t_{n-p,0}} e^{\omega D} \frac{\partial}{\partial t_{p+1,m+1}} \right) \]. \]

(158)

Here

\[ D \equiv \delta - \delta \bar{\delta} - \delta \delta + \bar{\delta} \delta, \]

(159)

and \( \delta \) and \( \bar{\delta} \) are the shift operators defined as follows:

\[ \delta \left( \frac{\partial}{\partial t_{m,n}} \right) \equiv \frac{\partial}{\partial t_{m+1,n}}, \]

(160)

\[ \bar{\delta} \left( \frac{\partial}{\partial t_{m,n}} \right) \equiv \frac{\partial}{\partial t_{m,n+1}}. \]

(161)

The arrows above operators \( \delta, \bar{\delta} \) in (159) indicate the direction in which these operators apply. The operators \( e^{-\omega D} \) in (157) and (158) act within the brackets encompassing them. Note that the first terms in the expressions (157) and (158) for the operators \( W_{mn} \) and \( \overline{W}_{mn} \) coincide with the “classical” \( Diff(C) \) generators (152). The additional “anomalous” terms in the curly brackets of expressions for \( W_{mn} \) and \( \overline{W}_{mn} \) appear from the variation of the measure in the integral (5).

One can check directly that operators (157) and (158) satisfy the commutation relations (153), thus providing us with the \( Diff(C) \) constraints for the NMM. It follows from (153) that operators \( W_{m,0} \) and \( \overline{W}_{n,0} \) generate two commuting copies of the subalgebra of Virasoro algebra. Thus the partition function of NMM model obeys Virasoro constraints as well. There is a nice feature of Virasoro generators \( W_{m,0} \) and \( \overline{W}_{n,0} \) following from their representation (157) and (158) : the tedious terms involving integrals over \( \omega \) are absent in this case, so the expressions for Virasoro generators \( W_{m,0} \) and \( \overline{W}_{n,0} \) are similar to the ones appearing in the conventional matrix models (cf. [33]):

\[ W_{m,0} = \sum_{k,l \geq 0} k t_{kl} \frac{\partial}{\partial t_{m+k,l}} - \frac{1}{2} (m+1) \frac{\partial}{\partial t_{m,0}} + \frac{1}{2} \sum_{p=0}^{m} \frac{\partial}{\partial t_{m-p,0}} \frac{\partial}{\partial t_{p,0}}, \]

(162)

\[ \overline{W}_{n,0} = \sum_{k,l \geq 0} l t_{kl} \frac{\partial}{\partial t_{k,n+l}} - \frac{1}{2} (n+1) \frac{\partial}{\partial t_{0,n}} + \frac{1}{2} \sum_{p=0}^{n} \frac{\partial}{\partial t_{0,n-p}} \frac{\partial}{\partial t_{p,0}}. \]

(163)

The generators of \( Diff(C) \) given by (143) and (150) can be presented in the form \( \tilde{z}^n l_m \) and \( \overline{\pi}_{m,n} = z^n \overline{\pi}_m \), where \( l_n, \overline{\pi}_m \) generate two commuting copies of Virasoro algebra. In this representation the invariance with respect to both Virasoro algebras implies \( Diff(C) \)-invariance. One might hope that, correspondingly, the invariance of the partition function with respect to \( W_{m,0} \) and \( \overline{W}_{n,0} \) of equations (162) and (163) implies the invariance with respect to \( Diff(C) \) generated by (157) and (158). Presently we do not have any results which could justify such a hope.

In conclusion of the present Section we would like to make the following comment. Usually the derivation of Ward identities in the conventional matrix models deals with the partition
function written as an integral over the subset of the set of all matrices. Unfortunately, such a representation of the partition function of the NMM is unknown due to the non-linearity of the space of normal matrices. However, we have managed to derive $Diff(C)$-constraints for NMM starting from its the eigenvalue form (1). Actually, this approach has its own advantages. For example we are now able to apply similar considerations to analyze the structure of Ward identities in the models which are genuinely non-matrix.

For example, consider a one-dimensional $\beta$-model, which was originally introduced in [12], see [30] for a review. This and similar models reappeared in recent literature in quite different contexts such as study of chaotic systems [45] and fractional statistics [14], [24]. The partition function of the model is

$$Z_N(\beta, t) = \int dx_1 \prod_{i=1}^{N} dx_i |\Delta(x_i)\|^\beta e^{-\sum_{j=1}^{N} \sum_{k=1}^{\infty} t_k x_j^k},$$

(164)

where $\beta$ is any non-negative number and the integration goes over $\mathbb{R}^N$. The orthogonal, hermitian and quaternionic matrix models are the particular cases of the $\beta$-model for the values of $\beta$ equal to 1, 2 and 4 correspondingly. In general (164) can be thought of as a classical partition of the two-dimensional Coulomb beads threaded on a string; $\beta$ is identified in this case with the charge of a bead. The expression in the exponent has a meaning of the external potential. Exploiting the invariance of the integral (164) under the change of variables $x_i \to x_i + \epsilon x_i^{n+1}$, $n \geq -1$, we obtain the following set of Virasoro constraints satisfied by the model:

$$L_n^\beta Z_N(\beta, t) = 0, n \geq -1,$$

(165)

$$L_n^\beta \equiv \frac{\beta}{2} \sum_{k=0}^{n} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k}} - (1 - \frac{\beta}{2})(n + 1) \frac{\partial}{\partial t_n} + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}}.$$  

(166)

To our best knowledge this simple observation was not done before.

8 Conclusion

We see that NMM is a highly tractable yet nontrivial model possessing rich structures. The correlation functions of NMM with an axially symmetric potential can be expressed in terms of a holomorphic function of one variable. This holomorphic property leads to the universality of the correlation functions in the scaling limit. In the particular case of monomial potentials this holomorphic function is expressible in terms of degenerate hypergeometric functions. The NMM admits a free fermion representation. Using it we find that the behavior of the partition function of the NMM with respect to an arbitrary variation of the potential is governed by a completely integrable system of non-linear differential equations. This integrable system constitutes a multidimensional extension of the $KP(N)$ hierarchy. In the simplest case when $N = 1$ it contains $(2+1)$-d Burgers equations. The partition function of the NMM is subject to $Diff(C)$-constraints which reflect the symmetries of the model.
From the results of this paper we can foresee the following possible generalizations and/or developments of. For example there is an integrable hierarchy corresponding to an appropriate subset $U \subset \mathbb{Q}$ which is different from the set $\mathcal{U}_N$ used in this paper. By “appropriate” we mean that the ideal of the ring $\mathcal{B}$ consisting of the operators annihilating the set $h_g(t), g \in U$ is finitely generated. An example of such a set is served by $\{g \in \mathbb{Q} | \text{deg}(g) \leq n\}$.

Another way to generalize our construction is to start with an arbitrary finitely generated $\mathbb{Z}$-graded abelian group $\mathbb{G}$ and fix a semigroup $\mathbb{Q} \subset \mathbb{G}$ in such a way that the $\mathbb{Z}$-gradation of $\mathbb{G}$ induces $\mathbb{Z}_+$-gradation of $\mathbb{Q}$. Let us take for example $\mathbb{G} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, \mathbb{Q} = \{(m, n, p) \in \mathbb{G} | m, n, p \geq 0\}$. In the simplest case analogous to the case $N = 1$ considered in the paper we will have three wave operators $W_i = \partial_i + a_i(t), i = 1, 2, 3$ subject to constraints $[W_i, W_j] = 0$. These constraints have a solution $a_i(t) = \partial_i \Psi(t)$ and one of the simplest equations of corresponding hierarchy of equations written in terms of the potential $\Psi$ reads

$$\frac{\partial \Psi}{\partial t_{(1,1,1)}} + (\partial_x \Psi)(\partial_y \partial_z \Psi) + (\partial_y \Psi)(\partial_x \partial_z \Psi) + (\partial_z \Psi)(\partial_x \partial_y \Psi) = (\partial_x \partial_y \partial_z \Psi) + (\partial_x \Psi)(\partial_y \Psi)(\partial_z \Psi).$$

This equation has the following solution:

$$\Psi = \ln \left( \int dudvdw \Phi(u, v, w)e^{xu+yv+zw+uvw_{(1,1,1)}} \right),$$

which can be verified by direct substitution.

It is worth mentioning that integrable systems in $2 + 1$ dimensions which are different from KP hierarchy have been investigated recently using Lax formalism (see e. g. \cite{1} and \cite{48} ): generalized Schroedinger equations, Davey- Stewartson equation (a generalization of non-linear Schroedinger equation to (2+1) dimensions) and their close relatives. It will be interesting to study the Lax structure of 2 + 1-dimensional Burgers hierarchy ( extended-KP(1)) introduced in Section 6.

The following question seems to be extremely interesting to explore: is there a possible connection between NMM and multidimensional gravity analogous to the well-known relations between HMM and two-dimensional gravity? To address this question we need to formulate an analog of Feynman rules for the computation of the coefficients of the asymptotic expansion of the partition function of NMM in the case when coupling constants are large. Corresponding analysis is complicated by the non-linearity of the space of normal matrices, but there are no principle obstructions for the progress.

The asymptotic expansion of the partition function of the hermitian matrix model (with a non-polynomial potential) carries topological information about the moduli space of Riemann surfaces \cite{37}. What can be said in this respect about the partition function of NMM?

As far as algebraic geometry is concerned it will be interesting to study an analog of quasiperiodic solutions to KP hierarchy \cite{24} in the context of extended-KP($N$) hierarchies. Such solutions could correspond to algebraic surfaces on the one hand and to points of $Gr(N, \infty)$ on the other, thus providing us with a sort of multidimensional Krichever construction, \cite{14}. 

34
Appendix

The proof of Lemma 1

Recall that wave operator $W_p$ defined by (95) is of order $N$ in $\partial_x$ while $W_q$ from (96) is of the first order in $\partial_y$. Any operator $O \in \mathcal{B}$ can be presented in the form

$$O = \sum_{m_1=0}^{M_1} \sum_{m_2=0}^{M_2} C_{m_1,m_2} \partial_x^{m_1} \partial_y^{m_2};$$

where coefficients $C$’s are elements of $\mathbb{C}[[t]]$. Therefore,

$$O = \sum_{m_1} C_{m_1} \partial_x^{m_1} \partial_y^{M_2} + \text{terms of lower order in } \partial_y.$$

Or,

$$O = \sum_{m_1} C_{m_1} \partial_x^{m_1} \partial_y^{M_2-1} W_q + \text{terms of lower order in } \partial_y.$$

So, applying induction we see that

$$O = b_q W_q + O',$$

where $O' \in \mathcal{B}$ is of zeroth order in $\partial_y$. If the degree of $O'$ with respect to $\partial_x$ is less than $N$ we are done. If not, we have $O' = C_K \partial_x^K + \text{lower order terms} = C_K \partial_x^{K-N} W_p + \text{lower order terms}$. If the order of the omitted terms is greater than $N$ we continue the process. After a finite number of steps we will arrive at the following representation of the operator $O'$:

$$O' = b_p W_p + \sum_{g \in U_N} c_g(t) \vec{\partial}^g,$$

where $b_p$ is a differential operators in $\partial_x$ only. Therefore an operator $O$ can be presented as follows:

$$O = b_p W_p + b_q W_q + \sum_{g \in U_N} c_g(t) \vec{\partial}^g. \quad (167)$$

Applying equality (167) to the function $h(t)$ with $h \in U_N$ we see that

$$\sum_{g \in U_N} c_g(t) (\vec{\partial}^g h) = 0, \quad h \in U_N. \quad (168)$$

But (168) is a system of linear algebraic homogeneous equations with respect to $\{c_g\}_{g \in U_N}$ with the determinant which is an invertible formal power series. Thus $c_g = 0, g \in U_N$ and Lemma 1 is proved.
9.2 The proof of Lemma 2

First of all let us introduce the grading of the ring $\mathcal{B}$ by setting $deg(\partial_x) = 1$, $deg(\partial_y) = N$. The degree of an element of $\mathcal{B}$ is then by definition the maximal degree of monomials in the linear combination constituting this element. We will prove Lemma 2 basing on the following proposition:

**Lemma 4** Any element $O \in \mathcal{B}$ admits a unique representation of the form

$$O = \sum_{n_1, n_2 \geq 0} c_{n_1, n_2} W_q^{n_2} W_p^{n_1},$$

where $deg(c_{n_1, n_2}) < N$ and $c_{n_1, n_2} = 0$ if $n_1 >> 0$ or $n_2 >> 0$.

We will prove Lemma 4 later. Now let us show how the statement of Lemma 2 can be deduced from Lemma 4.

Consider the following sequence of maps:

$$0 \to \mathcal{B} \overset{\alpha}{\to} \mathcal{B} \oplus \mathcal{B} \overset{\beta}{\to} I \to 0,$$

where $\alpha(b) = \left( -b(W_q + Y), bW_p \right)$, $b \in \mathcal{B}$ and $\beta(c_1, c_2) = c_1 W_p + c_2 W_q$, $(c_1, c_2) \in \mathcal{B} \oplus \mathcal{B}$. Lemma 2 is equivalent actually to the statement of exactness of this sequence. So, let us check the exactness of (170).

First, we see that the map $\alpha$ is monomorphism. Really, $\alpha(b) = 0$ implies in particular that $bW_p = 0$ which yields $b = 0$, as the ring $\mathcal{B}$ is an integral domain. Thus, $Ker(\alpha) = 0$. Second, the map $\beta$ is epimorphism. Really, it follows from Lemma 1 that for any $O \in I$ there are $c_1, c_2 \in \mathcal{B}$ such that $O = c_1 W_p + c_2 W_q = \beta(c_1, c_2)$. Thus, $\beta$ is onto.

Finally, we have to verify that $Im(\alpha) = Ker(\beta)$. It is easy to see that $Im(\alpha) \subset Ker(\beta)$. Really, $\beta \cdot \alpha(b) = \beta\left( (-b \cdot (W_q + Y), b \cdot W_p) \right) = b \cdot ([W_p, W_q] - YW_p)$, which is zero in virtue of the relation (132). Let us prove the opposite inclusion, $Ker(\beta) \subset Im(\alpha)$. If $(c_1, c_2) \in Ker(\beta)$ then for any $b \in \mathcal{B}$ $(c_1, c_2) - \alpha(b) = \left( c_1 + b(W_q + Y), c_2 - bW_p \right) \in Ker(\beta)$. By Lemma 4, $c_2 = \sum_{n_1, n_2 \geq 0} q_{n_1, n_2} W_q^{n_2} W_p^{n_1}$. Choose $b = \sum_{n_1, n_2 \geq 0} q_{n_1, n_2} W_q^{n_2} W_p^{n_1}$. Then, $(c_1, c_2) - \alpha(b) = \left( c_1 + b(W_q + Y), \sum_{n \geq 0} q_{0, n} W_q^n \right) \in Ker(\beta)$. Again, by Lemma 4, $c_1 + b(W_q + Y) = \sum_{n_1, n_2 \geq 0} p_{n_1, n_2} W_q^{n_2} W_p^{n_1}$. So,

$$\sum_{n_1, n_2 \geq 0} p_{n_1, n_2} W_q^{n_2} W_p^{n_1} + \sum_{n \geq 0} q_{0, n} W_q^{n+1} = 0$$

But the l.h.s. of the equation above is a representation of 0 in the form (169), it follows then from the uniqueness of such representation that $p_{n_1, n_2} = 0$ for $n_1, n_2 \geq 0$ and $q_{0, n} = 0$ for $n \geq 0$. Going back we conclude that $(c_1, c_2) - \alpha(b) = 0$, which proves that $Ker(\beta) \in Im(\alpha)$.

It remains to prove Lemma 4 which states the existence and uniqueness of decomposition (169). Take any element $O \in \mathcal{B}$ of order less than $(n + 1) \cdot N$ with $n \geq 0$. It can be presented in the following form:

$$O = \sum_{k=0}^{n} q_k \partial_x^{k \cdot N} \partial_y^{-k} + O,'$$

(171)
where the degree of $q_k$ is less than $N$ and the degree of $O'$ is less than $k \cdot N$. Thus,

$$O - O' = q_0 W_q^n + q_0 (\partial_y^n - W_q^n) + \cdots + q_n \partial_x^n + q_1 \partial_y^{n-1} \partial_x^n + \cdots + q_n \partial_x^n + D_1,$$

(172)

where $\text{deg}(q_k') < N$ and $\text{deg}(D_1) < n \cdot N$.

Suppose that we have proved that for some $m$ such that $0 \leq m < n$,

$$O - O' = c_0 W_q^n + \cdots + c_m W_q^n - m W_p^n + d_{m+1} \partial_y^{m-1} \partial_x^{(m+1)} - N + \cdots + d_n \partial_x^n + D_{m+1},$$

(173)

where degrees of operators $c_i$ and $d_j$ are less than $N$ and $\text{deg}(D_{m+1}) < N$. Then we see that

$$O - O' = c_0 W_q^n + \cdots + c_m W_q^n - m W_p^n + c_{m+1} W_q^n - m W_p^n + (m+1) - N + \cdots + d_n \partial_x^n + D_{m+1} + d_{m+2} \partial_y^{m-2} \partial_x^{(m+2)} - N + \cdots + d_n \partial_x^n + D_{m+2},$$

where degrees of operators $c_i$ and $d_j$ are less than $N$ and $\text{deg}(D_{m+1}) < N$. At some point we defined $c_{m+1} = d_{m+1}$. Thus we proved by induction, the base of which is (172) and the induction hypothesis is (173), that

$$O = \sum_{k=0}^{n} c_k W_q^k W_p^{n-k} + O'',$$

(174)

where $\text{deg}(O'') < n \cdot N$. Note that if $\text{deg}(O) < N$, then the existence of (169) is clear. So, using (174) to generate induction in degree we verify the existence of representation (169) for any $O \in B$.

To prove the uniqueness of (169) we must show that the equality

$$\sum_{n_1, n_2 \geq 0} c_{n_1, n_2} W_q^{n_2} W_p^{n_1} = 0$$

(175)

implies $c_{n_1, n_2} = 0$ for all $n_1, n_2 \geq 0$. We assume of course that $\text{deg}(c_{n_1, n_2}) < N$. Suppose that $n \cdot N \leq \text{deg}(\sum_{n_1, n_2 \geq 0} c_{n_1, n_2} W_q^{n_2} W_p^{n_1}) < (n+1) \cdot N$ with $n > 0$. Then

$$W_q^{n_2} W_p^{n_1} + \sum_{n_1+n_2 = n} c_{n_1, n_2} W_q^{n_2} W_p^{n_1} + O'' = 0,$$

with $\text{deg}(O'') < n \cdot N$.

Thus $c_{0, n} \partial_y + \text{terms of lesser degree in } \partial_y) = 0$. Therefore, $c_{0, n} = 0$. Continuing step-by-step considerations of terms of the highest degree we see that $c_{n_1, n_2} = 0$ for $n_1 + n_2 = n$. But this contradicts the assumption that $\text{deg}(\sum_{n_1, n_2 \geq 0} c_{n_1, n_2} W_q^{n_2} W_p^{n_1}) \geq n \cdot N$ with $n > 0$. Thus all coefficients $c_{n_1, n_2}$ in the l.h.s. of (173) for $n_1 + n_2 > 0$ are equal to 0 and we are left with the equality $c_{0, 0} = 0$. The uniqueness of (169) is thus proved and so is Lemma 4.
9.3 The proof of Lemma 3

First we will construct \( N \) linearly independent elements of \( \mathbb{C}[t] \) annihilated by \( W_p \) and \( W_q \) and satisfying condition (137). Then we will prove that any element of \( \mathbb{C}[t] \) annihilated by \( W_p \) and \( W_q \) is a linear combination of these elements with coefficients independent of \( x, y \).

Let \( \tilde{h}_1, \ldots, \tilde{h}_N \) be \( N \) linearly independent elements of \( \mathbb{C}[t] \) generating \( \text{Ker} W_p \). They can be chosen to have the following form: \( \tilde{h}_k = x^{k-1} + \) (higher order terms in \( x \)) with \( k = 1, \ldots, N \), so that the condition (137) is satisfied. Applying (132) to these elements, we see that \( W_q \tilde{h}_i(t) \in \text{Ker} W_p \). Therefore, there exists an \( N \times N \) matrix \( E(t) \) independent from \( x \) such that

\[
W_q \tilde{h}_i(t) = \sum_j E(t)^j_j \tilde{h}_j(t) \tag{176}
\]

Consider an invertible matrix \( F(t) \) solving the equation

\[
\left( \partial_y F(t) \right) \cdot F(t)^{-1} = E(t). \tag{177}
\]

Such a solution always exists in the formal category: it is easy to verify that \( F(t) = I + F_1(t) \cdot y + F_2(t) \cdot y^2 + \cdots \), where \( I \) is an \( N \times N \) identity matrix and \( F_m(t), m > 0 \) are \( N \times N \) matrices independent of \( y \). They are determined one-by-one from the recursion relation \( F_{m+1}(t) = \frac{1}{m+1} \sum_{i+j=m} E_i(t) \cdot F_j(t) \) with \( m \geq 0 \), \( E(t) = \sum_{i \geq 0} E_i(t)y^i \) and \( F_0(t) \equiv I \). The evaluation of our solution on complex numbers for \( y \) can be written as usual in the form of path-ordered exponent.

Consider now a set of functions \( h_1, \ldots, h_N \) defined from: \( \tilde{h}_i(t) = \sum_j [F(t)]^j_i \tilde{h}_j(t), \) \( i = 1, \ldots, N \). Then \( W_q h_i(t) = 0, i = 1, \ldots, N \) due to (176) and the fact that the matrix \( F(t) \) solves equation (177); and \( W_p h_i(t) = 0, i = 1, \ldots, N \) due to the fact that \( F(t) \) is independent from \( x \). Thus \( h_1, \ldots, h_N \in \text{Ker} W_p \cap \text{Ker} W_q \). Due to the non-degeneracy of the matrix \( F(t) \) and the fact that the elements \( \tilde{h}_1, \ldots, \tilde{h}_N \) were chosen to satisfy the condition (137) it follows that elements \( h_1, \ldots, h_N \) also satisfy the condition (137).

Moreover, these elements generate the intersection of the kernels: suppose there is \( f(t) \in \mathbb{C}[t] \) such that \( f(t) \in \text{Ker} W_p \cap \text{Ker} W_q \). Then \( f(t) \in \text{Ker} W_p \) as well. Therefore, there are coefficients \( d_1(t), \ldots, d_N(t) \) independent of \( x \) such that

\[
f(t) = \sum_i d_i(t) h_i(t). \tag{178}
\]

Thus all we have to prove is that coefficients \( d_i \)'s are independent from \( y \). Applying \( W_q \) to (178) we obtain that \( \sum_i \partial_y \left( d_i(t) \right) h_i(t) = 0 \). Then by linear independence of \( h_i \)'s over formal power series depending on \( y \) and \( t_g, g \in \mathbb{Q} \setminus \{ p, q \} \) we conclude \( \partial_y (d_i(t)) = 0, i = 1, \ldots, N \). Lemma 3 is proved.

9.4 The proof of bosonization relation

Here we present the proof of (137). First, consider operators \( P_+(-t) \equiv e^{-H(t)}P_+e^{H(t)} \) and \( P_-(-t) \equiv e^{-H(t)}P_-e^{H(t)} \). Following [20], we will call them Clifford operators. The identities
below express the properties of the Clifford operators which will be important for us:

\[ \langle \text{vac}|P_+(t) = \langle \text{vac}|e^{H(t)} = \langle \text{vac}|P_-(t). \]  

The proof of (179) is not completely straightforward and was not explained in [8]. Therefore let us outline the proof.

Consider the state

\[ \langle \epsilon | = \langle \text{vac}|e^{H(t)}P_+(t). \]  

We wish to compute \( \frac{d}{d\epsilon} \langle \epsilon |. \) Note that \( H(t) \equiv e^{-H(t)}H(t)e^{H(t)} = \sum_{g \in \mathbf{G}_+} t_g J_g(-\epsilon t), \) where we used the definition (56) and the agreement (59). Therefore

\[ \frac{d}{d\epsilon} \langle \text{vac}|e^{H(t)}P_+(t) = \sum_{g \in \mathbf{G}_+} t_g \sum_{h \in \mathbf{G}_+} \langle \text{vac}|e^{H(t)}\psi_h(-\epsilon t)\bar{\psi}_{g+h}(-\epsilon t)P_+(t) = 0, \]

where the first equality is due to the properties (59)-[11] of projector operators and the last equality follows from (52). Therefore the state \( \langle \epsilon | \) is in fact independent of \( \epsilon. \) Finally, observing that \( \langle \text{vac}|P_+ = \langle \text{vac}| \) we obtain the desired result:

\[ \langle \text{vac}|e^{H(t)} = \langle \text{vac}|e^{H(t)}P_+(t) = \langle \epsilon = 1 | = \langle \epsilon = 0 | = \langle \text{vac}|P_+(t). \]

The first equality in (179) is proved. The second one can be verified along the same lines.

Next we need to prove that

\[ \langle U_N|e^{H(t)}\psi(z)AP_+ = X(t, z)z^{-(N-1)}P_U_N|\psi(N-1)p|e^{H(t)}AP_+. \]  

The proof is the following. Using (56) we can perform the following computation:

\[ [P_+(t)\hat{U}_N(t), \psi(z)]|AP_+ = e^{V(t, z)}e^{-H(t)}[P_+\hat{U}_N, \psi(z)]|e^{H(t)}AP_+ + \sum_{n \geq 0} \bar{z}^{-n}\bar{p}[P_+\hat{U}_N, \psi(N-1-n)p]e^{H(t)}AP_+ + \sum_{n \geq 0} \bar{z}^{-n}\bar{p}[P_+\hat{U}_N, \psi(N-1-n)p]e^{H(t)}AP_+ \]

where \( \prod_{g \in U_N} \bar{\psi}_g = \hat{U}_N \) and \( \mu_g = \frac{1}{n} \) if \( g = n \cdot p \) with \( n > 0 \) and \( \mu_g = 0 \) otherwise. A calculation shows that

\[ P_+(-\mu) = P_+ e^{\left(\bar{z}^{-p}\sum_{g \in \mathbf{G}_+} \psi_g \bar{\psi}_{g+p}(-\mu)\right)} \]

Using this result and observing that \( \psi_h P_+ = 0, h \in \mathbf{G}_- \) we find that

\[ [P_+(t)\hat{U}_N(t), \psi(z)]|AP_+ = X(t, z)z^{-(N-1)}P_+ e^{-H(t)}P_+|\hat{U}_N(-\mu), \psi(N-1)p|e^{H(t)}AP_+. \]  

39
Applying the operator equality (182) to the left vacuum and using the property (179) of Clifford operators we see that
\[
\langle \text{vac} | \hat{U}_N e^{H(t)} \psi(\vec{z}) \rangle P_+ = X(t, \vec{z}) \vec{z}^{(N-1)} \cdot p \langle \text{vac} | \hat{U}_N \phi(\vec{z}) \rangle P_+ e^{H(t)} \cdot p,
\]
where we used that \( \langle \text{vac} | \hat{U}_N (-\mu) \phi(\vec{z}) \rangle_p = \langle \text{vac} | \hat{U}_N \phi(\vec{z}) \rangle_P \), which can be verified directly. Taking into account that \( \langle \text{vac} | \hat{U}_N \equiv \langle U_N \rangle \), we obtain (181).

Finally let us apply (181) to \( |U_N \rangle \) and use the fact that \( P_+ |U_N \rangle = |U_N \rangle \). Dividing the result by \( \tau(U_N, A, t) \) we arrive at the equation (147).

10 Acknowledgments

We are grateful to D. Fuchs, J. Hunter, A. Konechny, G. Kuperberg, M. Mulase, A. Schwarz and P. Vanhaecke for numerous discussions and reading the manuscript. This research was partially supported by the U. S. Department of Energy and National Science Foundation.

References

[1] M. Ablowitz and A. Fokas, Method of solution for a class of multidimensional non-linear evolution equations, Phys. Rev. Lett. 51 (1), p. 7 (1983);

[2] E. Brezin and A. Zee, Universality of the correlations between eigenvalues of large random matrices, Nucl. Phys. B402 p. 613 (1993);

[3] E. Brezin and A. Zee, Universal relation between Green’s functions in random matrix theory, e-print archive: cond-mat/9507032, Nucl. Phys. B453, p. 531 (1995);

[4] J. Burgers, The non-linear diffusion equation: asymptotic solutions and statistical problems, Dordrecht-Holland; Boston: D. Reidel Pub. Co., 1974;

[5] L. -L. Chau and Y. Yu, Unitary polynomials in normal matrix model and wave functions for the fractional quantum Hall effect, Phys. Lett. A167, p. 452 (1992);

[6] L. -L. Chau and O. Zaboronsky, Normal matrix model, Toda lattice hierarchy, and the two-dimensional electron gas in the strong magnetic field, Proceedings in memory of professor Wolfgang Kroll, ed. J. P. Hsu et. al., World Scientific, Singapore, 1997;

[7] J. Cole, Quart. Appl. Math. 9 p. 225 (1951);

[8] E. Date, M. Kashiwara and T. Miwa, Transformation groups for soliton equations. II, Proc. Japan Acad. 57, Ser. A, p.387 (1981);

[9] E. Date, M. Jimbo M. Kashiwara and T. Miwa, Transformation groups for soliton equations. III, J. Phys. Soc. Jpn., 50, Ser. A, p. 342 (1981);
[10] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, *Integrable systems*, Proc. RIMS Symp. “Non-linear integrable systems - classical and quantum theory”, p. 39, 1983;

[11] B. Dubrovin, A. Fomenko, S. Novikov, *Modern geometry - methods and applications*, New-York: Springer-Verlag, 1990;

[12] F. Dyson, *Statistical theory of energy levels of complex systems*, I, II and III, Jour. Math. Phys. 3, p. 140, p. 157 and p. 166 (1962);

[13] A. Fomenko, D. Fuchs, and V. Gutenmacher, *Homotopic topology*, Budapest : Akademia Kiado : [Distributors, Kultura], 1986;

[14] L. Friedlander and A. Schwarz, *Grassmannian and elliptic operators*, e-print archive: funct-an/9704003;

[15] M. Fukuma, H. Kawai and R. Nakayama, *Continuum Schwinger-Dyson equations and universal structures in two-dimensional quantum gravity*, Int. J. Mod. Phys. A6, p. 1385 (1991);

[16] G. Gasper, M. Rahman, *Basic hypergeometric series*, Cambridge [England] ; New York : Cambridge University Press, 1990,

[17] S. Gurbatov, A. Malakhov, A. Saichev, *Nonlinear random waves and turbulence in nondispersive media: waves, rays, particles*, New-York: Manchester University press, 1991;

[18] E. Hopf, Comm. Pure Appl. Math. 3 p. 201 (1950);

[19] H. Itoyama and Y. Matsuo, *Noncritical Virasoro algebra of the d < 1 matrix model and the quantized string field*, Phys. Lett. B255 p. 202 (1991);

[20] M. Kashiwara and T. Miwa, *Transformation groups for soliton equations. I*, Proc. Japan Acad. , 57, Ser. A, p. 342, 1981;

[21] V. Kazakov, *The appearance of matter fields from quantum fluctuations of 2D-gravity*, Mod. Phys. Lett. A4, p. 2125 (1989);

[22] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Generalized Kontsevich model versus Toda hierarchy and discrete matrix models*, Nucl. Phys. B397 p. 339, 1993;

[23] Yu. Makenko, *Loop equations in matrix models and in 2D gravity*, Mod. Phys. Lett. A6, (no. 21), p.1901 (1991);

[24] I. Kogan, G. Semenoff, *Fractional spin, magnetic moment and the short range interactions of anyons*, Nucl. Phys. B368 p. 718 (1992);

[25] A. Kolmogorov, S. Fomin, *Introductory real analysis*, New-York: Dover Publications Inc., 1975;
[26] I. Krichever, Russ. Math. Surveys, 32 p. 185;

[27] N. Kuznetsov and B. Rozhdestvensky, ZhVMMF 1 (2), p. 217 (1961);

[28] L. Landau and E. Lifshitz, *Fluid Mechanics*, London: Pergamon Press, 1987;

[29] A. Leznov and M. Saveliev, *Theory of group representations and integration of non-linear systems* \( X_{a,z} \bar{z} = \exp (kx)_a \), Physica 3D, p. 62 (1981);

[30] M. Mehta, *Random matrices*, San-Diego: Academic Press, 1991;

[31] A. Mironov and A. Morozov, *On the origin of Virasoro constraints in matrix models: lagrangian approach*, Phys. Lett. B252 p. 47, 1990;

[32] F. Mohling, *Statistical mechanics: methods and applications*, Jamaica, Queens, N. Y.: Publishers Creative services Inc., 1982;

[33] A. Morozov, *Integrability and matrix models*, e-print archive: hep-th/9303139, Phys. Usp. 1 p. 1, 1994;

[34] M. Mulase, Math. Sci. 228 (1982) and private communication;

[35] Y. Ohta, J. Satsuma, D. Takakashi and T. Tokihiro, *An elementary introduction to Sato theory*, Progress of Theoretical Physics Supplement 94, p. 210 (1988);

[36] G. Paffuti and P. Rossi, *A solution to Wilson’s loop equation in lattice QCD* , Phys. Lett B92 p. 321 (1980);

[37] R. Penner, *Perturbative series and the moduli space of Riemann surfaces*, Comm. Math. Phys. 113, p. 229 (1987);

[38] A. Polyakov, *The theory of turbulence in two dimensions*. Nucl. Phys. B396 p. 367 (1993);

[39] A. Polyakov, *Turbulence without pressure*, PUPT-1546, Jun 1995. 13pp., e-print archive: hep-th/9506189;

[40] M. Sato, T. Miwa and M. Jimbo, *Holonomic quantum fields*, Publ. RIMS, Kyoto Univ., 14, p. 223 (1978); 15, pp. 201, 577, 871 (1979); 16, p. 531 (1980);

[41] M. Sato, M. Jimbo, T. Miwa and Y. Mori, *Holonomic quantum fields: the unanticipated link between deformation theory of differential equations and quantum fields*, RIMS-305 (1979); Lausanne Math. Phys., 119 (1979);

[42] M. Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifold*, RIMS Kokyuroku, 439, p. 30 (1981);

[43] M. Sato and Y. Sato (Mori), *Nonlinear partial differential equations in Applied Science*, ed. H. Fujita, P. Lax and G. Strang, Kinokuniya/North-Holland, Tokyo, 1983, p. 259;
[44] G. Semenoff, *Anyons and Chern-Simons theory: a review*, PRINT-91-0208 (BRITISH-COLUMBIA), Feb 1991. 32pp. Presented at Karpacz Winter School for Theoretical Physics, Karpacz, Poland, Feb 18 - Mar 1, 1991.

[45] B. Simons, P. Lee and B. Altshuler, *Matrix models, one-dimensional fermions, and quantum chaos*, Phys. Rev. Lett., 72 (1) p. 64 (1994);

[46] Ya. Sinai, *Two results concerning asymptotic behavior of solutions of the Burgers equation with force*, Jour. Stat. Phys., 64, p.1 (1991).

[47] K. Ueno and K. Takasaki, *Toda lattice hierarchy. I and II*, Proceedings of the Japan Academy, Ser. A, 59 (5), p. 167 and *ibid.*, 59 (6), p. 215 (1983);

[48] V. Zakharov and S. Manakov, *Construction of the multidimensional integrable systems and their solutions*, Funktz. Anal Prilozh., 19 (2), p. 11 (1985).