On Vertex Operator Construction of Quantum Affine Algebras

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Abstract

We describe a construction of the quantum deformed affine Lie algebras using vertex operators in the free field theory. We prove Serre relations for the quantum deformed Borel subalgebras of affine algebras, namely \( \hat{sl}_2 \) case is considered in detail. We provide some formulas for the generators of affine algebra.

Keywords: quantum groups, quantum affine algebras, free fields theory, minimal models, Serre relations

1 Introduction

Vertex operator algebras [1, 2, 3] in the free field theories [4, 5, 6] possess a rich mathematical structure. Their connection with the representation theory of affine Lie algebras has been found in [7] and with the theory of quantum groups in [8]. Various aspects of the vertex operator realization of quantum groups were studied in [9, 10].

In this paper we study vertex operator construction of the quantum deformed Borel subalgebras of finite-dimensional and affine algebras in the theory of the free scalar fields \( \phi^i, i = 1, \ldots, n \). Free field theory is described by the action

\[
S = \int d^2 z \left( \partial \phi^i \overline{\partial} \phi^i + i \alpha_0 R \rho^i \phi^i \right),
\]

where \( R \) is a two-dimensional curvature for a background metric, \( \alpha_0 \) is a background charge and \( \rho^i \) is some constant vector. We assume that \( \phi^i \) take values in some appropriate torus. The root lattice associated with the torus gives rise to a Lie algebra \( g \) [2]. We consider vertex operators of the conformal dimension one and investigate their commutation relations. They form an algebra which is the quantum deformation of the original one. Strictly speaking, if the underlying algebra has a Cartan decomposition \( g = n_- \oplus h \oplus n_+ \) then the vertex

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operators corresponding to the simple roots satisfy the identities of $U_q(n_+)$ with some deformation parameter $q = q_+$. In the same way the $U_q(n_-)$ quantum group (with another deformation parameter $q_-$) can be obtained.

This type of construction for finite dimensional simply-laced algebras was suggested in [6] and studied in [9] (see [10] for a review). The aim of this work is to generalize the construction for the case of affine quantum algebra. We proof the Serre relations and give explicit formulas for generators in affine case.

Serre relations in quantum affine algebras have been studied also in [11, 12].

The vertex operators in the theory (1) has a natural classical limit $\alpha_0 \to 0$, which should lead to the classical vertex operators of [2]. In this paper we use the approach in which classical limit $\alpha_0 \to 0$ can be directly obtained (compare with [9, 10, 13], where it is not so explicit).

There is a well known close connection of the conformal minimal models and quantum conformal Toda theory at the special values of the parameters. One may hope that the construction proposed may be helpful for the understanding of the quantization of conformal affine Toda theory [14].

The paper is organized as follows. In section 2 we describe the construction of $U_q(sl_3)$. In section 3 we prove the Serre relations in this case. The construction of the quantum deformed affine algebra $U_q(\hat{sl}_2)$ is presented in sec. 4; details of the calculation of Serre relations and formulas for the generators in the affine case are collected in the appendix. Let us note that the proof of the Serre relations for $U_q(sl_3)$ is completely the same as for any $ADE$ type algebra, so the construction can be straightforwardly generalized to an arbitrary simply laced algebra.

### 2 Construction of $U_q(sl_3)$

The energy-momentum tensor corresponding to (1) is given by:

$$T(z) = -\frac{1}{2}(\partial \varphi(z), \partial \varphi(z)) + i\alpha_0(\rho, \partial^2 \varphi(z)), \quad (2)$$

where $(,)$ stands for the inner product, so $\varphi^i \varphi^i = (\varphi, \varphi)$. $T(z)$ generates Virasoro algebra with central charge $c = n - 12\alpha_0^2(\rho, \rho)$. Consider the vertex operators $\int dz V^\pm_\alpha(\varphi) = \int dz : e^{i\alpha_\pm (\alpha, \varphi)} :$, integrated over suitable contours. Here $: \ldots :$ denotes a normal ordering prescription and $\alpha_\pm$ are some constants obtained from the following condition on the conformal dimension $\Delta$ of the vertex:

$$\Delta(V^\pm_\alpha) = \frac{1}{2} \alpha^2_\pm(\alpha, \alpha) - \alpha_0 \alpha_\pm(\rho, \alpha) = 1, \quad (3)$$

$$\alpha_\pm = \frac{\alpha_0}{2} \pm \sqrt{\left(\frac{\alpha_0}{2}\right)^2 + 1}. \quad (4)$$

In this formula we have assumed that $\alpha$ and $\beta$ are the simple roots of $sl_3$ and $\rho^i = \frac{1}{2} \sum_+ \alpha^i$ is the Weyl vector, i. e. $(\alpha, \alpha) = 2, (\alpha, \beta) = -1$ and $(\rho, \alpha) = (\rho, \beta) = 1$. Consider the algebra with two generators:

$$E_\alpha = \int dz : e^{i\alpha_+(\alpha, \varphi)} :, \quad E_\beta = \int dz : e^{i\alpha_+(\beta, \varphi)} :,$$
where the integration contour in the complex plane $z$ goes counterclockwise from the point $z = 1$ to the same point $z = 1$ encircling the point $z = 0$. For the product of $n$ vertex operators we obtain the following expression:

$$
[V_{\alpha_1} \ldots V_{\alpha_n}] := \int_{\Gamma_n} dz_1 \ldots dz_n : e^{i\alpha_+ (\alpha_1, \varphi)}(z_1) : \ldots : e^{i\alpha_+ (\alpha_n, \varphi)}(z_n) :
$$

where we use the Wick rule. The integration contour $\Gamma_n$ (fig.1.) is chosen according to [6]. Namely, it consists of $n$ nested circles of unit radius oriented counterclockwise from the point $z = 1$ to $z = 1$ around zero. The integral (6) defined in this way contains singularities, when $z_k = z_l$. We regularize it by the point-splitting prescription with the parameter $\varepsilon$; the removal of the regularization takes place in the limit $\varepsilon \to 0$.

![Fig.1. Integration contour $\Gamma_n$.](image)

The generator corresponding to the root $\alpha + \beta$ is given by the $q$-deformed commutator:

$$
E_{\alpha + \beta} = -[E_\alpha, E_\beta]_{q^{-1}} := -[V_\alpha V_\beta - q^{-1} V_\beta V_\alpha] ; \quad q = q_+ = e^{i \pi \alpha_+^2}.
$$

This generator has a nonlocal form (see sec. 3). All other $q$-commutators (of $E_\alpha, E_\beta$ with $E_{\alpha + \beta}$) are equal to zero, which is the statement of Serre relation for the quantum group $U_q(sl_3)$:

$$
[E_\alpha, [E_\alpha, E_\beta]_{q^{-1}}]_{q} = [[V_\alpha V_\beta - (q + q^{-1}) V_\beta V_\alpha + V_\beta V_\alpha V_\alpha] = 0.
$$

The proof is presented in the next section and is based on the regularization of the expression (8) by the point splitting prescription. The main observation in the course of computation of (8) is that it is satisfied in all orders in $\frac{1}{\varepsilon}$.

The formulas (7,8) define the commutation relations for the upper-triangular part of the quantum group $U_q(sl_3)$ with the deformation parameter $q_+$.

There is the "classical" limit of this construction $\alpha_0 \to 0$ ($q \to -1$), when the action (1) contains only the kinetic term. This limit does not coincide with the limit $q \to 1$, when the quantum group reduces to a Lie algebra. Taking this limit we obtain the so-called Frenkel-Kac-Segal (FKS) construction [1, 2] in the bosonic string theory. The $q$-commutation relation (7) goes to

$$
E_\alpha E_\beta - (-1)^{(\alpha, \beta)} E_\beta E_\alpha,
$$

and the factor $(-1)^{(\alpha, \beta)}$ is removed by appropriate cocycles (see [2] for details).

### 3 Serre relation for $U_q(sl_3)$

In this section we prove the Serre relation (8)
Let us introduce the following notation for the integral (6) with the ordering of the variables:

\[ I_{\alpha_1 \ldots \alpha_n} = \int_{\text{arg } z_k < \text{arg } z_{k+1} - \varepsilon} dz_1 \ldots dz_n : e^{i\alpha_1 + (\alpha_1, \varphi)}(z_1) : \ldots : e^{i\alpha_n + (\alpha_n, \varphi)}(z_n) :, \]  

(10)

where \(1 \leq k \leq n - 1\). We can construct the third generator explicitly, using \(q = e^{i\pi \alpha_+^2}\):

\[ E_{\alpha + \beta} = -[V_\alpha V_\beta - q^{-1}V_\beta V_\alpha] \]

\[ = - \int_{\Gamma_2} dz_1 dz_2 (z_1 - z_2)^{-\alpha_+^2} : e^{i\alpha + (\alpha, \varphi)}(z_1) e^{i\alpha + (\beta, \varphi)}(z_2) : + q^{-1} \cdot (\alpha \leftrightarrow \beta). \]  

(11)

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{contours.png}
\end{array} \]

Fig. 2. Contours of integration in (11).

Change of the variables \(z_1 \leftrightarrow z_2\) in the second integral produces compensating multiple factor \(q\) and replaces the integration contours as shown above on fig. 2. One could integrate separately around the \(\varepsilon\)-circle \((z_1 - z_2 = \varepsilon e^{i\psi})\) and rewrite the remaining part in the form (10):

\[ E_{\alpha + \beta} = -e^{1 - \alpha_+^2} \int d\psi dz_2 e^{i(1 - \alpha_+^2)\psi} : e^{i\alpha + (\alpha, \varphi)}(z_2 + \varepsilon e^{i\psi}) e^{i\alpha + (\beta, \varphi)}(z_2) : -(1 - q^{-2}) I_{\alpha \beta}. \]  

(12)

Here the factor \((1 - q^{-2})\) appears due to \((z_1 - z_2)^{-\alpha_+^2}\) in the integrand. Note that in the classical limit \(\alpha_0 \to 0\) (or, equivalently \(q \to -1\)) the coefficient at the nonlocal term \(I_{\alpha \beta}\) vanishes and we get the "classical" definition of the third generator \([1, 2]\): \(E_{\alpha + \beta} = \int dz : e^{i(\alpha_+ + \beta, \varphi)} :\).

The crucial point in our consideration is the Serre identity (8) for \(U_q(sl_3)\) quantum group. Let us rewrite it in the following way:

\[ [V_\alpha V_\beta - q^{-1}V_\beta V_\alpha] - q[V_\alpha V_\beta V_\alpha - q^{-1}V_\beta V_\alpha V_\alpha] \]

\[ = \int_{C_1} dz_1 dz_2 dz_3 (z_1 - z_2)^{2\alpha_+^2} (z_1 - z_3)^{-\alpha_+^2} (z_2 - z_3)^{-\alpha_+^2} : e^{i\alpha + (\alpha, \varphi)}(z_1) \]

\[ e^{i\alpha + (\alpha, \varphi)}(z_2) e^{i\alpha + (\beta, \varphi)}(z_3) : - q \int_{C_2'} dz_1 dz_2 dz_3 (z_1 - z_2)^{-\alpha_+^2} (z_1 - z_3)^2 \alpha_+^2 \]

\[ (z_2 - z_3)^{-\alpha_+^2} : e^{i\alpha + (\alpha, \varphi)}(z_1) e^{i\alpha + (\beta, \varphi)}(z_2) e^{i\alpha + (\alpha, \varphi)}(z_3) :. \]  

(13)

Then changing the variables \(z_1 \to z_2, z_2 \to z_3, z_3 \to z_1\) in the second integral (after that contour \(C_2'\) is changing on \(C_2\) and multiple \(q^{-1}\) appears) and integrating over \(\varepsilon\)-circles:
\( z_1 - z_3 = \varepsilon e^{i\theta}, \quad z_2 - z_3 = \varepsilon e^{i\psi} \) we obtain (see fig.3):

\[
\varepsilon^2 \int d\theta \, d\psi \, dz_3 \left( e^{i\theta} - e^{i\psi} \right)^2 e^{i(1-\alpha_s)} \left( \beta + \psi \right) : e^{i\alpha_+}(\beta \varphi)(z_3 + \varepsilon e^{i\theta}) e^{i\alpha_+}(\alpha \varphi)(z_3 + \varepsilon e^{i\psi})
\]

\[
e^{i\alpha_+}(\beta \varphi)(z_3) : + \varepsilon^2 \int d\psi \, dz_1 \, dz_3 \left( z_1 - z_3 - \varepsilon e^{i\psi} \right)^2 e^{i(1-\alpha_s)} \psi
\]

\[
e^{i\alpha_+}(\alpha \varphi)(z_1) e^{i\alpha_+}(\alpha \varphi)(z_3) : + \varepsilon^2 \int d\theta \, dz_2 \, dz_3 \left( z_2 - z_3 + \varepsilon e^{i\theta} \right)^2 e^{i(1-\alpha_s)} \theta
\]

\[
e^{i\alpha_+}(\beta \varphi)(z_3) : + \varepsilon^2 \int d\psi \, dz_2 \, dz_3 (z_1 - z_2)^2 \alpha \phi (z_1 - z_3)^{-\alpha^2} \psi (z_2 - z_3)^{-\alpha^2}
\]

\[
e^{i\alpha_+}(\beta \varphi)(z_2) e^{i\alpha_+}(\alpha \varphi)(z_2) e^{i\alpha_+}(\beta \varphi)(z_3) : .
\]

\[Fig.3. \text{Contours of integration in (13,14)(all contours taken counterclockwise).}\]

The first term in (14), which came from integration around the two \( \varepsilon \)-circles, vanishes in the limit \( \varepsilon \to 0 \). Divergent terms (second and third) cancel each other. The contour \( C \) in the fourth integral is the contour \( C \) without \( \varepsilon \)-circles. It contains the integration over circle \( |z_1 - z_2| = \varepsilon \), which is vanishing in the limit \( \varepsilon \to 0 \). The remaining integral might be rewritten in the basis of \( I_{\alpha_1...\alpha_n} \) integrals (10):

\[
I_{\alpha \beta} + q^2 q^{-1} I_{\alpha \beta} - q^2 q^{-4} I_{\alpha \beta} - q^2 q^2 I_{\alpha \beta} = 0.
\]

(15)

This completes the proof of the Serre identity.

4 **Quantum group** \( U_q(\widehat{sl}_2) \)

In order to generalize the construction above to the affine case one should introduce two additional light-cone fields \( \varphi_+ \) and \( \varphi_- \) and deform the action (1) in the following way

\[
S = \int d^2z \left( \partial \varphi \partial \varphi + \partial \varphi_+ \partial \varphi_- + i\alpha \rho \varphi \right)
\]

(16)

Let \( \beta_0 \) and \( \beta_1 \) be the simple roots of the affine Lie algebra \( \widehat{sl}_2 \) with the Cartan matrix

\[
a_{ij} = \begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
\]

(17)

Vertex operators of the conformal dimension 1 corresponding to the simple roots are given by

\[
E_\alpha = \int dz \, e^{i\alpha_+}((\beta_0, \varphi) + \varphi_+), \quad E_\alpha = \int dz \, e^{i\alpha_+}((\beta_1, \varphi).
\]

(18)
Let us write down those commutation relations of $U_q(\hat{sl}_2)$, which contain only simple root generators:

\[ e_\delta = [E_0, E_1] q^{-2}, \]
\[ e_n = A(n, q)[E_1, [E_0, [E_0, [E_0, [E_1, [E_0, E_1] q^{-2}]]]]] q^2, \]
\[ e_{n\delta} = B(n, q)[E_0, e_{n-1}] q^2, \]

where $A(n, q)$ and $B(n, q)$ are some normalizing constants. The generator $e_n$ corresponds to the root $\beta_1 + n\delta$. The Serre relations read:

\[ [e_{n+1}, e_m] q^2 + [e_{m+1}, e_n] q^2 = 0, \]
\[ [E_1, [E_1, [E_0, E_1]]] q^{-2} = 0, \quad (m = n = 0). \]

The proof of the Serre relation (23) is similar to the case of $U_q(sl_3)$ and is presented in the Appendix. We provide also the formulas for the generators (20, 21).

In the classical limit ($\alpha_0 \to 0$) the generators (18) reduce to:

\[ E_0 = \int dz e^{i((\beta_0, \varphi)+(\varphi+)), \quad E_1 = \int dz e^{i(\beta_1, \varphi)}, \]

which coincides with the FKS construction of $\hat{sl}_2$ [1, 2].

5 Concluding remarks

We have described the vertex operator construction of the quantum group $U_q(n^+_+)$, where $n^+_+$ is the Borel subalgebra of some Lie algebra $g$. By analogy one can construct $U_q(n^-_-)$. It still remains unclear how to combine these two parts into one object.

This type of the construction (using only scalars $\varphi^i$) also can be applied to the following underlying algebras: finite-dimensional $A, D, E$-series, affine $A(1), D(1), E(1)$ and, possibly, to some hyperbolic algebras.

We have discussed the relationship between the vertex operator realization of quantum groups and the FKS construction. Possible applications to the quantization of conformal affine Toda theories will be studied elsewhere.

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Appendix
In this appendix we present the proof of the Serre relations for $U_q(\hat{sl}_2)$. We also provide some formulas for the generators corresponding to the positive roots of $U_q(\hat{sl}_2)$. First, let us compute the light-like root generator (19) (see appendix A for the details of calculation):

$$e_\delta = [E_0, E_1]_{q^{-2}} = \varepsilon^{1-2\alpha_+^2} \int \varepsilon^{(1-2\alpha_+^2)} \psi : e^{i\alpha_+(\beta_0, \varphi) + \varphi_+} (z + i\varepsilon \psi)$$

(25)

In the classical limit ($\alpha_0 \to 0$) the second term vanishes and the first integral reduces to $e_\delta = \int dz \partial \varphi e^{\varphi_+}$.

The proof of Serre relation (23):

$$[E_1, [E_1, [E_1, E_0]]_{q^2}]_{q^{-2}} = 0,$$

(26)

reduces to the proof that the following integral is vanishing:

$$\int \Gamma dz_1 dz_2 dz_3 dz_4 (z_1 - z_2)^2 \alpha_+^2 (z_1 - z_3)^2 \alpha_+^2 (z_4 - z_3)^2 \alpha_+^2 (z_2 - z_3)^2 \alpha_+^2 (z_2 - z_4)^2 \alpha_+^2$$

(27)

The contour $\Gamma$ is constructed as follows. Each variable $z_1, z_2, z_3$ is integrated from a point 1 to 1 encircling point $z_4$, the variable $z_4$ is integrated afterwards over the unit circle. The contours of integration around $z_1, z_2, z_3$ are nested.

It is convenient to introduce the following notation. Let $\delta_j$ (1 < $j$ < 3) denote the contour of integration for the variable $z_j$ that consists of an $\varepsilon$-circle around $z_4$: $z_j - z_4 = \varepsilon e^{i\theta_j}$. Also denote by $\gamma_j$ a contour of integration for variable $z_j$ that consists of two parts: first, we integrate counterclockwise around the arc $0 < \arg z_j < \arg z_4 - \varepsilon$, $|z_j| > |z_4|$ and secondly, clockwise around the arc $0 < \arg z_j < \arg z_4 - \varepsilon$, $|z_j| < |z_4|$.

According to this notations the integration contour $\Gamma$ in (27) is separated into 8 pieces:

1. Integration over $\delta_1, \delta_2, \delta_3$ and $z_4$. This integral is proportional to $\varepsilon^3$ and vanishes in the limit $\varepsilon \to 0$.
2. $\delta_1, \gamma_2, \gamma_3$ and $z_4$.
3. $\gamma_1, \delta_2, \gamma_3$ and $z_4$.
4. $\gamma_1, \gamma_2, \delta_3$ and $z_4$.

Each of these integrals is proportional to the following integral:

$$\varepsilon^{2(1-\alpha_+^2)} \int \varepsilon^{i\chi} dx dy dz (x - y)^{2\alpha_+^2} (x - z - \varepsilon e^{i\chi})^{2\alpha_+^2} (x - z)^{-2\alpha_+^2} (y - z - \varepsilon e^{i\chi})^{-2\alpha_+^2}$$

$$\alpha_+^2 (y - z)^{-2\alpha_+^2} e^{i(1-2\alpha_+^2) \chi} : e^{i\alpha_+(\beta_0, \varphi)(x)} e^{i\alpha_+(\beta_0, \varphi)(y)} e^{i\alpha_+(\beta_0, \varphi)(z)} e^{i\alpha_+(\beta_0, \varphi)(z)} (z + \varepsilon e^{i\chi}) e^{i\alpha_+(\beta_0, \varphi)(w)}.$$ 

(28)

Here the integration over $x, y, z$ is ordered, as in (10). The sum of the proportionality coefficients is equal to zero. Similarly the integrals over the:

5. $\delta_1, \delta_2, \gamma_3$ and $z_4$.
6. $\delta_1, \gamma_2, \delta_3$ and $z_4$.
7. $\gamma_1, \delta_2, \delta_3$ and $z_4$.
8. $\gamma_1, \gamma_2, \gamma_3$ and $z_4$.

The integration over splitting points $z_1, z_2$ and $z_3$ does not contribute in the limit $\varepsilon \to 0$. The remaining integral can be rewritten in the basis of $I_{\alpha_1 \ldots \alpha_n}$ (10) and also vanishes.
Let us perform here the computations of some generators in the affine case. They are based on the following formulas:

\[ [E_0, I_{\epsilon_0 \epsilon_1 \ldots \epsilon_n}] = (1 - q^{-4}) \sum_{i=0}^{n} [a_i + 1]q^i I_{\epsilon_0 \epsilon_1 \ldots \epsilon_{i-1}0 \epsilon_i \ldots \epsilon_{n-1} \epsilon_n}, \]  

\[ [E_1, I_{\epsilon_0 \epsilon_1 \ldots \epsilon_n}] = (q^2 - q^{-2}) \sum_{i=0}^{n} [b_i]q^i I_{\epsilon_0 \epsilon_1 \ldots \epsilon_{i-1}1 \epsilon_i \ldots \epsilon_{n-1} \epsilon_n}, \]

where:

\[ a_i = \frac{1}{2} \sum_{k=0}^{i-1} (\beta_k, \beta_0), \quad b_i = \frac{1}{2} \sum_{k=0}^{i-1} (\beta_k, \beta_1). \]

In these formulas \( I_{\epsilon_0 \epsilon_1 \ldots \epsilon_n} \) stands for \( I_{\beta_0 \beta_1 \beta_2 \ldots \beta_n} \) (\( \epsilon_i = 0 \) or 1) (see (10)).

The expressions for \( e_n \) and \( e_{n\delta} \) can be written in the same manner as (6) and (14). We can analytically continue the integrands in these expressions (regarding \( \alpha^2 \) as a complex parameter, see e. g. [15]) from the region, where the integrand is well-defined, so that \( \varepsilon \)-terms are not significant. Doing this and using the formulas above the generators (20) can be written down in the basis of \( I_{\epsilon_0 \epsilon_1 \ldots \epsilon_n} \)-integrals:

\[ e_n = A(n, q)q^{2n}(1 - q^{-4})^{2n} \sum_{\epsilon_1 + \ldots + \epsilon_{2n-1} = n} c_{\epsilon_0 \epsilon_1 \ldots \epsilon_{2n-1}} I_{\epsilon_0 \epsilon_1 \ldots \epsilon_{2n-1}}, \]

where summation goes over all nontrivial permutations of \( \epsilon_1, \ldots, \epsilon_{2n-1} \). The coefficients \( c_{\epsilon_0 \epsilon_1 \ldots \epsilon_{2n-1}} \) are given by the following expression:

\[ c_{\epsilon_0 \epsilon_1 \ldots \epsilon_{2n-1}} = \sum_{j_1(\epsilon_j = 1) j_2 \neq j_1(\epsilon_j = 0)} \ldots \sum_{j_{2n-1} \neq j_{2n-2} \ldots \neq j_1(\epsilon_{j_{2n-1}} = 1)} [b_{j_1}]q^i [a'_{j_2} + 1]q^j [a'_{j_{2n-2}} + 1]q^k [b'_{j_{2n-1}}]q^l, \]

where we use the notations:

\[ a'_{j_{2l}} = \frac{1}{2} \sum_{k=0}^{j_{2l}-1} (\beta_k, \beta_0), \quad b'_{j_{2l-1}} = \frac{1}{2} \sum_{k=0}^{j_{2l}-1} (\beta_k, \beta_1). \]

The symbol \( \sum_{j_k(\epsilon_{j_k} = 1)} \) denotes summation over all \( j_k \) (\( 1 \leq j_k \leq 2n - 4 \)) such that \( \epsilon_{j_k} = 1 \). The formulas for \( e_{n\delta} \) can be written down, using (21,29). For example, using (29,30,32,33) one can obtain:

\[ e_{\delta} = (1 - q^{-4})I_{01}, \]

\[ e_1 = [E_1, [E_0, E_1]q^{-2}] = -q^2(1 - q^{-4})^2I_{011}, \]

\[ e_{2\delta} = [E_0, [E_1, [E_0, E_1]q^{-2}]]q^2 = -q^2(1 - q^{-4})^3(I_{0011} + (1 + [2]q^2)I_{0011}), \]

\[ e_2 = [E_1, [E_0, [E_1, [E_0, E_1]q^{-2}]]q^2] = q^4(1 - q^{-4})^4(I_{01011} + (2 + [2]q^2)I_{01011} + (1 + [2]q^2)^2I_{00111}), \]

8
\[ e_{34} = [E_0, [E_1, [E_0, [E_0, E_1]_{q^{-2}}]_{q^2}]]_{q^2} = q^4(1 - q^{-4})^5((1 + [2]_{q^2})(2 + [2]_{q^2})I_{001101} \\
+ (3 + [2]_{q^2})I_{010101} + I_{011001} + (1 + [2]_{q^2})^2(2 + [2]_{q^2})I_{010011} + \\
(1 + [2]_{q^2})(2 + [2]_{q^2} + [2]_{q^2}(2 + [2]_{q^2}))I_{001011} + (1 + [2]_{q^2})^2(1 + [2]_{q^2} + [3]_{q^2})I_{000111}), \] (39)

up to constants \(A(n, q), B(n, q)\).

References

[1] I. Frenkel and V. Kac, Inv. Math. 62 (1980) 23
G. Segal, Comm. Math. Phys. 80 (1981) 301
[2] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303
[3] M. Halpern, Phys. Rev. D12 (1975) 1684; Phys. Rev. D13 (1976) 337
T. Banks, D. Horn and H. Neuberger Nucl. Phys. B108 (1976) 119
[4] V. Dotsenko, Adv. Stud. in Pure Math. 16 (1988) 123
[5] A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetsky and S. Shatashvili, Int. J. Mod. Phys. A5 (1990) 2495
[6] G. Felder, Nucl. Phys. B317 (1989) 215; erratum. Nucl. Phys.324 (1989) 548
[7] B. Feigin, E. V. Frenkel, Usp. Mat. Nauk 43 No.5 (263) (1988) 227-228
[8] A. Tsuchiya and Y. Kanie, Lett. Math. Phys. 13 (1987) 303
[9] P. Bouwknegt, J. McCarthy and K. Pilch, Comm. Math. Phys. 131 (1990) 125
[10] P. Bouwknegt, J. McCarthy and K. Pilch, Prog. Theor. Phys. Suppl. 102 (1990) 67
[11] A. Varchenko and V. Schechtman, Integral Representations of N-Point Conformal Correlators in the WZW Model, Bonn, Max-Planck Institute (1989) 1-22
[12] A. Varchenko, Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups, Advanced Series in Mathematical Physics, vol. 21, World Scientific (1995)
[13] B. Feigin and E. Frenkel, Lect. Notes. in Math. 1620 (1995), hep-th/9310022
[14] O. Babelon and L. Bonora, Phys. Lett. B244 (1990) 220
[15] V. Dotsenko and V. Fateev, Nucl. Phys. B240 (1984) 312
V. Dotsenko and V. Fateev, Nucl. Phys. B251 (1985) 691