A Remarkably Efficient Method for Eliminating Renormalization Scale Ambiguities for Perturbative QCD

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We identify a property of renormalizable SU(N)/U(1) gauge theories, the Intrinsic Conformality (iCF), which underlies the scale invariance of physical observables and leads to a remarkably efficient method to solve the conventional renormalization scale ambiguity at every order in pQCD: the PMC∞. This new method reflects the underlying conformal properties displayed by pQCD at NNLO, eliminates the scheme dependence of pQCD predictions and is consistent with the general properties of the PMC (Principle of Maximum Conformality). We introduce a new method to identify conformal and $\beta$-terms which can be applied either to numerical or to theoretical calculations. We illustrate the PMC∞ for the thrust and C-parameter distributions in $e^+e^-$ annihilation and then we show how to apply this new method to general observables in QCD. We point out how the implementation of the PMC∞ can significantly improve the precision of pQCD predictions; its implementation in multi-loop analysis also simplifies the calculation of higher orders corrections in a general renormalizable gauge theory.

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A. Introduction

A key issue in making precise predictions in QCD is the uncertainty in setting the renormalization scale $\mu_R$ in order to determine the correct running coupling $\alpha_s(\mu_R^2)$ in the perturbative expansion of a scale invariant quantity. The Conventional practice of simply guessing the scale $\mu_R$ of the order of a typical momentum transfer $Q$ in the process, and then varying the scale over a range $Q/2$ and $2Q$, gives predictions which depend on the renormalization scheme, retains dependence on the initial scale choice, leads to a non-conformal series which diverges as $\sim \alpha_s^2 \beta_0^2 n!$, and is even invalid for QED [1]. In fact, a physical process will depend on many invariants and thus have multiple renormalization scales which depend on the dynamics of the process. Other proposals for renormalization scale setting such as PMS [2] or FAC [3] not only have the same difficulties, but they also lead to incorrect and unphysical results [4]. It has been shown recently how all the theoretical constraints can be satisfied at once, leading to accurate results by using the PMC (the Principle of Maximum Conformality) [5–7]. The primary purpose of the PMC method is to solve the scale-setting ambiguity, it has been extended to all orders [8, 9] and it determines the correct running coupling and the correct momentum flow accordingly to the RGE invariance [10, 11]. This leads to results that are invariant respect to the initial renormalization scale in agreement with the requirement of scale invariance of an observable in pQCD [12]. We show here, how the implementation at all orders of the PMC simplifies in many cases by identifying only the $\beta_0$-terms at each order of accuracy due to the presence of a new property of the perturbative corrections. First we recall that there is no ambiguity in setting the renormalization scale in QED. The standard Gell-Mann–Low scheme determines the correct renormalization scale identifying the scale with the virtuality of the exchanged photon [13]. For example, in electron–muon elastic scattering, the renormalization scale is the virtuality of the exchanged photon, i.e. the spacelike momentum transfer squared $\mu_R^2 = q^2 = t$. Thus

$$\alpha(t) = \frac{\alpha(t_0)}{1 - \Pi(t, t_0)} \tag{1}$$

where

$$\Pi(t, t_0) = \frac{\Pi(t) - \Pi(t_0)}{1 - \Pi(t_0)}$$

From Eq.1 it follows that the renormalization scale $\mu_R = t$ can be determined by the $\beta_0$-term at the lowest order. This scale is sufficient to sum all the vacuum polarization contributions into the dressed photon propagator, both proper and improper at all orders. Again in QED, considering the case of two photon exchange a new different scale is introduced in order to absorb all the $\beta$-terms related to the new subset/subprocess into the scale. Also in this case the scale can be determined by identifying

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the lowest order $\beta_0$-term alone. This term identifies the virtuality of the exchanged momenta causing the running of the scale in that subprocess. This scale again would sum all the contributions related to the $\beta$-function into the renormalization scale and no further corrections need to be introduced to the scale at higher orders. Given that the pQCD and pQED predictions match analytically in the $N_C \to 0$ limit where $C_F\alpha_{QCD} \to \alpha_{QED}$ (see ref. [14]) we extend the same procedure to pQCD. In fact in many cases in QCD the $\beta_0$ terms alone can determine the pQCD renormalization scale at all orders [15] eliminating the renormalon contributions $\alpha^u_s / \beta^m_0 n!$. Though in non-Abelian theories other diagrams related to the three- and four-gluon vertices arise, these terms do not necessarily spoil this procedure. In fact, in QCD, the $\beta_0$ terms arising from the renormalization of the three-gluon or four-gluon vertices as well as from gluon wavefunction renormalization determine the renormalization scales of the respective diagrams and no further corrections to the scales need to be introduced at higher orders. In conclusion, if we focus on a particular class of diagrams we can fix the PMC scale by determining the $\beta_0$-term alone and we show this to be connected to the general scale invariance of an observable in a gauge theory. We introduce in this article a parametrization of the observables which stems directly from the analysis of the perturbative QCD corrections and which reveals interesting properties like scale invariance independently from the process or from the kinematics. We point out that this parametrization can be an intrinsic general property of gauge theories and we define this property as Intrinsic Conformality (iCF$^1$). We also show how this property directly indicates what is the correct renormalization scale $\mu_R$ at each order of calculation and we define this new method PMC$_\infty$: Infinite-Order Scale-Setting using the Principle of Maximum Conformality. We discuss the iCF property and the PMC$_\infty$ for the case of the thrust and C-parameter distributions in $e^+e^- \to 3jets$ and we show the results.

In general a normalized IR safe single variable observable, such as the thrust distribution for $e^+e^- \to 3jets$ [16, 17], is the sum of pQCD contributions calculated up to NNLO at the initial renormalization scale $\mu_0$:

\[
\frac{1}{\sigma_0} \frac{d\sigma}{dO} = \left\{ \frac{\alpha_s(\mu_0)}{2\pi} \frac{dA_O(\mu_0)}{dO} + \left( \frac{\alpha_s(\mu_0)}{2\pi} \right)^2 \frac{dO B_O(\mu_0)}{dO} \right. \\
+ \left. \left( \frac{\alpha_s(\mu_0)}{2\pi} \right)^3 \frac{dO C_O(\mu_0)}{dO} + O(\alpha_s^4) \right\}, \tag{2}
\]

where the $\sigma_0$ is tree-level hadronic cross section, $A_O$, $B_O$, $C_O$ are respectively the LO, NLO and NNLO coefficients, $O$ is the selected non-integrated variable. For sake of simplicity we will refer to the differential coefficients as to implicit coefficients and we drop the derivative symbol, i.e.

\[
A_O(\mu_0) = \frac{dA_O(\mu_0)}{dO}, \quad B_O(\mu_0) = \frac{dO B_O(\mu_0)}{dO}, \\
C_O(\mu_0) = \frac{dO C_O(\mu_0)}{dO}. \tag{3}
\]

We define here the Intrinsic Conformality as the property of a renormalizable SU(N)/U(1) gauge theory, like QCD, which yields to a particular structure of the perturbative corrections that can be made explicit representing the perturbative coefficients using the following parametrization$^2$:

\[
A_O(\mu_0) = A_{Conf}, \quad B_O(\mu_0) = B_{Conf} + \frac{1}{2} \beta_0 \ln \left( \frac{\mu^2}{\mu_0^2} \right) A_{Conf}, \\
C_O(\mu_0) = C_{Conf} + \beta_0 \ln \left( \frac{\mu^2}{\mu_0^2} \right) B_{Conf} + \\
+ \frac{1}{4} \left[ \beta_1 + \beta_0^2 \ln \left( \frac{\mu^2}{\mu_0^2} \right) \right] \ln \left( \frac{\mu^2}{\mu_0^2} \right) A_{Conf}. \tag{4}
\]

where the $A_{Conf}, B_{Conf}, C_{Conf}$ are the scale invariant Conformal Coefficients (i.e. the coefficients of each perturbative order not depending on the scale $\mu_0$) while we define the $\mu_N$ as Intrinsic Conformal Scales and $\beta_0, \beta_1$ are the first two coefficients of the $\beta$-function. We remind that the implicit coefficients are defined at the scale $\mu_0$ and that they change according to the standard RG equations under a change of the renormalization scale according to:

\[
A_O(\mu_R) = A_O(\mu_0), \quad B_O(\mu_R) = B_O(\mu_0) + \frac{1}{2} \beta_0 \ln \left( \frac{\mu_R^2}{\mu_0^2} \right) A_O(\mu_0), \\
C_O(\mu_R) = C_O(\mu_0) + \beta_0 \ln \left( \frac{\mu_R^2}{\mu_0^2} \right) B_O(\mu_0) + \\
+ \frac{1}{4} \left[ \beta_1 + \beta_0^2 \ln \left( \frac{\mu_R^2}{\mu_0^2} \right) \right] \ln \left( \frac{\mu_R^2}{\mu_0^2} \right) A_O(\mu_0). \tag{5}
\]

It can be shown that the form of Eq.4 is scale invariant and it is preserved under a change of the renormalization scale from $\mu_0$ to $\mu_R$ by standard RG equations Eq.5, i.e.:

\[
A_O(\mu_R) = A_{Conf}, \quad B_O(\mu_R) = B_{Conf} + \frac{1}{2} \beta_0 \ln \left( \frac{\mu_R^2}{\mu_0^2} \right) A_{Conf},
\]

$^1$ Here the Conformality has to be intended as RG invariance only.

$^2$ We are neglecting here other running parameters such as the mass terms.
\[ C_O(\mu_R) = C_{\text{Conf}} + \beta_0 \ln \left( \frac{\mu_R^2}{\mu_I^2} \right) B_{\text{Conf}} + \]
\[ + \frac{1}{4} \left[ \beta_1 + \beta_0^2 \ln \left( \frac{\mu_R^2}{\mu_I^2} \right) \right] \ln \left( \frac{\mu_R^2}{\mu_I^2} \right) A_{\text{Conf}} \] (6)

We notice that the form of Eq. 4 is invariant and that the initial scale dependence is exactly removed by \( \mu_R \). Extending this parametrization to all orders we achieve a scale invariant quantity: the iCF-parametrization is a sufficient condition in order to obtain a scale invariant observable.

In order to show this property we collect together the terms identified by the same conformal coefficient, we name each set as conformal subset and we extend the property to the order \( n \):
\[
\sigma_I = \left\{ \frac{\alpha_s(\mu_0)}{2\pi} \right\}
\]
\[
+ \frac{1}{4} \left[ \beta_1 + \beta_0^2 \ln \left( \frac{\mu^2}{\mu_I^2} \right) \right] \ln \left( \frac{\mu^2}{\mu_I^2} \right) \alpha_s(\mu_0) + \ldots \right\} A_{\text{Conf}}
\]
\[
\sigma_{II} = \left\{ \frac{\alpha_s(\mu_0)}{2\pi} \right\}
\]
\[
+ \beta_0 \ln \left( \frac{\mu^2}{\mu_I^2} \right) \alpha_s(\mu_0) + \ldots \right\} B_{\text{Conf}}
\]
\[
\sigma_{III} = \left\{ \frac{\alpha_s(\mu_0)}{2\pi} \right\}
\]
\[
+ \ldots \right\} C_{\text{Conf}}
\]
\[
\vdots
\]
\[
\sigma_n = \left\{ \frac{\alpha_s(\mu_0)}{2\pi} \right\} \ldots \right\} L_{\text{n Conf}},
\]
(7)
in each subset we have only one intrinsic scale and only one conformal coefficient and the subsets are disjoint, then no mixing terms among the scales or the coefficients are introduced in this parametrization. Besides the structure of the subsets remains invariant under a global change of the renormalization scale as shown from Eq.6. The structure of each conformal set \( \sigma_I, \sigma_{II}, \sigma_{III} \ldots \) and consequently the iCF are preserved also if we fix a different renormalization scale for each conformal subset, i.e.:
\[
\left( \frac{\partial}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \sigma_n = 0.
\]
(8)

We define here this property of Eq. 7 of separating an observable in the union of ordered scale invariant disjoint subsets \( \sigma_I, \sigma_{II}, \sigma_{III} \ldots \) as ordered scale invariance.

In order to extend the iCF to all orders we first define a partial limit \( J_{/n} \rightarrow \infty \) as the limit obtained including the higher order corrections relative only to those \( \beta_0, \beta_1, \beta_2, \ldots, \beta_{n-2} \) terms that have been determined already at the order \( n \) for each subset and then we perform the complementary \( n \) limit which consists in including all the remaining terms. For the \( J_{/n} \) limit we have:
\[
\lim_{J_{/n} \rightarrow \infty} \sigma_I \rightarrow \left( \frac{\alpha_s(\mu_I)^{n-2}}{2\pi} \right) A_{\text{Conf}}
\]
\[
\lim_{J_{/n} \rightarrow \infty} \sigma_{II} \rightarrow \left( \frac{\alpha_s(\mu_{II})^{n-3}}{2\pi} \right) ^2 B_{\text{Conf}}
\]
\[
\lim_{J_{/n} \rightarrow \infty} \sigma_{III} \rightarrow \left( \frac{\alpha_s(\mu_{III})^{n-4}}{2\pi} \right) ^3 C_{\text{Conf}}
\]
\[
\vdots
\]
\[
\lim_{J_{/n} \rightarrow \infty} \sigma_n \equiv \left( \frac{\alpha_s(\mu_0)}{2\pi} \right) ^n L_{n \text{Conf}} \rightarrow \text{Conformal Limit}
\]
(9)

where the \( \alpha_s(\mu_I)^{n-2} \) is the coupling calculated up to the \( \beta_{n-2} \) at the intrinsic scale \( \mu_I \). Given the particular ordering of the powers of the coupling, in each conformal subset we have the coefficients of the \( \beta_0, \ldots, \beta_{n-k-1} \) terms, where \( k \) is the order of the conformal subset and the \( n \) is the order of the highest subset with no \( \beta \)-terms.

We notice that the limit of each conformal subset is finite and scale invariant up to the \( \sigma_{n-1} \). The remaining scale dependence is confined in the coupling of the \( n^{th} \) term. Any combination of the \( \sigma_1, \ldots, \sigma_{n-1} \) subsets is finite and scale invariant. We can now extend the iCF to all orders performing the \( \bar{n} \) limit. In this limit we include all the remaining higher order corrections. For the calculated conformal subsets this leads to define the coupling at the same scales but including all the missing \( \beta \) terms. Thus each conformal subset remains scale invariant. We point out that we are not making any assumption on the convergence of the series for this limit. Then we have:
\[
\lim_{\bar{n} \rightarrow \infty} \sigma_1 \rightarrow \left( \frac{\alpha_s(\mu_I)}{2\pi} \right) A_{\text{Conf}}
\]
\[
\lim_{\bar{n} \rightarrow \infty} \sigma_{II} \rightarrow \left( \frac{\alpha_s(\mu_{II})}{2\pi} \right) ^2 B_{\text{Conf}}
\]
\[
\lim_{\bar{n} \rightarrow \infty} \sigma_{III} \rightarrow \left( \frac{\alpha_s(\mu_{III})}{2\pi} \right) ^3 C_{\text{Conf}}
\]
\[
\vdots
\]
\[
\lim_{\bar{n} \rightarrow \infty} \sigma_n \equiv \lim_{n \rightarrow \infty} \left( \frac{\alpha_s(\mu_0)}{2\pi} \right) ^n L_{n \text{Conf}} \rightarrow \text{Conformal Limit}
\]
(10)

where here now \( \alpha_s(\mu_I) \) is the complete coupling determined at the same scale \( \mu_I \). The Eq.10 shows that the whole renormalization scale dependence has been completely removed. In fact both the intrinsic scales \( \mu_N \) and the conformal coefficients \( A_{\text{Conf}}, B_{\text{Conf}}, C_{\text{Conf}}, \ldots, L_{n \text{Conf}}, \ldots \) are not depending on the particular choice of the initial scale. The only term with a residual \( \mu_0 \) dependence is the \( n \)-term, but this dependence cancels in the limit \( n \rightarrow \infty \). The scale dependence is totally confined in the coupling \( \alpha_s(\mu_0) \) and its behavior doesn’t depend on the particular choice of any scale \( \mu_0 \) in the perturbative region, i.e. \( \lim_{n \rightarrow \infty} \alpha_s(\mu_0)^n \sim a^n \) with \( a < 1 \). Hence the limit of \( \lim_{n \rightarrow \infty} \sigma_n \) depends only on the properties of the theory and not on the scale of the coupling in the perturbative regime. The proof given here shows that the iCF
is sufficient to have a scale invariant observable and it is not depending on the particular convergence of the series. In order to show the necessary condition we separate the two cases of a convergent series and an asymptotic expansion. For the first case the necessary condition stems directly from the uniqueness of the iCF form, since given a finite limit and the scale invariance any other parametrization can be reduced to the iCF by means of appropriate transformations in agreement with the RG equations. For the second case, we have that an asymptotic expansion though not convergent, can be truncated at a certain order \( n \), which is the case of Eq.\( \ref{eq:3} \). Given the particular structure of the iCF we can perform the first partial limit \( J_1/n \) and we would achieve a finite and scale invariant prediction, \( \sigma_{N-1} = \sum_{i=1}^{n-1} \sigma_i \), for a truncated asymptotic expansion, as shown in Eq.\( \ref{eq:4} \). Given the truncation of the series in the region of maximum of convergence the \( n \)-th term would be reduced to lowest value and so the scale dependence of the observable would reach its minimum. Given the finite and scale invariant limit \( \sigma_{N-1} \) we conclude that the iCF is unique and then necessary for an ordered scale invariant truncated asymptotic expansion up to the \( n \)-th order. We point out that in general the iCF form is the most general and irreducible parametrization which leads to the scale invariance, other parametrization are forbidden since if we introduce more scales\(^3\) in the logarithms of one subset we would spoil the invariance under the RG transformation and we could not achieve Eq.\( \ref{eq:6} \), while on the other hand no scale dependence can be introduced in the intrinsic scales since it would remain in the observable already in the first partial limit \( J_1/n \) and it could not be eliminated. The conformal coefficients are conformal by definition at each order, thus they don’t depend on the renormalization scale and they don’t have a perturbative expansion. Hence the iCF is a necessary and sufficient condition for scale invariance.

B. The iCF and the ordered scale invariance

The iCF-parametrization can stem either from an inner property of the theory, the iCF, or from direct parametrization of the scale-invariant observable. In both cases the iCF-parametrization makes the scale dependence of the observable explicit and it exactly preserves the scale invariance. Once we have defined an observable in the iCF-form, we have not only the scale invariance of the entire observable, but also the ordered scale invariance (i.e. the scale invariance of each subset \( \sigma_n \) or \( \sigma_{N-1} \)). The latter property is crucial in order to obtain scale invariant observables independently from the particular kinematical region and independently from the starting order of the observable or the order of the truncation of the series. Since in general, a theory is blind respect to the particular observable/process that we are going to investigate, the theory should preserve the ordered scale invariance in order to define always scale invariant observables. Hence if the iCF is an inner property of the theory, it leads to implicit coefficients that are neither independent nor conformal. This is made explicit in the Eq.\( \ref{eq:4} \), but it is hidden in the perturbative calculations in case of the implicit coefficients. For instance the presence of the iCF clearly reveals when a particular kinematical region is approached and the \( A_O \) becomes null. This would cause a break of the scale invariance since a residual initial scale dependence would remain in the observable in the higher orders coefficients. The presence of the iCF solves this issue by leading to the correct redefinition of all the coefficients at each order preserving the correct scale invariance exactly. Thus in the case of a scale invariant observable \( O \) defined, according to the implicit form (Eq.\( \ref{eq:2} \)), by the coefficients \( \{ A_O, B_O, C_O, ..., O_O, \ldots \} \), it cannot simply undergo the change \( \rightarrow \{ 0, B_O, C_O, ..., O_O, \ldots \} \), since this would break the scale invariance. In order to preserve the scale invariance we must redefine the coefficients \( \{ A_O = 0, B_O, C_O, ..., O_O, \ldots \} \) cancelling out all the initial scale dependence originated from the LO coefficient \( A_O \) at all orders. This is equivalent to subtract out a whole invariant conformal subset \( \sigma_i \) related to the coefficient \( A_{Conf} \) from the scale invariant observable \( O \). This mechanism is clear in the case of the explicit form of the iCF, Eq.\( \ref{eq:4} \), where if \( A_{Conf} = 0 \) then the whole conformal subset is null and the scale invariance is preserved. We underline that the conformal coefficients can acquire all the possible values without breaking the scale invariance, they contain the essential information on the physics of the process, while all the correlation factors can be reabsorbed in the renormalization scales as shown by the PMC method\([5-9]\). Hence if a theory has the property of the ordered scale invariance it preserves exactly the scale invariance of observables independently from the process, the kinematics and the starting order of the observable. We put forward that if a theory has the Intrinsic Conformality all the renormalized quantities, such as cross sections, can be parametrized with the iCF-form. This property should be preserved by the renormalization scheme or by the definition of IR safe quantities and it should be preserved also in observables defined in effective theories.

C. The PMC\( _{\infty} \)

We introduce here a new method to eliminate the scale-setting ambiguity in single variable scale invariant distributions named PMC\( _{\infty} \). This method is based on the original PMC principle\([5-9]\) and agrees with all the PMC

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\( ^3 \) Here we refer to the form of Eq.\( \ref{eq:4} \). In principle it is possible to write other parametrizations preserving the scale invariance, but these can be reduced to the iCF by means of appropriate transformations in agreement with the RG equations.
different formulations for the PMC-scales at the lowest order. Essentially the core of the PMC∞ is the same of all the BLM-PMC prescriptions[18], i.e. the correct running coupling value and hence its renormalization scale at the lowest order is identified by the β0-term at each order, or equivalently by the intrinsic conformal scale µN. The PMC∞ preserves the iCF and then the scale invariance absorbing an infinite set of β-terms at all orders. This method differs from the other PMC prescriptions since, due to the presence of the Intrinsic Conformality, no perturbative correction in αs needs to be introduced at higher orders in the PMC-scales. Given that all the β-terms of a single conformal subset are included in the renormalization scale already with the definition at lowest order, no initial scale or scheme dependence are left due to the unknown β-terms in each subset. The PMC∞-scale of each subset can be unambiguously determined by β0-term of each order, we underline that all logarithms of each subset have the same argument and all the differences arising at higher orders have to be included only in the conformal coefficients. Reabsorbing all the β-terms into the scale also the β0 n! terms (related to the renormalons [19]) are eliminated, thus the precision is improved and the perturbative QCD predictions can be extended to a wider range of values. The initial scale dependence is totally confined in the unknown PMC∞ scale of the last order of accuracy (i.e. up to NNLO case in the αs(μ0)3). Thus if we fix the renormalization scale independently to the proper intrinsic scale for each subset µN, we end up with a perturbative sum of totally conformal contributions up to the order of accuracy:

\[
\frac{1}{\sigma_0} \frac{O_d(\mu_1, \mu_{II}, \mu_{III})}{dO} = \left\{ \frac{\alpha_s(\mu_1)}{2\pi} \frac{O_dA_{Conf}}{dO} \right\} + \frac{\alpha_s(\mu_{II})}{2\pi} \frac{2OdB_{Conf}}{dO} + \frac{\alpha_s(\mu_{III})}{2\pi} \frac{3OdC_{Conf}}{dO}
\]

\[+ O(\alpha_s^2), \quad (11)\]

at this order µIII = µ0.

D. iCF coefficients and scales

We describe here how all the coefficients of Eq. 4 can be identified from a numerical either theoretical perturbative calculation. We will use as template the NNLO thrust distribution results calculated in Refs. [20, 21]. Since the leading order is already (AConf) void of β-terms we start with NLO coefficients. A general numerical/theoretical calculation keeps tracks of all the color factors and the respective coefficients:

\[
B_O(N_f) = C_F \left[ C_A B_O^N + C_F B_O^{CF} + T_F N_f B_O^{N_f} \right]; \quad (12)
\]

where \[C_F = \frac{(N_f^2 - 1)}{2N_f}, \quad C_A = N_c \quad \text{and} \quad T_F = 1/2. \]

The dependence on N_f is made explicit here for sake of clarity. We can determine the conformal coefficient \[B_{Conf}\] of the NLO order straightforwardly, by fixing the number of flavors \[N_f\] in order to kill the \[β_0\] term:

\[
B_{Conf} = B_O \left( \frac{N_f}{\frac{33}{2}} \right),
\]

\[
B_{β_0} \equiv \log \left( \frac{\mu_0^2}{\mu_I^2} \right) = 2 \frac{B_O - B_{Conf}}{β_0 A_{Conf}}, \quad (13)
\]

we would achieve the same results in the usual PMC way, i.e. identifying the \[N_f\] coefficient with the \[β_0\] term and then determining the conformal coefficient. Both methods are consistent and results for the intrinsic scales and the coefficients are in perfect agreement. At the NNLO a general coefficient is made of the contribution of six different color factors:

\[
C_O(N_f) = \frac{C_F}{4} \left( N_f^2 C_O^N + C_O^{N_0} + \frac{1}{N_c^2} C_O^{N_f} \right)
\]

\[+ N_f N_c \cdot C_O^{N_f N_c} + \frac{N_f}{N_c} C_O^{N_f N_c} + N_f^2 C_O^{N_f} \right\}. \quad (14)
\]

In order to identify all the terms of Eq.4 we notice first that the coefficients of the terms \[β_0^2 \text{ and } β_1 \] are already given by the NLO coefficient \[B_{β_0}\], then we need to determine only the \[β_0\]- and the conformal \[C_{Conf}\]-terms. In order to determine the latter coefficients we use the same procedure we used for the NLO, i.e. we set the number of flavors \[N_f = \frac{33}{2}\] in order to drop off all the \[β_0\] terms. We have then:

\[
C_{Conf} = C_O \left( \frac{N_f}{\frac{33}{2}} \right) - \frac{1}{4} \bar{β}_1 B_{β_0} A_{Conf},
\]

\[
C_{β_0} \equiv \log \left( \frac{\mu_0^2}{\mu_{II}^2} \right) = \frac{1}{β_0 A_{Conf}} \left( C_O - C_{Conf} \right)
\]

\[- \frac{1}{4} \bar{β}_1 B_{β_0} A_{Conf} - \frac{1}{4} \bar{β}_1 B_{β_0} A_{Conf}, \quad (15)\]

with \[\bar{β}_1 \equiv β_1(N_f = \frac{33}{2}) = -107. \]

This procedure can be extended at every order and one may decide whether to cancel the \[β_0, β_1 \text{ or } β_2\] by fixing the appropriate number of flavors. The results can be compared leading to determine exactly all the coefficients. We point out that extending the Intrinsic Conformality to all orders we can predict at this stage the coefficients of all the color factors of the higher orders related to the β-terms except those related to the higher order conformal coefficients \[β_0\]-terms (e.g. at NNNLO the \[D_{Conf}\] and \[D_{β_0}\]). The β-terms are coefficients that stem from UV-divergent diagrams connected with the running of the coupling constant and not from UV-finite diagrams. UV-finite \[N_F\] terms may arise but would not contribute to the β-terms. While \[N_F\] terms coming from UV-divergent diagrams, depending dynamically on the virtuality of the underlying quark and gluon subprocesses have to be considered as β-terms and they would determine the intrinsic conformal scales. In general, each \[μ_N\] is an independent function of the \[\sqrt{s}\] of the selected variable \[O\] and it varies with the number.
of colors $N_c$ mainly due to $ggg$- and $gggg$-vertices. The latter terms arise at higher orders only in non-Abelian theory but they do not spoil the iCF-form. We underline that iCF applies to scale invariant single variable distributions, in case one is interested in the renormalization of a particular diagram, e.g. the $ggg$-vertex, contributions from different $\beta$-terms should be singled out in order to identify the respective intrinsic conformal scale consistently with the renormalization of the non-Abelian $ggg$-vertex, as shown in [22].

E. The PMC$_\infty$ scales at LO and NLO

According to the PMC$_\infty$ prescription we fix the renormalization scale to $\mu_N$ at each order absorbing all the $\beta$ terms into the coupling. We notice a small mismatch between the zeroes of the conformal coefficient $B_\text{Conf}$ and those of the remainder $\beta_0$-term at the numerator (formula is shown in Eq. 15). Due to our limited knowledge of the strong coupling at low energies, in order to avoid singularities in the NLO-scale $\mu_{II}$, we introduce a regularization which leads to a finite scale $\tilde{\mu}_{II}$. These singularities stem from a rather logarithmic behavior of the conformal coefficients when low values of the variable $1 - T$ are approached. Large logarithms arise from the IR divergences cancellation procedure and they can be resummed in order to restore a predictive perturbative regime[23–28]. We point out that IR cancellation should not spoil the iCF property and a IR cancellation Monte Carlo technique consistent with the iCF would be required. Whether this is an actual deviation from the iCF-form has to be further investigated. However, since the discrepancies between the coefficients are rather small, we introduce a regularization method based on re-definition of the norm of the coefficient $B_\text{Conf}$ in order to cancel out these singularities in the $\mu_{II}$-scale. This regularization is consistent with the PMC principle and up to the accuracy of the calculation it does not introduce any bias effect in the results and no ambiguity in the NLO-PMC$_\infty$ scale. All the differences introduced by the regularization would start at the NNNLO order of accuracy and they can be absorbed after in the higher order PMC$_\infty$ scales. Thus the first two PMC$_\infty$ scales result:

$$\mu_{II} = \sqrt{s} \cdot e^{-\frac{1}{2}B_{\beta_0}}; \quad (1-T)<0.33 \quad (16)$$

$$\tilde{\mu}_{II} = \begin{cases} \sqrt{s} \cdot e^{-\frac{1}{2}C_{\beta_0}} & \frac{B_{\text{Conf}}}{A_{\text{Conf}}+B_{\beta_0}A_{\beta_0}}, \quad (1-T)<0.33 \\ \sqrt{s} \cdot e^{-\frac{1}{2}C_{\beta_0}} & \left(\frac{C_1}{11B_{\beta_0} - \frac{2}{3}B_{\gamma_0}}\right), \quad (1-T)>0.33 \end{cases} \quad (17)$$

where $\sqrt{s} = M_{Z_0}$, and the value of $\eta = 3.51$ has been fixed by matching the zeroes of numerator and denominator of $C_{\beta_0}$. We have to point out that in the region $(1 - T) > 0.33$ we have a clear example of Intrinsic Conformality-iCF where the kinematical constraints set the $A_{\text{Conf}} = 0$. According to the Eq.6 setting the $A_{\text{Conf}} = 0$ the whole conformal subset $\delta_1$ becomes null. In this case all the $\beta$ terms at NLO and NNLO disappear except the $\beta_0$-term at NNLO which determines the $\mu_{II}$ scale. The surviving $N_f$ terms at NLO or the $N_f^2$ at NNLO are related to the finite $N_f$-term at NLO and to the mixed $N_f \cdot N_f$ term arising from $B_0 \cdot \beta_0$ at NNLO. Using the following parametrization:

$$A_0 = 0, \quad B_0 = B_0 + B_1 \cdot N_f, \quad C_0 = C_0 + C_1 \cdot N_f + C_2 \cdot N_f^2. \quad (18)$$

we can determine the $\tilde{\mu}_{II}$ for the region $(1 - T) > 0.33$ as shown in Eq.17:

$$\tilde{\mu}_{II} = \sqrt{s} e^{-\frac{1}{2} \left(\frac{C_1}{11B_{\beta_0} - \frac{2}{3}B_{\gamma_0}}\right)} \quad (19)$$

by identifying the $\beta_0$-term at NNLO. The LO and NLO PMC$_\infty$ scales are shown in Fig.1. We notice that the two PMC$_\infty$ scale have similar behaviors in the range $(1-T) < 0.33$ and the LO-PMC$_\infty$ scale agrees with the PMC scale used in [29]. This method totally eliminates both the ambiguity in the choice of the renormalization scale and the scheme dependence at all orders in QCD.

![FIG. 1: The LO-PMC$_\infty$ (Solid Red) and the NLO-PMC$_\infty$ (Dashed Black) scales for thrust.](image)

F. NNLO Thrust distribution results

We use here the results of Ref. [20, 21] and for the running coupling $\alpha_s(Q)$ we use the RunDec program [30]. In order to normalize consistently the thrust distribution we expand the denominator in $\alpha_0 \equiv \alpha_s(\mu_0)$ while the numerator has the couplings renormalized at different PMC$_\infty$-scales $\alpha_1 \equiv \alpha_s(\mu_{II})$, $\alpha_{II} \equiv \alpha_s(\tilde{\mu}_{II})$. We point out here that the proper normalization would be given by the integration of the total cross section after renormalization with the PMC$_\infty$ scales, nonetheless since the PMC$_\infty$ prescription involves only absorption of higher order terms into the scales the difference would be within
the accuracy of the calculations, i.e. \( \sim O(\alpha_s^4(\mu_0)) \). The experimentally measured thrust distribution is normalized to the total hadronic cross section \( \sigma_{\text{tot}} \) as follows:

\[
\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dO} = \{ \Sigma_I + \Sigma_{II} + \Sigma_{III} + O(\alpha_s^3) \},
\]

where

\[
\sigma_{\text{tot}} = \sigma_0 \left( 1 + \frac{\alpha_s(\mu_0)}{2\pi} A_{\text{tot}} + \left( \frac{\alpha_s(\mu_0)}{2\pi} \right)^2 B_{\text{tot}} + O(\alpha_s^3) \right)
\]

is the total integrated cross section and the \( A_{\text{tot}}, B_{\text{tot}} \) are:

\[
A_{\text{tot}} = \frac{3}{2} C_F;
\]

\[
B_{\text{tot}} = \frac{C_F}{4} N_c + \frac{3}{4} C_F \frac{\beta_0}{2} (11 - 8\xi(3)) - \frac{3}{8} C_F^2.
\]

The normalized subsets in the region \( (1 - T) < 0.33 \) are then:

\[
\Sigma_I = A_{\text{Conf}} \frac{\alpha_I}{2\pi}
\]

\[
\Sigma_{II} = (B_{\text{Conf}} + \eta A_{\text{Conf}} A_{\text{Conf}}) \left( \frac{\alpha_{\text{Conf}}}{2\pi} \right)^2 - \eta A_{\text{Conf}} A_{\text{Conf}} \left( \frac{\alpha_I}{2\pi} \right)^2
\]

\[
\Sigma_{III} = (C_{\text{Conf}} - A_{\text{Conf}} B_{\text{Conf}}) \left( \frac{\alpha_{\text{Conf}}}{2\pi} \right)^3
\]

Normalized subsets for the region \( (1 - T) > 0.33 \) can be achieved simply by setting \( A_{\text{Conf}} = 0 \) in the Eq. 23. Within the numerical precision of these calculations there is no evidence of the presence of spurious terms, such as UV-finite \( N_f \) terms up to NNLO [31]. These terms must be rather small or comparable with numerical fluctuations. Besides we notice a small rather constant difference between the iCF-predicted and the calculated coefficient for the \( N_f^2 \) color factor of Ref. [20] which might be addressed to a \( N_f^2 \) UV-finite coefficient or possibly to statistics. This small difference must be included in the conformal coefficient and it has a complete negligible impact on the total thrust distribution. In Fig. 2 we show the thrust distribution at NLO and at NNLO with the use of the \( \text{PMC}_\infty \) method. We notice that the \( \text{PMC}_\infty \) prescription significantly improves the theoretical predictions, if compared with the results using the Conventional Scale Setting shown in [20] and [32][33]. Results are in remarkable agreement with the experimental data in a wider range of values \( 0.015 \leq 1 - T \leq 0.33 \) and they show a significant improvement of the PMC results of [29] when the two-jets and the multi-jets regions are approached, i.e. the region of the peak and the region \( (1 - T) > 0.33 \) respectively. The use of the iCF-PMC\( \infty \) approach on perturbative thrust QCD-calculations restores the correct behavior of the thrust distribution in the region \( (1 - T) > 0.33 \) and this is a clear effect of the iCF property. Comparison with the experimental data has been improved all over the spectrum and the introduction of the \( N^3 \) order correction would improve this comparison especially in the multi-jet \( 1 - T > 0.33 \) region. In Fig. 2, the theoretical errors for the thrust distribution at NNLO are also shown (the shaded area). Conformal quantities are not affected by a change of renormalization scale. Thus the errors shown give an evaluation of the level of conformality achieved up to the order of accuracy and they have been calculated using standard criteria, i.e. varying the remaining initial scale value in the range \( \sqrt{s}/2 \leq \mu_0 \leq 2\sqrt{s} \). Results show an average error of \( \delta \approx 3.97 \% \) (we use the same definition of \( \delta \) as in Ref. [32] for comparison). In the \( \text{PMC}_\infty \) method theoretical errors are given by the unknown intrinsic conformal scale of the last order of accuracy. We expect this scale not to be significantly different from those of the previous orders. In this particular case, as shown in Eq. 23, we have also a dependence on the initial scale \( \alpha_s(\mu_0) \) left due to the normalization and to the regularization terms. These errors represent the 40% of the whole theoretical error and in order to be eliminated the correct normalization would be required.

![FIG. 2: The thrust distribution under the \( \text{PMC}_\infty \) at NLO (DotDashed Blue) and at NNLO (Solid Red). The experimental data points are taken from the ALEPH, DELPHI, OPAL, L3, SLD experiments [36–40]. The shaded area shows theoretical errors for the \( \text{PMC}_\infty \) predictions at NNLO.](image)

**G. NNLO C-parameter distribution results**

The same analysis applies straightforwardly to the C-parameter distribution including the regularizing \( \eta \) parameter which has been set to the same value 3.51. The same scales of Eq. 16 and Eq. 17 apply to the C-parameter distribution in the region \( 0 < C < 0.75 \) and in the region \( 0.75 < C < 1 \). In fact, due to kinematical constraints that set the \( A_{\text{Conf}} = 0 \), we have the same iCF effect also for the C-parameter. Results for the C-parameter scales and distribution are shown in Fig. 3 and Fig. 4 respectively. The theoretical calculations at NNLO (Blue) with \( \mu_0 = M_Z \) and the interpolating functions
This property leads to the PMC\(_\infty\) : an infinite-order scale setting based on the Principle of Maximum Conformality method. The PMC\(_\infty\) method preserves the iCF form and is void of ambiguities. The absence of ambiguities in PMC\(_\infty\) is clearly shown hereby simply noticing that the new scale at each order is fixed by considering the \(\beta_0\)-i.e. \(N_f\)-term and no other term is needed in agreement with the iCF parametrization. Thus PMC\(_\infty\) scale fixing procedure is constrained by the iCF form and the choice of the scales is totally free of any ambiguity. The only unknown scale which remains unfixed apparently, is the one given by the last order of accuracy. In this article this scale has been fixed to the \(\mu_{III} = \mu_0 = M_{Z_0}\) for sake of a consistent comparison with the Conventional method. We remark that the last ”unknown” PMC scale can be fixed to the last known one. As we have also shown in this article, in general the differences between two consecutive scales are rather small. In the case of the thrust and C-parameter distributions at NNLO, the differences between the \(\mu_{III} = M_{Z_0}\) or \(\mu_{III} = \mu_{II}\) are also negligible and these differences are totally confined within the errors shown for the distributions in the perturbative region. Given the Conformal Limit of the Eq. 10, the last term in the iCF determines the level of conformality reached by the expansion and the whole scale dependence is confined in the coupling. As previously shown, this term is the main source of the uncertainties (as also reported in the Ref. [34]) and a further investigation regarding the regularization method applied here would be necessary.

I. CONCLUSIONS

We have introduced here the iCF Intrinsic Conformality, a parametrization which preserves exactly the scale invariance of an observable. We have shown a new method to solve the conventional renormalization scale ambiguity in QCD named PMC\(_\infty\) which preserves the iCF and leads to conformal renormalization scales. This method agrees with the PMC at LO and and it applies to observables especially in cases of a clearly manifested iCF, as for example in the case of the event shape variables. We have presented a new procedure to identify the iCF/PMC\(_\infty\)-coefficients at all orders which can be applied to either numerical or analytical calculations. PMC\(_\infty\) has been applied to the NNLO thrust and C-parameter distributions and the results show perfect agreement with the experimental data predicting an average theoretical error of 3.97\% and 4.1\%. The PMC\(_\infty\) method eliminates the renormalon growth \(\alpha_s^n \beta_0^m n!\), the scheme dependence and all the uncertainties related to the scale ambiguity up to the order of accuracy. The iCF/PMC\(_\infty\) scales are identified by the lowest order logarithm related to the \(\beta_0\)-term at each order and all the physics of the process is contained in the conformal coefficients. This is in complete agreement with QED and with the Gell-Mann and Low scheme [13, 14]. We have

(Dashed Lines) with errors (Shaded Area) have been also shown together with the experimental data (Green). Theoretical errors have been calculated, as in the previous case, using standard criteria and results indicate an average error over the whole spectrum of \(\delta \simeq 4.1\%\). The NNLO QCD predictions for the C-parameter distribution has been improved [41] and PMC\(_\infty\) results show a remarkable agreement with the experimental data away from the regions \(C < 0.05\) and \(C \simeq 0.75\). A further improvement of the perturbative calculations could be obtained also by introducing the large logarithms resummation technique in these regions.

FIG. 3: The LO-PMC\(_\infty\) (Solid Red) and the NLO-PMC\(_\infty\) (Dashed Black) scales for C-parameter.

FIG. 4: The C-parameter distribution under the PMC\(_\infty\) at NNLO (Blue) with \(\mu_{III} = \mu_0 = M_{Z_0}\). The experimental data points (Green) are taken from the ALEPH experiment [36]. Dashed lines show fits of the theoretical calculations with interpolating functions for the values of the remaining initial scale \(\mu_0 = M_{Z_0}, M_{Z_0}/2\) and \(2M_{Z_0}\) (the Red Dashed refers to \(M_{Z_0}\)). The shaded area shows theoretical errors for the PMC\(_\infty\) predictions at NNLO calculated varying the remaining initial scale value in the range \(\sqrt{s}/2 \leq \mu_0 \leq 2\sqrt{s}\).

H. Comment on the last PMC\(_\infty\) scale

We have shown in this article a property of the perturbative QCD corrections which is consistent with the MC results of single variable distributions for \(e^+e^- \rightarrow 3\text{jets}\).
discussed why iCF should be considered a strict requirement for a theory in order to preserve the scale invariance of the observables and we have shown that iCF is consistent with the single variable thrust and C-parameter distributions. We point out that other conformal aspects of QCD resulting from different sectors such as Conformally Invariant Scale Relations-CSR [35] or dual theories as AdS/CFT [42] might also be related to the Intrinsic Conformality. We underline that the iCF property in a theory would have two main remarkable consequences: first, it shows what is the correct coupling constant at each order as a function of the conformal intrinsic scale $\mu_N$, and second, since only the conformal and the $\beta_0$ coefficients need to be identified in the observables at each order, by means of the PMC$_\infty$ method the iCF would reveal its predictive feature for the coefficients of the higher order color factors. We point out that in many cases the implementation of the iCF in a multi-loop calculations procedure would lead to a significant reduction of the color factors coefficients and it would speed up the calculations for higher order corrections.

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