Spectral curves for the rogue waves

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Abstract

Here we find the spectral curves, corresponding to the known rational or quasi-rational solutions of AKNS hierarchy equations, ultimately connected with the modeling of the rogue waves events in the optical waveguides and in hydrodynamics. We also determine spectral curves for the multi-phase trigonometric, hyperbolic and elliptic solutions for the same hierarchy. It seems that the nature of the related spectral curves was not sufficiently discussed in existing literature.

Introduction

The quasi-rational multi-parametric solutions for the NLS equation, explaining a phenomena of the multiple rogue waves generation, were first obtained in [12] in 2010, (see also [13], and [14]), - using a slightly modified technique of work [15]. Recently this approach was extended by two of the authors in [30, 34] to the whole AKNS hierarchy.

It was mentioned in [12] that it is possible to attend the same goal by using a properly generalized Darboux transformation method in spirit of the works [28] and [29], - to the focusing NLS equation. This was done in [19]. See also [23] where an iterative application of onset Darboux transformation was used.

At the same time, Hirota method was successfully applied in [31] to the same problem. Let us emphasize that three aforementioned approaches were developed without any use of the spectral curves, associated with the finite gap solutions of the NLS equation.

The 4-th approach, also suggested in [12], is to start from the finite-gap solutions of the focusing NLS equation associated with nonsingular hyperelliptic curves, and to consider an appropriate passage to the limit, corresponding to the confluence of several branch points of the spectral curve. To some extent, this later approach was first considered in [12,22], resulting in explicit description of various kinds of modulation instability for the focusing NLS model, described by means of elliptic or trigonometric functions. To obtain the aforementioned quasi-rational solutions it was necessary to go from the results of [22] to a further degeneration of spectral curves and related solutions, taking care to keep a maximal number of free parameters at the end. This was realized in a number of works by P.Gaillard, (see for instance [16–18], see also further comments about these works in [14]).

Here in this work we explain, how to solve the inverse problem, - i.e. how, for various kinds of explicit solutions, obtained without any apriory contact with spectral curves, - to restore these spectral curves. Thus, using quite elementary methods, valid for the whole AKNS hierarchy. we reply
to the frequently posed question: what spectral curves corresponds to rank \( n \) multiple rogue waves solutions, or higher Peregrin breathers, or trigonometric breathers and multi-breathers solutions?

Assuming the solutions are expressed by means of trigonometric, hyperbolic or rational functions, we show that the related spectral curves have multiple branch points. The goal of the work is to derive and analyze equations of the spectral curves for known solutions, finding the relations between parameters of solutions and parameters of the spectral curves. This will allow us to carry out the degeneration process of generic algebro-geometric multi-phase solutions to solutions, expressed by means of elementary or elliptic functions and investigate new solutions of the AKNS hierarchy equation. It will also help to establish correspondence between parametrizations of the multi-phase solutions of AKNS hierarchy equations, obtained by different methods.

1 Brief introduction to AKNS hierarchy

For further details see [1] and especially [30, 34] in our context. AKNS hierarchy equations follow from compatibility conditions of the systems:

\[
\begin{aligned}
\Psi_x &= U \Psi, \\
\Psi_{t_k} &= V_k \Psi,
\end{aligned}
\]

\[ U := \lambda J + U^0, \quad V_1 := 2\lambda U + V_1^0, \quad V_{k+1} := 2\lambda V_k + V_{k+1}^0, \quad k \geq 1 \\
J := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U^0 := \begin{pmatrix} 0 & ip \\ -iq & 0 \end{pmatrix}.
\]

The conditions \((\Psi_x)_{t_k} = (\Psi_{t_k})_x\) implies \(V_k\), recursive relations for off-diagonal elements of matrices \(V^0_k\), and the relations between diagonal and off-diagonal elements of these matrices.

\[ [J, V_1^0] = 2(U^0)_x, \quad [J, V_{k+1}^0] = 2(V_k^0)_x + 2[V_k^0, U^0], \quad k \geq 1. \]

In particular,

\[ V_1^0 = \begin{pmatrix} -ip & -px \\ -qx & ip \end{pmatrix}, \quad V_2^0 = \begin{pmatrix} pxq - qx^2 & 2ip^2 q - ipxx \\ -2iq^2 p + iqxx & qx^2 - pxq \end{pmatrix}. \]

The integrable nonlinear equations from AKNS hierarchy are derived from the recursion relations

\[ (U^0)_{t_k} = (V_k^0)_x + [V_k^0, U^0] = \frac{1}{2} [J, V_{k+1}^0]. \quad (\ast) \]

Let us list first five members of AKNS hierarchy and their reduced versions, starting the RAKNS hierarchy:

\[
\begin{aligned}
ip_{t_1} + p_{xx} - 2p^2 q &= 0, \\
-iq_{t_1} + q_{xx} - 2q^2 p &= 0.
\end{aligned}
\]

When \( q = \mp p^* \) it reduces to a focusing nonlinear Schrödinger equation (NLS):

\[ ip_{t_1} + p_{xx} + 2|p|^2 p = 0, \quad (\dagger) \]
or to its defocusing version:

\[ ip_t + p_{xx} - 2|p|^2 p = 0, \]

Next member of AKNS hierarchy is called coupled modified Korteweg-de Vries (cmKdV) system:

\[
\begin{aligned}
p_{t_2} + p_{xxx} - 6pqp_x &= 0, \\
q_{t_2} + q_{xxx} - 6pqq_x &= 0.
\end{aligned}
\] (2)

Setting \( q = -p^* \) it reduces to complex modified Korteweg de Vries (cmKdV) equation:

\[ p_{t_2} + p_{xxx} + 6|p|^2 p_x = 0. \] (2a)

Taking \( q = \pm p \) in coupled modified Korteweg-de Vries system, we get usual (real) mKdV equation:

\[ p_{t_2} + p_{xxx} \pm 6p^2 p_x = 0. \]

The third member of AKNS hierarchy:

\[
\begin{aligned}
-ip_{t_3} - p_{xxxx} + 8pqp_{xx} + 2p^2 q_{xx} + 6p^2 q + 4pp_x q_x - 6p^3 q^2 &= 0, \\
-iq_{t_3} - q_{xxxx} + 8pqq_{xx} + 2q^2 p_{xx} + 6pq^2 q_x - 6p^2 q^3 &= 0.
\end{aligned}
\] (3)

Under the constraints \( q = -p^* \), \( t_3 = -t \) it reduces to Lakshmanan-Porsezian-Daniel (LPD) equation [11][25][33]:

\[ ip_t + p_{xxxx} + 8|p|^2 p_{xx} + 2p^2 p^*_{xx} + 6p^2 p^* + 4p|p_x|^2 + 6|p|^4 p = 0. \] (3a)

The fourth member of AKNS hierarchy:

\[
\begin{aligned}
p_{t_4} - p_{5x} + 10pqp_{xxx} + 20qp_x p_{xx} + 10pp_x q_{xx} + 10pp_x q_{xx} + 10p^2 q_x - 30q^2 p^2 p_x &= 0, \\
q_{t_4} - q_{5x} + 10qpp_{xxx} + 20qp_x q_{xx} + 10q_{xxx} p_{xx} + 10q_{xxx} p_{xx} + 10q^2 p_x - 30q^2 p^2 q_x &= 0
\end{aligned}
\] (4)

Taking \( q = -p^* \) and \( t_4 = -t \), it reduces to:

\[ p_t + p_{5x} + 10|p|^2 p_{xxx} + 20p_{xxx} p_x p^* + 10(|p_x|^2 p)_x + 30|p|^4 p_x = 0. \] (4a)

Surprisingly as short as (3a).

The fifth member of AKNS hierarchy:

\[
\begin{aligned}
-ip_{t_5} + p_{6x} - 12pqp_{xxxx} - 2p^2 q_{xxxx} - 30p_x q_{xxx} - 18pq_x p_{xxx} - 8pp_x q_{xxx} - 8pp_x q_{xxx} - 8pp_x q_{xxx} - 8pp_x q_{xxx} \\
-50p_x q_{xxx} + 50q^2 p_x^2 p_{xx} - 20p_{xxx} q_x - 22pq_x p_{xxx} + 20q_{xxx} q^3 + 10p^3 q_x^2 + 70q^2 p_x^2 + 60q^2 p_x q_x - 20q^3 p^4 &= 0,
\end{aligned}
\]

\[
\begin{aligned}
- iq_{t_5} + q_{6x} - 12qpq_{xxxx} - 2q^2 p_{xxxx} - 30q_x p_{xxx} - 18qp_x q_{xxx} - 8qq_x p_{xxx} - 8qq_x p_{xxx} - 8qq_x p_{xxx} - 8qq_x p_{xxx} \\
-50q_x p_{xxx} + 50q^2 q_x^2 q_{xx} - 20q_{xxx} p - 22qp_{xxx} q_x - 20p_{xxx} q_x^2 + 20p_{xxx} q^3 + 10q^3 p_x^2 + 70q^2 p_x^2 q_x - 60q^2 p_x q_x - 20q^3 p^4 &= 0.
\end{aligned}
\]
For $q = -p^*$, it reduces to a single scalar equation:

$$
ip_t + p_{6x} + 12|p|^2 p_{xxxxx} + 2p^2 p_{xxxx}^* + 30p_{xxx}p_x p^* + 18p_{xxx} p_{x}^* + 8(p_x p_{xxx}^* + 
+ 50p_{xx} p_x^2 + 50p_{xx} p|^4 + 20p_{xx}^2 p^* + 22|p_{xx}|^2 p + 20p^2 p_{xx}^* + 20|p|^2 p_{xx}^* + 
+ 10p^3 (p_x^*)^2 + 70p_x|^2 p^* + 60|p|^2 |p_x|^2 p + 20|p|^6 p = 0.
$$

The system, corresponding to the $k$-th member of AKNS hierarchy, can be written as follows:

$$
\begin{cases}
p_{k} = i^k H_k(p, q),
q_{k} = (-i)^{k} H_k(q, p).
\end{cases}
$$

Explicit formulas for the functions $H_k(p, q)$ for $k = 1, 2, \ldots 5$ are given above. For higher values of $k$ they can be easily obtained by use of the symbolic computation from [7], and for $k = 6, 7$ they can be found in [6] where $K_j(x, t) = H_{j+1}(p, -p^*)$. All equations of the reduced AKNS hierarchy (RAKNS hierarchy) and, in particular, (1a), (2a), (3a), (4a) etc. can be written as

$$
p_{6x} = i^k H_k(p, -p^*).
$$

All the members of the AKNS and RAKNS hierarchies are covariant with respect to space and time translations, generalized Galilean and scaling transformations (see [34]). All members of AKNS and RAKNS hierarchies have a well known common feature: for any integer $k$ there exist functions

$$
p(x, t_1, \ldots, t_k),
$$

satisfying all equations of the hierarchy simultaneously.

Another integrable equations can be obtained using the function $p(x, t_1, \ldots, t_k)$ and substituting the variable $t$ directly into several phases $t_k$ simultaneously. For example, the integrable Hirota equation [7, 10, 20, 21, 26] has the form

$$
ip_t + \alpha H_1(p, -p^*) - i\beta H_2(p, -p^*) = 0.
$$

It is easy to see that this equation has a solution in the form $p(x, \alpha t, -\beta t, \ldots, t_k)$, where $p(x, t_1, \ldots, t_k)$ is an arbitrary solution of the equations of the AKNS hierarchy. The more complex model, used in [5, 34], is described by the equation

$$
ip_t + \alpha H_1(p, -p^*) - i\beta H_2(p, -p^*) + \gamma_1 H_3(p, -p^*) = 0.
$$

It is easy to understand that it has solutions in the form $p(x, \alpha t, -\beta t, -\gamma_1 t, \ldots, t_k)$. Next in order equations are:

$$
ip_t + \alpha H_1(p, -p^*) - i\beta H_2(p, -p^*) + \gamma_1 H_3(p, -p^*) - i\gamma_2 H_4(p, -p^*) = 0
$$

and

$$
ip_t + \alpha H_1(p, -p^*) - i\beta H_2(p, -p^*) + \gamma_1 H_3(p, -p^*) - i\gamma_2 H_4(p, -p^*) + \gamma_3 H_5(p, -p^*) = 0.
$$

The coefficients of the spectral curves equations do not depend on times $t_k$. Therefore, the spectral curves equations depend only on the “parent” solution $p(x, t_1, \ldots, t_k)$: it does not depend on the choice of the selected member of hierarchy.
2 Appel equation and spectral curves

The equation $\Psi_x = U \Psi$ has the following scalar form

$$
\psi_x = -i\lambda \psi + ip \phi, \\
\phi_x = i\lambda \phi - iq \psi,
$$

or

$$
\psi_{xx} - \frac{p_x}{p} \psi_x + \left( \lambda^2 - i\lambda \frac{p_x}{p} - pq \right) \psi = 0.
$$

Let $\psi_1$ and $\psi_2$ are two linearly independent solutions of the equation

$$
\psi_{xx} + P(x) \psi_x + Q(x) \psi = 0.
$$

Then the function $Y = \psi_1 \psi_2$ satisfies Appel equation (\[36\], Part II, Chapter 14, Example 10; \[8\])

$$
Y_{xxx} + 3PY_{xx} + (P' + 4Q + 2P^2)Y_x + (2Q' + 4PQ)Y = 0.
$$

Taking the coefficients of (7) as in (6) from Appel equation we get:

$$
Y_{xxx} - 3\frac{p_x}{p} Y_{xx} + \left( 4\lambda^2 - 4i\lambda \frac{p_x}{p} + \frac{3p_x^2 - p_{xx}p}{p^2} - 4pq \right) Y_x - \\
- \left( 4\lambda^2 \frac{p_x}{p} + 2i\lambda \frac{p_{xx}p - 3p_x^2}{p^2} + 2pq_x - 2qp_x \right) Y.
$$

Assume, that (9) admits the solution of the form

$$
Y = \sum_{j=0}^{g} \gamma_j(x) \lambda^{g-j}.
$$

Substituting (10) into (9) and equating the coefficients at the same powers of $\lambda$, we obtain for the coefficients $\gamma_j$. The first two equations have the form

$$
4\gamma_0' - 4\frac{p_x}{p} \gamma_0 = 0, \\
4\gamma_1' - 4\frac{p_x}{p} \gamma_1 - 4i\frac{p_x}{p} \gamma_0' - 2i\frac{p_{xx}p - 3p_x^2}{p^2} \gamma_0 = 0.
$$

From these two equations we get

$$
\gamma_0(x) = c_0 p(x), \\
\gamma_1(x) = \frac{i}{2}(c_0 p_x + c_1 p).
$$

Next equations have the form

\[1\] Recall that in the case $P = 0$, $Q = q(x) - \lambda$, the existence of solution polynomial in $\lambda$ of Appel equation allows to isolate a very wide class of integrable potentials, including all finite-gap periodic and almost periodic potentials, all reflectionless potentials and more generally all Bargmann potentials.
4\gamma_j'' - 4\frac{p_x}{p_x} \gamma_j + 2i \frac{p_x p - 3 p^2}{p^2} \gamma_j + \gamma_j' - 2(pq_x - qp_x) \gamma_j = 0. \quad (11)

Equations (11) allow to find the remaining coefficients using the recursive relations:

\[
\gamma_{j+2} = c_{j+2} p + \int \left( \frac{p_x}{p} \gamma_j' + \frac{p_{xx} p - 3 p^2}{2 p^3} \gamma_j - \frac{1}{4p} \gamma_j + \frac{3 p_x}{4 p^2} \gamma_j' + \left( \frac{p_{xx} p - 3 p^2}{4 p^3} + q \right) \gamma_j' - \frac{p x q - q x p}{2 p} \gamma_j \right) dx.
\]

Assuming that \( c_0 = 1 \), we obtain from (12) the following equalities

\[
\gamma_2 = -\frac{1}{4}(p_{xx} - 2 p^2 q + c_1 p_x + c_2 p), \quad \gamma_3 = -\frac{i}{8}(p_{xxx} - 6 p q p_x + c_1 (p_{xx} - 2 p^2 q) + c_2 p_x + c_3 p).
\]

Of course, since \( \gamma_j \equiv 0 \) for \( j > g \), equations (11) and (12) can only be used if \( j \leq g - 2 \). It can be shown that for \( \gamma_j \) when \( j \geq 3 \) we get:

\[
\gamma_j = \left( \frac{i}{2} \right)^j \left( H_{j-1}(p, q) + \sum_{k=1}^{j-2} c_k H_{j-1-k}(p, q) + c_{j-1} p_x + c_j p \right).
\]

For \( j = g - 1 \) and \( j = g \) equation (11) takes the form

\[
-4i \frac{p_x}{p} \gamma_j' - 2i \frac{p_{xx} p - 3 p^2}{p^2} \gamma_g + \gamma_g' - 3 \frac{p_x}{p} \gamma_g' - \left( 4pq + \frac{p_{xx} p - 3 p^2}{p^2} \right) \gamma_g' - 2(p q_x - q p_x) \gamma_g = 0 \quad (13)
\]

and

\[
\gamma_g'' - 3 \frac{p_x}{p} \gamma_g'' - \left( 4pq + \frac{p_{xx} p - 3 p^2}{p^2} \right) \gamma_g' - 2(p q_x - q p_x) \gamma_g = 0. \quad (14)
\]

Knowing \( p \) and \( q \), we can find from equations (13), (14) the values of the constants \( c_k \).

Since the Wronskian \( W[\psi_1, \psi_2] := (\psi_2)_x \psi_1 - (\psi_1)_x \psi_2 \) of any two solutions of (7) satisfy the differential equation \( W_x = -P(x)W \), and, in (8) \( P(x) = -p_x p^{-1}(x) \), we get

\[
W[\psi_1, \psi_2] = -2i \nu(\lambda)p(x),
\]

where \( \nu(\lambda) \) is \( x \)-independent function of \( \lambda \). Knowing the product of solutions \( Y := \psi_1 \psi_2 \) and their Wronskian \( W \), we obtain

\[
\frac{\psi'_1}{\psi_1} = \frac{Y' + W}{2Y}, \quad \frac{\psi'_2}{\psi_2} = \frac{Y' - W}{2Y}.
\]

Hence

\[
\psi_{1,2} = \sqrt{Y} \exp \left( \pm i \nu(\lambda) \int \frac{p(x)dx}{Y(x)} \right) . \quad (15)
\]
Substituting (15) in (6) and simplifying, we obtain an equation of the spectral curve

\[ \nu^2(\lambda) = \frac{Y^2}{p^2} \lambda^2 - \frac{4}{p^3} Y^2 - \frac{2pY_{xx}Y + pY_x^2 + 2p_Y Y}{4p^3}. \]  

(16)

The right-hand side of the equation (16) is a polynomial of degree \( g + 2 \). Its coefficients are integrals of motion for the NLS equation. These integrals can be found by substituting (10) in (16) and simplifying.

3 Examples

In this section, we will assume that \( \text{Im } a = 0, \text{Im } b = 0 \).

3.1 Plane, solitary and “dnoidal” waves solutions

Let us consider solution of the nonlinear Schrödinger equation (1a) in the form of a plane wave

\[ p(x, t) = ae^{2i(a^2 - 2\beta^2)t - 2ibx}. \]

The spectral curve of this solution \( \Gamma_0 \) has both topological and arithmetic genus \( g = g_a = 0 \):

\[ \Gamma_0 = \{ (\nu, \lambda) : \nu^2 = (\lambda - b)^2 + a^2 \}. \]

The well known soliton solution of the NLS equation is described by the formula

\[ p(x, t) = 2ae^{4i(a^2 - \nu^2)t - 2ibx} \cdot \cosh(2ax + 8abt). \]

The spectral curve \( \Gamma_1s \) of this solution is a degenerate (singular) elliptic curve of topological genus \( g = 0 \) and arithmetic genus \( g_a = 1 \) Here we have \( c_1 = 4ib \).

\[ \Gamma_1s = \{ (\nu, \lambda) : \nu^2 = \left( (\lambda - b)^2 + a^2 \right)^2 \}. \]  

(17)

The one-phase solution of the nonlinear Schrödinger equation with periodic amplitude has the form of a dnoidal wave:

\[ p(x, t) = 2ae^{4i(2a^2 - k^2a^2 - b^2)t - 2ibx} \cdot \text{dn}(2ax + 8abt; k), \]

where \( \text{dn}(x; k) \) is an elliptic Jacobi function [2]. The spectral curve of the “dnoidal” wave also has both topological and arithmetic genus 1 i.e. \( g = 1 = g_a \), and \( (c_1 = 4ib) \). Contrary to the previous example its spectral curve \( \Gamma_{1dn} \) is a non-singular elliptic curve i.e. it has only simple branch points:

\[ \Gamma_{1dn} = \{ (\nu, \lambda) : \nu^2 = \left( (\lambda - b)^2 + a^2(1 - k_1^2) \right) \left( (\lambda - b)^2 + a^2(1 + k_1^2) \right) \}. \]  

(18)

where \( k^2 + k_1^2 = 1 \).

Thus, a plane wave is the null-phase solution of the nonlinear Schrödinger equation, and the solitary and “dnoidal” waves are one-phase. The phase of solitary and “dnoidal” waves are determined by equality

\[ X = 2ax + 8abt. \]

It is easy to see that in the limit at \( k_1 \to 0, k \to 1 \) the spectral curve of the “dnoidal” wave goes into the spectral curve of a solitary wave, and the “dnoidal” wave goes into a solitary wave.
3.2 Peregrine soliton

Let us consider the well-known Peregrine soliton [32]

\[ p(x, t) = \left( 1 - \frac{4(1 + iT)}{X^2 + T^2 + 1} \right) e^{2it}, \quad X \equiv 2x, \quad T \equiv 4t. \]

Performing the Galilean and scaling transformations [34], we obtain the general form of the Peregrine soliton

\[ p(x, t) = a \left( 1 - \frac{4(1 + iT)}{X^2 + T^2 + 1} \right) e^{2i(\sigma^2 - 2\kappa^2)t - 2i\kappa x}, \]

\[ X \equiv 2ax + 8abt, \quad T \equiv 4a^2t. \] (19)

Of course the function (19) satisfies the NLS equation.

Substituting (19) and \( q = -p^* \) into (13) for \( g = 2 \), we get

\[ c_1 = 6ib, \quad c_2 = -6a^2 - 12b^2. \]

The function (19) satisfies equation (14) for \( g = 2 \). Therefore the function (19) is a degenerate two-gap solution of the nonlinear Schrödinger equation. Calculating the spectral curve \( \Gamma_p \) of the Peregrine soliton (19), we get

\[ \Gamma_p := \{ (\nu, \lambda) : \nu^2 = (\lambda - b)^3 + a^3 \} \]

or

\[ \Gamma_p := \{ (\nu, \lambda) : \nu^2 = (\lambda - b - ia)^3(\lambda - b + ia)^3 \} . \]

Therefore a solution (19) with \( a = \text{Im}(\lambda_1) \), \( b = \text{Re}(\lambda_1) \) corresponds to a degenerated spectral curve \( \Gamma_{per} \):

\[ \Gamma_{per} := \{ (\nu, \lambda) : \nu^2 = (\lambda - \lambda_1)^3(\lambda - \lambda_1^*)^3 \} . \]

In the case of the canonical form of the Peregrine soliton, i.e. for \( X = 2x, T = 4t \), the constants \( c_k \) are equal to

\[ c_1 = 0, \quad c_2 = -6, \]

and the spectral curve becomes \( \Gamma_p \):

\[ \Gamma_p := \{ (\nu, \lambda) : \nu^2 = (\lambda^2 + 1)^3 \} . \] (20)

It is clear that all spectral curves connected with Peregrine soliton are singular curves of arithmetic genus \( g_a = 2 \) and of topological genus \( g = 0 \).

3.3 The Kuznetsova-Ma soliton and the Akhmediev breather

The Kuznetsova-Ma soliton [24, 27] is a two-phase solution periodic in \( x \). Let us write it in the form [30]

\[ p(x, t) = \left( 1 - \frac{2k(k \cosh(kxT) + i \kappa \sinh(kxT))}{\sqrt{x^2 + k^2 \cosh(kxT) - \kappa \cos(kX)}} \right) e^{2i(kx^2 + k^2)t}, \]

\[ X \equiv 2x, \quad T \equiv 4t, \quad k = \sin \theta, \quad \kappa = \cos \theta \] (\( \theta \) is a parameter of the solution).
From (13) and (14) for \( g = 2 \) we get

\[ c_1 = 0, \quad c_2 = -2 - 4 \cos^2 \theta. \]

The spectral curve of the Kuznetsov-Ma soliton (21) is given by the equation

\[ \nu^2 = (\lambda^2 + 1)(\lambda^2 + \cos^2 \theta)^2. \]

For the case of the Kuznetsov-Ma soliton (21) with arguments \( X \equiv 2ax + 8abt, \ T \equiv 4a^2t \) the constant \( c_1 \) and \( c_2 \) are equal

\[ c_1 = 6ib, \quad c_2 = -12b^2 - (2 + 4 \cosh^2 \theta)a^2, \]

and the spectral curve is given by the equation

\[ \nu^2 = (\lambda^2 - 2b\lambda + b^2 + a^2) (\lambda^2 - 2b\lambda + b^2 + a^2 \cos^2 \theta)^2 \]

or

\[ \nu^2 = (\lambda - b)^2 + a^2\ (\lambda - b)^2 + a^2 \cos^2 \theta)^2. \quad (22) \]

The Akhmediev breather [3] is a two-phase solution periodic in \( t \). It can be obtained from the Kuznetsov-Ma soliton (21) by substitution \( \theta \rightarrow i\theta \) [30]:

\[ p(x, t) = \left( 1 + \frac{2k(k \cos(kxT) + i\kappa \sin(kxT))}{\sqrt{\kappa^2 - k^2 \cos(kxT) - \kappa \cosh(kX)}} \right) e^{2i(\kappa^2 - k^2)t}, \quad (23) \]

where \( \kappa = 2x, \ T = 4t, \ k = \sinh \theta, \ \kappa = \cosh \theta. \) Correspondingly, the constants \( c_1 \) and \( c_2 \) are equal

\[ c_1 = 0, \quad c_2 = -2 - 4 \cosh^2 \theta, \]

and the spectral curve of Akhmediev breather (23) is given by equation

\[ \nu^2 = (\lambda^2 + 1)(\lambda^2 + \cosh^2 \theta)^2. \]

Let us remark that for \( X \equiv 2ax + 8abt, \ T \equiv 4a^2t \) the constants \( c_1 \) and \( c_2 \) are equal

\[ c_1 = 6ib, \quad c_2 = -12b^2 - (2 + 4 \cosh^2 \theta)a^2, \quad (24) \]

and the spectral curve is given by equation

\[ \nu^2 = (\lambda^2 - 2b\lambda + b^2 + a^2) (\lambda^2 - 2b\lambda + b^2 + a^2 \cosh^2 \theta)^2 \]

or

\[ \nu^2 = (\lambda - b)^2 + a^2\ (\lambda - b)^2 + a^2 \cosh^2 \theta)^2. \]

For all exemples of this subsection the related spectral curves are singular algebraic curves of arithmetic genus \( g_a = 2 \) and of topological genus \( g = 0 \). The spectral curves considered here have a couple of simple branch points and a couple of double branch points each.
3.4 Rank-2 rogue wave solution

Rank 2 rogue waves solution reads \[30\]

\[
\Psi_2(X, T_1, T_2, T_3) := \left( 1 - 12 \frac{G(X, T_1, T_2, T_3) + iH(X, T_1, T_2, T_3)}{Q(X, T_1, T_2, T_3)} \right) e^{2it_1 - 6it_3 + 20it_5 + \ldots},
\]  

(25)

where

\[
G(X, T_1, T_2, T_3) = (X^2 + 3T_1^2 + 3)^2 - 4T_1^4 + 2XT_2 + 2T_1T_3 - 12,
\]

\[
H(X, T_1, T_2, T_3) = T_1(X^2 + T_1^2 + 1)^2 + 2XT_1T_2 + T_3(T_1^2 - X^2 - 1) - 8T_1(X^2 + 2),
\]

\[
Q(X, T_1, T_2, T_3) = (X^2 + T_1^2 + 1)^3 + T_2^2 + 2XT_2(3T_1^2 - X^2 + 3) + T_3^2 + 2T_1T_3(T_1^2 - 3X^2 + 9) + 24T_1^4 - 4T_1^2X^2 + 96T_1^2 + 24X^2 + 8.
\]

Here

\[
X = 2x - 12t_2 + 60t_4 + \ldots,
\]

\[
T_1 = 4t_1 - 24t_3 + 120t_5 + \ldots,
\]

\[
T_2 = -48t_2 + 480t_4 + \ldots,
\]

\[
T_3 = -96t_3 + 960t_5 + \ldots.
\]

It is easy to see, that this solution is four-phase. First phase is \(X, \ldots\), fourth phase is \(T_3\). Hence, for arithmetic genus \(g_a\) of the related spectral curve we have \(g_a = 4\). Calculating the constants \(c_k\), we get

\[
c_4 = -6c_2 - 30, \quad c_3 = -6c_1, \quad c_2 = -10, \quad c_1 = 0.
\]

The spectral curve \(\Gamma_2\) for the solution \[25\] is

\[
\Gamma_2 := \{ (\nu, \lambda) : \nu^2 = (\lambda^2 + 1)^5 \}
\]  

(26)

3.5 Rank-3 rogue wave and its spectral curve

A “freak wave” of rank 3 is defined by the following equalities

\[
\Psi_3(X, T_1, \ldots, T_5) = \left( 1 - 24 \frac{G(X, T_1, \ldots, T_5) + iH(X, T_1, \ldots, T_5)}{Q(X, T_1, \ldots, T_5)} \right) e^{2it_1 - 6it_3 + 20it_5 + \ldots},
\]

(27)

where

\[
G(X, T_1, \ldots, T_5) = X^{10} + \sum_{j=0}^{8} g_j X^j,
\]

\[
g_8 = 15T_1^2 + 15, \quad g_7 = 0,
\]

\[
g_6 = 50T_1^4 - 60T_1^2 - 80T_1T_3 + 210,
\]

\[
g_5 = 120T_2^2T_2 + 120T_2 + 18T_4,
\]

\[
g_4 = 70T_1^4 - 150T_1^2 - 200T_1T_3 + 450T_1^2 - (600T_3 - 30T_5)T_1 + 150T_2^2 - 50T_3^2 - 450,
\]

10
\[ g_3 = 400T_4^4T_2 + (2400T_2 + 60T_4)T_1^2 + 800T_1T_2T_3 - 1200T_2 + 60T_4, \]

\[ g_2 = 45T_1^8 + 420T_4^6 + 6750T_4^4 + (2400T_3 + 180T_5)T_3^3 - \\
- (300T_2^2 - 900T_1^4 + 13500)T_1^7 - (7200T_3 - 180T_5)T_1 - 300T_2^2 - 300T_1^2 - 675, \]

\[ g_1 = 280T_4^6T_2 - (600T_2 + 150T_4)T_4 + 800T_1^3T_2T_3 + (1800T_2 + 540T_4)T_2^2 - \\
- (2400T_2T_3 - 120T_2T_5 + 120T_3T_4)T_1 - 200T_2^3 - 200T_2T_3^2 - 1800T_2 + 90T_5, \]

\[ g_0 = 11T_1^{10} + 495T_1^8 + 120T_1^7T_3 + 2190T_1^6 + (2040T_3 - 42T_5)T_1^5 + (350T_2^2 + 150T_3^2 - 7650)T_1^4 + \\
+ (1800T_3 - 420T_5)T_1^3 + (300T_2^2 - 120T_2T_4 + 300T_3^2 - 120T_3T_5 - 2025)T_1^2 - \\
- (200T_2^2T_3 + 200T_3^3 - 1800T_3 + 90T_5)T_1 + 750T_2^2 - 120T_2T_4 + 2550T_3^2 - \\
- 240T_3T_5 + 6T_2^4 + 6T_3^2 + 675; \]

\[
 H(X, T_1, \ldots, T_5) = T_1X^{10} + \sum_{j=0}^{8} h_jX^j,
\]

\[
h_8 = 5T_1^3 - 15T_1 - 5T_3, \quad h_7 = 0, \quad h_6 = 10T_1^3 - 140T_1^3 - 40T_2T_3 - 150T_1 + 40T_3 - 5T_5, \]

\[
h_5 = 40T_2^2T_3 - 6(20T_2 - 3T_4)T_1 - 40T_2T_3, \quad h_4 = 10T_1^7 - 210T_1^5 - 50T_1T_3 - 450T_3^3 - 15(20T_3 - T_5)T_1^2 + 50(3T_2^2 - T_2^2 - 27)T_1 + \\
+ 15(10T_3 - T_5), \quad h_3 = 80T_2^2T_2 + 20(40T_2 + T_4)T_1^3 + 400T_2T_2T_3 - 60(20T_2 + T_4)T_1 - 20(20T_2 - T_2T_5 + T_3T_4), \]

\[
h_2 = 5T_1^9 - 60T_1^7 + 1710T_1^5 + 15(80T_3 + 3T_5)T_1^4 - 100(T_2^2 - 3T_3^2 + 63)T_1^3 - 90T_1^2T_3 + \\
+ 75(4T_2^2 + 4T_1^3 + 63)T_1 + 5(20T_2^2T_3 + 20T_3^2 + 720T_3 - 27T_5), \quad h_1 = 40T_1^7T_2 - 30(28T_2 + T_4)T_3 - 200T_1T_2T_3 - 60(30T_2 + T_4)T_3^3 - 60(20T_2T_3 - T_2T_5 + T_3T_4)T_1^2 - \\
- 50(4T_2^2 + 4T_2T_3^2 + 108T_2 - 9T_4)T_1 + 60(10T_2T_3 - T_2T_5 + T_3T_4), \]

\[
h_0 = T_1^{11} + 25T_1^9 + 15T_1T_3 - 870T_1^7 + (100T_3 - 7T_5)T_1^6 + 10(T_2^2 + 3T_3^2 - 963)T_1^5 - \\
- 75(8T_3 + T_5)T_1^4 - 5(100T_2^2 + 87T_2T_4 + 100T_3^2 + 8T_3T_5 + 495)T_1^3 - \\
- 5(20T_2^2T_3 + 20T_3^3 + 1980T_3 - 99T_5)T_2^2 - 3(350T_2^2 - 40T_2T_4 + 550T_3^2 - 2T_2^2 - 2T_5^2 - 1575)T_1 + \\
+ 5(20T_2^2T_3 - 4T_2T_3 + 8T_2T_3T_4 - 60T_3^2 + 4T_3^2T_3 + 315T_3 - 9T_5); \]

\[
 Q(X, T_1, \ldots, T_5) = (X^2 + T_1^2 + 1)^6 - 20T_2X^9 + \sum_{j=0}^{8} q_jX^j, \]

\[
 q_8 = -120T_1^2 - 60T_1T_3 + 120, \quad q_7 = -12T_4, \]

\[
 q_6 = -240T_1^2 - 160T_1T_3 + 480T_1^2 + (960T_3 - 60T_5)T_1 + 60T_2^2 + 140T_3^2 + 2320, \]

\[
 q_5 = 120T_1^4T_2 - (720T_2 - 108T_4)T_1^2 - 480T_1T_2T_3 + 1080T_2 + 108T_4, \]

\[
 q_4 = -120T_1^4T_2 - 1440T_1^2 - (360T_3 - 60T_5)T_1^3 + (900T_2^2 - 300T_3^2 + 13440)T_1^2 - \\
- (5400T_3 - 540T_5)T_1 + 900T_2^2 + 120T_2T_4 - 1500T_3^2 + 120T_3T_5 + 3360, \]

\[ 11 \]
\[ q_3 = 160T_1^8T_2 + (7200T_2 + 60T_4)T_1^4 + 1600T_1^3T_2T_3 + (21600T_2 - 360T_4)T_1^2 + 
+ (4800T_2T_3 + 240T_2T_5 - 240T_3T_4)T_1 + 400T_2^3 + 400T_2T_3^2 - 7200T_2 + 540T_4, \]
\[ q_2 = 240T_1^7 + 13440T_1^5 + (4320T_3 + 108T_5)T_1^3 - (300T_2^2 - 900T_3^2 - 78240)T_1^4 + 
+ (43200T_3 + 1080T_5)T_1^2 + (1800T_2^2 + 16200T_3^2 - 36480)T_1^3 + 
+ (1200T_2^2T_3 + 1200T_3^2 - 64800T_3 + 2700T_5)T_1 - 2700T_2^2 + 900T_3^2 - 720T_5T_8 + 
+ 36T_4^2 + 36T_5^2 + 12144, \]
\[ q_1 = 60T_1^6T_2 + (240T_2 - 60T_4)T_1^5 - 480T_2^3T_5T_3 - (5400T_2 + 1620T_4)T_1^4 - 
- (14400T_2T_3 - 240T_2^2T_5 + 240T_3^3)T_1^3 - (1200T_2^3 + 1200T_2T_3^2 - 54000T_2 + 5940T_4)T_1^2 - 
- (21600T_2T_3 - 2160T_2^2T_5 + 2160T_3^3)T_1 - 1200T_2^3 + 240T_2^2T_4 - 600T_2T_3^2 + 
+ 480T_2T_3T_5 - 240T_4T_3^2 + 13500T_2 - 540T_4, \]
\[ q_0 = 120T_1^10 + 20T_1^9T_3 + 3720T_1^8 + (1200T_3 - 12T_3)T_1^7 + (140T_2^2 + 60T_3^2 + 15280)T_1^6 + 
+ (5400T_3 - 612T_3)T_1^5 + (900T_2^2 - 120T_2T_4 - 1500T_3^2 - 120T_3T_5 + 143760)T_1^4 - 
- (400T_2^3T_3 + 400T_3^3 - 82800T_3 - 540T_3)T_1^3 + 
+ (8100T_2^2 + 720T_2T_4 + 18900T_3^2 + 36T_2^2 + 36T_3^2 + 93144)T_1^2 + 
+ (6000T_2^3T_3 - 240T_2^2T_5 + 480T_2T_3T_4 + 1200T_3^3 + 240T_2^2T_5 + 83700T_3 - 2700T_5)T_1 + 400T_2^2 + 
+ 800T_3T_2^2 + 400T_3^3 + 9900T_2^2 - 1080T_2T_4 + 24300T_3^2 - 1800T_3T_5 + 36T_4^2 + 36T_5^2 + 2024. \]

Here

\[
X = 2x - 12t_2 + 60t_4 + \ldots,
\]
\[ T_1 = 4t_3 - 24t_4 + 120t_5 \ldots. \]
\[ T_2 = -48t_2 + 480t_4 + \ldots. \]
\[ T_3 = -96t_3 + 960t_5 + \ldots. \]
\[ T_4 = -3840t_4 + \ldots. \]
\[ T_5 = -7680t_5 + \ldots. \]

The rogue wave solution of the rank 3 is a six-phase solution of the AKNS hierarchy equations of the aritmetic genus \(g_a = 6\). Calculating the constants \(c_k\), we get

\[
c_6 = -6c_4 - 30c_2 - 140, \quad c_5 = -6c_3 - 30c_1, \quad c_4 = -10c_2 - 70,
\]
\[
c_3 = -10c_1, \quad c_2 = -14, \quad c_1 = 0. \]

It follows from (16) that the spectral curve \(\Gamma_3\), corresponding to solution (27) is

\[ \Gamma_3 = \{ (\nu, \lambda) : \nu^2 = (\lambda^2 + 1)^7 \}, \quad (28) \]

i.e. it represents a singular algebraic curve of the arithmetic genus 6.
Concluding remarks

- The spectral curves $\Gamma_N$, corresponding to Matveev-Dubard-Smirnov \cite{13,14,30} quasi-rational rank $N$ solutions of AKNS hierarchy equations are:

$$\Gamma_N := \{ (\nu, \lambda) : \nu^2 = (\lambda^2 + 1)^{2N+1} \},$$

i.e. it represents singular algebraic curve of the arithmetic genus $2N$ with 2 branch points: $(0,i)$ and $(0,-i)$ of multiplicity $2N+1$ each.

- Polynomials in the RHS of the spectral curves related with solutions, containing trigonometric or hyperbolic functions, have one pair of simple complex conjugate roots and double complex conjugate roots. Solutions of this type can be obtained by the Darboux transformation of a plane wave.

- Polynomials in the RHS of the spectral curves equations of multi-solitons solutions have only double complex conjugate roots. Multi-solitons solutions can be obtained by the Darboux transformation of the zero seed solution. They can also be obtained by passage to the limit in theta functional formulas related with nonsingular hyperelliptic spectral curves.

- It will be interesting to investigate the solutions of whole AKNS hierarchy equation for the spectral curve $\Gamma_r$ defined by

$$\Gamma_r = \left\{ (\nu, \lambda) : \nu^2 = (\lambda^2 - 1)(\lambda^2 - k^{-2}) \prod_{j=1}^{N} (\lambda^2 - \lambda_j^2)^2 \right\};$$

$$1 < \lambda_1 < \lambda_2 < \ldots < \lambda_N < k^{-1}.$$

For the NLS equation it was done in \cite{4} and in the section 4.5 of \cite{9} in the context of studying multiphase modulations of the dnoidal wave solution. The related spectral curve obviously has a topological genus $g = 1$ and the arithmetic genus $g_a = 2N + 1$. It has 4 simple branch points and $2N$ double branch points.

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