NON-HOPFIAN RELATIVELY FREE GROUPS

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Abstract. To solve problems of Gilbert Baumslag and Hanna Neumann, posed in the 1960’s, we construct a nontrivial variety of groups all of whose noncyclic free groups are non-hopfian.

1. Introduction

Recall that a group $G$ is called hopfian if every epimorphism $G \to G$ is an automorphism. Let $F_m$ be a free group of finite rank $m > 1$, $N$ be a normal subgroup of $F_m$ and $V(N)$ a verbal subgroup of $N$ defined by a set of words $V$. In [2], Baumslag proved that if both quotients $F_m/N$, $N/V(N)$ are residually finite then the group $F_m/V(N)$ is also residually finite. In this connection, Baumslag [2] posed the following problem. Is $F_m/V(N)$ a hopfian group if the group $F_m/N$ is hopfian? In particular, if $F_m = N$ then $F_m/N$ is trivial and hopfian and so the Baumslag problem asks about the hopfian property of relatively free groups $F_m/V(F_m)$, where $V(F_m)$ is a verbal (or fully invariant) subgroup of $F_m$. The problem on the hopfian property of finitely generated relatively free groups was independently stated by H. Neumann [12, Problem 15]. Recall that a finitely generated residually finite group is hopfian (e.g., see Corollary 41.44 [12]). Hence, if one could show that relatively free groups are residually finite then H. Neumann’s problem would be solved in the affirmative. However, this is not the case in general and it follows from Novikov–Adian results [13], [11] (see also [14], [10], [5], [11], [6]) on the Burnside problem for odd $n \geq 665$ and Kostrikin–Zelmanov results [8], [9], [19] on the restricted Burnside problem for $n = p^k$, where $p$ is prime, that the free $m$-generator Burnside group $B(m, n) = F_m/F_m^{p^n}$ of exponent $n$ is not residually finite if $m > 1$ and $n$ is odd, $n = p^k > 665$ (in fact, $B(m, n)$ is not residually finite for all $m > 1$ and $n > 1$ as follows from results of [8], [9], [19], [20], [1], [13], [5] and the classification of finite simple groups). Whether the group $B(m, n)$ is hopfian (or, more generally, whether there is a hopfian non-residually finite relatively free group) is still unknown (the problem on the hopfian property of $B(m, n)$ for odd $n > 1$ is stated in [10, Problem 11.36(c)]).

In this article we construct a variety of groups of exponent 0 all of whose noncyclic free groups are non-hopfian providing thereby negative solutions to the foregoing problems of Baumslag and H. Neumann. To construct the identities that define such a variety of groups, we let $[a, b] = aba^{-1}b^{-1}$ be the commutator of $a$, $b$ and set

\begin{align*}
  v_0(x, y) &= x, & v_1(x, y) &= [(x^d y^d)^d, x^d]^d y, & v_2(x, y) &= [v_1(x, y)^d, x^d].
\end{align*}

(1)

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Now we define the words $w_1(x, y)$, $w_2(x, y)$ by following formulas

$$w_1(x, y) = x^{ε_1}v_1(x, y)^{n}x^{ε_2}v_1(x, y)^{n+2} \ldots x^{ε_h/2}v_1(x, y)^{n+h-6}$$

$$x^{ε_{h/2-1}}v_1(x, y)^{n+h-4}x^{ε_{h/2}}v_1(x, y)^{(n+h-2)+h/2}x^{ε_1}v_1(x, y)^{-(n+1)}$$

$$x^{ε_2}v_1(x, y)^{-(n+3)} \ldots x^{ε_h/2}v_1(x, y)^{-(n+h-3)x^{ε_h/2}}v_1(x, y)^{-(n+h-1)}, \quad (2)$$

$$w_2(x, y) = yv_2(x, y)^{n^2+1}v_1(x, y)^{ε_2}v_2(x, y)^{n^2+2}v_1(x, y)^{ε_3}v_2(x, y)^{n^2+3} \ldots$$

$$\ldots v_1(x, y)^{ε_h-1}v_2(x, y)^{n^2+h-1}v_1(x, y)^{ε_n}v_2(x, y)^{n^2+h}, \quad (3)$$

where $h \equiv 0 \pmod{20}$,

$$ε_{10k+1} = ε_{10k+2} = ε_{10k+3} = ε_{10k+5} = ε_{10k+6} = 1,$$

$$ε_{10k+4} = ε_{10k+7} = ε_{10k+8} = ε_{10k+9} = ε_{10k+10} = -1$$

for $k = 0, 1, \ldots, h/10 - 1$, and $h, d, n$ are sufficiently large positive integers (with $n \gg d \gg h \gg 1$). Note that the sums of exponents on $x, y$ in both (2) and (3) are zeros.

Now we can state our main result (note that this result was announced in \[7, \text{ Theorem 5}]); for the sake of simplicity of proofs we changed the identities $w_1(x, y) \equiv 1$, $w_2(x, y) \equiv 1$ presented in \[7].

**Theorem.** Let $\mathfrak{M}$ be the variety of groups defined by identities $w_1(x, y) \equiv 1$, $w_2(x, y) \equiv 1$, where the words $w_1(x, y)$, $w_2(x, y)$ are given by formulas (2), (3). Then any free group of rank $m > 1$ in the variety $\mathfrak{M}$ is not hopfian.

To prove this Theorem, in Sect. 2, we inductively construct a group presentation by means of generators and defining relations. In Sect. 3, we apply the geometric machinery of graded diagrams, developed by Ol’shanskii \[15, 16], to study sub-presentations of this presentation and to prove a number of technical lemmas. In particular, we use the notation and terminology of \[15, 16] and all notions that are not defined in this paper can be found in \[15, 16]. In Sect. 4, we show that the group presentation constructed in Sect. 2 defines a free group of rank $m > 1$ in the variety $\mathfrak{M}$ and this group is non-hopfian.

### 2. Inductive Construction

As in \[16, \text{ we will use numerical parameters}]

$$\alpha > \beta > γ > δ > ε > ζ > η > i$$

and $h = δ^{-1}$, $d = η^{-1}$, $n = i^{-1}$ ($h, d, n$ were already used in (2)–(3)) and employ the least parameter principle (LPP) (according to LPP a small positive value for, say, $ζ$ is chosen to satisfy all inequalities whose smallest (in terms of the relation $\succ$) parameter is $ζ$).

Let $A = \{a_1, \ldots, a_m\}$ be an alphabet, $m > 1$, and $F(A)$ be the free group in $A$. Elements of $F(A)$ are referred to as words in $A^{±1} = A \cup A^{-1}$ or just words. Denote $G(0) = F(A)$ and let the set $R_0$ be empty. To define the group $G(i)$ by induction on $i \geq 1$, assume that the group $G(i - 1)$ is already constructed by its presentation

$$G(i - 1) = \langle A \parallel R = 1, R ∈ R_{i-1} \rangle.$$

Let $X_i$ be a set of words (in $A^{±1}$) of length $i$, called *periods of rank* $i$, which is maximal with respect to the following two properties:
(A1) If $A \in X_i$ then $A$ (that is, the image of $A$ in $G(i - 1)$) is not conjugate in $G(i - 1)$ to a power of a word of length $|A| = i$.

(A2) If $A$, $B$ are distinct elements of $X_i$ then $A$ is not conjugate in $G(i - 1)$ to $B$ or $B^{-1}$.

If the images of two words $X$, $Y$ are equal in the group $G(i - 1)$, $i \geq 1$, then we will say that $X$ is equal in rank $i - 1$ to $Y$ and will write $X \equiv Y$. Analogously, we will say that two words $X$, $Y$ are conjugate in rank $i - 1$ if their images are conjugate in the group $G(i - 1)$. As in \cite{16}, a word $A$ is called simple in rank $i - 1$, $i \geq 1$, if $A$ is conjugate in rank $i - 1$ neither to a power $B^k$, where $|B| < |A|$ nor to a power of period of some rank $\leq i - 1$. We will also say that two pairs $(X_1, X_2)$, $(Y_1, Y_2)$ of words are conjugate in rank $i - 1$, $i \geq 1$, if there is a word $W$ such that $X_1 \equiv W Y_1 W^{-1}$ and $X_2 \equiv W Y_2 W^{-1}$.

Consider the set of all possible pairs $(X, Y)$ of words in $A^{\pm 1}$ and pick $z^* \in \{1, 2\}$. This set is partitioned by equivalence $z^*$-classes $\mathcal{C}_\ell(z^*)$, $\ell = 1, 2, \ldots$, of the equivalence relation $\sim$ defined as follows: $(X_1, Y_1) \sim (X_2, Y_2)$ if and only if the pairs $(v_z, (X_1, Y_1), w_z, (X_1, Y_1))$ and $(v_z, (X_2, Y_2), w_z, (X_2, Y_2))$ are conjugate in rank $i - 1$. It is convenient to enumerate (in some way)

$$\mathcal{C}_{A^+, \ell}(z^*), \mathcal{C}_{A^-, \ell}(z^*), \ldots$$

all $z^*$-classes of pairs $(X, Y)$ such that $w_z(X, Y) \neq 1$ and $v_z(X, Y)$ is conjugate in rank $i - 1$ to some power $A^\prime$, where $A \in X_i$ and $f$ are fixed.

It follows from definitions that every class $\mathcal{C}_{A^+, \ell}(z^*)$ contains a pair

$$(X_{A^+, j, z^*}, Y_{A^+, j, z^*})$$

with the following properties. The word $X_{A^+, j, z^*}$ is graphically equal (that is, letter-by-letter) equal to a power of $B_{A^+, j, z^*}$, where $B_{A^+, j, z^*}$ is simple in rank $i - 1$ or a period of rank $\leq i - 1$; $Y_{A^+, j, z^*} \equiv Z_{A^+, j, z^*} Y_{A^+, j, z^*} Z_{A^+, j, z^*}^{-1}$, where the symbol $\equiv$ means the graphical equality, $Y_{A^+, j, z^*}$ is graphically equal to a power of $C_{A^+, j, z^*}$, where $C_{A^+, j, z^*}$ is simple in rank $i - 1$ or a period of rank $\leq i - 1$. We can also assume that if $D_1 \in \{A, B_{A^+, j, z^*}, C_{A^+, j, z^*}\}$ is conjugate in rank $i - 1$ to $D_2^{\pm 1}$, where $D_2 \in \{A, B_{A^+, j, z^*}, C_{A^+, j, z^*}\}$, then $D_1 \equiv D_2$. Finally, the word $Z_{A^+, j, z^*}$ is picked for fixed $X_{A^+, j, z^*}, Y_{A^+, j, z^*}$ so that the length $|Z_{A^+, j, z^*}|$ is minimal (and the pair $X_{A^+, j, z^*}, Z_{A^+, j, z^*} Y_{A^+, j, z^*} Z_{A^+, j, z^*}^{-1}$ belongs to $\mathcal{C}_{A^+, \ell}(z^*)$). Similar to \cite{15, 10, 17}, the triple $(X_{A^+, j, z^*}, Y_{A^+, j, z^*}, Z_{A^+, j, z^*})$ is called an $(A^+, j, z^*)$-triple corresponding to the class $\mathcal{C}_{A^+, \ell}(z^*)$ (in rank $i - 1$).

Now for every class $\mathcal{C}_{A^+, \ell}(z^*)$ we pick a corresponding $(A^+, j, z^*)$-triple

$$(X_{A^+, j, z^*}, Y_{A^+, j, z^*}, Z_{A^+, j, z^*})$$

in rank $i - 1$ and construct a defining word $R_{A^+, j, z^*}$ of rank $i$ as follows. Pick a word $W_{A^+, j, z^*}$ of minimal length so that

$$v_z(X_{A^+, j, z^*}, Y_{A^+, j, z^*}) \equiv W_{A^+, j, z^*} A^\prime W_{A^+, j, z^*}^{-1}.$$ 

Let $T_{A^+, j, z^*}$, $U_{A^+, j, z^*}$ be words of minimal length such that

$$T_{A^+, j, z^*} \equiv W_{A^+, j, z^*}^{-1} v_z(X_{A^+, j, z^*}, Y_{A^+, j, z^*}) W_{A^+, j, z^*}$$

$$U_{A^+, j, z^*} \equiv W_{A^+, j, z^*}^{-1} Y_{A^+, j, z^*} W_{A^+, j, z^*}.$$
Lemma 1. The presentation \( \varepsilon \) of Lemmas 30.3, 30.4, 30.5 \([16]\) (in rank \( i \))

Proof. This proof is quite similar to the proof of Lemma 29.4 \([16]\). Inductive references to Lemma 29.3 \([16]\), we can conclude that the defining relators \( R, R' \) correspond to the same value of \( \varepsilon \). Therefore, Lemma 12 enables us to finish the proof of the analog of Lemma 29.3 as in \([16]\) (note that we need Lemma 12 only when \( \varepsilon = 2 \)).
Lemma 2. Suppose that $X_1$, $X_2$ are some words and $k \neq 0$ is an integer. Then
(a) If $[X_1^k, X_2^k] \neq 1$, then $[X_1, X_2] \neq 1$;
(b) If $X_1^k \neq X_2^k$, then $X_1 \neq X_2$. In particular, the group $G(i)$, defined by \(\mathcal{L}\), is torsion-free.

Proof. (a) It follows from definitions and Lemma 25.2 \[16\] that the group $G(i)$, defined by \(\mathcal{L}\), is torsion-free. Hence, we can apply Lemma 25.12 \[16\] to equality $[X_1^k, X_2^k] \neq 1$ and obtain that $[X_1, X_2] \neq 1$, as required.

(b) The equality $X_1^k = X_2^k$ implies that $[X_1^k, X_2^k] = 1$ and, by part (a), we have $[X_1, X_2] = 1$. Hence, $(X_1X_2^{-1})^k = 1$ and, as above, it follows from definitions and Lemma 25.2 \[16\] that $X_1X_2^{-1} = 1$. Lemma 2 is proved. \(\square\)

Recall that a subgroup $H$ of a group $G$ is called antinormal if for every $g \in G$ the inequality $gHg^{-1} \cap H \neq \{1\}$ implies that $g \in H$.

Lemma 3. (a) Every word is conjugate in rank $i$ to a power of either a simple in rank $i$ word or a period of rank $\leq i$.

(b) Suppose that each of $A$, $B$ is either a simple in rank $i$ word or a period of rank $\leq i$ and $A^k$ is conjugate in rank $i$ to $B^\ell$, $\ell \neq 0$. Then $A$ is conjugate in rank $i$ to $B$ or to $B^{-1}$.

(c) Let $A$ be a simple in rank $i$ word or a period of rank $\leq i$. Then the cyclic subgroup $\langle A \rangle$, generated by $A$, of the group $G(i)$, defined by \(\mathcal{L}\), is antinormal.

Proof. Part (a) follows from definitions (see also Lemma 18.1 \[16\]). Since the group $G(i)$ is torsion-free by Lemma 2 we can argue as in the proof of Lemma 25.17 \[16\] (see also the proof of Theorem 19.4 \[16\]) to prove part (b) and obtain that an equality of the form $ZA^kZ^{-1} = A^\ell$, $\ell \neq 0$, implies that $Z \in \langle A \rangle$, as required in part (c). Lemma 3 is proved. \(\square\)

In addition to $A$, $B$, . . ., $H$-maps which are introduced and investigated in \[16\], we will need $I$-maps (cf. \[17\]) defined as follows. A $B$-map $\Delta$ is called an $I$-map if following properties (I1)–(I5) hold.

(I1) $\Delta$ is a map on a sphere punctured at least once and at most thrice.

(I2) The cyclic sections of the boundary $\partial \Delta$ of $\Delta$ are products of sections (some of which or all can be cyclic) of two types: long sections and short sections.

(I3) If $s$ is a long section of $\partial \Delta$ then $s$ is smooth of rank $r(s)$ and $|s| > 10 \zeta^{-r(s)}$.

(I4) If $t$ is a short section of $\partial \Delta$ then $|t| < \zeta|s_0|$, where $s_0$ is a long section of minimal length.

(I5) If $L(\partial \Delta)$ and $S(\partial \Delta)$ are the numbers of long and short sections of $\partial \Delta$, respectively, then $S(\partial \Delta) \leq 2L(\partial \Delta)$ and $1 \leq L(\partial \Delta) \leq 10$.

Lemma 4. Suppose that $\Delta$ is an $I$-map. Then there is a system of pairwise disjoint regular contiguity submaps of long sections in $\Delta$ such that no two distinct contiguity submaps of a long section $s_1$ to a long section $s_2$ are contained in any larger contiguity submap of $s_1$ to $s_2$ and the sum of contiguity arcs of contiguity submaps of the system is greater than $(1 - \alpha^{1/4})L_s$, where $L_s$ is the sum of lengths of all long sections of $\partial \Delta$.

Proof. Without loss of generality, we can assume that every short section $t$ of $\partial \Delta$ is geodesic in $\Delta$ (and that if $t$ is cyclic then $t$ is cyclically geodesic; note that we can always replace $t$ by a homotopic to $t$ in $\Delta$ geodesic path).
This proof is analogous to the proof of Lemma 24.6 [16] (see also Lemmas 23.15, 24.2 [16] on C- and D-maps). Repeating arguments of the proof of Lemma 24.6 [16], we can establish similar estimates for an I-map \( \Delta \). Note that we need to make straightforward corrections of the number of distinguished contiguity submaps between \( p \) and \( q \), where \( p, q \) are sections of \( \partial \Delta \) or \( \partial \Pi \) and \( \Pi \) is a 2-cell of \( \Delta \), and of the number of distinguished contiguity submaps between sections of \( \partial \Delta \). As in the proof of Lemma 24.6 [16], we obtain the estimate \( M < \alpha \nu(\Delta) \), where \( M \) is the sum of weights of all inner edges of \( \Delta \) and \( \nu(\Delta) \) is the total weight of \( \Delta \).

Now we can argue as in the proof of Lemma 23.15 [16] to derive that the sum of lengths of outer arcs of long sections of \( \partial \Delta \) is greater than \( (1 - \alpha^{1/4})L_\alpha \). \( \square \)

**Lemma 5.** Suppose that each of \( A_1, A_2, B \) is a simple in rank \( i \) word or a period of rank \( \leq i \), \( [A_1^{k_1}, Z A_2^{k_2} Z^{-1}] \neq 1 \) for some word \( Z \), and \( B^\ell \) is conjugate in rank \( i \) either to \([A_1^{d_{k_1}}, Z A_2^{d_{k_2}} Z^{-1}] \) or to \([A_1^{d_{k_1}}] Z A_2^{d_{k_2}} Z^{-1} \). Then \( 0 < \ell \leq \alpha(\xi^{-1})^{-1} \) and either 

\[
\max(\{ A_1^{1/d_{k_1}}, A_2^{1/d_{k_2}} \}) \leq \xi^{-1} B^\ell \text{ if } B^\ell \text{ is conjugate in rank } i \text{ to } [A_1^{d_{k_1}}, Z A_2^{d_{k_2}} Z^{-1}] \text{ or }
\max(\{ A_1^{1/d_{k_1}}, A_2^{1/d_{k_2}} \}) \leq \xi^{-2} B^\ell \text{ if } B^\ell \text{ is conjugate in rank } i \text{ to } [A_1^{d_{k_1}}] Z A_2^{d_{k_2}} Z^{-1}.
\]

**Proof.** Without loss of generality (see also Lemma 5), we can assume that if \( B_1 \in \{ A_1^{\pm 1}, A_2^{\pm 1}, B^{\pm 1} \} \) is conjugate in rank \( i \) to \( B_2 \in \{ A_1^{\pm 1}, A_2^{\pm 1}, B^{\pm 1} \} \), then \( B_1 = B_2 \).

If \( \ell = 0 \), that is, either \( [A_1^{d_{k_1}}, Z A_2^{d_{k_2}} Z^{-1}] \equiv 1 \) or \( [A_1^{d_{k_1}}] Z A_2^{d_{k_2}} Z^{-1} \equiv 1 \) then, by Lemma 2 we have that either \( [A_1^{k_1}, Z A_2^{k_2} Z^{-1}] \equiv 1 \) or \( A_1^{k_1} Z A_2^{k_2} Z^{-1} \equiv 1 \), contrary to lemma’s hypothesis \( [A_1^{k_1}, Z A_2^{k_2} Z^{-1}] \neq 1 \). Hence, \( \ell \neq 0 \).

First assume that

\[
[A_1^{d_{k_1}}, Z A_2^{d_{k_2}} Z^{-1}] \equiv W_B B^\ell W_B^{-1}
\]

for some word \( W_B \). Then there is a reduced diagram \( \Delta_1 \) of rank \( i \) on a thrice punctured sphere the labels of 3 cyclic sections of whose boundary \( \partial \Delta_1 \) are \( A_1^{d_{k_1}}, A_1^{-d_{k_1}}, B^\ell \) (\( \Delta_1 \) can be constructed from a simply connected diagram \( \Delta_0 \) of rank \( i \) for equality (8) by identifying sections of \( \partial \Delta_0 \) labelled by \( Z A_2^{d_{k_2}} Z^{-1} \) and \( Z A_2^{-d_{k_2}} Z^{-1} \), \( W_B \) and \( W_B^{-1} \)). If \( \ell > \alpha(\xi^{-1})^{-1} \), then \( \Delta_1 \) is a G-map (see [16] Sect. 24.2) and, as in the proof of 25.19 [16], it follows from Lemma 24.8 [16] that \( B^\ell \equiv 1 \), contrary to Lemma 2 and \( \ell \neq 0 \). Hence, \( \ell \leq \alpha(\xi^{-1})^{-1} \), as desired.

Suppose that \( |B^\ell| < \xi|A_1^{d_{k_1}}| \). Then \( \Delta_1 \) is an E-map (see [16] Sect. 24.2) and it follows from Lemmas 24.6, 25.10 [16] that cyclic sections of \( \partial \Delta_1 \), labelled by \( A_1^{d_{k_1}}, A_1^{-d_{k_1}} \), are \( A_1 \)-compatible. Now it is easy to see that \( B^\ell \equiv 1 \), whence \( \ell = 0 \) by Lemma 2. This contradiction to \( \ell \neq 0 \) shows that \( |A_1^{d_{k_1}}| \leq \xi^{-1}|B^\ell| \), as desired. Analogously, \( |A_2^{d_{k_2}}| \leq \xi^{-1}|B^\ell| \) (using equality (8), we can construct a similar diagram \( \Delta_2 \) the labels of 3 cyclic sections of whose boundary \( \partial \Delta_2 \) are \( A_2^{d_{k_2}}, A_2^{-d_{k_2}}, B^\ell \), and then argue as before).

Now assume that

\[
A_1^{d_{k_1}} Z A_2^{d_{k_2}} Z^{-1} \equiv W_B B^\ell W_B^{-1}
\]

for some word \( W_B \). Then there is a reduced diagram \( \Delta \) of rank \( i \) on a thrice punctured sphere the labels of 3 cyclic sections of whose boundary \( \partial \Delta \) are \( A_1^{d_{k_1}}, A_2^{d_{k_2}}, B^{-\ell} \) (\( \Delta \) can be constructed from a simply connected diagram \( \Delta_0 \) of rank \( i \) for equality (9) by identifying the sections of \( \partial \Delta_0 \) labelled by \( Z \) and \( Z^{-1} \), \( W_B \) and \( W_B^{-1} \)). If \( |\ell| > \alpha(\xi^{-1})^{-1} \), then \( \Delta \) is a G-map and, using Lemmas 24.8, 25.10 [16], we can conclude that one of cyclic sections of \( \partial \Delta \) is compatible with another cyclic section.
Proof. If \( k_D = 0 \), that is, \( X^d Y^d \equiv 1 \), then, by Lemma 19, \( XY \equiv 1 \) and, therefore, \( [X, Y] \equiv 1 \), contrary to inequality (12). Hence \( k_D \) \( \neq 0 \).
In view of inequality 12, we can apply Lemma 3 to the pair \( X \equiv B^{k_E}, Y \equiv ZC^{k_c}Z^{-1} \) which yields that \(|k_D| \leq 100\zeta^{-1}\) and \(|B^{dk_B}|, |C^{dk_c}| \leq \zeta^{-2}|D^{k_D}|\). Inequalities 13 are proved.

In view of equality \( B^{dk_B}ZC^{dk_c}Z^{-1} \equiv W_D^{k_D}W^{-1}_D \), there is a reduced diagram \( \Delta \) of rank \( i \) on a thrice punctured sphere the labels of three cyclic sections of whose boundary \( \partial \Delta \) are \( B^{dk_B}, C^{dk_c}, D^{-k_D} \). It follows from Lemmas 22.2, 24.9 10 that

\[
|Z| < (1 + 4\gamma)(|B^{dk_B}| + |C^{dk_c}| + |D^{k_D}|).
\]

In view of inequalities 13, we have

\[
|Z| < (1 + 4\gamma)(2\zeta^{-2} + 1)|D^{k_D}| < 3\zeta^{-2}|D^{k_D}|
\]
as claimed in 14.

By estimates 10 and already proven inequality \( |Z| < 3\zeta^{-2}|D^{k_D}| \), we have

\[
|X^dY^d| = |B^{dk_B}| + |C^{dk_c}| + 2|Z| < 8\zeta^{-2}|D^{k_D}|
\]

Hence, it follows from Lemmas 1 and 22.1 10 that

\[
|W_D| < (\gamma + \frac{1}{2})(|X^dY^d| + |D^{k_D}|) < 5\zeta^{-2}|D^{k_D}|
\]

and inequalities 14 are proved.

Now assume that \( k_E = 0 \). Then \( (X^dY^d)^dX^d \equiv 1 \) which implies that

\[
[(X^dY^d)^d, X^d] \equiv 1.
\]

Hence, by Lemma 2 we have \([X, Y] \equiv 1\), contrary to inequality 12.

Consider the equality

\[
W_D^{k_D}W^{-1}_D B^{dk_B} \equiv W_E^{k_E} W^{-1}_E.
\]

In view of 15,

\[
|B|, |C| \leq d^{-1}\zeta^{-2}|D^{k_D}| \leq d^{-1}\zeta^{-2} \cdot 100\zeta^{-1}|D|
\]

and so \(|B|, |C| < |D|\) (LPP: \( \zeta > \eta = d^{-1} \)). Hence, \([W_D^{k_D}W^{-1}_D, B^{dk_B}] \equiv 0 \) (otherwise, we would have a contradiction to Lemma 3). This last inequality enables us to apply Lemma 4 to equality 11 and conclude that \(|k_E| \leq 100\zeta^{-1}, |D^{k_D}| \leq \zeta^{-2}|E^{k_E}|\). Inequalities 16 are proved.

By 15, 14, 16, we obtain

\[
|W_D^{k_D}W^{-1}_D B^{dk_B}| < (1 + 11\zeta^{-2}d^{-1})|D^{dk_B}| < 2\zeta^{-2}|E^{k_E}|
\]

(LPP: \( \zeta > \eta = d^{-1} \)). Therefore, it follows from Lemmas 1 and 22.1 10 that

\[
|W_E| < (\gamma + \frac{1}{2})(|W_D^{k_D}W^{-1}_DB^{dk_B}| + |E^{k_E}|) < 2\zeta^{-2}|E^{k_E}|
\]
as claimed in 16.

Next assume that \( k_F = 0 \). Then \(((X^dY^d)^dX^d)^d, X^d) \equiv 1 \). Hence, by Lemma 2 we obtain \([X, Y] \equiv 1\), which contradicts inequality 12.

Now consider the equality \([W_E^{k_E} W^{-1}_E, B^{dk_B}] \equiv W_F^{k_F} W^{-1}_F \). By Lemma 5 \(|k_F| \leq 100\zeta^{-1}, |E^{k_E}| \leq \zeta^{-1}|F^{k_F}|\), and estimates 17 are proved.

In view of equality \([W_E^{k_E} W^{-1}_E, B^{dk_B}] \equiv W_F^{k_F} W^{-1}_F \), it follows from Lemma 22.1 10 that

\[
|W_F| < (\gamma + \frac{1}{2}) \cdot 2(2|W_E| + |E^{k_E}| + |B^{dk_B}| + \frac{1}{2}|F^{k_F}|).
\]
Hence, by estimates (13), (15), (16), (17), we get

\[ |W^i_F| < (1 + 2\gamma)((2\zeta^{-2}d^{-1} + 1)|E^{dk_F}| + \zeta^{-4}d^{-2}|E^{dk_F}| + \frac{1}{2}|F^{dk_F}|) < \]
\[ < (1 + 2\gamma)((2\zeta^{-2}d^{-1} + 1) + \zeta^{-4}d^{-2})\zeta^{-1} + \frac{1}{2})|F^{dk_F}|) < 2\zeta^{-1}|F^{dk_F}| \]

\[ \text{LPP: } \gamma \succ \zeta \succ \eta = d^{-1}, \text{ as required. Lemma 6 is proved.} \]

**Lemma 7.** In the foregoing notation, the following inequalities hold

\[ 0 < |k_G| \leq 10\zeta^{-1}, \]
\[ |F^{dk_F}| \leq \zeta^{-1}|G^{k_G}|, \]
\[ |W^i_G| < \zeta^{-1}|G^{k_G}|. \]

\[ \text{(20)} \]
\[ \text{(21)} \]
\[ \text{(22)} \]

**Proof.** If \( k_G = 0 \) then \( |((Xd^{d^d})dXd^d, Xd^d)Y \| = 1 \) which implies that \( F^{dk_F} \)

is conjugate in rank \( i \) to \( C^{-kC} \). It follows from definitions and Lemma 6 that \( F \equiv C \).

However, it follows from Lemma 6 that

\[ |C| \leq \zeta^{-5}d^{-3}|F^{dk_F}| \leq 100\zeta^{-6}d^{-3}|F| < |F| \]

\[ \text{(23)} \]

(LPP: \( \zeta \succ \eta = d^{-1} \)). Hence \( k_G \neq 0 \).

By definitions, we have

\[ W^i_F F^{dk_F} W^{-1}_F ZC^{kC} Z^{-1} F = W^i_G G^{k_G} W^{-1}_G. \]

\[ \text{(24)} \]

By Lemma 6

\[ |W^{-1}_F ZC^{kC} Z^{-1} W^i_F| < (4\zeta^{-1} + 6\zeta^{-5}d^{-2} + \zeta^{-5}d^{-3})|F^{dk_F}| < \]
\[ < 5\zeta^{-1}d^{-1}|F^{dk_F}| < \zeta|F^{dk_F}| \]

\[ \text{(25)} \]

(LPP: \( \zeta \succ \eta = d^{-1} \)).

If \( |G^{k_G}| < \zeta|F^{dk_F}| \) then a reduced annular diagram of rank \( i \) for conjugacy of

words \( F^{dk_F} W^{-1}_F ZC^{kC} Z^{-1} W^i_F \) and \( G^{k_G} \) (see (24)) is an \( F \)-map (see Sect. 24.2

16) whose existence contradicts Lemma 4 and Lemmas 24.7, 25.10 16. Therefore,

\[ |G^{k_G}| \geq \zeta|F^{dk_F}| \text{ and (24) is proved.} \]

In view of equality (24), there is a reduced diagram \( \Delta \) of rank \( i \) on a thrice

punctured sphere the labels of whose cyclic sections are \( F^{dk_F}, C^{kC}, G^{-kG} \). It

follows from Lemma 6 that

\[ |C^{kC}| < \zeta^{-5}d^{-4}|F^{dk_F}| < \zeta^2|F^{dk_F}| \]

(LPP: \( \zeta \succ \eta = d^{-1} \)). Hence, by (25), \( |C^{kC}| < \zeta \min(|F^{dk_F}|, |G^{k_G}|) \).

If \( |k_G| > 10\zeta^{-1} \), then \( \Delta \) is an \( E \)-map (see Sect. 24.2 16) and we can argue as in the proof

of Lemma 25.19 16 to show that \( F \equiv G \) and the cyclic sections of \( \Delta \) labelled by

\( F^{dk_F}, G^{-kG} \) are \( F \)-compatible. Then \( C^{kC} \) is conjugate in rank \( i \) to a power of \( F \).

This, however, is impossible by Lemma 6 and inequality (24). Hence, \( |k_G| \leq 10\zeta^{-1} \)

and inequalities (20) are proved.

By Lemma 11 we can apply Lemma 22.1 16 to a reduced annular diagram of

rank \( i \) for conjugacy of words \( W^i_F F^{dk_F} W^{-1}_F ZC^{kC} Z^{-1} \) and \( G^{k_G} \) to obtain, using

estimates (21), (25), that

\[ |W^i_G| < (\gamma + \frac{1}{2})(\zeta + 1)(|F^{dk_F}| + |G^{k_G}|) \leq (\gamma + \frac{1}{2})(\zeta^{-1} + 1)|G^{k_G}| < \zeta^{-1}|G^{k_G}|, \]

as claimed in (22). Lemma 7 is proved. \( \square \)
Lemma 8. In the foregoing notation, the following inequalities hold

$$0 < |k_H| \leq 100\zeta^{-1}, \quad |G^{dkc}| \leq \zeta^{-1}|H^{kH}|,$$

(26)

$$|W_H| < 2\zeta^{-1}|H^{kH}|.$$

(27)

Proof. Assume that $$\left[\left((X^dY^d)X^dY^dY, X^d\right), X^d\right] \neq 1.$$ Then, by Lemma 2 we also have $$\left[\left((X^dY^d)X^dY^dY, X^d\right), X^d\right] \neq 1.$$ Since $$X = B^{kH},$$ it follows from Lemma 3 that $$\left[\left((X^dY^d)X^dY^dY, X^d\right), X^d\right] \neq 1$$ and so $$G^{kc}$$ is conjugate in rank $$i$$ to $$B^i.$$ By Lemmas 6 and 7

$$|B^{kH}| \leq \zeta^{-5}d^{-3}|F^{kF}| \leq \zeta^{-6}d^{-4}|G^{kc}| \leq 10\zeta^{-7}d^{-4}|G| < |G|$$

(LPP: $$\zeta \succ \eta = d^{-1}),$$ whence $$|B| < |G|.$$ This, however, contradicts Lemma 9.

Hence, $$\left[\left((X^dY^d)X^dY^dY, X^d\right), X^d\right] = 1$$ and so $$k_H \neq 0.$$

By definitions, $$[W_GG^{dkc}G_W^{-1}, B^{dkb}] = W_HH^{kH}W_H^{-1}.$$ By Lemma 5 $$|k_H| \leq 100\zeta^{-1},$$ $$|G^{kc}| \leq \zeta^{-1}|H^{kH}|$$ and estimates (20) are proved.

As in proofs of Lemmas 6, 7 we have from Lemmas 11 and 22.1 that

$$|W_H| < (\gamma + \frac{1}{2})\cdot 2(2|W_G| + |G^{dkc}| + |B^{dkb}| + \frac{1}{2}|H^{kH}|).$$

Hence, by Lemmas 6, 7 and estimates (26),

$$|W_H| < (1 + 2\gamma)((2\zeta^{-1}d^{-1} + 1)|G^{dkc}| + \zeta^{-6}d^{-4}|G^{dkc}| + \frac{1}{2}|H^{kH}|) <

< (1 + 2\gamma)((2\zeta^{-1}d^{-1} + 1 + \zeta^{-6}d^{-4})\zeta^{-1} + \frac{1}{2})|H^{kH}| < 2\zeta^{-1}|H^{kH}|$$

(LPP: $$\gamma \succ \zeta \succ \eta = d^{-1}),$$ as required in (27). Lemma 8 is proved. \(\square\)

Lemma 9. Let $$R_{A^{i,j,z^*}}$$ be a defining word of rank $$i + 1$$ defined by (4) if $$z^* = 1$$ or by (3) if $$z^* = 2.$$ Then $$0 < |f| \leq 100\zeta^{-1},$$ $$|A| > d,$$ the words $$T_{A^{i,j,z^*}}, U_{A^{i,j,z^*}}$$ do not belong to the cyclic subgroup $$\langle A \rangle,$$ generated by $$A,$$ of $$G(i)$$ and

$$\max(T_{A^{i,j,z^*}}, |U_{A^{i,j,z^*}}|) < d|A|.$$

Proof. First we let $$z^* = 1.$$ It follows from definitions that, in the foregoing notation, we can assume that

$$A \equiv G, \quad T_{A^{i,j,1}} \equiv W_G^{-1}B^{kH}W_G, \quad U_{A^{i,j,1}} \equiv W_G^{-1}ZC^{kc}Z^{-1}W_G,$$

and $$f = f(A^{i,j,1})$$ is $$k_G.$$ Hence, in view of Lemmas 5, 7

$$0 < |f| \leq 100\zeta^{-1},$$

$$|A| \geq 10^{-1}\zeta^2|F^{dkF}| \geq 10^{-1}\zeta^7d^3|B^{dkb}| \geq 10^{-1}\zeta^7d^3 > d$$

(LPP: $$\zeta \succ \eta = d^{-1}),$$ and

$$|T_{A^{i,j,1}}, U_{A^{i,j,1}}| \leq 2(|W_G| + |Z|) + |B^{kH}| + |C^{kc}| <

< (2(\zeta^{-1} + 3\zeta^{-6}d^{-3}) + 2\zeta^{-6}d^{-4})|G^{kc}| < 3\zeta^{-1}|G^{kc}| \leq 30\zeta^{-2}|G| < d|A|$$

(LPP: $$\zeta \succ \eta = d^{-1}).$$

Assume that one of $$T_{A^{i,j,1}}, U_{A^{i,j,1}}$$ belongs to $$< G(i).$$ Then one of $$B^{kH},$$ $$C^{kc}$$ is conjugate in rank $$i$$ to a power of $$G.$$ However, by Lemmas 6, 7

$$|B^{kH}|, |C^{kc}| \leq \zeta^{-6}d^{-4}|G^{kc}| \leq 10\zeta^{-7}d^{-4}|G| < |G|$$

(LPP: $$\zeta \succ \eta = d^{-1}),$$ whence $$|B|, |C| < |G|$$ which is a contradiction to Lemma 9.
Now we let \( z^* = 2 \). It follows from definitions that, in the foregoing notation, we can assume that
\[
A \equiv H, \quad T_{A^f,j,2} \equiv W_H^{-1}W_G^{k_G}W_G^{-1}W_H, \quad U_{A^f,j,2} \equiv W_H^{-1}ZG^{k_G}Z^{-1}W_H,
\]
and \( f = f(A^f, j, 2) \) is \( k_H \). Hence, in view of Lemmas 6, 7, 8, we have
\[
0 < |f| \leq 100\zeta^{-1},
\]
\[
|A| \geq 10^{-2}\zeta^2|G^{k_G}| \geq 10^{-2}\zeta^2d|F^{dk_f}| \geq 10^{-2}\zeta^2d^2 > d
\]
(LPP: \( \zeta \succ \eta = d^{-1} \)) and
\[
|T_{A^f,j,2}|, |U_{A^f,j,2}| \leq 2(|W_H| + |W_G| + |Z|) + |G^{k_G}| + |C^{k_C}| < \\
< (2(2\zeta^{-1} + \zeta^{-2}d^{-1} + 3\zeta^{-7}d^{-4}) + \zeta^{-1}d^{-1} + \zeta^{-7}d^{-5})|H^{k_H}| < \\
< 5\zeta^{-1}|H^{k_H}| \leq 500\zeta^{-2}|H| < d|A|
\]
(LPP: \( \zeta \succ \eta = d^{-1} \)), as required.

Assume that one of \( T_{A^f,j,2}, U_{A^f,j,2} \) belongs to \( \langle H \rangle \subseteq G(i) \). Then one of \( G^{k_G}, C^{k_C} \) is conjugate in rank \( i \) to a power of \( H \). However, we saw above that \( |B|, |C| < |G| \) and it follows from Lemma 9 that
\[
|G| \leq \zeta^{-1}d^{-1} \cdot 100\zeta^{-1}|H| < |H|
\]
(LPP: \( \zeta \succ \eta = d^{-1} \)). Therefore, \( |G|, |C| < |H| \) and, as before, we have a contradiction to Lemma 3, Lemma 9 is proved.

**Lemma 10.** Suppose that \((X_1, Y_1)\) and \((X_2, Y_2)\) are two pairs of words such that
\[
(X_1^dY_1^d)^dX_1^d
\]
is conjugate in rank \( i \) to \((X_2^dY_2^d)^dX_2^d\) and \(|X_1, Y_1| \neq 1, |X_2, Y_2| \neq 1\). Then the pairs \((X_1, Y_1)\) and \((X_2, Y_2)\) are conjugate in rank \( i \).

**Proof.** Without loss of generality we can assume that
\[
X_1 \equiv B_1^{k_1}, \quad X_2 \equiv B_2^{k_2}, \quad Y_1 \equiv Z_1C_1^{k_C}, Z_1^{-1}, \quad Y_2 \equiv Z_2C_2^{k_C}, Z_2^{-1},
\]
\[
X_1^dY_1^d \equiv W_D, X_1^{dk_1}W_D^{-1}, \quad X_2^dY_2^d \equiv X_2^{dk_2}X_2^{-1},
\]
where \( B_1, B_2, C_1, C_2, D_1, D_2 \) are words simple in rank \( i \) or periods of rank \( \leq i \), the words \( Z_1, Z_2, W_D, W_{D_2} \) have the minimal lengths among all words satisfying the corresponding equalities, and if \( A_1 \in \{ B_1, B_2, C_1, C_2, D_1, D_2 \} \) is conjugate in rank \( i \) to \( A_2^{\pm 1} \), where \( A_2 \in \{ B_1, B_2, C_1, C_2, D_1, D_2 \} \), then \( A_1 \equiv A_2 \).

Consider a reduced annular diagram \( \Delta \) of rank \( i \) for conjugacy of the words
\[
W_D, D_1^{dk_1}D_1^{dk_2}W_D^{-1}, W_D, D_2^{dk_2}W_D^{-1}D_2^{dk_2}.\]
Denote two cyclic sections of the boundary \( \partial \Delta \) of \( \Delta \) by \( p_1q_1 \) and \((p_2q_2)^{-1}\), where
\[
\varphi(p_1) \equiv D_1^{dk_1}, \quad \varphi(q_1) \equiv W_D^{-1}B_1^{dk_1}W_D, \quad \varphi(p_2) \equiv D_2^{dk_2}, \quad \varphi(q_2) \equiv W_D^{-1}B_2^{dk_2}W_D.
\]
It follows from Lemma 9 that
\[
|W_D^{-1}B_1^{dk_1}W_D| < 11\zeta^{-2}|D_1^{dk_1}| \leq 11\zeta^{-2}d^{-1}|D_1^{dk_1}| < \zeta^3|D_1^{dk_1}|
\]
(LPP: \( \zeta \succ \eta = d^{-1} \)). Analogously, \( |W_D^{-1}B_2^{dk_2}W_D| < \zeta^3|D_2^{dk_2}| \). Hence,
\[
|q_1| < \zeta^3|p_1|, \quad |q_2| < \zeta^3|p_2|.
\]
(28)
Assume that $|q_1| \geq \zeta |p_2|$. Then it follows from (28) that

$$|p_2q_2| < (1 + \zeta^3)|p_2| \leq \zeta^{-1}(1 + \zeta^3)|q_1| < \zeta^2(1 + \zeta^3)|p_1| < \zeta |p_1|.$$  

Hence, $|p_2q_2| < \zeta |p_1|$ and, in view of (28), $\Delta$ is an $F$-map whose existence contradicts Lemma 1 and Lemmas 24.7, 25.10 [16].

Therefore, we can assume that $|q_1| < \zeta |p_2|$ and, similarly, $|q_2| < \zeta |p_1|$. Now we can see that $\Delta$ is an $I$-map. It follows from Lemmas 4 and 25.10 [16] that $D_1 \equiv D_2 \equiv D$ and the sections $p_1, p_2$ are $D$-compatible. By Lemma 6, $|k_{D_1}| \leq 100 \zeta^{-1}$. Hence, using estimate (28), we get

$$|q_1| < \zeta^3 |p_1| = \zeta^3 |D_1^{dk_{D_1}}| \leq 100 \zeta^2 |D_1^d| < \zeta |D_1^d| = \zeta |D^d|.$$  

Analogously, $|q_2| < \zeta |D^d|$. If $k_{D_1} \neq k_{D_2}$, then cutting $\Delta$ along a simple path, that makes $p_1$ and $p_2$ $D$-compatible, we could turn $\Delta$ into an $F$-map whose existence contradicts Lemma 1 and Lemmas 24.7, 25.10 [16]. Hence, it is shown that $k_{D_1} = k_{D_2} = k$. Cutting $\Delta$ along a simple path, that makes $p_1$ and $p_2$ $D$-compatible, we can see that

$$W_{D_2}D^{\ell_D}W_{D_1}^{\ell_{D_1}}D_1^{dk_{D_1}}W_{D_1}D^{-\ell_D}W_{D_2}^{-1} \equiv B_2^{dk_{D_2}}$$

for some integer $\ell_D$. By Lemma 8, $B_1$ and $(B_2)^{\pm 1}$ are conjugate in rank $i$. Hence, $B_1 \equiv B_2 \equiv B$. It also follows from Lemma 8 that $k_{B_1} = k_{B_2}$ and $W_{D_2}D^{\ell_D}W_{D_1}^{-1} \equiv B_{\ell_B}$ for some integer $\ell_B$. This implies that $W_{D_1} \equiv B^{-\ell_B}W_{D_2}D^{\ell_D}$ and so

$$X_1^{\ell_D}X_1^d \equiv W_{D_1}D^{\ell_D}W_{D_1}^{-1} \equiv B^{-\ell_B}W_{D_2}D^{\ell_D}D^{\ell_D}D^{-\ell_D}W_{D_2}^{-1}B_{\ell_B} \equiv$$

$$\equiv B^{-\ell_B}X_2^{\ell_D}X_2^dB_{\ell_B} \equiv X_2^{\ell_D}B^{-\ell_B}X_2^dB_{\ell_B}.$$  

Since $X_1 \equiv X_2 \equiv B^{k_{D_1}}$, we obtain that $Y_1^i \equiv B^{-\ell_B}Y_2^dB_{\ell_B}$. By Lemma 2, $Y_1^i \equiv B^{-\ell_B}Y_2^dB_{\ell_B}$ and we see that the pair $(X_2, Y_2)$ is conjugate in rank $i$ to $(X_1, Y_1)$ by $B_{\ell_B}$. Lemma 10 is proved.

**Lemma 11.** Suppose that $(X_1, Y_1)$ and $(X_2, Y_2)$ are two pairs of words such that $(X_1^iY_1^d)^iX_1^i$ is conjugate in rank $i$ to $((X_2^dY_2^d)^iX_2^d)^{-1}$ and $|X_1, Y_1|^i \neq 1$, $|X_2, Y_2|^i \neq 1$. Then the pairs $(X_1, Y_1)$ and $(X_2^{-1}, Y_2^{-1})$ are conjugate in rank $i$.

**Proof.** Observe that $((X_2^dY_2^d)^iX_2^d)^{-1} \equiv ((X_2^{-1})^d(Y_2^{-1})^d)^i(X_2^{-1})^d$. Therefore, if $(X_1^iY_1^d)^iX_1^d$ is conjugate in rank $i$ to $((X_2^dY_2^d)^iX_2^d)^{-1}$, then $(X_1^iY_1^d)^iX_1^d$ is conjugate in rank $i$ to $((X_2^{-1})^d(Y_2^{-1})^d)^i(X_2^{-1})^d$. Hence, our claim follows from Lemma 10.

**Lemma 12.** Suppose that $(X_1, Y_1)$ and $(X_2, Y_2)$ are two pairs of words such that $v_1(X_1, Y_1)$ is conjugate in rank $i$ to $v_1(X_2, Y_2)$ and $|X_1, Y_1|^i \neq 1$, $|X_2, Y_2|^i \neq 1$. Then the pairs $(X_1, Y_1)$ and $(X_2, Y_2)$ are conjugate in rank $i$.

**Proof.** Conjugating the pairs $(X_1, Y_1)$, $(X_2, Y_2)$ in rank $i$ if necessary, we can assume that

$$X_1 \equiv D_1^{k_{B_1}}, \ Y_1 \equiv Z_1C_1^{k_{C_1}}Z_1^{-1}, \ X_2 \equiv D_2^{k_{B_2}}, \ Y_2 \equiv Z_2C_2^{k_{C_2}}Z_2^{-1},$$

where each of $B_1, B_2, C_1, C_2$ is either simple in rank $i$ or a period of some rank $\leq i$ and, when $D_1^{k_{B_1}}$, $D_2^{k_{B_2}}$, $C_1^{k_{C_1}}$, $C_2^{k_{C_2}}$ are fixed, the words $Z_1, Z_2$ are picked to
have minimal length. Furthermore, consider the following equalities
\[
X_1^d Y_1^d \overset{i}{\equiv} W_{D_1} D_1^{k_{D_1}} W_{D_1}^{-1}, \quad X_2^d Y_2^d \overset{i}{\equiv} W_{D_2} D_2^{k_{D_2}} W_{D_2}^{-1},
\]
\[
(X_1^d Y_1^d)^d X_1^d \overset{i}{\equiv} W_{E_1} E_1^{k_{E_1}} W_{E_1}^{-1}, \quad (X_2^d Y_2^d)^d X_2^d \overset{i}{\equiv} W_{E_2} E_2^{k_{E_2}} W_{E_2}^{-1},
\]
\[
[((X_1^d Y_1^d)^d X_1^d)^d X_1^d] \overset{i}{\equiv} W_{F_1} F_1^{k_{F_1}} W_{F_1}^{-1}, \quad [((X_2^d Y_2^d)^d X_2^d)^d X_2^d] \overset{i}{\equiv} W_{F_2} F_2^{k_{F_2}} W_{F_2}^{-1},
\]
where each of \(D_1, E_1, F_1, D_2, E_2, F_2\) is either a simple in rank \(i\) word or a period of some rank \(\leq i\) and the conjugating words \(W_{D_1}, W_{E_1}, W_{F_1}, W_{D_2}, W_{E_2}, W_{F_2}\) are picked (when \(D_1, E_1, F_1, D_2, E_2, F_2\) are fixed) to have minimal lengths.

We can also assume that if \(A_1 \in \{B_1, C_1, D_1, E_1, F_1, B_2, C_2, D_2, E_2, F_2\}\) is conjugate in rank \(i\) to \(A_2^\pm 1\), where \(A_2 \in \{B_1, C_1, D_1, E_1, F_1, B_2, C_2, D_2, E_2, F_2\}\), then \(A_1 \equiv A_2\). Consider a reduced annular diagram \(\Delta\) of rank \(i\) for conjugacy of the words \(W_{F_1} F_1^{dk_{F_1}} W_{F_2}^{-1} Z_1 C_1^{k_{C_1}} Z_1^{-1}\) and \(W_{F_2} F_2^{dk_{F_2}} W_{F_2}^{-1} Z_2 C_2^{k_{C_2}} Z_2^{-1}\). Denote two cyclic sections of the boundary \(\partial \Delta\) of \(\Delta\) by \(p_1\) and \(p_2\), where
\[
\varphi(p_1) \equiv F_1^{dk_{F_1}}, \quad \varphi(q_1) \equiv W_{F_1}^{-1} Z_1 C_1^{k_{C_1}} Z_1^{-1} W_{F_1},
\]
\[
\varphi(p_2) \equiv F_2^{dk_{F_2}}, \quad \varphi(q_2) \equiv W_{F_2}^{-1} Z_2 C_2^{k_{C_2}} Z_2^{-1} W_{F_2}.
\]
It follows from Lemma 6 that
\[
|W_{F_1}^{-1} Z_1 C_1^{k_{C_1}} Z_1^{-1} W_{F_1}| < (4\zeta^{-1} + 7\zeta^{-5}d^{-2})|F_1^{dk_{F_1}}| \leq (4\zeta^{-1} + 7\zeta^{-5}d^{-2})d^{-1}|F_1^{dk_{F_1}}| < \zeta^3|F_1^{dk_{F_1}}|
\]
(LPP: \(\zeta \succ \eta = d^{-1}\)). Analogously, \(|W_{F_2}^{-1} Z_2 C_2^{k_{C_2}} Z_2^{-1} W_{F_2}| < \zeta^3|F_2^{dk_{F_2}}|\). Hence,
\[
|q_1| < \zeta^3|p_1|, \quad |q_2| < \zeta^3|p_2|.
\]
(29)

Assume that \(|q_1| \geq \zeta|p_2|\). Then it follows from (29) that
\[
|p_2q_2| < (1 + \zeta^3)|p_2| \leq \zeta^{-1}(1 + \zeta^3)|q_1| < \zeta^2(1 + \zeta^3)|p_1| \leq \zeta|p_1|.
\]
Hence, \(|p_2q_2| < \zeta|p_1|\) and, in view of (29), \(\Delta\) is an \(F\)-map whose existence contradicts Lemma 1 and Lemmas 24.7, 25.10 10.

Therefore, we can assume that \(|q_1| < \zeta|p_2|\) and, similarly, \(|q_2| < \zeta|p_1|\). Now we can see that \(\Delta\) is an \(I\)-map. By Lemmas 3 and 25.10 10, \(F_1 \equiv F_2 \equiv F\) and sections \(p_1\) and \(p_2\) are \(F\)-compatible. By Lemma 6 \(|k_{F_1}| \leq 100\zeta^{-1}\). Hence, using estimate (29), we have
\[
|q_1| < \zeta^3|p_1| = \zeta^3|F_1^{dk_{F_1}}| \leq 100\zeta^2|F_1^d| < \zeta|F_1^d| = \zeta|F^d|.
\]
Analogously, \(|q_2| < \zeta|F^d|\).

If \(k_{F_1} \neq k_{F_2}\), then, cutting \(\Delta\) along a simple path, that makes \(p_1\) and \(p_2\) \(F\)-compatible, we could turn \(\Delta\) into an \(F\)-map whose existence contradicts Lemma 1 and Lemmas 24.7, 25.10 10. Hence, \(k_{F_1} = k_{F_2}\) and, cutting \(\Delta\) along a simple path, that makes \(p_1\) and \(p_2\) \(F\)-compatible, we can see that \(Y_1\) and \(Y_2\) are conjugate in rank \(i\).

Since \(k_{F_1} = k_{F_2}\), it follows from definitions that the word \([W_{E_1} E_1^{dk_{E_1}} W_{E_1}^{-1}, B_1^{dk_{B_1}}]\) is conjugate in rank \(i\) to \([W_{E_2} E_2^{dk_{E_2}} W_{E_2}^{-1}, B_2^{dk_{B_2}}]\). Let \(\Delta\) be a reduced annular
the boundary \( \partial \Delta \) of \( \Delta \) by \( p_1q_1p_2q_2 \) and \((p_3q_3p_4q_4)^{-1}\), where

\[
\varphi(p_1) \equiv \varphi(p_2)^{-1} \equiv E_1^{dk_{E_1}}, \quad \varphi(q_1) \equiv \varphi(q_2)^{-1} \equiv W_{E_1}^{-1}B_1^{-dk_{E_1}}W_{E_1} \\
\varphi(p_3) \equiv \varphi(p_4)^{-1} \equiv E_2^{dk_{E_2}}, \quad \varphi(q_3) \equiv \varphi(q_4)^{-1} \equiv W_{E_2}^{-1}B_2^{-dk_{E_2}}W_{E_2}.
\]

It follows from Lemma 6 that

\[
|W_{E_1}^{-1}B_1^{-dk_{E_1}}W_{E_1}| < (4 \zeta^{-2} + \zeta^{-4}d^{-1})|E_1^{dk_{E_1}}| \leq (4 \zeta^{-2} + \zeta^{-4}d^{-1}) d^{-1}|E_1^{dk_{E_1}}| < \zeta^3|E_1^{dk_{E_1}}|
\]

(LPP: \( \zeta > \eta = d^{-1} \)). Therefore,

\[
|q_1|, |q_2| < \zeta^3|p_1| \quad \text{and} \quad |q_1|, |q_2| < \zeta^3|p_2|.
\]  \tag{30}

Similarly,

\[
|q_3|, |q_4| < \zeta^3|p_3| \quad \text{and} \quad |q_3|, |q_4| < \zeta^3|p_4|.
\]  \tag{31}

Assume that \( |q_1| \geq \zeta|p_4| \). Then, by (30)–(31), we have

\[
|p_3q_3p_4q_4| < 2(1 + \zeta^3)|p_3| \leq 2\zeta^{-1}(1 + \zeta^3)|q_1| < 2\zeta^2(1 + \zeta^3)|p_1| < \zeta|p_1|.
\]

In view of estimates (30)–(31), we can turn \( \Delta \) into an \( E \)-map \( \Delta' \) by pasting together \( q_1 \) and \( q_2 \). It follows from Lemmas 24.6, 25.10 that the images of \( p_1 \) and \( p_2 \) in \( \Delta' \) are \( E_1 \)-compatible. This implies that \( k_{E_2} = 0 \), contrary to Lemma 6.

Therefore, we can assume that \( |q_1| < \zeta|p_3| \) and, similarly, \( |q_3| < \zeta|p_1| \). Now we can see that \( \Delta \) is an \( I \)-map. If \( p_1 \) and \( p_2 \) are \( E_1 \)-compatible in \( \Delta \), then \( k_{E_2} = 0 \), contrary to Lemma 6. Hence, \( p_1 \) and \( p_2 \) may not be \( E_1 \)-compatible in \( \Delta \). Similarly, \( p_3 \) and \( p_4 \) are not \( E_2 \)-compatible in \( \Delta \). Therefore, it follows from Lemmas 6 and 25.10 that \( E_1 \equiv E_2 \equiv E \) and either \( p_1 \) is \( E \)-compatible with \( p_3 \) and \( p_2 \) is \( E \)-compatible with \( p_4 \) or \( p_1 \) is \( E \)-compatible with \( p_4 \) and \( p_2 \) is \( E \)-compatible with \( p_3 \). Let \( t_1 \) and \( t_2 \) be simple disjoint paths in \( \Delta \) that make corresponding pairs of paths \( p_1, p_2, p_3^{-1}, p_4^{-1} \) \( E \)-compatible. Let us cut \( \Delta \) along \( t_1 \) and \( t_2 \) and then paste the two resulting diagrams along the images of \( q_3 \) and \( q_4 \) (recall that \( \varphi(q_3) \equiv \varphi(q_4)^{-1} \)). Let \( \Delta_0 \) denote the simply connected diagram of rank \( i \) thus obtained from \( \Delta \). Observe that

\[
\varphi(\partial \Delta_0) \equiv E_1^{dk_{E_1}}W_{E_1}^{-1}B_1^{-dk_{E_1}}W_{E_1}E_1^{dke_{E_1}}W_{E_1}^{-1}B_1^{-dk_{E_1}}W_{E_1} \equiv 1,
\]  \tag{32}

where either \( \ell_E = d(k_{E_1} - k_{E_2}) \) in the case when \( p_1 \) is \( E_1 \)-compatible in \( \Delta \) with \( p_3 \) and \( p_2 \) is \( E_1 \)-compatible with \( p_4 \) or \( \ell_E = d(k_{E_1} + k_{E_2}) \) in the case when \( p_1 \) is \( E_1 \)-compatible in \( \Delta \) with \( p_4^{-1} \) and \( p_2 \) is \( E_1 \)-compatible with \( p_3 \). If \( k_{E_1} \neq \pm k_{E_2} \) then it follows from Lemma 6 equality (32) and definitions that \( B_1 \equiv E_1 \). On the other hand, it follows from Lemma 6 that

\[
|B_1| \leq \zeta^{-4}d^{-2}|E_1^{dk_{E_1}}| < 100\zeta^{-5}d^{-2}|E_1| < |E_1|
\]

(LPP: \( \zeta > \eta = d^{-1} \)). This contradiction shows that \( k_{E_1} = \pm k_{E_2} \) and so \( E_1^{dk_{E_1}} \equiv E_2^{\pm k_{E_2}} \). By definitions, this means that the word \((X_t^iY_{t}^d)^iX_t^i\) is conjugate in rank \( i \) to \((X_t^iY_{t}^d)^dX_t^i \equiv 1\).

Assume that \((X_t^iY_{t}^d)^iX_t^i\) is conjugate in rank \( i \) to \((X_t^iY_{t}^d)^dX_t^i \equiv 1\). Then, by Lemma 11 \( Y_1 \) is conjugate in rank \( i \) to \( Y_2^{-1} \). On the other hand, as we saw above, \( Y_1 \) is conjugate in rank \( i \) to \( Y_2 \). Hence, \( Y_2 \) is conjugate in rank \( i \) to \( Y_2^{-1} \). This, however, by Lemmas 3 and 2 implies that \( Y_2 \equiv 1 \). This contradiction to \([X_1,Y_1] \neq 1 \) proves
that \((X_i^Y Y_i^i X_i^d)^d X_i^d\) is conjugate in rank \(i\) to \((X_i^Y Y_i^i X_i^d)^d X_i^d\). Now a reference to Lemma 11 completes the proof of Lemma 12. \(\square\)

4. PROOF OF THEOREM

Our Theorem is immediate from the following below Lemmas 13

**Lemma 13.** The group \(G(\infty)\), defined by presentation (4), is naturally isomorphic to the free group \(F(A)/W_{1,2}(F(A))\) of the variety \(\mathcal{V}\) in the alphabet \(A\).

**Proof.** It follows from the definition of defining words of the group \(G(\infty)\) that each of them is in \(W_{1,2}(F(A))\) and so there is a natural epimorphism

\[G(\infty) \to F(A)/W_{1,2}(F(A)).\]

Suppose that \(X, Y\) are some words in \(A^{\pm 1}\) and

\[w_{z^*}(X, Y) \neq 1\] (33)

in the group \(G(\infty)\), where \(z^* \in \{1, 2\}\).

Let \(A\) be a period of some rank such that \(A^f\) for some \(f\) is conjugate in \(G(\infty)\) to \(v_{z^*}(X, Y)\). (The existence of such an \(A\) follows from definitions and Lemma 8 see also Lemma 18.1 [16].) Note that, in view of (33), \([X, Y] \neq 1\) in \(G(\infty)\). Hence, by Lemmas 8, 10, we can replace the pair \((X, Y)\) by a conjugate in the group \(G(\infty)\) pair \((X', Y')\) such that \(X' \equiv B^k A, Y' \equiv Z C^k Z^{-1}\), and \(v_{z^*}(X, Y) = W_A A^f W_{A^{-1}}\) in \(G(\infty)\), where \(B, C\) are some periods, \(|f| > 0\) and

\[|X^d| + |Y^d| = |B| d^k + |C^d| + 2|Z| < 8\zeta^{-5} d^{-2}|F|^k \leq 8\zeta^{-6} d^{-3}|A| f.\]

Hence, the length \(|v_{z^*}(X, Y)|\), which is either \(||(X^d X^d) d X^d|, X^d| d X^d|\) if \(z^* = 1\) or \(||(X^d Y^d) d X^d|, X^d| d X^d|\) if \(z^* = 2\), can be estimated as follows

\[|v_{z^*}(X, Y)| \leq 2d(2d(|X^d| + |Y^d|) + |X^d| + 2|X^d| + |Y^d| + 2|X^d| <
\]

\[< 5d^4(|X^d| + |Y^d|) < 40d^4 |A| f < 10^4 |A| f < 10^4 |A|.\]

for \(0 < |f| \leq 100\zeta^{-1}\) by Lemmas 4, 5. Consider a reduced annular diagram \(\Delta\) of some rank \(i\) for conjugacy of \(v_{z^*}(X, Y)\) and \(A^f\). By Lemmas 1 and 22.1 [16], \(\Delta\) can be cut into a simply connected diagram \(\Delta_1\) along a simple path \(t\) which connects points on distinct components of \(\partial \Delta\) with \(|t| < \gamma|\partial \Delta|\). Therefore,

\[|\partial \Delta_1| < (1 + 2\gamma)|\partial \Delta| < (1 + 2\gamma)(10^4 d^4 \zeta^{-7} + 100 \zeta^{-1}) |A| < \frac{1}{2} n |A|\]

(LPP: \(\gamma \geq \zeta \geq \eta = d^{-1} \geq \iota = n^{-1}\)). Then, by Lemmas 11, 20.4 and 23.16 [16] applied to \(\Delta_1\), the diagram \(\Delta_1\) contains no 2-cells of rank \(\geq |A|\), whence \(\Delta_1, \Delta\) are diagrams of rank \(|A|^{-1}\). Since \(A \in X_{|A|}\), it follows from the construction of defining words of rank \(|A|\) that there will be a defining word in \(S_{|A|}\) which guarantees that \(w_{z^*}(X, Y) \equiv 1\). A contradiction to the assumption [16] proves that \(G(\infty)\) is in \(\mathcal{V}\) and Lemma 13 is proved. \(\square\)

**Lemma 14.** The group \(G(\infty)\), defined by presentation (4), is not hopfian.

**Proof.** By Lemma 13 \(G(\infty)\) is the free group of \(\mathcal{V}\) in \(A\) and, therefore, every map \(a_1 \to U_1, \ldots, a_m \to U_m\), where \(U_1, \ldots, U_m\) are words in \(A^{\pm 1}\), extends to a homomorphism \(G(\infty) \to G(\infty)\).

Consider a homomorphism \(\psi_{\infty} : G(\infty) \to G(\infty)\) defined by \(\psi_{\infty}(a_j) = a_j\) if \(j \neq 2\) and \(\psi_{\infty}(a_2) = v_1(a_1, a_2)\). Observe that the relation \(w_2(a_1, a_2) = 1\) in \(G(\infty)\) and
the definition \([4]\) of the word \(w_2(x,y)\) ensure that \(a_2 \in \langle a_1, v_1(a_1, a_2) \rangle \subseteq G(\infty)\). Hence, \(\psi\) is an epimorphism. Assume that \(\psi\) is an automorphism. Then it follows from the relation \(w_1(a_1, a_2) = 1\) in \(G(\infty)\) that the word

\[
U \equiv a_1^n a_2 a_1^s a_2 a_2 + \ldots a_1^s h/2 - 2 a_2 a_1^{n-h} a_2 a_1^{n-h-4} a_2^{h/2} (n+h-2) + h/2 \\
+ a_2 a_1^{n-h-1} a_2 a_2 a_2 a_2 a_2 a_2 a_2 a_2 (n+h-1)
\]

is equal to 1 in \(G(\infty)\). Note that \(U\) is cyclically reduced and \(|U| < (n + h)h\).

Consequently,

\[
|U| < (n + h)h < (1 - \alpha)(h - 1)nd
\]

(LPP: \(\alpha \succ \zeta = h^{-1} \succ \eta = d^{-1} \succ \iota = n^{-1}\)).

On the other hand, since \(U = 1\) in \(G(\infty)\) and \(U \neq 1\) in the free group \(F(A)\), it follows from Lemmas \([4]\) and 23.16 \([16]\) that

\[
|U| > (1 - \alpha)|\partial \Pi|,
\]

where \(\Pi\) is a 2-cell of positive rank. Furthermore, it follows from Lemmas \([9]\) and 20.4 \([16]\) (together with condition \(B\), see Sect. 20.4 \([16]\)) that

\[
|\partial \Pi| > (h - 1)nr(\Pi) > (h - 1)nd
\]

for an arbitrary 2-cell \(\Pi\) of rank \(r(\Pi) \geq 1\). However, inequalities \([35]\)–\([36]\) contradict \([4]\). This contradiction shows that \(U \neq 1\) in \(G(\infty)\). Thus, \(U\) is in the kernel of \(\psi\) and \(G(\infty)\) is not hopfian, as required.

In conclusion, we remark that, for every \(i \geq 0\), the group \(G(i)\), given by presentation \([7]\), is finitely presented (this follows from definitions and Lemmas \([6]\)–\([9]\)) and satisfies a linear isoperimetric inequality (this can be proved similar to Lemma 21.1 \([5]\)). Therefore, \(G(i)\) is a torsion-free Gromov hyperbolic group (see \([3]\)) and, so, by Sela’s results \([18]\), \(G(i)\) is a hopfian group. Therefore, the non-hopfian group \(G(\infty)\), given by presentation \([7]\), is a limit of hopfian groups \(G(i)\), \(i = 0, 1, \ldots\).

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