SAM Lectures  
on Extremal Black Holes  
in $d = 4$ Extended Supergravity

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Abstract

We report on recent results in the study of extremal black hole attractors in $N = 2$, $d = 4$ ungauged Maxwell-Einstein supergravities.

For homogeneous symmetric scalar manifolds, the three general classes of attractor solutions with non-vanishing Bekenstein-Hawking entropy are discussed. They correspond to three (inequivalent) classes of orbits of the charge vector, which sits in the relevant symplectic representation $R_V$ of the $U$-duality group. Other than the $\frac{1}{2}$-BPS one, there are two other distinct non-BPS classes of charge orbits, one of which has vanishing central charge.

The complete classification of the $U$-duality orbits, as well as of the moduli spaces of non-BPS attractors (spanned by the scalars which are not stabilized at the black hole event horizon), is also reviewed.

Finally, we consider the analogous classification for $N \geq 3$-extended, $d = 4$ ungauged supergravities, in which also the $\frac{1}{N}$-BPS attractors yield a related moduli space.
1 Introduction

In the framework of ungauged Einstein supergravity theories in $d = 4$ space-time dimensions, the fluxes of the two-form electric-magnetic field strengths determine the charge configurations of stationary, spherically symmetric, asymptotically flat extremal black holes (BHs). Such fluxes sit in a representation $R_V$ of the $U$-duality group $G_4$ of the underlying $d = 4$ supergravity, defining the embedding of $G$ into the larger symplectic group $Sp(2n, \mathbb{R})$. Moreover, after the study of [2], for symmetric scalar manifolds $\frac{G_4}{H_4}$ (see Eq. (2.1.1) below) the fluxes belong to distinct classes of orbits of the representation $R_V$, i.e. the $R_V$-representation space of $G_4$ is actually “stratified” into disjoint classes of orbits. Such orbits are defined and classified by suitable constraints on the (lowest order, actually unique) $G$-invariant $I$ built out of the symplectic representation $R_V$.

For all $N \geq 3$, $d = 4$ supergravities the scalar manifold of the theory is an homogeneous symmetric space $\frac{G_4}{H_4}$. Thus, for such theories some relations between the coset expressions of the aforementioned orbits and different real (non-compact) forms of the stabilizer $H_4$ can be established [3]. It is here worth remarking that the “large” charge orbits (having $I \neq 0$) support the Attractor Mechanism [4]–[8], whereas the “small” ones (having $I = 0$) do not.

Recently, a number of papers have been devoted to the investigation of extremal BH attractors (see e.g. [9]–[90]; for further developments and Refs., see also e.g. [91]–[95]), essentially because new classes of solutions to the so-called Attractor Equations were (re)discovered. Such new solutions have been found to determine non-BPS (Bogomol’ny-Prasad-Sommerfeld) BH horizon geometries, breaking all supersymmetries (if any).

The present report, originated from lectures given at the School on Attractor Mechanism (SAM2007), held on June 18-22 2007 at INFN National Laboratories in Frascati (LNF), Italy, is devoted to an introduction to the foundations of the theory of $U$-duality orbits in the theory of

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1Here $U$-duality is referred to as the “continuous” version, valid for large values of the charges, of the $U$-duality groups introduced by Hull and Townsend [1].
extremal BH attractors in $N = 2$, $d = 4$ MESGT’s based on symmetric manifolds. Also $N \geq 3$-extended, $d = 4$ supergravities, as well as the issue of moduli spaces of attractor solutions, will be briefly considered. Our review incorporates some of the more recent developments that have taken place since SAM2007.

The plan of the report is as follows.

Sect. 2 is devoted to the treatment of $N = 2$, $d = 4$ MESGT’s based on symmetric scalar manifolds.

In Subsect. 2.1 we review some basic facts about such theories, concerning their “large” charge orbits and the relations with the extremal BH solutions to the corresponding Attractor Eqs.

Thence, Subsect. 2.2 reports the general analysis, performed in [3], of the three classes of extremal BH attractors of $N = 2$, $d = 4$ symmetric magic MESGT’s, and of the corresponding classes of “large” charge orbits in the symplectic representation space of the relevant $d = 4$ U-duality group. In particular, the $1/2$-BPS solutions are treated in Subsubsect. 2.2.1 while the two general species of non-BPS $Z \neq 0$ and non-BPS $Z = 0$ attractors are considered in Subsubsect. 2.2.2.

The splittings of the mass spectra of $N = 2$, $d = 4$ symmetric magic MESGT’s along their three classes of “large” charge orbits [3], and the related issues of massless Hessian modes and moduli spaces of attractor solutions, are considered in Subsect. 2.3.

Subsect. 2.4 deals with the crucial result that the massless Hessian modes of the effective BH potential $V_{BH}$ of $N = 2$, $d = 4$ MESGT’s based on symmetric scalar manifolds at its critical points actually correspond to “flat” directions. Such “flat” directions are nothing but the scalar degrees which are not stabilized at the event horizon of the considered $d = 4$ extremal BH, thus spanning a moduli space associated to the considered attractor solution. Nevertheless, the BH entropy is still well defined, because, due to the existence of such “flat” directions, it is actually independent on the unstabilized scalar degrees of freedom.

Actually, moduli spaces of attractors solutions exist at least for all ungauged supergravities based on homogeneous scalar manifolds. The classification of such moduli spaces (and of the corresponding supporting “large” orbits of U-duality for $N \geq 3$, $d = 4$ supergravities is reported in Sect. 3.

Sect. 4 concludes the present report, with some final comments and remarks.

2 $N = 2$, $d = 4$ Symmetric MESGT’s

2.1 U-Duality “Large” Orbits

The critical points of the BH effective potential $V_{BH}$ for all $N = 2$ symmetric special geometries in $d = 4$ are generally referred to as attractors. These extrema describe the “large” configurations (BPS as well as non-BPS) of $N = 2, 6, 8$ supergravities, corresponding to a finite, non-vanishing quartic invariant $I_4$ and thus to extremal BHs with classical non-vanishing entropy $S_{BH} \neq 0$. The related orbits in the $R_V$ of the $d = 4$ U-duality group $G_4$ will correspondingly be referred to as “large” orbits. The attractor equations for BPS configurations were first studied in [4]-[7], and flow Eqs. for the general case were given in [8].

2These theories were called "magical" MESGT’s in the original papers. In some of the recent literature they are referred to as "magic" MESGT’s which we shall adopt in this review.

3In the present report we do not consider the other $N = 2$, $d = 4$ MESGT’s with symmetric scalar manifolds, given by the two infinite sequences $SU(1,1+n) \times SU(1+n)$ and $SU(1,1) \times SO(2+2+n)$ $SU(2+2+n)$. These theories are treated in detail in the two Appendices of [3].
Attractor solutions and their “large” charge orbits in \( d = 5 \) have been recently classified for the case of all rank-2 symmetric spaces in [24].

In [3] the results holding for \( N = 8, \ d = 4 \) supergravity were obtained also for the particular class of \( N = 2, \ d = 4 \) symmetric Maxwell-Einstein supergravity theories (MESGT’s) [96, 97, 98], which we will now review. Such a class consists of \( N = 2, \ d = 4 \) symmetric Maxwell-Einstein supergravity theories (MESGT’s) [96, 97, 98], which we will now review. Such a class consists of \( N = 2, \ d = 4 \) supergravities sharing the following properties:

i) beside the supergravity multiplet, the matter content is given only by a certain number \( n_V \) of Abelian vector multiplets;

ii) the space of the vector multiplets’ scalars is an homogeneous symmetric special Kähler manifold, i.e. a special Kähler manifold with coset structure

\[
G_4 \equiv \frac{G}{H_4} \equiv \frac{G}{H_0 \times U(1)},
\]

where \( G \equiv G_4 \) is a semisimple non-compact Lie group and \( H_4 \equiv H_0 \times U(1) \) is its maximal compact subgroup (mcs) (with symmetric embedding, as understood throughout);

iii) the charge vector in a generic (dyonic) configuration with \( n_V + 1 \) electric and \( n_V + 1 \) magnetic charges sits in a real (symplectic) representation \( R_V \) of \( G \) of \( \text{dim} (R_V) = 2(n_V + 1) \).

By exploiting such special features and relying on group theoretical considerations, in [3] the coset expressions of the various distinct classes of “large” orbits (of dimension \( 2n_V + 1 \)) in the \( R_V \)-representation space of \( G \) were related to different real (non-compact) forms of the compact group \( H_0 \). Correspondingly, the \( N = 2, \ d = 4 \) Attractor Eqs. were solved for all such classes, also studying the scalar mass spectrum of the theory corresponding to the obtained solutions.

The symmetric special Kähler manifolds of \( N = 2, \ d = 4 \) MESGT’s have been classified in the literature (see e.g. \[99, 100\] and Refs. therein). All such theories can be obtained by dimensional reduction of the \( N = 2, \ d = 5 \) MESGT’s that were constructed in \[96, 97, 98\]. The MESGT’s with symmetric manifolds that originate from \( d = 5 \) all have cubic prepotentials determined by the norm form of the Jordan algebra of degree three that defines them \[96, 97, 98\].

The unique exception is provided by the infinite sequence \((n \in \mathbb{N} \cup \{0\}, \ n_V = n + r = n + 1)\) \[101\]

\[
I_n : \frac{SU(1, 1 + n)}{U(1) \times SU(1 + n)}, \ r = 1,
\]

where \( r \) stands for the rank of the coset throughout. This is usually referred to as minimal coupling sequence, and it is endowed with quadratic prepotential. It should be remarked that the \( N = 2 \) minimally coupled supergravity is the only (symmetric) \( N = 2, \ d = 4 \) MESGT which yields the pure \( N = 2 \) supergravity simply by setting \( n = -1 \) (see e.g. \[76\] and Refs. therein).

Only another infinite symmetric sequence exists, namely \((n \in \mathbb{N} \cup \{0, -1\}, \ n_V = n + r = n + 3)\)

\[
II_n : \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2 + n)}{SO(2) \times SO(2 + n)}, \ r = 3,
\]

This one has a \( d = 5 \) origin and its associated Jordan algebras are not simple. It is referred to as the “generic Jordan family” since it exists \( \forall n \in \mathbb{N} \cup \{0, -1\} \). The first elements of such sequences \([2.1.2]\) and \([2.1.3]\) correspond to the following manifolds and holomorphic prepotential functions
in special coordinates:

\[ I_0 : \frac{SU(1,1)}{U(1)}, \quad F(t) = -\frac{i}{2} (1 - t^2); \]  

\[ II_{-1} : \frac{SU(1,1) \times SO(2,1)}{U(1) \times SO(2)} = \left( \frac{SU(1,1)}{U(1)} \right)^2, \quad F(s, t) = st^2; \]  

\[ III_0 : \frac{SU(1,1) \times SO(2,2)}{U(1) \times SO(2) \times SO(2)} = \left( \frac{SU(1,1)}{U(1)} \right)^3, \quad F(s, t, u) = stu; \]  

It is here worth remarking that the so-called \( t^3 \) model, corresponding to the following manifold and holomorphic prepotential function in special coordinates:

\[ III_0 : \frac{SU(1,1)}{U(1)}, \quad F(t) = t^3, \]  

is an isolated case in the classification of symmetric SK manifolds (see e.g. \cite{102}; see also \cite{103} and Refs. therein), but it can be thought also as the “\( t^3 \) degeneration” of the \( stu \) model (see e.g. \cite{50}; see also Subsect. 2.4 for a treatment of models \( I_0 \) and \( t^3 \)).

As mentioned, all manifolds of type \( I \) correspond to quadratic prepotentials (\( C_{ijk} = 0 \)), and all manifolds of type \( II \) correspond to cubic prepotentials (in special coordinates \( F = \frac{1}{3!} d_{ijk} t_i t_j t_k \) and therefore \( C_{ijk} = e^K d_{ijk} \), where \( K \) denotes the Kähler potential and \( d_{ijk} \) is a completely symmetric rank-3 constant tensor). The 3-moduli case \( III_0 \) is the well-known \( stu \) model \cite{104, 105} (see also e.g. \cite{77} and Refs. therein), whose noteworthy triality symmetry has been recently related to quantum information theory \cite{106, 107, 108, 109, 110, 111}.

Beside the infinite sequence \( II \), there exist four other MESGT’s defined by simple Euclidean Jordan algebras of degree three with the following rank-3 symmetric manifolds:

\[ III : \frac{E_{7(-25)}}{E_6 \times U(1)}; \]  

\[ IV : \frac{SO^*(12)}{U(6)}; \]  

\[ V : \frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}; \]  

\[ VI : \frac{Sp(6,\mathbb{R})}{U(3)} \]  

The \( N = 2, d = 4 \) MESGT’s whose geometry of scalar fields is given by the manifolds \( III-VI \) are called “magic”, since their symmetry groups are the groups of the famous Magic Square of Freudenthal, Rozenfeld and Tits associated with some remarkable geometries \cite{112, 113}. The four \( N = 2, d = 4 \) magic MESGT’s \( III-VI \), as their \( d = 5 \) versions, are defined by four simple Euclidean Jordan algebras \( J_3^O, J_3^H, J_3^C \) and \( J_3^R \) of degree 3 with irreducible norm forms, namely by the Jordan algebras of Hermitian \( 3 \times 3 \) matrices over the four division algebras, i.e. respectively over the octonions \( \mathbb{O} \), quaternions \( \mathbb{H} \), complex numbers \( \mathbb{C} \) and real numbers \( \mathbb{R} \) \cite{96, 97, 98, 114, 115, 116, 117}.

By denoting with \( n_V \) the number of vector multiplets coupled to the supergravity one, the total number of Abelian vector fields in the considered \( N = 2, d = 4 \) MESGT is \( n_V + 1 \); correspondingly, the real dimension of the corresponding scalar manifold is \( 2n_V = \text{dim} (G) - 4 \).
Table 1: Data of the two sequences of symmetric $N = 2, d = 4$ MESGT’s

|       | $I$                                                                 | $II$                                                                 |
|-------|----------------------------------------------------------------------|----------------------------------------------------------------------|
| $G$   | $SU(1, 1 + n)$                                                      | $SU(1, 1) \times SO(2, 2 + n)$                                       |
| $H_0$ | $SU(1 + n)$                                                        | $SO(2) \times SO(2 + n)$                                            |
| $r$   | 1                                                                   | 3                                                                   |
| $\dim \left( \frac{H_0 G}{SU(1, 1)} \right)$ | $2(n + 1)$             | $2(n + 3)$             |
| $n_V$ | $n + r = n + 1$                                                     | $n + r = n + 3$                                                     |
| $R_V$ | $(2(n + 2))_R$                                                      | $(2(n + 4))_R$                                                      |
| $R_{H_0}$ | $(n + 1)_C$                                                    | $(n + 2 + 1)_C$                                                    |
| $\dim \left( R_V \right)$ | $2(n + 2)$             | $2(n + 4)$             |
| $\dim \left( R_{H_0} \right)$ | $2(n + 1)$             | $2(n + 3)$             |
| $R_V$ | $(2(n + 2))_R$                                                      | $(2(n + 4))_R$                                                      |
| $R_{H_0} + 1_c +$ | $(n + 1)_C + 1_c +$                                                   | $(n + 2 + 1)_C + 1_c +$                                                   |
|       | + c.c.                                                             | + c.c.                                                             |

$\dim \left( H_0 \right) - 1$. Since the $2(n_V + 1)$-dim. vector of extremal BH charge configuration is given by the fluxes of the electric and magnetic field-strength two-forms, it is clear that $\dim \left( R_V \right) = 2(n_V + 1)$.

Since $H_0$ is a proper compact subgroup of the duality semisimple group $G$, one can decompose the $2(n_V + 1)$-dim. real symplectic representation $R_V$ of $G$ in terms of complex representations of $H_0$, obtaining in general the following decomposition scheme:

\[
R_V \rightarrow R_{H_0} + \overline{R_{H_0}} + 1_c + \overline{1_c} = R_{H_0} + 1_c + \text{c.c.},
\]

where “c.c.” stands for the complex conjugation of representations throughout, and $R_{H_0}$ is a certain complex representation of $H_0$.

The basic data of the cases $I$-$VI$ listed above are summarized in Tables 1 and 2.

It was shown in [2] that $\frac{1}{2}$-BPS orbits of $N = 2, d = 4$ symmetric MESGT’s are coset spaces of the form

\[
\mathcal{O}_{\frac{1}{2}-\text{BPS}} = \frac{\mathcal{G}}{H_0},
\]

\[
\dim \left( \mathcal{O}_{\frac{1}{2}-\text{BPS}} \right) = \dim \left( G \right) - \dim \left( H_0 \right) = 2n_V + 1 = \dim \left( R_V \right) - 1.
\]

We need to consider the $N = 2$ Attractor Eqs.; these are nothing but the criticality conditions for the $N = 2$ BH effective potential [118, 6]

\[
V_{BH} \equiv |Z|^2 + G^{ij}D_iZD_jZ
\]

in the corresponding special Kähler geometry [8]:

\[
\partial_i V_{BH} = 0 \iff 2ZD_iZ + iC_{ijk}G^{ij}G^{kl}\overline{D_kZD_lZ} = 0, \forall i = 1, \ldots, n_V.
\]

$C_{ijk}$ is the rank-3, completely symmetric, covariantly holomorphic tensor of special Kähler geometry, satisfying (see e.g. [119])

\[
\overline{D_i}C_{ijk} = 0, \quad D_iC_{ijk} = 0,
\]

\[
(2.1.15)
\]
Table 2: Data of the four magic symmetric $N = 2$, $d = 4$ MESGT’s. $14^\prime_\mathbb{R}$ is the rank-3 antisymmetric tensor representation of $Sp(6, \mathbb{R})$. In $(3, 3')_C$ the prime distinguishes the representations of the two distinct $SU(3)$ groups

where the square brackets denote antisymmetrization with respect to the enclosed indices.

For symmetric special Kähler manifolds the tensor $C_{ijk}$ is covariantly constant:

$$D_i C_{jkl} = 0,$$

which further implies $[97, 99]$:

$$G^{k_0} G^{r_0} C_{r(pq)k} G^{r_k} = \frac{4}{3} G_{(q)k} C_{ijp}.$$

This equation is simply the $d = 4$ version of the “adjoint identity” satisfied by all (Euclidean) Jordan algebras of degree three that define the corresponding MESGT’s in $d = 5$ $[97, 99]$: $d_r(pq) d_{ij} d_{kl} = \frac{4}{3} \delta_r^{ij} d_{ijp}$.

$Z$ is the $N = 2$ “central charge” function, whereas $\{D_i Z\}_{i=1,...,n_V}$ is the set of its Kähler-covariant holomorphic derivatives, which are nothing but the “matter charge” functions of the system. Indeed, the set $\{q_0, q, p^0, p^i\} \in \mathbb{R}^{2n_V + 2}$ and $\{Z, D_i Z\} \in \mathbb{C}^{n_V + 1}$ (when evaluated at purely $(q, p)$-dependent critical values of the moduli) are two equivalent basis for the charges of the system, and they are related by a particular set of identities of special Kähler geometry $[118, 21, 22]$. The decomposition (2.1.12) corresponds to nothing but the splitting of the sets $\{q_0, q, p^0, p^i\}$ (\{Z, D_i Z\}) of $2n_V + 2$ ($n_V + 1$) real (complex) charges (“charge” functions) in $q_0, p^0$ ($Z$) (related to the graviphoton, and corresponding to $1_C + c.c.$) and in $\{q, p^i\}$ ($\{D_i Z\}$) (related to the $n_V$ vector multiplets, and corresponding to $R_{H_0} + c.c.$).

In order to perform the subsequent analysis of orbits, it is convenient to use “flat” $I$-indices by using the (inverse) $n_V$-bein $e_I^T$ of \(G_{H_0 \times U(1)}\):

$$D_I Z = e_I^T D_i Z.$$
By switching to “flat” local $I$-indices, the special Kähler metric $G_\mathcal{J}$ (assumed to be regular, i.e. strictly positive definite everywhere) will become nothing but the Euclidean $n_V$-dim. metric $\delta_\mathcal{I}\mathcal{J}$. Thus, the attractor eqs. (2.1.15) can be “flattened” as follows:

$$\partial_I V_{BH} = 0 \iff 2Z D_I Z + iC_{IJK} \delta^{J\mathcal{J}} \delta^K_{\mathcal{K}} D_{\mathcal{J}} Z D_{\mathcal{K}} Z = 0, \forall I = 1, ..., n_V. \tag{2.1.21}$$

Note that $C_{IJK}$ becomes an $H_0$-invariant tensor \[120\]. This is possible because $C_{ijk}$ in special coordinates is proportional to the invariant tensor $d_{IJK}$ of the $d = 5$ U-duality group $G_5$. $G_5$ and $H_0$ correspond to two different real forms of the same Lie algebra \[87\].

As it is well known, $\frac{1}{2}$-BPS attractors are given by the following solution \[8\] of attractor eqs. (2.1.15) and (2.1.21):

$$Z \neq 0, \quad D_I Z = 0 \iff D_I Z = 0, \forall I = 1, ..., n_V. \tag{2.1.22}$$

Since the “flattened matter charges” $D_I Z$ are a vector of $R_{H_0}$, Eq. (2.1.22) directly yields that $\frac{1}{2}$-BPS solutions are manifestly $H_0$-invariant. In other words, since the $N = 2$, $\frac{1}{2}$-BPS orbits are of the form $\frac{G}{H_0}$, the condition for the $(n_V + 1)$-dim. complex vector $(Z, D_i Z)$ to be $H_0$-invariant is precisely given by Eq. (2.1.22), defining $N = 2, \frac{1}{2}$-BPS attractor solutions.

Thus, as for the $N = 8$, $d = 4$ attractor solutions (see e.g. \[3\] and Refs. therein), also for the $N = 2, d = 4 \frac{1}{2}$-BPS case the invariance properties of the solutions at the critical point(s) are given by the maximal compact subgroup (mcs) of the stabilizer of the corresponding charge orbit, which in the present case is the compact stabilizer itself. Thus, at $N = 2 \frac{1}{2}$-BPS critical points the following enhancement of symmetry holds:

$$S \longrightarrow H_0, \tag{2.1.23}$$

where here and below $S$ denotes the compact symmetry of a generic orbit of the real symplectic representation $R_V$ of the $d = 4$ duality group $G$.

However, all the scalar manifolds of $N = 2, d = 4$ symmetric MESGT’s have other species of regular critical points $V_{BH}$ (and correspondingly other classes of “large” charge orbits).

Concerning the $N = 2, d = 4$ symmetric MESGT’s, the rank-1 sequence $I$ has one more, non-BPS class of orbits (with vanishing central charge), while all rank-3 aforementioned cases $II$-$V I$ have two more distinct non-BPS classes of orbits, one of which with vanishing central charge.

The results about the classes of “large” charge orbits of $N = 2, d = 4$ symmetric MESGT’s are summarized in Table \[4\].

2.2 Classification of Attractors

The three classes of orbits in Table 3 correspond to the three distinct classes of solutions of the $N = 2, d = 4$ Attractor Eqs. (2.1.15) and (2.1.21).

\footnote{We should note that the column on the right of Table 2 of \[2\] is not fully correct. Indeed, that column coincides with the central column of Table 3 of the present paper (by disregarding case $I$ and shifting $n \rightarrow n - 2$ in case $II$), listing the non-BPS, $Z \neq 0$ orbits of $N = 2, d = 4$ symmetric MESGT’s, which are all characterized by a strictly negative quartic $E_7$-invariant $\mathcal{I}_4$. This does not match what is claimed in \[2\], where such a column is stated to list the particular class of orbits with $\mathcal{I}_4 > 0$ and eigenvalues of opposite sign in pair.

Actually, the statement of \[2\] holds true only for the case $I$ (which, by shifting $n \rightarrow n - 1$, coincides with the last entry of the column on the right of Table 2 of \[2\]). On the other hand, such a case is the only one which cannot be obtained from $d = 5$ by dimensional reduction. Moreover, it is the only one not having non-BPS, $Z \neq 0$ orbits, rather it is characterized only by a class of non-BPS orbits with $Z = 0$ and $\mathcal{I}_4 > 0$.}
As already mentioned, the class of \(\frac{1}{2}\)-BPS orbits corresponds to the solution (2.1.22) determining \(N = 2, \frac{1}{2}\)-BPS critical points of \(V_{BH}\). Such a solution yields the following value of the BH scalar potential at the considered attractor point(s) [8]:

\[
V_{BH, \frac{1}{2}-BPS} = |Z|^{2}_{\frac{1}{2}-BPS} + \left[ G^{t} D_{i} Z D_{i} Z \right]^{\frac{1}{2}-BPS} = |Z|^{2}_{\frac{1}{2}-BPS}. \tag{2.2.1.1}
\]

The overall symmetry group at \(N = 2\), \(\frac{1}{2}\)-BPS critical point(s) is \(H_0\), stabilizer of \(O_{\frac{1}{2}-BPS} = \frac{G}{H_0}\). The symmetry enhancement is given by Eq. (2.1.23). For such a class of orbits

\[
I_{\frac{1}{2}-BPS} = |Z|^{1}_{\frac{1}{2}-BPS} > 0. \tag{2.2.1.2}
\]
2.2.2 Non-BPS

The two classes of \( N = 2 \) non-BPS “large” charge orbits respectively correspond to the following solutions of \( N = 2 \) attractor eqs. (2.1.15):

\[
\begin{cases}
\text{non-BPS, } Z \neq 0: & \\
Z \neq 0,
D_i Z \neq 0 \text{ for some } i \in \{1, \ldots, n_V\},
\end{cases}
\]

\[
I_{4,\text{non-BPS},Z \neq 0} = - \left( |Z|_{\text{non-BPS},Z \neq 0}^2 + \left( G \delta ^{ij} D_i Z \overline{D_j Z} \right)_{\text{non-BPS},Z \neq 0} \right)^2 = -16 |Z|_{\text{non-BPS},Z \neq 0}^4 < 0;
\]

\( (2.2.2.1) \)

\[
\begin{cases}
\text{non-BPS, } Z = 0: & \\
Z = 0,
D_i Z \neq 0 \text{ for some } i \in \{1, \ldots, n_V\},
\end{cases}
\]

\[
I_{4,\text{non-BPS},Z = 0} = \left( G \delta ^{ij} D_i Z \overline{D_j Z} \right)_{\text{non-BPS},Z = 0}^2 > 0.
\]

\( (2.2.2.2) \)

In the treatment given below, we will show how the general solutions of Eqs. (2.1.15), respectively determining the two aforementioned classes of \( N = 2 \) non-BPS extremal BH attractors, can be easily given by using “flat” local \( I \)-coordinates in the scalar manifold.

In other words, we will consider the “flattened” attractor eqs. (2.1.21), which can be specialized in the “large” non-BPS cases as follows:

\[
\begin{cases}
\text{non-BPS, } Z \neq 0: & \\
2 \overline{D}_1 D_1 Z = -i C_{IJK} \delta ^{IJ} \delta ^{K\bar{\rho}} \overline{D}_j Z \overline{D}_l Z;
\end{cases}
\]

\( (2.2.2.3) \)

\[
\begin{cases}
\text{non-BPS, } Z = 0: & \\
C_{IJK} \delta ^{IJ} \delta ^{K\bar{\rho}} \overline{D}_j Z \overline{D}_l Z = 0.
\end{cases}
\]

\( (2.2.2.4) \)

Thus, by respectively denoting with \( \hat{H} (\bar{H}) \) the stabilizer of the \( N = 2 \), non-BPS, \( Z \neq 0 \) \( (Z = 0) \) classes of orbits listed in Table 3, our claim is the following: the general solution of Eqs. (2.2.2.3) \( (2.2.2.4) \) is obtained by retaining a complex charge vector \( (Z, D_i Z) \) which is invariant under \( \hat{h} \) \( (\bar{h}) \), where \( \hat{h} \) \( (\bar{h}) \) is the mcs \( 6 \) of \( \hat{H} \) \( (\bar{H}) \).

As a consequence, the overall symmetry group of the \( N = 2 \), non-BPS, \( Z \neq 0 \) \( (Z = 0) \) critical point(s) is \( \hat{h} \) \( (\bar{h}) \). Thus, at \( N = 2 \), non-BPS, \( Z \neq 0 \) \( (Z = 0) \) critical point(s) the following enhancement of symmetry holds:

\[
N = 2, \text{non-BPS, } Z \neq 0 : \mathcal{S} \longrightarrow \hat{h} = \text{mcs} \left( \hat{H} \right);
\]

\( (2.2.2.5) \)

\[
N = 2, \text{non-BPS, } Z = 0 : \mathcal{S} \longrightarrow \frac{\hat{h}}{\overline{U}(1)} = \frac{\text{mcs}(\hat{H})}{\overline{U}(1)}.
\]

\footnote{Indeed, while \( H_0 \) is a proper compact subgroup of \( G \), the groups \( \hat{H}, \bar{H} \) are real (non-compact) forms of \( H_0 \), as it can be seen from Table 3 (see also \( \text{[121, 122]} \)). Therefore in general they admit a mcs \( \hat{h}, \bar{h} \), which in turn is a (non-maximal) compact subgroup of \( G \) and a proper compact subgroup of \( H_0 \).

It is interesting to notice that in all cases (listed in Table 3) \( G \) always admits only 2 real (non-compact) forms \( \hat{H}, \bar{H} \) of \( H_0 \) as proper subgroups (consistent with the required dimension of orbits). The inclusion of \( \hat{H}, \bar{H} \) in \( G \) is such that in all cases \( \hat{H} \times SO(1,1) \) and \( \bar{H} \times U(1) \) are different maximal non-compact subgroups of \( G \).}
\[ \begin{array}{|c|c|c|c|c|}
\hline
 & \hat{H}_0 & \tilde{H} & \hat{H} & \tilde{h} \\
\hline
I & SU(n+1) & SU(n) & SU(1,n) & SU(n) \\
\hline
II & SO(2) \times SO(2 + n) & SO(1, 1 + n) & SO(1 + n) & SO(2) \times SO(n) \\
\hline
III & E_6 \equiv E_6(-78) & E_6(-26) & E_6(-14) & F_4 \equiv F_4(-52) \times SO(10) \\
\hline
IV & SU(6) \times SU(2) & SU(4, 2) & SU(4) \times SU(2) & SU(3) \times SU(2) \times U(1) \\
\hline
V & SU(3) \times SU(3) & SL(3, \mathbb{C}) & SU(2, 1) \times SU(1, 2) & SU(3) \times SU(2) \times U(1) \\
\hline
VI & SU(3) & SL(3, \mathbb{R}) & SU(2, 1) & SO(3) \times SU(2) \\
\hline
\end{array} \]

Table 4: Stabilizers and corresponding maximal compact subgroups of the “large” classes of orbits of \( N = 2, d = 4 \) symmetric MESGT's. \( \hat{H} \) and \( \tilde{H} \) are real (non-compact) forms of \( H_0 \), the stabilizer of \( \frac{1}{2} \)-BPS orbits.

It is worth remarking that the non-compact group \( \hat{H} \) stabilizing the non-BPS, \( Z \neq 0 \) class of orbits of \( N = 2, d = 4 \) symmetric MESGT's, beside being a real (non-compact) form of \( H_0 \), is isomorphic to the duality group \( G_5 \) of \( N = 2, d = 5 \) symmetric MESGT's.

Since the scalar manifolds of \( N = 2, d = 5 \) symmetric MESGT's are endowed with a real special geometry \([96, 97, 98]\), the complex representation \( R_{H_0} \) of \( H_0 \) decomposes in a pair of irreducible real representations \( (R_{\hat{h}} + 1)_\mathbb{R} \)'s of \( \hat{h} = mcs(\hat{H}) \varsubsetneq H_0 \) (see Subsubsect. 2.2.2 and in particular Eq. (2.2.2.6)). As we will see below, such a fact crucially distinguishes the non-BPS, \( Z \neq 0 \) and \( Z = 0 \) cases.

The stabilizers (and the corresponding \( mcs \)'s) of the non-BPS, \( Z \neq 0 \) and \( Z = 0 \) classes of orbits of \( N = 2, d = 4 \) symmetric MESGT's are given in Table 4.

**Non-BPS, \( Z \neq 0 \)** Let us start by considering the class of non-BPS, \( Z \neq 0 \) orbits of \( N = 2, d = 4 \) symmetric MESGT's.

As mentioned, the “flattened matter charges” \( D_I Z \) are a vector of \( R_{H_0} \). In general, \( R_{H_0} \) decomposes under the \( mcs \) \( \hat{h} \subset H \) as follows:

\[ R_{H_0} \rightarrow (R_{\hat{h}} + 1)_\mathbb{C}, \quad (2.2.2.6) \]

where the r.h.s. is made of the complex singlet representation of \( \hat{h} \) and by another non-singlet representation.
real representation of \( \hat{h} \), denoted above with \( R_{\hat{h}} \). As previously mentioned, despite being complex, \((R_{\hat{h}} + 1)_C\) is not charged with respect to \( U(1) \) symmetry because, due to the 5-dimensional origin of the non-compact stabilizer \( \hat{H} \) whose mcs is \( \hat{h} \), actually \((R_{\hat{h}} + 1)_C\) is nothing but the complexification of its real counterpart \((R_{\hat{h}} + 1)_R\). The decomposition \((2.2.2.6)\) yields the following splitting of “flattened matter charges”:

\[
D_I Z \longrightarrow \left( D_I Z, D_{\hat{l}_0} Z \right),
\]

where \( \hat{I} \) are the indices along the representation \( R_{\hat{h}} \), and \( \hat{l}_0 \) is the \( \hat{h} \)-singlet index.

By considering the related attractor eqs., it should be noticed that the rank-3 symmetric tensor \( C_{IJK} \) in Eqs. \((2.2.2.3)\) corresponds to a cubic \( H_0 \)-invariant coupling \((R_{H_0})^3\). By decomposing \((R_{H_0})^3\) in terms of representations of \( \hat{h} \), one finds

\[
(R_{H_0})^3 \longrightarrow (R_{\hat{h}})^3 + (R_{\hat{h}})^2 1_C + (1_C)^3.
\]

Notice that a term \( R_{\hat{h}}(1_C)^2 \) cannot be in such a representation decomposition, since it is not \( \hat{h} \)-invariant, and thus not \( H_0 \)-invariant. This implies that components of the form \( C_{IJK} \) cannot exist. Also, a term like \((1_C)^3\) can appear in the r.h.s. of the decomposition \((2.2.2.3)\) since as we said the \( \hat{h} \)-singlet \( 1_C \), despite being complex, is not \( U(1) \)-charged.

It is then immediate to conclude that the solution of \( N = 2, d = 4 \) non-BPS, \( Z \neq 0 \) extremal BH attractor eqs. in “flat” indices \((2.2.2.3)\) corresponds to keeping the “flattened matter charges” \( D_I Z \) \( \hat{h} \)-invariant. By virtue of decomposition \((2.2.2.8)\), this is obtained by putting

\[
D_I Z = 0, D_{\hat{l}_0} Z \neq 0,
\]

i.e. by putting all “flattened matter charges” to zero, except the one along the \( \hat{h} \)-singlet (and thus \( \hat{h} \)-invariant) direction in scalar manifold. By substituting the solution \((2.2.2.9)\) in Eqs. \((2.2.2.3)\), one obtains

\[
2Z\text{D}_{\hat{l}_0} Z = -iC_{\hat{l}_0\hat{l}_0\hat{l}_0} \left( \text{D}_{\hat{l}_0} \hat{Z} \right)^2 \overset{\text{Z} \neq 0}{\longrightarrow} \text{D}_{\hat{l}_0} Z = -i \frac{C_{\hat{l}_0\hat{l}_0\hat{l}_0}}{2Z} \left( \text{D}_{\hat{l}_0} \hat{Z} \right)^2
\]

\[
\downarrow
\]

\[
\left| \text{D}_{\hat{l}_0} Z \right|^2 \left( 1 - \frac{1}{4} \frac{C_{\hat{l}_0\hat{l}_0\hat{l}_0}}{|Z|^2} \left| \text{D}_{\hat{l}_0} Z \right|^2 \right) = 0
\]

\[
\downarrow
\]

\[
\left| \text{D}_{\hat{l}_0} Z \right|^2 = 4 \frac{|Z|^2}{\left| C_{\hat{l}_0\hat{l}_0\hat{l}_0} \right|^2};
\]

this is nothing but the general criticality condition of \( V_{BH} \) for the 1-modulus case in the locally “flat” coordinate \( \hat{l}_0 \), which in this case corresponds to the \( \hat{h} \)-singlet direction in the scalar manifold. Such a case has been thoroughly studied in non-flat \( i \)-coordinates in [21].

All \( N = 2, d = 4 \) symmetric MESGT’s (disregarding the sequence \( I \) having \( C_{ijk} = 0 \)) have a cubic prepotential \((F = \frac{1}{4} d_{ijk} t^i t^j t^k \) in special coordinates), and thus in special coordinates it holds that \( C_{ijk} = e^K d_{ijk} \), with \( K \) and \( d_{ijk} \) respectively denoting the Kähler potential and the completely symmetric rank-3 constant tensor that is determined by the norm form of the
underlying Jordan algebra of degree three \[97\]. In the cubic \(n_V = 1\)-modulus case, by using Eq. (2.1.18) it follows that
\[
\left(G_{\text{TT}}^1\right)^3 |C_{1,1,11}|^2 = |C_{1,1,11}|^2 = \frac{4}{3}, \tag{2.2.2.12}
\]
where the subscripts “s” and “f” respectively stand for “special” and “flat”, denoting the kind of coordinate system being considered. By substituting Eq. (2.2.2.12) in Eq. (2.2.2.11) one obtains the result
\[
|D_{\tilde{h}}Z|^2 = 3|Z|^2. \tag{2.2.2.13}
\]
Another way of proving Eq. (2.2.2.13) is by computing the quartic invariant along the \(\tilde{h}\)-singlet direction, then yielding
\[
I_{4,\text{non-BPS},Z\neq 0} = -16 |Z|_{\text{non-BPS},Z\neq 0}^2. \tag{2.2.2.14}
\]

The considered solution (2.2.2.9)-(2.2.2.11), (2.2.2.13) is the \(N = 2\) analogue of the \(N = 8, d = 4\) non-BPS “large” solution discussed in \[23\], and it yields the following value of the BH scalar potential at the considered attractor point(s) \[21\] \[23\]:
\[
V_{\text{BH,non-BPS},Z\neq 0} = 4 |Z|_{\text{non-BPS},Z\neq 0}^2. \tag{2.2.2.15}
\]

Once again, as for the non-BPS \(N = 8\) “large” solutions, we find the extra factor 4.

From the above considerations, the overall symmetry group at \(N = 2\) non-BPS, \(Z \neq 0\) critical point(s) is \(\tilde{h}\), mcs of the non-compact stabilizer \(\tilde{H}\) of \(\mathcal{O}_{\text{non-BPS},Z\neq 0}\).

**Non-BPS, \(Z = 0\)** Let us now move to consider the other class of non-BPS orbits of \(N = 2, d = 4\) symmetric MESGT’s.

It has \(Z = 0\) and it was not considered in \[2\] (see also Footnote 4). We will show that the solution of the \(N = 2, d = 4\), non-BPS, \(Z = 0\) extremal BH attractor eqs. (2.2.2.4) are the “flattened matter charges” \(D_1Z\) which are invariant under \(\tilde{h}/U(1)\), where \(\tilde{h}\) is the mcs of \(\tilde{H}\), the stabilizer of the class \(\mathcal{O}_{\text{non-BPS},Z=0} = \mathcal{G}/H\).

Differently from the non-BPS, \(Z \neq 0\) case, in the considered non-BPS, \(Z = 0\) case there is always a \(U(1)\) symmetry acting, since the scalar manifolds of \(N = 2, d = 4\) symmetric MESGT’s all have the group \(\tilde{h}\) of the form
\[
\tilde{h} = \tilde{h}' \times U(1), \quad \tilde{h}' \equiv \frac{\tilde{h}}{U(1)}. \tag{2.2.2.16}
\]

The compact subgroups \(\tilde{h}'\) for all \(N = 2, d = 4\) symmetric MESGT’s are listed in Table 4. In the case at hand, we thence have to consider the decomposition of the previously introduced complex representation \(R_{\tilde{h}_0}\) under the compact subgroup \(\tilde{h}' \subset H_0\). In general, \(R_{\tilde{h}_0}\) decomposes under \(\tilde{h}' \subset \tilde{H}\) as follows:
\[
R_{\tilde{h}_0} \longrightarrow (W_{\tilde{h}'}, Y_{\tilde{h}'}, 1)_C, \tag{2.2.2.17}
\]
where in the r.h.s. the complex singlet representation of \(\tilde{h}'\) and two complex non-singlet representations \(W_{\tilde{h}'}, Y_{\tilde{h}'}\) of \(\tilde{h}'\) appear. In general, \(W_{\tilde{h}'}\), \(Y_{\tilde{h}'}\), and \(1_C\) are charged (and thus not invariant) with respect to the \(U(1)\) explicit factor appearing in (2.2.2.16). The decomposition (2.2.2.17) yields the following splitting of “flattened matter charges”:
\[
D_1Z \longrightarrow \left(D_{\tilde{i}_W}Z, D_{\tilde{i}_Y}Z, D_{\tilde{i}_0}Z\right), \tag{2.2.2.18}
\]
where \(\tilde{i}_W\) and \(\tilde{i}_Y\) respectively denote the indices along the complex representations \(W_{\tilde{h}'}, \ Y_{\tilde{h}'}, \) and \(\tilde{i}_0\) is the \(\tilde{h}'\)-singlet index.
Once again, the related $N = 2$, $d = 4$ non-BPS, $Z = 0$ extremal BH attractor eqs. \[2.2.2.4\] contain the rank-3 symmetric tensor $C_{IJK}$, corresponding to a cubic $H_0$-invariant coupling $(R_{H_0})^3$. The decomposition of $(R_{H_0})^3$ in terms of representations of $\tilde{h}'$ yields

\[(R_{H_0})^3 \longrightarrow (W_{\tilde{h}'})^2 Y_{\tilde{h}'} + (Y_{\tilde{h}'})^2 1_C. \tag{2.2.2.19}\]

When decomposed under $\tilde{h}'$, $(R_{H_0})^3$ must be nevertheless $\tilde{h}$-invariant, and therefore, beside the $\tilde{h}'$-invariance, one has to consider the invariance under the $U(1)$ factor, too. Thus, terms of the form $(W_{\tilde{h}'})^3$, $(Y_{\tilde{h}'})^3$, $W_{\tilde{h}'}(1_C)^2$, $Y_{\tilde{h}'}(1_C)^2$ and $(1_C)^3$ cannot exist in the $\tilde{h}$-invariant r.h.s. of decomposition \[2.2.2.19\].

Notice also that the structure of the decomposition \[2.2.2.19\] implies that components of the cubic coupling of the form $C_{I_0I_0I_0}$, $C_{I_0I_0I_0}$ and $C_{I_0I_0I_0}$ cannot exist. For such a reason, it is immediate to conclude that the solution of $N = 2$, $d = 4$ non-BPS, $Z = 0$ extremal BH attractor eqs. in “flat” indices \[2.2.2.4\] corresponds to keep the “flattened matter charges” $D_I Z$ $\tilde{h}'$-invariant. By virtue of decomposition \[2.2.2.19\], this is obtained by putting

\[D_{I_0} Z = 0 = D_{I_0} Z, \quad D_{I_0} Z \neq 0, \tag{2.2.2.20}\]

i.e., by putting all “flattened matter charges” to zero, except the one along the $\tilde{h}'$-singlet (and thus $\tilde{h}'$-invariant, but not $U(1)$-invariant and therefore not $\tilde{h}$-invariant) direction in the scalar manifold.

The considered solution \[2.2.2.20\] does not have any analogue in $N = 8$, $d = 4$ supergravity, and it yields the following value of the BH scalar potential at the considered attractor point(s):

\[V_{BH, non-BPS, Z=0} = |Z|^2_{non-BPS, Z=0} + \left[G^2 D_I Z \overline{D_I Z}\right]_{non-BPS, Z=0} =
\]

\[= \left|D_{I_0} Z\right|^2_{non-BPS, Z=0}. \tag{2.2.2.21}\]

It is here worth remarking that in the $stu$ model it can be explicitly computed that \[50\] \[77\]

\[V_{BH, non-BPS, Z=0} = \left|D_{I_0} Z\right|^2_{non-BPS, Z=0} = |Z|^2_{\frac{1}{2}-BPS} = V_{BH, \frac{1}{2}-BPS}. \tag{2.2.2.22}\]

From above considerations, the overall symmetry group at $N = 2$ non-BPS, $Z = 0$ critical point(s) is $\tilde{h}' = \tilde{h}$, $\tilde{h}$ being the mcs of the non-compact stabilizer $\tilde{H}$ of $O_{non-BPS, Z=0}$.

The general analysis carried out above holds for all $N = 2$, $d = 4$ symmetric magic MESGT’s, namely for the irreducible cases $III-VI$ listed in Tables 2 and 3. The cases of irreducible sequence $I$ and of generic Jordan family $II$ deserve suitable, slightly different treatments, respectively given in Appendices I and II of \[3\].

### 2.3 Critical Spectra and Massless Hessian Modes of $V_{BH}$

The effective BH potential $V_{BH}$ gives different masses to the different BPS-phases of the considered symmetric $N = 2$, $d = 4$ MESGT’s. The fundamental object to be considered in such a framework is the moduli-dependent $2n_V \times 2n_V$ Hessian matrix of $V_{BH}$, which in complex basis
\[ \mathbf{H}^{V_{BH}} = \begin{pmatrix} D_i D_j V_{BH} & D_i \overline{D}_j V_{BH} \\ D_j \overline{D}_i V_{BH} & \overline{D}_i \overline{D}_j V_{BH} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{M}_{ij} & \mathcal{N}_{i\overline{j}} \\ \mathcal{N}_{j\overline{i}} & \overline{\mathcal{M}}_{ij} \end{pmatrix}; \] (2.3.1)

\[ \mathcal{M}_{ij} \equiv D_i D_j V_{BH} = D_j D_i V_{BH} = \]
\[ = 4 i Z C_{ij k} G^{k \overline{k}} D_k Z + i G^{k \overline{k}} C^{(i} k l D_{i} \overline{D}_{j} Z D_{l} Z; \] (2.3.2)

\[ \mathcal{N}_{i\overline{j}} \equiv D_i \overline{D}_j V_{BH} = \overline{D}_j D_i V_{BH} = \]
\[ = 2 \left[ G_{\overline{j} \overline{i}} |Z|^2 + D_{i} \overline{Z} D_{j} Z + G^{m \overline{m}} G^{n \overline{n}} C_{i k l} \overline{C}_{j m n} \overline{D}_{k} Z D_{m} Z \right]; \] (2.3.3)

\[ \mathcal{M}^T = \mathcal{M}, \mathcal{N}^\dagger = \mathcal{N}. \] (2.3.4)

By analyzing \( \mathbf{H}^{V_{BH}} \) at critical points of \( V_{BH} \), it is possible to formulate general conclusions about the mass spectrum of the corresponding extremal BH solutions with finite, non-vanishing entropy, i.e. about the mass spectrum along the related classes of “large” charge orbits of the symplectic real representation \( R_{V} \) of the \( d = 4 \) duality group \( G \).

Let us start by remarking that, due to its very definition (2.1.14), the \( N = 2 \) effective BH potential \( V_{BH} \) is positive for any (not necessarily strictly) positive definite metric \( G_{i\overline{j}} \) of the scalar manifold. Consequently, the stable critical points (i.e. the attractors in a strict sense) will necessarily be minima of such a potential. As already pointed out above and as done also in [21][22], the geometry of the scalar manifold is usually assumed to be regular, i.e. endowed with a metric tensor \( G_{i\overline{j}} \) being strictly positive definite everywhere.

### 2.3.1 \( \frac{1}{2} \)-BPS

It is now well known that regular special Kähler geometry implies that all \( N = 2 \) \( \frac{1}{2} \)-BPS critical points of all \( N = 2, d = 4 \) MESGT’s are stable, and therefore they are attractors in a strict sense. Indeed, the Hessian matrix \( \mathbf{H}^{V_{BH}}_{\frac{1}{2}-BPS} \) evaluated at such points is strictly positive definite:

\[ \mathcal{M}_{ij,\frac{1}{2}-BPS} = 0, \]
\[ \mathcal{N}_{i\overline{j},\frac{1}{2}-BPS} = 2 G_{ij} \bigg|_{\frac{1}{2}-BPS} |Z|^2_{\frac{1}{2}-BPS} > 0, \] (2.3.1.5)

where the notation “\( > 0 \)” is clearly understood as strict positive definiteness of the quadratic form related to the square matrix being considered. Notice that the Hermiticity and strict positive definiteness of \( \mathbf{H}^{V_{BH}}_{\frac{1}{2}-BPS} \) are respectively due to the Hermiticity and strict positive definiteness of the Kähler metric \( G_{i\overline{j}} \) of the scalar manifold.

By switching from the non-flat \( i \)-coordinates to the “flat” local \( I \)-coordinates by using the
(inverse) Vielbein $e_i^j$ of the scalar manifold, Eqs. (2.3.1.5) can be rewritten as

$$\mathcal{M}_{IJ,\frac{1}{2}-BPS} = 0,$$

$$\mathcal{N}_{I\mathcal{T},\frac{1}{2}-BPS} = 2\delta_{I\mathcal{T}}|Z|^2_{\frac{1}{2}-BPS} > 0.$$  

Thus, one obtains that in all $N = 2, d = 4$ MESGT’s the $\frac{1}{2}$-BPS mass spectrum in “flat” coordinates is monochromatic, i.e. that all “particles” (i.e. the “modes” related to the degrees of freedom described by the “flat” local $I$-coordinates) acquire the same mass at $\frac{1}{2}$-BPS critical points of $V_{BH}$.

### 2.3.2 Non-BPS, $Z \neq 0$

In this case the result of [13] should apply, namely the Hessian matrix $H^V_{non-BPS,Z\neq0}$ should have $n_V + 1$ strictly positive and $n_V - 1$ vanishing real eigenvalues. By recalling the analysis performed in Sect. 2.2, it is thence clear that such massive and massless non-BPS, $Z \neq 0$ “modes” fit distinct real representations of $\hat{h} = mcs(\hat{H})$, where $\hat{H}$ is the non-compact stabilizer of the class $O_{non-BPS,Z\neq0} = \frac{G}{\hat{H}}$ of non-BPS, $Z \neq 0$ “large” charge orbits.

This is perfectly consistent with the decomposition (2.2.2.6) of the complex representation $R_{H_0}$ $(\dim_{\mathbb{R}} R_{H_0} = 2n_V)$ of $H_0$ in terms of representations of $\hat{h}$:

$$R_{H_0} \rightarrow (R_{\hat{h}} + 1)_C = (R_{\hat{h}} + 1 + R_{\hat{h}} + 1)_\mathbb{R}, \quad \dim_{\mathbb{R}} (R_{\hat{h}})_\mathbb{R} = n_V - 1.$$  

(2.3.2.7)

As yielded by the treatment given in Subsect. 2.2.2, the notation “$(R_{\hat{h}} + 1)_C = (R_{\hat{h}} + 1 + R_{\hat{h}} + 1)_\mathbb{R}$” denotes nothing but the decomplexification of $(R_{\hat{h}} + 1)_C$, which is actually composed by a pair of real irreducible representations $(R_{\hat{h}} + 1)_\mathbb{R}$ of $\hat{h}$.

Therefore, the result of [13] can be understood in terms of real representations of the $mcs$ of the non-compact stabilizer of $O_{non-BPS,Z\neq0}$: the $n_V - 1$ massless non-BPS, $Z \neq 0$ “modes” are in one of the two real $R_{\hat{h}}$’s of $\hat{h}$ in the r.h.s. of Eq. (2.3.2.7), say the first one, whereas the $n_V + 1$ massive non-BPS, $Z \neq 0$ “modes” are split in the remaining real $R_{\hat{h}}$ of $\hat{h}$ and in the two real $\hat{h}$-singlets. The resulting interpretation of the decomposition (2.3.2.7) is

$$R_{H_0} \rightarrow \begin{pmatrix} (R_{\hat{h}})_\mathbb{R} \\ n_V - 1 \text{ massless} \end{pmatrix} + \begin{pmatrix} (R_{\hat{h}})_\mathbb{R} + 1_R + 1_R \\ n_V + 1 \text{ massive} \end{pmatrix}.$$  

(2.3.2.8)

It is interesting to notice once again that there is no $U(1)$ symmetry relating the two real $R_{\hat{h}}$’s (and thus potentially relating the splitting of “modes” along $O_{non-BPS,Z\neq0}$, since in all symmetric $N = 2, d = 4$ MESGT’s $\hat{h}$ never contains an explicit factor $U(1)$ (as instead it always happens for $\hat{h}$!); this can be related to the fact that the non-compact stabilizer is $\hat{H}$ whose $mcs$ is $h$.

### 2.3.3 Non-BPS, $Z = 0$

For the class $O_{non-BPS,Z=0}$ of “large” non-BPS, $Z = 0$ orbits the situation changes, and the result of [13] no longer holds true, due to the local vanishing of $Z$.

In all magic $N = 2, d = 4$ MESGT’s the complex representation $R_{H_0}$ of $H_0$ decomposes under $\hat{h}' = mcs(\hat{H})/U(1)$ in the following way (see Eq. (2.2.2.17)):

$$R_{H_0} \rightarrow W_{\hat{h}'} + Y_{\hat{h}'} + 1_C,$$  

(2.3.3.9)
where in the r.h.s. the complex $\tilde{h}'$-singlet and the complex non-singlet representations $W_{\tilde{h}'}$ and $Y_{\tilde{h}'}$ of $\tilde{h}'$ appear. Correspondingly, the decomposition of the $H_0$-invariant representation $(R_{H_0})^3$ in terms of representations of $\tilde{h}'$ reads (see Eq. (2.2.2.19))

$$(R_{H_0})^3 \rightarrow (W_{\tilde{h}'})^2 \gamma_{\tilde{h}'} + (Y_{\tilde{h}'})^2 1_C.$$  

(2.3.3.10)

Let us now recall that $\dim \mathbb{R} R_{H_0} = 2n_V$ and $\dim \mathbb{R} 1_C = 2$, and let us define

$$\dim \mathbb{R} W_{\tilde{h}'} \equiv W_{\tilde{h}'};$$
$$\dim \mathbb{R} Y_{\tilde{h}'} \equiv Y_{\tilde{h}'};$$

(2.3.3.11)

Thus, it can generally be stated that the mass spectrum along $O_{\text{non-BPS}, Z=0}$ of all magic $N = 2, d = 4$ symmetric MESGT’s splits under $\tilde{h}'$ as follows:

- the mass “modes” fitting the $W_{\tilde{h}'}$ real degrees of freedom corresponding to the complex $(U(1)^{\text{charged}})$ non-$\tilde{h}'$-singlet representation $W_{\tilde{h}'}$, (which does not couple to the complex $\tilde{h}'$-singlet in the $H_0$-invariant decomposition $\gamma_{\tilde{h}'}$) remain massless;
- the mass “modes” fitting the $Y_{\tilde{h}'} + 2$ real degrees of freedom corresponding to the complex $(U(1)^{\text{charged}})$ non-$\tilde{h}'$-singlet representation $Y_{\tilde{h}'}$ and to the $(U(1)^{\text{charged}})$ $\tilde{h}'$-singlet $1_C$ all become massive.

The resulting interpretation of the decomposition (2.3.3.9) is

$$R_{H_0} \rightarrow \begin{pmatrix} W_{\tilde{h}'} \\ Y_{\tilde{h}'} \end{pmatrix} \begin{pmatrix} \text{massless} \\ \text{massive} \end{pmatrix} + \begin{pmatrix} \gamma_{\tilde{h}'} + 1_C \\ \text{massive} \end{pmatrix}.$$  

(2.3.3.12)

The interpretations (2.3.3.9) and (2.3.3.12) show that, even though the complex representations $W_{\tilde{h}'}$, $Y_{\tilde{h}'}$, and $1_C$ of $\tilde{h}$ are charged with respect to the explicit factor $U(1)^{\text{charged}}$ always appearing in $\tilde{h}$, this fact does not affect in any way the splitting of the non-BPS, $Z = 0$ mass “modes”.

The critical mass spectra of the irreducible sequence $SU(1,1+n)_{(1)} \times SO(2+2n)_{(2)\times SO(2+2n)}$ are treated in Appendices I and II of [3], respectively.

Generally, the Hessian $H^{V_{BH}}$ at regular $N = 2$, non-BPS critical points of $V_{BH}$ exhibits the following features: it does not have “repeller” directions (i.e., strictly negative real eigenvalues), it has a certain number of “attractor” directions (related to strictly positive real eigenvalues), but it is also characterized by some vanishing eigenvalues, corresponding to massless non-BPS “modes”.

A priori, in order to establish whether the considered $N = 2$, non-BPS critical points of $V_{BH}$ are actually attractors in a strict sense, i.e. whether they actually are stable minima of $V_{BH}$ in the scalar manifold, one should proceed further with covariant differentiation of $V_{BH}$, dealing (at least) with third and higher-order derivatives.

The detailed analysis of the issue of stability of both classes of regular non-BPS critical points ($Z \neq 0$ and $Z = 0$) of $V_{BH}$ in $N = 2, d = 4$ (symmetric) MESGT’s was performed in [12]. In that paper it was found that, for all supergravities with homogeneous (not necessarily symmetric) scalar manifolds the massless Hessian modes are actually “flat” directions of $V_{BH}$, i.e. that the Hessian massless modes persist, at the critical points of $V_{BH}$ itself, at all order in covariant differentiation of $V_{BH}$. This is reported in the next Section.
Table 5: Moduli spaces of non-BPS $Z \neq 0$ critical points of $V_{BH,N=2}$ in $N = 2, d = 4$ symmetric supergravities ($\hat{h}$ is the maximal compact subgroup of $\hat{H}$). They are the $N = 2, d = 5$ symmetric real special manifolds [42].

2.4 From Massless Hessian Modes of $V_{BH}$ to Moduli Spaces of Attractors

In $N = 2$ homogeneous (not necessarily symmetric) and $N > 2$-extended (all symmetric), $d = 4$ supergravities the Hessian matrix of $V_{BH}$ at its critical points is in general semi-positive definite, eventually with some vanishing eigenvalues (massless Hessian modes), which actually are flat directions of $V_{BH}$ itself [39, 42]. Thus, it can be stated that for all supergravities based on homogeneous scalar manifolds the critical points of $V_{BH}$ which correspond to “large” black holes (i.e. for which one finds that $V_{BH} \neq 0$) all are stable, up to some eventual flat directions.

As pointed out above, the Attractor Equations of $N = 2, d = 4$ MESGT with $n_V$ Abelian vector multiplets may have flat directions in the non-BPS cases [39, 42], but not in the $\frac{1}{2}$-BPS one [8] (see Eqs. (2.3.1.5) and (2.3.1.6) above).

Tables 5 and 6 respectively list the moduli spaces of non-BPS $Z \neq 0$ and non-BPS $Z = 0$ attractors for symmetric $N = 2$, $d = 4$ special Kähler geometries, for which a complete classification is available [42] (the attractor moduli spaces should exist also in homogeneous non-symmetric $N = 2$, $d = 4$ special Kähler geometries, but their classification is currently unknown). The general “rule of thumb” to construct the moduli space of a given attractor solution in the considered symmetric framework is to coset the stabilizer of the corresponding charge orbit by its mcs. By such a rule, the $\frac{1}{2}$-BPS attractors do not have an associated moduli space simply because the stabilizer of their supporting BH charge orbit is compact. On the other hand, all attractors supported by BH charge orbits whose stabilizer is non-compact exhibit a non-vanishing moduli space. Furthermore, it should be noticed that the non-BPS $Z \neq 0$ moduli spaces are nothing but the symmetric real special scalar manifolds of the corresponding $N = 2$, $d = 5$ supergravity.
Table 6: Moduli spaces of non-BPS $Z = 0$ critical points of $V_{BH,N=2}$ in $N = 2, d = 4$ symmetric supergravities ($\tilde{h}$ is the maximal compact subgroup of $\tilde{H}$). They are (non-special) symmetric Kähler manifolds [22].

Nevertheless, it is worth remarking that some symmetric $N = 2, d = 4$ supergravities have no non-BPS flat directions at all.

The unique $n_V = 1$ symmetric models are the so-called $t^2$ and $t^3$ models; they are based on the rank-1 scalar manifold $SU(1, n \in \mathbb{N} \cup \{0\})$, but with different holomorphic prepotential functions.

The $t^2$ model is the first element $(n = 0)$ of the sequence of irreducible symmetric special Kähler manifolds $SU(1,n+1)$, but with different holomorphic prepotential. Its bosonic sector is given by the $(U(1))^6 \rightarrow (U(1))^2$ truncation of Maxwell-Einstein-axion-dilaton (super)gravity, i.e. of pure $N = 4, d = 4$ supergravity (see e.g. [71] and Refs. therein).

On the other hand, the $t^3$ model has cubic prepotential; as pointed out above, it is an isolated case in the classification of symmetric SK manifolds (see e.g. [102]; see also [103] and Refs. therein), but it can be thought also as the $s = t = u$ degeneration of the $stu$ model. It is worth pointing out that the $t^2$ and $t^3$ models are based on the same rank-1 SK manifold, with different constant scalar curvature, which respectively can be computed to be (see e.g. [35] and Refs. therein)

\[
\frac{SU(1,1)}{U(1)}, \quad t^2 \text{ model} : R = -2; \\
\frac{SU(1,1)}{U(1)}, \quad t^3 \text{ model} : R = -\frac{2}{3}.
\]
Beside the $\frac{1}{2}$-BPS attractors, the $t^2$ model admits only non-BPS $Z = 0$ critical points of $V_{BH}$ with no flat directions. Analogously, the $t^3$ model admits only non-BPS $Z \neq 0$ critical points of $V_{BH}$ with no flat directions.

For $n_V > 1$, the non-BPS $Z \neq 0$ critical points of $V_{BH}$, if any, all have flat directions, and thus a related moduli space (see Table 5). However, models with no non-BPS $Z = 0$ flat directions at all and $n_V > 1$ exist, namely they are the first and second element ($n = -1, 0$) of the sequence of reducible symmetric special Kähler manifolds $SU(1,1) \times SU(2,n+2) \times SU(3,n+2)$ ($n_V = n + 3, n \in \mathbb{N}\cup\{0, -1\}$) (see e.g. [3] and Refs. therein), i.e. the so-called $st^2$ and $stu$ models, respectively. The $stu$ model ([104][105], see also e.g. [77] and Refs. therein) has two non-BPS $Z \neq 0$ flat directions, spanning the moduli space $SO(1,1) \times SO(1,1)$ (i.e. the scalar manifold of the $stu$ model in $d = 5$), but no non-BPS $Z = 0$ massless Hessian modes at all. On the other hand, the $st^2$ model (which can be thought as the $t = u$ degeneration of the $stu$ model) has one non-BPS $Z \neq 0$ flat direction, spanning the moduli space $SO(1,1)$ (i.e. the scalar manifold of the $st^2$ model in $d = 5$), but no non-BPS $Z = 0$ flat direction at all. The $st^2$ is the “smallest” symmetric model exhibiting a non-BPS $Z \neq 0$ flat direction.

Concerning the “smallest” symmetric models exhibiting a non-BPS $Z = 0$ flat direction they are the second ($n = 1$) element of the sequence $SU(1,n+1)$ and the third ($n = 1$) element of the sequence $SU(1,1) \times SU(2,n+2) \times SU(3,n+2)$. In both cases, the unique non-BPS $Z = 0$ flat direction spans the non-BPS $Z = 0$ moduli space $SU(1,1) \times SU(2,1) \sim SO(2,1)$ (see Table 6), whose local geometrical properties however differ in the two cases (for the same reasons holding for the $t^2$ and $t^3$ models treated above).

We conclude by recalling that in [123][124][125] it was shown that the $N = 2, d = 5$ magic
In $N \geq 3$-extended, $d = 4$ supergravities, whose scalar manifold is always symmetric, there are flat directions of $V_{BH}$ at both its BPS and non-BPS critical points. As mentioned above, from a group-theoretical point of view this is due to the fact that the corresponding supporting BH charge orbits always have a non-compact stabilizer [42, 61]. The BPS flat directions can be interpreted in terms of left-over hypermultiplets’ scalar degrees of freedom in the truncation down to the $N = 2, d = 4$ theories [126, 39]. In Tables 7 and 8 all (classes of) “large” charge orbits and the corresponding moduli spaces of attractor solution in $N \geq 3$-extended, $d = 4$ supergravities are reported [61].
4 Conclusions

In the present report we dealt with results holding at the classical, Einstein supergravity level. It is conceivable that the flat directions of classical extremal BH attractors will be removed (i.e. lifted) by quantum (perturbative and non-perturbative) corrections (such as those coming from higher-order derivative contributions to the gravity and/or gauge sector) to the classical effective BH potential $V_{BH}$. Consequently, at the quantum level, moduli spaces for attractor solutions may not exist at all (and therefore also the actual attractive nature of the critical points of $V_{BH}$ might be destroyed). However, this may not be the case for $N=8$.

In the presence of quantum lifts of classically flat directions of the Hessian matrix of $V_{BH}$ at its critical points, in order to answer the key question: “Do extremal BH attractors (in a strict sense) survive at the quantum level?”, it is thus crucial to determine whether such lifts originate from Hessian modes with positive squared mass (corresponding to attractive directions) or with negative squared mass (i.e. tachyonic, repeller directions).

The fate of the unique non-BPS $Z \neq 0$ flat direction of the $st^2$ model in presence of the most general class of quantum perturbative corrections consistent with the axionic-shift symmetry has been studied in [80], showing that, as intuitively expected, the classical solutions get lifted at the quantum level. Interestingly, in [80] it is found that the quantum lift occurs more often towards repeller directions (thus destabilizing the whole critical solution, and destroying the attractor in a strict sense), than towards attractive directions. The same behavior may be expected for the unique non-BPS $Z = 0$ flat direction of the $n = 2$ element of the quadratic irreducible sequence and the $n = 3$ element of the cubic reducible sequence (see above).

Generalizing it to the presence of more than one flat direction, this would mean that only a (very) few classical attractors do remain attractors in a strict sense at the quantum level; consequently, at the quantum (perturbative and non-perturbative) level the “landscape” of extremal BH attractors should be strongly constrained and reduced.

Despite the considerable number of papers written on the Attractor Mechanism in the extremal BHs of the supersymmetric theories of gravitation in past years, still much remains to be discovered along the way leading to a deep understanding of the inner dynamics of (eventually extended) space-time singularities in supergravities, and hopefully of their fundamental high-energy counterparts, such as $d = 10$ superstrings and $d = 11$ $M$-theory.

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