UNIFORM MATROIDS ARE EHRHART POSITIVE

LUIS FERRONI

Abstract. In [1] De Loera et al. conjectured that the Ehrhart polynomial of the basis polytope of a matroid has positive coefficients. We prove this conjecture for all uniform matroids. In other words, we prove that every hypersimplex is Ehrhart positive. In order to do that, we introduce the notion of weighted Lah numbers and study some of their properties. Then we provide a formula for the coefficients of the Ehrhart polynomial of a hypersimplex in terms of these numbers.

1. Introduction

The basis polytope (also known as the matroid polytope) of a matroid is defined as the convex hull of the indicator functions of its basis (cf. [2]). It encodes all the information about the matroid, hence providing a geometric point of view on matroidal notions and problems. Moreover, this polytope allows to define new invariants of the matroid, see for instance [3].

An important invariant of a polytope $P$ whose vertices lie in $\mathbb{Z}^d$, is the so-called Ehrhart polynomial, introduced in [4]. It is defined as the polynomial $p \in \mathbb{Q}[t]$ such that $p(t) = |\mathbb{Z}^d \cap tP|$ for $t \in \mathbb{Z}_{\geq 0}$, being $tP$ the dilation of $P$ with respect to the origin by the factor $t$. In particular, since the basis polytope of a matroid has vertices with 0/1 coordinates, we can consider its Ehrhart polynomial, which we will sometimes refer as the Ehrhart Polynomial of the matroid itself.

In the paper [1 Conjecture 2(B)], the authors posed the following conjecture:

Conjecture 1.1 (De Loera et. al.). Let $P(M)$ be the matroid polytope of a matroid $M$. Then the coefficients of the Ehrhart polynomial of $P(M)$ are positive.

Also in [1 Lemma 29] they give a proof of the following result:

Proposition 1.2. The coefficients of the Ehrhart polynomial of the matroid polytope of the uniform matroid $U_{2,n}$ are positive.

Their proof is based on inequalities, using a result of Katzman in [5] regarding a formula Ehrhart polynomials of uniform matroids, in the context of algebras of Veronese type. We will restate this result in the following section and exploit it to prove the main result of this article.

Theorem 1.3. The coefficients of the Ehrhart polynomial of the matroid polytope of all uniform matroids $U_{k,n}$ are positive.

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It is worth noting that the polytope $\mathcal{P}(U_{k,n})$ is known in the literature (see for example [6]) as the hypersimplex $\Delta_{n,k}$. So our main result can be rephrased so to say that all hypersimplices are Ehrhart positive.

In the paper [7] the Conjecture [1.1] has been strengthened and reformulated in a more general setting, asserting that indeed all generalized permutohedra are Ehrhart positive. Also, in [8] there is a survey on the families of polytopes that are known to be Ehrhart positive, and those that are conjectured to also have this property.

2. The Ehrhart Polynomial of Uniform Matroids

We recall from [9] that a matroid is a pair $M = (E, \mathcal{B})$ where $E$ is a finite set and $\mathcal{B}$ is a family of subsets of $E$ called the set of basis that satisfy the following properties:

1. $\mathcal{B} \neq \emptyset$.
2. If $A$ and $B$ are distinct members of $\mathcal{B}$ and $a \in A \setminus B$, then there exists an element $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

This notion provides a useful generalization of both the concept of graph and of linear independence in vector spaces. Also, recall that a uniform matroid $U_{k,n}$ is defined as the matroid on the set $E = \{1, 2, \ldots, n\}$ (which for short we will denote by $[1, n]$) with $\mathcal{B} = \{B \subseteq E : |B| = k\}$.

Recalling [2], the basis polytope (or matroid polytope) of a matroid $M = ([1, n], \mathcal{B})$ is defined as:

$$\mathcal{P}(M) := \text{convex hull}\{e_B : B \in \mathcal{B}\},$$

where $e_B$ is a vector in $\mathbb{R}^n$ defined as a sum of canonical vectors, $e_B := \sum_{i \in B} e_i$, i.e. it is an indicator of the elements in the set $B$.

The polytopes that can arise as the basis polytope of a matroid are characterized in [2]. These are precisely the polytopes that have vertices with 0/1 coordinates and have edges with directions $e_i - e_j$. Conversely, every such a polytope comes from a matroid.

For a uniform matroid $U_{k,n}$, where $k$ denotes the rank and $n$ the number of elements in the underlying set, we will denote $E_{k,n}$ the Ehrhart Polynomial of its basis polytope.

Remark 2.1. For a given uniform matroid $U_{k,n}$ where $1 \leq k \leq n - 1$, its basis polytope $\Delta_{n,k}$ is $(n - 1)$-dimensional, so its Ehrhart polynomial has degree $n - 1$.

Also, observe that, since the matroids $U_{k,n}$ and $U_{n-k,n}$ are the dual of each other, they have the same Ehrhart polynomial (this can be seen directly from the polytopes: there is a reflection that interchanges the zeros and ones of the vertices that maps $\mathcal{P}(U_{k,n}) \rightarrow \mathcal{P}(U_{n-k,n})$).

We will base our computations on a formula found by Katzman in [5, Corollary 2.2] for $E_{k,n}$, in the language of Veronese algebras. For the sake of completeness we state this result here, adapting it to our notation.

Theorem 2.2. The Ehrhart Polynomial $E_{k,n}(t)$ of the uniform matroid $U_{k,n}$ is given by:

$$E_{k,n}(t) = \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} \binom{(k-j)t + n - 1 - j}{n-1}.$$
It is not at all apparent from this formula that the coefficients of the polynomial $E_{k,n}$ are positive. Indeed the alternating factor $(-1)^j$ and the fact that the variable $t$ appears inside a binomial coefficient which in turn for $j > 1$ is a polynomial with some negative coefficients don’t permit us to see this fact directly.

We will establish two supplementary results that are not necessary to prove Theorem 1.3 but are of some independent interest and are not at all evident.

First, recall that if one has a (formal) power series $f(x) := \sum_{j=0}^{\infty} a_j x^j$, it is customary to use the notation $[x^t] f(x) := a_t$.

**Proposition 2.3.** If $1 \leq k \leq n - 1$, and $t \geq 0$ the coefficient of $x^{kt}$ in the polynomial $(1 + x + x^2 + \ldots + x^t)^n$ is exactly $E_{k,n}(t)$.

**Proof.** We will use generating functions to compute the coefficient of $x^{kt}$ in $(1 + x + \ldots + x^t)^n$ and will compare it to the formula (2.1). Note that:

$$[x^{kt}] (1 + x + \ldots + x^t)^n = [x^{kt}] \left( \frac{1-x^{t+1}}{1-x} \right)^n$$

$$= [x^{kt}] \left( 1 - x^{t+1} \right)^n \cdot \frac{1}{(1-x)^n}$$

So writing $\left( 1 - x^{t+1} \right)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{(t+1)j}$ and $\frac{1}{(1-x)^n} = \sum_{j=0}^{\infty} \binom{n-1+j}{n-1} x^j$, the coefficient of $x^{kt}$ in this product can be computed as a convolution:

$$\sum_{j=0}^{k-1} (-1)^j \binom{n}{j} \binom{n-1 + (k-j)t - j}{n-1}$$

where the sum ends in $k-1$ since in the first of our two formal series we have $x^{(t+1)j}$ and we are computing the coefficient of $x^{kt}$. Also, the second binomial coefficient in our expression comes from the fact that $(t+1)j + ((k-j)t - j) = kt$.

Since the formula we obtained is exactly (2.1), the result follows. □

As a corollary of the previous result we have a rather unexpected consequence regarding the Ehrhart polynomials of all the uniform matroids on a given fixed set.

**Corollary 2.4.** Let $n$ be a positive integer. Then:

$$E_{1,n}(t) + E_{2,n}(t) + \ldots + E_{n-1,n}(t) = \frac{(t+1)^n - t - 1}{t}.$$

**Proof.** Let’s fix $t$. By Proposition 2.3 we have that $E_{1,n}(t)$ is the coefficient of $x^t$ in $(1 + x + \ldots + x^t)^n$, $E_{2,n}(t)$ is the coefficient of $x^{2t}$, and so on. The sum on the left hand side of the statement is simply the sum of all the coefficients of the terms with an exponent multiple of $t$.

However, notice that:

$$(1 + x + \ldots + x^t)^n = (1 + x + \ldots + x^t) \cdot \ldots \cdot (1 + x + \ldots + x^t)_{\text{n times}}.$$
up a multiple of $t$. We can choose the first $n-1$ numbers in all $(t+1)^{n-1}$ possible ways, and we can always with the last choose one to make a multiple of $t$, being careful with the fact that if we already had a multiple of $t$, then the last one has two possibilities, namely: 0 or $t$. In particular, we get a recurrence:

$$T_n = (t+1)^{n-1} + T_{n-1},$$

where $T_i$ is the number of ways of choosing $i$ numbers in $\{0, 1, \ldots, t\}$ that add up a multiple of $t$. Iterating this recurrence we get:

$$T_n = (t+1)^{n-1} + (t+1)^{n-2} + \ldots + (t+1).$$

Since this is just a geometric series, this gives us:

$$T_n = \frac{(t+1)^n - t - 1}{t}.$$

Also, since $t$ is immaterial at this point, we can just equate coefficients to get the result of the statement. □

3. Weighted Lah Numbers

In this section we develop some useful tools to prove Theorem 1.3. We recall the definition of Lah numbers (also known as Stirling Numbers of the 3rd kind) which have been extensively studied (see for example [10, 11, 12]).

**Definition 3.1.** The Lah number $L(n, m)$ is defined as the number of ways of partitioning the set $\{1, 2, \ldots, n\}$ into exactly $m$ linearly ordered blocks.

**Example 3.2.** $L(3, 2) = 6$ because we have the following possible partitions:

- $\{(1, 2), (3)\}, \{(2, 1), (3)\}$,
- $\{(1, 3), (2)\}, \{(3, 1), (2)\}$,
- $\{(2, 3), (1)\}, \{(3, 2), (1)\}$.

If $\pi$ is a partition of $\{1, \ldots, n\}$ into $m$ linearly ordered blocks, for any of these blocks $b$, we will write $b \in \pi$. So, for example $(2, 3) \in \{(2, 3), (1)\}$.

**Remark 3.3.** We have the equality $L(n, m) = \frac{n!}{m!} \binom{n-1}{m-1}$. This can be proven easily by a combinatorial argument as follows. Order the $n$ numbers on the set in any fashion. To get the partition we can put $m-1$ divisions in any of the $n-1$ spaces between two consecutive numbers. Then divide by $m!$, the number of ways of ordering all the blocks.

There are several generalizations of these numbers such as the $r$-Lah numbers, and the associated Lah numbers (cf. [13]). We will introduce a new generalization that we will call weighted Lah numbers.

**Definition 3.4.** Let $\pi$ be a partition of the set $\{1, \ldots, n\}$ into $m$ linearly ordered blocks. We define the weight of $\pi$ by the following formula:

$$w(\pi) := \sum_{b \in \pi} w(b),$$

where $w(b)$ is the number of elements in $b$ that are smaller (as positive integers) than the first element in $b$. 

Example 3.5. Among the 6 partitions that we have seen that exist of \{1, 2, 3\} into 2 blocks, we have:
\[
\begin{align*}
  w(\{(1, 2), (3)\}) &= 0 + 0 = 0, \quad w(\{(2, 1), (3)\}) = 1 + 0 = 1, \\
  w(\{(1, 3), (2)\}) &= 0 + 0 = 0, \quad w(\{(3, 1), (2)\}) = 1 + 0 = 1, \\
  w(\{(2, 3), (1)\}) &= 0 + 0 = 0, \quad w(\{(3, 2), (1)\}) = 1 + 0 = 1.
\end{align*}
\]
Note that there are exactly 3 of these partitions of weight 0 and exactly 3 of weight 1.

Definition 3.6. We define the weighted Lah Numbers \(W(\ell, n, m)\) as the number of partitions of weight \(\ell\) of \(\{1, \ldots, n\}\) into exactly \(m\) linearly ordered blocks.

Example 3.7. Rephrasing the conclusion of the Example 3.5 we have that \(W(0, 3, 2) = 3\) and \(W(1, 3, 2) = 3\).

The set of all partitions of \(\{1, \ldots, n\}\) into \(m\) linearly ordered blocks and weight \(\ell\) will be denoted by \(W(\ell, n, m)\).

Remark 3.8. Observe that \(W(\ell, n, m) \neq 0\) only for \(0 \leq \ell \leq n - m\). Since the maximum weight can be obtained by ordering every block in such a way that its maximum element is on the first position. Also, we have the following:
\[
W(0, n, m) = \left[ \begin{array}{c} n \\ m \end{array} \right]
\]
where the brackets denote the Stirling numbers of the first kind. This can be proven combinatorially by noticing that for every permutation with exactly \(m\) cycles, we can present it in a unique way as a partition of weight \(n - m - \ell\).

Remark 3.9. We have symmetry, namely:
\[
W(\ell, n, m) = W(n - m - \ell, n, m).
\]
This equality is a consequence of the fact that for \(\pi \in W(\ell, n, m)\) we can associate bijectively an element \(\pi' \in W(n - m - \ell, n, m)\) as follows. In \(\pi\) interchange the positions of 1 and \(n\), of 2 and \(n - 1\), and so on. What one gets is exactly a partition of weight \(n - m - \ell\).

It is possible to obtain a bunch of recurrences to compute \(W(\ell, n, m)\) recursively. For instance we include the following:

| \(m\) | 1  | 2  | 3  | 4  | 5  |
|---|---|---|---|---|---|
| 1 | 24 | 24 | 24 | 24 | 24 |
| 2 | 50 | 70 | 70 | 50 |   |
| 3 | 35 | 50 | 35 |   |   |
| 4 | 10 | 10 |   |   |   |
| 5 | 1  | 1  |   |   |   |

Table 1. \(W(\ell, 5, m)\)

| \(m\) | 1  | 2  | 3  | 4  | 5  |
|---|---|---|---|---|---|
| 1 | 120| 120| 120| 120| 120|
| 2 | 274| 404| 444| 404| 274|
| 3 | 225| 375| 375| 225|   |
| 4 | 85 | 130| 85 |   |   |
| 5 | 15 | 15 |   |   |   |
| 6 | 1  |   |   |   |   |

Table 2. \(W(\ell, 6, m)\)
Proposition 3.10. The following recurrence holds for $n, m \geq 2$:

$$W(\ell, n, m) = (n - 1)W(\ell - 1, n - 1, m) + \sum_{j=0}^{n-1} \frac{(n - 1)}{j} j!W(\ell, n - j - 1, m - 1).$$

Proof. Every $\pi \in W(\ell, n, m)$ has the number 1 inside a block. If this number is not the first element of his block, this means that if we remove it from $\pi$ we end up getting an element of $W(\ell - 1, n - 1, m)$ (with every element shifted by one). Analogously, we can pick an element of $W(\ell, n - 1, m)$ (which we think of as having every element shifted by one) and reconstruct an element of $W(\ell, n, m)$ but adjoining the element 1 in such a way that it is not a first element of his block. There are $n - 1$ possibilities of where to put the number $n$ to get an element of $W(\ell, n, m)$. So we get the first summand.

The remaining cases to consider are those on which 1 is the first element of his block. In this case we choose $j$ elements to be in this block, and in every possible order of these elements, the block will always have weight 0. So the remaining $n - j - 1$ elements will have to be arranged in $m - 1$ blocks of weight $\ell$. \hfill \Box

Remark 3.11. The last proposition tells us that if we make the subtraction $W(\ell, n, m) - (n - 1)W(\ell, n - 1, m)$ we end up getting an expression for which the sum cancels out to give just the recurrence:

$$W(\ell, n, m) = (n - 1)W(\ell - 1, n - 1, m) + (n - 1)W(\ell, n - 1, m) - (n - 1)(n - 2)W(\ell - 1, n - 2, m) + W(\ell, n - 1, m - 1).$$

We establish now a bivariate generating function for $W(\ell, n, m)$ for a fixed $m$.

Lemma 3.12. We have the equality:

$$W(\ell, n, m) = \frac{n!}{m!} [x^n s^\ell] \left( \frac{1}{1-s} \right)^m \left( \log \left( \frac{1}{1-x} \right) - \log \left( \frac{1}{1-sx} \right) \right)^m.$$

Proof. Recall that the exponential generating function of the Lah Numbers gives us:

$$L(n, m) = \frac{n!}{m!} [x^n] \left( \frac{x}{1-x} \right)^m.$$

Observe that $\frac{1}{1-x} = \sum_{k=1}^{\infty} x^k$ is the exponential generating function for the factorials. We want to introduce a new variable $s$ as a marker for the weight in order to get a bivariate generating function. For this purpose, we can replace in the previous sum $x^k$ by $\frac{x^k}{k} (1 + s + \ldots + s^{k-1})$. This comes from the fact that $\frac{(k-1)!}{k!}$ of these orderings are such that the element $j + 1$ is in the first position, so the weight of the block is exactly $j$. We have:

$$W(\ell, n, m) = \frac{n!}{m!} [x^n s^\ell] \left( \sum_{k=1}^{\infty} \frac{x^k}{k} (1 + s + \ldots + s^{k-1}) \right)^m.$$

Notice that using the formula for the geometric series, the sum in the parentheses can be rewritten as $\frac{1}{1-s} \left( \sum_{k=1}^{\infty} x^k - \sum_{k=1}^{\infty} \frac{(sx)^k}{k} \right)$ which in turn is just

$$\frac{1}{1-s} \left( \log \left( \frac{1}{1-x} \right) - \log \left( \frac{1}{1-sx} \right) \right),$$

which gives the desired result. \hfill \Box
Corollary 3.13. For all $\ell, n, m$ one has:

$$W(\ell, n, m) = \sum_{j=0}^{\ell} \sum_{i=0}^{n-m} (-1)^{j+i} \binom{n}{j} \binom{m + \ell - j - 1}{m - 1} \binom{n - j}{m + i - j}.$$ 

Proof. Recall that from the exponential generating function of the Stirling numbers of the first kind one has:

$$\left[ \frac{\alpha}{\beta} \right] = \frac{\alpha!}{\beta!} [x^\alpha] \left( \log \left( \frac{1}{1-x} \right) \right)^\beta$$

Now, using Lemma 3.12 we have the chain of equalities:

$$W(\ell, n, m) = \frac{n!}{m!} [x^n s^\ell] \left( \frac{1}{1-s} \right)^m \left( \log \left( \frac{1}{1-x} \right) - \log \left( \frac{1}{1-sx} \right) \right)^m$$

$$= \frac{n!}{m!} [x^n s^\ell] \left( \frac{1}{1-s} \right)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left( \log \left( \frac{1}{1-x} \right) \right)^{m-k} \left( \log \left( \frac{1}{1-sx} \right) \right)^k$$

$$= n! [x^n s^\ell] \left( \frac{1}{1-s} \right)^m \sum_{k=0}^{m} (-1)^k \sum_{j=0}^{n} \left[ \frac{\log \left( \frac{1}{1-x} \right)^{m-k}}{(m-k)!} \right] \left[ \frac{\log \left( \frac{1}{1-sx} \right)^k}{k!} \right]$$

$$= n! [s^\ell] \left( \frac{1}{1-s} \right)^m \sum_{k=0}^{m} (-1)^k \sum_{j=0}^{n} \left[ \frac{\log \left( \frac{1}{1-x} \right)^{m-k}}{(m-k)!} \right] \left[ \frac{\log \left( \frac{1}{1-sx} \right)^k}{k!} \right]$$

$$= n! [s^\ell] \left( \frac{1}{1-s} \right)^m \sum_{k=0}^{m} (-1)^k \sum_{j=0}^{n} \left[ \frac{\log \left( \frac{1}{1-x} \right)^{m-k}}{(m-k)!} \right] \left[ \frac{\log \left( \frac{1}{1-sx} \right)^k}{k!} \right]$$

$$= \sum_{k=0}^{m} \binom{n}{j} \binom{m - 1 + \ell - j}{m - 1} \binom{n - j}{m - k} \binom{j}{j-i}$$

which is the expression we looked for. Notice that the upper limits of both sums can be set to be $\ell$ and $n - m$ to avoid summing zeros. \qed

4. The Proof of Theorem 1.3

For $0 \leq m \leq n - 1$, we will call $e_{k,n,m}$ the coefficient of $t^m$ in the polynomial $E_{k,n}(t)$. Our aim is to show that all these $e_{k,n,m}$ are positive.

Lemma 4.1. The following formula holds:

$$e_{k,n,m} = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} \sum_{i=0}^{n-m-1} (-1)^{j+i} \binom{n}{j} (k-j)^m \binom{n-j}{m+1+i-j} \binom{j}{j-i}$$

Proof. We will work with the formula (2.1). Observe that:

$$[t^m] \left( \begin{array}{c} (k-j)t + n - 1 - j \\ n-1 \end{array} \right) = \frac{1}{(n-1)!} [t^m] \left( (k-j)t + n - 1 - j \right) \cdot \ldots \cdot \left( (k-j)t + 1 - j \right)$$
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\[
\frac{1}{(n-1)!} (k-j)^m P_{1-j,n-1-j}^{n-1-m},
\]

where the notation \( P_{a,b}^u \) stands the sum of all possible products of \( u \) numbers chosen in the interval of integers \([a,b]\). Observe that one has the following equality:

\[
P_{1-j,n-1-j}^{n-1-m} = \sum_{i=0}^{n-m-1} P_{1-j,-1}^i P_{1,n-1-j}^{n-1-m-i}
\]

\[
= \sum_{i=0}^{n-m-1} (-1)^i P_{1,j-1}^i P_{1,n-1-j}^{n-1-m-i}.
\]

Since it is a well known fact that \( P_{a,b}^u = \lfloor b+1 \rfloor \), we have that

\[
P_{1-j,n-1-j}^{n-1-m} = \sum_{i=0}^{n-m-1} (-1)^i \left[ \begin{array}{c} j \\ j-i \end{array} \right] \left[ \begin{array}{c} n-j \\ m+1+i-j \end{array} \right].
\]

so, in particular,

\[
[k^m] \left( \frac{(k-j) t^n - n - 1 - j}{n-1} \right) = \frac{1}{(n-1)!} \sum_{i=0}^{n-m-1} (-1)^i (k-j)^m \left[ \begin{array}{c} j \\ j-i \end{array} \right] \left[ \begin{array}{c} n-j \\ m+1+i-j \end{array} \right].
\]

The result follows easily from (2.1) and this last identity. \( \square \)

Notice that if we write \( f_{j,n,m} \) as a short name for

\[
\sum_{n-j=0}^{n-m-1} (-1)^i \left[ \begin{array}{c} j \\ j-i \end{array} \right] \left[ \begin{array}{c} n-j \\ m+1+i-j \end{array} \right],
\]

we can rewrite the formula of Lemma 4.1 as follows:

\[
(4.1) \quad e_{k,n,m} = \frac{1}{(n-1)!} \sum_{j=0}^{k} (-1)^j \left( \begin{array}{c} n \\ j \end{array} \right) (k-j)^m f_{j,n,m}.
\]

We changed the upper limit of the sum, given for \( j = k \) we are adding 0. With all these been said, let’s state the result from which our main theorem follows.

**Theorem 4.2.** For all \( k, n, m \) as above, we have that:

\[
e_{k,n,m} = \frac{1}{(n-1)!} \sum_{\ell=0}^{k-1} W(\ell, n, m+1) A(m, k-\ell-1),
\]

where \( A(m, k-\ell-1) \) stands for the Eulerian numbers (cf. \[10\]). In particular \( e_{k,n,m} \) is positive.

**Proof.** From equation (4.1) we can see that:

\[
e_{k,n,m} = \frac{1}{(n-1)!} [x^k] F_{n,m}(x) \cdot G_m(x),
\]

where \( F_{n,m}(x) := \sum_{j=0}^{n} (-1)^j \left[ \begin{array}{c} n \\ j \end{array} \right] f_{j,n,m} x^j \) and \( G_m(x) = \sum_{j=0}^{\infty} j^m x^j \). It is a well known consequence of the Worpitzky Identity (cf. \[10\]) that:

\[
G_m(x) = \frac{1}{(1-x)^{m+1}} \sum_{j=0}^{m} A(m, j) x^{j+1},
\]

where \( A(m, j) \) is an Eulerian number (the number of permutations of \( m \) elements with exactly \( j \) descents, cf. \[12\]).
So we have that the product $F_{n,m}(x) \cdot G_m(x)$ is equal to:

$$\frac{1}{(1-x)^{m+1}} F_{n,m}(x) \sum_{j=0}^{m} A(m,j) x^{j+1}.$$ 

we can try to compute the product of the first two factors.

$$C_{n,m}(x) := \frac{1}{(1-x)^{m+1}} F_{n,m}(x)$$

Observe that:

$$[x^\ell] C_{n,m}(x) = [x^\ell] \left( \frac{1}{(1-x)^{m+1}} F_{n,m}(x) \right)$$

$$= \sum_{j=0}^{\ell} (-1)^j \binom{n}{j} f_{j,n,m} \left( m + \ell - j \right)$$

$$= \sum_{j=0}^{\ell} \sum_{i=0}^{n-m-1} (-1)^{i+j} \binom{n}{j} \binom{m + \ell - j}{m} \binom{j}{i} \binom{n-j}{m+1+i-j}$$

$$= W(\ell, n, m + 1).$$

where in the last step we used Corollary 3.13. In particular $C_{n,m}(x)$ is a polynomial, and the result now follows performing the product $C_{n,m}(x) \cdot \sum_{j=0}^{m} A(m,j) x^{j+1}$ to get the identity of the statement.

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Università di Bologna, Dipartimento di Matematica, Piazza di Porta San Donato, 5, 40126 Bologna BO - Italia
E-mail address: ferroniluis@gmail.com