Superconformal Field Theory In Six Dimensions
And Supertwistor

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We studied the quantum dynamics of six dimensional $N = (2,0)$ superconformal field theory (the QNG theory). We developed the spinor technique for six-dimensional quantum field theories. By combining this technique with the canonical quantization procedure, we can overcome the subtlety of the chiral nature of $N = (2,0)$ free tensor multiplet and work out its quantum mechanical theory. We then studied the $T^2$ compactification of the QNG theory and argued that the resulting four dimensional quantum field theory is indeed the $N = 4$ super-Yang-Mills theory with the $SL(2,\mathbb{Z})$ duality coming from the mapping class symmetries of $T^2$. We also investigated the BPS self-dual string excitations by proposing a CFT description to their world sheet theory. At last, we constructed the super-twistor space $\hat{PT}$ that corresponded to the superconformal group $U^*Sp(4|2,\mathbb{H}) \subset OSp(8|4,\mathbb{C})$, encoded the full free tensor multiplet into it and made some speculations on the possible super-twistor formulation of the QNG theory.

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1. Introduction

The six dimensional $\mathcal{N} = (2, 0)$ superconformal quantum field theory – named as the quantum non-Abelian gerbe (QNG) theory recently [1] – is an important but still mysterious theory. This theory was originally found [2] by considering the Type-IIB superstring theory at an A – D – E singularity of the K3 compactification. The QNG theory is believed to be a well defined quantum field theory in high spacetime dimensions and can serve as a starting point to understand the dynamics of various four-dimensional gauge theories (see the references in [3][4]). In particular, the $\mathbf{T}^2$ compactification of it turns out to be the four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory with complex coupling constant $\tau$ determined by the complex structure of $\mathbf{T}^2$, and the Montonen-Olive duality of the four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory [5] was interpreted as the $SL(2, \mathbb{Z})$ transformations of $\mathbf{T}^2$(for the simply-laced gauge groups) [2]. Besides, the QNG theory is also crucial for M theory. This theory is believed to be the exact theory that controls the world volume dynamics of almost coincident parallel $M_5$-branes [6][7]. And, this theory is conjectured to be the dual Matrix theory [8] describing M theory compactified on $\mathbf{T}^4$ [9]. Moreover, the QNG theory is also the dual conformal field theory(CFT) of M theory on $AdS_7 \times S^4$ [10] (see [3] for an introduction).

The QNG theory has many exotic but fascinating properties. Firstly, this theory is in some certain senses unique. It is the unique superconformal quantum field theory
that lives in maximum spacetime dimension \((D = 6)\) with maximum supersymmetries – according to Nahm’s classification to the superconformal algebra \([11]\). And it is expected to be an isolated interacting fixed point theory of the six-dimensional renormalization group flow (see, for example, \([12]\)). The superconformal symmetry group of this theory is \(U^*Sp(4|2, H) \subset OSp(8|4, \mathbb{C})\), where \(OSp(8|4, \mathbb{C})\) is the complexification of \(U^*Sp(4|2, H)\).

Secondly, the QNG theory is a chiral theory with the chiral supersymmetry \(N = (2, 0)\), this chiral nature makes its Lagrangian description to be a quite subtle problem. In fact, even in Abelian case an ordinary Lagrangian description for the tensor field \(H\) does not exist \([1]\). Thirdly, the theory’s non-Abelian nature introduces more subtleties to its mysterious quantum dynamics. For example, after compactifying on \(T^2\), the theory’s non-Abelian gerbe group \(G\) will become the non-Abelian gauge group \(G\) of the resulting four-dimensional \(N = 4\) super-Yang-Mills theory. Here, \(G\) must be the tensor product of some simple laced Lie groups (in types as \(A – D – E\)) and some copies of \(U(1)\) group.

The purpose of the present paper is to try to develop a systematical method to approach the QNG theory. Our method is based on the union of four different elements. The first two elements are based on a careful analysis to the superconformal symmetry \(U^*Sp(4|2, H) \subset OSp(8|4, \mathbb{C})\) (and its chiral primary representations and operators), and on a combination of the spinor method and the canonical quantization procedure. By using this combination extensively, we can overcome the chiral nature of the free tensor multiplet and get its quantum mechanical theory successfully.

The third element is to get a glance at the non-Abelian nature of the QNG theory, by perturbing it down to four spacetime dimensions \([1]\). Our arguments go as follows: Firstly, we deform the QNG theory to a generic point at the moduli space. We then compactify the theory on a torus \(T^2\) and demonstrate explicitly that the Mantonen-Olive dualities of the four-dimensional Abelian gauge theory come from the \(SL(2, \mathbb{Z})\) mapping class symmetries of \(T^2\). We also show that the four dimensional superconformal symmetry \(PSU(2, 2|4) \subset PSL(4|4, \mathbb{C})\) and its central extensions come from the \(T^2\) compactification of \(U^*Sp(4|2, H) \subset OSp(8|4, \mathbb{C})\) and its central extensions. Especially, the BPS states which become massless at the singularities of the moduli spaces of the two theories match. Thus, we argued that the enhanced non-Abelian gauge symmetry of the four-dimensional theory

\[1\] Nevertheless, the quantum mechanical theory of the Abelian gerbe theory does exist and a beautiful approach to construct its partition function – by identifying the right theta function – has been proposed \([13]\).
should be a manifestation of the corresponding non-Abelian nature of the QNG theory, which is the conjectured connection between the QNG theory and the four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory.

The last element is to study the self-dual string excitations and their tensionless limits. After perturbing the QNG theory to a generic point of the moduli space. There are self-dual string excitations which are coupled to the tensor multiplets. The tensions of these self-dual strings are proportional to the expectation values of the scalars of tensor multiplet. At the moduli space singularities, these self-dual strings become tensionless, the interactions between these tensionless strings and the tensor multiplets may be the origins of the non-Abelian nature of the QNG theory.

In the present paper, motivated by the elegance of the spinor technique and the success of the twistor-string theory for the four-dimensional $\mathcal{N} = 4$ perturbative super-Yang-Mills theory, we try to unify the above four elements into a unique formulation of the QNG theory in terms of the variables of $\hat{T}$. Here, $\hat{T}$ is the supertwistor space that corresponds to the supergroup $OSp(8|4, \mathbb{C})$. We constructed $\hat{T}$ and studied the action of the superconformal symmetry group $OSp(2,6|2) \subset OSp(8|4, \mathbb{C})$ on it. And we also made some efforts to translate the information concerning the tensor fields and the self-dual strings in spacetime to the related data in $\hat{T}$. Finally, we argue that all the information of the QNG theory can be appropriately encoded into the supertwistor space, although the specific formulation is presently unknown to the author.

Our tentative approach may provide a supplement to the beautiful previous ones, which include, for examples, the proposed Matrix theory description for the discrete light cone quantization (DLCQ) of the QNG theory, and the recent proposals to investigate the interactions between a free tensor multiplet and the self-dual strings.

The paper is organized as follows. In section 2, we study the superconformal algebra of $OSp(2,6|2) \subset OSp(8|4, \mathbb{C})$ (see [20] for an investigation with ordinary spacetime indices) and its chiral primary representations. Especially, we include a discussion to the spectrum of scalar chiral primary operators and their dimensions/R-charges relationship. We present the quantum mechanical theory of the Abelian gerbe theory in section 3. At the same section, we also calculate the quantum anomalies of the quantum Abelian gerbe theory, with the results agreeing with known results. Section 4 is devoted to giving a first look at the non-Abelian nature of the QNG theory by investigating the six-dimensional origin of the four-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory in terms of the Hamiltonian formalism. In section 5, we study the self-dual string excitations and their tensionless limits. Section 6 towards a super-twistor formulation for the QNG theory.
2. Superconformal Algebra And Chiral Primary Operators

$\mathcal{N} = (2, 0)$ Superconformal Algebra

The $\mathcal{N} = (2, 0)$ six-dimensional superconformal field theory has a symmetry of supergroup $U^*Sp(4|2, H) \simeq OSp(2, 6|2)$ whose bosonic part is $Spin(2, 6) \times Sp(2, H)_{\mathbb{R}}$. $Spin(2, 6)$ is the two-fold covering of the conformal group $SO(2, 6)$ of $(1+5)$-dimensional Minkowski space $W$. $Sp(2, H)_{\mathbb{R}} \simeq Spin(5)_{\mathbb{R}}$ is the $\mathbb{R}$-symmetry group. And the fermionic generators of $OSp(2, 6|2)$ are in the $(8, 4)$ representation of $Spin(2, 6) \times Sp(2, H)_{\mathbb{R}}$.

One can view $Sp(2, H)_{\mathbb{R}}$ as the intersection of $SU(4)_{\mathbb{R}}$ and $Sp(4, \mathbb{C})$, $Sp(2, H)_{\mathbb{R}} \simeq SU(4)_{\mathbb{R}} \cap Sp(4, \mathbb{C})$, where $Sp(4, \mathbb{C})$ is the subgroup – which commutes with a symplectic form $\omega^{AB}$ – of the volume preserving automorphism $SL(4, \mathbb{C})$ of a four-dimensional complex space $\mathbb{C}^4_{\mathbb{R}}$. Thus, in the $(4)$ representation, the generators of $Sp(2, H)_{\mathbb{R}}$ can be written as $R^{(AB)} = \frac{1}{2} \left( R^A_C \omega^{CB} + R^B_C \omega^{CA} \right)$, where the traceless Hermitian matrixes $R^A_B$ are the generators of $SU(4)_{\mathbb{R}}$ in fundamental representation. The isomorphism between the $(4)$ representation and its dual can be achieved by raising and lowing the indices with $\omega^{AB}$ and its inverse $\omega_{AB}$. The volume form $\epsilon_{ABCD}$ of $\mathbb{C}^4_{\mathbb{R}}$ turns out to be an invariant of $Sp(2, H)_{\mathbb{R}} \subset Sp(4, \mathbb{C})$ and can be given as $\epsilon_{ABCD} = \frac{1}{2} (\omega_{AC} \omega_{BD} - \omega_{BC} \omega_{AD})$.

One can decompose the conformal group $SO(2, 6)$ as the direct product of $(1+5)$-dimensional Lorentz group $SO(1, 5)$ and the $SO(1, 1)$ group generated by the dilation $D$. Accordingly, the fermionic generators $(8, 4)$ of $OSp(2, 6|2)$ can be decomposed into the $Q$-charges $Q^A_\alpha$ and $S$-charges $S_{\alpha A}$, with

$$i[D, Q^A_\alpha] = + \frac{1}{2} Q^A_\alpha,$$

$$i[D, S_{\alpha A}] = - \frac{1}{2} S_{\alpha A}. \quad (2.1)$$

Clearly, $Q^A_\alpha$ transforms as a chiral spinor of $Spin(1, 5)$ and $S_{\alpha A}$ transforms as an anti-chiral spinor.

Now, we’d like to recall some familiar properties of the spinors of $(1+5)$-dimensional spacetime. Firstly, the complexification $Spin(6, \mathbb{C})$ of the Lorentz group $Spin(1, 5)$ has the four dimensional chiral spinor and anti-chiral spinor representations, denoted as $S^+$ and $S^-$. Secondly, $S^+$ and $S^-$ are dual to each other. Thus, given a chiral spinor $\lambda^\alpha \in S^+$ and an anti-chiral spinor $\mu_\alpha \in S^-$, one can form an invariant

$$(\mu, \lambda) = \mu_\alpha \lambda^\alpha. \quad (2.2)$$
Here, we have used the superscription $\alpha$ to label the indices of the spinor in $S^+$, and the subscription $\alpha$ to label the indices of the spinor in $S^-$. Finally, by noticing that Spin$(6, \mathbb{C}) \simeq SL(S^+)=SL(4, \mathbb{C})$, one can impose a real structure $\tau$ on $S^+$ to get the real form Spin$(1,5) \simeq SL(2, \mathbb{H})$. The condition can be explicitly written as, $\tau O = \overline{O} \tau$, where the real structure $\tau$ is represented as a skew matrix $\tau^{\alpha}_{\beta}$ that satisfies $\tau^2 = -1$, and $O = \exp\left(\frac{1}{2} \theta\right) = \frac{1}{2} \theta_{\mu\nu} \gamma^\mu \wedge \gamma^\nu$, is an arbitrary element of Spin$(1,5)$. Here, $\gamma^\mu$ are the bases of the $(1+5)$-dimensional Clifford algebra, $\gamma^\mu \wedge \gamma^\nu = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$, and $\theta_{\mu\nu}$ are the parameters of the Lorentz group Spin$(1,5) \simeq SL(2, \mathbb{H})$.

One can then pick up a real structure $\hat{\tau}$ on the definition super vector space $\mathbb{C}^{8|4} = (S^+, S^-, \mathbb{C}^5_\mathbb{A})$ of the complexified superconformal group $OSp(8|4, \mathbb{C})$. $\hat{\tau}$ acts on the fermionic generators $Q^\alpha_A$ and $S_{\alpha A}$ as follows (the symplectic-Majorana-Weyl conditions)

$$Q^\alpha_A = \tau^{\alpha}_{\beta} \omega_{AB} \overline{Q}^B \quad S_{\alpha A} = \tau^{\beta}_{\alpha} \omega_{AB} \overline{S}^B ,$$

(2.3)

where $\tau^{\alpha}_{\beta}$ is the inverse of $\tau^{\alpha}_{\beta}$.

Some further relationships between the representations of Spin$(6, \mathbb{C})$ will be helpful for us. Firstly, one recalls that the vector representation $V_{C}$ of Spin$(6, \mathbb{C})$ can be constructed as the wedge product of two chiral (or anti-chiral) spinors $V_{C} \simeq S^+ \wedge S^+$ ($V_{C} \simeq S^- \wedge S^-$). Thus, an arbitrary Spin$(6, \mathbb{C})$ vector $A^\mu$ (or $A_\mu$) can be rewritten as $A^{\alpha \beta}$ (or $A_{\alpha \beta}$), with $A^{\alpha \beta} = \gamma^{\alpha \beta A} A^A$, $A_{\alpha \beta} = A^\mu \gamma^\mu_{\alpha \beta}$ (the $\alpha \beta$ superscriptions of $\gamma^{\alpha \beta}_A$ and the $\alpha \beta$ subscriptions of $\gamma^{\mu}_{\alpha \beta}$ both are antisymmetric), where $\gamma^{\mu}_{\alpha \beta}$ are the gamma matrices that map the anti-chiral spinors to the chiral spinors, $\gamma^{\mu}_{\alpha \beta}$ are gamma matrices that map chiral spinors to anti-chiral spinors, and the six-dimensional Clifford algebra is realized as $\{ \gamma^{\alpha \xi}_A \gamma_{\nu \xi \beta} + \gamma^{\alpha \xi}_B \gamma_{\mu \xi \beta} \} = 2 \delta^{\alpha}_{\beta} \eta_{\mu \nu}$, where $\eta_{\mu \nu}$ is the metric of $(1+5)$-dimensional Minkowski space $W$ with signature $(-+---+)$.

Further more, from the volume form of $SL(4, \mathbb{C}) \simeq Spin(6, \mathbb{C})$, one has two natural Spin$(6, \mathbb{C})$ invariant tensors $\epsilon_{\alpha \beta \gamma \delta}$ and $\epsilon^{\alpha \beta \gamma \delta}$, which are just the spinor notational correspondences of the metric tensor $\eta_{\mu \nu}$ and $\eta^{\mu \nu}$, respectively. Hence, one can lower and raise the spinor indices of an arbitrary Spin$(6, \mathbb{C})$ vector by using $\epsilon_{\alpha \beta \gamma \delta}$ and $\epsilon^{\alpha \beta \gamma \delta}$. For examples, $A_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} A^{\gamma \delta}$, $A^{\alpha \beta} = \frac{1}{2} \epsilon^{\alpha \beta \gamma \delta} A_{\gamma \delta}$, and $\eta_{\mu \nu} A^\mu A^\nu = (1/4) A_{\alpha \beta} A^{\alpha \beta}$.

The $Q - Q$ anti-commutators and the $S - S$ anti-commutators of the superconformal algebra can now be written as

$$\{ Q^\alpha_A, Q^\beta_B \} = \omega_{AB} P^{\alpha \beta} ,$$

$$\{ S_{\alpha A}, S_{\beta B} \} = \omega_{AB} K_{\alpha \beta} ,$$

(2.4)
where \( P^{\alpha\beta} = P^{\mu\gamma}_{\alpha\beta} \) are the translations, and \( K^{\alpha\beta} = K^{\mu\gamma}_{\alpha\beta} \) are the special conformal transformations. One can also write out the \( P - S \) and \( K - Q \) commutators

\[
\begin{align*}
\{P^{\alpha\beta}, S_{\gamma A}\} &= Q^{\alpha}_{A} \delta_{\gamma}^{\beta} - Q^{\beta}_{A} \delta_{\gamma}^{\alpha} \\
\{K^{\alpha\beta}, Q_{A}\} &= S_{\alpha A} \delta_{\gamma}^{\gamma} - S_{\beta A} \delta_{\gamma}^{\gamma}.
\end{align*}
\]

(2.5)

Clearly, \( P^{\alpha\beta} \) are the operators with conformal weight 1 and \( K^{\alpha\beta} \) are the operators with weight \(-1\), that is \( i[D, P^{\alpha\beta}] = P^{\alpha\beta} \) and \( i[D, K^{\alpha\beta}] = -K^{\alpha\beta} \). \( \hat{\tau} \) naturally acts on \( P^{\alpha\beta} \) and \( K^{\alpha\beta} \) as

\[
\begin{align*}
P^{\alpha\beta} &= \tau^{\alpha}_{\gamma} \tau^{\beta}_{\delta} P^{\gamma\delta} \\
K^{\alpha\beta} &= \tau^{\gamma}_{\alpha} \tau^{\delta}_{\beta} K^{\gamma\delta}.
\end{align*}
\]

The other nontrivial commutators or anti-commutators of the superconformal algebra involve the generators \( J_{\mu\nu} \) of the Lorentz group \( \text{Spin}(1,5) \). In terms of the spinor notations, one get a traceless matrix \( J^{\alpha}_{\beta} \) which acts on the chiral spinor space \( S^{+} \) as

\[
J^{\alpha}_{\beta} = \frac{1}{2} J_{\mu\nu} (\gamma^{\mu} \wedge \gamma^{\nu})_{\beta}^{\alpha}.
\]

The real condition that should be imposed on \( J^{\alpha}_{\beta} \) is obvious. The conformal weight of \( J^{\alpha}_{\beta} \) is zero, which means that \( [D, J^{\alpha}_{\beta}] = 0 \). The \( P - K \) commutators can be given as

\[
\begin{align*}
i\{P^{\alpha\beta}, K_{\gamma\xi}\} &= \delta^{\alpha}_{\gamma} \gamma^{\beta}_{\xi} - \delta^{\beta}_{\gamma} \gamma^{\alpha}_{\xi} + \delta^{\alpha}_{\xi} \gamma^{\beta}_{\gamma} - \delta^{\beta}_{\xi} \gamma^{\alpha}_{\gamma} - \delta^{\alpha\beta}_{\gamma\xi} D,
\end{align*}
\]

(2.6)

where \( \delta^{\alpha\beta}_{\gamma\xi} = \frac{1}{2} (\delta^{\alpha}_{\gamma} \delta^{\beta}_{\xi} - \delta^{\beta}_{\gamma} \delta^{\alpha}_{\xi}) \). And the \( J - K \) commutators are \( i[J^{\alpha}_{\beta}, K_{\gamma\xi}] = \delta^{\alpha}_{\gamma} K_{\beta\xi} - \delta^{\alpha}_{\xi} K_{\beta\gamma} \). Other commutators involving \( J^{\alpha}_{\beta} \) form the familiar algebra of (1+5)-dimensional Lorentz group.

Since \( Q^{\alpha}_{A} \) and \( S_{\alpha A} \) transform as the chiral and anti-chiral spinors of \( \text{Spin}(6,\mathbb{C}) \cong SL(4,\mathbb{C}) \), one has \( i[J^{\alpha}_{\beta}, Q^{\gamma}_{A}] = \delta^{\gamma}_{\beta} Q^{\alpha}_{A} \) and \( i[J^{\alpha}_{\beta}, S_{\gamma A}] = -\delta^{\alpha}_{\gamma} S_{\beta A} \). The \( Q - S \) anti-commutators can now be written as

\[
\{Q^{\alpha}_{A}, S_{\beta B}\} = \frac{1}{2} J^{\alpha}_{\beta} - i\delta^{\alpha}_{\beta} \left( \frac{1}{2} \omega_{AB} D - \mathcal{R}_{(AB)} \right),
\]

(2.7)

where the relative factors of the various terms of the right hand side may be fixed by using the Jacobi identities.

**Chiral Primary Operators**

One can organize the local operators of the QNG theory as various irreducible representations of the superconformal group \( \text{OSp}(2,6|2) \). This may be achieved by performing the radial quantization, classifying the quantum states according to their different \( \text{OSp}(2,6|2) \) symmetries, and then using the state-operator correspondence of conformal field theory to map these states to their corresponding operators. In some details, one continues the (1+5)-dimensional theory from the Minkowski space \( \mathbb{W} \) to a six-dimensional
Euclidean space $\mathbb{R}^6$, the superconformal group is then rotated to $OSp(1,7|2)$. One then pick out a point of $\mathbb{R}^6$ as the origin and cut out a tiny hole of infinite small radius around it. This procedure breaks the conformal symmetry $Spin(1,7)$ into $Spin(1,1) \times Spin(6)$, where $Spin(1,1)$ is the group of the dilations along the radial direction and $Spin(6)$ is the rotation group around the origin. Finally, one takes the generator of $Spin(1,1)$ as the Hamiltonian, quantizes the theory along the radial direction, and constructs the Hilbert space on $S^5$ surrounding the origin. State-operator correspondence tells us that an operator inserted at the origin will create a quantum state in the Hilbert space, and inversely, the shrinking of a given state on $S^5$ (into the origin) will define an operator at the origin. And in the present paper, with the hope of not confusing the reader, we’ll use the same notations to denote the generators of $OSp(2,6|2)$ and the generators of $OSp(1,7|2)$. Thus, $D$ is used to denote the Hamiltonian in the radial quantization. And, since what we are considering is a unitary quantum theory, $K_{\alpha\beta}$ and $S^A_{\alpha}$ are the Hermitian conjugations of $P^{\alpha\beta}$ and $Q^\alpha_A$.

In radial quantization, by repeatedly acting the raising operators $Q^\alpha_A$, $P^{\alpha\beta}$ on the state of highest weight (superconformal primary), which is annihilated by the lowering operators $S^A_{\alpha}$, $K_{\alpha\beta}$, one can construct the full superconformal module associating with the primary state. Some of the states of the superconformal module are $Q$ descendants only, but not $P$ descendants. These states are conformal primaries of the conformal group $Spin(2,6)$. By using the state-operator correspondence, the superconformal primary states will be mapped to superconformal primary operators, and the conformal primaries will be mapped to conformal primary operators.

For the ordinary superconformal small representations, at least 8 of the 16 $Q$-charges will annihilate the superconformal primary state. Hence, these representations include $2^8 = 256$ conformal primaries at most. The primary operators of these small representations are called chiral primary operators. A special property of chiral primary operators is that their conformal weights are uniquely determined by their $R$-symmetries.

Both the $R$-symmetries of the scalar chiral primary operators for the QNG theory and their dimensions/\(R\)-charges relationship are well known, and can be determined as follows. One considers a scalar chiral primary operator $O^{(n_1,n_2)}$, which transforms as the representation of $Sp(2,H)_R$ with highest weight $(n_1,n_2)$, where $n_1, n_2 \geq 0$ are integers. We denote the correspond state of $O$ as $|\psi_O\rangle$. By using the algebra (2.7), one then has

\[
\langle \psi_O| S_{A\beta} Q^\alpha_B |\psi_O\rangle = -i \left( \frac{1}{2} \omega_{AB} \Delta_O - \langle \psi_O| R_{(AB)} |\psi_O\rangle \right) \delta^\alpha_\beta,
\] 

(2.8)
where $\Delta_{\mathcal{O}}$ is the eigenvalue of operator $D$ on $|\psi_{\mathcal{O}}\rangle$ (the dimension of the chiral primary operator $\mathcal{O}$). The right hand side of (2.8) tells us that the $Sp(2,\mathbb{H})_{\mathbb{R}}$ weight of $\mathcal{O}(n_1,n_2)$ is $(n_1 - \frac{1}{2}\Delta_{\mathcal{O}}, n_2 - \frac{1}{2}\Delta_{\mathcal{O}})$. For chiral operator, this weight must vanish since some combinations of $Q_\alpha^A$ will vanish the left hand side of (2.8). Thus, we arrive at the conclusion that the nontrivial scalar chiral primary operators must be in the $(n,n), n \geq 1$ representations of $Sp(2,\mathbb{H})_{\mathbb{R}}$, and the dimensions of these operators are given as [21]

$$\Delta_{\mathcal{O}} = 2n.$$ (2.9)

On the other hand, for $G = U(N)$ case, one can combine the predictions, to the spectrum of chiral primary operators, of the AdS$_7 \times$ S$^4$/CFT$_6$ correspondence and of the DLCQ description of $SU(N)$ QNG theory [18], which will tell us that the scalar chiral primary operator $\mathcal{O}^{(n)}, 1 \leq n \leq N$ falls into the $n-$th order symmetric traceless irreducible representation of $SO(5)$ with conformal weight $\Delta_{\mathcal{O}} = 2n$. These results agree with the results of the above paragraph, since $SO(5) \simeq Sp(2,\mathbb{H})_{\mathbb{R}}/\mathbb{Z}_2$ and the fundamental representation of $SO(5)$ is identical to the $(1,1)$ representation of $Sp(2,\mathbb{H})_{\mathbb{R}}$.

We now consider two notable examples of $\mathcal{O}^{(n)}$ with $n = 1$ and $n = 2$, in some more details. In these cases the small representations are even more shorter. For the $n = 1$ case. $\mathcal{O}^{(1)}$ and its descendants form the free tensor multiplet of $N = (2,0)$ superalgebra, which takes value in the decoupled diagonal $U(1)$ part of $U(N)$. This free tensor multiplet consists of a scalar fields $\Phi^{AB}$, four chiral fermions $\Psi_\alpha^A$, and a self-dual tensor $H^{(\alpha\beta)}$, where the primary operator $\Phi^{AB}$ takes the $(1,1)$ representation of $Sp(2,\mathbb{H})_{\mathbb{R}} \simeq Spin(5)_{\mathbb{R}}$, $\Psi_\alpha^A$ and $H^{(\alpha\beta)}$ are the $Q$-descendant and $Q^2$ descendant of $\Phi^{AB}$, respectively. More properties of these fields and their descendants will be discussed in next section, explicitly.

The free tensor multiplet is itself a somewhat subtle theory for its chiral nature. On the other hand, the quantum Abelian gerbe theory – consists of several copies of free tensor multiplet – is the IR effective theory of the QNG theory at a generic point of the moduli space. Thus, in a certain sense the free tensor multiplets can be viewed as one of the two elements of QNG theory – the other elements may be the self-dual strings. And before going into QNG theory, we will firstly work out the quantum mechanics theory of the free tensor multiplet in section 3.

For the $n = 2$ case. The chiral primary operators $T^{(AB,CD)}$ have conformal weight 4, where the $AB$ (and $CD$) are antisymmetric, while between the pairs of $AB$ and $CD$ are symmetric. And the traceless condition imposes constraint $T^{(AB,CD)} \epsilon_{ABCD} = 0$. 
These operators are the only relevant deformations of the interacting QNG theory and they preserve some of the supersymmetries. The associated representation is ultra short since its conformal primaries are descendants of $T^{(AB,CD)}$ with no more than five-raising operators $Q$. The bosonic conformal primaries include the $R$-symmetry currents $J^\alpha_{\mu(AB)} = J^\mu_{\alpha(AB)} \gamma^\alpha_{\mu}$ of the QNG theory, the currents $J^{(\alpha\beta)}_{AB} = \frac{1}{3!} J^{\mu\nu\rho}_{AB} (\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho)^{\alpha\beta}$ of self-dual strings which satisfy $J^{(\alpha\beta)}_{AB} \omega^{AB} = 0$, and the energy-momentum tensor $T^{(\alpha\beta,\gamma\delta)} = \frac{1}{2} T^{\mu\nu} (\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\nu \gamma_\alpha \gamma_\mu)$ of the QNG theory. $J^\alpha_{\mu(AB)}$ and $J^{(\alpha\beta)}_{AB}$ are dimension five operators. They are $Q^2$ descendants of $T^{(AB,CD)}$. And the dimension six operators $T^{(\alpha\beta,\gamma\delta)}$ are $Q^4$ descendants of $T^{(AB,CD)}$. All these operators should commute with appropriate action of the real structure $\hat{\tau}$.

**The Moduli Space**

A general QNG theory is an intrinsically strong interacting conformal field theory—with superconformal group $OSp(2,6|2)$ that we have studied. The theory has a rank $r$ gerbe group $G$, which is in type of $A$ – $D$ – $E$ series of the Lie groups. QNG theory lies on the singularity points of the moduli space $M_r$.

$M_r$ is isomorphic to the $W_G$ orbifold $R^{5r}/W_G$ of $R^{5r}$, where $W_G$ is the Weyl group of $G$. Locally, $M_r$ can be parameterized by the vacuum expectations of the scalars $\Phi^{AB}$ that take values in the Cartan subalgebra of the Lie algebra of $G$. And globally, $M_r$ should be parameterized by the vacuum expectations of scalar chiral primary operators $\mathcal{O}^{(n)}$, $1 \leq n \leq r$ that we have studied.

The singularity of $M_r$, that defines the QNG theory, deserves some extra treatments. There, arguably, the vacuum expectations of $\mathcal{O}^{(n)}$ should vanish. This vanishing is due to their positive dimensions, which is guaranteed by (2.9), since in a CFT only the dimension zero operator can get the vacuum expectation.

One can perturb the QNG theory away from the singularity to a generic point of $M_r$. This will break the non-Abelian group $G$ down to their maximal torus $T_G \subset G$. Constrained by the sixteen-supercharges of the theory, the resulting low energy effective theory, ignoring the high derivative corrections, will be described by $r$ free tensor multiplets of the $N = (2,0)$ supersymmetries, which form a quantum Abelian gerbe theory with Abelian gerbe group $T_G$. This theory is a free field theory with superconformal group $OSp(2,6|2)$. 

9
3. Quantum Abelian Gerbe Theory

We now turn to investigate the quantum mechanics of the Abelian gerbes, which is one of the crucial elements of the QNG as we have argued in section 2. There are two subtleties, one is the unconventional and somewhat complicated kinematics due to the unusual high spacetime dimensions, the other is the lack of an ordinary Lagrangian description for the theory due to its chiral nature.

We'll overcome the first subtlety by intensively utilizing the spinor techniques (A parallel spin-helicity technique has been developed and apply to the calculation of scattering amplitude of the six-dimensional Yang-Mills field [22].) This technique will be developed in the classical field theory context firstly, and will then be applied to the quantization of the tensor multiplet. The overcoming of the second subtlety is by utilizing the Hamiltonian formalism which has been used to the quantization of the free tensor multiplet by [23] [24].

In this section, we'll show that, by combining the spinor technique with the Hamiltonian formalism, the quantum mechanical theory of the tensor multiplet will be elegantly worked out. As an illustration, we'll calculate the OPE of the fields in the tensor multiplet. We'll also calculate the anomalies of the tensor multiplet, compare the results with the well known results from using other methods and find agreements.

We'll mainly focus on the case of one free tensor multiplet. The generalization to the full $r$ tensor multiplets associated with the maximal torus $T_G \subset G$ will be addressed in the last subsection of the present section, where we also try to get an understanding – from the field theory viewpoint – of the somewhat mysterious $A - D - E$ classification of the QNG theory.

3.1. The Classical Theory Of The Tensor Multiplet

The Tensor Multiplet

As we have mentioned in the last section, each tensor multiplet [25] of the $N = (2, 0)$ supersymmetries contains, one antisymmetric two-form field $B^2$ with self-dual three-form strength $H = dB$, with $H = H$, four symplectic-Majorana-Weyl fermions $\Psi^\alpha_A$ that transform as $S^+ \otimes S_R = (4, 4)$ representation under $\text{Spin}(1, 5) \times \text{Sp}(2, H)_{3R} \subset SL(4, \mathbb{C}) \times \mathbb{R}^2$. However, in terms of mathematical language, such a field $B$ is named as a connection of a gerbe [26].
$Sp(4, \mathbb{C})$, and five scalars $Φ_{AB}$ with antisymmetric superscripts $A, B$. The fermions $Ψ^α_A$ should satisfy the real condition,

$$Ψ^α_A = τ^α_β ω_{AB} \bar{Ψ}^B_β.$$  \hspace{1cm} (3.1)

One can see that (3.1) is the natural extension of the real structure $\hat{r}$ on the configuration space of the fields. To see this point explicitly, one defines the appropriate action of $\hat{r}$ on $Ψ^α_A$ as $\hat{r}(Ψ)^α_A = τ^α_β ω_{AB} \bar{Ψ}^B_β$. Clearly, (3.1) just means that $Ψ^α_A$ commute with $\hat{r}$, $Ψ^α_A = \hat{r}(Ψ)^α_A$. The real structure $\hat{r}$ can also be naturally acted on the self-dual tensor $H^{(αβ)}$ and the scalars $Φ_{AB}$ by imposing the real conditions

$$H^{(αβ)} = τ^α_γ τ^β_δ H^{(γδ)}$$

$$Φ_{AB} = ω_{AC} ω_{BD} Φ^{CD}.$$  \hspace{1cm} (3.2)

Since $Φ_{AB}$ transform as a direct sum (1) $⊕$ (5) of the (1) and (5) dimensional representations of $Sp(2, H)_R$, thus to describe the five scalars of the tensor multiplet, one should impose a further condition to project the (1) representation out. This condition is $Φ_{AB} ω^{AB} = 0$.

In terms of the spinor notation, self-duality means that $H$ can be described as a chiral field $H^{(αβ)}$ with symmetric superscripts $α, β$, $H^{(αβ)} = H^{(βα)}$. The connection between the spinor notation $H^{(αβ)}$ and the ordinary expression of the self-dual three-form is through $H^{(αβ)} = \frac{1}{3!} H_{μνρ} (γ^μ \wedge γ^ν \wedge γ^ρ)^{αβ}$, where $γ^μ \wedge γ^ν \wedge γ^ρ$ denotes the totally antisymmetrical product of $γ^μ, γ^ν$ and $γ^ρ$.

The connection $B$ of the self-dual gerbe field $H$ can now be written as $B^α_β$, with $B^α_β = \frac{1}{2} B_{μν} (γ^μ \wedge γ^ν)^{αβ}$. $H^{(αβ)}$ can then be given as $H^{(αβ)} = ∂^αγ B^γ_β + ∂^βγ B^α_γ$, where the partial differential operator $∂^αγ = ∂/∂x^α$. Here, $x^α_β$ are the spinor notations of the spacetime coordinates $x^μ$, with $x^α_β = x^μ γ^α_μ$ satisfying $x^α_β = τ^α_γ τ^β_δ x^γδ$. The self-duality of $H^{(αβ)}$ means that $B^α_β$ should satisfy $∂_αγ B^γ_β + ∂_βγ B^α_γ = 0$. And the gauge transformations $B → B + dA$ is now $B^α_β → B^α_β + (∂^αγ A^γ_β - \frac{1}{4} δ^α_β γ^δ A^γ_δ)$, where $A^α_β = A^μ γ^μ_α β$.

Under the action of the supersymmetry transformation $ε^A α Q^α_A$, the component fields of the $N = (2, 0)$ tensor multiplet transform as

$$δΦ_{AB} = (ε^A α Q^α_B + ε_B α Q^α_A - \frac{1}{2} ω_{AB} ε^C γ Q^Cγ)$$

$$δΨ^α_A = ε^B β \left( \frac{1}{2} H^{(αβ)} ω_{AB} + ∂^αβ Φ_{AB} \right)$$

$$δH^{(αβ)} = ε^A γ \left( ∂^αγ Ψ^β_A + ∂^βγ Ψ^α_A \right).$$  \hspace{1cm} (3.3)
where the infinite small fermionic parameters $\epsilon^A_\alpha$ satisfy the real conditions $\epsilon^A_\alpha = \omega^{AB}_\alpha \tau^\alpha_\beta \tau^\beta_{\beta B}$. From (3.3) one can explicitly see that $\Psi^\alpha_A$ is the $Q$-descendant of $\Phi_{AB}$, $H^{(\alpha\beta)}$ is the $Q^2$-descendant of $\Phi_{AB}$, and the descendants with more than two $Q$ charges acting are not conformal primaries, just as we have mentioned in last section.

A kind of super-coordinates expressions of the supersymmetrical transformations will be suggestive. Thus, we’d like to present it in this paragraph, although there are no superspace formulation for the QNG theory known to the author. To do this, one can take the super-coordinates of the (1+5)-dimensional $N = (2, 0)$ superspace as $(x_\alpha^\beta, \theta^A_\alpha)$, where $\theta^A_\alpha$ are the fermionic coordinates that satisfy the real conditions $\theta^A_\alpha = \omega^{AB}_\alpha \tau^\alpha_\beta \theta^\beta_{\beta B}$. On the operators defined on the superspace, the translations $P^\alpha_\beta$ now act as the differential operators

$$Q^\alpha_A = \frac{\partial}{\partial \theta^A_\alpha} + \theta^A_\alpha \frac{\partial}{\partial x^\alpha_\xi}.$$  

(3.4)

**Dirac Equation And Kinematics In (1+5)-Dimensional Spacetime**

To simplify the kinematics associated with the classical or quantum mechanical theory of the $N = (2, 0)$ tensor multiplet, we’ll start to develop the spinor techniques in this subsubsection by studying an auxiliary Dirac equation

$$\partial_\alpha^\beta \Psi^\beta(x) = 0.$$  

(3.5)

Besides the application to $N = (2, 0)$ tensor multiplet, the same technique may also be useful to many other field theories defined in (1+5)-dimensional spacetime.

To analyze the solutions, we Fourier transform (3.5) to an algebraic equation defined on momentum space

$$p_\alpha^\beta \Psi^\beta(p) = 0,$$  

(3.6)

where $p_\alpha^\beta = p_\mu \gamma^\mu_\alpha^\beta$. By using the Clifford algebra, one has $p^{\alpha\gamma} p_\beta^\gamma = \delta^\alpha_\beta p^2$. Hence, (3.6) implies $p^2 \Psi^\alpha(p) = 0$, which means that the wave function $\Psi^\alpha(p)$ must support on the light cone $L_C : p_\alpha^\beta p^{\alpha\beta} = 0$. We denote the maximal compact subgroup of Spin(1, 5) that preserves the light-like vector $p_\alpha^\beta$ as $W_p$ (the little group). Clearly, $W_p \simeq \text{Spin}(4) \simeq SU(2) \times SU(2)$. To classify the solutions of the Dirac equation, we shall use the maximal torus $H_p = U(1) \times U(1)$ of $W_p$ to label helicities. The two solutions $\lambda^\alpha(p), \tilde{\lambda}^\beta(p)$ of (3.6) have helicities $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0)$, respectively, which means that under $e^{i\theta} \times e^{i\phi} \in H_p$

$$\lambda^\alpha(p) \rightarrow e^{i\frac{1}{2} \phi} \lambda^\alpha(p), \quad \tilde{\lambda}^\beta(p) \rightarrow e^{-i\frac{1}{2} \phi} \tilde{\lambda}^\beta(p).$$  

(3.7)
Here, the convention is for positive energy waves, the helicities of negative energy waves are reverse to it.

It is now possible to represent the light-like momentum $p^{\alpha\beta}$ in terms of the wave functions $\lambda^\alpha(p)$ and $\tilde{\lambda}^\alpha(p)$ (by noticing that $V_C = S^+ \wedge S^+$), with

$$p^{\alpha\beta} = 2(\lambda^\alpha(p) \wedge \tilde{\lambda}^\beta(p)) = \left(\lambda^\alpha(p) \tilde{\lambda}^\beta(p) - \lambda^\beta(p) \tilde{\lambda}^\alpha(p)\right).$$

(3.8)

The null condition $p^2 = (1/4)\frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} p^{\alpha\beta} p^{\gamma\delta} = 0$ of the light like vector is satisfied, obviously. To see that (3.8) fulfills the Dirac equation (3.6) with plane-wave solutions $\lambda^\alpha(p)$ and $\tilde{\lambda}^\alpha(p)$, one rewrites (3.6) as

$$p^{\alpha\beta} \Psi^\gamma(p) = p^{[\alpha\beta} \Psi^{\gamma]}(p) = 0.$$  

(3.9)

Since $p^{[\alpha\beta} \Psi^{\gamma]}(p) = 0 \Leftrightarrow \epsilon_{\alpha\beta\gamma\delta} p^{[\alpha\beta} \Psi^{\gamma]}(p) = 0 \Leftrightarrow p^{\gamma\delta} \Psi^\gamma(p) = 0$, the equivalence between (3.8) and (3.9) follows. Obviously, (3.8) satisfies (3.9). Hence, $p_\alpha \beta \lambda^\beta = p_\alpha \beta \tilde{\lambda}^\beta = 0$.

Now, given some $\lambda^\alpha$ and $\tilde{\lambda}^\beta$, the corresponding light-like momentum $p^{\alpha\beta}$ is automatically fixed through the following equation

$$p^{\alpha\beta} = 2(\lambda^\alpha \wedge \tilde{\lambda}^\beta),$$

(3.10)

where – noticing the real condition $p^{\alpha\beta} = \tau^{\alpha}_\gamma \tau^{\beta}_\delta \bar{p}^{\gamma\delta}$ satisfied by real momentum – $\tilde{\lambda}^\alpha$ is the complex conjugate of $\lambda^\alpha$, with $\tilde{\lambda}^\alpha = \tau^{\alpha}_\beta \bar{\lambda}^\beta$. Conversely, given the light-like momentum $p$, $\lambda$ and $\tilde{\lambda}$ cannot be specified through (3.10) only. The additional information needed is equivalent to a choice of the plane wave solutions of the chiral Dirac equation (3.9) (or its equivalent form (3.6)), as we have seen.

In fact, one needs to view $\lambda^\alpha$ (and $\tilde{\lambda}^\alpha$) as the elements of vector bundle over the space of light-like momentum, and view $p^{\alpha\beta} \rightarrow (\lambda^\alpha(p), \tilde{\lambda}^\beta(p))$ as the promotion of $p^{\alpha\beta}$, with (3.10) being satisfied. This promotion can be related to the second Hopf fibration $S^7 \rightarrow S^4$. To see this connection, one picks a time direction $\iota^{\alpha\beta}$ with $\iota^2 = (1/4)\iota_\alpha \iota_\beta \iota^{\alpha\beta} = 1$ and keeps the time component $p^0$ of $p$ fixed, where $p^0 = p \cdot \iota = (1/4)p_{\alpha\beta} \iota^{\alpha\beta}$. Then, all the null momentums along different spatial directions and with fixed $|\vec{p}| = p^0$ will form a $S^4$. Correspondingly, the space of possible chiral spinors are determined by $\frac{1}{2} \iota_\alpha \lambda^\alpha \wedge \tilde{\lambda}^\beta = p^0$ modulo a scaling $\lambda \rightarrow t \lambda$, $\tilde{\lambda} \rightarrow t^{-1} \tilde{\lambda}$, where $t \in \mathbb{C}^\ast$. This determines a quadratic surface $S^7$ in the complex four-dimensional space $S^+\!$ of chiral spinors. Equation (3.10) then gives us the Hopf map from $S^7$ to $S^4$. 

13
These discussions can be summarized as follows. Firstly, one Fourier transforms (3.9) to the coordinates space, and gets an equivalent form of the conventional chiral Dirac equation (3.5),

$$\partial^{[\alpha\beta}\Psi^{\gamma]}(x) = 0.$$  (3.11)

One can then see that the two plane-wave solutions of (3.11) with the same momentum $p^{\alpha\beta}$ and opposite helicities can be chosen as

$$\lambda^\alpha \exp \left( i \frac{1}{2} x_{\alpha\beta} \lambda^\alpha \wedge \tilde{\lambda}^\beta \right),$$

$$\tilde{\lambda}^\alpha \exp \left( i \frac{1}{2} x_{\alpha\beta} \lambda^\alpha \wedge \tilde{\lambda}^\beta \right).$$  (3.12)

Obviously, the momentum $p^{\alpha\beta}$ and $\lambda^\alpha, \tilde{\lambda}^\alpha$ are connected through $p^{\alpha\beta} = 2(\lambda^\alpha \wedge \tilde{\lambda}^\beta)$.

**Classical Field Theory and Plane Waves**

Now, we start to study the classical field theory and plane wave expansions of the free $\mathcal{N} = (2, 0)$ tensor multiplet by utilizing the kinematics that we just studied. To study the classical theory of tensor multiplet, all we need to do is to solve the equations of motion of various component fields $\Phi_{AB}, \Psi^\alpha_A,$ and $H^{(\alpha\beta)}$. These equations are $\partial_{\alpha\beta} \partial^{\alpha\beta} \Phi_{AB} = 0$, $\partial_{\alpha\beta} \Psi^\alpha_A = 0$, and $\partial_{\alpha\gamma} H^{(\gamma\beta)} = 0$, respectively.

We begin from $\Phi_{AB}(x)$ which satisfies the Klein-Gordon equation

$$\partial_{\alpha\beta} \partial^{\alpha\beta} \Phi_{AB}(x) = 0.$$  (3.13)

By Fourier transforming this equation to momentum space and noticing that $p^2 \delta(p^2) = 0$ ($\delta(x)$ is the Dirac delta function), one can solve (3.13) as

$$\Phi_{AB}(x) = (2\pi)^4 \int_{p} \delta(p^2) \left( \frac{\phi_{AB}(p)}{2z_p} + \frac{\tilde{\phi}_{AB}(p)}{2\pi_p} \right),$$  (3.14)

where $z_p = \exp(ip \cdot x)$ is the plane wave with momentum $p$ propagating along spacetime $W$ (the form of (3.14) suggests that field $\Phi_{AB}$ may be appropriate continued to the complexification of $W$), the integration $\int_p$ over momentum space is given by $\int d^6 p / (2\pi)^6$, and the Lorentz invariant factor $\delta(p^2)$ is inserted to restrict the integration on the light-cone $L_C = L_C^+ \cup L_C^-$, which is the union of positive patch $L_C^+$ ($p^0 > 0$) and negative patch $L_C^-$ ($p^0 < 0$). The real conditions (3.2) can then be satisfied by identifying the amplitude $\phi_{AB}(p)$ with the amplitude $\tilde{\phi}_{AB}(p)$ through

$$\tilde{\phi}_{AB}(p) = \omega_{AC} \omega_{BD} \phi^{CD}(p).$$  (3.15)
And the condition $\Phi_{AB}(x)\omega^{AB} = 0$ can be fulfilled by requiring $\phi_{AB}(p)\omega^{AB} = 0$.

Moreover – noticing the specific form of the plane wave $z_p = \exp(ip\cdot x)$ on spacetime $W$ – one should identify the modes $\phi_{AB}(-p)$ on $L^-_C$ with the modes $\tilde{\phi}_{AB}(p)$ on $L^+_C$ and vice versa, $\phi_{AB}(-p) = \tilde{\phi}_{AB}(p)$, $\tilde{\phi}_{AB}(-p) = \phi_{AB}(p)$, where $p \in L^+_C$. Hence, on real spacetime $W$, (3.14) can be rewritten as more familiar form

$$
\Phi_{AB}(x) = (2\pi) \int_{p \in L^+_C} \left( \phi_{AB}(p)e^{-ip\cdot x} + \phi_{AB}(-p)e^{ip\cdot x} \right),
$$

(3.16)

with splitting positive frequency part and negative frequency part.

We now turn to the equation of motion of $\Psi^\alpha_A$, which can be written as

$$
p^{[\alpha\beta}\Psi^\gamma_A](p) = 0.
$$

(3.17)

This equation is equivalent to $p_{\alpha\beta}\Psi^\beta_A(p) = 0$ with the same reason that has been explained in last subsubsection. Equation (3.17) have four solutions $u^\alpha_A(p)$ with helicity $(+\frac{1}{2},0)$ and four solutions $\tilde{u}^\alpha_A(p)$ with helicity $(-\frac{1}{2},0)$, both $u^\alpha_A(p)$ and $\tilde{u}^\alpha_A(p)$ transform as (4) representation of $SP(2,H)_R$.

Clearly, after promoting the light-like momentum $p^{\alpha\beta}$ to $\lambda^\alpha$ and $\bar{\lambda}^\alpha$ with $p^{\alpha\beta} = 2(\lambda^\alpha \wedge \bar{\lambda}^\beta)$, wave functions $u^\alpha_A(p)$ can be factorized as, $u^\alpha_A(p) = \lambda^\alpha \Psi^+_A$. Wave function $\tilde{u}^\alpha_A(p)$ can be likewise factorized as $\tilde{u}^\alpha_A(p) = \bar{\lambda}^\alpha \tilde{\Psi}^-_A$. One can then expand $\Psi^\alpha_A(x)$ as

$$
\Psi^\alpha_A(x) = (2\pi) \int p \delta(p^2) \left[ (\lambda^\alpha \Psi^+_A + i\bar{\lambda}^\alpha \tilde{\Psi}^-_A)/(2z_p) + (\bar{\lambda}^\alpha \tilde{\Psi}^+_A + i\lambda^\alpha \Psi^-_A)/(2\bar{\Psi}_p) \right],
$$

(3.18)

where $\tilde{\Psi}^+_A(p)$ can be set as $\tilde{\Psi}^+_A(p) = \omega_{AB} \tilde{\Psi}^B_+(p)$ and $\tilde{\Psi}^-_A(p)$ can be set as $\tilde{\Psi}^-_A(p) = \omega_{AB} \tilde{\Psi}^-_B(p)$, to fulfill the real conditions (3.1).

Obviously, one can separate the free field $\Psi^\alpha_A$ as a helicity $+\frac{1}{2}$ part $\Psi^{\alpha+}_A$ plus a helicity $-\frac{1}{2}$ part $\Psi^{\alpha-}_A$, with $\Psi^\alpha_A(x) = \Psi^{\alpha+}_A(x) + \Psi^{\alpha-}_A(x)$. Here,

$$
\Psi^{\alpha+}_A(x) = (2\pi) \int p \delta(p^2) \lambda^\alpha(p) \left( \frac{\Psi^+_A(p)}{2z_p} + i\frac{\tilde{\Psi}^-_A(p)}{2\bar{\Psi}^p} \right),
$$

$$
\Psi^{\alpha-}_A(x) = (2\pi) \int p \delta(p^2) \bar{\lambda}^\alpha(p) \left( i\frac{\Psi^-_A(p)}{2z_p} + \frac{\tilde{\Psi}^+_A(p)}{2\bar{\Psi}^p} \right).
$$

(3.19)

The real condition (3.1) now means that $\Psi^{\alpha+}_A(x)$ and $\Psi^{\alpha-}_A(x)$ are complex conjugate to each other, with $\Psi^{\alpha+}_A(x) = \tau^\alpha_{\beta\lambda} \omega_{AB} \tilde{\Psi}^B_-(x)$.
In separating the positive and negative energy modes of the fields $\Psi_{\alpha}^+(\tau)$ and $\Psi_{\alpha}^-(\tau)$, one should make use of the relationships of $\Psi_{\alpha}^+(\tau) = \tilde{\Psi}_{\alpha}^-(\tau)$, $\tilde{\Psi}_{\alpha}^-(\tau) = -\Psi_{\alpha}^+(\tau)$ and $\Psi_{\alpha}^+(\tau) = -\tilde{\Psi}_{\alpha}^+(\tau)$, $\tilde{\Psi}_{\alpha}^-(\tau) = -\Psi_{\alpha}^-(\tau)$, where $\tau \in L^+_C$. Here, we have used $\lambda^\alpha(-\tau) = i\lambda^\alpha(\tau)$, $\tilde{\lambda}^\alpha(-\tau) = i\tilde{\lambda}^\alpha(\tau)$ to preserve the relation $p^{\alpha\beta} = 2(\lambda^\alpha \wedge \tilde{\lambda}^\beta)$ under $\tau \rightarrow -\tau$. One can then have $\Psi_{\alpha}^+(\tau) = (2\pi) \int_{p \in L^+_C} \lambda^\alpha(p)[\Psi_{\alpha}^+(p)e^{-ip\cdot x} + i\tilde{\Psi}_{\alpha}^+(p)e^{ip\cdot x}]$, and $\Psi_{\alpha}^-(\tau) = (2\pi) \int_{p \in L^+_C} i\tilde{\lambda}^\alpha(p)[\Psi_{\alpha}^-(p)e^{ip\cdot x} + i\tilde{\Psi}_{\alpha}^-(p)e^{-ip\cdot x}]$.

Let’s turn to the equations of motion of chiral field $H^{(\alpha\beta)}$. We Fourier transform the equation $\partial_{\alpha\beta}H^{(\beta\gamma)}(x) = 0$ to momentum space,

$$p_{\alpha\beta}H^{(\beta\gamma)}(p) = 0. \quad (3.20)$$

By multiplying $p^{\alpha\alpha}$ to left hand side of $(3.20)$ and using Clifford algebra, one can easily see that the wave function $H^{(\alpha\beta)}(p)$ is supported on light-cone $L^+_C$. Equation $(3.20)$ has three solutions, $H_{+}^{\alpha\beta}(p)$, $H_{0}^{\alpha\beta}(p)$ and $H_{-}^{\alpha\beta}(p)$, labeled according to their different helicities. Under $e^{i\phi} \times e^{i\varphi} \in H_p$, these solutions transform as

$$H_{+}^{\alpha\beta}(p) \rightarrow e^{i\theta} H_{+}^{\alpha\beta}(p)$$

$$H_{0}^{\alpha\beta}(p) \rightarrow H_{0}^{\alpha\beta}(p)$$

$$H_{-}^{\alpha\beta}(p) \rightarrow e^{-i\theta} H_{-}^{\alpha\beta}(p). \quad (3.21)$$

To employ the kinematic relation $p^{\alpha\beta} = 2(\lambda^\alpha \wedge \tilde{\lambda}^\beta)$, one first rewrites $(3.20)$ as $p^{\alpha\beta}H^{(\gamma\delta)} = 0 \ (\text{One can also transform this equation to coordinate space and get } \partial^{\alpha\beta}H^{(\gamma\delta)} = 0)$. One can then appropriately factorize the wave functions as $H_{+}^{\alpha\beta} = \lambda^\alpha \lambda^\beta H_+ + H_{0}^{\alpha\beta} = (\lambda^\alpha \tilde{\lambda}^\beta + \lambda^\beta \tilde{\lambda}^\alpha)H_0$ and $H_{-}^{\alpha\beta} = \tilde{\lambda}^\alpha \lambda^\beta H_-$. We now can expand $H^{(\alpha\beta)}(x)$ as the summation of a helicity +1 field $H_{+}^{(\alpha\beta)}(x)$, a helicity −1 field $H_{-}^{(\alpha\beta)}(x)$ and a helicity 0 field $H_{0}^{(\alpha\beta)}(x)$. Here, $H_{+}^{(\alpha\beta)}(x)$ and $H_{-}^{(\alpha\beta)}(x)$, which should be complex conjugate to each other $H_{+}^{(\alpha\beta)}(x) = \tau_\gamma \tau_\delta H_{-}^{(\gamma\delta)}(x)$ to fulfill the real condition $(3.22)$, are given by the expansions

$$H_{+}^{(\alpha\beta)}(x) = (2\pi) \int_{p \in L^+_C} \lambda^\alpha \lambda^\beta \left( H_+(p)e^{-ip\cdot x} + \bar{H}_-(p)e^{ip\cdot x} \right)$$

$$H_{-}^{(\alpha\beta)}(x) = (2\pi) \int_{p \in L^+_C} \lambda^\alpha \tilde{\lambda}^\beta \left( H_-(p)e^{-ip\cdot x} + \bar{H}_+(p)e^{ip\cdot x} \right), \quad (3.22)$$

and the self conjugate part $H_{0}^{(\alpha\beta)}(x)$, with $H_{0}^{(\alpha\beta)}(x) = \tau_\gamma \tau_\delta \bar{H}_{0}^{(\gamma\delta)}(x)$, is given as

$$H_{0}^{(\alpha\beta)}(x) = (2\pi) \int_{p \in L^+_C} i\left( \frac{\lambda^\alpha \tilde{\lambda}^\beta + \lambda^\beta \tilde{\lambda}^\alpha}{2} \right) \left( H_0(p)e^{-ip\cdot x} + \bar{H}_0(p)e^{ip\cdot x} \right). \quad (3.23)$$

16
3.2. The Quantum Theory Of Tensor Multiplet And OPE

The quantization of the free tensor multiplet is subtle due to its chiral nature. With \( \Psi^\alpha_A \) and \( \Phi_{AB} \), there is no problem, but – as is well known – an ordinary Lagrangian description of \( H \) field is unavailable due to its self-duality. One can understand this point by noticing that there is no quadratic Lorentz invariant concerning chiral field \( H^{(\alpha\beta)} \) only. For this reason, we’ll adopt Hamiltonian formalism and perform canonical quantization to the free tensor multiplet.

To present the Hamiltonian formalism, we pick out an arbitrary time-like Killing vector \( \iota^{\alpha\beta} \partial_{\alpha\beta} \), with \( \iota^2 = (1/4)\iota_{\alpha\beta}\iota^{\alpha\beta} = 1 \), take spacetime \( W \) as the form of \( W = R^1 \times X \), where \( X \) is a five-dimensional spatial slice and \( R^1 \) is the time direction generated by \( \iota^{\alpha\beta} \partial_{\alpha\beta} \). The associated Hamiltonian \( H \) of the system is then given by

\[
H = \int_X T^{(\alpha\beta,\gamma\delta)} \iota_{\alpha\beta} \iota_{\gamma\delta},
\]

where \( T^{(\alpha\beta,\gamma\delta)} \) is the energy-momentum tensor, which belongs to the \( n = 2 \) chiral primary multiplet of \( OSp(2,6|2) \) as we have discussed in section 2.

Canonical quantization is not an obvious Lorentz covariant procedure. Especially, it depends on some time direction \( \iota^{\alpha\beta} \) chosen arbitrarily. But, we argue that the quantum mechanical theory, got by canonically quantizing the free tensor multiplet, is in fact Lorentz covariant, since the correlation functions and operator product expansions turn out to be independent of \( \iota^{\alpha\beta} \) as one will see.

Now, we try to write out the energy-momentum tensor \( T^{(\alpha\beta,\gamma\delta)} \) of the free tensor multiplet, explicitly. Up to quadratic terms of \( H^{(\alpha\beta)} \), \( T^{(\alpha\beta,\gamma\delta)} \) must be given as

\[
T^{(\alpha\beta,\gamma\delta)} \sim \frac{c}{2} (H^{(\alpha\gamma)} H^{(\beta\delta)} - H^{(\beta\gamma)} H^{(\alpha\delta)}) + \ldots,
\]

where the normalization constant \( c \) will be fixed as \( 1/4\pi \), and the ellipsis stands for the terms that concerning \( \Phi_{AB} \) and \( \Psi^\alpha_A \) (these terms are omitted since they are a little complicated and only contribute usual quadratic terms – of \( \Phi_{AB} \) and \( \Psi^\alpha_A \) – to \( H \)). One can check this expression by noticing the symmetries of the indices of \( T^{(\alpha\beta,\gamma\delta)} \) and the traceless conditions

\[
T = T^{(\alpha\beta,\gamma\delta)} \epsilon_{\alpha\beta\gamma\delta} = 0,
\]

\textsuperscript{3} With a single auxiliary scalar field, one can construct a Lorentz-covariant Lagrangian formulation of the M5 brane effective action, see [27]. For a non-covariant M5-brane action in flat D=11 superspace, see [28]. The Hamiltonian formulation was considered in [29].
which is enforced by the conformal invariance (But, as we’ll discuss, the Weyl rescaling of
the $N = (2, 0)$ tensor multiplet is an anomalous symmetry, thus, for a general background
$T$ is in general quantum mechanically non-vanishing. In that situations, the conformal
invariance will be destroyed.).

The full Hamiltonian of the free tensor multiplet can now be written as

$$
H = \frac{1}{4\pi} \int_X \left( \iota_{\alpha\beta} \iota_{\gamma\delta} H^{(\alpha\gamma)} H^{(\beta\delta)} + \Psi^\alpha A \bar{\iota}_{\alpha\beta} \Psi^\beta A \right) + \frac{1}{16\pi} \int_X \left( \Pi^{\alpha\beta} \Pi_{\alpha\beta} - \frac{1}{4} \bar{\iota}^{\alpha\beta} \Phi^{A\beta} \bar{\iota}_{\alpha\beta} \Phi_{A\beta} \right),
$$

(3.26)

where the terms concerning $\Psi^\alpha A$ and scalars $\Phi^{A\beta}$ are usual. $\Pi^{\alpha\beta}$ is given by time derivative of $\Phi^{A\beta}$, $\Pi^{\alpha\beta} = \iota^{\alpha\beta} \partial \Phi^{A\beta} = (1/4) \iota^{\alpha\beta} \partial \Phi^{A\beta}$, and $\bar{\iota}^{\alpha\beta}$ satisfies $\iota^{\alpha\beta} \bar{\iota}_{\alpha\beta} = 0$. Here, the normalization has been appropriately chosen to make the theory consistent, quantum mechanically.

We now substitute the modes expansions of the free tensor multiplet, studied in sub-
section 3.1, into the Hamiltonian $H$ (3.26). The terms concerning five scalars will be, by
substituting (3.14),

$$
\int_{\vec{p}} |(p \cdot \iota)| (1/4) \bar{\Phi}^{A\beta} (p) \phi_{A\beta} (p),
$$

(3.27)

where $\int_{\vec{p}} = \int d^5 \vec{p}/[(2\pi)^6(2p^0)]$ is given by carrying out the $p^0 = p \cdot \iota$ integration in $\int_p \delta(p^2) \theta(p^0)$, which will enforce $p^0 = |\vec{p}| > 0$. And we have dropped the zero point energy of the scalar fields since the total zero point energy of the full tensor multiplet will be exactly canceled out, due to the boson-fermion pairing of the supersymmetry. Clearly, in the canonical quantization, $\phi_{A\beta} (p)$ and $\bar{\Phi}^{A\beta} (p)$ should be identified as the annihilation and creation operators, respectively, with the quantization conditions

$$
[\phi_{A\beta} (p), \bar{\Phi}^{C\delta} (q)] = (2\pi)^6 (2p^0) \delta(p - q) \delta^{C\delta}_{A\beta},
$$

(3.28)

where $\delta^{A\beta}_{C\delta} = 1/2 (\delta^{A} C \delta^{B} D - \delta^{B} C \delta^{A} D)$. One can check (3.28) by using it to calculate the commutators $[\phi_{A\beta} (p), H]$. The result is $[\phi_{A\beta} (p), H] = (p \cdot \iota) \phi_{A\beta} (p)$, which is just the Fourier transformations of the Heisenberg equation $i(d\Phi_{A\beta}/dt) = [\Phi_{A\beta}, H]$.

Similarly, after substituting the mode expansion of the fermions into $H$, and with the help of the relation $p^{\alpha\beta} = \lambda^\alpha \bar{\lambda}^\beta - \lambda^\beta \bar{\lambda}^\alpha$, one will have the terms of the fermion modes

$$
\int_{\vec{p}} |(p \cdot \iota)| \left( \bar{\psi}^4 A \psi^+ A + \bar{\psi}^4 A \psi^- A \right),
$$

(3.29)

18
Clearly, this term means that $\Psi^+_A(\vec{p})$ and $\Psi^-_A(\vec{p})$ should be quantized as the annihilation operators and their complex conjugate are the creation operators, with

$$\{\Psi^+_A(\vec{p}), \overline{\Psi}^+_B(\vec{q})\} = \{\Psi^-_A(\vec{p}), \overline{\Psi}^-_B(\vec{q})\} = (2\pi)^6(2p^0)\delta(\vec{p} - \vec{q})\delta^B_A. \quad (3.30)$$

Applying the same procedure to the tensor field $H^{(\alpha\beta)}$, one will get the terms of the expansion modes of $H^{(\alpha\beta)}$, with

$$\int_{\vec{p}} |(p \cdot \iota)|\left(\overline{H}_+H_+ + \overline{H}_-H_- + \overline{H}_0H_0\right), \quad (3.31)$$

which means that $H_+$, $H_-$ and $H_0$ are the annihilation operators and their complex conjugates are the creation operators, with the quantization relations

$$[H_+(p), \overline{H}_+(q)] = (2\pi)^6(2p^0)\delta(\vec{p} - \vec{q})$$
$$[H_-(p), \overline{H}_-(q)] = (2\pi)^6(2p^0)\delta(\vec{p} - \vec{q}) \quad (3.32)$$
$$[H_0(p), \overline{H}_0(q)] = (2\pi)^6(2p^0)\delta(\vec{p} - \vec{q}).$$

In the present conventions, all the modes are defined on the positive light cone $L^+_C$. One can use various relations between negative energy modes and the conjugate of positive modes with opposite helicity to extend the quantization relations to the full modes algebra. For example, by using $\Psi^+_A(-p) = \overline{\Psi}^-_A(p)$, one can have

$$\{\Psi^+_A(p), \overline{\Psi}^+_B(q)\} = (2\pi)^6(2p \cdot \iota)\delta(\vec{p} + \vec{q})\omega_{AB}. \quad (3.33)$$

By acting the creation operators on the Fock vacuum $|0\rangle$, one can construct the full Fock space $\mathcal{H}_F$ of the oscillating modes of free tensor multiplet. For example, the one particle Hilbert space can be constructed as follows. $|AB, \vec{p}\rangle = \overline{\phi}^{AB}(\vec{p})|0\rangle$ are the one particle states of the scalars, which transformed as (5) representation under the $\mathbb{R}$ symmetry group $Sp(2, H)_\mathbb{R}$, and $|\frac{1}{2}, A, \vec{p}\rangle = \overline{\Psi}^+_A(\vec{p})|0\rangle$ are the one particle states of the helicity $+\frac{1}{2}$ fermions, which transform as $|\frac{1}{2}, A, \vec{p}\rangle \rightarrow e^{i\frac{1}{2}\iota}|\frac{1}{2}, A, \vec{p}\rangle$ under the helicity transformation $e^{i\theta} \times e^{i\varphi} \subset H_p$, and transform as (4) representation under $Sp(2, H)_\mathbb{R}$. Similarly, $| -\frac{1}{2}, A, \vec{p}\rangle = \overline{\Psi}^-_A(\vec{p})|0\rangle$ are the one particle states of the helicity $-\frac{1}{2}$ fermions. Acting the creation operators $\overline{\Psi}^+_A(\vec{p})$, $\overline{\Psi}^-_A(\vec{p})$, and $\overline{H}_0(\vec{p})$ on $|0\rangle$ once will create the one particle states $|+, \vec{p}\rangle$, $|-, \vec{p}\rangle$ and $|0, \vec{p}\rangle$, respectively. Under $e^{i\theta} \times e^{i\varphi} \in H_p$, $|+, \vec{p}\rangle$ is helicity $+1$ with $|+, \vec{p}\rangle \rightarrow e^{i\theta}|+, \vec{p}\rangle$, $|-, \vec{p}\rangle$ is helicity $-1$ with $|-, \vec{p}\rangle \rightarrow e^{-i\theta}|-, \vec{p}\rangle$, and $|0, \vec{p}\rangle$ is helicity 0 with $|0, \vec{p}\rangle \rightarrow |0, \vec{p}\rangle$, all these states are $Sp(2, H)_\mathbb{R}$ singlets. These one particle states,
$|AB, \vec{p}\rangle$, $|\pm \frac{1}{2}, A, \vec{p}\rangle$, $|\pm, \vec{p}\rangle$ and $|0, \vec{p}\rangle$ consist of a short representation of the $N = (2,0)$ supersymmetries.

We now can calculate out the modes expression of $Q^\alpha_A$. This can be achieved by substituting the mode expansions of free tensor multiplet into the supercharges

$$Q^\alpha_A = \frac{1}{2\pi} \int_X \left( H^{(\alpha\beta)}\epsilon_{\beta\gamma} \Psi^\gamma_A + \partial^{\alpha\beta} \Phi_{AB} \epsilon_{\beta\gamma} \Psi^\gamma_B \right). \quad (3.34)$$

The result is (by using the relation $p^{\alpha\beta} = \lambda^\alpha \tilde{\lambda}^\beta - \lambda^\beta \tilde{\lambda}^\alpha$)

$$Q^\alpha_A = \int_{\vec{p}} (p \cdot \iota) \left[ \lambda^\alpha ((\underline{H} - \Psi^+_A + \tilde{H} - \tilde{\Psi}^+_A)) + \tilde{\lambda}^\alpha ((\underline{H} - \tilde{\Psi}^+_A - \tilde{H} + \tilde{\Psi}^+_A)) \right] + i \int_{\vec{p}} (p \cdot \iota) \left[ \lambda^\alpha ((\overline{\Phi}^+ \Psi^+_B + \Phi_{AB} \overline{\Psi}^+_B) + \tilde{\lambda}^\alpha ((\overline{\Phi}^- \Psi^-_B - \Phi_{AB} \overline{\Psi}^-_B)) \right]. \quad (3.35)$$

One can check that this expression satisfies real conditions (2.3). As a further check, one can also calculate the $Q - Q$ anti-commutators by using the modes algebra (3.28), (3.30) and (3.32), the results just give out the terms (3.27), (3.29) and (3.32) of the Hamiltonian.

However, the calculations in previous paragraph are restricted to oscillating modes. In more general case, the $Q - Q$ anti-commutators will contain additional terms $Z^{\alpha\beta}_{AB}$, which come from the anti-commutator between the first and the second term of the right hand side of (3.34) and serve as central extensions of $N = (2,0)$ supersymmetry algebra (This mechanism is familiar in four-dimensional supersymmetrical gauge theory [30]). By noticing the equation of motion $\partial_{\alpha\beta} H^{(\beta\gamma)} = 0$ of $H^{(\alpha\beta)}$, $Z^{\alpha\beta}_{AB}$ can be written as

$$Z^{\alpha\beta}_{AB} = \frac{1}{2\pi} \int_X \iota^{\beta\gamma} \partial_{\gamma\delta} (H^{(\delta\beta)} \Phi_{AB}) \quad (3.36)$$

where $\partial X$ stands for the spatial boundary at the infinity, and $n_{\alpha\beta}$ is the unit normal vector of $\partial X$, with $n^2 = 1$ and $n \cdot \iota \sim n_{\alpha\beta} \iota^{\alpha\beta} = 0$. Therefore, if the scalars $\Phi_{AB}$

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4 This can be got by integrating the supercurrent $J^{\alpha\gamma\delta}_{A} = J^{\alpha\mu}_{A} \gamma^{\delta}_{\mu}$ over $X$, with $Q^\alpha_A = \int_X \iota^{\alpha\gamma} \partial_{\gamma\delta} (H^{(\delta\beta)} \Phi_{AB})$. For free tensor multiplet, the supercurrent $J^{\alpha\delta}_{A}$ can be schematically written as $J^{\alpha\delta}_{A} \sim H^{(\alpha\delta)} \Psi^\gamma_A - H^{(\alpha\gamma)} \Psi^\delta_A + ...$. We note that this expression satisfies the trace condition $J^{\alpha}_{A\alpha\beta} = 0$ enforced by the conformal invariance, since $H^{(\alpha\delta)} \Psi^\gamma_A \epsilon_{\delta\gamma\alpha\beta} = H^{(\alpha\gamma)} \Psi^\delta_A \epsilon_{\delta\gamma\alpha\beta} = 0$, automatically.
have nonzero vacuum expectation $\Phi_0 \psi_{AB}$ at the infinity, where $\psi_{AB}$ is a unit vector of $\text{Sp}(2,\mathbb{H})\mathbb{R} \simeq \text{Spin}(5)\mathbb{R}$ with $(1/4)\psi_{AB}\psi^{AB} = 1$, $\mathcal{Z}_{AB}^{\alpha\beta}$ will serve as a measurement of the $H$-flux through $\partial X$, as one can see from (3.30). In fact, the integrand of (3.30) is just the operator $J_{AB}^{(\alpha\beta)}$ in $n=2$ chiral primary multiplet of $\text{OSp}(2,6|2)$, with $J_{AB}^{(\alpha\beta)} \sim H^{(\alpha\beta)}\Phi_{AB}$. Thus, $\mathcal{Z}_{AB}^{\alpha\beta}$ can be viewed as the conservation charges of the current $J_{AB}^{(\alpha\beta)}$ of self-dual strings. In section 5, we’ll discuss the self-dual string excitations extending along some $l^{\alpha\beta}$ direction. There, $\Phi_0$ will be interpreted as the tension of the string, $H$-fluxes will measure the winding number $w$ of the self-dual string, $\mathcal{Z}_{AB}^{\alpha\beta}$ will then be given as $w|\Phi_0|\psi_{AB}l^{\alpha\beta}L$, where $L$ is the string length along $l^{\alpha\beta}$ direction.

Operator Product Expansions

Having quantized the free tensor multiplet, we can now calculate the operator product expansion (OPE) of the theory. Since what we are considering is a free field theory, we only need to consider the $\Phi - \Phi$ OPE, the $\Psi - \Psi$ OPE and the $H - H$ OPE, the OPE of other local operators will then follows.

Let’s begin from the $\Phi - \Phi$ OPE. By using the mode expansions of the scalar fields and the mode algebra (3.28), one can get the most singular term of the $\Phi - \Phi$ OPE, with

$$\Phi_{AB}(x)\Phi_{CD}(0) \sim (2\pi)\epsilon_{ABCD} \frac{i}{4\pi^3|x|^4} + \ldots$$

(3.37)

where the factor $1/(4\pi^3|x|^4)$ can be got by evaluating the propagator $\Delta(x) = \int_p \frac{e^{-ip\cdot x}}{p^2}$. Since we are considering a free field theory, the ellipsis of (3.37) is nothing but an operator given by the normal ordering $: \Phi_{AB}(x)\Phi_{CD}(0) :$, which is nonsingular and can be expanded as the Taylor series $\sum \frac{1}{n!}x^{\mu_1}...x^{\mu_n}\partial_{\mu_1}...\partial_{\mu_n}\Phi_{AB}\Phi_{CD}(0)$.

Similarly, by using the mode expansions (3.18) and the mode algebra (3.30), one can calculate the OPE of the fermions $\Psi^\alpha_A$, with the help of $p^{\alpha\beta} = \lambda^\alpha \bar{\lambda}^\beta - \lambda^\beta \bar{\lambda}^\alpha$, the result is

$$\Psi^\alpha_A(x)\Psi^\beta_B(0) = (2\pi)\omega_{AB} \frac{x^{\alpha\beta}}{\pi^3|x|^6} + : \Psi^\alpha_A(x)\Psi^\beta_B(0) :$$

(3.38)

Here, the factor $x^{\alpha\beta}/(\pi^3|x|^6)$ is given by evaluating $\partial^{\alpha\beta}\Delta(x)$. This result can also be checked by noticing that $\langle \Psi^\alpha(x)\Psi^\beta(0) \rangle \sim \langle [Q^\alpha, \Phi(x)][Q^\beta, \Phi(0)] \rangle$, the supersymmetry invariance of the vacuum state then tells us that $\langle [Q^\alpha, [Q^\beta, \Phi(x)]]\Phi(0) \rangle \sim \langle Q^\alpha, [Q^\beta, \Phi(x)] \rangle \Phi(0) \rangle \sim$.  

\[ \int_0^\infty d\tau \frac{1}{(4\pi\tau)^{\frac{3}{4}}e^{-\frac{x^2}{4\tau}}} \]

\[ = \frac{1}{(4\pi|x|)^{\frac{3}{4}}}. \]

21
\[ \langle [Q^\alpha, Q^\beta], \Phi(x) \Phi(0) \rangle \sim \partial^{\alpha\beta} \langle \Phi(x) \Phi(0) \rangle, \] where we have omitted all the R-symmetry indices in this sketchy calculation.

The $H - H$ OPE can also be calculated by using the mode algebra (3.32) and the transformation $p^{\alpha\beta} = \lambda^\alpha \tilde{\lambda}^\beta - \lambda^\beta \tilde{\lambda}^\alpha$, with

\[ H^{(\alpha\beta)}(x) H^{(\gamma\delta)}(0) = \left( 2\pi \right)^3 \frac{6}{\pi^3 |x|^8} (x^{\alpha\gamma} x^{\beta\delta} + x^{\beta\gamma} x^{\alpha\delta}) + : H^{(\alpha\beta)}(x) H^{(\gamma\delta)}(0) : \] (3.39)

where the factor $6(x^{\alpha\gamma} x^{\beta\delta} + x^{\beta\gamma} x^{\alpha\delta})/(\pi^3 |x|^8)$ is got by evaluating $\left( \partial^{\alpha\gamma} \partial^{\beta\delta} + \partial^{\beta\gamma} \partial^{\alpha\delta} \right) \Delta(x)$.

In fact, up to a normalization constant, all these OPE’s can be fixed by superconformal symmetries. We have related the OPE of free fermions to the OPE of the scalars by using supersymmetries. The relationship between the $H - H$ OPE and the OPE of the scalars is similar, since the free field operators form a chiral multiplet of $OSp(2|6) \to OSp(2|6 \to n = 1$ chiral primary representations, as we have noted in section 2. The conformal weight of $\Phi_{AB}$ is 2, thus the most singular term (multiplying to weight 0 unit operator) of the $\Phi - \Phi$ OPE must be $\sim 1/|x|^4$, just as (3.37) shows us. Hence, our explicit calculations of the OPEs of free tensor multiplet can be viewed as a consistency check to our quantum theory of the Abelian gerbe.

### 3.3. The Direct Calculations Of Anomalies

By utilizing the OPEs of free tensor multiplet, we can now calculate the R symmetry anomaly and the gravitational anomaly of the tensor multiplet. This is achieved by calculating the four-point functions of the R symmetry currents – with schematic form $\langle J^R_{\beta\gamma} J^R_{\beta'\gamma'} \rangle$ – and four-point functions of the energy-momentum tensor – with schematic form $\langle TTTT \rangle$. The mixed anomalies may also be calculated by evaluating various four-point functions concerning both the energy-momentum tensor and R symmetry current. We then compare our direct calculations to the results got by utilizing index theorems and find agreements. We’ll not consider the Weyl anomaly separately, since it is related to the R symmetry anomaly by supersymmetry\(^6\).

By noticing that only the chiral fermions $\Psi^\alpha_A$ of the tensor multiplet contribute to the R symmetry anomaly, we’ll focus on the terms of $J^R_{\beta\gamma} (AB)$ that concern $\Psi^\alpha_A$ only

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\(^6\) One can see this as follows, firstly $\partial^{\alpha\beta} J^R_{\beta\gamma} \sim [P^{\alpha\beta}, J^R_{\beta\gamma}]$, using the superalgebra $\sim [\{Q^\alpha, Q^\beta\}, J^R_{\beta\gamma}]$, and by noticing that $\{Q^\alpha (Q^\beta, J^R_{\beta\gamma}) \} \sim \epsilon_{\alpha'\beta'\gamma'\delta'} T^{(\alpha'\beta',\gamma'\delta')} \delta^\alpha_{\gamma'}$, one finally has $\partial^{\alpha\beta} J^R_{\beta\gamma} \sim T^\delta_{\gamma'}$. 
\( J^{\alpha\beta}_{R(AB)} \sim (\Psi^\alpha_A \Psi^\beta_B + \Psi^\alpha_B \Psi^\beta_A) + \ldots \), where the ellipsis stands for the terms contributed by the scalars (roughly \( \omega^{CD}(\Phi_{AC}\partial^{\alpha\beta}\Phi_{DB} + \Phi_{BC}\partial^{\alpha\beta}\Phi_{DA}) \)).

By using the \( \Psi - \Psi \) OPE (3.38), the four-point function of the Spin(5)\( _R \) currents can be calculated out, with schematic terms

\[
\langle J^{\alpha_1\beta_1}_{R(AB_1)}(x_1) J^{\alpha_2\beta_2}_{R(AB_2)}(x_2) J^{\alpha_3\beta_3}_{R(AB_3)}(x_3) J^{\alpha_4\beta_4}_{R(AB_4)}(x_4) \rangle \\
\sim D_{AB} \frac{(x_1 - x_2)^{\beta_1\alpha_2}(x_2 - x_3)^{\beta_2\alpha_3}(x_3 - x_4)^{\beta_3\alpha_4}(x_4 - x_1)^{\beta_4\alpha_1}}{|x_1 - x_2|^6 |x_2 - x_3|^6 |x_3 - x_4|^6 |x_4 - x_1|^6},
\]

(3.40)

where \( D_{AB} \) is the abbreviation of \( D_{A_1B_1A_2B_2A_3B_3A_4B_4} \sim (\omega_{B_1A_2}\omega_{B_2A_3}\omega_{B_3A_4}\omega_{B_4A_1}) + \) \textit{permutations}. Other terms can be got by noticing the symmetries of the spinor indices and \( R \) symmetry indices.

We now couple \( J^{\alpha\beta}_{R(AB)} \) to a nontrivial background \( Sp(2,H)_R \) gauge field with potential \( A^{(AB)}_{\alpha\beta} \). One can view \( A^{(AB)}_{\alpha\beta} \) as the connection of a nontrivial \( Sp(2,H)_R \) vector bundle \( S_R \rightarrow W \) and the configurations of \( \Psi^\alpha_A \) as the sections of \( S_R \). Now, we have

\[
\epsilon^{AB} \langle \partial_{\alpha\beta} J^{\alpha\beta}_{R(AB)}(x) \rangle_A \sim \epsilon^{AB}(x) \int_2 \int_3 \int_4 \langle \partial_{\alpha\beta} J^{\alpha\beta}_{R(AB)}(x) \rangle J^R \cdot A(x_2) J^R \cdot A(x_3) J^R \cdot A(x_4)),
\]

where \( J^R \cdot A(x) \sim J^{\alpha\beta}_{R(AB)} A^{\alpha\beta}_{\alpha\beta}(x) \) and \( \epsilon^{AB}(x) \) are the parameters of a infinite small \( Sp(2,H)_R \) gauge transformation. By using \( \partial_{\alpha\beta} \left( \frac{1}{|x|^4} \right) \sim x_{\alpha\beta}/|x|^6 \), \( \partial_{\alpha\beta} \left( x^\gamma |x|^6 \right) \sim \delta^\gamma_{\alpha} \delta(x) \), and after some tedious calculations, one finally has

\[
\epsilon^{AB} \langle \partial_{\alpha\beta} J^{\alpha\beta}_{R(AB)} \rangle \sim -\frac{1}{3} \text{Tr}_R \left[ (dA)^{\alpha}_{\beta} (dA)^{\beta}_{\gamma} (dA)^{\gamma}_{\alpha} \epsilon \right] + \frac{1}{4} \text{Tr}_R \left[ (dA)^{\alpha}_{\beta} \epsilon \right] \text{Tr}_R \left[ (dA)^{\beta}_{\gamma} (dA)^{\gamma}_{\alpha} \right],
\]

(3.41)

where \( (dA)^{\alpha}_{\beta} = \frac{i}{2} (\partial^{\mu} A^\nu - \partial^{\nu} A^\mu) (\gamma^{\mu} \wedge \gamma^{\nu})^{\alpha}_{\beta} \), and we have omitted the \( R \) symmetry indices since we are tracing over them. The Wess-Zumino consistency conditions, combing equation (3.41), tell us that the full \( R \)-symmetry anomalies can be got from the descendents of the terms \( I^R_A \sim -\frac{1}{3} \text{Tr}_R (F' F' F' F') \) and \( \frac{1}{4}[\text{Tr}_R (F' F')]^2 \) (by expanding it to the first order in \( \epsilon \)), where \( F' = F + \epsilon \). By using the Pontrjagin classes of the \( Sp(2,H)_R \simeq \text{Spin}(5)_R \) vector bundle \( S_R \rightarrow W \), \( p_1(S_R) \sim -\frac{1}{2} \text{Tr}_R (F'^2) \) and \( p_2(S_R) \sim -\frac{1}{4} \text{Tr}_R (F'^4) + \frac{1}{8}[\text{Tr}_R (F'^2)]^2 \), one can then reorganize the terms of \( I^R_A \) as

\[
I^R_A \sim p_2(S_R) + \frac{1}{4} p_1(S_R)^2.
\]

(3.42)

Up to an overall numerical factor that we did not try to fix in our heuristic derivation, (3.42) agrees with the results of [31]. A careful calculation may give out the correct numerical factor \( \frac{1}{48} \).
For completeness, we now include the index theoretical derivation of the $\mathcal{R}$ symmetry anomaly (following [31]). The beautiful arguments of [32] tell us that this anomaly is given by descending the fourth order terms of the Chern character $\text{Ch}(S_{\mathcal{R}})$ of $S_{\mathcal{R}} \to W$. To get $\text{Ch}(S_{\mathcal{R}})$, one picks out a maximal torus $T_{\mathcal{R}} \subset Sp(2,H)_{\mathcal{R}} \simeq \text{Spin}(5)_{\mathcal{R}}$, $T_{\mathcal{R}} = \text{Spin}(2)_{\mathcal{R}} \times \text{Spin}(2)_{\mathcal{R}}$. Letting $(t_1,t_2)$ denote the pull back of the two generators of cohomology ring $H^*(BT_{\mathcal{R}},\mathbb{Q})$, where $BT_{\mathcal{R}}$ is the classifying space of $T_{\mathcal{R}}$ vector bundles. Since $\Psi^\alpha_A$ transform as the spinor representation of $\text{Spin}(5)_{\mathcal{R}}$, the four weights of $\Psi^\alpha_A$ are $(+\frac{1}{2},+\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2}),(+\frac{1}{2},-\frac{1}{2}),(-\frac{1}{2},+\frac{1}{2})$. $\text{Ch}(S_{\mathcal{R}})$ can then be calculated as $\text{Ch}(S_{\mathcal{R}}) = \frac{1}{2}(e^{\frac{1}{2}t_1} + e^{-\frac{1}{2}t_1})(e^{\frac{1}{2}t_2} + e^{-\frac{1}{2}t_2})$, where the factor $\frac{1}{2}$ is included since $\Psi^\alpha_A$ satisfy the real condition (3.1). Finally, one expands $\text{Ch}(S_{\mathcal{R}})$ to fourth order and gets $rac{1}{192}(t_1^4 + t_2^4) + \frac{1}{32}t_1^2t_2^2$. Hence, the $\mathcal{R}$ symmetry anomaly is given by the descendant of

$$I_{\mathcal{R}}^A = \frac{1}{48}(p_2(S_{\mathcal{R}}) + \frac{1}{4}p_1(S_{\mathcal{R}})^2).$$

Here, the Pontrjagin classes are given as $p_1(S_{\mathcal{R}}) = t_1^2 + t_2^2$ and $p_2(S_{\mathcal{R}}) = t_1^2t_2^2$.

Gravitational Anomaly And Mixed Anomaly

The gravitational anomaly can also be calculated by computing the four point functions of the energy-momentum tensor $T^{(\alpha\beta,\gamma\delta)}$. But the result can be more directly got by evaluating the Feynman diagrams. Figure 1 is one of the typical anomalous diagram, where the interior loop is for the chiral fermions $\Psi^\alpha_A$ or for the chiral tensor $H^{(\alpha\beta)}$. The interaction between these fields and the background gravity is through $(1/4)^2h^{(\alpha\beta,\gamma\delta)}T^{(\alpha\beta,\gamma\delta)}$, where the energy-momentum tensor of the chiral tensor $H^{(\alpha\beta)}$ is given as $T^{(\alpha\gamma,\beta\delta)} \sim H^{(\alpha\beta)}H^{(\gamma\delta)} - H^{(\gamma\beta)}H^{(\alpha\delta)} + ..., h^{(\alpha\beta,\gamma\delta)} = h^{\mu\nu}\gamma^\alpha_\mu\beta_\gamma\gamma^\delta_\nu$, and the background metric is given as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. At every vertex of Figure 1 but the dotted one one inserts $(1/2)h^{\mu\nu}T_{\mu\nu}$, but at the dotted vertex, one should use $\varepsilon^{\mu}\partial^\nu T_{\mu\nu}$.

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7 For a detailed discussion to other anomalous diagrams, especially the diagrams with seagull vertices where two gravitons are simultaneously attached, see [32].
We now focus on the contributions of the chiral tensor $H^{(\alpha\beta)}$ since the contributions of $\Psi^{\alpha}_A$ are quite usual. The propagator of $H^{(\alpha\beta)}$ can be easily calculated by using the quantization relations or from (3.39) directly,

$$\langle H^{(\alpha\beta)}(x)H^{(\gamma\delta)}(0) \rangle \sim \int e^{-ip\cdot x} \frac{p^{\alpha\gamma}p^{\beta\delta} + p^{\beta\gamma}p^{\alpha\delta}}{p^2 + i\epsilon}.$$  \hspace{1cm} (3.44)

One can then evaluate the Feynman diagrams by using (3.44). Fortunately, the calculations that we need to perform have been essentially carried out in the original calculations of the gravitational anomaly [32]. The underlying reason is because that the spinor techniques have been employed there – to simplify the algebra – in calculating gravitational anomaly of the self-dual field. In fact, one can identify our chiral field $H^{(\alpha\beta)}$ as the symmetrical part of the chiral projection of the field $\phi_{\alpha\beta}$ introduced in [32] (restricted on six-dimensions). Moreover, upon truncating $\phi_{\alpha\beta}$ to $H^{(\alpha\beta)}$, the momentum independent term – which is irrelevant for the gravitational anomaly – of [32]'s formula (48) automatically disappears. The $\phi_{\alpha\beta}$ propagator there then becomes our $H^{(\alpha\beta)}$ propagator (3.44). And the energy-momentum tensor of projected $\phi_{\alpha\beta}$ (formula (51) of [32]) becomes the energy-momentum tensor of $H^{(\alpha\beta)}$. Thus, one can borrow the calculations of [32]. It turns out that the anomaly terms of the chiral tensor are the descendents of $I^H_A$, which is given
by $-\frac{1}{8}$ times the power 4 terms of the expansion of $L$-genus,

$$I^H_A = \frac{1}{360} \left(p_1(TW)^2 - 7p_2(TW)\right). \quad (3.45)$$

We now include the contributions of the chiral fermions $\Psi^\alpha_A$. According to the calculations of \[\text{[31]}\], the corresponding anomaly terms are given by the power 4 terms of the expansion of $\frac{1}{2}\hat{A}(W) \times 4$, where $\hat{A}(W)$ is the $\hat{A}$-genus, $\frac{1}{2}$ comes from the real condition (3.43) and the factor 4 comes for four chiral fermions. By adding these contributions and the terms $I^H_A$ of the chiral tensor $H^{(\alpha\beta)}$, one has the total gravitational anomaly $I^G_A$

$$I^G_A = \frac{1}{48} \left(4p_1(TW)^2 - p_2(TW)\right). \quad (3.46)$$

The mixed terms $I^M_A$ of the $R$ symmetry anomaly and the gravitational anomaly are contributed by the chiral fermions and can also be calculated by evaluating the relevant Feynman diagrams. According to \[\text{[31]}\]'s results, $I^M_A$ can be extracted from the mixed terms of the expansion of $\frac{1}{2}\hat{A}(W)\text{Ch}(S_R)$, where the factor $1/2$ comes from the real condition imposed on $\Psi^\alpha_A$,

$$I^M_A = -\frac{1}{4} \frac{p_1(S_R)p_1(TW)}{24}. \quad (3.47)$$

The total terms $I_A$ of the anomalies are then given by the summation of the $R$-symmetry anomaly $I^R_A$ (3.43), the gravitational anomaly $I^G_A$ (3.46) and the mixed anomaly $I^M_A$ (3.47), $I_A = I^G_A + I^R_A + I^M_A$.

**Digress To The Anomaly Cancelation Of M5 Brane**

We now digress to discuss the implications of the anomalies of quantum Abelian gerbe theory when it is applied to the $M_5$ brane dynamics. In what follows we will connect anomaly $I_A$ to the gravitational anomaly of $M_5$ brane. And, for completeness, we’ll also include a treatment to the subtle cancelation mechanism (due to \[\text{[31]}\] [33] [34]) of this anomaly.

It is well known that, the six-dimensional quantum Abelian gerbe theory can be realized as the low energy world-volume theory of $M_5$ brane. The five scalars $\Phi^{AB}$ extend to the five coordinates $x^{\perp AB}_\perp$ (in Plank unit $2\pi l_p = 1$) of the transverse position of five-brane world volume $W$. The Spin(5)$_R$ vector bundle $N_R \to W$ serves as the normal bundle of the five-brane $W$ in eleven-dimensional spacetime $M$, $W \hookrightarrow M$. And the $R$ symmetries Spin(5)$_R \subset GL(5, \mathbb{R})$ act as local diffeomorphisms along the transverse directions.
Hence, both the anomalies of the diffeomorphisms along $W$ and the $R$ symmetry anomaly are gravitational anomalies of $M$-theory and must be canceled to preserve the general covariances.

It turns out that this cancelation is achieved by including the anomalies that inflow from the bulk. The inflowing anomalies come from the Chern-Simons terms $I_{CS}$ of the bulk theory. These terms concern the three-form potential $C$ and its four-form strength $G$ of the eleven-dimensional supergravity multiplet. After normalizing the kinetic term of $C$ as $I_k = -\frac{1}{4\pi} \int_M G \wedge *G$, $I_{CS}$ can be written as

$$I_{CS} = -\frac{1}{6} \int_M C \wedge \frac{G}{2\pi} \wedge \frac{G}{2\pi} + \int_M C \wedge *J,$$  \hspace{1cm} (3.48)

where $J$ is given by $*J = \frac{1}{48} [p_2(TM) - \frac{1}{4} p_1(TM)^2]$ and can be viewed as the current of dissolved $M_2$ branes.

One then views the $M_5$ brane as the magnetic source of the potential $C$, its presence will modify the Bianchi identity of the four-form strength $G$ to $dG = 2\pi [\delta W]$, where $[\delta W]$ is the Poincare dual of $W$.

To represent $[\delta W]$ in terms of the Thom class $\Phi_W$ of the normal bundle $N_R \to W$, one picks out a tubular neighborhood $T_\epsilon$ of $W$ in $M$ by attaching to each point of $W$ a five-dimensional open sphere of sufficiently small radius $\epsilon$ perpendicular to $W$ at the center. $T_\epsilon$ is diffeomorphic to the normal bundle $N_R$ of $W$ in $M$. And the boundary of $T_\epsilon$’s closure forms a sphere bundle $S_\epsilon \to W$ with four-dimensional sphere surface $S^4_\epsilon$ (of radius $\epsilon$) as its fiber. $[\delta W]$ can then be taken as the extension of $\Phi_W$ in $M$. By using the Euler class $e(S_\epsilon)$ of $S_\epsilon$, this extension can be explicitly written as $[\delta W] = \frac{1}{2} e(S_r) \wedge d\delta(r - \epsilon) = \frac{1}{2} d(e(S_r) \theta(r - \epsilon))$, where the factor $\frac{1}{2}$ is inserted by noticing that the integration of $e(S_r)$ along fiber $S^4_\epsilon$ is $\int_{S^4_\epsilon} e(S_r) = 2$. Thus, the modified Bianchi identity can be satisfied by modifying the four-form strength $G$ as

$$G = dC + 2\pi \theta(r - \epsilon)e(S_r)/2.$$ \hspace{1cm} (3.49)

The term – due to the current of the dissolved $M_2$ branes (the second term of (3.48)) – of the inflowing anomaly is given by the descendant of $(G \wedge *J)|_{S^4_\epsilon+0}$. By substituting (3.49), integrating over $S^4_\epsilon+0$ and taking the $\epsilon \to 0$ limit, one will have $I_C = (2\pi) * J|_W$.\hspace{1cm}8

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8 By noticing that $TM|_W = TW \oplus N_R$, one has $p_1(TM) = p_1(TW) + p_1(N_R), p_2(TM) = p_2(TW) + p_2(N_R) + p_1(TW)p_1(N_R)$. Hence, $*J|_W$ can be given as $*J|_W = \frac{1}{48}[p_2(TW) - \frac{1}{4} p_1^2(TW) + p_2(N_R) - \frac{1}{4} p_1^2(N_R) + \frac{1}{2} p_1(TW)p_1(N_R)]$.
And the anomaly inflows from the first terms of (3.48) is given by the descendant of 
\[-\frac{1}{24\pi}(G \wedge G \wedge G)|_{S_{+0}}.
\]
By substituting the expression (3.43) of \(G\), one has 
\[-(2\pi)\frac{1}{48}e(S_\epsilon) \wedge e(S_\epsilon) \wedge e(S_\epsilon).
\]
After integrating over the fiber \(S_\epsilon^4\), one will have \(I_S = -(2\pi)\frac{1}{24}p_2(N_R)\).

A little algebra shows us that all the anomalies cancel out after summing up the anomalous terms \(I_A\), \(I_C\) and \(I_S\), that is
\[I_A + I_C + I_S = 0.\tag{3.50}\]

3.4. Quantization Of Fluxes

The purpose of the present subsection is to discuss the quantum mechanics of the zero modes of the free tensor multiplet defined on a more general spacetime manifold. As one will see that these modes should be classified by the cohomology group (or compact cohomology group) of the (1+5)-dimensional spacetime \(W\). Thus, if \(W\) is topologically trivial, for example \(W = \mathbb{R}^{1,5}\) as the most parts of the present paper are focusing on, these zero modes are absent, and one needs not to discuss their quantization. But, in applying QNG theory to describe \(M\) theory compactified on \(T^4\), in the matrix theory context, \(W\) may be \(T^5 \times \mathbb{R}^1\), so that its cohomology is nontrivial. Hence, it is deserved to quantize the zero modes of \(H^{(\alpha\beta)}\) canonically. These zero modes are the static solutions of the following equations with nontrivial fluxes
\[
\tilde{\partial}_{\alpha\beta} \tilde{H}^{(\beta\gamma)} = 0 \\
\iota^{\beta\gamma} \tilde{\partial}_{\alpha\beta} \tilde{G}^{(\gamma\delta)} = 0, \tag{3.51}
\]
the accurate definition of the magnetic field \(\tilde{H}^{(\alpha\beta)}\) and electric field \(\tilde{G}^{(\alpha\beta)}\) will be given later. On a curved \(X\), the partial differential operator \(\tilde{\partial}_{\alpha\beta}\) in equation (3.51) should be understood as a covariant derivative with some spin connection.

It would be convenient to pick out an axial gauge. We denote the connection of the \(U(1)\) gerbe in this gauge as \(\tilde{B}^\alpha_{\beta}\) which satisfies the conditions \(\tilde{B}^\alpha_{\gamma\beta} = \tilde{B}^\beta_{\gamma\alpha}\) and \(\iota_{\alpha\gamma} \tilde{B}^\gamma_{\beta} = \iota_{\beta\gamma} \tilde{B}^\gamma_{\alpha}\). We then decompose the tensor \(H^{(\alpha\beta)}\) as time derivative part \(\tilde{\Pi}^{(\alpha\beta)}\) plus spatial derivative part \(\tilde{H}^{(\alpha\beta)}\). Here, \(\tilde{\Pi}^{(\alpha\beta)} = 2(\iota \cdot \partial)\iota^{\alpha\gamma} \tilde{B}^{\beta}_{\gamma}\) and \(\tilde{H}^{(\alpha\beta)} = \tilde{\partial}^{\alpha\gamma} \tilde{B}^{\beta}_{\gamma} + \tilde{\partial}^{\beta\gamma} \tilde{B}^{\alpha}_{\gamma}\).

\(^9\) To see this result, one notices \(S_\epsilon^4 \simeq \text{Spin}(5)_R / \text{Spin}(4)_R\), and views \(N_R\) as the vector bundle of this \(\text{Spin}(4)_R\), hence, \(p_2(N_R) = e^2(N_R)\), which can be identified with the pull back of \(e^2(S_\epsilon)\). By further using \(\int_{S_\epsilon^4} e(S_\epsilon) = 2\), the result follows. For a rigorous treatment of the related mathematical facts see [38] Lemma. 2.1.
And the self dual condition $\partial_{\alpha\gamma} B^\gamma_{\beta} + \partial_{\beta\gamma} B^\gamma_{\alpha} = 0$, acting on the configuration space of $B^\alpha_{\beta}$, now becomes $\bar{\partial}_{\alpha\gamma} \bar{B}^\gamma_{\beta} + \bar{\partial}_{\beta\gamma} \bar{B}^\gamma_{\alpha} = -2(\cdot \partial)_{\alpha\gamma} \bar{B}^\gamma_{\beta} = -t_{\alpha\gamma} t_{\beta\delta} \bar{\Pi}^{(\gamma\delta)}$. This can be rewritten as

$$\bar{G}_{(\alpha\beta)} = \bar{\Pi}^{(\gamma\delta)} t_{\gamma\alpha} t_{\delta\beta},$$

(3.52)

where we have denoted $\bar{\partial}_{\alpha\gamma} \bar{B}^\gamma_{\beta} + \bar{\partial}_{\beta\gamma} \bar{B}^\gamma_{\alpha}$ as $-\bar{G}_{(\alpha\beta)}$. Now the equation of motion $\partial_{\alpha\beta} H^{(\gamma\alpha)} = 0$ can be rewritten as the form of (3.51).

To perform the canonical quantization procedure, we will view (3.52) as a constraint equation that acts on the Hilbert space after finishing quantization. In doing so, we effectively enlarge the chiral theory to a theory of ordinary tensor field and then throw out the anti-self-dual part after quantizing it.

Now, we naturally identify $\bar{\Pi}^{(\alpha\beta)}$ with the conjugate momentum operator of $\bar{B}^\alpha_{\beta}$. By using the canonical commutator between $\bar{B}^\alpha_{\beta}$ and $\bar{\Pi}^{(\alpha\beta)}$, one can calculate the commutator between $\bar{H}^{(\alpha\beta)}$ and $\bar{\Pi}^{(\alpha\beta)}$. For convenience, we view $\bar{H}^{(\alpha\beta)}$ as a three form $\bar{H}$ and $\bar{\Pi}^{(\alpha\beta)}$ as a two form $\bar{\Pi}$ on $X$, then the $\bar{H} - \bar{\Pi}$ commutator can be elegantly written as

$$\left[ \int_X \frac{\bar{H}}{2\pi} \wedge \alpha, \int_X \frac{\bar{\Pi}}{2\pi} \wedge \beta \right] = \frac{1}{2\pi i} \int_X \beta \wedge d\alpha$$

(3.53)

where $\alpha$ and $\beta$ are two arbitrary 2-forms on $X$.

Since we are focusing on the quantization of (3.51)’s static solutions with non-trivial fluxes, we can eliminate $\bar{\Pi}^{(\alpha\beta)}$ by making use of equation (3.52) and viewing $\bar{G}$ as the three form $\star \bar{\Pi}$. The fluxes will consist of a system with Hamiltonian

$$\frac{1}{4\pi} \int_X \left( t_{\alpha\beta} t^{\gamma\delta} \bar{G}_{(\alpha\gamma)} \bar{G}_{(\beta\delta)} + \bar{H}^{(\alpha\gamma)} \bar{H}^{(\beta\delta)} t_{\alpha\beta} t^{\gamma\delta} \right) + \frac{1}{2\pi} \int_X \bar{H}^{(\alpha\beta)} \bar{G}_{(\alpha\beta)},$$

(3.54)

and with nontrivial $\bar{H} - \bar{G}$ commutators,

$$\left[ \int_X \frac{\bar{H}}{2\pi} \wedge \alpha, \int_X \frac{\bar{G}}{2\pi} \wedge \beta \right] = \frac{L(\alpha, \beta)}{2\pi i},$$

(3.55)

where the link number $L(\alpha, \beta)$ between $\alpha$ and $\beta$ is given by $\int_X \beta \wedge d\alpha$, which is antisymmetric about $\alpha$ and $\beta$, $L(\alpha, \beta) = -L(\beta, \alpha)$.

For the compactness of the Abelian group $U(1)$, the magnetic fluxes $\int \frac{\bar{H}}{2\pi}$ on some three cycles will be topologically quantized, and the electric fluxes $\int \frac{\bar{G}}{2\pi}$ on three cycles will be canonically quantized. Thus, roughly speaking, both these fluxes will be classified by $H^3(X, \mathbb{Z})$, or by $H^2(X, \mathbb{R}/2\pi \mathbb{Z}) = \text{Hom}(H^3(X, \mathbb{Z}), U(1))$ through Pontrjagin-Poincare duality. Then, the phase space of the fluxes may be identified as $H^2(X, \mathbb{R}/2\pi \mathbb{Z}) \oplus$
\(H^2(X, \mathbb{R}/2\pi\mathbb{Z})\). And the commutator (3.53) gives us a symplectic form \(L(\alpha, \beta)\) on this phase space, which will enable us to properly quantize these fluxes. The final results may be identical to the analysis to the quantum self-duality of the fluxes of \(H\) \([13]\).

**Generalization**

It is interesting to generalize the above discussions of \(U(1)\) Abelian gerbe theory to more general case with Abelian gerbe group \(T_G\). The generalization of the quantization of oscillating modes is trivial. We now focus on the generalization of the quantization of \(H\)-fluxes. To specify the periods of the \(r\) tensors of \(T_G\) theory, one should pick out an imbedding of \(U(1)\) into \(T_G\) firstly. All possible imbedding will define a magnetic charge space \(\mathbb{M} = \text{Hom}(U(1), T_G)\). The periods of the \(r\) tensor fields should take value in \(H^3(X, \mathbb{Z}) \otimes \mathbb{M}\). (3.56)

To specify \(\mathbb{M}\), one puts a coordinates system \((z_1, z_2, ..., z_r), |z_i|^2 = 1, (i = 1, 2...r)\) on the maximal torus \(T_G\). The imbedding \(U(1) \to T_G\) can then be characterized by

\[z \to (z^{k_1}, z^{k_2}, ..., z^{k_r}),\] (3.57)

where \(z\) is an element of \(U(1)\) group. Thus, each imbedding is specified by a sequence of \(r\) integers \((k_1, k_2, ......., k_r)\). The collection of all possible imbedding form a \(r\)-dimensional lattice \(\Lambda_{\text{cochar}}\), which is the cocharacter space of \(G\). Hence, \(\mathbb{M} = \Lambda_{\text{cochar}}\).

The Poincare and Pontrjagin duality of \(H^3(X, \mathbb{Z}) \otimes \mathbb{M}\) is \(H^2(X, T_G)\), where \(T_G\) is the maximal torus of the Langlands dual group \(\hat{G}\) of \(G\). Further more, the flat connections are parameterized by \(H^2(X, T_G)\), which is the Poincare and Pontrjagin duality of the electric fluxes \(H^3(X, \mathbb{Z}) \otimes \mathbb{E}\). Thus, the phase space of the \(H\)-fluxes may be identified as

\[H^2(X, T_G) \simeq H^2(X, T_G).\] (3.58)

To study the quantum mechanical theory of these fluxes, one should impose an appropriate symplectic form on this phase space. This symplectic form should be some kind of natural generalization of \(L(\alpha, \beta)\). It seems that this can be achieved only when \(T_G\) is identical to \(T_{\hat{G}}\). Clearly, the A – D – E series of Lie groups satisfy this requirement.

As an analog of ’t Hooft’s discussions for the Abelian fluxes of a non-Abelian gauge theory \([36]\), one can naturally guess that the Hilbert space of Abelian flux states of a QNG theory with gerbe group \(G_{\text{ad}}\) is given by \(H^3(X, \mathbb{Z}(\hat{G}))\), where \(\hat{G}\) is the universal covering of \(G\), \(\mathbb{Z}(\hat{G})\) is \(\hat{G}\)'s center, and \(G_{\text{ad}} = \hat{G}/\mathbb{Z}(\hat{G})\). This result may be checked by firstly reducing the QNG theory on a \(S^1\), and then comparing \(H^3(X, \mathbb{Z}(\hat{G}))\) to the Hilbert space of the Abelian fluxes of the resulting \((4 + 1)\)-dimensional gauge theory \([37]\).
4. $N = 4$ Gauge Theory From QNG Theory

In last section, to study the QNG theory, we perturbed it into quantum Abelian gerbe theory. In this section we will try to get a glance at the non-Abelian nature of QNG theory by perturbing it into a four-dimensional $N = 4$ super-Yang-Mills theory with gauge group $G$. To do this, we compactify the six dimensional theory on a torus $T^2$, which is characterized by two parameters $(\tau, A_T)$, where $\tau$ is its complex structure and $A_T$ is its world volume. We’ll see that $\tau$ will become the complex coupling constant of the gauge theory, and the Mantonen Olive duality of $N = 4$ super-Yang-Mills theory can be explained as the mapping class symmetry of $T^2$. We’ll understand how to get the four-dimensional superconformal group $PSU(2, 2|4)$ (and its central extensions) from $OSp(2, 6|2)$ (and its central extensions), under the compactification.

All these results have been known (conjectured) in literatures. But, by using the formalism in the present paper, we can get them quite directly. Especially, for the Abelian theories, we can explicitly write down the dimensional reduction. We further argue that the six dimensional origin of $N = 4$ super-Yang-Mills theories and their $SL(2, \mathbb{Z})$ dualities can be extended to the non-Abelian cases.

4.1. The Dimension Reduction Of Superconformal Symmetries And Dualities

In this subsection we’ll try to match the $T^2$ reduction of the algebra of superconformal group $OSp(2, 6|2)$ with the algebra of $PSU(2, 2|4)$ which is the superconformal group of four-dimensional $N = 4$ super-Yang-Mills theory. The benefits of this matching are two folds. Firstly, it is required by the matching between dimension reduced QNG theory and the $N = 4$ super-Yang-Mills theory. Moreover, with the six-dimensional interpretation of Mantonen Olive duality in mind, one can conveniently derive the $SL(2, \mathbb{Z})$ transformations of the supersymmetries with the results agreeing with [38].

The Dimension Reduction Of Superconformal Symmetries

To see the dimension reduction of the $(1+5)$-dimensional superconformal symmetry, we let the compactified $T^2$ be along the $x^5, x^6$ directions and decompose the Lorentz group $Spin(1, 5)$ into $Spin(1, 3) \times Spin(2)$, where $Spin(1, 3)$ is the Lorentz group of four-dimensional theory and $Spin(2)$ is the rotation of $(5, 6)$ directions. The chiral spinor of $Spin(1, 5) \simeq SL(2, \mathbb{H}) \subset SL(4, \mathbb{C})$ can then be decomposed into irreducible representations of $Spin(1, 3) \times Spin(2)$, with the decomposing of indexes

\[
(\alpha) \rightarrow (a, +\frac{1}{2}) \oplus (\dot{a}, -\frac{1}{2}),
\]  

(4.1)
where \(a = 1, 2\) are the chiral spinor indexes and \(\dot{a} = 1, 2\) are the anti-chiral spinor indexes of the four-dimensional Lorentz group \(\text{Spin}(1, 3) \subset \text{Spin}(4, \mathbb{C}) \simeq SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})\), \(+1/2, -1/2\) are the weights (the eigenvalues of the generator \(S_{56} = \frac{1}{2} \gamma^5 \gamma^6\)) of \(\text{Spin}(2)\).

We then decompose the \(Q\)-charges and \(S\)-changes into

\[
Q^\alpha_A \rightarrow (Q^a_A, \omega_{AB} Q^{B}_{\dot{a}}) \\
S^A_{\dot{a}} \rightarrow (\omega^{AB} \overline{S}_{B\dot{a}}, S^{aA}) ,
\]

according to their \(\text{Spin}(2)\) weights, where \(Q^a_A\) weights \(+1/2\) and \(S^{aA}\) weights \(-1/2\). The real structure \(\hat{\tau}\) now acts as the complex conjugation that exchanges \(Q^a_A, S^{aA}\) with \(Q^{B}_{\dot{a}}, S^{B}_{\dot{a}}\), respectively.

The six-dimensional momentum \(P^{\alpha\beta}\) can be decomposed as,

\[
P^{\alpha\beta} \rightarrow P^{ab} \oplus P_{\dot{a}\dot{b}} \oplus Z \oplus \overline{Z} ,
\]

where \(P_{\dot{a}\dot{b}}\) is the conjugate of \(P^{\dot{a}\dot{b}}\) with \(P_{\dot{a}\dot{b}} = \epsilon_{bd} \epsilon_{\dot{a}\dot{c}} P^{dc}\), and \(Z, \overline{Z}\) are the momentums around the internal space \(T^2\) with \(\text{Spin}(2)\) weight \(+1\) and \(-1\), respectively. \(K_{\alpha\beta}\) can also be likewise decomposed.

One can then derive some of the superalgebras of \(\text{PSU}(2, 2|4)\) by dimensional reducing the superalgebras of \(\text{OSp}(2, 6|2)\) which have been explicitly written out in section \(\S\). For examples, the \(Q - Q\) anti-commutators are

\[
\{Q^a_A, Q^{B}_{\dot{a}}\} = P^{a\dot{a}} \delta^B_A .
\]

Likewise, one can have the \(S - S\) anti-commutators \(\{S^{aA}, S^{B}_{\dot{a}}\} = K^{a\dot{a}} \delta^A_B\). The \(K - S\) commutators and \(P - Q\) commutators are

\[
i[K^{a\dot{a}}, S^B_\dot{b}] = \delta^a_b Q^{B\dot{a}}, \quad i[P^{a\dot{a}}, Q^B_\dot{b}] = \delta^a_b S^{B\dot{a}}
\]

and their complex conjugates.

Further more, one can decompose the angular momentum \(J^\alpha_{\dot{a}}\) as,

\[
J^\alpha_{\dot{a}} \rightarrow J_{ab} \oplus J_{\dot{a}b} \oplus Z^{a\dot{a}} \oplus \overline{Z}_{\dot{a}a} ,
\]

where \(J_{ab}\) and \(J_{\dot{a}b}\), which are complex conjugate to each other, are the selfdual and anti-selfdual parts of the four-dimensional angular momentums, and \(Z^{a\dot{a}}\) is a vector of \(\text{Spin}(4, \mathbb{C})\) (the complexification of the \((1+3)\)-dimensional Lorentz group \(\text{Spin}(1, 3)\)) \(\overline{Z}_{\dot{a}a}\) is the complex conjugate of \(Z^{a\dot{a}}\). One can have the \(Q - S\) anti-commutators (by using the results in section \(\S\)),

\[
\{Q^a_A, S^B_\dot{b}\} = J_{ab} - i \epsilon_{ab} (D \delta^B_A - \mathcal{R}^B_A) .
\]
The $\overline{Q} - S$ anti-commutators are the complex conjugation of (4.6).

Concerning the terms of $Z$, one will have the additional $Q - Q$ anti-commutators, 
\[ \{ Q^a_A, Q^b_B \} = \epsilon^{ab}_{\alpha\beta} \omega_{AB} Z \] and its complex conjugate \( \{ \overline{Q}^a_A, \overline{Q}^b_B \} = \epsilon^{ab}_{\dot{\alpha}\dot{\beta}} \omega^{AB} \overline{Z} \). States carrying nonzero $Z$ charge will break the $SU(4)_R$ symmetry down to the $Spin(5) \simeq Sp(2,H)_R$ subgroup that preserves $\omega_{AB}$. The corresponding BPS states will satisfy the condition $|P|^2 - |Z|^2 = 0$. In terms of six-dimensional theory, this condition is just the on-shell condition of massless particles, $P^{\alpha\beta} P_{\alpha\beta} = 0$. (Concerning $Z^a_{\dot{a}}, Z^\dot{a}_{a}$, they will contribute additional $Q - S$ anti-commutators \( \{ Q^a_A, S^\dot{a}_B \} = Z^a_{\dot{a}} \omega_{AB} \) and its complex conjugate \( \{ \overline{Q}^a_A, S^\dot{a}_B \} = \overline{Z}^\dot{a}_{a} \omega^{AB} \).

To see the meaning of $Z$, one takes $T^2$ as $S \times S'$ with radius $U$ and $V$ respectively. Let $u, v$ denote the coordinates of $S$ and $S'$ with $(u \sim u + 2\pi U, v \sim v + 2\pi V)$. $Z$ can then be written as

$$ Z = \frac{n}{V} + i \frac{m}{U} , \quad (4.7) $$

where $n \in \mathbb{Z}, m \in \mathbb{Z}, n/V$ are the momentum modes around the $v$ cycle and $m/U$ are the momentum modes around the $u$ cycle. To generalize to the case of a general $T^2$ with complex structure $\tau$, one introduce complex coordinates $dz = dv + i(V/U)du, d\overline{z} = dv - i(V/U)du$ to the special cases that we just considered, and identify $V/U$ as $\text{Im} \tau$ (in these special cases $\text{Re} \tau = 0$). The general cases correspond to set $dz = dv + \tau du, d\overline{z} = dv + \overline{\tau} du$, and make the replacement of $iV/U \rightarrow \tau$ and $VU \rightarrow A_T$, where $A_T = (i/2) \int_{T^2} (dz \wedge d\overline{z})/\text{Im} \tau$ is the volume of $T^2$. $Z$ can then be generalized as

$$ Z = \frac{(n + \tau m)}{\sqrt{\text{Im} \tau A_T}} . \quad (4.8) $$

To rewrite the parameter $1/\sqrt{A_T}$ of (4.8) in terms of the variables of the resulting four-dimensional gauge theory. We first recall that the quantum states of the fluxes $H^2(X, T_G) \simeq H^2(T^2, T_G) \simeq T_G$, thus, for a $T_G$ quantum Abelian gerbe theory compactified on $T^2$, we will have $r$ additional scalars coming from the Wilson-t’Hooft surfaces of $r$ two forms $B^i, i = 1, ..., r$. Now, we set $r = 1$ for simplicity (this case will be further discussed in more details in next section) with the additional scalar $\Phi \sim B/\sqrt{A_T}$. Noticing that the flux quantization means $BA_T \sim 1$, we have $\Phi \sim 1/\sqrt{A_T}$. In another words, the central charge $Z$ is proportional to the vacuum expectation of the additional scalar.

\[ ^{10} \] Since one can easily carry out the integration $(i/2) \int_{T^2} dz \wedge d\overline{z}$ to get $(i/2)(\oint_v dz \oint_u d\overline{z} - \oint_u dz \oint_v d\overline{z}) = (\text{Im} \tau) VU.$
Further more, the $SU(4)_{\mathbb{R}} \subset SL(4, \mathbb{C})_{\mathbb{R}}$ invariance of the underlying four-dimensional $N = 4$ superconformal field theory will tells us that general central terms $Z_{AB}$ of four-dimensional $N = 4$ superalgebra must take the form of

$$Z_{AB} = \langle \Phi_{AB} \rangle (n + \tau m) / \sqrt{\text{Im} \tau}, \quad (4.9)$$

where $\langle \Phi_{AB} \rangle$ stand for the vacuum expectations of the scalars $\Phi_{AB}$ of the $N = 4$ gauge theory. These scalars are the dimensional reduction of the original five scalars in $N = (2, 0)$ tensor multiplet plus an additional scalar that comes from the tensor field. We shall discuss these in details in next subsection.

Now, the problem is what are the six dimensional explanations about the terms of $Z_{AB}$ other than $Z$? It turns out that they come from the dimensional reduction of the central extension of six-dimensional $N = (2, 0)$ superalgebra.

As we have calculated (for free tensor multiplet, by using the expressions (3.34)), the $Q - \bar{Q}$ anti-commutators of the six dimensional $N = (2, 0)$ theory have a central extension $Z^\alpha{}^\beta_{AB}$ contributed by the self-dual strings,

$$\{ Q^\alpha_A, Q^\beta_B \} = \omega_{AB} P^{\alpha\beta} + Z^\alpha{}^\beta_{AB}, \quad (4.10)$$

where $Z^\alpha{}^\beta_{AB}$ is carried by a infinite long string extended along some direction of $X$ or by a winding string wound around a super-symmetric 1-cycle of $X$. $Z^\alpha{}^\beta_{AB}$ transform as the 5 representation of $Sp(2, \mathbb{H})_{\mathbb{R}}$, thus

$$Z^\alpha{}^\beta_{AB} \omega^{AB} = 0. \quad (4.11)$$

We now decompose $Z^\alpha{}^\beta_{AB} \rightarrow \epsilon^{ab} Z_{AB} \oplus \epsilon^{\dot{a}\dot{b}} \overline{Z}_{AB} \oplus Z^{a\dot{c}}_{AB} \oplus \overline{Z}^{\dot{a}}{}^{\dot{c}}_{AB}$, with the constraint (4.11) acts on them respectively. Combing $\epsilon^{ab} Z_{AB}$ with $\omega_{AB} \epsilon^{ab} Z$ that comes from the decomposing of $P^{\alpha\beta}$, one has a $SU(4)_{\mathbb{R}} \subset SL(4, \mathbb{C})_{\mathbb{R}}$ vector $Z_{AB}$, with the constraint (4.11) now dropped (One can likewise form the 6 representation $\overline{Z}_{AB}$ of $SU(4)_{\mathbb{R}} \subset SL(4, \mathbb{C})_{\mathbb{R}}$). Moreover, the combinations of the $Sp(2, \mathbb{H})_{\mathbb{R}} \subset Sp(4, \mathbb{C})_{\mathbb{R}}$ 5 representation $Z^{a\dot{c}}_{AB}$ and singlet representation $\omega_{AB}$ will form a 6 representation $Z^{a\dot{c}}_{AB}$ of $SU(4)_{\mathbb{R}} \subset SL(4, \mathbb{C})_{\mathbb{R}}$, and one can likewise form the complex conjugation $\overline{Z}^{\dot{a}}{}_{\dot{c}}^{AB}$ of $Z^{a\dot{c}}_{AB}$. Thus, we have

$$\{ Q^a_A, Q^b_B \} = \epsilon^{ab} Z_{AB}, \quad \{ Q^a_A, \overline{S}^{\dot{b}}_B \} = Z^{a\dot{c}}_{AB}, \quad (4.12)$$
and their complex conjugations. Charges $Z^{\hat{a} \hat{b}}_{AB}$ are carried by the self-dual strings along the four dimensions $\mathbb{R}^3 \times \mathbb{R}^1$.

**The Duality Transformations Of The Superalgebra**

Having reduced the superconformal group $OSp(2,6|2)$ of QNG theory into the superconformal group $PSU(2,2|4)$ of $N=4$ gauge theory, and matched their central extensions, we now turn to derive the $SL(2,\mathbb{Z})$ transformations of the four-dimensional supersymmetries, with the six-dimensional explanation of Montonen-Olive duality in mind.

Firstly, one notice that the transformation $S : u \to v, v \to -u$ acts as $dz \to -\frac{1}{\tau} dz$, $d\bar{z} \to -\frac{1}{\tau} d\bar{z}$ and $\tau \to -\frac{1}{\tau}$, that is, $dz/\sqrt{\tau} \to idz/\sqrt{\tau}$ and $d\bar{z}/\sqrt{\tau} \to -id\bar{z}/\sqrt{\tau}$. Clearly, this transformation implies a Spin(2) rotation $\exp(i\phi_S) = -|\tau|/\tau$. Further more the six-dimensional origin (4.2) tells us that $Q^a_A$ is weight $1/2$ under Spin(2), thus one has the $S$-duality transformation of $Q^a_A$, $Q^a_A \to e^{i\phi_S/2}Q^a_A = \sqrt{-|\tau|/\tau}Q^a_A$. By noticing that $T : \tau \to \tau + 1$ is trivially acting on $Q^a_A$, one can get the transformations of $Q^a_A$ under full $SL(2,\mathbb{Z})$ mapping class symmetries, $\tau \to (a\tau + b)/(c\tau + d), ad - bc = 1, with$

$$Q^a_A \to \left( \frac{|c\tau + d|}{c\tau + d} \right)^{1/2} Q^a_A.$$ (4.13)

This result is exactly identical to the corresponding results of [38] (section 2).

Further more, one can consider the central extension of the $N=4$ superalgebra by the anti-commutators $\{Q^a_A, Q^b_B\} = \epsilon^{ab} Z_{AB}$, which breaks the $SU(4)_R$ symmetry. The six-dimensional explanation of this extension has been given in last subsection. We can see that (4.13) implies the $SL(2,\mathbb{Z})$ transformation of $Z_{AB}$

$$Z_{AB} \to \left( \frac{|c\tau + d|}{c\tau + d} \right) Z_{AB},$$ (4.14)

and $|Z|$ is $SL(2,\mathbb{Z})$ invariant.

In next section, we’ll see that $\tau$ should be identified as the complex coupling constant of $N=4$ super-Yang-Mills theory. And the mapping class symmetry of $T^2$ should be identified as the $SL(2,\mathbb{Z})$ Montonen-Olive duality. Thus, the quantum numbers $(m,n)$ of equation (4.13) should be interpreted as the quantum numbers of magnetic-electric charges.

By noticing the transformation law of $\tau, \tau \to (a\tau + b)/(c\tau + d)$, and $\text{Im}\tau \to \text{Im}\tau/|\tau + d|^2$, equation (4.14) will tell us that the $SL(2,\mathbb{Z})$ duality transformations act on the magnetic and electric charges as

$$(m,n) \to (m,n) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = (m,n) \left( \begin{array}{cc} d & -b \\ -c & a \end{array} \right).$$ (4.15)
4.2. N = 4 Super-Yang-Mills Theory From QNG Theory And SL(2, \mathbb{Z}) Dualities

We now compactify tensor multiplet on $T^2$, $X = R^3 \times T^2$. We’ll show that the reduced Hamiltonian of the $U(1)$ quantum Abelian gerbe theory is identical to the one of the four-dimensional $N = 4 U(1)$ gauge theory. The self-duality of $H$ plus the mapping class transformation of $T^2$ lead to the $S$ duality of four dimensional gauge theory which is found through a five dimensional gauge theory argument \[1\] [2].

For simplicity, we’ll take $T^2 = S \times S’$. $S$ and $S’$ are circles with radius $U$ and $V$, respectively. Let $u$ and $v$ denote the coordinates of $S$ and $S’$ with $(u \sim u + 2\pi U, v \sim v + 2\pi V)$. We now can pull the tensor field $\vec{H}$ down to $R^3$ by assuming

$$\vec{H} = F du/2\pi U + \vec{F} dv/2\pi V + g du \wedge dv + *f,$$

where $\vec{F} = \frac{1}{2} F_{ij} dx^i \wedge dx^j (i = 1, 2, 3)$ stands for the electric 2-form on the three dimensional space $R^3$, and $F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$ stands for the magnetic 2-form.

We’ll focus on the $\vec{F}$ and $F$ fields. By substituting (4.16) into the Hamiltonian $H$ and integrating over $T^2$, one can get

$$H = \frac{1}{4\pi} \int_{R^3} \left( \frac{V}{U} ||F||^2 + \frac{U}{V} ||\vec{F}||^2 + \ldots \right),$$

where $||F||^2$ and $||\vec{F}||^2$ stand for $F \wedge *F$ and $\vec{F} \wedge *\vec{F}$, respectively, and the ellipses refer to terms involving other fields of the theory.

One can compare (4.17) with the Hamiltonian of four dimensional electro-magnetic field. Here, we normalize the action of the electro-magnetic field as $\frac{1}{e^2} \int (\vec{E}^2 - \vec{B}^2)$. In terms of the canonical variables, the associated Hamiltonian is of the form $H = \int_{R^3} \left( e^2 \frac{1}{4} \vec{E}^2 + \frac{1}{e^2} \vec{B}^2 \right)$. Clearly, this Hamiltonian can be identified with (4.17) by setting $F = \frac{1}{2} B_{ij} dx^i \wedge dx^j, \vec{F} = *(\partial_i dx^i)$ and identifying the electric coupling constant as

$$\frac{e^2}{4\pi} = \frac{U}{V}.$$

From (4.16) and (4.18), one can see that the transformation $u \rightarrow v, v \rightarrow -u$ acts as the $S$ duality of the four dimensional theory, which interchanges the magnetic field $F$ with the electric field $\vec{F}$ and maps the coupling constant $e^2$ to $g^2$,

$$e^2 \rightarrow g^2 = (4\pi)^2/e^2.$$
Generalized To The Full Supersymmetrical Theory

To study the dimensional reduction of the 4-fermions $\Psi_\alpha^A$, one decomposes $\Psi_\alpha^A$ as $\Psi_\alpha^{(a,+1/2)} \oplus \Psi_\alpha^{(\dot{a}, -1/2)}$. One can then make the following identifications

$$\frac{i}{2} \int_{T^2} dz \wedge d\bar{z} \bar{\Psi}^{(\dot{a}, -\frac{1}{2})}_A L_{(\dot{a}, -\frac{1}{2})(a,+\frac{1}{2})} \Psi_\alpha^{(a,+\frac{1}{2})}_A \rightarrow \bar{\chi}^A \bar{\theta}_\alpha \chi_\alpha^A$$

(4.20)

where, $\chi_\alpha^A$ and $\bar{\chi}^A$ are the chiral and anti-chiral fermions of the four-dimensional theory, and $L$ stands for some operator acting on the fermions. Since there is no real problem for the Lagrangian description of the chiral fermions, we’ll not explicitly write down the reduction in terms of the Hamiltonian, but write out the terms of action of $\chi_\alpha^A$ and $\bar{\chi}_\dot{a}$

$$I_F = \frac{1}{4\pi} \int_{R^3 \times R^1} \left( \bar{\chi}^A \partial_\dot{\alpha} \chi_\alpha^A \right).$$

(4.21)

These terms can be got by Lergende transforming the Hamiltonian got from the dimensional reduction.

Now, we can study the action of $S$-duality on $\chi_\alpha^A$ and $\bar{\chi}^A$ by utilizing their six dimensional origins that we have just explored. Firstly, by viewing (4.20), one can see that $\chi_\alpha^A$ is roughly identified with $\Psi_\alpha^{(a,+1/2)}$, and $\bar{\chi}^A$ is roughly identified with $\bar{\Psi}^{(\dot{a}, -\frac{1}{2})}$. One then notices that the transformations $u \rightarrow v, v \rightarrow -u, dz \rightarrow -\frac{1}{\tau} dz, d\bar{z} \rightarrow -\frac{1}{\tau} d\bar{z}$ imply a Spin(2) rotation $\exp(i\phi_S) = -|\tau|/\tau$. Under this rotation, $dz \rightarrow \exp(i\phi_S) dz, d\bar{z} \rightarrow \exp(-i\phi_S) d\bar{z}$, and $\Psi_\alpha^{(a,+1/2)} \rightarrow e^{i\phi_S/2} \Psi_\alpha^{(a,+1/2)}$, $\Psi_\alpha^{(a,-1/2)} \rightarrow e^{-i\phi_S/2} \Psi_\alpha^{(a,-1/2)}$, thus

$$\chi_\alpha^A \rightarrow \exp(+i\frac{1}{2} \phi_S) \chi_\alpha^A$$

$$\bar{\chi}^A \rightarrow \exp(-i\frac{1}{2} \phi_S) \bar{\chi}^A.$$

(4.22)

The full mapping class symmetries $SL(2, \mathbb{Z})$ are generated by the $S$ transformation, which we have just investigated, and the $T$ transformation $\tau \rightarrow \tau + 1$ which is trivially acting on these fermions. Thus, a general transformation $\tau \rightarrow (a\tau + b)/(c\tau + d)$ must act as

$$\chi_\alpha^A \rightarrow \left( \frac{|c\tau + d|}{c\tau + d} \right)^{1/2} \chi_\alpha^A$$

$$\bar{\chi}^A \rightarrow \left( \frac{|c\tau + d|}{c\tau + d} \right)^{1/2} \bar{\chi}^A.$$

(4.23)

This may be compared to the transformations of $\chi_\alpha^A$ and $\bar{\chi}^A$ under Montonen-Olive dualities.
We now turn to the dimension reduction of the tensor field and the scalar fields. One first decompose $H^{(\alpha\beta)}$ as

$$
+(1/\sqrt{A_T}) \left( \mathcal{F}^{ab}_+ d\bar{z}/\sqrt{\tau} + \mathcal{F}^{\dot{a}\dot{b}}_- d\bar{z}/\sqrt{\tau} \right)
$$

$$
-i(1/\sqrt{A_T}) \left( F^{ab}_+ \sqrt{\tau} d\bar{z} - F^{\dot{a}\dot{b}}_- \sqrt{\tau} d\bar{z} \right)
$$

(4.24)

plus the terms concerning the derivatives of an additional scalar field $\Phi$, which is connected with the one form $g$ in (4.16) through $g \sim d\Phi$, with $\partial^{\alpha\beta} \Phi + \partial^{\dot{a}\dot{b}} \Phi$. The real condition now tells us that $F^{\dot{a}\dot{b}}_-$ and $\mathcal{F}^{\dot{a}\dot{b}}_-$ are the complex conjugations of $F^{ab}_+$ and $\mathcal{F}^{ab}_+$, respectively.

By substituting (4.24) into the Hamiltonian $H$ (3.26), one has the terms concerning the electro-magnetic fields of the resulting four dimensional theory

$$
\frac{i}{4\pi} \int_{R^3} \left( \tau \| F_+ \|^2 - \tau \| F_- \|^2 \right)
$$

$$
- \frac{i}{4\pi} \int_{R^3} \left( \frac{1}{\tau} \| \mathcal{F}_+ \|^2 - \frac{1}{\tau} \| \mathcal{F}_- \|^2 \right),
$$

(4.25)

where $\| F_+ \|^2$ and $\| F_- \|^2$ stand for $F^{ab}_+ F^{ab}_+$ and $F^{\dot{a}\dot{b}}_- F^{\dot{a}\dot{b}}_-$, the meaning of the notations $\| \mathcal{F}_\pm \|^2$ are likewise. Comparing to the Hamiltonian of four dimensional $U(1)$ gauge theory one can see that $F_+$, $F_-$ are the selfdual and anti-selfdual part of the magnetic field $F$, and $\mathcal{F}_+$, $\mathcal{F}_-$ are the selfdual and anti-selfdual part of the electric field $\mathcal{F}$. Thus, $\tau$ will be identified as the complex coupling constant of the gauge theory

$$
\tau = \frac{4\pi i}{e^2} + \frac{\theta}{2\pi},
$$

(4.26)

where $\theta$ is the theta angle. And, by noticing the decomposition (4.24), one will see that the $v \rightarrow -u, u \rightarrow v$ transformation will exchange the electric field and the magnetic field

$$
\mathcal{F}_+ \leftrightarrow \tau F_+, \mathcal{F}_- \leftrightarrow \tau F_-.
$$

(4.27)

Thus, this transformation can indeed be identified with the $S$-duality.

Now, we turn to the dimensional reduction of the scalars. We have five-scalars $\Phi_{AB}$ which transform as the $5$ representation of $Sp(2, H)_R$. The dimensional reduction of the tensor field $H^{(\alpha\beta)}$ will give us an additional scalar $\omega_{AB} \Phi$ as we have mentioned. The combination of $\omega_{AB} \Phi$ and the previous five scalars forms the $6$ representation of the $R$-symmetry
SU(4)_R ⊂ SL(4, C)_R of the resulting four dimensional N = 4 gauge theory. Without confusing, we’ll denote these scalars as Φ_{AB} with the constraint equation Φ_{AB}ω^{AB} = 0 now dropped. These six scalars should satisfy the following real conditions
\[ \Phi_{AB} = \frac{1}{2} \epsilon_{ABCD} \Phi^{CD}, \] (4.28)
where ε_{ABCD} is the volume form of the complex space C^4_R which represents SL(4, C)_R.

The dimensional reduction of Φ_{AB} is achieved through
\[ i \tfrac{1}{2} (\text{Im} \tau)^{-1} \int dz \wedge d\bar{z} \Phi^{AB} \Phi_{AB} \rightarrow \Phi^{AB} \Phi_{AB}. \] With an appropriate normalization, the corresponding terms of the Hamiltonian concerning these scalars are
\[ \frac{1}{2\pi} \int_{R^3} \left( \Pi^{AB} \Pi_{AB} + \partial \Phi^{AB} \partial \Phi_{AB} \right). \] (4.29)
The mapping class symmetry \( u \rightarrow v, v \rightarrow -u \) trivially acts on these scalars, \( \Phi_{AB} \rightarrow \Phi_{AB} \).

Now, we would like to transform the Hamiltonian, got from the dimensional reduction of the six dimensional theory, into the action expressions, \( I = I_F + I_B \). Here, \( I_F \) of the fermions has been written out (4.21), and \( I_B \) of the bosonic fields can be written as
\[ I = \int_{R^3 \times R^1} \left( \frac{i\tau}{4\pi} F^{ab} F_{ab} - \frac{i\tau}{4\pi} F^{ab} F_{\dot{a}\dot{b}} \right) + \frac{1}{4\pi} \partial_{ab} \Phi^{AB} \partial^{ab} \Phi_{AB}, \] (4.30)
where the self dual part \( F^{ab} \) of four-dimensional gauge field is \( F^{ab} = F^{ab}_+ + i \mathcal{F}^{ab}_+ \), and the anti-self dual part \( F^{\dot{a}\dot{b}} \) is \( F^{\dot{a}\dot{b}}_+ - i \mathcal{F}^{\dot{a}\dot{b}}_+ \). As one can see that \( I \) is indeed the action of four-dimensional N = 4 U(1) gauge theory.

We have shown that the dimensional reduction on T^2 of the quantum U(1) gerbe theory exactly reproduce the Hamiltonian of four dimensional N = 4 super-symmetrical U(1) gauge theory. And the mapping class symmetry of T^2 is transmuted into the SL(2, Z) duality of the four-dimensional N = 4 gauge theory. These results can be trivially generalized to the case of T_G Abelian gerbe theory, which is the low energy effective theory of six-dimensional N = (2, 0) theory at a generic point of the moduli space \( R^5 / W_G \). Thus, by including the additional r scalars coming from the dimensional reduction of r H fields, the reduced theory can be reasonably identified with the low energy effective theory of four-dimensional N = 4 super-Yang-Mills theory at a generic point of the moduli space \( R^{6r} / W_G \). Further more, as it is well known that at the singularities of \( R^{6r} / W_G \) some BPS states with nontrivial magnetic-electric charges become massless, and the corresponding theory becomes an interacting N = 4 superconformal field theory with some non-Abelian
gauge group $G' \subset G$. By noticing the matching between the superconformal symmetries of the four-dimensional theory and of the six-dimensional theory, one naturally expects that this four-dimensional interacting superconformal field theory may be the dimensional reduction of a six-dimensional QNG theory with the non-Abelian gerbe group $G' \subset G$. The symmetries are enhanced due to some BPS self-dual strings becoming tensionless at the singularities. In next section, we will turn to investigate these self-dual string excitations, which consist another ingredient of the QNG theory.

5. The Self-Dual Strings

In the present section, we’ll study the self-dual string excitations of the $N = (2, 0)$ theory and their tensionless limit. We’ll begin from the classical self-dual string solution of the Bogomolny equation, and then we’ll try to propose a world sheet $N = (4, 4)$ superconformal field theory to describe their long wavelength oscillations and their coupling with the free tensor multiplets. We’ll argue that under the tensionless limit these self-dual strings will become some objects moving in supertwistor space $\tilde{T}$, which should be discussed in more details in the next section.

Bogomolnyi Equations And Self-Dual Strings

We begin from the Bogomolnyi equation that should be fulfilled by the configurations of the BPS states,

$$\delta \Psi^\alpha_A = 0 \Leftrightarrow \left( \frac{1}{2} H^{(\alpha\beta)} e^A_\beta + \partial^{\alpha\beta} \Phi^A_B e^B_\beta \right) = 0. \quad (5.1)$$

Here, we have used the second equation of (3.3).

A straight self-dual string configurations will break the $(1+5)$-dimensional Lorentz symmetry to $SO(1, 1) \times SO(4)_\perp$, where $SO(1, 1)$ is generated by the boost along the string and $SO(4)_\perp$ is the rotation around it. Moreover, this self-dual string will also break the $\mathbb{R}$ symmetry down to $SO(4)_\mathbb{R}$. Thus, the static configurations of such straight strings are parameterized by the imbedding of $SO(4)_\perp \hookrightarrow SO(5)$ (the spatial rotation group) and $SO(4)_\mathbb{R} \hookrightarrow SO(5)_\mathbb{R}$.

To specify a static straight BPS string, one can pick out a $SO(5)_\mathbb{R} \simeq Sp(2, H)_\mathbb{R}$ vector $\psi_{AB}$ with $\psi_{AB} \omega^{AB} = 0$, $\psi_{B}^A \psi_{C}^B = \delta^A_C$, and a spatial direction $l_{\alpha\beta}$ with $l_{\alpha\beta} t^{\alpha\beta} = 0$. Now, let the string lie along $l_{\alpha\beta}$ direction, and set the scalars to take the form of $\Phi_{AB} = \psi_{AB} \Phi$ which will break the $SO(5)_\mathbb{R}$ symmetry down to $SO(4)_\mathbb{R}$, here $\Phi$ is a scalar.
function of $l_{\alpha\beta}$, $t_{\alpha\beta}$ and the transverse coordinates $x_\perp$. It is convenient to take the $\sigma_+, \sigma_-$ coordinates along the world sheet of the string, with $\sigma_+ = l + t$ and $\sigma_- = l - t$ ($l$ and $t$ are the vectors whose components are $l_{\alpha\beta}$ and $t_{\alpha\beta}$, respectively). On these coordinates, the $SO(1,1)$ boost acts as $\sigma_+ \rightarrow e^{\theta} \sigma_+, \sigma_- \rightarrow e^{-\theta} \sigma_-$. 

Clearly, $SO(4)_\perp$ preserves $l_{\alpha\beta}$, and $SO(4)_R$ preserves $\psi_{AB}$. One then decomposes the $SO(1,5)$ anti-chiral spinor $\epsilon^A_{\alpha}$ into $\epsilon^A_{-\alpha} \oplus \epsilon^A_{+\dot{\alpha}}$ according to $SO(1,5) \rightarrow SO(1,1) \times SO(4)_\perp$, where $a, \dot{a} = 1, 2$ are used to label the chiral and anti-chiral spinors of $SO(4)_\perp$, and $+ (-)$ labels the $+1/2 (-1/2)$ weight of $SO(1,1)$. One can further decompose the tensor fields $H^{(\alpha\beta)}$ into $H^{(a\dot{c})}_{+1} \oplus H^{(\dot{a}\dot{c})}_{-1} \oplus H^{\dot{a}\dot{c}}$, where the subscriptions $+1$ and $-1$ are the weights of $SO(1,1)$ ($H^{\dot{a}\dot{c}}$ is invariant under $SO(1,1)$). Thus (5.1) can be rewritten as

\begin{equation}
\frac{1}{2} H^{(ab)}_{+1} \epsilon^-_{-b} + \epsilon^{ab} \partial^+ \Phi \psi^A_{-b} + 0
\end{equation}

which are a pair of equations concerning the derivatives along the string world sheet. Here, $\epsilon^{ab}$ and $\epsilon^{\dot{a}\dot{b}}$ are the symplectic forms of the $SO(4)_\perp \subset SO(4,\mathbb{C}) \simeq SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ chiral and anti-chiral spinors, respectively, and we'll use them to raise and lower the $SO(4)_\perp$ spinor indexes. The decomposition of (5.1) also gives us a pair of equations concerning the derivatives of the transverse coordinates,

\begin{equation}
\frac{1}{2} H^{\dot{a}\dot{b}}_{+1} \epsilon^+_{+\dot{b}} + \epsilon^{\dot{a}\dot{b}} \partial^- \Phi \psi^A_{+\dot{b}} = 0,
\end{equation}

In getting these equations, we have used the decompositions of $\partial^{\alpha\beta} \rightarrow \partial^{\dot{a}\dot{b}} \oplus \epsilon^{ab} \partial^+ \oplus \epsilon^{\dot{a}\dot{b}} \partial^-$, according to their representations under $SO(1,5) \rightarrow SO(1,1) \times SO(4)_\perp$.

For the static configurations, $\partial_+ \Phi = \partial_- \Phi = 0$. If the considered configuration preserve some supersymmetries, the pair of equations (5.2) will enforce $H^{(ab)}_{+1}$ vanish or $H^{(\dot{a}\dot{b})}_{-1}$ vanish or both. To satisfy the pair of equations (5.3), one set

\begin{equation}
H^{\dot{a}\dot{b}}/2 = \partial^{\dot{a}\dot{b}} \Phi.
\end{equation}

By substituting (5.4) into (3), one can get

\begin{equation}
\epsilon^A_{-a} + \psi^A_{-b} \epsilon^-_{-a} = 0
\end{equation}

\begin{equation}
\epsilon^A_{+\dot{a}} - \psi^A_{+\dot{b}} \epsilon^+_{+\dot{a}} = 0.
\end{equation}

41
Combining the equations of motion $\partial_{\alpha\beta} H^{(\beta\gamma)} = 0$, (5.4) implies that $\Phi$ should be harmonic function of $x_\perp$, and can be solved as $\Phi(x_\perp) = \Phi_0 + Q/\pi x_\perp^2$, where $Q$ is the charge of the self-dual string, with $\lim_{x_\perp \to \infty} \int_{S^3(x_\perp)} \frac{H}{2\pi} = \int_{S^3(x_\perp)} \star \frac{H}{2\pi} = Q$, $S^3(x_\perp)$ is the 3-dimensional sphere surrounding the string at transversal radius $|x_\perp|$.

**World Sheet Theory Of The Self-Dual String**

The $\frac{1}{2}$-BPS states correspond to the case of $H_{+1}^{(ab)} = 0$, $H_{-1}^{(ab)} = 0$ of the static configurations. In this case, half of the super-symmetries that satisfy (5.5) can be preserved. By viewing $\psi^A_B$ as $\gamma_5^R$ of the remaind $\mathcal{R}$ symmetry $SO(4)_R$, one can take that the preserved super-symmetries as anti-chiral and chiral spinors of $SO(4)_R$, with $\gamma_5^R \epsilon^{A}_{-a} = -\epsilon^{A}_{-a}$, $\gamma_5^R \epsilon^{A}_{+a} = \epsilon^{A}_{+a}$, where $A, \hat{A} = 1, 2$ are the indexes of $SO(4)_R$ chiral and anti-chiral spinors (should not confuse with the indexes of 4-components of $Sp(2, H)_R$ which are labeled as $A$). And the broken super-symmetries are $\epsilon^{A}_{-a}$ and $\epsilon^{A}_{+a}$.

We’ll denote the generators of the preserved supersymmetries as $G^a_{-\frac{1}{2}A}$, and $G^{\hat{a}}_{+\frac{1}{2}A}$, which are left-moving charges and right-moving charges along the string. Here the subscription $-1/2$ is used to denote the $1/2$ dimension of these supercharges.

Further more, we suppose that the global superconformal group of the IR superconformal field theory, describing the long wavelength oscillating of the string world sheet, is given by the dimensional reduction of $OSp(2, 6|2) \subset OSp(8|4, \mathbb{C})$. Thus, the world sheet IR theory is a $N = (4, 4)$ superconformal field theory. Its global superconformal group is $PSU(1, 1|2) \subset PSL(2|2, \mathbb{C}) \hookrightarrow OSp(8|4, \mathbb{C}) \supset OSp(2, 6|2)$. Here we are focusing on the left-moving part of this superconformal field theory, the investigation to the right-moving part is quite similar, and we will leave it to the reader. In this reduction, the $SU(2)$ $\mathcal{R}$-symmetry group of $PSU(1, 1|2)$ is identified as $SU(2)_L^R \subset SO(4)_R \hookrightarrow SO(5)_R$ acting on the left-moving modes, and the $SU(2)$ exterior automorphism of the algebra of $PSU(1, 1|2)$ is identified as $SU(2)_L^L \subset SO(4)_L$. We’ll denote the generators of $SU(2)_L^R$ as $R^{AB}_A$, which is the reduction of $\mathcal{R}_{(AB)}$. The reduction of the S-charges give us the dimension $-1/2$ fermionic generators $G^a_{+\frac{1}{2}A}$, with the $G - G$ anti-commutators

$$\{G^a_{-\frac{1}{2}A}, G^{c}_{+\frac{1}{2}B}\} = \epsilon^{ac}_{\hat{A}B} L_0 + \epsilon^{ac}_{\hat{A}B} R^{AB}_{\hat{A}B};$$

where $L_0$ is the dimension 0 generator of the left-moving conformal group $SL(2, \mathbb{R})$.

The localization, on the string, of the broken left moving fermionic transformations $\epsilon^{A}_{+a}$ gives out 4-fermionic left moving fields $\psi^{\hat{A}}_a$, and $\partial x_{\hat{a}a}$ are their super-partners. Also, the broken right moving fermionic transformations $\epsilon^{A}_{-a}$ give us $\overline{\partial} x_{\hat{a}a}$ and $\overline{\psi}^{\hat{A}}_a$. These fields
localize on the self-dual string and form the collective coordinates describing its long scale oscillations. Now, we can write out the most simple possible action of the world sheet theory,

\[ I = \frac{1}{4\pi} \int_{\Sigma} |\Phi_0| \left( \partial x^{\hat{a}\hat{a}} \bar{\partial} x_{\hat{a}\hat{a}} + \psi^{\hat{a}}_{\dot{A}} \bar{\partial} \psi_{\dot{A}} + \tilde{\psi}^{\hat{a}A} \partial \tilde{\psi}^A \right), \tag{5.7} \]

where the string tension \( \Phi_0 \) is the expectation value of \( \Phi \) at the infinity of the string, and we have rotated the world sheet coordinates into complex coordinates \( z, \bar{z} \) with \( \sigma_+ \rightarrow z \), \( \sigma_- \rightarrow \bar{z} \). This is a \( c = 6 \ N = (4, 4) \) superconformal field theory with level \( k = 1 \) current algebra \( R_{\dot{A}\dot{B}}(z) \). One can work out the explicit expressions of various currents, for example, the left-moving supercurrents are given by

\[ G^{\hat{a}}_{\dot{A}}(z) = |\Phi_0| \partial x^{\dot{a}a} \psi_{\dot{a}\dot{A}}(z). \tag{5.8} \]

And one can calculate the OPE of these currents, for example, the \( G - G \) OPE is

\[ G^{\hat{a}}_{\dot{A}}(z) G^{\hat{b}}_{\dot{B}}(0) \sim \frac{2}{z^3} \epsilon^{ab} \epsilon_{\dot{A}\dot{B}} + \epsilon^{ab} \epsilon_{\dot{A}\dot{B}} \frac{1}{z} T(0) \]

\[ + \epsilon^{ab} \frac{2}{z^2} R_{\dot{A}\dot{B}}(0) + \epsilon^{ab} \frac{1}{z} \partial R_{\dot{A}\dot{B}}(0). \tag{5.9} \]

The interactions between the self-dual string and the free tensor multiplets can be given by calculating the correlations of various appropriate operators inserted in the path integration of the world sheet theory. The operators that correspond to the incident waves of the tensor field are \( \exp(i|\Phi_0|^{-1} \int_{\Sigma} H^{\hat{a}\hat{c}} x_{\hat{a}\hat{c}}) \), \( \exp(i|\Phi_0|^{-1/2} \int_{\Sigma} H^{(ac)} \tilde{\psi}^a_{\dot{A}} \tilde{\psi}^c_{\dot{A}}) \), or \( \exp(i|\Phi_0|^{-1/2} \int_{\Sigma} H^{(\hat{a}\hat{c})} \psi_{\hat{a}\dot{A}} \psi_{\hat{c}\dot{A}}) \) according to their polarizations. The operators that correspond to the incident waves of \( \Psi^{\hat{a}}_{\dot{A}} \) are \( \exp(i|\Phi_0|^{-3/4} \int_{\Sigma} \Psi^a_{\dot{A}} \tilde{\psi}^a_{\dot{A}}) \), \( \exp(i|\Phi_0|^{-3/4} \int_{\Sigma} \Psi^a_{\dot{A}} \psi_{\hat{a}\dot{A}} \partial x_{\hat{a}\hat{a}}) \), \( \exp(i|\Phi_0|^{-1/4} \int_{\Sigma} \Psi^a_{\dot{A}} \psi_{\hat{a}\dot{A}} \partial x_{\hat{a}\hat{a}}) \), \( \exp(i|\Phi_0|^{-1/4} \int_{\Sigma} \Psi^a_{\dot{A}} \psi_{\hat{a}\dot{A}} \bar{\partial} x_{\hat{a}\hat{a}}) \), where we have decomposed the incident waves of \( \Psi^{\hat{a}}_{\dot{A}} \) as \( \Psi^a_{\dot{A}} \oplus \Psi^\hat{a}_{\dot{A}} \oplus \Psi^\dot{a}_{\dot{A}} \oplus \Psi^{\hat{a}}_{\dot{A}} \). And the operators that correspond to the incident waves of the scalars can be discussed likewise.

Now, we want to understand the tensionless limit \( |\Phi_0| \rightarrow 0 \) of the self-dual string. The author did not have a clear idea about how to take this limit appropriately. But there are some clues that may be notable. Firstly, by noticing the imbedding \( PSL(2|2, \mathbb{C}) \hookrightarrow OSp(8|4, \mathbb{C}) \) that we have defined, especially by noticing the actions of the R-symmetries, one can see that the world sheet \( N = (4, 4) \) superconformal field theory may be identified as the IR conformal field theory of the Coulomb branch of some vector and hypermultiplets with \( N = (4, 4) \) supersymmetry. Thus, one may try to identify the tensionless limit of the world sheet theory with the IR conformal field theory of the Higgs branch,
with the transverse rotation $SO(4)_{\perp} \simeq SU(2)^L_{\perp} \otimes SU(2)^R_{\perp}$ acting as the R-symmetry. The situation is quite similar with the Matrix-String proposal [1] describing the dynamics of IIA $NS_5$ branes (little string theory). But, in the present situation, the tensionless strings cannot move along the space-time since the transverse rotations act as the R-symmetries. Furthermore, as we have argued in last section that under taking the tensionless string limit, we will have the full QNG theory which realizes the superconformal symmetry $OSp(2, 6|2) \subset OSp(8|4, \mathbb{C})$. Thus, we naturally expect that the world sheet theory should be explicitly $OSp(2, 6|2) \subset OSp(8|4, \mathbb{C})$ invariant. A conjecture is that these tensionless strings are in fact moving in the supertwistor space $\widehat{T}$ of $OSp(8|4, \mathbb{C})$.

6. Toward A Formulation In Supertwistor Space

We now try to unify the various elements that we have investigated in the previous sections into a unique framework to formulate the QNG theory in terms of the variables of supertwistor space. We haven’t finished this program, but some results may be notable. In this section, we’ll construct the supertwistor space $\widehat{PT}$ corresponding to the QNG theory, and implement the superconformal symmetry $U^*Sp(4|2, H) \subset OSp(8|4, \mathbb{C})$ in it. We’ll then encode the information of the full free tensor multiplet into a superfield $\Psi$ in $\widehat{PT}$, by utilizing the Penrose transformation. And we’ll propose a super-twistor space effective action of the $\Psi$ field.

Some speculations concerning the possible non-Abelian generalization and the supertwistor formulation of the QNG theory are also presented in this section.

6.1. The Twistor and Supertwistor Space

Twistor Space And Conformal Symmetry

We can construct the twistor space $\mathbb{T}$ of the $(1 + 5)$-dimensional spacetime $\mathbb{W}$ as follows. Firstly, we complexify $\mathbb{W}$ as $\mathbb{W}$. We then conformally compactify $\mathbb{W}$ as the standard quadric $\mathbb{Q}$ of the 7-dimensional complex projective space $\mathbb{P}$ of 8-dimensional vector space $\mathbb{V}$. In terms of the homogeneous coordinates $v^I \{I = 1, 2...8\}$ of $\mathbb{P}$, $\mathbb{Q}$ can be given as

$$(v^1)^2 + (v^2)^2 + ... + (v^8)^2 = 0. \quad (6.1)$$

The subgroup of the automorphism of $\mathbb{P}$ that preserves $\mathbb{Q}$ is $\text{Spin}(8, \mathbb{C})$, which is the complexification of the conformal symmetry $\text{Spin}(2, 6)$.
On the local patch $W$ of $Q$, one can take the local coordinates $\{x^\mu, \mu = 1, 2, \ldots, 6\}$. To see the action of Spin$(8, \mathbb{C})$ on these local coordinates explicitly, we set $u = v^7 + iv^8$, $v = v^7 - iv^8$, and rewrite the equation (6.1) as

$$(v^1)^2 + (v^2)^2 + \ldots + (v^6)^2 + uv = 0.$$  (6.2)

Let $W$ be the $u \neq 0$ local patch, and let the local coordinates $x^\mu$ be $x^\mu = v^\mu/u$. In terms of these coordinates, $W$ can be described as $x^2 + u/v = 0$.

On $W$, the generator $i(v^7 \frac{\partial}{\partial v^8} - v^8 \frac{\partial}{\partial v^7}) = -(u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v})$ of the rotation in $\{7, 8\}$ plane acts as $D = -x \cdot \frac{\partial}{\partial x}$. And the symmetry subgroup Spin$(6, \mathbb{C})$ that commutes with $D$ acts as the complexified Lorentz group of $W$. Finally, the translation $P^\mu$ and the special conformal translation $K^\mu$ can be determined as $P^\mu = (u \frac{\partial}{\partial v^\mu} - 2v^\mu \frac{\partial}{\partial v^\mu})$ and $K^\mu = (v \frac{\partial}{\partial v^\mu} - 2v^\mu \frac{\partial}{\partial u})$. Obviously, $[D, P^\mu] = P^\mu$, $[D, K^\mu] = -K^\mu$.

The twistor space $T$ is the 8-dimensional complex space $S^+$ of the chiral spinors of Spin$(8, \mathbb{C})$. The associated 7-dimensional projective space $\mathbb{P}T$ is the associated projective twistor space. One can equip $T$ with a symmetrical quadratic form $(\phi, \phi')_T$ that preserves Spin$(8, \mathbb{C})$, where $\phi \in T$ and $\phi' \in T$ are two arbitrary chiral spinors of Spin$(8, \mathbb{C})$. This determines a 6-dimensional quadric $Q$ in projective twistor space $\mathbb{P}T$, with $\phi^\sigma \in Q$ satisfying $(\phi, \phi)_T = 0$.

The standard correspondence between $Q$ and $Q$ (due to the triality of Spin$(8, \mathbb{C})$) tells us that the points of complex spacetime $Q$ are parameterizing the 3-plane in twistor quadric $Q$ (notice that any 6-dimensional complex quadric contains 3-plane) and the 3-planes in $Q$ are parameterized by the points of $Q$.

We now decompose the chiral spinor $\phi^\sigma$ into $\mu_\alpha$ and $\lambda^\alpha$ according to the different eigenvalues under $D$, $\phi^\sigma = (\lambda^\alpha, \mu_\beta)$, where

$$[D, \lambda^\alpha] = \frac{1}{2} \lambda^\alpha, \quad [D, \mu_\alpha] = -\frac{1}{2} \mu_\alpha.$$  (6.3)

And under this decomposition the quadratics $Q$ will be described as

$$\lambda^\alpha \mu_\alpha = 0.$$  (6.4)

Under the action of the Spin$(6, \mathbb{C})$, $\lambda^\alpha$ transforms as a chiral spinor, while $\mu_\alpha$ transforms as an anti-chiral spinor. (6.3) indicates that the dilation generator $D$ should act as

$$D = \frac{1}{2} \left( \lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} - \mu_\alpha \frac{\partial}{\partial \mu_\alpha} \right).$$  (6.5)
The relationship between $W$ and the twistor quadric $Q'$ can be explicitly formulated as a generalized Penrose equation

$$\mu_\alpha + x_{\alpha\beta}\lambda^\beta = 0. \quad (6.6)$$

By noticing that the $\alpha\beta$ indexes of $x_{\alpha\beta}$ are antisymmetric, one can easily see that for a given $x^\mu$, (6.6) satisfies (6.4) automatically. This means that the 3-plane $D_x$ of $PT$ described by (6.6) are in fact the 3-plane of twistor quadric $Q'$. On the other hand, given $(\lambda^\alpha, \mu_\alpha)$ fixed, the correspond $x_{\alpha\beta}$ is determined by (6.6) up to a shift $V_{\alpha\beta}$ that satisfies

$$V_{\alpha\beta}\lambda^\beta = 0. \quad (6.7)$$

This equation always have three linear independent solutions since any chiral spinor $\lambda^\alpha$ of $\text{Spin}(6, \mathbb{C})$ is a pure spinor. Thus, any point $(\lambda^\alpha, \mu_\beta)$ of $Q$ corresponds to a 3-plane of $Q$, such a 3-plane can be named as a $\alpha$-plane, in terms of the terminology of twistor theory.

By utilizing (6.6), one can write out the expressions of the other generators of the conformal symmetry group in terms of the twistor coordinates. For examples,

$$P^{\alpha\beta} = \lambda^\alpha \frac{\partial}{\partial \mu_\beta} - \lambda^\beta \frac{\partial}{\partial \mu_\alpha},$$

$$K_{\alpha\beta} = \mu_\alpha \frac{\partial}{\partial \lambda^\beta} - \mu_\beta \frac{\partial}{\partial \lambda^\alpha}. \quad (6.8)$$

The $\text{Spin}(6, \mathbb{C})$ rotations act as

$$J^\alpha_\beta = \left(\lambda^\alpha \frac{\partial}{\partial \lambda^\beta} - \mu_\beta \frac{\partial}{\partial \mu_\alpha}\right) - \frac{1}{4} \delta^\alpha_\beta \left(\lambda^\delta \frac{\partial}{\partial \lambda^\delta} - \mu_\delta \frac{\partial}{\partial \mu_\delta}\right). \quad (6.9)$$

To return to the real $(1 + 5)$-dimensional spacetime $W$, one should impose an appropriate real structure $\tau$ on the twistor quadric $Q$. The action of $\tau$ can be explicitly written as

$$\tau : \lambda^\alpha \rightarrow \tilde{\lambda}^\alpha = \tau^\alpha_\beta \lambda^\beta, \quad \mu_\alpha \rightarrow \tilde{\mu}_\alpha = \tau^\beta_\alpha \mu_\beta. \quad (6.10)$$

By using Penrose equation (6.6), one can get the action of $\tau$ on $x^{\alpha\beta}$. All the $x^{\alpha\beta}$ that commute with $\tau$ form the real slice $W$ of the complex spacetime $Q$. In terms of the homogeneous coordinates, $W$ is described by

$$-(v^1)^2 + (v^2)^2 + \ldots + (v^7)^2 - (v^8)^2 = 0. \quad (6.11)$$
To generalize above construction of the twistor space \( \mathbb{Q} \) to the super-symmetrical case, with the super-conformal symmetry group \( OSp(2,6|2) \), we will introduce four fermionic homogenous coordinates \( \psi^A \) to the \( \mathbb{C}_4^4\), which appear in section 3. The construction in section 3 indicates that \( \psi^A \) transforms as the 4 representation under \( Sp(2,H)\).

The full super-twistor space \( \hat{T} \) now can be constructed as the super-linear space \( \mathbb{C}^8|4 \) whose bosonic part is coordinated as \( (\lambda^\alpha, \mu_\beta) \) and whose fermionic part is coordinated as \( \psi^A \). The projective space of \( \hat{T} \) is denoted as \( \hat{\mathbb{P}T} \). The correspond super-twistor quadric \( \hat{Q} \) can be given as

\[
\lambda^\alpha \mu_\alpha + \frac{1}{2} \psi^A \psi_A = 0. \tag{6.12}
\]

Besides the previous generators (6.5), (6.8), (6.9), there are some additional generators that preserve (6.12). The additional bosonic generators are \( \mathcal{R}^A_B = \psi^A \frac{\partial}{\partial \psi^B} - \psi_B \frac{\partial}{\partial \psi^A} \), the additional fermionic \( Q \)-generators are

\[
Q^A_A = \psi_A \frac{\partial}{\partial \mu_\alpha} + \lambda^\alpha \frac{\partial}{\partial \psi_A}, \tag{6.13}
\]

and the additional fermionic \( S \)-generators are \( S_{\alpha A} = \mu_\alpha \frac{\partial}{\partial \psi^A} - \psi_A \frac{\partial}{\partial \lambda^\alpha} \). Here, we have used the same symbols for the superconformal generators and the generators of the super-twistor symmetries since we’ll identified them in what follows.

Such an identification can be achieved by generalizing the Penrose equation (3.6) to the fermionic coordinates

\[
\psi_A + \theta_{A\alpha} \lambda^\alpha = 0, \tag{6.14}
\]

as one can see by comparing (6.13) to (3.4).

Given \( \theta_{A\alpha} \) satisfying the symplectic-Majorana condition, (6.14) is preserved by the real structure \( \hat{\tau} \) which acts as \( \psi_A \rightarrow \hat{\psi}_A = \omega_{AB} \bar{\psi}^B, \lambda^\alpha \rightarrow \bar{\lambda}^\alpha, \mu_\alpha \rightarrow \bar{\mu}_\alpha \). Clearly, \( \hat{\tau} \) preserves the twistor quadric \( \hat{Q} \). This gives a natural real structure to \( \hat{\mathbb{P}T} \) that gives out the real superspace of the 5 + 1 dimensional spacetime. And the \( \mathcal{R}^A_B \) generators that commute with this real structure naturally give out \( \mathcal{R}^{(AB)} \). Hence, we get the full \( OSp(2,6|2) \) superconformal symmetries from the supertwistor space \( \hat{Q} \).
6.2. Towards A Formulation Of The QNG Theory

Encoding The Tensor Multiplet

Having constructed the supertwistor space of the $N = (2, 0)$ super-conformal field theory, we now try to transform the free tensor multiplet into twistor space by using the six-dimensional Penrose transformation, which has been developed in [42][43].

According to the Penrose transformation, solutions of the wave equations for helicity $h$ are equivalent to the element of the sheaf cohomology group $H^3(Q', \mathcal{O}(-2h - 4))$. To employ it, one picks out an element $g(\lambda, \mu)$ of $H^3(Q', \mathcal{O}(-2h - 4))$, restricts $g(\lambda, \mu)$ to be an element of the restricted cohomology group $H^3(D_x, \mathcal{O}(-2h - 4))$ for a fixed spacetime point $x$, and notices the natural measurement $\mu(D)$ on the 3-plane $D_x$, with $\mu(D) = \epsilon_{\alpha\beta\gamma\delta}\lambda^\alpha d\lambda^\beta \wedge d\lambda^\gamma \wedge d\lambda^\delta = \langle \lambda, d\lambda, d\lambda, d\lambda \rangle$. The Penrose transformations can then be given out by appropriate contour integrations over $D_x$.

Now, we apply the Penrose transformation to the linear equations of motion of the tensor multiplet that we have studied in section 3. For the five scalars, $g(\lambda, \mu)$ can be written as $\Phi^{AB}(\lambda, \mu) \in H^3(Q', \mathcal{O}(-4))$, with

$$
\Phi^{AB}(x) = \frac{1}{(2\pi i)^3} \int_{D_x} \mu(D) \cdot \Phi^{AB}(\lambda, \mu) \cdot \omega^{AB} \sim \frac{1}{|x|^4} \omega^{AB} ,
$$

(6.15)

As an example, we set $\Phi^{AB}(\lambda, \mu) \sim \omega^{AB} \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle / [\langle \mu, \lambda_1 \rangle \langle \mu, \lambda_2 \rangle \langle \mu, \lambda_3 \rangle \langle \mu, \lambda_4 \rangle] \in H^3(Q', \mathcal{O}(-4))$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are four different chiral spinors of $\mathbb{PT}$, $\langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle = \epsilon_{\alpha\beta\gamma\delta}\lambda^\alpha_1 \lambda^\beta_2 \lambda^\gamma_3 \lambda^\delta_4$. By performing the contour integration we can get

$$
\Phi^{AB}(x) = \int_{D_x} \frac{\langle \lambda, d\lambda, d\lambda, d\lambda \rangle}{(2\pi i)^3} \frac{\langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle}{\langle \mu, \lambda_1 \rangle \langle \mu, \lambda_2 \rangle \langle \mu, \lambda_3 \rangle \langle \mu, \lambda_4 \rangle} \omega^{AB} \sim \frac{1}{|x|^4} \omega^{AB} ,
$$

(6.16)

which is a wave function $\Phi^{AB}(x)$ on spacetime $W$, with a delta function source supported on the origin.

For the chiral fermion $\Psi^\alpha_A$, we need to decompose it into the $+1/2$ helicity part $\Psi^\alpha_+ A$ and the $-1/2$ helicity part $\Psi^\alpha_- A$ as we have done in section 3.1. For the positive helicity part, the Penrose transformation can be simply given as

$$
\Psi^\alpha_+(x) = \frac{1}{(2\pi i)^3} \int_{D_x} \mu(D) \cdot \lambda^\alpha \Psi^\alpha_+(\lambda, \mu) ,
$$

(6.17)

where $\Psi^\alpha_+(\lambda, \mu) \in H^3(Q', \mathcal{O}(-5))$ is the correspond twistor wave function. To get the Penrose transformation of the negative helicity part $\Psi^\alpha_- A$, one notices that on $D_x$ the
plane wave $\tilde{\lambda}^\alpha \exp(i \frac{1}{2} \lambda^\alpha \wedge \tilde{\lambda}^\beta x_{\alpha \beta})$ can be rewritten as $\tilde{\lambda}^\alpha \exp(\mu_\beta \tilde{\lambda}^\beta) \sim (\partial/\partial \mu_\alpha) \exp(\mu_\beta \tilde{\lambda}^\beta)$ by using the Penrose equation (6.6). Thus, the Penrose transformation can be written as

$$
\Psi_{-A}(x) = \frac{1}{(2\pi i)^3} \int_{D_x} \mu(\mathbb{D}) \cdot \frac{\partial}{\partial \mu_\alpha} \Psi_{-A}(\lambda, \mu),
$$

(6.18)

where the twistor wave function $\Psi_{-A}(\lambda, \mu)$ is an element of $H^3(Q', \mathcal{O}(-3))$.

Quite similar arguments will convince us that the Penrose transformations for the chiral tensor field $H^{(\alpha\beta)}$ should be given as follows. Firstly, the transformations for the $+1$ helicity part $H^+_{(\alpha\beta)}$ and $-1$ helicity part $H^-_{(\alpha\beta)}$ can be written as

$$
H^+_{(\alpha\beta)}(x) = \frac{1}{(2\pi i)^3} \int_{D_x} \mu(\mathbb{D}) \cdot \lambda^\alpha \lambda^\beta H^+(\lambda, \mu)
$$

and

$$
H^-_{(\alpha\beta)}(x) = \frac{1}{(2\pi i)^3} \int_{D_x} \mu(\mathbb{D}) \cdot \frac{\partial}{\partial \mu_\alpha} \frac{\partial}{\partial \mu_\beta} H^-(\lambda, \mu),
$$

(6.19)

where the twistor wave functions $H^+(\lambda, \mu)$ and $H^-(\lambda, \mu)$ are the elements of $H^3(Q', \mathcal{O}(-6))$ and $H^3(Q', \mathcal{O}(-2))$, respectively. Secondly, the Penrose transformation of the helicity 0 part $H^0_{(\alpha\beta)}$ can be written as

$$
H^0_{(\alpha\beta)}(x) = \frac{1}{(2\pi i)^3} \int_{D_x} \mu(\mathbb{D}) \cdot \left( \lambda^\alpha \frac{\partial}{\partial \mu_\beta} + \lambda^\beta \frac{\partial}{\partial \mu_\alpha} \right) H^0(\lambda, \mu),
$$

(6.20)

where $H^0(\lambda, \mu) \in H^3(Q', \mathcal{O}(-4))$.

Now, we can encode the full free tensor multiplet as a superfield $\Psi$ defined on the supertwistor space $\hat{\mathbb{PT}}$,

$$
\Psi = H_ - + \psi^A \Psi_A + \frac{1}{2} \psi^A \psi_A H_0 + \frac{1}{2} \psi^A \psi^B \Phi_{AB} + \frac{1}{3!} \epsilon_{ABCD} \psi^A \psi^B \psi^C \Phi^D + \frac{1}{4!} \epsilon_{ABCD} \psi^A \psi^B \psi^C \psi^D H_+.
$$

(6.21)

Clearly, $\Psi(\phi, \psi)$ is an element of the cohomology group $H^3(\hat{\mathbb{PT}}', \mathcal{O}(-2))$. By viewing the sheaf cohomology group $H^3(\hat{\mathbb{PT}}')$ as the $\hat{\mathcal{D}}$ cohomology group $H^3_{\hat{\mathcal{D}}}(\hat{\mathbb{PT}}')$, one can identify the wave function $\Psi$ as a $(0, 3)$ form on $\hat{\mathbb{PT}}$.

There is a natural holomorphic measure $\hat{\Omega} = \phi^1 d\phi^2 \wedge d\phi^2 \wedge \ldots \wedge d\phi^8 \wedge d\psi^1 \wedge d\psi^2 \wedge \ldots \wedge d\psi^4$ on $\hat{\mathbb{PT}}$. $\hat{\Omega}$ takes value in the line bundle $\mathcal{O}(4)$ since under $(\phi, \psi) \rightarrow t(\phi, \psi)$, $t \in \mathbb{C}^*$, $\hat{\Omega} \rightarrow t^4 \hat{\Omega}$. Thus, at the linear level, one can write out a well defined effective action for the twistor wave function $\Psi$,

$$
I_\Psi = \frac{1}{2} \int_{\hat{\mathbb{PT}}} \hat{\Omega} \wedge (\Psi \hat{\mathcal{D}} \Psi).
$$

(6.22)
Towards A Super-twistor Formulation Of The QNG Theory

We now suppose that for the non-Abelian cases with non-Abelian gerbe group \( G = U(N) \subset G_C = GL(N, \mathbb{C}) \) the correspond supertwistor wave function \( \Psi \) (which is a \((0, 3)\) form) takes value in \( \text{ad}(E) \otimes \mathcal{O}(-2) \), where \( \text{ad}(E) \) denotes the adjoint bundle of the \( GL(N, \mathbb{C}) \) holomorphic vector bundle \( E \rightarrow \widehat{\mathbb{TP}} \).

One may generalize (6.22) to \( \frac{1}{2} \int_{\widehat{\mathbb{PT}}} \Omega \wedge \text{Tr} (\Psi \overline{\mathcal{D}} \Psi) \), but this cannot be the correct action since it is essentially a linear theory (and we have known that the QNG can not be a linear theory). To cure this problem, the simplest thing that we can do is to introduce a \((0, 1)\) connection \( \mathcal{A} \) which takes value in \( \text{ad}(E) \otimes \mathcal{O} \), and generalize the effective action \( I_{\Psi} \) to some \( I_{\Psi, \mathcal{A}} \). One of the terms of \( I_{\Psi, \mathcal{A}} \), that corresponds to the term of \( I_{\Psi} \), may be

\[
\frac{1}{2} \int_{\widehat{\mathbb{PT}}} \Omega \wedge \text{Tr} (\Psi \overline{\mathcal{D}} \mathcal{A} \Psi),
\]

(6.23)

where \( \overline{\mathcal{D}} \mathcal{A} \Psi = \overline{\mathcal{D}} \Psi + [\mathcal{A}, \Psi] \). The equation of motion of \( \Psi \) will give us \( \overline{\mathcal{D}} \mathcal{A} \Psi = 0 \), which will define an element of \( H^{3}(\widehat{\mathbb{PT}'}, \text{ad}(E) \otimes \mathcal{O}(-2)) \). And one may need to study holomorphic 3-planes \( \widehat{\mathbb{D}} \) (and integrate over the moduli space of them) of \( \widehat{\mathbb{PT}} \), with additional terms

\[
\frac{1}{2} \int_{\widehat{\mathbb{D}}} \mu(\widehat{\mathbb{D}}) \wedge \text{Tr} (\mathcal{A} \overline{\mathcal{D}} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}).
\]

(6.24)

Here, \( \mu(\widehat{\mathbb{D}}) \) is the natural holomorphic measure on \( \widehat{\mathbb{D}} \), which takes value in the holomorphic line bundle \( \mathcal{O} \) of \( \widehat{\mathbb{PT}} \). One can get the equation of motion \( \overline{\mathcal{D}} \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \), which defines an element of \( H^{1}(\widehat{\mathbb{D}}, \text{ad}(E) \otimes \mathcal{O}) \) on each holomorphic 3-plane.

The investigation in section 5 tells us that the contributions of \( \mathcal{A} \) may come from the objects that correspond to the tensionless strings. Presently, the author is totally ignorant to these potential objects.

To shed light on this obscure situation, trying to extract out the twistor string theory – proposed to capture the perturbative structure of \( N = 4 \) super-Yang-Mills theory – from some known aspects of the super-twistor formulation for the QNG theory may be desired, since the four-dimensional theory has a natural QNG theory origin as we have seen in section 4. The similarity between Hodges’s momentum-twistor coordinates \( (Z^\alpha, W_\beta) \), introduced in [44] to cure the problem of spurious poles in gauge-theoretical scattering amplitudes, and our twistor coordinates \( (\lambda^\alpha, \mu_\beta) \) may suggest some interesting connections, we’ll leave it to the future [45].
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