Parafermions in Hierarchical Fractional Quantum Hall States

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Motivated by recent theoretical progress demonstrating the existence of non-Abelian parafermion zero modes in domain walls on interfaces between two dimensional Abelian topological phases of matter, we investigate the properties of gapped interfaces of hierarchical fractional quantum Hall states, in the lowest Landau level, characterized by the Hall conductance \( \sigma_{xy}(m,p) = \frac{v^p e^2}{2\pi m_p+1} \frac{c^2}{h} \), for integer numbers \((m,p)\) with \(m,p \geq 1\). The case \(m = 1\) corresponds to the experimentally well established sequence of fractional quantum Hall states with \( \sigma_{xy} = \frac{1}{2} \frac{c^2}{h} , \frac{2}{3} \frac{c^2}{h} , \frac{3}{4} \frac{c^2}{h} , \ldots \), which has been observed in many two dimensional electron gases. Exploring the mechanism by which the \((m,p+1)\) hierarchical state is generated from the condensation of quasiparticles of the “parent” state \((m,p)\), we uncover a remarkably rich sequence of parafermions in hierarchical interfaces whose quantum dimension \(d_{m,p}\) depends both upon the total quantum dimension \(D_{m,p} = \sqrt{2mp+1}\) of the bulk Abelian phase, as well as on the parity of the “hierarchy level” \(p\), which we associate with the \(\mathbb{Z}_2\) stability of Majorana zero modes in one dimensional topological superconductors. We show that these parafermions reside on domain walls separating segments of the interface where the low energy modes are gapped by two distinct mechanisms: (1) a charge neutral backscattering process or (2) an interaction that breaks \(U(1)\) charge conservation symmetry and stabilizes a condensate whose charge depends on \(p\). Remarkably, this charge condensate corresponds to a clustering of quasiparticles of fractional charge \(\frac{p}{2mp+1} e\), allowing us to draw a correspondence between these fractionalized condensates and Read-Rezayi non-Abelian fractional quantum Hall cluster states.

I. INTRODUCTION

Topological phases of matter are promising systems to realize fault-tolerant quantum computation due to the long-range entanglement of the quantum many-body state. Emergent quasiparticles are referred to as parafermions. Non-Abelian phases have been theoretically investigated in a variety of contexts, from fractional quantum Hall (FQH) systems to quantum spin liquids and recent years have seen exciting experimental progress to detect signatures of non-Abelian quasiparticles.

In the last two decades it has been noticed that superconductivity is an important mechanism to stabilize emergent low energy excitations with non-Abelian character. A well known example is Kitaev’s one-dimensional (1D) p-wave superconductor supporting Majorana zero modes at the edges. In this context, the edge of the finite system behaves as a domain wall interpolating between a non-trivial superconductor and a charge neutral insulator, i.e., vacuum.

Recent theoretical breakthroughs in topological phases have demonstrated that emergent non-Abelian extrinsic defects can be stabilized as domain wall states in edges and interfaces or boundaries of 2D topological phases whose bulk quasiparticles obey solely Abelian statistics. In certain cases previously considered, domain walls represent twist defects of an anyonic symmetry, which is a transformation that permutes the anyons without changing their fundamental statistical properties.

In the presence of such domain walls, the system can encode a non-trivial ground state degeneracy. The zero modes in question, which constitute a generalization of Majorana fermions, are referred to as parafermions. Parafermions have been introduced to describe phase transitions of 2D classical clock models with \(\mathbb{Z}_n\) symmetry and in recent years, there has been a renewed interest surrounding the relationship between parafermions and topological systems. A system with \(2N\) parafermion zero modes furnishes a ground state manifold with \(d^{2N}\) states, where \(d\) represents the quantum dimension of the parafermion. Majorana fermions correspond to the special case \(d = \sqrt{2}\), which occur as zero energy excitations in the edges of 1D topological superconductors and at the vortex core of 2D
chiral p-wave superconductors. \cite{27,28}

A 2D electron gas under external magnetic field provides a rich realization of Abelian phases of matter in the form of the FQH effect. \cite{29} In addition to the Laughlin states \cite{30}, a plethora of FQH plateaus with quantized Hall conductance are observed upon changing the magnetic field or the electron density. A remarkable aspect of these topological phases, particularly when in the first Landau level, is their hierarchical organization \cite{31,32} into a sequence on incompressible states characterized by the quantized Hall conductance

$$\sigma_{xy}(m, p) = \frac{e^2}{h} \frac{p}{2mp + 1}, \quad (1.1)$$

where $e$ is the electron’s charge, $h$ is the Planck’s constant, $m$ and $p$ are integer numbers greater or equal than one, which characterize the sequence of FQH states with filling fraction $\nu(m, p) = \frac{p}{2mp + 1} < 1$. The index $p$ labels the “position” of the hierarchical state, whose primary state ($p = 1$ and fixed value of $m$) is the Laughlin state with filling fraction $\nu(m, 1) = 1/(2m + 1)$. We shall refer to each of the hierarchical states above by a pair of integer numbers $(m, p)$. For instance, in the FQH plateaus of Hall conductance $\sigma_{xy}/(e^2/h) = 1/3, 2/5, 3/7, 4/9, ...$, the condensation of quasiparticles of the primary Laughlin state with $\sigma_{xy}/(e^2/h) = 1/3$ yields the first hierarchical state with $\sigma_{xy}/(e^2/h) = 2/5$; which in turn gives rise to the second hierarchical state with $\sigma_{xy}/(e^2/h) = 3/7$, and so on. Even though the investigation of bulk topological properties of the hierarchy of Abelian FQH in the lowest Landau level has a long history, \cite{32,38} the relationship between the bulk anyon condensation and the properties of non-Abelian parafermion zero modes supported by these phases Abelian phases remains an open problem. \textit{The goal of this work is to address this problem, thus establishing a correspondence between the hierarchy of FQH states and the local interactions on their interfaces capable of stabilizing non-Abelian parafermion zero modes.}

We also note that, by means of a “folding transformation,” the results obtained here for the non-chiral interfaces of hierarchical time-reversal symmetry breaking FQH states straightforwardly apply to the non-chiral edge states pertaining to the hierarchy of time-reversal symmetric Abelian Fractional Topological insulators. \cite{39,40}

In interfaces of Laughlin states with filling fraction $\nu = \frac{1}{2m+1}$ ($m$ integer), a domain wall between a segment gapped by the charge neutral backscattering $H_{bs} = \psi_L^\dagger \psi_R + H.c.$ – where $\psi_L$ and $\psi_R$ are, respectively, the fermionic operators on the left and right edges of the interface – and another region gapped by the charge 2 condensate $H_{pair} = \psi_L^2 + H.c.$, supports a $\mathbb{Z}_{2(2m+1)}$ parafermion with quantum dimension $d = \sqrt{2(2m + 1)}$. \cite{41,42} The $m = 0$ case corresponds to Majorana fermions on domain walls at the interface of the $\nu = 1$ integer quantum Hall state, akin to the 1D p-wave superconductor. \cite{27} The parafermion zero modes that occur for $m \geq 1$ then represent a fractionalization of the Majorana fermion. A noteworthy aspect of the quantum dimension of the parafermions $d = d_{1D} \times d_{bulk}$, is that it receives a contribution from the fractionalized bulk Abelian state, $d_{bulk} = \sqrt{2m + 1}$, as well as a contribution $d_{1D} = \sqrt{2}$ stemming from the 1D physics. This property of Laughlin-type interfaces raises some important questions regarding the remaining sequence of hierarchical states: (1) What is the nature of local interactions and domain walls responsible for parafermions? (2) Specifically, does a charge 2 pairing interaction play an important role for generic hierarchical states, in complete analogy to Laughlin states? (3) What is the interplay between bulk topology and 1D physics in determining the quantum dimension of parafermions?

In this work, we will address these questions using an effective Luttinger liquid theory to describe the low energy modes of the homogeneous interface between hierarchical FQH states. The presence of chiral edge modes is a well-known consequence of the topological order of the bulk hierarchical state described by the bulk Chern-Simons gauge theory. \cite{41} In this effective edge/interface theory, local operators that open an energy gap are expressed as generalized sine-Gordon (local) operators, which satisfy a compatibility condition (or, null condition) to ensure a stable gapped fixed point. \cite{42} Since the number of low energy modes at the interface grows with the level of the hierarchy $p$, any attempt of a general understanding of the parafermion problem that applies to the entire hierarchical sequence would seem a hopeless question. Nevertheless, much to the contrary, we shall demonstrate that an general and comprehensive understanding of the universal properties of parafermions is possible because of the relationship established between states of the hierarchy via anyon condensation. Specifically, the anyon condensation mechanism translates into a hierarchy of $K$ matrices describing the bulk anyons, which, in turn, will allow us to systematically study the local interactions that gap the homogeneous interfaces.

By an explicit analysis of the sine-Gordon gap opening operators at the homogeneous interface, we shall precisely identify local interactions that stabilize parafermion zero modes at domain walls separating gapped segments where charge conservation is preserved from other segments where the interactions break $U(1)$ charge conservation symmetry and give rise to a condensate of charge

$$Q_{m,p} = \begin{cases} 2p & p \in \text{odd} \\ p & p \in \text{even} \end{cases}, \quad (1.2)$$

which, notably, depends on the level $p$ of the hierarchy. In particular, it shows an even-odd effect as a function of $p$. Moreover, we observe that, in general, $Q_{m,p} > 2$, which departs from the Laughlin interface. \cite{41,42} Thus for $p \geq 2$, we identify new forms of local interactions that require a clustering mechanism beyond the conventional BCS pairing. \cite{43} Explicit form of these local interactions will be discussed in Sections IV and V.
Our main result is summarized in Fig. 1. By studying the low energy properties of the domain walls separating the \( U(1) \) symmetry preserving and broken regions of the interface, we will show that they support parafermion zero modes with quantum dimension

\[
d_{m,p} = \begin{cases} \sqrt{2} D_{m,p} & p \in \text{odd} \\ D_{m,p} & p \in \text{even} \end{cases},
\]

where \( D_{m,p} = \sqrt{2mp + 1} \) is the total quantum dimension of the bulk state, which is related to the topological entanglement entropy \([34, 35]\) of the hierarchical state \((m,p)\) via \( \gamma_{m,p} = \log D_{m,p} \) and to the minimum quasi-hole charge \( e^*_{m,p} = D_{m,p}^2 \). We again notice the quantum dimension of the parafermions reflect an even-odd effect in terms of \( p \), similarly to Eq. (1.2). Furthermore, the emergence of non-Abelian zero modes with quantum dimensions given by Eq. (1.3) suggest a different mechanism than that considered in Refs. [4, 13, 14, 18, 21], where the defects with non-Abelian character relate to twist defects of a symmetry of the anyon group of the Abelian state. For instance, in Section IV we show that, in the first and second hierarchical states, twist defects of the charge conjugation anyonic symmetry behave as Majorana fermions \((d = \sqrt{2})\) and twist defects associated with a “layer permutation” anyonic symmetry \([13]\) are trivial \((d = 1)\), in contrast with Eq. (1.3).

Equations (1.2) and (1.3) embody a rich fractionalization phenomenon at the gapped interface. As shall be demonstrated here, a segment of the interface where the charge condensate Eq. (1.2) is realized is associated with the expectation value of an operator of charge \( \nu_{m,p} = \frac{p}{2mp + 1} \) (recall \( e = 1 \) unit), which shows that the charge \( p \) operator (for \( p \) even) is realized by a cluster of \((2mp + 1)\) quasiparticles. It turns out the appearance of \( \mathbb{Z}_{2(2mp + 1)} \) parafermions with quantum dimension \( d_{m,p} = \sqrt{2mp + 1} \) is a direct consequence of this clustered state, as shall be explained later. In the odd \( p \) case, the same type of charge condensate is formed, however, according to Eq. (1.3) domain walls support \( \mathbb{Z}_{2(2mp + 1)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2mp + 1} \) parafermions with quantum dimension \( d_{m,p} = \sqrt{2} \times \sqrt{2mp + 1} \), where the extra \( \mathbb{Z}_2 \) structure is reminiscent of Majorana zero models in 1D topological superconductors.

We argue that this even-odd effect is a manifestation of the \( \mathbb{Z}_2 \) classification of 1D topological superconductors, where the integer index \( p \) plays the role of the number of stacked copies of 1D topological superconductors. The interpretation of this result is natural in the hydrodynamical Abelian Chern-Simons Abelian theory of the hierarchical FQH states, where the universal information of the hierarchical states \((m,p)\) is represented by a square integer valued \( K \) matrix of dimension \( p \). \([34, 35]\) Under a suitable \( \text{SL}(p, \mathbb{Z}) \) transformation, the \( K \) matrix can be interpreted as a \( p \)-layer FQH system, which reduces to an integer quantum Hall system of \( p \) filled Landau when \( m = 0 \). This interpretation of the Chern-Simons hydrodynamical theory will enable contact with the \( \mathbb{Z}_2 \) classification of 1D topological superconductors and support the validity of Eq. (1.3) which will be explicitly derived in Sec. V.

The anyon cluster state realized at the homogeneous interface of hierarchical FQH states bears a remarkable resemblance with the Read-Rezayi non-Abelian states where the ground state is build from clusters of \( k \) electrons which yield a gapped bulk with non-Abelian quasiparticles and an edge that supports a chiral neutral \( \mathbb{Z}_k \) parafermion mode. \([10]\) The case \( k = 2 \) corresponds to the Moore-Read paired state \([8]\) where electrons (or composite fermions) form a paired state whose neutral sector is described by an effective chiral p-wave superconductor. \([27]\) Reference \([15]\) has shown that the \( \nu = 2/3 \) FQH coupled to a superconductor can support \( \mathbb{Z}_3 \) parafermion zero modes on domain walls. Quite remarkably, hybridization of the \( \mathbb{Z}_3 \) parafermions modes throughout the bulk can give rise to a non-Abelian phase with properties similar to the \( \mathbb{Z}_3 \) Read-Rezayi FQH states that supports non-Abelian anyons capable of realizing universal quantum computation. \([15, 47]\) By the same token, and given the generality of the results established here, we expect that the deconfinement of parafermion zero modes realized in the hierarchy of Abelian FQH states to give rise to a rich class of 2D non-Abelian phases, thus unveiling fresh connections between families of Abelian and non-Abelian phases.

This paper is organized as follows. In Section II we give an overview of the 2D hydrodynamical Chern-Simons theory for the bulk hierarchical Abelian FQH states. \([34, 35]\) The topological information about the hierarchical Abelian state \((m,p)\) is encoded by the \( K_{m,p} \) matrix and charge vector \( q_{m,p} \). An fundamental point of this discussion is that the hierarchy of Abelian states, related by anyon condensation, establishes a useful mapping between \( K \) matrices of the elements of the hierarchy: \( K_{m,1} \rightarrow K_{m,2} \rightarrow \cdots \rightarrow K_{m,p-1} \rightarrow K_{m,p} \rightarrow \cdots \), with a corresponding mapping for the charge vectors. This recursive form of the \( K \) matrix and charge vector will be explored in order to establish the properties expressed in Eq. (1.3) and Eq. (1.2). In Section III we provide a general discussion of the properties of domain walls and parafermion zero modes on the interface of hierarchical states, where we shall consider two sets of local interactions, one that preserves and one that breaks \( U(1) \) charge conservation. An important take home message of this general discussion is that the quantum dimension of parafermions can be efficiently calculated for any hierarchical \((m,p)\) FQH state, despite the fact that the number of edge modes scales with the hierarchy level \( p \). Then, in Section IV we apply this formalism to the Laughlin primary states \((p = 1)\) and the first three hierarchical states \( p = 2, 3, 4 \). (In the case \( m = 1 \), these represent the FQH states with filling fractions \( 2/5, 3/7 \) and \( 4/9 \).) This explicit analysis will be crucial in pointing to the general hierarchy of parafermions, which will be worked out in Section V. Finally, in Section VI we shall summarize and discuss our results, as well as present perspectives.
II. OVERVIEW OF THE HIERARCHY OF ABELIAN FQH STATES

The stability of the sequence of incompressible hierarchical FQH states characterized by quantized bulk filling \( \nu_{xy}(m,p) = \frac{e^2}{2\pi m^2 + 1} \) can be understood in terms of the effective nucleation of an even number \( 2m \) of flux quanta per electron – implemented by a Chern-Simons gauge field [38] –, giving rise to composite fermions [31, 49], which, at mean field level, occupy an integral number \( p \) of filled effective Landau levels.

An alternative description to the composite fermion approach employs a Chern-Simons hydrodynamical theory to capture the universal properties of the hierarchical sequence. [34–38] This hydrodynamical approach explains the sequence of Abelian FQH states in terms of sequential anyon condensations. Given the \( (2+1) \)-dimensionality of the problem, electron and quasiparticle conserved currents each can be parametrized by a \( U(1) \) gauge field. The condensation of quasiparticles in a given plateau state labeled by \( (m, p) \) then gives rise to the hierarchical state \( (m, p+1) \), whose effective theory contains an additional Chern-Simons field. The resulting effective theory of \( (m, p) \) hierarchical state then corresponds to an Abelian Chern-Simons theory that depends upon the integral square and symmetric \( K \) matrix of dimension \( p \), which characterizes the Abelian topological order of the hierarchical FQH state \( (m, p) \).

In addition to the Abelian statistics of bulk quasiparticles, the hydrodynamical Chern-Simons theory yields direct information about the low energy properties of the edge states, which form a chiral Luttinger liquid. [11] The enlargement of the dimension of the \( K \) matrix as a function of the hierarchical parameter \( p \) signals the increase in the number of chiral edge modes. Therefore, when considering local interactions among the modes of such an interface, it is seen that opening of an energy gap is achieved by generalized sine-Gordon local operators whose conserved fluxes are encoded in the integral square and symmetric \( K \) matrix. Hence, the mean field state, which in turn provides a potential link between local operators at the interface and the bulk topological order.

The above-mentioned correspondence between the bulk phase and local edge operators will be explored to establish a correspondence between the hierarchical Abelian states and the parafemions zero modes in their interfaces. Given the importance of this formalism, in Section II A we review essential elements of the hydrodynamical Chern-Simons theory of 2D hierarchical FQH states [34, 38] leading to the recursive form of the \( K \) matrix and charge vector and, later in Section II B, we make contact the chiral 1D Luttinger liquid theory governing the low energy physics of the edge states. [11]

A. Abelian Chern-Simons theory of the hierarchical FQH states

Throughout the rest of the paper, we work in units where \( e = \hbar = 1 \), unless when we present formulas for the Hall conductance where the fundamental constants will be explicitly displayed.

Let us begin with Laughlin states at filling fraction \( \nu_{m,1} = 1/(2m+1) \), which are the primary states \((m,1)\) of the hierarchical sequence. Their effective low energy theory is captured by the 2D bulk Chern-Simons Lagrangian

\[
L^2\text{D}_{m,1} = - \frac{2m + 1}{4\pi} \varepsilon^{\alpha\beta\gamma} a_\alpha^{\dagger} \partial_\beta a_\gamma + \frac{1}{2\pi} \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta a_\gamma \tag{2.1}
\]

where \( a_\mu^{\dagger} \) is a dynamical Chern-Simons gauge field, \( A_\mu \) is the external electromagnetic field, Greek indices account for space-time coordinates \( \{0,1,2\} = \{t,x,y\} \) and the conserved electric current is \( J^\mu = \frac{1}{2\pi} \varepsilon^{\alpha\beta\gamma} \partial_\alpha a_\gamma \). Furthermore, here and throughout, repeated indices are summed over. Integrating out the Chern-Simons gauge field \( a_\mu^{\dagger} \) yields the electromagnetic response

\[
L^2\text{D}_{m,1}\text{ response} = \frac{1}{4\pi} \frac{2m + 1}{2m+1} \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \tag{2.2}
\]

that encodes the Hall conductance \( \sigma_{xy}(m,1) = \frac{e^2}{\pi(2m+1)} \).

Expressing the quasiparticle’s conserved current by \( J^\mu_2 = \frac{1}{2\pi} \varepsilon^{\alpha\beta\gamma} \partial_\beta a_\gamma^{\dagger} \), the effective theory of the first hierarchical state \((m,2)\) is given by

\[
L^2\text{D}_{m,2} = - \frac{2m + 1}{4\pi} \varepsilon^{\alpha\beta\gamma} a_\alpha^{\dagger} \partial_\beta a_\gamma + \frac{1}{2\pi} \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta a_\gamma + \frac{1}{4\pi} \varepsilon^{\alpha\beta\gamma} a_\alpha^{\dagger} \partial_\beta a_\gamma^{\dagger} + \frac{1}{4\pi} \varepsilon^{\alpha\beta\gamma} a_\alpha^{\dagger} \partial_\beta a_\gamma - \frac{2}{4\pi} \varepsilon^{\alpha\beta\gamma} a_\alpha^{\dagger} a_\alpha a_\gamma^{\dagger} a_\gamma, \tag{2.3}
\]

where the first two terms of the second line of Eq. 2.3 come from the minimal coupling \( j^\mu_2 a_\alpha^{\dagger} \) and the last term captures the property that, in the mean field state, the density of quasiparticles \( j^0 \) satisfies \( j^0 = \frac{1}{4\pi} \nabla \cdot a_\alpha^{\dagger} \), implying condense forming a bosonic Laughlin state with filling \( 1/2 \).

By introducing the Chern-Simons doublet \( a_\mu^T = (a_\mu^1, a_\mu^2) \), Eq. 2.3 reads

\[
L^2\text{D}_{m,2} = - \frac{1}{4\pi} \varepsilon^{\alpha\beta\gamma} a_\alpha^T K_{m,2} \partial_\beta a_\gamma + \frac{q T}{2\pi} \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta a_\gamma, \tag{2.4}
\]

where

\[
K_{m,2} = \begin{pmatrix} 2m + 1 & 1 \\ -1 & 2 \end{pmatrix}, \quad q_{m,2} = (1,0)^T.
\]

Then, integrating out the Chern-Simons fields yields the electromagnetic response

\[
L^2\text{D}_{m,2}\text{ response} = \frac{1}{4\pi} \frac{2}{4m+1} \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma, \tag{2.5}
\]

which encodes the Hall conductance \( \sigma_{xy}(m,2) = \frac{e^2}{\pi(4m+1)} \) of the first hierarchical state.
Carrying out these previous steps sequentially generates the hydrodynamical Chern-Simons theory of the hierarchy of Abelian FQH states \[43-38\]

\[ L^{2D}_{m,p+1} = -\frac{1}{4\pi} \epsilon^{\alpha \beta \gamma} K_{m,p+1} \partial_\gamma a_\alpha + q^T \frac{1}{2\pi} \epsilon^{\alpha \beta} A_\alpha \partial_\beta a_\alpha^T, \quad (2.6a) \]

\[ K_{m,p+1} = \begin{pmatrix}
0 \\
n_{m,p} \\
0 \\
0 \ldots 0 \quad -1 \\
-1 \\
2
\end{pmatrix}, \quad (2.6b) \]

\[ q_{m,p+1} = (\mathbf{q}_{m,p} \quad 0)^T. \quad (2.6c) \]

The physical mechanism by which the \((m,p+1)\) daughter state is generated from the condensation of anyons of the \((m,p)\) parent state is mathematically manifested in the recursive form of the \(K\) matrix Eq. 2.6b and the charge vector Eq. 2.6c. Moreover,

\[ \det(K_{m,p}) = 2mp + 1, \quad (2.6d) \]

gives the torus ground state degeneracy of the FQH state and measures the total quantum dimension of the Abelian topological phase

\[ D_{m,p} = \sqrt{|\det(K_{m,p})|} = \sqrt{2mp + 1}. \quad (2.6e) \]

Finally, the Hall conductance of the \((m,p)\) state, obtained from integrating out the Chern-Simons, is given by

\[ \sigma_{xy}(m,p) = \frac{e^2}{h} q_{m,p} K_{m,p}^{-1} q_{m,p} = \frac{e^2}{h} \frac{p}{2mp + 1} \quad (2.7) \]

It turns out that, for purpose of studying the properties of parafermion zero modes at the interface of hierarchical states, the recursive structure embodied in the \(K\) matrix and charge vector in Eq. 2.6 will play a central role as shall be discussed in Sections IV and V.

An important consideration is that the topological field theory Eq. 2.6 is only defined up to an SL\((p, \mathbb{Z})\) transformation \(a_\mu \rightarrow (W^T)^{-1} a_\mu, \quad K_{m,p} \rightarrow W K_{m,p} W^T\) and \(q_{m,p} \rightarrow W q_{m,p}\), which represents a relabeling of the quasiparticles that leaves their statistics unchanged. This freedom can be exploited to represent the hierarchical FQH state in the alternative basis \[36\]

\[ K_{m,p} = W_p K_{m,p} W_p^T \]

\[ \begin{pmatrix}
2m + 1 \\
2m + 1 \\
2m + 1 \\
2m + 1 \\
\vdots \\
2m + 1
\end{pmatrix}, \quad (2.8a) \]

\[ q_{m,p} = W_p q_{m,p} = (1, 1, \ldots, 1)^T, \quad (2.8b) \]

where

\[ W_p = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \ldots & 1
\end{pmatrix} \in \text{SL}(p, \mathbb{Z}). \quad (2.8c) \]

The \(K\) matrix Eq. 2.8a has the following appealing interpretation: the diagonal odd integers \(2m + 1\) alone represent a system of \(p\) layers of Laughlin \(\nu = 1/(2m + 1)\) states. The even off-diagonal factors of 2, on the other hand, represent bosonic correlations among the \(p\) fermionic layers. Notice the charge vector Eq. 2.8a denotes that the each layer carries unit charge under the external electromagnetic field. It will prove useful to explore both representations Eq. 2.6 and Eq. 2.8 when describing the properties of parafermions.

**B. Luttinger liquid theory of the hierarchical edge states**

According to the bulk-boundary correspondence, the bulk topological phase given by Eq. 2.6 supports a chiral edge Luttinger liquid \[41\]

\[ \mathcal{L}^R_{m,p} = -\frac{1}{4\pi} \partial_\gamma \Phi^T R_{m,p} \cdot \partial_x \Phi_R - \frac{1}{4\pi} \partial_\gamma \Phi^T R_{m,p} \cdot V_{m,p}^R \cdot \partial_x \Phi_R + \frac{1}{2\pi} q_{m,p} \epsilon^{\alpha \beta} A_\alpha \partial_\beta \Psi_R, \quad (2.9) \]

where \(V^R_{m,p}\) is a positive-definite matrix ensuring a bounded edge spectrum. Furthermore, since all the \(p\) eigenvalues of \(K_{m,p}\) are positive, the edge contains \(p\) right-moving modes described by the fields \(\Phi_R\), which are depicted at the bottom part of the interface shown in Fig. 2-(a). The top part of the interface supports left-moving modes that are described a similar Lagrangian as in Eq. 2.9 albeit with an appropriate sign change of the \(K\) matrix that reflects the opposite orientation of the edge modes. Therefore, the Luttinger liquid theory of the interface reads

\[ \mathcal{L}_{m,p} = -\frac{1}{4\pi} \partial_\gamma \Phi^{T} L_{m,p} \cdot \partial_x \Phi - \frac{1}{4\pi} \partial_\gamma \Phi^{T} L_{m,p} \cdot V_{m,p} \cdot \partial_x \Phi + \frac{1}{2\pi} q_{m,p} \epsilon^{\alpha \beta} A_\alpha \partial_\beta \Psi_L, \quad (2.10a) \]

where

\[ \Psi_R \quad \text{and} \quad \Psi_L \quad \text{are, respectively, the right- and left-moving} \quad p\text{-tuplet of bosonic edge fields,} \]

\[ Q_{m,p}^T = (q_{m,p}^T - q_{m,p}^T) = (1, 0, \ldots, 0, -1, 0, \ldots, 0) \quad (2.10b) \]
is the charge vector and

\[ \mathbf{K}_{m,p} = \begin{pmatrix} K_{m,p} & 0 \\ 0 & -K_{m,p} \end{pmatrix} \]  \hspace{1cm} (2.10d)

is the K matrix of the interface. The equal-time commutation relations of the edge fields reads

\[ [\partial_x \Phi_a(x), \Phi_b(x')] = 2\pi i \left( \mathbf{K}_{m,p}^{-1} \right)_{ab} \delta(x - x'). \]  \hspace{1cm} (2.11)

Gapping the p pairs of counter propagating modes at the interface is achieved with a set of p commuting sine-Gordon local interactions

\[ U[\Lambda_i] = \cos(\Lambda_i^T \mathbf{K}_{m,p} \Phi), \quad i = 1, \ldots, p, \]  \hspace{1cm} (2.12)

where \( \Lambda_i \) are 2p-component integer vectors representing correlated backscattering processes between right- and left-moving local quasiparticles at the interface. Furthermore, owing to the non-trivial commutation relations, Eq. 2.11, satisfied by the edge fields, the integer vectors \( \Lambda_i \) are required to satisfy the null condition

\[ \Lambda_i^T \mathbf{K}_{m,p} \Lambda_j = 0, \quad i, j = 1, \ldots, p \]  \hspace{1cm} (2.13)

in order for the local interactions Eq. 2.12 to form a compatible set of mutually bosonic operators. Moreover, the integer U(1) charge of the operator \( U[\Lambda_i] \) is

\[ Q[\Lambda_i] = \Lambda_i^T Q_{m,p}. \]  \hspace{1cm} (2.14)

Clearly, integer vectors continue to satisfy Eq. 2.13 upon rescaling by an integer greater than one. Then with the respect to the Luttinger liquid fixed point Eq. 2.10, these rescaled null vectors describe local operators with larger scaling dimensions, which are then less relevant at low energy and can be disregarded. Therefore, in the remaining of this paper, we shall only focus our attention on null vectors that are primitive. As shown in that work, a single integer vector is primitive when the greatest common divisor of its entries is 1. A set of integer vectors is primitive if and only if the greatest common divisor of the set of minors of the \( p \times 2p \) integer matrix \( \mathcal{M}[[\Lambda]] \) is 1, where \( \mathcal{M}[[\Lambda]] \) is the integer matrix whose rows are formed by the null vectors \( \Lambda_i \).

In general, the low energy modes of the interface can become gapped due to distinct types of local interactions, each one associated with a primitive null set \( \{ \Lambda_i \} \) satisfying Eq. 2.13. As the number \( p \) of counter-propagating modes at the interface grows (i.e., as one moves “deeper” into the hierarchical sequence), one expects a corresponding increase in the number of gapping channels of the interface, as a consequence of more available backscattering channels amongst the counter-propagating modes. As such, the investigation of the low energy properties of interfaces of hierarchical FQH states poses a very rich physics problem. In the following discussion, we shall concentrate on certain classes of local interactions leading to gapped interfaces, whose properties will be described in generality in Section 3 and, more specifically, in Section 4 and 5.

### III. Domain Walls in Hierarchical Interfaces: General Properties

We now discuss the properties of domain walls and parafermion zero modes associated with hierarchical interfaces. One of the central points of this Section is the ansatz Eq. 3.4 that describes the \( U(1) \) symmetry broken interactions at the interface and which will permit us to determine, efficiently, the quantum dimension of the parafermions localized on the domain walls for an interface that holds \( p \) counter propagating modes.

As described in Section 2.13, the number of chiral edge modes grows with the hierarchy index \( p \), which increases the number of gap opening channels. In this context, addressing all possible forms of gapped interfaces seems a formidable task, which is beyond the scope of this work. Instead, we shall focus on a specific class of local interactions, which will be shown to stabilize parafermion zero modes on domain walls along the interface. We shall consider two types of gapped interfaces. The first one is...
formed by charge neutral backscattering, while the second one breaks charge conservation. Our focus is then on the low energy properties of domain walls separating charge conserving and the non-conserving gapped segments. We note that superconducting pairing correlations have been recently induced in integer quantum Hall edges, [50] which represents a promising step to create superconductor/FQH heterostructures.

The homogeneous interface described by Eq. 2.10 admits local charge neutral backscattering that gap the interface and heal the bulk states, as represented by the red segments in Fig. 2-(b). This interface, which allows Abelian anyons to hop across and propagate as bona fide deconfined bulk quasiparticles, is created by the backscattering terms

\[ U[\Lambda_i^{(0)}] = \cos (\Lambda_i^{(0)} K_{m,p}) \Phi, \]  

where

\[ \Lambda_i^{(0)} = (e_i, 0, \ldots, 0, 1, 0, \ldots, 0)^T \]  

for \( i = 1, \ldots, p \) is a set of integer vectors. Charge conservation obeyed by the interactions Eq. 3.1a follows from

\[ Q[\Lambda_i^{(0)}] = \Lambda_i^{(0)} Q_{m,p} = \left( e_i^T e_i^T \right) \left[ q_{m,p} \right] = 0, \]  

and the null condition of the integer vectors Eq. 3.1b

\[ \Lambda_i^{(0)} K_{m,p} \Lambda_i^{(0)} = \left( e_i^T e_i^T \right) \left[ K_{m,p} \Phi \right] \left( e_j^T e_j^T \right) = e_i^T K_{m,p} e_j - e_i^T K_{m,p} e_j = 0 \]  

is verified \( \forall i, j = 1, \ldots, p \).

We now consider another set of local interactions

\[ U[\Lambda_i] = \cos (\Lambda_i K_{m,p}) \Phi \]  

that break charge conservation and depend upon the integer vectors

\[ \Lambda_i = \begin{cases} \Lambda_i^{(0)} & \text{if } \Lambda_i^{(0)} \neq 0 \\ \Lambda_i^{(0)} & \text{if } \Lambda_i^{(0)} \neq 0 \end{cases} \]  

for \( i = 1, \ldots, p \).

The interaction \( U[\Lambda_i] \) breaks charge conservation, while the \( U[\Lambda_i^{(0)}] \) conserve charge. Despite its simple form, this ansatz will be shown to embody a non-trivial charge condensate that gaps the interface and stabilizes parafermions on domain walls between segments of the interface gapped by the interactions Eq. 3.4a from those segments gapped by the charge neutral interactions Eq. 3.1a. Furthermore it permits an analytical understanding of the mechanism behind the formation of domain wall parafermions. Since the subset of \( p - 1 \) null vectors \( \{ \Lambda_i = \Lambda_i^{(0)} \} \) in Eq. 3.4b satisfies the null condition, Eq. 2.13 reduces to \( p \) independent equations that can be solved exactly, as we shall demonstrate in the following Sections [3V and 3W].

In order to determine the quantum dimension of the parafermion zero modes, we consider a series of domain walls at the interface that separate segments \( S_0 = \cup \left( x_{2i} + \varepsilon, x_{2i+1} - \varepsilon \right) \) gapped by the interactions Eq. 3.1a from the segments \( S = \cup \left( x_{2i-1} + \varepsilon, x_{2i} - \varepsilon \right) \) gapped by the interactions Eq. 3.4a, where \( \varepsilon = 0^+ \) is a positive regulator for the domain walls. In the strong coupling limit, the ground state is obtained by locking the sine-Gordon terms Eq. 3.1a and Eq. 3.4a to their minima on the respective segments \( S^{(0)} \) and \( S \). The ground state degeneracy can be obtained by constructing a set of operators with support on these gapped segments

\[ \Gamma_{2i-1,2i} = \exp \left( i \frac{\pi}{N_{m,p}} \int_{x_{2i-1}}^{x_{2i}+\varepsilon} dx \Lambda_i K_{m,p} \Phi \right), \]  

\[ \Gamma_{2i,2i+1} = \exp \left( i \frac{\pi}{N_{m,p}} \int_{x_{2i}}^{x_{2i+1}+\varepsilon} dx \Lambda_i K_{m,p} \Phi \right), \]  

where

\[ N_{m,p} = \Lambda_i^{(0)} K_{m,p} \Lambda_i \in \mathbb{Z}^+ \]  

It follows from the commutation relations Eq. 2.11 that operators defined in Eq. 3.5a and Eq. 3.5b commute with the Hamiltonian along the interface and satisfy the algebra

\[ \Gamma_{2i-1,2i} \Gamma_{2j,2j+1} = \exp \left( i \frac{\pi}{N_{m,p}} \delta_{i,j} \delta_{i-1,j} \right) \Gamma_{2j,2j+1} \Gamma_{2i-1,2i} \]  

\[ \Gamma_{2i,2i+1} \Gamma_{2k-1,2k} = \Gamma_{2k,2k+1+1} = 1. \]  

The dimension of the minimum representation of this algebra clearly corresponds to the ground state degeneracy. \( \Gamma_{2i,2i+1} \) act as raising or lowering operator to its neighbors \( \Gamma_{2i-1,2i} \) and \( \Gamma_{2i+1,2i+2} \) as \( \mathbb{Z}_{N_{m,p}} \) clock operators. For a configuration with \( 2n_{dw} \) domain walls, Eq. 3.6 conveys the ground state degeneracy \( |N_{m,p}|^{n_{dw}} \) and the quantum dimension of the parafermion

\[ d_{m,p} = \sqrt{|N_{m,p}|}. \]  

Therefore, the quantum dimension of the parafermions depends upon a single integer given by Eq. 3.5c.

Finally, the ground state degeneracy stems from the existence of parafermion zero modes on the domain walls

\[ \alpha_{2i} = e^{\frac{2\pi i}{N_{m,p}} \Lambda_i K_{m,p} \Phi(x_{2i} + \varepsilon) + \Lambda_i^{(0)} K_{m,p} \Phi(x_{2i} + \varepsilon)} \]  

\[ \alpha_{2i+1} = e^{\frac{2\pi i}{N_{m,p}} \Lambda_i^{(0)} K_{m,p} \Phi(x_{2i+1} + \varepsilon) + \Lambda_i K_{m,p} \Phi(x_{2i+1} + \varepsilon)} \]  

which satisfy the \( \mathbb{Z}_{N_{m,p}} \) parafermion algebra

\[ \alpha_i \alpha_j = e^{\frac{2\pi i}{N_{m,p}} \sin(i-j)} \alpha_j \alpha_i, \]  

(3.9)
and are related to the operators in Eq. 3.5 by
\[ \alpha_{2i}^{{\dagger}} \alpha_{2i+1} \sim \Gamma_{2i+1,2i}, \quad (3.10) \]
with similar relations holding for the other segments on the interface. This establishes that the bilinear terms constructed out of the parafermion operators commute with the Hamiltonian at the interface.

In the following Section we shall impose the null and primitive conditions to the interactions given by the ansatz Eq. 3.4 and use it to obtain the local \( U(1) \) symmetry broken interactions and parafermion zero modes for the first few hierarchical states. (For completeness we shall also revisit the primary Laughlin state studied in Refs. [9–12].) The analysis of these explicit cases will point to the general formulation valid for all hierarchical states, which we shall present in Section V.

IV. HIERARCHY OF PARAFERMIONS: EXAMPLES

In this Section we apply the formalism introduced in Section III from the primary Laughlin states up until the third hierarchical state.

A. Primary Laughlin state \((m, 1)\)

The Laughlin state with filling fraction \( \nu_{m,1} = \frac{1}{2m+1} = \frac{1}{3}, \frac{1}{5}, \ldots \) is described by a one-component quantum fluid with \( K_{m,1} = 2m + 1 \) and \( g_{m,1} = 1 \). The interface Luttinger liquid theory Eq. 2.10 has
\[ K_{m,1} = \begin{pmatrix} 2m + 1 & 0 \\ 0 & -(2m + 1) \end{pmatrix}, \quad Q_{m,1} = (1, -1)^T, \quad (4.1) \]
with electron operators given by \( \psi_{L/R} = e^{i(2m+1)\phi_{L/R}/2} \), where \( \phi_{L/R} \) are the chiral boson fields at the interface.

The charge neutral backscattering Eq. 3.1 for the Laughlin interface reads
\[ \Lambda_{1}^{(0)} = (1, 1), \quad (4.2) \]
\[ U[\Lambda_{1}^{(0)}] = \frac{\lambda_1}{2} (\psi_{L}^{\dagger} \psi_{R} + \text{H.c.}) \]
\[ = \lambda_1 \cos [(2m + 1)(\phi_{R} - \phi_{L})]. \quad (4.3) \]
Alternatively, the interface can be gapped via charge 2 electron pairing.
\[ \Lambda_{1} = (1, -1), \quad (4.4) \]
\[ U[\Lambda_{1}] = \frac{\lambda_1'}{2} (\psi_{L}^{\dagger} \psi_{R} + \text{H.c.}) \]
\[ = \lambda_1' \cos [(2m + 1)(\phi_{R} + \phi_{L})]. \quad (4.5) \]

Then, according to Eq. 3.5c
\[ N_{m,1} = \Lambda_{1}^{(0)} K_{m,1} \Lambda_{1} = 2(2m + 1) \quad (4.6) \]
enables \( Z_{2(2m+1)} \cong Z_2 \oplus Z_{2m+1} \) parafermions
\[ \alpha_i \alpha_j = e^{i \frac{2\pi}{2(2m+1)} \text{sgn}(i-j)} \alpha_j \alpha_i \quad (4.7) \]
with quantum dimension
\[ d_{m,1} = \sqrt{2} \times \sqrt{2m + 1} = \sqrt{2} \times D_{m,1}, \quad (4.8) \]
which are localized on the domain walls between regions gapped by the charge neutral local backscattering Eq. 4.3 and regions gapped by the charge 2 paring Eq. 4.5. Notice that the quantum dimension of the parafermion shows a contribution from the bulk topological order through the total quantum dimension \( D_{m,1} = \sqrt{2m + 1} \) of the Laughlin state, as well as a contribution \( \sqrt{2} \) reminiscent of Majorana zero modes in a 1D topological superconductor. [7]. Because of this 1D effect, even in the absence of deconfined bulk anyons, which corresponds to the \( \nu = 1 \) IQH state where \( m = 0 \), there is one Majorana zero mode localized on each domain wall of the IQH interface.

The presence of \( Z_{2(2m+1)} \) parafermions on the domain walls, as seen by Eq. 3.1, manifests that the charge \( 1/m \) operator
\[ \langle O_{\frac{1}{m+1}} \rangle = \langle e^{i \frac{(2m+1)(\phi_{R} + \phi_{L})}{2}} \rangle = \langle e^{i \frac{(\phi_{R} + \phi_{L})}{2}} \rangle \neq 0 \quad (4.9) \]
acquires a non-zero expectation on the segments of the interface that are gapped by the interaction Eq. 4.3.

B. First Hierarchical State \((m, 2)\)

The interface of the first hierarchical state with filling fraction \( \nu_{m,2} = \frac{2}{3m+1} = \frac{2}{5}, \frac{2}{7}, \ldots \) contains two pairs of counter-propagating fields \( \Phi^{\dagger} = (\phi_{R}^{\dagger}, \phi_{L}^{\dagger}, \phi_{R}, \phi_{L}) \). The Luttinger liquid Lagrangian of the interface, Eq. 2.10 has
\[ K_{m,2} = K_{m,2} \oplus (-K_{m,2}), \quad Q_{m,2} = (q_{m,2}^{T} - q_{m,2}^{T}), \quad (4.10) \]
where we adopt the hierarchical representation Eq. 2.6b for the \( K \) matrix of the bulk state.

According to Eq. 3.1, the pair of local interactions
\[ U[\Lambda_{1}^{(0)}] = \frac{\lambda_1}{2} (\psi_{L}^{\dagger} \psi_{R} + \text{H.c.}) \]
\[ = \lambda_1 \cos [(2m + 1)(\phi_{R} - \phi_{L}) - (\phi_{2R} - \phi_{2L})] \quad (4.11a) \]
and
\[ U[\Lambda_2^{(0)}] = \frac{\lambda_2}{2} \left( \psi_{2L}^* \psi_{2R} + \text{H.c.} \right) = \lambda_2 \cos [-\phi_{1R} + \phi_{1L}] + 2(\phi_{2R} - \phi_{2L}) , \]
associated with the null vectors
\[ \Lambda_1^{(0)} = (1, 0, 1, 0), \quad \Lambda_2^{(0)} = (0, 1, 0, 1) , \tag{4.11c} \]
gap the interface without breaking charge conservation. The local operators at the interface correspond to
\[ \psi_{a,R/L} = e^{i \sum_{j} (K_{R/L})_{ab} \phi_{a,R/L}} \],
for \( a = 1, 2 \). Eq. \( 4.11 \) represents local charge neutral backscattering that localizes the interface low energy modes.
We now seek the null vectors and corresponding charge non-conserving interactions that gap the modes of the interface. Following Eq. \( 3.4 \) we consider
\[ \Lambda_1 = (x_1, y_1, x_2, y_2), \quad \Lambda_2 = (0, 1, 0, 1) , \tag{4.12} \]
where \( x_1, x_2, y_1, y_2 \) are integers. Notice that the interaction
\[ U[\Lambda_1] = \cos(\Lambda_1 K_{m,2} \Phi) \]
is charge neutral, while
\[ U[\Lambda_2] \]
represents local charge neutral backscattering that localizes the interface of the primary Laughlin states. Our analysis shows the presence of \( \mathbb{Z}_{4m+1} \) parafermions. In this case, parafermion operators are constructed from operators that create fractional charge \( 2/5 \) and charge zero quasiparticle pairs on each side to the domain walls. Interestingly, the \( 2 \) condensate results from the coalescence of a quintuplet of quasiparticles of charge \( 2/5 \).
This interaction represents a charge 2 condensate, where \( \psi_{1R} \) and \( \psi_{1L} \) are both charge 1 fermionic operators, and \( \psi_{2R} \) accounts for a charge zero operator with bosonic self-statistics.
According to Eq. \( 3.5c \)
\[ N_{m,2} = \Lambda_1^{(0)} K_{m,2} \Lambda_1 = 4m + 1 \tag{4.17} \]
shows the presence of \( \mathbb{Z}_{4m+1} \) parafermions
\[ \alpha_i \alpha_j = e^{i \frac{2\pi}{4m+1} \Phi} \alpha_j \alpha_i \tag{4.18} \]
with quantum dimension
\[ d_{m,2} = \sqrt{4m+1} = D_{m,2} , \tag{4.19} \]
which are localized on the domains of the interface. This result shows that the quantum dimension of the parafermion is a direct manifestation of bulk topological order of the first hierarchical state \( (m, 2) \) through its total quantum dimension \( D_{m,2} = \sqrt{4m+1} \).
As an application of our analysis, the homogeneous interface of \( \nu = 2/5 \) FQH states (where \( m = 1 \) and \( p = 2 \)) supports \( \mathbb{Z}_5 \) parafermions. This case, parafermion operators are constructed from operators that create fractional charge \( 2/5 \) and charge zero quasiparticle pairs on each side to the domain walls. Interestingly, the \( 2 \) condensate results from the coalescence of a quintuplet of quasiparticles of charge \( 2/5 \).

The generalization of the \( \nu = 2/5 \) state to other first hierarchical states with arbitrary values of \( m \) shows that the \( \mathbb{Z}_{4m+1} \) parafermion results from the formation of a \( (4m+1) \)-tuple of charge \( 2/(4m+1) \) fractional quasiparticles giving rise to a charge 2 condensate. This, in turn, is a manifestation of the non-zero expectation value of the charge \( 2 \) operator
\[ \langle \mathcal{O}_{2/(4m+1)} \rangle \equiv \langle e^{i \frac{\Lambda_1 K_{m,2} \Phi}{4m+1}} \rangle \neq 0 \tag{4.20} \]
on the segments of gapped by the interaction Eq. \( 4.16 \).

Compared with the primary Laughlin states discussed in Section \( IV \), the quantum dimension of the parafermions in the first hierarchical state does not manifest a \( \sqrt{2} \) contribution expected for the 1D topological superconductor. To shed light on this result, we make use of the representation of the first hierarchical state Eq. \( 2.8 \)
\[ \tilde{K}_{m,2} = W_2 K_{m,2} W_2^T = \begin{pmatrix} 2m+1 & 2m \\ 2m & 2m+1 \end{pmatrix} \tag{4.21} \]
\[ \tilde{q}_{m,2} = (1, 1) , \]
where the first hierarchical state can be thought of as a coupled FQH bilayer, with each layer carrying \( U(1) \) charge \( q = 1 \). Therefore, the degrees of freedom at the interface constitute two pairs of counter-propagating fermion modes, twice the number of degrees of freedom in the interface of the primary Laughlin states. Our
analysis then shows that adding an extra pair of counterpropagating modes renders the Majorana zero mode unstable, which is an indication of the $\mathbb{Z}_2$ stability of Majorana fermions in 1D topological superconductors.

We now draw an important comparison between the parafermion zero modes discussed in our setup and the theory of extrinsic defects associated with anyonic symmetries of the Abelian phase. In Ref. [13], the $\mathbb{Z}_2$ twisted defects associated with the layer permutation of $\mathbb{Z}$ symmetries of the Abelian phase. In Ref. [13], the $\mathbb{Z}_2$ twist defect of the first hierarchical state is trivial, since $d_{\mathbb{Z}_2} = 1$. Furthermore, it can be demonstrated [2] that the twist defects associated with charge conjugation anyonic symmetry correspond Majorana fermions, but not the $\mathbb{Z}_{2m+1}$ parafermions discussed here.

These preliminary findings regarding the primary and first hierarchical states point to the existence of an outstanding even-odd effect that ties the stability of a Majorana zero mode to the parity of the hierarchical index $p$, as shown in Fig. [1]. In the next two Subsections we shall validate this even-odd effect by explicitly showing that the parafermions of second hierarchical state ($p = 3$) possess a $\sqrt{2}$ contribution to their quantum dimension, similar to the primary states ($p = 1$) in Section IV A on the other hand, parafermions of third hierarchical state ($p = 4$) repeat the same behavior as those of the first hierarchical state ($p = 2$).

C. Second Hierarchical State $(m,3)$

The low energy modes of the interface of the second hierarchical state are described in terms of the fields $\Phi^T = (\phi^{R}_{1}, \phi^{R}_{2}, \phi^{R}_{3}, \phi^{L}_{1}, \phi^{L}_{2}, \phi^{L}_{3})$ and the Lagrangian Eq. 2.10 has

$$K_{m,3} = K_{m,3} \oplus (-K_{m,3}), \quad Q_{m,3}^T = (q_{m,3}^T, -q_{m,3}^T),$$

$$K_{m,3} = \begin{pmatrix} 2m + 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad q_{m,3}^T = (1, 0, 0),$$

where we adopt the hierarchical representation Eq. 2.6b.

The local charge neutral interactions Eq. 3.1 take the form

$$U[\Lambda_i^{(0)}] = (\lambda_i/2)(\psi^R_{iR}\psi_{iL} + H.c.),$$

$$= \cos (\Lambda_i^{(0)} K_{m,3} \Phi), \quad \text{for } i = 1, 2, 3,$$

associated with the null vectors

$$\Lambda_1^{(0)} = (1, 0, 0, 1, 0, 0)$$
$$\Lambda_2^{(0)} = (0, 1, 0, 0, 1, 0)$$
$$\Lambda_3^{(0)} = (0, 0, 1, 0, 0, 1).$$

Following Eq. 3.4, we now consider another set of null vectors

$$\Lambda_1 = (x_1, y_1, z_1, x_2, y_2, z_2)$$
$$\Lambda_2 = \Lambda_3 = (0, 1, 0, 0, 1, 0)$$

(4.26)

(where $x_1, ..., z_2$ are integers) corresponding to the local interactions

$$U[\Lambda_i] = \cos (\Lambda_i K_{m,3} \Phi) \quad i = 1, 2, 3,$$

where $U[\Lambda_2]$ and $U[\Lambda_3]$ are charge neutral and $U[\Lambda_1]$ carries charge $x_1 - x_2 \neq 0$. Imposing the null condition results in three equations

$$\Lambda_1 K_{m,3} \Lambda_1 = (2m + 1)(x_1^2 - x_2^2) - 2x_1y_1$$
$$+ 2x_2y_2 + 2(y_1^2 - y_1z_1 - y_2^2 + y_2z_2 + z_1^2 - z_2^2) = 0,$$

$$\Lambda_1 K_{m,3} \Lambda_2 = -x_1 + x_2 + 2(y_1 - y_2) + z_1 + z_2 = 0,$$

$$\Lambda_1 K_{m,3} \Lambda_3 = -y_1 + y_2 + 2(z_1 - z_2) = 0.$$

Eqs. 4.28a and 4.28c result in

$$z_1 - z_2 = \frac{x_1 - x_2}{3} = \frac{y_1 - y_2}{2},$$

which gives

$$[2x_2 + 3(z_1 - z_2)](z_1 - z_2) = 0$$

(4.30)

upon substitution onto Eq. 4.28a.

A non-trivial solution of Eq. 4.29 and Eq. 4.30 yields the null vector $\Lambda_1 = (3t, y_2 + 4t, z_2 + 2t, -3t, y_2, z_2)$ for $t \neq 0$ and $y_2, z_2 \in \mathbb{Z}$. Furthermore, the non-zero minors of $\mathcal{M}[[\Lambda_1, \Lambda_2, \Lambda_3]]$ belong in the set $\{\pm 2t, \pm 3t, \pm 4t\}$ from which the primitive condition follows for $t = \pm 1$. Finally, setting $t = 1, y_2 = -2$ and $z_2 = -1$, gives the null vector

$$\Lambda_1 = (3, 2, 1, -3, -2, -1),$$

and the corresponding local interaction

$$U[\Lambda_1] = \cos (\Lambda_1 K_{m,3} \Phi) = \cos [(6m + 1)(\phi^{R}_{1} + \phi^{L}_{1})].$$

It follows from Eq. 4.31 and Eq. 3.5c that

$$N_{m,3} = \Lambda_1^{(0)} K_{m,3} \Lambda_1 = 2(6m + 1).$$

(4.33)
which establishes the presence of $\mathbb{Z}_{2(6m+1)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{6m+1}$ parafermions

$$\alpha_i \alpha_j = e^{i \frac{2\pi}{6m+1} \text{sgn}(i-j)} \alpha_j \alpha_i \quad (4.34)$$

with quantum dimension

$$d_{m,3} = \sqrt{2} \times \sqrt{6m+1} = \sqrt{2} \times D_{m,3}. \quad (4.35)$$

This explicit calculation, therefore, confirms that the structure of the parafermions in the second hierarchical state is similar to that observed in the primary Laughlin states discussed in Section [V.A] with the quantum dimension of the parafermion being a manifestation of both the bulk Abelian order and the non-trivial 1D superconductor. Nevertheless, an important distinction emerges in this case, for the interaction Eq. [4.32] represents a condensate of charge $Q[\Lambda_1] = Q_{m,3}\Lambda_3 = 6$. The stability of parafermions, as seen in Eq. [3.8], is captured by the expectation value of the charge $Z_{6m+1}$ operator

$$\langle \mathcal{O}_{6m+1} \rangle \equiv \langle e^{i \frac{2\pi}{6m+1} \mathcal{K}_{m,3}\Lambda_1} \rangle = \langle e^{i \frac{2\pi \nu_m + \theta}{2}} \rangle \neq 0, \quad (4.36)$$

on the segments of the interface gapped by this interaction. As an example, the interface between two FQH states at filling fraction $\nu = 3/7$ (corresponding to $m = 1$, $p = 3$), can give rise to $\mathbb{Z}_{13} \cong \mathbb{Z}_2 \otimes \mathbb{Z}_7$ parafermions along the domain walls described here. The charge 6 condensate, in this case, is formed by condensation of fractional charge $3/7$.

It is instructive to seek an understanding of this charge 6 gapped interface in the the representation Eq. [2.8]

$$\tilde{K}_{m,3} = \begin{pmatrix} 2m + 1 & 2m & 2m \\ 2m & 2m + 1 & 2m \\ 2m & 2m & 2m + 1 \end{pmatrix}, \quad \tilde{q}_{m,3} = (1, 1, 1) \quad (4.37)$$

where the $K$ matrix and charge vectors resemble a tri-layer FQH state, where each layer carries unit charge. The $\text{SL}(3, \mathbb{Z})$ transformation to this new basis

$$W_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad (4.38)$$

changes the null vector to

$$\tilde{\Lambda}_1 = \left[ (W_3^{-1})^T \oplus (W_3^{-1})^T \right] \Lambda_1 = (1, 1, 1, -1, -1, -1)^T. \quad (4.39)$$

Then the interaction

$$U[\tilde{\Lambda}_1] = \tilde{\lambda} \cos \left( \tilde{\Lambda}_1 \tilde{K}_{m,3} \tilde{\Phi} \right)$$

$$\sim \tilde{\lambda} \tilde{\psi}_1 \tilde{\psi}_2 \tilde{\psi}_3 \tilde{\psi}_L \tilde{\psi}_2 \tilde{\psi}_3 \tilde{\psi}_L + \text{H.c.} \quad (4.40)$$

is manifestly a charge 6 operator involving pairing of 3 local fermions on each side of the interface. This interaction is a generalization of the charge 2 pairing at the interface of Laughlin states.

Similarly to the discussion of the first hierarchical state, we compare our set up with the theory of extrinsic defects associated with anyonic symmetries of the Abelian phase. In Ref. [13], the $\mathbb{Z}_3$ twist defects associated with the layer permutations of the Abelian phase characterized by the $K$ matrix

$$K_{m,\ell} = \begin{pmatrix} m & \ell & \ell \\ \ell & m & \ell \\ \ell & \ell & m \end{pmatrix} \quad (4.41)$$

where studied and it was shown that domain walls separating two distinct gapping charge conserving gap terms support parafermions with quantum dimension $d_{\mathbb{Z}_3} = |m - \ell|$. Comparing Eq. [4.37] and Eq. [4.41] shows that that $\mathbb{Z}_3$ twist defect of the second hierarchical state is trivial, since $d_{\mathbb{Z}_3} = 1$. Furthermore, it can be shown [52] that twist defects associated with charge conjugation anyonic symmetry are associated with Majorana fermions, in contrast with the $\mathbb{Z}_{2(6m+1)}$ parafermions discussed here.

## D. Third hierarchical state ($m, 4$)

The interface of the third hierarchical FQH state ($m, 4$) with filling fraction $\nu_{m,4} = \frac{m}{6m+1}$ supports the low energy mode fields $(\psi_{1L}^R, \psi_{2L}^R, \psi_{3L}^R, \phi_{1L}^R, \phi_{2L}^R, \phi_{3L}^R, \phi_{4L}^R)$ where the Lagrangian Eq. [2.10] has

$$K_{m,4} = K_{m,4} \oplus (-K_{m,4}), \quad Q_{m,4}^T = (q_{m,4}, -q_{m,4})$$

$$K_{m,4} = \begin{pmatrix} 2m + 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad q_{m,4}^T = (1, 0, 0, 0) \quad (4.42)$$

Charge neutral null vectors parametrizing the gap
opening interactions Eq. 3.1a read
\[
\begin{align*}
\Lambda_1^{(0)} &= (1, 0, 0, 0, 1, 0, 0, 0)^T \\
\Lambda_2^{(0)} &= (0, 1, 0, 0, 0, 1, 0, 0)^T \\
\Lambda_3^{(0)} &= (0, 0, 1, 0, 0, 0, 1, 0)^T \\
\Lambda_4^{(0)} &= (0, 0, 0, 1, 0, 0, 0, 1)^T.
\end{align*}
\tag{4.43}
\]

Carrying out an analysis similar to that discussed in Sections [IVB, IVC] and [IVD], we find that the null independent on the parity of the index :p:

\[
\begin{align*}
\Lambda_i = \begin{cases} 
(2, 3, 2, 1, -2, 0, 0, 0)^T & i = 1 \\
\Lambda_i^{(0)} & i = 2, 3, 4
\end{cases}
\tag{4.44}
\]

Notice that the interaction
\[
U[\Lambda] = \cos (\Lambda_1 K_{m,p} \Phi) \\
\sim \psi_{1R}^3 \psi_{1L}^2 + \psi_{1R}^2 \psi_{1L}^3 + \sim H.c
\tag{4.45}
\]
is a charge 4 cluster operator where \(\psi_{1R}\) and \(\psi_{1L}\) are charge 1 local operators with fermionic statistics and \(\psi_{2R}, \psi_{3R}, \psi_{4R}\) are charge zero local operators with bosonic statistics.

From Eq. 4.44 and Eq. 3.5c we get
\[
N_{m,q} = \Lambda_1^{(0)} K_{m,q} \Lambda_1 = (8m + 1),
\tag{4.46}
\]
which establishes the presence of \(\mathbb{Z}_{8m+1}\) parafermions
\[
\alpha_i \alpha_j = e^{i \frac{2 \pi}{8m+1} \text{sgn}(i-j)} \alpha_j \alpha_i
\tag{4.47}
\]
with quantum dimension
\[
d_{m,q} = \sqrt{8m + 1} = D_{m,q}.
\tag{4.48}
\]

This result shows, that the parafermions in the third hierarchical state behave similarly to the first hierarchical state discussed in Section [IVB]. This non-trivial condensate is manifested in the expectation value of the charge 4 \(\frac{4}{8m+1}\) operator
\[
\langle O_{\frac{4}{8m+1}} \rangle \equiv \langle e^{i \frac{\Lambda_1^{(0)} K_{m,q} \Lambda_1}{8m+1}} \rangle \neq 0.
\tag{4.49}
\]
So, for instance, the \(\nu = 4/9\) FQH state \((m = 1, p = 4)\) is seen to support \(\mathbb{Z}_q\) parafermions along domain walls at its gapped edge. The charge 4 condensate is formed by a cluster of 9 quasiparticles with charge 4/9.

V. HIERARCHY OF PARAFERMIONS: GENERAL CASE

The properties of the parafermions zero modes stabilized on domain walls of the \(p = 1, 2, 3, 4\) hierarchical states discussed in Section [IV] reveal a remarkable dependence on the parity of the index \(p\), which labels the depth of the hierarchy. This dependence reflects an interplay between the bulk topological order, which gives rise to quasiparticle fractionalization, and the 1D SPT order that stabilizes Majorana zero modes in non-trivial topological superconductors. An appealing mechanism to account for such an even-odd dependence emerges when upon expressing the bulk topological order of the hierarchical state in the representation Eq. 2.8, where the K-matrix gives an interpretation of the bulk topological order as a series of \(p\) Laughlin-type layers (as indicated in the diagonal odd integers \(2m+1\) coupled to each other by bosonic correlations (indicated by the off-diagonal even integers \(2m\)). As such, the number of pairs of counter-propagating fermion modes in interfaces of hierarchical states \((m, p)\) and \((m, p + 2)\) differ by 2. In the limit, where these Laughlin FQH layers are decoupled (which would correspond to \(K = (2m + 1) \operatorname{diag}(1, ..., 1)\)), the even-odd effect associated with the stability of Majorana zero modes is a direct consequence of the \(\mathbb{Z}_2\) stability (instability) associated with an odd (even) number of Majorana zero modes per domain. Remarkably, our analysis will show that this structure persists even when the layers are coupled, according to the K-matrix Eq. 2.8a.

One of the goals of this Section is to show that this \(\mathbb{Z}_2\) pattern indeed persists for all hierarchical states, whose bulk topological order are represented by Eq. 2.6 or, equivalently, Eq. 2.8. We will show that the interactions require breaking of charge conservation in such a way that the charge of the condensate depends on the hierarchical level \(p\). Notably, while the primary Laughlin \((p = 1)\) admit a charge 2 condensate that, in principle, can be induced by a weak-pairing mechanism (or proximity to a superconductor), for generic states of the hierarchical sequence, the \(\mathbb{Q}_{m,p} > 2\) charge of the condensate signals that a non-BCS strong coupling mechanism is at play. This situation departs significantly from the stability of Majorana zero modes in superconducting wires [7] as well as in interfaces of Laughlin-type states. [9,12]

An interesting property of the condensate is that it involves clustering of \(2mp + 1\) quasiparticles of charge \(\nu_{m,p} = \frac{p}{2mp+1}\). This scenario of parafermions being stabilized by clustering of quasiparticles is analogous to clustering property of non-Abelian Read-Rezayi FQH states. [46] where electrons (or composite fermions) forming an order-\(k\) cluster, give rise to an incompressible state that supports non-Abelian bulk excitations and chiral charge neutral \(\mathbb{Z}_k\) parafermions on the boundary. (The special case \(k = 2\) corresponds to the Moore-Read states with a chiral Majorana fermions at the boundary, which is a candidate topological order for the \(\nu = 5/2\) FQH.)

In Sections [VA] and [VB] we discuss, respectively, the \(p = \text{odd}\) and \(p = \text{even}\) hierarchical states in generality, where we shall provide explicit expressions for the null vectors and, consequently, the local interactions that gap the interface and give rise to parafermion zero modes localized on domain walls.
A. Hierarchical states: \( p = \text{odd integer} \)

Consider the interface between hierarchical FQH states with filling fraction

\[
\nu_{m,p} = \frac{p}{2mp + 1}, \quad m \in \mathbb{Z}_+, \quad p = 1, 3, 5, \ldots
\]  

(5.1)

As discussed in Section III, this interface admits a gapped region realized by the local charge neutral interactions Eq. 5.1. We now demonstrate that the interface formed by the states in Eq. 5.1 admits a set of local gap opening interactions that breaks \( U(1) \) charge conservation and represent a charge \( Q_{m,p} \) condensate where

\[
Q_{m,p} = 2p, \quad m \in \mathbb{Z}_+, \quad p = 1, 3, 5, \ldots
\]  

(5.2)

In the representation Eq. 2.6, this condensate is realized by the local interactions

\[
U[\Lambda_i] = \cos (\Lambda_i K_{m,p} \Phi), \quad i = 1, \ldots, p,
\]  

(5.3)

represented by the integer vectors

\[
\Lambda_i = \begin{pmatrix} 1 \\ v_{p} \\ -v_{p} \end{pmatrix}, \quad v_{p} = \begin{pmatrix} p \\ p-1 \\ \vdots \\ 1 \end{pmatrix}
\]  

(5.4)

\[
\Lambda_i = \begin{pmatrix} 1 \\ v_{p} \end{pmatrix}, \quad i = 2, \ldots, p,
\]

which satisfy the null condition

\[
\Lambda_i K_{m,p} \Lambda_j, \quad i, j = 1, \ldots, p.
\]  

(5.5)

It follows immediately from Eq. 5.4 that the interaction corresponds to a condensate of charge \( Q[\Lambda_i] = \Lambda_i Q_{m,p} = 2p \), in accordance with Eq. 5.2. (Note that for \( 2 \leq i \leq p \): \( Q[\Lambda_i] = \Lambda_i Q_{m,p} = 0 \).) Moreover, the non-zero minors of \( \mathcal{M}(\{\Lambda_i\}) \) can be shown to form the set \( \{\pm(p-1), \pm 2, \pm (p+1)\} \) whose greatest common divisor is one, which shows that the integer vectors in Eq. 5.4 are primitive. Furthermore, to establish the null condition Eq. 5.5, we first realize that this condition is clearly satisfied for \( i, j = 2, \ldots, p \) as a consequence of Eq. 3.3. Then, the remaining non-trivial conditions we need to show are for \( i = 1 \) and \( j = 1, \ldots, p \). To establish this result, all we need is the identity

\[
K_{m,p} v_p = \begin{pmatrix} 2mp + 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (2mp + 1)e_1,
\]  

(5.6)

where \( v_p \) is \( p \) dimensional integer vector defined in Eq. 5.4. To demonstrate this result, let \( K_{m,p} v_p = \sum_{k=1}^{p} a_k e_k \). Except for the first and last rows, the remaining rows of \( K_{m,p} \) are formed by consecutive entries \(-1, 2, 1\) and the remaining ones equal to zero. The first row has \((K_{m,p})_{11} = 2m+1\) and \((K_{m,p})_{12} = -1\). The last row has \((K_{m,p})_{p-1} = -1\) and \((K_{m,p})_{p,p} = 2\). Putting all together,

\[
a_1 = p(2m+1) + (-1) \times (p-1) = 2mp + 1
\]

\[
a_k = (-1) \times (k+1) + 2 \times (k) + (-1) \times (k-1)
\]

\[
= 0, \quad 2 \leq k \leq p-1
\]

\[
a_p = (-1) \times (2) + 2 \times (1) = 0,
\]

which proves Eq. 5.9. Finally, by taking into account the orthonormal basis vectors \( e_i \), it is straightforward to verify the null condition Eq. 5.5.

The form of the charge 2\( p \) interaction, which follows directly from Eq. 5.6, is

\[
U[\Lambda_i] = \cos (\Lambda_i K_{m,p} \Phi) = \cos [(2mp + 1)(\phi_i^R + \phi_i^T)]
\]  

(5.8)

In the basis given by Eq. 2.8 the integer vectors \( \Lambda_i \) transforms to

\(\bar{\Lambda}_i = (W^{-1})^T \Lambda_i = (1, 1, \ldots, 1, -1, -1, \ldots, -1)^T \). (5.9)

The meaning of the charge 2\( p \) interaction in this representation is manifestly given by

\[
U[\bar{\Lambda}_i] = \cos (\bar{\Lambda}_i \bar{K}_{m,p} \bar{\Phi})
\]  

\[
\sim (\tilde{\psi}_1^T \tilde{\psi}_1^R) \cdots (\tilde{\psi}_p^T \tilde{\psi}_p^R) + H.c.
\]  

(5.10)

which represents the cluster of 2\( p \) fermions, each one carrying charge \( q = 1 \). Eq. 5.10 generalizes, to every odd value of \( p \), the charge 6 interaction that gaps the interface of the hierarchical state (\( m, 3 \)) depicted in Fig. 3.

Finally, the quantum dimension of the parafermion localized at the domain wall between the segments of the interface that are gapped by interactions Eqs. 3.1 and Eq. 5.3 follows from

\[
N_{m,p} = \Lambda_i^{(0)} K_{m,p} \Lambda_i = (e_1^T e_1^T) \left(\frac{2mp + 1}{2mp + 1}\right) e_1
\]  

(5.11)

\[
= 2(2mp + 1),
\]

which establishes the existence of parafermions of quantum dimension

\[
d_{m,p} = \sqrt{2} \times \sqrt{2mp + 1},
\]  

(5.12)

at the interface of hierarchical states filling fraction

\[
\nu_{m,p} = \frac{p}{2mp + 1}, \quad m \in \mathbb{Z}_+, \quad p = 1, 3, 5, \ldots
\]

(5.13)

B. Hierarchical states: \( p = \text{even integer} \)

Consider the interface between hierarchical FQH states with filling fraction

\[
\nu_{m,p} = \frac{p}{2mp + 1}, \quad m \in \mathbb{Z}_+, \quad p = 2, 4, 6, \ldots
\]  

(5.13)
As discussed in Section [III] this interface admits a gapped region realized by the local charge neutral interactions Eq. [3.1] We now prove that the interface formed by the states in Eq. [5.13] can be gapped by local interactions that break charge conservation symmetry and give rise to a condensate of charge

$$Q_{m,p} = p, \quad m \in \mathbb{Z}_+ , \quad p = 2, 4, 6, ...$$  \hspace{1cm} (5.14)

In the representation Eq. [2.6] this charge $p$ condensate is realized by the local interactions

$$U[A_i] = \cos (\Lambda_i K_{m,p} \Phi)$$  \hspace{1cm} (5.15)

where the integer vectors read

$$\Lambda_i = \begin{cases} (\frac{p}{2} - 1, -p, 0, 0, \ldots, 0)^T & \text{for } i = 1, 2, \ldots, p \text{ in Eq. [5.16]} \\ (0) & \text{and satisfy the null condition} \\ (\Lambda_i K_{m,p} \Lambda_j = 0, \quad \forall i, j = 1, \ldots, p. \text{ (5.17)}

With the null vectors Eq. [5.16] we directly find that this corresponds to a condensate of charge $Q[A_1] = \Lambda_1 Q_{m,p} = p$, as given by Eq. [5.14]. (Notice that for $2 \leq i \leq p$ : $Q[A_i] = \Lambda_i Q_{m,p} = 0$.) Moreover, from the explicit form of the integer vector $\Lambda_1$ in Eq. [5.16] one verifies that the charge $p$ interaction

$$U[A_1] = \cos (\Lambda_1 K_{m,p} \Phi)$$  \hspace{1cm} (5.18)

represents a cluster operator where $\psi_{1R}$ and $\psi_{1L}$ are charge $1$ local operators with fermionic statistics and $\psi_{2R}, ..., \psi_{pR}$ are charge zero local operators with bosonic statistics.

We now demonstrate the validity of Eq. [5.17]. Since $\{ \Lambda_i^{(o)} \}, \quad i = 2, \ldots, p$ forms, by construction, a subset of null vectors, the only non-trivial relations left to be verified are $\Lambda_i K_{m,p} \Lambda_i = 0$ for $i = 1, ..., p$. In order to establish these conditions, we directly calculate

$$K_{m,p} \Lambda_1 = \begin{pmatrix} \frac{p}{2} (2m - 1) + 1 \\ 0 \\ \vdots \\ 0 \\ \frac{p}{2} (2m + 1) \end{pmatrix},$$  \hspace{1cm} (5.19)

leading to

$$\Lambda_1 K_{m,p} \Lambda_1 = \frac{p}{2} \left[ \frac{p}{2} (2m - 1) + 1 \right] + (p - 1) \frac{p}{2} - \frac{p}{2} \left[ \frac{p}{2} (2m + 1) \right] = 0$$

$$\Lambda_2 K_{m,p} \Lambda_1 = \frac{p}{2} - \frac{p}{2} = 0$$

$$\Lambda_i K_{m,p} \Lambda_1 = 0, \quad i = 3, ..., p,$$

where the last equation is a consequence of $e^T \cdot e_{1,2} = 0$ for $i = 3, ..., p$. This then establishes the null condition Eq. [5.17].

Finally, the quantum dimension of parafermion localized at the domain wall between the segments of the interface that are gapped by interactions Eqs. [3.1a] and Eq. [5.15] follows from

$$N_{m,p} = \Lambda_1^{(0)} K_{m,p} \Lambda_1 = 2mp + 1,$$  \hspace{1cm} (5.21)

which shows establishes the existence of parafermions with quantum dimension

$$d_{m,p} = \sqrt{2mp + 1}$$  \hspace{1cm} (5.22)

at the interface with filling fraction $\nu_{m,p} = \frac{p}{2mp + 1}$, with $m \in \mathbb{Z}_+$ and $p = 2, 4, 6, ...$.

We have then explicitly demonstrated, in Sections [VA] and [VB] the existence of a local charge condensate, Eq. [1.2] that stabilizes non-Abelian parafermions with quantum dimensions given by Eq. [1.3] and depicted in Fig. [1]

VI. SUMMARY AND OUTLOOK

In this work, we have established a correspondence between the sequence of Abelian hierarchical FQH states in the first Landau level and a class of extrinsic non-Abelian zero modes localized on domain walls that separate charge neutral and $U(1)$ symmetry broken gapped segments of the interfaces. Our analysis of the low energy properties of the bulk hierarchical state employed the hydrodynamical Chern-Simons theory by which the FQH system with Hall conductance $\sigma_{xy}(m,p) = \frac{e^2}{h} \frac{p}{2mp+1}$ is represented in terms of a $p$-component $U(1)$ Chern-Simons gauge theory parametrized by an integer valued $K$ matrix. \textsuperscript{34-38} The edge of such hierarchical state, in turn, supports $p$ chiral low energy modes described by a $p$-component chiral boson field whose commutation relations are determined by the $K$ matrix. As such, we have studied gap opening processes in a homogeneous interface with $p$ pairs of counter-propagating modes, as depicted in Fig. [2] Gapping these modes at the interface is realized by $p$ local sine-Gordon type operators.

Through a detailed examination of the locality and frustration free conditions of sine-Gordon operators on
the homogeneous interface, we have found that the
hierarchical states admit $U(1)$ symmetry breaking inter-
actions that give rise to a condensate whose charge is
a function of the hierarchical index $p$, as per Eq. 1.2
Therefore, our results show that a charge 2 condensate
only occurs for $p = 1$ and $p = 2$, i.e., for the primary
Laughlin states with filling fraction $\nu = \frac{1}{2m+1}$ and the first
hierarchical states with filling fraction $\nu = \frac{2}{4m+1}$, for
integer $m > 1$. (We note that charge 2 condensates formed
the basis of earlier studies of parafermions in interfaces
and trenches of Laughlin states and the particle-hole
conjugate of the $\nu = 1/3$ Laughlin state at filling
$\nu = 2/3$. For the general $p > 2$ case investigated
in this work, on the other hand, we have found that the
local $U(1)$ symmetry breaking interactions involve a non-
conventional charge clustering mechanism whereby more
than two electrons are glued together. Our findings then
open the interesting possibility of exploring these gapped
interfaces as a basis for constructing families of uncon-
ventional $U(1)$ symmetry broken phases in 2D by pro-
moting the interfaces to a “wire network”, in the spirit
of Ref. 63.

One of the main results of this work was establishing
that the properties of parafermions zero modes stem from
the existence of a cluster state of charge given by Eq. 1.2
which translates into a cluster of fractionalized quasiparticles
of charge $p/(2mp + 1)$. This state bears a striking
resemblance with the Read-Rezayi FQH states that rep-
resent non-Abelian FQH states where electrons form cluster
states. In fact, this correspondence has been explored in
Ref. 15, where it was shown that superconducting is-
lands in the $\nu = 2/3$ FQH state harbor $\mathbb{Z}_3$ parafermions
that are closely related to the neutral parafermion excita-
tions of the $\mathbb{Z}_3$ Read-Rezayi FQH, whose ground state
wavefunction encodes clustering of 3 electrons. From
this perspective, the results obtained here for the en-
tire sequence of Abelian hierarchical FQH states establish
a rich connection between two distinct families of
Abelian and non-Abelian topological orders, and suggest,
in particular, a route to describe the hierarchy of non-
Abelian phases via the deconfinement of extrinsic
parafermion zero modes in corresponding Abelian phases
of matter, which is an important topic worth of further
investigation. Furthermore, since the parafermions in the
setting considered here do not manifest any direct rela-
tionship with anyonic symmetries, the deconfinement of
these non-Abelian defects may require a theoretical treat-
ment that differs from those of Refs. 55 and 56, which
dealt with the deconfinement of twist defects related to
symmetries of the anyon group.

We have found an appealing dependence of the charge
condensate and the quantum dimension of parafermions
on the parity of the level hierarchy $p$, as shown in Fig. 1.
Borrowing from insights related to the multi-layer rep-
resentation of the K matrix of the hierarchical state –
outside the fact that each hierarchical state we studied
is understood to be realized in a monolayer system in
the lowest Landau level –, we have argued that the even-
odd dependence on $p$ is indicative of the $\mathbb{Z}_2$ stability of
Majorana zero modes in 1D topological superconductors,
where the parity of $p$ matches the parity of Majorana zero
models per domain, according to the quantum dimension
Eq. 1.3. According to this result, the quantum dimension
of the parafermions depends both on the bulk Abelian
topological order via the total quantum dimension $D_{m,p}$
of the hierarchical Abelian bulk phase, as well as on the
Majorana modes stabilized by conservation of fermion
parity in 1D gapped fermionic phases of matter. 57–59

We close by pointing to a relation between the non-zero
charge condensate at the interface of hierarchical FQH
states and the entanglement entropy associated with an
entanglement cut across the interface. 60–62 In that re-
gard, it is possible to show that the interactions giving
rise to the charge condensate in the hierarchical states are
invariant under a global $\mathbb{Z}_k \times \mathbb{Z}_k$ symmetry [where $k = p
(= p/2)$ for odd (even) values of $p$], which correspond to
transformations on the local operators having support on
each side of the interface. It was shown in Ref. 64 that
the existence of local gap opening interactions possessing
such discrete symmetry gives rise to a non-trivial log ($k$)
correction to the bulk universal value of the topological
entanglement entropy, which, in non-chiral bulk Abelian
phases, characterize the onset of a 1D gapped SPT chain
along the interface. (See also Ref. 65 for an early discus-
sion of entanglement corrections in 2D Abelian phases
of matter and Ref. 66 for a relationship between such
entanglement corrections and string order parameters.)
We stress, however, that hierarchical states studied here
are chiral phases, which implies that the parafermion zero
modes are not protected by the emergent $\mathbb{Z}_k \times \mathbb{Z}_k$ sym-
metry of the local interactions that stabilize the condensate.
Nevertheless, we note that the local interactions we have
discussed here by no means exhaust the possible classes
of gapped interfaces that can be formed in hierarchical
states. It is then an important open question whether
such hierarchical interfaces can support also genuine 1D
SPT phases of matter protected by other classes of dis-
crete symmetries, as recently discussed in Ref. 64.

In summary, leveraging on the anyon condensation
mechanism that gives rise to the hierarchical sequence
of Abelian FQH states, we have discovered a remarkably
rich sequence of non-Abelian parafermions that are sta-
bilized by clustered states of electrons and quasiparticles
on their interfaces. Our study opens an exciting possibil-
ity to gain a deeper understanding of the structure of 2D
non-Abelian states by exploring more familiar and well
understood Abelian phases of matter.

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