Views of the Chiral Magnetic Effect

Kenji Fukushima

Abstract  My personal views of the Chiral Magnetic Effect are presented, which starts with a story about how we came up with the electric-current formula and continues to unsettled subtleties in the formula. There are desirable features in the formula of the Chiral Magnetic Effect but some considerations would lead us to even more questions than elucidations. The interpretation of the produced current is indeed very non-trivial and it involves a lot of confusions that have not been resolved.

1 Introduction – Discovery of the Chiral Magnetic Effect

The Chiral Magnetic Effect (CME) is concisely summarized in the following handy formula;

\[ j = N_c \sum_f \frac{q_f^2 \mu_5}{2\pi^2} B, \]  

which represents an electric current associated with the non-zero chirality and the external magnetic field \( B \). Here \( N_c \) stands for the number of colors in quantum chromodynamics (QCD) and \( q_f \) represents the electric charge carried by the quark flavor \( f \) where \( f \) runs over up, down, strange, etc. Equation (1) looks simple, but the physical meaning of this CME current is far from simple. Let me begin with telling some historical remarks on the discovery of the CME-current formula, hoping that it may be instructive and even inspiring to some readers.

When we, Harmen Warringa, Dima Kharzeev, and I, started working on the computation of \( j \), we had no a priori idea about the final answer, hence we did not really expect that the final result should be such elegant and beautiful. For several years

Kenji Fukushima
Department of Physics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama-shi, Kanagawa 223-8522, JAPAN, e-mail: fuku@rk.phys.keio.ac.jp
Harmen and Dima had been working on the implication of axion physics in the context of the relativistic heavy-ion collision [I will come back to the relevance of the CME to axion physics later.] At that time, around the year of 2007, I was thinking of a different (but related) physics problem, i.e. color-superconducting states in a strong $B$ inspired by a pioneering work \cite{8}. Harmen and I just chatted in the corridor of the RIKEN BNL Research Center (RBRC) about $B$-effects on color superconductivity, which was soon promoted to intriguing discussions, and a fruitful collaboration after all. Nearly simultaneously with the successful completion of our project on color superconductivity in $B$ \cite{16} (see also Ref. \cite{34} for an accidental coincidence of the research interest with our Ref. \cite{16}), a monumental paper by Harmen, Dima, and Larry McLerran appeared \cite{28}. While we were finalizing the color-superconductivity paper (or struggling with referees, probably), Harmen excitedly explained the idea of the Chiral Magnetic Effect to me. Also, I clearly remember that Larry came over mischievously (as always) to ask about the strength of my $B$ in the neutron-star environment ($eB \sim 10^{15}$ gauss at most on the magnetar surface). As compared to their $B$ produced in the relativistic heavy-ion collision where $eB \sim 10^{20}$ gauss could be reached, mine was only negligible...

Indeed, historically speaking, the recognition of such $B$ as strong as the QCD energy scale $\Lambda_{\text{QCD}}$ in realistic circumstances was an important turning point to get the $B$-physics research into gear. In other words, physics researches at $eB \sim \Lambda_{\text{QCD}}^2$ have come to make pragmatic sense rather than purely academic one since this turning point in 2007. There was really a tremendous change in the attitude of researchers.

One year later, Harmen invited me to his continued project with Dima on the Chiral Magnetic Effect. In their first paper the formula was given in a different style from what is known today, namely, it was not the current but the charge separation $Q$ expressed as \cite{28}

$$Q = 2Q_w \sum_f |q_f \gamma(2|q_f \Phi)|.$$  

(2)

Here $Q_w$ is the topological charge (i.e. counter part of $\mu_5$ in Eq. (1)) and $\gamma(x)$ is a function dependent on the microscopic dynamics of quark matter. According to the analysis in Ref. \cite{28} one can approximate $\gamma(x)$ by a simple function; $\gamma(x \leq 1) = x, \gamma(x \geq 1) = 1$. This means that, if the magnetic flux per unit topological domain, $\Phi$, is large enough, $Q \approx 2Q_w \sum_f |q_f|$. This result is naturally understood from the index theorem, i.e. $2Q_w = N_5 = N_l - N_r$. Under such strong $B$, all the spin directions should completely align in parallel with $B$, and thus the momentum directions are uniquely determined in accord with the chirality. All produced chirality should contribute to the charge separation, leading to $Q \approx N_5 \sum_f |q_f|$ that is nothing but $2Q_w \sum_f |q_f|$. In the weak field case, on the other hand, $Q \approx 4\Phi Q_w \sum_f q_f^2$ was the theoretical estimate.

Equation (2) is as a meaningful formula as Eq. (1), but the determination of $\gamma(x)$ requires some assumptions. Besides, since the formula involves $Q_w$, it is unavoidable to think of topologically non-trivial gauge configurations. As a matter of fact, Harmen and I once tried to compute $Q$ concretely on top of the real-time topological configuration, namely, the Lüscher-Schechter classical solution \cite{31-34}, which turned out to be too complicated to be of any practical use. Then, Harmen hit on
a brilliant idea to deal with $Q_w$, or strictly speaking, an idea to skirt around $Q_w$. [He invented another nice trick later to treat $Q_w$ more directly. I will come to this point later.] The crucial point is the following; it is not the topological charge $Q_w$ but the chirality $N_5$ that causes the charge separation. It is tough to think of $Q_w$, then what about starting with $N_5$ not caring too much about its microscopic origin? If one wants to fix a value of some number, one should introduce a chemical potential conjugate to the number. In this case of $N_5$, the necessary ingredient is the chiral chemical potential $\mu_5$ that couples the chiral-charge operator $\bar{\psi} \gamma^5 \gamma^5 \psi$. In my opinion the introduction of $\mu_5$ was a simple and great step to make the CME transparent to everybody. In this way the CME has eventually gotten equipped with enough simplicity and clarity.

The remaining task was to answer the following question; what is $j$ in a system with both $\mu_5$ and $B$? Harmen and I were first going to calculate the expectation value of the current operator $\bar{\psi} \gamma^\mu \psi$ directly (see the derivation A in Ref. [11]). To this end we had to solve the Dirac equation in the presence of $\mu_5$ and $B$ to construct the propagator. Now I am very familiar with the way how to do this explicitly, but when we started working on this project, we had not had enough expertise yet, apart from some straightforward calculations in color superconductivity. Some years later Harmen, Dima, and I wrote a paper in which we reported the diagrammatic method to derive Eq. (1) (see Appendix A in Ref. [12]). Let me briefly explain this derivation here; the electric current in the $z$-axis direction is written in terms of the propagator as

$$j_z = N_c \sum_f \frac{q_f |q_f B|}{2\pi^2} \sum_n \int_T^0 \frac{dp_0}{2\pi} \int \frac{dp_z}{2\pi} \int dx \left[ \frac{i}{\tilde{p} \gamma^\mu + \mu_5 \gamma^5 \gamma^5 - M_f} P_n(x) \right],$$

(3)

where the $p_0$-integration is either at $T = 0$ or the Matsubara sum at $T \neq 0$. If we choose the gauge as $A_0 = A_x = A_z = 0$ and $A_y = Bx$, the tilde momentum in the denominator is $\tilde{p} = (p_0, 0, -\text{sgn}(qB) \sqrt{2qB} n, p_z)$. We do not need the explicit form of the Landau wave-functions $P_n(x)$ that take a $4 \times 4$ matrix structure in Dirac space. Because we are interested in $j \parallel B$ here, $\gamma^\mu$ commutes with $P_n(x)$ and thus we need only $P_n(x)^2$ which equals 1 for $n > 0$ and $(1 + i\text{sgn}(qB) \gamma^0 \gamma^5) / 2$ for $n = 0$. After some calculations one can confirm that Eq. (3) is reduced to Eq. (1) regardless of the temperature $T$ and the flavor-dependent mass $M_f$. Let me make a comment on this rather naïve calculation. In most cases the proper-time method is the best way to proceed in theoretical calculations [42, 22] and the above form of the quark propagator is not widely known. For the purpose of calculating a finite quantity like the CME current, I would like to stress that the above quark propagator should be equally useful. Actually it is almost obvious in Eq. (3) that any contributions from the Landau non-zero modes are vanishing and the current arises from the Landau zero-mode only.

Coming back to the story of our first attempt to discover $j$, I remember that Harmen and I came to the office and brought different answers every morning and had
the hottest discussions all the day. It took us a few days until we eventually convinced ourselves to arrive at the right answer. Later on, Harmen had great efforts to dig out several independent derivations of Eq. (1) while preparing for our paper. Among various derivations we first found the one based on the thermodynamic potential (i.e. the derivation C in Ref. [11]). Because this calculation plays some role in later discussions on the physical interpretation of the CME current, let us take a closer look at the detailed derivation using the thermodynamic potential.

The most essential ingredient is the quasi-particle energy dispersion relation in the presence of $B$ and $\mu_s$. For $B$ along the $z$-axis, one can solve the Dirac equation

$$\omega^2_{p.s} = \left[ (p_z^2 + 2|q_f B|n)^{1/2} + \text{sgn}(p_z) s \mu_s \right]^2 + M_f^2,$$

where $s$ is the spin, $q_f$ and $M_f$ are the electric charge and the mass of quark flavor $f$. Once the one-particle energy is given, one can immediately write the thermodynamic potential down as

$$\Omega = N_c \sum_f \frac{|q_f B|}{2\pi} \sum_{s=\pm} \sum_n \alpha_{n,s} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \left[ \omega_{p,s} + T \sum_{\pm} \ln \left( 1 + e^{-\left( \omega_{p,s} \pm \mu_s / T \right)} \right) \right]$$

at finite temperature $T$ and quark chemical potential $\mu$. The spin factor, $\alpha_{n,s}$, is defined as $\alpha_{n,s} = 1$ ($n > 0$), $\delta_{s+}$ ($n = 0, q_f B > 0$), $\delta_{s-}$ ($n = 0, q_f B < 0$). This factor is necessary to take care of the fact that the Landau zero-mode ($n = 0$) exists for one spin state only. The current $j_z$ is obtained by differentiating $\Omega$ with respect to the vector potential $A_z$. Because the vector potential in the matter sector resides only through the covariant derivative, the following replacement is possible inside of the $p_z$-integration,

$$\frac{\partial}{\partial A_z} = \frac{q}{dp_z}.$$  

The combination of this derivative and the $p_z$-integration ends up with the surface terms. It is the characteristic feature of the quantum anomaly that a finite answer results from the ultraviolet edges in the momentum integration. That is, the CME current reads,

$$j_z = N_c \sum_f \frac{|q_f B|}{4\pi^2} \left[ q_f (p_z = \Lambda) - q_f (p_z = -\Lambda) \right]$$

$$= N_c \sum_f \frac{|q_f B|}{2\pi^2} \left[ \omega_{p,\pm} (p_z = \Lambda) - \omega_{p,\pm} (p_z = -\Lambda) \right]$$

$$= N_c \sum_f \frac{|q_f B|}{4\pi^2} \left[ (\Lambda \pm \mu_s) - (\Lambda \mp \mu_s) \right] = N_c \sum_f \frac{q_f^2 \mu_s}{2\pi^2} B.$$  

Here, in the second and the third lines, $\pm$ appears from the Landau zero-mode allowed by $\alpha_{n,s}$, i.e. $\pm$ amounts to $\text{sgn}(q_f B)$ which cancels the modulus of $|q_f B|$,
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and the matter part drops off for infinitely large $\omega_{p_z}(p_z = \pm \Lambda)$. It would be just a several-line calculation to make sure that Eq. (3) is equivalent with Eq. (7) and they are calculations at the one-loop level. It is also a common character of the quantum anomaly that the one-loop calculation would often give the full quantum answer. Although I do not know any explicit check of the higher-order loop effects, the above method at the one-loop level is my favorite derivation of Eq. (1), all the calculation procedures are so elementary and transparent.

2 Chiral Separation Effect

Soon later, Harmen and I found that a very similar topological current had been discovered in the neutron-star environment, that is, the axial current associated with the quark chemical potential $\mu$ and the magnetic field $B$:

$$j_S = N_c \sum_f \frac{q_f^2 \mu}{2\pi^2} B.$$

This is a chiral dual version of Eq. (1). Nowadays people call Eq. (8) the Chiral Separation Effect (CSE) in contrast to Eq. (1) referred to as the Chiral Magnetic Effect. When we learned the fact that Eq. (8) had been known earlier, our excitement got cooled down a bit. Also, three years later, we came to know that the CME formula had been discovered further earlier. Now there is a consensus in the community that the CME formula (1) was first derived by Alex Vilenkin [47]. It was an embarrassment for me to have overlooked his work until he brought our attention to his old papers. In fact an equivalent of Eq. (1) has been rediscovered over and over again [20, 19, 1] and I would not be surprised even if Eq. (1) is still buried in further unknown works. [I am not talking about the recent activities to derive Eq. (1) from a deeper insight into physics such as Berry’s curvature [45, 50], hydro or kinetic approaches [25, 18, 46, 24], and so on, which really deserve more investigations.]

The derivation of Eq. (8) is worth discussing here. The topological effects in quantum electrodynamics (QED) from $N_c \times N_f$ quarks add terms in the action as

$$\delta S = \int d^4 x \theta(x) \left[ \partial_\mu j_5^\mu(x) + N_c \sum_f \frac{q_f^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) \right],$$

associated with an axial rotation by $\theta(x)$. In this way we see that the axial current is not conserved but anomalous. With the replacement of $A_0 = \mu$ and $\epsilon^{0ijk} \partial_i A_k = B^i$, one can transform this expression using the integration by parts into

$$\delta S = \int d^4 x \partial_\mu \theta(x) \left[ -j_5^\mu(x) - N_c \sum_f \frac{q_f^2}{2\pi^2} \epsilon^{0ijk} A_0(x) \partial_j A_k(x) \right]$$
\[
\int d^4x \partial_1 \theta(x) \left[ -j_5^2(x) + N_c \sum_f \frac{q_f^2}{2\pi^2} \mu B'(x) \right],
\]
from which Eq. (3) immediately follows. This derivation also tells us that the \(B\)-induced current in the right-hand side of Eq. (3) is nothing but a part of the Chern-Simons current \(\sim \epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma\) in QED. It should be mentioned that the derivation presented above is a little bit cooked up by me for the illustration purpose and one should refer to the original paper [32] for more careful treatments of the surface integral.

Before going on our discussions, let me point out that the above derivation implicitly assumes massless quarks. If quarks are massive, Eq. (9) should be modified with an additional term \(2iM_f \langle \bar{\psi}_f \gamma^5 \psi_f \rangle\). This modification would be harmless as long as the pseudo-scalar condensate is vanishing, but in principle, Eq. (8) could be dependent on \(M_f\) unlike Eq. (1) as argued explicitly in Ref. [32]. In fact it is quite subtle whether Eq. (8) is sensitive to \(M_f\) or not, and I will address this question in an explicit way soon later.

It would be an interesting question how to derive Eq. (8) microscopically just like the ways addressed in the previous section. In fact I have once tried to prove Eq. (8) based on the thermodynamic potential by inserting an axial gauge field. There must be a way along this line, but I could not solve it (or I would say that I did not have enough time to find it out...). Instead, here, let me introduce another derivation based on the propagator as in Eq. (3).

The axial current is expressed as
\[
j^A_z = N_c \sum_f \frac{q_f |q_f B|}{2\pi} \sum_n \int_T^T \frac{dp_0}{2\pi} \int \frac{dp_z}{2\pi} \int dx \frac{1}{L_x} \times \text{tr} \left[ \gamma^z \gamma^5 \tilde{P}_n(x) \frac{i}{\bar{P} \gamma^\mu + \mu \gamma^0 - M_f} P_n(x) \right]
\]
at finite quark chemical potential \(\mu\). It is easy to see that any contributions from \(n \neq 0\) vanish due to the Dirac trace. Only the Landau zero-mode produces a term involving \(\gamma^z \gamma^5\) which makes \(\text{tr}(\gamma^0 \gamma^\mu \gamma^\nu \gamma^e) = -4i \neq 0\). Then, the above expression simplifies as
\[
j^A_z = -N_c \sum_f \frac{q_f^2 B}{2\pi} \int_T^T \frac{dp_0}{2\pi} \int \frac{dp_z}{2\pi} \text{tr} \left[ \gamma^z \gamma^5 \frac{\bar{P} \gamma^\mu + \mu \gamma^0 + M_f}{(p_0 + \mu)^2 - p_z^2 - M_f^2} \gamma^\nu \gamma^e \right]
\]
\[
= 4iN_c \sum_f \frac{q_f^2 B}{2\pi} \int_T^T \frac{dp_0}{2\pi} \int \frac{dp_z}{2\pi} \frac{p_0 + \mu}{(p_0 + \mu)^2 - p_z^2 - M_f^2}
\]
\[
= N_c \sum_f \frac{q_f^2 B}{2\pi} \frac{\partial Z(\mu)}{\partial \mu},
\]
where \(Z(\mu)\) denotes the partition function at finite density in (1+1)-dimensional theory (as a result of the dimensional reduction with the Landau zero-mode), and
thus the $\mu$-derivative leads to the quark density $n$. In the second line we used

$$2(p_0 + \mu)/[(p_0 + \mu)^2 - p_z^2 - M_f^2] = (\partial/\partial \mu)\ln[(p_0 + \mu)^2 - p_z^2 - M_f^2].$$

One might have thought that it is a simple exercise to evaluate $Z(\mu)$ with the (1+1)-dimensional integration. The fact is, however, that the finite-$\mu$ system in (1+1) dimensions is by no means simple.

In Ref. [32] one can find exactly the same expression as above in a slightly different calculation and the density is written as (see Eq. (37) in Ref. [32]),

$$n_f(T, \mu) = \int \frac{dp_z}{2\pi} \frac{1}{e^{(\omega_f - \mu)/T} + 1} - \frac{1}{e^{(\omega_f + \mu)/T} + 1}$$ (13)

with $\omega_f = \sqrt{p_z^2 + M_f^2}$. This result is certainly $M_f$-dependent as suggested in the paragraph below Eq. (10), and this would make a sharp contrast to the CME current (1).

We know, however, that the density in the (1+1)-dimensional fermionic theory arises from the anomaly [41] and the density (13) is not the right answer. In fact, in view of the second line of Eq. (12), it seems at a glance that the $\mu$-dependence could be absorbed in the $p_0$-integration, which already gives us an impression that something non-natural should be happening. To see this, let us take one-step back to the microscopic expression, i.e., the (1+1)-dimensional partition function reads,

$$Z = i \int d\rho_0 \int \frac{dp_z}{2\pi} \frac{1}{e^{\rho_0 + \mu} - e^{\rho_0 - \mu}}$$ (14)

from which the $\mu$-dependence could be eliminated by the chiral rotation (for the zero-mode basis only),

$$\psi_0 = e^{i\gamma^\mu p_z} \psi_0^\prime$$ (15)

leading to (here, we shall show results at $T = 0$ for simplicity, but nothing is changed even at finite $T$),

$$Z = i \int d\rho_0 \int \frac{dp_z}{2\pi} \text{tr}[e^{i\gamma^\mu p_z} (\gamma^\rho (i\partial_0 + \mu) - \gamma^\rho i\partial_z - M_f)] e^{i\gamma^\mu p_z}$$

$$= i \int d\rho_0 \int \frac{dp_z}{2\pi} \text{tr}[\gamma^\rho i\partial_0 - \gamma^\rho i\partial_z - M_f]$$

$$= \int_{\Lambda - \mu}^{\Lambda + \mu} \frac{dp_z}{2\pi} \frac{1}{2} \partial_f + \int_{-\Lambda - \mu}^{-\Lambda + \mu} \frac{dp_z}{2\pi} \frac{1}{2} \partial_f$$ (16)

with $\partial_f = \sqrt{p_z^2 + |M_f|^2}$, where $M_f = M_f e^{2i\gamma^\mu p_z}$ is the chirally tilted mass. The momentum integration is shifted according to the chiral rotation (15). The first (second) integral corresponds to the particle (anti-particle, respectively) contribution. Thus, one can extract the $\mu$-dependent piece from the surface terms as follows;
\[ Z = \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi} \hat{\phi}_f + \left( \int_{-\Lambda}^{\Lambda+\mu} + \int_{-\Lambda}^{\Lambda-\mu} \right) \frac{dp}{2\pi} \hat{\phi}_f \]
\[ = \mu^2 \frac{d^2}{dx^2} \int_{-\Lambda}^{\Lambda+\mu} \frac{dp}{2\pi} \hat{\phi}_f + (\mu\text{-independent terms}) \]
\[ = \frac{\mu^2}{2\pi} + (\mu\text{-independent terms}), \tag{17} \]

which results in the density \( n_f = \mu / \pi \) that is independent of \( M_f \). It is clear from the second line of the above calculation that the density originates from the ultraviolet edges, which is the characteristic feature of the anomaly. The full quantum answer is then given as

\[ n_f = \frac{\partial Z(\mu)}{\partial \mu} = \frac{\mu}{\pi} \Rightarrow \begin{cases} \begin{aligned} j_z^A &= N_c \sum_f q_f^2 \mu, \\ \text{(18)} \end{aligned} \end{cases} \]

Equivalently, if one is interested in deriving the same answer from Eq. (12) directly, one should split the composite operator as \( \bar{\psi}(x) \gamma^5 \psi(x) \rightarrow \bar{\psi}(x + \epsilon) \gamma^5 \psi(x) \) and insert the infinitesimal gauge connection from \( x \) to \( x + \epsilon \). Interestingly, contrary to Ref. \[32\], the Chiral Separation Effect \( (8) \) is presumably insensitive to the quark mass just like the Chiral Magnetic Effect \( (1) \). Whether Eq. (8) is robust or not regardless of \( M_f \) is an important question particularly in the context of the Chiral Magnetic Wave (CMW) \[30\]. The anomalous nature of the density \( (18) \) implies that the CMW can exist also in the chiral-symmetry breaking phase where quarks acquire substantial mass dynamically.

I would not insist that I could prove the non-renormalization of Eq. (8) since the above is just a one-loop perturbation and non-perturbative interactions may change the story; I would like to thank Igor Shovkovy for raising this unanswerable but unforgettable question. The interested readers may consult Refs. \[21, 15\] for some examples of non-non-renormalization. Anyway, I can at least say with confidence that, if \( B \) is super-strong, the reduction to the \( (1+1) \)-dimensional system should be strict, and then Eq. (13) must be altered, conceivably as \( n_f = \mu / \pi \). \[9\].

Although the interpretation of Eq. (15) may swerve a bit from our main stream, I would emphasize that Eq. (15) is extremely interesting and it would be definitely worth revisiting its profound meaning. Actually, Eq. (15) has an impact on the structure of the QCD vacuum. Let us consider the hadronic phase with spontaneous breakdown of chiral symmetry. After the rotation \( (15) \), apart from the anomalous term \( \mu^2 / (2\pi) \), the system is reduced to that at zero density, which means that \( \chi = \langle \bar{\psi}_0 \psi_0 \rangle \) should take a finite value. Therefore, in terms of the original fields \( \psi_0 \), the chiral condensates form a spiral structure,

\[ \langle \bar{\psi}_0 \psi_0 \rangle = \chi \cos(2\mu z), \quad \langle \bar{\psi}_0 \gamma^5 \psi_0 \rangle = \chi \sin(2\mu z), \tag{19} \]

which is called the chiral spiral or the dual chiral-density wave (if \( \gamma^5 \) is involved) \[7\] \[33\]. In particular, if the above type of the inhomogeneous ground state is caused by \( B \), it is sometimes called the chiral magnetic spiral \[5\].
I should emphasize that Eq. (18) does not really require the dimensional reduction, while the chiral magnetic spiral needs the pseudo (1+1)-dimensional nature under sufficiently strong $B$. This point might be a bit puzzling. As long as $j^i$ is concerned, only the Landau zero-mode remains non-vanishingly for any $B$, and the momentum integration is purely (1+1)-dimensional. The chiral condensate is, however, not spiral but homogeneous for small $B$ because of contributions from all non-zero Landau levels. That is, the genuine chiral condensate is $\langle \bar{\psi} \psi \rangle = \sum_n \langle \bar{\psi}_n \psi_n \rangle$, among which only the Landau zero-mode has a special structure as in Eq. (19). I would conjecture, hence, that there is no sharp phase transition from the homogeneous chiral condensate at $B = 0$ to the chiral magnetic spiral at $B \neq 0$, but it may be possible that the inhomogeneous zero-mode contribution gradually develops, which exhibits a smooth crossover to the chiral spirals with increasing $B$.

3 What is the chiral chemical potential?

Equation (8) is very similar to the CME current (1), so that one might have thought at a first glance that Eq. (1) emerges trivially from the insertion of $\gamma^5$ in both sides of Eq. (8). The relation between Eqs. (1) and (8) is not such simple, though. As a matter of fact, this point was a major source of confusions about the validity of Eq. (1). One can readily extend the field-theoretical derivation of Eq. (8) using Eq. (9) in order to obtain Eq. (1) by introducing the axial vector fields $A_5^R$ or the chiral gauge fields, $A_R = (A_\mu + A_5^R)/2$ and $A_L = (A_\mu - A_5^L)/2$. Then, in the same manner as in the previous section, one can formulate the counterpart of Eq. (9) associated with a vector rotation by $\beta(x)$, that is,

$$\delta S = \int d^4 x \beta(x) \left[ \partial_\mu j^\mu(x) + N_c \sum_f q_f^2 \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^R(x) F_{\rho\sigma}^R(x) 
+ N_c \sum_f q_f^2 \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^L(x) F_{\rho\sigma}^L(x) \right].$$

(20)

This leads to $-j^i - N_c \sum_f (q_f^2/2\pi^2) \epsilon^{ijk}(A_5^R - A_5^L) \partial_j A_k = 0$ just as in Eq. (10), and this is nothing but Eq. (1) after the identification of $A_5^R$ as $\mu_5$ (see the derivation D in Ref. [11]). Although the derivation may look flawless, it triggered suspicious views of Eq. (1), which was first addressed by Toni Rebhan, Andreas Schmitt, and Stefan Stricker using the Sakai-Sugimoto model [38]. It should be noted that the CME current had been exactly reproduced in the holographic models [49].

Obviously, one has to deal with the chiral gauge theory with both $A_R$ and $A_L$ to introduce $\mu_5$ in the above way, and it is well-known that the anomaly in the chiral gauge theory has a more complicated structure than that in the vector gauge theory. Roughly speaking, the anomaly is a consequence from the inconsistency between chiral invariance and gauge symmetry. In the vector gauge theory, usually, the vector current is strictly conserved due to adherence to gauge symmetry, and the anomaly
is seen in the axial vector channel only (see Eq. (9)). In the case in the chiral gauge theory, however, there is no such strict demand from the theory and it should be prescription dependent how the anomaly may appear in the vector and the axial vector currents. Indeed we can clearly see from Eq. (20) that the vector current is also anomalous. There are two representative results known as the covariant anomaly and the consistent anomaly, and they can coincide only when the anomaly cancellation holds, as is the case in the Standard Model. The authors of Ref. [38] claimed that the vector current should be free from the anomaly and the theory should accommodate the Bardeen counter-terms to cancel the anomalous terms in Eq. (20). Then, needless to say, the CME current is vanishing!

This argument scared Harmen and me very much. In 2009 when Ref. [38] came out, Harmen was a postdoc in Frankfurt and I was also there as a visitor. Harmen’s face is always very white, but he got even more whity, and we had a lot of discussions on Ref. [38] in Frankfurt with a fear that we might have made a big steaming mistake... At that time, neither Harmen nor I was 100% confident in Eq. (1) (maybe Dima was?), and the necessity of the Bardeen counter-terms sounded plausible. This puzzle was one of the issues discussed in a RBRC workshop, “P- and CP-odd Effects in Hot and Dense Matter” in May, 2010. One of the invited participants, Valery Rubakov, wrote a note to clarify this issue based on the discussions in the workshop [39]. The essence in his argument is the following. If one introduces \( \mu_5 \) as the zeroth component of the axial gauge field, the CME current is gone indeed. However, QCD and QED are not the chiral gauge theory. One should then introduce \( \mu_5 \) in a different way as a conjugate to the Chern-Simons charge. Therefore, instead of adding a term \( \mu_5 \bar{\psi} \gamma^0 \gamma^5 \psi \) in a form of the covariant derivative in the Lagrangian, one should think of the Chern-Simons current \( K^\mu \) which is deduced from

\[
N_c \sum_f \frac{q_f^2}{16\pi^2} e^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x) = \partial_\mu \left[ N_c \sum_f \frac{q_f^2}{4\pi^2} e^{\mu\nu\rho\sigma} A_\nu(x) \partial_\rho A_\sigma(x) \right] = \partial_\mu K^\mu(x)
\]

in the QED sector. The term to be added in the Lagrangian is,

\[
S_{cs} = - \int d^4x \mu_5 K^0(x) = \partial_0 \left[ -N_c \sum_f \frac{q_f^2}{4\pi^2} \int d^4x e^{ijjk} A_i(x) \partial_j A_k(x) \right],
\]

from which Eq. (1) immediately follows as a result of the derivative, \( j^i = \delta S_{cs} / \delta A_i(x) \). One may worry about gauge invariance in the above prescription. It would be then more convenient to rewrite \( S_{cs} \) in the following way after the integration by parts, that is manifestly gauge invariant,

\[
S_{cs} = \int d^4x \theta(t) N_c \sum_f \frac{q_f^2}{16\pi^2} e^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x),
\]

where \( \partial_0 \theta(t) = \mu_5 \). In other words, we can say that \( \mu_5 \) is the time derivative of the \( \theta \) angle in the QED sector, which was pointed out already in Ref. [11] and the idea of the charge separation driven by inhomogeneous \( \theta \) can be traced back to Ref. [26].
A subsequent question naturally arises; what happens if \( \theta(x) \) has not only temporal but also spatial dependence in general? The Chern-Simons-Maxwell theory with general \( \theta(x) \) provides us with the following modified Maxwell equations:

\[
\nabla \cdot \mathbf{E} = \rho + N_c \sum_{f} \frac{q_f^2}{2\pi^2} (\nabla \theta) \cdot \mathbf{B} ,
\]

\[
\nabla \times \mathbf{B} - \partial_0 \mathbf{E} = j + N_c \sum_{f} \frac{q_f^2}{2\pi^2} \left[ (\partial_0 \theta) \mathbf{B} - (\nabla \theta) \times \mathbf{E} \right] ,
\]

and Faraday’s law and Gauß’s law are not altered. We see that the CME current appears in the right-hand side of Eq. (25) as if it is a part of the external current. In this manner we can conclude from Eq. (24) that an electric-charge density is induced by spatially inhomogeneous \( \theta(x) \) in the presence of \( \mathbf{B} \). To the best of my knowledge Eqs. (24) and (25) are the quickest derivation of the Chiral Magnetic Effect, as discussed first in Ref. [27].

[After I finished writing this article, I was informed by Toni, one of the authors of Ref. [38], that the confusion about the CME in the holographic context seems to continue. I am not able enough to make any judgment here, and those who want to dive into this confusion can consult the recent analysis in Ref. [3].]

4 What really flows?

To tell the truth, I have never gotten any satisfactory answer to the following question; what really flows? I have had various discussions with people who have various backgrounds, but those discussions ended up with more confusions than before. Thanks to useful conversations, nevertheless, my eyes have been open to various views of Eq. (1). People (including me) say that the CME current is an electric current induced by \( \mathbf{B} \) just like Ohm’s law with the electric field \( \mathbf{E} \). Let me begin by suspecting this interpretation that people just take for granted.

In classical electrodynamics Eq. (25) is usually written in a slightly different way, i.e.,

\[
\nabla \times \mathbf{B} = j + \partial_0 \mathbf{E} + N_c \sum_{f} \frac{q_f^2}{2\pi^2} \left[ (\partial_0 \theta) \mathbf{B} - (\nabla \theta) \times \mathbf{E} \right] ,
\]

and \( \partial_0 \mathbf{E} \) is called the displacement current. We see that the CME current should be a genuine current if \( \partial_0 \mathbf{E} \) can be regarded as a real electric current, for they enter Ampère’s law on equal footing. In other words, if the displacement current is not a real current, the CME current is not, either. Now, we know from our experience that \( \partial_0 \mathbf{E} \) is only the time derivative of the electric field and no electric charge flows associated with the displacement current. The displacement current certainly plays the equivalent role as \( j \) as a source to create \( \mathbf{B} \), but it is clear that there is no movement of electric charge at all. It would be therefore a legitimate claim to insist that the
charge separation from the CME current might be an illusion. I would emphasize the importance to distinguish the current and the charge in the argument here. For example, the most well-known example of the displacement current is the problem of the capacitor that is composed of two separate conductors. Let a capacitor be connected to the wire with finite electric current. More and more electric charge accumulates on the conductors and produces stronger and stronger electric field inside as the time goes. Then, even though two conductors are physically separate and no electric current flows between them, the displacement current flows as if the electric current flowed along the wire without the capacitor. The distribution of the electric charge stored on the conductors is, however, totally different depending on the situation with and without the capacitor. In this sense, thus, the charge itself may not flow and the charge separation may not occur with the CME current also.

A related criticism against the CME current is that the current computed in Eq. (7) for example is the expectation value of the current operator, \( \bar{\psi} \gamma^\mu \psi \), and it is not necessarily the current. In fact, there are some studies on the Chiral Magnetic Effect in the lattice gauge theory; the correlation functions of the chirality and the current were measured in Ref. [6], and later Eq. (1) was checked directly on the lattice [48]. It is not so straightforward, however, to interpret these lattice results properly. A system with a finite electric current could be steady but is out of equilibrium. What one can calculate in the thermal system in equilibrium like the situation of the lattice simulation in Euclidean space-time is the electric-current conductivity according to the Kubo formula. It is a tricky question what \( \langle \bar{\psi} \gamma^\mu \psi \rangle \) really represents in the lattice simulation. Let me take one example for concreteness. If the system has a condensate of the omega meson, \( \omega^\mu \), the interpolation field of \( \omega^\mu \) is \( \sim \bar{\psi} \gamma^\mu \psi \) and then \( j^\mu = \langle \bar{\psi} \gamma^\mu \psi \rangle \neq 0 \), but this does not necessarily mean that the system has a persistent current. To make this point clearer, the spin operator in terms of the Dirac matrices is \( S^z = \frac{i}{2} [\gamma^\rho, \gamma^\sigma] = \frac{i}{2} \text{diag}(\sigma^3, \sigma^3) \), so that the spin expectation value is \( S^z = \langle \bar{\psi} S^z \psi \rangle = \frac{1}{2} \langle \phi_R^\dagger \sigma^3 \phi_L \rangle + \frac{1}{2} \langle \phi_L^\dagger \sigma^3 \phi_R \rangle \), while the current expectation value is \( j^z = \langle \bar{\psi} \gamma^z \psi \rangle = \langle \phi_R^\dagger \sigma^3 \phi_R \rangle - \langle \phi_L^\dagger \sigma^3 \phi_L \rangle \), where \( \phi_L \) and \( \phi_R \) are two-component spinors in the left-handed and right-handed chirality, respectively. Here, the similarity between \( S^z \) and \( j^z \) implies that we can regard \( j^z \) as a static quantity like the spin \( S^z \), which may well be the most appropriate interpretation of the lattice measurement.

From the point of view of the theoretical treatment of the electric current, the formulation based on the linear response theory must be a good starting point. I believe that the work along this line in Ref. [29] should be one of the most important literature to think of physics of the Chiral Magnetic Effect. They computed the one-loop diagram on top of the \( \mu_5 \) background to find the chiral magnetic conductivity \( \sigma_x(\omega, p) \). The result is consistent with Eq. (11) in a particular limit; \( \sigma_x(\omega = 0, p \to 0) = \lim_{p \to 0} \sigma_x(0, p) = N_c \sum_i (q_i^2 \mu_5 / 2\pi^2) \) which correctly reproduces the CME current. In view of the result of Ref. [29], on the other hand, it seems \( \sigma_x(\omega \to 0, p = 0) = 0 \). [This latter limit is not manifestly addressed in Ref. [29].] It seems to be vanishing from Eq. (38) and Fig. 1 of Ref. [29].] This is a problem because the latter limit rather than the former one is more relevant to the real-time dynamics. The fact that the former limit (\( \omega = 0 \) first and \( p = 0 \) next) gives the CME
current (1) suggests that the CME current should be a static quantity just as measured in the lattice simulation and thus not a genuine electric current!? One may still consider that the intuitive argument leading to Eq. (2) should work anyway. My impression is also that all above-mentioned problems are just on the conceptual level (though I have no idea how to reconcile them) and in practice the CME current flows according to Eq. (1) after all. Indeed if there are almost massless quarks in a quark-gluon plasma and a strong $B$ is imposed on a topological domain, an electric current must be induced for sure. An example of the real-time calculation of the CME current with not $\mu_5$ but a topological domain is quite instructive in this sense [13]. The central innovation in Harmen’s idea (as discussed in Ref. [13]) was to mimic the topological domain by putting $E$ and $B$ parallel to each other, with which $\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \neq 0$. Then, the particle production occurs via the Schwinger mechanism and the produced particles are accelerated by the fields, and the electric current is generated. The current is time dependent and the current-generation rate can be analytically written down. In this setup the physical origin of the CME current is crystal-clear! So, if anything is fishy in physics of the Chiral Magnetic Effect, it should have something to do with technical defects of $\mu_5$ in equilibrium circumstances.

Supposing that physics of the Chiral Magnetic Effect should be robust, let us admit the CME current (1) as it is to proceed to the next question, that is actually the central question in this section; what really flows?

An intuitive explanation tells us that quarks simply flow in a quark-gluon plasma. It is, however, based on a classical picture, and such a picture misses quantum character that is indispensable for phenomena related to the quantum anomaly. Look at the derivation of the CME current in Eq. (7). If this derivation captures the underlying physics of the CME current, the origin of the current comes from quarks with infinitely large momenta. Where are such fast-moving quarks in the real quark-gluon plasma? They may spill out from the vacuum through quantum processes, but how is it possible to retrieve particles with infinite momenta? Usually the quantum anomaly involves ultraviolet regions of the momentum integration as a loop of virtual particles, meanwhile ultraviolet particles directly participate in the physical observable in the CME problem. It is very hard (at least for me) to imagine that the current generation in such a way really happens in a physical plasma. This deliberation brings me a further doubt about the static evaluation of the CME current.

A natural extension of this question about the origin of the CME current is whether it exists in the hadronic phase and, if it does, how the current appears in terms of hadronic degrees of freedom. Actually this question has been something in mind for a long time since when we published Ref. [11]. In the hadronic phase an electric current should be attributed to charged pions, but pions are insensitive to chirality and thus $\mu_5$ or the strong $\theta$ angle. One possible answer would be that there is no CME in the hadronic phase, and if so, it would be fantastic; the CME current can be a signature for quark deconfinement, as implied in Ref. [11]. I had heard that Harmen wanted to analyze the CME using the chiral perturbation theory, though he never worked it out.
Recently I have clarified what would happen in the hadronic phase and wrote a paper with one of my students, Kazuya Mameda [14]. Our conclusion was a very natural one, and a very perplexing one at the same time.

The CME current is unchanged even in the hadronic phase, which is very natural since the CME current has the anomalous origin that arises from ultraviolet fluctuations. At low energies the anomaly should be saturated by infrared degrees of freedom, which is sometimes referred to as the anomaly matching. This idea is formulated as the Wess-Zumino-Witten action and the current should be given by the derivative of the total effective action with respect to the gauge field. In this way we found that the leading-order term in the chiral Lagrangian leads to the current,

$$j^\mu_L = -i e f_\pi^2 4 \text{tr} \left[ (\Sigma^\mu - \tilde{\Sigma}^\mu) \tau^3 \right] \simeq e(\pi^- i\partial^\mu \pi^+ - \pi^+ i\partial^\mu \pi^-) + \ldots \quad (27)$$

where $\Sigma^\mu = U^\dagger \partial^\mu U$, $\tilde{\Sigma}^\mu = (\partial^\mu U) U^\dagger$, and $U = e^{i\pi^a \tau^a / f_\pi}$ are the standard notation in the chiral Lagrangian. The physical meaning of the above expression is plain as seen from the expansion in terms of the pion fields. It is a common form of the probability flow in Quantum Mechanism representing the electric current associated with the flow of the charged pions.

A more non-trivial contribution comes from the Wess-Zumino-Witten part, which leads to the current associated with the $\pi^0$ domain-wall [44], i.e.,

$$j^\mu_{WZW} = N_c \sum_f \frac{q_f}{8\pi^2 f_\pi} e^{\mu \nu \rho \sigma} (\partial_\nu \pi^0) F_\rho\sigma . \quad (28)$$

This current is very similar to the CME current (1) and $\theta(x)$ is just replaced by $\pi^0(x)/(4\pi^2 f_\pi)$. Although Eq. (28) is not the Chiral Magnetic Effect, it would give us a clue to think about the physical meaning of the CME current. Finally, the CME current appears from the so-called contact part of the Wess-Zumino-Witten action [23];

$$S_P = N_c \sum_f \frac{q_f^2}{8N_f\pi^2} e^{\mu \nu \rho \sigma} \int d^4x A_\mu(x)/(\partial_\nu A_\rho(x)) \partial_\sigma \theta(x) , \quad (29)$$

which is just equivalent to the Chern-Simons action already discussed in Eq. (22). [\theta in Eq. (29) has a different normalization by $2N_f$ by convention.] Naturally the current derived from Eq. (29) should reproduce the CME current (1). This is how one can get the CME current in the hadronic phase and my surprise lies in the fact that the pion dynamics is completely decoupled from the CME current.

Because the $\pi^0$ domain-wall looks a bit more intuitive than the mystical $\theta$ angle, we shall consider a possible interpretation of the current (28). This is certainly a current, but no charged pions, $\pi^\pm$, are involved in the formula. Then, it is as puzzling in Eq. (28) how the current can flow and what really flows.

To answer this question, let me emphasize a very useful analogue of the Josephson current in superconductivity. The Josephson junction consists of superconducting materials and a thin layer of insulator (S-I-S) or non-superconducting metal (S-)
N-S) sandwiched by them. There was a big debate about whether the super-current can flow or not through the insulating barrier. Of course, there is no Cooper pair inside of the insulator, and thereby there is nothing that takes care of the super-current. It should have been a natural attitude to get skeptical about such a current [4]. This situation, a current without current carriers, is quite reminiscent of our problem of the CME current or the current accompanied by the $\pi^0$ domain-wall. Everyone knows that the Josephson current is the experimental fact today [2]. For the Josephson current, the coherence is the most important; in superconductor the quantum state is characterized by a wave-function just like a problem in Quantum Mechanics. In the QCD case, also, such a coherent state is realized by the condensation of fields, namely, $\pi^0(x)$ in Eq. (28) is to be regarded as a macroscopic wave-function. One may then raise a question; the current of the $\pi^0$ domain-wall may be okay, but what about the CME current? There is no coherent field but only $\theta(x)$ that is not a dynamical field but just a space-time dependent parameter! This is perfectly a sensible question. To answer this, I would say that $\theta(x)$ could be promoted to the dynamical field without mentioning on a possibility of axion, at least in the hadronic phase. If the system has a pseudo-scalar (and iso-scalar) condensate such as the condensate of the $\eta^0$ meson (that forms $\eta'$ with a mixture with $\eta^\prime$), it could be mapped to $\theta(x)$ in the chiral Lagrangian approach. Once this mapping is noticed, there is no longer a big conceptual difference between the current in Eq. (28) and the CME current in Eq. (1). The analogy to the Josephson current may support the reality of the CME current, but this argument does not tell us anything about the microscopic constituent of the current yet.

Equation (28) means that the current can exist just with the $\pi^0(x)$ profile and the magnetic field, and then the only possible carrier of the current should be the quark content inside of $\pi^0$. Therefore, even in the hadronic phase, I must think that charged quarks flow to produce the electric current. Contrary to the intuition, there is no inconsistency with the notion of quark confinement. Regardless of the presence of the flow of quarks, these flowing quarks can be still confined in a big wave-function of the $\pi^0(x)$ profile. In this way, confined quarks can flow without breaking confinement because of the coherent background of the meson fields.

In reality it is next to impossible to achieve such an environment with abundant $\pi^0$ that forms a condensate to test Eq. (28) because $\pi^0$ quickly decays into photons via the anomalous QED process. This implies that the CME current may be also diminished by the photon production. Indeed, Eq. (29) exactly describes such a process of $\theta(x)$ decaying into $2\gamma$. It is interesting, besides, that one $\gamma$ can be provided from $B$ in the case with background fields. More specifically, one injected $\gamma$ and another $\gamma$ from $B$ can produce a $\theta$ (or the $\eta^0$ meson), that is nothing but the Primakoff effect [36]. The Primakoff effect has an application as a tool to detect the axion [35, 37], which is understandable from the above argument once $\theta(x)$ is augmented as a dynamical axion field. Because physics of the Chiral Magnetic Effect has a connection to axion physics through $\theta(x)$ (that was actually the very beginning of the path toward Ref. [28], as I mentioned), it should be naturally motivated to think of some application of the Primakoff effect in the context of the Chiral Magnetic Effect too. Then, the reverse process of the Primakoff effect, namely,
\( \gamma(B) + \theta \rightarrow \gamma \), should be the most relevant to the experimental opportunity. In the relativistic heavy-ion collision, the profile of the magnetic field \( B(x) \) can be estimated by the simulation, and the precise measurement of \( \gamma \) with subtraction of the background from the \( \pi^0 \) decay is available nowadays. The unknown piece in the reverse Primakoff effect is the profile of the \( \theta(x) \) distribution. Needless to say, nothing is more important and ambiguous than the concrete distribution of \( \theta(x) \) for any attempt to perform serious computation of the CME-related phenomena. This is why most of works on the Chiral Magnetic Effect address only qualitative predictions.

I think that it must be a very interesting challenge to find a condensed-matter counterpart in which the Chiral Magnetic Effect may be visible and testable experimentally. This is not an unrealistic desire; axion physics can be discussed in the so-called topological magnetic insulator [35], and why not the Chiral Magnetic Effect? In fact, recently, there are appearing some works one after another along this line.

5 My Outlook

Many people (including me) are still working on the theoretical aspects of the Chiral Magnetic Effect and its relatives such as the Chiral Separation Effect, the Chiral Magnetic Wave, etc. It is highly demanded to make some firm theoretical estimation about the experimental observables affected by the CME and related phenomena. To this end, however, one needs to know the early-time dynamics even before the formation of the quark-gluon plasma. In fact, the most interesting regime that has the greatest impact on the CME happens to be the most difficult regime to describe theoretically.

A missing theoretical link between the coherent wave-function right after the collision and the thermalized plasma is the last piece of the jigsaw puzzle. Particles should be produced from quantum fluctuations on top of coherent fields (i.e. the Color Glass Condensate; CGC), that translates into the entropy production. It is known that the coherent background fields accommodate topological flux-tubes that play a role of \( Q_w \) in Eq. (2). Produced particles inside of those flux-tubes under a strong \( B \) should have a characteristic momentum distribution and this would embody the Chiral Magnetic Effect in a quantitative way. In my opinion, thus, the early-thermalization problem must be resolved even before talking about the observational possibility of the Chiral Magnetic Effect, or they may have something to do with each other. This is because, as I stated in the previous section, the real-time dynamics of the Chiral Magnetic Effect should involve the Schwinger process of particle production, and the particle production as in the Lund string model should be responsible for the entropy production from fields, and thus thermalization ultimately.

One may claim that the complete isotropization and thermal equilibrium should be no longer required to account for the experimental observation. This is true indeed, and this is a good news for the CME, for the thermalization means that the
system should lose any memory and have only one information, i.e. the temperature. If the thermalization is incomplete, it would enhance the chance that the observed distribution of particles may still remember the early-time environment like the presence of the strong $B$ and/or the topological flux-tubes. For the purpose of testing the idea as compared to the experimental data, it is indispensable to perform some serious simulation of the early-time dynamics of the heavy-ion collisions.

Unfortunately, there is no successful simulation starting from the CGC initial condition to achieve some reasonable input for the hydrodynamics within a reasonable time scale (See Ref. [17] for a latest attempt). There are so many theoretical efforts in this direction including mine [10] and it should be definitely worth discussing them, but not here and on another occasion maybe. In this article I have discussed physics of the Chiral Magnetic Effect and presented my views on the physical interpretation. Now my story has become a bit too diverging, and I should stop here, with a hope that some readers may find my views useful for future investigations.

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References

1. Alekseev, A.Y., Cheianov, V.V., Frohlich, J.: Universality of transport properties in equilibrium, Goldstone theorem and chiral anomaly. Phys. Rev. Lett. 81, 3503–3506 (1998). DOI 10.1103/PhysRevLett.81.3503
2. Anderson, P.W., Rowell, J.M.: Probable observation of the josephson superconducting tunneling effect. Phys. Rev. Lett. 10, 230 (1963). DOI 10.1103/PhysRevLett.10.230. URL http://link.aps.org/doi/10.1103/PhysRevLett.10.230
3. Ballon-Bayona, A., Peeters, K., Zamaklar, M.: A chiral magnetic spiral in the holographic Sakai-Sugimoto model (2012)
4. Bardeen, J.: Tunneling into superconductors. Phys. Rev. Lett. 9, 147 (1962). DOI 10.1103/PhysRevLett.9.147. URL http://link.aps.org/doi/10.1103/PhysRevLett.9.147
5. Basar, G., Dunne, G.V., Kharzeev, D.E.: Chiral magnetic spiral. Phys. Rev. Lett. 104, 232,002 (2010). DOI 10.1103/PhysRevLett.104.232002
6. Buividovich, P., Chernodub, M., Luschevskaya, E., Polikarpov, M.: Numerical evidence of chiral magnetic effect in lattice gauge theory. Phys. Rev. D80, 054,503 (2009). DOI 10.1103/PhysRevD.80.054503
7. Deryagin, D., Grigoriev, D.Y., Rubakov, V.: Standing wave ground state in high density, zero temperature QCD at large N(c). Int. J. Mod. Phys. A7, 659–681 (1992). DOI 10.1142/S0217751X92000302
8. Ferrer, E.J., de la Incera, V., Manuel, C.: Magnetic color flavor locking phase in high density QCD. Phys. Rev. Lett. 95, 152,002 (2005). DOI 10.1103/PhysRevLett.95.152002
9. Fukushima, K.: QCD matter in extreme environments. J. Phys. G39, 013,101 (2012). DOI 10.1088/0954-3899/39/1/013101
10. Fukushima, K., Gelis, F.: The evolving Glasma. Nucl. Phys. A874, 108–129 (2012). DOI 10.1016/j.nuclphysa.2011.11.003
11. Fukushima, K., Kharzeev, D.E., Warringa, H.J.: The Chiral magnetic effect. Phys. Rev. D78, 074,033 (2008). DOI 10.1103/PhysRevD.78.074033
12. Fukushima, K., Kharzeev, D.E., Warringa, H.J.: Electric-current susceptibility and the chiral magnetic effect. Nucl. Phys. A836, 311–336 (2010). DOI 10.1016/j.nuclphysa.2010.02.003
13. Fukushima, K., Kharzeev, D.E., Warringa, H.J.: Real-time dynamics of the Chiral Magnetic Effect. Phys. Rev. Lett. 104, 212.001 (2010). DOI 10.1103/PhysRevLett.104.212001
14. Fukushima, K., Mameda, K.: Wess-Zumino-Witten action and photons from the Chiral Magnetic Effect (2012)
15. Fukushima, K., Ruggieri, M.: Dielectric correction to the Chiral Magnetic Effect. Phys. Rev. D82, 054,001 (2010). DOI 10.1103/PhysRevD.82.054001
16. Fukushima, K., Warringa, H.J.: Color superconducting matter in a magnetic field. Phys. Rev. Lett. 100, 032,007 (2008). DOI 10.1103/PhysRevLett.100.032007
17. Gale, C., Jeon, S., Schenke, B., Tribedy, P., Venugopalan, R.: Event-by-event anisotropic flow in heavy-ion collisions from combined Yang-Mills and viscous fluid dynamics (2012)
18. Gao, J.H., Liang, Z.T., Pu, S., Wang, Q., Wang, X.N.: Chiral Anomaly and Local Polarization Effect from Quantum Kinetic Approach (2012)
19. Giovannini, M., Shaposhnikov, M.: Primordial hypermagnetic fields and triangle anomaly. Phys. Rev. D57, 2186–2206 (1998). DOI 10.1103/PhysRevD.57.2186
20. Giovannini, M., Shaposhnikov, M.: Primordial magnetic fields, anomalous isocurvature fluctuations and big bang nucleosynthesis. Phys. Rev. Lett. 80, 22–25 (1998). DOI 10.1103/PhysRevLett.80.22
21. Gorbar, E., Miransky, V., Shovkovy, I.: Chiral asymmetry of the Fermi surface in dense relativistic matter in a magnetic field. Phys. Rev. C80, 032,801 (2009). DOI 10.1103/PhysRevC.80.032801
22. Gusynin, V., Miransky, V., Shovkovy, I.: Dimensional reduction and dynamical chiral symmetry breaking by a magnetic field in (3+1)-dimensions. Phys. Lett. B349, 477–483 (1995). DOI 10.1016/0370-2693(95)00232-A
23. Kaiser, R.: Anomalies and WZW term of two flavor QCD. Phys. Rev. D63, 076,010 (2001). DOI 10.1103/PhysRevD.63.076010
24. Kalaydzhyan, T.: Chiral superfluidity of the quark-gluon plasma (2012)
25. Kalaydzhyan, T., Kirsch, I.: Fluid/gravity model for the chiral magnetic effect. Phys. Rev. Lett. 106, 211,601 (2011). DOI 10.1103/PhysRevLett.106.211601
26. Kharzeev, D.: Parity violation in hot QCD: Why it can happen, and how to look for it. Phys. Lett. B633, 260–264 (2006). DOI 10.1016/j.physletb.2005.11.075
27. Kharzeev, D.E.: Topologically induced local P and CP violation in QCD x QED. Annals Phys. 325, 205–218 (2010). DOI 10.1016/j.aop.2009.11.002
28. Kharzeev, D.E., McLerran, L.D., Warringa, H.J.: The Effects of topological charge change in heavy ion collisions: 'Event by event P and CP violation'. Nucl. Phys. A803, 227–253 (2008). DOI 10.1016/j.nuclphysa.2008.02.298
29. Kharzeev, D.E., Warringa, H.J.: Chiral Magnetic conductivity. Phys. Rev. D80, 034,028 (2009). DOI 10.1103/PhysRevD.80.034028
30. Kharzeev, D.E., Yee, H.U.: Chiral magnetic wave. Phys. Rev. D83, 085,007 (2011). DOI 10.1103/PhysRevD.83.085007
31. Luscher, M.: SO(4) symmetric solutions of Minkowskian Yang-Mills field equations. Phys. Lett. B70, 321 (1977). DOI 10.1016/0370-2693(77)90668-2
32. Metlitski, M.A., Zhitnitsky, A.R.: Anomalous axion interactions and topological currents in dense matter. Phys. Rev. D72, 045,011 (2005). DOI 10.1103/PhysRevD.72.045011
33. Nakano, E., Tatsumi, T.: Chiral symmetry and density wave in quark matter. Phys. Rev. D71, 114,006 (2005). DOI 10.1103/PhysRevD.71.114006
34. Noronha, J.L., Shovkovy, I.A.: Color-flavor locked superconductor in a magnetic field. Phys. Rev. D76, 105,030 (2007). DOI 10.1103/PhysRevD.76.105,030,10.1103/PhysRevD.86.049901
35. Ooguri, H., Oshikawa, M.: Instability in magnetic materials with dynamical axion field. Phys. Rev. Lett. 108, 161,803 (2012). DOI 10.1103/PhysRevLett.108.161803
36. Primakoff, H.: Photoproduction of neutral mesons in nuclear electric fields and the mean life of the neutral meson. Phys. Rev. 81, 899 (1951). DOI 10.1103/PhysRev.81.899
37. Raffelt, G., Stodolsky, L.: Mixing of the Photon with Low Mass Particles. Phys. Rev. D37, 1237 (1988). DOI 10.1103/PhysRevD.37.1237
38. Rebhan, A., Schmitt, A., Stricker, S.A.: Anomalies and the chiral magnetic effect in the Sakai-Sugimoto model. JHEP 1001, 026 (2010). DOI 10.1007/JHEP01(2010)026
39. Rubakov, V.: On chiral magnetic effect and holography (2010)
40. Schechter, B.M.: Yang-Mills theory on the hypertorus. Phys. Rev. D16, 3015 (1977). DOI 10.1103/PhysRevD.16.3015
41. Schon, V., Thies, M.: 2-D model field theories at finite temperature and density (2000)
42. Schwinger, J.S.: On gauge invariance and vacuum polarization. Phys. Rev. 82, 664–679 (1951). DOI 10.1103/PhysRev.82.664
43. Sikivie, P.: Experimental Tests of the Invisible Axion. Phys. Rev. Lett. 51, 1415 (1983). DOI 10.1103/PhysRevLett.51.1415
44. Son, D., Stephanov, M.: Axial anomaly and magnetism of nuclear and quark matter. Phys. Rev. D77, 014,021 (2008). DOI 10.1103/PhysRevD.77.014021
45. Son, D.T., Yamamoto, N.: Berry curvature, triangle anomalies, and chiral magnetic effect in Fermi liquids (2012)
46. Stephanov, M., Yin, Y.: Chiral Kinetic Theory (2012)
47. Vilenkin, A.: Equilibrium parity violating current in a magnetic field. Phys. Rev. D22, 3080–3084 (1980). DOI 10.1103/PhysRevD.22.3080
48. Yamamoto, A.: Chiral magnetic effect in lattice QCD with a chiral chemical potential. Phys. Rev. Lett. 107, 031,601 (2011). DOI 10.1103/PhysRevLett.107.031601
49. Yee, H.U.: Holographic Chiral Magnetic Conductivity. JHEP 0911, 085 (2009). DOI 10.1088/1126-6708/2009/11/085
50. Zahed, I.: Anomalous chiral Fermi surface. Phys. Rev. Lett. 109, 091,603 (2012). DOI 10.1103/PhysRevLett.109.091603