The constitutive tensor of linear elasticity: its decompositions, Cauchy relations, null Lagrangians, and wave propagation

Yakov Itin
Inst. Mathematics, Hebrew Univ. of Jerusalem,
E.J. Safra Campus, Givat Ram, Jerusalem 91904, Israel,
and Jerusalem College of Technology, 21 Havaad Haleumi, POB 16031,
Jerusalem 91160, Israel, email: itin@math.huji.ac.il

Friedrich W. Hehl
Inst. Theor. Physics, Univ. of Cologne, 50923 Köln, Germany,
and Dept. Physics & Astron., Univ. of Missouri,
Columbia, MO 65211, USA, email: hehl@thp.uni-koeln.de

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Abstract

In linear anisotropic elasticity, the elastic properties of a medium are described by the fourth rank elasticity tensor $C$. The decomposition of $C$ into a partially symmetric tensor $M$ and a partially antisymmetric tensors $N$ is often used in the literature. An alternative, less well-known decomposition, into the completely symmetric part $S$ of $C$ plus the reminder $A$, turns out to be irreducible under the 3-dimensional general linear group. We show that the $SA$-decomposition is unique, irreducible, and preserves the symmetries of the elasticity tensor. The $MN$-decomposition fails to have these desirable properties and is such inferior from a physical point of view. Various applications of the $SA$-decomposition are discussed: the Cauchy relations (vanishing of $A$), the non-existence of elastic null Lagrangians, the decomposition of the elastic energy and of the acoustic wave propagation. The acoustic or Christoffel tensor is split in a Cauchy and a non-Cauchy part. The Cauchy part governs the longitudinal wave propagation. We provide explicit examples of the effectiveness of the $SA$-decomposition. A complete class of anisotropic media is proposed that allows pure polarizations in arbitrary directions, similarly as in an isotropic medium.

Key index words: anisotropic elasticity tensor, irreducible decomposition, Cauchy relations, null Lagrangians, acoustic tensor
1 Introduction and summary of the results

Consider an arbitrary point \( P_0 \), with coordinates \( x^i_0 \), in an undeformed body. Then we deform the body and the same material point is named \( P \), with coordinates \( x^i \). The position of \( P \) is uniquely determined by its initial position \( P_0 \), that is, \( x^i = x^i(x^j_0) \). The displacement vector \( u = u^i \partial_i \) is then defined as

\[
  u^i(x^j_0) = x^i(x^j_0) - x^i_0 \quad (i, j, \cdots = 1, 2, 3) .
\]

(1)

We distinguish between lower (covariant) and upper (contravariant) indices in order to have the freedom to change to arbitrary coordinates if necessary.

1.1 Strain and stress

The deformation and the stress state of an elastic body is, within linear elasticity theory, described by means of the strain tensor \( \varepsilon_{ij} \) and the stress tensor \( \sigma^{ij} \). The strain tensor as well as the stress tensor are both symmetric, that is, \( \varepsilon_{ij} = \frac{1}{2}(\varepsilon_{ij} - \varepsilon_{ji}) = 0 \) and \( \sigma^{ij} = 0 \), see Love (1927) [25], Landau & Lifshitz (1986) [22], Haussühl (2007) [16], Marsden & Hughes (1983) [26], and Podio-Guidugli (2000) [29]; thus, \( \varepsilon_{ij} \) and \( \sigma^{ij} \) both have 6 independent components.

The strain tensor can be expressed in terms of the displacement vector via

\[
  \varepsilon_{ij} = g_{ki} \partial_j u^k + g_{kj} \partial_i u^k =: 2 g_{k(i} \partial_{j)} u^k ,
\]

(2)

whereupon \( g_{ij} \) is the metric of the three-dimensional (3d) Euclidean background space and \( \partial_i := \partial/x^i_0 \). In Cartesian coordinates, we have \( \varepsilon_{ij} = 2 \partial_i u_j \), with \( u_i := g_{ik} u^k \).

The stress tensor fulfills the momentum law

\[
  \partial_j \sigma^{ij} + \rho b^i = \rho \ddot{u}^i ;
\]

(3)

here \( \rho \) is the mass density, \( b^i \) the body force density, and a dot denotes the time derivative.

1.2 Elasticity tensor

The constitutive law in linear elasticity for a homogeneous anisotropic body, the generalized Hooke law, postulates a linear relation between the two second-rank tensor fields, the stress \( \sigma^{ij} \) and the strain \( \varepsilon_{kl} \):

\[
  \sigma^{ij} = C^{ijkl} \varepsilon_{kl} .
\]

(4)

The elasticity tensor \( C^{ijkl} \) has the physical dimension of a stress, namely force/area. Hence, in the International System of Units (SI), the frame components of \( C^{ijkl} \) are measured in pascal \( P \), with \( P := N/m^2 \).

In 3d, a generic fourth-order tensor has 81 independent components. It can be viewed as a generic 9 \( \times \) 9 matrix. Since \( \varepsilon_{ij} \) and \( \sigma^{ij} \) are symmetric, certain symmetry relations hold also for the elasticity tensor. Thus,

\[
  \varepsilon_{[kl]} = 0 \quad \Longrightarrow \quad C^{ijkl} = 0 .
\]

(5)
This is the so-called right minor symmetry. Similarly and independently, the symmetry of the stress tensor yields the so-called left minor symmetry,

\[ \sigma^{[ij]} = 0 \quad \Rightarrow \quad C^{[ijkl]} = 0. \]  

(6)

Both minor symmetries, (5) and (6), are assumed to hold simultaneously. Accordingly, the tensor \( C^{ijkl} \) can be represented by a \( 6 \times 6 \) matrix with 36 independent components.

The energy density of a deformed material is expressed as \( W = \frac{1}{2} \sigma^{ij} \varepsilon_{ij} \). When the Hooke law is substituted, this expression takes the form

\[ W = \frac{1}{2} C^{ijkl} \varepsilon_{ij} \varepsilon_{kl}. \]  

(7)

The right-hand side of (7) involves only those combinations of the elasticity tensor components which are symmetric under permutation of the first and the last pairs of indices. In order to prevent the corresponding redundancy in the components of \( C^{ijkl} \), the so-called major symmetry,

\[ C^{ijkl} - C^{klij} = 0 \]  

(8)

is assumed. Therefore, the \( 6 \times 6 \) matrix becomes symmetric and only 21 independent components of \( C^{ijkl} \) are left over.

As a side remark we mention that the components of a tensor are always measured with respect to a local frame \( e_{\alpha} = e^i_{\alpha} \partial_i \); here \( \alpha = 1, 2, 3 \) numbers the three linearly independent legs of this frame, the triad. Dual to this frame is the coframe \( \vartheta^\beta = e^j_\beta dx^j \), see Schouten (1989) [33] and Post (1997) [30]. For the elasticity tensor, the components with respect to such a local coframe are \( C^{\alpha\beta\gamma\delta} := e^i_\alpha e^j_\beta e^k_\gamma e^l_\delta C^{ijkl} \). They are called the physical components of \( C \). For simplicity, we will not set up a frame formalism since it doesn’t provide additional insight in the decomposition of the elasticity tensor, that is, we will use coordinate frames \( \partial_i \) in the rest of our article.

Because of (5), (6), and (8), we have the following

**Definition:** A 4th rank tensor of type \( (1^4) \) qualifies to describe anisotropic elasticity if

(i) its physical components carry the dimension of force/area (in SI pascal),
(ii) it obeys the left and right minor symmetries,
(iii) and it obeys the major symmetry.

It is then called elasticity tensor (or elasticity or stiffness) and, in general, denoted by \( C^{ijkl} \).

In Secs. 2.1 and 2.2 we translate our notation into that of Voigt, see Voigt (1928) [39] and Love (1927) [25], and discuss the corresponding 21-dimensional vector space of all elasticity tensors, see also Del Piero (1979) [12].

Incidentally, in linear electrodynamics, see Post (1962) [30], Hehl & Obukhov (2003) [18], and Itin (2009) [20], we have a 4-dimensional constitutive tensor \( \chi^{\mu\nu\sigma\kappa} = -\chi^{\nu\mu\sigma\kappa} = -\chi^{\mu\kappa\nu\sigma} = \chi^{\sigma\kappa\mu\nu} \), with \( \mu, \nu, ..., = 0, 1, 2, 3 \). Surprisingly, this tensor corresponds also to a \( 6 \times 6 \) matrix, like \( C^{ijkl} \) in elasticity. The major symmetry is the same, the minor symmetries are those of an antisymmetric pair of indices.
1.3 Decompositions of the 21 components elasticity tensor

In Sec. 2.3, we turn first to an algebraic decomposition of $C^{ijkl}$ that is frequently discussed in the literature: the elasticity tensor is decomposed into the sum of the two tensors $M^{ijkl} := C^{i(jk)}l$ and $N^{ijkl} := C^{i[jk]l}$, which are symmetric or antisymmetric in the middle pair of indices, respectively. We show that $M$ and $N$ fulfill the major symmetry but not the minor symmetries and that they can be further decomposed. Accordingly, this reducible decomposition does not correspond to a direct sum decomposition of the vector space defined by $C$, as we will show in detail. Furthermore we show that the vector space of $M$ is 21-dimensional and that of $N$ 6-dimensional.

Often, in calculation within linear elasticity, the tensors $M$ and $N$ emerge. They are auxiliary quantities, but due to the lack of the minor symmetries, they are not elasticities. Consequently, they cannot be used to characterize a certain material elastically. These quantities are placeholders that are not suitable for a direct physical interpretation.

Subsequently, in Sec. 2.4, we study the behavior of the physical components of $C$ under the action of the general linear 3d real group $GL(3, \mathbb{R})$. The $GL(3, \mathbb{R})$ commutes with the permutations of tensor indices. This fact yields the well-known relation between the action of $GL(3, \mathbb{R})$ and the action of the symmetry group $S_p$. Without restricting the generality of our considerations, we will choose local coordinate frames $\partial_i$ for our considerations.

Figure 1. A tensor $T^{ijkl}$ of rank 4 in 3-dimensional (3d) space has $3^4 = 81$ independent components. The 3 dimensions of our image represent this 81d space. The plane $\mathbb{C}$ depicts the 21 dimensional subspace of all possible elasticity (or stiffness) tensors. This space is span by its irreducible pieces, the 15d space of the totally symmetric elasticity $S$ (a straight line) and the 6d space of the difference $A = \mathbb{C} - S$ (also depicted, I am sad to say, as a straight line). — Oblique to $\mathbb{C}$ is the 21d space $\mathbb{M}$ of the reducible $M$-tensor and the 6d space of the reducible $N$-tensor. The $\mathbb{C}$ “plane” is the only place where elasticities (stiffnesses) are at home. The spaces $\mathbb{M}$ and $\mathbb{N}$ represent only elasticities, provided the Cauchy relations are fulfilled. Then, $A = N = 0$ and $\mathbb{M}$ and $\mathbb{C}$ cut in the 15d space of $S$. Notice that the spaces $M$ and $C$ are intersecting exactly on $S$. 
In this way, we arrive at an alternative decomposition of $C$ into two pieces $S$ and $A$, which is irreducible under the action of the $GL(3, \mathbb{R})$. The known device to study the action of $S_p$ is provided by Young’s tableaux technique. For the sake of completeness, we will be briefly describe Young’s technique in the Appendix. In an earlier paper, see Hehl & Itin (2002)\cite{Hehl2002}, we discussed this problem already, but here we will present rigorous proofs of all aspects of this irreducible decomposition. It turns out that the space of the $S$-tensor is 15-dimensional and that of the $A$-tensor 6-dimensional.

In Sec. 2.5, we compare the reducible $MN$-decomposition of Sec. 2.3 with the irreducible $SA$-decomposition of Sec. 2.4 and will show that the latter one is definitely to be preferred from a physical point of view. The formulas for the transition between the $MN$- and the $SA$-decomposition are collected in the Propositions 9 and 11. The irreducible 4th rank tensor $A_{ijkl}$ can alternatively be represented by a symmetric 2nd rank tensor $\Delta_{ij}$ (Proposition 10). We visualized our main results with respect to the reducible and the irreducible decompositions of the elasticity tensor in Figure 1, for details, please see Sec.2.5.

The group $GL(3, \mathbb{R})$, which we are using here, provides the basic, somewhat coarse-grained decomposition of the elasticity tensor. For finer types of irreducible decompositions under the orthogonal subgroup $SO(3)$ of $GL(3, \mathbb{R})$, see, for example, Rychlewski (1984)\cite{Rychlewski1984}, Walpole (1984)\cite{Walpole1984}, Surrel (1993)\cite{Surrel1993}, Xiao (1998)\cite{Xiao1998}, and the related work of Backus (1970)\cite{Backus1970} and Baerheim (1993)\cite{Baerheim1993}. As a result of our presentation, the great number of invariants, which emerge in the latter case, can be organized into two subsets: one related to the $S$ piece and the other one to the $A$ piece of $C$.

1.4 Physical applications and examples

Having now the irreducible decomposition of $C$ at our command, Sec. 3 will be devoted to the physical applications. In Sec.3.1, we discuss the Cauchy relations and show that they correspond to the vanishing of one irreducible piece, namely to $A = 0$ or, equivalently, to $\Delta = 0$. As a consequence, the totally symmetric piece $S$ can be called the Cauchy part of the elasticity tensor $C$, whereas the $A$ piece measures the deviation from this Cauchy part; it is the non-Cauchy part of $C$. The reducible pieces $M$ and $N$ defy such an interpretation and are not useful for applications in physics. In this sense, we can speak of two kinds of elasticity, a Cauchy type and a non-Cauchy type.

In Sec. 3.2, this picture is brought to the elastic energy and the latter decomposed in a Cauchy part and its excess, the “non-Cauchy part”. In other words, this distinction between two kinds of elasticity is also reflected in the properties of the elastic energy.

A null Lagrangian is such an Lagrangian whose Euler-Lagrange expression vanishes, see Crampin & Saunders (2005)\cite{Crampin2005}; we also speak of a “pure divergence”. In Sec. 3.3, elastic null Lagrangians are addressed, and we critically evaluate the literature. We show—in contrast to a seemingly widely held view—that for an arbitrary anisotropic medium there doesn’t exist an elastic null Lagrangian. The expressions offered in the corresponding literature are worthless as null Lagrangians, since they still depend on all components of the elasticity tensor. We collect these results in Proposition 12.

In Sec. 3.4, we define for acoustic wave propagation the Cauchy and non-Cauchy parts
of the Christoffel (or acoustic) tensor $\Gamma^{ij} = C^{kij} n_k n_l / \rho$ \textit{(}unit wave covector, $\rho =$ mass density). We find some interesting and novel results for the Christoffel tensor\textit{(}, see Propositions 13 and 14). In Sec. 3.5, we investigate the polarizations of the elastic wave. We show that the longitudinal wave propagation is completely determined by the Cauchy part of the Christoffel tensor, see Proposition 15. In Proposition 16 a new result is presented on the propagation of purely polarized waves; we were led to these investigations by following up some ideas about the interrelationship of the symmetry of the elasticity tensor and the Christoffel tensor in the papers of Alshits and Lothe (2004)\textit{[1]} and Bóna et al. (2004, 2007, 2010)\textit{[5 6 7]}. In Sec. 4, we investigate examples, namely isotropic media (Sec. 4.1) and media with cubic symmetry (Sec. 4.2). Modern technology allows modeling composite materials with their effective elastic properties, see Tadmor & Miller (2011)\textit{[37]}. Irreducible decomposition of the elasticity tensor can be used as a guiding framework for prediction of certain features of these materials. As an example, we presented in Proposition 17 a complete new class of anisotropic materials that allow pure polarizations to propagate in arbitrary directions, similarly as in isotropic materials.

1.5 Notation

We use here tensor analysis in 3d Euclidean space with explicit index notation, see Sokolnikoff (1951)\textit{[34]} and Schouten (1954, 1989)\textit{[32 33]}. Coordinate (holonomic) indices are denoted by Latin letters $i, j, k, \ldots$; they run over $1, 2, 3$. Since we allow arbitrary curvilinear coordinates, covariant and contravariant indices are used, that is, those in lower and in upper position, respectively, see Schouten (1989)\textit{[33]}. Summation over repeated indices is understood. We abbreviate symmetrization and antisymmetrization over $p$ indices as follows:

$$ (i_1 i_2 \ldots i_p) := \frac{1}{p!} \{ \sum \text{ over all permutations of } i_1 i_2 \ldots i_p \}, $$

$$ [i_1 i_2 \ldots i_p] := \frac{1}{p!} \{ \sum \text{ over all even perms.} - \sum \text{ over all odd perms.} \}. $$

The Levi-Civita symbol is given by $\epsilon_{ijk} = +1, -1, 0$, for even, odd, and no permutation of the indices $123$, respectively; the analogous is valid for $\hat{\epsilon}_{ijk}$. The metric $ds^2 = g_{ij} dx^i \otimes dx^j$ has Euclidean signature. We can raise and lower indices with the help of the metric. In linear elasticity theory, tensor analysis is used, for example, in Love (1927)\textit{[25]}, Sokolnikoff (1956)\textit{[35]}, Landau & Lifshitz (1986)\textit{[22]}, and Haussühl (2007)\textit{[16]}. For a modern presentation of tensors as linear maps between corresponding vector spaces, see Marsden & Hughes (1983)\textit{[26]}, Podio-Guidugli (2000)\textit{[29]}, and Hetnarski & Ignaczak (2011)\textit{[19]}. In Hehl & Itin (2002)\textit{[17]}, we denoted the elasticity moduli differently. The quantities of our old paper translate as follows into the present one: (1)$c^{ijkl} \equiv S^{ijkl}$, (2)$\hat{c}^{ijkl} \equiv A^{ijkl}$, (2)$\hat{c}^{ijkl} \equiv \hat{N}^{ijkl}$, and $c^{ijkl} \equiv M^{ijkl}$. 
2 Algebra of the decompositions of the elasticity tensor

2.1 Elasticity tensor in Voigt’s notation

The standard “shorthand” notation of $C^{ijkl}$ is due to Voigt, see Voigt (1928) and Love (1927). One identifies a symmetric pair $\{ij\}$ of 3d indices with a multi-index $I$ that has the range from 1 to 6:

\[ 11 \rightarrow 1, \ 22 \rightarrow 2, \ 33 \rightarrow 3, \ 23 \rightarrow 4, \ 31 \rightarrow 5, \ 12 \rightarrow 6. \] (9)

Then the elasticity tensor is expressed as a symmetric $6 \times 6$ matrix $C_{IJ}$. Voigt’s notation is only applicable since the minor symmetries (5) and (6) are valid. Due to the major symmetry, this matrix is symmetric, $C_{[IJ]} = 0$. Explicitly, we have

\[
\begin{bmatrix}
C^{1111} & C^{1122} & C^{1133} & C^{1123} & C^{1131} & C^{1112} \\
* & C^{2222} & C^{2233} & C^{2223} & C^{2231} & C^{2212} \\
* & * & C^{3333} & C^{3323} & C^{3331} & C^{3312} \\
* & * & * & C^{2323} & C^{2331} & C^{2312} \\
* & * & * & * & C^{3131} & C^{3112} \\
* & * & * & * & * & C^{1212}
\end{bmatrix}
\equiv
\begin{bmatrix}
C^{11} & C^{12} & C^{13} & C^{14} & C^{15} & C^{16} \\
* & C^{22} & C^{23} & C^{24} & C^{25} & C^{26} \\
* & * & C^{33} & C^{34} & C^{35} & C^{36} \\
* & * & * & C^{44} & C^{45} & C^{46} \\
* & * & * & * & C^{55} & C^{56} \\
* & * & * & * & * & C^{66}
\end{bmatrix}. \] (10)

For general anisotropic materials, all the displayed components are nonzero and independent of one another. The stars in both matrices denote those entries that are dependent due to the symmetries of the matrices.

2.2 Vector space of the elasticity tensor

The set of all generic elasticity tensors, that is, of all fourth rank tensors with the minor and major symmetries, builds up a vector space. Indeed, a linear combination of two such tensors, taken with arbitrary real coefficients, is again a tensor with the same symmetries. We denote this vector space by $\mathcal{C}$.

**Proposition 1.** For the space of elasticity tensors,

\[ \dim \mathcal{C} = 21. \] (11)

**Proof.** The basis of $\mathcal{C}$ can be enumerated by the elements of the matrix $C^{IJ}$. For instance, the element $C^{11} = C^{1111}$ is related to the basis vector

\[ E_1 = \partial_1 \otimes \partial_1 \otimes \partial_1 \otimes \partial_1; \] (12)

the element $C^{12} = C^{1122}$ corresponds to the basis vector

\[ E_2 = \frac{1}{2} (\partial_1 \otimes \partial_1 \otimes \partial_2 \otimes \partial_2 + \partial_2 \otimes \partial_2 \otimes \partial_1 \otimes \partial_1); \] (13)
the element \( C^{13} = C^{1133} \) corresponds to the basis vector
\[
E_3 = \frac{1}{2} (\partial_1 \otimes \partial_1 \otimes \partial_3 \otimes \partial_3 + \partial_3 \otimes \partial_3 \otimes \partial_1 \otimes \partial_1) .
\] (14)

To the element \( C^{14} = C^{1123} \), we relate the basis vector
\[
E_4 = \frac{1}{4} (\partial_1 \otimes \partial_1 \otimes \partial_2 \otimes \partial_3 + \partial_1 \otimes \partial_1 \otimes \partial_3 \otimes \partial_2 + \partial_2 \otimes \partial_3 \otimes \partial_1 \otimes \partial_1 + \partial_3 \otimes \partial_2 \otimes \partial_1 \otimes \partial_1) .
\] (15)

In this way, a set of vectors \( \{ E_1, \cdots, E_{21} \} \) is constructed. Since there are no relations between the 21 components \( C^{IJ} \), all these vectors are linearly independent. Moreover every elasticity tensor can be expanded as a linear combination of \( \{ E_1, \cdots, E_{21} \} \). Thus a basis of of the vector space \( \mathcal{C} \) consists of 21 vectors.

\section{Reducible decomposition of \( C^{ijkl} \)}

\subsection{Definitions of \( M \) and \( N \) and their symmetries}

In the literature on elasticity, a special decomposition of \( C^{ijkl} \) into two tensorial parts is frequently used, see, for example, Cowin (1989)\cite{9}, Campanella & Tonton (1994)\cite{8}, Podio-Guidugli (2000)\cite{22}, Weiner (2002)\cite{12}, and Haussühl (2007)\cite{16}. It is obtained by symmetrization and antisymmetrization of the elasticity tensor with respect to its two middle indices:

\[
M^{ijkl} := C^{i[jk]l} , \quad N^{ijkl} := C^{i[jk][l]} , \quad \text{with} \quad C^{ijkl} = M^{ijkl} + N^{ijkl} .
\] (16)

Sometimes the same operations are applied for the second and the fourth indices. Due to the symmetries of the elasticity tensor, these two procedures are equivalent to one another.

We recall that the elasticity tensor fulfills the left and right minor symmetries and the major symmetry:

\[
C^{i[jkl]} = 0 , \quad C^{ij[kl]} = 0 ; \quad C^{ijkl} - C^{klij} = 0 .
\] (17)

\textbf{Proposition 2.} The major symmetry holds for both tensors \( M^{ijkl} \) and \( N^{ijkl} \).

\textbf{Proof.} We formulate the left-hand side of the major symmetries and substitute the definitions given in (16):

\[
M^{ijkl} - M^{klij} = C^{i(jk)l} - C^{k(ji)l} = \frac{1}{2} \left( C^{ijkl} + C^{ikjl} - C^{klij} - C^{kilj} \right) = 0 ,
\] (18)

\[
N^{ijkl} - N^{klij} = C^{i[jk][l]} - C^{i[kj][l]} = \frac{1}{2} \left( C^{ijkl} - C^{ikjl} - C^{klij} + C^{kilj} \right) = 0 .
\] (19)

\textbf{Proposition 3.} In general, the minor symmetries do not hold for the tensors \( M^{ijkl} \) and \( N^{ijkl} \).
Proof. We formulate the left minor symmetries for \( M \) and \( N \) and use again the definitions from (16):

\[
M^{ijkl} = \frac{1}{2} (C^{ijkl} + C^{[ijkl]}) = \frac{1}{2} C^{k[ij]}l = \frac{1}{2} N^{kijl}, \tag{20}
\]

\[
N^{ijkl} = \frac{1}{2} (C^{ijkl} - C^{[ijkl]}) = -\frac{1}{2} C^{k[ij]}l = -\frac{1}{2} N^{kijl}; \tag{21}
\]

here indices that are excluded from the (anti)symmetrization are enclosed by vertical bars. Both expressions don’t vanish in general. Moreover, we are immediately led to \( M^{ijkl} = -N^{ijkl} \neq 0 \).

Using the major symmetry of Proposition 2, we recognize that the right minor symmetries \( M^{ijkl} = 0 \) and \( N^{ijkl} = 0 \) do not hold either, since \( M^{ijkl} = M^{[kl]ij} \) and \( N^{ijkl} = N^{[kl]ij} \).

Consequently, the tensors \( M^{ijkl} \) and \( N^{ijkl} \) do not belong to the vector space \( \mathcal{C} \) and cannot be written in Voigt’s notation. Thus, these partial tensors \( M \) and \( N \) themselves cannot serve as elasticity tensors for any material.

2.3.2 Vector spaces of the \( N- \) and \( M\)-tensors

We will denote the set of all \( N \)-tensors by \( \mathcal{N} \). It is a vector space. Indeed, \( N^{ijkl} \) is defined as a fourth rank tensor which is skew-symmetric in the middle indices and constructed from the elasticity tensor. A linear combination of such tensors \( \alpha N^{ijkl} + \beta \tilde{N}^{ijkl} \) will be also skew-symmetric. Moreover, it can be constructed from the tensor \( \alpha C^{ijkl} + \beta \tilde{C}^{ijkl} \), which satisfies the basic symmetries of the elasticity tensor.

A simplest way to describe a finite dimensional vector space, is to write-down its basis. In the case of a tensor vector space, it is enough to enumerate all the independent components of the tensor.

**Proposition 4.** For the space of \( N \)-tensors, \( \dim \mathcal{N} = 6 \).

Proof. We can write-down explicitly six components of the “antisymmetric” tensor \( N^{ijkl} \) as

\[
N^{1122} = \frac{1}{2} (C^{12} - C^{66}), \quad N^{1133} = \frac{1}{2} (C^{13} - C^{55}), \quad N^{1123} = \frac{1}{2} (C^{14} - C^{56}),
\]

\[
N^{2233} = \frac{1}{2} (C^{23} - C^{44}), \quad N^{2231} = \frac{1}{2} (C^{25} - C^{46}), \quad N^{1233} = \frac{1}{2} (C^{36} - C^{45}). \tag{22}
\]

All other components vanish or differ from the components given in (22) only in sign. Since all the components \( C^{IJ} \) with \( I \leq J \) are assumed to be independent, then the components of \( N^{ijkl} \) of (22) are also independent.

The set of all \( M \)-tensors is also a vector space, which we denote by \( \mathcal{M} \).

**Proposition 5.** For the space of \( M \)-tensors, \( \dim \mathcal{M} = 21 \).
Proof. The dimension of the vector space $\mathcal{M}$ can also be calculated by considering the independent components of a generic tensor $M_{ijkl}$. We find,

$$
\begin{align*}
M_{1111} &= C_{11}^{11}, & M_{1113} &= C_{15}^{11}, & M_{1112} &= C_{16}^{11}, & M_{2222} &= C_{22}^{22}, \\
M_{2221} &= C_{26}^{26}, & M_{3333} &= C_{33}^{33}, & M_{3332} &= C_{34}^{33}, & M_{3331} &= C_{35}^{33}, \\
M_{2331} &= C_{45}^{45}, & M_{3221} &= C_{46}^{46}, & M_{3113} &= C_{55}^{55}, & M_{3112} &= C_{56}^{55}, \\
M_{1122} &= \frac{1}{2} (C_{12}^{12} + C_{66}^{66}), & M_{1133} &= \frac{1}{2} (C_{13}^{13} + C_{55}^{55}), & M_{2223} &= C_{24}^{24}, \\
M_{1123} &= \frac{1}{2} (C_{14}^{14} + C_{56}^{56}), & M_{2233} &= \frac{1}{2} (C_{23}^{23} + C_{44}^{44}), & M_{2332} &= C_{44}^{44}, \\
M_{2231} &= \frac{1}{2} (C_{25}^{25} + C_{46}^{46}), & M_{1233} &= \frac{1}{2} (C_{36}^{36} + C_{45}^{45}), & M_{1221} &= C_{66}^{66}.
\end{align*}
$$

All these 21 components are linearly independent, thus the dimension of the vector space $\mathcal{M}$ is at least 21. However, since every element of $\mathcal{M}$ is defined in terms of 21 independent elastic constants $C^{ij}$, the dimension of $\mathcal{M}$ cannot be greater than 21. □

### 2.3.3 Algebraic properties of $M$ and $N$ tensors

Observe some principal features of the tensors $M_{ijkl}$ and $N_{ijkl}$:

(i) Inconsistency. In general, a certain component of $C_{ijkl}$, say $C^{1223}$, can be expressed in different ways in terms of the components of $M_{ijkl}$ and $N_{ijkl}$:

$$
C^{1223} \equiv \frac{1}{2} (M_{1223} + N_{1223}) = M_{2312}^{\text{maj}} = M_{2312}^{\text{sym}} = M_{2132}^{(23)} = C_{46}^{46}.
$$

On the other hand, we have

$$
C^{1223} = C^{2123} \equiv M_{2123}^{2123} + N_{2123}^{2123}.
$$

With

$$
M_{2123}^{2123} = \frac{1}{2} (C_{46}^{46} + C_{25}^{25}) \quad \text{and} \quad N_{2123}^{2123} = \frac{1}{2} (C_{46}^{46} - C_{25}^{25}) \neq 0,
$$

we recover the result in (24), but is was achieved with the help of a non-vanishing component of $N$.

(ii) Reducibility. Since, in general, the tensor $M_{ijkl}$ is not completely symmetric, a finer decomposition is possible,

$$
M_{ijkl} = M^{ijkl} + K^{ijkl}.
$$

Accordingly, $C^{ijkl}$ can be decomposed into three tensorial pieces:

$$
C^{ijkl} = M^{ijkl} + K^{ijkl} + N^{ijkl}.
$$

(iii) Vector spaces. The “partial” vector spaces, $\mathcal{M}$ and $\mathcal{N}$, are not subspaces of the vector space $\mathcal{C}$ and their sum $\mathcal{M} + \mathcal{N}$ is not equal to $\mathcal{C}$.

Thus, the $MN$-decomposition is problematic from an algebraic point of view. Our aim is to present an alternative irreducible decomposition with better algebraic properties.
2.4 Irreducible decomposition of $C^{ijkl}$

2.4.1 Definitions of $S$ and $A$ and their symmetries

In Eq. (146) of the Appendix, we decomposed a fourth rank tensor irreducibly. Let us apply it to the elasticity tensor $C^{ijkl}$. Since the dimension of 3d space is less than the rank of the tensor, the last diagram in (146), representing $C^{ijkl}$, is identically zero. Also the minor symmetries remove some of the diagrams. Dropping the diagrams which are antisymmetric in the pairs of the indices $(i_1, i_2)$ and $(i_3, i_4)$, we are eventually left with the decomposition

$$i_1 \otimes i_2 \otimes i_3 \otimes i_4 = \frac{1}{4!} i_1 i_2 i_3 i_4 + \beta \left( i_1 i_2 i_3 i_4 \right) + \gamma i_1 i_2 i_3 i_4.$$  \hspace{1cm} (29)

Let us now apply the major symmetry. It can be viewed as pair of simultaneous permutations $i_1 \leftrightarrow i_3$ and $i_2 \leftrightarrow i_4$. The first and the last diagrams in (29) are invariant under those transformations. The two diagrams in the middle change their signs and are thus identically zero. Thus, we are left with irreducible parts:

$$i_1 \otimes i_2 \otimes i_3 \otimes i_4 = \frac{1}{4!} i_1 i_2 i_3 i_4 + \gamma i_1 i_2 i_3 i_4.$$ \hspace{1cm} (30)

In correspondence with the first table, the first subtensor of $C^{ijkl}$ is derived by complete symmetrization of its indices

$$S^{ijkl} = (I + (i_1, i_2) + (i_1, i_3) + \cdots + (i_1, i_2, i_3) + \cdots + (i_1, i_2, i_3, i_4)) C^{ijkl}.$$ \hspace{1cm} (31)

Here the parentheses denote the cycles of permutations. Consequently,

$$S^{ijkl} := C^{ijkl} = \frac{1}{3!} \left( C^{ijkl} + C^{ikjl} + C^{kjil} + \cdots + C^{kijl} + \cdots + C^{lijk} \right).$$ \hspace{1cm} (32)

On the right hand side, we have a sum of $4! = 24$ terms with all possible orders of the indices. If we take into account the symmetries (5), (6), and (8) of $C^{ijkl}$, we can collect the terms:

$$S^{ijkl} = C^{ijkl} = \frac{1}{3} \left( C^{ijkl} + C^{ikjl} + C^{lijk} \right).$$ \hspace{1cm} (33)

According to (30), the second irreducible piece of $C^{ijkl}$ can be defined as

$$A^{ijkl} := C^{ijkl} - C^{ijkl} = C^{ijkl} - S^{ijkl}.$$ \hspace{1cm} (34)

This result can also be derived by evaluating the last diagram in (30). Substitution of (33) into the right-hand side of (34) yields

$$A^{ijkl} = \frac{1}{3} \left( 2C^{ijkl} - C^{ikjl} - C^{kijl} \right).$$ \hspace{1cm} (35)

If we totally symmetrize the left- and the right-hand sides of (34), an immediate consequence is

$$A^{ijkl} = 0.$$ \hspace{1cm} (36)

If we symmetrize (33) with respect to the indices $jkl$, we recognize that its right-hand side vanishes. Accordingly, we have the
Proposition 6. The tensor $A$ fulfills the additional symmetry
\[
A^{ijkl} = 0 \quad \text{or} \quad A^{ijkl} + A^{iklj} + A^{iljk} = 0.
\] (37)

2.4.2 Vector spaces of the $S$- and $A$-tensors

We denote the 21-dimensional vector space of $C$ by $\mathcal{C}$. The irreducible decomposition of $\mathcal{C}$ signifies the reduction of $\mathcal{C}$ to the direct sum of its two subspaces, $\mathcal{S} \subset \mathcal{C}$ for the tensor $S$, and $\mathcal{A} \subset \mathcal{C}$ for the tensor $A$,
\[
\mathcal{C} = \mathcal{S} \oplus \mathcal{A}. \tag{38}
\]
The vector spaces $\mathcal{S}$ and $\mathcal{A}$ have only zero in their intersection and the decomposition of the corresponding tensors is unique. According to Proposition 1, the sum of the dimensions of the subspaces is equal to 21. The two irreducible parts $S^{ijkl}$ and $A^{ijkl}$ preserve their symmetries under arbitrary linear frame transformations. In particular, they fulfill the minor and major symmetries of $C^{ijkl}$ likewise. Accordingly, $S$ and $A$, or $S$ alone (but not $A$ alone, as we will see later) can be elasticity tensors for a suitable material—in contrast to $M$ and $N$.

The dimensions of the vector spaces of $\mathcal{S}$ and $\mathcal{A}$ can now be easily determined.

Proposition 7. For the vector space $\mathcal{S}$ of the tensors $S^{ijkl}$,
\[
\dim \mathcal{S} = 15. \tag{39}
\]
Proof. The number of independent components of a totally symmetric tensor of rank $p$ in $n$ dimensions is $\binom{n+p-1}{p} = \binom{n-1+p}{n-1}$ or, for dimension 3 and rank 4, $\binom{6}{2} = 15$, see Schouten (1954) [32].

According to Proposition 1, we have $\dim \mathcal{C} = 21$. Because of (38) and (39), we have

Proposition 8. For the vector space $\mathcal{A}$ of the tensors $A^{ijkl}$,
\[
\dim \mathcal{A} = 6. \tag{40}
\]

2.4.3 Irreducible parts in Voigt’s notation

In Voigt’s 6d notation we have
\[
C^{IJ} = S^{IJ} + A^{IJ} \quad \text{with} \quad C^{[IJ]} = S^{[IJ]} = A^{[IJ]} = 0. \tag{41}
\]
The $6 \times 6$ matrix $S^{ij}$ has 15 independent components. We choose the following ones [see Voigt (1928), Eq. (36) on p.578],
\[
\begin{align*}
S^{11} &= C^{11}, & S^{22} &= C^{22}, & S^{33} &= C^{33}, \\
S^{15} &= C^{15}, & S^{16} &= C^{16}, & S^{26} &= C^{26}, \\
S^{24} &= C^{24}, & S^{34} &= C^{34}, & S^{35} &= C^{35}, \\
S^{12} &= \frac{1}{3}(C^{12} + 2C^{66}), & S^{13} &= \frac{1}{3}(C^{13} + 2C^{55}), \\
S^{14} &= \frac{1}{3}(C^{14} + 2C^{56}), & S^{23} &= \frac{1}{3}(C^{23} + 2C^{44}), \\
S^{25} &= \frac{1}{3}(C^{25} + 2C^{46}), & S^{36} &= \frac{1}{3}(C^{36} + 2C^{45}).
\end{align*} \tag{42}
\]
The $6 \times 6$ matrix of $A^{ij}$ has 6 independent components. We choose the following ones,

$$A^{12} = \frac{2}{3} (C^{12} - C^{66}), \quad A^{13} = \frac{2}{3} (C^{13} - C^{55}), \quad A^{14} = \frac{2}{3} (C^{14} - C^{56}),$$

$$A^{23} = \frac{2}{3} (C^{23} - C^{44}), \quad A^{25} = \frac{2}{3} (C^{25} - C^{46}), \quad A^{36} = \frac{2}{3} (C^{36} - C^{45}) \quad (43)$$

The decomposition (41) can be explicitly presented as

$$\begin{bmatrix}
    C^{11} & C^{12} & C^{13} & C^{14} & C^{15} & C^{16} \\
    * & C^{22} & C^{23} & C^{24} & C^{25} & C^{26} \\
    * & * & C^{33} & C^{34} & C^{35} & C^{36} \\
    * & * & * & C^{44} & C^{45} & C^{46} \\
    * & * & * & * & C^{55} & C^{56} \\
    * & * & * & * & * & C^{66}
\end{bmatrix}
= \begin{bmatrix}
    S^{11} & S^{12} & S^{13} & S^{14} & S^{15} & S^{16} \\
    * & S^{22} & S^{23} & S^{24} & S^{25} & S^{26} \\
    * & * & S^{33} & S^{34} & S^{35} & S^{36} \\
    * & * & * & S^{44} & S^{45} & S^{46} \\
    * & * & * & * & S^{55} & S^{56} \\
    * & * & * & * & * & S^{66}
\end{bmatrix}
+ \begin{bmatrix}
    0 & A^{12} & A^{13} & A^{14} & 0 & 0 \\
    * & 0 & A^{23} & 0 & A^{25} & 0 \\
    * & * & 0 & 0 & 0 & A^{36} \\
    * & * & * & -\frac{1}{2}A^{23} & -\frac{1}{2}A^{36} & -\frac{1}{2}A^{25} \\
    * & * & * & * & -\frac{1}{2}A^{13} & -\frac{1}{2}A^{14} \\
    * & * & * & * & * & -\frac{1}{2}A^{12}
\end{bmatrix} \quad (44)$$

Here, we use boldface for the independent components of the tensors. Note that all three matrices are symmetric.

2.5 Comparing the SA- and the MN-decompositions with each other

We would now like to compare the two different decompositions:

$$C^{ijkl}_{21} = S^{ijkl}_{15} + A^{ijkl}_{16} = M^{ijkl}_{21} + N^{ijkl}_{6} \quad (45)$$

The dimensions of the corresponding vector spaces are displayed explicitly. This makes it immediately clear that $A$ can be expressed in terms of $N$ and vice versa. Take the antisymmetric part of (45) with respect to $j$ and $k$ and find:

**Proposition 9.** The auxiliary quantity $N^{ijkl}$ can be expressed in terms of the irreducible elasticity $A^{ijkl}$ as follows:

$$N^{ijkl} = A^{i[jk]l} \quad (46)$$

Its inverse reads,

$$A^{ijkl} = \frac{1}{4} N^{ij(kl)} \quad (47)$$
Proof. Resolve (46) with respect to $A$. For this purpose we recall that $A$ obeys the right minor symmetry: $A_{ijkl} = A_{ij(kl)}$. This suggests to take the symmetric part of (46) with respect to $k$ and $l$. Then,

$$N_{ij(kl)} = \frac{1}{4} \left( 2A_{ijkl} - A_{iklj} - A_{ilkj} \right) = \frac{1}{4} \left[ 2A_{ijkl} - \left( -A_{klj} - A_{ijkl} \right) \right] = A_{ijkl}.$$

(48)

Both, $A$ and $N$ have only 6 independent components. In 3d this means that it must be possible to represent them as a symmetric tensor of 2nd rank. With the operator $\frac{1}{2} \epsilon_{mij}$, we can always map an antisymmetric index pair $ij$ to a corresponding vector index $m$. The tensor $A_{ijkl}$ has 4 indices, that is, we have to apply the $\epsilon$ operator twice. Since $A_{ijkl}$ obeys the left and right minor symmetries, that is, $A_{[ij]kl} = A_{[ij][kl]} = 0$, the $\epsilon$ has always to transvect one index of the first pair and one index of the second pair. This leads, apart from trivial rearrangements, to a suitable definition.

**Proposition 10.** The irreducible elasticity $A_{ijkl}$ can be equivalently described by a symmetric 2nd rank tensor

$$\Delta_{mn} := \frac{1}{4} \epsilon_{mik} \epsilon_{njl} A_{ijkl},$$

with the inverse

$$N_{ij(kl)} = \epsilon_{ikm} \epsilon_{jln} \Delta_{mn} \quad \text{or} \quad A_{ijkl} = \frac{4}{3} \epsilon_{i(km} \epsilon_{j]ln} \Delta_{mn}.$$

(50)

Proof. The symmetry of $\Delta_{mn}$ can be readily established:

$$\Delta_{[mn]} = \frac{1}{8} \left( \epsilon_{mik} \epsilon_{njl} A_{ijkl} - \epsilon_{nik} \epsilon_{mjl} A_{ijkl} \right) = \frac{1}{8} \epsilon_{mik} \epsilon_{njl} \left( A_{ijkl} - A_{ijkl} \right) = 0.$$

(51)

Eq. (50)$_1$ can be derived by substituting (49) into its right-hand side and taking care of (46). Eq. (50)$_2$ then follows by applying (17).

The symmetric 2nd rank tensor $\Delta_{mn}$ (differing by a factor 2) was introduced by Haussühl (1983, 2007), its relation to the irreducible piece $A$ was found by Hehl & Itin (2002). By means of (34), $\Delta_{mn}$ can be calculated directly from the undecomposed elasticity tensor:

$$\Delta_{mn} = \frac{1}{4} \epsilon_{mik} \epsilon_{njl} C_{ijkl}.$$

(52)

The tensor $M$, like $N$, can also be expressed in terms of irreducible pieces: We substitute (46) into (45),

$$C_{ijkl} = S_{ijkl} + A_{ijkl} = M_{ijkl} + A_{[ijkl]}^i,$$

and resolve it with respect to $M$:

$$M_{ijkl} = S_{ijkl} + A_{ijkl} - A_{[ijkl]}^i$$

$$= S_{ijkl} + \frac{1}{2} \left( 2A_{ijkl} - A_{ijkl} + A_{ijk} \right).$$

(54)

We collect the terms and find
Proposition 11. The auxiliary quantity \( M^{ijkl} \) can be expressed in terms of the irreducible elasticities as follows:

\[
M^{ijkl} = S^{ijkl} + A^{(ijkl)}.
\]  

(55)

According to Proposition 3, the reducible parts \( M \) and \( N \) don’t obey the left and the right minor symmetries. Consequently—in contrast to \( S \) and \( A \), which both obey the major and the minor symmetries—\( M \) and \( N \) cannot be interpreted as directly observable elasticity tensors. Therefore, in all physical applications we are forced to use eventually the irreducible pieces \( S \) and \( A \). The reducible parts \( M \) and \( N \) are no full-fledged substitutes for them and can at most been used for book keeping.

3 Physical applications of the irreducible decomposition

3.1 Cauchy relations, two kinds of elasticity

Having the \( SA \)-decomposition at our disposal, it is clear that we can now classify elastic materials. The anisotropic material with the highest symmetry is that for which the \( A \) elasticity vanishes:

\[
A^{ijkl} = 0 \quad \text{or} \quad \Delta_{mn} = 0 \quad \text{or} \quad N^{ijkl} = 0.
\]  

(56)

The last equation, in accordance with the definition (16) of \( N \), can also be rewritten as

\[
C^{ijkl} = C^{ikjl}.
\]  

(57)

These are the so-called Cauchy relations, for their history, see Todhunter (1960)\(^{38}\). The representation in (57) is widely used in elasticity literature, see, for example, Haussühl (1983)\(^{15}\), Cowin (1989)\(^{9}\), Cowin & Mehrabadi (1992)\(^{10}\), Campanella & Tonon (1994)\(^{8}\), Podio-Guidugli (2000)\(^{29}\), Weiner (2002)\(^{42}\), and Hehl & Itin (2002)\(^{17}\).

In Voigt’s notation, we can use \( A^{IJ} = 0 \) in (13) and find the following form of the Cauchy relations,

\[
C^{12} = C^{66}, \quad C^{13} = C^{55}, \quad C^{14} = C^{56}, \quad C^{23} = C^{44}, \quad C^{25} = C^{46}, \quad C^{36} = C^{45},
\]  

(58)

see Love (1927)\(^{25}\) and Voigt (1928)\(^{39}\). Of course, the same result can also be read off from (22) for \( N = 0 \).

Let us first notice that for most materials the Cauchy relations do not hold even approximately. In fact, the elasticity of a generic anisotropic material is described by the whole set of the 21 independent components \( C^{IJI} \) and not by a restricted set of 15 independent components obeying the Cauchy relations, see Haussühl (2007)\(^{16}\). This fact seems to nullify
the importance of the Cauchy relations for modern solid state theory and leave them only as historical artifact.

However, a lattice-theoretical approach to the elastic constants shows, see Leibfried (1962) [23], that the Cauchy relations are valid provided (i) the interaction forces between the atoms or molecules of a crystal are central forces, as, for instance, in rock salt, (ii) each atom or molecule is a center of symmetry, and (iii) the interaction forces between the building blocks of a crystal can be well approximated by a harmonic potential, see also Perrin (1979) [28]. In most elastic bodies this is not fulfilled at all, see the detailed discussion in Haussühl (2007) [16]. Accordingly, a study of the violations of the Cauchy relations yields important information about the intermolecular forces of elastic bodies. One should look for the deviation of the elasticity tensor from its Cauchy part. Recently, Elcoro & Etxerbarria (2011) [13] pointed out that the situation is more complex than thought previously, for details we refer to their article.

This deviation measure, being a macroscopic characteristic of the material, delivers important information about the microscopic structure of the material. It must be defined in terms of a unique proper decomposition of the elasticity tensor. Apparently, the tensor $N^{ijkl}$ cannot serve as a such deviation, in contrast to the stipulations of Podio-Guidugli (2000), for example, because it is not an elasticity tensor and because its co-partner $M^{ijkl}$ has 21 components, that is, as many as the elasticity tensor $C^{ijkl}$ itself. Only when $N^{ijkl} = 0$ is assumed, the tensor $M^{ijkl}$ is restricted to 15 independent components. The problem of the identification of the deviation part is solved when the irreducible decomposition is used. In this case, we can define the main or Cauchy part, as given by the tensor $S^{ijkl}$ with 15 independent components, and the deviation or non-Cauchy part, presented by the tensor $A^{ijkl}$ with 6 independent components.

We can view the elasticity tensor as being composed of two independent parts, $S$ and $A$. Due to the irreducible decomposition the set of elastic constants can be separated into two subsets which are components of two independent tensors, $C = S \oplus A$. In this way, the completely symmetric Cauchy part of the elasticity tensor $S^{ijkl}$ has an independent meaning. The additional part $A^{ijkl}$ can be referred than as as a non-Cauchy part. Why do pure Cauchy materials ($S \neq 0, A = 0$) and pure non-Cauchy materials ($S = 0, A \neq 0$) not exist in nature? Can such pure types of materials be designed artificially? Or does some principal fact forbid the existence of pure Cauchy and pure non-Cauchy materials? These questions seem to be important for elasticity theory and even for modern material technology. Note that in the framework of the reducible $MN$-decomposition such questions cannot even be raised. It is because the reducible $M$ and $N$ parts themselves cannot serve as independent elasticity tensors. We will address subsequently the reason why pure non-Cauchy materials are forbidden.

### 3.2 Elastic energy

In linear elasticity, using the generalized Hooke law, the elastic energy is given by

$$W = \frac{1}{2} \sigma^{ij} \varepsilon_{ij} = \frac{1}{2} C^{ijkl} \varepsilon_{ij} \varepsilon_{kl}.$$  \hfill (59)
Because of the irreducible decomposition $C = S + A$, we can split this energy in a Cauchy part and a non-Cauchy part:

$$ W = \langle C \rangle W + \langle nC \rangle W, \quad \text{with} \quad \langle C \rangle W := \frac{1}{2} S^{ijkl} \varepsilon_{ij} \varepsilon_{kl}, \quad \langle nC \rangle W := \frac{1}{2} A^{ijkl} \varepsilon_{ij} \varepsilon_{kl}. \quad (60) $$

We denote the strain $\varepsilon_{ij}$ can be expressed in terms of the displacement gradients according to $\varepsilon_{ij} = 2 \partial_i u_j$, we can do the analogous for the elastic energy:

$$ W = \frac{1}{2} C^{ijkl} \varepsilon_{ij} \varepsilon_{kl} = 2 C^{ijkl} \partial_i u_j \partial_k u_l = 2 C^{ijkl} \partial_i u_j \partial_k u_l. \quad (61) $$

We can drop both pairs of parentheses because the corresponding symmetries are imprinted already in the elasticity tensor.

3.3 Null Lagrangians in linear elasticity?

We would like now to discuss a proposal on null Lagrangians by Podio-Guidugli (2000) [29], see also literature cited by him. We substitute the decomposition (16) into (61):

$$ W = 2 C^{ijkl} \partial_i u_j \partial_k u_l = 2 M^{ijkl} \partial_i u_j \partial_k u_l + 2 N^{ijkl} \partial_i u_j \partial_k u_l. \quad (62) $$

We turn our attention now to the last term and to the antisymmetry of $N$, namely to $N^{ijk} = 0$. There is a subtlety involved. We want to integrate partially in the last term. For exploiting the antisymmetry of $N$ in $j$ and $k$, we need the partial differentials $\partial_j$ and $\partial_k$. Therefore, we use the left minor symmetry of $C$ and rewrite (62) as

$$ W = 2 C^{ijkl} \partial_j u_i \partial_k u_l = 2 M^{ijkl} \partial_j u_i \partial_k u_l + 2 N^{ijkl} \partial_j u_i \partial_k u_l. \quad (63) $$

Of course, the sum of the two terms has not changed, but each single term did change since $M$ and $N$ do not obey the left minor symmetry. Now we can partially integrate in the last term:

$$ 2 N^{ijkl} \partial_j u_i \partial_k u_l = 2 N^{ijkl} \left[ \partial_j (u_i \partial_k u_l) - u_i \partial_j \partial_k u_l \right]. \quad (64) $$

Since $N^{ijk} = 0$ and $\partial_j \partial_k = 0$, the last term drops out and we are left with

$$ W = 2 M^{ijkl} \partial_j u_i \partial_k u_l + 2 N^{ijkl} \partial_j (u_i \partial_k u_l). \quad (65) $$

The second $N$-term is a total derivative, as was already shown by Lancia et al. (1995) [21]. Thus, only the $M$-term is involved in the variational principle for deriving the equations of motion. This result was interpreted by Lancia et al. (1995) [21] as solving the null Lagrangian problem for linear elasticity theory, since the additional $M$-tensor is the only one that is involved in the equilibrium equation. However, this statement does not have an invariant meaning. Indeed, the remaining part with the $M$-tensor has exactly the same set of 21
independent components as the initial $C$-tensor, see Proposition 5. Moreover, this tensor allows a successive decomposition which can include additional null Lagrangian terms.

We find from (65), using the Propositions 9 and 11 and subsequently Proposition 6,

$$W = (2S_{ijkl} - A^{ijkl}) \partial_j u_i \partial_k u_l + 2A^{ijkl} \partial_j (u_i \partial_k u_l) .$$

Consequently the $A$-tensor is included in the total derivative term, which vanishes if the Cauchy relations hold. However the same $A$-tensor appears together with the $S$-tensor also in the first part of the energy functional.

For static configurations, (59) can play a role of a Lagrangian functional whose variation with respect to the displacement field generates the equilibrium equation. The null-Lagrangian is defined as that part of the strain energy functional that does not contribute to the equilibrium equation. The problem is to identify the null-Lagrangian part of the elasticity Lagrangian and consequently to establish which set of the independent components of the elasticity tensor contributes to the equilibrium equation.

As a cross-check for our considerations, we determine the equilibrium conditions for the Lagrangian (66). Up to a total derivative term, the variation of the Lagrangian (66) reads

$$\delta W = (2S_{ijkl} - A^{ijkl}) (\partial_j u_i \partial_k u_l + \partial_j \delta u_i \partial_k u_l) .$$

Since the minor and the major symmetries hold for the $S$ and $A$ tensors, the last two terms can be summed up:

$$\delta W = 2 (2S_{ijkl} - A^{ijkl}) \partial_j \delta u_i \partial_k u_l .$$

We integrate partially,

$$\delta W = 2 (2S_{ijkl} - A^{ijkl}) \partial_j (\delta u_i \partial_k u_l) - 2 (2S_{ijkl} - A^{ijkl}) \delta u_i \partial_j \partial_k u_l ,$$

and can read off the equilibrium conditions as

$$(2S_{ijkl} - A^{ijkl}) \partial_j \partial_k u_l = 0 .$$

Only at a first glance, this equation seems to be new. Indeed, we can rewrite it by using the Propositions 6 and 11,

$$(2S_{ijkl} + A^{ijkl} + A^{iklj}) \partial_j \partial_k u_l = 2 (S_{ijkl} + A^{ijkl}) \partial_j \partial_k u_l = 2M_{ijkl} \partial_j \partial_k u_l = 0 ,$$

Since $M_{ijkl} = C^{ijkl}$, we can substitute it and, because of $\partial_j \partial_k] = 0$, we have the standard equilibrium equation

$$C^{ijkl} \partial_j \partial_k u_l = 0 .$$

Our calculations confirms the following:

**Proposition 12.** Any total derivative term in the elastic energy functional can be regarded as only a formal expression. It does not remove any subset of the elastic constants from the equilibrium equation, that is, for an arbitrary anisotropic material an elastic null Lagrangian does not exist.
3.4 Wave equation

The wave propagation in linear elasticity for anisotropic media is described by the following equation:

\[ \rho g^{il} \ddot{u}_l - C^{ijkl} \partial_j \partial_k u_l = 0. \] (73)

Here the displacement covector \( u_l \) is assumed to be a function of the time coordinate and of the position of a point in the medium. All other coefficients, the mass density \( \rho \), the elasticity tensor \( C^{ijkl} \), and the metric tensor \( g^{il} \), are assumed to be constant; moreover, we use Cartesian coordinates, that is, the Euclidean metric reads \( g^{ij} = \text{diag}(1, 1, 1) \).

We make a plane wave ansatz, with the notation as in Nayfeh (1985)[27]:

\[ u_l = U_i e^i(\zeta n_j x^j - \omega t). \] (74)

Here \( U_i \) is the covector of a complex constant amplitude, \( \zeta \) the wave-number, \( n_j \) the propagation unit covector, \( \omega \) the angular frequency, and \( i^2 = -1 \). Substituting (74) into (73), we obtain a system of three homogeneous algebraic equations

\[ (\rho \omega^2 g^{il} - C^{ijkl} \zeta^2 n_j n_k) U_l = 0. \] (75)

It has a non-trivial solution if and only if the characteristic equation holds,

\[ \det (\rho \omega^2 g^{il} - C^{ijkl} \zeta^2 n_j n_k) = 0. \] (76)

With the definitions of the Christoffel tensor

\[ \Gamma^{il} := \frac{1}{\rho} C^{ijkl} n_j n_k \] (77)

and of the phase velocity \( v := \omega/\zeta \), the system (75) takes the form

\[ (v^2 g^{il} - \Gamma^{il}) U_l = 0, \] (78)

whereas the characteristic equation reads

\[ \det (v^2 g^{il} - \Gamma^{il}) = 0. \] (79)

Due to the minor and major symmetries of the elasticity tensor, the Christoffel tensor turns out to be symmetric

\[ \Gamma^{[ij]} = 0. \] (80)

For a symmetric matrix, the characteristic equation has only real solutions. Every positive real solution corresponds to an acoustic wave propagating in the direction of the wave covector \( n_i \). Thus, in general, for a given propagation covector \( n_i \), three different acoustic waves are possible. The cases with zero solutions (null modes) and negative solutions (standing waves) must be treated as unphysical because they do not satisfy the causality requirements.

A symmetric tensor, by itself, cannot be decomposed directly under the action of the group \( GL(3, \mathbb{R}) \). However, the irreducible decomposition of the elasticity tensor generates
the corresponding decomposition of the Christoffel tensor. Substituting the irreducible SA-decomposition into (77), we obtain
\[ \Gamma^{ij} = S^{ij} + A^{ij}, \] (81)
where
\[ S^{il} := S^{ijkl} n_j n_k = S^{li}, \quad \text{and} \quad A^{il} := A^{ijkl} n_j n_k = A^{li}. \] (82)
These two symmetric tensors correspond to the Cauchy and non-Cauchy parts of the elasticity tensor. We will call \( S^{ij} \) Cauchy Christoffel tensor and \( A^{ij} \) non-Cauchy Christoffel tensor.

Substituting (33) and (35) into (82), we find the explicit expressions
\[ S^{il} = \frac{1}{3\rho} \left( C^{ijkl} + C^{iklj} + C^{iljk} \right) n_j n_k \] (83)
and
\[ A^{il} = \frac{1}{3\rho} \left( 2C^{ijkl} - C^{iklj} - C^{ilkj} \right) n_j n_k. \] (84)

If we transvect \( n_l \) with \( A^{il} \), we observe a generic fact:

**Proposition 13.** For each elasticity tensor \( C^{ijkl} \) and for each wave covector \( n_i \),
\[ A^{il} n_l = 0. \] (85)

**Proof.** We have \( A^{il} n_l = A^{ijkl} n_j n_k n_l = A^{ijkl} n_j n_k n_l \) (37) = 0.

**Proposition 14.** The determinant of the non-Cauchy Christoffel tensor is equal to zero,
\[ \det(A^{ij}) = 0. \] (86)

**Proof.** Equation (85) can be viewed as a linear relation between the rows of the matrix \( A^{il} \); this proves its singularity.

The wave equation (78) can be rewritten as
\[ (v^2 g^{il} - S^{il} - A^{il}) U_l = 0, \] (87)
with the characteristic equation
\[ \det(v^2 g^{il} - S^{il} - A^{il}) = 0. \] (88)
We can now recognize the reason why a pure non-Cauchy medium is forbidden. In the case \( S^{il} = 0 \), the characteristic equation (88) takes the form
\[ \det(v^2 g^{il} - A^{il}) = 0. \] (89)
Since \( \det(A^{il}) = 0 \), at least one of its eigenvalues is zero. Such a null-mode wave is forbidden because of causality reasons.

The Christoffel tensor is real and symmetric, thus all its eigenvalues are real and the associated eigenvectors are orthogonal. In order to have three real positive eigenvalues, we need to satisfy the condition of positive definiteness of the matrix \( \Gamma^{ij} \). In this case, the following possibilities arise:
(i) All the eigenvalues are distinct \( v_1^2 > v_2^2 > v_3^2 \),

(ii) two eigenvalues coincide \( v_1^2 > v_2^2 = v_3^2 \) or \( v_1^2 = v_2^2 > v_3^2 \), or

(iii) three eigenvalues coincide \( v_1^2 = v_2^2 = v_3^2 \).

The Christoffel matrix depends on the propagation vector \( \Gamma^{ij}(\vec{n}) \), thus the conditions indicated above determine the directions in which the wave can propagate.

### 3.5 Polarizations

Equation (78), or its decomposed form (87), represents acoustic wave propagation in an elastic medium. It is an eigenvector problem in which eigenvalues \( v^2 \) are the solutions of (88). In general, three distinct real positive solutions correspond to three independent waves \((1) U_l, (2) U_l, \) and \((3) U_l\), called acoustic polarizations, see Nayfeh (1985)[27].

For isotropic materials, there are three pure polarizations: one longitudinal or compression wave with
\[
\vec{U} \times \vec{n} = 0,
\]
that is, the polarization is directed along the propagation vector, and two transverse or shear waves with
\[
\vec{U} \cdot \vec{n} = 0,
\]
that is, the polarization is normal to the propagation vector. In general, for anisotropic materials, three pure modes do not exist. The identification of the pure modes and the condition for their existence is an interesting problem.

Let us see how the irreducible decomposition of the elasticity tensor, which we applied to the Christoffel tensor, can be used here. For a chosen direction vector \( \vec{n} \), we introduce a vector and a scalar according to
\[
S^i := \Gamma^{ij} n_j, \quad S := \Gamma^{ij} n_i n_j.
\]
This notation is consistent since \( S^i \) and \( S \), due to (85), depend only on the Cauchy part of the elasticity tensor,
\[
S^i = S^{ij} n_j, \quad S = S^{ij} n_i n_j.
\]

**Proposition 15.** Let \( n^i \) denotes an allowed direction for the propagation of a compression wave. Then the velocity \( v_L \) of this wave in the direction of \( n^i \) is determined only by the Cauchy part of the elasticity tensor:
\[
v_L = \sqrt{S}.
\]

**Proof.** For the longitudinal wave, \( u_j = \alpha n_j \). Thus, (78) becomes
\[
(v^2 g^{ij} - \Gamma^{ij}) n_j = 0 \quad \text{or} \quad v^2 g^{ij} n_j = S^{ij} n_j.
\]
Transvecting both sides of the last equation with \( n_i \), we obtain (94). \( \square \)
Let us now discuss in which directions the three pure polarizations can propagate.

**Proposition 16.** For a medium with a given elasticity tensor, all three purely polarized waves (one longitudinal and two transverse) can propagate in the direction $\vec{n}$ if and only if

$$S^i = Sn^i.$$ (96)

**Proof.** Since the Christoffel matrix is symmetric, it has real eigenvalues and three orthogonal eigenvectors. We have three pure polarizations if and only if one of these eigenvectors points in the direction of $\vec{n}$. Consequently,

$$\Gamma_{ij}n_j = v_L^2 n^i.$$ (97)

Since $\Gamma_{ij}n_j = S^{ij}n_j = S^{i}$, we have $S^i = v_L^2 n^i$ or, after substituting (94), we obtain (96). \qed

Accordingly, for a given medium, the directions of the purely polarized waves depend on the Cauchy part of the elasticity tensor alone. In other words, two materials, with the same Cauchy parts $S^{ijkl}$ of the elasticity tensor but different non-Cauchy parts $A^{ijkl}$, have the same pure wave propagation directions and the same longitudinal velocity.

### 4 Examples

#### 4.1 Isotropic media

In order to clarify the results discussed above, consider as an example an isotropic elastic medium. Then, the elasticity tensor can be expressed in terms of the metric tensor $g^{ij}$ as

$$C^{ijkl} = \lambda g^{ij}g^{kl} + \mu \left( g^{ik}g^{lj} + g^{il}g^{jk} \right) = \lambda g^{ij}g^{kl} + 2\mu g^{i(k}g^{l)j},$$ (98)

with the Lamé moduli $\lambda$ and $\mu$, see Marsden & Hughes (1983)[26]. The irreducible decomposition of this expression involves the terms

$$S^{ijkl} = (\lambda + 2\mu) g^{ij}g^{kl} = \frac{\lambda + 2\mu}{3} \left( g^{ij}g^{kl} + g^{ik}g^{lj} + g^{il}g^{jk} \right),$$ (99)

and

$$A^{ijkl} = \frac{\lambda - \mu}{3} \left( 2g^{ij}g^{kl} - g^{ik}g^{lj} - g^{il}g^{jk} \right).$$ (100)

The symmetric second rank tensor $\Delta$ of equation (49), which is equivalent to $A$, reads

$$\Delta_{mn} = \frac{1}{4} \epsilon_{mik} \epsilon_{njl} A^{ijkl} = \frac{\lambda - \mu}{12} \epsilon_{mik} \epsilon_{njl} \left( 2g^{ij}g^{kl} - g^{ik}g^{lj} - g^{il}g^{jk} \right)$$ (101)

or

$$\Delta_{ij} = \frac{\lambda - \mu}{2} g_{ij}.$$ (102)

Of course, (102) is much more compact than (100).
Accordingly, the irreducible decomposition reveals two fundamental combinations of the isotropic elasticity constants. We denote them by

$$\begin{align*}
\alpha & := \frac{\lambda + 2\mu}{3}, \\
\beta & := \frac{\lambda - \mu}{3}.
\end{align*}$$

Then we can rewrite the canonical representation (98) of the elasticity tensor as

$$C^{ijkl} = \alpha (g^{ij}g^{kl} + g^{ik}g^{lj} + g^{il}g^{jk}) + \beta (2g^{ij}g^{kl} - g^{ik}g^{lj} - g^{il}g^{jk}) = (\alpha + 2\beta) g^{ij}g^{kl} + 2(\alpha - \beta) g^{ik}g^{lj}. \quad (104)$$

The alternative MN-decomposition can also be derived straightforwardly. We can read off from (98) directly that

$$M^{ijkl} = C^{i(jk)l} = (\lambda + \mu) g^{i(j}g^{k)l} + \mu g^{il}g^{jk}. \quad (105)$$

In a similar way we can find for $N^{ijkl}$ the expression

$$N^{ijkl} = C^{i[jk]l} = (\lambda - \mu) g^{i[j}g^{k]l}. \quad (106)$$

The key fact is here that this decomposition is characterized by three parameters

$$\alpha' = \frac{\lambda + \mu}{2}, \quad \beta' = \mu, \quad \gamma' = \frac{\lambda - \mu}{2}. \quad (107)$$

In both decompositions, the Cauchy relations are described by the equation

$$\lambda = \mu. \quad (108)$$

Only in the special case when the Cauchy relations hold, the tensors $S^{ijkl}$ and $M^{ijkl}$ are equal.

Consider now the energy functional and the problem of the identification of the null Lagrangian. When the SA-decomposition is substituted into the energy functional, we remain with two irreducible parts with their leading coefficients $\alpha$ and $\beta$. Thus, the number of parameters is not reduced.

As for the MN-decomposition, the $N$ term, with its leading coefficient $\gamma'$, being a total derivative, does not contribute to the equilibrium equation. We still remain with two terms with their leading parameters $\alpha'$ and $\beta'$. Moreover, as it was shown above, these terms can be recovered in the initial functional. It proves that there is no such thing as a null Lagrangian in elasticity, not even in the simplest isotropic case.

In this isotropic case, the main difference between the SA- and the MN-decomposition becomes manifest. The irreducible SA-decomposition dictates the existence of two independent fundamental parameters of the isotropic medium, whereas the reducible MN-decomposition deals with three linearly dependent parameters.

Let us calculate now the characteristic velocities of the acoustic waves. Two two Christoffel matrices are

$$S^{ij} = \alpha \left( g^{ij} + 2n^i n^j \right). \quad (109)$$
and
\[ A^{ij} = \beta \left( g^{ij} - n^i n^j \right) \] \hspace{1cm} (110)

Hence,
\[ S^i = 3\alpha n^i, \hspace{0.5cm} S = 3\alpha \] \hspace{1cm} (111)

Observe that the condition \((\ref{96})\) of pure polarization is satisfied now identically. Thus, we recover the well known fact that in isotropic media every direction allows propagation of purely polarized waves.

In terms of the moduli \(\alpha\) and \(\beta\), the characteristic equation for the acoustic waves takes the form
\[ \det \left[ (v^2 - \alpha + \beta)g^{ij} - (2\alpha + \beta)n^i n^j \right] = 0 \] \hspace{1cm} (112)

The longitudinal wave velocity
\[ v_1^2 = S = 3\alpha = \lambda - 2\mu \] \hspace{1cm} (113)

is one solution for this equation. Indeed, in this case, the characteristic equation \((\ref{112})\) turns into the identity \(\det (g^{ij} - n^i n^j) = 0\), which proves that \((\ref{113})\) is an eigenvalue.

We immediately find additional solutions for this equation. For
\[ v_2^2 = \alpha - \beta = \mu \] \hspace{1cm} (114)

\((\ref{112})\) turns into the identity \(\det (n^i n^j) = 0\). Moreover, the rank of this matrix is equal to one: \(\text{rank}(n^i n^j) = 1\). Consequently, the multiplicity of the root \((\ref{114})\) is equal to two and the third eigenvalue is the same one:
\[ v_3^2 = \alpha - \beta = \mu \] \hspace{1cm} (115)

With the use of the irreducible decomposition, we can answer now the following question: Does there exist an anisotropic medium in which purely polarized waves can propagate in every direction? It is clear that media with the same Cauchy part of the elasticity tensor (and arbitrary non-Cauchy part) have the same property in this respect. As a consequence we have

**Proposition 17.** The most general type of an anisotropic medium that allows propagation of purely polarized waves in an arbitrary direction has an elasticity tensor of the form

\[ C^{ijkl} = \begin{bmatrix} \alpha & \alpha/3 + 2\rho_1 & \alpha/3 + 2\rho_2 & 2\rho_3 & 0 & 0 \\ * & \alpha & \alpha/3 + 2\rho_4 & 0 & 2\rho_5 & 0 \\ * & * & \alpha & 0 & 0 & 2\rho_6 \\ * & * & * & \alpha/3 - \rho_4 & -\rho_6 & -\rho_5 \\ * & * & * & * & \alpha/3 - \rho_2 & -\rho_3 \\ * & * & * & * & * & \alpha/3 - \rho_1 \end{bmatrix}, \] \hspace{1cm} (116)

where \(\rho_1, \ldots, \rho_6\) are arbitrary parameters. The velocity of the longitudinal waves in this medium is \(v_L = \sqrt{3\alpha} = \sqrt{\lambda - 2\mu}\).
4.2 Cubic media

Cubic crystals are described by three independent elasticity constants. In a properly chosen coordinate system, they can be put, see Nayfeh (1985), into the following Voigt matrix:

\[
C_{ijkl} = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1144} & C_{1222} & C_{1233} & C_{1244} & C_{1333} & C_{1344} & C_{1444} \\
C_{2222} & C_{2233} & C_{2244} & C_{2333} & C_{2344} & C_{2444} \\
C_{3333} & C_{3344} & C_{3444} \\
C_{4444} \\
\end{bmatrix}
\]

We decompose it irreducibly by using (42) and (43) and find the Cauchy part

\[
S_{ijkl} = \begin{bmatrix}
\tilde{\alpha} & \tilde{\beta} & 0 & 0 \\
* & \tilde{\alpha} & \tilde{\beta} & 0 \\
* & * & \tilde{\alpha} & 0 \\
* & * & * & \tilde{\beta} \\
\end{bmatrix}, \quad \tilde{\alpha} = C_{11}, \quad \tilde{\beta} = \frac{1}{3} (C_{12} + 2C_{66}), \quad (118)
\]

and the non-Cauchy part

\[
A_{ijkl} = \begin{bmatrix}
0 & 2\tilde{\gamma} & 2\tilde{\gamma} & 0 & 0 & 0 \\
* & 0 & 2\tilde{\gamma} & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & * & * & -\tilde{\gamma} & 0 & 0 \\
* & * & * & * & -\tilde{\gamma} & 0 \\
\end{bmatrix}, \quad \tilde{\gamma} = \frac{1}{3} (C_{12} - C_{66}), \quad (119)
\]

Accordingly, the elasticity tensor is expressed in terms of the three new elastic constants \(\tilde{\alpha}, \tilde{\beta}, \text{ and } \tilde{\gamma}\). For the Cauchy relation we have

\[
\tilde{\gamma} = 0, \quad \text{or} \quad C_{12} = C_{66}. \quad (120)
\]

The Cauchy part of the Christoffel tensor takes the form

\[
S_{i} = \begin{bmatrix}
\widetilde{\alpha} n_1^2 + 3\tilde{\beta}(n_2^2 + n_3^2) \\
2\tilde{\beta} n_1 n_2 \\
2\tilde{\beta} n_1 n_3 \\
\end{bmatrix} \quad (121)
\]

The corresponding vector and scalar invariants (93) read

\[
S_{i} = \begin{bmatrix}
n_1((\tilde{\alpha} - 3\tilde{\beta}) n_1^2 + 3\tilde{\beta}) \\
n_2((\tilde{\alpha} - 3\tilde{\beta}) n_2^2 + 3\tilde{\beta}) \\
n_3((\tilde{\alpha} - 3\tilde{\beta}) n_3^2 + 3\tilde{\beta}) \\
\end{bmatrix} \quad (122)
\]
and
\[ S = (\tilde{\alpha} - 3\tilde{\beta})(n_1^4 + n_2^4 + n_3^4) + 3\tilde{\beta}, \] (123)
respectively. The non-Cauchy part of the Christoffel tensor turns out to be
\[ A^{ij} = \tilde{\gamma} \begin{bmatrix} -n_2^2 - n_3^2 & n_1n_2 & n_1n_3 \\ n_1n_2 & -n_1^2 - n_3^2 & n_2n_3 \\ n_1n_3 & n_2n_3 & -n_1^2 - n_2^2 \end{bmatrix}. \] (124)
The corresponding invariants \( A^{ij}n_j \) and \( A^{ij}n_in_j \) are zero.

Let us use the pure polarization condition \( S^i = Sn^i \) in order to derive the purely polarized propagation directions. We can easily derive all solutions of this equation
- Edges: \( n_1 = 1, n_2 = n_3 = 0 \), etc. The longitudinal velocity is
  \[ v_L = \sqrt{\tilde{\alpha}} = \sqrt{C^{11}}. \] (125)
- Face diagonals: \( n_1 = n_2 = \frac{1}{2}\sqrt{2}, n_3 = 0 \), etc. The longitudinal velocity is
  \[ v_L = \frac{1}{2} \sqrt{2(\tilde{\alpha} + 3\tilde{\beta})} = \frac{1}{2} \sqrt{2(C^{11} + C^{12} + 2C^{66})}. \] (126)
- Space diagonals: \( n_1 = n_2 = n_3 = \frac{1}{3}\sqrt{3} \), etc. The longitudinal velocity is
  \[ v_L = \frac{1}{3} \sqrt{3(\tilde{\alpha} + 6\tilde{\beta})} = \frac{1}{3} \sqrt{3(C^{11} + 2C^{12} + 4C^{66})}. \] (127)

Due to Propositions 15 and 16, a wide class of materials with the same \( S \)-tensor and an arbitrary \( A \)-tensor will have exactly the same directions and the same velocities of the longitudinal waves. The elasticity tensor for such materials can be written as
\[ C^{ijkl} = \begin{bmatrix} \tilde{\alpha} & \tilde{\beta} + 2\rho_1 & \tilde{\beta} + 2\rho_2 & 2\rho_3 & 0 & 0 \\ * & \tilde{\alpha} & \tilde{\beta} + 2\rho_4 & 0 & 2\rho_5 & 0 \\ * & * & \tilde{\alpha} & 0 & 0 & 2\rho_6 \\ * & * & * & \tilde{\beta} - \rho_4 & -\rho_6 & -\rho_5 \\ * & * & * & * & \tilde{\beta} - \rho_2 & -\rho_3 \\ * & * & * & * & * & \tilde{\beta} - \rho_1 \end{bmatrix}. \] (128)

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A Irreducible decomposition of tensors of rank \( p \)

In order to understand what should be considered as the proper decomposition of a tensor, which is of rank \( p \) in its covariant or contravariant indices, that is, \( T^{ijkl} \) or \( T_{ijkl} \), we must look closer on its precise algebraic meaning. Let us start with a vector space \( V \) over the number field \( F \). This construction comes together with the group of general linear transformations \( GL(\text{dim}V, F) \). This group includes all invertible square matrices of the size \( \text{dim}V \times \text{dim}V \) with the entries in \( F \). In our case, \( F = \mathbb{R} \) and \( V = \mathbb{R}^3 \). Consequently, we are dealing with the group \( GL(3, \mathbb{R}) \) that can be considered as a group of transformations between two bases \( \{e_i\} \) and \( \{e'_i\} \) of \( V \):

\[
e'_i = L_{i'}^i e_i, \quad L_{i'}^i \in GL(3, \mathbb{R}).
\]  

(129)

The vector space \( V \) cannot be decomposed invariantly into a sum of subspaces. Indeed, every two nonzero vectors of \( V \) can be transformed each other by the use of certain matrix of \( GL(3, \mathbb{R}) \). Thus, the group \( GL(3, \mathbb{R}) \) acts transitively on \( V \).

A tensor of rank \( p \) is defined as a multi-linear map from the Cartesian product of \( p \) copies of \( V \) into the field \( F \),

\[
T : V \times \cdots \times V \rightarrow F.
\]  

(130)

The set of all tensors \( T \) of the rank \( p \) compose a vector space by itself, say \( \mathcal{T} \). The dimension of this tensor space is equal to \( n^p \). Thus, for rank 4 in 3d we have \( 3^4 = 81 \). As a basis in \( \mathcal{T} \), we can take tensor products of basis elements in \( V \),

\[
e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}.
\]  

(131)

Accordingly, an arbitrary contravariant tensor of rank \( p \) can be expressed as

\[
T = T^{i_1i_2\cdots i_p} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}.
\]  

(132)

Under a transformation (129) of a basis of the vector space, the basis of the tensor space is multiplied by a product of the matrices \( L'_{i'}^i \). This is a ‘derived transformation’ in the sense of Littlewood (1944)\[24\]. An important fact is that the tensor space \( \mathcal{T} \) is decomposed to a direct sum of subspaces which are invariant under general linear transformations. For instance, the span of the basis

\[
e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}
\]  

(133)

describes the subspace of all tensors which are symmetric under the permutation of the two first indices. Another subspace is obtained as the span of all basis tensors of the form

\[
e_{[i_1} \otimes e_{i_2]} \otimes \cdots \otimes e_{i_p},
\]  

(134)

which is the subspace of all tensors antisymmetric under the permutation of two first indices.

By taking different permutations of the basis vectors we can obtain different subtensors (elements of the subspace of \( \mathcal{T} \) of a tensor of arbitrary rank. There are all together \( p! \) permutations for the system of \( p \) objects. In fact, every \( GL(3, \mathbb{R}) \)-invariant subtensor can
be obtained by the operator which permutes the indices, see Littlewood (1944)\cite{24}. Since we need the sums of all the tensors with permuted indices, we must extend the group of permutations to its group algebra (Frobenius algebra). This construction involves formal sums, in an addition to the group multiplication.

In this way, each tensor of rank two or greater can be \textit{irreducibly decomposed} under the action of the linear group $GL(3,\mathbb{R})$. The original tensor is written as a linear combination of simpler tensors of rank $p$, which, under the action of $GL(3,\mathbb{R})$, transform only under themselves. These partial tensors obey their specific symmetries \textit{in addition} to the symmetries of the initial tensor. Possible types of the irreducible tensors are determined by the use of Young’s tableaux.

\subsection{Example: Young’s decomposition of second rank tensors}

Consider a generic second rank tensor $T_{ij}$ which is decomposed into a sum of its symmetric and antisymmetric parts,

$$T^{ij} = T^{(ij)} + T^{[ij]} .$$

(135)

It means that one starts with the tensor space

$$\mathcal{T} = \text{Span} \{ e_i \otimes e_j \}$$

(136)

The span of the linear combination of basis tensors

$$\mathcal{S} = \text{Span}\{ (e_{(i} \otimes e_{j)}) \}$$

(137)

composes the subspace of tensors symmetric under the permutation of two indices. Indeed, an arbitrary tensor in $\mathcal{S}$ is decomposed as

$$S^{ij} = S^{ij} e_{(i} \otimes e_{j)} .$$

(138)

Thus we have $S^{[ij]} = 0$. Another subspace $\mathcal{A}$ is obtained as a span of the linear combination

$$\mathcal{A} = \text{Span} \{ e_{[i} \otimes e_{j]} \} .$$

(139)

This is a subspace of antisymmetric tensors,

$$A = A^{[ij]} e_{[i} \otimes e_{j]} .$$

(140)

Here $A^{(ij)} = 0$. In this way, the tensor space is represented as a direct sum of its subspaces $\mathcal{T} = \mathcal{S} \oplus \mathcal{A}$. These subspaces are invariant under the transformations of $GL(3,\mathbb{R})$ while further decomposition into smaller subspaces is impossible. Hence we have an irreducible decomposition.

In Young’s description, this decomposition is given by two diagrams that are graphical representations of the permutation group $S_2$

$$\square \otimes \square = \square \oplus \square .$$

(141)
The left-hand side here denotes the tensor product of two vectors, i.e., a generic asymmetric second order tensor. The right-hand side is given as a sum of two second order tensors. The symmetric tensor is represented by the row diagram while the antisymmetric tensor is given by the column diagram. In our simplest example, only two tables given in (141) are allowed. The next step is to fill in the tables with the different indices of the tensor. In particular, we write

$$i \otimes j = \left( I + (ij) \right) A_{ij} = A_{ij} + A_{ji}, \quad (142)$$

and

$$i \otimes j = \left( I - (ij) \right) A_{ij} = A_{ij} - A_{ji}. \quad (143)$$

Here $I$ denotes the identity operator, $(ij)$ is the permutation operator, $(I+(ij))$ and $(I-(ij))$ are respectively Young’s symmetrizer and antisymmetrizer operators. Thus, we have

$$i \otimes j = \alpha i \otimes j \oplus \beta \frac{i}{j}. \quad (144)$$

The decomposition (135) is then obtained by inserting suitable leading coefficients $\alpha$ and $\beta$. In general, these coefficients are calculated by using a combinatorial formula. It is easy to see that, for the completely symmetric and completely antisymmetric diagrams of $n$ cells, the coefficients are equal to $1/n!$. Thus, the coefficients in (144) are equal to $1/2$ and the decomposition (135) is recovered.

### A.2 Young’s decomposition of fourth rank tensors

The first step is to construct Young’s tableaux corresponding to a fourth rank tensor.

**Rule 1.** The four cells representing the indices of the tensor must be glued into tables of all possible shapes. The only restriction is that the number of cells in any row must be less or equal to the number of cells in the previous row. The number of irreducible subtensors of a given tensor is equal to the number of the tables of all possible shapes.

Due to this rule, a generic fourth order tensor can be irreducibly decomposed into the sum of five independent parts. These parts are described by the following Young’s diagrams, which are the graphical representation of the permutation group $S_4$, see Boerner (1970) or Hamermesh (1989),

$$\begin{align*}
\begin{array}{cccc}
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\end{array}
& =
\begin{array}{cccc}
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\end{array}
\oplus
\begin{array}{cccc}
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\end{array}
\oplus
\begin{array}{cccc}
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\end{array}
\oplus
\begin{array}{cccc}
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\end{array}
\oplus
\begin{array}{cccc}
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\boxdot & \boxdot & \boxdot & \boxdot \\
\end{array}
\end{align*} \quad (145)$$

The left-hand side describes a generic fourth order tensor. On the right-hand side, the first diagram represents the completely symmetric tensor. The middle diagrams are for the tensors which are partially symmetric and partially antisymmetric. The last diagram represents a completely antisymmetric tensor.

The next step of Young’s procedure is to fill in the tables with the indices. In order to avoid the repetitions, the following rule is used:
Rule 2. In each row and each column of Young’s table, the positions of the indices, i.e., the numbers \( i_1, i_2, \ldots, i_p \), are inserted in the increasing order. The symmetrization operators correspond to the rows of the table. Since different rows contain no common indices, the corresponding symmetrization operators commute. Different antisymmetrization operators correspond to the columns of the table. They also commute with each other and can be taken in an arbitrary order. The symmetrization and antisymmetrization operators act on the same indices, hence they do not commute and thus must be taken in some fixed order.

Thus, due to graphical representation (145) of the permutation group \( S_4 \), we have the following parts of a generic three dimensional tensor \( T_{i_1i_2i_3i_4} \)

\[
\begin{align*}
  i_1 \otimes i_2 \otimes i_3 \otimes i_4 &= \alpha \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \end{pmatrix} + \beta \left( \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ i_3 & i_4 \end{pmatrix} + \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ i_2 & i_4 \end{pmatrix} + \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ i_3 & i_4 \end{pmatrix} + \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ i_2 & i_4 \end{pmatrix} \right) + \\
  & \quad \gamma \left( \begin{pmatrix} i_1 & i_2 \\ i_3 & i_4 \end{pmatrix} + \begin{pmatrix} i_1 & i_3 \\ i_2 & i_4 \end{pmatrix} \right) + \delta \left( \begin{pmatrix} i_1 & i_2 \\ i_3 & i_4 \end{pmatrix} + \begin{pmatrix} i_1 & i_3 \\ i_2 & i_4 \end{pmatrix} + \begin{pmatrix} i_1 & i_4 \\ i_2 & i_3 \end{pmatrix} \right) + \varepsilon \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \end{pmatrix} \quad (146)
\end{align*}
\]

Rule 3. The coefficients \( \alpha, \beta, \gamma, \delta, \varepsilon \) are determined by the combinatorial formula, see Hamermesh (1989)[14]. They can be calculated using the Mathematica package ’Combinatorics’. The first and the last coefficients are especially simple. For an \( n \)-order tensor,

\[
\alpha = \varepsilon = \frac{1}{n!}. \quad (147)
\]

For a generic 4th rank tensor, all these diagrams are relevant. In an \( n \)-dimensional space, it has \( n^4 \) independent components, which are distributed between the diagrams (146). The explicit form of the corresponding terms can be found in Wade (1941)[40].

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