Covariance-engaged Classification of Sets via Linear Programming

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Abstract: Set classification aims to classify a set of observations as a whole, as opposed to classifying individual observations separately. To formally understand the unfamiliar concept of binary set classification, we first investigate the optimal decision rule under the normal distribution, which utilizes the empirical covariance of the set to be classified. We show that the number of observations in the set plays a critical role in bounding the Bayes risk. Under this framework, we further propose new methods of set classification. For the case where only a few parameters of the model drive the difference between two classes, we propose a computationally-efficient approach to parameter estimation using linear programming, leading to the Covariance-engaged Linear Programming Set (CLIPS) classifier. Its theoretical properties are investigated for both independent case and various (short-range and long-range dependent) time series structures among observations within each set. The convergence rates of estimation errors and risk of the CLIPS classifier are established to show that having multiple observations in a set leads to faster convergence rates, compared to the standard classification situation in which there is only one observation in the set. The applicable domains in which the CLIPS performs better than competitors are highlighted in a comprehensive simulation study. Finally, we illustrate the usefulness of the proposed methods in classification of real image data in histopathology.

Key words and phrases: Bayes risk, $\ell_1$-minimization, Quadratic discriminant analysis, Set classification, Sparsity.
1. Introduction

Classification is a useful tool in statistical learning with applications in many important fields. A classification method aims to train a classification rule based on the training data to classify future observations. Some popular methods for classification include linear discriminant analyses, quadratic discriminant analyses, logistic regressions, support vector machines, neural nets and classification trees. Traditionally, the task at hand is to classify an observation into a class label.

Advances in technology have eased the production of a large amount of data in various areas such as healthcare and manufacturing industries. Oftentimes, multiple samples collected from the same object are available. For example, it has become cheaper to obtain multiple tissue samples from a single patient in cancer prognosis (Miedema et al., 2012). To be explicit, Miedema et al. (2012) collected 348 independent cells, each contains observations of varying numbers (tens to hundreds) of nuclei. Here, each cell, rather than each nucleus, is labelled as either normal or cancerous. Each observation of nuclei contains 51 measurements of shape and texture features. A statistical task herein is to classify the whole set of observations from a single set (or all nuclei in a single cell) to normal or cancerous group. Such a problem was coined as set classification by Ning and Karypis (2009), studied in Wang et al. (2012) and Jung and Qiao (2014), and was seen in the image-based pathology literature (Samsudin and Bradley, 2010, Wang et al., 2010, Cheplygina et al., 2015, Shifat-E-Rabbi et al., 2020) and in face recognition based on pictures obtained from multiple cameras, sometime called image set classification (Arandjelovic and Cipolla, 2006, Wang et al., 2012). The set classification is not identical to the multiple-instance learning (MIL) (Maron and
Lozano-Pérez (1998), Chen et al. (2006), Ali and Shah (2010), Carbonneau et al. (2018) as seen by Kuncheva (2010). A key difference is that in set classification a label is given to sets whereas observations in a set have different labels in the MIL setting.

While conventional classification methods predict a class label for each observation, care is needed in generalizing those for set classification. In principle, more observations should ease the task at hand. Moreover, higher-order statistics such as variances and covariances can now be exploited to help classification. Our approach to set classification is to use the extra information, available to us only when there are multiple observations. To elucidate this idea, we illustrate samples from three classes in Fig. 1. All three classes have the same mean, and Classes 1 and 2 have the same marginal variances. Classifying a single observation near the mean to any of these distributions seems difficult. On the other hand, classifying several independent observations from the same class should be much easier. In particular, a set classification method needs to incorporate the difference in covariances to differentiate these classes.

In this work, we study a binary set classification framework, where a set of observations $X = \{X_1, \ldots, X_M\}$ is classified to either $Y = 1$ or $Y = 2$. In particular, we propose set classifiers that extend quadratic discriminant analysis to the set classification setting, and are designed to work well in set-classification of high-dimensional data whose distributions are similar to those in Fig. 1.

To provide a fundamental understanding of the set classification problem, we establish the Bayesian optimal decision rule under normality and homogeneity (i.i.d) assumptions. This Bayes rule utilizes the covariance structure of the testing set of future observations.
Figure 1: A 2-dimensional toy example showing classes with no difference in the mean or the marginal variance.

We show in Section 2 that it becomes much easier to make accurate classification for a set when the set size, \( m_0 \), increases. In particular, we demonstrate that the Bayes risk can be reduced exponentially in the set size \( m_0 \). To the best of our knowledge, this is the first formal theoretical framework for set classification problems in the literature.

Built upon the Bayesian optimal decision rule, we propose new methods of set classification in Section 3. For the situation where the dimension \( p \) of the feature vectors is much smaller than the total number of training samples, we demonstrate that a simple plug-in classifier leads to satisfactory risk bounds similar to the Bayes risk. Again, a large set size plays a key role in significantly reducing the risk. In high-dimensional situations where the number of parameters to be estimated (\( \approx p^2 \)) is large, we make an assumption that only a few parameters drive the difference of two classes. With this sparsity assumption, we propose to estimate the parameters in the classifier via linear programming, and the resulting classifiers
are called Covariance-engaged LInear Programming Set (CLIPS) classifiers. Specifically, the quadratic and linear parameters in the Bayes rule can be efficiently estimated under the sparse structure, thanks to the extra observations in the training set due to having sets of observations. Our estimation approaches are closely related to and built upon the successful estimation strategies in Cai et al. (2011) and Cai and Liu (2011). In estimation of the constant parameter, we perform a logistic regression with only one unknown, given the estimates of quadratic and linear parameters. This allows us to implement CLIPS classifier with high computation efficiency.

We provide a thorough study of theoretical properties of CLIPS classifiers and establish an oracle inequality in terms of the excess risk, in Section 4. In particular, the estimates from CLIPS are shown to be consistent, and the strong signals are always selected with high probability in high dimensions. Moreover, the excess risk can be reduced by having more observations in a set, one of the new phenomena for set classification, which are different from that obtained by naively having pooled observations.

In the conventional classification problem where $m_0 = 1$, a special case of the proposed CLIPS classifier becomes a new sparse quadratic discriminant analysis (QDA) method (cf. Fan et al., 2015, 2013; Li and Shao, 2015; Jiang et al., 2018; Qin, 2018; Zou, 2019; Gaynanova and Wang, 2019; Pan and Mai, 2020). As a byproduct of our theoretical study, we show that the new QDA method enjoys better theoretical properties compared to state-of-the-art sparse QDA methods such as Fan et al. (2015).

The advantages of our set classifiers are further demonstrated in comprehensive simulation studies. Moreover, we provide an application to histopathology in classifying sets of
nucleus images to normal and cancerous tissues in Section 5. Proofs of main results and technical lemmas can be found in the supplementary material. Also present in the supplementary material is a study on the case where observations in a set demonstrate certain spatial and temporal dependent structures. There, we utilize various (both short- and long-range) dependent time series structures within each set by considering a very general vector linear process model.

2. Set Classification

We consider a binary set-classification problem. The training sample \( \{(X_i, Y_i)\}_{i=1}^N \) contains \( N \) sets of observations. Each set, \( X_i = \{X_{i1}, X_{i2}, \ldots, X_{iM_i}\} \subset \mathbb{R}^p \), corresponds to one object, and is assumed to be from one of the two classes. The corresponding class label is denoted by \( Y_i \in \{1, 2\} \). The number of observations within the \( i \)th set is denoted by \( M_i \) and can be different among different sets. Given a new set of observations \( (X^\dagger, Y^\dagger) \), the goal of set classification is to predict \( Y^\dagger \) accurately based on \( X^\dagger \) using a classification rule \( \phi(\cdot) \in \{1, 2\} \) trained on the training sample.

To formally introduce set classification problem and study its fundamental properties, we start with a setting in which the sets in each class are homogeneous in the sense that all the observations in a class, regardless of the set membership, follow the same distribution independently. Specifically, we assume both the \( N \) sets \( \{(X_i, Y_i)\}_{i=1}^N \) and the new set \( (X^\dagger, Y^\dagger) \) are generated in the same way as \( (X, Y) \) independently. To describe the generating process of \( (X, Y) \), we denote the marginal class probabilities by \( \pi_1 = \text{pr}(Y = 1) \) and \( \pi_2 = \text{pr}(Y = 2) \), and the marginal distribution of the set size \( M \) by \( p_M \). We assume that the random variables
2.1 Covariance-engaged Set Classifiers

Suppose that there are $M^\dagger = m$ observations in the set $X^\dagger = \{X_1^\dagger, \ldots, X_m^\dagger\}$ that is to be classified (called testing set), and its true class label is $Y^\dagger$. The Bayes optimal decision rule classifies the set $X^\dagger = \{x_1, \ldots, x_m\}$ to Class 1 if the conditional class probability of Class 1 is greater than that of Class 2, that is, $\mathsf{pr}(Y^\dagger = 1 \mid M^\dagger = m, X_j^\dagger = x_j, j = 1, \ldots, m) > 1/2$. This is equivalent to

$$
g(x_1, \ldots, x_m) = \frac{1}{m} \log \left( \frac{\pi_1 p_M(m) \prod_{j=1}^{m} f_1(x_j)}{\pi_2 p_M(m) \prod_{j=1}^{m} f_2(x_j)} \right)
= \frac{1}{m} \log(\pi_1/\pi_2) - \frac{1}{2} \log(|\Sigma_1|/|\Sigma_2|) - \frac{1}{2} \mu_1^T \Sigma_1^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma_2^{-1} \mu_2
+ (\Sigma_1^{-1} \mu_1 - \Sigma_2^{-1} \mu_2)^T \bar{x} + \frac{1}{2} \bar{x}^T (\Sigma_2^{-1} - \Sigma_1^{-1}) \bar{x} + \frac{1}{2} \text{tr}(\Sigma_2^{-1} - \Sigma_1^{-1}) S. \tag{2.1}
$$

Here $|\Sigma_k|$ denotes the determinant of the matrix $\Sigma_k$ for $k = 1, 2$, $\bar{x} = \sum_{j=1}^{m} x_j/m$ and $S = \sum_{j=1}^{m} (x_j - \bar{x})(x_j - \bar{x})^T / m$ are the sample mean and sample covariance of the testing set. Note that the realization $X^\dagger = \{x_1, x_2, \ldots, x_m\}$ implies both the number of observations $m$ and $Y$ are independent. In other words, the class membership $Y$ can not be predicted just based on the set size $M$. Conditioned on $M = m$ and $Y = y$, observations $X_1, X_2, \ldots, X_M$ in the set $X$ are independent and each distributed as $f_y$. This is equivalent to

$$
\text{Here } |\Sigma_k| \text{ denotes the determinant of the matrix } \Sigma_k \text{ for } k = 1, 2, \text{ and } \bar{x} = \sum_{j=1}^{m} x_j/m \text{ and } S = \sum_{j=1}^{m} (x_j - \bar{x})(x_j - \bar{x})^T / m \text{ are the sample mean and sample covariance of the testing set. Note that the realization } X^\dagger = \{x_1, x_2, \ldots, x_m\} \text{ implies both the number of observations } m \text{ and } Y \text{ are independent. In other words, the class membership } Y \text{ can not be predicted just based on the set size } M. \text{ Conditioned on } M = m \text{ and } Y = y, \text{ observations } X_1, X_2, \ldots, X_M \text{ in the set } X \text{ are independent and each distributed as } f_y. \text{ This is equivalent to } \pi_1 p_M(m) \prod_{j=1}^{m} f_1(x_j) > \pi_2 p_M(m) \prod_{j=1}^{m} f_2(x_j), \text{ due to Bayes theorem and the independence assumption among } Y \text{ and } M^\dagger. \text{ Let us now assume that the conditional distributions are both normal, that is, } f_1 \sim N(\mu_1, \Sigma_1) \text{ and } f_2 \sim N(\mu_2, \Sigma_2). \text{ Then the Bayes optimal decision rule depends on the quantity } g(x_1, \ldots, x_m) = \frac{1}{m} \log \left( \frac{\pi_1 p_M(m) \prod_{j=1}^{m} f_1(x_j)}{\pi_2 p_M(m) \prod_{j=1}^{m} f_2(x_j)} \right)
= \frac{1}{m} \log(\pi_1/\pi_2) - \frac{1}{2} \log(|\Sigma_1|/|\Sigma_2|) - \frac{1}{2} \mu_1^T \Sigma_1^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma_2^{-1} \mu_2
+ (\Sigma_1^{-1} \mu_1 - \Sigma_2^{-1} \mu_2)^T \bar{x} + \frac{1}{2} \bar{x}^T (\Sigma_2^{-1} - \Sigma_1^{-1}) \bar{x} + \frac{1}{2} \text{tr}(\Sigma_2^{-1} - \Sigma_1^{-1}) S. \tag{2.1}
$$
2.1 Covariance-engaged Set Classifiers

and the i.i.d. observations $x_j$ for $j = 1, \ldots, m$. The Bayes rule can be expressed as

$$
\phi_B(X^\dagger) = 2 - \mathbb{1}\{g(x_1, \ldots, x_m) > 0\}, \quad \text{(2.2)}
$$

where

$$
g(x_1, \ldots, x_m) = \frac{1}{m} \log(\pi_1/\pi_2) + \beta_0 + \beta^T \bar{x} + \bar{x}^T \nabla \bar{x}/2 + \text{tr}(\nabla S)/2,
$$

in which the constant coefficient $\beta_0 = \{-\log(|\Sigma_1|/|\Sigma_2|) - \mu_1^T \Sigma_1^{-1} \mu_1 + \mu_2^T \Sigma_2^{-1} \mu_2\}/2 \in \mathbb{R}$,

the linear coefficient vector $\beta = \Sigma_1^{-1} \mu_1 - \Sigma_2^{-1} \mu_2 \in \mathbb{R}^p$ and the quadratic coefficient matrix $
abla = \Sigma_2^{-1} - \Sigma_1^{-1} \in \mathbb{R}^{p \times p}$. The Bayes rule $\phi_B$ under the normal assumption in (2.2) uses the

summary statistics $m$, $\bar{x}$ and $S$ of $X^\dagger$.

We refer to (2.2) and any estimated version of it as a covariance-engaged set classifier. In

Section 3 several estimation approaches for $\beta_0$, $\beta$ and $\nabla$ will be proposed. In this section,

we further discuss a rationale for considering (2.2).

The covariance-engaged set classifier (2.2) resembles the conventional QDA classifier. As

a natural alternative to (2.2), one may consider the sample mean $\bar{x}$ as a representative of

the testing set and apply QDA to $\bar{x}$ directly to make a prediction. In other words, one is

about to classify this single observation $\bar{x}$ to one of the two normal distributions, that is,

$f'_1 \sim N(\mu_1, \Sigma_1/m)$ and $f'_2 \sim N(\mu_2, \Sigma_2/m)$. This simple idea leads to

$$
\phi_{B,\bar{x}}(X^\dagger) = 2 - \mathbb{1}\{g_{\text{QDA}}(\bar{x}) > 0\}, \quad \text{(2.3)}
$$

where

$$
g_{\text{QDA}}(\bar{x}) = \frac{1}{m} \log(\pi_1/\pi_2) + \beta'_0 + \beta'^T \bar{x} + \bar{x}^T \nabla \bar{x}/2,
$$

in which $\beta'_0 = \{-\frac{1}{m} \log(|\Sigma_1|/|\Sigma_2|) - \mu_1^T \Sigma_1^{-1} \mu_1 + \mu_2^T \Sigma_2^{-1} \mu_2\}/2$. One major difference between

(2.2) and (2.3) is that the term $\text{tr}(\nabla S)/2$ is absent from (2.3). Indeed, the advantage of

(2.2) over (2.3) comes from the extra information in the sample covariance $S$ of $X^\dagger$. In the

regular classification setting, (2.2) coincides with (2.3) since $\text{tr}(\nabla S)/2$ vanishes when $X^\dagger$ is
a singleton.

Given multiple observations in the testing set, another natural approach is a majority vote applied to the QDA decisions of individual observations:

$$\phi_{MV}(x^t) = 2 - \mathbb{1}\left\{ \frac{1}{m} \sum_{j=1}^{m} \text{sign}[g_{QDA}(x_j)] > 0 \right\},$$

(2.4)

where \(\text{sign}(t) = 1, 0, -1\) for \(t > 0\), \(t = 0\) and \(t < 0\) respectively. In contrast, since \(g(x^t) = \frac{1}{m} \sum_{j=1}^{m} g_{QDA}(x_j)\), our classifier (2.2) predicts the class label by a weighted vote of individual QDA decisions. In this sense, the majority voting scheme (2.4) can be viewed as a discretized version of (2.2). In Section 3, we demonstrate that our set classifier (2.2) performs significantly better than (2.4).

**Remark 1.** We have assumed that \(M\) and \(Y\) are independent in the setting. In fact, this assumption is not essential and can be relaxed. In a more general setting, there can be two different distributions of \(M\), \(p_{M1}(m)\) and \(p_{M2}(m)\) conditional on \(Y = 1\) and \(Y = 2\) respectively. Our analysis throughout the paper remains the same except that they would replace two identical factors \(p_M(m)\) in the first equality of (2.1). If \(p_{M1}(m)\) and \(p_{M2}(m)\) are dramatically different, then the classification is easier as one can make decision based on the observed value of \(m\). In this paper, we only consider the more difficult setting where \(Y\) and \(M\) are independent.

### 2.2 Bayes Risk

We show below an advantage of having a set of observations for prediction, compared to having a single observation. For this, we suppose for now that the parameters \(\mu_k\) and \(\Sigma_k\),
2.2 Bayes Risk

$k = 1, 2,$ are known and make the following assumptions. Denote $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ as the greatest and smallest eigenvalues of a symmetric matrix $A$.

**Condition 1.** The spectrum of $\Sigma_k$ is bounded below and above: there exists some universal constant $C_e > 0$ such that $C_e^{-1} \leq \lambda_{\text{min}}(\Sigma_k) \leq \lambda_{\text{max}}(\Sigma_k) \leq C_e$ for $k = 1, 2$.

**Condition 2.** The support of $p_M$ is bounded between $c_m m_0$ and $C_m m_0$, where $c_m$ and $C_m$ are universal constants and $m_0 = \mathbb{E}(M)$. In other words, $p_M(a) = 0$ for any integer $a < c_m m_0$ or $> C_m m_0$. The set size $m_0$ can be large or growing when a sequence of models are considered.

**Condition 3.** The prior class probability is bounded away from 0 and 1: there exists a universal constant $0 < C_{\pi} < 1/2$ such that $C_{\pi} \leq \pi_1, \pi_2 \leq 1 - C_{\pi}$.

We denote $R_{Bk} = \text{pr}(\phi_B(X^\dagger) \neq k \mid Y^\dagger = k)$ as the risk of the Bayes classifier (2.2) given $Y^\dagger = k$. Let $\delta = \mu_2 - \mu_1$. For a matrix $B \in \mathbb{R}^{p \times p}$, we denote $\|B\|_F = (\sum_{i=1}^{p} \sum_{j=1}^{p} B_{ij}^2)^{1/2}$ as its Frobenius norm, where $B_{ij}$ is its $ij$th element. For a vector $a \in \mathbb{R}^p$, we denote $\|a\| = (\sum_{i=1}^{p} a_i^2)^{1/2}$ as its $\ell_2$ norm. The quantity $D_p = (\|\nabla\|_F^2 + \|\delta\|^2)^{1/2}$ plays an important role in deriving a convergence rate of the Bayes risk $R_B = \pi_1 R_{B1} + \pi_2 R_{B2}$. Although the Bayes risk does not have a closed form, we show that under mild assumptions, it converges to zero at a rate on the exponent.

**Theorem 1.** Suppose that Conditions 1-3 hold. If $D_p^2 m_0$ is sufficiently large, then $R_B \leq 4 \exp\left(-c' m_0 D_p^2\right)$ for some small constant $c' > 0$ depending on $C_e$, $c_m$ and $C_{\pi}$ only. In particular, as $D_p^2 m_0 \to \infty$, we have $R_B \to 0$. 
The significance of having a set of observations is illustrated by this fundamental theorem. When \( p_M(1) = 1 \), which implies \( M^\dagger = 1 \) and \( m_0 = 1 \), Theorem 1 provides a Bayes risk bound \( R_B \leq 4 \exp(-c'D_p^2) \) for the theoretical QDA classifier in the regular classification setting.

To guarantee a small Bayes risk for QDA, it is clear that \( D_p^2 \) must be sufficiently large. In comparison, for the set classification to be successful, we may allow \( D_p^2 \) to be very close to zero, as long as \( m_0D_p^2 \) is sufficiently large. The Bayes risk of \( \phi_B \) can be reduced exponentially in \( m_0 \) because of the extra information from the set.

We have discussed an alternative classifier via using the sample mean \( \bar{x} \) as a representative of the testing set, leading to \( \phi_{B,\bar{x}} \) \( (2.3) \). The following proposition quantifies its risk, which has a slower rate than that of Bayes classifier \( R_B \).

**Proposition 1.** Suppose that Conditions 1-3 hold. Denote the risk of classifier \( \phi_{B,\bar{x}} \) in \( (2.3) \) as \( R_{\bar{x}} \). Assume \( \|\nabla\|^2_{F} + m_0\|\delta\|^2 \) is sufficiently large. Then \( R_{\bar{x}} \leq 4 \exp(-c'(\|\nabla\|^2_{F} + m_0\|\delta\|^2)) \) for some small constant \( c' > 0 \) depending on \( C_e, c_m \) and \( C_\pi \) only. In addition, the rate on the exponent cannot be improved in general, i.e., \( R_{\bar{x}} \geq \exp(-c''(\|\nabla\|^2_{F} + m_0\|\delta\|^2)) \) for some small constant \( c'' > 0 \).

**Remark 2.** Compared to the result in Theorem 1, the above proposition implies that classifier \( \phi_{B,\bar{x}} \) needs a stronger assumption but has a slower rate of convergence when the mean difference \( m_0\|\delta\|^2 \) is dominated by the covariance difference \( \|\nabla\|^2_{F} \). After all, this natural \( \bar{x} \)-based classification rule only relies on the first moment of the data set \( \mathcal{X}^\dagger \) while the sufficient statistics, the first two moments, are fully used by the covariance-engaged classifier in \( (2.2) \).
3. Methodologies

We now consider estimation procedures for $\phi_B$ based on $N$ training sets $\{(X_i, Y_i)\}_{i=1}^N$. In Section 3.1, we first consider a moderate-dimensional setting where $p \leq c_0 m_0 N$ with a sufficiently small constant $c_0 > 0$. In this case we apply a naive plug-in approach using natural estimators of the parameters $\pi_k, \mu_k$ and $\Sigma_k$. A direct estimation approach using linear programming, suitable for high-dimensional data, is introduced in Section 3.2. Hereafter, $p = p(N)$ and $m_0 = m_0(N)$ are considered as functions of $N$ as $N$ grows.

3.1 Naive Estimation Approaches

The prior class probabilities $\pi_1$ and $\pi_2$ can be consistently estimated by the class proportions in the training data, $\hat{\pi}_1 = N_1/N$ and $\hat{\pi}_2 = N_2/N$, where $N_k = \sum_{i=1}^N 1\{Y_i = k\}$. Let $n_k = \sum_{i=1}^N M_i 1\{Y_i = k\}$ denote the total sample size for Class $k = 1, 2$. The set membership is ignored at the training stage, due to the homogeneity assumption. Note $n_k$, $n_1 + n_2$ and $N_k$ are random while $N$ is deterministic. One can obtain consistent estimators of $\mu_k$ and $\Sigma_k$ based on the training data and plug them in (2.2). It is natural to use the maximum likelihood estimators given $n_k$,

$$\hat{\mu}_k = \sum_{(i,j):Y_i=k} X_{ij}/n_k \text{ and } \hat{\Sigma}_k = \sum_{(i,j):Y_i=k} \{(X_{ij} - \hat{\mu}_k)(X_{ij} - \hat{\mu}_k)^T\}/n_k. \quad (3.5)$$

For classification of $X^\dagger = \{X_1^\dagger, \ldots, X_{M^\dagger}^\dagger\}$ with $M^\dagger = m$, $X_i^\dagger = x_i$, the set classifier (2.2) is estimated by

$$\hat{\phi}(X^\dagger) = 2 - 1 \left\{ \frac{1}{m} \log(\hat{\pi}_1/\hat{\pi}_2) + \hat{\beta}_0 + \hat{\beta}^T \bar{x} + \bar{x}^T \nabla \bar{x} / 2 + \text{tr}(\nabla S) / 2 > 0 \right\}, \quad (3.6)$$
where \( \hat{\beta}_0 = -\frac{1}{2}\left\{ \log(|\hat{\Sigma}_1|/|\hat{\Sigma}_2|) - \hat{\mu}_1^T\hat{\Sigma}_1^{-1}\hat{\mu}_1 + \hat{\mu}_2^T\hat{\Sigma}_2^{-1}\hat{\mu}_2 \right\} \), \( \hat{\beta} = \hat{\Sigma}_1^{-1}\hat{\mu}_1 - \hat{\Sigma}_2^{-1}\hat{\mu}_2 \) and \( \hat{\nabla} = \hat{\Sigma}_2^{-1} - \hat{\Sigma}_1^{-1} \). In (3.6) we have assumed \( p < n_k \) so that \( \hat{\Sigma}_k \) is invertible.

The generalization error of set classifier (3.6) is \( \hat{R} = \pi_1\hat{R}_1 + \pi_2\hat{R}_2 \) where \( \hat{R}_k = \text{pr}(\hat{\phi}(X') \neq k \mid Y' = k) \). The classifier itself depends on the training data \( \{(X_i, Y_i)\}_{i=1}^N \) and hence is random. In the equation above, \( \text{pr} \) is understood as the conditional probability given the training data. Theorem 2 reveals a theoretical property of \( \hat{R} \) in a moderate-dimensional setting which allows \( p, N, m_0 \) to grow jointly. This includes the traditional setting in which \( p \) is fixed.

**Theorem 2.** Suppose that Conditions 1-3 hold. For any fixed \( L > 0 \), if \( D_p^2m_0 \geq C_0 \) for some sufficiently large \( C_0 > 0 \) and \( p \leq c_0Nm_0 \), \( p^2/(Nm_0D_p^2) \leq c_0 \), \( \log p \leq c_0N \) for some sufficiently small constant \( c_0 > 0 \), then with probability at least \( 1 - O(p^{-L}) \) we have \( \hat{R} \leq 4\exp\left(-c'm_0D_p^2\right) \) for some small constant \( c' > 0 \) depending on \( C_x, c_m, L \) and \( C_e \).

In Theorem 2, large values of \( m_0 \) not only relax the assumption on \( D_p \) but also reduce the Bayes risk exponentially in \( m_0 \) with high probability. A similar result for QDA, where \( M_i = M^T \equiv 1 \) and \( m_0 = 1 \), was obtained in Li and Shao (2015) under a stronger assumption \( p^2/(ND_p^2) \to 0 \).

For the high-dimensional data where \( p = p(N) \gg Nm_0 \) and hence \( p > n_k \) with probability 1 for \( k = 1, 2 \) by Condition 2, it is problematic to plug in the estimators (3.5) since \( \hat{\Sigma}_k \) is rank deficient with probability 1. A simple remedy is to use a diagonalized or enriched version of \( \hat{\Sigma}_k \), defined by \( \hat{\Sigma}_{k(d)} = \text{diag}\{\hat{\sigma}_{k,ii}\}_{i=1,\ldots,p} \) or \( \hat{\Sigma}_{k(e)} = \hat{\Sigma}_k + \delta I_p \), where \( \delta > 0 \) and \( I_p \) is a \( p \times p \) identity matrix. Both \( \hat{\Sigma}_{k(d)} \) and \( \hat{\Sigma}_{k(e)} \) are invertible. However, to our best knowledge, no theoretical guarantee has been obtained without some structural assumptions.
3.2 A Direct Approach via Linear Programming

To have reasonable classification performance in high-dimensional data analysis, one usually has to take advantage of certain extra information of the data or model. There are often cases where only a few elements in $\nabla = \Sigma_2^{-1} - \Sigma_1^{-1}$ and $\beta = \Sigma_1^{-1} \mu_1 - \Sigma_2^{-1} \mu_2$ truly drive the difference between the two classes. A naive plug-in method proposed in Section 3.1 has ignored such potential structure of the data. We assume that both $\nabla$ and $\beta$ are known to be sparse such that only a few elements of those are nonzero. In light of this, the Bayes decision rule (2.2) implies the dimension of the problem can be significantly reduced, which makes consistency possible even in the high-dimensional setting.

We propose to directly estimate the quadratic term $\nabla$, the linear term $\beta$ and the constant $\beta_0$ coefficients respectively, taking advantage of the assumed sparsity. As the estimates are efficiently calculated by linear programming, the resulting classifiers are called Covariance-engaged Linear Programming Set (CLIPS) classifiers.

We first deal with the estimation of the quadratic term $\nabla = \Sigma_2^{-1} - \Sigma_1^{-1}$, which is the difference between the two precision matrices. We use some key techniques developed in the literature of precision matrix estimation (cf. Meinshausen and Bühlmann, 2006; Bickel and Levina, 2008; Friedman et al., 2008; Yuan, 2010; Cai et al., 2011; Ren et al., 2015). These methods estimate a single precision matrix with a common assumption that the underlying true precision matrix is sparse in some sense. For the estimation of the difference, we propose to use a two-step thresholded estimator.

As the first step, we adopt the CLIME estimator (Cai et al., 2011) to obtain initial estimators $\hat{\Omega}_1$ and $\hat{\Omega}_2$ of the precision matrices $\Sigma_1^{-1}$ and $\Sigma_2^{-1}$. Let $\|B\|_1 = \sum_{i,j} |B_{ij}|$ and
3.2 A Direct Approach via Linear Programming

\[ \|B\|_\infty = \max_{i,j} |B_{ij}| \] be the vector \( \ell_1 \) norm and vector supnorm of a \( p \times p \) matrix \( B \) respectively.

The CLIME estimators are defined as

\[ \tilde{\Omega}_k = \arg\min_{\Omega} \|\Omega\|_1 \text{ subject to } \|\hat{\Sigma}_k\Omega - I\|_\infty < \lambda_{1,N}, \; k = 1, 2, \tag{3.7} \]

for some \( \lambda_{1,N} > 0 \).

Having obtained \( \tilde{\Omega}_1 \) and \( \tilde{\Omega}_2 \), in the second step, we take a thresholding procedure on their difference, followed by a symmetrization to obtain our final estimator \( \tilde{\nabla} = (\tilde{\nabla}_{ij}) \) where

\[ \tilde{\nabla}_{ij} = \min\{\nabla_{ij}, \nabla_{ji}\}, \nabla_{ij} = (\tilde{\Omega}_{2,ij} - \tilde{\Omega}_{1,ij})I\left\{|\tilde{\Omega}_{2,ij} - \tilde{\Omega}_{1,ij}| > \lambda'_{1,N}\right\}, \tag{3.8} \]

for some thresholding level \( \lambda'_{1,N} > 0 \).

Although this thresholded CLIME difference estimator is obtained by first individually estimating \( \Sigma_k^{-1} \), we emphasize that the estimation accuracy only depends on the sparsity of their difference \( \nabla \) rather than the sparsity of either \( \Sigma_1^{-1} \) or \( \Sigma_2^{-1} \) under a relatively mild bounded matrix \( \ell_1 \) norm condition. We will show in Theorem 3 in Section 4 that if the true precision matrix difference \( \nabla \) is negligible, \( \tilde{\nabla} = 0 \) with high probability. When \( \tilde{\nabla} = 0 \), our method described in (3.12) becomes a linear classifier adaptively. The computation of \( \tilde{\nabla} \) (3.8) is fast, since the first step (CLIME) can be recast as a linear program and the second step is a simple thresholding procedure.

Remark 3. As an alternative, one can also consider a direct estimation of \( \nabla \) that does not rely on individual estimates of \( \Sigma_k^{-1} \). For example, by allowing some deviations from the identity \( \Sigma_1 \nabla \Sigma_2 - \Sigma_1 + \Sigma_2 = 0 \), Zhao et al. (2014) proposed to minimize the vector \( \ell_1 \) norm of \( \nabla \). Specifically, they proposed \( \tilde{\nabla}^{ZCL} \in \arg\min_B \|B\|_1 \text{ subject to } \|\hat{\Sigma}_1 B \hat{\Sigma}_2 - \hat{\Sigma}_1 + \hat{\Sigma}_2\|_\infty \leq \lambda''_{1,n} \), where \( \lambda''_{1,n} \) is some thresholding level. This method, however, is computationally expensive.
(as it has \(O(p^2)\) number of linear constraints when casted to linear programming) and can only handle relatively small size of \(p\). See also Jiang et al. (2018). We chose to use (3.8) mainly because of fast computation.

Next we consider the estimation of the linear coefficient vector \(\beta = \beta_1 - \beta_2\), where \(\beta_k = \Sigma_k^{-1}\mu_k\), \(k = 1, 2\). In the literature of sparse QDA and sparse LDA, typical sparsity assumptions are placed on \(\mu_1 - \mu_2\) and \(\Sigma_1 - \Sigma_2\) (see Li and Shao (2015) or placed on both \(\beta_1\) and \(\beta_2\) (see, for instance Cai and Liu 2011, Fan et al. 2015). In the latter case, \(\beta\) is also sparse as it is the difference of two sparse vectors. For the estimation of \(\beta\), we propose a new method which directly imposes sparsity on \(\beta\), without specifying the sparsity for \(\mu_k\), \(\Sigma_k\) or \(\beta_k\) except for some relatively mild conditions (see Theorem 4 for details.)

The true parameter \(\beta_k\) satisfies \(\Sigma_k\beta_k - \mu_k = 0\). However, due to the rank-deficiency of \(\hat{\Sigma}_k\), there are either none or infinitely many \(\theta_k\)'s that satisfy an empirical equation \(\hat{\Sigma}_k\theta_k - \hat{\mu}_k = 0\). Here, \(\hat{\mu}_k\) and \(\hat{\Sigma}_k\) are defined in (3.5). We relax this constraint and seek a possibly non-sparse pair \((\theta_1, \theta_2)\) with the smallest \(\ell_1\) norm difference. We estimate the coefficients \(\beta\) by \(\tilde{\beta} = \tilde{\beta}_1 - \tilde{\beta}_2\), where

\[
(\tilde{\beta}_1, \tilde{\beta}_2) = \arg\min_{(\theta_1, \theta_2): \|\theta_k\|_1 \leq L_1} \|\theta_1 - \theta_2\|_1 \text{ subject to } \|\hat{\Sigma}_k\theta_k - \hat{\mu}_k\|_\infty < \lambda_{2,N}, \ k = 1, 2, \ (3.9)
\]

where \(L_1\) is some sufficiently large constant introduced only to ease theoretical evaluations. In practice, the constraint \(\|\theta_k\|_1 \leq L_1\) can be removed without affecting the solution. Note that Jiang et al. (2018) proposed to estimate \((\Sigma_1^{-1} + \Sigma_2^{-1})(\mu_1 - \mu_2)\) rather than \(\beta = \Sigma_1^{-1}\mu_1 - \Sigma_2^{-1}\mu_2\).

The direct estimation approach for \(\beta\) above shares some similarities with that of Cai and Liu (2011), especially in the relaxed \(\ell_\infty\) constraint. However Cai and Liu (2011) focused on
3.2 A Direct Approach via Linear Programming

A direct estimation of \( \Sigma^{-1}(\mu_2 - \mu_1) \) for linear discriminant analysis in which \( \Sigma = \Sigma_1 = \Sigma_2 \), while we target on \( \Sigma_2^{-1}\mu_2 - \Sigma_1^{-1}\mu_1 \) instead. Our procedure \( (3.9) \) can be recast as a linear programming problem (see, for example, Candes and Tao 2007; Cai and Liu 2011) and is computationally efficient.

Finally, we consider the estimation of the constant coefficient \( \beta_0 \). The conditional class probability \( \eta(x_1, \ldots, x_m) = \text{pr}(Y = 1 \mid M = m, X_i = x_i, i = 1, \ldots, m) \) that a set belongs to Class 1 given \( X = \{x_1, \ldots, x_m\} \) can be evaluated by the following logit function,

\[
\log \left( \frac{\eta(x_1, \ldots, x_m)}{1 - \eta(x_1, \ldots, x_m)} \right) = \log \frac{\pi_1}{\pi_2} + \log \left\{ \frac{\prod_{i=1}^{m} f_1(x_i)}{\prod_{i=1}^{m} f_2(x_i)} \right\} = \log(\pi_1/\pi_2) + m(\beta_0 + \bar{x}^T \tilde{\beta} + \frac{1}{2} \bar{x}^T \nabla \bar{x} + \frac{1}{2} \text{tr}(\nabla S)),
\]

where \( \bar{x} \) and \( S \) are the sample mean and covariance of the set \( \{x_1, \ldots, x_m\} \) respectively. Having obtained our estimators \( \tilde{\nabla} \) and \( \tilde{\beta} \) from \( (3.8) \) and \( (3.9) \), and estimated \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) by \( N_1/N \) and \( N_2/N \) from the training data, we have only a scalar \( \beta_0 \) undecided. We may find an estimate \( \tilde{\beta}_0 \) by conducting a simple logistic regression with dummy independent variable \( M_i \) and offset \( \log(\hat{\pi}_1/\hat{\pi}_2) + M_i \left( \bar{X}_i^T \tilde{\beta} + \bar{X}_i^T \tilde{\nabla} \bar{X}_i/2 + \text{tr}(\tilde{\nabla} S_i)/2 \right) \) for the \( i \)th set of observations in the training data, where \( M_i, \bar{X}_i, \) and \( S_i \) are sample size, sample mean, and sample covariance of the \( i \)th set. In particular, we solve

\[
\tilde{\beta}_0 = \arg\min_{\theta_0 \in \mathbb{R}} \ell(\theta_0 \mid \{(X_i, Y_i)\}_{i=1}^{N}, \tilde{\beta}, \tilde{\nabla}), \text{ where the negative log-likelihood is } \ell(\theta_0 \mid \{(X_i, Y_i)\}_{i=1}^{N}, \tilde{\beta}, \tilde{\nabla}) = \frac{1}{N} \sum_{i=1}^{N} \left( (Y_i - 2)M_i \left( \theta_0 + \frac{\log(\hat{\pi}_1/\hat{\pi}_2)}{M_i} + \bar{X}_i^T \tilde{\beta} + \bar{X}_i^T \tilde{\nabla} \bar{X}_i/2 + \text{tr}(\tilde{\nabla} S_i)/2 \right) \right) + \log \left[ 1 + \exp \left\{ M_i \left( \theta_0 + \frac{\log(\hat{\pi}_1/\hat{\pi}_2)}{M_i} + \bar{X}_i^T \tilde{\beta} + \bar{X}_i^T \tilde{\nabla} \bar{X}_i/2 + \text{tr}(\tilde{\nabla} S_i)/2 \right) \right\} \right].
\]
Since there is only one independent variable in the logistic regression above, the optimization can be easily and efficiently solved.

For the purpose of evaluating theoretical properties, we apply the sample splitting technique (Wasserman and Roeder, 2009; Meinshausen and Bühlmann, 2010). Specifically, we randomly choose the first batch of $N_1/2$ and $N_2/2$ sets from two classes in the training data to obtain estimators $\tilde{\nabla}$ and $\tilde{\beta}$ using (3.8) and (3.9). Then $\tilde{\beta}_0$ is estimated based on the second batch along with $\tilde{\nabla}$ and $\tilde{\beta}$ using (3.10). We plug all the estimators in (3.8), (3.9) and (3.10) into the Bayes decision rule (2.2) and obtain the CLIPS classifier,

$$
\tilde{\phi}(X^t) = 2 - \mathbb{1}\left\{ \frac{\log(\hat{\pi}_1/\hat{\pi}_2)}{m} + \tilde{\beta}_0 + \tilde{\beta}^T \bar{x} + \bar{x}^T \tilde{\nabla} \bar{x}/2 + \text{tr}(\tilde{\nabla}S)/2 > 0 \right\},
$$

where $\bar{x}$ and $S$ are sample mean and covariance of $X^t$ and $M^t = m$ is its size.

4. Theoretical Properties of CLIPS

In this section, we derive the theoretical properties of the estimators from (3.8)–(3.10) as well as generalization errors for the CLIPS classifier (3.12). In particular, we demonstrate the advantages of having sets of independent observations in contrast to classical QDA setting with individual observations under the homogeneity assumption of Section 2. Parallel results under various time series structures can be found in the supplementary material.

To establish the statistical properties of the thresholded CLIME difference estimator $\tilde{\nabla}$ defined in (3.8), we assume that the true quadratic parameter $\nabla = \Sigma_2^{-1} - \Sigma_1^{-1}$ has no more than $s_q$ nonzero entries,

$$
\nabla \in \mathcal{F}\mathcal{M}_0(s_q) = \{ A = (a_{ij}) \in \mathbb{R}^{p \times p}, \text{symmetric}: \sum_{i,j=1}^{p} \mathbb{1}\{a_{ij} \neq 0\} \leq s_q \}. \quad (4.13)
$$
Denote $\text{supp}(A)$ as the support of the matrix $A$. We summarize the estimation error and a subset selection result in the following theorem.

**Theorem 3.** Suppose Conditions 1-3 hold. Moreover, assume $\nabla \in \mathcal{F}\mathcal{M}_0(s_q), \|\Sigma_k^{-1}\|_{\ell_1} \leq C_{\ell_1}$ with some constant $C_{\ell_1} > 0$ for $k = 1, 2$ and $\log p \leq c_0 N$ with some sufficiently small constant $c_0 > 0$. Then for any fixed $L > 0$, with probability at least $1 - O(p^{-L})$, we have that

$$
\|\hat{\nabla} - \nabla\|_\infty \leq 2\lambda_{1,N}',
$$

$$
\|\hat{\nabla} - \nabla\|_F \leq 2\sqrt{s_q}\lambda_{1,N}',
$$

$$
\|\hat{\nabla} - \nabla\|_1 \leq 2s_q\lambda_{1,N}',
$$

as long as $\lambda_{1,N} \geq CC_{\ell_1}\sqrt{\frac{\log p}{Nm_0}}$ and $\lambda_{1,N}' \geq 8C_{\ell_1}\lambda_{1,N}$ in (3.8), where $C$ depends on $L, C_e, C_\pi$ and $c_m$ only. Moreover, we have $\text{pr}(\text{supp}(\hat{\nabla}) \subset \text{supp}(\nabla)) = 1 - O(p^{-L})$.

**Remark 4.** The parameter space $\mathcal{F}\mathcal{M}_0(s_q)$ can be easily extended into an entry-wise $\ell_q$ ball or weak $\ell_q$ ball with $0 < q < 1$ (Abramovich et al., 2006) and the estimation results in Theorem 3 remain valid with appropriate sparsity parameters. The subset selection result also remains true and the support of $\hat{\nabla}$ contains those important signals of $\nabla$ above the noise level $\sqrt{(\log p)/Nm_0}$. To simplify the analysis, we only consider $\ell_0$ balls in this work.

**Remark 5.** Theorem 3 implies that both the error bounds of estimating $\nabla$ under vector $\ell_1$ norm and Frobenius norm rely on the sparsity $s_q$ imposed on $\nabla$ rather than those imposed on $\Sigma_2^{-1}$ or $\Sigma_1^{-1}$. Therefore, even if both $\Sigma_2^{-1}$ and $\Sigma_1^{-1}$ are relatively dense, we still have an accurate estimate of $\nabla$ as long as $\nabla$ is very sparse and $C_{\ell_1}$ is not large.

The proof of Theorem 3, provided in the supplementary material, partially follows from Cai et al. (2011).
Next we assume $\beta = \beta_1 - \beta_2$ is sparse in the sense that it belongs to the $s_l$-sparse ball,

$$\beta \in \mathcal{F}_0(s_l) = \{ \alpha = (a_j) \in \mathbb{R}^p : \sum_{j=1}^{p} \mathbb{1}\{\alpha_j \neq 0\} \leq s_l \}. \quad (4.14)$$

Theorem 4 gives the rates of convergence of the linear coefficient estimator $\tilde{\beta}$ in (3.9) under the $\ell_1$ and $\ell_2$ norms. Both depend on the sparsity of $\beta$ only rather than that of $\beta_1$ or $\beta_2$.

**Theorem 4.** Suppose Conditions 1-3 hold. Moreover, assume that $\beta \in \mathcal{F}_0(s_l)$, $\log p \leq c_0 N$, $\|\beta_k\|_1 \leq C_\beta$ and $\|\mu_k\| \leq C_\mu$ with some constants $C_\beta, C_\mu > 0$ for $k = 1, 2$ and some sufficiently small constant $c_0 > 0$. Then for any fixed $L > 0$, with probability at least $1 - O(p^{-L})$, we have that

$$\|\tilde{\beta} - \beta\|_1 \leq C'' C_{\ell_1} s_l \lambda_{2,N},$$

$$\|\tilde{\beta} - \beta\| \leq C'' C_{\ell_1} \sqrt{s_l \lambda_{2,N}},$$

as long as $\lambda_{2,N} \geq C' \sqrt{\frac{\log p}{N m_0}}$ in (3.9), where $\max\{\|\Sigma_1^{-1}\|_1, \|\Sigma_2^{-1}\|_1\} \leq C_{\ell_1}$ and $C'', C'$ depend on $L, c_e, c_m, C_\pi, C_\beta$ and $C_\mu$ only.

**Remark 6.** The parameter space $\mathcal{F}_0(s)$ can be easily extended into an $\ell_q$ ball or weak $\ell_q$ ball with $0 < q < 1$ as well and the results in Theorem 4 remain valid with appropriate sparsity parameters. We only focus on $\mathcal{F}_0(s)$ in this paper to ease the analysis.

Lastly, we derive the rate of convergence for estimating the constant coefficient $\beta_0$. Since $\tilde{\beta}_0$ is obtained by maximizing the log-likelihood function after plugging $\tilde{\beta}$ and $\tilde{\nabla}$ in (3.10), the behavior of our estimator $\tilde{\beta}_0$ critically depends on the accuracy for estimating $\beta$ and $\nabla$. Theorem 5 provides the result for $\tilde{\beta}_0$ based on certain general initial estimators $\tilde{\beta}$ and $\tilde{\nabla}$ with the following mild condition.
Condition 4. The expectation of the conditional variance of class label $Y$ given $X$ is bounded below, that is, $E(\text{Var}(Y \mid X)) > C_{\log} > 0$, where $C_{\log}$ is some universal constant.

Theorem 5. Suppose Conditions 1-4 hold, $\log p \leq c_0 N$ with some sufficiently small constant $c_0 > 0$ and $\|\mu_k\| \leq C_{\mu}$ with some constant $C_{\mu} > 0$ for $k = 1, 2$. Besides, we have some initial estimators $\hat{\beta}$, $\hat{\nabla}$, $\hat{\pi}_1$ and $\hat{\pi}_2$ such that $m_0(1 + \sqrt{(\log p)/m_0})\|\hat{\beta} - \beta\| + m_0(1 + (\log p)/m_0)\|\hat{\nabla} - \nabla\|_1 + \max_{k=1,2} |\pi_k - \hat{\pi}_k| \leq C_{p}$ for some sufficiently small constant $C_{p} > 0$ with probability at least $1 - O(p^{-L})$. Then, with probability at least $1 - O(p^{-L})$, we have

$$|\hat{\beta}_0 - \beta_0| \leq C_{\delta} \left(1 + \sqrt{\frac{\log p}{m_0}}\|\hat{\beta} - \beta\| + (1 + \frac{\log p}{m_0})\|\hat{\nabla} - \nabla\|_1 + \max_{k=1,2} \frac{|\pi_k - \hat{\pi}_k|}{m_0} + \sqrt{\frac{\log p}{Nm_0^2}}\right),$$

where constant $C_{\delta}$ depends on $L, C_e, C_\pi, C_{\log}, C_{\mu}, C_m$ and $c_m$.

Remark 7. Condition 4 is determined by our data generating process stated in Section 2.1. It is satisfied when the classification problem is non-trivial. For example, it is valid if $\text{pr}\{C' < \text{pr}(Y = 1 \mid X) < 1 - C'\} > C$ with some constants $C$ and $C' \in (0, 1)$. As a matter of fact, Condition 4 is weaker than the typical assumption: $C_{\log} < \text{pr}(Y = 1 \mid X) < 1 - C_{\log}$ with probability 1 for $X$, which is often seen in the literature of logistic regression. See, for example, Fan and Lv (2013) and Fan et al. (2015).

Theorems 3, 4 and 5 demonstrate the estimation accuracy for the quadratic, linear and constant coefficients in our CLIPS classifier (3.12) respectively. We conclude this section by establishing an oracle inequality for its generalization error via providing a rate of convergence of the excess risk. To this end, we define the generalization error of CLIPS classifier as $\tilde{R} = \pi_1 \tilde{R}_1 + \pi_2 \tilde{R}_2$, where $\tilde{R}_k = \text{pr}(\tilde{\phi}(X^\dagger) \neq k \mid Y^\dagger = k)$ is the probability that a new
set observation from Class $k$ is misclassified by the CLIPS classifier $\tilde{\phi}(\mathcal{X}')$. Again $pr$ is the conditional probability given the training data $\{(\mathcal{X}_i, \mathcal{Y}_i)\}_{i=1}^N$ which $\tilde{\phi}(\mathcal{X}')$ depends on.

We introduce some notation related to the Bayes decision rule in (2.2). Recall that given $M^r = m$, the Bayes decision rule $\phi_B(\mathcal{X}')$ solely depends on the sign of the function $g(\mathcal{X}') = \frac{1}{m} \log(p_{1}/p_2) + \beta_0 + \beta^T \bar{x} + \bar{x}^T \nabla \bar{x}/2 + \text{tr}(\nabla S)/2$. We define by $F_{k,m}$ the conditional cumulative distribution function of the oracle statistic $g(\mathcal{X}')$ given that $M^r = m$ and $\mathcal{Y}^r = k$. The upper bound of the first derivatives of $F_{1,m}$ and $F_{2,m}$ for all possible $m$ near 0 is denoted by $d_N$,

$$d_N = \max_{m \in [c_m m_0, C_m m_0]} \left\{ \sup_{t \in [-\delta_0, \delta_0]} \left| F'_{k,m}(t) \right| \right\},$$

where $\delta_0$ is any sufficiently small constant. The value of $d_N$ is determined by the generating process and is usually small whenever the Bayes rule performs reasonably well. According to Theorems 3, 4 and 5 with probability at least $1 - O(p^{-L})$, our estimators satisfy that

$$\Xi_N := \left( 1 + \sqrt{\frac{\log p}{m_0}} \right) \| \bar{\beta} - \beta \| + \left( 1 + \frac{\log p}{m_0} \right) \| \bar{\nabla} - \nabla \|_1 + \max_{k=1,2} \left| \frac{\hat{\pi}_k - \pi_k}{m_0} \right| + \left| \frac{\hat{\beta}_0 - \beta_0}{m_0} \right| = O(\kappa_N),$$

where $\kappa_N := (1 + (\log p)/m_0)s_q \lambda_1 + (1 + (\log p)/m_0)C_{\ell_1} \hat{s}_l \lambda_2 + \sqrt{(\log p)/(Nm_0^2)}$. It turns out the quantity $\kappa_N d_N$ is the key to obtain the oracle inequality. Condition 5 below guarantees that the assumptions of Theorem 5 are satisfied with high probability in our settings.

**Condition 5.** Suppose $\kappa_N m_0 \leq c_0$ and $\kappa_N d_N \leq c_0$ with some sufficiently small constant $c_0 > 0$.

Theorem 6 below reveals the oracle property of CLIPS classifier and provides a rate of convergence of the excess risk, that is, the generalization error of CLIPS classifier less the
Bayes risk $R_B$ defined in Section 2.2.

**Theorem 6.** Suppose that the assumptions of Theorems 3 and 4 hold and that Conditions 4–5 also hold. Then with probability at least $1 - O(p^{-L})$, we have the oracle inequality

$$
\tilde{R} \leq R_B + C_g(\kappa_N d_N + p^{-L}),
$$

where constant $C_g$ depends on $L, C_e, C_\pi, C_{\log}, C_\beta, C_m, c_m$ and $C_\mu$ only. In particular, we have $\tilde{R}$ converges to the Bayes risk $R_B$ in probability as $N$ goes to infinity.

Theorem 6 implies that with high probability, the generalization error of CLIPS classifier is close to the Bayes risk with rate of convergence no slower than $\kappa_N d_N$. In particular, whenever the the quantities $d_N$ and $C_\ell 1$ are bounded by some universal constant, the thresholding levels $\lambda_{1,N} = O(\sqrt{\log p/(m_0 N)})$ and $\lambda_{2,N} = O(\sqrt{\log p/(m_0 N)})$ yield the rate of convergence $\kappa_N d_N$ in the order of

$$
(1 + (\log p/m_0)\sqrt{\log p/(m_0 N)})\sqrt{s_l} + (1 + (\log p/m_0)\sqrt{\log p/(m_0 N)s_q}).
$$

(4.15)

The advantage of having large $m_0$ can be understood by investigating (4.15) as a function of $m_0$. Indeed, the leading term of (4.15) is

$$
\frac{\log p}{m_0^{3/2}} \sqrt{\frac{\log p}{N} s_q}, \text{ if } m_0 \leq \log p \cdot \min\{1, \frac{s_q^2}{s_l}\};
$$

$$
\frac{\sqrt{\log p}}{m_0} \sqrt{\frac{\log p}{N} \sqrt{s_l}}, \text{ if } \log p \cdot \frac{s_q^2}{s_l} \leq m_0 \leq \log p;
$$

$$
\sqrt{\frac{1}{m_0}} \sqrt{\frac{\log p}{N}} (\sqrt{s_l} + s_q), \text{ if } \log p \leq m_0.
$$

To illustrate the decay rate, we assume $s_l \geq s_q^2$. Then as $m_0$ increases, the error decreases at the order of $m_0^{3/2}$ up to certain point $\log p \cdot \frac{s_q^2}{s_l}$, and then decreases at the order of $m_0$ up
to another point log \( p \). When \( m \) is large enough so that \( m_0 \geq log p \), then the error decreases at the order of \( \sqrt{m_0} \).

To further emphasize the advantage of having sets of observations, we compare a general case \( m_0 = m^* \) where \( \log p \leq m^* \) with the special case that \( m_0 = 1 \), i.e., the regular QDA situation. Then the quantity \( \kappa_N \) with \( m^* \) has a faster decay rate with a factor of order between \( \sqrt{m^* \log p} \) and \( \sqrt{m^* \log p} \) (depending on the relationship between \( s_l \) and \( s_q \)) compared to the \( m_0 = 1 \) case, thanks to the extra observations within each set.

**Remark 8.** The above discussion reveals that in high-dimensional setting the benefit of the set-classification cannot be simply explained by having \( N^* = Nm_0 \) independent observations instead of having only \( N \) individual observations as in the classical QDA setting. Indeed, if we have \( N^* \) individual observations in the classical QDA setting, then the implied rate of convergence would be either \( \log p \sqrt{\frac{\log p}{Nm_0}} s_q \) (if \( \log p \cdot s_q^2 \geq s_l \)) or \( \sqrt{\log p} \sqrt{\frac{\log p}{Nm_0}} s_l \) (otherwise), which is slower than the one provided in equation (4.15).

**Remark 9.** It is worthwhile to point out that even in the special QDA situation where \( m_0 = 1 \), due to the sharper analysis, our result is still new and the established rate of convergence \( (\log p)/N^{1/2} \sqrt{s_l} + (\log p)^{3/2}/N^{1/2} s_q \) in Theorem 6 is at least as good as the one \( (\log p)^{3/2}/N^{1/2} (s_q + s_l) \) derived in the oracle inequality of Fan et al. (2015) under similar assumptions. Whenever \( s_l > s_q \), our rate is even faster with a factor of order \( \sqrt{s_l \log p} \) than that in Fan et al. (2015).

**Remark 10.** Results in this section, including Theorem 6, demonstrate the full advantages of the set classification setting in contrast to the classical QDA setting. When multiple observations within each set have short-range dependence, the rates of convergence for estimating
key parameters as well as the oracle inequality resemble the results under independent assumption. However, the results significantly change when there is a long-range dependence structure among multiple observations.

5. Numerical Studies

In this section we compare various versions of covariance-engaged set classifiers with other set classifiers adapted from traditional methods. In addition to the CLIPS classifier, we use the diagonalized and enriched versions of $\hat{\Sigma}_k$ respectively (labeled as Plugin(d) and Plugin(e)) introduced at the end of Section 3.1, and plug them in the Bayes rule (2.2), as done in (3.6). For comparisons, we also supply the estimated $\hat{\beta}_0$, $\beta$ and $\nabla$ from the CLIPS procedure to a QDA classifier which is applied to all the observations in a testing set, followed by a majority voting scheme (labeled as QDA-MV). Lastly, we calculate the sample mean and variance of each variable in an observation set to form a new feature vector as done in Miedema et al. (2012); then support vector machine (SVM; Cortes and Vapnik, 1995) and distance weighted discrimination (DWD; Marron et al., 2007; Wang and Zou, 2018) are applied to the features to make predictions (labeled as SVM and DWD respectively). We use R library clime to calculate the CLIME estimates, R library e1071 to calculate the SVM classifier, and R library sdwd (Wang and Zou, 2016) to calculate the DWD classifier.

5.1 Simulations

Three scenarios are considered for simulations. In each scenario, we consider a binary setting with $N = 7$ sets in a class, and $M = 10$ observations from normal distribution in each set.
Scenario 1 We set the precision matrix for Class 1 to be $\Sigma_1^{-1} = (1 + \sqrt{p})I_p$. For Class 2, we set $\Sigma_2^{-1} = \Sigma_1^{-1} + \tilde{\nabla}$, where $\tilde{\nabla}$ is a $p \times p$ symmetric matrix with 10 elements randomly selected from the upper-triangular part whose values are $\zeta$ and other elements being zeros. For the mean vectors, we set $\mu_1 = \Sigma_1(u, u, 0, \ldots, 0)^T$ and $\mu_2 = (0, \ldots, 0)^T$. Note that this makes the true value of $\beta = \Sigma_1^{-1}\mu_1 - \Sigma_2^{-1}\mu_2 = (u, u, 0, \ldots, 0)^T$, that is, only the first two covariates have linear impacts on the discriminant function if $u \neq 0$. In this scenario, the true difference in the precision matrices has some sparse and large non-zero entries, whose magnitude is controlled by $\zeta$. Note that while the diagonals of the precision matrices are the same, the diagonals of the covariance matrices are different between the two classes.

Scenario 2 We set the covariance matrices for both classes to be the identity matrix, except that for Class 1 the leading 5 by 5 submatrix of $\Sigma_1$ has its off-diagonal elements set to $\rho$. The rest of the setting is the same as in Scenario 1. In this scenario, both the difference in the covariance and the difference in the precision matrix are confined in the leading 5 by 5 submatrix, so that the majority of matrix entries are the same between the two classes. The level of difference is controlled by $\rho$: when $\rho = 0$, the two classes have the same covariance matrix.

Scenario 3 We set the precision matrix $\Sigma_1$ for Class 1 to be a Toeplitz matrix whose first row is $(1 - \rho^2)^{-1}(\rho^0, \rho^1, \rho^2, \ldots, \rho^{p-1})$. The covariance for Class 2, $\Sigma_2$, is a diagonal matrix with the same diagonals as those of $\Sigma_1$. It can be shown that the precision matrix for Class 1 is a band matrix with degree 1, that is, a matrix whose nonzero entries are
confined to the main diagonal and one more diagonal on both sides. Since the precision matrix for Class 2 is a diagonal matrix, the difference between the precision matrix has up to \( p + 2(p - 1) \) nonzero entries. The magnitude of the difference is controlled by the parameter \( \rho \). The rest of the setting is the same as in Scenario 1.

We consider different comparisons where we vary the magnitude of the difference in the precision matrices (\( \zeta \) or \( \rho \)), the magnitude of the difference in mean vectors (\( u \)), or the dimensionality (\( p \)), when the other parameters are fixed.

**Comparison 1 (varying \( \zeta \) or \( \rho \))** We vary \( \zeta \) or \( \rho \) but fix \( p = 100 \) and \( u = 0 \), which means that the mean vectors have no discriminant power since the true value of \( \beta \) is a zero vector. It shows the performance with different potentials in the covariance structure.

**Comparison 2 (varying \( u \))** We vary \( u \) while fixing \( p = 100 \) and \( \zeta = 0.55 \) in Scenario 1 or \( \rho = 0.5 \) and 0.3 in Scenarios 2 and 3. This case illustrates the potentials of the mean difference when there is some useful discriminative power in the covariance matrices.

**Comparison 3 (varying \( p \))** We let \( p = 80, 100, 120, 140, 160 \) while fixing \( \zeta \) or \( \rho \) in the same way as in Comparison 2 and fixing \( u = 0.05, 0.025 \) and 0.025 in Scenarios 1, 2 and 3 respectively.

Figure 2 shows the performance for Scenario 1. In the left panel, as \( \zeta \) increases, the difference between the true precision matrices increases. The proposed CLIPS classifier performs the best among all methods under consideration. It may be surprising that the Plugin(d) method, which does not consider the off-diagonal elements in the sample covariance, can
Figure 2: Set classification for Scenario 1. The three panels are corresponding to varying $\zeta$, varying $u$ and varying $p$ respectively. The CLIPS classifier performs very well when the effect of covariance dominates that of the mean difference.

work reasonably well in this setting where the major mode of variation is in the off-diagonal of the precision matrices. However, since large values in the off-diagonal of the precision matrix can lead to large values of some diagonal entries of the covariance matrix, the good performance of Plugin(d) has some partial justification.

In the middle panel of Figure 2, the mean difference starts to increase. While every method more or less gets some improvement, the DWD method has gained the most (it is even the best performing classifier when the mean difference $u$ is as large as 1.) This may be due to the fact that the mean difference on which DWD relies, instead of the difference in the precision matrix, is sufficiently large to secure a good performance in separating sets between two classes.

Figure 3 shows the results for Scenario 2. In contrast to Scenario 1, there is no difference in the diagonals of the covariances between the two classes (the precision matrices are still
Figure 3: Set classification for Scenario 2. The three panels are corresponding to varying $\rho$, varying $u$ and varying $p$ respectively. The classifiers that do not engage covariance perform poorly when there is no mean difference signal.

Figure 4: Set classification for Scenario 3. The three panels are corresponding to varying $\rho$, varying $u$ and varying $p$ respectively. As in Scenario 2, the classifiers that do not engage covariance perform poorly when there is no mean difference signal.
not read the off-diagonal of the sample covariances and hence both classes have the same precision matrices from its viewpoint.) As a matter of fact, all these methods perform as badly as random-guess. The CLIPS classifier always performs the best in this scenario in the left panel. Similar to the case in Scenario 1, as the mean difference increases (see the middle panel), the DWD method starts to get some improvement.

The results for Scenario 3 (Figure 4) are similar to Scenario 2, except that, this time the advantage of two covariance-engaged set classification methods, CLIPS and Plugin(e), seems to be more obvious when the mean difference is 0 (see left panel). Moreover, the QDA-MV method also enjoys some good performance, although not as good as the CLIPS classifier.

In all three scenarios, it seems that the test classification error is linearly increasing in the dimension $p$, except for Scenario 3 in which the signal level depends on $p$ too (greater dimensions lead to greater signals.)

### 5.2 Data Example

One of the common procedures used to diagnose hepatoblastoma (a rare malignant liver cancer) is biopsy. A sample tissue of a tumor is removed and examined under a microscope. A tissue sample contains a number of nuclei, a subset of which is then processed to obtain segmented images of nuclei. The data we analyzed contain 5 sets of nuclei from normal liver tissues and 5 sets of nuclei from cancerous tissues. Each set contains 50 images. The data set is publicly available [http://www.andrew.cmu.edu/user/gustavor/software.html](http://www.andrew.cmu.edu/user/gustavor/software.html) and was introduced in [Wang et al. (2011, 2010)](http://www.andrew.cmu.edu/user/gustavor/software.html).

We tested the performance of the proposed method on the liver cell nuclei image data
set. First, the dimension was reduced from 36,864 to 30 using principal component analysis. Then, among the 50 images of each set, 16 images are retained as training set, 16 are tuning set and another 16 are test set. In other words, for each of the training, tuning, and testing data sets, there are 10 sets of images, five from each class, with 16 images in each set.

Table 1 summarizes the comparison between the methods under consideration. All three covariance-engaged set classifiers (CLIPS, Plugin(d) and Plugin(e)), along with the QDA-MV method, perform better than methods which do not take the covariance matrices much into account, such as DWD and SVM (note that they do look into the diagonal of the covariance matrix.)

To get some insights to the reason that covariance-engaged set classifiers work and traditional methods fail, we visualize the data set in Figure 5. Subfigure (1) shows the scatter plot of the first two principal components of all the elementary observations (ignoring the set memberships) in the data sets, in which different colors (blue versus violet) depict the

| Method  | number of misclassified sets | standard error |
|---------|-----------------------------|----------------|
| CLIPS   | 0.01/10                     | 0.0104         |
| Plugin(d) | 0.74/10                   | 0.0450         |
| Plugin(e) | 0.97/10                   | 0.0178         |
| QDA-MV  | 0.08/10                     | 0.0284         |
| DWD     | 3.24/10                     | 0.1164         |
| SVM     | 3.13/10                     | 0.1130         |

Table 1: Classification performance for the liver cell nucleus image data.
two different classes. Observations in the same set are shown in the same symbol. The first strong impression is that there is no mean difference between the two classes on the observation level. In contrast, it seems that it is the second moment such as the variance that distinguishes the two classes.

Figure 5: PCA scatter plots for the liver cell nucleus image data. Both classes are shown in different colors. (1): the elementary observations in the raw space; different sets are shown in different symbols. (2) and (3): the augmented space seen by the DWD and SVM methods. (4) is a zoomed-in version of (3). It is shown that traditional multivariate methods have a fundamental difficulty for this data set.
One may argue that DWD and SVM should theoretically work here because they work on the augmented space where the mean and variance of each variable are calculated for each observation set, leading to a $2p$-dimensional feature vector for each set. However, Subfigures (2)–(4) invalidate this argument. We plot the augmented training data in the space formed by the first two principal components (Subfigure (2)). The augmented test data are shown in the same space in Subfigure (3) with a zoomed-in version in Subfigure (4). Note that the scales for Subfigures (2) and (3) are the same. These figures show that there are more than just the marginal mean and variance that are useful here, and our covariance-engaged set classification methods have used the information in the right way.

**Supplementary Materials**

The online supplementary materials contain additional theoretical arguments and proofs of all results.

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S1. Theoretical Properties under Time Series Structures

We consider the performance of CLIPS classifier when observations within each set are allowed to follow various time series structures, and extend the results obtained in Theorems 3, 4, 5 and 6 in these dependent settings.

We follow the assumption in Section 2 that both the $N$ sets $\{(X_i, Y_i)\}_{i=1}^{N}$ and the new set $(X^\dagger, Y^\dagger)$ are generated in the same way as $(X, Y)$ independently. In this section, the generating process of $(X, Y)$ is generalized to allow both short-range and long-range dependent time series. Specifically, while we still assume $Y$ and $M$ are independent with class probabilities $\pi_k$ ($k = 1, 2$) and distribution $p_M$ respectively, here we assume that conditioned on $M = m$ and $Y = y$, observations $X_1, X_2, \ldots, X_m$ in the set $X$ follow a vector linear process,

$$X_i = \mu_y + \sum_{t=0}^{\infty} A_{yt} \xi_{i-t},$$

where $A_{yt}$ are $p \times p$ dimensional coefficient matrices in class $Y = y$ and $\xi_t = (\xi_{t1}, \ldots, \xi_{tp})^T$ with $(\xi_{tj})_{t \in \mathbb{Z}, j = 1, \ldots, p}$ being i.i.d. standard normal variables. Note that the covariance matrices of individual observation from two classes are $\Sigma_y := \Sigma_{y0} = \sum_{t=0}^{\infty} A_{yt} A_{yt}^T$ for $y = 1, 2$. In general, the auto-covariance matrices at lag $k$ of all observations within each set, that is
Cov(\(X_i, X_{i+k}\)) := \Sigma_{yk} = \sum_{t=0}^{\infty} A_{yt} A_{y(t+k)}^T \text{ for } y = 1, 2. \text{ The above vector linear process is flexible since the coefficient matrices } A_{yt} \text{ can capture both spatial and temporal dependences. One important example is the vector auto-regression (VAR) model. It has been widely used in many fields, including functional Magnetic Resonance Imagine (fMRI) and microarray data (Dinov et al., 2005; Posekany et al., 2011).}

To characterize the dependence relationship of the time series, we impose conditions on the coefficient matrices. Set \(A_{yt} = (a_{yt,ij})_{1\leq i,j\leq p}. \text{ Then we assume the Gaussian linear process satisfies the following decay condition on } A_{yt} \text{ for both classes } y = 1, 2, \text{ and all } t \geq 0,

\[
\max_{1\leq i\leq p} \left( \sum_{j=1}^{p} a_{yt,ij}^2 \right)^{1/2} \leq C_{TS} (1 + t)^{-\nu},
\]

where \(C_{TS} > 0 \text{ is some constant and } \nu > 1/2 \text{ reflects the decay rate. The requirement } \nu > 1/2 \text{ is needed to guarantee that the covariance matrix } \Sigma_y = \sum_{t=0}^{\infty} A_t A_t^T \text{ is finite. In particular, in the time series literature, when } \nu > 1, \text{ the corresponding linear process is said to have a short-range dependence (SRD) because rows of the the corresponding auto-covariance matrices } \Sigma_{yk} \text{ are absolutely summable, which yields relatively weak dependence among all observations within each set. When } 1/2 < \nu < 1, \text{ the corresponding auto-covariance matrices may not be absolutely summable and thus the linear process is said have a long-range dependence (LRD). See, for example Beran (2017); Wu et al. (2010) for more details.}

We investigate generalization errors for the CLIPS classifier \(\tilde{\phi}(\mathcal{X}^\dagger) \text{ in (3.12) under the vector linear process model for both short-rang and long-range dependence. It is worthwhile pointing out that } \phi_B \text{ in (2.2) is no longer the Bayes decision rule due to the time series structure. In contrast, the full Bayes decision rule for model (S1.1) requires the knowledge}}

\(\ldots\)
of all coefficient matrices $A_{yt}$ for $t \in \mathbb{Z}, y = 1, 2$. However, in high-dimensional situations, it is difficult to estimate all coefficient matrices $A_{yt}$ accurately if not impossible at all. With the decay condition (S1.2), it is still reasonable to apply some simplified quadratic classifier such as $\phi_B(X^\dagger)$ in (2.2) to predict $Y^\dagger$ as if all observations in the test set $X^\dagger$ are independent.

Indeed, under the independence case in which $A_{yt} = 0$ for all $t \geq 1$, $\phi_B(X^\dagger)$ is the oracle of our CLIPS classifier $\tilde{\phi}(X^\dagger)$. With the general time series structure (S1.1), we need to define the oracle of our CLIPS first.

$$\tilde{\phi}(X^\dagger) = 2 - 1 \left\{ \frac{\log(\hat{\pi}_1/\hat{\pi}_2)}{m} + \tilde{\beta}_0 + \tilde{\beta}^T \bar{x} + \bar{x}^T \tilde{\nabla} \bar{x}/2 + \text{tr}(\tilde{\nabla} S)/2 > 0 \right\}. \quad (S1.3)$$

Recall that the key estimation in our CLIPS classifier displayed above include quadratic term $\tilde{\nabla}$, linear coefficient $\tilde{\beta}$ and an intercept coefficient $\tilde{\beta}_0$. While the estimations $\tilde{\nabla}$ in (3.8) and $\tilde{\beta}$ in (3.9) are proposed to estimate their counterparts in our CLIPS classifier $\nabla = \Sigma_2^{-1} - \Sigma_1^{-1}$ and $\beta = \beta_1 - \beta_2$ where $\beta_y = \Sigma_y^{-1} \mu_y$ respectively, the constant coefficient estimator $\tilde{\beta}_0$ in (3.10) is obtained via a logistic regression model. Therefore, the oracle $\beta_{0,TS}$ of $\tilde{\beta}_0$ in the current setting is defined as the minimizer of the following population loss function, that is,

$$\beta_{0,TS} = \arg\min_{\theta_0 \in \mathbb{R}} \mathbb{E}\ell(\theta_0 \mid \{(X_i, Y_i)\}_{i=1}^N, \beta, \nabla), \quad (S1.3)$$

where $\ell(\theta_0 \mid \{(X_i, Y_i)\}_{i=1}^N, \beta, \nabla)$ is defined in (3.11). We point out the interpretation of $\ell(\cdot)$ is no longer the negative log-likelihood function and thus $\beta_{0,TS}$ is not always equal to the quantity $\beta_0 = \{-\log(|\Sigma_1/|\Sigma_2|) - \mu_1^T \Sigma_1^{-1} \mu_1 + \mu_2^T \Sigma_2^{-1} \mu_2\}/2$ defined in (2.2). However, the oracle classifier $\phi_{B,TS}$ of CLIPS defined below is always no worse (i.e., has the same or smaller generalization error) than $\phi_B$ in (2.2) due to its definition (S1.3). Again, for the independence case, we have $\phi_{B,TS} = \phi_B$.

$$\phi_{B,TS}(X^\dagger) = 2 - 1 \left\{ \frac{\log(\pi_1/\pi_2)}{m} + \beta_{0,TS} + \beta^T \bar{x} + \bar{x}^T \nabla \bar{x}/2 + \text{tr}(\nabla S)/2 > 0 \right\}. \quad (S1.4)$$
From now on, we denote by $R_{B,TS}$ the oracle risk although the subscript $B$ no longer implies the Bayes decision rule.

We first extend Theorem 3 and establish the statistical properties of the thresholded CLIME difference estimator $\tilde{\nabla}$ defined in (3.8). Again, we assume that the true quadratic parameter $\nabla = \Sigma_2^{-1} - \Sigma_1^{-1} \in \mathcal{F}\mathcal{M}_0(s_q)$ has sparsity no more than $s_q$ defined in (4.13).

**Theorem 1.** Consider the vector linear process defined in (S1.1) that satisfies the decay condition (S1.2). Suppose Conditions 1-3 hold. Moreover, assume $\nabla \in \mathcal{F}\mathcal{M}_0(s_q)$, $\|\Sigma_k^{-1}\|_{\ell_1} \leq C\ell_1$ with some constant $C\ell_1 > 0$ for $k = 1, 2$ and $\log p \leq c_0 N$ with some sufficiently small constant $c_0 > 0$. Then for any fixed $L > 0$, with probability at least $1 - O(p^{-L})$, we have that

\[
\|\tilde{\nabla} - \nabla\|_\infty \leq 2\lambda'_{1,N}, \\
\|\tilde{\nabla} - \nabla\|_F \leq 2\sqrt{s_q}\lambda'_{1,N}, \\
\|\tilde{\nabla} - \nabla\|_1 \leq 2s_q\lambda'_{1,N},
\]

as long as $\lambda'_{1,N} \geq 8C\ell_1\lambda_{1,N}$ in (3.8) and

\[
\lambda_{1,N} \geq \begin{cases} 
CC\ell_1\sqrt{\frac{\log p}{Nm_0}} & \text{if } \nu > 3/4 \\
CC\ell_1\sqrt{\frac{\log p \log m_0}{Nm_0^2}} & \text{if } 1/2 < \nu < 3/4
\end{cases},
\]

where $C$ depends on $L, C_e, C_\pi, C_{TS}$ and $c_m$. Moreover, we have $\Pr(\text{supp}(\tilde{\nabla}) \subset \text{supp}(\nabla)) = 1 - O(p^{-L})$.

**Remark 1.** The choice of tuning parameter $\lambda_{1,N}$ and the rates of convergence on the boundary case $\nu = 3/4$ can also be dealt. In particular, we require $\lambda_{1,N} \geq CC\ell_1\sqrt{\frac{\log p \log m_0}{Nm_0}}$ if $\nu = 3/4$. See the proof of Theorem 1 for further details.

The results in Theorem 1 critically depend on the estimation accuracy of the sample covariance matrix under the supnorm in various time series dependence structures within each
set. Such technical results are detailed in Lemma 6 in Appendix, where the corresponding analysis requires an application of Hanson-Wright inequality. In particular, if $\nu > 3/4$, then the rates of convergence for estimating $\nabla$ are the same as those under the independence assumption. If $n < 3/4$, that is, the vector linear process has a long-range dependence, then the rates can be affected and reduced correspondingly.

We turn to the statistical properties of the linear coefficient estimator $\tilde{\beta}$ defined in (3.9) under time series structure. The following theorem is an extension of Theorem 4, in which we assume that $\beta = \beta_1 - \beta_2$ belongs to the $s_l$-sparse ball defined in (4.14).

**Theorem 2.** Consider the vector linear process defined in (S1.1) that satisfies the decay condition (S1.2). Suppose Conditions 1-3 hold. Moreover, assume that $\beta \in \mathcal{F}_0(s_l)$, $\log p \leq c_0 N$, $\|\beta_k\|_1 \leq C_\beta$ and $\|\mu_k\| \leq C_\mu$ with some constants $C_\beta, C_\mu > 0$ for $k = 1, 2$ and some sufficiently small constant $c_0 > 0$. Then for any fixed $L > 0$, with probability at least $1 - O(p^{-L})$, we have that

$$\|\tilde{\beta} - \beta\|_1 \leq C'' C_l s_l \lambda_{2,N},$$

$$\|\tilde{\beta} - \beta\| \leq C'' C_l \sqrt{s_l} \lambda_{2,N},$$

as long as the tuning parameter $\lambda_{2,N}$ in (3.9) satisfies

$$\lambda_{2,N} \geq \begin{cases} 
C' \sqrt{\frac{\log p}{N m_0}} & \text{if } \nu > 1 \\
C' \sqrt{\frac{\log p}{N m_0^{2\nu - 1}}} & \text{if } 1/2 < \nu < 1 
\end{cases},$$

where $\max\{\|\Sigma_1^{-1}\|_{\ell_1}, \|\Sigma_2^{-1}\|_{\ell_1}\} \leq C_{\ell_1}$ and $C''$, $C'$ depend on $L, C_e, c_m, C_\pi, C_\beta, C_{TS}$ and $C_\mu$.

**Remark 2.** The choice of tuning parameter $\lambda_{2,N}$ and the rates of convergence on the boundary case $\nu = 1$ can also be dealt. In particular, we require $\lambda_{2,N} \geq C' \sqrt{\frac{\log p \log^2 m_0}{N m_0}}$ if $\nu = 1$. See the proof of Theorem 2 for further details.
At a high level, the estimation accuracy of linear coefficients are determined by both estimation accuracy of the sample mean and that of the sample covariance matrix under the supnorm. While under the short-range dependence structure both rates of convergence are equal to $\sqrt{(\log p/(Nm_0))}$, the rate of convergence of sample mean dominates that of sample covariance matrix when there is a long-range dependence among multiple observations within each set.

Next, we derive the rate of convergence for estimating the oracle constant coefficient $\beta_{0,TS}$ defined in (S1.3) under the general time series structure. The accuracy of our estimator $\tilde{\beta}_0$ critically depends on the accuracy for estimating $\beta$ and $\nabla$. Theorem 3 extends Theorem 5 from the independent case to the general time series structure. We need one mild condition, the population strong convexity of the loss function $\ell(\beta_{0,TS} | \{(X_i, Y_i)\}_{i=1}^N, \beta, \nabla)$ at the oracle point $\beta_{0,TS}$.

**Condition 1.** Set $\bar{X}$ and $S$ as the sample mean and variance of the set of observations $(X, Y)$ with set size $M$. Define $Z_i = \log(\pi_1/\pi_2)/M + X^T\beta + X^T\nabla X/2 + \text{tr}(\nabla S)/2$. The expectation of the variable $\frac{\exp(M(\beta_0+Z))}{(1+\exp(M(\beta_0+Z)))^2}$ is bounded below by $C_{\log} > 0$, where $C_{\log}$ is some universal constant.

**Remark 3.** Strong convexity Condition 1 coincides with Condition 4 for the independent case. Indeed, for the independent case we have $\text{Var}(Y | X) = \frac{\exp(M(\beta_0+Z))}{(1+\exp(M(\beta_0+Z)))^2}$.

**Theorem 3.** Consider the vector linear process defined in (S1.1) that satisfies the decay condition (S1.2). Suppose Conditions 1-4 and 1 hold, $\log p \leq c_0 N$ with some sufficiently small constant $c_0 > 0$ and $\|\mu_k\| \leq C_\mu$ with some constant $C_\mu > 0$ for $k = 1, 2$. Besides, we have some initial estimators $\hat{\beta}, \hat{\nabla}, \hat{\pi}_1$ and $\hat{\pi}_2$ such that $m_0(\|\hat{\beta} - \beta\|_1)(1 + U_\beta) + m_0(\|\hat{\nabla} -$
\[ \nabla \|1 \nabla_{1} (1 + U_{\nabla}) + \max_{k=1,2} |\pi_{k} - \hat{\pi}_{k}| \leq C_{p} \text{ for some sufficiently small constant } C_{p} > 0 \text{ with probability at least } 1 - O(p^{-L}). \]

Then, with probability at least \(1 - O(p^{-L})\), we have

\[ |\tilde{\beta}_{0} - \beta_{0}| \leq C_{\delta} \left( (\|\tilde{\beta} - \beta\|_{1})(1 + U_{\beta}) + (\|\nabla - \nabla\|_{1})(1 + U_{\nabla}) + \max_{k=1,2} |\pi_{k} - \hat{\pi}_{k}|/m_{0} + \sqrt{\log p / N m_{0}^{2}} \right), \]

where \(U_{\beta}\) satisfies

\[
U_{\beta} = \begin{cases} 
\sqrt{\frac{\log p}{m_{0}}} & \text{if } \nu > 1 \\
\sqrt{\frac{\log p}{m_{0}^{\frac{\nu-1}{2}}}} & \text{if } 1/2 < \nu < 1 
\end{cases}
\]

\(U_{\nabla}\) satisfies

\[
U_{\nabla} = \begin{cases} 
\frac{\log p}{m_{0}} & \text{if } \nu > 1 \\
\frac{\log p}{m_{0}^{\frac{\nu-1}{2}}} & \text{if } 1/2 < \nu < 1 
\end{cases}
\]

and constant \(C_{\delta}\) depends on \(L, C_{e}, C_{\pi}, C_{\log}, C_{\mu}, C_{TS}, C_{m}, c_{m}\).

**Remark 4.** The rates of convergence on the boundary case \(\nu = 1\) can also be dealt. In particular, we require \(U_{\beta} = \sqrt{\frac{\log p \log^{2} m_{0}}{m_{0}}}\) and \(U_{\nabla} = \frac{\log p \log^{2} m_{0}}{m_{0}}\) if \(\nu = 1\). See the proof of Theorem 3 for further details.

We point out that the rate of convergence for estimating \(\beta_{0,TS}\) depends on the estimation accuracy of the linear coefficient through a term \(\|\tilde{\beta} - \beta\|_{1}\) in Theorem 3 while it relies on a potentially smaller term \(\|\tilde{\beta} - \beta\|_{2}\) in Theorem 5 under independent assumption in Section 4. This is due to a technical reason and the result cannot be improved (i.e., replacing \(\|\tilde{\beta} - \beta\|_{1}\) by \(\|\tilde{\beta} - \beta\|_{2}\)) if we only assume the decay condition (S1.2).

Theorems 1, 2 and 3 extend Theorems 3, 4 and 5 respectively, and demonstrate the estimation accuracy for the quadratic, linear and constant coefficients in our CLIPS classifier (3.12) under the general time series structure. Finally, we establish an oracle inequality for its generalization error via providing a rate of convergence of the excess risk. Recall the
S1. THEORETICAL PROPERTIES UNDER TIME SERIES STRUCTURES

generalization error of CLIPS classifier is \( \hat{R} = \pi_1 \hat{R}_1 + \pi_2 \hat{R}_2 \), where \( \hat{R}_k = \Pr(\hat{\phi}(X^\dagger) \neq k \mid Y^\dagger = k) \). Again \( \Pr \) is the conditional probability given the training data \( \{(X_i, Y_i)\}_{i=1}^N \) which \( \hat{\phi}(X^\dagger) \) depends on. In addition, we define the generalization error of the oracle classifier \( \phi_{B,TS} \) as \( R_{B,TS} = \pi_1 R_{1,TS} + \pi_2 R_{2,TS} \), where \( R_{k,TS} = \Pr(\phi_{B,TS}(X^\dagger) \neq k \mid Y^\dagger = k) \).

We need to introduce some notation \( d_{N,TS} \) related to the oracle classifier in (S1.4), which is similar to \( d_N \) defined in Section 4 for independence case. Recall the oracle classifier \( \phi_{B,TS}(X^\dagger) \) solely depends on the sign of the function \( g_{TS}(X^\dagger) = \frac{1}{m} \log(\pi_1/\pi_2) + \beta_{0,TS} + \beta^T \bar{x} + \bar{x}^T \nabla \bar{x}/2 + \text{tr}(\nabla S)/2 \). We define by \( F_{k,m,TS} \) the conditional cumulative distribution function of the oracle statistic \( g_{TS}(X^\dagger) \) given that \( M^\dagger = m \) and \( Y^\dagger = k \), and define by \( d_{N,TS} \) the upper bound of their first derivatives for all possible \( m \) near 0,

\[
d_{N,TS} = \max_{m \in [c_m m_0, C_m m_0]} \left\{ \sup_{t \in [-\delta_0, \delta_0]} \left| F'_{k,m,TS}(t) \right| \right\},
\]

where \( \delta_0 \) is any sufficiently small constant. The value of \( d_{N,TS} \) is determined by the vector linear process (S1.1) and performance of the oracle classifier. We define the counterparts of \( \Xi_N \) and its statistical order \( \kappa_N \) defined in Section 4 under the general time series structure below, which critically determine the excess risk. Indeed, one can show that Theorems 1, 2 and 3 imply that with probability at least \( 1 - O(p^{-L}) \),

\[
\Xi_{N,TS} := (1 + U_\beta)\|\hat{\beta} - \beta\|_1 + (1 + U_\nabla)\|\hat{\nabla} - \nabla\|_1 + \max_{k=1,2} \frac{|\hat{\pi}_k - \pi_k|}{m_0} + |\hat{\beta}_0 - \beta_{0,TS}| = O(\kappa_{N,TS}),
\]

where \( \kappa_{N,TS} := (1 + U_\nabla)s q_{1,N} + (1 + U_\beta)C_{t1}s_{1,N} + \sqrt{(\log p)/(Nm_0^2)} \), and the key quantities \( U_\beta, U_\nabla, \lambda_{1,N} \) and \( \lambda_{2,N} \) are specified in the statement of Theorems 1, 2 and 3 for various value of \( \mu \). The quantity \( \kappa_{N,TS} d_{N,TS} \) is the leading rate of convergence in the oracle inequality.

We need one more condition to guarantee the assumptions of Theorem 3 are satisfied with high probability, which is similar to Condition 5 for independence case.
Condition 2. Suppose $m_0 \kappa_{N,TS} \leq c_0$ and $\kappa_{N,TS}d_{N,TS} \leq c_0$ with some sufficiently small constant $c_0 > 0$.

Theorem 4 below reveals the oracle property of CLIPS classifier under the general time series structure.

**Theorem 4.** Suppose that the assumptions of Theorems 1 and 2 hold and that Conditions 1–2 also hold. Then with probability at least $1 - O(p^{-L})$, we have the oracle inequality

$$\tilde{R} \leq R_{B,TS} + C_g(\kappa_{N,TS}d_{N,TS} + p^{-L}),$$

where constant $C_g$ depends on $L, C_\epsilon, C_\pi, C_\log, C_\beta, C_m, c_m, C_{TS}$ and $C_\mu$ only.

## S2. Proofs of Main Results

### Proof of Theorem 1

**Proof.** We only prove that $R_{B1} \to 0$ and the proof of $R_{B2} \to 0$ is similar. In addition note that

$$R_{Bk} = \Pr(\phi_B(X^\dagger) \neq k \mid Y^\dagger = k) = \sum_{m = c_m m_0}^{C_m m_0} \Pr(\phi_B(X^\dagger) \neq k \mid Y^\dagger = k, M^\dagger = m) \cdot p_M(m)$$

$$= \sum_{m = c_m m_0}^{C_m m_0} R_{Bk,m} \cdot p_M(m),$$

where the last equality is due to independence of $Y^\dagger$ and $M^\dagger$, and Condition 2. Hence it is sufficient for us to focus on any fixed $m \in [c_m m_0, C_m m_0]$.

Given that the set is from Class 1, we have $X_i^\dagger \sim N(\mu_1, \Sigma_1), i = 1, \ldots, m$. The Bayes decision rule classifies the set to Class 2, i.e., $\phi_B(X^\dagger) = 2$ in (2.2) if $g(X_1^\dagger, \ldots, X_m^\dagger) < 0$, 

which is equivalent to

$$\sum_{i=1}^{m} \left( X_i^\dagger - \mu_1 \right)^T \nabla \left( X_i^\dagger - \mu_1 \right) - 2m\delta^T \Sigma_2^{-1}(\bar{X} - \mu_1) + m\delta^T \Sigma_2^{-1}\delta - m \log \left( \frac{|\Sigma_1|}{|\Sigma_2|} \right) + 2 \log \left( \frac{\pi_1}{\pi_2} \right) < 0,$$

(S2.5)

where $\bar{X} = \sum_{i=1}^{m} X_i^\dagger/m$ is the sample mean.

Define $V = \Sigma_1^{-1/2} \Sigma_2^{-1/2} - I$ where $I$ is the identity matrix. We set $Z_i = \Sigma_1^{-1/2} (X_i^\dagger - \mu_1) \sim N(0, I)$, $A_{m,p} = \sum_{i=1}^{m} Z_i^T V Z_i - 2m\delta^T \Sigma_2^{-1/2} \tilde{Z}$ with $\tilde{Z} = \sum_{i=1}^{m} Z_i/m$. Then the Bayes risk $R_{B1,m}$ can be written as, following from (S2.5),

$$R_{B1,m} = \Pr \left( A_{m,p} - \mathbb{E} A_{m,p} < -\alpha \right),$$

where $\alpha = m\text{tr}(V) + m\delta^T \Sigma_2^{-1}\delta - m \log \left( \frac{|\Sigma_1|}{|\Sigma_2|} \right) + 2 \log \left( \frac{\pi_1}{\pi_2} \right)$ since $\mathbb{E} A_{m,p} = m\text{tr}(V)$.

The strategy to bound $R_{B1,m}$ is to show that $|A_{m,p} - \mathbb{E} A_{m,p}|$ concentrates on $\sqrt{mD_p}$ but $\alpha > 0$ diverges at a faster rate of $mD_p^2$.

We first give an upper bound of the magnitude of $A_{m,p} - \mathbb{E} A_{m,p}$. Write the eigen-decomposition of $V$ as $U \Lambda U^T$ and the diagonal matrix $\Lambda = \text{diag}(\lambda_j)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Moreover, set $\tilde{Z}_i = U^T Z_i \sim N(0, I)$ with $\tilde{Z}_{i,j}$ its $j$th entry. Note that

$$A_{m,p} - \mathbb{E} A_{m,p} = \sum_{i=1}^{m} \sum_{j=1}^{p} \lambda_j (\tilde{Z}_{i,j}^2 - 1) - 2m\delta^T \Sigma_2^{-1/2} \tilde{Z}.$$

The tail probability of normal distribution implies

$$\Pr(|2m\delta^T \Sigma_2^{-1/2} \tilde{Z}| > t) \leq 2 \exp \left\{ -\frac{1}{2} \left( \frac{t}{2\sqrt{m}\|\delta^T \Sigma_2^{-1/2}\|} \right)^2 \right\} \leq 2 \exp \left( -\frac{C_e^{-3} t^2}{8m\|\delta\|^2} \right),$$

(S2.6)

where the last inequality is due to Condition 1. Since $\tilde{Z}_{i,j}^2 - 1$ is sub-exponential, Bernstein’s inequality (e.g. Vershynin, 2012, Proposition 5.16) implies that there exists some universal
constant $c_1 > 0$ such that
\[
\Pr\left(|\sum_{i=1}^{m} \sum_{j=1}^{p} \lambda_j (\tilde{Z}_{i,j}^2 - 1) | > t \right) \leq 2 \exp \left( -c_1 \min\left( \frac{t^2}{m\|A\|_F^2}, \frac{t}{\max\{|\lambda_1|, |\lambda_p|\}} \right) \right). \tag{S2.7}
\]

Now we focus on the lower bound of $\alpha$. First of all, notice that $m\delta^T\Sigma^{-1}_2\delta \geq mC_2^{-1}\|\delta\|^2$ by Condition 1. Moreover, there exists some constant $c_2 > 0$ depending on $C_e$ only such that
\[
m\text{tr}(V) - m \log\{\|\Sigma_1|/|\Sigma_2|\} = m (\text{tr}(V) - \log |I + V|) = m \sum_{j=1}^{p} (\lambda_j - \log(1 + \lambda_j)) \geq c_2 m\|A\|_F^2. \tag{S2.8}
\]
where the last inequality follows from that $\lambda_j + 1 \in [C_e^{-2}, C_e^2]$ according to Condition 1. Note that $\|A\|_F = \|V\|_F = \|\Sigma_1^{1/2}\nabla\Sigma_1^{1/2}\|_F$ and $C_e^{-1} \leq \|V\|_F/\|\nabla\|_F \leq C_e$ according to Condition 1. Therefore by combining the above two results we conclude $\alpha \geq c_3 mD_p^2 + 2\log(\pi_1/\pi_2)$ with $c_3 = \min(c_2 C_e^{-2}, C_e^{-1}) > 0$.

Note that by Conditions 1 and 3, $\lambda_1$ in equation (S2.7) and $2\log(\pi_1/\pi_2)$ in the expression of $\alpha$ are bounded. When $mD_p^2$ is large enough, we can pick $t = cmD_p^2$ for small enough $c > 0$ in equations (S2.6) and (S2.7) such that $A_{m,p} - \mathbb{E}A_{m,p} > -\alpha$ with probability at least $1 - 4\exp\left(-c'mD_p^2\right)$. Therefore we complete our proof by seeing that for each fixed $m$, $R_{B_1,m} \leq 4\exp\left(-c'mD_p^2\right)$ for some small constant $c' > 0$, together with the fact $m \in [c_m m_0, C_m m_0]$ from Condition 2.

\hspace{1cm} \Box

**Proof of Proposition 1**

*Proof.* Note that instead of $m$ observations with i.i.d. $N(\mu_k, \Sigma_k)$ from either class $k = 1, 2$, in the current case, we only have one representative $\bar{x} \sim N(\mu_k, \Sigma_k/m)$ with $k = 1$ or 2. Therefore, the proof of upper bound, i.e., $R_{\bar{x}} \leq 4\exp\left(-c'(\|\nabla\|_F^2 + m_0\|\delta\|^2)\right)$ for some small constant $c' > 0$, simply follows from the proof of Theorem 1 by replacing $m_0$ and $\Sigma_k$ by 1.
and $\Sigma_k/m_0$ respectively.

To show the rate on the exponent cannot be further improved in general, we need a little more efforts. Following the proof procedures for Theorem 1, it is sufficient to show the same lower bound on each $R_{xk,m} := \Pr(\phi_{B,x}(X^t) \neq k \mid Y^t = k, M^t = m)$ where $m \in [c_m m_0, C_m m_0]$. Given that the set is from Class 1, we have $\bar{\alpha}$ where $\alpha$ is a two small constants $c > 0$. Assume the support of vector $(\lambda_1, \ldots, \lambda_p)^T$ and the diagonal matrix $\Lambda = \text{diag}(\lambda_j)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. Moreover, set $Z = \sqrt{m}U \Sigma^{-1/2}_2 (\bar{X}^t - \mu_1) \sim N(0, I)$ with $Z_j$ its $j$th entry, and $A_{m,p} = \sum_{j=1}^{p} \lambda_j Z_j^2 - 2\sqrt{m} \delta^T \Sigma^{-1}_2 \Sigma^{-1/2}_1 U Z$. Then the risk $R_{x1,m}$ can be written as $R_{x1,m} = \Pr(A_{m,p} - \mathbb{E}A_{m,p} < -\alpha)$, where $\alpha = \sum_{j=1}^{p} \lambda_j + m \delta^T \Sigma^{-1}_2 \delta - \log|\Sigma_1|/|\Sigma_2|) + 2 \log(\pi_1/\pi_2)$ since $\mathbb{E}A_{m,p} = \sum_{j=1}^{p} \lambda_j$. We first upper bound the value of $\alpha$. Notice that $mC^2_e \delta^2 \leq m \delta^T \Sigma^{-1}_2 \delta \leq mC_e \delta^2$ by Condition 1. Moreover, a similar argument to (S2.8) also provides an upper bound, i.e., for two small constants $c_2 < c'_2 < 1$, we have $c_2 \|\Lambda\|^2_F \leq \sum_{j=1}^{p} \lambda_j - \log|\Sigma_1|/|\Sigma_2| \leq c'_2 \|\Lambda\|^2_F$. By Condition 3, $2 \log(\pi_1/\pi_2)$ in the expression of $\alpha$ is bounded. Therefore, under our assumption on sufficiently large $\|\nabla\|^2_F + m_0 \|\delta\|^2$, we have that $0 < \alpha < c(\|\nabla\|^2_F + m_0 \|\delta\|^2)$ with some small $c > 0$.

We show the rate on exponent cannot be further improved by showing a lower bound for some special cases of $\mu_1, \mu_2, \Sigma_1, \Sigma_2$. Assume the support of vector $(\lambda_1, \ldots, \lambda_p)^T$ and the support of vector $\delta^T \Sigma^{-1}_2 \Sigma^{-1/2}_1 U$ are disjoint (e.g., both $\Sigma_k$ are diagonal matrices with difference on the first $p/2$ diagonal entries, and only the last $p/2$ coordinates on mean difference $\delta$.
are nonzero). For this scenario, the first term $T_1 := \sum_{j=1}^{p} \lambda_j Z_j^2$ and second term $T_2 := -2\sqrt{m \delta^T \Sigma_2^{-1} \Sigma_1^{1/2} U Z}$ in $A_{m,p}$ are independent. To show that the term $T_1$ is non-positive with probability away from zero, we apply Proposition 2.4 in Johnstone (2001) to obtain that $\text{pr}(T_1 < 0) > \gamma > 0$ with some absolute constant $\gamma > 0$ by noting that the first term is a weighted Chi-square variable. By tail probability of normal distribution and the upper bound of $\alpha$, we further obtain that $\text{pr}(T_2 < -\alpha) > \exp(c(\|\nabla\|^2_F + m_0 \|\delta\|^2))$ with some small $c > 0$. In the end, by independence, we obtain that $R_{x1,m} = \text{pr}(A_{m,p} - E A_{m,p} < -\alpha) > \exp(c'(\|\nabla\|^2_F + m_0 \|\delta\|^2))$ with some small $c'' > 0$, which completes our proof. \hfill \Box

**Proof of Theorem 2**

*Proof.* We only prove that $\hat{R}_1 \to 0$ with high probability and $\hat{R}_2 \to 0$ can be shown by symmetry. The strategy of the proof is similar to that for Theorem 1. We further focus on each fixed $m \in [c_m m_0, C_m m_0]$ since

$$\hat{R}_k = \sum_{m=c_m m_0}^{C_m m_0} \text{pr}(\hat{\phi}(\mathcal{X}^\dagger) \neq k \mid \mathcal{Y}^\dagger = k, \mathcal{M}^\dagger = m) \cdot p_M(m)$$

$$= \sum_{m=c_m m_0}^{C_m m_0} \hat{R}_{k,m} \cdot p_M(m). \quad (S2.9)$$

The quadratic set classifier classifies the set to 2, that is, $\hat{\phi}(\mathcal{X}^\dagger) = 2$ in (3.6) if

$$\sum_{i=1}^{m} \left( X_i^\dagger - \hat{\mu}_1 \right)^T \hat{\nabla} \left( X_i^\dagger - \hat{\mu}_1 \right) - 2m \hat{\delta}^T \hat{\Sigma}_2^{-1} (\bar{X} - \hat{\mu}_1) + m \hat{\delta}^T \hat{\Sigma}_2^{-1} \hat{\delta} - m \log \left( \frac{\hat{\Sigma}_1}{\hat{\Sigma}_2} \right) + 2 \log \left( \frac{\hat{\pi}_1}{\hat{\pi}_2} \right) < 0,$$

where $\hat{\delta} = \hat{\mu}_2 - \hat{\mu}_1$ and $\bar{X} = \sum_{i=1}^{m} X_i^\dagger / m$. Define

$$\hat{A}_{m,p} = \sum_{i=1}^{m} \left( X_i^\dagger - \hat{\mu}_1 \right)^T \hat{\nabla} \left( X_i^\dagger - \hat{\mu}_1 \right) - 2m \hat{\delta}^T \hat{\Sigma}_2^{-1} (\bar{X} - \hat{\mu}_1) := \hat{A}_{1,m,p} + \hat{A}_{2,m,p}.$$
Then the generalization error $\hat{R}_{1,m}$, which is a random variable as a function of $\{(X_i, Y_i)\}_{i=1}^N$, can be written as

$$\hat{R}_1 = \hat{R}_1((X, Y)) = \text{pr}(\hat{A}_{m,p} - \mathbb{E}\hat{A}_{m,p} < -\hat{\alpha}),$$  

(S2.10)

where \(\text{pr}\) and \(\mathbb{E}\) are understood as the conditional expectation given $\{(X_i, Y_i)\}_{i=1}^N$ and

$$\hat{\alpha} = \mathbb{E}(\hat{A}_{1,m,p} + \hat{A}_{2,m,p}) + m\hat{\delta}^T\hat{\Sigma}_2^{-1}\hat{\delta} - m \log \left( |\hat{\Sigma}_1| / |\hat{\Sigma}_2| \right) + 2 \log \left( \frac{\hat{\pi}_1}{\hat{\pi}_2} \right).$$

The following lemma facilitates our analysis.

**Lemma 1.** For any fixed $L > 0$, under the assumptions $p \leq c_0 N m_0$ and $\log p \leq c_0 N$ with sufficiently small $c_0 > 0$, we have that (i) $C'^{-1} \leq \lambda_{\min}(\hat{\Sigma}_k) \leq \lambda_{\max}(\hat{\Sigma}_k) \leq C'$; (ii) $\|\mu_k - \hat{\mu}_k\| \leq C'\sqrt{\frac{p}{Nm_0}}$; (iii) $\|\Sigma_k - \hat{\Sigma}_k\|_F \leq C'\sqrt{\frac{p^2}{Nm_0}}$ and (iv) $|\pi_k - \hat{\pi}_k| \leq C'\sqrt{\frac{\log p}{N}}$, $k = 1, 2$ with probability at least $1 - O(p^{-L})$, where positive constant $C'$ depend on $C_e, c_m, L$ and $C_\pi$ only.

From now on, we condition on the event $E$ in which results (i)-(iv) of Lemma 1 hold for training data $\{(X_i, Y_i)\}_{i=1}^N$. All positive constants used hereafter only depend on $C_e$ and $c_0$. Clearly, since $p^2/(N m_0 D_p^2)$ is sufficiently small, Lemma 1 (ii) and (iii) imply that

$$\hat{D}_p = \left( \|\hat{\Sigma}\|_F^2 + \|\hat{\delta}\|^2 \right)^{1/2} \asymp D_p.$$  

(S2.11)

We show the concentration radius of $\hat{A}_{m,p} - \mathbb{E}\hat{A}_{m,p}$ is much smaller than $\hat{\alpha}$ under our assumptions.

First of all, we analyze the left side $\hat{A}_{m,p} - \mathbb{E}\hat{A}_{m,p} = \Sigma_{k=1}^2(\hat{A}_{k,m,p} - \mathbb{E}\hat{A}_{k,m,p})$. Note that

$$\hat{A}_{2,m,p} - \mathbb{E}\hat{A}_{2,m,p} = -2\sum_{i=1}^m \hat{\delta}^T\hat{\Sigma}_2^{-1}\Sigma_1^{1/2}Z_i,$$

where $Z_i = \Sigma_1^{-1/2}(X_i^\dagger - \mu_1) \overset{i.i.d.}{\sim} N(0, I)$. Note Lemma 1 implies the spectral norm $\left\|\hat{\Sigma}_2^{-1}\Sigma_1^{1/2}\right\|_{\ell_2} \leq C'C_e^{1/2}$. The tail probability of normal distribution implies (similarly as in equation (S2.6)) there exists some constant $C_1 > 0$ such...
that,
\[ \text{pr}(|\hat{A}_{2,m,p} - \mathbb{E}\hat{A}_{2,m,p}| > t) \leq 2 \exp \left( - \frac{C_1 t^2}{m\|\hat{\delta}\|^2} \right). \]  
(S2.12)

Besides, \( \hat{A}_{1,m,p} - \mathbb{E}\hat{A}_{1,m,p} = W_1 + W_2, \) where
\[
W_1 : = \text{tr}[\nabla \left( \sum_{i=1}^{m} (X_i^\dagger - \mu_1) (X_i^\dagger - \mu_1)^T \right)] - \text{tr}[\nabla m \Sigma_1],
\]
\[
W_2 := 2 (\mu_1 - \hat{\mu}_1)^T \nabla \Sigma_1^{1/2} \sum_{i=1}^{m} Z_i.
\]

Set \( \hat{V} = \Sigma_1^{1/2} \nabla \Sigma_1^{1/2} \) and its eigen-values \{\hat{\lambda}_j\}_{j=1}^{p}. \) By a similar argument using Bernstein’s inequality like (S2.7), we have that there exists some constant \( c_1 > 0 \) such that
\[ \text{pr}(|W_1| > t) \leq 2 \exp \left( -c_1 \min\left( \frac{t^2}{m\|\hat{V}\|_F^2}, \frac{t}{\max\{\|\hat{\lambda}_1\|, \|\hat{\lambda}_p\|\}} \right) \right). \]  
(S2.13)

To control \( W_2, \) we apply again the tail probability of normal distribution to obtain that for some constants \( C_2, C_3 > 0, \)
\[ \text{pr}(|W_2| > t) \leq 2 \exp \left( - \frac{C_2 t^2}{m\|\hat{\delta}\|_F^2 \cdot \|\mu_1 - \hat{\mu}_1\|^2} \right) \leq 2 \exp \left( - \frac{C_3 t^2}{m\|\hat{V}\|_F^2} \right), \]  
(S2.14)

since \( \|\mu_1 - \hat{\mu}_1\| \leq C' \sqrt{\frac{p}{Nm_0}} \leq C' c_0^{1/2} \) by Lemma 1. Therefore equations (S2.12)-(S2.14), together with (S2.11), imply that for some \( C_4 > 0, \)
\[ \text{pr}(|\hat{A}_{m,p} - \mathbb{E}\hat{A}_{m,p}| > t) \leq 6 \exp \left( - \frac{C_4 t^2}{mD_p^2} \right). \]  
(S2.15)

Now we lower bound the right side \( \hat{\alpha}. \) This term can be decomposed into six terms.
\[
\hat{\alpha} = m\hat{\delta}^T \hat{\Sigma}_2^{-1} \hat{\delta} + \left[ \text{mtr}(\nabla \hat{\Sigma}_1) - m \log \left( \left| \hat{\Sigma}_1 \right| / \left| \hat{\Sigma}_2 \right| \right) \right] + 2 \log \left( \frac{\hat{\pi}_1}{\hat{\pi}_2} \right) + 
\]
\[
mtr(\nabla (\Sigma_1 - \hat{\Sigma}_1)) - 2m\hat{\delta}^T \hat{\Sigma}_2^{-1} (\mu_1 - \hat{\mu}_1) + m (\mu_1 - \hat{\mu}_1) \nabla (\mu_1 - \hat{\mu}_1)^T.
\]

These terms have the following bounds respectively with some constant \( C_5, C_6, C_7, C_8, C_9 > \)
\[ m \hat{\delta}^T \hat{\Sigma}^{-1}_2 \hat{\delta} \geq C_5 m \| \hat{\delta} \|^2, \quad (S2.16) \]
\[ \text{mtr}(\hat{\nabla}(\Sigma_1 - \hat{\Sigma}_1)) \leq C_6 m \| \hat{\nabla} \|_F (\Sigma_1 - \hat{\Sigma}_1) \leq C_7 m \| \hat{\nabla} \|_F (p^2/Nm_0)^{1/2}, \quad (S2.18) \]
\[ |2 \log (\hat{\pi}_1/\hat{\pi}_2)| \leq C_6, \quad (S2.20) \]
\[ m (\mu_1 - \hat{\mu}_1) \hat{\nabla} (\mu_1 - \hat{\mu}_1)^T \leq C_9 m (p/Nm_0). \quad (S2.21) \]

Equations (S2.16) and (S2.17) are due to (i) of Lemma 1. In particular, (S2.17) follows from a similar argument as (S2.8). Equations (S2.18) and (S2.19) follow from (iii) and (ii) of Lemma 1 respectively while equation (S2.20) is due to (iv) of Lemma 1 and Condition 3. Equation (S2.21) follows from (i) and (ii) of Lemma 1. Furthermore, notice that \[ p^2/(Nm_0D_p^2) \] is sufficiently small and \( m_0D_p^2 \) is sufficiently large, equations (S2.16)-(S2.21) as well as (S2.11) yield that \( \hat{\alpha} \geq C_{10} mD_p^2 \) for some small constant \( C_{10} > 0 \).

Finally, the lower bound of \( \hat{\alpha} \) and concentration of \( \hat{A}_{m,p} - \mathbb{E}\hat{A}_{m,p} \) in (S2.15) with \( t = c''mD_p^2 \) for small enough \( c'' > 0 \), together with the assumption \( D_p^2 m \) is sufficiently large, imply that the generalization error of the quadratic set classification rule \( \hat{R}_{1,m} \leq 2 \exp (-c' mD_p^2) \) for each \( m \in [c_m, m_0, C_m m_0] \) on the event \( \mathcal{E} \). Hence we complete our proof by applying Lemma 1 and equation (S2.9), that is, \( \hat{R} \leq 4 \exp (-c' m_0D_p^2) \) with probability at least \( 1 - O(p^{-L}) \).

**Proof of Theorem 3**

**Proof.** First we show that \( \Sigma_k^{-1} \) is feasible for the optimization problem (3.7), that is \( \| \hat{\Sigma}_k \Sigma_k^{-1} - I \|_\infty < \lambda_{1,N} \). It suffices to show that \( \| \hat{\Sigma}_k - \Sigma_k \|_\infty < C_{1L}^{-1} \lambda_{1,N} \) because \( \| \hat{\Sigma}_k \Sigma_k^{-1} - I \|_\infty \leq \lambda_{1,N} \).
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∥\hat{\Sigma}_k - \Sigma_k\|_\infty \|\Sigma_k^{-1}\|_{\ell_1} \leq \|\hat{\Sigma}_k - \Sigma_k\|_\infty C_{\ell_1}. The following lemma establishes this result, given our choice of \(\lambda_{1,N} \geq CC_{\ell_1} \sqrt{\log p} / (Nm_0)\) and the assumption \(\log p \leq c_0 N\) with some sufficiently small \(c_0 > 0\).

**Lemma 2.** Recall the number of set from class \(k\) is denoted as \(N_k = \sum_{i=1}^{N} \mathbb{1}\{Y_i = k\}\). Then given any positive integer \(N_1\) and \(N_2\), we have that with probability at least \(1 - O(p^{-L})\) (i) \(\|\hat{\mu}_k - \mu_k\|_\infty \leq C' \sqrt{\log p} / (N_km_0)\) and (ii) \(\|\hat{\Sigma}_k - \Sigma_k\|_\infty \leq C'(\sqrt{\log p} / (N_km_0) + \log p) / (N_km_0)\), \(k = 1, 2\), where positive constant \(C'\) depends on \(C_e, c_m\) and \(L\) only. Under the assumption \(\log p \leq c_0 N\) with some sufficiently small \(c_0 > 0\), we further have that (i) \(\|\hat{\mu}_k - \mu_k\|_\infty \leq C \sqrt{\log p} / (N_km_0)\) and (ii) \(\|\hat{\Sigma}_k - \Sigma_k\|_\infty \leq C \sqrt{\log p} / (N_km_0)\), \(k = 1, 2\) with probability at least \(1 - O(p^{-L})\), where the constant \(C\) also depends \(C_\pi\) besides \(C_e, c_m, L\).

From now on, we condition on the event in which both results of the second part in Lemma 2 hold. We next control the supnorm bound \(\|\Sigma_k^{-1} - \tilde{\Omega}_k\|_\infty\). Since both \(\Sigma_k^{-1}\) and \(\tilde{\Omega}_k\) are feasible for (3.7), we have \(\|\hat{\Sigma}_k(\Sigma_k^{-1} - \tilde{\Omega}_k)\|_\infty = \|\hat{\Sigma}_k\Sigma_k^{-1}I - (\hat{\Sigma}_k\tilde{\Omega}_k - I)\|_\infty \leq 2\lambda_{1,N}\). Moreover,

\[
\|\Sigma_k(\Sigma_k^{-1} - \tilde{\Omega}_k)\|_\infty \leq \|(\hat{\Sigma}_k - \Sigma_k)(\Sigma_k^{-1} - \tilde{\Omega}_k)\|_\infty + \|\Sigma_k^{-1} - \tilde{\Omega}_k\|_\ell_1 \|\Sigma_k - \Sigma_k\|_\infty + 2\lambda_{1,N} \\
\leq \|\Sigma_k^{-1} - \tilde{\Omega}_k\|_\ell_1 \|\Sigma_k - \Sigma_k\|_\infty + 2\lambda_{1,N} \\
\leq \left(\|\Sigma_k^{-1}\|_{\ell_1} + \|\tilde{\Omega}_k\|_{\ell_1}\right) C_{\ell_1}^{-1} \lambda_{1,N} + 2\lambda_{1,N} \\
\leq 2C_{\ell_1} C_{\ell_1}^{-1} \lambda_{1,N} + 2\lambda_{1,N} = 4\lambda_{1,N},
\]

where we have used the fact \(\tilde{\Omega}_k\) is the solution of CLIME which implies for each \(j = 1, \ldots, p\), \(\|(\tilde{\Omega}_k)_j\|_1 \leq \|(\Sigma_k^{-1})_j\|_1\) and hence \(\tilde{\Omega}_k\|_{\ell_1} \leq \|\Sigma_k^{-1}\|_{\ell_1}\), where \((\tilde{\Omega}_k)_j\) and \((\Sigma_k^{-1})_j\) denote the \(j\)th column of \(\tilde{\Omega}_k\) and \(\Sigma_k^{-1}\) respectively. We conclude with \(\|\Sigma_k^{-1} - \tilde{\Omega}_k\|_\infty \leq \|\Sigma_k^{-1}\|_{\ell_1} \|\Sigma_k(\Sigma_k^{-1} - \tilde{\Omega}_k)\|_\infty \leq 4M_0\lambda_{1,N}\).
Based on the supnorm bound obtained above, we have

$$\|\tilde{\Omega}_2 - \tilde{\Omega}_1 - \nabla\|_{\infty} \leq \|\Sigma^{-1}_1 - \tilde{\Omega}_1\|_{\infty} + \|\Sigma^{-1}_2 - \tilde{\Omega}_2\|_{\infty} \leq 8C_{\ell_1} \lambda_{1,N}. \quad (S2.22)$$

Recall that supp($\nabla$) is the support of the matrix $\nabla$. The thresholding step (3.8), together with (S2.22), guarantees that $\tilde{\nabla}_{ij} = 0$ for any $(i,j) \notin$ supp($\nabla$), noting that $\lambda'_{1,N} \geq 8C_{\ell_1} \lambda_{1,N}$.

Therefore we have shown the subset selection result, that is, $\text{pr}(\text{supp}(\tilde{\nabla}) \subset \text{supp}(\nabla)) = 1 - O(p^{-L})$. Moreover, we have that $\|\tilde{\nabla} - \nabla\|_{\infty} \leq 8C_{\ell_1} \lambda_{1,N} + \lambda'_{1,N} \leq 2\lambda'_{1,N}$. In the end, we complete the proof by noting that the Frobenius norm bound and vector $\ell_1$ norm bound are the consequences of supnorm bound and subset selection result, that is, $\text{pr}(\|\tilde{\nabla} - \nabla\|_F \leq 2\lambda'_{1,N} \sqrt{s_q}) = 1 - O(p^{-L})$ and $\text{pr}(\|\tilde{\nabla} - \nabla\|_1 \leq 2\lambda'_{1,N} s_q) = 1 - O(p^{-L})$. □

**Proof of Theorem 4**

*Proof*. We first show that $(\beta_1, \beta_2) = (\Sigma^{-1}_1 \mu_1, \Sigma^{-1}_2 \mu_2)$ is feasible in (3.9) with the constant $L_1$ set as $C_\beta$. Note since $\|\beta_k\|_1 \leq C_\beta$, The pair $(\beta_1, \beta_2)$ satisfies the $\ell_1$ norm constraint. This fact, together with the following lemma, implies that $(\beta_1, \beta_2)$ is feasible with probability at least $1 - O(p^{-L})$ and hence $\|\tilde{\beta}\|_1 \leq \|\beta\|_1$.

**Lemma 3.** Under the assumption $\log p \leq c_0 N$ with some sufficiently small constant $c_0 > 0$, we have that $\text{pr}(\|\Sigma_k \beta_k - \tilde{\mu}_k\|_{\infty} \geq C \sqrt{\log p/N_{m_0}}) \leq C' p^{-L}$, $k = 1, 2$, where $C' > 0$ is some universal constant and constant $C > 0$ depends on $C_e, c_m, C_\pi, C_\beta, C_\mu$ and $L$ only.

Next we show that $\|\tilde{\beta} - \beta\|_{\infty} \leq 6C_{\ell_1} \lambda_{2,N}$. Notice that for $k = 1, 2$, there exists some
constant $C > 0$ such that with probability at least $1 - O(p^{-L})$,

$$
\| \Sigma_k (\beta_k - \beta_k) \|_\infty \leq \| \hat{\Sigma}_k (\beta_k - \beta_k) \|_\infty + \| (\Sigma_k - \hat{\Sigma}_k) (\beta_k - \beta_k) \|_\infty \\
\leq \| \hat{\Sigma}_k \beta_k - \mu_k \|_\infty + \| \hat{\Sigma}_k \beta_k - \mu_k \|_\infty + \| \Sigma_k - \hat{\Sigma}_k \|_\infty (\| \beta_k \|_1 + \| \beta_k \|_1) \\
\leq 2\lambda_{2,N} + 2C\epsilon \sqrt{\frac{\log p}{Nm_0}} \leq 3\lambda_{2,N},
$$

where we have used supnorm bound (S2.23) in the first and third equations and the fact

where we have used supnorm bound (S2.23) in the first and third equations and the fact

Therefore we further have,

$$
\| \tilde{\beta} - \beta \|_\infty \leq 2 \sum_{k=1}^{2} \| \tilde{\beta}_k - \beta_k \|_\infty \leq 2 \sum_{k=1}^{2} \| \Sigma^{-1}_k \| \| \Sigma_k (\tilde{\beta}_k - \beta_k) \|_\infty \leq 6C\epsilon \lambda_{2,N}. \quad (S2.23)
$$

In the end, we condition on the event in which both (S2.23) and the fact that $(\beta_1, \beta_2)$ is feasible hold. The arguments above imply this event holds with probability at least $1 - O(p^{-L})$. We are ready to prove the rates of convergence of $\tilde{\beta}$ under $\ell_1$ and $\ell_2$ norm losses. Denote the support of $\beta$ by $T$. Set $t = 6C\epsilon \lambda_{2,N}$ and the thresholded version of $\tilde{\beta}$ as $\tilde{\beta}^{thr} = (\tilde{\beta}^{thr}_j)$, where $\tilde{\beta}^{thr}_j = \tilde{\beta}_j 1\{ |\tilde{\beta}_j| \geq 2t \}$. Since $\beta = \beta_1 - \beta_2$ is feasible, we have that

$$
\| \beta \|_1 \geq \| \tilde{\beta}^{thr} \|_1 + \| \tilde{\beta} - \tilde{\beta}^{thr} \|_1 \geq \| \tilde{\beta} - \tilde{\beta}^{thr} \|_1 + \| \beta \|_1 - \| \tilde{\beta}^{thr} - \beta \|_1.
$$

Therefore we obtain that $\| \tilde{\beta} - \tilde{\beta}^{thr} \|_1 \leq \| \tilde{\beta}^{thr} - \beta \|_1$, which further implies that $\| \tilde{\beta} - \beta \|_1 \leq 2\| \tilde{\beta}^{thr} - \beta \|_1$. To show the bound of $\| \tilde{\beta} - \beta \|_1$, it suffices to bound $\| \tilde{\beta}^{thr} - \beta \|_1$. Indeed, we bound its $\ell_2$ norm as an intermediate step,

$$
\| \tilde{\beta}^{thr} - \beta \|^2 = \| (\tilde{\beta}^{thr} - \beta)_T \|^2 \\
= \sum_{j \in T} (\tilde{\beta}^{thr}_j - \beta_j)^2 1\{ \tilde{\beta}^{thr}_j = 0 \} + \sum_{j \in T} (\tilde{\beta}_j - \beta_j)^2 1\{ \tilde{\beta}_j^{thr} \neq 0 \} \\
\leq \sum_{j \in T} \beta_j^2 1\{ \beta_j \leq 3t \} + st^2 \leq 10st^2, \quad (S2.24)
$$

where we have used supnorm bound (S2.23) in the first and third equations and the fact
|T| \leq s_l due to \beta \in \mathcal{F}_0(s_l) in the third and fourth equations. Consequently,
\[ \| \tilde{\beta}^{thr} - \beta \|_1 = \left\| \left( \tilde{\beta}^{thr} - \beta \right) \right\|_1 \leq \sqrt{s_l}\| \tilde{\beta}^{thr} - \beta \| = \sqrt{10}s_l t, \]
which completes our first desired result \[ \| \tilde{\beta} - \beta \|_1 \leq 2\sqrt{10}s_l t = 12\sqrt{10}C_{t_1}s_l\lambda_{2,N}. \]

To show the bound of \[ \| \tilde{\beta} - \beta \| \leq \| \tilde{\beta}^{thr} - \beta \| + \| \tilde{\beta} - \tilde{\beta}^{thr} \|, \]
it suffices to bound \[ \| \tilde{\beta} - \tilde{\beta}^{thr} \| \]
given (S2.24). To this end, we note \[ \| \beta \|_1 \geq \| \tilde{\beta} \|_1 \] implies that \[ \| \tilde{\beta}^{thr} \|_1 \leq \| \tilde{\beta} - \beta \|_1 \leq 2\sqrt{10}s_l t. \]
Moreover,
\[ \| \tilde{\beta} - \tilde{\beta}^{thr} \|^2 = \left\| \left( \tilde{\beta}^{thr} - \beta \right) \right\|^2 + \left\| \left( \tilde{\beta}^{thr} - \tilde{\beta} \right) \right\|^2 \leq 4t^2s_l + \sum_{j \in T^c} \tilde{\beta}^2_j \mathbb{1}\{|\tilde{\beta}_j| < 2t\} \leq 4t^2s_l + \| \tilde{\beta}^{thr} \|_1 \max_{j \in T^c}\{|\tilde{\beta}_j| \mathbb{1}\{|\tilde{\beta}_j| < 2t\} \} \leq (4 + 4\sqrt{10})t^2s_l, \quad (S2.25) \]
where the first inequality follows from \[ |\tilde{\beta}^{thr}_j - \tilde{\beta}_j| < 2t \] and \[ |T| \leq s_l, \] and the second one is due to Hölder’s inequality. Therefore combining (S2.24) and (S2.25), we obtained the second desired result \[ \| \tilde{\beta} - \beta \| \leq \sqrt{s_l}(\sqrt{10} + (4 + 4\sqrt{10})^{1/2}). \]

**Proof of Theorem 5**

*Proof.* Since we use sample splitting technique, estimators \( \tilde{\beta} \) and \( \tilde{\nabla} \) are independent with the second batch of the training data used in (3.10). We assume fixed \( \tilde{\beta} \) and \( \tilde{\nabla} \), which satisfy our assumptions throughout the analysis. With a slight abuse of notation, we still use \( N \) to denote the number of sample sets, although only half of the sample sets are applied to count \( n_k \) and \( \hat{\pi}_k, \ k = 1, 2. \)

Recall that \( \bar{X}_i \) and \( S_i \) are the sample mean and variance of the \( i \)th set of observations. Define \( \bar{Z}_i = \log(\hat{\pi}_1/\hat{\pi}_2)/M_i + \bar{X}_iT\tilde{\beta} + \bar{X}_iT\nabla\bar{X}_i/2 + \text{tr}(\nabla S_i)/2 \), which is used to approximate \( Z_i = \log(\pi_1/\pi_2)/M_i + \bar{X}_iT\beta + \bar{X}_iT\nabla\bar{X}_i/2 + \text{tr}(\nabla S_i)/2 \). To facilitate analysis, we denote
\( \ell(\theta_0|\{(X_i, Y_i)\}_{i=1}^N, \tilde{\beta}, \tilde{\nabla}) \) as \( \ell(\theta_0) \) for short. Rewrite our estimator in the following way,

\[
\tilde{\beta}_0 = \arg\min_{\theta_0 \in \mathbb{R}} \ell(\theta_0), \text{ where } \ell(\theta_0) = \frac{1}{N} \sum_{i=1}^N \left[ \log \left( 1 + \exp \left( M_i(\theta_0 + \tilde{Z}_i) \right) \right) - (2 - Y_i)M_i(\theta_0 + \tilde{Z}_i) \right].
\]

We start our analysis by conditioning on \( \{X_i\}_{i=1}^N \). Define \( \ell_0(\theta_0, \tilde{Z}) = \mathbb{E}(\ell(\theta_0)|\{X_i\}_{i=1}^N) \) where the expectation is understood as the conditional expectation given \( \{X_i\}_{i=1}^N \). Note that the function \( \ell_0(\theta_0, \tilde{Z}) \) depends on \( \theta_0, \{M_i\}_{i=1}^N \) and \( \{\tilde{Z}_i\}_{i=1}^N \) only. Then the difference \( \ell(\theta_0) - \ell_0(\theta_0, \tilde{Z}) = \frac{1}{N} \sum_{i=1}^N (Y_i - \mathbb{E}(Y_i|X_i))M_i(\theta_0 + \tilde{Z}_i) := E_0 \). Recall \( \beta_0 \) is the true constant coefficient. Since \( \tilde{\beta}_0 \) is the minimizer, we have \( \ell(\tilde{\beta}_0) \leq \ell(\beta_0) \), i.e.,

\[
\ell_0(\tilde{\beta}_0, \tilde{Z}) \leq \ell_0(\beta_0, \tilde{Z}) + E_{\beta_0} - E_{\tilde{\beta}_0} \leq \ell_0(\beta_0, \tilde{Z}) + m_0R_1 \left| \tilde{\beta}_0 - \beta_0 \right|. \tag{S2.26}
\]

In the end, we need to bound the term \( R_1 = |\frac{1}{Nm_0} \sum_{i=1}^N (Y_i - \mathbb{E}(Y_i|X_i))M_i| \). By applying Hoeffding’s inequality (e.g. Vershynin, 2012, Proposition 5.10), we obtain \( R_1 \leq C_r \sqrt{(\log p)/N} \) with probability at least \( 1 - O(p^{-L}) \), where constant \( C_r \) depends on \( L \) and \( C_m \) only, noting that \( M_i \leq C_m m_0 \) by Condition 2. This probabilistic statement on bounding \( R_1 \) is valid conditioning on any realization of \( \{X_i\}_{i=1}^N \) and thus is also valid unconditionally.

Next we apply the Taylor expansion to the function \( \ell_0(\theta_0, \tilde{Z}) \) to analyze our estimator. Here due to misspecified values \( \tilde{Z}_i \), we need a refined version of Taylor expansion (Bach et al., 2010, Proposition 1).

**Lemma 4** (Bach et al. (2010)). Let \( g(t) : \mathbb{R} \to \mathbb{R} \) be a convex three times differentiable function such that it satisfies for all \( t \in \mathbb{R} \), \( |g'''(t)| \leq Lg''(t) \) for some \( L > 0 \). Then we have
for any \( t \) and \( v \in \mathbb{R} \),

\[
g(t + v) \geq g(t) + vg'(t) + \frac{g''(t)}{L^2} (e^{-L|v|} + L|v| - 1).
\]

It is not hard to see that the third derivative of \( \ell_0(\theta_0, \tilde{Z}) \) w.r.t. \( \theta_0 \) is bounded by its second derivative up to a multiplicative factor \( \max_i M_i \), i.e.,

\[
\max_{\theta_0} \left| \ell''_0(\theta_0, \tilde{Z})/\ell''_0(\theta_0, \tilde{Z}) \right| \leq \max_i M_i,
\]

where hereafter \( \ell'_0(\cdot, \cdot) \), \( \ell''_0(\cdot, \cdot) \) and \( \ell'''_0(\cdot, \cdot) \) are defined as the first, second and third derivative of \( \ell_0(\cdot, \cdot) \) w.r.t. the first argument respectively. Applying Lemma 4 to \( \ell_0(\theta_0, \tilde{Z}) \) at point \( \beta_0 \) and by Condition 2, we obtain that

\[
\ell_0(\tilde{\beta}_0, \tilde{Z}) - \ell_0(\beta_0, \tilde{Z}) \geq \ell'_0(\beta_0, \tilde{Z})(\tilde{\beta}_0 - \beta_0) + \frac{\ell''_0(\beta_0, \tilde{Z})}{C_m^2 m_0^2} (e^{-C_m m_0 |\tilde{\beta}_0 - \beta_0|} + C_m m_0 |\tilde{\beta}_0 - \beta_0| - 1).
\]

(S2.27)

Note that with misspecified values \( \tilde{Z}_i \), in general \( \ell'_0(\beta_0, \tilde{Z}) \neq 0 \). To finish our proof, we need an upper bound for \( \ell'_0(\beta_0, \tilde{Z}) \) and a lower bound for \( \ell''_0(\beta_0, \tilde{Z}) \) with misspecified values \( \tilde{Z}_i \).

Thus the term \( |\tilde{Z}_i - Z_i| \) critically determines the estimation accuracy. The following bound of \( |\tilde{Z}_i - Z_i| \) is helpful for our later analysis.

**Lemma 5.** Under the assumptions of Theorem 5, there exists some constant \( C_z > 0 \) depending on \( c_m, C_m, C_\pi, C_\mu \) and \( C_e \) such that with probability at least \( 1 - O(p^{-L}) \) we have uniformly for all \( i = 1, \ldots, N \)

\[
|\tilde{Z}_i - Z_i| \leq \frac{1}{M_i} \left| \log \left( \frac{\hat{\pi}_1 \pi_2}{\hat{\pi}_2 \pi_1} \right) \right| + \left| X_i^T (\tilde{\beta} - \beta) \right| + \frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T (\hat{\nabla} - \nabla) X_{ij} / 2 \right|
\]

\[
\leq C_z \left( (1 + \sqrt{\frac{\log p}{m_0}}) \| \tilde{\beta} - \beta \| + (1 + \frac{\log p}{m_0}) \| \hat{\nabla} - \nabla \|_1 + \max_{k=1,2} \frac{|\pi_k - \hat{\pi}_k|}{m_0} \right) \quad \text{(S2.28)}
\]

Indeed, the conclusion (S2.28) is valid with the same probability \( 1 - O(p^{-L}) \) conditioning on any realization of \( \{Y_i\}_{i=1}^N \) and \( \{M_i\}_{i=1}^N \).
Lemma 5 and our assumption imply that with probability at least $1 - O(p^{-L})$ we have

$$m_0 \max_i |\tilde{Z}_i - Z_i| := R_2$$

is sufficiently small.

Note that the expectation of the score function $\ell'_0(\beta_0, Z) = 0$ where $\ell'_0(\beta_0, Z)$ is obtained by replacing $\tilde{Z}_i$ by $Z_i$ in $\ell'_0(\beta_0, \tilde{Z})$, $i = 1, \ldots, N$. We are ready to bound the magnitude of $\ell'_0(\beta_0, \tilde{Z})$,

$$\left| \ell'_0(\beta_0, \tilde{Z}) \right| = \left| \frac{1}{N} \sum_{i=1}^{N} \left( \frac{M_i \exp(M_i(\beta_0 + \tilde{Z}_i))}{1 + \exp(M_i(\beta_0 + Z_i))} - \frac{M_i \exp(M_i(\beta_0 + Z_i))}{1 + \exp(M_i(\beta_0 + Z_i))} \right) \right| \\
\leq \frac{1}{N} \sum_{i=1}^{N} M_i^2 |\tilde{Z}_i - Z_i| \\
\leq C_m^2 m_0 R_2,$$

(S2.29)

where the first inequality follows from that the derivative of $\exp(M_i(\beta_0 + \tilde{Z}_i))$ w.r.t. $\tilde{Z}_i$ is always bounded by $M_i$ and the second inequality is due to Condition 2, $M_i \leq C_m m_0$ and definition of $R_2$.

Moreover, by Condition 4, we have that the expectation of the i.i.d. bounded random variable $\text{Var}(Y_i | X_i) = \frac{\exp(M_i(\beta_0 + Z_i))}{(1 + \exp(M_i(\beta_0 + Z_i)))^2}$, $i = 1, \ldots, N$, is bounded away from $C_{\log}$. We apply Hoeffding’s inequality and the fact $\log p \leq c_0 N$ to obtain that with probability at least $1 - O(p^{-L})$, we have

$$\frac{1}{N} \sum_{i=1}^{N} M_i^2 \left( \frac{\exp(M_i(\beta_0 + Z_i))}{1 + \exp(M_i(\beta_0 + Z_i))} \right) \left( \frac{1}{1 + \exp(M_i(\beta_0 + Z_i))} \right) \geq C_{\log}^2 m_0^2,$$

where the positive constant $C_{\log}^2 > 0$ depends on $C_{\log}$ and $L$. Since $m_0 \max_i |\tilde{Z}_i - Z_i| := R_2$ is sufficiently small with probability at least $1 - O(p^{-L})$, the union bound argument further implies that

$$\ell''_0(\beta_0, \tilde{Z}) = \frac{1}{N} \sum_{i=1}^{N} M_i^2 \left( \frac{\exp(M_i(\beta_0 + \tilde{Z}_i))}{1 + \exp(M_i(\beta_0 + \tilde{Z}_i))} \right) \left( \frac{1}{1 + \exp(M_i(\beta_0 + \tilde{Z}_i))} \right) \\
\geq C_{\log}^2 m_0^2,$$

(S2.30)
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with probability at least $1 - O(p^{-L})$ for some positive constant $C_{\text{low}} > 0$.

In the end, plugging (S2.26), (S2.29) and (S2.30) into (S2.27) and applying the union bound argument, we obtain that with probability $1 - O(p^{-L})$,

$$C_{\text{low}}C_m^{-2}(e^{-C_m m_0 |\tilde{\beta}_0 - \beta_0|} + C_m m_0 |\tilde{\beta}_0 - \beta_0| - 1) \leq m_0 \left(C_m^2 R_2 + R_1\right) |\tilde{\beta}_0 - \beta_0|. \quad (S2.31)$$

We apply the following fact

$$e^{-2\gamma/(1-\gamma)} + (1 - \gamma) \frac{2\gamma}{1 - \gamma} - 1 \geq 0 \text{ for } \gamma \in (0, 1),$$

to (S2.31) and obtain that

$$C_m m_0 |\tilde{\beta}_0 - \beta_0| \leq \frac{2C_m(C_m^2 R_2 + R_1)/C_{\log}}{1 - C_m(C_m^2 R_2 + R_1)/C_{\log}}.$$

Since $C_m^2 R_2 + R_1$ are sufficiently small, we have that $C_m(C_m^2 R_2 + R_1)/C_{\log} < 1/2$ which implies $C_m m_0 |\tilde{\beta}_0 - \beta_0| < 2$. This fact itself further implies that $(e^{-C_m m_0 |\tilde{\beta}_0 - \beta_0|} + C_m m_0 |\tilde{\beta}_0 - \beta_0| - 1) \geq (C_m m_0 |\tilde{\beta}_0 - \beta_0|)^2/2$. Consequently, (S2.31) implies that

$$|\tilde{\beta}_0 - \beta_0| \leq 2C^{-1}m_0^{-1}(C_m^2 R_2 + R_1),$$

which further completes our proof, together with Lemma 5 (bound of $R_2$) and the bound of $R_1$,

$$|\tilde{\beta}_0 - \beta_0| \leq C_{\delta}\left((1 + \sqrt{\frac{\log p}{m_0}})\|	ilde{\beta} - \beta\| + (1 + \frac{\log p}{m_0})\|\tilde{\nabla} - \nabla\|_1 + \max_{k=1,2} \frac{|\pi_k - \hat{\pi}_k|}{m_0} + \sqrt{\frac{\log p}{Nm_0^2}}\right),$$

where the constant $C_{\delta} = 2C_{\text{low}}^{-1}(C_m^2 C_z + C_r)$. \hfill \qed
Proof of Theorem 6

Proof. Recall that for each \( k = 1, 2 \), the corresponding Bayes risk and generalization error of CLIPS classifier can be decomposed as

\[
R_{B_k} = \sum_{m = c_i m_0}^{C_i m_0} \Pr(\phi_B(X^\dagger) \neq k \mid Y^\dagger = k, M^\dagger = m) p_M(m) := \sum_m R_{B_k, m} p_M(m),
\]

\[
\hat{R}_k = \sum_{m = c_i m_0}^{C_i m_0} \Pr(\tilde{\phi}(X^\dagger) \neq k \mid Y^\dagger = k, M^\dagger = m) p_M(m) := \sum_m \hat{R}_{k, m} p_M(m).
\]

Therefore, it is sufficient to bound the difference \( \hat{R}_{k, m} - R_{B_k, m} \) for each fixed \( k = 1, 2 \) and fixed \( m \in [c_i m_0, C_i m_0] \).

Recall that \( \Xi_N = (1 + \sqrt{\frac{\log(p)}{m_0}})\|\tilde{\beta} - \beta\| + (1 + \frac{\log(p)}{m_0})\|\tilde{\nabla} - \nabla\|_1 + \max_{k=1,2} |\tilde{\pi}_k - \pi_k| + |\tilde{\beta}_0 - \beta_0|. \)

Define the event \( \mathcal{E}_0 = \{\Xi_N \leq C_\Xi \kappa_N\} \), where \( \kappa_N = (1 + \frac{\log(p)}{m_0}) s_q^2 + (1 + \sqrt{\frac{\log(p)}{m_0}}) C_\ell \sqrt{s_i} \lambda_2 + \sqrt{\frac{\log(p)}{N m_0}} \), the constant \( C_\Xi = 2(2 + C'')(C_{\delta} + 1) \) and other constants \( C'', C_{\delta} \) can be tracked back from Theorems 3-5. We first show that our estimators satisfy that \( \Pr(\mathcal{E}_0) = 1 - O(p^{-L}) \) by Theorems 3-5. Indeed, Theorems 3 and 4 provides bounds of \( \|\tilde{\beta} - \beta\| \) and \( \|\tilde{\nabla} - \nabla\|_1 \) respectively. The estimation error of \( \max_{k=1,2} |\tilde{\pi}_k - \pi_k|/m_0 \) follows from Lemma 1. Assuming these bounds hold, the first part of Condition 5 implies that the assumption in Theorem 5 is satisfied with the initial estimators being our quadratic and linear estimators. Thus Theorem 5 further implies the upper bound for \( |\tilde{\beta}_0 - \beta_0| \). Hereafter, we assume event \( \mathcal{E}_0 \) holds.

We follow the notation introduced in the proof of Theorem 5 on the set of observations \( (X^\dagger, Y^\dagger) \) and define \( \tilde{Z} = \log(\tilde{\pi}_1/\tilde{\pi}_2)/M^\dagger + \tilde{x}^T \tilde{\beta} + \tilde{x}^T \tilde{\nabla} \bar{x}/2 + \text{tr}(\tilde{\nabla} S)/2 \), which is used to approximate \( Z = \log(\pi_1/\pi_2)/M^\dagger + x^T \beta + x^T \nabla \bar{x}/2 + \text{tr}(\nabla S)/2 \), where \( \bar{x} \) and \( S \) are the sample mean and covariance of the set \( X^\dagger \). Then we define the event \( \mathcal{E}_z = \{|\tilde{Z} - Z| \leq C_Z \Xi_N\} \). Lemma 5 applied to \( (X^\dagger, Y^\dagger) \) and the second part of Condition 5 imply that on event \( \mathcal{E}_0 \) uniformly for all \( k = 1, 2 \) and \( m \in [c_i m_0, C_i m_0] \), we have \( \Pr(\mathcal{E}_z | Y^\dagger = k, M^\dagger = m) \geq 1 - C'_g p^{-L} \).
Without loss of generality, we focus on the case \( k = 1 \). Recall \( \tilde{R}_{k,m} \) relies on the estimators \( \tilde{\beta}_0, \beta, \vec{\beta}, \tilde{\pi}_1 \) and \( \hat{\pi}_2 \) and hence is random. On the event \( E_0 \), we have that

\[
\tilde{R}_{1,m} = \text{pr} \left( \tilde{Z} + \tilde{\beta}_0 \leq 0 \mid \mathcal{Y}^\dagger = 1, M^\dagger = m \right)
\]

\[
= \text{pr} \left( Z + \beta_0 \leq Z - \tilde{Z} + \beta_0 - \tilde{\beta}_0 \mid \mathcal{Y}^\dagger = 1, M^\dagger = m \right)
\]

\[
= \text{pr} \left( Z + \beta_0 \leq Z - \tilde{Z} + \beta_0 - \tilde{\beta}_0, \mathcal{E}_z \mid \mathcal{Y}^\dagger = 1, M^\dagger = m \right) + \text{pr} ( \mathcal{E}_z^c \mid \mathcal{Y}^\dagger = 1, M^\dagger = m )
\]

\[
\leq C_g p^{-L} + \text{pr} ( Z + \beta_0 \leq (C_z + 1) \Xi_N, \mathcal{E}_z \mid \mathcal{Y}^\dagger = 1, M^\dagger = m )
\]

\[
\leq C_g p^{-L} + \text{pr} ( Z + \beta_0 \leq (C_z + 1) C \Xi_N \mathcal{Y}^\dagger = 1, M^\dagger = m )
\]

\[
= C_g p^{-L} + F_{1,m}( (C_z + 1) C \Xi_N ),
\]

where the first inequality follows from the conditional probability \( \text{pr} ( \mathcal{E}_z \mid \mathcal{Y}^\dagger = k, M^\dagger = m ) \geq 1 - C_g p^{-L} \) and the definition of the event \( \mathcal{E}_z \), the second inequality is due to the event \( \mathcal{E}_0 \), and the last equality follows from the definition of the cumulative distribution function \( F_{1,m} ( t ) \).

In addition, by the definition of the deterministic value \( R_{B_{k,m}} \), we have

\[
R_{B1,m} = \text{pr} ( Z + \beta_0 \leq 0 \mid \mathcal{Y}^\dagger = 1, M^\dagger = m ) = F_{1,m}(0).
\]

By our assumption, the quantity \( (C_z + 1) C \Xi_N \) is sufficiently small and hence less than \( \delta_0 \). It follows from (S2.32)-(S2.33) and definition of \( d_N \) that on the event \( \mathcal{E}_0 \), we have that

\[
\tilde{R}_{1,m} - R_{B1,m} \leq C_g p^{-L} + \sup_{t \in [-\delta_0, \delta_0]} \left| F_{1,m}'(t) \right| (C_z + 1) C \Xi_N
\]

\[
\leq C_g p^{-L} + (C_z + 1) C \Xi_N d_N .
\]

Similarly we can show that same upper bound applies to \( \tilde{R}_{2,m} - R_{B2,m} \) uniformly for all \( m \in [c_m m_0, C_m m_0] \). Therefore on the event \( \mathcal{E}_0 \), we obtain that \( \tilde{R} \leq R_B + C_g p^{-L} + (C_z + 1) C \Xi_N d_N \), which completes our proof.
Proof of Theorem 1

Proof. By inspecting the proof of Theorem 3, one realizes that the proof of Theorem 1 is almost identical to that of Theorem 3 except that the role of Lemma 2 is replaced by the following important lemma under time series structures for different values of \( \nu \). More specifically, one only need to show that \( \| \hat{\Sigma}_k - \Sigma_k \|_\infty < C_{\ell_1}^{-1} \lambda_{1,N} \) with probability at least \( 1 - O(p^{-L}) \) and the choice of \( \lambda_{1,N} \) under time series structures is determined by (result (ii) of the second part in) Lemma 6. Therefore, we omit the proof details.

Lemma 6. Consider the vector linear process defined in \((S1.1)\) that satisfies the decay condition \((S1.2)\). Suppose Conditions 1-3 hold. Recall the number of set from class \( k \) is denoted as \( N_k = \sum_{i=1}^{N} \mathbb{1}\{Y_i = k\} \). Then given any positive integer \( N_1 \) and \( N_2 \), we have that (i)

\[
\| \hat{\mu}_k - \mu_k \|_\infty \leq \begin{cases} 
C' \sqrt{\frac{\log p}{N_k m_0}} & \text{if } \nu > 1 \\
C' \sqrt{\frac{\log p \log^2 m_0}{N_k m_0}} & \text{if } \nu = 1 \\
C' \sqrt{\frac{\log p}{N_k m_0^{\nu - 1}}} & \text{if } 1/2 < \nu < 1
\end{cases},
\]

and (ii)

\[
\| \hat{\Sigma}_k - \Sigma_k \|_\infty \leq \begin{cases} 
C' \left( \sqrt{\frac{\log p}{N_k m_0}} + \frac{\log p}{N_k m_0} \right) & \text{if } \nu > 1 \\
C' \left( \sqrt{\frac{\log p}{N_k m_0}} + \frac{\log p \log^2 m_0}{N_k m_0} \right) & \text{if } \nu = 1 \\
C' \left( \sqrt{\frac{\log p}{N_k m_0}} + \frac{\log p}{N_k m_0^{\nu - 1}} \right) & \text{if } 3/4 < \nu < 1 \\
C' \left( \sqrt{\frac{\log p \log m_0}{N_k m_0} + \frac{\log p}{N_k m_0^{\nu - 1}}} \right) & \text{if } \nu = 3/4 \\
C' \left( \sqrt{\frac{\log p \log^2 m_0}{N_k m_0} + \frac{\log p}{N_k m_0^{\nu - 1}}} \right) & \text{if } 1/2 < \nu < 3/4
\end{cases},
\]

for \( k = 1, 2 \) with probability at least \( 1 - O(p^{-L}) \), where positive constant \( C' \) depends on \( C_e, c_m, C_{TS} \) and \( L \) only.

In addition, under the assumption \( \log p \leq c_0 N \) with some sufficiently small \( c_0 > 0 \), we
have that (i)

\[ \| \hat{\mu}_k - \mu_k \|_\infty \leq \begin{cases} 
\frac{C \sqrt{\log p}}{Nm_0} & \text{if } \nu > 1 \\
\frac{C \sqrt{p \log^2 m_0}}{Nm_0} & \text{if } \nu = 1 \\
\frac{C \sqrt{p \log 2}}{Nm_0} & \text{if } \nu = 1/2 < \nu < 1 \\
\end{cases} \]

and (ii)

\[ \| \hat{\Sigma}_k - \Sigma_k \|_\infty \leq \begin{cases} 
\frac{C \sqrt{\log p}}{Nm_0} & \text{if } \nu > 3/4 \\
\frac{C \sqrt{p \log m_0}}{Nm_0} & \text{if } \nu = 3/4 \\
\frac{C \sqrt{\log p}}{Nm_0^{2\nu-2}} & \text{if } 1/2 < \nu < 3/4 \\
\end{cases} \]

for \( k = 1, 2 \) with probability at least \( 1 - O(p^{-L}) \), where positive constant \( C \) also depends on \( C_\pi \) besides \( C_e, c_m, C_{TS} \) and \( L \).

\[ \square \]

**Proof of Theorem 2**

*Proof.* By inspecting the proof of Theorem 4, one realizes that the proof of Theorem 2 is almost identical to that of Theorem 4 except that the role of Lemma 3 is replaced by the following important lemma (Lemma 7) under time series structures for different values of \( \nu \).

More specifically, the results follow from some algebra (deterministically) on the event that both \( \| \tilde{\beta} - \beta \|_\infty \leq 6C_{t1} \lambda_{2,N} \) and that \( (\beta_1, \beta_2) \) is feasible hold. To this end, Lemma 7 implies that \( (\beta_1, \beta_2) \) is feasible with probability at least \( 1 - O(p^{-L}) \). In addition, we show that the choice of \( \lambda_{2,N} \) and Lemma 7 together imply that \( \| \tilde{\beta} - \beta \|_\infty \leq 6C_{t1} \lambda_{2,N} \) with probability
at least $1 - O(p^{-L})$. Indeed, on the event that $(\beta_1, \beta_2)$ is feasible, we have

$$\|\Sigma_k (\hat{\beta}_k - \beta_k)\|_\infty \leq \|\hat{\Sigma}_k (\beta_k - \beta_k)\|_\infty + \| (\Sigma_k - \hat{\Sigma}_k) (\hat{\beta}_k - \beta_k)\|_\infty$$

$$\leq \|\hat{\Sigma}_k \beta_k - \hat{\mu}_k\|_\infty + \|\hat{\Sigma}_k \hat{\beta}_k - \hat{\mu}_k\|_\infty + \|\Sigma_k - \hat{\Sigma}_k\|_\infty \left(\|\beta_k\|_1 + \|\hat{\beta}_k\|_1\right)$$

$$\leq 2\lambda_{2,N} + 2C_\beta C_\Sigma \kappa_{\Sigma} \leq 3\lambda_{2,N},$$

where $\kappa_{\Sigma} = \sqrt{\frac{\log p}{Nm_0}} \left(\sqrt{\frac{\log p \log m_0}{Nm_0}}, \sqrt{\frac{\log p}{Nm_0^{1/\nu - 3}}}ight)$ when $\nu > 3/4$ ($\nu = 3/4, 1/2 < \nu < 3/4$) respectively. In the above derivation, we have used assumption on $\|\beta_k\|_1$, constraints on estimators, the choice of our $\lambda_{2,N}$ (i.e., $2C_\beta C_\Sigma \kappa_{\Sigma} \leq \lambda_{2,N}$) and the result (ii) of the second part in Lemma 6. Therefore we further have,

$$\|\hat{\beta} - \beta\|_\infty \leq \sum_{k=1}^{2} \|\hat{\beta}_k - \beta_k\|_\infty \leq \sum_{k=1}^{2} \|\Sigma_k^{-1}\|_\infty \|\Sigma_k (\hat{\beta}_k - \beta_k)\|_\infty \leq 6C_{\ell_1} \lambda_{2,N}. $$

Therefore, we complete the proof.

Lemma 7. Consider the vector linear process defined in (S1.1) that satisfies the decay condition (S1.2). Suppose Conditions 1-3 hold. Under the assumptions $\|\beta_k\|_1 \leq C_\beta$, $k = 1, 2$ with some constants $C_\beta > 0$ and $\log p \leq c_0 N$ with some sufficiently small constant $c_0 > 0$, we have that

$$\|\hat{\Sigma}_k \beta_k - \hat{\mu}_k\|_\infty \leq \begin{cases} C \sqrt{\frac{\log p}{Nm_0}} & \text{if } \nu > 1 \\ C \sqrt{\frac{\log p \log^2 m_0}{Nm_0}} & \text{if } \nu = 1 \\ C \sqrt{\frac{\log p}{Nm_0^{\nu - 3}}} & \text{if } 1/2 < \nu < 1 \end{cases},$$

for $k = 1, 2$ with probability at least $1 - O(p^{-L})$, where constant $C > 0$ depends on $C_e, c_m, C_\pi, C_\beta, C_\mu, C_{TS}$ and $L$ only.
S2. PROOFS OF MAIN RESULTS

Proof of Theorem 3

Proof. By inspecting the proof of Theorem 5, one realizes that the proof of Theorem 3 is very similar to that of Theorem 4. The major differences are that $\beta_0$ is replaced by $\beta_{0,TS}$ and that the role of Lemma 5 is replaced by Lemma 8 under time series structures for different values of $\nu$, which is provided at the end of this proof.

We only highlight the differences from the proof of Theorem 3 below briefly.

We still define $\tilde{Z}_i = \log(\hat{\pi}_1/\hat{\pi}_2)/M_i + \bar{X}_i^T \tilde{\beta} + \bar{X}_i^T \tilde{\nabla} \bar{X}_i/2 + \text{tr}(\tilde{\nabla} S_i)/2$, which is used to approximate $Z_i = \log(\pi_1/\pi_2)/M_i + \bar{X}_i^T \beta + \bar{X}_i^T \nabla \bar{X}_i/2 + \text{tr}(\nabla S_i)/2$. Note that under the general time series structures, $\beta_{0,TS}$ is the population minimizer of the loss function. Thus, we can still obtain the inequality similar to (S2.26), i.e.,

$$\ell_0(\tilde{\beta}_0, \tilde{Z}) \leq \ell_0(\beta_{0,TS}, \tilde{Z}) + E_{\beta_{0,TS}} - E_{\tilde{\beta}_0}$$

where $R_1 = \frac{1}{N m_0} \sum_{i=1}^N \{Y_i - \bar{X}_i \times (Y_i | X_i)\} M_i \leq C r \sqrt{\frac{\log p}{N}}$ with probability at least $1 - O(p^{-L})$ by Hoeffding’s inequality. Again, this statement is valid conditioning on any realization of $\{X_i\}_{i=1}^N$ and thus is also valid unconditionally.

In addition, applying Lemma 4 to $\ell_0(\theta_0, \tilde{Z})$ at point $\beta_{0,TS}$ and by Condition 2, we still have (S2.27) with $\beta_0$ being replaced by $\beta_{0,TS}$, i.e.,

$$\ell_0(\tilde{\beta}_0, \tilde{Z}) - \ell_0(\beta_{0,TS}, \tilde{Z}) \geq \ell_0'(\beta_{0,TS}, \tilde{Z})(\tilde{\beta}_0 - \beta_{0,TS}) + \frac{\ell_0''(\beta_{0,TS}, \tilde{Z})}{C_m^2 m_0^2} (e^{-C_m m_0} |\tilde{\beta}_0 - \beta_{0,TS}| + C_m m_0 |\tilde{\beta}_0 - \beta_{0,TS}| - 1).$$

We next bound $\ell_0'(\beta_{0,TS}, \tilde{Z})$ from above and bound $\ell_0''(\beta_{0,TS}, \tilde{Z})$ from below.

By applying Lemma 8 and our assumption, we have that with probability at least $1 - O(p^{-L})$, $m_0 \max_i |\tilde{Z}_i - Z_i| := R_2$ is sufficiently small. Therefore, with the fact that $\ell_0'(\beta_{0,TS}, Z) =
0 where \( \ell'_0(\beta_{0,TS}, Z) \) is obtained by replacing \( \tilde{Z}_i \) by \( Z_i \) in \( \ell'_0(\beta_{0,TS}, \tilde{Z}) \), \( i = 1, \ldots, N \), we can still obtain an upper bound \( |\ell'_0(\beta_{0,TS}, \tilde{Z})| \) similar to (S2.29), i.e.,

\[
\left| \ell'_0(\beta_{0,TS}, \tilde{Z}) \right| = \left| \frac{1}{N} \sum_{i=1}^{N} \left( \frac{M_i \exp(M_i(\beta_{0,TS} + \tilde{Z}_i))}{1 + \exp(M_i(\beta_{0,TS} + \tilde{Z}_i))} - \frac{M_i \exp(M_i(\beta_{0,TS} + Z_i))}{1 + \exp(M_i(\beta_{0,TS} + Z_i))} \right) \right| \\
\leq C^2 m_0 R_2, \tag{S2.36}
\]

Moreover, by Condition 1, we have that the expectation of the i.i.d. bounded random variable \( \frac{\exp(M_i(\beta_{0} + Z_i))}{(1 + \exp(M_i(\beta_{0} + Z_i)))^2} \), \( i = 1, \ldots, N \), is bounded away from \( C_{\log} \). Following a similar argument, we are able to obtain a similar result to (S2.30), i.e., with probability at least \( 1 - O(p^{-L}) \),

\[
\ell''_0(\beta_{0,TS}, \tilde{Z}) = \frac{1}{N} \sum_{i=1}^{N} M_i^2 \left( \frac{\exp(M_i(\beta_{0,TS} + \tilde{Z}_i))}{1 + \exp(M_i(\beta_{0,TS} + \tilde{Z}_i))} \right) \left( \frac{1}{1 + \exp(M_i(\beta_{0,TS} + \tilde{Z}_i))} \right) \\
\geq C_{\text{low}} m_0^2, \tag{S2.37}
\]

In the end, plugging (S2.34), (S2.36) and (S2.37) into (S2.35) and applying the union bound argument, we obtain that with probability \( 1 - O(p^{-L}) \),

\[
C_{\text{low}} C_{m}^{-2} (e^{-C_{m} m_0 |\tilde{\beta}_0 - \beta_{0,TS}|} + C_{m} m_0 |\tilde{\beta}_0 - \beta_{0,TS}| - 1) \leq m_0 TS \left( C_{m}^2 R_2 + R_1 \right) |\tilde{\beta}_0 - \beta_0|. \tag{S2.38}
\]

Then following a similar deterministic argument, we obtain that with probability \( 1 - O(p^{-L}) \),

\[
|\tilde{\beta}_0 - \beta_{0,TS}| \leq 2C_{\text{low}}^{-1} m_0^{-1} \left( C_{m}^2 R_2 + R_1 \right),
\]

which further completes our proof, together with Lemma 8 and the bound of \( R_1 \),

\[
|\tilde{\beta}_0 - \beta_{0,TS}| \leq C_{5} \left( (1 + U_\beta) \| \tilde{\beta} - \beta \|_1 + (1 + U_\Sigma) \| \tilde{\nabla} - \nabla \|_1 + \max_{k=1,2} \frac{|\pi_k - \hat{\pi}_k|}{m_0} + \sqrt{\frac{\log p}{Nm_0^2}} \right).
\]

**Lemma 8.** Under the assumptions of Theorem 3, there exists some constant \( C_2 > 0 \) depending on \( c_m, C_m, C_\pi, C_\mu, C_{TS} \) and \( C_e \) such that with probability at least \( 1 - O(p^{-L}) \) we have
uniformly for all $i = 1, \ldots, N$

$$
|\tilde{Z}_i - Z_i| \leq \frac{1}{M_i} \log \left( \frac{\hat{\pi}_1 \hat{\pi}_2}{\tilde{\pi}_2 \tilde{\pi}_1} \right) + \left| \tilde{X}_i^T (\tilde{\beta} - \beta) \right| + \frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T (\tilde{\nabla} - \nabla) X_{ij} / 2 \right|

\leq C_z \left( (1 + U_\beta) \|\tilde{\beta} - \beta\|_1 + (1 + U_\nabla) \|\tilde{\nabla} - \nabla\|_1 + \max_{k=1,2} \frac{|\pi_k - \hat{\pi}_k|}{m_0} \right), \quad (S2.39)
$$

where $U_\beta$ satisfies

$$
U_\beta = \begin{cases} 
\sqrt{\log p / m_0} & \text{if } \nu > 1 \\
\sqrt{\log p \log^2 m_0 / m_0} & \text{if } \nu = 1 \\
\sqrt{\log p / m_0} & \text{if } 1/2 < \nu < 1
\end{cases}
$$

and $U_\nabla$ satisfies

$$
U_\nabla = \begin{cases} 
\log p / m_0 & \text{if } \nu > 1 \\
\log p \log^2 m_0 / m_0 & \text{if } \nu = 1 \\
\log p / m_0 & \text{if } 1/2 < \nu < 1
\end{cases}
$$

Indeed, the conclusion (S2.39) is valid with the same probability $1 - O(p^{-L})$ conditioning on any realization of $\{\mathcal{Y}_i\}_{i=1}^N$ and $\{M_i\}_{i=1}^N$.

Proof of Theorem 4

Proof. By inspecting the proof of Theorem 6, one realizes that the proof of Theorem 4 is almost identical to that of Theorem 6 with $\phi_B, \beta_0, \Xi_N, \kappa_N, F_{k,m}$ and $R_B$ being replaced by their counterparts $\phi_{B,TS}, \beta_{0,TS}, \Xi_{N,TS}, \kappa_{N,TS}, F_{k,m,TS}$ and $R_{B,TS}$ under the time series structure respectively. Therefore, we omit the proof details.
S3. Proofs of Supporting Lemmas

Proof of Lemma 1

Proof. Recall $n_1 = \sum_{i=1}^{N} M_i \mathbb{1}\{Y_i = 1\}$ with $\mathbb{1}\{Y_i = 1\}$ i.i.d. Bernoulli with probability $\pi_1 \in [C_{\pi}, 1 - C_{\pi}]$ and $M_i \in [c_m m_0, C_{m} m_0]$ with probability 1. Hoeffding’s inequality (e.g. Vershynin, 2012, Proposition 5.10) implies that there exists some constant $C'$ depending on $C_{\pi}$ and $L$ only such that (iv) holds, i.e. $|\pi_1 - \hat{\pi}_1| \leq C' \sqrt{\frac{\log p}{N}}$ with probability at least $1 - p^{-L}$. Consequently, $n_1 \geq c N m_0$ for some constant $c$ depending on $c_m, C_{\pi}$ and $L$ with probability at least $1 - p^{-L}$ given $\log p \leq c_0 N$ and Condition 3. Similar results apply to $\hat{\pi}_2$ and $n_2$. From now on, we condition on the above event and only need to show (i)-(iii) hold with probability at least $1 - p^{-L}$.

Since $\Sigma_k^{-1/2}(\hat{\mu}_k - \mu_k) \sim N(0, \frac{1}{n_k} I_p)$, the tail probability of Chi-squared distribution (Laurent and Massart, 2000, e.g.) implies that for any $0 < t < 1$, $\Pr(\|\sqrt{n_k} \Sigma_k^{-1/2}(\hat{\mu}_k - \mu_k)\|^2/p - 1 \geq t) \leq 2 \exp(pt^2/8)$. Hence, by picking a small $t$ (e.g. $t = 0.1$) as well as Condition 1 and $n_k > c N m_0$, we obtain the result (ii) holds with probability at least $1 - O(p^{-L})$.

In addition, it follows from the Davidson-Szarek bound (e.g. Davidson and Szarek, 2001, Theorem II.7) that for each $k$, there exists some constant $C > 0$ depending on $C_{e}, L$ such that $\|\Sigma_k - \hat{\Sigma}_k\|_{\ell^2} < C \sqrt{p/(Nm_0)}$ with probability at least $1 - 2p^{-L}$, given Condition 1 and the fact $p < c_0 N m_0$ with a sufficiently small $c_0$. Here $\|\cdot\|_{\ell^2}$ denotes the matrix spectral norm. Consequently, the assumption $p < c_0 N m_0$ and Condition 1, together with a union bound argument, implies the result (i). Result (iv) also follows, noting that $\|\cdot\|_F \leq \sqrt{p}\|\cdot\|_{\ell^2}$.
Proof of Lemma 2

Proof. Recall that \( n_k = \sum_{i=1}^{N} M_i \mathbb{1}\{Y_i = k\} \) denote the total sample size for Class \( k = 1, 2 \).

From now on, we condition on \( n_1 \) and \( n_2 \). Write \( X_{ij} = \mathbb{E}X_{ij} + U_{ij} \), where \( U_{ij} \sim N(0, \Sigma_{Y_i}) \).

Then we have \( \hat{\Sigma}_k = \left( \frac{1}{n_k} \sum_{(i,j):Y_i=k} U_{ij}U_{ij}^T \right) - (\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)^T \). Since \( \hat{\mu}_k - \mu_k \sim N(0, \frac{1}{n_k}\Sigma_k) \), tail probability of normal distribution with union bound implies that for any \( L > 0 \), there exists some constant \( C_1 > 0 \) depending on \( L \) only such that for \( k = 1, 2 \),

\[
\text{pr}\left( \|\hat{\mu}_k - \mu_k\|_{\infty} \geq C_1 \sqrt{\left( \max_j \sigma_{k,jj} \right) \log p / n_k} \right) \leq p^{-L}. \tag{S3.40}
\]

Moreover, since \( \mathbb{E}\frac{1}{n_k} \sum_{(i,j):Y_i=k} U_{ij}U_{ij}^T = \Sigma_k \) and each entry of \( U_{ij}U_{ij}^T \) is sub-exponentially distributed, Bernstein’s inequality (e.g. Vershynin, 2012, Proposition 5.16) with union bound implies that there exists some constant \( C_2 > 0 \) depending on \( L \) such that

\[
\text{pr}\left( \left\| \frac{1}{n_k} \sum_{(i,j):Y_i=k} U_{ij}U_{ij}^T - \Sigma_k \right\|_{\infty} \geq C_2 \max_j \sigma_{k,jj} \left( \sqrt{\frac{\log p}{n_k}} + \frac{\log p}{n_k} \right) \right) \leq p^{-L}. \tag{S3.41}
\]

Combining (S3.40) and (S3.41) and the fact that \( M_i \in [c_m m_0, C_m m_0] \), we have obtained both results (i) and (ii) of the first part of Lemma 2 with probability at least \( 1 - 4p^{-L} \), where the constant \( C' > 0 \) depends on \( c_m, C_e \) and \( L \) only.

We move to the second part of Lemma 2. Note the distribution of each \( X_{ij} \) is independent of \( N_k \) and \( n_k \). We follow the same argument on bounding \( n_1 \) and \( n_2 \) as that at the beginning of the proof of Lemma 1. In particular, given \( \log p \leq c_0 N \), we have \( \text{pr}(n_k \geq cN m_0) = 1 - p^{-L} \) for \( k = 1, 2 \) and some constant \( c > 0 \). Then both results (i) and (ii) of the second part of Lemma 2 immediately follow from the first part of Lemma 2 and a union bound argument. \( \square \)
Proof of Lemma 3

Proof. We follow the same argument on bounding $n_1$ and $n_2$ as that at the beginning of the proof of Lemma 1. In particular, given $\log p \leq c_0 N$, we have $\Pr(n_k \geq c N m_0) = 1 - p^{-L}$ for $k = 1, 2$ and some constant $c > 0$.

Write $X_{ij} = \mathbb{E}X_{ij} + U_{ij}$, where $U_{ij} \sim N(0, \Sigma_{y_i})$. We have $\hat{\Sigma}_k = (\frac{1}{n_k} \sum_{(i,j): Y_{ij} = k} U_{ij} U_{ij}^T) - (\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)^T$. Result (i) of Lemma 2 implies that there exists some constant $C_1 > 0$ such that

$$\Pr(\|\hat{\mu}_k - \mu_k\|_\infty \geq C_1 \sqrt{\frac{\log p}{N m_0}}) = O(p^{-L}). \quad (S3.42)$$

According to our assumptions, we have $\|\Sigma_k^{-1} \mu_k\| \leq \lambda_{\min}(\Sigma_k)^{-1}\|\mu_k\| \leq C_\epsilon C_\mu$. We condition on $n_1$ and $n_2$. Then the normality of $\hat{\mu}_k - \mu_k \sim N(0, \Sigma_k/n_k)$ yields that for $k = 1, 2$ and some constant $C''$ depending on $L$ only, we have

$$\left| (\mu_k - \hat{\mu}_k)^T \Sigma_k^{-1} \mu_k \right| \geq C'' \lambda_{\max}(\Sigma_k) C_\epsilon C_\mu \sqrt{\frac{\log p}{n_k}}$$

with probability at most $p^{-L}$. Taking union bound with the event $n_k \geq c N m_0$, we obtain that there exists some constant $C_2' > 0$ such that

$$\Pr\left(\left| (\mu_k - \hat{\mu}_k)^T \Sigma_k^{-1} \mu_k \right| \geq C_2' \sqrt{\frac{\log p}{N m_0}} \right) \leq 2p^{-L}. \quad (S3.43)$$

Therefore, equations (S3.42)-(S3.43) imply that there exists some constant $C_2 > 0$ such that with probability $1 - O(p^{-L})$,

$$\| (\mu_k - \hat{\mu}_k)(\mu_k - \hat{\mu}_k)^T \beta_k \|_\infty < C_2 \frac{\log p}{N m_0}. \quad (S3.44)$$

By our choice of $\lambda_{2,N}$, we have that $\lambda_{2,N}/2 > (C_1 + C_2 + C_2')\sqrt{(\log p)/(N m_0)}$. Consequently, given equations (S3.42)-(S3.44), decomposition of $\Sigma_k$ and $\log p = o(N)$, to conclude $(\beta_1, \beta_2)$ is feasible, i.e. $\left\| \hat{\Sigma}_k \beta_k - \hat{\mu}_k \right\|_\infty < \lambda_{2,N}$, $k = 1, 2$, we only need to show with probability
1 − O(p−L) that
\[ \| \left( \frac{1}{n_k} \sum_{(i,j): Y_i = k} U_{ij} U_{ij}^T \right) \Sigma^{-1} \mu_k - \mu_k \|_\infty < \frac{1}{2} \lambda_{2,N}. \]  
(S3.45)

Note that the rth coordinate is 
\[ \frac{1}{n_k} \sum_{(i,j): Y_i = k} \left( U_{ij,r} U_{ij}^T \right) \Sigma^{-1} \mu_k - \mu_{k,r} \],
the sum of i.i.d. centered sub-exponential variable since each summand is the product of two normal variables \( U_{i,j} \) and \( U_{i,j}^T \Sigma^{-1} \mu_k \). Moreover, the sub-exponential variable has constant parameter since \( U_{ij} \) and \( U_{ij,r} \) have bounded variance. Thus Bernstein’s inequality (e.g. Vershynin, 2012, Proposition 5.16) with union bound over all coordinates and the event \( n_k \geq cN m_0 \) implies that there exists some constant \( C_3 > 0 \) such that (we also used that \( \log p \leq c_0 N \) when applying the Bernstein’s inequality)
\[ \text{pr}(\| \left( \frac{1}{n_k} \sum_{(i,j): Y_i = k} U_{ij} U_{ij}^T \right) \Sigma^{-1} \mu_k - \mu_k \|_\infty > C_3 \sqrt{\frac{\log p}{N m_0}}) \leq 2p^{-L}. \]  
(S3.46)

By picking a large constant \( C' \) in our choice of \( \lambda_{2,N} \), we obtain \( \lambda_{2,N}/2 > C_3 \sqrt{(\log p)/(N m_0)} \), which completes the proof of (S3.45).

\[ \square \]

**Proof of Lemma 5**

Proof. It is sufficient to show that for any realization of \( \{Y_i\}_{i=1}^N \) and \( \{M_i\}_{i=1}^N \), equation (S2.28) is valid for each \( i \) with probability at least \( 1 - O(p^{-L-1}) \). Indeed, this fact, together with the union bound argument and \( p \geq N \) implies the desired result. The first inequality of (S2.28) follows from the definitions of \( \tilde{Z}_i \) and \( Z_i \) directly. We show the second inequality holds in the remaining of proof with probability at least \( 1 - O(p^{-L-1}) \) for the fixed \( i \). Without loss of generality, we assume \( Y_i = 1 \) and \( M_i = m_0 c_m \).

Recall that the initial estimators satisfy \( \max_{k=1,2} |\pi_k - \hat{\pi}_k| \leq C_p \) with a sufficiently small constant \( C_p \). Consequently, we have that \( \hat{\pi}_1, \hat{\pi}_2 \in [C_\pi/2, 1 - C_\pi/2] \) by Condition 3, which
further yields \( \frac{1}{m_{0c_m}} \log \left( \frac{\hat{\pi}_1 \hat{\pi}_2}{\hat{\pi}_2 \hat{\pi}_1} \right) \leq C_{z1} \max_{k=1,2} |\pi_k - \hat{\pi}_k|/m_0 \) with some universal constant \( C_{z1} \) depending on \( c_m \) and \( C_{\pi} \) only by the boundedness of \( \hat{\pi}_1/\hat{\pi}_2 \).

To deal with the term \( |\bar{X}_i^T(\tilde{\beta} - \beta)| \), we note that \( \bar{X}_i \sim N(\mu_1, \Sigma_1/(m_0c_m)) \), which implies that \( |\bar{X}_i^T(\tilde{\beta} - \beta)| \leq \|\tilde{\beta} - \beta\|\|\mu_1\| + \|\tilde{\beta} - \beta\| (C_e/(m_0c_m))^{1/2} |D| \), where \( D \sim N(0, 1) \). According to the tail probability of standard normal distribution, we obtain that with probability at least \( 1 - O(p^{-L-1}) \), that \( |D| \leq C'_z \sqrt{\log p} \) where \( C'_z \) only depends on \( L \). This fact, together with the assumption \( \|\mu_1\| \leq C_\mu \) further implies that \( |\bar{X}_i^T(\tilde{\beta} - \beta)| \leq C_{z2}\|\tilde{\beta} - \beta\|(1 + \sqrt{(\log p)/m_0}) \) with probability \( 1 - O(p^{-L-1}) \), where \( C_{z2} = ((C_e/c_m)^{1/2} C'_z + C_\mu) \).

Finally, we provide an upper bound for \( \frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T(\tilde{\nabla} - \nabla) X_{ij}/2 \right| \). Since \( X_{i1}, \ldots, X_{iM_i} \) are i.i.d. copies of \( N(\mu_1, \Sigma_1) \), we naturally decompose it into three terms as follows with \( U_{ij} := X_{ij} - \mu_1 \sim N(0, \Sigma_1) \)

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T(\tilde{\nabla} - \nabla) X_{ij}/2 \right| \\
\leq \frac{1}{M_i} \left| \sum_{j=1}^{M_i} U_{ij}^T(\tilde{\nabla} - \nabla) U_{ij}/2 \right| + \left| \mu_1^T (\tilde{\nabla} - \nabla) \mu_1/2 \right| + \frac{1}{M_i} \left| \sum_{j=1}^{M_i} \mu_1^T (\tilde{\nabla} - \nabla) U_{ij} \right| \quad (3.47)
\]

We deal with these three terms individually. First of all, \( |\mu_1^T (\tilde{\nabla} - \nabla) \mu_1/2| \leq C_\mu^2 \|\tilde{\nabla} - \nabla\|_{1/2} \) by the assumption \( \|\mu_1\| \leq C_\mu \). Second, the term \( \left( \sum_{j=1}^{M_i} \mu_1^T (\tilde{\nabla} - \nabla) U_{ij} \right)/M_i \) follows a distribution of \( N(0, \mu_1^T (\tilde{\nabla} - \nabla) \Sigma_1 (\tilde{\nabla} - \nabla) \mu_1/(m_0c_m)) \), which yields that with probability at least \( 1 - O(p^{-L-1}) \) that

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} \mu_1^T (\tilde{\nabla} - \nabla) U_{ij} \right| \leq \left( \mu_1^T (\tilde{\nabla} - \nabla) \Sigma_1 (\tilde{\nabla} - \nabla) \mu_1/(m_0c_m) \right)^{1/2} C''_z \sqrt{\log p} \\
\leq C_\mu C'_z (C_e/c_m)^{1/2} \|\tilde{\nabla} - \nabla\|_1 \sqrt{\frac{\log p}{m_0}},
\]

where we have used tail probability of standard normal distribution and the last inequality
follows from Condition 1. Third, by Hölder’s inequality, we have

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} U^T_{ij} \left( \tilde{\nabla} - \nabla \right) U_{ij}/2 \right| = \left| \text{tr} \left( \left( \tilde{\nabla} - \nabla \right) \sum_{j=1}^{M_i} U^T_{ij} U_{ij}/M_i \right)/2 \right| \\
\leq \frac{1}{2} \left\| \tilde{\nabla} - \nabla \right\|_1 \left\| \sum_{j=1}^{M_i} U^T_{ij} U_{ij}/M_i \right\|_{\infty}.
\]

Since each entry of \( \sum_{j=1}^{M_i} U^T_{ij} U_{ij}/M_i - \Sigma_1 \) is the sum of centered sub-exponential variable with bounded parameter. The Bernstein’s inequality (e.g. Vershynin, 2012, Proposition 5.16) with union bound over all \( p^2 \) entries implies that there exists some constant \( C''_z > 0 \) depending on \( L \) and \( C_e \) only such that \( \| \sum_{j=1}^{M_i} U^T_{ij} U_{ij}/M_i - \Sigma_1 \|_{\infty} \leq C''_z \left( \sqrt{\frac{\log p}{c_m m_0}} + \frac{\log p}{c_m m_0} \right) \) with probability at least \( 1 - O(p^{-L-1}) \). Therefore, we obtain that with probability \( 1 - O(p^{-L-1}) \),

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} U^T_{ij} \left( \tilde{\nabla} - \nabla \right) U_{ij}/2 \right| \leq \left( C''_z \left( \sqrt{\frac{\log p}{c_m m_0}} + \frac{\log p}{c_m m_0} \right) + C_e \right) \| \tilde{\nabla} - \nabla \|_1/2,
\]

where we have used \( \| \Sigma_1 \|_{\infty} \leq C_e \) by Condition 1. Combining the upper bounds of three terms above, we finally obtain that with probability \( 1 - O(p^{-L-1}) \),

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} X^T_{ij} \left( \tilde{\nabla} - \nabla \right) X_{ij}/2 \right| \leq C_{z3} \left( \frac{\log p}{m_0} + \frac{\log p}{m_0} + 1 \right) \| \tilde{\nabla} - \nabla \|_1 \\
\leq C_{z3} \left( \frac{\log p}{m_0} + 1 \right) \| \tilde{\nabla} - \nabla \|_1,
\]

where constant \( C'_{z3} = C'_\mu/2 + C\mu C'_z (C_e/c_m)^{1/2} + (C_e + C''_z/\sqrt{c_m} + C''_z/c_m)/2 \) and \( C_{z3} = 2C'_{z3} \).

To complete our proof, we combine all bounds for \( \frac{1}{n_0 c_m} \left| \log \left( \frac{2 \pi x_1}{p_2 \pi_1} \right) \right|, |X_i^T (\tilde{\beta} - \beta)| \) and \( \frac{1}{M_i} \left| \sum_{j=1}^{M_i} X^T_{ij} (\tilde{\nabla} - \nabla) X_{ij}/2 \right| \) with \( C_z = C_{z1} + C_{z2} + C_{z3} \). \( \square \)

**Proof of Lemma 6**

*Proof.* We show the first part of Lemma 6 in this proof. The second part of Lemma 6 immediately follows from the first part, that \( \log p \leq c_0 N \), and the fact that \( \text{pr}(n_k \geq c N m_0) = \)
1 − p−L for k = 1, 2 and some constant c > 0, which is obtained from the argument at the beginning of the proof of Lemma 1.

In this proof, we need the following technical result, which is a direct consequence of Lemma VI.1 in Chen et al. (2016).

**Lemma 9** (Chen et al. (2016)). Let ν > 1/2 and \((a_t)_{t \in \mathbb{Z}}\) be a real sequence such that 
\[a_t \leq C_{TS}(1 + t)^{-\nu}\] for \(t \geq 0\) and \(a_t = 0\) if \(t < 0\). Let \(\gamma_l = \sum_{t=0}^{\infty} |a_t a_{t+l}|\). Then (i) \(\gamma_l = O(l^{-\nu})\) \((O(l^{-1} \log l)\) and \(O(l^{1-2\nu})\) \) and \(\sum_{k=0}^{l} \gamma_k = O(1)\) \((O(\log^2 l)\) and \(O(l^{2-2\nu})\) \) hold for \(\nu > 1\) \((\nu = 1\) and \(1/2 < \nu < 1\) respectively); (ii) \(\sum_{k=0}^{l} \gamma_k^2 = O(1)\) \((O(\log l)\) and \(O(l^{3-4\nu})\) \) hold for \(\nu > 3/4\) \((\nu = 3/4\) and \(1/2 < \nu < 3/4\) respectively).

Without loss of generality, we assume that the first \(N_1\) sets are from Class 1 (i.e., \(\mathcal{Y}_i = 1\) for \(i = 1, ..., N_1\)) and only prove results (i)-(ii) for Class 1. We first show result (i), i.e., bound the term \(\|\mu_1 - \hat{\mu}_1\|_\infty\). In the following, we bound each entry of \(\mu_1 - \hat{\mu}_1\) and then take a union bound argument to finish the proof. To bound the \(l\)th entry \((l = 1, ..., p)\), i.e., \(|\mu_{1l} - \hat{\mu}_{1l}|\), we collect the \(l\)th entry \(X_{ij,l}\) of each observation \(X_{ij}, i = 1, ..., N_1, j = 1, ..., M_i\) and observe that its centered version can be denoted according to the vector linear process (S1.1) as

\[
(X_{1M_1,l}, ..., X_{11,l}; X_{2M_2,l}, ..., X_{21,l}; ...; X_{N_1M_{N_1},l}, ..., X_{N_11,l})^T - (\mu_{1l}, ..., \mu_{1l})^T = A(l)\xi, \quad (S3.48)
\]

where \(\xi = (\xi_{1M_1}, \xi_{1(M_1-1)}; \xi_{2M_2}, \xi_{2(M_2-1)}; ...; \xi_{N_1M_{N_1}}, \xi_{N_1(M_{N_1}-1)})^T\) with i.i.d. \(N(0, 1)\) en-
tries, and $A^{(l)}$ is a block diagonal matrix,

$$
A^{(l)} = \begin{bmatrix}
A^{(l),1} & 0 & 0 & 0 \\
0 & A^{(l),2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & A^{(l),N_1}
\end{bmatrix},
$$

in which the $i$ the block ($i = 1, ..., N_1$) $A^{(l),i}$ has the following form

$$
A^{(l),i} = \begin{bmatrix}
A_{10,l} & A_{11,l} & A_{12,l} & \ldots & A_{1(M_i-1),l} & A_{1M_i,l} & \ldots \\
0 & A_{10,l} & A_{11,l} & \ldots & A_{1(M_i-2),l} & A_{1(M_i-1),l} & \ldots \\
0 & 0 & A_{10,l} & \ldots & A_{1(M_i-3),l} & A_{1(M_i-2),l} & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_{10,l} & A_{11,l} & \ldots
\end{bmatrix}.
$$

In the above representation, $A_{1t,l}$ denotes the $l$th row of the coefficient matrix $A_{1t}$ defined in our vector linear process (S1.1). Given (S3.48), one immediately obtains that

$$
\mu_1 - \hat{\mu}_1 \sim N \left(0, 1^T A^{(l)} (A^{(l)})^T 1 / n_1^2 \right),
$$

(S3.49)

where $n_1 = \sum_{i=1}^{N_1} M_i$ denote the total sample size for Class 1, and $1$ denotes the $n_1$-dimensional vector with each entry being 1.

It remains to bound the variance in (S3.49) for different value of $\nu > 1/2$. To this end, we note that

$$
1^T A^{(l)} (A^{(l)})^T 1 = \sum_{i=1}^{N_1} 1^T A^{(l),i} (A^{(l),i})^T 1 := \sum_{i=1}^{N_1} 1^T \Gamma^{(i),i} 1,
$$

where we set $\Gamma^{(i),i} = A^{(l),i} (A^{(l),i})^T$ and $1$ in the $i$th summand denotes the $M_i$-dimensional vector with each entry being 1 respectively. Due to the time series structure, the matrix

$$
\Gamma^{(i),i} = \begin{bmatrix}
\Gamma^{(1),1} & 0 & 0 & \ldots \\
0 & \Gamma^{(2),2} & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \Gamma^{(N_1),N_1}
\end{bmatrix},
$$

in which the $i$ block ($i = 1, ..., N_1$) $\Gamma^{(i),i}$ has the following form

$$
\Gamma^{(i),i} = \begin{bmatrix}
\Gamma_{10,1} & \Gamma_{11,1} & \Gamma_{12,1} & \ldots & \Gamma_{1(M_i-1),1} & \Gamma_{1M_i,1} & \ldots \\
0 & \Gamma_{10,2} & \Gamma_{11,2} & \ldots & \Gamma_{1(M_i-2),2} & \Gamma_{1(M_i-1),2} & \ldots \\
0 & 0 & \Gamma_{10,3} & \ldots & \Gamma_{1(M_i-3),3} & \Gamma_{1(M_i-2),3} & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \Gamma_{10,N_1} & \Gamma_{11,N_1} & \ldots
\end{bmatrix}.
$$

In the above representation, $\Gamma_{1t,1}$ denotes the $1$th row of the coefficient matrix $\Gamma_{1t}$ defined in our vector linear process (S1.1). Given (S3.48), one immediately obtains that

$$
\mu_1 - \hat{\mu}_1 \sim N \left(0, 1^T \Gamma^{(l)} (\Gamma^{(l)})^T 1 / n_1^2 \right),
$$

(S3.49)
is a $M_i$-dimensional Toeplitz matrix with elements $\gamma_{l,j}^{(i)}$, where

$$\left| \gamma_{l,j}^{(i)} \right| = \left| \sum_{t=0}^{\infty} A_{1l,t} \left( A_{1(t+j),l}^{(i)} \right)^T \right|$$

$$\leq \sum_{t=0}^{\infty} \left( \sum_{k=1}^{p} a_{1t,lk}^2 \right)^{1/2} \left( \sum_{k=1}^{p} a_{1(t+j),lk}^2 \right)^{1/2}$$

$$\leq C \zeta_j,$$

(S3.50)

where $C > 0$ is some constant, $\zeta_j = j^{-\nu} (j^{-1} \log j$ and $j^{1-2\nu})$ for $\nu > 1$ ($\nu = 1$ and $1/2 < \nu < 1$ respectively). The first inequality above follows from Cauchy-Schwarz inequality and the second inequality is due to the decay condition of the coefficient matrix in (S1.2) and Lemma 9 (i). Consequently, we can bound the variance as follows, noting that $n_1 > c_m N_1 m_0$ by Condition 3,

$$\frac{1^T A(l) \left( A(l)^T \right) 1}{n_1^2} \leq \frac{\sum_{i=1}^{N_1} M_i \sum_{j=0}^{M_i-1} \zeta_j}{(c_m N_1 m_0)^2}$$

$$\leq \frac{C_m N_1 m_0 \sum_{j=0}^{c_m m_0-1} \zeta_j}{(c_m N_1 m_0)^2} \leq \begin{cases} C \frac{1}{N_1 m_0} & \text{if } \nu > 1 \\ C \frac{\log^2 m_0}{N_1 m_0} & \text{if } \nu = 1 \\ C \frac{1}{N_1 m_0^{2\nu-1}} & \text{if } 1/2 < \nu < 1 \end{cases}$$

where the last inequality follows from Lemma 9 (i). In the end, the result (i) of the first part immediately follows from the above variance bound and the the tail probability of normal distribution with a union bound argument.

Now we turn to the result (ii). In the following, we bound each entry of $\Sigma_1 - \hat{\Sigma}_1$ and then take a union bound argument to finish the proof. To bound the $lk$th entry ($l, k = 1, \ldots, p$), i.e., $|\sigma_{1,lk} - \hat{\sigma}_{1,lk}|$, we note that

$$\sigma_{1,lk} - \hat{\sigma}_{1,lk} = \frac{1}{n_1} \left( \xi^T \left( A(l)^T A^{(k)} \right) \xi - \mathbb{E} \xi^T \left( A(l)^T A^{(k)} \right) \xi \right) - (\mu_{1l} - \hat{\mu}_{1l})(\mu_{1k} - \hat{\mu}_{1k}) =: T_1 + T_2,$$

(S3.51)
where the second term can be bounded with probability at least $1 - O(p^{-(L+2)})$ using the result (i) shown above, that is,

$$|T_2| \leq \begin{cases} 
C \frac{\log p}{N_1 m_0} & \text{if } \nu > 1 \\
C \frac{\log p \log^2 m_0}{N_1 m_0} & \text{if } \nu = 1 \\
C \frac{\log p}{N_1 m_0^{\nu-1}} & \text{if } 1/2 < \nu < 1 
\end{cases}.$$ 

It remains to bound the first term $|T_1|$. To this end, we apply the Hanson-Wright inequality (e.g. Rudelson and Vershynin, 2013, Theorem 1.1) since $\xi$ contains i.i.d $N(0,1)$ entries. Note that by Condition 3, we have $c_m m_0 \leq M_i \leq C_m m_0$. Therefore,

$$\text{pr}(|T_1| \geq x) \leq 2 \exp \left(-C \min \left\{ \| (A^{(l)})^T A^{(k)} \|_F^2 x^2 N_1^2 m_0^2, \lambda_{\max}^{-1} \left( (A^{(l)})^T A^{(k)} \right) x N_1 m_0 \right\} \right),$$

where $\lambda_{\max}(\cdot)$ denotes the largest singular value. In what follows, we bound $\| (A^{(l)})^T A^{(k)} \|_F^2$ and $\lambda_{\max} \left( (A^{(l)})^T A^{(k)} \right)$ separately.

To bound the first term, we note that by Cauchy-Schwarz inequality,

$$\| (A^{(l)})^T A^{(k)} \|_F^2 = \text{trace} \left( (A^{(l)})^T A^{(l)} (A^{(k)})^T A^{(k)} \right) \leq \| \Gamma^{(l)} \|_F \| \Gamma^{(k)} \|_F,$$

where we set $\Gamma^{(l)} = A^{(l)} (A^{(l)})^T$. In addition, we have

$$\| \Gamma^{(l)} \|_F^2 = \sum_{i=1}^{N_1} \| (\Gamma^{(l)})^i \|_F^2$$

$$= \sum_{i=1}^{N_1} (M_i (\gamma_i^l)^2 + 2(M_i - 1) (\gamma_1^l)^2 + \ldots + 2(\gamma_{M_i-1}^l)^2)$$

$$\leq C N_1 m_0 \sum_{j=0}^{C_m m_0 - 1} (\gamma_j^l)^2 \leq \begin{cases} 
C N_1 m_0 & \text{if } \nu > 3/4 \\
C N_1 m_0 \log m_0 & \text{if } \nu = 3/4 \\
C N_1 m_0^{4-4\nu} & \text{if } 1/2 < \nu < 3/4 
\end{cases},$$ 

(S3.54)
where the last inequality follows from Lemma 9 (ii).

To bound the second term, we note that

$$\lambda_{\text{max}} \left( (A^{(l)})^T A^{(k)} \right) \leq \lambda_{\text{max}} \left( \Gamma^{(l)} \right)^{1/2} \lambda_{\text{max}} \left( \Gamma^{(k)} \right)^{1/2}. \quad (S3.55)$$

In addition, due to the block structure of $\Gamma^{(l)}$, we have

$$\lambda_{\text{max}} \left( \Gamma^{(l)} \right) = \max_{i = 1, \ldots, N_1} \left\{ \lambda_{\text{max}} \left( \Gamma^{(l), i} \right) \right\} \leq 2 \sum_{j = 0}^{C m_0} \gamma_j^{(l)} \leq \begin{cases} C & \text{if } \nu > 1 \\ C \log m_0 & \text{if } \nu = 1 \\ C m_0^{2-2\nu} & \text{if } 1/2 < \nu < 1 \end{cases}, \quad (S3.56)$$

where the last inequality is due to Lemma 9 (i).

Plugging equations (S3.53)-(S3.56) into equation (S3.51), we obtain that with probability at least $1 - O(p^{-(L+2)})$,

$$|T_1| \leq \begin{cases} C \left( \sqrt{\log p \over N_k m_0} + \log p \over N_k m_0 \right) & \text{if } \nu > 1 \\ C \left( \sqrt{\log p \over N_k m_0} + \log p \log^2 m_0 \over N_k m_0 \right) & \text{if } \nu = 1 \\ C \left( \sqrt{\log p \log m_0 \over N_k m_0} + \log p \over N_k m_0 \right) & \text{if } 3/4 < \nu < 1 \\ C \left( \sqrt{\log p \log m_0 \over N_k m_0} + \log p \over N_k m_0 \right) & \text{if } \nu = 3/4 \\ C \left( \sqrt{\log p \log m_0 \over N_k m_0} + \log p \over N_k m_0 \right) & \text{if } 1/2 < \nu < 3/4 \end{cases}.$$  

In the end, the result (ii) of the first part immediately follows from a union bound argument by plugging the bounds of $T_1$ and $T_2$ above into equation (S3.51). We point out that the upper bound of $T_1$ dominates that of $T_2$. Therefore, we complete the proof. \qed
Proof of Lemma 7

Proof. The proof of this lemma is essentially similar to that of Lemma 3. Recall that \( \| \beta_k \|_1 \leq C_\beta \) for \( k = 1, 2 \). Therefore, by H"older's inequality we have

\[
\left\| \hat{\Sigma}_k \beta_k - \hat{\mu}_k \right\|_\infty \leq \left\| \hat{\Sigma}_k - \Sigma_k \right\|_\infty \| \beta_k \|_1 + \| \mu_k - \hat{\mu}_k \|_\infty \leq \left\| \hat{\Sigma}_k - \Sigma_k \right\|_\infty C_\beta + \| \mu_k - \hat{\mu}_k \|_\infty.
\]

Consequently, the fact that \((\beta_1, \beta_2)\) is feasible with probability at least \( 1 - O(p^{-L}) \) immediately follows from our choice of \( \lambda_{2,N} \), the fact that \( \log p \leq c_0 N \) and results (i)-(ii) in the second part of Lemma 6. It is worthwhile to point out that according to the fact \( \log p \leq c_0 N \) and the bounds provided in Lemma 6, \( \left\| \hat{\Sigma}_k \beta_k - \hat{\mu}_k \right\|_\infty \) is dominated by the term \( \| \mu_k - \hat{\mu}_k \|_\infty \).

Proof of Lemma 8

Proof. The proof of this lemma is similar to that of Lemma 5. We only highlight the main differences briefly below. The first inequality follows from the definitions of \( \tilde{Z}_i \) and \( Z_i \) directly.

We show the second inequality holds below with probability at least \( 1 - O(p^{-(L+1)}) \) for the fixed \( i \). Without loss of generality, we assume \( \mathcal{Y}_i = 1 \) and \( M_i = m_0 c_m \).

Following the lines in the proof of Lemma 5, we still can show that

\[
\frac{1}{m_0 c_m} \left| \log \left( \frac{\pi_1 \pi_2}{\pi_2 \pi_1} \right) \right| \leq C_{z_1} \max_{k=1,2} \left| \pi_k - \hat{\pi}_k \right|/m_0 \text{ with some constant } C_{z_1}.
\]

To deal with the term \( |\tilde{X}_i^T (\tilde{\beta} - \beta)| \), we note that with probability at least \( 1 - O(p^{-L-1}) \)

\[
|\tilde{X}_i^T (\tilde{\beta} - \beta)| \leq \| \tilde{X}_i \|_\infty \| \tilde{\beta} - \beta \|_1 \\
\leq (\| \mu_1 \|_\infty + U_\beta) \| \tilde{\beta} - \beta \|_1 \\
\leq C(1 + U_\beta) \| \tilde{\beta} - \beta \|_1,
\]

(S3.57)
where the first inequity is due to Cauchy-Schwarz inequality, the second one follows from result (i) in the first part of Lemma 6 with $N_1 = 1$, and the last one is due to the fact that $\|\mu_1\| \leq C_\mu$.

Finally, we provide an upper bound for $\frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T (\tilde{\nabla} - \nabla) X_{ij}/2 \right|$. Set $U_{ij} := X_{ij} - \mu_1$. We still decompose it as we did in the proof of Lemma 5,

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T (\tilde{\nabla} - \nabla) X_{ij}/2 \right| \\
\leq \frac{1}{M_i} \left| \sum_{j=1}^{M_i} U_{ij}^T (\tilde{\nabla} - \nabla) U_{ij}/2 \right| + \left| \mu_1^T (\tilde{\nabla} - \nabla) \mu_1/2 \right| + \frac{1}{M_i} \left| \sum_{j=1}^{M_i} \mu_1^T (\tilde{\nabla} - \nabla) U_{ij} \right|.
\]

The second term still can be bounded as $|\mu_1^T (\tilde{\nabla} - \nabla) \mu_1/2| \leq C_\mu^2 \|\tilde{\nabla} - \nabla\|_1/2$ by the assumption $\|\mu_1\| \leq C_\mu$. To bound the first term ($\sum_{j=1}^{M_i} \mu_1^T (\tilde{\nabla} - \nabla) U_{ij}/M_i$, we note that with probability at least $1 - O(p^{-L-1})$,

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} \mu_1^T (\tilde{\nabla} - \nabla) U_{ij} \right| \leq \frac{1}{M_i} \sum_{j=1}^{M_i} U_{ij} \|\mu_1\|_{\infty} \|\mu_1^T (\tilde{\nabla} - \nabla) \|_1 \\
\leq C U_\beta C_\mu \|\tilde{\nabla} - \nabla\|_1,
\]

where we used result (i) in the first part of Lemma 6 with $N_1 = 1$ during the last inequality above. To bound the third term, by Hölder’s inequality, we have with probability at least
1 − O(\(p^{-L-1}\)),

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} U_{ij}^T \left( \tilde{\nabla} - \nabla \right) U_{ij}/2 \right| = \left| \operatorname{tr}(\left( \tilde{\nabla} - \nabla \right) \sum_{j=1}^{M_i} U_{ij}^T U_{ij}/M_i)/2 \right|
\]

\[
\leq \frac{1}{2} \left\| \tilde{\nabla} - \nabla \right\|_1 \left\| \sum_{j=1}^{M_i} U_{ij}^T U_{ij}/M_i \right\|_{\infty}
\]

\[
\leq \begin{cases} 
C \left\| \tilde{\nabla} - \nabla \right\|_1 \left( \sqrt{\log p/m_0} + \frac{\log p}{m_0} \right) & \text{if } \nu > 1 \\
C \left\| \tilde{\nabla} - \nabla \right\|_1 \left( \sqrt{\log p/m_0} + \frac{\log p \log^2 m_0}{m_0} \right) & \text{if } \nu = 1 \\
C \left\| \tilde{\nabla} - \nabla \right\|_1 \left( \sqrt{\log p \log m_0/m_0} + \frac{\log p}{m_0^{1/2}} \right) & \text{if } \nu = 3/4 \\
C \left\| \tilde{\nabla} - \nabla \right\|_1 \left( \sqrt{\log p \log m_0/m_0^{\nu-2}} + \frac{\log p}{m_0^{\nu-1}} \right) & \text{if } 1/2 < \nu < 3/4
\end{cases}
\]

where we have applied the bound of \(|T_1|\) in the proof of Lemma 6 with \(N_1 = 1\) in the last inequality above.

Combining the upper bounds of three terms above, we finally obtain that with probability

\[
1 − O(\(p^{-L-1}\)),
\]

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T \left( \tilde{\nabla} - \nabla \right) X_{ij}/2 \right| \leq C(1 + U_\nabla)\| \tilde{\nabla} - \nabla \|_1.
\]

To complete our proof, we combine all bounds for \(\frac{1}{m_0^{\epsilon_m}} \left| \log \left( \frac{\bar{\pi}_1}{\bar{\pi}_2 \pi_1} \right) \right|\), \(|\bar{X}_i^T (\tilde{\beta} - \beta)|\) and

\[
\frac{1}{M_i} \left| \sum_{j=1}^{M_i} X_{ij}^T (\tilde{\nabla} - \nabla) X_{ij}/2 \right|.
\]

\[
\]
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