Unitarity Boomerang

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For the three family quark flavor mixing, the best parametrization is the original Kobayashi-Maskawa matrix, \(V_{KM}\), with four real parameters: three rotation angles \(\theta_1, \theta_2, \theta_3\) and one phase \(\delta\). A popular way of presentation is by the unitarity triangle which, however, explicitly displays only three, not four, independent parameters. Here we propose an alternative presentation which displays simultaneously all four parameters: the unitarity boomerang.

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Introduction

As is well known, there are different ways of parameterizing the Kobayashi-Maskawa\(^1\) quark mixing matrix, \(V_{KM}\). For three generations of quarks, \(V_{KM}\) is a \(3 \times 3\) unitary mixing matrix with three rotation angles \((\theta_1, \theta_2, \theta_3)\) and one CP violating phase \(\delta\). The magnitudes of the elements \(V_{ij}\) of \(V_{KM}\) are physical quantities which do not depend on parametrization. However, the value of \(\delta\) does. For example, in the Particle Data Group (PDG) parametrization\(^2\), adopted from Ref.\(^3\), \(\delta \sim 70^\circ\), whereas the phase in the original KM parametrization has a different value, \(\delta \sim 90^\circ\). Care must be exercised in quoting a value of \(\delta\), as it depends on how the matrix is parameterized. For example, the statement made after Eq. (11.3) in the current edition of PDG is misleading, because it identifies, incorrectly, the phase \(\delta\) of Ref.\(^1\).
It can therefore be more useful to employ only physically-measurable quantities. To this end, it has long ago been suggested that a unitarity triangle (UT) be used as a useful presentation for the quark flavor mixing, especially of CP violation. Because of the unitary nature of the KM matrix, one has
\[ \sum_i V_{ij} V_{ik}^* = \delta_{jk} \] and
\[ \sum_j V_{ji} V_{kj}^* = \delta_{ik}, \]
where the first and second indices of \( V_{ij} \) take the values \( u, c, t, \ldots \) and \( d, s, b, \ldots \), respectively. For three generations of quarks, when \( j \neq k \), these equations form closed triangles in a plane, the UTs. Six UTs can be formed with all of them having the same area. \( A(UT) \), which is equal to half of the value of the Jarlskog determinant \( J \), so that \( A(UT) = \frac{1}{2} J \). The inner angles of a given UT are therefore closely related to the CP violating measure \( J \). When the inner angles are measured independently, their sum, whether it turns out to be consistent with precisely 180°, provides a test for the unitarity of the KM matrix. The unitarity triangle is also a popular way, to present CP violation, with three generations of quarks. A UT, however, does not contain all the information encoded in the KM matrix, \( V_{KM} \). Although a UT has three inner angles and three sides, it contains only three independent parameters. The three parameters can be chosen to be two of the three inner angles and the area, or the three sides, or some combination thereof. One needs an additional parameter fully to represent the physics: this is hardly surprising, as the original UT idea of involved only two, of the three, rows or columns of the \( 3 \times 3 \) matrix, \( V_{KM} \).

An improved presentation is thus rendered desirable, in order better to present the KM matrix, \( V_{KM} \), diagrammatically. In this Letter, we propose such a new diagram, the unitarity boomerang.

The unitarity boomerang contains information from a pair of UTs. The different ways of choosing the pair contain, of course, equivalent information. Nevertheless, the specific choice, in the next section, was made judiciously, such as to maximize the minimum vertex angle in the unitarity boomerang. This choice is, we believe, the most convenient.

**Unitarity Boomerang**

We indicate the KM matrix and its elements by \( V_{KM} = (V_{KM})_{ij} \), with \( i = u, c, t \) and \( j = b, s, d \). The unitarity of this matrix implies \( \sum_i V_{ij} V_{ik}^* = \delta_{jk} \) and \( \sum_j V_{ij} V_{kj}^* = \delta_{ik} \). The
$j \neq k$ and $i \neq k$ cases form, respectively, the six possible different $UT$ presentations for $V_{KM}$ in a convenient two-dimensional plane. There are, thus, a total of 18 inner angles in the six $UT$s. However, only 9 are different because, by Euclidean geometry, each angle, in any particular $UT$, must have its equal counterpart in another, different, $UT$. This coincides with the fact that there are 9 different phase expressions of the KM matrix for different parameterizations \cite{8}. To understand this simple but crucial discussion consider the two $UT$s defined by

$$UT(a) \quad (V_{KM})_{ud}(V_{KM})_{ub}^* + (V_{KM})_{cd}(V_{KM})_{cb}^* + (V_{KM})_{td}(V_{KM})_{tb}^* = 0$$

$$UT(b) \quad (V_{KM})_{ud}(V_{KM})_{tb}^* + (V_{KM})_{us}(V_{KM})_{ts}^* + (V_{KM})_{ub}(V_{KM})_{tb}^* = 0 \quad (1)$$

The inner angles defined by $UT$ (a), in Eq. (1), are

$$\phi_1(\beta) = \arg\left(-\frac{(V_{KM})_{cd}(V_{KM})_{cb}^*}{(V_{KM})_{td}(V_{KM})_{tb}^*}\right)$$

$$\phi_2(\alpha) = \arg\left(-\frac{(V_{KM})_{td}(V_{KM})_{tb}^*}{(V_{KM})_{ud}(V_{KM})_{ub}^*}\right)$$

$$\phi_3(\gamma) = \arg\left(-\frac{(V_{KM})_{ud}(V_{KM})_{ub}^*}{(V_{KM})_{cd}(V_{KM})_{cb}^*}\right) \quad (2)$$

Correspondingly, the unitarity triangle, $UT(b)$ in Eq. (1), defines another three inner angles

$$\phi_1'(\beta') = \arg\left(-\frac{(V_{KM})_{us}(V_{KM})_{ts}^*}{(V_{KM})_{ub}(V_{KM})_{tb}^*}\right)$$

$$\phi_2'(\alpha') = \arg\left(-\frac{(V_{KM})_{ub}(V_{KM})_{tb}^*}{(V_{KM})_{td}(V_{KM})_{td}^*}\right)$$

$$\phi_3'(\gamma') = \arg\left(-\frac{(V_{KM})_{td}(V_{KM})_{td}^*}{(V_{KM})_{us}(V_{KM})_{ts}^*}\right) \quad (3)$$

It is clear that $\phi_2' = \phi_2$.

Since all the six $UT$s have the same area $J/2$, not all the different 9 angles are independent. For example $J = |(V_{KM})_{td}(V_{KM})_{tb}^*||(V_{KM})_{ud}(V_{KM})_{ub}^*| \sin \phi_2 = |(V_{KM})_{td}(V_{KM})_{tb}^*||(V_{KM})_{cd}(V_{KM})_{cb}^*| \sin \phi_1 = |(V_{KM})_{us}(V_{KM})_{ts}^*||(V_{KM})_{ub}(V_{KM})_{tb}^*| \sin \phi_1' = |(V_{KM})_{ud}(V_{KM})_{td}^*||(V_{KM})_{us}(V_{KM})_{ts}^*| \sin \phi_3'$. It can be shown that only 4 independent parameters are needed to parameterize the six $UT$s, and two different $UT$s contain the needed 4 parameters.

The values for the angles in $UT(a)$, of Eq. (1), derived from various experiments given by PDG are\cite{2}: $\phi_1 = (21.46 \pm 0.98)^\circ$ (derived from data on $\sin(2\phi_1) = 0.681 \pm 0.025$), and
the values for $\phi_2$ and $\phi_3$ are $(88^{+6}_{-5})^\circ$ and $(77^{+30}_{-32})^\circ$, respectively. These values are consistent with the unitarity of the KM matrix within error bars, and therefore also with a choice of presentation which we now formulate in terms of a novel combination of two different unitarity triangles (a) and (b). $UT(a)$, defined by Eq. (1), is almost a right triangle, by virtue of $\phi_2$. Numerically, the angles $\phi_1'$ and $\phi_3'$ are close to $\phi_1$ and $\phi_2$, respectively. All the angles in the two $UTs$ are sizable, making experimental determination of them merely challenging, while for the other four choices of $UT$ there is always, at least, one small angle where measurement may be exceptionally difficult. It is therefore easiest to work with the two $UTs$, $UT(a)$ and $UT(b)$, for practical purposes. We now show that, by combining information from these two $UTs$, into the boomerang diagram \(^1\) displayed in Fig. 1, all information needed to specify the KM matrix, $V_{KM}$, can be extracted.

\[\text{FIG. 1: The unitarity boomerang. The sides are: } AC = |(V_{KM})_{ud}(V_{KM})_{ub}^*|, \quad AC' = |(V_{KM})_{ub}(V_{KM})_{tb}^*|, \quad AB = |(V_{KM})_{td}(V_{KM})_{tb}^*|, \quad AB' = |(V_{KM})_{ud}(V_{KM})_{td}^*|, \quad BC = |(V_{KM})_{cd}(V_{KM})_{cb}^*| \quad \text{and } B'C' = |(V_{KM})_{us}(V_{KM})_{ts}^*|.\]

The unitarity boomerang is formed by locating the common angle $\phi_2' = \phi_2$ from the two $UTs$ of $UT(a)$ and $UT(b)$ at the top point $A$ and the shortest sides, $AC = |(V_{KM})_{ud}(V_{KM})_{ub}^*|$ and $AC' = |(V_{KM})_{ub}(V_{KM})_{tb}^*|$, on the opposite sides. The other sides are: $AB = |(V_{KM})_{td}(V_{KM})_{tb}^*|$, $AB' = |(V_{KM})_{ud}(V_{KM})_{td}^*|$, $BC = |(V_{KM})_{cd}(V_{KM})_{cb}^*|$ and $B'C' = |(V_{KM})_{us}(V_{KM})_{ts}^*|$. We emphasize that Fig. \(^1\) is drawn with the central experimental values of

\(^1\) The name arises from resemblance to the hunting instrument.
\[ AC = 3.50 \times 10^{-3}, \quad AC' = 3.59 \times 10^{-3}, \quad AB = 8.73 \times 10^{-3}, \quad AB' = 8.51 \times 10^{-3}, \quad BC = 9.36 \times 10^{-3} \text{ and } B'C' = 9.19 \times 10^{-3}. \]

One can choose the area \((J/2)\) of the triangles, two inner angles from one of the UTs (for example \(\phi_1\) and \(\phi_2\)), and a third angle from the other UT (for example \(\phi_3'\)) as the four independent parameters.

**Original KM parametrization and Unitarity Boomerang**

To show explicitly how the unitarity boomerang can provide all information needed to specify the quark flavor mixing, we work with a specific parametrization, \(V_{KM}\), originally given by Kobayashi and Maskawa\textsuperscript{1}.

\[
V_{KM} = \begin{pmatrix}
c_1 & -s_1c_3 & -s_1s_3 \\
 s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\
 s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta}
\end{pmatrix}. \tag{4}
\]

One can also work with other parameterizations, such as that adopted by the PDG. But we find an interesting feature of the original KM parametrization which turns out to be very convenient for the discussions of the unitarity boomerang.

Using experimental values\textsuperscript{2} for \((V_{KM})_{us} = 0.2257\pm 0.0010, (V_{KM})_{ub} = 0.00359\pm 0.00016\) and \((V_{KM})_{td} = 0.00874^{+0.00026}_{-0.00037}\), one finds that \(s_2s_3 << 1\). At a few percent level, one has \((V_{KM})_{ub} = (c_1s_2s_3 - c_2c_3e^{-i\delta}) \approx -c_2c_3e^{-i\delta}\).

Then

\[
\phi_2 = \arg\left(-\frac{s_1s_2* (c_1s_2s_3 - c_2c_3e^{-i\delta})}{c_1* (-s_1s_3)}\right) \\
\approx \arg\left(\frac{s_1s_2* (-c_2c_3e^{-i\delta})}{c_1* s_1s_3}\right) = \pi - \delta. \tag{5}
\]

The CP violating phase \(\delta\), in this parametrization, is equal to \(\pi - \phi_2\), to a good approximation\textsuperscript{9}.

The fact that \(\phi_2 = (88^{+6}_{-5})^\circ\) implies \(\delta \approx 90^\circ\). The approximate right angle at the top of the boomerang diagram may indicate that CP, from a deeper perspective, is maximally
violated [10, 11]. Kobayashi and Maskawa, with remarkable prescience, made an excellent choice of parametrization. We suggest that the original parametrization of Kobayashi-Maskawa matrix be used as the standard parametrization. A parametrization suggested by Fritzsch and Xing [10], which also has its phase close to $\phi_2$, is another alternative interesting parametrization. From the unitarity boomerang, one can easily obtain approximation solutions for the four physical parameters. One first notices that the relation in Eq. (5) allows one to read off the $\delta$ from the top angle in the diagram. Taking the ratio, of the two sides $AC/AC'$ or $AB/AB'$, one obtains $|(V_{KM})_{ud}/(V_{KM})_{ub}^*| \approx c_1$ since $|(V_{KM})_{ub}|$ is very close to 1. With $c_1$ and therefore $s_1$ known, the length of the sides $AB$ and $AC'$ then provide the values for $s_2$ and $s_3$.

One can obtain more precise solutions by using the following information from four sides, $AC = a$, $BC = b$, $AB = c$ and $AB' = d$ of the unitarity boomerang:

$$a = |(V_{KM})_{ud}(V_{KM})_{ub}^*| = c_1 s_1 s_3, \quad b = |(V_{KM})_{cd}(V_{KM})_{cb}^*| = s_1 c_2 |c_1 c_2 s_3 + s_2 c_3 e^{-i\delta}|,$$

$$c = |(V_{KM})_{ld}(V_{KM})_{lb}^*| = s_1 s_2 |c_1 s_2 s_3 - c_2 c_3 e^{-i\delta}|, \quad d = |(V_{KM})_{ud}(V_{KM})_{ld}^*| = c_1 s_1 s_2. \quad (6)$$

Using the above, one can express $s_{1,2,3}$ and $\delta$ as functions of $a$, $b$, $c$ and $d$. The KM parameters can be determined. For example

$$a^2 - c_1^2 + c_1^4 \left(\frac{c^2}{d^2} - \frac{b^2}{c_1^4 - c_1^2 d^2}\right) = 0. \quad (7)$$

Solving for the roots of the above equations, the $c_1^2$ is determined up to four possible discrete solutions. Restricting to real positive solutions with magnitude less than 1, one can further limit the choices.

The other angles, and the phase, can be determined from the following relations

$$s_2 = \frac{d}{c_1 s_1}, \quad s_3 = \frac{a}{c_1 s_2},$$

$$\cos \delta = \frac{b^2/s_1 c_2^2 - (c_1^2 c_2^2 s_3^2 + s_2^2 c_3^2)}{2 c_1 c_2 s_2 c_3 s_3} = \frac{c_1^2 s_2^2 s_3^2 + c_2^2 c_3^2 - c^2/s_1^2 s_2^2}{2 c_1 c_2 s_2 c_3 s_3}. \quad (8)$$

After applying the constraint on $c_2^2$, that they satisfy $0 \leq c_{2,3}^2 \leq 1$, the solution is even more restricted. Putting in numerical values, for the sides, and comparing with the approximate solution above, we find that a unique solution survives.
Numerically, with the current central values for \( a, b, c \) and \( d \), we obtain

\[
c_1 = 0.97419, \quad s_2 = 0.0387, \quad s_3 = 0.0162, \quad \delta = 88.83^\circ.
\] (9)

and these numbers are self consistent.

One should be aware, that there remain errors, on the sides and angles of the boomerang. This leads to distortion of the \( UB \) away from the true one. When constructing the \( UB \), one can first use measurable quantities without assuming unitarity to form one of the \( UT \), say, the \( UT \) defined by triangle \( ABC \) in Fig. 1. This can be achieved by using the measured \( \alpha \) and \( \beta \) and also the length of side \( AB \), \( c = |(V_{KM})_{td}(V_{KM})_{tb}| \). The major error comes from the uncertainty in \( |(V_{KM})_{td}(V_{KM})_{tb}| \) measured from \( B_b - \bar{B}_d \) mixing. Assuming \( |(V_{KM})_{tb}| \) is almost one, then \[ 2 \], \( |(V_{KM})_{td}| = (8.09 \pm 0.6) \times 10^{-3} \). One then uses information on the values of \( |(V_{KM})_{ud}| \) and \( |(V_{KM})_{ub}| \) to construct the sides \( AB' \) and \( AC' \) to complete the boomerang. The error in \( |(V_{KM})_{td}| \) will cause uncertainty in the side \( AB' \) of the \( UB \) with \( d = (7.88 \pm 0.58) \times 10^{-3} \). At present within error bars, one cannot be sure which side, \( AB \) or \( AB' \), is longer. Further reduce the errors in \( |(V_{KM})_{td}(V_{KM})_{tb}| \) can be achieved by better understanding of the bag factor in \( B_d - \bar{B}_d \) mixing \[ 2 \]. Another way to improve the situation is to note that the value \( |(V_{KM})_{tb}|/|(V_{KM})_{ud}| \) plays an important role which also determine the ratio of \( AC \) and \( AC' \). Therefore precise measurement of \( |(V_{KM})_{tb}| \) is crucial in constructing an accurate \( UB \). Future studies of top quark decay and single top quark production at colliders, such as the LHC, will provide useful information.

To give a quantitative feeling, we have carried out an estimate assuming that the errors in \( a, b, c \) and \( d \) are given by the current PDG data with Gaussian errors to obtain the resultant errors in the KM angles. We obtain \( \Delta c_1 = 0.046 \), an error which is reasonably small. But errors on \( s_{2,3} \) are large with \( \Delta s_2 = 0.032 \) and \( \Delta s_3 = 0.077 \). Such a larger error bolsters preference for the boomerang, to disentangle, most perspicuously, the quark flavor mixing. Note that errors, on \( s_{2,3} \), are due to empirically-generated uncertainties on \( (V_{KM})_{td}, (V_{KM})_{tb} \) and \( (V_{KM})_{ub} \).

Indeed, when we look more closely at Eq. (7), it does turn out that the quantity \( c \) enters that equation, only in a combination \( (c^2/d^2) \), just so that \( (V_{KM})_{td} \) cancels out. If one takes into account, the errors are reduced to \( \Delta c_1 = 0.032, \Delta s_2 = 0.023 \) and \( \Delta s_3 = 0.055 \).
If uncertainties on all four sides can be reduced, say by another factor of three, we project that errors can be reduced to $\Delta c_1 = 0.011$, $\Delta s_2 = 0.076$ and $\Delta s_3 = 0.018$, thus illustrating how the chosen boomerang may, in the foreseeable future, return to increase human knowledge. Our proposal, to move from a single triangle to a boomerang combination, therefore reflects, more than anything else, the increase in precision which is justifiably anticipated from the high-energy experiments.

**Discussion**

The most popular way to present the flavor mixing for three generations of quarks is by a unitarity triangle which, however, explicitly displays only three of four independent parameters. To have a diagrammatical representation for the full four independent parameters, we have proposed improvement to the unitarity boomerang.

By studying the unitarity boomerang, one can obtain all the information enshrined in KM matrix. We find that the original parametrization by Kobayashi and Maskawa is particularly convenient for this purpose. The angle $\phi_2$ in the boomerang diagram, to a good approximation, can be identified with the phase $\delta$ in the original KM parametrization \[1\]. The fact that $\phi_2 = (88^{+6}_{-5})^\circ$ implies $\delta \approx 90^\circ$, so that this parametrization may be the right one to study assiduously, in order to probe further the connection to the origin of, possibly maximal, CP violation. We, therefore, humbly submit that the original parametrization of KM matrix be kept as the standard, and that the unitarity boomerang shown in FIG.1 be used unambiguously to present the experimental information.

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