HOMOTOPICAL ALGEBRA FOR $C^*$-ALGEBRAS

OTGONBAYAR UUYE

Abstract. Category of fibrant objects is a convenient framework to do homotopy theory, introduced and developed by Ken Brown. In this paper, we apply it to the category of $C^*$-algebras. In particular, we get a unified treatment of (ordinary) homotopy theory for $C^*$-algebras, $KK$-theory and $E$-theory, as all of these can be expressed as the homotopy theory of a category of fibrant objects.

Contents

0. Introduction 1
1. Abstract Homotopy Theory 3
   1.1. Categories of Fibrant Objects 3
   1.2. Fibre and Homotopy Fibre 5
   1.3. Homotopy Category 6
   1.4. Homology Theories and Localizations 9
   1.5. Example of $\pi_0$-Top 10
2. Applications to the Category of $C^*$-algebras 12
   2.1. Ordinary Homotopy Theory 14
   2.2. $C^*$-Invariant Homotopy Theory 16
   2.3. Topological $K$-Theory 16
   2.4. $KK$-Theory 18
   2.5. Universal Homology Theories 19
Appendix A. No Quillen Model Structure (following Andersen-Grodal) 21
References 23

0. Introduction

Basic homotopy theory for $C^*$-algebras can be developed in an analogous way to the homotopy theory for topological spaces, using the Gelfand-Naimark duality between pointed compact Hausdorff spaces and abelian $C^*$-algebras. This is carried out, for example, by Rosenberg in [Ros82] and

Date: March 23, 2012.
2010 Mathematics Subject Classification. Primary (46L85); Secondary (55U35).
Key words and phrases. category of fibrant objects, abstract homotopy theory, $C^*$-algebras, $KK$-theory, $K$-theory.
Schochet in \[\text{Sch84}\]. Thus, for instance, we have a version of the Puppe exact sequence, with essentially the same proof (cf. \[\text{Sch84}, \text{Proposition 2.6}\]).

There is one big difference: the homotopy theory for $C^*$-algebras does not admit a Quillen model category structure, as first pointed out by Andersen-Grodal (see Appendix A). This is unfortunate, since model categories provide a standard and powerful framework to study various aspects of homotopy theories. However, it turns out that not everything is lost: the category of $C^*$-algebras behave as if it was the subcategory of the fibrant objects in a model category, and this is enough for many purposes, because many proofs in model category theory start by reducing to the case of (co)fibrant objects.

The notion of a “category of fibrant objects” is abstracted and developed by Brown in \[\text{Bro74}\]. In this paper, we apply Brown’s theory to the category of $C^*$-algebras. In Section \[1\] we review some basic facts about abstract homotopy theory in the setting of category of fibrant objects.

In Section \[2\] we first apply the abstract theory of Section \[1\] to the ordinary homotopy theory for $C^*$-algebras (this essentially recovers \[\text{Sch84}\]). We also show that the Meyer-Nest’s UCT category (cf. \[\text{MN06}\]), Kasparov’s $KK$-theory (cf. \[\text{Kas80, Kas88}\]), and Connes-Higson’s $E$-theory (cf. \[\text{Hig90, CH90}\]) can be described as the homotopy category of a category of fibrant objects. As a corollary, we get a unified treatment of the triangulated structures on these categories.

In addition to ordinary homotopy theory, we also have shape theories for (separable) $C^*$-algebras (cf. \[\text{EK86, Bla85}\]). In \[\text{Dăd94}\], Dădărlat constructed the strong shape category and showed that it is equivalent to the asymptotic homotopy category of separable $C^*$-algebras of Connes-Higson (cf. \[\text{CH90}\]).

Unfortunately and unlike the commutative case (cf. \[\text{Cat81, CH81}\]), we do not (yet) have a category of fibrant objects whose homotopy category describes the strong shape category. However, as we show in subsection \[2.5\] the suspension-stable version considered by Thom (cf. \[\text{Tho03}\]) does arise as the stable homotopy category of a category of fibrant objects. We also show that Thom’s connective $K$-theory category fits well in this framework (cf. loc.cit.).

Needless to say, Brown’s theory of category of fibrant objects is not the only way to approach the homotopy theory for $C^*$-algebras. The main “reason” for the failure for the existence of a model structure on the category of $C^*$-algebras is that the category is too small, so an alternative approach would be to enlarge the category of $C^*$-algebras. Joachim and Johnson produced a model category structure for $KK$-theory by enlarging the category of $C^*$-algebras to a suitable category of topological algebras (cf. \[\text{JJ06}\]). Østvær developed a homotopy theory by enlarging the category of $C^*$-algebras to the category of $C^*$-spaces (cf. \[\text{Ost10}\]). Cuntz described an alternative construction of bivariant $K$-theories in \[\text{Cun98}\].

We also note that Voigt computed the $K$-theory of free orthogonal quantum groups in \[\text{Voi11}\] using Meyer-Nest’s triangulated category approach to
the Baum-Connes conjecture (cf. \cite{MN06}). This seems to be the first concrete results in the theory of operator algebras, which can be proved only using abstract homotopy theoretic methods.

Applications of the framework developed in this paper will appear elsewhere.

Acknowledgments. This research was supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation at the University of Copenhagen. I thank the referee for many useful suggestions.

1. Abstract Homotopy Theory

For the convenience of the reader we recall some basic notions and results from abstract homotopy theory. See \cite{Qui67, Bro74, Hel68, KP97, GJ99} for details.

1.1. Categories of Fibrant Objects. The following is our main definition.

Definition 1.1 \((\text{Brown} \ \text{Bro74})\). Let \(C\) be category with terminal object \(\ast\) and let \(F \subseteq C\) and \(W \subseteq C\) be distinguished subcategories. We say that \(C\) is a category of fibrant objects if the following conditions (F0) - (FW2) hold.

(F0) The class \(F\) is closed under composition.
(F1) Isomorphisms of \(C\) are in \(F\).
(F2) The pullback in \(C\) of a morphism in \(F\) exists and is in \(F\).
(F3) For any object \(B\) of \(C\), the morphism \(B \to \ast\) is in \(F\).

Morphisms of \(F\) are called fibrations and denoted \(\rightarrow\).

(W1) Isomorphisms of \(C\) are in \(W\).
(W2) If two of \(f, g\) and \(gf\) are in \(W\), then so is the third.

Morphisms of \(W\) are called weak equivalences and denoted \(\sim\).

(FW1) The pullback in \(C\) of a morphism in \(W \cap F\) is in \(W \cap F\).

Morphisms of \(W \cap F\) are called trivial fibrations and denoted \(\sim\).

(FW2) For any object \(B\) of \(C\), the diagonal map \(B \to B \times B\) admits a factorization

\[
\begin{align*}
\begin{array}{c}
B \sim \rightarrow \\
B^I \rightarrow \\
B \times B,
\end{array}
\end{align*}
\]

where \(s \in W\) is a weak equivalence, \(d = (d_0, d_1) \in F\) is a fibration.

The object \(B^I\) or more precisely the tuple \((B^I, s, d_0, d_1)\) is called a path-object of \(B\).

If there is no risk for confusion, we simply say that \(C\) is a category of fibrant objects. If the terminal object is also an initial object, we say that \(C\) is a pointed category of fibrant objects.
Remark 1.2. (1) The condition (F0) is superfluous since $F$ is assumed to be a subcategory. But it is convenient to have a notation for this property.
(2) The conditions (F1) and (W1) imply that $F$ and $W$ contain all objects of $C$.
(3) The conditions (F2) and (F3) imply that $C$ is has finite products.

Remark 1.3. Dually there is a notion of a category of cofibrant objects.

The following is the motivating example.

Example 1.4. For any model category $M$, the full subcategory $M_f$ consisting of the fibrant objects in $M$ is naturally a category of fibrant objects, by restricting the weak equivalences and the fibrations to $M_f$.

In particular, if $\text{Top}$ denote the category of compactly generated weakly Hausdorff topological spaces and continuous maps, then

(1) $\text{Top}$, homotopy equivalences, Hurewicz fibrations;
(2) $\text{Top}$, weak homotopy equivalences, Serre fibrations;

are examples of categories of fibrant objects.

A more algebraic example is the following: let $R$ be a ring and let $\text{Ch}(R)$ denote the category of chain complexes of left $R$-modules and chain maps. Then

(3) $\text{Ch}(R)$, quasi-isomorphisms, degreewise epimorphisms

is a category of fibrant objects. In these three examples, all objects are fibrant i.e. $M_f = M$.

Definition 1.5. A functor between categories of fibrant objects is said to be exact if it preserves all the relevant structure: it sends the terminal object to the terminal object, fibrations to fibrations, weak equivalences to weak equivalences and pullbacks (of fibrations) to pullbacks.

Example 1.6. Let $C$ be a category of fibrant objects and let $A \subseteq C$ be a full reflective subcategory i.e. the inclusion $i: A \rightarrow C$ is a right-adjoint. Suppose that for any $B \in A$, a path-object $B^I$ can be chosen to lie in $A$. Then $A$ is a category of fibrant objects by restricting weak equivalences and fibrations, since limits in $A$ can be computed in $C$; and the inclusion $i: A \rightarrow C$ is exact.

Occasionally, we find it convenient to isolate the notions of weak equivalences and fibrations.

Definition 1.7. Let $C$ be a category. A subcategory of weak equivalences is a subcategory $W \subseteq C$ satisfying (W1) and (W2). If $C$ has a terminal object, a subcategory of fibrations is a subcategory $F \subseteq C$ satisfying (F0) - (F3).
1.2. **Fibre and Homotopy Fibre.**

**Lemma 1.8** (Factorization Lemma). Let \( f : A \to B \) be a morphism in a category of fibrant objects. Consider the diagram

\[
\begin{array}{ccc}
Nf & \xrightarrow{p} & B \\
\downarrow & & \downarrow \\
Nf & \xrightarrow{d_0(f)} & B^I \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\]  

(1.2)

where \((B^I, s, d_0, d_1)\) is a path-object for \( B \) and \( Nf \) is the pullback \( A \times_B B^I \) and \( p \) is the composition \( d_1 \circ d_0^*(f) \) and \( i \) is the map determined by the section \( s \).

Then \( p \) is a fibration and \( i \) is a right inverse to a trivial fibration (in particular, a weak equivalence) and \( f = p \circ i \).

*Proof.* [Bro74, Factorization Lemma].

**Definition 1.9.** We call \( Nf \) a mapping path-object of \( f \).

**Corollary 1.10.** Let \( C \) be a category of fibrant objects and let \( D \) be a category with weak equivalences. Let \( F : C \to D \) be a functor that sends trivial fibrations to weak equivalences. Then \( F \) send weak equivalences to weak equivalences.

*Proof.* [Bro74, Factorization Lemma].

Now we consider pointed categories.

**Definition 1.11.** Let \( p \) be a fibration in a pointed category of fibrant objects. The **fibre** \( F \) of \( f \) is the pullback

\[
\begin{array}{ccc}
F & \xrightarrow{i} & E \\
\downarrow & & \downarrow \\
* & \xrightarrow{p} & B
\end{array}
\]  

(1.3)

We express this situation by the diagram

\[
F \xrightarrow{i} E \xrightarrow{p} B 
\]  

(1.4)

**Definition 1.12.** Let \( f : A \to B \) be a morphism in a pointed category of fibrant objects. The **homotopy fibre** \( Ff \) of \( f \) is the fibre of \( Nf \xrightarrow{p} B \), where \( p \) is as in the Factorization Lemma (Lemma 1.8).

**Lemma 1.13.** Let \( p \) be a fibration in a pointed category of fibrant objects with fibre \( F \). Then the natural map

\[
F \to Fp
\]  

is a weak equivalence.
Proof. Apply [Bro74, Lemma 4.3] to

\[ \begin{array}{ccc}
F & \longrightarrow & E \\
\downarrow & & \downarrow p \\
Fp & \longrightarrow & Np \\
\end{array} \]

(1.6)

\[ \begin{array}{ccc}
\downarrow & & \downarrow \\
B & \longrightarrow & B \\
\end{array} \]

\[ \begin{array}{ccc}
\downarrow & & \downarrow \wr \downarrow \\
B & \longrightarrow & B \\
\end{array} \]

1.3. Homotopy Category.

Notation 1.14. If \( C \) is a category, we write \( \text{Ob} C \) for the objects of \( C \) and write \( \text{Mor}_C(A, B) \) for the space of morphisms from \( A \) to \( B \), for \( A, B \in C \).

Definition 1.15. The homotopy category of a category \( C \) of fibrant objects with weak equivalences \( W \) is the localization

\[ \text{Ho}(C) := C[W^{-1}] \].

(1.7)

In other words, there is given a functor \( \gamma : C \rightarrow \text{Ho}(C) \), called the localization functor, with the property that for any functor \( k : C \rightarrow D \) such that \( k(t) \) is invertible in \( D \) for all \( t \in W \), there exist a unique functor \( \text{Ho}(C) \rightarrow D \) making the diagram

\[ \begin{array}{ccc}
\text{Ho}(C) & \longrightarrow & D \\
\downarrow & & \downarrow \gamma \\
C & \longrightarrow & D \\
\end{array} \]

(1.8)

commute.

Often we write \([A, B]_C\) for \( \text{Mor}_{\text{Ho}(C)}(A, B) \). Note that there is no guarantee that \([A, B]_C\) is a small set (see Corollary 1.19).

Definition 1.16. Let \( C \) be a category of fibrant objects. Two morphisms \( f_0, f_1 : A \Rightarrow B \)

(1.9)

are said to be right-homotopic if for some path-object \((B^I, s, d_0, d_1)\) of \( B \), there is a morphism \( h : A \rightarrow B^I \) such that \( f_0 = d_0 h \) and \( f_1 = d_1 h \).

The two are said to be homotopic if there is a weak equivalence \( t : A' \rightarrow A \) such that \( f_0 t, f_1 t : A' \Rightarrow B \) are right-homotopic.

Right-homotopy and homotopy are equivalence relations, and moreover, homotopy is compatible with the composition in \( C \) (cf. [Bro74, Section 2]).

Definition 1.17. Let \( C \) be a category of fibrant objects. We denote the category of homotopy classes in \( C \) by \( \pi C \) and let \( \pi : C \rightarrow \pi C \) denote the quotient functor.

The following is the fundamental result of Brown.
Theorem 1.18 (Brown [Bro74, Theorem 2.1]). Let $\mathbb{C}$ be a category of fibrant objects. Then $\pi W \subseteq \pi \mathbb{C}$ admits a calculus of right fractions. It follows that, for $A, B \in \mathbb{C}$,

$$[A, B]_\mathbb{C} \cong \colim_{A' \to A} \text{Mor}_{\pi \mathbb{C}}(A', B)$$  \hspace{1cm} (1.10)

and hence if $\gamma: \mathbb{C} \to \text{Ho}(\mathbb{C})$ is the localization functor, then

(1) any morphism in $[A, B]_\mathbb{C}$ can be written as a right-fraction

$$A \xrightarrow{\gamma(t)^{-1}} A' \xrightarrow{\gamma(f)} B$$  \hspace{1cm} (1.11)

where $t \in W$ is a weak equivalence, and

(2) if $f_0, f_1$ are morphisms in $\text{Mor}_\mathbb{C}(A, B)$, then $\gamma(f_0) = \gamma(f_1)$ if and only if $f_0$ and $f_1$ are homotopic i.e. $\pi(f_0) = \pi(f_1)$.

□

Corollary 1.19. Let $\mathbb{C}$ be a category of fibrant objects and let $A$ be an object in $\mathbb{C}$. Suppose that the category $W_A$ of weak equivalences over $A$ is "coinitially small" i.e there exists a set $S_A$ of objects in $\mathbb{C}$ such that for any $A' \to A$, there is a $A'' \to A'$ such that $A'' \in S_A$, then $[A, B]_\mathbb{C}$ is a small set for every $B \in \mathbb{C}$.

□

Proof. See [GZ67, Proposition 2.4].

Now we consider pointed categories.

Definition 1.20. Let $B$ be an object of a pointed category of fibrant objects. A loop-object of $B$ is the fibre $\Omega B$ of $(d_0, d_1): B^I \to B \times B$, where $(B^I, s, d_0, d_1)$ is a path-object of $B$.

Lemma 1.21. Let $\mathbb{C}$ be a pointed category of fibrant objects. Then $\Omega$ defines a functor

$$\Omega: \text{Ho}(\mathbb{C}) \to \text{Ho}(\mathbb{C})$$  \hspace{1cm} (1.12)

called the loop-object functor.

(1) For any $B \in \mathbb{C}$, the object $\Omega B$ is naturally a group object in $\text{Ho}(\mathbb{C})$ and $\Omega^2 B$ is naturally an abelian group object in $\text{Ho}(\mathbb{C})$.

(2) For any fibration $p: E \to B$ with fibre $F$, there is a natural right-action $F \times \Omega B \to F$ in $\text{Ho}(\mathbb{C})$. In particular, we have a natural map $\Omega B \to F$ in $\text{Ho}(\mathbb{C})$.

Proof. See [Bro74, Section 4].

Note that while $\text{Ho}(\mathbb{C})$ depends only on the weak equivalences, the loop-object functor $\Omega$ depends also on the fibrations.

Definition 1.22. Let $\mathbb{C}$ be a pointed category of fibrant objects. We define the stable homotopy category of $\mathbb{C}$ as the category

$$\text{SHo}(\mathbb{C}) := \text{Ho}(\mathbb{C})[\Omega^{-1}]$$  \hspace{1cm} (1.13)

obtained from $\text{Ho}(\mathbb{C})$ by inverting the endofunctor $\Omega$. 
Objects of $\text{SHo}(C)$ are $(A, n)$ with $A \in \text{Ho}(C)$ and $n \in \mathbb{Z}$ and the morphisms are given by

$$\text{Mor}_{\text{SHo}(C)}((A, n), (B, m)) := \text{colim}_{k \to \infty} [\Omega^{n+k} A, \Omega^{n+k} B]_C.$$  \hfill (1.14)

For $n \in \mathbb{Z}$, we have natural functors, also denoted $\Omega^n$,

$$\Omega^n : \text{Ho}(C) \to \text{SHo}(C), \quad A \mapsto (A, n),$$  \hfill (1.15)

which sends morphisms in $\text{Mor}_{\text{Ho}(C)} A, B$ to the corresponding element in $\text{Mor}_{\text{SHo}(C)}((A, n), (B, n))$.

**Theorem 1.23.** Let $C$ be a pointed category of fibrant objects. Then the stable homotopy category $\text{SHo}(C)$ is a triangulated category with the shift

$$\Sigma = \Omega^{-1} : \text{SHo}(C) \to \text{SHo}(C)$$  \hfill (1.16)

given by $(A, n) \mapsto (A, n-1)$ and the distinguished triangles given by triangles isomorphic to triangles of the form

$$\xymatrix{ (\Omega B, n) \ar[r] & (F, n) \ar[r] & (E, n) \ar[r] & (B, n),}$$  \hfill (1.17)

where $n \in \mathbb{Z}$ and $E \to B$ is a fibration, $F \to E$ is the fibre inclusion and $\Omega B \to F$ is the morphism obtained from Lemma 1.21.

**Proof.** See [Hel68] or [Hov99, May01]. \hfill $\square$

**Remark 1.24.** We note that for any $f \in [A, B]_C$ and $n \in \mathbb{Z}$, we have a natural distinguished triangle

$$\xymatrix{ (\Omega B, n) \ar[r] & (F f, n) \ar[r] & (A, n) \ar[r] & (B, n).}$$  \hfill (1.18)

**Definition 1.25.** We say that a pointed category of fibrant objects $C$ is stable, if the loop functor $\Omega : \text{Ho}(C) \to \text{Ho}(C)$ is invertible.

**Remark 1.26.** If $C$ is a stable pointed category of fibrant objects, then

$$\Omega^0 : \text{Ho}(C) \to \text{SHo}(C)$$  \hfill (1.19)

is an equivalence of categories. In particular, $\text{Ho}(C)$ is naturally a triangulated category with shift $\Sigma = \Omega^{-1} : \text{Ho}(C) \to \text{Ho}(C)$.

**Example 1.27.** Let $M$ be a pointed Quillen model category and let $M_f$ be the full subcategory of fibrant objects in $M$, considered a category of fibrant objects as in Example 1.4. Then the inclusion $M_f \to M$ induces an equivalence $\text{Ho}(M_f) \cong \text{Ho}(M)$, with compatible loop-objects and fibration sequences. Compare [Bro74] and [Qui67].
1.4. Homology Theories and Localizations.

**Definition 1.28.** A homology theory on a pointed category of fibrant objects $C$ is a homology theory on $\text{SHo}(C)$ i.e. an exact functor $\mathcal{H} : \text{SHo}(C) \to \text{Ab}$.

**Definition 1.29.** Let $C$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $C$.

A morphism $t : A \to B$ is said to be an $\mathcal{H}$-equivalence if the induced maps

$$(\Omega^n t)_* : \mathcal{H}(A, n) \to \mathcal{H}(B, n)$$

are isomorphisms for all $n \in \mathbb{Z}$.

An object $F \in C$ is said to be $\mathcal{H}$-acyclic if $\mathcal{H}(F, n) = 0$ for all $n \in \mathbb{Z}$.

Note that since homology theories are homotopy invariant by definition, weak equivalences in $C$ are $\mathcal{H}$-equivalences.

**Lemma 1.30.** Let $C$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $C$. Then a morphism $t$ in $C$ is an $\mathcal{H}$-equivalence if and only if its homotopy fibre $F_t$ is $\mathcal{H}$-acyclic.

**Proof.** Clear from the long-exact sequence associated to the distinguished triangle of Remark 1.24. $\square$

**Corollary 1.31.** Let $C$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $C$. Then a fibration $p \in C$ with fibre $F$ is an $\mathcal{H}$-equivalence if and only if $F$ is $\mathcal{H}$-acyclic.

**Proof.** By Lemma 1.13 the natural map $F \to Fp$ is a weak equivalence, hence an $\mathcal{H}$-equivalence. The proof is complete by Lemma 1.30. $\square$

**Theorem 1.32.** Let $C$ be a pointed category of fibrant objects and let $\mathcal{H}$ be a homology theory on $C$. Then $\mathcal{H}$-equivalences and fibrations define a pointed category of fibrant objects on $C$, denoted $R_\mathcal{H}C$, with the same path and loop objects as in $C$.

**Proof.** It is clear that $\mathcal{H}$-equivalences form a subcategory of weak equivalences. Hence we need to show the compatibility conditions (FW1) and (FW2) are satisfied.

(FW1) Let $p : E \to B$ be a fibration which is also an $\mathcal{H}$-equivalence. We need to show that for any $f : A \to B$, the pullback $f^*(p)$ is again an $\mathcal{H}$-equivalence. But this is immediate from Corollary 1.31 applied to the diagram:

$$
\begin{array}{ccc}
F & \to & E \times_B A \\
\downarrow & & \downarrow \mathstrut_{f^*(p)} \\
F & \to & A
\end{array}
$$

where $F$ is the fibre of $p$.

(FW2) Since weak equivalences are $\mathcal{H}$-equivalences, path-objects in $C$ also give path-objects in the new category of fibrant objects $R_\mathcal{H}C$. $\square$
**Definition 1.33.** Let $C$ be a pointed category of fibrant objects and let $S \subseteq C$ be a class of morphisms. We say that a morphism $t \in \text{Mor}_C(A, B)$ is a $S^{-1}$-weak equivalence if for any homology theory $\mathcal{H} : \text{SHo}(C) \to \text{Ab}$ such that every $s \in S$ is an $\mathcal{H}$-equivalence, $t$ is an $\mathcal{H}$-equivalence.

**Theorem 1.34.** Let $C$ be a pointed category of fibrant objects and let $S \subseteq C$ be a class of morphisms. Then $S^{-1}$-weak equivalences and fibrations define a pointed category of fibrant objects, denoted $R_S C$. The stable homotopy category $\text{SHo}(R_S C)$ is naturally equivalent to the Verdier localization $\text{SHo}(C)[(\Omega^0 S)^{-1}]$ as a triangulated category.

**Proof.** Considering all homology theories $\mathcal{H} : \text{SHo}(C) \to \text{Ab}$ in which every $t \in S$ is an $\mathcal{H}$-equivalence in Theorem 1.32, we see that $R_S C$ is indeed a category of fibrant objects.

Now consider the natural triangulated functor $Q : \text{SHo}(C) \to \text{SHo}(R_S C)$ induced by $C \to R_S C$. Since any $s \in S$ is a $S^{-1}$-weak equivalence, we see that $Q(\Omega^0 s)$ is invertible in $\text{SHo}(R_S C)$.

We show that $Q$ is the universal triangulated functor that invert $\Omega^0 S \subseteq \text{SHo}(C)$. Indeed, let $R : \text{SHo}(C) \to (P, \Omega^{-1})$ be a triangulated functor such that morphisms in $R(\Omega^0 S) \subseteq \text{Morph}(R(A, 0), R(B, 0))$ are all invertible.

Let $t \in \text{Morph}(A, B)$ be a $S^{-1}$-weak equivalence. Then for any $D \in P$, $\mathcal{H} : \text{SHo}(C) \to \text{Ab}$, $(A, n) \mapsto \text{Morph}(D, R(A, n))$ is a homology theory by [Tho03, Theorem 2.3.8] and every $s \in S$ is an $\mathcal{H}$-equivalence, hence we see that $t$ too is an $\mathcal{H}$-equivalence. By Yoneda’s lemma, $R(\Omega^0 t)$ is invertible in $P$. Thus $R$ induces a functor $R_* : \text{Ho}(R_S C) \to P$ which is easily seen to intertwine the $\Omega$’s, hence induces a functor $\tilde{R} : \text{SHo}(R_S C) \to P$. Since $R$ is a triangulated homology theory, $\tilde{R}$ is a triangulated functor and $R = \tilde{R} \circ Q$. The uniqueness of $\tilde{R}$ is clear.

In other words, $\text{SHo}(R_S C)$ is the universal triangulated homology theory for which all morphisms of $S$ are equivalences (cf. [Tho03, Definition 2.3.3]).

**1.5. Example of $\pi_0$-Top.** Now we construct a simple, but very useful, example of a category of fibrant objects. Let $\text{Top}$ denote the category of compactly generated weakly Hausdorff topological spaces and continuous maps.

**Definition 1.35.** A map $p : E \to B$ is called a $\pi_0$-fibration if it satisfies the following path-lifting property:

\[
\begin{array}{ccc}
\{0\} & \longrightarrow & E \\
\downarrow & \nearrow & \downarrow p \\
[0, 1] & \longrightarrow & B
\end{array}
\]  \hspace{1cm} (1.22)

A map $t : A \to B$ is called a $\pi_0$-equivalence if

\[t_* : \pi_0(A) \to \pi_0(B)\] \hspace{1cm} (1.23)
is a bijection.

A $\pi_0$-trivial fibration is a $\pi_0$-fibration which is also a $\pi_0$-equivalence.

**Lemma 1.36.** All $\pi_0$-trivial fibrations are surjective.

**Proof.** Let $p : E \to B$ be a $\pi_0$-trivial fibration let $b \in B$. Then there is a diagram

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & E \\
\downarrow & & \downarrow p \\
[0, 1] & \underset{h}{\longrightarrow} & B
\end{array}
$$

with $h(1) = b$. Lifting $h$ to a path in $E$, and evaluating at 1, we get $e \in E$ such that $p(e) = b$. Hence $p$ is surjective. \hfill \Box

**Proposition 1.37.** The $\pi_0$-fibrations and $\pi_0$-equivalences give the structure of a category of fibrant objects on $\text{Top}$.

**Proof.** The $\pi_0$-fibrations form a subcategory of fibrations essentially because the path-lifting property is a right-lifting property:

(F0) If $E \to D$ and $D \to B$ are $\pi_0$-fibrations, so is their composition:

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & E \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & D \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & B
\end{array}
$$

(F1) If $A \to B$ is a homeomorphisms, then it is a $\pi_0$-fibration:

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & A \\
\downarrow & & \downarrow \cong \\
\{0\} & \longrightarrow & B
\end{array}
$$

(F2) If $E \to B$ is a $\pi_0$-fibration, then for any $A \to B$, the map $A \times_B E \to A$ is a $\pi_0$-fibration:

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & A \times_B E \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & A \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & B
\end{array}
$$
(F3) For any $B$, the map $B \to \ast$ is a $\pi_0$-fibration:

$$
\begin{array}{ccc}
\{0\} & \to & B \\
\downarrow & & \downarrow \\
[0,1] & \to & \ast
\end{array}
$$

(1.27)

The properties (W1) and (W2) are obvious.

(FW1) Let $p: E \sim \to B$ be a $\pi_0$-trivial fibration and let $f: A \to B$ be an arbitrary map. Consider the pullback

$$
\begin{array}{ccc}
A \times_B E & \to & E \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array}
$$

(1.28)

Then we need to show that $f^*(p)$ is a $\pi_0$-equivalence. The injectivity of $\pi_0(f^*(p))$ follows from the fact that it is detected by the right-lifting property with respect to $\{0,1\} \hookrightarrow [0,1]$. The surjectivity of $\pi_0(f^*(p))$ follows from Lemma 1.36 since the pullback of a surjection is again a surjection and surjections are surjective on $\pi_0$.

(FW2) Let $[a,b]$ be a compact interval, $a < b$, and let

$$B_{[a,b]} := \text{Mor}_{\text{Top}}([a,b], B)$$

(1.29)

denote the space of continuous maps $[a,b] \to B$. Then the constant-path map $s: B \to B_{[a,b]}$ is a $\pi_0$-equivalence (in fact, a homotopy equivalence).

Let $e_c: B_{[a,b]} \to B$ denote the evaluation at $c \in [a,b]$. Then the map $(e_a,e_b): B_{[a,b]} \to B \times B$ is a $\pi_0$-fibration, since the rectangle $[0,1] \times [a,b]$ retracts to the union of its three sides ⊑.

Thus $(B_{[a,b]}, s, e_a, e_b)$ is a path-object for $B$. For fixed $a$ and $b$, this is functorial.

□

2. Applications to the Category of $C^*$-algebras

Let $C^*$ denote the category of $C^*$-algebras and $\ast$-homomorphisms. It is complete and cocomplete and pointed – the zero object is the zero algebra $0$ – symmetric monoidal category with respect to the maximal tensor product. We refer to [Mey08] for the details.

The category $C^*$ is naturally enriched over $\text{Top}$, the Cartesian closed category of compactly generated weakly Hausdorff topological spaces. Indeed, since $C^*$-algebras are normed, they are compactly generated and weakly Hausdorff as spaces, hence there is a forgetful functor $C^* \to \text{Top}$. For $C^*$-algebras $A$ and $B$, we give $\text{Mor}_{C^*}(A,B)$ the subspace topology from $\text{Mor}_{\text{Top}}(A,B)$ via the forgetful functor. It is easy to see that $\text{Mor}_{C^*}(A,B)$
is a closed subspace of $\text{Mor}_{\text{Top}}(A, B)$, hence itself a compactly generated weakly Hausdorff space.

Let $A^* \subset C^*$ denote the full subcategory of abelian $C^*$-algebras. By the Gelfand-Naimark duality, $A^*$ is equivalent to the opposite category of the category $\text{CH}_*$ of pointed, compact Hausdorff topological spaces and pointed continuous maps. If $X$ is a compact Hausdorff space, we write $C(X)$ for the (unital) $C^*$-algebra of continuous functions on $X$. If in addition $X$ has a base point, we write $C_0(X)$ for the $C^*$-algebra of continuous functions on $X$ vanishing at the base point.

If we enrich $\text{CH}_*$ over $\text{Top}$ by the inclusion $\text{CH}_* \subset \text{Top}$, the Gelfand-Naimark duality becomes an equivalence of enriched categories.

**Remark 2.1.** The category $C^*$ of $C^*$-algebras is also enriched over the category of Hausdorff spaces, using the compact-open topology on morphism spaces. However, in order to facilitate the connection to algebraic topology, we use the compactly generated compact-open topology. Note that if $A$ is separable, then the compact-open topology on $\text{Mor}_{C^*}(A, B)$ is metrizable, hence compactly generated.

**Lemma 2.2.** Let $B$ be a $C^*$-algebra and let $X$ be a compact Hausdorff space. Then $\text{Mor}_{\text{Top}}(X, B)$ is naturally a $C^*$-algebra isomorphic to $C(X) \otimes B$.

**Proof.** By [Str, Proposition 2.13] the topology on $\text{Mor}_{\text{Top}}(X, B)$ coincides with the topology given by the norm $||f|| := \sup_{x \in X} ||f(x)||_B$. The rest is standard (cf. [WO93, Corollary T.6.17]). □

The following is the main property of the enrichment that we use. See also [JJ06, Proposition 3.4] and [Mey08, Proposition 24].

**Lemma 2.3.** Let $A$ and $B$ be $C^*$-algebras and let $X$ be a compact Hausdorff space. Then there is an identification

$$\text{Mor}_{\text{Top}}(X, \text{Mor}_{C^*}(A, B)) \cong \text{Mor}_{C^*}(A, C(X) \otimes B)$$

natural in $A$, $B$ and $X$.

**Proof.** Since $A$ and $B$ are compactly generated weakly Hausdorff spaces, we have a natural identification

$$\text{Mor}_{\text{Top}}(X, \text{Mor}_{\text{Top}}(A, B)) \cong \text{Mor}_{\text{Top}}(A, \text{Mor}_{\text{Top}}(X, B)),$$

by [Str, Proposition 2.12]. Hence by Lemma 2.2

$$\text{Mor}_{\text{Top}}(X, \text{Mor}_{\text{Top}}(A, B)) \cong \text{Mor}_{\text{Top}}(A, C(X) \otimes B).$$

Now it is easy to check that this restricts to the identification in (2.1).

□

Often we will make this identification implicitly.

**Remark 2.4.** Note that there are pointed analogues of Lemma 2.2 and Lemma 2.3.
Corollary 2.5. For any $D \in C^*$, the functor $\text{Mor}_{C^*}(D, -): C^* \to \text{Top}$ preserves pullbacks.

Proof. Let $D$ be fixed and let $F := \text{Mor}_{C^*}(D, -)$.

Consider a pullback diagram
\begin{equation}
\begin{array}{ccc}
A \times_B E & \longrightarrow & E \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\end{equation}
in $C^*$. We need to prove that the natural map
\begin{equation}
\Phi: F(A \times_B E) \to F(A) \times_{F(B)} F(E)
\end{equation}
is a homeomorphism. It is clear that $\Phi$ is a continuous bijection. Hence it suffices to prove that for any $X$ compact Hausdorff, a map $X \to F(A \times_B E)$ is continuous if the compositions $X \to F(A)$ and $X \to F(E)$ are continuous. However, this follows from Lemma 2.3 and its proof. □

2.1. Ordinary Homotopy Theory. The (ordinary) homotopy category of $C^*$-algebras is the category of $C^*$-algebras and homotopy classes of $*$-homomorphisms, denoted $\pi_0 C^*$ for the time being:
\begin{equation}
\text{Mor}_{\pi_0 C^*}(A, B) := \pi_0(\text{Mor}_{C^*}(A, B)).
\end{equation}

We now give $C^*$ the structure of a category of fibrant objects, whose homotopy category is $\pi_0 C^*$.

Definition 2.6. A $*$-homomorphism $t: A \to B$ is called a homotopy equivalence if the induced map
\begin{equation}
t_*: \text{Mor}_{C^*}(D, A) \to \text{Mor}_{C^*}(D, B)
\end{equation}
is a $\pi_0$-equivalence, in the sense of Definition 1.35, for all $D \in C^*$.

Remark 2.7. By Yoneda’s Lemma, $t \in C^*$ is a homotopy equivalence if and only if $\pi_0(t) \in \pi_0 C^*$ is invertible.

Definition 2.8. A $*$-homomorphism $p: E \to B$ is called a Schochet fibration if the induced map
\begin{equation}
p_*: \text{Mor}_{C^*}(D, E) \to \text{Mor}_{C^*}(D, B)
\end{equation}
is a $\pi_0$-fibration, in the sense of Definition 1.35, for all $D \in C^*$.

Remark 2.9. Schochet called these maps cofibrations in [Sch84], because, under the Gelfand-Naimark duality, the condition in Definition 2.8 for a $*$-homomorphism of abelian algebras corresponds to the homotopy extension property for the corresponding map of (pointed compact Hausdorff) spaces.

In a similar way, it is customary that $\text{Mor}_{\text{Top}^*}(S^1, B) \cong C_0(S^1) \otimes B$ is called the suspension of $B$, since $C_0(S^1) \otimes C_0(X) \cong C_0(S^1 \wedge X)$ for $B = C_0(X)$. Here $X$ is a pointed compact Hausdorff space and $C_0(X)$ is the continuous functions vanishing at the base point. See also Remark A.3.
However, for the sake of consistency, in this paper we will keep our notations and terminologies compatible with that of Section 1.

**Proposition 2.10.** All Schochet fibrations are surjective.

*Proof.* Let \( p: E \to B \) be a Schochet fibration. Consider the universal algebra generated by a positive contraction:

\[
C := C^*([0, 1]) = C_0(0, 1). \tag{2.9}
\]

Then for any \( b \in B, 0 \leq b \leq 1 \), there is a path

\[
[0, 1] \ni r \mapsto (x \mapsto rb) \in \text{Mor}_{C^*}(C, B), \tag{2.10}
\]

which lifts to \( 0 \in \text{Mor}_{C^*}(C, E) \) at \( r = 0 \). Lifting the path to \( \text{Mor}_{C^*}(C, E) \), we get \( e \in E, 0 \leq e \leq 1 \), such that \( p(e) = b \). It follows that \( p \) is surjective. \( \square \)

The following theorem is contained in \([Sch84]\).

**Theorem 2.11.** The category of \( C^* \)-algebras \( C^* \) is a pointed category of fibrant objects with weak equivalences the homotopy equivalences and fibrations the Schochet fibrations, whose homotopy category is the ordinary homotopy category i.e. \( \text{Ho}(C^*) = \pi_0 C^* \).

*Proof.* Everything follows from Proposition 1.37: we need to use Corollary 2.5 for (F2) and (FW1), and Lemma 2.3 with \( X = [a, b] \) for (FW2).

It follows from the construction of the path-object in \( C^* \) that two \( * \)-homomorphisms \( f_0, f_1 \in C^* \) are right-homotopic if and only if \( \pi_0(f_0) = \pi_0(f_1) \) in \( \pi_0 C^* \) and by Remark 2.7 this happens if and only if \( f_0, f_1 \) are homotopic in the sense of Definition 1.16. Hence

\[
\text{Ho}(C^*) = \pi C^* = \pi_0 C^*. \tag{2.11}
\]

\( \square \)

Note that \( C^* \) has a functorial path-object, given by \( C[0, 1] \otimes B \), hence also a functorial loop-object \( \Omega B := C_0(0, 1) \otimes B \).

The stable homotopy category \( \text{SHo}(C^*) \) is the suspension-stable homotopy category of \( C^* \)-algebras studied by Rosenberg \([Ros82]\) and Schochet \([Sch84]\).

**Remark 2.12.** Let \( sC^* \) denote the category of separable \( C^* \)-algebras. Then considering only \( D \) separable in Definitions 2.6 and 2.8 we get a structure of a category of fibrant objects on \( sC^* \).

**Remark 2.13.** The following are well-known and/or easy to see.

\[\footnote{\text{usually called suspension in the } C^*\text{-context}}\]
(1) The localization $\mathcal{C}^* \to \text{Ho}(\mathcal{C}^*)$ preserves arbitrary coproducts and arbitrary products:
\[
\prod_{i \in I} [A_i, B][\mathcal{C}^*] = \prod_{i \in I} [A_i, B][\mathcal{C}^*],
\]
\[
[A, \prod_{i \in I} B_i][\mathcal{C}^*] = \prod_{i \in I} [A, B_i][\mathcal{C}^*].
\]
(2) The loop functor $\Omega: \text{Ho}(\mathcal{C}^*) \to \text{Ho}(\mathcal{C}^*)$ preserves finite products:
\[
\Omega(B_1 \times B_2) \cong \Omega B_1 \times \Omega B_2,
\]
but not finite coproducts (for example, the natural map $\Omega \prod C \to \Omega(\prod C)$ is not a homotopy equivalence).
(3) The loop functor $\Omega: \text{Ho}(\mathcal{C}^*) \to \text{Ho}(\mathcal{C}^*)$ does not preserve infinite products, and in particular does not admit a left-adjoint; see Appendix A.
(4) The “stable homotopy functor” $\Omega^0: \text{Ho}(\mathcal{C}^*) \to \text{SHo}(\mathcal{C}^*)$ preserves finite products, but not finite coproducts.

2.2. $\mathcal{C}^*$-Invariant Homotopy Theory. Let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space.

**Proposition 2.14.** Defining the weak equivalences to be
\[
\{ t \in \mathcal{C}^* \mid t \otimes \text{id}_\mathcal{K} \text{ is a homotopy equivalence} \}
\]
and the fibrations to be
\[
\{ p \in \mathcal{C}^* \mid p \otimes \text{id}_\mathcal{K} \text{ is a Schochet fibration} \}
\]
defines a category of fibrant objects on $\mathcal{C}^*$, denoted $\mathcal{M}$.

**Proof.** This is clear since $- \otimes \text{id}_\mathcal{K}$ preserves pullbacks. \qed

Let $e_{11}: \mathcal{C} \to \mathcal{K}$ denote a rank-one projection. Then for any $B \in \mathcal{M}$, the morphism $\text{id}_B \otimes e_{11}$ is a weak equivalence in $\mathcal{M}$.

It follows that $\text{Ho}(\mathcal{M})$ is the “monoidal” localization $\text{Ho}(\mathcal{C}^*)[\otimes e_{11}^{-1}]$:
\[
[A, B]_\mathcal{M} \cong [A, B \otimes \mathcal{K}][\mathcal{C}^*],
\]
\[
\cong [A \otimes \mathcal{K}, B \otimes \mathcal{K}][\mathcal{C}^*].
\]

In the notation of [Hig90], the categories $\text{Ho}(\mathcal{M})$ and $\text{SHo}(\mathcal{M})$ are the not necessarily separable analogues of $\text{TH}$ and $\text{TS}$ respectively. When restricted to the abelian algebras, $\text{SHo}(\mathcal{M})$ gives the $kk$ groups of Dădărlat-McClure [DM00].

2.3. Topological $K$-Theory. Taking $\mathcal{H}$ to be topological $K$-theory in Theorem 1.32 we get a category $\mathcal{K} = R_K \mathcal{C}^*$ of fibrant objects whose weak equivalences are $K$-equivalences and fibrations are Schochet fibrations. Compare [JJ06] and [MN06]. It follows from Theorem 2.15 that $\text{Ho}(\mathcal{K})$ has small hom sets.
Let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space and let $e_{11}: \mathbb{C} \to \mathcal{K}$ denote a rank-one projection. Then

$$\text{id}_A \otimes e_{11}: A \to A \otimes \mathcal{K}$$

is a $K$-equivalence. We also have a natural isomorphism $\Omega^2 A \to A \otimes \mathcal{K}$ in $\text{Ho}(\mathcal{K})$, since Bott periodicity can be implemented by a boundary map associated to a Toeplitz type extension. It follows that

$$\Omega: \mathcal{K} \to \mathcal{K}$$

is invertible. Hence $\mathcal{K}$ is stable and the natural functor $\text{Ho}(\mathcal{K}) \to \text{SHo}(\mathcal{K})$ is an equivalence of categories. In particular, $\text{Ho}(\mathcal{K})$ is a triangulated category in a natural way, and $\text{SHo}(\mathcal{C}^*) \to \text{Ho}(\mathcal{K})$ is a triangulated functor.

The following is a version of the Universal Coefficient Theorem of Rosenberg and Schochet (cf. [RS87]). It can be deduced from results in [MN06], but we give a self-contained proof.

**Theorem 2.15.** For $B \in \mathcal{K}$, we have

$$[\mathbb{C}, B]_\mathcal{K} \cong K_0(B).$$

More generally, for $A, B \in \mathcal{K}$, there is a natural short exact sequence

$$\text{Ext}(K_{s+1}(A), K_s(B)) \to [A, B]_\mathcal{K} \to \text{Hom}(K_s(A), K_s(B)),$$

where

$$\text{Hom}(K_s(A), K_s(B)) := \bigoplus_{i=0,1} \text{Hom}_Z(K_i(A), K_i(B)) \quad \text{and}$$

$$\text{Ext}(K_{s-1}(A), K_s(B)) := \bigoplus_{i=0,1} \text{Ext}_Z^1(K_i(A), K_i(B)).$$

**Proof.** We have a natural (additive) map

$$[A, B]_\mathcal{K} \to \text{Hom}_Z(K_s(A), K_s(B)).$$

We claim that this is an isomorphism if $K_s(A)$ is free — for $A = \mathbb{C}$ we get (2.21).

Indeed, suppose that $K_s(A)$ is free. First note that we have natural isomorphisms

$$K_0(D) = [q\mathbb{C}, D \otimes \mathcal{K}]_{\mathcal{C}^*},$$

$$K_1(D) = [\Omega \mathbb{C}, D \otimes \mathcal{K}]_{\mathcal{C}^*},$$

where $q\mathbb{C}$ is the kernel of the folding map $(\mathbb{C} \amalg \mathbb{C} \to \mathbb{C})$. We have a $K$-equivalence $q\mathbb{C} \cong \mathbb{C}$. 

Then it is clear that any map $K_*(A) \rightarrow K_*(B)$ can be implemented by an element of the form

$$
(\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega \mathbb{C}) \rightarrow B \otimes K
$$

in $\text{Ho}(K)$. Hence (2.25) is surjective. To see injectivity of (2.25), let

$$
A \sim A' \rightarrow B
$$

be a morphism in $[A, B]_K$ that maps to $0 \in \text{Hom}(K_*(A), K_*(B))$. Then we can complete (2.29) to a homotopy-commutative diagram

$$
(\coprod_I q\mathbb{C}) \coprod (\coprod_J \Omega \mathbb{C}) \sim A' \otimes K \rightarrow B \otimes K
$$

in $\text{Ho}(C^*)$. Then the top horizontal map is null-homotopic, i.e. zero in $\text{Ho}(C^*)$, hence zero in $\text{Ho}(K)$. In other words, (2.25) is injective if $K_*(A)$ is free.

The general case follows using a geometric resolution of $K_*(A)$. See for instance [Uuy11]. □

2.4. **KK-Theory.** In the next two subsections, we will concentrate on the category $\text{sC}^*$ of separable $C^*$-algebras.

**Definition 2.16.** A $\ast$-homomorphisms $t: A \rightarrow B$ in $\text{sC}^*$ is called a **KK-equivalence** if

$$
t_*: \text{Mor}_{\text{C}^*}(qD, A \otimes K) \rightarrow \text{Mor}_{\text{C}^*}(qD, B \otimes K)
$$

is a $\pi_0$-equivalence for all $D \in \text{sC}^*$, where $qD$ is the kernel of the map $\text{id}_D \coprod \text{id}_D: D \coprod D \rightarrow D$.

**Theorem 2.17.** The category of separable $C^*$-algebras forms a category of fibrant objects with weak equivalences the KK-equivalences and fibrations the Schochet fibrations, denoted $\text{KK}$, whose homotopy category $\text{Ho}(\text{KK})$ is equivalent to the KK-category of Kasparov. It follows that Kasparov’s KK-category is a stable triangulated category.
Proof. The proof of Theorem 2.11 works in this case as well and proves that \( \text{KK} \) is a category of fibrant objects.

We use the following Cuntz type picture of \( \text{KK} \)-theory:

\[
\text{KK}(A, B) := [qA \otimes \mathcal{K}, qB \otimes \mathcal{K}]_{\text{C}^*}. \tag{2.32}
\]

Consider the functor \( \Phi: \text{KK} \to \text{KK} \) that send \( f: A \to B \) to the composition

\[
q(f) \otimes \text{id}_{\mathcal{K}}: qA \otimes \mathcal{K} \to qB \otimes \mathcal{K}. \tag{2.33}
\]

Then \( \Phi \) is indeed functor which is additive and \( \Phi(f) \) is a homotopy equivalence if and only if \( f \) is a \( \text{KK} \)-equivalence. Hence the induced functor \( \text{Ho}(\Phi): \text{Ho}(\text{KK}) \to \text{KK} \) is faithful. Moreover, by \([\text{Cun87}, \text{Theorem 1.6}]\), \( \Phi(A) \) is homotopy equivalent to \( \Phi(\Phi(A)) \), hence \( \text{Ho}(\Phi) \) is full. It follows that \( \text{Ho}(\Phi) \) is an equivalence of categories.

Stability follows from Bott Periodicity. \( \square \)

Remark 2.18. Note that in Theorem 2.17, we can take the semi-split surjections, i.e. surjections with a completely positive contractive splitting, to be the fibrations. Indeed, the only nontrivial part is (FW1): if \( p: E \to B \) is a semi-split surjection which is also a \( \text{KK} \)-equivalence and \( f: A \to B \) is arbitrary, then the pullback \( f^*(p) \) is also a \( \text{KK} \)-equivalence. However, this is clear since if \( p \) is a semi-split surjection with kernel \( F \), then \( F \to Fp \) is a \( \text{KK} \)-equivalence (see \([\text{Bla98}, \text{Theorem 19.5.5}]\)), hence \( p \) is a \( \text{KK} \)-equivalence if and only \( F \) is \( \text{KK} \)-contractible if and only if \( f^*(p) \) is a \( \text{KK} \)-equivalence (see Diagram (1.21)).

Note also that Schochet fibrations and semi-split surjections give rise to the same class of distinguished triangles in \( \text{Ho}(\text{KK}) \cong \text{SHo}(\text{KK}) \).

2.5. Universal Homology Theories. We consider \( s\text{C}^* \) as a category of fibrant objects with weak equivalences the homotopy equivalences and fibrations the Schochet fibrations. In this subsection, we identify various localizations of \( s\text{C}^* \).

Definition 2.19. A fibre homology theory on \( s\text{C}^* \) is a homology theory the pointed category of fibrant objects \( s\text{C}^* \) in the sense of Definition 1.28 i.e. a homological functor on the triangulated category \( \text{SHo}(s\text{C}^*) \) to \( \text{Ab} \).

Definition 2.20. We say that a fibre homology theory \( \mathcal{H} \) on \( s\text{C}^* \) is excisive with respect to a surjection \( p \), if the inclusion \( \ker(p) \to Fp \) is an \( \mathcal{H} \)-equivalence. A homology theory on \( s\text{C}^* \) is a fibre homology theory excisive with respect to all surjections.

Definition 2.21. We say that a morphism \( t \in s\text{C}^* \) is a weak equivalence if it is an \( \mathcal{H} \)-equivalence for all homology theories \( \mathcal{H} \) on \( s\text{C}^* \).

Remark 2.22. Note that homotopy equivalences are weak equivalences.

Theorem 2.23. The category \( s\text{C}^* \) forms a pointed category of fibrant objects with weak equivalences as in Definition 2.21 and fibrations the Schochet
fibrations, whose stable homotopy category is triangulated equivalent to the stable homotopy category of \([\text{Tho03, Theorem 3.3.5}]\).

By \([\text{Dăd94}]\), the stable homotopy category mentioned above is equivalent to the suspension-stable version of the strong shape category.

**Proof.** It follows from Theorem 1.34 that the stable homotopy category is a *universal* triangulated homology theory in the sense of \([\text{Tho03, Definition 2.3.3}]\). Then \([\text{Tho03, Theorem 3.3.6}]\) finishes the proof. \(\square\)

For a Hilbert space \(H\), let \(e_H : C \rightarrow K(H)\) denote a rank-one projection.

**Definition 2.24.** A (fibre) homology theory \(H\) on \(sC^*\) is said to be

1. *matrix-invariant* if \(\text{id}_B \otimes e_H\) is an \(H\)-equivalence for all \(B \in sC^*\) and \(H\) finite dimensional and
2. \(C^*\)-invariant if \(\text{id}_B \otimes e_H\) is an \(H\)-equivalence for all \(B \in sC^*\) and \(H\) separable.

**Definition 2.25.** A morphism \(t \in sC^*\) is said to be

1. an *\(bu\)-equivalence* if it induces isomorphism on all matrix-invariant homology theories and
2. an *\(E\)-equivalence* if it induces isomorphism on all \(C^*\)-invariant homology theories.

**Theorem 2.26.** (1) The category \(sC^*\) forms a pointed category of fibrant objects with weak equivalences the \(bu\)-equivalences and fibrations the Schochet fibrations, whose stable homotopy category is triangulated equivalent to the category \(bu\) of \([\text{Tho03, Theorem 4.2.1}]\).

(2) The category \(sC^*\) forms a stable pointed category of fibrant objects with weak equivalences the \(E\)-equivalences and fibrations the Schochet fibrations, whose homotopy category is a triangulated category, equivalent to the \(E\)-theory of Higson.

**Proof.** Follows from Theorem 1.34 and the universal properties of \(bu\) and \(E\) (cf. \([\text{Tho03}]\)). \(\square\)

**Remark 2.27.** (1) Let \(p : E \rightarrow B\) be a surjection with kernel \(F\). Then \(p\) is a weak equivalence in the sense of Definition 2.21 if and only if \(F\) is \(H\)-acyclic for all homology theories \(H\) on \(sC^*\). Indeed, we have a map of extensions where the vertical maps are all weak equivalences:

\[
\begin{array}{c}
0 \rightarrow F \\
\downarrow \\
0 \rightarrow F_p \rightarrow N_p \rightarrow B \\
\downarrow \\
0 \rightarrow B
\end{array}
\]

Hence the claim follows from the naturality of the long exact sequence associated to homology theories. It follows that in Theorem 2.26 and Theorem 2.26 we can take the fibrations to be all...
surjections. However, the distinguished triangles in the stable homotopy category would be the same (see the diagram (2.34)).

(2) We can also describe the $KK$-category of Kasparov as the universal split-exact triangulated homology theory in a similar way.

Appendix A. No Quillen Model Structure (following Andersen-Grodal)

As noted in the introduction, the homotopy theory of $C^*$-algebras does not come from a Quillen model structure. This was perhaps first pointed out as part of a 1997 preprint by Andersen-Grodal [AG97], where they also established a Baues fibration category structure [Bau89] on $C^*$-algebras (a notion very similar to a category of fibrant objects; see [Bau89, Rem. I.1a.6]). Since their work however remains unpublished, we, by permission of the authors, reproduce their non-existence argument in this appendix.

Recall that if $M$ is a Quillen model category, then the full subcategory $M_f$ of fibrant objects in $M$ is a category of fibrant objects (cf. Example 1.4).

**Theorem A.1.** Let $C^*$ denote the pointed category of fibrant objects of Theorem 2.11. Then $C^*$ is not the full subcategory of fibrant objects of a Quillen model category.

The essential part of the proof is to see that the loop functor does not admit a left adjoint, as already remarked on in Remark 2.13(3).

**Lemma A.2.** Let $M_f$ be the full subcategory fibrant objects of a Quillen model category $M$, considered as a category of fibrant objects as in Example 1.4. Then the loop-functor

$$ \Omega: \text{Ho}(M_f) \to \text{Ho}(M_f) $$

admits a left-adjoint.

**Proof.** Follows from Theorem I.1.1 and Theorem I.2.2 of [Qui67] and the definitions. $\square$

The following Lemma is clear.

**Lemma A.3.** Let $A^* \subseteq C^*$ denote the full subcategory consisting of abelian $C^*$-algebras. Then $A^*$ is a reflective (monoidal) subcategory of $C^*$ – the left-adjoint of the inclusion $i: A^* \to C^*$ is the abelianization $(-)^{ab}: C^* \to A^*$:

$$ \text{Mor}_{A^*}(D^{ab}, B) \cong \text{Mor}_{C^*}(D, iB), $$

for $D \in C^*$, $B \in A^*$. $\square$

In particular, $A^*$ is a pointed category of fibrant objects (cf. Example 1.6).

**Corollary A.4.** The homotopy category $\text{Ho}(A^*)$ is a full reflective subcategory of $\text{Ho}(C^*)$ and the loop-functor

$$ \Omega: \text{Ho}(A^*) \to \text{Ho}(A^*) $$

is the restriction of $\Omega: \text{Ho}(C^*) \to \text{Ho}(C^*)$ to $\text{Ho}(A^*)$. 

Proof. The adjunction \( A.2 \) descends to the homotopy categories and gives an adjunction:
\[
[D^\text{ab}, B]_{A^*} \cong [D, iB]_{C^*}, \tag{A.4}
\]
for \( D \in C^* \), \( B \in A^* \). See also \([Bro74, \text{Adjoint functor lemma}]\). The rest of the statements are clear.

Consequently, we see that \( \text{S}\text{Ho}(A^*) \) is a full triangulated subcategory of \( \text{S}\text{Ho}(C^*) \).

**Lemma A.5.** The loop-functor \( \Omega: \text{Ho}(A^*) \to \text{Ho}(A^*) \) does not admit a left-adjoint.

**Proof.** By Gelfand-Naimark duality, the category \( \text{CM}_* \) of pointed compact Hausdorff spaces is contravariantly equivalent to \( A^* \), hence form a category of cofibrant objects. We need to show that the functor
\[
\Sigma = S^1 \wedge - : \text{Ho}(\text{CM}_*) \to \text{Ho}(\text{CM}_*) \tag{A.5}
\]
does not admit a right-adjoint. We show that, in fact, the functor
\[
\text{Ho}(\text{CM}_*) \to \text{Set}_*, \quad X \mapsto [\Sigma X, S^1]_{\text{CM}_*} \tag{A.6}
\]
is not representable, where \( \text{Set}_* \) denote the category of pointed sets. Indeed, suppose that for some \( Y \in \text{Ho}(\text{CM}_*) \) we have a natural identification
\[
[\Sigma X, S^1]_{\text{CM}_*} \cong [X, Y]_{\text{CM}_*}. \tag{A.7}
\]
Let \( \text{Top}_* \) denote the category of pointed compactly generated weakly Hausdorff topological spaces. Then \( \text{CM}_* \) is a full (reflective) subcategory of \( \text{Top}_* \) and \( \text{Ho}(\text{CM}_*) \) is a full subcategory of \( \text{Ho}(\text{Top}_*) \). Moreover, the functor \( \Sigma \) of \( \text{A.5} \) is the restriction of
\[
\Sigma = S^1 \wedge - : \text{Ho}(\text{Top}_*) \to \text{Ho}(\text{Top}_*), \tag{A.8}
\]
which does have a right-adjoint
\[
\Omega = \text{Mor}_{\text{Top}_*}(S^1, -) : \text{Ho}(\text{Top}_*) \to \text{Ho}(\text{Top}_*). \tag{A.9}
\]
Hence for \( X \in \text{CM}_* \), we have
\[
[X, Y]_{\text{Top}_*} \cong [X, Y]_{\text{CM}_*} \cong [\Sigma X, S^1]_{\text{CM}_*} \cong [X, \Omega S^1]_{\text{Top}_*}. \tag{A.10}
\]
Moreover, by Yoneda’s Lemma, the natural identification above must be induced by a map \( f: Y \to \Omega S^1 \) of \( \text{Top}_* \). This is a contradiction, because, since \( Y \) is compact \( f \) cannot be surjective on \( \pi_0 \).

**Corollary A.6.** The loop-functor \( \Omega: \text{Ho}(C^*) \to \text{Ho}(C^*) \) does not admit a left-adjoint.
Proof. Suppose that $\Sigma: \text{Ho}(C^*) \to \text{Ho}(C^*)$ is a left-adjoint of $\Omega$. It follows that the composition
\[
(\sim)_{ab} \circ \Sigma \circ i: \text{Ho}(A^*) \to \text{Ho}(C^*) \to \text{Ho}(C^*) \to \text{Ho}(A^*) \quad (A.13)
\]
is a left-adjoint of $\Omega: \text{Ho}(A^*) \to \text{Ho}(A^*)$, contradicting Lemma A.5 \hfill \square

Now Theorem A.1 follows from Lemma A.2 and Corollary A.6.

References

[AG97] Kasper Andersen and Jesper Grodal, A Baues fibration category structure on Banach and $C^*$-algebras, \url{http://www.math.ku.dk/~jg/papers/fibcat.pdf}, 1997.

[Bau89] Hans Joachim Baues, Algebraic homotopy, Cambridge Studies in Advanced Mathematics, vol. 15, Cambridge University Press, Cambridge, 1989. MR 985099 (90i:55016)

[Bla85] Bruce Blackadar, Shape theory for $C^*$-algebras, Math. Scand. 56 (1985), no. 2, 249–275. MR 813640 (87b:46074)

[Bla98] Bruce Blackadar, $K$-theory for operator algebras, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR 1656031 (99g:46104)

[Bro74] Kenneth S. Brown, Abstract homotopy theory and generalized sheaf cohomology, Trans. Amer. Math. Soc. 186 (1974), 419–458. MR 0341469 (49 #6220)

[Cat81] Fritz Cathey, Strong shape theory, Shape theory and geometric topology (Dubrovnik, 1981), Lecture Notes in Math., vol. 870, Springer, Berlin, 1981, pp. 215–238. MR 643532 (83d:55008)

[CH81] Allan Calder and Harold M. Hastings, Realizing strong shape equivalences, J. Pure Appl. Algebra 20 (1981), no. 2, 129–156. MR 601680 (82g:55010)

[CH90] Alain Connes and Nigel Higson, Déformations, morphismes asymptotiques et $K$-théorie bivariante, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), no. 2, 101–106. MR 1065438 (91m:46114)

[Cun87] Joachim Cuntz, A new look at $KK$-theory, $K$-Theory 1 (1987), no. 1, 31–51. MR 899916 (89a:46142)

[Cun98] Joachim Cuntz, A general construction of bivariant $K$-theories on the category of $C^*$-algebras, Operator algebras and operator theory (Shanghai, 1997), Contemp. Math., vol. 228, Amer. Math. Soc., Providence, RI, 1998, pp. 31–43. MR 1667652 (2000b:46125)

[Däd94] Marius Dădărlat, Shape theory and asymptotic morphisms for $C^*$-algebras, Duke Math. J. 73 (1994), no. 3, 687–711. MR 1262931 (95c:46117)

[DM00] Marius Dădărlat and James McClure, When are two commutative $C^*$-algebras stably homotopy equivalent?, Math. Z. 235 (2000), no. 3, 499–523. MR 1800209 (2001k:46080)

[EK86] E. G. Effros and J. Kaminker, Homotopy continuity and shape theory for $C^*$-algebras, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., vol. 123, Longman Sci. Tech., Harlow, 1986, pp. 152–180. MR 866493 (88a:46082)

[GJ99] Paul G. Goerss and John F. Jardine, Simplicial homotopy theory, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR 1711612 (2001d:55012)

[GZ67] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967. MR 0210125 (35 #1019)

[Hel68] Alex Heller, Stable homotopy categories, Bull. Amer. Math. Soc. 74 (1968), 28-63. MR 0224090 (36 #7137)
