ON L.S.-CATEGORY OF A FAMILY OF RATIONAL ELLIPTIC SPACES

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Abstract. Let $X$ be a finite type simply connected rationally elliptic CW-complex with Sullivan minimal model $(\Lambda V, d)$ and let $k \geq 2$ the biggest integer such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$.

In [11] the authors showed that if $(\Lambda V, d_k)$ is moreover elliptic then $\text{cat}(\Lambda V, d) = (k - 2) \text{dim} V_{\text{even}} + \text{dim} V_{\text{odd}}$.

Our work focuses on the estimation of L.S.-category of such spaces in the case when $k = 3$ and when $(\Lambda V, d_3)$ is not necessarily elliptic.

1. Introduction

Let $X$ be a finite type simply connected CW-complex with Sullivan minimal model $(\Lambda V, d)$ and let $k \geq 2$ the biggest integer such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$ and $\text{dim}(V) < \infty$.

Consider on $(\Lambda V, d)$ the filtration given by

$$F^p = \Lambda^{(k-1)p} V = \bigoplus_{i = (k-1)p}^{\infty} \Lambda^i V.$$

$F^p$ is preserved by the differential $d$ and satisfies $F^p(\Lambda V) \otimes F^q(\Lambda V) \subseteq F^{p+q}(\Lambda V)$, $\forall p, q \geq 0$, so it is a filtration of differential graded algebras. Also, since $F^0 = \Lambda V$ and $F^{p+1} \subseteq F^p$ this filtration is decreasing and bounded, so it induces a convergent spectral sequence. Its $0^{th}$-term is

$$E_0^{p,q} = \left( \frac{F^p}{F^{p+1}} \right)^{p+q} = \left( \frac{\Lambda^{(k-1)p} V}{\Lambda^{(k-1)(p+1)} V} \right)^{p+q}.$$
Hence, we have the identification:

\[ E^p,q_0 = (\Lambda^{p(k-1)}V \oplus \Lambda^{p(k-1)+1}V \oplus \ldots \oplus \Lambda^{p(k-1)+k-2}V)^{p+q} \]

In this general situation, the \(1^{th}\)-term is the graded algebra \(\Lambda V\) proveded with a differential \(\delta\), which is\’t necessarily a derivation on the set \(V\) of generators (see §3). That is \((\Lambda V, \delta)\) is a commutative differential graded algebra, but it is not a Sullivan algebra. The spectral sequence is therefore:

\[ H^p,q(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d) \]

Hence if \(\dim(V) < \infty\) and \((\Lambda V, \delta)\) has finite dimensional cohomology, then \((\Lambda V, d)\) is elliptic. This gives a new family of rationally elliptic spaces for which \(d = \sum d_i\).

Recall first that in [11] the authors gives the explicit formula \(\text{cat}(\Lambda V, d) = \dim V_{\text{odd}} + (k-1)\dim V_{\text{even}}\) of L.-S. category for a minimal Sullivan model \((\Lambda V, d)\) satisfying the restrictive condition: \((\Lambda V, d_k)\) is also elliptic.

It is important to note also that their algorithm that induces the fundamental class of \((\Lambda V, d)\) from that of \((\Lambda V, d_k)\) corresponds to the progress of a cocycle that survive to term \(E_\infty\) (cf. Remark 1).

The main result of this work is a project of determination of an explicit formula for \(\text{cat}(\Lambda V, d)\) with \((\Lambda V, d)\) being elliptic and \((\Lambda V, d_k)\) not elliptic, completing the formula given by L. Lechuga and A. Murillo in [11].

In what follow, we consider the case where \(d = \sum_{i \geq 3} d_i\), that is where \(k = 3\) and \(N\) designate the formal dimension of \((\Lambda V, d)\). With the notation as above, our first result reads:

**Theorem 1.1.** If \((\Lambda V, d)\) is elliptic and \(H^N(\Lambda V, \delta) = \mathbb{Q} \alpha\) is one dimensionial, then \(\text{cat}_0(X) = \text{cat}(\Lambda V, d) = \sup\{k \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq k}V\}\).

Let \((\Lambda W, d)\) a minimal Sullivan model of \((\Lambda V, d)\). If \(\dim(W) < \infty\) then \((\Lambda W, d)\) is a Gorenstein algebra and so is \((\Lambda V, d)\). If additionaly \(\dim H(\Lambda V, \delta) < \infty\), then \((\Lambda W, d)\) is elliptic and so its L.S. category is finite. It follows [2 Th. 29.15] that \(M\text{cat}(\Lambda V, \delta) < \infty\). Hence [1 Th. 3.6] \(H(\Lambda V, \delta)\) is a Poincaré Duality algebra. There follow the

**Corollary 1.** Let \((\Lambda W, d)\) a minimal Sullivan model of \((\Lambda V, d)\). If \(\dim(W) < \infty\) and \(\dim H(\Lambda V, \delta) < \infty\) then \(\text{cat}_0(X) = \sup\{k \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq k}V\}\).

**Remark 1.** Now if \(\dim H^N(\Lambda V, \delta) > 1\), the technique used to show Theorem 1.1 can be adapted to have a similar result under this general hypothesis. The procedure is as follows:

Note first that in the proof of Theorem 1.1, the algorithm applied to the representative \(\omega_0\) of the generating class of \(H^N(\Lambda V, \delta)\) resulted in one of the fundamental
class of $H(ΛV, d)$ because $ω_0$ is a cocycle which survives to $E_∞$ in the spectral sequence.

On the other hand, since $\dim(V) < \infty$, we have $\dim H^N(ΛV, δ) < \infty$, with $N$ being the formal dimension of $(ΛV, d)$. Since the filtration induces on cohomology a graduation such that $H^N(ΛV, δ) = \bigoplus_{p+q=N} H^{p,q}(ΛV, δ)$, there is a basis $\{α_1, ..., α_m\}$ of $H^N(ΛV, δ)$ with $α_i \in H^{p_i,q}(ΛV, δ)$, $(1 ≤ i ≤ m)$. That is, $α_i = [(ω^i_0, ω^i_1)]$, where $(ω^i_0, ω^i_1) ∈ Λ^{2p_i} V ⊕ Λ^{2p_i+1} V$. Also since $(ΛV, d)$ is elliptic, there exist a unique $j$ such that some $α_j ∈ H^{p_j,q_j}(ΛV, δ)$ survives to $E_∞$ and consequently induces an representative of the fundamental class of $(ΛV, d)$. Explicitly, the corresponding obstructions $[a^0_2] = 0$, $[a^3_3] = 0$, ..., $[a_{t_j+l_j-2}^t] = 0$ are necessary satisfied.

Now, applying to $(ω^0_1, ω^1_1) ∈ Λ^{2p_1} V ⊕ Λ^{2p_1+1} V$, the same role as that applied to $ω_0$ in the case of the first inequality (see §4) we obtain an $ω_{t_j+t_j-1} ∈ Λ^{2p_1} V$ (resp. $ω_{t_j+t_j-1} ∈ Λ^{2p_1+1} V$) if $ω^i_0 \neq 0$ (resp. if $ω^i_0 = 0$) representing the fundamental class of $(ΛV, d)$. It follows that $e_0(ΛV, d) ≥ 2p_j$ (resp. $e_0(ΛV, d) ≥ 2p_j + 1$).

For the other inequality, any representative $ω ∈ Λ^s V$ (where $s = e_0(ΛV, d)$) of the fundamental class of $(ΛV, d)$ induces by the same way, a representative $(ω_0, ω_1) ∈ Λ^s V$ of a certain non zero class in $H^N(ΛV, δ) = \bigoplus_{p+q=N} H^{p,q}(ΛV, δ)$. By convergence of the spectral sequence [ω] correspond to a basis element of $E^{∞,∞}_{∞}−s$ which is one-dimensional, by ellipticity. It follow that in $E^{∞,∞}_{2}−s$ there is an element which survives to $E^{∞,∞}_{∞}−s$. Hence with notations above, $s = 2p_j$ or $s = 2p_j + 1$. Therefore $e_0(ΛV, d) = 2p_j$ or $e_0(ΛV, d) = 2p_j + 1$.

With the notation of the previous remark we can therefore state the following generalization of the previous theorem

**Theorem 1.2.** If $(ΛV, d)$ is elliptic and $\dim H^N(ΛV, d) = m$ with basis $\{α_1, ..., α_m\}$. Then $\text{cat}_0(X) = \text{cat}(ΛV, d) = r_j$ with $r_j = 2p_j$ or else $r_j = 2p_j + 1$.

2. Basic facts and properties

Let $K$ be a field of characteristic $≠ 2$.

A Sullivan algebra is a free commutative differential graded algebra (cdga for short) $(ΛV, d)$ (where $ΛV = \text{Exterior}(V^{even}) ⊕ \text{Symmetric}(V^{odd})$) generated by the graded $K$-vector space $V = \bigoplus_{i=0}^{∞} V^i$ which has a well ordered basis $\{x_α\}$ such that $dx_α ∈ ΛV_{<α}$. Such algebra is said minimal if $\text{deg}(x_α) < \text{deg}(x_β)$ implies $α < β$. If $V^0 = V^1 = 0$ this is equivalent to saying that $d(V) ⊆ \bigoplus_{i=2}^{∞} Λ^i V$.

A Sullivan model for a commutative differential graded algebra $(A, d)$ is a quasi-isomorphism (morphism inducing isomorphism in cohomology) $(ΛV, d) \rightarrow (A, d)$ with source, a Sullivan algebra. If $H^0(A) = K$, $H^1(A) = 0$ and $\dim(H^i(A, d)) < ∞$ for all $i ≥ 0$, then [7] Th.7.1, this minimal model exists. If $X$ is a topological
space any (minimal) model of the algebra $C^*(X, \mathbb{K})$ is said a Sullivan (minimal) model of $X$.

The differential $d$ of any element of $V$ is a "polynomial" in $\Lambda V$ with no linear term. A model $(\Lambda V, d)$ is elliptic if both $V$ and $H^*(\Lambda V, d)$ are finite dimensional spaces (see for example [2]).

For an elliptic space with model $(\Lambda V, d)$ the formal dimension $N$, i.e., the largest $n$ for which $H^n(\Lambda V, d) \neq 0$, is given by [5]

$$N = \dim V^{\text{even}} - \sum_{i=1}^{\dim V} (-1)^{|x_i|} |x_i|$$

An element $0 \neq \omega \in H^N(\Lambda V, d)$ is called a fundamental or top class of $(\Lambda V, d)$.

In [7] S. Halperin associated to any minimal model $(\Lambda V, d)$ a pure model $(\Lambda V, d^{\sigma})$ defined as follows:

If $Q = V^{\text{even}}$ and $P = V^{\text{odd}}$ then

$$(\Lambda V, d^{\sigma}) = (\Lambda Q \otimes \Lambda P, d^{\sigma}); \quad d^{\sigma}(Q) = 0 \quad \text{and} \quad (d - d^{\sigma})(P) \subseteq \Lambda Q \otimes \Lambda^+ P$$

This model is related to $(\Lambda V, d)$ via the odd spectral sequence

$$H^{p,q}(\Lambda V, d^{\sigma}) \Rightarrow H^{p+q}(\Lambda V, d)$$

The main result using this algebra and due to S. Halperin ([5]) shows that in the rational case, if $\dim(V) < \infty$, then:

$$\dim(H(\Lambda V, d)) < \infty \iff \dim(H(\Lambda V, d^{\sigma})) < \infty$$

If $X$ is a topological space, $\text{cat}(X)$ is the least integer $n$ such that $X$ is covered by $n+1$ open subset $U_i$, each contractible in $X$. It is an invariant of homotopy type (c.f. [2]). In [3] Y. Félix, S. Halperin and J.M. Lemaire showed that for Poincaré duality spaces, the rational LS-category coincide with the rational Toomer invariant denoted $e_0(X)$.

By [1, Lemma 10.1] the Toomer invariant of a minimal model $e_0(\Lambda V, d)$ is the largest integer $s$ for which there is a non trivial cohomology class in $H^s(\Lambda V, d)$ represented by a cycle in $\Lambda^{\geq s} V$. As usual, $\Lambda^s V$ denotes the elements in $\Lambda V$ of "wordlength" $s$. For more details [2], [6], [12] are standard references.

In [8] A. Murillo gave an expression of the fundamental class of $H(\Lambda V, d)$ in the case where $(\Lambda V, d)$ is a pure model. We recall it here: Assume $\dim V < \infty$, choose homogeneous basis $\{x_1, ..., x_n\}, \{y_1, ..., y_m\}$ of $V^{\text{even}}$.
and $V^{odd}$ respectively, and write
\[ dy_j = a_j^1 x_1 + a_j^2 x_2 + \ldots + a_j^{n-1} x_{n-1} + a_j^n x_n \quad j = 1, 2, \ldots, m, \]
where each $a_j^i$ is a polynomial in the variables $x_i, x_{i+1}, \ldots, x_n$, and consider the matrix,
\[
A = \begin{pmatrix}
a_1^1 & a_1^2 & \ldots & a_1^n \\
a_2^1 & a_2^2 & \ldots & a_2^n \\
\vdots & \vdots & \ddots & \vdots \\
a_m^1 & a_m^2 & \ldots & a_m^n
\end{pmatrix}
\]

For any $1 \leq j_1 < \ldots < j_n \leq m$, denote by $P_{j_1 \ldots j_n}$ the determinant of the matrix of order $n$ formed by the columns $i_1, i_2, \ldots, i_n$ of $A$:
\[
\begin{pmatrix}
a_{j_1}^1 & \ldots & a_{j_1}^n \\
\vdots & \ddots & \vdots \\
a_{j_n}^1 & \ldots & a_{j_n}^n
\end{pmatrix}
\]

Then (see [8]) if $\dim H^*(\Lambda V, d) < \infty$ the element $\omega \in \Lambda V$
\[
\omega = \sum_{1 \leq j_1 < \ldots < j_n \leq m} (-1)^{j_1 + \ldots + j_n} P_{j_1 \ldots j_n} y_{j_1} \ldots y_{j_n} y_{1} \ldots y_{m},
\]
is a cycle representing the fundamental class of the cohomology algebra.

3. The spectral sequence

In what follows, we give the expression for $\delta$ in the case where $k=3$.

As mentioned in the introduction, our filtration is one of filtered differential graded algebras, hence in this case the identification (II) becomes:
\[
E_0^{p,q} = (\Lambda^{2p} V \oplus \Lambda^{2p+1} V)^{p+q}
\]
with the product given by:
\[
(u, v) \otimes (u', v') = (uu', uv' + vu'), \quad \forall (u, v) \in E_0^{p,q}, \forall (u', v') \in E_0^{p', q'}.
\]

On the other hand, since $d_1 = d_2 = 0$ the differential on $E_0$ is zero, hence $E_0^{p,q} = E_0^{p,q}$ and so the identification above gives the following diagram
\[ E_1^{p,q} \xrightarrow{\cong} (\Lambda^2pV \oplus \Lambda^{2p+1}V)^{p+q} \]

\[ E_1^{p+1,q} \xrightarrow{\cong} (\Lambda^{2(p+1)}V \oplus \Lambda^{2(p+1)+1}V)^{p+q+1} \]

with \( \delta \) defined as follows,

\[ \delta(u, v) = (d_3u, d_3v + d_4u) \]

Let \( E_1^p = E_1^{p,*} = \bigoplus_{q \geq 0} E_1^{p,q} \) and \( E_1^* = \bigoplus_{p \geq 0} E_1^{p,*} \). This gives a commutative differential graded algebra \( (E_1^*, \delta) \) which is the first term of our spectral sequence:

\[ E_2^{p,q} = H^{p,q}(\Lambda V, \delta) \Rightarrow H^{p+q}(\Lambda V, d). \]

4. Proof of the theorem 1.1

Recall that we restrict ourself to the case \( k = 3 \). The approch used here is inspered by that used in [11]. Note also that the subsequent notations imposed us to replace certain somes by pairs and vice-versa.

**For the first inequality**

We note first that since by hypothesis, \( \dim H^N(\Lambda V, d) = 1 \), the class \( \alpha \in E_2^{*,*} \) must survive to \( E_\infty \).

In what follow we put : \( r = \sup\{k \geq 0, \alpha = [\omega_0] \text{ with } \omega_0 \in \Lambda^{\geq k}V\} \).

Let then \( \omega_0 \in \Lambda^{\geq r}V \). We may suppose that \( r = 2p \) is even (indeed, if \( r = 2p+1 \) is odd, it suffice to rewrite \( \omega_0 \) with the coordinate in \( \Lambda^{2p}V \) being 0). More explicily \( \omega_0 \in (\Lambda^{2p}V \oplus \Lambda^{2p+1}V) \oplus (\Lambda^{2p+2}V \oplus \Lambda^{2p+3}V) \oplus ... \).

Since \( |\omega_0| = N \), there is an integer \( l \) such that:

\[ \omega_0 = \omega_0^0 + \omega_0^1 + ... + \omega_0^l \]

with \( \omega_0^i = (\omega_0^{i,1}, \omega_0^{i,2}) \in \Lambda^{2(p+i)}V \oplus \Lambda^{2(p+i)+1}V \)

We hace successivly:

\[ \delta(\omega_0^i) = \delta(\omega_0^{i,1}, \omega_0^{i,2}) = (d_3\omega_0^{i,1}, d_3\omega_0^{i,2} + d_4\omega_0^{i,1}) \]

\[ \delta(\omega_0) = \sum_{i=0}^{l} \delta(\omega_0^{i,1}, \omega_0^{i,2}) = \sum_{i=0}^{l} (d_3\omega_0^{i,1}, d_3\omega_0^{i,2} + d_4\omega_0^{i,1}) \]

Also, we have \( d\omega_0 = d\omega_0^0 + d\omega_0^1 + ... + d\omega_0^l \), with:

\[ d\omega_0^0 = d(\omega_0^{0,1}, \omega_0^{0,2}) = (d_3\omega_0^{0,1}, d_3\omega_0^{0,2} + d_4\omega_0^{0,1}) + ... \in (\Lambda^{2p+2}V \oplus \Lambda^{2p+3}V) \oplus ... \]
Therefore

\[ d\omega_0 = d(\omega_0^{1,1}, \omega_0^{1,2}) = (d_3\omega_0^{1,1}, d_3\omega_0^{1,2} + d_4\omega_0^{1,1}) + \ldots \in (\Lambda^{2p+4}V \oplus \Lambda^{2p+5}V) \oplus \ldots \]

\[ d\omega_0^i = d(\omega_0^{i,1}, \omega_0^{i,2}) = (d_3\omega_0^{i,1}, d_3\omega_0^{i,2} + d_4\omega_0^{i,1}) + \ldots \in (\Lambda^{2p+2i}V \oplus \Lambda^{2p+2i+1}V) \oplus \ldots \]

Therefore

\[ d\omega_0 = (d_3(\omega_0^{0,1} + \omega_0^{1,1} + \omega_0^{2,1} + \ldots) + d_4\omega_0^{0,2} + d_5\omega_0^{0,1} + \ldots, d_3(\omega_0^{0,2} + \omega_0^{1,2} + \ldots) + d_4(\omega_0^{0,1} + \omega_0^{1,1} + \omega_0^{2,1} + \ldots) + d_5\omega_0^{0,2} + d_6\omega_0^{0,1} + \ldots) \]

that is: \( d\omega_0 = \delta(\omega_0) + (d_4\omega_0^{0,2} + d_5\omega_0^{0,1} + \ldots, d_5\omega_0^{0,2} + d_6\omega_0^{0,1} + \ldots) \). As \( \delta(\omega_0) = 0 \) we can rewrite:

\[ d\omega_0 = a_0^0 + a_3^0 + \ldots + a_{t+1}^0 \quad \text{with} \quad a_i^0 = (a_i^{0,1}, a_i^{0,2}) \in \Lambda^{2(p+i)}V \oplus \Lambda^{2(p+i)+1}V \]

Note also that \( t \) is a fixed integer. Indeed the degree of \( a_{t+1}^0 \) is greater or equal than \( 2(2(p + t + l) + 1) \) and it coincides with \( N + 1 \), being \( N \) the formal dimension. Then

\[ N + 1 \geq 2(2(p + t + l) + 1) \]

Hence

\[ t \leq \frac{1}{4}(N - 4p - 4l - 1). \]

In what follows, we take \( t \) the largest integer satisfying this inequality.

Now, we have:

\[ d^2\omega_0 = d(a_2^0) + da_3^0 + \ldots + da_{t+1}^0 \]

\[ = d(a_2^{0,1}, a_2^{0,2}) + d(a_3^{0,1}, a_3^{0,2}) + \ldots + d(a_{t+1}^{0,1}, a_{t+1}^{0,2}) \]

with;

\[ d(a_2^{0,1}, a_2^{0,2}) = d_3(a_2^{0,1}, a_2^{0,2}) + d_4(a_2^{0,1}, a_2^{0,2}) + d_5(a_2^{0,1}, a_2^{0,2}) + \ldots \]

\[ = (d_3a_2^{0,1}, d_3a_2^{0,2} + d_4a_2^{0,1}) + (d_5a_2^{0,1} + d_4a_2^{0,2} + da_2^{0,1} + d_5a_2^{0,2}) + \ldots \]

\[ d(a_3^{0,1}, a_3^{0,2}) = d_3(a_3^{0,1}, a_3^{0,2}) + d_4(a_3^{0,1}, a_3^{0,2}) + d_5(a_3^{0,1}, a_3^{0,2}) + \ldots \]

\[ = (d_3a_3^{0,1}, d_3a_3^{0,2} + d_4a_3^{0,1}) + (d_5a_3^{0,1} + d_4a_3^{0,2} + da_3^{0,1} + d_5a_3^{0,2}) + \ldots \]

\[ \ldots \]

It follows that:

\[ d^2\omega_0 = (d_3a_2^{0,1}, d_3a_2^{0,2} + d_4a_2^{0,1}) + (d_5a_2^{0,1} + d_4a_2^{0,2} + d_3a_3^{0,1}, d_6a_2^{0,1} + d_5a_2^{0,2} + d_4a_3^{0,1} + d_3a_3^{0,2}) + \ldots \]

Since \( d^2\omega_0 = 0 \), we have \( (d_3a_2^{0,1}, d_3a_2^{0,2} + d_4a_2^{0,1}) = \delta(a_2^0) = 0 \) with \( a_2^0 \in \Lambda^{2(p+2)}V \oplus \Lambda^{2(p+2)+1}V \). Hence \( a_2^0 \) is a \( \delta \)-boundary, i.e., there is \( b_2 \in \Lambda^{2(p+2)-2}V \oplus \Lambda^{2(p+2)-1}V \) such that \( a_2^0 = \delta(b_2) \). Otherwise the cocycle will not survive to \( E_3 \) and a fortiori to \( E_\infty \).
Consider $\omega_1 = \omega_0 - b_2$ and reconsider the previous calculation:

$$d\omega_1 = d\omega_0 - db_2$$

$$= (a_2^0 + a_3^0 + ... + a_{t+l}^0) - (d_3b_2 + d_4b_2 + ... + d_{t+3}b_2)$$

With

$$d_3b_2 = d_3(b_2^1, b_2^2) = (d_3b_2^1, d_3b_2^2) \in \Lambda^{2p+4}V \oplus \Lambda^{2p+5}V$$

$$d_4b_2 = d_4(b_2^1, b_2^2) = (d_4b_2^1, d_4b_2^2) \in \Lambda^{2p+5}V \oplus \Lambda^{2p+6}V$$

This implies that:

$$d\omega_1 = a_2^0 + a_3^0 + ... + a_{r+t}^0 - (d_3b_2^1, d_3b_2^2 + d_4b_2^1) + ...$$

$$= a_2^0 - d_3b_2 + a_3^0 + ... + a_{r+t}^0 - (d_5b_2^1 + d_4b_2^2, d_5b_2^2 + ...) - ...$$

$$= a_3^0 - (d_5b_2^1 + d_4b_2^2, d_5b_2^2 + ...) + ...$$

and then:

$$d\omega_1 = a_3^1 + a_4^1 + ... + a_{t+l}^1, \quad \text{with} \quad a_i^1 \in \Lambda^{2(p+i)}V \oplus \Lambda^{2(p+i)+1}V$$

So,

$$d^2\omega_1 = da_3^1 + da_4^1 + ... + da_{t+l}^1$$

$$= d(a_3^{1,1}, a_3^{1,2}) + d(a_4^{1,1}, a_4^{1,2}) + ... + d(a_{t+l}^{1,1}, a_{t+l}^{1,2})$$

$$= (d_3a_3^{1,1}, d_3a_3^{1,2} + d_4a_3^{1,1}) + (d_5a_3^{1,1} + d_4a_3^{1,2} + d_5a_4^{1,1}, d_5a_3^{1,2} + ...) + ...$$

Since $d^2\omega_1 = 0$, by wordlength reasons, $(d_3a_3^{1,1}, d_3a_3^{1,2} + d_4a_3^{1,1}) = \delta(a_3^1) = 0$. Hence (for the same reason as before) $a_3^1$ is a $\delta$-boundary, i.e., there is $b_3 \in \Lambda^{2(p+3)-2}V \oplus \Lambda^{2(p+3)-1}V$ such that $\delta(b_3) = a_3^1$.

Consider $\omega_2 = \omega_1 - b_3$.

By the same way we show that

$$d\omega_2 = a_4^2 + a_5^2 + ... + a_{t+l}^2, \quad \text{with} \quad a_i^2 \in \Lambda^{2(p+i)}V \oplus \Lambda^{2(p+i)+1}V$$

We continue this process defining inductively $\omega_j = \omega_{j-1} - b_{j+1}$, $j < r + l$ such that:

$$d\omega_j = a_{j+2}^j + a_{j+3}^j + ... + a_{t+l}^j, \quad \text{with} \quad a_i^j \in \Lambda^{2(p+i)}V \oplus \Lambda^{2(p+i)+1}V$$

Also, we have:

$$\omega_{t+l-2} = \omega_{t+l-3} - b_{t+l-1}, \quad \text{with} \quad b_{t+l-1} \in \Lambda^{2(p+t+l-1)-2}V \oplus \Lambda^{2(p+t+l-1)-1}V$$

$$d\omega_{t+l-2} = a_{t+l}^{t+l-2} = \delta(b_{t+l-1}) \in \Lambda^{2(p+t+l)}V \oplus \Lambda^{2(p+t+l)+1}V$$

$$d^2\omega_{t+l-2} = da_{t+l}^{t+l-2} = (d_3a_{t+l}^{t+l-2,1}, d_3a_{t+l}^{t+l-2,2} + d_4a_{t+l}^{t+l-2,1}) + ...$$

Since $d^2\omega_{t+l-2} = 0$, by wordlength reasons,

$$(d_3a_{t+l}^{t+l-2,1}, d_3a_{t+l}^{t+l-2,2} + d_4a_{t+l}^{t+l-2,1}) = \delta(a_{t+l}^{t+l-2}) = 0$$
Hence $a_{t+1}^{l-2}$ is a $\delta$-boundary, i.e., there is $b_{t+l} \in \Lambda^{2(p+t+l)-2}V \oplus \Lambda^{2(p+t+l)-1}V$ such that $\delta(b_{t+l}) = a_{t+1}^{l-2}$.

Consider $\omega_{t+l-1} = \omega_{t+1} - b_{t+l}$.

Note that $|d(\omega_{t+l-1})| = |d\omega_{t+l-2}| = N + 1$, but by the hypothesis on $t$, we have:

$|d(\omega_{t+l-2} - b_{t+l})| = |a_{t+1}^{l-2} - \delta(b_{t+l}) - (d - \delta)b_{t+l}| = |(d - \delta)b_{t+l}| > N + 1$.

Then $d\omega_{t+l-1} = 0$ and so $\omega_{t+l-1}$ can't be a $d$-boundary. Indeed suppose that $\omega_{t+l-1} = (\omega_0^0 + \omega_1^0 + \ldots + \omega_r^0) - (b_2 + b_3 + \ldots + b_{t+l})$ were a $d$-boundary. By wordlength reasons, $\omega_0^0$ would be a $\delta$-boundary, i.e., there is $x \in \Lambda^{2p-2}V \oplus \Lambda^{2p-1}V$ such that $\delta(x) = \omega_0^0$. Then

$$\omega_0 = \delta(x) + \omega_0^1 + \ldots + \omega_r^l$$

Since $\delta(\omega_0^0) = 0$, we would have $\delta(\omega_0^0 + \ldots + \omega_r^0) = 0$, but $\omega_0^0 + \ldots + \omega_r^l$ is not a $\delta$-boundary.

Thus $\omega_{t+l-1}$ is a non trivial cocycle of degree $N$, the formal dimension, and therefore it represents the fundamental class.

Finally, since $\omega_{t+l-1} \in \Lambda^{2r}V$ we have:

$$e_0(\Lambda V, d) \geq r$$

For the second inequality

Denote $s = e_0(\Lambda V, d)$ and let $\omega \in \Lambda^{2s}V$ be a cocycle representing the generating class $\alpha$ of $H^*(\Lambda V, d)$. Write $\omega = \omega_0 + \omega_1 + \ldots + \omega_t$, $\omega_i \in \Lambda^{s+i}V$. We deduce that:

$$d\omega = (d_3\omega_0 + d_3\omega_1 + \ldots + d_3\omega_r) + (d_4\omega_0 + d_4\omega_1 + \ldots + d_4\omega_t) + \ldots$$

$$= \delta(\omega_0, \omega_1) + \ldots$$

Since $d\omega = 0$, by wordlength reasons, it follows that $\delta(\omega_0, \omega_1) = 0$.

If $(\omega_0, \omega_1)$ were a $\delta$-boundary, i.e., $(\omega_0, \omega_1) = \delta(x)$, then

$$\omega - dx = (\omega_0, \omega_1) + \ldots + \omega_t - (d_3x + d_4x + \ldots)$$

$$= (\omega_0, \omega_1) - \delta(x) + (\omega_2 + \omega_3 + \ldots + \omega_t) - \ldots$$

so $\omega - dx \in \Lambda^{2s+2}V$ which contradicts the fact $s = e_0(\Lambda V, d)$.

Hence $(\omega_0, \omega_1)$ represents the generating class of $H^N(\Lambda V, \delta)$.

Since $(\omega_0, \omega_1) \in \Lambda^{2s}V$ we will have $s \leq r$

Hence

$$e_0(\Lambda V, d) \leq r$$
We conclude that
\[ e_0(\Lambda V, d) = r \]

5. SOME EXAMPLES AND REMARKS

Example 1. Let \((\Lambda V, d)\) be the pure model defined by \(V^{\text{even}} = \langle x_2, x_6 \rangle\), \(V^{\text{odd}} = \langle y_5, y_{15}, y_{23} \rangle\), \(dx_2 = dx_6 = 0\), \(dy_5 = x_3^2\), \(dy_{15} = x_2^2 x_6^3\) and \(dy_{23} = x_6^4\).

Clearly, we have \(\dim H(\Lambda V, d_3) = \infty\) and \(\dim H(\Lambda V, d) < \infty\).

We note also that, since \(N = 37\) is odd, then any representative of the fundamental class of \((\Lambda V, d)\) will be of the form: \(n_1 x_2^k x_6^l y_5 + n_2 x_2^k' x_6^l' y_{15} + n_3 x_2^{k''} x_6^{l''} y_{23}\), with \(n_1, n_2\) and \(n_3 \in \mathbb{N}\).

Using A. Murillo’s algorithm (cf. §2) the matrix determining the fundamental class is:
\[
A = \begin{pmatrix}
x_2^2 & 0 \\
x_2 x_6^3 & 0 \\
0 & x_6^3
\end{pmatrix}
\]

So \(\omega_0 = -x_2^2 x_6^3 y_{15} + x_2 x_6^5 y_5 \in \Lambda^{\geq 0} V\) is an generator of this fundamental cohomology class. As in the first example, it is straightforward to verify that there is only two representatives, with \(\omega_1 = x_2^4 x_6 y_{23} - x_2^3 x_6^3 y_{15}\) being the second one. It follow that \(e_0(\Lambda V, d) = 6\).

Remark also that for this model, \((\omega_0, \omega_0) = (-x_2^2 x_6^3 y_{15}, x_2 x_6^5 y_5) \in \Lambda^6 V \oplus \Lambda^7 V\) is a \(\delta\)-cocycle and in fact \([[(\omega_0, \omega_0)]]) \in H^N(\Lambda V, \delta)\) is non zero. \([[x_2^2 x_6^3 y_{23}, 0]]\) is another generating class, hence \(\dim H^N(\Lambda V, \delta) > 1\). The algorithm described in remark 1 is applied to \((\omega_0^0, \omega_0^1)\).

On the other hand, \(\omega_0\) is not an \(d_3\)-cocycles, but \(0 \neq [\omega_1] \in H^N(\Lambda V, d_3)\). Also \(0 \neq [x_2 x_6^5 y_{23}]\) is another generating class of \(H^N(\Lambda V, d_3)\), hence \(\dim H^N(\Lambda V, d_3) > 1\). Application of the algorithm in the proof of Theorem 5 in \(\boxplus\) to \(\omega_1\) (which is a homogenous \(d_3\)-cocycle) gives immediately \(\omega_1\) as a representative of the fundamental class of \((\Lambda V, d)\).

Finally we also that \(e_0(\Lambda V, d) = 6 \neq (k - 2) \dim V^{\text{even}} + \dim V^{\text{odd}} = 5\).

Example 2. Let \((\Lambda V, d)\) be the pure model defined by \(V^{\text{even}} = \langle x_2, x_6 \rangle\), \(V^{\text{odd}} = \langle y_5, y_{13}, y_{23} \rangle\), \(dx_2 = dx_6 = 0\), \(dy_5 = x_3^2\), \(dy_{13} = x_2 x_6^2\) and \(dy_{23} = x_6^4\).

Clearly, we have \(\dim H(\Lambda V, d_3) = \infty\) and \(\dim H(\Lambda V, d) < \infty\).

We note also that, since \(N = 35\) is odd, then any representative of the fundamental class of \((\Lambda V, d)\) will be of the form: \(n_1 x_2^k x_6^l y_5 + n_2 x_2^k' x_6^l' y_{13} + n_3 x_2^{k''} x_6^{l''} y_{23}\), with \(n_1, n_2\) and \(n_3 \in \mathbb{N}\).
Using A. Murillo’s algorithm (cf. §2) the matrix determining the fundamental class is:

\[
A = \begin{pmatrix}
 x_2^2 & 0 \\
 x_6^2 & 0 \\
 0 & x_6^3 \\
\end{pmatrix}
\]

So \(\omega_0 = -x_2^2 x_6^3 y_13 + x_6^5 y_5 \in \Lambda^{\geq 6} V\) is a generator of this fundamental cohomology class. Another representative of this class is \(\omega_1 = -x_2^2 x_6 y_23 + x_6^2 x_6^3 y_13\). It is a straightforward calculation to prove that they are the unique representatives.

We conclude that \(e_0(\Lambda V, d) = 6\).

On the other hand \(H^N(\Lambda V, d)\) has at least two generators: \((\omega_0, 0) \in \Lambda^6 V \oplus \Lambda^7 V\) and \([[0, x_6^2 y_23]]\), hence \(\dim H^N(\Lambda V, \delta) > 1\). We have also \(\dim H^N(\Lambda V, d_3) > 1\) with \([\omega_0]\) and \([x_6^2 y_23]\) being two generators of \(H^N(\Lambda V, d_3)\). Here the algorithm is applied to \((\omega_0, 0)\) and the one of [11] is applied to \([\omega_0]\).

Note finally that \(e_0(\Lambda V, d) = 6 \neq (k - 2) \dim V^{even} + \dim V^{odd} = 5\).

**Remark 2.** It should be noted that the algorithms that are described in [11] and in Remark 1 are both valid in the previous examples. The privilege of one or the other depends on \(\dim H^N(\Lambda V, \delta)\) and \(\dim H^N(\Lambda V, d_3)\) and also in the expressions of there basis.

On the other hand all the lower bounds for \(e_0(\Lambda V, d)\) known up to now can be used to relax the application of the algorithm.

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