The Fuzzy Arithmetic Operations of Transmission Average on Pseudo-Hexagonal Fuzzy Numbers and Its Application in Fuzzy System Reliability Analysis

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\section*{ABSTRACT}
In most natural science fields, triangular and trapezoidal fuzzy numbers are commonly used while in engineering and social science fields such as sociology and psychology while treating the uncertainties these numbers are not applicable and fuzzy numbers with more parameters and clear definitions of their arithmetic operations are needed. In order to fill this gap in the literature, we propose the new fuzzy arithmetic operations based on transmission average on pseudo-hexagonal fuzzy numbers, which was already implied in [1] in its rudimentary form and was finally presented in its fully fledged form in [2]. Several illustrative examples were given to show the accomplishment and ability of the proposed method. We present a new method for fuzzy system reliability analysis based on the fuzzy arithmetic operations of transmission average, where the reliabilities of the components of a system are represented by pseudo-hexagonal fuzzy numbers defined in the universe of discourse [0, 1]. The proposed method has the advantages of modeling and analyzing fuzzy system reliability in a more flexible and more intelligent manner. Finally, a marine power plant is considered in fuzzy environment. The reliability of components of the proposed model is considered as pseudo-hexagonal fuzzy numbers.

\section*{1. Introduction}
In most of the cases in social sciences triangular and trapezoidal fuzzy numbers [3,4] may not be enough to measure the attributes usually associated with opinions leading to an ordinal information which can be represented by more than four different points on the real line. Therefore, even the trapezoidal fuzzy numbers can not be enough to represent such cases arising from social science measurements. In this study, we propose the notion of pseudo-octagonal fuzzy numbers in order to fill this gap in literature.

As regards fuzzy arithmetic operations using the extension principle (in the domain of the membership function) or the interval arithmetics (in the domain of the $\alpha$-cuts), we have some problem in subtraction operator, division operator and obtaining the membership
functions of operators. Although with the revised definitions on subtraction and division, usage of interval arithmetic for fuzzy operators have been permitted, because it always exists, but it’s not efficient, it means that result’s support is a major agent (dependence effect) and also complex calculations of interval arithmetic in determining the membership function of operators based on the extension principle, are not yet resolved. Therefore, we eliminated such deficiency with the new fuzzy arithmetic operations based on transitional spaces as TA [1,2]. In this paper, we propose the new fuzzy arithmetic operations based on TA on pseudo-hexagonal fuzzy numbers.

The aim of this study is twofold: At first, providing the fuzzy arithmetic operations of transmission average on pseudo-hexagonal fuzzy numbers and then use such fuzzy numbers with the proposed operations in fuzzy system reliability analysis. Reliability analysis is an important topic in engineering. Reliability is the probability that a component will not fail to perform within specified limits in a given time. In order to handle the insufficient information, the fuzzy approach is used to evaluate. Therefore fuzzy theory opened the way for facilitating the modeling and fuzziness aspect of system reliability. Several investigators pay attention to applying the fuzzy sets theory to reliability analysis [5–10].

In this paper, we have considered the failure distribution of the components with the pseudo-hexagonal fuzzy numbers based on the fuzzy arithmetic operations of TA. This is more flexible and more general than above mentioned methods including the interval arithmetic and $\alpha$-cuts are used to evaluate fuzzy system reliability.

The paper is organized as follows. In Section 2, we present the fuzzy arithmetic operations of transmission average on pseudo-hexagonal fuzzy numbers. The analysis of fuzzy system reliability is investigated in Section 3. A technical example to illustrate applying the method and comparing the results of the new method with the previous methods are given in Section 4. Finally, conclusions and future research are drawn in Section 5.

2. The Fuzzy Arithmetic Operations of TA on Pseudo-Hexagonal Fuzzy Numbers

At first, we propose the new fuzzy arithmetic operations based on transmission average on pseudo-hexagonal fuzzy numbers, which was already implied in [1] in its rudimentary form and was finally presented in its fully-fledged form in [2]. The properties of these proposed operations and their fundamental qualities are discussed. Several illustrative examples were given to show the accomplishment and ability of the proposed method.

2.1. Preliminaries and Notations

In this subsection, some notations and background about the concept are brought.

**Definition 2.1.1**: [11] Let $X = X_1 \times \cdots \times X_n$ be the Cartesian product of universes, and $A = A_1 \times \cdots \times A_n$ be fuzzy sets in each universe respectively. Also let $Y$ be another universe and $B \in Y$ be a fuzzy set such that $B = f(A_1 \times \cdots \times A_n)$, where $f : X \to Y$ is a monotonic mapping. Then Zadeh’s EP is defined as follows:

\[
\mu_B(y) = \sup \min (\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_1)) \quad (x_1 \times \cdots \times x_n) \in f^{-1}(y) \tag{1}
\]

where $f^{-1}(y)$ is the inverse function of $y = f(x_1 \times \cdots \times x_n)$. 
**Definition 2.1.2:** [11] (α -cut Representation) A fuzzy set, \( A \) can be represented (decomposed) as,

\[
A = \bigcup_{\alpha \in (0,1]} \alpha A_{\alpha}, \quad A_{\alpha} = [A_{\alpha}^l, A_{\alpha}^u].
\]  

(2)

where

\[
A_{\alpha} = \{x | \mu_A(x) \geq \alpha\},
\]

\[
\alpha A_{\alpha} = \{(x, \alpha) | x \in A_{\alpha}\}.
\]

(3)

It is easy to check that the following holds:

\[
\mu_A(x) = \sup_{x \in A_{\alpha}} \alpha.
\]

(4)

**Definition 2.1.3:** [12] (Fuzzy number) A fuzzy set \( A \) in \( \mathbb{R} \) is called a fuzzy number if it satisfies the following conditions:

(i) \( A \) is normal,

(ii) \( A_{\alpha} \) is a closed interval for every \( \alpha \in (0, 1] \),

(iii) the support of \( A \) is bounded.

According to the definition of fuzzy number mentioned above and our emphasis on pseudo-geometric fuzzy numbers, we define a pseudo- hexagonal fuzzy number as follows:

**Definition 2.1.4:** (Pseudo- hexagonal fuzzy numbers) A fuzzy number \( \tilde{A} \) is called a Pseudo-hexagonal fuzzy number if its membership function \( \mu_{\tilde{A}}(x) \) is given by

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
 l_{\tilde{A}^1}(x), & a_1 \leq x \leq a_2, \\
 l_{\tilde{A}^2}(x), & a_2 \leq x \leq \tilde{a}, \\
 1, & \tilde{a} \leq x \leq a_3, \\
 r_{\tilde{A}^2}(x), & \tilde{a} \leq x \leq a_3, \\
 r_{\tilde{A}^1}(x), & a_3 \leq x \leq a_4, \\
 0, & \text{otherwise}.
\end{cases}
\]

(5)

Where the pair of functions \((l_{\tilde{A}^1}(x), l_{\tilde{A}^2}(x))\) and \((r_{\tilde{A}^2}(x), r_{\tilde{A}^1}(x))\) are nondecreasing and nonincreasing functions, respectively. Also,

\[
l_{\tilde{A}^1}(a_2) = l_{\tilde{A}^2}(a_2) = \frac{1}{2},
\]

\[
r_{\tilde{A}^2}(a_3) = r_{\tilde{A}^1}(a_3) = \frac{1}{2}.
\]

(6)

The pseudo- hexagonal fuzzy number \( \tilde{A} \) is denoted by

\[
(a_1, a_2, \tilde{a}, a_3, a_4, (l_{\tilde{A}^1}(x), l_{\tilde{A}^2}(x)), (r_{\tilde{A}^2}(x), r_{\tilde{A}^1}(x)),
\]

(7)
and the hexagonal fuzzy number by
\[(a_1, a_2, \bar{a}, a_3, a_4, (\bar{a} - a_1), (\bar{a} - a_3), (\bar{a} - a_4))\],
that, \((- ,\cdot), (\cdot ,\cdot), (\cdot ,\cdot)\) means \(l_{\bar{A}_1}(x), l_{\bar{A}_2}(x), r_{\bar{A}_1}(x)\) and \(r_{\bar{A}_1}(x)\) are linear.

**Definition 2.1.5:** [13] (Fuzzy arithmetic operations based on interval arithmetic(\(\alpha\)-cut)) A popular way to carry out fuzzy arithmetic operations is by way of interval arithmetic. This is possible because any \(\alpha\)-cut of a fuzzy number is always an interval. Therefore, any fuzzy number may be represented as a series of intervals. Let us consider two interval numbers \([a,b]\) and \([c,d]\) where \(a \leq b\) and \(c \leq d\). Then the following arithmetic operations proceed as shown below:

(i) addition: \([a, b] + [c, d] = [a + c, b + d]\),
(ii) subtraction: \([a, b] - [c, d] = [a - d, b - c]\),
(iii) multiplication: \([a, b] \cdot [c, d] = [m \in [a, c], a d, b c, b d], m a x [a c, a d, b c, b d]\),
(iv) division: \([a, b]/[c, d] = [a, b] \cdot [1/d, 1/c]\).

### 2.2. The new Fuzzy Arithmetic Operations based on TA

As regards fuzzy arithmetic operations using of the extension principle (in the domain of the membership function) or the interval arithmetics (in the domain of the \(\alpha\)-cuts), we have some problem in subtraction operator, division operator and obtaining the membership functions of operators. Although with the revised definitions on subtraction and division, usage of an interval arithmetic for fuzzy operators have been permitted, because it always exists, it means that results support is major agent (dependence effect) and also complex calculations of interval arithmetic in determining the membership function of operators based on the extension principle, are not yet resolved. Therefore, we eliminated such deficiency with the fuzzy arithmetic operations based on TA [1].

We define fuzzy arithmetic operations based on TA for addition, subtraction, multiplication and division on pseudo-hexagonal fuzzy numbers as follows:

Consider two pseudo- hexagonal fuzzy numbers,

\[
\bar{A} = (a_1, a_2, \bar{a}, a_3, a_4, (l_{\bar{A}_1}(x), l_{\bar{A}_2}(x)), (r_{\bar{A}_1}(x), r_{\bar{A}_2}(x)), a = \frac{a + \bar{a}}{2},
\]

\[
\bar{B} = (b_1, b_2, \bar{b}, b_3, b_4, (l_{\bar{B}_1}(x), l_{\bar{B}_2}(x)), (r_{\bar{B}_1}(x), r_{\bar{B}_2}(x)), b = \frac{b + \bar{b}}{2},
\]

with the following \(\alpha\)-cut forms:

\[
\bar{A} = \bigcup_{\alpha \in (0, \frac{1}{2})} \alpha A_{11\alpha} \bigcup \left( \bigcup_{\alpha \in \left[\frac{1}{2}, 1\right]} \alpha A_{22\alpha} \right),
\]

\[
A_{11\alpha} = [l_{\bar{A}_1}^{-1}(\alpha), r_{\bar{A}_1}^{-1}(\alpha)], A_{22\alpha} = [l_{\bar{A}_2}^{-1}(\alpha), r_{\bar{A}_2}^{-1}(\alpha)],
\]

\[
\bar{B} = \bigcup_{\alpha \in (0, \frac{1}{2})} \alpha B_{11\alpha} \bigcup \left( \bigcup_{\alpha \in \left[\frac{1}{2}, 1\right]} \alpha B_{22\alpha} \right), B_{11\alpha} = [l_{\bar{B}_1}^{-1}(\alpha), r_{\bar{B}_1}^{-1}(\alpha)], B_{22\alpha} = [l_{\bar{B}_2}^{-1}(\alpha), r_{\bar{B}_2}^{-1}(\alpha)].
\]
Then, 

(i) addition,

\[
\widetilde{A + B} = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha \cdot (A + B)_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in \left[\frac{1}{2}, 1\right]} \alpha \cdot (A + B)_{22\alpha} \right),
\]

where

\[
(A + B)_{11\alpha} = \left[ \frac{a + b}{2} + \left( \frac{l_{A1}^{-1} (\alpha) + l_{B1}^{-1} (\alpha)}{2} \right), \frac{a + b}{2} + \left( \frac{r_{A1}^{-1} (\alpha) + r_{B1}^{-1} (\alpha)}{2} \right) \right],
\]

\[
(A + B)_{22\alpha} = \left[ \frac{a + b}{2} + \left( \frac{l_{A2}^{-1} (\alpha) + l_{B2}^{-1} (\alpha)}{2} \right), \frac{a + b}{2} + \left( \frac{r_{A2}^{-1} (\alpha) + r_{B2}^{-1} (\alpha)}{2} \right) \right],
\]

(ii) subtraction,

Firstly,

\[
\widetilde{B} = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha \cdot (-B)_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in \left[\frac{1}{2}, 1\right]} \alpha \cdot (-B)_{22\alpha} \right),
\]

where

\[
(-B)_{11\alpha} = [-2b + l_{B1}^{-1} (\alpha), -2b + r_{B1}^{-1} (\alpha)],
\]

\[
(-B)_{22\alpha} = [-2b + l_{B2}^{-1} (\alpha), -2b + r_{B2}^{-1} (\alpha)],
\]

finally,

\[
\widetilde{A - B} = \widetilde{A} + \widetilde{-B},
\]

\[
\widetilde{A - B} = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha \cdot (A - B)_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in \left[\frac{1}{2}, 1\right]} \alpha \cdot (A - B)_{22\alpha} \right),
\]

\[
(A - B)_{11\alpha} = \left[ \frac{a - 3b}{2} + \left( \frac{l_{A1}^{-1} (\alpha) + l_{B1}^{-1} (\alpha)}{2} \right), \frac{a - 3b}{2} + \left( \frac{r_{A1}^{-1} (\alpha) + r_{B1}^{-1} (\alpha)}{2} \right) \right],
\]

\[
(A - B)_{22\alpha} = \left[ \frac{a - 3b}{2} + \left( \frac{l_{A2}^{-1} (\alpha) + l_{B2}^{-1} (\alpha)}{2} \right), \frac{a - 3b}{2} + \left( \frac{r_{A2}^{-1} (\alpha) + r_{B2}^{-1} (\alpha)}{2} \right) \right],
\]

(iii) multiplication,

\[
\widetilde{A \cdot B} = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha \cdot (A - B)_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in \left[\frac{1}{2}, 1\right]} \alpha \cdot (A - B)_{22\alpha} \right),
\]
where

\[
\begin{align*}
(A.B)_{11\alpha} &= \begin{cases} 
\left[ \frac{b}{2} r^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) f^{-1}_{B1} (\alpha), \left( \frac{b}{2} \right) r^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B1} (\alpha) \right], & a \geq 0, b \geq 0 \\
\left[ \frac{b}{2} r^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) f^{-1}_{B1} (\alpha), \left( \frac{b}{2} \right) r^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B1} (\alpha) \right], & a \geq 0, b < 0, \\
\left[ \frac{b}{2} r^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B1} (\alpha), \left( \frac{b}{2} \right) r^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) l^{-1}_{B1} (\alpha) \right], & a \leq 0, b \leq 0, \\
\left[ \frac{b}{2} l^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B1} (\alpha), \left( \frac{b}{2} \right) l^{-1}_{A1} (\alpha) + \left( \frac{a}{2} \right) l^{-1}_{B1} (\alpha) \right], & a \leq 0, b \geq 0,
\end{cases}
\end{align*}
\]

(17)

\[
\begin{align*}
(A.B)_{22\alpha} &= \begin{cases} 
\left[ \frac{b}{2} r^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) f^{-1}_{B2} (\alpha), \left( \frac{b}{2} \right) r^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B2} (\alpha) \right], & a \geq 0, b \geq 0 \\
\left[ \frac{b}{2} r^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) f^{-1}_{B2} (\alpha), \left( \frac{b}{2} \right) r^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B2} (\alpha) \right], & a \geq 0, b < 0, \\
\left[ \frac{b}{2} r^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B2} (\alpha), \left( \frac{b}{2} \right) l^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) l^{-1}_{B2} (\alpha) \right], & a \leq 0, b \leq 0, \\
\left[ \frac{b}{2} l^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) r^{-1}_{B2} (\alpha), \left( \frac{b}{2} \right) l^{-1}_{A2} (\alpha) + \left( \frac{a}{2} \right) l^{-1}_{B2} (\alpha) \right], & a \leq 0, b \geq 0,
\end{cases}
\end{align*}
\]

(18)

(iv) division,

Firstly,

\[
\overline{B}^{-1} = \left( \bigcup_{\alpha \in (0, 1]} \alpha \cdot (B^{-1})_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha \cdot (B^{-1})_{22\alpha} \right),
\]

(19)

where

\[
(B^{-1})_{11\alpha} = \left[ \frac{1}{b^2} f^{-1}_{B1} (\alpha), \frac{1}{b^2} r^{-1}_{B1} (\alpha) \right],
\]

(20)

\[
(B^{-1})_{22\alpha} = \left[ \frac{1}{b^2} f^{-1}_{B2} (\alpha), \frac{1}{b^2} r^{-1}_{B2} (\alpha) \right],
\]

(21)

finally,

\[
\overline{A \cdot B}^{-1} = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha \cdot (A \cdot B^{-1})_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha \cdot (A \cdot B^{-1})_{22\alpha} \right),
\]

(22)

\[
\begin{align*}
(\overline{A \cdot B}^{-1})_{11\alpha} &= \begin{cases} 
\left[ \frac{1}{2b} f^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) f^{-1}_{B1} (\alpha), \left( \frac{1}{2b} \right) r^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) r^{-1}_{B1} (\alpha) \right], & a \geq 0, b > 0 \\
\left[ \frac{1}{2b} r^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) f^{-1}_{B1} (\alpha), \left( \frac{1}{2b} \right) r^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) r^{-1}_{B1} (\alpha) \right], & a \geq 0, b < 0, \\
\left[ \frac{1}{2b} r^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) r^{-1}_{B1} (\alpha), \left( \frac{1}{2b} \right) l^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) l^{-1}_{B1} (\alpha) \right], & a \leq 0, b < 0, \\
\left[ \frac{1}{2b} l^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) r^{-1}_{B1} (\alpha), \left( \frac{1}{2b} \right) l^{-1}_{A1} (\alpha) + \left( \frac{a}{2b^2} \right) l^{-1}_{B1} (\alpha) \right], & a \leq 0, b > 0,
\end{cases}
\end{align*}
\]

(23)
Remark 2.2.1: The division operation on the pseudo-hexagonal fuzzy number

\[ \tilde{0} = (a_1, a_2, -a, a, a_3, a_4, l_{01}(x), l_{02}(x), r_{01}(x), r_{02}(x)), a > 0, \]

is not able to be defined.

Remark 2.2.2: Since the pseudo-hexagonal fuzzy numbers are a special case of pseudo-geometric fuzzy numbers, we have the lemma and theorems from the ref. [1] for the pseudo-hexagonal fuzzy numbers.

2.3. Numerical Examples

In this subsection, we provided several numerical samples to illustrate the application of the proposed method on pseudo-hexagonal and hexagonal fuzzy numbers. We also compared the results of the new method with the previous methods.

Example 2.3.1: In this example, we compare the results TA method with EP (\( \alpha \)-cut) method. Let

\[ A = (1, 2, 4, 6, 7, 9, (-, -), (-, -)), \]
\[ B = (4, 5, 8, 9, 10, 12, (-, -), (-, -)), \]

with the following \( \alpha \)-cut forms: (See Figure 1)

\[ A = \left( \bigcup_{\alpha \in [0, 1]} \alpha A_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in [1, 2]} \alpha A_{22\alpha} \right), A_{11\alpha} = [2\alpha + 1, -4\alpha + 9], \]
\[ A_{22\alpha} = [4\alpha, -2\alpha + 8], \]
\[ B = \left( \bigcup_{\alpha \in [0, 1]} \alpha B_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in [1, 2]} \alpha B_{22\alpha} \right), B_{11\alpha} = [2\alpha + 4, -4\alpha + 12], \]
\[ B_{22\alpha} = [6\alpha + 2, -2\alpha + 11]. \]

Then using the elementary fuzzy arithmetic operations based on the EP (\( \alpha \)-cut) and TA, we get:
1. Based on the EP ($\alpha$ -cut):

$$A + B = (5, 7, 12, 15, 21, (-, -), (-, -)), -$$

$$B = (-12, -10, -9, -8, -5, -4, (-, -), (-, -)), A - B$$

$$= (-11, -8, -5, -2, 2, 5, (-, -), (-, -)),$$

$$A.B = (4, 10, 32, 54, 70, 108(-, -), (-, -)), B^{-1}$$

$$= \left( \frac{1}{12}, \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{5}, \frac{1}{4}, (-, -), (-, -) \right),$$

$$A.B^{-1} = \left( \frac{1}{12}, \frac{2}{10}, \frac{4}{9}, \frac{6}{8}, \frac{7}{9}, \frac{1}{4}, (-, -), (-, -) \right).$$

2. Based on the TA:

$$A + B = \left( \frac{37}{4}, \frac{41}{4}, \frac{51}{4}, \frac{57}{4}, \frac{61}{4}, \frac{69}{4}, (-, -), (-, -) \right), -$$

$$B = (-13, -12, -9, -8, -7, -5, (-, -), (-, -)), A - B$$

$$= \left( \frac{-31}{4}, \frac{-27}{4}, \frac{-27}{4}, \frac{-7}{4}, \frac{-1}{4}, (-, -), (-, -) \right), A.B$$

$$= \left( \frac{57}{4}, \frac{21}{4}, \frac{37}{4}, \frac{27}{4}, \frac{27}{4}, (-, -), (-, -) \right), B^{-1}$$

$$= \left( \frac{16}{172}, \frac{20}{172}, \frac{32}{172}, \frac{36}{172}, \frac{40}{172}, \frac{48}{172}, (-, -), (-, -) \right), A.B^{-1}$$

$$= \left( \frac{57}{172}, \frac{84}{172}, \frac{148}{172}, \frac{192}{172}, \frac{219}{172}, \frac{273}{172}, (-, -), (-, -) \right).$$

The graphical comparison is shown in Figures 2-5.

**Example 2.3.2:** Let

$$A = \left( \bigcup_{\alpha \in \left(0,1\right]} \alpha \cdot A_{11\alpha} \right) \cup \left( \bigcup_{\alpha \in \left[\frac{1}{2},1\right]} \alpha \cdot A_{22\alpha} \right),$$

$$A_{11\alpha} = \left[ 2 - 4\sqrt{4 - 4\alpha}, 4 + \sqrt{16 - 16\alpha} \right], A_{22\alpha} = \left[ 2 - 8\sqrt{2(1 - \alpha)}, 4 - 4\sqrt{2}(\alpha - 1) \right],$$

$$B = \left( \bigcup_{\alpha \in \left(0,1\right]} \alpha \cdot B_{11\alpha} \right) \cup \left( \bigcup_{\alpha \in \left[\frac{1}{2},1\right]} \alpha \cdot B_{22\alpha} \right),$$

$$B_{11\alpha} = \left[ \frac{5}{2} \alpha - 1, \frac{33}{4} - \frac{7}{2} \left( \alpha - \frac{1}{2} \right) \right], B_{22\alpha} = \left[ (\alpha + 1)^2 - 2, (3 - \alpha)^2 + 2 \right].$$

Then using the elementary fuzzy arithmetic operations based on the TA, we get:

$$\widetilde{A + B} = \left( \bigcup_{\alpha \in \left(0,1\right]} \alpha \cdot (A + B)_{11\alpha} \right) \cup \left( \bigcup_{\alpha \in \left[\frac{1}{2},1\right]} \alpha \cdot (A + B)_{22\alpha} \right),$$

(25)
Figure 1. The hexagonal fuzzy numbers of example 2.3.1.

Figure 2. The red graph based on the extension principle ($\alpha$-cut). The black graph is based on the transmission average.

where

$$\begin{align*}
(A + B)_{11\alpha} &= \left[ \frac{3 + 4}{2} + \left( \frac{2 - 4\sqrt{4 - 4\alpha + \frac{3}{2}\alpha - 1}}{2} \right), \frac{3 + 4}{2} \\
&+ \left( \frac{4 + \sqrt{16 - 16\alpha + \frac{33}{4} - \frac{7}{2}(\alpha - \frac{1}{2})}}{2} \right) \right],
\end{align*}$$

(26)
\( (A + B)_{22\alpha} = \left[ \frac{3 + 4}{2} + \left( \frac{2 - 8\sqrt{2}(1 - \alpha) + (\alpha + 1)^2 - 2}{2} \right), \frac{3 + 4}{2} + \left( \frac{4 - 4\sqrt{2}(\alpha - 1) + (3 - \alpha)^2 + 2}{2} \right) \right] \) \hspace{0.5cm} (27)

\( \tilde{B} = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha \cdot (B)_{11\alpha} \right) \bigcup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha \cdot (B)_{11\alpha} \right) \), \hspace{0.5cm} (28)

where

\( (B)_{11\alpha} = \left[ -2 \times 4 + \frac{5}{2} \alpha - 1, -2 \times 4 + \frac{33}{4} - \frac{7}{2} \left( \alpha - \frac{1}{2} \right) \right] \), \hspace{0.5cm} (29)
Figure 5. The red graph based on the extension principle ($\alpha$-cut). The black graph is based on the transmission average.

\[
(-B)_{22\alpha} = [-2 \times 4 + (\alpha + 1)^2 - 2, -2 \times 4 + (3 - \alpha)^2 + 2], \tag{30}
\]

\[
\tilde{A} - B = A + (-B),
\]

\[
\tilde{A} - B = \left( \bigcup_{\alpha \in (0, 1]} \alpha.\tilde{(A - B)_{11\alpha}} \right) \bigcup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha.\tilde{(A - B)_{22\alpha}} \right), \tag{31}
\]

where

\[
(A - B)_{11\alpha} = \left[ \frac{3 - 3 \times 4}{2} + \left( \frac{2 - 4\sqrt{4 - 4\alpha} + 5\alpha - 1}{2} \right), \frac{3 - 3 \times 4}{2} \right.
\]

\[
+ \left( \frac{4 + \sqrt{16 - 16\alpha} + \frac{33}{4} - \frac{7}{2} (\alpha - \frac{1}{2})}{2} \right), \tag{32}
\]

\[
(A - B)_{22\alpha} = \left[ \frac{3 - 3 \times 4}{2} + \left( \frac{2 - 8\sqrt{2(1 - \alpha)} + (\alpha + 1)^2 - 2}{2} \right), \frac{3 - 3 \times 4}{2} \right.
\]

\[
+ \left( \frac{4 - 4\sqrt{2(\alpha - 1) + (3 - \alpha)^2 + 2}}{2} \right) \right], \tag{33}
\]

\[
\tilde{A}.B = \left( \bigcup_{\alpha \in (0, 1]} \alpha.\tilde{(A.B)_{11\alpha}} \right) \bigcup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha.\tilde{(A.B)_{22\alpha}} \right), \tag{34}
\]

where

\[
(\tilde{A}.B)_{11\alpha} = \left[ \left( \frac{4}{2} \right) \left( 2 - 4\sqrt{4 - 4\alpha} \right) + \left( \frac{3}{2} \right) \left( \frac{5}{2} \alpha - 1 \right), \left( \frac{4}{2} \right) \left( 4 + \sqrt{16 - 16\alpha} \right) \right.
\]

\[
+ \left( \frac{3}{2} \right) \left( \frac{33}{4} - \frac{7}{2} (\alpha - \frac{1}{2}) \right), \tag{35}
\]
\[(\widetilde{A}B)_{22\alpha} = \left[\left(\frac{4}{2}\right)\left(2 - 8\sqrt{2}(1 - \alpha)\right) + \left(\frac{3}{2}\right)((\alpha + 1)^2 - 2), \left(\frac{4}{2}\right)\left(4 - 4\sqrt{2}(\alpha - 1)\right) + \left(\frac{3}{2}\right)((3 - \alpha)^2 + 2)\right]\]

\[B^{-1} = \left(\bigcup \alpha \in (0, \frac{1}{2}]^\alpha (B^{-1})_{11\alpha}\right) \bigcup \left(\bigcup \alpha \in [\frac{1}{2}, 1]^\alpha (B^{-1})_{22\alpha}\right),\]

where

\[(B^{-1})_{11\alpha} = \left[\frac{1}{4^2}\left(\frac{5}{2}\alpha - 1\right), \frac{1}{4^2}\left(\frac{33}{4} - \frac{7}{2}\left(\alpha - \frac{1}{2}\right)\right)\right],\]

\[(B^{-1})_{22\alpha} = \left[\frac{1}{4^2}\left((\alpha + 1)^2 - 2\right), \frac{1}{4^2}\left((3 - \alpha)^2 + 2\right)\right],\]

\[\widetilde{A}_B^{-1} = \left(\bigcup \alpha \in (0, \frac{1}{2}]^\alpha (A_B^{-1})_{11\alpha}\right) \bigcup \left(\bigcup \alpha \in [\frac{1}{2}, 1]^\alpha (A_B^{-1})_{22\alpha}\right),\]

\[
\begin{align*}
(\widetilde{A}_B^{-1})_{11\alpha} &= \left\{\left[\left(\frac{1}{2 \times 4}\right)\left(2 - 4\sqrt{4 - 4\alpha}\right) + \left(\frac{3}{2 \times 4^2}\right)\left(\frac{5}{2}\alpha - 1\right), \left(\frac{1}{2 \times 4}\right)\left(4 + \sqrt{16 - 16\alpha}\right) + \left(\frac{3}{2 \times 4^2}\right)\left(\frac{33}{4} - \frac{7}{2}\left(\alpha - \frac{1}{2}\right)\right)\right]\right\} \\
+ \left[\left(\frac{3}{2 \times 4^2}\right)\left(\frac{33}{4} - \frac{7}{2}\left(\alpha - \frac{1}{2}\right)\right)\right]
\end{align*}
\]

\[
(\widetilde{A}_B^{-1})_{22\alpha} = \left\{\left[\left(\frac{1}{2 \times 4}\right)\left(2 - 8\sqrt{2}(1 - \alpha)\right) + \left(\frac{3}{2 \times 4^2}\right)((\alpha + 1)^2 - 2), \left(\frac{1}{2 \times 4}\right)\left(4 - 4\sqrt{2}(\alpha - 1)\right) + \left(\frac{3}{2 \times 4^2}\right)((3 - \alpha)^2 + 2)\right]\right\}
\]

3. Reliability Analysis of Fuzzy System Using TA-based Arithmetic Operations

Using TA-based fuzzy number arithmetic operations, a new procedure for analyzing fuzzy system reliability is shown in this section; the reliability of each system component is denoted by a pseudo-hexagonal fuzzy number. This is a more flexible and more generic method than all the aforementioned methods (including the interval arithmetic), and \(\alpha\)-cuts are utilized for assessing fuzzy system reliability.

3.1. Fault Tree Analysis

A fault tree usually includes the top event, the basic events and the logic gates. Gates indicate relationships of events. While doing the system-design, fault tree (the logic diagram) is outlined for analysis of the potential factors in system failure; factors like hard-ware, soft-ware, environment, human factor. Based on the known combinations and probabilities of basic events, we calculate the probabilities of system failure.
3.2. Fuzzy Operators based on TA of Fault Tree Analysis

During the fuzzy fault tree analysis, the probabilities of basic events are described as fuzzy numbers and the traditional logic gate operators are replaced by fuzzy logic gate operators to obtain the fuzzy probability of the top event.

In this subsection, we present a new method for analyzing fuzzy system reliability based on TA, where the reliability of the components of a system is represented by pseudo-hexagonal fuzzy number.

Lemma 3.2.1: Let $A_1, A_2, \ldots, A_n$ be pseudo-hexagonal fuzzy numbers as follows:

$$A_i = (a_i, a_i, a_j, a_i, a_i, (l_{A_1}(x), l_{A_2}(x)), (r_{A_1}(x), r_{A_1}(x)), a_i = \frac{a_i + \tilde{a}_i}{2}, (a_i > 0)$$

with the following $\alpha$-cut form:

$$A_i = \left( \bigcup_{\alpha \in (0, 1]} \alpha A_{11i} \right) \cup \left( \bigcup_{\alpha \in [1, 2]} \alpha A_{22i} \right),$$

$$A_{11i} = \left[ l_{A_1}^{-1}(\alpha), r_{A_1}^{-1}(\alpha) \right], A_{22i} = \left[ l_{A_2}^{-1}(\alpha), r_{A_2}^{-1}(\alpha) \right],$$

then, (1)

$$\prod_{i=1}^{n} A_i = \left( \bigcup_{\alpha \in (0, 1]} \alpha A_{11i} \right) \cup \left( \bigcup_{\alpha \in [1, 2]} \alpha A_{22i} \right),$$

$$A_{11i} = \left[ l_{A_1}^{-1}(\alpha), r_{A_1}^{-1}(\alpha) \right], A_{22i} = \left[ l_{A_2}^{-1}(\alpha), r_{A_2}^{-1}(\alpha) \right],$$

where

$$l_{A_1}^{-1} (\alpha) = \sum_{k=0}^{n-2} \left( \frac{\prod_{i=1, j\neq n-k} a_i}{2^{k+1}} \right) l_{A_1}^{-1}(\alpha),$$

$$r_{A_1}^{-1} (\alpha) = \sum_{k=0}^{n-2} \left( \frac{\prod_{i=1, j\neq n-k} a_i}{2^{k+1}} \right) r_{A_1}^{-1}(\alpha),$$

$$l_{A_2}^{-1} (\alpha) = \sum_{k=0}^{n-2} \left( \frac{\prod_{i=1, j\neq n-k} a_i}{2^{k+1}} \right) l_{A_2}^{-1}(\alpha),$$

$$r_{A_2}^{-1} (\alpha) = \sum_{k=0}^{n-2} \left( \frac{\prod_{i=1, j\neq n-k} a_i}{2^{k+1}} \right) r_{A_2}^{-1}(\alpha).$$

$$1 - A_i = \left( \bigcup_{\alpha \in (0, 1]} \alpha \cdot (1 - A)_{11i} \right) \cup \left( \bigcup_{\alpha \in [1, 2]} \alpha \cdot (1 - A)_{22i} \right),$$

$$= [l_{(1 - A)_{11i}}^{-1}(\alpha), r_{(1 - A)_{11i}}^{-1}(\alpha)], (1 - A)_{22i} = [l_{(1 - A)_{22i}}^{-1}(\alpha), r_{(1 - A)_{22i}}^{-1}(\alpha)].$$
where

\[
I^{-1}_{(1-A)1}(\alpha) = \frac{1}{2} (2 - 3 a_i + I^{-1}_{A1}(\alpha)), r^{-1}_{(1-A)1}(\alpha) = \frac{1}{2} (2 - 3 a_i + r^{-1}_{A1}(\alpha)), I^{-1}_{(1-A)2i}(\alpha) = \frac{1}{2} (2 - 3 a_i + r^{-1}_{A2}(\alpha)).
\]

**Proof:** We have the above cases, by mathematical induction and according to the fuzzy arithmetic operations of TA on pseudo-hexagonal fuzzy numbers.

Consider a serial system shown in Figure 6, where the reliability \( R_i \) of component \( x_i \) is represented by a pseudo-hexagonal fuzzy number defined in the universe of discourse \([0, 1]\):

\[
R_i = (r_{i1}, r_{i2}, l_i, r_i, r_{i4}, (l_{R1i}(x), l_{R2i}(x)), (r_{R2i}(x), r_{R1i}(x)), r_i = \frac{l_i + r_i}{2}, (r_i > 0)
\]

or,

\[
R_i = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha R_{11i} \right) \cup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha R_{22i} \right), R_{11i} = \left[ I^{-1}_{R1i}(\alpha), r^{-1}_{R1i}(\alpha) \right], R_{22i} = \left[ I^{-1}_{R2i}(\alpha), r^{-1}_{R2i}(\alpha) \right].
\]

Then, the reliability \( R \) of the serial system can be evaluated by the (3.1) lemma as follows:

\[
R = R_1 \cdot R_2 \ldots R_n = \prod_{i=1}^{n} R_i = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha R_{11i} \right) \cup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha R_{22i} \right), R_{11i} = \left[ I^{-1}_{R1i}(\alpha), r^{-1}_{R1i}(\alpha) \right], R_{22i} = \left[ I^{-1}_{R2i}(\alpha), r^{-1}_{R2i}(\alpha) \right],
\]

where

\[
I^{-1}_{\prod_{i=1}^{n} R1i}(\alpha) = \sum_{k=0}^{n-2} \left( \prod_{i=1}^{n} R_{1i} \right) I^{-1}_{R1(n-k)}(\alpha) + \left( \prod_{i=2}^{n} r_i \right) I^{-1}_{R11}(\alpha),
\]

\[
r^{-1}_{\prod_{i=1}^{n} R1i}(\alpha)) = \sum_{k=0}^{n-2} \left( \prod_{i=1}^{n} R_{1i} \right) r^{-1}_{R1(n-k)}(\alpha) + \left( \prod_{i=2}^{n} r_i \right) r^{-1}_{R11}(\alpha),
\]

\[
I^{-1}_{\prod_{i=1}^{n} R2i}(\alpha) = \sum_{k=0}^{n-2} \left( \prod_{i=1}^{n} R_{2i} \right) I^{-1}_{R2(n-k)}(\alpha) + \left( \prod_{i=2}^{n} r_i \right) I^{-1}_{R21}(\alpha),
\]

\[
r^{-1}_{\prod_{i=1}^{n} R2i}(\alpha)) = \sum_{k=0}^{n-2} \left( \prod_{i=1}^{n} R_{2i} \right) r^{-1}_{R2(n-k)}(\alpha) + \left( \prod_{i=2}^{n} r_i \right) r^{-1}_{R21}(\alpha).
\]
The reliability of the parallel system can be evaluated as follows:

\[ R = 1 - \prod_{i=1}^{n} (1 - R_i) = \left( \bigcup_{\alpha \in (0,1]} \alpha \cdot R_{11\alpha} \right) \cup \left( \bigcup_{\alpha \in [1,2]} \alpha \cdot R_{22\alpha} \right), R_{11\alpha} \]

\[ = \left\{ \frac{1}{2} \left( 2 - 3 \left( 1 - \prod_{i=1}^{n} r_j \right) + \frac{r_1^{-1}}{\prod_{i=1}^{n} (1-R)1i} (\alpha) \right) \right\}, \frac{1}{2} \left( 2 - 3 \left( 1 - \prod_{i=1}^{n} r_j \right) + \frac{r_1^{-1}}{\prod_{i=1}^{n} (1-R)1i} (\alpha) \right) \right\}, R_{22\alpha} \]

Then, the reliability \( R \) of the parallel system can be evaluated as follows:

\[ R = 1 - \prod_{i=1}^{n} (1 - R_i) = \left( \bigcup_{\alpha \in (0,1]} \alpha \cdot R_{11\alpha} \right) \cup \left( \bigcup_{\alpha \in [1,2]} \alpha \cdot R_{22\alpha} \right), R_{11\alpha} \]

\[ = \left\{ \frac{1}{2} \left( 2 - 3 \left( 1 - \prod_{i=1}^{n} r_j \right) + \frac{r_1^{-1}}{\prod_{i=1}^{n} (1-R)1i} (\alpha) \right) \right\}, \frac{1}{2} \left( 2 - 3 \left( 1 - \prod_{i=1}^{n} r_j \right) + \frac{r_1^{-1}}{\prod_{i=1}^{n} (1-R)1i} (\alpha) \right) \right\}, R_{22\alpha} \]

where

\[ \prod_{i=1}^{n} (1 - R_i) = \left( \bigcup_{\alpha \in (0,1]} \alpha \cdot \prod_{i=1}^{n} (1 - R)_{11\alpha} \right) \cup \left( \bigcup_{\alpha \in [1,2]} \alpha \cdot \prod_{i=1}^{n} (1 - R)_{22\alpha} \right), R_{11\alpha} \]

\[ = \left\{ \frac{1}{2} \left( 2 - 3 \left( 1 - \prod_{i=1}^{n} r_j \right) + \frac{r_1^{-1}}{\prod_{i=1}^{n} (1-R)1i} (\alpha) \right) \right\}, \frac{1}{2} \left( 2 - 3 \left( 1 - \prod_{i=1}^{n} r_j \right) + \frac{r_1^{-1}}{\prod_{i=1}^{n} (1-R)1i} (\alpha) \right) \right\}, R_{22\alpha} \]

\[ l^{-1}_{\prod_{i=1}^{n} (1-R)1i} (\alpha) = \sum_{k=0}^{n-2} \left( \prod_{i=1, i\neq n-k}^{n} (1 - r_i) \right) \left( \prod_{i=1}^{n} (1-R)1(n-k) (\alpha) \right) \]

\[ r^{-1}_{\prod_{i=1}^{n} (1-R)1i} (\alpha) = \sum_{k=0}^{n-2} \left( \prod_{i=1, i\neq n-k}^{n} (1 - r_i) \right) \left( \prod_{i=1}^{n} (1-R)1(n-k) (\alpha) \right) \]
4. A Technical Example

A marine power plant [4] has two generators G1 and G2 one located at the stern and the other at the bow. Each generator is connected to its respective micro switch board-1 and micro switch board-2.

The distributive switchboard receives the supply from the switchboards through cables C1 and C2 and respective junction boxes D and E. The two micro switchboards are interconnected through a long cable C3 and the junction boxes A and B. The schematic diagram is shown in Figure 8.

Let us assume that basic components subjected to failure are

(a) Generators G1 and G2.
(b) Microswitch board-1 (MSB-1) and Microswitch board-2 (MSB-2).
(c) Interconnecting cable C3 and junction boxes A and B, all are treated as one unit.
(d) Junction boxes D and E.
(e) Distributive switchboard (DSB).
In this example, we show a failure of the marine power plant in the form of a fuzzy number for the more comprehensive analysis to improve the educational process. A fault tree for the top event ‘failure of the marine power plant’ is shown in Figure 9.

Due to a more accurate estimate of each failure event and generalized reliability analysis of the system shown in Figure 9, let us assume the basic events of this fault tree have the following pseudo-hexagonal fuzzy number defined in the universe of discourse $[0, 1]$:

$$R_i = \left( r_{i1}, r_{i2}, l_R, \tilde{l}_R, r_{i3}, r_{i4}, (l_{R_{11}}(x), l_{R_{22}}(x)), (r_{R_{12}}(x), r_{R_{11}}(x)), r_i = \frac{r_j + \tilde{r}_i}{2} \right),$$

or,

$$R_i = \left( \bigcup_{\alpha \in (0, \frac{1}{2}]} \alpha.R_{11i\alpha} \right) \cup \left( \bigcup_{\alpha \in [\frac{1}{2}, 1]} \alpha.R_{22i\alpha} \right), R_{11i\alpha} = \left[ l_{R_{11}}^{-1}(\alpha), r_{R_{11}}^{-1}(\alpha) \right],$$

$$R_{22i\alpha} = \left[ l_{R_{22}}^{-1}(\alpha), r_{R_{22}}^{-1}(\alpha) \right], i = 1, 2, \ldots, 22,$$

where

- $R_1$, represents the unreliability of the distributive switchboard.
- $R_2$, represents the reliability of the event that no power is coming to distributive switchboard.
- $R_3$, represents the reliability of the event that there is no power supply from the junction box D.
Figure 9. Fault tree of marine power plant.

$R_4$, represents the reliability of the event that there is no power supply from the junction box E.

$R_5$, represents the unreliability of the junction box D.

$R_6$, represents the reliability of the event that there is no power supply to the junction box D.

$R_7$, represents the unreliability of micro switchboard-1.

$R_8$, represents the reliability of the event that there is no power supply to micro switchboard-1.

$R_9$, represents the unreliability of generator G1.

$R_{10}$, represents the reliability of the event that there is no power supply through the junction boxes A and B.

$R_{11}$, represents the unreliability of generator G2.

$R_{12}$, represents the unreliability of the junction boxes A and B.

$R_{13}$, represents the unreliability of micro switchboard-2.
$R_{14}$ represents unreliability of the junction box E.

$R_{15}$ represents the reliability of the event that there is no power supply to the junction box E.

$R_{16}$ represents the unreliability of micro switchboard-2.

$R_{17}$ represents the reliability of the event that there is no power supply to micro switchboard-2.

$R_{18}$ represents the unreliability of generator G2.

$R_{19}$ represents the reliability of the event that there is no power supply through the junction boxes D and E.

$R_{20}$ represents the unreliability of generator G1.

$R_{21}$ represents the unreliability of the junction boxes A and B.

$R_{22}$ represents the unreliability of micro switchboard-1.

Based on the previous discussion (the reliability of the serial and parallel systems), we get a failure of the marine power plant ($R$) as follows:

$$R = 1 - [(1 - R_1)(1 - R_2)],$$

where

the calculation of $R_3$:

$$R_3 = 1 - [(1 - R_5)(1 - R_6)],$$

$$R_6 = 1 - [(1 - R_7)(1 - R_8)],$$

$$R_8 = R_9 \cdot R_{10},$$

$$R_{10} = 1 - [(1 - R_{11})(1 - R_{12})(1 - R_{13})],$$

the calculation of $R_4$:

$$R_4 = 1 - [(1 - R_{14})(1 - R_{15})],$$

$$R_{15} = 1 - [(1 - R_{16})(1 - R_{17})],$$

$$R_{17} = R_{18} \cdot R_{19},$$

$$R_{19} = 1 - [(1 - R_{20})(1 - R_{21})(1 - R_{22})].$$

Finally, we can calculate the system reliability $R$ by the (3.2.1) lemma. If we required the system to have a fault probability of $x_0$ as a limit, then, $\alpha \geq \alpha_0$ is necessary, where $\alpha_0 = \inf\{\alpha | x_0 \notin R_\alpha\}$. In this case, we allow the system to be uncertain and flexible to an extent that the fault probabilities be in the $R_{\alpha_0}$.

It is worth mentioning, the proposed model is applicable for every marine power plant with having the statistical data.

5. Conclusion and Future Research

Fuzzy reliability is based on the concept of fuzzy set. When the failure rate is fuzzy, according to Zadeh’s extension principle, the reliability measure will be fuzzy as well. In this paper, the
The use of the concept of pseudo-hexagonal fuzzy numbers and the component failure probabilities are considered as a new type of fuzzy number as pseudo-hexagonal to incorporate the uncertainties in the parameter, due to a more realistic estimate of them.

We used the new TA-operations [1,2], because of smaller results, easier computations and some particular properties. The developed method has been used to analyze the fuzzy reliability of a marine power plant. The major advantage of using the pseudo-hexagonal fuzzy numbers and the new operations of transmission average (TA), is the smaller results support, easier calculations and special properties than fuzzy arithmetic operations based on the extension principle (in the domain of the membership function) and the interval arithmetic (in the domain of the $\alpha$-cuts). The proposed methodology can be used for a more general problem when systems are distributed according to other fuzzy numbers. The future work of this study will focus on the fuzzy arithmetic operations based on TA for $n$-polygonal fuzzy numbers and its application in fuzzy system reliability analysis.

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No potential conflict of interest was reported by the authors.

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