Finite quotients of surface braid groups and
double Kodaira fibrations

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Abstract Let $\Sigma_b$ be a closed Riemann surface of genus $b$. We give an account of some results obtained in the recent papers [CaPol21, Pol20, PolSab21] and concerning what we call here pure braid quotients, namely non-abelian finite groups appearing as quotients of the pure braid group on two strands $P_2(\Sigma_b)$. We also explain how these groups can be used in order to provide new constructions of double Kodaira fibrations.

1 Introduction

A Kodaira fibration is a smooth, connected holomorphic fibration $f_1 : S \to B_1$, where $S$ is a compact complex surface and $B_1$ is a compact complex curve, which is not isotrivial (this means that not all its fibres are biholomorphic to each others). The genus $b_1 := g(B_1)$ is called the base genus of the fibration, whereas the genus $g := g(F)$, where $F$ is any fibre, is called the fibre genus. If a surface $S$ is the total space of a Kodaira fibration, we will call it a Kodaira fibred surface. For every Kodaira fibration we have $b_1 \geq 2$ and $g \geq 3$, see [Kas68, Theorem 1.1]. Since the fibration is smooth, the condition on the base genus implies that $S$ contains no rational or elliptic curves; hence it is minimal and, by the sub-additivity of the Kodaira dimension, it is of general type, hence algebraic.
Examples of Kodaira fibrations were originally constructed in [Kod67, At69] in order to show that, unlike the topological Euler characteristic, the signature $\sigma$ of a real manifold is not multiplicative for fibre bundles. In fact, every Kodaira fibred surface $S$ satisfies $\sigma(S) > 0$, see for example the introduction of [LLR20], whereas $\sigma(B_1) = \sigma(F) = 0$, and so $\sigma(S) \neq \sigma(B_1)\sigma(F)$.

A double Kodaira surface is a compact complex surface $S$, endowed with a double Kodaira fibration, namely a surjective, holomorphic map $f: S \rightarrow B_1 \times B_2$ yielding, by composition with the natural projections, two Kodaira fibrations $f_i: S \rightarrow B_i$, $i = 1, 2$.

The purpose of this article is to give an account of recent results, obtained in the series of papers [CaPol21, Pol20, PolSab21], concerning the construction of some double Kodaira fibrations (that we call diagonal) by means of group-theoretical methods. Let us start by introducing the needed terminology. Let $b_2$ and $n_2$ be two positive integers, and let $P_2(\Sigma_b)$ be the pure braid group on two strands on a closed Riemann surface of genus $b$. We say that a finite group $G$ is a pure braid quotient of type $b - n$ if there exists a group epimorphism $\varphi: P_2(\Sigma_b) \rightarrow G$ (1) such that $\varphi(A_{12})$ has order $n$, where $A_{12}$ is the braid corresponding, via the isomorphism $P_2(\Sigma_b) = \pi_1(\Sigma_b \times \Sigma_b - \Delta)$, to the homotopy class in $\Sigma_b \times \Sigma_b - \Delta$ of a loop in $\Sigma_b \times \Sigma_b$ “winding once” around the diagonal $\Delta$. Since $A_{12}$ is a commutator in $P_2(\Sigma_b)$ and $n \geq 2$, it follows that every pure braid quotient is a non-abelian group, see Remark 2.

By Grauert-Remmert’s extension theorem together with Serre’s GAGA, the existence of a pure braid quotient as in (1) is equivalent to the existence of a Galois cover $f: S \rightarrow \Sigma_b \times \Sigma_b$, branched over $\Delta$ with branching order $n$. After Stein factorization, this yields in turn a diagonal double Kodaira fibration $f: S \rightarrow \Sigma_{b_1} \times \Sigma_{b_2}$. We have $f = f_i$, i.e. no Stein factorization is needed, if and only if $G$ is a strong pure braid quotient, an additional condition explained in Definition 3.

We are now in a position to state our first results, see Theorems 1, 2, 3:

- If $b \geq 2$ is an integer and $p \geq 5$ is a prime number, then both extra-special $p$-groups of order $p^{5b+1}$ are non-strong pure braid quotients of type $(b, p)$.
- If $b \geq 2$ is an integer and $p$ is a prime number dividing $b + 1$, then both extra-special $p$-groups of order $p^{2b+1}$ are pure braid quotients of type $(b, p)$.
- If a finite group $G$ is a pure braid quotient, then $|G| \geq 32$, with equality holding if and only if $G$ is extra-special. Moreover, in the last case, we can explicitly compute the number of distinct quotients maps of type (1), up to the natural action of $\text{Aut}(G)$.

We believe that such results are significant because, although we know that $P_2(\Sigma_b)$ is residually $p$-finite for all $p \geq 2$ (see [BarBel09, pp. 1481-1490]), it is usually tricky to explicitly describe its non-abelian finite quotients.

The geometrical counterparts of the above group-theoretical statements allow us to construct infinite families of double Kodaira fibrations with interesting numerical
properties, for instance having slope greater than $2 + 1/3$ or signature equal to 16, see Theorems 4, 5, 6:

- Let $f: S_p \rightarrow \Sigma_b \times \Sigma_b$ be the diagonal double Kodaira fibration associated with a non-strong pure braid quotient $\varphi: P_2(\Sigma_2) \rightarrow G$ of type $(2, p)$, where $G$ is an extra-special $p$-group $G$ of order $p^9$ and $b' = p^3 + 1$. Then the maximum slope $\nu(S_p)$ is attained for precisely two values of $p$, namely

$$\nu(S_3) = \nu(S_7) = 2 + \frac{12}{35}.$$  

Furthermore, $\nu(S_p) > 2 + 1/3$ for all $p \geq 5$. More precisely, if $p \geq 7$ the function $\nu(S_p)$ is strictly decreasing and

$$\lim_{p \rightarrow +\infty} \nu(S_p) = 2 + \frac{1}{3}.$$  

- Let $\Sigma_b$ be any closed Riemann surface of genus $b$. Then there exists a double Kodaira fibration $f: S \rightarrow \Sigma_b \times \Sigma_b$. Moreover, denoting by $\kappa(b)$ the number of such fibrations, we have

$$\kappa(b) \geq \omega(b + 1),$$

where $\omega: \mathbb{N} \rightarrow \mathbb{N}$ stands for the arithmetic function counting the number of distinct prime factors of a positive integer. In particular,

$$\limsup_{b \rightarrow +\infty} \kappa(b) = +\infty.$$  

- Let $G$ be a finite group and $f: S \rightarrow \Sigma_b \times \Sigma_b$ be a Galois cover, with Galois group $G$, branched over the diagonal $\Delta$ with branching order $n$. Then $|G| \geq 32$, and equality holds if and only if $G$ is extra-special. If $G$ is extra-special of order 32 and $(b, n) = (2, 2)$, then $f: S \rightarrow \Sigma_2 \times \Sigma_2$ is a diagonal double Kodaira fibration such that

$$b_1 = b_2 = 2, \quad g_1 = g_2 = 41, \quad \sigma(S) = 16.$$  

As a consequence of the last result, we obtain a sharp lower bound for the signature of a diagonal double Kodaira fibration, see Theorem 7:

- Let $f: S \rightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be a diagonal double Kodaira fibration, associated with a pure braid quotient $\varphi: P_2(\Sigma_{b_1}) \rightarrow G$ of type $(b, n)$. Then $\sigma(S) \geq 16$, and equality holds precisely when $(b, n) = (2, 2)$ and $G$ is an extra-special group of order 32.

Note that our methods show that every curve of genus $b$ (and not only some special curve with extra automorphisms) is the basis of a (double) Kodaira fibration and that, in addition, the number of distinct Kodaira fibrations over a fixed base can be arbitrarily large. Furthermore, every curve of genus 2 is the base of a (double) Kodaira fibration with signature 16 and this provides, to our knowledge, the first example of positive-dimensional family of (double) Kodaira fibrations with small signature.
The aforementioned examples with signature 16 also provide new “double solutions” to a problem, posed by G. Mess and included in Kirby’s problem list in low-dimensional topology, see [Kir97, Problem 2.18 A], asking what is the smallest number $b$ for which there exists a real surface bundle over a real surface with base genus $b$ and non-zero signature. We actually have $b = 2$, also for double Kodaira fibrations, see Theorem 8:

- Let $S$ be double Kodaira surface, associated with a pure braid quotient $\varphi : P_2(\Sigma_b) \rightarrow G$ of type $(2, 2)$, where $G$ is an extra-special group of order 32. Then the real manifold $X$ underlying $S$ is a closed, orientable $4$-manifold of signature 16 that can be realized as a real surface bundle over a real surface of genus 2, with fibre genus 41, in two different ways.

In fact, it is an interesting question whether 16 and 41 are the minimum possible values for the signature and the fibre genus of a (not necessarily diagonal) double Kodaira surface $f : S \rightarrow \Sigma_2 \times \Sigma_2$, but we will not develop this point here.

The above results paint a rather clear picture regarding pure braid quotients and the relative diagonal double Kodaira fibrations when $|G| \leq 32$. It is natural then to investigate further this topic for $|G| > 32$, and indeed this paper also contains the following new result, see Theorem 9:

- If $G$ is a finite group with $32 < |G| < 64$, then $G$ is not a pure braid quotient.

We provide only a sketch of the proof, which is based on calculations performed by means of the computer algebra system GAP4, see [GAP4]; the details will appear in a forthcoming paper.

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**Notation and conventions.** The order of a finite group $G$ is denoted by $|G|$. If $x \in G$, the order of $x$ is denoted by $o(x)$. The subgroup generated by $x_1, \ldots, x_n \in G$ is denoted by $\langle x_1, \ldots, x_n \rangle$. The center of $G$ is denoted by $Z(G)$ and the centralizer of an element $x \in G$ by $C_G(x)$. If $x, y \in G$, their commutator is defined as $[x, y] = xyx^{-1}y^{-1}$. We denote both the cyclic group of order $p$ and the field with $p$ elements by $\mathbb{Z}_p$. We use sometimes the IdSmallGroup(G) label from GAP4 list of small groups. For instance, $S_4 = G(24, 12)$ means that $S_4$ is the twelfth group of order 24 in this list.

## 2 Pure surface braid groups and finite braid quotients

Let $\Sigma_b$ be a closed Riemann surface of genus $b \geq 2$, and let $\mathcal{P} = (p_1, p_2)$ be an ordered pair of distinct points on $\Sigma_b$. A **pure geometric braid** on $\Sigma_b$ based at $\mathcal{P}$ is a pair $(\alpha_1, \alpha_2)$ of paths $\alpha_i : [0, 1] \rightarrow \Sigma_b$ such that

- $\alpha_i(0) = \alpha_i(1) = p_i$ for all $i \in \{1, 2\}$
- the points $\alpha_1(t), \alpha_2(t) \in \Sigma_b$ are pairwise distinct for all $t \in [0, 1]$, 


Definition 1 The pure braid group on two strands on $\Sigma_b$ is the group $P_2(\Sigma_b)$ whose elements are the pure braids based at $\mathcal{P}$ and whose operation is the usual concatenation of paths, up to homotopies among braids.

It can be shown that $P_2(\Sigma_b)$ does not depend on the choice of the set $\mathcal{P} = (p_1, p_2)$, and that there is an isomorphism

$$P_2(\Sigma_b) \cong \pi_1(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P})$$

where $\Delta \subset \Sigma_b \times \Sigma_b$ is the diagonal.

The group $P_2(\Sigma_b)$ is finitely presented for all $b$, and explicit presentations can be found in [Bel04, Bir69, GG04, S70]. Here we follow the approach in [GG04, Sections 1-3], referring the reader to that paper for further details.

Proposition 1 ([GG04, Theorem 1]) Let $p_1, p_2 \in \Sigma_b$, with $b \geq 2$. Then the map of pointed topological spaces given by the projection onto the first component

$$(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P}) \rightarrow (\Sigma_b, p_1)$$

induces a split short exact sequence of groups

$$1 \rightarrow \pi_1(\Sigma_b - \{p_1\}, p_2) \rightarrow P_2(\Sigma_b) \rightarrow \pi_1(\Sigma_b, p_1) \rightarrow 1. \quad (3)$$

For all $j \in \{1, \ldots, b\}$, let us consider now the $2b$ elements

$$\rho_{1j}, \tau_{1j}, \rho_{2j}, \tau_{2j}$$

of $P_2(\Sigma_b)$ represented by the pure braids shown in Figure 2.
If $\ell \neq i$, the path corresponding to $\rho_{ij}$ and $\tau_{ij}$ based at $p_\ell$ is the constant path. Moreover, let $A_{12}$ be the pure braid shown in Figure 3. In terms of the isomorphism (2), the generators $\rho_{ij}$, $\tau_{ij}$ correspond to the generators of $\pi_1(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P})$ coming from the usual description of $\Sigma_b$ as the identification space of a regular 2b-gon, whereas $A_{12}$ corresponds to the homotopy class in $\Sigma_b \times \Sigma_b - \Delta$ of a topological loop in $\Sigma_b \times \Sigma_b$ that “winds once” around $\Delta$.

**Proposition 2 ([GG04, Theorem 7])** The group $P_2(\Sigma_b)$ admits the following presentation.

Generators

$$\rho_{11} \cdots, \rho_{1b}, \tau_{11} \cdots, \tau_{1b}, A_{12}$$

Relations
Surface relations:

\[
\begin{align*}
\left[ \rho_{1b}, \tau_{1b} \right] \cdots \left[ \rho_{1b}, \tau_{1b-1} \right] = A_{12} \\
\left[ \rho_{2b}, \tau_{2b} \right] \cdots \left[ \rho_{2b}, \tau_{2b-1} \right] = A_{12}^{-1}
\end{align*}
\]

Action of \( \rho_{1j} \):

\[
\begin{align*}
[\rho_{1j}, \rho_{2k}] &= 1 & \text{if } j < k \\
[\rho_{1j}, \rho_{2j}] &= 1 \\
[\rho_{1j}, \rho_{2k}] &= A_{12}^{-1} \rho_{2k} \rho_{2j} \rho_{2k}^{-1} A_{12} & \text{if } j > k \\
[\rho_{1j}, \tau_{2k}] &= 1 & \text{if } j < k \\
[\rho_{1j}, \tau_{2j}] &= A_{12}^{-1} \\
[\rho_{1j}, \tau_{2k}] &= [A_{12}^{-1}, \tau_{2k}] & \text{if } j > k \\
[\rho_{1j}, A_{12}] &= [\rho_{2j}, A_{12}]
\end{align*}
\]

Action of \( \tau_{1j} \):

\[
\begin{align*}
[\tau_{1j}, \rho_{2k}] &= 1 & \text{if } j < k \\
[\tau_{1j}, \rho_{2j}] &= \tau_{2j}^{-1} A_{12} \tau_{2j} \\
[\tau_{1j}, \rho_{2k}] &= \tau_{2j}^{-1}, A_{12} & \text{if } j > k \\
[\tau_{1j}, \tau_{2k}] &= 1 & \text{if } j < k \\
[\tau_{1j}, \tau_{2j}] &= [\tau_{2j}^{-1}, A_{12}] \\
[\tau_{1j}, \tau_{2k}] &= \tau_{2j}^{-1} A_{12} \tau_{2j} A_{12} \tau_{2k} \tau_{2j}^{-1} A_{12}^{-1} \tau_{2j}^{-1} A_{12} \tau_{2k}^{-1} & \text{if } j > k \\
[\tau_{1j}, A_{12}] &= [\tau_{2j}^{-1}, A_{12}]
\end{align*}
\]

Remark 1 The inclusion map \( \iota : \Sigma_b \times \Sigma_b \rightarrow \Delta \rightarrow \Sigma_b \times \Sigma_b \) induces a group epimorphism \( \iota_* : \pi_1(\Sigma_b \times \Sigma_b, \mathcal{P}) \rightarrow \pi_1(\Sigma_b \times \Sigma_b, \mathcal{P}) \), whose kernel is the normal closure of the subgroup generated by \( A_{12} \). Thus, given any group homomorphism \( \varphi : \pi_1(\Sigma_b \times \Sigma_b, \mathcal{P}) \rightarrow G \), it factors through \( \pi_1(\Sigma_b \times \Sigma_b, \mathcal{P}) \) if and only if \( \varphi(A_{12}) \) is trivial.

Tedious but straightforward calculations show that the presentation given in Proposition 2 is invariant under the substitutions

\[
A_{12} \leftrightarrow A_{12}^{-1}, \quad \tau_{1j} \leftrightarrow \tau_{2b+1-j}, \quad \rho_{1j} \leftrightarrow \rho_{2b+1-j},
\]

where \( j \in \{1, \ldots, b\} \). These substitutions correspond to the involution of \( \mathbb{P}_2(\Sigma_b) \) induced by a reflection of \( \Sigma_b \) switching the \( j \)-th handle with the \((b+1-j)\)-th handle.
for all \( j \). Hence we can exchange the roles of \( p_1 \) and \( p_2 \) in (3), and see \( \mathcal{P}_2(\Sigma_b) \) as the middle term of a split short exact sequence of the form
\[
1 \rightarrow \pi_1(\Sigma_b - \{p_2\}, p_1) \rightarrow \mathcal{P}_2(\Sigma_b) \rightarrow \pi_1(\Sigma_b, p_2) \rightarrow 1,
\]
induced by the projection onto the second component
\[
(\Sigma_b \times \Sigma_b - \Delta, \mathcal{P}) \rightarrow (\Sigma_b, p_2).
\]
The elements
\[
\rho_{11}, \ldots, \rho_{1b}, \tau_{11}, \ldots, \tau_{1b}, A_{12}
\]
can be seen as generators of the kernel \( \pi_1(\Sigma_b - \{p_2\}, p_1) \) in (7), whereas the elements
\[
\rho_{21}, \ldots, \rho_{2b}, \tau_{21}, \ldots, \tau_{2b}
\]
yield a complete system of coset representatives for \( \pi_1(\Sigma_b, p_2) \).

We can now define the objects studied in this paper.

**Definition 2** Take positive integers \( b, n \geq 2 \). A finite group \( G \) is called a pure braid quotient of type \( (b, n) \) if there exists a group epimorphism
\[
\varphi: \mathcal{P}_2(\Sigma_b) \rightarrow G
\]
such that \( \varphi(A_{12}) \) has order \( n \).

**Remark 2** Since we are assuming \( n \geq 2 \), the element \( \varphi(A_{12}) \) is non-trivial and so the epimorphism \( \varphi \) does not factor through \( \pi_1(\Sigma_b \times \Sigma_b, \mathcal{P}) \), see Remark 1. The geometrical relevance of this condition will be explained in Section 4. The same condition also shows that a pure braid quotient is necessarily non-abelian, because \( \varphi(A_{12}) \) is a non-trivial commutator in \( G \), see (6).

Sometimes we will use the term pure braid quotient in order to indicate the full datum of the quotient homomorphism (8), instead of the quotient group \( G \) alone.

If \( G \) is a pure braid quotient, then the two subgroups
\[
K_1 := \langle \varphi(\rho_{11}), \varphi(\tau_{11}), \ldots, \varphi(\rho_{1b}), \varphi(\tau_{1b}), \varphi(A_{12}) \rangle
\]
\[
K_2 := \langle \varphi(\rho_{21}), \varphi(\tau_{21}), \ldots, \varphi(\rho_{2b}), \varphi(\tau_{2b}), \varphi(A_{12}) \rangle
\]
are both normal in \( G \), and hence there are two short exact sequences
\[
1 \rightarrow K_1 \rightarrow G \rightarrow Q_2 \rightarrow 1
\]
\[
1 \rightarrow K_2 \rightarrow G \rightarrow Q_1 \rightarrow 1,
\]
in which the elements \( \varphi(\rho_{21}), \varphi(\tau_{21}), \ldots, \varphi(\rho_{2b}), \varphi(\tau_{2b}) \) yield a complete system of coset representatives for \( Q_2 \), whereas the elements \( \varphi(\rho_{11}), \varphi(\tau_{11}), \ldots, \varphi(\rho_{1b}), \varphi(\tau_{1b}) \) yield a complete system of coset representatives for \( Q_1 \).

Let us end this section with the following definition, whose geometrical meaning will become clear later, see Remark 5 of Section 4.
Definition 3 A pure braid quotient $\varphi: P_2(\Sigma_b) \longrightarrow G$ is called strong if $K_1 = K_2 = G$.

3 Extra-special groups as pure braid quotients

We know that $P_2(\Sigma_b)$ is residually $p$-finite for all prime number $p \geq 2$, see [BarBel09, pp. 1481-1490]. This implies that, for every $p$, we can find a non-abelian finite $p$-group $G$ that is a pure braid quotient of type $(b, q)$, where $q$ is a power of $p$. However, it can be tricky to explicitly describe some of these quotients.

In this section we will present a number of results in this direction, obtained in the series of articles [CaPol21, Pol20, PolSab21]; our exposition here will closely follow the treatment given in these papers. Let us start by introducing the following classical definition, see for instance [Gor07, p. 183] and [Is08, p. 123].

Definition 4 Let $p$ be a prime number. A finite $p$-group $G$ is called extra-special if its center $Z(G)$ is cyclic of order $p$, and the quotient $V = G/Z(G)$ is a non-trivial, elementary abelian $p$-group.

An elementary abelian $p$-group is a finite-dimensional vector space over the field $\mathbb{Z}_p$, hence it is of the form $V = (\mathbb{Z}_p)^{\dim V}$ and $G$ fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow G \longrightarrow V \longrightarrow 1. \quad (10)$$

Note that, $V$ being abelian, we must have $[G, G] = \mathbb{Z}_p$, namely the commutator subgroup of $G$ coincides with its center. Furthermore, since the extension (10) is central, it cannot be split, otherwise $G$ would be isomorphic to the direct product of the two abelian groups $\mathbb{Z}_p$ and $V$, which is impossible because $G$ is non-abelian. It can be also proved that, if $G$ is extra-special, then $\dim V$ is even and so $|G| = p^{\dim V + 1}$ is an odd power of $p$.

For every prime number $p$, there are precisely two isomorphism classes $M(p)$, $N(p)$ of non-abelian groups of order $p^3$, namely

$$M(p) = \langle t, z \mid t^p = t^p = 1, [t, z] = 1, [t, t] = z^{-1} \rangle$$

$$N(p) = \langle t, z \mid t^p = z^p = 1, [t, z] = 1, [t, t] = z^{-1} \rangle$$

and both of them are in fact extra-special, see [Gor07, Theorem 5.1 of Chapter 5].

If $p$ is odd, then the groups $M(p)$ and $N(p)$ are distinguished by their exponent, which equals $p$ and $p^2$, respectively. If $p = 2$, the group $M(p)$ is isomorphic to the dihedral group $D_8$, whereas $N(p)$ is isomorphic to the quaternion group $Q_8$.

We can now provide the classification of extra-special $p$-groups, see [Gor07, Section 5 of Chapter 5].

Proposition 3 If $b \geq 2$ is a positive integer and $p$ is a prime number, there are exactly two isomorphism classes of extra-special $p$-groups of order $p^{2b+1}$, that can be described as follows.
• The central product $H_{2b+1}(\mathbb{Z}_p)$ of $b$ copies of $M(p)$, having presentation

$$H_{2b+1}(\mathbb{Z}_p) = \langle t_1, t_2, \ldots, t_b, z \mid t_j^p = t_j^p = z^p = 1, $$
$[t_j, z] = [t_j, z] = 1, $$
$[t_j, t_k] = [t_j, t_k] = 1, $$
$[t_j, t_k] = z^{\delta_jk} \rangle.$$

If $p$ is odd, this group has exponent $p$.

• The central product $G_{2b+1}(\mathbb{Z}_p)$ of $b - 1$ copies of $M(p)$ and one copy of $N(p)$, having presentation

$$G_{2b+1}(\mathbb{Z}_p) = \langle t_1, t_1, \ldots, t_b, z \mid t_j^p = t_j^p = z, $$
$t_1^p = t_1^p = \ldots = t_{b-1}^p = t_{b-1}^p = z^p = 1, $$
$[t_j, z] = [t_j, z] = 1, $$
$[t_j, t_k] = [t_j, t_k] = 1, $$
$[t_j, t_k] = z^{\delta_jk} \rangle.$$

If $p$ is odd, this group has exponent $p^2$.

We are now in a position to state our first two results.

**Theorem 1** ([CaPol21, Section 4], [Pol20]) If $b \geq 2$ is an integer and $p \geq 5$ is a prime number, then both extra-special $p$-groups of order $p^{3b+1}$ are pure braid quotients of type $(b, p)$. All these quotients are non-strong, in fact $K_1$ and $K_2$ have index $p^{2b}$ in $G$.

**Theorem 2** ([CaPol21, Section 4], [Pol20]) If $b \geq 2$ is an integer and $p$ is a prime number dividing $b + 1$, then both extra-special $p$-groups of order $p^{2b+1}$ are strong pure braid quotients of type $(b, p)$.

Theorems 1 and 2 were originally proved by the first author and A. Causin in [CaPol21], but only in the case $G = H_{4b+1}(\mathbb{Z}_p)$ and $G = H_{2b+1}(\mathbb{Z}_p)$, respectively, by using some group-cohomological results related to the structure of the cohomology algebra $H^*(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z}_p)$. Let us give here a sketch of the argument, referring the reader to the aforementioned paper for full details.

Assuming $p \geq 3$, we identified $H_{4b+1}(\mathbb{Z}_p)$ with the symplectic Heisenberg group $\text{Heis}(V, \omega)$, where

$$V = H_1(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z}_p) \cong H_1(\Sigma_b \times \Sigma_b, \mathbb{Z}_p) \cong (\mathbb{Z}_p)^{4b}$$

and $\omega$ is a symplectic form on $V$. This group is the central extension

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow \text{Heis}(V, \omega) \longrightarrow V \longrightarrow 1$$

(11) of the additive group $V$ given as follows: the underlying set of $\text{Heis}(V, \omega)$ is $V \times \mathbb{Z}_p$, endowed with the group law
\[(v_1, t_1) (v_2, t_2) = \left(v_1 + v_2, t_1 + t_2 + \frac{1}{2} \omega(v_1, v_2)\right). \tag{12}\]

By basic linear algebra, all symplectic forms on \((\mathbb{Z}_p)^4b\) are equivalent to the standard symplectic form; thus, given two symplectic forms \(\omega_1, \omega_2\) on \(V\), the two Heisenberg groups \(\text{Heis}(V, \omega_1), \text{Heis}(V, \omega_2)\) are isomorphic. Moreover, the center of the Heisenberg group coincides with its commutator subgroup and is isomorphic to \(\mathbb{Z}_p\).

Now, let

\[\phi: P_2(\Sigma_b) \rightarrow V\]

be the group epimorphism given by the composition of the reduction mod \(p\) map \(H_1(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z}) \rightarrow V\) with the abelianization map \(P_2(\Sigma_b) \rightarrow H_1(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z})\). We have a commutative diagram

\[
\begin{array}{ccc}
P_2(\Sigma_b) & \xrightarrow{\phi} & V \\
\downarrow{\phi} & & \downarrow{\phi} \\
\mathbb{Z}_p & \rightarrow & \text{Heis}(V, \omega) \\
1 & \rightarrow & V \\
\end{array}
\]

and we denote by \(u \in H^2(V, \mathbb{Z}_p)\) the cohomology class corresponding to the bottom Heisenberg extension. Then a lifting \(\phi: P_2(\Sigma_b) \rightarrow \text{Heis}(V, \omega)\) of \(\phi\) exists if and only if \(\phi^* u = 0 \in H^2(P_2(\Sigma_b), \mathbb{Z}_p)\).

The next step is to provide an interpretation of the cohomological condition \(\phi^* u = 0\) in terms of the symplectic form \(\omega\), and this is achieved by using the following facts:

- we have a natural identification

\[H^2(V, \mathbb{Z}_p) \cong \Lambda^2(V^\vee) \oplus V^\vee \tag{13}\]

under which the extension class \(u\) giving the Heisenberg central extension (11) corresponds to \((\omega, \epsilon)\). Here \(\epsilon: V \rightarrow \mathbb{Z}_p\) stands for the linear functional on \(V\) defined by \(\epsilon(v) = w^p\), where \(w\) is any preimage of \(v\) in \(\text{Heis}(V, \omega)\);

- we have natural identifications

\[V^\vee \cong H^1(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z}_p) \cong H^1(\Sigma_b \times \Sigma_b, \mathbb{Z}_p)\]

and there is a commutative diagram

\[
\begin{array}{ccc}
\Lambda^2(V^\vee) & \xrightarrow{\epsilon} & H^2(\Sigma_b \times \Sigma_b, \mathbb{Z}_p) \\
\eta & & \downarrow{\eta} \\
H^2(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z}_p) & \rightarrow & H^2(\Sigma_b \times \Sigma_b, \mathbb{Z}_p) \\
\end{array}
\]
where the vertical map is the quotient by the 1-dimensional vector subspace of $H^2(\Sigma_b \times \Sigma_b, \mathbb{Z}_p)$ generated by the class $\delta$ of the diagonal, whereas $\eta$ and $\xi$ stand for the cup product maps;

- $\Sigma_b \times \Sigma_b - \Delta$ is an aspherical space, namely all its higher homotopy group vanish, and so for all $i \geq 1$ there is a natural isomorphism

$$H^i(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z}_p) \cong H^i(\mathcal{P}_2(\Sigma_b), \mathbb{Z}_p)$$

(14)

where $\mathbb{Z}_p$ is endowed, as an abelian group, with the structure of trivial $\mathcal{P}_2(\Sigma_b)$-module.

Combining all this, we infer that there is a commutative diagram

$$\begin{array}{ccc}
\wedge^2 V^\vee \oplus V^\vee & \xrightarrow{=} & H^2(\Sigma_b, \mathbb{Z}_p) \\
\downarrow & & \downarrow \\
\text{Alt}^2(V) & \xrightarrow{=} & H^2(\mathcal{P}_2(\Sigma_b), \mathbb{Z}_p)
\end{array}$$

where the isomorphism on the left is (13), the vertical map on the left is the projection onto the first summand and the vertical map on the right is (14). Since the projection of the extension class $u \in H^2(\Sigma_b, \mathbb{Z}_p)$ can be naturally identified with $\omega \in \text{Alt}^2(V)$, we have proved the following

**Proposition 4** The obstruction class $\phi^* u \in H^2(\mathcal{P}_2(\Sigma_b), \mathbb{Z}_p)$ can be naturally interpreted as the image $\eta(\omega) \in H^2(\Sigma_b \times \Sigma_b - \Delta, \mathbb{Z}_p)$ of the symplectic form $\omega \in \text{Alt}^2(V)$ via the cup-product map $\eta$.

As a consequence, we obtain the following lifting criterion, that we believe is of independent interest.

**Proposition 5** A lifting $\varphi: \mathcal{P}_2(\Sigma_b) \longrightarrow \text{Heis}(V, \omega)$ of $\phi: \mathcal{P}_2(\Sigma_b) \longrightarrow V$ exists if and only if $\eta(\omega) = 0$. Furthermore, if $\varphi$ exists, then $\varphi(A_{12})$ has order $p$ if and only if $\xi(\omega) \in H^2(\Sigma_b \times \Sigma_b, \mathbb{Z}_p)$ is a non-zero integer multiple of the diagonal class $\delta$. In this case, $\varphi$ is necessarily surjective.

Inspired by Proposition 5, we say that a symplectic form $\omega \in \text{Alt}^2(V)$ is of Heisenberg type if $\xi(\omega)$ is a non-zero integer multiple of $\delta$; equivalently, $\omega$ is of Heisenberg type if $\eta(\omega) = 0$ and $\xi(\omega) \neq 0$. By the previous discussion it follows that, if $\omega$ is of Heisenberg type, $\text{Heis}(V, \omega)$ is a pure braid quotient of type $(b, p)$.

We are therefore left with the task of constructing symplectic forms of Heisenberg type on $V$. We denote by $\alpha_1, \beta_1, \ldots, a_b, \beta_b$ the images in $H^1(\Sigma_b, \mathbb{Z}_p) = H^1(\Sigma_b, \mathbb{Z}) \otimes \mathbb{Z}_p$ of the elements of a basis of $H^1(\Sigma_b, \mathbb{Z})$ which is symplectic with respect to the cup product; then, we can choose for $V$ the ordered basis

$$r_{11}, t_{11}, \ldots, r_{1b}, t_{1b}, r_{21}, t_{21}, \ldots, r_{2b}, t_{2b}$$

(15)

where, under the isomorphism $V \cong H_1(\Sigma_b \times \Sigma_b, \mathbb{Z}_p)$ induced by the inclusion $\iota: \Sigma_b \times \Sigma_b - \Delta \longrightarrow \Sigma_b \times \Sigma_b$, the elements $r_{ij}, t_{ij}, r_{2j}, t_{2j} \in V$ are the duals of
the elements $\alpha_j \otimes 1, \beta_j \otimes 1, 1 \otimes \alpha_j, 1 \otimes \beta_j \in H^1(\Sigma_b \times \Sigma_b, \mathbb{Z}_p) \cong H^1(\Sigma_b, \mathbb{Z}_p) \otimes H^1(\Sigma_b, \mathbb{Z}_p)$, respectively.

Since $p \geq 5$, we can find non-zero scalars $\lambda_1, \ldots, \lambda_b, \mu_1, \ldots, \mu_b \in \mathbb{Z}_p$ such that

$$1 - \lambda_i\mu_1 \neq 0 \text{ for all } i \in \{1, \ldots, b\}$$

Then we consider the alternating form $\omega: V \times V \to \mathbb{Z}_p$ represented, with respect to the ordered basis (15), by the skew-symmetric matrix

$$\Omega_b = \begin{pmatrix} L_b & J_b \\ J_b & M_b \end{pmatrix} \in \text{Mat}(4b, \mathbb{Z}_p)$$

where the blocks are the elements of $\text{Mat}(2b, \mathbb{Z}_p)$ given by

$$L_b = \begin{pmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ -\lambda_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_b & 0 \\ 0 & \cdots & 0 & -\lambda_b & 0 \end{pmatrix}$$

$$M_b = \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 \\ -\mu_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_b & 0 \\ 0 & \cdots & 0 & -\mu_b & 0 \end{pmatrix}$$

$$J_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Standard Gaussian elimination shows that

$$\det \Omega_b = (1 - \lambda_1\mu_1)^2 (1 - \lambda_2\mu_2)^2 \cdots (1 - \lambda_b\mu_b)^2 > 0$$

and so $\omega$ is non-degenerate. Moreover, a direct computation yields $\xi(\omega) = \delta$, that is, $\omega$ is of Heisenberg type. The calculation of the indices of $K_1$ and $K_2$ in $G$ is now straightforward, and this completes the proof of Theorem 1 in the case $G = H_{4b+1}(\mathbb{Z}_p)$.

Now, let us assume that $p$ divides $b + 1$, so that $-b = 1$ holds in $\mathbb{Z}_p$, and take

$$\lambda_1 = \ldots = \lambda_b = \mu_1 = \ldots = \mu_b = -1 \in \mathbb{Z}_p.$$
\[
\Omega_b = \begin{pmatrix} J_b & J_b \\ J_b & J_b \end{pmatrix} \in \text{Mat}(4b, \mathbb{Z}_p)
\]

has rank \(2b\) and, subsequently, \(\omega\) has a \(2b\)-dimensional kernel \(V_0\), namely

\[
V_0 = \langle r_{11} - r_{21}, t_{11} - t_{21}, \ldots, r_{1b} - r_{2b}, t_{1b} - t_{2b} \rangle.
\]

The set \(V \times \mathbb{Z}_p\), with the operation (12), is a group whose center equals \(V_0 \times \mathbb{Z}_p\) and that, with slight abuse of notation, we denote again by \(\text{Heis}(V, \omega)\). Furthermore, the argument in Proposition 5 still applies, providing the existence of a lifting \(P_2(\Sigma_b) \rightarrow \text{Heis}(V, \omega)\). Setting \(W = V/V_0\), the alternating form on \(V\) descends to a symplectic form on \(W\), that we denote it again by \(\omega\); so \(\text{Heis}(W, \omega)\) is a genuine Heisenberg group, endowed with a group epimorphism \(\text{Heis}(V, \omega) \twoheadrightarrow \text{Heis}(W, \omega)\). Composing this epimorphism with the lifting \(P_2(\Sigma_b) \rightarrow \text{Heis}(V, \omega)\), we obtain a group epimorphism \(\varphi: P_2(\Sigma_b) \rightarrow \text{Heis}(W, \omega)\) such that \(\varphi(A_{12})\) is non-trivial and central, hence of order \(p\).

Since \(W\) is a \(\mathbb{Z}_p\)-vector space of dimension \(2b\), the group \(\text{Heis}(W, \omega)\) is isomorphic to \(H_{2b+1}(\mathbb{Z}_p)\); finally, a simple computation based on the expression (18) for \(\ker \Omega_b\) yields \(K_1 = K_2 = G\), and this shows Theorem 2 in the case \(G = H_{2b+1}(\mathbb{Z}_p)\).

The proof of Theorems 1 and 2 in full generality (i.e., for all extra-special groups) was given in [Pol20], using a completely algebraic technique that avoided the use of symplectic geometry and of group cohomology. It is based on the following

**Definition 5** Let \(G\) be a finite group. A **diagonal double Kodaira structure** of type \((b, n)\) on \(G\) is an ordered set of \(4b + 1\) generators

\[
\mathfrak{S} = (t_{11}, t_{11}, \ldots, r_{1b}, t_{1b}, t_{21}, \ldots, r_{2b}, t_{2b}, z),
\]

with \(o(z) = n\), that are images of the ordered set of generators

\[
(\rho_{11}, \tau_{11}, \ldots, \rho_{1b}, \tau_{1b}, \rho_{21}, \tau_{21}, \ldots, \rho_{2b}, \tau_{2b}, A_{12})
\]

via a pure braid quotient \(\varphi: P_2(\Sigma_b) \rightarrow G\) of type \((b, n)\). The structure is called **strong** if

\[
\langle t_{11}, t_{11}, \ldots, r_{1b}, t_{1b}, t_{21}, \ldots, r_{2b}, t_{2b} \rangle = \langle t_{21}, t_{21}, \ldots, r_{2b}, t_{2b} \rangle = G.
\]

Therefore, checking whether \(G\) is a pure braid quotient of type \((b, n)\) is equivalent to checking whether it admits a diagonal double Kodaira structure \(\mathfrak{S}\) of type \((b, n)\). Moreover, by definition, \(\varphi: P_2(\Sigma_b) \rightarrow G\) is strong if and only if \(\mathfrak{S}\) is.

Let us refer now to the presentations for extra-special \(p\)-groups given in Proposition 3. Assuming that \(p\) divides \(b + 1\), in both cases \(G = H_{2b+1}(\mathbb{Z}_p)\) and \(G = G_{2b+1}(\mathbb{Z}_p)\) we can obtain a strong diagonal double Kodaira structure \(\mathfrak{S}\) on \(G\) by setting

\[
r_{1j} = r_{2j} = r_j, \quad t_{1j} = t_{2j} = t_j
\]

for all \(j = 1, \ldots, b\). The divisibility condition is necessary to ensure that the element of \(\mathfrak{S}\) satisfy the two relations coming from the surface relations in Proposition 2. This proves Theorem 2.
In order to prove Theorem 1, it is convenient to consider the following alternative presentation of extra-special $p$-groups. Consider any non-degenerate, skew-symmetric matrix $A = (a_{jk})$ of order $2b$ over $\mathbb{Z}_p$, and consider the finitely presented groups

$$H(A) = \langle x_1, \ldots, x_{2b}, z \mid x_1^p = \ldots = x_{2b}^p = z^p = 1, \quad [x_1, z] = \ldots = [x_{2b}, z] = 1, \quad [x_j, x_k] = z^{a_{jk}} \rangle,$$

(19)

$$G(A) = \langle x_1, \ldots, x_{2b}, z \mid x_1^p = \ldots = x_{2b-2}^p = z^p = 1, \quad x_{2b-1}^p = x_{2b}^p = z, \quad [x_1, z] = \ldots = [x_{2b}, z] = 1, \quad [x_j, x_k] = z^{a_{jk}} \rangle,$$

(20)

where the exponent in $z^{a_{jk}}$ stands for any representative in $\mathbb{Z}$ of $a_{jk} \in \mathbb{Z}_p$. Standard computations show that $H(A) \cong H_{2b+1}(\mathbb{Z}_p)$ and $G(A) \cong G_{2b+1}(\mathbb{Z}_p)$. Now we can take as $A$ the matrix $\Omega_b \in \text{Mat}(4b, \mathbb{Z}_p)$ given in (17). Setting $G = H(\Omega_b)$ or $G = G(\Omega_b)$, the group $G$ is generated by a set of $4b + 1$ elements

$$\mathcal{S} = \{t_1, t_{11}, \ldots, t_{1b}, t_{1b}, t_{21}, t_{21}, \ldots, t_{2b}, t_{2b}, z\}$$

subject to the relations (19) or (20), respectively. One can check that $\mathcal{S}$ provides a diagonal double Kodaira structure of type $(b, p)$ on $G$, and so a diagonal double Kodaira structure of the same type on the isomorphic group $H_{4b+1}(\mathbb{Z}_p)$ or $G_{4b+1}(\mathbb{Z}_p)$. This proves Theorem 1.

**Remark 3** In particular, the pure braid quotients of smallest order detected by the methods detailed so far are the extra-special groups of order $2^7 = 128$, corresponding to the case $(b, p) = (3, 2)$ in Theorem 2.

Recently, in the paper [PolSab21], we were able to significantly lower the value of $|G|$, actually providing a sharp lower bound for the order of a pure braid quotient.

**Theorem 3 ([PolSab21])** Assume that $G$ is a finite group that is a pure braid quotient. Then $|G| \geq 32$, with equality if and only if $G$ is extra-special. In this case, the following holds.

- There are precisely $2211840 = 1152 \cdot 1920$ distinct group epimorphisms $\varphi : P_2(\Sigma_2) \rightarrow G$, and all of them make $G$ a strong pure braid quotient of type $(2, 2)$.
- If $G = G(32, 49) = H_5(\mathbb{Z}_2)$, these epimorphisms form 1920 orbits under the natural action of $\text{Aut}(G)$.
- If $G = G(32, 50) = G_5(\mathbb{Z}_2)$, these epimorphisms form 1152 orbits under the natural action of $\text{Aut}(G)$.
The proof of Theorem 3 is obtained again by looking at the diagonal double Kodaira structures on $G$, see Definition 5.

Remark 4 A key observation is that if $G$ is a CCT-group, namely $G$ is not abelian and commutativity is a transitive relation on the set of the non-central elements, then $G$ admits no diagonal double Kodaira structures and, subsequently, it cannot be a pure braid quotient.

A long but straightforward analysis shows that there are precisely eight non-CCT groups with $G \leq 32$, namely $G = S_4$ and $G = G(32, t)$ with $t \in \{6, 7, 8, 43, 44, 49, 50\}$.

These case are handled separately, and a refined analysis proves that only $G(32, 49)$ and $G(32, 50)$, i.e. the two extra-special groups, admit diagonal double Kodaira structures.

Finally, the number of such structures in each case is computed by using the same techniques as in [Win72]; more precisely, we exploit the fact that $\mathcal{V} = G/\mathcal{Z} G$ can be endowed with a natural structure of 4-dimensional symplectic vector space over $\mathbb{Z}_2$, and that $\text{Out}(G)$ embeds in $\text{Sp}(4, \mathbb{Z}_2)$ as the orthogonal group associated with the quadratic form $q$ on $\mathcal{V}$ related to the symplectic form $(\cdot, \cdot)$ by $q(\vec{x}, \vec{y}) = q(\vec{x}) + q(\vec{y}) + (\vec{x}, \vec{y})$.

4 Geometrical application: diagonal double Kodaira fibrations

Recall that a Kodaira fibration is a smooth, connected holomorphic fibration $f_1: S \rightarrow B_1$, where $S$ is a compact complex surface and $B_1$ is a compact complex curve, which is not isotrivial (this means that not all fibres are biholomorphic each other). The genus $b_1 := g(B_1)$ is called the base genus of the fibration, whereas the genus $g := g(F)$, where $F$ is any fibre, is called the fibre genus.

Definition 6 A double Kodaira surface is a compact complex surface $S$, endowed with a double Kodaira fibration, namely a surjective, holomorphic map $f: S \rightarrow B_1 \times B_2$ yielding, by composition with the natural projections, two Kodaira fibrations $f_i: S \rightarrow B_i, i = 1, 2$.

With a slight abuse of notation, in the sequel we will use the symbol $\Sigma_b$ to indicate both a closed Riemann surface of genus $b$ and its underlying real surface. If a finite group $G$ is a pure braid quotient of type $(b, n)$ then, by using Grauert-Remmert’s extension theorem together with Serre’s GAGA, the group epimorphism $\varphi: \mathbb{P}_2(\Sigma_b) \rightarrow G$ yields the existence of a smooth, complex, projective surface $S$ endowed with a Galois cover

$$f: S \rightarrow \Sigma_b \times \Sigma_b$$

with Galois group $G$ and branched precisely over $\Delta$ with branching order $n$, see [CaPol21, Proposition 4.4]. Composing the group monomorphisms $\pi_1(\Sigma_b - \{p_i, p_j\}) \rightarrow \mathbb{P}_2(\Sigma_b)$ with $\varphi: \mathbb{P}_2(\Sigma_b) \rightarrow G$, we get two homomorphisms
Finite quotients of surface braid groups and double Kodaira fibrations

\[ \varphi_1: \pi_1(\Sigma_b - \{p_2\}, p_1) \to G, \quad \varphi_2: \pi_1(\Sigma_b - \{p_1\}, p_2) \to G, \]

whose images are the normal subgroups \( K_1 \) and \( K_2 \) defined in (9).

By construction, these are the homomorphisms induced by the restrictions \( f_i: \Gamma_i \to \Sigma_b \) of the Galois cover \( f: S \to \Sigma_b \times \Sigma_b \) to the fibres of the two natural projections \( \pi_i: \Sigma_b \times \Sigma_b \to \Sigma_b \). Since \( \Delta \) intersects transversally at a single point all the fibres of the natural projections, it follows that both such restrictions are branched at precisely one point, and the number of connected components of the smooth curve \( \Gamma_i \subset S \) equals the index \( m_i := [G: K_i] \) of \( K_i \) in \( G \).

So, taking the Stein factorizations of the compositions \( \pi_i \circ f: S \to \Sigma_b \) as in the diagram below

\[
\begin{array}{ccc}
S & \xrightarrow{\pi_i \circ f} & \Sigma_b \\
\downarrow f_i & & \downarrow \theta_i \\
\Sigma_b & & \\
\end{array}
\]

we obtain two distinct Kodaira fibrations \( f_i: S \to \Sigma_b_i \), hence a double Kodaira fibration by considering the product morphism

\[ f = f_1 \times f_2: S \to \Sigma_b_1 \times \Sigma_b_2. \]

**Definition 7** We call \( f: S \to \Sigma_b_1 \times \Sigma_b_2 \) the diagonal double Kodaira fibration associated with the pure braid quotient \( \varphi: \mathbb{P}(\Sigma_b) \to G \). Conversely, we will say that a double Kodaira fibration \( f: S \to \Sigma_b_1 \times \Sigma_b_2 \) is of diagonal type \((b, n)\) if there exists a pure braid quotient \( \varphi: \mathbb{P}(\Sigma_b) \to G \) of the same type such that \( f \) is associated with \( \varphi \).

Since the morphism \( \theta_i: \Sigma_b_i \to \Sigma_b \) is étale of degree \( m_i \), by using the Hurwitz formula we obtain

\[ b_1 - 1 = m_1(b - 1), \quad b_2 - 1 = m_2(b - 1). \]  

Moreover, the fibre genera \( g_1, g_2 \) of the Kodaira fibrations \( f_1: S \to \Sigma_b_1, f_2: S \to \Sigma_b_2 \) are computed by the formulae

\[ 2g_1 - 2 = \frac{|G|}{m_1} (2b - 2 + n), \quad 2g_2 - 2 = \frac{|G|}{m_2} (2b - 2 + n), \]

where \( n := 1 - 1/n \). Finally, the surface \( S \) fits into a diagram

\[
\begin{array}{ccc}
S & \xrightarrow{f} & \Sigma_b \times \Sigma_b \\
\downarrow f & & \downarrow \theta_1 \times \theta_2 \\
\Sigma_b_1 \times \Sigma_b_2 & & \\
\end{array}
\]
so that the diagonal double Kodaira fibration $f: S \rightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ is a finite cover of degree $\frac{|G|}{m_1m_2}$ branched precisely over the curve

$$(\theta_1 \times \theta_2)^{-1}(\Delta) = \Sigma_{b_1} \times \Sigma_{b_2}.$$ 

Such a curve is always smooth, being the preimage of a smooth divisor via an étale morphism. However, it is reducible in general, see [CaPol21, Proposition 4.11].

**Remark 5** By definition, the pure braid quotient $\varphi: P_2(\Sigma_b) \rightarrow G$ is strong (see Definition 3) if and only if $m_1 = m_2 = 1$, that in turn implies $b_1 = b_2 = b$, i.e., $f = f$. In other words, $\varphi$ is strong if and only if no Stein factorization as in (21) is needed or, equivalently, if and only if the Galois cover $f: S \rightarrow \Sigma_b \times \Sigma_b$ induced by $\varphi$ is already a double Kodaira fibration, branched on the diagonal $\Delta \subset \Sigma_b \times \Sigma_b$.

We can now compute the invariants of $S$ as follows, see [CaPol21, Proposition 4.8].

**Proposition 6** Let $f: S \rightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be a diagonal double Kodaira fibration, associated with a pure braid quotient $\varphi: P_2(\Sigma_b) \rightarrow G$ of type $(b, n)$. Then we have

$$c_1^2(S) = |G| (2b - 2)(4b - 4 + 4n - n^2)$$
$$c_2(S) = |G| (2b - 2)(2b - 2 + n)$$

where $n = 1 - 1/n$. As a consequence, the slope and the signature of $S$ can be expressed as

$$\nu(S) = \frac{c_1^2(S)}{c_2(S)} = 2 + \frac{2n - n^2}{2b - 2 + n}$$
$$\sigma(S) = \frac{1}{3} \left( c_1^2(S) - 2c_2(S) \right) = \frac{1}{3} |G| (2b - 2) \left( 1 - \frac{1}{n^2} \right).$$

(23)  

**Remark 6** Not all double Kodaira fibrations are of diagonal type. In fact, if $S$ is of diagonal type, then its slope satisfies $\nu(S) = 2 + s$, where $0 < s < 6 - 4\sqrt{2}$, see [Pol20].

We can now specialize these results, by taking as $G$ an extra-special $p$-group and using what we have proved in Section 3. Fix $b = 2$ and let $p \geq 5$ be a prime number. Then every extra-special $p$-group $G$ of order $p^{4b+1} = p^{9}$ is a non-strong pure braid quotient of type $(2, p)$ and such that $m_1 = m_2 = p^{2b}$, see Theorem 1. Setting $b' := p^4 + 1$, cf. equations (22), by [CaPol21, Proposition 4.11] the associated diagonal double Kodaira fibration $f: S \rightarrow \Sigma_{b'} \times \Sigma_{b'}$ is a cyclic cover of degree $p$, branched over a reduced, smooth divisor $D$ of the form

$$D = \sum_{c \in (\mathbb{Z}/p)^{ib}} D_c$$

where the $D_c$ are pairwise disjoint graphs of automorphisms of $\Sigma_{b'}$. 
By using Proposition 6, we can now construct infinitely many double Kodaira fibrations with slope strictly higher than $2 + 1/3$.

**Theorem 4** ([CaPol21, Proposition 4.12], [Pol20]) Let $f : S_p \rightarrow \Sigma_b \times \Sigma_b$ be the diagonal double Kodaira fibration associated with a non-strong pure braid quotient $\varphi : P_2(\Sigma_2) \rightarrow G$ of type $(2, p)$, where $G$ is an extra-special $p$-group $G$ of order $p^9$. Then the maximum slope $v(S_p)$ is attained for precisely two values of $p$, namely

$$v(S_3) = v(S_7) = 2 + \frac{12}{35}.$$  

Furthermore, $v(S_p) > 2 + 1/3$ for all $p \geq 5$. More precisely, if $p \geq 7$ the function $v(S_p)$ is strictly decreasing and

$$\lim_{p \rightarrow +\infty} v(S_p) = 2 + \frac{1}{3}.$$  

**Remark 7** The original examples by Atiyah, Hirzebruch and Kodaira have slope lying in the interval $(2, 2 + 1/3)$, see [BHPV03, p. 221]. Our construction provides an infinite family of Kodaira fibred surfaces such that $2 + 1/3 < v(S) \leq 2 + 12/35$, maintaining at the same time a complete control on both the base genus and the signature. By contrast, the ingenious “tautological construction” used in [CatRol09] yields a higher slope than ours, namely $2 + 2/3$, but it involves an étale pullback “of sufficiently large degree”, that completely loses control on the other quantities. Note that [LLR20] gives (at least in principle) an effective version of the pullback construction.

If $p$ is a prime number dividing $b + 1$, by Theorem 2 every extra-special $p$-group $G$ of order $p^{2b+1}$ is a strong pure braid quotient of type $(b, p)$, and this gives in turn a diagonal double Kodaira fibration $f : S \rightarrow \Sigma_b \times \Sigma_b$, see Remark 5. If $\omega : \mathbb{N} \rightarrow \mathbb{N}$ stands for the arithmetic function counting the number of distinct prime factors of a positive integer, see [HarWr08, p. 335], we obtain

**Theorem 5** ([CaPol21, Corollary 4.18], [Pol20]) Let $\Sigma_b$ be any closed Riemann surface of genus $b$. Then there exists a double Kodaira fibration $f : S \rightarrow \Sigma_b \times \Sigma_b$. Moreover, denoting by $\kappa(b)$ the number of such fibrations, we have

$$\kappa(b) \geq \omega(b + 1).$$

In particular,

$$\limsup_{b \rightarrow +\infty} \kappa(b) = +\infty.$$  

As far as we know, this is the first construction showing that all curves of genus $b \geq 2$ (and not only some special curves with extra automorphisms) appear in the base of at least one double Kodaira fibration $f : S \rightarrow \Sigma_b \times \Sigma_b$. In addition, two Kodaira fibred surfaces corresponding to two distinct prime divisors of $b + 1$ are non-homeomorphic, because the corresponding signatures are different: just use (23) with $n = p$ and $|G| = p^{2b+1}$ and note that, for fixed $b$, the function expressing $\sigma(S)$
is strictly increasing in \( p \). This shows that the number of topological types of \( S \), for a fixed base \( \Sigma_b \), can be arbitrarily large.

Let us now consider Theorem 3, whose geometrical translation is

**Theorem 6** ([PolSab21]) Let \( G \) be a finite group and \( f: S \to \Sigma_b \times \Sigma_b \) be a Galois cover, with Galois group \( G \), branched over the diagonal \( \Delta \) with branching order \( n \). Then \( |G| \geq 32 \), and equality holds if and only if \( G \) is extra-special. In this case, the following holds.

1. There exist \( 2211840 = 1152 \cdot 1920 \) distinct \( G \)-covers \( f: S \to \Sigma_2 \times \Sigma_2 \), and all of them are diagonal double Kodaira fibrations such that
   \[
   b_1 = b_2 = 2, \quad g_1 = g_2 = 41, \quad \sigma(S) = 16.
   \]

2. If \( G = G(32, 49) = H_5(\mathbb{Z}_2) \), these \( G \)-covers form 1920 equivalence classes up to cover isomorphisms.
3. If \( G = G(32, 50) = H_5(\mathbb{Z}_2) \), these \( G \)-covers form 1152 equivalence classes up to cover isomorphisms.

As a consequence, we obtain a sharp lower bound for the signature of a diagonal double Kodaira fibration. In fact, the second equality in (23) together with Theorem 6 imply that, for every such fibration, we have

\[
\sigma(S) = \frac{1}{3} |G| (2b - 2) \left( 1 - \frac{1}{n^2} \right) \geq \frac{1}{3} \cdot 32 \cdot (2 \cdot 2 - 2) \left( 1 - \frac{1}{2^2} \right) = 16,
\]

and this in turn establishes the following result.

**Theorem 7** ([PolSab21]) Let \( S \) be a double Kodaira surface, associated with a pure braid quotient \( \varphi: \mathcal{P}_2(\Sigma_b) \to G \) of type \((b, n)\). Then \( \sigma(S) \geq 16 \), and equality holds precisely when \((b, n) = (2, 2)\) and \( G \) is an extra-special group of order 32.

**Remark** If \( S \) is a double Kodaira fibration, corresponding to a pure braid quotient \( \varphi: \mathcal{P}_2(\Sigma_b) \to G \) of type \((b, n)\), then, using the terminology in [CatRol09], it is very simple. Let us denote by \( \mathcal{M}_S \) the connected component of the Gieseker moduli space of surfaces of general type containing the class of \( S \), and by \( \mathcal{M}_b \) the moduli space of smooth curves of genus \( b \). Thus, by applying [Rol10, Thm. 1.7] and using the fact that \( \Delta \subset \Sigma_b \times \Sigma_b \) is the graph of the identity \( \text{id}: \Sigma_b \to \Sigma_b \), we infer that every surface in \( \mathcal{M}_S \) is still a very simple double Kodaira fibration and that there is a natural map of schemes

\[
\mathcal{M}_b \to \mathcal{M}_S,
\]

which is an isomorphism on geometric points. Roughly speaking, since the branch locus \( \Delta \subset \Sigma_b \times \Sigma_b \) is rigid, all the deformations of \( S \) are realized by deformations of \( \Sigma_b \times \Sigma_b \) preserving the diagonal, hence by deformations of \( \Sigma_b \), cf. [CaPol21, Proposition 4.22]. In particular, this shows that \( \mathcal{M}_S \) is a connected and irreducible component of the Gieseker moduli space.
Every Kodaira fibred surface $S$ has the structure of a real surface bundle over a real surface, and so $\sigma(S)$ is divisible by 4, see [Mey73]. If, in addition, $S$ has a spin structure, i.e. its canonical class is 2-divisible in Pic($S$), then $\sigma(S)$ is a positive multiple of 16 by Rokhlin’s theorem, and examples with $\sigma(S) = 16$ are constructed in [LLR20]. It is not known if there exists a Kodaira fibred surface with $\sigma(S) \leq 12$.

Constructing (double) Kodaira fibrations with small signature is a rather difficult problem. As far as we know, before the present work the only examples with signature 16 were the ones listed in [LLR20, Table 3, Cases 6.2, 6.6, 6.7 (Type 1), 6.9]. The examples in Theorem 7 are new, since both the base genera and the fibre genera are different from the ones in the aforementioned cases. Our results also show that every curve of genus 2 is the base of a double Kodaira fibration with signature 16. Thus, we obtain two families of dimension 3 of such fibrations that, to our knowledge, provides the first examples of positive-dimensional families of double Kodaira fibrations with small signature.

Theorem 7 also provide new “double solutions” to a problem, posed by G. Mess and included in Kirby’s problem list in low-dimensional topology, see [Kir97, Problem 2.18 A], asking what is the smallest number $b$ for which there exists a real surface bundle over a real surface with base genus $b$ and non-zero signature. We actually have $b = 2$, also for double Kodaira fibrations.

**Theorem 8 ([PolSab21])** Let $S$ be a double Kodaira surface, associated with a pure braid quotient $\varphi : P_2(\Sigma_b) \to G$ of type $(2, 2)$, where $G$ is an extra-special group of order 32. Then the real manifold $X$ underlying $S$ is a closed, orientable 4-manifold of signature 16 that can be realized as a real surface bundle over a real surface of genus 2, with fibre genus 41, in two different ways.

It is an interesting question whether 16 and 41 are the minimum possible values for the signature and the fibre genus of a (non necessary diagonal) double Kodaira fibration $f : S \to \Sigma_2 \times \Sigma_2$; however, this topic exceeds the scope of this paper.

### 5 Beyond $|G| = 32$

This last section contains the new result of this article. As we already observed, so far we have detailed a rather clear picture regarding pure braid quotient groups $G$ and the relative diagonal double Kodaira fibrations for $|G| \leq 32$. Indeed, there is no pure braid quotient of order strictly less than 32 and for $|G| = 32$ we have only the two extra-special groups, see Theorem 3. Furthermore, Theorem 2 provides examples of pure braid quotients starting with order equal to 128, see Remark 3, and they are extra-special groups, too. It seems then natural to investigate this matter further for $|G| > 32$, for instance, in order to look for non extra-special examples. In this direction we have obtained the following result that highlights the existence of a gap between orders 32 and 64.

**Theorem 9** If $G$ is a finite group with $32 < |G| < 64$, then $G$ is not a pure braid quotient.
Here we just give a sketch of the proof, while the full details will appear elsewhere. We know that the group $G$ cannot be abelian, see Remark 2, and we also mentioned that CCT-groups cannot be pure braid quotients, see Remark 4. On the other hand, by [PolSab21] we know that, if $G$ is a pure braid quotient and admits no proper quotients that are pure braid quotients, then $G$ must be monolithic, i.e., the intersection $\text{soc}(G)$ of its non-trivial normal subgroups is non-trivial. In fact, consider the epimorphism $\varphi: P_2(\Sigma_b) \to G$ and assume that there is a non-trivial, normal subgroup $N$ of $G$ such that $\varphi(A_{12}) \notin N$. Then, composing the projection $G \to G/N$ with $\varphi$, we obtain a pure braid quotient $\tilde{\varphi}: P_2(\Sigma_b) \to G/N$, which leads to a contradiction. It follows that $\varphi(A_{12}) \in \text{soc}(G)$, in particular $G$ is monolithic.

By Theorem 3 this implies that, if a pure braid quotient $G$ satisfies our assumptions on the order, then $G$ must be monolithic. A straightforward computer calculation with GAP4 now shows that there are precisely two non-abelian groups $G$ with $32 < |G| < 64$ that are both non-CCT and monolithic, namely $G(54, 5)$ and $G(54, 6)$; by the remarks above, they are the only possible candidates to be pure braid quotients in that range for $|G|$. Finally, a brute force check (again by using GAP4) shows that these groups admit no diagonal double Kodaira structure, proving our assertion.

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