THE A PRIORI TANΘ THEOREM FOR EIGENVECTORS

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ABSTRACT. Let $A$ be a self-adjoint operator on a Hilbert space $H$. Assume that the spectrum of $A$ consists of two disjoint components $\sigma_0$ and $\sigma_1$ such that the convex hull of the set $\sigma_0$ does not intersect the set $\sigma_1$. Let $V$ be a bounded self-adjoint operator on $H$ off-diagonal with respect to the orthogonal decomposition $H = H_0 \oplus H_1$ where $H_0$ and $H_1$ are the spectral subspaces of $A$ associated with the spectral sets $\sigma_0$ and $\sigma_1$, respectively. It is known that if $\|V\| < \sqrt{2d}$ where $d = \text{dist}(\sigma_0, \sigma_1) > 0$ then the perturbation $V$ does not close the gaps between $\sigma_0$ and $\sigma_1$. Assuming that $f$ is an eigenvector of the perturbed operator $A + V$ associated with its eigenvalue in the interval $(\min(\sigma_0) - d, \max(\sigma_0) + d)$ we prove that under the condition $\|V\| < \sqrt{2d}$ the (acute) angle $\theta$ between $f$ and the orthogonal projection of $f$ onto $H_0$ satisfies the bound $\tan \theta \leq \frac{\sqrt{d}}{2}$ and this bound is sharp.

1. INTRODUCTION

Given a self-adjoint operator $A$ on a Hilbert space $H$ assume that $\sigma_0$ is an isolated part of its spectrum, that is,

$$d = \text{dist}(\sigma_0, \sigma_1) > 0,$$

(1.1)

where $\sigma_1 = \text{spec}(A) \setminus \sigma_0$ is the rest of the spectrum of $A$. In this case we say that there are open gaps between the sets $\sigma_0$ and $\sigma_1$. It is well known (see, e.g., [8, §135]) that a sufficiently small self-adjoint perturbation $V$ of $A$ does not close these gaps which allows one to think of the corresponding disjoint spectral components $\sigma_0'$ and $\sigma_1'$ of the perturbed operator $L = A + V$ as a result of the perturbation of the spectral sets $\sigma_0$ and $\sigma_1$, respectively.

Assuming (1.1), in this note we are concerned with the perturbations $V$ that are off-diagonal with respect to the partition $\text{spec}(A) = \sigma_0 \cup \sigma_1$, i.e., with perturbations that anticommute with the difference $E_A(\sigma_0) - E_A(\sigma_1)$ of the spectral projections $E_A(\sigma_0)$ and $E_A(\sigma_1)$ associated with the spectral sets $\sigma_0$ and $\sigma_1$, respectively. In general, it is known (see [8, Theorem 1]) that such perturbations do not close the gaps between the sets $\sigma_0$ and $\sigma_1$ (which means that the inequality $\text{dist}(\sigma_0', \sigma_1') > 0$ holds) whenever

$$\|V\| < \frac{\sqrt{3}}{2}d.$$

(1.2)

Moreover, if no assumptions are made about the location of $\sigma_0$ and $\sigma_1$ except the assumption (1.1) then condition (1.2) is sharp (see [8, Example 1.5]).

However there are two important particular mutual dispositions of the spectral sets $\sigma_0$ and $\sigma_1$ that ensure the disjointness of the perturbed spectral sets $\sigma_0'$ and $\sigma_1'$ under conditions on $\|V\|$ much weaker than the general one of (1.2). The first of these two dispositions is the one where the sets $\sigma_0$ and $\sigma_1$ are subordinated, say

$$\sup \sigma_0 < \inf \sigma_1.$$

(1.3)

\footnotesize

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The second disposition corresponds to the case where one of the sets \(\sigma_0\) and \(\sigma_1\) is lying in a finite gap of the other set, say \(\sigma_0\) lies in a finite gap of \(\sigma_1\), which means that

\[
\text{conv}(\sigma_0) \cap \sigma_1 = \emptyset, \tag{1.4}
\]

where \(\text{conv}(\sigma)\) denotes the convex hull of a set \(\sigma \subseteq \mathbb{R}\). (We recall that by a finite gap of a closed Borel set \(\Sigma\) on \(\mathbb{R}\) one understands an open finite interval belonging to the complement \(\mathbb{R} \setminus \Sigma\) of \(\Sigma\) such that both of its end points belong to \(\Sigma\).)

It is known that if (1.3) holds then for any bounded off-diagonal perturbation \(V\) the interval \((\sup \sigma_0, \inf \sigma_1)\) belongs to the resolvent set of the perturbed operator \(L = A + V\), and hence \(\sigma'_0 \subset (-\infty, \sup \sigma_0]\) and \(\sigma'_1 \subset [\inf \sigma_1, +\infty)\) (see [11], [8]; cf. [4]). In the case of the disposition (1.4), it has been proven in [5] (see also [4]) that the gaps between \(\sigma_0\) and \(\sigma_1\) remain open if the off-diagonal perturbation \(V\) satisfies the (sharp) condition

\[
\|V\| < \sqrt{2}d. \tag{1.7}
\]

Under this condition the spectrum of \(L = A + V\) consists of two disjoint components \(\sigma'_0\) and \(\sigma'_1\) such that

\[
\sigma'_0 \subset (\inf \sigma_0 - d, \sup \sigma_0 + d) \quad \text{and} \quad \sigma'_1 \subset \mathbb{R} \setminus \Delta,
\]

where \(\Delta\) denotes the gap of \(\sigma_1\) that contains \(\sigma_0\). Notice that the norm bound \(\|V\| < \sqrt{2}d\) is also sharp in the sense that, if it is violated, the spectrum of \(L\) in the gap \(\Delta\) may be empty at all (see [5] Example 1.6).

Now assume that the perturbed spectral set \(\sigma'_0\) contains an eigenvalue of the operator \(L = A + V\) and let \(f, f \neq 0\), be an eigenvector of \(L\) corresponding to this eigenvalue. Denote by \(\theta\) the (acute) angle between the vector \(f\) and its projection \(f_0 = E_A(\sigma_0)f\) onto the spectral subspace \(\mathcal{H}_0 = \text{Ran}E_A(\sigma_0)\) of \(A\) associated with the unperturbed spectral set \(\sigma_0\).

It is known that under the subordination condition (1.3) for any bounded off-diagonal perturbation \(V\) the angle \(\theta\) can not exceed \(\pi/4\). Moreover, the following sharp estimate holds

\[
\theta \leq \frac{1}{2} \arctan \left( \frac{2\|V\|}{d} \right) \quad \left( < \frac{\pi}{4} \right). \tag{1.5}
\]

This bound is a simple corollary to the celebrated Davis–Kahan \(\tan 2\Theta\) Theorem [3] (also see [2] Theorem 6.1 and [6] Theorem 2.4).

In the case of the spectral disposition (1.4) an \textit{a posteriori} bound on the angle \(\theta\) under condition \(\|V\| < \sqrt{2}d\) follows from [5] Theorem 2.4. This bound reads

\[
\theta \leq \arctan \left( \frac{\|V\|}{\delta} \right), \tag{1.6}
\]

where \(\delta\) denotes the distance between the perturbed spectral set \(\sigma'_0\) and unperturbed spectral set \(\sigma_1\). Since \(\delta\) may be arbitrarily small (see Example 2.5 below), the bound (1.6) gives in general no \textit{a priori} uniform estimate for \(\theta\) except that \(\theta < \pi/2\).

The present note is aimed just at giving an \textit{a priori} sharp bound on the angle \(\theta\) in the case of the disposition (1.4). In particular, we will prove that under condition \(\|V\| < \sqrt{2}d\) this angle is strictly separated from \(\pi/2\). Our main result is as follows

\textbf{Theorem 1.} Given a self-adjoint operator \(A\) on the Hilbert space \(\mathcal{H}\) assume that

\[
\text{spec}(A) = \sigma_0 \cup \sigma_1, \quad \text{dist}(\sigma_0, \sigma_1) = d > 0, \quad \text{and} \quad \text{conv}(\sigma_0) \cap \sigma_1 = \emptyset.
\]

Let \(V\) be a bounded self-adjoint operator on \(\mathcal{H}\) off-diagonal with respect to the decomposition \(\mathcal{H} = \text{Ran}E_A(\sigma_0) \oplus \text{Ran}E_A(\sigma_1)\). Assume in addition that

\[
\|V\| < \sqrt{2}d \tag{1.7}
\]
and that the operator $L = A + V$ possesses an eigenvector $f$ associated with an eigenvalue $z \in (\inf \sigma_0 - d, \sup \sigma_0 + d)$. Then the (acute) angle $\theta$ between the vector $f$ and its projection $E_A(\sigma_0)f$ onto the sub-space $\text{Ran} E_A(\sigma_0)$ satisfies the bound

$$\theta \leq \arctan\left( \frac{\|V\|}{d} \right).$$

(1.8)

**Remark 2.** The bound (1.8) implies that under condition (1.7) the angle $\theta$ can never exceed the value of $\arctan \sqrt{2}$, i.e.

$$\theta < \arctan \sqrt{2} \approx 0.304 \pi.$$

We also remark that for $\|V\| < d$ the bound (1.8) follows from [7, Theorem 2.4].

Throughout the paper by $\Xi(D, d, b)$ we will denote a function of three real variables $D$, $d$, and $b$ defined on the set $\Omega = \{(D, d, b) | D > 0, \ 0 < d \leq D/2, \ 0 \leq b < \sqrt{dD}\}$ by the following expressions

$$\Xi(D, d, b) = \begin{cases} \tan^2\left( \frac{1}{2} \arctan\frac{2b}{d} \right) & \text{if } b^2 \leq d\sqrt{D} - \sqrt{2}d, \\ 1 + \frac{2b^2}{D^2} - \frac{2}{D^2} \sqrt{(dD - b^2)((D - d)D - b^2)} & \text{if } d\sqrt{D} - \sqrt{2}d < b^2 < dD \end{cases}$$

(1.9)

Here and further on by $\tan^2 \theta, \theta \in \mathbb{R}$, we understand the square of the tangent of $\theta$, that is, $\tan^2 \theta = (\tan \theta)^2$.

Theorem 1 appears to be a corollary to a more general statement (Theorem 3.2) that is proven under a weaker than (1.7) but more specific condition $\|V\| < \sqrt{d|\Delta|}$ where $\Delta$ again denotes the (finite) gap of the set $\sigma_1$ that contains $\sigma_0$ and $|\Delta|$ stands for the length of the interval $\Delta$. If this condition holds then the off-diagonal perturbation $V$ does not close the gaps between $\sigma_0$ and $\sigma_1$ (see [4, Theorem 1 (i)]). The claim of Theorem 3.2 is that under the condition $\|V\| < \sqrt{d|\Delta|}$ the following inequality holds

$$\tan \theta \leq (\Xi(|\Delta|, d, \|V\|))^{1/2}.$$ 

(1.10)

In particular, from formula (1.9) defining the function $\Xi$ one can see that if $|\Delta| > 2d$ then for $V$ small enough, namely for $V$ such that

$$\|V\|^2 \leq d\sqrt{|\Delta|} \sqrt{\frac{\sqrt{|\Delta|} - \sqrt{2}d}{2}},$$

the bound on $\theta$ is the same as the bound (1.5) prescribed by the $\tan 2\theta$ Theorem.

The paper is organized as follows. In Section 2 we consider a three-dimensional version of the problem and prove the bound (1.10) in the case of $3 \times 3$ matrices. The general infinite-dimensional case is studied in Section 3. In the proof of the central result of this section, the one of Theorem 3.2, we essentially rely on Lemma 2.2 of Section 2.

Throughout the paper we use the standard notation $M^\top$ for the transpose of a matrix $M$. 
2. A THREE-DIMENSIONAL CASE

We start our consideration with the case where $H = C^3$ and the operators $A$ and $V$ are $3 \times 3$ matrices. Assume that

$$ A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \gamma_- & 0 \\ 0 & 0 & \gamma_+ \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & b_- & b_+ \\ b_- & 0 & 0 \\ b_+ & 0 & 0 \end{pmatrix}, $$

where $\lambda, \gamma_\pm, b_\pm \in \mathbb{R}$, and $\gamma_+ > \gamma_-$. The matrices $A$ and $V$ are symmetric. Moreover, under the assumption that $\lambda \neq \gamma_\pm$ the matrix $V$ is off-diagonal with respect to the partition $\text{spec}(A) = \sigma_0 \cup \sigma_1$ of the spectrum of $A$ into the disjoint sets

$$ \sigma_0 = \{ \lambda \} \quad \text{and} \quad \sigma_1 = \{ \gamma_-, \gamma_+ \}. $$

It is convenient for us to write the matrix $L = A + V$ in the following $2 \times 2$ block form

$$ L = \begin{pmatrix} \lambda & B \\ B^* & A_1 \end{pmatrix}, $$

(2.1)

where $B$ and $A_1$ are $1 \times 2$ and $2 \times 2$ matrices given by

$$ B = \begin{pmatrix} b_- & b_+ \end{pmatrix}, \quad A_1 = \begin{pmatrix} \gamma_- & 0 \\ 0 & \gamma_+ \end{pmatrix}, $$

(2.2)

respectively. Clearly, $\|V\| = \|B\| = \sqrt{|b_-|^2 + |b_+|^2}$.

Throughout this section by $\Delta$ we will denote the spectral gap of the operator $A_1$ between its eigenvalues $\gamma_-$ and $\gamma_+$, i.e.

$$ \Delta = (\gamma_-, \gamma_+). $$

**Lemma 2.1.** Given a matrix $L$ of the form (2.1), (2.2), assume that $\lambda \in \Delta$ and

$$ \|B\| < \sqrt{d|\Delta|}, $$

(2.3)

where $|\Delta| = \gamma_+ - \gamma_-$ stands for the length of the interval $\Delta$ and $d = \text{dist}(\sigma_0, \sigma_1) = \min\{\gamma_+ - \lambda, \lambda - \gamma_-\}$. Then $L$ has a unique eigenvalue $z$ in the interval $\Delta$ and this eigenvalue is simple. Moreover,

$$ \gamma_- < z_{\min} \leq z \leq z_{\max} < \gamma_+, $$

where

$$ z_{\min} = \lambda - \|B\| \tan \left( \frac{1}{2} \arctan \frac{2\|B\|}{\gamma_+ - \lambda} \right), \quad (2.4) $$

$$ z_{\max} = \lambda + \|B\| \tan \left( \frac{1}{2} \arctan \frac{2\|B\|}{\lambda - \gamma_-} \right). \quad (2.5) $$

**Proof.** Lemma 2.1 is an elementary corollary to [4, Theorem 3.2].

**Lemma 2.2.** Assume that the hypothesis of Lemma 2.1 holds. Let $z$ be the eigenvalue of the matrix $L$ in the interval $\Delta$ and $f, f \neq 0$, the corresponding eigenvector, $Lf = zf$. Then the (acute) angle $\theta$ between the vectors $f$ and $f_0 = (1, 0, 0)^T$ satisfies the following bound

$$ \tan^2 \theta \leq \Xi(|\Delta|, d, \|B\|), $$

(2.6)

where the function $\Xi$ is given by [1,9].
Assume, without loss of generality, that \( \gamma_+ = -\gamma_- = \gamma > 0 \). Otherwise one can simply make the corresponding shift of the origin of the spectral parameter axis. Assume, in addition, that \( B \neq 0 \) and \( \lambda \geq 0 \). (There is no loss of generality in the latter assumption since, for \( \lambda < 0 \), instead of \( L \) one may consider the matrix \(-L\).

Thus, in the proof we will assume that

\[
\Delta = (\gamma, \gamma), \quad 0 < \lambda < \gamma, \quad \text{and} \quad d = \min(\gamma - \lambda, \lambda + \gamma).
\]

Under the hypothesis that \( \|B\| < \sqrt{d|\Delta|} (= \sqrt{2d\gamma}) \), from \([4, \text{Theorem 1 (i)}]\) it follows that if the eigenvalue \( z \) of \( L \) is in \( \Delta \) then the corresponding eigenvector \( f, \lambda f = zf \), may be chosen in the form

\[
f = (1, x_-, x_+)^T,
\]

with \( x_\pm \in \mathbb{C} \) such that the matrix \( X = (x_--x_+)^T \) satisfies the Riccati equation

\[
\lambda X - A_1X + XBX = B^*.
\]

Moreover,

\[
z = \lambda + BX.
\]

Taking into account \([2.2]\) equations \([2.7]\) and \([2.8]\) imply

\[
x_- = \frac{b_-}{\gamma + z} \quad \text{and} \quad x_+ = \frac{b_+}{-\gamma + z}.
\]

Hence

\[
\|X\|^2 = \frac{|b_-|^2}{(\gamma + z)^2} + \frac{|b_+|^2}{(-\gamma + z)^2}.
\]

In addition, from \([2.8]\) and \([2.9]\) one concludes that \( z \) is the solution to equation

\[
z = \lambda + \frac{|b_-|^2}{\gamma + z} + \frac{|b_+|^2}{-\gamma + z}.
\]

Let \( t \in [0, 1] \) be such that

\[
|b_+|^2 = t \|B\|^2
\]

and, hence,

\[
|b_-|^2 = (1-t) \|B\|^2.
\]

Notice that under the assumptions we use, the bounds \( z_{\text{min}} \) of \([2.4]\) and \( z_{\text{max}} \) of \([2.5]\) can be written in the form

\[
z_{\text{min}} = \frac{\gamma + \lambda}{2} - \sqrt{\frac{(\gamma - \lambda)^2}{4} + \|B\|^2},
\]

\[
z_{\text{max}} = \frac{-\gamma - \lambda}{2} + \sqrt{\frac{(\gamma + \lambda)^2}{4} + \|B\|^2}.
\]

It is easy to see that, given the value of \( \|B\| \), for \( t \) in \([2.12]\) and \([2.13]\) varying between 0 and 1 the solution \( z \) to equation \([2.11]\) fills the whole interval \([z_{\text{min}}, z_{\text{max}}]\). Moreover, with \( t \) decreasing from 1 to 0 the value of \( z \) is continuously and monotonously increasing from \( z_{\text{min}} \) to \( z_{\text{max}} \).

On the other hand one can express \( t \) through \( z \). With \( |b_\pm| \) given by \([2.12]\) and \([2.13]\) from \([2.11]\) it follows that

\[
t = \frac{1}{2\gamma \|B\|^2}[(z - \lambda)(z^2 - \gamma^2) - \|B\|^2(z - \gamma)].
\]
Taking this into account, we rewrite expression (2.10) in the form
\[ \|X\|^2 = \varphi(z), \]
where the function \( \varphi \) is given by
\[ \varphi(z) = \frac{\|B\|^2 + 2(\lambda - z)z}{\gamma^2 - z^2}. \]  
(2.17)

That is, given the value of \( \|B\| \), the norm of the solution \( X \) to the Riccati equation (2.7) may be considered as a function of the only variable \( z \) that runs through the interval \([z_{\text{min}}, z_{\text{max}}]\).

There is a single point \( z_0 \) within the interval \((-\gamma, \gamma)\) where the derivative of the function \( \varphi(z) \) is zero. This point reads
\[ z_0 = \begin{cases} 
0 & \text{if } \lambda = 0, \\
\frac{2\gamma^2 - \|B\|^2}{2\lambda} - \frac{\sqrt{\left(\frac{2\gamma^2 - \|B\|^2}{2\lambda}\right)^2 - \gamma^2}}{\gamma} & \text{if } \lambda > 0.
\end{cases} \]  
(2.18)

It provides the function \( \varphi(z) \) with a maximum.

One concludes by inspection that inequality (2.3) (along with the assumptions \( \lambda \geq 0 \) and \( B \neq 0 \)) implies
\[ z_0 < z_{\text{max}}. \]

At the same time \( z_0 \leq z_{\text{min}} \) if \( 0 < \|B\| \leq \beta \) and \( z_0 > z_{\text{min}} \) if \( \beta < \|B\| < \sqrt{2d\gamma} \) where
\[ \beta = \left[(\gamma - \lambda)\sqrt{\gamma\gamma - \sqrt{\gamma\gamma}} - \lambda\right]^{1/2} = \left[d\sqrt{|\Delta|\sqrt{|\Delta| - \sqrt{2d}}} \right]^{1/2}. \]  
(2.19)

Therefore,
\[ \max_{z \in [z_{\text{min}}, z_{\text{max}}]} \varphi(z) = \varphi(z_{\text{min}}) \quad \text{if } 0 < \|B\| \leq \beta \]  
(2.20)

and
\[ \max_{z \in [z_{\text{min}}, z_{\text{max}}]} \varphi(z) = \varphi(z_0) \quad \text{if } \beta < \|B\| < \sqrt{|\Delta|}. \]

By substituting (2.14) and (2.15) into (2.17) one arrives with
\[ \varphi(z_{\text{min}}) = \frac{d^2}{2\|B\|^2} \left(1 + \frac{2\|B\|^2}{d^2} - \sqrt{1 + \frac{4\|B\|^2}{d^2}}\right) = \tan^2 \left(\frac{1}{2} \arctan \frac{2\|B\|}{d}\right) \]  
(2.21)

and
\[ \varphi(z_0) = 1 + \frac{2\|B\|^2}{|\Delta|^2} - \frac{2}{|\Delta|^2} \sqrt{(d|\Delta| - \|B\|^2)(|\Delta| - d)||\Delta| - \|B\|^2)}, \]  
(2.22)

respectively. To get (2.6), it only remains to observe that \( \tan \theta = \|X\| \).

The proof is complete. \( \square \)

Remark 2.3. The bound (2.6) is optimal in the sense that given the values of \( |\Delta| > 0, \) \( d \in (0, |\Delta|/2), \) and \( \|B\| < \sqrt{d|\Delta|}, \) it is possible to choose a matrix \( L \) of the form (2.1), (2.2) such that for the eigenvector \( f = (1, x_-, x_+)^T \) associated with the (only) eigenvalue \( z \) of \( L \) within the interval \((-\gamma_-, \gamma_+)\) inequality (2.6) turns into equality.
To prove this statement set \( \gamma = \frac{\lambda}{z}, \gamma_\pm = \pm \gamma, \) and \( \lambda = \gamma - d. \) If \( \|B\| \leq \beta \) where \( \beta \) is given by (2.19) then choose \( b_- = 0 \) and \( b_+ = \|B\| \). Observe that in this case \( z = z_{\min} \) and hence by (2.20) such a choice of \( b_\pm \) just provides \( \|X\|^2 = x_\pm^2 \) with its maximal possible value, i.e. the equalities \( \tan^2 \theta = \phi(z_{\min}) = \Xi(|\Delta|, \|B\|) \) hold. If \( \|B\| > \beta \), first compute \( t \) by formula (2.13) for \( z = z_0 \) with \( z_0 \) given by (2.13). Then introduce \( b_+ = \sqrt{t\|B\|} \) and \( b_- = \sqrt{T-t\|B\|}. \) In such a case \( z = z_0 \) is the eigenvalue of the matrix \( L \) in \( \Delta \) and we have the equality \( \tan^2 \theta = \phi(z_0), \) that is, again the equality \( \tan^2 \theta = \Xi(|\Delta|, \|B\|) \) holds.

**Example 2.4.** Again assume that \( \gamma_+ = -\gamma_- = \frac{A}{\Xi} > 0. \) Assume in addition that \( \lambda = 0 \) and \( b_+ = b_- = \frac{b}{\sqrt{2}} \) for some \( b \geq 0. \) From (2.11) it is easy to see that in this case \( z = 0 \) is the (only) eigenvalue of the matrix \( L \) within the interval \( \Delta. \) Moreover, for the corresponding eigenvector \( f = (1, x_-, x_+)^T \) by (2.9) one infers that \( x_- = -\frac{b}{\sqrt{2}d} \) and \( x_+ = \frac{b}{\sqrt{2}d} \) taking into account that \( \gamma_- = -d \) and \( \gamma_+ = d. \) Since \( \|B\| = b, \) the equality \( \tan \theta = \sqrt{|x_-|^2 + |x_+|^2} \) yields

\[
\tan \theta = \frac{\|B\|}{d}.
\]

Notice that in this example \( \Xi(|\Delta|, \|B\|) = \Xi(2d, \|B\|) = \frac{\|B\|^2}{d^2} \) and, thus, the equality \( \tan^2 \theta = \Xi(|\Delta|, \|B\|) \) holds, too.

**Example 2.5.** Consider a matrix \( L \) of the form (2.1) with \( \gamma_- \), \( \gamma_+ \), and \( \lambda \) like in Example 2.4 that is, with \( \gamma_+ = -\gamma_- = d > 0 \) and \( \lambda = 0. \) Set \( b_+ = 0 \) and let \( b_- \) satisfy inequalities \( 0 \leq b_- < \sqrt{d|\Delta|}. \) Obviously, \( \|V\| = b_- \), \( |\Delta| = 2d \) and, thus, we have \( \|V\| < \sqrt{2d}. \) The eigenvalue \( z \) of the matrix \( L \) in the interval \( \Delta \) (which is the corresponding solution to (2.11)) simply coincides with \( z_{\min} \) (cf. formula (2.13)),

\[
z = -\frac{d}{2} + \sqrt{\frac{d^2}{4} + \|V\|^2}.
\]

Clearly, \( z \to d \) as \( \|V\| \to \sqrt{2d}. \) That is, in this case the distance \( \delta = \text{dist}(\sigma_0', \sigma_1) \) between the perturbed spectral set \( \sigma_0' = \{z\} \) and unperturbed spectral set \( \sigma_1 = \{-d, d\} \) can be done arbitrarily small.

### 3. General case

Recall that by a finite spectral gap of a self-adjoint operator \( T \) one understands an open finite interval on \( \mathbb{R} \) lying in the resolvent set of \( T \) and being such that both of its end points belong to the spectrum of \( T. \)

In the sequel, we adopt the following hypothesis.

**Hypothesis 3.1.** Let the Hilbert space \( \mathcal{H} \) be decomposed into the orthogonal sum of two subspaces, i.e.

\[
\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1.
\]

Assume that a self-adjoint operator \( L \) on \( \mathcal{H} \) reads with respect to the decomposition (3.1) as a \( 2 \times 2 \) operator block matrix

\[
L = \begin{pmatrix} A_0 & B \\ B^* & A_1 \end{pmatrix}, \quad \text{Dom}(L) = \mathcal{H}_0 \oplus \text{Dom}(A_1),
\]

where \( A_0 \) is a bounded self-adjoint operator on \( \mathcal{H}_0, A_1 \) a possibly unbounded self-adjoint operator on \( \mathcal{H}_1, \) and \( B \) a bounded operator from \( \mathcal{H}_1 \) to \( \mathcal{H}_0. \) Assume in addition, that \( A_1 \) has
a finite spectral gap $\Delta = (\gamma_-, \gamma_+)$, $\gamma_- < \gamma_+$, the spectrum of $A_0$ lies in $\Delta$, i.e. $\text{spec}(A_0) \subset \Delta$, and

$$\|B\| < \sqrt{d|\Delta|},$$

(3.2)

where

$$d = \text{dist}(\text{spec}(A_0), \text{spec}(A_1)).$$

If $f$ is a non-zero element of the Hilbert space $\mathfrak{H}$ and $\mathfrak{K}$ is a subspace of $\mathfrak{H}$, by the angle between $f$ and $\mathfrak{K}$ we understand the acute angle $\theta$ between $f$ and its orthogonal projection $f_{\mathfrak{K}}$ onto $\mathfrak{K}$, that is, $\theta = \arccos(||f_{\mathfrak{K}}||/||f||)$.

**Theorem 3.2. Assume Hypothesis 3.1** Assume in addition that the operator $L$ has an eigenvalue lying in the gap $\Delta$. Let $f$ be an eigenvector of $L$ associated with this eigenvalue. Then the (acute) angle $\theta$ between the vector $f$ and the subspace $\mathfrak{K}_0$ satisfies the bound

$$\tan^2 \theta \leq \Xi(||\Delta||, d, \|B\|),$$

(3.3)

where the function $\Xi$ is given by (1.9).

**Proof.** Assume that the eigenvector $f = f_0 \oplus f_1$, $f_0 \in \mathfrak{K}_0$, $f_1 \in \text{Dom}(A_1)$, of the operator $L$ is associated with an eigenvalue $\lambda \in \Delta$. Then the following equalities hold

$$A_0 f_0 + B f_1 = z f_0$$

(3.4)

$$B^* f_0 + A_1 f_1 = z f_1$$

(3.5)

Taking into account that $f$ is in the resolvent set of $A_1$, from (3.5) it follows that

$$f_1 = -(A_1 - z)^{-1} B^* f_0.$$  

(3.6)

Hence, $f_0 \neq 0$ (otherwise, for $f_0 = 0$, one would have $f_1 = 0$ and then $f = 0$). Equations (3.4) and (3.6) yield

$$A_0 f_0 - B(A_1 - z)^{-1} B^* f_0 = z f_0,$$

which implies

$$\langle A_0 f_0, f_0 \rangle - \langle B(A_1 - z)^{-1} B^* f_0, f_0 \rangle = z \|f_0\|^2$$

(3.7)

From now on suppose that

$$\|f_0\| = 1$$

(3.8)

and set $\lambda = \langle A_0 f_0, f_0 \rangle$. Clearly,

$$\lambda \in [\inf \text{spec}(A_0), \sup \text{spec}(A_0)].$$

(3.9)

By the spectral theorem we have

$$\langle B(A_1 - z)^{-1} B^* f_0, f_0 \rangle = \int_{\mathbb{R} \setminus (\gamma_- \gamma_+)} \frac{\langle dE_{A_1}(\mu) B^* f_0, B^* f_0 \rangle}{\mu - z},$$

(3.10)

where $E_{A_1}(\mu)$, $\mu \in \mathbb{R}$, denotes the spectral family of $A_1$. Let

$$\Delta_- = (-\infty, \gamma_-] \quad \text{and} \quad \Delta_+ = [\gamma_+, \infty).$$

By the mean value theorem there are real numbers $\mu_- \leq \gamma_-$ and $\mu_+ \geq \gamma_+$ such that

$$\int_{\Delta_{\pm}} \frac{\langle dE_{A_1}(\mu) B^* f_0, B^* f_0 \rangle}{\mu - z} = \frac{\langle E_{A_1}(\Delta_{\pm}) B^* f_0, B^* f_0 \rangle}{\mu_{\pm} - z} = \frac{\|E_{A_1}(\Delta_{\pm}) B^* f_0\|^2}{\mu_{\pm} - z},$$

(3.11)

respectively. Introduce the non-negative numbers $b_{\pm}$ by

$$b_{\pm} = \sqrt{\Delta_{\pm} \|E_{A_1}(\Delta_{\pm}) B^* f_0\|},$$

(3.12)
where
\[ \alpha_\pm = \frac{|\gamma_\pm - z|}{|\mu_\pm - z|} \leq 1. \] (3.13)

Obviously,
\[ \int_{\Delta_\pm} \frac{(dE_{A_1}(\mu)B^*f_0,B^*f_0)}{(\mu - z)^2} = \frac{b_\pm^2}{\gamma_\pm - z}. \] (3.14)

Thus, taking into account (3.9), (3.10), and (3.11), equation (3.7) turns into
\[ \lambda - b_\pm^2 - \frac{b_\pm^2}{\gamma_\pm - z} = 0 \] (3.15)

At the same time, by (3.6) we have
\[ \|f_1\|^2 = \int_{B(\gamma_-,\gamma_+)} \frac{(dE_{A_1}(\mu)B^*f_0,B^*f_0)}{(\mu - z)^2}. \] (3.16)

The contributions of the intervals \((-\infty, \gamma_-]\) and \([\gamma_+, \infty)\) to the integral on the r.h.s. part of (3.16) are estimated separately. For the first interval one derives
\[ \int_{\Delta_-} \frac{(dE_{A_1}(\mu)B^*f_0,B^*f_0)}{(\mu - z)^2} \leq \frac{1}{\gamma_- - z} \int_{\Delta_-} \frac{(dE_{A_1}(\mu)B^*f_0,B^*f_0)}{z - \mu}, \] which by (3.14) means
\[ \int_{\Delta_-} \frac{(dE_{A_1}(\mu)B^*f_0,B^*f_0)}{(\mu - z)^2} \leq \frac{b_-^2}{(\gamma_- - z)^2}. \] (3.17)

In a similar way one concludes that
\[ \int_{\Delta_+} \frac{(dE_{A_1}(\mu)B^*f_0,B^*f_0)}{(\mu - z)^2} \leq \frac{b_+^2}{(\gamma_+ - z)^2}. \] (3.18)

Then by combining (3.16), (3.17), and (3.18) one infers that
\[ \|f_1\|^2 \leq x_-^2 + x_+^2, \] (3.19)

where
\[ x_\pm = -\frac{b_\pm}{\gamma_\pm - z}. \] (3.20)

From (3.15), (3.20) it follows that the vector \(y = (1, x_-, x_+)^T\) is an eigenvector of the 3 \times 3 matrix
\[ \tilde{L} = \begin{pmatrix} \lambda & b_- & b_+ \\ b_- & \gamma_- & 0 \\ b_+ & 0 & \gamma_+ \end{pmatrix} \]
associated with the eigenvalue \(z\), that is, \(\tilde{L}y = \lambda y\). By (3.9) for \(\delta = \text{dist}(\lambda, \{\gamma_-, \gamma_+\})\) we have
\[ d \leq \delta \leq \frac{|\Delta|}{2}. \] (3.21)

In addition, by (3.12) the square of the norm \(\|\tilde{B}\| = \sqrt{b_-^2 + b_+^2}\) of the 1 \times 2 matrix-row \(\tilde{B} = (b_-, b_+)^T\) reads
\[ \|\tilde{B}\|^2 = \alpha_-^2 (E_{A_1}(\Delta_-)B^*f_0,B^*f_0) + \alpha_+^2 (E_{A_1}(\Delta_+)B^*f_0,B^*f_0) \]
and hence
\[ ||\tilde{B}||^2 \leq (\mathbb{E}_{A_1}(\Delta_-)B^*f_0,B^*f_0) + (\mathbb{E}_{A_1}(\Delta_+)B^*f_0,B^*f_0) \]
\[ = (B^*f_0,B^*f_0) = ||B^*f_0||^2 \]
\[ \leq ||B||^2, \quad (3.22) \]
taking into account first (3.13) and then (3.8). By the hypothesis inequality (3.2) holds. Combining (3.21) and (3.22) with (3.2) implies
\[ ||\tilde{B}||^2 < \sqrt{\delta|\Delta|}. \quad (3.23) \]
By Lemma 2.2 one then concludes that \( x^2 + x^2_{\perp} \leq \Xi(\|\Delta\|,\delta,\|\tilde{B}\|) \) which by (3.8) and (3.13) implies that
\[ \tan^2 \theta \leq \Xi(\|\Delta\|,\delta,\|\tilde{B}\|). \quad (3.24) \]
Given \( |\Delta| > 0, d \in (0,|\Delta|/2], \) and \( ||\tilde{B}|| \) satisfying (3.22), it is easy to see that the function \( \Xi(\|\Delta\|,\delta,\|\tilde{B}\|) \) is monotonously increasing with increasing \( ||\tilde{B}||, ||\tilde{B}|| \leq ||B|| \). For \( d < |\Delta|/2 \) it also monotonously increases if \( \delta \) decreases from \( \frac{|\Delta|}{2} \) to \( d \). Therefore, from (3.24) it follows that \( \tan^2 \theta \leq \Xi(\|\Delta\|,d,||B||) \), completing the proof. \( \Box \)

**Remark 3.3.** The bound (3.3) is optimal. This follows from Remark 3.2.

**Remark 3.4.** Notice that under condition \( ||B|| < \sqrt{d(|\Delta| - d)} \) by (7) Theorem 5.3] the operator angle \( \Theta \) between the unperturbed and perturbed spectral subspaces \( \text{Ran} E_A(\sigma_0) \) and \( \text{Ran} E_L(\sigma_0^*) \) satisfies the following (sharp) estimate:
\[ \Theta \leq \frac{1}{2} \arctan \kappa(||B||), \quad (3.25) \]
where the function \( \kappa(b) \) is defined for \( 0 \leq b < \sqrt{d(|\Delta| - d)} \) by
\[ \kappa(b) = \begin{cases} 
\frac{2b}{d} & \text{if } b \leq \sqrt{\frac{d}{2} (\frac{|\Delta|}{2} - d)} \\
\frac{|\Delta|}{2} + \sqrt{d(|\Delta| - d) \left[ \left( \frac{|\Delta|}{2} - d + b \right)^2 \right]} \quad & \text{if } b > \sqrt{\frac{d}{2} (\frac{|\Delta|}{2} - d)}.
\end{cases} \]
Surely, the bound (3.25) implies the corresponding estimate for the angle \( \theta \):
\[ \theta \leq \frac{1}{2} \arctan \kappa(||B||) \quad \text{whenever} \quad ||B|| < \sqrt{d(|\Delta| - d)}. \quad (3.26) \]
One observes by inspection that \( \Xi(|\Delta|,d,b) \leq \tan^2 \left( \frac{1}{2} \arctan \kappa(b) \right), \) \( 0 \leq b < \sqrt{d(|\Delta| - d)} \).

Moreover, if \( |\Delta| > 2d \) then for \( \sqrt{\frac{d}{2} (\frac{|\Delta|}{2} - d)} < b < \sqrt{d(|\Delta| - d)} \) the strict inequality \( \Xi(|\Delta|,d,b) < \tan^2 \left( \frac{1}{2} \arctan \kappa(b) \right) \) holds. Therefore, the bound (3.26) is not optimal in the case of eigenvectors.

Now we are in position to prove Theorem 1. This theorem appears to be a simple corollary to Theorem 3.2.

**Proof of Theorem 1.** Set \( \delta_0 = \text{Ran} E_A(\sigma_0) \) and \( \delta_0 = \text{Ran} E_A(\sigma_1) \). With respect to the orthogonal decomposition \( \delta_0 = \delta_0 \oplus \delta_1 \) the operators \( A \) and \( V \) read as \( 2 \times 2 \) block operator matrices,
\[ A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \]
where \( B = V|_{\mathcal{H}_0} \); \( \text{Dom}(A) = \mathcal{H}_0 \oplus \text{Dom}(A_1) \) and \( \text{Dom}(L) = \text{Dom}(A) \). Assume that \( \Delta \) is a gap of the set \( \sigma_1 \) that contains the whole set \( \sigma_0 \). Surely, the length \( |\Delta| \) of this gap satisfies the estimate \( |\Delta| \geq 2d \) and the bound \( 1.7 \) implies the inequality \( \|B\| < \sqrt{d|\Delta|} \). Then by Theorem 3.2, we have
\[ \tan^2 \theta \leq \Xi(|\Delta|, d, \|V\|), \]

taking into account that \( \|V\| = \|B\| \). Now it only remains to observe that \( \Xi(D, d, \|V\|) \) is a non-increasing function of the variable \( D, D \geq 2d \). For \( D \) varying in the interval \( [2d, \infty) \) it achieves its maximal value just at \( D = 2d \) and this value equals
\[ \max_{D, D \geq 2d} \Xi(|\Delta|, d, \|V\|) = \frac{\|V\|^2}{d^2}. \]

Thus, the following inequality holds
\[ \tan \theta \leq \frac{\|V\|}{d}. \]

The proof is complete. \( \square \)

**Remark 3.5.** Example 2.4 shows that the bound \( 1.8 \) is sharp.

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**REFERENCES**

[1] V. Adamyan, H. Langer, and C. Tretter, *Existence and uniqueness of contractive solutions of some Riccati equations*, J. Funct. Anal. **179** (2001) 448 – 473.

[2] C. Davis, *The rotation of eigenvectors by a perturbation. I and II*, J. Math. Anal. Appl. **6** (1963), 159 – 173; **11** (1965), 20 – 27.

[3] C. Davis and W. M. Kahan, *The rotation of eigenvectors by a perturbation. III*, SIAM J. Numer. Anal. **7** (1970), 1 – 46.

[4] V. Kostrykin, K. A. Makarov, and A. K. Motovilov, *On the existence of solutions to the operator Riccati equation and the tan \( \Theta \) theorem*, Int. Eq. Op. Th. **51** (2005), 121–140; arXiv: math.SP/0210032v2.

[5] V. Kostrykin, K. A. Makarov, and A. K. Motovilov, *Perturbation of spectra and spectral subspaces*, Tran. Amer. Math. Soc. (to appear); arXiv: math.SP/0306023v1.

[6] V. Kostrykin, K. A. Makarov, and A. K. Motovilov, *A generalization of the tan2\( \Theta \) Theorem*, Operator Theory: Adv. Appl. **149** (2004), 349 – 372; arXiv: math.SP/0302020v1.

[7] A. K. Motovilov and A. V. Selin, *Some sharp norm estimates in the subspace perturbation problem*, arXiv: math.SP/0409558v1.

[8] F. Riesz and B. Sz.-Nagy, *Leçons d’analyse fonctionelle*, 2nd ed., Académia Kiado, Budapest, 1953.

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