Single-Peaked Opinion Updates

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Abstract

We consider opinion diffusion for undirected networks with sequential updates when the opinions of the agents are single-peaked preference rankings. Our starting point is the study of preserving single-peakedness. We identify voting rules that, when given a single-peaked profile, output at least one ranking that is single peaked w.r.t. a single-peaked axis of the input. For such voting rules we show convergence to a stable state of the diffusion process that uses the voting rule as the agents’ update rule. Further, we establish an efficient algorithm that maximises the spread of extreme opinions.

1 Introduction

Ahead of elections, but also for competing products on a market, finding a way to maximally spread one particular opinion through exposure of contents to targeted agents has become of increasing interest. Advances in understanding the diffusion of opinions help to grasp the extent to which opinions can be manipulated and spread in networks. We study opinion diffusion [Grandi, 2017] in a setting where agents’ opinions are modelled as single-peaked rankings over a set of candidates. In each update step one agent observes the preferences of all their neighbours in the network, aggregates these by a given voting rule and changes their opinion accordingly. For issues where preferences are naturally single-peaked, it seems reasonable to assume that also the updated preferences of an agent in a diffusion process remain single-peaked. We investigate which voting rules are applicable in this sense, which lead to converging diffusion dynamics, and whether it is tractable to find update sequences that maximally spread an extreme opinion.

Research has found that computing an optimal sequence of updates to spread a specific opinion is easy for two competing opinions [Bredereck and Elkind, 2017] but it turns out to be hard in most cases involving multiple independent opinions [Auletta et al., 2019; Auletta et al., 2020]. However, for other scenarios, notably elections, the agents’ opinions are better modeled by rankings over some candidates than by independent opinions. In this paper, we explore this additional structure on opinions which allows us to consider various known voting rules (such as Kemeny, (weak) Dodgson, and Minimax Condorcet) as update rules. Since different opinions are not necessarily treated equally under some voting rules, our work significantly differs from work studying opinion diffusion of multiple independent opinions which often use some simple threshold function for the updating process. For example, under some voting rule one opinion (i.e. ranking) might only be adopted if and only if the majority of the agent’s neighbours bares this opinion, whereas the same might not be true for other opinions under the same rule.

Originally motivated by preference aggregation in context of economic phenomena such as prices or quantities [Black, 1948], single-peakedness is probably the most prominent restricted preference domain in social choice [Brandt et al., 2015; Faliszewski et al., 2014; Faliszewski et al., 2011]. This domain restriction solves many computational and conceptual issues of preference aggregation: among other inviting properties, the aggregation of rankings becomes tractable (e.g. for Kemeny), Condorcet cycles cannot exist, and Arrow’s impossibility theorem does not apply anymore. While political elections are often not (perfectly) single-peaked, preferences in other settings often depend on some one-dimensional criterion, such as when voting on the temperature in a room, choosing the starting time of some event, fixing the voting age for an election, or considering an adequate price for a product. For Kemeny’s rule (and thus the many other rules that coincide with Kemeny in the single-peaked domain), it is easy to see that, given single-peaked preferences, also at least one Kemeny outcome ranking is single-peaked. We investigate whether the same can be said for other ranking rules.

The paper first provides basic definitions in Section 2. The main part revolves around the following key questions which are discussed in Sections 3 to 5 and for each of which we briefly discuss related work here. We conclude in Section 6.

Which rules preserve single-peakedness? As we consider opinion diffusion of single-peaked preferences, we want to identify ranking rules that allow agents’ preferences to remain single-peaked after updates. Under the single-peaked domain, it is known that Condorcet winners exist such that ranking adaptions of Slater’s rule [Slater, 1961], Ranked Pairs/Tideman [Tideman, 1987], Beat Path/Schulze [Schulze, 2011], and Split Cycle [Holliday and Pacuit, 2020] – which repeatedly pick Condorcet winners – coincide with Ke-
Is it tractable to find a sequence of updates that promotes the maximal spread of an (extreme) opinion? In line with many works on opinion diffusion we study the maximal spread of one particular opinion. While for the case of two opinions a simple greedy approach guarantees a stable state with a maximal spread of one opinion [Frischknecht et al., 2013], the problem of finding such a sequence when three or more opinions are present becomes NP-hard for strict majority updates even on very restricted graph structures [Bredereck et al., 2020]. In addition, Auletta et al. [2019] show that this problem is hard for three opinions when using weak majority updates but identify some tractable graph structures. As we consider single-peaked preference rankings as opinions, we focus on the spread of extreme opinions, i.e., the opinions that rank the left- or rightmost candidates on the single-peaked axis the highest. Here we show that under voting rules for which an extreme ranking can only be adopted if and only if a (weak) majority of neighbours hold this opinion, a greedy approach similar to the approach for two opinions and majority updates can guarantee a maximum number of stable agents with the extreme opinion (but the remaining agents do not necessarily become stable). This case applies to Kemeny and Minimax Condorcet updates as well as updates based on weak Dodgson (as defined in [Fishburn, 1977]). Note that these results only hold since only updates to extreme opinions must follow the weak majority of their neighbours, in contrast to the general hardness results for majority updates on more than two opinions for converging sequences established by Auletta et al. [2019] and Auletta et al. [2020].

### 2 Preliminaries

A preference profile \( \mathcal{P} = (C, V) \) consists of a set \( C \) of \( m \) candidates and a set \( V \) of \( n \) voters. Each voter \( v \) is associated with a preference list (or ranking) \( >_v \) over the candidates. If, for instance, \( C = \{a, b, c\} \), the order \( c >_v b >_v a \) means that voter \( v \) prefers \( c \) the most and \( a \) the least. We omit the subscript from \( > \) if it is clear from context. We omit the set of rankings from the profile \( \mathcal{P} \) for convenience.

**Restricted Preferences.** Central to our work is the concept of single-peakedness, a domain restriction that assumes that voters’ preferences are mainly influenced by the position of the candidates on a one-dimensional axis.

**Definition 1.** We call a preference profile \( \mathcal{P} = (C, V) \) single-peaked if there is some ordering \( \triangleright \) of the candidates \( C \), the so called single-peaked axis, such that for all \( c_1, c_2, c_3 \in C \) with \( c_1 \triangleright c_2 \triangleright c_3 \) or \( c_3 \triangleright c_2 \triangleright c_1 \) each voter \( v \in V \) satisfies \( c_j \triangleright_v c_k \) if \( c_j \triangleright_v c_k \). We call the most preferred candidate of voter \( v \)'s preference its peak \( \triangleright (v) \).

In the remainder of the paper, when talking about a single-peaked preference profile \( \mathcal{P} = (C, V) \) without further specification, we assume \( |C| = m \) and that \( \mathcal{P} \) is single-peaked with respect to the axis \( c_1 \triangleright \cdots \triangleright c_m \). We denote by \( \mathcal{L} \) the set of all rankings over candidates \( C \) and by \( \mathcal{L}_{sp}(C) \) the set of single-peaked rankings over \( C \) w.r.t. the single-peaked axis \( \triangleright \). The extreme opinion \( (c_1 \triangleright \cdots \triangleright c_m) \in \mathcal{L}_{sp}(C) \) is denoted by \( r^1 \) and the opposite extreme \( (c_m \triangleright \cdots \triangleright c_1) \in \mathcal{L}_{sp}(C) \) by \( r^2 \).

| Rule                        | CWC | CLC | SPP | EMC |
|-----------------------------|-----|-----|-----|-----|
| Kemeny                      | ✓   | ✓   | ✓   | ✓   |
| Minimax Condorcet           | ✓   | ✓   | ✓   | ✓   |
| Weak Dodgson                | ✓   | ✓   | ✓   |     |
| Dodgson                     | ✓   | ✓   |     |     |
| Copeland                    | ✓   |     |     |     |
| Single Transferable Vote    |     |     | ✓   | ✓   |
| Borda                       |     |     |     |     |

Table 1: An overview which ranking rules are weak Condorcet winner (CWC) and loser (CLC) consistent, single-peaked preserving (SPP), and extremist majority consistent (EMC). See Sections 2 and 5.1 for definitions of the properties.
Condorcet Winners and Losers. For a preference profile \( P = (C, V) \), a (weak) Condorcet winner \( c \) is a candidate that is not defeated by another candidate in a pairwise comparison, i.e., for all \( c' \in C \setminus \{c\} \) we have \( \{v \in V \mid c \succ_v c'\} \geq \{v \in V \mid c' \succ_v c\} \). A strict Condorcet winner is a candidate \( c \) that strictly defeats all other candidates in a pairwise comparison, i.e., for all \( c' \in C \setminus \{c\} \) we have \( \{v \in V \mid c \succ_v c'\} > \{v \in V \mid c' \succ_v c\} \). The notions of (weak) Condorcet loser and strict Condorcet loser are defined analogously. While this is not true for general profiles, for single-peaked profiles a weak Condorcet winner and loser always exist and the set of weak Condorcet winners can be determined as indicated by the Median Voter Theorem.

Lemma 1 ([Black, 1948]). Let \( P \) be a single-peaked preference profile. Consider the order of the voters’ top-ranked candidates along the single-peaked axis. Then the set of median peaks of the voters’ preferences and any candidate between median peaks is the set of (weak) Condorcet winners.

Ranking Rules. A ranking rule is an aggregator function \( R: L^n \rightarrow \mathcal{P}(L) \) (where \( \mathcal{P} \) denotes the power set) that aggregates the preference rankings of voters into an output set of preference rankings. If ranking rule \( R \) always outputs rankings that rank a (weak) Condorcet winner highest whenever such a candidate exists, we call \( R \) (weak) Condorcet winner consistent. Similarly, a ranking rule \( R \) is called (weak) Condorcet loser consistent, if \( R \) always outputs rankings that rank a (weak) Condorcet loser lowest whenever one exists. Note that while it is usually desired to aggregate the preference rankings of voters into a single output ranking, a property often called resoluteness, most ranking rules fail to achieve this. In this case, one often uses tie-breaking rules to resolve irresoluteness. Specific ranking rules that we consider more closely and a general tie-breaking rule that is applied throughout this paper are deferred at the end of this section.

Diffusion Process. For a subset of voters \( V' \subseteq V \) of a preference profile \( P = (C, V) \) we define the preference profile induced by \( V' \) as \( P[V'] = (C, V') \). A preference network is a graph \( G = (V, E) \) with some profile \( (C, V) \) where the voters coincide with the vertices.

Given a preference network \( G = (V, E) \) with preference profile \( P = (C, V) \), we consider the following opinion diffusion process. At each update step, a voter \( v \) (also called the active voter) applies a ranking rule \( R \) on the preference profile induced by their neighbourhood \( N(v) \) in the preference network and takes the aggregated ranking as their new opinion. Formally, we replace the preference ranking of \( v \) by a ranking in \( R(P[N(v)]) \). We speak of update rules instead of ranking rules when considering opinion diffusion processes.

If an active voter \( v \) does not change their opinion, i.e., a ranking in \( R(P[N(v)]) \) coincides with \( v \)'s current preference ranking, then we call the voter \( v \) stable. A preference network \( G = (V, E) \) with preference profile \( P = (C, V) \) is in a stable state if all the voters in \( V \) are stable. We call a sequence of voters \( (v_1, \ldots, v_k) \) an update cycle in preference network \( G \) with preference profile \( P \) and ranking rule \( R \), when updating voters’ opinions along the voter sequence leads to the same preference profile \( P \) for \( G \). A ranking rule \( R \) is said to converge if any sequence of voter updates in any preference network \( G \) with profile \( P \) is finite, i.e., results in a stable state. That is, if \( R \) converges there cannot exist any update cycles.

Kemeny’s rule. Kemeny’s rule returns those rankings that minimise Kendall’s tau distance to all voters’ preference rankings. For a pair of rankings \( \succ, \succ_1 \), Kendall’s tau distance between \( \succ \) and \( \succ_1 \) is defined as \( d_K(\succ, \succ_1) = \sum_{c \in C} d_{\succ, \succ_1}(c, c') \), where \( d_{\succ, \succ_1}(c, c') \) is set to 0 if \( c \succ c' \) and \( c' \succ_1 c \) rank and \( c \) and \( c' \) consistently, and is set to 1 otherwise.

The Kemeny score of a ranking \( r \) with respect to a profile \( P = (C, V) \) is defined as \( d_K(r, P) := \sum_{v \in V} d_K(r, v) \). A ranking \( r \) with a minimal Kemeny score is called a Kemeny ranking of \( P \) and its score is the Kemeny score of the profile.

Brill et al. [2016] propose an update procedure by which voters swap the positions of two adjacent candidates in their ranking whenever the (strict) majority of their neighbours orders the two candidates in this way. When all voters’ preferences are single-peaked, their update rule can replicate Kemeny’s rule by making every voter execute such updates repeatedly until they reach a fixed preference. However, as the work of Brill et al. concerns opinion diffusion in directed graphs (that are acyclic or simple cycles), their results are not transferable to our setting. Further details on this relation are deferred to the full version of this paper.\(^1\)

Minimax Condorcet. Minimax Condorcet (MMC), also known as Simpson’s rule, ranks candidates by their worst defeat in pairwise comparisons. Let \( d_{\text{pm}}(c', c) := |\{v \in V \mid c \succ_v c'\} - |\{v \in V \mid c' \succ_v c\}| \) be the popularity margin between \( c, c' \in C \). Then MMC ranks candidates \( c \in C \) in non-decreasing order by their MMC score \( \text{MMC}(c) := \max_{c' \in C} d_{\text{pm}}(c', c) \). MMC differs from Kemeny’s rule even in the single-peaked domain, e.g., by violating Condorcet loser consistency.

Observation 1 (\( \star \)). MMC is not Condorcet loser consistent even in the single-peaked domain.

Tie-Breaking. Since the active voter in the opinion diffusion process can change only to a single new opinion we introduce a tie-breaking rule for irresolutive ranking rules. First, since the focus of this paper is single-peaked preferences, we always restrict the set of winning rankings to those that are single-peaked w.r.t. the given single-peaked axis, if existent. Further, we envision the active voter to tend to the single-peaked ranking closest to their own, i.e., we break possible further ties by minimising Kendall’s tau distance to the current opinion of the active voter. This assumption models a reluctance of agents to change their opinions and is needed in the proofs of the presented results. Any remaining ties are broken arbitrarily (deterministic). Note that a random tie-breaking rule would hinder our analysis of convergence, as stability cannot be guaranteed.

3 Preserving Single-Peakedness

In this section, we consider the question of whether a ranking rule preserves single-peakedness, i.e., outputs at least one single-peaked ranking when given a single peaked profile.\(^1\)

\(^1\)Available at https://arxiv.org/abs/2204.14094. Proofs of results marked with \( \star \) are deferred to the full version.
This is not only relevant for our later analysis of diffusion of single-peaked opinions, but also extends the existing research on this domain restriction.

**Definition 2.** A ranking rule $\mathcal{R}$ preserves single-peakedness if, given a single-peaked preference profile $\mathcal{P} = (C, V) \in \mathcal{L}_{sp}(C)$ with axis $\succ$, it always outputs at least one single-peaked ranking w.r.t. $\succ$, i.e., $\mathcal{R}(\mathcal{P}) \cap \mathcal{L}_{sp}(C) \neq \emptyset$.

We next investigate which ranking rules preserve single-peakedness. Intriguingly, our results show that preserving single-peakedness is independent of being (weak) Condorcet winner or loser consistent: Copeland’s rule is weak Condorcet winner and loser consistent but does not preserve single-peakedness. Conversely, Minimax Condorcet preserves single-peakedness but is not Condorcet loser consistent. It is easy to construct rules that preserve single-peakedness but are not Condorcet winner consistent. One such example is the rule that always returns the ranking $r^+$. Further, $r_1, \ldots, r_h$ are exactly those single-peaked rankings where the candidates $c_1, \ldots, c_i$ are the peaks.

**Proof sketch.** We construct the ordering iteratively. For $m = 3$ choose $(c_1 \succ c_2 \succ c_3)$, $c_2 \succ c_1 \succ c_3$, $c_2 \succ c_3 \succ c_1$, $c_3 \succ c_1 \succ c_2$. We then iteratively include candidates at the end of the single-peaked axis. For this we take the rankings of the previous ordering (over $m - 1$ candidates) and replace them with an ordering of rankings where the new candidate $c_m$ is included in every possible position (not violating single-peakedness, starting with the highest position possible. Lastly we append the sequence with $c_m \succ c_{m-1} \succ \cdots \succ c_1$.□

We can now prove that MMC always outputs at least one single-peaked ranking if the input profile is single-peaked. Assuming that this is not the case, we reach a contradiction by analysing the MMC scores of three neighbouring candidates that are creating a dip with the help of the order of single-peaked rankings constructed in Lemma 3.

**Theorem 1 (★).** MMC preserves single-peakedness.

**3.3 Other Rules**

We show that many other ranking rules that do not coincide with the above-mentioned rules fail to preserve single-peakedness; see the full version for details. We identify counterexamples for Dodgson’s and Copeland’s rule (with 5 candidates), Single Transferable Vote (with 3 candidates), and Borda’s rule (with 4 candidates), which all can be easily extended to examples with more candidates.

**Proposition 1 (★).** Dodgson’s, Copeland’s and Borda’s rule and Single Transferable Vote do not preserve single-peakedness.

However, we can show that on 3-candidate profiles Borda’s rule preserves single-peakedness. Copeland’s rule with 3 candidates coincides with Kemeny’s rule due to weak Condorcet loser and winner consistency. It thus preserves single-peakedness on these profiles. Note that any ranking on 2 candidates is single-peaked; thus under the here discussed rules the only open cases are for Copeland’s rule on profiles with 4 candidates and for Dodgson’s rule with 3 or 4 candidates.

We remark that weak Dodgson [Fishburn, 1977], in which one ranks candidates by the number of pairwise swaps needed to make a candidate a weak Condorcet winner, does preserve single-peakedness; see the full version for details.

**4 Convergence of Single-Peaked Updates**

We identified in Section 3 that Kemeny’s rule (or its equivalents on the single-peaked domain) and MMC preserve single peakedness. In this section, we show first for Kemeny updates and then for MMC updates (with tie-breaking as described in Section 2) that any sequence of updates is finite, i.e., ends in a stable state after finitely many updates. Note that convergence of weak Dodgson remains an open problem.

**4.1 Convergence of Kemeny Updates**

In the following, we outline how a stable state can be found in any network with single-peaked voter preferences that use Kemeny’s rule as an update rule (while maintaining single-peakedness and using the above mentioned tie-breaking rule).
Theorem 2. Let $G = (V, E)$ be a network with single-peaked preference profile $\mathcal{P} = (C, V)$. Then any update sequence using Kemeny’s rule is of length at most $|E| \cdot \binom{|C|}{2}$. That is, Kemeny’s rule converges.

Proof. We show that any update to $G$ decreases the sum $\sum_{u, v \in E} d_{\mathcal{P}}(v, u) \cdot r_v$ of the distances by at least 1. Consider an update by Kemeny’s rule for a voter $v \in V$ with opinion $r_v$, and let $\sigma$ be the newly obtained opinion of $v$, i.e., the Kemeny ranking of the profile $\mathcal{P}[N(v)]$ of $v$’s neighbours in $G$. Let $\kappa := \sum_{u \in N(v)} d_{\mathcal{P}}(u, v)$ and let $\kappa^* := \sum_{u \in N(v)} d_{\mathcal{P}}(u, \sigma)$. As $\kappa$ is an opinion minimizing Kendall’s tau distance to the opinions of $N(v)$, we have $\kappa^* \leq \kappa$. Further, only $v$ changes its opinion, so the sum of Kendall’s tau distances cannot increase. Thus we can estimate how much the opinion of $v$ changes.

5 Maximally Spreading Extreme Opinions

Especially in the political context, studying extreme opinions and their spread on social networks is often seen as more important than the spread of more central views. Platforms such as Facebook, Youtube or Instagram invest vast resources to combat the spread of hateful and extreme opinions.

Recall that for single-peaked axis $c_1 \succ \cdots \succ c_m$ the rankings $r^\uparrow = c_1 \succ \cdots \succ c_m$ and $r^\downarrow = c_m \succ \cdots \succ c_1$ are called the extreme opinions, as they rank the extreme candidates on the single-peaked axis highest. In the following, we consider which conditions an extreme opinion can be maximally spreading in a single-peaked preference network.

For this purpose we define the notion of extremist majority consistency for ranking rules that preserve single-peakedness – a property that holds for both Kemeny’s rule and MMC. We then formulate a general algorithm for extremist majority consistent rules that maximally spreads extreme opinions.

5.1 Extremist Majority Consistency

Informally, extremist majority consistency states that an extreme opinion can only be adopted by an agent when a majority of the agent’s neighbours hold this opinion.

Definition 3. Let $\mathcal{P} = (C, V) \in \mathcal{L}_{sp}^\infty(C)$. We call a ranking rule $\mathcal{R}$ that preserves single-peakedness extremist majority consistent if for any extreme opinion $r^* \in \{r^\uparrow, r^\downarrow\}$ we have $r^* \in \mathcal{R}(\mathcal{P}) \iff$ (a weak) majority in $V$ has opinion $r^*$.

One might think that for extremist majority consistency it is sufficient that a ranking $\mathcal{R}$ preserves single-peakedness and is weak Condorcet winner consistent. However, this is not the case since these two properties alone do not enforce that there exists an output ranking that is single-peaked and ranks the desired weak Condorcet winner ($c_1$ or $c_m$) highest.

We proceed by showing that Kemeny’s update rule and MMC are extremist majority consistent.

Theorem 4. Kemeny’s rule is extremist majority consistent.

Proof. Since the output of Kemeny’s rule is a weak Condorcet winner, it is already extremist majority consistent. Further, since Kemeny’s rule converges, any output ranking is also extremist majority consistent.
\[ d_{k}(r^\uparrow, r) \geq d_{k}(r', r) + 1 \text{ for all single-peaked rankings } r \in L_{sp}^\uparrow(C) \setminus \{r^\uparrow\}. \]

Also:

\[ d_{k}(r', P) = \sum_{v \in V} d_{k}(r', v) \]

\[ = |\{v \in V \mid v \succ r^\uparrow\}| + \sum_{v \in V \setminus \{v : v \not\succ r^\uparrow\}} d_{k}(r', v) \leq |\{v \in V \mid v \succ r^\uparrow\}| + \sum_{v \in V \setminus \{v : v \not\succ r^\uparrow\}} d_{k}(r^\uparrow, v) - 1 \leq |\{v \in V \mid v \succ r^\uparrow\}| + d_{k}(r^\uparrow, P) - |\{v \in V \mid v \not\succ r^\uparrow\}| \leq |\{v \in V \mid v \succ r^\uparrow\}| + d_{k}(r', P) - |\{v \in V \mid v \not\succ r^\uparrow\}| \]

and hence, \(|\{v \in V \mid v \succ r^\uparrow\}| \geq |\{v \in V \mid v \not\succ r^\uparrow\}|.

Note that for non-extreme opinions this equivalence need not hold, i.e., a Kemeny ranking need not be supported by a majority of the voters. However, if the majority of voters has opinion \(r\), then \(r\) is a Kemeny ranking; see the full version of this paper for details.

MMC, just like Kemeny’s rule, outputs an extreme opinion if and only if a (weak) majority of voters has this opinion.

**Theorem 5 (★).** MMC is extremist majority consistent.

### 5.2 Maximally Spreading Extreme Opinions for Extremist Majority Consistent Update Rules

We will next prove that for an extremist majority consistent ranking rule \(R\) and an extremist opinion \(r^*\) the following greedy update sequence \(\sigma^*\) reaches a stable state that maximises the number of voters with opinion \(r^*\). Note that due to step (3) below this sequence is only well defined if there exists a finite sequence of updates w.r.t. \(R\) that leads to a stable state for any network.

**Greedy Sequence \(\sigma^*\) for Extreme Opinion \(r^*\)**

1. Update every non-stable voter with opinion \(r' \neq r^*\) to opinion \(r^*\) if possible.
2. Update every non-stable voter with opinion \(r^*\).
3. Stabilise network: update non-stable voters with opinions \(r' \neq r^*\).

A similar algorithm was used before in the setting with two competing opinions in [Auletta et al., 2015; Bredereck and Elkind, 2017] and in the setting of discrete preference games in [Chierichetti et al., 2013]. In particular, Bredereck and Elkind [2017, Proposition 1] show that in the binary model with only two possible opinions, first updating every non-stable voter with the second opinion to the first opinion, and then updating every non-stable voter with the first opinion to the second opinion yields a stable outcome with a maximum number of voters having the first opinion. Similarly, we derive the following result.

**Lemma 6 (★).** Let \(G = (V, E)\) be a preference network with single-peaked preference profile \(P\) and let \(R\) be an extremist majority consistent ranking rule. The subsequence \(\sigma\) of updates in \(\sigma^*\) that only runs steps (1) and (2) satisfies:

1. The length of \(\sigma\) is at most \(2|V|\).
2. Each voter changes their opinion at most twice on \(\sigma\).
3. Computing \(\sigma\) requires \(O(|V|^2)\) executions of \(R\).

(iv) After \(\sigma\), the set \(V^*\) of voters that still have opinion \(r^*\) is stable.

(v) For every sequence \(\sigma'\) that maximises the number of voters with opinion \(r^*\) such that every such voter is stable, the set of voters with opinion \(r^*\) is equal to \(V^*\).

If the ranking rule in use converges, then we are guaranteed that step (3) has a finite number of updates. What remains to show is that, in this case, running step (3) does not affect the stability of the voters with opinion \(r^*\) and that the set \(V^*\) of voters with opinion \(r^*\) after completing \(\sigma^*\) is the same as for any other sequence that reaches a stable state and maximises the spread of opinion \(r^*\).

**Proposition 2 (★).** Let \(G = (V, E)\) be a network with single-peaked preference profile \(P\) and an extremist majority consistent and converging ranking rule \(R\), and let \(r^*\) be an extreme opinion. Let \(V^*\) be the set of voters with opinion \(r^*\) after completing steps (1) and (2) of \(\sigma^*\). The sequence \(\sigma^*\) satisfies:

1. After completing \(\sigma^*\), every voter is stable.
2. During step (3) of \(\sigma^*\), \(V^*\) remains unchanged.
3. For every sequence \(\sigma'\) that maximises the number of voters with opinion \(r^*\) such that every voter (independent of its opinion) becomes stable, the set of voters with opinion \(r^*\) is equal to \(V^*\).

**Corollary 1.** Let \(G = (V, E)\) be a network with preference profile \(P = (C, V) \in L_{sp}^\uparrow(C)\). Then one can compute a sequence of Kemeny updates that maximises the number of voters with opinion \(r^*\) in \(O(|V|^3 \cdot |C|^3)\) time.

**Corollary 2.** Let \(G = (V, E)\) be a network with preference profile \(P = (C, V) \in L_{sp}^\uparrow(C)\). Then there exists a finite sequence of MMC updates that maximises the number of voters with opinion \(r^*\) in \(\{r^\uparrow, r^\uparrow\}\) and ends in a stable state.

### 6 Conclusion

Motivated by the question of how single-peaked opinions propagate in social networks we studied opinion diffusion processes in this domain. We investigated which rules admit a single-peaked outcome given single-peaked preferences. For these rules we then established convergence for arbitrary update sequences and provided an algorithm that outputs an update sequence to optimally spread an extreme opinion in any preference network.

We conclude by suggesting future research directions. While single-peakedness is a well-established domain restriction, the study of opinion diffusion under single-crossing or single-caved domains is also well motivated. Further, our initial study on opinion diffusion may be extended: can we efficiently compute update sequences to optimally spread non-extreme opinions? Lastly, while many ranking rules coincide with Kemeny’s rule in the single-peaked domain, this is not necessarily true for preference profiles with bounded single-peaked width [Cornaz et al., 2012]. In terms of opinion diffusion, while a Kemeny ranking can be computed efficiently whenever the single-peaked width is small [Cornaz et al., 2013], it is not clear whether, e.g., our greedy sequence is applicable in this scenario.
Acknowledgements

Anne-Marie George was supported under the NRC Grant No 302203 “Algorithms and Models for Socially Beneficial Artificial Intelligence”. Jonas Israel was supported by the DFG grant BR 4744/2-1.

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