Ghost and singularity free theories of gravity

Candidate: Luca Buoninfante
Registration Number: 0522600065

Supervisors:
Prof. Gaetano Lambiase
Dr. Anupam Mazumdar

Co-Supervisor:
Prof. Massimo Blasone

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L. Buoninfante

Dipartimento di Fisica “E.R. Caianiello” Università di Salerno, I-84084 Fisciano (SA), Italy.

Abstract

Albert Einstein’s General Relativity (GR) from 1916 has become the widely accepted theory of gravity and been tested observationally to a very high precision at different scales of energy and distance. At the same time, there still remain important questions to resolve. At the classical level cosmological and black hole singularities are examples of problems which let us notice that GR is incomplete at short distances (high energy). Furthermore, at the quantum level GR is not ultraviolet (UV) complete, namely it is not perturbatively renormalizable. Most of the work try to solve these problems modifying GR by considering finite higher order derivative terms. Fourth Derivative Gravity, for example, turns out to be renormalizable, but at the same time it introduces ghost. To avoid both UV divergence and presence of ghost one could consider sets of infinite higher derivative terms that can be expressed in the form of entire functions satisfying the special property do not introduce new poles other than GR graviton one. By making a special choice for these entire functions, one could show that such a theory describes a gravity that, at least in the linear regime, can avoid both the presence of ghost and classical singularities (both black hole and cosmological singularities).

In this master’s thesis we review some of these aspects regarding gravitational interaction, focusing more on the classical level. Most of the calculations are done in detail and an extended treatment of the formalism of the spin projector operators is presented.
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Convention and notation

Natural units

We are going to list all the conventions and the notations we shall use in this thesis. We shall also follow Ref. [1].

In relativistic quantum field theory, it is standard to set

\[ c = 2.998 \times 10^8 \text{ m s}^{-1} = 1 \]

which turns meters into seconds and

\[ \hbar = \frac{\hbar}{2\pi} = 1.054572 \times 10^{-34} \text{ J s} = 1 \]

which turns joules into inverse seconds.

The use \( \hbar = 1 = c \) units (natural units) can simplify particle physics notation considerably. Since one typically deals with particles that are both relativistic and quantum mechanical, a lot of \( h' \)'s and \( c' \)'s will encumber the equations if natural units are not adopted. This makes all quantities have dimensions of energy (or mass, using \( E = mc^2 \)) to some power. Quantities with positive mass dimension, (e.g. momentum \( p \)) can be thought of as energies and quantities with negative mass dimension (e.g. position \( x \)) can be thought of as lengths.

Some examples are:

\[
\begin{align*}
[dx] &= [x] = [t] = M^{-1}, \\
[\partial_\mu] &= [p_\mu] = [k_\mu] = M^1, \\
[\text{velocity}] &= \frac{[x]}{[t]} = M^0.
\end{align*}
\]

Thus

\[ [dx^4] = M^{-4}. \]

The action is a dimensionless quantity

\[ [S] = [\int d^4x \mathcal{L}] = M^0, \]
and, consequently, it implies that 

\[[\mathcal{L}] = M^4.\]

For example, a free scalar field has Lagrangian\(^1\) \(\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi\) so 

\[[\phi] = M^1\]

and so on. In general bosons, whose kinetic terms have two derivatives, have mass dimension 1 and fermions, whose kinetic terms have one derivative, have mass dimension \(\frac{3}{2}\).

Since \(\hbar = 1\) then \(p_\mu = k_\mu\), so in this thesis we will use the four-wave vector to indicate the four-momentum.

2\(\pi\) factors

Often people get confused because they don’t know whether consider or not the factors of 2\(\pi\). The origin of all 2\(\pi\)’s is the relation

\[\delta(x) = \int_{-\infty}^{+\infty} dp e^{\pm 2\pi ikx}.\]

This identity holds with either sign. To remove the 2\(\pi\) from the exponent, we can rescale either \(x\) or \(p\). Since position generally is not an angular coordinate, it makes sense to rescale \(p\). Then

\[2\pi \delta(x) = \int_{-\infty}^{+\infty} dp e^{\pm ikx}.\]

Our convention for Fourier transforms is that the momentum space integrals have \(\frac{1}{2\pi}\) factors while position space integrals have no 2\(\pi\) factors:

\[f(x) = \int \frac{d^4 p}{(2\pi)^4} \tilde{f}(p)e^{\pm ikx}, \quad \tilde{f}(p) = \int d^4 x f(x)e^{\pm ikx}\]

Since we adopt the convention that the Fourier transform of the partial derivative \(\partial_\mu\) is \(ik_\mu\),

\[\partial_\mu f(x) = ik_\mu f(x),\]

we choose the “+” sign for \(f(x)\) and the “−” sign for \(\tilde{f}(p)\).

\(^1\)To be more rigorous we should call \(\mathcal{L}\) Lagrangian density. While the Lagrangian is defined as \(\mathcal{L} = \int d^3 x \mathcal{L}\). However in this thesis we shall just use the word Lagrangian.
Metric signature

In the Minkowski space-time the metric has signature

\[ \eta_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \]

Because of this convention one has \( k^2 = k_0^2 - \bar{k}^2 = m^2 > 0 \). The alternative choice, \( \eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1) \), would make \( k^2 < 0 \). Note that with the symbol “\( \bar{\cdot} \)” we indicate the three-spatial vectors. Then, the norm of a three-vector \( \bar{a} \) is indicated by \( |\bar{a}| \).

The sign of terms in Lagrangians is set so that they have positive energy density. In fact, given the Lagrangian \( \mathcal{L} = \mathcal{L}_{\text{kin}} - V \), we know that the potential energy \( V \) is positive in a stable system. For example, for a scalar field the mass term \( \frac{1}{2}m^2\phi^2 \) gives positive energy, so \( V = \frac{1}{2}m^2\phi^2 \) and \( \mathcal{L} = \mathcal{L}_{\text{kin}} - \frac{1}{2}m^2\phi^2 \). The kinetic term sign can then be chosen to obtain the correct dispersion relation. Thus, since \( \Box = \partial_\mu \partial^\mu \rightarrow -k^2 \) in momentum space and \( k^2 = m^2 \) on-shell, the field equations of motion is \((\Box + m^2)\phi = 0\). Therefore we have

\[ \mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2. \]

Since the propagator is defined as the inverse of the operator \(-(\Box + m^2)\), its form also depends on the convention sign:

\[ \mathcal{P}(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{i(x - y)} \frac{1}{k^2 - m^2 + i\epsilon}. \]

For other kind of Lagrangians, like photon and graviton ones, the signs are always chosen in a way to give positive defined energy and consistency with dispersion relation.

In all the equations we shall employ the modern summation convention where contracted indices can be raised or lowered without ambiguity:

\[ a \cdot b = a_\mu b^\mu = a^{\mu\nu}b_\mu = \eta_{\mu\nu}a^{\mu\nu}b_\nu. \]

Indices

In this thesis we shall use the index 0 for the temporal coordinate, and the other indices 1, 2, 3 for the spatial coordinates.

\[ ^2\text{If we consider the Fourier transform of this field equation we obtain } (-k^2 + m^2)\phi(k) = 0, \text{ that implies } k^2 = m^2, \text{ i.e. the dispersion relation is satisfied.} \]

\[ ^3\text{To obtain the second form for the Lagrangian we have integrated by parts and neglected the surface terms.} \]
CONVENTION AND NOTATION

Then, latin indices $i, j, k, l$ and so on generally run over three spatial coordinate labels, usually, 1, 2, 3 or $x, y, z$. Greek indices $\mu, \nu, \rho, \sigma$ and so on generally run over the four coordinate labels in a general coordinate system.

Note that in this thesis we shall frequently suppress the indices, especially when we work with the spin projector operators. Thus, for instance, $P^{2}_{\mu\nu\rho\sigma}$ will be just written as $P^{2}_{\mu\nu\rho\sigma}$, and in the same way also in the formulas that contain the spin projector operators there will be a suppression of the indices.

The indices are lowered and raised by the metric tensor $g_{\mu\nu}(x)$, that in the Minkowski space-time is represented by $\eta_{\mu\nu}$.

The adopted conventions for the geometric objects are given by:

Riemann tensor:

$$R_{\mu\lambda\nu\sigma} = \partial_{\lambda}\Gamma_{\mu\nu}^{\alpha} - \partial_{\nu}\Gamma_{\mu\lambda}^{\alpha} + \Gamma_{\lambda\rho}^{\alpha}\Gamma_{\mu\nu}^{\rho} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\lambda\mu}^{\rho};$$

Ricci tensor, $R_{\mu\nu} = R_{\mu\nu}^{\alpha\beta} = g^{\alpha\rho}R_{\alpha\mu\nu\rho}$:

$$R_{\mu\nu} = \partial_{\alpha}\Gamma_{\mu\nu}^{\alpha} - \partial_{\nu}\Gamma_{\mu\alpha}^{\alpha} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\nu\alpha}^{\beta} - \Gamma_{\nu\beta}^{\alpha}\Gamma_{\mu\alpha}^{\beta};$$

Then, we also have the curvature scalar $R = R_{\mu}^{\mu} = g^{\mu\nu}R_{\mu\nu}$.

By lowering the upper index with the metric tensor we can obtain the completely covariant Riemann tensor:

$$R_{\mu\nu\lambda\sigma} = \frac{1}{2} \left( \partial_{\lambda}\partial_{\sigma}g_{\mu\nu} + \partial_{\mu}\partial_{\sigma}g_{\nu\lambda} - \partial_{\sigma}\partial_{\nu}g_{\mu\lambda} - \partial_{\nu}\partial_{\lambda}g_{\mu\sigma} \right) + g_{\alpha\beta} \left( \Gamma_{\nu\lambda}^{\alpha}\Gamma_{\mu\sigma}^{\beta} - \Gamma_{\nu\sigma}^{\alpha}\Gamma_{\mu\lambda}^{\beta} \right).$$

It is worth to introduce the linearized forms for Riemann tensor, Ricci tensor and scalar curvature as we shall frequently use them. By performing the following perturbation around Minkowski metric,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$

the curvature tensors become

$$R_{\mu\nu\lambda\sigma} = \frac{1}{2} \left( \partial_{\lambda}h_{\mu\nu\sigma} + \partial_{\mu}h_{\lambda\nu\sigma} - \partial_{\sigma}h_{\lambda\nu\mu} - \partial_{\nu}h_{\lambda\mu\sigma} \right);$$

$$R_{\mu\nu} = \frac{1}{2} \left( \partial_{\rho}h_{\mu\nu}^{\rho} + \partial_{\rho}h_{\nu\mu}^{\rho} - \partial_{\mu}h_{\nu\rho} - \partial_{\nu}h_{\mu\rho} \right);$$

$$R = \partial_{\mu}\partial_{\nu}h^{\mu\nu} - \Box h.$$

Let us introduce a notation for the expressions containing either symmetric or antisymmetric terms. The indices enclosed in parentheses or brackets satisfy, respectively, the properties of symmetry or antisymmetry defined by the following rules:

$$T_{(\mu\nu)} \equiv \frac{1}{2} \left( T_{\mu\nu} + T_{\nu\mu} \right), \quad T_{[\mu\nu]} \equiv \frac{1}{2} \left( T_{\mu\nu} - T_{\nu\mu} \right).$$
Finally, let us discuss on the coupling constants appearing in the Einstein equations. The usual form of the field equation for General Relativity is given by

\[ G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \kappa \tau_{\mu\nu}, \]

where \( G_{\mu\nu} \) is the Einstein tensor and \( \tau_{\mu\nu} \) the energy-momentum tensor. It is important to explicit the form of the constant \( \kappa \) in natural units. Its expression in SI units is well known, and is given by \( \kappa = \frac{8\pi G}{c^4} \), where the value of the Newton constant is \( G = 6.67 \times 10^{-8} \text{ g}^{-1} \text{ cm}^3 \text{ s}^{-2} \). In natural units, since \( c = 1 \), one has \( \kappa = 8\pi G \). Often it is useful to display the Planck mass in the gravitational field equations. Indeed, the Planck mass is defined as

\[ m_P := \sqrt{\frac{\hbar c}{G}} \simeq 1.2 \times 10^{19} \text{ GeV}/c^2 \]

and in natural units \( G = \frac{1}{M_P^2} \). To get rid of the \( 2\pi \) factor is useful to introduce the reduced Planck mass that is defined as

\[ M_P := \sqrt{\frac{\hbar c}{8\pi G}} \simeq 2.4 \times 10^{18} \text{ GeV}/c^2 = 4.3 \times 10^{-9} \text{ kg}. \]

In this way the coupling constant reads as

\[ \kappa = \frac{1}{M_P^2}, \]

and the gravitational field equations turn out to be expressed in terms of the reduced Planck mass.

**Acronyms**

GR: General Relativity\(^4\).
H-E: Einstein-Hilbert.
ED: ElectroDynamics.
IDG: Infinite Derivative theories of Gravity.
UV: UltraViolet.

\(^4\)In this thesis we shall frequently use this acronym. Some authors use the expression General Relativity also to refer to modified Einstein-Hilbert action because the most of the fundamental principles of Einstein’s theory are still valid. Anyway, in this thesis every time we use the expression General Relativity (or its acronym) we mean Einstein’s GR, whose action is the Einstein-Hilbert action.
Albert Einstein’s General Relativity (GR) from 1916 has become the widely accepted theory of gravity and been tested observationally to a very high precision. Although a vast amount of observational data have made GR a remarkable theory, there still remain fundamental questions to resolve. At the classical level, cosmological and black hole singularities are examples of problems which let us suspect that the theory is incomplete at short distances (high energy). Furthermore, at the quantum level GR is not UV complete, namely it turns out to be perturbatively non-renormalizable.

One can easily do a power counting to understand whether GR, i.e. Einstein-Hilbert action, can be renormalizable. The superficial degree of divergence of a loop integral in GR turns out to be

\[ D = 4L - 2I + 2V, \]

where \( L \) is the number of loops, \( I \) is the number of internal propagators and \( V \) is the number of vertices. Using the well known topological relation

\[ L = 1 + I - V, \]

we get

\[ D = 2L + 2. \]

Thus, as the number of loops increases the superficial degree of divergence increases too, making the theory of GR perturbatively non-renormalizable.

In 1972, ’t Hooft and Veltman calculated the one-loop effective action of Einstein’s theory. They found that gravity coupled to a scalar field is non-renormalizable, but also showed how to introduce counter-terms to make pure GR finite at one-loop. In the following years the non-renormalizability of gravity coupled to various types of matter was also established. The crucial result was only obtained several years later by Goroff and Sagnotti and van de Ven, who showed the existence of a divergent term cubic in curvature in GR action at two loops.

All these works suggested that the perturbative treatment of Einstein’s theory as a quantum field theory, either on its own or coupled to generic matter fields, leads to the appearance of divergences that spoil the predictivity of the theory. There were

\[ ^5 \text{Note that the superficial degree of divergence in any dimension } d \text{ is given by } D = dL - 2I + 2V. \]
subsequently several attempts that tried to resolve this problem. Most of them emerge in the context of quantum field theory, while other attempts are based on different principles. Below we list different approaches that physicists used to follow and are still following [3].

• First, one could change the gravitational action, so the field equations. Examples of this approach are $f(\mathcal{R})$ theories, Fourth Derivative Gravity, Infinite Derivative Gravity (IDG) and so on. In 1977, Stelle proved that a theory containing four-derivative terms in the Lagrangian (i.e. terms quadratic in curvature) is perturbatively renormalizable [34]. Unfortunately it also appeared that this kind of Lagrangian leading to a renormalizable theory contain propagating ghosts\(^6\). These ghosts, would be physical particles and hence would violate the unitarity. On the other hand, a Lagrangian that do not contain ghosts turns out non-renormalizable. Thus, at the perturbative level, it seems to be present a problem of incompatibility between unitarity and renormalizability.

• A second attempt was based on the introduction of new particles and new symmetries, creating a new theory of gravity. So far the most important examples in this class are supergravity theories (SUGRA), whose pioneers are Freedman, Ferrara and van Nieuwenhuizen [7]. Supersymmetric theories are very special because the balance of bosonic and fermionic degrees of freedom leads to cancellation of divergences in loop diagrams and indeed even the simplest SUGRAs do not have the two-loop divergence that is present in GR. Besides the improved quantum behavior, these theories have other kind of either theoretical and experimental difficulties that thwarted this hope.

• A third possibility is that the non-renormalizability is an intrinsic pathology of the perturbative approach, and not of gravity itself. There have been more than one way of implementing this idea. The Hamiltonian approach to quantum gravity can be viewed as falling in this broad category. Examples of this subapproach are Geometrodynamics and Loop Quantum Gravity. There was also the covariant formalism, in which most work has been based, more or less explicitly, on the Feynman “sum over histories” approach. Misner was one of the pioneer [8]. Different versions of the gravitational functional integral were developed, like the Euclidean version and the lattice approach. Then, there was the non-perturbative way out of the issue of the UV divergences that is known as “non-perturbative renormalizability” and originates from the work of Wilson on the renormalization group.

• A very popular attempt that doesn’t follow the principles of quantum field theory is given by String Theory [9]. This is the main approach to construct an unifying quantum framework of all interaction. The quantum aspect of the gravitational

\(^6\)See Appendix C for a discussion on “good” and “bad” ghosts.
INTRODUCTION

field only emerges in a certain limit in which the different interactions can be distinguished from each other. All particles have their origin in excitations of fundamental strings. The fundamental scale is given by the string length; it is supposed to be of the order of the Planck length.

This thesis is focused on the first category of attempts. Most of the work on modifying GR has focused upon studying finite higher order derivative gravity and, as we have already mentioned, an example is Fourth Order Derivative Theory of Gravity by Stelle which is quadratic in curvatures. In 1977, Stelle considered the following action

\[ S = \int d^4x \sqrt{-g} \left( \alpha R + \beta R^2 + \gamma R^{\mu\nu} R_{\mu\nu} \right) \]  

and he proved that the theory is renormalizable for appropriate values of the coupling constants. Unfortunately, precisely for these values of the coupling constants the theory exhibits a bad behavior. It has a negative energy propagating degree of freedom that causes instability around the Minkowski vacuum and violation of unitary in the quantum regime. The spin-2 component of the UV modified graviton propagator is roughly given by

\[ \Pi = \Pi_{GR} - \frac{P^2}{k^2 - m^2}, \]  

and it shows the presence of the so called Weyl ghost in the spin-2 component that violates stability and unitarity conditions. Thus, in the UV regime the special form of modified Stelle propagator makes the loop integral appearing in the Feynman diagrams convergent at 1-loop and beyond but unfortunately this costs the presence of a massive spin-2 ghost.

Instead, as for \( f(R) \) we have the opposite situation. In fact, the theory is free-ghost but at the same time is non-renormalizable. It seems that one is lead to conclude that there is incompatibility between renormalizability and unitarity.

In 1989 Kuz’min [20] and in 1997 Tomboulis [21] noticed that if one considers a non-polynomial Lagrangian containing an infinite series in higher derivatives gauge theories and theories of gravity can be made perturbatively super-renormalizable. Following this

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7Such modified gravitational action has been also studied in Ref. [10],[11],[12],[13] and [14].
8Note that the square Riemann tensor doesn’t appear in Stelle action because of the existence of the so called Euler topological invariant. In fact, the following relation holds:
\[ R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 = \nabla_{\mu} K^\mu, \]

where the total derivative gives a zero contribute in the action. Thus, the Riemann tensor can be rewritten in terms of the Ricci tensor and curvature scalar, unless than a total derivative.
9It is a ghost because of the presence of the minus sign that comes from a negative kinetic energy in the Lagrangian. In Appendix C more details on ghost and unitarity are discussed.
direction, in 2006 the authors Biswas, Mazumdar and Siegel [16] argued that the absence of propagating ghosts and the renormalizability of the theory can only be realized if one considers an infinite number of derivative terms, and in particular it could be done by making use of the exponential function, that are allowed by the condition of general covariance [17]. In such a modified gravity they also argued that the theory could be asymptotically free. The infinite derivative action considered in [16] includes cosmological non-singular bouncing solutions, i.e. solutions that avoid the presence of Big Bang and Big Crunch. Based on the action introduced in [16] other progress was made. Cosmological perturbation analysis were performed in Ref. [18] which makes the bouncing model more robust, and tells us about some possible connection to inflationary cosmology. All these results suggest that it might be possible to make gravitational interaction weaker both at short distances and at early times, in a consistent way.

Indeed, in 2012 Biswas, Gerwick, Koivisto and Mazumdar [19] noticed that, by including an infinite number of derivative terms, the theory could be asymptotically free in the UV regime preserving general covariance and without violating fundamental physical principles, such as unitarity. The action considered by the authors in [19] is given by\(^{10,11,12}\)

\[
S = \int \sqrt{-g} \left( -\mathcal{R} + \mathcal{R}_1(\Box)\mathcal{R} + \mathcal{R}_{12}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}_{123}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} \right), \tag{6}
\]

where the \(\mathcal{F}_i(\Box)\)'s are functions of the D’Alambertian operator, \(\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu\), and contain an infinite set of derivatives:

\[
\mathcal{F}_i(\Box) = \sum_{n=0}^{\infty} f_{i,n} \Box^n, \quad i = 1, 2, 3. \tag{7}
\]

One requires that the \(\mathcal{F}_i(\Box)\)'s are analytic at \(\Box = 0\) so that one can recover GR in the infrared regime. Theories described by the action (6) are called Infinite Derivative theories of Gravity (IDG).

It is also interesting the fact that such infinite higher derivative actions appear in non-perturbative string theories.

Making an appropriate choice for the functions \(\mathcal{F}_i(\Box)\), the authors in Ref. [19]\(^{13}\) obtained the following modified propagator [32]

\[
\Pi = \frac{1}{a(k^2)}\Pi_{GR}, \tag{8}
\]

\(^{10}\)See Ref. [20], [21], [16], [22] and [23] for previous works in which the same idea to introduce infinitely many higher derivatives appears. See also [24] for a different interesting way to proceed.

\(^{11}\)See also [25], [26], [27], [28], [29], [30] and [31].

\(^{12}\)To be more precise we should write \(\mathcal{F}_i(\frac{\Box}{M^2})\) to have a dimensionless argument (\(\Box = [M]^2\)), but for simplicity we shall always write just \(\mathcal{F}_i(\Box)\), implying that a squared mass is present also in the denominator.

\(^{13}\)See also Ref. [23].
where
\[ a(k^2) = e^{\frac{k^2}{M^2}}, \] (9)
whose expression in coordinate space is given by
\[ a(\Box) = e^{-\frac{\Box}{M^2}}. \] (10)
Thus, the GR propagator is just modified by the multiplicative factor \( a(k^2) \). Since \( a(k^2) \) has no zeros on the complex plane\(^{14}\), this modification doesn’t introduce any new pole, in fact the only present pole is the massless transverse and traceless graviton. The parameter \( M \) corresponds to the scale at which the modification becomes important. Since this model is non-local because of the presence of an infinite set of higher derivatives, \( M \) could be the scale at which non-local effects emerge.

So far, good and promising results have been obtained in the linearized regime. In fact, the authors in Ref. \cite{19} have found that this models describe a singularity-free gravity, although their result only holds for mini black holes with a mass much smaller than the Planck mass\(^{15}\). Moreover, as we already mentioned, the theory also admits periodic cosmological solutions with bounce showing a scenario in which the singularity issue of the standard cosmology could be solved.

However, to say something more about classical singularities one should study the full theory in a generic curved background. In Ref. \cite{17} the authors have obtained the full field equations for the most general IDG action quadratic in the curvatures we wrote above and also verified the consistency with the already known results in the linear regime. One of the main aim of these IDG theories is to study astrophysical black holes in the framework of the full (i.e. not linearized) theory and try to understand whether singularities are present, but so far no new meaningful results have been obtained.

As for the quantum level, the current results from quantum loop computations in IDG theories seem very promising. In particular it seems that the presence of the exponential functions containing infinite derivatives could make convergent the loop integrals in the Feynman diagrams, giving a strong clue for the renormalizability of the theory. So far, just a toy scalar model has been considered to face the problem of renormalizability by the authors in Ref. \cite{33}. They got a modified superficial degree of divergence due to the presence of the exponential contribution; the relevant term is given by
\[ D = 1 - L. \] (11)

\(^{14}\)Typical functions with this kind of characteristic can be written as
\[ a(\Box) = e^{-\gamma(\Box)}, \]
where \( \gamma(\Box) \) is an analytic function of \( \Box \). Physically, as it has already said for the functions \( \mathcal{F}_i(\Box) \), the analyticity property of \( \gamma(\Box) \) is required to recover GR in the infrared limit. It is then easy to see that for any polynomial \( \gamma(\Box) \), as long as the highest power has positive coefficient, the propagator will be even more convergent than the exponential case \cite{15}, \cite{19}.

\(^{15}\)The result holds in the Newtonian approximation where the gravitational potential is very weak.
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Thus, if the number of loops is such that $L \geq 2$, the superficial degree of divergence, i.e. the power momenta in the integrals, becomes negative and the loop amplitudes are superficially convergent. Unlike what happens in GR, it seems that IDG theories can be perturbatively renormalizable.

For a toy scalar model it has been found that the 2-point function is divergent at one loop but, by adding appropriate counter terms, it can be made renormalized and the UV behavior of all other 1-loop diagrams as well as the 2-loop, 2-point function show the same renormalizable behavior.

Modesto in Ref. [23] (see also [35]) reconsiders the theory of Tomboulis [21] and states that by introducing special entire functions the theory of gravitational interaction can be made renormalizable at one loop and also at higher loops. In this way, since only a finite number of diagrams diverges in the UV limit the theory should be super-renormalizable.

Organization of this thesis

The goal of this thesis is to reach a satisfactory understanding of ghost and singularity free theories of gravity in the context of IDG models. Our study is mostly focused on the classical aspects, although something is said about the quantum level.

The work is structured as follows:

Chapter 1: We shall study the theory of photon field to worm up before dealing with the theory of graviton field. By starting from the photon Lagrangian we determine the main results, like the field equations, the counting of the degrees of freedom for both on-shell and off-shell photon and the photon propagator. We work by using the formalism of the spin projector operator by which the spin components of the photon field are more explicit. We also determine a set of polarization vectors in terms of which we rewrite the photon propagator. In the last subsection we perform the unitarity analysis verifying whether the theory contains ghosts.

Chapter 2: Once warming up with the theory of the photon field, we can easily approach the theory of the graviton field that corresponds to the linearized Einstein gravity. This chapter has the same structure of the previous one. Also here we make extensive use of the formalism of the spin projector operators. This formalism turns out to be very useful either to determine the graviton propagator and to distinguish the different graviton spin components. We shall see that the physical part of the propagator (saturated propagator) has a scalar component other than the spin-2 component. We also determine a set of polarization tensors in terms of which we rewrite the graviton propagator. In the last subsection we perform also for the graviton Lagrangian the unitarity analysis showing that “bad” ghosts are absent, but there is the
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presence of a “good” ghost, corresponding to the scalar graviton component, that is fundamental either to ensure the unitarity of the theory and to give the exact number of propagating mode components for the graviton, i.e. the two traceless and transverse ones.

Chapter 3 : We introduce the most general quadratic action of gravity. We consider the linear regime and determine field equations and propagator always by using the spin projector operators. Without specifying the form of the coefficients that appear in the theory we cannot say anything about either how many degrees of freedom propagates and unitarity. Indeed at the end of the chapter we make three special choices for the coefficients and we obtain GR, $f(R)$ theory and conformally invariant gravity as subclasses.

Chapter 4 : The starting point is the linear quadratic action obtained in the previous chapter. We are going to make a special choice for the coefficients to obtain a free-ghost theory of gravity. We also notice that the theory is singularity-free in the weak approximation (Newtonian approximation) and we are able to state that mini black holes don’t have any singularities. Indeed, we obtain a modified non-singular Newtonian potential that gives us the well known Newtonian potential in the infrared limit. At the end of the chapter we discuss about the parameter $M$ coming from the exponential factor, trying to understand its physical meaning and to obtain some constraints.

Conclusions: We summarize what we have done in this thesis and also give an overview of what physicists are working on.

Appendix A : We discuss on the concepts of on-shell and off-shell particles.

Appendix B : We treat in details the vector and tensor decompositions in terms of the spin projector operators. First we introduce the group representations and then the associated spin projector decompositions.

Appendix C : We introduce the concept of unitarity. We then give the definition of ghost field and discuss the difference between “good” and “bad” ghost in connection with the violation of the unitarity. We also give a prescription to verify whether ghosts violate the unitarity conditions, i.e whether bad ghosts are present. In the last section we discuss Fourth Derivative Gravity as an application.

Appendix D : We present the table of the Clebsch-Gordan coefficients because it turns to be useful to determine the graviton polarization tensors by starting from the photon polarization vectors.
Chapter 1

Vector field: photon

In this first chapter we are going to treat the theory of the vector field that describes the propagation of the electromagnetic wave (photon) in electrodynamics (ED). It will turns out to be a good exercise to warm up before discussing the linearized GR, i.e. the theory of the symmetric two-rank tensor, that we shall treat in the next chapter. We have organized both chapters in the same way, but, of course, for GR case the work will be harder as we have to deal with a tensor field and not with a simple vector field. In particular, for both chapters, we shall use the spin projectors formalism that will be very useful to calculate the propagator and understand which are the physical spin components [36], [37], [39].

1.1 Photon Lagrangian

The free real massless vector field is described by the Lagrangian

\[ \mathcal{L}_V = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu), \quad (1.1) \]

where \( A^\mu \) is a four-vector. We can rewrite \( \mathcal{L}_V \) as a quadratic form in the following way

\[ \mathcal{L}_V = \frac{1}{2} A_\mu \mathcal{O}^{\mu\nu} A_\nu, \quad (1.2) \]

where the symmetric operator \( \mathcal{O}^{\mu\nu} \) is given by

\[ \mathcal{O}^{\mu\nu} := \Box \left( \eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box} \right). \quad (1.3) \]

The Euler-Lagrange equations for \( A^\mu \) are given by

\[ \partial_\mu \frac{\partial \mathcal{L}_V}{\partial (\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}_V}{\partial A_\nu}, \]

\[ \partial_\mu \left( \frac{\partial \mathcal{L}_V}{\partial (\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}_V}{\partial A_\nu}, \]
and since \( \partial_{\mu} \frac{\partial L_V}{\partial (\partial_{\mu} A_\nu)} = (\Box \eta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) A^\nu \) and \( \frac{\partial L_V}{\partial A_\nu} = 0 \) we obtain the field equations

\[
(\Box \eta_{\mu\nu} - \partial_{\mu} \partial_{\nu}) A^\nu = 0. \tag{1.4}
\]

We can decompose (see Appendix B) every four-vector in terms of spin-1 and spin-0 components under the rotation group \( SO(3) \), i.e. \( A^\mu \in 0 \oplus 1 \), by introducing a complete set of projectors:

\[
\{\theta, \omega\} : \begin{cases} 
\theta_{\mu\nu} := \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box}, \\
\omega_{\mu\nu} := \frac{\partial_{\mu} \partial_{\nu}}{\Box}
\end{cases}, \tag{1.5}
\]

that in momentum space becomes

\[
\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}. \tag{1.6}
\]

It is easy to show that the following properties hold

\[
\theta + \omega = \mathbb{I} \Leftrightarrow \theta_{\mu\nu} + \omega_{\mu\nu} = \eta_{\mu\nu}
\]

\[
\theta^2 = \theta, \quad \omega^2 = \omega, \quad \theta \omega = 0
\]

\[
\Leftrightarrow \theta_{\mu\nu} \theta^\nu_{\rho} = \theta_{\mu\rho}, \quad \omega_{\mu\nu} \omega^\nu_{\rho} = \omega_{\mu\rho}, \quad \theta_{\mu\nu} \omega^\nu_{\rho} = 0,
\]

so we can decompose the four-vector \( A^\mu \) as

\[
A_\mu = \theta^\nu_{\mu} A_\nu + \omega^\nu_{\mu} A_\nu. \tag{1.8}
\]

This special decomposition corresponds to that in which \( A^\mu \) decomposes in transverse and longitudinal components. In fact, if \( k^\mu \) is the 4-momentum associated to the electromagnetic wave (or photon) we can immediately see that

\[
k^\mu \theta_{\mu\nu} = 0, \quad k^\mu \omega_{\mu\nu} = k_\nu;
\]

hence \( \theta \) and \( \omega \) project along the transverse and longitudinal components respectively. Furthermore, we can also verify that the transverse component has spin-1 and the longitudinal one spin-0 by calculating the trace of the two projectors:

\[
\eta^{\mu\nu} \theta_{\mu\nu} = 3 = 2(1) + 1 \text{ (spin-1)},
\]

\[
\eta^{\mu\nu} \omega_{\mu\nu} = 1 = 2(0) + 1 \text{ (spin-0)};
\]

By means this formalism of the spin projector operators in the space of four-vectors, one can make more clear which are the spin components of the vector field and, also, make
1.2. PHOTON DEGREES OF FREEDOM

the discussion more elegant.
The field equations (1.4) can be easily recast in terms of the projectors:

\[ \Box \theta_{\mu\nu} A^\nu = 0. \tag{1.11} \]

In momentum space the last equations become

\[ - k^2 \theta_{\mu\nu} A^\nu = 0 \Rightarrow k^2 = 0 \Rightarrow E^2 = |\vec{k}|^2, \tag{1.12} \]

namely the vector field \( A^\mu \) is such that only the spin-1 massless component propagates. Note that the equation (1.11) tells us that the transverse and longitudinal components decouple by means the field equations. If we want to speak in terms of spin components, we can say that only the spin-1 component propagates. In the next section we will see how to keep only the two physical degrees of freedom by imposing a gauge.

To conclude this section we shall introduce the gauge transformation under which the Lagrangian (1.1) is invariant.

Gauge invariance of photon Lagrangian

Let us observe that the Lagrangian (1.1), and the field equations (1.4), are invariant under gauge transformations

\[ A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha, \tag{1.13} \]

where \( \alpha \) is any differentiable function. In fact, by rewriting the field equations (1.4) in terms of the transformed vector field \( A'_\mu \), we obtain

\[ (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) A'_\nu = 0. \tag{1.14} \]

\[ \Leftrightarrow (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu + (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) \partial^\nu \alpha = 0, \]

and

\[ (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) \partial^\nu \alpha = 0 \Rightarrow (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu = 0, \tag{1.15} \]

i.e. the field equations don’t change under the gauge transformation (1.13). This invariance preserves the masslessness of the field.

1.2 Photon degrees of freedom

Now we want to determine the number of degrees of freedom of a vector field. We shall see that by using field equations and gauge invariance we can get rid of the unphysical degrees of freedom. At the end we shall see that an on-shell photon has only two degrees of freedom, instead an off-shell photon\(^1\) three degrees of freedom.

\(^1\)For a discussion on on-shell and off-shell photons see Appendix A. Note also that as synonyms of on-shell and off-shell we shall also use real and virtual respectively.
1.2. PHOTON DEGREES OF FREEDOM

1.2.1 On-shell photon

Let us start working with on-shell photon. First of all let us expand $A_\mu(k)$ in the basis of four-vectors $\{k_\nu, \tilde{k}_\nu, \varepsilon_1^\mu, \varepsilon_2^\mu\}$:

$$
k_\mu \equiv (k^0, \tilde{k}), \quad \tilde{k}_\mu \equiv (\tilde{k}^0, -\tilde{k}), \quad \varepsilon_i^\mu \equiv (0, \varepsilon_i), \quad i = 1, 2, \quad (1.16)
$$

thus

$$
A_\mu(k) = a k_\mu + b \tilde{k}_\mu + c^i \varepsilon_{i,\mu} \quad (1.17)
$$

By substituting (1.17) in the field equations (1.4) written in momentum space, we obtain

$$
(k^2 A_\mu(k) - k_\mu k^\nu A_\nu(k)) = 0
$$

$$
\Leftrightarrow \quad a k^2 k_\mu + b k^2 \tilde{k}_\mu + c^i k^2 \varepsilon_{i,\mu} - a k_\mu k^2 - b k_\mu k \cdot \tilde{k} - c^i k_\mu k^\nu \varepsilon_{i,\nu} = 0
$$

and by using the orthonormality relations (1.16) one has

$$
b k^2 \tilde{k}_\mu + c^i k^2 \varepsilon_{i,\mu} - b k_\mu k \cdot \tilde{k} = 0.
$$

Then

$$
k \cdot \tilde{k} = \eta_{\mu\nu} k^\mu \tilde{k}^\nu = (k^0)^2 + (\tilde{k})^2 \neq 0
$$

$$
\Rightarrow \quad b k^2 \tilde{k}_\mu + c^i k^2 \varepsilon_{i,\mu} - b k_\mu ((k^0)^2 + (\tilde{k})^2) = 0. \quad (1.18)
$$

If we consider $\mu = 0$ we have

$$
b k^2 \tilde{k}_\mu - b k_\mu ((k^0)^2 + (\tilde{k})^2) = 0
$$

$$
\Leftrightarrow \quad -2b |\tilde{k}|^2 k^0 = 0 \Rightarrow b = 0; \quad (1.19)
$$

the field equations allow to get rid of one degree of freedom, so now we have $4 - 1 = 3$ degrees of freedom.

If $b = 0$ then (1.18) becomes $k^2 c^i \varepsilon_{i,\mu} = 0$, and its spatial part gives us

$$
k^2 c^i \varepsilon_i = 0 \Rightarrow k^2 c^i = 0 \Leftrightarrow k^2 = 0 \lor c^i = 0. \quad (1.20)
$$

Hence we have two possible solutions that satisfy the last equation.

If $k^2 = 0$ one has

$$
A_\mu(k) = a k_\mu + c^i \varepsilon_{i,\mu}, \quad (1.21)
$$

i.e. $A^\mu$ describes a massless particle.

While, if $c = 0$ one has

$$
A_\mu(k) = a k_\mu. \quad (1.22)
$$
1.2. PHOTON DEGREES OF FREEDOM

Now, by choosing a gauge, we will see that only one of them is physically admissible. The gauge transformation (1.13) in momentum space is

\[ A'_\mu(k) = A_\mu(k) + ik_\mu \alpha(k), \] (1.23) and in the basis (1.16) we have chosen becomes

\[ (a'k_\mu + c'\varepsilon_{i,\mu}) = (ak_\mu + c\varepsilon_{i,\mu}) + ik_\mu \alpha(k) \]

\[ \Leftrightarrow \quad c^i = c^i, \quad a'(k) = a(k) + i\alpha(k). \]

By choosing \( \alpha(k) = -\frac{1}{i}a(k) \) we can eliminate the coefficient \( a \) in (1.17), so we get rid of another unphysical degree of freedom. At this point, we can immediately notice that if we choose \( c^i = 0 \) as the solution of the equation (1.20) we will obtain \( A_\mu = 0 \), but this is not a physical solution. Thus the solution of the equation (1.20) is \( k^2 = 0 \) that tells us again that the photon is massless. Hence, by means the constraint of the field equations and the freedom of choosing a gauge, we obtain that for an on-shell photon the vector field \( A_\mu \) assumes the following form:

\[ a = 0 = b \Rightarrow A_\mu(k) = c\varepsilon_{i,\mu}, \quad k^2 = 0. \] (1.24)

In conclusion we got rid of the unphysical degrees of freedom, keeping only the two physical one, whose information is held in the two coefficients \( c^1 \) and \( c^2 \).

1.2.2 Off-shell photon

The above discussion holds just for on-shell gauge field. As for off-shell photon we can’t impose the field equations, so we can eliminate just one degree of freedom via gauge symmetry. It means that we cannot eliminate the coefficient \( b \), that will turns out to be different from zero giving the third degree of freedom in the counting. Hence, we are able to state that an off-shell photon has three degrees of freedom.

We can summarize the two cases with the following expression:\(^2\)

\[
\text{4-dimension : } \begin{cases} 
\text{off-shell : } & 3 \text{ d.o.f.} \\
\text{on-shell : } & 2 \text{ d.o.f.}
\end{cases} \quad (1.25)
\]

\(^2\)More generally, if we are in \( D \)-dimensions we have:

\[
D\text{-dimension : } \begin{cases} 
\text{off-shell : } & (D - 1) \text{ d.o.f.} \\
\text{on-shell : } & (D - 2) \text{ d.o.f.}
\end{cases}
\]

as particular case we can see that in \( D = 2 \)

\[
\text{2-dimension : } \begin{cases} 
\text{off-shell : } & 1 \text{ d.o.f.} \\
\text{on-shell : } & 0 \text{ d.o.f.}
\end{cases}
\]
1.3 Photon propagator

Now our aim is to obtain the photon propagator; we can do it by working with the spin projector operators. In general given a Lagrangian written as a quadratic form in terms of an operator $O$, the propagator is defined as the inverse operator $O^{-1}$. A generic operator $O$ acting in the four-vectors space can be expanded in the basis $\{\theta, \omega\}$:

$$O = a\theta + b\omega;$$  \hspace{1cm} (1.26)

one can say the same for its inverse

$$O^{-1} = c\theta + d\omega.$$  \hspace{1cm} (1.27)

In general the coefficients $a, b, c, d$ can be complex numbers. Imposing that $OO^{-1} = I$, or equivalently $O^{\mu\nu}O^{-1}_{\mu\nu} = \delta^{\mu}_{\nu}$, we can obtain the propagator once we know the form of operator $O$; in fact:

$$(a\theta + b\omega)(c\theta + d\omega) = I$$

$$\Leftrightarrow ac\theta + bd\omega = I \Leftrightarrow c = \frac{1}{a}, \ d = \frac{1}{b}$$

$$\Rightarrow O^{-1} = \frac{1}{a}\theta + \frac{1}{b}\omega.$$  \hspace{1cm} (1.28)

In the case of the Lagrangian (1.1) we have $O = -k^2\theta$, i.e. $a = -k^2$ and $b = 0$. We notice that, since $b = 0$ we cannot directly invert the operator $O$. Thus the operator $O$ we have defined for the Lagrangian (1.1) is not invertible\(^3\).

We encounter the same problem also starting from the field equations. In fact, by considering the presence of a source $J^\mu$ we have to add the term $-A_\mu J^\mu$ to the Lagrangian and the field equations become

$$(\Box - \partial_\mu \partial_\nu) A^\nu = J_\mu \Leftrightarrow \Box A^\nu = (\theta_{\mu\nu} + \omega_{\mu\nu}) J^\nu \Leftrightarrow O_{\mu\nu} A^\nu = (\theta_{\mu\nu} + \omega_{\mu\nu}) J^\nu.$$  \hspace{1cm} (1.29)

Again to obtain the propagator we have to invert the same operator $O$ and in principle we could do it by acting on both members with spin projection operators\(^4\):

$$\theta \rightarrow -k^2\theta A = \theta J,$$  \hspace{1cm} (1.30)

namely on-shell photons don’t exist; then in $D = 3$

3-dimension : \[
\begin{cases}
\text{off-shell} : & 2 \text{ d.o.f.} \\
\text{on-shell} : & 1 \text{ d.o.f.}
\end{cases}
\]

\(^3\)Physically we can interpret this result saying that the fact that $b = 0$ implies that the spin-0 component (longitudinal component) doesn’t propagate, so it won’t appear in the physical part of the propagator.

\(^4\)We are suppressing the indices for simplicity.
1.3. PHOTON PROPAGATOR

\[ \omega \rightarrow 0 A = \omega J \Rightarrow \omega J = 0. \]  

(1.31)

We can notice that it’s impossible to invert both equations, indeed from (1.30) we can obtain the transverse component \( \theta A = -\frac{\theta}{\kappa} J \), but we are not able to do the same with (1.31) because we can’t invert the zero at the first member. Hence we have found that the spin-0 projection is undetermined and it means that there is a gauge freedom and concurrently a restriction on the source, i.e. the equation (1.31), \( \omega J = 0 \).\(^5\)

This mathematical obstacle can be overcome by adding a gauge fixing term to the Lagrangian (1.1). Moreover, we know that in any gauge theory the choice a gauge is needed to get rid of because of the spurious degrees of freedom. Thus we have seen that, already at classical level, we need a gauge fixing term because of both mathematical and physical reasons.

Although we have this problem of inversion for the operator \( O \), we can always obtain the physical part of the propagator. Indeed, the propagator always contains a gauge dependent part that is not physical and a gauge independent part that is physical, namely the part of the propagator appearing when we want to calculate, for example, the scattering amplitudes. The physical part of the propagator is often called saturated propagator, or sandwiched propagator, because it corresponds to the sandwich of the propagator between two conserved currents.\(^6\)

Now, first we are going to invert the operator \( O \) by introducing a gauge fixing term so we can obtain the propagator that, of course, will turn out to be gauge dependent. Secondly we are going to determine the saturated propagator, that doesn’t need the introduction of a gauge fixing term.

\(^5\)It is worth observing the connection between undetermined spin component and gauge freedom. In the case of ED, from (1.2), we can see that the Lagrangian is composed only by the spin-1 component \( L_V = \frac{1}{2} A_\mu \Box \theta^{\mu\nu} A_\nu \). \(\)\(^\text{(1.32)}\)

One can notice the Lagrangian in the last equation is invariant under spin-0 transformation, \( \delta A_\mu \sim \omega^{\mu\nu} A_\nu \), in fact \( \theta \omega = 0 \); and we also know that there is a restriction on the source in terms of the spin-0 component, \( \omega J = 0 \). It means that for the Lagrangian (1.32) there is a gauge symmetry that corresponds to the gauge invariance under transformations \( \delta A_\mu = \partial_\mu \alpha \). The arbitrary function \( \alpha \) is the scalar associated to the spin-0 symmetry.

\(^6\)In general, given a propagator \( P \) and two conserved currents \( J_1, J_2 \), the saturated propagator is given by the sandwich

\[ J_1 \overset{P}{\longrightarrow} J_2. \]  

(1.33)

Note that we are not writing the indices neither for the propagator \( P \) nor for the conserved currents \( J_1, J_2 \) for simplicity, but in general \( P \) can have two indices, for example two for photon case (\( D_{\mu\nu} \)) and four indices for graviton case (\( \Pi_{\mu\nu\rho\sigma} \)), as we shall see below.
1.3. PHOTON PROPAGATOR

1.3.1 Gauge fixing term for photon Lagrangian

By introducing the Lorenz gauge fixing term \(-\frac{1}{2\alpha} (\partial_\mu A^\mu)^2\), the total Lagrangian is

\[
\tilde{L}_V = L_v - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2
\]

(1.34)

where \(\tilde{O}^{\mu\nu} := \Box \eta^{\mu\nu} + \left(\frac{1}{\alpha} - 1\right) \partial^\mu \partial^\nu\), or in terms of the spin projector operators in momentum space

\[
\tilde{O}^{\mu\nu} = -k^2 \left( \theta^{\mu\nu} + \frac{1}{\alpha} \omega^{\mu\nu} \right).
\]

(1.35)

We notice that (1.35) corresponds to the operator (1.3) plus an additive term given by the gauge fixing term. Now we can invert the operator in (1.2) once we go into momentum space, in fact \(b = \frac{1}{\alpha} \neq 0\), i.e. the longitudinal component is present too. Thus, the photon propagator in a generic gauge is\(^7\)

\[
D^{\mu\nu}(k) \equiv \tilde{O}^{-1\mu\nu} = -\frac{1}{k^2} (\theta^{\mu\nu} + \alpha \omega^{\mu\nu}) = -\frac{1}{k^2} \left[ \eta^{\mu\nu} + (\alpha - 1) \frac{k_\mu k_\nu}{k^2} \right].
\]

(1.36)

Particular nomenclatures associated with \(\alpha\) are

\[
\begin{cases}
\alpha = 1 : \text{Feynman gauge} \\
\alpha = 0 : \text{Landau gauge}
\end{cases}
\]

(1.37)

as we have already observed the physics is unaffected by the value of \(\alpha\).

Note that the propagator in (1.36) contains a part that depends on the coefficient \(\alpha\), i.e. a gauge dependent part. If we consider the sandwich between two currents \(J(-K)\) and \(J(k)\) we can already notice that, because of the conservation law

\[
\partial^\mu J_\mu = 0 \iff k^\mu J_\mu = 0,
\]

(1.38)

\(^7\)Note that often the propagator is also defined as the vacuum expectation value of the time ordered product:

\[
\langle T \{ A_\mu(-k)A_\nu(k) \} \rangle = iD_{\mu\nu}(k) = -\frac{i}{k^2} (\theta_{\mu\nu} + \alpha \omega_{\mu\nu}).
\]

The expression \(T \{ A_\mu(-k)A_\nu(k) \}\) is the Fourier transform of the time ordered product that in coordinate space is defined as

\[
T \{ A_\mu(x)A_\nu(y) \} := A_\mu(x)A_\nu(y)\Theta(x_0 - y_0) + A_\nu(y)A_\mu(x)\Theta(y_0 - x_0),
\]

where the function \(\Theta(x_0 - y_0)\) is equal to 1 if \(x_0 > y_0\), and to 0 otherwise.
the saturated propagator is gauge independent:

\[ J^\mu(-k) D_{\mu\nu}(k) J^\nu(k) = J^\mu(-k) \eta_{\mu\nu} \frac{k^2}{k^2} J^\nu(k). \] (1.39)

This last equation will be the result of the next calculation.

### 1.3.2 Saturated photon propagator

Consider again the field equations (1.29) in momentum space and add to the both members the term \(-k^2 \omega_{\mu\nu} A^\nu\) :

\[ -k^2 (\theta_{\mu\nu} + \omega_{\mu\nu}) A^\nu(k) = J_\mu(k) - k^2 \omega_{\mu\nu} A^\nu(k). \] (1.40)

Now by multiplying with \(\theta^\mu_\rho\) we obtain

\[ -k^2 (\theta^\mu_\rho \theta_{\mu\nu} + \theta^\mu_\rho \omega_{\mu\nu}) A^\nu(k) = \theta^\mu_\rho J^\mu(k) - k^2 \theta^\mu_\rho \omega_{\mu\nu} A^\nu(k), \]

and, since \(\theta^\mu_\rho \theta_{\mu\nu} = \theta^\rho_\nu\) and \(\theta^\mu_\rho \omega_{\mu\nu} = 0\), we have

\[ -k^2 \theta_{\mu\nu} A^\nu(k) = \theta_{\mu\nu} J^\nu(k) \Leftrightarrow \theta_{\mu\nu} A^\nu(k) = -\frac{1}{k^2} \theta_{\mu\nu} J^\nu(k) \] (1.41)

Then by multiplying with \(\omega^\mu_\rho\) we obtain

\[ -k^2 (\omega^\mu_\rho \theta_{\mu\nu} + \omega^\mu_\rho \omega_{\mu\nu}) A^\nu(k) = \omega^\mu_\rho J^\mu(k) - k^2 \omega^\mu_\rho \omega_{\mu\nu} A^\nu(k) \]

\[ \Leftrightarrow -k^2 \omega_{\mu\nu} A^\nu(k) = \omega_{\mu\nu} J^\nu(k) - k^2 \omega_{\mu\nu} A^\nu(k). \] (1.42)

The equation (1.31) says that \(\omega_{\mu\nu} J^\mu = 0\), so by multiplying (1.42) with \(J^\nu(-k)\) we obtain

\[ J^\mu(-k) \omega_{\mu\nu} A^\nu(k) = -\frac{1}{k^2} J^\mu(-k) \omega_{\mu\nu} J^\nu(k), \] (1.43)

and doing the same for (1.41) one has

\[ J^\mu(-k) \theta_{\mu\nu} A^\nu(k) = -\frac{1}{k^2} J^\mu(-k) \theta_{\mu\nu} J^\nu(k). \] (1.44)

By combining the last two equations we obtain

\[ J^\mu(k) (\theta_{\mu\nu} + \omega_{\mu\nu}) A^\nu(k) = J^\mu(-k) \frac{-1}{k^2} (\theta_{\mu\nu} + \omega_{\mu\nu}) J^\nu(k) = J^\mu(-k) \frac{-\theta_{\mu\nu}}{k^2} J^\nu(k), \] (1.45)

or, without writing the indices,

\[ J(-k) (\theta + \omega) A(k) = J(-k) \frac{-1}{k^2} (\theta + \omega) J(k) = J(-k) \frac{-\theta}{k^2} J(k). \] (1.46)
We can notice that we have calculated the saturated propagator:

\[ J^\mu(-k) D_{\mu\nu}(k) J^{\nu}(k) = J^\mu(-k) \frac{-\theta_{\mu\nu}}{k^2} J^{\nu}(k); \quad (1.47) \]

hence the quantity on the right side between the two conserved currents is the physical part of the photon propagator. We can notice that a virtual (off-shell) photon has only the spin-1 component, namely the two transverse components and the longitudinal one.

### 1.4 Photon propagator and polarization sums

#### 1.4.1 Polarization vectors

In the section 1.2 we have seen that a virtual photon is a spin-1 particle, namely it has three components; instead a real photon has only two components, the longitudinal part is absent, and we showed that the difference between off-shell and on-shell is that for the latter we can also impose the field equations in addition to the gauge condition. By rewriting the saturated propagator in terms of the polarization vectors of the photon we can see explicitly that the two transverse and the longitudinal components for the off-shell photon and only the two transverse for the on-shell one are present.

To construct the set of polarization vectors we have to specify if we are considering either massless photon \((m = 0)\) or massive photon \((m \neq 0)\), [46]. Since the longitudinal component of a massive photon can be also chosen for the longitudinal component of a massless off-shell photon, we will study both massive and massless cases. In both cases the set of polarization vectors \(\{\epsilon^{(0)}, \epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)}\}\) has to form a 4-dimensional orthonormal and complete basis satisfying

\[ \epsilon^{(\lambda),\mu} \epsilon^{\mu}_{(\lambda')} = \eta_{\lambda\lambda'} \quad \text{(orthonormality)} \quad (1.48) \]

and

\[ \sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon^{(\lambda),\mu} \epsilon_{(\lambda),\nu} = \eta_{\mu\nu} \quad \text{(completeness).} \quad (1.49) \]

We shall start with the construction of the set in the case of massive photon.

---

\(^8\)We have seen that the saturated propagator, that corresponds to the physical part (gauge-independent), is invertible. Note that we cannot say that \(-\frac{\theta}{k^2}\) is the propagator or the inverse of the operator \(O\) : in fact \(O\theta = -k^2 \theta + \frac{1}{k^2} \theta = \theta \neq \mathbb{1}\). But we can say that \(J(-k)O(k)J(k)\) is invertible and the inverse operator turns out to be \(J(-k)\frac{\theta}{k^2} J(k)\).

\(^9\)We shall also follow Ref. [46].
1.4. PHOTON PROPAGATOR AND POLARIZATION SUMS

Massive photon

We choose a frame of reference in which the plane wave has spatial momentum $\mathbf{k}$. Now we choose two space-like transverse polarization vectors

$$\epsilon_{(1)}^\mu \equiv (0, \bar{\epsilon}_{(1)}) , \quad \epsilon_{(2)}^\mu \equiv (0, \bar{\epsilon}_{(2)})$$

(1.50)

imposing the conditions

$$\bar{\epsilon}_{(1)} \cdot \mathbf{k} = 0 = \bar{\epsilon}_{(2)} \cdot \mathbf{k}$$

(1.51)

and

$$\bar{\epsilon}_{(i)} \cdot \bar{\epsilon}_{(j)} = \delta_{ij}.$$  

(1.52)

The third polarization vector is chosen such that its spatial component points in the direction of the momentum $\mathbf{k}$, that is normalized according to (1.48). We will adopt the further condition that the four-vector $\epsilon_{(3)}^\mu$ is orthogonal to the four-momentum $k^\mu$,

$$k^\mu \epsilon_{(3), \mu} = 0.$$  

(1.53)

Taking this equation and the normalized condition (1.48) for $\lambda = 3 = \lambda'$, we found the components of the thus constructed longitudinal polarization vector:

$$\epsilon_{(3)}^\mu \equiv \left( \frac{\mathbf{k}}{m}, \frac{\mathbf{k}}{m} \right).$$

(1.54)

The normalization condition $\epsilon_{(3)}^2 = -1$ is satisfied since $\epsilon_{(3)}^2 = \frac{\mathbf{k}}{m} - \frac{\mathbf{k}^2 (k^0)^2}{m^2} = \frac{\mathbf{k}}{m} - \frac{(k^0)^2}{m} = -1$. It is worth noting that the longitudinal polarization vector (1.54) is not well defined in the case of massless photon because we have the mass $m$ at the denominator. This problem will be addressed below.

To complete the vector basis in Minkowski space we need to introduce a fourth time-like polarization vector with index $\lambda = 0$. We can simply use the 4-momentum $k^\mu$, namely

$$\epsilon_{(0)}^\mu := \frac{k^\mu}{m},$$

(1.55)

where the factor $\frac{1}{m}$ ensures the normalization condition according to (1.48). Also, it is obvious that the four-vector in (1.55) is orthogonal to the three space-like polarization vectors. Let us write down the 4-dimensional scalar product of our set of polarization vectors with the momentum vector:

$$k^\mu \epsilon_{(0), \mu} = k, \quad k^\mu \epsilon_{(\lambda), \mu} = 0, \quad \lambda = 1, 2, 3.$$  

(1.56)

One can also check that the completeness relation (1.49) is satisfied for the set we have just constructed.
1.4. PHOTON PROPAGATOR AND POLARIZATION SUMS

We know that the Lorenz condition \( k \cdot \epsilon = 0 \) has to hold\(^{10}\) but the vector polarization \( \epsilon^{(0)} \) doesn’t satisfy it. We have only three physical polarization vectors: in fact for a massive photon we have three degrees of freedom. A virtual massless photon, as we have already showed in (1.25) with \( D = 4 \), has three degrees of freedom too, and we shall see that for it we can choose the same longitudinal component, with the only difference that we cannot use the mass \( m \) as vector component (see below).

**Massless photon**

To construct the polarization states of the massless photon we can begin as in the massive case and introduce two transverse polarization vectors, \( \lambda = 1, 2 \), as in (1.50). However, now the momentum \( k^\mu \) can no longer be used as a basis vector: it cannot be normalized to 1 since the dispersion relation now reads \( k^2 = 0 \). In addition, as we have already noticed above, the longitudinal polarization vector (1.54) is not defined for \( k^2 = 0 \). In the massless case is impossible to construct a third polarization vector which is normalizable and at the same time such that the scalar product with \( k^\mu \) is zero.

To avoid this problem we arbitrarily define a time-like unit vector and choose it as time-like polarization vector, which in the chosen Lorentz frame simply is given by

\[
\epsilon^{\mu}_{(0)} := n^\mu \equiv (1, 0, 0, 0), \quad n^2 = +1. \tag{1.57}
\]

The longitudinal polarization vector can be then written in covariant form as\(^{11}\)

\[
\epsilon^{\mu}_{(3)} := k^\mu - n^\mu (n \cdot k) \sqrt{(n \cdot k)^2 - k^2}. \tag{1.58}
\]

This vector indeed has the correct normalization

\[
\epsilon^{\mu}_{(3)} \epsilon_{(3), \mu} = \frac{k^\mu k_\mu - 2(n \cdot k)^2 + n^2 (n \cdot k)}{(n \cdot k)^2 - k^2} = -1 \tag{1.59}
\]

We can easily verify that in this special Lorentz frame, where \( n^0 = 1 \) and \( n^2 = +1 \), the longitudinal polarization vector becomes

\[
\epsilon^{\mu}_{(3)} \equiv \left(0, \frac{\vec{k}}{k} \right). \tag{1.60}
\]

The 4-dimensional scalar product of the basis vectors and the momentum vector reads

\[
k \cdot \epsilon^{(1)} = 0 = k \cdot \epsilon^{(2)}, \quad k \cdot \epsilon^{(0)} = -k \cdot \epsilon^{(3)} = k \cdot n, \tag{1.61}
\]

---

\(^{10}\)For a massive photon the condition \( k \cdot \epsilon = 0 \) is a consistency relation that holds once we impose the field equations and the current conservation.

\(^{11}\)Since we are also writing \( k^2 \) we are considering the more general off-shell case. To obtain the on-shell third component we have just to impose \( k^2 = 0 \).
1.4. PHOTON PROPAGATOR AND POLARIZATION SUMS

which, of course, is valid in any frame of reference.
One can easily verify that the set we have just constructed satisfies the orthonormality and completeness relations (1.48) and (1.49).

Finally one also show that if we choose the spatial momentum \( \vec{k} \) along the third direction in the Minkowski space, \( \bar{k} = |\vec{k}| \hat{z} \), the set of polarization vectors reduces in the simple form

\[
\epsilon^{(0)} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(1)} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^{(3)} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.62}
\]

From this set we can easily go to a new set of polarization vectors whose transverse polarizations describe states with helicity \( j_z = +1 \) and \( j_z = -1 \). They can be easily introduced in the following way

\[
\begin{align*}
\epsilon^{\mu}_{(1,+1)} &= \frac{1}{\sqrt{2}} \left( \epsilon^{\mu}_{(1)} + i \epsilon^{\mu}_{(2)} \right), \\
\epsilon^{\mu}_{(1,-1)} &= \frac{1}{\sqrt{2}} \left( \epsilon^{\mu}_{(1)} - i \epsilon^{\mu}_{(2)} \right). \tag{1.63}
\end{align*}
\]

We can check that they correspond to the two helicity states by acting with the rotation matrix around the third axis

\[
R^{(z)\mu}_\nu (\vartheta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta & 0 \\ 0 & -\sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.64}
\]

In fact we obtain

\[
R^{(z)\mu}_\nu (\vartheta) \epsilon^{\nu}_{(1,+1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta & 0 \\ 0 & -\sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos \vartheta + i \sin \vartheta \\ -\sin \vartheta + i \cos \vartheta \\ 0 \end{pmatrix} = e^{i\vartheta} \epsilon^{\mu}_{(1,+1)}; \tag{1.65}
\]
1.4. PHOTON PROPAGATOR AND POLARIZATION SUMS

and

\[ R^{(z)}_{\nu\mu}(\vartheta)\epsilon^{(1,-1)}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta & 0 \\ 0 & -\sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos \vartheta + i \sin \vartheta \\ -\sin \vartheta - i \cos \vartheta \\ 0 \end{pmatrix} \]

\[ = e^{-i\vartheta} \begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix} = e^{-i\vartheta} \epsilon^{(1,-1)}_{\mu}. \]  

(1.66)

Finally let us introduce the three polarization vectors for a off-shell massless photon. We learned, from (1.25) with \( D = 4 \), that a virtual massless photon has three degrees of freedom that correspond to the three component of a spin-1 vector, indeed by studying the photon propagator we saw that only the spin-1 component, \( j_z = 1 \), is present (see (1.47)). The two transverse polarization vectors \( \epsilon^{(1,+1)}_{\mu} \) and \( \epsilon^{(1,-1)}_{\mu} \) correspond to the \( j_z = +1 \) and \( j_z = -1 \) helicity spin-1 components. To complete the spin-1 components it remains to define the longitudinal one with helicity \( j_z = 0 \), namely such that \( R^{(z)}_{\nu\mu}(\vartheta)\epsilon^{(1,0)}_{\nu} = \epsilon^{(1,0)}_{\mu} \) (scalar component). We can define as longitudinal polarization the same polarization vector used for the massive photon, with the only difference that we cannot use \( m \) as part of the components:

\[ \epsilon^{(1,0)}_{\mu} \equiv \frac{1}{k} \begin{pmatrix} k^3 \\ 0 \\ 0 \end{pmatrix}, \quad R^{(z)}_{\nu\mu}(\vartheta)\epsilon^{(1,0)}_{\nu} = \epsilon^{(1,0)}_{\mu} \]

(1.67)

Hence, the three polarization vectors for an off-shell massless photon are

\[ \epsilon^{(1,+1)}_{\mu} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \quad \epsilon^{(1,-1)}_{\mu} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}, \quad \epsilon^{(1,0)}_{\mu} \equiv \frac{1}{k} \begin{pmatrix} k^3 \\ 0 \\ 0 \end{pmatrix}. \]

(1.68)

1.4.2 Photon propagator in terms of polarization vectors

Now we can come back to what we anticipated at the beginning of the subsection 1.4.1. Our aim is to rewrite the saturated propagator in terms of the polarization vectors: we shall see that for a real (on-shell) photon only the two transverse polarization vectors \( \epsilon^{(1,+1)}_{\mu} \) and \( \epsilon^{(1,-1)}_{\mu} \) are present; instead for a virtual (off-shell) photon also the longitudinal polarization vector \( \epsilon^{(1,0)}_{\mu} \) is present. Hence we have another confirmation that the saturated propagator reads the interaction of a spin-1 particle with two conserved currents\(^{12}\).

\(^{12}\) In fact in (1.47) one realized that the physical part of the photon propagator has just the spin-1 component \( \theta \); while the spin-0 component \( \omega \) is absent.
Let us start studying the off-shell case.

**Off-shell photon**

Let us consider the saturated photon propagator in (1.47) or (1.78):

\[
J^\mu(-k)D_{\mu\nu}(k)J^\nu(k) = -\frac{1}{k^2} J^\mu(-k)\eta_{\mu\nu}J^\nu(k)
\]

\[
= -\frac{1}{k^2} [-J^0(-k)J^0(k) + J^1(-k)J^1(k) + J^2(-k)J^2(k) + J^3(-k)J^3(k)]
\]

and a virtual photon of four-momentum \( k^\mu \equiv (k^0, 0, 0, k^3) \). Current conservation implies that

\[
k^0J^0 = k^3J^3.
\]

Note that \(^13\)

\[
J^\mu(-k) \left( \sum_{j_2=+1,-1} \epsilon_{(1,j_2),\mu}\epsilon^*_{(1,j_2),\nu} \right) J^\nu(k) = J^\mu(-k)\epsilon_{(1,+1),\mu}\epsilon^*_{(1,+1),\nu}J^\nu(k)
\]

\[
+ J^\mu(-k)\epsilon_{(1,-1),\mu}\epsilon^*_{(1,-1),\nu}J^\nu(k)
\]

\[
= \frac{1}{2} \left[ J^1(-k) + iJ^2(-k) \right] \left[ J^1(k) - iJ^2(k) \right]
\]

\[
+ \frac{1}{2} \left[ J^1(-k) - iJ^2(-k) \right] \left[ J^1(k) + iJ^2(k) \right]
\]

\[
= J^1(-k)J^1(k) + J^2(-k)J^2(k),
\]

\(^13\)We are considering the complex conjugation because now the polarization vectors have also complex components.
and also that
\[ J^\mu(-k) \left( \epsilon_{(1,0),\mu} \epsilon^*_{(1,0),\nu} \right) J^\nu(k) = \frac{1}{k^2} \left[ J^0(-k)k_3 + J^3(-k)k_0 \right] \left[ J^0(k)k_3 + J^3(k)k_0 \right] \]
\[ = \frac{1}{k^2} \left[ -J^0(-k)k^3 + J^3(-k)k^0 \right] \left[ -J^0(k)k^3 + J^3(k)k^0 \right] \]
\[ = \frac{1}{k^2} \left[ J^0(-k)J^0(-k)(k^3)^2 - J^0(-k)J^3(-k)k^0k^3 \right. \]
\[ \left. - J^3(-k)J^0(-k)k^0k^3 + J^3(-k)J^3(-k)(k^0)^2 \right]. \quad (1.72) \]

Then, since the conservation relation (1.70) holds, we obtain
\[ J^\mu(-k) \left( \epsilon_{(1,0),\mu} \epsilon^*_{(1,0),\nu} \right) J^\nu(k) = -J^0(-k)J^0(k) + J^3(-k)J^3(k) \quad (1.73) \]

Hence, from the relations (1.70), (1.71) and (1.73), we deduce that in the off-shell case the saturated propagator can be rewritten as
\[ J^\mu(-k)D_{\mu\nu}(k)J^\nu(k) = -\frac{1}{k^2} J^\mu(-k) \left( \sum_{jz=+1,-1,0} \epsilon_{(1,jz),\mu} \epsilon^*_{(1,jz),\nu} \right) J^\nu(k). \quad (1.74) \]

As we have already seen in the section 1.2, we have had another confirmation that a virtual photon has three degrees of freedom.

**On-shell photon**

As for on-shell photon \((k^0 = k^3)\) the conservation law (1.70) becomes
\[ J^0 = J^3, \quad (1.75) \]
and so equation (1.69) reduces to
\[ J^\mu(-k)D_{\mu\nu}(k)J^\nu(k) = -\frac{1}{k^2} \left[ J^1(-k)J^1(k) + J^2(-k)J^2(k) \right]. \quad (1.76) \]

Hence, from (1.71) the saturated propagator for on-shell photon can be rewritten as
\[ J^\mu(-k)D_{\mu\nu}(k)J^\nu(k) = -\frac{1}{k^2} J^\mu(-k) \left( \sum_{jz=+1,-1} \epsilon_{(1,jz),\mu} \epsilon^*_{(1,jz),\nu} \right) J^\nu(k), \quad (1.77) \]
and this expression explicitly shows that an on-shell photon has only two degrees of freedom, i.e. the two transverse components, as we have already shown in the section 1.2.
1.5 Ghosts and unitarity analysis in Electrodynamics

In the Appendix C we show a method by which we can verify whether ghosts\footnote{In this case we mean “bad” ghost, i.e. ghosts whose presence violates the unitarity of the theory. Instead the “good” ghosts are ghosts whose presence is necessarily required to preserve the unitarity (see Appendix C).} and tachyons are absent, and, so, whether the theory preserves the unitarity. The method states that to verify whether ghosts and tachyons are absent in a given Lagrangian, one has to require that the propagator has only first order poles at \( k^2 - m^2 = 0 \) with real masses \( m \) (no tachyons) and with positive residues (no ghosts) \footnote{Note that the positivity of the imaginary part of the amplitude residue is just a necessary condition to ensure the unitarity condition, it is not a sufficient condition.} [36], [38]. Therefore, to verify that the presence of ghosts doesn’t violate the unitarity, we couple the propagator to external conserved currents, \( J^\mu \), compatible with the symmetry of the theory, and afterward we verify the positivity of the residue of the current-current amplitude\footnote{We have used the reality condition \( J(-k) = J^*(k) \).}.

In ED \( m = 0 \), since we are considering massless photons, so we know that tachyons are absent. Now let us consider the following tree level amplitude\footnote{See Ref. [41] to see how to calculate the residues of complex functions, or any other books on complex analysis.} [39]:

\[
\mathcal{A} = J^{*\mu}(k) \langle T (A_\mu(-k)A_\nu(k)) \rangle J^\nu(k) = iJ^\mu(-k)D_{\mu\nu}(k)J^\nu(k)
\]

\[
= -iJ^\mu(-k) \frac{1}{k^2} \left[ \eta_{\mu\nu} + (\alpha - 1) \frac{k_\mu k_\nu}{k^2} \right] J^\nu(k)
\]

\[
= -i \frac{1}{k^2} \eta_{\mu\nu} J^\mu(-k) J^\nu(k).
\]

We are going to study the residue of the amplitude to check whether the unitarity is preserved and at same time ghosts are absent. Hence let us calculate the residue of the amplitude (1.78) at \( k^2 = 0 \) :\footnote{Since we want to verify the positivity of the imaginary part of the residue in \( k^2 = 0 \), we can valuate the current conservation for \( k^0 = k^3 \).}

\[
\text{Res}_{k^2=0} \{ \mathcal{A} \} = \text{Res}_{k^2=0} \left\{ \frac{-i}{k^2} \eta_{\mu\nu} J^\mu(-k) J^\nu(k) \right\}
\]

\[
= \lim_{k^2 \to 0} k^2 \left( \frac{-i}{k^2} \eta_{\mu\nu} J^\mu(-k) J^\nu(k) \right) = -i\eta_{\mu\nu} J^\mu(-k) J^\nu(k).
\]
\[ 0 = k^\mu J_\mu = k^0 J_0 + k^3 J_3 = k^0 (J_0 + J_3) = 0 \Rightarrow J_0 = -J_3 = J^3; \quad (1.80) \]

thus by substituting in the residue (1.79) we find
\[
\text{Res}_{k^2 = 0} \left\{ \frac{-i}{k^2} \eta_{\mu\nu} J^{\mu*}(k) J^{\nu}(k) \right\} = -i J^0 J^0 + i J^1 J^1 + i J^2 J^2 + i J^3 J^3
\]
\[
= i (|J^1|^2 + |J^2|^2) \Rightarrow \text{Im} \text{Res}_{k^2 = 0} \{ \mathcal{A} \} > 0. \quad (1.81)
\]

Finally we have showed that the imaginary part of the residue is positive, so ghosts are absent and the unitarity is preserved.
Chapter 2

Symmetric two-rank tensor field: graviton

In the previous section we have studied the ED case where the main character was a vector field $A^\mu$. Now our aim is to study the same topics but in the context of linearized GR where the main physical quantity is a symmetric two-rank tensor, $h_{\mu\nu}$.

In my opinion the most elegant approach to work on linearized GR is the geometrical interpretation which gives us the physical meaning of the theory. This geometrical perspective is based on the fact that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is the metric of the space-time, where $h_{\mu\nu}$ represents a perturbation of Minkowski background. In this approach, the gravitational interaction is described by geometric tools such as the equation of geodesic deviation, curvature tensors, and so, it is interpreted as deformation of space-time. This deformation is associated to the propagation of gravitational waves which are determined by examining how $h_{\mu\nu}$ contributes to the curvature of the background space-time. This first way to proceed is also called top down approach.

General Relativity can also be seen as a classical\(^1\) field theory in its linearized form, and from this point of view one can directly introduce the concept of quantization. In this approach we can apply all standard field-theoretical methods; we shall treat linearized gravity as a classical field theory of the symmetric field $h_{\mu\nu}$ living in a flat space-time with Minkowski metric $\eta_{\mu\nu}$. In this way we forget that $h_{\mu\nu}$ has an interpretation in terms of a space-time metric, and instead we treat it as any other field living in Minkowski space-time. This second way to proceed is also called bottom up approach.

The geometrical and the field-theoretical approaches are complementary; some aspects of gravitational waves physics can be better understood from the former pro, some from the latter, and together they give us a deeper overall understanding.

In this Chapter we shall start from the point of view of geometrical approach and then by

\(^1\)With the word “classical” we mean “not quantum”.
linearizing the theory we shall proceed using tools of classical field theory. Afterwards we will be able to discuss linearized gravity from the point of view of quantum field theory, and we will obtain the graviton propagator. We will be able to interpret the graviton, the particle associated to the gravitational wave, as a particle mediator of gravitational interaction.

In our approach we will make use of the spin projector operators, by which we can better understand the number and which degrees of freedom propagate in General Relativity. In our approach we will take inspiration from\(^2\) \cite{36, 39}.

2.1 Graviton Lagrangian

Our starting point is the Lagrangian for any symmetric two-rank tensor field. We can obtain it in more ways: for example we can consider all the possible invariants quadratic in the tensor field \(h_{\mu\nu}\) and by imposing the field equations we can find the value of the coefficients for each terms. We shall proceed in a different way, i.e. we want to linearize starting from the geometrical approach to GR.

Once we have perturbed the metric

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu},
\]

we can obtain the Lagrangian by considering the quadratic part in \(h_{\mu\nu}\) of Hilbert-Einstein action\(^3\)

\[
S_{HE} = -\int d^4x \sqrt{-g} R.
\]

By performing the perturbation around Minkowski background one has

\[
S_{HE}(g_{\mu\nu}) = S_{HE}(\eta_{\mu\nu} + \delta g_{\mu\nu}) = S_{HE}(\eta_{\mu\nu}) + \frac{\delta S_{HE}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + O((\delta g)^2),
\]

where we are neglecting cubic terms in the perturbation \(h_{\mu\nu}\). Note that, since \(R(\eta_{\mu\nu}) = 0\), then also \(S_{HE}(\eta_{\mu\nu}) = 0\); moreover terms linear in the perturbation \(\delta g^{\mu\nu}\) do not appear. Thus, the linearized H-E action, quadratic in \(\delta g^{\mu\nu}\) is given by

\[
S_{HE} = \frac{\delta S_{HE}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = -\int d^4x (\delta g^{\mu\nu}) \left( R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \right)
\]

\(^2\) See also Ref. \cite{40} and \cite{61} for examples in which the formalism of the spin projector operators is used.

\(^3\) Let us note that the coupling constant \(\kappa = \frac{1}{M^2}\) doesn’t appear in H-E action (2.2). According to our convention, the coupling constant is introduced when the interaction term with a matter source is considered (see below).
2.1. GRAVITON LAGRANGIAN

Once we note that $\delta g^{\mu\nu} = -h^{\mu\nu}$, by using the linearized forms of the Riemann tensor, Ricci tensor and scalar tensor:

$$
\mathcal{R}_{\mu\nu\lambda\sigma} = \frac{1}{2}\left(\partial_\sigma \partial_\lambda h_{\mu\nu} + \partial_\mu \partial_\sigma h_{\nu\lambda} - \partial_\sigma \partial_\nu h_{\mu\lambda} - \partial_\mu \partial_\lambda h_{\nu\sigma}\right),
$$

$$
\mathcal{R}_{\mu\nu} = g^{\alpha\rho}\mathcal{R}_{\alpha\mu\nu} = \frac{1}{2}\left(\partial_\rho \partial_\nu h^\rho_\mu + \partial_\mu \partial_\rho h^\rho_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right),
$$

$$
\mathcal{R} = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h,
$$

the perturbed action in (2.4) becomes

$$
S_{HE} = -\int d^4x(-h^{\mu\nu})\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R}\right)
$$

$$
= \int d^4x h^{\mu\nu}\left[\frac{1}{2}\left(\partial_\rho \partial_\nu h^\rho_\mu + \partial_\mu \partial_\rho h^\rho_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right)\right. \\
- \frac{1}{2}\eta_{\mu\nu}\left(\partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h\right]
$$

$$
= \int d^4x \left(h^{\mu\nu}_\sigma \partial^\sigma \partial^\rho h_{\mu\nu} - h\partial^{\mu\sigma} \partial^\rho h_{\mu\nu} - \frac{1}{2}h_{\mu\nu} \Box h^{\mu\nu} + \frac{1}{2}h \Box h\right)
$$

$$
= \int d^4x L_{HE},
$$

thus the Lagrangian for any symmetric two-rank tensor is

$$
L_{HE} := h^{\mu\nu}_\sigma \partial^\sigma \partial^\rho h_{\mu\nu} - h\partial^{\mu\sigma} \partial^\rho h_{\mu\nu} - \frac{1}{2}h_{\mu\nu} \Box h^{\mu\nu} + \frac{1}{2}h \Box h.
$$

By raising and lowering the indices with the metric tensor $\eta_{\mu\nu}$, we can rewrite the Lagrangian (2.7) in the following way:

$$
L_{HE} = h^{\mu\nu}_\sigma \left(\partial^\rho \partial^\sigma \eta^{\mu\rho}\right) h_{\rho\sigma} - h^{\mu\nu}_\sigma \left(\partial^\rho \partial^\sigma \eta^{\mu\sigma}\right) h_{\rho\sigma}
$$

$$
- \frac{1}{2}h_{\mu\nu} \left(\eta^{\mu\rho} \eta^{\nu\sigma} \Box\right) h_{\rho\sigma} + \frac{1}{2}h_{\mu\nu} \left(\eta^{\mu\nu} \eta^{\rho\sigma} \Box\right)
$$

$$
= \frac{1}{2}h_{\mu\nu} \left[2\partial^\rho \partial^\sigma \eta^{\mu\rho} - 2\partial^\rho \partial^\sigma \eta^{\nu\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} \Box + \eta^{\mu\sigma} \eta^{\nu\rho} \Box\right] h_{\rho\sigma}
$$

$$
= \frac{1}{2}h_{\mu\nu} \left[\left(\eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\rho} \eta^{\nu\sigma}\right) \Box - \eta^{\mu\sigma} \partial^\rho \partial^\sigma - \eta^{\nu\rho} \partial^\mu \partial^\nu
$$

$$
+ \frac{1}{2} \left(\eta^{\nu\rho} \partial^\sigma \partial^\rho + \eta^{\rho\sigma} \partial^\mu \partial^\rho + \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\mu\rho} \partial^\nu \partial^\sigma\right)\right] h_{\rho\sigma}.
$$

\footnote{We started from the geometrical point of view to find $L_{HE}$, and in this case $h_{\mu\nu}$ is interpreted as the metric perturbation; but from the point of view of field theory it is only seen as a generic field and we can’t say that it is related to any metrics at this level.}
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Thus, the Lagrangian (2.7) can be recast as

\[ L_{HE} = \frac{1}{2} h_{\mu \nu} O^{\mu \nu \rho \sigma} h_{\rho \sigma}, \]  

(2.9)

where \( O^{\mu \nu \rho \sigma} \) is given by

\[ O^{\mu \nu \rho \sigma} := - \left( \frac{1}{2} \eta^{\mu \rho} \eta^{\nu \sigma} + \frac{1}{2} \eta^{\mu \sigma} \eta^{\nu \rho} - \eta^{\mu \nu} \eta^{\rho \sigma} \right) \Box - \eta^{\mu \nu} \partial^\rho \partial^\sigma - \eta^{\rho \sigma} \partial^\mu \partial^\nu \]

\[ + \frac{1}{2} \left( \eta^{\nu \rho} \partial^\mu \partial^\sigma + \eta^{\nu \sigma} \partial^\mu \partial^\rho + \eta^{\mu \rho} \partial^\nu \partial^\sigma + \eta^{\mu \sigma} \partial^\nu \partial^\rho \right) \],

(2.10)

and satisfies the symmetries

\[ O^{\mu \nu \rho \sigma} = O^{\nu \mu \rho \sigma} = O^{\mu \nu \sigma \rho} = O^{\rho \sigma \mu \nu}. \]  

(2.11)

By varying the linearized action, we can obtain the Euler-Lagrange equations for the symmetric two-rank tensor \( h_{\mu \nu} \).

In order to do this, it is more convenient to rewrite the Lagrangian (2.7) only in terms of the first derivatives of \( h_{\mu \nu} \) by means integration by parts:

\[ L_{HE} = -\partial_\rho h^\rho_{\alpha \beta} \partial_\alpha h^\beta_{\mu \nu} + \partial_\alpha h \partial_\beta h^\beta_{\alpha \mu} + \frac{1}{2} \partial_\mu h^\alpha_{\alpha \beta} \partial_\beta h_{\alpha \beta} - \frac{1}{2} \partial_\rho h \partial^\rho h. \]

(2.12)

The field equations are given by

\[ \partial_\sigma \frac{\partial L_{HE}}{\partial (\partial_\sigma h_{\mu \nu})} = \frac{\partial L_{HE}}{\partial h_{\mu \nu}}, \]

(2.13)

thus by computing the derivatives with respect to \( h_{\mu \nu} \) and \( \partial_\sigma h_{\mu \nu} \) of the Lagrangian in (2.12) we obtain

\[ \frac{\partial L_{HE}}{\partial h_{\mu \nu}} = 0 \]  

(2.14)

and

\[ \frac{\partial L_{HE}}{\partial (\partial_\sigma h_{\mu \nu})} = -\eta^{\mu \sigma} \partial_\rho h^\nu_{\rho \mu} - \eta^{\nu \sigma} \partial_\rho h^{\rho \mu} + \partial^\rho h^{\mu \nu} \]

\[ + \eta^{\mu \nu} \partial_\rho h^{\sigma \rho} + \eta^{\nu \sigma} \partial^\mu h - \eta^{\mu \nu} \partial^\sigma h \]

\[ \Rightarrow \partial_\rho \frac{\partial L_{HE}}{\partial (\partial_\rho h_{\mu \nu})} = -\partial^\mu \partial_\rho h^\nu_{\rho \mu} - \partial^\nu \partial_\rho h^{\rho \mu} + \Box h^{\mu \nu} \]

\[ + \eta^{\mu \nu} \partial_\rho \partial_\sigma h^{\rho \sigma} + \partial^\mu \partial^\nu h - \eta^{\mu \nu} \Box h. \]

(2.15)

Hence, substituting (2.14) and (2.15) into (2.13) we obtain the field equations in the vacuum:

\[ \partial^\mu \partial_\rho h^\nu_{\rho \mu} + \partial^\nu \partial_\rho h^{\rho \mu} - \Box h^{\mu \nu} + \eta^{\mu \nu} \partial_\rho \partial_\sigma h^{\rho \sigma} - \partial^\mu \partial^\nu h + \eta^{\mu \nu} \Box h = 0, \]

(2.16)

\footnote{From the geometrical point of view we can obtain the same field equations by linearizing Einstein equations.}
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or equivalently, raising and lowering the indices with $\eta_{\mu\nu}$,

$$\left(\eta_{\mu\rho}\Box + \eta_{\rho\sigma}\partial_{\mu}\partial_{\sigma} - \eta_{\mu\sigma}\partial_{\nu}\partial_{\rho} + \eta_{\nu\rho}\partial_{\mu}\partial_{\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}\Box\right) h^{\rho\sigma} = 0. \quad (2.17)$$

Now multiply for $\eta^{\mu\nu}$ one gets

$$(2\Box \eta_{\rho\sigma} - 2\partial_{\rho}\partial_{\sigma}) h^{\rho\sigma} + 4 (\partial_{\rho}\partial_{\sigma} - \Box \eta_{\rho\sigma}) = 0 \quad (2.18)$$

$$\Leftrightarrow (\partial_{\rho}\partial_{\sigma} - \Box \eta_{\rho\sigma}) h^{\rho\sigma} = 0.$$ 

Since (2.18) holds, the field equations in the vacuum assume a simplified form

$$(\eta_{\mu\rho}\Box + \eta_{\rho\sigma}\partial_{\mu}\partial_{\sigma} - \eta_{\mu\sigma}\partial_{\nu}\partial_{\rho} + \eta_{\nu\rho}\partial_{\mu}\partial_{\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}\Box) h_{\rho\sigma} = 0 \quad (2.19)$$

In presence of a source $\tau^{\mu\nu}$ we have to add the term $-\kappa h_{\alpha\beta}\tau^{\alpha\beta}$ to the Lagrangian $L_{HE}$, and the field equations become

$$\Box h^{\mu\nu} - \partial^\rho \partial_\rho h^{\mu\nu} - \partial^\nu \partial_\nu h^{\mu\nu} + \eta^{\mu\nu} \partial_\sigma \partial_\rho h^{\rho\sigma} + \partial^\mu \partial^\nu h - \eta^{\mu\nu} \Box h = -\kappa \tau^{\mu\nu}, \quad (2.20)$$

since

$$\frac{\partial L_{HE}}{\partial h_{\mu\nu}} = -\kappa \tau^{\mu\nu}. \quad (2.21)$$

Also in this case, in analogy to the case of vector fields, we can introduce the spin projector operators in the space of the symmetric two-rank tensors. We can decompose (see Appendix $B$) a symmetric two-rank tensor in terms of spin-2, spin-1 and two spin-0 components under the rotation group $SO(3)$, i.e. $h_{\mu\nu} \in 0 \oplus 0 \oplus 1 \oplus 2$, by introducing the following set of operators

$$P^2_{\mu\nu\rho\sigma} = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho}) - \frac{1}{3} \theta_{\mu\nu} \theta_{\rho\sigma},$$

$$P^1_{\mu\nu\rho\sigma} = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}),$$

$$P^0_{s,\mu\nu\rho\sigma} = \frac{1}{3} \theta_{\mu\nu} \theta_{\rho\sigma}, \quad P^0_{w,\mu\nu\rho\sigma} = \omega_{\mu\nu} \omega_{\rho\sigma}, \quad (2.22)$$

$$P^0_{sw,\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}} \theta_{\mu\nu} \omega_{\rho\sigma}, \quad P^0_{ws,\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}} \omega_{\mu\nu} \theta_{\rho\sigma}.$$  

\[\footnote{From the geometrical point of view it means that $\mathcal{R} = 0$ in the vacuum. In fact the linearized form of the Ricci scalar is $\mathcal{R} = \partial_{\rho}\partial_{\sigma}h^{\rho\sigma} - \Box h = (\partial_{\rho}\partial_{\sigma} - \eta_{\rho\sigma}\Box) h^{\rho\sigma}.$} \]

\[\footnote{We have the minus sign because starting from GR in a generic background we have: $S = S_{HE} + S_m$, where $S_{HE}$ was already examined in (2.6), and the matter part whose variation with respect to the metric is $\delta S_m = \kappa \int d^4x g^{\mu\nu} \tau_{\mu\nu} = \kappa \int d^4x (\delta g^{\mu\nu} g^{\rho\sigma} \delta g_{\rho\sigma}) \tau_{\mu\nu}$, and since $\delta g_{\alpha\beta} = h_{\alpha\beta}$, from the point of view of the field theory approach we consider $-\kappa h_{\alpha\beta} \tau^{\alpha\beta}.$} \]

\[\footnote{In Appendix $B$ we discuss more in detail the basis of spin projector operators, taking into account also the possibility to have antisymmetric operators.} \]
where the projectors $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$ have been already defined in the previous Chapter (see eqs. (1.5)-(1.7)). It is worth recalling their expressions in momentum space:

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}.$$ 

The operators defined in (2.22) satisfy the following orthogonality relations

$$P^i_a P^j_b = \delta_{ij} \delta_{ab}, \quad P^0_{ab} P^i_c = \delta_{i0} \delta_{ac} P^0_{ab},$$

$$P^0_{ab} P^0_{cd} = \delta_{ad} \delta_{bc} P^0_{a}, \quad P^i_c P^0_{ab} = \delta_{i0} \delta_{ac} P^0_{a},$$

where $i, j = 2, 1, 0$ and $a, b, c, d = s, w, \text{absent}, 9$ and also the completeness property

$$P^2 + P^1 + P^0_s + P^0_w = 1 \iff (P^2 + P^1 + P^0_s + P^0_w)_{\mu\nu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}).$$

As we have done for the projectors $\theta$ and $\omega$ in the previous chapter, we can show what is the associated spin for each spin projector operators in (2.22) (see Appendix B):

$$\eta^{\mu\rho} \eta^{\nu\sigma} P^2_{\mu\nu\rho\sigma} = 5 = 2(2) + 1 \quad \text{(spin-2)},$$

$$\eta^{\mu\rho} \eta^{\nu\sigma} P^1_{\mu\nu\rho\sigma} = 3 = 2(1) + 1 \quad \text{(spin-1)},$$

$$\eta^{\mu\rho} \eta^{\nu\sigma} P^0_{s,\mu\nu\rho\sigma} = 1 = 2(0) + 1 \quad \text{(spin-0)},$$

$$\eta^{\mu\rho} \eta^{\nu\sigma} P^0_{w,\mu\nu\rho\sigma} = 1 = 2(0) + 1 \quad \text{(spin-0)}.$$

Moreover one can easily show that the following relations hold:

$$\eta^{\mu\nu} P^2_{\mu\nu\rho\sigma} = 0 = \eta^{\mu\nu} P^2_{\mu\nu\rho\sigma} \quad \text{(traceless)},$$

$$k^\mu P^2_{\mu\nu\rho\sigma} = 0 = k^\mu P^1_{\mu\nu\rho\sigma} \quad \text{(transverse)}.$$

Note that we have introduced six operators of that form the basis

$$\{ P^2, P^1, P^0_s, P^0_w, P^0_{sw}, P^0_{ws} \}$$

in terms of which the symmetric four-rank tensor $O^{\mu\nu\rho\sigma}$ can be expanded. At the same time, from the properties (2.23)-(2.24) we notice that four out of six form a complete set of spin projector operators,

$$\{ P^2, P^1, P^0_s, P^0_w \}.$$
in terms of which a symmetric two-rank tensor can be decomposed in one spin-2, one spin-1 and two spin-0 components. The operators $\mathcal{P}_{sw}^0$ and $\mathcal{P}_{ws}^0$ are not projectors as we can see from the relations (2.23), but are necessary to close the algebra and form a basis of symmetric four-rank tensors in terms of which the operator space of the gravitational field equations can be spanned. They can potentially mix the two scalar multiplets $s$ and $w$ (See Appendix B for more details).

This basis of projectors represents six field degrees of freedom. The other four fields in a symmetric tensor field, as usual, represent the gauge (unphysical) degrees of freedom. $\mathcal{P}^2$ and $\mathcal{P}^1$ represent transverse and traceless spin-2 and spin-1 degrees, accounting for four degrees of freedom, while $\mathcal{P}_s^0$ and $\mathcal{P}_w^0$ represent the spin-0 scalar multiplets. In addition we need also to consider the transition operators $\mathcal{P}_{sw}^0$ and $\mathcal{P}_{ws}^0$ which are not projectors as we can see from (2.23), but are necessary to close the algebra and form a basis; they can mix the two scalar multiplets.

In terms of the spin projector operators $h_{\mu\nu}$ decomposes as

$$h_{\mu\nu} = \mathcal{P}_{\rho\sigma}^2 h_{\rho\sigma} + \mathcal{P}_{\rho\sigma}^1 h_{\rho\sigma} + \mathcal{P}_{s,\rho\sigma}^0 h_{\rho\sigma} + \mathcal{P}_{w,\rho\sigma}^0 h_{\rho\sigma}. \quad (2.29)$$

In momentum space, taking into account the presence of the source, the field equations (2.17) assume the form

$$\left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\rho\sigma} k_{\mu} k_{\nu} - \eta_{\mu\sigma} k_{\nu} k_{\rho} - \eta_{\nu\sigma} k_{\rho} k_{\mu} + \eta_{\mu\nu} k_{\rho} k_{\sigma} - \eta_{\mu\sigma} k_{\nu} k_{\rho} \right) h^{\rho\sigma} = \frac{\kappa}{k^2} \tau_{\mu\nu} \quad \Leftrightarrow \quad \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\rho\sigma} \omega_{\mu\nu} - \eta_{\mu\sigma} \omega_{\nu\rho} - \eta_{\nu\sigma} \omega_{\mu\rho} + \eta_{\mu\nu} \omega_{\rho\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho} \right) h^{\rho\sigma} = \frac{\kappa}{k^2} \tau_{\mu\nu}. \quad (2.30)$$

This equation can be expressed in terms of the spin projector operators. In fact by manipulating (2.30) we obtain

$$\left[ \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) + (\eta_{\rho\sigma} \omega_{\mu\nu} + \eta_{\mu\nu} \omega_{\rho\sigma}) - (\eta_{\mu\nu} \eta_{\rho\sigma}) + \frac{1}{2} (\eta_{\rho\sigma} \omega_{\nu\mu} + \eta_{\nu\alpha} \omega_{\mu\rho} + \eta_{\nu\rho} \omega_{\mu\sigma}) \right] h^{\rho\sigma} = \frac{\kappa}{k^2} \tau_{\mu\nu}, \quad (2.31)$$
and since, as shown in Appendix B.2, the following relations hold

\[
\frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) = (P^2 + P^1 + P^0_s + P^0_w)_{\mu\nu\rho\sigma},
\]

\[
\eta_{\mu\nu} \omega_{\sigma\rho} + \eta_{\rho\sigma} \omega_{\mu\nu} = (\sqrt{3} (P^0_{sw} + P^0_{ws}) + 2P^0_w)_{\mu\nu\rho\sigma}, \tag{2.32}
\]

\[
\frac{1}{2} (\eta_{\mu\rho} \omega_{\nu\sigma} + \eta_{\mu\sigma} \omega_{\nu\rho} + \eta_{\nu\sigma} \omega_{\mu\rho} + \eta_{\nu\rho} \omega_{\mu\sigma}) = (P^1 + 2P^0_w)_{\mu\nu\rho\sigma},
\]

\[
\eta_{\mu\nu} \eta_{\rho\sigma} = (3P^0_s + P^0_w + \sqrt{3} (P^0_{sw} + P^0_{ws}))_{\mu\nu\rho\sigma},
\]

the equations (2.31) become

\[
\left[ (P^2 + P^1 + P^0_s + P^0_w) + (\sqrt{3} (P^0_{sw} + P^0_{ws}) + 2P^0_w) \right. \\
- (3P^0_s + P^0_w + \sqrt{3} (P^0_{sw} + P^0_{ws})) - (P^1 + 2P^0_w) \right]_{\mu\nu\rho\sigma} h^{\rho\sigma} = \frac{\kappa}{k^2} \tau_{\mu\nu}. \tag{2.33}
\]

In terms of the spin projector operators \(P^2\) and \(P^0_s\) (2.33) read

\[
(P^2 - 2P^0_s)_{\mu\rho\rho\sigma} h^{\rho\sigma} = \frac{\kappa}{k^2} \tau_{\mu\nu}. \tag{2.34}
\]

For reasons of simplicity, often we will write the equations, involving the spin projector operators, without writing the indices, so (2.34) can be also written as

\[
(P^2 - 2P^0_s) h = \frac{\kappa}{k^2} \tau. \tag{2.35}
\]

Note that to rewrite the field equations in terms of the spin projector operators, we have also rewritten the operator \(O\) in (2.10) in terms of them. Indeed, from the Lagrangian (2.9), the associated field equations in momentum space turn out to be

\[
O_{\mu\nu\rho\sigma} h^{\rho\sigma} = \kappa \tau_{\mu\nu}. \tag{2.36}
\]

By comparing the equation (2.36) with (2.34) we notice that

\[
O_{\mu\nu\rho\sigma} = k^2 (P^2 - 2P^0_s)_{\mu\rho\rho\sigma}. \tag{2.37}
\]

Remark 1. Let us observe that not only the spin-2 component is present but also a spin-0 component, while the spin-1 component is absent. Thus, in total we have six degree of freedom: five spin-2 and one spin-0 components. The other four degrees of freedom drop out because of the presence of gauge invariance, as we shall see below. Studying the graviton propagator we can appreciate the importance of the spin-0 component; we shall discuss on it in the subsection 2.4.2.
2.1. GRAVITON LAGRANGIAN

Gauge invariance of graviton Lagrangian

At the end of Chapter 1 we have seen that \( \delta A_\mu (x) = \partial_\mu \alpha(x) \) is a gauge symmetry for the Lagrangian \( \mathcal{L}_V \) and for the associated field equations. We could ask what is the gauge symmetry for \( \mathcal{L}_{HE} \) and its field equations. Let us consider the one-parameter group of diffeomorphisms \( x'\mu \equiv x^\mu (x) \) and in particular its infinitesimal form

\[
x'\mu = x^\mu + \xi^\mu(x) \iff x^\mu = x'^\mu - \xi^\mu(x)
\] (2.38)

Now we want to study how the metric tensor \( g_{\mu\nu}(x) \) transforms under (2.38):

\[
g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) = (\delta^\alpha_\mu - \partial_\mu \xi^\alpha) (\delta^\beta_\nu - \partial_\nu \xi^\beta) g_{\alpha\beta}(x)
\] (2.39)

Moreover, by expanding in Taylor series \( g'_{\mu\nu}(x') \) one gets

\[
g'_{\mu\nu}(x') \simeq g_{\mu\nu}(x) + \xi^\alpha \partial_\alpha g_{\mu\nu}(x).
\] (2.40)

Substituting the latter equation in (2.39), we obtain

\[
\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = - (\partial_\mu \xi^\alpha) g_{\alpha\nu} - (\partial_\nu \xi^\alpha) g_{\mu\alpha} - \xi^\alpha \partial_\alpha g_{\mu\nu}
\]

\[
= -\partial_\mu \xi^\nu - \partial_\nu \xi^\mu + \xi^\alpha (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})
\]

\[
= -\partial_\mu \xi^\nu - \partial_\nu \xi^\mu + 2\Gamma^\alpha_{\mu\nu} \xi^\alpha
\]

\[
\iff \delta g_{\mu\nu}(x) = -\nabla_\mu \xi^\nu - \nabla_\nu \xi^\mu.
\] (2.41)

For Minkowski background we have

\[
\nabla_\mu \to \partial_\mu \Rightarrow \delta h_{\mu\nu} = -\partial_\mu \xi^\nu - \partial_\nu \xi^\mu,
\]

and finally, if we redefine \( -\xi^\mu \to \xi^\mu \) the variation of \( h_{\mu\nu} \) reads as

\[
\delta h_{\mu\nu} = \partial_\mu \xi^\nu + \partial_\nu \xi^\mu.
\] (2.42)

We can easily verify that the transformation (2.42) is a gauge symmetry for the Lagrangian \( \mathcal{L}_{HE} \) and for the associated field equations. Indeed, substituting (2.42) in the equations (2.19) one has

\[
\square \partial_\mu \xi^\nu + \square \partial_\nu \xi^\mu + 2\partial_\mu \partial_\nu \partial_\alpha \xi^\alpha - \partial_\mu \partial_\alpha \partial^\alpha \xi^\nu - \partial_\nu \partial_\alpha \partial^\alpha \xi^\mu - 2\partial_\nu \partial_\mu \partial_\alpha \xi^\alpha = 0,
\]

\[\text{Note that we are using the fact that the transformation is infinitesimal, namely } \partial_\alpha g'_{\mu\nu}(x) = \partial_\alpha g_{\mu\nu}(x) + \mathcal{O}(\xi^2) \text{ and } \partial_\alpha \xi^\alpha(x') = \partial_\alpha \xi^\alpha(x) + \mathcal{O}(\xi^2).\]
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i.e. the field equations \((2.19)\) are invariant under gauge transformation. By means the gauge symmetry \((2.42)\) we can transform the tensor field \(h_{\mu\nu}\), so that we are able to choose a special gauge in which, for example, the field equations simplify. In the case of vector field we considered the Lorenz gauge \(\partial_{\mu}A_{\mu} = 0\). Analogously for the symmetric tensor field a possible gauge is the De Donder gauge

\[
\partial_\alpha h_\mu^\alpha - \frac{1}{2} \partial_\alpha h = 0 \Leftrightarrow \partial_\alpha \left( h_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha h \right) = 0. \tag{2.43}
\]

By choosing this gauge the field equations in the vacuum reduce to the famous wave equation (i.e. the same equations of ED that we obtain in the Lorenz gauge):

\[
\Box h_{\mu\nu} + \partial_{\mu}\partial_{\nu}h - \partial_{\mu}\partial_\alpha h_\nu^\alpha - \partial_{\nu}\partial_\alpha h_\mu^\alpha = 0
\]

\[
\Leftrightarrow \Box h_{\mu\nu} - \partial_{\mu} \left( \partial_\alpha h_\nu^\alpha - \frac{1}{2} \partial_\nu h \right) - \partial_{\nu} \left( \partial_\alpha h_\mu^\alpha - \frac{1}{2} \partial_\mu h \right) = 0,
\]

\[
\Rightarrow \Box h_{\mu\nu} = 0. \tag{2.44}
\]

Going to the momentum space \((2.44)\) becomes

\[- k^2 h_{\mu\nu} = 0 \Rightarrow k^2 = 0, \tag{2.45}\]

namely \(h_{\mu\nu}\) describes a massless graviton.

### 2.2 Graviton degrees of freedom

As we have done for the vector field case, now we want to understand how many physical degrees of freedom the graviton has, and we shall see that by imposing the field equations and the gauge symmetry we can get rid of the spurious degrees of freedom. Since the two-rank tensor \(h_{\mu\nu}\) is symmetric it has only 10 independent components; our aim is to find and eliminate the unphysical ones. Also for the graviton case we shall distinguish the on-shell and off-shell case. We are going to start studying the on-shell case \([39]\).

#### 2.2.1 On-shell graviton

Let us consider the field equations and the gauge symmetry in momentum space

\[
\begin{cases}
  k^2 h_{\mu\nu} + k_{\mu}k_{\nu} h - k_{\mu}k_{\alpha} h_{\nu}^\alpha - k_{\nu}k_{\alpha} h_{\mu}^\alpha = 0 \\
  \delta h_{\mu\nu} = i (k_{\mu}\xi_{\nu} + k_{\nu}\xi_{\mu})
\end{cases} \tag{2.46}
\]
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and the basis of four-vectors \( \left\{ k^\mu, \tilde{k}^\mu, \varepsilon_1^\mu, \varepsilon_2^\mu \right\} \), such that the relations (1.16) hold

\[
k^\mu \equiv (k^0, \vec{k}), \quad \tilde{k}^\mu \equiv (\tilde{k}^0, -\vec{k}), \quad \varepsilon_1^\mu \equiv (0, \vec{\varepsilon}_1), \quad \varepsilon_2^\mu \equiv (0, \vec{\varepsilon}_2), \quad i = 1, 2.
\]

\[
h_{\mu\nu}(k) = a(k)k_\mu k_\nu + b(k)k_\mu \tilde{k}_\nu + c_i(k)k_\mu \varepsilon_{i\nu}^\mu \tag{2.47}
\]

\[
+ d(k)\tilde{k}_\mu \tilde{k}_\nu + e_i(k)\tilde{k}_\mu \varepsilon_{i\nu}^\mu + f_{ij}(k)\varepsilon_{(i\nu}^\mu \varepsilon_{j)}^\nu,
\]

where the coefficients \( \{a, b, c_1, c_2, d, e_1, e_2, f_{11}, f_{12}, f_{21}, f_{22}\} \) take into account the graviton degrees of freedom. By substituting (2.47) in the first of (2.46) \( a, b \) and \( c \) cancel each other disappearing from the equation and what remains is

\[
d(k)k^2\tilde{k}_\mu \varepsilon_{\mu\nu}^\nu + e_i(k)k^2\tilde{k}_\mu \varepsilon_{i\nu}^\nu + f_{ij}(k)k^2\varepsilon_{(i\nu}^\mu \varepsilon_{j)}^\mu + d(k)\tilde{k}^2k_\mu k_\nu + e_i(k)k_\mu k_\nu \vec{k} \cdot \varepsilon^i
\]

\[
+ f_{ij}(k)k_\mu k_\nu (\varepsilon^i \cdot \varepsilon^j) - d(k)k_\mu \left(k \cdot \tilde{k}\right) \tilde{k}_\nu - e_i(k)k_\mu k_\alpha \tilde{k}(\mu \varepsilon^i_\nu) - f_{ij}(k)k_\mu k_\alpha \varepsilon^i_\nu \varepsilon^{\alpha j}_{\mu} - d(k)k_\mu k_\alpha \tilde{k}(\alpha \varepsilon^i_\mu) - f_{ij}(k)k_\nu k_\alpha \varepsilon^{i j}_{\mu \alpha} = 0 \tag{2.48}
\]

Because of the (1.16) it turns out that

\[
e_i(k)k_\mu k_\nu \vec{k} \cdot \varepsilon^i = f_{ij}(k)k_\mu k_\alpha \varepsilon^i_\alpha \varepsilon^j_{\mu} = f_{ij}(k)k_\mu k_\alpha \varepsilon^i_\mu \varepsilon^{\alpha j}_{\nu} = 0,
\]

\[
e_i(k)k_\nu k^\alpha \tilde{k}(\mu \varepsilon^i_\alpha) = \frac{1}{2} e_i(k)k_\nu \left(k \cdot \tilde{k}\right) \varepsilon^i_\mu, \tag{2.49}
\]

\[
f_{ij}(k)k_\mu k_\nu (\varepsilon^i \cdot \varepsilon^j) = - f_{ij}(k)k_\mu k_\nu \delta^{ij} = f_{ii}k_\mu k_\nu,
\]

where \( f_{ii} \equiv \sum_{i=1,2} f_{ii} = f_{11} + f_{22} \) is the trace of \( f \). Hence, the relations (2.49) reduce the expansion (2.48) as

\[
d(k)k^2\tilde{k}_\mu \varepsilon_{\mu\nu}^\nu + e_i(k)k^2\tilde{k}_\mu \varepsilon_{i\nu}^\nu + f_{ij}(k)k^2\varepsilon_{(i\nu}^\mu \varepsilon_{j)}^\mu + d(k)\tilde{k}^2k_\mu k_\nu - f_{ii}k_\mu k_\nu
\]

\[
- d(k)k_\mu \left(k \cdot \tilde{k}\right) \tilde{k}_\nu - d(k)k_\nu \left(k \cdot \tilde{k}\right) \tilde{k}_\mu - \frac{1}{2} e_i(k)k_\nu \left(k \cdot \tilde{k}\right) \varepsilon^i_\mu - \frac{1}{2} e_i(k)k_\mu \left(k \cdot \tilde{k}\right) \varepsilon^i_\nu = 0 \tag{2.50}
\]

\[\text{Note that we have only } 11 \text{ coefficients and not } 16 \text{ because the expansion (2.47) takes already into account the symmetry of } h_{\mu\nu}, \text{ in fact only symmetrized products are present.}\]
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Let us consider the $\mu = 0 = \nu$ component of (2.50):

\[
dk^2(k^0)^2 + d\tilde{k}^2(k^0)^2 - f_{ii}(k^0)^2 - 2d(k^0)^2 \left( k \cdot \tilde{k} \right) = 0
\]

\[
\Rightarrow 2d \left( k^2 - k \cdot \tilde{k} \right) = f_{ii} \Rightarrow f_{ii} = -2d\tilde{k}^2.
\]

Now contract (2.50) with $\eta^{\mu\nu}$ and use (2.51):

\[
2dk^2\tilde{k}^2 - 2f_{ii}k^2 - 2d \left( k \cdot \tilde{k} \right)^2 = 0 \Leftrightarrow d \left( k^2\tilde{k}^2 - (k \cdot \tilde{k})^2 + 2\tilde{k}^2k^2 \right) = 0
\]

\[
\Rightarrow -2d\tilde{k}^2 \left( k \cdot \tilde{k} \right) = 0 \Rightarrow d = 0,
\]

namely, since (2.51) holds, $f_{ij}$ is traceless, $f_{ii} = 0$; thus (2.50) becomes

\[
e_i k^2 \tilde{k}(\mu \varepsilon^i_{\nu}) + f_{ij}k^2(\mu \varepsilon^i_{\nu}) - \frac{1}{2} e_i k^0 \left( k \cdot \tilde{k} \right) \varepsilon^i_{\nu} - \frac{1}{2} e_i k^0 \left( k \cdot \tilde{k} \right) \varepsilon^i_{\mu} = 0.
\]

Noting that $\varepsilon^0_i = 0$, if now we consider the components $\mu = 0$ and a generic $\nu$ in (2.53), we obtain

\[
\frac{1}{2} e_i k^0 \varepsilon^i_{\nu} - \frac{1}{2} e_i k^0 \left( k \cdot \tilde{k} \right) \varepsilon^i_{\nu} = 0
\]

\[
\Leftrightarrow e_i k^0 \left( k^2 - k \cdot \tilde{k} \right) \varepsilon^i_{\nu} = 0 \Leftrightarrow e_i = 0, \ i = 1, 2,
\]

since the four-vectors $\varepsilon^i_{\nu}$ are linearly independent and $k^0 \left( k^2 - k \cdot \tilde{k} \right) k^0 (-2\tilde{k}^2) \neq 0$.

Hence the field equations get rid of five coefficients: $d, f_{ii}, e_1, e_2$ and the expansion (2.47) reads as

\[
h_{\mu\nu}(k) = a(k)k_\mu k_\nu + b(k)k_\mu \tilde{k}_{\nu} + c_i(k)k_{(\mu} \varepsilon^i_{\nu)} + f_{ij}\varepsilon^i_{(\mu} \varepsilon^j_{\nu)}),
\]

with condition $f_{ii} = 0$.

Now we want to verify that the coefficients $a, b, c_1, c_2$ can be eliminated by using the gauge symmetry $\delta h_{\mu\nu} = i \left( k_{\mu} \xi_{\nu} + k_{\nu} \xi_{\mu} \right)$. First of all let us expand $\xi_{\mu}$ in the basis (1.16)

\[
\xi_{\mu}(k) = a(k)k_\mu + \beta(k)\tilde{k}_\mu + \gamma_i(k)\varepsilon^i_{\mu},
\]

thus the gauge symmetry relation becomes

\[
\delta h_{\mu\nu} = i2 \left( a k_{\mu} k_\nu + k_{(\mu} k_{\nu)} \right) \varepsilon^i_{\mu}.
\]

By the gauge transformation corresponding to (2.57) we can go from the tensor field in (2.55) to a new one $h'_{\mu\nu}$:
2.2. GRAVITON DEGREES OF FREEDOM

\[ h'_{\mu\nu} = h_{\mu\nu} + \delta h_{\mu\nu} \]

\[ \Leftrightarrow a'k_{\mu}k_{\nu} + b'k_{(\mu}k_{\nu)} + c'_i\varepsilon^i_{(\mu}\varepsilon^j_{\nu)} + f'_{ij}\varepsilon^i_{(\mu}\varepsilon^j_{\nu)} = (a + i2\alpha)k_{\mu}k_{\nu} + (b + i2\beta)k_{(\mu}k_{\nu)} \quad (2.58) \]

\[ + (c_i + i2\gamma_i)k_{(\mu}\varepsilon^i_{\nu)} + f'_{ij}\varepsilon^i_{(\mu}\varepsilon^j_{\nu)} \]

The last equation implies that the following relations hold\(^{14}\)

\[ a' = a + i2\alpha, \quad b' = b + i2\beta, \quad c'_i = c_i + i2\gamma_i, \quad f'_{ij} = f_{ij}. \quad (2.59) \]

We can immediately notice that by choosing

\[ \alpha = \frac{a}{2}, \quad \beta = \frac{b}{2}, \quad \gamma_i = \frac{c_i}{2}, \quad (2.60) \]

we get rid of the coefficients \(a, b\) and \(c_i\).

Hence the only remaining coefficients are \(f_{12} = f_{21}\) and \(f_{11} = -f_{22}\). Finally we can state that the on-shell massless graviton has only \(two\) physical degrees of freedom.

2.2.2 Off-shell graviton

If we consider the off-shell case, since we cannot use the constraint of the field equations, we won’t be able to eliminate the coefficients \(d, e_1, e_2, f_{ii}\) and so an off-shell graviton will have \(six\) degrees of freedom.

We can summarize the counting of degrees of freedom for on-shell and off-shell photon in the following expression\(^{15}\):

\[^{14}\text{Don’t get confused! We are indicating with the letter “i” both the imaginary unit and the index component.}\]

\[^{15}\text{In general, if we are in } D\text{-dimensions the following rules hold:}\]

\[ D\text{-dimension : } \begin{cases} 
\text{off-shell : } & \frac{D(D-1)}{2} \text{ d.o.f.} \\
\text{on-shell : } & \frac{D(D-3)}{2} \text{ d.o.f.} 
\end{cases} \quad (2.61) \]

as particular case we can see that in \(D = 2\)

\[ 2\text{-dimension : } \begin{cases} 
\text{off-shell : } & 1 \text{ d.o.f.} \\
\text{on-shell : } & -1 \text{ d.o.f.} 
\end{cases} \]

but we can notice that for the real (on-shell) graviton we have a negative number of degrees of freedom, so this case is not considerable; then in \(D = 3\)

\[ 3\text{-dimension : } \begin{cases} 
\text{off-shell : } & 3 \text{ d.o.f.} \\
\text{on-shell : } & 0 \text{ d.o.f.} 
\end{cases} \]

namely the physical graviton doesn’t propagate.
2.3 Graviton Propagator

Now our aim is to obtain the propagator for the graviton; we will use the formalism of the spin projector operators. We shall proceed in the same way we did for the vector field case, but since we are dealing with two-rank tensor the propagator will have four indices. The graviton propagator is defined as the inverse operator of the operator $O$ in (2.10). We will present two equivalent methods:

1. To obtain the propagator one proceed straightforwardly with the inversion of the operator $O$, finding the operator $O^{-1}$ such that $OO^{-1} = I$;

2. To obtain the propagator one always inverts the operator $O$, but by acting with the spin projector operators on the field equations.

First we are going to consider the general case by studying the symmetric operator $O$ associated to any symmetric two-rank field Lagrangian. Then, we will specialize to the case of GR where the operator $O$ is defined in (2.10).

Method 1

Given a symmetric two-rank tensor field Lagrangian, its associated operator $O$ can be expanded in terms of the basis of spin projector operators\(^\text{16}\) (2.22)\(^\text{17}\):

$$O = A \mathcal{P}^2 + B \mathcal{P}^1 + C \mathcal{P}_s^0 + D \mathcal{P}_w^0 + E \mathcal{P}_{sw}^0 + F \mathcal{P}_{ws}^0,$$

(2.63)

and, similarly, for its inverse we have

$$O^{-1} = X \mathcal{P}^1 + Y \mathcal{P}^0 + Z \mathcal{P}_s^0 + W \mathcal{P}_w^0 + R \mathcal{P}_{sw}^0 + S \mathcal{P}_{ws}^0,$$

(2.64)

By imposing $OO^{-1} = I$ and using the orthogonality relations (2.23) we can find the relations among the two sets of coefficients $\{A, B, C, D, E, F\}$ and $\{X, Y, Z, W, R, S\}$ :

\(^{16}\)We have to note that we are doing an abuse of nomenclature, in fact we are using the word “projector” also for $\mathcal{P}_{sw}^0$ and $\mathcal{P}_{ws}^0$ that are not projectors.

\(^{17}\)Note that we are still suppressing the indices for simplicity.
The equation is satisfied if, and only if, the following relations hold:

\[ AX = 1, \quad BY = 1, \quad CZ + ES = 1, \]
\[ DW + FR = 1, \quad CR + EW = 0, \quad DS + FZ = 0; \]

namely

\[ X = \frac{1}{A}, \quad Y = \frac{1}{B}, \quad Z = \frac{D}{CD - EF}, \]
\[ W = \frac{C}{CD - EF}, \quad R = \frac{F}{EF - CD}, \quad S = \frac{E}{EF - CD}. \]

The inverse operator \( \mathcal{O}^{-1} \) turns out to be

\[
\mathcal{O}^{-1} = \frac{1}{A} \mathcal{P}^2 + \frac{1}{B} \mathcal{P}^1 + \frac{D}{CD - EF} \mathcal{P}^0_s + \frac{C}{CD - EF} \mathcal{P}^0_w
\]
\[+ \frac{F}{EF - CD} \mathcal{P}^0_{sw} + \frac{E}{EF - CD} \mathcal{P}^0_{ws}. \]

Now let us come back to GR case, specializing Method 1 to the H-E Lagrangian (2.9).

First of all we need the operator \( \mathcal{O} \) in (2.10) expressed in terms of the spin projector operators. We have already gotten its expression in (2.37) and, by suppressing the indices for simplicity, is given by

\[ \mathcal{O} = k^2 \left( \mathcal{P}^2 - 2 \mathcal{P}^0_s \right). \]

Comparing with (2.63) we notice that only two coefficients are not zero:

\[ A = k^2, \quad B = 0, \quad C = -2k^2; \]
\[ D = 0, \quad E = 0, \quad F = 0. \]

We notice that, as we have already seen for vector field, the operator \( \mathcal{O} \) defined in (2.10) is not invertible, in fact we cannot invert something equal to zero.\(^{18}\) Hence we have the same mathematical obstacle already met with the vector field case, and we learned

\(^{18}\)Physically we can interpret this result saying that the spin-1 and spin-0 \( w \) components don’t propagate, so they won’t appear in the physical part of the propagator.
that it can be got over by adding a gauge fixing term to the Lagrangian (2.7). We have also learned that the saturated propagator in which appears only the physical (gauge-independent) part of the propagator is invertible.

Before introducing the gauge fixing term and inverting the propagator let us introduce the second method to determine the propagator.

Method 2

Also with this second method, first we shall consider a general case and then specialize to GR case. The field equations, in terms of the operator $\mathcal{O}$, in the presence of a matter source $\tau_{\mu\nu}$ read as

$$\mathcal{O}^{\mu\nu\rho\sigma} h_{\rho\sigma} = \kappa \tau^{\mu\nu},$$

or, without specifying the indices,

$$\mathcal{O} h = \kappa \tau.$$  

(2.70)

(2.71)

To derive the propagator the prescription is the following:

- We go to the momentum space and express the field equations (2.71) in terms of the spin projector operators$^{19}$:

$$ (A \mathcal{P}^2 + B \mathcal{P}^1 + C \mathcal{P}_s^0 + D \mathcal{P}_w^0 + E \mathcal{P}_{sw}^0 + F \mathcal{P}_{ws}^0) h = \kappa (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0) \tau;$$

(2.72)

- By acting with each spin projector operators on (2.72) one can decompose the field equations into a decoupled set of new equations corresponding to the relevant degrees of freedom;

- Then these equations will be invertible and we can obtain the propagator.

Hence, following the above prescription, let us act with each spin projector operators on (2.72):

$$ \mathcal{P}^2 \rightarrow \mathcal{P}^2 h = \frac{1}{A} \kappa \mathcal{P}^2 \tau;$$

(2.73)

$$ \mathcal{P}^1 \rightarrow \mathcal{P}^1 h = \frac{1}{B} \kappa \mathcal{P}^1 \tau;$$

(2.74)

$$ \mathcal{P}_s^0 \rightarrow (C \mathcal{P}_s^0 + E \mathcal{P}_{sw}^0) h = \kappa \mathcal{P}_s^0 \tau;$$

(2.75)

$$ \mathcal{P}_w^0 \rightarrow (D \mathcal{P}_w^0 + F \mathcal{P}_{ws}^0) h = \kappa \mathcal{P}_w^0 \tau;$$

(2.76)

$^{19}$Let us note again that we use the basis of six operators to expand the operator $\mathcal{O}$; while we need just the four projectors to decompose the symmetric two-rank tensor $\tau$. 

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\[ \mathcal{P}_{sw}^0 \rightarrow (D\mathcal{P}_{sw}^0 + F\mathcal{P}_s^0) \, h = \kappa \mathcal{P}_{sw}^0 \tau; \] (2.77)
\[ \mathcal{P}_{ws}^0 \rightarrow (C\mathcal{P}_{ws}^0 + E\mathcal{P}_w^0) \, h = \kappa \mathcal{P}_{ws}^0 \tau. \] (2.78)

We can see that the equations for \( \mathcal{P}^2 \) and \( \mathcal{P}^1 \) are decoupled, but for the scalar components seems to be coupled. Consider the system composed of the equations (2.75) and (2.77):

\[
\begin{cases}
(C\mathcal{P}_s^0 + E\mathcal{P}_{sw}^0) \, h = \kappa \mathcal{P}_s^0 \tau \\
(D\mathcal{P}_{sw}^0 + F\mathcal{P}_s^0) \, h = \kappa \mathcal{P}_{sw}^0 \tau
\end{cases}
\] (2.79)

or, equivalently,
\[
\begin{pmatrix} C & E \\ F & D \end{pmatrix} \begin{pmatrix} \mathcal{P}_s^0 h \\ \mathcal{P}_{sw}^0 h \end{pmatrix} = \kappa \begin{pmatrix} \mathcal{P}_s^0 \tau \\ \mathcal{P}_{sw}^0 \tau \end{pmatrix}.
\] (2.80)

By noting that the inverse of the matrix
\[ M = \begin{pmatrix} C & E \\ F & D \end{pmatrix} \] (2.81)
is
\[ M^{-1} = \frac{1}{CD - EF} \begin{pmatrix} D & -F \\ -E & C \end{pmatrix}, \] (2.82)
we are able to invert the system (2.80) by multiplying with (2.82)

\[
\begin{pmatrix} \mathcal{P}_s^0 h \\ \mathcal{P}_{sw}^0 h \end{pmatrix} = \kappa \frac{1}{CD - EF} \begin{pmatrix} D & -F \\ -E & C \end{pmatrix} \begin{pmatrix} \mathcal{P}_s^0 \tau \\ \mathcal{P}_{sw}^0 \tau \end{pmatrix},
\] (2.83)
or, equivalently,
\[
\begin{cases}
\mathcal{P}_s^0 h = \kappa \frac{1}{CD - EF} (D\mathcal{P}_s^0 - F\mathcal{P}_{sw}^0) \tau \\
\mathcal{P}_{sw}^0 h = \kappa \frac{1}{CD - EF} (-E\mathcal{P}_s^0 + C\mathcal{P}_{sw}^0) \tau
\end{cases}
\] (2.84)

We can do the same for the equations (2.76) and (2.78) and obtain

\[
\begin{cases}
\mathcal{P}_w^0 h = \kappa \frac{1}{CD - EF} (C\mathcal{P}_w^0 - E\mathcal{P}_{ws}^0) \tau \\
\mathcal{P}_{ws}^0 h = \kappa \frac{1}{CD - EF} (-F\mathcal{P}_w^0 + D\mathcal{P}_{sw}^0) \tau
\end{cases}
\] (2.85)
We have seen that also the scalar components decouple. Hence, using (2.73), (2.74) and the firsts of (2.84) and (2.85) we are able to obtain the inverse operator $O^{-1}$:

$$(\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0) h = \kappa \left[ \frac{1}{A} \mathcal{P}^2 + \frac{1}{B} \mathcal{P}^1 + \frac{D}{CD - EF} \mathcal{P}_s^0 + \frac{C}{CD - EF} \mathcal{P}_w^0 \right] \tau.$$  

We can observe that the expression in brackets in the equation (2.86) is the inverse operator $O^{-1}$, i.e. the propagator. Note also that, as we expected, the result in (2.86) coincides with the expression in the equation (2.67) obtained with the first method.

Now let us come back to GR case. In (2.34) we have already obtained the field equations in terms of the spin projector operators:

$$k^2 (\mathcal{P}^2 - 2\mathcal{P}_s^0) h = \kappa (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0) \tau.$$  

By acting with $\mathcal{P}^2$ we have

$$-\frac{k^2}{2} \mathcal{P}^2 h = \kappa \mathcal{P}^2 \tau \Rightarrow \mathcal{P}^2 h = \kappa \left( \frac{\mathcal{P}^2}{k^2} \right) \tau;$$  

and by acting with $\mathcal{P}_s^0$

$$-k^2 \mathcal{P}_s^0 h = \kappa \mathcal{P}_s^0 \tau \Rightarrow \mathcal{P}_s^0 h = \kappa \left( -\frac{\mathcal{P}_s^0}{2k^2} \right) \tau.$$  

By acting with $\mathcal{P}^1$ and $\mathcal{P}_w^0$ we have:

$$0h = \kappa \mathcal{P}^1 \tau \Rightarrow \mathcal{P}^1 \tau = 0,$$  

$$0h = \kappa \mathcal{P}_w^0 \tau \Rightarrow \mathcal{P}_w^0 \tau = 0,$$  

so it’s impossible to obtain the components $\mathcal{P}^1 h$ and $\mathcal{P}_w^0 h$ because we have zero on the left side. Recall that in ED case we had the spin-0 component undetermined, instead in GR case we have the spin-1 and spin-0 $w$ components that are undetermined.20

Now, we are going to proceed as already done in the previous Chapter for photon propagator: firstly we want to introduce a gauge fixing term to invert the operator $O$, secondly we want to determine the saturated propagator without the choice of any gauge fixing term.

20The equations (2.90) and (2.91) impose the constraints on the matter source and correspond to a gauge freedom. The concept is the same that we have mentioned in the previous chapter for the vector field $A$. In fact, if we analyze the GR Lagrangian:

$$\mathcal{L}_{HE} = -\frac{1}{2} h_{\mu\nu} \Box (\mathcal{P}^2 - 2\mathcal{P}_s^0)^{\mu\nu\rho\sigma} h_{\rho\sigma},$$  

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2.3. GRAVITON PROPAGATOR

2.3.1 Gauge fixing term for graviton Lagrangian

We are going to introduce the following gauge fixing term, called De Donder gauge (see (2.43)):

\[ L_{gf} := -\frac{1}{2\alpha} \left( \partial_{\rho} h_{\mu}^{\rho} - \frac{1}{2} \partial_{\mu} h \right) \left( \partial_{\sigma} h^{\mu\sigma} - \frac{1}{2} \partial^{\mu} h \right); \]  

(2.94)

where \( \alpha \) is called gauge parameter. The total GR Lagrangian becomes

\[ \tilde{L}_{HE} = L_{HE} + L_{gf}. \]  

(2.95)

As we have done for the Lagrangian \( L_{HE} \) in (2.9) raising and lowering the indices with the metric tensor \( \eta_{\mu\nu} \), we can easily rewrite the gauge fixing term (2.94) as

\[ L_{gf} = \frac{1}{2} h_{\mu\nu} O_{gf}^{\mu\nu\rho\sigma} h_{\rho\sigma}, \]  

(2.96)

where the operator \( O_{gf}^{\mu\nu\rho\sigma} \) is defined as

\[ O_{gf}^{\mu\nu\rho\sigma} := -\frac{2}{\alpha} \eta^{\mu\rho} \partial^{\nu} \partial^{\sigma} + \frac{1}{\alpha} \eta^{\rho\sigma} \partial^{\mu} \partial^{\nu} + \frac{1}{\alpha} \eta^{\mu\nu} \partial^{\rho} \partial^{\sigma} - \frac{1}{2\alpha} \eta^{\mu\nu} \eta^{\rho\sigma} \Box. \]  

(2.97)

Now we can rewrite also the total Lagrangian (2.95) as a quadratic form

\[ \tilde{L}_{HE} = \frac{1}{2} h_{\mu\nu} (O + O_{gf})^{\mu\nu\rho\sigma} h_{\rho\sigma}, \]  

(2.98)

or, defining \( \tilde{O} := O + O_{gf} \), in a more compact form we read

\[ \tilde{L}_{HE} = \frac{1}{2} h_{\mu\nu} \tilde{O}^{\mu\nu\rho\sigma} h_{\rho\sigma}, \]  

(2.99)

with

\[ \tilde{O}^{\mu\nu\rho\sigma} = -\left( \frac{1}{2} \eta^{\mu\rho} \eta^{\nu\sigma} + \frac{1}{2} \eta^{\mu\sigma} \eta^{\nu\rho} - \left( 1 - \frac{1}{2\alpha} \right) \eta^{\mu\nu} \eta^{\rho\sigma} \right) \Box \]

\[ + \left( \frac{1}{\alpha} - 1 \right) \left( \eta^{\mu\nu} \partial^{\rho} \partial^{\sigma} + \eta^{\rho\sigma} \partial^{\mu} \partial^{\nu} \right) \]

\[ + \frac{1}{2} \left( 1 - \frac{1}{\alpha} \right) \left( \eta^{\nu\rho} \partial^{\mu} \partial^{\sigma} + \eta^{\nu\sigma} \partial^{\mu} \partial^{\rho} + \eta^{\rho\mu} \partial^{\nu} \partial^{\sigma} + \eta^{\mu\sigma} \partial^{\nu} \partial^{\rho} \right). \]  

(2.100)

It is invariant under spin-1 transformation as

\[ \delta h_{\mu\nu} \sim P_{\mu\nu\rho\sigma}^{1} h^{\rho\sigma}, \]  

(2.93)

because of the orthogonality relations, \( P_{1}^{1} P_{2} = P_{1}^{1} P_{s}^{0} = 0 \), and there are also the restrictions (2.90) and (2.91) on the source. It means that for the Lagrangian (2.92) there is a gauge symmetry that corresponds to the gauge invariance under transformations \( \delta h_{\mu\nu} = \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu} \). The arbitrary four-vector \( \xi_{\mu} \) is the vector field associated to the spin-1 symmetry.
2.3. GRAVITON PROPAGATOR

Now we want to rewrite the operator $\tilde{O}$ in terms of the spin projectors operators and, again, since it is symmetric we need only the six operator introduced above. By proceeding in the same way we have done without gauge fixing term, going into momentum space, we obtain:

$$\tilde{O} = k^2 \left[ p^2 + \frac{1}{\alpha} p^1 + \left( \frac{3}{2\alpha} - 2 \right) p^0_s + \frac{1}{2\alpha} p^0_w - \frac{\sqrt{3}}{2} p^0_{sw} - \frac{\sqrt{3}}{2} p^0_{ws} \right]. \quad (2.101)$$

We notice that this operator is invertible, in fact no coefficients is equal to zero (see (2.63)):

$$A = k^2, \quad B = \frac{k^2}{\alpha}, \quad C = k^2 \left( \frac{3}{2\alpha} - 2 \right),$$

$$D = \frac{k^2}{2\alpha}, \quad E = -k^2 \frac{\sqrt{3}}{2\alpha}, \quad F = -k^2 \frac{\sqrt{3}}{2\alpha}. \quad (2.102)$$

The coefficients in (2.66) specialized to the operator (2.101) are

$$X = \frac{1}{k^2}, \quad Y = \frac{\alpha}{k^2}, \quad Z = -\frac{1}{2k^2},$$

$$W = \frac{4\alpha - 3}{2k^2}, \quad R = -\frac{\sqrt{3}}{2k^2}, \quad S = -\frac{\sqrt{3}}{2k^2}, \quad (2.103)$$

and the general form of the propagator with arbitrary coefficients (2.67) in this case reads as

$$\Pi_{GR} \equiv \tilde{O}^{-1} = \frac{1}{k^2} \left[ p^2 + \alpha p^1 - \frac{1}{2} p^0_s + \frac{4\alpha - 3}{2} p^0_w - \frac{\sqrt{3}}{2} p^0_{sw} - \frac{\sqrt{3}}{2} p^0_{ws} \right]. \quad (2.104)$$

---

Note that often the propagator is also defined as the vacuum expectation value of the time ordered product:

$$\langle T \{ h_{\mu\nu}(-k) h_{\rho\sigma}(k) \} \rangle = i \Pi_{GR,\mu\nu\rho\sigma}(k)$$

$$= i \frac{k^2}{k^2} \left[ p^2 + \alpha p^1 - \frac{1}{2} p^0_s + \frac{4\alpha - 3}{2} p^0_w - \frac{\sqrt{3}}{2} p^0_{sw} - \frac{\sqrt{3}}{2} p^0_{ws} \right]_{\mu\nu\rho\sigma}. \quad (2.104)$$

The expression $T \{ h_{\mu\nu}(-k) h_{\rho\sigma}(k) \}$ is the Fourier transform of the time ordered product that in coordinate space is defined as

$$T \{ T \{ h_{\mu\nu}(x) h_{\rho\sigma}(y) \} \} := h_{\mu\nu}(x) h_{\rho\sigma}(y) \Theta(x_0 - y_0) + h_{\rho\sigma}(y) h_{\mu\nu}(x) \Theta(y_0 - x_0),$$

where the function $\Theta(x_0 - y_0)$ is equal to 1 if $x_0 > y_0$, and to 0 otherwise.
2.3. GRAVITON PROPAGATOR

As we have already seen for the vector field case, a special gauge is called *Feynman gauge*, corresponding to the choice \( \alpha = 1 \):

\[
\Pi_{GR,\mu\nu\rho\sigma} = \frac{1}{k^2} \left[ \mathcal{P}^2 + \mathcal{P}^1 - \frac{1}{2} \mathcal{P}^0_s + \frac{1}{2} \mathcal{P}^0_w - \frac{\sqrt{3}}{2} \mathcal{P}^0_{sw} - \frac{\sqrt{3}}{2} \mathcal{P}^0_{ws} \right]_{\mu\nu\rho\sigma}
\]

\[
= \frac{1}{k^2} \left[ (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0_s + \mathcal{P}^0_w) 
- \left( \frac{3}{2} \mathcal{P}^0_s + \frac{1}{2} \mathcal{P}^0_w \frac{\sqrt{3}}{2} (\mathcal{P}^0_{sw} + \mathcal{P}^0_{ws}) \right) \right]_{\mu\nu\rho\sigma}
\]

and since the following relations hold

\[
(\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0_s + \mathcal{P}^0_w)_{\mu\nu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\rho} \eta_{\mu\sigma}),
\]

\[
\left( \frac{3}{2} \mathcal{P}^0_s + \frac{1}{2} \mathcal{P}^0_w \frac{\sqrt{3}}{2} (\mathcal{P}^0_{sw} + \mathcal{P}^0_{ws}) \right)_{\mu\nu\rho\sigma} = \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma},
\]

we obtain a very simple form for the propagator in the Feynman gauge:

\[
\Pi_{GR,\mu\nu\rho\sigma} = \frac{1}{2k^2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\rho} \eta_{\mu\sigma} - \eta_{\mu\nu} \eta_{\rho\sigma}).
\]

2.3.2 Saturated graviton propagator

Our starting point are the field equations (2.34) that we write again for convenience

\[
k^2 (\mathcal{P}^2 - 2\mathcal{P}^0_s) h(k) = \kappa (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0_s + \mathcal{P}^0_w) \tau(k).
\]

Add to the both members of the last equations the terms \( k^2 (\mathcal{P}^1 + 3\mathcal{P}^0_s + \mathcal{P}^0_w) h : \)

\[
k^2 (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0_s + \mathcal{P}^0_w) h(k) = \kappa (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0_s + \mathcal{P}^0_w) \tau(k)
\]

\[
+ k^2 (\mathcal{P}^1 + 3\mathcal{P}^0_s + \mathcal{P}^0_w) h(k); \tag{2.108}
\]

then, multiply for \( \tau(-k) \) on the left side

\[
k^2 \tau(-k) (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0_s + \mathcal{P}^0_w) h(k) = \kappa \tau(-k) (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0_s + \mathcal{P}^0_w) \tau(k)
\]

\[
+ k^2 \tau(-k) (\mathcal{P}^1 + 3\mathcal{P}^0_s + \mathcal{P}^0_w) h(k). \tag{2.109}
\]
2.3. GRaviton Propagator

Because of the equations (2.88)-(2.91) the last equation reduces to

\[ \tau(-k) (\mathcal{P}^2 + \mathcal{P}_s^0 + \mathcal{P}_w^0) h(k) = \kappa \tau(-k) \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{1}{2} \mathcal{P}_s^0 \right) \tau(k), \]

(2.110)

namely we have derived the saturated propagator for the massless graviton:

\[ \tau(-k) \Pi_{GR}(k) \tau(k) \equiv \tau(-k) \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{1}{2} \mathcal{P}_s^0 \right) \tau(k), \]

(2.111)

or, if we want to write the equations with the indices, one has

\[ \tau^{\mu\nu}(-k) \Pi_{GR,\mu\nu\rho\sigma}(k) \tau^{\rho\sigma}(k) \equiv \tau^{\mu\nu}(-k) \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{1}{2} \mathcal{P}_s^0 \right)_{\mu\nu\rho\sigma} \tau^{\rho\sigma}(k). \]

(2.112)

Now we want to rewrite (2.112) in the following by making explicit the form of the projectors \( \mathcal{P}^2 \) and \( \mathcal{P}_s^0 \). By keeping in mind that \( k_{\mu} \tau^{\mu\nu} = 0 \), one has

\[ \tau^{\mu\nu}(-k) \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{1}{2} \mathcal{P}_s^0 \right)_{\mu\nu\rho\sigma} \tau^{\rho\sigma}(k) = \tau^{\mu\nu}(-k) \frac{1}{k^2} \left( \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho}) - \frac{1}{2} \theta_{\mu\nu} \theta_{\rho\sigma} \right) \tau^{\rho\sigma}(k) \]

\[ = \tau^{\mu\nu}(-k) \frac{1}{k^2} \left[ \theta_{\mu\rho} \theta_{\nu\sigma} - \frac{1}{2} \theta_{\mu\nu} \theta_{\rho\sigma} \right] \tau^{\rho\sigma}(k) \]

\[ = \tau^{\mu\nu}(-k) \frac{1}{k^2} \left[ (\eta_{\mu\rho} - \omega_{\mu\rho}) (\eta_{\nu\sigma} - \omega_{\nu\sigma}) \right. \]

\[ - \frac{1}{2} (\eta_{\mu\nu} - \omega_{\mu\nu}) (\eta_{\rho\sigma} - \omega_{\rho\sigma}) \right] \tau^{\rho\sigma}(k) \]

\[ = \tau^{\mu\nu}(-k) \frac{1}{k^2} \left[ (\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\rho} \omega_{\nu\sigma} - \omega_{\mu\rho} \eta_{\nu\sigma} + \omega_{\mu\rho} \omega_{\nu\sigma}) \right. \]

\[ - \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\rho} \omega_{\nu\sigma} - \omega_{\mu\rho} \eta_{\nu\sigma} + \omega_{\mu\rho} \omega_{\nu\sigma}) \right] \tau^{\rho\sigma}(k) \]

\[ = \tau^{\mu\nu}(-k) \frac{1}{2k^2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\rho} \eta_{\mu\sigma} - \eta_{\mu\nu} \eta_{\rho\sigma} \right) \tau^{\rho\sigma}(k). \]

(2.113)

The equation (2.113) tells us that the physical (gauge-independent) part of the propagator can be easily express as a product of metric tensors. Note also that the physical part of the propagator coincides with the propagator in the Feynman gauge (\( \alpha = 1 \)) we have determined in (2.106).

Moreover, it is worth observing that, since in the saturated propagator just the physical...
2.3. GRAVITON PROPAGATOR

part appears (see (2.112)), if we consider the sandwich between two conserved currents of the propagator in De Donder gauge (2.104) that has both physical and gauge dependent parts, the latter should vanishes. To show this, first notice that all the spin projector operators proportional to the momentum $k^\mu$, i.e. $\mathcal{P}_1^1$, $\mathcal{P}_w^0$, $\mathcal{P}_{sw}^0$ and $\mathcal{P}_{ws}^0$, acting on the source give us a null contribution because of the conservation law of the source (see (2.22) for the expression of the projectors):

$$\mathcal{P}_1^1_{\mu\nu\rho\sigma} \tau^{\rho\sigma} = \frac{1}{2} \left( \theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho} \right) \tau^{\rho\sigma},$$

$$= \frac{1}{2} \left( \frac{k_{\nu} k_{\sigma}}{k^2} + \frac{k_{\mu} k_{\rho}}{k^2} + \frac{k_{\mu} k_{\sigma}}{k^2} + \frac{k_{\nu} k_{\rho}}{k^2} \right) \tau^{\rho\sigma} = 0,$$

$$\mathcal{P}_w^0_{\mu\nu\rho\sigma} \tau^{\rho\sigma} = \omega_{\mu\nu} \omega_{\rho\sigma} \tau^{\rho\sigma} = \frac{k_{\nu} k_{\sigma}}{k^2} \tau^{\rho\sigma} = 0,$$

$$\mathcal{P}_{sw}^0_{\mu\nu\rho\sigma} \tau^{\rho\sigma} = \frac{1}{\sqrt{3}} \tau^{\mu\nu} \omega_{\mu\nu} \omega_{\rho\sigma} \tau^{\rho\sigma} = \frac{1}{\sqrt{3}} \frac{k_{\nu} k_{\sigma}}{k^2} \tau^{\rho\sigma} = 0,$$

$$\tau^{\mu\nu} \mathcal{P}_{ws}^0_{\mu\nu\rho\sigma} \tau^{\rho\sigma} = \frac{1}{\sqrt{3}} \tau^{\mu\nu} \omega_{\mu\nu} \theta_{\rho\sigma} = \frac{1}{\sqrt{3}} \frac{k_{\nu} k_{\sigma}}{k^2} \theta_{\rho\sigma} = 0.$$

While for $\mathcal{P}_s^2$ and $\mathcal{P}_s^0$ the contribution are not zero,

$$\tau^{\mu\nu} (-k) \mathcal{P}_s^2_{\mu\nu\rho\sigma} \tau^{\rho\sigma} (k) \neq 0, \quad \tau^{\mu\nu} (-k) \mathcal{P}_s^0_{\mu\nu\rho\sigma} \tau^{\rho\sigma} (k) \neq 0. \quad (2.115)$$

We are now able to calculate the sandwich between two conserved currents of (2.104):

$$\tau^{\mu\nu} (-k) \frac{1}{k^2} \left[ \mathcal{P}^2 + \alpha \mathcal{P}_1^1 - \frac{1}{2} \mathcal{P}_s^0 + \frac{4\alpha - 3}{2} \mathcal{P}_w^0 - \frac{\sqrt{3}}{2} \mathcal{P}_{sw}^0 - \frac{\sqrt{3}}{2} \mathcal{P}_{ws}^0 \right]_{\mu\nu\rho\sigma} \tau^{\rho\sigma} (k) =$$

$$= \tau^{\mu\nu} (-k) \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{1}{2} \mathcal{P}_s^0 \right)_{\mu\nu\rho\sigma} \tau^{\rho\sigma} (k). \quad (2.116)$$

What we found is that also starting from a generic gauge, we have just had the confirmation that the physical part of the graviton propagator is\(^{22}\)

$$\Pi_{GR} = \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{1}{2} \mathcal{P}_s^0 \right). \quad (2.117)$$

In the next chapters, especially in Chapter 3 and 4, our discussions will concern the physical part of the propagator, i.e. (2.117) and its modification in the framework of special theories of modified gravity.

\(^{22}\)We have to observe that (2.117) doesn’t represent the propagator but its physical part that remain in the saturated propagator, i.e. in (2.116). Sometime it can happen that we make an abuse of nomenclature calling it *just* with word propagator.
2.4 Graviton propagator and polarization sums

2.4.1 Polarization tensors

In the subsection 1.4.1 we constructed a set of spin-1 polarization vector for the photon, $e_{(j=1,j_2)}^\mu$. Our aim now is to do the same with graviton, i.e. to introduce a set of polarization tensors. In particular we want the spin-2 polarization tensors, $\epsilon_{(j,j_2)}^{\mu\nu}$. An easy way to construct $j=2$ polarization tensors is to take products of the $j=1$ polarization vectors given in (1.68), that we write down again for convenience:

$$\epsilon_{(1,+1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \epsilon_{(1,-1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad \epsilon_{(1,0)} = \frac{1}{k} \begin{pmatrix} k^3 \\ 0 \\ 0 \\ k^0 \end{pmatrix}. \quad (2.118)$$

The right products can be obtained using Clebsch-Gordan coefficients (Appendix E) corresponding to the product of two spin-1 : $1 \otimes 1$. In fact, looking at Appendix E, the product of two $j=1$ gives $j=2$, $j=1$ and $j=0$ polarization tensors.

$j=2$ gives us five polarization tensors, $j_2 = +2, +1, 0, -1, -2$ :

$$\epsilon_{(2,+2)}^{\mu\nu} = \epsilon_{(1,+1)}^{\mu} \otimes \epsilon_{(1,+1)}^{\nu} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.119)$$

$$\epsilon_{(2,-2)}^{\mu\nu} = \epsilon_{(1,-1)}^{\mu} \otimes \epsilon_{(1,-1)}^{\nu} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.120)$$

$$\epsilon_{(2,+1)}^{\mu\nu} = \frac{1}{\sqrt{2}} \left( \epsilon_{(1,+1)}^{\mu} \otimes \epsilon_{(1,0)}^{\nu} + \epsilon_{(1,0)}^{\mu} \otimes \epsilon_{(1,+1)}^{\nu} \right) = \frac{1}{2k} \begin{pmatrix} 0 & k^3 & i k^3 & 0 \\ k^3 & 0 & 0 & k^0 \\ i k^3 & 0 & 0 & i k^0 \\ 0 & k^0 & i k^0 & 0 \end{pmatrix}. \quad (2.120)$$


\[
\epsilon_{(2,-1)}^{\mu\nu} = \frac{1}{\sqrt{2}} \left( \epsilon_{(1,-1)}^{\mu} \otimes \epsilon_{(1,0)}^{\nu} + \epsilon_{(1,0)}^{\mu} \otimes \epsilon_{(1,-1)}^{\nu} \right)
\]

\[
= \frac{1}{2k} \begin{pmatrix}
0 & k^3 & -ik^3 & 0 \\
k^3 & 0 & 0 & k^0 \\
-ik^3 & 0 & 0 & -ik^0 \\
0 & k^0 & -ik^0 & 0
\end{pmatrix}, \tag{2.121}
\]

\[
\epsilon_{(2,0)}^{\mu\nu} = \frac{1}{\sqrt{6}} \left( \epsilon_{(1,1)}^{\mu} \otimes \epsilon_{(1,-1)}^{\nu} + \epsilon_{(1,-1)}^{\mu} \otimes \epsilon_{(1,1)}^{\nu} + 2 \epsilon_{(1,0)}^{\mu} \otimes \epsilon_{(1,0)}^{\nu} \right)
\]

\[
= \frac{1}{\sqrt{6}} \begin{pmatrix}
2 \frac{(k^3)^2}{k^2} & 0 & 0 & 2 \frac{k^3 k^0}{k^2} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
2 \frac{k^3 k^0}{k^2} & 0 & 0 & 2 \frac{(k^0)^2}{k^2}
\end{pmatrix}. \tag{2.122}
\]

Then the polarization tensor corresponding to \( j = 0 \) is

\[
\epsilon_{(0,0)}^{\mu\nu} = \frac{1}{\sqrt{3}} \left( \epsilon_{(1,1)}^{\mu} \otimes \epsilon_{(1,-1)}^{\nu} + \epsilon_{(1,-1)}^{\mu} \otimes \epsilon_{(1,1)}^{\nu} - \epsilon_{(1,0)}^{\mu} \otimes \epsilon_{(1,0)}^{\nu} \right)
\]

\[
= \frac{1}{\sqrt{3}} \begin{pmatrix}
-\frac{(k^3)^2}{k^2} & 0 & 0 & -\frac{k^3 k^0}{k^2} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{k^3 k^0}{k^2} & 0 & 0 & -\frac{(k^0)^2}{k^2}
\end{pmatrix}. \tag{2.123}
\]

In principle we also have the spin-1 polarization tensor \( \epsilon_{(1,1)}^{\mu\nu}, \epsilon_{(1,0)}^{\mu\nu} \) and \( \epsilon_{(1,-1)}^{\mu\nu} \):

\[
\epsilon_{(1,1)}^{\mu\nu} = \frac{1}{\sqrt{2}} \left( \epsilon_{(1,1)}^{\mu} \otimes \epsilon_{(1,0)}^{\nu} - \epsilon_{(1,0)}^{\mu} \otimes \epsilon_{(1,1)}^{\nu} \right), \tag{2.124}
\]

\[
\epsilon_{(1,-1)}^{\mu\nu} = \frac{1}{\sqrt{2}} \left( \epsilon_{(1,0)}^{\mu} \otimes \epsilon_{(1,-1)}^{\nu} - \epsilon_{(1,-1)}^{\mu} \otimes \epsilon_{(1,0)}^{\nu} \right), \tag{2.125}
\]

\[
\epsilon_{(1,0)}^{\mu\nu} = \frac{1}{\sqrt{2}} \left( \epsilon_{(1,1)}^{\mu} \otimes \epsilon_{(1,-1)}^{\nu} - \epsilon_{(1,-1)}^{\mu} \otimes \epsilon_{(1,1)}^{\nu} \right). \tag{2.126}
\]

But (2.124), (2.125) and (2.126) say that the polarization tensor with \( j = 1 \) are antisymmetric, so we shall not consider them.\(^{23}\)

One can easily compute the value of the helicity of each polarization tensor by acting

\(^{23}\)Even if we consider them, since they appear sandwiched between two conserved currents, they give a null contribution when are multiplied with symmetric tensors, \( \tau^{\mu\nu} \epsilon_{(1,jz),\mu\nu} = 0 \).
twice with the rotation matrix around the third axis (see (1.64) for its definition). The following relations hold:

\[
R_{\rho}^{(z)} \mu (\vartheta) R_{\sigma}^{(z)} \nu (\vartheta) \epsilon^{\rho \sigma}_{(2,+2)} = e^{i \vartheta} \epsilon^{\mu \nu}_{(2,+2)},
\]

\[
R_{\rho}^{(z)} \mu (\vartheta) R_{\sigma}^{(z)} \nu (\vartheta) \epsilon^{\rho \sigma}_{(2,-2)} = e^{-i \vartheta} \epsilon^{\mu \nu}_{(2,-2)},
\]

\[
R_{\rho}^{(z)} \mu (\vartheta) R_{\sigma}^{(z)} \nu (\vartheta) \epsilon^{\rho \sigma}_{(2,+1)} = e^{i \vartheta} \epsilon^{\mu \nu}_{(2,+1)},
\]

\[
R_{\rho}^{(z)} \mu (\vartheta) R_{\sigma}^{(z)} \nu (\vartheta) \epsilon^{\rho \sigma}_{(2,-1)} = e^{-i \vartheta} \epsilon^{\mu \nu}_{(2,-1)},
\]

\[
R_{\rho}^{(z)} \mu (\vartheta) R_{\sigma}^{(z)} \nu (\vartheta) \epsilon^{\rho \sigma}_{(2,0)} = \epsilon^{\mu \nu}_{(2,0)},
\]

\[
R_{\rho}^{(z)} \mu (\vartheta) R_{\sigma}^{(z)} \nu (\vartheta) \epsilon^{\rho \sigma}_{(0,0)} = \epsilon^{\mu \nu}_{(0,0)}.
\]

Hence we have seen that we have six polarization tensors, (2.118)-(2.123), one per each degree of freedom of a virtual graviton. For a real graviton we expect that only two polarization tensors are physical; indeed in the next subsection we shall see that the physical ones are those corresponding to the helicity states \(j_z = +2, -2\).

### 2.4.2 Graviton propagator in terms of polarization tensors

As we have done for the photon propagator, we want to rewrite the saturated propagator in terms of the polarization tensors and see which components are present for either on-shell and off-shell graviton. Unlike the case of the virtual photon, which presented only a spin-1 component, we will have confirmation that a virtual graviton in addition to the spin-2 component has also a spin-0 components. We will be able to write these two different components in terms of the polarization tensors. The fact that the graviton has a \(j = 0\) component, as we have verified in the section 2.3, doesn’t violate any fundamental principle, but its presence is very important because cancels the \(j = 2, j_z = 0\) component when we consider a on-shell graviton. We are also taking inspiration from [42].

#### Off-shell graviton

Our starting point is the saturated propagator in (2.112). By expliciting the spin projector operators in terms of \(\eta_{\mu \nu}\) and \(k_\mu\), using the relations (2.105) and implementing the conservation law of the source, \(k_\mu \tau^{\mu \nu} = 0\), one gets

\[
\tau^{\mu \nu} (-k) \Pi_{GR, \mu \nu \rho \sigma} (k) \tau^{\rho \sigma} (k) = \tau^{\mu \nu} (-k) \frac{1}{2k^2} (\eta_{\mu \rho} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \rho} - \eta_{\mu \nu} \eta_{\rho \sigma}) \tau^{\rho \sigma} (k). \tag{2.28}
\]

The explicit expression of (2.28), once we compute the products between \(\tau^{\mu \nu}\) and the
metric tensor $\eta_{\mu\nu}$, is
\[
\tau^{\mu\nu} (-k) \Pi_{GR,\mu\nu;\rho\sigma}(k) \tau^{\rho\sigma}(k) = \frac{1}{k^2} \left\{ \frac{1}{2} \tau^{00} (-k) \left[ \tau^{00}(k) + \tau^{11}(k) + \tau^{22}(k) + \tau^{33}(k) \right] \right.
\]
\[
+ \frac{1}{2} \tau^{11} (-k) \left[ \tau^{00}(k) + \tau^{11}(k) - \tau^{22}(k) - \tau^{33}(k) \right]
\]
\[
+ \frac{1}{2} \tau^{22} (-k) \left[ \tau^{00}(k) - \tau^{11}(k) + \tau^{22}(k) - \tau^{33}(k) \right]
\]
\[
+ \frac{1}{2} \tau^{33} (-k) \left[ \tau^{00}(k) - \tau^{11}(k) - \tau^{22}(k) + \tau^{33}(k) \right]
\]
\[
- 2\tau^{01} (-k) \tau^{01}(k) - 2\tau^{02} (-k) \tau^{02}(k) - 2\tau^{03} (-k) \tau^{03}(k)
\]
\[
+ 2\tau^{12} (-k) \tau^{12}(k) + 2\tau^{13} (-k) \tau^{13}(k) + 2\tau^{23} (-k) \tau^{23}(k) \right\}.
\]
(2.129)

The symmetry of the source, $\tau^{\mu\nu} = \tau^{\nu\mu}$, has been used to obtain the last expression.

Let us suppose that the virtual graviton has the four-momentum $k^\mu \equiv (k^0, 0, 0, k^3)$. In the off-shell case the conservation law $k_\mu \tau^{\mu\nu} = 0$ assumes the following form:
\[
k^0 \tau^{0\nu} = k^3 \tau^{3\nu}.
\]
(2.130)

By performing easy calculations, using symmetry and (2.130), we can rewrite (2.129) in a more convenient form:
\[
\tau^{\mu\nu} (-k) \Pi_{GR,\mu\nu;\rho\sigma}(k) \tau^{\rho\sigma}(k) = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4,
\]
(2.131)

where
\[
\Pi_1 = \frac{1}{k^2} \left\{ \frac{1}{2} \left[ (\tau^{11} (-k) - \tau^{22}(-k)) \left( \tau^{11}(k) - \tau^{22}(k) \right) \right] + 2\tau^{12} (-k) \tau^{12}(k) \right\},
\]
\[
\Pi_2 = \frac{2}{k^2} \left[ \tau^{13}(-k) \tau^{13}(k) + \tau^{23}(-k) \tau^{23}(k) - \tau^{01}(-k) \tau^{01}(k) - \tau^{02}(-k) \tau^{02}(k) \right],
\]
\[
\Pi_3 = \frac{1}{6k^2} \left[ 2 \left( \tau^{00}(-k) - \tau^{33}(-k) \right) + \tau^{11}(-k) + \tau^{22}(-k) \right]
\]
\[
\times \left[ 2 \left( \tau^{00}(k) - \tau^{33}(k) \right) + \tau^{11}(k) + \tau^{22}(k) \right],
\]
\[
\Pi_4 = \frac{1}{6k^2} \left[ -\tau^{00}(-k) + \tau^{33}(-k) + \tau^{11}(-k) + \tau^{22}(-k) \right]
\]
\[
\times \left[ -\tau^{00}(k) + \tau^{33}(k) + \tau^{11}(k) + \tau^{22}(k) \right].
\]
(2.132)
Now we want to rewrite each $\Pi_i$ in terms of the polarization tensors. Always using symmetry and conservation law, we notice that:

$$\tau^{\mu\nu}(-k) \left( \sum_{j_z = +2, -2} \epsilon_{(2,j_z),\mu\nu} \epsilon^*_{(2,j_z),\rho\sigma} \right) \tau^{\rho\sigma}(k) =$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tau^{\rho\sigma}(k)$$

$$+ \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tau^{\rho\sigma}(k) =$$

$$= \frac{1}{4} [\tau^{11}(-k) + 2i\tau^{12}(-k) - \tau^{22}(-k)] [\tau^{11}(k) - 2i\tau^{12}(k) - \tau^{22}(k)]$$

$$+ \frac{1}{4} [\tau^{11}(-k) - 2i\tau^{12}(-k) - \tau^{22}(-k)] [\tau^{11}(k) + 2i\tau^{12}(k) - \tau^{22}(k)]$$

$$= \frac{1}{2} \left[ (\tau^{11}(-k) - \tau^{22}(-k)) (\tau^{11}(k) - \tau^{22}(k)) \right] + 2\tau^{12}(-k)\tau^{12}(k) = \Pi_1. \quad (2.133)$$

So the first line of (2.131) corresponds to $j_z = +2, -2$. In the same way we can connect $\Pi_2, \Pi_3, \Pi_4$ to the other polarization tensors. Although we are not going to do all the tedious calculations, one can see that:

$$\tau^{\mu\nu}(-k) \left( \sum_{j_z = +1, -1} \epsilon_{(2,j_z),\mu\nu} \epsilon^*_{(2,j_z),\rho\sigma} \right) \tau^{\rho\sigma}(k) =$$

$$2 [\tau^{13}(-k)\tau^{13}(k) + \tau^{23}(-k)\tau^{23}(k) - \tau^{01}(-k)\tau^{01}(k) - \tau^{02}(-k)\tau^{02}(k)] = \Pi_2; \quad (2.134)$$

$$\tau^{\mu\nu}(-k) \epsilon_{(2,0),\mu\nu} \epsilon^*_{(2,0),\rho\sigma} \tau^{\rho\sigma}(k) =$$

$$= \frac{1}{6} \left[ 2 (\tau^{00}(-k) - \tau^{33}(-k)) + \tau^{11}(-k) + \tau^{22}(-k) \right]$$

$$\times [2 (\tau^{00}(k) - \tau^{33}(k)) + \tau^{11}(k) + \tau^{22}(k)] = \Pi_3; \quad (2.135)$$

$$\tau^{\mu\nu}(-k) \epsilon_{(0,0),\mu\nu} \epsilon^*_{(0,0),\rho\sigma} \tau^{\rho\sigma}(k) =$$

$$= \frac{1}{3} \left[ -\tau^{00}(-k) + \tau^{33}(-k) + \tau^{11}(-k) + \tau^{22}(-k) \right]$$

$$\times [-\tau^{00}(k) + \tau^{33}(k) + \tau^{11}(k) + \tau^{22}(k)] = \Pi_4. \quad (2.136)$$

So $\Pi_2$ corresponds to $j = 2, j_z = \pm 1$ component, $\Pi_3$ to $j = 2, j_z = 0$ and $\Pi_4$ to the spin-0 component, $j = 0.$
Now we need to rewrite the saturated propagator in terms of the polarization tensors; in fact from (2.133)-(2.136) we obtain
\[
\tau^{\mu\nu}(-k) \Pi_{GR,\mu\nu\rho\sigma}(k) \tau^{\rho\sigma}(k) = \tau^{\mu\nu}(-k) \frac{1}{k^2} \left( \sum_{j_z = -2}^{+2} \epsilon_{(2,j_z),\mu\nu} \epsilon^*_{(2,j_z),\rho\sigma} \right) \tau^{\rho\sigma}(k),
\]
where the sum runs on over the five \( j = 2 \) polarization tensors. Equation (2.137) confirms that a virtual graviton has also a scalar component,
\[
\tau^{\mu\nu}(-k) \frac{1}{2} \epsilon_{(0,0),\mu\nu} \epsilon^*_{(0,0),\rho\sigma} \tau^{\rho\sigma}(k).
\]
By comparing the saturated propagator in (2.137) with its form in (2.112) we can read the spin projector operators \( P^2 \) and \( P^0_s \) in terms of the polarization tensors:
\[
P^2_{\mu\nu\rho\sigma} \equiv \sum_{j_z = -2}^{+2} \epsilon_{(2,j_z),\mu\nu} \epsilon^*_{(2,j_z),\rho\sigma}
\]
\[
P^0_{s,\mu\nu\rho\sigma} \equiv \epsilon_{(0,0),\mu\nu} \epsilon^*_{(0,0),\rho\sigma}.
\]

**On-shell graviton**

As for the real graviton the saturated propagator (2.131) reduces in a more simple form when on-shell condition is imposed. In fact, in the on-shell case \( k^0 = k^3 \) the conservation law 2.130 becomes
\[
\tau^{0\nu} = \tau^{3\nu}.
\]
But (2.139) implies also \( \tau^{00} = \tau^{03} = \tau^{33} \). Hence because of this last relation and (2.128) the saturated propagator for a real graviton becomes
\[
\tau^{\mu\nu}(-k) \Pi_{GR,\mu\nu\rho\sigma}(k) \tau^{\rho\sigma}(k) = \frac{1}{k^2} \left\{ 2 \tau^{12}(-k) \tau^{12}(k) + \frac{1}{2} \left[ (\tau^{11}(-k) - \tau^{22}(-k)) (\tau^{11}(k) - \tau^{22}(k)) \right] \right\}
\]
\[
= \tau^{\mu\nu}(-k) \frac{1}{k^2} \left( \sum_{j_z = +2,-2} \epsilon_{(2,j_z),\mu\nu} \epsilon^*_{(2,j_z),\rho\sigma} \right) \tau^{\rho\sigma}(k).
\]

(2.140)
The last equation confirms that an on-shell graviton has only two degrees of freedom, indeed only the two polarization tensors with helicity $j_z = +2$ and $j_z = -2$ are present.

We must also notice that when we go on-shell, it happens that in equation (2.131) the component $\Pi^4$ cancels with $\Pi^3$, namely the scalar component cancels with the $j = 2$, $j_z = 0$ component (longitudinal component). Thus we have shown what we have anticipated at the beginning of this subsection, i.e. that the $j = 0$ component is necessary in order to ensure that a real graviton has no $j = 2$, $j_z = 0$ component.

### 2.5 Ghosts and unitarity analysis in General Relativity

In this section we want to check whether ghosts are absent, so, whether the unitarity is preserved in GR\(^\text{[39],[38]}\). As we have done for the vector field case we shall follow the method discussed in Appendix C, i.e. we will verify the positivity of the imaginary part of the amplitude residue at the pole $k^2 = 0$.

Let us consider the propagator in (2.104) with a generic parameter $\alpha$. As we have already seen above, the choice of De Donder gauge, corresponding to the symmetry $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, implies the conservation of the energy-momentum tensor $\partial_\mu \tau^{\mu\nu} = 0$, or equivalently in momentum space $k_\mu \tau^{\mu\nu} = 0$.

The amplitude of a process in which a graviton is created at a point of the space-time and is annihilated at another point is described by an amplitude of the following type (see (2.112) or (2.116)):

$$A = \tau^{\mu\nu}(k) \langle T (h_{\mu\nu}(-k) h_{\rho\sigma}(k)) \rangle \tau^{\rho\sigma}(k)$$

$$= i \tau^{\mu\nu}(k) \Pi_{GR,\mu\nu\rho\sigma} \tau^{\rho\sigma}(k), \quad \text{(2.141)}$$

$$= i \tau^{\mu\nu}(k) \frac{1}{k^2} \left( P^2 - \frac{1}{2} P^0 \right) \tau^{\rho\sigma}(k).$$

where $k^\mu$ is the four-momentum of the graviton. By following the calculations in (2.113) we can also recast the equation (2.141) just in terms of the Minkowski metric tensor:

$$A = i \tau^{\mu\nu}(k) \frac{1}{2k^2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\nu\rho} \eta_{\mu\sigma} - \eta_{\mu\nu} \eta_{\rho\sigma}) \tau^{\rho\sigma}(k)$$

$$= \frac{i}{k^2} \left( \tau^{\mu\nu}(k) \tau_{\mu\nu}(k) - \frac{1}{2} \tau^{\rho\sigma}(k) \tau_{\rho\sigma}(k) \right) \quad \text{(2.142)}$$

$$= \frac{i}{k^2} \left( |\tau^{\mu\nu}|^2 - \frac{1}{2} |\tau_{\sigma\sigma}|^2 \right).$$
2.5. GHOSTS AND UNITARITY ANALYSIS IN GENERAL RELATIVITY

The residue of the amplitude (2.142) in \( k^2 = 0 \) is

\[
\text{Res}_{k^2=0} A = \lim_{k^2 \to 0} k^2 \frac{i}{k^2} \left( |\tau^{\mu\nu}|^2 - \frac{1}{2} |\tau^\sigma|^2 \right) = i \left( |\tau^{\mu\nu}|^2 - \frac{1}{2} |\tau^\sigma|^2 \right)
\]  

(2.143)

\[\Rightarrow \text{Im} \{\text{Res}_{k^2=0} A\} = |\tau^{\mu\nu}|^2 - \frac{1}{2} |\tau|^2.\]  

(2.144)

To understand if the imaginary part of the residue is either positive or negative we have to analyze the structure of the current \( \tau^{\mu\nu} \). Let us consider again the basis (1.16), \( \{k^{\mu}, \tilde{k}^{\mu}, \varepsilon_1^\mu, \varepsilon_2^\mu\} \), in the space of the four-vectors, and expand the current \( \tau^{\mu\nu} \) as

\[
\tau^{\mu\nu}(k) = a(k)k_\mu k_\nu + b(k)k_\mu(\tilde{k}_\nu) + c_i(k)k_\mu \varepsilon_i^\nu
\]

\[+ d(k)\tilde{k}_\mu \tilde{k}_\nu + e_i(k)\tilde{k}_\mu \varepsilon_i^\nu + f_{ij}(k)\varepsilon_i^\mu \varepsilon_j^\nu.\]

(2.145)

Impose the conservation law for \( \tau^{\mu\nu} \):

\[
[k^{\mu}\tau^{\mu\nu}(k)]_{k^2=0} = \frac{1}{2}b(k) \left( k^{\mu}\tilde{k}_\mu \right) k_\nu + d(k) \left( k^{\mu}\tilde{k}_\mu \right) \tilde{k}_\nu + \frac{1}{2}e_i(k) \left( k^{\mu}\tilde{k}_\mu \right) \varepsilon_i^\nu = 0.
\]  

(2.146)

Note that we have valuated the last equation in \( k^2 = 0 \) and we shall also do the same with the following formulas, because we are interested in the residue of the amplitude that is calculated in \( k^2 = 0 \).

Since \( k \cdot \tilde{k} = k^\mu k_\mu = (k^0)^2 + \tilde{k}^2 = 2\tilde{k}^2 \neq 0 \), (2.146) reads as

\[
\frac{1}{2}b(k)k_\nu + d(k)\tilde{k}_\nu + \frac{1}{2}e_i(k)\varepsilon_i^\nu = 0.
\]

(2.147)

If we consider the component \( \nu = 0 \) of the the last equation we obtain

\[
\frac{1}{2}b(k)k_0 + d(k)k_0 = 0, \quad k_0 \neq 0 \Rightarrow b(k) = -2d(k),
\]  

(2.148)

so (2.147) becomes

\[
b(k) \left( k_\nu - \tilde{k}_\nu \right) + \frac{1}{2}e_i(k)\varepsilon_i^\nu = 0,
\]

(2.149)

and multiplying for \( \varepsilon^{j, \nu} \) we have

\[
\frac{1}{2}e_i(k)\varepsilon^{j, \nu}\varepsilon_i^\nu = \frac{1}{2}e_i(k)\delta^{ji} = 0 \Leftrightarrow e_i(k) = 0, \quad i = 1, 2.
\]

(2.150)

Hence \( k^2 = 0 \) implies \( d(k) = 0 \) and \( e_1(k) = 0 = e_2(k) \), and the expansion for \( \tau^{\mu\nu} \) simplifies as

\[
[\tau^{\mu\nu}(k)]_{k^2=0} = a(k)k_\mu k_\nu + c_i(k)k_\mu \varepsilon_i^\nu + f_{ij}(k)\varepsilon_i^\mu \varepsilon_j^\nu;
\]
2.5. GHOSTS AND UNITARITY ANALYSIS IN GENERAL RELATIVITY

Instead for the trace $\tau^\sigma(k)$ valuated in $k^2 = 0$ one has\footnote{Note that we are always following the Einstein convention. So, with $f = f_{ii}$ we mean the trace $f = \sum_i f_{ii}$, with $|f|^2$ we mean $|f|^2 = (\sum_i f_{ii}) (\sum_j f_{jj})$ and with $|f_{ij}|^2$ we mean $|f_{ij}|^2 = \sum_{ij} f_{ij}^* f_{ij}$.}

$$[\tau^\sigma(k)]_{k^2=0} = -\delta^{ij} f_{ij} = - f_{ii}$$  \hfill (2.151)

Let us now calculate the quantities present in the imaginary part of the residue. We can start from $\tau^\mu\nu(k)\tau_{\mu\nu}(k)$:

$$[\tau^\mu\nu(k)\tau_{\mu\nu}(k)]_{k^2=0} = \frac{1}{4} f_{ij}^*(k) f_{kl}(k) (\varepsilon^{i\mu} \varepsilon^{j\nu} + \varepsilon^{i\mu} \varepsilon^{k\nu}) (\varepsilon^{k\mu} \varepsilon^{l\nu} + \varepsilon^{k\mu} \varepsilon^{l\nu})$$

$$= \frac{1}{2} f_{ij}^*(k) f_{kl}(k) (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) = f_{ij}^*(k) f_{kl}(k)$$  \hfill (2.152)

$$= |f_{ij}|^2$$

instead for the trace part $\tau^*(k)\tau(k)$ we obtain:

$$[\tau^*(k)\tau(k)]_{k^2=0} = f_{ii}^* f_{jj} = |f_{ii}|^2.$$  \hfill (2.153)

Now, by using the formulas (2.152) and (2.153) we are able to rewrite the imaginary part of the residue in terms of the coefficients $f_{ij}$:

$$\text{Im}\{\text{Res}_{k^2=0} A\} = |f_{ij}|^2 - \frac{1}{2} |f|^2.$$  \hfill (2.154)

Going ahead with the calculations we obtain:

$$|f_{ij}|^2 - \frac{1}{2} |f|^2 = f_{ij}^* f_{ij} - f_{ii}^* f_{jj}$$

$$= f_{11}^* f_{11} + 2 f_{12}^* f_{12} + f_{22}^* f_{22}$$

$$- \frac{1}{2} (f_{11}^* f_{11} + f_{11}^* f_{22} + f_{22}^* f_{11} + f_{22}^* f_{22})$$  \hfill (2.155)

$$= \frac{1}{2} (f_{11} - f_{22})^* (f_{11} - f_{22}) + 2 f_{12}^* f_{12}$$

$$= \frac{1}{2} |f_{11} - f_{22}|^2 + 2 |f_{12}|^2 > 0$$

$$\Rightarrow \text{Im}\{\text{Res}_{k^2=0} A\} > 0,$$  \hfill (2.156)

i.e. in GR the unitarity is not violated.
Remark 2. The spin-0 part of the propagator, having a minus sign, can let us suspect the presence of a ghost that could violate the unitarity of the theory. We have just showed that the unitarity is preserved, so it means that the spin-0 component corresponds to a “good” ghost that is fundamental to ensure the unitarity. In the previous chapter we have seen why its presence is so important, in fact we pointed out that the spin-0 component is equal and opposite to the helicity-0 component of the spin-2 part of the propagator, so they cancel out.
Chapter 3

Quadratic gravity

In the Introduction, we have emphasized that Einstein’s General Relativity is the best theory we have to describe the gravitational interaction, although we need to face serious questions when one considers the behavior at short distances (or high energy). Indeed, problems of divergence emerge because of the presence of black hole and cosmological singularities at the classical level and because of the UV incompleteness at the quantum level. Because of these problems modification of Hilbert-Einstein action is demanded, but one must take care of preserving the well known and valid behavior in the infrared regime (or large distances) that is consistent with the experiments.

In this chapter we shall consider one of the most intuitive modification of GR, i.e. in addition to the Einstein-Hilbert term we are going to consider all the possible terms quadratic in the curvatures \( R_{\mu_1\nu_1\lambda_1\sigma_1} \).

\[ \mathcal{S}_q = \frac{1}{2} \int d^4x \sqrt{-g} R_{\mu_1\nu_1\lambda_1\sigma_1} D^\mu_1{}_{\nu_1}{}_{\lambda_1}{}_{\sigma_1} R^\mu_2{}_{\nu_2}{}_{\lambda_2}{}_{\sigma_2}, \]

(3.1)

where the operator \( D \) is made in such a way to preserve the general covariance, so it must be a differential operator containing only covariant derivatives and the metric tensor \( g_{\mu\nu} \).

Note that it can happen that a differential operator acts on the left of the Riemann tensor as well, but one can always recast that into the above expression (3.1) integrating \( \sqrt{-g} \).
3.1. MOST GENERAL QUADRATIC CURVATURE ACTION OF GRAVITY

By part.

By considering all possible independent combinations of covariant derivatives and metric tensors in the operator $\mathcal{D}$, one can write down the most general quadratic action $S_q$ (parity-invariant and torsion free) explicitly:

$$S_q = \int d^4x \sqrt{-g} \left[ \mathcal{R} F_1(\Box) \mathcal{R} + \mathcal{R} F_2(\Box) \nabla_\mu \nabla_\nu \mathcal{R}^{\mu\nu} + \mathcal{R} F_3(\Box) \mathcal{R}^{\mu\nu} + \mathcal{R} F_4(\Box) \nabla_\nu \nabla_\lambda \mathcal{R}^{\mu\lambda} + \mathcal{R} F_5(\Box) \nabla_\mu \nabla_\nu \nabla_\sigma \mathcal{R}^{\mu\nu\sigma} \mathcal{R}^{\mu\nu\sigma} + \mathcal{R} F_6(\Box) \nabla_\nu \nabla_\lambda \mathcal{R}^{\mu\nu\lambda} \mathcal{R}^{\mu\nu\lambda} + \mathcal{R} F_7(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\rho \mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} + \mathcal{R} F_8(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\rho \mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} + \mathcal{R} F_9(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\sigma \mathcal{R}^{\mu\nu\sigma\sigma} \mathcal{R}^{\mu\nu\sigma\sigma} + \mathcal{R} F_{10}(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\rho \mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} + \mathcal{R} F_{11}(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\sigma \mathcal{R}^{\mu\nu\sigma\sigma} \mathcal{R}^{\mu\nu\sigma\sigma} + \mathcal{R} F_{12}(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\sigma \mathcal{R}^{\mu\nu\sigma\sigma} \mathcal{R}^{\mu\nu\sigma\sigma} + \mathcal{R} F_{13}(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\sigma \mathcal{R}^{\mu\nu\sigma\sigma} \mathcal{R}^{\mu\nu\sigma\sigma} + \mathcal{R} F_{14}(\Box) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\sigma \mathcal{R}^{\mu\nu\sigma\sigma} \mathcal{R}^{\mu\nu\sigma\sigma} \right]$$

$$\equiv \int d^4x \sqrt{-g} L_q$$

Note that the order of the covariant derivatives does not matter because their commutator is proportional to another curvature leading to $O(\mathcal{R}^3)$ modifications. The most general quadratic action is captured by 14 arbitrary functions, the $F_i(\Box)$'s, that are functions of the D'Alambertian operator, $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$. For now we do not make other assumptions on the functions $F_i(\Box)$'s, but in the next chapter their form will be fundamental for our results. The Lagrangian introduced in the last line of (3.2) is the quadratic curvature Lagrangian of the modified theory of gravity we are considering.

We notice that not all the 14 terms are independent. Indeed by using the antisymmetry properties of the Riemann tensor,

$$\mathcal{R}^{(\mu\nu)}_{\lambda\sigma} = \mathcal{R}^{\mu\nu}_{(\lambda\sigma)} = 0,$$

and

$$\nabla_\alpha \mathcal{R}^{\mu\nu\lambda\sigma} + \nabla_\sigma \mathcal{R}^{\mu\nu\alpha\lambda} + \nabla_\lambda \mathcal{R}^{\mu\nu\sigma\alpha} = 0,$$
the action (3.2) can be reduced to the following simpler form:

\[ S_q = \int d^4x \sqrt{-g} \left[ \mathcal{R} \mathcal{F}_1(\Box) \mathcal{R} + \mathcal{R}_{\mu\nu} \mathcal{F}_2(\Box) \mathcal{R}_{\mu\nu} + \mathcal{R}_{\mu\nu\lambda\sigma} \mathcal{F}_3(\Box) \mathcal{R}_{\mu\nu\lambda\sigma} + \mathcal{R}_{\mu\nu} \mathcal{F}_4(\Box) \nabla_\mu \nabla_\nu \nabla_\sigma \mathcal{R}_{\mu\nu\lambda\sigma} + \mathcal{R}_{\mu\nu} \mathcal{F}_5(\Box) \nabla_\mu \mathcal{R}_{\nu\lambda} + \mathcal{R}_{\mu\nu\lambda\sigma} \mathcal{F}_6(\Box) \nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\sigma \mathcal{R}_{\mu\nu\lambda\sigma} \right], \tag{3.3} \]

where the functions \( \mathcal{F}_i(\Box) \)'s are new functions depending on the \( F_1(\Box) \)'s. So we got rid of 8 of the 14 terms already in a curved background.

Remember that for a complete and consistent description, we also need to include the contribution from the well known Einstein-Hilbert term. Thus, the full action we need to consider is

\[ S = -\int d^4x \sqrt{-g} \mathcal{R} + S_q. \tag{3.4} \]

### 3.1.1 Linearized quadratic action

We shall follow by following the same steps and prescriptions we have already seen for GR in the previous Chapter. Hence, our first task is to obtain the quadratic (in \( h_{\mu\nu} \)) free part of the above action (3.4). We can note that the covariant derivatives must be taken on the Minkowski space-time, and we can commute them freely as they are simple partial derivatives. Thus, the terms with \( \mathcal{F}_4(\Box), \mathcal{F}_5(\Box) \) and \( \mathcal{F}_6(\Box) \) in the action (3.3) do not contribute in the limit of flat background because of the vanishing value of the symmetric-antisymmetric products between partial derivatives and Riemann tensor, whose index pairs are antisymmetric. For example, one has

\[ \mathcal{R} \mathcal{F}_4(\Box) \nabla_\mu \nabla_\lambda \nabla_\nu \nabla_\sigma \mathcal{R}_{\mu\nu\lambda\sigma} = \mathcal{R} \mathcal{F}_4(\Box) \partial_\mu \partial_\lambda \partial_\nu \partial_\sigma \mathcal{R}_{\mu\nu\lambda\sigma} = \mathcal{R} \mathcal{F}_4(\Box) \partial_\mu \partial_\nu \partial_\lambda \partial_\sigma \mathcal{R}_{\mu\nu\lambda\sigma} = 0 \tag{3.5} \]

and one can do the same with \( \mathcal{F}_4 \) and \( \mathcal{F}_5 \). At the end the linearized form of the action (3.4) reads as

\[ S = \int d^4x \left[ -\mathcal{R} + \mathcal{R} \mathcal{F}_1(\Box) \mathcal{R} + \mathcal{R}_{\mu\nu} \mathcal{F}_2(\Box) \mathcal{R}_{\mu\nu} + \mathcal{R}_{\mu\nu\lambda\sigma} \mathcal{F}_3(\Box) \mathcal{R}_{\mu\nu\lambda\sigma} \right]. \tag{3.6} \]
3.1. MOST GENERAL QUADRATIC CURVATURE ACTION OF GRAVITY

We recall for convenience the linearized forms of the Riemann tensor, the Ricci tensor and the scalar curvature, already introduced in the previous chapter:

\[
R_{\mu\nu\lambda\sigma} = \frac{1}{2} (\partial_\nu \partial_\lambda h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\lambda} - \partial_\sigma \partial_\nu h_{\mu\lambda} - \partial_\mu \partial_\lambda h_{\nu\sigma}),
\]

(3.7)

\[
R_{\mu\nu} = \frac{1}{2} (\partial_\rho \partial_\nu h_{\mu\rho} + \partial_\rho \partial_\mu h_{\nu\rho} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}),
\]

(3.8)

\[
R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h.
\]

(3.9)

Substitute now the linearized curvatures (3.7), (3.8) and (3.9) in the action (3.6). Then, ignoring surface terms one has:

\[
RF_1(\Box)F_1(\Box) = (\partial_\mu \partial_\nu h^{\mu\nu} - \Box h) \frac{1}{2} h^2 + h^{\mu\nu} \partial_\nu \partial_\alpha \partial_\beta h^{\alpha\beta} - h^{\mu\nu} F_1(\Box) \Box \partial_\mu \partial_\nu h.
\]

(3.10)

Since \( F_1(\Box) \)'s are functions of covariant D'Alambertian (on Minkowski space) we can always use the integration by parts and so, ignoring surface terms again, we have:

\[
RF_1(\Box)F_1(\Box) = \frac{1}{2} h^2 + h^{\mu\nu} \partial_\nu \partial_\alpha \partial_\beta h^{\alpha\beta} - 2 h \Box \partial_\mu \partial_\nu h^{\mu\nu}.
\]

(3.11)

For the two other terms we have:
3.1. MOST GENERAL QUADRATIC CURVATURE ACTION OF GRAVITY

\[ R_{\mu\nu} F_2(\Box) R^{\mu\nu} = \frac{1}{2} \left( \partial_\rho \partial_\sigma h^\rho_{\mu} + \partial_\rho \partial_\mu h^\rho_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} \right) F_3(\Box) \]

\[ \times \frac{1}{2} \left( \partial_\alpha \partial^\nu h^{\alpha\mu} + \partial_\alpha \partial^\mu h^{\alpha\nu} - \partial^\nu \partial^\mu h - \Box h^{\mu\nu} \right) \]

\[ = \frac{1}{4} F_2(\Box) \left[ \left( \partial_\mu \partial_\nu h^{\rho\sigma} \partial^\rho \partial^\sigma \right) + \left( \Box h_{\mu\nu} \Box h^{\mu\nu} \right) \right] \]

\[ + \left( -\partial_\rho \partial_\nu h^\rho_{\mu} \partial^\nu h^{\rho\sigma} - \partial_\rho \partial_\mu h^\rho_\nu \partial^\nu h^{\rho\sigma} \right) + \left( \partial_\rho \partial_\nu h^\rho_{\mu} \partial^\nu h^{\rho\sigma} - \partial_\rho \partial_\mu h^\rho_\nu \partial^\nu h^{\rho\sigma} \right) \]

\[ \left( \partial_\mu \partial_\nu h^{\rho\sigma} + \partial_\mu \partial_\mu h^{\rho\sigma} \right) \]

\[ \left( \partial_\rho \partial_\nu h^\rho_{\mu} \partial^\nu h^{\rho\sigma} - \partial_\rho \partial_\mu h^\rho_\nu \partial^\nu h^{\rho\sigma} \right) \]

(3.12)

\[ = F_2(\Box) \left[ \frac{1}{4} h \Box^2 h + \frac{1}{4} h_{\mu\nu} \Box^2 h^{\mu\nu} + \frac{1}{4} \left( -2 h \Box \partial_\mu \partial_\nu h^{\mu\nu} \right) \right] \]

\[ + \frac{1}{4} \left( -2 h^\rho_{\mu} \Box \partial_\rho \partial_\nu h^{\mu\nu} \right) + \frac{1}{4} \left( 2 h^{\lambda\sigma} \partial_\sigma \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} \right) \]

\[ = F_2(\Box) \left[ \frac{1}{4} h \Box^2 h + \frac{1}{4} h_{\mu\nu} \Box^2 h^{\mu\nu} - \frac{1}{2} h \Box \partial_\mu \partial_\nu h^{\mu\nu} \right] \]

\[ - \frac{1}{2} h^\rho_{\mu} \Box \partial_\rho \partial_\nu h^{\mu\nu} + \frac{1}{2} h^{\lambda\sigma} \partial_\sigma \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} \right] \]

Finally, following the same steps the last term becomes:

\[ R_{\mu\nu\lambda\sigma} F_3(\Box) R^{\mu\nu\lambda\sigma} = \frac{1}{2} \left( \partial_\nu \partial_\lambda h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\lambda} - \partial_\nu \partial_\sigma h_{\mu\lambda} - \partial_\mu \partial_\lambda h_{\nu\sigma} \right) F_3(\Box) \]

\[ \times \frac{1}{2} \left( \partial^\nu \partial^\lambda h^{\mu\sigma} + \partial^\mu \partial^\sigma h^{\nu\lambda} - \partial^\nu \partial^\sigma h^{\mu\lambda} - \partial^\mu \partial^\lambda h^{\nu\sigma} \right) \]

\[ = F_3(\Box) \left[ h_{\mu\nu} \Box^2 h^{\nu\mu} + h^{\lambda\sigma} \partial_\sigma \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} - 2 h^\rho_{\mu} \Box \partial_\rho \partial_\nu h^{\mu\nu} \right] \]

(3.13)

Furthermore we have to consider the quadratic \( h \)-terms due to the Hilbert-Einstein action that we introduced in (2.2.1):

\[ S_{HE} = \int d^4 x \left( h^\rho_{\mu} \partial^\sigma \partial^\nu h^\rho_{\mu} - h^{\lambda\sigma} \partial_\sigma \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + \frac{1}{2} h \Box \sqrt{-g} \right) \]

(3.14)

It remains to write the whole \( h \)-quadratic action with all the terms due to the Hilbert-Einstein action and the quadratic terms in the curvatures; we can quickly see that we
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have only 5 different combination terms:

\[
S_q = - \int d^4x \left\{ \frac{1}{2} h_{\mu\nu} \Box \left[ 1 - \frac{1}{2} \mathcal{F}_2(\Box) - 2 \mathcal{F}_3(\Box) \right] h^{\mu\nu} \\
+ h_\mu^\sigma \left[ -1 + \frac{1}{2} \mathcal{F}_2(\Box) + 2 \mathcal{F}_3(\Box) \right] \partial_\sigma \partial_\nu h^{\mu\nu} \\
+ \frac{1}{2} h \left[ 1 + 2 \mathcal{F}_1(\Box) + \frac{1}{2} \mathcal{F}_2(\Box) \right] \partial_\mu \partial_\nu h^{\mu\nu} \\
+ \frac{1}{2} h \Box \left[ -2 \mathcal{F}_1(\Box) - \frac{1}{2} \mathcal{F}_2(\Box) - 1 \right] h \\
+ \frac{1}{2} h^{\lambda\sigma} \left[ -2 \mathcal{F}_1(\Box) - \mathcal{F}_2(\Box) - 2 \mathcal{F}_3(\Box) \right] \partial_\nu \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} \right\}.
\]

(3.15)

Remark. It is important to observe that this last action \( S_q \) is not the starting action (3.1). Indeed (3.1) the letter "q" means that the action is quadratic in the curvatures, instead now it means that the action is quadratic in \( h \). In the last action we have written there is also the Hilbert-Einstein contribution, linear in the curvature.

We can write the action in a more compact form defining the following coefficients:

\[
a(\Box) := 1 - \frac{1}{2} \mathcal{F}_2(\Box) - 2 \mathcal{F}_3(\Box), \\
b(\Box) := -1 + \frac{1}{2} \mathcal{F}_2(\Box) + 2 \mathcal{F}_3(\Box), \\
c(\Box) := 1 + 2 \mathcal{F}_1(\Box) + \frac{1}{2} \mathcal{F}_2(\Box), \\
d(\Box) := -1 - 2 \mathcal{F}_1(\Box) - \frac{1}{2} \mathcal{F}_2(\Box), \\
f(\Box) := -2 \mathcal{F}_1(\Box) - \mathcal{F}_2(\Box) - 2 \mathcal{F}_3(\Box),
\]

(3.16)

so we obtain

\[
S_q = - \int d^4x \left\{ \frac{1}{2} h_{\mu\nu} \Box \left[ a(\Box) h^{\mu\nu} + h_\mu^\sigma b(\Box) \partial_\sigma \partial_\nu h^{\mu\nu} \\
+ h c(\Box) \partial_\mu \partial_\nu h^{\mu\nu} + \frac{1}{2} h \Box d(\Box) h + \frac{1}{2} h^{\lambda\sigma} f(\Box) \partial_\nu \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} \right] \right\}.
\]

(3.17)
3.1. MOST GENERAL QUADRATIC CURVATURE ACTION OF GRAVITY

or, equivalently the Linearized quadratic Lagrangian is

\[ L_q = -\frac{1}{2} h_{\mu\nu} \Box a(\Box) h^{\mu\nu} - h^\sigma \Box b(\Box) \partial_\sigma \partial_\nu h^{\mu\nu} - h c(\Box) \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{2} h \Box d(\Box) h - \frac{1}{2} h^{\lambda\sigma} f(\Box) \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} \]

(3.18)

The function \( f(\Box) \) appears only in higher order theories.

From the above expressions (3.16) of the coefficients we easily deduce the following interesting relations

\[
\begin{aligned}
    a(\Box) + b(\Box) &= 0 \\
    c(\Box) + d(\Box) &= 0 \\
    b(\Box) + c(\Box) + f(\Box) &= 0
\end{aligned}
\]

(3.19)

so that we are really left with only two independent arbitrary functions: a number much smaller than the beginning one, 14. Below we shall see that this can be better understood as a consequence of the Bianchi identities.

As we have already done with ED and GR cases, by raising and lowering the space-time indices with the metric tensor \( \eta_{\mu\nu} \), we can rewrite the Lagrangian (3.18) in the following form

\[ L_q = \frac{1}{2} h_{\mu\nu} O_q^{\mu\nu\rho\sigma} h_{\rho\sigma}, \]

(3.20)

where the operator \( O_q^{\mu\nu\rho\sigma} \) is defined as

\[
O_q^{\mu\nu\rho\sigma} := -\left( \frac{a(\Box)}{2} \eta^{\mu\rho} \eta^{\nu\sigma} + \frac{a(\Box)}{2} \eta^{\mu\sigma} \eta^{\nu\rho} + d(\Box) \eta^{\mu\nu} \eta^{\rho\sigma} \right) - b(\Box) \left( \eta^{\mu\nu} \partial^\rho \partial^\sigma + \eta^{\rho\sigma} \partial^\mu \partial^\nu \right) \\
- \frac{c(\Box)}{2} \left( \eta^{\mu\rho} \partial^\nu \partial^\sigma + \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\nu\rho} \partial^\mu \partial^\sigma + \eta^{\nu\sigma} \partial^\mu \partial^\rho \right) - f(\Box) \partial^\mu \partial^\nu \partial^\rho \partial^\sigma.
\]

(3.21)

3.1.2 Field equations

We want to derive the field equations associated to the action (3.17). In this case, since the Lagrangian contains higher orders, we also need to consider the functional derivatives with respect to the second derivatives of the field \( h_{\mu\nu} \) in the Euler-Lagrange equations. Let us determine the Euler-Lagrange equations for the linearized Lagrangian \( L_q \),

\[
\partial_\alpha \frac{\partial L_q}{\partial (\partial_\alpha h_{\mu\nu})} - \partial_\alpha \partial_\beta \frac{\partial L_q}{\partial (\partial_\alpha \partial_\beta h_{\mu\nu})} = \frac{\partial L_q}{\partial h_{\mu\nu}}.
\]

(3.22)
To apply (3.22) we have to rewrite the term with \( f(\Box) \) in (3.18) in the following more convenient form:

\[
\mathcal{L}_q = -\frac{1}{2} h_{\mu\nu} \Box a(\Box) h^{\mu\nu} - h_\mu^\sigma b(\Box) \partial_\sigma \partial_\nu h^{\mu\nu} - h c(\Box) \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{2} h \Box d(\Box) h - \frac{1}{2} \partial_\sigma \partial_\lambda h^{\lambda\sigma} \frac{f(\Box)}{\Box} \partial_\mu \partial_\nu h^{\mu\nu}.
\]

(3.23)

Now we can proceed with the computation of the derivatives. First of all we notice that in (3.23) there are not terms containing first derivatives of the field \( h_{\mu\nu} \):

\[
\partial_\sigma \frac{\partial \mathcal{L}_q}{\partial (\partial_\alpha \partial_\beta h^{\mu\nu})} = 0.
\]

(3.24)

As for the the second term on the left side of (3.22) we have:

\[
\frac{\partial \mathcal{L}_q}{\partial (\partial_\alpha \partial_\beta h^{\mu\nu})} = -\frac{1}{2} a(\Box) \eta^{\alpha\beta} h_{\mu\nu} - b(\Box) h_\mu^\alpha \delta^\beta_\nu - c(\Box) h_{\rho\sigma} \eta^{\rho\sigma} \delta^\alpha_\mu \delta^\beta_\nu
\]

\[
- \frac{1}{2} d(\Box) h_{\rho\sigma} \eta^{\rho\sigma} \eta_{\mu\nu} - f(\Box) \Box^{-1} \partial_\mu \partial_\nu h^{\alpha\beta}
\]

\[
\Rightarrow \partial_\alpha \partial_\beta \frac{\partial \mathcal{L}_q}{\partial (\partial_\alpha \partial_\beta h^{\mu\nu})} = -\frac{1}{2} a(\Box) \Box h_{\mu\nu} - b(\Box) \partial_\rho \partial_\sigma h_\mu^\alpha h_\nu^\alpha - c(\Box) \partial_\mu \partial_\nu h_{\rho\sigma} \eta^{\rho\sigma}
\]

\[
- \frac{1}{2} d(\Box) \eta_{\mu\nu} \eta^{\rho\sigma} \Box h_{\rho\sigma} - f(\Box) \Box^{-1} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma h^{\alpha\beta}.
\]

(3.25)

Instead the derivative with respect to the field \( h^{\mu\nu} \) gives us

\[
\frac{\partial \mathcal{L}_q}{\partial h^{\mu\nu}} = -\frac{1}{2} a(\Box) \Box h_{\mu\nu} - b(\Box) \partial_\rho \partial_\sigma h_\mu^\alpha
\]

\[
- c(\Box) \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} - \frac{1}{2} d(\Box) \eta_{\mu\nu} \eta^{\rho\sigma} \Box h_{\rho\sigma}.
\]

(3.26)

Hence from (3.24), (3.25) and (3.26) we deduce that the field equations for the quadratic Lagrangian (3.23) are

\[
a(\Box) \Box h_{\mu\nu} + b(\Box) \left( \partial_\rho \partial_\sigma h_\mu^\alpha + \partial_\alpha \partial_\nu h_\mu^\alpha \right) + c(\Box) \left( \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \partial_\mu \partial_\nu h \right)
\]

\[
+ \eta_{\mu\nu} d(\Box) \Box h + f(\Box) \Box^{-1} \partial_\sigma \partial_\rho \partial_\mu \partial_\nu h^{\alpha\beta} = 0
\]

(3.27)

If there is also the matter contribution in the action, described by energy-momentum tensor of matter \( \tau_{\mu\nu} \), we have to add the term \(-\kappa \tau_{\rho\sigma} h^{\rho\sigma}\) to the Lagrangian (3.23) and when we
compute the derivative $\frac{\partial L}{\partial h_{\mu\nu}}$ we have also to consider the contribution $-\kappa \tau_{\mu\nu}$. Thus the field equations in presence of matter will read as
\[
a(\Box) \Box h_{\mu\nu} + b(\Box) (\partial_\mu \partial_\alpha h_\nu^\alpha + \partial_\alpha \partial_\nu h_\mu^\alpha) + c(\Box) (\eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \partial_\mu \partial_\nu h)
+ \eta_{\mu\nu} d(\Box) \Box h + f(\Box) \Box^{-1} \partial_\beta \partial_\alpha \partial_\mu \partial_\nu h^{\alpha\beta} = -\kappa \tau_{\mu\nu}.
\] (3.28)

One can demonstrate that the energy-momentum tensor is conserved because of the generalized Bianchi identity \[45\] due to diffeomorphism invariance: $\nabla_\mu \tau^\mu_\nu = 0$. Thus, by acting with the covariant derivative on the field equation
\[
a(\Box) \Box \partial_\mu h^\nu_\nu + b(\Box) (\Box \partial_\mu h^\nu_\nu + \partial_\alpha \partial_\nu \partial_\mu h^{\alpha\mu}) + c(\Box) (\partial_\nu \partial_\alpha \partial_\beta h^{\alpha\beta} + \Box \partial_\nu h)
+ d(\Box) \Box \partial_\nu h + f(\Box) \Box^{-1} \partial_\beta \partial_\alpha \partial_\mu \partial_\nu h^{\alpha\beta} = 0
\]
\[
\Leftrightarrow (c + d) \Box \partial_\nu h + (a + b) \Box \partial_\mu h^\nu_\nu + (b + c + f) \partial_\nu \partial_\alpha \partial_\mu h^{\alpha\mu} = 0
\] (3.29)

Now it is clearer why the relations (3.19) must be valid. In fact the last equation (3.29) holds if, and only if, the quantities in brackets are zero, i.e. if the relations (3.19) are satisfied.

### 3.2 Propagator for quadratic Lagrangian

Now we want to derive the form of the physical (gauge independent) part of the propagator for the quadratic Lagrangian (3.18). Remember that we have introduced two different methods to calculate the propagator in the previous Chapter. In this section we shall refer to the second one.

The first step is to rewrite the field equations (3.28) in terms of the spin projector operators (2.22). The field equations (3.28) in momentum space read as
\[
a(-k^2) h_{\mu\nu} + b(-k^2) (k_\mu k_\alpha h^\alpha_\nu + k_\alpha k_\nu h^\alpha_\mu) + \frac{c(-k^2)}{k^2} (\eta_{\mu\nu} k_\alpha k_\beta h^{\alpha\beta} + k_\mu k_\nu h)
+ \eta_{\mu\nu} \frac{d(-k^2)}{k^2} h + \frac{f(-k^2)}{k^4} k_\beta k_\alpha k_\mu k_\nu h^{\alpha\beta} = \kappa \frac{\tau_{\mu\nu}}{k^2}.
\] (3.30)

Now we can write every term of the field equations (3.30) in the following way (use the relations (B.54)):
\[
a(-k^2) h_{\mu\nu} = a(-k^2) (\mathcal{P}^2 + \mathcal{P}^1 + \mathcal{P}^0 + \mathcal{P}_s^0 + \mathcal{P}_w^0) h;
\] (3.31)
3.2. PROPAGATOR FOR QUADRATIC LAGRANGIAN

\[ b(-k^2) (k_\nu k_\alpha h_\mu^\alpha + k_\alpha k_\mu h_\nu^\alpha) = b(-k^2) k^2 (\omega_{\sigma\nu} h_\mu^\sigma + \omega_{\sigma\mu} h_\nu^\sigma) \]

\[ = b(-k^2) k^2 (\eta_{\mu\rho} \omega_{\nu\sigma} + \eta_{\nu\rho} \omega_{\mu\sigma}) h^\rho_\sigma \]

\[ = b(-k^2) k^2 \frac{1}{2} (\eta_{\nu\rho} \omega_{\mu\sigma} + \eta_{\mu\sigma} \omega_{\nu\rho} + \eta_{\nu\sigma} \omega_{\mu\rho} + \eta_{\mu\rho} \omega_{\nu\sigma}) h^\rho_\sigma \]

\[ = b(-k^2) k^2 (P^1 + 2P^0_w) h; \]  \hspace{1cm} (3.32)

\[ c(-k^2) (\eta_{\mu\nu} k_\rho k_\sigma h^\rho_\sigma + k_\mu k_\nu h) = c(-k^2) k^2 (\eta_{\mu\nu} \omega_{\rho\sigma} h^\rho_\sigma + \omega_{\mu\nu} \eta_{\rho\sigma} h^\rho_\sigma) \]

\[ = c(-k^2) k^2 (\theta_{\mu\nu} \omega_{\rho\sigma} + \omega_{\mu\nu} \theta_{\rho\sigma} + \omega_{\mu\nu} \omega_{\rho\sigma}) h^\rho_\sigma \]

\[ = c(-k^2) k^2 (2P^0_w + \sqrt{3}(P^0_{sw} + P^0_{ws})) h; \]  \hspace{1cm} (3.33)

\[ d(-k^2) \eta_{\mu\nu} \eta^{\rho\sigma} h^\rho_\sigma = d(-k^2) (\theta_{\mu\nu} + \omega_{\mu\nu}) (\theta_{\rho\sigma} + \omega_{\rho\sigma}) h^\rho_\sigma \]

\[ = d(-k^2) (\theta_{\mu\nu} \theta_{\rho\sigma} + \theta_{\mu\nu} \omega_{\rho\sigma} + \omega_{\mu\nu} \theta_{\rho\sigma} + \omega_{\mu\nu} \omega_{\rho\sigma}) h^\rho_\sigma \]  \hspace{1cm} (3.34)

\[ = d(-k^2) (3P^0_s + P^0_w + \sqrt{3}(P^0_{sw} + P^0_{ws})) h; \]

\[ f(-k^2) k^\rho k_\sigma k_\mu h^\rho_\sigma = f(-k^2) k^4 \omega_{\mu\nu} \omega^\rho_\sigma h^\rho_\sigma = f(-k^2) P^0_w h. \]  \hspace{1cm} (3.35)

For the above calculations we have used a lot the relations (2.32) (see in Appendix B.2) already used for GR. Furthermore, we are still suppressing the space-time indices for convenience.

Hence, from the relations (3.31)-(3.35) the field equations in terms of the spin projector operators read as:

\[ \left[ a(-k^2) (P^2 + P^1 + P^0_s + P^0_w) + b(-k^2) (P^1 + 2P^0_w) + c(-k^2) \left( 2P^0_w + \sqrt{3} (P^0_{sw} + P^0_{ws}) \right) \right. \]

\[ + d(-k^2) \left( 3P^0_s + P^0_w + \sqrt{3} (P^0_{sw} + P^0_{ws}) \right) + f(-k^2) P^0_w \]  \hspace{1cm} h = \frac{k}{\kappa} \left( \frac{P^2 + P^1 + P^0_s + P^0_w}{k^2} \right) \hspace{1cm} (3.36)

Now we are ready to invert the field equations and then obtain the corresponding propagator: we shall proceed following the prescription introduced in the section 2.3—Method 2.

By acting with the projector \( P^2 \) on (3.36) and using the orthogonality relations (2.23)
we find
\[ \mathcal{P}^2 h = \kappa \left( \frac{\mathcal{P}^2}{a(-k^2)k^2} \right) \tau; \quad (3.37) \]
by acting with \( \mathcal{P}^1 \), one finds
\[ \left( a(-k^2) + b(-k^2) \right) \mathcal{P}^1 h = \kappa \frac{\mathcal{P}^1}{k^2} \tau, \quad (3.38) \]
but since \( a + b = 0 \), then there are no vector degrees of freedom, and accordingly the stress-energy tensor must have no vector part:
\[ 0 \mathcal{P}^1 h = \kappa \frac{\mathcal{P}^1}{k^2} \tau \Rightarrow \mathcal{P}^1 \tau = 0. \quad (3.39) \]
Next let us look at the scalar multiplets. By acting with \( \mathcal{P}_s^0 \) we find
\[ \left( a + 3d \right) \mathcal{P}_s^0 + \sqrt{3} \left( c + d \right) \mathcal{P}_{sw}^0 \right) h = \kappa \frac{\mathcal{P}_s^0}{k^2} \tau, \]
but since \( c + d = 0 \), then we obtain
\[ \mathcal{P}_s^0 h = \kappa \left( \frac{\mathcal{P}_s^0}{(a - 3c)k^2} \right) \tau; \quad (3.40) \]
instead by acting with \( \mathcal{P}_w^0 \)
\[ \left( a + 2b + 2c + d + f \right) \mathcal{P}_w^0 = \kappa \frac{\mathcal{P}_w^0}{k^2} \tau, \quad (3.41) \]
but
\[ a + 2b + 2c + d + f = (a + b) + b + c + (c + d) + f \]
\[ = b + c + f = 0 \]
\[ \Rightarrow 0 \mathcal{P}_w^0 h = \kappa \frac{\mathcal{P}_w^0}{k^2} \tau \Rightarrow \mathcal{P}_w^0 \tau = 0, \quad (3.42) \]
so there is no \( w \)–multiplet that contributes to the degrees of freedom in the propagator.
We can also note that, in principle, the scalar multiplets are coupled, but by acting with the spin projector operators the scalars decouple.
As it happens in GR the spin components \( \mathcal{P}^1 h \) and \( \mathcal{P}_w^0 h \) of the tensor field \( h_{\mu\nu} \) are undetermined and two restrictions of the source \( \tau_{\mu\nu} \) hold, i.e. (3.39) and (3.42). Remember that these two source constraints are associated to some gauge freedom which in turn is associated to an invariance of the Lagrangian. Indeed also the quadratic Lagrangian (3.23) is invariant under spin-1 transformation: \( \delta h \sim \mathcal{P}^1 h \).
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Because of (3.37) and (3.40) we expect that the physical part of the propagator contains only the spin-2 and spin-0 components, \( P^2 \) and \( P^0 \). We can easily show this last statement by proceeding as we did for the saturated GR propagator following the steps from (2.108) to (2.111). Doing this we obtain the saturated propagator for the quadratic Lagrangian (3.18):

\[
\tau(-k)\Pi(k)\tau(k) \equiv \tau(-k) \left( \frac{P^2}{ak^2} + \frac{P^0_s}{(a-3c)k^2} \right) \tau(k),
\]

and the physical (gauge independent) part of the propagator reads as

\[
\Pi(k) = \frac{P^2}{ak^2} + \frac{P^0_s}{(a-3c)k^2}.
\]

3.3 Ghosts and unitarity analysis in quadratic gravity

Once we have determined the propagator associated to the Lagrangian of the theory we can study whether ghosts and tachyons are absent and whether the unitarity condition is preserved. Hence we have to study the positivity of the imaginary part of the current-current amplitude residue. The amplitude is given by

\[
\mathcal{A} = i\tau^{\mu'\nu'}(k) \frac{1}{k^2} \left( \frac{P^2}{ak^2} + \frac{P^0_s}{(a-3c)k^2} \right)_{\mu'\nu'} \tau^{\rho\sigma}(k).
\]

We know that the coefficients \( a \equiv a(-k^2) \) and \( c \equiv c(-k^2) \) depend on the square momentumin \( k^2 \), thus their special form could bring other poles in the propagator and so new states in addition to the graviton one. For example, if the coefficients \( a \) and \( b \) are polynomial in \( k^2 \) we will have new poles for sure. Hence without specifying the particular dependence of the coefficients on \( k^2 \) we are not able to state anything neither about the presence of ghosts and tachyon, nor about unitarity condition.

In the next section and in the next Chapter we shall make special choices for the coefficients \( a \) and \( c \).

3.4 Applications to special cases

3.4.1 General Relativity

In this chapter we have been studying a case of modified, or extended, theories of gravity because we want to try to solve problems that GR is not able to explain. A good new
3.4. APPLICATIONS TO SPECIAL CASES

theory must give us GR when we consider some limits. In fact, since we want to recover the right infrared behavior of GR, we require from any viable theory that for $k^2 \to 0$

$$a(0) = c(0) = -b(0) = -d(0) = 1,$$  \hspace{1cm} (3.46)

corresponding to the GR values (in GR these functions are the same constants for any Fourier mode). The condition (3.46) ensures that as $k^2 \to 0$, we have only the physical graviton propagator

$$\lim_{k^2 \to 0} \Pi = \frac{P^2}{k^2} - \frac{P_s^0}{2k^2} \equiv \Pi_{GR}.$$  \hspace{1cm} (3.47)

We have already seen that the GR is a free-ghost theory, and so the choice (3.46) gives us a theory that preserves the unitarity, without ghosts and tachyons. Thus, we conclude that $k^2 = 0$ pole just describes the physical graviton state, i.e. there are no new states, but just the state predicted by GR. Secondly, we can also note that we are left with only a single arbitrary function, $a = 1$.

3.4.2 $f(\mathcal{R})$ theory

The $f(\mathcal{R})$ gravity is actually a family of theories, each one defined by a different function of the Ricci scalar. The general action of the this theory is

$$S_f = \int d^4x \sqrt{-g} f(\mathcal{R}).$$  \hspace{1cm} (3.48)

For our aims here, one can just consider the expansion of the Lagrangian around flat space ($\mathcal{R} = 0$):

$$f(\mathcal{R}) = f(0) + f'(0)\mathcal{R} + \frac{1}{2} f''(0)\mathcal{R}^2 + \cdots.$$  \hspace{1cm} (3.49)

The zero order term identifies with the cosmological constant, $f(0) = -\kappa \Lambda$, and the first order term should reduce to the Einstein-Hilbert term in a healthy theory, $f'(0) = -1$. The relevant modification of the theory is contained in the quadratic part, i.e. the second order term in the expansion. Since now only the function $F_1(\Box)$ is nonzero in (3.2), the
relations (3.16) become

\[ a(\Box) = 1, \]
\[ b(\Box) = -1, \]
\[ c(\Box) = 1 + 2F_1(\Box) = 1 + f''(0), \]  
\[ d(\Box) = -1 - 2F_1(\Box) = -1 - f''(0), \]
\[ f(\Box) = -2F_1(\Box) = + f''(0). \]  

The physical part of the propagator (3.44) specialized to \( f(R) \) is given by

\[ \Pi_f(k) = \frac{\mathcal{P}^2}{k^2} + \frac{\mathcal{P}_s^0}{k^2(1 - 3 + 3f''(0)k^2)} \]
\[ = \Pi_{GR} + \frac{1}{2} \frac{\mathcal{P}_s^0}{k^2 - m^2}, \]  

where \( m^2 := \frac{2}{3} f''(0) \). The scalar part of the propagator is modified as we expected. In fact, since these theories are a particular class of scalar-tensor theories, an extra scalar degree of freedom must be taken into account. Hence, the \( f(R) \) modification of GR introduces an additional spin-0 particle which is not a ghost and moreover is non-tachyonic as long as \( f''(0) > 0 \).

### 3.4.3 Conformally invariant gravity

We are now going to consider the Weyl squared gravity as an example of theory with presence of ghost. The Weyl tensor is defined as

\[ C_{\mu\nu\rho\sigma} := \mathcal{R}_{\mu\nu\rho\sigma} + \frac{\mathcal{R}}{6} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) - \frac{1}{2} (g_{\mu\rho}\mathcal{R}_{\nu\sigma} - g_{\mu\sigma}\mathcal{R}_{\nu\rho} - g_{\nu\rho}\mathcal{R}_{\mu\sigma} + g_{\nu\sigma}\mathcal{R}_{\mu\rho}). \]

The theory is then specified by the conformally invariant Weyl-squared term,

\[ \mathcal{L}_C = - \left( \mathcal{R} + \frac{1}{m^2} C^2 \right), \]  

where

\[ C^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = \frac{1}{3} \mathcal{R}^2 - 2 \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}; \]

\(^{2}\)Here we must be careful because we are using the letter \( f \) both for the coefficient and for the Lagrangian function term in \( f(R) \) theory. So don’t get confused!
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hence the action is

\[
S_C = - \int d^4x \sqrt{-g} \left[ R + \frac{1}{m^2} \left( \frac{1}{3} R^2 - 2 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \right]. \tag{3.54}
\]

From (3.54) we can easily see that the relations (3.16) become

\[
a(\Box) = 1 + \frac{1}{m^2} \Box,
\]
\[
b(\Box) = -1 - \frac{1}{m^2} \Box,
\]
\[
c(\Box) = 1 + \frac{1}{3m^2} \Box,
\]
\[
d(\Box) = -1 - \frac{1}{3m^2} \Box,
\]
\[
f(\Box) = + \frac{2}{3m^2} \Box. \tag{3.55}
\]

Going into the momentum space and using the relations (3.55), from (3.44) we obtain a propagator with a double pole in the spin-2 component:

\[
\Pi_C = \frac{p^2}{(1 - \frac{1}{m^2}k^2) k^2} + \frac{p_s^0}{(1 - \frac{1}{m^2}k^2 - 3 + \frac{1}{m^2}k^2) k^2}
\]
\[
= \frac{p^2}{(1 - \frac{1}{m^2}k^2) k^2} - \frac{p_s^0}{2k^2} = \Pi_{GR} - \frac{p^2}{k^2 - m^2}. \tag{3.56}
\]

From the latter form of the propagator one can notice the presence of an extra spin-2 degree of freedom with respect to GR. Moreover, the new contribution comes with the wrong sign: this is the Weyl ghost. In particular we can show that this is a “bad” ghost that violates the unitarity. As we have already done for GR, we can follow the same prescription calculating the imaginary part of the current-current amplitude residue. Indeed:

\[
A = i \tau^{\mu\nu}(k) \Pi_{C,\mu\nu\rho\sigma}(k) \tau^{\rho\sigma}(k)
\]
\[
= i \tau^{\mu\nu}(k) \Pi_{GR,\mu\nu\rho\sigma}(k) \tau^{\rho\sigma}(k) - i \tau^{\mu\nu}(k) \frac{p^2}{k^2 - m^2} \tau^{\rho\sigma}(k)
\]
\[
= A_{GR} + A_2, \tag{3.57}
\]
where
\[ \mathcal{A}_{GR} = i \tau^{\mu \nu}(k) \Pi_{GR, \mu \nu \rho \sigma}(k) \tau_{\rho \sigma}(k), \]
\[ \mathcal{A}_2 = -i \tau^{\mu \nu}(k) \frac{\mathcal{P}^2}{k^2 - m^2} \tau_{\rho \sigma}(k). \] (3.58)

One can show that \( \text{Im} \text{Res}_{k^2 = 0} \{ \mathcal{A}_{GR} \} > 0 \) (see for the GR part eq. (2.144)), and that \( \text{Im} \text{Res}_{k^2 = m^2} \{ \mathcal{A}_2 \} < 0 \). So the last inequality says that the presence of the spin-2 massive ghost violates the unitarity condition (see also Appendix C.4.).

In the next chapter we shall consider a special choice for the coefficients \( a, b, c, d \) and \( f \) that will give us a particular theory in which the form of the graviton propagator is modified without adding new physical states other than the spin-2 massless, traceless and transverse graviton of GR.
Chapter 4

Infinite derivative theories of gravity

4.1 Consistency conditions on $a(\Box)$, $c(\Box)$

In the previous chapter we have taken the most general covariant quadratic free-torsion action for gravity (3.2); we have considered the linearized form (3.17) and calculated the physical propagator (3.44). At the end we examined different choices of the coefficients $a, b, c, d,$ and $f$ that gave us different (sub-)theories: GR, $f(\mathcal{R})$ gravity and conformally invariant gravity. The situation is the following: $f(\mathcal{R})$ theories can be ghost-free but they are not able to improve the UV behavior, while modifications involving $\mathcal{R}_{\mu \nu \rho \sigma}$ can improve the UV behavior but, on the other hands, they suffer from the presence of the Weyl ghost. Before seeing how to overcome this problem, it is worth listing the conditions needed to have a stable and ghost free theory around the Minkowski background [15]:

- As we would expect from a healthy modified theory of gravity, we want to recover GR at large distances (infrared limit); this request implies that $a(\Box), c(\Box)$ must be analytic around $\Box = 0 (k^2 = 0)$;

- To avoid the presence of the Weyl ghost must we need to require that $a(\Box)$ cannot have any zeroes, so that the only part of the spin-2 component is the GR graviton one;

- We also demand that the scalar mode does not have any ghosts other than the the scalar component of the GR propagator (i.e. the pole $k^2 = 0$). So, $a - 3c$ in the propagator (3.44) can at most have one zero. To satisfy these conditions we should be able to express $c(\Box)$ in the following general form

$$c(\Box) = \frac{a(\Box)}{3} \left[ 1 + 2 \left( 1 - \frac{\Box}{m^2} \right) \tilde{c}(\Box) \right], \quad (4.1)$$

where $\tilde{c}(\Box)$ must be analytic around $\Box = 0$ and cannot have any zeros;
Moreover, the fact that we do not want any tachyonic modes implies $m^2 > 0$ in (4.1).

From the above conditions, by making different choices for the coefficients $a(\Box)$ and $c(\Box)$, different types of theories emerge. The difference relies on the number of degrees of freedom included in the physical part of the propagator, and whether they are either massive or massless. We will make the choice $m^2 \to 0$ and $\tilde{c}(\Box) = 0$, so that the resulting theory only contains the GR graviton as a propagating degree of freedom. Furthermore, we have only one independent function, $a(\Box) = c(\Box)$, that controls the modification in the UV regime. We shall see that this type of theories is a good candidate for a ghost free and renormalizable theory of gravity.

### 4.2 Ghost and singularity free theories of gravity

In the previous section we have seen that by making the choice of having only the GR graviton pole in the modified propagator, the expression (4.1) reduces to $a(\Box) = c(\Box)$. Moreover, because of (3.19) the following relations hold

$$a(\Box) = -b(\Box) = c(\Box) = -d(\Box) \Rightarrow f(\Box) = 0. \quad (4.2)$$

This means that we are essentially left with just a single free function

$$a(\Box) := 1 - \frac{1}{2} F_2(\Box) \Box - 2 F_3(\Box) \Box, \quad (4.3)$$

and from the relations (3.16) we are able to write $F_3(\Box)$ as

$$F_3(\Box) = - \left( F_1(\Box) + \frac{F_2}{2}(\Box) \right). \quad (4.4)$$

Now, since the function $F_3$ satisfies (4.4), the action (3.6) becomes

$$S = \int d^4x \left[ -R + R F_1(\Box) R + R_{\mu \nu} F_2(\Box) R^{\mu \nu} - R_{\mu \nu \lambda \sigma} \left( F_1(\Box) + \frac{F_2}{2}(\Box) \right) R^{\mu \nu \lambda \sigma} \right]. \quad (4.5)$$

By working around Minkowski space-time, since the covariant derivatives become simple partial derivatives, we can move the function $F_i(\Box)$ on the left of the curvatures by integrating by parts. In this way, we are able to get rid of the product of two Riemann tensor by implementing the Euler topological invariant relation:

$$R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma} - 4 R^{\mu \nu} R_{\mu \nu} + R^2 = \nabla_i K_i, \quad (4.6)$$
4.2. GHOST AND SINGULARITY FREE THEORIES OF GRAVITY

where $\nabla_\mu K^\mu$ is a four-divergence (surface term) that doesn’t contribute to the action variation. Thus, the action (4.5) becomes

$$S = \int d^4x \left[ -\mathcal{R} + \mathcal{R}_1(\Box)\mathcal{R} + \mathcal{R}_{\mu\nu}\mathcal{F}_2(\Box)\mathcal{R}^{\mu\nu} \right]; \quad (4.7)$$

while in terms of the coefficient $a(\Box)$, by assuming $\mathcal{F}_3(\Box) = 0$, we have

$$S = \int d^4x \left[ -\mathcal{R} - \frac{1}{2} a(\Box)\Box \mathcal{R} + \mathcal{R}_{\mu\nu} a(\Box)\Box \mathcal{R}^{\mu\nu} \right]. \quad (4.8)$$

Then, since $a(\Box)$ is the only function remaining, by using the linearized form of the curvatures (see (2.5)) the linearized action and the linearized field equations can be obtained:

$$S_q = -\int d^4x \left[ \frac{1}{2} h_{\mu\nu} a(\Box) \Box h^{\mu\nu} - h_\mu^\sigma a(\Box) \partial_\sigma \partial_\nu h^{\mu\nu} + h a(\Box) \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{2} h a(\Box) \Box h \right] \quad (4.9)$$

and

$$a(\Box) \left[ \Box h_{\mu\nu} - (\partial_\mu \partial_\alpha h_\alpha^\nu + \partial_\alpha \partial_\nu h_\mu^\alpha) + (\eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \partial_\mu \partial_\nu h) - \eta_{\mu\nu} \Box h \right] = -\kappa \tau_{\mu\nu}. \quad (4.10)$$

Our theory has to contain only the graviton pole, so the spin-2 and spin-0 component cannot have any other pole (zeros). The physical propagator (3.44) because of the choice (4.2) becomes

$$\Pi(k) = \frac{1}{a(-k^2)} \left( \frac{p^2}{k^2} - \frac{p_0^2}{2k^2} \right) = \frac{1}{a(-k^2)} \Pi_{GR}(k), \quad (4.11)$$

i.e. we obtain the the GR propagator modified by the factor $\frac{1}{a(-k^2)}$. The next step is choosing a special form for the coefficient $a(\Box)$ that suits our aims. We require that there are no gauge-invariant poles other than the transverse and traceless massless physical graviton pole, thus the coefficient $a(\Box)$ cannot vanish in the complex plane. Special functions that satisfies these characteristics are the exponentials of entire functions: indeed they do not have poles in the complex plane and vanish only at infinity.

Hence, for the coefficient $a(\Box)$ can be done the following choice:

$$a(\Box) = e^{-\frac{\Box}{M^2}}, \quad (4.12)$$

where $M$ is a parameter that makes dimensionless the exponent of the exponential and, physically, corresponds to the scale at which the modification to GR made by this theory should appear. We can note that expanding in Taylor series the exponential in (4.12) one has

$$a(\Box) = e^{-\frac{\Box}{M^2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\Box}{M^2} \right)^n. \quad (4.13)$$
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The meaning of (4.13) is that we are dealing with a theory containing an infinite set of derivatives expressed in the form of an exponential function\footnote{The most general quadratic action for IDG theories is given by}

\begin{equation}
S = \int \sqrt{-g} \left( -\mathcal{R} + \mathcal{F}_1(\square) \mathcal{R} + \mathcal{R}^{\mu\nu} \mathcal{F}_2(\square) \mathcal{R}_{\mu\nu} + \mathcal{R}^{\mu\nu\rho\sigma} \mathcal{F}_3(\square) \mathcal{R}_{\mu\nu\rho\sigma} \right),
\end{equation}

where the $\mathcal{F}_i(\square)$'s are functions of the D’Alambertian operator, $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$, and contain an infinite set derivatives:

\begin{equation}
\mathcal{F}_i(\square) = \sum_{n=0}^{\infty} f_{i,n} \square^n, \quad i = 1, 2, 3.
\end{equation}

One requires that the $\mathcal{F}_i(\square)$'s are analytic at $\square = 0$ so that one can recover GR in the infrared regime.

\begin{equation}
\frac{1}{M^2},
\end{equation}

For this reason these particular theories of gravity are called Infinite Derivative Theories of Gravity (IDG). In momentum space one has

\begin{equation}
a(-k^2) = e^{\frac{k^2}{M^2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{k^2}{M^2} \right)^n,
\end{equation}

so the physical propagator read as

\begin{equation}
\Pi(k) = e^{\frac{k^2}{M^2}} \left( \frac{\mathcal{P}^2}{k^2} - \frac{\mathcal{P}_0^2}{2k^2} \right) = e^{\frac{-k^2}{M^2}} \Pi_{GR}(k),
\end{equation}

and we notice that the only difference from GR case is that the graviton propagator turns out to be modified by a multiplicative transcendental function.

\subsection{Non-singular Newtonian potential}

In this subsection we shall see how the potential modifies in this special theory of gravity when one considers the Newtonian approximation (or weak field approximation). We will see that the resulting potential is non-singular for values of the radius equal to zero, i.e. is not divergent like Newtonian potential $\frac{1}{r}$ but it is finite.

Hence, we are going to focus particularly on the classical short-distance behavior. As is usual, we want to solve the linearized modified field equations (4.10) for a point source:

\begin{equation}
\tau_{\mu\nu} = \rho \delta_\mu^\alpha \delta_\nu^\beta = m_g \delta^3(x) \delta_\mu^0 \delta_\nu^0,
\end{equation}

where $m_g$ is the mass of the object that is generating the gravitational potential. In Newtonian approximation the metric, besides to have small perturbation, is stationary and the sources are static (or, anyway, with neglecting velocities). Hence the metric reduces to

\begin{equation}
ds^2 = (1 + 2\Phi) dt^2 - (1 - 2\Psi) |d\vec{x}|^2,
\end{equation}
4.2. GHOST AND SINGULARITY FREE THEORIES OF GRAVITY

with $|\Phi|, |\Psi| << 1$, since in Newtonian approximation the potentials are weak.

Now our aim is to determine a form for the potentials $\Phi$ and $\Psi$, i.e. to find the metric.

Let us consider the trace and the 00 component of the equations (4.10) keeping in mind that in Newtonian approximation $\partial_0 h_{\mu\nu} = 0$:

Trace:

$$a(\Box) \left[ \Box h - (2 \partial_\alpha \partial_\beta h^{\alpha\beta}) + (4 \partial_\alpha \partial_\beta h^{\alpha\beta} + \Box h) - 4 \Box h \right] = -\kappa \tau$$

$$\Leftrightarrow 2a(\Box) \left[ - \Box h + \partial_\alpha \partial_\beta h^{\alpha\beta} \right] = -\kappa \rho; \quad (4.18)$$

00 component:

$$a(\Box) \left[ \Box h_{00} + \partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h \right] = -\kappa \rho. \quad (4.19)$$

Note that the metric (4.17) can be rewritten making explicit the perturbation $h_{\mu\nu}$:

$$ds^2 = dt^2 - |d\vec{x}|^2 + (2\Phi dt^2 + 2\Psi |d\vec{x}|^2)$$

$$= \eta_{\mu\nu} + h_{\mu\nu}, \quad (4.20)$$

with

$$h_{\mu\nu} \equiv \begin{pmatrix} 2\Phi & 0 & 0 & 0 \\ 0 & 2\Psi & 0 & 0 \\ 0 & 0 & 2\Psi & 0 \\ 0 & 0 & 0 & 2\Psi \end{pmatrix}. \quad (4.21)$$

From (4.20) and the assumption of static potential we have:

$$h = \eta^{\mu\nu} h_{\mu\nu} = 2 (\Phi - 3\Psi), \quad (4.22)$$

$$h_{00} = 2\Phi, \quad (4.23)$$

$$\partial^\mu \partial^\nu h_{\mu\nu} = \partial^i \partial^j h_{ij} = 2\nabla^2 \Psi, \quad (4.24)$$

$$\Box \rightarrow -\nabla^2 = -\delta^{ij} \partial_i \partial_j. \quad (4.25)$$

Applying the (4.22)-(4.25), the trace and 00 component equations become respectively:

$$2a(\nabla^2) \left[ 2\nabla^2 (\Phi - 3\Psi) + 2\nabla^2 \Psi \right] = -\kappa \rho \quad (4.26)$$

and

$$a(\nabla^2) \left[ 4\nabla^2 \Psi \right] = \kappa \rho. \quad (4.27)$$

By comparing these last equations, at the end, we notice that the two potentials $\Phi$ and $\Psi$ satisfy the same equation:

$$4a(\nabla^2) \nabla^2 \Phi = 4a(\nabla^2) \nabla^2 \Psi = \kappa \rho = \kappa m_g \delta^3(x). \quad (4.28)$$
We can find the solutions of the last equations by going into the momentum space and then going back to the coordinate space. Since the two potentials satisfy the same equations, let us consider the potential $\Phi$:

$$4a(\nabla^2)\nabla^2\Phi = \kappa m_g \delta^3(\vec{x}) \rightarrow -4a(\vec{k}^2)\vec{k}^2\Phi(k) = \kappa m_g$$  \hspace{1cm} (4.29)

$$\Rightarrow \Phi(r) = -\frac{\kappa m_g}{4} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2a(-\vec{k}^2)}$$

$$= -\frac{\kappa m_g}{4(2\pi)^3} \int_0^\infty d||\vec{k}|| \int_{-1}^{+1} d(\cos \theta) \int_0^{2\pi} d\varphi \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2a(-\vec{k}^2)}$$  \hspace{1cm} (4.30)

$$= -\frac{\kappa m_g}{8\pi^2} \frac{1}{r} \int_0^\infty d||\vec{k}|| \frac{e^{-\frac{k^2}{M^2}} \sin |\vec{k}|r}{|k|}.$$  \hspace{1cm} (4.31)

From the last equation we can already notice that the potential is modified, in fact apart from the Newtonian contribution $\frac{1}{r}$ we also have a new factor given by the integral. This integral corresponds to one of the special functions, i.e. the Error Function:

$$\int_0^\infty d||\vec{k}|| \frac{e^{-\frac{k^2}{M^2}} \sin |\vec{k}|r}{|k|} = \frac{\pi}{2} \text{Erf} \left( \frac{rM}{2} \right).$$

Hence the solutions of the equations (4.28) are given by ([19], [23], [25])

$$\Phi(r) = \Psi(r) = -\frac{1}{16\pi} \frac{m_g}{M_p^2} \frac{1}{r} \text{Erf} \left( \frac{rM}{2} \right),$$  \hspace{1cm} (4.32)

where we have used the fact that $\kappa = \frac{1}{M_p^2}$, with $M_p \simeq 1.2 \times 10^{18}$ GeV reduced Planck mass.

We observe that as $r \rightarrow \infty$, Erf $\left( \frac{rM}{2} \right) \rightarrow 1$, and we recover the GR limit (infrared limit), i.e. the usual Newtonian potential $\Phi(r) = \Psi(r) \sim -\frac{1}{r}$. On the other hand, as $r \rightarrow 0$, Erf $\left( \frac{rM}{2} \right) \rightarrow \frac{rM}{2}$, namely

$$r \rightarrow 0 \Rightarrow \Phi(r) = \Psi(r) \sim -\frac{m_g M}{M_p^2}.$$  \hspace{1cm} (4.33)

We have found that there are no divergences. The potential, in fact, converges to a finite value as shown in (4.33). Thus, although the matter source has a delta function singularity, the potentials remain finite. Further, since we are working in the approximation of weak potentials, the whole discussion holds as long as

$$\frac{m_g M}{M_p^2} \ll 1 \Leftrightarrow m_g M \ll M_p^2.$$  \hspace{1cm} (4.34)
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It is clear that for small masses our theory provides a very different description of space-time as compared to GR. In fact, according to our model there are no black-hole like solutions (no horizon and no singularity) as long as the mass source satisfies the condition (4.34). Unfortunately, our analysis cannot say anything about large mass astrophysical black holes because the Newtonian potentials become too large for us to be able to trust the perturbative calculations.

Below we have made more explicit how Newton potential is modified in this model by plotting the both singular and non-singular functions:

![Graph showing modified Newton potentials](image)

Figure 4.1: In this plot both singular and non-singular functions are drawn: the blue line represents the non-singular potential, the dashed line the singular Newton potential. The plot was obtained by using Wolfram Mathematica 9.0.

We can also study what is the gravitational force $\vec{F}$ that a test mass $m$ undergoes in the gravitational potential (4.32) generated by the point source $m_g$. Once we have the potential (4.32) we can obtain the force by calculating its derivative with respect to $r$:

$$F = m\ddot{r} = -m \frac{\partial \phi(r)}{\partial r}$$

$$= \frac{mm_g}{M_p^2} \left[ \frac{M}{\sqrt{\pi}} \frac{e^{-\left(\frac{Mr}{2}\right)^2}}{r} - \frac{Erf\left(\frac{Mr}{2}\right)}{r^2} \right]. \quad (4.35)$$
4.2. GHOST AND SINGULARITY FREE THEORIES OF GRAVITY

In the following plot we have plotted the gravitational force \((4.35)\) as a function of \(\frac{Mr}{2}\). We can notice that there is a minimum at \(^2\)

\[
\frac{Mr_{\text{min}}}{2} = 0.9678 \simeq 1. \tag{4.36}
\]

It means that for value of the coordinate \(r < r_{\text{min}} \simeq \frac{2}{M}\) the gravitational force starts decreasing, until it vanishes for \(r = 0\). The scale at which this happens is dictated by the parameter \(M\) (see next section).

We can state that this model describe a **classic asymptotic free** theory of gravity\(^3\).

![Graph showing gravitational force as a function of Mr/2 with minimum at Mr_{\text{min}}/2 = 0.9678 \simeq 1.](image)

Figure 4.2: In this plot both gravitational force deriving from the non-singular potential \((4.32)\) and Newton force are drawn: the blue line represents the non-singular gravitational force, the dashed line the singular Newton force. The axis \(x\) defines the variable \(x = \frac{Mr}{2}\). The plot was obtained by using Wolfram Mathematica 9.0.

\(^2\)The minimum was calculated by using Wolfram Mathematica 9.0.

\(^3\)A theory is “asymptotic free” when the perturbation approach gets better and better at higher energies, and in the infinite momentum limit, the coupling constant vanishes. It means that particles become asymptotically weaker as energy increase and distance decrease. An example is the theory of Quantum Chromodynamics (QCD). The author in Ref. [19],[33] and [15] believe that in the framework of IDG theories the gravitational interaction could behave like QCD interaction. Since at the classical level we obtain a gravitational force the decrease with the distance, we can use the expression “classical asymptotic freedom” and so the non-singular potential behavior is a strong clue in favour of an “asymptotically free quantum theory of gravity”.

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Remark 3. We have seen that an UV modification involving the exponential function $e^{-\frac{\Box}{M^2}}$ is able to eliminate the Newtonian singularity. This is also a promising clue that let us believe in the realization of asymptotic freedom within IDG models [19],[33],[15]. It could be possible that a non-singular behavior (“classical” asymptotic freedom) can be connected to a “quantum” asymptotic freedom behavior. In the above discussion we have made use of one special transcendental function given by (4.12). Note that a more general choice could be
\[ a(\Box) = e^{-\gamma(\Box)}, \]  
where $\gamma(\Box)$ is an analytic function of $\Box$. It is then easy to see that for any polynomial $\gamma(\Box)$, as long as the highest power has positive coefficient, we will have a potentially asymptotic free theory and the propagator will be even more convergent than the exponential case (4.12).

### 4.3 Parameter M

In the previous section we have seen that by making an appropriate choice for the only independent coefficient $a(\Box)$, we are able to obtain a non-singular gravitational potential that in the UV limit gives us the GR behavior. So far we haven’t talked yet about the kind of modification we are considering at the physical level, but we have just done mathematical consideration.

We know that GR is recovered in the low energy regime (infrared limit), so it means that modifications of GR should be visible for higher energy (short distances). The energy scale at which the modification becomes noticeable is given by the mass $M$ appearing in the exponent of the exponential $a(\Box)$. It would be interesting to constrain the parameter by putting some bounds. We know for sure that we have an upper bound given by the reduced mass Planck, $M < M_P \simeq 2.4 \times 10^{18} \text{GeV}$, but it is too high to get interesting physical results.

It is worth noting that $M$ corresponds to a scale of non-locality, and this is due to the fact that IDG theories describe a non-local gravitational interactions. Let us clarify why infintive derivative theories of gravity are non-local.

From Cauchy theorem on differential equation we know that the higher the number of derivatives is the more initial data you have to provide to find a solution. If you have some Lagrangian that contains an infinite number of derivatives (or derivatives appearing non-polynomially, such as one over derivative) then you have to provide an infinite amount of initial data which amounts to non-local info, in the sense we now explain.

\[ \text{Informally, locality means that physics over here is independent of physics over there; we don’t have to have the wavefunction of the universe to see what happens in our lab [1].} \]

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If we have a theory with second derivatives it means that we have just to provide the field and its first derivative as initial values, at a specific point. So we don’t need to know the whole function, but just its value in the neighborhood of a point of its domain, i.e. we just need to know the function *locally*. Instead, if we have infinite derivative, for example if we think in terms of Taylor expansions around an initial value, then you have to provide the full function (and thus non-local information).

In Ref. [49] the authors put a bottom bound on the parameter $M$ by using the results obtained by the tests of the fall of Newtonian gravity, that has been tested in the laboratory up to $5.6 \times 10^{-5} m$ [50]$^5$, which implies that the scale of non-locality should be bigger than $M > 2 \times 10^4 m^{-1} = 0.01 eV$. This gives us also an upper limit for the minimum radius in equation (4.36):

$$r_{\text{min}} < \frac{2}{2 \times 10^4 m^{-1}} = 10^{-4} m.$$  \hspace{1cm} (4.38)

Hence the current experimental limits put the bound on non-locality to be around greater than $0.01 eV$ and smaller than $10^{18} GeV$. It means that any modification from Newtonian potential can occur in the gulf of scales spanning some 30 orders of magnitude. Our knowledge about the gravitational interaction at short distances is really limited!

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$^5$ The experiment discussed in Ref. [50] was conducted with torsion-balance and the inverse-square law was tested with 95% confidence.
Conclusions

In the last ten years, a lot of work was made by several groups. Today there are promising clues and a lot of confidence to continue working on infinitive derivative theories of gravity. In particular we have seen that they can be free from instabilities and ghosts around Minkowski background, but at the same time resolve the singularity problem for static mini-black holes. IDG theories could be also able to resolve the cosmological singularity problem, but so far only partially. In Ref. [19] and [15] the authors resolve the problem in the linearized regime, obtaining sinusoidal (periodic) solutions for the scale factor $a(t)$, i.e. bounce solutions. These solutions hold only for vanishing energy-momentum tensors. They also notice that the singularity can be avoided only if one introduces an additional massive scalar degree of freedom, whose corresponding mass will be connected to the frequency of expansion and contraction of the universe. Furthermore, in Ref. [16], [15] and [18] the authors also face the cosmological singularity problem in the full (not linearized) theory by using the following action

$$S = \int d^4x \left[ -\mathcal{R} + \mathcal{R}\mathcal{F}_1(\Box)\mathcal{R} \right];$$

i.e. a sub-case of the general action (3.6), $\mathcal{F}_2(\Box) = 0 = \mathcal{F}_3(\Box)$. Thus the information about the non-locality, and so about the presence of an infinite set of higher derivatives, is contained in the entire function $\mathcal{F}_1(\Box) = \sum_{n=0}^{\infty} f_{1,n} \Box^n$.

These classical results are in contrast with extended theories of gravity with finite higher derivatives, which show be either ghost-free or singularity-free, but not both. In this thesis we haven’t faced the quantum aspects of IDG theories but we have given a little overview in the previous chapter. Several groups have addressed the question of renormalizability and obtained good and promising results so far. The fundamental role is played by the exponential function $e^{-\frac{1}{\mathcal{M}^2}}$ that makes the theory softened in the UV regime and gives a first strong clue that the theory could describe an asymptotically free gravity.

As we can notice, there are still several questions that remain to be faced and better understood. Let us conclude listing some of these remaining challenges by following the authors in Ref. [15]:
• **Black hole singularity:** we have seen that IDG theories solve the singularity problem in the weak field approximation as shown above by the form of the modified non-singular Newtonian potential. At the classical level, one of the remaining aim is to understand whether the singularity can be also avoided in the strong field regime, i.e. for astrophysical black holes.

• **Cosmological singularity:** as we have already mentioned, in the linear regime, the cosmological singularity problem was solved in absence of matter source. To conduct a perturbative study for generic matter sources, the quadratic terms in the perturbation are not sufficient and we need to include cubic interactions. Furthermore, the exact cosmological solutions were only obtained in the presence of a cosmological constant [16]. A realistic cosmological scenario must also give an explanation for the inflationary phase. People are still trying to find a way to include and describe this primordial transition.

• **Extension to different backgrounds:** while IDG models have been investigated around the Minkowski space-time, one needs to extend this way of proceeding to de Sitter and FLRW backgrounds. Some progress has already been done in Ref. [51] and [52].

• **Unitarity:** The absence of "bad" ghost in the bare propagator implies that the unitarity condition is preserved at the tree level but it says nothing when loops are involved. While unitarity and causality of non-local theories have been formally argued for example in Ref. [53], [54] and [21], one should check that the several techniques used to calculate the loop integrals do not violate the unitarity condition once one or more loops are considered.

• **UV behavior:** as we have already emphasized, for these models the quantum UV behavior seems to be improved, but so far computations have only been carried out up to 2 loops. So far just a toy scalar model [33] has been considered, but obviously as a next step one has to deal with gravitational interaction in the framework of IDG theories. It could also happen that the exponential choice we have considered is not the most suitable to get a renormalizable and unitary theory. If this is the case, there might be different entire functions (satisfying the same properties we mentioned several times) that could solve this diatribe.
Appendix A

Off-shell and on-shell particles

In quantum field theory, off-shell (virtual) particles don’t satisfy the Einstein dispersion relation; instead, on-shell (real) particles do satisfy this relation. In formula it means that:

\[
\text{off-shell : } E^2 - |\vec{p}|^2 \neq m^2 \\
\text{on-shell : } E^2 - |\vec{p}|^2 = m^2.
\]

Naturally, if we deal with massless particles the last relations have to be considered with \( m = 0 \).

Let us consider a massive scalar field for simplicity. We know that the Lagrangian for a massive scalar field is given by

\[
\mathcal{L} = -\frac{1}{2} \phi (\Box + m^2) \phi,
\]

and the Euler-Lagrange equations are given by the Klein-Gordon equations:

\[
(\Box + m^2) \phi = 0,
\]

that in momentum space becomes

\[
(-p^2 + m^2) \phi = 0 \Rightarrow p^2 - m^2 = 0,
\]

where \( p^2 = p^\mu p_\mu = E^2 - |\vec{p}|^2 \). Hence a scalar field satisfying the field equations satisfies also the Einstein dispersion relation and so it will describe on-shell particles. Note that the Noether theorem is an on-shell theorem, in fact in the demonstration we have to impose the validity of the field equations to obtain the conservation law of the current, \( \partial_\mu J^\mu = 0 \), that, in turn, is an on-shell equation.

Note that the virtual particles corresponding to internal propagators in a Feynman
diagram are in general allowed to be off-shell. In fact, given a virtual particle with 4–momentum $q$, if the Einstein dispersion relation held ($q^2 = m^2$) we would have a singularity in the propagator:

$$\mathcal{P}(q) = \frac{1}{q^2 - m^2}.$$ 

Note also that, since the field equations are a constraint for the field, it means that off-shell particles turn out to be “more free” than on-shell particles in terms of degrees of freedom (see sections 1.2 and 2.2).
Appendix B

Spin projector operators decomposition

B.1 Tensor decomposition

We have used a lot the formalism of the spin projector operators to derive most of the results of this thesis. Especially we have found them very useful to understand which are the spin components of photon and graviton, and for the calculation of the propagators. In this appendix our aim is to understand why we can decompose either a vector or a two-rank tensor in terms of the spin projector operators. First of all we shall study the tensor representations of the Lorentz Group, especially the irreducible tensor representations under $SO(3)$; then we shall introduce special projector operators, called spin projector operators, by which we can decompose our tensor in two scalar, one transverse vector and one transverse and traceless tensor components. As for the part on tensor representation we shall also take inspiration from Ref. [47].

B.1.1 Lorentz tensor representation

Let us consider a two-rank tensor $\varphi^{\mu\nu}$ with two contravariant indices in Minkowski space. By definition $\varphi^{\mu\nu}$ is an object that under Lorentz transformations transforms as

$$\varphi'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma \varphi^{\rho\sigma}. \quad (B.1)$$

Tensors are examples of representations of the Lorentz group. For instance a generic two-rank tensor $\varphi^{\mu\nu}$ has 16 components and (B.1) shows that these components transform among themselves, i.e. they form a basis for a 16-dimensional representation of the Lorentz group.

In Group Theory the irreducible representations of any group turn out to be very important; for example, they are very useful when we want to decompose a tensor object
in its several spin components. We can notice that the 16-dimensional representation, we have just introduced, is reducible in different irreducible parts. First of all we easily understand that if $\varphi$ is symmetric (antisymmetric) then also $\varphi'$ will be symmetric (antisymmetric), so the symmetric and antisymmetric parts of a tensor $\varphi^{\mu\nu}$ don’t mix, and the 16-dimensional representation is for sure reducible into a 6-dimensional antisymmetric representation $\psi^{\mu\nu}$ and a 10-dimensional symmetric representation $h^{\mu\nu}$. One can explicitly see this decomposition in symmetric and antisymmetric parts in the following way:

$$\varphi^{\mu\nu} = h^{\mu\nu} + \psi^{\mu\nu},$$

$$\begin{align*}
   h^{\mu\nu} &:= \frac{1}{2} (\varphi^{\mu\nu} + \varphi^{\nu\mu}) \\
   \psi^{\mu\nu} &:= \frac{1}{2} (\varphi^{\mu\nu} - \varphi^{\nu\mu}).
\end{align*}$$

(B.2)

Furthermore, the trace of a symmetric tensor can be also isolated. Indeed, it is invariant under Lorentz transformation:

$$h' = \eta_{\mu\nu} h^{\mu\nu} = \eta_{\mu\nu} \Lambda^{\rho}_{\mu} \Lambda^{\sigma}_{\nu} h^{\rho\sigma} = \eta_{\rho\sigma} h^{\rho\sigma} = h;$$

so a traceless tensor remains traceless after a Lorentz transformation, and thus the 10-dimensional symmetric representation decomposes into a 9-dimensional irreducible symmetric traceless representation and a 1-dimensional scalar representation. In formula this means that

$$h^{T\mu\nu} := h^{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} h, \quad h = \eta_{\mu\nu} h^{\mu\nu},$$

(B.3)

where the apex “$T$” means “traceless”, in fact $\eta_{\mu\nu} h^{T\mu\nu} = h - \frac{4}{4} h = 0$.

In representation theory the following notation is commonly used: an irreducible representation is denoted by its dimensionality (the number of components), written in boldface. Thus the scalar representation is denoted as $\mathbf{1}$, the four-vector representation as $\mathbf{4}$, the antisymmetric representation as $\mathbf{6}$ and the traceless symmetric representation as $\mathbf{9}$.

The tensor representation (B.1) sees the action of two Lorentz matrices. It means that the representation (B.1) is a tensor product of two four-vector representations, namely each of two contravariant indices of $\varphi^{\mu\nu}$ transforms separately as a four-vector index. The tensor product of two representation is denoted by the symbol $\otimes$. Since we have found that a two-rank tensor can be decomposed into the direct sum of three irreducible representations, denoting the direct sum by $\oplus$, we can express the irreducible representation in terms of the dimensionality introduced above:

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}.$$  

(B.4)

Analogously one can obtain the tensor decomposition into irreducible parts when more than two indices are present. The most general irreducible representation of the Lorentz group are found starting from a generic tensor with an arbitrary number of indices, removing first all traces, and then symmetrizing or antisymmetrizing over all pairs of
indices. Note also that, by raising and lowering the indices with the Minkowski metric tensor $\eta_{\mu\nu}$, we can always restrict to contravariant tensors. Thus, for instance, $V^\mu$ and $V_\mu$ are equivalent representations.

All tensor representations are in a sense derived from the four-vector representation, since the transformation law of a tensor is obtained applying separately on each Lorentz index the matrix $\Lambda^\mu_\nu$ that defines the transformation law of a four-vector. This means that tensor representations are tensor product of four-vector representations and for this reason, the four-vector plays a fundamental role.

### B.1.2 Decomposition of Lorentz tensors under $SO(3)$

We know how a tensor behaves under a generic Lorentz transformation. Now, we are going to focus particularly on the transformation properties of a tensor under the $SO(3)$ rotation subgroup, and we can therefore ask what is the angular momentum $j$ of the various tensor representations. We will be able to decompose a generic two-rank tensor in terms of its spin components.

Let us recall that the representations of $SO(3)$ are labeled by an index $j$ which assumes integer values $j = 0, 1, 2 \ldots$; while the dimension of the representation, labeled by $j$, is defined by $2j + 1$. Then within each representation, there are $2j + 1$ states labeled by $j_z = -j, \ldots, j$. Note that for $SO(3)$ it is more common to denote the representation as $j$, i.e. to label it with the associated angular momentum rather than with the dimension of the representation, $2j + 1$. Hence in this notation, $0$ is the scalar (singlet, spin-0), $1$ is a triplet (spin-1) with components $j_z = -1, 0, 1$, while $2$ is a representation of dimension 5 (spin-2), and so on with higher dimensionality.

A Lorentz scalar is of course also scalar under rotations, so it has $j = 0$. A four-vector $V^\mu = (V_0, \vec{V})$ is an irreducible representation of the Lorentz group, since a generic Lorentz transformation mixes all four components, but under $SO(3)$ it is reducible. Indeed, spatial rotations do not mix $V^0$ with $\vec{V} : V^0$ is invariant under spatial rotations, so it has $j = 0$, while the three spatial components $V^i$ form an irreducible 3-dimensional representations of $SO(3)$, with $j = 1$.

By adopting the above convention according to which the representations are indicated by the associated angular momentum, the decomposition of a four-vector into the direct sum of a scalar and a spin-1 representation under $SO(3)$ can be written as

$$V^\mu \in 0 \oplus 1.$$  \hspace{1cm} (B.5)

While in terms of their dimensions we should write

$$V^\mu \in 4 = 1 \oplus 3.$$  \hspace{1cm} (B.6)

Now we would like to understand which are the spin components of a two-rank tensor $\varphi^{\mu\nu}$, i.e. what angular momenta appear. By definition we know that $\varphi^{\mu\nu}$ transforms
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as the tensor product of two four-vector representations. Since, from a point of view of $SO(3)$, a four-vector decomposes as $0 \oplus 1$, a generic two-rank tensor has the following decomposition in angular momenta

$$ \varphi^{\mu\nu} \in (0 \oplus 1) \otimes (0 \oplus 1) = (0 \otimes 0) \oplus (0 \otimes 1) \oplus (1 \otimes 0) \oplus (1 \otimes 1) $$

$$ = 0 \oplus 1 \oplus 1 \oplus (0 \oplus 1 \oplus 2). \quad (B.7) $$

In (B.7) we have used the usual rule to compose angular momenta, according to which the composition of two angular momentum $j_1$ and $j_2$ is given by all angular momentum between $|j_1 - j_2|$ and $j_1 + j_2$:

$$ 0 \otimes 0 = 0, \quad 0 \otimes 1 = 1 \otimes 0 = 1, \quad 1 \otimes 1 = 0 \oplus 1 \oplus 2. \quad (B.8) $$

Thus, under the rotation group $SO(3)$, $\varphi^{\mu\nu}$ decomposes as two spin-0 representations, three spin-1 representations and one spin-2 representation.

It would be interesting to see how these representations are shared between the symmetric traceless, the trace and the antisymmetric part of the tensor $\varphi^{\mu\nu}$, since these are irreducible Lorentz representations. So, let us see how these two different irreducible decompositions match to each other.

The trace is a Lorentz scalar, so it is in particular scalar under $SO(3)$ and therefore is a $0$ representation. An antisymmetric tensor $\psi^{\mu\nu}$ has six components, which can be written as the direct sum of the two three-vectors $\psi^{0i}$ and $\frac{1}{2} \epsilon^{ijk} \psi^{jk}$. These are two spatial vectors (two triplets) and so

$$ \psi^{\mu\nu} \in 1 \oplus 1. \quad (B.9) $$

Since we have identified the trace $h$ with $0$ and $\psi^{\mu\nu}$ with $1 \oplus 1$, by comparing (B.4) and (B.7) we can see that the nine components of a symmetric traceless tensor $h^{T\mu\nu}$ decompose, under the subgroup $SO(3)$, as

$$ h^{T\mu\nu} \in 0 \oplus 1 \oplus 2. \quad (B.10) $$

Remark 4. We have seen that a generic two-rank tensor can be written as a tensor product of two four-vectors. So let us observe that when we write $\varphi^{\mu\nu}$ as $(0 \oplus 1) \otimes (0 \oplus 1)$, the first $0$ corresponds to taking the index $\mu = 0$, the first $1$ corresponds to taking the index $\mu = i$, and similarly for the second factor $(0 \oplus 1)$ and the index $\nu$. Therefore in equation (B.7) we have the following correspondence:

$$ 0 \otimes 0 \rightarrow \varphi^{00}, \quad 0 \otimes 1 \rightarrow \varphi^{0i}, \quad 1 \otimes 0 \rightarrow \varphi^{i0}, \quad 1 \otimes 1 \rightarrow \varphi^{ij}. $$

It is clear that, under spatial rotations $SO(3)$, $\varphi^{00}$ behaves like a scalar, while $\varphi^{0i}$ and $\varphi^{i0}$ like spatial vectors. As for the spatial components $\varphi^{ij}$, its antisymmetric part

---

1In Electrodynamics one has an important example of antisymmetric tensor, i.e the tensor field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this case the two vectors are $E^i = -F^{0i}$ and $B^i = -\frac{1}{2} \epsilon^{ijk} F^{jk}$, i.e. the electric and magnetic fields.
\( \psi_{ij} = \frac{1}{2} (\varphi_{ij} - \varphi_{ji}) \) has only three independent component and so it turns out to be a spatial vector, giving a 3-dimensional representation 1; while its symmetric part can be separated into its trace, which gives the second 0 representation, and the traceless symmetric part, which must have \( j = 2 \).

In general, a symmetric tensor with \( N \) indices contains angular momentum up to \( j = N \). In four dimensions, higher antisymmetric tensors with four indices, \( \varphi^{\mu\nu\rho\sigma} \), has only one independent component \( \varphi^{0123} \), so it must be a Lorentz scalar. An antisymmetric tensor with three indices, \( \varphi^{\mu\nu\rho} \), has \( \frac{4 \times 3 \times 2}{3!} = 4 \) components and it has the same transformation properties of a four-vector.

### B.1.3 Tensor decomposition in curved space-time

Up to now we have only considered the behavior of a Lorentz tensor and we managed to obtain its decomposition in terms of spin-0, spin-1 and spin-2 components. We can also do the same with a generic two-rank tensor (or any rank) defined by means diffeomorphism group transformations. More generally a two-rank tensor \( \varphi^{\mu\nu} \) is an object that transforms in the following way

\[
x^{\prime\mu} \equiv x^{\mu}(x) \Rightarrow \varphi^{\prime\mu\nu}(x') = \frac{\partial x^{\prime\mu}}{\partial x^{\mu}} \frac{\partial x^{\prime\nu}}{\partial x^{\nu}} \varphi^{\rho\sigma}(x).
\]  

(B.11)

Also here we can easily see that symmetric and antisymmetric parts don’t mix between them, so first of all \( \varphi^{\mu\nu} \) decomposes in symmetric part \( h^{\mu\nu} \) and antisymmetric part \( \psi^{\mu\nu} \) as in (B.2). Then we can isolate the trace component in the symmetric part \( h^{\mu\nu} \) because of \( h \) being invariant under diffeomorphism group, in fact

\[
h(x) \equiv g_{\mu\nu}(x) h^{\mu\nu}(x) \Rightarrow h'(x') = g_{\mu\nu}(x') h^{\mu\nu}(x')
\]

\[
= g_{\mu\nu}(x') \frac{\partial x^{\prime\mu}}{\partial x^{\mu}} \frac{\partial x^{\prime\nu}}{\partial x^{\nu}} h^{\rho\sigma}(x) = g_{\rho\sigma}(x) h^{\rho\sigma}(x)
\]

(B.12)

so \( h \) turns out to be an invariant also under diffeomorphism transformations. We have learned that also for a generic tensor, that transforms as in (B.11), we can have a decomposition in trace, antisymmetric and symmetric traceless components.

Now we want to study the behavior of \( \varphi^{\mu\nu}(x) \) from the point of view of \( SO(3) \), trying to obtain a kind of decomposition as in (B.7). We know that a generic rotation transformation doesn’t mix time and space components, i.e. we can define a rotation in the

\[\text{In General Relativity an important example is given by the physical graviton. It can be described by a traceless symmetric spatial tensor (transverse to the propagation direction) corresponding to spin-2.}\]
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following way
\[
\begin{align*}
  x^0 &\rightarrow x'^0 = x^0 \\
  \bar{x} &\rightarrow \bar{x}' = f(\bar{x})
\end{align*}
\]  
(B.13)

Let us study separately the antisymmetric and the symmetric parts (see (B.2)):
\[
\varphi^{\mu\nu}(x) = h^{\mu\nu}(x) + \psi^{\mu\nu}(x).
\]
Recall that \(\psi^{\mu\nu}(x)\) has 6-independent component and \(h^{\mu\nu}(x)\) has 10-independent components because of their antisymmetric and symmetric nature. First we can notice that \(\psi^{0i}\) and \(\psi^{ij}\) are two three-vectors. In fact, by implementing the transformation (B.13) one gets
\[
\psi^{0i}(x') = \frac{\partial x'^0}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^\sigma} \varphi^{\rho\sigma}(x) = \frac{\partial x'^i}{\partial x^0} \varphi^{00}(x) = \frac{\partial x'^i}{\partial x^0} \psi^{0j}(x);
\]
then we can introduce the other vector in the following way
\[
\psi^k := \frac{1}{2} \varepsilon^{ijk} \psi^{ij}.
\]
(B.15)

As for the symmetric part we can notice that it decomposes into the scalar trace component, the scalar component \(h^{00}\), the three-vector component \(h^{0i}\) and the three-traceless tensor component \(h^{ij}\). In fact
\[
\begin{align*}
  h^{00}(x') &= \frac{\partial x'^0}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^\beta} h^{\alpha\beta}(x) = \frac{\partial x'^0}{\partial x^0} \frac{\partial x^0}{\partial x^0} h^{00}(x) = h^{00}(x) \quad \text{(scalar)}, \\
  h^{0i}(x') &= \frac{\partial x'^0}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^\beta} h^{\alpha\beta}(x) = \frac{\partial x'^0}{\partial x^0} \frac{\partial x^0}{\partial x^j} h^{0j}(x) = \frac{\partial x'^i}{\partial x^j} h^{0j}(x) \quad \text{(3-vector)}, \\
  h^{ij}(x') &= \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^j} h^{\alpha\beta}(x) = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x^j} h^{kl}(x) \quad \text{(3-vector)}.
\end{align*}
\]
(B.16)

Then we can easily see that the trace \(\bar{h}^3\) of the three-tensor \(h^{ij}\) is an invariant from the point of view of \(SO(3)\),
\[
\bar{h}' = -g^{ij}(x') h'_{ij}(x') = -g^{ij}(x') \frac{\partial x'k}{\partial x^i} \frac{\partial x'l}{\partial x^j} h_{kl}(x)
\]
\[
= -g^{kl}(x) h_{kl}(x) = \bar{h};
\]
(B.17)

\footnote{We are using the symbol \(\bar{h}\) to not create confusion with trace \(h\) in four dimensions. Recall that, how we can see in [48], in 3-dimensions we have to consider the spatial metric tensor defined as \(\gamma_{ij} = -g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}}\) and \(\gamma^{ij} = -g^{ij}\), to define and calculate the trace \(\bar{h}\) in 3-dimensions. Hence \(\bar{h} = \gamma_{ij} h^{ij} = \gamma^{ij} h_{ij}\).}
where in the last step, since the metric tensor \( g_{\mu\nu}(x) \) is symmetric, we have considered \( g_{ij}(x) \) as a three-tensor. Hence we can define a three-traceless tensor as

\[
h^{ij} := h^{ij} - \frac{g^{ij}}{g_{kl}g^{kl}} \bar{h}; \tag{B.18}
\]

we can also verify that it is traceless:

\[
h^T = -g^{ij} h^T_{ij} = -g^{ij} h_{ij} + \frac{g_{ij}g^{ij}}{g_{kl}g^{kl}} \bar{h} = \bar{h} - \bar{h} = 0. \tag{B.19}
\]

Hence, finally, because of the equations (B.14) and (B.15) for the antisymmetric part, and the equations (B.16)-(B.19) for the symmetric part, the two-rank tensor \( \varphi^{\mu\nu} \) decomposes into two scalar components, three three-vector components and one three-traceless tensor component.

This kind of decomposition is very important in the theory of cosmological perturbations where the two-rank tensor that decomposes is the metric tensor perturbation that, being symmetric, has two scalar, one vector and one tensor components.

### B.2 Spin projector operators

So far we have seen how to decompose a generic two-rank tensor (or more generally a \( N \)-rank tensor) into scalar, vector and tensor components. At this point one question that we can ask could be: can we define a complete set of projector operators by which we are able to project the tensor \( \varphi^{\mu\nu} \) along its scalar, vector and tensor components? The answer is “yes” and in this section we shall introduce these useful operators.

Furthermore, we are going to introduce also a basis in terms of which one can decompose any four-rank tensor operator\(^4\) \( \mathcal{O}^{\mu\nu\rho\sigma} \) appearing in a given parity-invariant Lagrangian

\[
\mathcal{L} = \frac{1}{2} \varphi_{\mu\nu} \mathcal{O}^{\mu\nu\rho\sigma} \varphi_{\rho\sigma}, \tag{B.20}
\]

or in the associated field equations once we consider the presence of a source \( J^{\mu\nu} \),

\[
\mathcal{O}^{\mu\nu\rho\sigma} \varphi_{\rho\sigma} = \lambda J^{\mu\nu}, \tag{B.21}
\]

where \( \lambda \) is the coupling constant. In other words we can say that, the operator space of the field equations (B.21) can be spanned in the basis mentioned above.

Note that in the case of GR we have symmetric tensors \( \varphi_{\mu\nu} \rightarrow h_{\mu\nu} \) and \( J_{\mu\nu} \rightarrow \tau_{\mu\nu} \), the coupling constant is given by \( \lambda \rightarrow \kappa \) and the operator \( \mathcal{O}^{\mu\nu\rho\sigma} \) is symmetric.

---

\(^4\)In this Appendix, when we say “four-rank tensor operator” we refer to the operator \( \mathcal{O}^{\mu\nu\rho\sigma} \) that appears in a given parity-invariant Lagrangian for a two-rank tensor field.
B.2. SPIN PROJECTOR OPERATORS

B.2.1 Four-vector decomposition

A generic 4-vector $V^\mu$ can be projected along transverse and longitudinal components, $V^\mu \in 0 \oplus 1$, and we can perform the projection by introducing a set of two projectors, $\{\theta, \omega\}$, in the following way:

$$V^\mu = \theta^\mu_\nu V^\nu + \omega^\mu_\nu V^\nu, \quad (B.22)$$

where the operators $\theta^\mu_\nu$ and $\omega^\mu_\nu$ are defined as

$$\left\{ \begin{array}{l}
\theta^\mu_\nu := \eta^\mu_\nu - \frac{\partial^\mu \partial_\nu}{k^2} \\
\omega^\mu_\nu := \frac{\partial^\mu \partial_\nu}{k^2}
\end{array} \right. \quad (B.23)$$

These projectors in momentum space are given by

$$\theta^\mu_\nu = \eta^\mu_\nu - \frac{k^\mu k_\nu}{k^2}, \quad \omega^\mu_\nu = \frac{k^\mu k_\nu}{k^2}. \quad (B.24)$$

It is easy to show that the following properties hold\(^5\)

$$\theta + \omega = I \Leftrightarrow \theta^\mu_\nu + \omega^\mu_\nu = \eta^\mu_\nu$$

$$\theta^2 = \theta, \quad \omega^2 = \omega, \quad \theta \omega = 0$$

$$\Leftrightarrow \theta^\mu_\nu \theta^\nu_\rho = \delta^\mu_\rho, \quad \omega^\mu_\nu \omega^\nu_\rho = \delta^\mu_\rho, \quad \theta^\mu_\nu \omega^\nu_\rho = 0, \quad (B.25)$$

namely the set $\{\theta, \omega\}$ turns out to be complete and orthogonal.

One can also verify that this special decomposition corresponds to that in which $V^\mu$ decomposes in transverse and longitudinal components. In fact, if $k^\mu$ is the 4-momentum associated to the electromagnetic wave (or photon) we can immediately see that

$$k^\mu \theta^\nu_\mu = 0, \quad k^\mu \omega^\nu_\mu = k^\nu; \quad (B.26)$$

hence $\theta$ and $\omega$ project along the transverse and longitudinal components respectively.

Furthermore, we notice that the transverse component has spin-1 and the longitudinal one spin-0 by calculating the trace of the two projectors:

$$\eta^{\mu\nu} \theta^\mu_\nu = 3 = 2(1) + 1 \quad \text{(spin-1)}, \quad (B.27)$$

$$\eta^{\mu\nu} \omega^\mu_\nu = 1 = 2(0) + 1 \quad \text{(spin-0)}.$$

The relations (B.27) tell us that (B.22) corresponds to the decomposition of a four-vector in terms of spin-1 and spin-0 components under the rotation group $SO(3)$, i.e. $V^\mu \in 0 \oplus 1$ (see the Subsection B.1.2).

\(^5\)As we have already said more times, we shall often write the projectors suppressing the indices.
B.2. SPIN PROJECTOR OPERATORS

B.2.2 Two-rank tensor decomposition

We know that a two-rank tensor behaves like the tensor product of two four-vector, so we can find the projector operators for $\varphi^{\mu\nu}$ by decomposing each index separately:

$$\varphi^{\mu\nu} \equiv V^\mu \otimes U^\nu. \quad (B.28)$$

Moreover, we know that we can decompose $\varphi^{\mu\nu}$ in its symmetric and antisymmetric parts as done in (B.2), so one can study the two parts separately.

**Symmetric decomposition**

Let us start with the symmetric part $h_{\mu\nu} \in 0 \oplus 0 \oplus 1 \oplus 2$. By seeing $h_{\mu\nu}$ as a symmetric tensor product of two four-vectors, the decomposition can be performed as follow:

$$h_{\mu\nu} = (\theta_{\mu\rho} + \omega_{\mu\rho}) (\theta_{\nu\sigma} + \omega_{\nu\sigma}) h^{\rho\sigma} = (\theta_{\mu\rho} h_{\nu\sigma} + \theta_{\mu\sigma} h_{\nu\rho} + \theta_{\nu\rho} h_{\sigma\mu} + \theta_{\nu\sigma} h_{\rho\mu}) h^{\rho\sigma} = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho}) h^{\rho\sigma} - \frac{1}{3} \theta_{\mu\rho} \theta_{\rho\sigma} h^{\rho\sigma} + \frac{1}{3} \theta_{\mu\rho} \theta_{\rho\sigma} h^{\rho\sigma} + \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}) h^{\rho\sigma}. \quad (B.29)$$

Now we can define the spin projector operators:

$$P^2_{\mu\nu\rho\sigma} = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho}) - \frac{1}{3} \theta_{\mu\rho} \theta_{\rho\sigma},$$

$$P^1_{m,\mu\nu\rho\sigma} = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}) h^{\rho\sigma}, \quad (B.30)$$

$$P^0_{s,\mu\nu\rho\sigma} = \frac{1}{3} \theta_{\mu\rho} \theta_{\rho\sigma}, \quad P^0_{w,\mu\nu\rho\sigma} = \omega_{\mu\rho} \omega_{\nu\sigma}. \quad (B.31)$$

The set

$$\mathcal{O}^i \equiv \{ P^2, P^1_m, P^0_s, P^0_w \}, \quad i = 1, 2, 3, 4, \quad (B.32)$$

forms a complete set of spin projector operators in terms of which a symmetric two-rank tensor can be decomposed. In fact, one can easily verify that:

$$\mathcal{O}^i \mathcal{O}^j = \delta_{ij} \mathcal{O}^i, \quad \mathcal{O}^1 + \mathcal{O}^2 + \mathcal{O}^3 + \mathcal{O}^4 = \mathbb{I}, \quad (B.32)$$

---

6We are labeling the spin-1 projector operator also with the letter $m$. When we consider only symmetric tensors we can avoid it and write directly $P^1$, as it has been done in this thesis (see Chapter 2).

7We are suppressing the indices, but if we want to be more precise we should write $O^i_{\mu\nu\rho\sigma} = \delta_{ij} O^i_{\mu\nu\rho\sigma}$ or $P^i_{a,\mu\nu\rho\sigma} = \delta_{ij} \delta_{ab} P^i_{a,\mu\nu\rho\sigma}.$
or in terms of $P$'s

\[
P^i_a P^j_b = \delta_{ij} \delta_{ab} P^i_a, \quad P^2 + P^1 + P^0_s + P^0_w = \mathbb{I}. \tag{B.33}
\]

The second property of (B.32) (or (B.33)) has been already showed when we constructed and defined the set of operators in (B.29), but we can also show it explicitly:

\[
P^2 + P^1 + P^0_s + P^0_w = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho})
+ \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho})
= \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) + \frac{1}{2} (\eta_{\nu\rho} \theta_{\mu\sigma} + \frac{1}{2} \eta_{\nu\sigma} \omega_{\mu\rho})
- \frac{1}{2} \eta_{\nu\sigma} \omega_{\mu\rho} - \frac{1}{2} \eta_{\nu\rho} \omega_{\mu\sigma}
= \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) = \mathbb{I}.
\]

Hence, we have found a complete set of projector operators to decompose $h_{\mu\nu}$:

\[
h_{\mu\nu} = P^2_{\mu\nu\rho\sigma} h^{\rho\sigma} + P^1_{\mu\nu\rho\sigma} h^{\rho\sigma} + P^0_{s, \mu\nu\rho\sigma} h^{\rho\sigma} + P^0_{w, \mu\nu\rho\sigma} h^{\rho\sigma}
= (P^2 + P^1 + P^0_s + P^0_w)_{\mu\nu\rho\sigma} h^{\rho\sigma}. \tag{B.34}
\]

Note that to form a basis in terms of which any symmetric four-rank tensor can be expanded we also need to introduce other two operators that mix the two scalar components $s$ and $w$. They are required to close the algebra of the spin projector operators.

These two new operators are defined as follow

\[
P^0_{sw, \mu\nu\rho\sigma} = \frac{1}{\sqrt{3}} \theta_{\mu\rho} \omega_{\nu\sigma}, \quad P^0_{ws, \mu\nu\rho\sigma} = \frac{1}{\sqrt{3}} \omega_{\mu\sigma} \theta_{\rho\nu}. \tag{B.35}
\]

Now, the orthogonality relations in (B.33) can be extended to the operators $P^0_{sw}$ and $P^0_{ws}$, so that we obtain (when $a \neq b$ and $c \neq d$)

\[
P^i_a P^j_b = \delta_{ij} \delta_{ab} P^i_a, \quad P^0_{ab} P^i_c = \delta_{i0} \delta_{bc} P^i_c,
P^0_{cd} P^0_{ab} = \delta_{ad} \delta_{bc} P^0_{a}. \tag{B.36}
\]

where $i, j = 2, 1, 0$ and $a, b, c, d = m, s, w$, absent.

Hence, the set $\{P^2, P^1, P^0_s, P^0_w, P^0_{sw}, P^0_{ws}\}$ forms a basis of symmetric four-rank tensors.

---

8 Note that we are doing an abuse because we are calling $P^0_{sw}$ and $P^0_{sd}$ projectors, but they are not like that. In fact this becomes very clear by looking at the orthogonality relations below, (B.36). Often we will make this abuse of nomenclature.

9 Note that the projector $P^2$ does not have any lower index, so it can happen that $a, b, c, d$ are absent.
B.2. SPIN PROJECTOR OPERATORS

Remark 5. Since we have applied the formalism of the spin projector operators to gravity theories, it is worth observing that the basis of projectors represents six field degrees of freedom. The other four fields in a symmetric tensor field, as usual, represent the gauge (unphysical) degrees of freedom. $\mathcal{P}^2$ and $\mathcal{P}^1$ represent transverse and traceless spin-2 and spin-1 degrees, accounting for four degrees of freedom, while $\mathcal{P}^0_s$ and $\mathcal{P}^0_w$ represent the spin-0 scalar multiplets. In addition to the four projectors we also need to introduce the operators $\mathcal{P}^0_{sw}$ and $\mathcal{P}^0_{ws}$ which are necessary to close the algebra and form a basis of operators acting in the space of the symmetric two-rank tensors. From the relations (B.36) we notice that $\mathcal{P}^0_{sw}$ and $\mathcal{P}^0_{ws}$ are not projector operators, but transition operators that mix the two spin-0 projector operators, $s$ and $w$.

We can also verify that $\mathcal{P}^2$ is traceless and transverse, in fact:

$$
\eta^{\mu\nu} \mathcal{P}^2_{\mu\nu\rho\sigma} h^{\rho\sigma} = \left[ \frac{1}{2} \left( \eta^{\mu\nu} \theta_{\mu\rho} \theta_{\nu\sigma} + \eta^{\mu\nu} \theta_{\mu\sigma} \theta_{\nu\rho} \right) - \frac{1}{3} \eta^{\mu\nu} \theta_{\mu\rho} \theta_{\rho\sigma} \right] h^{\rho\sigma} = 0
$$

Then

$$
k^{\mu} \mathcal{P}^2_{\mu\nu\rho\sigma} h^{\rho\sigma} = \left[ \frac{1}{2} \left( k^{\mu} \theta_{\mu\rho} \theta_{\nu\sigma} + k^{\mu} \theta_{\mu\sigma} \theta_{\nu\rho} \right) - \frac{1}{3} k^{\mu} \theta_{\mu\rho} \theta_{\rho\sigma} \right] h^{\rho\sigma} = 0
$$

Since $k^{\mu} \theta_{\mu\rho} = k^{\mu} \theta_{\mu\sigma} = k^{\mu} \theta_{\mu\rho} = 0$, (B.38) becomes $k^{\mu} \mathcal{P}^2_{\mu\nu\rho\sigma} h^{\rho\sigma} = 0$. With this choice, none among the operators $\mathcal{P}^0_s$, $\mathcal{P}^0_w$, $\mathcal{P}^0_{sw}$ and $\mathcal{P}^0_{ws}$ corresponds to the trace operator: we shall show how to construct an operator that acting on the tensor field $h_{\mu\nu}$ gives us the trace $h$ at the end of the section B.3.

Antisymmetric decomposition

Let us now work on the antisymmetric part $\psi^{\mu\nu} \in 1 \oplus 1$. By proceeding as we have already done for the symmetric part, we get:

$$
\psi_{\mu\nu} = (\theta_{\mu\rho} + \omega_{\mu\rho}) (\theta_{\nu\sigma} + \omega_{\nu\sigma}) \psi^{\rho\sigma}
= (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\rho} \omega_{\nu\sigma} + \omega_{\mu\rho} \theta_{\nu\sigma} + \omega_{\mu\rho} \omega_{\nu\sigma}) \psi^{\rho\sigma}
= \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} - \theta_{\mu\sigma} \theta_{\nu\rho}) \psi^{\rho\sigma}
+ \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} - \theta_{\mu\sigma} \omega_{\nu\rho} - \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}) \psi^{\rho\sigma}
$$

(B.39)
We can define the spin projector operators for the antisymmetric part as follow:

\[ P_{b,\mu\nu\rho\sigma}^1 = \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} - \theta_{\mu\sigma} \theta_{\nu\rho}) , \]
\[ P_{e,\mu\nu\rho\sigma}^1 = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} - \theta_{\mu\sigma} \omega_{\nu\rho} - \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}) . \]

Thus, we obtain

\[ \psi_{\mu\nu} = P_{b,\mu\nu\rho\sigma}^1 \psi_{\rho\sigma} + P_{e,\mu\nu\rho\sigma}^1 \psi_{\rho\sigma} = (P_b^1 + P_e^1)_{\mu\nu\rho\sigma} \psi_{\rho\sigma} . \]

The set \( \{ P_b^1, P_e^1 \} \) is complete and allows us to project every antisymmetric tensor along its two vector components. Observe that the letters \( b \) and \( e \) refer, respectively, to magnetic spin-1 and electric spin-1, due to the fact that in electrodynamics the same happens with the antisymmetric tensor \( F_{\mu\nu} = \partial_{\mu} A_\nu - \partial_{\nu} A_\mu \)

Note that, in the antisymmetric case, the completeness relation is given by

\[ (P_b^1 + P_e^1)_{\mu\nu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho}) . \]

**Full decomposition**

We are now able to decompose any two-rank tensor \( \varphi^{\mu\nu} \) along the spin components corresponding to the irreducible representations of the group \( SO(3) \),

\[ \varphi^{\mu\nu} \in 1 \oplus 1 \oplus 0 \oplus 0 \oplus 1 \oplus 2 , \]

in terms of the spin projector operators. Indeed, we can extend the symmetric set \( \{ P^2, P_m^1, P_s^0, P_w^0 \} \) by including the antisymmetric part \( \{ P_b^1, P_e^1 \} \). Thus any two-rank tensor can be decomposed in terms of the complete set of spin projectors operators,

\[ \mathcal{O}_i = \{ P^2, P_m^1, P_s^0, P_w^0, P_b^1, P_e^1 \} , \quad i = 1, 2, 3, 4, 5, 6 , \]

in the following way:

\[ \varphi^{\mu\nu} = P^2_{\mu\rho\sigma} \varphi^{\rho\sigma} + P_{m,\mu\rho\sigma}^1 \varphi^{\rho\sigma} + P_{s,\mu\rho\sigma}^0 \varphi^{\rho\sigma} \\
+ P_{w,\mu\rho\sigma}^0 \varphi^{\rho\sigma} + P_{b,\mu\rho\sigma}^1 \varphi^{\rho\sigma} + P_{e,\mu\rho\sigma}^1 \varphi^{\rho\sigma} \]

\[ = (P^2 + P_m^1 + P_s^0 + P_w^0 + P_b^1 + P_e^1)_{\mu\rho\sigma} \varphi^{\rho\sigma} . \]

We are also interested to form a basis in terms of which any four-rank tensor can be expanded. We have already seen that for the symmetric part we needed to define two
operators that mix the scalar components. To complete the full basis we need to introduce other two spin-1 operator that mix the spin-1 components. They are defined as

$$\mathcal{P}^1_{me,\mu\nu\rho\sigma} = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} - \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} - \theta_{\nu\sigma} \omega_{\mu\rho}),$$  \hspace{1cm}  \text{(B.45)}

$$\mathcal{P}^1_{em,\mu\nu\rho\sigma} = \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} - \theta_{\nu\rho} \omega_{\mu\sigma} - \theta_{\nu\sigma} \omega_{\mu\rho}).$$  

In this way we have closed the algebra and formed the basis of four-rank tensors

$$\{ \mathcal{P}^2, \mathcal{P}^1_m, \mathcal{P}^0_s, \mathcal{P}^0_w, \mathcal{P}^1_e, \mathcal{P}^0_b, \mathcal{P}^0_{sw}, \mathcal{P}^1_{em}, \mathcal{P}^1_{me} \}. \hspace{1cm}  \text{(B.46)}$$

It easy to show that the following orthogonal relations hold (when $a \neq b$ and $c \neq d$)

$$\mathcal{P}^i_a \mathcal{P}^j_b = \delta_{ij} \delta_{ab} \mathcal{P}^j_a, \quad \mathcal{P}^0_{ab} \mathcal{P}^i_c = \delta_{i0} \delta_{bc} \mathcal{P}^i_a, \quad \mathcal{P}^i_c \mathcal{P}^0_{ab} = \delta_{i0} \delta_{ac} \mathcal{P}^0_{ab}, \quad \mathcal{P}^0_{ab} \mathcal{P}^0_{cd} = \delta_{ad} \delta_{bc} \mathcal{P}^0_a, \hspace{1cm}  \text{(B.47)}$$

where $i, j = 2, 1, 0$ and $a, b, c, d = m, s, w, b, e,$ absent. Hence, the introduction of the four operators $\mathcal{P}^0_{ws}, \mathcal{P}^0_{sw}, \mathcal{P}^1_{em}, \mathcal{P}^1_{me}$ is important to satisfy the relations (B.47) that define the algebra of the operators.

Note that for the full decomposition the completeness relations takes into account both symmetric and antisymmetric part, and it is given by

$$\left( \mathcal{P}^2 + \mathcal{P}^1_m + \mathcal{P}^0_s + \mathcal{P}^0_w + \mathcal{P}^1_e + \mathcal{P}^1_b \right)_{\mu\nu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$$

$$+ \frac{1}{2} (\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$$

$$= \eta_{\mu\rho}\eta_{\nu\sigma}. \hspace{1cm}  \text{(B.48)}$$

\[^{10}\text{No operators which connect electric and magnetic spin-1 spaces (} \mathcal{P}^1_{eb} \text{ and } \mathcal{P}^1_{be} \text{), nor operators which connect the third pair of spin-1 spaces (} \mathcal{P}^1_{bm} \text{ and } \mathcal{P}^1_{mb} \text{). To understand why these operators are not needed we have to observe that the four-rank tensor operators we want to expand in the full basis is present in the Lagrangians and so in the associated field equations. For instance, given the following free Lagrangian}

$$\mathcal{L} = \frac{1}{2} \varphi_{\mu\nu} O^{\mu\nu\rho\sigma} \varphi_{\rho\sigma},$$

\[^{10}\text{we need to expand the operator } O^{\mu\nu\rho\sigma} \text{ in terms of the full basis (B.46). Now, if the Lagrangian is invariant under parity transformations the presence of such transition operators is excluded. While, a parity-violation case would bring to the presence of terms like } \epsilon_{\mu\nu\rho\sigma} \varphi^\mu \varphi^\rho \varphi^\sigma \text{ or } \epsilon_{\mu\nu\rho\sigma} \partial^\mu \varphi^\rho \varphi^\sigma \text{ in the Lagrangian and so the operators } \mathcal{P}^1_{eb}, \mathcal{P}^1_{be}, \mathcal{P}^1_{mb} \text{ and } \mathcal{P}^1_{bm} \text{ would appear. See Ref. [36] for more details.}

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We can also find the spin value of each spin projector operators by contracting with the identity matrix $\eta_{\mu\rho}\eta_{\nu\sigma}$. Indeed the following relation holds:

$$ (\eta^{\mu\rho}\eta^{\nu\sigma}) P^j_{\mu\nu\rho\sigma} = 2(j) + 1, \quad (B.49) $$

where $j$ is the spin associated to the spin projector operator $P^j$. Note that, because of the symmetry, the product can also read as $\eta^{\mu\rho}\eta^{\nu\sigma} P^j$. Hence we can easily verify that:

$$\eta^{\mu\rho}\eta^{\nu\sigma} P^2_{\mu\nu\rho\sigma} = 5 = 2(2) + 1 \quad \text{(spin-2)}, $$

$$\eta^{\mu\rho}\eta^{\nu\sigma} P^1_{m,\mu\nu\rho\sigma} = 3 = 2(1) + 1 \quad \text{(spin-1)}, $$

$$\eta^{\mu\rho}\eta^{\nu\sigma} P^0_{s,\mu\nu\rho\sigma} = 1 = 2(0) + 1 \quad \text{(spin-0)}, $$

$$\eta^{\mu\rho}\eta^{\nu\sigma} P^0_{w,\mu\nu\rho\sigma} = 1 = 2(0) + 1 \quad \text{(spin-0)}, $$

$$\eta^{\mu\rho}\eta^{\nu\sigma} P^1_{b,\mu\nu\rho\sigma} = 3 = 2(1) + 1 \quad \text{(spin-1)}, $$

$$\eta^{\mu\rho}\eta^{\nu\sigma} P^1_{e,\mu\nu\rho\sigma} = 3 = 2(1) + 1 \quad \text{(spin-1)}. $$

Note that the relations (B.50) don’t hold for the operators $P^0_{sw}, P^0_{ws}, P^1_{me}, P^1_{em}$ because they are not projectors as we have already pointed out.

The relations (B.50) tell us that (B.34) corresponds to the decomposition of a symmetric two-rank tensor in terms of one spin-2, one spin-1 and two spin-0 components under the rotation group $SO(3)$, i.e. $h^{\mu\nu} \in 0 \oplus 1 \oplus 0 \oplus 2$; while (B.41) corresponds to the decomposition of an antisymmetric tensor in terms of two spin-1 components, i.e. $h^{\mu\nu} \in 1 \oplus 1$ (see the Subsection B.1.2).

Remark 6. The basis (B.46) is important, for instance, when we want to determine the propagator associated to any Lagrangian. We have used only the symmetric space in this thesis, but in general we can have a general Lagrangian that required the use of the complete basis containing also the antisymmetric operators [36]. Thus in the general case, to invert the operators $O$ of any Lagrangian (see Chapter 2) we need to expand it in terms of the symmetric and the antisymmetric spin projector operators, and to invert the field equations we have to act with both kind of operators, i.e. with the full basis (B.46). In formula, the equations (B.20) and (B.21) can be recast in terms of the spin projector operators as follow. First of all let us recall the full set of the projectors as

$$ O^i \equiv \{ P^2, P^1_m, P^0_s, P^0_w, P^1_b, P^1_e, P^0_{sw}, P^0_{ws}, P^1_{sw}, P^1_{sw} \}, \quad i = 1, \ldots, 10. \quad (B.51) $$

As we have already seen considering symmetric and antisymmetric decompositions, the identity matrix in the symmetric (antisymmetric) case can be rewritten as $\frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho})$ ($\frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho})$).
B.2. SPIN PROJECTOR OPERATORS

We are now able to expand the operator $O$ in a compact way:

$$\mathcal{L} = \frac{1}{2} \varphi_{\mu\nu} O^{\mu\nu\rho\sigma} \varphi_{\rho\sigma}$$

$$= \frac{1}{2} \varphi_{\mu\nu} \left( \sum_{i=1}^{10} C_i O^{i,\mu\nu\rho\sigma} \right) \varphi_{\rho\sigma}, \quad (B.52)$$

or, in other words, we can say that the operator space of the field equations can be spanned as

$$O^{\mu\nu\rho\sigma} \varphi_{\rho\sigma} = \lambda J^{\mu\nu}$$

$$\Leftrightarrow \left( \sum_{i=1}^{10} C_i O^{i,\mu\nu\rho\sigma} \right) \varphi_{\rho\sigma} = \lambda \left( \sum_{i=1}^{6} C_i O^{i,\mu\nu\rho\sigma} \right) J_{\rho\sigma}, \quad (B.53)$$

where the coefficients $C_i$ are defined by the specific Lagrangian we are considering.

Now, for instance, one could determine the propagator for the generic Lagrangian (B.52) by following the prescription we introduced in Chapter 2. This case would be more general because the Lagrangian is neither symmetric or antisymmetric, but it has both parts. See Ref. [36] for more details.

**Useful relations**

Now we want to list some important relations between spin projector operators, among which some of them turned out to be very useful to rewrite the Lagrangians and the field equations in terms of the spin projector operators. These relations are:

$$\frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) = (P^2 + P^1 + P^0_s + P^0_w)_{\mu\nu\rho\sigma},$$

$$\eta_{\mu\nu} \omega_{\rho\sigma} + \eta_{\rho\sigma} \omega_{\mu\nu} = \sqrt{3} (P^0_{sw} + P^0_{ws}) + 2P^0_w)_{\mu\nu\rho\sigma},$$

$$\frac{1}{2} (\eta_{\mu\rho} \omega_{\nu\sigma} + \eta_{\mu\sigma} \omega_{\nu\rho} + \eta_{\nu\rho} \omega_{\mu\sigma} + \eta_{\nu\sigma} \omega_{\mu\rho}) = (P^1 + 2P^0_w)_{\mu\nu\rho\sigma},$$

$$\eta_{\mu\nu} \eta_{\rho\sigma} = (3P^0_s + P^0_w + \sqrt{3} (P^0_{sw} + P^0_{ws}))_{\mu\nu\rho\sigma},$$

$$P^2_{\mu\nu\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) - \frac{1}{3} \eta_{\mu\nu} \eta_{\rho\sigma} - \left[ P^1_s + \frac{2}{3} P^0_w - \frac{1}{\sqrt{3}} (P^0_{sw} + P^0_{ws}) \right]_{\mu\nu\rho\sigma},$$

$$P^0_{s,\mu\nu\rho\sigma} = \frac{1}{3} \eta_{\mu\nu} \eta_{\rho\sigma} - \frac{1}{3} \left[ P^0_w + \sqrt{3} (P^0_{sw} + P^0_{ws}) \right]_{\mu\nu\rho\sigma}.$$
B.3 Examples of metric decompositions

In this section we shall see examples of tensor decomposition, in particular the metric perturbation decomposition, so we shall deal only with symmetric tensor.

Let us find the decomposition for the choice, (B.30) and (B.35), we have made for the complete set of spin projector operators. We can note that the following metric decomposition can be written in terms of our set of spin projector operators matching metric components with spin projector components. This metric decomposition is

\[ h_{\mu\nu} = h_{\mu\nu}^{TT} + \frac{1}{2} \left( \partial_{\mu} \xi_{\nu}^T + \partial_{\nu} \xi_{\mu}^T \right) + \left( \nabla^2 \eta_{\mu\nu} - \partial_{\mu} \partial_{\nu} \right) s + \partial_{\mu} \partial_{\nu} w, \]  

with \( h_{\mu\nu}^{TT} \) transverse and traceless with respect to the Lorentz indices,

\[ \partial^\mu h_{\mu\nu}^{TT} = 0, \quad \eta^{\mu\nu} h_{\mu\nu}^{TT} = 0; \]  

and \( \partial^\mu \xi_{\mu}^T = 0. \)

To match with our choice of spin projector operators (B.30) and (B.35) we have to choose:

\[ \xi_{\mu}^T := 2 \left( \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\nabla^2} \right) \partial_{\sigma} h^{\nu\sigma}, \]

\[ s := \frac{1}{3} \left( \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\nabla^2} \right) h^{\mu\nu}, \]

\[ w := \frac{\partial_{\mu} \partial_{\nu}}{\nabla^2} h^{\mu\nu}; \]  

\[ h_{\mu\nu}^{TT} := h_{\mu\nu} - \frac{\partial_{\mu}}{\nabla^2} \left( \eta_{\nu\rho} - \frac{\partial_{\nu} \partial_{\rho}}{\nabla^2} \right) \partial_{\sigma} h^{\rho\sigma} - \frac{\partial_{\nu}}{\nabla^2} \left( \eta_{\mu\rho} - \frac{\partial_{\mu} \partial_{\rho}}{\nabla^2} \right) \partial_{\sigma} h^{\rho\sigma} - \frac{1}{3} \left( \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\nabla^2} \right) \left( \eta_{\rho\sigma} - \frac{\partial_{\rho} \partial_{\sigma}}{\nabla^2} \right) h^{\rho\sigma} - \partial_{\mu} \partial_{\nu} \frac{\partial_{\rho} \partial_{\sigma}}{\nabla^2} h^{\rho\sigma}. \]

Let us rewrite the equations (B.55)-(B.57) in momentum space:

\[ h_{\mu\nu} = h_{\mu\nu}^{TT} + \frac{i}{2} \left( k_{\mu} \xi_{\nu}^T + k_{\nu} \xi_{\mu}^T \right) + \left( -k^2 \eta_{\mu\nu} + k_{\mu} k_{\nu} \right) s - k_{\mu} k_{\nu} w; \]  

where \( k_{\mu} \) and \( k_{\nu} \) are the momentum components.
B.3. EXAMPLES OF METRIC DECOMPOSITIONS

\[ \xi^T_\mu = -\frac{2i}{k^2} \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) k_\sigma h^\nu_\sigma, \]

\[ s = -\frac{1}{3} \frac{1}{k^2} \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) h_{\mu\nu}, \]

\[ w = \frac{k_\mu k_\nu}{k^2} h_{\mu\nu}, \]

\[ h^{TT}_{\mu\nu} = h_{\mu\nu} - \frac{k_\mu}{k^2} \left( \eta_{\nu\rho} - \frac{k_\nu k_\rho}{k^2} \right) k_\sigma h^\rho_\sigma - \frac{k_\nu}{k^2} \left( \eta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2} \right) k_\sigma h^\rho_\sigma \]

\[ + \frac{1}{3} \left( -\frac{k^2}{k^2} \eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) \left( \eta_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2} \right) h^\rho_\sigma - \frac{k_\mu k_\nu k_\rho k_\sigma}{k^2} h^\rho_\sigma, \]

Now we shall identify each term in \( h_{\mu\nu} \) with the corresponding spin projector operator.

\[ \frac{1}{2} \left( \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu \right) \rightarrow \frac{k_\mu}{k^2} \left( \eta_{\nu\rho} - \frac{k_\nu k_\rho}{k^2} \right) k_\sigma h^\rho_\sigma + \frac{k_\nu}{k^2} \left( \eta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2} \right) k_\sigma h^\rho_\sigma \]

\[ = (\theta_\mu \omega_{\nu\sigma} + \theta_\nu \omega_{\mu\sigma}) h^\rho_\sigma \]

\[ = \frac{1}{2} (\theta_\mu \omega_{\nu\sigma} + \theta_\nu \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma}) h^\rho_\sigma \]

\[ = \mathcal{P}^{1}_{\mu\nu\rho\sigma} h^\rho_\sigma; \]

\[ (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) s \rightarrow -\frac{1}{3} \left( -\frac{k^2}{k^2} \eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) \left( \eta_{\rho\sigma} - \frac{k_\rho k_\sigma}{k^2} \right) h^\rho_\sigma \]

\[ = \frac{1}{3} \theta_\mu \theta_{\rho\sigma} h^\rho_\sigma \]

\[ = \mathcal{P}^{0}_{s,\mu\nu\rho\sigma} h^\rho_\sigma; \]

\[ \partial_\mu \partial_\nu r \rightarrow \frac{k_\mu k_\nu k_\rho k_\sigma}{k^2} h^\rho_\sigma = \omega_{\mu\nu} \omega_{\rho\sigma} h^\rho_\sigma = \mathcal{P}^{0}_{w,\mu\nu\rho\sigma} h^\rho_\sigma; \]

and as for \( h^{TT}_{\mu\nu} \), of course, we have

\[ h^{TT}_{\mu\nu} \rightarrow \mathcal{P}^{2}_{\mu\nu\rho\sigma} h^\rho_\sigma. \]

Hence we have decomposed the metric perturbation (B.55) in terms of the spin projection operators and we matched each of them with the corresponding scalar, vector and tensor components:

\[ \mathcal{P}^{2} \leftrightarrow h^{TT}_{\mu\nu}, \quad \mathcal{P}^{1} \leftrightarrow \xi^T_\mu, \quad \mathcal{P}^{0}_{s} \leftrightarrow s, \quad \mathcal{P}^{0}_{w} \leftrightarrow w. \]
Let us consider another example of decomposition:

\[
h_{\mu\nu} = h_{\mu\nu}^{TT} + \frac{1}{2} \left( \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu \right) + \partial_\mu \partial_\nu \alpha + \frac{1}{3} \eta_{\mu\nu} \beta,
\]

where \(h_{\mu\nu}^{TT}\) is always traceless and transverse and \(\xi^T_\mu\) transverse. The definition of \(\xi^T_\mu\) is the same of the previous example as in (B.57), instead the two scalar components are defined as

\[
\alpha := \left( \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \right) h_{\mu\nu},
\]

\[
\beta := -\frac{1}{3} \left( \eta_{\mu\nu} - 4 \frac{\partial_\mu \partial_\nu}{\Box} \right) h_{\mu\nu}.
\]

By following the procedure showed in the first example we can obtain

\[
h_{\mu\nu}^{TT} \to P^2_{\mu\nu\rho\sigma} h^{\rho\sigma}, \quad \frac{1}{2} \left( \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu \right) \to P^1_{\mu\nu\rho\sigma} h^{\rho\sigma},
\]

\[
\partial_\mu \partial_\nu \alpha \to \left( P^0_w - \frac{\sqrt{3}}{3} P^0_{ws} \right)_{\mu\nu\rho\sigma} h^{\rho\sigma}, \quad \frac{1}{3} \eta_{\mu\nu} \beta \to \left( P^0_s + \frac{\sqrt{3}}{3} P^0_{ws} \right)_{\mu\nu\rho\sigma} h^{\rho\sigma}.
\]

We can also have a metric perturbation decomposition in which one of the two scalar components corresponds to the trace \(h\) of \(h_{\mu\nu}\), and in this case we need to know how to write the trace in terms of the spin projector operators. One can easily check that the trace assume the following form

\[
h = \eta^{\mu\nu} \left( P^0_s + P^0_w \right)_{\mu\nu\rho\sigma} h^{\rho\sigma}.
\]
Appendix C

Ghosts and unitarity

In this appendix we want to define and show the validity of some tools we used in this thesis to verify whether the unitarity is preserved. We saw that a necessary condition for a theory to be unitary is that (bad) ghosts are absent. To check that ghosts field doesn’t violate the unitarity condition we made use of a prescription by which we verified the positivity of the imaginary part of the residue current-current amplitude. First of all we are going to define the concept of unitarity, then to define ghost fields and at the end to show why the prescription we used is valid.

C.1 Unitarity condition

In a quantum theory, we expect that the sum of all probabilities is equal to 1. This means that the norm of a state $|s\rangle$ at time $t = 0$ should be the same at a later time $t$:

$$\langle s, t = 0 | s, t = 0 \rangle = \langle s, t | s, t \rangle .$$  \hspace{1cm} (C.1)

This means that the Hamiltonian should be Hermitian, $H^\dagger = H$, because

$$|s, t\rangle = e^{iHt} |s, t = 0\rangle ,$$  \hspace{1cm} (C.2)

but it also means that $S$–matrix should be unitary, since by definition one has $S = e^{iHt}$. Thus

$$\langle s, t | s, t \rangle = \langle s, t = 0 | e^{-iHt} e^{iHt} | s, t = 0 \rangle = \langle s, t = 0 | S^\dagger S | s, t = 0 \rangle$$

$$\Rightarrow S^\dagger S = 1 .$$  \hspace{1cm} (C.3)

The unitarity of the $S$–matrix is equivalent to conservation of probability, which seems to be a property of our universe as far as anyone can tell.
C.1. UNITARITY CONDITION

Optical theorem

One of the most important implications of unitarity is the optical theorem that we are going to discuss below.

Let us write the $S-$matrix as

$$ S = 1 + iT, $$

where $T$ is called transfer matrix and its elements are defined as

$$ \langle f | T | i \rangle = (2\pi)^4 \delta^4(p_f - p_i)\mathcal{M}(i \rightarrow f), $$

with $\mathcal{M}(i \rightarrow f)$ scattering amplitude. The matrix $T$ is not hermitian, in fact from (C.4) we have

$$ 1 = S^\dagger S = (1 - iT^\dagger)(1 + iT) $$

$$ \Rightarrow i (T^\dagger - T) = T^\dagger T, $$

that is an equivalent form to express the unitarity condition.

Let us now introduce the following orthonormal and complete set of states $|n\rangle$:

$$ \langle n | m \rangle = \delta_{nm}, \sum_n |n\rangle\langle n| = 1. $$

We can also write each state $|n\rangle$ in terms of the momenta $k_i$ of the particles in it, so in this way the completeness relation reads as

$$ 1 = \sum_n \int d\Pi_n |n\rangle\langle n| $$

$$ = \sum_n \prod_{j \in n} \int \frac{dk_j}{(2\pi)^3 2E_j} \frac{1}{|k_1, k_2, \ldots, k_n\rangle\langle k_1, k_2, \ldots, k_n|}. $$

The left-side of (C.5) is\(^2\)

$$ \langle f | i(T^\dagger - T)| i \rangle = i(2\pi)^4 \delta^4(p_f - p_i) \left( \mathcal{M}^\dagger(i \rightarrow f) - \mathcal{M}(i \rightarrow f) \right); $$

instead, the left side, by using the relation (C.7), reads as

$$ \langle f | T^\dagger T | i \rangle = \sum_n \int d\Pi_n \langle f | T^\dagger | n \rangle \langle n | T | i \rangle = $$

\(^1\)See Ref. [1] for more details.
\(^2\)Don’t get confused because of the presence of two “$i$” : one is the imaginary unit and the other one represents a generic initial state!
\[
\sum_n \hat{d}_n \Pi_n M^\dagger(n \rightarrow f) M(i \rightarrow n). \tag{C.9}
\]

Then the unitarity condition (C.5) implies
\[
\mathcal{M}^\dagger(i \rightarrow f) - \mathcal{M}(i \rightarrow f) = -i \sum_n d\Pi_n (2\pi)^4 \delta^4(p_n - p_i) \mathcal{M}^\dagger(n \rightarrow f) M(i \rightarrow n), \tag{C.10}
\]
that represents the generalized optical theorem.

Note that the relation (C.10) must work order by order in perturbation theory. But while the left hand side is matrix elements, the right hand side is matrix elements squared. This means that at order $\lambda^2$ in some coupling the left hand side must be a loop to match a tree-level calculation on the right hand side. Thus, the imaginary parts of loop amplitudes are determined completely by tree-level amplitudes. In particular, we must have the loops otherwise without loops unitarity would be violated.

To the extent that trees represent classical physics and loops represent quantum effects, the optical theorem implies that the quantum theory is uniquely determined by the classical theory because of unitarity [1].

The most important case is when $|f\rangle = |i\rangle = |X\rangle$ for some state $X$. In particular, when $|X\rangle$ is a 1-particle state, (C.10) becomes
\[
\text{Im}\mathcal{M}(X \rightarrow X) = m_X \sum_n \Gamma(X \rightarrow n). \tag{C.11}
\]

Here $\mathcal{M}(X \rightarrow X)$ is a 2-point function, i.e. a propagator. So (C.11) says that the imaginary part of the propagator is equal to the sum of the decay rates into every possible particle.

If $|X\rangle$ is a 2-particle state, then (C.10) becomes
\[
\text{Im}\mathcal{M}(X \rightarrow X) = 2E_{CM} |\vec{p}_{CM}| \sum_n \sigma(X \rightarrow n). \tag{C.12}
\]

This says that the imaginary part of the forward scattering amplitude is proportional to the total cross section, which is the optical theorem from optics.

## C.2 Ghost fields

In this section we define a ghost field and we shall see what its presence implies both at the classical level and at the quantum level. We will only consider a scalar field for simplicity but the results also hold for vector and tensor field. We shall follow Ref. [56] in the first subsection.
Let us consider the following Lagrangian:

\[ \mathcal{L} = \frac{a}{2} \partial_\mu \phi \partial^\mu \phi - \frac{b}{2} m^2 \phi^2, \]

where \( a = \pm 1 \) and \( b = \pm \). The momentum conjugate to \( \phi \) is defined by

\[ \pi := \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} = a \partial_0 \phi \equiv a \dot{\phi}, \]

We can obtain the Hamiltonian density by performing the following Legendre transformation

\[ \mathcal{H} = \pi \dot{\phi} - \mathcal{L} \]

\[ = \pi \dot{\phi} - \frac{a}{2} \partial_\mu \phi \partial^\mu \phi + \frac{b}{2} m^2 \phi^2 \]

\[ = \frac{a}{2} (\dot{\phi}^2 + (\nabla \phi)^2) + \frac{b}{2} m^2 \phi^2 \]

in terms of which the Hamiltonian is defined as

\[ H = \int d^3 x \left[ \frac{a}{2} (\dot{\phi}^2 + (\nabla \phi)^2) + \frac{b}{2} m^2 \phi^2 \right]. \]

Note that

- \( a = b = +1 \) : the Hamiltonian is positive semi-definite and therefore bounded from below;
- \( a = b = -1 \) : the Hamiltonian is negative semi-definite and therefore bounded from above;
- \( a = -b \) : the Hamiltonian is indefinite and so it is not bounded either from below or from above.

If \( a = b = -1 \), the field \( \phi \) is called ghost field, if \( a = +1 \) and \( b = -1 \) one has a tachyon field, finally if \( a = -1 \) and \( b = +1 \) it is called tachyonic ghost field. More generally one gives the following definition:

“A ghost field is defined as a field which has negative kinetic energy”.

In the next two subsections we shall see what happens if ghosts are present in the Lagrangian. We shall see that the presence of ghost has implications either if we perform a
C.2. GHOST FIELDS

classical study or if we study the quantum aspects of the theory. Indeed, at the classical level the presence of ghost could cause instabilities of the vacuum since the energy is not bounded from below, while at the quantum level ghosts correspond to states with negative norm and they could violate the unitarity condition. However, the presence of ghost does not always violate fundamental principles. In fact, we have to distinguish good ghosts from bad ghosts. A good ghost doesn’t violate any fundamental principles since they never appear as observable physical state; while a bad ghost does violate fundamental principles because it is associated with physical particles.

C.2.1 Ghosts at the classical level

If a Hamiltonian is unbounded from below (like in the cases $a = b = +1$ and $a = -b$) instabilities can emerge in the system. However, if a ghost field $\phi$ is free, namely if interactions are absent, one can easily see that the system will be still stable as the energy is conserved, independently of its sign. In fact, any constant that multiplies a (classical) Lagrangian does not change the physics, since it does not appear in the equations of motion. Thus, the choices $a = b = +1$ and $a = b = -1$ are completely equivalent at the classical level. Translating the previous words in formula, the field equation for the Lagrangian (C.3) is given by

$$ (a \Box + b m^2) \phi(x) = 0. \quad (C.17) $$

So, if we take the cases $a = b = +1$ and $a = b = -1$ we notice that the field equations are the same.

In momentum space the last equation becomes

$$ (-a k^2 + b m^2) \phi(k) = 0 \Rightarrow a(k^0)^2 - a \bar{k}^2 = bm^2 > 0. \quad (C.18) $$

We can easily verify what we have already said above, namely that for $a = -b$ we have also a tachyonic solution. In fact (C.18) gives $(k^0)^2 - \bar{k}^2 = -m^2 > 0$ that implies $m^2 < 0$, i.e. complex masses.

If we consider the following Fourier decomposition

$$ \phi(\bar{x}, t) = \int \frac{d^3k}{(2\pi)^3} \phi_k(t) e^{i \bar{k} \cdot \bar{x}}, \quad (C.19) $$

we obtain that, for $a = b = \pm 1$, every mode $\phi_k(t)$ evolves independently from the others and satisfies the equation

$$ \ddot{\phi}_k(t) + (m^2 + \bar{k}^2) \phi_k(t) = 0, \quad (C.20) $$

which exhibits oscillatory solutions of frequency given by $\omega(\bar{k}) = \sqrt{m^2 + \bar{k}^2}$. A small perturbation at $t = t_0$ from the configuration $\phi = 0$ is described by small Fourier coefficients $\phi_k(t_0)$, and the oscillatory behavior ensures that there is no exponential enhancement, i.e. the perturbation remains small for $t > t_0$. Instead, if $a = -b$, the frequency
\[ \omega(\vec{k}) = \sqrt{\vec{k}^2 - m^2} \] turns out to be imaginary when \( \vec{k}^2 < m^2 \) holds and so the Fourier modes suffer from an exponential growth, implying the presence of an instability in the theory. However, the situation changes if there is interaction, and so energy exchanges, between a ghost field and a normal (non-ghost) field.

Let us, in fact, consider the following interacting Lagrangian
\[ \mathcal{L} = \frac{a}{2} \partial_\mu \phi \partial^\mu \phi - \frac{a}{2} m_\phi^2 \phi^2 + \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} m_\psi \psi^2 - V_{\text{int}}(\phi, \psi), \] (C.21)

where for hypothesis the potential \( V_{\text{int}} \) does not contain derivative interaction terms, but only depends on the two fields, and admits the solution \( \phi = \psi = 0 \) as a local minimum. Performing the Legendre transformation we obtain the Hamiltonian
\[ \mathcal{H} = \frac{a}{2} \left( \dot{\phi}^2 + (\nabla \phi)^2 \right) + \frac{a}{2} m_\phi^2 \phi^2 + \frac{1}{2} \left( \dot{\psi}^2 + (\nabla \psi)^2 \right) + \frac{1}{2} m_\psi \psi^2 + V_{\text{int}}(\phi, \psi). \] (C.22)

First, note that, if \( V_{\text{int}}(\phi, \psi) = 0 \), the state \( \phi = \psi = 0 \) is still stable independently from the sign of \( a \). The stability is preserved as the conservation energy law can be applied separately for the two non-interacting fields \( \phi \) and \( \psi \). However, we have to point out that already at this level there is a difference between the cases \( a = +1 \) and \( a = -1 \). Although the system is stable, the choice \( a = -1 \) corresponds to an infinite number of different states with \( E = 0 \) which cannot be associated to small perturbations of the vacuum (minimum) \( \phi = \psi = 0 \).

Secondly, in the case \( V_{\text{int}}(\phi, \psi) \neq 0 \), the minimum configuration is still a solution of the field equation and one can show that the Hamiltonian can be bounded from below for constant values of the dynamical fields. Note that, since now the interaction term is not vanishing, the configurations cannot have zero energy anymore. However, by perturbing the vacuum configuration, one can construct states with energy values very close to zero. Therefore, if \( a = -1 \), the available volume of the momentum space turns out to be infinite with an infinite number of excited states. Thus, since for entropy reasons the total energy is redistributed into the largest possible class of states, the decay towards these excited states is extremely favoured, and this can be summarized saying that the system is unstable for small oscillations\(^3\).

As we shall see in the section C.3, a Lagrangians of the type (C.21) with \( a = -1 \) is equivalent to a Lagrangian containing higher derivatives. In 1850, Ostrogradsky demonstrated a theorem in which he states that the Hamiltonian of a non-degenerate higher derivative theory is unbounded from below, and also from above [57], so instabilities are present. For this reason, the classical instability due to the presence of a negative kinetic term, that we have treated in this subsection, is called Ostrogradskian instability.

\(^3\)See Ref. [56] for more details.
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C.2.2 Ghosts at the quantum level

We have seen that instability problems are associated to the presence of a ghost at the classical level. At the quantum level the presence of a (bad) ghost is even more problematic. Let us just consider the Lagrangian (C.3) with the ghost choice for the coefficients, \( a = b = -1 \):

\[
L = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} m^2 \phi^2 = \frac{1}{2} \phi (\Box + m^2) \phi.
\]  

(C.23)

We can see that for a ghost field the propagator in momentum space is given by

\[
P(k) = -\frac{1}{k^2 - m^2},
\]

(C.24)

i.e. it turns out to have a minus sign of difference with respect to an ordinary field, thus its residue is negative\(^4\).

To quantize the (ghost) scalar field theory we need to impose the following commutation relations:

\[
\begin{align*}
[\phi(\vec{x}, t), \pi(\vec{x}', t)] &= i\delta^3(\vec{x} - \vec{x}'), \\
[\phi(\vec{x}, t), \phi(\vec{x}', t)] &= 0, \\
[\pi(\vec{x}, t), \pi(\vec{x}', t)] &= 0,
\end{align*}
\]

(C.25)

where

\[
\phi(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \left( a_k e^{i(k\cdot\vec{x} - \omega_k t)} + a_k^\dagger e^{-i(k\cdot\vec{x} - \omega_k t)} \right)
\]

(C.26)

and

\[
\pi(\vec{x}, t) := \frac{\partial L}{\partial \dot{\phi}} = -\dot{\phi}.
\]

(C.27)

The coefficients \( a_k \) and \( a_k^\dagger \) are the usual annihilation and creation operators, respectively.

Because of the minus sign in the definition of the conjugate momentum (C.27) and to have consistency with the commutation relations (C.25), the commutation relations for the annihilation and creation operators must be

\[
\begin{align*}
[a_k, a_{k'}^\dagger] &= -\delta^3(\vec{k} - \vec{k}'), \\
[a_k, a_{k'}] &= 0, \\
[a_k, a_{k'}] &= 0.
\end{align*}
\]

(C.28)

\(^4\)Often the negativity of the propagator residue is taken to define a ghost field. Furthermore, some authors define the propagator including the imaginary unit “\(i\)” and so they define a ghost field in terms of the imaginary part of its residue.
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The problem of states with negative norms associated to the presence of a ghost field becomes evident if we try to construct the Fock space for the ghost scalar field. Let us assume that a normalized vacuum state $|0\rangle$ exists which has the property

$$a_k |0\rangle = 0 \quad \forall k, \quad \langle 0|0 \rangle = 1.$$  \hfill (C.29)

As usual the eigenstates of the occupation number can be constructed by applying the creation operators $a_k^\dagger$ to the vacuum. The state vector containing $n_k$ ghost fields reads

$$|n_k\rangle = \frac{1}{\sqrt{n_k!}} \left(a_k^\dagger\right)^{n_k} |0\rangle.$$  \hfill (C.30)

Now, if we calculate the norm of the one-particle state $|1_k\rangle$, we obtain

$$\langle 1_k|1_k \rangle = \langle 0|a_k a_k^\dagger|0 \rangle = \langle 0| -\delta^3(\vec{k} - \vec{k}) + a_k^\dagger a_k |0 \rangle \quad \Rightarrow \quad \langle 0|0 \rangle < 0,$$

where we have used the commutation relation (C.28). Hence, we have just showed that the state with one ghost field has negative norm.

Another example of ghost field one has in Electrodynamics where the time component of the four-vector $A_\mu$ has a negative kinetic term $g_{mn}$. Also in this case one can show that state associated to the scalar time component have negative norm caused by a minus sign in the definition of the commutation relation for the 0-component. In Electrodynamics these ghost field states don’t appear as physical states, but they are very important because are necessary to cancel out the longitudinal component of the vector field. Since it doesn’t violate the unitarity condition is a good ghost. We have also seen an example of good ghost in GR, where we have a scalar graviton component that behave as a ghost. In gauge quantum field theory we have another important example of good ghost, the Faddev-Popov ghost. It was firstly introduced to maintain the consistency between gauge invariance and path integral formulation. Secondly, as Feynman noticed, it is very necessary to preserve unitarity, in fact its absence wouldn’t satisfy the optical theorem that is an implication of the unitarity condition. It is remarkable, doubtless of profound significance, that good ghosts solve, simultaneously, the problem of unitarity and gauge invariance [55].

Instead, an example of bad ghost is given by the Pauli-Villars ghost that they introduce to define a regulation scheme to solve the divergence problem in quantum field theory. They add a particle with very large mass $M$ whose propagator has a minus sign:

$$\mathcal{P}_P-V(k) = -\frac{1}{k^2 - M^2}.$$  

One can show that the presence of this propagator in quantum loop doesn’t preserve the unitarity, in fact the optical theorem is not satisfied.

Then, we have also already encountered another example of bad ghost in subsection 3.4.3, i.e the Weyl ghost (see eq. (3.56), (3.58) and also Appendix C.4.).
C.2.3 Ghost-unitarity analysis

So far in this appendix we have introduced the concept of unitarity and the optical theorem, see as one of the unitarity implications, and defined the concept of ghost field analyzing its classical and quantum nature. We learned that good ghosts preserve the unitarity condition and that bad ghosts violate it. Now, we are going to introduce a method by which we can verify whether the presence of a ghost preserve the unitarity or not, namely whether it is a good or a bad ghost [38], [59], [58]. To reach our aim we need to work with the path integral formulation.

From the path integral formulation of the quantum field theory we know that the vacuum-vacuum transition amplitude in the presence of a source $J$ corresponds to the generating function $Z_0[J]$, where the subscript 0 means that we are dealing with the free theory. As we can see in many books of quantum field theory, like [1] and [55], one can shows that the vacuum-vacuum amplitude is given by

$$Z_0[J] = \langle 0, \infty | 0, -\infty \rangle^J = \exp \left\{ i \int d^4x \int d^4y \frac{1}{2} J(x) \mathcal{P}(x - y) J(y) \right\}, \quad (C.32)$$

with the normalization choice $Z_0[0] = 1$.

Now, we want to recast the integrals in (C.32) in terms of integrals on the momentum $k$. To do this we need to rewrite the integrand as Fourier transforms:

$$\mathcal{P}(x - y) = \int \frac{d^4k}{(2\pi)^4} \mathcal{P}(k) e^{ik \cdot (x - y)} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} e^{ik \cdot (x - y)},$$

$$J(x) = \int \frac{d^4k_1}{(2\pi)^4} J(k_1) e^{ik_1 \cdot x}, \quad (C.33)$$

$$J(y) = \int \frac{d^4k_2}{(2\pi)^4} J(k_2) e^{ik_2 \cdot y}.$$  

Thus, the integral in (C.32) becomes:

$$\int d^4x \int d^4y J(x) \mathcal{P}(x - y) J(y) =$$

$$= \int \frac{d^4k}{(2\pi)^4} \int d^4k_1 \int d^4k_2 J(k_1) \mathcal{P}(k) J(k_2) \left[ \int \frac{d^4x}{(2\pi)^4} e^{ix \cdot (k + k_1)} \right] \left[ \int \frac{d^4y}{(2\pi)^4} e^{-iy \cdot (k - k_2)} \right]. \quad (C.34)$$

By using the integral representation in momentum space of the delta function in four dimensions

$$\delta^4(k + k_1) = \int \frac{d^4x}{(2\pi)^4} e^{-i(k + k_1) \cdot x}, \quad \delta^4(k - k_2) = \int \frac{d^4y}{(2\pi)^4} e^{-i(k - k_2) \cdot y}, \quad (C.35)$$
C.2. GHOST FIELDS

we obtain

\[ \int d^4x \int d^4y J(x) \mathcal{P}(x - y) J(y) \sim \int d^4k J(-k) \mathcal{P}(k) J(k). \]  

(C.36)

Thus, the vacuum-vacuum amplitude in the presence of a source \( J \) becomes

\[ \langle 0, \infty | 0, -\infty \rangle^J \sim \exp \left\{ i \int d^4k J(-k) \mathcal{P}(k) J(k) \right\}. \]  

(C.37)

Note that we can decompose the integral on the momentum \( k \) as

\[ i \int d^4k J(-k) \mathcal{P}(k) J(k) = \int d^3k \left[ \int dk_0 i J(-k) \mathcal{P}(k) J(k) \right] \]  

(C.38)

and we can calculate the integral on the time component \( k_0 \) by using the residue theorem of Cauchy:

\[ \int d^4k J(-k) \mathcal{P}(k) J(k) = \int d^3k \left[ 2\pi i \text{Res} \left\{ iJ(-k) \mathcal{P}(k) J(k) \right\}_{k^2 = 0} \right]. \]  

(C.39)

Hence, the equation (C.37) can be recast in the following way

\[ \langle 0, \infty | 0, -\infty \rangle^J \sim \exp \left\{ \int d^3k \left[ 2\pi i \text{Res} \left\{ iJ(-k) \mathcal{P}(k) J(k) \right\}_{k^2 = m^2} \right] \right\}. \]  

(C.40)

Let us note that the integrand represents the current-current amplitude in momentum space: \( \mathcal{A}(k) = iJ(-k) \mathcal{P}(k) J(k) \). We can easily notice that the sign of the imaginary part of the residue is crucial: if it is positive we obtain a negative exponent but if it is negative the exponential is positive giving a vacuum-vacuum amplitude greater than 1. The quantity \( \langle 0, \infty | 0, -\infty \rangle^J \) is the transition amplitude to go from the initial state \( |0, -\infty \rangle \) to the final state \( |0, \infty \rangle \). The probability to find the system in the initial state is given by \( P \) and it has to be less than 1, \( P < 1 \); while the probability to transit to the final state is given by \( 1 - P \) and it has to be less than one too. We notice that if the imaginary part of the amplitude residue in (C.40) gives a negative value we have that \( 1 - P > 1 \Rightarrow P < 0 \), i.e. as result we obtain negative probabilities that makes the theory non-unitary.

In this thesis we have applied this analysis to photon and graviton case and we verified that the unitarity is non violated.

We can conclude this section saying that we have found a method to verify whether the presence of ghosts violate the unitarity condition. The only thing we need to do is to check the sign of the imaginary part of the amplitude residue: if it is positive the unitarity condition is preserved; if it is negative the unitarity condition is violated.

---

5 We are ignoring the \( 2\pi \) factors.

6 In the equation (C.40) we have written \( k^2 = m^2 \) meaning that the residue is calculate at the pole \( m^2 \), but in this thesis we have considered massless photon and massless graviton with poles \( k^2 = 0 \).

7 Note that if the integrand \( 2\pi i \text{Res} \left\{ iJ(-k) \mathcal{P}(k) J(k) \right\}_{k^2 = m^2} \) is positive (negative), the integral \( \int d^3k \left[ 2\pi i \text{Res} \left\{ iJ(-k) \mathcal{P}(k) J(k) \right\}_{k^2 = m^2} \right] \) will be positive (negative) too.
C.3 Ghosts in higher derivative theories

We have already said many times that when one considers theories with higher derivatives ghost fields appear in the theory. The more important example is the Fourth Derivative Gravity in which the propagator has a double pole at $k^2 = 0$ suggesting the appearance of a ghost in the Hilbert space and signifying that either unitarity or positivity of the energy spectrum might be violated \[34\].

In this section we want to show that a Lagrangian with higher derivatives is equivalent to a Lagrangian with lower derivatives but with the presence of ghost fields. We shall do this just for a quadratic fourth derivative Lagrangian \[62\], \[63\].

Let us consider a scalar field with fourth derivative Lagrangian given by

\[
L = -\frac{1}{2} \phi \left( \Box + m^2 \right) \left( \Box + M^2 \right) \phi, \tag{C.41}
\]

where the masses $m$ and $M$ can be also equal to zero, but we are going to consider the massive case\(^8\) as in Ref. \[62\]. So, let us suppose $M > m$. Defining the following two new fields

\[
\psi_1 := \frac{(\Box + M^2) \phi}{\sqrt{(M^2 - m^2)}}, \quad \psi_2 := \frac{(\Box + m^2) \phi}{\sqrt{(M^2 - m^2)}}, \tag{C.42}
\]

the Lagrangian \(C.41\) can be rewritten as

\[
L = -\frac{1}{2} \psi_1 \left( \Box + m^2 \right) \psi_1 + \frac{1}{2} \psi_2 \left( \Box + M^2 \right) \psi_2, \tag{C.43}
\]

where the term corresponding to the field $\psi_2$ has the wrong sign, i.e. it is a ghost field. We are considering a theory without interaction but, naturally, the study can be extend to Lagrangians with interaction potential $V_{\text{int}}(\phi)$. We know that at the classical level if the potential is set to zero ghost fields don’t violate any fundamental principles. At the quantum level, on the other hand, one could be in trouble even in the absence of interaction, as can be seen by looking at the free field propagator for $\phi$. In momentum space, this is the inverse of a fourth order expression in $k$, \(\left((-k^2 + m^2)(-k^2 + M^2)\right)^{-1}\), which can be expanded as

\[
\mathcal{P}(k) = \frac{1}{(M^2 - m^2)} \left( \frac{1}{k^2 - m^2} - \frac{1}{k^2 - M^2} \right). \tag{C.44}
\]

This is just the difference of the propagators for $\psi_1$ and $\psi_2$. The important point, that we have already mentioned other times, is that the propagator for $\psi_2$ appears with a negative sign and it could violate the unitarity condition.

\(^8\)See Ref. \[63\] for a more rigorous treatment of both massless and massive cases.
C.4. FOURTH DERIVATIVE GRAVITY

We can also consider other cases, for example the case in which one has only one has a massless ordinary field and a massive ghost field with mass $m$. The starting Lagrangian, in this case, is given by

$$
\mathcal{L} = -\frac{1}{2} \phi \Box (\Box + m^2) \phi \quad (C.45)
$$

and it is equivalent to the following Lagrangian

$$
\mathcal{L} = -\frac{1}{2} \psi_1 \Box \psi_1 + \frac{1}{2} \psi_2 \Box (\Box + m^2) \psi_2, \quad (C.46)
$$

where $\psi_1$ is an ordinary scalar field with positive kinetic term, while $\psi_2$ a massive ghost field. The authors in Ref. [63] show that the Lagrangians (C.45) and (C.46) describe equivalent theories by showing that the respective generating functional are the same apart from a multiplicative factor if we impose that $J_1 = -J_2 = \frac{d}{m}$. They also do the same analysis for a gauge field theory.

Although we have neither gone into details nor been very rigorous, we have have seen why higher derivative terms in the Lagrangian correspond to ghost fields. In the next section we shall consider Higher (or Fourth) Derivative Gravity and shall show that it is affected by the presence of bad ghost in the spin-2 sector.

C.4 Fourth Derivative Gravity

The action for higher derivative gravity is:

$$
S = \int d^4x \sqrt{-g} \left[ -\mathcal{R} + \alpha \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \beta \mathcal{R}^2 \right]. \quad (C.47)
$$

The linearized action, quadratic in the perturbation $h_{\mu\nu}$ is given by (see eq. (3.17))

$$
S_q = -\int d^4x \left[ \frac{1}{2} h_{\mu\nu} \Box a(\Box) h^{\mu\nu} + h_{\mu}^{\sigma} b(\Box) \partial_\sigma \partial_\nu h^{\mu\nu} \\
+ h c(\Box) \partial_\mu \partial_\nu h^{\mu\nu} + \frac{1}{2} h \Box d(\Box) h + \frac{1}{2} h^{\lambda\sigma} f(\Box) \partial_\sigma \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} \right], \quad (C.48)
$$
where the coefficients (3.16) in this case are

\[ a(\Box) := 1 - \frac{1}{2} \alpha \Box, \]

\[ b(\Box) := -1 + \frac{1}{2} \alpha \Box, \]

\[ c(\Box) := 1 + 2 \beta \Box + \frac{1}{2} \alpha \Box, \] \hspace{1cm} (C.49)

\[ d(\Box) := -1 - 2 \beta \Box - \frac{1}{2} \alpha \Box, \]

\[ f(\Box) := -2 \beta - \alpha. \]

One can show that the physical part (gauge-independent) of the propagator in momentum space obtained in (3.44),

\[ \Pi(k) = \frac{\mathcal{P}^2}{ak^2} + \frac{\mathcal{P}_s^0}{(a - 3c)k^2}, \] \hspace{1cm} (C.50)

in the case of the action (C.49) (or (C.48)) assumes the following expression

\[ \Pi(k) = \frac{\mathcal{P}^2}{[1 + \frac{1}{2} \alpha k^2] k^2} + \frac{\mathcal{P}_s^0}{[-2 + (2 \alpha + 6 \beta) k^2] k^2}. \] \hspace{1cm} (C.51)

By playing with the fractions one can obtain the following form for the propagator:

\[ \Pi(k) = \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{\mathcal{P}_s^0}{2} \right) - \frac{\mathcal{P}^2}{k^2 - m_2^2} + \frac{1}{2} \frac{\mathcal{P}_s^0}{k^2 - m_0^2}, \] \hspace{1cm} (C.52)

where \( m_2 = -\left(\frac{1}{2} \alpha\right)^{-1} \) and \( m_0 = (\alpha + \beta)^{-1} \). Note that

\[ \Pi_{GR}(k) = \frac{1}{k^2} \left( \mathcal{P}^2 - \frac{\mathcal{P}_s^0}{2} \right) \] \hspace{1cm} (C.53)

is the GR graviton propagator corresponding to the Hilbert-Einstein linearized action (quadratic in \( h_{\mu \nu} \)) obtained in Chapter 2. While the second and the third terms in (C.52) correspond to a massive spin-2 ghost with mass \( m_2 \) and a normal massive scalar with mass \( m_0 \), respectively. We want to understand whether the presence of the massive spin-2 ghost violates the unitarity.

Let us consider the amplitude

\[ \mathcal{A} = i \tau^{\mu \nu}(k) \Pi(\kappa)_{\mu \nu \rho \sigma} \tau^{\rho \sigma}(k) = \mathcal{A}_{GR} + \mathcal{A}_2 + \mathcal{A}_0, \] \hspace{1cm} (C.54)
where
\[ A_{GR} = i\tau^{\mu\nu}(k)\Pi_{GR}(k)_{\mu\nu\rho\sigma}\tau^{\rho\sigma}(k), \]
\[ A_2 = -i\tau^{\mu\nu}(k)\frac{\mathcal{P}^2_{\mu\nu\rho\sigma}}{k^2 - m_2^2}\tau^{\rho\sigma}(k), \]
\[ A_0 = i\tau^{\mu\nu}(k)\frac{1}{2}\frac{\mathcal{P}^0_{s,\mu\nu\rho\sigma}}{k^2 - m_0^2}\tau^{\rho\sigma}(k). \]

We need to calculate the imaginary part of the residue of the full amplitude in (C.54), that corresponds to the sum of the residue of the three amplitudes in (C.55). By using the definition of the spin projector operators \( \mathcal{P}^2 \) and \( \mathcal{P}^0_s \) one can show that
\[ \text{ImRes}_{k^2=0}\{A_{GR}\} = |\tau^{\mu\nu}(0)|^2 - \frac{1}{2}|\tau(0)|^2 \]
\[ \text{ImRes}_{k^2=m_2^2}\{A_2\} = -\left(|\tau^{\mu\nu}(m_2)|^2 - \frac{1}{2}|\tau(m_2)|^2\right) \]
\[ \text{ImRes}_{k^2=m_0^2}\{A_0\} = \frac{1}{6}|\tau(m_0)|^2. \]

Now, let us consider the following expansion of the source \( \tau(k) \) already used in Chapter 2:
\[ \tau_{\mu\nu}(k) = a(k)k_\mu k_\nu + b(k)k_{(\mu}k_{\nu)} + c_i(k)k_{(\mu}\varepsilon_i^{\nu)} \]
\[ + d(k)\tilde{k}_\mu \tilde{k}_\nu + e_i(k)\tilde{k}_{(\mu}\varepsilon_i^{\nu)} + f_{ij}(k)\varepsilon_i^{\mu}(\varepsilon_j^{\nu)}, \]
where the basis of the expansion is \( \{k^\mu, \tilde{k}^\mu, \varepsilon_1^\mu, \varepsilon_2^\mu\} \), such that
\[ k^\mu \equiv (k^0, \tilde{k}), \quad \tilde{k}^\mu \equiv (\tilde{k}^0, -\tilde{k}), \quad \varepsilon_i^\mu \equiv (0, \varepsilon_i), \]
\[ k^\mu \varepsilon_{i,\mu} = 0 = \tilde{k}^\mu \varepsilon_{i,\mu}, \quad \varepsilon_i^\mu \varepsilon_{j,\mu} = -\varepsilon_i \cdot \varepsilon_j = -\delta_{ij}, \quad i = 1, 2. \]
C.4. FOURTH DERIVATIVE GRAVITY

One can show that
\[
\text{ImRes}^k_{k^2 = 0} \{ \mathcal{A}_{\text{GR}} \} = \left| f_{ij}(0) \right|^2 - \frac{1}{2} \left| f(0) \right|^2 = \frac{1}{2} |f_{11}(0) - f_{22}(0)|^2 + 2 |f_{12}(0)|^2 > 0
\]
\[
\text{ImRes}^k_{k^2 = m_2^2} \{ \mathcal{A}_2 \} = -\left[ \frac{2}{3} (a(m_2) - d(m_2))^2 m_2^4 + \frac{m_2^2}{2} (|c_i(m_2)|^2 - |e_i(m_2)|^2) \right.
\]
\[
+ |f_{ij}(m_2)|^2 - \frac{1}{2} |f(m_2)|^2 - \frac{2}{3} (a(m_2) - d(m_2)) m_2^2 f_{ii}(m_2) \right]
\]
\[
\text{ImRes}^k_{k^2 = m_0^2} \{ \mathcal{A}_0 \} = \frac{1}{6} \left[ (a(m_0) - d(m_0))^2 m_0^4 + |f_{ii}(m_0)|^2 \right.
\]
\[
- 2 (d(m_0) - a(m_0)) m_0^2 f_{ii}(m_0) \left| > 0 \right.
\]

As usual, we consider $\tau > 0$, thus $f_{ii} < 0$ [38]. Since the source $\tau_{\mu\nu}(k)$ is arbitrary, its Fourier modes (i.e. the coefficients $a(k), b(k), \ldots$) can be freely chosen and we can make choices such that only one of the three residues contributes at a time. We quickly notice that the massless and the massive scalar poles (first and second lines in (C.59)) are well defined physical state. While, if we choose the source $\tau_{\mu\nu}(k)$ such that only the pole $m_2^2$ contributes one can see that the second line in eq. (C.59) is not positive defined. For example, if $|c_i(m_2)|^2 - |e_i(m_2)|^2 > 0$ and $a(m_2) - d(m_2) > 0$ we get
\[
\text{ImRes}^k_{k^2 = m_2^2} \{ \mathcal{A}_2 \} < 0
\]
that violates the unitarity at the tree level.
We could imagine to make special choices for the coefficients in the source expansion (C.57) to obtain a positive value for the sum of the three residues and so a ghost-free theory\(^9\). But, in this way we would restrict $\tau_{\mu\nu}(k)$ by hand to get a ghost-free sum and it does not mean that the theory is healthy because interactions can always generate the $\tau_{\mu\nu}(k)$-configurations that have not been considered [64].

Hence, we have seen that Higher Derivative Gravity is not a healthy theory because of the presence of the massive spin-2 (bad) ghost that violates the unitarity at the tree level.

\(^9\)Keep in mind that with the nomenclature “ghost-free” we refer to a theory free from “bad” ghost.
Appendix D

Clebsch-Gordan coefficients

In the subsection 2.4.1 we calculate the graviton polarization tensors by starting from the photon polarization vectors. We saw every polarization tensor as the composition of two spin-1 polarization. In this way we obtained five polarization tensors of spin-2, three polarization tensors of spin-1 that we didn’t considered because of their antisymmetric nature, and one polarization tensor of spin-0. Since we have constructed the set of polarization tensors by composition of two spin-1 vectors, we made use of the table of Clebsch-Gordan coefficients.

Below we report the table of some Clebsch-Gordan coefficients. The composition \( 1 \otimes 1 \) we have used is indicated with red arrows.
Figure D.1: This Clebsch-Gordan coefficients table was taken from PDG (Particle Data Group) listings. The product $1 \otimes 1$ is the composition we are interested in and it is indicated by two red arrows.
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