ANALYTICAL ANALYSIS OF LYOT CORONAGRAPHS RESPONSE

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ABSTRACT

We derive an analytical solution to the computation of the output of a Lyot coronagraph for a given complex amplitude on the pupil plane. This solution, which does not require any simplifying assumption, relies on an expansion of the entrance complex amplitude on a Zernike base. According to this framework, the main contribution of the paper is the expression of the response of the coronagraph to a single base function. This result is illustrated by a computer simulation that describes the classical effect of propagation of a tip-tilt error in a coronagraph.

Subject headings: instrumentation: adaptive optics — techniques: high angular resolution

1. INTRODUCTION

The discovery of extrasolar planets is at the origin of a renewed interest in stellar coronagraphy. Considering the ambition of the targeted objectives, many authors have pointed out the necessity for a very accurate analysis of the system in order to study various undesired effects. For example, the specific properties of the light intensity measured by a system based on an extreme adaptive optics system and a coronagraph imply that neither the residuals of the turbulence nor the ideal coronagraphed point-spread function can be neglected with respect to the faint object (planet). Aime & Soummer (2004) analyzed the fact that the wave front amplitudes associated with these two contributions will interfere, leading to the so-called pinned speckles. Another example is given by Lloyd & Sivaramakrishnan (2005), who pointed out that a small misalignment of the star with the center of the stop can result in a fake source. A related problem is also present in Soummer (2005), which derives the optimal apodization for an arbitrary shaped aperture using an algorithm proposed independently in Guyon & Roddier (2000) that relies on iterated simulations of the coronagraph response.

More generally, an intense activity aims to optimize the different coronagraph parameters (mask size, apodization shape, etc.) for a number of projects dedicated to devising high dynamic range imaging on the VLT (Sphere), Gemini (GPI), or the Subaru telescope (HiCIAO; see, e.g., Aime & Vakili 2006). The input/output relation of a coronagraph is in this case simulated by numerical computations based on discrete Fourier transforms. However, such a numerical technique suffers from the well-known problems related to the choice of the extent of the sampled surface and the sampling frequency, which both define the sampling in the transformed domain. Note that this compromise is coupled with the difficulty of numerically evaluating the simulation errors.

This work focuses on the analytical characterization of the response of a Lyot coronagraph. The objective is obviously also to gain deeper insight into the behavior of the system. This problem has already been studied in the literature, and analytical results were obtained under various assumptions. In the one-dimensional case, Lloyd & Sivaramakrishnan (2005) assume that the Lyot stop is band limited and the phase on the telescope aperture is small. This last hypothesis is removed in Sivaramakrishnan et al. (2005), where the computation is carried out for a rectangular pupil assuming again that the Lyot stop is band limited. The development presented herein for a circular pupil differs from these approaches, substituting these simplifying assumptions with an expansion of the complex amplitude on an orthogonal basis.

Section 2 recalls the general formalism of Lyot coronagraphy and justifies the choice of an expansion of the complex amplitude on a Zernike base. Section 3 contains the main results of the paper; the response of the coronagraph to a Zernike polynomial is computed. The result involves an infinite sum. A bound on the truncation error is then derived. Section 4 presents two simulations. First, the response of the coronagraph to the first six Zernike functions is computed. Then the formalism derived in this paper is used to illustrate the effect of a tip-tilt error in a coronagraph. A short Appendix containing the material required for the mathematical derivations of § 2 is included at the end of the paper.

2. NOTATIONS AND HYPOTHESIS

2.1. Coronagraph Formalism

We follow the notations of Aime et al. (2002) and Soummer et al. (2003). The successive planes of the coronagraph are denoted by A, B, C, and D, where A is the entrance aperture, B denotes the focal plane with the mask (without loss of generality we assume that the amplitude of the mask is $1 - \epsilon$, where $\epsilon = 1$ corresponds to the classical Lyot coronagraph and $\epsilon = 2$ to the Roddier coronagraph), C is the image of the aperture with the Lyot stop, and D is the image in the focal plane after the coronagraph. The aperture transmission function is $p(x, y)$, and the wave front complex amplitude in A is $\Psi(x, y)$. In the case of an apodized pupil, we assume that the apodization function is included in $\Psi(x, y)$. In order to simplify the notations, the mask function in B is defined with coordinates proportional to $1/\lambda f$ and decomposed as

$$1 - \epsilon m \left( \frac{x}{\lambda f}, \frac{y}{\lambda f} \right),$$  \hspace{1cm} (1)$$

where the function $m(\cdots)$ equals 1 inside the coronagraphic mask and 0 outside.

We make the usual approximations of paraxial optics. Moreover we neglect the quadratic phase terms associated with the propagation of the waves or assume that the optical layout is properly designed to cancel it (Aime 2004). The expressions in Cartesian coordinates of the complex amplitude in the successive planes are

$$\Psi_A(x, y) = \Psi(x, y)p(x, y),$$ \hspace{1cm} (2)

$$\Psi_B(x, y) = \frac{1}{j \lambda f} \Psi_A \left( \frac{x}{\lambda f}, \frac{y}{\lambda f} \right) \left[ 1 - \epsilon m \left( \frac{x}{\lambda f}, \frac{y}{\lambda f} \right) \right],$$ \hspace{1cm} (3)
\[
\Psi_C(x, y) = \frac{1}{j\lambda f} \hat{\Psi}_B \left( \frac{x}{\lambda f}, \frac{y}{\lambda f} \right) p(-x, -y) \\
= -\left\{ \Psi_A(-x, -y) - e^{i\Psi_A(-u, -v)} \ast \hat{p}(u, v) \right\} p(-x, -y),
\]

(4)

\[
\Psi_D(x, y) = -\frac{1}{j\lambda f} \hat{\Psi}_A \left( \frac{x}{\lambda f}, \frac{y}{\lambda f} \right) + \frac{1}{j\lambda f} \left[ \hat{\Psi}_A - x, -y \right] \left( \frac{x}{\lambda f}, \frac{y}{\lambda f} \right) \times m(-x, -y) \ast \hat{p}(-x, -y),
\]

(5)

where \( \hat{f} \) is the Fourier transform of \( f \) and the asterisk denotes convolution. Equations (5) and (6) assume that the Lyot stop is the same as the pupil. However, for classical “unapodized” Lyot coronagraphs, the residual intensity in plane \( C \) is concentrated at the edges of the pupil, and a reduction of the Lyot stop size is needed in order to improve the rejection. The case of a reduced Lyot stop, which consists of convolving equation (6) by the pupil apodization, will be discussed in § 3. It is important to note that the reduction of the Lyot stop can be avoided using a prolate apodized entrance pupil, which will optimally concentrate the residual amplitude in \( C \) (see, e.g., Aime et al. 2002).

For the coronagraph response being derived herein for a circular pupil, the use of polar coordinates is preferred. Transcription of previous equations to polar coordinates is straightforward. Moreover, as long as the aperture transmission function and the stop have a circular symmetry, their Fourier transform will verify the same symmetry, as proved by equation (A3) with \( m = 0 \), i.e., the Hankel transform. This leads to the expression of the complex amplitude in \( D \),

\[
\Psi_D(r, f, \theta) = -\frac{1}{j\lambda f} \hat{\Psi}_A(r, \theta + \pi) + \frac{e^{i\Psi_A}}{j\lambda f} \times \left\{ \hat{\Psi}_A(r, \theta + \pi) m(r) \ast \hat{p}(r) \right\}(r, \theta),
\]

(7)

where the convolution of the two functions is still computed with respect to the Cartesian coordinates \( (x, y) \).

2.2. Choice of a Base

As mentioned in § 1, the analytical computation of the coronagraph response proposed herein relies on the expansion of the complex amplitude in \( A \) on an orthogonal basis. Equation (7) shows that the coronagraph acts linearly on the complex amplitude; consequently, the problem simplifies to the computation of the response of each basis function. The retained solution consists of the expansion of the complex amplitude in \( A \) on Zernike polynomials. Basic properties of the Zernike polynomials required in the paper are recalled in the Appendix.

Adopting the usual ordering of the Zernike circle polynomial (Mahajan 1994), we can write

\[
\Psi_A(r, \theta) = \sum_{(m,n)} a_{m,n} U_n^m(r/R, \theta) = \sum_k a_k Z_k(r/R, \theta),
\]

(8)

where \( R \) is the radius of the aperture. This expansion is rather unusual, the Zernike polynomials being generally used for the expansion of the wave front. However, it is worthy to note that, as equation (5) shows, a coronagraphic system will always introduce amplitude aberration. Hence, even in the case of a perfect wave with no aberration in \( A \), an expansion of only the phase in \( C \) will not be appropriate. Finally, equation (9) can also be justified by the fact that it coincides (up to a linear transform) with the classical approximation of the complex amplitude in the case of sufficiently small phase errors assuming a first-order development of the exponential function.

We illustrate the expansion in equation (9) in the case of tip-tilt error with an apodized pupil,

\[
\Psi_A(r, R, \theta) = a(r) \Pi(r)e^{j\beta \cos \theta},
\]

(10)

where \( a(r) \) denotes the pupil apodization and \( \Pi(r) = 1 \) for \( r \in [0, 1) \) and \( 0 \) if \( r \geq 1 \). Computation of the projection of \( \Psi_A(r, \theta) \) on \( U_n^m(r/R, \theta) \) is straightforward using the definition of the Bessel functions of integer order (Abramowitz & Stegun 1972),

\[
\int_0^{2\pi} \int_0^R \Psi_A(r, \theta) U_n^m(r/R, \theta) r \, dr \, d\theta = R^2 \int_0^{2\pi} R_n^m(r) \cos(m\theta)a(r)e^{j\beta \cos \theta} r \, dr \, d\theta = 2\pi R^2 \int_0^1 a(r) R_n^m(r) J_m(\beta r) r \, dr.
\]

(11)

The projection of \( \Psi_A(r, \theta) \) on \( U_n^{-m}(r/R, \theta) \) equals 0.

1. In the unapodized case, \( a(r) = 1 \), and the integral in equation (12) can be computed using equation (A2),

\[
2\pi R^2 \int_0^1 a(r) R_n^m(r) J_m(\beta r) r \, dr = 2\pi R^2 j^m(-1)^{(n-m)/2} J_{n+1}(\beta) / \beta.
\]

(13)

The coefficient \( a_k \) is then obtained by dividing this quantity by the energy of \( U_n^m(r/R, \theta) \) (see Born & Wolf 1991), leading to

\[
a_k = j^m(-1)^{(n-m)/2}(n+1) J_{n+1}(\beta) / (1 + \delta(n)) \beta.
\]

(14)

2. A particularly important case is that in which \( a(r) \) is proportional to the circular prolate function \( \varphi_0(c, r) \) (Soummer et al. 2003). In this case, the integral in equation (12) can be computed using the expansion of \( \varphi_0(c, r) \) derived in Slepian (1964),

\[
\varphi_0(c, r) = \sum_{k=0}^{\infty} a_k^{0,0} (c) \sqrt{r} F(k + 1, -k; 1; r^2).
\]

(15)

The function \( F(k + 1, -k; 1; r^2) \) defined in equation (A8) reduces to a polynomial of order \( 2k \), which, as mentioned in Slepian (1964), “is closely related to the Zernike polynomials.” Indeed, using equation (A7) and the results below, it can be easily checked that \( F(k + 1, -k; 1; r^2) = (-1)^k R_{2k}^2(r) \). Inserting this expansion into equation (12) and integrating term by term leads to integrals that generalize equation (A2). These integrals can be computed, for example, by using integrals of the type \( \int_0^1 r^s J_n(\beta r) r \, dr \) (Gradshteyn et al. 2000). This derivation will not be presented herein for the sake of brevity.

Finally, for more complicated complex amplitudes, the \( a_k \) can be, of course, computed numerically. This problem has been addressed in Pawlik & Liao (2002) using a piecewise approximation of \( \Psi_A(x, y) \) over a lattice of squares with size \( \Delta \times \Delta \) and
centered on point \((x_i, y_j)\). In this case, the estimation of \(a_k\) is given by

\[
\hat{a}_k = \sum_{(x_i, y_j) \in D} \Psi_A(x_i, y_j) w_n^m(x_i, y_j)^*,
\]

where \(w_n^m(x, y)\) is the integral of the Zernike polynomial \(U_n^m(r/R, \phi)\) over the square centered on \((x, y)\). Pawlak & Liao (2002) give bounds for the mean integrated squared error on the reconstruction of \(\Psi_A(x, y)\) when the coefficients are given by equation (16). This analysis is particularly important in our case, because it quantifies the dependence of the error on the smoothness of \(\Psi_A(x, y)\), the sampling rate \(\Delta\), and the geometrical error due to the circular geometry of the pupil.

3. CORONAGRAPH RESPONSE

3.1. Response of the Coronagraph to a Zernike Polynomial

The purpose of this section is to compute the complex amplitude in \(D\) when the complex amplitude in \(A\) is the Zernike polynomial with radial degree \(n\) and azimuthal frequency \(m\). In this case, the complex amplitude \(\Psi_D(r, \theta)\) is denoted as \(D_n^m(r, \theta)\). According to equation (7), the difficulty in the computation of \(D_n^m(r, \theta)\) lies in the evaluation of the convolution

\[
\Xi(r, \theta) = \{ [\hat{\Psi}_A(r, \theta + \pi) m(r)] \ast \hat{\rho}(r) \}(r, \theta).
\]

In this expression, \(m(r)\) is an “annular” mask of radius \(d\), which, with the definition adopted in equation (3), is defined as

\[
m(r) = \Pi \left( \frac{r}{d} \right).
\]

The computation of the convolution in \(\Xi(r, \theta)\) is sketched in Figure 1. Using equation (17), \(\Xi(r, \theta)\) simplifies to

\[
\Xi(r, \theta) = \int_{0}^{2\pi} \int_{0}^{\frac{d}{2}} \hat{\Psi}_A(\rho, \phi + \pi) \hat{\rho}(\sqrt{r^2 + \rho^2 - 2r \rho \cos(\theta - \phi)}) \rho d\rho d\phi.
\]

The next step consists of substituting into this equation:

1. \(\hat{\rho}(r)\) by the Fourier transform of \(\rho(r) = \Pi(r/R)\),

\[
\hat{\rho}(r) = \frac{R J_1(2\pi Rr)}{r};
\]

2. \(\Psi_A(\rho, \phi)\) by \(U_n^m(\rho R, \phi)\) and consequently \(\hat{\Psi}_A(\rho, \phi)\) by \(R^2 \hat{U}_n^m(\rho R, \phi)\), where \(\hat{U}_n^m(\rho, \phi)\) is given in equation (A5).

In order to simplify the notations, we define the new “standardized” integral \(\tilde{\Xi}(r, \theta, \xi)\) as

\[
\tilde{\Xi}(r, \theta, \xi) = \int_{0}^{\xi} \int_{0}^{2\pi} \cos(m\phi) J_{n+1}(\rho) \rho d\rho d\phi.
\]

It can be easily checked in this case that

\[
\Xi(r, \theta) = \int_{0}^{r} \int_{0}^{2\pi} \cos(m\phi) J_{n+1}(\rho) \rho d\rho d\phi.
\]

It is straightforward using equation (A9),

\[
\int_{0}^{2\pi} \cos(m\phi) C_k^{(1)}(\cos(\theta - \phi)) d\phi = \cos(m\theta) \sum_{q=0}^{k} \delta(m - k + 2q).
\]

2. Computation of the integral on \(\phi\) relies on recursion formulas on indefinite integrals of products of Bessel functions with \(k \neq n\) (Abramowitz & Stegun 1972),

\[
\int_{0}^{\xi} \frac{J_m(\rho) J_n(\rho)}{\rho} d\rho = \frac{\xi J_{m-1}(\xi) J_n(\xi) - \xi J_m(\xi) J_{n-1}(\xi) + (n-k) J_m(\xi) J_n(\xi)}{k^2 - n^2},
\]

\[
\int_{0}^{\xi} \frac{J_n(\rho)^2}{\rho} d\rho = \frac{1}{2n} \left[ 1 - J_0(\xi)^2 - 2 \sum_{q=1}^{n-1} J_q(\xi)^2 - J_n(\xi)^2 \right].
\]

After computation of the integral of equation (23), substitution of equation (22) into equation (7) gives the complex amplitude in \(D\) for a single basis function \(\Psi_A(r, \theta) = U_n^m(r/R, \theta)\),

\[
D_n^m(r, \theta) = j^{m-1}(-1)^{(n-m)/2} R \cos(m\theta) \times \left[ \frac{J_{n+1}(2\pi \mu r)}{r} + \epsilon \sum_{k=0}^{n-m} \eta_{m,n,k}(2\pi \mu d) J_{k+1}(2\pi \mu r) \right],
\]
Fig. 2.—Complex amplitude in $A$ and squared root of the amplitude in $D$, i.e., $|D^*(r, \theta)|$. The parameters used in the simulation are $\lambda f = 1$, $R = 1$, $d = 3$, and $\epsilon = 1$ (Lyot coronagraph).
Fig. 3.— Same as Fig. 2, but for different \((n, m)\).
be redeveloped by replacing \( \hat{p}(r) \) with \( \alpha^2 \hat{p}(\alpha r) \), and straightforward computation shows that:

1. Similarly to equation (26), the convolution in equation (17) will expand in an infinite sum of functions \( \cos(m \theta) J_{k+1}(2 \pi \alpha \mu r)/r \). However, the “radial contribution” to the coefficients weighting these functions (see eq. [27]) becomes

\[
\int_0^\infty J_{k+1}(\rho) J_{k+1}(\alpha^{-1}\rho) \frac{d \rho}{\rho},
\]

which cannot be computed straightforwardly, as in equations (24) and (25).

2. The first term in equation (7) is now replaced by the Fourier transform of \( U_m^n(\rho; R, \phi) \Pi(r/\alpha R) \), which cannot be anymore calculated using equation (A2).

### 3.2. Bound for the Truncation Error of \( D_n^m(r, \theta) \)

As we are interested in the computation of \( C_n^m(r, \theta) \) or \( D_n^m(r, \theta) \) from the implementation of equation (26), the errors produced when the infinite sum is truncated must be studied. In order to reduce mathematical developments, we only present herein the results for \( D_n^m(r, \theta) \) when the size of the Lyot stop equals the size of the pupil.

We define the truncation error on \( D_n^m(r, \theta) \),

\[
\mathcal{E}_N(r, \theta; m, n, \mu, d) = eR \left| \cos(m \theta) \sum_{k=N+1}^\infty \eta_{m,n,k}(2\pi \mu d) \frac{J_{k+1}(2 \pi \mu r)}{r} \right|.
\]

Computation of a bound on the truncation error relies on the classical upper bound for the Bessel functions of integer order (Abramowitz & Stegun 1972),

\[
|J_{k+1}(r)| \leq \frac{(r/2)^{k+1}}{k!}, \quad r \geq 0.
\]

Substitution of this result into equation (27) gives

\[
\eta_{m,n,k}(\xi) \leq (k+1) \sum_{q=0}^k \delta(m - k + 2q) \frac{1}{k + n + 2 + k! n!} \left( \frac{\xi}{2} \right)^{k+n+2} \tag{32}
\]

which leads to the bound for the truncation error,

\[
\mathcal{E}_N(r, \theta; m, n, \mu, d) \leq eR(\pi \mu)^{3+n} d^{2+n} \frac{1}{n!} \sum_{k=N+1}^\infty \frac{k + 1}{(k!)^2} \left( \frac{[\pi \mu]^2 rd}{2} \right)^k. \tag{34}
\]

The above series is absolutely convergent for \( r > 0 \). As a consequence, the expansion in equation (26) converges uniformly for \( (r, \theta) \in [0, \infty) \times [0, 2\pi] \). Finally, it is worth noting that the
computation of the infinite sum in the upper bound in equation (34) can be avoided by using the equality

$$\sum_{k=0}^{\infty} \frac{k + 1}{(k!)^2} x^k = I_0(2\sqrt{x}) + \sqrt{x}I_1(2\sqrt{x}),$$

(35)

where $I_n(x)$ is the modified Bessel function.

4. SIMULATION RESULTS

4.1. Response of the Coronagraph to the First Zernike Function

Figures 2 and 3 give the intensity in the $D$ plane of the coronagraph when the complex amplitude in the $A$ plane is one of the first six Zernike polynomials. The complex amplitudes have been computed using equation (26). Each row contains $U_{nm}(r, \theta)$ and $D_{nm}(r, \theta)$ for a given $(n, m)$. These plots have been obtained by truncating the infinite summation of equation (26) after the first 40 terms.

The relevance of the truncation error bound is verified in Figure 4. This plot shows the error bound in equation (34) as a function of $r$ for the parameters used in Figures 2 and 3. The increase of the bound with $r$ is simply due to the fact that the majoration of $|J_{k+1}(r)|$ given by equation (31) is only relevant for small values of $r$ as long as $|J_{k+1}(r)|$ is bounded on $[0, \infty)$. It is important to note that this plot justifies, at least for this configuration, the validity of a truncation at $N = 40$ for the computation of $D_{nm}(r, \theta)$. In this case, the truncation error is in fact always less than $10^{-10}$.

4.2. Application to Tip-Tilt Error Analysis

The effects of a tip-tilt error in Lyot coronagraphs has been extensively studied by Lloyd & Sivaramakrishnan (2005) and
Sivaramakrishnan et al. (2005). The aim of the simulation presented here is only to validate the results derived in § 2, simulating the particular case in which there is a misalignment of the star with the center of the stop. According to the previous notations, the complex amplitude in \( D \) decomposes as equation (29). In the case of a tip-tilt error in \( A \), the values of the coefficients \( a_{(m,n)} \) are given by equation (13).

Figure 5 shows \( |\Psi_D(r, \theta)| \) for different values of \( \beta > 0 \) (the case \( \beta = 0 \) is given in the first row of Fig. 2). The truncation in the summation from equation (29) has been chosen taking into account that equation (14) implies

\[
|a_{(m,n)}| \sim \frac{4}{\sqrt{2\pi|1+\delta(m)|}} \sqrt{\beta} \left( \frac{e^{\beta}}{2n} \right)^n,
\]

when \( n \to \infty \). According to the notations of equation (9), \( \Psi_D(r, \theta) \) equals equation (20) shifted by \( -\beta \sqrt{r}/(2\pi R) \). Consequently, the star is behind the focal stop in the first two images and outside in the last one.

5. CONCLUSION

In this paper we have presented a theoretical formalism for the analytical study of the Lyot coronagraph response. The main purposes of this work are of course to assist coronagraph design but also to improve data processing performances for the detection and characterization of extrasolar planets.

1. The first application is the computation of the response of the coronagraph to a planet at a given position. This is achieved, for example, in the case of a classical Lyot coronagraph using equations (29) and (13). This point is essential for the derivation of an optimal decision scheme to test the presence of a planet at a given location.

2. This formalism can also be applied to fully characterize the statistical properties of the complex amplitude in the \( D \) plane. For a given spatial covariance in \( A \), which is fixed through the covariance of coefficients \( a_k \), the spatial covariance in \( D \) becomes

\[
\text{Cov}[\Psi_D(r, \theta)\Psi_D(r', \theta')] = \sum_{k,l} \text{Cov}[a_k, a_l] D_k(r, \theta) D_l(r', \theta').
\]

Although detection algorithms based solely on the marginal distribution of the complex amplitude can be developed, as in Ferrari et al. (2006), the use of an accurate model for the spatial correlation of the complex amplitude is essential in order to derive detection algorithms with optimal performances, as demonstrated in Chatelain et al. (2006).

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APPENDIX

This appendix presents some facts about Fourier transform in polar coordinates and Zernike and Gegenbauer polynomials.

Among the various available possibilities that can be used to define an orthogonal set of functions on the unit radius disk, a central position is held by the Zernike polynomials (see, e.g., Mahajan 1994 and references therein). They are defined for \( n \geq m \) by

\[
U^m_n(r, \theta) = R^m_n(r)\cos(m\theta) \Pi(r), \quad U^{-m}_n(r, \theta) = R^m_n(r)\sin(m\theta) \Pi(r), \quad (A1)
\]

when \( n \) and \( m \) share the same parity. The \( R^m_n(r) \) are the radial polynomials. Different normalizations exist for \( R^m_n(r) \), we retain herein the definition of Born & Wolf (1991), \( R^m_n(1) = 1 \). Among the many properties verified by these polynomials, we focus on (see Born & Wolf [1991], Appendix VII for the proof)

\[
\int_0^1 r R^m_n(r) J_m(\nu r) \, dr = ( -1 )^{(m-n)/2} J_{m+1}(\nu) \nu. \quad (A2)
\]

This equality allows straightforward computation of the Fourier transform of the Zernike polynomials. In fact, recall first that when \( f(r, \theta) = g(r)\cos(m\theta) \) and \( m \in \mathbb{Z} \), a simple change of variables in the Fourier transform integral leads to

\[
\hat{f}(\rho, \phi) = 2\pi (-1)^m \cos(m\phi) \int_0^{\infty} r g(r) J_m(2\pi r \rho) \, dr. \quad (A3)
\]

An analog result for the inverse Fourier transform of \( \hat{f}(\rho, \phi) = h(\rho)\cos(m\phi) \) is

\[
f(r, \theta) = 2\pi j^m \cos(m\theta) \int_0^{\infty} \rho h(\rho) J_m(2\pi r \rho) \, d\rho. \quad (A4)
\]

Applying the result of equation (A3) with equation (A2) immediately gives

\[
\tilde{U}^m_n(\rho, \phi) = j^m (-1)^{(n+m)/2} \cos(m\phi) \frac{J_{n+1}(2\pi \rho)}{\rho}, \quad (A5)
\]

\[
\tilde{U}^{-m}_n(\rho, \phi) = j^m (-1)^{(n+m)/2} \sin(m\phi) \frac{J_{n+1}(2\pi \rho)}{\rho}. \quad (A6)
\]
The previous equation gives the inverse Fourier transform of \( \cos(m\phi)J_{n+1}(2\pi \rho)/\rho \) when \( n \geq m \geq 0 \) and \( n \) and \( m \) share the same parity. In the general case when \( n \geq 0 \) and \( m \geq 0 \), this inverse Fourier transform, denoted as \( f(r, \theta) \), must be computed independently. If we substitute \( h(r) \) with \( J_{n+1}(2\pi \rho)/\rho \) in equation (A4), the resulting integral is a Weber-Schafheitlin integral (Abramowitz & Stegun 1972). This results in \( f(r, \theta) = j^m \cos(m\theta)R_m^n(r) \), where if \( r < 1 \),

\[
R_m^n(r) = r^m \frac{\Gamma((n+m)/2+1)}{\Gamma((m+1)/2+1)} F\left(\frac{n+m}{2}+1, \frac{m-n}{2}, m+1, r^2\right)
\]  

\(\text{(A7)}\)

and \( F(a, b; c; z) \) is the Gauss hypergeometric function (see Gradshteyn et al. 2000)

\[
F(a, b; c; z) = 1 + \frac{ab}{1!} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \ldots .
\]  

\(\text{(A8)}\)

It is interesting to note from equations (A7) and (A8) that if \( b = (m-n)/2 \in \mathbb{Z}^- \), the sum in equation (A8) reduces to a polynomial in \( z \) of order \(- (m-n)/2\). Consequently, \( R_m^n(r) \) reduces to a polynomial with degree \( n \), which of course coincides up to \((-1)^{(m-n)/2} \) with \( R_m^n(r) \) for \( r \leq 1 \). For this reason, \( R_m^n(r) \) can be considered as a natural generalization of the Zernike polynomials. Note that, contrary to the generalization proposed in Myrick (1966) or Wünsche (2005), this generalization is not a polynomial.

We now briefly give the principal results related to the Gegenbauer polynomials. (See, e.g., Andrews et al. [1999] or Abramowitz & Stegun [1972] for detailed properties.) The Gegenbauer (or ultraspherical) polynomials, denoted as \( \phi C_{\kappa}^{(\ell)}(t) \), are defined as the coefficients of the power series expansion of \( rC(1-2rt+r^2)^{-\nu} \),

\[
\frac{1}{(1-2rt+r^2)^\nu} = \sum_{k=0}^{\infty} \phi C_{\kappa}^{(\ell)}(t)r^k.
\]

For example, \( C_{\kappa}^{(1)}(t) \) gives the Chebyshev polynomial of the second kind \( U_{\kappa}(t) \),

\[
C_{\kappa}^{(1)}(\cos \psi) = \sum_{q=0}^{\kappa} \cos [(k-2q)\psi].
\]  

\(\text{(A9)}\)

Among the numerous beautiful properties of the Gegenbauer polynomials, we focus on the expansion

\[
J_{\nu}(w) \frac{w}{w^2} = 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (k+\nu) \frac{J_{\kappa+\nu}(r) J_{\kappa+\nu}(\rho)}{r^\nu} \phi C_{\kappa}^{(\ell)}(\cos \gamma),
\]  

\(\text{(A10)}\)

where \( w = \sqrt{r^2 + \rho^2 - 2r\rho \cos \gamma} \).

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