ON FINITE QUOTIENT AUBRY SET FOR GENERIC GEODESIC FLOWS

GONZALO CONTRERAS AND JOSÉ ANTÔNIO G. MIRANDA

Abstract. We study the structure of the Mather and Aubry sets for the family of lagrangians given by the kinetic energy associated to a riemannian metric $g$ on a closed manifold $M$. In this case the Euler-Lagrange flow is the geodesic flow of $(M, g)$. We prove that there exists a residual subset $G$ of the set of all conformal metrics to $g$, such that, if $\mathcal{G} \in G$ then the corresponding geodesic flow has a finitely many ergodic $c$-minimizing measures, for each non-trivial cohomology class $c \in H^1(M, \mathbb{R})$. This implies that, for any $c \in H^1(M, \mathbb{R})$, the quotient Aubry set for the cohomology class $c$ has a finite number of elements for this particular family of lagrangian systems.

1. Introduction

Let $M$ be a closed smooth manifold endowed with a $C^\infty$-riemannian metric $g$ on $M$ and let $L : TM \to \mathbb{R}$ be the lagrangian defined as the kinetic energy corresponding to $g$, i.e,

$$L(x, v) = \frac{1}{2}g_x(v, v).$$

Then the Euler-Lagrange equation for this particular class of Tonelli lagrangian, that will be called riemannian lagrangian, coincides with the equation of the geodesics on $(M, g)$:

$$\frac{D}{dt} \gamma(t) = 0.$$

Therefore the corresponding Euler-Lagrange flow $\phi_t : TM \to TM$ is the geodesic flow of the riemannian manifold $(M, g)$.

In this paper, we study generic properties of $c$-minimizing measures for the class of riemannian lagrangians by performing conformal perturbations of the metric $g$, that do not depend of the non-trivial cohomology class $c \in H^1(M, \mathbb{R})$. Recall that a metric $\tilde{g}$ is conformally equivalent to $g$, if there exist a smooth function $f : M \to \mathbb{R}$, such that $\tilde{g}_x(\cdot, \cdot) = e^{f(x)}g_x(\cdot, \cdot)$ for all $x \in M$.

The concept of minimizing measures was introduced by Mather in [Mat91] for time periodic Tonelli lagrangian (see section 2.1). In this setting, in [Mañ96], Mañé proved the existence of a residual set $O(c)$ in $C^\infty(M)$ (that depends on $c \in H^1(M, \mathbb{R})$), such that, $u \in O(c)$ implies that the perturbed lagrangian $L - u$ has a unique $c$-minimizing measure, and he asked the question about the existence of a residual set $O$ in $C^\infty(M)$.
such that any lagrangian \( L - u \), with \( u \in \mathcal{O} \), has a unique \( c \)-minimizing measure for each \( c \in H^1(M, \mathbb{R}) \). In that direction, Bernard and Contreras in [BC08] obtained a residual \( \mathcal{O} \subset C^\infty(M) \), such that any lagrangian \( L - u \), with \( u \in \mathcal{O} \), has at most \( 1 + \text{dim } H^1(M, \mathbb{R}) \) ergodic \( c \)-minimizing measures for every \( c \in H^1(M, \mathbb{R}) \).

In the case where \( L \) is a mechanical lagrangian, that is

\[
L(x, v) = \frac{1}{2} g_x(v, v) - u(x),
\]

it follows from Carneiro's theorem (see [Car95, Theorem 1]), that the support of a \( c \)-minimizing measure of \( L \) is contained in the level of value \( k = \alpha(c) \) of the energy function \( E_L = \partial_v L \cdot v - L \). Applying the Maupertuis' principle (see for example [Arn89, pg 247]) to the mechanical lagrangian \( L \), we have that the restriction of the Euler-Lagrange flow of \( L \) to the energy level \( E_L^{-1}(k) \), for

\[
k > \min \{ \alpha(c) : c \in H^1(M, \mathbb{R}) \} = \alpha(0) = \max \{ u(x) : x \in M \},
\]

is a reparametrization of a new geodesic flow for a metric conformally equivalent to \( g \), but its corresponding conformal factor obligatorily changes when we change the energy levels of \( k \).

So, fixing a cohomology \( c \in H^1(M, \mathbb{R}) \), from the Carneiro-Maupertuis arguments applied to the family of mechanical lagrangians obtained through perturbations of \( L \) by summing potentials in the residual subset given by Mañé’s theorem, it follows that we can obtain a residual set \( \mathcal{G}(c) \subset C^\infty(M) \), such that, if \( f \in \mathcal{G}(c) \) then the lagrangian \( \tilde{L}(x, v) = e^{f(x)} L(x, v) \) has a unique \( c \)-minimizing measure. Note that, if we want to study a property of \( c \)-minimizing measures for a mechanical lagrangian, that is independent of \( c \in H^1(M, \mathbb{R}) \), we necessarily need to consider an uncountable set of energy values. Thus, we will not be able to obtain a residual set of metrics conformally equivalent to \( g \) using the arguments of Carneiro-Maupertuis type on the generic family of mechanical lagrangians given by the Bernard-Contreras’ theorem.

We will prove a analogous result of [BC08] for the class of riemannian lagrangian by defining an appropriate setting where we are able to apply an abstract result, proved in [BC08], and to show a separability property (see Lemma 5) for non-trivial cohomology classes. Hence we will prove the following theorem.

**Theorem 1.** Let \((M, g)\) be a closed riemannian manifold with \( \text{dim}(H_1(M, \mathbb{R})) \geq 1 \). There exists a residual subset \( \mathcal{G} \subset C^\infty(M) \), such that, if

\[
\tilde{L}(x, v) = \frac{1}{2} e^{f(x)} g_x(v, v)
\]

with \( f \in \mathcal{G} \), then for each non-trivial cohomology class \( c \in H^1(M, \mathbb{R}) \) there exist at most \( 1 + \text{dim } H^1(M, \mathbb{R}) \) ergodic \( c \)-minimizing measures for \( \tilde{L} \).

In Mather’s studies on the dynamics of Tonelli lagrangian systems and the existence of Arnold diffusion, he found that understanding certain aspects of the Aubry sets seems
to help in the construction of orbits with interesting behavior. In particular, it seems useful to know that what we call the quotient Aubry set for cohomology $c$, and denote by $(\mathcal{A}_L(c), \delta_c)$, is a totally disconnected metric space, that is, every connected component consists of a single point. In [Mat04], Mather showed examples of mechanical lagrangians on $T\mathbb{T}^d$, with $d \geq 2$ and the potential $u \in C^{2d-3,1-\epsilon}(\mathbb{T}^d)$, whose a quotient Aubry set is isometric to an closed interval. In [Sor08], Sorrentino proved that Mather’s counterexamples are optimal, he proved that, for every mechanical lagrangian on a closed manifold with a sufficiently regular potential, the quotient Aubry set corresponding to the zero cohomology class is totally disconnected. Analogous result had been proved independently by Fathi, Figalli and Rifford in [FFR09]. Note that, the Bernard-Contreras’ theorem imply that for generic lagrangian, in the Mañé sense, the quotient Aubry set $(\mathcal{A}_L(c), \delta_c)$ has at most a finite number of elements, for every $c \in H^1(M, \mathbb{R})$. This follows from the fact that the elements of the quotient Aubry set, that are usually called (projected) static classes, are disjoint subsets and support at least one ergodic minimizing measure (see [CDI97]).

The Arnold diffusion question is usually stated as perturbation with time periodic potentials and looking for orbits which “visit” prescribed homology classes and whose energy goes to infinity. But the question still holds for autonomous or riemannian lagrangians if the diffusion is restricted to a fixed energy level. A result of Contreras-Paternain [CP02], obtaining heteroclinics between static classes may be useful in that context. The hypothesis in [CP02] is, precisely, the finitness of the quotient Aubry set.

In the case of lagrangians given by the kinetic energies of riemannian metrics on $M$, it is easy to see that, for the zero cohomology class the associated (projected) Aubry set is $M$ and the quotient Aubry set is a single point. Mather proved in [Mat03, Prop. 2] that, if dim $M \leq 3$ and $c \in H^1(M, \mathbb{R})$ then $(\mathcal{A}_L(c), \delta_c)$ is totally disconnected. Then, by Theorem 1 and the above observations, we have the following corollary.

**Corollary 2.** Let $(M, g)$ be a closed Riemannian Manifold. There exists a residual subset $\mathcal{G} \subset C^\infty(M)$, such that, if

$$\tilde{L}(x, v) = \frac{1}{2} \ e^{f(x)} \ g_x(v, v)$$

with $f \in \mathcal{G}$, then, for any $c \in H^1(M, \mathbb{R})$, the quotient Aubry set $(\mathcal{A}_L(c), \delta_c)$ has a finite number of elements.

2. Preliminary results

2.1. The Aubry-Mather theory for Tonelli lagrangian systems. Let us recall the concepts of the Mather and Aubry sets for (autonomous) Tonelli lagrangian introduced by J. Mather in [Mat91], [Mat93] and Mañé in [Mañ95] respectively.

Let $L : TM \to \mathbb{R}$ be a smooth Tonelli lagrangian defined in the tangent bundle $TM$ of a closed smooth riemannian manifold $(M, g)$, i.e., $L(x, v)$ satisfy the two conditions:

- Convexity: for each fiber $T_x M$, the restriction $L(x, \cdot) : T_x M \to \mathbb{R}$ has positive defined Hessian, and,
• Superlinearity: \( \lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = \infty \), uniformly in \( x \in M \).

The action of \( L \) over an absolutely continuous curve \( \gamma: [a, b] \to M \) is defined by:

\[
A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt.
\]

The extremal curves for the action are given by solutions of the Euler-Lagrange equation that in local coordinates can be written as:

\[
\left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} \right) = 0.
\]

Since \( L \) is convex and \( M \) is compact, the Euler-Lagrange equation defines a complete flow \( \phi_t: TM \to TM \), that is called lagrangian flow of \( L \) and is defined by

\[
\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t)),
\]

where \( \gamma: \mathbb{R} \to M \) is the solution of (2) with initial conditions \( \gamma(0) = x \) and \( \dot{\gamma}(0) = v \).

The energy function \( E_L: TM \to \mathbb{R} \) is defined as

\[
E_L(x, v) := \frac{\partial L}{\partial v}(x, v) \cdot v - L(x, v).
\]

Any non-empty energy level \( E^{-1}_L(k) \subset TM \) is compact and invariant by the lagrangian flow.

We denote by \( \mathfrak{B}(L) \) the set of all Borel probability measures that are invariant by the Euler-Lagrange flow of \( L \). Given a closed 1-form \( \omega \) on \( M \), consider the deformed lagrangian \( L_\omega: TM \to \mathbb{R} \) defined by

\[
L_\omega(x, v) = L(x, v) - \omega_x(v).
\]

Since \( d\omega = 0 \), the lagrangians \( L_\omega \) and \( L \) have the same Euler Lagrange flow. We say that \( \mu \in \mathfrak{B}(L) \) is a c-minimizing measure of \( L \) if

\[
\int_{TM} L - \omega \, d\mu = \min \left\{ \int_{TM} L - \omega \, d\nu : \nu \in \mathfrak{B}(L) \right\},
\]

where \( c = [\omega] \in H^1(M, \mathbb{R}) \). Let \( \mathfrak{M}_L(c) \subset \mathfrak{B}(L) \) be the set of all c-minimizing measures (it only depends on the cohomology class \( c \)). The ergodic components of a c-minimizing measure are also c-minimizing measures, so the set \( \mathfrak{M}_L(c) \) is a simplex whose extremal measures are ergodic c-minimizing measures. The \( \alpha \)-function, defined as

\[
\alpha(c) = -\min \left\{ \int_{TM} L - \omega \, d\nu : \nu \in \mathfrak{B}(L) \right\},
\]

is a convex and superlinear function \( \alpha: H^1(M, \mathbb{R}) \to \mathbb{R} \). For each \( c \in H^1(M, \mathbb{R}) \), we define the Mather set of cohomology class \( c \) as:

\[
\tilde{\mathcal{M}}_L(c) = \bigcup_{\mu \in \mathfrak{M}_L(c)} \text{Supp}(\mu).
\]
We set $p(\mathcal{M}_L(c)) = \mathcal{M}_L(c)$, and call it the \textit{projected Mather set}, where $p : TM \rightarrow M$ denotes the canonical projection. The celebrated \textit{graph theorem} proved by Mather in [Mat91], asserts that $\mathcal{M}(c)$ is non-empty, compact, invariant by the Euler-Lagrange flow and $p|_{\mathcal{M}_L(c)} : \mathcal{M}_L(c) \rightarrow \mathcal{M}_L(c)$ is a bi-Lipschitz homeomorphism. In [Car95] M. J. Carneiro proved (in the autonomous case), that this set is contained in the energy level $\mathcal{E}(c) := E_{\mathcal{L}}^{-1}(\alpha(c))$.

Following Mather in [Mat93], for $t > 0$ and $x, y \in M$, define the \textit{action potential} for the lagrangian deformed by a closed 1-form $\omega$ as:

$$h_\omega(x, y, t) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) - \omega_{\gamma(s)}(\dot{\gamma}(s)) \, ds \right\},$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \rightarrow M$ such that $\gamma(0) = x$ and $\gamma(t) = y$. The infimum is in fact a minimum by Tonelli’s theorem.

We define the \textit{Peierls barrier} for the lagrangian $L - \omega$ as the function $h_\omega : M \times M \rightarrow \mathbb{R}$ given by:

$$h_\omega(x, y) = \liminf_{t \rightarrow +\infty} \{ h_\omega(x, y, t) + \alpha([\omega])t \}.$$

Define the \textit{projected Aubry set} for the cohomology class $c = [\omega] \in H^1(M, \mathbb{R})$ as the set

$$\mathcal{A}_L(c) = \{ x \in M : h_\omega(x, y) = 0 \}.$$

By symmetrizing $h_\omega$, we define the \textit{semidistance} $\delta_c$ on $\mathcal{A}_L(c)$:

$$\delta_c(x, y) = h_\omega(x, y) + h_\omega(y, x).$$

This function $\delta_c$ is non-negative and satisfies the triangle inequality.

Finally, we define the \textit{quotient Aubry set} of the cohomology class $c = [\omega] \in H^1(M, \mathbb{R})$, denoted by $(\mathcal{A}_L(c), \delta_c)$, to be the metric space obtained by identifying two points $x, y \in \mathcal{A}_L(c)$ if their semidistance $\delta_c(x, y)$ vanishes.

2.2. \textbf{Abstract Results.} In order to state the abstract theorem proved by Bernard and Contreras in [BC08, Theorem 5], we need to be given:

- three topological vector spaces $E, F, G$;
- a continuous linear map $\pi : F \rightarrow G$;
- a bi-linear pairing, $\langle \cdot, \cdot \rangle : E \times G \rightarrow \mathbb{R}$;
- two metrizable convex compact subsets $H \subset F$ and $K \subset G$, such that $\pi(H) \subset K$.

And we suppose that:

(i) The map $E \times K \ni (u, \nu) \mapsto \langle u, \nu \rangle$ is continuous.

(ii) The compact $K$ is separated by $E$. This means that, if $\mu$ and $\nu$ are two different elements of $K$, then there exists $u \in E$ such that $\langle u, \mu \rangle \neq \langle u, \nu \rangle$.

(iii) $E$ is a Frechet space. It means that $E$ is a topological vector space whose topology is defined by a translation invariant metric, and that $E$ is complete for this metric.
So, given a linear functional \( L : H \to \mathbb{R} \), we denote by
\[
M_H(L) := \left\{ \mu \in H : L(\mu) = \min_{\nu \in H} L(\nu) \right\}
\]
and by \( M_K(L) \) the image \( \pi(M_H(L)) \). These are compact convex subsets of \( H \) and \( K \).

In this setting Bernard and Contreras proved the following abstract result:

**Theorem 3** (Th. 5 in [BC08]). For every finite dimensional affine subspace \( A \in H^* \), there exists a residual subset \( \mathcal{O}(A) \subset E \) such that, for all \( u \in \mathcal{O}(A) \) and all \( L \in A \), we have that
\[
\dim M_K(L - u) \leq \dim A.
\]

### 2.3. Holonomic measures

Let \( C_q \) be the set of continuous functions \( f : TM \to \mathbb{R} \) with growth at most quadratic, i.e.
\[
\|f\|_q := \sup_{(x,v) \in TM} \frac{|f(x,v)|}{1 + g_x(v,v)^2} < \infty.
\]
edowed with the norm defined above. Let \( C_q^* \) be the dual space of \( C_q \) endowed with the weak* topology. If \( u \in C^0(M) \) then \( u \) is bounded and hence the function \( (x,v) \mapsto u(x) g_x(v,v) \) is in \( C_q \). Therefore the function \( C_q^* \to \mathbb{R} \) defined by
\[
\mu \mapsto \int u(x) g_x(v,v) \, d\mu
\]
is continuous.

For each \( C^1 \) curve \( \gamma : \mathbb{R} \to M \) periodic, with period \( T > 0 \), we define the probability \( \mu_\gamma \) on the Borel \( \sigma \)-algebra of \( TM \) by
\[
\int f \, \mu_\gamma = \frac{1}{T} \int_0^T f(\gamma(t), \dot{\gamma}(t)) \, dt,
\]
for all continuous functions \( f : TM \to \mathbb{R} \) with compact support. Let \( \Gamma \) be the set of probabilities of this form. Observe that \( \dot{\gamma}(t) \) is bounded and hence \( \Gamma \) is naturally embedded in \( C_q^* \). Finally, the set of \( L^2 \) holonomic probabilities \( \mathcal{H}_2 \) is the closure of \( \Gamma \) in \( C_q^* \).

Let \( L : TM \to M \) be a Tonelli lagrangian and \( \omega \) be a closed 1-form on \( M \). It follows from Birkhoff Theorem (cf. [Mañ96, prop. 1.1(b)]) and Carneiro’s result (cf. [Car95, cor. 1]), that \( \mathcal{M}_L(\omega) \subset \mathcal{H}_2 \).

Mañé defines the set of \( L^1 \) holonomic measures \( \mathcal{H}_1 \) in the same fashion as above but using the norm
\[
\|f\|_1 := \sup_{(x,v) \in TM} \frac{|f(x,v)|}{1 + \sqrt{g_x(v,v)}} < \infty.
\]
In [Mañ96, prop. 1.2 and 1.3]) Mañé proved the existence of measures \( \mu \in \mathcal{H}_1 \) such that:
\[
\int_{TM} L - \omega \, d\mu = \min \left\{ \int_{TM} L - \omega \, d\nu : \nu \in \mathcal{H}_1 \right\}
\]
and that these measures are invariant by the Euler-Lagrange flow. Thus they are in $\mathcal{M}_L(\omega) \subset H_2$. Therefore, the concept of c-minimizing measures can be reformulated by taking the minimum of the action on the set of the $L^2$ holonomic probabilities. It is

$$\mathcal{M}_L(c) = \{ \mu \in H_2 : \int_{TM} L - \omega \ d\mu = \min_{\nu \in H} \int_{TM} L - \omega \ d\nu \}.$$  

The important goal is that the set $H_2$ does not depend on the lagrangian.

3. On the minimizing measures of geodesic flows

We will apply Theorem 3 in the family of lagrangians given by all kinetic energies corresponding to all $C^\infty$-riemannian metrics on $M$, that are conformally equivalent to $g$. We consider the sets $C_q, C^*_q$ and $H_2$, as in the section 2.3, and we define the following setting:

- $E = (C^\infty(M), d)$ the metric space, where
  $$d(u, v) = \|u - v\|_\infty = \sum_{k\in \mathbb{N}} \frac{1}{2^k} \arctan(\|u - v\|_{C^k}).$$
  This metric space is complete, hence any residual subset of $E$ is dense.

- $F = C^*_q$ the vector space of continuous linear functionals $\mu : C_q \to \mathbb{R}$ endowed with the weak* topology. Observe that $F = C^*_q$ is the set of finite Borel signed measures on $TM$ such that
  $$\int_{TM} g(v, v) \ d\mu < \infty.$$  
  We have that $H_2 \subset F$.

- $G = (C^0(M))^*$ the vector space of continuous linear functions $\nu : C^0(M) \to \mathbb{R}$ endowed with the weak* topology.
- The bilinear pairing $\langle \cdot, \cdot \rangle : E \times G \to \mathbb{R}$ is defined by $\langle u, \nu \rangle = \nu(u)$.
- The linear map $\pi_g : F \to G$ is given by
  $$\pi_g(\mu) : C^0(M) \to \mathbb{R}$$
  $$u \mapsto \pi_g(\mu)(u) = \int_{TM} u(x) \ g_x(v, v) \ d\mu.$$  
  Note that
  $$|\langle u, \pi_g(\mu) \rangle| \leq \|u\|_{C^0} \int_{TM} g_x(v, v) \ d\mu,$$
  then $\pi_g(\mu)$ is continuous and hence $\pi_g(\mu) \in G$. The argument in (6) shows that for any $u \in E$, the function $F \ni \mu \mapsto \langle u, \pi_g(\mu) \rangle$ is continuous. Therefore the map $\pi_g$ is continuous in the weak* topology of $G$. Also the bilinearity implies that the map $(u, \mu) \mapsto \langle u, \pi_g(\mu) \rangle$ is continuous.

- $n \in \mathbb{N}$.  


The compact $H_n \subset H_2 \subset F$ is the set of holonomic probability measures which are supported on the compact $B_n = \{(x, v) \in TM : g_x(v, v) \leq 2n\}$.

The compact $K_n = \pi_g(H_n) \subset G$. By duality we have that $K_n$ is separated by $E$.

Let $L : TM \to \mathbb{R}$ be the riemannian lagrangian corresponding to $g$ and let $\omega$ be a closed 1-form on $M$. Then $L - \omega$ defines a continuous linear functional $A_c : H_2 \to \mathbb{R}$ given by integration, it is

$$\nu \mapsto A_c(\nu) = \int_{TM} L - \omega \ d\nu,$$

where $[\omega] = c \in H^1(M, \mathbb{R})$.

Let $A_{c,u}^n : H_n \to \mathbb{R}$ be the action defined as

$$A_{c,u}^n(\mu) = A_c(\nu) + \langle u, \pi_g(\mu) \rangle = \int_{TM} (L - \omega) + \pi_g^*(u) \ d\mu$$

where $\pi_g^*(u)(x, v) := u(x) \ g_x(v, v)$. We denote by $M_{H_n}(c, u)$ the set of measures $\mu \in H_n$ which minimize the action $A_{c,u}^n : H_n \to \mathbb{R}$.

Let $A$ be the affine subspace of continuous linear functionals on $H_2$ of the kind $A_c$, with $c \in H^1(M, \mathbb{R})$. By applying Theorem 3, we obtain a residual subset $\mathcal{O}(A, n) \subset E$ such that $A_c \in A$ and $u \in \mathcal{O}(A, n)$ imply that $\dim \pi_g(M_{H_n}(c, u)) \leq \dim A$.

We set

$$\mathcal{O}(A) = \bigcap_{n \in \mathbb{N}} \mathcal{O}(A, n).$$

By the Baire property $\mathcal{O}(A)$ is residual subset of $E$ and we have that, if $A_c \in A$, $n \in \mathbb{N}$ and $u \in \mathcal{O}(A)$, then:

$$\dim \pi_g(M_{H_n}(c, u)) \leq \dim A.$$

Note that, if $u(x) > -1/2$ for all $x \in M$, then $\tilde{L}(u) := L + \pi_g^*(u)$ is the lagrangian corresponding to kinetic energy for the perturbed riemannian metric

$$\tilde{g}(u) := e^{\ln(1+2u)}g = (1 + 2u)g.$$ 

Then, given a closed 1-form $\omega$, the lagrangian flow of $(L - \omega) + \pi_g^*(u) = \tilde{L}(u) - \omega$ is the geodesic flow of the metric $\tilde{g}(u)$.

We recall that by the Mather’s graph theorem [Mat91], the union of the supports of all $c$-minimizing measures of a Tonelli lagrangian is a compact set. Therefore, by (7), there exists $m = m(g, u, c) \in \mathbb{N}$ such that

$$\mathfrak{M}_{\tilde{L}(u)}(c) = M_{H_n}(c, u), \ \forall n \geq m.$$ 

We need to compare the dimension of the sets $\mathfrak{M}_{\tilde{L}(u)}(c)$ of minimizing measures for the Tonelli lagrangian $\tilde{L}(u) - \omega$ (with $[\omega] = c$) with the dimension of the sets $\pi_g(M_{H_n}(c, u))$, for $A_c \in A$ and $u \in \mathcal{G}(A) := \mathcal{O}(A) \cap \{ u \in C^\infty(M) : u(x) > -1/2 \}$. 
For the zero cohomology class, it is easy to see that $ML(0)$ contain all the Dirac measures supported in a point of the zero section of $TM$, i.e $\delta_{(x,0)} \in ML(0)$ for all $x \in M$. On the other hand, for each measure $\delta_{(x,0)} \in ML(0)$, by the definition of the map $\pi_g$, for any $u \in E$ we have
\[
\langle u, \pi_g(\delta_{(x,0)}) \rangle = \int_{TM} u(x) g_x(v, v) \, d\delta_{(x,0)} = u(x)g(0,0) = 0.
\]
Then $\pi_g(ML) = \pi_g(M_{H_\alpha}(0, u)) = \{0\} \subset G = C^0(M)^*$. For any non-trivial cohomology class, we prove the following two lemmas that complete the proof of the Theorem 1.

**Lemma 4.** For $L(x, v) = \frac{1}{2} g_x(v, v)$ we have that $\alpha(c) > 0$ if $c \neq 0$.

**Proof.** Let $c \neq 0$ be a non-trivial cohomology class and $\omega$ a closed 1-form with $[\omega] = c$. Then there exists a $C^1$ curve $\gamma$ on $M$ such that $\oint_{\gamma} \omega \neq 0$. Parametrizing $\gamma$ in opposite direction if necessary, we can assume that $\oint_{\gamma} \omega = -b < 0$. Since $\gamma$ is not a fixed point, we have that $a = A_L(\gamma) > 0$. For $s > 0$ define $\gamma_s(t) := \gamma(st)$. Then
\[
A_{L-\omega}(\gamma_s) = s^2 A_L(\gamma) + s \oint_{\gamma} \omega = as^2 - bs < 0 \quad \text{for} \quad 0 < s < \frac{b}{a}.
\]
Let $s = \frac{b}{2a}$, then from (5) we have that
\[
\alpha(c) \geq -\int (L - \omega) \, d\mu_{\gamma_s} = -A_L(\gamma_s) + \oint_{\gamma_s} \omega = -as^2 + bs > 0.
\]
\[\square\]

**Lemma 5.** Let $L$ be the lagrangian of a riemannian metric $g$. Let $c \in H^1(M, \mathbb{R})$ be a non-trivial cohomology class. Then the linear map
\[
\pi_g|_{ML(c)} : ML(c) \longrightarrow G
\]
is injective.

**Proof.** We need to show that if $\mu \neq \nu \in ML(c)$, then $\pi_g(\mu) : E \rightarrow \mathbb{R}$ and $\pi_g(\nu) : E \rightarrow \mathbb{R}$ are two different linear functionals, i.e there exist an element $u \in C^\infty(M)$ such that:

\[
\langle u, \pi_g(\mu) \rangle = \int_{TM} \pi_g^\mu(u) \, d\mu \neq \int_{TM} \pi_g^\nu(u) \, d\nu = \langle u, \pi_g(\nu) \rangle
\]

Let $p : TM \rightarrow M$ be the canonical projection. Let us recall that the Mather set
\[
\mathcal{M}_L(c) = \bigcup_{\mu \in ML(c)} \text{supp}(\mu)
\]
is compact and is contained in the energy level
\[
\mathcal{E}(c) = E_L^{-1}(\alpha(c)) = \{(x, v) \in TM : g_x(v, v) = 2\alpha(c) \}.
\]
see (3) and (5). We also recall that, by the graphic property [Mat91], the restriction
\[ p|_{\mathcal{M}_L(c)} : \tilde{\mathcal{M}}_L(c) \to M \]
is injective and its inverse \((p|_{\mathcal{M}_L(c)})^{-1}\) is Lipschitz on the projected Mather set \(\mathcal{M}_L(c)\).

Let \(\tilde{B} \subset TM\) be a Borel set such that \(\mu(\tilde{B}) \neq \nu(\tilde{B})\). Consider
\[ B := p(\tilde{B}) \subset M. \]

Observing that \(\tilde{B} \subset p^{-1}(B)\), we have \(\mu(\tilde{B}) \leq \mu(p^{-1}(B))\). Since \(\text{supp} \mu \subset \mathcal{M}_L(c)\), by the graphic property,
\[ \mu(p^{-1}(B)) = \mu((p|_{\mathcal{M}_L(c)})^{-1}B) = \mu(\tilde{B} \cap \mathcal{M}_L(c)) \leq \mu(\tilde{B}). \]

Therefore \(\mu(\tilde{B}) = \mu(p^{-1}(B))\). Repeating the same argument to the minimizing measure \(\nu\), we have that \(\nu(\tilde{B}) = \nu(p^{-1}(B))\). Then
\[ \mu(p^{-1}(B)) \neq \nu(p^{-1}(B)) \]

Let \(\chi_B : M \to \mathbb{R}\) be the characteristic function of the set \(B\). Then
\[ \int_{TM} \chi_B(x)g_x(v, v) \, d\mu = \int_{p^{-1}(B)} g_x(v, v) \, d\mu = \int_{p^{-1}(B) \cap \mathcal{M}_L(c)} 2\alpha(c) \, d\mu = 2\alpha(c)\mu(p^{-1}(B)) \]
and similarly
\[ \int_{TM} \chi_B(x)g_x(v, v) \, d\nu = \int_{p^{-1}(B)} g_x(v, v) \, d\nu = \int_{p^{-1}(B) \cap \mathcal{M}_L(c)} 2\alpha(c) \, d\nu = 2\alpha(c)\nu(p^{-1}(B)) \]

Since by Lemma 4, \(\alpha(c) \neq 0\), we can choose a \(C^\infty\)-function \(u : M \to \mathbb{R}^+\), satisfying
\[ \int_{TM} u(x)g_x(v, v) \, d\mu \neq \int_{TM} u(x)g_x(v, v) \, d\nu. \]
This implies (8) and completes the proof of the lemma.

\[ \square \]

Acknowledgments

J. A. G. Miranda is very grateful to CIMAT for the hospitality and to CNPq-Brazil for the partial financial support. G. Contreras was partially supported by CONACYT, Mexico, grant 178838. G. Contreras is very grateful to MSRI, Berkeley, NFS Grant No. DMS-1440140 for the hospitality during Fall 2018 and to the Simons Foundation for financial support.
References

[Arn89] V. I. Arnold, Mathematical methods of classical mechanics, second ed., Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 1989, Translated from the Russian by K. Vogtmann and A. Weinstein. MR 997295

[BC08] P. Bernard and G. Contreras, A generic property of families of Lagrangian systems, Ann. of Math. (2) 167 (2008), no. 3, 1099–1108. MR 2415395 (2009d:37113)

[Car95] M. J. D. Carneiro, On minimizing measures of the action of autonomous Lagrangians, Nonlinearity 8 (1995), no. 6, 1077–1085. MR 1363400 (96j:58062)

[CDI97] G. Contreras, J. Delgado, and R. Iturriaga, Lagrangian flows: the dynamics of globally minimizing orbits. II, Bol. Soc. Brasil. Mat. (N.S.) 28 (1997), no. 2, 155–196. MR 1479500 (98i:58093)

[CP02] Gonzalo Contreras and Gabriel P. Paternain, Connecting orbits between static classes for generic Lagrangian systems, Topology 41 (2002), no. 4, 645–666. MR 1905833 (2003i:37059)

[FFR09] A. Fathi, A. Figalli, and L. Rifford, On the Hausdorff dimension of the Mather quotient, Comm. Pure Appl. Math. 62 (2009), no. 4, 445–500. MR 2492705

[Mai95]———, Lagrangian flows: the dynamics of globally minimizing orbits, International Conference on Dynamical Systems (Montevideo, 1995), Longman, Harlow, 1996, Reprinted in Bol. Soc. Brasil. Mat. (N.S.) 28 (1997), no. 2, 141–153., pp. 120–131. MR 1479499 (98i:58092)

[Mai96] R. Mañé, Generic properties and problems of minimizing measures of Lagrangian systems, Nonlinearity 9 (1996), no. 2, 273–310. MR 1384478 (97d:58118)

[Mat91] J. N. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z. 207 (1991), no. 2, 169–207. MR 1109661 (92m:58048)

[Mat93]———, Variational construction of connecting orbits, Ann. Inst. Fourier (Grenoble) 43 (1993), no. 5, 1349–1386. MR 1275203 (95c:58075)

[Mat03]———, Total disconnectedness of the quotient Aubry set in low dimensions, Comm. Pure Appl. Math. 56 (2003), no. 8, 1178–1183, Dedicated to the memory of Jürgen K. Moser. MR 1989233

[Mat04]———, Examples of Aubry sets, Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1667–1723. MR 2104599

[Sor08] A. Sorrentino, On the total disconnectedness of the quotient Aubry set, Ergodic Theory Dynam. Systems 28 (2008), no. 1, 267–290. MR 2380310

CIMAT, A.P. 402, 36.000, Guanajuato. GTO, México.

E-mail address: gonzalo@cimat.mx

Universidade Federal de Minas Gerais, Av. Antônio Carlos 6627, 31270-901, Belo Horizonte, MG, Brasil.

E-mail address: jan@mat.ufmg.br