Classical Light Beams and Geometric Phases

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We present a study of geometric phases in classical wave and polarisation optics using the basic mathematical framework of quantum mechanics. Important physical situations taken from scalar wave optics, pure polarisation optics, and the behaviour of polarisation in the eikonal or ray limit of Maxwell’s equations in a transparent medium are considered. The case of a beam of light whose propagation direction and polarisation state are both subject to change is dealt with, attention being paid to the validity of Maxwell’s equations at all stages. Global topological aspects of the space of all propagation directions are discussed using elementary group theoretical ideas, and the effects on geometric phases are elucidated.

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I. INTRODUCTION

The quantum mechanical geometric phase was discovered by Berry in 1983–84 [1]. The context was unitary evolution governed by the Schrödinger equation in the adiabatic approximation, i.e., with a hermitian Hamiltonian possessing a ‘gentle’ time-dependence. Assuming that as an operator the Hamiltonian is cyclic, i.e., it returns to its original form after a certain interval of time (during which there are no level crossings), the approximate solutions to the Schrödinger equation are also cyclic. The geometric phase is then seen explicitly in these solutions at the end of the cycle.

Immediately after Berry’s discovery, it was pointed out by Barry Simon [2] that the geometric phase expresses the non-integrability, or anholonomy, of a natural ‘rule of parallel transport’ (a connection) in a principal fibre bundle, with structure group $U(1)$, which occurs in the framework of quantum mechanics. This was therefore a characterisation of this phase in the language of differential geometry.

The ensuing years saw two streams of work relating to the geometric phase. One consisted of extensions of Berry’s original work, in the sense of relaxing the conditions under which the phase is definable. The other consisted of interesting earlier results which could be reinterpreted as instances of this phase, and so as precursors to it. We recall three significant efforts of the first kind. Aharonov and Anandan [3] showed that the adiabatic condition is not necessary—given a cyclic solution to the Schrödinger equation involving any (time-dependent) Hamiltonian, one can reconstruct a corresponding geometric phase. This was followed by the work of Samuel and Bhandari [4], in which the cyclic condition on a solution was also dispensed with. Given a solution of the Schrödinger equation involving any (time-dependent) Hamiltonian, over any stretch of time, one can in a simple way extend it to a closed or cyclic solution, then use the Aharonov-Anandan method to identify a geometric phase. Both these extensions of Berry’s original framework used quantum mechanical notions, specifically the Schrödinger equation. The third step in the direction of increasing generality was taken by Mukunda and Simon [5, 6]: the geometric phase is entirely kinematical in content, not requiring a Hamiltonian operator and associated Schrödinger equation. It is determined once one is given a (sufficiently smooth) curve of unit vectors in any complex Hilbert space, without reference to any specifically quantum mechanical notions. (The relevant expressions and definitions are recalled below).

Turning to the efforts of the second kind, within quantum mechanics we may cite the Bohm-Aharonov effect (already dealt with by Berry in his original work), and the clarification of the connection between Bargmann invariants and geometric phases [7]. Beyond these, it is interesting that many instances of the geometric phase have been identified within classical (wave) optics—the Gouy phase from 1890 [8, 9]; the work by Rytov, and Vladimirskii [11, 12] in 1938 and 1941 on the behavior of light polarisation in the short wavelength limit of wave optics; and Pancharatnam’s

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studies in 1956 [13] on light polarisation and an associated phase. These classical optics examples will be reviewed briefly in the next Section. It will be seen that even in these situations the basic idea of a complex linear vector space carrying a hermitian inner product, usually regarded as characteristic of quantum mechanics, is essential to be able to identify ‘classical geometric phases’. In this way, the deep link between Bargmann invariants and geometric phases, and the connection to Berry’s original discovery, are always kept in evidence.

With this background, we now describe briefly the kinematic approach to the geometric phase. Let $\mathcal{H}$ be a complex Hilbert space of any dimension, with vectors $\psi, \phi, \cdots$ and inner product $(\psi, \phi)$ [14]. In quantum mechanics $|\langle \psi, \phi \rangle|^2$ is related to a probability; in classical wave optics $\| \psi \|^2 = (\psi, \psi)$ generally stands for light intensity which can, but need not be, normalised. We next denote by $\mathcal{B}$ the unit sphere in $\mathcal{H}$:

$$\mathcal{B} = \{ \psi \in \mathcal{H} \mid \| \psi \|^2 = (\psi, \psi) = 1 \} \subset \mathcal{H}. \quad (1.1)$$

The group $U(1)$ of complex phase factors acts on $\mathcal{B}$ (also on $\mathcal{H}$) in a natural way:

$$\psi \in \mathcal{B} \rightarrow \psi' = e^{i\alpha} \psi \in \mathcal{B}, \quad 0 \leq \alpha < 2\pi. \quad (1.2)$$

The quotient $\mathcal{B}/U(1)$, i.e., collections or equivalence classes of vectors $\{ e^{i\alpha} \psi, \psi \}$ differing only by phases, forms the ‘ray space’ $\mathcal{R}$:

$$\mathcal{R} = \mathcal{B}/U(1) = \{ \rho(\psi) = \psi \psi^\dagger \mid \psi \in \mathcal{B} \}. \quad (1.3)$$

Whereas $\mathcal{B}$ is a subset of $\mathcal{H}$, the ray space $\mathcal{R}$ is not: the $\mathcal{B} - \mathcal{R}$ relationship is that there is a projection map $\pi$ from the former to the latter:

$$\pi : \mathcal{B} \rightarrow \mathcal{R} : \psi \in \mathcal{B} \rightarrow \pi(\psi) = \rho(\psi) \in \mathcal{R}. \quad (1.4)$$

Referring to an earlier comment, $\mathcal{B}$ is a $U(1)$ principal fibre bundle over the base $\mathcal{R}$. In the quantum mechanics context, points in $\mathcal{R}$ correspond one-to-one to physical pure states.

In this framework, given any continuous piecewise once differentiable parametrised curve $C$ in $\mathcal{B}$,

$$C = \{ \psi(s) \in \mathcal{B} \mid s_1 \leq s \leq s_2 \} \subset \mathcal{B}, \quad (1.5)$$

with image $C$ in $\mathcal{R}$,

$$C = \pi[C] = \{ \rho(s) = \rho(\psi(s)) = \psi(s) \psi(s)^\dagger \mid s_1 \leq s \leq s_2 \} \subset \mathcal{R}, \quad (1.6)$$

the geometric phase $\varphi_g[C]$ is defined:

$$\varphi_g[C] = \varphi_{\text{tot}}[C] - \varphi_{\text{dyn}}[C],$$

$$\varphi_{\text{tot}}[C] = \arg(\psi(s_1), \psi(s_2)),$$

$$\varphi_{\text{dyn}}[C] = \Im \int_{s_1}^{s_2} ds \left( \psi(s), \frac{d \psi(s)}{ds} \right). \quad (1.7)$$

As indicated, $\varphi_g[C]$ is a functional of $C \subset \mathcal{R}$, while $\varphi_{\text{tot}}$ and $\varphi_{\text{dyn}}$ are both functionals of $C \subset \mathcal{B}$.

An important consequence of this definition is a result involving the so-called Bargmann invariants [5]. The simplest such invariant involves three pairwise nonorthogonal vectors $\psi_1, \psi_2, \psi_3 \in \mathcal{B}$ and is the expression

$$\Delta_3(\psi_1, \psi_2, \psi_3) = (\psi_1, \psi_2)(\psi_2, \psi_3)(\psi_3, \psi_1), \quad (1.8)$$

which is in general complex. The fact that $\Delta_3(e^{i\alpha_1} \psi_1, e^{i\alpha_2} \psi_2, e^{i\alpha_3} \psi_3) = \Delta_3(\psi_1, \psi_2, \psi_3)$ for all real $\alpha_1, \alpha_2, \alpha_3$ shows that $\Delta_3$ lives in $\mathcal{R}$ rather than in $\mathcal{B}$. Indeed Bargmann, during the course of his famous proof of Wigner’s theorem, introduced $\Delta_3$ simply to point out this gauge-invariance property and to indicate that it could be used to distinguish between unitaries and antiunitaries: while $\Delta_3$ is invariant under unitaries, its argument changes signature under antiunitaries. It is in [5, 6] that this object introduced by Bargmann almost in passing was elevated to become the basis of a complete kinematic theory of geometric phase.

To relate $\arg(\Delta_3(\psi_1, \psi_2, \psi_3))$ to a geometric phase, it is necessary to connect the ‘vertices’ $\psi_1, \psi_2, \psi_3$ pairwise in some way, so as to construct a closed continuous piecewise once-differentiable loop in $\mathcal{B}$ reminiscent of cyclic quantum evolution. This can be done using the idea of geodesics in $\mathcal{R}$. Given two nonorthogonal vectors $\psi, \phi \in \mathcal{B}$, and assuming
for definiteness that \((\psi, \phi)\) is real positive, the (shorter) geodesic in \(\mathcal{R}\) connecting \(\rho(\psi)\) to \(\rho(\phi)\) is the image under \(\pi\) of the curve \(C = \{\psi(s)\} \subset \mathcal{B}\) described as follows:

\[
\begin{align*}
(\psi, \phi) &= \cos \theta, \quad 0 < \theta < \pi/2 : \\
\psi(s) &= \psi \cos s + \phi^{-1} \frac{\sin s}{\sin \theta}, \quad 0 \leq s \leq \theta, \\
\phi^{-1} &= \phi - \psi \cos \theta.
\end{align*}
\]

Along this \(C\) one has

\[
(\psi(s'), \psi(s)) = \cos(s' - s), \quad 0 \leq s', s \leq \theta.
\]

Then the connection between Bargmann invariants and geometric phases is:

\[
\begin{align*}
\arg(\Delta_3(\psi_1, \psi_2, \psi_3)) &= -\varphi_g[C], \\
C &= \text{triangle in } \mathcal{R} \text{ with vertices } \rho(\psi_1), \rho(\psi_2), \rho(\psi_3) \\
\text{and connecting geodesics as sides.}
\end{align*}
\]

A very far-reaching generalisation of this relation, when \(\dim \mathcal{H} \geq 3\), has been developed more recently [15–18]. It is worth emphasizing that the framework described above, based on the triplet of spaces \(\mathcal{H}, \mathcal{B}, \mathcal{R}\), supports the geometric phase concept in a simple and direct way. Though suggested by the formal (complex linear space) structure of quantum mechanics, it can be used in other situations as well, such as classical wave optics. We adopt this viewpoint in trying to define geometric phases in various physical, particularly classical optical, situations.

Before we outline the organization of the material of this paper, it may be useful to add an extended remark by way of pointing to the precise context of this work. There has been considerable interest in recent times to understand the interplay between the spatial degree of freedom (coherence) and polarisation degree of electromagnetic beams [see, for instance, [19] and references therein]. It is equally important to understand the behaviour of this interplay as the Maxwell beam passes through an optical system. Indeed, it turns out that the defining properties of the age old Mueller matrix cannot be correctly enumerated without consideration of this interplay or entanglement [20, 21].

A lens of focal length \(f\) relates the output field amplitude \(\psi_{\text{out}}(x_1, x_2)\) (just after the lens plane) to the input \(\psi_{\text{in}}(x_1, x_2)\) (just before the lens plane), \((x_1, x_2)\) being Cartesian variables in the transverse plane, through

\[
\psi_{\text{out}}(x_1, x_2) = \exp \left( -i \frac{x_1^2 + x_2^2}{2\lambda f} \right) \psi_{\text{in}}(x_1, x_2).
\]

But when it comes to vector waves, it is clear that the same transformation applied to every (Cartesian) component of the electric field vector \(\mathbf{E}(x_1, x_2)\) will not map solutions of Maxwell equations at the input plane to solutions at the output, for such a democratic action on the electric field components does not respect the transversality condition \(\nabla \cdot \mathbf{E} = 0\). Since this condition is a constraint connecting the spatial degrees of freedom to the polarization degree, it would be respected only if the spatial modulation \(\exp \left( -i \frac{x_1^2 + x_2^2}{2\lambda f} \right)\) is accompanied by ‘appropriate’ local rotations of the electric field components (local polarization) [22].

Let us arrange the components of the electric and magnetic field amplitude vectors \(\mathbf{E}(x_1, x_2), \mathbf{B}(x_1, x_2)\) in a transverse plane \(z = \text{constant}\) into a six-component electromagnetic vector

\[
\mathbf{A}(x_1, x_2) = \left( \begin{array}{c} \mathbf{E}(x_1, x_2) \\ \mathbf{B}(x_1, x_2) \end{array} \right).
\]

The approach of [22] rooted at the very Poincaré symmetry of the Maxwell system of equations led to this fundamental result: if \(T(x_1, x_2)\) is the amplitude transmittance function of an optical system in scalar Fourier optics [a lens, for instance, has \(T(x_1, x_2) = \exp \left( -i \frac{x_1^2 + x_2^2}{2\lambda f} \right)\)], then the action

\[
T : \mathbf{A}_{\text{in}}(x_1, x_2) \rightarrow \mathbf{A}_{\text{out}}(x_1, x_2) = T(Q_1, Q_2)\mathbf{A}_{\text{in}}(x_1, x_2),
\]

\[
Q_1 = x_1 \mathbb{I}_{6 \times 6} + \lambda G_1, \quad Q_2 = x_2 \mathbb{I}_{6 \times 6} + \lambda G_2,
\]

where \(G_1, G_2\) are a pair of \(6 \times 6\) numerical matrices arising from the structure of the Poincaré group [22], does take solutions \(\mathbf{A}_{\text{in}}(x_1, x_2)\) of Maxwell’s equations to solutions \(\mathbf{A}_{\text{out}}(x_1, x_2)\). That is, the matrices \(G_1, G_2\) effect on the components of \(\mathbf{A}(x_1, x_2)\) the correct local rotations alluded to above [22]. This result readily leads to Fourier optics for Maxwell beams [23] and to electromagnetic Gaussian beams [24], resulting in a straightforward description of not
only the longitudinal component but also the cross-polarisation component [25]. It is in respect of this result that the late Henri Bacry anticipated: “it is highly probable that a rigorously gauge theory will be developed in a near future”, the local rotations referred to above constituting “an SO(3) gauge group” [26].

The work presented here is only the first step of an ambitious programme constituting our attempt towards a possible realization of this anticipation. While the earlier formulation of Fourier optics for Maxwell beams [23] concentrated on paraxial propagation about a fixed direction, the present work aims at laying a global and structurally robust skeleton in the space of directions, handling satisfactorily the well known topological obstructions. In the sequel, we plan to adapt suitably the methods of [23] in local patches of the space of directions, and then ‘stitch’ together the patches in a smooth manner to arrive at the general case.

The contents of this paper are organised as follows. Section II gives brief accounts of three applications of the geometric phase concept to classical optical situations: the Gouy phase in scalar paraxial wave optics; the Pancharatnam study of phases in pure polarization optics with fixed propagation direction; and the behaviour of polarisation in the eikonal or ray limit of Maxwell’s equations in a transparent medium with given refractive index function. The Pancharatnam case uses the Poincaré sphere $S^2_{pol}$ of polarisation states, for a fixed direction of propagation, while the ray case uses the sphere of propagation directions $S^2_{dir}$. In all these cases, the use of the basic quantum mechanical framework is highlighted. Section III builds on the last example of Section II in two ways—the generalisation from the unphysical case of a single ray to a physical beam of finite cross-sectional area made up of a narrow bundle of nearly parallel rays; and the inclusion of polarization gadgets in the path of the beam. Once again the quantum mechanical framework proves adequate, and now both spheres $S^2_{pol}$, $S^2_{dir}$ come into the picture. Section IV takes up certain global features of the sphere of directions $S^2_{dir}$, and builds on a recent suggestion [27] that passage to the complex extension of the tangent planes to $S^2_{dir}$ removes an obstruction which exists in the real domain. (The work of [27] was motivated in part by earlier works of [28–30].) Using elementary group theoretical arguments, based on the groups $SO(3)$ and $SU(2)$, a particularly simple global basis of complex orthonormal vector fields tangent to $S^2_{dir}$ is constructed. Section V uses the constructions of Section IV to study again the beams of Section III and their geometric phases: in appropriate situations, the complete geometric phase separates into a contribution from $S^2_{pol}$ and another from $S^2_{dir}$. The final Section VI contains some concluding remarks, while the Appendix compares the present framework for handling geometric phases with that proposed in [27].

II. EXAMPLES OF CLASSICAL OPTICAL GEOMETRIC PHASES

In this Section we review three situations in classical optics displaying geometric phases, presenting only the essential details. The first concerns scalar wave optics, the other two include polarisation. Quantum mechanical notation is used when convenient [31].

A. The case of the Gouy phase

We deal with the scalar optical wave field in free space, with fixed (angular) frequency $\omega$, wave number $k = \omega/c$ and wavelength $\lambda = 2\pi/k$. In the paraxial approximation to the Helmholtz equation, with the positive z-axis as the propagation direction, we obtain the paraxial wave equation in two transverse dimensions:

$$i\lambda \frac{\partial}{\partial z} \psi(x, y; z) = \frac{\lambda^2}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y; z), \quad (2.1)$$

where $\lambda = \lambda/(2\pi) = k^{-1}$, reminiscent of $\hbar$. (The exponential factor $e^{i(kz - \omega t)}$ has been omitted in $\psi$). This is formally similar to the Schrödinger equation in quantum mechanics for a free nonrelativistic particle of unit mass in two dimensions, with $\lambda$ in place of $\hbar$, and with the longitudinal variable $z$ playing the role of ‘time’. The “Hamiltonian operator” $H$ for Eq. (2.1) is

$$H = \frac{1}{2} \left( p_x^2 + p_y^2 \right), \quad p_x = -i\lambda \frac{\partial}{\partial x}, \quad p_y = -i\lambda \frac{\partial}{\partial y}. \quad (2.2)$$

If we restrict to one transverse dimension, we have the simpler paraxial wave equation

$$i\lambda \frac{\partial}{\partial z} \psi(x; z) = H \psi(x; z), \quad H = \frac{1}{2} p_x^2 = -\frac{\lambda^2}{2} \frac{\partial^2}{\partial x^2}. \quad (2.3)$$
Then the centred Gaussian solution to Eq. (2.3), with width $w$ so that

$$\text{Im } q < 0 : \psi_0(x; q) = \left( \frac{-\text{Im } q}{\pi \lambda |q|^2} \right)^{1/4} \exp \left( \frac{i x^2}{2 \lambda q} \right),$$

$$\int_{-\infty}^{\infty} dx \left| \psi_0(x; q) \right|^2 = 1. \quad (2.4)$$

Then the centred Gaussian solution to Eq. (2.3), with width $w$ in the ‘waist’ plane $z = 0$, is:

$$\psi(x; 0) = \psi_0 \left( x; \frac{i w^2}{2\lambda} \right) = \left( \frac{2}{\pi w^2} \right)^{1/4} \exp \left( -\frac{x^2}{w^2} \right)$$

$$\rightarrow \psi(x; z) = e^{i\varphi_G(z)} \psi_0 \left( x; q(z) \right);$$

$$\varphi_G(z) = -\frac{1}{2} \tan^{-1}\left( \frac{z}{z_R} \right); \quad q(z) = z - iz_R,$$

$$z_R = \text{Rayleigh range} = \frac{w^2}{2\lambda} = \frac{\pi w^2}{\lambda}. \quad (2.5)$$

Here, $\varphi_G(z)$ is the evolving Gouy phase. It is the argument of $\psi(0; z)$ (on-axis phase at the plane $z = \text{constant}$) and ‘jumps’ by $-\pi/2$ (by $-n\pi/2$ for $n$ transverse dimensions) across the waist plane:

$$\varphi_G(z) = \text{arg } \psi(0; z),$$

$$\varphi_G(\infty) - \varphi_G(-\infty) = -\pi/2. \quad (2.6)$$

That the parameter $q$ lives in the lower half plane is a consequence of our taking monochromatic time-dependence in the usual form $\exp(-i\omega t)$. Had it been taken in the form $\exp(\omega t)$, as some authors do, then $q$ would live in the upper half plane. The evolution of Gaussian beams through first order systems described by abcd-matrix is governed by the well known Kogelnik abcd-law [32],

$$q_{\text{in}} \rightarrow q_{\text{out}} = \frac{a q_{\text{in}} + b}{c q_{\text{in}} + d}, \quad (2.7)$$

of which the particular case $q(z_1) \rightarrow q(z_2) = q(z_1) + (z_2 - z_1)$, corresponding to free propagation from $z_1$ to $z_2$ [i.e., $(a, b, c, d) = (1, z_2 - z_1, 0, 1)$], is already quoted in Eq. (2.5). It may be noted in passing that the abcd-law has been generalized to partially coherent Gaussian beams, the so-called Gaussian Schell-model beams, in [33] and to arbitrary beams in [34].

Our aim now is to show that $\varphi_G(z)$ is essentially a geometric phase. For this we need the extension of the relation (1.11) to the four-vertex Bargmann invariant, and then specialize it in a particular way. For the moment we use quantum mechanical notation, with $\psi$ denoting a Hilbert space vector. The generalisation of the connection (1.11) is $[\psi_1, \cdots, \psi_4$ are unit vectors$]:$

$$\Delta_4(\psi_1, \psi_2, \psi_3, \psi_4) = \begin{pmatrix} \psi_1, & \psi_2, & \psi_3, & \psi_4 \end{pmatrix} \begin{pmatrix} \psi_3, & \psi_4, & \psi_1, & \psi_2 \end{pmatrix},$$

$$\text{arg } \Delta_4(\psi_1, \psi_2, \psi_3, \psi_4) = -\varphi_G[C],$$

$C =$ quadrilateral in $\mathcal{R}$ with vertices $\rho(\psi_1), \cdots, \rho(\psi_4)$

and geodesics connecting $\rho(\psi_1)$ to $\rho(\psi_2)$, $\cdots$, $\rho(\psi_4)$ to $\rho(\psi_1)$ as sides. \quad (2.8)

Now to the specialisation of this relation. Let $s$ be an evolution parameter, and $H_0$ a ‘Hamiltonian operator’ independent of $s$; and let $\psi_0(s)$ obey the ‘Schrödinger equation’

$$\frac{d}{ds} \psi_0(s) = H_0 \psi_0(s), \quad (2.9)$$

so that

$$\psi_0(s_2) = e^{-i(s_2 - s_1)H_0} \psi_0(s_1). \quad (2.10)$$
In the relation (2.8) we now choose $\psi_1$ to be a convenient ‘reference vector’ $\psi_R$, which allows the measurement of the phase $\varphi(s)$ of $\psi_0(s)$ with respect to it in the Pancharatnam sense (i.e., through an inner-product):

$$\varphi(s) = \arg(\psi_R, \psi_0(s)).$$

(2.11)

Further, we choose $\psi_2 = \psi_0(s_1), \psi_3 = \text{a ‘zero energy’ vector } \psi_E \text{ obeying } H_0 \psi_E = 0, \text{ and } \psi_4 = \psi_0(s_2)$. Then, using also Eq. (2.10), the connection (2.8) becomes:

$$\varphi(s_2) - \varphi(s_1) = \varphi_g[\text{quadrilateral in } R \text{ with vertices}$$

$$\rho(\psi_R), \rho(\psi_0(s_1)), \rho(\psi_E), \rho(\psi_0(s_2)),$$

and geodesic sides].

(2.12)

To apply Eq. (2.12) to the present case, we set $s = \lambda^{-1}z$ and $H_0 = \frac{i}{\hbar} \frac{\partial}{\partial q}$, so (2.9) becomes (2.3); and associate ‘wave functions’ $\psi_R(x), \psi(x; z), \psi_E(x)$ with the Hilbert space vectors $\psi_R, \psi_0(s), \psi_E$ respectively. To arrange $\varphi(s)$ in Eq. (2.11) to be the Gouy phase $\varphi_G(z)$, recalling the ‘on-axis’ identification in Eq. (2.6), the wave function $\psi_R(x)$ must become essentially $\delta(x)$. Next, to obey the condition $H_0 \psi_E = 0$ the wave function $\psi_E(x)$ must become $x$-independent, i.e., a plane wave with wave vector strictly along the $z$-axis (recall that we have dropped, following (2.1), a factor $\exp[i(kz - \omega t)])$. With these clues we take $\psi_R(x)$ and $\psi_E(x)$ to be particular limiting forms of $\psi_0(x; q)$ (and as our interest is in phases alone we disregard real factors which diverge or vanish in the limits):

$$\psi_R(z) = \lim_{q_1=0, q_2\to-0} \psi_0(x; q_1 + iq_2)$$

$$= \lim_{q_2\to-0} \frac{1}{(-\pi \lambda q^2)^{1/4}} \exp \left( \frac{x^2}{2\lambda q^2} \right) \sim \delta(x);$$

$$\psi_E(x) = \lim_{q_1=0, q_2\to-\infty} \psi_0(x; q_1 + iq_2)$$

$$= \lim_{q_2\to-\infty} \frac{1}{(-\pi \lambda q^2)^{1/4}} \exp \left( \frac{x^2}{2\lambda q^2} \right) \sim \text{constant in } x.$$

(2.13)

Then indeed, with $s = z/\lambda$,

$$\varphi(s) = \arg(\psi_R, \psi_0(s)) = \arg \left( \int_{-\infty}^{\infty} dx \, \psi_R(x) \psi(x; z) \right)$$

$$= \arg \psi(0; z) = \varphi_G(z),$$

(2.14)

so Eq. (2.12) becomes:

$$\varphi_G(z_2) - \varphi_G(z_1)$$

$$= \varphi_g[\text{quadrilateral in } R \text{, vertices } \rho(\psi_R(x) \sim \delta(x)),$$

$$\rho(\psi(x; z_1)), \rho(\psi_E(x) \sim \text{constant}), \rho(\psi(x; z_2))$$

and geodesic sides].

(2.15)

This already shows that (differences of) Gouy phases are certain geometric phases. However, for improved understanding, we can analyse the right hand side further as follows.

The argument of $\varphi_g$ on the right hand side in the result (2.15) is a quadrilateral in the ‘ray space’ $R$, with geodesic sides, as it should be. The geodesics needed here are to be constructed in the manner of Eq. (1.9) at the vector space or wave amplitude level, followed by projection $\pi$ to $R$. A quite subtle analysis [35] (here omitted) shows that in the present instance (and some others of interest) we can use ‘geodesics’ drawn within the manifold of centred Gaussian amplitudes (which may be called ‘constrained geodesics’), and the basic connection (1.11), (2.8) between Bargmann invariants and geometric phases continues to be valid. Next, as the definition (1.7) of geometric phases shows, for practical calculations one can choose any convenient ‘lift’ $C$ of $\varphi$ at the level of Hilbert space vectors or wave amplitudes, obeying $C = \pi[C]$. In particular if $C$ is chosen to be a closed loop (the quadrilateral $C$ in $R$ is of course closed) the piece $\varphi_{\text{rot}}[C]$ in Eq. (1.7) vanishes and we are left with $\varphi_g[C] = -\varphi_{\text{dyn}}[C]$. Beyond this, one can use the phase freedom at each point along $C$ to assume, in the present case, that $C$ is a closed loop within the space of centred Gaussian wave functions $\psi_0(x; q)$. It can then be pictured or drawn as a closed curve in the lower half of the complex $q$ plane. One must only ensure that the ‘vertices’ are chosen properly, so as to project onto the vertices specified in $R$ in (2.15), and the connecting curves represent ‘constrained geodesics’ properly. When all this is done, the result is as shown in Fig. 1.
The arcs connecting $q = 0$ to $q = -i z_R + z_1$ (i.e., $\psi_1$ to $\psi_2$) and $q = -i z_R + z_2$ to $q = 0$ (i.e., $\psi_4$ to $\psi_1$) are both circular, with centres on the $q_1$ axis. The straight lines connecting $q = -i z_R + z_1$ to $q = -i \infty$ (i.e., $\psi_2$ to $\psi_3$) and $q = -i \infty$ to $q = -i z_R + z_2$ (i.e., $\psi_3$ to $\psi_4$) are both vertical, parallel to the $q_2$ axis. All of them taken in sequence ‘represent’ the closed $C$:

$$C \sim q = 0 \rightarrow q = -i z_R + z_1 \rightarrow q = -i \infty \rightarrow q = -i z_R + z_2 \rightarrow q = 0 \quad (2.16)$$

and Eq. (2.15) takes the more explicit form

$$\varphi_G(z_2) - \varphi_G(z_1) = -\varphi_{\text{dyn}}[C]
= -\text{Im} \oint_C \left\{ \left( \psi_0(x; q), \frac{\partial}{\partial q_1} \psi_0(x; q) \right) dq_1 
+ \left( \psi_0(x; q), \frac{\partial}{\partial q_2} \psi_0(x; q) \right) dq_2 \right\}. \quad (2.17)$$

The integration in the $q$ half plane is along the curve (2.16), while the inner products in Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ (integrations with respect to $x$) are left implicit. With some effort one can confirm that the integral over $C$ on the right in Eq. (2.17) indeed reproduces the difference between Gouy phases on the left, as determined by Eq. (2.5).

Alternatively, the line integral in Eq. (2.17) equals \cite{10} the negative of one-fourth of the (hyperbolic) area of the enclosed quadrilateral, the abed-law being a signature of the natural Lobachevskian hyperbolic geometry with metric

$$d\tilde{q}^2 = (dq_1^2 + dq_2^2)/q_2^2 \quad (2.18)$$

underlying the manifold of Gaussian states, the lower half $q$-plane. The area itself is given by the ‘hyperbolic deficiency’ which, for a (geodesic) quadrilateral, equals $2\pi$ minus sum of the interior angles.

The interior angle vanishes at $R$ as well as at $E$. For the other two angles $\alpha_1, \alpha_2$ we see from Fig. 1 that $\alpha_j = 2(\pi/2 - \theta_j)$ and $\tan \theta_j = z_j/z_R$. Thus the deficiency equals $2(\theta_1 + \theta_2)$, leading to a geometric phase of $-(\theta_1 + \theta_2)/2$. One thus recovers

$$\varphi_G(z_2) - \varphi_G(z_1) = -\frac{1}{2}(\text{arctan}(z_2/z_R) - \text{arctan}(z_1/z_R)), \quad (2.19)$$

well known in the context of laser beams, now as a geometric phase.

We appreciate that this demonstration of the link between Gouy and geometric phases is fully within the $\mathcal{H} - \mathcal{B} - \mathcal{R}$ framework of quantum mechanics used in the quantum kinematic approach \cite{5, 6} to geometric phase, briefly recapitulated in Section 1.
B. The Pancharatnam case

Now we include the polarization degree of freedom, and to begin with consider the extreme case when it is the only variable. With given frequency \( \omega \) and wave number \( k = \omega / c \), we fix also the direction of propagation to be the positive \( z \)-axis, and consider plane waves in various states of pure polarization. The analysis again falls perfectly into the quantum mechanical \( \mathcal{H} - \mathcal{B} - \mathcal{R} \) scheme. Dropping the standard factor \( e^{i(kz - \omega t)} \), at each \( z \) the electric field is a complex two-component vector in the transverse x-y plane, \( \mathbf{E} = (\mathbf{E}_x,\mathbf{E}_y) \). The Hilbert space \( \mathcal{H} \) for this case is then \( \mathcal{H} = \mathbb{C}^2 \) of dimension two.

As is well known, the spaces \( \mathcal{B} \) and \( \mathcal{R} \) are \( S^3 \) and \( S^2_{pol} \) respectively, unit spheres in real four and three-dimensional Euclidean spaces, the latter being the Poincaré sphere of pure polarization states:

\[
\mathcal{H} = \mathbb{C}^2 \rightarrow \mathcal{B} = S^3, \quad \mathcal{R}_2 = B_3/U(1) = S^2_{pol}. \tag{2.20}
\]

Given \( \mathbf{E} \) at some \( z \), the corresponding pure polarization state is represented by a point \( \mathbf{n} \in S^2_{pol} \) computed as follows:

\[
\mathbf{E} \rightarrow \mathbf{n} = (\mathbf{E} \cdot \mathbf{n})^{-1} \mathbf{E} \cdot \mathbf{n} \in S^2_{pol},
\]

\[
\mathbf{n} = (n_1, n_2, n_3) = (\sigma_3, \sigma_1, \sigma_2), \tag{2.21}
\]

where the \( \sigma \)'s are the standard quantum mechanical Pauli matrices. Under free propagation governed by the free Maxwell equations, the amplitude \( \mathbf{E} \) and the polarization state \( \mathbf{n} \) are both constant: \( \mathbf{n} \) is stationary on \( S^2_{pol} \).

More generally, we imagine the plane wave passing through transparent linear intensity preserving polarization gadgets which act on \( \mathbf{E} \) and alter the polarization state \( \mathbf{n} \). These are placed at various locations (lumped) or over various stretches (distributed) along the \( z \)-axis, separated from one another by intervals of free propagation. The effect of such gadgets on \( \mathbf{E} \) is again governed by Maxwell’s equations for propagation of the field through suitable transparent material media. As our interest is only in the behaviour of the polarization state \( \mathbf{n} \), the intensity being held constant, we can represent each polarization gadget by a corresponding element of the two-dimensional unitary unimodular group \( SU(2) \) —the additional \( U(1) \) phase in the full unitary group \( U(2) \) is not relevant for this purpose.

With this physical picture in place, let us write \( \mathbf{E}(z) \) and \( \mathbf{n}(z) \) for the field and the polarization state at position \( z \) along the propagation axis:

\[
\mathbf{E}(z) \in \mathbb{C}^2 \rightarrow \mathbf{n}(z) = (\mathbf{E}(z) \cdot \mathbf{n}(z))^{-1} \mathbf{E}(z) \cdot \mathbf{n}(z) \in S^2_{pol}. \tag{2.22}
\]

Then \( \mathbf{E}(z) \) evolves according to the Schrödinger-like equation

\[
i \frac{d\mathbf{E}(z)}{dz} = H(z)\mathbf{E}(z), \quad H(z) = \frac{1}{2} \mathbf{n} \cdot \mathbf{a}(z), \tag{2.23}
\]

where \( \mathbf{a}(z) \) is a real three-dimensional vector and \( H(z) \) is the ‘Hamiltonian’. Correspondingly for \( \mathbf{n}(z) \) we have

\[
\frac{d\mathbf{n}(z)}{dz} = \mathbf{a}(z) \times \mathbf{n}(z). \tag{2.24}
\]

Thus while \( \mathbf{E}(z) \) undergoes a gradually unfolding \( SU(2) \) transformation, \( \mathbf{n}(z) \) experiences a gradual rotation belonging to \( SO(3) \) [37]. (Free propagation stretches correspond to \( H(z) = 0 \), and hence to \( \mathbf{a}(z) = 0 \); for lumped elements like a quarter or half wave plate \( \mathbf{a}(z) \) is a Dirac delta function). Over a finite stretch \( z_1 \) to \( z_2 \), we have:

\[
\mathbf{E}(z_2) = \mathcal{U}(z_2, z_1)\mathbf{E}(z_1), \quad \mathcal{U}(z_2, z_1) \in SU(2); \]

\[
\mathbf{n}(z_2) = \mathcal{R}(z_2, z_1)\mathbf{n}(z_1), \quad \mathcal{R}(z_2, z_1) \in SO(3), \tag{2.25}
\]

with \( \mathcal{U}(z_2, z_1) \) determining \( \mathcal{R}(z_2, z_1) \) through the well known \( SU(2) \) to \( SO(3) \) homomorphism [38].

If \( H(z) \) and \( \mathbf{a}(z) \) are constant from \( z_1 \) to \( z_2 \), say \( H \) and \( \mathbf{a} \) respectively, we have

\[
\mathcal{U}(z_2, z_1) = e^{-i(z_2 - z_1)H}, \quad \mathcal{R}(z_2, z_1) = \mathcal{R}(\mathbf{a}, \ (z_2 - z_1)|\mathbf{a}|), \quad \mathbf{a} = \mathbf{a}/|\mathbf{a}|, \tag{2.26}
\]

where \( \mathcal{R}(\mathbf{a}, \alpha) \) is the right handed rotation about axis \( \mathbf{a} \) by amount \( \alpha \) [37]. Then over such a stretch

\[
\mathbf{E}(z)^\dagger H \mathbf{E}(z) = \frac{1}{2} \mathbf{E}(z)^\dagger \mathbf{E}(z) \mathbf{a} \cdot \mathbf{n}(z) = \text{constant}, \tag{2.27}
\]
and as $E(z)^\dagger E(z) = \text{constant}$ as well, we see that $\hat{n}(z)$ moves on a latitude circle in a plane perpendicular to $a$. In case $a \cdot \hat{n}(z) = 0$, $\hat{n}(z)$ moves on the great circle perpendicular to $\hat{a}$, the equator with respect to $\hat{a}$; and as then $E(z)^\dagger HE(z) = 0$, such stretches contribute zero dynamical phases.

A cyclic evolution in this Pancharatnam situation carries the electric field over some curve $C \subset B_3 = S^3$ (assuming for simplicity $E(z)^\dagger E(z) = 1$), say from $E^{(1)}$ at $z_1$ to $E^{(2)} = e^{i\theta}E^{(1)}$ at $z_2$. Then $\hat{n}(z)$ describes a closed loop $C_{\text{pol}} \subset S^2_{\text{pol}}$. By Eq. (1.7), the associated geometric phase can be readily computed, and it turns out to be very simply related to the geometry of $S^2_{\text{pol}}$:

$$\varphi_g[C_{\text{pol}}] = \varphi_{\text{tot}}[C] - \varphi_{\text{dyn}}[C]$$

$$= \arg E^{(1)} E^{(2)} - \text{Im} \int_{z_1}^{z_2} dz \frac{dE(z)}{dz} \int_{z_1}^{z_2} dz E(z)^\dagger H(z)E(z)$$

$$= 1/2 \Omega[C_{\text{pol}}] \ ,$$

(2.28)

where $\Omega[C_{\text{pol}}]$ is the solid angle (with sign) subtended by $C_{\text{pol}}$ at the origin of $S^2_{\text{pol}}$.

In the original Pancharatnam analysis, $C_{\text{pol}}$ is a spherical triangle on $S^2_{\text{pol}}$ with sides being great circle arcs (i.e., geodesics) [39], leading as mentioned above to $\varphi_{\text{dyn}}[C] = 0$ if piecewise constant ‘Hamiltonians’ are used. And $\varphi_g[C_{\text{pol}}]$ reduces to the negative of the phase of a three-vertex Bargmann invariant, a special simple instance of Eq. (1.11): if fields $E^{(1)}, E^{(2)}, E^{(3)}$ lead via Eq. (2.21) to the vertices $\hat{n}_1, \hat{n}_2, \hat{n}_3$ of $C_{\text{pol}}$ then

$$C_{\text{pol}} = \text{spherical triangle on } S^2_{\text{pol}} :$$

$$\varphi_g[C_{\text{pol}}] = 1/2 \Omega[C_{\text{pol}}]$$

$$= - \arg(E^{(1)} E^{(2)} E^{(2)} E^{(3)} E^{(3)} E^{(1)}) \ .$$

(2.29)

C. Polarisation in the eikonal limit

The third situation we consider from the geometric phase perspective is one that has been studied for a long time on account of its obvious physical relevance. It is the short wave length—or eikonal or ray—limit of Maxwell’s equations, leading to differential equations for light rays in a given transparent medium, plus the law for evolution of the electric field along them [40]. We first recall the basic equations resulting from the eikonal limit, then some important previous work, and finally consider the situation from the geometric phase perspective.

In comparison to the previous Pancharatnam case, in the eikonal limit the propagation (ray) direction is allowed to vary while, in a sense to be clarified later, the polarisation state stays constant. We consider Maxwell’s equations for propagating electric (and magnetic) fields in a transparent nonconducting non-magnetic material medium characterised by a time-independent isotropic refractive index function $n(x)$. To leading order, the eikonal limit gives a system of second order ordinary differential equations whose solutions are rays in the medium:

$$\frac{d}{ds} \left( n(x) \frac{dx}{ds} \right) = \nabla n(x) \ .$$

(2.30)

Each solution $x(s)$ (for given initial conditions) determines a ray $\Gamma$, a curve in physical three-dimensional Euclidean space. Here $s$ is arc length measured along $\Gamma$ from some starting point on $\Gamma$. We hereafter work with some definite $\Gamma$.

As a space curve, $\Gamma$ is characterized by the following vectors and scalars defined pointwise along it, the dot denoting derivative with respect to $s$:

$$v(s) = \dot{x}(s) = \text{unit tangent;}$$

$$\mathbf{n}(s) = \dot{v}(s)/|\dot{v}(s)| = \text{unit principal normal;}$$

$$\mathbf{b}(s) = v(s) \times n(s) = \text{unit binormal;}$$

$$\kappa(s) = |\dot{\mathbf{v}}(s)| = \text{curvature},$$

$$\tau(s) = b(s) \cdot \dot{n}(s) = \text{torsion} \ .$$

(2.31)
At each \( x(s) \in \Gamma, (v(s), n(s), b(s)) \) is a right handed orthonormal triad, unique at generic points with nonzero curvature; it is locally determined by \( \dot{x}(s) \) and \( \dot{x}(s) \). Formally these vectors obey the ‘equations of motion’

\[
\begin{align*}
\dot{v} &= \kappa b_v v = (v \cdot \dot{v})_v v, \\
\dot{n} &= (\kappa b + \tau v)_v n, \\
\dot{b} &= \tau v \wedge b.
\end{align*}
\]

(2.32)

The first equation (which is actually trivial) means that \( v \) obeys the minimal Fermi-Walker transport law [41], while \( n \) and \( b \) do not do so.

Next we consider the evolution of the electric field \( E(x(s)) \equiv E(s) \) along \( \Gamma \). This comes from the next to leading order terms in the eikonal limit of Maxwell’s equations, and when expressed in terms of the normalised electric field \( \Psi(s) \) we have again the Fermi-Walker transport law along with the transversality condition:

\[ \Psi(s) = E(s)/\sqrt{E(s) \cdot E(s)} : \dot{\Psi}(s) = \kappa(s) b(s) \cdot \Psi(s), \]

\( v(s) \cdot \Psi(s) = 0. \)

(2.33)

That both \( v(s) \) and \( \Psi(s) \) obey the Fermi-Walker law is consistent with the need to maintain the transversality condition \( v(s) \cdot \Psi(s) = 0 \) along \( \Gamma \).

At each \( x(s) \in \Gamma, n(s) \) and \( b(s) \) span the transverse plane perpendicular to \( v(s) \) there. If we introduce another orthonormal basis in this plane, \( e_a(s), a = 1, 2 \), obeying the Fermi-Walker transport law like \( \Psi(s) \), we have the evolution equations

\[ e_a(s) \cdot e_b(s) = \delta_{ab}, \quad e_1(s) \cdot e_2(s) = v(s), \quad e_a(s) \cdot v(s) = 0; \]

\[ \dot{e}_a(s) = \kappa(s) b(s) \cdot e_a(s), \quad a = 1, 2. \]

(2.34)

As initial condition we take

\[ e_1(s_1) = n(s_1), \quad e_2(s_1) = b(s_1) \]

(2.35)

at some \( s = s_1 \). Then the pair \((e_1, e_2)\) rotates steadily with respect to the pair \((n, b)\) at a rate given by the torsion:

\[ \frac{d}{ds} \begin{pmatrix} e_a(s) \cdot n(s) \\ e_a(s) \cdot b(s) \end{pmatrix} = \begin{pmatrix} 0 & \tau(s) \\ -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} e_a(s) \cdot n(s) \\ e_a(s) \cdot b(s) \end{pmatrix}, \quad a = 1, 2; \]

\[ \begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix} = \begin{pmatrix} \cos \chi(s) & -\sin \chi(s) \\ \sin \chi(s) & \cos \chi(s) \end{pmatrix} \begin{pmatrix} n(s) \\ b(s) \end{pmatrix}, \]

\[ \chi(s) = \int_{s_1}^{s} ds' \tau(s'). \]

(2.36)

Now \( e_a(s) \cdot \Psi(s) \) are constants along \( \Gamma \):

\[ \Psi(s) = z_a e_a(s), \quad z_a = e_a(s) \cdot \Psi(s) = \text{constant}, \]

\[ z^\dagger z = (z_1^2 + z_2^2)(z_1 z_2) = 1. \]

(2.37)

All the three-dimensional vectors \( x, v, n, b, E, \Psi, e_a \) have corresponding components with respect to some fixed global Cartesian frame in space. The representation (2.37) identifies \( \Psi(s) \) at each \( x(s) \in \Gamma \) with a ‘vector’ \( z \) in the two-dimensional complex linear space \( \mathbb{C}^2 \). Using this we can represent the polarization state at \( x(s) \in \Gamma \) by a point \( \hat{n}(z) \) on the Poincaré sphere \( S^2_{pol} \):

\[ \Psi(s) \rightarrow \hat{n}(z) = z^\dagger \tau z \in S^2_{pol}. \]

(2.38)

As long as no polarization gadgets are placed anywhere on \( \Gamma \), the \( z_a \) are constants, so the polarization state represented by \( \hat{n}(z) \in \mathbb{S}^2_{pol} \) is also constant: only the propagation direction \( v(s) \) varies. This is to be compared with the Pancharatnam situation: under free propagation, both propagation direction \( k \) and polarization state \( \hat{n} \in \mathbb{S}^2_{pol} \) are constant. If polarization gadgets are placed along the axis, \( k \) (by definition) stays constant, while \( \hat{n}(z) \) moves on \( \mathbb{S}^2_{pol} \).
In the present context we can say the cyclic case occurs when the choice of a ‘later’ point $x(s_2) \in \Gamma$ is such that $v(s_2)$, $n(s_2)$, $b(s_2)$ are the same as $v(s_1)$, $n(s_1)$, $b(s_1)$ respectively at the initial point $x(s_1) \in \Gamma$. This happens if

$$\text{cyclic case : } \dot{x}(s_2) = \dot{x}(s_1), \quad \ddot{x}(s_2) = \ddot{x}(s_1).$$

(2.39)

The behaviours of input linear and circular polarizations are then particularly simple. The linear case corresponds to real $z_0$; then $\Psi(s)$ is a vector in space with real Cartesian components all along $\Gamma$. From Eqs. (2.35,2.36) we can relate $\Psi(s_2)$ to $\Psi(s_1)$ as follows:

**Linear polarisation:**

$$\Psi(s_1) = \cos \theta \, e_1(s_1) + \sin \theta \, e_2(s_1),$$

$$\Psi(s_2) = \cos \theta \, e_1(s_2) + \sin \theta \, e_2(s_2),$$

$$\chi(s_2) = \int_{s_1}^{s_2} ds \, \tau(s).$$

(2.40)

The two transverse planes at $x(s_2)$, $x(s_1)$ on $\Gamma$ are parallel to one another, and $\Psi(s_2)$ is obtained from $\Psi(s_1)$ by a right handed rotation by angle $\chi(s_2)$ about $v(s_1)$. A detailed calculation shows that $\chi(s_2)$ has a geometrical meaning. Over the range $s_1 \leq s \leq s_2$, the unit tangent $v(s)$ to $\Gamma$ describes a closed loop $C_{\text{dir}}$ on the sphere of directions $S^2_{\text{dir}}$:

$$C_{\text{dir}} = \{v(s) \in S^2_{\text{dir}} \mid s_1 \leq s \leq s_2\} \subset S^2_{\text{dir}}, \quad v(s_2) = v(s_1).$$

(2.41)

Then we have the result that the integrated torsion is (the negative of) the solid angle subtended by $C_{\text{dir}}$ at the centre of $S^2_{\text{dir}}$:

$$\chi(s_2) = \int_{s_1}^{s_2} ds \, \tau(s) = -\Omega[C_{\text{dir}}].$$

(2.42)

In the cases of circular polarisations, we get phase shifts rather than a rotation in space. In these cases, $\Psi(s)$ is a complex three-vector at all $x(s)$ on $\Gamma$:

**RCP/LCP**:

$$\Psi(s_1) = \frac{1}{\sqrt{2}} (e_1(s_1) \pm i e_2(s_1)) \longrightarrow$$

$$\Psi(s) = \frac{1}{\sqrt{2}} (e_1(s) \pm i e_2(s))$$

$$= \frac{1}{\sqrt{2}} e^{\pm i \chi(s)} (n(s) \pm i b(s)),$$

$$\Psi(s_2) = e^{\pm i \chi(s_2)} \Psi(s_1) = e^{\mp i \Omega[C_{\text{dir}}]} \Psi(s_1).$$

(2.43)

These results on the behaviours of polarization in the ray limit of Maxwell’s equations were obtained very early by Rytov and by Vladimirskii [11, 12]. In particular, Rytov showed that the phase difference between RCP and LCP evolves at a rate proportional to the torsion; while Vladimirskii showed that the spatial rotation experienced in the cyclic case for linear polarization is essentially by the solid angle $\Omega[C_{\text{dir}}]$.

To cast the above discussion into the geometric phase format of Section I, it is useful to write the evolution equation (2.33) for the (normalised) electric field in a Schrödinger-like form with a suitable hermitian Hamiltonian operator. We view $\Psi(s)$ (referred to axes fixed in space) as a (normalised) element of $\mathcal{H} = \mathbb{C}^3$, which is the Hilbert space in the present context, and find:

$$i \frac{d}{ds} \Psi(s) = H(s)\Psi(s),$$

$$H(s) = i \kappa(s) (n(s)v(s)^T - v(s)n(s)^T).$$

(2.44)

Thus $H(s)$ is a pure imaginary antisymmetric $3 \times 3$ matrix. The transversality condition $v(s)^T \Psi(s) = 0$ is to be added as a constraint consistent with the evolution. The definition (1.7) allows us to define a geometric phase for any $s_1$ and $s_2$, and we find that due to transversality the dynamical phase always vanishes. Bringing in the spaces
\( B_5 \approx S^5 \) and \( R_4 = CP^2 \), the complex two-dimensional projective space appropriate to \( \mathcal{H} = \mathbb{C}^3 \), we have:

\[
C = \{ \Psi(s) \in \mathbb{C}^3 | s_1 \leq s \leq s_2 \} \subset B_5,
\]

\[
\pi[C] = C \subset R_4 :
\]

\[
\varphi_g[C] = \varphi_{\text{tot}}[C] - \varphi_{\text{dyn}}[C],
\]

\[
\varphi_{\text{tot}}[C] = \text{arg}(\Psi(s_1)\dagger\Psi(s_2)),
\]

\[
\varphi_{\text{dyn}}[C] = \text{Im} \int_{s_1}^{s_2} ds \Psi(s)\dagger \frac{d\Psi(s)}{ds}
\]

\[
= \text{Im} \left( -i \int_{s_1}^{s_2} ds \Psi(s)\dagger H(s)\Psi(s) \right) = 0,
\]

i.e.,

\[
\varphi_g[C] = \varphi_{\text{tot}}[C].
\] (2.45)

Here we recognize that \( C \) cannot be drawn freely in \( B_5 \) because of the transversality condition, so in this way it is constrained by \( \Gamma \).

To illustrate the above, let us quote some particular cases in a table, Eq. (2.46):

| Choices of \( s_1, s_2 \) | Polarisation | Behaviour of \( \Psi(s) \) | \( \varphi_g[C] \) |
|---------------------------|--------------|----------------|------------------|
| Free                      | Linear       | Real           | 0 or \( \pi \)   |
| \( \Gamma \) cyclic, Eq. (2.39) | RCP/LCP     | \( \Psi(s_2) = e^{\mp i\Omega[C_{\text{dir}}]}\Psi(s_1) \) | \( \mp \Omega[C_{\text{dir}}] \) |

The distinction between \( C \subset R_4 \) and \( C_{\text{dir}} \subset S^2_{\text{dir}} \) should be kept in mind.

### III. COMBINED PATH AND POLARISATION GEOMETRIC PHASES

The brief reviews presented in the previous Section show that the Pancharatnam situation and the ray optic limit are mutually complementary. In the former only the polarization state changes, while in the latter only the propagation direction changes. Now we try to cover the (important) middle ground between them. We endeavour to build up a physical picture, based ultimately on Maxwell’s equations, with the motivation to arrive at geometric phases in the framework of Section I.

As in the eikonal limit, we consider light traveling through a transparent non-magnetic stationary medium with refractive index \( n(x) \). We recall that the concept of a single ray is not physically meaningful and cannot be realised. The eikonal limit of Maxwell’s equations leads at first to a first order partial differential equation in three-dimensional space for the eikonal, a function \( S(x) \). A particular eikonal \( S(x) \) leads to a continuous family or succession of wave fronts over each of which \( S(x) \) is constant, and which taken together cover some region of physical space. Rays are then lines drawn in this region, orthogonal at each point to the wave front passing through that point. These rays are solutions to Eq. (2.30). Thus one eikonal \( S(x) \) determines a corresponding succession of wave fronts and in turn one family of rays. There is only one wavefront, and only one ray belonging to it, through each point in the relevant region.

It is in this sense that single rays are not directly physically realisable. The best that we can do is to consider a narrow or well collimated (i.e., nearly parallel) bundle of nearby rays with some nonzero cross-sectional area which may vary along the bundle [Consequently the wavefronts along the bundle, correspondingly limited in their spatial extent, are nearly planar]. Calling this a beam, at each point along it we have some finite spatially limited wavefront. In this picture we have in mind some \( \Gamma \) obeying Eq. (2.30) acting as the ‘backbone’ of the beam. At each location \( x(s) \in \Gamma \), we have a propagation direction \( v(s) \), a spatially limited ‘plane wave’ perpendicular to \( v(s) \), and a transverse electric field \( E(x(s)) = E(s) \). Thus we arrive at a physical picture of a continuous succession of limited plane waves each at a spatial location \( x(s) \), with propagation direction \( v(s) \) and in some polarization state. Now we can go a step further and allow the wavelength to be finite, as long as it is much smaller than all other physically relevant dimensions, including the linear dimensions of the limited plane wave elements.

In this way we motivate the passage from a physically unrealisable ray to a realisable beam by a process of ‘thickening’ of the former. In the sequel, the spatial locations \( x(s) \) of successive plane wave elements of the beam will sometimes be omitted. The parameter \( s \) continues to be distance measured along the beam from some initial point, increasing at each location in the direction of \( v(s) \).

For a beam propagating ‘freely’ in the medium in this way, the evolution equation for \( \Psi(s) \) is Eq. (2.33). This, as we have seen, is a consequence of Maxwell’s equations in the medium, and can be put into the Schrödinger-like form
(2.44) with a hermitian Hamiltonian operator. The solution Eq. (2.37) with constant \( z \) implies a constant polarization state \( \hat{n}(z) \in S^2_{pol} \) given in Eq. (2.38).

We can now go another step further and imagine placing various polarisation gadgets over (short) stretches of the beam, equivalently of \( \Gamma \), where \( z \) varies as function of \( s \), governed by a ‘polarisation Hamiltonian’ as in Eq. (2.23). Thus we arrive at new evolution equations for \( \Psi(s) \) based on the following ingredients:

\[
\Psi(s) = z_a(s) e_a(s) : \\
i \frac{d}{ds} e_a(s) = H^{(dir)}(s) e_a(s), \\
H^{(dir)}(s) = i \kappa(s) (n(s) v(s)^T - v(s) n(s)^T) ; \\
i \frac{d}{ds} z(s) = H^{(pol)}(s) z(s), \\
H^{(pol)}(s) = \frac{1}{2} \tau \cdot a(s), \quad a(s) \text{ real} .
\] (3.1)

We have now written \( H^{(dir)}(s) \) for the ‘direction’ part of the Hamiltonian, appearing in Eq. (2.44); it is completely determined by the local geometrical properties of \( \Gamma \). The other contribution to the evolution of \( \Psi(s) \) is from the ‘polarization’ part of the Hamiltonian, as in Eq. (2.23), written now as \( H^{(pol)}(s) \). This controls the evolution of the local two-component transverse description of \( \Psi(s) \) resolved along \( e_a(s) \). The complete evolution equation for \( \Psi(s) \) is easily found to be Schrödinger-like, with a Hamiltonian which is a (complex) hermitian 3 \times 3 matrix:

\[
i \frac{d}{ds} \Psi(s) = \left( H^{(dir)}(s) + H^{(pol)}(s)^T \right) \Psi(s), \\
H^{(dir)}(s) = i \kappa(s) (n(s) v(s)^T - v(s) n(s)^T) , \\
H^{(pol)}(s) = \frac{1}{2} a_j(s) (\tau_j)_{ab} e_a(s) e_b(s)^T .
\] (3.2)

The implied evolution equation for \( \hat{n}(z) \in S^2_{pol} \) is as in Eq. (2.24):

\[
\hat{n}(z(s)) \equiv \hat{n}(s) : \quad \frac{d \hat{n}(s)}{ds} = a(s) \cdot \hat{n}(s) .
\] (3.3)

Over portions of the beam free of polarization gadgets, where \( a(s) = 0 \), the local properties of \( \Gamma \) determine the propagation, and the polarization state is constant. Passage through gadgets leads to changing \( z(s) \) and \( \hat{n}(s) \). Both kinds of changes in \( \Psi(s) \) are ultimately traced back to Maxwells’ equations; and the complete evolution equation (3.2) respects the transversality condition \( v(s)^T \Psi(s) = 0 \). In all of this, the separation of effects due to change in beam direction and those due to presence of polarisation gadgets, is essentially unambiguous.

Let us now bring in geometric phase considerations. As in the ray case in Section II(C), we are able to use the basic quantum mechanical \( \mathcal{H} = \mathcal{B} - \mathcal{R} \) framework with \( \mathcal{H} = \mathbb{C}^2, \mathcal{B}_5 = S^5, \) and \( \mathcal{R}_4 = CP^2 \) which is of real dimension four. For the calculation of dynamical phases we need the result

\[
\text{Im} \left( \Psi(s), \frac{d \Psi(s)}{ds} \right) = \text{Im} \left( -i(\Psi(s),(H^{(dir)}(s) + H^{(pol)}(s))^T)\Psi(s) ) \right) = -\frac{1}{2} a(s) \cdot \hat{n}(s) ,
\] (3.4)

so there is a contribution only from the presence of polarisation gadgets [This was to be expected since we have arranged the ‘evolution in direction’ to be of vanishing dynamical phase]. For general \( s_1 \) and \( s_2 \) with initial and final spatial positions \( x(s_1), x(s_2) \) on the beam we define:

\[
\mathcal{C} = \{ \Psi(s) \in \mathcal{H} \mid s_1 \leq s \leq s_2 \} \subset \mathcal{B}_5, \\
\pi[C] = C \subset \mathcal{R}_4.
\] (3.5)

(It is implicit that \( \Psi(s) \) is located in space at \( x(s) \) and is transverse, so as in Section II it cannot be drawn arbitrarily
and then the geometric phase becomes,

\[ \varphi_g[C] = \varphi_{tot}[C] - \varphi_{dyn}[C], \]
\[ \varphi_{tot}[C] = \arg(\Psi(s_1)\Psi^*(s_2)), \]
\[ \varphi_{dyn}[C] = \Im \int_{s_1}^{s_2} ds \left( \frac{d}{ds} \Psi(s) \right) \]
\[ = -\frac{1}{2} \int_{s_1}^{s_2} ds \mathbf{a}(s) \cdot \hat{n}(s). \]  

Eq. (3.6)

We illustrate this result in a special situation, where a connection to the results in Section II in the Pancharatnam case (B) can be made. Let us firstly choose \( s_1 \) and \( s_2 \) so that this stretch of \( \Gamma \) is ‘cyclic’ in the sense of Eq. (2.39). Then we make an independent additional assumption that the polarisation gadgets placed along the beam between \( x(s_1) \) and \( x(s_2) \) are such that (for a particular initial \( \Psi(s_1) \)) \( z(s_2) \) turns out to be a phase times \( z(s_1) \). This then means that the curve traced by \( \hat{n}(s) \in S^2_{\text{pol}} \) is a closed loop. In all the conditions assumed are:

\[ \hat{x}(s_2) = \hat{x}(s_1), \quad \hat{x}(s_2) = \hat{x}(s_1); \]
\[ z(s_2) = e^{i\theta} z(s_1), \quad \hat{n}(s_2) = \hat{n}(s_1); \]
\[ C_{\text{pol}} = \{ \hat{n}(s) \in S^2_{\text{pol}} | s_1 \leq s \leq s_2 \} \subset S^2_{\text{pol}}; \quad \text{closed}. \]

Eq. (3.7)

By Eqs. (2.36,2.42) we relate \( \mathbf{e}_a(s_2) \) to \( \mathbf{e}_a(s_1) \):

\[ \begin{pmatrix} e_1(s_2) \\ e_2(s_2) \end{pmatrix} = \begin{pmatrix} \cos \Omega[C_{\text{dir}}] & \sin \Omega[C_{\text{dir}}] \\ -\sin \Omega[C_{\text{dir}}] & \cos \Omega[C_{\text{dir}}] \end{pmatrix} \begin{pmatrix} e_1(s_1) \\ e_2(s_1) \end{pmatrix}, \]

(3.8)

where \( \Omega[C_{\text{dir}}] \) is the solid angle subtended at the centre of \( S^2_{\text{dir}} \) by \( C_{\text{dir}} \) defined in Eq. (2.41). We see that with the conditions (3.7) we deal with two closed loops, \( C_{\text{dir}} \subset S^2_{\text{dir}} \) and \( C_{\text{pol}} \subset S^2_{\text{pol}} \), on the sphere of directions and on the Poincaré sphere respectively. Now we can calculate the geometric phase for this situation using Eq. (3.6):

\[ \varphi_g[C] = \arg \left( z(s_1)\Psi(s_1) - \sin \Omega[C_{\text{dir}}] \right) \]
\[ \cos \Omega[C_{\text{dir}}] \sin \Omega[C_{\text{dir}}] \cos \Omega[C_{\text{dir}}] \]
\[ \arg \left( \cos \Omega[C_{\text{dir}}] + 2i \sin \Omega[C_{\text{dir}}] \Im z_1(s_1)z_2(s_1)^* \right). \]

Eq. (3.8)

On comparing Eqs. (2.27,2.28) of the Pancharatnam situation with Eq. (3.1), we see that the first two terms here add up to \( \frac{1}{2} \Omega[C_{\text{pol}}] \):

\[ \theta + \frac{1}{2} \int_{s_1}^{s_2} ds \mathbf{a}(s) \cdot \hat{n}(s) = \frac{1}{2} \Omega[C_{\text{pol}}]. \]

(3.9)

If we finally specialize to input circular polarisations, the third term also simplifies:

\[ \text{RCP/LCP : } z_1(s_1) = \frac{1}{\sqrt{2}}, \quad z_2(s_1) = \pm \frac{i}{\sqrt{2}}; \]
\[ \arg \left( \cos \Omega[C_{\text{dir}}] + 2i \sin \Omega[C_{\text{dir}}] \Im z_1(s_1)z_2(s_1)^* \right) \]
\[ = -\Omega[C_{\text{dir}}], \]

(3.10)

and then the geometric phase becomes,

\[ \varphi_g[C] = \frac{1}{2} \Omega[C_{\text{pol}}] - \Omega[C_{\text{dir}}]. \]

(3.11)

We may remark finally that while the geometric phase in the present physical situation is always defined by Eq. (3.6), it is only in a quite special situation that we get a simple final expression (3.12), in a way combining the Pancharatnam result (2.28) and the pure ray result (2.46). What needs to be stressed however is that the separation of the contributions from the sphere of directions \( S^2_{\text{dir}} \) and from the Poincaré (polarization) sphere \( S^2_{\text{pol}} \) is essentially unambiguous. In the Pancharatnam limit, \( C_{\text{dir}} \) shrinks to a point and we recover (2.28); while in the pure ray limit with no polarisation gadgets, it is \( C_{\text{pol}} \) that shrinks to a point and we get back (2.46) for circular polarisations.
IV. SOME GLOBAL ASPECTS OF THE SPHERE OF DIRECTIONS

The situation analysed in the previous Section from the geometric phase point of view is that of a (narrow well collimated) light beam of fixed frequency travelling in physical space through a given transparent medium, encountering various polarisation gadgets on its way. The path of the beam is based on a ray $\Gamma = \{ \mathbf{x}(s) \} \subset \mathbb{R}^3$ obeying Eq. (2.30) for a given refractive index function $n(\mathbf{x})$. From $\Gamma$ we obtain a particular one-dimensional curve $C_{\text{dir}} = \{ \mathbf{v}(s) = \hat{\mathbf{x}}(s) \} \subset S_{\text{dir}}^2$, the two-dimensional sphere of directions.

The ray $\Gamma$ also gives a preferred choice of a real orthonormal basis $\{ \mathbf{e}_a(s) \}$ in the transverse plane at each $\mathbf{x}(s) \in \Gamma$, perpendicular to $\mathbf{v}(s)$ there. By resolving the normalised complex transverse electric field $\Psi(\mathbf{x}(s))$ with respect to this basis, we are able to describe it by a normalized complex two-component column vector $\mathbf{z}(s)$, leading to the representation of the polarization state by a point $\hat{\mathbf{z}}(s) \in S_{\text{pol}}^2$. In particular, real $\Psi(\mathbf{x}(s))$ implies real $\mathbf{z}(s)$ and vice versa, corresponding to linear polarisations.

The choice of $\Gamma$ thus provides both $C_{\text{dir}} \subset S_{\text{dir}}^2$, and $\{ \mathbf{e}_a(s) \}$. We can regard the latter as a preferred real orthonormal basis in the real tangent plane $T_{\mathbf{v}(s)} S_{\text{dir}}^2 \simeq \mathbb{R}^2$, for each $\mathbf{v}(s) \in C_{\text{dir}}$. As a result, the geometric phase contributions from beam direction and beam polarisation are essentially unambiguously separated.

Let us now view the problem from another more global perspective, not immediately related to a ray or to a picture embedded in physical space. We take the sphere $S_{\text{dir}}^2$ of plane wave propagation directions as starting point, writing $\hat{\mathbf{k}}$ for points on it (instead of $\mathbf{v}(s)$ obtained from $\Gamma$ as up to now). Each $\hat{\mathbf{k}}$ is the unit vector in the direction of a wave vector $\mathbf{k}$ associated with a possible (spatially limited) propagating plane wave. We now ask if there is a way to choose a real orthonormal basis $\{ \mathbf{e}_a(\hat{\mathbf{k}}) \}$ in the real tangent plane $T_{\hat{\mathbf{k}}} S_{\text{dir}}^2 \simeq \mathbb{R}^2$, well defined and varying smoothly with $\hat{\mathbf{k}}$ for all $\hat{\mathbf{k}} \in S_{\text{dir}}^2$.

Since this question is posed prior to the possible choice of a ray $\Gamma$, even if such $\{ \mathbf{e}_a(\hat{\mathbf{k}}) \}$ exist, it need have nothing to do with the $\{ \mathbf{e}_a(s) \}$ later supplied by a ray $\Gamma$ at a point on it where $\mathbf{v}(s) = \hat{\mathbf{k}}$. As we have seen, it is $\{ \mathbf{e}_a(s) \}$ which has specific advantages from a physical point of view, which may be absent with $\{ \mathbf{e}_a(\hat{\mathbf{k}}) \}$.

It is however a known fact from differential geometry that such choices of $\{ \mathbf{e}_a(\hat{\mathbf{k}}) \}$ for all $\hat{\mathbf{k}} \in S_{\text{dir}}^2$ do not exist. This is expressed by saying that the sphere $S_{\text{dir}}^2$ is not parallelizable [42] as a real four-dimensional manifold the tangent bundle $TS_{\text{dir}}^2$ is not (homeomorphic to) the product $S_{\text{dir}}^2 \times \mathbb{R}^2$. A useful way to display this circumstance, suited for further developments, is as follows.

In real three-dimensional Euclidean space let us choose a right handed Cartesian system of axes with origin $O$, and with $\mathbf{e}_j$, $j = 1, 2, 3$, the unit vectors along the coordinate axes. Points on the unit sphere $S_{\text{dir}}^2$ with centre at $O$ will be written $\hat{\mathbf{k}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. It is necessary to define two subsets $S_{\text{dir}}^2$, $S_{\text{dir}}^2$ of $S_{\text{dir}}^2$ whose union gives $S_{\text{dir}}^2$ but which have a nontrivial (indeed, substantial) overlap:

\begin{align}
S_{\text{dir}}^2 &= \{ \hat{\mathbf{k}}(\theta, \phi) \in S_{\text{dir}}^2 | 0 \leq \theta < \pi, 0 \leq \phi < 2\pi \}, \\
S_{\text{dir}}^2 &= \{ \hat{\mathbf{k}}(\theta, \phi) \in S_{\text{dir}}^2 | 0 < \theta \leq \pi, 0 \leq \phi < 2\pi \}, \\
S_{\text{dir}}^2 &\cup S_{\text{dir}}^2 = S_{\text{dir}}^2, \\
S_{\text{dir}}^2 &\cap S_{\text{dir}}^2 = \{ \hat{\mathbf{k}}(\theta, \phi) \in S_{\text{dir}}^2 | 0 < \theta < \pi, 0 \leq \phi < 2\pi \}. \quad (4.1)
\end{align}

We need the action of proper rotations, elements of the rotation group $SO(3)$, on $S_{\text{dir}}^2$. The right handed rotation about axis $\hat{\mathbf{a}} \in S^2$ by angle $\alpha$ corresponds to the $3 \times 3$ matrix

\begin{align}
R_{jk}(\hat{\mathbf{a}}, \alpha) &= \delta_{jk} \cos \alpha + \epsilon_{jkl} a_k (1 - \cos \alpha) - \epsilon_{jkl} a_l \sin \alpha, \\
0 \leq \alpha \leq 2\pi. \quad (4.2)
\end{align}

For any $\hat{\mathbf{k}} \in S_{\text{dir}}^2$, there are infinitely many rotations carrying $\hat{\mathbf{e}}_3$ to $\hat{\mathbf{k}}$. However there is no way to choose one such rotation for each $\hat{\mathbf{k}}$, such that it is globally well-defined and varies smoothly with $\hat{\mathbf{k}}$ for all $\hat{\mathbf{k}} \in S_{\text{dir}}^2$. Over $S_{\text{dir}}^2$, which is $S_{\text{dir}}^2$ with just one point (the south pole) removed, a convenient choice does exist:

\begin{align}
A'(\hat{\mathbf{k}}) &= R(\hat{\mathbf{e}}_3, \phi) R(\hat{\mathbf{e}}_2, \theta) R(\hat{\mathbf{e}}_1, \phi)^{-1} \\
&= R(\hat{\mathbf{e}}_2 \cos \phi - \hat{\mathbf{e}}_1 \sin \phi, \theta), \\
&0 \leq \theta < \pi, 0 \leq \phi < 2\pi; \\
A'(\hat{\mathbf{k}}) \hat{\mathbf{e}}_3 &= \hat{\mathbf{k}}. \quad (4.3)
\end{align}

This is well defined at $\theta = 0$ but not at $\theta = \pi$. Acting on $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ at the North pole, we get a real orthonormal basis
for the tangent plane $T_k S^2_{\text{dir}}$ when $\theta < \pi$:

$$e'_1(k) = A'(k) \hat{e}_1 = \begin{pmatrix} \sin^2 \phi + \cos \theta \cos^2 \phi \\ (\cos \theta - 1) \sin \phi \cos \phi \\ - \sin \theta \cos \phi \end{pmatrix},$$

$$e'_2(k) = A'(k) \hat{e}_2 = \begin{pmatrix} (\cos \theta - 1) \sin \phi \cos \phi \\ \cos^2 \phi + \cos \theta \sin^2 \phi \\ - \sin \theta \sin \phi \end{pmatrix}, \ k \in S^2_N. \tag{4.4}$$

Over $S^2_S$ a similar choice is:

$$A''(k) = R(\hat{e}_3, \phi) R(\hat{e}_2, \theta) R(\hat{e}_3, \phi),$$

$$0 < \theta \leq \pi, \ 0 \leq \phi < 2\pi :$$

$$A''(k) \hat{e}_3 = \hat{k}. \tag{4.5}$$

Now this is well defined at $\theta = \pi$ but not at $\theta = 0$. Acting on $\hat{e}_1, \hat{e}_2$ at the North pole we get a different real orthonormal basis for $T_k S^2_{\text{dir}}$ when $\theta > 0$:

$$e''_1(k) = A''(k) \hat{e}_1 = \begin{pmatrix} \cos \theta \cos^2 \phi - \sin^2 \phi \\ (1 + \cos \theta) \sin \phi \cos \phi \\ - \sin \theta \cos \phi \end{pmatrix},$$

$$e''_2(k) = A''(k) \hat{e}_2 = \begin{pmatrix} -(1 + \cos \theta) \sin \phi \cos \phi \\ \cos^2 \phi - \cos \theta \sin^2 \phi \\ \sin \theta \sin \phi \end{pmatrix}, \ k \in S^2_S. \tag{4.6}$$

In the overlap, which is all of $S^2_{\text{dir}}$ with just the north and south poles removed, we have connecting or ‘transition’ formulae:

$$\hat{k} \in S^2_N \cap S^2_S :$$

$$A''(\hat{k}) = A'(\hat{k}) R(\hat{e}_3, 2\phi) = R(\hat{k}, 2\phi) A'(\hat{k});$$

$$e''_a(k) = R(\hat{k}, 2\phi) e'_a(\hat{k}), \ a = 1, 2. \tag{4.7}$$

There are now two equally good ways to express the nonparallelizable nature of $S^2_{\text{dir}}$: (i) it is not possible to extend the definition of $A'(k)$ (respectively $A''(k)$) to cover the South pole $\theta = \pi$ (respectively the North pole $\theta = 0$) possessing smooth behaviour for all $\hat{k} \in S^2_{\text{dir}}$; (ii) the real orthonormal bases $\{e'_a(\hat{k})\}$, $\{e''_a(\hat{k})\}$ for $T_k S^2_{\text{dir}}$ over $S^2_N$, $S^2_S$ respectively cannot be modified in any way to yield a real orthonormal basis for $T_k S^2_{\text{dir}}$ varying smoothly with $\hat{k}$ all over $S^2_{\text{dir}}$. A more formal statement is this: it is impossible to find two smoothly varying angles $\chi'(\hat{k}), \chi''(\hat{k})$ over $S^2_N$, $S^2_S$ respectively such that the transition group element $R(\hat{e}_3, 2\phi)$ in Eq. (4.7) can be factorised as

$$R(\hat{e}_3, 2\phi) = R(\hat{e}_3, \chi'(\hat{k})) R(\hat{e}_3, \chi''(\hat{k}))^{-1}, \ \forall \hat{k} \in S^2_N \cap S^2_S. \tag{4.8}$$

For, if such choices were possible, then $A'(k) R(\hat{e}_3, \chi'(\hat{k})) = A''(k) R(\hat{e}_3, \chi''(\hat{k}))$ would carry $\hat{e}_3$ to $\hat{k}$ and be smoothly defined for all $\hat{k} \in S^2_{\text{dir}}$.

Now the ‘topological obstruction’ described above is in the real domain, i.e., viewing each tangent plane $T_k S^2_{\text{dir}}$ as a real two-dimensional vector space $\mathbb{R}^2$. It has however been pointed out recently that if one complexifies each $T_k S^2_{\text{dir}}$ into a complex two-dimensional vector space $(T_k S^2_{\text{dir}})^* \cong \mathbb{C}^2$, then the obstruction vanishes [27]: it is possible to choose orthonormal bases for these complexified tangent spaces in a globally smooth manner. There is naturally considerable freedom in such choices; we describe now a group theory based choice which seems natural and minimal in some sense. This requires the use of the group $SU(2)$ (which is a double cover of $SO(3)$, though this property is not used in the $SU(2)$ version of Eq. (4.8) established below). What we will show is that the factorisation attempted in Eq. (4.8) is possible if on the right hand side we allow for elements from $SU(2)$.

The defining representation of $SU(2)$ is

$$SU(2) = \{U = 2 \times 2 \text{ complex matrices} \mid U^\dagger U = I_{2 \times 2}, \det U = 1\}, \tag{4.9}$$
Here, the group SU is a globally well-defined and smoothly varying matrix in SU matrices. The subset see that \( \sigma \) cyclic changes. Now the overlap transition rule (4.7) involves the subgroup of elements \( \mathcal{U}(\hat{a}, \alpha) \in SU(2) \rightarrow R(\hat{a}, \alpha) \in SO(3) \).

The rotations \( A'(\hat{k}), A''(\hat{k}) \) defined in Eqs. (4.3, 4.5) are images, in the sense of this mapping, of elements \( \mathcal{U}'(\hat{k}), \mathcal{U}''(\hat{k}) \) in SU(2) respectively:

\[
\mathcal{U}(\hat{k}) = e^{\frac{i}{2} \phi \sigma_3} e^{-\frac{i}{2} \theta \sigma_2} e^{i \phi \sigma_3}, \quad \hat{k} \in S^2_N;
\]
\[
\mathcal{U}''(\hat{k}) = e^{\frac{i}{2} \phi \sigma_3} e^{-\frac{i}{2} \theta \sigma_2} e^{-i \phi \sigma_3}, \quad \hat{k} \in S^2_S. \tag{4.12}
\]

Now the overlap transition rule (4.7) involves the subgroup of elements \( R(\hat{e}_3, 2\phi) \in SO(2) \subset SO(3) \), which happen to ‘coincide’ with elements \( \mathcal{U}(\hat{e}_2, 4\phi) \in SU(2) \) in the following sense:

\[
R(\hat{e}_3, 2\phi) = \left( \begin{array}{ccc} \mathcal{U}(\hat{e}_2, 4\phi) & 0 & 0 \\ 0 & 0 & 1 \end{array} \right),
\]
\[
\mathcal{U}(\hat{e}_2, 4\phi) = e^{-2i\phi \sigma_2} = \left( \begin{array}{cc} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{array} \right). \tag{4.13}
\]

It now turns out that within SU(2) a factorisation of the form (4.8) is possible:

\[
\mathcal{U}(\hat{e}_2, 4\phi) = V'(\hat{k})^{-1} V''(\hat{k}),
\]
\[
V'(\hat{k}) = e^{-i\phi \sigma_2} e^{-i \phi \sigma_3} e^{i \phi \sigma_3}, \quad \hat{k} \in S^2_N;
\]
\[
V''(\hat{k}) = e^{-i \phi \sigma_2} e^{-i \phi \sigma_3} e^{-i \phi \sigma_3}, \quad \hat{k} \in S^2_S. \tag{4.14}
\]

The structures of \( V'(\hat{k}), V''(\hat{k}) \) are suggested by those of \( \mathcal{U}'(\hat{k}), \mathcal{U}''(\hat{k}) \) in Eq. (4.12): in the latter we make the cyclic changes \( \sigma_1 \rightarrow \sigma_3 \rightarrow \sigma_2 \rightarrow \sigma_1 \), and replace \( \phi \) by \( 2\phi \). If we use Eq. (4.14) in Eq. (4.13) and then in Eq. (4.7) we see that

\[
A''(\hat{k}) = A'(\hat{k}) \left( \begin{array}{ccc} V'(\hat{k})^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{array} \right),
\]

i.e., \( A(\hat{k}) \rightarrow A'(\hat{k}) \left( \begin{array}{ccc} V'(\hat{k})^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) = A''(\hat{k}) \left( \begin{array}{ccc} V''(\hat{k})^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \tag{4.15} \)

is a globally well-defined and smoothly varying matrix in SU(3) with the property

\[
A(\hat{k})\hat{e}_3 = \hat{k}, \quad \forall \ \hat{k} \in S^2_{\text{dir}}. \tag{4.16}
\]

Here, the group SU(3) is the three-dimensional extension of SU(2) in Eq. (4.9), and consists of 3×3 unitary unimodular matrices. The subset \( \text{not subgroup} \) of SU(3) carrying \( \hat{e}_3 \) to \( \hat{k} \) is easy to characterise:

\[
A \in SU(3) : \ A\hat{e}_3 = \hat{k} \iff A = A \left( \begin{array}{ccc} \mathcal{U} & 0 \\ 0 & 0 & 1 \end{array} \right),
\]
\[
A \in SO(3), \ \mathcal{U} \in SU(2), \ A\hat{e}_3 = \hat{k}. \tag{4.17}
\]
This shows that $H$ is the tensor product
\[ \mathcal{H} = L^2(S^2_{\text{dir}}) \otimes \mathcal{H}^{(2)}, \]

(4.25)
where $L^2(S^2_{\text{pol}})$ is the Hilbert space of (scalar) complex square integrable functions over $S^2_{\text{pol}}$, and $\mathcal{H}^{(2)}$ is the two-dimensional complex Hilbert space (appropriate for the ‘polarization qubit’).

If $E(\hat{k})$ is a transverse electric field amplitude of a plane wave with propagation direction $\hat{k}$, using the expansion (4.20) we may attempt to represent its polarisation state by a point on the Poincaré sphere $S^2_{\text{pol}}$ in the ‘usual’ way:

$$\hat{k} \cdot \mathbf{E}(\hat{k}) = 0 : \mathbf{E}(\hat{k}) = z_\alpha g_\alpha(\hat{k}), \quad z_\alpha = g_\alpha(\hat{k})^* \cdot \mathbf{E}(\hat{k})$$
$$\rightarrow \mathbf{n}(z) = (z^\dagger z)^{-1} z^\dagger \tau z \in S^2_{\text{pol}}.$$ (4.26)

However this is not in general the ‘usual representation’ of polarisation states in the sense that, for instance, linear polarisations corresponding to real $\mathbf{E}(\hat{k})$ (upto overall phases) need not imply real $z$, so $\mathbf{n}(z)$ may not lie on the equator of $S^2_{\text{pol}}$ in the 1-2 plane. Indeed, for $\hat{k} \in S^2_{\text{dir}}$, from Eqs. (4.7,4.19) we have:

$$\begin{pmatrix} e_1'(\hat{k}) \\ e_2'(\hat{k}) \end{pmatrix} = \mathcal{U}_0(\theta, \phi) \begin{pmatrix} g_1(\hat{k}) \\ g_2(\hat{k}) \end{pmatrix},$$

$$\mathcal{U}_0(\theta, \phi) = \begin{pmatrix} \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \sin 2\phi & -i \sin \frac{\theta}{2} \cos 2\phi \\ -i \sin \frac{\theta}{2} \cos 2\phi & \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \sin 2\phi \end{pmatrix};$$

$$\frac{1}{\sqrt{2}}(e_1'(\hat{k}) + ie_2'(\hat{k})) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos \frac{\theta}{2} + e^{2i\phi} \sin \frac{\theta}{2} \\ \sin 2\phi \end{pmatrix} g_1(\hat{k})$$
$$+ \frac{i}{\sqrt{2}}\begin{pmatrix} \cos \frac{\theta}{2} - e^{2i\phi} \sin \frac{\theta}{2} \\ \sin 2\phi \end{pmatrix} g_2(\hat{k});$$

$$\frac{1}{\sqrt{2}}(e_1'(\hat{k}) - ie_2'(\hat{k})) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos \frac{\theta}{2} - e^{-2i\phi} \sin \frac{\theta}{2} \\ \sin 2\phi \end{pmatrix} g_1(\hat{k})$$
$$- \frac{i}{\sqrt{2}}\begin{pmatrix} \cos \frac{\theta}{2} + e^{-2i\phi} \sin \frac{\theta}{2} \\ \sin 2\phi \end{pmatrix} g_2(\hat{k}).$$ (4.27)

Using these in Eq. (4.26) we find that states of RCP and LCP are represented on $S^2_{\text{pol}}$ by the diametrically opposite points $\pm(\sin \theta \cos 2\phi, \sin \theta \sin 2\phi, \cos \theta)$, not by the usual North and South poles $(0,0,\pm 1)$. Correspondingly linear polarization states lie on the great circle on $S^2_{\text{pol}}$ in the plane orthogonal to $(\sin \theta \cos 2\phi, \sin \theta \sin 2\phi, \cos \theta)$.

V. GLOBAL BASES AND GEOMETRIC PHASES

We consider applications of the global results of the previous Section to the calculation of geometric phases. In the treatment in Sections II, III the starting point was a ray $\Gamma$ in a given transparent medium, based on which a beam passing through polarisation gadgets was then considered. From this, a curve $C_{\text{dir}} \subset S^2_{\text{dir}}$ was obtained, as in Eq. (2.41). The normalised electric field along the beam was then used to define a curve $C \subset B_5$ in the quantum mechanical $\mathcal{H} - \mathcal{B} - \mathcal{R}$ framework, Eqs. (2.45, 3.5), with $\mathcal{H} = \mathbb{C}^3$. At all stages the validity of Maxwell’s equations was kept in mind.

In [27], however, a curve $C_{\text{dir}} \subset S^2_{\text{dir}}$ is taken as the starting point for the discussion of geometric phases for beams with varying direction and polarisation state. From the point of view developed by us, this would mean that in principle, for a chosen $C_{\text{dir}} \subset S^2_{\text{dir}}$ to be physically realisable, we must imagine a transparent medium with suitable refractive index function $n(x)$, and a ray $\Gamma$ in this medium, such that a beam traveling along $\Gamma$ reproduces $C_{\text{dir}}$ as we follow $\mathbf{v}(s) = \dot{x}(s)$ along $\Gamma$. All this as well as the validity of Maxwell’s equations will be implicitly assumed in what follows.

In the notation of Section IV, then, we imagine being given a curve $C_{\text{dir}} = \{ \mathbf{k}(s) \in S^2_{\text{dir}} \} \subset S^2_{\text{dir}}$, and at each value of $s$ a normalised transverse electric field $\Psi(s)$:

$$\Psi(s) \in \mathbb{C}^3, \quad \Psi(s)^* \cdot \Psi(s) = 1, \quad \dot{k}(s) \cdot \Psi(s) = 0.$$ (5.1)

(Though not explicitly stated, the parameter $s$ could be the distance measured from some starting point on a beam in physical space $\mathbb{R}^3$). For calculating geometric phases we again use the $\mathcal{H} - \mathcal{B} - \mathcal{R}$ framework with $\mathcal{H} = \mathbb{C}^3$, (the framework used in [27] is different and is briefly recounted in the Appendix), and define

$$\mathcal{C} = \{ \Psi(s) \in \mathcal{H} \mid \Psi(s)^\dagger \Psi(s) = 1, \quad \dot{k}(s) \cdot \Psi(s) = 0, \quad s_1 \leq s \leq s_2 \} \subset B_5,$$

$$\pi[\mathcal{C}] = C \subset \mathcal{R}_4.$$ (5.2)
We expand $\Psi(s)$ in the complex global basis for $(T^S_{\hat{k}})_{\text{dir}}$ described in Eq. (4.19):

$$\Psi(s) = z_a(s) g_a(\hat{k}(s)), \quad z_a(s) = g_a(\hat{k}(s))^* \cdot \Psi(s),$$

and their derivatives $\dot{z}_a(s)$.

Thus $\hat{h}$

With some algebra the elements of $h$ can be calculated in terms of

$$h_{ab}(s) = h_{ba}(s)^* = -i g_a(\hat{k}(s))^* \cdot \dot{g}_b(\hat{k}(s)),$$

and then we have:

$$\varphi_g[C] = \varphi_{\text{tot}}[C] - \varphi_{\text{dyn}}[C],$$

$$\varphi_{\text{tot}}[C] = \arg(\langle \Psi(s_1) \rangle^* \cdot \langle \Psi(s_2) \rangle),$$

$$\varphi_{\text{dyn}}[C] = \text{Im} \int_{s_1}^{s_2} ds \langle \Psi(s) \rangle \langle \dot{\Psi}(s) \rangle$$

$$= \text{Im} \left\{ \int_{s_1}^{s_2} ds z(s)^\dagger \dot{z}(s) + i \int_{s_1}^{s_2} ds z(s)^\dagger h(s) z(s) \right\}.$$ (5.6)

With some algebra the elements of $h(s)$ can be calculated in terms of $\theta(s), \phi(s)$, the spherical polar angles of $\hat{k}(s)$, and their derivatives $\dot{\theta}(s), \dot{\phi}(s)$:

$$h_{11}(s) = -h_{22}(s)$$

$$= -\frac{1}{2} \left[ \dot{\theta}(s) \sin 2\phi(s) + \dot{\phi}(s) \cos 2\phi(s) \sin 2\theta(s) \right],$$

$$h_{12}(s) = h_{21}(s)^*$$

$$= \frac{1}{2} \left[ \dot{\theta}(s) \cos 2\phi(s) - (\sin 2\theta(s) \sin 2\phi(s)$$

$$+ i (1 - \cos 2\theta(s)) \dot{\phi}(s) \right].$$ (5.7)

It is interesting that the elements of the matrix $h(s)$, which arise from the dependences of $g_a(\hat{k})$ on $\hat{k}$, have rather elementary forms, which can be ascribed to the group theoretical arguments that led to the construction of $\{g_a(\hat{k})\}$.

As an illustration, let us consider the case where $C_{\text{dir}}$ is a closed loop, i.e., $\hat{k}(s_2) = \hat{k}(s_1)$. Let us further assume that $\Psi(s_2)$ differs from $\Psi(s_1)$ just by a phase $\theta$ so that $C$ is closed. Since in any case $g_a(\hat{k})$’s are determined by $\hat{k}$, these assumptions mean that

$$g_a(\hat{k}(s_2)) = g_a(\hat{k}(s_1));$$

$$\Psi(s_2) = e^{i\theta} \Psi(s_1) \Rightarrow z(s_2) = e^{i\theta} z(s_1), \quad \dot{n}(s_2) = \dot{n}(s_1).$$ (5.8)

Thus $\dot{n}(s) \equiv \dot{n}(z(s))$ describes a closed loop $C_{\text{pol}} \subset S^2_{\text{pol}}$, and the geometric phase (5.6) becomes:

$$\varphi_g[C] = \theta - \text{Im} \int_{s_1}^{s_2} ds z(s)^\dagger \dot{z}(s) - \int_{s_1}^{s_2} ds z(s)^\dagger h(s) z(s).$$ (5.9)

Comparing the first two terms with Eq. (2.28) we see that they reproduce exactly $\frac{1}{2} \Omega[C_{\text{pol}}]$, and the net result is

$$\varphi_g[C] = \frac{1}{2} \Omega[C_{\text{pol}}] - \int_{s_1}^{s_2} ds z(s)^\dagger h(s) z(s).$$ (5.10)
Further simplification of the second term seems not possible on general grounds, unless one has some information on the way $\Psi(s)$ varies with $s$ as $\hat{k}(s)$ traces the loop $C_{\text{dir}}$.

The separation of $\varphi_s[C]$ into the two terms on the right in Eq. (5.10) corresponds to the use of the $\{g_a(\hat{k})\}$ as a basis for $(T_{\hat{k}}S^2_{\text{dir}})^\circ$ at each $\hat{k}$. A change from $\{g_a(\hat{k})\}$ to some other globally smooth basis would alter both terms, while preserving the value of $\varphi_s[C]$. This could possibly limit the direct physical meaning we may ascribe to, say, $\frac{1}{i} \Omega[C_{\text{pol}}]$ on the right hand side.

VI. CONCLUDING REMARKS

We hope to have shown that in all geometric phase considerations in the domain of classical optics, the mathematical framework of quantum mechanics is adequate and flexible enough to provide a basis for the entire analysis. This is so in scalar wave, pure polarisation, as well as beam propagation problems. We have attempted to provide a clear picture of the situations being considered, fully tracing the phenomena ultimately to Maxwell’s equations in every case. The relevance of global topological aspects when discussing propagation direction and polarisation state simultaneously was pointed out in [27]. In our treatment we have addressed these using elementary group theoretical arguments relevant to the situation — leading, in our view, to particularly simple and elegant results.

The approach of this work now needs to be extended to other situations where, in place of a narrow beam endowed with polarisation properties, an extended polarised wave field in space is contemplated. This and other similar extensions will be taken up elsewhere.

Appendix: Comparison with the approach in [27]

Throughout this paper we have tried to show that the standard $\mathcal{H} - B - R$ structure of quantum mechanics, with $B$ a $U(1)$ principal fibre bundle over base $\mathcal{R}$, can be used under all circumstances to handle geometric phases in classical optical situations. In [27] a somewhat different structure has been used. We describe here briefly the connection between the two approaches.

For geometric phases associated with light beams we have used the complex three-dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}^3$ with inner product; the unit sphere $S^5 \simeq S^5$ of real dimension five; and the ray space $\mathcal{R}_4 \simeq \mathbb{C}P^2$ of real dimension four. Here $B_5$ is a $U(1)$ principal fibre bundle over base $\mathcal{R}_4$. We now define characteristic subsets of these spaces as follows:

\[ \hat{k} \in S^3_{\text{dir}} : \]
\[ \mathcal{H}_k = \{ E \in \mathcal{H} | \hat{k} \cdot E = 0 \} \simeq \mathbb{C}^2 ; \]
\[ B_k = B_5 \cap \mathcal{H}_k = \{ E \in \mathcal{H} | E^1 E = 1, \ \hat{k} \cdot E = 0 \} ; \]
\[ \mathcal{R}_k = B_k / U(1) = \{ \rho(E) = EE^\dagger \in \mathcal{R}_4 | E \in B_k \} ; \]
\[ B_k \simeq S^3, \ \mathcal{R}_k \simeq S^2, \text{ for each } \hat{k} \in S^3_{\text{dir}}. \] (A.1)

In more detail in the case of $\mathcal{R}_k$ we have:

\[ \rho \in \mathcal{R}_k \iff \rho = 3 \times 3 \text{ complex matrix,} \]
\[ \rho^1 = \rho^2 = \rho \geq 0 , \ \text{Tr} \rho = 1, \ \rho \hat{k} = 0 . \] (A.2)

For two points $\hat{k}, \hat{k}' \in S^3_{\text{dir}}$, we find easily:

\[ \hat{k}, \hat{k}' \neq 0 : B_k \cap B_{k'} = \{ E = q \hat{k} \wedge \hat{k}' | \hat{k} \wedge \hat{k}' | , | q | = 1 \} , \] (A.3)

consisting of essentially real $E$ corresponding to linear polarisations.

In contradistinction, the total and base spaces used in [27] are $\mathcal{T}, \mathcal{L}$ defined as:

\[ \mathcal{T} = \bigcup_{\hat{k} \in S^3_{\text{dir}}} B_k ; \] (a)

\[ \mathcal{L} = \bigcup_{\hat{k} \in S^3_{\text{dir}}} \mathcal{R}_k . \] (b)

(A.4)
These too are of real dimensions five and four respectively, and $T$ is a $U(1)$ principal fibre bundle over base $L$.

It is easy to see that the second statement in Eq. (4.22) leads to related Cartesian product structures for $T$ and $L$:

$$T = S^2_{\text{dir}} \times S^3, \quad L = S^2_{\text{dir}} \times S^2_{\text{pol}}.$$  \hspace{1cm} (A.5)

In our treatment, as mentioned above, we use uniformly $B_5$ (the sphere of normalised vectors in $\mathbb{C}^3$) rather than $T$, and the associated projective space $R_4 \equiv CP^2$ rather than $L$. It is important to recognize that $T \neq B_5$, and $L \neq R_4$. Writing $p$ for general points in $T$:

$$p \in T \Leftrightarrow p = (\hat{k}, E), \quad \hat{k} \in S^2_{\text{dir}}, \quad E \in B_k.$$  \hspace{1cm} (A.6)

Since $B_k \subset B_5$, the map $T \to B_5$ is well-defined:

$$p = (\hat{k}, E) \in T \to E \in B_5.$$  \hspace{1cm} (A.7)

However this is a many-to-one map. Given $E \in B_5$, $p = (\hat{k}, E)$ is not unique as:

- if $E \wedge E^* \neq 0$ : $\hat{k}$ is fixed up to a sign, resulting in a two-fold ambiguity;
- if $E \wedge E^* = 0$ : $\hat{k}$ is fixed up to an $SO(2)$ rotation, more precisely an $O(2)$ rotation, resulting in a continuous ambiguity involving linear polarisation states.

Thus, while both $T$ and $B_5$ are real five-dimensional manifolds, we do not have a one-to-one map between them, so they are not identical spaces. In a similar way, it can be checked that $L$ and $R_4$ are nonidentical.

In [27], geometric phases are defined for smooth closed curves $C_0 \subset T$, with images $C_0 \subset L$. Such a curve $C_0$ in parametrised form may be written as

$$C_0 = \left\{ p(s) = (\hat{k}(s), E(s)) \in T \mid s_1 \leq s \leq s_2 \right\} \subset T,$$  \hspace{1cm} (A.9)

with suitable end point conditions. In our approach, since as seen in Eq. (A.7) the map $T \to B_5$ is well-defined, we can pass from $C_0 \subset T$ to $C \subset B_5$ in an unambiguous manner:

$$C = \left\{ E(s) \in B_k(s) \mid s_1 \leq s \leq s_2 \right\} \subset B_5,$$  \hspace{1cm} (A.10)

and then use Eq. (1.7) to define the geometric phase in the kinematic approach. This is similar to the way in which in Sections 2 and 3 we take the electric field vector along a ray or a beam and use it to obtain a smooth curve in $B_5$ for which a geometric phase can be defined using the kinematic approach. The expression for the phase given in [27] is the same as in our treatment, which stays entirely within the standard $\mathcal{H} - B - R$ structure of quantum mechanics.

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See, for instance, E.M. Rabei et al. in Ref. [15].

The definition of this group is given later, in Eq. (4.9).

The group $SO(3)$ is defined in the axis-angle description later in Eq. (4.2).

The relationship between $SU(2)$ and $SO(3)$ is described, using axis-angle variables, later in Eq. (4.11) below.

It may be useful to recall that for a spherical triangle on $S^2$, the corresponding solid angle (subtended at the centre of the sphere) is the ‘spherical excess’, i.e., the amount by which the sum of the three internal angles exceeds $\pi$. This excess occurs because $S^2$ possesses positive curvature.

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