Curvature types of planar curves for gauges

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Abstract. In this paper results from the differential geometry of curves are extended from normed planes to gauge planes which are obtained by neglecting the symmetry axiom. Based on the gauge analogue of the notion of Birkhoff orthogonality from Banach space theory, we study all curvature types of curves in gauge planes, thus generalizing their complete classification for normed planes. We show that (as in the subcase of normed planes) there are four such types, and we call them analogously Minkowski, normal, circular, and arc-length curvature. We study relations between them and extend, based on this, also the notions of evolutes and involutes to gauge planes.

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1 Introduction

Although the concept of normed (or real Banach) planes describes the setting of Finsler geometry locally, the knowledge on existing types of curvatures of curves in such planes is widespread and not systematized in the literature; only very recently this gap was filled by the paper [1]. Using the notion of Birkhoff orthogonality from Banach space theory in a meaningful way, the

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The authors of [1] gave the first complete representation of all four curvature types making sense for curves in such planes. They also studied geometric properties of these types and their relations to each other. In the present paper, we extend this framework from normed to gauge planes, i.e., to planes satisfying all the axioms of two-dimensional real Banach spaces except for the symmetry axiom. For this purpose, we introduce an analogously generalized orthogonality type (as for norms, we stay with the name "Birkhoff orthogonality"), and the same is done with other needed notions. It turns out that we obtain precisely four natural analogues of the curvature types for norms, and correspondingly we call them Minkowski, normal, circular, and arc-length curvature. We present basic geometric relations between these curvature types, and as an application the concepts of evolutes and involutes as well as reverse evolutes and reverse involutes are introduced and investigated.

The geometry of finite dimensional real Banach (or normed) spaces, also called Minkowski geometry, is studied in the basic references [1], [6], [7], [8], [9] and [10], and recently it has strong relations to fields like optimization, discrete and computational geometry, convexity, convex and functional analysis, approximation theory and so on. Also from the viewpoint of differential geometry it is natural to develop geometric concepts for norms. Regarding curvature types of planar curves this was, as already mentioned, done in [1], and for surfaces in higher dimensions we refer to [2] and [6]. Deleting the symmetry axiom (i.e., going from norms to gauges), one can find almost no analogous references. We could locate the single paper [5] which contains various results on curves and hypersurfaces derived explicitly for gauge planes and spaces in the spirit of differential geometry, and in [6] certain multifocal hypersurfaces (e.g., polyellipsoids) are studied for norms and gauges.

A function $F : \mathbb{R}^n \to \mathbb{R}$ on the $n$-dimensional linear space $\mathbb{R}^n$ is called a convex distance function, or a gauge, if it satisfies the following conditions:

(i) $F(x) \geq 0$ for $x \in \mathbb{R}^n$, and $F(x) = 0 \iff x = 0$,

(ii) $F(\lambda x) = \lambda F(x)$ for $x \in \mathbb{R}^n, \lambda > 0$,

(iii) $F(x + y) \leq F(x) + F(y)$ for $x, y \in \mathbb{R}^n$.

Having this notion, the space $(\mathbb{R}^n, F)$ is called an $n$-dimensional generalized Minkowski space or gauge space (cf. [6]), hence being a direct extension of the concept of normed or Minkowski space (or linear Finsler space). We expect that the study of generalized Minkowski spaces will give new insights important also for Finsler geometry (cf. [3] and [4]). Clearly, for $n = 2$ generalized Minkowski planes or gauge planes are obtained.

Our paper is organized as follows. In Section 2 we give some basic notions for and facts on generalized Minkowski planes. In particular, the notions of associated gauge and Birkhoff orthogonality are introduced. In Section 3 we
directly generalize the main results from [1], namely by deriving the four announced curvature types of curves in gauge planes. Also relations between these four curvature types are presented there. Applying then our concepts, we prove in Section 4 the bi-directional relation between evolutes and involutes of given curves in gauge planes. All these results differ from those published on curvatures, evolutes, and involutes in [5], see the following Remark.

Remark. In [5] Guggenheimer discussed two curvature types of curves in generalized Minkowski planes which correspond to the Minkowski and the normal curvature in our setting. But different to his approach, our formulation is based on the notion of associated gauge, and explicit computational formulas for curvatures are obtained. Also, Guggenheimer defined a type of evolutes which is different from ours. For his setting, only a one-sided relation between evolutes and involutes can be shown.

2 Basic facts

Let \((\mathbb{R}^2, F)\) be a generalized Minkowski plane whose unit disk \(B\) and whose unit circle \(S\) are defined by

\[ B = \{ x \in \mathbb{R}^2 ; F(x) \leq 1 \} , \quad S = \{ x \in \mathbb{R}^2 ; F(x) = 1 \} ; \]

here \(B\) is a compact, convex set having the origin 0 as interior point and \(S\) as its boundary. This is equivalent to the property that the considered plane is equipped with a convex distance function \(F\) as defined in the introduction.

Let \([\cdot, \cdot] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\) be a map given by

\[ [x, y] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \]

where \(x = (x_1, x_2), y = (y_1, y_2)\) and \(\mid \mid\) denotes the usual determinant. Based on this, let

\[ F_a(x) = \sup \{ [y, x] ; F(y) = 1 \}, \]

which is also a gauge. We call \(F_a\) the gauge associated to \(F\). Then \([y, x] \leq F(y)F_a(x)\). Considering the orientation, we say that \(x\) is Birkhoff orthogonal to \(y\) (denoted by \(x \perp_B y\)) if \([x, y] = F(x)F_a(y)\) (cf. [7] for the subcase of normed planes).

We set

\[ S_a = \{ x \in \mathbb{R}^2 ; F_a(x) = 1 \} \]
and let 

\[
F_{a,a}(x) = (F_a)_a(x) = \sup\{[y, x]; F_a(y) = 1\}.
\]

**Example 1.** We consider a Randers norm on \(\mathbb{R}^2\) given by 

\[
F(x) = F(x_1, x_2) = \sqrt{x_1^2 + x_2^2 + bx_1},
\]

where \(|b| < 1\). The equation \(F(y) = 1\) is equivalent to 

\[
\{(1 - b^2)y_1 + b\}^2 + (1 - b^2)y_2^2 = 1.
\]

We can compute that 

\[
F_a(x) = \frac{1}{1 - b^2} \sqrt{(1 - b^2)x_1^2 + x_2^2} - \frac{b}{1 - b^2}x_2,
\]

which is also a Randers norm. The equation \(F_a(y) = 1\) is equivalent to 

\[
y_1^2 + (y_2 - b)^2 = 1,
\]

and we can show that 

\[
F_{a,a}(x) = \sqrt{x_1^2 + x_2^2 - bx_1} = F(-x).
\]

**Example 2.** Let \(D\) be a triangle in \(\mathbb{R}^2\) containing the origin as interior point and having the vertices \((p_1, p_2), (q_1, q_2)\) and \((r_1, r_2)\). Then we can interpret \(D\) as unit disk of a gauge \(F\). We can see that the unit disk \(B_a\) for \(F_a\) is represented by 

\[
p_1x_2 - p_2x_1 \leq 1, \quad q_1x_2 - q_2x_1 \leq 1, \quad r_1x_2 - r_2x_1 \leq 1,
\]

with vertices 

\[
\frac{1}{p_1q_2 - p_2q_1}(q_1 - p_1, q_2 - p_2), \quad \frac{1}{q_1r_2 - q_2r_1}(r_1 - q_1, r_2 - q_2), \quad \frac{1}{r_1p_2 - r_2p_1}(p_1 - r_1, p_2 - r_2).
\]

Similarly computing, we find that the unit disk \(B_{a,a}\) for \(F_{a,a}\) is the triangle with vertices \(-(p_1, p_2), -(q_1, q_2)\) and \(-(r_1, r_2)\). This implies that 

\[
F_{a,a}(x) = F(-x).
\]

In general we get the following
Proposition 2.1. Let $F$ be a gauge on $\mathbb{R}^2$. Then we have:

(i) $F_{a,a}(x) = F(-x)$ for any $x \in \mathbb{R}^2$.

(ii) $x \vdash_B y$ is equivalent to the fact that $y$ is Birkhoff orthogonal to $-x$ with respect to $F_a$ (denoted by $y \vdash_B^a (-x)$).

Proof. (i) Since 

$$[y,x] = [-x,y] \leq F(-x)F_a(y),$$

we have $F_{a,a}(x) \leq F(-x)$. Conversely, for any $x$ we can choose $\tilde{y} \neq 0$ so that 

$$[\tilde{y}, -x] = F(-x)F_a(\tilde{y}).$$

Then

$$F_{a,a}(x) \geq \left[ \frac{\tilde{y}}{F_a(\tilde{y})}, x \right] = \frac{1}{F_a(\tilde{y})}[-x, \tilde{y}] = F(-x).$$

Thus we have $F_{a,a}(x) = F(-x)$.

(ii) By (i), the Birkhoff orthogonality condition $[x,y] = F(x)F_a(y)$ is rewritten as $[y,-x] = F_a(y)F_{a,a}(-x)$. So $x \vdash_B y$ is equivalent to the fact that $y$ is Birkhoff orthogonal to $-x$ with respect to $F_a$. □

Remark. As in Example 2, for a convex polygon containing the origin as interior point we can show that $F_{a,a}(x) = F(-x)$. Thus, suitable approximation of general convex compact domains containing the origin as interior point by convex polygons yields that we have $F_{a,a}(x) = F(-x)$ also for this general setting. This gives another proof of Prop. 2.1(i).

Remark. When $F$ is symmetric (that is, $F(x) = F(-x)$), then we have $F_{a,a}(x) = F(x)$ (cf. [7]).

3 Curvature types

Let $(\mathbb{R}^2, F)$ be a generalized Minkowski plane. In the following, we assume that its unit circle $S$ is smooth and strictly convex. Let $\gamma(s)$ be an oriented smooth curve in $(\mathbb{R}^2, F)$ with arc-length parameter $s$ such that $F(\gamma'(s)) = 1$.

Let $\varphi(t)$ be a counter-clockwise parametrization of $S$ such that $[\varphi(t), \varphi'(t)] > 0$. Set

$$u(t) = \int_{t_0}^t [\varphi(\tau), \varphi'(\tau)]d\tau,$$

and let $\varphi(u)$ be a parametrization of $S$ by $u$. Then we have

$$\varphi(u) \vdash_B \varphi'(u), \quad F_a(\varphi'(u)) = 1.$$
We can write $\gamma'(s) = \varphi(u(s))$, and we define the Minkowski curvature by $k_m(s) := u'(s)$. We set $n_\gamma(s) = \varphi'(u(s)) \in S_a$, which we call the right normal vector field. Then we have

$$\gamma'(s) \perp_B n_\gamma(s), \quad [\gamma'(s), n_\gamma(s)] = 1, \quad \gamma''(s) = k_m(s)n_\gamma(s).$$

Let $\psi(v)$ be a counter-clockwise parametrization of the unit circle $S_a$ for $F_a$ such that $F_{a.a}(\psi'(v)) = F(-\psi'(v)) = 1$. Then we can write $n_\gamma(s) = \psi(v(s))$. We define the normal curvature by $k_n(s) := \psi'(v(s))$.

**Lemma 3.1.** $\gamma'(s) = -\psi'(v(s))$.

Proof. By the condition, we have $\psi(v) \perp_B \psi'(v)$. By (ii) of Prop. 2.1, this is equivalent to $-\psi'(v) \perp_B \psi(v)$. Thus we have

$$\gamma'(s) \perp_B n_\gamma(s), \quad -\psi'(v(s)) \perp_B n_\gamma(s).$$

Since $F(\gamma'(s)) = F(-\psi'(v(s))) = 1$ and $S$ is strictly convex, we get $\gamma'(s) = -\psi'(v(s))$. □

By Lemma 3.1, we have

$$n'_\gamma(s) = -k_n(s)\gamma'(s).$$

Let $\varphi(t)$ be a counter-clockwise parametrization of $S$ by the arc-length parameter $t$ such that $F(\varphi'(t)) = 1$. Then we can write $\gamma'(s) = \varphi'(t(s))$, and we define the circular curvature by $k_c(s) := t'(s)$. We call $\varphi(t(s)) \in S$ the left normal vector field, which satisfies $[\gamma'(s), \varphi(t(s))] < 0$. Now we define the evolute $E$ of $\gamma$ by

$$E(s) = \gamma(s) - \frac{1}{k_c(s)}\varphi(t(s)).$$

Then

$$E'(s) = -\left(\frac{1}{k_c(s)}\right)'\varphi(t(s)).$$

So $E$ is the envelop of left normal lines.

For the counter-clockwise parametrization $\varphi(t)$ of $S$ by the arc-length parameter $t$ with $F(\varphi'(t)) = 1$, we can also write $\gamma'(s) = \varphi(\tilde{t}(s))$, and we define the arc-length curvature by $k_l(s) := \tilde{t}'(s)$. Thus we can define the involute $I$ of $\gamma$ by

$$I(s) = \gamma(s) + (c - s)\gamma'(s) = \gamma(s) + (c - s)\varphi(\tilde{t}(s)).$$

Next we discuss relations between the four obtained types of curvatures.
Proposition 3.1. \( k_l(s) = F(n_\gamma(s))k_m(s) \).

Proof. Since \( \gamma''(s) = k_m(s)n_\gamma(s) = k_l(s)\varphi'({\hat{t}}(s)), \)
\( \gamma'(s) \vdash_B n_\gamma(s), \quad \gamma'(s) = \varphi({\hat{t}}(s)) \vdash_B \varphi'({\hat{t}}(s)), \)
\( n_\gamma(s) \in S_a, \quad \varphi'({\hat{t}}(s)) \in S, \)
we have
\[ \varphi'({\hat{t}}(s)) = \frac{n_\gamma(s)}{F(n_\gamma(s))}, \quad k_l(s) = F(n_\gamma(s))k_m(s). \]

Proposition 3.2. The circular curvature \( k_c \) is the normal curvature \( k_a \) with respect to the associated gauge \( F_a \).

Proof. Let \( s_a \) be the arc length of \( \gamma \) with respect to \( F_a \). Let \( n_\gamma^a \) be the right normal vector field to \( \gamma \) with respect to \( F_a \). Since \( n_\gamma^a \in S_{a,a} = (S_a)_a \), we have \( F_{a,a}(n_\gamma^a) = F(-n_\gamma^a) = 1 \). Then
\[ \frac{d\gamma}{ds_a}(s_a) \vdash_B n_\gamma^a(s_a), \]
and \( \gamma'(s) \vdash_B n_\gamma^a(s_a(s)). \) By (ii) of Prop. 2.1 and \( \gamma'(s) = \varphi'(t(s)), \) we have
\[ -n_\gamma^a(s_a(s)) \vdash_B \varphi'(t(s)). \]
On the other hand,
\[ \varphi(t(s)) \vdash_B \varphi'(t(s)). \]
Since \(-n_\gamma^a(s_a(s)) \in S, \varphi(t(s)) \in S \) and \( S \) is strictly convex, we find that
\[ -n_\gamma^a(s_a(s)) = \varphi(t(s)). \]
Differentiation with respect to \( s \) yields
\[ k_a^a(s_a(s))\frac{d\gamma}{ds_a}(s_a(s)) \cdot \frac{ds_a}{ds} = \varphi'(t(s))t'(s) \]
and
\[ k_a^a(s_a(s))\gamma'(s) = k_c(s)\gamma'(s). \]
Thus we get \( k_c(s) = k_a^a(s_a(s)). \) \qed

We assume that \( F \) is smooth on \( \mathbb{R}^2 \setminus \{0\} \) and give formulas for curvatures. Let \( (x_1, x_2) \) be the standard coordinates on \( (\mathbb{R}^2, F) \). From the positive homogeneity of \( F \), we have
(a) \[ \sum_{i=1}^{2} t_i F_{x_i}(t_1, t_2) = F(t_1, t_2) \]
and
(b) \[ \sum_{i=1}^{2} t_i F_{x_i x_i}(t_1, t_2) = 0 \]
for \((t_1, t_2) \in \mathbb{R}^2 \setminus \{0\}\) (cf. Chap. 1 of [3]).

Let \(\gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau))\) be an oriented smooth curve in \((\mathbb{R}^2, F)\) with arbitrary parameter \(\tau\). Let \(s\) be the arc-length parameter. Then
\[
\frac{d\gamma}{ds} = \frac{1}{F(\gamma_1', \gamma_2')} (\gamma_1', \gamma_2').
\]

We note that the vector
\[
(-F_{x_2}(\gamma_1', \gamma_2'), F_{x_1}(\gamma_1', \gamma_2'))
\]
is parallel to the tangential line of \(S\) at \(d\gamma/ds\). By the definition of the associated gauge \(F_a\), we have
\[
F_a(-F_{x_2}(\gamma_1', \gamma_2'), F_{x_1}(\gamma_1', \gamma_2')) = 1,
\]
where we use (a). So the right normal vector field \(n_\gamma\) is given by
\[
n_\gamma = (-F_{x_2}(\gamma_1', \gamma_2'), F_{x_1}(\gamma_1', \gamma_2')).
\]

Using (a), we can compute
\[
\frac{d^2\gamma}{ds^2} = \frac{1}{(F(\gamma_1', \gamma_2'))^3} \left( \gamma_1'' \{ F(\gamma_1', \gamma_2') - \gamma_1' F_{x_1}(\gamma_1', \gamma_2') \} - \gamma_1' \gamma_2'' F_{x_2}(\gamma_1', \gamma_2') \right)
\]
\[
- \gamma_2'' \{ F(\gamma_1', \gamma_2') - \gamma_2' F_{x_2}(\gamma_1', \gamma_2') \} - \gamma_1'' \gamma_2' F_{x_1}(\gamma_1', \gamma_2')
\]
\[
= \frac{\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2'}{(F(\gamma_1', \gamma_2'))^3} (-F_{x_2}(\gamma_1', \gamma_2'), F_{x_1}(\gamma_1', \gamma_2')) = \frac{\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2'}{(F(\gamma_1', \gamma_2'))^3} \cdot n_\gamma.
\]
Thus we get

**Theorem 3.1.** The Minkowski curvature is computed by
\[
k_m = \frac{\gamma_1' \gamma_2'' - \gamma_1'' \gamma_2'}{(F(\gamma_1', \gamma_2'))^3}.
\]

And by Proposition 3.1 and Theorem 3.1, we obtain
Corollary 3.1. The arc-length curvature is given by

\[ k_l = \frac{\gamma''_1 \gamma'_2 - \gamma''_2 \gamma'_1}{(F(\gamma'_1, \gamma'_2))^3} \cdot F(-F_{x_2}(\gamma'_1, \gamma'_2), F_{x_1}(\gamma'_1, \gamma'_2)). \]

Next we have

\[ \frac{dn_\gamma}{ds} = \frac{1}{F(\gamma'_1, \gamma'_2)}(-F_{x_2}(\gamma'_1, \gamma'_2))', (F_{x_1}(\gamma'_1, \gamma'_2))'. \]

Differentiating the relation

\[ \gamma'_1 F_{x_1}(\gamma'_1, \gamma'_2) + \gamma'_2 F_{x_2}(\gamma'_1, \gamma'_2) = F(\gamma'_1, \gamma'_2) \]

from (a), we have

\[ \gamma'_1(F_{x_1}(\gamma'_1, \gamma'_2))' + \gamma'_2(F_{x_2}(\gamma'_1, \gamma'_2))' = 0. \]

Therefore \( \gamma'_1 \neq 0 \) yields

\[ \frac{dn_\gamma}{ds} = -\frac{(F_{x_2}(\gamma'_1, \gamma'_2))'}{\gamma'_1 F(\gamma'_1, \gamma'_2)}(\gamma'_1, \gamma'_2) = -\frac{(F_{x_2}(\gamma'_1, \gamma'_2))'}{\gamma'_1} \cdot \frac{d\gamma}{ds}. \]

Similarly, when \( \gamma'_2 \neq 0 \), then we obtain

\[ \frac{dn_\gamma}{ds} = \frac{(F_{x_1}(\gamma'_1, \gamma'_2))'}{\gamma'_2} \cdot \frac{d\gamma}{ds}. \]

Thus we have the following theorem.

Theorem 3.2. (i) If \( \gamma'_1 \neq 0 \), then we get

\[ k_n = \frac{(F_{x_2}(\gamma'_1, \gamma'_2))'}{\gamma'_1}. \]

(ii) If \( \gamma'_2 \neq 0 \), then we have

\[ k_n = -\frac{(F_{x_1}(\gamma'_1, \gamma'_2))'}{\gamma'_2}. \]

Remark. The above computation on \( k_n \) is related to (b).

By Proposition 3.2 and Theorem 3.2, the following corollary is obtained.
Corollary 3.2. (i) If $\gamma' \neq 0$, then we obtain
$$k_c = \frac{((F_a)_{x_2}(\gamma'_1, \gamma'_2))'}{\gamma'_1}.$$

(ii) If $\gamma'_2 \neq 0$, then we see that
$$k_c = -\frac{((F_a)_{x_1}(\gamma'_1, \gamma'_2))'}{\gamma'_2}.$$

Remark. In general, the formula for $k_c$ is not explicit because $F_a$ is used.

Remark. (i) In the Euclidean case, those formulas reduce to the classical one.
(ii) The analogous discussion in normed planes can be seen in Section 5 of [1], where radial coordinates are used.

4 Evolutes and involutes

Let $\gamma(s)$ be an oriented smooth curve in $(\mathbb{R}^2, F)$ with arc-length parameter $s$ so that $F(\gamma'(s)) = 1$, as in Section 3. For an oriented curve $\alpha(t)$, we define the reverse curve $\alpha^{-}$ by $\alpha^{-}(t) = \alpha(-t)$.

Theorem 4.1. (i) If $(c - s)k_l(s) > 0$, then the evolute of the involute $I$ of $\gamma$ is $\gamma$.

(ii) If $(c - s)k_l(s) < 0$, then the reverse evolute of the reverse involute $I^{-}$ of $\gamma$ is $\gamma$.

Proof. (i) First we have
$$I'(s) = (c - s)k_l(s)\varphi'(\hat{t}(s)).$$

Let $s^*$ denote the arc length of $I$. Since $(c - s)k_l(s) > 0$, we have
$$\frac{ds^*}{ds} = F(I'(s)) = (c - s)k_l(s), \quad \frac{dI}{ds^*} = \varphi'(\hat{t}(s)).$$

The circular curvature $k^*_c$ of $I$ is given by
$$k^*_c = \frac{d\hat{t}}{ds^*} = \frac{d\hat{t}}{ds} \frac{ds}{ds^*} = \frac{1}{c - s}.$$

So the evolute $E_I$ of $I$ satisfies
$$E_I(s) = I(s) - \frac{1}{k^*_c(s)}\varphi(\hat{t}(s)) = \gamma(s).$$
(ii) Since \((c-s)k_l(s) < 0\) and letting \(s = -\sigma\), we have \((c+\sigma)k_l(-\sigma) < 0\). The reverse involute \(I^-\) is given by

\[
I^-(\sigma) = I(-\sigma) = \gamma(-\sigma) + (c+\sigma)\varphi(\hat{t}(-\sigma)).
\]

Then

\[
(I^-)'(\sigma) = -(c+\sigma)k_l(-\sigma)\varphi'(\hat{t}(-\sigma)).
\]

Let \(s^*\) denote the arc length of \(I^-\). Then

\[
\frac{ds^*}{ds} = F((I^-)'(\sigma)) = -(c+\sigma)k_l(-\sigma), \quad \frac{dI^-}{ds^*} = \varphi'(\hat{t}(-\sigma)).
\]

The circular curvature \(k^*_c\) of \(I^-\) is given by

\[
k^*_c = \frac{d}{ds^*}(\hat{t}(-\sigma)) = \frac{d}{d\sigma}(\hat{t}(-\sigma))\frac{d\sigma}{ds^*} = \frac{1}{c+\sigma}.
\]

So the evolute \(E_{(I^-)}\) of \(I^-\) satisfies

\[
E_{(I^-)}(\sigma) = I^-(\sigma) - \frac{1}{k^*_c(\sigma)}\varphi(\hat{t}(-\sigma)) = \gamma(-\sigma).
\]

Now, by \(\sigma = -s\) (that is, reversing the orientation), we get

\[
E_{(I^-)}(s) = \gamma(s).
\]

\(\square\)

**Theorem 4.2.** \(\text{(i) If } k'_c(s) > 0, \text{ then an involute of the evolute } E \text{ of } \gamma \text{ is } \gamma.\)

\(\text{(ii) If } k'_c(s) < 0, \text{ then a reverse involute of the reverse evolute } E^- \text{ of } \gamma \text{ is } \gamma.\)

Proof. (i) We have

\[
E'(s) = -\left(\frac{1}{k_c(s)}\right)' \varphi(t(s)) = \frac{k'_c(s)}{(k_c(s))^2} \varphi(t(s)).
\]

Let \(s^*\) denote the arc length of \(E\). Since \(k'_c(s) > 0\), we have

\[
\frac{ds^*}{ds} = F(E'(s)) = \left(\frac{1}{k_c(s)}\right)' = \frac{k'_c(s)}{(k_c(s))^2},
\]

\[
\frac{dE}{ds^*} = \varphi(t(s)),
\]

and

\[
s^* = \int \left(\frac{1}{k_c(s)}\right)' ds = -\frac{1}{k_c(s)} + c.
\]
So an involute $I_E$ of $E$ satisfies

$$I_E(s) = E(s) + (c - s^*) {dE \over ds^*} = \gamma(s).$$

(ii) Since $k'_c(s) < 0$ and letting $s = -\sigma$, we have $k'_c(-\sigma) < 0$. The reverse evolute $E^-$ is given by

$$E^-(\sigma) = E(-\sigma) = \gamma(-\sigma) - \frac{1}{k_c(-\sigma)} \varphi(t(-\sigma)).$$

Then

$$(E^-)'(\sigma) = - \left( \frac{1}{k_c(-\sigma)} \right)' \varphi(t(-\sigma)) = - \frac{k'_c(-\sigma)}{(k_c(-\sigma))^2} \varphi(t(-\sigma)).$$

Let $s^*$ denote the arc length of $E^-$. Then

$$\frac{ds^*}{d\sigma} = F((E^-)'(\sigma)) = - \left( \frac{1}{k_c(-\sigma)} \right)' = - \frac{k'_c(-\sigma)}{(k_c(-\sigma))^2},$$

and

$$\frac{dE^-}{ds^*} = \varphi(t(-\sigma)),$$

and

$$s^* = - \int \left( \frac{1}{k_c(-\sigma)} \right)' d\sigma = - \frac{1}{k_c(-\sigma)} + c.$$

So an involute $I_{(E^-)}$ of $E^-$ satisfies

$$I_{(E^-)}(\sigma) = E^-(\sigma) + (c - s^*) {dE^- \over ds^*} = \gamma(-\sigma).$$

Reversing the orientation by $\sigma = -s$, we get

$$I_{(E^-)}(-s) = \gamma(s).$$

\[ \square \]

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