This paper concerns the construction and analysis of a new family of exact general relativistic shock waves. The construction resolves the open problem of determining the expanding waves created behind a shock-wave explosion into a static isothermal sphere with an inverse square density and pressure profile. The construction involves matching two self-similar families of solutions to the perfect fluid Einstein field equations across a spherical shock surface. The matching is accomplished in Schwarzschild coordinates where the shock waves appear one derivative less regular than they actually are. Separately, both families contain singularities, but as matched shock-wave solutions, they are singularity free. There was no guarantee ahead of time that the matching of the two families could be achieved within the regions where both families are nonsingular. Indeed, for pure radiation equations of state, the matching occurs near the sonic point of the interior expanding wave and this makes the analysis quite delicate, both numerically and formally. It is for this reason the construction is accompanied by a rigorous existence proof in the pure radiation case. The analysis is extended to demonstrate Lax stability in the pure radiation case and provide a criterion for Lax stability in all other cases. These shock-wave solutions represent an intriguing new mechanism in General Relativity for exhibiting accelerations in asymptotically Friedmann spacetimes, analogous to the accelerations modelled by the cosmological constant in the Standard Model of Cosmology. However, unlike in the Standard Model of Cosmology, these shock-wave solutions solve the Einstein field equations in the absence of a cosmological constant, opening up the question of whether a purely mathematical mechanism could account for the cosmic acceleration observed today, rather than dark energy.

**Keywords**  General Relativity · Shock Wave · Cosmology · Dark Energy

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1 Introduction

Consider a static isothermal sphere with an inverse square density profile held in gravitational equilibrium by its own pressure. These spherically symmetric solutions of the perfect fluid Einstein field equations form a subset of the Tolman-Oppenheimer-Volkoff (TOV) spacetimes, some of which represent idealised models of stars. The models with an inverse square density profile are physically unstable due to the singularity in pressure and density at the radial centre. In this paper, the singular core is removed and replaced by an expanding wave, separated from the rest of the static spacetime by a spherical shock surface. The resulting model is the general relativistic analogue of a shock-wave explosion into a static isothermal sphere. In particular, this paper determines the physically relevant family of interior expanding waves that can be matched to an exterior static sphere to form a global general relativistic shock wave.

These shock waves conserve mass-energy and momentum across the shock surface and produce no delta function sources in Schwarzschild coordinates. Moreover, there exists a local coordinate transformation for which the solution has optimal metric regularity, which is $C^{1,1}$. Such properties make these shock waves true weak solutions of the perfect fluid Einstein field equations, with the expanding waves given as exact solutions of an explicitly defined system of ODE. Most importantly, this paper determines the expanding wave created behind an outgoing shock wave in the case of pure radiation, the state of matter in the Radiation Dominated Epoch of the Early Universe.

The physically relevant family of expanding waves is a subset of a larger family of spherically symmetric self-similar spacetimes first considered by Cahill and Taub [1], but the problem of determining whether members of this family are the actual expanding waves of outgoing shock waves has remained open. This paper resolves this open problem for shock waves expanding outward into a static singular isothermal sphere such that the global solution regularises the singularity at the radial centre. The physically relevant family consists of spacetimes that are asymptotically Friedmann approaching the radial origin at some fixed time or approaching the distant future at some fixed radius. This family contains the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, representing the current Standard Model of Cosmology, with all other members corresponding to self-similar perturbations of this spacetime. This family was first identified by Carr and Yahil [3] and later independently identified by Smoller and Temple [10]. Carr and Yahil describe these perturbations as density perturbations, whereas Smoller and Temple describe these perturbations as perturbations to the magnitude of accelerated expansion of the spacetime.

Smoller and Temple show that the parameter corresponding to the magnitude of accelerated expansion, known as the acceleration parameter, exhibits a property similar to the cosmological constant in a flat FLRW spacetime. Given that the incorporation of a shock wave into a cosmological model is a simple and natural way to bound the total mass of a big bang, Temple conjectures that there might be a connection between the anomalous acceleration incurred in the expansion wave created behind a big bang shock wave and the acceleration supposedly induced by dark energy. A dual-state model in the form of a general relativistic shock wave is essential for this conjectured mechanism of accelerated expansion as the limited parameter freedom in spherically symmetric self-similar perfect fluid spacetimes determines the value of the acceleration parameter when the equations of state each side of the shock surface are fixed. It would be remarkable if the accelerations specified by the model at the centre of expansion matched the accelerations currently accounted for by dark energy, although this would be a model for which the Milky Way would be close to the centre of expansion.

Such a Big Wave or Big Shock cosmological model places the present observable Universe within the spherical shock surface, which will eventually come into view. This paper works towards such a model by constructing an analogous shock wave for which the shock surface lies within the Hubble radius, that is, the shock surface would already be observable if taken as a cosmological model. The motivation for considering shock waves in the Radiation Dominated Epoch is that the accelerated expansion exhibited in this epoch influences the accelerated expansion found in the Matter Dominated Epoch that we inhabit today. This connection is considered by Smoller, Temple and Vogler [11] and an interesting consequence of this work is the realisation that the flat FLRW spacetime in the Matter Dominated Epoch is unstable to spherical perturbations. This opens up the question of what alternative spacetimes in the Matter Dominated Epoch we are more likely to find ourselves in, and in particular, what spacetimes in the Radiation Dominated Epoch give rise to them.

The construction of a shock wave in the Radiation Dominated Epoch with an asymptotically Friedmann interior and a shock surface beyond the Hubble radius, along with formalising the details of the transition into the Matter Dominated Epoch, are topics of current research for Alexander, Temple and Vogler. It is useful to note that Smoller and Temple [8] have already identified a method for matching an FLRW and a static spacetime beyond the Hubble radius, with the details provided in [9]. This method involves placing the whole Universe, including the shock surface, within the Schwarzschild radius of a time reversed black hole. However, Smoller and Temple did not fully resolve the expansion wave behind the shock for pure radiation equations of state each side of the shock surface.
Construction of the family of general relativistic shock waves considered in this paper builds upon the work of a number of authors, beginning with Cahill and Taub \[1 \]. In both [1] and this paper, solutions of the perfect fluid Einstein field equations are assumed to be spherically symmetric and self-similar in the variable:

\[ \xi = \frac{r}{t} \]

that is, self-similarity of the first kind. These two assumptions reduce the Einstein field equations, a system of nonlinear partial differential equations, to a system of nonlinear ordinary differential equations in the single variable \( \xi \). It is in this setting that Cahill and Taub establish criteria for the uniqueness of solutions, along with a method to form shock waves. The flat FLRW and TOV spacetimes are explicit solutions to these equations when admitting barotropic equations of state, which for self-similar solutions of the first kind, are restricted to the form:

\[ p = \sigma \rho \]

for some constant \( \sigma \), referred to as the equation of state parameter. It is also shown in [1] that the TOV spacetimes form the unique family of static perfect fluid spacetimes that are spherically symmetric and self-similar of the first kind. This family forms the exterior of all general relativistic shock waves considered in this paper with shock waves admitting an asymptotically Friedmann interior being referred to as Friedmann-Static shock waves.

A Friedmann-static shock wave is constructed by Cahill and Taub in [1] by matching a pure radiation \( (\sigma = \frac{1}{2}) \) flat FLRW spacetime to a TOV spacetime across a spherical shock surface. Cahill and Taub claimed the existence of a two-parameter family of self-similar pure radiation spacetimes that could be matched to a TOV spacetime to form a shock wave in a subsequent paper that was not published and possibly never completed. This paper partially resolves this open problem for the one-parameter subfamily of asymptotically Friedmann pure radiation spacetimes.

Despite the large influence of Cahill and Taub’s work, this paper primarily builds upon the work of Smoller and Temple, who did not initially consider self-similarity. In [5], a number of useful theorems concerning the regularity of spherically symmetric general relativistic shock waves are given, and in [7], a criteria for determining the Lax stability of these shock waves is introduced, that is, stability in the gas dynamical sense. Furthermore, Smoller and Temple [7] generalise Cahill and Taub’s shock wave to a one-parameter family of Friedmann-static shock waves, with the free parameter corresponding to either the interior equation of state parameter or the exterior equation of state parameter, but not both. This is an important detail as it implies an additional parameter is needed to construct a general relativistic shock wave in the Radiation Dominated Epoch, that is, a shock wave with a pure radiation equation of state on both the interior and exterior side of the shock surface.

This additional parameter is identified by Smoller and Temple in [10]. In [10] the family of asymptotically Friedmann spacetimes are derived in self-similar Schwarzschild coordinates local to the radial centre. This family is a one-parameter or two-parameter family depending on whether the equation of state parameter is included in the count. It is common in the literature to exclude this parameter from the count but this paper does not follow this convention and instead makes clear whether the equation of state parameter is included in each instance. This additional parameter is the aforementioned acceleration parameter and it is in [10] that the relation between the acceleration parameter and the cosmological constant is formalised.

Unbeknown to Smoller and Temple at the time, Carr and Yahil had already identified the family of asymptotically Friedmann spacetimes in self-similar comoving coordinates, the same convention employed by Cahill and Taub. However, unlike the exact local form derived by Smoller and Temple, Carr and Yahil instead derived an asymptotic local form. The complete classification of spherically symmetric self-similar in \( \xi \) solutions to the perfect fluid Einstein field equations was later completed by Carr and Coley [2]. In addition to determining the number of free parameters present in each family of solutions, Carr and Coley provide a detailed discussion of the physical relevance of each of these families. It was Carr and Coley that provided the inspiration to using a phase-space approach in this paper.

There is a fair amount of preliminary theory that needs to be introduced prior to the construction and analysis of the new family of Friedmann-static shock waves and this is the purpose of Section [2]. The ODE introduced by Cahill and Taub, and used by Carr and others, are derived using self-similar comoving coordinates, whereas the ODE introduced by Smoller and Temple [10] are derived using self-similar Schwarzschild coordinates. There are advantages to both approaches, but the latter approach is more useful in the construction of shock-wave solutions, so this approach is adopted. Once Smoller and Temple’s system of ODE are introduced, the TOV, FLRW, and asymptotically FLRW solutions are derived in self-similar Schwarzschild coordinates. Proceeding these derivations, the shock wave construction process is introduced, followed by important regularity and Lax stability results.

Section [3] begins with an alternative derivation of the explicit one-parameter family of Friedmann-static shock waves originally derived in [7]. This warm-up derivation introduces Lemma [2] which is central to the construction of the more general two-parameter family of Friedmann-static shock waves, with both parameter counts including \( \sigma \). Since the asymptotically Friedmann spacetimes are not known explicitly, numerical approximations are used to construct...
the two-parameter family of Friedmann-static shock waves. The acceleration parameter and shock position are then approximated in the pure radiation case.

Section 4 begins with Lemma 3, which generalises the Lax characteristic conditions to a broad family of outgoing general relativistic shock waves. Following this, and an alternative derivation of the stability theorem from [7], an analysis of the Rankine-Hugoniot jump conditions results in Theorem 6, which establishes the Lax stability of all outgoing spherically symmetric self-similar in general relativistic shock waves with a TOV exterior.

Section 5 begins by introducing Lemma 5, a monotonicity lemma that bounds the family of asymptotically Friedmann spacetimes within a broad region of the ODE system phase space. The remainder of this section is dedicated to the proof of Theorem 7, which is a rigorous existence proof of the Friedmann-static pure radiation shock wave. This is the unique Friedmann-static shock wave that models a perfect fluid with a pure radiation equation of state both sides of the shock. Theorem 7 is the main result of this paper and the first result to establish the existence of a non-explicit global weak solution of the ODE derived by Smoller and Temple. The proof is complicated by the fact that the matching of the interior and exterior spacetimes occur near the sonic surface of the ODE system phase space.

Finally, Section 6 introduces a conjecture regarding the rigorous existence of the full family of Friedmann-static shock waves and discusses some open problems that remain.

2 Preliminaries

2.1 Spherically Symmetric Self-Similar Field Equations

Consider first the Einstein field equations:

$$ G = \kappa T $$

where $G$ is the Einstein curvature tensor, $T$ is the stress-energy-momentum tensor and:

$$ \kappa = \frac{8\pi G}{c^4} $$

where $c$ is the speed of light and $G$ is the gravitational constant. When modelling a perfect fluid, the stress-energy-momentum tensor takes the form:

$$ T = \left( \rho + \frac{p}{c^2} \right) u \otimes u + pg $$

where $g$ is the metric tensor, $\rho$ is the fluid density, $p$ is the fluid pressure and $u$ is the fluid four-velocity.

**Definition 1.** A *similarity solution* is a solution that is spherically symmetric and self-similar of the first kind, that is, the self-similar variable takes the form:

$$ \xi = \frac{r}{t} $$

where $t$ and $r$ are the temporal and radial coordinates respectively.

From now onwards, the constants $c$ and $G$ are set to unity and all solutions are assumed to be similarity solutions of the perfect fluid Einstein field equations. Any metric of a similarity solution may be written, without loss of generality, in the following self-similar Schwarzschild coordinate form:

$$ ds^2 = -B(\xi)dt^2 + \frac{1}{A(\xi)}dr^2 + r^2 d\Omega^2 $$

where $A, B > 0$ and $d\Omega^2$ denotes the standard metric on the unit two-sphere, that is:

$$ d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2 $$

Under the assumption of spherical symmetry the fluid four-velocity may also be written without loss of generality as:

$$ u = (u^0, u^1, 0, 0) $$

and under the normalisation condition:

$$ g(u, u) = -1 $$

the fluid four-velocity has only one independent component. The normalisation condition means that the fluid four-velocity can be fully specified through a single variable.
Definition 2. The *Schwarzschild coordinate velocity* is defined by:

\[ v = \frac{1}{\sqrt{AB}} u^1 \]

Together with \( A, B, \rho \) and \( p \), the Schwarzschild coordinate velocity \( v \) is one of five unknown variables that completely specify a similarity solution. As there are only four independent components of the Einstein field equations for our metric ansatz, an equation of state is required to close the system. In this light we assume that all solutions have a barotropic equation of state, that is, one of the form:

\[ p = p(\rho) \]

Under this assumption, and in addition to the similarity assumption, it is demonstrated in [1] that a barotropic equation of state takes the more restricted form:

\[ p = \sigma \rho \]

for some constant \( \sigma \). Note that this only applies to spherically symmetric solutions of the self-similar variable \( \xi \). Self-similar solutions of the second kind, which include those that admit the self-similar variable:

\[ \xi_\lambda = \frac{r}{l\lambda} \]

for some non-zero constant \( \lambda \), has a different restricted form for the barotropic equation of state. Physically, \( \sigma \) represents the square root of the sound speed in the fluid, so for a strictly positive pressure and subluminal sound speed we require:

\[ 0 < \sigma < 1 \]

Definition 3. The special case \( \sigma = \frac{1}{3} \) corresponds to the extreme relativistic limit of free particles and the state of matter known as pure radiation.

A perfect fluid stress-energy-momentum tensor with a pure radiation equation of state models the matter in the Radiation Dominated Epoch of the Universe. This equation of state also yields a trace free stress-energy-momentum tensor. Now the similarity assumption reduces the complexity of the Einstein field equations, which without such an assumption form a system of nonlinear PDE when written in Schwarzschild coordinates. By substituting (2) and (3) into (1), Smoller and Temple [10] demonstrate that the Einstein field equations reduce to the following system of nonlinear ODE:

\[ \xi \frac{dA}{d\xi} = -\frac{(3 + 3\sigma)(1 - A)v}{\{\cdot\}_S} \]

(5)

\[ \xi \frac{dG}{d\xi} = -G \left[ \left( \frac{1 - A}{A} \right) \left( \frac{3 + 3\sigma}{2} \right) \left( 1 + \frac{4Gv}{3 - \sigma} \right) - 1 \right] \]

(6)

\[ \xi \frac{dv}{d\xi} = -\left( \frac{1 - v^2}{2} \right) \left[ 3\sigma \{\cdot\}_S + \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)^2}{4} \{\cdot\}_N \right] \]

(7)

in addition to the constraint:

\[ \rho = \frac{3(1 - v^2)(1 - A)G}{\kappa r^2 \{\cdot\}_S} \]

(8)

where \( G \), not to be confused with the Einstein tensor, is defined by:

\[ G = \frac{\xi}{\sqrt{AB}} \]

(9)

and:

\[ \{\cdot\}_S = 3(G - v) - 3\sigma v(1 - Gv) \]

\[ \{\cdot\}_N = -3(G - v)^2 + 3\sigma v^2(1 - Gv)^2 \]

\[ \{\cdot\}_D = \frac{3}{4}(3 + 3\sigma) \left[ (G - v)^2 - \sigma(1 - Gv)^2 \right] \]

Note that the original derivation provided in [10] is only completed in the pure radiation case. It is not difficult to modify this derivation to yield equations (5)-(8) for general \( \sigma \), but this derivation will not be given here. Note also that under the change of variable:

\[ \xi = e^s \]

(10)
the equations become explicitly autonomous, since:

\[ \xi \frac{d}{d\xi} = \frac{d}{ds} \]  

(11)

The autonomous nature of these equations distinguish them from the self-similar ODE derived by Cahill and Taub [1], which are derived in self-similar comoving coordinates. It is worth noting that in self-similar comoving coordinates, the variable \( G \) is the Schwarzschild coordinate analogue of the variable \( V \), defined in [1], which represents surfaces of constant \( \xi \) relative to the fluid.

2.2 Tolman-Oppenheimer-Volkoff Spacetimes

The Tolman-Oppenheimer-Volkoff (TOV) spacetimes are the family of static spherically symmetric perfect fluid spacetimes. It is demonstrated in [1] that the self-similar subset of TOV spacetimes which solve the perfect fluid Einstein field equations with a barotropic equation of state form the unique family of static similarity spacetimes. In the context of equations (5)-(8) these spacetimes are distinguished by having a Schwarzschild coordinate velocity that is identically zero.

**Proposition 1.** Perfect fluid similarity spacetimes are static if and only if \( v \equiv 0 \).

**Proof.** Since it is known that the family of static perfect fluid similarity spacetimes are unique, it is sufficient to demonstrate that solutions with zero Schwarzschild coordinate velocity are static. In this light, substituting \( v \equiv 0 \) into equation (5) implies \( A \equiv A_0 \) for some constant \( A_0 \). Furthermore, substituting \( v \equiv 0 \) into equation (7) requires:

\[
9\sigma - \frac{1}{4}(3 + 3\sigma)^2 \left( \frac{1 - A_0}{A_0} \right) = 0
\]

to ensure \( \nu' \equiv 0 \). This means \( A_0 \) can be given as a function of \( \sigma \) as so:

\[
A_0(\sigma) = 1 - 2M(\sigma)
\]

where:

\[
M(\sigma) = \frac{2\sigma}{1 + 6\sigma + \sigma^2}
\]

Now substituting both \( v \equiv 0 \) and \( A \equiv A_0 \) into equation (6) and solving for \( G(\xi) \) yields:

\[
G(\xi) = C_1 \xi^{\frac{\sigma}{1 + \sigma}}
\]

for some positive constant \( C_1 \). Putting these results together yields the following metric in self-similar Schwarzschild coordinates:

\[
ds^2 = -C_2 \xi^{\frac{2\sigma}{1 + \sigma}} dt^2 + \frac{1}{1 - 2M(\sigma)} dr^2 + r^2 d\Omega^2
\]

for some positive constant \( C_2 \). The density is given by:

\[
\rho = \frac{2M(\sigma)}{\kappa r^2}
\]

Note that because \( v \equiv 0 \), the coordinate frame is comoving with the fluid. Finally, making the temporal transformation:

\[
\tilde{t} = \frac{1 + \sigma}{1 - \sigma} \frac{1}{1 + \sigma} \quad \tilde{r} = r
\]

puts the metric in the following explicitly static form:

\[
d\tilde{s}^2 = -C_2 \tilde{r}^{\frac{2\sigma}{1 + \sigma}} d\tilde{t}^2 + \frac{1}{1 - 2M(\sigma)} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2
\]

noting that the density also remains static. \( \square \)

The TOV(\( \sigma \)) spacetimes are a remarkably convenient and simple set of solutions to the perfect fluid Einstein field equations, mostly because they can be placed in a coordinate system which is comoving, explicitly self-similar and in Schwarzschild form simultaneously.
Definition 4. A scale transformation is a transformation of the form:
\[ \tilde{t} = T_0 t \]
\[ \tilde{r} = R_0 r \]
where $T_0$ and $R_0$ are constants. A parameter that appears in a solution is essential if it cannot be removed by a scale transformation and inessential if it can.

Essential and inessential parameters are discussed in more detail in [1]. In counting the number of essential parameters in a solution, Cahill and Taub and Carr and others first fix a value for $\sigma$ and count the remaining essential parameters. This convention is not adopted in this paper and instead makes clear whether the equation of state parameter is included in each instance.

Proposition 2. The one-parameter family of TOV spacetimes, denoted by $\text{TOV}(\sigma)$, are given in self-similar comoving Schwarzschild coordinates as:
\[ ds^2 = -\alpha^2 \xi^{\frac{4\sigma}{1+3\sigma}} dt^2 + \frac{1}{1 - 2M(\sigma)} dr^2 + r^2 d\Omega^2 \]
\[ \rho = \frac{2M(\sigma)}{\kappa r^2} \]
\[ p = \sigma \rho \]
where $\alpha$ is an inessential parameter and:
\[ M(\sigma) = \frac{2\sigma}{1 + 6\sigma + \sigma^2} \]

Proof. This follows from the proof of Proposition [1] \qed

2.3 Friedmann-Lemaître-Robertson-Walker Spacetimes

The flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes are the family of spatially homogeneous spherically symmetric spacetimes. Following [3], the self-similar in $\xi$ subset of FLRW spacetimes which solve the perfect fluid Einstein field equations with a barotropic equation of state take the following form in self-similar comoving coordinates:
\[ ds^2 = -e^{2\varphi} d\tilde{t}^2 + e^{2\psi} d\tilde{r}^2 + \tilde{\alpha}^2 \tilde{r}^2 d\Omega^2 \]
\[ \rho = \frac{2\tilde{\xi}^2}{\kappa \tilde{r}^2} \]
\[ p = \sigma \rho \]
where:
\[ e^{2\varphi} = \beta^2 \]
\[ e^{2\psi} = \gamma^{-2} \xi^{-\frac{4\sigma}{1+3\sigma}} \]
\[ \tilde{\alpha}^2 = \tilde{\xi}^{-\frac{4\sigma}{1+3\sigma}} \]
and:
\[ \beta = \sqrt{6} \]
\[ \gamma = \frac{3 + 3\sigma}{1 + 3\sigma} \]

Note that the density is independent of $r$ and that the metric can also be put into an explicitly spatially homogeneous form through the purely radial transformation:
\[ \tilde{t} = \tilde{t} \]
\[ \tilde{r} = \tilde{r} \tilde{\xi}^{\frac{1+3\sigma}{1+3\sigma}} \]
to yield:
\[ ds^2 = -\beta^2 d\tilde{t}^2 + \tilde{t}^{\frac{4\sigma}{1+3\sigma}} (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2) \]
Proposition 3. The one-parameter family of similarity perfect fluid FLRW spacetimes with barotropic equations of state, denoted by \( \text{FLRW}(0, \sigma, 1) \), are given in self-similar Schwarzschild coordinates as:

\[
\begin{align*}
&ds^2 = -\delta^{-2} \left[ 1 + \frac{1}{3}(1 + 3\sigma)\xi^{\frac{2+4\delta}{3+3\sigma}} \right]^{-\frac{1+3\delta}{3+3\sigma}} \left[ 1 - \frac{2}{3}\xi^{\frac{2+4\delta}{3+3\sigma}} \right]^{-1} dt^2 + \left[ 1 - \frac{2}{3}\xi^{\frac{2+4\delta}{3+3\sigma}} \right]^{-1} dr^2 + r^2 d\Omega^2 \\
&v = \frac{2}{\sqrt{6}} \xi^{1+3\sigma} \\
&\rho = \frac{3v^2}{k\sigma^2} \\
&p = \sigma \rho
\end{align*}
\]

where \( \delta \) is an inessential parameter and:

\[
\xi = \frac{1}{\sqrt{6}} \delta^{-1}(3 + 3\sigma)\xi^{\frac{1+3\sigma}{3+3\sigma}} \left[ 1 + \frac{1}{3}(1 + 3\sigma)\xi^{\frac{2+4\delta}{3+3\sigma}} \right]^{-\frac{3+3\delta}{3+3\sigma}} \tag{12}
\]

The zero in the first argument of \( \text{FLRW}(0, \sigma, 1) \) corresponds to the flat subset of FLRW spacetimes, that is, those with \( k = 0 \) in reduced circumference Schwarzschild coordinates. The one in the third argument corresponds to the lack of perturbation, which is defined in the next subsection.

Proof. To change \( \text{FLRW}(0, \sigma, 1) \) from self-similar comoving coordinates to self-similar Schwarzschild coordinates, the following coordinate transformation, given in \([1]\), is used:

\[
\begin{align*}
&dt = e^{-\mu} \left( e^\varphi \cosh \omega \ dt + e^\psi \sinh \omega \ d\hat{r} \right) \tag{13} \\
&dr = e^{-\nu} \left( e^\varphi \sinh \omega \ d\hat{r} + e^\psi \cosh \omega \ d\hat{r} \right) \tag{14}
\end{align*}
\]

where:

\[
\tanh \omega = e^{\psi - \varphi} \frac{\partial_t(\mathcal{R})}{\partial_r(\mathcal{R})} \tag{15}
\]

\[
e^{-2\nu} = e^{-2\psi} \left( \frac{\partial_t(\mathcal{R})}{\partial_r(\mathcal{R})} \right)^2 - e^{-2\varphi} \left( \frac{\partial_t(\mathcal{R})}{\partial_r(\mathcal{R})} \right)^2 \tag{16}
\]

and \( \mu \) is such that \( dt \) is a perfect differential. Relations \([15]\) and \([16]\) come from setting \( r = \mathcal{R} \hat{r} \). The resulting self-similar Schwarzschild form of the metric is then given by:

\[
\begin{align*}
&ds^2 = -e^{2\mu} dt^2 + e^{2\nu} dr^2 + r^2 d\Omega^2
\end{align*}
\]

To begin, the \( \tanh \omega \) term is computed as so:

\[
\tanh \omega = e^{\psi - \varphi} \frac{\partial_t(\mathcal{R})}{\partial_r(\mathcal{R})}
\]

\[
= \beta^{-1} \gamma^{-1} \xi^{-\frac{2+4\delta}{3+3\sigma}} \left( 1 - \gamma^{-1} \xi^{-\frac{2+4\delta}{3+3\sigma}} \right)^{-\frac{1}{2}}
\]

and this yields:

\[
\cosh \omega = (1 - \tanh^2 \omega)^{-\frac{1}{2}} = \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+4\delta}{3+3\sigma}} \right]^{-\frac{1}{2}}
\]

\[
\sinh \omega = \tanh \omega (1 - \tanh^2 \omega)^{-\frac{1}{2}} = 2(1 + 3\sigma)^{-1} \beta^{-1} \gamma^{-1} \xi^{-\frac{2+4\delta}{3+3\sigma}} \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+4\delta}{3+3\sigma}} \right]^{-\frac{1}{2}}
\]

The \( e^{-2\nu} \) term is computed similarly:

\[
e^{-2\nu} = e^{-2\psi} \left( \frac{\partial_t(\mathcal{R})}{\partial_r(\mathcal{R})} \right)^2 - e^{-2\varphi} \left( \frac{\partial_t(\mathcal{R})}{\partial_r(\mathcal{R})} \right)^2
\]

\[
= \gamma^2 \xi^{\frac{2+4\delta}{3+3\sigma}} \left( \xi^{-\frac{2+4\delta}{3+3\sigma}} + \xi^{\gamma^{-1}} \left( 1 - \gamma^{-1} \xi^{-\frac{2+4\delta}{3+3\sigma}} \right)^{-\frac{1}{2}} \right)^2 - \beta^{-2} \xi^2 \left( 1 - \gamma^{-1} \xi^{-\frac{2+4\delta}{3+3\sigma}} \right)\xi^{-\frac{2+4\delta}{3+3\sigma}}
\]

\[
= (1 + 3\sigma)^2 (3 + 3\sigma)^{-2} \gamma^2 \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+4\delta}{3+3\sigma}} \right]
\]

\[
\]
and this results in:
\[
\begin{align*}
 dt &= \beta e^{-\mu} \left[ 1 - 4(1 + 3\sigma)^{-2}(2 - 2\gamma - 2\frac{\xi 2 + 6\sigma}{3 + 3\sigma})^{-\frac{1}{2}} \left[ dt + 2(1 + 3\sigma)^{-1} \beta^{-2}(2 - 2\gamma - 2\frac{\xi 2 + 6\sigma}{3 + 3\sigma}) \delta r \right] \right] \\
 dr &= 2(3 + 3\sigma)^{-1} \xi \frac{\delta r}{t^\gamma} \left[ dt + 1 + 2(1 + 3\sigma)^{-1} \xi \frac{\delta r}{t^\gamma} \right]
\end{align*}
\]

Now given that \( \mu \) is such that the right hand side of (13) is a perfect differential, then:
\[
\frac{\partial}{\partial r} e^{-\eta} = \frac{\partial}{\partial r} \left[ 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} e^{-\eta} \right]
\]
where:
\[
e^{-\eta} = e^{-\mu} \left[ 1 - 4(1 + 3\sigma)^{-2}(2 - 2\gamma - 2\frac{\xi 2 + 6\sigma}{3 + 3\sigma})^{-\frac{1}{2}} \right]
\]

The equation for \( \eta \) can be solved as so:
\[
\frac{\partial}{\partial r} e^{-\eta} = \frac{\partial}{\partial r} \left[ 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} e^{-\eta} \right]
\]
\[
\Leftrightarrow e^{-\eta} = \frac{1}{d\xi} d\xi e^{-\eta} = -\frac{e^{-\eta}}{t^2} \frac{d}{d\xi} \left[ 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} e^{-\eta} \right]
\]
\[
\Leftrightarrow \eta' = 2(1 + 3\sigma)^{-1} \beta^{-2}(2 - 2\gamma - 2\frac{\xi 2 + 6\sigma}{3 + 3\sigma} \eta e^{-\eta} + (3\sigma - 1)(3 + 3\sigma)^{-1} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} e^{-\eta})
\]
\[
\Leftrightarrow \eta' = 2(1 + 3\sigma)^{-1}(3\sigma - 1)(3 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right]^{-1}
\]
\[
\Leftrightarrow \eta = (3\sigma - 1)(2 + 6\sigma)^{-1} \log \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right] + C_3
\]
where \( C_3 \) is a constant. This then yields:
\[
e^{-\eta} = \delta \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right]^{\frac{1 - 3\sigma}{2 + 3\sigma}}
\]
for some positive constant \( \delta \), thus:
\[
\begin{align*}
 dt &= \delta \beta \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right]^{\frac{1 - 3\sigma}{2 + 3\sigma}} \left[ dt + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \delta r \right]
\end{align*}
\]

Because the transformation is taking the metric from one self-similar form to another, let:
\[
t = \mathcal{F}(\xi) \tilde{t}
\]
so that:
\[
\begin{align*}
 \frac{\partial t}{\partial \tilde{t}} &= \mathcal{F}(\xi) \frac{\partial \tilde{t}}{\partial \tilde{t}} = \mathcal{F}(\xi) - \mathcal{F}'(\xi) = \delta \beta \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right]^{\frac{1 - 3\sigma}{2 + 3\sigma}} \\
 \frac{\partial \tilde{t}}{\partial \tilde{r}} &= \mathcal{F}'(\xi) = 2\delta (1 + 3\sigma)^{-1} \beta^{-1} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right]^{-1}
\end{align*}
\]

Solving these equations yields the same function for \( \mathcal{F}(\xi) \) only when the integration constant is zero, thus:
\[
\mathcal{F}(\xi) = \delta \beta \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right]^{\frac{1 - 3\sigma}{2 + 3\sigma}}
\]

Now the fluid four-velocity \( \mathbf{u} \) is given in self-similar comoving coordinates as:
\[
\mathbf{u} = (u^0, u^1, u^2, u^3) = (e^{-\mu}, 0, 0, 0) = (\beta^{-1}, 0, 0, 0)
\]
and in self-similar Schwarzschild coordinates as:
\[
\mathbf{u} = (u^0, u^1, u^2, u^3) = \left( u^0 \frac{\partial}{\partial t}, u^1 \frac{\partial}{\partial \tilde{t}}, u^2 \frac{\partial}{\partial \tilde{r}}, u^3 \frac{\partial}{\partial \tilde{r}} \right) = \left( \beta^{-1} \frac{\partial}{\partial t}, \beta^{-1} \frac{\partial}{\partial \tilde{t}}, 0, 0 \right)
\]
\[
= \left( \delta \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \frac{\xi 2 + 6\sigma}{3 + 3\sigma} \right]^{\frac{1 - 3\sigma}{2 + 3\sigma}}, 2(3 + 3\sigma)^{-1} \beta^{-1} \xi \frac{1 - 3\sigma}{3 + 3\sigma}, 0, 0 \right)
\]
Therefore, by Definition 2:

\[ v = e^{\nu - \mu} \frac{u^1}{u^0} = 2(3 + 3\sigma)^{-1}\beta^{-1}\xi^{1+3\sigma} \]

Finally, by substituting in \( \beta \) and \( \gamma \) and noting that:

\[ \xi = \frac{r}{t} = \frac{T(\xi)}{\xi} = \frac{R(\xi)}{\xi} \]

the rest follows.

Spacetimes that solve equations (5)-(7) can be denoted by the triple \( (A, G, v) \), which specifies the metric through \( A \) and \( G \), the fluid four-velocity through \( v \) and the density through the constraint (8).

**Proposition 4.** \( \text{FLRW}(0, \sigma, 1) \) is given implicitly by:

\[ A = 1 - v^2 \quad (17) \]
\[ G = \frac{1}{2}(3 + 3\sigma)v \left(1 + \frac{1}{2}(1 + 3\sigma)v^2\right)^{-1} \quad (18) \]
\[ v = \frac{2}{\sqrt{6}}\xi^{1+3\sigma} \quad (19) \]

**Proof.** First note that (19) is immediately obtained from Proposition 3. By definition:

\[ A = e^{-2\nu} = 1 - \frac{2}{3}\xi^{2+6\sigma} \]
\[ G = \xi e^{-\nu} = \frac{1}{\sqrt{6}}(3 + 3\sigma)^{\xi^{1+3\sigma}} \left[1 + \frac{1}{3}(1 + 3\sigma)\xi^{2+6\sigma}\right]^{-1} \]

which then yields (17) and (18).

To check that (17)-(19) satisfies equations (5)-(7), it is recommended to first show:

\[ \xi \frac{d}{d\xi} = \xi \frac{d^2}{d\xi^2} = \frac{(3 + 3\sigma)^2}{2 + 6\sigma} \frac{v}{AG} \xi \frac{d}{d\xi} \]

and secondly show:

\[ \xi \frac{dA}{d\xi} = \frac{2 + 6\sigma v^2}{3 + 3\sigma} \]
\[ \xi \frac{dG}{d\xi} = \frac{2 + 6\sigma}{(3 + 3\sigma)^2} \left(1 - \frac{1}{2}(1 + 3\sigma)v^2\right) \frac{G^2}{v} \]
\[ \xi \frac{dv}{d\xi} = \frac{1 + 3\sigma}{3 + 3\sigma}v \]

Then by Proposition 4 it is not difficult to confirm that (17)-(19) solve equations (5)-(7).

**Corollary 1.** \( \text{FLRW}(0, \frac{1}{3}, 1) \) is given in self-similar Schwarzschild coordinates as:

\[ ds^2 = \frac{1}{1 - v^2} \left(-\delta^{-2}dt^2 + dv^2\right) + r^2d\Omega^2 \]
\[ v = \frac{1 - \sqrt{1 - \delta^2\xi^2}}{\delta} \]
\[ \rho = \frac{3v^2}{\kappa r^2} \]
\[ p = \sigma \rho \]

where \( \delta \) is an inessential parameter.
Proof. From Proposition 3 in the case $\sigma = \frac{1}{3}$, relation (12) can be inverted to yield $v$. The metric then follows from using this inversion and some algebraic manipulation.

Corollary 2. FLRW(0, $\frac{1}{3}$, 1) is given implicitly by:

$$A = 1 - v^2$$  \hspace{1cm} (20)

$$G = \frac{2v}{1 + v^2}$$  \hspace{1cm} (21)

$$G = \delta \xi$$  \hspace{1cm} (22)

Proof. Relations (20) and (21) follow immediately from Proposition 4 and relation (22) follows from (9) and the identity $B = \delta^{-2}A^{-2}$ from Corollary 1.

2.4 Asymptotically Friedmann-Lemaître-Robertson-Walker Spacetimes

Let $(A, G, v)$ denote a solution of equations (5)-(7). Since these equations are autonomous, solutions can be represented by non-intersecting trajectories in $(A, G, v)$ space. The FLRW(0, $\sigma$, 1) spacetimes solve equations (5)-(7) and constraint (8) with the trajectories emanating from the point:

$$(A, G, v) = (1, 0, 0)$$

The nature of equations (5)-(7) suggests that to analyse this point, it is helpful to rewrite these equations as functions of $v$, $A$ and $H$, where $H$ is defined as the ratio:

$$H = \frac{G}{v}$$

This is completed in [10] for $\sigma = \frac{1}{3}$, however it is not difficult to reproduce these equations for general $\sigma$, especially when working from equations (5)-(7), although this will not be done here. Recalling (10) and (11), equations (5)-(8) are given in autonomous form as functions of $v$, $A$ and $H$ as so:

$$\frac{dv}{ds} = -v \left(1 - v^2\right) \left[3\sigma\{\cdot\}_S^* + \left(\frac{1 - A}{A}\right) \frac{(3 + 3\sigma)^2\{\cdot\}_N^*}{4\{\cdot\}_S^*} \right]$$  \hspace{1cm} (23)

$$\frac{dA}{ds} = -\frac{(3 + 3\sigma)(1 - A)}{\{\cdot\}_S^*}$$  \hspace{1cm} (24)

$$\frac{dH}{ds} = -H \left[\left(\frac{1 - A}{A}\right) \frac{(3 + 3\sigma)[(1 + v^2)H - 2]}{2\{\cdot\}_S^*} - 1\right] - \frac{H dv}{v ds}$$  \hspace{1cm} (25)

with:

$$\rho = \frac{3(1 - v^2)(1 - A)H}{\kappa r^2\{\cdot\}_S^*}$$  \hspace{1cm} (26)

and where:

$$\{\cdot\}_S^* = -(3 + 3\sigma) + (3 + 3\sigma v^2)H$$

$$\{\cdot\}_N^* = -(3 - 3\sigma) + 2(3 - 3\sigma v^2)H - (3 - 3\sigma v^4)H^2$$

$$\{\cdot\}_D = \frac{1}{4}(3 + 3\sigma)(3\sigma - 3v^2) - \frac{3}{2}(3 - 3\sigma^2)Hv^2 + \frac{1}{4}(3 + 3\sigma)(3 - 3\sigma v^2)H^2v^2$$

In variables $v$, $A$ and $H$, the point of interest is given by:

$$(v, A, H) = \left(0, 1, \frac{1}{2}(3 + 3\sigma)\right)$$

and it is not difficult to check that this is a fixed point of the system of equations (23)-(25). Following [10], a linear analysis of this fixed point is achieved by first representing equations (23)-(25) as:

$$v' = F_1(v, A, H)$$
$$A' = F_2(v, A, H)$$
$$H' = F_3(v, A, H)$$

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and then denoting these equations by:

\[ U' = F(U) \]

where:

\[ U = (v, A, H)^T \]
\[ F = (F_1(U), F_2(U), F_3(U))^T \]

Next, the Jacobian of \( F \) at the fixed point is calculated. Note that the Jacobian is calculated in [10], however there is a small error in this calculation, so a brief new derivation will be produced. To begin, denote the fixed point by \( U_0 \) and note that:

\[ dF_2(U_0) = \left[ \frac{\partial F_2}{\partial v}, \frac{\partial F_2}{\partial A}, \frac{\partial F_2}{\partial H} \right] \bigg|_{U_0} = (0, 2, 0) \]

Neglecting terms second order in \( v \) and second order in terms that vanish at \( U_0 \) on the right hand side of (23) gives:

\[ dF_1(U_0) = d \left[ -v \left( -\frac{2}{3\sigma(3 + 3\sigma)} \right) \left[ 9\sigma H - 3\sigma(3 + 3\sigma) \right] \right]_{U_0} = (1, 0, 0) \]

and similarly for (25) gives:

\[ dF_3(U_0) = d \left[ \left( -\frac{2}{3\sigma(3 + 3\sigma)} \right) \left[ 9\sigma H - 3\sigma(3 + 3\sigma) + \left( 1 - \frac{A}{A} \right) (3 + 3\sigma)(1 - H) H \right] \frac{6\sigma}{12H - 3(3 + 3\sigma)} \right]_{U_0} = \left( 0, -\frac{(1 + 3\sigma)(3 + 3\sigma)^2}{24\sigma^2}, -3 \right) \]

Thus the Jacobian is given by:

\[ dF(U_0) = \begin{pmatrix} dF_1(U_0) \\ dF_2(U_0) \\ dF_3(U_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & N(\sigma) & -3 \end{pmatrix} \]

where:

\[ N(\sigma) = -\frac{(1 + 3\sigma)(3 + 3\sigma)^2}{24\sigma^2} \]

This means \( U_0 \) is a hyperbolic rest point of the system of equations (23)-(25) with eigenvalues:

\[ \lambda_1 = 1 \]
\[ \lambda_2 = 2 \]
\[ \lambda_3 = -3 \]

Therefore the solutions:

\[ U(s) = U_0 + V(s) \]

where \( V(s) \) solves the linearised equations:

\[ V' = dF(U_0) \cdot V \]

lie in the two-dimensional unstable manifold \( M_0 \) of \( U_0 \), given by:

\[ M_0 = \left\{ \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}(3 + 3\sigma) \end{pmatrix} \right\} + \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^s + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2s} \right\} \]
In particular:

\[
U(s) = \left( \begin{array}{c}
C_4 e^s \\
1 + C_5 e^{2s} \\
\frac{1}{2}(3 + 3\sigma)
\end{array} \right)
\]

for arbitrary constants \(C_4\) and \(C_5\). In the variable \(\xi\), the solutions are given by:

\[
A_1(\xi) = 1 + C_5 \xi^2 \\
G_1(\xi) = \frac{1}{2}(3 + 3\sigma)C_4 \xi \\
v_1(\xi) = c\xi
\]

with the subscript denoting the fact that \((A_1, G_1, v_1)\) represents a solution to the linearised version of equations (5)-(7).

Now for small \(\xi\), functions \(A, G\) and \(v\) of FLRW\((0, \sigma, 1)\) are given to leading order as:

\[
A \approx 1 - \frac{4}{(3 + 3\sigma)^2} \delta^2 \xi^2 \\
G \approx \delta \xi \\
v \approx \frac{2}{3 + 3\sigma} \delta \xi
\]

Comparing (27)-(29) to (30)-(32) suggests setting \(C_4\) and \(C_5\), without loss of generality, as so:

\[
C_4 = \frac{2}{3 + 3\sigma} \delta \\
C_5 = -\frac{4}{(3 + 3\sigma)^2} \delta^2 a^2
\]

where \(a\) is an essential parameter. Including \(\sigma\), we have that (27)-(29) is a two-parameter family of solutions originating from the fixed point \(U_0\) with the leading order approximations of FLRW\((0, \sigma, 1)\) as a one-parameter subset. The FLRW\((0, \sigma, 1)\) spacetimes correspond to \(a = 1\) and any other value of \(a\) represents a self-similar perturbation from FLRW\((0, \sigma, 1)\). From this point onwards the value of \(\delta\) is fixed as:

\[
\delta = \frac{1}{4}(3 + 3\sigma)
\]

so as to simplify calculations and match the notation and proceeding definition found in [10].

**Definition 5.** The asymptotically Friedmann spacetimes, denoted by FLRW\((0, \sigma, a)\), are defined as the two-parameter family of solutions to (5)-(8) with the following leading order form as \(\xi \to 0\):

\[
A(\xi) \approx 1 - \frac{1}{4} a^2 \xi^2 \\
G(\xi) \approx \frac{1}{4}(3 + 3\sigma) \xi \\
v(\xi) \approx \frac{1}{2} \xi
\]

Furthermore, the parameter \(a\) is referred to as the acceleration parameter.

It is demonstrated by Smoller and Temple [10] that the FLRW\((0, \sigma, a)\) spacetimes are distinct from the FLRW\((k, \sigma, 1)\) spacetimes, and in particular, that the FLRW\((k, \sigma, 1)\) spacetimes are not self-similar in the variable \(\xi\). The asymptotic form of the FLRW\((0, \sigma, a)\) spacetimes was first found by Carr and Yahil [3], although this asymptotic form is given in comoving coordinates. The FLRW\((0, \sigma, a)\) spacetimes are exact solutions of Einstein’s field equations, even though they are not known explicitly. Despite this, we can still give a leading order approximation of the FLRW\((0, \sigma, a)\) solutions local to the centre of expansion.

**Proposition 5.** The FLRW\((0, \sigma, a)\) spacetimes are given in self-similar Schwarzschild coordinates to leading order as \(\xi \to 0\) as so:

\[
ds^2 \approx -\frac{16}{(3 + 3\sigma)^2} \left( 1 + \frac{1}{4} a^2 \xi^2 \right) dt^2 + \left( 1 + \frac{1}{4} a^2 \xi^2 \right) dr^2 + r^2 d\Omega^2 \\
v \approx \frac{1}{2} \xi \\
\rho \approx \frac{3a^2 \xi^2}{4KR^2} \\
p = \sigma \rho
\]
Proof. This follows from Proposition 3 and Definition 5 by noting that:

\[ B = \frac{\xi^2}{\mathcal{A}G^2} \]

As a cosmological model, the closer the acceleration parameter is to one, the more spatially homogeneous the associated universe is, with homogeneity increasing the closer an observer is to the centre of expansion. Despite this, any FLRW(0, σ, a) spacetime with \( a \neq 1 \) is still inhomogeneous and thus violates the Cosmological Principle. The acceleration parameter is also responsible for the rate of acceleration of the spacetime. In the Radiation Dominated Epoch, observational measurements suggest cosmic acceleration was small, which corresponds to an acceleration parameter only slightly larger than one. An FLRW(0, \( \frac{1}{3} \), a) universe with an acceleration parameter close to one is thus close to the Standard Model of Cosmology in the Radiation Dominated Epoch, since the spacetime appears homogeneous close to the centre of expansion and induces a small acceleration. A more in-depth consideration of the cosmological applications of FLRW(0, σ, a) spacetimes is given in [10].

2.5 Shock Wave Construction

Suppose that we have two spherically symmetric, although not necessarily self-similar, solutions to the perfect fluid Einstein field equations. Let us denote these solutions by the triples \((g, p, u)\) and \((\bar{g}, \bar{p}, \bar{u})\) and assume also that these solutions have equations of state \( p = \rho(p) \) and \( \bar{p} = \bar{\rho}(\bar{p}) \) respectively. Since we are assuming spherical symmetry, when specifying a set of coordinates \((t, r, \theta, \phi)\), it is sufficient to only consider the coordinates \((t, r)\). In this light, let metrics \(g\) and \(\bar{g}\) be given in Schwarzschild coordinates \((t, r)\) and \((\bar{t}, \bar{r})\) respectively as so:

\[
\begin{align*}
\text{ds}^2 &= -B(t, r)dt^2 + \frac{1}{A(t, r)} dr^2 + r^2 d\Omega^2 \\
\text{d}\bar{s}^2 &= -\bar{B}(\bar{t}, \bar{r})d\bar{t}^2 + \frac{1}{\bar{A}(\bar{t}, \bar{r})} d\bar{r}^2 + \bar{r}^2 d\bar{\Omega}^2
\end{align*}
\]

where the coordinate variables \((\theta, \phi)\) and \((\bar{\theta}, \bar{\phi})\) have been identified.

Definition 6. We say that two metrics can be matched on a spherical surface \(\bar{r} = \Phi(\bar{t})\) if there exists a common set of coordinates \((\bar{t}, \bar{r})\) such that the coefficients of the metrics agree on this surface when written in these coordinates.

It is not required that the metrics be given in Schwarzschild coordinates in order to be matched, but it does provide a convenient set of coordinates from which the metrics can be compared. For metrics \(g\) and \(\bar{g}\), we may simply take \((t, r)\) as our common set of coordinates and ask which transformation of the form:

\[
\begin{align*}
\bar{t} &= \bar{t}(t, r) \\
\bar{r} &= \bar{r}(t, r)
\end{align*}
\]

is required in order to match these metrics. The reason Schwarzschild coordinates are so useful is because we automatically match the \(d\Omega^2\) coefficients through the identification \(\bar{r} = r\). This identification means that in order to avoid introducing \(dt dr\) terms, the most general transformation that can be applied takes the form \(\bar{t} = \bar{t}(t)\). Thus for two metrics given in Schwarzschild coordinates, the process of matching these metrics reduces to the existence of a spherical surface \(r = \Phi(t)\) and a coordinate transformation \(\bar{t} = \bar{t}(t)\) that satisfy the following algebraic-differential equations:

\[
\begin{align*}
B(t, \Phi(t)) &= \bar{B}(\bar{t}(t), \Phi(t))[\bar{\xi}(t)]^2 \\
A(t, \Phi(t)) &= \bar{A}(\bar{t}(t), \Phi(t))
\end{align*}
\]

If these equations can be solved, then metrics \(g\) and \(\bar{g}\) can be matched along the surface \(r = \Phi(t)\). However, such a matching does not automatically imply that mass-energy and momentum are conserved across the surface. With this in mind, let us assume that there exists a set of coordinates \((t, r)\) for which the metrics match on the spherical surface \(r = \Phi(t)\) and define:

\[
\Sigma = \{(t, r, \theta, \phi) : r = \Phi(t), t > 0\}
\]

Definition 7. We say that the spacetime given by the matched metric \(g \sqcup \bar{g}\), along with the associated hydrodynamic variables, forms a shock-wave solution of the perfect fluid Einstein field equations if the Rankine-Hugoniot jump conditions hold across the surface \(\Sigma\). Furthermore, the spherical surface \(\Sigma\) is known as the shock surface or simply the shock.
As like in classical shock-wave theory, the Rankine-Hugoniot jump conditions express the weak form of the conservation of mass-energy and momentum across the shock-surface.

**Proposition 6.** Let $p \in \Sigma$ and $U$ be a neighbourhood of $p$, then the weak form of the conservation of mass-energy and momentum across $\Sigma \cap U$ is given by:

$$
\int_U T^{\mu\nu} \nabla_\nu \varphi \, d\mathbf{x} = 0 \quad \forall \varphi \in C^\infty_c(U) \tag{34}
$$

or equivalently:

$$
\int_U G^{\mu\nu} \nabla_\nu \varphi \, d\mathbf{x} = 0 \quad \forall \varphi \in C^\infty_c(U) \tag{35}
$$

**Proof.** If each component of the stress-energy-momentum tensor $T$ is differentiable in $U$, then the conservation of mass-energy and momentum in $U$ is given by:

$$
\nabla_\nu T^{\mu\nu} = 0
$$

These conditions are equivalent to:

$$
\int_U \varphi \nabla_\nu T^{\mu\nu} \, d\mathbf{x} = 0 \quad \forall \varphi \in C^\infty_c(U)
$$

and by using the identity:

$$
\varphi \nabla_\nu T^{\mu\nu} = \nabla_\nu (\varphi T^{\mu\nu}) - T^{\mu\nu} \nabla_\nu \varphi
$$

are then equivalent to:

$$
\int_U \nabla_\nu (\varphi T^{\mu\nu}) \, d\mathbf{x} - \int_U T^{\mu\nu} \nabla_\nu \varphi \, d\mathbf{x} = 0 \quad \forall \varphi \in C^\infty_c(U)
$$

Now since $\varphi$ is compactly supported within $U$, the divergence theorem implies:

$$
\int_U \nabla_\nu (\varphi T^{\mu\nu}) \, d\mathbf{x} = \int_{\partial U} \varphi T^{\mu\nu} n_\nu \, d\mathbf{y} = 0 \quad \forall \varphi \in C^\infty_c(U)
$$

where $n$ denotes the outward normal vector and $y$ denotes the restriction of the $x$ coordinates to $\Sigma$. Thus (35) yields the weak form of the conservation of mass-energy and momentum across $\Sigma \cap U$. Conditions (35) then follow from equation (1).

The following proposition specifies the general relativistic form of the Rankine-Hugoniot jump conditions.

**Proposition 7.** The Rankine-Hugoniot jump conditions are given by:

$$
[G^{\mu\nu}] n_\nu = 0
$$

where:

$$
[G^{\mu\nu}] n_\nu = G^{\mu\nu}(g) n_\nu - G^{\mu\nu}(\bar{g}) n_\nu
$$

**Proof.** Let $U = U_1 \cup U_2$ where $\partial U_1 \cap \partial U_2 = \Sigma \cap U$ and assume that $g$ and $\bar{g}$ are sufficiently regular on their respective side of $\Sigma$, then:

$$
\int_U G^{\mu\nu} \nabla_\nu \varphi \, d\mathbf{x} = \int_{U_1} G^{\mu\nu}(g) \nabla_\nu \varphi \, d\mathbf{x} + \int_{U_2} G^{\mu\nu}(\bar{g}) \nabla_\nu \varphi \, d\mathbf{x}
$$

$$
= \int_{U_1} \nabla_\nu (\varphi G^{\mu\nu}(g)) \, d\mathbf{x} - \int_{U_1} \varphi \nabla_\nu G^{\mu\nu}(g) \, d\mathbf{x}
$$

$$
+ \int_{U_2} \nabla_\nu (\varphi G^{\mu\nu}(\bar{g})) \, d\mathbf{x} - \int_{U_2} \varphi \nabla_\nu G^{\mu\nu}(\bar{g}) \, d\mathbf{x}
$$

$$
= \int_{\partial U_1} \varphi G^{\mu\nu}(g) n_\nu \, d\mathbf{y} - \int_{\partial U_2} \varphi G^{\mu\nu}(\bar{g}) n_\nu \, d\mathbf{y}
$$

$$
= \int_{\Sigma} \varphi G^{\mu\nu}(g) n_\nu \, d\mathbf{y} - \int_{\Sigma} \varphi G^{\mu\nu}(\bar{g}) n_\nu \, d\mathbf{y}
$$

$$
= \int_{\Sigma} \varphi [G^{\mu\nu}] n_\nu \, d\mathbf{y} \quad \forall \varphi \in C^\infty_c(U)
$$

Thus:

$$
[G^{\mu\nu}] n_\nu = 0 \iff \int_U G^{\mu\nu} \nabla_\nu \varphi \, d\mathbf{x} = 0 \quad \forall \varphi \in C^\infty_c(U)
$$
2.6 Regularity

As like in Subsection 2.5, consider the solution triples \((g, \rho, u)\) and \((\bar{g}, \bar{\rho}, \bar{v})\). Assume that these solutions can be matched Lipschitz continuously along a spherical surface \(\Sigma\) with a spacelike normal vector \(n\) to form the matched metric \(g \cup \bar{g}\). Furthermore, let \(g \cup \bar{g}\) satisfy the Rankine-Hugoniot jump condition across \(\Sigma\) so that \(g \cup \bar{g}\) forms a shock-wave solution. For the rest of this subsection, the matched metric is to be referred to simply as the metric.

It is reasonable to be concerned with the regularity of such a solution, since a Lipschitz continuous shock wave has discontinuities in the first-order derivatives of the metric and delta function sources in the second-order derivatives. The Einstein tensor comprises second-order derivatives of the metric, so this too is expected to harbour delta function sources. On the other side of the Einstein field equations, the hydrodynamic variables \(\rho, p\) and \(u\), along with the metric, form the stress-energy-momentum tensor, and since the hydrodynamic variables are expected to be at worst discontinuous at the shock, so too is the stress-energy-momentum tensor. This is problematic, since the Einstein field equations cannot have different levels of regularity on the left and right hand sides of the equation. However, it turns out that even though delta function sources may appear in the second-order derivatives of the metric at the shock, with such being coordinate dependent, the Einstein tensor does not have any delta function sources, that is, the delta function sources cancel in the Einstein tensor. This result is summarised in the following theorem of [7].

**Theorem 1.** Let \(\Sigma\) denote a smooth, three-dimensional surface with a spacelike normal vector \(n\). Assume that the components of the metric are continuous on each side of \(\Sigma\) and Lipschitz continuous across \(\Sigma\) in some fixed coordinate system. Then the following statements are equivalent:

1. \([K]\) = 0 at each point of \(\Sigma\), where \(K\) is the second fundamental form of the metric.
2. The Riemann curvature and Einstein tensors, viewed as second-order operators on the metric components, produce no delta function sources on \(\Sigma\).
3. For each point \(p \in \Sigma\) there exists a \(C^{1,1}\) coordinate transformation defined in a neighbourhood of \(p\) such that in the new coordinates, which can be taken to be the Gaussian normal coordinates for the surface, the metric components are \(C^{1,1}\) functions of these coordinates.
4. For each point \(p \in \Sigma\) there exists a coordinate frame that is locally Lorentzian at \(p\) and can be reached from the original coordinates by a \(C^{1,1}\) coordinate transformation.

Moreover, if any one of these statements hold, then the Rankine-Hugoniot jump conditions:

\[ [G^{\mu\nu}]n_\nu = 0 \]

hold at each point of \(\Sigma\).

This theorem provides a criterion for the removal of the delta function sources and also a coordinate system for which the shock-wave solution can achieve optimal regularity, that is, when the metric has a Lipschitz continuous derivative at the shock. The following theorem, also from [7], provides convenient criteria for satisfying one of the equivalent statements of Theorem 1.

**Theorem 2.** Assume the following:

1. That \(g\) and \(\bar{g}\) are two spherically symmetric metrics that match across a three-dimensional surface \(\Sigma\) to form the matched metric \(g \cup \bar{g}\).
2. The matched metric is Lipschitz continuous across \(\Sigma\).
3. The normal \(n\) to \(\Sigma\) is non-null.

Then the following are equivalent:

1. \([G^{\mu\nu}]n_\nu = 0\)
2. \([G^{\mu\nu}]n_\mu n_\nu = 0\)
3. \([K]\) = 0 at each point of \(\Sigma\), where \(K\) is the second fundamental form of the metric.
4. The components of the matched metric in any Gaussian-normal coordinate system are \(C^{1,1}\) functions of these coordinates across \(\Sigma\).
If the conditions of Theorem 2 are satisfied, then it is clear that the weak form of mass-energy and momentum
conservation across the shock surface is equivalent to the single condition:

\[ T^{\mu\nu} n_\mu n_\nu = 0 \]

Thus the Rankine-Hugoniot jump conditions reduce to the single equivalent condition:

\[ G^{\mu\nu} n_\mu n_\nu = 0 \]

Therefore a shock-wave solution, which satisfies the Rankine-Hugoniot jump conditions by definition, only requires the
metric to be continuous on each side of \( \Sigma \) and Lipschitz continuous across \( \Sigma \) to satisfy the equivalent statements of
Theorems 1 and 2. The proofs of these theorems can be found in [6].

2.7 Lax Stability

The Lax stability of general relativistic shock waves is considered by Smoller and Temple in [7]. This stability is
determined in the gas dynamical sense, that is, a shock is considered stable when characteristics in the same family as
the shock impinge on the shock from both sides, see for example [4] and [5]. The conditions required for this, known
as the Lax characteristic conditions, are derived in the same manner as done in [7]. Note that the Lax characteristic
conditions lead to the time irreversibility of solutions, since characteristics impinge on the shock, entropy increases and
information is lost. In classical gas dynamics, the density and pressure are always larger behind stable shock waves,
which means spherically symmetric shock waves with a greater pressure and density on the interior are expected to
expand.

Consider again the solution triples \((g, \rho, \bar{u})\) and \((\bar{g}, \bar{\rho}, \bar{v})\) with equations of state \( p = p(\rho) \) and \( \bar{p} = \bar{p}(\bar{\rho}) \) respectively,
and assume that these solutions form the shock-wave solution \( g \cup \bar{g} \). As a spherical surface has an interior and exterior,
let \( g \) represent the interior metric, which is given in comoving coordinates \((\hat{t}, \hat{r})\) as so:

\[
 ds^2 = -e^{2\psi} \, d\hat{t}^2 + e^{2\hat{\psi}} \, d\hat{r}^2 + \mathcal{R}^2 \, \hat{r}^2 \, d\Omega^2
\]

Finally, let \( \bar{g} \) represent the exterior metric, with the associated comoving coordinates denoted by \((\bar{\hat{t}}, \bar{\hat{r}})\). The objective is
to determine the Lax characteristic conditions at the shock surface.

**Lemma 1.** The shock speed relative to the interior fluid is given by:

\[
 e^{\psi - \hat{\psi}} \Phi
\]

where \( \hat{\Phi} \) is the position of the shock in coordinates comoving with the interior fluid.

**Proof.** This proof largely follows an analogous proof provided in [7]. To begin, recall that the speed of a shock is a
coordinate dependent quantity that can be interpreted in a special relativistic sense at a point \( p \) in coordinate systems for
which:

\[
 ds^2 = -d\bar{t}^2 + d\bar{r}^2 + \bar{\mathcal{R}}^2 \, d\Omega^2
\]

where \( \bar{\mathcal{R}}_0 \) is the value of \( \bar{\mathcal{R}} \) at \( p \). In a locally Minkowskian coordinate frame, a speed at \( p \) transforms according to the
special relativistic velocity transformation law when a Lorentz transformation is performed. The shock speed at a point \( p \)
on the shock in a locally Minkowskian frame that is comoving with the interior fluid will now be determined. To this
end, let \( \hat{\bar{\Phi}} = \Phi(\hat{\Phi}) \) be the position of the shock in \((\hat{t}, \hat{r})\) coordinates and let \((\hat{t}, \hat{r})\) coordinates correspond to a locally
Minkowskian system at \( p \) obtained from \((\hat{t}, \hat{r})\) by a transformation of the form:

\[
 \hat{t} = \bar{\Phi}(\hat{\Phi}) \\
 \hat{r} = \bar{\Phi}(\hat{\Phi})
\]

so that, in \((\hat{t}, \hat{r})\) coordinates:

\[
 ds^2 = -e^{2\psi} \left( \frac{d\hat{t}}{d\hat{t}} \right)^2 d\hat{t}^2 + e^{2\hat{\psi}} \left( \frac{d\hat{r}}{d\bar{r}} \right)^2 d\bar{r}^2 + \mathcal{R}^2 \bar{r}^2 d\Omega^2
\]

Choose \((\hat{t}, \hat{r})\) so that:

\[
 \frac{d\hat{t}}{d\hat{t}} = e^{\varphi} \\
 \frac{d\hat{r}}{d\bar{r}} = e^{\hat{\psi}}
\]
Then in $(\tilde{t}, \tilde{r})$ coordinates at $p$ the metric takes the form of (37). The $(\tilde{t}, \tilde{r})$ coordinates represent the class of locally Minkowskian coordinate frames that are fixed relative to the fluid particles of the interior spacetime at the point $p$, that is, any two members of this class of coordinate frames differ only by higher order terms that do not affect the calculation of radial velocities at $p$. Thus the speed $\dot{\tilde{r}}$ of a particle in $(\tilde{t}, \tilde{r})$ coordinates gives the value of the speed of the particle relative to the interior fluid in the special relativistic sense. If the speed of a particle in $(\tilde{t}, \tilde{r})$ coordinates is $\dot{\tilde{r}}$, then its geometric speed relative to observers fixed with the interior fluid, and hence also fixed relative to the radial coordinate $\tilde{r}$ of the metric $g$ because the fluid is comoving, is equal to:

$$e^{\psi-\varphi} \dot{\tilde{r}}$$

since:

$$\frac{d\tilde{r}}{dt} = \frac{d\tilde{r}}{d\tilde{t}} \frac{d\tilde{t}}{dt} \frac{d\tilde{t}}{dt} = e^{\varphi-\psi} \frac{d\tilde{r}}{d\tilde{t}}$$

(38)

Now considering the shock wave moves with speed $\Phi$, therefore by (38) the speed of the shock relative to the interior fluid particles must be given by (36), which completes the proof. □

Let $\tilde{\lambda}_{nt}^+$ and $\tilde{\lambda}_{nt}^-$ denote the speeds of the interior characteristics in $(\tilde{t}, \tilde{r})$ coordinates. Since the characteristic speeds on the interior side of the shock equal the sound speeds in locally Minkowskian coordinates, we have:

$$\tilde{\lambda}_{nt}^\pm = \pm \sqrt{\frac{dp}{d\rho}}$$

The $-,+$ characteristics refer to the 1,2-characteristic families respectively. In the 1+1 dimensional theory of conservation laws, the Lax characteristic conditions state that the characteristic curves in the family of the shock impinge upon the shock from both sides. Since we are considering shocks that are outward moving with respect to $\tilde{r}$ and $\tilde{r}$, it follows that on the interior side, only the 2-characteristic can impinge on the shock, and thus the shock must be identified as a 2-shock. Alternatively, the Lax characteristic conditions can be thought of as selecting whether it is the outgoing or ingoing shock that is stable. For more details on $n$-shocks, see [5]. Let $\tilde{\lambda}_{Ext}^+$ and $\tilde{\lambda}_{Ext}^-$ denote the speeds of the exterior characteristics in $(\tilde{t}, \tilde{r})$ coordinates. Since the shock has been identified as a 2-shock, the Lax characteristic conditions are given as the following inequalities:

$$\tilde{\lambda}_{Ext}^+ < s < \tilde{\lambda}_{nt}^+$$

(39)

where $s$ is the speed of the shock in $(\tilde{t}, \tilde{r})$ coordinates.

**Proposition 8.** For an expanding shock wave, the Lax characteristic conditions are given as the following inequalities:

$$\frac{\tilde{w} + \sqrt{\frac{dp}{d\rho}}}{1 + \tilde{w} \sqrt{\frac{dp}{d\rho}}} < e^{\psi-\varphi} \Phi < \sqrt{\frac{dp}{d\rho}}$$

(40)

where:

$$\tilde{w} = e^{\psi-\varphi} \frac{\partial \tilde{r}}{\partial \tilde{t}} \left( \frac{\partial \tilde{t}}{\partial \tilde{r}} \right)^{-1}$$

**Proof.** This proof largely follows an analogous proof provided in [7]. Since the shock wave is expanding, it is a 2-shock, so the Lax characteristic conditions are given by (35), and by Lemma 7 $s$ is given by (36). As we are working in $(\tilde{t}, \tilde{r})$ coordinates, $\tilde{\lambda}_{nt}^+$ is already known, so it remains to determine $\tilde{\lambda}_{Ext}^+$. Let $\tilde{v}$, $\tilde{v}$ and $\tilde{v}$ denote the exterior fluid four-velocity given in interior comoving, exterior comoving and interior locally Minkowskian coordinates respectively. Since the aim is to compute the characteristic speed, which is a ratio of two vector components, a tangent vector of any length is sufficient. By writing $\tilde{x} = (\tilde{t}, \tilde{r})$ and $\bar{x} = (\tilde{t}, \tilde{r})$, then:

$$\dot{\bar{\tilde{X}}}^\mu = \frac{\partial \tilde{X}^\mu}{\partial \tilde{x}^\nu} \tilde{v}^\nu = \frac{\partial \tilde{X}^\mu}{\partial \tilde{x}^\nu} \tilde{v}^0 = \frac{\partial \tilde{X}^\mu}{\partial \tilde{x}^0}$$

In light of this, the speed of the exterior fluid as measured in the interior coordinates $(\tilde{t}, \tilde{r})$ is given by:

$$\tilde{w} = \tilde{w}^1 \tilde{v}^1 = \frac{\partial \tilde{x}^1}{\partial \tilde{x}^0} \left( \frac{\partial \tilde{x}^0}{\partial \tilde{x}^0} \right)^{-1} = \frac{\partial \tilde{r}}{\partial \tilde{t}} \left( \frac{\partial \tilde{t}}{\partial \tilde{r}} \right)^{-1}$$
and so, by (38):
\[ \tilde{w} = e^{\psi - \varphi} \tilde{w} \]
This gives the exterior fluid speed in \((\tilde{t}, \tilde{r})\) coordinates, and since the sound speed in the exterior spacetime is given by:
\[ \sqrt{\frac{dp}{d\tilde{r}}} \]
the relativistic addition of velocities formula yields:
\[ \tilde{\lambda}^{+}\text{ext} = \frac{\tilde{w} + \sqrt{\frac{dp}{d\tilde{r}}}}{1 + \tilde{w} \sqrt{\frac{dp}{d\tilde{r}}}} \]
which completes the proof.

3 Friedmann-Static Shock Waves

3.1 Analytical Results

The objective of this section is the construction of the family of FLRW \((0, \sigma, a)\)-TOV \((\bar{\sigma})\) shock waves. As remarked previously, the triple \((A,G,v)\) is used to denote a solution of the spherically symmetric self-similar perfect fluid Einstein field equations (5)-(7).

Definition 8. A shock-wave solution with an FLRW \((0, \sigma, a)\) spacetime on the interior and a TOV \((\bar{\sigma})\) spacetime on the exterior is referred to as a Friedmann-static shock wave and denoted by FLRW \((0, \sigma, a)\)-TOV \((\bar{\sigma})\).

Since all Friedmann-static shock waves share a TOV \((\bar{\sigma})\) exterior, the following lemma is of great utility in their construction.

Lemma 2. Let \((A,G,v)\) denote a spherically symmetric self-similar solution to the perfect fluid Einstein field equations with equation of state \(p = \sigma \rho\). If there exists a \(\xi_0 > 0\) such that:
\[ A(\xi_0) = 1 - 2M(\bar{\sigma}) \] (41)
then \((A,G,v)\) can be matched to TOV(\bar{\sigma}) on the surface \(\xi = \xi_0\) and the Rankine-Hugoniot jump condition is given by:
\[ \frac{[\sigma + v^2(\xi_0)]G(\xi_0) - (1 + \sigma)G^2(\xi_0)v(\xi_0)}{[1 + \sigma v^2(\xi_0)]G(\xi_0) - (1 + \sigma)v(\xi_0)} = \bar{\sigma} \] (42)

Proof. Let the metric of the \((A,G,v)\) solution in self-similar Schwarzschild coordinates be given by:
\[ ds^2 = -B(\xi)d\tau^2 + \frac{1}{A(\xi)}dr^2 + r^2d\Omega^2 \]
and recall by Proposition 2 that TOV(\bar{\sigma}) is given in self-similar comoving Schwarzschild coordinates as:
\[ ds^2 = -\bar{\xi}^2d\bar{\tau}^2 + \frac{1}{1 - 2M(\bar{\sigma})d\bar{\rho}^2 + \bar{\rho}^2d\bar{\Omega}^2} \]
\[ \bar{\rho} = \frac{2M(\bar{\sigma})}{\bar{\rho}^2} \]
where the inessential parameter has been set to one and:
\[ M(\bar{\sigma}) = \frac{2\bar{\sigma}}{1 - 6\bar{\sigma} + \bar{\sigma}^2} \]
Because both metrics are specified in Schwarzschild coordinates, the \(d\Omega^2\) components automatically match under the identification \(\bar{\tau} = r\). Matching the \(dr^2\) components implies that the shock surface is defined by \(\xi = \xi_0\), with the constant \(\xi_0\) given implicitly by (41). This also implies that \(B\) is constant on the surface. A temporal rescaling of the form:
\[ \tilde{\tau} = \alpha \tau \]
With the matching in place, recall from Subsection 2.5 that the Rankine-Hugoniot jump condition is equivalent to:

\[ \alpha \xi = \xi = \xi_0 \]

and matches the \( dt^2 \) coefficients providing \( \alpha \) satisfies:

\[ \alpha^2 \xi_0^{-\frac{2\sigma}{\xi_0}} = B(\xi_0) \tag{43} \]

With the matching in place, recall from Subsection 2.5 that the Rankine-Hugoniot jump condition is equivalent to:

\[ [T^\mu_\nu]n_\mu n_\nu = T^\mu_\nu(g, \rho, p, u_{FLRW})n_\mu n_\nu - T^\mu_\nu(g, \bar{\rho}, \bar{p}, u_{TOV})n_\mu n_\nu = 0 \]

where \( n \) is the outward normal to the shock surface. Using this and \( p = \sigma \rho \) we obtain:

\[ (1 + \sigma)\rho u_{FLRW}^\mu u_{FLRW}^\nu n_\mu n_\nu + \sigma \rho |n|^2 - (1 + \bar{\sigma})\bar{\rho} u_{TOV}^\mu u_{TOV}^\nu n_\mu n_\nu - \bar{\sigma} \bar{\rho} |n|^2 = 0 \]

Now since the surface is defined by \( \xi = \xi_0 \), which is equivalent to:

\[ r - \xi_0 t = 0 \]

then the components of the normal satisfy:

\[ n_\mu dx^\mu = d(r - \xi_0 t) = -\xi_0 dt + dr \]

and so:

\[ n_0 = -\xi_0 \]
\[ n_1 = 1 \]

Noting that the metric components are identified on the surface, the following identities are obtained:

\[ |n|^2 = A(\xi_0) - \xi_0^2 B^{-1}(\xi_0) \]
\[ u_{FLRW}^0 = [1 - v^2(\xi_0)]^{-\frac{1}{2}} B^{-\frac{1}{2}}(\xi_0) \]
\[ u_{FLRW}^1 = v(\xi_0)[1 - v^2(\xi_0)]^{-\frac{1}{2}} A^\frac{1}{2}(\xi_0) \]
\[ u_{FLRW}^\mu u_{FLRW}^\nu n_\mu n_\nu = [1 - v^2(\xi_0)]^{-1} \left[v(\xi_0)A^\frac{1}{2}(\xi_0) - \xi_0 B^{-\frac{1}{2}}(\xi_0)\right]^2 \]
\[ u_{TOV}^\mu u_{TOV}^\nu n_\mu n_\nu = \xi_0^2 B^{-1}(\xi_0) \]

Applying these identities puts the Rankine-Hugoniot jump condition in the following form:

\[ 0 = (1 + \sigma)[1 - v^2(\xi_0)]^{-1} \left[v(\xi_0)A^\frac{1}{2}(\xi_0) - \xi_0 B^{-\frac{1}{2}}(\xi_0)\right]^2 \rho \]

\[ - (1 + \bar{\sigma})\xi_0^2 B^{-1}(\xi_0)\bar{\rho} \]

\[ + \left[A(\xi_0) - \xi_0^2 B^{-1}(\xi_0)\right] (\sigma \rho - \bar{\sigma} \bar{\rho}) \]

Dividing by \( A(\xi_0) \) and substituting \( B(\xi_0) \) for \( G(\xi_0) \) then yields:

\[ 0 = (1 + \sigma)[1 - v^2(\xi_0)]^{-1} \left[v(\xi_0) - G(\xi_0)\right]^2 \rho \]

\[ - (1 + \bar{\sigma})G^2(\xi_0)\bar{\rho} \]

\[ + \left[1 - G^2(\xi_0)\right] (\sigma \rho - \bar{\sigma} \bar{\rho}) \]

Finally, applying (8) and (41) gives (42), which completes the proof. \( \Box \)

As FLRW\((0, \sigma, 1)\) is known explicitly, it is possible to construct an explicit FLRW\((0, \sigma, 1)\)-TOV(\(\bar{\sigma}\)) shock wave. Such a construction has been achieved by Cahill and Taub \(1\) for the case \( \sigma = \frac{1}{\alpha} \) and in full generality by Smoller and Temple \(2\). The result can instead be derived directly from Lemma 2.

**Theorem 3.** For each \( 0 < \sigma < 1 \), FLRW\((0, \sigma, 1)\) can be matched to TOV(\(\bar{\sigma}\)) to form a general relativistic shock wave providing:

\[ \bar{\sigma} = H(\sigma) \]

where:

\[ H(\sigma) = \frac{1}{2} \sqrt{9\sigma^2 + 54\sigma + 49} - \frac{3}{2} \sigma - \frac{7}{2} \]  \(\tag{44}\)
Proof. The matching follows similarly to the matching completed in the proof of Lemma 2, but with (41) and (43) replaced with:

\[ 1 - \frac{2}{3} \xi_0^{\frac{2+6\alpha}{2+6\beta}} = 1 - 2M(\bar{\sigma}) \]

and:

\[ \varrho^2 \xi_0^{\frac{2+6\alpha}{2+6\beta}} = \frac{16}{(3 + 3\sigma)^2} \left[ 1 + \frac{1}{3} (1 + 3\sigma) \xi_0^{\frac{2+6\alpha}{2+6\beta}} \right]^{-\frac{2-6\alpha}{2+6\beta}} \left[ 1 - \frac{2}{3} \xi_0^{\frac{2+6\alpha}{2+6\beta}} \right]^{-1} \]

respectively, where the inessential parameter is set by (33) and \( \xi \) is defined implicitly by:

\[ \xi = \frac{4}{\sqrt{6}} \varepsilon^{1+3\sigma} \left[ 1 + \frac{1}{3} (1 + 3\sigma) \xi_0^{\frac{2+6\alpha}{2+6\beta}} \right]^{-\frac{3+3\sigma}{2+6\beta}} \]

Note that this matching is Lipschitz continuous, as any \( 0 < \sigma < 1 \) and \( 0 < \bar{\sigma} < 1 \) imply that the components of the interior and exterior metrics are continuous in a neighbourhood of the surface when given in \( (t, r) \) coordinates. Thus it remains to show that the condition \( \bar{\sigma} = H(\sigma) \) is equivalent to the Rankine-Hugoniot jump condition, which we know by Lemma 2 is given by:

\[ \frac{[\sigma + v^2(\xi_0)]G(\xi_0) - (1 + \sigma)G^2(\xi_0)v(\xi_0)}{[1 + \sigma v^2(\xi_0)]G(\xi_0) - (1 + \sigma)v(\xi_0)} = \bar{\sigma} \]

By Proposition 4, \( G(\xi_0) \) can be substituted for \( v(\xi_0) \), which in turn can be substituted for \( A(\xi_0) \) to yield:

\[ \frac{(3\sigma + 3[1 - A(\xi_0)])(2 + (1 + 3\sigma)[1 - A(\xi_0)]) - (3 + 3\sigma)^2[1 - A(\xi_0)]}{A(\xi_0)(2 + (1 + 3\sigma)[1 - A(\xi_0)])} = \bar{\sigma} \]

Finally, substituting \( A(\xi_0) \) for \( 1 - 2M(\bar{\sigma}) \) yields:

\[ \sigma = \frac{\bar{\sigma}(7 + \bar{\sigma})}{3(1 - \bar{\sigma})} \]

which is equivalent to \( \bar{\sigma} = H(\sigma) \). \( \square \)

Definition 9. The Rankine-Hugoniot curve, denoted by:

\[ v = \Gamma_{RH}(G; \sigma, \bar{\sigma}) \]

is the curve in \((A, G, v)\) space that is generated by constraints (41) and (42).

3.2 Numerical Results

We are now in a position to extend the family of FLRW(0, \( \sigma, 1 \))-TOV(\( \bar{\sigma} \)) shock waves to the family of FLRW(0, \( \sigma, a \))-TOV(\( \bar{\sigma} \)) shock waves. Even though the FLRW(0, \( \sigma, a \)) spacetimes are exact solutions, they are not known explicitly away from \( \xi = 0 \), so these solutions need to be approximated numerically. One way of describing FLRW(0, \( \sigma, a \)) solutions is to numerically generate their trajectories in \((A, G, v)\) space, such as in Figure 1.

The FLRW(0, \( \frac{1}{3}, 1 \)) trajectory obeys the implicit relationship given by Corollary 2 that is, as \( \xi \) increases from zero, \( G \) increases linearly with \( \xi \), \( v \) increases according to (21) and \( A \) decreases according to (20). General FLRW(0, \( \frac{1}{3}, a \)) trajectories are similar to the FLRW(0, \( \frac{1}{3}, 1 \)) trajectory for small \( \xi \) but differ as \( \xi \) increases. One property that remains similar for larger \( \xi \) is the near linear dependence of \( G \) on \( \xi \). Note that because equations (23)-(27) are autonomous, all trajectories, the Rankine-Hugoniot curve and surfaces \{\( s \) = 0\} and \{\( D \) = 0\} are all independent of \( \xi \). Thus it is often easier to think of \( G \) as the independent variable and consider the trajectory as a function of \( G \).

The TOV(\( \bar{\sigma} \)) trajectories are simple to represent in \((A, G, v)\) solution space because they are the lines defined by \( A = 1 - 2M(\bar{\sigma}) \) and \( v = 0 \). Since:

\[ \min_{0 \leq \bar{\sigma} \leq 1} \{1 - 2M(\bar{\sigma})\} = \frac{1}{2} \] (45)

the TOV(\( \bar{\sigma} \)) trajectories span the surface:

\[ \frac{1}{2} < A < 1 \]

\[ v = 0 \]
General Relativistic Shock Waves that Exhibit Cosmic Acceleration

The reason considering solutions in \((A, G, v)\) space is so useful, is because of the immediate implication that any trajectory that crosses the \(A = 1 - 2M(\bar{\sigma})\) plane Lipschitz continuously can be matched to the TOV(\(\bar{\sigma}\)) solution. Furthermore, if the solution trajectory crosses the \(A = 1 - 2M(\bar{\sigma})\) plane and intersects the Rankine-Hugoniot curve, which lies in this plane, then the solution can be matched to the TOV(\(\bar{\sigma}\)) solution to form a general relativistic shock wave. We already know from Theorem 3 that for \(a = 1\) the relationship between \(\sigma\) and \(\bar{\sigma}\) obeys \(\bar{\sigma} = H(\sigma)\). For \(a \neq 1\), trajectories can be generated numerically and the parameters \(a, \sigma\) and \(\bar{\sigma}\) can be adjusted to achieve the intersection. Since the intersection imposes a single constraint on the parameters \(a, \sigma\) and \(\bar{\sigma}\), we conclude that the family of FLRW(0, \(\sigma, a\))-TOV(\(\bar{\sigma}\)) shock waves is a one-parameter family for each \(\sigma\). Fixing \(\sigma = \frac{1}{3}\), the resulting family partially answers a claim given in [1] by determining a subset of the pure radiation similarity spacetimes that can be matched to TOV(\(\bar{\sigma}\)) to form a general relativistic shock wave.

A physically important Friedmann-static shock wave is the one for which the equation of state each side of the shock models pure radiation, since these shock waves may have been present during the Radiation Dominated Epoch. As demonstrated in Figure 2 for \(\sigma = \bar{\sigma} = \frac{1}{3}\), the value of \(a\) can be varied in order to achieve an intersection and thus form the Friedmann-static pure radiation shock wave. In Figure 2 the leftmost trajectory overshoots the curve and rightmost trajectory undershoots it. The leftmost, centre and the rightmost trajectories are generated for:

\[ a = 2.8 \]
\[ a = 2.58 \]
\[ a = 2.4 \]

respectively. Therefore, the value of the acceleration parameter for the Friedmann-static pure radiation shock wave is approximated by:

\[ a \approx 2.58 \]

with the corresponding point of intersection approximated by:

\[ \xi_0 \approx 0.706 \]

We know from Subsection 2.4 that the FLRW(0, \(\sigma, a\)) spacetimes can exhibit an accelerated expansion similar to the accelerated expansion found in the Standard Model of Cosmology when \(a \approx 1\). It is conjectured by Temple that the accelerated expansion observed today is possibly not the result of dark energy, but instead from being within a vast primordial shock wave with an FLRW(0, \(\sigma, a\)) interior. By vast, we mean a shock wave with a shock surface that lies beyond the Hubble radius, that is, not presently observable.
What makes this proposal particularly interesting is that the magnitude of acceleration, parameterised by $a$, is determined purely mathematically by the equation of state parameter each side of the shock, assuming a TOV$(\bar{\sigma})$ exterior. However, with $a \approx 2.58$, the Friedmann-static pure radiation shock wave exhibits a cosmic acceleration many orders of magnitude larger than what is observed today, and observational measurements suggest that cosmic acceleration has only increased since the Radiation Dominated Epoch. Furthermore, the Friedmann-static pure radiation shock surface lies within the Hubble radius. Each of these properties rule out the Friedmann-static pure radiation shock wave as a cosmological model, but does not rule out a shock-wave cosmological model consisting of an interior FLRW$(0, \sigma, a)$ spacetime matched to a non-TOV$(\bar{\sigma})$ exterior. For $a = 1$, one such shock wave is constructed with the shock surface lying beyond the Hubble radius by Smoller and Temple in [8], with the full details of this construction provided in [9]. It remains an open problem to construct shock waves with $a \neq 1$ FLRW$(0, \sigma, a)$ interiors for which the resulting shock surface lies beyond the Hubble radius.

4 Lax Stability of Shock Waves with Static Exteriors

4.1 The Lax Characteristic Conditions

As remarked in Subsection 2.7, the Lax characteristic conditions provide a notion of stability for general relativistic shock waves and provide an entropy argument for determining whether the shock wave is expected to expand or contract. The Lax characteristic conditions considered in this section apply to expanding shock waves, meaning that shock waves that satisfy these conditions are expected to expand.

Lemma 3. Let $(A, G, v)$ denote a spherically symmetric self-similar solution to the perfect fluid Einstein field equations with equation of state $p = \sigma \rho$. If there exists a $\xi_0 > 0$ such that $(A, G, v)$ can be matched to TOV$(\bar{\sigma})$ to form a shock-wave solution, then the Lax characteristic conditions are given by:

\[
\frac{\sqrt{\bar{\sigma}} - v(\xi_0)}{1 - \sqrt{\bar{\sigma}}v(\xi_0)} < \frac{G(\xi_0) - v(\xi_0)}{1 - G(\xi_0)v(\xi_0)} < \frac{\sqrt{\sigma}}{2} \tag{46}
\]

Proof. As a reverse to the coordinate transformation introduced in the proof of Proposition 3, we begin by transforming a general solution given in self-similar Schwarzchild coordinates, to a solution given in self-similar comoving coordinates. Noting that $B$ and $u$ are given implicitly by the triple $(A, G, v)$, we can write this solution in self-similar Schwarzchild coordinates as so:

\[
\begin{align*}
\frac{ds^2}{-B(\xi)d\xi^2 + \frac{1}{A(\xi)}dr^2 + r^2d\Omega^2} &= (u^0, u^1, 0, 0)
\end{align*}
\]
where $p$ and $\rho$ are determined by [4] and [8] respectively. In self-similar comoving coordinates, the solution can be written as:

$$d\tilde{s}^2 = -e^{2\phi} dt^2 + e^{2\psi} dr^2 + A^2 \hat{r}^2 d\Omega^2$$

$$\hat{u} = (\hat{u}^0, 0, 0, 0)$$

To eliminate the radial component of the four-velocity, the transformation from Schwarzschild to comoving coordinates must satisfy:

$$\hat{u}^1 = u^0 \frac{\partial \hat{r}}{\partial t} + u^1 \frac{\partial \hat{r}}{\partial r} = 0$$

which is equivalent to:

$$\frac{\partial \hat{r}}{\partial t} = -\xi v \frac{\partial \hat{r}}{\partial r} \quad (47)$$

Now given that:

$$d\hat{t} = \frac{\partial \hat{t}}{\partial t} dt + \frac{\partial \hat{t}}{\partial r} dr$$

$$d\hat{r} = \frac{\partial \hat{r}}{\partial t} dt + \frac{\partial \hat{r}}{\partial r} dr$$

then:

$$d\hat{t} = \left( \frac{\partial \hat{t}}{\partial t} \frac{\partial \hat{r}}{\partial r} - \frac{\partial \hat{t}}{\partial r} \frac{\partial \hat{r}}{\partial t} \right)^{-1} \left( -\frac{\partial \hat{r}}{\partial r} \frac{\partial \hat{t}}{\partial r} + \frac{\partial \hat{r}}{\partial t} \frac{\partial \hat{r}}{\partial t} \right)$$

$$d\hat{r} = \left( \frac{\partial \hat{r}}{\partial t} \frac{\partial \hat{r}}{\partial r} - \frac{\partial \hat{r}}{\partial r} \frac{\partial \hat{r}}{\partial t} \right)^{-1} \left( -\frac{\partial \hat{r}}{\partial t} \frac{\partial \hat{r}}{\partial t} + \frac{\partial \hat{r}}{\partial r} \frac{\partial \hat{r}}{\partial r} \right)$$

Thus to keep the metric diagonal, the following condition is also needed:

$$B \frac{\partial \hat{r}}{\partial r} \frac{\partial \hat{t}}{\partial r} - \frac{1}{A} \frac{\partial \hat{r}}{\partial t} \frac{\partial \hat{r}}{\partial t} = 0$$

which by (47) is equivalent to:

$$\frac{\partial \hat{t}}{\partial r} = -\frac{G v \hat{t}}{\xi} \quad (48)$$

The most general transformation that preserves self-similarity takes the form:

$$\hat{t} = T(\xi) t$$

$$\hat{r} = R(\xi) r$$

and conditions (47) and (48) determine the functions $T(\xi)$ and $R(\xi)$. In self-similar Schwarzschild coordinates the shock speed is given by $\xi = \xi_0$, so in self-similar comoving coordinates the shock speed is given by $\hat{\xi} = \hat{\xi}_0$, where:

$$\hat{\xi} = \frac{\hat{r}}{\hat{t}} = \frac{R(\xi) r}{T(\xi) t}$$

Thus by Lemma[1] the shock speed is given in interior locally Minkowskian coordinates by:

$$e^{\psi - \phi} \hat{\xi}_0$$

By Proposition[8] it remains to determine $e^{\psi - \phi}$ and $\hat{w}$. In this light:

$$e^{2\phi} = \frac{1}{A} \left( \frac{\partial \hat{r}}{\partial t} \right)^2 - \frac{\xi^2}{A G^2} \left( \frac{\partial \hat{r}}{\partial r} \right)^2$$

$$= \frac{\xi^2 (1 - v^2)}{A G^2} \left( \frac{\partial \hat{r}}{\partial r} \right)^2$$
and:

\[ e^{2\psi} = \frac{1}{A} \left( \frac{\partial \hat{t}}{\partial t} \right)^2 - \frac{\xi^2}{AG^2} \left( \frac{\partial \hat{r}}{\partial r} \right)^2 \]

Now:

\[ \frac{\partial \hat{r}}{\partial t} = -\xi^2 R'(\xi) \]
\[ \frac{\partial \hat{r}}{\partial r} = R(\xi) + \xi R'(\xi) \]

so (47) yields:

\[ -\xi^2 R' = -\frac{\xi v}{G} (R + \xi R') \]
\[ \Leftrightarrow \xi R' = \frac{v}{G} - \frac{\xi v}{R} \]

Similarly:

\[ \frac{\partial \hat{t}}{\partial t} = T(\xi) - \xi T'(\xi) \]
\[ \frac{\partial \hat{t}}{\partial r} = T'(\xi) \]

to which (48) yields:

\[ T' = -\frac{Gv}{\xi} (T - \xi T') \]
\[ \Leftrightarrow \xi T' = -\frac{Gv}{1 - Gv} T \]

Therefore the shock speed is given in interior locally Minkowskian coordinates by:

\[ e^{\psi - \varphi} \hat{\xi}_0 = G(\xi_0) \frac{\partial \hat{t}}{\partial t} \left( \frac{\partial \hat{r}}{\partial r} \right)^{-1} \frac{R(\xi_0)}{T(\xi_0)} \]
\[ = G(\xi_0) - v(\xi_0) \frac{1}{1 - G(\xi_0) v(\xi_0)} \]

By Proposition 2, TOV(\bar{\sigma}) is comoving in Schwarzschild coordinates, and given that TOV(\bar{\sigma}) is matched to (A, G, v) in (t, r) coordinates, then the (\bar{t}, \bar{r}) coordinates of Proposition 8 are identified with (t, r), so:

\[ \hat{w} = e^{\psi - \varphi} \hat{\xi}_0 \]
\[ = e^{\psi - \varphi} \hat{r} \left( \frac{\partial \hat{t}}{\partial t} \right)^{-1} \]
\[ = e^{\psi - \varphi} \hat{r} \left( \frac{\partial \hat{t}}{\partial t} \right)^{-1} \]
\[ = G(\xi_0) \frac{\partial \hat{t}}{\partial t} \left( \frac{\partial \hat{r}}{\partial r} \right)^{-1} \]
\[ = -v(\xi_0) \]

Finally, substituting \( e^{\psi - \varphi} \hat{\xi}_0, \hat{w}, \) and the equations of state into (40) yields (46).

The following theorem was first proved in [7], but can instead be obtained directly from Lemma 3. In [7], the value of \( \sigma_1 \) is approximated, however it is now possible to obtain an exact value.

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**Theorem 4.** The expanding FLRW(0, σ, 1)-TOV(σ) shock-wave solutions satisfy the Lax characteristic conditions for:

\[ 0 < \sigma < \sigma_1 \]

where:

\[ \sigma_1 = \frac{1 + \sqrt{10}}{9} \approx 0.462 \]

**Proof.** By Lemma 3 and Proposition 4 we know that FLRW(0, σ, 1) satisfies (42) and:

\[ G(\xi_0) = \frac{1}{2} (3 + 3 \sigma) v(\xi_0) \left( 1 + \frac{1}{2} (1 + 3 \sigma) v^2(\xi_0) \right)^{-1} \]  \hspace{1cm} (49)

at the point of intersection with the shock surface. Solving (42) and (49) for \( G(\xi_0) \) and \( v(\xi_0) \) yields:

\[ G(\xi_0) = \frac{1}{2} (3 + \bar{\sigma}) v(\xi_0) \]  \hspace{1cm} (50)

\[ v(\xi_0) = \sqrt{\frac{2(3\sigma - \bar{\sigma})}{(1 + 3\sigma)(3 + \bar{\sigma})}} \]  \hspace{1cm} (51)

Thus using (44), (50) and (51), the left hand inequality of (46) is found to be satisfied for \( 0 < \sigma < 1 \) and the right hand inequality is found to be satisfied for \( 0 < \sigma < \sigma_1 \).

---

**4.2 Lax Stability Theorem**

**Lemma 4.** Let \((A, G, v)\) denote a spherically symmetric self-similar solution to the perfect fluid Einstein field equations with equation of state \( p = \sigma \rho \). If there exists a \( \xi_0 > 0 \) such that \((A, G, v)\) can be matched to TOV(σ) to form a shock-wave solution, then the shock speed is subluminal if:

\[ G(\xi_0) < 1 \]  \hspace{1cm} (52)

and in such a case the Lax characteristic conditions reduce to:

\[ G(\xi_0) > \sqrt{\sigma} \]  \hspace{1cm} (53)

\[ \{\cdot\}_D(\xi_0) < 0 \]  \hspace{1cm} (54)

**Proof.** By Lemma 3 the shock speed is subluminal if:

\[ \frac{G(\xi_0) - v(\xi_0)}{1 - G(\xi_0)v(\xi_0)} < 1 \]

which for \( 0 < v < 1 \) is equivalent to (52). For \( G < 1 \) it is then not difficult to check that the left hand inequality of (46) is equivalent to (53). Thus it remains to demonstrate that the right hand inequality is equivalent to (54). In this light, we have:

\[ \{\cdot\}_D = \frac{3}{4} (3 + 3\sigma) \left[ (G - v)^2 - \sigma (1 - Gv)^2 \right] \]

\[ = \frac{3}{4} (3 + 3\sigma) \left[ G - v + \sqrt{\sigma}(1 - Gv) \right] \left[ G - v - \sqrt{\sigma}(1 - Gv) \right] \]

and for \( 0 < v < G < 1 \) we see that \( \{\cdot\}_D = 0 \) is equivalent to:

\[ \frac{G - v}{1 - Gv} = \sqrt{\sigma} \]

which completes the proof.

The following theorem, also first proved in [7], demonstrates that even though FLRW(0, σ, 1)-TOV(σ) shock waves can be constructed mathematically, their physical applicability may be limited to \( \sigma < \sigma_2 \).

**Theorem 5.** The FLRW(0, σ, 1)-TOV(σ) shock-wave solutions have subluminal shock speeds for:

\[ 0 < \sigma < \sigma_2 \]

where:

\[ \sigma_2 = \frac{\sqrt{5}}{3} \approx 0.745 \]
The following definition is consistent with the sonic surface definition given in [2], that is, the surfaces have the same physical interpretation.

**Definition 10.** The *singular surface* and *sonic surface* are defined in $(A, G, v)$ space by \( \{ \cdot \}_S = 0 \) and \( \{ \cdot \}_D = 0 \) respectively. Moreover, the *subsonic region* and *supersonic region* are defined by \( \{ \cdot \}_D < 0 \) and \( \{ \cdot \}_D > 0 \) respectively. Furthermore, solutions whose trajectories remain in the subsonic region are referred to as *subsonic*, those that remain in the supersonic region as *supersonic* and those that pass through the sonic surface as *transonic*.

As a consequence of Lemma 4, the sonic surface serves as a convenient indicator for the Lax stability of an expanding general relativistic shock wave with a TOV $(\bar{\sigma})$ exterior. This is particularly useful for numerical approximations, since if the intersection with the Rankine-Hugoniot curve is in the subsonic region and to the right of the \( G = \sqrt{\sigma} \) plane, the resulting shock wave is stable in the Lax sense. For \( \sigma \neq \bar{\sigma} \), condition (53) is not automatically satisfied, since the Rankine-Hugoniot jump condition curve does not intersect the \( v = 0 \) plane at \( G = \sqrt{\sigma} \), as Figure 3 demonstrates.

In Figure 3 the leftmost, centre and rightmost dashed curves correspond to:

\[
\begin{align*}
\sigma < \bar{\sigma} &= \frac{2}{3} \\
\sigma = \bar{\sigma} &= \frac{1}{3} \\
\sigma > \bar{\sigma} &= \frac{1}{6}
\end{align*}
\]

respectively. In the \( \sigma < \bar{\sigma} \) case, the Rankine-Hugoniot curve always touches the singular surface at \((G, v) = (0, 0)\) and \((G, v) = (1, 1)\). In the \( \sigma = \bar{\sigma} \) case, the Rankine-Hugoniot curve always touches the sonic surface at \((G, v) = (\sqrt{\bar{\sigma}}, 0)\) and \((G, v) = (1, 1)\). In the \( \sigma > \bar{\sigma} \) case, the Rankine-Hugoniot curve also touches the sonic surface at \((G, v) = (1, 1)\) and intersects it at:

\[
G = \sqrt{\sigma(1 + \bar{\sigma}) + \sqrt{[\sigma - \bar{\sigma][1 - \sigma \bar{\sigma}]} \over 1 + \sigma}
\]

\[
v = \sqrt{\frac{\sigma - \bar{\sigma}}{1 - \sigma \bar{\sigma}}}
\]

Thus for an expanding Friedmann-static shock wave to be unstable in the Lax sense, the solution trajectory must either intersect the Rankine-Hugoniot curve before the \( G = \sqrt{\sigma} \) plane or after passing through the sonic surface. Since conditions (53) and (54) are always satisfied for \( \sigma = \bar{\sigma} \), then the expanding Friedmann-static shock waves for which \( \sigma = \bar{\sigma} \) are always stable in the Lax sense, as the following theorem summarises.
**Theorem 6.** Let \((A, G, v)\) denote a spherically symmetric self-similar solution to the perfect fluid Einstein field equations with equation of state \(p = \sigma \rho\). If there exists a \(\xi_0 > 0\) such that \((A, G, v)\) can be matched to TOV(\(\sigma\)) to form a shock-wave solution with a subluminal shock speed, then the Lax characteristic conditions are satisfied if:

1. \(\sigma = \bar{\sigma}\) or
2. \(\sigma < \bar{\sigma}\) and \(G(\xi_0) > \sqrt{\sigma}\) or
3. \(\sigma > \bar{\sigma}\) and \(\{\cdot\}_D(\xi_0) < 0\).

**Proof.** This is an immediate consequence of Lemma 4 and the discussion proceeding Definition 10.

---

### 5 Existence of Friedmann-Static Pure Radiation Shock Waves

#### 5.1 Monotonicity Lemma

We know from Proposition 4 that FLRW(0, \(\sigma, 1\)) solutions have a certain structure that allow us to determine if and where the solution trajectory crosses the singular or sonic surfaces. However, even though FLRW(0, \(\sigma, a\)) solutions can be expected to behave similar to FLRW(0, \(\sigma, 1\)) solutions for \(a \approx 1\), there is no guarantee that they remain similar as \(\xi\) increases or for larger values of \(a\). In particular, all FLRW(0, \(\sigma, 1\)) and TOV(\(\sigma\)) solutions are transonic, whereas this is not the case for general FLRW(0, \(\sigma, a\)) solutions, most of which are only subsonic. A Friedmann-static shock wave is always transonic, since either the interior FLRW(0, \(\sigma, a\)) or exterior TOV(\(\sigma\)) spacetime must pass through the sonic surface. We know from Figure 2 that the FLRW(0, \(\frac{1}{3}, 2.4\)) trajectory differs significantly from the FLRW(0, \(\frac{1}{3}, 1\)) trajectory as \(\xi\) increases, since it encounters a singularity in equation (7) by hitting the sonic surface. The following lemma helps to predict the behaviour of FLRW(0, \(\sigma, a\)) trajectories. Note from Figure 3 that \(\{\cdot\}_D < 0\) implies the trajectory remains to the left of the sonic surface and \(\{\cdot\}_S > 0\) implies that the trajectory remains below the singular surface. The monotonicity of \(A\) and \(G\) implies that the trajectory advances to the right whilst simultaneously approaching the \(A = 1 - 2M(\sigma)\) surface.

**Lemma 5.** Let \(0 < \sigma < 1, a > 0\) and \(\xi > 0\). Then so long as FLRW(0, \(\sigma, a\)) satisfies:

\[
\begin{align*}
A &> 1 - 2M(\sigma) \\
\{\cdot\}_D &< 0
\end{align*}
\]

it also satisfies:

\[
\begin{align*}
A' &< 0 \\
G' &> 0 \\
v &> 0 \\
\{\cdot\}_S &> 0
\end{align*}
\]

**Proof.** From Definition 5, for sufficiently small \(\xi > 0\) we have the following inequalities:

\[
\begin{align*}
1 - A &> 0 \\
A' &< 0 \\
G &> 0 \\
G' &> 0 \\
v &> 0 \\
\{\cdot\}_S &> 0
\end{align*}
\]

and so the FLRW(0, \(\sigma, a\)) trajectory begins by satisfying inequalities (55)-(58). It is thus sufficient to show that each one of the four inequalities is implied by the other three as \(\xi\) increases. In this light, assume \(v > 0\) and \(\{\cdot\}_S > 0\), then equation (5) and the small \(\xi\) inequality \(1 - A > 0\) implies inequality (55). For inequality (56), assume \(v > 0\) and \(\{\cdot\}_S > 0\) and note that \(\{\cdot\}_S > 0\) and \(\{\cdot\}_D < 0\) imply \(v < 1\). Given these constraints, equation (6) implies:

\[
\begin{align*}
\frac{\xi dG}{d\xi} &= -G \left[ \frac{1 - A}{A} \right] \frac{(3 + 3\sigma)(1 + v^2)G - 2v}{2\{\cdot\}_S} \\
&= G \left[ 1 - \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)(1 + v^2)G - (6 + 6\sigma)v}{(6 + 6\sigma v^2)G - (6 + 6\sigma)v} \right] \\
&> G \left[ 1 - \left( \frac{1 - A}{A} \right) \right] \\
&> 0
\end{align*}
\]
with the last line following from (45) and the small $\xi$ inequality $G' > 0$. Now it is sufficient to demonstrate inequality (57) in the interval $0 < G < \sqrt{\sigma}$, since the sonic surface intersects the $v = 0$ plane at $G = \sqrt{\sigma}$ and we are assuming that the trajectory stays off the sonic surface. In this light, assume $A' < 0, G' > 0$ and $\{\cdot\}_{S} > 0$ and note that $A' < 0$ implies $A < 1$ and $G' > 0$ implies $G > 0$. By equation (7), the sign of $v'$ on the plane $v = 0$ in the region bounded by $1 - 2M(\sigma) < A < 1$ and $0 < G < \sqrt{\sigma}$ is strictly positive, since:

$$\xi \frac{dv}{d\xi} = -\left(1 - \frac{v^2}{2(\{\cdot\}_{D})^2}\right)\left[3\sigma\{\cdot\}_{S} + \left(1 - \frac{A}{\sigma}\right)\frac{(3 + 3\sigma)^2\{\cdot\}_{N}}{4\{\cdot\}_{S}}\right]$$

$$= \left(\frac{2G}{3(3 + 3\sigma)(\sigma - G^2)}\right)\left[9\sigma - \frac{(3 + 3\sigma)^2}{4} \left(1 - \frac{A}{\sigma}\right)\right]$$

$$> 0$$

Thus any trajectory that begins above the $v = 0$ plane remains above the plane. The small $\xi$ inequality $v > 0$ along with this result then implies inequality (57). Note that such a result still holds when $1 - 2M(\bar{\sigma}) < A < 1$ for $0 < \bar{\sigma} < \sigma < 1$, since:

$$9\sigma - \frac{(3 + 3\sigma)^2}{4} \left(1 - \frac{A}{\sigma}\right) > 9\sigma - \frac{(3 + 3\sigma)^2}{4} \left(\frac{2M(\sigma)}{1 - 2M(\sigma)}\right)$$

$$= 9\sigma - \frac{9\bar{\sigma}(3 + 3\sigma)^2}{(3 + 3\sigma)^2}$$

$$= (3 + 3\sigma)^2 \left(\frac{9\sigma}{(3 + 3\sigma)^2} - \frac{9\bar{\sigma}}{(3 + 3\sigma)^2}\right)$$

$$\geq 0$$

Finally, inequality (58) is demonstrated in a similar manner to inequality (57) by showing that trajectories stay away from the surface $\{\cdot\}_{S} = mv$ for some $0 < m < \frac{3}{2}$. The upper bound for $m$ ensures FLRW$(0, \sigma, a)$ trajectories initially satisfy inequality (58). Now assume inequalities (55)-(57) hold and note that these inequalities additionally imply $A < 1$ and $G > 0$. Since $\{\cdot\}_{S} = mv$ is equivalent to:

$$G = \frac{(3 + 3\sigma + m)v}{3 + 3\sigma v^2}$$

then by equation (6) and (59), we have:

$$q_A(v; \sigma, m) = \xi \frac{dG}{d\xi} \bigg|_{\{\cdot\}_{S} = mv}$$

$$= \left(\frac{dG}{d\xi} - \frac{(3 + 3\sigma + m)(3 - 3\sigma v^2)}{(3 + 3\sigma v^2)^2}\xi \frac{dv}{d\xi}\right) \bigg|_{\{\cdot\}_{S} = mv}$$

$$= \frac{(3 + 3\sigma + m)v}{3 + 3\sigma v^2} \left[1 - \left(1 - \frac{A}{\sigma}\right)\frac{3\sigma + m(1 + v^2) - 2(3 + 3\sigma v^2)}{2m(3 + 3\sigma v^2)}\right]$$

$$\frac{(3 + 3\sigma + m)(3 - 3\sigma v^2)}{(3 + 3\sigma v^2)^2} \left(\frac{1 - v^2}{2(\{\cdot\}_{D})^2}\right)\left[3\sigma mv + \left(1 - \frac{A}{\sigma}\right)\frac{(3 + 3\sigma)^2\{\cdot\}_{N}}{4mv}\right]$$

$$= \frac{(3 + 3\sigma + m)v}{3 + 3\sigma v^2} \left[3 + 3\sigma v^2 + \frac{3\sigma m(1 - v^2)(3 - 3\sigma v^2)}{2(\{\cdot\}_{D})^2} + \left(1 - \frac{A}{\sigma}\right)\{\cdot\}_{N}\right]$$

$$\left\{\cdot\}_{A} + \{\cdot\}_{B} + \{\cdot\}_{C}\right]$$

where:

$$\{\cdot\}_{A} = \frac{(3 + 3\sigma)(3 - 3\sigma - m)(1 - v^2)}{2m}$$

$$\{\cdot\}_{B} = \frac{(3 + 3\sigma)^2(1 - v^2)(3 - 3\sigma v^2)\{\cdot\}_{N}}{8mv^2(\{\cdot\}_{D})^2}$$

$$\{\cdot\}_{C} = -(3 + 3\sigma)v^2$$

The objective for this part is to find an $m$ such that $q_A(v; \sigma, m) > 0$ for all $0 < \sigma < 1$ and $0 < v < v_a$ for arbitrary $v_a < 1$. Note that it is always possible to choose an $m$ small enough to ensure $v_a < v_I(\sigma, m)$, where $v_I(\sigma, m)$ is the intersection of surfaces (59) and $\{\cdot\}_{D} = 0$, since:

$$\lim_{m\to 0} v_I(\sigma, m) = 1$$
Now even though it can be shown that \( \{ \cdot \}_A + \{ \cdot \}_B + \{ \cdot \}_C > 0 \) for a certain interval of \( v \), it is easier to show \( \{ \cdot \}_A + \{ \cdot \}_B > 0 \) for the whole interval \( 0 < v < v_I \). This the case since:

\[
\{ \cdot \}_A + \{ \cdot \}_B = \frac{(3 + 3\sigma)(1 - v^2)}{2m} \left[ 3 - 3\sigma - m + \frac{(3 + 3\sigma)(3 - 3\sigma v^2)}{4v^2\{ \cdot \}_D} \right]
\]

\[
= \frac{8m(\{ \cdot \}_D)^2}{3(3 + 3\sigma v^2)^2} \left[ 4(3 - 3\sigma - m)(\{ \cdot \}_B)v^2 - (3 + 3\sigma)(3 - 3\sigma v^2)\{ \cdot \}_N \right]
\]

\[
= \frac{3\sigma(3 + 3\sigma)^2(1 - v^2)(3 - 3\sigma v^2 + n)}{8(-\{ \cdot \}_D)(3 + 3\sigma v^2)^2} \left[ 3 - 3v^2 + \sigma(9 + n)v^2 - \sigma(9 + n\sigma)v^4 \right] > 0
\]

where \( m = n\sigma \) for some \( 0 < n < \frac{3}{2} \). With \( \{ \cdot \}_A + \{ \cdot \}_B > 0 \) and \( \{ \cdot \}_C < 0 \), then for \( \frac{1}{2} < A < 1 \) we have:

\[
\left( \frac{1 - A}{A} \right) \{ \cdot \}_A + \{ \cdot \}_B + \{ \cdot \}_C > \left( \frac{1 - A}{A} \right) \{ \cdot \}_C > \{ \cdot \}_C
\]

Thus for any \( 0 < \sigma < 1 \) and \( 0 < v < v_* \):

\[
\lim_{m \to 0} q_A(v; \sigma, m) = \lim_{n \to 0} q_A(v; \sigma, n\sigma)
\]

\[
> \lim_{n \to 0} \frac{(3 + 3\sigma + n\sigma)(1 - v^2)}{(3 + 3\sigma v^2)^2} \left[ 3 + 3\sigma v^2 + \frac{3n\sigma^2(1 - v^2)(3 - 3\sigma v^2)}{2\{ \cdot \}_D} \right] > 0
\]

Therefore, for any interval \( 0 < v < v_* \), with \( v_* < 1 \), there exists an \( 0 < n < \frac{3}{2} \) such that the surface \( \{ \cdot \}_S = n\sigma v \) cannot be crossed. Now assume for contradiction that a trajectory crosses the \( \{ \cdot \}_S = 0 \) surface. Because \( v_1(\sigma, 0) = 1 \) and we assume \( \{ \cdot \}_D < 0 \), the trajectory cannot cross the surface \( \{ \cdot \}_S = 0 \) at \( v = 1 \), so it must intersect at some point \( 0 < v_* < 1 \). Given that FLRW\((0, \sigma, a)\) satisfies \( \{ \cdot \}_S > n\sigma v \) initially for any \( 0 < n < \frac{3}{2} \) and we can pick a \( v_* \) such that \( v_* < v_* < 1 \), we know that the surface \( \{ \cdot \}_S = n\sigma v \) cannot be crossed in the interval \( 0 < v < v_* \), which is a contradiction. Thus under our assumptions, FLRW\((0, \sigma, a)\) satisfies inequality \( (58) \) and completes the proof.

5.2 Existence Theorem

In Section[3] the Friedmann-static pure radiation shock wave is constructed numerically. Now with Lemma[5]in place, we are in a position to provide a rigorous proof of this construction.

**Theorem 7.** There exists an \( a > 1 \) such that FLRW\((0, \frac{1}{3}, a)\) can be matched to TOV\((\frac{1}{3})\) to form a pure radiation general relativistic shock wave that satisfies the Lax characteristic conditions.

5.2.1 Outline of Proof

From Lemma[5]we know that any FLRW\((0, \sigma, a)\) trajectory inevitably hits either the sonic surface or the matching surface \( A = 1 - 2M(\sigma) \). The first part of the proof argues that it is sufficient to find a \( b > 1 \) such that the FLRW\((0, \frac{1}{3}, b)\) trajectory hits the matching surface and overshoots the Rankine-Hugoniot curve. This is because we know the explicitly given FLRW\((0, \frac{1}{3}, 1)\) trajectory undershoots the Rankine-Hugoniot curve, so the continuity of the parameter \( a \) guarantees an intersection for some \( 1 < a < b \).

The next part of the proof also follows from Lemma[5]in the sense that the monotonicity of \( G \) means that \( A \) and \( v \) can be considered as functions of \( G \) rather than functions of \( \xi \). The remainder of the proof consists of the computationally difficult task of building a trapping region around the FLRW\((0, \frac{1}{3}, b)\) trajectory to guarantee that it overcuts the Rankine-Hugoniot curve.

The trapping region is constructed of perturbations of high-order Taylor polynomials of \( A \) and \( v \). These polynomials are functions of \( G \) and bound \( A \) and \( v \) from above and below all the way to the matching surface. There are multiple factors involved in constructing such a region, such as the magnitude of perturbation of the Taylor polynomials, the choice of \( b \) and the minimum order of the Taylor polynomials. Each factor effects the others so a careful balance must be found between them.

Due to the complexity of equations \( (5) \)-(\(7)\), the high-order Taylor polynomials have coefficients that are too large to be written in any useful way, as doing so would double the length of this paper. Instead, a procedure is given for generating
these explicit polynomials using symbolic manipulation software, from which they can be analysed and graphed as necessary.

In parallel to choosing the perturbations of the polynomials, the value of \( b \) and the minimum order of the polynomials, the perturbed Taylor region must also satisfy a set of inequalities that demonstrate vectors on the boundary of the region point into the region, thus trapping any trajectory that begins inside. This process is completed through trial and error and is made difficult by virtue of the fact that the matching for \( \sigma = \tilde{\sigma} = \frac{1}{3} \) occurs near the sonic surface, where the Taylor polynomials attempt to approximate a potentially singular function.

The final step involves showing that the choice of perturbed Taylor region is in fact a trapping region that contains the \( \text{FLRW}(0, \frac{1}{3}, b) \) trajectory from \( G = 0 \) to the matching surface. The Lax characteristic conditions then follow by Theorem 6.

5.2.2 The Shooting Argument

By Lemma 2 and Definition 9 for \( \text{FLRW}(0, \frac{1}{3}, a) \) to match with TOV(\( \frac{1}{3} \)) to form a general relativistic shock wave, then \( \text{FLRW}(0, \frac{1}{3}, a) \) must satisfy:

\[
A(\xi_0) = \frac{4}{7} \tag{60}
\]

\[
v(\xi_0) = \Gamma_{RH} \left( G(\xi_0); \frac{1}{3}, \frac{1}{3} \right) \tag{61}
\]

for some positive constant \( \xi_0 \). We know from Theorem 5 that \( \text{FLRW}(0, \frac{1}{3}, 1) \) cannot form a general relativistic shock wave with TOV(\( \frac{1}{3} \)), since \( \sigma = \tilde{\sigma} = \frac{1}{3} \) is not a solution of \( \tilde{\sigma} = H(\sigma) \). Instead, when the \( \text{FLRW}(0, \frac{1}{3}, 1) \) trajectory hits the \( A = \frac{4}{7} \) plane, then:

\[
v(\xi_0) < \Gamma_{RH} \left( G(\xi_0); \frac{1}{3}, \frac{1}{3} \right)
\]

That is, the \( \text{FLRW}(0, \frac{1}{3}, 1) \) trajectory passes under the Rankine-Hugoniot curve. Note that the explicitly known \( \text{FLRW}(0, \frac{1}{3}, 1) \) solution is able to cross the sonic surface without becoming singular due to the cancellation of \( \{ \cdot \}_{D} \) in equation (7) when on the sonic surface. General \( \text{FLRW}(0, \sigma, a) \) solutions typically become singular at the sonic point, that is, the point of intersection with the sonic surface. Now suppose that there exists a \( b > 1 \) such that the \( \text{FLRW}(0, \frac{1}{3}, b) \) trajectory hits the plane \( A = \frac{4}{7} \) with:

\[
v(\xi_0) > \Gamma_{RH} \left( G(\xi_0); \frac{1}{3}, \frac{1}{3} \right) \tag{62}
\]

then providing the transition of the \( \text{FLRW}(0, \frac{1}{3}, 1) \) trajectory to the \( \text{FLRW}(0, \frac{1}{3}, b) \) trajectory crosses the Rankine-Hugoniot curve, there exists an \( 1 < a < b \) such that (60) and (61) are satisfied. An example of this process is demonstrated numerically in Figure 2 Lemma 9 establishes the fact that if the \( \text{FLRW}(0, \frac{1}{3}, a) \) trajectory remains in the subsonic region, then it must eventually hit the \( A = \frac{4}{7} \) plane. The continuous dependence of \( \text{FLRW}(0, \frac{1}{3}, a) \) on the parameter \( a \) means that there is a continuous transition from \( \text{FLRW}(0, \frac{1}{3}, 1) \) to \( \text{FLRW}(0, \frac{1}{3}, b) \), at least up until the trajectory hits the \( A = \frac{4}{7} \) plane or hits the sonic surface. This continuous transition, along with Lemma 9, guarantees the crossing of the Rankine-Hugoniot curve in the \( \sigma = \tilde{\sigma} = \frac{1}{3} \) case, since the transition from hitting the sonic surface to hitting the \( A = \frac{4}{7} \) plane occurs on the intersection of the sonic surface with the \( A = \frac{4}{7} \) plane, which lies under the Rankine-Hugoniot curve. Thus it is sufficient to rigorously demonstrate the existence of an \( \text{FLRW}(0, \frac{1}{3}, b) \) solution that satisfies (60) and (62). We know from Figure 2 that a numerical approximation of the \( \text{FLRW}(0, \frac{1}{3}, \frac{14}{5}) \) trajectory passes above the Rankine-Hugoniot curve, so existence is considered for \( b = \frac{14}{5} \).

5.2.3 Monotonicity Simplification

Because the \( \text{FLRW}(0, \frac{1}{3}, a) \) trajectories originate from the fixed point of an unstable manifold, the vector field generated by the system of equations (5)-(7) points toward the \( \text{FLRW}(0, \frac{1}{3}, a) \) trajectories when moving away from the fixed point. This fact allows for the construction of a trapping region around the trajectory and this is how the \( \text{FLRW}(0, \frac{1}{3}, \frac{14}{5}) \) trajectory is shown to overshoot the Rankine-Hugoniot curve. Since Lemma 9 establishes the monotonicity of \( G \) as a
function of \( \xi, A \) and \( v \) can be considered as functions of \( G \), with equations \( (5)-(7) \) becoming:

\[
\frac{dA}{dG} = -\left( \xi \frac{dG}{d\xi} \right)^{-1} \frac{(3 + 3\sigma)(1 - A)v}{1} \{s\} S \\
\xi \frac{dG}{d\xi} = -G \left[ \frac{(1 - A)}{A} \right] \left( \frac{3 + 3\sigma}{2} \right) (1 + v^2) G - 2v - 1 \\
\frac{dv}{dG} = -\left( \xi \frac{dG}{d\xi} \right)^{-1} \left( \frac{1 - v^2}{2} \{d\} \right) \left[ 3\sigma \{s\} S + \frac{(1 - A)}{A} \frac{(3 + 3\sigma)^2 \{n\}}{4} \right] 
\]

(63)

In this sense, the trajectory of FLRW(0, \( \frac{1}{3}, \frac{14}{5} \)) can be represented as \((A(G), v(G))\), with \( G \) parameterising the progress of the trajectory towards the \( A = \frac{4}{7} \) plane. This step provides a considerable simplification, since the trapping region now only needs to contain \( A \) and \( v \).

### 5.2.4 Perturbations of the Taylor Polynomials

The next step is to construct a trapping region using the Taylor polynomials of \( A \) and \( v \) about \( G = 0 \). In this light, define:

\[
P_{2N+1}(G) = \sum_{n=0}^{N} \frac{A^{(2n)}(0)}{(2n)!} G^{2n} \\
Q_{2N+1}(G) = \sum_{n=0}^{N} \frac{v^{(2n+1)}(0)}{(2n+1)!} G^{2n+1}
\]

noting that \( A \) and \( v \) have even and odd Taylor polynomials respectively. Furthermore, define:

\[
A_M(G) = P_{2N-1}(G) + M_A G^{2N} \\
A_m(G) = P_{2N-1}(G) + m_A G^{2N} \\
v_M(G) = Q_{2N-1}(G) + M_v G^{2N+1} \\
v_m(G) = Q_{2N-1}(G) + m_v G^{2N+1}
\]

where \( M_A, m_A, M_v \) and \( m_v \) are chosen so that:

\[
m_A < \frac{A^{(2N)}(0)}{(2N)!} < M_A \\
m_v < \frac{v^{(2N+1)}(0)}{(2N+1)!} < M_v
\]

The functions \( A_M \) and \( A_m \) are used to bound \( A \) from above and below respectively, with \( v_M \) and \( v_m \) providing analogous bounds for \( v \). The objective is to show:

\[
v_m(G_0) > \Gamma_{RH} \left( G_0; \frac{1}{3}, \frac{1}{3} \right)
\]

(65)

where \( G_0 \) is found implicitly through:

\[
A_M(G_0) = \frac{4}{7}
\]

(66)

This is so the lowest point of the Taylor trapping region of \( v \) remains above the Rankine-Hugoniot curve for the most conservative value of \( G \), which is given by the intersection of the highest point of the Taylor trapping region of \( A \) with the \( A = \frac{4}{7} \) plane.

### 5.2.5 Generation of the Taylor Polynomials

To generate the Taylor polynomials of \( A(G) \) and \( v(G) \) from equations \( (63) \) and \( (64) \), we begin with Definition\( 5 \) which gives FLRW(0, \( \sigma, a \)) to leading order in \( \xi \) as:

\[
A(\xi) \approx 1 - \frac{1}{4} a^2 \xi^2 \\
G(\xi) \approx \frac{1}{4} (3 + 3\sigma) \xi \\
v(\xi) \approx \frac{1}{2} \xi
\]

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Thus to leading order in $G$, we have:

$$A(G) \approx 1 - \frac{4a^2}{(3 + 3\sigma)^2}G^2$$

$$v(G) \approx \frac{2}{3 + 3\sigma}G$$

Now to obtain the third-order polynomials, we first substitute:

$$A(G) = 1 - \frac{4a^2}{(3 + 3\sigma)^2}G^2$$

$$v(G) = \frac{2}{3 + 3\sigma}G + \frac{v^{(3)}(0)}{3!}G^3$$

into equations (63) and (64) and then solve for $v^{(3)}(0)$ to eliminate the leading order term. This gives:

$$v^{(3)}(0) = \frac{3! \cdot 2(1 - a^2 + 14\sigma - 4a^2\sigma + 33\sigma^2 - 3a^2\sigma^2)}{5\sigma(3 + 3\sigma)^3}$$

The same procedure can be used to determine $A^{(4)}(0)$, which is given by:

$$A^{(4)}(0) = -\frac{4! \cdot 4a^2(3 - 3a^2 + 42\sigma - 22a^2\sigma + 39\sigma^2 + 21a^2\sigma^2)}{5\sigma(3 + 3\sigma)^4}$$

Now this process can be continued to determine $v^{(5)}(0)$, then $A^{(6)}(0)$ and so on. However, it is recommended to use symbolic manipulation software, since for $v^{(5)}(0)$ we see that it is already difficult to fit this term on one line:

$$v^{(5)}(0) = \frac{-5\sigma^2(3 + 3\sigma)^5}{2 \cdot 5!}v^{(5)}(0) = 1 - 40\sigma - 642a^2 - 2568a^3 - 3087\sigma^4 - a^4 + 48a^2\sigma - 8a^4\sigma + 348a^2\sigma^2$$

$$+ 768a^2\sigma^3 + 468a^2\sigma^4 + 14a^4\sigma^2 + 120a^4\sigma^3 + 99a^4\sigma^4$$

By the time we reach $v^{(33)}(0)$, we require over 29 000 characters to write this single coefficient down. This is clearly next to impossible to do by hand but straightforward to do using symbolic manipulation software.

### 5.2.6 Determining the Perturbed Taylor Region

For large enough $N$, it is possible to find values for $M_A, m_A, M_v$ and $m_v$ such that (65) and (66) are satisfied and inequalities:

$$A_m(G) < A < A_M(G)$$

$$v_m(G) < v < v_M(G)$$

hold for $0 < G < G_0$. Although, as will be seen in the last step, the values are actually selected to satisfy (65), (66) and (75) instead, which then imply (65)–(68). This is done through extensive trial and error, using a numerical approximation of $A$ and $v$ as a guide. Note that the Taylor polynomials of $A$ and $v$ converge quicker for smaller values of $b$, but larger values of $b$ allow for (65) to be more easily satisfied, this is why $b = \frac{14}{5}$ is chosen, as it provides a good compromise. In this light, and using a numerical approximation of $A$ and $v$ as a guide, it is found that $N = 16$ and the following values satisfy (65), (66) and (75):

$$M_A = (1 + 2^{-7})\frac{A^{(32)}(0)}{(32)!}$$

$$m_A = 2^{-1}\frac{A^{(32)}(0)}{(32)!}$$

$$M_v = 2^{-1}\frac{v^{(33)}(0)}{(33)!}$$

$$m_v = 2^{5}\frac{v^{(33)}(0)}{(33)!}$$

noting that $M_v$ and $m_v$ are chosen in the knowledge that $v^{(33)}(0)$ is negative.
Figure 4: This figure depicts $A_M(G)$ and $A_m(G)$ by the top and bottom dotted curves respectively. Note that these curves are almost indistinguishable until they cross the $A = \frac{3}{4}$ plane, which is given by the unbroken line at the bottom.

Figure 5: This figure depicts $v_M(G)$ and $v_m(G)$ by the top and bottom dotted curves respectively. The Rankine-Hugoniot curve is given by the dashed curve and the singular and sonic surfaces are given as unbroken curves.

With $M_A$, $m_A$, $M_v$, and $m_v$ specified, the Taylor polynomials of $A$ and $v$ can be computed and $A_M$, $A_m$, $v_M$ and $v_m$ become known explicitly. The graphs of these bounding functions are given in Figure 4 and Figure 5.

Even at 33rd order, Figure 5 shows that $v_M$ and $v_m$ noticeably diverge after passing the Rankine-Hugoniot curve. This is due to the trajectory approaching the sonic surface, where the solution is likely to become singular, resulting in a slower convergence of the Taylor polynomials. With $A_M$ known explicitly, relation (66) can be solved for $G_0$, at least approximately, to yield:

$$G_0 \approx 0.601$$

and this results in inequality (65) being satisfied, since $v_m$ is also known explicitly.
5.2.7 Showing the Perturbed Taylor Region is a Trapping Region

The final step is to show that inequalities (67) and (68) hold in the interval $0 < G < G_0$. To do this, the structure of equations (63) and (64) can be exploited, that is, it is possible to show:

$$\frac{\partial}{\partial v} \frac{dA}{dG} < 0$$  \hspace{1cm} (69)

$$\frac{\partial}{\partial A} \frac{dv}{dG} > 0$$  \hspace{1cm} (70)

within the region given by (67) and (68). Starting with (69), we have:

$$\frac{\partial}{\partial v} \frac{dA}{dG} = \frac{4(1 - A) v}{\xi dG} \frac{dG}{d\xi} \left( \xi dG \right)^{-1} \frac{dG}{d\xi}^{-1} \frac{\partial}{\partial v} \left( \xi dG \right)^{-1} \frac{\partial}{\partial v} (1 - A) v$$

$$= -\frac{4(1 - A) G^2}{\xi dG} \left( \xi dG \right)^{-2} \left[ 4v^2(3 - v^2) \left( \frac{1}{A} \right) + (3 - v^2)^2 \right]$$

$$< 0$$

which holds in the more general region described by $\frac{2}{5} < A < 1$, $v > 0$, $\{\cdot\}_S > 0$ and $\{\cdot\}_D < 0$. For (70) we have:

$$\frac{\partial}{\partial A} \frac{dv}{dG} = -\frac{(1 - v^2)}{2\{\cdot\}_D} \left[ \{\cdot\}_S + 4 \left( \frac{1 - A}{A} \right) \right] \frac{dA}{dG} \left( \xi dG \right)^{-1}$$

$$- \left( \frac{dG}{d\xi} \right)^{-1} \frac{(1 - v^2)}{2\{\cdot\}_D} \frac{dA}{d\xi} \left[ \{\cdot\}_S + 4 \left( \frac{1 - A}{A} \right) \right]$$

$$= \frac{G}{A^2} \left( \xi dG \right)^{-2} \left[ (2(1 + v^2)G - 4v) \{\cdot\}_S + 4 \{\cdot\}_N \right]$$

$$> 0$$

which holds in the region described by $v > 0$, $\{\cdot\}_D < 0$ and:

$$(2(1 + v^2)G - 4v) \{\cdot\}_S + 4 \{\cdot\}_N < 0$$

This region is slightly smaller than the region described by $v > 0$, $\{\cdot\}_D < 0$ and $\{\cdot\}_S > 0$, but includes the region given by (67) and (68) nonetheless. Now by construction, we know that (67) and (68) are satisfied in the interval $0 < G < G_c$ for some small $G_c > 0$, so to demonstrate (67) and (68) in the interval $0 < G < G_0$, it is sufficient to demonstrate:

$$\frac{d}{dG} (A_M - A)|_{A = A_M} \geq 0$$  \hspace{1cm} (71)

$$\frac{d}{dG} (A - A_m)|_{A = A_m} \geq 0$$  \hspace{1cm} (72)

$$\frac{d}{dG} (v_M - v)|_{v = v_M} \geq 0$$  \hspace{1cm} (73)

$$\frac{d}{dG} (v - v_m)|_{v = v_m} \geq 0$$  \hspace{1cm} (74)

in the interval $G_c \leq G < G_0$. Note that the left hand sides of (71)-(74) are functions of $A$, $v$ and $G$, so (69) can be used to determine the most conservative value of $v$ in (71) and (72), and (70) can be used to determine the most conservative value of $A$ in (73) and (74). In particular, the most conservative choice out of $v_M$ and $v_m$ for (71) is $v_M$, and the most conservative choice for (72) is $v_M$. Likewise, the most conservative choice out of $A_M$ and $A_m$ for (73) is $A_M$, and the most conservative choice for (74) is $A_m$. This can be interpreted as remaining within the right wall of the trapping region implies remaining below the ceiling, and remaining below the ceiling implies remaining within the left wall and so on. Such an interpretation can be summarised as so:

$$A < A_M \Rightarrow v < v_M$$

$$\uparrow \qquad \downarrow$$

$$v > v_m \Leftarrow A > A_m$$
Figure 6: This figure depicts $\frac{d}{dG}(A_M - A)|_{A=A_M}$ and $\frac{d}{dG}(A - A_m)|_{A=A_m}$ as unbroken and dashed curves respectively.

Figure 7: This figure depicts $\frac{d}{dG}(v_M - v)|_{v=v_M}$ and $\frac{d}{dG}(v - v_m)|_{v=v_m}$ as unbroken and dashed curves respectively.

Now using these conservative choices, the left hand sides of (71)-(74) become explicitly known functions of $G$ and thus the interval for which they remain positive can be calculated, at least approximately. From Figure 6 and Figure 7, the intervals for which (71)-(74) hold are given by:

$0 < G < G_1$
$0 < G < G_2$
$0 < G < G_3$
$0 < G < G_4$

respectively, where:

$G_1 > G_I$
$G_2 \approx 0.627$
$G_3 > G_I$
$G_4 \approx 0.612$

and $G_I$ is the value of $G$ for which $v_m$ intersects the sonic surface. Since (65) has already been established, then $G_0 < G_I$ and thus:

$G_0 < \min\{G_1, G_2, G_3, G_4\}$

(75)

Therefore (67) and (68) hold in the interval $0 < G < G_0$ and since the Lax characteristic conditions follow by Theorem 6, the proof is complete.
6 Concluding Remarks

Throughout the preceding sections, the necessary machinery has been introduced, and in some places developed, to enable the construction of a new family of exact general relativistic shock waves that exhibit a cosmic acceleration. This construction resolves the open problem of determining the expanding waves created behind a shock-wave explosion into a static isothermal sphere with an inverse square density profile, and in doing so, partially resolves a long-standing problem posed by Cahill and Taub. We saw in Section 2 that this family of shock waves is one derivative less regular in Schwarzschild coordinates than it actually is and that any delta function sources are cancelled within the Einstein tensor. Section 3 built the machinery required for a phase-space analysis and determined the cosmic acceleration inherent in asymptotically Friedmann pure radiation spacetimes. Section 4 and Section 5 brought all this machinery together and established the existence and Lax stability of this new family of general relativistic shock waves, and in doing so, provided a mechanism for exhibiting an accelerated expansion whilst removing the central singularity from the static isothermal sphere.

Given that it is possible to rigorously demonstrate the existence of a Friedmann-static pure radiation shock wave, the obvious follow-up question is whether it is possible to rigorously demonstrate the existence of the full two-parameter family of Friedmann-static shock waves, with σ included in this parameter count. For a certain range of values of σ and σ there is every reason to expect this to be possible.

**Conjecture 1.** For \(0 < \sigma \leq \frac{1}{3}\), there exists an \(a > 0\) such that \(FLRW(0, \sigma, a)\) can be matched to \(TOV(\sigma)\) to form a general relativistic shock wave.

The resolution of this conjecture is one avenue of future research. The continuous dependence of the solution trajectories on the parameters means that existence is all but guaranteed for \(\sigma, \sigma \approx \frac{1}{3}\) and \(\sigma \approx H(\sigma)\). Moreover, the existence proof in the pure radiation case is readily modified to demonstrate existence for any fixed pair \(0 < \sigma \leq \sigma \leq \frac{1}{3}\). The difficulty arises when generalising the proof from fixed parameter values to two-dimensional parameter spaces, since conservative estimates need to be satisfied for all values of \(\sigma\) and \(\sigma\) in such spaces. It is likely to be possible to construct such a proof by patching together many subproofs demonstrating existence in small two-dimensional parameter spaces, although this method may be rather tedious.

Another avenue of future research is in regard to the possible cosmological applications of Friedmann-static shock waves. It is shown in Section 5 that the Friedmann-static pure radiation shock wave yields an acceleration parameter value of \(a \approx 2.58\). For reference, the acceleration parameter that would be expected in the Radiation Dominated Epoch would likely satisfy \(a \approx 1\). However, the discussions of [9] and [10] imply that Friedmann-static shock waves have shock fronts that lie within the Hubble radius, so these shock waves are not viable cosmological models in the Radiation Dominated Epoch. As mentioned in the introduction, Smoller and Temple demonstrate in [9] that it is possible to construct a general relativistic shock wave, with a shock surface beyond the Hubble radius, by modelling the entire Universe as a finite mass explosion within the Schwarzschild radius of a time-reversed black hole. However, this is only completed for an explicitly known flat FLRW spacetime on the interior and not for a pure radiation equation of state each side of the shock surface.

The possibility remains to construct a general relativistic pure radiation shock wave with a shock surface beyond the Hubble radius and determine the resulting rate of expansion. If after transition into the Matter Dominated Epoch the predicted rate of expansion lies within current estimates, then this mechanism offers a mathematically independent derivation for the cosmic acceleration observed today without a cosmological constant, and thus, without dark energy.

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References

[1] Michael Cahill and Abraham Taub. Spherically Symmetric Similarity Solutions of the Einstein Field Equations for a Perfect Fluid. *Communications in Mathematical Physics*, 21(1):1–40, 1971.

[2] Bernard Carr and Alan Coley. Complete Classification of Spherically Symmetric Self-Similar Perfect Fluid Solutions. *Physical Review D*, 62(4):044023, 2000.
[3] Bernard Carr and Amos Yahil. Self-Similar Perturbations of a Friedmann Universe. *The Astrophysical Journal*, 360:330–342, 1990.

[4] Peter Lax. Hyperbolic Systems of Conservation Laws II. *Communications on Pure and Applied Mathematics*, 10(4):537–566, 1957.

[5] Joel Smoller. *Shock Waves and Reaction-Diffusion Equations, Second Edition*. Springer, 1994.

[6] Joel Smoller and Blake Temple. Shock-Wave Solutions of the Einstein Equations: The Oppenheimer-Snyder Model of Gravitational Collapse Extended to the Case of Non-Zero Pressure. *Archive for Rational Mechanics and Analysis*, 128(3):249–297, 1994.

[7] Joel Smoller and Blake Temple. Astrophysical Shock-Wave Solutions of the Einstein Equations. *Physical Review D*, 51(6):2733–2743, 1995.

[8] Joel Smoller and Blake Temple. Shock-Wave Cosmology Inside a Black Hole. *Proceedings of the National Academy of Sciences*, 100(20):11216–11218, 2003.

[9] Joel Smoller and Blake Temple. Cosmology, Black Holes and Shock Waves Beyond the Hubble Length. *Methods and Applications of Analysis*, 11(1):77–132, 2004.

[10] Joel Smoller and Blake Temple. General Relativistic Self-Similar Waves that Induce an Anomalous Acceleration into the Standard Model of Cosmology. *Memoirs of the American Mathematical Society*, 218(1025), 2012.

[11] Joel Smoller, Blake Temple, and Zeke Vogler. An Instability of the Standard Model of Cosmology Creates the Anomalous Acceleration Without Dark Energy. *Proceedings of the Royal Society A*, 473(2207):20160887, 2017.