The effect of the quantization of the centrifugal stretching on the analysis of rotational spectra of even-even deformed nuclei in rare earth and actinide regions

Abdurahim A. Okhunov
Namangan Institute of Engineering and Technology, 160115 Namangan, Uzbekistan and
Department of Science in Engineering, KOE, International Islamic
University Malaysia, P.O Box 10, 50728 Kuala Lumpur, Malaysia

Mohd Kh. M. Abu El Sheikh
Department of Physics, University of Malaya, 50603 Kuala Lumpur, Malaysia
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Deviation from the \( I (I + 1) \) rule in the even–even isotopes \( \text{Sm, Gd, Dy, Er, Yb, Hf} \), \( \text{W and Os} \) nuclei have been studied with a new approach based on the idea that the rotational nucleus being stretched out along the symmetry axis, where such stretching is treated according to quantum mechanics. This approach led to a new formula that describes the dependence of the moment of inertia on the angular momentum, and such formula works well for all even-even nuclei in the actinide and rare-earth region in the range \( 2.9 < R = \frac{A}{2I} < 3.33 \). The calculations were carried out with the stretched model where the stretching of the nucleus with the rotation is quantized. The obtained results with a new derived formula which gives a reasonable agreement with the experimental data for all \( I \) up to 16.

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I. INTRODUCTION

Bohr in his model of "Coupling between the motion of the single particle to the nuclear surface oscillation" predicted that, in the special case of the strong-coupling where the nucleus is well deformed, the nuclear spectrum is rotational with spin sequence \( I = 0, 2, 4, 6, \ldots \), even parity and with energy levels follows the simple formula \( E(I) = \frac{\hbar^2}{2I} (I + 1), \) where \( I \) is the moment of inertia which is expected to be constant for such nuclei. According to such model, nuclei having these characteristics are expected to be in the region where a large number of particles are outside the closed shell. However, it was shown later that the energy spacing experimentally measured of the states of the ground band increases less rapidly than that is predicted by the formula Eq. (1) when \( I \) is constant. Originally, Bohr and Mottelson \( ^{1} \) suggested that this decrease in the energy spacing may be understood within the context of the coupling between rotational and vibrational modes of motion which contributes a term of the form \(-B[I(I+1)]^2\), and Eq. (1) becomes

\[
E(I) = AI(I + 1) - B[I(I + 1)]^2
\]  

where \( A, B, C, D, \ldots \) are parameters, which is can be determined by fitting this equation with experimental results. The problem here is that, for a feasible comparison, we require as many parameters as there are experimental data to be fitted. Gupta \( ^{10, 11, 12} \) introduces a model with a rotational-vibrational interaction that relates all parameters \( C, D, E, \ldots \) to the first two basic parameters \( A, B \). This model is known as non-rigid rotator model and the energy states are expressed as a power series in terms of \( B/A \) and the weight factor \( I(I+1) \) as:

\[
E(I) = AI(I + 1)\left[1 - \frac{A}{B}(I + 1)\right] + 3\left(\frac{A}{B}(I + 1)\right)^2 - ...
\]  

The two parameters \( A \) and \( B \) in this equation are related to the moment of inertia and the rotation-vibration coefficient respectively. However, such correction was found to be insufficient to represent the experimental spectra obtained with high spin \( I \)-values or even for low lying states of transitional nuclei \( ^{8} \). One can generalize Eq. (2) to a power series form \( ^{9} \)

\[
E(I) = AI(I + 1) - B[I(I + 1)]^2 + C[I(I + 1)]^3 - D[I(I + 1)]^4 + ...
\]  

where \( A, B, C, D, \ldots \) are parameters. The obtained results with a new derived formula which gives a reasonable agreement with the experimental date for all \( I \) up to 16.

\[\text{Electronic address: } aaokhunov@gmail.com \]

\[\text{Electronic address: } mhsr70@gmail.com \]

\[\text{Corresponding author: } mhsr70@gmail.com \]
is not sufficient to describe the experimental spectrum. Instead he suggested that the phenomenological sum an infinite series in \( I(I+1) \) to gate the following compact expression for the energy \( E(I) \)

\[
E(I) = A(I)I(I+1),
\]

where \( A(I) = A \left[ 1 - \frac{\hbar^2\beta}{2\pi I} \right] \), which could be consistent with that derived by Bohr for describing the experimental spectrum with the provision that \( A(I) \) in Eq. (5) is variable and it is a decreasing function of \( I \). Although this two-parameter model by Sood \([9]\) fitted with high accuracy the experimental spectrum of rare-earth nuclei including the \( Os \) isotopes and \( N = 90 \) nuclei, there are many questions about the parameter \( N \) has been left open such as; what is the physical quantity that the parameter \( N \) represents? Or what are the factors that \( N \) depends on? The answer to these questions is very important to specify the factors which control the increase of \( \mathfrak{g} \) with the increase in \( I \).

An alternative approach to interpret such decreasing in the energy space was introduced first by Morinaga \([13]\) and later by Diamond, Stephen and Swiatecki \([16]\). Morinaga \([15]\) was the first who suggested that the decrease in energy spacing is due to the increase in the moment of inertia \( \mathfrak{I} \). Diamond et al. \([16]\), attribute this increase in \( \mathfrak{I} \) to some sort of centrifugal stretching. This model was known as the beta-stretching model. Let us assume the suggestion of Bohr that the total energy \( E_I \) of the state \( I \) is defined as

\[
E(I) = \frac{\hbar^2}{2\mathfrak{I}}I(I+1) + \frac{1}{2}C(\beta_I - \beta_0)^2,
\]

where \( C \) is the stiffness of the nucleus. The authors in reference \([16]\) assume the hydrodynamical formula \( 3\mathfrak{I}\beta^2 \) for the moment of inertia. Where \( \beta_I \) in Eq. (6) is the value of \( \beta \) which satisfy the condition

\[
\frac{\partial E(I)}{\partial \beta} = 0,
\]

so that the total energy is minimized. This approach gives good fit only for strongly deformed nuclei. Bands outside this region cannot be fitted with reasonable accuracy.

M.A.J. Mariscotti et al. \([17]\) thought that the moment of inertia \( \mathfrak{I} \) depends, not only on the deformation parameter \( \beta \) but also on Coriolis pairing effect, therefore, they suggested \( \beta \) in Eq. (7) should be replaced by a general variable \( t \) which might represents not only the deformation parameter, but also all other microscopic features like the effective pairing. The dependence of \( \mathfrak{I} \) from \( t \) can be expressed as \( \mathfrak{I} = \text{const} \cdot t^n \), \( n \) being integer. Since the best fit for all ground state bands ranging \( 2.34 \leq R \leq 3.33 \) were obtained at \( n = 1 \), then \( \mathfrak{I} \) itself, can be considered as a general variable and the equation of the total energy for the ground state bands, Eq. (7), takes the form,

\[
E(I) = \frac{\hbar^2}{2\mathfrak{I}}I(I+1) + \frac{1}{2}C(\beta_I - \beta_0)^2,
\]

with the equilibrium condition

\[
\frac{\partial E(I)}{\partial \beta} = 0,
\]

must be satisfied for each state of \( I \). The model of Mariscotti and et. al. \([17]\) is known as the variable moment of inertia (VMI) model. In spite of the great success of this model in getting excellent fit for all ground-state bands ranging \( 2.34 \leq R \leq 3.33 \), but, like Sood, the authors considered the moment of inertia as a general variable and they did not clarify precisely how this general variable depends on the deformation parameter and the microscopic features and also what is the ratio that each parameter contribute.

Harris suggested an expansion of both; total energy and the moment of inertia in power of \( \omega^2 \) instead of \( I(I+1) \) expansion of Bohr and Mottelson, Eq. (3). It was found excellent agreement with energies in the ground state bands of deformed nuclei \([19]\). In reference \([20]\), a theoretical analysis of the deviation from the adiabatic theory (i.e. the case of strong coupling mentioned before) was presented on the basis of phenomenological model \( \text{(Phen.M)} \)[21], take into account the Coriolis mixture of low-lying state bands. Authors \([20]\) calculated the values of moment of inertia \( \beta_0 \), \( \beta_1 \) using Harris parametrization \([19]\), and energy spectra of rotational ground \( (gr) \) state band of even-even \( 152-156 Sm, 156-160 Gd, 160-166 Dy, 166-176 Yb, 170-180 Hf \) and \( 174-184 W \) nuclei. The obtained results show a very good agreement with experimental data. Furthermore, recently for the even-even deformed \( 154,156 Sm, 156-160 Gd, 160-164 Dy, 164-170 Er, 170-176 Yb, 176-180 Hf, 182-186 W, 186-190 Os, 194-198 Pt, 222-228 Ra, 226-234 Th, 230-238 U \) and \( 238,240 Pu \) nuclei the moment of inertia has been calculated by the theoretical extension of hydrodynamical model in view of the contributions arising from higher order terms of radial distribution \([22]\). Such calculated values are found to be in better agreement than the original model for both axially deformed and triaxial nuclei. This highlights the crucial approximation involved in the irrotational picture of liquid droplet in terms of small amplitude vibrations and further supports the large amplitude vibrations at the nuclear surface.

A nuclear rotation-vibration model based on the crank Bohr-Mottelson Hamiltonian has been successfully applied to normal rotational bands of only the well deformed even-even nuclei in reference \([23]\). Recently, a three parameters formula were derived by H. X. Huang, C. S. Wu, and J. Y. Zeng \([24]\). This formula was analyzed the rotational spectra up to \( I = 20 \). It shows an excellent agreement with experimental data.
agreement with experiment for all actinide and rare earth nuclei only in well-deformed region i.e. $3.2 < R < 3.33$. Like Diamond and Gupta, this formula doesn’t work well for nuclei outside of the strongly deformed region.

The present work aimed to show that the idea of a spinning nucleus being stretched out along the symmetry axis that is mentioned in reference [16], when such stretching is quantized, leads to the derivation of a formula for the moment of inertia that is capable to reproduce level energies in the ground state bands up to $I = 16$ for all actinide and rare earth nuclei in well-deformed as well as in transitional regions in the range $2.9 < R < 3.33$. As a consequence, one can prove that the moment of inertia depends mainly on the deformation parameter of the nucleus. The results of this work showed that the effect of all microscopic features like Coriolis forces and pairing effect is so small and it can be neglected in many cases. A brief description of the formulation of this formula will be displayed in section (2).

Application of this formula to the ground state bands of nuclei only in well-deformed region i.e. $3.3$. In section 4 the important conclusions will be summarized.

II. FORMALISM OF THE MODEL

In the present model, we shall follow the suggestion of Diamond et al. [16], that the increase in the moment of inertia can be interpreted on the basis of the idea that spinning nucleus exhibits some sort of centrifugal stretching along the symmetric axis, the very simple classical form of the moment of inertia of the mass element $dm$ is

$$d\mathfrak{I}_I = r^2 dm,$$  \hspace{1cm} (10)

where $r$ is the effective radius of rotation. According to the hydrodynamical model, a very little of the nuclear matter is actually taking part in the effective rotational motion or, in other words, the rotational motion can be pictured as a motion of wave around the nuclear surface, so

$$dm = \frac{M}{A} R^2 d\Omega = \frac{M}{4\pi} d\Omega, \hspace{1cm} (11)$$

where $A \approx 4\pi R^2$ is the total area of the surface, $R^2 d\Omega \approx R^2 \sin \theta d\theta d\phi$ is the surface element, $R$ is the distance of the mass element $dm$ from the center of the nucleus. Where $r$ in Eq. (10) can be written as $r_0 + \Delta r$ where $\Delta r$ is the stretched in the nucleus due to rotation. It follows

$$d\mathfrak{I} = (r_0 + \Delta r)^2 \frac{M}{4\pi} d\Omega = r_0^2 \left(1 + \frac{\Delta r}{r_0}\right)^2 \frac{M}{4\pi} d\Omega. \hspace{1cm} (12)$$

We assume that the nucleus has symmetry axis which is perpendicular to its rotational axis then the stretched can be expanded in a complete set of spherical harmonics as $\Delta r = \sum_{\lambda,\mu} a_{\lambda,\mu} Y_{\lambda,\mu}(\theta, \phi)$. Putting this definition in to Eq. (12) we get

$$d\mathfrak{I} = r_0^2 \left(1 + \sum_{\lambda,\mu} a_{\lambda,\mu}^* Y_{\lambda,\mu}(\theta, \phi) \right)^2 \frac{M}{4\pi} d\Omega \hspace{1cm} (13)$$

$$= r_0^2 \left(1 + 2 \sum_{\lambda,\mu} a_{\lambda,\mu}^* Y_{\lambda,\mu}(\theta, \phi) \right)$$

$$+ \sum_{\lambda,\mu,\mu'} a_{\lambda,\mu}^* a_{\lambda,\mu'}^* Y_{\lambda,\mu'} Y_{\lambda,\mu'}^* \frac{M}{4\pi} d\Omega.$$ 

Integrating both sides of Eq. (13) over the whole surface of the nucleus, one can find

$$\mathfrak{I} = \mathfrak{I}_0 \left(1 + \frac{1}{4\pi} \sum_{\lambda,\mu} (-1)^\mu a_{\lambda,\mu} a_{\lambda,-\mu} \right), \hspace{1cm} (14)$$

where $\mathfrak{I}_0$ is the surface moment of inertia in space-fixed coordinates. The second term in Eq. (15) represents the departure from the spherical equilibrium shape in term of complete set of spherical harmonic functions $Y_{\lambda,\mu}$. It is found that even-even nuclei can be accurately described in terms of a deformation of order $\lambda = 2$. In Eq. (15) the suffix index $\mu$ which represents the orientation of the nucleus in space-fixed coordinates runs from $-\lambda$ to $\lambda$. In our case $\mu$ runs $-2$ to $2$. We drop the subscript $\lambda = 2$ from all deformation parameters henceforth

$$\mathfrak{I} = \mathfrak{I}_0 \left(1 + \frac{1}{4\pi} \sum_{\mu} (-1)^\mu a_{-\mu} a_{-\mu} \right). \hspace{1cm} (16)$$

In the usual way $a_{\mu}$ can be written in term of the annihilation and creation operators of phonon $\xi_{\mu}$ and $\xi_{\mu}^T$ as

$$a_{\mu} = \sqrt{\frac{\hbar\omega}{2C}} (\xi_{\mu} + (-1)^\mu \xi_{\mu}^T), \hspace{1cm} (17)$$

where $\hbar\omega$ is the effective radius of rotation.
where \( \omega = \sqrt{C/B} \) is the angular frequency, \( B, C \) are the inertial and stiffness parameters respectively. Putting Eq. (17) into Eq. (16) we get

\[
\mathcal{I}_I = \mathcal{I}_0 \left[ 1 + \frac{1}{2\pi C} \left( \sum_{\mu} (1 + 2\hat{\xi}_\mu) \right) \right] (18)
\]

\[
= \mathcal{I}_0 \left[ 1 + \frac{\hbar \omega}{2\pi C} \left( \sum_{\mu} (1 + 2\hat{n}_\mu) \right) \right],
\]

where \( \hat{n}_\mu = \hat{\xi}_\mu \hat{\xi}_\mu \) is the number operator. Since Eq. (18) (17) include a parameter represents the number operator it can be used not only for the ground state bands but also for beta bands and the first gamma bands. In the case of the ground state band the number of operator is zero. Replacing \( \hbar \omega \) by \( \hbar^2 I(I+1)/2\mathcal{I}_I \) in Eq. (18) is reduced to

\[
\mathcal{I}_I = \mathcal{I}_0 \left[ 1 + \frac{5}{16\pi} \frac{\hbar^2 I(I+1)}{\mathcal{I}_0 C} \right] (19)
\]

\[
= \mathcal{I}_0 \left[ 1 + \frac{5\hbar^2}{16\pi} \frac{I(I+1)}{\mathcal{I}_0 C} \right] \left[ \frac{1}{1 + \frac{5\hbar^2}{16\pi} \frac{I(I+1)}{\mathcal{I}_0 C}} \right]
\]

Eq. (19) is a recursion relation of \( \mathcal{I}_I \) (i.e., \( \mathcal{I}_I \) are defined in terms of itself), and it represents the moment of inertia of the nucleus at an angular momentum \( I \), approximated to the second order of perturbation. The factor 5 in Eq. (19) arises because the summation over \( \mu \) runs from \( \lambda = -2 \) to \( \lambda = 2 \) through \( \mu = 0 \) as mentioned above. Finally, the energy levels in the ground state bands is casted into the following form,

\[
E_{rot}(I) = \frac{\hbar^2 I(I+1)}{2\mathcal{I}_I} (20)
\]

\[
= \frac{\hbar^2 I(I+1)}{2\mathcal{I}_0} \left[ \frac{1}{1 + \frac{5\hbar^2}{16\pi} \frac{I(I+1)}{\mathcal{I}_0 C}} \right] A
\]

\[
= \frac{A}{1 + \frac{B I(I+1)}{1+BI(I+1)}} I(I+1),
\]

where, \( A \equiv \frac{\hbar^2}{2\mathcal{I}_0} \) and \( B \equiv \frac{5\hbar^2}{8\pi C} \). We will call this Eq. (20) (20) quantized \( \beta \)-stretching equation.

These parameters embed in them the intrinsic moment of inertia \( \mathcal{I}_0 \) and stiffness of the nucleus \( C \). They are constant for a particular nucleus, but their values differ from one nucleus to another. In the present model, they are adjustable parameters and evaluated by fitting Eq. (20) against experimentally measured energy levels.

In the following section we discuss the validation of the model by comparing its predictions against experimental data.

III. RESULTS AND DISCUSSION

The present investigation is dedicated to study of the energy levels of the ground band states in even-even isotopes \( ^{62}Sm, \ ^{64}Gd, \ ^{66}Dy, \ ^{68}Er, \ ^{70}Yb, \ ^{72}Hf, \ ^{74}W \) and \( ^{76}Os \) with the neutron numbers ranging from 90 to 114, by the quantizing of stretching model for 40 nuclei ranging from atomic number 152 to 190 and having the energy ratio \( 2.9 < R < 3.33 \).

The simple expression Eq. (20), has been used to evaluate the level energies up to spin \( I = 16 \). The parameters \( A \) and \( B \) in Eq. (20) has been determined by the least squares method with fitting first three energy values of ground state band, which is obtained experimentally ref. [29].

Comparison between the calculated results obtained by Eq. (20) with the experimental data [29] taken from the decay data [29] which is taken from the decay data [29] (http://www.nndc.bnl.gov/ndat2/) and calculated values according to the VMI model [17] for the energy levels \( E(I) \) of the ground state bands for a large collection of even-even nuclei are given in Table 1.

It can be seen from Table 1 that there is an overall agreement with the experimental data with errors not more than 0.5% for most of the nuclei. Very few cases are found to display an error of 5% which arises because of the spread of the experimental points which increases rapidly with \( I \). The present model has the advantage of being simpler in the form of the energy levels than the VMI model.

To understand the anomaly in ground state bands, we have carried out calculations and compared with the experimental data [29] which is taken from the decay data [29] (http://www.nndc.bnl.gov/ndat2/) and also with the other theoretical results by the VMI model [17], phenomenological model Phen.M [20] takes into account the Coriolis mixture of low-lying state bands for \( ^{170-176}Yb, \ ^{170-180}Hf, \) and \( ^{174-184}W \) nuclei, respectively as a detailed example, shown in Figure 1-3.

We can see from the Figure 1-3 that this comparison showed the results of our calculation are coincide with experimental date and also with the results of VMI and phenomenological Phen.M models. And it can be seen that the calculated energy levels reproduce the energies of the yrast levels qualitatively describe for all even-even nuclei.

A graphical comparison of the calculated and experimental values of the energy ratios \( E_I/E_2 \) of excited states \( E_2 \) as a function of the ratio \( R = E_I/E_2 \) in the range 2.90 to 3.33 for all \( I \) up to 16 is shown in Figure 4. It is clearly obvious that the coincidence between experimental data and the predictions of the current work is satisfactory. Specifically, we observe that at energy levels where \( I < 14 \) the theoretical curve represent with high
accuracy the experimental points. That is the theoretical curve passes nearly through all experimental points. For higher energy levels the spread of the experimental points increases rapidly with \( I \) and no longer a smooth curve to pass through all points.

Comparison of the ratio \( E_{10}/E_2 \) is plotted against \( E_4/E_2 \) which is obtained in the current model with the experimental data [29] and other three models, namely the Gupta [ Eq. (4)], Bohr-Mottelson 2-parameter models [ Eq. (2)] and Sood [ Eq. (5)] are illustrated in Figure 5. While the prediction of our work compares nicely with the experimental data for all the region under consideration of the nuclei, both Gupta and Bohr-Mottelson 2 parameter model diverge from the experimental data except at the proximity around the \( E_4/E_2 \approx 3.33 \) regime. This infers that the Gupta and Bohr-Mottelson 2 parameter model can hold only a few nuclei which is confined in the range of \( 3.25 < R < 3.33 \). Although the success of Sood is impressive some shortcomings nonetheless. For example, a parameter \( N \) is introduced [see Eq. (5)] without scientific justification, nothing has been mentioned about the physical interpretation or identity of this parameter in spite of its importance in fitting the experimental points.

IV. SUMMARY AND CONCLUSION

The energy spectra of the ground state bands of even-even \(^{62}\text{Sm}, ^{64}\text{Gd}, ^{66}\text{Dy}, ^{68}\text{Er}, ^{70}\text{Yb}, ^{72}\text{Hf}, ^{74}\text{W} \) and \(^{76}\text{Os} \) isotopes have been systematically studied using the theoretical framework of the stretching model where the stretching of the nucleus with the rotation is quantized. A semi-classical formula for the ground states bands of even-even nuclei is obtained. This derivation is made within the framework of the stretching model where the stretching of the nucleus with the rotation is quantized.

- The quantization of the stretching that spinning nuclei exhibit led to a formula for the moment of inertia that can be applied successfully for all nuclei \( E_4/E_2 \geq 2.9 \). While the classical treatment of such stretching in References [16] works well only for nuclei in the region of strongly deformed nuclei where \( E_4/E_2 \geq 3.2 \).

- Unlike VMI where the moment of inertia \( \mathfrak{I} \) was considered as a general variable, our work has a characteristic that the two parameters in our formula are clear and their physical meanings are well known.

- The full coincidence of our formula which is derived on the basis that the moment of inertia depends only on the deformation parameter with the results presented in references [17] [20] which have taken into consideration, in addition to deformation parameter, the pairing effect and Coriolis interaction means that the contributions of the latter two parameters, i.e. pairing effect and Coriolis interaction are so small and it can be neglected.

- The excellent fitting that this formula with the experimental results especially in levels with \( I \leq 14 \) insure that the low-lying energy levels represent a collective motion of the nuclear surface. For high energy levels there are small deviations of the experimental value from the theoretical curve due to small contribution from microscopic features. This means that microscopic considerations should be taken into account in order to get a very accuracy results at high energy levels.

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FIG. 1: (Color online) Comparison between calculated results and experimental data [29], moreover with the theoretical results obtained by the VMI [17] and Phen.M [20] of the energy spectra of ground state bands in isotopes Yb.
FIG. 2: (Color online) Comparison between calculated results and experimental data [29], moreover with the theoretical results obtained by the VMI [17], and Phen.M [20] of the energy spectra of ground state bands in isotopes Hf.
FIG. 3: (Color online) Comparison between calculated results and experimental data [29], moreover with the theoretical results obtained by the $VMI$ [17], and $Phen.M$ [20] of the energy spectra of ground state bands in isotopes $W$. 
FIG. 4: (Color online) Comparison calculated results obtained by Eq. (15) with the experimental data [17] for energy ratio $E_1/E_2$ as a function of $R$ for a different values of $I$.

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FIG. 5: (Color online) Comparison of calculated results with the experimental data and results of other models for the energy ratio $E_{10}/E_2$ as a function of $R = E_4/E_2$. The solid curves are computed according Eq. (15) and dot is experimental data [17].

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TABLE I: The energy levels \(E(I)\) of the ground state band of even-even nuclei. For each nucleus, the first row is the experimental values [http://www.nndc.bnl.gov/nudat2/], the second and third rows the calculated values according to this work and that of the VMI model [17].

| Nuclei | A | B | Results | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
|--------|---|---|---------|---|---|---|---|----|----|----|----|
| \(^{152}\text{Sm}\) | 21.76 | 0.021 | Exp. VMI Current work | 121.8 | 366.5 | 706.9 | 1125.4 | 1609.3 | 2165.8 | 2406.8 | 3365.0 |
| \(^{154}\text{Sm}\) | 14.47 | 0.0088 | Exp. VMI Current work | 82.0 | 266.8 | 544.1 | 902.8 | 1330.0 | 1825.9 | 2373.0 | 2968.2 |
| \(^{152}\text{Gd}\) | 21.99 | 0.0206 | Exp. VMI Current work | 123.1 | 371.0 | 717.7 | 1144.4 | 1637.1 | 2184.7 | 2777.3 | 3404.5 |
| \(^{154}\text{Gd}\) | 15.87 | 0.0105 | Exp. VMI Current work | 89.0 | 288.2 | 584.7 | 965.1 | 1416.1 | 1924.2 | 2475.8 | 3095.3 |
| \(^{156}\text{Gd}\) | 13.82 | 0.0058 | Exp. VMI Current work | 79.5 | 261.5 | 539.4 | 901.0 | 1349.0 | 1865.0 | 2377.3 | 3007.1 |
| \(^{154}\text{Dy}\) | 12.74 | 0.0051 | Exp. VMI Current work | 73.5 | 264.5 | 539.2 | 903.1 | 1349.0 | 1865.0 | 2377.3 | 3007.1 |
| \(^{156}\text{Dy}\) | 15.37 | 0.0084 | Exp. VMI Current work | 79.6 | 261.4 | 537.8 | 909.5 | 1338.0 | 1845.4 | 2415.3 | 3042.1 |
| \(^{158}\text{Dy}\) | 24.60 | 0.0240 | Exp. VMI Current work | 334.3 | 746.8 | 1223.7 | 1747.2 | 2304.1 | 2892.6 | 3508.7 | 4172.8 |
| \(^{160}\text{Dy}\) | 17.81 | 0.0129 | Exp. VMI Current work | 79.6 | 261.4 | 537.8 | 909.5 | 1338.0 | 1845.4 | 2415.3 | 3042.1 |
| \(^{162}\text{Dy}\) | 15.37 | 0.0084 | Exp. VMI Current work | 86.8 | 283.8 | 581.1 | 966.9 | 1428.0 | 1950.5 | 2513.8 | 3089.8 |
| \(^{164}\text{Dy}\) | 14.02 | 0.0056 | Exp. VMI Current work | 80.7 | 265.7 | 549.2 | 921.5 | 1374.8 | 1901.6 | 2493.5 | 3150.2 |
| \(^{166}\text{Dy}\) | 12.74 | 0.0051 | Exp. VMI Current work | 73.4 | 242.2 | 501.3 | 843.7 | 1261.3 | 1745.9 | 2290.6 | 2887.1 |
| \(^{168}\text{Er}\) | 12.74 | 0.0051 | Exp. VMI Current work | 74.6 | 243.5 | 501.2 | 842.3 | 1262.0 | 1755.4 | 2318.2 | 2946.1 |
| \(^{170}\text{Er}\) | 18.32 | 0.0115 | Exp. VMI Current work | 102.9 | 329.6 | 666.7 | 1096.7 | 1602.8 | 2165.1 | 2745.7 | 3292.4 |
| \(^{172}\text{Er}\) | 16.09 | 0.0076 | Exp. VMI Current work | 91.4 | 299.4 | 614.4 | 1024.6 | 1518.1 | 2082.8 | 2702.6 | 3411.2 |
| \(^{174}\text{Er}\) | 14.25 | 0.0078 | Exp. VMI Current work | 80.6 | 265.9 | 545.5 | 911.2 | 1349.5 | 1846.5 | 2389.3 | 2967.3 |
| \(^{176}\text{Er}\) | 13.68 | 0.0036 | Exp. VMI Current work | 79.8 | 264.1 | 548.9 | 928.9 | 1398.0 | 1950.2 | 2579.6 | 3281.1 |
| \(^{178}\text{Er}\) | 13.47 | 0.0036 | Exp. VMI Current work | 78.6 | 260.1 | 540.7 | 915.0 | 1376.6 | 1918.6 | 2537.2 | 3225.7 |
| \(^{180}\text{Os}\) | 24.04 | 0.0197 | Exp. VMI Current work | 126.9 | 400.3 | 794.0 | 1277.9 | 1812.0 | 2364.1 | 2840.7 | 3320.1 |
| \(^{182}\text{Os}\) | 21.17 | 0.0110 | Exp. VMI Current work | 119.8 | 383.7 | 774.1 | 1274.8 | 1871.2 | 2457.6 | 3261.4 | 4046.5 |
| \(^{184}\text{Os}\) | 24.41 | 0.0133 | Exp. VMI Current work | 137.2 | 434.1 | 868.9 | 1429.0 | 2068.0 | 2781.3 | 3504.8 | 4419.5 |
| \(^{186}\text{Os}\) | 28.04 | 0.0183 | Exp. VMI Current work | 155.0 | 478.0 | 940.0 | 1514.8 | 2170.1 | 2856.3 | 3562.6 | 4326.5 |
| \(^{188}\text{Os}\) | 33.44 | 0.0238 | Exp. VMI Current work | 186.7 | 547.9 | 1050.4 | 1666.5 | 2357.0 | 3096.9 | 3874.8 | 4734.4 |
| \(^{190}\text{Os}\) | 18.00 | 0.0046 | Exp. VMI Current work | 185.3 | 554.6 | 1052.4 | 1648.5 | 2325.0 | 3069.8 | 3874.8 | 4734.4 |
| \(^{192}\text{Os}\) | 17.91 | 0.0046 | Exp. VMI Current work | 179.1 | 546.3 | 1054.3 | 1666.7 | 2356.0 | 3100.4 | 3883.4 | 4691.9 |