A note on the free energy of the coupled system in the Sherrington-Kirkpatrick model.

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Abstract

In this paper we consider a system of spins that consists of two configurations $\sigma^1, \sigma^2 \in \Sigma_N = \{-1, +1\}^N$ with Gaussian Hamiltonians $H^1_N(\sigma^1)$ and $H^2_N(\sigma^2)$ correspondingly, and these configurations are coupled on the set where their overlap is fixed $\{R_{1,2} = N^{-1} \sum_{i=1}^N \sigma^1_i \sigma^2_i = u_N\}$. We prove the existence of the thermodynamic limit of the free energy of this system given that $\lim_{N \to \infty} u_N = u \in [-1,1]$ and give the analogue of the Aizenman-Sims-Starr variational principle that describes this limit via random overlap structures.

Key words: spin glasses, Sherrington-Kirkpatrick model.

1 Introduction and main results.

In this paper we will consider a system that consists of two configurations of spins that are coupled by fixing their overlap. Our main goal is to prove the existence of the thermodynamic limit of the free energy of this system and to give the characterization of this limit via random overlap structures in the sense of Aizenman-Sims-Starr [1]. Let us start by introducing all necessary notations and definitions.

For any $N \geq 1$, let us consider a space $\Sigma_N = \{-1, +1\}^N$ and consider two Hamiltonians $H^\ell_N(\sigma)$ for $\ell = 1, 2$ on $\Sigma_N$ given by

$$H^\ell_N(\sigma) = N^{1/2} \sum_{p \geq 1} \frac{a^\ell_p}{N^{p/2}} \sum_{i_1,\ldots,i_p} g_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}, \quad (1.1)$$

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where \((g_{i_1,\ldots,i_p})\) are standard Gaussian random variables independent for all \(p \geq 1\) and all \((i_1, \ldots, i_p)\), and the sequences \((a_p^\ell)_{p \geq 1}\) are such that
\[
\sum_{p \geq 1} (a_p^\ell)^2 < \infty.
\] (1.2)

For \(\ell, \ell' \in 1, 2\), let us define the functions \(\xi_{\ell,\ell'}: [-1, 1] \rightarrow \mathbb{R}\) by
\[
\xi_{\ell,\ell'}(x) = \sum_{p \geq 1} a_p^\ell a_p^{\ell'} x^p
\] (1.3)
so that
\[
\frac{1}{N} \mathbb{E} H_N^\ell(\sigma^1) H_N^{\ell'}(\sigma^2) = \xi_{\ell,\ell'}(R_{1,2}),
\] (1.4)
where the overlap
\[
R_{1,2} = R(\sigma^1, \sigma^2) = \frac{1}{N} \sum_{i \leq N} \sigma_i^1 \sigma_i^2.
\]

The condition \(1.2\) implies that the functions \(\xi_{\ell,\ell'}\) are well-defined and smooth on \([-1, 1]\). From now on we will also assume that the sequences \((a_p^\ell)\) are such that the functions \(\xi_{\ell,\ell'}\) are convex on \([-1, 1]\). For example, this holds if \(a_p^\ell = 0\) for \(p\) odd and \(a_p^\ell \geq 0\) for \(p\) even. We define the functions,
\[
\theta_{\ell,\ell'}(x) = x\xi_{\ell,\ell'}(x) - \xi_{\ell,\ell'}(x).
\] (1.5)
The convexity of \(\xi_{\ell,\ell'}\) implies that for any \(x, y \in [-1, 1]\) we have
\[
\xi_{\ell,\ell'}(x) - x\xi_{\ell,\ell'}(y) + \theta_{\ell,\ell'}(y) \geq 0.
\] (1.6)

Given \(u \in [-1, 1]\), let us consider a sequence \((u_N)_{N \geq 1}\) such that for each \(N\) we have \(u_N = k/N\) for some integer \(-N \leq k \leq N\) and such that \(\lim_{N \rightarrow \infty} u_N = u\). Given the external fields \(h_1, h_2 \in \mathbb{R}\), we define,
\[
F_N(u_N) = \frac{1}{N} \mathbb{E} \log Z_N(u_N),
\] (1.7)
where
\[
Z_N(u_N) = \sum_{R_{1,2}=u_N} \exp \left( \sum_{\ell \leq 2} H_N^\ell(\sigma^\ell) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq N} \sigma_i^\ell \right).
\] (1.8)
The quantity \(F_N(u_N)\) represents the free energy of the set of configurations \(\{R_{1,2} = u_N\}\). The main reason that \(u_N\) was chosen of the type \(k/N\) is that this set be not empty.

Our first goal will be to prove the following.

\textbf{Theorem 1} The limit
\[
\lim_{N \rightarrow \infty} F_N(u_N) = \mathcal{P}(u)
\] (1.9)
exists and depends on \(u\) but not on the sequence \((u_N)\).
The main idea in the proof of this Theorem is the interpolation method of Guerra-Toninelli which was developed by authors in [4] to prove the existence of the thermodynamic limit of the free energy of one copy of the system with Hamiltonian $H^k_N(\sigma)$. They also extended their method in [5] to prove the existence of the thermodynamic limit in a variety of mean field models. In fact, a part of the proof of Theorem 1 is very similar to the proof of the main result in [5] which was motivated by the idea of restricting to the set of configurations with given overlap introduced by Michel Talagrand in [8]. However, the situation considered in Theorem 1 is slightly different, mainly, due to the fact that we consider the set $\{R_{1,2} = u_N\}$ of configurations with overlap exactly equal to $u_N$ rather than being in the neighborhood of $u_N$. This will require some additional approximation result, Lemma 1 below. We will prove that the sequence $F_N(u_N)$ can be approximated by a superadditive sequence over the restricted range of indices and apply the following Proposition due to DeBruijn-Erdős [2] (see also Theorem 1.9.1 in [6]).

**Proposition 1 (DeBruijn-Erdős)** If the sequence $(a_N)$ of real numbers satisfies the superadditivity condition

$$a_{m+n} \geq a_m + a_n \text{ over the restricted range } \frac{1}{2} n \leq m \leq 2n,$$

then $\lim_{n \to \infty} a_n / n = \sup a_n / n$.

Next, we will characterize the limit $\mathcal{P}(u)$ in (1.9) via the analogue of Aizenman-Sims-Starr variational principle [1]. This characterization is motivated by the following idea. In [13] Michel Talagrand proved a certain replica symmetry breaking upper bound on $F_N(u_N)$ and conjectured that the bound should be precise in the limit, i.e. should be equal to $\mathcal{P}(u)$ in (1.9). He also emphasized that the computation of this limit is a natural approach to solving the so called chaos problem. It is interesting to note that the formula conjectured by Talagrand can be written via Derrida-Ruelle probability cascades as in the case of the Parisi formula in the Sherrington-Kirkpatrick model. On the other hand, the Parisi formula in the SK model written via Derrida-Ruelle cascades can be included in a broader variational principle described in [1]. This connection motivates us to give a variational characterization of the limit $\mathcal{P}(u)$ in terms of random overlap structures in the sense of Aizenman-Sims-Starr [1]. We hope that this characterization will provide some insight into what should be the correct Parisi ansatz for $\mathcal{P}(u)$ and whether the formula conjectured by Talagrand indeed holds.

Given a parameter $\delta > 0$, we define the random overlap structure (ROSt) as the following collection of:

1. a countable set $\mathcal{A}$;
2. a sequence $(q_{\alpha,\beta}^{\ell,\ell'})$ for $\alpha, \beta \in \mathcal{A}, \ell, \ell' \in \{1, 2\}$ such that
   $$|q_{\alpha,\beta}^{\ell,\ell'}| \leq 1, \quad q_{\alpha,\alpha}^{\ell,\ell} = 1 \quad \text{and} \quad |q_{\alpha,\alpha}^{1,2} - u| \leq \delta; \quad (1.10)$$
3. an arbitrary random sequence $(w_\alpha)_{\alpha \in \mathcal{A}}$ such that
   $$w_\alpha \geq 0 \quad \text{and} \quad \sum_{\alpha \in \mathcal{A}} w_\alpha = 1 \text{ a.s.;} \quad (1.11)$$
(4) Gaussian sequences \((z^{1}(\alpha), z^{2}(\alpha))_{\alpha \in A}\) and \((y^{1}(\alpha), y^{2}(\alpha))_{\alpha \in A}\) independent of each other and of the sequence \((w_{\alpha})_{\alpha \in A}\) with the following covariance operators
\[
\mathbb{E} z^{\ell}(\alpha) z^{\ell'}(\beta) = \xi^{\ell,\ell'}_{\alpha,\beta} \quad \text{and} \quad \mathbb{E} y^{\ell}(\alpha) y^{\ell'}(\beta) = \theta^{\ell,\ell'}_{\alpha,\beta}. \tag{1.12}
\]
Let \((z^{1}_{i}(\alpha), z^{2}_{i}(\alpha))_{\alpha \in A}\) be a sequence of independent copies of \((z^{1}(\alpha), z^{2}(\alpha))_{\alpha \in A}\) for \(i \geq 1\). We also assume that all random variables here are independent of the Hamiltonians \(H_{N}^{\ell}(\sigma)\). Let us denote such generic collection (1) - (4) as \(\Omega_{\delta}\), where we will make the dependence of \(\Omega_{\delta}\) on the parameter \(\delta\) in (1.10) explicit.

One could try to describe conditions on the sequence \((q^{\ell,\ell'}_{\alpha,\beta})\) that would guarantee the existence of the Gaussian sequences with the covariance structure (1.12). Instead, we will simply assume that we consider only random overlap structures \(\Omega_{\delta}\) that such sequences exist.

One reason why we are not interested in the general case is because, as in [1], one particular ROS will play a special role in characterization of the limit \(P(u)\) in (1.9) and it will be constructed explicitly. Given a ROS \(\Omega_{\delta}\), let us now consider the quantity
\[
G_{N}(u_{N}, \Omega_{\delta}) = \frac{1}{N} \mathbb{E} \log \sum_{\alpha \in A} w_{\alpha} \sum_{R_{1,2} = u_{N}} \exp \sum_{\ell \leq 2} \sum_{i \leq N} \sigma^{\ell}_{i}(z^{\ell}_{i}(\alpha) + h_{\ell}) - \frac{1}{N} \mathbb{E} \log \sum_{\alpha \in A} w_{\alpha} \exp \sqrt{N} \sum_{\ell \leq 2} y^{\ell}(\alpha). \tag{1.13}
\]

The following theorem holds.

**Theorem 2** There exists a sequence \((u'_{N})\) such that \(\lim_{N \to \infty} u'_{N} = u\) and such that the limit in (1.9) holds
\[
P(u) = \lim_{N \to \infty} \lim \inf_{\delta \to 0} G_{N}(u'_{N}, \Omega_{\delta}). \tag{1.14}
\]

## 2 Proof of Theorem 1.

Given \(\varepsilon > 0\), let us consider a set
\[
U_{N,\varepsilon} = \{(\sigma^{1}, \sigma^{2}) : R_{1,2} \in [u_{N} - \varepsilon, u_{N} + \varepsilon]\}. \tag{2.1}
\]
and define
\[
F_{N}(U_{N,\varepsilon}) = \frac{1}{N} \mathbb{E} \log \sum_{U_{N,\varepsilon}} \exp \left( \sum_{\ell \leq 2} H_{N}^{\ell}(\sigma^{\ell}) + \sum_{\ell \leq 2} h_{\ell} \sum_{i \leq N} \sigma^{\ell}_{i}\right). \tag{2.2}
\]
In order to utilize the ideas of Guerra and Toninelli in [4] and [5], we first need to prove the following approximation result.

**Lemma 1** There exists a constant \(L\) independent of \(N\) such that for all \(\varepsilon \in [0, 1]\)
\[
F_{N}(U_{N,\varepsilon}) \leq F_{N}(u_{N}) + L \sqrt{\varepsilon}. \tag{2.3}
\]
Proof. For each $\sigma^1 \in \Sigma_N$ let us consider the sets

$$U_ε(\sigma^1) = \{\sigma^2 : R_{1,2} \in [u_N - ε, u_N + ε]\}, \quad U(\sigma^1) = \{\sigma^2 : R_{1,2} = u_N\}.$$  

For each $\sigma^2 \in U_ε(\sigma^1)$ we can find an element $\pi(\sigma^1, \sigma^2) \in U(\sigma^1)$ such that the Hamming distance

$$d(\sigma^2, \pi(\sigma^1, \sigma^2)) = \frac{1}{N} \sum_{i \leq N} I(\sigma^2_i \neq \pi(\sigma^1, \sigma^2)_i) \leq \frac{ε}{2}, \quad (2.4)$$

Indeed, since $R_{1,2} = 1 - 2d(\sigma^1, \sigma^2)$, for $\sigma^2 \in U_ε(\sigma^1)$ we have

$$\frac{1 - u_N}{2} - \frac{ε}{2} \leq \frac{1}{N} \sum_{i \leq N} I(\sigma^2_i \neq \pi(\sigma^1, \sigma^2)_i) \leq \frac{1 - u_N}{2} + \frac{ε}{2}.$$  

Therefore, by changing at most $Nε/2$ coordinates of the vector $\sigma^2$ we can obtain a vector $\pi(\sigma^1, \sigma^2)$ such that

$$\frac{1}{N} \sum_{i \leq N} I(\sigma^2_i \neq \pi(\sigma^1, \sigma^2)_i) = \frac{1 - u_N}{2},$$

which means that $\pi(\sigma^1, \sigma^2) \in U(\sigma^1)$ and $d(\sigma^2, \pi(\sigma^1, \sigma^2)) \leq ε/2$. If we write

$$H^2_N(\sigma^2) + h_2 \sum_{i \leq N} \sigma^2_i = H^2_N(\pi(\sigma^1, \sigma^2)) + h_2 \sum_{i \leq N} \pi(\sigma^1, \sigma^2)_i + H^2_N(\sigma^2) - H^2_N(\pi(\sigma^1, \sigma^2)) + h_2 \sum_{i \leq N} (\sigma^2_i - \pi(\sigma^1, \sigma^2)_i)$$

then, clearly, $F_N(U_{N, ε}) \leq I + II$, where

$$I = \frac{1}{N} \mathbb{E}_{U_{N, ε}} \max(H^2_N(\sigma^2) - H^2_N(\pi(\sigma^1, \sigma^2)) + h_2 \sum_{i \leq N} (\sigma^2_i - \pi(\sigma^1, \sigma^2)_i))$$

$$\leq \frac{1}{N} \mathbb{E}_{U_{N, ε}} (H^2_N(\sigma^2) - H^2_N(\pi(\sigma^1, \sigma^2))) + |h_2| ε$$

and

$$II = \frac{1}{N} \mathbb{E}_{U_{N, ε}} \log \sum \exp\left(H^1_N(\sigma^1) + \sum_{i \leq N} h_1 \sigma^1_i + H^2_N(\pi(\sigma^1, \sigma^2)) + h_2 \sum_{i \leq N} \pi(\sigma^1, \sigma^2)_i\right).$$

To estimate the first term in $I$ we use Slepian’s inequality that implies (see [4])

$$\mathbb{E}_{U_{N, ε}} \max(H^2_N(\sigma^2) - H^2_N(\pi(\sigma^1, \sigma^2)))$$

$$\leq 3 \sqrt{\log \text{card} U_{N, ε}} \max(H^2_N(\sigma^2) - H^2_N(\pi(\sigma^1, \sigma^2)))^{1/2}$$

$$\leq 6N \sqrt{\log 2} \max(\xi_{2,2}(1) - \xi_{2,2}(R(\sigma^2, \pi(\sigma^1, \sigma^2)))^{1/2},$$

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where we used (1.4) and an estimate \( \text{card}U_{N,\varepsilon} \leq 2^{2N} \). By (2.4)

\[
R(\sigma^2, \pi(\sigma^1, \sigma^2)) = 1 - 2d(\sigma^2, \pi(\sigma^1, \sigma^2)) \geq 1 - \varepsilon.
\]

Therefore,

\[
\left| \xi_{2,2}(1) - \xi_{2,2}(R(\sigma^2, \pi(\sigma^1, \sigma^2))) \right| \leq \max_{x \in [-1,1]} |\xi_{2,2}'(x)| \varepsilon
\]

and, thus, \( I \leq L\sqrt{\varepsilon} \). To estimate II we will simply count how many elements \( \sigma \in U_\varepsilon(\sigma^1) \) are projected onto an element \( \sigma^2 \in U(\sigma^1) \), i.e. for \( \sigma^2 \in U(\sigma^1) \) we consider

\[
\ell(\sigma^1, \sigma^2) = \text{card}\{\sigma \in U_\varepsilon(\sigma^1) : \pi(\sigma^1, \sigma) = \sigma^2\}.
\]

Then, obviously,

\[
II = \frac{1}{N} E \log \sum_{R_1,2 = u_N} \ell(\sigma^1, \sigma^2) \exp \left( H^1_N(\sigma^1) + h_1 \sum_{i \leq N} \sigma^1_i + H^2_N(\sigma^2) + h_2 \sum_{i \leq N} \sigma^2_i \right)
\]

\[
\quad \leq F_N(u_N) + \frac{1}{N} \max_{R_1,2 = u_N} \log \ell(\sigma^1, \sigma^2).
\]

Since by (2.4), \( d(\sigma^2, \pi(\sigma^1, \sigma^2)) \leq \varepsilon/2 \), we have

\[
\ell(\sigma^1, \sigma^2) \leq \text{card}\{\sigma \in \Sigma_N : d(\sigma, \sigma^2) \leq \varepsilon/2\} = \text{card}\{\sigma \in \Sigma_N : \sum_{i \leq N} I(\sigma_i \neq 1) \leq N\varepsilon/2\}
\]

\[
= \text{card}\{\sigma \in \Sigma_N : \sum_{i \leq N} \sigma_i \geq N(1 - \varepsilon)\} \leq 2^N \exp(-NI(1 - \varepsilon)),
\]

where \( I(x) = \frac{1}{2} ((1 + x) \log(1 + x) + (1 - x) \log(1 - x)) \). In the last inequality we used a large deviation estimate for the Bernoulli r.v. (see, for example, A.9 in [3]). Hence,

\[
\frac{1}{N} \max_{R_1,2 = u_N} \log \ell(\sigma^1, \sigma^2) \leq \log 2 - I(1 - \varepsilon) = \log \left( 1 + \frac{\varepsilon}{2 - \varepsilon} \right) + \frac{\varepsilon}{2} \log \frac{2 - \varepsilon}{\varepsilon} \leq L\sqrt{\varepsilon}
\]

for \( \varepsilon \in [0,1] \). This finishes the proof of Lemma 11.

Clearly, Lemma 11 implies that

\[
|F_N(u_N) - F_N(u'_N)| \leq L|u_N - u'_N|^{1/2}
\]

for \( |u_N - u'_N| \leq 1 \) and, therefore, in order to prove the existence of the limit \( \lim_{N \to \infty} F_N(u_N) \) for any sequence \( (u_N) \) such that \( \lim_{N \to \infty} u_N = u \) it is enough to prove it for one such sequence. Therefore, from now on we will make a specific choice of \( (u_N) \) that satisfies the following condition,

\[
|u_N - u| \leq \frac{1}{N}.
\]

Clearly, it is possible to take \( u_N \) of the type \( u_N = k/N \) that satisfies this condition.

The next Lemma is similar to the techniques in [5].
Lemma 2 If \((u_N)\) satisfies (2.3) then there exists a constant \(A\) independent of \(N\) such that the sequence 
\[
a_N = NF_N(u_N) - AN^{1/2}
\]
satisfies superadditivity condition 
\[
a_{M+N} \geq a_M + a_N \text{ over the restricted range } \frac{1}{2} N \leq M \leq 2N.
\]

Proof: Given \(N, M \geq 1\), let us consider a space \(\Sigma_{M+N}\) and for each \(\sigma \in \Sigma_{N+M}\) we will write
\[
\sigma = (\rho, \tau) \text{ where }
\]
\[
\rho = (\rho_1, \ldots, \rho_M) = (\sigma_1, \ldots, \sigma_M) \in \Sigma_M, \quad \tau = (\tau_1, \ldots, \tau_N) = (\sigma_{M+1}, \ldots, \sigma_{M+N}) \in \Sigma_N. \quad (2.6)
\]

For \(\sigma^1 = (\rho^1, \tau^1)\) and \(\sigma^2 = (\rho^2, \tau^2)\) we define 
\[
R_{1,2} = R(\rho^1, \rho^2) = \frac{1}{M} \sum_{i \leq M} \rho_i^1 \rho_i^2 \quad \text{and} \quad R_{1,2} = R(\tau^1, \tau^2) = \frac{1}{N} \sum_{i \leq N} \tau_i^1 \tau_i^2.
\]

Let us write the overlap \(R_{1,2} = R(\sigma^1, \sigma^2)\) as 
\[
R_{1,2} = \frac{M}{M+N} R_{1,2} + \frac{N}{M+N} R_{1,2}.
\]

Then we have,
\[
U_{M,N} := \{ R_{1,2} = u_M, R_{1,2} = u_N \} \subseteq \{ R_{1,2} = u'_{M+N} := \frac{M}{M+N} u_M + \frac{N}{M+N} u_N \}. \quad (2.7)
\]

If we define \(\varepsilon = |u'_{M+N} - u_{M+N}|\) then 
\[
\{ R_{1,2} = u'_{M+N} \} \subseteq U_{M+N,\varepsilon} = \{ R_{1,2} \in [u_{M+N} - \varepsilon, u_{M+N} + \varepsilon] \}
\]

and, therefore, \(U_{M,N} \subseteq U_{M+N,\varepsilon}\). This together with Lemma 1 implies, 
\[
F_{M+N}(u_{M+N}) \geq F_{M+N}(U_{M+N,\varepsilon}) - L\sqrt{\varepsilon} \geq F_{M+N}(U_{M,N}) - L\sqrt{\varepsilon},
\]

where
\[
F_{M+N}(U_{M,N}) = \frac{1}{M+N} \mathbb{E} \log \sum_{\sigma_{M+N}} \exp \left( \sum_{\ell \leq 2} H_{M+N}^{\ell}(\sigma^{\ell}) + \sum_{\ell \leq 2} h_{\ell} \sum_{i \leq M+N} \sigma_i^{\ell} \right).
\]

Condition (2.3) implies that \(\varepsilon \leq 3/(M+N)\) and, therefore,
\[
F_{M+N}(u_{M+N}) \geq F_{M+N}(U_{M,N}) - \frac{L}{(M+N)^{1/2}}. \quad (2.8)
\]

Given \(t \in [0,1]\), let us consider an interpolating Hamiltonian
\[
H_t(\sigma^1, \sigma^2) = \sqrt{t} \sum_{\ell \leq 2} H_{M+N}^{\ell}(\sigma^{\ell}) + \sqrt{1-t} \sum_{\ell \leq 2} \left( H_{M}^{\ell}(\rho^{\ell}) + H_{N}^{\ell}(\tau^{\ell}) \right)
\]
where the Hamiltonians \( H_\ell^M, H_\ell^N \) and \( H_\ell^{M+N} \) are independent of each other, and define a function \( \varphi(t) \) by

\[
(M + N) \varphi(t) = \mathbb{E} \log \sum_{U_{M,N}} \exp \left( H_t(\sigma^1, \sigma^2) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M+N} \sigma_i^\ell \right). \tag{2.9}
\]

It is easy to see that

\[
\varphi(1) = F_{M+N}(U_{M,N}) \quad \text{and} \quad \varphi(0) = \frac{M}{M+N} F_M(u_M) + \frac{N}{M+N} F_N(u_N).
\]

We will show below that for some constant \( L \),

\[
\varphi'(t) \geq -\frac{L}{N + M}. \tag{2.10}
\]

This control of the derivative will imply that \( \varphi(1) \geq \varphi(0) - L/(M + N) \) and, combining this with (2.8), we get

\[
(M + N)F_{M+N}(u_{M+N}) \geq MF_M(u_M) + NF_N(u_N) - L(M + N)^{1/2}.
\]

If given \( A > 0 \) we consider a sequence \( a_N = NF_N(u_N) - AN^{1/2} \) then this can be written equivalently as,

\[
a_{M+N} \geq a_M + a_N + AM^{1/2} + AN^{1/2} - (A + L)(M + N)^{1/2}.
\]

When \( N/2 \leq M \leq 2N \), we have

\[
M^{1/2} + N^{1/2} \geq (M + N)^{1/2} \left( \sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}} \right)
\]

and, thus,

\[
AM^{1/2} + AN^{1/2} - (A + L)(M + N)^{1/2} \geq \left( \left( \sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}} - 1 \right) A - L \right)(M + N)^{1/2} \geq 0,
\]

if \( A \) is large enough. This proves that

\[
a_{M+N} \geq a_M + a_N \quad \text{over the restricted range} \quad \frac{1}{2} N \leq M \leq 2N,
\]

which is precisely the statement of Lemma. Hence, it remains to prove (2.10).

Let us denote by \( \langle \cdot \rangle_t \) the average with respect to the Gibbs’ measure \( G_{M,N} \) on \( U_{M,N} \) with Hamiltonian

\[
H_t(\sigma^1, \sigma^2) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M+N} \sigma_i^\ell.
\]

Then the standard computation utilizing Gaussian integration by parts gives (see, for example, [4] or Theorem 2.10.1 in [8]),

\[
(M + N) \varphi'(t) = \frac{1}{2} \sum_{\ell, \ell' \leq 2} \mathbb{E} \left( (M + N) \xi_{\ell,\ell'}(R_{\ell,\ell'}) - M \xi_{\ell,\ell'}(R_{\ell,\ell'})^1 - N \xi_{\ell,\ell'}(R_{\ell,\ell'})^2 \right)_t
\]

\[
- \frac{1}{2} \sum_{\ell, \ell' \leq 2} \mathbb{E} \left( (M + N) \xi_{\ell,\ell'}(R(\sigma^\ell, \sigma^{\ell'})) - M \xi_{\ell,\ell'}(R(\rho^\ell, \rho^{\ell'})) - N \xi_{\ell,\ell'}(R(\tau^\ell, \tau^{\ell'})) \right)_t,
\]

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where \((\sigma^1, \sigma^2)\) is an independent copy of \((\sigma^1, \sigma^2)\) with respect to the Gibbs' measure \(G_{M,N}\). Since
\[
R(\sigma^\ell, \sigma'^\ell) = \frac{M}{M+N} R(\rho^\ell, \rho'^\ell) + \frac{N}{M+N} R(\tau^\ell, \tau'^\ell),
\]
the convexity of \(\xi_{\ell,\ell'}\) implies that
\[
(M + N)\xi_{\ell,\ell'}(R(\sigma^\ell, \sigma'^\ell)) \leq M\xi_{\ell,\ell'}(R(\rho^\ell, \rho'^\ell)) + N\xi_{\ell,\ell'}(R(\tau^\ell, \tau'^\ell)),
\]
and, therefore,
\[
(M + N)\varphi'(t) \geq \frac{1}{2} \sum_{\ell,\ell' \leq 2} \mathbb{E}\left( (M + N)\xi_{\ell,\ell'}(R_{\ell,\ell'}) - M\xi_{\ell,\ell'}(R_{\ell,\ell}') - N\xi_{\ell,\ell'}(R_{\ell,\ell}') \right).
\]
For \(\ell = \ell'\) we have \(R_{\ell,\ell} = R^1_{\ell,\ell} = R^2_{\ell,\ell} = 1\). Also since the average \(\langle \cdot \rangle_t\) is defined on \(U_{M,N}\) we have \(R^1_{1,2} = u_M, R^2_{1,2} = u_N\) and by (2.7) \(R_{1,2} = u'_{M+N}\). Thus,
\[
(M + N)\varphi'(t) \geq (M + N)\xi_{1,2}(u'_{M+N}) - M\xi_{1,2}(u_M) - N\xi_{1,2}(u_N).
\]
Condition (2.5) implies that for all \(N\) we have \(|\xi_{1,2}(u_N) - \xi_{1,2}(u)| \leq L/N\) and this, clearly, implies (2.10).

Combining Lemma 2 and Proposition 1 proves that the limit \(\lim_{N \to \infty} a_N/N\) exists and it is, obviously, equal to the limit \(\lim_{N \to \infty} F_N(u_N)\), which finishes the proof of Theorem 1.

3 Proof of Theorem 2.

In this section we will assume that the sequence \((u_N)\) satisfies (2.5). Let us start by proving the following upper bound.

**Lemma 3** For some constant \(L\) independent of \(N\) we have,
\[
F_N(u_N) \leq \inf_{\Omega_\delta} G_N(u_N, \Omega_\delta) + L\delta + LN^{-1}. \tag{3.1}
\]

**Proof.** Consider an arbitrary random overlap structure \(\Omega_\delta\). Given \(t \in [0, 1]\), let us consider a Hamiltonian \(H_t(\alpha, \sigma^1, \sigma^2)\) on the set \(A \times \{R_{1,2} = u_N\}\) given by
\[
H_t(\alpha, \sigma^1, \sigma^2) = \sqrt{t} \left( \sum_{\ell \leq 2} H^\ell_N(\sigma^\ell) + \sqrt{N} \sum_{\ell \leq 2} y^\ell(\alpha) \right)
+ \sqrt{1-t} \sum_{\ell \leq 2} \sum_{i \leq N} \sigma^\ell_i z^\ell_i(\alpha) + \sum_{\ell \leq 2} \sum_{i \leq N} h^\ell_i \sigma^\ell_i,
\]
and consider a function
\[
\varphi(t) = \frac{1}{N} \mathbb{E} \log \sum_{\alpha \in A} w_\alpha \sum_{R_{1,2} = u_N} \exp H_t(\alpha, \sigma^1, \sigma^2).
\]


Clearly, the statement of lemma is then equivalent to
\[ \varphi(1) \leq \varphi(0) + L\delta + LN^{-1}. \]
We will prove this by showing that the derivative \(\varphi'(t) \leq L\delta + LN^{-1}\). Let us denote by \(\langle \cdot \rangle_t\) the average with respect to the Gibbs' measure on \(A \times \{R_{1,2} = u_N\}\) with Hamiltonian \(H_t(\alpha, \sigma^1, \sigma^2)\). Then the standard computation utilizing Gaussian integration by parts and covariance structure (1.12) gives
\[
\varphi'(t) = \frac{1}{2} \sum_{\ell, \ell' \leq 2} \mathbb{E}\left( \xi_{\ell,\ell'}(R_{\ell,\ell'} - R_{\ell,\ell'}\xi_{\ell,\ell'}(q^1_{\alpha,\alpha}) + \theta_{\ell,\ell'}(q^1_{\alpha,\alpha}) \right)_{t}
\]
\[
- \frac{1}{2} \sum_{\ell, \ell' \leq 2} \mathbb{E}\left( \xi_{\ell,\ell'}(R(\sigma^\ell, \sigma^{\ell'})) - R(\sigma^\ell, \sigma^{\ell'})\xi_{\ell,\ell'}(q^1_{\alpha,\alpha}) + \theta_{\ell,\ell'}(q^1_{\alpha,\alpha}) \right)_{t},
\]
where \((\beta, \bar{\sigma}^1, \bar{\sigma}^2)\) is an independent copy of \((\alpha, \sigma^1, \sigma^2)\). Using the fact that the average \(\langle \cdot \rangle_t\) is taken over the set where \(R_{1,2} = u_N\), the first sum on the right hand side is equal to
\[
\mathbb{E}\left( \xi_{1,2}(u_N) - u_N\xi_{1,2}(q^1_{\alpha,\alpha}) + \theta_{1,2}(q^1_{\alpha,\alpha}) \right)_{t} \leq L\delta + LN^{-1},
\]
where the last inequality follows from the fact that by (1.10) we have \(|q^1_{\alpha,\alpha} - u| \leq \delta\) and by (2.5) we have \(|u_N - u| \leq N^{-1}\). The second line in (3.2) is negative by (1.6) and this finishes the proof.

To prove the lower bound, let us start with a couple of simple lemmas.

**Lemma 4** If a sequence \((a_N)\) is such that \(\lim_{N \to \infty} a_N/N = \gamma\) then for any \(N \geq 1\) we have
\[
\frac{1}{N} \liminf_{M \to \infty} (a_{M+N} - a_M) \leq \gamma.
\]

**Proof.** Suppose that for some \(N \geq 1\) and for some \(\varepsilon > 0\)
\[
\frac{1}{N} \liminf_{M \to \infty} (a_{M+N} - a_M) \geq \gamma + \varepsilon.
\]
Then there exists \(M_0 \geq 1\) such that for all \(M \geq M_0\)
\[
\frac{1}{N} (a_{M+N} - a_M) \geq \gamma + \varepsilon/2
\]
and, therefore, for \(k \geq 0\)
\[
\frac{1}{N} (a_{M+(k+1)N} - a_{M+kN}) \geq \gamma + \varepsilon/2.
\]
Adding these inequalities for \(0 \leq k \leq m - 1\) we get
\[
\frac{1}{N} (a_{M+mN} - a_M) \geq m\left(\gamma + \varepsilon/2\right) \quad \text{or} \quad \frac{1}{mN} (a_{M+mN} - a_M) \geq \gamma + \varepsilon/2.
\]
Letting \(m \to \infty\) yields that \(\liminf_{N \to \infty} a_N/N \geq \gamma + \varepsilon/2\) and this contradicts the fact that \(\lim_{N \to \infty} a_N/N = \gamma\).  

Lemma 5  Consider a sequence \((u_N)\) such that (2.5) holds. Then there exists a sequence \((u'_N)\) such that for each \(N \geq 1\),

\[ \frac{M}{M+N}u_M + \frac{N}{M+N}u'_N = u_{M+N} \]

(3.3)

for infinitely many \(M \geq 1\).

**Proof.** For a fixed \(N\), consider a sequence \(u'_N(M)\) defined by (3.3), i.e.

\[ Nu'_N(M) = (M+N)u_{M+N} - Mu_M. \]

We have

\[ N(u'_N(M) - u) = (M+N)(u_{M+N} - u) - M(u_M - u) \]

and, therefore, (2.5) implies that \(N|u'_N(M) - u| \leq 2\). Since \(Nu'_N(M)\) is an integer between \(-N\) and \(N\), it can take a finite number of values and, thus, we can find an infinite subsequence \((M_k)_{k \geq 1}\) such that \(u'_N(M_k) = u'_N(M_1)\). Take \(u'_N = u'_N(M_1)\).

\[ \square \]

**Theorem 3**  There exists a sequence \((u'_N)\) such that for all \(N \geq 1\),

\[ P(u) \geq \lim_{\delta \to 0} \inf_{\Omega_{\delta}} G_N(u'_N, \Omega_{\delta}). \]

**Proof.** If we consider a sequence \(a_N = NF_N(u_N)\) then, by Theorem 1, we have that the limit \(\lim_{N \to \infty} a_N/N = P(u)\). Lemma 3 then implies that for any \(N \geq 1\),

\[ \frac{1}{N} \liminf_{M \to \infty} \left( (M+N)F_{M+N}(u_{M+N}) - MF_M(u_M) \right) \leq P(u). \]

(3.4)

We can write

\[ \frac{1}{N} \left( (M+N)F_{M+N}(u_{M+N}) - MF_M(u_M) \right) = \frac{1}{N} \mathbb{E} \log Z_{M+N}(u_{M+N}) - \frac{1}{N} \mathbb{E} \log Z_M(u_M), \]

(3.5)

where \(Z_N(u_N)\) was defined in (1.8). For \(\sigma \in \Sigma_{M+N}\) we will write \(\sigma = (\rho, \tau)\) as in (2.6). Consider the sequence \((u'_N)\) as in Lemma 5. Then as in (2.7) the condition (3.3) implies that for infinitely many \(M \geq 1\) we have

\[ \{R_{1,2} = u_{M+N}\} \supseteq U'_{M,N} := \{R_{1,2}^1 = u_M, R_{1,2}^2 = u'_N\}. \]

For simplicity of notations let us assume that this holds for all \(M \geq 1\) rather than a subsequence \((M_k)\). Therefore,

\[ Z_{M+N}(u_{M+N}) \geq Z_{M,N}(u_M, u_N) := \sum_{u'_{M,N}} \exp \left( \sum_{i \leq 2} H_{M+N, \tau}(\sigma^i) + \sum_{i \leq 2} h_\ell \sum_{i \leq M+N} \sigma^i_i \right). \]
Let us decompose the Hamiltonian in $Z_{M,N}(u_M, u_N)$ as,

$$
\sum_{\ell \leq 2} H_{M+N}^\ell (\sigma^\ell) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M+N} \sigma_i^\ell = \sum_{\ell \leq 2} H_{M+N}^\ell (\rho^\ell) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M} \rho_i^\ell + \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (Z_i^\ell (\rho^\ell) + h_\ell) + R(\sigma^1, \sigma^2).
$$

(3.6)

The first two terms on the right hand side represent the part of the Hamiltonian that depends on the first $M$ coordinates $\rho$ only, i.e. here

$$
H_{M+N}^\ell (\rho^\ell) = (M + N)^{1/2} \sum_{p \geq 1} \frac{q_p^\ell}{(M + N)^{p/2}} \sum_{i_1, \ldots, i_p \leq M} g_{i_1, i_2, \ldots, i_p} \rho_{i_1}^\ell \cdots \rho_{i_p}^\ell.
$$

(3.7)

The third term consists of the terms in the Hamiltonian that depend only on one of the last $N$ coordinates $(\tau_1^\ell, \ldots, \tau_N^\ell)$ of $\sigma^\ell$, i.e.

$$
Z_i^\ell (\rho^\ell) = (M + N)^{1/2} \sum_{p \geq 1} \frac{q_p^\ell}{(M + N)^{p/2}} \sum_{i_1, \ldots, i_p \leq M} g_{i_1, i_2, \ldots, i_p} \rho_{i_1}^\ell \cdots \rho_{i_{p-1}}^\ell,
$$

where

$$
g_{i_1, \ldots, i_p} = g_{i_1, i_2, \ldots, i_p} + g_{i_2, i_3, \ldots, i_p} + \cdots + g_{i_{p-1}, i_p}.
$$

Finally, the last term $R(\sigma^1, \sigma^2)$ consists of all the terms of the Hamiltonian that depend on at least two coordinates in $\tau^\ell$. Note that $R(\sigma^1, \sigma^2)$ is independent of all other terms in (3.6) and, therefore, Hölder’s inequality implies that

$$
\frac{1}{N} \mathbb{E} \log Z_{M,N}(u_M, u_N) \geq \frac{1}{N} \mathbb{E} \log \sum_{U'_{M,N}} \exp \left( \sum_{\ell \leq 2} H_{M+N}^\ell (\rho^\ell) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M} \rho_i^\ell + \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (Z_i^\ell (\rho^\ell) + h_\ell) \right).
$$

(3.8)

For each $(\rho^1, \rho^2)$ let us denote

$$
W(\rho^1, \rho^2) = \exp \left( \sum_{\ell \leq 2} H_{M+N}^\ell (\rho^\ell) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M} \rho_i^\ell \right)
$$

so that (3.8) becomes

$$
\frac{1}{N} \mathbb{E} \log Z_{M,N}(u_M, u_N) \geq \frac{1}{N} \mathbb{E} \log \sum_{U'_{M,N}} W(\rho^1, \rho^2) \exp \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (Z_i^\ell (\rho^\ell) + h_\ell).
$$

(3.9)

The sequences $(Z_i^\ell (\rho^\ell))$ are independent for different indices $i$, and the covariance operator of $(Z_i^\ell (\rho^\ell))$ is given by

$$
\mathbb{E} Z_i^\ell (\rho^\ell) Z_j^\ell (\rho^{\ell'}) = \sum_{p \geq 1} \left( \frac{M}{M + N} \right)^{p-1} a_p^\ell a_p^{\ell'} \mathbb{E} (R(\rho^\ell, \rho^{\ell'}))^p = \xi_{\ell, \ell'} (R(\rho^\ell, \rho^{\ell'})) + o_M(1)
$$
as $M \to \infty$, uniformly over $R(\rho^1, \rho^2) \in [-1, 1]$. Therefore, one can substitute (up to a small error) the random variables $Z_i^\ell(\rho^\ell)$ in (3.9) with the random variables

$$z_i^\ell (\rho^\ell) = M^{1/2} \sum_{p \geq 1} \frac{a_p}{M^{p/2}} \sum_{i_1, \ldots, i_p \leq M} g_{i_1, \ldots, i_p}^i \rho_{i_1} \cdots \rho_{i_p} \quad (3.10)$$

with covariance operator

$$\mathbb{E} z_i^\ell (\rho^\ell) z_i'^\ell (\rho'^\ell) = \xi_{\ell, \ell'} (R(\rho^\ell, \rho'^\ell)). \quad (3.11)$$

Namely, we have,

$$\frac{1}{N} \mathbb{E} \log \sum_{U_{M,N}'} W(\rho^1, \rho^2) \exp \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (Z_i^\ell(\rho^\ell) + h_\ell)$$

$$= \frac{1}{N} \mathbb{E} \log \sum_{U_{M,N}'} W(\rho^1, \rho^2) \exp \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (z_i^\ell(\rho^\ell) + h_\ell) + o_M(1),$$

when $M \to \infty$. This is easy to show by interpolating between $Z_i^\ell$ and $z_i^\ell$ via

$$z_i^\ell (\rho^\ell, t) = \sum_{p \geq 1} a_p \left( \frac{t}{M^{(p-1)/2}} + \frac{1 - t}{(M + N)^{(p-1)/2}} \right) \sum_{i_1, \ldots, i_p \leq M} g_{i_1, \ldots, i_p}^i \rho_{i_1} \cdots \rho_{i_p}$$

and considering

$$\varphi(t) = \frac{1}{N} \mathbb{E} \log \sum_{U_{M,N}'} W(\rho^1, \rho^2) \exp \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (z_i^\ell(\rho^\ell, t) + h_\ell).$$

Then it is a straightforward calculation to show that $\varphi'(t) = o_M(1)$ uniformly for $t \in [0, 1]$. Thus, we finally get,

$$\frac{1}{N} \mathbb{E} \log Z_{M,N}(u_M, u_N) \geq \frac{1}{N} \mathbb{E} \log \sum_{U_{M,N}'} W(\rho^1, \rho^2) \exp \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (z_i^\ell(\rho^\ell) + h_\ell) - o_M(1)$$

$$= \frac{1}{N} \mathbb{E} \log \sum_{R_{1,2} = u_M} W(\rho^1, \rho^2) \sum_{R_{1,2} = u_N} \exp \sum_{\ell \leq 2} \sum_{i \leq N} \tau_i^\ell (z_i^\ell(\rho^\ell) + h_\ell) - o_M(1) \quad (3.12)$$

where $z_i^\ell(\rho^\ell)$ are defined in (3.10). Next, let us consider

$$Z_M(u_M) = \sum_{R_{1,2} = u_M} \exp \left( \sum_{\ell \leq 2} H_M^\ell (\rho^\ell) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M} \rho_i^\ell \right).$$

Comparing

$$H_M^\ell (\rho^\ell) = M^{1/2} \sum_{p \geq 1} \frac{a_p}{M^{p/2}} \sum_{i_1, \ldots, i_p \leq M} g_{i_1, \ldots, i_p}^i \rho_{i_1} \cdots \rho_{i_p},$$

13
with $H_{M+N}^f(\rho^f)$ in (3.7) we can write

$$H_M^f(\rho^f) \overset{d}{=} H_{M+N}^f(\rho^f) + \sqrt{N} Y^f(\rho^f)$$

(3.13)

where

$$Y^f(\rho^f) = \frac{1}{\sqrt{N}} \sum_{p=1}^{\infty} \left( \frac{1}{M^{p-1}} - \frac{1}{(M+N)^{p-1}} \right)^{1/2} a_p \sum_{i_1, \ldots, i_p \leq M} \tilde{g}_{i_1, \ldots, i_p} \rho_{i_1} \cdots \rho_{i_p},$$

where $(\tilde{g}_{i_1, \ldots, i_p})$ are i.i.d. Gaussian r.v independent of the Hamiltonians $H_{M+N}^f(\rho^f)$. Using (3.13), we can write

$$\frac{1}{N} \mathbb{E} \log Z_M(u_M) = \frac{1}{N} \mathbb{E} \log \sum_{R_{1,2} = u_M} \exp \left( \sum_{\ell \leq 2} H_{M+N}^f(\rho^f) + \sum_{\ell \leq 2} h_\ell \sum_{i \leq M} \rho_i^f + \sqrt{N} \sum_{\ell \leq 2} Y^f(\rho^f) \right)$$

$$= \frac{1}{N} \mathbb{E} \log \sum_{R_{1,2} = u_M} W(\rho^f, \rho^f^2) \exp \sqrt{N} \sum_{\ell \leq 2} Y^f(\rho^f).$$

(3.14)

It is easy to compute that the covariance operator of $(Y^f(\rho^f))$ satisfies

$$\mathbb{E} Y^f(\rho^f) Y^f(\rho^{f'}) = \theta_{f,f'}(R(\rho^f, \rho^{f'})) + o_M(1)$$

as $M \to \infty$. Therefore, as above one can substitute (up to a small error) the random variables $(Y^f(\rho^f))$ in (3.14) with the random variables

$$y^f(\rho^f) = \sum_{p=1}^{\infty} (p-1) a_p \sum_{i_1, \ldots, i_p \leq M} \tilde{g}_{i_1, \ldots, i_p} \rho_{i_1} \cdots \rho_{i_p},$$

(3.15)

with covariance operator

$$\mathbb{E} y^f(\rho^f) y^{f'}(\rho^{f'}) = \theta_{f,f'}(R(\rho^f, \rho^{f'})).$$

(3.16)

(3.14) then gives,

$$\frac{1}{N} \mathbb{E} \log Z_M(u_M) = \frac{1}{N} \mathbb{E} \log \sum_{R_{1,2} = u_M} W(\rho^f, \rho^f^2) \exp \sqrt{N} \sum_{\ell \leq 2} y^f(\rho^f) + o_M(1)$$

(3.17)

as $M \to \infty$. Plugging (3.12) and (3.17) into (3.2) and (3.4) we get

$$P(u) \geq \liminf_{M \to \infty} G_{M,N}$$

(3.18)

where

$$G_{M,N} = \frac{1}{N} \mathbb{E} \log \sum_{R_{1,2} = u_M} W(\rho^f, \rho^f^2) \sum_{R_{1,2} = u_N} \exp \sum_{\ell \leq 2} \sum_{i \leq N} t_{i,\ell}(z_i^f(\rho^f) + h_\ell)$$

$$- \frac{1}{N} \mathbb{E} \log \sum_{R_{1,2} = u_M} W(\rho^f, \rho^f^2) \exp \sqrt{N} \sum_{\ell \leq 2} y^f(\rho^f).$$

(3.19)
If we define \( \alpha = (\varrho^1, \varrho^2) \), define
\[
A = \{ (\varrho^1, \varrho^2) : R_{1,2}^{1} = R(\varrho^1, \varrho^2) = u_M \},
\]
(3.20)

let
\[
w_{\alpha} = W(\varrho^1, \varrho^2) / \sum_{R_{1,2}^1 = u_M} W(\varrho^1, \varrho^2),
\]
(3.21)

and let \( z_\ell^f(\alpha) = z_\ell^f(\varrho^f) \) and \( y^f(\alpha) = y^f(\varrho^f) \), then (3.19) can be rewritten as
\[
G_{M,N} = \frac{1}{N} \mathbb{E} \log \sum_{\alpha \in A} w_{\alpha} \sum_{R_{1,2}^1 = u_M^N} \exp \sum_{\ell \leq 2} \sum_{i \leq N} \tau_\ell(z_\ell^f(\alpha) + h_\ell)
\]
\[
- \frac{1}{N} \mathbb{E} \log \sum_{\alpha \in A} w_{\alpha} \exp \sqrt{N} \sum_{\ell \leq 2} y^f(\alpha).
\]
(3.22)

Clearly, \( G_{M,N} \) is written in the form of (3.13), i.e. \( G_{M,N} = G_N(u_N^\prime, \Omega_\delta) \), where the random overlap structure \( \Omega_\delta \) is the collection of (3.20), (3.21), (3.10) and (3.16). Equations (3.11) and (3.16) imply that the conditions (1) - (4) in the definition of ROSt are satisfied with \( q_{\alpha,\beta} = R(\varrho^\ell, \varrho^{\ell'}) \) where \( \beta = (\bar{\varrho}^1, \bar{\varrho}^2) \). Since \( q_{\alpha,\alpha}^{1,2} = R(\varrho^1, \varrho^2) = u_M \) for \( \alpha = (\varrho^1, \varrho^2) \in A \), we can take \( \delta = |u_M - u| \) which goes to 0 as \( M \to \infty \). Equation (3.18), therefore, implies
\[
P(u) \geq \lim_{\delta \to 0} \inf_{\Omega_\delta} G_N(u_N^\prime, \Omega_\delta).
\]

This finishes the proof of Theorem 3.

Lemma 3 and Theorem 3, of course, imply Theorem 2.

In conclusion, we would like to note that the analogue of the Aizenman-Sims-Starr variational principle is particularly interesting because of the specific representations of the random overlap structures (3.19) or (3.22). We hope that the analysis of these structures will direct toward what should be the Parisi ansatz for the limit \( P(u) \) in (1.9) or, at least, will provide some ideas in this direction.

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