Convolutional Codes with Optimum Bidirectional Distance Profile
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Abstract—In this paper we present tables of convolutional codes with the optimum bidirectional distance profile (OBDP), defined as the minimum of the distance profiles of the code and its corresponding "reverse" code. Such codes minimize the average complexity of bidirectional sequential decoding algorithms. The computer search is accelerated by the facts that optimum distance profile (ODP) codes of larger memory must have ODP codes of smaller memory as their "prefixes", and that OBDP codes can be obtained by "concatenating" ODP and reverse ODP codes of smaller memory.

Index Terms—Convolutional codes, distance profile, bidirectional decoding.

I. INTRODUCTION

Let

\[
G(D) = \begin{bmatrix}
g_{11}(D) & \cdots & g_{1n}(D) \\
\vdots & \ddots & \vdots \\
g_{k1}(D) & \cdots & g_{kn}(D)
\end{bmatrix}
\]

(1)

be the generator matrix of a rate \(k/n\) convolutional encoder, where \(g_{ij}(D) \in \mathbb{F}_2[D]\) are binary generator polynomials and let \(m = \max_{i,j} \deg(g_{ij}(D))\) be its memory.

The \(l\)th value of the column distance function (CDF) of the code \([1]\) is

\[
d_l = \min_{u\cdot v = 0} w_H(v_{[0,l]}),
\]

(2)

where \(w_H(\cdot)\) is the Hamming weight of a sequence, \(u = (u_0, u_1, \ldots)\) and \(v = (v_0, v_1, \ldots)\) are the sequences of \(k\)-dimensional information and \(n\)-dimensional code binary vectors, respectively, and for a sequence \(x, x_{[a,b]} = (x_a, x_{a+1}, \ldots, x_b)\). If \(d = (d_0, d_1, \ldots)\) and \(d' = (d_0', d_1', \ldots)\) are two CDFs and \(d_0 = d_0', d_1 = d_1', \ldots, d_l = d_l'\) for some \(l\), then we say that \(d\) is better than \(d'\), i.e., \(d > d'\). A code with an optimum CDF, \(d^*\), for given parameters \(k, n\) and \(m\), is one for which there is no other code with the same parameters whose CDF \(d > d^*\). The distance profile (DP) of a code \([1]\) is its truncated CDF, \(d_{[0,m]}\), with comparison and optimality defined in the same way.

For most often used sequential decoding algorithms, such as the stack algorithm \([2, 3]\), and the Fano algorithm \([4]\), optimum distance profile (ODP) codes are used in order to minimize the average number of code tree node extensions.

The influence of the CDF and DP on the decoding complexity has been studied when binary symmetric channel \([5]\) and additive white Gaussian noise channel \([6]\) are used. In both cases, it is desirable that values \(d_l\) grow as quickly as possible at the beginning of the CDF, hence the definition of its optimality, with its earlier values having a more profound influence. ODP codes and codes optimized with respect to other criteria have been investigated extensively, e.g., \([7]-[13]\).

Although the complexity of standard unidirectional decoding algorithms is minimized when ODP codes are used, that is not the case with bidirectional decoding algorithms, e.g., \([14]-[17]\). The number of visited nodes in the forward code tree depends on the DP of the original code, but in the backward code tree it depends on the DP of the “reverse” code.

II. BIDIRECTIONAL DISTANCE PROFILE

Let \(\bar{G}(D) = D^mG(D^{-1})\) be the generator matrix of the reverse code of the code defined by \(G(D)\), and let \(\bar{d} = (\bar{d}_0, \bar{d}_1, \ldots)\) be its CDF. In order to take into account both decoding directions, we define the bidirectional CDF (BCDF), \(d\), as the sequence of values \(d_l = \min(d_l, \bar{d}_l)\), and the bidirectional DP (BDP) as \(d_{[0,m]}\), as well as the corresponding optimum BCDF and BDP (OBCDF and OBDP) codes.

Let \(g_{ij}(D)\) denote the \(i\)th coefficient of the generator polynomial \(g_{ij}(D)\), and let

\[
G^{(l)} = \begin{bmatrix}
g_{11}^{(l)} & \cdots & g_{1n}^{(l)} \\
\vdots & \ddots & \vdots \\
g_{k1}^{(l)} & \cdots & g_{kn}^{(l)}
\end{bmatrix},
\]

(3)

and

\[
G^{[a,b]}(D) = \sum_{l=a}^{b} G^{(l)} D^l,
\]

(4)

\[
G^{[a,b]}(D) = \sum_{l=a}^{b} G^{(m-l)} D^l,
\]

(5)

so \(G(D) = G^{[0,m]}(D)\) and \(\bar{G}(D) = G^{[0,m]}(D)\). Also for \(G(D) = G^{[0,p]}(D) + G^{[p+1,m]}(D)\), we say that \(G^{[0,p]}(D)\) is a prefix of \(G(D)\), and that \(G(D)\) is obtained by concatenation of \(G^{[0,p]}(D)\) and \(G^{[p+1,m]}(D)\). It is easy to see that if the code defined by \(G(D)\) is ODP, then the code defined by \(G^{[0,p]}(D)\) must also be ODP.

If the code memory, \(m\), is odd, we can similarly decompose the generator matrix as \(G(D) = G^F(D) + D^{m} G^R(D^{-1})\), with \(G^F(D) = G^{[0,(m-1)/2]}(D)\) and \(G^B(D) = G^{[0,(m-1)/2]}(D)\). Here we call \(G^F(D)\) the forward half and \(G^B(D)\) the backward half of \(G(D)\), and we can express the reverse code generator...
matrix as $\tilde{G}(D) = G^B(D) + D^mG^F(D^{-1})$. From the definitions of the corresponding CDFs, we see that $d_{(0,m-1)/2}$ depends only on $G^F(D)$, and $d_{(0,m-1)/2}$ depends only on $G^B(D)$. Since $G^F(D)$ and $G^B(D)$ can be chosen independently, the first half of the BDP, $d_{(0,m-1)/2}$, can be made equal to that of an ODP code, if and only if both $G^F(D)$ and $G^B(D)$ define ODP codes of memory $(m-1)/2$.

If the code memory, $m$, is even, we can decompose the generator matrix as $G(D) = G^F(D) + G^B(\tilde{G}(D))$, where now $G^F(D) = G^{[0,m/2-1]}(D)$ and $G^B(D) = G^{[m/2,1]}(D)$. In this case $d_{(0,m-1)/2}$ can be optimized in the same way.

In order to describe the procedure for finding OBCDF and OBDP codes, a few definitions are necessary. In $\mathbb{F}_2$, let $0 < 1$. Binary polynomials are compared lexicographically, i.e., for $g(D), h(D) \in \mathbb{F}_2[D]$, let $g(D) < h(D)$ if $g^{(j)}(h^{(j)})$ for some $l$. Vectors of binary polynomials are compared lexicographically, i.e., for $x, y \in (\mathbb{F}_2[D])^n$, let $x < y$ if $x_1 = y_1, \ldots, x_{l-1} = y_{l-1}, x_l < y_l$ for some $l$. If $g_{(i)}(g_{(2)}), \ldots, g_{(n)}(g_{(2)})$ is the $i$th row $(j$th column) of a polynomial $k \times n$ matrix $G(D)$, we say that it has sorted rows (columns) if $g_{(i)} \leq \cdots \leq g_{(i)}(g_{(2)}), \ldots, g_{(n)}(g_{(2)})$. Let $\pi$ denote the set of all permutations of elements $\{1, \ldots, l\}$.

For given $k$ and $n$, let $U_m$ denote the set of ODP codes of memory $m$, and let $B_m$ denote the set of OBCDF (OBDP) codes of memory $m$. The OBCDF (OBDP) codes can be

| $m$ | $G^{OBCDF}(D)$ | $d_{\infty}$ |
|-----|----------------|--------------|
| 1   | 1 6 3          | 3            |
| 2   | 5 7 5          | 5            |
| 3   | 54 64 6       | 6            |
| 4   | 46 62 6       | 6            |
| 5   | 57 75 8       | 8            |
| 6   | 564 774 8     | 8            |
| 7   | 452 756 10    | 10           |
| 8   | 477 635 10    | 10           |
| 9   | 5414 6006 10  | 10           |
| 10  | 4522 6006 9   | 9            |
| 11  | 4417 6171 12  | 12           |
| 12  | 5446 60014 11 | 11           |
| 13  | 57276 76572 12| 12           |
| 14  | 40375 71637 12| 12           |

| $m$ | $G^{OBDP}(D)$ | $d_{\infty}$ |
|-----|----------------|--------------|
| 1   | 2 4 6          | 4            |
| 2   | 5 7 7          | 7            |
| 3   | 44 54 74       | 9            |
| 4   | 52 66 76       | 12           |
| 5   | 45 51 77       | 12           |
| 6   | 434 564 704    | 13           |
| 7   | 446 616 722    | 14           |
| 8   | 533 575 665    | 15           |
| 9   | 4674 6754 7544 | 15           |
| 10  | 5772 6056 7296 | 18           |
| 11  | 4135 5057 7263 | 23           |
| 12  | 51624 66234 71154| 22          |
| 13  | 59256 65126 72552| 20          |
| 14  | 40701 53765 67273| 22          |
| 15  | 403402 517712 703156| 22        |
| 16  | 421765 531607 706321| 29        |
| 17  | 4303404 5060254 6501424| 23      |
| 18  | 4763236 6143606 7454762| 26      |
| 19  | 5704623 6231075 7432617| 31      |
| 20  | 65342714 55412944 62027664| 32     |
| 21  | 40666602 53634752 67375666| 34     |
| 22  | 51275623 66500617 71740675| 37     |
| 23  | 551571614 616366264 770378264| 34    |
| 24  | 527061652 641577756 737773026| 36    |

TABLE II

| $m$ | $G^{OBCDF}(D)$ | $d_{\infty}$ |
|-----|----------------|--------------|
| 1   | 2 4 6          | 4            |
| 2   | 5 7 7          | 7            |
| 3   | 44 54 74       | 9            |
| 4   | 52 66 76       | 12           |
| 5   | 45 51 77       | 12           |
| 6   | 434 564 704    | 13           |
| 7   | 446 616 722    | 14           |
| 8   | 533 575 665    | 15           |
| 9   | 4674 6754 7544 | 15           |
| 10  | 5772 6056 7296 | 18           |
| 11  | 4135 5057 7263 | 23           |
| 12  | 51624 66234 71154| 22          |
| 13  | 59256 65126 72552| 20          |
| 14  | 40701 53765 67273| 22          |
| 15  | 403402 517712 703156| 22        |
| 16  | 421765 531607 706321| 29        |
| 17  | 4303404 5060254 6501424| 23      |
| 18  | 4763236 6143606 7454762| 26      |
| 19  | 5704623 6231075 7432617| 31      |
| 20  | 65342714 55412944 62027664| 32     |
| 21  | 40666602 53634752 67375666| 34     |
| 22  | 51275623 66500617 71740675| 37     |
| 23  | 551571614 616366264 770378264| 34    |
| 24  | 527061652 641577756 737773026| 36    |
TABLE III

OBCDF and OBDP codes, $R = 1/4$

| $m$ | $G_{OBCDF}(D)$ | $d_\infty$ |
|-----|----------------|------------|
|  1  | 2  4  6  6  6 | 1          |
|  2  | 5  5  6  6  10| 2          |
|  3  | 44 54 64 74 12| 3          |
|  4  | 46 50 62 72 14| 4          |
|  5  | 45 51 67 73 16| 5          |
|  6  | 434 564 614 704 17| 6        |
|  7  | 406 536 602 752 18| 7        |
|  8  | 471 525 603 727 21| 8        |
|  9  | 4314 5704 6174 7024 22| 9        |
| 10  | 4102 5756 6106 7372 24| 10       |
| 11  | 4633 5647 6631 7135 30| 11       |
| 12  | 41204 52524 62074 74114 35| 12       |
| 13  | 47516 57666 66772 71362 32| 13       |
| 14  | 41057 52225 60503 75041 27| 14       |
| 15  | 435314 503024 632704 760174 34| 15       |
| 16  | 467516 545322 661066 713662 38| 16       |
| 17  | 442753 564627 657211 723135 40| 17       |
| 18  | 4665544 5440434 6604154 7121234 38| 18       |
| 19  | 4733366 5746156 6755652 7306372 38| 19       |
| 20  | 4502051 5535655 6465507 7055313 45| 20       |
| 21  | 41256354 52575164 62006044 74145334 44| 21       |
| 22  | 40153002 51135112 63027276 76564146 46| 22       |
| 23  | 47015547 57277275 66030633 71554071 44| 23       |
| 24  | 43316304 52546524 61675614 74616147 48| 24       |
| 25  | 422044512 512204242 605716076 7607178206 48| 25       |
| 26  | 404671717 510005045 621451423 747473101 50| 26       |

TABLE IV

OBCDF and OBDP codes, $R = 2/3$

| $m$ | $G_{OBCDF}(D)$ | $d_\infty$ |
|-----|----------------|------------|
|  1  | 2  6  6  3 | 1          |
|  2  | 5  7  2  5 | 2          |
|  3  | 04 60 74 6 5 6 | 3          |
|  4  | 00 46 62 6 6 | 4          |
|  5  | 05 40 73 6 6 | 5          |
|  6  | 024 664 770 7 | 6          |
|  7  | 130 482 642 7 | 7          |
|  8  | 020 673 757 7 | 8          |

TABLE V

OBCDF and OBDP codes, $R = 2/4$

| $m$ | $G_{OBCDF}(D)$ | $d_\infty$ |
|-----|----------------|------------|
|  1  | 0  2  6  6 | 5          |
|  2  | 1  3  4  6 | 6          |
|  3  | 14 34 54 74 | 8          |
|  4  | 00 44 26 46 | 9          |
|  5  | 04 35 52 77 | 12         |
|  6  | 064 540 654 210 | 13         |
|  7  | 024 226 540 632 | 14         |

Obtained using the following two-stage procedure:

1) Find all ODP codes for the desired values of $m$.
   1. Set $U_{-1} = \{0_{k \times n}\}$.
      Set $m = 0$.
   2. For all prefixes $G(D) \in U_{m-1}$:
      For all binary $k \times n$ matrices $G(m)$,
      Form $G(D) = G'(D) + G(m)D^m$.
      If $G(D)$ has sorted rows and columns, calculate its DP.
      Retain in $U_m$ only the matrices whose DPs are
   the best of all the calculated ones.
   3. If more sets are needed, set $m = m + 1$ and go to
      step 2.

II) Find all OBCDF (OBDP) codes for the desired values
    of $m$.
    1. Set $m = 1$.
    2. Set $B_m = \{\}$.
       Set $p = \lfloor (m - 1)/2 \rfloor$.
In general, for given $k$, $n$, and $m$, OBCDF and OBDP codes are not unique. In order to optimize their bit error rate performance, additional selection is performed with respect to their information distance spectra, $c = (c_1, c_2, \ldots)$, where $c_d$ is the sum of Hamming weights of information sequences of code trellis paths which start in the zero state, depart from it at the beginning, return to the zero state only at their termination, and whose codewords have Hamming weight $d$. As before, we define $c' < c$ if $c_1 < c_1'$, $\ldots$, $c_{l-1} = c_{l-1}'$, $c_l < c_l'$ for some $l$, and we search for the lowest $c$. 

In tables [VI] generator matrices of OBCDF and OBDP codes for rates $1/2$, $1/3$, $1/4$, $2/3$, $2/4$, and $3/4$ are given in octal notation (using the convention in [7], i.e., left-aligned), along with their free distances ($d_\infty$) and the indications when OBCDF and OBDP codes differ. All the codes have optimized information distance spectra. The matrices are unique up to their reversal and permutations of their rows or columns, except for the cases $R = 2/4$, $m = 4$, OBDP, and $R = 2/4$, $m = 6$, OBCDF, where the codes found have identical information distance spectra, but their generators are nontrivially different.

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