On computational complexity of length embeddability of graphs

Mikhail Tikhomirov

Abstract

A graph $G$ is embeddable in $\mathbb{R}^d$ if vertices of $G$ can be assigned with points of $\mathbb{R}^d$ in such a way that all pairs of adjacent vertices are at the distance 1. We show that verifying embeddability of a given graph in $\mathbb{R}^d$ is NP-hard in the case $d > 2$ for all reasonable notions of embeddability.

1 Introduction

The distance graph of $S \subset \mathbb{R}^d$ is defined as the graph $G = (V, E)$, where $V = S$ and $E$ is the set of all pairs of points $x, y \in S$ such that $x$ and $y$ are at the distance 1. A graph is a distance graph in $\mathbb{R}^d$ if it is isomorphic to the distance graph of some set $S \subset \mathbb{R}^d$. Some famous problems concerning distance graphs are the Erdős’ unit distance problem on the maximal number of unit distances between $n$ points in $\mathbb{R}^2$ (see [1], [2], [3]), the Hadwiger–Nelson problem on the chromatic number of $\mathbb{R}^2$ (see [1], [4], [5]), etc.; surveys of various results about distance graphs can be found at [6], [7].

We also consider a similar notion of embeddability in $\mathbb{R}^d$ (see, e.g., [8]). A graph $G = (V, E)$ is embeddable in $\mathbb{R}^d$ if there exists a mapping $\varphi : V \to \mathbb{R}^d$ such that $||\varphi(u) - \varphi(v)||_{\mathbb{R}^d} = 1$ for all pairs $(u, v) \in E$. It is clear that any distance graph in $\mathbb{R}^d$ is embeddable in $\mathbb{R}^d$ but the converse does not always hold. These two notions differ in the following:

- Different vertices of an embeddable graph may be assigned with the same point in $\mathbb{R}^d$ while all vertices of a distance graph should be assigned with pairwise distinct points.

- Non-adjacent vertices of an embeddable graph can be located at the distance 1 while non-adjacent vertices of a distance graph are forbidden to be placed at distance 1.

We will say that an embedding $\varphi : V \to \mathbb{R}^d$ is strict if $\forall u, v \in V (u, v) \in E \iff ||\varphi(u) - \varphi(v)||_{\mathbb{R}^d} = 1$; we will say that an embedding $\varphi : V \to \mathbb{R}^d$ is injective if $\forall u, v \in V v \neq u \Rightarrow \varphi(v) \neq \varphi(u)$. It is clear that a graph $G$ is a distance graph in $\mathbb{R}^d$ iff there exists a strict and injective embedding of $G$ in $\mathbb{R}^d$. Thus we obtain four different notions of embeddability (strict/non-strict, injective/non-injective) which include two notions described above.
For each of the four notions of embeddability in \( \mathbb{R}^d \) we can pose the computational decision problem of determining embeddability of chosen type for the given graph; we shall call this problem \( \mathbb{R}^d \)-UNIT-DISTANCE-(STRICT)-(INJECTIVE)-EMBEDDABILITY depending on the embeddability type. The computational complexity of these problems is studied in [8], [9]. In [9] it is shown that \( \mathbb{R}^d \)-UNIT-DISTANCE-(STRICT)-(INJECTIVE)-EMBEDDABILITY is NP-hard for each type of embeddability and each value of \( d \geq 2 \). Unfortunately, the proof in [9] for the case \( d > 2 \) is false as it is based on the result [10] due to Lovász which states the upper bound \( d + 1 \) for the chromatic number of the \( d \)-dimensional sphere circumscribed about a regular simplex on \( d + 1 \) vertices with unit length edges. In [11], [12] Raigorodskii points out that this bound is wrong and proves an exponential lower bound of this value; thus a new proof is needed for the case \( d > 2 \), which is the point of this paper.

The main result is

**Theorem 1.** Computational problems \( \mathbb{R}^d \)-UNIT-DISTANCE-EMBEDDABILITY, \( \mathbb{R}^d \)-UNIT-DISTANCE-STRIC-EMBEDDABILITY, \( \mathbb{R}^d \)-UNIT-DISTANCE-INJECTIVE-EMBEDDABILITY, \( \mathbb{R}^d \)-UNIT-DISTANCE-STRIC-INJECTIVE-EMBEDDABILITY are NP-hard for each \( d > 2 \).

To prove this result we construct a reduction of the classic NP-complete problem of graph vertex 3-coloring (3-COLORING) (see [13]) to each of the four embeddability problems: for any given graph \( G \) we explicitly construct a graph \( H = 3 \)-COLORING-\( \mathbb{R}^d \)-UNIT-DISTANCE-EMBEDDABILITY-REDUCTION(\( G \)) such that the size of \( H \) is linear in the size of \( G \) (for every fixed \( d \)) and the following conditions hold:

- If no valid vertex 3-coloring of \( G \) exists, then there is no embedding of \( H \) in \( \mathbb{R}^d \);
- If a valid vertex 3-coloring of \( G \) exists, then there is a strict injective embedding of \( H \) in \( \mathbb{R}^d \).

The possibility of such construction implies NP-hardness of all four mentioned problems. It should be mentioned that the question whether the described problems lie in NP is open.

## 2 Notion of rod

Let us introduce some necessary definitions.

A *weighted graph* \( G = (V, E, w) \) is an ordered triple such that \( (V, E) \) is a graph and \( w : E \to \mathbb{R}_+ \) is a function that assigns a positive number to each element of \( E \); for every edge \( e \in E \) we will say that \( w(e) \) is the *length* of the edge \( e \). If \( w \equiv 1 \), the weighted graph \( G \) is called a *unit distance graph*. A *length embedding* (or, more simply, an *embedding*) of the weighted graph \( G = (V, E, w) \) in \( \mathbb{R}^d \) is a map \( \varphi : V \to \mathbb{R}^d \) such that \( \forall u, v \in V \ (u, v) \in E \Rightarrow ||\varphi(u) - \varphi(v)||_{\mathbb{R}^d} = w((u, v)) \).

**Remark:** In the sequel, we will identify vertices of the graph with points of \( \mathbb{R}^d \) — their images under the embedding if that doesn’t cause confusion.

An embedding \( \varphi \) of the weighted graph \( G = (V, E, w) \) in \( \mathbb{R}^d \) is called *non-critical* if the following conditions hold:
Lemma 1. Let \( G = (V_G, E_G, w_G) \), \( H = (V_H, E_H, w_H) \) be weighted graphs. Suppose \( V_G \cap V_H = \{u, v\} \), \( e = (u, v) \in E(G) \), \( w_G(e) = l \), and \( H \) is a \((u, v)\)-rod of length \( l \). Let \( G' = (V_{G'}, E_{G'}, w_{G'}) \), where \( V_{G'} = V_G \cup V_H \), \( E_{G'} = (E_G \setminus \{e\}) \cup E_H \), \( w_{G'} = w_G(E_G \setminus \{e\}) + w_H(E_H) \) (informally, we replace the edge \( e \) in \( G \) by the subgraph \( H \) to obtain \( G' \)). Then:

- If there is no embedding of \( G \) in \( \mathbb{R}^d \), then there is no embedding of \( G' \) in \( \mathbb{R}^d \).
- If there exists a non-critical embedding of \( G \) in \( \mathbb{R}^d \), then there exists a non-critical embedding of \( G' \) in \( \mathbb{R}^d \).

Proof. In any embedding of \( G' \) the distance between vertices \( u \) and \( v \) is equal to \( l \). Suppose we have an embedding of \( G' \); we can erase all vertices outside \( V_G \) to obtain an embedding of \( G \). The first claim is thus proven.

Now consider a non-critical embedding \( \varphi_G \) of the weighted graph \( G \) in \( \mathbb{R}^d \). Construct an embedding \( \varphi_{G'} \) of \( G' \) as follows:

- Let \( \varphi_{G'}(x) = \varphi_G(x) \) for all \( x \in V_G \);
- Choose a non-critical embedding \( \varphi_H \) of the weighted graph \( H \) such that \( \varphi_H(u) = \varphi_G(u) \), \( \varphi_H(v) = \varphi_G(v) \) (such embedding exists since \( ||\varphi_G(u) - \varphi_G(v)|| = l \) and \( H \) is a \((u, v)\)-rod of length \( l \); let \( \varphi_{G'}(y) = \varphi_H(y) \) for all \( y \in V_H \).
It is clear that this definition of $\varphi_{G'}$ is consistent. However, it is possible that $\varphi_{G'}$ is not a non-critical embedding. Note that no vertex of $V_{G'} \setminus \{u, v\}$ lies on the straight line $uv$ since the embeddings $\varphi_G$ and $\varphi_H$ are non-critical.

Let $S$ denote the set of all rotations of $\mathbb{R}^d$ about the line $uv$. $S$ is isomorphic to the $(d - 2)$-dimensional sphere (each rotation can be assigned with the image of some point which doesn’t lie on $uv$). For any $\psi \in S$ let $\psi \ast_H \varphi_{G'}$ denote the mapping from $V_{G'}$ in $\mathbb{R}^d$ such that $\psi \ast_H \varphi_{G'}(x) = \varphi_{G'}(x)$ for every $x \in V_G$ and $\psi \ast_H \varphi_{G'}(y) = \psi(\varphi_{G'}(y))$ for every $y \in V_H$; clearly, this definition is consistent. It is also clear that for every rotation $\psi \in S$ the mapping $\psi \ast_H \varphi_{G'}$ is an embedding of $G'$ in $\mathbb{R}^d$.

We now show that there exists a rotation $\psi \in S$ such that $\psi \ast_H \varphi_{G'}$ is a non-critical embedding of $G'$ in $\mathbb{R}^d$. Consider all $\psi \in S$ such that the embedding $\psi \ast_H \varphi_{G'}$ is not non-critical for some reason. In that case, one of the following conditions must hold:

- The embedding $\psi \ast_H \varphi_{G'}$ places two vertices of $G'$ (denote them $x$ and $y$) at the same point. It follows from the non-criticality of $\varphi_G$ and $\varphi_H$ that $x$ and $y$ cannot lie both in $V_G$ or both in $V_H$. Thus WLOG $x \in V_G \setminus \{u, v\}, y \in V_H \setminus \{u, v\}$.

  The vertex $y$ does not lie on the line $uv$ and no two rotations place $y$ at the same point. Therefore for every pair of vertices $x, y$ there is at most one rotation $\psi \in S$ that superposes $x$ and $\psi(y)$, thus the set of all rotations $\psi$ such that the embedding $\psi \ast_H \varphi_{G'}$ places some two vertices in the same point is finite and its spherical measure in $S$ is zero.

- The embedding $\psi \ast_H \varphi_{G'}$ places two non-adjacent vertices of $G'$ (denote them $x$ and $y$ once more) at the distance 1. Once again, $x, y \in V_G$ or $x, y \in V_H$ leads to a contradiction; thus WLOG $x \in V_G \setminus \{u, v\}, y \in V_H \setminus \{u, v\}$.

  Let $P_y$ denote the $(d - 2)$-dimensional sphere — the locus of the point $\psi(y)$ for all $\psi \in S$; the radius of $P_y$ is non-zero since $y$ does not lie on the line $uv$. If $||x - \psi(y)||_{\mathbb{R}^d} = 1$, then $\psi(y)$ lies on the $(d - 1)$-dimensional sphere of radius 1 centered at $x$; denote it $P_x$. We assume that the intersection of $P_x$ and $P_y$ is not empty.

  If $P_x$ contains $P_y$ as a subset, then $x$ must lie on the line $uv$; that would contradict the non-criticality of $\varphi_G$. Otherwise, the intersection of $P_x$ and $P_y$ is a $(d - 3)$-dimensional sphere (possibly, of zero radius).

  In any case, the set of rotations that place $x$ and $\psi(y)$ at the distance 1 has zero measure in $S$. Thus the set of rotations that place some two non-adjacent vertices at the distance 1 has zero measure in $S$.

- The embedding $\psi \ast_H \varphi_{G'}$ places some three vertices on a straight line; denote these vertices $x, y, z$. Similarly to previous cases, if we assume $x, y, z \in V_G$ or $V_H$ we arrive at a contradiction.

  WLOG, let $x, y \in V_G \setminus \{u, v\}, z \in V_H \setminus \{u, v\}$. Since the point $z$ can not lie on the line $uv$, the sphere $\psi(z)$ for $\psi \in S$ has non-zero radius and the line $xy$ passes through the point $\psi(z)$ for at most two values of $\psi$.

  Now let $x \in V_G \setminus \{u, v\}$ and $y, z \in V_H \setminus \{u, v\}$. The rotation $\psi \in S$ places the point $x$ on the line $\psi(yz)$ iff the point $\psi^{-1}(x)$ lies on the line $yz$ (here $\psi^{-1}$ means
the inverse rotation of $\psi$), therefore in this case the line $yz$ must cross the locus of $\psi^{-1}(x)$ for all $\psi \in S$. Clearly, the locus is a sphere of non-zero radius, thus line $\psi(yz)$ passes through the point $x$ for at most two values of $\psi$.

It follows from the above that the set of rotations $\psi \in S$ such that $\psi \ast_H \varphi_{G'}$ places some three vertices on a straight line is finite.

To sum up, the set of rotations $\psi$ such that the embedding $\psi \ast_H \varphi_{G'}$ is not non-critical has zero measure in the $(d-2)$-dimensional sphere of all possible rotations about the line $uv$. Therefore almost every rotation $\psi \in S$ yields a non-critical embedding $\psi \ast_H \varphi_{G'}$ of the graph $G'$ in $\mathbb{R}^d$.

\[\square\]

3 Construction of rods

Let $h = \sqrt{\frac{d+1}{2d}}$ denote the altitude length of a regular $d$-dimensional simplex with the edge length 1; denote $D = 2h$. Clearly, $D > \sqrt{2}$.

**Lemma 2.** Let $G$ and $H$ be unit distance rods of length $a$ and $b$ respectively. Then there exists a unit distance rod of length $ab$.

**Proof.** It suffices to make lengths of all edges of $G$ be equal to $b$ and successively apply Lemma 1 to every edge of the resulting graph and the graph $H$. \[\square\]

Consider a graph $M_d$ on a set of vertices $V_{M_d} = K_1 \cup K_2 \cup \{A, B, C\}$, $|K_1| = |K_2| = d$. Add the following edges of unit length to $M_d$:

- make cliques on $K_1$ and $K_2$;
- connect the vertices $A$ and $B$ with every vertex of $K_1$;
- connect the vertices $A$ and $C$ with every vertex of $K_2$;
- finally, connect the vertices $B$ and $C$.

The graph $M_d$ is called a $d$-dimensional Moser spindle (the figure 1 illustrates a 5-dimensional Moser spindle). Is it easy to see that $M_d$ is a unit distance $d$-dimensional $(A, B)$-rod of length $D$. Repeatedly applying Lemma 2 to copies of $M_d$, we arrive at

**Corollary 1.** For every non-negative integer $k$ there exists a unit distance $d$-dimensional rod of length $D^k$.

**Lemma 3.** For all numbers $a, b$ such that $0 < a < b < 1$ there exists a number $l$ satisfying $a < l < b$ and a graph $G$ such that $G$ is a unit-distance $d$-dimensional rod of length $l$. 
Figure 1: 5-dimensional Moser spindle

Proof. Construct $G$ as follows. Choose a set of vertices $K$ of size $d - 1$ and connect its elements pairwise by unit length edges. Then, take a sequence of vertices $v_1, \ldots, v_n$ (the exact number of vertices $n$ will be determined later) and connect every vertex of the sequence $v_i$ with every vertex of $K$ by a unit length edge. If the location of vertices of $K$ is fixed, then all vertices $v_1, \ldots, v_n$ must lie on some circle centered at $O$, where $O$ is the center of the regular simplex with vertices in $K$. The radius of the circle is equal to the altitude length of the $(d - 1)$-face of the regular $d$-simplex with the side length 1, i.e. $\sqrt{\frac{d}{2(d-1)}} = r$. Let $\pi$ denote the plane containing this circle.

For each $i$ from 1 to $n - 1$ connect the vertices $v_i$ and $v_{i+1}$ by an edge of length 1; also for each $i$ from 1 to $n - 2$ connect the vertices $v_i$ and $v_{i+2}$ by an edge of length $D$. Now in every embedding of the graph $G$ the angle $\angle v_i O v_{i+1}$ is equal to the dihedral angle of a regular $d$-simplex; denote this angle $\alpha = \arccos \frac{1}{d}$. Additionally, the least rotation of the plane $\pi$ about the point $O$ that moves the point $v_i$ to $v_{i+1}$ has the same direction for every $i$. It is clear that no three vertices of $G$ lie on a straight line.

Let us introduce an angular coordinate system $\psi$ on $\pi$ centered at $O$ such that $\psi(v_1) = 0$, $\psi(v_2) = \alpha$. Clearly, $\psi(v_i) = (i - 1)\alpha \mod 2\pi$ (by $\alpha \mod 2\pi$ we mean $\alpha + k \times 2\pi$ for an integer $k$ such that $0 \leq \alpha + k \times 2\pi < 2\pi$). By Niven’s theorem (see [14], Corollary 3.12), $\alpha/2\pi$ can not be a rational number when $d \geq 3$, therefore the infinite sequence $x_i = (i - 1)\alpha \mod 2\pi$ is dense in $[0; 2\pi]$. Thus there exists a positive integer $N$ such that $x_N \in (2 \arcsin \frac{a}{2R}; 2 \arcsin \frac{b}{2R})$ and $||v_1 - v_N|| \in (a; b)$.

It follows from the above that the graph $G$ is a $d$-dimensional $(v_1, v_N)$-rod. Finally, successively apply Lemma 2 to each $D$-length edge of the graph $G$ and the graph $M_d$; the resulting graph is a unit distance $d$-dimensional $(v_1, v_N)$-rod that satisfies all the conditions.

\[\square\]

Theorem 2. For all numbers $a, b$ such that $0 < a < b$ there exists a number $l$ satisfying $a < l < b$ and a graph $G$ such that $G$ is a unit-distance $d$-dimensional rod of length $l$.

Proof. Choose a non-negative integer $k$ such that $D^k > b$ and denote $G'$ the rod obtained by applying Lemma 3 for numbers $\frac{a}{D^k}$ and $\frac{b}{D^k}$. Now apply Lemma 2 to the graph $G'$ and the rod of length $D^k$. 

6
Let RodLength($a, b$) denote the number $l$ produced by Theorem 2 for given numbers $a$ and $b$, and Rod($a, b$) denote the rod of corresponding length.

4 The reduction setup

Consider a graph $G = (V_G, E_G)$ — the input of the 3-COLORING problem. We now construct a weighted graph $H = (V_H, E_H, w_H) = 3$-COLORING-$\mathbb{R}^d$-EMBEDDABILITY-REDUCTION($G$) such that the embeddability of $H$ in $\mathbb{R}^d$ is equivalent to the existence of a solution to the 3-COLORING for the graph $G$. We shall identify the elements of $V_G$ and the integers from 1 to $|V_G|$ for the sake of convenience.

To establish properties of the following setup we will need the following Lemma 4.

Let $0 \leq l < L \leq R < r$, $\delta = \min(L-l, r-R)$. Let also $G = (V_G, E_G, w_G)$, $H = (V_H, E_H, w_H)$ — weighted graphs, $v, u \in V_G$, $V_H = V_G \cup \{z\}$, $E_H = E_G \cup \{(v, z), (u, z)\}$, $w_H(e) = w_G(e)$ for all $e \in E_G$, $w_H((v, z)) = a \in \left[\frac{L+R}{2} - \frac{\delta}{3}; \frac{L+R}{2} + \frac{\delta}{3}\right)$, $w_H((u, z)) = b \in \left[\frac{R-L}{2} + \frac{\delta}{3}; \frac{R-L}{2} + \frac{\delta}{2}\right)$.

Then:

• In every embedding of the graph $H$ the inequalities $l < \|v-u\| < r$ hold.

• If there exists a non-critical embedding of $G$ such that $L \leq \|v-u\| \leq R$, then there exists a non-critical embedding of $H$.

Proof. First of all, let us show that $l < a - b < L \leq R < a + b < r$. Indeed,

\[ a - b > \left(\frac{L + R}{2} - \frac{\delta}{3}\right) - \left(\frac{R - L}{2} + \frac{\delta}{2}\right) = L - \delta \geq l; \]

\[ a - b < \left(\frac{L + R}{2} + \frac{\delta}{3}\right) - \left(\frac{R - L}{2} + \frac{\delta}{3}\right) = L; \]

\[ a + b > \left(\frac{L + R}{2} - \frac{\delta}{3}\right) - \left(\frac{R - L}{2} + \frac{\delta}{3}\right) = R; \]

\[ a + b < \left(\frac{L + R}{2} + \frac{\delta}{3}\right) + \left(\frac{R - L}{2} + \frac{\delta}{2}\right) = R + \delta \leq r. \]

Consider any embedding of the graph $H$. It follows from the triangle inequality applied to vertices $v, z, u$ that $l < a - b \leq \|v-u\| \leq a + b < r$. The first claim is thus proven.

Now, consider a non-critical embedding of the graph $G$ such that $\|u-v\| \in [L; R]$. It follows from $a - b < L \leq \|u-v\| \leq R < a + b$ that it is possible to place the vertex $z$ in such a way that $\|v-z\| = a$, $\|u-z\| = b$ and $z$ does not lie on the line $vu$. We have obtained an embedding of the graph $H$; it is possible to modify this embedding to obtain a non-critical embedding by choosing an appropriate rotation of $z$ about the
line \( uu \); the proof of the existence of such rotation copies the similar proof from Lemma 1 almost entirely.

Denote \( r_0 = \sqrt{\frac{d}{2(d-1)}} \), \( \text{chord}(\alpha) = 2r_0 \sin \alpha/2 \) — the length of the chord which contracts an \( \alpha \)-measured arc of a circle of radius \( r_0, \varepsilon = \frac{\pi}{24} \).

Let us introduce additional notation as follows:
\[
\begin{align*}
\delta_{uu} & = \min(\text{chord}(2\pi/3) - \text{chord}(2\pi/3 - \varepsilon/2), \text{chord}(2\pi/3 + \varepsilon/2) - \text{chord}(2\pi/3)), \\
a_{uu} & = \text{RodLength}(\text{chord}(2\pi/3) - \delta_{uu}/3, \text{chord}(2\pi/3) + \delta_{uu}/3), \\
b_{uu} & = \text{RodLength}(\delta_{uu}/3, \delta_{uu}/2), \\
\delta_{uv} & = \text{chord}(\pi/3 - \varepsilon/2) - \text{chord}(\pi/3 - \varepsilon), \\
a_{uv} & = \text{RodLength}\left(\frac{2r_0 + \text{chord}(\pi/3 - \varepsilon/2)}{2} - \delta_{uv}/3, \frac{2r_0 + \text{chord}(\pi/3 - \varepsilon/2)}{2} + \delta_{uv}/3\right), \\
b_{uv} & = \text{RodLength}\left(\frac{2r_0 - \text{chord}(\pi/3 - \varepsilon/2)}{2} + \delta_{uv}/3, \frac{2r_0 - \text{chord}(\pi/3 - \varepsilon/2)}{2} + \delta_{uv}/2\right), \\
\delta_{vv} & = \text{chord}\left(\frac{2\pi}{3} - \varepsilon\right) - \text{chord}\left(\frac{5\pi}{6}\varepsilon\right), \\
a_{vv} & = \text{RodLength}\left(\frac{2r_0 + \text{chord}(2\pi/3 - \varepsilon)}{2} - \delta_{vv}/3, \frac{2r_0 + \text{chord}(2\pi/3 - \varepsilon)}{2} + \delta_{vv}/3\right), \\
b_{vv} & = \text{RodLength}\left(\frac{2r_0 - \text{chord}(2\pi/3 - \varepsilon)}{2} + \delta_{vv}/3, \frac{2r_0 - \text{chord}(2\pi/3 - \varepsilon)}{2} + \delta_{vv}/2\right).
\end{align*}
\]

Construct \( H = 3\text{-COLORING-R}^d\text{-EMBEDDABILITY-REDUCTION}(G) \) as follows:

\[
V_H = K \cup U \cup V \cup Aux, E_H = E_K \cup E_{KV} \cup E_U \cup E_{VU} \cup E_V.
\]

Here:

- \( Aux \) is the set of all auxiliary vertices used in the sequel of the description (\( aux \)...);
- \( K \) — the set of vertices of size \( d - 1 \);
- \( E_K \) — the set of edges connecting all pairs of vertices of \( K \);
- \( U = \{u_0, u_1, u_2\} \);
- \( E_{KV} \) — the set of edges connecting every vertex of \( U \) with every vertex of \( K \);
- \( \{u_0, aux_{u_0,u_1}, (aux_{u_0,u_1}, u_1), (u_0, aux_{u_0,u_2}), (aux_{u_0,u_2}, u_2), (u_1, aux_{u_1,u_2}), (aux_{u_1,u_2}, u_2)\};
- \( V = \{v_1, ..., v_{|V|}\} \);
- \( E_{KV} \) — the set of edges connecting every vertex of \( V \) with every vertex of \( K \);
- \( E_{U} = \bigcup_{v \in V} \bigcup_{u \in U} \{v, aux_{v,u}, (aux_{v,u}, u)\};
- \( E_{V} = \bigcup_{v_i, v_j \in E_{V}} \{v_i, aux_{v_i,v_j}, (aux_{v_i,v_j}, v_j)\}.$

The edge lengths are assigned as follows:

- \( e \in E_K \cup E_{KV} \cup E_{KV} \Rightarrow w_H(e) = 1; \)
- \( w_H(u_0, aux_{u_0,u_1}) = w_H(u_0, aux_{u_0,u_2}) = w_H(u_1, aux_{u_1,u_2}) = a_{uu}, \)
- \( w_H(aux_{u_0,u_1}, u_1) = w_H(aux_{u_0,u_2}, u_2) = w_H(aux_{u_1,u_2}, u_2) = b_{uu}; \)
Consider an embedding of the graph \( G \) of vertices of \( \text{REDUCTION} \). Let both vertices \( v \) points \( c \) arc between \( r \) some circle of radius \( r \) between \( \pi/3 \) and \( \pi/2 \); denote this circle \( \rho \) and its center \( O \).

Successively apply the first part of Lemma 4 with the following parameters.

Let \( v = u_0, u = u_1, z = aux_{u_0,u_1}, l = \text{chord}(2\pi/3 - \epsilon/2), L = R = \text{chord}(2\pi/3), r = \text{chord}(2\pi/3 + \epsilon/2). \)

We obtain that chord\((2\pi/3 - \epsilon/2) < ||u_0 - u_1|| > \text{chord}(2\pi/3 + \epsilon/2)\), which is equivalent to \( 2\pi/3 - \epsilon/2 < \angle u_0Ou_1 < 2\pi/3 + \epsilon/2 \). We can establish similar inequalities for \( u_0, u_2 \) and \( u_1, u_2 \).

Let \( v \in V, u \in U, z = aux_{v,u}, l = \text{chord}(\pi/3 - \epsilon), L = \text{chord}(\pi/3 - \epsilon/2), R = 2r_0, r = \infty. \)

Then chord\((\pi/3 - \epsilon) < ||v - u||\), which is equivalent to \( \pi/3 - \epsilon < \angle vOu \).

Let \( v_i, v_j \in V, (i, j) \in E_G: v = v_i, u = v_j, z = aux_{v_i,v_j}, l = \text{chord}\left(\frac{\epsilon}{2}\right), L = \text{chord}(2\pi/3 - \epsilon), R = 2r_0, r = \infty. \)

Then chord\(\left(\frac{\epsilon}{2}\right) < ||v_i - v_j||\), which is equivalent to \( \frac{\epsilon}{2} < \angle v_iOv_j \).

Construct the coloring of vertices of \( G \) as follows: if the vertex \( v_i \) lies on the shortest arc between \( u_0 \) and \( u_1 \) in the embedding of \( H \), the vertex \( i \in V_G \) is assigned with the color \( c(i) = 2 \); if \( v_i \) lies on the shortest arc between \( u_0 \) and \( u_2 \), then \( c(i) = 1 \); otherwise, \( c(i) = 0 \). It is clear that this coloring is unambiguously defined for any embedding of \( H \). We now prove that this coloring of vertices of \( G \) is valid, that is, for every edge \( (i, j) \in E(G) \) we have \( c(i) \neq c(j) \).

Let us show that for every edge \( (i, j) \in E_G \) the shortest arc of \( \rho \) between the points \( v_i \) and \( v_j \) contains at least one vertex of \( U \). Assume the contrary, then WLOG both vertices \( v_i \) and \( v_j \) lie on the shortest arc between \( u_0 \) and \( u_1 \), and \( \angle u_0Ou_1 = \angle u_0Ov_i + \angle v_iOv_j + \angle v_jOu_1 > (\pi/3 - \epsilon) + \frac{5}{2} \epsilon + (\pi/3 - \epsilon) = 2\pi/3 + \epsilon/2 \). But that contradicts with \( \angle u_0Ou_1 < 2\pi/3 + \epsilon/2 \), thus at least one vertex of \( U \) must lie between \( v_i \) and \( v_j \). In that case the colors of \( i \) and \( j \) are different; therefore the coloring is valid.

The first part of Theorem 3 is thus proven.

Denote \( H' = (V_{H'}, E_{H'}, w_{H'}) \), where \( V_{H'} = K \cup U \cup V, \ E_{H'} = E_K \cup E_{KU} \cup E_{KV}, \ w_{H'} \equiv 1 \). Clearly, \( H' \) is a subgraph of \( H \).

Now consider a valid vertex 3-coloring of \( G \); let us construct a non-critical embedding of \( H \). First, construct a non-critical embedding of \( H' \) as follows:
• choose an arbitrary regular \((d - 2)\)-simplex with edge length 1 and identify its vertices with vertices of \(K\); let \(O\) denote the center of the simplex and \(\rho\) denote the locus of all points at the distance 1 from all vertices of the simplex; clearly, \(\rho\) is a circle of radius \(r_0\);

• choose an arbitrary equilateral triangle inscribed in \(\rho\); place the vertices \(u_0, u_1, u_2\) at the vertices of the triangle; denote \(\gamma_0\) the set of all points \(x \in \rho\) such that \(\angle u_0Ox > \pi - \varepsilon/2\); clearly, \(\gamma_0\) is an open arc of angular measure \(\varepsilon\); similarly define sets \(\gamma_1, \gamma_2\);

• suppose the vertex \(i \in V_G\) is assigned with color \(c(i) \in \{0, 1, 2\}\) in the given 3-coloring; place every vertex \(v_i \in V\) in such a way that \(v_i\) lies on the arc \(\gamma_{c(i)}\) for every \(i \in V_G\) and no two vertices of \(V\) are at the same point; since the arcs \(\gamma_0, \gamma_1, \gamma_2\) have non-zero angular measure, such arrangement of vertices of \(V\) is possible.

It can be easily verified that the arrangement of vertices of \(K \cup V \cup U\) described above yields a non-critical embedding of the graph \(H'\).

Now let us add vertices of the set \(Aux\) one by one and successively apply the second part of Lemma 4 to show the existence of a non-critical embedding for every new graph. When all vertices of \(Aux\) are added, we obtain a non-critical embedding of the graph \(H\) since every vertex of \(Aux\) is adjacent to exactly two vertices of \(V \cup U\).

Successively apply the second part of Lemma 4 with the following parameters.

• Let \(v = u_0, u = u_1, z = aux_{u_0,u_1}, l = \text{chord}(2\pi/3 - \varepsilon/2), L = R = \text{chord}(2\pi/3), r = \text{chord}(2\pi/3 + \varepsilon/2)\).

  The points \(u_0\) and \(u_1\) are at the vertices of an equilateral triangle inscribed in the circle \(\rho\), thus \(||u_0 - u_1|| = \text{chord}(2\pi/3)\) and the conditions of the lemma are satisfied.

  Apply the lemma in a similar way to \(u_0, u_2, aux_{u_0,u_2}\) and \(u_1, u_2, aux_{u_1,u_2}\).

• Let \(v \in V, u \in U, z = aux_{v,u}, l = \text{chord}(\pi/3 - \varepsilon), L = \text{chord}(\pi/3 - \varepsilon/2), R = 2r_0, r = \infty\).

  There is at least one vertex \(u_i \in U\) such that \(\angle vOu_i > \pi - \varepsilon/2\), thus \(\angle vOu \geq |\angle vOu_i - \angle u_iOu| > \pi/3 - \varepsilon/2\) and \(||v - u|| > \text{chord}(\pi/3 - \varepsilon/2)\); the conditions of the lemma are satisfied.

• Let \(v_i, v_j \in V, (i, j) \in E_G\): \(v = v_i, u = v_j, z = aux_{v_i,v_j}, l = \text{chord}(\frac{\varepsilon}{2}), L = \text{chord}(2\pi/3 - \varepsilon), R = 2r_0, r = \infty\).

  The points \(v_i\) and \(v_j\) lie on different arcs \(\gamma_{c(i)}, \gamma_{c(j)}\). Let \(u_k\) denote the vertex of \(U\) that lies on the shortest arc between \(v_i\) and \(v_j\). Then \(\angle v_iOv_j = \angle v_iOu_k + \angle u_kOv_j > 2(\pi/3 - \varepsilon/2) = 2\pi/3 - \varepsilon\), and \(||v_i - v_j|| > \text{chord}(2\pi/3 - \varepsilon)\); the conditions of the lemma are satisfied.

After all applications of Lemma 4 we obtain a non-critical embedding of the graph \(H\). 

\(\square\)
The constructed graph $H$ has $O(|V_G| + |E_G|)$ vertices and edges (we recall that the dimension $d$ is a fixed constant), but it contains edges of non-unit length; however, for every such edge its length is equal to $\text{RodLength}(a, b)$ for some $a, b$; moreover, the set of possible pairs $(a, b)$ is finite and independent on the input graph $G$. Thus, upon multiple applications of Lemma 2 each edge of non-unit length can be replaced by a subgraph that is isomorphic to $\text{Rod}(a, b)$ for some $(a, b)$; the size of the graph will increase by at most $K$ times, where $K$ is the maximal size of $\text{Rod}(a, b)$ for all used pairs of $(a, b)$; clearly, the value of $K$ depends only on $d$. Therefore the resulting graph $H' = 3$-COLORING-$\mathbb{R}^d$-UNIT-DISTANCE-EMBEDDABILITY-REDUCTION$(G)$ has $O(|V_G| + |E_G|)$ vertices and edges as well. Finally, we obtain

**Theorem 4.** Let the graph $H' = 3$-COLORING-$\mathbb{R}^d$-UNIT-DISTANCE-EMBEDDABILITY-REDUCTION$(G)$ be constructed by a given graph $G = (V_G, E_G)$ as described above. Then:

- If there is no valid 3-coloring of vertices of $G$, then there is no embedding of $H'$ in $\mathbb{R}^d$.

- If a valid 3-coloring of vertices of $G$ exists, then there exists a non-critical embedding of $H'$ in $\mathbb{R}^d$.

From Theorem 4, the linearity of the size of $H'$, and the fact that the problem of vertex 3-coloring is NP-hard (see [13]) Theorem 1 eventually follows.
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