Reduced Theoretical Error for QED Tests with $^4$He$^+$ Spectroscopy

C.P. Burgess, P. Hayman, Markus Rummel, and László Zalavári

Physics & Astronomy, McMaster University, Hamilton, ON, Canada, L8S 4M1
Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada

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We apply point-particle effective field theory (PPEFT) to electronic and muonic $^4$He$^+$ ions, and use it to identify linear combinations of spectroscopic measurements for which the theoretical uncertainties are much smaller than for any particular energy levels. The error is reduced because these combinations are independent of all short-range physics effects up to a given order in the expansion in the small parameters $R/a_B$ and $Z\alpha$ (where $R$ and $a_B$ are the ion's nuclear and Bohr radii). In particular, the theory error is not limited by the precision with which nuclear matrix elements can be computed, or compromised by the existence of any novel short-range interactions, should these exist. These combinations of $^4$He measurements therefore provide particularly precise tests of QED. The restriction to $^4$He arises because our analysis assumes a spherically symmetric nucleus, but the argument used is more general and extendable to both nuclei with spin, and to higher orders in $R/a_B$ and $Z\alpha$.

I. INTRODUCTION

Recently, laser spectroscopy of muonic atoms $^1$ has opened the door to new and unique precision tests of Quantum Electrodynamics (QED). However, the small size of most QED corrections to atomic levels $^2$ makes them compete with more mundane energy shifts, such as those due to the finite size of the nucleus. This makes uncertainties in computing nuclear contributions to atomic energy shifts important components of the theoretical error budget when comparing with experiments.

These theoretical uncertainties are made worse by the 'proton-radius' problem $^3$, in which the root-mean-squared charge radius inferred from the leading nuclear contributions to atomic energy shifts appears to depend on the flavour of the orbiting lepton. Until it is understood whether this problem is solved by a better understanding of the experimental errors or through the existence of new physics (such as a new short-range force coupling differently to muons and electrons $^4$), this discrepancy must be treated as an unknown unknown when assessing the theory error.

A better understanding of the nature of short-distance nucleus/lepton interactions is therefore an important prerequisite for exploiting the precision of spectroscopic measurements to test QED. This is where the point-particle effective field theory (PPEFT) framework can help $^6$ $^8$. This formalism allows one to write a small set of effective interactions that capture the effects of all short-distance contributions to atomic energy levels (including both nuclear-scale physics and any hypothetical new short-range forces), order-by-order in powers of the relevant small size, $R$, of the physics in question. For nuclear physics $R$ would be of order the nuclear radius, while for a new short-range force it would instead be the force’s range. The existence of these effective interactions allows a robust parameterization of the contributions of short-distance physics to atomic energy levels, without having to understand the details of its microscopic origin.

Of course, knowing the underlying microscopic physics in question (such as the structure of the relevant nucleus) it becomes possible to compute the size of these effective interactions from first principles. In this language the uncertainties in nuclear-structure calculations enter into predictions through any inaccuracy in the values so inferred for the effective interactions.

Instead of trying to reduce the inaccuracy of these effective couplings, in this paper we identify combinations of spectroscopic measurements from which all of the relevant short-distance effective couplings drop out to a fixed order in the expansion in $R/a_B = mR\alpha$ and $Z\alpha$, where $m$ is the mass of the orbiting particle, $Z$ is the nuclear charge, $\alpha$ is the fine-structure constant and $a_B$ is the relevant Bohr radius. These combinations are particularly interesting because the absence of short-distance contributions to them means that the theoretical error for these observables is controlled by powers of $R/a_B$ or $Z\alpha$ rather than by the larger uncertainties arising from (say) nuclear physics. A similar approach has been used to cancel dependence on nuclear effects for the hyperfine splitting in hydrogen $^6$ (as well as to highlight nuclear isotope dependence, among other reasons $^2$ $^12$), however our approach has the advantage of being systematic, and can be applied in principle to any spinning or spinless nucleus. We can also extend our results to higher orders, as we illustrate by identifying nuclear-free combinations to higher order in $Z\alpha$ than has been done previously.

II. PPEFT OF $^4$He

As applied to atoms, PPEFT exploits the hierarchy of scales between the size, $R$, of a small source (the nucleus) and the much larger size, $a_B$, of the Bohr radius. The expansion of observables in powers of $R/a_B$ reveals them not to depend on all of the nuclear details, but only on a set of 'generalized multipole moments', similar to the way that ordinary multipole moments control the expansion of the electrostatic field of a compact charge distribution.
The required moments are identified by writing the first-quantized action for the nucleus that includes all possible local interactions between its centre-of-mass coordinate, \( y^\mu(\tau) \), and the ‘bulk’ fields \( A_\mu(x) \) and \( \Psi(x) \), describing the electromagnetic potential and the Dirac field of the orbiting particle respectively. For simplicity, we assume a spherically symmetric nucleus, specialize to helium (or other doubly magic nuclei) and restrict to parity-preserving interactions, leading to (1):

\[
S_p = - \int_\mathcal{W} d\tau \left[ M - Ze A_\mu \dot{y}^\mu + c_s \nabla \cdot \Psi + ic_o \nabla \gamma_\mu \Psi \dot{y}^\mu + \hbar \gamma^\mu \partial^\tau F_{\mu\nu} \right. \\
+ \left. d_s \dot{y}^\mu \nabla D_\mu \Psi + id_v \dot{y}^\mu \gamma^\nu \nabla \gamma_\mu D_\nu \Psi \\
+ \left( d_s + d_v ( \dot{y}^\mu \gamma^\nu F_{\mu\nu} \right. + \left. 1/2 c_v F_{\mu\nu} F^\mu\nu + \cdots \right].
\]

Here \( \mathcal{W} \) denotes the world-line \( y^\mu(\tau) \) of the nucleus — along which \( \tau \) is its proper time with derivative \( \dot{y}^\mu := dy^\mu/d\tau \) — at which all bulk fields are evaluated; \( \gamma^\mu \) denotes the Dirac gamma matrices and \( D_\mu \Psi := (\partial_\mu + ieA_\mu)\Psi \), as appropriate for fermions of charge \(-e\).

The first line describes the physics of a point source — with mass \( M \) and charge \( Ze \). The couplings \( c_s, c_v \), and \( \hbar \) in the second line have dimensions of \([\text{length}]^2\), and so are expected to be of order \( R^2 \) in size, up to dimensionless \( O(1) \) coefficients. Similarly the couplings \( d_s, d_v, d_B \) and \( d_B \) have dimension \([\text{length}]^3\) and should be of order \( R^3 \) and so on, with the ellipses containing all terms suppressed by more than three powers of \( R \).

For simplicity of presentation we neglect nuclear recoil effects and work with a static nucleus: \( y^\mu = \delta^\mu_0 \). (This amounts to dropping powers of \( m/M \), though these corrections are easily included). The above action then becomes

\[
S_p = - \int_\mathcal{W} dt \left[ M - Ze A_0 \right. \\
+ \left. c_s \nabla \cdot \Psi + ic_o \nabla \gamma_0 \Psi - \hbar \nabla \cdot E \\
+ \left. d_s \nabla D_0 \Psi + id_v \nabla \gamma_0 D_0 \Psi + d_s E^2 + d_v B^2 + \cdots \right].
\]

This reveals \( \hbar = \frac{1}{2} Ze r_2^2 \) to be the charge-radius, \( d_s \) and \( d_v \) to be related to the order-\( R^3 \) Friar moment contributions to the nuclear electrostatic form-factor \( \hat{F} \), and \( d_B \) to encode the contribution of nuclear polarizability to atomic energy levels \( \hat{\nu} \), and so on.

To study atomic helium in this framework we solve for \( \psi(x) = \langle 0 | \Psi(x) | n \rangle \) away from the nucleus, which is given in the usual way by QED. Neglecting radiative corrections and \( R \)-dependent effects means solving the Dirac equation with a Coulomb potential:

\[
(\hat{\mathcal{D}} + m)\psi = \left[ -\gamma^0 \left( \omega + \frac{Z\alpha}{r} \right) + \gamma^r \nabla + m \right] \psi = 0,
\]

for energy eigenstates \( \psi \propto e^{-i\omega t} \). This has well-known solutions of definite parity and total angular momentum given by:

\[
\psi^\pm = \left( f_{\pm}(r) U_{jj}^\pm(\theta, \phi) + ig_{\pm}(r) U_{jj}^\mp(\theta, \phi) \right),
\]

where \( \psi^+ (\psi^-) \) denotes the positive (negative) parity eigenstate, \( U_{jj}^\pm \) are the Dirac spinor harmonics with definite total angular momentum \( j = \ell \pm \frac{1}{2} \) and parity \( \Pi \), and hence \( f_+(r) \) and \( g_+(r) \) solve the radial parts of the Dirac equation.

In this language the entire influence of nuclear-scale physics on the orbiting fermion arises through the boundary condition implied by the point-particle action \( \mathcal{W} \) for the bulk fields \( \Psi \) and \( A_\mu \) near the origin \( \mathcal{R} \). Since these fields diverge when taken to \( r = 0 \) ignoring the nuclear interior (such as does the Coulomb potential), this boundary condition must be imposed at a small (but arbitrary) distance \( r = \epsilon \) outside the nucleus: \( R \leq \epsilon \ll a_B \) (with \( R \) the smallest radius where an external extrapolation is valid). For instance, the leading source contribution to the Dirac boundary condition turns out to be

\[
\frac{c_s + \epsilon v}{4\pi e^2} = \left( \frac{g_+}{f_+} \right)|_{r=\epsilon} \quad \text{and} \quad \frac{c_s - \epsilon v}{4\pi e^2} = \left( \frac{f_-}{g_-} \right)|_{r=\epsilon},
\]

where \( \epsilon v = c_s + \frac{1}{2} Ze^2 r_2^2 \).

However, the details of this boundary condition are not important for the purposes of this paper, since in what follows it suffices to parameterize the ratios \( (g_+/f_+)|_{r=\epsilon} \) and \( (f_-/g_-)|_{r=\epsilon} \) evaluated at \( \epsilon = R \) in terms of their expansion in the two small quantities \( R/a_B \) and \( (Z\alpha)^2 \). This is most conveniently done by writing:

\[
\left( \frac{g_+}{f_+} \right)_{r=R} = \xi_g (Z\alpha), \quad \text{and} \quad X \left( \frac{f_-}{g_-} \right)_{r=R} = \frac{\xi_f}{2n},
\]

where \( X := (m - \omega)/(m + \omega) \) is included for later notational simplicity, and \( n \) is the state’s principal quantum number. At the atomic energy levels, \( \omega = m - (Z\alpha)^2 m/(2n^2) + \cdots \), the quantities \( \xi_f \) and \( \xi_g \) have the expansions

\[
\xi_g := \hat{g}_1 \left( \frac{1}{6n^3} \right) (mR\alpha) + \hat{g}_3 (Z\alpha)^2 + \cdots, \quad \xi_f := \hat{f}_1 (mR\alpha) + \hat{f}_2 (mR\alpha)^2 + \hat{f}_3 (Z\alpha)^2 + \cdots,
\]

where the ellipses involve terms including more powers of \( (mR\alpha) \) and/or \( (Z\alpha)^2 \) than those written, and the dependence on \( n \) follows directly from the \( \omega \)-dependence of the radial Dirac equation.

The physics of the nucleus is parameterized by the dimensionless quantities \( \hat{g}_1 \) and \( \hat{f}_1 \), and the normalizations of \( \mathcal{W} \) are chosen so that \( \hat{g}_1 \) is determined by \( c_s, c_v \), and \( \hbar \) and so defines the charge radius, \( \langle r^2 \rangle \), relevant to atomic energy levels. For instance, if the nucleus is modelled as a charge distribution then \( R \) is the radius where the charge vanishes, and \( \langle r^2 \rangle = r_2^2 = 3(1+2\hat{g}_1)/R^2 \). Similarly, the
quantities \( \hat{g}_2 \) and \( \hat{f}_1 \) are the first ones that are sensitive to the couplings \( d_s, d_u, d_p, \) and \( d_{ps}, \) and so contain the nuclear moment and polarization contributions.

Finally, employing the known solutions for \( f_{\pm} \) and \( g_{\pm} \) in the bulk (i.e., the standard Dirac-Coulomb solutions), and imposing normalizability in addition to the boundary conditions above leads to formulae for the perturbations to the Dirac-Coulomb energy levels, parameterized by the couplings \( \hat{f}_i \) and \( \hat{g}_i \). For \( j = 1/2, \) the nucleus-dependent shift to the positive- and negative-parity energy levels is:

\[
\delta E^+ \simeq \frac{m^3 R^2 (Z\alpha)^4}{n^3} \left\{ 2(1 + 2\hat{g}_1) \right. \\
+ \left[ 2\hat{g}_2 - \frac{8}{3} - 4\hat{g}_1(\hat{g}_1 + 2) \right] (mRZ\alpha) \\
+ \left[ 4\hat{g}_3 + 5 + 8\hat{g}_1 - 2\hat{g}_1^2 + (1 + 2\hat{g}_1) \left\{ \frac{12n^2 - n - 9}{2n^2(1 + 1)} \\
- 2 \ln \left( \frac{2mRZ\alpha}{n} \right) - 2H_{n+1} - 2\gamma \right\} (Z\alpha)^2 + \ldots \}
\]

where the ellipses contain terms that are down relative to those shown by \((mRZ\alpha)^2\) within the parentheses, and

\[
\delta E^- \simeq -\frac{(n^2 - 1)}{n^5} m^4 R^3 (Z\alpha)^5 \left\{ \hat{f}_1 - \frac{2}{3} \right. \\
- \frac{(n^2 - 1)}{n^5} m^3 R^2 (Z\alpha)^4 \left\{ O((Z\alpha)^2) \\
+ O\left( (mRZ\alpha)^2 \right) + \ldots \right\}
\]

where the ellipses contain terms suppressed by higher powers of \((mRZ\alpha)\) and \((Z\alpha)^2\). Here \( H_m \) are the harmonic numbers \( H_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}, \) and \( \gamma \) is the Euler-Mascheroni constant. For muons, \((mRZ\alpha) \approx (Z\alpha)^3\) but for electrons \((mRZ\alpha) \approx (Z\alpha)^2\). It is important to note here that for a nucleus described simply by a static charge distribution, \( \hat{f}_1 \) is generically \( 2/3 \), and so the leading term in \((Z\alpha)^2\) generically vanishes. However, some sort of new physics could modify the value of \( \hat{f}_1 \) and make this term relevant, so from an agnostic effective field theory point of view, it must not be discarded.

For nuclei modelled as simple static charge distributions, these expressions reduce to the well-known formulæ for finite-size corrections to the Dirac-Coulomb energies \([8, 14]\). However, the real power of the expressions \([8]\) and \([9]\) is in their generality. Because the parameters \( \hat{f}_i \) and \( \hat{g}_i \) are interchangeable with the effective couplings to any order in \( R \), and these incorporate all possible short-distance interactions consistent with the particle content and symmetries, the above energy-shift formulæ capture all possible small-scale physics that affects the orbiting fermion at this order in \( R \)—regardless of the details of what that physics might be.

### III. ELIMINATING SHORT-RANGE PHYSICS

Of particular note, all uncertainties to do with nuclear matrix elements and short-distance interactions enter into atomic energy shifts through the uncertainties in the coefficients \( \hat{g}_i \) in eq. \((8)\) and the coefficients \( \hat{f}_i \) in eq. \((9)\). Rather than trying to reduce the theoretical error by computing these coefficients more accurately, we now identify combinations of observables from which these coefficients cancel. How many observables are required to do so depends on how accurate a cancellation is required. At leading order only \( \hat{g}_1 \) enters into any energy shift, so it can be eliminated given expressions for any two energy levels. At subleading order in \((mRZ\alpha)\) there are three relevant parameters: \( \hat{g}_1, \hat{g}_2 \) and \( \hat{f}_1 \), so four energies are required in order to find a nucleus-independent observable, and so on. In general, if \( N \) of the \( \hat{g}_i \) and \( \hat{f}_i \) coefficients enter into \((8)\) and \((9)\) to a desired precision, then \( N + 1 \) measurements are required to obtain a nuclear parameterization-free combination. In certain circumstances, however, that number can be reduced. For instance if a difference of parity-even energies at the same \( m \) and \( Z \) is taken, then to order \((mRZ\alpha)\), \( \hat{f}_1 \) is irrelevant, and only one combination of \( \hat{g}_1 \) and \( \hat{g}_2 \) arises, so only two pieces of data are necessary to make a prediction free of nuclear uncertainties. Our second and third examples below both improve on the \( N + 1 \) bound in such a way.

To formalize this we write the energy levels of hydrogenic atoms as:

\[
E_{n,j,\pm} = E_{n,j}^{\text{Dirac}} \mp \delta E_{n,j,\pm}^{\text{PP}} \mp \delta E_{n,j,\pm}^{\text{QED}} \mp \delta E_{n,j,\pm}^{\text{EM}}, \quad (10)
\]

where \( E_{n,j}^{\text{Dirac}} \) is the energy eigenvalue predicted by the Dirac-Coulomb solution, and \( \delta E_{n,j}^{\text{QED}} \) contains all QED radiative corrections in the limit of a point nucleus. \( \delta E_{n,j}^{\text{PP}} \) is the nucleus-dependent contribution given above, and \( \delta E_{n,j}^{\text{QED}} \) contains all QED radiative corrections to that contribution, both of which suffer from systematic uncertainties arising from model-dependent charges distributions, and the proton radius problem.

However, only differences \( \Delta E_{n,j} \) := \( E_{n,j,\pm} \mp E_{n,j,\pm} \) between energy levels can be measured spectroscopically. For these quantities we write:

\[
\Delta E_{n,j} = \Delta E_{n,j}^{\text{PP}} + \Delta E_{n,j}^{\text{QED}} + \Delta E_{n,j}^{\text{EM}}, \quad (11)
\]

with the Dirac-Coulomb and point-source QED effects labelled together as "EM". Our goal is to identify linear combinations of these observables from which the \( \hat{g}_i \) and \( \hat{f}_i \) cancel, and we do so explicitly here for muonic atoms at a relative accuracy of \((mRZ\alpha)\) required to see the Friar moment. For electrons we go to the same accuracy, which is slightly more involved due to the necessity of keeping terms at relative order \((Z\alpha)^2\).

Consider first the case of muonic \(^4\text{He}^+\). Here, measurements of the \( 2S_{1/2} \rightarrow 2P_{3/2} \) and \( 2S_{1/2} \rightarrow 2P_{1/2} \) transitions have already been made \([1]\). To relative order \((mRZ\alpha)\), the negative-parity energy shift is negligible.
when \( j = 3/2 \) (for any \( n \)), and so the first transition offers

\[
\Delta E_{2S_{1/2} \rightarrow 2P_{3/2}}^{\text{PP}} \simeq \frac{1}{8} m^3 R^2 (Z\alpha)^4 \left\{ 2(1 + 2\hat{g}_1) \right\},
\]

which can be used to solve for either \( \hat{g}_1 \) or \( \hat{g}_2 \). The second transition contains the same short-range effects apart from the non-negligible contribution to the \( j = 1/2 \) negative-parity state, and so the difference between these two transitions,

\[
\Delta E_{2S_{1/2} \rightarrow 2P_{3/2}}^{\text{PP}} - \Delta E_{2S_{1/2} \rightarrow 2P_{1/2}}^{\text{PP}} = -\frac{3m^4 R^3 (Z\alpha)^5}{32}\left( \hat{f}_1 - \frac{2}{3} \right),
\]

fixes \( \hat{f}_1 \). Nuclear effects cannot be eliminated using only these two transitions. (Notice however that when \( \hat{f}_1 = 2/3 \), nuclear effects do happen to vanish to order \((mRZ\alpha)\)).

The situation changes if measurements of the same transition with higher \( n \)-values can be made. For instance, if the difference (13) is available with \( n = 3 \), then the linear combination

\[
\frac{243}{8}\left( \Delta E_{3S_{1/2} \rightarrow 3P_{3/2}}^{\text{PP}} - \Delta E_{3S_{1/2} \rightarrow 3P_{1/2}}^{\text{PP}} \right) - \frac{32}{3}\left( \Delta E_{2S_{1/2} \rightarrow 2P_{3/2}}^{\text{PP}} - \Delta E_{2S_{1/2} \rightarrow 2P_{1/2}}^{\text{PP}} \right) = 0
\]

vanishes identically, and so the same linear combination of transition energies satisfies

\[
\frac{243}{8}\left( \Delta E_{3S_{1/2} \rightarrow 3P_{3/2}}^{\text{PP}} - \Delta E_{3S_{1/2} \rightarrow 3P_{1/2}}^{\text{PP}} \right) - \frac{32}{3}\left( \Delta E_{2S_{1/2} \rightarrow 2P_{3/2}}^{\text{EM}} - \Delta E_{2S_{1/2} \rightarrow 2P_{1/2}}^{\text{EM}} \right) = 0
\]

which is a direct probe of QED contributions into which nuclear effects first enter at relative order \((Z\alpha)^2\) (note that here, the radiative corrections \(\delta E^{\text{PP}} \text{QED}\) are negligible since to this order they do not distinguish between the \( j = 1/2 \) and \( j = 3/2 \) states \([18]\)). This is effectively as if the nuclear uncertainty for this observable were of order \((Z\alpha)^2\).

Another example arises when one notes that \(\delta\) implies all non-QED nucleus-dependent contributions up to relative \(O(mRZ\alpha)\) are independent of \(n\) apart from the overall factor of \(1/n^3\). This observation is especially useful for the \(nS_{1/2} \rightarrow nP_{3/2}\) transitions where the parity-odd energy shift is negligible at that order, since it implies, e.g., the linear combination

\[
3\Delta E_{3S_{1/2} \rightarrow 3P_{3/2}}^{\text{EM}} - 2\Delta E_{2S_{1/2} \rightarrow 2P_{3/2}}^{\text{EM}} = 0,
\]

vanishes identically. For the muon, QED corrections to finite-size effects first arise at this order (relative order \(\alpha\)), and are proportional to the squared proton charge radius \([18]\), which in our language means \(\delta E^{\text{PP}} \text{QED} \propto (1 + 2\hat{g}_1) R^2\). Consequently the same linear combination of transition energies satisfies

\[
3\Delta E_{3S_{1/2} \rightarrow 3P_{3/2}}^{\text{EM}} - 2\Delta E_{2S_{1/2} \rightarrow 2P_{3/2}}^{\text{EM}} = 3\Delta E_{3S_{1/2} \rightarrow 3P_{3/2}}^{\text{EM}} - 2\Delta E_{2S_{1/2} \rightarrow 2P_{1/2}}^{\text{EM}}
\]

where \(Q_{n=1,j=2}^{3}\) is a numerical coefficient calculable from QED predictions \([18]\). A direct probe of nuclear-free QED contributions similar to \([15]\) can therefore be obtained with one further measurement—such as if the \(4S_{1/2} \rightarrow 4P_{3/2}\) transition can be observed. So with five measurements, we have two tests of QED for which all small-distance uncertainties (including those due to nuclear calculations) drop out, down to relative order \((Z\alpha)^2\).

Similar arguments apply to measurements of \(S\)-wave to \(S\)-wave transitions in electronic helium, although here we must also be sure that the nuclear properties drop out for both relative \((mRZ\alpha)\) and \((Z\alpha)^2\) orders (since for electrons these are similar in size). To that order, the short-range energy shift can be separated into a piece with trivial \(n\)-dependence, and a piece with non-trivial \(n\)-dependence:

\[
\Delta E_{n_1S_{1/2} \rightarrow n_2S_{1/2}}^{\text{PP}} = \left( \frac{1}{n_1} - \frac{1}{n_2} \right) C + F(n_1, n_2)(1 + 2\hat{g}_1),
\]

where

\[
C := m^3 \alpha R^2 (Z\alpha)^4 \left\{ 2(1 + 2\hat{g}_1) \right\},
\]

\[
+ \left[ 2\hat{g}_2 - \frac{8}{3} - 4\hat{g}_1(\hat{g}_1 + 2) \right](mRZ\alpha)
\]

\[
+ \left[ 4\hat{g}_3 + 5 + 8\hat{g}_1 - 2\hat{g}_1^2 - 2(\gamma + \text{ln}(mRZ\alpha))(1 + 2\hat{g}_1) \right](Z\alpha)^2,
\]

contains all terms in \(\delta\) independent of \(n\), apart from the overall \(1/n^3\) which has been factored out, and \(F(n_1, n_2)\) contains the rest of the difference (a complicated function of \(n_1\) and \(n_2\)) with the common \((1 + 2\hat{g}_1)\) factored out:

\[
F(n_1, n_2) := 2m^3 \alpha R^2 (Z\alpha)^6 \left\{ \frac{12n_2^2 - n_1 - 9}{4n_1^3(n_1 + 1)} \right\}
\]

\[
- \frac{12n_2^2 - n_2 - 9}{4n_2^3(n_2 + 1)} \frac{\text{ln}n_1}{n_1^3} - \frac{\text{ln}n_2}{n_2^3} - \frac{H_{n_2 + 1} - H_{n_1 + 1}}{n_2^3} + \frac{Q_{n=1,j=2}^{n_1,n_2}}{Z}
\]

One can then use, e.g., the measurements \(2S \rightarrow 1S\) and \(3S \rightarrow 1S\) to write:

\[
\frac{27}{26}\Delta E_{3S \rightarrow 1S} - \frac{8}{7}\Delta E_{2S \rightarrow 1S} = \frac{27}{26}\Delta E_{3S \rightarrow 1S} - \frac{8}{7}\Delta E_{2S \rightarrow 1S},
\]

\[
+ \left( \frac{27}{26}F(3, 1) - \frac{8}{7}F(2, 1) \right)(1 + 2\hat{g}_1)
\]
which can be used to solve for $\hat{g}_1$ in terms of measurements and Dirac-Coulomb/QED predictions. This expression for $\hat{g}_1$ can then be substituted into the analogous linear combination of the $4S \rightarrow 1S$ and $2S \rightarrow 1S$ transitions (for example), which would then constitute a pure test of QED. Explicitly, for this example one would find:

$$
\frac{64}{63} (\Delta E_{4S \rightarrow 1S} - \Delta E_{4S \rightarrow 1S}^{EM}) = \frac{27}{26} N (\Delta E_{3S \rightarrow 1S} - \Delta E_{3S \rightarrow 1S}^{EM})
+ \frac{8}{7} (1 - N) (\Delta E_{2S \rightarrow 1S} - \Delta E_{2S \rightarrow 1S}^{EM}), \quad (22)
$$

where $N = (\frac{64}{63} F(4, 1) - \frac{8}{7} F(2, 1))/(\frac{27}{26} F(3, 1) - \frac{8}{7} F(2, 1))$ is a numerical coefficient that can be computed once the QED coefficients $Q_{n_1,n_2}^{S_{1/2},S_{1/2}}$ [18] are known.

IV. CONCLUSION

We here apply the PPEFT framework to muonic and electronic atoms with spinless nuclei, which produce systematic parameterizations of the energy level shifts due to all short-range physics, including (but not limited to) uncertainties in evaluating nuclear contributions. Our parameterization cleanly identifies the number of parameters required to eliminate dependence on nuclear physics to any given measurement precision, and moreover, we identify several simple linear combinations of observables from which these parameters drop out altogether. We find the number of unique measurements required to assemble a parameterization-free theoretical prediction at any order is at most $N + 1$, where $N$ is the number of $\hat{g}_i$ and $\hat{f}_i$ coefficients relevant to the measurement precision. This bound need not necessarily be saturated. In certain circumstances, some degrees of freedom become redundant (e.g., in (17), $\hat{f}_1$ is irrelevant) or degenerate (e.g., only one combination of $\hat{g}_2$ and $\hat{g}_3$ appears in (22)), and so reduce the total number of measurements required to make a nucleus-independent statement.

For these observables our formulae reduce the theoretical error in tests of QED by eliminating any uncertainties arising from explicit models of the nucleus. The same combinations are also independent of any potential short-range new physics (should this prove to be responsible for the proton-radius problem) allowing tests of QED using only muonic $^4$He whose validity is undiminished by the existence of such forces.

Although we here address only spinless nuclei, it is certainly possible to include nuclei with spin in the PPEFT framework, and work is ongoing to do so. Though nuclear spin changes the counting of parameters in the energy shift formulae above, the principle remains exactly the same and we expect in this case also to be able to build observables from which short-range contributions completely drop out.

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[1] A. Antognini, et. al., Can. J. Phys. 89(1) (2011) 47-57, Parthey, C. G. and Matveev, A. and Alnis, J. and Bernhardt, A. and Beyer, A. and Holzwarth, R. and Maistrou, A. and Pohl, R. and Predelhe, K. and Udem, T. and Wilken, T. and Kolachesky, N. and Abgrall, M. and Rovera, D. and Saloman, C. and Laurent, P. and Hänsch, T. W., Phys. Rev. Lett. 107 (2011) 203001 [arXiv:1107.3101] [physics.atom-ph].
N. T. Amaro, F. D. Antognini, Hyperfine Interact (2012) 212: 195.
A. Antognini, et al., [arXiv:1509.03235] [physics.atom-ph], R. Pohl et al., “Laser Spectroscopy of Muonic Atoms and Ions,” JPS Conf. Proc. 18, 011021 (2017) doi:10.7566/JPSCP.18.011021 [arXiv:1609.03440] [physics.atom-ph].

[2] See e.g., Peter J. Mohr, David B. Newell and Barry C. P. Burgess, P. Hayman, M. Rummel and L. Zalavari, [hep-ph]; V. Barger, C. W. Chiang, W. Y. Keung and D. Marfa- A.C. Zemach, Phys. Rev. [hep-ph].

[3] K. A. Woodle et al., Phys. Rev. A 41 (1990) 93.
A. Pineda, Phys. Rev. C 71 (2005) 065205 [hep-ph/0412142].
R. Pohl, et al., Nature 466 (2010) 213-216.
R. J. Hill and G. Paz, Phys. Rev. D 82 (2010) 113005 [arXiv:1008.4619] [hep-ph];
R. Pohl, R. Gilman, G. A. Miller and K. Pachucki, Ann. Rev. Nucl. Part. Sci. 63 (2013) 175 [arXiv:1301.0095] [physics.atom-ph];
A. Antognini, et al., Science 339 (2013) 417-420.
R. Pohl, R. Gilman, G. A. Miller and K. Pachucki, Annu. Rev. Nucl. Part. Sci. 63 (2013) 175-204.

[4] V. Barger, C. W. Chiang, W. Y. Keung and D. Marfatia, Phys. Rev. Lett. 106 (2011) 153001 [arXiv:1011.3519] [hep-ph];
D. Tucker-Smith and I. Yavin, Phys. Rev. D 83 (2011) 101702 [arXiv:1011.4922] [hep-ph];
B. Batell, D. McKeen and M. Pospelov, Phys. Rev. Lett. 107 (2011) 011803 [arXiv:1103.0721] [hep-ph];
A. De Rujula, Phys. Lett. B 697 (2011) 26-31.
C. E. Carlson and B. C. Rislow, Phys. Rev. D 86 (2012) 035013 [arXiv:1206.3587] [hep-ph];
R. Onofrio, EPL 104 (2013) 20020, Li-Bang Wang and Wei-Tou Ni, Mod. Phys. Lett. A 28 (2013) 1350094.

[5] W. E. Caswell and G. P. Lepage, Phys. Lett. 167B (1986) 437.

[6] J. L. Friar and J. W. Negele, Advances in Nuclear Physics, 8 (1975) 219-376.

[7] S. G. Karshenboim, D. McKeen and M. Pospelov, Phys. Rev. D 90 (2014) no.7, 073004 Addendum: [Phys. Rev. D 90 (2014) no.7, 079905 [arXiv:1401.6154] [hep-ph]]
D. Robson, Int. J. Mod. Phys. E 23 (2015) no.12, 1450090 [arXiv:1305.4552] [nucl-th].
P. Brax and C. Bur rage, Phys. Rev. D 91 (2015) 043515,

[8] M. Weitz, A. Huber, F. Schmidt-Kaler, D. Leibfried, W. Vassen, C. Zimmermann, K. Pachucki, T. W. Hänsch, L. Julien, and F. Biraben. Phys. Rev. A, 52:2664–2681, Oct 1995.
Savely G Karshenboim. Physics Reports, 422(1):1–63, 2005.
[18] Eides M.I., Grotch H., Shelyuto V.A. (2007) Nuclear Size and Structure Corrections. In: Theory of Light Hydrogenic Bound States. Springer Tracts in Modern Physics (Chapter 6), and Lamb Shift in Light Muonic Atoms. In: Theory of Light Hydrogenic Bound States (Chapter 7). Springer Tracts in Modern Physics, vol 222. Springer, Berlin, Heidelberg