NOETHER-LEFSCHETZ FOR $K_1$ OF A SURFACE, REVISITED

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Abstract. Let $Z \subset \mathbb{P}^3$ be a general surface of degree $d \geq 5$. Using a Lefschetz pencil argument, we give a elementary new proof of the vanishing of a regulator on $K_1(Z)$.

1. Statement of result

Let $Z$ be a smooth quasiprojective variety over $\mathbb{C}$, and for given nonnegative integers $k, m$, let $\text{CH}^k(Z, m)$ be the higher Chow group as introduced in [Blo1]. In [Blo2], Bloch constructs a cycle class map into any suitable cohomology theory. In our setting, the corresponding map is:

$$\text{cl}_{k,m} : \text{CH}^k(Z, m) \to H^{2k-m}_D(Z, \mathbb{Q}(k)),$$

where $H^{2k-m}_D(Z, \mathbb{Q}(k))$ is Deligne-Beilinson cohomology, which fits in a short exact sequence

$$0 \to H^{2k-m-1}_F(Z, \mathbb{C}) \to H^{2k-m}_D(Z, \mathbb{Q}(k)) \to H^{2k-m}_F(Z, \mathbb{C}) \to 0.$$  

Our primary interest is when $Z$ is also complete, and $m = 1$. Thus one has the corresponding map:

$$\text{cl}_{k,1} : \text{CH}^k(Z, 1) \to \frac{H^{2k-2}(Z, \mathbb{C})}{F^k H^{2k-2}(Z, \mathbb{C}) + H^{2k-2}(Z, \mathbb{Q}(1)).}$$

Let $H_{g-1} := H^{2k-2}(Z, \mathbb{Q}(k-1)) \cap F^{k-1} H^{2k-2}(Z, \mathbb{C})$ be the Hodge group. Then one has an induced map

$$\text{cl}_{k,1} : \text{CH}^k(Z, 1) \to \frac{H^{2k-2}(Z, \mathbb{C})}{F^k H^{2k-2}(Z, \mathbb{C}) + H_{g-1}(Z) \otimes \mathbb{C} + H^{2k-2}(Z, \mathbb{Q}(1)).}$$

It is known that $\text{cl}_{k,1}$ is trivial for $Z$ a sufficiently general complete intersection and of sufficiently high multidegree. This is a consequence of the

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work of Nori [No], together with a technique similar to that given in [G-S]. The argument is presented in [MS]. Further, it is noted in [MS], based on an effective bound in [Pa], that
\[
\text{cl}^2_1 : \text{CH}^2(Z, 1) \to H^2(Z, \mathbb{C}) + \text{H}^1(\mathbb{Z}) \otimes \mathbb{C} + H^2(Z, \mathbb{Q}(2))
\]
is trivial for sufficiently general surfaces $Z \subset \mathbb{P}^3$ of degree $d \geq 5$. The method of Nori involves passing to the universal family of complete intersections of a given multidegree, in a given projective space. A similar point of view appears in [Na]. In this paper, we give an elementary and direct proof of the triviality of $\text{cl}^2_1$ for a general surface $Z \subset \mathbb{P}^3$ of degree $d \geq 5$, by working with a Lefschetz pencil of degree $d \geq 5$ surfaces in $\mathbb{P}^3$. Thus our main theorem is an elementary new proof of the following:

**Main Theorem.** For a sufficiently general surface $Z \subset \mathbb{P}^3$ of degree $d \geq 5$, the map $\text{cl}^2_1$ is trivial.

We remark that the theorem is trivially true, without the generic hypothesis, if $\text{deg} Z \leq 3$, as $H^2(Z)$ is algebraic. From the works of Collino, Voisin, S. M"uller-Stach, et al, and more recently the authors [C-L], it is false if $\text{deg} Z = 4$. Since our method requires only a Lefschetz pencil as opposed to the universal family of surfaces of degree $d$ in $\mathbb{P}^3$, and that it provides a rather simple proof of a counterexample of the Hodge-$D$-conjecture of Beilinson [Bei1], we believe that this approach has some merit. In particular, we believe that this argument is potentially useful in other settings.

2. **Some definitions**

1. **Deligne cohomology.** We assume that the reader is familiar with Deligne cohomology, such as can be found in [Bei] and [EV]. In the case of a smooth projective variety $Z$, and if we put $Q(j) = \mathbb{Q}(2\pi\sqrt{-1})^j$, one introduces the Deligne complex
\[
\mathbb{Q}(j)_D : \quad Q(j) \to O_Z \to \Omega^1_Z \to \cdots \to \Omega^{i-1}_Z,
\]
and defines $H^i_D(Z, Q(j)) := H^i(\mathbb{Q}(j)_D)$ (hypercohomology). This gives rise to a short exact sequence
\[
0 \to H^{i-1}(Z, \mathbb{C}) \to F^jH^{i-1}(Z, \mathbb{C}) + H^{i-1}(Z, Q(j)) \to H^i_D(Z, Q(j)) \to 0.
\]
A similar exact sequence holds quasiprojective $Z$ that are not necessarily smooth.

2. **Higher Chow groups.** For a quasiprojective $Z$, the following abridged definition of $\text{CH}^k(Z, 1)$ will suffice [La] (cf. [MS]).
Definition. $\text{CH}^k(Z, 1)$ is the homology of the middle term in the complex

$$
\prod_{\text{cd}_2 Y = k-2} K_2(C(Y)) \xrightarrow{\text{Tame}} \prod_{\text{cd}_2 Y = k-1} K_1(C(Y)) \xrightarrow{\text{div}} \prod_{\text{cd}_2 Y = k} K_0(C(Y)),
$$

where we recall that $K_1(F) = F^\times$ and $K_0(F) = \mathbb{Z}$, for a field $F$, and Tame, div are respectively the Tame symbol and divisor maps.

Note: For the most part, we will identify $\text{CH}^k(-, m)$ with $\text{CH}^k(-, m) \otimes \mathbb{Q}$, unless there is a specific reason to work with $\text{CH}^k(-, m)$ (and in which case the interpretation will be clear).

(3) Horizontal displacement. Let $h : W \to S$ be a proper smooth morphism of quasiprojective varieties over $\mathbb{C}$, with smooth projective fiber $W_t := h^{-1}(t)$. Fix a reference point $t_0 \in S$ and consider a disk $\Delta$ centered at $t_0$. It is well known that there is a diffeomorphism $h^{-1}(\Delta) \approx \Delta \times W_{t_0}$. Thus for a cohomology class $\gamma := \gamma_{t_0} \in H^*(W_{t_0})$, one can talk about its horizontal displacement $\gamma_t \in H^*(W_t)$, for $t \in \Delta$ and more generally for $t \in S$. Consider the Hodge decomposition $H^*(W_t, \mathbb{C}) = \bigoplus_{p+q=\bullet} H^{p,q}(W_t)$, $\gamma_t = \bigoplus_{p+q=\bullet} \gamma_t^{p,q}$. We say that the Hodge $(p, q)$ components deform horizontally if $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in \Delta$. By analytic considerations of Hodge subbundles, this is equivalent to saying that $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in S$.

3. Proof of the main theorem

Let $\{X_t\}_{t \in \mathbb{P}^1}$ be a Lefschetz pencil of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$, i.e. the general fiber $X_t$ is smooth, and each singular fiber has an ordinary double point singularity. We will think of this pencil in the form $X \subset \mathbb{P}^3 \times \mathbb{P}^1$, i.e. where $X$ is the blowup of $\mathbb{P}^3$ along the base locus $\cap_{t \in \mathbb{P}^1} X_t$. Suppose that for a general $t \in \mathbb{P}^1$, the cycle class map $\text{cl}_{2,1} : \text{CH}^2(X_t, 1) \to H^3_D(X_t, \mathbb{Q}(2))$ is nontrivial. We can assume that $X$ is defined over an algebraically closed field $L$ of finite transcendence degree over $\mathbb{Q}$, i.e. $X/\mathbb{C} = X_L \times \mathbb{C}$. Let $\eta$ be the generic point of $\mathbb{P}^1_L$. For some finite algebraic extension $K \supset L(\eta)$, and via a suitable embedding $K \hookrightarrow \mathbb{C}$, there is a class $\xi_K \in \text{CH}^2(X_K := X_\eta \times K, 1)$ such that $\text{cl}_{2,1}(\xi_K) \neq 0$ in $H^3_D(X_K(\mathbb{C}), \mathbb{Q}(2))$. [The situation here is not unlike that found in [Lew1], p. 191.] There is a smooth projective curve $\Gamma_L$ with function field $L(\Gamma) = K$. Then after a base change $Y = X \times_{\mathbb{P}^1} \Gamma$, $\xi_K$ defines a cycle in $\xi \in \text{CH}^2(Y_U, 1)$, where $U \subset \Gamma$ is a Zariski open subset of $\Gamma$ and $Y_U = \sqcup_{t \in U} Y_t$. This uses the fact that

$$\text{CH}^2(X_K, 1) = \text{CH}^2(Y_{\bar{\eta}}, 1) = \lim_{\rightarrow U} \text{CH}^2(Y_U, 1),$$

where $Y_{\bar{\eta}}$ is the generic fiber of $Y$ over $\Gamma_L$. We want to spread $\xi$ to all of $\Gamma$. However, there is obstruction preventing us to do it; rather we can extend it after a suitable modification of $\xi$. That is, we will show that there exists
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Let us first extend (3.1) \( CH^2(1) = CH^1(Y_B) \to CH^2(Y) \to CH^2(Y_U) \to 0 \) over \( Q \), where \( B = \Gamma \setminus U \) and \( Y_B = \cup_{t \in B} Y_t \).

Note that the map \( CH^1(Y_B) \to CH^2(Y) \) might not be injective if \( |B| > 1 \), so there is obstruction to extend \( \xi \) directly.

Let \( H \) be a plane in \( \mathbb{P}^3 \) and \( \pi^*H \subset Y \) be the pullback of \( H \) under the projection \( \pi : Y \to \mathbb{P}^3 \). Let \( C_b = \pi^*H \cap Y_b \) for \( b \in B \) and \( C_B = \cup_{b \in B} C_b \).

Let us first extend \( \xi \) to \( Y \setminus C_B \). We look at the localization sequence

(3.2) \( CH^2(Y \setminus C_B, 1) \to CH^2(Y_U, 1) \to CH^1(Y_B \setminus C_B) \to CH^2(Y \setminus C_B) \)

Note that

(3.3) \( CH^1(Y_B \setminus C_B) = \bigoplus_{b \in B} CH^1(Y_b \setminus C_b) \)

We claim that \( CH^1(Y_t \setminus C_t) \otimes Q = 0 \) for every \( t \in \Gamma \).

The classical Noether-Lefschetz theorem tells us that a general surface of degree \( d \geq 4 \) in \( \mathbb{P}^3 \) has Picard rank 1. This statement was refined by Mark Green [4] to the following. Let \( M = \mathbb{P}^N \) be the space parameterizing surfaces of degree \( d \) in \( \mathbb{P}^3 \) and \( M_2 \subset M \) be the subset parameterizing surfaces with Picard rank \( \geq 2 \). Then \( \text{codim} M_2 = d - 3 \). So when \( d \geq 5 \), \( M_2 \) has codimension at least 2 in \( M \) and a general pencil will avoid this locus. Thus \( \text{Pic}(Y_t) \otimes Q = Q \) for every \( t \in \Gamma \). Note that \( Y_t \) might be singular, i.e., \( Y_t \) has an ordinary double point. Since an ordinary double point is a quotient singularity, every Weil divisor of \( Y_t \) is \( Q \)-Cartier. Therefore, \( CH^1(Y_t) \otimes Q = \text{Pic}(Y_t) \otimes Q \). In any case, we have

(3.4) \( CH^1(Y_t) \otimes Q = \text{Pic}(Y_t) \otimes Q = \text{Pic}(\mathbb{P}^3) \otimes Q = Q \).

Obviously, \( CH^1(Y_t) \) is generated by \( C_t = \pi^*H \cap Y_t \) over \( Q \). Consequently,

(3.5) \( CH^1(Y_t \setminus C_t) \otimes Q = 0 \)

and there is no obstruction to extend \( \xi \) to \( Y \setminus C_B \). So we may regard \( \xi \) as a class in \( CH^2(Y \setminus C_B, 1) \) from now on.

There might be obstruction to further extend \( \xi \) to all of \( Y \) by the localization sequence

(3.6) \( CH^2(Y, 1) \to CH^2(Y \setminus C_B, 1) \xrightarrow{\phi} CH^0(C_B) \xrightarrow{\gamma} CH^2(Y) \)

where

(3.7) \( CH^0(C_B) = \bigoplus_{b \in B} CH^0(C_b) = Q^{|B|} \)

with \( \beta = |B| \).

\(^1\)Strictly speaking, we don’t really need the localization sequence in this paper. Rather, it is used out of convenience.
Let $\xi = \sum_\alpha (f_\alpha, D_\alpha)$ where $D_\alpha$ is a divisor on $Y \setminus C_B$ and $f_\alpha$ is a rational function on $D_\alpha$. We have
\begin{equation}
\sum_\alpha \text{div}(f_\alpha) = 0.
\end{equation}
Let $\overline{D_\alpha}$ be the closure of $D_\alpha$ in $Y$ and $f_\alpha$ naturally extends to a rational function $\overline{f}_\alpha$ on $\overline{D_\alpha}$. Let $\bar{\xi} = \sum_\alpha (f_\alpha, \overline{D_\alpha})$. We no longer have (3.8). Instead,
\begin{equation}
\sum_\alpha \text{div}(\overline{f}_\alpha) = \sum_{b \in B} m_b C_b
\end{equation}
for some $m_b \in \mathbb{Z}$. Actually, the RHS of (3.9) is exactly the image of $\xi$ under the map $\phi : \text{CH}^2(Y \setminus C_B, 1) \to \text{CH}^0(C_B)$ in (3.6), i.e.,
\begin{equation}
\phi(\xi) = \sum_{b \in B} m_b C_b.
\end{equation}
Note that $\phi(\xi)$ lies in the kernel of $\gamma : \text{CH}^0(C_B) \to \text{CH}^2(Y)$ and there is a natural map $\text{CH}^0(C_B) \to \text{CH}^1(\Gamma)$ via
\begin{equation}
\text{CH}^0(C_B) \to \text{CH}^2(Y) \to \text{CH}^3(\mathbb{P}^3 \times \Gamma) \to \text{CH}^1(\Gamma).
\end{equation}
Note that the map $\text{CH}^3(\mathbb{P}^3 \times \Gamma) \to [\mathbb{P}^1] \otimes \text{CH}^1(\Gamma) = \text{CH}^1(\Gamma)$, comes from the projective bundle formula. Of course, the map $\text{CH}^0(C_B) \to \text{CH}^1(\Gamma)$ simply sends $C_b$ to $Nb$, where $N = d$. And $\phi(\xi)$ maps to zero under this map, i.e. the divisor $\sum m_b C_b$ is $N$-torsion in $\text{CH}^1(\Gamma) = \text{Pic}(\Gamma)$.

Note that $\pi^* H$ is a fibration of curves over $\Gamma$. So the fact $\sum m_b C_b$ is torsion in $\text{CH}^1(\Gamma)$ implies that $\sum m_b C_b$ is $N$-torsion in $\text{CH}^1(\pi^* H)$. Consequently, there exists a rational function $f_H$ on $\pi^* H$ such that
\begin{equation}
\text{div}(f_H) = N \sum_{b \in B} m_b C_b.
\end{equation}
So we may simply modify $\bar{\xi}$ as follows
\begin{equation}
\xi' = \bar{\xi} - \frac{1}{N} (f_H, \pi^* H).
\end{equation}
Now $\xi' \in \text{CH}^2(Y, 1)$ and $\text{cl}_{2,1}(\xi'_t) = \text{cl}_{2,1}(\xi_t)$ for all $t \in U$, where we recall that
\begin{equation}
\text{cl}_{2,1} : \text{CH}^2(Y_t, 1) \to \frac{H^3_D(Y_t, \mathbb{Q}(2))}{H^1_G(Y_t) \otimes (\mathbb{C}/\mathbb{Q}(1))}
\end{equation}
is the induced map. This is due to the fact that the restrictions $f_H$ to $Y_t$ are obviously constants. Thus we can now replace $\xi$ by $\xi'$. Next observe that even though $Y$ is complete, it may be singular. It is worthwhile pointing out that we can further pull back $\xi$ to a desingularization $\tilde{Y}$ of $Y$. More precisely,

Claim. There exists $\tilde{\xi} \in \text{CH}^2(\tilde{Y}, 1)$ such that $\tilde{\xi}$ and $\xi$ agree on the open set where $\tilde{Y}$ and $Y$ are isomorphic.
The usefulness of this claim is as follows. The (cohomological) cycle class map \( \text{cl}_{2,1} : \text{CH}^2(Y, 1) \to H^3_D(Y, \mathbb{Q}(2)) \) is only defined if \( Y \) is smooth. Granting the existence of this cycle class map, the remaining argument only requires the completeness of \( Y \). There is a short exact sequence:

\[
0 \to \frac{H^2(Y, \mathbb{C})}{F^2H^2(Y, \mathbb{C}) + H^2(Y, \mathbb{Q}(2))} \to H^3_D(Y, \mathbb{Q}(2)) \to F^2 \cap H^3(Y, \mathbb{Q}(2)) \to 0.
\]

But since \( Y \) is complete, a weight argument gives \( F^2 \cap H^3(Y, \mathbb{Q}(2)) = 0 \). Thus for \( t \in U \), \( \text{cl}_{2,1}^t(\xi_t) \) is given by the restriction \( \text{cl}_{2,1}^t(\xi) \big|_{Y_t} \), i.e., induced by the restriction

\[
\frac{H^2(Y, \mathbb{C})}{F^2H^2(Y, \mathbb{C}) + H^2(Y, \mathbb{Q}(2))} \to \frac{H^2(Y_t, \mathbb{C})}{F^2H^2(Y_t, \mathbb{C}) + H^2(Y_t, \mathbb{Q}(2))}.
\]

Thus as \( t \in U \) varies, the class \( \text{cl}_{2,1}^t(\xi_t) \) varies by horizontal displacement; further, the restriction \( H^2(Y) \to H^2(Y_t) \) is a morphism of mixed Hodge structures. Thus \( \text{cl}_{2,1}^t(\xi_t) \) is induced by a class in \( H^2(Y_t) \), whose Hodge \((p, q)\) components displace horizontally, i.e., preserving the given Hodge type. But over the set where \( \Gamma \to \mathbb{P}^1 \) ramifies, one can find open sets \( \Delta_\Gamma \subset U \subset \Gamma \), \( \Delta \subset \mathbb{P}^1 \), in the strong topology, such that \( \Delta_\Gamma \simeq \Delta \). Thus \( \text{cl}_{2,1}^t(\xi_t) = 0 \), by virtue of:

**Lemma.** Consider a Lefschetz pencil \( \{Z_t\}_{t \in \mathbb{P}^1} \) of surfaces in \( \mathbb{P}^3 \) of degree \( d \geq 1 \), and let \( U_0 \subset \mathbb{P}^1 \) be the smooth set. Further, let \( \Delta \subset U_0 \) be a disk, and assume given \( \gamma_1 \in H^2(Z_t, \mathbb{C}) \), a horizontal displacement of a class \( \gamma \) for \( t \in \Delta \). If the \((p, q)\) components of \( \gamma_t \) also horizontally displace, then \( \gamma_t \in H^1_{\text{lg}}(Z_t) \).

**Proof.** This follows from a standard monodromy argument, together with the analyticity of Hodge subbundles.

Finally, we attend to:

**Proof of claim.** It turns out that the singularities of \( Y \) are quite mild. Note that the singularities of \( Y \) are introduced during the base change \( \Gamma \to \mathbb{P}^1 \); \( Y \) becomes singular when the map \( \Gamma \to \mathbb{P}^1 \) ramifies over a point \( t \in \mathbb{P}^1 \) where \( X_t \) is singular, i.e., it has an ordinary double point. Therefore, the singularities of \( Y \) have the type of \( x^2 + y^2 + z^2 + t^m = 0 \). Let \( p \in Y \) be such a singularity. We may solve \( p \) by a sequence of blowups:

\[
Y = Y_\mu \xrightarrow{\varphi_\mu} Y_{\mu-1} \xrightarrow{\varphi_{\mu-1}} \ldots \xrightarrow{\varphi_1} Y_0 = Y
\]

where \( \mu = \lfloor m/2 \rfloor \). The exceptional divisor \( E_k \subset Y_k \) of \( \varphi_k \) is a quadric in \( \mathbb{P}^3 \); it is a cone over a conic curve if \( 2k < m \) and it is a smooth quadric if \( m = 2k \). Let \( p_0 = p \) and \( p_k \in E_k \) be the vertex of the cone \( E_k \) for \( 2k < m \). It is obvious that \( Y_k \) is locally given by \( x^2 + y^2 + z^2 + t^{m-2k} = 0 \) at \( p_k \) and \( \varphi_{k+1} : Y_{k+1} \to Y_k \) is the blowup of \( Y_k \) at \( p_k \).
In order to pull back $\xi$ to $\tilde{Y}$, we do it step by step, i.e., we first pull it back to $Y_1$, then $Y_2$ and so on. We will show that there exists a sequence of cycles $\{\xi_k \in \text{CH}^2(Y_k, 1)\}$ with all of them agreeing on the open set $Y \setminus \{p\}$.

By induction, it suffices to pull back the cycle $\xi_{k-1} \in \text{CH}^2(Y_{k-1}, 1)$ to $\xi_k \in \text{CH}^2(Y_k, 1)$. Since $\varphi_k: Y_k \to Y_{k-1}$ is the blowup of $Y_{k-1}$ at $p_{k-1}$,

\[(3.15) \quad Y_k \setminus E_k \cong Y_{k-1} \setminus \{p_{k-1}\}.\]

So the question is again to extend a class in $\text{CH}^2(Y_k \setminus E_k, 1)$ to $\text{CH}^2(Y_k, 1)$. We look at the localization sequence

\[(3.16) \quad \text{CH}^2(Y_k, 1) \to \text{CH}^2(Y_k \setminus E_k, 1) \to \text{CH}^1(E_k) \to \text{CH}^2(Y_k) \]

If $E_k$ is a cone over a conic curve, then $\text{CH}^1(E_k) = \mathbb{Q}$ (see [Ha, Appendix A, Example 1.1.2, p. 428]) and $\gamma: \text{CH}^1(E_k) \to \text{CH}^2(Y_k)$ is obviously injective.

Suppose that $E_k$ is a smooth quadric. This happens in the last step of blowups, i.e., when $k = \mu$ and $m = 2\mu$ is even. Now

\[(3.17) \quad \text{CH}^1(E_k) = \text{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}.\]

Let $L_1, L_2 \subset E_k$ be the two rulings of $E_k$ which generate $\text{CH}^1(E_k)$. We claim that $L_1$ and $L_2$ are numerically independent on $Y_k$, i.e., there exist divisors $D_1, D_2 \subset Y_k$ such that $D_i \cdot L_j = 0$ if $i = j$ and $D_i \cdot L_j \neq 0$ if $i \neq j$. This certainly implies that $\gamma$ is injective.

Note that $Y_{k-1}$ has an ordinary double point $x^2 + y^2 + z^2 + t^2 = 0$ at $p_{k-1}$. It is well known that there exist two small resolutions of $Y_{k-1}$. That is, we may blow down $Y_k$ along either of the two rulings $L_1$ and $L_2$. Let $g: Y_k \to Y'_k$ be the blowdown of $Y_k$ along $L_1$. Let $D$ be an ample divisor on $Y'_k$. Then $g^*D \cdot L_2 \neq 0$ since $D$ is ample on $Y'_k$ and $g^*D \cdot L_1 = 0$ since $g_*L_1 = 0$. We are done.

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