Spontaneous Emission from a Two-Level Atom in a Rectangular Waveguide

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(Dated: May 22, 2014)

Quantum mechanical treatment of light inside dielectric media is important to understand the behavior of an optical system. In this paper, a two-level atom embedded in a rectangular waveguide surrounded by a perfect electric conductor is considered. Spontaneous emission, propagation, and detection of a photon are described by the second quantization formalism. The quantized modes for light are divided into two types: photonic propagating modes and localized modes with exponential decay along the direction of waveguide. Though spontaneous emission depends on all possible modes including the localized modes, detection far from the source only depends on the propagating modes. This discrepancy of dynamical behaviors gives two different decay rates along space and time in the correlation function of the photon detection.

PACS numbers: 14.70.Bh,42.50.Pq,79.60.Jv

I. INTRODUCTION

The photon is one of the fascinating objects that has been researched since Einstein’s introduction. Usually, the currently accepted definition of photon is a monochromatic Fourier mode for electromagnetic waves in the vacuum [1]. Since arbitrary shapes of photonic modes can be generated by superposition of modes, this usually accepted definition has been successfully used to describe many different physical processes [2].

Recently, the technology of dielectric material fabrication is able to handle small sample sizes, comparable to the wavelengths of light [3,4]. Generation of a single photon inside the dielectric media and propagation through waveguide has been considered [5]. In that investigation, a creation (annihilation) operator of the photon was used, which propagates to the right or left. Since this system is a one-dimensional waveguide, \(e^{ikx}\)-type of modes are enough to handle problems for the generation and propagation of photons. Although an integrated photonic crystal circuit operates at the single-photon level and is approximated as one-dimensional system, the actual phenomena occur in three dimensions. So, we need to treat the problem quantum-mechanically with Maxwell’s equations in three dimensions, or the second quantization method to describe photon’s behavior inside the dielectric material. Furthermore, from several formalisms [7], we also need to discern a proper description for photon inside dielectric material by comparing experimentally verifiable quantities.

To initiate such investigation, this paper has considered a simple model system to treat the generation, propagation, and detection of light inside a dielectric material. We have found that the spontaneous decay can induce exponentially localized modes, which are usually ignored in the propagation problem, and that there are two different decay rates for the space and time in their correlation function. Though the exponentially localized modes satisfy the transversality condition, \(\nabla \cdot E = 0\), they do not propagate. So only the nonlocalized modes are propagating modes, which are present in the far-field region. However, in near field region the photon detector measures the whole electric field, and the distinction of propagating modes from the localized modes is not possible through photon detection.

The remainder of the paper is organized as follows. In Sec. II we derive the electric field operator using the second quantization method according to Glauber’s approach [8]. In Sec. III, we treat spontaneous emission in a two-level atom using the electric field operator derived in Sec. II. Finally, we calculate the spatio-temporal correlation function, and in Sec. IV we summarize our conclusions.

II. ELECTRIC FIELD OPERATOR IN RECTANGULAR PEC WAVEGUIDE

To see the effect of dielectric materials on photon emission, propagation, and detection, let’s consider a rectangular waveguide filled by a dielectric material with constant electric permittivity \(\epsilon\) and constant magnetic permeability \(\mu\). Here, we choose a rectangular waveguide since it has a simple though nontrivial geometry. For simplicity, we consider that the dielectric material inside the waveguide is surrounded by a perfect electric conductor (PEC) so that the tangential electric field at boundary is zero. Let’s put the waveguide along the \(z\)-axis, and the center of coordinates at the vertex of the rectangles with the length along \(x\)-axis as \(a\) and that along \(y\)-axis as \(b\) with \(a > b\), as shown in FIG. [1]
The Maxwell equations in source free region are
\[ \nabla \cdot \mathbf{D} = 0, \]  
\[ \nabla \cdot \mathbf{B} = 0, \]  
\[ \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \]  
\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \]  
with constituent equations \( \mathbf{D} = \varepsilon \mathbf{E} \) and \( \mathbf{B} = \mu \mathbf{H}. \)

To get the field operator using the second quantization method, we need complete eigenmodes from the classical Maxwell’s wave equations \([8]\). By assuming a harmonic time dependence \( e^{-i\omega t} \) with frequency \( \nu \), which is considered as eigenvalue, Eqs. (3) and (4) become
\[ \nabla \times \mathbf{E} = i\nu \mathbf{B} = i\nu \mu \mathbf{H}, \]  
\[ \nabla \times \mathbf{H} = -i\nu \mathbf{D} = -i\nu \varepsilon \mathbf{E}. \]

The Maxwell’s equations \([5]\) and \([6]\) lead to wave equations for \( \mathbf{E} \) and \( \mathbf{H} \):
\[ \nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0, \]  
\[ \nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0, \]
with \( k = \nu \sqrt{\mu/\varepsilon}. \) If we assume the propagation along z-axis as \( e^{-\gamma z} \), the longitudinal field components \( (E_x, E_y, H_x \) and \( H_y) \) can be written in terms of the transverse field components \( (E_z \) and \( H_z) \) as follows:
\[ E_x = \frac{1}{k^2} \left( -\gamma \frac{\partial E_z}{\partial x} - i\nu \mu \frac{\partial H_z}{\partial y} \right), \]  
\[ E_y = \frac{1}{k^2} \left( -\gamma \frac{\partial E_z}{\partial y} + i\nu \mu \frac{\partial H_z}{\partial x} \right), \]  
\[ H_x = \frac{1}{k^2} \left( i\nu \frac{\partial E_z}{\partial y} - \gamma \frac{\partial H_z}{\partial x} \right), \]  
\[ H_y = \frac{1}{k^2} \left( -i\nu \frac{\partial E_z}{\partial x} - \gamma \frac{\partial H_z}{\partial y} \right) \]
with \( h^2 = \gamma^2 + k^2. \) Since \( \gamma = \sqrt{h^2 - k^2} \), \( \gamma \) will be pure imaginary for \( \nu > \nu_c \) or pure real otherwise. Here, \( \nu_c = h/\sqrt{\mu} \) is the cutoff-frequency.

The solutions of the wave-equations are divided into two types of modes: the transverse magnetic (TM) modes with \( H_z = 0 \), and the transverse electric (TE) modes with \( E_z = 0 \). The field components for the TM modes are
\[ E_{z\text{TM}} = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ E_{x\text{TM}} = -\frac{\gamma}{h^2} \left( \frac{m\pi}{a} \right) E_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ E_{y\text{TM}} = -\frac{\gamma}{h^2} \left( \frac{n\pi}{b} \right) E_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ H_{x\text{TM}} = i\nu \frac{m\pi}{b} \left( \frac{n\pi}{b} \right) E_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ H_{y\text{TM}} = -i\nu \frac{m\pi}{a} \left( \frac{n\pi}{a} \right) E_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma z}, \]
with positive integers \( m \) and \( n \), and for TE modes are
\[ H_{z\text{TE}} = H_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ E_{z\text{TE}} = \frac{i\nu \mu}{h^2} \left( \frac{n\pi}{b} \right) H_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ E_{y\text{TE}} = -\frac{i\nu \mu}{h^2} \left( \frac{m\pi}{a} \right) H_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ H_{x\text{TE}} = \frac{\gamma}{h^2} \left( \frac{m\pi}{a} \right) H_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-\gamma z}, \]  
\[ H_{y\text{TE}} = \frac{\gamma}{h^2} \left( \frac{n\pi}{b} \right) H_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma z}, \]
with non-negative integers \( m \) and \( n \) without simultaneous \( m = 0 \) and \( n = 0 \). The boundary conditions suggest that allowable \( \gamma_{mn} = \sqrt{h_{mn}^2 - k^2} \) with \( h_{mn} = \sqrt{(m\pi/a)^2 + (n\pi/b)^2} \). The propagating mode with the lowest cutoff frequency is a TE mode with \( m = 1 \) and \( n = 0 \).

In the usual treatment of the propagation problem with injected light as input along the waveguide, the modes with exponential behavior \( (\gamma > 0) \) have been excluded since they diverge at \( |z| \to \infty \), and only the oscillating eigenmodes \( (\gamma = i\beta) \) are considered, since they can propagate into the far-field region, where the detector is located. However, in our system the light is generated inside the dielectric material, and we want to solve a photon-generation problem in quantum mechanics. So, the usually ignored exponential decay modes can operate as one of the quantized modes, which can interact with the embedded atom. For a single atom located at \( z = 0 \), the replacement of the exponential function \( e^{-\gamma z} \) into \( e^{-\gamma |z|} \) is an acceptable eigenmode with a singularity in first derivative at \( z = 0 \). One thing to note is that all eigenmodes satisfy the transversality condition, that is \( \nabla \cdot \mathbf{E}_{mn}^{\text{TE(TM)}} = 0 \). Since the corresponding wave vector \( \mathbf{k} \) for the localized modes has a complex number in component along the z-axis, these modes do not propagate and only oscillate in time near the source.

![FIG. 1. Diagram for a rectangular waveguide along z-axis. The waveguide is filled with a dielectric material with electric permittivity \( \varepsilon \) and magnetic permeability \( \mu \). We assume that \( a > b \). (Color online)\]
The corresponding electric field operator can be written as [8]

\[ \hat{E} = \sum_{\nu} \sum_{K} \{ \mathbf{E}^\nu_K \hat{a}^\nu_K e^{-i\nu t} + (\mathbf{E}^\nu_K)^* \hat{a}^\nu_K e^{i\nu t} \} \]  

(23)

where \( K = \{ \text{TE( or TM)}, m, n \} \) denotes the set of quantum numbers for eigenmodes, and \( \hat{a}^\nu_K \) (\( \hat{a}^\nu_K \) adjoint) is annihilation (creation) operator for the corresponding mode. \( \mathbf{E}^\nu_K \) are the eigenmodes for the electric field for a fixed frequency with orthonormality

\[ \int d\mathbf{r} \mathbf{E}_K^\nu \cdot (\mathbf{E}_K^\nu)^* = 0 \quad \text{for } K \neq K'. \]  

(24)

The orthogonality condition for modes with different frequency is still ambiguous. However, since the frequency is given as the eigenvalue of Maxwell’s equation, the summation over \( \nu \) might be similarly interpreted as that of continuous quantum states such as position states [10]. From the creation \( \hat{a}^\nu_K \) and annihilation \( \hat{a}^\nu_K \) operators in the \( K \) mode with frequency \( \nu \), we can define the number operator \( \hat{n}^\nu_K \equiv \hat{a}^\nu_K \hat{a}^\nu_K \) with a photon number state \( |n^\nu_K\rangle \) as the corresponding eigenstate, such that \( \hat{a}^\nu_K |n^\nu_K\rangle = n^\nu_K |n^\nu_K\rangle \). The general photon number state can be constructed by applying the corresponding creation operator:

\[ |n^\nu_K\rangle \equiv \frac{(\hat{a}^\nu_K)^{\dagger n}}{\sqrt{n!}} |0\rangle. \]  

(25)

Similarly to continuous modes, such as position modes or momentum modes in quantum mechanics [10], we can assume the completeness and orthonormality of these modes for different frequencies:

\[ \langle n^\nu'|n^\nu\rangle = \delta_{K,K'} \delta(\nu - \nu'), \]  

(26)

and

\[ \sum_{\nu} \sum_{K} |n^\nu_K\rangle \langle n^\nu_K| = 1. \]  

(27)

According to Eqs. (26) and (27), we can treat the states over the continuous variable \( \nu \) as independent to each other, similarly to ones over discrete variables.

To find the unknown coefficients \( E_0 \) and \( H_0 \), let’s assign \( h \nu \) as the energy of the photonic modes:

\[ \frac{1}{2} \int d^3 r \{ \epsilon |\mathbf{E}_K^\nu|^2 + \mu |\mathbf{H}_K^\nu|^2 \} = h \nu, \]  

(28)

where \( \hbar \) is Dirac’s constant. Since the integration along \( z \)-axis diverges for the propagating modes, we should be careful when carrying out the above integration along \( z \)-axis. Similar to the vacuum mode, we might consider a periodicity along \( z \)-axis, and restrict the integration range \([-L/2, L/2]\) for propagating modes. The normalization gives for a propagating TM mode (\( \gamma > 0 \))

\[ |E_0|^2 = \frac{4\gamma h \hbar^2_{mn}}{e^2 \mu \nu LS}, \]  

(30)

for a propagating TE mode (\( \gamma = i\beta \))

\[ |H_0|^2 = \frac{4\hbar^2_{mn}}{\nu \epsilon \mu^2 LS}, \]  

(31)

and for a localized TE mode (\( \gamma > 0 \))

\[ |H_0|^2 = \frac{4\gamma h \hbar^2_{mn}}{\nu \epsilon \mu^2 S}, \]  

(32)

with the cross-sectional area of the waveguide \( S = ab \). Since the periodicity \( L \) is arbitrary, we need another condition for the summation over frequency for the propagating modes.

For a description of photons in vacuum, the summation index over states is wave-number \( k \), which satisfy periodicity \( k = 2\pi n/L \) with integers \( n \). The transition from discrete summation to continuous integration is given by [2]

\[ \sum_k \rightarrow \left( \frac{L}{2\pi} \right) \int dk. \]  

(33)

Similarly, we can consider the wave-number \( \beta \) along \( z \)-axis for the propagating modes. The summation over \( \nu \) is divided into two parts by the cut-off frequency \( \nu_c \) for given \( K = \{ \text{TE( or TM)}, m, n \} \):

\[ \sum_{\nu} \rightarrow \sum_{\nu < \nu_c} + \sum_{\nu > \nu_c} = \sum_{\nu < \nu_c} + \sum_{\beta}. \]  

(34)

Then, the summation over \( \beta \) can be transformed into continuous integration similarly as is done in vacuum

\[ \sum_{\beta} \rightarrow \left( \frac{L}{2\pi} \right) \int \frac{dk_z}{d\beta} d\beta = \left( \frac{L}{2\pi} \right) \int \frac{d\beta}{v_{gz}}, \]  

(35)

where \( v_{gz} = \frac{d\beta}{dk_z} \) is the group velocity along the \( z \)-axis. In our case, \( v_{gz} = 1/\sqrt{\epsilon_0} \), which is same to the phase velocity. Then the arbitrary quantization length \( L \), shown in Eqs. (29) and (31), will be canceled out by the \( L \) in Eq. (35) in the calculation of any physical observables.

The localized modes show exponential decay along the \( z \)-direction, and depend on the position of the two-level atom. However, along the \( x \)- and \( y \)-axes they show an oscillatory behavior. Furthermore, we emphasize again that these localized modes are also quantized modes since they are the eigenmodes of the two wave equations, Eq. (7) and Eq. (8).

### III. SPONTANEOUS EMISSION

#### A. Hamiltonian

Let’s consider a spontaneous emission from a two-level atom embedded inside the PEC rectangular waveguide,
shown in FIG. [2] With the dipole approximation and rotating wave approximation, the Hamiltonian [2] is

$$H = H_P + H_A + H_{\text{int}}$$

where

$$H_P = \sum_\nu \sum_K \hbar \nu \left( (\hat{a}_K^\nu)\dagger \hat{a}_K^\nu + \frac{1}{2} \right),$$

$$H_A = \hbar \omega_a |a\rangle\langle a| + \hbar \omega_b |b\rangle\langle b|,$$

$$H_{\text{int}} = -e\mathbf{r} \cdot \mathbf{E} = \hbar \sum_\nu \sum_K g_K^\nu \hat{\sigma}_+ \hat{a}_K^\nu + \text{H.C.}.$$  

Here, H.C. denotes Hermitian Conjugate. The atomic transition operators are

$$\hat{\sigma}_+ = |a\rangle\langle b|,$$

$$\hat{\sigma}_- = (\hat{\sigma}_+)^\dagger = |b\rangle\langle a|,$$

and the interaction coefficient is

$$g_K^\nu = -\frac{\varphi_{ba} \cdot \mathbf{E}_K^\nu}{\hbar}$$

with dipole transition matrix element $\varphi_{ba} = e |b\rangle |a\rangle$.

### B. Decay rate and shift of transition energy

In the interaction picture, the Hamiltonian is

$$\mathcal{V} = \hbar \sum_\nu \sum_K [g_K^\nu \hat{\sigma}_+ \hat{a}_K^\nu e^{i(\omega - \nu)t} + \text{H.C.}]$$

with $\omega = \omega_a - \omega_b$. Here we ignore the zero-point energy $\frac{1}{2} \hbar \nu$, since it just shifts the energy level of Hamiltonian [2]. From the dressed state for the two-level atom with the field

$$|\psi(t)\rangle = c_a(t) |a\rangle|0\rangle + \sum_\nu \sum_K c_{b,\nu,K} |b, 1^\nu_K\rangle,$$

the corresponding Schrödinger equation,

$$\dot{|\psi(t)\rangle} = -\frac{i}{\hbar} \mathcal{V} |\psi(t)\rangle,$$

gives the following equations of motion for the coefficients $c_a(t)$ and $c_{b,\nu,K}(t)$

$$\dot{c}_a(t) = -i \sum_\nu \sum_K g_K^\nu e^{i(\omega - \nu)t} c_{b,\nu,K}(t),$$

$$\dot{c}_{b,\nu,K}(t) = -i (g_K^\nu)^* e^{-i(\omega - \nu)t} c_a(t),$$

with initial conditions $c_a(0) = 1$ and $c_{b,\nu,K}(0) = 0$, which means that the two-level atom is prepared in an excited state. Integration of Eq. (47) over $t$ and substitution into Eq. (46) gives the following integro-differential equation for $c_a(t)$:

$$\dot{c}_a(t) = -\sum_\nu \sum_K |g_K^\nu|^2 \int_0^t dt' e^{i(\omega - \nu)(t-t')} c_a(t')$$

$$\approx -\sum_\nu \sum_K |g_K^\nu|^2 \int_0^t dt' e^{i(\omega - \nu)(t-t')} c_a(t)$$

$$= -i \sum_\nu \sum_K |g_K^\nu|^2 \left[ \frac{P}{\omega - \nu} - i\pi \delta(\omega - \nu) \right] c_a(t)$$

$$= -\left( \frac{\Gamma_{\text{eff}}}{2} - i \delta \omega \right) c_a(t),$$

where the Markovian approximation for slow varying $c_a(t)$ is used to get this term out of the time integration [2], and $P$ denotes the Cauchy principal value [11]. Here, $\Gamma_{\text{eff}}$ is the decay constant and $\delta \omega$ is the level shift of the transition from the upper level to lower one:

$$\Gamma_{\text{eff}} = \pi \sum_\nu \sum_K |g_K^\nu|^2 \delta(\omega - \nu),$$

$$\delta \omega = \sum_\nu \sum_K P |g_K^\nu|^2 \frac{1}{\omega - \nu}.$$  

The summation goes over all possible states including the localized modes.

### C. Propagation through the waveguide

The temporal behavior of the excited state is given by

$$c_a(t) = e^{-(\Gamma_{\text{eff}}/2 - i \delta \omega)t} c_a(0) = e^{-(\Gamma_{\text{eff}}/2 - i \delta \omega)t}$$

$$c_{b,\nu,K}(t) = -i (g_K^\nu)^* \int_0^t dt' e^{-i(\nu - \omega)t'} e^{-\Gamma_{\text{eff}}' t'/2}$$

$$= (g_K^\nu)^* \left[ \frac{1 - e^{i(\nu - \omega)(t - \Gamma_{\text{eff}}'/2)}}{\nu - \omega + i\Gamma_{\text{eff}}'/2} \right],$$

with shifted frequency $\tilde{\omega} = \omega - \delta \omega$. So all the quantized modes are induced by the de-excitation of the atom.

Since we put the detector far away from the atom, the measured modes are just the propagating modes, because the localized modes decay exponentially along the wave-guide axis and can be ignored in a region near the detector. In addition, $c_a(t)$ is also exponentially decaying.
in time. After \( t \gg \Gamma_{\text{eff}}^{-1} \), the dressed state becomes

\[
|\psi(t)\rangle \approx \sum_{\nu} \sum_{K'} c_{b,\nu,K} |b, 1_K^r\rangle = |b\rangle \otimes |\gamma\rangle
\]

(53)

where the positive frequency part in electric field operator from Eq. (23) is

\[
\hat{E}^{(+)}(r, t) = \sum_{\nu} \sum_{K} E_K \hat{a}_K e^{-i\nu t}. 
\]

(57)

For the single photonic mode case, Eq. (55), we obtain the following expression through a contour integral, which is explicitly derived in the Appendix:

\[
\langle 0 | \hat{E}^{(+)}(r, t) | \gamma \rangle_{\text{sm}} = i \frac{\hat{w}}{\pi^2 \epsilon S} \frac{\sqrt{\epsilon S}}{\sqrt{(\pi/\alpha)^2 - \hat{w}^2}} 
\times \sin \frac{\pi x_0}{\alpha} \sin \frac{\pi x_0}{\alpha} \Theta(t - \sqrt{\epsilon \mu} z) 
\times e^{(i\beta_r - i\beta_i) \Delta z - i(\hat{w} - i\Gamma_{\text{eff}}/2) t},
\]

(58)

where the distance \( \Delta z \) is from the source atom to the field operator evaluation point along \( z \)-direction, and \( \Theta \) is the step function. Explicit expressions for \( \beta_r \) and \( \beta_i \) are shown in the Appendix. One thing to note is that the above equation is evaluated at the shifted resonance frequency \( \hat{w} = \omega - \delta \omega \).

\[\text{FIG. 3. Profile of correlation function in (z, t) plane. The distant detector at z can detect after some time t allowable by the causality relation, shown as black solid line on (z, t) plane. In a time t along z axis, the detection probability shows exponential behavior with decay constant } \Gamma_{\text{spa}}, \text{ drawn by red dashed line. Similarly, the detection probability at fixed position is maximum at the initial time and shows exponential decay along t with decay constant rate } \Gamma_{\text{eff}}. \text{ Generally, } \Gamma_{\text{eff}} \text{ and } \Gamma_{\text{spa}} \text{ are different. The maximum value of correlation at z shows also the exponential behavior. The values of parameters are } \sqrt{\epsilon \mu} = 1.2 \text{ and } \Gamma_{\text{spa}}/\Gamma_{\text{eff}} = 0.8. \text{ (Color online)}}\]

Then, the first-order correlation function is

\[
G_{\text{sm}}^{(1)}(r, t) = \left( \frac{\hat{w}}{\pi^2 \epsilon S} \right)^2 \frac{\hat{w}^2}{(\pi/\alpha)^2 - \hat{w}^2} \sin^2 \frac{\pi x_0}{\alpha} \sin^2 \frac{\pi x_0}{\alpha} \Theta(t - \sqrt{\epsilon \mu} \Delta z) e^{\Gamma_{\text{spa}} \Delta z - \Gamma_{\text{eff}} t},
\]

(59)
where the spatial decay rate is

$$\Gamma_{\text{spa}} = 2\beta_0. \tag{60}$$

One thing to note is that the ratio between the spatial decay rate $\Gamma_{\text{spa}}$ and the temporal decay rate $\Gamma_{\text{eff}}$ is generally different from the velocity $\sqrt{\epsilon\mu}$. In Fig. 3, we show the profile of the correlation function in $(z,t)$ plane. The distant detector at $z$ from the source atom, located at $z_0$, can detect after a propagation time $t = \sqrt{\epsilon\mu}|z-z_0|$ by the causality relation. Along the $z$ axis, the detection probability shows exponential behavior with decay constant $\Gamma_{\text{spa}}$. Similarly, the detection probability at a fixed position is maximum at the arrival of the photon and shows exponential decay along $t$ with decay constant rate $\Gamma_{\text{eff}}$. It also shows the exponential behavior along the line with $t - \sqrt{\epsilon\mu}z = \text{const.}$, since the ratio is different from $\sqrt{\epsilon\mu}$.

In Eq. (60), the spatial decay and temporal decay are different in the first order correlation function due to the geometrical boundary of the waveguide. The ratio of two decay rates $\Gamma_{\text{spa}}/\Gamma_{\text{eff}}$ is equal to $\sqrt{\epsilon\mu}$ at $\tilde{\omega} = \tilde{\omega}_d$ with

$$\tilde{\omega}_d^2 = \frac{\epsilon^2\mu^2\Gamma_{\text{spa}}^2 + 12\epsilon\mu^2\Gamma_{\text{eff}}^2(\pi/a)^2 + 4(\pi/a)^4}{8(\epsilon\mu\Gamma_{\text{eff}}^2 + 2(\pi/a)^2)\epsilon\mu}, \tag{61}$$

which is derived from Eq. (70) in the Appendix. In the limit $\omega \to \infty$, the spatial decay rate approaches the temporal decay rate. In the $a \gg b$ limit (that is, like a shallow slab), the temporal decay rate is larger than the spatial one when $\tilde{\omega}^2 > \Gamma_{\text{eff}}^2/8$. The correlation function is maximum in the cross-section of the waveguide when the detector and the embedded atom are in the middle of the waveguide along the $x$ axis.

In comparison, the correlation function in free space is

$$G^{(2)}_{\text{free}}(r,t) = \frac{|E_0|^2}{|r-r_0|^2} \Theta \left( t - \frac{|r-r_0|}{c} \right) e^{-\Gamma t} \left( t - \frac{|r-r_0|}{c} \right), \tag{62}$$

where $c$ is the speed of light, and the vacuum decay rate

$$\Gamma = \frac{1}{4\pi\epsilon_0} \frac{4\omega^3\nu_{ab}^2}{3\hbar c^3}. \tag{63}$$

with the dipole moment $\nu_{ab}$. Here,

$$E_0 = -\frac{\omega^2\nu_{ab}\sin^2\eta}{4\pi\epsilon_0 \epsilon_0 c^3}, \tag{64}$$

and $\eta$ is the angle of the dipole moment from $z$-axis.

IV. CONCLUSION

A simple model for the spontaneous emission problem is reconsidered in terms of the second quantization formalism. It has been revealed that the usually ignored localized modes should be considered as one of the quantized modes to calculate the decay rate for spontaneous emission. However, the description for propagation and detection of photon is similar to classical treatment since localized modes cannot contribute to these phenomena. One thing to notice is that the decay rates in space and time are different with smaller spatial decay rate usually. The problem with dispersive dielectric material will be investigated later.

Furthermore, since we get the quantized electric field operator in this simple system, we can describe any quantum optical phenomena also. However, to extend Glauber’s method to other systems might be difficult since it is usually difficult to calculate the complete set of eigenfunctions. To overcome this obstacle, approximate methods to the quantized field should be developed, such as to numerically evaluate the eigenmodes for a description of the behavior of the optical system.

APPENDIX: EVALUATION OF THE FIRST-ORDER CORRELATION FUNCTION

The calculation for the transition probability amplitude at the detection from a propagating photonic mode to vacuum mode is explicitly shown. From Eqs. (55) and (57),

$$\langle 0|E^{(+)}(r,t)|\gamma\rangle_{\text{sm}} = \sum_{\nu} \epsilon_{\text{TE,10}} e^{-i\nu t} \frac{\theta_{\text{TE,10}}(r_0)}{(\nu - \tilde{\omega}) + i\Gamma_{\text{eff}}/2}$$

$$\approx -\frac{2\sqrt{\epsilon\mu}}{\pi\epsilon S^2} \sin \frac{\pi x_0}{a} \sin \frac{\pi x}{a} \int d\beta \frac{\nu_{ab}^* e^{-i\beta (z-z_0)}-i\nu t}{(\nu - \tilde{\omega}) + i\Gamma_{\text{eff}}/2}$$

$$\approx -\frac{2\tilde{\omega} \sqrt{\epsilon\mu}\nu_{ab}}{\pi\epsilon S} \sin \frac{\pi x_0}{a} \sin \frac{\pi x}{a} \int d\beta \frac{e^{-i\beta (z-z_0)}-i\nu t}{(\nu - \tilde{\omega}) + i\Gamma_{\text{eff}}/2} \tag{65}$$

where the frequency $\nu$ is approximated by the shifted resonance frequency $\tilde{\omega}$.

![FIG. 4. Profile of contour integral in complex $\nu$-plane. (Color online)](image-url)
where the pole is located,
\[
\int d\beta \frac{e^{i\beta(z_0-z)-i\nu t}}{(\nu-\tilde{\omega}) + i\Gamma_{\text{eff}}/2} \approx \sqrt{\frac{i\mu}{\pi}} \int \frac{\nu}{\beta} d\nu \frac{e^{i\beta(z_0-z)-i\nu t}}{(\nu-\tilde{\omega}) + i\Gamma_{\text{eff}}/2} \approx \sqrt{\frac{i\mu}{\pi}} \frac{\tilde{\omega}}{\nu} \int d\nu \frac{e^{i\beta(z_0-z)-i\nu t}}{(\nu-\tilde{\omega}) + i\Gamma_{\text{eff}}/2}. \tag{66}
\]
Since the pole is at lower-half plane in complex \(\nu\)-plane, the contour \(\mathcal{C}\) is as shown in Fig. 4 to get the nonzero result.
\[
\int d\nu \frac{e^{i\beta(z_0-z)-i\nu t}}{(\nu-\tilde{\omega}) + i\Gamma_{\text{eff}}/2} = \frac{1}{2\pi i} \int_{\mathcal{C}} d\nu \frac{e^{i\beta(z_0-z)-i\nu t}}{(\nu-\tilde{\omega}) + i\Gamma_{\text{eff}}/2} = \frac{1}{2\pi i} \Theta(t - \sqrt{\epsilon \mu}) e^{i(\beta_r + i\beta_i)\Delta z - i(\tilde{\omega} - i\Gamma_{\text{eff}}/2)t}, \tag{67}
\]
where \(\Delta z = |z - z_0|\) is the distance between the source atom and a detector, and \(\beta_r\) and \(\beta_i\) are
\[
\beta_r + i\beta_i = \beta(\nu = \tilde{\omega} - i\Gamma_{\text{eff}}/2)
\]
\[
= \sqrt{\frac{i\mu}{\pi}} \left(\tilde{\omega} - i\frac{\Gamma_{\text{eff}}}{2}\right)^2 - \left(\frac{\pi}{a}\right)^2
\]
\[
= \sqrt{\frac{i\mu}{\pi}} \left(\tilde{\omega}^2 - \frac{\Gamma_{\text{eff}}^2}{4}\right) - \left(\frac{\pi}{a}\right)^2 - i\sqrt{\epsilon \mu} \tilde{\omega} \Gamma_{\text{eff}}. \tag{68}
\]
After a simple evaluation, we get
\[
\beta_r = \sqrt{\frac{A + \sqrt{A^2 + 4\Gamma_{\text{eff}}^2}}{2}} > 0, \tag{69}
\]
\[
\beta_i = -\sqrt{\frac{A^2 + 4\Gamma_{\text{eff}}^2 - A}{2}} \tag{70}
\]
with \(A = \sqrt{\epsilon \mu} (\tilde{\omega}^2 - \Gamma_{\text{eff}}^2/4) - \pi^2/a^2\), and \(B = \sqrt{\epsilon \mu} \tilde{\omega} \Gamma_{\text{eff}}/2\). Here, we choose \(\beta_r\) as positive.

Substituting Eq. (67) into Eq. (69) we obtain the transition probability amplitude
\[
\langle 0|\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)|\gamma\rangle_{\text{sm}} = i\tilde{\omega} \sqrt{\epsilon S} \sqrt{\frac{\tilde{\omega}}{(\pi/a)^2 - \omega^2}} \times \sin \frac{\pi x}{a} \sin \frac{\pi \rho}{a} \Theta(t - \sqrt{\epsilon \mu} \tilde{\omega}) \times e^{i(\beta_r - i\beta_i)\Delta z - i(\tilde{\omega} - i\Gamma_{\text{eff}}/2)t}. \tag{71}
\]

ACKNOWLEDGMENTS

This work was supported by the Air-Force office of Scientific Research and the National Science Foundation.

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