The Hamiltonians of Linear Quantum Fields:
II. Classically Positive Hamiltonians

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Abstract

For linear bose field theories, I show that if a classical Hamiltonian function is strictly positive in a suitable sense, the classical evolution must be conjugate, by a symplectic motion, to a strongly continuous one-parameter orthogonal group. This can be viewed as an infinite-dimensional analog of the existence of action-angle coordinates.

This result is used to show that there is an intimate connection between unitarity of the quantum evolution and boundedness-below of the quantum Hamiltonians. (Recent work has shown that generically, the Hamiltonian operators of quantum fields in curved space–time are not bounded below and do not generate unitary evolutions.) Precisely, whenever the quantum Hamiltonians exist as self-adjoint operators, they are bounded below. Lower bounds on the normal–ordered quantum Hamiltonian operators are computed.

Finally, it is shown that there is a broad class of “quantum inequalities:” if \( f^{ab} \) is a smooth compactly-supported future-directed test function, then the operator \( \int f^{ab} T_{ab} \, d\text{vol} \) is bounded below.

1 Introduction

This is the second of a three-part series analyzing the Hamiltonians of linear quantum fields. A general introduction will be found in the first paper, Helfer (1999), hereafter called Part I.

When the evolution of the field operators in a linear quantum field theory does not preserve their decomposition into creation and annihilation parts, the analysis of the theory can be difficult and is not yet wholly understood. This situation can arise for several reasons:

- It occurs naturally in the presence of time-dependent external potentials; in particular, it is the generic situation for quantum fields in curved space–time.

- It occurs when several linear fields are linearly coupled, for instance, in models of the quantum electromagnetic field in dispersive media.
It occurs when the evolution is not a perfect symmetry. This happens, for example, when the corresponding operator is an inhomogeneous spatial or temporal average of the stress–energy. Such operators are the subject of the quantum inequalities, which bound the persistence of negative energy densities, and are important in quantum measurement issues.

In such situations, phenomena which are at least naïvely very pathological can occur. In the case of quantum fields in curved space–time, it has been shown that the quantum Hamiltonian operators have only a restricted existence, being defined only as quadratic forms. (That is, there is a dense family of states for which the expectations \( \langle \Psi | \hat{H} | \Psi \rangle \) are defined, but there are no known non-zero states for which \( \hat{H} | \Psi \rangle \) exists as an element in the physical Hilbert space.) The quantum Hamiltonians’ expectations are unbounded below, and the corresponding evolutions are not unitarily implementable. This means that the algebra of field operators does not evolve by unitary motions. This is distinct from the evolution of the state vectors, which is unitary (and, in the usual “relativistic Heisenberg” picture, trivial except for reductions). Any sort of non-unitarity in quantum theory should be taken seriously, and the physical significance of that discovered recently is not yet clear.

One of the main aims of these papers is to get a firm enough mathematical control on the phenomenon that progress on a physical understanding will be possible.

It may be helpful to comment on how these issues are related to the “abstract algebraic approach.” In general, “algebraic approaches” seek to formulate quantum field theory, as much as possible, in terms of the algebras of observables. They seek to avoid, or treat as derived concepts, the realization of those observables as operators on Hilbert spaces.

While such approaches have demonstrable power, the construction of a physical representation is necessary to fully define the quantum theory, at least in a conventional sense. For example, the set of allowable \( n \)-point functions (with the correct asymptotics) depends on the physical choice of representation, not just on the algebra of observables. In general, while some of the physical content is specified in the algebraic structure alone, the full physical content depends on knowing the representation. For instance, Hawking’s (1974, 1975) prediction of black-hole evaporation relies strongly on the choice of representation. (See Helfer 2003 for a critique of this prediction.)

The unboundedness-below and failure of unitary implementability are problems which are much more apparent in an approach based on the construction of representations than in an abstract algebraic approach. Since the aim of these papers is to develop a framework in which the significances of these issues can be assessed, the approach here is very much based on the study of representations.

Which sort of approach will ultimately — when the significances of these issues are understood — be most appropriate remains to be seen. Should the apparent pathologies turn out to be simply mathematical fine points not of real physical consequence, an algebraic approach might well be the most suitable. On the other hand, unitarity is such a basic concept of quantum theory, and boundedness-below of energy such a fundamental concern in any physical theory, that it may well turn out that these issues are important and are best understood within a representation.

In Part I of this series, I determined under what conditions an infinitesimal symmetry of the classical phase space gave rise to a self-adjoint quantum operator. In that paper, no special properties of the symmetry were used. In the present paper, I specialize to those symmetries corresponding to positive classical Hamiltonian functions. These arise in particular for those which are energy operators, in the sense that they correspond to
evolution forward in time.

**Structure Theory**  The first aim of this paper is to classify the different possible structures such classical Hamiltonians might have. This will then be used to analyze their quantizations.

Theorem 1, then, is a classification of the possible classical Hamiltonians. Essentially, it states that a classically positive Hamiltonian function generates a family of motions which is similar, by a canonical transformation, to a one-parameter orthogonal group. (However, the theorem as stated and proved here requires an additional technical hypothesis, which is believed not to be necessary. See the discussion preceding the theorem.) This comes about because even in this infinite-dimensional context there is something like a compactness of the constant-energy surfaces in the classical phase space. This is a delicate and remarkable result, which is presumably of interest in the theory of infinite-dimensional dynamical systems. It implies an infinite-dimensional analog of the existence of action-angle variables, for example. It also implies that the classical evolution remains uniformly bounded in time. This extends the structure theory for classical Hamiltonian field theories that was developed earlier (see Chernoff and Marsden 1974).

**Self-Adjointness and Boundedness-Below**  Using the classical structure theorem to get a handle on the quantum theory, we find (Theorem 2) that a classically positive Hamiltonian is self-adjointly implementable iff the similarity effecting its transformation to a generator of orthogonal motions corresponds to a restricted Bogoliubov transformation, and, in this case, the quantum Hamiltonian must be bounded below.1 Thus one has a very strong connection between self-adjointness and boundedness-below, for energy operators. It should be emphasized that these results make no presupposition about what renormalization prescription is to be used.

There is an old folk-theorem in quantum field theory: “A Hamiltonian determines its quantization,” meaning that a formal expression for a Hamiltonian should have a unique (modulo c-numbers) implementation as an operator. Theorem 2 can be viewed as allowing one to make this statement precise, for linear field theories. It shows that classically positive Hamiltonians have certain mathematically allowed quantizations, which may or may not be physically acceptable, according to whether the Bogoliubov transformation is restricted. One might think that the correct interpretation of the present results is simply that, when the Bogoliubov transformations turn out not to be restricted, one picked the “wrong” original set of canonical variables, and one should choose another, leading to an allowed quantization. However, at least for the case of quantum fields in curved space–time, this does not seem to be the correct interpretation. There, the acceptable choices of canonical variables for quantization are determined by the “Hadamard” condition, and in general the Hamiltonians are not compatible with this.

In other words, the physical considerations leading to a choice of canonical variables, and those leading to the choice of Hamiltonian, conflict in that the Hamiltonian cannot be self-adjointly realized. It should also be mentioned that even if one chooses a mathematically allowed quantization giving rise to a self-adjoint Hamiltonian on one hypersurface in space–time, one would need inequivalent representations for the Hamiltonians at nearby hypersurfaces.

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1Recall that the restricted Bogoliubov transformations are those which lead to unitarily equivalent quantizations.
Normal-Ordering and Lower Bounds

The results described so far are general statements about when a classical Hamiltonian is self-adjointly implementable at the quantum level, without presupposing a specific renormalization scheme. One would like to know what relation these results bear to standard renormalization theory. A choice of renormalization prescription is necessary, too, to move beyond the statement that the Hamiltonians are bounded below and be able to speak of their lower bounds (since renormalization in particular determines the c-number contribution to the Hamiltonians).

Almost universally for linear fields, normal-ordering (or an equivalent prescription, like point-splitting) is used as the prescription. We show that, for classically positive operators, normal ordering may not suffice to renormalize the Hamiltonian, and that finiteness of the normally-ordered ground-state energy is equivalent to the existence of the normally-ordered Hamiltonian as a self-adjoint operator (Theorems 3 and 4). We also compute some lower bounds for the normally-ordered ground state energy (Theorem 5).

Quantum Inequalities

For some time now, it has been conjectured that suitably temporally averaged measures of the energy density operator for quantum fields in curved space–time should be bounded below. Such bounds are known as quantum inequalities, following pioneering work of Ford (1978). We establish the existence (but not the precise numerical bounds) of a wide class of these: for any smooth, compactly-supported test function $f^{ab}$ which is future-directed in the sense that it is at each point a symmetrized product $t^a u^b + u^a t^b$ of future-directed vectors, the operator

$$\int \hat{T}_{ab} f^{ab} \, d\text{vol}$$

is bounded below (Theorem 7). The significance of this result is discussed more fully at the beginning of section 6.

Unfortunately, in order to establish these inequalities, we cannot make direct use of the earlier structure results. This is because the Hamiltonians corresponding to averages of components of the stress–energy over compact regions of space–time, do not satisfy the strong form of classical positivity needed for those theorems. The quantum inequalities are proved from a partial result on the structure theory for these “weakly positive” classical Hamiltonians, Theorem 8.

The organization of the paper is this. Section 2 contains some preliminaries. Section 3 goes over the main structure of classically positive Hamiltonians; in Section 4, these results are applied to quantum field theory. Section 5 works out the connection with normal ordering, and Section 6 the structure necessary to establish the existence of the quantum inequalities. The last section contains some discussion.

Summary of Notation. Here is a summary of the notation used. Unfortunately, there are quite a few things denoted conventionally by similar symbols.

- $H$ is the space of solutions of the classical field equations, a real separable Hilbertable space equipped with a symplectic form $\omega$.
- $H_C$ is the space $H$ equipped with the complex structure defined by $J$, and so made into a complex Hilbert space.
- $\mathcal{H}$ is the Hilbert space on which the representation acts.
- $\| \cdot \|_{\text{op}}$ is the operator norm.
- $\| \cdot \|_{\text{HS}}$ is the Hilbert–Schmidt norm.
- $A$ is the field algebra.
- $A$ is the Hamiltonian vector field on the space of classical solutions.
\( A \) is the Lie adjoint of \( A \), that is, the derivative of conjugation by \( g(t) = e^{tA} \).

Note. Since \( H \) is not canonically a Hilbert space, I will generally emphasize the dependence of properties on the choice of norm, that is, of \( J \). Thus one has \( J \)-linear, rather than complex-linear, transformations. Similarly, there are \( J \)-symmetric, \( J \)-orthogonal, etc., transformations.

2 Preliminaries

Throughout, we shall let \( H \) be a complex infinite-dimensional separable Hilbert space. The complex inner product on \( H \) will be denoted \( \langle \cdot, \cdot \rangle \). We shall let \( H \) be the underlying real Hilbert space. Then we write \( J: H \to H \) for the real-linear map given by \( v \mapsto iv \), and

\[
(v, w) = \Re \langle v, w \rangle \\
\omega(v, w) = \Im \langle v, w \rangle
\]

Then \( \langle \cdot, \cdot \rangle \) is the canonical real inner product on \( H \) and \( \omega \) is a symplectic form on \( H \) which is non-degenerate in that it defines isomorphisms from \( H \) to its dual. Note that

\[
(v, w) = \omega(v, Jw).
\]

Thus any two of \( \omega \), \( J \) and \( \langle \cdot, \cdot \rangle \) determine the third.

Throughout, the real adjoint of a real-linear operator (perhaps only densely defined) \( L \) will be denoted \( L^* \). Thus the defining relation is \( \langle v, L^*w \rangle = (Lv, w) \) with domain \( D(L^*) = \{ w \in H \mid (v, L^*w) = (Lv, w) \text{ for some } L^*w \text{ for all } v \in D(L) \} \).

Definition 1. The symplectic group of \( H \) is

\[ \text{Sp}(H) = \{ g: H \to H \mid g \text{ is linear, continuous, and preserves } \omega \} \]

Its elements are the symplectomorphisms.

The symplectic group does not depend on the real inner product on \( H \) (or on the complex structure); it depends only on \( \omega \) and the structure of \( H \) as a Hilbertable space. It has naturally the structure of a Banach group, using the operator norm to define the topology.

Definition 2. The restricted symplectic group of \( H \) is

\[ \text{Sp}_{\text{rest}}(H) = \{ g \in \text{Sp}(H) \mid g^{-1}Jg - g \text{ is Hilbert–Schmidt} \} \]

We recall that a strongly continuous one-parameter subgroup of \( \text{Sp}(H) \) is a one-parameter subgroup \( t \mapsto g(t) \) such that, for each \( v \in H \), the map \( t \mapsto g(t)v \) is continuous. (In general, one can also consider semigroups, defined for \( t \geq 0 \), but as every symplectomorphism is invertible, in our case every semigroup extends to a group, which is strongly continuous iff the semigroup is.) According to the Hille–Yoshida–Phillips Theorem, such groups have the form \( g(t) = e^{tA} \), where \( A \) is a densely-defined operator on \( H \) (with certain spectral properties), and \( \|g(t)\|_{\text{op}} \leq Me^{\beta|t|} \) for some \( M, \beta \geq 0 \). The spectrum of \( A \) is confined to the strip \( |\Re \lambda| \leq \beta \).
3 Classically Positive Hamiltonians

The analysis so far has been concerned with general symmetries of the phase space. In the case of time evolution, there are important additional properties. The most fundamental of these is that, in the classical context, the energy cannot be negative. Indeed, this fact plays a key role in establishing the existence and stability of temporal evolution from initial data. In this section, we shall investigate this extra structure.

A key result is that when the energy function is positive, the evolution must be conjugate to a unitary group. This is quite remarkable, even in the case of finite dimensions, since the eigenvalues of a general Hamiltonian vector may be complex. On the other hand, the result is (in finite dimensions) essentially an extension of the proof of the existence of action–angle variables.

The argument in finite dimensions is this. Let $A$ be the generator of a one-parameter group of symplectic transformations, and suppose its energy function $\frac{1}{2}\omega(v, Av)$ is a positive-definite quadratic form. Since evolution by $g(t) = e^{tA}$ preserves this form, we see that $g(t)v$ remains bounded for all $t$, for any $v$. This means that the eigenvalues of $A$ must be purely imaginary, and that its Jordan form (over the complex) must be purely diagonal. Thus $A$ must be conjugate to an anti-Hermitian matrix.

In infinite dimensions, the result is more difficult for several reasons. In the first place, since the operator $A$ is unbounded, the form $\omega(v, Av)$ is not defined everywhere. This means that for a dense family of $v$’s, the form has the value $+\infty$, and the fact that this value remains constant in $t$ does not allow us to conclude that the orbits $g(t)v$ remain bounded. Also, it is not known a priori that $A$ (or even $g(t)$) has anything like a Jordan normal form. (Indeed, the required property, known as “spectrality,” is in general a very delicate thing to establish. An example of a Hamiltonian with $A$ nonspectral was given in Paper I.) In fact, our argument turns on a recently established hyperfunctional analog of a theorem of Bochner, and it is possible that the lack of adequate analytic tools prevented an earlier proof.

Definition 3. The generator $A$ of a strongly-continuous one-parameter subgroup of the symplectic group is called classically positive if the form $\frac{1}{2}\omega(v, Av) \geq c\|v\|^2$ for some $c > 0$.

The requirement that $c$ be strictly positive will be used essentially in what follows. Its effect is to rule out certain potential infrared problems. (The analysis could be modified to accommodate a finite number of zero modes of $A$, however.)

The next theorem is one of our main results. It asserts that (with one technical proviso) classically positive Hamiltonians are in fact similar, by bounded symplectomorphisms, to the generators of orthogonal motions on phase space. Thus this is a general structure theorem, which can be thought of as an analog of the statement that action–angle variables exist for linear systems with positive Hamiltonians and finitely many degrees of freedom.

As mentioned above, the theorem contains a technical proviso, which is that $A_-$, the $J$-antilinear part of $A$, be bounded. (So the theorem would apply to any $A$ such that an some positive complex structure $J$ could be found for which $A_-$ is bounded.) This condition is verified in all examples known to me, and holds in particular for quantum fields in curved space–time (Helfer 1996). Still, it would be be more satisfying to remove this hypothesis, and I believe this can be done. However, the arguments if unbounded $A_-$ are allowed are much more technically complicated, will be pursued elsewhere.
Proposition 1. Let $A$ be classically positive. Then its spectrum lies on the imaginary axis.

Proof. Let $\mathcal{D}(A)$ be the domain of $A$. The spectrum of $A$ is the set of points $\lambda$ at which the map $\lambda - A : \mathcal{D}(A) \to H$ is not invertible. More precisely, since $H$ is not canonically a complex vector space, we work with the complexification. This will be done in the usual way, without introducing unnecessary notations for complexifications. Then $\omega(\tau, Av)$ is a Hermitian form bounded away from zero (as a form).

Suppose $\lambda - A$ is not one-to-one. Then $A$ has an eigenvector $v$ with eigenvalue $\lambda$. Then
\[
\|e^{\lambda t}v\|^2 = \|g(t)v\|^2 \\
\leq (2c)^{-1}\omega(g(t)v, Ag(t)v) \\
= (2c)^{-1}\omega(\tau, Av).
\]

This can hold for all $t$ only if $\lambda$ is purely imaginary (or zero).

Now suppose that $\lambda - A$ is not onto, and its image lies in some hyperplane $\{x \mid (v, x) = 0\}$. This means that $v$ is an eigenvector of $AT$ with eigenvalues $\lambda$. Now $AT$ is the generator of $g^T(t) = -Jg(-t)J$. The domain of $AT$ is $J\mathcal{D}(A)$, and the energy function is $(1/2)\omega(y, A^*y) = (1/2)\omega(y, JAJy) = -(1/2)\omega((Jy, A(Jy))$. As in the previous paragraph, we find
\[
\|e^{\lambda t}v\|^2 \leq -(2c)^{-1}\omega(\tau, ATv),
\]
forcing the real part of $\lambda$ to vanish.

We now take up the more delicate case, where $\lambda - A$ is one-to-one but not onto, but its image is dense. In this case, we may find a sequence $\{v_n\}$ of unit vectors in $H$ which are elements of $\mathcal{D}(A)$ such that $\|v_n\| + \|Av_n\| = 1 + \|Av_n\|$ is bounded and $(\lambda - A)v_n$ tends to zero. We have
\[
(g(t) - e^{\lambda t})v = e^{\lambda t} \int_0^t g(u)e^{-\lambda u}du (A - \lambda)v
\]
for any $v \in \mathcal{D}(A)$. Thus
\[
\|g(t) - e^{\lambda t}\|v \leq M(e^{\beta t} - e^{\Re\lambda t})(\beta - \Re\lambda)^{-1}\|(A - \lambda)v\|
\]
where we have taken $t \geq 0$ for simplicity (and we have used the Hille–Yoshida–Phillips bound $\|g(t)\|_{op} \leq Me^{\beta|t|}$).

Now, suppose $\Re\lambda > 0$. We may choose $T > 0$ so that $e^{\lambda T}$ is as large as desired. For such a $T$, and any $\epsilon > 0$, we may choose $n$ large enough so that $\|g(t) - e^{\lambda t}\|v_n < \epsilon$ for all $0 \leq t \leq T$. In this case we will have
\[
\|e^{\lambda t}v_n\|^2 \leq (\|g(t)v_n\| + \epsilon)^2 \leq (\sqrt{(2c)^{-1}\omega(\tau, Av_n)} + \epsilon)^2.
\]
However, this is a contradiction, for the right-hand side is uniformly bounded, and $\|e^{\lambda t}v_n\| = e^{\Re\lambda t}$ can be made as large as desired. Thus $\Re\lambda \leq 0$.

Consideration of $t < 0$ similarly rules out the case $\Re\lambda < 0$, and so we must have $\Re\lambda = 0$.

\[\square\]

Theorem 1. Let $A$ be classically positive, and suppose its $J$-antilinear part $A_-$ is bounded. Then there is a positive-definite bounded $J$-symmetric symplectomorphism $\gamma$, and a $J$-real-anti-self-adjoint $J$-linear closed (possibly unbounded) operator $\sigma$, such that
\[
g(t) = \gamma^{-1}e^{\sigma t}\gamma.
\]
There is a $J$-orthogonal $J$-invariant projection-valued measure $dF(\theta)$ supported on $[\theta_0, \infty)$ for some $\theta_0 > 0$ such that
\[
e^{\sigma t} = \int_{\mathbb{R}} e^{i\theta J} dF(\theta).
\]

Proof. The idea will be to define a positive complex structure $J_A$ relative to which $g(t)$ is orthogonal. The conclusions will follow almost directly from this. At a formal level, one has
\[
J_A = \pi^{-1} \int_{-\infty}^{\infty} (\lambda - A)^{-1} d\lambda.
\]
However, the sense in which this integral converges as $\lambda \to \pm\infty$ needs to be made precise. Even if $A$ were known to be bounded, the existence of this integral would be a bit delicate. In the present case, there are a number of technicalities, which arise because we need to gradually establish enough properties of $J_A$ to show that it exists as a bounded operator. Once this is done, the remainder of the proof will be routine algebraic computations.

We begin by showing that $J_A$ exists (as a potentially unbounded operator) on the dense domain $D(A)$.

The integral for $J_A$ converges strongly on $D(A)$ in the sense of a Cauchy principal value. To see this, first note that since $\lambda^{-1}$ converges near infinity as a Cauchy principal value, it is enough to show that
\[
P \int_{|\lambda| \geq \lambda_0} [(\lambda - A)^{-1} - \lambda^{-1}] v d\lambda
\]
converges, where “P” indicates the principal value and $v \in D(A)$. The integrand can be re-written:
\[
((\lambda - A)^{-1} - \lambda^{-1}) v = \lambda^{-1}(\lambda - A)^{-1} Av.
\]
On the other hand, the Hille–Yoshida–Phillips estimate $\|g(t)\|_{op} \leq Me^\beta |t|$ implies
\[
\|(\lambda - A)^{-1}\|_{op} \leq M(\Re \lambda - \beta)^{-1}
\]
for sufficiently large $\Re \lambda$ (and similarly for negative $\Re \lambda$). From this estimate, we have, for $v \in D(A)$,
\[
((\lambda - A)^{-1} - \lambda^{-1}) v = \lambda^{-1}(\lambda - A)^{-1} Av
\]
of order $O(\lambda^{-2})$ as $\Re \lambda \to \infty$. This is integrable at infinity, and so $J_A v$ exists for $v \in D(A)$.

It is straightforward to verify that $J_A^2 = -1$ in a suitable sense, namely on $D(A^2)$. This is a singular-integral computation using the definition of $J_A$. We first re-write the principal-value integral:
\[
J_A = \pi^{-1} \int_{0}^{\infty} \left[(\lambda - A)^{-1} + (-\lambda - A)^{-1}\right] d\lambda
\]
(understood strongly on $D(A)$). Now we have
\[
J_A^2 = \pi^{-2} \int_{0}^{\infty} d\lambda \int_{0}^{\infty} d\mu \left[(\lambda - A)^{-1} + (-\lambda - A)^{-1}\right] \left[(\mu - A)^{-1} + (-\mu - A)^{-1}\right]
\]
(understood strongly on $D(A^2)$). After some algebra, this can be re-written as
\[
J_A^2 = \pi^{-2} \int_{0}^{\infty} d\lambda \int_{0}^{\infty} d\mu \left[(-\lambda^2 + A^2)^{-1} - (-\mu^2 + A^2)^{-1}\right] \frac{A^2}{\lambda^2 - \mu^2}.
\]
While the existence of this as a Riemann integral requires the cancellation of \((-\lambda^2 + A^2)^{-1}\) against \((-\mu^2 + A^2)^{-1}\) in order to compensate for the singularity in the denominator \(\lambda^2 - \mu^2\) at \(\lambda = \mu\), we may break the integral into two if each is interpreted as a principal value:

\[
J^2_A = \pi^{-2} \mathcal{P} \int_0^\infty d\lambda \int_0^\infty d\mu (-\lambda^2 + A^2)^{-1} \frac{A^2}{\lambda^2 - \mu^2} - \pi^{-2} \mathcal{P} \int_0^\infty d\lambda \int_0^\infty d\mu (-\mu^2 - A^2)^{-1} \frac{A^2}{\lambda^2 - \mu^2}.
\]

(This will of course be compatible with definition of the Riemann integral, since it simply corresponds to a particular way of forming the limit which is the integral.) If we do this, then we can use the distributional identity

\[
\int_0^\infty (\lambda^2 - \mu^2)^{-1} d\mu = -(\pi^2/2) \delta(\lambda)
\]
on the first integral, and its counterpart for integration over \(\lambda\) on the second, to obtain

\[
J^2_A = -1
\]
immediately, strongly on \(D(A^2)\).

In the next stage of the analysis, we shall want to consider the \(J\)-linear and antilinear portions of \(J_A\). Formally, these are given by \((J_A)_\pm = (1/2)(J_A \pm JJ_A J)\). However, in order to make sense of this, we must show that \(J_A\) and \(JJ_A J\) have a common dense domain.

We shall show that the \(J\)-invariant dense set \(D(A^2) + JD(A^2) \subset D(A)\); then \((J_A)_\pm\) are naturally defined on this domain. (Here \(D(A^2) = \{v \in H \mid v = A^{-2}w \text{ for some } w \in H\}\).) It is clear that \(D(A^2) \subset D(A)\); we must show that \(JD(A^2) \subset D(A)\). So let \(v = JA^{-2}w \in JD(A^2)\). Then

\[
v = A^{-1}(AJ)A^{-2}w = A^{-1}(JA - 2JA_-)A^{-2}w = A^{-1}(JA^{-1} - 2JA_-A^{-2})w,
\]

which is an element of \(D(A)\). The same sort of argument shows \(D(A^n) + JD(A^n) \subset D(A^{n-1})\) for \(n \geq 1\).

Now in fact \((J_A)_-\) exists as a bounded operator. To see this, we will show that the \(J\)-antilinear part \(((\lambda - A)^{-1})_-\) of the resolvent is \(O(\lambda^{-2})\) in operator norm. This will be accomplished by estimating

\[
((\lambda - A)^{-1})_- = \int_0^\infty g_-(t)e^{-\lambda t} dt.
\]

(Using the symmetry \(t \mapsto -t\), \(A \mapsto -A\), it is enough to consider the case of positive \(\Re \lambda\).)

In section 4 of paper I, we considered a quantity \(L(t) = 2g_-(t)Jg(t)\) and showed

\[
L(t) = \int_0^t G(u)(2A_- J) du
\]

where

\[
G(u)Q = g(u)Qg(-u).
\]
Using this, we find that for $\Re \lambda$ sufficiently large
\[
\int_0^\infty g_{-}(t)e^{-\lambda t} dt = (-1/2) \int_0^\infty L(t)g(-t)e^{-\lambda t} dt
\]
\[
= (-1/2) \int_0^\infty L(t)(-\lambda - A)^{-1} \frac{d}{dt}g(-t)e^{-\lambda t} dt
\]
\[
= (1/2) \int_0^\infty (G(t)(2A_{-}J))(-\lambda - A)^{-1}g(-t)e^{-\lambda t} dt .
\]
This Hille–Yoshida–Phillips estimates, and the boundedness of $A_-$, now imply this is $O(\lambda^{-2})$ at infinity in operator norm.

We remark that the boundedness of $(J_A)_-$ implies that $J_A$ extends naturally to exist on $D(A) + JD(A)$, for we can define $J_AJ = J(A)_+ - J(J_A)$. However, we shall see shortly that in fact $J_A$ is itself bounded.

We next note that $\omega(\cdot, (J_A)^{-})$ is a positive-definite symmetric form on $D(A)$. (This follows easily from the fact that $A$ is a generator of symplectomorphisms and that it is classically positive.) Similarly, conjugating by $J$, we have $\omega(\cdot, -JJA^{-})$ a positive-definite symmetric form on $JD(A)$ (or $D(A)$). Thus $(J_A)_+$ also defines a positive-definite symmetric form, and hence $-J(J_A)_+$ is a $J$-symmetric operator defining a positive-definite form $(\cdot, -J(J_A)_+^{-}) = \omega(\cdot, (J_A)_+^{-})$ on $D(A)$. Thus $-J(J_A)_+$ has a canonical extension to a positive self-adjoint operator on a dense domain in $H$. (We shall continue to denote this operator by $-J(J_A)_+$.)

We now re-write the equation $J_A^2 = -1$ in terms of the self-adjoint operators $-J(J_A)_\pm$. We have
\[
((J_A)_+ + (J_A)_-)^2 = ((J_A)_+)^2 + ((J_A)_-)^2 + (J_A)_+(J_A)_- + (J_A)_-(J_A)_+ = -1 ,
\]
and the $J$-linear and $J$-antilinear parts of this are
\[
((J_A)_+)^2 + ((J_A)_-)^2 = -1 \quad \text{and} \quad (J_A)_+(J_A)_- + (J_A)_-(J_A)_+ = 0 .
\]
Multiplying through by $(-J)^2$, we get
\[
(-J(J_A)_+)^2 - (-J(J_A)_-)^2 = 1 \quad \text{and} \quad (-J(J_A)_+)(-J(J_A)_-) - (-J(J_A)_-)(-J(J_A)_+) = 0 .
\]
From the first equation and the boundedness of $-J(J_A)_-$, we see that $-J(J_A)_+$ (and hence $J_A$) is bounded. According to the second, the operators $-J(J_A)_+, -J(J_A)_-$ commute. Thus, again using the first equation, we may find a $J$-symmetric, $J$-antilinear operator $\Theta$ such that
\[
-J(J_A)_- = \sinh 2\Theta \quad \text{and} \quad -J(J_A)_+ = \cosh 2\Theta .
\]
(The factor of two is for later convenience.) It will be useful to rewrite these. Note that
\[
(J_A)_- = J \sinh 2\Theta = \sinh(2J\Theta) \quad \text{and} \quad (J_A)_+ = J \cosh 2\Theta = J \cosh(2J\Theta) ,
\]
where the $J$-antilinearity of $\Theta$ has been used (note that $(J\Theta)^2 = + (\Theta)^2$).

We now let
\[
\gamma = \cosh \Theta - J \sinh \Theta = \exp(-J\Theta) .
\]
Since
\[
\omega(v, J\Theta w) = \omega(\Theta v, Jw) = -\omega(J\Theta v, w) ,
\]

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the operator $J\Theta$ is a generator of symplectomorphisms and $\gamma$ is a symplectomorphism. We also have

$$\gamma^{-1}J = e^{t\Theta}Je^{-Jt} = Je^{-Jt}e^{t\Theta} = J(\cosh(2\Theta) - J \sinh(2\Theta)) = JA.$$ 

Now we shall show that $\gamma g(t)\gamma^{-1}$ is $J$-orthogonal:

$$\omega(\gamma g(t)\gamma^{-1}v, J\gamma g(t)\gamma^{-1}w) = \omega(v, \gamma g(-t)\gamma^{-1}J\gamma g(t)\gamma^{-1}w) = \omega(v, \gamma g(-t)JA\gamma^{-1}w) = \omega(v, JA\gamma^{-1}w) = \omega(v, J w).$$

(Here we have used the fact that, by construction, the operator $JA$ is invariant under conjugation by $g(t)$.) Since $\gamma g(t)\gamma^{-1}$ is $J$-orthogonal and a symplectomorphism, it is $J$-linear. We may thus set

$$\sigma = \gamma^{-1}A\gamma.$$ 

\section{Application to Quantum Field Theory}

We now apply the structure theorems of the previous section to quantum field theory.

**Theorem 2.** In order that a classically positive operator $A$ with bounded antilinear part generate a one-parameter group of restricted symplectic motions, it is necessary and sufficient that the operator $\gamma$ be a restricted symplectomorphism. In this case the corresponding Hamiltonian operator is bounded below.

**Proof.** The $J$-antilinear part of $g(t)$ is

$$J \sinh \Theta e^{t\sigma} \cosh \Theta + \cosh \Theta e^{t\sigma}(-J \sinh \Theta) = J(\sinh \Theta e^{t\sigma} \cosh \Theta - \cosh \Theta e^{t\sigma} \sinh \Theta).$$

Multiplying on the left by $-J \sech \Theta$ and on the right by $\sech \Theta$ (both bounded operators with bounded inverses), we see that the antilinear part of $g(t)$ is Hilbert–Schmidt iff

$$\tanh \Theta e^{t\sigma} - e^{t\sigma} \tanh \Theta$$

is, or equivalently if

$$\tanh \Theta - e^{t\sigma} \tanh \Theta e^{-t\sigma}$$

is.

The idea now will be to think of $\tanh \Theta$ as a vector in the space of symmetric bounded operators, and consider the action of $e^{t\sigma}$ on this space by conjugation. However, this space is not a Hilbert space, and in order to take advantage of the spectral theory of operators on Hilbert space, it is more convenient to regard $\tanh \Theta$ as a sort of unbounded form on the Hilbert–Schmidt operators.
Let $V$ be the space of Hilbert–Schmidt $J$-antilinear, $J$-symmetric endomorphisms of $H$. This space is naturally a complex Hilbert space, with complex structure given by $L \mapsto JL$ and inner product

$$\langle [M, L] \rangle = \text{tr}(ML) + i \text{tr}(LM).$$

Conjugation by $\exp \sigma t$ is a strongly continuous unitary map on this space. We may apply the usual spectral theory of one-parameter unitary groups on Hilbert space to this.

In fact, we can work out the spectral resolution explicitly in terms of that for $\exp \sigma t$. For we have

$$e^{\sigma t} Le^{-\sigma t} = \int e^{\theta t} J dF(\theta) Le^{-\phi t} J dF(\phi)$$

$$= \int e^{(\theta + \phi) t} J dF(\theta) L dF(\phi)$$

$$= \int e^{\xi t} J dE(\xi) L$$

where we have defined $dE(\xi) L = \int_{(\theta, \phi) | \theta + \phi = \xi} dF(\theta) L dF(\phi)$. One can check that $dE$ is a projection-valued measure on $V$, and the equation above provides the spectral resolution of conjugation by $e^{\sigma t}$. Note that since $dF(\theta)$ is supported for $\theta \geq \theta_0 > 0$, the measure $dE(\xi)$ is supported for $\xi \geq 2\theta_0 > 0$. (This may be counterintuitive, as one thinks of the generator of a one-parameter group of conjugations as having eigenvalues which are differences of the eigenvalues of the generator of the original group. However, in the present case there is a very curious interaction between the fact that the spectrum is imaginary and the antilinearity of the elements of $V$. This is in some sense the central point of the proof.)

The above analysis does not quite apply directly to $\tanh \Theta$ or $\Theta$, since $\Theta$ may not be Hilbert–Schmidt. However, the operator $\Theta$ is bounded, and so can be regarded as a linear functional on the space $V_0$ of trace-class elements of $V$. The space $V_0$ is dense in $V$ and invariant under conjugation by $e^{\sigma t}$, and so the spectral resolution derived above for this conjugation can be applied, by duality, to $\Theta$, and similarly to $\tanh \Theta$.

That $g_-(t)$ be Hilbert–Schmidt is thus equivalent to requiring

$$\int_{[2\theta_0, \infty)} (e^{\xi t} J - 1) dE(\xi) \tanh \Theta$$

to be so. Since $dE(\xi)$ resolves $V$ into the orthogonal direct integral of Hilbert–Schmidt operators, the integral above, restricted to any compact interval of $\xi$-values, must be Hilbert–Schmidt. This may be seen to be equivalent to the requirement $\tanh \Theta$ be Hilbert–Schmidt by elementary arguments. And since $\Theta$ is $J$-symmetric, this is equivalent to $\Theta$ being Hilbert–Schmidt.

We now take up the boundedness-below. If $\Theta$ is Hilbert–Schmidt, then $\exp J \Theta$ provides a restricted symplectomorphism taking $A$ to $\sigma$. The image of the restricted symplectomorphism is unitarily implementable, and the quantum Hamiltonian induced by $\exp t \sigma$ is $-(J\sigma)_{a}^{b} Z^{a} \partial_{b}$, which is bounded below.

This results also implies that the self-adjoint implementation of the Hamiltonian is essentially unique.
Corollary 1. If a classically positive operator $A$ with bounded antilinear part is self-adjointly implementable in the representations determined by $J_1$ and $J_2$, then its implementations are unitarily equivalent (modulo an additive constant).

Corollary 2. A classically positive operator $A$ with bounded antilinear part fixes a distinguished complex structure $J_A$, relative to which $A$ is $J_A$-linear and $J_A$-anti-self-adjoint. The operator $A$ is self-adjointly implementable in the representation determined by a second complex structure $J'$ if and only if $J_A - J'$ is Hilbert–Schmidt.

5 Connection with Normal Ordering

As is well-known, even the simplest linear quantum field theories in Minkowski space contain divergent terms. For example, the vacuum energy in a region is $\{1/2\} \sum \hbar \omega$, where $\omega$ runs over all the independent modes. The standard prescription for dealing with these divergences is normal ordering, that is, writing all creation operators before annihilation operators, thus eliminating the infinite c-number terms. Of course, normal ordering will not distinguish between two operators differing by a finite c-number, but will reduce them to the same operator. For this reason, normal ordering operators may lose certain important physical information, for example, Casimir-type effects.

In this section, though, we are concerned with a more severe question: supposing that one knows that a self-adjoint Hamiltonian exists, can it necessarily be given by normal-ordering the classical Hamiltonian (modulo a finite c-number term)? We shall find that the answer may be No. In other words, there are at least in principle linear quantum field theories which require more than normal ordering to be successfully renormalized. This points up the delicacy of the issues involved in analyzing the quantum Hamiltonian.

While in elementary examples, there is no ambiguity in what is meant by normal ordering an operator, in the present, very general, context, some care is needed to make this precise. This will now be explained.

Let $A$ be a classically positive Hamiltonian, and let $\sigma$ and $\Theta$ be as in the previous section: $\sigma$ is a $J$-skew, $J$-linear, $J$-self-adjoint map and $\Theta$ is a $J$-antilinear, $J$-symmetric bounded operator with $\sigma = e^{-J\Theta} A e^{J\Theta}$. We put $D = -J\sigma$. Then, using the representation by creation operators $Z^\alpha$ and destruction operators $\partial^\alpha$ as described in the previous paper, the quantum Hamiltonian corresponding to $A$ is the image of $D_{\alpha\beta} Z^\alpha \partial^\beta$ under conjugation by the Bogoliubov transformation induced by $\exp J\Theta$. Notice that with this definition, the Hamiltonian has the same spectrum as $D_{\alpha\beta} Z^\alpha \partial^\beta$, and in particular has zero as its minimum.

At a formal level, the normal ordering is accomplished in the usual way, and one finds
\[ \hat{H} = \hat{H}_{\text{normal}} + E_0 \] (5)
where the normal-ordered Hamiltonian is
\[ \hat{H}_{\text{normal}} = \overline{C}_{\alpha\beta} Z^\alpha Z^\beta + B_{\alpha\beta} Z^\alpha \partial^\beta + C^{\alpha\beta} \partial^\alpha \partial^\beta, \] (6)
with
\[ \overline{C} = J \cosh \Theta D \sinh \Theta \] (7)
\[ B = \cosh \Theta D \cosh \Theta + \sinh \Theta D \sinh \Theta \] (8)
\[ E_0 = \text{tr} \sinh \Theta D \sinh \Theta \] (9)
in matrix form.

The difficulty in making the relations (6)–(9) precise is not merely in the fact that it is hard to analyze the individual quantities $B, C, E_0$. One has to decide what sort of properties are required of these in order to say that one has successfully renormalized the Hamiltonian by normal ordering. In the simplest case, one could require that $E_0$ is finite, and that $H_{\text{normal}}$ be well-defined by term-by-term action on (at least) a dense family of polynomials. However, one could also imagine a more general situation, where the domain consisted of functions $\Psi(Z)$ such that, while the actions of the individual terms in (6) did not give elements in the Hilbert space, there were nevertheless cancellations so that the net result was indeed an element of the Hilbert space. Indeed, there are even more extreme possibilities. One could envisage situations in which $E_0 = \pm \infty$, but the elements in the domain are chosen so that, with a proper limiting procedure, the quantity $H|\Psi\rangle$ is well-defined even though $H_{\text{normal}}|\Psi\rangle$ is not separately! Thus at some point one must decide what sort of regularity the notion of “normal ordering” requires; otherwise, saying that normal ordering suffices to renormalize the Hamiltonian becomes a statement with no force. At present, we shall assume it requires $E_0$ to be finite. This is very weak.

**Theorem 3.** Let $A$ be classically positive with bounded antilinear part. The minimum of the normal-ordered Hamiltonian is

$$-E_0 = - \text{tr} \sinh \Theta D \sinh \Theta$$

if this is finite. If it is finite, it is negative. If this is infinite, normal ordering does not suffice to renormalize the Hamiltonian.

**Proof.** We have mentioned everything except the negativity. But $\sinh \Theta D \sinh \Theta$ is a positive symmetric form.

Since one can arrange for $\Theta$ to be Hilbert–Schmidt but $E_0$ divergent, this implies that in principle at least that there are linear quantum field theories for which self-adjoint Hamiltonians exist, but they cannot be realized by normal-ordered operators; some more sophisticated renormalization is required. There would be no general reason for rejecting such Hamiltonians as unphysical, although in a particular system one might have physical arguments that normal ordering should suffice to regularize the theory.

For classically positive Hamiltonians, finiteness of the normal-ordered ground-state energy implies existence of the classical Hamiltonian as a self-adjoint operator:

**Theorem 4.** Let $A$ be classically positive with bounded antilinear part, and suppose the normal-ordered ground state energy $E_0$ is finite. Then the Hamiltonian is a self-adjoint operator.

**Proof.** Let $v_j$ be a $J$-orthonormal basis, and let $D = \int_{(\lambda_0, \infty)} \lambda dE(\lambda)$, where $\lambda_0 > 0$. Then we are given that

$$\sum_j (v_j, \sinh \Theta \int_{(\lambda_0, \infty)} \lambda dE(\lambda) \sinh \Theta v_j)$$

\(^2\text{Just this sort of thing would occur if one defined } H|\Psi\rangle \text{ by a sequence of formal operations which amounted to conjugation by the Bogoliubov transformation } \exp J\Theta.\)
converges. The sum and integration are of non-negative terms, so the convergence is absolute. This quantity dominates \( \text{tr} \sinh \Theta \lambda_0 I \sinh \Theta \) (where \( I \) is the identity). Since \( \lambda_0 > 0 \), this implies \( \sinh \Theta \) is Hilbert–Schmidt, which implies \( \Theta \) is.

In general, it is hard to work out \( \Theta \) from the normal-ordered Hamiltonian. In principle, one can work out the ground-state energy exactly as \( (1/4) \text{tr}_R(|A| - B) \), where \( |A| = \sqrt{-A^2} \); this expression does not require a knowledge of \( \Theta \). However, this quantity is usually too awkward to work with directly in applications. It is useful to have some approximate formulas in terms of \( B \) and \( C \), which can be worked out directly from \( A \) and \( J \).

**Theorem 5.** If the Hamiltonian is classically positive with bounded antilinear part and the normal-ordered ground-state energy is finite, it is bounded by

\[
-E_0 \geq - \text{tr} \, CB^{-1} C
\]

and also by

\[
-E_0 \geq - \text{tr} \, \sqrt{C^T C}.
\]

If \( A \) is classically positive with bounded antilinear part and either of the quantities on the right is finite, then the Hamiltonian is self-adjoint and can be renormalized by normal ordering.

**Proof.** We have \( B \geq \cosh \Theta D \cosh \Theta \) as (densely-defined) symmetric forms. Thus \( B^{-1} \leq (\cosh \Theta D \cosh \Theta)^{-1} \) as symmetric forms. Thus

\[
\text{tr} \, CB^{-1} C \leq \text{tr} \, J \sinh \Theta D \cosh \Theta (\cosh \Theta D \cosh \Theta)^{-1} \sinh \Theta D \sinh \Theta J
= -E_0.
\]

We have

\[
C^T C = \sinh \Theta D \cosh^2 \Theta D \sinh \Theta
= \sinh \Theta D^2 \sinh \Theta + (\sinh \Theta D \sinh \Theta)^2.
\]

However, since the first form is positive and symmetric we have

\[
\sqrt{C^T C} \geq \sinh \Theta D \sinh \Theta
\]

as forms, and so

\[
-E_0 \geq \text{tr} \, \sqrt{C^T C}.
\]

\[\square\]

6 Quantum Inequalities

In this section, I show how to adapt the previous arguments to a certain important family of situations where \( \omega(v, Av) \) is a positive indefinite form. I will begin by discussing the class of operators to be considered and its significance. Then I shall give the results.
The proofs of the theorems in this section are fairly lengthy. This is ultimately bound up with technical problems at the boundary of the space–time region whose energy–momentum content is to be measured. A brief discussion of this is given at the end of this section, after theorem 4 but some readers may want to look at this before the proofs of theorems 6 and 7.

6.1 The Sorts of Results Sought

Ford (1978) was the first to show that temporal averaging could bound some of the local negative energies encountered in quantum field theory. In the case of the Klein–Gordon field on Minkowski space, he and Roman (1997) proved that

$$\langle \Psi | \int_{-\infty}^{\infty} \hat{T}_{00}(t, 0, 0, 0) b(t) dt | \Psi \rangle / \langle \Psi | \Psi \rangle \geq -\frac{3}{32\pi^2} \frac{\hbar c}{(ct_0)^4},$$

(10)

where

$$b(t) = \frac{t_0/\pi}{t_0^2 + t^2}$$

(11)

is a sampling function of area unity and characteristic scale $\sim t_0$. Following Ford, lower bounds on energy operators for relativistic quantum field theories are known as quantum inequalities.

In the past years, quantum inequalities of increasing generality have been established.

- For the massless field in two-dimensional Minkowski space, there is a broad class of elegant results (Flanagan 1997). However, the divergences of this theory are significantly softer than in four dimensions. It is especially problematic to draw conclusions about the boundedness of four-dimensional energies from two-dimensional results.

- For the energy density measured by static observers in static space–times, Pfenning and Ford (1997, 1998) established quantum inequalities for Lorentzian sampling functions, and Fewster and Teo (1999) for more general sampling functions.

- The results contained in the present papers were released (Helfer 1999a,b).

- A general (applicable to a Klein–Gordon field in a globally hyperbolic space–time) quantum inequality for the energy density was given by Fewster (2000). Similar results were established for the Dirac field in curved space–time (Fewster 2002) and for the spin-one field in curved space–time (Fewster and Pfenning 2003).

- Very little is known about bounds on the four-momentum density of the quantum field. There is one result, in Minkowski space (Helfer 1998).

- It was shown by Fewster and Roman (2003) that “null energy inequalities” do not exist, that is, that averages of the component $\hat{T}_{a0}l^a l^b$ of the stress–energy operator along a null geodesic with tangent $l^a$ are unbounded below, even in Minkowski space.

One would like to generalize these results to apply not just to the energy density, but to other components of the stress–energy tensor. This is because classical matter fields satisfy not only the Weak Energy Condition (which requires the energy density to be positive) but also the Dominant Energy Condition (which requires the four-momentum density to
be future-pointing). One would like to know what bounds there are on violations of the Dominant Energy Condition.

I think that the approach used by Fewster and co-workers can be extended to give such results. I also believe that that approach gives a very useful “hands-on” understanding of the quantum inequalities in terms of the geometry of space–time as reflected in the ultraviolet Hadamard asymptotics. On the other hand, the present paper gives a rather different perspective.

The previous two sections can be interpreted as showing that classical positivity enforces boundedness-below of the quantum Hamiltonian. This is very close to saying that classically positive measures of energy (including classically positive averages of the stress–energy) should have quantum-inequality counterparts; the difference lies in technicalities associated with the finite extent in space–time of the regions over which the averages are taken. Leaving aside these technicalities for the moment, then, we have a suggestive argument for a very broad class of quantum inequalities which emphasizes the general algebraic properties of the classical measures of energy rather than the ultraviolet asymptotics of the quantum field theory. (The ultraviolet asymptotics do play an important if brief role in showing that $A_-$ is bounded.)

For generic space–times, there is no preferred vacuum state and no preferred associated quantization. Rather, one has a family of unitarily equivalent (modulo infrared issues) Hadamard quantizations. Different choices of such quantization lead to different normal-ordering prescriptions and normal-ordered Hamiltonians which differ by c-numbers. This is bound up with the well-known ambiguities in fixing the c-number part of the stress–energy operator in generic space–times. Thus, a specific numerical lower bound on the Hamiltonian is only meaningful given choices which resolve these ambiguities. Given our current lack of understanding of how to effect these resolutions, such numerical values would be data of no clear significance.

There are, however, two sorts of results which would be of immediate significance. One of these would be asymptotic formula for lower bounds, as the sampling function becomes more and more localized. (For example, as $t_0 \downarrow 0$ in (10).) In such cases, because the energy densities diverge, the c-number ambiguities in the stress–energy become insignificant. While I believe that such results can be derived using the techniques of this paper, they require lengthy computations which are too far out of the main line of argument. I shall give such results elsewhere.

The second sort of presently-useful result would be a general proof that temporally-averaged Hamiltonians are bounded below. This is a statement which is meaningful even in the face of the c-number ambiguities, and it is this result which will be proved in the remainder of this section.

In order to treat such very general cases, it is probably necessary to pass to compactly supported sampling functions. (For otherwise, with no assumptions about the space–time, one has no control about how small data from initially spatially distant regions propagate inwards and are amplified by the space–time geometry.) I shall not use the stress–energy localized to a world-line, but rather to a compact four-volume. Probably one can obtain parallel results for world-lines, but the proofs would be longer. For four-volumes, we can appeal to very general results in distribution theory (cf. the proof of Proposition below).
6.2 Structure of the Hamiltonians; Boundedness-Below

Let \((M, g_{ab})\) be an oriented, time-oriented space–time, globally hyperbolic with compact Cauchy surfaces. (The restriction to compact Cauchy surfaces is a technical device to simplify the analysis and remove infrared ambiguities. It is not physically significant. The analysis is all local, and the spatial dimensions can be arbitrarily large.) Consider the quantum theory of a real scalar field governed by the equation

\[
(\nabla^a \nabla_a + m^2)\phi = 0,
\]

where \(m^2 \geq 0\). The associated classical stress–energy is

\[
T_{ab} = \nabla_a \phi \nabla_b \phi - (1/2) g_{ab} (\nabla_c \phi \nabla^c \phi - m^2 \phi^2).
\]

We shall say a smooth symmetric compactly-supported tensor field \(f_{ab}\) is future-directed if it is at any point a sum of terms \(t^a u^b + u^a t^b\) with \(t^a, u^a\) future-pointing. The content of the classical dominant energy condition (which holds for this classical field) is that \(\int f_{ab} T_{ab} d\text{vol} \geq 0\) for any future-directed \(f_{ab}\).

Letting \(A\) be the generator corresponding to this Hamiltonian, we have

\[
(1/2) \omega(\phi, A\phi) = \int f_{ab} T_{ab} d\text{vol}.
\]

The operator \(A\) is not in general classically positive. In the first place, the test function may be supported in an arbitrarily small volume, and so \(A\) may have a large kernel. A more severe problem is that the smooth fall-off of the test function generally leads to a spectrum including points arbitrarily close to zero. However, a good fraction of the structure deduced for classically positive Hamiltonians still applies.

It is appropriate to review what the correspondence between the rather general approach adopted so far in this paper and the specifics of the present scalar field are.

The space \(H\) consists of classical solutions to the field equation whose restrictions to any Cauchy surface have Sobolev regularity \(1/2\). (This choice of regularity makes all integrals for the symplectic form and the inner product converge as they should.) The symplectic form is given by

\[
\omega(\phi, \psi) = \int_{\Sigma} (\psi^* d\phi - \phi^* d\psi)
\]

where \(\Sigma\) is any Cauchy surface. It is independent of the surface chosen by virtue of the field equation.

It should be emphasized that, while our ultimate interest is in quantum fields, the formulas given so far in this subsection are entirely classical. (Thus \(A\) is an operator on the classical space of solutions — a Hamiltonian vector field —, not a quantum operator.) The connection with quantum field theory is made through the specification of a complex structure \(J\), or equivalently, a two-point function. Let us review how this comes about.

As mentioned above, there is in general no canonical vacuum state. However, there is a class of states which is preferred, the Hadamard states, those whose highest-order ultraviolet asymptotics agree with those of the Fock representation in Minkowski space. Any one of these can be used as a “vacuum” for the construction of the field theory, by following the usual mathematical pattern of the Fock construction. To see this, note that

\[
(0|\hat{\phi}(x)\hat{\phi}(y)|0) = (0|\hat{\phi}^+(x)\hat{\phi}^-(y)|0)
\]
in the usual construction. On the other hand, if one is given the two-point function, one can take this equation as defining the positive- and negative-“frequency” projections of the field, and hence the annihilation and creation operators. (The term “frequency” is used here only to bring out the correspondence with the usual Fock construction. “Frequency” here is not the Fourier transform with respect to any obvious time variable.) Representations constructed from different Hadamard states will be unitarily equivalent. Thus specification of a Hadamard state allows one to determine a mathematical structure analogous to the usual Fock one, the key portion being a mathematical analog of the decomposition into positive and negative “frequencies,” which allows the definition of annihilation and creation operators. Contrariwise, such a positive-/negative-frequency decomposition of the field operators would determine a Fock-type quantization.

The decomposition into positive and negative “frequencies” is naturally encoded in a complex structure $J$, setting $J\phi = i(\phi_+ - \phi_-)$. Thus the quantization is specified by the complex structure $J$. In the case of Hadamard states, it turns out that $J$ can be expressed as a pseudodifferential operator (cf. section 6 of paper I).

**Proposition 2.** Let $A$ be the generator associated to a smooth compactly-supported test field (not necessarily future-directed). Then $A_-$, the $J$-antilinear part of $A$, is represented on initial data by an operator with smooth kernel. In particular, the operator $A_-$ is compact.

**Proof.** We make use of the microlocal properties of the two-point functions, discovered by Radzikowski (1992) and Junker (1995). A summary adequate for understanding this proof is in the paper of Brunetti et al. (1996). For the general theory of wave-front sets, see Hörmander (1983).

The antilinear part of $A$ is got by projecting its positive-to-negative and negative-to-positive “frequency” parts using the two point function; it corresponds to the two-point kernel

$$A_-(y, z) = \int f^{ab}(x)((\nabla_x a)K(x, y))(\nabla_x b)K(x, z))$$

$$- (1/2)g_{ab}(\nabla_x a)K(x, y))(\nabla_x b)K(x, z))$$

$$+ (1/2)g_{ab}K(x, y)K(x, z) \text{dvol}(z).$$

Here $K(x, y)$ is the two-point function. The wave–front set of $K(x, y)$ is

$$\text{WF}(K) = \{(x, k; y, l) \mid (x, k) \sim (y, l), k \text{ is future-pointing}\}.$$  

Here and in what follows, it is understood that $(x, k), (y, l) \in T^*M - \{0\}$; and $(x, k) \sim (y, l)$ iff there is a null geodesic from $x$ to $y$ with covector $k$ at $x$ and $l$ at $y$. Thus

$$\text{WF}(K(x, y)K(x, z)) \subset \{(x, k, y, l, z, m) \mid (x, k) \sim (y, l) \text{ or } (x, k) \sim (z, m)\}$$

$$\cup \{(x, k, y, l, z, m) \mid (x, k_1) \sim (y, l) \text{ and } (x, k_2) \sim (z, m) \text{ for some } k_1, k_2 \text{ with } k = k_1 + k_2\}.$$  

When this is integrated against $f^{ab}(x)$ to form $A_-(y, z)$, the result has

$$\text{WF} (A_-(y, z)) \subset \{(y, l, z, m) \mid (x, 0, y, l, z, m) \in \text{WF}(K(x, y)K(x, z)) \text{ for some } x\},$$

which is empty. Since the two-point function has a smooth kernel, so does its restriction to act on initial data.
Proposition 3. Let \( A \) be the generator associated to smooth compactly-supported future directed test field. Then the spectrum of \( A \) lies on the imaginary axis.

**Proof.** Consider \( A = A_+ + A_- \) as a perturbation of its \( J \)-linear part. The term \( A_+ \) is \( J \)-real anti-self-adjoint and so has purely imaginary spectrum. On the other hand, the compactness of \( A_- \) means that any element of \( \text{spec} A - \text{spec} A_+ \) must be an eigenvalue. (To see this, suppose one has \( \lambda \in \text{spec} A - \text{spec} A_+ \) and a sequence \( v_n \) of unit vectors in the domain of \( A \) with \( (\lambda - A)v_n \to 0 \). Multiplying by \( (\lambda - A_+)^{-1} \) we have \( v_n - (\lambda - A_+)^{-1}A_-v_n \to 0 \). Since \( (\lambda - A_+)^{-1}A_- \) is compact, this implies \( (\lambda - A_+)^{-1}A_- \) has an eigenvalue unity. However, this implies \( \lambda \) is an eigenvalue of \( A \).) However, then the positivity of \( A \) implies, as in the proof of Proposition 4, that any such eigenvalue is imaginary.

\[ \square \]

Proposition 4. Let \( A \) be the generator associated to smooth compactly-supported future-directed test field. Then for any \( t \in \mathbb{R} \), the spectrum of \( g(t) = \exp t A \) lies on the unit circle.

**Proof.** We consider \( g(t) \) as a perturbation of \( \exp tA_+ \); the latter is \( J \)-unitary and so has spectrum on the unit circle. It follows from standard perturbation theory (Dunford and Schwartz 1988, Theorem VIII.1.19) that \( g(t) - \exp tA_+ \) is compact. Applying the argument of the previous proof, we see that any element of the spectrum of \( g(t) \) not already in the spectrum of \( \exp tA_+ \) must be an eigenvalue; but then that \( g(t) \) be a symplectomorphism implies that eigenvalue lies on the unit circle.

\[ \square \]

Theorem 6. Let \( A \) be the generator associated to smooth compactly-supported future-directed test field. Then there is a (strong) projection-valued distribution \( dE(l) \) which provides a spectral resolution of \( A \) in the sense of Fourier transforms:

\[ \int_{-\infty}^{\infty} \phi(l) \, dE(l) = \int_{-\infty}^{\infty} g(t) \hat{\phi}(t) \, dt \]

strongly, for suitable test functions \( \phi \). As \( \phi(l) \) approaches \( l \) in a suitable sense, we have \( A = \int \) \( l dE(l) \) strongly on \( D(A) \).

The distribution \( dE(l) \) is locally a measure, and this measure is locally integrable except possibly at zero; the quantity \( l dE(l) \) is integrable at zero. (Here and throughout, these statements are to be understood strongly, that is, the operators applied to elements of \( H \).)

**Proof.** This proof is somewhat technical, and makes use of the theory of Fourier hyperfunctions. This is the class of generalized functions \( F' \) dual to \( F = \{ \phi \mid \phi, \hat{\phi} \text{ are both smooth of exponential decay } \} \). (Here \( \hat{\phi} \) denotes the Fourier transform of \( \phi \).) The results we use are contained in the papers of Chung and Kim (1995) and Chung et al. (1994). The topology on \( F \) is given by the family of seminorms \( \| \phi \|_{k,h} = \sup_{x,n} |\phi^{(n)}(x)| e^{k|x|}/(h^n n!) \). A sequence \( \phi_j \to 0 \) in \( F \) if for some \( k, h > 0 \) one has \( \| \phi_j \|_{k,h} \to 0 \) as \( j \to \infty \).

Put \( \lambda = il \), where \( l \) is a real parameter (which may acquire a small imaginary part). Let \( [(il - A)^{-1}] \) denote the jump in \( (il - A)^{-1} \) from above the real \( l \)-axis to below; this jump is by definition a hyperfunction. (One defines its pairing with a test function by integrating slightly above and slightly below the axis; cf. Cerezo et al. 1975.) We shall interpret this hyperfunction in the strong sense, that is, as applied to any vector in \( H \). All integrals of operators in what follows are also to be interpreted in the strong sense.
We now show that \([(il - A)^{-1}]\) is not just a hyperfunction, but in fact a Fourier hyperfunction. This means that for any \(\phi \in \mathcal{F} = \{\phi \mid \phi, \hat{\phi} \text{ are both smooth of exponential decay}\}\), the integral \(\int_{-\infty}^{\infty}[(il - A)^{-1}] \phi(l) \, dl\) is defined and varies continuously with \(\phi\). We have

\[
\int_{-\infty}^{\infty}[(il - A)^{-1}] \phi(l) \, dl = \int_{-\infty}^{\infty}[(il + \epsilon - A)^{-1} - (il - A)^{-1}] \phi(l) \, dl
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{0}^{\infty} g(t) e^{-t(\epsilon + il)} \, dt + \int_{-\infty}^{0} g(t) e^{t(\epsilon - il)} \, dt \right] \phi(l) \, dl
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t) e^{-|t|i} \, dt \phi(l) \, dl
\]

\[
= \int_{-\infty}^{\infty} g(t) e^{-|t|i} \hat{\phi}(t) \, dt
\]

\[
= \int_{-\infty}^{\infty} g(t) \tilde{\phi}(t) \, dt,
\]

the limit \(\epsilon \downarrow 0\) being understood. This is well-defined, since the class \(\mathcal{F}\) is invariant under Fourier transform. To see that it depends continuously on \(\phi\), we note that if \(\phi_j\) is a sequence tending to zero in \(\mathcal{F}\), then there exists \(h > 0\) such that \(\sup \hat{\phi}_j(t) \exp h \rho_t\) tends to zero as \(j \to \infty\). However, since \(\|g(t)\|_{op} \leq M \exp h |t|/2\) (say — using the Hille–Yoshida–Phillips estimate and the fact that the spectrum of \(A\) is purely imaginary), the integrals tend to zero as \(j \to \infty\).

Some comments are in order at this point: (a) We have just shown that the jump in the resolvent is the Fourier transform of \(g(t)\). (b) The integral displayed above must lie in the domain of \(A\) (since the class \(\mathcal{F}\) is invariant under differentiation). (c) These results are independent of positivity properties of \(A\). In general, then, groups of type zero (in the semigroup sense) have generators which can be analyzed in terms of Fourier hyperfunctions. These generalized functions admit a useful microlocalization.

A direct argument to establish positivity is probably possible, but partly for technical reasons and partly for its future utility, I shall give another. Every Fourier hyperfunction can be viewed as an initial datum for the heat equation. If the corresponding solution for \(t > 0\) is everywhere non-negative, then the hyperfunction is a measure. This result is due to Chung et al. (1995), and can be thought of as an extension of the Bochner–Schwartz theorem. (In this context, the initial datum is called an Aronszajn trace.)

The solution to the heat equation with initial value \([(il - A)^{-1}]\) is

\[
U(x,t) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-t)^2/(4t)} [(il - A)^{-1}] \, dl
\]

\[
= 2^{-1/2} \int_{-\infty}^{\infty} g(s) e^{-ixs-ts^2} \, ds.
\]

For \(t > 0\), this maps to the domain of \(A\). We also note the identities

\[
U(x,t)U(x,t) = (\pi/4t)^{1/2} U(x,t/2)
\]

and

\[
U(x,t) = 2^{-1/2} \int_{-\infty}^{\infty} g(-s) e^{-ixs-ts^2} \, ds,
\]

The limit \(\epsilon \downarrow 0\) being understood. This is well-defined, since the class \(\mathcal{F}\) is invariant under Fourier transform. To see that it depends continuously on \(\phi\), we note that if \(\phi_j\) is a sequence tending to zero in \(\mathcal{F}\), then there exists \(h > 0\) such that \(\sup \hat{\phi}_j(t) \exp h \rho_t\) tends to zero as \(j \to \infty\). However, since \(\|g(t)\|_{op} \leq M \exp h |t|/2\) (say — using the Hille–Yoshida–Phillips estimate and the fact that the spectrum of \(A\) is purely imaginary), the integrals tend to zero as \(j \to \infty\).

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\]

and

\[
U(x,t) = 2^{-1/2} \int_{-\infty}^{\infty} g(-s) e^{-ixs-ts^2} \, ds,
\]
where the overline indicates the complex conjugate. Using these, we have
\[
\omega(v, AU(x, t)v) = (4t/\pi)^{1/2} \omega(v, AU(x, 2t)U(x, 2t)v) \\
= (4t/\pi)^{1/2} \omega(\overline{U(x, 2t)v}, AU(x, 2t)v),
\]
and this is positive.

We know at this point that for any \( v \), the quantity \( \omega(v, A[(il-A)^{-1}]v) \) is an exponentially tempered measure. It is easily verified that the form \( \omega(\cdot, A[(il-A)^{-1}]\cdot) \) is Hermitian, so by polarization \( A[(il-A)^{-1}] \) is an operator-valued exponentially tempered measure (in the strong sense). If we could divide this by \( il \), we could conclude that \( [(il-A)^{-1}] \) is a measure. However, this is not obviously possible.

For any test function \( \phi(l) \in \mathcal{F} \), write \( \phi(l) = \phi(0)e^{-l^2} + (\phi(l) - \phi(0)e^{-l^2}) \). Then \( (\phi(l) - \phi(0)e^{-l^2}) \) is \( l \) times a smooth function. Using this decomposition, it is easy to see that \( [(il-A)^{-1}] \) extends to a linear form on those continuous functions of exponential decay which are \( C^1 \) at the origin. In particular, the Fourier hyperfunction \( [(il-A)^{-1}] \) is a distribution. Since \( [(il-A)^{-1}] \) is a measure, we must have \( [(il-A)^{-1}] = \alpha \delta(l) + \mu' \) for some \( \alpha \) and some measure \( \mu' \). (We are following the convention where distributions are represented by “generalized functions” under the integral sign, and so the measure is represented as \( \mu' \) or \( \mu'dl \). This is only a symbolic representation, and is not meant to assert any regularity of the measure with respect to Lebesgue measure. The prime is present so as to have the formal correspondence \( du = \mu'dl \), but we are not interested in trying to construct \( \mu \) or giving \( \mu' \) meaning as a derivative. We should more carefully absorb the \( dl \)'s into \( [(il-A)^{-1}] \) and \( \delta(l) \) and write simply \( d\mu(l). \) We caution that \( \mu'dl \) is known to be locally finite only on \( \mathbb{R} \) minus \{0\}.

The coefficient \( \alpha \) must vanish. To see this, note that we have (strongly)
\[
\alpha = -\lim_{a \downarrow 0} \int_{-\infty}^{\infty} [(il-A)^{-1}]e^{-l^2/(2a^2)}dl
\]
and hence
\[
A\alpha = -A\lim_{a \downarrow 0} \int_{-\infty}^{\infty} [(il-A)^{-1}]e^{-l^2/(2a^2)}dl \\
= -\lim_{a \downarrow 0} \int_{-\infty}^{\infty} i[(il-A)^{-1}]l^2e^{-l^2/(2a^2)}dl \\
= 0.
\]
Passage from the first to the second line is justified by use of the Fourier transform for \( [(il-A)^{-1}] \), or by the identity \( A[(il-A)^{-1}] = -1 + il[(il-A)^{-1}] \); passage from the second to the last by the fact that \( il[(il-A)^{-1}] \) is a measure. Now let \( v \in H \) and let \( w = \alpha v \). It is easy to see that as a distribution in space–time, the quantity \( w \) vanishes on \( M \) minus \( \text{supp } f^{ab} \). On the other hand, since \( Aw = 0 \), the local positivity of the form \( T^{ab}f_{ab} \) implies that \( w \) vanishes on the interior of \( \text{supp } f^{ab} \). (To see this, note that we may replace \( [(il-A)^{-1}]l \) in the integrand with \( [(iA)^{-1}](iA) \), which will annihilate anything supported outside of the support of \( f^{ab} \).) Thus if \( w \) were known to be smooth, we would have \( w = 0 \). However, it is easy to check that for any \( u \in H \) we have \( \omega(u, w) = -\omega(\alpha u, v) \). By the arguments just given, this vanishes for smooth \( u \); since the smooth \( u \)'s are dense in \( H \), it vanishes always and \( w = 0 \). Hence \( \alpha = 0 \).
We shall now write \( dE(l) = [(il - A)^{-1}] dl \). It is a projection-valued distribution which is locally a projection-valued measure. To see that it is projection-valued, note that for any test function \( \phi(l) \) we have
\[
\left( \int_{-\infty}^{\infty} [(il - A)^{-1}] \phi(l) \, dl \right)^2 = \int_{-\infty}^{\infty} g(t)g(s)\tilde{\phi}(t)\tilde{\phi}(s) \, dt \, ds
\]
\[
= \int_{-\infty}^{\infty} g(u)\tilde{\phi}(u - s)\tilde{\phi}(s) du \, ds
\]
\[
= \int_{-\infty}^{\infty} [(il - A)^{-1}] \phi(l)^2 \, dl .
\]

To establish the compatibility of the decompositions with the symplectic structure, notice that
\[
\omega(\int \phi(l)dE(l)v, \int \psi(k)dE(k)w) = \omega(v, \int \phi(-l)dE(l) \int \psi(k)dE(k)w)
\]
\[
= \omega(v, \int \phi(-l)\psi(l)dE(l)w) .
\]
For a subspace \( H_S = \int_S dE(l)H \) to be real, the set \( S \) must be symmetric (up to terms of \( dE \)-measure zero). For symmetric sets, the equation above shows that \( H_S \cap H_{S'} = \{0\} \) if \( S \cap S' = \emptyset \). Thus the spectral decomposition by \( dE \) respects the symplectic structure. That \( \omega \) must be strongly non-degenerate on each \( H_S \) follows.

To establish \( A = \int ildE(L) \) strongly on \( D(A) \), we consider
\[
\int_{-\infty}^{\infty} [(il - A)^{-1}] \phi(l) \, dl = \int_{-\infty}^{\infty} g(t)\tilde{\phi}(t) \, dt .
\]
Take, for instance, the function \( \phi = l\exp(-\epsilon l^2/2) \) and \( v \in D(A) \). Then
\[
\tilde{\phi} = -(2\pi\epsilon)^{-1/2} \frac{d}{dt} e^{-t^2/(2\epsilon)} du .
\]
Substituting this into the previously displayed equation and integrating by parts, we get
\[
\int_{-\infty}^{\infty} [(il - A)^{-1}] e^{-\epsilon l^2/2} \, dlv = (2\pi\epsilon)^{-1/2} \int_{-\infty}^{\infty} Ag(t)e^{-t^2/(2\epsilon)} \, dt .
\]
Taking \( \epsilon \downarrow 0 \) gives the required result.

It is natural to wonder about the integrability of \( dE(l) \) at infinity. This is a delicate issue, on account of the non-scalar nature of the measure. I believe it is possible to get fairly general results, but I shall not attempt these here. We saw above that we can get useful specific results by exploiting the Fourier transform relation
\[
\int_{-\infty}^{\infty} [(il - A)^{-1}] \phi(l) \, dl = \int_{-\infty}^{\infty} g(t)\tilde{\phi}(t) \, dt \quad (17)
\]
for specific suitable functions \( \phi(l) \). In particular, if \( \phi \) is a constant (or exponential of pure frequency), then \( \tilde{\phi} \) is a delta-function and the above equation defines the left-hand
side strongly. Similarly, for \( \phi(l) = t^{ikl} \) the Fourier transform exists as the derivative of a delta-function, and the equation defines the left-hand side strongly on \( D(A) \). Thus the class of \( \phi \)'s for which \( \int_{-\infty}^{\infty} [(il - A)^{-1}] \phi(l) dl \) exists strongly may be extend to include (for example) those which are sums of constants plus continuous compactly supported functions; and the integral may be defined strongly on \( D(A) \) for sums of linear functions plus continuous compactly supported functions.

The delicate issues involved in developing a very general theory of this have to do with clarifying precisely the set of admissible \( \phi \)'s and its topology. However, the observations we have made will be enough for this paper.

While we are not guaranteed the sort of canonical form we had for classically positive operators, we may still draw some conclusions by considering \( H \) as a limit of spaces.

**Proposition 5.** Let \( A \) be the generator associated to smooth compactly-supported future directed test field. Then the associated normally-ordered Hamiltonian operator is bounded below if \( A_\downarrow \) is trace-class; in this case, there is a bound

\[
-E_0 \geq -\text{tr} \sqrt{C^T C}.
\]

**Proof.** In order to make use of the results of previous sections on classically positive Hamiltonians, we shall introduce a modified family of operators \( A_\epsilon \) which are classically positive and tend to \( A \) as \( \epsilon \downarrow 0 \).

Let \( b(l) \) be a continuous bump function, supported on \([-1, 1]\), identically unity in a neighborhood of the origin, and symmetric. Let \( A_\epsilon = \int \text{R} il(1 - b(l/\epsilon)) dE(l) \). We may regard \( A_\epsilon \) either as an operator on \( H_\epsilon \) or, when convenient, on the subspace \( H(\epsilon) \int \text{R}(1 - b(l/\epsilon)) dE(l) H \). The discussion of the compatibility between \( dE(l) \) and \( \omega \) at the end of the last proof shows that \( \omega \) restricts to be non-degenerate on \( H(\epsilon) \), and then \( A_\epsilon \) is classically positive on \( (H(\epsilon), \omega) \). We have \( A_\epsilon v \to Av \) as \( \epsilon \downarrow 0 \) for \( v \in D(A) \), since \( ldE(l)v \) is a measure (and the mass of \( \{0\} \), that is, the coefficient \( \alpha \) in the previous proof, is zero).

Now let \( |\Psi\rangle \) be any Hadamard state of norm unity. This means that in the holomorphic representation \( \Psi(Z) \) is a polynomial whose coefficients are represented by smooth fields on space–time; in particular, these coefficients lie in (tensor products of) \( D(A) \). Thus we may compute

\[
(\langle \Psi| \hat{H} |\Psi\rangle = \lim_{\epsilon \downarrow 0} \langle \Psi| \hat{H}_\epsilon |\Psi\rangle ,
\]

where \( \hat{H}_\epsilon \) is the Hamiltonian defined from \( A_\epsilon \) by normal ordering. (Brunetti et al. 1996 showed that \( \hat{H} \) may be defined by normal ordering.) But we know a lower bound of \( \hat{H}_\epsilon \) is \( -\text{tr} \langle A_\epsilon \rangle \). Now in fact the lower bounds are monotonically decreasing with \( \epsilon \). This follows from the fact that for any fixed \( \epsilon_0 > 0 \), a fixed \( \Theta \) can be found which simultaneously provides a similarity of all \( A_\epsilon \) with \( \epsilon > \epsilon_0 \) to generators of orthogonal groups. It follows that

\[
\inf \hat{H} \geq \lim \inf \hat{H}_\epsilon,
\]

where \( \inf \hat{H}_\epsilon \) denotes the infimum of the spectrum.

Similarly, it follows from the formula for \( C^T C \) in theorem [5] that \( C^T C_\epsilon \) is a family of symmetric positive forms, which are (as forms) increasing as \( \epsilon \downarrow 0 \). But as we know that \( C_\epsilon \to C \) strongly, we have

\[
\lim_{\epsilon \downarrow 0} \text{tr} \sqrt{C^T C_\epsilon} \leq \text{tr} \sqrt{C^T C}.
\]

\( \square \)
We are now in a position to establish the existence of a very large class of quantum inequalities.

**Theorem 7.** Let $A$ be the generator associated to smooth compactly-supported future-directed test field. Then the associated quantum Hamiltonian is bounded below.

**Proof.** It only remains to note that since $A_-$ has a smooth kernel, it is trace-class. (See e.g. Treves 1980.)

The argument for this result has been very technical, and I wish to comment here on why this is.

First, it must be emphasized that because in general the operators we are dealing with are not self-adjoint (nor unitary), merely having some control over their spectrum tells us very little. For example, suppose we have an operator with a discrete set of eigenvalues:

$$\sum_j \lambda_j E_j,$$

(18)

where the $E_j$'s are projections. If the operator is self-adjoint, then the $E_j$'s are orthogonal projections, and in particular, uniformly bounded as operators. However, in the more general case, the $E_j$'s may have diverging bounds. (That is, we may be able to find a unit vector $v_j$ so that $\|E_j v_j\| \to \infty$.) Thus it is quite possible to have $\lambda_j \to 0$ but still have the sum above represent an unbounded operator, or to have the bounds of $\lambda_j E_j$ not tend to zero.

Just these sorts of concerns are present in the regime $l \approx 0$ for the operator $A$. This can be understood by considering its interpretation in space–time, as follows. Since $l$ is the Fourier transform variable to $t$, we may expect that the behavior of $A$ near the spectral parameter $l = 0$ is related to the $t \to \pm \infty$ asymptotics of $g(t)$. In space–time, this corresponds to flowing along the Hamiltonian vector field determined by $\int f^{ab} T_{ab} dvol$ for very long times. Now, if we start with some general solution $\phi$ and flow along this vector field, whatever oscillations $\phi$ has within the region $f^{ab} \neq 0$ will tend to pile up on the future and past boundaries of that region. Thus as $t \to \pm \infty$, the quantity $g(t) \phi$ will be approximately some average value in the interior of $f^{ab} \neq 0$, but quite scrunched up near the boundary. It is very possible that this results in $\text{d} E(l)$ not being integrable at $l = 0$.

A second difficulty is that we do not have very good control over the quantization of $A$, compared to that for classically positive operators. We know, from the work of Brunetti et al., that $A$ is self-adjointly implementable by normal ordering, but we do not have the sorts of explicit control over its lower bound that we had in the previous section. This is related to the first difficulty, in that what prevents us from having this control is the fact that the operator $\Theta$ may not be bounded, which is again due to the $l \approx 0$ behavior of $A$.

To get around this lack of control of the quantization, we approximated the operator $A$ by operators $A_\epsilon$. This approximation, though, was rather weak, necessitating some further, indirect steps.

From a physical point of view, the differences between $A$ and the limit of the $A_\epsilon$'s, and $\hat{H}$ and the $\hat{H}_\epsilon$'s, are measures of the importance of effects the boundary of the region $f^{ab} \neq 0$, which one would hope are unimportant. After all, the point of having $f^{ab}$ approach zero smoothly at the boundaries was precisely to try to minimize edge effects. However, at least the present argument does not show, for example, that $\lim_{\epsilon \to 0} \inf \hat{H}_\epsilon = \inf \hat{H}$.  

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7 Summary and Conclusions

These papers were motivated by the desire to understand some surprising and at least apparently pathological results for quantum fields in curved space–time. The worst of these is that, in generic circumstances, the Hamiltonians are unbounded below. This is absolutely counter to one’s expectations. If in fact these field theories do describe the real world, then one must explain why these pervasive arbitrarily negative energies do not lead to instabilities.

As emphasized in the introduction to Paper I, the present analysis is only a step to understanding these properties. We have aimed here to get a clear statement of what the mathematical structure of the theory actually is. We have seen that the assumption that the classical energy function of a Hamiltonian system is strictly positive provides a very strong restriction on its structure, somewhat analogous to the compactness of the energy surfaces in the finite-dimensional case. This gives one good mathematical control, and one can say under what conditions a self-adjoint quantum Hamiltonian exists. We have seen that there is an intimate connection between self-adjointness (or, at the level of finite evolutions, unitarity), and boundedness below. All self-adjoint quantizations are unitarily equivalent (modulo additive constants), and all are bounded below.

These positive mathematical results throw the pathological features into stronger relief. In generic circumstances, the Hamiltonians for temporal evolution of quantum field theories are neither self-adjoint nor bounded below. Typically, it is only temporally-averaged energy operators which are bounded below (and are self-adjoint): this is the force of the quantum inequalities, proved in the previous section.

The resolution of the pathologies will require physical input. I have shown earlier that, at least in many circumstances, there are limits from quantum measurement theory on the detection of negative energy densities (Helfer 1998). However, this is at present far from an explanation of why the predictedly generic arbitrarily negative energy densities seem to have no role in the world.

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