Kinetic exchange opinion model: solution in the single parameter map limit

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We study a recently proposed kinetic exchange opinion model (Lallouache et. al., Phys. Rev E 82, 056112 (2010)) in the limit of a single parameter map. Although it does not include the essentially complex behavior of the multiagent version, it provides us with the insight regarding the choice of order parameter for the system as well as some of its other dynamical properties. We also study the generalized two-parameter version of the model, and provide the exact phase diagram. The universal behavior along this phase boundary in terms of the suitably defined order parameter is seen.

I. INTRODUCTION

Dynamics of opinion and subsequent emergence of consensus in a society are being extensively studied recently [1–8]. Due to the involvement of many individuals, this type of dynamics in a society can be treated as an example of a complex system, thus enabling the use of conventional tools of statistical mechanics to model it [9, 10, 13]. Of course, it is not possible to capture all the diversities of human interaction through any model of this kind. But often it is our interest to find out the global perspectives of a social dynamics, like average opinion of all the individuals regarding an issue, where the intricacies of the interactions, in some sense, are averaged out. This is similar to the approach of kinetic theory, where the individual atoms, although following a deterministic dynamics, are treated as randomly moving objects and the macroscopic behaviors of the whole system are rather accurately predicted.

Indeed, there have been several attempts to realise the human interactions in terms of kinetic exchange of opinions between individuals [9, 10, 13]. Of course, there is no conservations in terms of opinion. Otherwise, this is similar to momentum exchange between the molecules of an ideal gas. These models were often studied using a finite confidence level, i.e., agents having opinions close to one another interacts. However, in a recently proposed model [14], an unrestricted interactions between all the agents were considered. The single parameter in the model described the ‘conviction’ which is a measure of an agents tendency to retain his opinion and also to convince others to take his opinion. It was found that beyond a certain value of this ‘conviction parameter’ the ‘society’, made up of N such agents, reaches a consensus, where majority shares similar opinion. As the opinion values could take any values between [-1:+1], a consensus means a spontaneous breaking of a discrete symmetry.

There have been subsequent studies to generalize this model, where the ‘conviction parameter’ and ‘ability to influence’ were taken as independent parameters [15]. In that two-parameter version, similar phase transitions were observed. However, the critical behaviors in terms of the usual order parameter, the average opinion, were found to be non-universal.

There have been other extensions in terms of a phase transition induced by negative interactions [16], an exact solution in a discrete limit [17], a generalized map version [18] and also the percolative transitions in a square lattices [19].

In the present study, we investigate the single parameter map version of the model, also proposed in Ref. [14]. In this limit, the map can be conceived as a random walk. Using standard random walk statistics, several static and dynamical quantities have been calculated. We show that the fraction of extreme opinion behaves like the actual order parameter for the system, and the average opinion shows unusual behavior near critical point. The critical behavior of the order parameter and its relaxation behavior near and at the critical point have been investigated both analytically and numerically.

II. MODEL AND ITS MAP VERSION

The kinetic exchange model of opinion pictures the opinion exchange between two agents like a scattering process in an ideal gas. However, unlike ideal gas, there is no conservation of the total opinion. This is similar to the kinetic exchange models of wealth distribution, where the conservation was also present [21]. The exchange equations of the model read

\[ O_i(t + 1) = \lambda O_i(t) + \epsilon O_j(t), \quad (1) \]

and a similar equation for \( O_j(t + 1) \), where \( O_i(t) \) is the opinion value of the \( i \)-th agent at time \( t \), \( \lambda \) is the ‘conviction parameter’ (considered to be equal for all agents for simplicity) and \( \epsilon \) is an annealed random number drawn from a uniform and continuous distribution between [0:1], which is the probability with which \( i \) and \( j \) interact (see [14]). Note that the choice of \( i \) and \( j \) are unrestricted.
making the effective interaction range to be infinite. The opinion values allowed are bounded between the limits $-1 \leq O_i(t) \leq +1$. So, whenever the opinion values are predicted to be greater (less) than +1 (-1) following Eq. (1), it is kept at +1 (-1). This bound, along with Eq. (1), defines the dynamics of the model.

This model shows a symmetry breaking transition at a critical value of $\lambda$ ($\lambda_c \approx 2/3$). The critical behaviors were studied using the average opinion $O_a = |\sum_{i=1}^{N} O_i(t \to \infty)|/N$ \[22\]. An alternative parameter was also defined in Ref. [14], which is the fraction of agents having extreme opinion values. This quantity also showed critical behaviors at the same transition point.

The model in its original form is rather difficult to tackle analytically (it can be solved in some special limits though [17]). However, as it is a fully connected model, a mean field approach would lead to the following evolution equation for the single parameter opinion value (cf. [14])

$$O(t+1) = \lambda(1+\epsilon)O(t). \quad (2)$$

This is, in fact, a stochastic map with the bound $|O(t)| \leq 1$. For all subsequent discussions, whenever an explicit time dependence of a quantity is not mentioned, it denotes the steady state value of that quantity and a subscript $a$ denotes the average over the randomness (i.e., ensemble average). As we will see from the subsequent discussions, this map can be conceived as a random walk with a reflecting boundary. As in the case of the multigagent version, the distribution of $\epsilon$ does not play any role in the critical behavior. We have considered two distributions, one is continuous in the interval [0:1] and the other is 0 and 1 with equal probability. Both of these give similar critical behavior.

We also briefly discuss the two-parameter model, where the 'conviction' of an agent and the ability to convince others were taken as two independent parameters \[15\]. In that context, the map would read

$$O(t+1) = (\lambda + \mu \epsilon)O(t), \quad (3)$$

where $\mu$ is the parameter determining an agent's ability to influence others. As before, $|O(t)| \leq 1$.

### III. RESULTS

#### A. Random walk picture

One can study the stochastic map described in Eq. (2) by describing it in terms of random walks. Writing $X(t) = \log(O(t))$, Eq. (2) can be written as

$$X(t+1) = X(t) + \eta, \quad (4)$$

where, $\eta(t) = \log[\lambda(1+\epsilon)]$. As is clear from the above equation, it actually describes a random walk with a reflecting boundary at $X = 0$ to take the upper cut-off of $O(t)$ into account. Depending upon the value of $\lambda$, the walk can be biased to either ways and is unbiased just at the critical point. As one can average independently over these additive terms in (4), this gives an easy way to estimate the critical point \[20\]. An unbiased random walk would imply $\langle \eta \rangle = 0$ i.e.,

$$\int_{0}^{1} \log[\lambda_c(1+\epsilon)]d\epsilon = 0 \quad (5)$$

giving $\lambda_c = \epsilon/4$, where we have considered an uniform distribution of $\epsilon$ in the limit [0:1]. This estimate matches very well with numerical results of this and earlier works \[14\]. In order to guess the $\lambda$ dependence of $O_a$ in the ordered region, we first estimate the "average return time" $T$ (return time is the time between two successive reflec-

![FIG. 1: The average return time $T$ of $O(t)$ to 1 in the map described in Eq. (2) is plotted with $(\lambda - \lambda_c)$. It shows a divergence with exponent 1 as is predicted from Eq. (4).](image1)

![FIG. 2: $O_a$ is plotted with $\lambda$. The data points are results of numerical simulations, which fits rather well with the solid line predicted from Eq. (3), with $k = 0.7$.](image2)
tions) as a function of bias of the walk. For this uniform
distribution of $\epsilon$, the average position to which the walker
goes following a reflection from the barrier is $(\lambda + 1)/2$.
The average amount of contribution in each step is given
by $\int \log[1 + \epsilon]/d\epsilon = \log(\lambda/\lambda_c)$. This, in fact, is a mea-
sure of the bias of the walk, which vanishes linearly with
$(\lambda - \lambda_c)$ as $\lambda \rightarrow \lambda_c$. So, in this map picture, one
would expect that on average by multiplying this $\lambda/\lambda_c$ factor $T$
times, $O(t)$ would reach 1 from $(\lambda + 1)/2$. Therefore,

$$\frac{\lambda + 1}{2} \left(\frac{\lambda}{\lambda_c}\right)^T = 1,$$

(6)
giving

$$T = -\frac{\log \lambda}{\log \lambda - \log \lambda_c} \approx -\frac{\log \lambda}{\lambda - \lambda_c}$$

(7)

for $\lambda \rightarrow \lambda_c$. Clearly, the average return time diverges
near the critical point obeying a power law: $T \sim (\lambda - \lambda_c)^{-1}$. In Fig. 1 we have plotted this average return time
as a function of $\lambda$. The power-law divergence agrees very
well with the prediction.

The steady state average value of $X(t)$ i.e., $X_a$ (and correspondingly $O_a$) is expected to be proportional to $\sqrt{T}$ in steps of log $\lambda$:

$$X_a \sim \sqrt{T} \log \lambda = k\sqrt{T} \log \lambda,$$

(8)

where $k$ is a constant. This gives

$$O_a = \exp(-k|\log \lambda|^{1/2}(\lambda - \lambda_c)^{-1/2}).$$

(9)

The above functional form fits quite well (see Fig. 2) with the numerical simulation results near the critical
point. It may be noted that the numerical results for the
kinetic opinion exchange (1) also fits quite well with this
expression (9). We note that $O_a$ increases from zero at the
critical point and eventually reaches 1 at $\lambda = 1$. But
its behavior close to critical point cannot be fitted with a
power-law growth usually observed for order-parameters.
Such peculiarity in the critical behavior of $O_a$ compels
us to exclude it as an order parameter though it satisfies some other good qualities of an order parameter.
Instead, we consider the average ‘condensation fraction’
$\rho_a$ as the order parameter. In the multi-agent version,
it was defined as the fraction of agents having extreme
opinion values i.e., -1 or +1. In this case it is defined as
the probability that $O(t) = 1$. We denote this quantity
by $\rho(t)$. As is clear from the definition, one must have

$$\rho_a \sim \frac{1}{T},$$

(10)

where, $T$ is the return time of the walker. As $T \sim (\lambda - \lambda_c)^{-1}$, $\rho_a \sim (\lambda - \lambda_c)^\beta$ with $\beta = 1$. This behavior is clearly
seen in the numerical simulations (see Fig. 3).

Also, the relaxation time shows a divergence as the critical point is approached. We argue that there is a
FIG. 3: The average condensation fraction (probability that $O = 1$) $\rho_a$ is plotted with $(\lambda - \lambda_c)$. A linear fit in the log-log scale gives the growth exponent 1, as predicted from Eq. (10). Inset shows the variation of $\rho$ with $\lambda$.
single time scale for both $O(t)$ and $\rho(t)$. So we calculate the divergence of relaxation time for $O(t)$ and numerically show that the results agree very well with the
relaxation time divergence for both $O(t)$ and $\rho(t)$. Consider the subcritical regime where the random walker is
biased away from the reflector and would have a probability distribution as

$$p(X) = \frac{A}{\sqrt{t}} \exp(-B(X - vt)^2)/t),$$

(11)

where $v \sim 1/T \sim (\lambda - \lambda_c)$ is the net bias and constants $A, B$ do not depend on $t$. One can therefore obtain the probability distribution $P$ of $O$ using $p(X)dX = P(O)dO$,

$$P(O) = \frac{A}{\sqrt{O}} \exp(-B(\log O - vt^2)/t).$$

(12)

Hence

$$O_a(t) = \int_0^1 OP(O)dO,$$

$$= \frac{A}{\sqrt{t}} \int_0^1 \exp(-B(\log O - vt^2)/t)dO$$

$$\sim \frac{A}{\sqrt{t}} \exp(-Bu^2t),$$

(13)
in the long time limit, giving a time scale of relaxation
$\tau \sim v^{-2} \sim (\lambda - \lambda_c)^{-2}$. We have fitted the relaxation of
$O_a(t)$, obtained numerically, with an exponential decay
and found $\tau$. As can be observed from Fig. 3 it shows a
clear divergence close to critical point with exponent 2.

We have obtained the relaxation time of $\rho_a(t)$ and it also shows similar divergence. Note that at $\lambda = \lambda_c, v = 0$ and it follows from Eq. (13) that $O_a(t) \sim t^{-1/2}$. This behavior is also confirmed numerically (see Fig. 5). The
average condensation fraction $\rho_a(t)$ too follows this scaling, giving $\delta = 1/2$ (as order parameter relaxes as $t^{-\delta}$ at critical point).

We have also investigated the effect of having an external field linearly coupled with $O(t)$. In the multiagent scenario, this can have the interpretation of the influence of media. The map equation now reads,

$$O(t+1) = \lambda(1+\epsilon t)O(t) + hO(t),$$

where $h$ is the field (constant in time). We have studied the response of $O_a$ and $\rho_a$ at $\lambda = \lambda_c$ due to application of small $h$. We find that (see Fig. 6) both grows linearly with $h$. One expects the order parameter to scale with external field at the critical point as $\rho_a \sim h^{1/\delta'}$. In this case $\delta' = 1$.

### B. Random walk with discrete step size

One can simplify the random walk mentioned above and make it a random walk with discrete step sizes. This can be done by considering the distribution of $\epsilon$ to be a double delta function, i.e., $\epsilon = 1$ or 0 with equal probability. This will make $\eta(t)$ in Eq. (2) to be $\log(\lambda)$ or $\log(2\lambda)$ with equal probability. Below critical point, both steps are in negative direction (away from reflector) and consequently taking the walker to $-\infty$. Exactly at critical point ($\lambda = \lambda_c$) the step sizes become equal and opposite i.e., $\log \lambda_c = -\log 2\lambda_c$ giving $\lambda_c = 1/\sqrt{2}$. Above critical point, one of the steps is positive and the other is negative. However, the magnitudes of the steps are different. This unbiased walker (probability of taking positive and negative steps are equal) with different step sizes can approximately be mapped to a biased walker with equal step size in both directions. To do that consider the probability $p(x,t)$ that the walker is at position $x$ at time $t$. One can then write the master equation

$$p(x,t+1) = \frac{1}{2}p(x+a,t) + \frac{1}{2}p(x+a+b,t),$$

where $a = \log(\lambda)$ and $b = \log(2)$. Clearly,

$$\frac{\partial p(x,t)}{\partial t} = \left( a + \frac{b}{2} \right) \frac{\partial p(x,t)}{\partial x} + \left( \frac{a^2}{2} + \frac{ab}{2} + \frac{b^2}{4} \right) \frac{\partial^2 p(x,t)}{\partial x^2}. \quad (16)$$

One can compare this equation with the usual biased random walker equation

$$\frac{\partial p(x,t)}{\partial t} = (p-q)a \frac{\partial p(x,t)}{\partial x} + \frac{(a')^2}{2} \frac{\partial^2 p(x,t)}{\partial x^2}, \quad (17)$$

where $a'$ is the field (constant in time).
where \( p \) and \( q \) are the probabilities of taking a positive and negative steps \( (p + q = 1) \) and \( a' \) is the (equal) step size in either direction. Comparing these two equations one gets

\[
p = \frac{1}{2} \left[ 1 + \frac{a + b/2}{a'} \right] \\
q = \frac{1}{2} \left[ 1 - \frac{a + b/2}{a'} \right] \\
a' = -\sqrt{(\log(\lambda) - \log(\lambda_c))^2 + (\log(\lambda_c))^2}. \quad (18)
\]

Therefore, as \( \lambda \to \lambda_c \), the bias \( (p - q) \sim (\lambda - \lambda_c)/a' \). These are consistent with the earlier calculations where we have taken the bias to be proportional to \( (\lambda - \lambda_c) \). To check if this mapping indeed works, we have simulated a biased random walk with above mentioned parameters and found it to agree with the original walk close to the critical point (see Fig. 7). Similar to the approach taken for the continuous step-size walk, one can find the return time of the walker. This time the walker is exactly located at \( \lambda \) when it is reflected from the barrier. The return time again diverges as \( (\lambda - \lambda_c)^{-1} \). Also \( O_a(t) \) will have a similar form up to some prefactors. Condensation fraction will increase linearly with \( (\lambda - \lambda_c) \) close to the critical point. All the other exponents regarding the relaxation time, time dependence at the critical point and dependence with external field are same as before. This shows that the critical behavior is universal with respect to changes in the distribution of \( \epsilon \).

### C. Two parameter map

As the multi-agent model was generalized in a two-parameter model [15], one can also study the map version of that two-parameter model. It would read

\[
O(t + 1) = (\lambda + \mu \epsilon) O(t).
\]

As before, one can take log of both sides and in similar notations

\[
X(t + 1) = X(t) + \log(\lambda + \mu \epsilon).
\]

This can also be seen as a biased random walk. If one makes \( \epsilon \) discrete, this is again a walk with unequal step sizes, which can again be mapped to a biased walk with equal step sizes. Therefore, in either case, some prefactors will be changed, but the critical behavior will be same as before. The critical behavior is, therefore, universal when studied in terms of the condensation fraction. For uniformly distributed \( \epsilon \) (in the range \([0:1]\)), one can get the expression for the phase boundary from the equation

\[
\int_0^1 \log(\lambda + \mu \epsilon) d\epsilon = 0,
\]

which gives

\[
\log(\lambda_c + \mu \epsilon) + \frac{\lambda_c}{\mu \epsilon} \log \left( \frac{\lambda_c + \mu \epsilon}{\lambda_c} \right) = 1. \quad (22)
\]

Of course, this gives back the \( \log \lambda_c = e/4 \) limit when \( \lambda = \mu \). The phase boundary is plotted in Fig. 8. It also agrees with numerical simulations.

### IV. SUMMARY AND CONCLUSIONS

In this paper we have studied the simplified map version of a recently proposed opinion dynamics model. The
map can be cast in a random walk picture with reflecting boundary. Then using the standard random walk statistics, some steady state as well as dynamical behaviors are calculated and these are compared with the corresponding numerical results. It is observed that the usual parameter of the system, i.e. the average value of the opinion \((O_a)\), does not follow any power-law scaling (see Eq. (9)). In fact, it is the condensation fraction, or in this case the probability \((\rho_a)\) that the opinion values touches the limiting value, turns out to be the proper order parameter, showing a power-law scaling near critical point.

The dynamical behaviors of the two quantities \((O_a(t))\) and \((\rho_a(t))\) are similar. We have found the power-law relaxation of these quantities at the critical point. Also, the divergence of the relaxation time on both sides of criticality shows similar behavior. We have also studied the effect of an external field in these models. At critical point, both the quantities grow linearly with the applied field. The average fluctuation in both of these quantities show a maximum near the critical point. These theoretical behaviour fits well with the numerical results.

In summary, we develop an approximate mean field theory for the dynamical phase transition one observed for the map (2) and find the average condensation fraction \(\rho_a \sim (\lambda - \lambda_c)^{\beta}\) with \(\beta = 1\) behave as the order parameter for the transition and it has typical relaxation time \(\tau \sim (\lambda - \lambda_c)^{-z}\) with \(z = 2\) and at critical point \(\lambda = \lambda_c(= e/4)\) decays as \(t^{-\delta}\) with \(\delta = 1/2\).