Holomorphic Continuation via Laplace-Fourier series

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Abstract

Let $B_R$ be the ball in the euclidean space $\mathbb{R}^n$ with center $0$ and radius $R$ and let $f$ be a complex-valued, infinitely differentiable function on $B_R$. We show that the Laplace-Fourier series of $f$ has a holomorphic extension which converges compactly in the Lie ball $\hat{B}_R$ in the complex space $\mathbb{C}^n$ when one assumes a natural estimate for the Laplace-Fourier coefficients.

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1 Introduction

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere where $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ denotes the euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Furthermore, let $Y_{k,l}(x)$, $l = 1, \ldots, a_k$, be a basis for the set of all harmonic homogeneous polynomials of degree $k \geq 0$ which are orthonormal with respect to the usual scalar product

$$\langle f, g \rangle_{S^{n-1}} = \int_{S^{n-1}} f(\theta)\overline{g(\theta)}d\theta,$$

see [3], [15]. For a function $f$ given on the ball $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ we have the the Laplace-Fourier series of $f$ given by

$$\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(r) Y_{k,l}(\theta)$$

with the Laplace-Fourier coefficients $f_{k,l}$, defined by

$$f_{k,l}(r) = \int_{S^{n-1}} f(r\theta) Y_{k,l}(\theta) d\theta;$$

where $f$ is a complex-valued, infinitely differentiable function on $B_R$.
This paper addresses the following question: Assume that \( f \) is in \( C^\infty (B_R) \), the set of all infinitely many times continuously differentiable functions \( f : B_R \to \mathbb{C} \). Under which conditions can we conclude that the Laplace-Fourier series of \( f \) provides a holomorphic extension to a natural domain \( G \) in \( \mathbb{C}^n \) (depending on \( B_R \)), say by imposing appropriate conditions on the Laplace-Fourier coefficients \( f_{kl} \) ?

Before stating our main result, let us recall some facts already observed in [4]:

(i) for \( f \in C^\infty (B_R) \) the Laplace-Fourier coefficient \( f_{k,l}(r) \) is infinitely many times differentiable on the interval \([0, R]\) and
\[
\frac{d^m}{dr^m} f_{k,l}(0) = 0 \quad \text{for} \quad m = 0, \ldots, k - 1; \tag{3}
\]

(ii) the function \( r \mapsto r^{-k} f_{k,l}(r) \) depends only on the variable \( r^2 \).

Clearly (i) and (ii) imply the existence of a function \( p_{k,l} \in C^\infty ([0, R^2]) \) such that
\[
p_{k,l}(r^2) = r^{-k} f_{k,l}(r). \tag{4}
\]

It is proved in [4] that a function \( f \) which is analytic on a neighborhood of 0 in \( \mathbb{R}^n \), has a holomorphic extension to a neighborhood of 0 in \( \mathbb{C}^n \) if and only if there exist \( t_0 > 0 \) and \( M > 0 \) such that for all \( k, m \in \mathbb{N}_0, l = 1, \ldots, a_k \)
\[
\sup_{t \in [0, t_0]} \left| \frac{d}{dt}^m p_{k,l}(t) \right| \leq M^{k+m+1} m! . \tag{5}
\]

Note that the last condition implies that all functions \( p_{k,l}(\zeta) \) are holomorphic for \( |\zeta| < 1/M \).

In this paper we want to generalize the result in [4], which is of local nature, to a global one. We associate to the ball \( B_R \) in \( \mathbb{R}^n \) a domain \( \widehat{B_R} \) in \( \mathbb{C}^n \), the so-called Lie ball (or the classical domain of E. Cartan of the type IV, see [1, p. 59], [10]) defined by
\[
\widehat{B_R} := \{ z \in \mathbb{C}^n : |z|^2 + \sqrt{|z|^4 - |q(z)|^2} < R^2 \}, \tag{6}
\]
where \( |z|^2 = |z_1|^2 + \ldots + |z_n|^2 \) for \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and
\[
q(z) = z_1^2 + \ldots + z_n^2.
\]

It is not assumed that the reader is acquainted with complex analysis for the Lie ball and we shall only need the definition given in (6). In particular, we shall not use the fact that the Lie ball \( \widehat{B_R} \) can be viewed as the harmonicity hull of the ball \( B_R \), see [11 p. 42] or [2]. The harmonicity hull of a domain \( G \) in \( \mathbb{R}^n \) is the unique largest domain in \( \mathbb{C}^n \) to which every polyharmonic function on \( G \) has a holomorphic continuation, see [11 p. 51]. According to [11 p. 54], the groundwork for the construction of harmonicity hulls was laid by N. Aronszajn
in 1935, and important contributions are due to P. Lelong [16]. In [13] Kiselman discusses hulls for elliptic operators with constant coefficients.

We are now able to formulate our two main results:

**Theorem 1** Let \( f \in C^\infty (BR) \) and let \( f_{k,l} \) and \( p_{k,l} \) be defined in (2) and (4). Then \( f \) has a holomorphic extension to the Lie ball \( \widehat{BR} \) if and only if the Laplace-Fourier coefficients \( f_{k,l} \) are holomorphic functions on the disc \( DR := \{ \zeta \in \mathbb{C} : |\zeta| < R \} \) for all \( k \in \mathbb{N}_0, l = 1, ..., a_k \), and for any \( 0 < \tau < \rho < R \) there exists a constant \( C_{\rho,\tau} > 0 \) such that for all \( k \in \mathbb{N}_0, l = 1, ..., a_k \)

\[
|p_{k,l} (\zeta)| \leq C_{\rho,\tau} \frac{1}{\rho^k} \text{ for all } |\zeta| \leq \tau^2.
\]

**Theorem 2** Suppose that \( f \) is holomorphic on \( \widehat{BR} \), and let \( f_{k,l} \) be the Laplace-Fourier coefficients of the restriction of \( f \) to \( BR \) and \( p_{k,l} (r^2) := r^{-k} f_{k,l} (r) \).

Then the series

\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} p_{k,l} (q(z)) Y_{k,l} (z)
\]

converges compactly and absolutely in \( \widehat{BR} \) to \( f(z) \).

As a byproduct of Theorem 1 we can prove the following well-known result by methods based on purely classical results for Laplace-Fourier series:

**(H)** Every harmonic function \( f : BR \to \mathbb{C} \) has a holomorphic extension to the Lie ball defined in (6).

Let us mention that the apparently most natural approach for a proof of (H), namely via multiple Taylor series, gives only a weaker result: every harmonic function \( f : BR \to \mathbb{C} \) has a holomorphic extension to the complex ball with center 0 and radius \( R/\sqrt{2} \), so to

\[
\{ z \in \mathbb{C}^n : |z| < R/\sqrt{2} \} \subset \widehat{BR}.
\]

see e.g. [8], and [5], [6] and [12] for related results. Of course, from the viewpoint of harmonicity hulls, (H) is a trivial consequence of the fact that the Lie ball defined in (6) is indeed the harmonicity hull of \( BR \). However, the proof of existence and description of a harmonicity hull in [1], depending on the seri-integral representation of a polyharmonic function, is far from being elementary. On the other hand, J. Siciak has noted in [21] that (H) is a simple consequence of the important fact (due to L.K. Hua) that for any homogeneous polynomial \( f \)

\[
\max_{x \in BR} |f(x)| = \max_{z \in \widehat{BR}} |f(z)|;
\]

for a proof of the latter result see [2, p. 115] or [17].
2 Proof of the results

For $z \in \mathbb{C}^n$ we write $z = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$. Let $\langle \xi, \eta \rangle = \sum_{j=1}^{n} \xi_j \eta_j$ be the usual scalar product on $\mathbb{R}^n$. Then $|z|^2 = |\xi|^2 + |\eta|^2$ and

$$q(z) := z_1^2 + \ldots + z_n^2 = |\xi|^2 - |\eta|^2 + 2 i \langle \xi, \eta \rangle.$$ \hfill (10)

A short computation shows that

$$|q(z)|^2 = \left( |\xi|^2 - |\eta|^2 \right)^2 + 4 \langle \xi, \eta \rangle^2 \leq |z|^4.$$ \hfill (11)

Let $P^n_k(t)$ be the Legendre polynomial of degree $k$ for dimension $n$ with the norming condition $P^n_k(1) = 1$. The addition theorem [18] says that for all $x, y \in \mathbb{R}^n$

$$\sum_{l=1}^{a_k} Y^*_{k,l}(x) Y^*_{k,l}(y) = |x|^k |y|^k \frac{a_k}{\omega_{n-1}} P^n_k \left( \left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle \right).$$ \hfill (12)

Here $Y^*_{k,l}$ is the polynomial defined by conjugating the coefficients of $Y_{k,l}$ and $a_k$ is the dimension of the space of all harmonic polynomials of degree $k$, or explicitly

$$a_k := (2k + n - 2) \frac{\Gamma (k + n - 2)}{\Gamma (k + 1) \Gamma (n - 1)}$$

where $\Gamma$ is the Gamma function. Further $\omega_{n-1}$ denotes the surface area of $S^{n-1}$.

**Theorem 3** Let $d_k$ be the leading coefficient of the Legendre polynomial $P^n_k$. For all $z \in \mathbb{C}^n$ with $|q(z)| \neq 0$ the identity

$$\sum_{l=1}^{a_k} |Y_{k,l}(z)|^2 = \frac{a_k}{\omega_{n-1}} |q(z)|^k P^n_k \left( \frac{|z|^2}{|q(z)|} \right)$$ \hfill (13)

holds, and for $q(z) = 0$

$$\sum_{l=1}^{a_k} |Y_{k,l}(z)|^2 = \frac{a_k}{\omega_{n-1}} d_k \cdot |z|^{2k}.$$ \hfill (14)

**Proof.** Let us consider the case that $k$ is even, say $k = 2k_1$. Then $P^n_k(t)$ contains only even powers in $t$, say $P^n_k(t) = \sum_{s=0}^{k_1} c_s t^{2s}$. The expression on the right hand side in (12) is equal to the polynomial

$$S_k(x,y) := \frac{a_k}{\omega_{n-1}} \sum_{s=0}^{k_1} c_s |x|^{2(k_1-s)} |y|^{2(k_1-s)} \left( \langle x, y \rangle \right)^{2s}.$$  

Clearly $z \mapsto q(z)$ is the holomorphic extension of $x \mapsto |x|^2$. Further $f(z,w) := \sum_{j=1}^{n} z_j w_j$ is the holomorphic extension of $(x,y) \mapsto \langle x, y \rangle$. Hence $S_k$ possesses the holomorphic extension

$$S_k(z,w) = \frac{a_k}{\omega_{n-1}} \sum_{s=0}^{k_1} c_s |q(z)|^{(k_1-s)} |q(w)|^{(k_1-s)} \left[ f(z,w) \right]^{2s}.$$ \hfill (15)
Clearly \( \sum_{i=1}^{a_k} Y_{k,l}(z) Y_{k,l}^*(w) \) is the holomorphic extension of the left hand side in (\ref{12}), so the latter expression is equal to (\ref{15}). Now take \( w := \xi \) and note that
\[
Y_{k,l}^*(\xi) = Y_{k,l}(z).
\]
Then
\[
\sum_{i=1}^{a_k} |Y_{k,l}(z)|^2 = \frac{a_k}{\omega_n-1} \sum_{s=0}^{k_1} c_s |q(z)|^{2(k_1-s)} |z|^{4s}.
\]
If \( q(z) = 0 \) we obtain (\ref{13}). If \( |q(z)| \neq 0 \) we can write
\[
\sum_{i=1}^{a_k} |Y_{k,l}(z)|^2 = \frac{a_k}{\omega_n} |q(z)|^{2k_1} \sum_{s=0}^{k_1} c_s \left( \frac{|z|^2}{|q(z)|} \right)^{2s} = \frac{a_k}{\omega_n} |q(z)|^k P_k^n \left( \frac{|z|^2}{|q(z)|} \right).
\]
If \( k \) is odd then \( P_k^n(t) \) contains only odd powers in \( t \), and one can employ similar techniques as in the even case. The proof is complete. \( \blacksquare \)

Lemma 4 Suppose that for a \( z \in \mathbb{C}^n \) holds \( |z|^4 - |q(z)|^2 \leq \tau^2 - |z|^2 \). Then
\[
|q(z)|^k P_k^n \left( \frac{|z|^2}{|q(z)|} \right) \leq \tau^{2k} \quad (16)
\]

**Proof.** We use the Laplace representation for the Legendre polynomial in [18, p. 21], showing that for real \( x \geq 1 \)
\[
P_k^n(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-2}} [x + \sqrt{x^2 - 1} \langle \xi_{n-1}, \eta_{n-1} \rangle]^{k} d\eta_{n-1}.
\]
It follows that \( P_k^n(x) \leq (x + \sqrt{x^2 - 1})^k \) for \( x \geq 1 \). We apply this to \( x := |z|^2 / |q(z)| \geq 1 \). Then
\[
|q(z)|^k P_k^n \left( \frac{|z|^2}{|q(z)|} \right) \leq \left( |z|^2 + \sqrt{|z|^4 - |q(z)|^2} \right)^k.
\]
Since \( \sqrt{|z|^4 - |q(z)|^2} \leq \tau^2 - |z|^2 \) we obtain the desired inequality. \( \blacksquare \)

**Theorem 5** Let \( p_{k,l} \) be holomorphic functions on the disc \( \mathbb{D}_{R^2} : = \{ \zeta \in \mathbb{C} : |\zeta| < R^2 \} \) for all \( k \in \mathbb{N}_0, l = 1, \ldots, a_k \), such that for any \( 0 < \tau < \rho < R \) there exists a constant \( C_{\rho,\tau} > 0 \) such that for all \( k \in \mathbb{N}_0, l = 1, \ldots, a_k \)
\[
|p_{k,l}(\zeta)| \leq C_{\rho,\tau} \frac{1}{\rho^l} \text{ for all } |\zeta| \leq \tau^2.
\]
Then
\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} p_{k,l}(q(z)) Y_{k,l}(z)
\]
converges compactly and absolutely in \( B_R \).

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Proof. We shall show that

\[ f_M (z) := \sum_{k=0}^{M} \sum_{l=1}^{a_k} p_{k,l} (q (z)) Y_{k,l} (z) \]  

(19)

converges compactly and absolutely on \( \hat{B}_R \) to a holomorphic function. Note that \( z \in \hat{B}_R \) implies \( |z| < R \), and (11) implies that \( |q (z)| < R^2 \). Hence (19) is well-defined and clearly \( f_M \) is holomorphic on \( \hat{B}_R \). Let now \( K \) be a compact subset of \( \hat{B}_R \). Recall that \( z \in \hat{B}_R \) if and only if \( \sqrt{|z|^4 - |q (z)|^2} < R^2 - |z|^2 \). Using the compactness of \( K \), we see that there exists \( \tau < R \) such that \( K \subset \hat{B}_\tau \). As above it follows that \( |q (z)| < \tau^2 \) for \( z \in K \). Take \( \rho \) with \( \tau < \rho < R \). Using (17) and the fact that \( q (z) \in \mathbb{D}_{\tau^2} \), we obtain for all \( z \in K \)

\[ |f_M (z)| \leq C_{\rho, \tau} \sum_{k=0}^{M} \rho^k \sum_{l=1}^{a_k} |Y_{k,l} (z)|. \]

Note that by the Cauchy-Schwarz inequality \( \sum_{l=1}^{a_k} |Y_{k,l} (z)| \leq \sqrt{a_k} \sqrt{\sum_{l=1}^{a_k} |Y_{k,l} (z)|^2} \). Then (13) and (16) show that

\[ |f_M (z)| \leq C_{\rho, \tau} \frac{1}{\sqrt{\omega_{n-1}}} \sum_{k=0}^{M} a_k \frac{\tau^k}{\rho^k}. \]

Since \( a_k / a_{k+1} \) converge to 1 we see that (19) converges. It follows that \( f_M \) converges uniformly to a holomorphic function. \( \blacksquare \)

The following result was proved in [4].

Lemma 6 Let \( f \in C^\infty (B_R) \). Then the Laplace-Fourier coefficients \( f_{k,l} \) are infinitely times differentiable at 0 and \( \frac{d^m}{dr^m} f_{k,l} (0) = 0 \) for \( m = 0, \ldots, k - 1 \).

Theorem 7 Let \( f \in C^\infty (B_R) \) and let \( p_{k,l} (r) := r^{-k} f_{k,l} (r) \). Suppose that there exists a continuous function \( \tilde{f} : \Delta_R \times \mathbb{S}^{n-1} \rightarrow \mathbb{C} \) such that

1) \( \tilde{f} (r, \theta) = f (r \theta) \) for all \( 0 \leq r < R \) and \( \theta \in \mathbb{S}^{n-1} \)

2) for each \( \theta \in \mathbb{S}^{n-1} \) the function \( \zeta \mapsto \tilde{f} (\zeta, \theta) \) is holomorphic for \( |\zeta| < R \).

Then \( f \) has a holomorphic extension to the Lie ball, the Laplace-Fourier series in (15) converges compactly to \( f \) and the coefficients \( p_{k,l} (\zeta) \) satisfy (17).

Proof. We want to apply Theorem 5 and we consider at first the Laplace-Fourier coefficients \( f_{k,l} \) of \( f \). We define the holomorphic extension of \( \tilde{f}_{k,l} (r) \) by

\[ f_{k,l} (\zeta) = \int_{\mathbb{S}^{n-1}} \tilde{f} (\zeta, \theta) Y_{k,l} (\theta) d\theta. \]  

(20)

Let us show that \( f_{k,l} \) is indeed holomorphic for \( |\zeta| < R \); let \( |\zeta_0| < R \) and \( \zeta_m \rightarrow \zeta_0 \) with \( |\zeta_m| < R \). Let \( \rho > 0 \) such that \( |\zeta_m| < \rho \) for all \( m \). Since \( \zeta \mapsto f (\zeta, \theta) \) is
holomorphic for $|\zeta| < R$ we can use Cauchy’s integral formula on the path $\gamma_\rho(t) = \rho e^{it}$, i.e.
\[
\tilde{f}(\zeta, \theta) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{\tilde{f}(\xi, \theta)}{\xi - \zeta} d\xi.
\]
We have to show that the differential quotient $D_m := \frac{f_{k,l}(\zeta_m) - f_{k,l}(\zeta_0)}{\zeta_m - \zeta_0}$ converges. Clearly
\[
D_m = \frac{1}{2\pi i} \frac{1}{\zeta_m - \zeta_0} \int_{\mathbb{S}^{n-1}} \int_{B_\rho} \left[ \frac{\tilde{f}(\xi, \theta)}{\xi - \zeta_m} - \frac{\tilde{f}(\xi, \theta)}{\xi - \zeta_0} \right] Y_{k,l}(\theta) d\theta
\]
and this is equal to
\[
\frac{1}{2\pi i} \int_{\mathbb{S}^{n-1}} \int_{B_\rho} \frac{\tilde{f}(\xi, \theta)}{(\xi - \zeta_m)(\xi - \zeta_0)} Y_{k,l}(\theta) d\theta.
\]
Since this expression has clearly a limit, $f_{k,l}$ is holomorphic. Further $p_{k,l}(r^2) := r^{-k} f_{k,l}(r)$ has a holomorphic extension to $\mathbb{D}_{R^2}$ by Lemma 5. Let now $0 < \tau < \rho < R$. By the Cauchy-Schwarz inequality and the orthonormality of $Y_{k,l}(\theta)$ one obtains from (20) for $|\zeta| = \rho$
\[
|f_{k,l}(\zeta)|^2 \leq \int_{\mathbb{S}^{n-1}} |f(\zeta, \theta)|^2 d\theta \cdot \int_{\mathbb{S}^{n-1}} |Y_{k,l}(\theta)|^2 d\theta \leq \omega_{n-1} \max_{t \in [0,2\pi], \theta \in \mathbb{S}^{n-1}} \left| f(e^{it}\rho \theta) \right|^2.
\]
The Cauchy estimate \(|g^{(s)}(0)| \leq \frac{\rho^s}{s!} \max_{|\zeta| = \rho} |g(\zeta)|\) applied to the function $g = f_{k,l}$, and $s = k + m$, and the last estimate imply
\[
\left| \frac{d^{m+k}}{dz^{m+k}} f_{k,l}(0) \right| \leq \sqrt{\omega_{n-1}} \max_{t \in [0,2\pi], \theta \in \mathbb{S}^{n-1}} \left| f(e^{it}\rho \theta) \right| \cdot \frac{(k+m)!}{\rho^{m+k}}. \tag{21}
\]
Let us write $f_{k,l}(r) = \sum_{m=k}^{\infty} \frac{1}{m!} \frac{d^{m+k}}{dz^{m+k}} f_{k,l}(0) \cdot r^m$ for $0 \leq r < R$, cf. 3. Since $r^{-k} f_{k,l}(r)$ is an even function we can write
\[
r^{-k} f_{k,l}(r) = \sum_{m=0}^{\infty} \frac{1}{(k+2m)!} \frac{d^{2m+k}}{dz^{2m+k}} f_{k,l}(0) \cdot r^{2m}.
\]
Hence
\[
p_{k,l}(t) = \sum_{m=0}^{\infty} \frac{1}{(k+2m)!} \frac{d^{2m+k}}{dz^{2m+k}} f_{k,l}(0) \cdot t^{2m}
\]
and using (21) we obtain the estimate for $|\zeta| \leq \tau$,
\[
|p_{k,l}(\zeta)| \leq C \sum_{m=0}^{\infty} \frac{\tau^{2m}}{\rho^{2m+k}} = C \frac{1}{\rho^k} \frac{1 - \tau^2}{\rho^2}
\]
where $C := \sqrt{\omega_{n-1}} \max_{t \in [0,2\pi], \theta \in \mathbb{S}^{n-1}} \left| f(e^{it}\rho \theta) \right|$ does not depend on $k \in \mathbb{N}_0$ and $l = 1, \ldots, a_k$. Now we apply Theorem 5 which gives that the Laplace-Fourier series in $\{\mathbb{R}^n\}$ converges compactly to a holomorphic function $g$. From
the uniform convergence of the Laplace-Fourier series of $g$ it is easy to see that $g$ has the same Laplace-Fourier coefficients as $f$, so we conclude that $f = g$. ■

**Proof.** of Theorem 2. Let $f$ be holomorphic on $B_R$. Clearly $g := f \mid B_R$ is a $C^\infty$-function. Let us define $\tilde{g} : \Delta_R \times S^{n-1} \to \mathbb{C}$ by $\tilde{g}(\zeta, \theta) := f(\zeta \theta)$ which is well-defined since $\zeta \theta \in \overline{B}_R$ for $|\zeta| < R$ and $\theta \in S^{n-1}$. Then $\tilde{g}$ is continuous, and for each fixed $\theta \in S^{n-1}$, the function $\zeta \mapsto \tilde{g}(\zeta, \theta)$ is holomorphic and $\tilde{g}(r \theta) = f(r \theta) = g(r \theta)$ for $0 \leq r < R$ and for all $\theta \in S^{n-1}$. The result follows from Theorem 7. ■

**Proof.** of Theorem 1. Let $f$ be holomorphic on $\overline{B}_R$. By the previous proof we can apply Theorem 7 so (7) is satisfied. For the converse, let $f \in C^\infty(B_R)$ and assume that (7) holds. By Theorem 5 the Laplace-Fourier series in (8) converges compactly to a holomorphic function $g$. Clearly $g$ has the same Laplace-Fourier coefficients as $f$, so we conclude that $f = g$. ■

For $f \in C^\infty(B_R)$ let $f_m$ be the $m$-th homogeneous Taylor polynomial, so

$$f_m(x) := \sum_{|\alpha| = m} \frac{D^\alpha f(0)}{\alpha!} x^\alpha$$

where we use the standard multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. We consider the space $A(B_R)$ of all $f \in C^\infty(B_R)$ such that

$$f(x) = \sum_{m=0}^{\infty} f_m(x) \quad (22)$$

converges absolutely and uniformly on compact subsets of $B_R$. The space $A(B_R)$ evolved naturally in the investigations in [19] for solving a conjecture of W. Hayman concerning sets of uniqueness for polyharmonic functions, see [9]. It is well known that every harmonic function $f : B_R \to \mathbb{C}$ is in $A(B_R)$. We shall now prove that each $f \in A(B_R)$ possesses a holomorphic extension to $\overline{B}_R$ (as we said in the introduction, this fact was already noticed in [21] with a different proof).

**Theorem 8** Let $f \in C^\infty(B_R)$, and suppose that the homogeneous Taylor series $\sum_{m=0}^{\infty} f_m$ converges compactly in $B_R$ to $f$. Then $f$ has a holomorphic extension to the Lie ball.

**Proof.** Let $f$ as described in the theorem. Write $x \in B_R$ in polar coordinates $x = r \theta$ with $\theta \in S^{n-1}$ and $r \geq 0$. Then $f_m(r \theta) = r^m f_m(\theta)$ and $\sum_{m=0}^{\infty} r^m |f_m(\theta)|$ converges uniformly for all $\theta \in S^{n-1}$ and $0 \leq r < R$. Let us define $\tilde{f} : \Delta_R \times S^{n-1} \to \mathbb{C}$ by

$$\tilde{f}(\zeta, \theta) := \sum_{m=0}^{\infty} \zeta^m f_m(\theta).$$

Clearly $\tilde{f}$ is continuous, and the map $\zeta \mapsto \tilde{f}(\zeta \theta)$ for $|\zeta| < R$ is clearly holomorphic for each fixed $\theta \in S^{n-1}$. The result follows from Theorem 7. ■
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