Uncertainty Quantification and Robust Design Optimization of Supersonic Biplane Airfoils

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Uncertainty analysis is receiving a lot of attention recently since high-fidelity computational fluid dynamics computations typically assume perfect knowledge of all parameters. In this research, uncertainty analyses of the Busemann’s biplane airfoil are performed with uncertain inputs of its freestream Mach number and angle-of-attack. The motivation of this research is to clarify the characteristics of several uncertainty analysis methods and to obtain more robust airfoil shapes than the original Busemann’s biplane airfoil from a robust design optimization. It was confirmed that the results obtained from using a divided difference filter, polynomial chaos and moment methods showed qualitative agreements with that of Monte Carlo simulation. The divided difference filter showed the highest efficiency in the present study. Robust design optimization was also performed using the divided difference filter, which explored optimal designs with better off-design performance than the original Busemann’s biplane airfoil.

Key Words: Supersonic Flow, Compressible Flow, Robust Design Optimization

1. Introduction

In recent years, uncertainty analysis has attracted attention in high-fidelity computational fluid dynamics (CFD) computations.1–10 All input variables are assumed to be completely known in general numerical analysis, while there are some uncertainties due to manufacturing error, deformation in usage conditions and so on. For a more sophisticated engineering design, it is important to evaluate the mean, standard deviation, probability density function (PDF) and cumulative distribution function (CDF) of performance metrics due to uncertainties of input variables. One of the most classical uncertainty analysis methods is Monte Carlo simulation (MC). In the MC, \(10^2–10^4\) sample points according to the PDF of the input variables are evaluated, and then the uncertainties of outputs are evaluated by examining the statistical properties of the results of these sample points. In inexpensive Monte Carlo simulation (IMC),11–13 MC is performed on a response surface model in an uncertain input space, and then uncertain outputs are obtained. The IMC can reduce the calculation cost dramatically compared to the MC, and its accuracy depends on that of the response surface model. The moment method (MM)14–16 is an uncertainty analysis method based on derivatives. The MM using first-order derivatives has often been used in robust design optimization (RDO) due to its simplicity and low computational cost. However, uncertain outputs of the MM are not satisfactory for highly non-linear functions or large variance in uncertain inputs.16 The polynomial chaos method (PC)1–10,17,18 is an approach that can efficiently treat the propagation of uncertainties in numerical analysis. In PC, uncertain variables are extended to include random variables using orthogonal polynomials. In the intrusive PC approach, not only input variables with uncertainties, but also all dependent variables are extended. Finally, governing equations are extended to each random mode, and then the propagation of uncertainties is analyzed efficiently by directly solving the extended governing equations. In the non-intrusive polynomial chaos (NIPC)5,6 approach, on the other hand, uncertain outputs are estimated from the results of given sample points. In the divided difference filter (DDF),14,16 several special sample points, that are often referred to as sigma points, are defined using kurtosis analysis of input variables with uncertainties. Then the mean and standard deviation of outputs are estimated from the weighted sum of the computational results of the sigma points. In the MC, IMC, NIPC and DDF, it is not necessary to modify the original simulation code and they can be applied to any complicated problem. Although the IMC and NIPC are suited for smooth outputs, users have to take into account the influence of the number and arrangement of sample points. On the other hand, the DDF can obtain the outputs without any consideration for that since the number and locations of the sigma points are theoretically determined. In addition, compared to the MM, derivative information is not needed in the DDF, which means that users don’t need to take into account appropriate step sizes for infinite difference evaluation of the derivatives.

Busemann’s biplane9–21 airfoils are proposed as one of the efficient supersonic airfoil topologies. Two isosceles triangle airfoils are arranged to realize successful shock interactions between the airfoils. The reduction rate of wave drag was nearly 90% compared to a diamond-wedge airfoil having the same sectional area. However, a choking phenomenon occurs at off-design flow conditions, and then the aerodynam-
namic drag increases discontinuously. For the two-dimen-
sional (2D) Busemann’s biplane airfoil, an uncertainty analysis
has been performed with an input uncertainty of the free-
stream Mach number \( \left( M_{\infty} \right) \). In this investigation, it was
confirmed that the standard deviation of aerodynamic drag
increased dramatically when the increase in aerodynamic
drag due to the choking phenomenon was taken into consid-
eration. According to Yamazaki and Suga,\(^{11}\) the calculation
error of PC was much lower than that of MC, which was only
demonstrated in a problem with one uncertain input. Under
real flight conditions, not only the fluctuation in \( M_{\infty} \), but also
that in the angle-of-attack (AoA), can be considered.

In this study, therefore, the uncertainty analysis of the
supersonic flow field around a 2D Busemann’s biplane airfoil is
performed using MC, MM, PC and DDF when uncertainties of normal distributions are given to \( M_{\infty} \) and AoA, and then the usefulness of each method is examined. In addi-
tion, the uncertainty analysis method suitable to be used in
RDO is selected from the viewpoint of accuracy and cal-
culation cost, and then a RDO of the 2D Busemann’s biplane
airfoil is also performed. To realize the construction of a
supersonic transport using the Busemann’s biplane airfoil, it is
important to minimize the standard deviation of aerody-
amic performance caused by the discontinuous change in
aerodynamic drag. Therefore, we investigate aerodynamic
mechanisms to improve the robustness of the Busemann’s
biplane airfoil.

2. Uncertainty Analysis Approach

In this section, uncertainty analysis approaches utilized in
this research are introduced.

2.1. MC

In MC, the mean \( \mu \) of function \( g \) with uncertainty input
variables \( \mathbf{\tilde{r}} \) is defined as follows,
\[
\mu(g(\mathbf{\tilde{r}})) = \int \cdots \int (g(\mathbf{\tilde{r}}) \Phi(\mathbf{\tilde{r}})) d\mathbf{\tilde{r}}_1 \cdots d\mathbf{\tilde{r}}_n
\]
(1)
where, \( n \) and \( \Phi \) are respectively the number of uncertain inputs
and PDF of \( \mathbf{\tilde{r}} \). When sample points \( \mathbf{\tilde{r}}_i \) are defined accord-
ing to the distribution of \( \Phi, \mu \) and the standard deviation \( \sigma \)
of \( g \) are estimated as follows,
\[
\mu(g(\mathbf{\tilde{r}})) = \frac{1}{N} \sum_{i=1}^{N} g(\mathbf{\tilde{r}}_i)
\]
(2)
\[
\sigma^2(g(\mathbf{\tilde{r}})) = \frac{1}{N} \sum_{i=1}^{N} [g(\mathbf{\tilde{r}}_i) - \mu(g(\mathbf{\tilde{r}}))]^2
\]
(3)
where, \( N \) is the number of sample points. In this research, the
\( N \) sample points are generated using the Latin Hypercube
Sampling (LHS) method.

2.2. MM

In MM, when PDFs of uncertain inputs are normal distribu-
tions, \( \mu \) and \( \sigma \) of \( g \) are defined as follows,
\[
\mu(g(\mathbf{\tilde{r}})) = g(\mathbf{\tilde{r}}_0) + \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\partial^2 g(\mathbf{\tilde{r}})}{\partial r_i^2} \right) \left| \frac{\partial^2 g(\mathbf{\tilde{r}})}{\partial r_i^2} \right| \left( r_i - r_{i0} \right)^2
\]
(4)
\[
\sigma^2(g(\mathbf{\tilde{r}})) = \sum_{i=1}^{n} \left( \frac{\partial^2 g(\mathbf{\tilde{r}})}{\partial r_i^2} \right) \left| \frac{\partial^2 g(\mathbf{\tilde{r}})}{\partial r_i^2} \right| \left( r_i - r_{i0} \right)^2
\]
(5)
where, \( \mathbf{\tilde{r}}_0 = (\mu_{r_1}, \mu_{r_2}, \cdots, \mu_{r_n}) \) is the mean value vector of
input variables. The first-order MM (MM 1) uses only the first terms of the right-hand sides of Eq. (4) and Eq. (5),
while the second-order MM (MM 2) uses all of the terms. Derivatives in Eq. (4) and Eq. (5) are calculated using the
second-order central difference method in this research.

2.3. PC

In PC, any variables with uncertainties are defined as follows,
\[
g(\mathbf{\tilde{X}}, \mathbf{\tilde{\xi}}) = \sum_{i=0}^{p} \omega_i (\mathbf{\tilde{X}}) \Psi_i (\mathbf{\tilde{\xi}})
\]
(6)
where, \( \omega_i, \Psi_i, \mathbf{\tilde{X}}, \text{and } \mathbf{\tilde{\xi}} \) are unknown coefficients, polynomial
chaos basis, deterministic variable vectors and random
variable vectors, respectively. There is a one-to-one corre-
spondence between the \( i \)-th components of \( \mathbf{\tilde{r}} \) and \( \xi \). \( P + 1 \)
is the number of terms and defined as follows,
\[
P + 1 = \frac{(n + p)!}{(n!p!)}
\]
(7)
where, \( p \) is the order of polynomial chaos. The kind of poly-
nomial chaos basis is determined from the PDFs of input vari-
ables with uncertainties. When the PDFs are normal distribu-
tions, the Hermite polynomial is used as the polynomial
chaos basis, which is defined as follows,
\[
\Psi_0 = 1, \quad \Psi_1 = \xi_1, \quad \Psi_2 = \xi_1^2 - 1, \cdots (n = 1)
\]
(8)
The ensemble average between the bases is defined as follows,
\[
\langle \Psi_i (\mathbf{\tilde{\xi}}) \Psi_j (\mathbf{\tilde{\xi}}) \rangle = \int \cdots \int \Psi_i (\mathbf{\tilde{\xi}}) \Psi_j (\mathbf{\tilde{\xi}}) W(\mathbf{\tilde{\xi}}) d\mathbf{\tilde{\xi}}
\]
(9)
From the orthogonality of the ensemble average between the
bases, the following equation is given.
\[
\langle \Psi_i (\mathbf{\tilde{\xi}}) \rangle = \langle \Psi_i^2 (\mathbf{\tilde{\xi}}) \rangle \delta_{ij}
\]
(10)
\( W(\mathbf{\tilde{\xi}}) \) and \( \delta_{ij} \) are a weight function and the delta of
Kronecker. From the following equations, \( \mu \) and \( \sigma \) of \( g \) are
calculated:
\[
\mu(g) = \omega_0
\]
(11)
\[
\sigma^2(g) = \sum_{i=1}^{p} \omega_i^2 \langle \Psi_i (\mathbf{\tilde{\xi}}) \rangle
\]
(12)
In addition, the PDF and CDF of arbitrary variables can be
evaluated by performing the MC for \( \mathbf{\tilde{\xi}} \) in Eq. (6). In the intrusive
PC, all input variables with uncertainties are extended by
Eq. (6), and then governing equations are also extended.
The number of governing equations is increased from \( n_{eq} \)
(Original number) to \( n_{eq} + P + 1 \). Then all unknown coefficients
can be calculated by solving the extended governing equations.
The details of the intrusive PC were given in Suga and
Yamazaki.\(^{10}\)
2.4. DDF

In DDF, $\mu$ and $\sigma$ of $g$ are defined as follows,

$$\mu(g_{\mathbf{r}}) = G_0 g(\bar{r}_0) + \sum_{i=1}^{n} G_i (g(\bar{r}_i) + g(\bar{r}_{-i}))$$

(13)

$$\sigma^2(g_{\mathbf{r}}) = \frac{1}{2} \sum_{i=1}^{n} [G_i (g(\bar{r}_i) - g(\bar{r}_{-i}))^2 + (G_i - 2G_i^2)(g(\bar{r}_i) + g(\bar{r}_{-i}) - 2g(\bar{r}_0)^2)]$$

(14)

where, $\bar{r}_i = (\mu_1, \mu_2, \ldots, \mu_i, \mu_{i+1}, \ldots, \mu_n)$ is defined by adding a variation of $\pm \Delta r_i$ to the $i$-th element of the mean value vector of input variables. $\bar{r}_0, \bar{r}_i$ and $\bar{r}_{-i}$ correspond to the locations of the sigma points. In addition, $G_0$ and $G_i$ are weight coefficients for each output. $G_0$, $G_i$ and $\Delta r_i$ are defined as follows,

$$G_0 = 1 - \sum_{i=1}^{n} \frac{1}{K_{r_i}}$$

(15)

$$G_i = \frac{1}{2K_{r_i}}$$

(16)

$$\Delta r_i = \sigma r_i \sqrt{K_{r_i}}$$

(17)

where, $\sigma r_i$ and $K_{r_i}$ are the standard deviation and kurtosis of the $i$-th uncertain input variable, respectively. $K_{r_i}$ is determined from the PDF of an uncertain input variable and is set to 3 for normal distributions. In the DDF, PDF and CDF cannot be evaluated while the number of sample points to be evaluated is $2n + 1$, that means the calculation cost is much smaller than MC.

2.5. Computational fluid dynamics method

In this research, governing equations are compressible Euler equations and the equations are solved using a gridless approach.22,23) The flow flux at the intermediate position of two computational points is evaluated by the approximate Riemann solver of Roe with second-order spatial accuracy. The temporal discretization is treated using a four-stage Runge-Kutta scheme. In the present study, the number of computational points is approximately $7.3 \times 10^4$, which is determined from a convergence study with respect to the computational points. The gridless approach is also used for the extended governing equations of the intrusive PC.1)

3. Comparison of Uncertainty Analysis Methods

In this section, uncertainty analysis methods are compared. In reality, $M_{\infty}$ and AoA can be varied when an aircraft is cruising. Thus, the uncertainty analysis considering uncertainties in both $M_{\infty}$ and AoA is performed. Normal distributions are assumed for both $M_{\infty}$ and AoA. With respect to $M_{\infty}$, its mean and standard deviation are respectively set to 1.7 and 0.0333. With respect to AoA, its mean and standard deviation are respectively set to 2.0 [deg] and 0.333 [deg]. In this study, 1,000 sample points generated using LHS are used for MC. With respect to MM, derivatives are evaluated using finite differencing in which the step sizes of $M_{\infty}$ and AoA are set to 0.0577 and 0.577 [deg], respectively. Actually, the results of MM vary with the settings of the step sizes and the step sizes were determined as to match with the $\Delta r_i$ of Eq. (17) in this study. In PC, the order of polynomial chaos $p$ is set to 1 or 2.

The result of MC using 1,000 sample points is referred to as MC (1000). Choking phenomena occurred at six sample points within the 1,000 points. The result of MC using the remaining 994 sample points is referred to as MC (994). The choking phenomena did not occur in all of the sigma points used for DDF. The mean and standard deviation distributions of static pressure obtained using each uncertainty analysis method are shown in Fig. 1 and Fig. 2, respectively. The distributions of MM 1 and MM 2 are omitted since they are very close to that of PC and DDF. In the mean distributions, there is no major difference between the methods, while there is a significant difference in the standard deviation distributions. Only in the result of MC (1000), the effect of the shock wave due to the choking phenomena is confirmed on the standard deviation distribution. In the present condition of input uncertainties, the probability of generating the choking phenomena is as small as 0.24%; therefore, the influence of the choking phenomena cannot be observed in MM, PC, and DDF. On the other hand, the standard deviation distributions obtained using MM, PC and DDF qualitatively agree well with that of MC (994).

The $\mu$ and $\sigma$ of the lift-to-drag ratio $L/D$ obtained using each uncertainty analysis method are shown in Table 1. The $\sigma$ of MC (1000) is larger than the others due to the effect of choking phenomena, while the $\sigma$ of MC (994) almost agrees with the others. Therefore, it is believed that the propagation of uncertainty without the effect of choking phenomena can be analyzed well in MM, PC, and DDF. In detail, although MM 1 gave the closest value of $\sigma$ towards that of MC (994), the $\mu$ of MM 1 was larger than the others. Therefore, MM 1 was not chosen as the uncertainty analysis method for RDO in this study. On the other hand, DDF pro-

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vides reasonable predictions for both $\mu$ and $\sigma$ when compared with that of MC (994). The calculation cost of each uncertainty analysis method is also shown in Table 1, in which the calculation cost of one deterministic CFD analysis is considered as a unit. The calculation costs of MM, PC and DDF are much smaller than that of MC (1000) and MC (994). Although the computational cost of PC ($p = 2$) is smaller than that of MC (1000) and MC (994), it is about 10 times larger than that of PC ($p = 1$). The number of governing equations extended by PC increases as the number of uncertain input variables $n$ and the order of polynomial chaos $p$ increase. When $n$ and $p$ are 2, it is extended from 4 to 24 in the case of 2D Euler equations (and then its RHS becomes more complicated). Therefore, the calculation cost of PC ($p = 2$) increases dramatically. On the other hand, DDF performs an uncertainty analysis with a very small calculation cost, which corresponds to 1/200 of MC (1000).

The influence of the choking phenomena cannot be considered in MM, PC and DDF when the probability of the choking phenomena is small (about 0.24% in this study). On the other hand, the propagation of uncertainty without the effect of choking phenomena is observed well in MM, PC and DDF. In addition, the uncertainty analysis can be performed in DDF reasonably with a much smaller calculation cost. From these results, the validity and effectiveness of the DDF uncertainty analysis method are confirmed.

### 4. Definition of RDO Problem

#### 4.1. Problem settings

In the RDO, normal distributions are assumed for both $M_\infty$ and AoA. With respect to $M_\infty$, its mean and standard deviation are respectively set to 1.7 and 0.0333. With respect to AoA, its mean and standard deviation are respectively set to 2.0 [deg] and 0.333 [deg]. DDF is used as the uncertainty analysis method in the RDO because it has the lowest calculation cost. The objectives of RDO are to maximize $\mu$ of $L/D$ and minimize $\sigma$ of $L/D$. Geometrical constraints are given as Eq. (18) and Eq. (19).

\[
S_{\text{new}} \geq S_{\text{original}} \tag{18}
\]

\[
S_{\text{upper,new}} \geq \frac{S_{\text{upper,original}}}{2} \tag{19}
\]

where, $S_{\text{new}}$, $S_{\text{original}}$, $S_{\text{upper,new}}$ and $S_{\text{upper,original}}$ are respectively the total sectional area of a newly designed airfoil, the total sectional area of the Busemann’s biplane airfoil, the sectional area of a newly designed upper airfoil, and the sectional area of the upper airfoil of the Busemann’s biplane. The reason why these constraints are given is to prevent very thin airfoils being designed.

### 4.2. Optimization methods

An optimization approach using a Kriging response surface approach\(^{13,24-26}\) and genetic algorithm (GA)\(^{27,28}\) is used for the RDO. Firstly, uncertainty analyses of 31 initial samples are performed by DDF. One sample corresponds to the Busemann’s biplane airfoil and the remaining 30 samples are generated by LHS. Then, approximate models of $\mu$ and $\sigma$ of $L/D$ are constructed in the design variables space. The search for global optimal solutions on the models is performed using GA. The number of individuals and generations of GA are both set to 50 (i.e., 2,500 evaluations). Two promising designs are respectively searched by maximizing the expected improvement (EI)\(^{24}\) of $\mu$ and $\sigma$ that corresponds to maximizing $\mu$ and minimizing $\sigma$. Another promising design is searched by maximizing expected hypervolume improvement (EHVI),\(^{25}\) which corresponds to finding non-dominated optimal designs. In conventional multi-objective optimization problems, there is no single solution capable of optimizing all objective functions simultaneously. One of objective functions can be improved by degrading some of the other objective functions. As a result, a number of optimal solutions called non-dominated optimal designs or Pareto optimal designs exist. The uncertainty analyses are performed for these three additional samples, and then the $\mu$ and $\sigma$ of $L/D$ of these samples are evaluated. The approximate models are updated using the information of these samples. RDO is performed by repeating these processes. In this study, RDO is performed until the number of samples reaches 200.

### 4.3. Airfoil shape parameterization

In RDO, the Busemann’s biplane airfoil is considered as the base design. Newly designed airfoils are represented using six Bezier curves. There are 13 control points to deform the airfoil and these points are moved using 15 design variables. A deformed airfoil and the range of design variables are shown in Fig. 3.

### 5. Results and Discussion

#### 5.1. Outline of optimization results

The relationship between objective functions of the initial and additional samples is shown in Fig. 4, in which the num-

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**Table 1. Mean and standard deviation of lift-to-drag ratio.**

| Method   | $\mu$ | $\sigma$ | Calculation cost |
|----------|-------|----------|------------------|
| MC (1000) | 22.3  | 3.21     | 1000             |
| MC (994)  | 22.4  | 2.77     | 994              |
| MM 1      | 23.1  | 2.66     | 5                |
| MM 2      | 22.4  | 2.97     | 9                |
| PC ($p = 1$) | 22.4  | 2.55     | 6                |
| PC ($p = 2$) | 22.1  | 2.58     | 59               |
| DDF       | 22.4  | 2.92     | 5                |

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**Fig. 3. Airfoil shape parameterization.**
number of sigma points where the choking phenomena occur ($N_{c}$) is also indicated. Except the designs with very small $\mu$ and $\sigma$ of $L/D$, the choking phenomena don’t occur on the sigma points of the non-dominated optimal designs. In addition, we confirmed that this RDO was sufficiently converged. The $\mu$ of $L/D$ increases as the $\sigma$ of $L/D$ increases, the trade-off relationship between $\mu$ and $\sigma$ of $L/D$ is confirmed. There are some designs whose values of $\mu$ and $\sigma$ of $L/D$ are both better than the Busemann’s biplane airfoil.

For a detailed discussion, four characteristic designs are selected and examined. These designs are respectively referred to as A, B, C and D. Design A has the smallest $\sigma$ of $L/D$ and the design D has the largest $\mu$ of $L/D$ in all designs except cases with $N_{c} = 5$. Since the designs with $N_{c} = 5$ have very low $\mu$ of $L/D$, these designs are not discussed in this study. Designs B and C are selected in the region where both $\mu$ and $\sigma$ of $L/D$ are better than the Busemann’s biplane airfoil. In this region, design B has the smallest $\sigma$ of $L/D$ and design C has the largest $\mu$ of $L/D$. For the purpose of comparison, another design optimization is performed using deterministic analyses, in which $M_{\infty}$ and AoA are respectively set to 1.7 and 2.0 [deg]. In this optimization, $L/D$ is maximized and the parameterization method and geometrical constraints are the same as RDO. The optimal design obtained using this optimization is referred to as deterministic optimal. $\mu$ and $\sigma$ of $L/D$ of the representative designs are shown in Table 2, in which $\mu$ and $\sigma$ of $L/D$ are evaluated using DDF. In the deterministic optimal, the $\sigma$ of $L/D$ is very large (i.e., it is out of range in Fig. 4). The aerodynamic performance of the representative designs obtained by deterministic analyses is shown in Table 3. In the deterministic analyses, $M_{\infty}$ and AoA are respectively set to 1.7 and 2.0 [deg]. The deterministic optimal has the largest $L/D$ under the mean flow condition. The shapes of the representative designs are shown in Fig. 5, in which $c$ indicates the chord length of the Busemann’s biplane airfoil. Designs B, C and D have thinner upper airfoils than the lower airfoils and are similar to the biplane airfoil proposed by Licher. In the biplane airfoil proposed by Licher, the lift force increases by designing the sectional area of the lower airfoil larger than that of the upper airfoil, which succeeds in improving aerodynamic performance. Therefore, these representative designs have larger $C_{l}$ than the Busemann’s biplane airfoil. The mean and standard deviation deviations of static pressure around the representative designs are respectively shown in Fig. 6 and Fig. 7. Only in the deterministic optimal, the influence of the choking phenomena is confirmed in Fig. 6 and Fig. 7 because the choking occurs at one of the sigma points. To investigate the aerodynamic performance of the representative designs, hyper surfaces of $C_{d}$ and $L/D$ are constructed from the information of 121 (i.e., $11 \times 11$ Cartesian grid points) deterministic CFD evaluations on the 2D space of $M_{\infty}$ and AoA that are shown in Fig. 8 and Fig. 9, respectively. With the Busemann’s biplane airfoil, the choking phenomena occur when $M_{\infty}$ is less than approximately 1.61. In the deterministic optimal, the choking phenomena do not occur, which confirms the influence of the choking phenomena.

![Fig. 4. Performance of initial and additional samples with the number of sigma points where the choking phenomena occur.](image)

Table 2. Mean and standard deviation of the lift-to-drag ratio.

| Airfoil                     | $\mu$   | $\sigma$ |
|-----------------------------|---------|----------|
| Busemann’s biplane airfoil  | 22.4    | 2.92     |
| Deterministic optimal       | 22.6    | 13.3     |
| A                           | 15.3    | 0.313    |
| B                           | 23.2    | 1.70     |
| C                           | 24.5    | 2.43     |
| D                           | 27.0    | 4.13     |

![Fig. 5. Sectional airfoil shapes.](image)

Table 3. Aerodynamic coefficients obtained using deterministic analysis ($M_{\infty} = 1.7$, AoA = 2.0 [deg]).

| Airfoil                     | $C_{d}$ | $C_{l}$ | $L/D$ |
|-----------------------------|---------|---------|-------|
| Busemann’s biplane airfoil  | $4.82 \times 10^{-3}$ | 0.111 | 23.1 |
| Deterministic optimal       | $4.41 \times 10^{-3}$ | 0.133 | 30.1 |
| A                           | $8.57 \times 10^{-3}$ | 0.139 | 15.5 |
| B                           | $6.81 \times 10^{-3}$ | 0.159 | 23.4 |
| C                           | $6.24 \times 10^{-3}$ | 0.156 | 24.9 |
| D                           | $4.89 \times 10^{-3}$ | 0.133 | 27.3 |
phenomena occur when $M_\infty$ is less than approximately 1.67. For robust optimal designs A, B, C and D, the choking phenomena occur at a smaller $M_\infty$ than the Busemann’s biplane airfoil. In addition, it is confirmed that the values of $C_d$ with the choking phenomena are reduced in the robust optimal designs. Design B has the smallest $M_\infty$ for the choking phenomena whose value of $C_d$ is also the smallest. In the deterministic optimal, which is the optimal at the condition of $M_1$ of 1.7 and AoA of 2.0 [deg], $L/D$ is larger than the other designs under the design condition, while $L/D$ decreases dramatically under off-design conditions, which means the deterministic optimal has lower robustness. For design A, which has the smallest $\sigma$ of $L/D$, a flat hyper-surface in the entire region is confirmed. The flat hyper-surface means that its performance shows little change with the variation in uncertain inputs, which means design A has higher robustness, while $L/D$ is small as approximately 15 in the entire region. For design D, which has the largest $\mu$ of $L/D$, higher $L/D$ values are confirmed under the design condition, while $L/D$ decreases under the off-design conditions. The hyper-surfaces of designs B and C have the intermediate characteristics of that of designs A and D.

From the results so far, it has been confirmed that the airfoil designs with smaller generation regions of choking phenomena have a smaller $\sigma$ in the $L/D$. Generation of the choking phenomena is believed to be related to the minimum cross-sectional area of the flow path, so the relationship between the minimum distance between the upper and lower airfoils and $N_c$ is investigated, as shown in Fig. 10. It was confirmed that $N_c$ increases as the minimum distance is reduced. Therefore, generation of the choking phenomena is believed to be suppressed by increasing the minimum distance, and then the variation in $C_d$ decreases.

### 5.2. Investigation of pressure distributions in design A

In Fig. 7, design A has a characteristic spatial distribution in which concentrated regions of larger $\sigma$ can be observed. In order to clarify the influence due to the variation in only one of AoA and $M_\infty$, the effect of AoA and $M_\infty$ are investigated separately. The standard deviation distributions of static pressure around design A are shown in Fig. 11, in which the characteristic spatial distribution appears when only $M_\infty$
is considered as the uncertain input. It is expected that the characteristic distribution improves the robustness, so that its deterministic static pressure distributions at three $M_\infty$ conditions of the sigma points are shown in Fig. 12. In addition, the standard deviation distributions of pressure coefficient ($\Delta Cp$) on the airfoils when only $M_\infty$ is considered as the uncertain input, are shown in Fig. 13. For the Busemann’s biplane airfoil, the shock wave generated at the leading-edge of the airfoil collides with the upstream side of the center vertex of the opposite airfoil and is reflected towards the trailing-edge of the airfoils at $M_\infty = 1.64$, while the shock wave collides with the downstream side of the center vertex at $M_\infty = 1.76$. The position where the static pressure rapidly increases due to the collision of the shock wave changes with the variation in $M_\infty$, so that the large $\sigma$ regions are confirmed around the center vertices of the airfoils in the Busemann’s biplane airfoil. In design A, on the other hand, the shock wave collides with the downstream side of the center vertex of the opposite airfoil and is reflected towards the downstream side under all conditions of $M_\infty$. Therefore, the large $\sigma$ regions are confirmed only on the downstream side of the center vertices of the airfoils. In addition, the direction of the aerodynamic force acting on the airfoils is greatly different between the upstream side and downstream side of the center vertices of the airfoils. The variation in the direction can be suppressed by concentrating the collision of the shock wave on the downstream side of the center vertices so that the robustness is greatly improved in design A.

5.3. Effect of airfoil vertex shapes

From Fig. 5, it is confirmed that the airfoils with smaller $\sigma$ have rounded vertices near $x/c = 0.5$. To investigate the effect of the rounded vertex on the $\mu$ and $\sigma$ of $L/D$, the vertices of design D are deformed to rounded vertices. The relationship between the radius $R$ of the rounded vertex of design D and $\mu$ and $\sigma$ of $L/D$ is shown in Fig. 4. $D_{Lr}$, $D_{Ur}$ and $D_{ULr}$ are respectively design D with a rounded vertex for the lower airfoil, for the upper airfoil and for both airfoils. Design $D_{Ul1.0}$ represents design D with a rounded vertex (i.e., $R$ of 1.0 [$x/c$]) for the upper airfoil. The vertices of de-

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**Fig. 11.** Standard deviation distributions of static pressure around design A.

**Fig. 12.** Deterministic pressure distributions around Busemann’s biplane airfoil and design A ($\alpha = 2.0$ [deg]).

**Fig. 13.** Airfoil shapes and standard deviation of $C_p$ on Busemann’s biplane airfoil and design A when uncertainty is given only for $M_\infty$. 
signs D, D_{ULr0.5} and D_{ULr1.0} are compared in Fig. 14. It is confirmed that both the $\mu$ and $\sigma$ of $L/D$ decrease along the Pareto front in Fig. 4 when $R$ of the rounded vertex increases. The values of $\sigma$ of designs D_{ULr} are smaller than that of designs D_{Lr} and D_{Ur}. Compared to designs D_{Lr}, the values of $\sigma$ for designs D_{Ur} are almost the same, while the values of $\mu$ of designs D_{Ur} are larger. The reason is that the lift force increases by making the sectional area of the lower airfoil larger than that of the upper airfoil. Therefore, $\mu$ of $C_l$ for designs D_{Ur} is larger than that of designs D_{Lr}. The static pressure distributions around designs D, D_{Ur1.0}, D_{Lr1.0} and D_{ULr1.0} under the sigma points conditions are shown in Fig. 15. In addition, the pressure difference between the mean static pressure calculated using DDF and the static pressure of each sigma point is shown in Fig. 16. High-pressure regions around the vertices of the airfoils are confirmed at $M_\infty = 1.64$ in design D. The regions appear because shock waves generated at the leading-edges collide at the upstream of the vertices of the airfoils. Reduction of the high-pressure regions is confirmed at $M_\infty = 1.64$ with a rounded vertex, which also contributes to reducing the difference in Fig. 16. Reduction of high-pressure regions around the trailing-edge on the opposite side of the airfoil is also confirmed. It is observed that the static pressure distributions around design D_{ULr1.0} are the most insensitive to the variation in $M_\infty$. For design D, expansion waves generate at the top of the vertex, while the expansion waves generate at wider region around the rounded vertex. Even if the position where the shock wave collides changes due to the variation in $M_\infty$, it is possible to suppress the sharp increase in pressure by generating expansion waves at the wider region. The increase in static pressure around the trailing-edge is also suppressed by the effect of the rounded vertex. This confirms that a rounded vertex suppresses the variation in pressure, thereby yielding suppression of the variation in $L/D$, which reduces the standard deviation of $L/D$.

6. Conclusion

In this study, an uncertainty analysis of the flow field around the 2D Busemann’s biplane airfoil was first performed using MC, MM, PC and DDF. These analysis methods were compared and investigated in detail while giving the uncertainties of both $M_\infty$ and AoA as normal distributions. Choking phenomena occurred at six sample points within 1,000 points used for MC. The influence of the choking phenomena could not be captured in MM, PC and DDF, while the propagation of other uncertainties was sufficiently captured. In addition, uncertainty analysis could be per-
formed reasonably, with a much smaller calculation cost for DDF than MC. From these results, the validity and effectiveness of the DDF uncertainty analysis method was confirmed.

RDO for the 2D Bussemann’s biplane airfoil was performed using DDF, in which the mean and standard deviation of $L/D$ were respectively maximized and minimized. The tradeoff relationship between the mean and standard deviation of $L/D$ was observed and optimal designs with smaller standard deviation, as well as a higher mean of $L/D$ than the Bussemann’s biplane airfoil, were obtained.

Then, aerodynamic mechanisms to improve the robustness were investigated in detail. As a result, the number of sigma points where the choking phenomena occurred became smaller as the minimum distance between the upper and lower airfoils became larger. In addition, airfoils with better robustness had rounded central vertices. The shock waves generated at the leading-edge collided with the vertices, increasing the static pressure around them. The position where the shock waves collided depends on the condition of $M_{\infty}$. A rounded vertex made expansion waves in wider regions, which resulted in reducing the static pressure around the rounded vertex within the specified range of uncertainty on $M_{\infty}$. Therefore, the sharp increase in static pressure due to the collision of shock waves was suppressed and the standard deviation of $C_p$ around the vertices was reduced, which resulted in improving the robustness.

As future work, the authors will perform RDO of a 3D Bussemann’s biplane wing and conduct a more detailed investigation. Since interactions between shock waves and the boundary layer may have a certain effect on RDO results, viscous flow simulations will be used for future RDO.

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