Abstract

We perform shell model calculations using a quadrupole-quadrupole interaction (Q.Q). We show results in single j shell spaces and the full S-D shell. We show that one gets useful results with Q.Q in both spaces. We emphasize the importance of the choice of single particle energies in order to obtain the results of Elliott using a Q.Q interaction without the momentum terms. We show a J(J+1) spectrum for a ground state band but with B(E2)'s different from the rotational model. We also show results not found in textbooks such as J (J+1), J(J-1) and J(J+3) excited bands. We find spectra starting with J=0 which have both even J and odd J members.

1 Introduction

Our goal is to systematically reexamine the Elliott SU3 model[1,2,3,4] with a few variations. We use a Q.Q interaction but without the momentum terms. When using the shell model to reproduce Elliott one has to introduce a specific single particle splitting, some of which includes interactions of the valence particle with the core [4,5]. But before doing all this we show that the Q.Q interaction is very useful in single j shell calculations and show an example where agreement with experiment is remarkable. Back to Elliott we look not only at spectra but systematics of quadrupole moments and B(E2)'s. A difference from the Elliott et al. papers above[1,2,3,4] is that they mainly emphasize the orbital parts of the wave functions–so spin or isospin quantum numbers are not shown. Examples are shown only for S=0 states i.e. J=L. In our shell model approach we get the full package-states with all possible spins and all possible isospins.
2 The Q.Q interaction in the single j shell.

The interaction we use is $-\chi \cdot Q \cdot Q = -\chi \sqrt{5 \left[(r^2 Y^3_x)(r^2 Y^3_y)\right]^0}$ In evaluating energies, unless specified otherwise, we set $\chi b^4$ to 1 MeV. Alternately one can say that the energy is in units of $\chi b^4$.

In the single j shell one definitely does not get a rotational spectrum. In Table I we compare the spectrum of even J states in $^{52}$Fe resulting from using Q.Q in a single j shell ($f_{7/2}$), (2 proton holes and 2 neutron holes in the $f_{7/2}$ shell). The strength of the interaction was adjusted so the energy of the J=2$^+$ state agreed with experiment. Given the simplicity of the interaction and the smallness of the model space the agreement is remarkable. Note that the energy of the J=12$^+$ state is lower than J=10$^+$ and this is reproduced in the calculation. Thus the J=12$^+$ is a very long lived isomeric state. Many other spin gap isomers are predicted using Q.Q in a single j shell. For $^{52}$Fe and $^{53}$Co the J=15/2$^+$,17/2$^+$ and 19/2$^+$ states are at 4.448,5.145 and 3.588 MeV respectively. Thus the J=19/2$^+$ state is isomeric. In $^{96}$Cd the J=14$^+$,15$^+$ and 16$^+$ states are at 4.138,4.201 and 3.483 MeV. Thus the J=16$^+$ state is predicted to be isomeric with Q.Q in single j. In $^{96}$Ag the J=10$^+$, 11$^+$ and 12$^+$ states are predicted to be at 2.535, 3.089 and 2.482 MeV respectively. Thus the J=12$^+$ state is predicted to be isomeric. These isomerisms have been verified experimentally.

Table I: Single j shell spectrum of $^{52}$Fe–Q.Q vs. experiment

| J   | Q.Q | EXP  |
|-----|-----|------|
| 0   | 0   | 0    |
| 2   | 0.849 | 0.849 |
| 4   | 2.094 | 2.384 |
| 6   | 3.982 | 4.325 |
| 8   | 5.996 | 6.360 |
| 10  | 7.389 | 7.382 |
| 12  | 7.168 | 6.958 |

Admittedly we have chosen the best example. In $^{44}$Ti (2 protons and 2 neutrons in the $f_{7/2}$ shell) the J=12$^+$ state is slightly above the J=10$^+$ state although it is still isomeric. Still, using the Q.Q in a wide variety of single j shell calculations gives a reasonable good start and is useful for orientation in regions where there is not enough data to get the 2 body matrix elements from experiment. For example in the $g_{9/2}$ shell the spectrum of the 2-hole system of $^{98}$In is not known.

For completeness we briefly mention some previous results involving the Q.Q interaction which so far are not well understood. In ref [6] it was noted that for identical particles in the $g_{9/2}$ shell seniority is in general not a good quantum number. However, it is for a limited set of interactions which do conserve seniority such as the delta interaction. If we compare the spectra of 3 neutrons in the $g_{9/2}$ shell with that of 5 neutrons we find E(21/2)-E(3/2) is the same in the 2 cases. Now the Q.Q interaction does not conserve seniority for identical
particles in the $g_{9/2}$ shell or beyond. What is of interest here is that the above splitting is equal in magnitude but opposite in sign for 3 and 5 particles.

Another unproven result by the same authors [8] for a system of 2 protons and 2 neutrons in a given shell - one finds when using a Q.Q interaction that some (but not all) T=2 states are degenerate in energy with some T=0 states. Of particular interest in the $g_{9/2}$ shell is the degeneracy of a unique J=4 T=2 state with seniority v=4 with a J=4 T=0 state also with seniority v=4. The J=4 T=2 v=4 state appears no matter what interaction is used even though in general seniority is not conserved for identical particles in the $g_{9/2}$ shell. Even more surprising is that with a Q.Q interaction there is a T=0 J=4 state with a definite seniority and it is degenerate with the unique J=4 T=2 state. In general, with any interaction, not just Q.Q, when one has mixed protons and neutrons seniority is not conserved in any shell.

For completeness we note that Zamick and Harper [9] showed that for 2 protons and 2 neutrons in a single j shell there is a very high overlap between the wave functions arising from a Q.Q interaction and properly symmetrized unitary 9j coefficients (U9j).

Another interesting feature of Q.Q in a single j shell calculation is that the spectrum of a particle-hole is “upside down” the spectrum of 2 particles. This is actually true for any multipole-multipole interaction, as noted by Talmi[10]. The empirical orders of the spins of energy levels, from low to high, for $^{42}$Sc (2 particles) is 0,1,7,3,5,2,4,6. For $^{48}$Sc (proton particle-neutron hole) the ordering is 6,5,4,3,7,1,2,0.

3 Elliott Model- Single Particle Energies and Degeneracies

In contrast to the previous section we here consider the use of Q.Q to produce rotational states in the shell model. We refer of course to the Elliott SU(3) model [1,2,3,4]. We now have to consider all configurations in a major shell. Although this model has been well studied we wish to emphasize certain aspects which are perhaps not so familiar, especially the choice of single particle energies in a formulation where we do not include the momentum terms in the interaction. We use the simple Q.Q interaction described above. The numbers are expressed in units of $\chi b^4$ where b is the oscillation length parameter (or if you like we set the value of $\chi b^4$ to one).

The Elliott formula for the energies is

$$E(SU(3)) = \chi'[-4(\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu))] + 3\chi'L(L+1)$$ (1)

where $\chi' = 5b^4/2(32\pi)$ $\chi$.

To get Elliott’s SU(3) results in the shell model one has to introduce a single particle energy splitting [4,5]

$$E(L_2) - E(L_1) = 3\chi'[L_2(L_2 + 1) - L_1(L_1 + 1)]$$ (2)
The splitting is 18 $\chi'$ in the S-D shell and 30 $\chi'$ in the P-F shell. Note that the bigger $L$ single particle level is at a higher energy than the smaller, i.e. D is higher than S in the S-D shell and F is at a higher energy than P in the P-F shell. This may go against experiment but if one wants to get Elliott’s results that is what one has to do. As noted by Zamick et al. [5] and by Moya de Deguerra et al. [6] when one uses the simple Q.Q interaction (without Elliott’s momentum terms) 2/3 of the splitting comes from the diagonal part of the Q.Q interaction and 1/3 comes from the particle core interaction. One can say that for the single nucleon configuration (e.g. $^{17}$O or $^{41}$Ca), one also has a rotational band consisting of 2 states $L=0$ and $L=2$ in the S-D shell and $L=1$ and $L=3$ in the P-F shell. See also discussions of momentum term removals by Talmi [10].

Before continuing we note that there have of course been many developments since the works of Elliott including higher configuration admixtures out of the S-D shell and works on higher shells e.g. P-F. Some selected works are refs[11-17]. Our intent here is quite different. We want to take a hard look at the Elliott model in it’s simplest form and see if there are some interesting features worth pointing out. The heading of the next section indicates that there are.

We will call this work SMQ.Q (SM=Shell Model), so as to make the distinction of working without and with momentum terms.

### 4 J (J+1), J(J-1) and J(J+3) Spectra

In Table II we show contrasting spectra. The familiar ground state band has a $J(J+1)$ spectrum with only even $J$’s. But the lowest excited bandhead at 5.073 MeV is multidegenerate. For $J=1$ there are 3 states at this energy, one with isospin $T=0$ and two with $T=1$. Here we show three bands that can be extracted. We will here consider only $T=0$ bands.

| J  | Ground Band | Excited Band 1 | Excited Band 2 | Excited Band 3 |
|----|-------------|----------------|----------------|----------------|
| 0  | 0           | 5.073          |                | 5.073          |
| 1  |             | 5.073          | 5.670          |                |
| 2  | 0.895       | 5.670          | 5.073          | 6.565          |
| 3  |             | 6.565          | 5.607          | 7.751          |
| 4  | 2.984       | 7.759          | 6.565          | 9.251          |
| 5  |             | 9.251          | 7.759          | 11.041         |
| 6  | 6.266       | 11.041         | 9.251          | 13.340         |
| 7  |             | 13.130         | 11.041         |                |
| 8  | 10.743      |                | 13.130         |                |

The ground band energies are given by $E(J) = 0.149 J(J+1)$. The energies for excited bands 1, 2, and 3 are given respectively by

$$E(J) = 4.772 + 0.149J(J + 1)$$  \hspace{1cm} (3)
\[ E(J) = 4.772 + 0.149J(J - 1) \quad (4) \]
\[ E(J) = 5.073 + 0.149J(J + 3) \quad (5) \]

Notice that for all bands the coefficient of \( J^2 \) is the same, namely 0.149. This means that these bands all have the same moments of inertia. The \( \lambda \mu \) values for the ground band are (8,0) and for the 3 excited bands (6,1).

One cannot help but notice that excited band 2 looks the same as excited band 1 except that \( J \) is shifted up by one unit. Likewise, band 3 has \( J \) shifted down by one unit relative to excited band 1. This suggests reorientations of \( L \) and \( S \) for these 3 bands. An unusual feature of band 3 is that it starts with \( J=0 \) but, unlike the ground state band it includes both even and odd \( J \)’s.

The pattern in Table 2 suggests that the \([L S]J\) configurations of the 3 excited bands are respectively \([L 1]J=L \), \([L 1]J=L+1 \) and \([L 1]J=L-1 \). We will examine electric quadrupole and magnetic dipole properties to verify these assignments.

5 \section*{B(E2)’s and Q(2^+) in the Elliott Model.}

In this work we take the effective charges to be 1.5 for the proton and 0.5 for the neutron. In Table III we list the B(E2)’s along the ground state band in the full space for nuclei in the S-D shell. They are in units of \( e^2 \text{fm}^4 \). We also show the same results in a reduced space (Table IV) where only \( s_{1/2} \) and \( d_{5/2} \) subshells are allowed (no \( d_{3/2} \)). This gives us a sense of how increasing configurations affects collectivity. The B(E2)’s in the full space (Table III) are substantially larger than in the reduced space. Whereas we get a perfect \( J(J+1) \) spectrum in Table III, we get a more compressed spectrum in Table IV and the \( J(J+1) \) fit is only approximate. Not surprisingly, the static quadrupole moments for \( J=2 \) and 4 are larger in magnitude in the full space than those in the reduced space.

In Vol 2 of their book Bohr and Mottelson give analytic formulas for Q and B(E2) [18].

\[
\langle K = 0, I_2 || \mu(2) || K = 0, I_1 \rangle = \left( \frac{5}{16\pi} \right)^{1/2} \frac{\hbar}{M\omega_o} (2\lambda+3)(2I_1+1)^{1/2} \langle I_1 020 | I_2 0 \rangle \times \left\{ \begin{array}{c} 1 \quad I_2 = I_1 \\ \left( 1 - \frac{(2I_1+3)}{2I_1+1} \right)^{1/2} \quad I_2 = I_1 + 2 \end{array} \right\} \quad (6) \]

We acknowledge early work on B(E2)’s with the Elliott model by Strottmann [19].

We note that the B(E2) from the lowest \( 2^+ \) state to the J=0 ground state is strong with the Q-Q interaction. The results are not dissimilar to what one obtains with realistic interactions. We make a comparison with the rotational model of Bohr and Mottelson[17] for which the following formulas hold:

\[
B(E2, KJ_2 \rightarrow KJ_1) = \frac{5}{16\pi} e^2 Q_0^2 <J_1 2K0|jJ_2K>^2 \quad (7) \]
Table III: Quadrupole moments (e fm$^2$) and B(E2)'s (e$^2$ fm$^4$) for the ground state band in the Elliott model-full S-D Space.

| Energy | J  | Q(J)  | B(E2) J→ J+2 |
|--------|----|-------|-------------|
| 0.8952 | 2  | -18.96 | 194         |
| 2.9841 | 4  | -24.13 | 132         |
| 6.2665 | 6  | -26.55 | 91          |
| 10.7426| 8  | -27.95 |             |

Table IV: Same as Table III but in reduced space - only s$_{1/2}$ and d$_{5/2}$ subshells included (no d$_{3/2}$).

| Energy | J  | Q(J)  | B(E2) J→ J+2 |
|--------|----|-------|-------------|
| 0.9983 | 2  | -14.75 | 121         |
| 2.9962 | 4  | -15.34 | 79          |
| 5.6076 | 6  | -13.46 | 44          |
| 8.091  | 8  | -11.81 |             |

$$Q(J) = (3K^2 - J(J+1))/((J + 1)(2J + 3))Q_0$$  \hspace{1cm} (8)

For J=2:
Q(K=0)= -2/7 $Q_0$
Q(K=2) = +2/7 $Q_0$ They are equal and opposite.

For J=0 K=0 → J=2 K=0 we have

$$-1.1039Q(J = 2)/\sqrt{B(E2)} = 1.013. \hspace{1cm} (9)$$

In the rotational model it would be one. How does the Elliott model compare with the rotational model?

In his second paper [2] Elliott says that the quadrupole moments in a "K=0 rotational band" are identical to those of the rotational model but the B(E2)'s are not. In Table III we confirm this for the ground state band of $^{20}$Ne.

In Table V we show selected quadrupole moments of 2+ states and B(E2)'s from the $0_1^+$ ground state to several 2+ states.

Table V: E(2n$^+$)MeV, Q(2n) e fm$^2$ and B(E2)0$1^+\rightarrow2n$ e$^2$fm$^4$ in $^{20}$Ne

| E (2$^+$)  | T  | Q(2n)  | B(E2)0$1^+\rightarrow2n$ |
|-----------|----|--------|-------------------------|
| 0.895     | 0  | -18.96 | 427.0                   |
| 5.073     | 0  | 8.05   | 0                       |
| 5.670     | 0  | -5.00  | 0                       |
| 6.565     | 0  | -7.38  | 0                       |
| 6.565     | 1  | -8.43  | 12.56                   |
| 8.356     | 0  | 0      | 0                       |
Note that although the strongest B(E2) is the intraband transition from 0^+_1 to 2^+_1 (427 e^2 fm^4), there is a finite, albeit weak, B(E2) to a T=1 state at 6.565 MeV. It should be pointed out that if we had chosen effective charges that were the same for the proton and the neutron, i.e. the isoscalar choice, there would not be any B(E2) strength to states at 6.565 MeV or for that matter to any states except the one at 0.895 MeV. Hence only 2 non-zero B(E2)’s, one to a T=0 one to a T=1 final state.

There are no other finite B(E2)’s from the 0^+_1 ground state, even to 2^+ states not shown. If we look at transitions from the 2^+_1 state to 0^+_n states there is only a single non-zero transition 2^+_1 to 0^+_1 (427.0/5=85.4 e^2 fm^4). The B(E2)’s to all other 0^+ states vanish. This is true even if the effective charges of the neutrons and protons are different.

As shown in Table V the quadrupole moment of the 2^+_1 state is negative, consistent with a prolate deformation for a K=0 band. There is a change of sign at 5.073 MeV consistent with a K=2 prolate band. Since there are 3 degenerate states at 5.670 MeV there is arbitrary as to how we distribute Q and B(E2) between the 2 T=0 degenerate states Clebsch-Gordan coefficients.

Table VI: B(E2)’s in excited band 1

| J_i [E_i] | J_f [E_f] | B(E2) | J_i [E_i] | J_f [E_f] | B(E2) |
|-----------|-----------|-------|-----------|-----------|-------|
| ∆ J = 2   | ∆ J = 1   |
| 1 [5.07]  | 3 [5.67]  | 53.11 | 1 [5.07]  | 2 [5.67]  | 10.99 |
| 3 [6.60]  | 86.13     |      | 2 [5.67]  | 3 [6.56]  | 41.85 |
| 2 [5.07]  | 4 [6.56]  | 18.09 | 3 [7.82]  | 15.17     |      |
| 4 [7.79]  | 76.4      |      | 3 [6.56]  | 4 [7.75]  | 11.65 |
| 3 [6.56]  | 97.39     |      | 3 [9.25]  | 1.3       |      |
| 5 [7.75]  | 2.13      |      | 4 [7.75]  | 11.65     |      |
| 4 [7.70]  | 2.78      |      | 4 [9.25]  | 10.85     |      |
| 6 [11.04] | 48.98     |      | 4 [7.79]  | 2.78      |      |
| 6 [11.08] | 49.98     |      | 5 [11.08] | 49.98     |      |
| 5 [9.25]  | 6 [11.08] | 3.04 | 5 [9.25]  | 6 [11.08] | 17.19 |
| 6 [11.08] | 7 [13.13] | 17.19| 6 [11.08] | 7 [13.13] | 17.19 |
Table VII: $B(E2)'s$ in excited band 2 $e^2$fm$^4$

| Band 2 | \( \Delta J = 2 \) | \( \Delta J = 1 \) |
|--------|----------------|----------------|
| \( J_i \left[ E_i \right] \) | \( J_f \left[ E_f \right] \) | \( B(E2) \) | \( J_i \left[ E_i \right] \) | \( J_f \left[ E_f \right] \) | \( B(E2) \) |
| 2 \[ 5.07 \] | 4 \[ 6.56 \] | 97.22 | 2 \[ 5.07 \] | 3 \[ 5.67 \] | 66.13 |
| 3 \[ 5.67 \] | 5 \[ 7.75 \] | 79.83 | 3 \[ 5.66 \] | 6 \[ 9.25 \] | 25.3 |
| 4 \[ 6.56 \] | 6 \[ 9.25 \] | 100 | 3 \[ 5.67 \] | 4 \[ 6.56 \] | 66.05 |
| 5 \[ 7.75 \] | 7 \[ 11.04 \] | 50.49 | 4 \[ 7.79 \] | 10.89 |
| 6 \[ 9.25 \] | 8 \[ 13.13 \] | 58.6 | 4 \[ 6.56 \] | 5 \[ 7.75 \] | 14.96 |
| \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] |
| \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] |
| \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] | \[ ] \[ ] |

Table VIII: $B(E2)'s$ in excited band 3

| Band 3 | \( \Delta J = 2 \) | \( \Delta J = 1 \) |
|--------|----------------|----------------|
| \( J_i \left[ E_i \right] \) | \( J_f \left[ E_f \right] \) | \( B(E2) \) | \( J_i \left[ E_i \right] \) | \( J_f \left[ E_f \right] \) | \( B(E2) \) |
| 0 \[ 5.07 \] | 2 \[ 5.67 \] | 116.7 | 1 \[ 5.67 \] | 2 \[ 6.56 \] | 55.28 |
| 1 \[ 5.67 \] | 3 \[ 6.56 \] | 28.68 | 3 \[ 6.56 \] | 3 \[ 7.75 \] | 13.50 |
| 2 \[ 6.56 \] | 3 \[ 7.75 \] | 91.45 | 4 \[ 9.25 \] | 5 \[ 11.04 \] | 3.22 |
| 3 \[ 7.75 \] | 4 \[ 9.25 \] | 108.4 | 5 \[ 11.04 \] | 6 \[ 13.34 \] | 18.11 |
| 4 \[ 9.25 \] | 5 \[ 11.04 \] | 53.25 | 5 \[ 11.04 \] | 6 \[ 13.34 \] | 18.11 |
| 5 \[ 11.04 \] | 6 \[ 13.34 \] | 61.13 | 6 \[ 13.34 \] | 7 \[ 15.57 \] | 5.22 |
In Tables VI, VII, and VIII we give results for B(E2)'s for the excited bands 1, 2, and 3. We show results only to final states of isospin $T_f=0$ but it should be noted that there are finite, albeit small, B(E2)'s to $T_f=1$ states. We use these as starting energies and angular momenta the ones shown in Table II—those of a $J(J-1)$ spectrum. The results for $\Delta J=2$ in band 2 show a simple behavior with strong B(E2)'s to higher J states and with concentration of the strength to one final states. However for delta J=1 the behavior is more complex. One gets branches to mainly 2 final states of the same angular momentum e.g. 3[5.67] to 4[6.56] and 2 4[7.79] with respective strengths of 66.05 and 10.89 e²fm⁴. For excited band 1 we get fragmentation for both delta J=2 and delta J=1.

We can reduce the expression for the B(E2) from configuration $[L_i J_i]$ to $[L_f J_f]$

$$B(E2) = \frac{(2J_f + 1)}{(2J_i + 1)(2L_f + 1))|U(2f Li J_f J_i; J_i J_f L_f)|^2}$$

(10)

Here $U$ is the unitary Racah coefficient. One can use this to get ratios of B(E2)'s of transitions which have the same $L_i$ and $L_f$. In those cases the reduced matrix elements drop out. For example in transitions from 5.67 MeV to 6.56 MeV, the value from $J=3$ to $J=4$ (Band 1) is 66.05 and from $J=2$ to $J=3$ (Band 2) is 41.85 so the ratio is 1.54. One can easily verify that one gets the same ratio from the above expression.

Table IX: Ratios in the ground state band of $^{20}\text{Ne}$—Q($J_f$/Q(2) and B(E2) $J_f$–$\rightarrow J_f$/B(E2)$0 \rightarrow 2$

| $J_f$ | 2 | 4 | 6 | 8 |
|------|----|----|----|----|
| Q($J_f$/Q(2) | 1.273 | 1.4 | 1.474 |
| Q($J_f$/Q(2) | 1.273 | 1.4 | 1.474 |
| B(E2)$J_f$/B(E2)$ | 0.456 | 0.310 | 0.167 |
| B(E2)$J_f$/B(E2)$ | 0.515 | 0.455 | 0.430 |

Table X: Comparison of quadrupole moments (e fm²) in the SMQ.Q model and the rotational model for the lowest "K=1" band.

| Energy MeV | J | Q(J) SMQ.Q | Q(J) rotational K=1 |
|-----------|---|------------|---------------------|
| 5.073 | 1 | -3.70 | 0.49 |
| 5.668 | 2 | -5.60 | -7.00 |
| 6.561 | 3 | -8.04 | -12.25 |
| 7.751 | 4 | -17.90 | -15.15 |
| 9.251 | 5 | -13.84 | -18.20 |
| 11.041 | 6 | -23.14 | -18.96 |
| 13.130 | 7 | -16.55 | -19.77 |
Table XI: Comparison of quadrupole moments (e fm$^2$) in the SMQ.Q model and the rotational model for the lowest "$K=2$" band

| Energy MeV | J  | Q(J) SMQ.Q | Q(J) rotational $K=2$ |
|------------|----|------------|-----------------------|
| 5.073      | 2  | 8.02       | 16.65                 |
| 5.668      | 3  | -10.77     | 0                     |
| 6.561      | 4  | -10.93     | -8.43                 |
| 7.751      | 5  | -21.02     | -13.46                |
| 9.251      | 6  | -15.20     | -16.66                |
| 11.041     | 7  | -24.91     | -18.86                |
| 13.130     | 8  | -17.46     | -20.46                |

Table XII: Comparison of quadrupole moments (e fm$^2$) in the SMQ.Q model and the rotational model for the lowest "$K=0$" band

| Energy MeV | J  | Q(J) SMQ.Q | Q(J) rotational $K=2$ |
|------------|----|------------|-----------------------|
| 5.073      | 0  | 0          | 0                     |
| 5.668      | 1  | -3.982     | -3.982                |
| 6.561      | 2  | -7.383     | -5.689                |
| 7.751      | 3  | -17.29     | -6.637                |
| 9.251      | 4  | -13.57     | -7.240                |
| 11.041     | 5  | -22.90     | -7.658                |
| 13.130     | 6  | -16.39     | -7.964                |

We confirm in Table IX the statement by Elliott[2] that the quadrupole moments in his model are identical to those of the rotational model for the ground band, which in the rotational model has $K=0$. We also confirm his statement that the B(E2)'s for the ground state are different. In fact, they are quite different. Elliott’s B(E2)’s drop off much faster with J than those of the rotational model. The same thing happens in shell model calculations with more realistic interactions [20].

In Table X we compare the quadrupole moments for excited band 1, as calculated in the SMQ.Q model with those of a K=1 rotional band. We show results for T=0 bands in Tables X through XII because there are less degeneracies for T=0 than for T=1. A least squares fit was made to minimize the deviations of the 2 models. In Table XI a similar comparison was made for excited and 2 with a K=2 rotational band. There are many differences. In the rotational model there is a monotonic decrease in the quadrupole moments with J but this is not the case in the SMQ.Q model. There are however some similarities such as the change of sign in Table XI as one goes from J=2 to higher J. The fact that the J=1 state of excited band 1 has a quadrupole moment of opposite sign to that of the J=2 state of band 2 favors an L=1 assignment to these to states rather than L=2.
6 Magnetic moments as identifiers of configurations in LS coupling

In this work much of the pre-Elliott work by E. P. Wigner comes into play [21]. We show results in Table XIII for magnetic moments of levels in excited bands 1, 2, and 3 using NuShellX. We can easily associate these with LS coupling wave functions $[LS]J$ by using the following expression applicable to $T=0$ states:

$$
\mu = (G_l[J(J+1)+L(L+1)-S(S+1)]+G_s[J(J+1)-L(L+1)+S(S+1)])/2*(J+1)
$$

(11)

| E  | Band 1 $[LS]J=L_0$ | Band 2 $[LS]J=L+1_0$ | Band 3 $[LS]J=L-1_0$ |
|----|-------------------|---------------------|---------------------|
| 5.07 | [11]1 0.69 | [11]2 1.38 | [11]0  |
| 5.70 | [21]2 1.126 | [21]3 1.88 | [21]1 0.310 |
| 6.56 | [31]3 1.524 | [31]4 2.38 | [31]2 0.747 |
| 7.75 | [41]4 2.076 | [41]5 2.88 | [41]3 1.215 |
| 9.26 | [51]5 2.563 | [51]6 3.38 | [51]4 1.696 |
| 11.04 | [61]6 3.547 | [61]7 4.28 | [61]6 2.675 |
| 13.13 | [71]7 4.042 | [71]8 4.28 | [71]6 2.675 |

For $T=0$ states the bare coupling values are $G_l=(1+0)/2 =0.5$ and $G_s=(5.586-3.826)/2=0.88$. For the ground state band $S$ is equal to zero, $([L 0] L)$ so the expression is simply $\mu = G_l L$ (with $J=L$). For the excited bands the above formula for $\mu$ agrees with the NuShellX results when we attribute to band 1 the configuration $[LS] J=L_0$ and to band 2 $[LS]J=L+1_0$. Note that band 2 is a stretched band so the formulas for $\mu$ is especially simple $\mu=2G_l L + G_s S=0.5(J-1)+0.88$.

The assigned configurations help to sharpen what was said in previous sections. The fact that band 2 has a $J(J-1)$ spectrum is due to the spin independence of the interaction that is here used, as well as the fact that $d_{3/2}$ and $d_{5/2}$ are degenerate. It costs no energy to take the spin orientation in band 1 and stretch it out to form band 2. Note that the results for the magnetic moments do not depend on the SU(3) quantum numbers-only on $L$, $S$, and $J$.

Some of the patterns of the B(E2)'s can also be explained. For the cases where $J_f=J_{i}+2$ there is a large fragmentation of B(E2) strength in Band 1 but not in Band 2. In band 1 one can go from $[LS]L$ to $[(L+1)S]L+2$ and $[(L+2)S]L+2$ but from band 2 one can only go from $[LS]L+1$ to $[L+2 S]L+3$.

7 CLOSING REMARKS

We start with a technical point. The high degeneracies resulting from the SMQ.Q (Elliott) model can lead to problems. One of the main ones is that...
the isospin assignments for degenerate states can get mixed up. We addressed this by raising the $d_{3/2}$ state 0.1 MeV above $d_{5/2}$. This removed most degeneracies so we could distinguish which states had isospins $T=0$. It also helps us keep a track of a band when for higher spin new bands pop up.

We have here considered various aspects of the Q.Q interaction. It serves as a reasonable interaction in small spaces where for example it has the feature of level inversion at high spins which is seen in many nuclei-e.g. $^{52}$Fe and other nuclei mentioned above. There are also early discussions of other spin gaps by Auerbach and Talmi [22,23].

In a full space i.e. Elliott’s SU(3) model [1,2,3] we feel the simplest aspects of the model deserve further attention. This model has sometimes been described as giving us rotations in the shell model. While this is true of the spectrum one has to qualify this statement when considering B(E2)’s and quadrupole moments, as noted by Elliott [1,2]. As one goes to high spins the B(E2)’s in the rotational model seem to flatten out but in the Elliott model, as indeed in the shell model after a certain J they fall off. In the rotational model the quadrupole moments are monotonically decreasing (i.e. becoming more negative) but in the Elliott model this is not the case. Perhaps the best thing to say is that the Elliott model gives us rotational behavior with shell model modifications.

As mentioned in the introduction the early papers of Elliott and Harvey [1,2,3,4] emphasize the orbital parts of the wave functions and although LS coupling is mentioned one does not see spin or isospin labels. In our shell model approach, SMQ.Q , using NUshellX [24 ] we necessarily get complete wave functions -orbital and spin combined , and the isospin quantum numbers as well. And interesting results come when we look at the behavior as a function of J rather than L.

By looking hard at this model, resisting the temptation to modify single particle energies to fit experiment, we uncover interesting new features of this such as the $J(J-1)$ and $J(J+3)$ spectra. We showed this for bands for which the lowest L value was one, but we get the same structure for any higher L. Perhaps the most interesting result is that we get spectra starting from J=0 which have both even J and odd J members i.e. $J=0,1,2,3,4,5,6$ for the configuration $[L S=1]J=L-1$. Although we have not explicitly tried to fit experimental data we have here suggestions for experiment, namely to look for these different patterns that we have found, or at least for remnants of these patterns. We hope both experimentalists and theorists will continue in this fascinating pursuit.

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