Polynomial KP and BKP $\tau$-functions and correlators

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Abstract

Lattices of polynomial KP and BKP $\tau$-functions labelled by partitions, with the flow variables equated to finite power sums, as well as associated multipair KP and multipoint BKP correlation functions, are expressed via generalizations of Jacobi’s bialternant formula for Schur functions and Nimmo’s Pfaffian ratio formula for Schur $Q$-functions. These are obtained by applying Wick’s theorem to fermionic vacuum expectation value representations in which the infinite group element acting on the lattice of basis states stabilizes the vacuum.

1 Lattices of KP and BKP $\tau$-functions: fermionic constructions and polynomial solutions

In [9] fermionic vacuum expectation value (VEV) representations were used to construct lattices of KP $\tau$-functions [24] labelled by integer partitions, and BKP $\tau$-functions [4] labelled by strict partitions. When the underlying infinite $GL(\infty)$ and $SO(\infty)$ group elements are related by an appropriately defined factorization, the KP $\tau$-functions, restricted to vanishing even flow variables, were shown to be expressible as finite sums over products of BKP $\tau$-functions. It was also shown that, choosing group elements that stabilize the vacuum, the resulting $\tau$-functions are symmetric polynomials, generalizing the “building block” solutions for the KP and BKP hierarchies consisting of Schur functions $s_\lambda$ and Schur $Q$ functions $Q_\alpha$, respectively, thereby giving a further perspective on the well-studied classes of KP and BKP $\tau$-functions of polynomial type [1–3,7,10,12,13,23,25–27].

In the present work, it is shown that when the flow variables are restricted to power sums in a finite number of auxiliary variables, Wick’s theorem implies finite determinantal

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and Pfaffian representations for such polynomial \( \tau \)-functions, as well as \( n \)-pair KP correlators and \( n \)-point BKP correlators, similar to Jacobi’s bialternant expression for Schur functions and Nimmo’s Pfaffian ratio formula for Schur’s \( Q \)-functions.

Following the approach developed by the Sato school \[4,5,11,24\], the basic framework of operators and flows on fermionic Fock space is recalled in Section 2, both for the charged fermion case and for a pair of commuting classes of neutral fermions. Charged fermions are used to construct VEV representations of a lattice of KP \( \tau \)-functions \( \{ \pi_{\lambda}(g)(t) \} \), where \( t = (t_1, t_2, \ldots) \) is the infinite sequence of KP flow variables, as well as \( n \)-pair correlators, labelled by pairs \((g, \lambda)\) consisting of an element \( g \in \text{GL}(\mathcal{H}) \) of the infinite group of invertible general linear transformations on an underlying Hilbert space \( \mathcal{H} \) and an integer partition \( \lambda \). Neutral fermions are similarly used to construct pairs of VEV representations of a lattice of BKP \( \tau \)-functions \( \{ \kappa_{\alpha}(h^\pm)(t_B) \} \), where \( t_B = (t_1, t_3, \ldots) \) is the infinite sequence of BKP flow variables, and \( 2n \)-point correlators, labelled by pairs \((h^\pm, \alpha)\) consisting of an element \( h^\pm \in \text{SO}(\mathcal{H}_{\phi^\pm}) \) of the infinite group of special orthogonal transformations on an underlying pair of mutually orthogonal, complementary subspaces \( \mathcal{H}_{\phi^\pm} \subset \mathcal{H} + \mathcal{H}^* \) of the direct sum of \( \mathcal{H} \) with its dual \( \mathcal{H}^* \), with respect to the natural scalar product, and a strict partition \( \alpha \).

It is known that when the group element \( g \in \text{GL}(\mathcal{H}) \) is chosen to be upper triangular, and hence its fermionic representation \( \hat{g} \) stabilizes the vacuum state \( |0\rangle \), the resulting KP \( \tau \)-functions are polynomials in the KP flow variable \( t = (t_1, t_2, \ldots) \) and, in fact, all polynomial KP \( \tau \)-functions are expressible in this way \[12, 14\]. In Section 3, it is shown that when the flow variables are restricted to finite power sums

\[
t_j = \langle x \rangle_j := \frac{1}{j} \sum_{a=1}^{n} x_j^a, \quad (1.1)
\]

in a set of \( n \) auxiliary variables \( x = (x_1, \ldots, x_n) \), Wick’s theorem leads to an expression for the KP \( \tau \)-functions as the ratio of a finite determinant of alternant form and the Vandermonde determinant, with the entries in the numerator alternant polynomials in the \( x \) variables, as in Jacobi’s bialternant formula for Schur functions \[16\]. The resulting formula for the KP \( \tau \)-function is thus a generalization of Jacobi’s formula. Although these \( \tau \)-functions are, in general, inhomogeneous symmetric polynomials, they nevertheless share many of the properties of Schur functions \[7, 16, 25\], such as the Giambelli identity \[16\].

It is also known, in the BKP case, that when the group elements \( h^\pm \in \text{SO}(\mathcal{H}_{\phi^\pm}) \) are similarly chosen to be upper triangular, so the fermionic representation \( \hat{h}^\pm \) again stabilizes the vacuum state \( |0\rangle \), the resulting BKP \( \tau \)-functions are again polynomials in the BKP flow variables \( t_B = (t_1, t_3, \ldots) \) and, in fact, this exhausts the full set of polynomial BKP \( \tau \)-functions \[13\]. In Section 3 it is shown that if the flow variables are restricted to power sums \( \langle x \rangle \) in the auxiliary variables \( (x_1, \ldots, x_n) \), Wick’s theorem leads to an expression for the BKP \( \tau \)-function as a ratio, in which the numerator is a finite Pfaffian whose entries are either polynomials in an even number of auxiliary variables \( \{x_a\}_{a=1,2n} \), or rational combinations \( \{M_{ab} := \frac{x_a - x_b}{x_a + x_b} \} \) of these, and the denominator is the Pfaffian of the
skew $2n \times 2n$ matrix $M$ with only these rational entries, thereby generalizing Nimmo’s formula [19] for Schur $Q$ functions, with which these, again, share many properties.

In Section 5 analogous finite determinantal formulae are deduced for the $n$-pair correlation functions associated to the lattice of KP $\tau$-functions $\{\pi_{\lambda}(g(t))\}$. For group elements $\hat{g}$ that stabilize the vacuum, these again reduce to polynomials in the coordinates of the points $\{x_a, y_a\}_{a=1,...,n}$ appearing in the correlation function, multiplied by an explicit rational factor. For the BKP case, the $2n$-point correlation functions are similarly expressed in terms of restrictions of the BKP $\tau$-functions $\{\kappa_{\alpha}(h^{\pm})(t_B)\}$ to power sum variables in the coordinates of the points. When the group elements $h^{\pm}$ stabilize the vacuum, the generalized Nimmo formula derived in Section 4 thus provides a Pfaffian ratio expression for the $2n$-point correlators as polynomials in the point coordinates multiplied by explicit rational factors.

Section 6 illustrates these results with some examples of polynomial KP and BKP $\tau$-functions, as well as with KP and BKP 2-point correlators.

2 Fermionic VEV representations of lattices of KP and BKP $\tau$-functions

We begin by recalling the construction [9] of lattices of KP $\tau$-functions $\{\pi_{\lambda}(g(t))\}$ and BKP $\tau$-functions $\{\kappa_{\alpha}(h)(t_B)\}$ labelled, respectively, by pairs $(g, \lambda)$ consisting, in the KP case, of an infinite group element $g \in \text{GL}(\mathcal{H})$ and an integer partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$, and in the BKP case by pairs $(h^{\pm}, \alpha)$ consisting of an infinite orthogonal group element $h^{\pm} \in \text{SO}(\mathcal{H}_{\phi^{\pm}})$ and a strict partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ with an even number $r$ of parts. (A survey of the use of fermionic methods in the theory of $\tau$-functions may be found in [18], Chapt. 3 and [6], Chapt. 5 and 7.)

2.1 Lattice of KP $\tau$-functions $\pi_{\lambda}(g)(t)$

The lattice of KP $\tau$-functions introduced in [9] may be expressed as fermionic vacuum state expectation values,

$$\pi_{\lambda}(g)(t) := \langle 0 | \hat{\gamma}_\lambda(t) \hat{g} | \lambda \rangle, \quad (2.1)$$

where $|0\rangle$ is the vacuum state in the charge zero sector of the fermionic Fock space $\mathcal{F}$, which is the semi-infinite exterior product space on a separable Hilbert space $\mathcal{H}$ with denumerable basis $\{e_j\}_{j \in \mathbb{Z}}$

$$\mathcal{F} := \Lambda^{\infty/2} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n, \quad (2.2)$$

$|\lambda\rangle$ is an orthonormal basis element in the zero fermionic charge sector $\mathcal{F}_0$, labelled by an integer partition $\lambda$, and

$$t = (t_1, t_2, \ldots) \quad (2.3)$$
denotes the KP flow variables. The latter may alternatively be interpreted as the evaluation of normalized power sums

\[ t = [x] := ([x]_1, [x]_2, \ldots, [x]_j, \ldots), \tag{2.4} \]

where

\[ [x]_j := \frac{1}{j} \sum_{a=1}^{n} x_a^j, \quad j \in \mathbb{N}^+ \tag{2.5} \]

in terms of a finite or infinite set of bosonic variables \( x = (x_1, \ldots, x_n) \).

The dual basis elements \( \{e^j\}_{j \in \mathbb{Z}} \) for \( \mathcal{H}^* \) are defined by

\[ e^j(e_k) = \delta_{jk}, \quad j, k \in \mathbb{Z} \tag{2.6} \]

and the vacuum state in the \( \mathcal{F}_0 \) sector is denoted

\[ |\emptyset; 0 \rangle = |0 \rangle := e_{-1} \wedge e_{-2} \wedge \cdots. \tag{2.7} \]

The charged fermionic creation and annihilation operators, \( \{\psi_j, \psi_j^\dagger\} \in \text{End}(\mathcal{F}) \) for \( j \in \mathbb{Z} \) are defined as the exterior and inner products with the basis elements, and dual basis elements, respectively

\[ \psi_j := e_j \wedge, \quad \psi^\dagger_j := i e_j. \tag{2.8} \]

They satisfy the usual anti-commutation relations

\[ [\psi_j, \psi_k]^+ = \delta_{jk}, \quad [\psi_j, \psi_k] = 0, \quad [\psi_j^\dagger, \psi_k^\dagger] = 0, \quad j, k \in \mathbb{Z} \tag{2.9} \]

and generate the fermionic representation of the Clifford algebra on \( \mathcal{H} + \mathcal{H}^* \) corresponding to the scalar product

\[ Q(v + \mu, w + \nu) := \nu(v) + \mu(w), \quad v, w \in \mathcal{H}, \quad \mu, \nu \in \mathcal{H}^*. \tag{2.10} \]

They also satisfy the vacuum annihilation conditions

\[ \psi_{-j}|0 \rangle = 0, \quad \psi_{-j}^\dagger|0 \rangle = 0, \tag{2.11} \]

\[ \langle 0 | \psi_{-j} = 0, \quad \langle 0 | \psi_{-j-1} = 0, \quad j \in \mathbb{N}^+. \tag{2.12} \]

For an integer partition \( \lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \) of length \( \ell(\lambda) \) with Frobenius indices \( [10], \tag{2.13} \)

\[ (\alpha|\beta) = (\alpha_1, \ldots, \alpha_r|\beta_1, \ldots, \beta_r), \]

the basis state \( |\lambda \rangle \) in the \( n = 0 \) sector \( \mathcal{F}_0 \) is (\cite{11}, Chapt. 3, \cite{6}, Chapt. 5)

\[ |\lambda \rangle := |\lambda; 0 \rangle = (-1)^{\sum_{j=1}^{r} \beta_j} \prod_{j=1}^{r} \psi_{\alpha_j} \psi_{\beta_j-1}^\dagger |0 \rangle. \tag{2.14} \]
When substituted in (2.1), this yields the VEV representation

$$\pi_\lambda(g)(t) := (-1)^{\sum_{j=1}^r \beta_j} \langle 0 | \hat{\gamma}_+(t) \hat{g} \prod_{j=1}^r \psi_{\alpha_j} \psi_{\beta_{j-1}}^\dagger | 0 \rangle,$$

(2.15)

More generally, an orthonormal basis \{\ket{\lambda; n}\} for the charge \(n\) subspace \(F_n \subset F\) is provided by

$$\ket{\lambda; n} = (-1)^{\sum_{j=1}^r \beta_j} \prod_{j=1}^r \psi_{\alpha_j+n} \psi_{\beta_{j+n-1}}^\dagger \ket{n},$$

(2.16)

where

$$l_i(n) := \lambda_i - i + n, \quad i \in \mathbb{N}^+$$

(2.18)

are the particle positions ([6], Chapt. 5) associated with the partition \(\lambda\) in the \(F_n\) sector (with \(\lambda_j := 0\) for \(j > \ell(\lambda)\)), and

$$\ket{n} := \ket{\emptyset; n} = e_{n-1} \wedge e_{n-2} \wedge \cdots$$

(2.19)

is the vacuum state in the \(F_n\) sector. The dual basis vectors, denoted \{\langle \lambda; n |\}, satisfy

$$\langle \lambda; n | \mu; m \rangle = \delta_{\lambda \mu} \delta_{nm}.$$

(2.20)

The fermionic representation \(\hat{g}\) of group elements \(g \in GL_0(\mathcal{H})\) in the identity component is

$$\hat{g} := \hat{g}(\tilde{A}) = e^{\hat{A}},$$

(2.21)

where

$$\hat{A} := \sum_{i,j \in \mathbb{Z}} \tilde{A}_{jk} \psi_j \psi_k^\dagger,$$

(2.22)

with \{\tilde{A}_{jk}\}_{j,k \in \mathbb{Z}} the elements of a doubly infinite matrix \(\tilde{A}\) such that, in the basis \{\ket{e_j}\}_{j \in \mathbb{Z}} \(g\) is represented by

$$g(\tilde{A}) = e^{\tilde{A}}.$$

(2.23)

Normal ordering of the product of a pair of linear elements

$$:\hat{L}_1 \hat{L}_2: = \hat{L}_1 \hat{L}_2 - \langle 0 | \hat{L}_1 \hat{L}_2 | 0 \rangle$$

(2.24)

is defined so the vacuum expectation value (VEV) vanishes.

The KP flows are generated by the infinite abelian subgroup \(\Gamma_+ \subset GL_0(\mathcal{H})\) of shift flows

$$\Gamma_+ = \{ \gamma_+(t) := e^{\sum_{j=1}^\infty t_j \Lambda^j}, \quad \Lambda(e_i) = e_{i-1}, \quad i \in \mathbb{Z} \}$$

(2.25)

whose elements \(\gamma_+(t)\) are represented fermionically as

$$\hat{\gamma}_+(t) := e^{\sum_{j=1}^\infty t_j \hat{L}_j},$$

(2.26)
where

\[ J_j := \sum_{k \in \mathbb{Z}} \psi_k \psi_{k+j}^\dagger, \quad j \in \mathbb{N}^+ \]

(2.27)

are the charged current components. These mutually commute

\[ [J_j, J_k] = 0, \quad \forall j, k \in \mathbb{N}^+ \]

(2.28)

and annihilate the vacuum state

\[ J_j |0 \rangle = 0, \quad \forall j \in \mathbb{N}^+. \]

(2.29)

The lattice of KP \( \tau \)-functions (2.1) can be extended to an infinite sequence of KP \( \tau \)-functions

\[ \pi_{\lambda,n}(g)(t) := \langle n| \hat{\gamma}_+(t) \hat{g}|\lambda; n \rangle = (-1)^{\sum_{j=1}^r \beta_j} \langle n| \hat{\gamma}_+(t) \hat{g} \rangle \prod_{j=1}^r \psi_{\alpha_j} \psi_{\beta_j-1}^\dagger |n \rangle, \]

(2.30)

defined in each sector \( \mathcal{F}_n \) which, for each pair \((g, \lambda)\), form an integer lattice of mKP \( \tau \)-functions. (See [5,11] or [6], Chapt. 7.)

We also introduce fermionic field operators

\[ \psi(z) := \sum_{j \in \mathbb{Z}} \psi_j z^j, \quad \psi_j^\dagger(z) := \sum_{j \in \mathbb{Z}} \psi_{j-1}^\dagger z^{j-1} \]

(2.31)

and “dressed” creation and annihilation operators

\[ \psi_j(A) := \hat{g}(A) \psi_j \hat{g}^{-1}(A) = \sum_{k \in \mathbb{Z}} g_{kj}(A) \psi_k \]

(2.32)

\[ \psi_j^\dagger(A) := \hat{g}(A) \psi_j^\dagger \hat{g}^{-1}(A) = \sum_{k \in \mathbb{Z}} g_{jk}^{-1}(A) \psi_k^\dagger. \]

(2.33)

In the special case where the matrix \( \hat{A} \) is strictly upper triangular

\[ \hat{A}_{ij} = 0 \text{ if } i \geq j \]

(2.34)

the \( \mathfrak{gl}(\mathcal{H}) \) algebra elements \( \hat{A} \) annihilate the vacuum state

\[ \hat{A}|0 \rangle = 0, \quad \hat{A}^\dagger|0 \rangle, \]

(2.35)

and the group elements \( \hat{g}(A) \) stabilize it

\[ \hat{g}(A)|0 \rangle. \]

(2.36)

We then have

\[ \psi_j(A) = \sum_{k=-\infty}^j g_{kj}(A) \psi_k, \quad g_{kk}(A) = 1, \quad \forall k \in \mathbb{Z}, \]

(2.37)

\[ \psi_j^\dagger(A) = \sum_{k=j}^\infty g_{jk}^{-1}(A) \psi_k^\dagger, \quad g_{kk}^{-1}(A) = 1, \quad \forall k \in \mathbb{Z}. \]

(2.38)
Now define two sequences of monic polynomials

\[ p_j(x|\tilde{A}) := \sum_{k=0}^{j} P_{kj}(\tilde{A})x^k, \quad P_{jj}(\tilde{A}) = 1, \quad j \in \mathbb{N}, \tag{2.39} \]

\[ p_j^*(y|\tilde{A}) := \sum_{k=0}^{j} P_{jk}^*(\tilde{A})y^k, \quad P_{jj}^*(\tilde{A}) = 1, \quad j \in \mathbb{N}, \tag{2.40} \]

where the upper triangular matrix of coefficients \( \{P_{kj}(A)\}_{j,k \in \mathbb{N}} \) is the \( \mathbb{N} \times \mathbb{N} \) block of the \( \mathbb{Z} \times \mathbb{Z} \) upper triangular matrix \( g(\tilde{A}) \) obtained by exponentiating the strictly upper triangular matrix \( \tilde{A} \) as in (2.23)

\[ P_{jk}(\tilde{A}) = g_{jk}(\tilde{A}), \quad P_{jj} = 1, \quad j \leq k, \quad j, k \in \mathbb{N}. \tag{2.41} \]

while the lower triangular matrix of coefficients \( \{P_{jk}^*(A)\}_{j,k \in \mathbb{N}} \) is the \( (-\mathbb{N}^+) \times (-\mathbb{N}^+) \) block of the inverse matrix \( g^{-1} \)

\[ P_{jk}^* = g_{j-k-1,-k-1}^{-1}, \quad P_{jj}^* = 1, \quad j \geq k, \quad j, k \in \mathbb{N}. \tag{2.42} \]

\textbf{Lemma 2.1.} The monic polynomials sequences \( \{p_j(x|\tilde{A})\}_{j \in \mathbb{N}} \) and \( \{p_j^*(y|\tilde{A})\}_{j \in \mathbb{N}} \) have the following VEV representations:

\[ p_j(x|\tilde{A}) = x^{-1} \langle 0|\psi^\dagger(x^{-1})\psi_j(\tilde{A})|0 \rangle \quad \forall \ j \in \mathbb{N}, \tag{2.43} \]

\[ p_j^*(y|\tilde{A}) = y^{-1} \langle 0|\psi(y^{-1})^\dagger\psi^*_j(\tilde{A})|0 \rangle \quad \forall \ j \in \mathbb{N}, \tag{2.44} \]

\textit{Proof.} Substitute expressions (2.37) for \( \psi_j(\tilde{A}) \) and (2.38) for \( \psi_j^*(\tilde{A}) \), and the series expansions (2.31) for \( \psi^\dagger(x^{-1}) \) and \( \psi(y^{-1})^\dagger \) into \( \langle 0|\psi^\dagger(x^{-1})\psi_j(\tilde{A})|0 \rangle \), and \( \langle 0|\psi(y^{-1})^\dagger\psi^*_j(\tilde{A})|0 \rangle \) and evaluate the terms in the sum using (2.11), (2.12), which imply

\[ \langle 0|\psi^\dagger(x^{-1})\psi_j|0 \rangle = x^{j+1}, \quad \text{for} \ j \geq 0 \tag{2.45} \]

\[ \langle 0|\psi(y^{-1})^\dagger\psi^*_j|0 \rangle = y^{j+1}, \quad \text{for} \ j \geq 0. \tag{2.46} \]

\[ \square \]

It follows [0,12] that the associated KP \( \tau \)-functions \( \pi_\lambda(g(\tilde{A}))(t) \) defined in (2.1) are polynomials in the flow variables \( t \). Besides polynomiality, they share many further properties with Schur functions \( s_\lambda(t) \), so if conditions (2.34)-(2.36) are satisfied, we denote these as

\[ s_\lambda(t|\tilde{A}) := \pi_\lambda(g(\tilde{A}))(t). \tag{2.47} \]

In particular, choosing \( \tilde{A} \) to vanish, so that \( g(\tilde{A}) \) is the identity element, we recover the Schur functions

\[ s_\lambda(t|0) = s_\lambda(t). \tag{2.48} \]

More generally, if \( \tilde{A} \) is upper triangular, we can extend the definition (2.47) of polynomial KP \( \tau \)-functions to an integer lattice of polynomial mKP \( \tau \)-functions for each \( \lambda \)

\[ s_{\lambda,n}(t|\tilde{A}) := \pi_{\lambda,n}(g(\tilde{A}))(t), \quad n \in \mathbb{Z}. \tag{2.49} \]
2.2 Lattice of BKP $\tau$-functions $\kappa_\alpha(h)(t_B)$

As in the KP case, a lattice of BKP $\tau$-functions was introduced in [9], which may also be expressed as fermionic VEV’s, but with the charged fermionic operators $\{\psi_j, \psi_j^\dagger\}_{j \in \mathbb{Z}}$ replaced by either of a pair of sequences $\{\phi_j^+, \phi_j^-\}_{j \in \mathbb{Z}}$ of mutually anti-commuting neutral fermionic operators $\{\phi_j^{\pm}\}_{j \in \mathbb{Z}}$, defined, as in [4, 11, 27], by

$$\phi_j^+ := \frac{1}{\sqrt{2}}(\psi_j + (-1)^j\psi_j^\dagger), \tag{2.50}$$
$$\phi_j^- := \frac{1}{\sqrt{2}}(\psi_j - (-1)^j\psi_j^\dagger). \tag{2.51}$$

These satisfy the anti-commutation relations

$$[\phi_j^+, \phi_k^+] = (-1)^j\delta_{j,-k}, \quad [\phi_j^-, \phi_k^-] = (-1)^j\delta_{j,-k}, \quad [\phi_j^+, \phi_k^-] = 0, \quad j, k \in \mathbb{Z} \tag{2.52}$$

and vacuum annihilation conditions

$$\phi_j^\pm|0\rangle = 0, \quad \langle 0|\phi_j^\pm = 0, \quad j > 0. \tag{2.53}$$

Their pairwise expectation values are:

$$\langle 0|\phi_j^+\phi_k^+|0\rangle = \langle 0|\phi_j^-\phi_k^-|0\rangle = \begin{cases} (-1)^k\delta_{j,-k} & \text{if } k > 0, \\ \frac{1}{2}\delta_{j,0} & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases} \tag{2.54}$$

$$\langle 0|\phi_j^+\phi_k^-|0\rangle = -\langle 0|\phi_j^-\phi_k^+|0\rangle = \frac{i}{2}\delta_{j,0}\delta_{k,0}. \tag{2.55}$$

The direct sum $\mathcal{H} + \mathcal{H}^*$ decomposes into an orthogonal direct sum with respect to the scalar product $Q$ defined in (2.10),

$$\mathcal{H} = \mathcal{H}_{\phi^+} \oplus \mathcal{H}_{\phi^-} \tag{2.56}$$

of two subspaces

$$\mathcal{H}_{\phi^+} = \text{span}\{f_j^+\}_{j \in \mathbb{Z}}, \quad \mathcal{H}_{\phi^-} = \text{span}\{f_j^-\}_{j \in \mathbb{Z}}, \tag{2.57}$$

where the bases $\{f_j^+\}_{j \in \mathbb{Z}}$ and $\{f_j^-\}_{j \in \mathbb{Z}}$, defined by

$$f_j^+ := \frac{1}{\sqrt{2}}(e_j + (-1)^je^{-j}), \quad f_j^- := \frac{i}{\sqrt{2}}(e_j - (-1)^je^{-j}), \tag{2.58}$$

satisfy the orthogonality relations

$$Q_\pm(f_j^\pm, f_k^\pm) = (-1)^k\delta_{j+k,0}, \quad \forall \ j, k \in \mathbb{Z} \tag{2.59}$$

with respect to the scalar products

$$Q_\pm := Q|_{\mathcal{H}_{\phi^\pm}} \tag{2.60}$$

on the subspaces $\mathcal{H}_{\phi^\pm}$ obtained by restriction of $Q$. 


The elements $h^\pm(A) \in \text{SO}(H_{\phi^\pm})$ of the corresponding mutually commuting orthogonal subgroups $\text{SO}(H_{\phi^\pm}) \subset \text{SO}(H + H^*, Q)$ have fermionic representations

$$\hat{h}^\pm := \hat{h}^\pm(A) := e^{A^\pm}, \quad (2.61)$$

that leave invariant the respective subspaces $F_{\phi^\pm} \subset F$, where

$$\hat{A}^\pm := \frac{1}{2} \sum_{j,k \in \mathbb{Z}} A_{jk} \phi_j^\pm \phi_k^\pm, \quad (2.62)$$

with $\{A_{jk}\}_{j,k \in \mathbb{Z}}$ the elements of a doubly infinite skew symmetric matrix $A$ determining the matrix representation $h$ of $h^\pm$ in the bases $\{f_j^\pm\}_{j \in \mathbb{Z}}$ by

$$h(A) = e^{A}, \quad (2.63)$$

where

$$\hat{A}_{jk} := (-1)^k A_{j,k}. \quad (2.64)$$

The BKP flows are generated by infinite abelian subgroups $\Gamma^{B\pm} \subset \text{SO}(H_{\phi^\pm})$ whose elements are represented fermionically as

$$\hat{\gamma}^{B\pm}(t_B) := e^{\sum_{j=1}^{\infty} t_{2j-1} J_{j}^{B\pm}}, \quad (2.65)$$

where the neutral current components $\{J_{j}^{B\pm}\}_{j \in \mathbb{N}^+}$ are defined as

$$J_{j}^{B+} := \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^{k+1} \phi_k^+ \phi_{-k-j}^+; \quad J_{j}^{B-} := \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^{k+1} \phi_k^- \phi_{-k-j}^-; \quad j \in \mathbb{N}^+. \quad (2.66)$$

Of these, the even ones $J_{2j}^{B\pm}$ vanish, while the odd ones mutually commute:

$$[J_{2j-1}^{B+}, J_{2k-1}^{B+}] = 0, \quad [J_{2j-1}^{B-}, J_{2k-1}^{B-}] = 0, \quad [J_{2j-1}^{B+}, J_{2k-1}^{B-}] = 0; \quad j, k \in \mathbb{N}^+. \quad (2.67)$$

and annihilate the vacuum state

$$J_{2j-1}^{B\pm}|0\rangle = 0, \quad \forall \ j \in \mathbb{N}^+. \quad (2.68)$$

They are related to the odd charged current components by

$$J_{2j-1} = J_{2j-1}^{B+} + J_{2j-1}^{B-}; \quad \forall \ j \in \mathbb{N}^+. \quad (2.69)$$

Following [4, 5, 11, 27], there are two types of neutral fermionic basis states

$$|\alpha^\pm\rangle := \phi_{\alpha_1}^+ \cdots \phi_{\alpha_r}^+|0\rangle, \quad (2.70)$$

spanning two subspaces $F_{\phi^\pm} \subset F$

$$F_{\phi^\pm} := \text{span}\{|\alpha^\pm\rangle\}. \quad (2.71)$$
Remark 2.1. Note that the subspaces $F_{\phi^\pm} \subset F$ are not mutually orthogonal. In fact, their intersection is infinite dimensional, as is their intersection with each of the fermionic charge sectors $F_n$. However they are invariant, respectively, under the two different infinite, mutually commuting subgroups $SO(\mathcal{H}_{\phi^\pm}) \subset SO(\mathcal{H} + \mathcal{H}^*, Q)$ represented fermionically by the elements $\{\hat{h}^\pm\}$. Since the two subgroups $SO(\mathcal{H}_{\phi^\pm})$ are isomorphic, as are their abelian subgroups $\Gamma^{B\pm}$ it is sufficient, in studying the resulting BKP $\tau$-functions, to consider only one of them. However, for consistency with earlier work [8, 9], in which these were related bilinearly to the corresponding lattices of KP $\tau$-functions (2.1), we retain here the notation for both types of operators $\{\phi_j^\pm\}_{j \in \mathbb{Z}}$ and fermionic Fock spaces $F_{\phi^\pm} \subset F$, although there is no difference in the resulting BKP $\tau$-functions or correlators constructed from them.

The lattice of BKP $\tau$-functions $\{\kappa_\alpha(h)(t_B)\}$ is defined [9] as:

$$\kappa_\alpha(h)(t_B) := \langle 0 | \hat{\gamma}^{B\pm}(t_B) \hat{h}^\pm | \alpha^\pm \rangle,$$

(2.72)

where

$$t_B = (t_1, t_3, \ldots)$$

(2.73)

denote the BKP flow variables, which may be restricted, as in (2.4), to evaluations on normalized power sums in an auxiliary (finite or infinite) set of bosonic variables $x = (x_1, x_2, \ldots, x_n)$

$$t_B = [x]_B, \quad [x]_B := ([x]_1, [x]_3, \ldots, [x]_{2j-1}, \ldots),$$

(2.74)

We also introduce fermionic field operators

$$\phi^\pm(z) := \sum_{j \in \mathbb{Z}} \phi_j^\pm z^j,$$

(2.75)

which are related to those defined in (2.31) by

$$\psi(z) = \frac{1}{\sqrt{2}} \left( \phi^+(z) - i\phi^-(z) \right), \quad \psi^\dagger(\frac{1}{z}) = \frac{1}{\sqrt{2}z} \left( \phi^+(z) + i\phi^-(z) \right)$$

(2.76)

$$\psi^\dagger(-z) \psi(z) = \frac{i}{z} \phi^+(z) \phi^-(z),$$

(2.77)

and the “dressed” operators

$$\phi_j^\pm(A) := \hat{h}^\pm(A) \phi_j^\pm(\hat{h}^\pm)^{-1}(A) = \sum_{k \in \mathbb{Z}} h_{kj}(A) \phi_k^\pm$$

(2.78)

In the special case where the matrix $A$ satisfies the antidiagonal triangularity condition

$$A_{jk} = 0 \quad \text{if} \quad j + k \geq 0,$$

(2.79)

and $\bar{A}$ the strictly upper triangular one

$$\bar{A}_{jk} = 0 \quad \text{if} \quad j \geq k,$$

(2.80)
the \( \mathfrak{so}(\mathcal{H}_{\phi^\pm}) \) algebra elements \( \hat{A}^\pm \) annihilate the vacuum state

\[
\hat{A}^\pm |0\rangle = 0, \quad (2.81)
\]

and the group elements \( \hat{h}^\pm(A) \) stabilize it

\[
\hat{h}^\pm(A) |0\rangle = |0\rangle. \quad (2.82)
\]

The matrix \( h(A) \) becomes upper triangular, with 1’s on the diagonal and therefore reduces to

\[
\phi_j^\pm(A) := \hat{h}^\pm(A)\hat{h}^\pm(\hat{h}^\pm)^{-1}(A) = \sum_{k=-\infty}^{j} h_{kj}(A)\phi_k^\pm, \quad (2.83)
\]

\[
h_{jj}(A) = 1, \quad \forall \ j \in \mathbb{Z}. \quad (2.84)
\]

Defining the upper triangular \( \mathbb{Z} \times \mathbb{Z} \) matrix \( P(A) \) with elements

\[
(P(A))_{jk} := (h(A))_{jk} - \frac{1}{2}\delta_{k0}(h(A))_{0j}, \quad (2.85)
\]

we again define a sequence of monic polynomials \( \{p_j(x|A)\}_{j \in \mathbb{N}^+} \),

\[
p_j(x|A) = \sum_{k=0}^{j} P_{kj}(A)x^k, \quad P_{jj} = 1, \quad \text{if} \ j \neq 0, \ P_{00} = \frac{1}{2}, \quad (2.86)
\]

and

\[
p_0(x|A) = \frac{1}{2}, \quad (2.87)
\]

such that the upper triangular matrix of coefficients \( \{P_{jk}(A)\}_{j,k \in \mathbb{N}} \) is the \( \mathbb{N} \times \mathbb{N} \) block \( P(A) \). We then have

**Lemma 2.2.** The polynomials \( \{p_j(x|A)\}_{j \in \mathbb{N}} \) have the following VEV representations:

\[
p_j(x|A) = \langle 0|\phi^\pm(-x^{-1})\phi_j^\pm(A)|0\rangle, \quad \forall \ j \in \mathbb{N}. \quad (2.88)
\]

**Proof.** This follows from substituting expression \( (2.83) \) for \( \phi_j^\pm(A) \) and the series expansion \( (2.75) \) for \( \phi^\pm(-x^{-1}) \) into \( \langle 0|\phi^\pm(x^{-1})\phi_j^\pm(A)|0\rangle \), and evaluating the terms in the sum using \( (2.54) \), which implies

\[
\langle 0|\phi^\pm(-x^{-1})\phi_j^\pm|0\rangle = \begin{cases} x^j & \text{for } j > 0, \\ \frac{1}{2} & \text{for } j = 0, \\ 0 & \text{for } j < 0. \end{cases} \quad (2.89)
\]
It follows \cite{9,13} that the associated BKP \( \tau \)-functions \( \kappa_\alpha(h(A))(t_B) \) are polynomials in the flow variables \( t_B \) which, besides polynomiality, share many properties with the (scaled) Schur \( Q \)-functions, which correspond to choosing \( A = 0 \). Therefore, if conditions \( (2.79)-(2.82) \) are satisfied, we denote these as

\[
Q_\alpha([x]_B|A) := \kappa_\alpha(h(A)([x]_B)), \tag{2.90}
\]

and similarly define \( Q_\alpha(x|A) \) by

\[
Q_\alpha(x|A) = 2^}\frac{r}{2} Q_\alpha(2[x]_B|A). \tag{2.91}
\]

In particular, choosing \( A \) to vanish, so \( h(A) \) is the identity element, we recover the Schur \( Q \)-functions

\[
Q_\alpha(x) = 2^}\frac{r}{2} Q_\alpha([2x]_B|0). \tag{2.92}
\]

### 3 Bialternant formula for polynomial KP tau functions

Setting \( t = [x] \), as in \cite{2.4}, with a finite number \( n \) of variables \( x = (x_1, \ldots, x_n) \), and assuming the length \( \ell(\lambda) \) of the partition \( \lambda \) satisfies \( \ell(\lambda) \leq n \), Jacobi's bialternant formula for Schur functions \cite{2.33} is

\[
\det \left( x_j^{\lambda_k - k + n} \right) \frac{1_{j,k \leq n}}{\Delta(x)} \tag{3.1}
\]

where we set \( \lambda_k = 0 \) for \( k > \ell(\lambda) \), and

\[
\Delta(x) = \prod_{1 \leq j < k \leq n} (x_j - x_k) \tag{3.2}
\]

is the Vandermonde determinant.

This can be generalized by replacing the monomials \( \{x^j\}_{j \in \mathbb{N}} \) by an arbitrary sequence of monic polynomials, as in \cite{7}

\[
p_j(x|\tilde{A}) := \sum_{k=0}^j P_{kj}(\tilde{A}) x^k, \quad P_{jj}(\tilde{A}) = 1, \quad j \in \mathbb{N}, \tag{3.3}
\]

where the upper triangular matrix of coefficients \( \{P_{jk}(A)\}_{j,k \in \mathbb{N}} \) is the \( \mathbb{N} \times \mathbb{N} \) block of the \( \mathbb{Z} \times \mathbb{Z} \) upper triangular matrix of coefficients \( P(\tilde{A}) \) obtained by exponentiating the strictly upper triangular matrix \( \tilde{A} \) as in \cite{2.23}

\[
P(\tilde{A}) = g(\tilde{A}) = e^{\tilde{A}}. \tag{3.4}
\]
Defining
\[ \tilde{s}_{\lambda,n}([x]|\tilde{A}) := \det \left( p_{\lambda_k-k+n}(x_j|\tilde{A}) \right)_{1 \leq j,k \leq n} / \Delta(x), \]  
(3.5)

it follows that this coincides with the KP $\tau$-function $s_{\lambda,n}(x|\tilde{A})$ defined in (2.49).

**Proposition 3.1.**
\[ \tilde{s}_{\lambda,n}([x]|\tilde{A}) = s_{\lambda,n}([x]|\tilde{A}). \]  
(3.6)

Although this result was proved in [7], and special cases have long been studied [2, 3, 20, 25], we provide here, for completeness, a self-contained proof.

**Proof.** Recall the following formula, related to the bosonization map [4, 11]:
\[ \langle 0 | \psi^\dagger(x_{n-1}^{-1}) \cdots \psi^\dagger(x_1^{-1}) = \left( \prod_{j=1}^{n} x_j \right) \Delta(x) \langle n | \tilde{\gamma}_+ \left( \sum_{a=1}^{n} [x_a] \right) \rangle. \]  
(3.7)

Eq. (3.6) is obtained by substituting (3.7) in (2.30) and (2.49), and choosing $t = [x]$. For upper triangular $\tilde{A}$ and partition $\lambda$, we have, from the expression (2.17) for the basis element $|\lambda; n\rangle$, the definition eq. (2.32) of $\psi_j(\tilde{A})$ and the fact that $\tilde{g}(\tilde{A})$ stabilizes the vacuum (2.36),
\[ s_{\lambda,n}([x]|\tilde{A}) = \langle n | \tilde{\gamma}_+ \left( \sum_{a=1}^{n} [x_a] \right) \psi_j(\tilde{A}) | \lambda; n \rangle \]
\[ = \frac{\langle 0 | \psi^\dagger(x_{n-1}^{-1}) \cdots \psi^\dagger(x_1^{-1}) \psi_{\lambda_1-1+n}(\tilde{A}) \cdots \psi_{\lambda_n}(\tilde{A}) | 0 \rangle}{\left( \prod_{j=1}^{n} x_j \right) \Delta(x)} \]
\[ = \frac{\det \left( \langle 0 | \psi^\dagger(x_{j-1}^{-1}) \psi_{\lambda_k-k+n}(\tilde{A}) | 0 \rangle \right)_{1 \leq j,k \leq n}}{\left( \prod_{j=1}^{n} x_j \right) \Delta(x)} \]
\[ = \frac{\det \left( p_{\lambda_k-k+n}(x_j|\tilde{A}) \right)_{1 \leq j,k \leq n}}{\Delta(x)} = \tilde{s}_{\lambda,n}([x]|\tilde{A}), \]  
(3.8)

where the third line follows from Wick’s theorem (Appendix A, eq. (A.5)) and the fourth from (2.43).

Besides the fact that the $s_{\lambda,n}([x]|\tilde{A})$’s are polynomial KP $\tau$-functions expressible via the bialternant formula (3.5), they also share with the Schur functions $s_{\lambda}([x])$ the property that, for $\lambda$ with Frobenius indices $(\alpha|\beta)$ as in (2.13), they satisfy the Giambelli identity [16], expressing them as determinants of the matrices whose elements are the functions $s_{(\alpha_i|\beta_j)}([x])$ corresponding to hook partitions for all pairs $(\alpha_i, \beta_j)$.

**Proposition 3.2** (Giambelli identity).
\[ s_{\lambda,n}([x]|\tilde{A}) = \det \left( s_{(\alpha_i|\beta_j)}([x]|\tilde{A}) \right)_{1 \leq i,j \leq r} \]  
(3.9)
Proof. From (2.1), (2.49) and 2.16, we have

\[ s_{\lambda,n}(\{x\}|\tilde{A}) = (-1)^{\sum_{j=1}^r \beta_j} (n|\hat{\gamma}_+(\{x\})\prod_{j=1}^r \psi_{\alpha+j+n}(\tilde{A})\psi_{-\beta_j+n-1}(\tilde{A})|n) \]

\[ = (-1)^{\sum_{j=1}^r \beta_j} (n|\hat{\gamma}_+(\{x\})\hat{\gamma}_-^{-1}(\{x\})\hat{\gamma}_+(\{x\})\psi_{-\beta_j+n-1}(\tilde{A})\hat{\gamma}_-^{-1}(\{x\})|n) \]

\[ = \det \left( (-1)^{\beta_j} (n|x\hat{\gamma}_+(\{x\})\psi_{\alpha+i+n}(\tilde{A})\psi_{-\beta_j+n-1}(\tilde{A})\hat{\gamma}_-^{-1}(\{x\})|n) \right)_{1 \leq i,j \leq r} \]

where the fact that \( \hat{\gamma}_-^{-1}(\{x\}) \) stabilizes the vacuum has been used in the second line and Wick’s theorem (see Appendix A) in the third. \( \square \)

4 Generalized Nimmo formula for polynomial BKP \( \tau \)-functions

Nimmo’s formula [19] similarly expresses Schur Q-functions \( Q_\alpha(x) \) associated to a strict partition \( \alpha = (\alpha_1, \ldots, \alpha_{2m}) \) of even cardinality (possibly including a vanishing part \( \alpha_{2m} = 0 \)), as the ratio of two Pfaffians:

\[ Q_\alpha(x) = 2^{2m} \frac{\text{Pf}(M_\alpha(x))}{\text{Pf}(M(x))}, \]

where \( x = (x_1, \ldots, x_{2n}) \) consists of an even number \( 2n \) of elements, \( M(x) \) is the \( 2n \times 2n \) skew symmetric matrix

\[ M_{ab}(x) := \frac{x_a - x_b}{x_a + x_b}, \quad 1 \leq a, b \leq 2n \]

and \( M_\alpha(x) \) is the \( 2(n + m) \times 2(n + m) \) block skew symmetric matrix

\[ M_\alpha(x) := \begin{pmatrix} M(x) & V_\alpha(x) \\ -V_\alpha(x)^T & 0 \end{pmatrix}, \]

with

\[ (V_\alpha(x))_{aj} := (x_a)^{\alpha_j}, \quad a = 1, \ldots, 2n, \quad j = 1, \ldots, 2m. \]

If the number of elements is odd \( (x_1, \ldots, x_{2n-1}) \), we just set \( x_{2n} = 0 \) in (4.1)-(4.4).

Now assume that the infinite matrix \( A \) appearing in eq. (2.62) satisfies the antidiagonal triangular conditions (2.79) or, equivalently, that \( \tilde{A} \) satisfies the strict upper triangular conditions (2.80), so that \( \hat{A}^\pm \) annihilates the vacuum (2.81) and \( \hat{h}^\pm(A) \) stabilizes it (2.82). It follows that the BKP \( \tau \)-function \( Q_\alpha(\{x\}|B|A) \) is a polynomial of degree \( \leq 2m \) (not
necessarily homogeneous) that is expressible via a generalized Nimmo formula. Define the $2m \times 2m$ skew symmetric matrix matrix $H_\alpha(A)$ with elements

$$H_\alpha(A)_{jk} := \begin{cases} \langle 0|\phi^\pm_{\alpha_j}(A)\phi^\pm_{\alpha_k}(A)|0\rangle, & 1 \leq j < k \leq 2m, \\ 0 & \text{if } j = k. \end{cases}$$

(4.5)

$$= -H_\alpha(A)_{kj}$$

(4.6)

**Proposition 4.1** (Generalized Nimmo formula). Assuming condition (2.79) to hold, we have

$$Q_\alpha(x|A) = 2^{2m}\frac{\text{Pf}(M_\alpha^H(x|A))}{\text{Pf}(M(x))},$$

(4.7)

where

$$M_\alpha^H(x|A) = \begin{pmatrix} M(x) & V_\alpha(x|A) \\ -V_\alpha(x|A)^T & 2H_\alpha(A) \end{pmatrix},$$

(4.8)

with

$$(V_\alpha(x|A))_{aj} = p_{\alpha_j}(x_a|A), \quad a = 1, \ldots, 2n, \quad j = 1, \ldots, 2m.$$  

(4.9)

**Remark 4.1.** Note that the numerator Pfaffian $\text{Pf}(M_\alpha^H(x|A))$ in (4.7) vanishes whenever any pair $x_a = x_b$ are equal, and hence we may factor out a Vandermonde determinant $\Delta(x_1, \ldots, x_{2n})$. The denominator Pfaffian is

$$\text{Pf}(M(x)) = \prod_{1 \leq a < b \leq 2n} \frac{x_a - x_b}{x_a + x_b},$$

(4.10)

so the $\Delta(x_1, \ldots, x_{2n})$ factors in the numerator and denominator cancel. This also places a factor $\prod_{1 \leq a < b \leq 2n}(x_a + x_b)$ in the numerator, which cancels the poles from the matrix elements $M_{ab}(x)$ at which $x_a + x_b$ vanishes for any distinct pair $(a, b)$. Therefore there are no poles, and the result is a polynomial which, since both the numerator and denominator reverse signs under any interchange $x_a \leftrightarrow x_b$, is symmetric.

**Proof.** (Proposition 4.1) We have the standard formula [4],

$$\langle 0|\phi^\pm(-x_{2n}^{-1}) \cdots \phi^\pm(-x_{1}^{-1}) = 2^{-n}\text{Pf}(M(x))\langle 0|\hat{\gamma}^B \pm(2[x]_B)$$

(4.11)

related to the bosonization map. From the fermionic VEV formula (2.88) for the polynomials $p_j(x_a|A)$, we have

$$(V_\alpha(x|A))_{aj} := \langle 0|\phi^\pm(-x_a^{-1})\phi^\pm_{j}(A)|0\rangle = p_j(x_a|A), \quad 1 \leq a \leq 2n, \quad j \in \mathbb{N},$$

(4.12)

and from (2.54)

$$\langle 0|\phi^\pm(-x_a^{-1})\phi^\pm(-x_b^{-1})|0\rangle = \frac{1}{2}\frac{x_a - x_b}{x_a + x_b}, \quad 1 \leq a, b \leq 2n.$$  

(4.13)
Substituting \((4.11)\) and \((2.78)\) into \((2.72)\), \((2.91)\) and \((2.90)\) and using the fact that \(\hat{h}(A)\) stabilizes the vacuum \((2.82)\) gives

\[
Q_\alpha(x|A) = 2^m \langle 0 | \hat{\gamma}^{B\pm}(2[x]_B)\hat{h}_\pm(A) | \alpha^\pm \rangle = 2^m \langle 0 | \hat{\gamma}^{B\pm}(2[x]_B)\phi_{\alpha_1}^\pm(A) \cdots \phi_{\alpha_{2m}}^\pm(A) | 0 \rangle
\]

\[
= 2^{n+m} \frac{\langle 0 | \phi(-x_2^{-1}) \cdots \phi(-x_1^{-1})\phi_{\alpha_1}^\pm(A) \cdots \phi_{\alpha_{2m}}^\pm(A) | 0 \rangle}{\text{Pf}(M(x))}
\]

\[
= 2^{n+m} \frac{\text{Pf} \left( \frac{1}{2} M(x) \quad V_\alpha(x|A) \right) \text{Pf}(M(x))}{\text{Pf}(M(x))} = 2^{2m} \frac{\text{Pf} \left( M(x) \quad V_\alpha(x|A) \right) \text{Pf}(M(x))}{\text{Pf}(M(x))}
\]

where Wick’s theorem (Appendix A, eq. \((A.1)\)) has been applied in the fourth line, and the matrix elements evaluated using relations \((4.6), (4.12)\) and \((4.13)\). \(\blacksquare\)

We also have an analog of Schur’s Pfaffian formula

\[
Q_\alpha(x) = \text{Pf} \left( Q_{ij}(x) \right)_{1 \leq i, j \leq 2m},
\]

for the functions \(Q_\alpha(x|A)\), where \(Q(x)\) is the skew-symmetric \(2m \times 2m\) matrix with elements

\[
Q_{ij}(x) = \begin{cases} 
Q_{(i,j)}([x]_B) & \text{if } i > j \\
0 & \text{if } i = j \\
-Q_{(j,i)}(x) & \text{if } i < j,
\end{cases}
\]

\(1 \leq i, j \leq 2m.\)

\[\text{Proposition 4.2.}\]

\[
Q_\alpha(x|A) = \text{Pf} \left( Q_{ij}(x|A) \right)_{1 \leq i, j \leq 2m},
\]

where \(Q(x|A)\) is the skew-symmetric \(2m \times 2m\) matrix with elements

\[
Q_{ij}(x|A) = \begin{cases} 
Q_{(i,j)}(x|A) & \text{if } i < j \\
0 & \text{if } i = j \\
-Q_{(i,j)}(x|A) & \text{if } i > j
\end{cases}, \quad 1 \leq i, j \leq 2m.
\]

\[\text{Proof.}\] This follows from substituting eq. \((2.70)\) into eqs. \((2.72), (2.90)\) and \((2.91)\) and using \((2.78)\) to express \(Q_\alpha(x|A)\), as in eq. \((4.15)\)

\[
Q_\alpha(x|A) = 2^m \langle 0 | \hat{\gamma}^{B\pm}(2[x]_B)\phi_{\alpha_1}^\pm(A) \cdots \phi_{\alpha_{2m}}^\pm(A) | 0 \rangle
\]

\(16\)
\[
\begin{aligned}
\langle 0|\hat{\gamma}^B\phi^\pm(A)(\hat{\gamma}^B)^{-1}(2[x]_B)\cdots\hat{\gamma}^B(2[x]_B)\phi^\pm_{\alpha}(A)(\hat{\gamma}^B)^{-1}(2[x]_B)|0\rangle \\
= \text{Pf} \left( 2\langle 0|\hat{\gamma}^B(2[x]_B)\phi^\pm_{\alpha}(A)\phi^\pm_{\alpha'}(A)|0\rangle \right)_{1\leq i,j\leq 2m} \\
= \text{Pf} \left( Q_{ij}(x|A) \right)_{1\leq i,j\leq 2m},
\end{aligned}
\]

(4.22)

where the fact that \(\hat{\gamma}^B([x]_B)\) stabilizes the vacuum has been used in the second line and Wick’s theorem (Appendix A, eq. (A.1) in the third.

If we further choose the matrix \(A\) to satisfy the vanishing conditions

\[A_{jk} = 0 \text{ if } j < 0, \ k \leq 0\]

(4.23)

or equivalently, \(\tilde{A}\) to satisfy

\[\tilde{A}_{jk} = 0 \text{ if } j < 0, \ k \geq 0,\]

(4.24)

we have

\[P_{jk}(A) = 0 \text{ if } j < 0, \ k \geq 0,\]

(4.25)

and it follows that, for \(j \geq 0\), \(\phi_j(A)\) is the finite triangular linear combination

\[\phi^{\pm}_j(A) = \sum_{k=0}^{j} P_{kj}(A) \phi^{\pm}_k\]

(4.26)

of \(\phi_k\)’s with \(k \geq 0\). This implies that

\[\langle 0|\phi^{\pm}_j(A)\phi^{\pm}_k(A)|0\rangle = 0, \ \forall \ j, k \geq 0, \ (j, k) \neq (0, 0),\]

(4.27)

so the skew matrix \(H_\alpha(A)\) defined in (4.6) vanishes and eq. (4.8) becomes

\[M_\alpha^0(x|A) = \left( \begin{array}{cc} M(x) & V_\alpha(x|A) \\ -V_\alpha(x|A)^T & 0 \end{array} \right).
\]

(4.28)

A particular case of such generalized Schur \(Q\)-functions, introduced in [10], corresponds to choosing the monic polynomials \(\{p_j(x|A)\}_{j \in \mathbb{N}}\) as

\[p_j(x|A_{\text{int}}(a)) := (x|a)^{(j)} = \prod_{i=1}^{j} (x - a_i), \ \ j \in \mathbb{N}^+, \ \ p_0(x, A_{\text{int}}(a)) := (x|a)^{(0)} = 1.
\]

(4.29)

where \(a := (a_1, a_2, \ldots)\) is an arbitrary sequence of shared roots. These were referred to as interpolation analogs of the Schur \(Q\)-functions in [10] and shown to be BKP \(\tau\)-functions in [23].

17
5 KP multipair and BKP multipoint correlators

5.1 KP \( n \)-pair correlators

When \( t \) is restricted to equal the difference between two finite (normalized) power sums

\[
t_j = \frac{1}{j} \sum_{a=1}^{n} (x_a^j - y_a^j), \quad j \in \mathbb{N}^+
\]  

in terms of two sets of \( n \) variables

\[
x := (x_1, \ldots, x_n), \quad y := (y_1, \ldots, y_n),
\]  

we denote it as

\[
t = [x, y] := [x] - [y].
\]  

For a partition \( \lambda \) and group element \( g(\vec{A}) \), define the \( n \)-pair correlation function

\[
K_{n,\lambda}(x, y | \vec{A}) := \langle 0 | \left( \prod_{a=1}^{n} \psi^\dagger(x_a^{-1}) \psi(y_a^{-1}) \right) \hat{g}(\vec{A}) | \lambda \rangle.
\]  

From the identity ([6], Chapt. 5.9)

\[
\langle 0 | \left( \prod_{a=1}^{n} \psi^\dagger(x_a^{-1}) \psi(y_a^{-1}) \right) = \left( \prod_{a=1}^{n} x_a y_a \right) \Delta(x, y) \langle 0 | \hat{\gamma} + ([x] - [y])\rangle,
\]  

where

\[
\Delta(x, y) := \det \left( \frac{1}{x_a - y_b} \right)_{1 \leq a, b \leq n} = \frac{(-1)^{n(n-1)} \Delta(x) \Delta(y)}{\prod_{1 \leq a, b \leq n} (x_a - y_b)}.
\]  

we have

\[
K_{n,\lambda}(x, y | \vec{A}) = \left( \prod_{a=1}^{n} x_a y_a \right) \Delta(x, y) \pi_{\lambda}(g(\vec{A}))( [x, y] ).
\]  

Now choose \( \vec{A} \) to be strictly upper triangular, so \( \hat{g}(\vec{A}) \) stabilizes the vacuum, as in [2.36]. Define the \( (n + r) \times (n + r) \) matrix

\[
N_{n,\lambda}^{G}(x, y | \vec{A}) := \begin{pmatrix} N_n(x, y) & W_{n,\alpha}(x | \vec{A}) \\ W_{n,\beta}^*(y | \vec{A})^T & G_{(\alpha|\beta)}(\vec{A}) \end{pmatrix},
\]  

consisting of four blocks: a left upper \( n \times n \) block \( N_n(x, y | \vec{A}) \) that is independent of \( \vec{A} \) and \( \lambda \):

\[
(N_{n,\alpha|\beta}^{G}(x, y | \vec{A}) \big)_{ab} = \langle 0 | \psi^\dagger(x_a^{-1}) \psi(y_b^{-1}) | 0 \rangle =: \frac{x_a y_b}{x_a - y_b},
\]  

18
a right lower \( r \times r \) block

\[
(G_{(a|\beta)}(\tilde{A}))_{ij} = (-1)^{\beta_i} \langle 0|\psi_{\alpha_j}(\tilde{A})\psi^{*}_{-\beta_{i-1}}(\tilde{A})|0\rangle, \quad 1 \leq i, j \leq r
\]

\[
= (-1)^{\beta_i} \sum_{k=0}^{r} P_{-k-1,a_j}(\tilde{A}) P^{*}_{\beta_{i-k}}(\tilde{A}),
\]

that is independent of \((x, y)\), and two rectangular \( n \times r \) and \( r \times n \) blocks given by the polynomials defined in eqs. (2.43), (2.44)

\[
(W_{n,\alpha}(x|\tilde{A}))_{aj} := \langle 0|\psi^{\dagger}(x^{-1})\psi_{\alpha_j}(\tilde{A})|x_0\rangle = x_a p_{\alpha_j}(x_a|\tilde{A}), \quad 1 \leq a \leq n,
\]

\[
(W^{*}_{n,\beta}(y|\tilde{A}))_{aj} = \langle 0|\psi(y^{-1})\psi^{*}_{-\beta_j}(\tilde{A})|y_0\rangle = y_a p^{*}_{\beta_j}(y_a|\tilde{A}), \quad 1 \leq j \leq r.
\]

Wick’s theorem then again implies a finite determinantal representation of \( \pi_\lambda(g(\tilde{A}))([x, y]) \) and hence of \( K_{n,\lambda}(x,y|\tilde{A}) \).

**Proposition 5.1.** The \( n \)-pair correlators are given by the determinantal formula

\[
K_{n,\lambda}(x,y|\tilde{A}) = \det \left( N^G_{n,\lambda}(x,y|\tilde{A}) \right),
\]

and therefore are rational functions, with simple poles at the points where \( x_a - y_b \) vanish,

**Remark 5.1.** Whenever two \( x_a \)'s or two \( y_a \)'s are equal, two of the rows or columns of \( \Delta(x)\Delta(y) \) coincide, and the determinant \( \Delta(x)\Delta(y) \) vanishes. Therefore, we can factor the product \( \Delta(x)\Delta(y) \) of Vandermonde determinants from the expression for \( K_{n,\lambda}(x,y|\tilde{A}) \). It follows from \( \Delta(x)\Delta(y) \) that

\[
\prod_{a=1}^{n} x_a y_a \tau_\lambda(g(\tilde{A}))([x, y]) = \frac{K_{n,\lambda}(x,y|\tilde{A})}{\Delta(x, y)}
\]

is a polynomial in the position variables \( \{x_a, y_a\}_{1 \leq a \leq n} \) of degree no greater than \( 2n + |\lambda| - r \).

**Proof.** (Proposition 5.1) Eqs. (5.4) and (2.16) imply

\[
K_{n,\lambda}(x,y|\tilde{A}) = (-1)^{\sum_{j=1}^{r} \beta_j} \langle 0|\psi^{\dagger}(x^{-1})\psi(y^{-1}) \prod_{a=1}^{n} \psi_{\alpha_j}(\tilde{A}) \psi^{\dagger}_{-\beta_j}(\tilde{A})|0\rangle
\]

\[
= \det \left( N^G_{n,\lambda}(x,y|\tilde{A}) \right),
\]

where Wick’s theorem (Appendix A eq. (A.5)) and Lemma 2.1 have been applied in the second line. \( \square \)
If we furthermore choose $\tilde{A}$ such that

$$\tilde{A}_{jk} = 0 \text{ if } j < 0, \ k \geq 0, \rightarrow P_{jk}(\tilde{A}) = 0 \text{ if } j < 0, \ k \geq 0,$$

(5.16)

which can always be done without changing the polynomials $\{p_j(x|\tilde{A})\}_{j \in \mathbb{N}}$ defined in (2.86), it follows from eq. (5.10), that

$$(G_{\lambda}(\tilde{A}))_{ij} = 0, \ \forall \ i, j \in \mathbb{N}.$$  

(5.17)

and hence

$$N_{n,\lambda}^0(x, y|\tilde{A}) = \left( \begin{array}{cc} N_n(x, y) & W_{n,\alpha}(x|\tilde{A}) \\ W_{n,\beta}(y|\tilde{A})^T & 0 \end{array} \right).$$

(5.18)

5.2 BKP $2n$-point correlators

Let $x = (x_1, \ldots, x_{2n})$ with all $x_a$’s distinct, and define the $2n$-point BKP correlation function

$$K_{2n,\alpha}^B(x) := \langle 0 | \phi^+(-x_{2n}^{-1}) \cdots \phi^+(-x_1^{-1}) \hat{h}^+(A)|\alpha^+ \rangle,$$

(5.19)

Eqs. (4.11) and (2.90) imply that $K_{2n,\alpha}^B(x)$ is related to $Q_{\alpha}([x]_B|A)$ by

$$K_{2n,\alpha}^B(x) = 2^{-2n-2m} \text{Pf}(M(x))Q_{\alpha}(x|A) = 2^{-2n-2m} \text{Pf}(M^G_{\alpha}(x|A)),$$

(5.20)

where $Q_{\alpha}([x]_B|A)$ is given by the generalized Nimmo formula (4.7) and $M^G_{\alpha}(x|A)$ is defined by (4.8) or, if the additional conditions (4.23) - (4.25) are fulfilled, by (4.28).

6 Examples

A large number of examples of lattices of polynomial KP Schur functions, labelled by partitions $\lambda$, that can be expressed in terms of systems of monic polynomials $\{p_j(x|\tilde{A})\}_{j \in \mathbb{N}}$ via formula (3.5) are detailed in [7, 17, 25]. These include: several of the various types of generalized Schur functions studied in [17], the factorial Schur functions [2, 3], and orthogonal and symplectic group characters [7, 25].

Two cases of particular interest are the generalized Laguerre multivariate polynomials [15, 20], given by the bialternant formula (3.5), with the system of monic polynomials $\{p_j(x|\tilde{A})\}_{j \in \mathbb{N}^+}$ chosen to be the associated Laguerre polynomials, and a 2-parameter generalization of these [9, 20], which we refer to as double Laguerre polynomials.

Example 6.1. Consider the special choice (see [9, 22])

$$\tilde{A}^r(p)_{jk} = p_{k-j}r_{j+1} \cdots r_k, \ k > j,$$

(6.1)

parametrized by two sets of infinite numbers of parameters $r := \{r_j\}_{j \in \mathbb{Z}}$ and $p = (p_1, p_2, p_3, \ldots)$. This gives rise to the following polynomial expression for the KP $\tau$-function defined in eqs. (2.7), (2.17) (see Example 4.4 in [9]),

$$s_\lambda(t|\tilde{A}^r(p)) = s_\lambda(t) + \sum_{\rho < \lambda} r_{\lambda/\rho} s_{\lambda/\rho}(p) s_\rho(t),$$

(6.2)
where \( s_{\lambda/\rho}(p) \) is the skew Schur function corresponding to the skew partition \( \lambda/\rho \), and

\[
r_{\lambda/\rho} := \prod_{i=1}^{\ell(\lambda)} \prod_{j=\rho_i+1}^{\lambda_i} r_{j-i} = \frac{r_{\lambda}}{r_{\rho}}
\]

(6.3)

Evaluating at \( t = [x] \), the \( \tau \)-function \( s_{\lambda}([x]|\bar{A}^r(p)) \) is given by the bialternant formula (3.5), in which the system of monic polynomials (2.39), (2.40) is:

\[
p_j(x|\bar{A}^r(p)) := x^{-1} \langle 0|\psi^\dagger(x^{-1})\hat{g}(\bar{A}^r(p))\psi_j|0 \rangle = x^j + \sum_{k=1}^j r_j \cdots r_{j-k+1} s_{(k)}(p) x^{j-k}
\]

(6.4)

\[
p_j^*(y|\bar{A}^r(p)) := y^{-1} \langle 0|\psi(y^{-1})\hat{g}(\bar{A}^r(p))\psi_{j-1}^\dagger|0 \rangle = y^j + \sum_{k=1}^j r_{j-k} \cdots r_{j-k-1} s_{(k)}(-p) y^{j-k}.
\]

(6.5)

We also have

\[
\langle 0|\hat{g}(\bar{A}^r(p))\psi_{j-1}^\dagger|0 \rangle = r_j r_{j-1} \cdots r_1 \langle 0|\hat{g}_+(p)\psi_j|0 \rangle = (-1)^j r_j r_{j-1} \cdots r_1 s_{(ij)}(p).
\]

(6.6)

Therefore, choosing \( t = [x] - [y] \), as in (3.3), the resulting \( n \)-pair correlation function \( K_{n,\lambda}(x,y|\bar{A}^r(p)) \) is given by (5.13), where the system of polynomials defining the rectangular matrices \( W_{n,\alpha}(x|\bar{A}^r(p)) \) and \( W_{n,\beta}^*(y|\bar{A}^r(p)) \) appearing in the \((n+m) \times (n+m)\) matrix \( N_{n,\lambda}(x,y|\bar{A}^r(p)) \) defined by (5.8) is (6.4), (6.5) and the matrix elements of \( G_{\alpha,\beta}(\bar{A}^r(p)) \) are

\[
\left( G_{\alpha,\beta}(\bar{A}^r(p)) \right)_{ij} := (-1)^{\beta_i} \langle 0|\hat{g}(\bar{A}^r(p))\psi_{\alpha_j}^\dagger|0 \rangle = r_{-\beta_1} \cdots r_{-\beta_j} s_{(\alpha_j|\beta_i)}(p).
\]

(6.7)

**Remark 6.1.** The multivariate Laguerre symmetric polynomials [15] are the special case of (6.2) obtained by choosing

\[
p = t_0 := (1,0,0,\ldots)
\]

(6.8)

\[
r_j = r_j(z) := -j(z+j).
\]

(6.9)

Then (6.4), (6.5) reduce to the usual associated Laguerre polynomials (normalized to be monic)

\[
p_j(x|\bar{A}^{r(z)}(t_0)) = \sum_{k=0}^j (-1)^{j-k} \binom{z+j}{z+k} \binom{j}{k} x^k
\]

\[
= (-1)^j j! L_j^{(z)}(x),
\]

(6.10)

\[
p_j^*(y|\bar{A}^{r(z)}(t_0)) = \sum_{k=0}^j (-1)^{j-k} \binom{-z+j}{-z+k} \binom{j}{k} y^k
\]

\[
= (-1)^j j! L_j^{(-z)}(x),
\]

(6.11)
\[
= p_j^*(y|\tilde{A}^{r(-z)}(t_0)).
\]  

(6.11)

For this case, since \( r_0 = 0 \), it follows from (6.7) that the matrix \( G_{(\alpha|\beta)}(\tilde{A}^{r(z)}(p)) \) vanishes.

The multivariate double Laguerre symmetric polynomial \([20]\) (see also Remark 4.2 in \([9]\)) is obtained from (6.2) by choosing

\[
p = t_0 := (1, 0, 0, \ldots)
\]

(6.12)

\[
r_j = r_j(z, z') := -(z + j)(z' + j),
\]

(6.13)

so that

\[
r_j(z) = r_j(z, 0).
\]

(6.14)

Then (6.4), (6.5) reduce to

\[
p_j(x|\tilde{A}^{r(z, z')}(t_0)) = \sum_{k=0}^{j} (-1)^{j-k} \binom{z + j}{z + k} \binom{z' + j}{z' + k} x^k,
\]

(6.15)

\[
p_j^*(y|\tilde{A}^{r(z, z')}(t_0)) = \sum_{k=0}^{j} (-1)^{j-k} \binom{-z + j}{-z + k} \binom{-z' + j}{-z' + k} y^k
\]

\[
= p_j(x|\tilde{A}^{r(-z, -z')}(t_0)).
\]

(6.16)

We also have the generalized Schur \( Q \)-functions \( Q_\alpha(x|A^r(p_B)) \), with (see \([9, 21]\))

\[
A^r(p_B)_{jk} = \begin{cases} 
(-1)^k p_{j-k} r_{j+1} \cdots r_{-k}, & j + k < 0 \text{ and odd} \\
0 & \text{if } j + k \text{ is even or } j + k \geq 0
\end{cases}
\]

(6.17)

given by the generalized Nimmo formula (4.7), with the rectangular matrix \( V_\alpha(x|A^r(p)) \) defined in (4.9) given by the polynomials

\[
p_j(x|(A^r(p_B))) := \langle 0|\phi_+^\pm(-x^{-1})\phi_+^\pm(A^r(p_B))|0 \rangle
\]

\[
= x^j + \sum_{k=1}^{i} r_j \cdots r_{j-k+1} x^{j-k} q_k(\frac{1}{2}p_B).
\]

(6.18)

which coincide with the \( p_j(x|\tilde{A}^r(p)) \)'s of eq. (6.5) when \( p = (p_1, 0, p_3, 0, \ldots, 0, p_{2j-1}, \ldots) \), and the matrix elements of \( H_\alpha(A^r(p_B)) \) are

\[
(H_\alpha(A^r(p_B)))_{ij} := \langle 0|\phi_{\alpha_i}(\tilde{A}^r(p_B))\phi_{\alpha_j}(\tilde{A}^r(p_B))|0 \rangle
\]

\[
= \frac{1}{2} r_1 \cdots r_{\alpha_i} r_1 \cdots r_{\alpha_j} Q_{(\alpha_i, \alpha_j)}(\frac{1}{2}p_B).
\]

(6.19)
Similarly to (6.2), this can be expressed as a sum over ordinary (scaled) Schur $Q$-functions $Q_{\beta}$, with $\beta \leq \alpha$

$$Q_{\alpha}(\{x\}|A^f(B)) = Q_{\alpha}([x]|B) + \sum_{\beta<\alpha} r_{\alpha/\beta}^B Q_{\alpha/\beta}(B) Q_{\beta}([x]|B),$$

(6.20)

where

$$Q_{\alpha/\beta}(B) = (\alpha|\gamma^B([p]|B)|\hat{\beta})$$

(6.21)

is the (scaled) skew Schur $Q$-function corresponding to the skew partition $\alpha/\beta$ and

$$r_{\alpha/\beta}^B := \prod_{i=1}^{2m} \prod_{j=\beta+1}^{\mu_i} r_j = \frac{r_{\alpha}}{r_{\beta}}$$

(6.22)

is the content product over nodes of the shifted skew Young diagram $[10]$ of $\alpha/\beta$.

**Example 6.2.** For KP single pair correlation functions with $N_{n,\lambda}^0(\{x,y|\tilde{A}_{fac}\})$ of the form (5.18), for $n = 1$, if the Frobenius rank $r$ of the partition function $\lambda$ is greater than 1 the determinant in (5.13) vanishes, so the only nonvanishing pair correlators are the trivial one, which is

$$K_{2,\theta}(x,y)|\tilde{A}) = \frac{xy}{y-x}$$

(6.23)

and the one for a hook partition, which is

$$K_{2,(\alpha_1|\beta_1)}(x,y)|\tilde{A}) = -p_{\alpha_1}(x|\tilde{A})p_{\beta_1}(y|\tilde{A}).$$

(6.24)

If we do not impose the null conditions (5.16) on $\tilde{A}$, so the matrix $G_{(\alpha|\beta)}(\tilde{A})$ is, in general, nonvanishing, we have

$$K_{2,(\alpha_1|\beta_1)}(x,y)|\tilde{A}) = \gamma \frac{xy}{y-x} - p_{\alpha_1}(x|\tilde{A})p_{\beta_1}(y|\tilde{A}).$$

(6.25)

where

$$\gamma = (G_{(\alpha_1,\beta_1)}(\tilde{A}))_{11}$$

(6.26)

**Example 6.3.** For BKP 2-point correlation functions $K_{2,\alpha}^B(\{x_1,x_2|A\})$ with $M^0_{\alpha}(\{x_1,x_2|A\})$ of the form (4.28), if the cardinality $r = 2m$ is greater than 2, $M^0_{\alpha}(\{x_1,x_2|A\})$, which can have rank at most 4, has vanishing Pfaffian. Therefore, the only nonvanishing 2-point correlations functions are the one for the trivial partition

$$K_{2,\theta}(x_1,x_2)|\tilde{A}) = \frac{x_1 - x_2}{x_1 + x_2}$$

(6.27)

and the one where $\alpha$ has two parts

$$K_{2,(\alpha_1,\alpha_2)}^B(x_1,x_2)|\tilde{A}) = p_{\alpha_1}(x_2)p_{\alpha_2}(x_1) - p_{\alpha_1}(x_1)p_{\alpha_2}(x_2).$$

(6.28)

If we do not impose the null conditions (4.23)-(4.25) on $A$, so the matrix $H_{\alpha}(\tilde{A})$ is, in general, nonvanishing, we have

$$K_{2,(\alpha_1,\alpha_2)}^B(x_1,x_2)|\tilde{A}) = \sigma \frac{x_1 - x_2}{x_1 + x_2} + p_{\alpha_1}(x_2)p_{\alpha_2}(x_1) - p_{\alpha_1}(x_1)p_{\alpha_2}(x_2),$$

(6.29)

where

$$\sigma = (H_{(\alpha_1,\alpha_2)}(\tilde{A}))_{12}.$$
A Appendix. Wick’s theorem

The following summarizes the implication of Wick’s theorem for fermionic VEV’s in a form that is used repeatedly in this work (see, e.g., [6], Chapt. 5.1).

**Theorem A.1** (Wick’s theorem). The vacuum expectation value of the product of an even number of linear elements \( \{w_i\}_{1 \leq i \leq 2n} \) of the fermionic Clifford algebra is

\[
\langle 0 | w_1 w_2 \cdots w_{2n} | 0 \rangle = \text{Pf} \left( W_{ij} \right)_{1 \leq i,j \leq 2n} \tag{A.1}
\]

where \( \{W_{ij}\}_{1 \leq i,j \leq 2n} \) are elements of the skew \( 2n \times 2n \) matrix defined by

\[
W_{ij} = \begin{cases} 
\langle 0 | w_i w_j | 0 \rangle & \text{if } i < j \\
0 & \text{if } i = j \\
-\langle 0 | w_i w_j | 0 \rangle & \text{if } i > j
\end{cases} \tag{A.2}
\]

whereas the VEV of the product of an odd number vanishes

\[
\langle 0 | w_1 w_2 \cdots w_{2n+1} | 0 \rangle = 0. \tag{A.3}
\]

In particular, if half the \( w_i \)'s consist of creation operators \( \{u_i\}_{i=1,\ldots,n} \) and the other half annihilation operators \( \{v_i\}^\dagger_{i=1,\ldots,n} \), so that

\[
\langle 0 | u_i u_j | 0 \rangle = 0, \quad \langle 0 | v_i^\dagger v_j^\dagger | 0 \rangle = 0, \quad 1 \leq i, j \leq n, \tag{A.4}
\]

then (A.1) reduces to

\[
\langle 0 | u_1 v_1^\dagger \cdots u_n v_n^\dagger | 0 \rangle = \det \left( \langle 0 | u_i v_j^\dagger | 0 \rangle \right)_{1 \leq i,j \leq n}. \tag{A.5}
\]

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**References**

[1] F. Balogh, T. Fonseca and J. Harnad, “Finite dimensional Kadomtsev-Petviashvili \( \tau \)-functions. I.” *J. Math. Phys.* 55, 083517 (2014) [1]

[2] L. Biedenharn and J. Louck, “A new class of symmetric polynomials defined in terms of tableaux”, *Adv. Appl. Math.*, 10, 396-438 (1989). [13] [20]

[3] L. Biedenharn and J. Louck, “Inhomogeneous basis set of symmetric polynomials defined by tableaux”, *Proc. Nat. Acad. Sci. U.S.A.*, bf 87, 1441-1445 (1990). [13] [20]
[4] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations IV. A new hierarchy of soliton equations of KP type”, *Physica* 4D, 343-365 (1982).

[5] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations”, In: *Nonlinear integrable systems - classical theory and quantum theory*, 39-120. World Scientific (Singapore), eds. M. Jimbo and T. Miwa (1983).

[6] J. Harnad and F. Balogh, “Tau functions and their applications”, Chapt. 5 and 7 and Appendix D, *Monographs on Mathematical Physics*, Cambridge University Press (in press, 2021).

[7] J. Harnad and E. Lee, “Symmetric polynomials, generalized Jacobi-Trudi identities and \( \tau \)-functions”, *J. Math. Phys.* 59, 091411 (2018).

[8] J. Harnad and A. Yu. Orlov, “Bilinear expansions of Schur functions in Schur Q-functions: a fermionic approach”, *Proc. Amer. Math. Soc.* (in press (2021)), arXiv:2008.13734 (2020).

[9] J. Harnad and A. Yu. Orlov, “Bilinear expansions of lattices of KP \( \tau \)-functions in BKP \( \tau \)-functions: a fermionic approach”, *J. Math. Phys.* 62, 013508 (2021).

[10] V.N. Ivanov, “Interpolation analogues of Schur Q-functions”, *Math. Sci.* 131, 5495-5507, (2005).

[11] M. Jimbo and T. Miwa, “Solitons and infinite-dimensional Lie algebras”, *Publ. Res. Inst. Math. Sci.*, 19 943-1001 (1983).

[12] V. G. Kac and J. van de Leur, “Equivalence of formulations of the MKP hierarchy and its polynomial tau-functions”, *Jap. J. Math.* 13, 235-271 (2018).

[13] V. G. Kac and J. van de Leur, “Polynomial tau-functions of BKP and DKP hierarchies”, *J. Math. Phys.* 60, 071702 (2019).

[14] V. G. Kac, N. Rozhkovskaya and J. van de Leur, “Polynomial Tau-functions of the KP, BKP, and the s-Component KP Hierarchies”, arXiv:2005.02665.

[15] M. Lasalle, “Polyphonmes de Laguerre gérénalisés”, *C. R. Acad. Sci. Paris, Ser. I*, 312, 725-728 (1991).

[16] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, (1995).

[17] I. G. Macdonald, “Schur Functions: Theme and Variations”, in *Actes 28e Séminaire Lotharingien* (Publication I.R.M.A., Strasbourg, pp. 5-39. (1992).

[18] T. Miwa, M. Jimbo and E. Date, “Solitons. Differential equations, symmetries and infinite dimensional algebras” *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, U.K. (2000).
[19] J. J. C. Nimmo, “Hall-Littlewood symmetric functions and the BKP equation”, *J. Phys. A*, 23, 751-60 (1990).

[20] G. Olshanski, “Laguerre and Meixner Orthogonal Bases in the Algebra of Symmetric Functions” *Int. Math. Res. Notices*, 2012, No. 16, 3615-3679 (2012).

[21] A. Yu. Orlov, “Hypergeometric Functions Related to Schur Q-Polynomials and the BKP Equation”. *Theor. Math. Phys.* 137 (2), 1574-1589 (2003).

[22] A. Yu. Orlov, D. M. Scherbin, “Multivariate hypergeometric functions as τ-functions of Toda lattice and Kadomtsev–Petviashvili equation” *Physica D: Nonlinear Phenomena* 152, 51-65 (2001).

[23] N. Rozhkovskaya, “Multiparameter Schur Q-Functions are solutions of the BKP hierarchy”, *SIGMA* 15, 065 (2019).

[24] M. Sato. “Soliton equations as dynamical systems on infinite dimensional Grassmann manifold”, *Kokyuroku, RIMS* 30-46, (1981).

[25] A. N. Sergeev and A. P. Veselov, “Jacobi-Trudi formula for generalized Schur polynomials”, *Moscow Math. J.* 14, 161-168 (2014).

[26] J. van de Leur and A. Yu. Orlov, “Pfaffian and Determinantal Tau Functions”, *Lett. Math. Phys.* 105 1499- 1531 (2015).

[27] Y. You, “Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups”, in: *Infinite-Dimensional Lie Algebras and Groups, Adv. Ser. Math. Phys.* 7, World Sci. Publ., Teaneck, NJ (1989).