Shortcuts in a Dynamical Universe and the Horizon Problem

Elcio Abdalla and Adenauer Girardi Casali

Instituto de Física, Universidade de São Paulo
C.P.66.318, CEP 05315-970, São Paulo, Brazil.

Abstract

We consider the dynamics of a FRW brane in a purely AdS background, from the point of view of the bulk, and explicitly construct the geodesical behaviour of gravitational signs leaving and subsequently returning to the brane. In comparison with photons following a geodesic inside the brane, we verify that shortcuts exist, though they are extremely small for today’s Universe. However, we show that at times just before nucleosynthesis, if high redshifts were available, the effect could be sufficiently large to solve the horizon problem. Assuming an inflationary epoch in the brane evolution, we argue that the influence of those signs through the extra dimension in the causal structure cannot be neglected. This effect could be relevant for probing the extra dimension in inflationary scenarios.

1 Introduction

As it has been recently argued in [1], although the standard model of particle physics is the uncontested theory of all interactions down to distances of $10^{-17}$m, there are good reasons to believe that new physics could be soon arising at the experimental level. On the other hand, on the purely theoretical side, string theory and further developments connected to it provide a background with immense appeal in order to solve long-standing problems of theoretical high energy physics. It is by now a widespread idea, from a general theoretical setup, that the so-called M-theory [2] is a reasonable description of our Universe: in the field theory limit, it is described by a solution of the (eventually 11-dimensional) Einstein equations with a cosmological constant, by means of a four dimensional membrane. In this picture
only gravity survives in the higher dimensions, while the remaining matter 
and gauge interactions are typically four dimensional.

If the picture above is the one physically realized in nature, a very large 
amount of new physics emerges. In particular, when membranes are solu-
tions of Einstein’s equations and matter fields reside inside the brane, the 
gravitational fields have to obey the Israel conditions at the sides of the 
brane. Thus, there is a possibility that gravitational fields propagating out 
of the brane speed up, reaching farther distances as compared to light prop-
agating inside the brane, a scenario that for a resident of the brane (such as 
ourselves) implies shortcuts.

It is our aim in this work to further develop these ideas in the case of 
a FRW brane Universe. The subject was developed until now from the 
point of view of the brane, where all time dependence is embedded in 
the bulk metric written in gaussian coordinates. The price payed for the 
knowledge of the position of the brane in the bulk is the complicated form 
of the bulk metric and, consequently, the complicated behaviour of geodesics 
in the bulk. However, if we treat the problem from the point of view of the 
bulk, where the brane evolves in a non-trivial way in a static AdS background, 
we can construct explicitly the causal structure of null geodesics leaving and 
subsequently returning to the brane. As it turns out, shortcuts are common, 
although harmless at the present days (the delay is vanishingly small), but 
could be large in the era before nucleosynthesis if high redshifts were available.

Moreover, one of the main goals of string theory nowadays is to prove itself 
able of coping with experimental evidences. Branes have been shown to be 
useful tools to understand the physics of strings and M-theory. As it has 
also recently been pointed out, brane Universes such as the one described 
above could imply the existence of relics of the extra dimensions in the cosmic 
microwave background. Unfortunately, recent developments with inflation 
guided by a scalar field in the brane indicate that the consistency equation is 
preserved. In this work however, we show that if inflation took part on 
the brane, the causal structure is definitely changed by those gravitational 
shortcuts, possibly leading to a non-usual period of causal evolution of scales. 
This could be responsible for distinct predictions in the cosmic microwave 
background structure for inflationary models.

The paper is organized as follows. Chapter two provides a short revision 
of the general setup adopted in this work: the view of the bulk. In chapter 
three we discuss how the brane evolves in the background and show the 
effects of the propagation of gravity trought the extra dimension. Finally, in
chapter 4 we explicitly construct the geodesical behaviour of the shortcuts found in the earlier fase of the evolution of the universe. A brief conclusion and discussion is also presented.

2 The General Setup

We consider a scenario where the bulk is a purely Anti-de-Sitter space-time of the form \[14\],

\[ ds^2 = h(a)dt^2 - \frac{da^2}{h(a)} - a^2 d\Sigma^2, \]

with \( h(a) = k + \frac{a^2}{l^2} \), \( l \sim 0.1 mm \) is the Randall-Sundrum length scale \[15\] and \( d\Sigma^2 \) represents the metric of the three dimensional spatial sections with \( k = 0, \pm 1 \),

\[ d\Sigma^2 = \frac{dr^2}{1 - k r^2} + r^2[ d\theta^2 + \sin^2(\theta) d\phi^2 ]. \]

The brane is localized at \( a_b(\tau) \), where \( \tau \) is the proper time on the brane. The unit vector normal to the brane is defined as (overdot and prime superscript denote differentiation with respect to \( \tau \) and \( a \) respectively)

\[ n = \dot{a}_b(\tau) dt - \dot{t}(\tau) da. \quad (1) \]

The normalization of \( n \) implies the relation between the bulk time \( t \) and the brane time \( \tau \),

\[ h(a_b)\dot{t}^2 - \dot{a}_b^2 h^{-1}(a_b) = 1 \quad (2) \]

and also the usual FRW expression for the distance in the brane ,

\[ ds^2 = d\tau^2 - a_b^2 d\Sigma^2. \]

Aiming at the Israel conditions in the brane \[3\] we compute the second fundamental form

\[ K_{ij} = e_i^\alpha e_j^\nu \nabla_\alpha n_\nu, \]

\[ K_{\tau \tau} = -\frac{a_b h}{(1 - k r^2)^2} \dot{\tau}, \quad (3) \]

\[ K_{\theta \theta} = -a_b r^2 \dot{\theta}, \quad (4) \]

\[ K_{\phi \phi} = -a_b r^2 \sin^2(\theta) \dot{\phi}. \quad (5) \]
and

\[ K_{\tau\tau} = \frac{1}{h't} \left( \dot{a}_b + \frac{h'}{2} \right) \]  

(7)

With these results, using the Israel conditions for a \( Z_2 \) symmetric configuration \[3\],

\[ K_{ij} = \frac{1}{2} \kappa^2 (S_{ij} - \frac{1}{3} h_{ij} S) \]  

(8)

we relate the discontinuity in the second fundamental form through the brane and the energy momentum tensor in the brane, \( S_{ij} \). For an isotropic distribution of matter, as given by

\[ S_{ij} = \epsilon_T u_i u_j - p_T (h_{ij} - u_i u_j) \]  

(9)

the following relations hold \[16\], \[17\], \[18\]

\[ \frac{d\epsilon_T}{d\tau} = -3 \frac{\dot{a}_b}{a_b} (\epsilon_T + p_T) \]

and

\[ \frac{\dot{a}_b^2}{a_b^2} + \frac{h}{a_b^2} = \frac{\kappa^4 (5) \epsilon_T^2}{36} \]  

(10)

Following \[19\] and \[20\], we introduce an intrinsic non-dynamical energy density \( \epsilon_0 \) defined by means of \( \epsilon_T = \epsilon_0 + \epsilon \), \( p_T = -\epsilon_0 + p \) where \( \epsilon \) and \( p \) corresponds to the brane matter. Thus the junction equations imply the usual energy conservation in the brane,

\[ \frac{d\epsilon}{d\tau} = -3 \frac{\dot{a}_b}{a_b} (\epsilon + p) \]  

(11)

and the modified Friedmann equation \[21\],

\[ H^2 = \left( \frac{\dot{a}_b}{a_b} \right)^2 = \frac{\Lambda_4}{3} + \frac{1}{M_{Pl}^2} \left( \frac{\epsilon}{3} + \frac{\epsilon^2}{6\epsilon_0} \right) - \frac{k}{a_b^2} \]  

(12)

with the hierarchy

\[ M_{Pl}^{-2} = \frac{\kappa^4 (5) \epsilon_0}{6} \]  

(13)

and the cosmological constant in the brane

\[ \frac{\Lambda_4}{3} = \left( \frac{\kappa^4 (5) \epsilon_0^2}{36} - \frac{1}{l^2} \right) \]  

(14)
The present density of the Universe is
\[ \epsilon(0) = \Omega_0 \epsilon_c = \Omega_0 3 M_{PL}^2 H_0^2, \]
where \(\Omega_0\) is the ratio of the density and the critical density of the Universe. Aiming at the energy conservation, the Friedmann equation can be written as
\[ H^2 = \frac{\Lambda_4}{3} + \Omega_0 H_0^2 a_b^2 \left(1 + \Omega_0 \frac{L_c^2}{a_b^2} \right) - \frac{k}{a_b^2}, \tag{15} \]
where \(L_c^2 = a_{b0}^2 H_0^2\).

We thus verify that there exist three phases in the evolution of the Universe. When \(a_b >>> L_c\) the linear term in the energy density prevails in the Friedmann equation, leading to the standard cosmology. Because \(H_0 l \sim 10^{-29}\) this happens in a redshift of \(a_{b0}/a_b \sim 10^{15}\), much earlier than the nucleosynthesis. For \(a_b << L_c\) the Universe expands at a slower pace as compared to the standard model, \(a_b \propto \tau^{1/q}\); and the quadratic term is the prevailing one. In an intermediate era where \(a_b \sim L_c\), both phases coexist.

We see that all the cosmological information is already known: the position of the brane in the extra dimension, \(a_b(\tau)\), is just the scale factor of the FRW metric and the junction conditions imply that the cosmological evolution of the brane is obtained by the usual energy conservation (11) and the modified Friedmann equations (12). However, in this work, we are particularly interested in the evolution of the brane with respect to the bulk. This relation can be obtained using (2) and transforming from the time of the brane to the time of the bulk. Thus the position of the brane can also be treated as a function of the bulk proper time satisfying the equation
\[ \frac{da_b}{dt} = \frac{da_b}{d\tau} \frac{\tau}{dt} = \dot{a}_b(\tau) - \frac{h(a_b)}{\sqrt{h(a_b) + \dot{a}_b(\tau)^2}}. \tag{16} \]

In the next section we solve the Friedmann equation (15) in order to obtain the evolution of the brane inside the bulk, namely (16).

3 The Brane evolving in the Bulk

The main reason to study the evolution of the brane from the point of view of the bulk is to simplify the analysis of gravitational signs leaving and subsequently returning to the brane. In fact in the static AdS background, the
equation for a null geodesic in the bulk, \( a = a(t) \), is particularly simplified \[4\],

\[
\ddot{a}(t) + \frac{\dot{a}^2(t)}{a^2} \left(1 - \frac{3h'(a)a}{2h(a)}\right) + \frac{h(a)}{a} \left(\frac{h'(a)}{2} - \frac{h}{a}\right) = 0 . \tag{17}
\]

On the other hand, the evolution of the brane in the bulk, \( a = a_b(t) \), at early times, \( a_b << L_c \), is dictated by

\[
\frac{da_b}{dt} = \frac{h(a_b)}{\sqrt{1 + \frac{h(a_b)}{a_b^2}}} = h(a_b) \left(1 + \frac{k + \frac{a_b^2}{l^2}}{\Omega_0^2 \frac{L_c^2}{4(1+l^2\Lambda_4)} L_c^{2q}}\right)^{-1/2} ,
\]

where we used the fact that the quadratic term in the energy prevails. Thus,

\[
\frac{da_b}{dt} = h(a_b) \left(1 + \frac{4(1 + l^2\Lambda_4) a_b^{2q}}{\Omega_0^2 \frac{L_c^2}{4(1+l^2\Lambda_4)} L_c^{2q}} (1 + kl^2/(a_b^2))\right)^{-1/2} ,
\]

and, for \( a_b << L_c \), since the observed cosmological constant is at most \( \Lambda_4 \sim H_0^2 \), \( \Lambda_4 l^2 << 1 \) we have

\[
\frac{da_b}{dt} \approx h(a_b) \left(1 - \frac{2(1 + l^2\Lambda_4) a_b^{2q}}{\Omega_0^2 \frac{L_c^2}{4(1+l^2\Lambda_4)} L_c^{2q}} (1 + kl^2/(a_b^2))\right) . \tag{18}
\]

Substituting the result for the evolution of the brane, \( \dot{a}(t) = h(a) \), in the geodesic equation \[17\], we verify that it is satisfied. Therefore, the trajectory of the brane differs from the trajectory of the null geodesic by a term of the order \( (\frac{a_b}{L_c})^{2q} \).

This means that for \( a_b << L_c \), the trajectory of the brane in the bulk is governed by

\[
\frac{da_b(t)}{dt} = k + \frac{a_b^2}{l^2} . \tag{19}
\]

Thus, if \( k = 0 \),

\[
a_b(t) = \frac{a_b(0)l^2}{l^2 + a_b(0)t} \tag{20}
\]

and, if \( k = -1 \),

\[
a_b(t) = \frac{2l(l + a_b(0))}{e^{2t/l}(l - a_b(0)) + l + a_b(0)} - l . \tag{21}
\]
In this last situation, if the Universe begins under the Randall-Sundrum scale, \( a(0) < l \), it will recollapse to the singularity in a finite time,

\[
t = \frac{l}{2} \ln\left(\frac{l + a_b(0)}{l - a_b(0)}\right).
\]

Indeed, there is an event horizon when \( a = l \) if \( k = -1 \).

In the case of an elliptic Universe,

\[
a_b(t) = l \tan\left(\frac{t}{l} + \tan^{-1}\left(\frac{a_b(0)}{l}\right)\right). \tag{22}
\]

Starting at the singularity \( t_0 = \tau_0 = a_b(t_0) = 0 \), we have

\[
a_b(t) = l \tan\left(\frac{t}{l}\right). \tag{23}
\]

Note that the evolution of the brane in the bulk is linear near the initial singularity \( a(t) \sim t \) for \( t << l \), diverging at the critical time \( t_c = \frac{\pi}{2} l \). In fact, the behaviour of all solutions is similar near the critical time

\[
t_c &= \frac{l^2}{a_b(0)}; \\
t_c &= \frac{l}{2} \ln\left(\frac{l + a_b(0)}{a_b(0) - l}\right); \\
t_c &= \frac{\pi}{2} l - l \tan^{-1}\left(\frac{a_b(0)}{l}\right) \tag{24}
\]

for \( k = 0, -1, +1 \) respectively.

As we approach the critical time \( a_b(t) \) increases quickly. When \( a_b(t) \sim L_c \) equation (19) is no longer valid, and the trajectory of the brane is no longer a geodesic. Thus, for a very short period, from the point of view of the bulk the brane undergoes a phase transition. In figure 4 this behaviour is shown in terms of the numerical solution of eq. (16) for a radiation dominated elliptic Universe.

Before the critical time, from the point of view of the bulk, there is no time left for the remaining graviton geodesics to reach the brane. This is in fact the kind of behaviour that we find in numerical studies of the complete equation for the evolution of the brane and null geodesics as in the example of figure 2.

On the other hand, for later times the evolution of the brane is softer, and shortcuts appear, as exhibited in fig. 3.
Figure 1: Evolution of the brane in the bulk at the intermediate epoch \( a_b \sim 0.01L_c \) until \( a_b \sim 200L_c \), radiation dominated era (RDE), \( \Omega_0 = 2 \).

### 3.1 The Effect of Shortcuts in Late Times Universes

The shortcuts just found for late times Universes could be used to probe the extra-dimensionality by the apparent violation of causality in the brane. Thus, suppose that in \( \tau = t = 0 \) an object could emit electromagnetic and gravitational waves and that we could be able to detect both signs in times \( \tau'_t \) and \( \tau'_g \). We compute now the order of magnitude of the advance in time of the graviton. Since the signals cover the same distance in the brane, the last integral reads

\[
\int_{0}^{\tau'_g} \frac{d\tau_g}{a_b(\tau_g)} = \int_{0}^{\tau'_g} \frac{dt_g}{a(t_g)} \sqrt{h(a) - \frac{\dot{a}(t_g)^2}{h(a)}} .
\]  

(25)

Here, \( a \) denotes the coordinate defining the geodesic in the bulk, and differs from the coordinate of the brane \( a_b \). In terms of the dimensionless parameters \( y \) and \( x \), \( a = Ly \) and \( t = Tx \) the last integral reads

\[
\int_{0}^{\tau'_g} \frac{dt_g}{a(t_g)} \sqrt{h(a) - \frac{\dot{a}(t)^2}{h(a)}} = \int_{0}^{\tau'_g} \frac{dt_g}{l} \sqrt{1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2 y^2} \frac{\dot{y}(x)^2}{T^2} .}
\]  

(26)
Using the relation between the time in the brane and that of the bulk we convert the above expressions in terms of the interval of time of the observer in the brane,

\[
\int_{\tau_g}^{\tau_g'} \frac{d\tau_g}{a(b(\tau_g))} = \int_{\tau_g}^{\tau_g'} \frac{d\tau_g}{l} \frac{L_\gamma}{h(a_b)} \sqrt{1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2 y^2} + y^4 L^2 T^2} \frac{\dot{\gamma}(x)^2}{1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2 y^2} + y^4 L^2 T^2}.
\]

The Friedmann equation implies

\[
\int_{\tau_g}^{\tau_g'} \frac{d\tau_g}{l} \frac{1}{h(a_b)} \sqrt{\frac{y_b^2 L^2}{l^2} + \Lambda_4 L^2 y_b^2 + \frac{\Omega_0}{y_b^{q-2}} \frac{L_c^q}{l^2 L^q-2} \frac{\dot{\gamma}(x)^2}{1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2 y^2} + y^4 L^2 T^2}.
\]

The cycle.

\[
= \int_{\tau_g}^{\tau_g'} \frac{d\tau_g}{l} \frac{y_b L}{h(a_b)} \frac{1}{l} \sqrt{1 + l^2 \Lambda_4 + \frac{\Omega_0}{y_b^{q-2}} \frac{L_c^q}{L^q y_b} \frac{\dot{\gamma}(x)^2}{1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2 y^2} + y^4 L^2 T^2}.
\]

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Figure 3: The trajectory of the brane in the bulk for $\Omega_0 = 2$, matter dominated era (MDE) with $k = 1$. Also null geodesics starting in the brane with various initial velocities.

\[
\int^\tau_g \frac{d\tau_g}{a_b(\tau_g)} \left[1 + \frac{l^2}{L^2 y_b^2} \right] \sqrt{1 + l^2 \Lambda_4 + \frac{\Omega_0 L_c^q}{y_b^q L^q}} \left[1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2} \frac{\dot{y}(x)^2}{y^4 L^2} \right].
\]

Therefore, if $L >> L_c >> l$ we obtain, at second order in $L_c/L$ and $l/L$,

\[
\int_0^{\tau_g} \frac{d\tau_g}{a_b(\tau_g)} \left[1 - \frac{l^2}{L^2 y_b^2} \right] \left[1 + \frac{1}{2} l^2 \Lambda_4 + \frac{1}{2} \frac{\Omega_0 L_c^q}{y_b^q L^q} \right] \left[1 + \frac{1}{2} \frac{l^2}{L^2 y^2} - \frac{1}{2} \frac{l^2}{T^2} \frac{\dot{y}(x)^2}{y^4 L^2} \right].
\]

Finally,

\[
\int_0^{\tau_g} \frac{d\tau_g}{a_b(\tau_g)} = \int_0^{\tau_g} \frac{d\tau_g}{a_b(\tau_g)} \left[1 + \frac{1}{2} \frac{\Omega_0 L_c^q}{y_b^q L^q} + \frac{1}{2} l^2 \Lambda_4 - \frac{l^2}{L^2 y_b^2} \right] + \frac{1}{2} \frac{l^2}{L^2 y^2} - \frac{1}{2} \frac{l^2}{T^2 L^2} \frac{\dot{y}(x)^2}{y^4 L^4}.
\]

Thus, at first order, the time difference between the photon and the graviton is corrected in the integrand by terms of the order $\frac{L_c^q}{y_b^q}$. Today this factor
is at most $10^{-58}$ and in the time of decoupling $10^{-46}$, showing that the time advance of the graviton can be safely neglected and is of no physical significance, in spite of the fact that the trajectory of the brane is distinctively different from the null geodesic.

3.2 The Effect of Shortcuts in Early Time Universe

From the analysis developed so far we learned that the periods of evolution of the Universe differ by the scale that defines the physical significance of the shortcuts. In an era where $a_b << L_c$, the trajectory of the brane in the bulk differs from the extreme geodesics by $(a_b/L_c)^2$ and the shortcuts do not appear, since the brane itself provides the graviton geodesic.

In the period when $a_b >> L_c$, the trajectory of the brane is far from a null geodesic and shortcuts appear, but they are not significant, since the skin depth of the graviton in the bulk is defined by the parameter $l << L_c << a_b$. The difference between the time intervals of the photon and the graviton is of the order $(L_c/a_b)^2$.

However, from the continuity of the evolution of the brane in the bulk, we expect that there is also an intermediate situation $a_b \sim L_c$ when physically important shortcuts could appear, since the evolution of the brane is far enough from a geodesic. Indeed, in figure 4 rescaling the geodesics in the bulk we can observe the behaviour of the brane as compared to the geodesics that start in it at a later time. It is clear that these shortcuts are serious mediators of homogenization of the matter in the brane in the era before nucleosynthesis.

From the evolution of the brane in the bulk in the intermediate epoch we thus conclude that there is a critical age $t_c$, after which the gravitational waves leaving the brane return before the arrival of the photons released at the same time as the gravitons. The behaviour of the geodesic in the bulk shows that any geodesic starting in the brane at a certain instant will be singular at a time later than the critical time, indicating that it will return to the brane. Thus, information leaks between regions which apparently are causally disconnected at times $t > t_c$.

In order to study the horizon problem, we now compare, at a certain time $\tau_n$ previous to nucleosynthesis, the graviton horizon $R_g$ with the observable proper distance of the universe (from the radiation decoupling until today)
Figure 4: The trajectory of the brane in the bulk leaving $a = 0.01L_c$ in the RDE. Geodesics in the bulk with different initial velocities are exhibited intercepting the brane after the critical time $t_c$.

Since the graviton evolves in a bulk geodesic,

$$R_g \equiv \int \frac{dr}{\sqrt{1 - kr^2}} = \int \frac{dt}{a} \sqrt{h(a) - \frac{\dot{a}^2}{h(a)}} = \int \frac{d\tau}{a} \sqrt{h(a_b) - \frac{\dot{a}_b^2}{h(a_b)}} \sqrt{h(a) - \frac{\dot{a}^2}{h(a)}}$$

$$= \int_0^{\tau_n} \frac{d\tau}{\frac{1}{a_b} \left(1 + \frac{l^2}{a_b^2}\right)} \sqrt{1 + \Lambda_4 l^2 + \frac{\Omega_0 L_c^2}{a_b^q} \left(1 + \frac{\Omega_0 L_c^2}{4a_b^q (1 + \Lambda_4 l^2)}\right) \times \frac{1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2 y^2} \frac{\dot{y}(x)^2}{L^2}}{1 + \frac{l^2}{L^2 y^2} - \frac{l^2}{T^2 y^2} + y^4 \frac{L^2}{T^2}}},$$

(27)

while in the brane, the size of the observable Universe is $R = \int \frac{dr}{u_b(\tau)}$.  

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We use the known results for the usual particle horizon

\[
R = \frac{1}{\sqrt{\Omega_0 H_0 a_{b0}}} \int_{z(\tau)}^{z(0)} z^{-q/2} dz = \frac{2}{(q - 2) \sqrt{\Omega_0 H_0 a_{b0}}} (z(\tau)^{1-q/2} - z(0)^{1-q/2})
\]

where \(z(\tau) = a_{b0}/a_b(\tau)\). Today,

\[
R \sim \frac{2}{\sqrt{\Omega_0 H_0 a_{b0}}} \equiv R_0 .
\]

For the computation of the graviton horizon, we work in a primordial era before nucleosynthesis. Thus, in the Friedmann equation we can neglect the usual cosmological term as well as the curvature term. At an epoch between Planck era and nucleosynthesis, \(l << a_b << H_0\), it is safe to neglect terms involving \(l^2/L^2\) in (27), and we find

\[
R_g \approx \int_0^{\tau_n} \frac{d\tau}{a_b} \sqrt{1 + \frac{\Omega_0 L_c^q}{a_b^q} + \frac{\Omega_c^q L_c^{2q}}{4 a_b^{2q}}} .
\]

Using the Friedmann equation we get

\[
R_g = \frac{1}{\Omega_0^{1/2} H_0 a_{b0}} \int_{z(\tau_n)}^{z(0)} \frac{dz}{z^2} \sqrt{1 + \frac{\Omega_0 H_0^2 l^2}{4} z^4} .
\] (28)

The integral diverges for arbitrarily high redshifts, proving that the horizon problem is potentially solvable. The behaviour of this integral can be determined. We have

\[
R_g \approx \frac{l}{a_{b0}} z(0) .
\]

Comparing with the size of the Universe today,

\[
\frac{R_g}{R_0} \sim \frac{\sqrt{\Omega_0}}{2 H_0} l z(0) .
\]

It looks like shortcuts are not enough to solve the horizon problem, since we would need to go back in time to \(z(0) \sim (H_0 l)^{-1} \sim 10^{29}, 10^{11}\) times higher than the redshift at the Planck time in the brane associated with the fundamental scale of gravity, \(\kappa_5\).

For the time being, let’s assume the validity of the theory in such high energies. In the next section we will argue that, in order to provide a solution
to the usual cosmological problems, inflationary models must use of those high redshifts. We may note, however, that there are actually two related time scales. In the primordial Universe the brane is evolving as a part of the bulk, with velocities close to that of light, and time intervals in the brane correspond to much longer intervals in the bulk. In fact, the relation between these scales is obtained from $t_0 = \tau_0 = a_b(\tau_0) = 0$ and from

$$t = \int_0^\tau d\tau \sqrt{\frac{h(a_b) + \dot{a}_b^2(\tau)}{h(a_b)}} .$$

For $a_b << L_c$ we have $\tau << l$, and one finds, as a consequence

$$t \approx \int_0^\tau d\tau \frac{\dot{a}_b(\tau)}{h(a_b)} = \int_0^{\Omega_0(\tau)} \frac{da_b}{k + a_b^2/l^2} .$$

This implies, for the example of a closed universe with $\Omega_0 = 2$,

$$t = l \arctan \left( \frac{a_b(\tau)}{l} \right) .$$

Therefore, when $a_b(\tau) \sim l$, that is, when the brane time is $\tau \sim 10^{-67} s$, the corresponding bulk time $t \sim \frac{\pi}{4} l \sim 10^{-11} s$ is much larger than the Planck scale, $t >> M_{Pl}^{-1}$.

If we assume that quantization is mandatory according to the bulk Planck energy scale, geodesics that start in the bulk with $z(0) \sim (H_0 l)^{-1}$ should be sufficient to homogenize the Universe before nucleosynthesis, reaching $R_g \sim 1$. In order to verify this, we have numerically integrated the expression for the distance reached by geodesics $a(t)$ starting in the brane at $a_b = l$, corresponding to a time $t = \pi/4l$, for $k = 1$ with $\Omega_0 \sim 2$ ($a_{b0} \sim H_0^{-1}$):

$$R_g = \int_{\pi/4}^{t_f} dt \frac{a}{\sqrt{h(a) - \frac{\dot{a}^2}{h(a)}}} . \quad (29)$$

The solution of the geodesic equation is given in figure 5. The brane evolves initially as a geodesic with speed $v_b \sim 2 (a_b \sim l)$ and we exhibit a geodesic with zero initial speed in the bulk. The point of inflexion in its trajectory is $t_f = 2.356 l > t_c = \frac{\pi}{2} l$. As we saw before, the brane evolves smoothly after the transition and the geodesics return to the brane if we
Figure 5: Null geodesic starting at \( a = l, \ t = \pi/4l \) with initial velocity \( v(\pi/4) = 0 \).

wait until \( t_f \), see (29). Although the trajectory diverges near this point, the integrand in (29) is finite and we can perform the integration, finding

\[ R_g = 2.526 \]

Thus, the observable Universe is smaller than the reach of the relic gravitons that start in a previous era.

4 Causal Gravitational Structure

In the last section we learned about the behaviour of null-geodesics for closed Universes. We can, however, study analytically the whole causal structure of the gravitational signs for a \( k = 0 \) Universe.

The geodesic equation, (17), is quite simple for purely AdS \( k = 0 \) space-times

\[
\frac{1}{a^2} \frac{da}{dt} = \frac{v(0)}{a(0)^2} .
\]
Thus, a geodesic that starts in the brane at \( a = a(0) \) and \( t = 0 \), with initial velocity \( v(0) \), returns to it when

\[
t_r \approx \frac{a(0)}{v(0)}.
\]

The expression for the gravitational horizon can also be integrated

\[
R_g = \int_{0}^{t_r} \frac{dt}{a} \sqrt{h(a) - \frac{\dot{a}^2}{h(a)}} = \int_{0}^{t_r} \frac{dt}{l} \sqrt{1 - \frac{l^4 \dot{a}^2}{a^4}}
\]

\[
= \frac{t_r}{l} \sqrt{1 - \frac{v(0)^2 l^4}{a(0)^4}}.
\]

Using the relation between the returning time and the initial velocity,

\[
R_g = \frac{t_r}{l} \sqrt{1 - \frac{l^4}{a(0)^2 l^4 r}}.
\]

In order to relate the returning time to the redshift, we must integrate the relation between time of the bulk and time of the brane \( \text{I(16)} \). We already know that from \( a(0) \) to the critical period of transition a time of \( t_c \approx \frac{l^2}{a(0)} \) has passed. After that, the evolution is dominated by the usual term in Friedmann equation and we can approximate

\[
t_r \approx \frac{l^2}{a(0)} + \int_{L_c}^{a_r} \frac{da_b}{a_b^2 H} \approx \frac{l^2}{a(0)} + \int_{L_c}^{a_r} \frac{da_b}{\sqrt{\Omega_0 H_0 a_b^q}}
\]

\[
\approx \frac{l^2}{a(0)} + \frac{2l}{(q-2)\sqrt{\Omega_0 H_0 a_0}} \left( z_r^{q/2+1} - 10^{-15} \right),
\]

where \( z_r \) must, of course, be greater than the redshift in the transition, \( z_{Lc} \approx 10^{15} \).

Substituting back in the expression of the gravitational horizon,

\[
R_g = \left[ \frac{l}{a(0)} + \frac{2}{(q-2)\sqrt{\Omega_0 H_0 a_0}} \left( z_r^{q/2+1} - 10^{-15} \right) \right]
\]

\[
\times \sqrt{1 - \left[ 1 + \frac{2a(0)}{(q-2)\sqrt{\Omega_0 H_0 a_0}} \left( z_r^{q/2+1} - 10^{-15} \right) \right]^2}.
\]
When considering the interesting situation of a high initial redshift \( z(0) = a_{b0}/a(0) > (H_0l)^{-1} \), this expression can be approximated by

\[
R_g \approx \frac{l}{a(0)} \left( \frac{4a(0)}{(q-2)\sqrt{\Omega_0 lH_0 a_{b0}}} \left( z_r^{-q/2+1} - 10^{-15} \right) \right). \tag{35}
\]

Comparing with the size of the horizon today, we find

\[
\left( \frac{R_g}{R_0} \right)_{z_r} \approx \sqrt{\frac{\Omega_0 lH_0}{(q-2)}} \sqrt{z(0)} \left( z_r^{-q/2+1} - 10^{-15} \right). \tag{36}
\]

We note, as we did in the last section, that if sufficiently large redshifts were available, the graviton horizon in a past epoch could be larger than the present size of the observable Universe.

We argue, however, that those high redshifts could be available even when the energy density in the brane is not quantized yet. Indeed, if inflation takes place in the brane high redshifts could be present in the beginning of the inflationary epoch.

Denoting the redshift when inflation ends by \( z_e \), if the size of the present Universe, \( R_0 \), is expected to be in causal contact during inflation, we must reach at least a redshift in the beginning of inflation, \( z(0) \), that solves

\[
\frac{a_{b0}R_0}{z(0)} = H^{-1}(z_e). \tag{37}
\]

The unusual results in brane-world cosmology are expected if inflation ends before the transition time, when the quadratic term in Friedmann equation dominates. In this case, we get

\[
\frac{a_{b0}R_0}{z(0)} = \frac{1}{\Omega_0 H_0^2 l z_e^4}
\]

and

\[
z(0) = 2\sqrt{\Omega_0 H_0 l z_e^4}. \tag{37}
\]

If \( H(z_e)l >> 1 \), it is simple to note, from equation (16), that the evolution of the brane in the bulk is not altered during inflation. Thus, we can substitute the result (37) in the complete expression for the causal gravitational horizon (36),

\[
\left( \frac{R_g}{R_0} \right)_{z_r} \approx \sqrt{\frac{\Omega_0}{(q-2)}} H_0 l z_e^2 \sqrt{z_r^{-q/2+1} - 10^{-15}}. \tag{38}
\]
This equation tells us that, in the time of nucleosynthesis, when $z_r \sim 10^{10}$, $R_g/R_0 \sim 1$ for a model with inflation ending just before what would be the Planck era, with $z_e \sim 10^{17}$ (where $z_{Pl} \sim 10^{18}$). This proves that successfully inflationary models ending before the transition time necessarily make relevant changes in the causal structure of the universe.

We are able to sketch the gravitational horizon for this kind of configuration. In figure 6 we show the behaviour of the Hubble horizon in comoving coordinates $H^{-1} a^{-1}/R_0$ and the graviton horizon $R_g/R_0$ for an inflationary model ending just before the Planck era, $z_e = 10^{17}$ and producing the necessary number of e-folds to solve the horizon problem. The fraction of the Universe in causal contact by gravitational signs in the nucleosynthesis epoch is just the present horizon $R_0$. Today, the gravitational horizon would be $10^5 R_0$.

5 Conclusions

Studying the shortcut problem in braneworld cosmology from the point of view of the bulk we have explicitly shown that shortcuts are indeed common in late time Universes, though they are extremely small and the time advance of the graviton can be safely neglected. However, we have also learned that gravitational signs may leave and subsequently return to the brane even in early times Universes. We have showed that those shortcuts exist and that the new scale of the model, $l$, implies in a minimum time scale for the reception of those signs by an observer in the brane. Before that critical time, the brane itself evolves like a null-geodesic in the bulk.

If high initial redshits were available the shortcuts just found could solve the horizon problem without inflation. More important however, may be the effect of those shortcuts with an inflationary epoch in the brane.

Brane-world models incorporate two changes in the cosmology, namely, the modified Friedmann equation and the possibility of leaking of gravity in the extra dimension. Using the first of those modifications, it was shown that, remarkably, the consistency equation is maintained in the brane-world formalism when the inflation is guided by a scalar field minimally coupled in the brane [13]. This consists in bad news for those who expect that brane cosmological configurations could probe the extra dimensionality of our Universe.

However, we have showed that if there was an inflationary epoch in the
Figure 6: We plot in log-log scale the evolution of the fraction of graviton horizon and the present observable size, $R_g/R_0$ (dashed line) and the same for the Hubble horizon $a_h^{-1}H^{-1}/R_0$ (solid line), for an inflationary model in the brane that ends after Planck time ($u_{PL} = -\log(z_{PL}) = -18$), and before transition time ($u_{TR} = -15$), in $u_{end} = -\log(z_{end}) = -17$. The present scale would be under the de-Sitter horizon if redshifts like $u(0) = -39$ were available. This imply in strong modifications of the causal structure for gravitational signs after the transition time.

brane evolution, the causal structure of the universe could be strongly modified. This could be a sign of an unusual evolution of the perturbations from the time they cross the de Sitter horizon, $H^{-1}$, during inflation, through the time they became causally connected again. In this case, there could be distinct predictions for the microwave background radiation structure even with the same consistency equation during inflation. Thus, further investigation
on the dynamics of perturbations in inflationary brane-world models may prove useful to probe the dimensionality of space-time.

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