Existence of global weak solutions to finitely extensible nonlinear bead–spring chain models for dilute polymers with variable density and viscosity

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Abstract
We prove the existence of global-in-time weak solutions to a general class of coupled bead–spring chain models that arise from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids with noninteracting polymer chains, with finitely extensible nonlinear elastic (FENE) spring potentials. The class of models under consideration involves the unsteady incompressible Navier–Stokes equations with variable density and density-dependent dynamic viscosity in a bounded domain in \( \mathbb{R}^d \), \( d = 2 \) or 3, for the density, the velocity and the pressure of the fluid, with an elastic extra-stress tensor appearing on the right-hand side in the momentum equation. The extra-stress tensor stems from the random movement of the polymer chains and is defined by the Kramers expression through the associated probability density function that satisfies a Fokker–Planck-type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term and a nonlinear density-dependent drag coefficient. We require no structural assumptions on the drag term in the Fokker–Planck equation; in particular, the drag term need not be corotational. With initial density \( \rho_0 \in [\rho_{\min}, \rho_{\max}] \) for the continuity equation, where \( \rho_{\min} > 0 \); a square-integrable and divergence-free initial velocity datum \( u_0 \) for the Navier–Stokes equation; and a nonnegative initial probability density function \( \psi_0 \) for the Fokker–Planck equation, which has finite relative entropy with respect to the Maxwellian \( M \) associated with the spring potential in the model, we prove, via a limiting procedure on certain regularization parameters, the existence of a global-in-time weak solution \( t \mapsto (\rho(t), u(t), \psi(t)) \) to the coupled Navier–Stokes–Fokker–Planck system, satisfying the
1. Introduction

This paper establishes the existence of global-in-time weak solutions to a large class of bead—spring chain models with finitely extensible nonlinear elastic (FENE) type spring potentials — a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymer solutions. The solvent is an incompressible, viscous, isothermal Newtonian fluid with variable density and viscosity confined to a bounded open Lipschitz domain \( \Omega \subset \mathbb{R}^d \), with boundary \( \partial \Omega \). For the sake of simplicity of presentation, we shall suppose that \( \Omega \) has a ‘solid boundary’ \( \partial \Omega \); the velocity field \( u \) will then satisfy the no-slip boundary condition \( u = 0 \) on \( \partial \Omega \). The polymer chains, which are suspended in the solvent, are assumed not to interact with each other. The equations of continuity, balance of linear momentum and incompressibility then have the form of the incompressible Navier–Stokes equations with variable density and viscosity (cf. Antontsev, Kazhikhov & Monakhov [2], Feireisl & Novotný [21], Lions [32] or Simon [46]) in which the elastic extra-stress tensor \( \tau \) (i.e., the polymeric part of the Cauchy stress tensor) appears as a source term in the conservation of momentum equation:

Given \( T \in \mathbb{R}_{>0} \), find \( \rho : (x,t) \in \Omega \times [0,T] \mapsto \rho(x,t) \in \mathbb{R} \), \( u : (x,t) \in \Omega \times [0,T] \mapsto u(x,t) \in \mathbb{R}^d \) and \( p : (x,t) \in \Omega \times (0,T] \mapsto p(x,t) \in \mathbb{R} \) such that

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad \text{in } \Omega \times (0,T],
\]

\[
\rho(x,0) = \rho_0(x) \quad \forall x \in \Omega,
\]

\[
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) - \nabla \cdot (\mu(\rho) D(u)) + \nabla \cdot \tau = \rho f + \nabla x \cdot \tau \quad \text{in } \Omega \times (0,T],
\]

\[
\nabla u = 0 \quad \text{in } \Omega \times (0,T],
\]

\[
u = 0 \quad \text{on } \partial \Omega \times (0,T],
\]

\[
(\rho u)(x,0) = (\rho_0 u_0)(x) \quad \forall x \in \Omega.
\]

It is assumed that each of the equations above has been written in its nondimensional form; \( \rho \) denotes a nondimensional solvent density, \( u \) a nondimensional solvent velocity, defined as the velocity field scaled by the characteristic flow speed \( U_0 \). Here \( D(u) := \frac{1}{2} (\nabla u + (\nabla u)^T) \) is the rate of strain tensor, with \( (\nabla u)(x,t) \in \mathbb{R}^{d \times d} \) and \( (\nabla u)_i \) the rate of strain tensor, with \( (\nabla u)(x,t) \in \mathbb{R}^{d \times d} \) and \( (\nabla u)_i \) the rate of strain tensor, with \( (\nabla u)_i = \frac{\partial v_i}{\partial x_j} \). The scaled dynamic viscosity of the solvent, \( \mu(\cdot) \in \mathbb{R}_{>0} \), is density-dependent; in addition, \( p \) is the nondimensional pressure and \( f \) is the nondimensional density of body forces.

In a bead—spring chain model, consisting of \( K + 1 \) beads coupled with \( K \) elastic springs to represent a polymer chain, the extra-stress tensor \( \tau \) is defined by the Kramers expression as a weighted average of \( \psi \), the probability density function of the (random) conformation vector \( q := (q_1^T, \ldots, q_K^T)^T \in \mathbb{R}^{Kd} \) of the chain (see Eq. (1.7) below), with \( q_i \) representing the \( d \)-component conformation/orientation vector of the \( i \)th spring. The Kolmogorov equation satisfied by \( \psi \) is a second-order parabolic equation, the Fokker–Planck equation, whose transport coefficients depend on the velocity field \( u \), and the hydrodynamic drag coefficient appearing in the Fokker–Planck equation is a linear function of the dynamic viscosity \( \mu \) (Stokes drag is assumed), which, in turn, is a nonlinear function of the density \( \rho \).
The domain $D$ of admissible conformation vectors $D \subset \mathbb{R}^{Kd}$ is a $K$-fold Cartesian product $D_1 \times \cdots \times D_K$ of balanced convex open sets $D_i \subset \mathbb{R}^d$, $i = 1, \ldots, K$; the term balanced means that $q_i \in D_i$ if, and only if, $-q_i \in D_i$. Hence, in particular, $0 \in D_i$, $i = 1, \ldots, K$. Typically $D_i$ is the whole of $\mathbb{R}^d$ or a bounded open $d$-dimensional ball centred at the origin $0 \in \mathbb{R}^d$ for each $i = 1, \ldots, K$. When $K = 1$, the model is referred to as the dumbbell model.

Let $\mathcal{O}_i \subset [0, \infty)$ denote the image of $D_i$ under the mapping $q_i \in D_i \mapsto \frac{1}{2}|q_i|^2$, and consider the spring potential $U_i \in C^1(\mathcal{O}_i; \mathbb{R}_{\geq 0}) \cap W_{\text{loc}}^2(\mathcal{O}_i; \mathbb{R}_{>0})$, $i = 1, \ldots, K$. Clearly, $0 \in \mathcal{O}_i$. We shall suppose that $U_i(0) = 0$ and that $U_i$ is unbounded on $\mathcal{O}_i$ for each $i = 1, \ldots, K$. The elastic spring force $F_i: D_i \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d$ of the $i$th spring in the chain is defined by

$$F_i(q_i) := U_i'\left(\frac{1}{2}|q_i|^2\right)q_i, \quad i = 1, \ldots, K. \quad (1.2)$$

The partial Maxwellian $M_i$, associated with the spring potential $U_i$, is defined by

$$M_i(q_i) := \frac{1}{Z_i}e^{-U_i(\frac{1}{2}|q_i|^2)}, \quad Z_i := \int_{D_i} e^{-U_i(\frac{1}{2}|q_i|^2)} \, dq_i, \quad i = 1, \ldots, K.$$

The (total) Maxwellian in the model is then

$$M(q) := \prod_{i=1}^{K} M_i(q_i) \quad \forall q := \left(q_1^T, \ldots, q_K^T\right)^T \in D := \prod_{i=1}^{K} D_i. \quad (1.3)$$

Observe that, for $i = 1, \ldots, K$,

$$M(q) \nabla_{q_i} \left[M(q)\right]^{-1} = -\left[M(q)\right]^{-1} \nabla_{q_i} M(q) = \nabla_{q_i} U_i \left(\frac{1}{2}|q_i|^2\right) = U_i'\left(\frac{1}{2}|q_i|^2\right)q_i, \quad (1.4)$$

and, by definition,

$$\int_D M(q) \, dq = 1. \quad \text{Example 1.1.} \quad \text{In the Hookean dumbbell model } K = 1, \text{ and the spring force is defined by } F(q) = q, \text{ with } q \in D = \mathbb{R}^d, \text{ corresponding to } U(s) = s, \ s \in \mathcal{O} = [0, \infty). \text{ More generally, in a Hookean bead--spring chain model, } K \geq 1, \ F_i(q_i) = q_i, \text{ corresponding to } U_i(s) = s, \ i = 1, \ldots, K, \text{ and } D \text{ is the Cartesian product of } K \text{ copies of } \mathbb{R}^d. \text{ The associated Maxwellian is}$

$$M(q) = M_1(q_1) \cdots M_K(q_K) = \frac{1}{Z} e^{-\frac{1}{2}|q|^2},$$

with $|q|^2 := |q_1|^2 + \cdots + |q_K|^2$ and $Z := Z_1 \cdots Z_K = (2\pi)^{Kd/2}$. Hookean dumbbell and Hookean bead--spring chain models are physically unrealistic as they admit arbitrarily large extensions. 

A more realistic class of models assumes that the springs in the bead--spring chain have finite extension: the domain $D$ is then taken to be a Cartesian product of $K$ bounded open balls $D_i \subset \mathbb{R}^d$, centred at the origin $0 \in \mathbb{R}^d$, $i = 1, \ldots, K$, with $K \geq 1$. The spring potentials $U_i: s \in [0, b_i/2) \mapsto U_i(s)$ in
[0, \infty), with \(b_i > 0, i = 1, \ldots, K\), are in that case nonlinear and unbounded functions, and the associated bead–spring chain model is referred to as a FENE (finitely extensible nonlinear elastic) model; in the case of \(K = 1\), the corresponding model is called a FENE dumbbell model.

Here we shall be concerned with finitely extensible nonlinear bead–spring chain models, with \(D := B(0, b_i^\frac{1}{2}) \times \cdots \times B(0, b_K^\frac{1}{2})\), where \(b_i > 0, i = 1, \ldots, K\), \(K \geq 1\), and \(B(0, b_i^\frac{1}{2})\) is a bounded open ball in \(\mathbb{R}^d\) of radius \(b_i^\frac{1}{2}\), centred at \(0 \in \mathbb{R}^d\). We shall adopt the following structural hypotheses on the spring potentials \(U_i\) and the associated partial Maxwellians \(M_i\), \(i = 1, \ldots, K\).

We shall suppose that for \(i = 1, \ldots, K\) there exist constants \(c_{ij} > 0, j = 1, 2, 3, 4\), and \(\gamma_i > 1\) such that the spring potential \(U_i\) satisfies

\[
\begin{align*}
(c_{i1}) \left[ \text{dist}(q_i, \partial D_i) \right]^{\gamma_i} &\leq M_i(q_i) \leq c_{i2} \left[ \text{dist}(q_i, \partial D_i) \right]^{\gamma_i} \quad \forall q_i \in D_i, \\
(c_{i3}) &\left[ \text{dist}(q_i, \partial D_i) \right] U_i \left( \frac{1}{2} |q_i|^2 \right) \leq c_{i4} \quad \forall q_i \in D_i.
\end{align*}
\]

Since \(\left[ U_i \left( \frac{1}{2} |q_i|^2 \right) \right]^2 = (-\log M_i(q_i) + \text{const.})^2\), it follows from \((c_{i1})\), \((c_{i2})\) that (if \(\gamma_i > 1\), as has been assumed here)

\[
\int_{D_i} \left[ 1 + \left[ U_i \left( \frac{1}{2} |q_i|^2 \right) \right]^2 + \left[ U_i \left( \frac{1}{2} |q_i|^2 \right) \right]^{\gamma_i} \right] M_i(q_i) \, dq_i < \infty, \quad i = 1, \ldots, K.
\]

**Example 1.2.** In the FENE (finitely extensible nonlinear elastic) dumbbell model, introduced by Warner [48], \(K = 1\) and the spring force is given by \(F(q) = (1 - |q|^2/b)^{-1}q, q \in D = B(0, b^\frac{1}{2})\), corresponding to \(U(s) = -\frac{b}{2} \log(1 - \frac{2s}{b^2})\), \(s \in \mathcal{C} = [0, \frac{b}{2})\), \(b > 2\). More generally, in a FENE bead–spring chain, one considers \(K + 1\) beads linearly coupled with \(K\) springs, each with a FENE spring potential. Direct calculations show that the partial Maxwellians \(M_i\) and the elastic potentials \(U_i, i = 1, \ldots, K\), of the FENE bead–spring chain satisfy the conditions \((c_{i1})\), \((c_{i2})\) with \(\gamma_i := b_i^\frac{1}{2}\), provided that \(b_i > 2, i = 1, \ldots, K\). Thus, \((1.6)\) also holds and \(b_i > 2, i = 1, \ldots, K\).

It is interesting to note that in the (equivalent) stochastic version of the FENE dumbbell model \((K = 1)\) a solution to the system of stochastic differential equations associated with the Fokker–Planck equation exists and has trajectorial uniqueness if, and only if, \(\gamma = \frac{b}{2} \geq 1\); (cf. Jourdain, Lelièvre & Le Bris [24] for details). Thus, in the general class of FENE-type bead–spring chain models considered here, the assumption \(\gamma_i > 1, i = 1, \ldots, K\), is the weakest ‘reasonable’ requirement on the decay-rate of \(M_i\) in \((1.5a)\) as \(\text{dist}(q_i, \partial D_i) \to 0\). See however the work of Liu & Shin [35] for a discussion about admissible boundary conditions in the case \(\gamma_i < 1\).

The governing equations of the general nonhomogeneous bead–spring chain models with centre-of-mass diffusion considered in this paper are \((1.1c), (1.1d)\), where the extra-stress tensor \(\tau_{\infty}\) is defined by the *Kramers expression*:

\[
\tau(x, t) = k \left( \sum_{i=1}^{K} \int_D \psi(x, q, t) q_i q_i^\top U_i \left( \frac{1}{2} |q_i|^2 \right) dq - K \varrho(x, t) I \right),
\]

with the density of polymer chains located at \(x\) at time \(t\) given by

\[
\varrho(x, t) = \int_D \psi(x, q, t) \, dq.
\]
(not to be confused with the solvent density $\rho(x, t)$). The probability density function $\psi$ is a solution of the Fokker–Planck (forward Kolmogorov) equation

$$\frac{\partial \psi}{\partial t} + (u \cdot \nabla_x) \psi + \sum_{i=1}^{K} \nabla_{q_i} \cdot (\sigma(u) q_i \psi) = \varepsilon \Delta_{x} \left( \frac{\psi}{\zeta(\rho)} \right) + \frac{1}{4\lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \nabla_{q_i} \cdot \left( M \nabla_{q_j} \left( \frac{\psi}{\zeta(\rho) M} \right) \right) \quad \text{in } \Omega \times D \times (0, T].$$

(1.9)

with $\sigma(\psi) \equiv \nabla_x \psi$ and a density-dependent scaled drag coefficient $\zeta(\cdot) \in \mathbb{R}_{>0}$. For a concise derivation of the Fokker–Planck equation (1.9) we refer the reader to the extended version of the present paper [10].

The nondimensional constant $k > 0$ featuring in (1.7) is a constant multiple of the product of the polymer number density (the number of polymer molecules per unit volume), the Boltzmann constant $k_B$, and the absolute temperature $T$. In (1.9), $\varepsilon > 0$ is the centre-of-mass diffusion coefficient defined as $\varepsilon := (\ell_0/L_0)^2/(4(K + 1)\lambda)$ with $L_0$ a characteristic length-scale of the solvent flow, $\ell_0 := \sqrt{k_B T/\mathcal{H}}$ signifying the characteristic microscopic length-scale and $\lambda := (\ell_0/4\lambda)(U_0/L_0)$, where $\ell_0 > 0$ is a characteristic drag coefficient and $\lambda > 0$ is a spring-constant. The nondimensional parameter $\lambda \in \mathbb{R}_{>0}$, called the Deborah number (and usually denoted by $De$), characterizes the elastic relaxation property of the fluid, and $A = (A_{ij})_{i,j=1}^{K}$ is the symmetric positive definite Rouse matrix, or connectivity matrix; for example, $A = \text{tridiag}[-1, 2, -1]$ in the case of a (topologically) linear chain; see, Nitta [39]. Concerning these scalings and notational conventions, we remark that the factor $1/(4\lambda)$ in Eq. (1.9) above appears as a factor $1/(2\lambda)$ in the Fokker–Planck equation in our earlier papers [9,11,8,12].

**Definition 1.1.** The collection of equations and structural hypotheses (1.1a)–(1.1f)–(1.9) will be referred to throughout the paper as model (P), or as the general nonhomogeneous FENE-type bead–spring chain model with centre-of-mass diffusion.

A noteworthy feature of Eq. (1.9) in the model (P) compared to classical Fokker–Planck equations for bead–spring models in the literature is the presence of the $x$-dissipative centre-of-mass diffusion term $\varepsilon \Delta_{x} \psi$ on the right-hand side of the Fokker–Planck equation (1.9). We refer to Barrett & Suli [6] for the derivation of (1.9) in the case of $K = 1$ and constant $\rho$; see also the article by Schieber [43] concerning generalized dumbbell models with centre-of-mass diffusion, and the recent paper of Gend & Liu [16] for a careful justification of the presence of the centre-of-mass diffusion term through asymptotic analysis. In standard derivations of bead–spring models the centre-of-mass diffusion term is routinely omitted on the grounds that it is several orders of magnitude smaller than the other terms in the equation. Indeed, when the characteristic macroscopic length-scale $L_0 \approx 1$ (for example, $L_0 = \text{diam}(\Omega)$), Bhave, Armstrong & Brown [13] estimate the ratio $\ell_0^2/L_0^2$ to be in the range of about $10^{-9} - 10^{-7}$. However, the omission of the term $\varepsilon \Delta_{x} \psi$ from (1.9) in the case of a heterogeneous solvent velocity $u(x, t)$ is a mathematically counterproductive model reduction. When $\varepsilon \Delta_{x} \psi$ is absent, (1.9) becomes a degenerate parabolic equation exhibiting hyperbolic behaviour with respect to $(x, t)$. Since the study of weak solutions to the coupled problem requires one to work with velocity fields $u$ that have very limited Sobolev regularity (typically $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)))$, one is then forced into the technically unpleasant framework of hyperbolically degenerate parabolic equations with rough transport coefficients (cf. Ambrosio [1], DiPerna & Lions [18], Mucha [38]). The resulting difficulties are further exacerbated by the fact that, when $D$ is bounded, a typical spring force $F(q)$ for a finitely extensible model (such as FENE) explodes as $q \to \partial D$; see Example 1.2 above. Thus, as in our earlier papers (cf. [6,7,9,11]), we shall retain the centre-of-mass diffusion term in (1.9); we also refer to these papers for a detailed survey of the relevant literature in the field, including in particular the works of Renardy [42], Lions & Masmoudi [33], E, Li & Zhang [20] and Li, Zhang &
In Barrett & Süli [6], we derived the coupled Navier–Stokes–Fokker–Planck model with centre-of-mass diffusion stated above, in the case of $K = 1$ and constant solvent-density $\rho$. We established the existence of global-in-time weak solutions to a mollification of the model for a general class of spring force-potentials including in particular the FENE potential. We justified also, through a rigorous limiting process, certain classical reductions of this model appearing in the literature that exclude the centre-of-mass diffusion term from the Fokker–Planck equation on the grounds that the diffusion coefficient is small relative to other coefficients featuring in the equation. In the case of a corotational drag term we performed a rigorous passage to the limit as the mollifiers in the Kramers expression and the drag term converge to identity operators.

In Barrett & Süli [7] we showed the existence of global-in-time weak solutions to the general class of noncorotational FENE type dumbbell models (including the standard FENE dumbbell model) with centre-of-mass diffusion, in the case of $K = 1$ and constant solvent-density $\rho$ with microscopic cut-off (cf. (1.11) and (1.12) below) in the drag term

$$\nabla q \cdot \left( \sigma(\sim u)q\psi \right) = \nabla q \cdot \left[ \sigma(\sim u)qM\zeta(\rho)\left( \frac{\psi}{\zeta(\rho)M} \right) \right].$$

(1.10)

Subsequently, in [9] and [11], we removed the presence of the cut-off by passing to the limit $L \to \infty$, with $K \geq 1$, and the solvent density $\rho$, the viscosity $\mu$ and the drag coefficient $\zeta$ kept constant.

In this paper we prove the existence of global-in-time weak solutions to FENE-type models without cut-off or mollification, in the general case of $K \geq 1$ and with variable solvent-density $\rho$, variable viscosity $\mu(\rho)$ and variable drag $\zeta(\rho)$. This is achieved by replacing the use of Dubinskiǐ’s compactness theorem in [9] with the application of the Div–Curl lemma in our proof of relative compactness of the sequence of approximating solutions to the Fokker–Planck equation in the Maxwellian-weighted $L^1$ space $L^1(0, T; L^1_M(\Omega \times D))$. Since the argument is long and technical, we give a brief overview of the main steps of the proof.

**Step 1.** Following the approach in Barrett & Süli [7,9,11] and motivated by recent papers of Jourdain, Lelièvre, Le Bris & Otto [25] and Lin, Liu & Zhang [31] (see also Arnold, Markowich, Toscani & Unterreiter [3], and Desvillettes & Villani [17]) concerning the convergence of the probability density function $\psi$ to its equilibrium value $\psi_\infty(\sim x, \sim q) := M(\sim q)$ (corresponding to the equilibrium value $u_\infty(\sim x) := 0$ of the velocity field in the case of constant density) in the absence of body forces $f$, we observe that if $\psi/(\zeta(\rho)M)$ is bounded above then, for $L \in \mathbb{R}_{>0}$ sufficiently large, the drag term (1.10) is equal to

$$\nabla q \cdot \left[ \sigma(\sim u)qM\zeta(\rho)\beta_L^L \left( \frac{\psi}{\zeta(\rho)M} \right) \right].$$

(1.11)

where $\beta_L^L \in C(\mathbb{R})$ is a cut-off function defined as

$$\beta_L^L(s) := \min(s, L).$$

(1.12)

More generally, in the case of $K \geq 1$, in analogy with (1.11), the drag term with cut-off is defined by

$$\sum_{i=1}^{K} \nabla q_i \cdot \left[ \sigma(\sim u_i)q_iM\zeta(\rho)\beta_L^L \left( \frac{\psi}{\zeta(\rho)M} \right) \right].$$

It then follows that, for $L \gg 1$, any solution $\psi$ of (1.9), such that $\psi/(\zeta(\rho)M)$ is bounded above by $L$, also satisfies
\[
\frac{\partial \psi}{\partial t} + (u \cdot \nabla)\psi + \sum_{i=1}^{K} \nabla q_i \cdot \left( \sigma(u)q_i M\zeta(\rho)\beta^L \left( \frac{\psi}{\zeta(\rho)M} \right) \right) \\
= \varepsilon \Delta_x \left( \frac{\psi}{\zeta(\rho)} \right) + \frac{1}{4\lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \nabla q_i \cdot \left( M \nabla q_i \left( \frac{\psi}{\zeta(\rho)M} \right) \right) \quad \text{in } \Omega \times D \times (0,T].
\]

(1.13)

Let \( \partial D_i := D_1 \times \cdots \times D_{i-1} \times \partial D_i \times D_{i+1} \times \cdots \times D_K \). We impose the following boundary and initial conditions:

\[
\left[ \frac{1}{4\lambda} \sum_{j=1}^{K} A_{ij} M \nabla q_i \left( \frac{\psi}{\zeta(\rho)M} \right) - \sigma(u)q_i M\zeta(\rho)\beta^L \left( \frac{\psi}{\zeta(\rho)M} \right) \right] \cdot \frac{q_i}{|q_i|} = 0
\]
on \( \Omega \times \partial D_i \times (0,T] \), for \( i = 1, \ldots, K \),

(1.14a)

\[
\varepsilon \nabla_x \left( \frac{\psi}{\zeta(\rho)} \right) \cdot n = 0 \quad \text{on } \partial \Omega \times D \times (0,T],
\]

(1.14b)

\[
\psi(\cdot,\cdot,0) = M(\cdot)\zeta(\rho_0(\cdot))\beta^L \left( \psi(\cdot,\cdot)/(\zeta(\rho_0(\cdot))M(\cdot)) \right) \geq 0 \quad \text{on } \Omega \times D,
\]

(1.14c)

where \( q_i \) is normal to \( \partial D_i \), as \( D_i \) is a bounded ball centred at the origin, and \( n \) is normal to \( \partial \Omega \).

The initial datum \( \psi_0 \) for the Fokker–Planck equation is nonnegative, defined on \( \Omega \times D \), with

\[
\int_{D} \psi_0(x,q) \, dq \in L^\infty(\Omega), \quad \int_{\Omega \times D} \psi_0(x,q) \, dq \, dx = 1.
\]

and assumed to have finite Kullback–Leibler relative entropy with respect to the Maxwellian \( M \); i.e.

\[
\int_{\Omega \times D} \psi_0(x,q) \log \frac{\psi_0(x,q)}{M(q)} \, dq \, dx < \infty.
\]

As we shall suppose throughout that the range of the function \( \zeta \) is a compact subinterval \( [\zeta_{\min},\zeta_{\max}] \) of \((0,\infty)\), the finiteness of the relative entropy with respect to the Maxwellian \( M \) is equivalent to demanding that

\[
\int_{\Omega \times D} \frac{\psi_0(x,q)}{\zeta(\rho_0(x))} \log \frac{\psi_0(x,q)}{\zeta(\rho_0(x))/M(q)} \, dq \, dx < \infty.
\]

Clearly, if there exists \( L > 0 \) such that \( 0 \leq \psi_0 \leq L\zeta(\rho_0)M \), then \( M\zeta(\rho_0)\beta^L(\psi_0/(\zeta(\rho_0)M)) = \psi_0 \). Henceforth \( L > 1 \) is assumed. In addition, the boundary conditions for \( \psi \) on \( \partial \Omega \times D \times (0,T] \) and \( \Omega \times \partial D \times (0,T] \) ensure that

\[
\int_{\Omega \times D} \psi(x,q,t) \, dq = \int_{\Omega \times D} \psi(x,q,0) \, dq = 1
\]

for a.e. \( t \in \mathbb{R}_{\geq0} \), in agreement with the requirement that \( \psi \) is a probability density function.
Definition 1.2. The coupled problem (1.1a)–(1.1f), (1.7), (1.8), (1.13), (1.14a)–(1.14c) will be referred to as model \((P_L)\), or as the general nonhomogeneous FENE-type bead–spring chain model with centre-of-mass diffusion and microscopic cut-off, with cut-off parameter \(L > 1\).

In order to highlight the dependence on \(L\), in subsequent sections the solution to (1.13), (1.14a)–(1.14c) will be labelled \(\psi_L\). The existence of a weak solution to model (P) under technical reasons, a further cut-off, now from below, is required, with a cut-off parameter \(d\) of diffusion and microscopic cut-off with step size depending of the cut-off parameter \(d\) of diffusion and microscopic cut-off with step size.

Step 2. Ideally, one would like to pass to the limit \(L \to \infty\) in problem \((P_L)\) to deduce the existence of solutions to (P). Unfortunately, such a direct attack at the problem is (except in the special case of \(d = 2\), or in the absence of convection terms from the model) fraught with technical difficulties. Instead, we shall first (semi)discretize problem \((P_L)\) by an implicit Euler scheme with respect to \(t\), with step size \(\Delta t\); this then results in a time-discrete version \((P^t_L)\) of \((P_L)\). By using Schauder’s fixed point theorem, we will show in Section 3 the existence of solutions to \((P^t_L)\). In the course of the proof, for technical reasons, a further cut-off, now from below, is required, with a cut-off parameter \(\delta \in (0, 1)\), which we shall let pass to 0 to complete the proof of existence of solutions to \((P^t_L)\) in the limit of \(\delta \to 0_+\) (cf. Section 3). Ultimately, of course, our aim is to show existence of weak solutions to the general nonhomogeneous FENE-type bead–spring chain model with centre-of-mass diffusion, (P), and that demands passing to the limits \(\Delta t \to 0_+\) and \(L \to \infty\); this then brings us to the next step in our argument.

Step 3. We shall link the time step \(\Delta t\) to the cut-off parameter \(L > 1\) by demanding that \(\Delta t = o(L^{-1})\), as \(L \to \infty\), so that the only parameter in the problem \((P^t_L)\) is the cut-off parameter (the centre-of-mass diffusion parameter \(\varepsilon\) being fixed). We shall show that \(\rho^L_{\Delta t}\) can be bounded, independent of the cut-off parameter \(L\), in \(L^\infty(0, T; L^\infty(\Omega))\). By using special energy estimates, based on testing the Fokker–Planck equation in \((P^t_L)\) with the derivative of the relative entropy with respect to the Maxwellian of the general nonhomogeneous FENE-type bead–spring chain model, we show that \(u^t_{\Delta t}\) can also be bounded, independent of \(L\). Specifically, \(u^t_{\Delta t}\) is bounded in the norm of the classical Leray space, independent of \(L\); also, the \(L^\infty\) norm in time of the relative entropy of \(\psi^t_{\Delta t}/\xi(\rho^L_{\Delta t})\) and the \(L^2\) norm in time of the Fisher information of \(\tilde{\psi}^t_{\Delta t} := \psi^t_{\Delta t}/(\xi(\rho^L_{\Delta t})M)\), bounded, independent of \(L\). We then use these \(L\)-independent bounds on the relative entropy and the Fisher information to derive an \(L\)-independent bound on a fractional-order in time Nikolskiǐ norm of \(u^t_{\Delta t}\).

Step 4. The collection of \(L\)-independent bounds from Step 3, then enables us to extract a weakly convergent subsequence of solutions to problem \((P^t_L)\) as \(L \to \infty\); and then further strongly convergent subsequences \(\{u^t_{\Delta t}\}_{L>1}\frac{}{}\) and \(\{\rho^L_{\Delta t}\}_{L>1}\frac{}{}\). The extraction of a strongly convergent subsequence from the weakly convergent sequence \(\{\tilde{\psi}^t_{\Delta t}\}_{L=1}\frac{}{}\) is considerably more complicated: after some technical preparation, we apply the Div–Curl lemma to obtain a weakly convergent sequence, from which we then extract a strongly convergent subsequence of solutions \(\{\rho^L_{\Delta t}, u^t_{\Delta t}, \tilde{\psi}^t_{\Delta t}\}\) to \((P^t_L)\) with \(\Delta t = o(L^{-1})\) as \(L \to \infty\), in \(L^\infty(0, T; L^p(\Omega)) \times L^2(0, T; L^p(\Omega)) \times L^p(0, T; L^p(M(\Omega \times D)))\) for any \(p \in [1, \infty)\); any \(r \in [1, \infty)\) when \(d = 2\) and any \(r \in [1, 6)\) when \(d = 3\); enabling us to pass to the limit with the microscopic cut-off parameter \(L\) in the model \((P^t_L)\), with \(\Delta t = o(L^{-1})\), as \(L \to \infty\), to deduce the existence of a weak solution to model (P), the general nonhomogeneous FENE-type bead–spring chain model with centre-of-mass diffusion.

The paper is structured as follows. We begin, in Section 2, by stating \((P_L)\), the coupled nonhomogeneous Navier–Stokes–Fokker–Planck system with centre-of-mass diffusion and microscopic cut-off for a general class of FENE-type spring potentials. In Section 3 we establish the existence of solutions to the time-discrete problem \((P^t_L)\). In Section 4 we derive an \(L\)-independent bound on the solvent density \(\rho^L_{\Delta t}\) in \(L^\infty(0, T; L^\infty(\Omega))\); we also derive a set of \(L\)-independent bounds on \(u^t_{\Delta t}\) in the classical Leray space, together with \(L\)-independent bounds on the relative entropy of \(\psi^t_{\Delta t}/\xi(\rho^L_{\Delta t})\) with respect to the Maxwellian \(M\), and the \(L^2\) norm in time of the Fisher information of \(\tilde{\psi}^t_{\Delta t} := \psi^t_{\Delta t}/(\xi(\rho^L_{\Delta t})M)\). We then use these \(L\)-independent bounds on spatial norms to show that the Nikol'skiǐ norm \(N'(0, T; L^2(\Omega))\) of \(u^t_{\Delta t}\) is bounded, independent of \(L\) and \(\Delta t = o(L^{-1})\), for a suitable value of \(\gamma \in (0, 1)\). This allows us to prove, via Simon’s extension of the Aubin–Lions principle, the convergence of the time-discrete approximations to a strong solution.
compactness theorem (cf. [45]), strong-convergence of the sequence \( \{u_{L}^{Δ\alpha}\}_{L > 1} \) in \( L^2(0, T; L'(Ω)) \) for \( r \in [1, \infty) \) when \( d = 2 \) and \( r \in [1, 6) \) when \( d = 3 \). We then use this strong convergence result together with the DiPerna–Lions theory of renormalized solutions to linear transport equations with nonsmooth transport velocities to deduce the strong convergence of the sequence of approximate densities \( \{ρ_{L}^{Δ\alpha}\}_{L > 1} \), and pass to the limit in our approximation to the continuity equation, as \( L \to \infty \), with \( Δ\alpha = o(L^{-1}) \). Weak convergence of the sequence \( \{\tilde{ψ_{L}^{Δ\alpha}}\}_{L > 1} \) in the Maxwellian-weighted \( L^1 \) space \( L^1(0, T; L^1_{M}(Ω \times D)) \) is an immediate consequence of our entropy estimate, via de la Vallée-Poussin’s theorem and the Dunford–Petits theorem. The proof of the strong convergence of the sequence is however considerably more complicated; it is established in Section 4.4, by first developing interior estimates in standard (unweighted) Lebesgue and Sobolev norms, exploiting the fact that on nonempty open relatively compact subsets of \( D \) the Maxwellian is bounded above and below by positive constants. We then use these interior estimates in conjunction with the Div–Curl lemma to deduce weak convergence of the sequence \( (1 + \frac{1}{2} t - 1)^{1/2} \) on nonempty open relatively compact subsets of \((0, T) \times Ω \times D\), where \( α ∈ (0, 1) \). Thus we can make use of the fact that the continuous functions \( s \in [0, \infty) \mapsto (1 + s)^{1/2} \) and \( s \in [0, \infty) \mapsto s^α \) are, respectively, strictly convex and strictly concave, and therefore weakly lower (respectively, upper) semicontinuous, to deduce that \( \{\tilde{ψ_{L}^{Δ\alpha}}\}_{L > 1} \) converges to a limiting function \( \tilde{ψ} \in L^1(0, T; L^1_{M}(Ω \times D)) \), almost everywhere on compact subsets of \((0, T) \times Ω \times D\); hence, by using a nested sequence of nonempty open relatively compact sets, we show that \( \{\tilde{ψ_{L}^{Δ\alpha}}\}_{L > 1} \) converges to \( ψ \) almost everywhere on \((0, T) \times Ω \times D\). Thanks to the fact that \( M(q) dq \) is a probability measure on \( D \), and therefore \( M(q) dq dx dt \) is a finite measure of \((0, T) \times Ω \times D\), Egoroff’s theorem then implies almost uniform convergence of the sequence, and therefore also convergence in measure; thus we can appeal to Vitali’s theorem to finally deduce strong convergence in \( L^1(0, T; L^1_{M}(Ω \times D)) \) of (a subsequence of) the sequence \( \{\tilde{ψ_{L}^{Δ\alpha}}\}_{L > 1} \) to problem \((P_{L}^{Δ\alpha})\), with \( Δ\alpha = o(L^{-1}) \), as \( L \to \infty \), to deduce the existence of a weak solution \( (\rho, u, ψ := M(\rho) \tilde{ψ}) \) to problem \((P)\), the general nonhomogeneous FENE-type bead–spring chain models with centre-of-mass diffusion. We shall operate within Maxwellian-weighted Sobolev spaces, which provide the natural functional-analytic framework for the problem. Our proofs require special density and embedding results in these spaces that are proved, respectively, in Appendix C and Appendix D of [8], which is an extended version of our paper [9] for FENE-type models in the special case of constant density, viscosity and drag.

2. The polymer model \((P_{L})\)

Let \( Ω ⊂ \mathbb{R}^d \) be a bounded open set with a Lipschitz-continuous boundary \( ∂Ω \), and suppose that the set \( D := D_1 × ⋯ × D_K \) of admissible conformation vectors \( q := (q_1, ⋯, q_K)^T \in (19) \) is such that \( D_i, i = 1, ⋯, K \), is an open ball in \( \mathbb{R}^d \), \( d = 2 \) or 3, centred at the origin with boundary \( ∂D_i \) and radius \( √b_i \), \( b_i > 2 \); let

\[
∂D := \bigcup_{i=1}^{K} ∂D_i, \quad \text{where} \quad ∂D_i := D_1 × ⋯ × D_{i-1} × ∂D_i × D_{i+1} × ⋯ × D_K. \tag{2.1}
\]

Collecting (1.1a)–(1.1f), (1.7), (1.8), (1.13) and (1.14a)–(1.14c), we then consider the following initial–boundary-value problem, dependent on the parameter \( L > 1 \). As has been already emphasized in the Introduction, the centre-of-mass diffusion coefficient \( ε > 0 \) is a physical parameter and is regarded as being fixed.

\((P_{L})\) Find \( ρ_{L} : (x, t) ∈ Ω × [0, T) → ρ_{L}(x, t) ∈ \mathbb{R}, \ u_{L} : (x, t) ∈ Ω × [0, T) → u_{L}(x, t) ∈ \mathbb{R}^d \) and \( p_{L} : (x, t) ∈ Ω × (0, T) → p_{L}(x, t) ∈ \mathbb{R} \) such that

\[
\frac{∂ρ_{L}}{∂t} + ∇_x \cdot (u_{L}ρ_{L}) = 0 \quad \text{in} \ Ω × (0, T], \tag{2.2a}
\]

\[
ρ_{L}(x, 0) = ρ_0(x) \quad ∀x ∈ Ω, \tag{2.2b}
\]
We impose the following boundary and initial conditions:

\[
\begin{align*}
\frac{\partial (\rho_L u_L)}{\partial t} + \nabla x \cdot (\rho_L u_L \otimes u_L) - \nabla x \cdot \left( \mu(\rho_L) D(u_L) \right) + \nabla_x p_L &= \rho_L f + \nabla_x \cdot \tau(\psi_L) \quad \text{in } \Omega \times (0, T], \\
\nabla_x \cdot u_L &= 0 \quad \text{in } \Omega \times (0, T], \\
u_L &= 0 \quad \text{on } \partial \Omega \times (0, T], \\
(\rho_L u_L)(\cdot, 0) = (\rho_0 u_0)(\cdot) &\quad \forall x \in \Omega.
\end{align*}
\]

(2.2c)

Here, for a given \( \psi_L : (x, q, t) \in \Omega \times D \times [0, T] \rightarrow \psi_L(x, q, t) \in \mathbb{R} \), and \( \tau(\psi_L) : (x, t) \in \Omega \times (0, T] \rightarrow \tau(\psi_L)(x, t) \in \mathbb{R}^{d \times d} \) is the symmetric extra-stress tensor defined as

\[
\tau(\psi_L) := k \left[ \sum_{i=1}^{K} C_i(\psi_L) - K \varrho(\psi_L) I \right].
\]

(2.3)

Here \( k \in \mathbb{R}_{>0} \), \( I \) is the unit \( d \times d \) tensor,

\[
C_i(\psi_L)(x, t) := \int_D \psi_L(x, q, t) U_i \left( \frac{1}{2} |q_i|^2 \right) q_i q_i^T \, dq, \quad \text{and}
\]

\[
\varrho(\psi_L)(x, t) := \int_D \psi_L(x, q, t) \, dq.
\]

(2.4a)

(2.4b)

The Fokker–Planck equation with microscopic cut-off satisfied by \( \psi_L \) is:

\[
\begin{align*}
\frac{\partial \psi_L}{\partial t} + (u_L \cdot \nabla x) \psi_L + \sum_{i=1}^{K} \nabla q_i \cdot \left[ \sigma(\psi_L) q_i M \xi(\rho_L) \beta^L \left( \frac{\psi_L}{\xi(\rho_L) M} \right) \right] &= \varepsilon \Delta_x \left( \frac{\psi_L}{\xi(\rho_L)} \right) + \frac{1}{4 \lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \nabla q_i \cdot \left( M \nabla q_j \left( \frac{\psi_L}{\xi(\rho_L) M} \right) \right) \quad \text{in } \Omega \times D \times (0, T].
\end{align*}
\]

(2.5)

Here, for a given \( L > 1 \), \( \beta^L \in C(\mathbb{R}) \) is defined by (1.12), \( \sigma(v) \equiv \nabla_x v \), and

\[
A \in \mathbb{R}^{K \times K} \quad \text{is symmetric positive definite with smallest eigenvalue } a_0 \in \mathbb{R}_{>0}.
\]

(2.6)

We impose the following boundary and initial conditions:

\[
\begin{align*}
\sum_{j=1}^{K} A_{ij} M \nabla q_j \left( \frac{\psi_L}{\xi(\rho_L) M} \right) - \sigma(\psi_L) q_i M \xi(\rho_L) \beta^L \left( \frac{\psi_L}{\xi(\rho_L) M} \right) \cdot \frac{q_i}{|q_i|} &= 0 \quad \text{on } \Omega \times \partial \bar{D}_i \times (0, T], \quad i = 1, \ldots, K,
\end{align*}
\]

(2.7a)

\[
\varepsilon \nabla_x \left( \frac{\psi_L}{\xi(\rho_L)} \right) \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega \times D \times (0, T],
\]

(2.7b)

\[
\psi_L(\cdot, t, 0) = M(\cdot) \xi(\rho_0(\cdot)) \beta^L (\psi_0(\cdot, \cdot)/(\xi(\rho_0(\cdot)) M(\cdot))) \geq 0 \quad \text{on } \Omega \times D,
\]

(2.7c)
where \( n \) is the unit outward normal to \( \partial \Omega \). The boundary conditions for \( \psi_L \) on \( \partial \Omega \times D \times (0, T] \) and \( \Omega \times \partial D \times (0, T] \) have been chosen so as to ensure that

\[
\int_{\Omega \times D} \psi_L(x, q, t) \, dq \, dx = \int_{\Omega \times D} \psi_L(x, q, 0) \, dq \, dx \quad \forall t \in (0, T]. \tag{2.8}
\]

Henceforth, we shall write

\[
\tilde{\psi}_L = \frac{\psi_L}{\zeta(\rho_L)M}, \quad \tilde{\psi}_0 = \frac{\psi_0}{\zeta(\rho_0)M}.
\]

Thus, (2.7c) in terms of this compact notation becomes: \( \tilde{\psi}_L(\cdot, \cdot, 0) = \beta^t(\tilde{\psi}_0(\cdot, \cdot)) \) on \( \Omega \times D \).

The notation \( | \cdot | \) will be used to signify one of the following. When applied to a real number \( x \), \( |x| \) will denote the absolute value of \( x \); when applied to a vector \( y \), \( |y| \) will stand for the Euclidean norm of the vector \( y \); and, when applied to a square matrix \( A \), \( |A| \) will signify the Frobenius norm, \( |\text{tr}(A^T A)|^{\frac{1}{2}} \), of the matrix \( A \), where, for a square matrix \( B \), \( \text{tr}(B) \) denotes the trace of \( B \).

3. Existence of a solution to the discrete-in-time problem

Let

\[
H := \{ w \in L^2(\Omega) : \nabla_x \cdot w = 0, (w \cdot n)|_{\partial \Omega} = 0 \} \quad \text{and} \quad V := \{ w \in H^1_0(\Omega) : \nabla_x \cdot w = 0 \}. \tag{3.1}
\]

where the divergence operator \( \nabla_x \cdot \) is to be understood in the sense of distributions on \( \Omega \). Let \( V' \) be the dual of \( V \).

For later purposes, we recall the following well-known Gagliardo–Nirenberg inequality. Let \( r \in [2, \infty) \) if \( d = 2 \), and \( r \in [2, 6] \) if \( d = 3 \) and \( \theta = d(\frac{1}{2} - \frac{1}{r}) \). Then, there is a constant \( C = C(\Omega, r, d) \), such that, for all \( \eta \in H^1(\Omega) \):

\[
\| \eta \|_{L^r(H^1(\Omega))} \leq C \| \eta \|_{L^2(\Omega)}^{1-\theta} \| \eta \|_{H^1(\Omega)}^\theta. \tag{3.2}
\]

Let \( F \in C(\mathbb{R}_{\geq 0}) \) be defined by \( F(s) := s(\log s - 1) + 1, s > 0 \). As \( \lim_{s \to 0^+} F(s) = 1 \), the function \( F \) can be considered to be defined and continuous on \( [0, \infty) \), where it is a nonnegative, strictly convex function with \( F(1) = 0 \). We assume the following:

\[
\partial \Omega \in C^{0,1}; \quad \rho_0 \in [\rho_{\min}, \rho_{\max}], \quad \text{with} \ \rho_{\min} > 0; \quad u_0 \in H; \quad \psi_0 \geq 0 \quad \text{a.e. on} \ \Omega \times D \quad \text{with} \ F(\tilde{\psi}_0) \in L^1_M(\Omega \times D) \quad \text{and} \quad \int_D \psi_0(x, q) \, dq \in L^\infty(\Omega); \quad \int_{\Omega \times D} \psi_0(x, q) \, dq \, dx = 1; \quad \text{the Rouse matrix} \ A \in \mathbb{R}^{K \times K} \text{ satisfies (2.6)};
\]

\[
\mu \in C([\rho_{\min}, \rho_{\max}], [\mu_{\min}, \mu_{\max}]), \quad \zeta \in C^1([\rho_{\min}, \rho_{\max}], [\zeta_{\min}, \zeta_{\max}]), \quad \text{with} \ \mu_{\min}, \zeta_{\min} > 0,
\]

\[
f \in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad D_i = B(0, b_i^\frac{1}{\gamma_i}), \quad \gamma_i > 1, \quad i = 1, \ldots, K \in (1.5a), (1.5b). \tag{3.3}
\]
where \( \kappa > 1 \) if \( d = 2 \) and \( \kappa = \frac{6}{5} \) if \( d = 3 \). For this range of \( \kappa \), we note from (3.2) that there exists a constant \( C_\kappa \in \mathbb{R}_{>0} \) such that
\[
\left| \int_\Omega w_1 \cdot w_2 \, dx \right| \leq C_\kappa \| w_1 \|_{L^2(\Omega)} \| w_2 \|_{H^1(\Omega)} \quad \forall w_1 \in L^2(\Omega), \ w_2 \in H^1_0(\Omega).
\]

(3.4)

In (3.3), \( L^p_M(\Omega \times D) \), for \( p \in [1, \infty) \), denotes the Maxwellian-weighted \( L^p \) space over \( \Omega \times D \) with norm
\[
\| \varphi \|_{L^p_M(\Omega \times D)} := \left\{ \int_{\Omega \times D} M|\varphi|^p \, dq \, dx \right\}^{\frac{1}{p}}.
\]

Similarly, we introduce \( L^p_M(D) \), the Maxwellian-weighted \( L^p \) space over \( D \). Letting
\[
\| \varphi \|_{H^1_M(\Omega \times D)} := \left\{ \int_{\Omega \times D} M[|\varphi|^2 + |\nabla_x \varphi|^2 + |\nabla_q \varphi|^2] \, dq \, dx \right\}^{\frac{1}{2}},
\]

we then set
\[
X \equiv H^1_M(\Omega \times D) := \{ \varphi \in L^1_{\text{loc}}(\Omega \times D) : \| \varphi \|_{H^1_M(\Omega \times D)} < \infty \}.
\]

(3.5)

(3.6)

It is shown in Appendix C of [8] (with the set \( X \) denoted by \( \tilde{X} \) there) that
\[
C^\infty(\Omega \times D) \text{ is dense in } X.
\]

(3.7)

We have from the Sobolev embedding theorem that
\[
H^1(\Omega; L^2_M(D)) \hookrightarrow L^s(\Omega; L^2_M(D)),
\]

where \( s \in [1, \infty) \) if \( d = 2 \) or \( s \in [1, 6] \) if \( d = 3 \). In addition, we note that the embeddings
\[
H^1_M(D) \hookrightarrow L^2_M(D),
\]

(3.9a)

\[
H^1_M(\Omega \times D) = L^2(\Omega; H^1_M(D)) \cap H^1(\Omega; L^2_M(D)) \hookrightarrow L^2_M(\Omega \times D) \equiv L^2(\Omega; L^2_M(D))
\]

(3.9b)

are compact if \( \gamma_i \geq 1 \), \( i = 1, \ldots, K \), in (1.5a), (1.5b); see Appendix D of [8].

We recall the Aubin–Lions–Simon compactness theorem, see, e.g., Simon [45, Theorem 5]. Let \( E_0, B \) and \( B_1 \) be Banach spaces, \( B_i, i = 0, 1 \), reflexive, with a compact embedding \( E_0 \hookrightarrow B \) and a continuous embedding \( B \hookrightarrow B_1 \). Then, any bounded closed subset \( E \) of \( L^2(0, T; E_0) \), such that
\[
\int_0^T \| \eta(t) - \eta(t - \theta) \|_{B_1}^2 \, dt \to 0 \quad \text{as } \theta \to 0, \text{ uniformly for } \eta \in E,
\]

(3.10)

is compact in \( L^2(0, T; B) \).

We shall also require the following two results, which are simple consequences of Lemma 1.1, Chapter III, Section 1, in [47]. For their proofs, see the proofs of Corollaries 3.1 and 3.2 in the extended version of the present paper [10].
**Proposition 3.1.** Suppose that \((a, b)\) is a bounded open subinterval of \(\mathbb{R}\) and let \(\mathcal{B}\) be a Banach space. Suppose further that \(u \in W^{1, 1}(a, b; \mathcal{B})\); then,

\[
 u(t) = u(a) + \int_a^t \frac{du}{ds}(s) \, ds \quad \text{for all } t \in [a, b].
\]

**Proposition 3.2.** Suppose that \(g \in L^1(a, b; \mathcal{B})\) and \(\eta \in \mathcal{B}'\). Then,

\[
 \left\langle \int_a^b g(t) \, dt, \eta \right\rangle = \int_a^b \left\langle g(t), \eta \right\rangle \, dt,
\]

where \(\langle \cdot, \cdot \rangle\) is the duality pairing between \(\mathcal{B}\) and \(\mathcal{B}'\).

Throughout we will assume that (3.3) hold, so that (1.6) and (3.9a), (3.9b) hold. We note for future reference that (2.4a) and (1.6) yield that, for \(\varphi \in L^2(M \times D)\),

\[
\int_{\Omega} \left| \int D M (\varphi) \right|^2 \, dx \leq C \left( \int_{\Omega \times D} M \left| \varphi \right|^2 \, dq \right),
\]

where \(C\) is a positive constant.

We state a simple integration-by-parts formula.

**Lemma 3.1.** Let \(\varphi \in H^1_M(D)\) and suppose that \(B \in \mathbb{R}^{d \times d}\) is a square matrix such that \(\text{tr}(B) = 0\); then,

\[
\int_D \sum_{i=1}^K (Bq_i) : \nabla x q_i \, dq = \int_D M \varphi \sum_{i=1}^K U_i \left( \frac{1}{2} |q_i|^2 \right) q_i q_i^T : B dq.
\]

**Proof.** See the proof of Lemma 3.1 in [9]. \(\square\)

We now formulate our discrete-in-time approximation of problem (P) for fixed parameters \(\varepsilon \in (0, 1]\) and \(L > 1\). For any \(T > 0\) and \(N \geq 1\), let \(\Delta t = T / N\) and \(t_n = n \Delta t\), \(n = 0, \ldots, N\). To prove existence of a solution under minimal smoothness requirements on the initial datum \(u_0 \in H\) (recall (3.3)), we assign to it the function \(u^0 = u^0(\Delta t) \in V\), defined as the unique solution of

\[
\int_{\Omega} \left[ \rho_0 u_0^0 \cdot \varphi + \Delta t \nabla x u_0^0 : \nabla x \varphi \right] \, dx = \int_{\Omega} \rho_0 u_0 \cdot \varphi \, dx \quad \forall \varphi \in V.
\]

Hence,
\[
\int_\Omega \left[ \rho_0 |u_0^0|^2 + \Delta t |\nabla_x u_0^0|^2 \right] \, dx \leq \int_\Omega \rho_0 |u_0^0|^2 \, dx \leq C. \tag{3.14}
\]

In addition, we have that \( \int_\Omega \rho_0 (u_0^0 - u_0) \cdot v \, dx \) converges to 0 for all \( v \in H \) in the limit of \( \Delta t \to 0_+ \).

Analogously to defining \( u_0^0 \in V \) for a given initial velocity field \( u_0 \in H \), we shall assign a certain 'smoothed' initial datum,

\[
\tilde{\psi}^0 = \tilde{\psi}^0(L, \Delta t) \in H^1_M(\Omega \times D),
\]
to the given initial datum \( \tilde{\psi}_0 = \psi_0 / (\zeta(\rho_0) M) \) such that

\[
\int_{\Omega \times D} M[\zeta(\rho_0) \psi^0 \varphi + \Delta t (\nabla_x \psi^0 \cdot \nabla_x \varphi + \nabla_q \psi^0 \cdot \nabla_q \varphi)] \, dq \, dx \\
= \int_{\Omega \times D} M \zeta(\rho_0) \beta^1(\tilde{\psi}_0) \varphi \, dq \, dx \quad \forall \varphi \in H^1_M(\Omega \times D). \tag{3.15}
\]

For \( p \in [1, \infty) \), let

\[
Z_p := \left\{ \varphi \in L^p_M(\Omega \times D): \varphi \geq 0 \text{ a.e. on } \Omega \times D \text{ and } \int_D M(q) \varphi(x, q) \, dq \in L^\infty(\Omega) \right\}. \tag{3.16}
\]

It is proved in the special case of \( \zeta(\rho_0(\cdot)) \equiv 1 \) in the appendix of [12] that there exists a unique \( \tilde{\psi}^0 \in H^1_M(\Omega \times D) \) satisfying (3.15); furthermore, \( \tilde{\psi}^0 \in Z_2 \).

\[
\int_{\Omega \times D} M \zeta(\rho_0) \mathcal{F}(\tilde{\psi}^0) \, dq \, dx + 4\Delta t \int_{\Omega \times D} M \left[ |\nabla_x \sqrt{\tilde{\psi}^0}|^2 + |\nabla_q \sqrt{\tilde{\psi}^0}|^2 \right] \, dq \, dx \\
\leq \int_{\Omega \times D} M \zeta(\rho_0) \mathcal{F}(\tilde{\psi}_0) \, dq \, dx, \tag{3.17a}
\]

\[
\text{ess.sup}_{\tilde{x} \in \Omega} \int_D M \tilde{\psi}^0 \, dq \leq \text{ess.sup}_{\tilde{x} \in \Omega} \int_D M \tilde{\psi}_0 \, dq \tag{3.17b}
\]

and

\[
\tilde{\psi}^0 = \beta^1(\tilde{\psi}_0) \to \tilde{\psi}_0 \text{ weakly in } L^1_M(\Omega \times D) \text{ as } L \to \infty, \Delta t \to 0_+. \tag{3.17c}
\]

In the case of variable \( \zeta(\rho_0(\cdot)) \) the same properties hold under the assumptions on \( \rho_0 \) and \( \zeta \) stated in (3.3). For example, the claim in (3.17c) that \( \tilde{\psi}^0 = \beta^1(\tilde{\psi}_0) \) a.e. on \( \Omega \times D \) follows from (3.15) on replacing \( \tilde{\psi}^0 \) by \( \tilde{\psi}_0 - L \) on the left-hand side of (3.15) and \( \beta^1(\tilde{\psi}_0) \) by \( \beta^1(\tilde{\psi}_0) - L \) on the right-hand side, which preserves the equality. We then take \( \varphi = [\tilde{\psi}_0 - L]_+ \) and note that \( \beta^1(\tilde{\psi}_0) - L \leq 0 \) to deduce, thanks to the positivity of \( \zeta(\rho_0) \) on \( \Omega \), that \( [\tilde{\psi}_0 - L]_+ = 0 \) a.e. on \( \Omega \times D \), which then implies that \( \tilde{\psi}^0 \leq L \) a.e. on \( \Omega \times D \). An analogous argument shows that \( \tilde{\psi}^0 \geq 0 \) a.e. on \( \Omega \times D \). In particular, \( \tilde{\psi}^0 = \beta^1(\tilde{\psi}_0) \) a.e. on \( \Omega \times D \) and \( \tilde{\psi}^0 \in Z_2 \), as was claimed in the line above (3.17a). In fact, \( \tilde{\psi}^0 \in L^\infty(\Omega \times D) \cap H^1_M(\Omega \times D) \).

The proof of (3.17b) is based on a similar cut-off argument. By defining, for \( \tilde{\psi}^0 = \tilde{\psi}_0(L, \Delta t) \), the function \( \chi_L^+ \) by
We shall denote the mapping
\[
\lambda^0_L(x) := \int_D M(q) \tilde{\psi}^0(x, q) \, dq, \quad x \in \Omega,
\]
and therefore, for each \( \omega \in \mathbb{R} \) and all \( \tilde{\psi} \in H^1(\Omega) \),
\[
\int_{\Omega} \left[ \zeta(\rho_0)(\lambda^0_L - \omega) \tilde{\psi} + \Delta t \nabla x (\lambda^0_L - \omega) \cdot \nabla \tilde{\psi} \right] \, dx
= \int_{\Omega} \zeta(\rho_0) \tilde{\psi} \left[ \int_D M(q) \beta^L(\tilde{\psi}_0) \, dq \right] \, dx \quad \forall \tilde{\psi} \in H^1(\Omega),
\]
applying (3.15) with \( \varphi(x, q) = \tilde{\varphi}(x) \otimes 1(q) \) and recalling Fubini’s theorem, we have that
\[
\int_{\Omega} \left[ \zeta(\rho_0)(\lambda^0_L - \omega) \tilde{\psi} + \Delta t \nabla x (\lambda^0_L - \omega) \cdot \nabla \tilde{\psi} \right] \, dx
= \int_{\Omega} \zeta(\rho_0) \tilde{\psi} \left( \int_D M(q) \beta^L(\tilde{\psi}_0) \, dq \right) \, dx - \omega \int_{\Omega} \tilde{\psi} \, dx.
\]
Now thanks to (3.3) we have that
\[
0 \leq \zeta(\rho_0) \int_D M(q) \beta^L(\tilde{\psi}_0) \, dq \leq \zeta(\rho_0) \int_D M(q) \tilde{\psi}_0 \, dq = \int_D \psi_0(x, q) \, dq \in L^\infty(\Omega).
\]
Therefore by selecting
\[
\omega := \text{ess.sup}_{x \in \Omega} \int_D M(q) \tilde{\psi}_0(x, q) \, dq = \text{ess.sup}_{x \in \Omega} \left( \frac{1}{\zeta(\rho_0(x))} \int_D \psi_0(x, q) \, dq \right)
\]
and by choosing \( \tilde{\psi} = [\lambda^0_L - \omega]_+ \) in (3.18), we deduce that
\[
\int_{\Omega} \left[ \zeta(\rho_0)([\lambda^0_L - \omega]_+)^2 + \Delta t |\nabla x ([\lambda^0_L - \omega]_+)^2 | \right] \, dx \leq 0.
\]
Hence, \( [\lambda^0_L - \omega]_+ = 0 \) a.e. on \( \Omega \). In other words, \( 0 \leq \lambda^0_L(x) \leq \omega \) a.e. on \( \Omega \), which then implies (3.17b).

We shall denote the mapping \( \tilde{\psi}_0 \mapsto \tilde{\psi}^0 \) by \( S_{\Delta t, L} \); i.e., \( \tilde{\psi}^0 = S_{\Delta t, L} \tilde{\psi}_0 \).

Let us define
\[
\mathcal{V} := \{ \eta \in L^\infty(\Omega) : \eta \in [\rho_{\text{min}}, \rho_{\text{max}}] \text{ a.e. on } \Omega \}.
\]
It follows for all \( \mathbf{v}, \mathbf{w} \in H^1(\Omega) \) that
\[
\mathbf{v} \otimes \mathbf{w} : \nabla \mathbf{x} \mathbf{w} = ( (\mathbf{v} \cdot \nabla \mathbf{x}) \mathbf{w} ) \cdot \mathbf{v} = -((\mathbf{v} \cdot \nabla \mathbf{x}) \mathbf{v}) \cdot \mathbf{w} + (\mathbf{v} \cdot \nabla \mathbf{x})(\mathbf{v} \cdot \mathbf{w}) \quad \text{a.e. in } \Omega.
\]
Noting the above, our discrete-in-time approximation of (P_L) is then defined as follows.

\((P_{\Delta t}^L)\) Let \( N \in \mathbb{N}_{\geq 1} \) and define \( \Delta t := T/N \); let, further, \( \rho^0_L := \rho_0 \in \mathcal{V} \), \( u^0_L := u^0 \in V \) and \( \tilde{\psi}_0^L := \tilde{\psi}_0 \in Z_2 \). For \( n = 1, \ldots, N \), and given \( (\rho^{n-1}_L, u^{n-1}_L, \tilde{\psi}^{n-1}_L) \in \mathcal{Y} \times V \times Z_2 \), find
It follows from (3.3) and (3.24) that

\[ \langle \cdot \rangle \]

where the symbol \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( W^{1, \frac{q}{q-1}}(\Omega) \) and \( W^{1, \frac{q}{q-1}}(\Omega) \), with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). We then define \( \rho^n_L := \rho^{[\Delta t]}|_{t_{n-1}, t_n}(\cdot, t_n) \in \mathcal{Y} \), and find \( (u^n_L, \tilde{\psi}^n_L) \in \mathcal{V} \times (X \cap Z_2) \) such that

\[
\int_{\Omega} \left[ \rho^n_L u^n_L - \rho^{n-1}_L u^{n-1}_L - \frac{1}{2} \rho^n_L - \rho^{n-1}_L u^n_L \right] \cdot w \, dx + \int_{\Omega} \mu(\rho^n_L) D(u^n_L) : D(w) \, dx \\
+ \frac{1}{2} \int_{t_{n-1}}^{t_n} \left[ \rho^{[\Delta t]} \right] [((u^n_L - \nabla_x) u^n_L) \cdot w - ((u^{n-1}_L - \nabla_x) w) \cdot u^n_L] \, dx \\
= \int_{\Omega} \rho^n_L f^n \cdot w \, dx - k \sum_{i=1}^{K} \int_{\Omega} C_i (M \zeta(\rho^n_L) \tilde{\psi}^n_L) : \nabla w \, dx \quad \forall w \in \mathcal{V},
\]

\[
\int_{\Omega \times D} \frac{M \zeta(\rho^n_L) \tilde{\psi}^n_L - \zeta(\rho^{n-1}_L) \tilde{\psi}^{n-1}_L}{\Delta t} \varphi \, dq \, dx \\
- \int_{\Omega \times D} M \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{[\Delta t]}_L) \right) u^{n-1}_L \cdot (\nabla_x \varphi) \tilde{\psi}^n_L \, dq \, dx \\
+ \int_{\Omega \times D} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} M \nabla q_i \tilde{\psi}^n_L - \sigma(u^n_L) q_i M \zeta(\rho^n_L) \beta^j(\tilde{\psi}^n_L) \cdot \nabla q_i \varphi \, dq \, dx \\
+ \int_{\Omega \times D} \varepsilon M \nabla_x \tilde{\psi}^n_L \cdot \nabla_x \varphi \, dq \, dx = 0 \quad \forall \varphi \in X;
\]

where, for \( t \in [t_{n-1}, t_n] \) and \( n = 1, \ldots, N \),

\[
\varphi^{\Delta t, +}(\cdot, t) = f^n(\cdot) := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} f(\cdot, t) \, dt \in L^\infty(\Omega).
\]

It follows from (3.3) and (3.24) that
\[ f^{\Delta t,+} \rightarrow f \] strongly in \( L^2(0, T; L^\infty(\Omega)) \) as \( \Delta t \rightarrow 0_+ \), \hspace{1cm} (3.25) 

where \( \lambda > 1 \) if \( d = 2 \) and \( \lambda = \frac{6}{5} \) if \( d = 3 \). Note that as the test function \( w \) in (3.23b) is chosen to be divergence-free, the term containing the density \( q \) of the polymer chains in the definition of \( \tau \) (cf. (2.3)) is eliminated from (3.23b).

For \( n \in \{1, \ldots, N\} \), and for the functions \( u^{n-1}_L \in V \) and \( \rho^{n-1}_L \in \bar{Y} \) fixed, the existence of a unique solution

\[ \rho^{\Delta t[L]}_L |_{[t_{n-1}, t_n]} \in L^\infty(t_{n-1}, t_n; L^\infty(\Omega)) \cap C\{[t_{n-1}, t_n]; L^2(\Omega)\} \hspace{1cm} (3.26) \]

to (3.23a) satisfying the initial condition \( \rho^{\Delta t[L]}_L |_{[t_{n-1}, t_n]}(\cdot, t_{n-1}) = \rho^{n-1}_L \) follows from Corollaries II.1 and II.2 and the discussion on p. 546 in DiPerna & Lions [18] (with our \( u^n_L \in V \) here extended from \( \bar{\Omega} \) to \( \mathbb{R}^d \) by 0). We refer to Appendix A in [10] for a justification that the notion of solution used in (3.22), (3.23a) is equivalent to the notion of distributional solution, used by DiPerna & Lions in [18].

The statement

\[ \rho^{\Delta t[L]}_L |_{[t_{n-1}, t_n]} \in W^{1,\infty}(t_{n-1}, t_n; W^{1,\frac{q}{q-1}}(\Omega)) \]

in (3.23a), with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), follows from the bound

\[
\left| \int_{t_{n-1}}^{t_n} \int_{\Omega} u^{n-1}_L \cdot \rho^{\Delta t[L]}_L \cdot \nabla \psi \, dx \, dt \right| \leq \left\| u^{n-1}_L \right\|_{L^3(\Omega)} \left\| \rho^{\Delta t[L]}_L \right\|_{L^\infty(t_{n-1}, t_n; L^\infty(\Omega))} \left\| \nabla \psi \right\|_{L^1(t_{n-1}, t_n; L^{\frac{q}{q-1}}(\Omega))}
\]

and the fact that \( \frac{\partial}{\partial t} \rho^{\Delta t[L]}_L + \nabla \cdot (u^{n-1}_L \rho^{\Delta t[L]}_L) = 0 \) in the sense of distributions on \( \Omega \times (t_{n-1}, t_n) \); hence (3.22).

A further relevant remark in connection with (3.23b) is that on noting that \( \rho^n_L = \rho^{\Delta t[L]}_L (\cdot, t_n) \) and \( \rho^{n-1}_L = \rho^{\Delta t[L]}_L (\cdot, t_{n-1}) \), the second term on its left-hand side can be rewritten as

\[
-\frac{1}{2} \int_\Omega \frac{\rho^n_L - \rho^{n-1}_L}{\Delta t} u^n_L \cdot w \, dx \, dt
\]

\[ = -\frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} \left( \frac{\partial \rho^{\Delta t[L]}_L}{\partial t} \cdot u^n_L \cdot w \right)_{W^{1,\frac{q}{q-1}}(\Omega)} \, dt \]

\[ = -\frac{1}{2} \int_\Omega \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \rho^{\Delta t[L]}_L \, dt \right) \left[ u^{n-1}_L \cdot \nabla \left( u^n_L \cdot w \right) \right] \, dx \quad \forall w \in H^1(\Omega), \hspace{1cm} (3.27) \]

where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). The first equality in (3.27) is a consequence of Proposition 3.1 and Proposition 3.2, on noting that \( \frac{\partial}{\partial t} \rho^{\Delta t[L]}_L \in L^\infty(t_{n-1}, t_n; W^{1,\frac{q}{q-1}}(\Omega)) \subset H^1(t_{n-1}, t_n; W^{1,\frac{q}{q-1}}(\Omega)) \), with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \).

The identities (3.21) and (3.27) now motivate the form of the expression in the second line of (3.23b). We note here that the requirements that \( q > 2 \) when \( d = 2 \) and \( q \geq 3 \) when \( d = 3 \) are the consequence of our demand that the scalar product \( u^n_L \cdot w \) of the functions \( u^n_L, w \in H^1(\Omega) \) belongs to \( W^{1,\frac{q}{q-1}}(\Omega) \), which is required in (3.27).
As $\rho_{t_0}^{-1} \in \mathcal{Y}$, $u_{t_0}^{-1} \in V$ (extended from $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^d$ by 0), $\rho^{[\Delta t]}_{t_0} \in C([t_{n-1}, t_n]; L^2(\Omega))$, $\xi \in C^1(\{\rho_{\min}, \rho_{\max}, \zeta_{\min}, \zeta_{\max}\})$, it follows from Corollary II.2 in the paper of DiPerna & Lions [18] that $\zeta(\rho_{[\Delta t]}^n)$ is a renormalized solution in the sense that

$$
\int_{t_{n-1}}^{t_n} \left( \frac{\partial \zeta(\rho^{[\Delta t]})}{\partial t}, \varphi \right)_{W^{-1, q}(\Omega)} \, dt
$$

$$
= \int_\Omega \left( \int_{t_{n-1}}^{t_n} \zeta(\rho^{[\Delta t]}) \, dt \right) [u^{n-1}_{[\Delta t]} \cdot \nabla_x \varphi] \, dx \quad \forall \varphi \in W^{1, q}_0(\Omega),
$$

(3.28)

where $q \in (2, \infty)$ when $d = 2$ and $q \in [3, 6]$ when $d = 3$. Hence, on observing that $\zeta(\rho_{[\Delta t]}^n) = \zeta(\rho^{[\Delta t]}_t, t_n)$ and $\zeta(\rho_{[\Delta t]}^{n-1}) = \zeta(\rho^{[\Delta t]}_t, t_{n-1})$, we have that

$$
\int_\Omega \frac{\zeta(\rho_{[\Delta t]}^n) - \zeta(\rho_{[\Delta t]}^{n-1})}{\Delta t} \varphi \, dx \, dt = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \left( \frac{\partial \zeta(\rho^{[\Delta t]})}{\partial t}, \varphi \right)_{W^{-1, q}(\Omega)} \, dt
$$

$$
= \int_\Omega \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{[\Delta t]}) \, dt \right) [u^{n-1}_{[\Delta t]} \cdot \nabla_x \varphi] \, dx \quad \forall \varphi \in W^{1, q}_0(\Omega),
$$

(3.29)

where $q \in (2, \infty)$ when $d = 2$ and $q \in [3, 6]$ when $d = 3$. The first equality in (3.29) is a consequence of Proposition 3.1 and Proposition 3.2, on noting that $\frac{\partial}{\partial t}\zeta(\rho^{[\Delta t]}_t) \in L^1(t_{n-1}, t_n; W^{1, q}_0(\Omega'))$, with $q \in (2, \infty)$ when $d = 2$ and $q \in [3, 6]$ when $d = 3$. Since $\tilde{\psi}_L^n \in X$, it follows from (3.29) that

$$
-\frac{1}{2} \int_\Omega \frac{\zeta(\rho_{[\Delta t]}^n) - \zeta(\rho_{[\Delta t]}^{n-1})}{\Delta t} \tilde{\psi}_L \varphi \, dx \, dt = -\frac{1}{2} \int_\Omega \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{[\Delta t]}) \, dt \right) [u^{n-1}_{[\Delta t]} \cdot \nabla_x (\tilde{\psi}_L \varphi)] \, dx
$$

$$
\quad \forall \varphi \in X, \quad \text{a.e. } q \in D,
$$

and therefore we can rewrite (3.23c) in the following equivalent form:

$$
\int_{\Omega \times D} M \left[ \frac{\zeta(\rho_{[\Delta t]}^n) \tilde{\psi}_L^n - \zeta(\rho_{[\Delta t]}^{n-1}) \tilde{\psi}_L^{n-1}}{\Delta t} - \frac{1}{2} \frac{\zeta(\rho_{[\Delta t]}^n) - \zeta(\rho_{[\Delta t]}^{n-1})}{\Delta t} \tilde{\psi}_L^n \right] \varphi \, dq \, dx
$$

$$
+ \frac{1}{2} \int_{\Omega \times D} M \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{[\Delta t]}) \, dt \right) [u^{n-1}_{[\Delta t]} \cdot (\nabla_x \tilde{\psi}_L^n) \varphi - (\nabla_x \tilde{\psi}_L^n) \tilde{\psi}_L^n] \, dq \, dx
$$

$$
+ \int_{\Omega \times D} \sum_{i=1}^{K} \sum_{j=1}^{4k} A_{ij} M \nabla q_i \tilde{\psi}_L^n \left[ \left( \sigma (u^n) q_i \right) M \zeta(\rho_{[\Delta t]}^n) \right] \cdot \nabla_{q_i} \varphi \, dq \, dx
$$

$$
+ \int_{\Omega \times D} \epsilon M \nabla_{x} \tilde{\psi}_L^n \cdot \nabla_x \varphi \, dq \, dx = 0 \quad \forall \varphi \in X.
$$

(3.30)

The following elementary result will play a crucial role; we refer to [10] for its proof.
Lemma 3.2. Suppose that $S \subset \mathbb{R}$ is an open interval and let $F \in W^{2,1}_{loc}(S)$. Let further $G$ denote the primitive function of $s \in S \mapsto sF''(s) \in \mathbb{R}$; i.e., $G'(s) = sF''(s)$, $s \in S$. Then, the following statements hold.

a) $s \in S \mapsto sF'(s) - F(s) - G(s) \in \mathbb{R}$ is a constant function on $S$; i.e., there exists $c_0 \in \mathbb{R}$ such that $sF'(s) - F(s) - G(s) = c_0$ for all $s \in S$.

b) The following identity holds for any $a, b \in S$ and any $A, B \in \mathbb{R}$:

$$
(Aa - Bb)F'(a) - (A - B)G(a) = A(F(a) + c_0) - B(F(b) + c_0) + B(b - a)^2 \int_0^1 F''(\theta a + (1 - \theta)b)\,d\theta.
$$

c) If, in addition, $B \geq 0$ and there exists a $d_0 \in \mathbb{R}$ such that $\text{ess}\inf_{\theta \in [0,1]} F''(\theta a + (1 - \theta)b) \geq d_0$, then

$$
(Aa - Bb)F'(a) - (A - B)G(a) \geq A(F(a) + c_0) - B(F(b) + c_0) + \frac{1}{2}d_0B(b - a)^2.
$$

In order to prove the existence of a solution to $(P_{\delta,L}^4)$, we require the following convex regularization $\mathcal{F}_\delta^4 \in C^{2,1}(\mathbb{R})$ of $\mathcal{F}$ defined, for any $\delta \in (0, 1)$ and $L > 1$, by

$$
\mathcal{F}_\delta^4(s) := \begin{cases} 
\frac{s^2-\beta^2}{2\delta} + s(\log\delta - 1) + 1 & \text{for } s \leq \delta, \\
\mathcal{F}(s) = s(\log s - 1) + 1 & \text{for } \delta \leq s \leq L, \\
\frac{s^2}{2L} - s^2 + s(\log L - 1) + 1 & \text{for } s \leq L.
\end{cases}
$$

Hence,

$$
\left[\mathcal{F}_\delta^4\right]'(s) = \begin{cases} 
\frac{1}{\delta} + \log\delta - 1 & \text{for } s \leq \delta, \\
\log s & \text{for } \delta \leq s \leq L, \\
\frac{1}{L} + \log L - 1 & \text{for } s \leq L,
\end{cases}
$$

$$
\left[\mathcal{F}_\delta^4\right]''(s) = \begin{cases} 
\delta^{-1} & \text{for } s \leq \delta, \\
1 & \text{for } \delta \leq s \leq L, \\
L^{-1} & \text{for } s \leq L.
\end{cases}
$$

We note that

$$
\mathcal{F}_\delta^4(s) \geq \begin{cases} 
\frac{s^2}{2\delta} & \text{for } s \leq 0, \\
\frac{s^2}{4L} - C(L) & \text{for } s \geq 0;
\end{cases}
$$

and that $\left[\mathcal{F}_\delta^4\right]''(s)$ is bounded below by $1/L$ for all $s \in \mathbb{R}$. Finally, we set

$$
\beta^4_\delta(s) := \left[\left[\mathcal{F}_\delta^4\right]''\right]^{-1}(s) = \max\{\beta^4(s), \delta\},
$$

and observe that $\beta^4_\delta(s)$ is bounded above by $L$ and bounded below by $\delta$ for all $s \in \mathbb{R}$. Note also that both $\beta^4$ and $\beta^4_\delta$ are Lipschitz continuous on $\mathbb{R}$, with Lipschitz constants equal to 1.
3.1. Existence of a solution to \((P^\Delta_L)\)

On recalling the discussion following (3.26), for \(n \in \{1, \ldots, N\}\) we define the function \(\rho^n_L := \rho^n_L|_{[t_{n-1}, t_n]}(\cdot, t_n) \in L^\infty(\Omega)\); it will be shown below that \(\rho^n_L \in \mathcal{Y}\), in fact. With \(\rho^n_L|_{[t_{n-1}, t_n]}\) thus fixed (and with its values \(\rho^n_L\) at \(t = t_{n-1}\) and \(t = t_n\), respectively, also fixed) we rewrite (3.23b) as

\[
\begin{align*}
 b(u^n_L, w) = & \ell_b(\overline{v};) (w) \quad \forall w \in \mathcal{V}; \\
\text{where, for all } w_i \in H^1_0(\Omega), \ i = 1, 2, \end{align*}
\]

\[
\begin{align*}
b(w_1, w_2) := & \int_{\Omega} \left[ \frac{1}{2} \left( \rho^n_L + \rho^n_L^{-1} \right) w_1 \cdot w_2 + \Delta t \mu(\rho^n_L) D(w_1) : D(w_2) \right] dx \\
& + \frac{1}{2} \Delta t \int_{t_{n-1}}^{t_n} \left[ \int_{\Omega} \rho^n_L^{[\Delta t]} dt \right] \left[ \left( u^{n-1}_L \cdot \nabla w_1 \right) \cdot w_2 - \left( u^{n-1}_L \cdot \nabla w_2 \right) \cdot w_1 \right] dx
\end{align*}
\]

(3.36a)

and, for all \(\varphi \in L^2_M(\Omega \times D)\) and \(w \in H^1_0(\Omega),\)

\[
\ell_b(\varphi) (w) := \int_{\Omega} \left[ \rho^n_L^{-1} u^{n-1}_L \cdot w + \Delta t \rho^n_L f^n \cdot w - \Delta t \sum_{i=1}^{K} \xi_i (M \xi (\rho^n_L) \varphi) \cdot \nabla x w \right] dx. \quad (3.36b)
\]

It follows from Korn's inequality

\[
\int_{\Omega} |D(w)|^2 dx \geq c_0 \| w \|_{H^1(\Omega)}^2 \quad \forall w \in H^1_0(\Omega), \quad (3.37)
\]

where \(c_0 > 0\), that, for \(u^{n-1}_L \in V\) and \(\rho^n_L^{-1}, \rho^n_L \in \mathcal{Y}\) fixed, \(b(\cdot, \cdot)\) is a continuous nonsymmetric coercive bilinear functional on \(H^1_0(\Omega) \times H^1_0(\Omega)\). In addition, for \(u^{n-1}_L \in V\) and \(\rho^n_L^{-1}, \rho^n_L \in \mathcal{Y}\) fixed, thanks to (3.4) and (3.11), \(\ell_b(\varphi)(\cdot)\) is a continuous linear functional on \(H^1_0(\Omega)\) for any \(\varphi \in L^2_M(\Omega \times D)\).

It is also convenient to rewrite (3.23c) (or, equivalently, (3.30)) as

\[
a(\overline{v};, \varphi) = \ell_a(u^n_L, \beta^L(\overline{v};))(\varphi) \quad \forall \varphi \in X, \quad (3.38)
\]

where, for all \(\varphi_i \in X, \ i = 1, 2,\)

\[
a(\varphi_1, \varphi_2) := \int_{\Omega \times D} M \left[ \xi (\rho^n_L) \varphi_1 \varphi_2 + \Delta t \varepsilon \nabla \varphi_1 \cdot \nabla \varphi_2 - \Delta t \left( \int_{t_{n-1}}^{t_n} \xi (\rho^n_L^{[\Delta t]}) dt \right) u^{n-1}_L \varphi_1 \cdot \nabla \varphi_2 \right. \\
+ \frac{\Delta t}{4 \lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \nabla q_i \varphi_1 \cdot \nabla q_j \varphi_2 \left. \right] dq dx \\
= \int_{\Omega \times D} M \left[ \frac{1}{2} (\xi (\rho^n_L) + \xi (\rho^n_L^{-1})) \varphi_1 \varphi_2 + \Delta t \varepsilon \nabla \varphi_1 \cdot \nabla \varphi_2 \right]
\]
\[ + \frac{1}{2} \Delta t \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta \left( \rho_L^{[\Delta t]} \right) dt \right) \left( u_L^{n-1} \varphi_2 \cdot \nabla \varphi_1 - u_L^{n-1} \varphi_1 \cdot \nabla \varphi_2 \right) \]

\[ + \frac{\Delta t}{4\lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} \nabla q_i \varphi_1 \cdot \nabla q_j \varphi_2 \right] dq dx, \]

(3.39a)

and, for all \( \psi \in H^1(\Omega) \), \( \eta \in L^\infty(\Omega \times D) \) and \( \varphi \in X \),

\[ \ell_a(\psi, \eta)(\varphi) := \int_{\Omega \times D} M \left[ \zeta \left( \rho_L^{n-1} \right) \tilde{\psi}_L^{n-1} \varphi + \Delta t \sum_{i=1}^{K} \sigma(\varphi) q_i \zeta \left( \rho_L^n \right) \eta \cdot \nabla q_i \varphi \right] dq dx. \]

(3.39b)

Hence, on noting (2.6), for \( u_L^{n-1} \in V \) and \( \rho_L^{[\Delta t]}_{|\{t_{n-1}, t_n\}} \) fixed (and therefore \( \rho_L^n \) and \( \rho_L^{n-1} \) also fixed), \( a(\cdot, \cdot) \) is a coercive bilinear functional on \( X \times X \). In order to show that \( a(\cdot, \cdot) \) is a continuous bilinear functional on \( X \times X \), we shall focus our attention on the case of \( d = 3 \); in the case of \( d = 2 \) the argument is completely analogous; we begin by noting that, by Hölder’s inequality and the Sobolev embedding theorem,

\[ \int_{\Omega \times D} M \left| u_L^{n-1} \right| \left| \nabla \varphi_1 \right| \left| \nabla \varphi_2 \right| dq dx \leq c(\Omega) \left\| u_L^{n-1} \right\|_{L^5(\Omega)} \left\| \varphi_1 \right\|_{H^4_0(\Omega \times D)} \left\| \varphi_2 \right\|_{H^4_0(\Omega \times D)}. \]

This, obvious applications of the Cauchy–Schwarz inequality, and the fact that the range of the function \( \zeta \) is the compact subinterval \([\xi_{\min}, \xi_{\max}]\) of \((0, \infty)\), then imply that \( a(\cdot, \cdot) \) is a continuous bilinear functional on \( X \times X \).

In addition, for all \( \psi \in H^1(\Omega) \), \( \eta \in L^\infty(\Omega \times D) \) and \( \varphi \in X \), we have that

\[ \left| \ell_a(\psi, \eta)(\varphi) \right| \leq \left\| \zeta \left( \rho_L^{n-1} \right) \tilde{\psi}_L^{n-1} \right\|_{L^5_0(\Omega \times D)} \left\| \varphi \right\|_{L^5_0(\Omega \times D)} \]

\[ + \Delta t \left( \int_{\Omega} M |q|^2 dq \right)^{1/2} \left\| \zeta \left( \rho_L^n \right) \eta \right\|_{L^\infty(\Omega \times D)} \left\| \nabla \varphi \right\|_{L^2(\Omega)} \left\| \nabla \psi \right\|_{L^2(\Omega \times D)}. \]

(3.40)

Therefore, by noting that \( \zeta \left( \rho_L^{n-1} \right) \tilde{\psi}_L^{n-1} \in Z_2 \), \( \zeta \left( \rho_L^n \right) \eta \in L^\infty(\Omega \times D) \) and recalling (1.3), it follows that \( \ell_a(\psi, \eta)(\cdot) \) is a continuous linear functional on \( X \) for any \( \psi \in H^1(\Omega) \) and \( \eta \in L^\infty(\Omega \times D) \).

Before we prove existence of a solution to the problem (3.23b), (3.23c), i.e., (3.35) and (3.38), let us first show by induction that the function \( \rho_L^{[\Delta t]}_{|\{t_{n-1}, t_n\}} \), whose existence and uniqueness in the function space \( L^\infty(t_{n-1}, t_n; L^\infty(\Omega)) \cap C([t_{n-1}, t_n]; L^2(\Omega)) \cap W^1,\infty(t_{n-1}, t_n; W^{1,\frac{q}{q-1}}(\Omega)^\prime) \) have already been established, with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), satisfies the two-sided bound \( \rho_{\text{min}} \leq \rho_L^{[\Delta t]}(x, t) \leq \rho_{\text{max}} \) for a.e. \( x \in \Omega \) and every \( t \in [t_{n-1}, t_n] \), i.e., that \( \rho_L^n \in \mathcal{Y} \) for all \( n \in \{0, \ldots, N\} \).

To this end, for \( \alpha \in (0, 1) \), we consider the regularized problem

\[ \int_{t_{n-1}}^{t_n} \left( \frac{\partial \rho_L^{[\Delta t]}_{\alpha}}{\partial t}, \eta \right)_{H^1(\Omega)} dt - \int_{t_{n-1}}^{t_n} \rho_L^{[\Delta t]} u_L^{n-1} \cdot \nabla \eta dx dt \]

\[ + \alpha \int_{t_{n-1}}^{t_n} \nabla \rho_L^{[\Delta t]}_{\alpha} \cdot \nabla \eta dx dt = 0 \quad \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)), \]

(3.41a)
subject to the initial condition

$$\rho_{L, \alpha}^{[\Delta t]}(t_{n-1}, t_n) = \rho_l^{n-1} \in \mathcal{Y},$$

(3.41b)

where \( \rho_l^{n-1} \in \mathcal{Y} \) was assumed for the purposes of our inductive argument; clearly, \( \rho_l^0 := \rho_0 \in \mathcal{Y} \), so the basis of the induction is satisfied. We begin by showing the existence and uniqueness of a solution \( \rho_{L, \alpha}^{[\Delta t]} \) to (3.41a), (3.41b) and that \( \rho_{\min} \leq \rho_{L, \alpha}^{[\Delta t]} \leq \rho_{\max} \) for a.e. \( x \in \Omega \) and for all \( t \in [t_{n-1}, t_n] \); we shall then pass to the limit \( \alpha \to 0_+ \) to deduce that the limiting function, which we shall show to coincide with \( \rho_l^{[\Delta t]} \), satisfies the two-sided bound \( \rho_{\min} \leq \rho_l^{[\Delta t]}(x, t) \leq \rho_{\max} \) for a.e. \( x \in \Omega \) and for all \( t \in [t_{n-1}, t_n] \); hence in particular we shall deduce that \( \rho_l^n = \rho_l^{[\Delta t]}(\cdot, t_n) \in \mathcal{Y} \).

The existence of a unique weak solution

$$\rho_{L, \alpha}^{[\Delta t]} \in C([t_{n-1}, T]; L^2(\Omega)) \cap L^2(\Omega) \cap H^1(\Omega),$$

(3.42b)

to (3.41a), (3.41b) is immediate; see, for example, Wloka [49, Theorem 26.1]. Further, on selecting, for \( s \in (t_{n-1}, t_n) \), the test function \( \eta = \chi_{[t_{n-1}, s]} \rho_{L, \alpha}^{[\Delta t]} \) in Eq. (3.41a), where for a set \( S \subset \mathbb{R} \), \( \chi_S \) denotes the characteristic function of \( S \), and noting that \( \eta n^{n-1} \in \mathcal{Y} \), we obtain the energy identity

$$\|\rho_{L, \alpha}^{[\Delta t]}(s)\|^2_{L^2(\Omega)} + 2\alpha \int_{t_{n-1}}^s \int_\Omega |\nabla_x \rho_{L, \alpha}^{[\Delta t]}(s)|^2 \, dx \, dt = \|\rho_l^{n-1}\|^2_{L^2(\Omega)}, \quad s \in (t_{n-1}, t_n),$$

which then implies that \( \{\rho_{L, \alpha}^{[\Delta t]}\}_{\alpha \in (0, 1)} \) is a bounded set in the function space \( L^\infty(t_{n-1}, t_n; L^2(\Omega)) \) and that \( \{\sqrt{\alpha} \nabla_x \rho_{L, \alpha}^{[\Delta t]}\}_{\alpha \in (0, 1)} \) is a bounded set in \( L^2(t_{n-1}, t_n; L^2(\Omega)) \). It then follows that

$$\left\{ \frac{\partial \rho_{L, \alpha}^{[\Delta t]}}{\partial t} \right\}_{\alpha \in (0, 1)}$$

is a bounded set in \( L^2(t_{n-1}, t_n; H^1(\Omega)) \).

Hence there exist an element \( \rho_{L, 0}^{[\Delta t]} \in L^\infty(t_{n-1}, t_n; L^2(\Omega)) \cap H^1(t_{n-1}, t_n; H^1(\Omega)) \) and a subsequence of \( \{\rho_{L, \alpha}^{[\Delta t]}\}_{\alpha \in (0, 1)} \) (not indicated) such that, as \( \alpha \to 0_+ \),

$$\rho_{L, \alpha}^{[\Delta t]} \to \rho_{L, 0}^{[\Delta t]} \quad \text{weak* in } L^\infty(t_{n-1}, t_n; L^2(\Omega)), \quad (3.42a)$$

$$\alpha \nabla_x \rho_{L, \alpha}^{[\Delta t]} \to 0 \quad \text{strongly in } L^2(t_{n-1}, t_n; L^2(\Omega)), \quad (3.42b)$$

$$\frac{\partial}{\partial t} \rho_{L, \alpha}^{[\Delta t]} \to \frac{\partial}{\partial t} \rho_{L, 0}^{[\Delta t]} \quad \text{weakly in } L^2(t_{n-1}, t_n; H^1(\Omega)). \quad (3.42c)$$

By a weak parabolic maximum principle based on a cut-off argument, we also have that

$$\rho_{\min} \leq \rho_{L, \alpha}^{[\Delta t]} \leq \rho_{\max} \quad \forall L > 1, \ \forall \alpha \in (0, 1),$$

and therefore,

$$\rho_{\min} \leq \rho_{L, 0}^{[\Delta t]} \leq \rho_{\max} \quad \forall L > 1, \ \forall \alpha \in (0, 1).$$
Thus, on passing to the limit in (3.41a) we deduce that
\[
\int_{t_{n-1}}^{t_n} \left( \frac{\partial \rho_L^{[\Delta t]}}{\partial t}, \eta \right)_{H^1(\Omega)} \, dt - \int_{t_{n-1}}^{t_n} \int_{\Omega} \rho_L^{[\Delta t]} u_L^{n-1} \cdot \nabla \eta \, dx \, dt = 0
\]
\[\forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)). \tag{3.43}\]

As
\[
\eta \in L^1(t_{n-1}, t_n; W^{1, q}(\Omega)) \mapsto \int_{t_{n-1}}^{t_n} \int_{\Omega} \rho_L^{[\Delta t]} u_L^{n-1} \cdot \nabla \eta \, dx \, dt \in \mathbb{R}
\]
is a continuous linear functional for all \( q \in (2, \infty) \) when \( d = 2 \) and all \( q \in [3, 6] \) when \( d = 3 \), the application of a density argument to (3.43) yields that
\[
\int_{t_{n-1}}^{t_n} \left( \frac{\partial \rho_L^{[\Delta t]}}{\partial t}, \eta \right)_{W^{1, q}(\Omega)} \, dt - \int_{t_{n-1}}^{t_n} \int_{\Omega} \rho_L^{[\Delta t]} u_L^{n-1} \cdot \nabla \eta \, dx \, dt = 0
\]
\[\forall \eta \in L^1(t_{n-1}, t_n; W^{1, q}(\Omega)), \tag{3.45a}\]
where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), and \( \rho_L^{[\Delta t]}(\cdot, t_{n-1}) = \rho_L^{n-1} \). As \( \rho_L^{[\Delta t]} \) is already known to be the unique weak solution to this problem by the argument from the beginning of this section, it follows that \( \rho_L^{n-1} = \rho_L^{[\Delta t]} \), and therefore
\[
\rho_{\min} \leq \rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]}(x, t) \leq \rho_{\max} \quad \text{for a.e. } x \in \Omega, \text{ for all } t \in [t_{n-1}, t_n] \text{ and } n = 1, \ldots, N. \tag{3.44}\]

In particular, for \( t = t_n \), \( \rho_{\min} \leq \rho^n_L(x) := \rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]}(x, t_n) \leq \rho_{\max} \) for a.e. \( x \in \Omega \); hence, \( \rho^n_L \in \mathcal{Y} \), as was claimed in the first sentence of this section.

Thus, for any given \( \rho_L^{n-1} \in \mathcal{Y} \) and \( u^{\alpha - 1}_L \in \mathcal{V}, n \in \{1, \ldots, N\} \), we have shown the existence of a unique function
\[
\rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]} \in L^\infty(t_{n-1}, t_n; L^\infty(\Omega)) \cap C([t_{n-1}, t_n]; L^2(\Omega)) \cap W^{1, \infty}(t_{n-1}, t_n; W^{1, q}(\Omega))
\]
such that \( \rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]}(\cdot, t_{n-1}) = \rho_L^{n-1} \), with \( \rho_L^0 := \rho_0 \in \mathcal{Y} \) when \( n = 1 \), and
\[
\int_{t_{n-1}}^{t_n} \left( \frac{\partial \rho_L^{[\Delta t]}}{\partial t}, \eta \right)_{W^{1, q}(\Omega)} \, dt - \int_{t_{n-1}}^{t_n} \int_{\Omega} \rho_L^{[\Delta t]} u_L^{n-1} \cdot \nabla \eta \, dx \, dt = 0
\]
\[\forall \eta \in L^1(t_{n-1}, t_n; W^{1, q}(\Omega)), \tag{3.45a}\]
where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), and \( \rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]} \in \mathcal{Y} \).

We now fix \( \rho_L^{n}(\cdot) := \rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]}(\cdot, t_n) \in \mathcal{Y} \), and we turn our attention to the proof of existence of solutions to (3.23b), (3.23c). To this end we consider the following regularized version of the system (3.23b), (3.23c), where we recall (3.35) and (3.38): for a given \( \delta \in (0, 1) \), find \( (u^{\alpha}_{L, \delta}, \psi^{\alpha}_{L, \delta}) \in \mathcal{V} \times X \) such that
In order to show that \((3.45a)\) decouples from \((3.45b), (3.45c)\); indeed, given \(\rho_L^{n-1} \in \mathcal{R}\) and \(u^n_{L,\delta} \in \mathcal{V}\), one can solve \((3.45a)\) uniquely for

\[
\rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]} \in L^\infty(t_{n-1}, t_n; L^\infty(\Omega)) \cap C([t_{n-1}, t_n]; L^2(\Omega)) \cap W^{1, \infty}(t_{n-1}, t_n; W^{1, \frac{2}{M}}(\Omega)'),
\]

where \(q \in (2, \infty)\) when \(d = 2\) and \(q \in [3, 6]\) when \(d = 3\); and \(\rho_L^{[\Delta t]}(\cdot, t_{n-1}) = \rho_L^{n-1}(\cdot)\); by defining \(\rho_L^k(\cdot) := \rho_L^{[\Delta t]}|_{[t_{n-1}, t_n]}(\cdot, t_n)\), we can then consider the system \((3.45b), (3.45c)\) for \((u^n_{L,\delta}, \tilde{\psi}^n_{L,\delta})\) independently of \((3.45a)\).

The existence of a solution to \((3.45b), (3.45c)\) will be proved by using a fixed-point argument. Given \(\tilde{\psi} \in \mathcal{L}^2_M(\Omega \times D)\), let \((u^*, \tilde{\psi}^*) \in \mathcal{V} \times \mathcal{X}\) be such that

\[
\begin{align*}
    b(u^*, w) &= \ell_b(\tilde{\psi})(w) \quad \forall w \in \mathcal{V}, \quad (3.46a) \\
    a(\tilde{\psi}^*, \varphi) &= \ell_a(u^*, \rho_L^L(\tilde{\psi}))(\varphi) \quad \forall \varphi \in \mathcal{X}. \quad (3.46b)
\end{align*}
\]

The Lax–Milgram theorem yields the existence of a unique solution \(u^* \in \mathcal{V}\) to \((3.46a)\) for a given \(\tilde{\psi} \in \mathcal{X}\), and the existence of a unique solution \(\tilde{\psi}^* \in \mathcal{X}\) to \((3.46b)\) for a given pair \((u^*, \tilde{\psi}) \in \mathcal{V} \times \mathcal{X}\). Therefore the overall procedure \((3.46a), (3.46b)\) maps a function \(\tilde{\psi} \in \mathcal{L}^2_M(\Omega \times D)\) into \(\tilde{\psi}^* \in \mathcal{X}\) is well defined.

**Lemma 3.3.** Let \(\mathcal{T} : \mathcal{L}^2_M(\Omega \times D) \to \mathcal{X} \subset \mathcal{L}^2_M(\Omega \times D)\) denote the nonlinear map that takes the function \(\tilde{\psi}\) to \(\tilde{\psi}^* = \mathcal{T}(\tilde{\psi})\) via the procedure \((3.46a), (3.46b)\). Then, the mapping \(\mathcal{T}\) has a fixed point. Hence, there exists a solution \((u^n_{L,\delta}, \tilde{\psi}^n_{L,\delta}) \in \mathcal{V} \times \mathcal{X}\) to \((3.45b), (3.45c)\).

**Proof.** The proof is an adaption of the proof of Lemma 3.2 in [9], where \(\rho_L^k = \rho_L^{n-1} = 1\) and \(\zeta(\cdot)\) is identically equal to a positive constant. Clearly, a fixed point of \(\mathcal{T}\) yields a solution of \((3.45b), (3.45c)\). In order to show that \(\mathcal{T}\) has a fixed point, we apply Schauder’s fixed-point theorem; that is, we need to show that: (i) \(\mathcal{T} : \mathcal{L}^2_M(\Omega \times D) \to \mathcal{L}^2_M(\Omega \times D)\) is continuous; (ii) \(\mathcal{T}\) is compact; and (iii) there exists a \(\mathcal{C}_* \in \mathbb{R}_{>0}\) such that

\[
\|\tilde{\psi}\|_{\mathcal{L}^2_M(\Omega \times D)} \leq \mathcal{C}_* \tag{3.47}
\]

for every \(\tilde{\psi} \in \mathcal{L}^2_M(\Omega \times D)\) and \(\kappa \in (0, 1]\) satisfying \(\tilde{\psi} = \kappa \mathcal{T}(\tilde{\psi})\).

(i) The proof can be found in the extended version of this paper [10].

(ii) Since the embedding \(\mathcal{X} \hookrightarrow \mathcal{L}^2_M(\Omega \times D)\) is compact, we directly deduce that the mapping \(\mathcal{T} : \mathcal{L}^2_M(\Omega \times D) \to \mathcal{L}^2_M(\Omega \times D)\) is compact.

(iii) Let us suppose that \(\kappa \mathcal{T}(\tilde{\psi})\); then, the pair \((\psi, \tilde{\psi}) \in \mathcal{V} \times \mathcal{X}\) satisfies

\[
\begin{align*}
    b(\psi, w) &= \ell_b(\tilde{\psi})(w) \quad \forall w \in \mathcal{V}, \quad (3.48a) \\
    a(\tilde{\psi}, \varphi) &= \kappa \ell_a(\psi, \rho_L^L(\tilde{\psi}))(\varphi) \quad \forall \varphi \in \mathcal{X}. \quad (3.48b)
\end{align*}
\]

Choosing \(w \equiv v\) in \((3.48a)\) yields that
\[
\frac{1}{2} \int_\Omega \left[ \rho_L^n |y|^2 + \rho_L^{n-1} |y - u_L^{n-1}|^2 - \rho_L^{n-1} |u_L^{n-1}|^2 \right] \, dx + \Delta t \int_\Omega \left( \mu(\rho_L^n) |D(y)|^2 \right) \, dx
\]
\[
= \Delta t \left[ \int_\Omega \rho_L^n f^n \cdot y \, dx - k \sum_{i=1}^K \int_\Omega \zeta_i (M \zeta(\rho_L^n) \tilde{\psi}) : \nabla x y \, dx \right].
\] (3.49)

Selecting \( \varphi = [F^L_\delta]'(\tilde{\psi}) \) in (3.48b), defining \( G^L_\delta(s) \in W^{1,1}_0(\mathbb{R}) \) by
\[
G^L_\delta(s) := \begin{cases} 
\frac{1}{2\varepsilon} (s^2 + \delta^2) - 1 & \text{if } s \leq \delta, \\
 s - 1 & \text{if } s \in [\delta, L], \\
\frac{1}{2\varepsilon} (s^2 + L^2) - 1 & \text{if } s \geq L;
\end{cases}
\] (3.50)

using that, thanks to (3.34), \([G^L_\delta]'(s) = s/\beta^L_\delta(s) = s[F^L_\delta]'(s)\); and that, by virtue of (3.29) with \( \varphi = [G^L_\delta]'(\tilde{\psi}) \), we have
\[
-Dt \int_{\Omega \times D} M \left( \frac{1}{Dt} \int_{t_{n-1}}^{t_n} \zeta(\rho_L^{[\Delta t]}(r)) \, dr \right) u_L^{n-1} \tilde{\psi} \cdot \nabla x [F^L_\delta]'(\tilde{\psi}) \, dq \, dx
\]
\[
= - \int_{\Omega \times D} M \left( \int_{t_{n-1}}^{t_n} \zeta(\rho_L^{[\Delta t]}(r)) \, dr \right) u_L^{n-1} \cdot \nabla x [G^L_\delta]'(\tilde{\psi}) \, dq \, dx
\]
\[
= - \int_{\Omega \times D} M(\zeta(\rho_L^n) - \zeta(\rho_L^{n-1}))(G^L_\delta(\tilde{\psi})) \, dq \, dx.
\] (3.51)

The convexity of \( F^L_\delta \) and Lemma 3.2, with \( c_0 = 0 \) on noting that \( s[F^L_\delta]'(s) - F^L_\delta(s) - G^L_\delta(s) = 0 \), then imply that
\[
\int_{\Omega \times D} M(\zeta(\rho_L^n)F^L_\delta(\tilde{\psi}) - \zeta(\rho_L^{n-1})F^L_\delta(\kappa \tilde{\psi}_L^{n-1})) \, dq \, dx
\]
\[
+ \frac{\Delta t}{4\varepsilon} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_{\Omega \times D} M \nabla y_j \tilde{\psi} \cdot \nabla q_i (\left[ F^L_\delta \right]'(\tilde{\psi})) \, dq \, dx + \varepsilon \Delta t \int_{\Omega \times D} M \nabla x \tilde{\psi} \cdot \nabla x (\left[ F^L_\delta \right]'(\tilde{\psi})) \, dq \, dx
\]
\[
\leq \kappa \Delta t \sum_{i=1}^K \int_{\Omega \times D} M \zeta(\rho_L^n) \sigma(y) q_i \cdot \nabla q_i \tilde{\psi} \, dq \, dx
\]
\[
= \kappa \Delta t \sum_{i=1}^K \int_\Omega \zeta_i (M \zeta(\rho_L^n) \tilde{\psi)) : \sigma(y) \, dx.
\] (3.52)

where in the transition to the final inequality we applied (3.12) with \( B := \sigma(y) \) (on account of it being independent of the variable \( q_i \)), together with the fact that \( \text{tr}(\sigma(y)) = \nabla x \cdot y = 0 \), and recalled (2.4a).
Combining (3.49) and (3.52), and noting (3.32b), (3.3), (3.4) and (3.37) yields that

\[
\frac{\kappa}{2} \int_{\Omega} \left[ \rho_L^n |\nabla| + \rho_L^n L |\nabla u_L^n - u_L^n|^{2} \right] dx + \kappa \Delta t \int_{\Omega} \mu \left( \rho_L^n \right) \|D(u_L^n)\|^2 dx \\
+ k \int_{\Omega \times D} M \left( \rho_L^n \right) \|\nabla \psi_L^n\|^2 dx + k L^{-1} \Delta t \int_{\Omega \times D} M \left( \rho_L^n \right) \left[ \varepsilon |\nabla \psi_L^n| + \frac{a_0}{4\lambda} |\nabla \psi_L^n| \right] dx \\
\leq \kappa \Delta t \int_{\Omega} \rho_L^n f_L^n \cdot \nu dx + \frac{\kappa}{2} \int_{\Omega} \rho_L^n |u_L^n|^{2} dx + k \int_{\Omega \times D} M \left( \rho_L^n \right) F_{\delta}^{\|} \left( \nabla \psi_L^n \right) dx \\
\leq \frac{\kappa \Delta t \mu_{\min}}{2} \int_{\Omega} \|D(u_L^n)\|^2 dx + \frac{\kappa \Delta t \mu_{\max} C_{\kappa}^2}{2 \mu_{\min} C_0} \|f_L^n\|^2_{L^\infty(\Omega)} \\
+ \frac{\kappa}{2} \int_{\Omega} \rho_L^n |u_L^n|^{2} dx + k \int_{\Omega \times D} M \left( \rho_L^n \right) F_{\delta}^{\|} \left( \nabla \psi_L^n \right) dx.
\]  

(3.53)

It is easy to show that \( F_{\delta}^{\|}(s) \) is nonnegative for all \( s \in \mathbb{R} \), with \( F_{\delta}^{\|}(1) = 0 \). Furthermore, for any \( \kappa \in (0, 1) \), \( F_{\delta}^{\|}(\kappa s) \leq F_{\delta}^{\|}(s) \) if \( s < 0 \) or \( 1 \leq \kappa s \), and also \( F_{\delta}^{\|}(\kappa s) \leq F_{\delta}^{\|}(0) \) if \( 0 \leq \kappa s \leq 1 \). Thus we deduce that

\[
F_{\delta}^{\|}(\kappa s) \leq F_{\delta}^{\|}(s) + 1 \quad \forall s \in \mathbb{R}, \forall \kappa \in (0, 1).
\]

(3.54)

Hence, the bounds (3.53) and (3.54), on noting (3.33), give rise to the desired bound (3.47) with \( C_{\kappa} \) dependent only on \( \delta, L, k, \mu_{\min}, \mu_{\max}, \xi, f, u_L^n \) and \( \psi_L^n \). Therefore (iii) holds, and so \( T \) has a fixed point, proving existence of a solution to (3.45b), (3.45c). \( \square \)

Choosing \( w \equiv u_L^n \) in (3.45b) and \( \varphi \equiv |\nabla \psi_L^n| \) in (3.45c), and combining and noting (3.3), then yields, as in (3.53), with \( C(L) \) a positive constant, independent of \( \delta \) and \( \Delta t \),

\[
\frac{1}{2} \int_{\Omega} \left[ \rho_L^n |u_L^n|^{2} + \rho_L^n |u_L^n - u_L^n|^{2} \right] dx + k \int_{\Omega \times D} M \left( \rho_L^n \right) F_{\delta}^{\|} \left( \nabla \psi_L^n \right) dx \\
+ \Delta t \left[ \frac{1}{2} \int_{\Omega} \mu \left( \rho_L^n \right) \|D(u_L^n)\|^2 dx + k L^{-1} \varepsilon \int_{\Omega \times D} M \left( \rho_L^n \right) \|\nabla \psi_L^n\|^2 dx \\
+ \frac{k L^{-1} a_0}{4\lambda} \int_{\Omega \times D} M \left( \rho_L^n \right) \|\nabla \psi_L^n\|^2 dx \right] \\
\leq \frac{1}{2} \int_{\Omega} \rho_L^n |u_L^n|^{2} dx + \frac{\Delta t \mu_{\max} C_{\kappa}^2}{2 \mu_{\min} C_0} \|f_L^n\|^2_{L^\infty(\Omega)} + k \int_{\Omega \times D} M \left( \rho_L^n \right) F_{\delta}^{\|} \left( \nabla \psi_L^n \right) dx \\
\leq C(L). 
\]  

(3.55)

Eq. (3.45a) being independent of \( \delta \), we are now ready to pass to the limit \( \delta \to 0^+ \) in (3.45b), (3.45c), to deduce the existence of a solution \( \{ (\rho_L^{(\Delta t)} |_{t_n-1, t_n}), u_L^n, \psi_L^n \}_{n=1}^N \) to (3.45), with \( \rho_L^n = \rho_L^{(\Delta t)}(\cdot, t_n) \in \mathbb{Y}, u_L^n \in V \) and \( \psi_L^n \in X \cap Z_2, n = 1, \ldots, N \). \( \square \)
Lemma 3.4. There exist a subsequence (not indicated) of \( \{ (u^n_{L, \delta}, \tilde{\psi}^n_{L, \delta}) \}_{\delta > 0} \), and functions \( u^n_L \in V \) and \( \tilde{\psi}^n_L \in X \cap Z_2 \), \( n \in \{1, \ldots, N\} \), such that, as \( \delta \to 0^+ \),

\[
\begin{align*}
  u^n_{L, \delta} & \to u^n_L \quad \text{weakly in } V, \quad (3.56a) \\
  u^n_{L, \delta} & \to u^n_L \quad \text{strongly in } L^r(\Omega), \quad (3.56b)
\end{align*}
\]

where \( r \in [1, \infty) \) if \( d = 2 \) and \( r \in [1, 6) \) if \( d = 3 \); and

\[
\begin{align*}
  M^{\frac{1}{2}} \tilde{\psi}^n_{L, \delta} & \to M^{\frac{1}{2}} \tilde{\psi}^n_{L} \quad \text{weakly in } L^2(\Omega \times D), \quad (3.57a) \\
  M^{\frac{1}{2}} \nabla_q \tilde{\psi}^n_{L, \delta} & \to M^{\frac{1}{2}} \nabla_q \tilde{\psi}^n_{L} \quad \text{weakly in } L^2(\Omega \times D), \quad (3.57b) \\
  M^{\frac{1}{2}} \nabla_x \tilde{\psi}^n_{L, \delta} & \to M^{\frac{1}{2}} \nabla_x \tilde{\psi}^n_{L} \quad \text{weakly in } L^2(\Omega \times D), \quad (3.57c) \\
  M^{\frac{1}{2}} \beta^2_{L, \delta} \tilde{\psi}^n_{L, \delta} & \to M^{\frac{1}{2}} \beta^2_{L} \tilde{\psi}^n_{L} \quad \text{strongly in } L^2(\Omega \times D), \quad (3.57d) \\
  M^{\frac{1}{2}} \tilde{\psi}^n_{L, \delta} & \to M^{\frac{1}{2}} \tilde{\psi}^n_{L} \quad \text{strongly in } L^2(\Omega \times D), \quad (3.57e)
\end{align*}
\]

for all \( s \in [1, \infty) \); and, for \( i = 1, \ldots, K \),

\[
C \in (M \zeta(\rho^n_L)^{\tilde{\psi}^n_{L, \delta}}) \to C \in (M \zeta(\rho^n_L)^{\tilde{\psi}^n_{L}}) \quad \text{strongly in } L^2(\Omega). \quad (3.57f)
\]

Furthermore, \( (\rho_L^{[\Delta t]} |_{t_{n-1}, t_n}, u^n_L, \tilde{\psi}^n_L) \) solves \((3.23a)-(3.23c)\) for \( n = 1, \ldots, N \). Hence, there exists a solution \((\rho_L^{[\Delta t]} |_{t_{n-1}, t_n}, u^n_L, \tilde{\psi}^n_L)\) to \((\rho^L)\), with \( \rho^n_L = \rho_L^{[\Delta t]}(t_n) \in Y \), \( u^n_L \in V \) and \( \tilde{\psi}^n_L \in X \cap Z_2 \) for all \( n = 1, \ldots, N \).

Proof. The weak convergence results \((3.56a)\), \((3.57a)\) and that \( \tilde{\psi}^n_L \geq 0 \) a.e. on \( \Omega \times D \) follow immediately from \((3.55)\), on noting \((3.37)\) and \((3.33)\). The strong convergence \((3.56b)\) for \( u^n_{L, \delta} \) follows from \((3.56a)\), on noting that \( V \subset H_0^1(\Omega) \) is compactly embedded in \( L^r(\Omega) \) for the stated values of \( r \).

The results \((3.57b)\), \((3.57c)\) follow from \((3.55)\); see the proof of Lemma 3.3 in [9] for details. The strong convergence result \((3.57d)\) for \( \tilde{\psi}^n_L \) follows directly from \((3.57a)-(3.57c)\) and \((3.9b)\). In addition, \((3.57e)\), \((3.57f)\) follow from \((3.57d)\), \((3.34)\), \((2.4a)\) and \((3.11)\).

It follows from \((3.56a)\), \((3.56b)\), \((3.57b)-(3.57f)\), \((3.36a)\), \((3.36b)\), \((3.39a)\), \((3.39b)\), \((3.40)\) and \((3.7)\) that we may pass to the limit \( \delta \to 0^+ \) in \((3.45b)\), \((3.45c)\) to obtain that \( (u^n_L, \tilde{\psi}^n_L) \in V \times X \) with \( \tilde{\psi}^n_L \geq 0 \) a.e. on \( \Omega \times D \) solves \((3.35), (3.38)\); that is, \((3.23b), (3.23c)\).

Next we shall show that

\[
\int\limits_{\Omega \times D} M(q) \tilde{\psi}^n_L(x, q) dq \ dx \in L^\infty(\Omega), \quad (3.58)
\]

uniformly with respect to \( L > 1 \) for all \( n \in \{1, \ldots, N\} \). Hence we shall deduce in particular that \( \tilde{\psi}^n_L \in Z_2 \). We begin by selecting \( \varphi(x, q) = \varphi(x) \otimes 1(q) \) in \((3.23c)\) with \( \tilde{\varphi} \in H^1(\Omega) \), which then yields that

\[
\int\limits_{\Omega \times D} M \frac{\zeta(\rho^n_L) \tilde{\psi}^n_L - \zeta(\rho^{n-1}_L) \tilde{\psi}^{n-1}_L}{\Delta t} q \ dx \ dx - \int\limits_{\Omega \times D} M \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{[\Delta t]}(t)) dt \right) u^{n-1}_L(x) \cdot (\nabla_x \varphi) \tilde{\psi}^n_L q \ dx \ dx
\]

\[
+ \varepsilon \int\limits_{\Omega \times D} M \nabla_x \tilde{\psi}^n_L \cdot \nabla_x \tilde{\varphi} q \ dx \ dx = 0 \quad \forall \tilde{\varphi} \in H^1(\Omega).
\]
We define
\[ \lambda^n_L(x) := \int_D M(q) \tilde{\psi}^n_L(x, q) \, dq, \quad n = 0, \ldots, N, \]
with \( \tilde{\psi}^0_L := \tilde{\psi}^0 = \beta^L(\tilde{\psi}^0) \), and note that \( \lambda^n_L \in H^1(\Omega) \). By Fubini’s theorem we then have that
\[
\int_\Omega \frac{\zeta(\rho^n_L \lambda^n_L - \zeta(\rho^{n-1}_L \lambda^{n-1}_L - \omega))}{\Delta t} \tilde{\phi}(x) \, dx - \int_\Omega \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{|\Delta t|}) \, dt \right) u^{n-1}_L \cdot (\nabla_x \tilde{\phi}) \lambda^n_L \, dx \\
+ \epsilon \int_\Omega \nabla_x \lambda^n_L \cdot \nabla_x \tilde{\phi} \, dx = 0 \quad \forall \tilde{\phi} \in H^1(\Omega).
\]
This in particular implies, using the identity (3.29) with \( \phi \) replaced by \( \omega \tilde{\phi}, \omega \in \mathbb{R}, \) that, for each \( \omega \in \mathbb{R}, \)
\[
\int_\Omega \frac{\zeta(\rho^n_L (\lambda^n_L - \omega) - \zeta(\rho^{n-1}_L (\lambda^{n-1}_L - \omega))}{\Delta t} \tilde{\phi}(x) \, dx \\
- \int_\Omega \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{|\Delta t|}) \, dt \right) u^{n-1}_L \cdot (\nabla_x \tilde{\phi}) (\lambda^n_L - \omega) \, dx \\
+ \epsilon \int_\Omega \nabla_x (\lambda^n_L - \omega) \cdot \nabla_x \tilde{\phi} \, dx = 0 \quad \forall \tilde{\phi} \in H^1(\Omega).
\]
On selecting \( \tilde{\phi} = [\lambda^n_L - \omega]_+ \) in this identity and omitting the (nonnegative) last term from the left-hand side of the resulting equality, we have that
\[
\int_\Omega \frac{\zeta(\rho^n_L (\lambda^n_L - \omega) - \zeta(\rho^{n-1}_L (\lambda^{n-1}_L - \omega))}{\Delta t} \,[\lambda^n_L - \omega]_+ \, dx \\
- \frac{1}{2} \int_\Omega \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{|\Delta t|}) \, dt \right) u^{n-1}_L \cdot \nabla_x ([\lambda^n_L - \omega]_+)^2 \, dx \leq 0. \tag{3.59}
\]
As, once again using (3.29), with \( \phi = ([\lambda^n_L - \omega]_+)^2 \) this time, we have for each \( \omega \in \mathbb{R} \) that
\[
\int_\Omega \frac{\zeta(\rho^n_L) - \zeta(\rho^{n-1}_L)}{\Delta t} ([\lambda^n_L - \omega]_+)^2 \, dx - \int_\Omega \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho^{|\Delta t|}) \, dt \right) u^{n-1}_L \cdot \nabla_x ([\lambda^n_L - \omega]_+)^2 \, dx = 0,
\]
we can rewrite the second term in (3.59) to deduce that
\[
\int_\Omega \frac{\xi(\rho_L^n) (\lambda^n_L - \omega) - \xi(\rho_L^{n-1}) (\lambda^{n-1}_L - \omega)}{\Delta t} [\lambda^n_L - \omega]_+ \, dx
\]
\[-\frac{1}{2} \int_\Omega \frac{\xi(\rho_L^n) - \xi(\rho_L^{n-1})}{\Delta t} \left( [\lambda^n_L - \omega]_+ \right)^2 \, dx \leq 0.
\]
\text{(3.60)}

The inequality (3.60) can be restated in the following equivalent form:

\[
\int_\Omega \frac{\xi(\rho_L^{n-1}) (\lambda^n_L - \omega) - (\lambda^{n-1}_L - \omega)}{\Delta t} [\lambda^n_L - \omega]_+ \, dx
+ \int_\Omega \frac{\xi(\rho_L^n) - \xi(\rho_L^{n-1})}{\Delta t} (\lambda^n_L - \omega) [\lambda^n_L - \omega]_+ \, dx
\]
\[-\frac{1}{2} \int_\Omega \frac{\xi(\rho_L^n) - \xi(\rho_L^{n-1})}{\Delta t} \left( [\lambda^n_L - \omega]_+ \right)^2 \, dx \leq 0,
\]
and hence, after simplifying the sum of the second and the third terms on the left-hand side,

\[
\int_\Omega \frac{\xi(\rho_L^{n-1}) (\lambda^n_L - \omega) - (\lambda^{n-1}_L - \omega)}{\Delta t} [\lambda^n_L - \omega]_+ \, dx
+ \frac{1}{2} \int_\Omega \frac{\xi(\rho_L^n) - \xi(\rho_L^{n-1})}{\Delta t} \left( [\lambda^n_L - \omega]_+ \right)^2 \, dx \leq 0.
\]
\text{(3.61)}

Since \( s \in \mathbb{R} \mapsto \frac{1}{2}((s - \omega)_+)^2 \in \mathbb{R}_{\geq 0} \) is a convex function, we have that

\[
\frac{(\lambda^n_L - \omega) - (\lambda^{n-1}_L - \omega)}{\Delta t} [\lambda^n_L - \omega]_+ \geq \frac{1}{2\Delta t}((\lambda^n_L - \omega)_+)^2 - ((\lambda^{n-1}_L - \omega)_+)^2).
\]

Using this inequality in the first term of (3.61), on noting that \( \xi(\rho_L^{n-1}) \geq \xi_{\text{min}} > 0 \) and multiplying the resulting inequality by \( 2\Delta t \), we get that

\[
\int_\Omega \xi(\rho_L^{n-1}) \left( ([\lambda^n_L - \omega]_+)^2 - ([\lambda^{n-1}_L - \omega]_+)^2 \right) \, dx + \int_\Omega (\xi(\rho_L^n) - \xi(\rho_L^{n-1})) \left( [\lambda^n_L - \omega]_+ \right)^2 \, dx \leq 0.
\]

Thus, on recalling that by hypothesis \( \rho_L^0 = \rho_0 \), we have that

\[
0 \leq \int_\Omega \xi(\rho_L^n) \left( [\lambda^n_L - \omega]_+ \right)^2 \, dx \leq \int_\Omega \xi(\rho_0) \left( [\lambda^n_L - \omega]_+ \right)^2 \, dx, \quad n = 1, \ldots, N.
\]
\text{(3.62)}

Now we choose \( \omega \) as in (3.19), which yields \([\lambda^0_L - \omega]_+ = 0 \) a.e. on \( \Omega \), and therefore, by (3.62), also \([\lambda^n_L - \omega]_+ = 0 \) a.e. on \( \Omega \) for all \( n \in \{1, \ldots, N\} \); in other words,
for a.e. \( x \in \Omega \) and all \( n = 0, \ldots, N \). Since \( \omega \) here is independent of \( L \), by recalling the definition of the function \( \lambda_n^L \) we thus deduce that \( \bar{\psi}_1^L \in Z_2 \), uniformly with respect to \( L \), as was claimed in the line below (3.58).

Finally, as \((\rho_0^L, u_1^L, \bar{\psi}_1^L) \in \mathcal{Y} \times \mathcal{V} \times Z_2 \), performing the above existence proof at each time level \( t_n \), \( n = 1, \ldots, N \), yields a solution \((\rho_{n}^L, u_{n}^L, \bar{\psi}_{n}^L)_{n=1}^N \) to \((P_{L}^{\Delta t}) \) with \( \rho_n^L = \rho_{n}^{L(\Delta t)}(\cdot, t_n) \), \( n = 1, \ldots, N \), by noting that \( \rho_{L}^{\Delta t} \) thus constructed is an element of \( C([0,T]; L^2(\Omega)) \). \( \square \)

4. Entropy estimates

Next, we derive bounds on the solution of \((P_{L}^{\Delta t}) \), independent of \( L \). Our starting point is Lemma 3.4, concerning the existence of a solution to the problem \((P_{L}^{\Delta t}) \). The model \((P_{L}^{\Delta t}) \) includes 'microscopic cut-off' in the drag term of the Fokker–Planck equation, where \( L > 1 \) is a (fixed, but otherwise arbitrary) cut-off parameter. Our ultimate objective is to pass to the limits \( L \to \infty \) and \( \Delta t \to 0 \) in the model \((P_{L}^{\Delta t}) \), with \( L \) and \( \Delta t \) linked by the condition \( \Delta t = o(L^{-1}) \), as \( L \to \infty \). To that end, we need to develop various bounds on sequences of weak solutions of \((P_{L}^{\Delta t}) \) that are uniform in the cut-off parameter \( L \) and thus permit the extraction of weakly convergent subsequences, as \( L \to \infty \), through the use of a weak compactness argument. The derivation of such bounds, based on the use of the relative entropy associated with the Maxwellian \( M \), is our main task in this section.

Let us introduce the following definitions, in line with (3.24):

\[
\begin{align*}
\bar{u}^{\Delta t,-}(\cdot,t) &:= u^n(\cdot), & \bar{u}^{\Delta t,+}(\cdot,t) &:= u^{n-1}(\cdot), & t \in (t_{n-1},t_n), & n = 1, \ldots, N. \tag{4.1b}
\end{align*}
\]

We shall adopt \( u_{L}^{\Delta t(\pm)} \) as a collective symbol for \( u_{L}^{\Delta t}, u_{L}^{\Delta t(\pm)} \). The corresponding notations \( \rho_{L}^{\Delta t}, \rho_{L}^{\Delta t(\pm)} \) and \( \bar{\psi}_{L}^{\Delta t}, \bar{\psi}_{L}^{\Delta t(\pm)} \) are defined analogously. In addition, we define the products \((\rho_{L}u_{L})^{\Delta t}, (\rho_{L}u_{L}^{\Delta t})^{\Delta t(\pm)} \) and \((\rho_{L}^{\Delta t} \bar{\psi}_{L}^{\Delta t}), (\zeta(\rho_{L}) \bar{\psi}_{L}^{\Delta t(\pm)}) \) and \((\zeta(\rho_{L}) \bar{\psi}_{L}^{\Delta t(\pm)}) \) similarly. The notation \( \rho_{L}^{\Delta t} \) signifying the piecewise linear interpolant of \( \rho_{L} \) with respect to the variable \( t \) is not to be confused with the notation \( \rho_{L}^{\Delta t} \), which denotes the function defined piecewise, over the union of time slabs \( \Omega \times [t_{n-1},t_n], n = 1, \ldots, N \), as the unique solution of Eq. (3.23a) subject to the initial condition \( \rho_{L}^{\Delta t}|_{(t_{n-1},t_n)} = \rho_{n}^{L-1}, n = 1, \ldots, N \), with \( \rho_{L}^{L} := \rho_0 \). Finally, we define the functions \( \rho_{L}^{\Delta t} \) and \( \zeta_{L}^{\Delta t} \) by

\[
\rho_{L}^{\Delta t}|_{(t_{n-1},t_n)} := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \rho_{L}^{\Delta t} \, dt, \quad \zeta_{L}^{\Delta t}|_{(t_{n-1},t_n)} := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \zeta(\rho_{L}^{\Delta t}) \, dt, \quad n = 1, \ldots, N. \tag{4.2}
\]

Using the above notation, (3.23a)-(3.23c) summed for \( n = 1, \ldots, N \) can be restated in the form: find \((\rho_{L}^{\Delta t}(t), u_{L}^{\Delta t}(t), \bar{\psi}_{L}^{\Delta t}(t)) \in \mathcal{Y} \times \mathcal{V} \times Z_2 \), \( t \in [0,T] \), such that

\[
\int_{0}^{T} \left( \frac{\partial \rho_{L}^{\Delta t}}{\partial t}, \eta \right)_{W^{-1,q}(\Omega)} \, dt - \int_{0}^{T} \int_{\Omega} \rho_{L}^{\Delta t} u_{L}^{\Delta t,-} \cdot \nabla \eta \, dx \, dt = 0,
\]

\[
\forall \eta \in L^1(0,T; W^{-1,q}(\Omega)), \tag{4.3a}
\]
\[
\int_0^T \int_\Omega \left[ \frac{\partial}{\partial t} (\rho_L u_L)^{\Delta t} - \frac{1}{2} \frac{\partial \rho_L^{\Delta t}}{\partial t} u_L^{\Delta t} - \frac{1}{2} \frac{\partial \rho_L^{\Delta t}}{\partial t} u_L^{\Delta t} \right] \cdot w \, dx \, dt + \int_0^T \int_\Omega \mu \left( \rho_L^{\Delta t} \right) D \left( u_L^{\Delta t} + \lambda (w) \right) \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega \rho_L^{\Delta t} \left[ \left( (u_L^{\Delta t} - \nabla x) u_L^{\Delta t} \right) \cdot w - \left( (u_L^{\Delta t} - \nabla x) w \right) \cdot u_L^{\Delta t} \right] \, dx \, dt \\
= \int_0^T \left[ \int_\Omega \rho_L^{\Delta t} \int_0^T \left[ M \xi(\rho_L^{\Delta t}) \tilde{\psi}_L^{\Delta t} + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^L A_{ij} M \nabla \tilde{\psi}_L^{\Delta t} \cdot \nabla \xi \right] \, dq \, dx \, dt \\
+ \int_0^T \int_{\Omega \times D} \left[ \varepsilon M \nabla \tilde{\psi}_L^{\Delta t} \cdot (u_L^{\Delta t} - \tilde{\psi}_L^{\Delta t}) \tilde{\psi}_L^{\Delta t} \right] \cdot \nabla \xi \phi \, dq \, dx \, dt \\
- \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[ \sigma (u_L^{\Delta t} + q_i) \xi \left( \rho_L^{\Delta t} \right) \beta_L \tilde{\psi}_L^{\Delta t} \right] \cdot \nabla \phi \, dq \, dx \, dt = 0
\]

\forall w \in L^1(0, T; \mathcal{V}),

\text{(4.3b)}

with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), subject to the initial conditions \( \rho_L^{\Delta t}(0) = \rho_0 \in \mathcal{V}, u_L^{\Delta t}(0) = u^0 \in \mathcal{V} \) and \( \tilde{\psi}_L^{\Delta t}(0) = \tilde{\psi}^0 \in X \cap Z_2 \), where we recall (3.13) and (3.15). We emphasize that (4.3a)-(4.3c) is an equivalent restatement of problem (P_{\lambda}^L), for which existence of a solution has been established (cf. Lemma 3.4).

In conjunction with \( \beta_L \), defined by (1.12), we consider the following cut-off version \( \mathcal{F}^L \) of the entropy function \( \mathcal{F} : s \in \mathbb{R}_{\geq 0} \mapsto \mathcal{F}(s) = s(\log s - 1) + 1 \in \mathbb{R}_{\geq 0} \):

\[
\mathcal{F}^L(s) := \begin{cases} 
 s(\log s - 1) + 1, & 0 \leq s \leq L, \\
 \frac{s^2 - L^2}{2L} + s(\log L - 1) + 1, & L \leq s.
\end{cases}
\]

Note that

\[
(\mathcal{F}^L)'(s) = \begin{cases} 
 \log s, & 0 < s \leq L, \\
 \frac{s}{L} + \log L - 1, & L \leq s
\end{cases}
\]

and

\[
(\mathcal{F}^L)''(s) = \begin{cases} 
 \frac{1}{s}, & 0 < s \leq L, \\
 \frac{1}{L}, & L \leq s
\end{cases}
\]

Hence,

\[
\beta^L(s) = \min(s, L) = \left[ (\mathcal{F}^L)''(s) \right]^{-1}, \quad s \in \mathbb{R}_{\geq 0},
\]
with the convention $1/\infty := 0$ when $s = 0$, and

$$ (\mathcal{F}^L)'(s) \equiv \mathcal{F}^L(s) = s^{-1}, \quad s \in \mathbb{R}_{>0}. \quad (4.8) $$

We shall also require the following inequality, relating $\mathcal{F}^L$ to $\mathcal{F}$:

$$ \mathcal{F}^L(s) \geq \mathcal{F}(s), \quad s \in \mathbb{R}_{>0}. \quad (4.9) $$

For $0 < s < 1$, (4.9) trivially holds, with equality. For $s \geq 1$, it follows from (4.8), with $s$ replaced by a dummy variable $\sigma$, after integrating twice over $\sigma \in [1, s]$, and noting that $(\mathcal{F}^L)'(1) = \mathcal{F}'(1)$ and $(\mathcal{F}^L)(1) = \mathcal{F}(1)$.

### 4.1. $L$-independent bounds on the spatial derivatives

We are now ready to embark on the derivation of the required bounds, uniform in the cut-off parameter $L$, on norms of $\rho_L^{A^+, \ast}(t) \in \mathcal{Y}$, $u_L^{A^+, \ast}(t) \in \mathcal{V}$ and $\tilde{\psi}_L^{A^+, \ast}(t) \in \mathcal{X} \cap \Omega_2$. $t \in (0, T]$. As far as $\rho_L^{A^+, \ast}$ is concerned, it follows by tracing the constants in the argument leading to inequality (17) in DiPerna & Lions [18] and recalling from (3.3) that $\rho_0 \in \mathcal{Y}$, that, for each $p \in [1, \infty]$,

$$ \|\rho_L^{A^+, \ast}(t)\|_{L^p(\Omega)} \leq \|\rho_0\|_{L^p(\Omega)}, \quad t \in (0, T], \quad (4.10a) $$

and therefore, for each $p \in [1, \infty]$,

$$ \|\rho_L^{A^+, \ast}(t)\|_{L^p(\Omega)} \leq \|\rho_0\|_{L^p(\Omega)}, \quad t \in (0, T]. \quad (4.10b) $$

Concerning $u_L^{A^+, \ast}$, we select $w = \chi_{[0, t]}u_L^{A^+, \ast}$ as test function in (4.3b), with $t$ chosen as $t_n$, $n \in \{1, \ldots, N\}$. We then deduce using the identity (3.49) with $v = u_L^{A^+, \ast}$ and $\tilde{\psi} = \tilde{\psi}_L^{A^+, \ast}$, on noting (3.3), (3.4), (3.37) and (3.14) that, with $t = t_n$,

$$ \int_{\Omega} \rho_L^{A^+, \ast}(t)|u_L^{A^+, \ast}(t)|^2 \, dx + \frac{\rho_{\min}}{\Delta t} \int_0^t \|u_L^{A^+, \ast} - u_L^{A^+, \ast}\|^2 \, ds + \int_0^t \int \mu(\rho_L^{A^+, \ast})|D(u_L^{A^+, \ast})|^2 \, dx \, ds \leq \int_{\Omega} \rho_0|u_0|^2 \, dx + \frac{2c_1^2}{\mu_{\min} \xi_0} \int_0^t \|f_{A^+, \ast}\|^2_{L^2(\Omega)} \, ds \leq -2k \int_0^t \int_{\Omega} \sum_{i=1}^K M(q) \left( \frac{1}{2} |q_i|^2 \right) \xi (\rho_L^{A^+, \ast}) \tilde{\psi}_L^{A^+, \ast} \chi_i \, \nabla \cdot u_L^{A^+, \ast} \, dq \, dx \, ds, \quad (4.11) $$

where $\|\cdot\|$ denotes the $L^2$ norm over $\Omega$. We intentionally did not bound the final term on the right-hand side of (4.11). As we shall see in what follows, this simple trick will prove helpful: our bounds on $\tilde{\psi}_L^{A^+, \ast}$ below will furnish an identical term with the opposite sign, so then by combining the bounds on $u_L^{A^+, \ast}$ and $\tilde{\psi}_L^{A^+, \ast}$ this pair of, otherwise dangerous, terms will be removed. This fortuitous cancellation reflects the balance of total energy in the system.

Having dealt with $u_L^{A^+, \ast}$, we now embark on the less straightforward task of deriving bounds on norms of $\tilde{\psi}_L^{A^+, \ast}$ that are uniform in the cut-off parameter $L$. The appropriate choice of test function in (4.3c) for this purpose is $\varphi = \chi_{[0, t]}(\mathcal{F}^L)'(\tilde{\psi}_L^{A^+, \ast})$ with $t = t_n$, $n \in \{1, \ldots, N\}$; this can be seen by noting
that with such a \( \varphi \), at least formally, the final term on the left-hand side of (4.3c) can be manipulated to become identical to the final term in (4.11), but with the opposite sign. While Lemma 3.4 guarantees that \( \tilde{\psi}_L^{\Delta t, +} (t) \) belongs to \( Z_2 \) for all \( t \in [0, T] \), and is therefore nonnegative a.e. on \( \Omega \times D \times [0, T] \), there is unfortunately no reason why \( \tilde{\psi}_L^{\Delta t, +} \) should be strictly positive on \( \Omega \times D \times [0, T] \), and therefore the expression \( (F^L)'(\tilde{\psi}_L^{\Delta t, +}) \) may in general be undefined; the same is true of \( (F^L)'(\tilde{\psi}_L^{\Delta t, +}) \), which also appears in the algebraic manipulations. We shall circumvent this problem by working with \( (F^L)'(\tilde{\psi}_L^{\Delta t, +} + \alpha) \) instead of \( (F^L)'(\tilde{\psi}_L^{\Delta t, +}) \), where \( \alpha > 0 \); since \( \tilde{\psi}_L^{\Delta t, +} \) is known to be nonnegative from Lemma 3.4, \( (F^L)'(\tilde{\psi}_L^{\Delta t, +} + \alpha) \) and \( (F^L)'(\tilde{\psi}_L^{\Delta t, +} + \alpha) \) are well-defined. After deriving the relevant bounds, which will involve \( F^L(\tilde{\psi}_L^{\Delta t, +} + \alpha) \) only, we shall pass to the limit \( \alpha \to 0_+ \), noting that, unlike \( (F^L)'(\tilde{\psi}_L^{\Delta t, +}) \) and \( (F^L)'(\tilde{\psi}_L^{\Delta t, +}) \), the function \( (F^L)'(\tilde{\psi}_L^{\Delta t, +}) \) is well-defined for any nonnegative \( \tilde{\psi}_L^{\Delta t, +} \). Thus, the core of the idea is to take any \( \varphi = \chi_{(0, t)}(F^L)'(\tilde{\psi}_L^{\Delta t, +} + \alpha) \), with \( t = t_n, n \in \{1, \ldots, N\} \), as test function in (4.3c), and then pass to the limit \( \alpha \to 0_+ \). An equivalent but slightly more transparent approach is to start from (3.23c) with the indices \( n \) and \( n - 1 \) in (3.23c) replaced by \( k \) and \( k - 1 \), respectively, choose \( \varphi = (F^L)'(\tilde{\psi}_L^{k} + \alpha) \) as test function, sum the resulting expressions through \( k = 1, \ldots, n \), with \( n \in \{1, \ldots, N\} \), and then pass to the limit \( \alpha \to 0_+ \). For reasons of clarity, we shall adopt the latter approach.

Thus, for \( k = 1, \ldots, n \) and \( n \in \{1, \ldots, N\} \), we arrive at the following identity

\[
\begin{align*}
\int_{\Omega \times D} M \frac{\xi(\rho_L^k)\tilde{\psi}_L^k - \xi(\rho_L^{k-1})\tilde{\psi}_L^{k-1}}{\Delta t} [(F^L)'(\tilde{\psi}_L^{k} + \alpha)] \, dq \, dx \\
- \int_{\Omega \times D} M \left( \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \xi(\rho_L^{[\Delta t]}) \, dt \right) u_L^{k-1} : (\nabla_x [(F^L)'(\tilde{\psi}_L^{k} + \alpha)]) \tilde{\psi}_L^k \, dq \, dx \\
+ \int_{\Omega \times D} \sum_{i=1}^{K} \frac{1}{4\lambda} \sum_{j=1}^{K} A_{ij} M \nabla_x \tilde{\psi}_L^k \cdot \left[ \frac{\sigma_\infty}{N} (u_L^k)^2 \right] - \sigma_{ij} \frac{M \xi(\rho_L^k) \beta_L^k (\tilde{\psi}_L^k)}{\varphi} \nabla_{ij} \left[ (F^L)'(\tilde{\psi}_L^{k} + \alpha) \right] \, dq \, dx \\
+ \int_{\Omega \times D} \varepsilon M \nabla_x \tilde{\psi}_L^k : \nabla_x [(F^L)'(\tilde{\psi}_L^{k} + \alpha)] \, dq \, dx = 0.
\end{align*}
\]

We shall manipulate each of the terms on the left-hand side of (4.12). We begin by considering

\[
T_1 := \int_{\Omega \times D} M \frac{\xi(\rho_L^k)\tilde{\psi}_L^k - \xi(\rho_L^{k-1})\tilde{\psi}_L^{k-1}}{\Delta t} [(F^L)'(\tilde{\psi}_L^{k} + \alpha)] \, dq \, dx
\]

and

\[
T_2 := - \int_{\Omega \times D} M \left( \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} \xi(\rho_L^{[\Delta t]}) \, dt \right) u_L^{k-1} : (\nabla_x [(F^L)'(\tilde{\psi}_L^{k} + \alpha)]) \tilde{\psi}_L^k \, dq \, dx
\]

in tandem. By noting that, thanks to (4.7),

\[
(\nabla_x [(F^L)'(\tilde{\psi}_L^{k} + \alpha)]) \tilde{\psi}_L^k = \tilde{\psi}_L^k [\beta_L^k (\tilde{\psi}_L^{k} + \alpha)]^{-1} \nabla_x \tilde{\psi}_L^k
\]

\[
= [\tilde{\psi}_L^{k} + \alpha] [\beta_L^k (\tilde{\psi}_L^{k} + \alpha)]^{-1} \nabla_x \tilde{\psi}_L^k + \alpha
\]

\[
= \nabla_x [G^k(\tilde{\psi}_L^{k} + \alpha) - \alpha [(F^L)'(\tilde{\psi}_L^{k} + \alpha)].
\]
\(G^L(s) := \begin{cases} 
  s - 1, & s \leq L, \\
  \frac{\gamma^2}{2} + \frac{t}{2} - 1, & L \leq s,
\end{cases} \)

we have that

\[ T_2 = - \int_\Omega \left( \frac{1}{\Delta t} \int_{t_k}^{t_{k+1}} \xi(\rho_L^{(\Delta t)}) \, dt \right) u_{L-1} \cdot \nabla_x \left[ \int_D M \left( G^L(\tilde{\psi}_L^k + \alpha) - \alpha \left[ (F^L)'(\tilde{\psi}_L^k + \alpha) \right] \right) \right] \, dx. \]

By applying (3.29) with \( \varphi = \int_D M[G^L(\tilde{\psi}_L^k + \alpha) - \alpha(F^L)'(\tilde{\psi}_L^k + \alpha)] \, dq \in W^{1, \frac{q}{\alpha}}(\Omega) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), we then have that

\[ T_2 = - \int_{\Omega \times D} M \frac{\xi(\rho^k_{L-1}) - \xi(\rho^k_{L-1})}{\Delta t} \left[ G^L(\tilde{\psi}_L^k + \alpha) - \alpha \left[ (F^L)'(\tilde{\psi}_L^k + \alpha) \right] \right] \, dq \, dx. \]

We note in passing that the statement \( \int_D M[G^L(\tilde{\psi}_L^k + \alpha) - \alpha(F^L)'(\tilde{\psi}_L^k + \alpha)] \, dq \in W^{1, \frac{q}{\alpha}}(\Omega) \) above follows, for all \( \alpha \in (0, 1) \), from the following considerations. Since \( \tilde{\psi}_L^k \in X \), also \( (F^L)'(\tilde{\psi}_L^k + \alpha) \in X \), and hence \( \int_D M(F^L)'(\tilde{\psi}_L^k + \alpha) \, dq \in H^1(\Omega) \subset W^{1, \frac{q}{\alpha}}(\Omega) \) for all \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). Furthermore, since \( \tilde{\psi}_L^k \in X \), also \( \Gamma := \int_D M[G^L(\tilde{\psi}_L^k + \alpha) \, dq \in L^1(\Omega) \) and, since \( X \subset H^1(\Omega; L^2_1(D)) \) and, by the Sobolev embedding (3.8), \( H^1(\Omega; L^2_0(D)) \subset L^q(\Omega; L^2_0(D)) \) for the range of \( q \) under consideration here, the definition of \( G^L \), a straightforward application of the Cauchy–Schwarz inequality to the integral over \( D \) involved in the definition of \( \Gamma \) and the application of Hölder’s inequality to the integral over \( \Omega \) involved in the definition of the \( L^{\frac{q}{\alpha}}(\Omega) \) norm, imply that \( \nabla_x \Gamma \in L^{\frac{q}{\alpha}}(\Omega) \). Finally, by a Gagliardo–Nirenberg inequality applied to the function \( \Gamma - f_\Omega \Gamma \, dx \) (cf. (2.9) and (2.10) on p. 45 in [27]; or inequality (2.19) on p. 73 of [28] in conjunction with Poincaré’s inequality in the \( L^{\frac{q}{\alpha}}(\Omega) \) norm; or Theorem 2.2 and Remark 2.1 in [26], where the proof of the Gagliardo–Nirenberg inequality can also be found):

\[ \left\| \Gamma - \frac{1}{\Omega} \int_\Omega \Gamma \, dx \right\| \leq C(q, d) \, \| \Gamma \|_{L^{\frac{q}{\alpha}}(\Omega)} \| \nabla_x \Gamma \|_{L^{\frac{q}{\alpha}}(\Omega)}; \]

hence \( \Gamma \in L^{\frac{q}{\alpha}}(\Omega) \), which, together with \( \nabla_x \Gamma \in L^{\frac{q}{\alpha}}(\Omega) \), implies that \( \Gamma \in W^{1, \frac{q}{\alpha}}(\Omega) \). Thus we deduce that \( \int_D M[G^L(\tilde{\psi}_L^k + \alpha) - \alpha(F^L)'(\tilde{\psi}_L^k + \alpha)] \, dq \in W^{1, \frac{q}{\alpha}}(\Omega) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), as was claimed above.

By rewriting \( T_1 \) as

\[ T_1 = \int_{\Omega \times D} M \frac{\xi(\rho^{k-1}_L) - \xi(\rho^{k-1}_L)}{\Delta t} \left[ (F^L)'(\tilde{\psi}_L^k + \alpha) \right] \, dq \, dx \]

and adding this to the expression for \( T_2 \) yields that
\[ T_1 + T_2 = \int_{\Omega \times D} M \zeta (\rho_{L}^{k-1}) \frac{(\tilde{\psi}_{L}^k + \alpha) - (\tilde{\psi}_{L}^{k-1} + \alpha)}{\Delta t} [(\mathcal{F})'(\tilde{\psi}_{L}^k + \alpha)] \, dq \, dx \]
\[ + \int_{\Omega \times D} M \frac{\zeta (\rho_{L}^k) - \zeta (\rho_{L}^{k-1})}{\Delta t} [(\tilde{\psi}_{L}^k + \alpha)] [(\mathcal{F})'(\tilde{\psi}_{L}^k + \alpha)] - \mathcal{G}^L(\tilde{\psi}_{L}^k + \alpha) \, dq \, dx \]
\[ = \int_{\Omega \times D} M \frac{\zeta (\rho_{L}^k) - \zeta (\rho_{L}^{k-1})}{\Delta t} [(\mathcal{F})'(\tilde{\psi}_{L}^k + \alpha)] \, dq \, dx \]
\[ - \int_{\Omega \times D} M \frac{\zeta (\rho_{L}^k) - \zeta (\rho_{L}^{k-1})}{\Delta t} \mathcal{G}^L(\tilde{\psi}_{L}^k + \alpha) \, dq \, dx. \]

By applying part c) of Lemma 3.2 with \( F(s) = \mathcal{F}^L(s) \), \( G(s) = \mathcal{G}^L(s) \), \( A = \zeta (\rho_{L}^k) \), \( B = \zeta (\rho_{L}^{k-1}) \), \( a = \tilde{\psi}_{L}^k + \alpha \), \( b = \tilde{\psi}_{L}^{k-1} + \alpha \), noting that \( s(\mathcal{F}^L)'(s) - \mathcal{F}^L(s) - \mathcal{G}^L(s) = 0 := c_0 \) for all \( s \in (0, \infty) \), and \( \text{ess} \inf_{s \geq 0} (\mathcal{F}^L)'(s) = 1/L := d_0 \), it follows that

\[ T_1 + T_2 \geq \frac{1}{\Delta t} \left[ \int_{\Omega \times D} M \zeta (\rho_{L}^k) \mathcal{F}^L(\tilde{\psi}_{L}^k + \alpha) \, dq \, dx \right] - \int_{\Omega \times D} M \zeta (\rho_{L}^{k-1}) \mathcal{F}^L(\tilde{\psi}_{L}^{k-1} + \alpha) \, dq \, dx \]
\[ + \frac{1}{2 \Delta t L} \int_{\Omega \times D} M \zeta (\rho_{L}^{k-1}) (\tilde{\psi}_{L}^k - \tilde{\psi}_{L}^{k-1})^2 \, dq \, dx. \] (4.13)

We now move on to the next term in (4.12): thanks to (2.6), we have that

\[ T_3 := \frac{1}{4\lambda} \int_{\Omega \times D} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} M \nabla q_i \tilde{\psi}_{L}^k \cdot \nabla q_i [(\mathcal{F})'(\tilde{\psi}_{L}^k + \alpha)] \, dq \, dx \]
\[ \geq \frac{d_0}{4\lambda} \int_{\Omega \times D} M [(\mathcal{F}^L)'(\tilde{\psi}_{L}^k + \alpha)] |\nabla q_i \tilde{\psi}_{L}^k|^2 \, dq \, dx. \] (4.14)

It is tempting to bound \([\mathcal{F}^L]'(\tilde{\psi}_{L}^k + \alpha)\) below further by \((\tilde{\psi}_{L}^k + \alpha)^{-1}\) using (4.8). We have refrained from doing so as the precise form of (4.14) will be required to absorb an extraneous term that the process of shifting \(\tilde{\psi}_{L}^k\) by the addition of \(\alpha > 0\) generates in term \(T_5\) below (cf. the last line in (4.16)). Once the extraneous term has been absorbed into the right-hand side of (4.14), we shall apply inequality (4.8) to the resulting expression to bound it below further.

The next term that has to be dealt with, this time by a direct use of (4.8), is

\[ T_4 := \varepsilon \int_{\Omega \times D} M \nabla x \tilde{\psi}_{L}^k \cdot \nabla x [(\mathcal{F})'(\tilde{\psi}_{L}^k + \alpha)] \geq \varepsilon \int_{\Omega \times D} M \frac{|\nabla x \tilde{\psi}_{L}^k|^2}{\tilde{\psi}_{L}^k + \alpha} \, dq \, dx. \] (4.15)

It remains to consider the critical final term

\[ T_5 := \int_{\Omega \times D} \sum_{i=1}^{K} \sigma (u_i L) q_i M \zeta (\rho_{L}^k) \beta^L (\tilde{\psi}_{L}^k) \cdot \nabla q_i [(\mathcal{F})'(\tilde{\psi}_{L}^k + \alpha)] \, dq \, dx \]
Thus, by applying the integration-by-parts formula (3.12) to the expression in the square brackets in the penultimate line of (4.16), we deduce that

\[
T_5 = - \int_{\Omega \times D} M \left( \frac{1}{2} |q_i|^2 \right) \xi(r_k^L) \psi_L^{k+1} \left[ (q_i q_i^T) : \nabla_x u_L^k \right] \, dq \, dx 
\]

\[
+ \int_{\Omega \times D} M \xi(r_k^L) \left[ 1 - \frac{\beta^L(\psi_L^k)}{\beta^L(\psi_L^k + \alpha)} \right] \sum_{i=1}^{K} \left[ (\nabla_x u_L^k) q_i \right] \cdot \nabla_q \psi_L^k \, dq \, dx. \tag{4.17}
\]

By summing (4.13), (4.14), (4.15) and (4.17) we obtain

\[
\frac{1}{\Delta t} \left[ \int_{\Omega \times D} M \xi(r_k^L) \mathcal{F}^L(\psi_L^{k+1} + \alpha) \, dq \, dx - \int_{\Omega \times D} M \xi(r_k^{L-1}) \mathcal{F}^L(\psi_L^{k-1} + \alpha) \, dq \, dx \right] 
\]

\[
+ \frac{1}{2\Delta t L} \int_{\Omega \times D} M \xi(r_k^{L-1})(\psi_L^k - \psi_L^{k-1})^2 \, dq \, dx 
\]

\[
+ \frac{a_0}{4\Delta t} \int_{\Omega \times D} M \left[ (\mathcal{F}^L)^\prime(\psi_L^{k+1} + \alpha) \right] \left| \nabla_q \psi_L^{k+1} \right|^2 \, dq \, dx + \varepsilon \int_{\Omega \times D} M \left| \nabla_x \psi_L^{k+1} \right|^2 \, dq \, dx 
\]

\[
\leq \int_{\Omega \times D} M \sum_{i=1}^{K} U \left( \frac{1}{2} |q_i|^2 \right) \xi(r_k^L) \psi_L^{k+1} \left[ (q_i q_i^T) : \nabla_x u_L^k \right] \, dq \, dx 
\]

\[
- \int_{\Omega \times D} M \xi(r_k^L) \left[ 1 - \frac{\beta^L(\psi_L^k)}{\beta^L(\psi_L^k + \alpha)} \right] \sum_{i=1}^{K} \left[ (\nabla_x u_L^k) q_i \right] \cdot \nabla_q \psi_L^k \, dq \, dx. \tag{4.18}
\]

As each term in (4.18) can be seen as the value of a piecewise constant function on the interval \((t_{k-1}, t_k)\), multiplication of (4.18) by \(\Delta t\) and summation over the indices \(k = 1, \ldots, n\), where \(n \in \{1, \ldots, N\}\), yields on noting that \(\psi_L^{\Delta t}(0) = \psi^0 = \beta^L(\psi^0)\), for \(t = t_n\), that

\[
\int_{\Omega \times D} M \xi(r_k^{\Delta t, +}(t)) \mathcal{F}^L(\psi_L^{\Delta t, +}(t) + \alpha) \, dq \, dx 
\]

\[
+ \frac{1}{2\Delta t L} \int_{0}^{t} \int_{\Omega \times D} M \xi(r_k^{\Delta t, -})(\psi_L^{\Delta t, +} - \psi_L^{\Delta t, -})^2 \, dq \, dx \, ds 
\]

\[
+ \frac{a_0}{4\Delta t} \int_{0}^{t} \int_{\Omega \times D} M \left[ (\mathcal{F}^L)^\prime(\psi_L^{\Delta t, +} + \alpha) \right] \left| \nabla_q \psi_L^{\Delta t, +} \right|^2 \, dq \, dx \, ds + \varepsilon \int_{0}^{t} \int_{\Omega \times D} M \left| \nabla_x \psi_L^{\Delta t, +} \right|^2 \, dq \, dx \, ds 
\]
\[
\begin{align*}
\leq & \int_{\Omega \times D} M \zeta(\rho_0) \mathcal{F}^L(\beta^L(\nabla 0) + \alpha) \, dq \, dx \\
+ & t \int_{\Omega \times D} M \sum_{i=1}^{K} q_i q_i^T U_i \left( \frac{1}{2} |q|^2 \right) \zeta(\rho_0^{\Delta t,+}) \nabla x u_i^{\Delta t,+} \, dq \, dx \, ds \\
- & t \int_{\Omega \times D} M \zeta(\rho_0^{\Delta t,+}) \left[ 1 - \frac{\beta^L(\nabla 0)}{\beta^L(\nabla 0) + \alpha} \right] \sum_{i=1}^{K} \left( \nabla x u_i^{\Delta t,+} q_i \right) \cdot \nabla x \nabla \psi_L^{\Delta t,+} \, dq \, dx \, ds. \tag{4.19}
\end{align*}
\]

We refer to [9, (4.14)–(4.18)] for the details of a similar, but somewhat simpler, argument in the case of \( \zeta \equiv 1 \). The denominator in the prefactor of the second integral motivates us to link \( \Delta t \) to \( L \) so that \( \Delta t = o(1) \) as \( \Delta t \to 0 \) (or, equivalently, \( \Delta t = o(L^{-1}) \) as \( L \to \infty \)), in order to drive the integral multiplied by the prefactor to the limit of \( L \to \infty \), once the product of the two has been bounded above by a constant, independent of \( L \).

Comparing (4.19) with (4.11) we see that after multiplying (4.19) by \( 2k \) and adding the resulting inequality to (4.11) the final term in (4.11) is cancelled by \( 2k \) times the second term on the right-hand side of (4.19). Hence, for any \( t = t_n \), with \( n \in \{1, \ldots, N\} \), we deduce that

\[
\begin{align*}
& \int_{\Omega} \rho_L^{\Delta t,+}(t) |u_L^{\Delta t,+}(t)|^2 \, dx + \frac{\rho_{\min}}{\Delta t} \int_{0}^{t} \| u_L^{\Delta t,+} - u_L^{\Delta t,-} \|^2 \, ds + \int_{0}^{t} \int_{\Omega} \mu(\rho_L^{\Delta t,+}) \left| D(u_L^{\Delta t,+}) \right|^2 \, dx \, ds \\
+ & 2k \int_{\Omega \times D} M \zeta(\rho_L^{\Delta t,+}(t)) \mathcal{F}^L(\nabla 0^{\Delta t,+}(t) + \alpha) \, dq \, dx \\
+ & \frac{\zeta_{\min}}{\Delta t} \int_{0}^{t} \int_{\Omega \times D} M(\nabla 0^{\Delta t,+} - \nabla 0^{\Delta t,-})^2 \, dq \, dx \, ds + 2k \varepsilon \int_{0}^{t} \int_{\Omega \times D} M \left| \nabla x \nabla \psi_L^{\Delta t,+} + \alpha \right|^2 \, dq \, dx \, ds \\
+ & \frac{\alpha_0 k}{2\lambda} \int_{0}^{t} \int_{\Omega \times D} M(\mathcal{F}^L)''(\nabla 0^{\Delta t,+} + \alpha) \left| \nabla q \nabla \psi_L^{\Delta t,+} \right|^2 \, dq \, dx \, ds \\
\leq & \int_{\Omega} \rho_0 |u_0|^2 \, dx + \frac{\rho_{\max} C_X}{\mu_{\min} C_0} \int_{0}^{t} \left\| \mathcal{F}^{\Delta t,+} \right\|^2_{L^2(\Omega)} \, ds + 2k \int_{\Omega \times D} M \zeta(\rho_0) \mathcal{F}^L(\beta^L(\nabla 0) + \alpha) \, dq \, dx \\
- & 2k \int_{0}^{t} \int_{\Omega \times D} M \zeta(\rho_0^{\Delta t,+}) \sum_{i=1}^{K} \left( \nabla x u_i^{\Delta t,+} q_i \right) \cdot \nabla x \nabla \psi_L^{\Delta t,+} \, dq \, dx \, ds. \tag{4.20}
\end{align*}
\]

Let \( b := (b_1, \ldots, b_K) \), recall (3.3), and \( b := |b|_1 := b_1 + \cdots + b_K \), then we can bound the magnitude of the last term on the right-hand side of (4.20) by

\[
\frac{\alpha_0 k}{4\lambda} \left( \int_{0}^{t} \int_{\Omega \times D} M(\mathcal{F}^L)''(\nabla 0^{\Delta t,+} + \alpha) \left| \nabla q \nabla \psi_L^{\Delta t,+} \right|^2 \, dq \, dx \, ds \right).
\]
\[
+ \alpha \frac{4\lambda kb \xi_{\max}^2}{a_0} \left( \int_0^t \int_\Omega \left| \nabla u_{L}^{\Delta t, +} \right|^2 \, dx \, ds \right),
\] (4.21)

see [11, (4.20)] for details in the case of \( \xi \equiv 1 \). Noting (4.21), (3.37), and using (4.8) to bound \((F^L)'(\tilde{\psi}_L^{\Delta t, +} + \alpha)\) from below by \(F''(\tilde{\psi}_L^{\Delta t, +} + \alpha) = (\tilde{\psi}_L^{\Delta t, +} + \alpha)^{-1}\) and (4.9) to bound \(F^L(\tilde{\psi}_L^{\Delta t, +} + \alpha)\) by \(F(\tilde{\psi}_L^{\Delta t, +} + \alpha)\) from below yields, for all \(t = t_n, n \in \{1, \ldots, N\}\), that

\[
\int_\Omega \rho_{L}^{\Delta t, +}(t) |u_{L}^{\Delta t, +}(t)|^2 \, dx + \frac{\rho_{\min}}{\Delta t} \int_0^t \left\| u_{L}^{\Delta t, +} - u_{L}^{\Delta t, -} \right\|^2 \, ds + \int_\Omega \mu(\rho_{L}^{\Delta t, +}) |D(u_{L}^{\Delta t, +})|^2 \, dx \, ds
\]

\[
+ 2k \int_{\Omega \times D} M(\rho_{L}^{\Delta t, +}(t)) \mathcal{F}(\tilde{\psi}_L^{\Delta t, +}(t) + \alpha) \, dq \, dx + \frac{\xi_{\min}k}{\Delta t L} \int_0^t \int_{\Omega \times D} M(\tilde{\psi}_L^{\Delta t, +} - \tilde{\psi}_L^{\Delta t, -})^2 \, dq \, dx \, ds
\]

\[
+ 2k \varepsilon \int_0^t \int_{\Omega \times D} \frac{\left| \nabla \tilde{\psi}_L^{\Delta t, +} \right|^2}{\tilde{\psi}_L^{\Delta t, +} + \alpha} \, dq \, dx \, ds + \frac{a_0 k}{4\lambda} \int_0^t \int_{\Omega \times D} M(\nabla \psi_L^{\Delta t, +})^2 \, dq \, dx \, ds
\]

\[
\leq \int_\Omega \rho_0 |u_0|^2 \, dx + \frac{\rho_{\max}^2 \varepsilon^2}{\mu_{\min} c_0} \int_0^t \left\| F^{\Delta t, +} \right\|_{L^2(\Omega)}^2 \, dx
\]

\[
+ 2k \int_{\Omega \times D} M(\rho_0) \mathcal{F}(\beta^L(\tilde{\psi}_0^L) + \alpha) \, dq \, dx + \alpha \frac{4\lambda kb \xi_{\max}^2}{a_0 c_0} \int_0^t \left\| D(u_{L}^{\Delta t, +}) \right\|^2 \, dx \, ds.
\] (4.22)

Next we note that an analogous argument to the one that was used to derive [9, (4.25)] yields that

\[
\int_{\Omega \times D} M(\rho_0) \mathcal{F}(\beta^L(\tilde{\psi}_0^L) + \alpha) \, dq \, dx \leq \frac{3\alpha}{2} \xi_{\max} |\Omega| + \int_{\Omega \times D} M(\rho_0) \mathcal{F}(\tilde{\psi}_0^L + \alpha) \, dq \, dx.
\] (4.23)

The only restriction we have imposed on \(\alpha\) so far is that it belongs to the open interval \((0, 1)\); let us now restrict the range of \(\alpha\) further by demanding that, in fact,

\[
0 < \alpha < \min\left(1, \frac{\mu_{\min} a_0 c_0}{4\lambda kb \xi_{\max}^2}\right).
\] (4.24)

where \(c_0\) is the constant appearing in the Korn inequality (3.37). Then, the last term on the right-hand side of (4.22) can be absorbed into the third term on the left-hand side, giving, on noting (3.37) and (4.23), for \(t = t_n\) and \(n \in \{1, \ldots, N\}\),

\[
\int_\Omega \rho_{L}^{\Delta t, +}(t) |u_{L}^{\Delta t, +}(t)|^2 \, dx + \frac{\rho_{\min}}{\Delta t} \int_0^t \left\| u_{L}^{\Delta t, +} - u_{L}^{\Delta t, -} \right\|^2 \, ds
\]

\[
+ \int_0^t \int_\Omega \left( \mu(\rho_{L}^{\Delta t, +}) - \alpha \frac{4\lambda kb \xi_{\max}^2}{a_0 c_0} \right) |D(u_{L}^{\Delta t, +})|^2 \, dx \, ds
\]
\[ + 2k \int_{\Omega \times D} M \zeta (\rho^{\Delta t,+}_L(t)) F (\tilde{\psi}_L^{\Delta t,+}(t) + \alpha) \, dq \, dx \]
\[ + \frac{\zeta_{\min}}{\Delta t L} \int_0^t \int_{\Omega \times D} M (\tilde{\psi}_L^{\Delta t,+} - \tilde{\psi}_L^{\Delta t,-})^2 \, dq \, dx \, ds \]
\[ + 2k \varepsilon \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \tilde{\psi}_L^{\Delta t,+}|^2}{\tilde{\psi}_L^{\Delta t,+} + \alpha} \, dq \, dx \, ds \]
\[ \leq \int_\Omega \rho_0 |u_0|^2 \, dx + \frac{\rho_{\max}^2 C^2}{\mu_{\min} C_0} \int_0^t \int_{\Omega \times D} \| f^{\Delta t,+} \|^2_{L^2(\Omega)} \, ds + 2k \int_{\Omega \times D} M \zeta (\rho_0) F (\tilde{\psi}_0) \, dq \, dx \quad (4.25) \]

The key observation at this point is that the right-hand side of (4.25) is completely independent of the cut-off parameter \( L \).

On noting [9, pp. 1243–44], we can pass to the limit \( \alpha \to 0^+ \) in (4.25) to obtain, for all \( t = t_n, n \in \{1, \ldots, N\} \), that

\[ \int_\Omega \rho^{\Delta t,+}_L(t) |u_L^{\Delta t,+}(t)|^2 \, dx + \frac{\rho_{\min}}{\Delta t} \int_0^t \int_\Omega \| u_L^{\Delta t,+} - u_L^{\Delta t,-} \|^2 \, ds + 2k \int_\Omega \zeta (\rho_L^{\Delta t,+}) |D (u_L^{\Delta t,+})|^2 \, dx \, ds \]
\[ + 2k \int_{\Omega \times D} M \zeta (\rho^{\Delta t,+}_L(t)) F (\tilde{\psi}_L^{\Delta t,+}(t)) \, dq \, dx \, ds \]
\[ + 8k \varepsilon \int_0^t \int_{\Omega \times D} M |\nabla_x \tilde{\psi}_L^{\Delta t,+}|^2 \, dq \, dx \, ds + \frac{a_0 k}{\lambda} \int_0^t \int_{\Omega \times D} M |\nabla_x \tilde{\psi}_L^{\Delta t,+}|^2 \, dq \, dx \, ds \]
\[ \leq \int_\Omega \rho_0 |u_0|^2 \, dx + \frac{\rho_{\max}^2 C^2}{\mu_{\min} C_0} \int_0^t \int_{\Omega \times D} \| f^{\Delta t,+} \|^2_{L^2(\Omega)} \, ds + 2k \int_{\Omega \times D} M \zeta (\rho_0) F (\tilde{\psi}_0) \, dq \, dx \quad (4.26a) \]
\[ \leq \int_\Omega \rho_0 |u_0|^2 \, dx + \frac{\rho_{\max}^2 C^2}{\mu_{\min} C_0} \int_0^T \int_{\Omega \times D} \| f \|^2_{L^2(\Omega)} \, ds + 2k \int_{\Omega \times D} M \zeta (\rho_0) F (\tilde{\psi}_0) \, dq \, dx \]
\[ =: \left[ B(u_0, f, \tilde{\psi}_0) \right]^2, \quad (4.26b) \]

where, in the last line, we used (3.17a) to bound the third term in (4.26a), and that \( t \in [0, T] \) together with the definition (3.24) of \( f^{\Delta t,+} \) to bound the second term.

4.2. \( L \)-independent bound on a fractional-order in time Nikol’skii norm of \( u_L^{\Delta t} \)

First, we have by (3.44) that \( \rho^{\Delta t}_L \in \mathcal{T} \) and therefore \( \| \rho^{\Delta t}_L \|_{L^\infty(0,T;L^\infty(\Omega))} \leq \rho_{\max} \). Thus, by (3.37), (3.3) and (4.26b), we then have that
\[
\| \nabla u^{\Delta t,+} \|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{c_0 \mu_{\min}} \int_0^T \int_\Omega \mu (\rho_{\Delta t,+}^L) D(u^{\Delta t,+})^2 \, dx \, ds \\
\leq \frac{1}{c_0 \mu_{\min}} [B(u_0, f, \tilde{\psi}_0)]^2, \tag{4.27}
\]
and, by (3.14), (4.26b) and (4.27), we have
\[
\| \nabla u^{\Delta t,-} \|^2_{L^2(0,T;L^2(\Omega))} = \Delta t \| \nabla u^0 \|^2 + \int_0^{T-\Delta t} \int_\Omega \| \nabla u^{\Delta t,+} \|^2 \, ds \\
\leq \int_\Omega \rho_0 |u_0|^2 \, dx + \| \nabla u^{\Delta t,+} \|^2_{L^2(0,T;L^2(\Omega))} \\
\leq \left(1 + \frac{1}{c_0 \mu_{\min}} \right) [B(u_0, f, \tilde{\psi}_0)]^2. \tag{4.28}
\]
We then have from (4.27), (4.28) and Poincaré's inequality that \( u_{\Delta t,-}^{\Delta t} \) \( L^2(0,T;H^1(\Omega)) \) \( C_\ast \), where \( C_\ast \) is a positive constant that depends solely on \( \varepsilon, \rho_{\min}, \rho_{\max}, \mu_{\min}, \xi_{\min}, \xi_{\max}, T, |A|, a_0, c_0, C_K, k, \lambda, K \) and \( b \), and is independent of \( L \) and \( \Delta t \). Hence, by Sobolev's embedding theorem,
\[
\| u_{\Delta t,-}^{\Delta t} \|_{L^2(0,T;L^1(\Omega))} \leq C_\ast, \quad \text{for all} \quad s \in [1, \infty) \quad \text{if} \quad d = 2, \\
\| u_{\Delta t,-}^{\Delta t} \|_{L^\infty(0,T;L^1(\Omega))} \leq C_\ast, \quad \text{for all} \quad s \in [1, 6] \quad \text{if} \quad d = 3, \tag{4.29}
\]
and hence also for all \( s = q \), with \( q \) as above, where \( C_\ast \) is independent of \( L \) and \( \Delta t \). In addition, it follows from \( \rho_{\Delta t,+}^L(t) \in Y \) for a.a. \( t \in (0,T) \), (4.26b), and as (3.14) implies that \( ||u^t||^2 \leq \frac{\rho_{\max}}{\rho_{\min}} ||u_0||^2 \), that
\[
\| u_{\Delta t,-}^{\Delta t} \|_{L^\infty(0,T;L^2(\Omega))} \leq C_\ast. \tag{4.30}
\]
Note that, by (3.44), \( \rho_{\min} \leq \rho_{\Delta t,-}^{\Delta t} \leq \rho_{\max} \) a.e. on \( \Omega \times [0,T] \) for all \( \Delta t \) and \( L \). Hence we have that
\[
\| \rho_{\Delta t,-}^{\Delta t} \|_{L^\infty(0,T;L^\infty(\Omega))} \leq \rho_{\max}. \quad \text{As}
\]
\[
(\rho_{\Delta t}^{\Delta t}, \cdot) = \rho_{\Delta t}^{\Delta t} \left[ \frac{t-t_n-1}{\Delta t} u_{\Delta t}^{n-1}(\cdot) + \frac{t_n-t}{\Delta t} u_{\Delta t}^{n-1}(\cdot) \right] + \frac{t_n-t}{\Delta t} \left( \rho_{\Delta t}^{n-1}(\cdot) - \rho_{\Delta t}^{n}(\cdot) \right) u_{\Delta t}^{n-1}(\cdot)
\]
for all \( t \in [t_{n-1}, t_n] \) and \( n = 1, \ldots, N \), which in turn implies that
\[
\| (\rho_{\Delta t}^{\Delta t}, t) \|_{L^1(\Omega)} \leq \| \rho_{\Delta t,+}^{\Delta t}, t \|_{L^\infty(\Omega)} \| u_{\Delta t}^{\Delta t}, t \|_{L^1(\Omega)} + \left( \| \rho_{\Delta t}, t \|_{L^\infty(\Omega)} + \| \rho_{\Delta t,+}, t \|_{L^\infty(\Omega)} \right) \| u_{\Delta t,-}^{\Delta t}, t \|_{L^1(\Omega)} \\
\leq \rho_{\max} \| u_{\Delta t,+}^{\Delta t}, t \|_{L^1(\Omega)} + 2 \rho_{\max} \| u_{\Delta t,-}^{\Delta t}, t \|_{L^1(\Omega)}
\]
for all \( t \in (t_{n-1}, t_n] \) and \( n = 1, \ldots, N \). By squaring both sides, using the algebraic–geometric mean inequality and integrating the resulting inequality over \( t \in [0, T] \), we deduce on noting (4.29) that

\[
\| (\rho_L u_L)^{\Delta t} \|_{L^2(0, T; L^1(\Omega))} \leq C_s, \quad \text{for } \begin{cases} s \in [1, \infty) & \text{if } d = 2, \\ s \in [1, 6] & \text{if } d = 3, \end{cases} \tag{4.31}
\]

where \( C_s \) is a positive constant, independent of \( L \) and \( \Delta t \). We shall use (4.3b) to improve the bound (4.31). To this end we first note that using (4.3a) in (4.3b) yields, for all \( \varepsilon > 0 \),

\[
\int_0^T \int_\Omega \left[ \frac{\partial}{\partial t} (\rho_L u_L)^{\Delta t} \cdot \varepsilon - \frac{1}{2} \rho_L^{[\Delta t]} u_L^{[\Delta t], -} \cdot \nabla_x (u_L^{[\Delta t], +} \cdot \varepsilon) \right] \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega \mu (\rho_L^{[\Delta t]}) D(u_L^{[\Delta t]}) : D(\varepsilon) \, dx \, dt
\]

\[
+ \frac{1}{2} \int_0^T \int_\Omega \rho_L^{[\Delta t]} \left[ \left( (u_L^{[\Delta t], -} \cdot \nabla_x) u_L^{[\Delta t], +} \right) \cdot \varepsilon - \left( (u_L^{[\Delta t], -} \cdot \nabla_x) w \right) \cdot u_L^{[\Delta t], +} \right] \, dx \, dt
\]

\[
= \int_0^T \left[ \int_\Omega \rho_L^{[\Delta t]} f^{[\Delta t], +} \cdot \varepsilon \, dx - k \sum_{i=1}^K \int_\Omega \zeta_i (M \xi (\rho_L^{[\Delta t]}, \tilde{\psi}_L^{[\Delta t], +}) : \nabla_x w) \, dx \right] \, dt, \tag{4.32}
\]

It follows that, for any \( t', t'' \in [0, T] \) such that \( 0 \leq t' < t'' \leq T \) and choosing \( \varepsilon = \chi_{[t', t'']} \), with \( \varepsilon \in \mathcal{V} \), where, as in the discussion following (3.41b), \( \chi_{[t', t'']} \) denotes the characteristic function of the interval \([t', t'']\), we have

\[
\int_{t'}^{t''} \int_\Omega \left[ \frac{\partial}{\partial t} (\rho_L u_L)^{\Delta t} \cdot \varepsilon - \frac{1}{2} \rho_L^{[\Delta t]} u_L^{[\Delta t], -} \cdot \nabla_x (u_L^{[\Delta t], +} \cdot \varepsilon) \right] \, dx \, dt
\]

\[
+ \int_{t'}^{t''} \int_\Omega \mu (\rho_L^{[\Delta t]}) D(u_L^{[\Delta t]}) : D(\varepsilon) \, dx \, dt
\]

\[
+ \frac{1}{2} \int_{t'}^{t''} \int_\Omega \rho_L^{[\Delta t]} \left[ \left( (u_L^{[\Delta t], -} \cdot \nabla_x) u_L^{[\Delta t], +} \right) \cdot \varepsilon - \left( (u_L^{[\Delta t], -} \cdot \nabla_x) w \right) \cdot u_L^{[\Delta t], +} \right] \, dx \, dt
\]

\[
= \int_{t'}^{t''} \left[ \int_\Omega \rho_L^{[\Delta t]} f^{[\Delta t], +} \cdot \varepsilon \, dx - k \sum_{i=1}^K \int_\Omega \zeta_i (M \xi (\rho_L^{[\Delta t]}, \tilde{\psi}_L^{[\Delta t], +}) : \nabla_x w) \, dx \right] \, dt \quad \forall \varepsilon \in \mathcal{V},
\]

and hence, equivalently (cf. (4.2) for the definition of \( \rho_L^{[\Delta t]} \)),

\[
\int_\Omega \left[ (\rho_L u_L)^{\Delta t} (t'') - (\rho_L u_L)^{\Delta t} (t') \right] \cdot \varepsilon \, dx
\]
We shall suppose that $t'' = t' + \delta$, where $\delta \in (0, T - t')$, and bound each of the terms $U_i$, $i = 1, \ldots, 5$, in turn. We note in particular that, by the definition (4.2) of $\rho_L^{[\Delta t]}$, the term $U_2 = 0$ when $t', t'' \in [0 = t_0, t_1, \ldots, t_{N-1}, t_N = T]$.

For $U_1$, by the Cauchy–Schwarz inequality, we have that

$$|U_1| \leq \mu_{\max} \left[ \int_{t'}^{t'+\delta} \left( \int_{\Omega} \left| D(u_L^{\Delta t_-}(t)) \right|^2 \, dx \right)^{\frac{1}{2}} \, dt \right] \| D(V) \|$$

$$\leq \mu_{\max} \delta^{\frac{1}{2}} \| D(u_L^{\Delta t_-}) \|_{L^2(t', t'+\delta; L^2(\Omega))} \| D(V) \| \quad \forall V \in \mathcal{V}. $$

Further, for any $q \in (2, \infty)$ when $d = 2$ and any $q \in [3, 6]$ when $d = 3$, we have that

$$|U_2| \leq \rho_{\max} \left[ \int_{t'}^{t'+\delta} \| u_L^{\Delta t_-}(t) \|_{L^q(\Omega)} \| \nabla u_L^{\Delta t_-}(t) \| \, dt \right] \| V \|_{L^q(\Omega)} \quad \forall V \in \mathcal{V}. $$

The Gagliardo–Nirenberg inequality (3.2) and Korn’s inequality (3.37) imply that

$$\| u_L^{\Delta t_-}(t) \|_{L^q(\Omega)} \leq C(d, q, \Omega) \| u_L^{\Delta t_-}(t) \|_{L^q(\Omega)}^{-\frac{1-d}{q}} \| D(u_L^{\Delta t_-}(t)) \|_{\frac{d}{q}} \quad t \in (0, T], $$

for all $q \in (2, \infty)$ when $d = 2$ and $q \in [3, 6]$ when $d = 3$. Therefore, by Korn’s inequality (3.37) again, and by Sobolev’s embedding theorem and Hölder’s inequality, we have that

$$|U_2| \leq C(d, q, \Omega) \rho_{\max} \| u_L^{\Delta t_-} \|_{L^{\frac{q}{q-1}}(\Omega)} \left[ \int_{t'}^{t'+\delta} \| D(u_L^{\Delta t_-}(t)) \|_{\frac{d}{q}} \| D(u_L^{\Delta t_-}(t)) \| \, dt \right] \| D(V) \|$$

$$\leq C(d, q, \Omega) \rho_{\max} \| u_L^{\Delta t_-} \|_{L^{\frac{q}{q-1}}(\Omega)} \delta^{\frac{q-d}{2q}} \| D(u_L^{\Delta t_-}) \|_{L^q(t', t'+\delta; L^2(\Omega))} \| D(u_L^{\Delta t_-}(t)) \|_{L^2(t', t'+\delta; L^2(\Omega))} \| D(V) \| \quad \forall V \in \mathcal{V}. $$
Hence,

\[ |U_2| \leq C(d, q, \Omega) \rho_{\max} \left\| u^\Delta t, - \right\|_{L^\infty(0, T; L^2(\Omega))} \left\| D(u^\Delta t, -) \right\|_{L^2(0, T; L^2(\Omega))} \times \delta \frac{\alpha d}{q} \right\| D(u^\Delta t, +) \left\|_{L^2(t', t'+\delta; L^2(\Omega))} \right\|_{D(y)} \right\| \forall y \in V, \]

for all \( q \in (2, \infty) \) when \( d = 2 \) and all \( q \in [3, 6] \) when \( d = 3 \). An identical argument yields that

\[ |U_3| \leq C(d, q, \Omega) \rho_{\max} \left\| u^\Delta t, - \right\|_{L^\infty(0, T; L^2(\Omega))} \left\| D(u^\Delta t, -) \right\|_{L^2(0, T; L^2(\Omega))} \times \delta \frac{\alpha d}{q} \right\| D(u^\Delta t, +) \left\|_{L^2(t', t'+\delta; L^2(\Omega))} \right\|_{D(y)} \right\| \forall y \in V, \]

for all \( q \in (2, \infty) \) when \( d = 2 \) and all \( q \in [3, 6] \) when \( d = 3 \).

For \( U_4 \), on noting (3.3), (3.4), Korn’s inequality (3.37) and Hölder’s inequality (with respect to the variable \( t' \)) yield that

\[ |U_4| \leq C(c_0, \kappa, \Omega) \rho_{\max} \delta \frac{1}{2} \left\| f^\Delta t, + \right\|_{L^2(t', t'+\delta; L^\infty(\Omega))} \right\|_{D(y)} \right\| \forall y \in V, \]

with \( \kappa > 1 \) if \( d = 2 \) and \( \kappa = \frac{6}{5} \) if \( d = 3 \).

Finally, for the term \( U_5 \), from (2.4a), (3.12) and the Cauchy–Schwarz inequality, we have that

\[ |U_5| = \left| k \sum_{i=1}^{K} \int_{t'}^{t'+\delta} \int_{\Omega \times D} M \zeta (\rho^\Delta t, + (t)) \left[ \sum_{i=1}^{K} \nabla \psi^\Delta t, + (t) \cdot (\nabla \times y) q_i \right] \nabla \cdot \nabla \times y \, dq \, dx \, dt \right| \]

\[ = \left| k \int_{t'}^{t'+\delta} \int_{\Omega \times D} M \zeta (\rho^\Delta t, + (t)) \left[ \sum_{i=1}^{K} \nabla \psi^\Delta t, + (t) \cdot (\nabla \times y) q_i \right] dq \, dx \, dt \right| \]

\[ = \left| -2k \int_{t'}^{t'+\delta} \int_{\Omega} M \zeta (\rho^\Delta t, + (t)) \int_{\Omega} M \sqrt{\nabla \psi^\Delta t, + (t)} \left[ \sum_{i=1}^{K} \nabla \psi^\Delta t, + (t) \cdot (\nabla \times y) q_i \right] dq \, dx \, dt \right| \]

\[ \leq 2k \zeta_{\max} \int_{t'}^{t'+\delta} \int_{\Omega} \left| \nabla \times y \right| \left[ \int_{\Omega} M \left| q \right|^2 \nabla \psi^\Delta t, + (t) \, dq \right]^{1/2} \left[ \int_{\Omega} M \left| \nabla q \right| \left| \nabla \psi^\Delta t, + (t) \right|^2 \, dq \right]^{1/2} \, dx \, dt \]

\[ \leq 2k \zeta_{\max} \int_{t'}^{t'+\delta} \int_{\Omega} \left| \nabla \times y \right| \left[ \int_{\Omega} M \left| q \right|^2 \nabla \psi^\Delta t, + (t) \, dq \right]^{1/2} \left[ \int_{\Omega} M \left| \nabla q \right| \left| \nabla \psi^\Delta t, + (t) \right|^2 \, dq \right]^{1/2} \, dx \, dt \]

\[ \times \text{ess.sup}_{(x, t) \in \Omega \times (0, T)} \left[ \int_{\Omega} M \left| q \right|^2 \nabla \psi^\Delta t, + (t) \, dq \right]^{1/2} \]

\[ \leq 2k \zeta_{\max} \delta \frac{1}{2} \left\| \nabla q \right\|_{L^2(t', t'+\delta; L^2(\Omega \times D))} \left\| \nabla \times y \right\| \]

\[ \times \text{ess.sup}_{(x, t) \in \Omega \times (0, T)} \left[ \int_{\Omega} M \left| q \right|^2 \nabla \psi^\Delta t, + (t) \, dq \right]^{1/2} \forall y \in V. \]
Since $|q|^2 = |q_1|^2 + \cdots + |q_K|^2 < b_1 + \cdots + b_K =: b$, the inequality (3.63) immediately implies that the final factor is bounded by $\sqrt{\alpha b}$. Korn’s inequality (3.37) then implies that

$$|U_5| \leq C(k, \omega, b, c_0)\delta^{\frac{1}{2}} \| \nabla \sqrt{\psi_L} \|_{L^2(t', t' + \delta; L^2(\Omega \times D))} \| D(\psi) \|_{V}$$. 

By collecting the upper bounds on the terms $U_i$, $i = 1, \ldots, 5$, and noting the upper bounds on the first and the third term on the left-hand side of (4.26b), we deduce that $\delta = t' - t < (0, T - t')$.

In what follows, we shall suppose that $t' < t''$, so that $\delta := t'' - t'$ is an integer multiple of $\Delta t$; then

$$\left( \rho_L u_L \right)^{\Delta t}(t') - \left( \rho_L u_L \right)^{\Delta t}(t') = \rho_L^{[\Delta t]}(t'') u_L^{\Delta t}(t'') - \rho_L^{[\Delta t]}(t') u_L^{\Delta t}(t') = \rho_L^{[\Delta t]}(t'') \left[ u_L^{\Delta t}(t'') - u_L^{\Delta t}(t') \right] + \left[ \rho_L^{[\Delta t]}(t'') - \rho_L^{[\Delta t]}(t') \right] u_L^{\Delta t}(t') \). \quad (4.35)$$

By selecting $\eta = \chi[t', t''] \left( u_L^{\Delta t}(t') \cdot v \right)$ in (4.3a) with $t'' = t' + \delta$, we have, using Korn’s inequality (3.37), that, for any $v \in V$,

$$\left( \rho_L^{[\Delta t]}(t' + \delta) - \rho_L^{[\Delta t]}(t') \right) \left( u_L^{\Delta t}(t') \cdot v \right) \leq \rho_{\max} \delta^{\frac{1}{2}} \| u_L^{\Delta t} \|_{L^2(t', t' + \delta; L^2(\Omega))} \| \nabla \chi (u_L^{\Delta t}(t') \cdot v) \|_{L^{\frac{q}{q-2}}(\Omega)}$$

$$\leq \rho_{\max} \delta^{\frac{1}{2}} \| u_L^{\Delta t} \|_{L^2(t', t' + \delta; L^2(\Omega))} \left( \| \nabla \chi (u_L^{\Delta t}(t') \cdot v) \|_{L^{\frac{q}{q-2}}(\Omega)} + \| \nabla \chi \|_{L^{\frac{q}{q-2}}(\Omega)} \right)$$

$$\leq C \delta^{\frac{1}{2}} \| u_L^{\Delta t} \|_{L^2(t', t' + \delta; L^2(\Omega))} \times \left( \| D(u_L^{\Delta t}(t')) \|_{V} \| u_L^{\Delta t}(t') \|_{L^2(\Omega)} + \| D(v) \|_{V} \| u_L^{\Delta t}(t') \|_{L^2(\Omega)} \right), \quad (4.36)$$

where $q \in [4, \infty)$ when $d = 2$ and $q \in [4, 6]$ when $d = 3$, and $C = C(c_0, \rho_{\max})$, where $c_0$ is the constant in Korn’s inequality (3.37).

By dotting (4.35) with $v \in V$ integrating over $\Omega$ and substituting (4.34) and (4.36) into the resulting identity, we deduce that
\[
\int_\Omega \rho_L^{[\Delta t]}(t' + \delta) [u_L^{\Delta t}(t' + \delta) - u_L^{\Delta t}(t')] \cdot \psi \, dx
\]
\[
\leq C \delta^{\frac{1}{2}} \|u_L^{\Delta t,-}\|_{L^2(t',t'+\delta;L^q(\Omega))} \times \left( \|D(u_L^{\Delta t,-})\|_{L^2(t',t'+\delta;L^q(\Omega))} \|D(u_L^{\Delta t,+})\|_{L^2(t',t'+\delta;L^q(\Omega))} \right)^{\frac{2}{q-2}} + C \|D(\psi)\|_{L^2(t',t'+\delta;L^q(\Omega))} \right) \]
\[
+ \delta^{\frac{1}{2}} \|\psi^{\Delta t,+}\|_{L^2(t',t'+\delta;L^q(\Omega \times D))},
\]
(4.37)

for all \( \psi \in \mathcal{V} \), where \( q \in [4, \infty) \) when \( d = 2 \) and \( q \in [4, 6] \) when \( d = 3 \), and \( C \) is a positive constant, independent of \( \Delta t, L \) and \( \delta \). Here \( \delta = \ell \Delta t \), where \( \ell = 1, \ldots, N-m \), and \( t' = m \Delta t \) for \( m = 0, \ldots, N-1 \). The symbol \( \delta \) will be understood to have the same meaning throughout the rest of this section, unless otherwise stated.

We now select \( \psi = u_L^{\Delta t}(t' + \delta) - u_L^{\Delta t}(t') \) in (4.37), sum the resulting collection of inequalities over \( t' \in \{0, t_1, \ldots, T - \delta\} \) (denoting the sum over all \( t' \) contained in this set by \( \sum_{t'=0}^{T-\delta} \)), and note the following obvious inequalities:

\[
\|u_L^{\Delta t,-}\|_{L^2(t',t'+\delta;L^q(\Omega))} = \|u_L^{\Delta t,-}\|_{L^2(t',t'+\delta;L^q(\Omega))} \leq C \|u_L^{\Delta t,-}\|_{L^2(0,T;L^q(\Omega))} \|D(u_L^{\Delta t,-})\|_{L^2(t',t'+\delta;L^q(\Omega))},
\]

\[
\|u_L^{\Delta t}(t' + \delta) - u_L^{\Delta t}(t')\|_{L^q}\right)^{\frac{2}{q-2}},
\]

\[
\|u_L^{\Delta t}(t')\|_{L^q} \leq \|u_L^{\Delta t}\|_{L^q(0,T;L^q(\Omega))} \quad \text{and} \quad \|u_L^{\Delta t}(t' + \delta)\|_{L^q} \leq \|u_L^{\Delta t}\|_{L^q(0,T;L^q(\Omega))},
\]

the first of which follows by Sobolev’s embedding theorem and Korn’s inequality (3.37), to deduce from (4.37) that

\[
\Delta t \sum_{t'=0}^{T-\delta} \rho_L^{[\Delta t]}(t' + \delta)\|u_L^{\Delta t}(t' + \delta) - u_L^{\Delta t}(t')\|^2 \, dx
\]
\[
\leq C \delta^{\frac{1}{2}} \|u_L^{\Delta t,-}\|_{L^2(0,T;L^q(\Omega))} \|u_L^{\Delta t}\|_{L^q(0,T;L^q(\Omega))} \times \left( \Delta t \sum_{t'=0}^{T-\delta} \|D(u_L^{\Delta t,-})\|_{L^2(t',t'+\delta;L^q(\Omega))} \|D(u_L^{\Delta t,+})\|_{L^2(t',t'+\delta;L^q(\Omega))} \right)^{\frac{2}{q-2}} + C \Delta t \sum_{t'=0}^{T-\delta} \|D(u_L^{\Delta t,+})\|_{L^2(t',t'+\delta;L^q(\Omega))} \|u_L^{\Delta t}(t' + \delta) - u_L^{\Delta t}(t')\|_{L^q} \right),
\]

\[
+ \Delta t \sum_{t'=0}^{T-\delta} \|D(u_L^{\Delta t,+})\|_{L^2(t',t'+\delta;L^q(\Omega))} \|D(u_L^{\Delta t}(t') - D(u_L^{\Delta t}(t')))\|_{L^q} \right) \]

\[
+ \Delta t \sum_{t'=0}^{T-\delta} \|D(u_L^{\Delta t}(t' + \delta) - D(u_L^{\Delta t}(t')))\|_{L^q} \right) \]

\[
+ C \Delta t \sum_{t'=0}^{T-\delta} \|D(u_L^{\Delta t}(t') - D(u_L^{\Delta t}(t')))\|_{L^q} \right) \]

\[
+ \delta^{\frac{1}{2}} \|u_L^{\Delta t,+}\|_{L^2(t',t'+\delta;L^q(\Omega \times D))} \right)
\]

\[
=: V_1(V_2 + V_3) + C \Delta t \sum_{t'=0}^{T-\delta} V_4(t')(V_5(t') + V_6(t') + V_7(t')), \quad (4.38)
\]
where $q \in [4, \infty)$ when $d = 2$ and $q \in [4, 6]$ when $d = 3$, and $C$ is a positive constant, independent of $\Delta t$, $L$ and $\delta$; $V_j$ denotes the expression in the first line on the right-hand side of (4.38); $V_2$ and $V_3$ denote the two terms in the bracketed expression multiplied by $V_1$; $V_4(t')$ is the factor in front of the bracket in the fourth line on the right-hand side of (4.38); and $V_5(t'), V_6(t')$ and $V_7(t')$ are the three terms in the bracketed expression multiplied by $V_4(t')$. We shall consider each of the terms $V_1, V_2, V_3, V_4(t'), \ldots, V_7(t')$ separately. We begin by noting that by (4.29) and (4.30)

$$V_1 = C\delta^\frac{1}{2} \left\|u_L^{\Delta t,-}\right\|_{L^2((0,T);\mathcal{L}^0(\Omega))}^{\frac{2}{q-2}} \left\|u_L^{\Delta t}\right\|_{L^\infty((0,T);\mathcal{L}^0(\Omega))}^{\frac{q-4}{2(q-2)}} \leq C\delta^\frac{1}{2}, \quad (4.39)$$

where $q$ is as above, and $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$. In the rest of this section we shall assume that $q \in (4, \infty)$ when $d = 2$ and $q \in (4, 6)$ when $d = 3$.

Next, for the term $V_2$, Hölder’s inequality with respective exponents $2(q-2)/(q-4)$, 2 and $q-2$ for the three factors under the summation sign in this term yields that

$$V_2 \leq \left(\Delta t \sum_{t'=0}^{T-\delta} \left\|D(u_L^{\Delta t,-})\right\|_{L^2(t', t'+\delta; L^2(\Omega))}^{\frac{q-4}{2(q-2)}} \left(\Delta t \sum_{t'=0}^{T-\delta} \left\|D(u_L^{\Delta t}(t'))\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \right)^{\frac{q-4}{2}} \times \left(\Delta t \sum_{t'=0}^{T-\delta} \left\|D(u_L^{\Delta t}(t')) - D(u_L^{\Delta t}(t'))\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{q-2}} \times \left(\Delta t \sum_{t'=0}^{T-\delta} \left\|D(u_L^{\Delta t}(t'))\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{q-2}} =: V_{21} V_{22} V_{23}.$$
where $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$. Analogously, since $u_{L}^{\Delta t}$ and $u_{L}^{\Delta t,+}$ coincide at all points $t^\prime = \ell \Delta t$, $\ell = 1, \ldots, N$, we have, again by (3.14) and (4.26b), that
\[
V_{22} = \left( \Delta t \|D(u^0)\|^2 + \Delta t \int_{0}^{T-\delta} \|D(u_{L}^{\Delta t,+}(s))\|^2 \, ds \right)^{\frac{1}{2}} \leq C,
\]
where $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$. For the term $V_{23}$, we have by the triangle inequality, shifting indices in the summation, and noting, once again, (3.14) and (4.26b), that
\[
V_{23} \leq 2^{\frac{q-d}{2}} \left( \Delta t \sum_{t^\prime=0}^{T} \|D(u_{L}^{\Delta t}(t^\prime))\|^2 \right)^{\frac{1}{2}} \leq C,
\]
where $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$. Thus we deduce that
\[
V_{2} = V_{21}V_{22}V_{23} \leq C \delta^{\frac{q-d}{2(q-2d)}},
\]
where $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$. An identical argument yields that
\[
V_{3} \leq C \delta^{\frac{q-d}{2(q-2d)}},
\]
and therefore
\[
V_{1}(V_{2} + V_{3}) \leq C \delta^{\frac{1}{2} + \frac{q-d}{2(q-2d)}}, \tag{4.40}
\]
where $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$.

Finally, by the Cauchy–Schwarz inequality applied to the sum starting in the fifth line of (4.38), the bound on the term $V_{23}$ above, using Lemma 4.1 with
\[
s \in (0, T) \mapsto g(s) = \left( \delta^\frac{1}{2} + \delta^{\frac{q-d}{2q}} \right) \|D(u_{L}^{\Delta t,+}(s))\|^2 + \delta \|f^{\Delta t,+}(s)\|_{L^\infty(\Omega)}^2 + \delta \|q\sqrt{V_{L}^{\Delta t,+}}\|_{L^2(\Omega \times D)},
\]
and the bound (4.26b), we deduce that
\[
C \Delta t \sum_{t^\prime=0}^{T-\delta} V_{4}(t^\prime)(V_{5}(t^\prime) + V_{6}(t^\prime) + V_{7}(t^\prime)) \leq C \left( \delta^\frac{1}{2} + \delta^{\frac{q-d}{2q}} \right)^{\frac{1}{2}} + C \delta^{\frac{1}{2} + \frac{1}{d}} + C \delta^{\frac{1}{2} + \frac{1}{d}}, \tag{4.41}
\]
where $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$. with $\delta = \ell \Delta t$, $\ell = 1, \ldots, N$.

On substituting (4.40) and (4.41) into (4.38), we thus have that
\[
\Delta t \sum_{t^\prime=0}^{T-\delta} \int_{\Omega} \rho_{L}^{\Delta t}(t^\prime + \delta)|u_{L}^{\Delta t}(t^\prime + \delta) - u_{L}^{\Delta t}(t^\prime)|^2 \, dx \leq C \delta^{\frac{1}{2} + \frac{q-d}{2q-2d}} + C \left( \delta^\frac{1}{2} + \delta^{\frac{q-d}{2q}} \right)^{\frac{1}{2}} + C \delta \leq C \delta^{1 - \frac{1}{q-2d}} \quad \text{with} \quad \begin{cases} \frac{q}{2} \in (4, \infty) & \text{if } d = 2, \\ \frac{1}{q} \in (4, 6] & \text{if } d = 3, \end{cases}
\]
where $C$ is a positive constant, independent of $L$, $\Delta t$ and $\delta$. with $\delta = \ell \Delta t$, $\ell = 1, \ldots, N$. 
On recalling that $\rho_L^{[\Delta t]}(t' + \delta) \geq \rho_{\min}$ for all $t' \in [0, T - \delta]$, we finally have that

$$
\Delta t \sum_{t'=0}^{T-\delta} \left\| u_L^{\Delta t}(t' + \delta) - u_L^{\Delta t}(t') \right\|^2 \leq C \delta^{1 - \frac{1}{\varphi(t)}} \quad \text{with} \quad \begin{cases} 
q \in (4, \infty) & \text{if } d = 2, \\
q \in (4, 6] & \text{if } d = 3,
\end{cases}
$$

where $C$ is a positive constant, independent of $\Delta t$, $L$, and $\delta$; with $\delta = \ell \Delta t$, $\ell = 1, \ldots, N$. As $u_L^{\Delta t}(\ell \Delta t) = u_L^{\Delta t, +}(\ell \Delta t)$ and $u_L^{\Delta t}((\ell - 1) \Delta t) = u_L^{\Delta t, -}(\ell \Delta t)$, $\ell = 1, \ldots, N$, and $u_L^{\Delta t, \pm}$ are piecewise constant functions on the partition $[0 = t_0, t_1, \ldots, t_{N - 1}, t_N = T]$, we thus deduce that

$$
\Delta t \sum_{t'=\Delta t}^{T-\delta} \left\| u_L^{\Delta t, \pm}(t' + \delta) - u_L^{\Delta t, \pm}(t') \right\|^2 \leq C \delta^{1 - \frac{1}{\varphi(t)}} \quad \text{with} \quad \begin{cases} 
q \in (4, \infty) & \text{if } d = 2, \\
q \in (4, 6] & \text{if } d = 3,
\end{cases}
$$

where $C$ is a positive constant, independent of $\Delta t$, $L$, and $\delta$, with $\delta = \ell \Delta t$, $\ell = 1, \ldots, N - 1$. By selecting $q$ as large as possible and taking the square root of the previous inequality, we have that

$$
\left\| u_L^{\Delta t, \pm}(t + \delta) - u_L^{\Delta t, \pm}(t) \right\|_{L^2(O, T - \delta; L^2(\Omega))} \leq C \delta^{\frac{\gamma}{2}}
$$

(4.43)

for all $\delta = \ell \Delta t$, $\ell = 1, \ldots, N - 1$, where $C$ is a positive constant independent of step size $\Delta t$, $L$ and $\delta$; $0 < \gamma < 1/2$ when $d = 2$ and $0 < \gamma < 3/8$ when $d = 3$.

We shall now extend the validity of (4.43) to values of $\delta \in (0, T]$ that are not necessarily integer multiples of $\Delta t$. We shall therefore at this point alter our original notational convention for $\delta$, and will consider $\delta = v \Delta t$, with $v \in (0, N]$. Let us define to this end $\ell = [v] := \max\{k \in \mathbb{N}; k < v\}$, $\delta := v - [v] \in (0, 1]$, and for $t \in (0, T]$ let $m \in \{0, \ldots, N - \ell - 2\}$ be such that $t \in (m \Delta t, (m + 1) \Delta t]$. Hence,

$$
u_L^{\Delta t, \pm}(t + v \Delta t) = \begin{cases} 
u_L^{\Delta t, \pm}(t + \ell \Delta t) & \text{if } t \in (m \Delta t, m \Delta t + (1 - \vartheta) \Delta t], \\
u_L^{\Delta t, \pm}(t + (\ell + 1) \Delta t) & \text{if } t \in (m \Delta t + (1 - \vartheta) \Delta t, (m + 1) \Delta t],
\end{cases}
$$

(4.44)

which then implies on noting that $s \in \mathbb{R}_{\geq 0} \mapsto s^\gamma \in \mathbb{R}_{\geq 0}$ is a concave function, that

$$
\left\| u_L^{\Delta t, \pm}(t + \delta) - u_L^{\Delta t, \pm}(t) \right\|^2_{L^2(O, T - \delta; L^2(\Omega))} = \int_0^{T - \vartheta \Delta t} \left\| u_L^{\Delta t, \pm}(t + v \Delta t) - u_L^{\Delta t, \pm}(t) \right\|^2 \mathrm{d} t \\
\leq (1 - \vartheta) \Delta t \sum_{t'=\Delta t}^{T-\ell \Delta t} \left\| u_L^{\Delta t, \pm}(t' + \ell \Delta t) - u_L^{\Delta t, \pm}(t') \right\|^2 \\
+ \vartheta \Delta t \sum_{t'=\Delta t}^{T-(\ell+1)\Delta t} \left\| u_L^{\Delta t, \pm}(t' + (\ell + 1) \Delta t) - u_L^{\Delta t, \pm}(t') \right\|^2 \\
\leq (1 - \vartheta) C (\ell \Delta t)^\gamma + \vartheta C ((\ell + 1) \Delta t)^\gamma \\
\leq C [(1 - \vartheta) \ell \Delta t + \vartheta (\ell + 1) \Delta t]^\gamma = C (v \Delta t)^\gamma = C \delta^\gamma,
$$

where $\delta = v \Delta t$, $v \in (0, N]$; $0 < \gamma < 1/2$ when $d = 2$ and $0 < \gamma < 3/8$ when $d = 3$; and $C$ is a positive constant, independent of $\Delta t$, $L$ and $\delta$. The second inequality in the chain of inequalities above follows by applying (4.42) first with $\delta = \ell \Delta t$ and then with $\delta = (\ell + 1) \Delta t$. 


Consequently,
\[
\| u_L^{\Delta t, \pm} (\cdot + \delta) - u_L^{\Delta t, \pm} (\cdot) \|_{L^2(0, T - \delta; L^2(\Omega))} \leq C \delta^\gamma
\]
for all \( \delta \in (0, T) \), where \( C \) is a positive constant independent of \( \Delta t \) and \( L \); \( 0 < \gamma < 1/2 \) when \( d = 2 \) and \( 0 < \gamma \leq 3/8 \) when \( d = 3 \).

Thus we have established the following Nikol'skiĭ norm estimate:
\[
\| u_L^{\Delta t, \pm} \|_{N^\gamma; 2(0, T; L^2(\Omega))} := \sup_{0 < \delta < T} \delta^{-\gamma} \| u_L^{\Delta t, \pm} (\cdot + \delta) - u_L^{\Delta t, \pm} (\cdot) \|_{L^2(0, T - \delta; L^2(\Omega))} \leq C,
\]
where \( C \) is a positive constant, independent of \( L \) and \( \Delta t \); \( 0 < \gamma < 1/2 \) when \( d = 2 \) and \( 0 < \gamma \leq 3/8 \) when \( d = 3 \).

**Remark 4.1.** We note in passing that in the special case of an incompressible Newtonian fluid with variable density, when the extra stress tensor appearing on the right-hand side of (1.1c) is identically zero, (4.45) continues to hold and improves the Nikol’skiĭ index \( \gamma = 1/4 \) obtained in the work of Simon [46, p. 1100, Proposition 8(ii)] and [46, p. 1103, Theorem 9(ii)] (under the hypothesis \( f \in L^1(0, T; L^2(\Omega)) \) compared with \( f \in L^2(0, T; L^k(\Omega)) \) assumed here with \( k > 1 \) when \( d = 2 \) and \( k > \frac{6}{5} \) when \( d = 3 \), and under the same assumptions on the initial data \( u_0 \) and \( \rho_0 \) as in (3.3) here).

4.3. *Strong convergence of the sequences \( \{\rho_L^{\Delta t, \pm}\}_{L>1}, \{u_L^{\Delta t, \pm}\}_{L>1} \) and weak convergence of \( \{\psi_L^{\Delta t, \pm}\}_{L>1} \)*

We begin by collecting a number of relevant bounds on the sequences \( \{\rho_L^{\Delta t, \pm}\}_{L>1}, \{u_L^{\Delta t, \pm}\}_{L>1} \), and \( \{\psi_L^{\Delta t, \pm}\}_{L>1} \).

First, we recall that \( \rho_L^{\Delta t}(t) \in \mathcal{Y} \) for all \( t \in [0, T] \) and from (4.10a) that
\[
\| \rho_L^{[\Delta t]}(t) \|_{L^p(\Omega), t \in (0, T]} \leq \rho_0 \|_{L^p(\Omega)}, \quad t \in (0, T],
\]
and therefore, for each \( p \in [1, \infty] \),
\[
\| \rho_L^{\Delta t, \pm}(t) \|_{L^p(\Omega)} \leq \rho_0 \|_{L^p(\Omega)}, \quad t \in (0, T].
\]

Next, noting (4.1a), (4.1b), a simple calculation yields that [see (6.32)–(6.34) in [8] for details]:
\[
\int_0^T \int_{\Omega \times D} M \left| \nabla_x \sqrt{\nu L} \Delta t \right|^2 \bar{d} x \, d t \leq 2 \int_0^T \int_{\Omega \times D} M \left[ \left| \nabla_x \sqrt{\nu L}^{\Delta t, +} \right|^2 + \left| \nabla_x \sqrt{\nu L}^{\Delta t, -} \right|^2 \right] \bar{d} x \, d t.
\]
and an analogous result with \( \nabla_x \) replaced by \( \nabla_q \). Then the bound (4.26b), on noting (3.3), (4.45), (3.37), (4.1a), (4.1b), (3.14), (3.17a), (4.47) and the convexity of \( \mathcal{F} \), imply the existence of a constant \( C > 0 \), depending only on \( B(u_0, f, \nu L) \) and on \( \varepsilon, \rho_{\text{min}}, \rho_{\text{max}}, \mu_{\text{min}}, \xi_{\text{min}}, \xi_{\text{max}}, T, |A|, a_0, c_0, C_{\xi}, \kappa, \lambda, K \) and \( b \), but not on \( L \) or \( \Delta t \), such that:
\[
\text{ess.sup}_{t \in [0, T]} \left\| u_L^{\Delta t, \pm}(t) \right\|^2 + \| u_L^{\Delta t, \pm} \|^2_{N^\gamma; 2(0, T; L^2(\Omega))} + \frac{1}{\Delta t} \int_0^T \left\| u_L^{\Delta t, +} - u_L^{\Delta t, -} \right\|^2 \, d t + \int_0^T \| \nabla_x u_L^{\Delta t, \pm} \|^2 \, d t
\]
where \( 0 < \gamma < 1/2 \) when \( d = 2 \) and \( 0 < \gamma \leq 3/8 \) when \( d = 3 \).

Henceforth, we shall assume that

\[
\Delta t = o(L^{-1}) \quad \text{as} \quad L \to \infty.
\]

Requiring, for example, that \( 0 < \Delta t \leq C_0/(L \log L) \), \( L > 1 \), with an arbitrary (but fixed) constant \( C_0 \) will suffice to ensure that (4.49) holds. The sequences \( \{\rho_{L}^{(\Delta t)}\}_{L \to 1}, \{\rho_{L}^{(\Delta t,\pm)}\}_{L \to 1}, \{\rho_{L}^{(\Delta t)}\}_{L > 1} \) and \( \{\tilde{\psi}_{L}^{(\Delta t,\pm)}\}_{L > 1} \) as well as all sequences of spatial and temporal derivatives of the entries of these sequences will thus be, indirectly, indexed by \( L \) alone, although for reasons of consistency with our previous notation we shall not introduce new, compressed, notation with \( \Delta t \) omitted from the superscripts. Instead, whenever \( L \to \infty \) in the rest of this section, it will be understood that \( \Delta t \) tends to 0 according to (4.49). We are now ready to embark on the passage to limit with \( L \to \infty \).

**Theorem 4.1.** Suppose that the assumptions (3.3) and the condition (4.49), relating \( \Delta t \) to \( L \), hold. Then, there exists a subsequence of \( \{(\rho_{L}^{(\Delta t)}, u_{L}^{(\Delta t)}, \tilde{\psi}_{L}^{(\Delta t)})\}_{L \to 1} \) (not indicated) with \( \Delta t = o(L^{-1}) \), and functions \( (\rho, u, \psi) \), with \( \tilde{\psi} \geq 0 \) a.e. on \( \Omega \times D \times [0, T] \), such that

\[
\rho \in L^{\infty}(0, T; \mathcal{Y}) \cap \mathcal{C}([0, T]; L^{p}(\Omega)), \quad u \in L^{\infty}(0, T; L^{2}(\Omega)) \cap L^{2}(0, T; V),
\]

where \( p \in [1, \infty) \), and

\[
\sqrt{\psi} \in L^{2}(0, T; H^{1}_{M}(\Omega \times D)).
\]

such that, as \( L \to \infty \) (and thereby \( \Delta t \to 0_{+} \)),

\[
\rho_{L}^{(\Delta t)} \to \rho \quad \text{weak* in} \quad L^{\infty}(0, T; L^{\infty}(\Omega)),
\]

\[
\rho_{L}^{(\Delta t)} \to \rho \quad \text{strongly in} \quad L^{\infty}(0, T; L^{p}(\Omega)),
\]

\[
\rho_{L}^{(\Delta t,\pm)}, \rho_{L}^{(\Delta t)} \to \rho \quad \text{strongly in} \quad L^{\infty}(0, T; L^{p}(\Omega)),
\]

\[
\mu(\rho_{L}^{(\Delta t,\pm)}), \mu(\rho_{L}^{(\Delta t)}) \to \mu(\rho) \quad \text{strongly in} \quad L^{\infty}(0, T; L^{p}(\Omega)),
\]

\[
\xi(\rho_{L}^{(\Delta t)}), \xi(\rho_{L}^{(\Delta t,\pm)}), \xi_{L}^{(\Delta t)} \to \xi(\rho) \quad \text{strongly in} \quad L^{\infty}(0, T; L^{p}(\Omega)),
\]

where \( p \in [1, \infty) \):

\[
u_{L}^{(\Delta t,\pm)} \to u \quad \text{weak* in} \quad L^{\infty}(0, T; L^{2}(\Omega)),
\]

\[
u_{L}^{(\Delta t,\pm)} \to u \quad \text{weakly in} \quad L^{2}(0, T; V),
\]

\[
u_{L}^{(\Delta t,\pm)} \to u \quad \text{strongly in} \quad L^{2}(0, T; L^{2}(\Omega)),
\]
where \( r \in [1, \infty) \) if \( d = 2 \) and \( r \in [1, 6) \) if \( d = 3 \); and

\[
\Psi_L^{\Delta t, (+)} \rightarrow \Psi_L \quad \text{weakly in } L^1(0, T; L^1_\mathcal{M}(\Omega \times D)),
\]

\[
M^{1/2} \nabla_x \Psi_L^{\Delta t, (+)} \rightarrow M^{1/2} \nabla_x \Psi \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)).
\]

\[
M^{1/2} \nabla_q \Psi_L^{\Delta t, (+)} \rightarrow M^{1/2} \nabla_q \Psi \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)).
\]

**Proof.** The weak convergence results (4.51a), (4.51b) follow directly from the first and fourth bounds in (4.48). We deduce the strong convergence result (4.51c) in the case of \( u_L^{\Delta t, +} \) on noting the second and fourth bounds in (4.48), (3.10), and the compact embedding of \( \Psi \) into \( L^r(\Omega) \cap H \), with the values of \( r \) as in the statement of the theorem. In particular, with \( r = 2 \), a subsequence of \( \{ u_L^{\Delta t, +} \}_{L > 1} \) converges to \( u \), strongly in \( L^2(0, T; L^2(\Omega)) \) as \( L \rightarrow \infty \), with \( \Delta t = o(L^{-1}) \). Then, by the third bound in (4.48), we deduce that the three corresponding subsequences \( \{ u_L^{\Delta t, (\pm)} \}_{L > 1} \) converge to \( u \), strongly in \( L^2(0, T; L^2(\Omega)) \) as \( L \rightarrow \infty \) (and thereby \( \Delta t \rightarrow 0 \)). Since these subsequences are bounded in \( L^2(0, T; H^1(\Omega)) \) (cf. the bound on the fourth term in (4.48)) and strongly convergent in \( L^2(0, T; L^2(\Omega)) \), it follows from (3.2) that (4.51c) holds, with the values of \( r \) as in the statement of the theorem. Thus we have proved (4.51a)–(4.51c).

The convergence result (4.50a) and the fact that \( \rho \in L^\infty(0, T; \mathcal{V}) \) follow immediately from (4.46a) and as \( \rho_L^{\Delta t}(t) \in \mathcal{V} \) for all \( t \in [0, T] \). Further (3.23a) implies that

\[
- \int_{t_{n-1}}^{t_n} \int_\Omega \rho_L^{[\Delta t]} \frac{\partial \eta}{\partial t} \, dx \, dt + \int_{t_{n-1}}^{t_n} \int_\Omega \rho_L^{[\Delta t]} u_L^{\Delta t, -} \cdot \nabla_x \eta \, dx \, dt = \int_{t_{n-1}}^{t_n} \int_\Omega \rho_L^{[\Delta t]} \eta \, dx,
\]

\( \forall \eta \in C^1([t_{n-1}, t_n]; W^{1, q/(q-2)}(\Omega)) \),

with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). Upon summation through \( n = 1, \ldots, N \), and noting that \( \rho_L^n = \rho_0 \), we then deduce that

\[
- \int_0^T \int_\Omega \rho_L^{[\Delta t]} \frac{\partial \eta}{\partial t} \, dx \, dt + \int_0^T \int_\Omega \rho_L^{[\Delta t]} u_L^{\Delta t, -} \cdot \nabla_x \eta \, dx \, dt = \int_\Omega \rho_0 \eta \, dx
\]

\( \forall \eta \in C^1([0, T]; W^{1, q/(q-2)}(\Omega)) \) s.t. \( \eta(\cdot, T) = 0 \),

with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). Hence, on letting \( L \rightarrow \infty \), with \( \Delta t = o(L^{-1}) \), and noting (4.50a) and (4.51c), we deduce that

\[
- \int_0^T \int_\Omega \rho \frac{\partial \eta}{\partial t} \, dx \, dt + \int_0^T \int_\Omega \rho u \cdot \nabla_x \eta \, dx \, dt = \int_\Omega \rho_0 \eta \, dx
\]

\( \forall \eta \in C^1([0, T]; W^{1, q/(q-2)}(\Omega)) \) s.t. \( \eta(\cdot, T) = 0 \),

(4.53)

with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). Thus we have shown that \( \rho \) is a weak solution to (1.1a), (1.1b). One can now apply the theory of DiPerna & Lions [18] to (1.1a), (1.1b). As \( u \in L^2(0, T; \mathcal{V}) \), it follows from Corollaries II.1 and II.2, and p. 546, in [18], that there exists a unique solution to (1.1a), (1.1b) for this given \( u \), which must therefore coincide with \( \rho \). In addition, \( \rho \in C([0, T], L^p(\Omega)) \) for \( p \in [1, \infty) \), and the following equality holds:
\[
\|\rho(t)\|_{L^p(\Omega)} = \|\rho_0\|_{L^p(\Omega)}, \quad t \in (0, T].
\] (4.54)

Thanks to (4.46a) and (4.50a), by the weak* lower semicontinuity of the norm function, and (4.54) with \( p = 2 \), we have that, for a.e. \( t \in [0, T] \),
\[
\|\rho(t)\|^2 \leq \liminf_{t \to \infty} \|\rho^{[\Delta t]}_L(t)\|^2 \leq \|\rho_0\|^2 = \|\rho(t)\|^2.
\]

This then implies, for a.e. \( t \in [0, T] \), that
\[
\|\rho(t)\|^2 = \lim_{t \to \infty} \|\rho^{[\Delta t]}_L(t)\|^2 = \|\rho_0\|^2.
\] (4.55)

Thus we have proved (4.50b) in the case of \( p = 2 \), which, on extracting a further subsequence, implies that
\[
\lim_{t \to \infty} \rho^{[\Delta t]}_L = \rho \quad \text{a.e. on } \Omega \times (0, T).
\] (4.56)

By recalling (3.44), we then deduce (4.50b) for all \( p \in [1, \infty) \) by Lebesgue’s dominated convergence theorem.

It follows from (3.45a), with \( \eta = \chi_{[t_{n-1}, t_n]}/\varphi, \quad t \in [t_{n-1}, t_n] \), and \( \eta = \chi_{[t_n, t]} \varphi, \quad t \in [t_{n-1}, t_n] \), and \( \varphi \in W^{1, \frac{q}{q-1}}(\Omega) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), that
\[
\|\rho^{[\Delta t]}_L - \rho^{[\Delta t, \pm]}_L\|_{L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega))} \leq C \max_{n=1, \ldots, N} \int_{t_{n-1}}^{t_n} \|\nabla \rho^{[\Delta t]}_L\|_{W^{1, \frac{q}{q-1}}(\Omega)} \, dt \leq C (\Delta t)^{\frac{1}{2}},
\] (4.57)

where we have noted (4.46a), (3.2) and (4.48). Similarly, on recalling (4.2), we obtain
\[
\|\rho^{[\Delta t]}_L - \rho^{[\Delta t]}_L\|_{L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega))} + \|\rho^{[\Delta t]}_L - \rho^{[\Delta t]}_L\|_{L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega))} \leq C (\Delta t)^{\frac{1}{2}},
\] (4.58)

and, on noting (3.28),
\[
\|\chi (\rho^{[\Delta t]}_L) - \chi (\rho^{[\Delta t]}_L)\|_{L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega))} \leq C (\Delta t)^{\frac{1}{2}}.
\] (4.59)

Applying (4.50b) with \( p = q \), we have that \( \rho^{[\Delta t]}_L \) converges to \( \rho \) strongly in \( L^\infty(0, T; L^q(\Omega)) = L^\infty(0, T; \frac{q}{q-1}(\Omega)) \subset L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega)) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in (3, 6] \) when \( d = 3 \); thus, \( \rho^{[\Delta t]}_L \) converges strongly to \( \rho \) in \( L^\infty(0, T; L^q(\Omega)) = L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega)) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). By means of a triangle inequality in the norm of \( L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega)) \) and noting (4.57), we thus deduce that \( \rho^{[\Delta t, \pm]}_L \) converges to \( \rho \) strongly in \( L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega)) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \); analogously, using (4.58) this time, \( \rho^{[\Delta t]}_L \) and \( \rho^{[\Delta t]}_L \) converge to \( \rho \) strongly in \( L^\infty(0, T; W^{1, \frac{q}{q-1}}(\Omega)) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \). Since, thanks to (4.46b), \( \{\rho^{[\Delta t, \pm]}_L\}_{L^{-1}} \), \( \{\rho^{[\Delta t]}_L\}_{L^{-1}} \) and \( \{\rho^{[\Delta t]}_L\}_{L^{-1}} \) are weak* compact in \( L^\infty(0, T; L^\infty(\Omega)) \subset L^{1, \infty}(0, T; W^{1, \frac{q}{q-1}}(\Omega)) \), where \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \), it follows that the weak* limits of the weak* convergent subsequences extracted from \( \{\rho^{[\Delta t, \pm]}_L\}_{L^{-1}} \), \( \{\rho^{[\Delta t]}_L\}_{L^{-1}} \) and \( \{\rho^{[\Delta t]}_L\}_{L^{-1}} \) have the same limit as \( \rho^{[\Delta t]}_L \): the element \( \rho \in L^\infty(0, T; L^\infty(\Omega)) \). The weak* lower semicontinuity of the norm function and inequality (4.46b) together imply that, for a.e. \( t \in [0, T] \),
Hence, proceeding as above in the case of $\rho^{[\Delta t]}$, we deduce (4.50c).

Concerning (4.50d) and (4.50e), these follow from (4.50c) and (4.59) via Lebesgue’s dominated convergence theorem by possibly extracting a further subsequence, thanks to our assumptions in (3.3) on $\mu$ and $\zeta$.

We complete the proof by establishing (4.52a)–(4.52c). According to (4.48) and (3.3),

$$2k\xi_{\min} \int_{\Omega \times D} M \mathcal{F}(\tilde{\psi}_{L}^{\Delta t(\pm)}(t)) \, dq \, dx \leq \left[ B(u_0, f, \psi_0) \right]^2$$

for all $t \in [0, T]$; hence, on noting that $s \log(s + 1) < 2|\mathcal{F}(s) + 1|$ for all $s \in \mathbb{R}_{>0}$, we have that

$$\max_{t \in [0, T]} \int_{\Omega \times D} M \tilde{\psi}_{L}^{\Delta t(\pm)}(t) \log(\tilde{\psi}_{L}^{\Delta t(\pm)}(t) + 1) \, dq \, dx \leq \frac{1}{k\xi_{\min}} \left[ B(u_0, f, \psi_0) \right]^2 + 2|\Omega|. \quad (4.60)$$

As $s \in \mathbb{R}_{>0} \mapsto s \log(s + 1) \in \mathbb{R}_{>0}$ is nonnegative, strictly monotonic increasing and convex, it follows from de la Vallée-Poussin’s theorem that the sequence of nonnegative functions $\{\tilde{\psi}_{L}^{\Delta t(\pm)}\}_{L>1}$, with $\Delta t = o(L^{-1})$ is uniformly integrable on $\Omega \times D \times (0, T)$ with respect to the measure $dv := M(q) \, dq \, dx \, dt$. Hence, by the Dunford–Pettis theorem, $\{\tilde{\psi}_{L}^{\Delta t(\pm)}\}_{L>1}$, with $\Delta t = o(L^{-1})$, is weakly relatively compact in $L^1(\Omega \times D \times (0, T); \nu) = L^1(0, T; L^1_{M}(\Omega \times D))$; i.e., there exist a nonnegative function $\tilde{\psi} \in L^1(0, T; L^1_{M}(\Omega \times D))$ and a subsequence (not indicated) such that

$$\tilde{\psi}_{L}^{\Delta t(\pm)} \rightarrow \tilde{\psi} \quad \text{weakly in } L^1(0, T; L^1_{M}(\Omega \times D)), \quad (4.61)$$

as $L \rightarrow \infty$, where $\Delta t = o(L^{-1})$. The fact that the limits of the subsequences of $\{\tilde{\psi}_{L}^{\Delta t(\pm)}\}_{L>1}$ are the same follows from the sixth bound in (4.48). Thus we have shown that (4.52a) holds, and that the limiting function $\tilde{\psi}$ is nonnegative.

The extraction of the convergence results (4.52b), (4.52c) from (4.48) can be found in Step 2 in the proof of Theorem 6.1 in [9]. \hfill $\square$

In the next section we shall strengthen (4.52a) by showing that (a subsequence of) the sequence $\{\tilde{\psi}_{L}^{\Delta t(\pm)}\}_{L>1}$ is strongly convergent to $\tilde{\psi}$ in $L^1(0, T; L^1_{M}(\Omega \times D))$ as $L \rightarrow \infty$, with $\Delta t = o(L^{-1})$.

4.4. Strong convergence of $\tilde{\psi}_{L}^{\Delta t(\pm)}$ in $L^1(0, T; L^1_{M}(\Omega \times D))$

In Section 4.2 we derived an $L$-independent bound on the Nikolskii norm, based on time-shifts, of the sequence $\{u_{L}^{\Delta t(\pm)}\}_{L>1}$ of approximate velocities. The Nikolskii norm bound (4.45) was used in the previous section in conjunction with the bounds on spatial derivatives of $\{u_{L}^{\Delta t(\pm)}\}_{L>1}$ established in Section 4.1 to deduce, via Simon’s extension of the Aubin–Lions theorem [45], strong convergence of $\{u_{L}^{\Delta t(\pm)}\}_{L>1}$ in $L^2(0, T; L^1(\Omega))$ as $L \rightarrow \infty$, with $\Delta t = o(L^{-1})$, for $1 \leq r < \infty$ when $d = 2$ and $1 \leq r < 6$ when $d = 2$, which we shall then use to pass to the limit in nonlinear terms in (4.3b) in conjunction with weak convergence results for the sequence, which suffice for passage to the limit in those terms in (4.3b) that depend linearly on $\{u_{L}^{\Delta t(\pm)}\}_{L>1}$.

In [9] we used a similar argument for the sequence of approximations to the solution of the Fokker–Planck equation, except that due to the form of the Kullback–Leibler relative entropy and the associated Fisher information in the bounds on spatial norms of the sequence resulting from our entropy-based testing (which, in turn, was motivated by the natural energy balance between the
Navier–Stokes and Fokker–Planck equations in the coupled system, that manifests itself in a fortu-
nitous cancellation of the extra-stress tensor in the Navier–Stokes equation with the drag term in the
Fokker–Planck equation in the course of the entropy-testing), we had to appeal to Dubinskii’s exten-
sion to seminormed cones in Banach spaces of the original Aubin–Lions theorem to deduce strong
convergence of the approximating sequence of probability density functions.

Unfortunately, in the present setting, the appearance of the nonlinear drag \( \zeta(\rho) \) in the Fokker–
Planck equation obstructs the application of Dubinskii’s compactness theorem, and the approach
based on Nikol’skii norm estimates, that was used in Section 4.2 in the density-dependent Navier–
Stokes equation, also fails, because — in order to compensate for the rather weak spatial control in
(4.26b) of the Kullback–Leibler relative entropy and the Fisher information — its application ultimately
requires a uniform \( L^\infty(0, T; L^\infty(\Omega \times D)) \) bound on the sequence of approximations to the probability
density function, which is not available. We shall therefore adopt a different approach here. Since
the argument below that finally delivers the desired compactness of the sequence \( \{\tilde{\psi}_L^{\Delta t(\pm)}\}_{L>1} \) (with
\( \Delta t = o(L^{-1}) \) as \( L \to \infty \)) in \( L^1(0, T; L^1_M(\Omega \times D)) \) is long and rather technical, we begin with a brief
overview of the key steps.

First, by (4.51a) (a subsequence of) the sequence \( \{\tilde{\psi}_L^{\Delta t(\pm)}\}_{L>1} \) is weakly convergent in the space
\( L^1(0, T; L^1_M(\Omega \times D)) \) to \( \tilde{\psi} \in L^1(0, T; L^1_M(\Omega \times D)) \) as \( L \to \infty \), with \( \Delta t = o(L^{-1}) \). We shall then make
use of the property that if \( \Phi \) is a strictly convex weakly lower-semicontinuous function defined on a
convex open set \( \mathbb{R} \), and the weak limit of \( \Phi(\tilde{\psi}_L^{\Delta t(\pm)}) \) is equal to \( \Phi(\tilde{\psi}) \), then the sequence
\( \{\tilde{\psi}_L^{\Delta t(\pm)}\}_{L>1} \) converges almost everywhere on \((0, T) \times \Omega \times D \) as \( L \to \infty \), with \( \Delta t = o(L^{-1}) \) (cf.
Theorem 10.20 on p. 339 of [21]). According to (4.48),

\[
\max_{t \in [0, T]} \int_{\Omega \times D} M_F(\tilde{\psi}_L^{\Delta t(\pm)}) \, dq \, dx
\]

is bounded, uniformly in \( L \) and \( \Delta t \); in addition \( F \) is strictly convex. Thus \( F \) may appear as a logi-
cal first candidate for the choice of the function \( \Phi \). Unfortunately, we do not know at this point if
the weak limit of the sequence \( \{F(\tilde{\psi}_L^{\Delta t(\pm)})\}_{L>1} \) in \( L^1(0, T; L^1_M(\Omega \times D)) \) is equal to \( F(\tilde{\psi}) \), and there-
fore the argument outlined at the beginning of this paragraph is not directly applicable with the
choice \( \Phi(s) = F(s) \). We shall therefore make a different choice: we select the strictly convex function
\( \Phi(s) = (1 + s)^{1+\alpha} \), \( s \geq 0 \), where \( \alpha \in (0, 1) \) is a suitable (small) positive real number. We note
in passing, as this will be important in the argument that will follow, that \( s \mapsto (1 + s)^\alpha \), \( s \geq 0 \), is a
strictly concave function on \( \mathbb{R}_{\geq 0} \) for \( \alpha \in (0, 1) \). Although we do not know at this point if, with
the latter choice of \( \Phi \), the weak limit of the sequence \( \{\Phi(\tilde{\psi}_L^{\Delta t(\pm)})\}_{L>1} \) in \( L^1(0, T; L^1_M(\Omega \times D)) \) is
equal to \( \Phi(\tilde{\psi}) \), and therefore this particular \( \Phi \) may seem no better than the original suggestion of
\( \Phi(s) = F(s) \), we note that by using estimates interior to \( \Omega \times D \) on subdomains \( \Omega_0 \times D_0 \subseteq \Omega \times D \) on
which the Maxwellian weight is bounded above and below by positive constants, and therefore the
function in bounded Maxwellian-weighted norms that result from (4.26b) become bounds in standard,
unweighted, Lebesgue and Sobolev norms, one can use function space interpolation between these
unweighted norms to deduce a uniform bound on the \( L^1(0, T; L^1_M(\Omega \times D)) \) norm of \( \tilde{\psi}_L^{\Delta t(\pm)} \),
for a suitable (small) value of \( \delta \), which, with \( \alpha \in (0, \delta) \) and an application of the Div–Curl lemma,
then implies that the weak limit in \( L^1(0, T; L^1(\Omega \times D)) \) of \( \Phi(\tilde{\psi}_L^{\Delta t(\pm)}) \) is equal to \( \Phi(\tilde{\psi}) \) as \( L \to \infty \),
with \( \Delta t = o(L^{-1}) \). Hence, by the argument, outlined in the beginning of this paragraph, we deduce
almost everywhere convergence of a subsequence on \((0, T) \times \Omega_0 \times D_0 \), and finally, using an increasing
sequence of nested Lipschitz subdomains \((0, T) \times \Omega_k \times D_k \), \( k = 1, 2, \ldots \), and extracting a diagonal sequence from
\( \tilde{\psi}_L^{\Delta t(\pm)} \), we arrive at a subsequence of \( \{\tilde{\psi}_L^{\Delta t(\pm)}\}_{L>1} \) that converges almost everywhere
on \((0, T) \times \Omega \times D \) to \( \tilde{\psi} \) as \( L \to \infty \), with \( \Delta t = o(L^{-1}) \). Since the set \( \mathcal{D} := (0, T) \times \Omega \times D \) has finite mea-
sure, according to Egoroff’s theorem (cf. Theorem 2.22 on p. 149 of [22]) almost everywhere conver-
gence implies almost uniform convergence, and in particular convergence in measure. Thus, by Vitali’s
convergence theorem (cf. Theorem 2.24 on p. 150 of [22]), and thanks to the uniform integrability of the
sequence \( \{\tilde{\psi}_L^{\Delta t(\pm)}\}_{L>1} \) in \( L^1(0, T; L^1_M(\Omega \times D)) \), we finally deduce the desired strong convergence
of the sequence \( \{ \tilde{\psi}_L^{\Delta t, (\pm)} \}_{L>1} \) in \( L^1(0, T; L^1_m(\Omega \times D)) \) as \( L \to \infty \), with \( \Delta t = o(L^{-1}) \). We will further strengthen this by using Lemma 4.2 below to strong convergence in the \( L^p(0, T; L^1_m(\Omega \times D)) \) norm, for any \( p \in [1, \infty) \).

We now embark on the programme outlined above by observing that, since on each compact subset of \( D \) the Maxwellian \( M \) is bounded above and below by positive constants (depending on the choice of the compact subset), it follows from (4.52a) that \( \{ \tilde{\psi}_L^{\Delta t, (\pm)} \}_{L>1} \), with \( \Delta t = o(L^{-1}) \), is weakly relatively compact in \( L^1_{\text{loc}}(\Omega \times D \times (0, T)) \). Hence, by uniqueness of the weak limit,

\[
\tilde{\psi}_L^{\Delta t, (\pm)} \to \tilde{\psi} \quad \text{weakly in } L^1_{\text{loc}}(0, T; L^1(\Omega \times D)).
\]  

We shall show that in fact

\[
\tilde{\psi}_L^{\Delta t, (\pm)} \to \tilde{\psi} \quad \text{a.e. on } (0, T) \times \Omega \times D.
\]  

It will be relevant in the argument below that the Cartesian product of two bounded open Lipschitz domains in \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively, is a bounded open Lipschitz domain in \( \mathbb{R}^{m+n} \), see, the footnote on page 56 of the extended version of this paper [10].

Let \( \Omega := \Omega \times D \), and suppose that \( \Omega_0 \) is a Lipschitz subdomain of \( \Omega \) such that \( \Omega_0 \Subset \Omega \). As \( s \log(s+1) + 1 > s \) for all \( s \in \mathbb{R}_{>0} \) we have from (4.60), the bounds on the seventh and the eighth term on the left-hand side of (4.48), and noting once again that \( M \) is bounded below on \( \Omega_0 \) by a positive constant (which may depend on \( \Omega_0 \)), that

\[
\max_{t \in [0,T]} \| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \|_{L^2(\Omega_0)}^2 + \int_0^T \| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \|_{H^1(\Omega_0)}^2 \, dt \leq C(\Omega_0),
\]  

where \( C(\Omega_0) \) is a positive constant, which may depend on \( \Omega_0 \) but is independent of \( L \) and \( \Delta t \). It then follows from the bound on the second term on the left-hand side of (4.64) by Sobolev's embedding theorem applied on the bounded Lipschitz domain \( \Omega_0 \Subset \Omega \subset \mathbb{R}^{(K+1)d} \) that

\[
\int_0^T \| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \|_{L^{(K+1)d}(\Omega_0)}^{2(K+1)d} \, dt \leq C(\Omega_0).
\]  

Interpolation between the first inequality in (4.64) and the inequality (4.65) then yields that

\[
\int_0^T \| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \|_{L^{2(K+1)d}(\Omega_0)}^{2(K+1)d} \, dt \leq C(\Omega_0).
\]  

By writing \( \tilde{\psi}_L^{\Delta t, (\pm)}(t) = \left[ \tilde{\psi}_L^{\Delta t, (\pm)}(t) \right]^2 \) and applying Hölder's inequality we then deduce for any \( p \in [1, 2) \) that

\[
\int_0^T \| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \|_{W^{1, p}(\Omega_0)}^p \, dt \leq 2^p \left( \int_0^T \| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \|_{H^1(\Omega_0)}^2 \, dt \right)^{p/2} \left( \int_0^T \| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \|_{L^{2p}(\Omega_0)}^{2p} \, dt \right)^{2-p/p} \leq C(\Omega_0),
\]  

(4.67)
provided that
\[
\frac{2p}{2 - p} \leq \frac{(K + 1)d + 2}{(K + 1)d}.
\] (4.68)

Let \( p_0 \) be the largest number in the range \([1, 2)\) that satisfies (4.68); we thus deduce from (4.67) that
\[
\int_0^T \left\| \tilde{\psi}_L^{\Delta t, (\pm)}(t) \right\|_{W^{1,p}(\mathcal{O}_0)}^p \, dt \leq C(\mathcal{O}_0), \quad \text{with } p_0 := \frac{(K + 1)d + 2}{(K + 1)d + 1} \in (1, 2). \] (4.69)

Thanks to (3.63) we also have that
\[
\left\| \tilde{\psi}_L^{\Delta t, (\pm)} \right\|_{L^\infty((0,T) \times \Omega_0; L^1(D_0))} \leq C,
\]
and therefore,
\[
\left\| \tilde{\psi}_L^{\Delta t, (\pm)} \right\|_{L^\infty((0,T) \times \Omega_0; L^1(D_0))} \leq C
\] (4.70)
for any two Lipschitz subdomains \( \Omega_0 \subsetneq \Omega \) and \( D_0 \subsetneq D \); here \( C \) is a positive constant, independent of \( L \) and \( \Delta t \). Fixing \( \mathcal{O}_0 = \Omega_0 \times D_0 \) and interpolating between (4.70) and (4.66), which states that
\[
\left\| \tilde{\psi}_L^{\Delta t, (\pm)} \right\|_{L^\infty((0,T) \times \Omega_0; L^1(D_0))} \leq C(\mathcal{O}_0),
\]
we deduce that for any two real numbers \( q_1 \) and \( q_2 \), with
\[
1 + \frac{2}{(K + 1)d} \leq q_1 < \infty \quad \text{and} \quad 1 < q_2 \leq 1 + \frac{2}{(K + 1)d}
\] (4.71)
and satisfying the relation
\[
q_1 \left( 1 - \frac{1}{q_2} \right) = \frac{2}{(K + 1)d},
\] (4.72)
we have that
\[
\left\| \tilde{\psi}_L^{\Delta t, (\pm)} \right\|_{L^{q_1}((0,T) \times \Omega_0; L^{q_2}(D_0))} \leq C(\mathcal{O}_0). \] (4.73)

Note further that since \( \rho_L^{\Delta t, +} \geq \rho_{\min} \) a.e. on \( \Omega \times [0, T] \) and \( \mu(\rho_L^{\Delta t, +}) \geq \mu_{\min} \) a.e. on \( \Omega \times [0, T] \), interpolation between the bounds (cf. (4.48))
\[
\max_{t \in [0,T]} \int_\Omega \left| u_L^{\Delta t, (\pm)}(t) \right|^2 \, dx \leq \frac{[B(u_0, f, \tilde{\psi})]^2}{\rho_{\min}};
\]
\[
\int_0^T \int_\Omega \left| D(u_L^{\Delta t, (\pm)})(t) \right|^2 \, dx \, dt \leq \frac{[B(u_0, f, \tilde{\psi})]^2}{\mu_{\min}}
\] (4.74)
yields, using Korn’s inequality (3.37), that
\[ \int_0^T \| u_L^{\Delta t} \|_{L^1(\Omega)}^s \, dt \leq C, \quad \text{where} \quad \begin{cases} s = \frac{2(d+2)}{d} = 4 & \text{when } d = 2, \\ s = \frac{2(d+2)}{d} = \frac{10}{3} & \text{when } d = 3, \end{cases} \tag{4.75} \]
and C is a positive constant, independent of L and \( \Delta t \). Hence, by Hölder’s inequality, (4.75) and (4.73), we get that
\[
\left[ \int_0^T \int_{\Omega_0 \times D_0} \left[ |u_L^{\Delta t} - \zeta_L^{\Delta t}\tilde{\psi}_L^{\Delta t} + |^{1+\delta} \, dq \, dx \, dt \right] \right]^{\frac{1}{1+\delta}} 
\leq \zeta_{\max} \left[ \int_0^T \int_{\Omega_0 \times D_0} \left[ |u_L^{\Delta t} - |^{(1+\delta)a} \, dx \, dt \right] \right]^{\frac{1}{1+\delta}} \left[ \int_0^T \int_{\Omega_0 \times D_0} \left( \int_{D_0} \left[ \tilde{\psi}_L^{\Delta t} \right]^{1+\delta} \, dq \right)^b \, dx \, dt \right]^{\frac{1}{1+\delta}b},
\tag{4.76}
\]
where \( \delta > 0 \) is to be chosen, and 1/a + 1/b = 1, 1 < a, b < \infty, with
\[ (1+\delta)a < \frac{2}{d}(d+2). \]
To this end, we define
\[ r := \frac{2}{(K+1)d} \]
and we select any
\[ q_1 > \left( 1 + \frac{d}{d+4} \right)(1+r). \]
Note that \( q_1 > 1 + r = 1 + 2/(d+1) \); and in particular \( q_1 > r \). We then define \( q_2 := 1+\delta \), where \( \delta := r/(q_1-r) \); hence \( q_1 > q_2 \). We let \( a := q_1/(q_1-q_2) \), \( b := q_1/q_2 \). Clearly, with such a choice of \( q_1 \) and \( q_2 \), we have that \( q_1(1 - (1/q_2)) = r \) and 1 < \( q_2 < 1 + r = 1 + 2/(d+1) \). We thus deduce from (4.76) using (4.75) and (4.73) that
\[ \| u_L^{\Delta t} - \zeta_L^{\Delta t}\tilde{\psi}_L^{\Delta t} \|_{L^{1+\delta}((0,T) \times \Omega_0)} \leq C(O_0), \tag{4.77} \]
where \( \delta > 0 \) is as defined above; \( O_0 = \Omega_0 \times D_0 \); and \( C(O_0) \) is a positive constant, independent of \( L \) and \( \Delta t \).
Analogously,
\[ \left\| \sum_{i=1}^K \left[ \sigma (u_L^{\Delta t}) \zeta (\rho_L^{\Delta t}) \right] \beta (\tilde{\psi}_L^{\Delta t}) \right\|_{L^{1+\delta}((0,T) \times \Omega_0)} \leq C(O_0). \tag{4.78} \]
this time with \(0 < \delta \leq r/(r + 2)\), where, as above, \(r = 2/((K + 1)d)\). This follows on noting the second inequality in (4.74), (3.37) and that, thanks to (4.73),

\[
\left\| \tilde{\psi}^{\Delta t, (+)}_L \right\|_{L^{31}(\Omega)} \leq \left\| \tilde{\psi}^{\Delta t, (+)}_L \right\|_{L^{31}(\Omega)} \leq C(\mathcal{O}_0),
\]

with \(\tilde{q}_1 = 2(1 + \delta)/(1 - \delta); \tilde{q}_2 = 1 + \delta\) with \(0 < \delta \leq r/(r + 2)\); \(q_2 = \tilde{q}_2; q_1\) related to \(q_2\) via (4.72), and noting that since \(0 < r/(r + 2) < r < 1\), we have \(1 + r < \tilde{q}_1 \leq q_1 = (1 + \delta)/\delta \ll \infty, 1 < \tilde{q}_2 = q_2 < 1 + r\). Here, again, \(C(\mathcal{O}_0)\) is a positive constant, independent of \(L\) and \(\Delta t\).

With the bounds we have established on \(\tilde{\psi}^{\Delta t, (+)}_L\), we are now in a position to apply the Div–Curl lemma, which we next state (cf., for example, [21, p. 343, Theorem 10.21]).

**Theorem 4.2.** Suppose that \(\mathcal{O} \subset \mathbb{R}^{N}\) is a bounded open Lipschitz domain and \(\mathfrak{N} \in N_{\geq 2}\). Let, for any real number \(s > 1, W^{-1,s}(\mathcal{O})\) and \(W^{-1,s}(\mathcal{O}; \mathbb{R}^{N\times N})\) denote the duals of the Sobolev spaces \(W^{1,s}(\mathcal{O})\) and \(W^{1,s}(\mathcal{O}; \mathbb{R}^{N\times N})\), respectively. Assume that

\[
\begin{align*}
H_n & \to H \quad \text{weakly in } L^p(\mathcal{O}; \mathbb{R}^N), \\
Q_n & \to Q \quad \text{weakly in } L^q(\mathcal{O}; \mathbb{R}^{N\times N}),
\end{align*}
\]

where \(\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1\). Suppose also that there exists a real number \(s > 1\) such that

\[
\begin{align*}
\text{div } H_n & \equiv \nabla \cdot H_n \text{ is precompact in } W^{-1,s}(\mathcal{O}), \quad \text{and} \\
\text{curl } Q_n & \equiv (\nabla Q_n - (\nabla Q_n)^T) \text{ is precompact in } W^{-1,s}(\mathcal{O}; \mathbb{R}^{N\times N}).
\end{align*}
\]

Then,

\[
H_n \cdot Q_n \to H \cdot Q \quad \text{weakly in } L^r(\mathcal{O}).
\]

Consider the following sequences of \(\mathfrak{N} = 1 + d + Kd\) component vector functions defined on the Lipschitz domain \(\mathcal{O} := (0, T) \times \Omega_0 \times D_0 \subset \mathbb{R}^{N}\),

\[
H_{\Delta t,L} := (H^{(t)}_{\Delta t,L}; H^{(x_1)}_{\Delta t,L}; \ldots; H^{(x_d)}_{\Delta t,L}; H^{(q_1,1)}_{\Delta t,L}; \ldots; H^{(q_1,d)}_{\Delta t,L}; \ldots; H^{(q_K,1)}_{\Delta t,L}; \ldots; H^{(q_K,d)}_{\Delta t,L})
\]

where

\[
H^{(t)}_{\Delta t,L} := M(\xi(\rho_\Delta L)\tilde{\psi}_L)^{\Delta t},
\]

\[
(H^{(x_1)}_{\Delta t,L}; \ldots; H^{(x_d)}_{\Delta t,L}) := u_{\Delta t}^{\Delta t,-}M_\Delta L^{\Delta t} \tilde{\psi}_L^{\Delta t,-} - \varepsilon M \nabla x \tilde{\psi}_L^{\Delta t,+},
\]

\[
(H^{(q_1,1)}_{\Delta t,L}; \ldots; H^{(q_1,d)}_{\Delta t,L}) := M[\sigma(\mu^{\Delta t,+}_L)q_1]^{\Delta t} \mu_1^{\Delta t,+} \tilde{\psi}_L^{\Delta t,+} - \frac{1}{4\lambda} \sum_{j=1}^K A_{ij}M \nabla q_j \tilde{\psi}_L^{\Delta t,+}, \quad i = 1, \ldots, K,
\]

and

\[
Q_{\Delta t,L} := ((1 + \tilde{\psi}_L^{\Delta t})^{\alpha}, 0, \ldots, 0) \quad \text{with } \alpha \in \left(0, \frac{1}{2}\right) \text{ fixed.}
\]
Thus, the sequence $Q_{\Delta t, L}$, with $\Delta t = o(L^{-1})$, is bounded in $L^p(\mathcal{O}, \mathbb{R}^{31})$; hence there exist an element $H \in L^p(\mathcal{O}, \mathbb{R}^{31})$ and a subsequence, not indicated, such that $H_{\Delta t, L} \to H$, weakly in $L^p(\mathcal{O}, \mathbb{R}^{31})$. Also, the sequence $\{Q_{\Delta t, L}\}_{L > 1}$, with $\Delta t = o(L^{-1})$, is bounded in $L^q(\mathcal{O}, \mathbb{R}^{31})$ with $q_* = 1/\alpha$; hence there exist a $Q \in L^q(\mathcal{O}, \mathbb{R}^{31})$ and a subsequence, not indicated, such that $Q_{\Delta t, L} \to Q$, weakly in $L^q(\mathcal{O}, \mathbb{R}^{31})$.

With our definition of $\{H_{\Delta t, L}\}_{L > 1}$, we have that

$$\text{div}_{t, x, q} H_{\Delta t, L} = 0.$$ 

Therefore the sequence $\{\text{div}_{t, x, q} H_{\Delta t, L}\}_{L > 1}$, with $\Delta t = o(L^{-1})$, is precompact in $W^{-1,s}(\mathcal{O})$ for all $s > 1$.

Further, since $\alpha \in (0, \frac{1}{2})$, it follows from (4.64) that $\{Q_{\Delta t, L}\}_{L > 1}$ satisfies

$$\int_{(0,T) \times \mathcal{O}_0} |\text{curl}_{t, x, q} Q_{\Delta t, L}|^2 dq \, dx \, dt \leq C \int_{(0,T) \times \mathcal{O}_0} |\nabla_{x, q}(1 + \tilde{\psi}_{t, x, q}^{\Delta t})|^2 dq \, dx \, dt \leq C \int_{(0,T) \times \mathcal{O}_0} |\nabla_{x, q} \sqrt{\tilde{\psi}_{t, x, q}^{\Delta t}}|^2 dq \, dx \, dt \leq C(\mathcal{O}_0).$$

Therefore, the sequence $\{\text{curl}_{t, x, q} Q_{\Delta t, L}\}_{L > 1}$, with $\Delta t = o(L^{-1})$, is precompact in the function space $W^{-1,2}(\mathcal{O}; \mathbb{R}^{31 \times 31})$.

We thus deduce from Theorem 4.2 that

$$H_{\Delta t, L} \cdot Q_{\Delta t, L} = \overline{H_{\Delta t, L} \cdot Q_{\Delta t, L}},$$

where the overline $\overline{\cdot}$ signifies the weak limit in $L^1(\mathcal{O})$ of the sequence appearing under the overline; thus,

$$\overline{\left(\xi(\rho_L)\tilde{\psi}_{t, x, q}^{\Delta t}\right)^{\Delta t}} = \left(\xi(\rho_L)\tilde{\psi}_{t, x, q}^{\Delta t}\right)^{\Delta t} \left(1 + \tilde{\psi}_{t, x, q}^{\Delta t}\right)^{\Delta t}.$$

As

$$\left(\xi(\rho_L)\tilde{\psi}_{t, x, q}^{\Delta t}\right)^{\Delta t}(\cdot, t) = \tilde{\psi}_{t, x, q}^{n}(\cdot) \left[\frac{t - t_{n-1}}{\Delta t} + \frac{t_n - t_{n-1}}{\Delta t}\right] + \left(\tilde{\psi}_{t, x, q}^{n-1}(\cdot) - \tilde{\psi}_{t, x, q}^{n}(\cdot)\right) \frac{t_n - t_{n-1}}{\Delta t}$$

for all $t \in [t_{n-1}, t_n]$ and $n = 1, \ldots, N$, which in turn implies that

$$\left(\xi(\rho_L)\tilde{\psi}_{t, x, q}^{\Delta t}\right)^{\Delta t} = \xi(\rho_L^{\Delta t, +})\tilde{\psi}_{t, x, q}^{\Delta t} + \left(\xi(\rho_L^{\Delta t, -}) - \xi(\rho_L^{\Delta t, +})\right)\tilde{\psi}_{t, x, q}^{\Delta t} - \theta_{\Delta t},$$

where $\theta_{\Delta t}$ is the nonnegative discontinuous piecewise linear function defined on $(0, T]$ by

$$\theta_{\Delta t}(t) = \frac{t_n - t}{\Delta t}, \quad t \in (t_{n-1}, t_n], \quad n = 1, \ldots, N,$

the fact that, by (4.50e),

$$\left\|\xi(\rho_L^{\Delta t, +}) - \xi(\rho_L^{\Delta t, -})\right\|_{L^\infty(0,T;L^p(\mathcal{O}))} \leq \left\|\xi(\rho_L^{\Delta t, -}) - \xi(\rho)\right\|_{L^\infty(0,T;L^p(\mathcal{O}))} + \left\|\xi(\rho_L^{\Delta t, +}) - \xi(\rho)\right\|_{L^\infty(0,T;L^p(\mathcal{O}))} \to 0.$$
as \( L \to \infty \) (with \( \Delta t = o(L^{-1}) \)), \( 1 < p < \infty \), implies, on noting (4.66), that
\[
\| (\zeta'(\rho_L^{\Delta t,-}) - \zeta'(\rho_L^{\Delta t,+})) \tilde{\psi}_L^{\Delta t,-} \|_{L^1(\mathcal{D})} \to 0,
\] (4.81)
as \( L \to \infty \) (with \( \Delta t = o(L^{-1}) \)). Further, by (4.50e), \( \zeta'(\rho_L^{\Delta t,+}) \to \zeta \in L^\infty(0, T; L^\infty(\Omega)) \) strongly in \( L^\infty(0, T; L^p(\Omega)) \), \( 1 \leq p < \infty \), and, by (4.62), \( \tilde{\psi}_L^{\Delta t} \to \tilde{\psi} \) weakly in \( L^\infty_{loc}(0, T; L^1(\Omega \times D)) \); thus we deduce, on noting (4.66), that \( \zeta'(\rho_L^{\Delta t,+}) \tilde{\psi}_L^{\Delta t} \) converges to \( \zeta'(\rho) \tilde{\psi} \), weakly in \( L^1(\mathcal{D}) \). Hence we have shown that
\[
\left( \zeta(\rho_L) \tilde{\psi}_L \right)^{\Delta t} = \zeta(\rho_L^{\Delta t,+}) \tilde{\psi}_L^{\Delta t} = \zeta(\rho) \tilde{\psi} \in L^1(\mathcal{D}).
\]
Consequently, we have from (4.79) that
\[
(\zeta(\rho_L) \tilde{\psi}_L)^{\Delta t} (1 + \tilde{\psi}_L^{\Delta t})^\alpha \to \zeta(\rho) \tilde{\psi} (1 + \tilde{\psi}_L^{\Delta t})^\alpha \text{ weakly in } L^1(\mathcal{D}).
\]
Noting (4.80) and (4.66) then yields that
\[
\zeta(\rho_L^{\Delta t,+}) \tilde{\psi}_L^{\Delta t} (1 + \tilde{\psi}_L^{\Delta t})^\alpha \to \zeta(\rho) \tilde{\psi} (1 + \tilde{\psi}_L^{\Delta t})^\alpha \text{ weakly in } L^1(\mathcal{D}).
\]
Thus, by the strong convergence \( \zeta(\rho_L^{\Delta t,+}) \to \zeta(\rho) \in L^\infty(0, T; L^\infty(\Omega)) \) in \( L^\infty(0, T; L^p(\Omega)) \), \( 1 < p < \infty \), which, thanks to our assumptions on the function \( \zeta \) stated in (3.3) implies that \( 1/\zeta(\rho_L^{\Delta t,+}) \to 1/\zeta(\rho) \in L^\infty(0, T; L^\infty(\Omega)) \) in \( L^\infty(0, T; L^p(\Omega)) \), \( 1 < p < \infty \), we finally have, on noting (4.66), that
\[
\tilde{\psi}_L^{\Delta t} (1 + \tilde{\psi}_L^{\Delta t})^\alpha \to \tilde{\psi} (1 + \tilde{\psi}_L^{\Delta t})^\alpha \text{ weakly in } L^1(\mathcal{D}).
\] (4.82)
As, by definition, \( (1 + \tilde{\psi}_L^{\Delta t})^\alpha \to (1 + \tilde{\psi}_L^{\Delta t})^\alpha \), weakly in \( L^1(\mathcal{D}) \), by adding this to (4.82) we have that
\[
(1 + \tilde{\psi}_L^{\Delta t})^{\alpha+1} = (1 + \tilde{\psi}_L^{\Delta t})(1 + \tilde{\psi}_L^{\Delta t})^\alpha \to (1 + \tilde{\psi})(1 + \tilde{\psi}_L^{\Delta t})^\alpha \text{ weakly in } L^1(\mathcal{D}).
\]
Thanks to the weak lower-semicontinuity of the continuous convex function \( s \in [0, \infty) \mapsto s^{\alpha+1} \in [0, \infty) \) it follows (cf. Theorem 10.20 on p. 339 of [21]) that
\[
(1 + \tilde{\psi})^{1+\alpha} \leq (1 + \tilde{\psi})(1 + \tilde{\psi}_L^{\Delta t})^\alpha.
\]
Consequently,
\[
(1 + \tilde{\psi})^\alpha \leq \frac{1}{(1 + \tilde{\psi}_L^{\Delta t})^\alpha}.
\] (4.83)
On the other hand, the function \( s \in [0, \infty) \mapsto s^\alpha \in [0, \infty) \) is continuous and concave, and therefore \( s \in [0, \infty) \mapsto -s^\alpha \in (-\infty, 0] \) is continuous and convex; thus, once again by the weak lower-semicontinuity of continuous convex functions, we immediately have (cf. Theorem 10.20 on p. 339 of [21]) that
\[
-(1 + \tilde{\psi})^\alpha \leq \frac{-1}{(1 + \tilde{\psi}_L^{\Delta t})^\alpha}.
\] (4.84)
We deduce from (4.83) and (4.84) that
and consequently, since the function $s \in [0, \infty) \mapsto -s^\alpha \in (-\infty, 0]$ is continuous and strictly convex, and its domain of definition, $[0, \infty)$, is a convex set, Theorem 10.20 on p. 339 of [21] implies that there exists a subsequence (not relabelled) such that

$$-(1 + \tilde{\psi})^\alpha = -(1 + \tilde{\psi}_L^\Delta t)^\alpha.$$ \hfill (4.85)

Next, we select an increasing nested sequence $\{\mathcal{D}_0^k\}_{k=1}^{\infty}$ of bounded open Lipschitz domains $\mathcal{D}_0^k = (0, T) \times \Omega_0^k \times D_0^k$, where $\{\Omega_0^k\}_{k=1}^{\infty}$ and $\{D_0^k\}_{k=1}^{\infty}$ are increasing nested sequences of bounded open Lipschitz domains in $\Omega$ and $D$, respectively, such that $\bigcup_{k=1}^{\infty} \mathcal{D}_0^k = (0, T) \times \Omega \times D$. Since for each $k$ we have pointwise convergence on $\mathcal{D}_0^k$ of a subsequence of $\{\tilde{\psi}_L^\Delta t\}_{l=1}^{\infty}$, by using a diagonal procedure, we can extract from $\{\tilde{\psi}_L^\Delta t\}_{l=1}^{\infty}$ a subsequence (which is, once again, not relabelled) such that

$$\tilde{\psi}_L^\Delta t \rightarrow \tilde{\psi} \quad \text{a.e. in } \mathcal{D}.$$ \hfill (4.86)

with respect to the Lebesgue measure on $(0, T) \times \Omega \times D$. Let, for any Borel subset $A$ of $(0, T) \times \Omega \times D$,

$$\nu(A) := \int_A M \, dq \, dx \, dt.$$

Since $M \in L^1((0, T) \times \Omega \times D)$, the measure $\nu$ is absolutely continuous with respect to the Lebesgue measure, which then implies that $\tilde{\psi}_L^\Delta t \rightarrow \tilde{\psi}$, almost everywhere with respect to the measure $\nu$ (or, briefly, $\nu$ almost everywhere) in $(0, T) \times \Omega \times D$. Since $\nu((0, T) \times \Omega \times D) < \infty$, according to Egoroff’s theorem (cf. Theorem 2.22 on p. 149 of [22]) $\nu$ almost everywhere convergence of $\tilde{\psi}_L^\Delta t$ to $\tilde{\psi}$ implies $\nu$ almost uniform convergence of $\tilde{\psi}_L^\Delta t$ to $\tilde{\psi}$, and in particular $\nu$ convergence in measure of $\tilde{\psi}_L^\Delta t$ to $\tilde{\psi}$. Finally, by Vitali’s convergence theorem (cf. Theorem 2.24 on p. 150 of [22]), the uniform integrability of the sequence $\{\tilde{\psi}_L^\Delta t\}_{l=1}^{\infty}$ in $L^1((0, T) \times \Omega \times D)$ and $\nu$ convergence in measure of $\tilde{\psi}_L^\Delta t$ to $\tilde{\psi}$ together imply that

$$\tilde{\psi}_L^\Delta t \rightarrow \tilde{\psi} \quad \text{strongly in } L^1(0, T; L^1_M(\Omega \times D)).$$ \hfill (4.87)

It follows from (4.87) and the sixth bound in (4.48) that

$$\tilde{\psi}_L^\Delta t(\pm) \rightarrow \tilde{\psi} \quad \text{strongly in } L^1(0, T; L^1_M(\Omega \times D)).$$ \hfill (4.88)

In fact, (4.88) can be further strengthened: it follows from Lemma 4.2 below and (4.88) that

$$\tilde{\psi}_L^\Delta t(\pm) \rightarrow \tilde{\psi} \quad \text{strongly in } L^p(0, T; L^1_M(\Omega \times D)) \quad \forall p \in [1, \infty).$$ \hfill (4.89)

**Lemma 4.2.** Suppose that a sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges in $L^1(0, T; L^1_M(\Omega \times D))$ to a function $\varphi \in L^1(0, T; L^1_M(\Omega \times D))$, and is bounded in $L^\infty(0, T; L^1_M(\Omega \times D))$; i.e., there exists $K_0 > 0$ such that $\|\varphi_n\|_{L^\infty(0, T; L^1_M(\Omega \times D))} \leq K_0$ for all $n \geq 1$. Then, $\varphi \in L^p(0, T; L^1_M(\Omega \times D))$ for all $p \in [1, \infty)$, and the sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges to $\varphi$ in $L^p(0, T; L^1_M(\Omega \times D))$ for all $p \in [1, \infty)$.

**Proof.** See the proof of Lemma 5.1 in [8]. \qed

This then completes our proof of strong convergence of the sequence $\{\tilde{\psi}_L^\Delta t(\pm)\}_{l=1}^{\infty}$ to the function $\tilde{\psi} \in L^\infty(0, T; L^1_M(\Omega \times D))$ in the norm of the space $L^p(0, T; L^1_M(\Omega \times D))$, for all $p \in [1, \infty)$. 

5. Passage to the limit $L \to \infty$: existence of weak solutions

We are now ready to pass to the limit with $L \to \infty$ and prove the existence of weak solutions to the FENE chain model with variable density and viscosity, which is the main result of the paper.

Theorem 5.1. Suppose that the assumptions (3.3) and the condition (4.49), relating $\Delta t$ to $L$, hold. Then, there exist a subsequence of $\{(\rho^L_t, \mathbf{u}_L^t, \varpi^L_t)\}_{L>1}$ (not indicated) with $\Delta t = o(L^{-1})$, and functions $(\rho, \mathbf{u}, \varpi)$ such that

$$
\rho \in L^\infty(0, T; \mathcal{Y}) \cap C([0, T]; L^p(\Omega)), \quad \mathbf{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V),
$$

where $p \in [1, \infty)$, and

$$
\varpi \in L^1(0, T; L_M^1(\Omega \times D)),
$$

with $\varpi \geq 0$ a.e. on $\Omega \times D \times [0, T]$, satisfying

$$
\int_D M(q)\varpi(x, q, t) \, dq \leq \text{ess} \sup_{x \in \Omega} \left( \frac{1}{\zeta(\rho(x))} \int_D \psi_0(x, q) \, dq \right) \text{ for a.e. } (x, t) \in \Omega \times [0, T],
$$

(5.1)

whereby $\varpi \in L^\infty(0, T; L_M^1(\Omega \times D))$; and finite relative entropy and Fisher information, with

$$
\mathcal{F}(\varpi) \in L^\infty(0, T; L_M^1(\Omega \times D)) \quad \text{and} \quad \sqrt{\varpi} \in L^2(0, T; H_M^1(\Omega \times D)),
$$

(5.2)

such that, as $L \to \infty$ (and thereby $\Delta t \to 0+$),

$$
\rho^{\Delta t}_L \to \rho \quad \text{weak* in } L^\infty(0, T; L^\infty(\Omega)),
$$

(5.3a)

$$
\rho^{\Delta t(\pm)}_L, \rho^{\Delta t(\pm)}_L \to \rho \quad \text{strongly in } L^\infty(0, T; L^p(\Omega)),
$$

(5.3b)

$$
\mu(\rho^{\Delta t(\pm)}_L) \to \mu(\rho) \quad \text{strongly in } L^\infty(0, T; L^p(\Omega)),
$$

(5.3c)

$$
\zeta(\rho^{\Delta t}_L), \zeta(\rho^{\Delta t(\pm)}_L), \zeta^{\Delta t}_L \to \zeta(\rho) \quad \text{strongly in } L^\infty(0, T; L^p(\Omega)),
$$

(5.3d)

where $p \in [1, \infty)$;

$$
u^{\Delta t(\pm)}_L \to \mathbf{u} \quad \text{weak* in } L^\infty(0, T; L^2(\Omega)),
$$

(5.4a)

$$
\nu^{\Delta t(\pm)}_L \to \mathbf{u} \quad \text{weakly in } L^2(0, T; V),
$$

(5.4b)

$$
\nu^{\Delta t(\pm)}_L \to \mathbf{u} \quad \text{strongly in } L^2(0, T; L^r(\Omega)),
$$

(5.4c)

where $r \in [1, \infty)$ if $d = 2$ and $r \in [1, 6)$ if $d = 3$; and

$$
M^{\frac{1}{2}} \nabla x \sqrt{\varpi^{\Delta t(\pm)}_L} \to M^{\frac{1}{2}} \nabla x \sqrt{\varpi} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)),
$$

(5.5a)

$$
M^{\frac{1}{2}} \nabla y \sqrt{\varpi^{\Delta t(\pm)}_L} \to M^{\frac{1}{2}} \nabla y \sqrt{\varpi} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)),
$$

(5.5b)

$$
\sqrt{\varpi^{\Delta t(\pm)}_L} \to \sqrt{\varpi} \quad \text{strongly in } L^p(0, T; L_M^1(\Omega \times D)),
$$

(5.5c)

$$
\beta^L(\sqrt{\varpi^{\Delta t(\pm)}_L}) \to \sqrt{\varpi} \quad \text{strongly in } L^p(0, T; L_M^1(\Omega \times D)),
$$

(5.5d)
for all \( p \in [1, \infty) \); and,

\[
\nabla_x \cdot \sum_{i=1}^{K} C_i(M(\rho(\Delta t)^+) \tilde{\psi}(\Delta t)^+) \rightarrow \nabla_x \cdot \sum_{i=1}^{K} C_i(M(\rho) \tilde{\psi}) \quad \text{weakly in } L^2(0, T; \mathcal{V}).
\]

(5.5e)

The triple \((\rho, u, \tilde{\psi})\) is a global weak solution to problem (P), in the sense that

\[
- \int_0^T \int_{\Omega} \rho \frac{\partial \eta}{\partial t} \, dx \, dt + \int_0^T \int_{\Omega} \left[ \mu(\rho)D(u) : D(w) - \rho(u \otimes u) : \nabla_x w \right] \, dx \, dt
\]

\[
= \int_{\Omega} \rho_0(x) u_0(x) \cdot w(x, 0) \, dx + \int_0^T \int_{\Omega} \left[ \rho f \cdot w - k \sum_{i=1}^{K} C_i(M(\rho) \tilde{\psi}) : \nabla_x w \right] \, dx \, dt
\]

\[
\forall w \in W^{1,1}(0, T; \mathcal{V}) \text{ s.t. } w(\cdot, T) = 0,
\]

(5.6a)

with \( q \in (2, \infty) \) when \( d = 2 \) and \( q \in [3, 6] \) when \( d = 3 \),

\[
- \int_0^T \int_{\Omega} \rho_0 \frac{\partial w}{\partial t} \, dx \, dt + \int_0^T \int_{\Omega} \left[ \mu(\rho)D(u) : D(w) - \rho(u \otimes u) : \nabla_x w \right] \, dx \, dt
\]

\[
= \int_{\Omega} \rho_0(x) u_0(x) \cdot w(x, 0) \, dx + \int_0^T \int_{\Omega} \left[ \rho f \cdot w - k \sum_{i=1}^{K} C_i(M(\rho) \tilde{\psi}) : \nabla_x w \right] \, dx \, dt
\]

\[
\forall w \in W^{1,1}(0, T; \mathcal{V}) \text{ s.t. } w(\cdot, T) = 0,
\]

(5.6b)

and

\[
- \int_0^T \int_{\Omega \times D} M(\rho(\Delta t)^+) \tilde{\psi} \frac{\partial \varphi}{\partial t} \, dq \, dx \, dt + \int_0^T \int_{\Omega \times D} M[\varepsilon \nabla_x \tilde{\psi} - u(\rho(\Delta t)^+) \tilde{\psi}] \cdot \nabla_x \varphi \, dq \, dx \, dt
\]

\[
+ \frac{1}{4\lambda} \int_0^T \int_{\Omega \times D} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij} M \nabla_{q_j} \tilde{\psi} \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt
\]

\[
- \int_0^T \int_{\Omega \times D} M \sum_{i=1}^{K} [\sigma(u) q_i] \zeta(\rho(\Delta t)^+) \tilde{\psi} \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt
\]

\[
= \int_{\Omega \times D} M(q) \zeta(\rho_0(x)) \tilde{\psi}_0(x, q) \varphi(x, q, 0) \, dq \, dx
\]

\[
\forall \varphi \in W^{1,1}(0, T; H^s(\Omega \times D)) \text{ s.t. } \varphi(\cdot, \cdot, T) = 0.
\]

(5.6c)

with \( s > 1 + \frac{1}{2}(K + 1)d \). In addition, the weak solution \((\rho, u, \tilde{\psi})\) satisfies for all \( t \in [0, T] \):

\[
\int_{\Omega} |\rho(t)|^p \, dx = \int_{\Omega} |\rho_0|^p \, dx,
\]

(5.7a)

for \( p \in [1, \infty) \), and the following energy inequality for a.e. \( t \in [0, T] \):
\[
\int_{\Omega} \rho(t) |u(t)|^2 \, dx + \int_0^t \int_{\Omega} \mu(\rho) |D(u)|^2 \, dx \, ds + 2k \int_{\Omega \times D} M \xi(\rho(t)) F(\widetilde{\psi}(t)) \, dq \, dx \\
+ 8k \epsilon \int_0^t \int_{\Omega \times D} M |\nabla \sqrt{\beta} \widetilde{\psi}|^2 \, dq \, dx \, ds + \frac{a_0 k}{\lambda} \int_0^t \int_{\Omega \times D} M |\nabla \sqrt{\beta} \widetilde{\psi}|^2 \, dq \, dx \\
\leq \int_{\Omega} \rho_0 |u_0|^2 \, dx + \frac{2^2 \max C^2_{\beta \lambda}}{\mu_{\min} C^2_{\lambda}} \int_0^t \|f\|^2_{L^p(\Omega)} \, ds + 2k \int_{\Omega \times D} M \xi(\rho_0) F(\widetilde{\psi}_0) \, dq \, dx \\
\leq \left[ B(u_0, f, \widetilde{\psi}_0) \right]^2,
\]

(5.7b)

with \( F(s) = s(\log s - 1) + 1, s \geq 0, \) and \( B(u_0, f, \widetilde{\psi}_0) \) as defined in (4.26b).

**Proof.** We split the proof into a number of steps.

**Step A.** The convergence results (5.3a)-(5.3d), (5.4a)-(5.4c) and (5.5a), (5.5b) were proved in Theorem 4.1. The strong convergence result (5.5c) was established in Section 4.4, see (4.89). Next from the Lipschitz continuity of \( \beta^L \), we obtain for any \( p \in [1, \infty) \) that

\[
\| \widetilde{\psi} - \beta^L(\widetilde{\psi}^L_{\Delta(t, \pm)}) \|_{L^p(0, T; L^1(\Omega \times D))} \\
\leq \| \widetilde{\psi} - \beta^L(\widetilde{\psi}) \|_{L^p(0, T; L^1(\Omega \times D))} + \| \beta^L(\widetilde{\psi}) - \beta^L(\widetilde{\psi}^L_{\Delta(t, \pm)}) \|_{L^p(0, T; L^1(\Omega \times D))} \\
\leq \| \widetilde{\psi} - \beta^L(\widetilde{\psi}) \|_{L^p(0, T; L^1(\Omega \times D))} + \| \widetilde{\psi} - \widetilde{\psi}^L_{\Delta(t, \pm)} \|_{L^p(0, T; L^1(\Omega \times D))}.
\]

(5.8)

The first term on the right-hand side of (5.8) converges to zero as \( L \to \infty \) on noting that \( \beta^L(\widetilde{\psi}) \) converges to \( \widetilde{\psi} \) almost everywhere on \( \Omega \times D \times (0, T) \) and applying Lebesgue’s dominated convergence theorem, the second term converges to 0 on noting (5.5c). That yields the desired result (5.5d).

As the sequences \( \{ \widetilde{\psi}^L_{\Delta(t, \pm)} \}_{L \geq 1} \) converge to \( \widetilde{\psi} \) strongly in \( L^p(0, T; L^1(\Omega \times D)) \), it follows (upon extraction of suitable subsequences) that they converge to \( \widetilde{\psi} \) a.e. on \( \Omega \times D \times (0, T) \). This then, in turn, implies that the sequences \( \{ F(\widetilde{\psi}^L_{\Delta(t, \pm)}) \}_{L \geq 1} \) converge to \( F(\widetilde{\psi}) \) a.e. on \( \Omega \times D \times [0, T] \); in particular, for a.e. \( t \in [0, T] \), the sequences \( \{ F(\widetilde{\psi}^L_{\Delta(t, \pm)}(\cdot, t)) \}_{L \geq 1} \) converge to \( F(\widetilde{\psi}(\cdot, t)) \) a.e. on \( \Omega \times D \). Since \( F \) is nonnegative, Fatou’s lemma then implies that, for a.e. \( t \in [0, T] \),

\[
\int_{\Omega \times D} M(q) F(\widetilde{\psi}(x, q, t)) \, dx \, dq \leq \liminf_{L \to \infty} \int_{\Omega \times D} M(q) F(\widetilde{\psi}^L_{\Delta(t, \pm)}(x, q, t)) \, dx \, dq \leq C_*,
\]

(5.9)

where the second inequality in (5.9) stems from the bound on the fifth term on the left-hand side of (4.48). Hence the first result in (5.2) holds, and the second was established in Theorem 4.1. Similarly, (5.1) is established on noting (3.63) and that \( \psi^L_{\Delta(t, \pm)} \geq 0 \). Analogously to (5.9), one can establish for a.e. \( t \in [0, T] \) that

\[
\int_{\Omega \times D} M(q) \xi(\rho(x)) F(\widetilde{\psi}(x, q, t)) \, dx \, dq \\
\leq \liminf_{L \to \infty} \int_{\Omega \times D} M(q) \xi(\psi^L_{\Delta(t, \pm)}(x)) F(\widetilde{\psi}^L_{\Delta(t, \pm)}(x, q, t)) \, dx \, dq.
\]

(5.10)
Finally, it is shown in Step 3.7 in the proof of Theorem 6.1 in [9] on noting (3.12), (5.5b), (5.5c) and (5.1), that in the case \( \zeta \equiv 1 \)

\[
k \int_0^T \sum_{i=1}^K \int_{\Omega} C_i (M \zeta (\rho_L^{\Delta t, +}) \widetilde{\psi}_L^{\Delta t, +}) : \nabla x w \, dx \, dt = k \int_0^T \sum_{i=1}^K \int_{\Omega} C_i (M \zeta (\rho) \psi) : \nabla x w \, dx \, dt \tag{5.11}
\]
as \( L \to \infty \), for any divergence-free function \( w \in C^1 ([0, T]; C_0^\infty (\Omega)) \). The proof there is easily generalized to the present variable \( \zeta \) on noting (5.3d). This implies (5.5e), thanks to the denseness of these smooth divergence functions in the function space \( L^2 (0, T; V) \), and on showing that the right-hand side of (5.11) is well-defined for \( w \in L^2 (0, T; V) \), on noting (3.12), (5.2) and (5.1).

Step B. We are now ready to return to (4.3a)-(4.3c) and pass to the limit \( L \to \infty \) (and thereby also \( \Delta t \to 0_+ \)). We shall discuss them one at a time, starting with Eq. (4.3a). We have already passed to the limit \( L \to \infty \) in (4.3a) using (5.3b) and (5.4c) to obtain (5.3) in the proof of Theorem 4.1. The desired result (5.6a) then follows from (5.43) on noting the denseness of the set of all functions contained in \( C^1 ([0, T]; W^{1, \frac{2}{3}} (\Omega)) \) and vanishing at \( t = T \) in the set of all functions contained in \( W^{1, 1} (0, T; W^{1, \frac{2}{3}} (\Omega)) \) and vanishing at \( t = T \). In addition, the energy equality (5.7a) was proved in the proof of Theorem 4.1, see (4.54).

Step C. Having dealt with (4.3a), we now turn to (4.3b), with the aim to pass to the limit with \( L \) (and \( \Delta t \)). We choose as our test function

\[
w \in C^1 ([0, T]; C_0^\infty (\Omega)) \quad \text{with } w ( \cdot, T) = 0, \quad \text{and } \nabla x \cdot w = 0 \quad \text{on } \Omega \text{ for all } t \in [0, T]. \tag{5.12}
\]

Clearly, any such \( w \) belongs to \( L^1 (0, T; V) \) and is therefore a legitimate choice of test function in (4.3b). Integration by parts with respect to \( t \) on the first term in (4.3b), and noting (3.27) and (4.2) for the second term yields that

\[
- \int_0^T \int_{\Omega} (\rho_L u_L)^{\Delta t} \cdot \frac{\partial w}{\partial t} \, dx \, dt - \frac{1}{2} \int_0^T \int_{\Omega} \rho_L^{\Delta t} [u_L^{\Delta t, -}] \cdot \nabla x (u_L^{\Delta t, +} \cdot w) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \mu (\rho_L^{\Delta t, +}) D (u_L^{\Delta t, +}) : D (w) \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_{\Omega} \rho_L^{\Delta t} \left[ [(u_L^{\Delta t, -} \cdot \nabla x) u_L^{\Delta t, +}] \cdot w - [(u_L^{\Delta t, -} \cdot \nabla x) w] \cdot u_L^{\Delta t, +} \right] \, dx dt \\
= \int_{\Omega} \rho_0 u_0^0 \cdot w (0) \, dx + \int_0^T \int_{\Omega} \rho_L^{\Delta t, +} f^{\Delta t, +} \cdot w \, dx \, dt \\
- k \sum_{i=1}^K \int_0^T \int_{\Omega} C_i (M \zeta (\rho_L^{\Delta t, +}) \widetilde{\psi}_L^{\Delta t, +}) : \nabla x w \, dx \, dt. \tag{5.13}
\]

Next we note from (4.1a), (4.1b) that for \( t \in (t_{n-1}, t_n], \ n = 1, \ldots, N, \)

\[
(\rho_L u_L)^{\Delta t} = \rho_L^{\Delta t} u_L^{\Delta t, +} - \frac{(t_n - t)}{\Delta t} \rho_L^{\Delta t, -} (u_L^{\Delta t, +} - u_L^{\Delta t, -}). \tag{5.14}
\]
It follows that we can pass to the limit \( L \to \infty \) (\( \Delta t \to 0 \)) in \((5.13)\), on noting \((5.14), (5.3b), (5.3c), (5.4b), (5.4c)\), \((5.5e), (3.25)\) and that \( u_0^0 \) converges to \( u_0 \) weakly in \( H \), to obtain

\[
- \int_0^T \int_\Omega \rho u \cdot \frac{\partial w}{\partial t} \, dx \, dt + \int_0^T \int_\Omega \mu(\rho) \frac{\partial D(w)}{\partial t} : D(w) \, dx \, dt - \frac{1}{2} \int_0^T \int_\Omega \rho u \cdot \nabla(x \cdot w) \, dx \, dt
\]

\[
+ \frac{1}{2} \int_0^T \int_\Omega \rho \left[ \left( u \cdot \nabla x \right) u \right] \cdot w - \left[ \left( u \cdot \nabla x \right) w \right] \cdot u \right] \, dx \, dt
\]

\[
\int_\Omega \rho_0 u_0 \cdot w(0) \, dx + \int_0^T \int_\Omega \rho f \cdot w \, dx \, dt - k \sum_{i=1}^K \int_0^T \int_\Omega \zeta_i(M \zeta(\rho) \psi) : \nabla w \, dx \, dt.
\]

The desired result \((5.6b)\) then follows from \((5.15)\) on noting \((3.21), \) the denseness of the test functions \((5.12)\) in \( W^{1,1}(0, T; V) \) and that all the terms in \((5.6b)\) are well-defined.

Step D. Similarly to \((5.13)\), we obtain from performing integration by parts with respect to time on the first term in \((4.3c)\) that

\[
\int_0^T \int_\Omega \left[ -M(\zeta(\rho_L) \psi_L) \frac{\partial \varphi}{\partial t} + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} M \nabla q_i \psi_L^{\Delta t_+} : \nabla q_j \varphi \right] \, dq \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega \left[ \varepsilon M \nabla q_i \psi_L^{\Delta t_+} - u_L^{\Delta t_-} M \zeta_L^{\Delta t} \psi_L^{\Delta t_+} \right] \cdot \nabla q_i \varphi \,dq \, dx \, dt
\]

\[
- \int_0^T \int_\Omega \left[ M \sum_{i=1}^K [\sigma(u_L^{\Delta t_+}) q_i] \zeta(\rho_L^{\Delta t_+}) \beta^L \psi_L^{\Delta t_+} : \nabla q_i \varphi \right] \, dq \, dx \, dt
\]

\[
= \int_\Omega M \zeta(\rho_0) \psi_0(0) \, dq \, dx \quad \forall \varphi \in C^1([0, T]; C^\infty(\Omega \times D)) \text{ s.t. } \varphi(\cdot, \cdot, T) = 0. \tag{5.16}
\]

We now pass to the limit \( L \to \infty \) (and \( \Delta t \to 0_+ \)) in \((5.16)\) to obtain \((5.6c)\) for the smooth \( \varphi \) of \((5.16)\) using the convergence results \((5.3d), (5.4b), (5.4c), (5.5a)-(5.5d)\) and \((3.17c)\). The desired result \((5.6c)\) then follows on noting the denseness of the test functions \( \varphi \in C^1(0, T; C^\infty(\Omega \times D)) \) with \( \varphi(\cdot, \cdot, T) = 0 \) in \( W^{1,1}(0, T; H^2(\Omega \times D)) \) with \( \varphi(\cdot, \cdot, T) = 0 \), for \( s > 1 + \frac{1}{2}(K + 1)d \), and that all the terms in \((5.6c)\) are well-defined.

Step E. The energy inequality \((5.7b)\) is a direct consequence of \((5.3a)-(5.3d), (5.4a), (5.4b)\) and \((5.5a), (5.5b),\) on noting \((3.25), (5.10)\) and the (weak) lower-semicontinuity of the terms on the left-hand side of \((4.26b)\). For example, it follows from \((4.26b)\) for a.e. \( t \in [0, T] \) that \( [\rho_L^{\Delta t_+}(t)]^2 u_L^{\Delta t_+}(t) \to g(t) \), weakly in \( L^2(\Omega) \) as \( L \to \infty \), and therefore \( \liminf_{L \to \infty} \int_\Omega \rho_L^{\Delta t_+}(t) u_L^{\Delta t_+}(t)^2 \, dx \geq \int_\Omega |g(t)|^2 \, dx \).

We then have from \((5.3b)\) and \((5.4a)\) that \( g(t) = [\rho(t)]^2 u(t) \). \( \square \)

Thus we have proved the existence of global-in-time weak solutions to a general class of coupled bead–spring chain models that arise from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids with noninteracting polymer chains, with finitely extensible nonlinear elastic (FENE) spring potentials, assuming that the solvent has variable density and viscosity, and that the hydrodynamic drag coefficient in the Fokker–Planck equation is density-dependent. In the special case when
the drag coefficient, the solvent density and the solvent viscosity are constant, Theorem 5.1 collapses to the main theorem in [9], where we also proved with \( f = 0 \), in this special case, the exponential equilibration of weak solutions to the Navier–Stokes–Fokker–Planck system.

The techniques developed in this paper can be modified quite straightforwardly in order to prove large-data global existence of weak solutions to kinetic models with configuration-dependent drag, in which, instead of being a nonlinear function of the unknown density \( \rho \), as in (19), the drag coefficient \( \zeta \) is a given \( C^1 \) function of \( q \), bounded above and below by positive constants (cf. de Gennes [15], Hinch [23], Larson [29], Schröder et al. [44]). The idea behind these models, which have been developed to explain physical mechanism by which large stresses rapidly build up in dilute polymer solutions, is that of a bead friction coefficient that depends strongly on the inter-bead distance through a nonlinear friction law. This principle of conformation-dependent hydrodynamic drag assumes that as a chain becomes extended by the flow, the strength of the hydrodynamic friction on the chain will also increase; see, [40] for a detailed survey.

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