GENERALIZED SHOCK SOLUTIONS FOR HYDRODYNAMIC BLACK HOLE ACCRETION

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1. INTRODUCTION

The process by which any gravitating, massive, astrophysical object captures its surrounding fluid is called accretion. Depending on the rotational energy content of the infalling material, accretion flows onto black holes (BHs) may be broadly classified into two different categories, i.e., nonrotating (spherical) and rotating (accretion disks) accretion. If the instantaneous dynamical velocity and local acoustic velocity of the accreting fluid, moving along a space curve parameterized by $r$, are $u(r)$ and $a(r)$, respectively, then the local Mach number $M(r)$ of the fluid can be defined as $M(r) = u(r)/a(r)$. The flow will be locally subsonic or supersonic according to $M(r) < 1$ or $M(r) > 1$, i.e., according to $u(r) < a(r)$ or $u(r) > a(r)$. The flow is transonic if at any moment it crosses $M = 1$. This happens when a subsonic-to-supersonic or supersonic-to-subsonic transition takes place either continuously or discontinuously. The point(s) where such crossing takes place continuously is (are) called the sonic point(s) and where such crossing takes place discontinuously are called shocks or discontinuities. It is generally argued that, in order to satisfy the inner boundary conditions imposed by the event horizon, accretion onto black holes exhibits transonic properties in general, which further indicates that formation of shock waves is possible in astrophysical fluid flows onto Galactic and extragalactic black holes. One also expects that shock formation in black hole accretion might be a general phenomenon because shock waves in rotating and nonrotating flows are convincingly able to provide an important and efficient mechanism for conversion of a significant amount of the gravitational energy (available from deep potential wells created by these massive compact accretors) into radiation by randomizing the directed infall motion of the accreting fluid. Hence, shocks possibly play an important role in governing the overall dynamical and radiative processes taking place in accreting plasma. Thus, the study of steady, stationary shock waves produced in black hole accretion has acquired a very important status in recent years, and it is expected that shocks may be an important ingredient in an accreting black hole system in general.

While the possibility of the formation of a standing spherical shock around compact objects was first conceived long ago (Bisnovatyi-Kogan, Zeldovich, & Sunyaev 1971), most of the works on shock formation in spherical accretion share more or less the same philosophy that one should incorporate shock formation to increase the efficiency of directed radial infall in order to explain the high luminosity of active galactic nuclei (AGNs) and quasi-stellar objects and to model their broadband spectrum (Jones & Ellison 1991). Considerable work has been done in this direction where several authors have investigated the formation and dynamics of standing shock in spherical accretion (Mészáros & Ostriker 1983; Protheroe & Kazanas 1983; Chang & Ostriker 1985; Kazanas & Ellison 1986; Babul, Ostriker, & Mészáros 1989; Park 1990a, 1990b). Ideas and formalisms developed in these works have been applied to study related interesting problems such as entropic-acoustic or various other instabilities in spherical accretion (Foglizzo & Tagger 2000; Blondin & Ellison 2001; Lai & Goldreich 2000; Foglizzo 2001; Kovalenko & Eremin 1998), production of high-energy cosmic rays from AGNs (Protheroe & Szabo 1992), study of the hadronic model of AGNs (Blondin & Königl 1987; Contopoulos & Kazanas 1995), highly energetic emission from relativistic particles in our Galactic center (Markoff, Melia, & Sarcevic 1999), explanation of high lithium abundances in the late-type, low-mass companions of soft X-ray transients (Guessoum & Kazanas 1999), and post-Newtonian (Das 1999, 2000, 2002) as well as complete general relativistic (Das 2001a) study of accretion-powered spherical winds emanating from Galactic and extragalactic black hole environments.

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ABSTRACT

For the first time, all available pseudo-Schwarzschild potentials are exhaustively used to investigate the possibility of shock formation in hydrodynamic, inviscid, black hole accretion disks. It is shown that a significant region of parameter space spanned by important accretion parameters allows shock formation for flow in all potentials used in this work. This leads to the conclusion that the standing shocks are essential ingredients in accretion disks around nonrotating black holes in general. Using a complete general relativistic framework, equations governing multitransonic black hole accretion and wind are formulated and solved, and the condition for shock formation in such flows is also derived in the Schwarzschild metric. Shock solutions for accretion flow in various pseudopotentials are then compared with such general relativistic solutions to identify which potential is the best approximation of Schwarzschild spacetime as far as the question of shock formation in black hole accretion disks is concerned.

Subject headings: accretion, accretion disks — black hole physics — hydrodynamics — relativity — shock waves

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With equal (if not more) importance and rigor, the question of shock formation in accretion disks around Schwarzschild black holes has been addressed by several authors. While the initial works in this direction can be attributed to Fukue (1983), Hawley, Wilson, & Smarr (1984), Ferrari et al. (1985), Sawada, Matsuda, & Hachisu (1986), and Spruit (1987), it was Fukue (1987) and Chakrabarti and his collaborators (Chakrabarti 1989, 1996a and references therein; Abramowicz & Chakrabarti 1990; Chakrabarti & Molteni 1993) who were the first to provide the satisfactory semianalytical or numerical global shock solution for transonic, inviscid, Keplerian, or sub-Keplerian rotating accretion around a Schwarzschild black hole. Consequently, their works were further supported and improved by several other independent works (Yang & Kafatos 1995, hereafter YK; Cuditz & Tsuruta 1998; Tóth, Keppens, & Botchev 1998). Because of the inner boundary conditions imposed by the event horizon, shocks form in BH accretion disks only if the flow has more than one real physical $X$-type sonic point (multitransonic flow). For a particular set of initial boundary conditions, some of the above mentioned works report multiplicity in shock location, but such a degeneracy can ultimately be removed by local stability analysis, allowing one to assert that only one stable shock location is possible. Hereafter, whenever we use the word “shock,” it is to be understood that we in general always refer to only the stable shock location unless otherwise mentioned.

The above mentioned works deserve attention because the shocked flows studied there are expected to explain the spectral properties of BH candidates. However, thus far in the astrophysical literature, the theoretical study of steady, standing shock formation in accretion disks around nonrotating BHs has suffered from two general limitations. First, the shock solutions were obtained either on a case-by-case basis or, even when successful attempts were made to provide a more complete analysis, the boundary of the parameter space responsible for shock formation was obtained only for global variation of the total specific energy $\epsilon$ (or accretion rate $\dot{M}$) and specific angular momentum $\lambda$ of the flow and not for variations of the polytropic constant $\gamma$ of the flow; rather, accretion was always considered to be ultrarelativistic, which may not always be a realistic assumption. Since $\gamma$ is expected to have great influence on the radiative properties of the flow in general, we think that ignoring the explicit dependence of shock solutions on $\gamma$ limits claims of generality. Second, except for YK, all available so-called global shock solutions have been discussed only in the context of one particular type of BH potential, namely, the Paczyński & Wiita (1980) potential $[\Phi_1(r)]$. Along with the $\Phi_1(r)$, recent studies (Das & Sarkar 2001 and references therein) enhance the importance of also considering three other pseudo-Schwarzschild BH potentials, one $[\Phi_2(r)]$ proposed by Nowak & Wagoner (1991) and two others $[\Phi_3(r)$ and $\Phi_4(r)]$ due to Artemova, Björnsson, & Novikov (1996, hereafter ABN), in mimicking the complete general relativistic spacetime for accretion around a Schwarzschild black hole. Hence, we believe that being restricted to only one specific pseudo-Schwarzschild BH potential does not guarantee the claimed “global” nature of so-called global shock solutions present in the literature; rather, one must study the transonic disk structure as well as shock formation in all available BH potentials to firmly assert the ubiquity of shock formation in a multitransonic accretion disk around a Schwarzschild BH. In this context, it is to be mentioned here that YK deserve special importance because the shock solution due to YK appears to be the only work available in the literature that provides the complete general relativistic description of shock formation exclusively for a nonrotating BH. Nevertheless, this work deals with isothermal accretion, but one understands that global isothermality in BH accretion is difficult to achieve for realistic flows, and a more general kind of BH accretion is expected to be governed by a polytropic equation of state. Also, YK do not provide the global parameter space dependence of shock solutions. A few authors claim to provide the full general relativistic shock solutions for Schwarzschild BHs as a limiting case of their results obtained in Kerr geometry (Chakrabarti 1996b, 1996c; Lu et al. 1997). In doing so, a number of assumptions are made, some of which, however, may not appear to be fully convincing. For example, either the disk is supposed to be in conical equilibrium (Lu et al. 1997), which should not be the case in reality because the realistic accretion flow should be in vertical equilibrium (Chakrabarti 1996a and references therein), or some results valid for isothermal accretion are directly applied to study the polytropic accretion in an ad hoc manner (Chakrabarti 1996b), or some of the Newtonian approximations are not very convincingly combined with complete general relativistic equations (Chakrabarti 1996c), which does not strengthen their claim for a full general relativistic treatment of shock formation. Hence, it is fair to say that although literature on general relativistic hydrodynamic BH accretion is well enriched by a number of important works (the following is an incomplete list of relevant papers on the subject: Novikov & Thorne 1973; Bardeen & Petterson 1975; Abramowicz, Jaroszynski, & Sikora 1978; Lu 1985, 1986; Karas & Mucha 1993; Björnsson 1995; Riflett & Herold 1995; Ipser 1996; Pariev 1996; Peitz & Appl 1997; Bao, Wiita, & Hadrava 1998; Gammie & Popham 1998; Gammie 1999), no well-accepted complete general relativistic global shock solution exclusively obtained for a hydrodynamic accretion disk around a Schwarzschild BH has yet appeared in the literature.

Motivated by the above mentioned limitations encountered by previous works in this field, the major aim of our work presented in this paper is to provide a generalized formalism that is expected to handle the formation of steady, standing Rankine- Hugoniot shock (RHS) in multitransonic hydrodynamic BH accretion flow and to identify which region of parameter space (spanned by every important accretion parameter, namely, $\epsilon$, $\lambda$, and $\gamma$) will be responsible for such shock formation for all available pseudo-Schwarzschild BH potentials. We would also like to compare the properties of multitransonic accretion in these BH potentials with complete general relativistic BH accretion as far as the issue of shock formation is concerned.

Hereafter, we will define the Schwarzschild radius $r_s$ as

\[
rs = \frac{2GM_{\text{BH}}}{c^2}
\]

(where $M_{\text{BH}}$ is the mass of the black hole and $G$ is the universal gravitational constant) so that the marginally bound circular orbit $r_h$ and the last stable circular orbit $r_s$ take the values $2rs$ and $3rs$, respectively, for a typical Schwarzschild

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2 By the terms “ultrarelativistic” and “purely nonrelativistic,” we mean a flow with $\gamma = 4/3$ and $\gamma = 5/3$, respectively, according to the terminology used in Frank, King, & Raine (1992).
black hole. Also, total mechanical energy per unit mass on \( r_s \) (sometimes called "efficiency," \( e \)) may be computed as \(-0.057\) for this case. Also, we will use a simplified geometric unit throughout this paper where radial distance is scaled in units of \( r_g \), radial dynamical velocity \( u \) and polytropic sound speed \( a \) of the flow are scaled in units of \( c \) (the velocity of light in vacuum), mass \( m \) is scaled in units of \( M_{BH} \), and all other derived quantities would be scaled accordingly. Also, for simplicity, we will use \( G = c = M_{BH} = 1 \). In the next section, we briefly describe a few important features of the four different pseudo-Schwarzschild "effective" BH potentials used in this work. In \( \S \) 3, we show how we formulate and solve the equations governing multitransonic BH accretion in these potentials that may have shocks. In \( \S \) 4, we study multitransonic BH accretion using the full general relativistic framework and argue which potential is expected to be the closest approximation of actual general relativistic solutions for which regions of parameter space spanned by \( \delta, \lambda, \) and \( \gamma \), as long as one concentrates only on shocked flows. Finally, in \( \S \) 5 we draw our conclusion by highlighting some of the possible important impacts of the study of shock formation on related fields.

2. PROPERTIES OF FOUR PSEUDO-SCHWARZSCHILD BLACK HOLE POTENTIALS

Rigorous investigation of the complete general relativistic multitransonic BH accretion disk structure is extremely complicated. At the same time, it is understood that, since relativistic effects play an important role in the regions close to the accreting black hole (where most of the gravitational potential energy is released), a purely Newtonian gravitational potential [in the form \( \Phi_N(r) = -1/r \) in the system of units used here] cannot be a realistic choice to describe transonic black hole accretion in general. To compromise between the ease of handling of a Newtonian description of gravity and the realistic situations described by complicated general relativistic calculations, a series of "modified" Newtonian potentials have been introduced to describe the general relativistic effects that are most important for accretion disk structure around Schwarzschild and Kerr black holes (see ABN for further discussion). Introduction of such potentials allows one to investigate the complicated physical processes taking place in disk accretion in a semi-Newtonian framework by avoiding pure general relativistic calculations, so that most of the features of spacetime around a compact object are retained and some crucial properties of the analogous relativistic solutions of disk structure can be reproduced with high accuracy. Hence, those potentials might be designated as "pseudo-Kerr" or "pseudo-Schwarzschild" potentials, depending on whether they are used to mimic the spacetime around a rapidly rotating or nonrotating/slowly rotating (Kerr parameter \( a \sim 0 \)) black hole, respectively. Below we describe four such pseudo-Schwarzschild potentials on which we concentrate in this paper. It is important to note that as long as one is not interested in astrophysical processes extremely close (within \( r_g \sim 2r_g \)) to a black hole horizon, one may safely use the following BH potentials to study accretion onto a Schwarzschild black hole with the advantage that the use of these potentials would simplify calculations by allowing one to use some basic features of flat geometry (additivity of energy or decoupling of various energy components, i.e., thermal \([a^2/(\gamma - 1)], \) kinetic \([u^2/2], \) or gravitational [\( \Phi, \) etc.; see subsequent discussions], which is not possible for calculations in a purely Schwarzschild metric (see \( \S \) 4). Also, one can study more complex many-body problems, such as accretion from an ensemble of donors, or overall efficiency of accretion onto an ensemble of black holes in a galaxy, or numerical hydrodynamic or magnetohydrodynamic accretion flows around a black hole, as simply as can be done in a Newtonian framework but with far better accuracy. So we believe that a comparative study of multitransonic accretion flow as well as shock formation using all these potentials might be quite useful in understanding some important features of various shock-related astrophysical phenomena, at least until one can have a complete and self-consistent theory of complete general relativistic shock formation exclusively for a Schwarzschild BH. However, one should be careful in using these potentials because none of the potentials discussed here are "exact" in the sense that they are not directly derivable from the Einstein equations. These potentials could be used only to obtain more accurate correction terms over and above the purely Newtonian results, and any "radically" new results obtained using these potentials should be cross-checked very carefully with the exact general relativistic theory.

Paczynski & Wiita (1980) proposed a pseudo-Schwarzschild potential of the form

\[
\Phi_1(r) = -\frac{1}{2(r-1)},
\]

which accurately reproduces the positions of \( r_s \) and \( r_h \) and gives the value of efficiency to be \(-0.0625\), which is in closest agreement with the value obtained in full general relativistic calculations. Also, the Keplerian distribution of angular momentum obtained using this potential is exactly the same as that obtained in pure Schwarzschild geometry. It is worth mentioning here that this potential was first introduced to study a thick accretion disk with super-Eddington luminosity. Also, it is interesting to note that although it had been thought of in terms of disk accretion, \( \Phi_1(r) \) is spherically symmetric with a scale shift of \( r_g \).

To analyze the normal modes of acoustic oscillations within a thin accretion disk around a compact object (slowly rotating black hole or weakly magnetized neutron star), Nowak & Wagoner (1991) approximated some of the dominant relativistic effects of the accreting black hole (slowly rotating or nonrotating) via a modified Newtonian potential of the form

\[
\Phi_2(r) = -\frac{1}{2r} \left[ 1 - \frac{3}{2r} + 12 \left( \frac{1}{2r} \right)^2 \right];
\]

\( \Phi_2(r) \) has the correct form of \( r_s \) as in the Schwarzschild metric but is unable to reproduce the value of \( r_h \). This potential has the correct general relativistic value of the angular velocity \( \Omega_s \) at \( r_s \). Also, it reproduces the radial epicyclic frequency \( \kappa_s \) (for \( r > r_s \)) close to its value obtained from general relativistic calculations, and among all BH potentials, \( \Phi_2(r) \) provides the best approximation for \( \Omega_s \) and \( \kappa_s \). However, this potential gives the value of efficiency as \(-0.064\), which is larger than that produced by \( \Phi_1(r) \); hence, the disk spectrum computed using \( \Phi_2(r) \) would be more luminous compared to a disk structure studied using \( \Phi_1(r) \).

Considering the fact that free-fall acceleration plays a very crucial role in Newtonian gravity, ABN proposed two
different BH potentials to study disk accretion around a nonrotating black hole. The first potential proposed by them produces exactly the same value of the free-fall acceleration of a test particle at a given value of \( r \) as is obtained for a test particle at rest with respect to the Schwarzschild reference frame and is given by

\[
\Phi_3(r) = -1 + \left(1 - \frac{1}{r}\right)^{1/2}. \tag{1c}
\]

The second one gives a value of free-fall acceleration that is equal to the value of the covariant component of the three dimensional free-fall acceleration vector of a test particle that is at rest in the Schwarzschild reference frame and is given by

\[
\Phi_4(r) = \frac{1}{2} \ln \left(1 - \frac{1}{r}\right). \tag{1d}
\]

Efficiencies produced by \( \Phi_3(r) \) and \( \Phi_4(r) \) are -0.081 and -0.078, respectively. The magnitude of efficiency produced by \( \Phi_3(r) \) being maximum, calculation of disk structure using \( \Phi_3(r) \) will give the maximum amount of energy dissipation, and the corresponding spectrum would be the most luminous one. Hereafter, we will refer to all these four potentials by \( \Phi_1(r) \) in general, where \( i = 1, 2, 3, 4 \) would correspond to \( \Phi_1 \) (eq. [1a]), \( \Phi_2 \) (eq. [1b]), \( \Phi_3 \) (eq. [1c]), and \( \Phi_4 \) (eq. [1d]), respectively. One should notice that while all other \( \Phi_i(r) \) have singularity at \( r = r_s \), only \( \Phi_3(r) \) has a singularity at \( r = 0 \). It can be shown that for \( r > r_s \), while \( \Phi_3 (r) \) is flatter compared to purely Newtonian potential \( \Phi_3(r) \), all other \( \Phi_i(r) \) are steeper to \( \Phi_3(r) \).

At any radial distance \( r \) measured from the accretor, one can define the effective potential \( \Phi_i(r) \) to be the summation of the gravitational potential and the centrifugal potential for matter accreting under the influence of the \( i \)th pseudopotential. The effective potential \( \Phi_i(r) \) can be expressed as

\[
\Phi_{\text{eff}}(r) = \Phi_i(r) + \frac{\lambda_i^2}{2r^2}, \tag{2a}
\]

where \( \lambda_i(r) \) is the nonconstant distance dependent specific angular momentum of accreting material. One then easily shows that \( \lambda_i(r) \) may have an upper limit

\[
\lambda_i^{\text{up}}(r) = r^{3/2} \sqrt{\Phi_i'(r)}, \tag{2b}
\]

where \( \Phi_i'(r) \) represents the derivative of \( \Phi_i(r) \) with respect to \( r \). For weakly viscous or inviscid flow, angular momentum can be taken as a constant parameter (\( \lambda \)), and equation (2a) can be approximated as

\[
\Phi_{\text{eff}}(r) = \Phi_i(r) + \frac{\lambda^2}{2r^2}. \tag{2c}
\]

For general relativistic treatment of accretion, the effective potential cannot be decoupled into its gravitational and centrifugal components. For a Schwarzschild metric of the form

\[
\text{ds}^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{1}{r})dt^2 + (1 - \frac{1}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

the world line of the accreting fluid is timelike, and the four-velocity of the fluid satisfies the normalization condition

\[
u_\mu u^\mu = -1,
\]

where \( u^\mu(u_\mu) \) is the contra(co)variant four-velocity of the fluid. The angular velocity \( \Omega(r) \) of the fluid can be computed as

\[
\Omega(r) = \frac{u^\theta}{u^t} = -\frac{\lambda g_{\theta t}}{g_{tt}} = \frac{\lambda(r-1)}{2r},
\]

where \( \lambda = -u_\phi/u_t \) is the specific angular momentum that is conserved for fluid dynamics as well as for particle dynamics for inviscid flow. The general relativistic effective potential \( \Phi_{\text{eff}}^\text{GR}(r) \) (excluding the rest mass) experienced by the fluid accreting onto a Schwarzschild BH can be expressed as

\[
\Phi_{\text{eff}}^\text{GR}(r) = r \sqrt{\frac{r-1}{r^3 - \lambda^2(1+r)}} - 1. \tag{2d}
\]

One can understand that the effective potentials in general relativity cannot be obtained by linearly combining its gravitational and rotational contributions because various energies in general relativity are combined together to produce nonlinearly coupled new terms.

In Figure 1, we plot \( \Phi_{\text{eff}}^\text{GR}(r) \) (obtained from eq. [2c]) and \( \Phi_{\text{eff}}^\text{GR}(r) \) as a function of \( r \) in logarithmic scale. The value of \( \lambda \) is taken to be 2 in units of \( 2\text{GM}/c^2 \). The \( \Phi_{\text{eff}}^\text{GR} \) curves for different \( \Phi_i(r) \) are marked exclusively in the figure, and the curve marked by \( \text{GR} \) represents the variation of \( \Phi_{\text{eff}}^\text{GR}(r) \) with \( r \). One can observe that \( \Phi_{\text{eff}}^\text{GR}(r) \) is in excellent agreement with \( \Phi_{\text{eff}}^\text{GR}(r) \); only for a very small value of \( r \) \( (r \to r_s) \), \( \Phi_{\text{eff}}^\text{GR}(r) \) starts deviating from \( \Phi_{\text{eff}}^\text{GR}(r) \), and this deviation keeps increasing as matter approaches closer and closer to the event horizon. All other \( \Phi_i(r) \) approach \( \Phi_{\text{eff}}^\text{GR}(r) \) at a radial distance (measured from the BH) considerably larger compared to the case for \( \Phi_1(r) \). If one defines \( \Delta_{\text{eff}}(r) \) to be the measure of the deviation of \( \Phi_{\text{eff}}^\text{GR}(r) \) with \( \Phi_{\text{eff}}^\text{GR}(r) \) at any point \( r \),

\[
\Delta_{\text{eff}}(r) = \Phi_{\text{eff}}^\text{GR}(r) - \Phi_{\text{eff}}^\text{GR}(r),
\]

\[\text{FIG. 1.--Effective BH potentials for general relativistic [\( \Phi_{\text{eff}}^\text{GR}(r) \)] as well as for pseudo--general relativistic [\( \Phi_{\text{eff}}^\text{GR}(r) \)] accretion disks as a function of the distance (measured from the event horizon in units of \( r_s \)) plotted in logarithmic scale. The specific angular momentum is chosen to be 2 in geometric units. See text for details.}\]
one observes that $\Delta_i^{\text{eff}}(r)$ is always negative for $\Phi_i^{\text{eff}}(r)$, but for other $\Phi_i^{\text{eff}}(r)$, it normally remains positive for low values of $\lambda$ but may become negative for a very high value of $\lambda$. If $|\Delta_i^{\text{eff}}(r)|$ be the modulus or the absolute value of $\Delta_i^{\text{eff}}(r)$, one can also see that, although only for a very small range of radial distance very close to the event horizon, $\Delta_i^{\text{eff}}(r)$ is maximum for the whole range of the distance scale while $\Phi_1(r)$ is the best approximation of general relativistic spacetime, $\Phi_2(r)$ is the worst approximation, and $\Phi_3(r)$ and $\Phi_3(r)$ are the second- and the third-best approximations as far as the total effective potential experienced by the accreting fluid is concerned. It can be shown that $|\Delta_i^{\text{eff}}(r)|$ nonlinearly anticorrelates with $\lambda$. The reason behind this is understandable. As $\lambda$ decreases, rotational mass as well as its coupling term with gravitational mass decreases for general relativistic accretion material, while for accretion in any $\Phi_i(r)$, centrifugal force becomes weak and gravity dominates; hence, deviation from the general relativistic case will be more prominent because general relativity is basically a manifestation of strong gravity close to the compact objects.

From the figure it is clear that for $\Phi_i^{\text{GR}}(r)$ as well as for all $\Phi_i^{\text{eff}}(r)$, a peak appears close to the horizon. The height of these peaks may roughly be considered as the measure of the strength of the centrifugal barrier encountered by the accreting material for respective cases. The deliberate use of the word “roughly” instead of “exactly” is due to the fact that here we are dealing with fluid accretion, and unlike particle dynamics, the distance at which the strength of the centrifugal barrier is maximum is located further away from the peak of the effective potential because here the total pressure contains the contribution due to fluid or “ram” pressure also. Naturally, the peak height for $\Phi_i^{\text{GR}}(r)$ as well as for $\Phi_i^{\text{eff}}(r)$ increases with increase of $\lambda$, and the location of this barrier moves away from the BH with higher values of angular momentum. If the specific angular momentum of accreting material lies between the marginally bound and marginally stable value, an accretion disk is formed. For inviscid or weakly viscous flow, the higher the value of $\lambda$, the higher the strength of the centrifugal barrier and the more the amount of radial velocity or the thermal energy that the accreting material must have to begin with so that it can be made to accrete onto the BH. In this connection, it is important to observe from the figure that accretion under $\Phi_2(r)$ will encounter a centrifugal barrier farthest away from the BH compared to other $\Phi_i(r)$. For accretion under all $\Phi_i(r)$ except $\Phi_1(r)$, the strength of the centrifugal barrier at a particular distance will be more compared to its value for full general relativistic accretion.

3. MULTITRANSONIC FLOW IN VARIOUS BLACK HOLE POTENTIALS AND SHOCK FORMATION

Following the standard literature, we consider a thin, rotating, axisymmetric, inviscid steady flow in hydrostatic equilibrium in the transverse direction. The assumption of hydrostatic equilibrium is justified for a thin flow because for such flows the infall timescale is expected to exceed the local sound crossing timescale in the direction transverse to the flow. The flow is also assumed to possess a considerably large radial velocity that makes the flow “advective.” The complete solutions of such a system require the dimensionless equations for conserved specific energy $\epsilon$ and angular momentum $\lambda$ of the accreting material, the mass conservation equations supplied by the transonic conditions at the sonic points, and the Rankine-Hugoniot conditions at the shock. The local half-thickness, $h_i(r)$, of the disk for any $\Phi_i(r)$ can be obtained by balancing the gravitational force by the pressure gradient and can be expressed as

$$h_i(r) = a\sqrt{r/(\gamma \Phi_i')},$$

where $\Phi_i' = d\Phi_i(r)/dr$. For a nonviscous flow obeying the polytropic equation of state $p = K\rho^n$ ($K$ is a measure of the specific entropy of the flow), integration of the radial momentum equation

$$u \frac{du}{dr} + \frac{1}{\rho} \frac{dP}{dr} + \frac{d}{dr} \left[ \Phi_i^{\text{eff}}(r) \right] = 0,$$

leads to the following energy conservation equation in steady state:

$$\epsilon = \frac{1}{2} u_0^2 + \frac{a_0^2}{\gamma - 1} + \frac{\lambda^2}{2r^2} + \Phi_i(r) = 0, \quad (3a)$$

and the continuity equation,

$$\frac{d}{dr} [u\rho h_i(r)] = 0,$$

can be integrated to obtain the barion number conservation equation:

$$\dot{M}_m = \sqrt{\frac{1}{\gamma} u_0 a_0 r^{3/2} (\Phi_i')^{-1/2}}. \quad (3b)$$

Following Chakrabarti (1989), one can define the entropy accretion rate $\dot{\mathcal{M}} = M_m K^{1/(\gamma - 1)} \gamma^{1/(\gamma - 1)}$, which undergoes a discontinuous transition at the shock location $r_{sh}$, where local turbulence generates entropy to increase $\dot{\mathcal{M}}$ for post-shock flows. For our purpose, the explicit expression for $\dot{\mathcal{M}}$ can be obtained as

$$\dot{\mathcal{M}} = \sqrt{\frac{1}{\gamma} u_0 a_0 r^{3/2} (\Phi_i')^{-1/2}}. \quad (3c)$$

In equations (3a)–(3c), the subscript $e$ indicates the values measured on the equatorial plane of the disk; however, we will drop $e$ hereafter if no confusion arises in doing so. One can simultaneously solve equations (3a)–(3c) for any particular $\Phi_i(r)$ and for a particular set of values of $\{\epsilon, \lambda, \gamma\}$. Hereafter, we will use the notation $[\mathcal{P}_i]$ for a set of values of $\{\epsilon, \lambda, \gamma\}$ for any particular $\Phi_i$.

For a particular value of $[\mathcal{P}_i]$, it is now quite straightforward to derive the space gradient of dynamical flow velocity $(du/dr)$ for flow in any particular $r$th BH potential $\Phi_i(r)$:

$$\frac{du}{dr} = \left[ \frac{\lambda^2}{r^3} + \Phi_i(r) - \frac{u^2}{(\gamma + 1)^2} \right] \left[ \frac{3}{r} + \Phi_i'(r)/\Phi_i'(r) \right]$$

where $\Phi_i'$ represents the derivative of $\Phi_i$. Since the flow is assumed to be smooth everywhere, if the denominator of equation (4a) vanishes at any radial distance $r$, the numerator must also vanish there to maintain the continuity of the flow. One therefore arrives at the so-called sonic-point
(alternately, the “critical-point”) conditions by simultaneously making the numerator and denominator of equation (4a) equal zero. The sonic-point conditions can be expressed as

\[ a_s' = \sqrt{\frac{1 + \gamma}{2}} a_s = \left[ \frac{\Phi'(r) + \gamma \Phi(r)}{r^2} \right] \frac{\lambda^2 + r^3 \Phi'(r)}{3 \Phi'(r) + r \Phi''(r)} , \]

where the subscript \( s \) indicates that the quantities are to be measured at the sonic point(s). For a fixed \( [\mathcal{P}] \), one can solve the following polynomial of \( r \) to obtain the sonic point(s) of the flow:

\[ \mathcal{E} = \left[ \frac{\lambda^2}{2r^2} + \Phi_i(r) \right] , \]

\[ -2 \frac{\gamma}{\gamma^2 - 1} \left[ \frac{\Phi'(r) + \gamma \Phi(r)}{r^2} \right] \frac{\lambda^2 + r^3 \Phi'(r)}{3 \Phi'(r) + r \Phi''(r)} = 0 . \]

Similarly, the value of \( \frac{d u}{d r} \), at its corresponding sonic point(s) can be obtained by solving the following equation:

\[ \frac{4 \gamma}{\gamma + 1} \left( \frac{d u}{d r} \right)^2_{s, i} - 2 u_s \frac{\gamma - 1}{\gamma + 1} \left[ \frac{\Phi_i'(r)}{\Phi_i(r)} \right]_{s, i} \left( \frac{d u}{d r} \right)_{s, i} + a_s^2 \left\{ \frac{\Phi_i''(r)}{\Phi_i'(r)} - \frac{2 \gamma}{(1 + \gamma)^2} \left[ \frac{\Phi_i'(r)}{\Phi_i(r)} \right]^2 + \frac{6 (\gamma - 1) \Phi_i'(r)}{(\gamma + 1)^2 \Phi_i(r)} \right. \]

\[ - \left. \frac{6 (2 \gamma - 1)}{\gamma^2 (\gamma + 1)^2} \right]_{s, i} + \frac{\Phi_{in}'}{\Phi_{in}} - \frac{3 \lambda^2}{r^2} = 0 , \]

where the subscript \( (s, i) \) indicates that the corresponding quantities for any \( i \)th potential are being measured at its corresponding sonic point(s), and \( \Phi_{in}'(r) = d^2 \Phi_i(r)/d r^3 \).

For all \( \Phi_i(r) \), we find a significant region of parameter space spanned by \( [\mathcal{P}] \) that allows the multiplicity of sonic points for accretion as well as for wind where two real physical inner and outer (with respect to the BH location) \( X \)-type sonic points \( r_{in} \) and \( r_{out} \) encompass one \( O \)-type unphysical middle sonic point \( r_{mid} \) in between. For a particular \( \Phi_i(r) \), if \( \mathcal{C}_i[\mathcal{P}] \) denotes the universal set representing the entire parameter space covering all values of \( [\mathcal{P}] \) and if \( \mathcal{B}_i[\mathcal{P}] \) represents one particular subset of \( \mathcal{C}_i[\mathcal{P}] \) that contains only the particular values of \( [\mathcal{P}] \) for which the above mentioned three sonic points are obtained, then \( \mathcal{B}_i[\mathcal{P}] \) can further be decomposed into two subsets \( \mathcal{C}_i^1[\mathcal{P}] \) and \( \mathcal{B}_i^1[\mathcal{P}] \) such that

\[ \mathcal{C}_i[\mathcal{P}] \subseteq \mathcal{B}_i[\mathcal{P}] \] for all \( \Phi_i(r) \) (marked in the figure) when \( \gamma = 4/3 \). While the specific energy \( \mathcal{E} \) is plotted along the \( Y \)-axis, the specific angular momentum \( \lambda \) is plotted along the \( X \)-axis. For \( \Phi_1(r) \), the shaded region PQR represents the parameter space spanned by \( \mathcal{E} \) and \( \lambda \) for which three sonic points will form in accretion \( (\mathcal{P} \equiv \mathcal{C}_1[\mathcal{P}]) \), while the wedge-shaped unshaded region PSR represents the parameter space for which three sonic points are formed in wind \( (\mathcal{P} \equiv \mathcal{D}_1[\mathcal{P}]) \). A similar kind of parameter-space division is shown for other \( \Phi_i(r) \) as well. A careful analysis of Figure 2 reveals the fact that, at least for ultrarelativistic flow, no region of parameter space common to all \( \Phi_i(r) \) is found for which \( [\mathcal{P}] \in \mathcal{C}_i[\mathcal{P}] \) or \( [\mathcal{P}] \in \mathcal{D}_i[\mathcal{P}] \) for \( \Phi_2(r) \) and \( \Phi_3(r) \), and a very small region of such a common zone in the parameter space is obtained (only for extremely low values of the energy and angular momentum of the accreting matter) for \( \Phi_2(r) \), \( \Phi_3(r) \), and \( \Phi_4(r) \). As the flow approaches its purely nonrelativistic limit, i.e., as we make \( \gamma \rightarrow 5/3 \), the tendency for such a mutual overlap of parameter space for \( \Phi_2(r) \), \( \Phi_3(r) \), and \( \Phi_4(r) \) increases. Nevertheless, \( \Phi_1(r) \) still remains “untouchable” by \( \Phi_2(r) \) and \( \Phi_3(r) \); only a particular region of parameter space (fairly low energy accretion with intermediate value of angular momentum) is commonly shared by \( \Phi_2(r) \) and \( \Phi_3(r) \).

One also observes that if \( \mathcal{E}_i^{max} \) and \( \lambda_i^{max} \) are the maximum available energy and angular momentum of the flow for any \( \Phi_i(r) \) for which \( [\mathcal{P}] \in \mathcal{C}_i[\mathcal{P}] \) or \( [\mathcal{P}] \in \mathcal{D}_i[\mathcal{P}] \), one can write

\[ \mathcal{E}_3^{max} > \mathcal{E}_4^{max} > \mathcal{E}_1^{max} > \mathcal{E}_2^{max} , \]

\[ \lambda_1^{max} > \lambda_2^{max} > \lambda_3^{max} > \lambda_4^{max} . \]

The above trend remains unaltered as \( \gamma \rightarrow 5/3 \), and we observe that both \( \mathcal{E}_i^{max} \) and \( \lambda_i^{max} \) nonlinearly anticorrelate with \( \gamma \).

If shock forms in accretion (in this work we do not study shock formation in wind), then \( [\mathcal{P}] \) responsible for shock formation must be somewhere from the region for which \( [\mathcal{P}] \in \mathcal{C}_i[\mathcal{P}] \), although not all \( [\mathcal{P}] \in \mathcal{C}_i[\mathcal{P}] \) will allow shock transition. Using equations (3a)–(3c), we combine the three standard Rankine-Hugoniot conditions (Landau & Lifshitz 1959) for vertically integrated pressure and density (see Matsumoto et al. 1984) to derive the following relation,
which is valid only at the shock location:

\[
(1 - \gamma) \left( \frac{\rho - \dot{\rho}}{\dot{\rho}} \right) \log \frac{E_{(i+th)}}{M} - \Theta (1 + \Theta - R_{\text{comp}})^{-1} + (1 + \Theta)^{-1} = 0 ,
\]

where \(E_{(i+th)}\) is the total specific thermal plus mechanical energy of the accreting fluid: \(E_{(i+th)} = \{ E - \langle \frac{1}{2}\eta^{2} + \Phi_{i} \rangle \}; R_{\text{comp}}\) and \(\beta\) are the density compression and entropy enhancement ratios, respectively, defined as \(R_{\text{comp}} = (\rho/\dot{\rho})\) and \(\beta = (M_{i}/M_{i})/\Theta = 1 - \Gamma (1 - \gamma)\) and \(\Gamma = \beta R_{\text{comp}}\), and “+” and “−” refer to the postshock and preshock quantities. The shock strength \(\mathcal{S}_{i}\) (ratio of the preshock to postshock Mach number of the flow) can be calculated as

\[
\mathcal{S}_{i} = R_{\text{comp}} (1 + \Theta).
\]

Equations (5) and (6) cannot be solved analytically because they are nonlinearly coupled. However, we have been able to simultaneously solve equations (3a)–(6) using iterative numerical techniques. We have developed an efficient numerical code that takes \(\mathcal{S}_{i}\) and \(\Phi_{i}\) as its input and can calculate \(r_{\text{sh}}\) along with any sonic or shock quantity as a function of \(\mathcal{S}_{i}\). It is to be noted that like the references cited in §1, we also obtain multiplicity in the shock location. We perform the local stability analysis and find that only one \(r_{\text{sh}}\) that forms in between \(r_{\text{out}}\) and \(r_{\text{mid}}\) is stable for all \(\Phi_{i}(r)\).

Let \([\mathcal{P}_{i}] \in \mathcal{F}_{i}[\mathcal{P}] \subseteq \mathcal{E}_{i}[\mathcal{P}]\) represent the region of parameter space for which a multitransonic supersonic flow is expected to encounter an RHS at \(r_{\text{sh}}\). At the shock, the supersonic flow becomes hotter, shock compressed, and subsonic. This subsonic subsonic flow will become supersonic again after passing through \(r_{\text{m}}\) and will ultimately cross the event horizon. For multitransonic flows with initial boundary conditions different from those discussed above, one can define \([\mathcal{P}_{i}] \in \mathcal{G}_{i}[\mathcal{P}_{i}]\), which is complement of \(\mathcal{F}_{i}[\mathcal{P}_{i}]\) related to \(\mathcal{E}_{i}[\mathcal{P}_{i}]\), so that for

\[
\{ \mathcal{F}_{i}[\mathcal{P}_{i}], [\mathcal{P}_{i}] \in \mathcal{E}_{i}[\mathcal{P}_{i}], [\mathcal{P}_{i}] \notin \mathcal{F}_{i}[\mathcal{P}_{i}] \} ,
\]

the shock location becomes imaginary in \(\mathcal{G}_{i}[\mathcal{P}_{i}]\); hence, no stable RHS forms in that region; rather, the shock keeps oscillating back and forth. We anticipate that \(\mathcal{G}_{i}[\mathcal{P}_{i}]\) is also an important zone that might be responsible for the quasi-periodic oscillation (QPO) of the BH candidates (see see §5).

Figure 3 demonstrates few typical flow topologies of the integral curves of motion for ultrarelativistic (\(\gamma = 4/3\)) shocked flows in various \(\Phi_{i}\) (indicated in the figure). While the distance from the event horizon of the central BH (scaled in units of \(r_{g}\) and plotted in logarithmic scale) is plotted along the \(X\)-axis, the local Mach number of the flow is plotted along the \(Y\)-axis. One can easily obtain such a set of figures for any \(\gamma\) and \(\{\mathcal{P}_{i}\}\) that allows shock formation. For all figures, ABCD represents the transonic accretion passing through the outer sonic point \(r_{\text{out}}\) (marked as B) if a shock would not form. However, since \(\mathcal{M}\) of the flow is higher at the inner sonic point \(r_{\text{in}}\) compared to \(\mathcal{M}\) at \(r_{\text{out}}\), the flow must encounter a shock at C (the vertical line CE marked by an arrowhead represents the shock transition), becomes subsonic, and jumps on the branch EF, which ultimately hits the event horizon supersonically after it passes through the inner sonic point \(r_{\text{in}}\), which is marked on EF by the small circle with a dot at the center. An asterisk in the figure indicates the location of the middle sonic point \(r_{\text{mid}}\). The corresponding values of \(r_{\text{in}}, r_{\text{mid}}, r_{\text{out}}\), the shock location \(r_{\text{sh}}\), and the shock strength \(S_{i}\) are indicated at the top of each figure, while the corresponding values of the total specific energy \(E\) and angular momentum \(\lambda\) for which the solutions are obtained are indicated inside each figure. GBH represents the “self-wind” of the flow, which, in the course of its motion away from the BH to infinity, becomes supersonic after passing through \(r_{\text{out}}\) at B. Collectively, ABCEF represents the real physical shocked accretion that connects infinity with the event horizon. The overall scheme for obtaining the above mentioned integral curves is as follows: First we compute \(r_{\text{in}}, r_{\text{mid}}, r_{\text{out}}\) by solving equation (4c). Then we obtain the dynamical velocity gradient of the flow at sonic points by solving equation (4d). For a chosen \(M_{\text{in}}\) (scaled in the units of the Eddington rate \(M_{\text{Edd}}\)), we then compute the local dynamical flow velocity \(u(r)\), the local polytropic sound speed \(a(r)\), the local radial Mach number \(M(r)\), the local fluid density \(\rho(r)\), and any other related dynamical or thermodynamic quantities by solving equations (4a)–(4d) from the outer sonic point using the fourth-order Runge-Kutta method. We start integrating from \(r_{\text{out}}\) in two different directions. Along BH, we solve only for \(u(r), a(r), \) and \(M(r)\) because shock does not form in subsonic flows. However, integration along BCD involves a different procedure. Along BCD, we not only compute \(u(r), a(r), \) and \(M(r)\) but also, at every integration step (with as small a step size as possible), we keep checking whether equation (5) is being satisfied at that point. To do so, at each and every point, we start with a suitable initial guess value of \(R_{\text{comp}}\) and \(\mathcal{S}_{i}\) and performs millions of iterations to check whether for any set of \([R_{\text{comp}}, \mathcal{S}_{i}]\) equation (5) is satisfied at that point and whether for such \([R_{\text{comp}}, \mathcal{S}_{i}]\), the value of \(\beta\) obtained from equation (5) becomes exactly equal to \(\mathcal{M}(r_{\text{in}}) / \mathcal{M}(r_{\text{out}})\), in other words, whether the entropy generated at that point (if any) becomes exactly equal to the difference between the entropies at the inner and the outer sonic points. If such conditions are satisfied at some particular point (point C in the figure), we argue that the shock forms at that point, and we can calculate any preshock and the postshock dynamical and thermodynamic quantities at the shock location \(r_{\text{sh}}\) (i.e., at C). Once a shock is formed, the flow jumps from its supersonic branch BCD to its subsonic branch EG. We again start calculating \(u(r), a(r), \) and \(M(r)\) and any other related flow quantities by solving equation (4a) using the fourth-order Runge-Kutta method (with the help of eqs. [3a]–[3c] and eq. [4d]), but this time from the inner sonic point \(r_{\text{in}}\) of the flow.

In Figure 4, we present the \(\mathcal{F}_{i}[\mathcal{P}_{i}]\) for all four \(\Phi_{i}\) (Fig. 4a), \(\Phi_{2}\) (Fig. 4b), \(\Phi_{3}\) (Fig. 4c), and \(\Phi_{4}\) (Fig. 4d). The specific energy \(\epsilon\), specific angular momentum \(\lambda\), and the polytropic index \(\gamma\) of the flow are plotted along the \(Z\), \(Y\), and \(X\)-axes, respectively. Each surface for a particular \(\Phi_{i}(r)\) is drawn for a particular value of \(\gamma\). While the first surfaces (which have the maximum surface areas) on the \(\epsilon-\gamma\) plane represent ultrarelativistic accretion (\(\gamma = 4/3\)), successive surfaces are also shown for higher values of \(\gamma\), taking a regular interval of \(\Delta \gamma = 0.025\). It is observed that as the flow approaches its purely nonrelativistic limit, the area of the \(\epsilon-\gamma\) surfaces responsible for shock formation starts shrinking. We find that the shock location correlates with \(\lambda\). This is obvious because the higher the flow angular momentum, the greater the rotational energy content of the flow and the higher the strength of the centrifugal barrier (which is responsible for...
breaking the incoming flow by forming a shock) as well as the further the location of such barrier from the event horizon. However, \( r_{sh} \) anticorrelates with \( \delta \) and \( \gamma \), which means that for the same \( \delta \) and \( \lambda \), shock in the purely nonrelativistic flow will form closer to the event horizon compared to the ultrarelativistic flow. We also observe that the shock strength \( \mathcal{S} \) nonlinearly anticorrelates with the shock location \( r_{sh} \), which indicates that the closer the shock forms to the BH, the higher is the strength \( \mathcal{S} \) and the entropy enhancement ratio \( \mathcal{C}_{12} \). The ultrarelativistic flows are supposed to produce the strongest shocks. The reason behind this is also easy to understand: the closer the shock forms to the event horizon, the higher the available gravitational potential energy to be released and the higher the radial advective velocity required to have a more vigorous shock jump. Compared to \( \Phi_2 \) and \( \Phi_3 \), \( \Phi_1 \) and \( \Phi_4 \) allow wider spans of \( \gamma \) as well as \( \lambda \) for shock formation. If \( \delta_{\text{max}}, \lambda_{\text{max}}, \) and \( \gamma_{\text{max}} \) represent the maximum values of the corresponding parameters for which shock formation is possible, we obtain \( \delta_{\text{max}}(\Phi_1) > \delta_{\text{max}}(\Phi_4) > \delta_{\text{max}}(\Phi_3) > \delta_{\text{max}}(\Phi_2), \lambda_{\text{max}}(\Phi_1) > \lambda_{\text{max}}(\Phi_4) > \lambda_{\text{max}}(\Phi_3) > \lambda_{\text{max}}(\Phi_2), \) and \( \gamma_{\text{max}}(\Phi_4) > \gamma_{\text{max}}(\Phi_1) > \gamma_{\text{max}}(\Phi_3) > \gamma_{\text{max}}(\Phi_2) \). Also, we observe that as the flow approaches its purely nonrelativistic limit more and more, shock may form for less and less angular momentum. For some \( \Phi_i(r) \), even a very small amount of angular momentum (\( \lambda < 1 \)) allows shock formation, which indicates that for purely nonrelativistic accretion, shock formation may take place even for quasi-spherical flow.

4. GENERAL RELATIVISTIC MULTITRANSONIC ACCRETION

Following the arguments provided by Novikov & Thorne (1973) and Chakrabarti (1996b), we derive the expressions
for the conserved total specific energy \( \dot{\epsilon}' \) (which includes the rest mass energy) and the entropy accretion rate \( \dot{S} \) as

\[
\dot{\epsilon}' = \frac{\gamma (\gamma - 1)}{\gamma - (1 + \alpha^2)} \sqrt{\frac{r-1}{1-r^2}} \left[ \lambda^2 (1 - \gamma) \right]^{-1/2}, \quad (7a)
\]

\[
\dot{S} = 5.657w^{1.25} \sqrt{\frac{r-1}{1-r^2}}
\times \left[ \frac{a^2 (\gamma - 1)}{\gamma - (1 + a^2)} \right]^{(\gamma+1)/2(\gamma-1)} \left[ \lambda^2 (1 - \gamma) \right]^{0.25}. \quad (7b)
\]

One can see from equation (7a) that the total specific energy, in this case, cannot be decoupled into various linearly additive contributions of separate physical origin (i.e., kinetic, thermal, rotational, or gravitational) as could be done for flows in any pseudopotential.

Following the procedure outlined in previous section, one can derive the dynamical flow velocity gradient for general relativistic accretion flow as

\[
\frac{du}{dr} = \left( \frac{2r}{2r} \left[ \frac{2r^3 - \lambda^2}{r^3 + \lambda^2 (1 - \gamma)} \right] - \frac{2r - 1}{2r(r - 1)} \right)
\times \left[ \frac{2a^2}{u(u^2 - 1)(\gamma + 1) + \frac{u}{1 - u^2}} \right]^{-1}, \quad (8a)
\]
from which the sonic point conditions comes out to be

\[ u_s = \frac{2}{\gamma + 1} \sqrt{2 - \frac{\gamma + 1}{2}} \times \left( \frac{1}{2r} \left[ \frac{2r^3 - \lambda^2}{r^3 + \lambda^2 (1 - \gamma)} - \frac{2r - 1}{2(r - 1)} \right] \right. \]

\[ \times \left. \left\{ \frac{5 - 7r_s}{4r(r - 1)} + \frac{\lambda^2 - 3r^2}{4(r^3 + \lambda^2 (1 - r))} \right\} \right) \cdot \quad (8b) \]

The sonic point(s) could be computed by solving the following equation:

\[ \delta^2 \left[ r_s^2 + \lambda^2 (1 - r_s) \right] - \frac{r_s - 1}{1 - \Psi(r_s, \lambda)} \left[ \frac{\gamma (\gamma - 1)}{\gamma - \eta(r_s, \lambda)} \right]^2 = 0, \quad (8c) \]

where

\[ \eta(r_s, \lambda) = 1 + \frac{\gamma + 1}{2} \Psi(r_s, \lambda), \]

\[ \Psi(r_s, \lambda) = \left\{ \frac{1}{2r} \left[ \frac{2r^3 - \lambda^2}{r^3 + \lambda^2 (1 - \gamma)} - \frac{2r - 1}{2(r - 1)} \right] \right. \]

\[ \times \left. \left\{ \frac{5 - 7r_s}{4r} + \frac{\lambda^2 - 3r^2}{4(r^3 + \lambda^2 (1 - r))} \right\} \right) \cdot \]

The dynamical flow velocity gradient at the sonic point(s) can be obtained by solving the following equation:

\[ \frac{2(2\gamma - 3a_s^2)}{(\gamma + 1)(u_s^2 - 1)} \left( \frac{du}{dr} \right)_s + 4\xi(r_s, \lambda) \left[ \frac{\gamma - (1 + a_s^2)}{u_s^2 - 1} \right] \left( \frac{du}{dr} \right)_s \]

\[ + \frac{2}{\gamma + 1} a_s^2 + 2 \xi(r_s, \lambda) \left[ 2 \xi(r_s, \lambda) \frac{\gamma - (1 + a_s^2)}{\gamma + 1} \right. \]

\[ - \frac{2r_s - 1}{r_s(r - 1)} \frac{3r^2 - \lambda^2}{r^2 + \lambda^2 (1 - r_s)} \]

\[ + \frac{40r^3 - 24r^2 - 12r^2 + 16r^2 - 13r + 5}{10r^4 - 8r^3 + \lambda^2 (8r^2 - 13r + 5)} \] = 0, \quad (8d)

where

\[ \xi(r_s, \lambda) = \frac{5 - 7r_s}{4r} + \frac{\lambda^2 - 3r^2}{4(r^3 + \lambda^2 (1 - r))} \cdot \]

We solve equation (8c) and find that like flows in various \( \Phi_i(r) \) here also a significant region of parameter space allows the multiplicity of sonic points for accretion as well as for wind where one O-type unphysical middle sonic point is flanked in between two X-type real physical sonic points \( r_{in} \) and \( r_{out} \). In Figure 5 we show the regions of parameter space for which multitransonic flow is obtained for both accretion and wind. The dimensionless conserved total specific energy \( \delta \) (excluding the rest-mass energy) is plotted along the \( Y \)-axis, whereas the specific angular momentum \( \lambda \) is plotted along the \( X \)-axis. In the region bounded by PQR and marked by \( \mathcal{C} \), three sonic points are formed in accretion, and in the region bounded by PRS and marked by \( \Psi \), three sonic points are formed in wind. While the figure is drawn for ultrarelativistic flows, the corresponding regions of parameter space can be obtained for any \( \gamma \). If \( \delta_{\text{max}} \) be the maximum value of the energy and if \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) be the maximum and minimum values of the angular momentum, respectively, for which three sonic points are formed in accretion for any particular \( \gamma \), we observe that \( \delta_{\text{max}}, \lambda_{\text{max}}, \lambda_{\text{min}} \) non-linearly anticorrelate with \( \gamma \). In other words, as the flow approaches its purely nonrelativistic limit, the area of the region involved in formation of multitransonic accretion decreases to a lower value.

In Figure 6, we show the integral curves of motion for general relativistic accretion of ultrarelativistic polytropic fluid. For a particular set of \( \delta, \lambda, \gamma \) shown in the figure, ABCD represents the accretion passing through the outer sonic point \( r_{out} \) (marked in the figure by B), the location of which can be found by solving equation (8c). EBI represents the self-wind. Flow along EFGH passes through the inner sonic point \( r_{in} \) (marked in the figure by F) and encompasses a middle sonic point \( r_{mid} \), the location of which is shown in the figure using an asterisk. As in Figure 3, here also we obtain the complete solution topology by integrating equation (8a) (with the help of eqs. [7a], [7b], and [8c]) using the fourth-order Runge-Kutta method.
If \( \Sigma \) and \( \Pi \) be the shock compression and the entropy enhancement ratio (at the shock location) for this case \((\Sigma = M_-/M_+, \Pi = \dot{M}_+/\dot{M}_-)\), one can show that the following equation will be satisfied when shock forms:

\[
\Pi \Sigma^{1/(1-\gamma)} \left( \frac{T}{T^+} \right)^{\gamma/(\gamma -1)} \left( 1 - u^2 \right)^{(1/4)(3-\gamma)/(\gamma -1)} = 1 ,
\]

where \( T(\pm) \) and \( u(\pm) \) are the preshock/postshock temperature and dynamical velocities of the flow, respectively. However, it is our limitation in this paper that we have not been able to formulate or solve any equation that can be used to calculate the shock location in general relativistic accretion onto Schwarzschild BHs. Nevertheless, if shock forms in such flow (which is, of course, expected), it is obvious that the set of \((\Sigma, \Pi)\) responsible for shock formation must belong to the region \(PQR \equiv [\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\); see § 3 of Figure 5 because shock will form only in multitransonic accretion. The above argument is useful for comparing accretion flows in various \(\Phi_i(r)\) with general relativistic accretion (at least as far as the question of shock formation in multitransonic flow is concerned) in the following way. Suppose that for ultrarelativistic flows, we take the region of parameter space \([\varphi_1] \in \varphi_1 [\varphi_1]\) for any \(\Phi_i(r)\) used in this paper (see Fig. 4) and then superpose that region with \(PQR\) of Figure 5 and study which \(\Phi_i(r)\) provides the maximum overlap between \([\varphi_1] \in \varphi_1 [\varphi_1]\) and \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\). That particular BH potential is then considered to be the most efficient pseudopotential in approximating the general relativistic, multitransonic, shocked BH accretion. However, such an “efficiency test” is not entirely unambiguous. We have yet to figure out the exact \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\). Hence, there may be some possibility that for any \(\Phi_i(r)\), although \([\varphi_i] \in \varphi_i [\varphi_i]\) will overlap with \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\), but instead of falling onto \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\), it will rather overlap with \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\) because the exact boundary between \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\) and \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\) could not be explored in our work. Nevertheless, we believe that still our arguments for the efficiency test are of some use, at least until one can find out the exact shock formation zone for general relativistic flow.

In Figure 7, we superpose Figure 5 on \([\varphi_1] \in \varphi_1 [\varphi_1]\) for all different \(\Phi_i(r)\) (marked in the figure) used in our work. Unlike other \([\varphi_i] \in \varphi_i [\varphi_i]\), \([\varphi_3] \in \varphi_3 [\varphi_3]\) is drawn using long-dashed lines to show its overlap with \([\varphi_2] \in \varphi_2 [\varphi_2]\). The figure is drawn for ultrarelativistic flow but can also be drawn for other values of \(\gamma\) as well. We observe that while \([\varphi_1] \in \varphi_1 [\varphi_1]\) has excellent overlap (except at very high energy) with \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\), no other \([\varphi_i] \in \varphi_i [\varphi_i]\) have any overlap with it. This leads to the conclusion that at least for ultrarelativistic flows, \(\Phi_1(r)\) is not only a very good approximation; rather, it is the only BH potential to approximate the general relativistic multitransonic shocked flow. However, as the flow approaches its purely nonrelativistic limit, we observe that the area of the overlapping zone for \(\Phi_1(r)\) decreases with higher \(\gamma\) and \([\varphi_1] \in \varphi_1 [\varphi_1]\) is pushed back to overlap rather with \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\); hence, unlike ultrarelativistic accretion, \(\Phi_1(r)\) may not be considered such a good approximation for purely nonrelativistic flows. Also, we find that a region of low-energy, high–angular momentum \([\varphi_4] \in \varphi_4 [\varphi_4]\) starts overlapping with \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\). So for high-\(\gamma\) flows, along with \(\Phi_1(r)\), \(\Phi_4(r)\) may also be considered as a plausible approximation for general relativistic accretion. Shocked flows in \(\Phi_2(r)\) and

\(\Phi_3(r)\) never show any overlap with \([\varphi_{GR}] \in \varphi_{GR} [\varphi_{GR}]\) for any value of \(\gamma\); hence, these potentials may not be considered to mimic the general relativistic multitransonic accretion flows.

5. CONCLUDING REMARKS

In this paper, we provide a generalized formalism that can formulate and solve the equations governing the advective, multitransonic, hydrodynamic BH accretion in all available pseudo-Schwarzschild potentials, which may contain steady, standing, Rankine-Hugoniot kinds of shocks. We have also formulated and solved the equations governing multitransonic, complete general relativistic BH accretion and wind in a Schwarzschild metric and compared our pseudo-Schwarzschild solutions with the general relativistic one. The main conclusions of this paper are the following:

1. We observe that a significant region of parameter space (spanned by the conserved total specific energy \(\epsilon\), the specific angular momentum \(\lambda\), and the polytropic index \(\gamma\) of the flow) allows shock formation for all potentials, which leads to the strong conclusion that stable, standing RSHs are inevitable ingredients in multitransonic accretion disks around nonrotating BHs. The same kind of conclusion was drawn by previous works in this field (see § 1) only for ultrarelativistic accretion in the Paczynski & Wiita (1980) potential, whereas we make this conclusion more general by incorporating all available BH potentials to study BH accretion for all possible values of \(\gamma\).

2. Since the shock forms at a particular radial distance, it is clear that self-similar solutions should not be invoked while studying real physical BH accretion and related phenomena.

3. It is sometimes argued that a nonstanding oscillating shock may modulate the disk spectrum in order to explain the dwarf novae outburst (Mauche, Raymond, & Mattei 1995) or QPO (Hua, Kazanas, & Titarchuk 1997). In this context, the region of parameter space, for which three sonic points are formed in accretion but still no steady, standing shock is found (see § 3), can be considered as quite an important zone because \([\varphi_i] \in \varphi_i [\varphi_i]\) may provide the relevant parameters responsible for such physical processes.
4. As far as the shock formation in ultrarelativistic black hole accretion is concerned, the Paczynski & Wiita (1980) potential $\Phi_1(r)$ is the only pseudopotential that can mimic the solutions of general relativistic accretion disks around nonrotating BHs in a very efficient way. However, in the purely nonrelativistic limit ($\gamma \approx 5/3$), along with $\Phi_1(r)$, another BH potential, $\Phi_2(r)$, proposed by ABN is also observed to mimic the general relativistic solutions, at least for low-energy, high-angular momentum flows. However, it is interesting to note one important feature of the Paczynski & Wiita potential $\Phi_1(r)$: like spherically symmetric accretion (see Das & Sarkar 2001), for an accretion disk also, $\Phi_1(r)$ is observed to be in excellent agreement with solutions for ultrarelativistic flow in a pure Schwarzschild metric; however, it starts loosing (albeit very slowly) its efficiency in mimicking a full general relativistic solution with higher values of $\gamma$, i.e., as the flow reaches its purely nonrelativistic limits, although the exact reason behind this is not quite clear to us.

Hot, dense, and exoentropic postshock regions in advective accretion disks are used as a powerful tool in understanding the spectral properties of BH candidates (Shrader & Titarchuk 1998 and references therein) and in theoretically explaining a number of diverse phenomena, including millisecond variability in the X-ray emission from low-mass X-ray binaries and the generation mechanism for high-frequency QPOs in general (Titarchuk, Lapidus, & Muslimov 1998 and references therein), high-energy emission from central engines of AGNs (Sivron, Caditz, & Tsuruta 1996), formation of heavier elements in BH accretion disks via nonexplosive nucleosynthesis (Mukhopadhyay & Chakrabarti 2000), formation and dynamics of accretion-powered Galactic and extragalactic jets, quiescent states of X-ray novae systems, and the outflow-induced low luminosity of our Galactic center (Das 1998, 2001b; Das & Chakrabarti 1999). A number of observational evidences are also present that are in close agreement with the theoretical predictions obtained from shocked accretion models (Rutledge et al. 1999; Munro, Morgan, & Remillard 1999; Webb & Malkan 2000; Rao, Yadav, & Paul 2000; Smith, Heindl, & Swank 2002). Thus, we believe that our present work may have far reaching consequences because of the following reasons:

1. Our generalized formalism assures that our model is not just an artifact of a particular type of potential only, and inclusion of every BH potential allows a substantially extended zone of parameter space allowing for the possibility of shock formation.

2. Of course, there is the possibility that in future someone may come up with a pseudo-Schwarzschild potential better than $\Phi_1(r)$, which will be the best approximation for complete general relativistic investigation of multitransonic shocked flow. In such a case, if one already formulates a generalized model for a multitransonic shocked accretion disk for any arbitrary $\Phi(r)$, exactly what we have done in this paper, then that generalized model will be able to readily accommodate that new $\Phi(r)$ without having any significant change in the fundamental structure of the formulation and solution scheme of the model, and we need not have to worry about providing any new scheme exclusively valid only for that new potential, if any.

3. Even if someone can provide a completely satisfactory model for shock formation in full general relativistic (Schwarzschild) BH accretion, still the utility of this work may not be completely irrelevant. Rigorous investigation of some of the shock-related phenomena is extremely difficult (if not completely impossible) to study using a full general relativistic framework. Hence, one is expected to always rely on these pseudopotentials because of the ease of handling them. For example, it was shown that (see § 4) the total energy of the general relativistic accretion flow cannot be decoupled into its constituent contributions, whereas for any kind of pseudopotential (see § 2), all individual energy components are linear under addition. This provides enough freedom and ease to simply add any extra component in the expression for energy to introduce any new physics in the system (radiative forces or magnetic fields, for example), which is certainly not possible while dealing with full general relativistic astrophysical flows around nonrotating BHs.

Thus, for the above mentioned reasons, we believe that compared to all previous works based solely on ultrarelativistic accretion in $\Phi_1$, our model is better equipped for handling various shock-related phenomena.

It is noteworthy that the idea of shock formation in advective BH accretion is contested by some authors (Narayan, Kato, & Honma 1997 and references therein). However, the fact that their claim against shock formation is, perhaps, inappropriate for many reasons has been shown (Molteni, Gerardi, & Valenza 2001) from energy considerations. One can understand that the problem of not finding shocks lies in the fact that nonshock advection-dominated accretion flow models are, perhaps, unable to produce multitransonic flows because only one inner sonic point close to the BH is explored by such works.

One can observe that flows characterized by $|\mathcal{P}| \in \mathcal{F}_1[\mathcal{P}]$ in our work may contain low intrinsic angular momentum for some cases (especially for purely nonrelativistic flow in some of the BH potentials). However, such weakly rotating flows are expected to be allowed by nature for various real physical situations like detached binary systems fed by accretion from OB stellar winds (Illarionov & Sunyaev 1975; Liang & Nolan 1984), semidetached, low-mass, nonmagnetic binaries (Bisikalo et al. 1998), and supermassive BHs fed by accretion from slowly rotating central stellar clusters (Illarionov 1987; Ho 1999 and references therein).

Even 28 years after the discovery of standard accretion disk theory (Shakura & Sunyaev 1973), exact modeling of viscous multitransonic BH accretion, including proper heating and cooling mechanisms, is still quite an arduous task, and we have not yet fully attempted this. However, our preliminary calculations show that the introduction of viscosity via a radius-dependent power-law distribution for angular momentum pushes the shock location closer to the BH; details of this work will be discussed elsewhere.

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