Counting Perfect Matchings in Dense Graphs Is Hard
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Abstract
We show that the problem of counting perfect matchings remains \#P-complete even if we restrict the input to very dense graphs, proving the conjecture in [5]. Here “dense graphs” refer to bipartite graphs of bipartite independence number \( \beta < 2 \), or general graphs of independence number \( \alpha \leq 2 \). Our proof is by reduction from counting perfect matchings in bipartite graphs, via elementary linear algebra tricks and graph constructions.

1 Notations and Technical Lemmas
First let us fix some notations.
- Problem \( \text{Permanent}(\mathcal{G}) \): How many perfect matchings are there for a given graph \( G \in \mathcal{G} \)?
- \( \mathcal{B} \): all bipartite graphs.
- \( \mathcal{B}^0 \): all complete bipartite graphs with potential parallel edges.
- \( \mathcal{B}^0' \): all \((n-3)\)-regular bipartite graphs.
- \( \mathcal{C} \): all graphs with independence number at most 2.
- \( f(n) := \begin{cases} (n-1)!! & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \) counts the number of perfect matchings in \( K_n \).

Towards our hardness results, we will study two special classes of square matrices. A main technical tool is the following theorem from linear algebra.

Theorem 1. (Schur product theorem) If matrices \( M_1, M_2 \) are positive definite, then their entry-wise product \( M_1 \circ M_2 \) is also positive definite.

Lemma 2. The matrix
\[
A_n = \begin{pmatrix}
0! & 1! & \cdots & n! \\
1! & 2! & \cdots & (n+1)! \\
\vdots & \vdots & \ddots & \vdots \\
n! & (n+1)! & \cdots & (2n)! \\
\end{pmatrix}
\]
is positive definite for all \( n \in \mathbb{N} \).

Proof. Let us index the rows and columns from 0 to \( n \). We pull out a factor of \( i! \) from each row \( i \) and a factor of \( j! \) from each column \( j \). This leaves us with a matrix \( A' \) where \( a'_{ij} = \binom{i+j}{j} = \binom{i+j}{i} \). It is a so-called symmetric Pascal matrix \([4]\). Observe that
\[
\binom{i+j}{j} = \sum_{k=0}^{n} \binom{i}{k} \binom{j}{j-k} = \sum_{k=0}^{n} \binom{k}{j} \binom{j}{k}
\]
where the first equality follows from a thought experiment. Suppose we are electing \( j \) leaders out from \( i+j \) candidates. To implement this, divide the candidates into two fixed groups of sizes \( i \) and \( j \), respectively. We elect \( k \) leaders from the first group and the remaining \( j-k \) from the second group, for a varying parameter \( k \).

Given the identity, it follows that
\[\begin{pmatrix}
0! & 1! & \cdots & n! \\
1! & 2! & \cdots & (n+1)! \\
\vdots & \vdots & \ddots & \vdots \\
n! & (n+1)! & \cdots & (2n)! \\
\end{pmatrix} = L L^T \quad \text{where} \quad \ell_{ik} := \binom{i}{k}.
\]
Clearly $L$ is a lower triangular matrix with an all-1 diagonal, which is invertible. So the matrix $A' = LL^T$ (and hence $A_n$) is positive definite. \hfill \qed

**Lemma 3.** The matrices

$$B_n := \begin{pmatrix} f(0) & f(2) & \cdots & f(n) \\ f(2) & f(4) & \cdots & f(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ f(n) & f(n+2) & \cdots & f(2n) \end{pmatrix} \quad \text{and} \quad C_n := \begin{pmatrix} f(2) & f(4) & \cdots & f(n) \\ f(4) & f(6) & \cdots & f(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ f(n) & f(n+2) & \cdots & f(2n-2) \end{pmatrix}$$

are positive definite for all $n \in 2\mathbb{N}$.

**Proof.** Note that $C_n$ is essentially $B_n$ without the first row and the last column. So it suffices to show $B_n$ is positive definite, and the property automatically transfers to $C_n$.

Let us index the rows and columns of $B_n$ from $0$ to $n/2$, thus $b_{ij} = f(2(i+j))$. Recall that for any $t \in \mathbb{N}$, we have

$$f(2t) = (2t-1)!! = \frac{(2t)!}{2^t t!} = \frac{t!}{2^t} \cdot \binom{2t}{t}.$$ 

This motivates us to split $B_n = U \circ V$, where

$$u_{ij} := \frac{(i+j)!}{2^i+j} \quad \text{and} \quad v_{ij} := \frac{(i+j)!}{2(i+j)}.$$

Observe that $U$ becomes the matrix $A_{n/2}$ in Lemma 2 if we multiply $2^i$ to each row $i$ and $2^j$ to each column $j$. Hence $U$ is positive definite.

Next we argue that $V$ is positive definite as well. First, we have identity

$$\binom{2(i+j)}{i+j} = \sum_{k=-n}^{n} \binom{2i}{i-k} \binom{2j}{j+k} = \sum_{k=-n}^{n} \binom{2i}{i-k} \binom{2j}{j-k}$$

due to the same thought experiment as before. Second, using the symmetry $(\binom{2i}{i-k}) = (\binom{2i}{i+k})$ and $(\binom{2j}{j-k}) = (\binom{2j}{j+k})$, we see that the terms for $k$ and $-k$ have the same value. So we may conclude

$$\binom{2(i+j)}{i+j} = \binom{2i}{i} \binom{2j}{j} + 2 \cdot \sum_{k=1}^{n} \binom{2i}{i-k} \binom{2j}{j-k},$$

and consequently

$$V = L \text{diag}(1, 2, \ldots, 2) L^T \quad \text{where} \quad \ell_{ik} := \frac{2i}{i-k}.$$

Because $L$ is a lower triangular matrix with an all-1 diagonal, $V$ must be positive definite. Finally, by Theorem 1 we see $B_n = U \circ V$ is positive definite. \hfill \qed

**Remark.** The matrix $V$ in the proof, among many other Hankel matrices starring binomial coefficients, are studied in combinatorics. See for example [1] and [2].

## 2 Hardness for bounded $\beta$

Now we are ready to prove our hardness results. The method is inspired by Okamoto, Uehara and Uno [6].

**Theorem 4.** Permanent$(B)$ reduces to Permanent$(B')$. As a result, the latter is $\mathbf{\#P}$-complete.

**Proof.** Given a bipartite $G \in B$ with $n$ vertices on each side, we construct a graph $G_i \in B'$ for each $i = 0, \ldots, n$ as follows:

- add $i$ vertices to the left part and $i$ vertices to the right part;
- then add an edge for each left-right vertex pair, even if they were connected in $G$.

Let $p_i$ be the number of perfect matchings in $G_i$, which is assumed efficiently computable.

|Section 2|
Denote by \( m_j \) the number of matchings \( M \subseteq E(G) : |M| = j \). Every such \( M \) extends to exactly \((n + i - j)! \) perfect matchings \( M' \subseteq E(G_i) \) with \( M' \cap E(G) = M \). Clearly different \( j, M \) contribute distinct \( M' \), and they cover all possible perfect matchings. Hence \( p_i = \sum_{j=0}^n (n + i - j)! \cdot m_j \). Writing in matrix form, we have \((p_0, \ldots, p_n) = (m_n, \ldots, m_0) A_n\), where \( A_n \) is exactly the invertible matrix in Lemma 2. Hence we could recover \((m_n, \ldots, m_0)\), in particular \( m_n\), from vector \((p_0, \ldots, p_n)\). \(\square\)

**Remark.** Our \( B' \) allows parallel edges, which is somewhat undesirable. Better reduction exists in the literature. Dagum and Luby [3] gave a purely combinatorial reduction to \( \text{Permanent}(B') \). Since any \( G \in B'' \) has \( \beta(G) \leq 3\), their result has similar philosophical implication.

### 3 Hardness for bounded \( \alpha \)

**Theorem 5.** \( \text{Permanent}(B) \) reduces to \( \text{Permanent}(C) \). As a result, the latter is \( \#P \)-complete.

**Proof.** Given a bipartite \( G \in B \) with \( n \) vertices on each side, we construct a graph \( G_i \in C \) for each \( i = 0, \ldots, n \) as follows:

- add \( i \) vertices to the left part and \( i \) vertices to the right part;
- then connect every pair of vertices in the left (resp. right) part.

By a mirror argument to Theorem 4, we establish a linear equation \( p_i = \sum_{j=0}^n f^2(n + i - j) \cdot m_j \). Writing in matrix form, we have \((p_0, \ldots, p_n) = (m_n, \ldots, m_0) Q \) where

\[
Q := \begin{pmatrix}
    f^2(0) & f^2(1) & \cdots & f^2(n) \\
    f^2(1) & f^2(2) & \cdots & f^2(n+1) \\
    \vdots & \vdots & \ddots & \vdots \\
    f^2(n) & f^2(n+1) & \cdots & f^2(2n)
\end{pmatrix}.
\]

It remains to prove that \( Q \) is invertible, so that we could recover \((m_n, \ldots, m_0)\), in particular \( m_n\), from vector \((p_0, \ldots, p_n)\).

As before, we index the rows and columns from 0 to \( n \). Observe that \( Q \) has a “checkerboard” pattern since \( q_{ij} = 0 \) iff \( i + j \) is odd. To clean up the picture, we lift even rows to the top, and then push even columns to the left. When \( n \) is even we derive

\[
Q' = \begin{pmatrix}
    B_n \circ B_n & 0 \\
    0 & C_n \circ C_n
\end{pmatrix},
\]

and similarly, when \( n \) is odd we derive

\[
Q' = \begin{pmatrix}
    B_{n-1} \circ B_{n-1} & 0 \\
    0 & C_{n+1} \circ C_{n+1}
\end{pmatrix}.
\]

By Theorem 1, both the top-left and bottom-right blocks are positive definite, hence invertible. Therefore \( \det(Q') \neq 0 \), showing the invertibility of \( Q' \) (and thus also \( Q \)). \(\square\)

### References

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