K-stability for varieties with a big anticanonical class

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To Claire Voisin on the occasion of the conference in her honor

Abstract. We extend the algebraic K-stability theory to projective klt pairs with a big anticanonical class. While in general such a pair could behave pathologically, it is observed in this note that the K-semistability condition will force them to have a klt anticanonical model, whose stability property is the same as that of the original pair.

Keywords. K-stability, finite generation, log Fano varieties

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1. Introduction

There has been tremendous progress in algebraic K-stability theory of log Fano pairs (see [Xu21] for a survey of the topic). In the recent works [DZ22] and [DR22], the Kähler–Einstein problem is considered for a Kähler manifold \((X,\omega)\) such that \(-K_X\) is big. More precisely, in [DZ22] the authors prove a transcendental Yau–Tian–Donaldson theorem for twisted big Kähler–Einstein metrics. As a consequence of their result, in the algebraic setting, uniform K-stability of \(X\) with a big anticanonical class implies the existence of a Kähler–Einstein metric. In this note we want to show that the K-stability theory in this setting, i.e. a projective klt pair with a big anticanonical class, essentially follows from the original (log) Fano case.

In general, there could be pathological examples in projective varieties \(X\) with a big anticanonical class \(-K_X\); e.g. the anticanonical ring \(R(X,-K_X) = \bigoplus_{m\in\mathbb{N}} H^0(X,-mK_X)\) is not necessarily finitely generated (see Example 3.8). However, we will show that the K-stability condition implies that \(X\) is of log Fano type.

**Theorem 1.1.** Let \((X,\Delta)\) be a klt projective pair with \(-K_X-\Delta\) big. Assume \(\delta(X,\Delta) \geq 1\). Then there exists an effective \(\mathbb{Q}\)-divisor \(\Gamma\) such that \((X,\Delta+\Gamma)\) is a log Fano pair, i.e. \((X,\Delta+\Gamma)\) is klt and \(-K_X-\Delta-\Gamma\) is ample. In particular,
\[
R(X,-r(K_X+\Delta)) = \bigoplus_{m\in\mathbb{N}} H^0(X,-m(K_X+\Delta))
\]
is finitely generated for any \(r\) such that \(r(K_X+\Delta)\) is Cartier.

Here \(\delta(X,\Delta)\) is defined in the exactly same fashion as in the case when \(-K_X-\Delta\) is ample (see [FO18,BJ20]), i.e.
\[
\delta(X,\Delta) = \inf_{E} \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)}.
\]
For the stronger and more precise statement, see Theorem 3.4. We note that the above finite generation is asked in [DR22].

The above observation makes it possible to use existing birational geometry techniques to study K-stability questions for \(X\) with a big anticanonical class. In fact, without too much difficulty, it reduces K-stability questions for \((X,\Delta)\) to K-stability questions for its anticanonical model \((Z,\Delta_Z)\), as we can see from the following statement.

**Theorem 1.2.** Let \((X,\Delta)\) be a klt projective pair with \(-K_X-\Delta\) big. Assume \(R = \bigoplus_{m\in\mathbb{N}} H^0(-m(K_X+\Delta))\) is finitely generated, and denote by \((Z,\Delta_Z)\) the anticanonical model. Then \((X,\Delta)\) is K-semistable (resp. K-stable, uniformly K-stable) if and only \((Z,\Delta_Z)\) is K-semistable (resp. K-stable, uniformly K-stable). In particular, uniform K-stability of \((X,\Delta)\) is the same as K-stability of \((X,\Delta)\).

**Remark 1.3.** In [DR22], Ding stability notions for a projective klt pair \((X,\Delta)\) with big \(-K_X-\Delta\) are developed. If one assumes \(R = \bigoplus_{m\in\mathbb{N}} H^0(-m(K_X+\Delta))\) is finitely generated and denotes by \((Z,\Delta_Z)\) the anticanonical model, then one can show a similar statement to Theorem 1.2; i.e. the Ding stability notions for \((X,\Delta)\) are equivalent to the notions for \((Z,\Delta_Z)\).
Notation and Convention.— Throughout this paper, we work over an algebraically closed field $k$ of characteristic 0. We follow the standard terminology from [KM98, Koll13].

For a normal log pair $(X, \Delta)$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and a divisor $E$ over $X$, we denote by $A_{X,\Delta}(E)$ the log discrepancy of $E$ with respect to $(X, \Delta)$.

We say a klt projective pair $(X, \Delta)$ is log Fano if $(X, \Delta)$ is klt and $-K_X - \Delta$ is ample, and a klt projective pair $(X, \Delta)$ is of log Fano type if there exists an effective $\mathbb{Q}$-divisor $D$ such that $(X, \Delta + D)$ is a log Fano pair.

We say an effective $\mathbb{Q}$-divisor $\Gamma$ on a projective log pair $(X, \Delta)$ is an $N$-complement for a positive integer $N$ if $N(K_X + \Delta + \Gamma) \sim 0$ and $(X, \Delta + \Gamma)$ is log canonical. A $Q$-complement is an $N$-complement for some $N$.

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2. $S$-invariants

Let $(X, \Delta)$ be an $n$-dimensional projective normal pair such that $-K_X - \Delta$ is big. For any prime divisor $E$ which appears on a birational model $\mu: Y \to X$, the $S$-invariant is defined as

$$S_{X,\Delta}(E) := \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^\infty \text{vol}(-\mu^*(K_X + \Delta) - tE) \, dt.$$  

Definition 2.1. If $(X, \Delta)$ is klt, we define

$$\delta(X, \Delta) := \inf_E \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)},$$

where $E$ runs through all valuations over $(X, \Delta)$. We say $(X, \Delta)$ is uniformly K-stable (resp. K-semistable), if $\delta(X, \Delta) > 1$ (resp. $\delta(X, \Delta) \geq 1$). We say $(X, \Delta)$ is K-stable if $A_{X,\Delta}(E) > S_{X,\Delta}(E)$ for any $E$ over $X$.

Remark 2.2. When $(X, \Delta)$ is log Fano, the equivalence between this way of defining K-stability notions using valuations and the original one using test configurations, called the Fujita–Li criterion, is proved in [Fuj19], [Lil7] and [BX19]. For $(X, \Delta)$ with a big anticanonical class, the current definition is formulated in [DZ22].

Remark 2.3. Theorem 1.2 says K-stability is indeed the same as uniform K-stability. For a log Fano pair, this is proved in [LXZ22] (see [XZ22] for a different proof).

Fix $m \in r \cdot \mathbb{N}$, let $R_m = H^0(X, -m(K_X + \Delta))$, and assume $N_m := \dim H^0(X, -m(K_X + \Delta)) > 0$. Following [FO18], we say a $Q$-divisor $D$ is an $m$-basis type divisor if

$$\frac{1}{m \cdot N_m} \text{ord}_E \left( \text{div}(s_1) + \cdots + \text{div}(s_{N_m}) \right)$$

for a basis $\{s_1, \ldots, s_{N_m} \}$ of $R_m$. In particular, $D \sim_Q -K_X - \Delta$.

We define $S_{X,\Delta,n}(E)$ (or $S_m(E)$ if $(X, \Delta)$ is clear) for any $E$ over $X$ as follows: $E$ yields a decreasing filtration $\mathcal{F}_E^\lambda$ ($\lambda \in \mathbb{R}$) on $R_m := H^0(X, -m(K_X + \Delta))$ by

$$\mathcal{F}_E^\lambda R_m = \left\{ s \in H^0(X, -m(K_X + \Delta)) \mid \text{ord}_E(s) \geq \lambda \right\},$$

and

$$S_m(E) = \frac{1}{m \cdot N_m} \text{ord}_E \left( \text{div}(s_1) + \cdots + \text{div}(s_{N_m}) \right)$$
Theorem 2.4. Keep the notation as above. Thus

\[
S_m(E) = \frac{1}{m \cdot N_m} \sum_{\lambda \in \mathbb{N}} \lambda \cdot \dim \text{Gr}_E^1 R_m,
\]

where \( \text{Gr}_E^1 R_m := \mathcal{F}_E^1 R_m / \mathcal{F}_E^{1+1} R_m \).

We also define

\[
\delta_m(X, \Delta) := \inf_E \frac{A_{X, \Delta}(E)}{S_m(E)}.
\]

The following are basic properties proved in [BJ20].

**Theorem 2.4.** Keep the notation as above.

1. We have \( \lim_{m \to \infty} S_m(E) = S(E) \).
2. For any \( \varepsilon > 0 \), there exists an \( m_0 \) such that for any \( E \) over \( X \) and \( m \geq m_0 \) with \( m \in r \cdot \mathbb{N} \),
   \[
   S_m(E) \leq (1 + \varepsilon) S(E).
   \]
3. We have \( \delta_m(X, \Delta) = \inf_D \text{lct}(X, \Delta; D) \), where \( D \) runs through all \( m \cdot \text{basis type divisors} \).
4. We have \( \lim_{m \to \infty} \delta_m(X, \Delta) = \delta(X, \Delta) \).

**Proof.** Statement (1) follows from [BJ20, Lemma 2.9] and (2) from [BJ20, Corollary 2.10]. Statement (3) is [BJ20, Proposition 4.3], and (4) is [BJ20, Theorem 4.4]. \( \square \)

We can consider more general filtrations.

**Definition 2.5.** By a (linearly bounded) filtration \( \mathcal{F} \) on \( R(X, r(K_X + \Delta)) = \bigoplus_{m \in r \cdot \mathbb{N}} R_m \), we mean the data of a family \( \mathcal{F}^1 R_m \subseteq R_m \) of \( k \)-vector subspaces of \( R_m \) for \( m \in r \cdot \mathbb{N} \) and \( \lambda \in \mathbb{R} \), satisfying

1. \( \mathcal{F}^1 R_m \subseteq \mathcal{F}^1 R_m \) when \( \lambda \geq \lambda' \);
2. \( \mathcal{F}^1 R_m = \bigcap_{\lambda' < \lambda} \mathcal{F}^1 R_m \) for any \( \lambda \);
3. there exist \( e, e' \in \mathbb{R} \) such that \( \mathcal{F}^{1^e} R_m = R_m \) and \( \mathcal{F}^{1^e'} R_m = 0 \) for any \( m \);
4. \( \mathcal{F}^1 R_m \cdot \mathcal{F}^1 R_m \subseteq \mathcal{F}^{1_{m+1}} R_{m+m} \).

For any filtration \( \mathcal{F} \) on \( R \), we can define \( S_m(\mathcal{F}) \) and \( S(\mathcal{F}) \) as in [BJ20, Sections 2.5 and 2.6, pp. 15-16], and we have

\[
(2.1) \quad \lim_{m \to \infty} S_m(\mathcal{F}) \to S(\mathcal{F});
\]

see [BJ20, Lemma 2.9].

**Lemma 2.6.** If \( A \) is an effective ample \( \mathbb{Q} \) divisor on \( X \) such that \( -K_X - \Delta - A \) is pseudoeffective, then \( S_{X, \Delta}(A) \geq \frac{1}{n+1} \).

**Proof.** Since \( -K_X - \Delta - A \) is pseudoeffective, for any \( t \geq 0 \), we have

\[
\text{vol}(-K_X - \Delta - tA) \geq \text{vol}((1-t)(-K_X - \Delta)).
\]

Thus

\[
S(A) = \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^{+\infty} \text{vol}(-K_X - \Delta - tA) dt \geq \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^1 \text{vol}((1-t)(-K_X - \Delta)) dt = \frac{1}{(n+1)}.
\]

\( \square \)
3. Finite generation

3.1. $\mathbb{Q}$-complements and finite generation

For a $\mathbb{Q}$-divisor $D$ with $|rD| \neq 0$ and any $m \in \mathbb{N}$, we denote by $\text{Bs}(|mrD|)$ the base ideal. We can define the log canonical threshold of the asymptotic linear series as follows:

$$\text{lct}(X, \Delta; ||-K_X-\Delta||) := \sup_{\ell} \text{lct} \left( X, \Delta; \frac{1}{\ell r} \text{Bs}[\ell r (-K_X-\Delta)] \right).$$

We can define a sequence of multiplier ideals

$$\mathcal{I} \left( X, \Delta; \frac{1}{r} \text{Bs}(|rD|) \right) \subseteq \mathcal{I} \left( X, \Delta; \frac{1}{2r} \text{Bs}(|2rD|) \right) \subseteq \cdots \subseteq \mathcal{I} \left( X, \Delta; \frac{1}{\ell r} \text{Bs}(|\ell r D|) \right) \subseteq \cdots.$$

By the ascending chain condition of ideals, this sequence will stabilize. We denote the maximal element by $\mathcal{I}(X, \Delta; ||-K_X-\Delta||)$ and call it the asymptotic multiplier ideal sheaf of $D$. For more background, see [Laz04, Section II.1]. Recall that for any ideal $a \subseteq \mathcal{O}_X$, we have $\text{lct}(X, \Delta; a) \geq 1$ if and only if $\mathcal{I}(X, \Delta; a) \geq 1$.

Lemma 3.1. Assume $(X, \Delta)$ is a projective pair with $-K_X-\Delta$ big. If

$$\text{lct}(X, \Delta; ||-K_X-\Delta||) > 1 \quad (\text{or equivalently } \mathcal{I}(X, \Delta; ||-K_X-\Delta||) = \mathcal{O}_X),$$

then $(X, \Delta)$ is of log Fano type.

Proof. From the assumption, there exists an effective $\mathbb{Q}$-divisor $D \sim_\mathbb{Q} -(K_X+\Delta)$ such that $(X, \Delta+D)$ is klt. Since $D$ is big, then $D \sim_\mathbb{Q} A+E$ for an ample $\mathbb{Q}$-divisor $A$ and an effective $\mathbb{Q}$-divisor $E$. Set

$$\Gamma := (1-\varepsilon)D + \varepsilon E$$

for $0 < \varepsilon \ll 1$; then $(X, \Delta+\Gamma)$ is klt, and $-K_X-\Delta-\Gamma \sim \varepsilon A$ is ample. Thus $(X, \Delta)$ is of log Fano type. \hfill \square

Definition 3.2. For any projective pair $(X, \Delta)$, we define the constant $a(X, \Delta)$ by

$$a(X, \Delta) = \sup_{t \in \mathbb{R}} \left\{ \begin{array}{ll}
\text{there exists an ample divisor } A \text{ such that } A-t(K_X+\Delta) \text{ is ample and } -K_X-\Delta-A \text{ is pseudoeffective} \end{array} \right\}.$$

If $-K_X-\Delta$ is big, then $a(X, \Delta) > 0$; if $-K_X-\Delta$ is ample, then $a(X, \Delta) = +\infty$.

Assumption 3.3. Let $(X, \Delta)$ be an $n$-dimensional klt projective pair with $-K_X-\Delta$ big. Assume

$$\delta(X, \Delta) > \frac{n+1}{n+1+a_0}, \quad \text{where } a_0 = a(X, \Delta).$$

Now we can show the following.

Theorem 3.4. Let $(X, \Delta)$ satisfy Assumption 3.3; then $(X, \Delta)$ is of log Fano type. In particular, any Cartier divisor $E$ on $X$ satisfies that $R(X, E) := \bigoplus_{m \in \mathbb{N}} H^0(X, mE)$ is finitely generated.

Proof. Let us first prove this when $\delta(X, \Delta) > 1$ as it is quite straightforward. By Theorem 2.4, we know that for a sufficiently large $m$ and any $m$-basis type divisor $D$,

$$\text{lct}(X, \Delta; D) \geq \delta_m(X, \Delta) > 1.$$

Thus we can apply Lemma 3.1.

In the general case, we may assume $\delta(X, \Delta) \leq 1$, and we need some perturbation argument. By our definition of $a(X, \Delta)$, for any $t \in (0, a(X, \Delta))$, there exists an ample $\mathbb{Q}$-divisor $A$ such that

$$-K_X-\Delta-A \sim_\mathbb{Q} E_1 \quad \text{and} \quad A-t(K_X+\Delta) \sim_\mathbb{Q} A_0,$$
where $E_1$ is an effective $\mathbb{Q}$-divisor and $A_0$ is an ample $\mathbb{Q}$-divisor. Moreover, by (3.2) we may assume

\begin{equation}
1 - \delta(X, \Delta) < \frac{t}{n + 1} \delta(X, \Delta).
\end{equation}

Fix $m_0 \in \mathbb{N}$ such that $|m_0A|$ is base-point-free. Then for any prime divisor $H \in |m_0A|$, by Lemma 2.6,

\[ S(H) = \frac{1}{\text{vol}(-K_X - \Delta)} \int \text{vol}(-K_X - \Delta - tH) dt = \frac{1}{m_0} S_{X, \Delta}(A) \geq \frac{1}{m_0(n + 1)}. \]

We can choose an $m$-basis type $\mathbb{Q}$-divisor $D_m$ compatible with $H$, so we can write $D_m = F_m + b_m H$, where

\begin{equation}
\lim_{m \to \infty} b_m = \lim_{m \to \infty} S_m(H) = S(H) \geq \frac{1}{m_0(n + 1)}.
\end{equation}

By (3.3), (3.4), and the equality $\lim_{m} \delta_m(X, \Delta) = \delta(X, \Delta)$, we can find a sufficiently large $m$ and a positive $\delta'$ such that $\delta' < \min\{\delta_m(X, \Delta), 1\}$ and

\begin{equation}
1 - \delta' < tm_0 b_m \delta'.
\end{equation}

Then $(X, \Delta + \delta' F_m)$ is klt, as $(X, \Delta + \delta' D_m)$ is klt and $D_m = F_m + b_m H$. Moreover,

\[-K_X - \Delta - \delta' F_m \sim_Q (1 - \delta')(K_X + \Delta) + \delta' b_m H,\]

which implies $(X, \Delta + \delta' F_m)$ is a log Fano pair since

\[-(1 - \delta')(K_X + \Delta) + \delta' b_m H \sim_Q (1 - \delta') \left( -(K_X + \Delta) + \frac{1}{t} A \right) + \left( \delta' b_m m_0 - \frac{1 - \delta'}{t} \right) A
\]

\[\sim_Q \frac{1}{t} \delta' A_0 + \left( \delta' b_m m_0 - \frac{1 - \delta'}{t} \right) A\]

is ample by (3.5).

The last statement then follows from [BCH+10].

**Corollary 3.5.** Let $(X, \Delta)$ satisfy Assumption 3.3. Let $r(K_X + \Delta)$ be Cartier and $Z := \text{Proj } R(X, -r(K_X + \Delta))$. Denote by $\Delta_Z$ the birational transform of $\Delta$ on $Z$; then $(Z, \Delta_Z)$ is a log Fano pair.

**Proof.** We know $f : X \to Z$ is a birational contraction; i.e. $\text{Ex}(f^{-1})$ does not contain any divisor, and $f_*(K_X + \Delta) = K_Z + \Delta_Z$ is antiample.

It follows from Theorem 3.4 that there exists a $\mathbb{Q}$-complement $\Gamma$ for $(X, \Delta)$ such that $(X, \Delta + \Gamma)$ is klt. Then $(Z, \Delta_Z + f_\Gamma)$ is klt as the pullbacks of $K_Z + \Delta_Z + f_\Gamma$ and $K_X + \Delta + \Gamma$ on a common resolution are equal. So $(Z, \Delta_Z)$ is klt. \qed

### 3.2. K-stability of the anticanonical model

Let $(X, \Delta)$ be a projective log pair with big $-K_X - \Delta$. Let $(Z, \Delta_Z)$ be its anticanonical model; i.e. $Z = \text{Proj } R(X, -r(K_X + \Delta))$, and $\Delta_Z$ is the birational transform of $\Delta$ on $Z$. Let $Y$ be a common resolution.

\begin{equation}
\begin{tikzcd}
Y \arrow{dr}{\pi} \arrow{ur}{\mu} & \\
(X, \Delta) \arrow{r}{f} & (Z, \Delta_Z).
\end{tikzcd}
\end{equation}

Then

\[\pi^*(K_Z + \Delta_Z) - \mu^*(K_X + \Delta) = B \geq 0\]
Lemma 3.6. Let \((X, \Delta)\) satisfy Assumption 3.3. Then for any prime divisor \(E\) over \(X\),
\[
A_{X,\Delta}(E) = A_{Z,\Delta_Z}(E) + \text{ord}_E(B)\quad\text{and}\quad S_{X,\Delta}(E) = S_{Z,\Delta_Z}(E) + \text{ord}_E(B).
\]

Proof. For the log discrepancy function, this follows directly from the definition since
\[
|m^*(\mathcal{K}_X + \Delta)| = |m^*(\mathcal{K}_Z + \Delta_Z)| + mB,
\]
we have \(S_{X,\Delta,m}(E) = S_{Z,\Delta_Z,m}(E) + \text{ord}_E(B)\). Therefore, the same is true for the \(S\)-function. □

Lemma 3.7. If \((Z, \Delta_Z)\) is klt, there exists a \(t > 0\) depending on \(Z\) (but not \(E\)) such that for any divisor \(E\) over \(X\)
\[
A_{Z,\Delta_Z}(E) \geq t \cdot \text{ord}_E(B).
\]

Proof. Since \((Z, \Delta_Z)\) is klt, we know that there exists a \(t > 0\) such that if we write \(\pi^*(\mathcal{K}_Z + \Delta_Z) = \mathcal{K}_Y + \Delta_1\), then \((\mathcal{K}_Y + \Delta_1 + tB)\) is sub-lc for some \(t > 0\); i.e. for any \(E\),
\[
A_{Z,\Delta_Z}(E) \geq t \cdot \text{ord}_E(B).
\]

Proof of Theorem 1.2. Since
\[
\delta(X, \Delta) = \inf_E A_{Z,\Delta_Z}(E) + \text{ord}_E(B)
\]
then it is clear that \(\delta(X, \Delta) \geq 1\) if and only if \(A_{Z,\Delta_Z}(E) \geq S_{Z,\Delta_Z}(E)\), i.e. \((Z, \Delta)\) is klt and \(\delta(Z, \Delta_Z) \geq 1\). Moreover,
\[
A_{X,\Delta}(E) = A_{Z,\Delta_Z}(E) + \text{ord}_E(B) > S_{Z,\Delta_Z}(E) + \text{ord}_E(B) = S_{X,\Delta}(E)
\]
if and only if \(A_{Z,\Delta_Z}(E) > S_{Z,\Delta_Z}(E)\).

Assume \(\delta(X, \Delta) > 1\); then \(\delta(Z, \Delta_Z) \geq \delta(X, \Delta)\). Conversely, if \(\delta(Z, \Delta_Z) > 1\), an easy calculation shows that
\[
\delta(X, \Delta) \geq \frac{\delta(Z, \Delta_Z)(t + 1)}{\delta(Z, \Delta_Z) + t} > 1,
\]
where \(t\) is the constant from Lemma 3.7. □

Example 3.8. This example has appeared in several works to present pathological phenomena, see e.g. [Gon12]: Let \(S\) be the blowup of \(\mathbb{P}^2\) at nine very general points. Then \(-K_S\) is known to be nef but not semiample. In fact, there will be a unique cubic curve passing through these nine points, and if we denote by \(E\) its birational transform on \(S\), then for any \(m \in \mathbb{N}\), \([-mK_S]\) has one element \(mE\).

Let \(H\) be an ample Cartier divisor on \(S\) and \(X = \mathbb{P}_S(E)\), where \(E := O_S + O_S(H)\). Denote by \(\pi: X \to S\) the natural morphism. We claim \(-K_X\) is big. In fact, since
\[
\omega_{X/S} = \wedge^2 O_{\mathbb{P}(E)}(-2),
\]
we have
\[
H^0(O_X(-mK_X)) = H^0(S, \pi_* (O_X(-mK_X)))
\]
\[
= H^0(S, \text{Sym}^{2m}(E) \otimes (\wedge^2 E)^{\otimes m} \otimes \wedge^2 S^{\otimes m})
\]
\[
= H^0\left(S, \bigoplus_{i=0}^{m} O_S(iH) \otimes O_S(-mH - mK_S)\right)
\]
\[
= H^0\left(S, \bigoplus_{i=0}^{m} O_S(iH - mK_S)\right),
\]
and since \(-K_S \sim E\) is nef, we have
\[
\text{vol}_X(-K_X) = 6 \int_0^1 \frac{1}{2}(tH - K_S)^2 = 3 \int_0^1 (t^2H^2 - 2tH(K_S))
\]
\[
= H^2 + 3H \cdot (-K_S) > 0.
\]
However, the algebra $\bigoplus_{i \leq m} H^0(iH - mK_S)$ is not finitely generated, since
$$\sum_{1 \leq j \leq m-1} H^0(O_S(-(jK_S))) \otimes H^0(O_S(H-(m-j)K_S)) \rightarrow H^0(O_S(H-mK_S))$$
is not surjective for any $m$. Thus we need generators from $H^0(O_S(H-mK_S))$ for every $m$.

By Theorem 1.1, we know $\delta(X) < 1$. Here we give a direct verification of this. We denote by $Y \subseteq X$ the section given by
$$E = O_S \oplus O_S(H) \rightarrow O_S.$$
Then similarly to before, we have
$$H^0(O_X(-mK_X - m_0 Y)) = H^0\left(S \bigoplus_{i=m_0}^{2m} O_S(iH) \otimes O_S(-mH - mK_S)\right),$$
where we follow the convention that if $m_0 > 2m$, then the direct sum is 0. Hence a direct calculation implies
$$\text{vol}(-K_X - tY) = \begin{cases} H^2 + 3H \cdot (-K_S) & \text{if } t \leq 1, \\ (2-t)((t^2-t+1)H^2 + 3tH \cdot (-K_S)) & \text{if } 1 \leq t \leq 2. \end{cases}$$
By an elementary calculation,
$$S_X(Y) = \frac{4H^2 + 5H \cdot (-K_S)}{H^2 + 3H \cdot (-K_S)} > \frac{5}{3} > 1 = A_X(Y),$$
which implies $\delta(X) < \frac{3}{2}$.

References

[BCH+10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[BJ20] H. Blum and M. Jonsson, Thresholds, valuations, and $K$-stability, Adv. Math. 365 (2020), 107062.

[BX19] H. Blum and C. Xu, Uniqueness of $K$-polystable degenerations of Fano varieties, Ann. of Math. (2) 190 (2019), no. 2, 609–656.

[DZ22] T. Darvas and K. Zhang, Twisted Kähler-Einstein metrics in big classes, preprint arXiv:2208.08324 (2022).

[DR22] R. Dervan and R. Reboulet, Ding stability and Kähler-Einstein metrics on manifolds with big anticanonical class, preprint arXiv:2209.08952 (2022).

[Fuj19] K. Fujita, A valuative criterion for uniform $K$-stability of $Q$-Fano varieties, J. reine angew. Math. 751 (2019), 309–338.

[FO18] K. Fujita and Y. Odaka, On the $K$-stability of Fano varieties and anticanonical divisors, Tohoku Math. J. (2) 70 (2018), no. 4, 511–521.

[Gon12] Y. Gongyo, On weak Fano varieties with log canonical singularities, J. reine angew. Math. 665 (2012), 237–252.

[Koll3] J. Kollár, Singularities of the minimal model program (with a collaboration of S. Kovács), Cambridge Tracts in Math., vol. 200, Cambridge Univ. Press, Cambridge, 2013.

[KM98] J. Kollár and S. Mori, Birational geometry of algebraic varieties (with the collaboration of C.H. Clemens and A. Corti; translated from the 1998 Japanese original), Cambridge Tracts in Math., vol. 134, Cambridge Univ. Press, Cambridge, 1998.

[Laz04] R. Lazarsfeld, Positivity in algebraic geometry. II, Ergeb. Math. Grenzgeb. (3), vol. 49, Springer-Verlag, Berlin, 2004.
[Li17] C. Li, *K-semistability is equivariant volume minimization*, Duke Math. J. 166 (2017), no. 16, 3147–3218.

[LXZ22] Y. Liu, C. Xu, and Z. Zhuang, *Finite generation for valuations computing stability thresholds and applications to K-stability*, Ann. of Math. (2) 196 (2022), no. 2, 507–566.

[Xu21] C. Xu, *K-stability of Fano varieties: an algebro-geometric approach*, EMS Surv. Math. Sci. 8 (2021), no. 1-2, 265–354.

[XZ22] C. Xu and Z. Zhuang, *Stable degenerations of singularities*, preprint arXiv:2205.10915 (2022).