TWISTED ALEXANDER POLYNOMIAL AND MATRIX-WEIGHTED ZETA FUNCTION

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Abstract. The twisted Alexander polynomial is an invariant of the pair of a knot and its group representation. Herein, we introduce a digraph obtained from an oriented knot diagram, which is used to study the twisted Alexander polynomial of knots. In this context, we show that the inverse of the twisted Alexander polynomial of a knot may be regarded as the matrix-weighted zeta function that is a generalization of the Ihara–Selberg zeta function of a directed weighted graph.

1. Introduction

The twisted Alexander polynomial introduced by Lin [5] is associated with a knot and a representation of the knot group in the 3-sphere. Wada [12] generalized this work for finitely presentable groups and showed how to define a twisted polynomial that is only given a presentation of a group and representation. In [4], Kitano showed that the twisted Alexander polynomial can be regarded as a Reidemeister torsion in the case of knot groups. Jiang and Wang studied the relationship between the twisted Alexander polynomial and the Lefschetz zeta function [3]. Various applications of the twisted Alexander polynomial have been obtained thereafter (see [7] for examples). The purpose of this article is to explicitly present the relationship between the twisted Alexander polynomial and a zeta function of a graph. Notably, the relationship between the classical Alexander polynomial and the Ihara–Selberg zeta function of a graph has been studied by Lin and Wang [6]. They showed a relationship between the classical Alexander polynomial and the colored Jones polynomial, which was called the Melvin–Morton conjecture. The result of this study will be a step forward in showing a relationship between the twisted Alexander polynomial and the colored Jones polynomial.

Ihara [1] introduced his zeta function as a Selberg-type zeta function for a discrete subgroup of the $p$-adic linear group of degree two and then Serre [9] interpreted Ihara’s work in terms of trees and graphs. It was shown that their reciprocals are explicit polynomials. A zeta function of a regular graph $G$ associated with a unitary representation of the fundamental group of $G$ was developed by Sunada [10, 11]. Watanabe and Fukumizu [13] defined the matrix-weighted zeta function of a graph and presented its determinant.

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expression. Then, the results were generalized by Sato, Mitsuhashi and Morita in their study [8].

To show the relationship between a zeta function and the twisted Alexander polynomial of a knot and a representation of the knot group, we introduce the notion of the knot graph $G$ which is obtained from a diagram $D$ with a base point of a knot $K$. Then, we consider the matrix-weighted zeta function associated with a knot and its representation of the knot group. Let $\zeta_{G,\rho}(t)$ be the matrix-weighted zeta function of the matrix-weighted knot graph $G$ associated with a representation $\rho$ of its knot group (see Sections 3 and 4 for details). We fix a base point on an arc of $D$ and denote the diagram by $D^*$. Let $\Delta_{K,D^*,\rho}(t)$ be the twisted Alexander polynomial of $K$ with a diagram $D^*$ and $\rho$ (see Definition 2.3 for the precise definition). The main result of this paper is the following.

**Theorem 1.1.** Under the above notation, we have

$$\Delta_{K,D^*,\rho}(t) = \zeta_{G,\rho}(t)^{-1}.$$  

The idea of the zeta function of knot graphs includes a much richer content than we have presented here. A more detailed study will be the subject of our future research.

2. Free differential calculus and twisted Alexander invariants

In this section, we give a short review of the twisted Alexander polynomial [5, 12].

Fix a diagram of an oriented knot $K$. We label the arcs in the knot diagram separated by crossings at the under-crossed strands using letters $x_1, x_2, \ldots, x_n$. Let $E_K$ be the exterior of $K$ in the 3-sphere. The knot group $G(K) = \pi_1(E_K, \ast)$ admits a Wirtinger presentation as follows. It has $x_1, x_2, \ldots, x_n$ as generators, and one relation for each crossing. Here, we assume that the base point $\ast$ of the fundamental group $G(K)$ is set in the upper part of the diagram of $K$. Then we have a presentation \( \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle \) as the Wirtinger presentation of the knot group $G(K)$ of a knot $K$, where $r_i : x_i = x_j^{\varepsilon}x_kx_j^{-\varepsilon}$ is according to the positive crossing ($\varepsilon = +1$) or the negative crossing ($\varepsilon = -1$) (Figure 1).

It is known that one relation can be obtained from the other relations, so that we may have a presentation of a knot group as $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle$.

The abelianization homomorphism

$$\alpha : G(K) \to H_1(E_K; \mathbb{Z}) = \langle t \rangle$$

is given by assigning to each generator $x_i$ the meridian element $t \in H_1(E_K; \mathbb{Z})$.

![Figure 1](image-url)
Let \( \rho \) be a homomorphism of \( G(K) \) into the special linear group \( \text{SL}(m, \mathbb{C}) \). The maps \( \rho \) and \( \alpha \) naturally induce two ring homomorphisms \( \tilde{\rho} : \mathbb{Z}[G(K)] \to M(m, \mathbb{C}) \) and \( \tilde{\alpha} : \mathbb{Z}[G(K)] \to \mathbb{Z}[t^{\pm 1}] \), where \( \mathbb{Z}[G(K)] \) is the group ring of \( G(K) \) and \( M(m, \mathbb{C}) \) is the matrix algebra of degree \( m \) over \( \mathbb{C} \). Then \( \tilde{\rho} \otimes \tilde{\alpha} \) defines a ring homomorphism \( \mathbb{Z}[G(K)] \to M(m, \mathbb{C}[t^{\pm 1}]) \). Let \( F_n \) denote the free group of generators \( x_1, \ldots, x_n \) and

\[
\Phi : \mathbb{Z}[F_n] \to M(m, \mathbb{C}[t^{\pm 1}])
\]

be the composition of the surjection \( \tilde{\phi} : \mathbb{Z}[F_n] \to \mathbb{Z}[G(K)] \) induced by the presentation of \( G(K) \) and the map \( \tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}[G(K)] \to M(m, \mathbb{C}[t^{\pm 1}]) \).

Let us consider the \((n-1) \times n\) matrix \( M \) whose \((i, j)\)th entry is the \( m \times m \) matrix

\[
\Phi\left( \frac{\partial r_i}{\partial x_j} \right) \in M(m, \mathbb{C}[t^{\pm 1}]),
\]

where \( \partial / \partial x \) denotes the free derivative. For \( 1 \leq \ell \leq n \), let us denote by \( M_\ell \) the \((n-1) \times (n-1)\) matrix obtained from \( M \) by removing the ‘\( \ell \)’th column. In particular, \( M_\ell \) is an \( m(n-1) \times m(n-1) \) matrix with entries in \( \mathbb{C}[t^{\pm 1}] \).

**Definition 2.1.** The twisted Alexander invariant of a knot \( K \) associated with a representation \( \rho : G(K) \to \text{SL}(m, \mathbb{C}) \) is defined to be the rational function

\[
\tilde{\Delta}_{K,\rho}(t) = \frac{\det M_\ell}{\det \Phi(1 - x_\ell)}
\]

and is well defined up to multiplication by \( \pm^k \) (\( k \in \mathbb{Z} \)).

The free differential is a homomorphism \( \partial / \partial x_i : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n] \) satisfying the following:

\[
\frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad (\text{Kronecker’s delta}),
\]

\[
\frac{\partial (pq)}{\partial x_j} = \frac{\partial p}{\partial x_j} + \frac{\partial q}{\partial x_j}, \quad \frac{\partial p^{-1}}{\partial x_j} = -p \frac{\partial p}{\partial x_j}.
\]

Then, by the straightforward calculations for \( x_i = x_j^sx_kx_j^{-s} \) (Figure 1), we have

\[
\frac{\partial}{\partial x_j} (x_jx_kx_j^{-1}) = 1 - x_jx_kx_j^{-1} = 1 - x_i \quad \text{and} \quad \frac{\partial}{\partial x_k} (x_jx_kx_j^{-1}) = x_j, \quad \text{if } j \neq k, \quad (2.1)
\]

\[
\frac{\partial}{\partial x_j} (x_jx_kx_j^{-1}) = 1, \quad \text{if } j = k, \quad (2.2)
\]

\[
\frac{\partial}{\partial x_j} (x_j^{-1}x_kx_j) = -x_j^{-1} + x_j^{-1}x_k \quad \text{and} \quad \frac{\partial}{\partial x_k} (x_j^{-1}x_kx_j) = x_j^{-1}, \quad \text{if } j \neq k, \quad (2.3)
\]

\[
\frac{\partial}{\partial x_j} (x_j^{-1}x_kx_j) = 1, \quad \text{if } j = k. \quad (2.4)
\]
Suppose $\rho(x_\ell) = X_\ell \in \text{SL}(m, \mathbb{C})$. Using (2.1)–(2.4), we have

$$\Phi\left( \frac{\partial (x_j x_k x_j^{-1})}{\partial x_j} \right) = I - t X_i \quad \text{and} \quad \Phi\left( \frac{\partial (x_j x_k x_j^{-1})}{\partial x_k} \right) = t X_j, \quad \text{if } j \neq k,$$

$$\Phi\left( \frac{\partial (x_j x_k x_j^{-1})}{\partial x_j} \right) = I, \quad \text{if } j = k,$$

$$\Phi\left( \frac{\partial (x_j^{-1} x_k x_j)}{\partial x_j} \right) = - \frac{1}{t} X_j^{-1} + X_j^{-1} X_k \quad \text{and} \quad \Phi\left( \frac{\partial (x_j^{-1} x_k x_j)}{\partial x_k} \right) = \frac{1}{t} X_j^{-1}, \quad \text{if } j \neq k,$$

$$\Phi\left( \frac{\partial (x_j^{-1} x_k x_j)}{\partial x_j} \right) = I, \quad \text{if } j = k.$$ (2.5) (2.6) (2.7) (2.8)

By making use of (2.5)–(2.8), we define an $mn \times mn$ matrix $N$ as follows. The matrix $N$ consists of $n \times n$ entries, each of which is a matrix of size $m \times m$. The $i$th row of $N$ has at most two non-zero matrix entries: for each relation $x_i = x_j^\varepsilon x_k x_j^{-\varepsilon}$, when $j \neq k$ and $\varepsilon = 1$, the $(i, k)$th entry is $i X_j$ and the $(i, j)$th entry is $I - t X_i$; when $j = k$ and $\varepsilon = 1$, the only non-zero matrix entry is the $(i, k)$th entry, which is equal to $I$. Similarly, when $j \neq k$ and $\varepsilon = -1$, the $(i, k)$th entry is $(1/t) X_j^{-1}$ and the $(i, j)$th entry is $-(1/t) X_j^{-1} + X_j^{-1} X_k$; when $j = k$ and $\varepsilon = -1$, the only non-zero matrix entry is the $(i, k)$th entry, which is equal to $I$.

Let $N_\ell$ be the $m(n - 1) \times m(n - 1)$ matrix obtained from $N$ by deleting the $\ell$th row and the $\ell$th column. Then we have the following.

**Proposition 2.2.** One has

$$\tilde{\Delta}_{K, \rho}(t) = \frac{\det(I - N_\ell)}{\det(1 - x_\ell)}.$$ (2.9)

Proposition 8.1 in [2] asserts the same result essentially by using the notions of quandle cocycle invariants.

**Definition 2.3.** Fix a base point of a diagram $D$ of an oriented knot $K$. We denote by $D^*$ the diagram with the base. Suppose the base point is put in just before an under-crossing. We label the arcs $x_1, \ldots, x_n$ according to the orientation starting from the base point so that each relation of the Wirtinger presentation can be seen as $r_i : x_i = x_j^\varepsilon x_{i+1} x_j^{-\varepsilon}$. Under this setting, we denote $\det(I - N_1)$ by $\Delta_{K, \rho_1}(t)$, and call it the twisted Alexander polynomial.

**Example 2.4.** Let $K$ be the figure eight knot and $D$ the diagram of $K$ as illustrated in Figure 2. The knot group $G(K)$ has the Wirtinger presentation

$$(x_1, x_2, x_3, x_4 \mid x_1 = x_4 x_2 x_4^{-1}, x_2 = x_1^{-1} x_3 x_1, x_3 = x_2 x_4 x_2^{-1}, x_4 = x_3^{-1} x_1 x_3).$$
Furthermore, this group $G(K)$ has the representation to $\text{SL}(2, \mathbb{C})$ such as

$$
\rho(x_1) = \begin{pmatrix}
\frac{1 + \sqrt{-3}}{2} & -1 \\
\frac{-1 - \sqrt{-3}}{2} & \frac{3 - \sqrt{-3}}{2}
\end{pmatrix} =: X_1, \quad \rho(x_2) = \begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix} =: X_2,
$$

$$
\rho(x_3) = \begin{pmatrix}
\frac{3 - \sqrt{-3}}{2} & 1 - \sqrt{-3}/2 \\
\frac{-1 + \sqrt{-3}}{2} & \frac{1 + \sqrt{-3}}{2}
\end{pmatrix} =: X_3, \quad \rho(x_4) = \begin{pmatrix}
1 & 0 \\
\frac{-1 + \sqrt{-3}}{2} & 1
\end{pmatrix} =: X_4.
$$

Then we have

$$
N = \begin{pmatrix}
O & tX_4 & O & I - tX_1 \\
-(1/t)X_1^{-1} + X_1^{-1}X_3 & O & (1/t)X_1^{-1} & O \\
O & I - tX_3 & O & tX_2 \\
(1/t)X_3^{-1} & O & -(1/t)X_3^{-1} + X_3^{-1}X_1 & O
\end{pmatrix}
$$

and

$$
\Delta_{K_{D^*, \rho}}(t) = \det(I - N_1) = \frac{(t - 1)^2(t^2 - 4t + 1)}{t^2}.
$$

It is worth noting that, in this case, $\det \Phi(1 - x_1) = (t - 1)^2$; hence

$$
\tilde{\Delta}_{K, \rho}(t) = \frac{t^2 - 4t + 1}{t^2}.
$$

3. Matrix-weighted zeta functions

In this section, we introduce the notion of a matrix-weighted zeta function of a graph $G$ according to [8].

Let $G = (V(G), E(G))$ be a connected finite graph (possibly with multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges $uv$ joining two vertices $u$ and $v$. For $uv \in E(G)$, an arc $(u, v)$ is the oriented edge from $u$ to $v$. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$.

A path $P$ of length $n$ in $G$ is a sequence $P = (e_1, \ldots, e_n)$ of $n$ arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})$ ($1 \leq i \leq n - 1$), where the indices are treated mod $n$. If $e_i = (v_i, v_{i+1})$ ($1 \leq i \leq n$), then we denote $P = (v_1, \ldots, v_{n+1})$. Set $|P| = n$, $o(P) = o(e_1)$ and
\( t(P) = t(e_n) \). Then \( P \) is called an \((\alpha(P), t(P))\)-path. We say that a path \( P = (e_1, \ldots, e_n) \) has a backtracking if \( e_i^{-1} = e_i \) for some \( i \) (\( 1 \leq i \leq n - 1 \)). A \((v, w)\)-path is called a cycle if \( v = w \). The inverse cycle of a cycle \( C = (e_1, \ldots, e_n) \) is the cycle \( C^{-1} = (e_n^{-1}, \ldots, e_1^{-1}) \).

We define an equivalence relation between cycles. Two cycles \( C_1 = (e_1, \ldots, e_m) \) and \( C_2 = (f_1, \ldots, f_m) \) are called equivalent if there exists \( k \) such that \( f_j = e_{j+k} \) for all \( j \). Let \([C]\) be the equivalence class that contains a cycle \( C \). Let \( B' \) be the cycle obtained by going \( r \) times around a cycle \( B \). Such a cycle is called a power of \( B \). A cycle \( C \) is said to be reduced if \( C \) has no backtracking, and a cycle is said to be prime if it is not a power of a strictly smaller cycle.

The Ihara zeta function of a graph \( G \) is a function of \( u \in \mathbb{C} \) with \(|u| \) sufficiently small, defined by

\[
\zeta(G, u) = \zeta_G(u) = \prod_{[C]} (1 - u^{[C]})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime, reduced cycles of \( G \) [1].

Then, we introduce the matrix-weighted zeta function of a graph \( G \). Let \( G \) be a connected graph with \( n \) vertices \( v_1, \ldots, v_n \) and \( m \) edges, and \( (a_1, \ldots, a_n) \in \mathbb{N}^n \). Set \( a_{v_i} = a_i \) (\( 1 \leq i \leq n \)). For each edge \( e = (v_i, v_j) \), let \( w = w(v_i, v_j) \) be an \( a_i \times a_j \) matrix. The set \( \{w(e)\} \) is called the matrix-weight of \( G \), and \( G \) is called the matrix-weighted graph.

**Definition 3.1.** [8] For each cycle \( C = (e_1, \ldots, e_k) \), we define the weight of the cycle \( C \) as the products of the weights of the edges:

\[
w(C) = w(e_1) \cdots w(e_k) .
\]

The matrix-weighted zeta function \( \zeta_G(w) \) of \( G \) is defined by

\[
\zeta_G(w) = \prod_{[C]} \det(I - w(C))^{-1},
\]

where \([C]\) runs over all equivalence classes of prime, reduced cycles of \( G \).

Let \( D(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\} \), and arrange arcs of \( G \) as follows: \( e_1, e_1^{-1}, e_2, e_2^{-1}, \ldots, e_m, e_m^{-1} \). Let

\[
U = \begin{pmatrix}
w(e_1) & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & w(e_m)
\end{pmatrix} .
\]

The matrix \( U \) is called the weight matrix of \( G \).

Set \( a = a_1 + \cdots + a_n \), \( b = \frac{1}{2} \sum_{e \in D(G)} (a_{o(e)} + a_{i(e)}) \). We define two \( b \times b \) matrices \( B = (B_{e,f})_{e,f \in D(G)} \) and \( J = (J_{e,f})_{e,f \in D(G)} \) as follows:

\[
B_{e,f} = \begin{cases}
I_{a(e)}, & \text{if } t(e) = \alpha(f), \\
O_{a(e),a(e)}, & \text{otherwise},
\end{cases} \quad J_{e,f} = \begin{cases}
I_{a(e)}, & \text{if } f = e^{-1}, \\
O_{a(e),a(f)}, & \text{otherwise},
\end{cases}
\]

\( (3.1) \)
where $B_{e,f}$ and $J_{e,f}$ are $a_{l(e)} \times a_{o(f)}$ matrices. The matrix $B - J$ is called the edge matrix of $G$.

Next, an $a \times a$ matrix $\hat{A} = \hat{A}(G) = (A_{xy})$ is defined as follows:

$$A_{xy} = \begin{cases} (I_{a_x} - w(x, y)w(y, x))^{-1}w(x, y), & \text{if } (x, y) \in D(G), \\ O_{a_x, a_y}, & \text{otherwise}. \end{cases}$$ (3.2)

The matrix $\hat{A}$ is called the adjacency matrix of $G$. Furthermore, an $a \times a$ matrix $\hat{D} = \hat{D}(G) = (D_{xy})$ is the diagonal matrix defined by

$$D_{xx} = \sum_{o(e)=x} w(e)(I - w(e^{-1})w(e))^{-1}w(e^{-1}).$$ (3.3)

A determinant expression for $\zeta_G(w)$ was given in [13] (see also Theorem 2 in [8]).

**Theorem 3.2.** Under the above notation, the reciprocal of the matrix-weighted zeta function of a matrix-weighted graph $G$ is given by

$$\zeta_G(w)^{-1} = \det(I_b - U(B - J)) = \det(I_a + \hat{D} - \hat{A}) \prod_{i=1}^m \det(I_{a_{o(e_i)}} - w(e_i)w(e_i^{-1})).$$

Thus, the graphs that have so far been considered are digraphs that are double (in the sense of replacing an unoriented edge by a pair of oppositely oriented edges) of undirected graphs. However, the graphs we will study in the following are single digraphs; thus we reduce the above results to fit our setting.

Suppose that $G$ is an oriented graph whose edges are oriented one-way, that is, there are no inverse edges. Let $D^o(G) = \{e_1, \ldots, e_m\}$ and

$$V = \begin{pmatrix} w(e_1) & \cdots & O \\ \vdots & & \vdots \\ O & \cdots & w(e_m) \end{pmatrix}.$$ According to (3.1), we define $B$ for $e, f \in D^o(G)$, namely, $B = (B_{e,f})_{e,f \in D^o(G)}$. We call $V$ the weight matrix and $B$ the edge matrix of $G$.

**Corollary 3.3.** Let $G$ be a matrix-weighted one-way oriented graph. Then we have

$$\zeta_G(w)^{-1} = \det(I_b/2 - VB) = \det(I_a - \hat{A}).$$

**Proof.** Suppose the weight of an inverse edge $e_i^{-1}$ is $O$. Then the weight of the cycle $C$ that contains $e_i^{-1}$ is equal to $O$ so that, by Definition 3.1, $\det(I - w(C)) = 1$. Thus, the condition that there does not exist edge $e_i^{-1}$ is equivalent to the condition that $w(e_i^{-1}) = O$ on $\zeta_G(w)$. In this condition, from (3.3), we have $\hat{D} = (D_{xx}) = O$ and $w(e_i)w(e_i^{-1}) = O$ in Theorem 3.2. Hence, we have $\zeta_G(w)^{-1} = \det(I_a - \hat{A})$. Moreover, in this case, there is no backtracking; thus we have $J = O$, so that $\zeta_G(w)^{-1} = \det(I_b/2 - VB)$. \hfill $\square$

**Remark 3.4.** For a matrix-weighted one-way oriented graph $G$, the adjacency matrix $\hat{A} = \hat{A}(G) = (A_{xy})$ is defined by

$$A_{xy} = \begin{cases} w(x, y), & \text{if } (x, y) \in D^o(G), \\ O_{a_x, a_y}, & \text{otherwise}, \end{cases}$$ (3.4)

from (3.2).
4. Knot graph

In this section, we introduce the notion of a knot graph and a matrix-weighted knot graph. The idea of a knot graph originated in [6], but it was not written explicitly. So we clarify it here.

Let \( K \) be an oriented knot in the 3-sphere and \( D \) be a diagram of \( K \). We define the knot graph according to the following steps.

**Step 1.** Fix a base point just before an under-crossing and cut \( D \) at the point; then we make a 1-string tangle \( T \). Let us set an orientation of \( T \).

**Step 2.** Set a point on the under-arc at a crossing of \( T \) successively according to the orientation. These points constitute vertices \( \{v_i\} \) of the knot graph.

**Step 3.** According to the arc on \( T \) from a point \( v_i \) to the next point \( v_{i+1} \), we connect \( v_i \) to \( v_{i+1} \) by an oriented edge, denoted by \( e_{i,i+1} \).

**Step 4.** Jump up from the point \( v_i \) to the over-arc of \( T \) and then go to the point \( v_j \) along the orientation of \( T \). According to the way, we connect the vertices \( v_i \) and \( v_j \) by an edge named \( e_{i,j} \). (Note that we have a loop if \( i = j \).)

**Step 5.** If \( j = i + 1 \), we have two edges from \( v_i \) to \( v_j \) \((= v_{i+1})\) via Steps 3 and 4. In this case, we reduce one of them, so that we connect the vertices \( v_i \) and \( v_j \) by one edge.

Step 5 corresponds to a kink, which can be reduced by Reidemeister move I. If the diagram has no kinks, this is not necessary.

**Example 4.1.** Figure 3 shows the 1-string tangle and the knot graph obtained from the diagram of the figure eight knot illustrated in Figure 2.

**Remark 4.2.** We suppose that the Wirtinger presentation is obtained by setting the base point of the knot group \( G(K) \) on the upper side of the diagram \( D \) of \( K \). In Step 4, we used jump ‘up’ because of the setting. We may consider a jump ‘down’ version if we suppose the base point is on the lower side.

![Figure 3](image-url)
Remark 4.3. It is worth noting that the diagram $D$ of $K$ can be uniquely reconstructed from $T$ so that any invariant of knots gives rise to a corresponding invariant of $T$.

The sign of a vertex $v_i$, denoted by $\text{sign}(v_i)$, is $+1$ ($-1$ respectively) if $v_i$ corresponds to a positive (negative respectively) crossing (see Figure 1).

Let us consider the Wirtinger presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ of $G(K)$ which is obtained from the diagram $D$ such that the generator $x_k$ corresponds to $e_{k-1}$. In this setting, $x_k$ in Figure 1 may be regarded as $x_{k+1}$. Further, suppose that $G(K)$ has a representation of $\rho(x_\ell) = X_\ell \in \text{SL}(m, \mathbb{C})$. The matrix-weighted knot graph associated with $\rho$ is defined as the knot graph with the following weight for each edge. These correspond to the results of the calculations (2.5)–(2.8).

For $i = 1, \ldots, n$, we have the following.

If $\text{sign}(v_i) = +1$ and $j \neq i + 1$,

$$
\begin{align*}
& w(v_i, v_{i+1}) = w(e_{i+1}) = tx_j, \\
& w(v_i, v_j) = w(e_{ij}) = 1 - tx_i.
\end{align*}
$$

(4.1)

If $\text{sign}(v_i) = +1$ and $j = i + 1$,

$$
\begin{align*}
& w(v_i, v_{i+1}) = w(e_{i+1}) = w(v_i, v_j) = w(e_{ij}) = 1.
\end{align*}
$$

(4.2)

If $\text{sign}(v_i) = -1$ and $j \neq i + 1$,

$$
\begin{align*}
& w(v_i, v_{i+1}) = w(e_{i+1}) = (1/t)x_j^{-1}, \\
& w(v_i, v_j) = w(e_{ij}) = - (1/t)x_j^{-1} + x_j^{-1}x_{i+1}.
\end{align*}
$$

(4.3)

If $\text{sign}(v_i) = -1$ and $j = i + 1$,

$$
\begin{align*}
& w(v_i, v_{i+1}) = w(e_{i+1}) = w(v_i, v_j) = w(e_{ij}) = 1.
\end{align*}
$$

(4.4)

See also Figure 4. It is worth noting that, if $i = n$, then we suppose $i + 1 = n + 1 = 1$, where $n$ is the number of the crossing of the diagram $D$ of $K$.

Suppose $G$ is a matrix-weighted knot graph associated with $\rho$. We denote by $\zeta_{G, \rho}(t)$ the matrix-weighted zeta function of $G$.

5. Proof of Theorem 1.1

Let $K$ be an oriented knot and $\rho : G(K) \to \text{SL}(m, \mathbb{C})$ a representation of the knot group. Fix an oriented diagram $D^*$ of $K$ with a base point and denote by $G$ the corresponding matrix-weighted knot graph associated with $\rho$. Then we can define the twisted Alexander polynomial $\Delta_{K, D^*, \rho}(t)$ (Definition 2.3) and the matrix-weighted zeta function $\zeta_{G, \rho}(t)$ of $G$. 

![Figure 4](image-url)
Following (3.4), set \( \hat{A} = \hat{A}(G) = (A_{ik}) \); then we have

\[
A_{ik} = \begin{cases} 
  w(v_i, v_k), & \text{if } (v_i, v_k) \in D^o(G), \\
  O, & \text{otherwise}.
\end{cases}
\]

Based on (4.1)–(4.4), we obtain the following condition: if \( \text{sign}(v_i) = +1 \) and \( j \neq i + 1 \), then \( A_{ii+1} = tX_j \) and \( A_{ij} = I - tX_i \). If \( \text{sign}(v_i) = +1 \) and \( j = i + 1 \), then \( A_{ii+1} = A_{ij} = I \).

Since there is no edge \( e \) such that \( t(e) = o(v_1) \) for a knot graph, we have \( w(e_{\ell 1}) = O \), i.e. the ‘first’ column of \( \hat{A} = (A_{ik}) \) is the \( mn \times m \) \( O \) matrix, where \( n \) is the number of vertices of \( G \). Let \( A \) be the matrix obtained from \( \hat{A} \) by deleting the ‘first’ row \((m \times mn)\) matrix) and ‘first’ column \((mn \times m)\) matrix). Then \( \det(I - \hat{A}) = \det(I - A) \). On the other hand, we have \( A = N_1 \) by the constructions, so \( \det(I - \hat{A}) = \det(I - N_1) \). Thus, from Definition 2.3 and Corollary 3.3, we obtain the conclusion.

**Example 5.1.** We use the figure eight knot as in Example 2.4. The knot graph \( G \) can be seen as illustrated in Figure 3. Let \( D^o(G) = \{ e_{12}, e_{14}, e_{23}, e_{34}, e_{43} \} \). As in the proof, by using the adjacency matrix of \( G \) in the calculation, we have the same conclusion as in Example 2.4. Moreover, the weights of the matrix-weighted knot graph are \( w(e_{12}) = tX_4, w(e_{14}) = I - tX_1, w(e_{23}) = t^{-1}X_1^{-1}, w(e_{32}) = I - tX_3, w(e_{34}) = tX_2 \) and \( w(e_{43}) = -t^{-1}X_1^{-1} + X_3^{-1}X_1 \), where \( X_i \) is the same matrix as in Example 2.4. Thus we have the following weight matrix and edge matrix of \( G \):

\[
V = \begin{pmatrix}
  tX_4 & & & & & \\
  & I - tX_1 & & & & \\
  & & t^{-1}X_1^{-1} & & & \\
  & & & I - tX_3 & & \\
  & & & & tX_2 & \\
  & & & & & -t^{-1}X_1^{-1} + X_3^{-1}X_1
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
  O & O & I & O & O & O \\
  O & O & O & O & O & I \\
  O & O & O & I & I & O \\
  O & O & I & O & O & O \\
  O & O & O & O & O & I \\
  O & O & O & I & I & O
\end{pmatrix}.
\]

By Example 2.4, Corollary 3.3 and straightforward calculation, we have

\[
\xi_{G, \rho}(t)^{-1} = \det(I - VB) = \frac{(t - 1)^2(t^2 - 4t + 1)}{t^2} = \Delta_{K_{G^*, \rho}}(t).
\]

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