A Sampling Technique of Proving Lower Bounds for Noisy Computations

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Abstract

We present a technique of proving lower bounds for noisy computations. This is achieved by a theorem connecting computations on a kind of randomized decision trees and sampling based algorithms. This approach is surprisingly powerful, and applicable to several models of computation previously studied.

As a first illustration we show how all the results of Evans and Pippenger (SIAM J. Computing, 1999) for noisy decision trees, some of which were derived using Fourier analysis, follow immediately if we consider the sampling-based algorithms that naturally arise from these decision trees.

Next, we show a tight lower bound of $\Omega(N \log \log N)$ on the number of transmissions required to compute several functions (including the parity function and the majority function) in a network of $N$ randomly placed sensors, communicating using local transmissions, and operating with power near the connectivity threshold. This result considerably simplifies and strengthens an earlier result of Dutta, Kanoria Manjunath and Radhakrishnan (SODA 08) that such networks cannot compute the parity function reliably with significantly fewer than $N \log \log N$ transmissions. The lower bound for parity shown earlier made use of special properties of the parity function and is inapplicable, e.g., to the majority function. In this paper, we use our approach to develop an interesting connection between computation of boolean functions on noisy networks that make few transmissions, and algorithms that work by sampling only a part of the input. It is straightforward to verify that such sampling-based algorithms cannot compute the majority function.

1 Introduction

We present a novel technique for analyzing randomized decision trees. This method does not depend upon the specific function being computed by the decision tree and can be applied for proving lower bounds in various models for a variety of functions.

We introduce the technique in the simplistic setting of $\epsilon$-noisy decision trees. $\epsilon$-noisy decision trees can be viewed as a simple kind of randomized decision trees. As an application of our technique, we show how it provides elementary and unified proofs of all the lower bounds of Evans and Pippenger [EP99] for average noisy decision tree complexity of several types of functions. Their work introduced the notion of noisy leaf complexity, which was analyzed using Fourier methods. However, as we show, our technique yields elementary arguments that places these results in a compact and unified framework.

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We then use our technique to derive the main result of this paper - a lower bound for wireless sensor networks. This simplifies and extends a lower bound of [DKmR08, DKmR15] for this model. A wireless sensor network consists of sensors that collect and cooperatively process data in order to compute some global function. The sensors interact with each other by transmitting wireless messages based on some protocol. The protocol is required to tolerate errors in transmissions since wireless messages typically are noisy.

In the problem we study, there are $n$ sensors, each with a boolean input, they are required to cooperatively compute some function of their inputs. The difficulty of this task, of course, depends on the noise and the connectivity of the network. In this paper, we assume that each bit sent is flipped (independently for each receiver) with probability $\epsilon > 0$ during transmission. As for connectivity, we adopt the widely used model of random planar networks. Here the sensors are assumed to be randomly placed in a unit square. Then each transmission is assumed to be received (with noise) by the sensors that are within some prescribed radius of the sender. The radius is determined by the amount of power used by the sensors, and naturally one wishes to keep the power used as low as possible, perhaps just enough to ensure that the entire network is connected. It has been shown by Gupta and Kumar [GK00] that the threshold of connectivity is $\theta \left( \sqrt{\frac{\ln n}{n}} \right)$ (with a radius much smaller than this the network will not be connected almost surely, and with radius much larger it will be connected almost surely).

It was shown by Dutta, Kanoria, Manjunath and Radhakrishnan [DKmR08, DKmR15], that computing the parity of the inputs requires $\Omega(n \log \log n)$ transmissions. This result showed that the protocol presented by Ying, Srikant and Dullerud [YSD06] for computing the sum of all the bits (and hence any symmetric functions of these bits) is optimal. The lower bound argument in [DKmR08, DKmR15] depended strongly on the properties of the parity function. In particular, the argument was not applicable for showing superlinear lower bounds for the majority and other symmetric functions in this model. Using the technique presented in this paper, we can now give optimal lower bounds for several functions, including the majority function.

The starting point of our method is the connection between noisy wireless network protocols, and certain randomized decision trees. This connection was derived in [DKmR08, DKmR15], and in turn made crucial use of a result in [GKS05]. The lower bound for parity in [DKmR08, DKmR15] was derived by rearranging the randomized decision trees obtained from wireless protocols computing parity. In this work, we show that computations on decision trees that arise in our context can be simulated by randomized algorithms that leave a non-trivial fraction of their inputs unread. Once this is established, it is relatively straightforward to conclude that several functions cannot be computed in this model.

In order to state our result formally we need a formal definition of the model of noisy communication networks, which we now reproduce from [DKmR08].

**Definition 1** (Noisy communication network, protocol). A communication network is an undirected graph $G$ whose vertices correspond to processors and edges correspond to communication links. A message sent by a processor is received by all its neighbors.

**Noise**: In an $\epsilon$-noise network, the messages are subjected to noise as follows. Suppose processor $v$ sends bit $b$ in time step $t$. Each neighbor of $v$ then receives an independent noisy version of $b$; that is, the neighbor $w$ of $v$ receives the bit $b \oplus \eta_{w,t}$, where $\eta_{w,t}$ is an $\epsilon$-noisy bit (that takes the value 1 with probability $\epsilon$ and 0 with probability $1 - \epsilon$), these noisy bits being mutually independent.

**Input**: An input to the network is an assignment of bits to the processors, and is formally an element of $\{0,1\}^{V(G)}$. 
Protocol: A protocol on $G$ for computing a function $f : \{0, 1\}^{V(G)} \rightarrow \{0, 1\}$ works as follows. The processors take turns to send single bit messages, which are received only by the neighbors of the sender. In the end, a designated processor $v^* \in V(G)$ declares the answer. The cost of the protocol is the total number of bits transmitted. A message sent by a processor is a function of the bits that it possesses until then. The protocol with cost $T$ is thus specified by a sequence of vertices $\langle v_1, v_2, \ldots, v_T \rangle$ and a sequence of $T$ functions $\langle g_1, g_2, \ldots, g_T \rangle$, where $g_t : \{0, 1\}^{j_t} \rightarrow \{0, 1\}$ and $j_t$ is the number of bits received by $v_t$ before time step $t$ (plus one if $v_t$ is an input processor). Furthermore, $v_T = v^*$, and the final answer is obtained by computing $g_T$. Note that in our model the number of transmissions is the same for all inputs.

Error: Such a protocol is said to be a $\delta$-error protocol, if for all inputs $x \in \{0, 1\}^{V(G)}$, $Pr[\text{output} = f(x)] \geq 1 - \delta$.

In this paper, we consider networks that arise out of random placement of processors in the unit square.

Definition 2 (Random planar network). A random planar network $\mathcal{N}(N, R)$ is a random variable whose values are undirected graphs. The distribution of the random variable depends on two parameters: $N$, the number of vertices, and $R$, the transmission radius. The vertex set of $\mathcal{N}(N, R)$ is $\{P_1, P_2, \ldots, P_N\}$. The edges are determined as follows. First, these processors are independently placed at random, uniformly in the unit square $[0, 1]^2$. Then, $E(\mathcal{N}) = \{(P_i, P_j) : \text{dist}(P_i, P_j) \leq R\}$.

Our main result is the following.

Theorem 3 (Lower bound for majority). Let $R \leq N^{-\frac{1}{3}}$. Let $\delta, \epsilon \in (0, \frac{1}{2})$. Then, with probability $1 - o(1)$ (over the placement of processors) every $\delta$-error protocol on $\mathcal{N}(N, R)$ with $\epsilon$-noise for computing the majority on $N$ bits requires $\Omega(N \log \log N)$ transmissions.

Remarks:

• It was conjectured in [DKmR08] that one cannot approximate the sum to within an additive error of $N^\alpha$ (for some $\alpha > 0$) using $O(N)$ transmissions. Using the techniques of this paper, we can prove this conjecture (details omitted from this paper).

• Typical protocols in wireless networks operate by computing using broadcasts in cells, where there is complete connectivity between the processors, and then aggregating the information across the cells (e.g. [KM07], [YSD06]). Here it makes sense to consider functions of the form $f(g(X_1), g(X_2), \ldots, g(X_k))$, where $X_i$ is the part of the input that falls in cell $i$, $f$ is some symmetric boolean function and $g$ is some function to be computed inside each cell. For example, if $f$ and $g$ are both parity, then this corresponds to the parity function on the entire input. Our techniques show that for most symmetric functions, and all $g$ with high sensitivity, one requires $\Omega(N \log \log N)$ transmissions. (Details omitted.)

As stated earlier, our technique yields simple proofs of previous lower bound results on noisy decision trees.
Definition 4 (Noisy decision trees). A boolean decision tree $T$ for input $x = \langle x_1, \ldots, x_n \rangle \in \{0, 1\}^n$ is a binary tree in which each internal node $v$ has a label $l(v) \in [1, n]$ and each leaf $\ell$ has a value $\text{val}(\ell) \in \{0, 1\}$. The two outgoing edges of each internal node are labelled by the value 0 and 1. The computation of $T$ on input $x$ is the unique path starting at the root of the tree and continuing up to a leaf as follows: at internal node $v$, the outgoing edge labelled with $x_l(v)$ is chosen to get to the next node. The result of the computation is $\text{val}(\ell)$ where $\ell$ is the leaf reached by the computation. In an $\epsilon$-noisy boolean decision tree, at each internal node, the incorrect outgoing edge is chosen with probability $\epsilon$ independent of former choices. Equivalently, each internal node $v$ is assigned a binary random variable $\eta_v$ that takes the value 1 with probability $\epsilon$ independently. Then, on reaching internal node $v$, the outgoing edge labelled $x_l(v) \oplus \eta_v$ is used to determine the next node. For each input $x$, the computation path and the value $T(x) = f(x)$ output by the tree is a random variable. Let $\text{depth}_{\epsilon, \delta}(f)$ be the minimum depth of an $\epsilon$-noisy decision tree $T$ such that $\Pr[T(x) \neq f(x)] \leq \delta$, where the probability is over the input $x$ chosen uniformly from $\{0, 1\}^n$ and the internal randomness of $T$.

The main results of Evans and Pippenger [EP99] were (a) $\text{depth}_{\epsilon, \frac{1}{3}}(\text{Parity}_n) = \Omega(n \log n/ \log(1/\epsilon))$, (b) $\text{depth}_{\epsilon, \frac{1}{3}}(f) = \Omega(n \log n/ \log(1/\epsilon))$ and for almost all functions; (c) $\text{depth}_{\epsilon, \frac{1}{3}}(f) = \Omega(n \log s)$ if $f$ is $n \left(1 - \frac{1}{s}\right)$-resilient.

In Section 2 we provide a simple proof of the lower bounds.

1.1 Related work

Noisy broadcast models have been studied in the past where all sensors receive all messages (with independent noise). Gallager [Gal88] showed a remarkable protocol to collect all input bits at one sensor using $O(N \log \log N)$ transmissions. Clearly, this give the same upper bound for computing any function of the input bits. Several other works have focussed on constructing protocols for specific functions in variants of the noisy broadcast model, e.g., Feige and Raghavan [FR00], Newman [New04], Kushilevitz and Mansour [KM98], and Goyal, Kindler and Saks [GKS05]. Using an insightful combination of information-theoretic and fourier based methods, Goyal, Kindler and Saks [GKS05] showed that Gallager’s protocol was the best possible for collecting all the bits.

In sensor networks, considerations of power impose stringent limits on the transmission radius. In this paper, we study networks arising from random placement of sensors with transmission radius around the threshold required to ensure connectivity. As mentioned above, in this model Ying, Srikant and Dullerud [YSD06] devised a protocol for computing the sum using $O(N \log \log N)$ transmissions. Kanoria and Manjunath [KM07] showed a protocol with $O(N)$ transmissions to compute the OR function. Making crucial use of a result of Goyal, Kindler and Saks [GKS05], it was shown in [DKmR08, DKmR15] that computing parity requires $\Omega(N \log \log N)$ transmissions which was then extended to several other functions in the initial presentation of this work [DR08].

Unlike in the model of noisy wireless networks, several lower bounds results have appeared in the literature on noisy decision trees. Reischuk and Schmeltz [RS91] showed that almost all boolean functions of $N$ arguments require $\Omega(N \log N)$ queries. Feige, Peleg, Raghavan, and Upfal [FRPU94] showed an $\Omega(N \log N)$ lower bound for the parity function. Evans and Pippenger [EP99] presented arguments to show that these results also hold in the average case.
1.2 Techniques

We now present an overview of the proof technique used to derive the lower bounds in this paper. A very detailed discussion of all the techniques can be found in the Phd thesis [Dut09].

The proof of our main result, like the proof in [DKmR08], first converts computations on noisy broadcast networks to computations on randomized decision trees.

Definition 5 (Randomized decision tree). A randomized decision tree is a model for processing inputs in \{0,1\}^n. For an internal node \( v \) of the tree let \( v_L \) be its left child and \( v_R \) its right child. Each internal node \( v \) of the tree is labelled by a pair \( (i_v,g_v) \), where \( i_v \in \{1,2,\ldots,n\} \), and \( g_v : \{0,1\} \to \{v_L,v_R\} \) is a noisy function, whose output depends on its input and some internal randomness that is independent for different noisy computations performed in the tree. Once an input \( x = (x_1,x_2,\ldots,x_n) \in \{0,1\}^n \) is fixed, the (random) output of the tree is determined by the following natural computation. We start at the root, and when we arrive at an internal node \( v \), we determine the next vertex by evaluating \( g_v(x_{i_v}) \). The (random) output of the tree on input \( x \in \{0,1\}^n \) is the 0-1 label of leaf reached.

Our lower bound for noisy broadcast networks will follow from a lower bound we show for randomized decision trees that arise from them. A central notion in our analysis, is the amount of uncertainty about each decision made at nodes that query other variables. We can now state the following crucial connection between broadcast protocols and randomized decision trees which follows from arguments in [DKmR08].

Proposition 8. If there is a broadcast protocol with \( Nd \) broadcasts for computing a the majority of \( N \) bits, then there is a randomized decision tree \( \mathcal{T} \) for computing the majority of \( N^d \) bits such that \( \beta(\mathcal{T}) \geq \exp(-\exp(O(d))) \). In particular, if \( d \leq \frac{1}{\log \log N} \) for some constant \( C \), then \( \beta(\mathcal{T}) \geq N^{-\frac{1}{16}} \).

The main contribution of this work is a technique for analyzing randomized decision trees with non-trivial uncertainty. Roughly, we show that if the uncertainty about a variable is non-trivial, then the computation can be performed by leaving the variable unread with some non-trivial probability. For example, consider a trivial tree with one root and two leaves. At the root we read a variable \( x \in \{0,1\} \), and moves to
the left child with probability \( \frac{1}{2} + (-1)^x \epsilon \). This tree can be simulated as follows. With probability \( 1 - 2\epsilon \), we do not read \( x \) at all, and move left or right with equal probability. With probability \( 2\epsilon \) we read \( x \) and move left if \( x = 0 \) and move right otherwise. If all nodes are of this kind and the tree has small depth, then it is not hard to see that we can simulate its computation leaving several variables unread. This is precisely the situation in the model of decision trees studied by Evans and Pippenger [EP99], which explains why our technique is effective there. However, applying this idea to the randomized decision trees guaranteed by Proposition 8 requires more careful analysis. The detailed argument is presented in Section 3.

2 The Evans-Pippenger lower bounds revisited

In this section, we show that the three average case lower bounds of Evans and Pippenger [EP99] follow immediately by considering sampling-based algorithms that arise naturally from noisy decision trees.

Definition 9 \(((r, \delta)\text{-sampling algorithm})\). We say that a randomized algorithm \( A \) is an \((r, \delta)\)-sampling algorithm if \( \Pr[A \text{ leaves at least } r \text{ variables unread}] \geq 1 - \delta \).

Definition 10 (Robust function). We say that \( f : \{0, 1\}^n \to \{0, 1\} \) is an \((r, \gamma)\)-robust function, if for every subcube \( L \) of dimension \( r \),

\[
\frac{1}{2} - \gamma \leq \Pr_{x \in L} [f(x) = 1] \leq \frac{1}{2} + \gamma.
\]

Once these definitions are in place, the proofs the results of Evans and Pippenger [EP99] follow easily from the following observations.

1. \( \epsilon \)-noisy boolean trees of small depth can be simulated by randomized algorithms that typically leave many variables unread.

**Lemma 11.** Suppose \( T \) is an \( \epsilon \)-noisy boolean decision tree with \( n \) variables and depth at most \( kn \). Then there is a randomized algorithm \( A \) that on all inputs simulates the computation on \( T \) (producing the same distribution on the leaves), and with probability at least \( 1 - \exp(-\frac{\epsilon^2 n}{16}) \), leaves at least \( \frac{\epsilon^2 n}{4} \) variables unread.

This lemma is the key to the analysis in this section. We present its elementary proof below.

2. Our next observation states that randomized algorithms of the kind promised by the above theorem cannot compute a robust function with small error.

**Lemma 12.** If \( A \) is an \((r, \delta)\)-sampling algorithm computing an \((r, \gamma)\)-robust function \( f \). Then,

\[
\Pr_{x \in \{0, 1\}^n} [A(x) = f(x)] \leq \frac{1}{2} + \delta + \gamma.
\]

3. The three types of functions considered by Evans and Pippenger are robust.

**Lemma 13.** (a) Almost all functions \( f : \{0, 1\}^n \to \{0, 1\} \) are \((6 \log \log n, o(1))\)-robust.

(b) The parity function is \((1, 0)\)-robust.

(c) A \( t \)-resilient function is \((n - t, 0)\)-robust.
From Lemma [11] it follows immediately that any $\epsilon$-noisy decision tree of depth at most $\frac{n}{\log(n/(8s))}$, can be simulated using an $(r, \frac{1}{8})$-sampling algorithm. Then, from Lemma [12] it follows that any such tree for an $(r, \delta)$-robust function makes error at least $\frac{1}{2} + \frac{1}{8} + \delta$. By combining this with Lemma [13] we obtain the following.

**Theorem 14** (Evans and Pippenger [EP99]).

1. $\text{depth}_{\epsilon, \frac{1}{4}}(f) = \Omega(n\log n / \log(1/\epsilon))$; for almost all functions;

2. $\text{depth}_{\epsilon, \frac{1}{4}}(\text{Parity}_n) = \Omega(n\log n / \log(1/\epsilon))$;

3. $\text{depth}_{\epsilon, \frac{1}{4}}(f) = \Omega(n \log s)$ if $f$ is $n \big(1 - \frac{1}{s}\big)$-resilient.

It remains to prove the lemmas claimed above.

**Proof of Lemma [12]** The randomized algorithm works by simulating the computation by the noisy boolean decision tree starting at the root. The algorithm has a boolean random variable $B_v$ for each internal node $v$ of the decision tree. Each $B_v$ takes the value 1 with probability $2\epsilon$ independently. At internal node $v$ of the tree, if $B_v = 1$, the sampling algorithm chooses one of the outgoing edges with probability half each, to get to the next node. If $B_v = 0$, the algorithm reads the value of the input variable $x_{l(v)}$ (without any error), and chooses the outgoing edge with that label. On reaching leaf $\ell(x)$, which is a random variable, the algorithm outputs $\text{val}(\ell(x))$ as the result. It is easy to see that for any input $x$, the distribution on leaves reached by the sampling algorithm is exactly the same as that reached by the noisy decision tree.

Fix an input $x$ and a leaf $\ell$ reached by the simulation. We will show that conditioned on arriving at this leaf, the algorithm leaves at least $\frac{2^k}{4} n$ variables unread with high probability. Suppose the variable $x_i$ appears $k_i$ times in $T$ on the path from the root to $\ell$. Then, the probability that $x_i$ is not read conditioned on the computation reaching $\ell$ is at least $\left(\frac{\epsilon}{1-\epsilon}\right)^{k_i} \geq \epsilon_i^k$. Since the depth of the tree is at most $nk$, there are at least $\frac{n}{4}$ variables that appear at most $2k$ times on the path to $\ell$. Each of these variables is independently left unread with probability at least $e^{2k}$. Using the Chernoff bound, with probability at least $1 - \exp(-\frac{e^{2k} n}{16})$, the algorithm leaves at least $\frac{e^{2k} n}{4}$ variables unread. Since this claim is true conditioned on each leaf, it also holds overall. □

**Proof of Lemma [12]** The probability that fewer than $r$ variables are left unread is at most $\delta$. Conditioned on the algorithm leaving $r$ variables unread, the probability that its output is correct is at most $\frac{1}{2} + \gamma$ because $f$ is $(r, \gamma)$-robust. □

**Proof of Lemma [12]** The second and third claims follow immediately from definitions. We justify the first claim using the following routine calculation. Consider a $d$-dimensional subcube of the boolean hypercube $\{0, 1\}^n$. Pick a random function $f$, and let $X$ be the random variable denoting the number of points in the subcube where $f$ takes the value 1. We have $E[X] = 2^d/2$. Let $t = \frac{1}{2\sqrt{d}}$. By the Chernoff bound, $\Pr[|X - E[X]| > t^2d^2] < 2\exp(-t^2d^4) = 2\exp(-2d^3/3)$. Taking the union bound over all subcubes of dimension $d$, the probability that $f$ has a bias of more than $t = \frac{1}{2\sqrt{d}}$ on any such subcube is at most $\binom{n}{d} \left(2\exp(-2d^3/3)\right)$. Thus, for $d \geq 6 \log \log n$, with probability $1 - o(1)$, $f$ has $o(1)$ bias on every subcube of dimension $d$. □
3 Proof of main Theorem

In this section we prove our lower bound on the number of transmissions needed to compute majority in a noisy wireless network. By Proposition 8 it is enough to show that randomized decision trees with high uncertainty cannot compute majority. We will first show how such trees can be simulated by sampling algorithms. The result follows from this because it is straightforward to verify that sampling algorithms that leave a super-constant number of variables unread cannot compute majority with low error.

Definition 15 (Sampling-based algorithm). By a sampling-based algorithm for computing a function \( f : \{0,1\}^n \rightarrow \{0,1\} \), we mean an algorithm of the following kind. In the sampling phase, the algorithm uses \( n \) sampling probabilities \( q_1, q_2, \ldots, q_n \in \{0,1\} \). Given an input \( x = \langle x_1, x_2, \ldots, x_n \rangle \), the algorithm constructs a string \( y \in \{0,1,*\}^n \) from \( x \), by independently replacing \( x_i \) by a \(*\) with probability \( 1 - q_i \). In the second phase, it declares its guess for \( f(x) \) based on \( y \) alone.

The main part of the argument is contained in the following theorem.

Theorem 16. Let \( T \) be a randomized decision tree with inputs from \( \{0,1\}^k \) that computes a function \( f \) with error at most \( \delta \). Then, there is a sampling algorithms \( A \) which independently samples the \( i \)-th variable with probability \( q_i = 1 - \beta_i(T) \), and computes \( f \) with error at most \( \delta \).

Preliminaries. First, we need some notation. For an internal node \( v \) of the tree, let \( v_L \) denote its left child, and \( v_R \) its right child. Suppose the input \( x_i \) is queried at a \( v \). The tree \( T \) specifies the probabilities for the computation to move to each child for each possible value of \( x_i \). For the node \( v \), let \( \chi_v \) be the event that the computation on \( T \) reaches the node \( v \). Clearly, \( \chi_v \) is the intersection of independent events \( \{ \chi_{v,i} : i = 1,2,\ldots,k \} \), where \( \chi_{v,i} \) is the event that the computation reaches node \( v \) assuming and the choices at all nodes not labelled \( i \) don’t leave the path. Note that the probability of \( \chi_{v,i} \) depends only on the value of \( x_i \); let \( p_{v,i}(z) \) be this probability. Then, the probability of the computation reaching the node \( v \) on input \( x = \langle x_1, \ldots, x_k \rangle \) is precisely \( \prod_{i=1}^k p_{v,i}(x_i) \).

Before formally stating the proof of the theorem, it will be useful to present a natural method for ‘computing’ \( \beta_i(T) \). This method works bottom up, assigning a value \( \beta_{v,i} \) to the node \( v \) of the tree. It will turn out that \( \beta_{v,i} = \beta_i(T) \). The intermediate values \( \beta_{v,i} \) produced in this algorithm, will be used crucially when our final sampling algorithms simulates the computation of \( T \). In fact, \( \beta_{v,i} \) has the following natural interpretation. Consider a tree \( T' \) as in the definition of \( \beta_i(T) \). Let \( D_0 \) and \( D_1 \) be the distributions on the leaves of \( T' \) when \( x_0 \) is set to 0 and 1. The minimum overlap (over all possible such \( T' \)) between \( D_0 \) and \( D_1 \) when restricted to the leaves in the subtree rooted at \( v \), is the quantity \( \beta_{v,i} \). With this interpretation, consider the following computation.

- for a leaf \( v \), let \( \beta_{v,i} = \min\{p_{v,i}(0), p_{v,i}(1)\} \).
- for an internal node \( v \) with children \( v_L \) and \( v_R \), where \( x_j \) is queried,

\[
\beta_{v,j} = \begin{cases} 
\min\{\beta_{vL,i}, \beta_{vR,i}\} & \text{if } i \neq j \\
\beta_{vL,i} + \beta_{vR,i} & \text{if } i = j
\end{cases}
\]

The following claim, which we state without a formal proof, is now immediate.

Proposition 17. For \( i = 1,2,\ldots,k \) we have

\[
\beta_{\text{root},i} = \beta_i(T) \\
\beta_{v,i} \leq \min\{p_{v,i}(0), p_{v,i}(1)\}
\]
Proof. Our goal is to simulate the computation of this tree using a randomized sampling algorithm, where input $x_i$ is read independently with probability $1 - \beta_i(T)$. We want to ensure that for every input the leaf reached in the end of this simulation has the same distribution as in the original tree $T$. To specify how this simulation is to be performed, we need to determine the following.

For each internal node of $T$, we need the transition probabilities for moving to each child when the input for that node is available in the sample, and when it is not. For this, we will specify for each internal node $v$, a function $\tilde{\ell}_v : \{0, 1\} \rightarrow [0, 1]$ and a value $\alpha_v \in [0, 1]$, which are to be used as follows. When the computation reaches node $v$, where $x_i$ is to be read and the value of $x_i$ is available, then the next node is $v_L$ with probability $\tilde{\ell}_v(x_i)$ and $v_R$ with probability $1 - \tilde{\ell}_v(x_i)$; if $x_i$ is not available then the next node is $v_L$ with probability $\alpha_v$, and $v_R$ with probability $1 - \alpha_v$.

Once $\tilde{\ell}_v$ and $\alpha_v$ have been specified, we may consider the events $\chi_v$ and $\chi_{v,i}$ as before. Note that the events in $\{\chi_{v,i} : i = 1, 2, \ldots, k\}$ are independent and their intersection is precisely $\chi_v$. Let $\tilde{p}_v(x)$ be the probability of $\chi_v$ in the simulation for input $x$, and similarly let $\tilde{p}_{v,i}(z)$ be the probability of the event $\chi_{v,i}$ when $x_i = z$ (note that the probability of $\chi_{v,i}$ depends only on $x_i$). Clearly, $\tilde{p}_v(x) = \prod_{i=1}^k \tilde{p}_{v,i}(x_i)$, and to show that our simulation is faithful to the original computation, it will suffice to verify that $p_{v,i} = \tilde{p}_{v,i}$ for all $i$. The rest of the proof consists of two steps.

Step 1: Using the values $\beta_{v,i}$ defined above, define $\tilde{\ell}_v$ and $\alpha_v$.

Step 2: Show that for $i \in \{1, 2, \ldots, k\}$ and each node $v \in T$ and $z \in \{0, 1\}$, we have $\tilde{p}_{v,i}(z) = p_{v,i}(z)$.

We now implement this two-step plan. Consider the first step. Recall the values of $\beta_{v,i}$ defined above using a bottom up computation on the tree $T$. We can now define $\alpha_v$ right away based on the $\beta_{v,i}$’s computed above. If $v$ has label $i$, then

$$\alpha_v = \frac{\beta_{vL,i}}{\beta_{v,i}}.$$  

Now, consider an internal node $v$. Let $v_1, v_2, \ldots, v_r = v$ be a path in the tree from the root $v_1$ to the node $v$. Let $\tilde{\beta}_{v,i}$ denote the probability that the computation reaches node $v$ and $x_i$ is not sampled, assuming that the choices at nodes not labelled $i$ do not cause the computation to leave the path. Thus,

$$\tilde{\beta}_{v,i} = \Pr[\chi_{v,i} \land -E_i] = \beta_{\text{root},i} \prod_{j=1}^{r-1} \gamma_j,$$

where $E_i$ is the event “$x_i$ is sampled,” and

$$\gamma_j = \begin{cases} 
1 & \text{if } x_i \text{ is not queried at } v_j \\
\alpha_{v_j} & \text{if } v_{j+1} \text{ is the left child of } v_j \\
1 - \alpha_{v_{j+1}} & \text{if } v_{j+1} \text{ is the right child of } v_j
\end{cases}.$$  

We will show that the following choice for $\tilde{\ell}_v$ ensures that the probability of reaching every node is preserved in our simulation:

$$\tilde{\ell}_v(z) = \frac{p_{vL,i}(z) - \tilde{\beta}_{vL,i}}{p_{v,i}(z) - \tilde{\beta}_{v,i}}.$$  

This completes Step 1.
Now, we move to Step 2 and verify that these definitions ensure that $p_{v,i}(z) = \tilde{p}_{v,i}(z)$. Clearly, the claim is true for the root, for both quantities are 1. Suppose the claim is true for a node $v$. We will now show that it is true of $v_L$ and $v_R$ as well. Consider, $v_L$. We have

$$p_{v_L,i}(z) = \Pr[\chi_{v_L,i} \land \mathcal{E}_i] + \Pr[\chi_{v_L,i} \land \neg \mathcal{E}_i],$$

where the probabilities are computed assuming that $x_i = z$. Using our assumption that the claim holds for $v$, we can compute the first term as

$$\Pr[\chi_{v,i} \land \mathcal{E}_i] \cdot \tilde{\ell}_{v,i}(z) = (p_{v,i}(z) - \tilde{\beta}_{v,i}) \cdot p_{v_L,i}(z) - \tilde{\beta}_{v_L,i}.$$

By definition, the second term is precisely $\tilde{\beta}_{v_L,i}$. It follows that $p_{v_L,i}(z) = \tilde{p}_{v_L,i}(z)$, and the claim holds for $v_L$. A similar calculation shows that the claim holds for $v_R$ as well. This completes Step 2.

Thus, the simulation induces the same distribution on the leaves of $T$ as the original computation, and therefore computes $f$ with the same probability of error.

Proposition 18. Suppose $N = 2k + 1$ is odd. Let $A$ be a sampling algorithm with inputs from $\{0, 1\}^N$, which leaves each variable unread with probability $N^{-k}$. Let $X$ be uniformly distributed on all strings with $k$ or $k + 1$ ones. Then, $\Pr[A(X) = \text{Parity}(X)] \leq \frac{1}{2} + o(1)$.

Proof. (Sketch.) By the Chernoff bound, with probability $1 - o(1)$, the size of the sample picked by $A$ is $N - \Theta(N^{4/5})$. We may assume that $A$ bases its decision only on the number of 1’s in the sample. Let $D_0$ be the distribution of the number of 1’s in the sample conditioned on the number of 1’s in $X$ being $k$ and let $D_1$ be the corresponding distribution conditioned on the number of 1’s in $X$ being $k + 1$. A direct computation shows that the relative entropy

$$S(D_0||D_1) = \sum_i D_0(i) \log \frac{D_0(i)}{D_1(i)} = o(1).$$

It follows that $\ell_1$ distance between $D_0$ and $D_1$ is $o(1)$. Our claim follows from this.

Proof of main theorem. Proposition 8 guarantees that if we have a protocol for computing majority with constant error that uses less than $\frac{1}{C}N \log \log N$ transmissions for some constant $C$, then we have a randomized decision tree to compute majority of $\sqrt{N}$ bits with constant error and $\beta(T) \geq N^{-1/2}$. Theorem 16 then guarantees we have a sampling algorithm $A$ that samples every variable with probability $1 - N^{-\frac{1}{10}}$, and yet manages to compute the majority of $\sqrt{N}$ bits with constant error. But this is impossible by Proposition 18.

4 Conclusions

In this paper, we presented a technique of converting computation on randomized decision tree model to computation on a model of sampling algorithms. We related the uncertainty of an input variable in the randomized decision tree model to the probability that the variable is left unread by the sampling algorithm.
We showed the power of this technique for proving lower bounds by providing elementary arguments to prove all the lower bounds on average noisy decision tree complexity for computing various functions presented by Evans and Pippenger [EP99].

Using our technique, we then presented lower bounds for wireless communication networks where there is a restriction on transmission power. Any bit sent by a transmitter is received (with channel noise) only by receivers which are within the transmission radius of the transmitter. We showed that to compute the parity and majority function of $N$ input bits with constant probability of error, we need $\Omega(N \log \log N)$ transmissions. This result simplifies and extends the same earlier lower bound for parity [DKmR08, DKmR15] and nicely complements the upper bound result of Ying, Srikant and Dullerud [YSD06], which showed that $O(N \log \log N)$ transmissions are sufficient for computing the sum of all the $N$ bits. Our result also implies that the sum of $N$ bits cannot be approximated up to a constant additive error by any constant error protocol for $N(N, R)$ using $o(N \log \log N)$ transmissions, if $R \leq N^{-\beta}$ for some $\beta > 0$.

References

[DKmR08] C. Dutta, Y. Kanoria, D. manjunath, and J. Radhakrishnan. A tight lower bound for parity in noisy communication networks. In Proc. of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1056–1065, January 2008.

[DKmR15] C. Dutta, Y. Kanoria, D. manjunath, and J. Radhakrishnan. How hard is computing parity with noisy communications? arXiv:1502.02290, 2015.

[DR08] C. Dutta and J. Radhakrishnan. Lower bounds for noisy wireless networks using sampling algorithms. In Proc. of the 49th Annual IEEE Symp. on Foundations of Computer Science (FOCS), pages 394–402, Philadelphia, USA, Oct 2008.

[Dut09] C. Dutta. Lower Bounds for Noisy Computations. PhD thesis, Tata Institute of Fundamental Research, Mumbai, India, 2009.

[EP99] W. Evans and N. Pippenger. Average-case lower bounds for noisy boolean decision trees. SIAM Journal on Computing, 28(2):433–446, April 1999.

[FK00] U. Feige and J. Kilian. Finding or in noisy broadcast network. Information Processing Letters, 73(1-2):69–75, January 2000.

[FRPU94] U. Feige, P. Raghavan, D. Peleg, and E. Upfal. Computing with noisy information. SIAM Journal on Computing, 23(5):1001–1018, October 1994.

[Gal88] R. G. Gallager. Finding parity in simple broadcast networks. IEEE Transactions on Information Theory, 34:176–180, 1988.

[GK00] P. Gupta and P. R. Kumar. The capacity of wireless networks. IEEE Transactions on Information Theory, 46(2):388–404, March 2000.

[GKS05] N. Goyal, G. Kindler, and M. E. Saks. Lower bounds for the noisy broadcast problem. In Proc. of the 46th Annual IEEE Symposium on Foundations of Computer Science; full version available at www.math.rutgers.edu/~saks/PUBS/nb-submitted.pdf, pages 40–52, October 2005.
[KM98] E. Kushilevitz and Y. Mansour. Computation in noisy radio networks. In *Proceedings of the 9th annual ACM-SIAM Symposium on Discrete Algorithms*, pages 236–243, 1998.

[KM07] Y. Kanoria and D. Manjunath. On distributed computation in noisy random planar networks. In *Proc. of the IEEE International Symposium on Information Theory*, France, June 2007.

[New04] I. Newman. Computing in fault tolerance broadcast networks. In *proc. of the 19th IEEE Annual Conference on Computational Complexity*, pages 113–122, 2004.

[RS91] R. Reischuk and B. Schmeltz. Reliable computation with noisy circuits and decision trees - a general $n \log n$ lower bound. In *Proc. of the 32nd Annual IEEE Symposium on Foundations of Computer Science*, pages 602–611, 1991.

[YSD06] L. Ying, R. Srikant, and G. Dullerud. Distributed symmetric function computation in noisy wireless sensor networks with binary data. In *Proc. of the 4th International Symposium on Modeling and Optimization in Mobile, Ad-Hoc and Wireless networks (WiOpt)*, pages 1–9, April 2006.