Periodic motion of a charge on a manifold in the magnetic fields

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Abstract

In this paper we prove the existence of a periodic motion of a charge on a large class of manifolds under the action of the magnetic fields. Our methods also give a class of closed manifolds whose cotangent bundles contain no the closed exact Lagrangian submanifolds.

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1 Introduction and results

1.1 The question

The periodic motion question of a charge on an Riemannian manifold \((N,g)\) in the magnetic field (abbreviated to “PMMQ” below) is a very important and difficult question in the mathematics and physics([Ar1][No]). It can be formulated as

\[
\text{PMMQ. Looking for the nonconstant periodic solutions of Hamiltonian system}
\]

\[
\dot{z} = X_{H_g}(z)
\]

on the energy level \(E_c = \{H_g = c\} \) with \(c > 0\), where \(H_g : T^*N \to \mathbb{R}\) is given by \(H_g(z) = \frac{1}{2}\|z\|^2_g\)
and \(X_{H_g}\) is the Hamiltonian vector field of \(H_g\) with respect to the twisted symplectic form \(\omega = \omega_{\text{can}} + \pi^*_N\Omega\) on \(T^*N\), the closed 2-form \(\Omega\) on \(N\) corresponds to the magnetic field.

In order to study it S.P.Novikov invented the variational principle of multi-valued functionals([No][GN][NT][T]), V.I.Arnold introduced the symplectic topology methods( [Ar2][Gi1]). On the detailed arguments of the history and progress of this question before 1995 the readers may refer to Ginzburg’s beautiful survey paper [Gi1]. In addition, as showed by Example 3.7 in [Gi1] or Example 4.2 in [Gi2] one cannot expect that the above question has always a solution. Thus it becomes very important to study some conditions under which the above question holds.

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1.2 The symplectic topology methods

A well-known question in symplectic geometry is Weinstein conjecture, which claims: every hypersurface $S$ of contact type in a symplectic manifold $(Q, \omega)$ carries at least a closed characteristic([W]). Here $S$ is said to be of contact type if there exists a transversal vectorfield $X$ defined on some open neighborhood $U$ of $S$ such that $L_X\omega = \omega([W])$. It is not difficult to check that the energy level $E_c = \{ H_g = c \}$ above cannot be of contact type in the sense if the magnetic field $\Omega$ is not exact (see Remark 1.6.A in §1.6 below). Thus PMMQ is different from the Weinstein conjecture in the symplectic manifold $(T^*N, \omega_{can} + \pi_N^*\Omega)$. Fortunately, motivated by the study of the latter Hofer and Zehnder introduced an important notion, Hofer-Zehnder symplectic capacity, which can not only be used to study the Weinstein conjecture but also PMMQ above. Let us first recall it. Given a symplectic manifold $(Q, \omega)$ we denote by $\mathcal{H}(Q, \omega)$ the subset of $C^\infty(Q, \mathbb{R})$ consisting of all smooth functionas with the following properties:

- There exists a compact subset $K \subset Q \setminus \partial Q$ depending on $H$ such that $H|_{Q \setminus K} \equiv m(H)$, i.e. a constant;
- There exists a nonempty open subset $U$ depending on $H$ such that $H|_U \equiv 0$;
- $0 \leq H(x) \leq m(H)$ for all $x \in Q$.

We shall call $H \in \mathcal{H}(Q, \omega)$ admissible (resp. C-admissible) if it has the property that all $T$-periodic (resp. contractible) solutions of $\dot{x} = X_H(x)$ on $Q$ having periods $0 < T \leq 1$ are constant solutions. Writing $\mathcal{H}_{ad}(Q, \omega)$ (resp. $\mathcal{H}_{cad}(Q, \omega)$) the sets of admissible (resp. C-admissible) $H \in \mathcal{H}(Q, \omega)$, we define

$$C_{HZ}(Q, \omega) = \sup \{ m(H) \mid H \in \mathcal{H}_{ad}(Q, \omega) \}, \quad \bar{C}_{HZ}(Q, \omega) = \sup \{ m(H) \mid H \in \mathcal{H}_{cad}(Q, \omega) \}.$$ 

Obviously, both are symplectic invariants, and it always holds that $C_{HZ}(Q, \omega) \leq \bar{C}_{HZ}(Q, \omega)$. $\bar{C}_{HZ}$ was first introduced in [Lu1]. We still call it Hofer-Zehnder symplectic capacity. As proved by Hofer and Zehnder([HZ]) and strengthened by Struwe ([St]) if $C_{HZ}(Q, \omega) < +\infty$ then for any compact hypersurface $S \subset Q$ and any embedding $\iota : S \times [0, 1] \to Q$ there exists a set $J \subset [0, 1]$ of parameters of measure 1 such that for every $s \in J$ the Hamiltonian flow on $\iota(S \times \{s\})$ carries a periodic orbit. We have showed in [Lu1][Lu2] that the similar conclusion for $\bar{C}_{HZ}$ still holds. More precisely saying, if $\bar{C}_{HZ}(\iota(S \times [0, 1]), \omega)$ is finite then for every parameter $s$ in a set $J$ of measure 1 the Hamiltonian flow on $\iota(S \times \{s\})$ carries a contractible (in $\iota(S \times \{s\})$) periodic orbit since $\iota(S \times \{s\})$ has the same homotopy type as $\iota(S \times [0, 1])$. However there exists a difference between $C_{HZ}$ and $\bar{C}_{HZ}$. That is, it only satisfies the following monotonicity axiom of weaker form:

For a symplectic embedding $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ of codimension zero, if either $M_1$ is simply connected or $\psi$ induces an injective homomorphism $\psi_* : \pi_1(M_1) \to \pi_2(M_2)$ then it holds that

$$\bar{C}_{HZ}(M_1, \omega_1) \leq \bar{C}_{HZ}(M_2, \omega_2).$$
Hence if we can prove that for a given \( c > 0 \) there exists a sufficiently small number \( \epsilon > 0 \) such that \( C_{HZ}(\{ c - \epsilon \leq H_g \leq c + \epsilon \}, \omega) < +\infty \) (resp. \( \bar{C}_{HZ}(\{ c - \epsilon \leq H_g \leq c + \epsilon \}, \omega) < +\infty \)) then for generic \( c' \) near \( c \) the level \( E_{c'} \) carries a (resp. contractible in \( E_{c'} \)) nonconstant periodic orbit of \( X_{H_g} \).

### 1.3 Some recent results

Before 1995, for the case of the nonexact magnetic field (i.e. \( \Omega \) a nonexact form) on the high dimension the general result is little. For the rational magnetic field, that is, a closed 2-form \( \Omega \) on \( N \) satisfying

\[
m(N, \Omega) = \inf \{ \langle [\Omega], \alpha \rangle > 0 \mid \alpha \in \pi_2(N) \} > 0,
\]

(1.1)

which is called the rationality index of \( \Omega \), we obtained a general result in Corollary E.2 of [Lu1].

Precisely speaking, for any rational closed two-form \( \Omega \) on a closed smooth manifold \( M \) and any Riemannian metric \( g \) on \( N := M \times \mathbb{R}/2\pi \mathbb{Z} \) we proved that

\[
C_{HZ}(\{ v \in T^*N \mid \| v \|_g \leq c \}, \omega_{\text{can}} + \pi^*_N(p_1^*\Omega)) < +\infty
\]

for sufficiently small \( c > 0 \), where \( p_1 : N \to M \) is the natural projection. This implies that there exists a nonconstant periodic orbit of Hamiltonian vector field \( X_{H_g} \) on the energy level \( E_c \) for almost all (resp. sufficiently small) \( c > 0 \) (in the sense of measure theory) if \( m(M, \Omega) = +\infty \) (resp. \( 0 < m(M, \Omega) < +\infty \)). For the periodic motions on the tours under the action of magnetic fields \( \Omega \) Mei-Yue Jiang proved that the generic levels of \( H_g \) carry a nonconstant periodic orbit provided that de Rham cohomology class \( [\Omega] \) is rational ([J1]). Recently, Ginzburg and Kerman removed the rationality assumption on \( \Omega \) in ([GiK]). It is very surprising that L. Polterovich used Hofer’s geometry approach to show that if for any nonzero magnetic field \( \Omega \) and Riemannian metric \( g \) on \( \mathbb{T}^n \) there exists a sequence of positive energy values \( c_k \) \( c_k(g, \Omega) \to 0 \) such that every level \( \{ H_g = c_k \} \) carries a nonconstant contractible closed orbit([P2]). For the case of the exact magnetic field readers may refer to [Vi2] [BT].

### 1.4 The exact Lagrangian embedding and normal submanifolds

Recall that a submanifold \( L \) of middle dimension in a symplectic manifold \( (Q, \omega) \) is called normal if there is a field of Lagrangian subspaces along \( L \) which is transversal to \( L([Si]) \). In [P1] it was proved that each Lagrangian submanifold \( L \), and those submanifolds which are sufficiently \( C^1 \)-close to \( L \), and each parallelizable totally real submanifold of \( Q \) with respect to some \( J \in \mathcal{J}(Q, \omega) \) are all normal.

**Example 1.4.A.** Every closed orientable 3-dimensional manifold is parallelizable totally real submanifold of \( C^3 \), and thus a normal submanifold in \( (C^3, \omega_0) \). It should be noted that \( S^3 \) is such a manifold and satisfies: \( H^1(S^3, \mathbb{R}) = H^2(S^3, \mathbb{R}) = 0 \), but it can not be embedded into \( (C^3, \omega_0) \) in the Lagrangian way because there is no any closed simply connected Lagrangian submanifold in
\((\mathbb{C}^n, \omega_0)\). On the other hand, the necessary and sufficient condition of \(n\)-dimensional totally real closed submanifold in \(\mathbb{C}^n\) was obtained in [A]( also refer to [Th 3.2.4, ALP]).

The following proposition is a key to proof of Theorem 1.5.A. We believe itself to have some independent importance.

**Proposition 1.4.B.** Let \(L\) be a closed normal submanifold in a symplectic manifold \((Q, \sigma)\), and \(g\) a Riemannian metric on \(L\). Assume that \(\sigma\) is exact near \(L\), i.e., \(\sigma = d\tau\) for some one-form near \(L\). Then for any \(c > 0\) there exists a \(0 < \delta_0(c) < 1\) such that for every \(0 < \delta \leq \delta_0(c)\) we have a symplectic embedding of codimension zero \(E_\delta\) from \((\{H_g \leq c\}, \omega^L_{\text{can}} + \pi^*_L(\sigma|_L))\) into \((Q, \frac{1}{\delta}\sigma)\). Consequently, for some \(0 < \delta_1 < \delta_0\) and all \(0 < \delta \leq \delta_1\) there also exists a symplectic embedding of codimension zero from \((\{H_g \leq c\}, \omega^L_{\text{can}})\) into \((Q, \frac{1}{\delta}\sigma)\). Moreover, for a given open neighborhood of \(L\) in \(Q\) we can require \(\delta > 0\) so small that the images of these symplectic embeddings are contained in this open neighborhood.

From Gromov’s striking theorem that there is no exact Lagrangian embedding of a closed manifold into \((\mathbb{R}^{2l}, \omega_0)([Gr])\) we immediately obtain the following corollary.

**Corollary 1.4.C.** For any closed normal submanifold in a symplectic manifold \((\mathbb{R}^{2l}, \omega_0)\) there is no exact Lagrangian embedding of a closed manifold into \(T^*L\).

For more results on the generalization of Gromov theorem the readers may refer to [V2][V3].

The monotonicity of the symplectic capacity directly leads to

**Corollary 1.4.D.** Let \(L\) be a closed normal submanifold in a symplectic manifold \((Q, \sigma)\). If there exists a neighborhood of \(L\), \(U\) such that \(\sigma\) is exact on it and \((U, \sigma)\) has finite symplectic capacity. Then for any Riemannian metric \(g\) on \(L\) and every \(c > 0\) the symplectic manifold \((\{H_g \leq c\}, \omega^L_{\text{can}})\) has finite capacity.

### 1.5 Main results

The manifolds in our main results below will be the product manifold \(N = M \times L\) of a closed smooth manifold \(M\) and a compact normal submanifold \(L\) without boundary of \((\mathbb{R}^{2l}, \omega_0)\). Denote by \(P_M : N \to M\) is the natural projection to the second factor and by \(P^*_M : H^2_{de}(M; \mathbb{R}) \to H^2_{de}(N; \mathbb{R})\) the homomorphism between their second de Rham cohomology groups induced by it. We shall omit the subscript “de” in the de Rham cohomology groups below. Then \(P^*_M(H^2(M, \mathbb{R}))\) is a subspace of \(H^2(N, \mathbb{R})\). Similarly, for any diffeomorphism \(\phi \in \text{Diff}(N)\) we denote by \(\phi^*\) the induced isomorphism between \(H^2(N; \mathbb{R})\) and itself. We get a subset of \(H^2(N; \mathbb{R})\) as follows

\[
\bigcup_{\phi \in \text{Diff}(N)} \phi^*(P^*_M(H^2(M, \mathbb{R}))). \tag{1.2}
\]

It seems to be strange. However, if \(H^2(L, \mathbb{R}) = 0\) and \(H^1(L, \mathbb{R}) = 0\) it directly follows from the Künneth formula that \(H^2(N, \mathbb{R})\) and \(H^2(M, \mathbb{R})\) is isomorphic. Thus in the case the set in (1.2) is equals to \(H^2(N, \mathbb{R})\).

Our first result to PMMQ above is
**Theorem 1.5.A.** Let $N = M \times L$ be a closed smooth manifold $M$ and a compact normal submanifold $L$ without boundary of $(\mathbb{R}^2, \omega_0)$ as above. $\Omega$ is a closed two-form on $N$ whose de Rham cohomology class $[\Omega]$ belongs to the set in (1.2). If $\Omega|_{\pi_2(N)} = 0$, then for every Riemannian metric $g$ on $N$ and $c > 0$ it holds that

$$C_{HZ}(\{H_g \leq c\}, \omega) < +\infty,$$

where $\omega := \omega_{can}^N + \pi_N^* \Omega$. Consequently, for generic $c > 0$ the level $E_c = \{H_g = c\}$ carries a nonconstant periodic orbit of $X_{H_g}$. Here $X_{H_g}$ is the Hamiltonian vector field determined by $i_{X_{H_g}} \omega = dH_g$. Specially, if $L$ is simply connected we can also guarantee such generic levels $E_c = \{H_g = c\}$ to carry a nonconstant periodic orbit with the contractible projection to $N$.

**Corollary 1.5.B.** If $H^2(L, \mathbb{R}) = 0$ and $\pi_1(M)$ is a finite group then for any Riemannian metric $g$ and a closed 2-form $\Omega$ on $N$ with $\Omega|_{\pi_2(N)} = 0$ the generic level $E_c = \{H_g = c\}$ carries a nonconstant periodic orbit of $X_{H_g}$.

If a submanifold $L$ of $(\mathbb{R}^2, \omega_0)$ is Lagrangian, rather than normal Theorem 1.5.A can be strengthened. The following is the second result to PMMQ.

**Theorem 1.5.C.** Let $N = M \times L$ be a product of a closed smooth manifold $M$ and a closed Lagrangian submanifold $L$ of $(\mathbb{R}^2, \omega_0)$. $\Omega$ is a rational closed two-form on $N$ whose de Rham cohomology class $[\Omega]$ belongs to the set in (1.2). Then for every Riemannian metric $g$ on $N$ there is an upper semi-continuous function $\Gamma^g_{\Omega} : [0, +\infty) \to [0, +\infty)$ (see (3.13)) such that for every $c > 0$ with $\Gamma^g_{\Omega}(c) < \sqrt{m(N, \Omega)}/\pi$ it holds that

$$\liminf_{c \to 0+} C_{HZ}(U(g, c, c), \omega) < \pi(\Gamma^g_{\Omega}(c))^2 < m(N, \Omega),$$

(1.3)

where $U(g, c, c) := \{z \in T^*N \mid c - \epsilon \leq H_g(z) \leq c + \epsilon\}$. Consequently, for almost all $c' > 0$ near $c$ the level $E_{c'} = \{H_g = c'\}$ carries a nonconstant periodic orbit of $X_{H_g}$, where $X_{H_g}$ is the Hamiltonian vector field of $H_g$ with respect to the symplectic form $\omega = \omega_{can}^N + \pi_N^* \Omega$.

**Remark 1.5.D.** In Corollary 1.5.C, if we denote by

$$c(\Omega, g) := \inf_{(\phi, \bar{\Omega}, \alpha)} \|\alpha\|_g,$$

where $\|\alpha\|_g = \sup_{z \in N} \sqrt{g(\alpha(z), \alpha(z))}$ and the infimum is taken over all possible $(\phi, \bar{\Omega}, \alpha)$ satisfying (2.8). Then when $c(\Omega, g)$ is small enough and $m(N, \Omega)$ is large enough the inequality

$$\Gamma^g_{\Omega}(c) < \sqrt{m(N, \Omega)/\pi}$$

(1.4)

always holds for sufficiently small $c > 0$.

In fact, for any given sufficiently small $\epsilon > 0$ we may choose $(\phi, \bar{\Omega}, \alpha)$ such that $\|\alpha\|_g < c(\Omega, g) + \epsilon$ and the image set of $\alpha$ (as a section) is contained in the image of $Y$ in (3.4). This is possible if $c(\Omega, g)$ is small enough. Notice that in this case $\Theta$ is a symplectomorphism from $(T^*N, \omega)$ to the symplectic manifold in (3.3) with $P_L^*(\lambda_0|_L) = 0$. Hence for sufficiently small $c > 0$ the set $\Theta(\{H_g = c\})$ is also contained in the image of $Y$. Denote by

$$r(c, g, \Theta, Y) := \inf\{r > 0 \mid P_L[\pi|_{\text{Im}(Y)}]^{-1} \circ \Theta(\{H_g = c\})] \subseteq U \cap F(Z^2(r)), \quad F \in \text{Symp}(\mathbb{R}^2, \omega_0)\},$$

r(c, g, \Theta, Y) := \inf\{r > 0 \mid P_L[\pi|_{\text{Im}(Y)}]^{-1} \circ \Theta(\{H_g = c\})] \subseteq U \cap F(Z^2(r)), \quad F \in \text{Symp}(\mathbb{R}^2, \omega_0)\},
then when \( m(N, \Omega) \) is so large that \( r(c, g, \Theta, \Upsilon) < \sqrt{m(N, \Omega)/\pi} \), or more general \( U \subset \mathbb{Z}^d(r_0) \) for some \( 0 < r_0 < \sqrt{m(N, \Omega)/\pi} \) it holds that

\[
\Gamma^g_{\Omega}(c) < \sqrt{m(N, \Omega)/\pi}.
\]

In some special cases we can get better results. For example, let \( L = S^1 = \mathbb{R}/2\pi\mathbb{Z} \) and \( \Omega \) a rational closed 2-form on \( N = M \times S^1 \) whose de Rham cohomology class belongs to the set in (1.2). For any Riemannian metric \( g \) on \( N \) we define the function \( \Xi^g_{\Omega} : [0, +\infty) \to [0, +\infty) \) by

\[
\Xi^g_{\Omega}(c) := \inf_{(\Theta, \hat{\Theta}, \epsilon)} \left| \int_{P_S(\Theta((U(g, c, \epsilon)))} \omega^S_{\text{can}} \right|,
\]

where the infimum is taken over all pairs \( (\Theta, \hat{\Theta}) \) satisfying Lemma 2.3 and all \( \epsilon > 0 \).

Our third result to PMMQ is given as follows.

**Theorem 1.5.E.** Under the above assumptions, if \( c > 0 \) is such that \( \Xi^g_{\Omega}(c) < m(N, \Omega) \) then

\[
\lim_{\epsilon \to 0^+} \inf \ C_{HZ}(U(g, c, \epsilon), \omega) < \pi(\Xi^g_{\Omega}(c))^2 < m(N, \Omega),
\]

and therefore for almost all \( c' > 0 \) near \( c \) the levels \( \{H_g = c'\} \) carries a nonconstant periodic orbit of \( X_{H_g} \), where \( X_{H_g} \) is the Hamiltonian vector field of \( X_{H_g} \) with respect to the symplectic form \( \omega = \omega_{\text{can}} + \pi^*_N \Omega \). In addition, the function \( \Xi^g_{\Omega} \) is upper semi-continuous, and also satisfies:

\[
\Xi^g_{\lambda \Omega}(\lambda c) = \lambda \Xi^g_{\Omega}(c) \quad \forall c > 0, \lambda > 0.
\]

This result can be generalized to the case that \( L = T^n \).

**Corollary 1.5.F.** Let \( \Omega \) be a rational closed 2-form on \( N = M \times T^n \) whose de Rham cohomology class belongs to the set in (1.2). Then for any Riemannian metric \( g \) on \( N \) there exists a nonnegative upper semi-continuous function \( \widehat{\Xi}^g_{\Omega} : [0, +\infty) \to [0, +\infty) \) such that for every \( c > 0 \) with \( \widehat{\Xi}^g_{\Omega}(c) < m(N, \Omega) \) and almost all \( c' > 0 \) near \( c \) the levels \( \{H_g = c'\} \) carries a nonconstant periodic orbit of \( X_{H_g} \), where \( X_{H_g} \) is the Hamiltonian vector field of \( H_g \) with respect to the symplectic form \( \omega = \omega_{\text{can}} + \pi^*_N \Omega \). Specially, the function \( \widehat{\Xi}^g_{\Omega} \) also satisfies

\[
\widehat{\Xi}^g_{\lambda \Omega}(\lambda c) = \lambda \widehat{\Xi}^g_{\Omega}(c) \quad \forall c > 0, \lambda > 0.
\]

With the method of the proof of Corollary 1.5.B and Theorems 1.5.C, 1.5.E we can easily arrive at the following corollary.

**Corollary 1.5.G.** In Theorem 1.5.C, 1.5.E, if \( H^2(L, \mathbb{R}) = 0 \) and \( \pi_1(M) \) is a finite group then for any Riemannian metric \( g \) and any rational closed 2-form \( \Omega \) on \( N \) the corresponding conclusions therein hold.

**Remark 1.5.H.** One may think that it is difficult to determine the values of \( \Gamma^g_{\Omega} \) and \( \Xi^g_{\Omega} \) in the Theorems and Corollaries above. But we affirm them to be finite numbers, and if \( \Omega_{|\pi_2(N)} = 0 \)
then \( m(N, \Omega) = +\infty \) and thus the conditions of these theorems are always satisfied in this case for all \( c > 0 \). On the other hand since the functions \( \Gamma^g_0 \) and \( \Xi^g_0 \) are upper semi-continuous the sets \( \{ c > 0 | \Gamma^g_0(c) < \sqrt{m(N, \Omega)/\pi} \} \) and \( \{ c > 0 | \Xi^g_0(c) < m(N, \Omega) \} \) are open. Hence if they are not empty then there exist nonconstant periodic solutions on the levels \( \{ H_g = c \} \) for all \( c \) in a positive measure set ([St]). On the other hand one may think that it is difficult to understand the meaning of functions \( \Gamma^g_0 \) and \( \Xi^g_0 \). Our starting points are to attempt using a series of symplectic embeddings of codimension zero to reduce our question to the case for which Theorem 2.1 may be applied, and to guarantee each step being optimal so that Theorem 2.1 is best applied. Both functions are to characterize the optimization in the way of the quantity. If the rationality condition \( m(M, \omega) > 0 \) in Theorem 2.1 can be removed then our arguments show that the generic energy levels carry a nonconstant Hamiltonian periodic orbit. However, it is regrettable for us not to be able to remove this assumption yet.

Our final result to PMMQ is about the case of tours \( T^n \). Let \( \Omega \) be a magnetic field (a closed 2-form) on it, and \( \omega = \omega_{\text{can}} + \pi^*_n \Omega \). As pointed out in §1.3 one had known that for any metric \( g \) on \( T^n \) generic levels \( E_c = \{ H_g = c \} \) in \( (T^*T^n, \omega) \) carries a nonconstant periodic orbit of \( X_{H_g}([\text{GiK}]) \). When \( \Omega \) is not exact we can furthermore obtain

**Theorem 1.5.I.** If \( n \geq 2 \) and the de Rham cohomology class \( [\Omega] \) is nonzero then for generic \( c > 0 \) the levels \( E_c = \{ H_g = c \} \) carries a nonconstant periodic orbit of \( X_{H_g} \) whose projection to the base \( T^n \) is contractible.

### 1.6 Two remarks

**Remark 1.6.A.** For a given Riemannian metric \( g \) on \( N \) and \( c > 0 \) we denote by

\[
\Sigma^g_c := \{ v \in T^*N | \| v \|_g = c \}.
\]

Let a closed two-form \( \Omega \) on \( T^*N \) be such that \( \omega_{\text{can}} + \Omega \) is a symplectic form on \( T^*N \). One may ask whether for sufficiently large \( c > 0 \) the hypersurface \( \Sigma^g_c \) is of contact type in the symplectic manifolds \( (T^*N, \omega_{\text{can}} + \Omega) \)? If this holds then our partial results may be derived from one in [Vi2].

The following proposition answers this question.

**Proposition 1.6.** If \( \Omega \) is not exact the hypersurface \( \Sigma^g_c \) cannot be of contact type in the symplectic manifolds \( (T^*N, \omega_{\text{can}} + \Omega) \).

**Proof.** Assume \( \Sigma^g_c \) to be of contact type for \( \omega_{\text{can}} + \Omega \), then there exists a sufficiently small \( \epsilon > 0 \) such that \( \omega_{\text{can}} + \Omega \) is exact on \( U(g, c, \epsilon) := \{ v \in T^*N | c - \epsilon < \| v \|_g < c + \epsilon \} \). Thus there is a one-form \( \alpha \) on \( U(g, c, \epsilon) \) such that

\[
\Omega = d\alpha \quad \text{on} \quad U(g, c, \epsilon)
\]

because \( \omega_{\text{can}} \) is always exact. Notice that \( \pi^*_N : T^*N \rightarrow N \) induces a natural isomorphism \( \pi^*_N : H^*(N, \mathbb{R}) \rightarrow H^*(T^*N, \mathbb{R}) \) and \( \pi^*_N([\Omega|_N]) = [\pi^*_N(\Omega|_N)] = [\Omega] \), we have \( \pi^*_N(\Omega|_N) - \Omega = d\beta \) for some one-form \( \beta \) on \( T^*N \). Hence

\[
\pi^*_N(\Omega|_N) = d(\alpha + \beta) \quad \text{on} \quad U(g, c, \epsilon)
\]
Take a smooth section \( F : N \to T^* N \) of \( \pi_N \) such that \( F(N) \subset U(g, c, \epsilon) \) (e.g., \( F \) is the zero section) then as a smooth map from \( N \) to \( T^* N \) it satisfies:

\[
F^* \circ \pi_N^*(\Omega|_N) = F^*d(\alpha + \beta) = d(F^*(\alpha + \beta)).
\]

But \( F^* \circ \pi_N^*(\Omega|_N) = (\pi_N \circ F)^*(\Omega|_N) \) and \( \pi_N \circ F = id_N \). Hence we get

\[
\Omega|_N = d(F^*(\alpha + \beta)).
\]

That is, \( \Omega|_N \) is a exact form on \( N \). Denote by \( \gamma := F^*(\alpha + \beta) \). Then

\[
\Omega = \pi_N^*(\Omega|_N) - d\beta = \pi_N^*(d\gamma) - d\beta = d(\pi_N^*\gamma - \beta),
\]

this leads to a contradiction. \( \square \)

**Remark 1.6.B.** Our results always deal with manifolds of product forms. These are due to the limitation of our methods. The following example shows that increasing a factor manifold in the base manifold will have a real influence on periodic orbits of magnetic fields.

Let \( M \) be a compact surface equipped with a metric \( g_0 \) of constant curvature \( K = -1 \) and \( \Omega \) the area form on \( M \). From Example 3.7 in [Gi1] we know that if \( c > 1 \) on the level \( E_c \) there are no closed characteristic with contractible projections with respect to the symplectic structure \( \omega_M = d\lambda_M + \pi_M^*\Omega \). Consider \( N := M \times S^3 \) and \( \omega = d\lambda_N + \pi_N^*(p_M^*\Omega) \). Here \( p_M : N \to M \) is the natural projection. Notice that \( m(M, \Omega) = +\infty \). By Corollary 1.6.C, for any Riemannian metric \( g_1 \) and the product metric \( g = g_0 \times g_1 \) on \( N \) and generic \( c > 0 \) the levels \( E_c := \{ H_g = c \} \) carries a nontrivial closed characteristic with the contractible projection to \( N \). On the other hand, Example 3.7 in [Gi1] showed that for \( c > 1 \) the Hamiltonian flow of \( X_{H_{g_0}} \) with respect to \( \omega_M \) on \( E_c^M := \{ H_{g_0} = c \} \) has no any closed characteristic with contractible projections to \( M \). Notice that \( E_c^M \times 0(S^3) \subset E_c \) for the zero section \( 0(S^3) \) of \( T^*S^3 \).

Our arguments are the symplectic topology methods. In §2 we give some lemmas and prove Proposition 1.4.B. The proofs of all theorems and corollaries are given in §3. Finally, some concluding remarks are given in §4.

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## 2 Some lemmas and proof of Proposition 1.4.B

Let us first recall the following theorem.
Theorem 2.1 ([Th C, Lu1]). Let $(M, \omega)$ be a strong geometrical bounded symplectic manifold with $m(M, \omega) > 0$. If $r \in (0, \sqrt{m(M, \omega)/\pi})$ then

$$C_{HZ}(M \times Z^{2n}(r), \omega \oplus \omega_0) \leq \pi r^2,$$ (2.1)

where $Z^{2n}(r) := \{(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n} | x_1^2 + y_1^2 \leq r^2\}$.

For $C_{HZ}$ this result was obtained in [FHV][HV] for $M$ closed, and in [Ma] for $(M, \omega) = (T^*Q, \omega_{can})$.

In view of our result mentioned in §1.3, a natural question is what conclusions one can get for a general closed two-form $\Omega$ on $N$ rather than on $M$. It was Professor Claude Viterbo who asked whether for a given closed two-form on $N$ there exist a closed two-form $\hat{\Omega}$ on $M$ and a $\Psi \in \text{Diff}(T^*N)$ such that

$$\omega_{can} + \pi_N^* \Omega = \Psi^*(\omega_{can} + \pi_N^*(P_M^* \hat{\Omega} + P_L^*(\omega_0|_L)))?$$ (2.2)

This idea motivates the studies of this paper.

The following lemma directly follows from the local coordinate arguments.

**Lemma 2.2.** For any closed 2-form $\Omega$ and 1-form $\alpha$ on a manifold $N$ the diffeomorphism $\Psi : T^*N \to T^*N$ given by

$$(m, v) \mapsto (m, v + \alpha(m))$$ (2.3)

satisfies:

$$\Psi^*(\omega_{can} + \pi_N^* \Omega) = \omega_{can} + \pi_N^* \Omega + \pi_N^*(d\alpha).$$ (2.4)

That is, $\Psi$ is a symplectomorphism from $(T^*N, \omega_1)$ to $(T^*N, \omega)$, where

$$\omega := \omega_{can} + \pi_N^* \Omega \quad \text{and} \quad \omega_1 := \omega_{can} + \pi_N^* \Omega + \pi_N^*(d\alpha).$$ (2.5)

**Lemma 2.3** Let $N = M \times L$ be as in Theorem 1.5.A. If the de Rham cohomology class $[\Omega]$ of a given closed two-form $\Omega$ on $N$ belongs to the set in (1.2) then there exists a closed two-form $\hat{\Omega}$ on $M$ with

$$m(N, \Omega) = m(M, \hat{\Omega})$$ (2.6)

such that $(T^*N, \omega)$ is symplectomorphic to $(T^*N, \tilde{\omega})$, where $\tilde{\omega}$ is given by

$$\tilde{\omega} := \omega_{can} + \pi_N^*(P_M^* \hat{\Omega} + P_L^*(\omega_0|_L)).$$ (2.7)

Specially, if $H^2(L, \mathbb{R}) = 0$ and either $H^1(M, \mathbb{R}) = 0$ or $H^1(L, \mathbb{R}) = 0$ then for every closed two-form $\Omega$ on $N$ the above conclusions hold.

**Proof.** Since $[\Omega]$ belongs to the set in (1.2) there exists a $\phi \in \text{Diff}(N)$ such that $[\phi^* \Omega] = \phi^*[\Omega]$ belongs to $P_M^*(H^2(M, \mathbb{R}))$, and thus there must exist a closed two-form $\hat{\Omega}$ on $M$ and a one-form $\alpha$ on $N$ such that

$$\phi^* \Omega - P_M^* \hat{\Omega} = d\alpha.$$ (2.8)
But such \( \phi, \Omega \) and \( \alpha \) may not be unique. Notice that \( \phi \) may lift to a symplectomorphism \( \Phi : (T^*N, \omega_{\text{can}}) \to (T^*N, \omega_{\text{can}}) \) by the formula

\[
\Phi(m, v) = (\phi(m), [d\phi(m)^{-1}]^*(v)).
\]

They satisfy that \( \pi_N \circ \Phi = \phi \circ \pi_N : T^*N \to N \) and therefore

\[
\Phi^* \circ \pi_N^* = \pi_N^* \circ \phi^* : \Omega^2(N) \to \Omega^2(T^*N).
\]

By the definition of \( \omega \) in (2.5) we get

\[
\Phi^* \omega = \omega_{\text{can}} + \pi_N^*(\phi^* \Omega) = \omega_{\text{can}} + \pi_N^*(P_M^* \Omega) + d(\pi_N^* \alpha) = \omega + d(\pi_N^*(\alpha - P_M^*(\lambda_0|_L)))
\]

because \( \omega_0 = d\lambda_0 \) is the standard symplectic form on \( \mathbb{R}^{2l} \). By Lemma 2.2 there exists a diffeomorphism \( \Psi \in \text{Diff}(T^*N) \) given by

\[
(m, v) \mapsto (m, v + \alpha(m) - P_M^*(\lambda_0|_L)(m))
\]

such that \( \Psi^* \hat{\omega} = \Phi^* \omega \) or \( (\Psi \circ \Phi^{-1})^* \hat{\omega} = \omega \). Then

\[
\Theta = \Theta(\Omega, \hat{\Omega}, \phi, \alpha) := \Psi \circ \Phi^{-1}
\]

given by

\[
(m, v) \mapsto \left( \phi^{-1}(m), d\phi(m)^*(v) + (\alpha - P_M^*(\lambda_0|_L)) (\phi^{-1}(m)) \right)
\]

is a symplectomorphic function from \( (T^*N, \omega) \) to \( (T^*N, \hat{\omega}) \). As to (2.6), notice that \( P_M \) induces a surjective homomorphism \( P_M^* : \pi_2(N) \to \pi_2(M) \) and that \( \phi \) induces an isomorphism \( \phi_* : \pi_2(N) \to \pi_2(N) \), it may follow from (1.1) and the equalities

\[
\langle [\Omega], \beta \rangle = \langle (\phi^{-1})^* \circ \phi^*([\Omega]), \beta \rangle = \langle \phi^*([\Omega]), \phi^{-1}_*(\beta) \rangle = \langle P_M^*([\hat{\Omega}]), (\phi^{-1})_*(\beta) \rangle = \langle [\hat{\Omega}], P_M^*(\phi^{-1}_*(\beta)) \rangle \quad \forall \beta \in \pi_2(N).
\]

The final claim is a direct consequence of Künneth formula because in this case \( P_M \) induces an isomorphism \( P_M^* : H^2(M, \mathbb{R}) \to H^2(N, \mathbb{R}) \). Consequently, Lemma 2.3 holds.

In order to prove Theorem 1.5.A we also need the following lemma, which perhaps goes back to the early work of Weinstein and Givental.

**Lemma 2.4**(cf.[P1, Prop.1.9] and [Si, Th. 2.4]). *For a closed normal submanifold \( L \) in a symplectic manifold \( (Q, \sigma) \) and the restriction \( \sigma|_L \) there exists an open neighbourhood \( U \) of \( L \) in \( Q \) and an embedding \( \varphi : U \to T^*L \) such that \( \varphi(L) \) coincides with the zero section of \( T^*L \) and \( \varphi^*(\omega_{\text{can}}^L + \pi^*_L(\sigma|_L)) = \sigma \).*

For a closed smooth manifold \( L \) of dimension \( l \) we choose an atlas \( \{(U_\alpha, \alpha)\} \) consisting of \( l \) local coordinate charts. We also require each \( \alpha(U_\alpha) \) to be equal to the unit ball \( B^l \) centred at origin in \( \mathbb{R}^l \). For \( q \in U_\alpha \) let \( \alpha(q) = (x_1^\alpha(q), \ldots, x_l^\alpha(q)) \). It induces an obvious bundle trivialization \( \Phi_\alpha : T^*U_\alpha \to B^l \times \mathbb{R}^l \) given by

\[
(q, v^s) \mapsto (x_1^\alpha(q), \ldots, x_l^\alpha(q); y_1^\alpha(q, v^s), \ldots, y_l^\alpha(q, v^s)),
\]

for \( q \in U_\alpha \) and \( v \in \mathbb{R}^l \). For \( q \) in the complement of \( U_\alpha \) let \( \alpha(q) = (x_1^\alpha(q), \ldots, x_l^\alpha(q)) \) and let \( \varphi(q) = (x_1^\varphi(q), \ldots, x_l^\varphi(q), y_1^\varphi(q, v^s), \ldots, y_l^\varphi(q, v^s)) \) be a point in \( T^*L \). Then there exists a unique \( \alpha \) such that \( \varphi \in U_\alpha \).
where \( y_\alpha^a(q, v^*) \) are determined by \( v^* \circ d\pi_L(q, v^*) = \sum_{j=1}^l y_j^a(q, v^*) dx_j^a(q) \). The following lemma is very key to the proof of Theorem 1.5.A.

**Lemma 2.5.** Let \( g \) be a Riemannian metric on \( L \) and \( \lambda \) a 1-form on \( L \). Then for given positive numbers \( a \) and \( \epsilon \leq 1 \) there exist \( b_\epsilon > 0 \) and a smooth function \( K_{a, \epsilon} : T^*L \rightarrow [0, 1] \) such that

(i) \( K_{a, \epsilon}(v^*) = 1 \) for \( \|v^*\|_g \leq a \), and \( K_{a, \epsilon}(v^*) = 0 \) for \( \|v^*\|_g \geq b_\epsilon \);

(ii) for \( K^\alpha_{a, \epsilon} := (\Phi^{-1}_a)^* K_{a, \epsilon} : B^l \times \mathbb{R}^l \rightarrow [0, 1] \) and \( \lambda^\alpha = (\alpha^{-1})^* \lambda = \sum_{i=1}^l \lambda_i^\alpha dx_i^\alpha \) it holds that

\[
\left| \sum_{i=1}^l \frac{\partial K^\alpha_{a, \epsilon}}{\partial y_i^\alpha} \lambda_i^\alpha \right| < \epsilon;
\]

(iii) \( A_{\text{can}}^L + Ad(\pi_L^* \lambda) + t(1 - A)d(K_{a, \epsilon} \pi_L^* \lambda) \) are symplectic forms on \( T^*L \) for every \( A > 1 \) and \( t \in [-1, 1] \).

**Proof.** Choose a smooth function \( \gamma : [0, \infty) \rightarrow [0, 1] \) such that (i) \( \gamma(t) = 1 \) for \( t \leq 1 \), (ii) \( \gamma(t) = 0 \) for \( t \geq 3 \), (iii) \( |\gamma'(t)| \leq 1 \). For each \( \epsilon > 0 \) we define

\[
\gamma_{a, \epsilon}(t) := \gamma(\epsilon t + 1 - a \epsilon).
\]

Then \( \gamma_{a, \epsilon}(t) = 1 \) for \( t \leq a \), \( \gamma_{a, \epsilon}(t) = 0 \) for \( t \geq a + 2/\epsilon \), and \( |\gamma_{a, \epsilon}'(t)| \leq \epsilon \) for all \( \epsilon > 0 \). Denote by \( H_{a, \epsilon}((q, v^*)) = \gamma_{a, \epsilon}(\|v^*\|_g) \). It is clearly smooth. Moreover, the local expression of it, \( H_{a, \epsilon}^\alpha := (\Phi^{-1}_a)^* H_{a, \epsilon} \), is given by

\[
H_{a, \epsilon}^\alpha(x_1^\alpha, \ldots, x_l^\alpha; y_1^\alpha, \ldots, y_l^\alpha) = \gamma_{a, \epsilon}\left( \sum_{i,j} g^\alpha_{ij}(x)y_i^\alpha y_j^\alpha \right).
\]

Thus

\[
\frac{\partial H_{a, \epsilon}^\alpha}{\partial y_i^\alpha}(x^\alpha, y^\alpha) = \gamma_{a, \epsilon}'\left( \sum_{i,j} g^\alpha_{ij}(x)y_i^\alpha y_j^\alpha \right) \frac{\partial}{\partial y_i^\alpha} \left( \sum_{i,j} g^\alpha_{ij}(x)y_i^\alpha y_j^\alpha \right) - \gamma_{a, \epsilon}\left( \sum_{i,j} g^\alpha_{ij}(x)y_i^\alpha y_j^\alpha \right) \frac{\partial}{\partial y_i^\alpha} \left( \sum_{i,j} g^\alpha_{ij}(x)y_i^\alpha y_j^\alpha \right).
\]

(2.14)

Denote by

\[
c(\lambda) := \max\{|\lambda_i^\alpha(x)| \mid x \in \alpha(U_\alpha), 1 \leq i \leq l, 1 \leq \alpha \leq m\}.
\]

Notice that there exists a constant \( C(g) > 0 \) such that

\[
\frac{\sum_{i,j} |g^\alpha_{ij}(x)| |y_i^\alpha y_j^\alpha|}{\sqrt{\sum_{i,j} g^\alpha_{ij}(x)y_i^\alpha y_j^\alpha}} \leq C(g)
\]

for all \( x^\alpha \in \alpha(U_\alpha) \) and \( y^\alpha \) and \( 1 \leq \alpha \leq m \). We get

\[
\left( \sum_{i=1}^l \frac{\partial H_{a, \epsilon}^\alpha}{\partial y_i^\alpha}(x^\alpha, y^\alpha) \lambda_i^\alpha(x^\alpha) \right) \leq \left( \sum_{i=1}^l \frac{\partial H_{a, \epsilon}^\alpha}{\partial y_i^\alpha}(x^\alpha, y^\alpha) |\lambda_i^\alpha(x^\alpha)| \right) \leq \varepsilon c(\lambda) C(g)
\]

(2.15)

for all \( x^\alpha \in \alpha(U_\alpha) \) and \( y^\alpha \) and \( 1 \leq \alpha \leq m \). Hence it suffices to choose \( \varepsilon_0 = \varepsilon_0(\epsilon) > 0 \) such that

\[
\varepsilon_0 C(g)c(\lambda) < \epsilon.
\]
Then \( K_{a,\varepsilon} := H_{a, \varepsilon_0} \) satisfies (i)(ii).

As to (iii), note that \( A \omega^L_{\text{can}} + \text{Ad}(\pi^*_L \lambda) + t(1 - A) d(K_{a,\varepsilon} \pi^*_L \lambda) \) has the following local expression

\[
A \left[ \sum_{i=1}^l dx_i^\alpha \wedge dy_i^\alpha + \sum_{i,j} \frac{\partial \lambda_j^\alpha}{\partial x_i^\alpha} dx_i^\alpha \wedge dx_j^\alpha \right] +
\]

\[
t(1-A) \left[ H_{a,\varepsilon_0}^\alpha \sum_{i,j} \frac{\partial \lambda_j^\alpha}{\partial x_i^\alpha} dx_i^\alpha \wedge dx_j^\alpha + \sum_{i,j} \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial x_i^\alpha} \lambda_j^\alpha dx_i^\alpha \wedge dx_j^\alpha - \sum_{i,j} \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial y_j^\alpha} \lambda_i^\alpha dx_i^\alpha \wedge dy_j^\alpha \right],
\]

whose matrix in the natural basis \( \partial/\partial x_1^\alpha, \cdots, \partial/\partial x_l^\alpha; \partial/\partial y_1^\alpha, \cdots, \partial/\partial y_l^\alpha \) is \( S = \begin{pmatrix} S_{xx} & S_{xy} \\ -S_{xy} & S_{yy} \end{pmatrix} \). Here \( S_{yy} = 0 \), \( S_{xx} = (a_{ij}) \) and \( S_{xy} = AI_t - t(1-A)(b_{ij}) \). The matrix elements \( a_{ij} \) and \( b_{ij} \) are given by

\[
a_{ij} = A \left( \frac{\partial \lambda_j^\alpha}{\partial x_i^\alpha} - \frac{\partial \lambda_i^\alpha}{\partial x_j^\alpha} \right) + t(1-A) \left[ H_{a,\varepsilon_0}^\alpha \left( \frac{\partial \lambda_j^\alpha}{\partial x_i^\alpha} - \frac{\partial \lambda_i^\alpha}{\partial x_j^\alpha} \right) + \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial x_i^\alpha} \lambda_j^\alpha - \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial y_j^\alpha} \lambda_i^\alpha \right],
\]

\[
b_{ij} = \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial y_j^\alpha} \lambda_i^\alpha.
\]

Notice that \( S \) is nonsingular if and only if \( S_{xy} \) is so. Assume that \( S_{xy} \zeta = 0 \) for some vector \( \zeta = (\zeta_1, \cdots, \zeta_l)^t \) in \( \mathbb{R}^l \). Then it holds that

\[
A \zeta = t(1-A)(\lambda_1^\alpha, \cdots, \lambda_l^\alpha)^t \left( \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial y_1^\alpha}, \cdots, \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial y_l^\alpha} \right) \zeta = t(1-A) \left( \sum_{i=1}^l \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial y_i^\alpha} \zeta_i \right) (\lambda_1^\alpha, \cdots, \lambda_l^\alpha)^t.
\]

It follows that \( \zeta = B(\lambda_1^\alpha, \cdots, \lambda_l^\alpha)^t \) for some \( B \in \mathbb{R} \). If \( \zeta \neq 0 \) then it holds that

\[
\frac{2A}{A - 1} = t \sum_{i=1}^l \frac{\partial H_{a,\varepsilon_0}^\alpha}{\partial y_i^\alpha} \lambda_i^\alpha.
\]

By (ii) the absolute value of the right hand of (2.18) is less than \( \epsilon < 1 \), and the left hand of it is more than 2 for every \( A > 1 \). This contradiction shows that \( S \) is nonsingular.

Having this key lemma we can prove Proposition 1.4.B as follows.

**Proof of Proposition 1.4.B.** Without loss of generality we may assume \((Q, \sigma)\) to be exact. By Lemma 2.4 there exists an open neighbourhood \( U \) of \( L \) in \( Q \) and an embedding \( \varphi : U \to T^*L \) such that \( \varphi(L) \) coincides with the zero section of \( T^*L \) and \( \varphi^*(\omega^L_{\text{can}} + \pi^*_L(\sigma|_L)) = \sigma \). Applying Lemma 2.5 to \( \lambda = \tau|_L, a = 2c \) and \( \epsilon = 1 \) we get a \( b_1 > 0 \) and a smooth function \( K_{a,1} \). Let us take \( \delta_0(c) > 0 \) so small that the diffeomorphism \( \Phi_\delta : T^*L \to T^*L \) given by

\[
(m, v) \mapsto (m, \delta_0 v)
\]

satisfies:

\[
\Phi_\delta_0(\{H_g \leq 2b_1\}) \subset \varphi(U).
\]

Notice that \( \Phi_\delta \) is a symplectomorphis from \((\{H_g \leq 2b_1\}, \omega^L_{\text{can}} + \pi^*_L(\sigma|_L))\) to

\[
(\Phi_\delta(\{H_g \leq 2b_1\}), \frac{1}{\delta}\omega^L_{\text{can}} + \pi^*_L(\sigma|_L))
\]
for any $0 < \delta \leq \delta_0$. Denote by $\omega_{\delta,t} := \frac{1}{\delta} \omega_{can}^L + \frac{1}{\delta} d(\pi_L^* \lambda) + t(1 - \frac{1}{\delta})d(K_{a,1}^\alpha \pi_L^* \lambda)$ for every $0 < \delta \leq \delta_0$ and $t \in [0,1]$. By Lemma 2.5 (iii) they are symplectic forms on $T^*L$. Moreover, $\omega_{\delta,1}$ and $\frac{1}{\delta} \omega_{can}^L + \pi_L^* (\sigma|_L)$ are same on $\Phi_\delta(\{H_g \leq 2c\})$. By the construction of $\omega_{\delta,t}$ in lemma 2.5 we have

$$\varphi^* \omega_{\delta,t} = \frac{1}{\delta} \sigma + t(1 - \frac{1}{\delta})d(\varphi^*(K_{a,1}^\alpha \pi_L^* (\tau|_L))).$$

(2.22)

They are all symplectic forms on $U$ and equal to $\frac{1}{\delta} \sigma$ near $\partial U$. Let us denote by $\hat{\omega}_{\delta,t} = \frac{1}{\delta} \sigma + t(1 - \frac{1}{\delta})d(\varphi^*(K_{a,1}^\alpha \pi_L^* (\tau|_L)))$ for $t \in [0,1]$. Since $\hat{\omega}_{\delta,t}$ may naturally be extended onto symplectic forms on $Q$ by assuming them being $\frac{1}{\delta} \sigma$ outside $U$ we still denote them by $\hat{\omega}_{\delta,t}$. Using Moser’s technique one can show that there exists a diffeomorphism $F \in \text{Diff}(Q)$ such that

$$F^*(\frac{1}{\delta} \sigma) = \hat{\omega}_{\delta,1}.$$  

(2.23)

Then the desired embedding $E_\delta$ is given by the composition

$$F \circ \varphi^{-1} \circ \Phi_\delta|_{\{H_g \leq c\}}.$$  

(2.24)

The second claim follows from the first one and Lemma 2.2. For the third claim note that $F$ may be identity outside $U$. \hfill \Box

## 3 Proof of the main results

### 3.1 Proofs of Theorem 1.5.A and Corollary 1.5.B

**Proof of Theorem 1.5.A.** Our idea of proof is to show that $C_{HZ}(\{H_g \leq c\}, \omega)$ is finite for any $c > 0$. For this purpose we first use Lemma 2.3 to get a symplectomorphism $\Theta$ from $(T^*N, \omega)$ to

$$(T^*M, \omega_{can}^M + \pi_M^* \hat{\Omega}) \times (T^*L, \omega_{can}^L + \pi_L^* (\omega_0|_L)).$$

(3.1)

Here $\hat{\omega}$ and $\hat{\Omega}$ are given by (2.7) (2.8) respectively. Since the symplectic capacity is symplectic invariant we only need to prove that $C_{HZ}(\{H_g \leq c\}, \hat{\omega})$ is finite for all $c > 0$. Fix a $c > 0$ and take the Riemannian metric $g_1$ on $M$ and $g_2$ on $L$. Then there exist positive numbers $c_1$ and $c_2$ such that $\{z_1 \in T^*M \mid H_{g_1}(z_1) \leq c_1\} \times \{z_2 \in T^*L \mid H_{g_2}(z_2) \leq c_2\}$ contains $\{H_g \leq c\}$. Therefore we only need to prove

$$C_{HZ}(\{H_{g_1} \leq c_1\} \times \{H_{g_2} \leq c_2\}, (\omega_{can}^M + \pi_M^* \hat{\Omega}) \oplus (\omega_{can}^L + \pi_L^* (\omega_0|_L))) < +\infty.$$  

(3.2)

By Proposition 2.6 there exist a $\delta > 0$ and a symplectic embedding of codimension zero $E_\delta$ from $\{(z_2 \in T^*L \mid H_{g_2}(z_2) \leq c_2), \omega_{can}^L + \pi_L^* (\omega_0|_L)\}$ into $(\mathbb{R}^2, \frac{1}{\delta} \omega_0)$. We may assume the image of it to be contained in $Z^2(R)$ for some large $R > 0$ since this image set is compact. Hence the monotonicity of the symplectic capacity implies that the left hand of (3.2) is less than

$$C_{HZ}(\{z_1 \in T^*M \mid H_{g_1} \leq c_1\} \times Z^2(R), (\omega_{can}^M + \pi_M^* \hat{\Omega}) \oplus \frac{1}{\delta} \omega_0)$$

$$< C_{HZ}(T^*M \times Z^2(R), (\omega_{can}^M + \pi_M^* \hat{\Omega}) \oplus \frac{1}{\delta} \omega_0)$$

$$\leq \frac{\pi R^2}{\delta} < +\infty.$$
Here in the final step we use Theorem 2.1 and fact that
\[ m(T^*M, \omega^M_{\text{can}} + \pi^*_M \bar{\Omega}) = m(M, \bar{\Omega}) = m(M, \Omega) = +\infty. \]

If \( L \) is simply connected it follows from the arguments above and remarks in [Lu2] that \( \tilde{C}_{HZ}(T^*N, \omega) < +\infty \). This completes proof of Theorem 1.5.A.

\[ \square \]

**Proof of Corollary 1.5.B.** Since \( \pi_1(M) \) is a finite group we choose a simply connected finite cover \( q_M : \tilde{M} \to M \). Denote by \( \tilde{N} := \tilde{M} \times L \). Notice that \( H^2(L, \mathbb{R}) = 0 \). It directly follows from the Künneth formula that \( H^2(\tilde{N}, \mathbb{R}) \) and \( H^2(\tilde{M}, \mathbb{R}) \) is isomorphic, and thus

\[ \bigcup_{\phi \in \text{Diff}(\tilde{N})} \phi^*(P^\ast_M(H^2(\tilde{M}, \mathbb{R}))) = H^2(\tilde{N}, \mathbb{R}). \]

That is, the requirement that the de Rham cohomology class of the related closed 2-form belongs to the set in (1.2) corresponding to \( \tilde{N} \) can always be satisfied.

Denote by \( \tilde{M} := q_M \times \text{id}_L \), and

\[ \tilde{Q}_M : T^*\tilde{N} \to T^*N : (m, v) \mapsto (\tilde{q}_M(m), [d\tilde{q}_M(m)^{-1}]^*(v)). \]

For \( \tilde{\Omega} := \tilde{q}_M^*\Omega \) it is easily checked that

\[ \tilde{\omega} := \omega^\tilde{N}_{\text{can}} + \pi^*_N \tilde{\Omega} = \tilde{Q}_M^*(\omega^N_{\text{can}} + \pi^*_N \Omega) \quad \text{and} \quad m(\tilde{N}, \tilde{\Omega}) = m(N, \Omega). \]

Applying Theorem 1.5.A to the symplectic manifold \((T^*\tilde{N}, \omega^\tilde{N}_{\text{can}} + \pi^*_N \tilde{\Omega})\) and the pullback metric \( \tilde{g} := \tilde{q}_M^*g \) we may get a nonconstant Hamiltonian periodic orbit on the generic levels \( \{z \in T^*\tilde{N}|H_{\tilde{g}}(z) = c\} \) of \( X_{H_{\tilde{g}}} \) which is the Hamiltonian vector field of \( H_{\tilde{g}} \) with respect to the symplectic form \( \tilde{\omega} \). Notice that the submersion \( \tilde{Q}_M \) maps a nonconstant Hamiltonian periodic orbit of \( X_{H_{\tilde{g}}} \) on \( \{z \in T^*\tilde{N}|H_{\tilde{g}}(z) = c\} \) to a nonconstant one on \( \{z \in T^*N|H_g(z) = c\} \) of \( X_{H_g} \) with respect to \( \omega \). Corollary 1.5.B is proved.

\[ \square \]

### 3.2 Proof of Theorem 1.5.C

The ideas are similar to that of Theorem 1.5.A. Since \( L \) is a closed Lagrangian submanifold of \((\mathbb{R}^{2l}, \omega_0)\) one can directly use Weinstein’s Lagrangian neighborhood theorem to get a symplectic embedding \( \varphi : (\mathcal{U}, \omega_0) \to (T^*L, \omega^L_{\text{can}}) \) with \( \varphi|_L = \text{id} \). Here \( \mathcal{U} \) is an open neighborhood of \( L \) in \( \mathbb{R}^{2l} \). As in the proof of Theorem 1.5.A we have a closed 2-form \( \bar{\Omega} \) on \( M \) determined by (2.8) and a symplectomorphism \( \Theta \) from \((T^*N, \omega)\) to

\[ (T^*N, \bar{\omega}) := (T^*M, \omega^M_{\text{can}} + \pi^*_M \bar{\Omega}) \times (T^*L, \omega^L_{\text{can}}). \tag{3.3} \]

Moreover, we have also a symplectic embedding of codimension zero

\[ \Upsilon : (T^*M, \omega^M_{\text{can}} + \pi^*_M \bar{\Omega}) \times (\mathcal{U}, \omega_0) \to (T^*M, \omega^M_{\text{can}} + \pi^*_M \bar{\Omega}) \times (T^*L, \omega^L_{\text{can}}) \tag{3.4} \]
given by \((z_1, z_2) \mapsto (z_1, \varphi(z_2))\), whose the image is an open neighborhood of zero section of \(T^*N\). For a given level \(E_c = \{z \in T^*N \mid H_g(z) = c\}\) with \(c > 0\) we can not guarantee that \(\Theta(E_c)\) is contained in the image of \(\Upsilon\) because \(\Theta\) does not necessarily map the zero section to the zero section. But we can always take a \(\delta > 0\) so small that the diffeomorphism \(\Psi_\delta : T^*N \to T^*N\) given by

\[
(m, v) = ((m_1, m_2), (v_1, v_2)) \mapsto ((m_1, m_2), (v_1, \delta v_2)),
\]

maps \(\Theta(E_c)\) into \(\text{Im}(\Upsilon)\). Let us denote by

\[
\delta_\delta(c, g, \Theta, \Upsilon) > 0
\]

the supreme of all such \(\delta > 0\). Then for every \(\delta \in (0, \delta)\) it holds that

\[
\Psi_\delta(\Theta(E_c)) \subset \text{Im}(\Upsilon).
\]

For such \(\delta\), the composition \((\Upsilon|_{\text{Im}(\Upsilon)})^{-1} \circ \Psi_\delta \circ \Theta\) is a symplectic embedding of codimension zero from an open submanifold of \((T^*N, \omega)\) containing \(E_c\) to

\[
(T^* M, \omega^M_{\text{can}} + \pi_M^* \tilde{\Omega}) \times (U, \omega_0/\delta).
\]

Denote by

\[
\Lambda(\Upsilon, \delta, \Theta, g, c) := P_U[(\Upsilon|_{\text{Im}(\Upsilon)})^{-1} \circ \Psi_\delta \circ \Theta(E_c)],
\]

where \(P_U : T^* M \times U \to U\) is the natural projection. It is a compact subset of open set \(U\) and is contained in \(U \cap Z^2(t)\) for some \(r > 0\). Let us define

\[
r(\delta, c, g, \Theta, \Upsilon)
\]

the infimum of all \(r > 0\) such that

\[
\Lambda(\Upsilon, \delta, \Theta, g, c) \subseteq U \cap F(Z^2(t))
\]

for some \(F \in \text{Symp}(\mathbb{R}^2, \omega_0)\}. \) We also define

\[
r(\bar{\delta}, c, g, \Theta, \Upsilon) = \inf_{0 < \delta < \bar{\delta}} r(\delta, c, g, \Theta, \Upsilon)/\sqrt{\delta},
\]

where \(Z^2(t)\) is as in Theorem 2.1, then we easily prove that \(0 < r(\delta, c, g, \Theta, \Upsilon) < +\infty\) since \(U\) is a bounded open subset of \(\mathbb{R}^2\). Moreover, for each \(R \in r(\delta, c, g, \Theta, \Upsilon)\) there exists a \(F \in \text{Symp}(\mathbb{R}^2, \omega_0)\) such that

\[
\Lambda(\Upsilon, \delta, \Theta, g, c) \subseteq U \cap F(Z^2(R)).
\]

Furthermore, we define

\[
r(c, g, \Omega) := \inf r(\delta, c, g, \Theta, \Upsilon),
\]

where the infimum is taken over all possible \((\Theta, \Upsilon)\) satisfying the above arguments. Then the function

\[
\Gamma^g_{\Omega} : [0, +\infty) \to [0, +\infty), \ c \mapsto r(c, g, \Omega)
\]
will satisfy the requirements of Theorem 1.5.C. To see these let \( c > 0 \) such that
\[
\Gamma^g_\Omega(c) < \sqrt{m(N, \Omega)/\pi}. \tag{3.14}
\]
Then by (3.12) (3.13) we have
\[
r(\bar{\delta}, c, g, \Theta, \Upsilon) < \sqrt{m(N, \Omega)/\pi} \tag{3.15}
\]
for some choice \((\Theta, \Upsilon)\), and therefore from (3.11) it follows that there exists a \( \delta \in (0, \bar{\delta}) \) such that
\[
r(\delta, c, g, \Theta, \Upsilon)/\sqrt{\delta} < \sqrt{m(N, \Omega)/\pi}. \tag{3.16}
\]
Let \( \varepsilon > 0 \) satisfy
\[
(r(\delta, c, g, \Theta, \Upsilon) + \varepsilon)/\sqrt{\delta} < \sqrt{m(N, \Omega)/\pi}. \tag{3.17}
\]
Then by the definition of \( r(\delta, c, g, \Theta, \Upsilon) \) in (3.10) there exists a \( F \in \text{Symp}(\mathbb{R}^{2l}, \omega_0) \) such that
\[
\Lambda(\Upsilon, \delta, \Theta, g, c) \subseteq \mathcal{U} \cap F(Z^{2l}(r(\delta, c, g, \Theta, \Upsilon) + \varepsilon)).
\]
Note that the left side is a compact subset and the right side is an open set. This implies that for sufficiently small \( \varepsilon > 0 \)
\[
P^U(\varphi_{\mathcal{U}}^{-1} \circ \Psi_{\delta} \circ \Theta(U(g, c, \varepsilon))] \subseteq \mathcal{U} \cap F(Z^{2l}(r(\delta, c, g, \Theta, \Upsilon) + \varepsilon)).
\]
Hence \( \mathcal{U} \varphi_{\mathcal{U}}^{-1} \circ \Psi_{\delta} \circ \Theta(U(g, c, \varepsilon)) \) contained in \( T^*M \times \mathcal{U} \cap F(Z^{2l}(r(\delta, c, g, \Theta, \Upsilon) + \varepsilon), \omega_0/\delta) \).
\[
(T^*M, \omega^M, \pi^* \Omega) \times (Z^{2l}(r(\delta, c, g, \Theta, \Upsilon) + \varepsilon), \omega_0/\delta). \tag{3.18}
\]
Using Theorem 2.1 and the fact that (3.18) is symplectomorphically to
\[
(T^*M, \omega^M, \pi^* \Omega) \times (Z^{2l}([r(\delta, c, g, \Theta, \Upsilon) + \varepsilon]/\sqrt{\delta}), \omega_0), \tag{3.19}
\]
we obtain that
\[
C_{HZ}(U(g, c, \varepsilon), \omega) \leq \pi \left( [r(\delta, c, g, \Theta, \Upsilon) + \varepsilon]/\sqrt{\delta} \right)^2.
\]
Hence (3.17) gives
\[
\lim_{\varepsilon \to 0^+} \inf C_{HZ}(U(g, c, \varepsilon), \omega) < m(N, \Omega).
\]
The monotonicity of symplectic capacity \( C_{HZ} \) leads to (1.3) directly.

Finally, we prove that the function \( \Gamma^g_\Omega \) is upper semi-continuous. To this goal we only need to prove that the function \( c \mapsto r(\delta, c, g, \Theta, \Upsilon) \) defined in (3.19) is upper semi-continuous. Fix a \( c > 0 \) and a real number \( \lambda > r(\delta, c, g, \Theta, \Upsilon) \), we wish to prove that if \( c' > 0 \) is sufficiently close to \( c \) then
\[
\lambda > r(\delta, c', g, \Theta, \Upsilon).
\]
Otherwise, assume that there exists a sequence of \( c_n > 0 \) such that
\[
c_n \to c \ (n \to \infty) \quad \text{and} \quad \lambda \leq r(\delta, c_n, g, \Theta, \Upsilon). \tag{3.20}
\]
Taking sufficiently small $\epsilon > 0$ such that

$$\lambda - 2\epsilon > r(\delta, c, g, \Theta, \Upsilon),$$

then by (3.10) we have

$$\Lambda(\Upsilon, \delta, \Theta, g, c_n) \not\subseteq U \cap F(Z^{2l}(\lambda - \epsilon)), \quad \forall F \in \text{Symp}(\mathbb{R}^2, \omega_0).$$

On the other hand when $n \to \infty$ the compact subsets $\Lambda(\Upsilon, \delta, \Theta, g, c_n)$ converges to the compact subset $\Lambda(\Upsilon, \delta, \Theta, g, c)$ in the Hausdorff metric (even stronger sense). Hence for every open neighborhood $V$ of $\Lambda(\Upsilon, \delta, \Theta, g, c)$ the sets $\Lambda(\Upsilon, \delta, \Theta, g, c_n)$ can be contained in $V$ for sufficiently large $n$. If we understand the cylinder $Z^{2n}(r)$ in Theorem 2.1 as the open cylinder, then for any fixed $F \in \text{Symp}(\mathbb{R}^2, \omega_0)$ the set $U \cap F(Z^{2l}(\lambda - 2\epsilon))$ is an open neighborhood of $\Lambda(\Upsilon, \delta, \Theta, g, c)$ and thus

$$\Lambda(\Upsilon, \delta, \Theta, g, c_n) \subseteq U \cap F(Z^{2l}(\lambda - 2\epsilon))$$

for sufficiently large $n$. This shows that

$$r(\delta, c_n, g, \Theta, \Upsilon) \leq \lambda - 2\epsilon,$$

which contradicts (3.20).

By the problem F.(d) on the page 101 of [K] it is easy to know that the function $c \mapsto r(\delta, c, g, \Theta, \Upsilon)$ and thus $c \mapsto r(c, g, \Omega)$ are upper semi-continuous. The proof of Theorem 1.5.C is completed. \qed

**Remark 3.2.A** From the proof of Theorem 1.5.C we may see that the condition $\Omega|_{\pi_2(N)} = 0$ may be weakened to the case that $m(N, \Omega)$ is only finite positive number. We here do not pursue it.

### 3.3 Proofs of Theorem 1.5.E and Corollary 1.5.F

**Proof of Theorem 1.5.E** Let $c > 0$ such that $\Xi^g_{\Omega}(c) < m(N, \Omega)$. For any $\epsilon > 0$ satisfying $\Xi^g_{\Omega}(c) + 2\epsilon < m(N, \Omega)$, by (1.5), we have $(\Theta, \Omega)$ and $\epsilon > 0$ such that

$$\left| \int_{P_S((U(g,c,\epsilon)))} \omega^S_{\text{can}} \right| < \Xi^g_{\Omega}(c) + \epsilon. \quad (3.21)$$

Since $P_S((U(g,c,\epsilon))) \subset \mathbb{S}^1 \times \mathbb{R}$ and the symplectomorphisms on 2-dimensional symplectic manifolds are equivalent to the diffeomorphisms preserving area we may find a symplectic embedding $F$ from $P_S((U(g,c,\epsilon)))$ into a disk $B^2$ of area $\Xi^g_{\Omega}(c) + 2\epsilon$ centred at origin in $\mathbb{R}^2$. Then $id \times F$ symplectically embeds $(U(g,c,\epsilon), \omega)$ into $(T^*M, \omega^M_{\text{can}} + \pi_M^*\Omega) \times (B^2, \omega_0)$. Thus by Theorem 2.1 it holds that

$$C_{HZ}(U(g,c,\epsilon), \omega) \leq \Xi^g_{\Omega}(c) + 2\epsilon < m(N, \Omega).$$

Using the same reason as in the proof of Theorem 1.5.C one can get conclusions.
In order to prove (1.6), we denote by \( \Theta_{\tilde{\Omega}}^g \) and \( \Theta_{\tilde{\Omega}}^{\lambda g} \) the corresponding symplectomorphisms to \((\Omega, \tilde{\Omega}, \alpha)\) and \((\lambda \Omega, \lambda \tilde{\Omega}, \lambda \alpha)\) constructed in (2.8) and (2.12) respectively, then it is easily checked that
\[
\Theta_{\tilde{\Omega}}^g(U(g, c, \epsilon)) = \{(\phi^{-1}(m), d\phi(m)^*(v) + \alpha(\phi^{-1}(m))) \mid (m, v) \in U(g, c, \epsilon)\},
\]
\[
\Theta_{\tilde{\Omega}}^{\lambda g}(U(g, \lambda c, \lambda \epsilon)) = \{(\phi^{-1}(m), \lambda [d\phi(m)^*(v) + \alpha(\phi^{-1}(m))] \mid (m, v) \in U(g, c, \epsilon)\}
\]
and thus
\[
\int_{P_S(\Theta_{\tilde{\Omega}}^{\lambda g}(U(g, \lambda c, \lambda \epsilon)))} \omega^S_{\text{can}} = \lambda \int_{P_S(\Theta(U(g, c, \epsilon)))} \omega^S_{\text{can}}.
\]
This can lead to (1.6). The upper semi-continuity of the function \( \Xi^H_{\tilde{\Omega}} \) may be proved similarly as in Theorem 1.5.C.

**Proof of Corollary 1.5.F** Under the assumptions of Corollary 1.5.F we have a closed two-form \( \tilde{\Omega}_1 \) on \( M \) with \( m(N, \Omega) = m(M, \tilde{\Omega}_1) \) and a diffeomorphism \( \Theta_1 \in \text{Diff}(T^*N) \) such that \( \omega = \Theta_1^*\tilde{\omega}_1 \), where
\[
\tilde{\omega}_1 := \omega_{\text{can}} + \pi^*_{N}(P^*_M \tilde{\Omega}_1).
\]
Writting \( M := M \times T^{n-1} \) and \( N = M \times S^1 \), it is easily checked that the closed two-form \( \Omega_2 := P^*_M \tilde{\Omega}_1 \) on \( N \) satisfies the requirements of Theorem 1.5.E. For \( \tilde{\omega}_1 = \omega_{\text{can}} + \pi^*_{N}\tilde{\Omega}_2 \) we can obtain a closed two-form \( \tilde{\Omega}_3 \) on \( M \) and a diffeomorphism \( \Theta_2 \in \text{Diff}(T^*N) \) such that \( \Theta_2^*\tilde{\omega}_2 = \tilde{\omega}_1 \), where
\[
\tilde{\omega}_2 := \omega_{\text{can}} + \pi^*_{N}(P^*_M \tilde{\Omega}_2).
\]

Denote by \( \Theta = \Theta_1 \circ \Theta_2 \) and set
\[
\tilde{\Xi}^g_{\tilde{\Omega}}(c) := \inf_{(\Theta, \tilde{\Omega}_1, \tilde{\Omega}_2, \epsilon)} \left| \int_{P_S(\Theta(U(g, c, \epsilon)))} \omega^S_{\text{can}} \right|,
\]  
where the infimum is taken over all triples \((\Theta, \tilde{\Omega}_1, \tilde{\Omega}_2)\) satisfying the above arguments and all \( \epsilon > 0 \). Then it is easily proved that for every \( c > 0 \)
\[
\lim_{\epsilon \to 0^+} C_{HZ}(U(g, c, \epsilon), \omega) \leq \tilde{\Xi}^g_{\tilde{\Omega}}(c) < m(N, \Omega)
\]  
if \( \tilde{\Xi}^g_{\tilde{\Omega}}(c) < m(N, \Omega) \), where we use that \( m(N, \Omega) = m(M, \tilde{\Omega}_1) = m(M, \tilde{\Omega}_2) \). Specially, similar to the proof of (1.6) in Theorem 1.5.E \( \tilde{\Xi}^g_{\tilde{\Omega}} \) also satisfies:
\[
\tilde{\Xi}^g_{\lambda \Omega}(\lambda c) = \lambda \tilde{\Xi}^g_{\tilde{\Omega}}(c).
\]  
for all \( \lambda > 0 \) and \( c > 0 \). The proof is completed.

### 3.4 Proof of Theorem 1.5.I

**Case 1.** \( n > 2 \).
Notice that \( T^*T^n = T^n \times \mathbb{R}^n \). We may denote by \((x_1, \cdots, x_n; y_1, \cdots, y_n)\) the coordinate in it. For the sake of clearness we give some reductions, which are, either more or less, contained in
[Gi1][GiK][J2]. Since \([dx_i \wedge dx_j](1 \leq i < j \leq n)\) form a basis of vector space \(H^2_{de}(T^n; \mathbb{R})\), and every closed two-form \(\Omega\) on \(T^n\) must have the following form:

\[
\Omega = \sum_{i,j} q_{ij}(x) dx_i \wedge dx_j,
\]

where the smooth function \(q_{ij}(x) = -q_{ji}(x)\) are 1-periodic for each variable \(x_i\). From them it follows that there exist constants \(c_{ij}\) such that

\[
[\Omega] = \sum_{i<j} c_{ij} [dx_i \wedge dx_j].
\]

In fact \(c_{ij} = \int_{T^n} q_{ij}(x)\). Setting \(b_{ij} = \frac{1}{2} c_{ij}\) then there exists a 1-form \(\alpha\) on \(T^n\) such that

\[
\Omega = \sum_{i,j} b_{ij} dx_i \wedge dx_j + d\alpha.
\]

Moreover, \([\Omega] \neq 0\) if and only if there at least exists a \(b_{ij} \neq 0\). We may write \(\alpha\) as

\[
\alpha = \sum_{i=1}^{n} a_i(x) dx_i,
\]

where the smooth functions \(a_i(x)\) are 1-periodic for each variable \(x_i\). It is easy to see that the transformation \(\psi: T^n \times \mathbb{R}^n \to T^n \times \mathbb{R}^n: (x, y) \mapsto (X, Y) = (x, y \cdot a(x))\) is a symplectomorphism from \((T^n \times \mathbb{R}^n, \omega)\) to \((T^n \times \mathbb{R}^n, \sum_{i=1}^{n} dX_i \wedge dY_i + \sum_{i,j} b_{ij} dX_i \wedge dX_j)\), where \(\omega = \sum_{i=1}^{n} dx_i \wedge dy_i + \Omega\). Now there at least exists a \(b_{ij} \neq 0\). It is this condition which leads to one to be able prove that \((T^n \times \mathbb{R}^n, \omega)\) is symplectomorphic to the product \((\mathbb{R}^{2k} \times W_1, \omega_0 \oplus \sigma)\) with \(k \geq 1([GiK])\), where \(\omega_0\) is the standard symplectic form on \(\mathbb{R}^{2k}\) and \(\sigma\) is a translation-invariant symplectic form on \(W_1 = \mathbb{R}^{n-2k} \times T^n\). Note that here the assumption \(n > 2\) is used. Let us denote the symplectomorphism by \(\phi\). For a given metric \(g\) on \(T^n\) and \(c > 0\) we also denote by

\[
U_c = \{(x, y) \in T^n \times \mathbb{R}^n | g(x)(y, y) \leq c^2\},
\]

then there exists a \(r_c > 0\) such that

\[
\phi(U_c) \subset W_1 \times B^{2r}(r_c).
\]

Denote by \(V_c = W_1 \times B^{2r}(r_c)\) and the inclusion maps

\[
I_{U_c}: U_c \hookrightarrow T^n \times \mathbb{R}^n, \quad I_{\phi(U_c)}: \phi(U_c) \hookrightarrow \mathbb{R}^{2k} \times W_1
\]

\[
I_{\phi(U_c) V_c}: \phi(U_c) \hookrightarrow V_c, \quad I_{V_c}: V_c \hookrightarrow \mathbb{R}^{2k} \times W_1.
\]

We have the following commutative diagram:

\[
\begin{array}{ccc}
T^n \times \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R}^{2k} \times W_1 \\
U_c \downarrow & & \downarrow I_{\phi(U_c)}
\end{array}
\]

\[
I_{U_c}: U_c \hookrightarrow T^n \times \mathbb{R}^n, \quad I_{\phi(U_c)}: \phi(U_c) \hookrightarrow \mathbb{R}^{2k} \times W_1
\]

\[
I_{\phi(U_c) V_c}: \phi(U_c) \hookrightarrow V_c, \quad I_{V_c}: V_c \hookrightarrow \mathbb{R}^{2k} \times W_1.
\]
Then the induced homomorphisms among their first homotopy groups satisfy:

\[ \phi_* \circ I_{U_c*} = I_{\phi(U_c)*} \circ (\phi|U_c)_*. \]

Since \( \phi_* \), \( I_{U_c*} \) and \( (\phi|U_c)_* \) are all isomorphisms the homomorphism \( I_{\phi(U_c)*} \) is also an isomorphism. But \( I_{\phi(U_c)} = I_{V_c} \circ I_{\phi(U_c)V_c} \). We get that \( I_{\phi(U_c)*} = I_{V_c*} \circ I_{\phi(U_c)V_c*} \). This implies that \( I_{\phi(U_c)V_c*} : \pi_1(\phi(U_c)) \to \pi_1(V_c) \) must be injective. Using Theorem 2.1 and the weak monotonicity of \( \tilde{C}_{HZ} \) we have:

\[ \tilde{C}_{HZ}(U_c, \omega) \leq \tilde{C}_{HZ}(V_c, \omega_0 + \sigma) \leq \tilde{C}_{HZ}(B^{2k}(r_c) \times W_1, \omega_0 + \sigma) \leq \pi r_c^2. \]

Hence, for generic \( c > 0 \) the level \( E_c \) carries a nonconstant periodic orbit \( z = z(t) \) of \( X_{H_g} \), which is contractible in \( U_c \). Since the fibre projection from \( U_c \) to \( T^n \) induces an isomorphism \( \pi_1(U_c) \to \pi_1(T^n) \) the projection of \( z = z(t) \) to the base \( T^n \) is contractible. This leads to our claim.

**Case 2.** \( n = 2 \).

Denote by \( \sigma_0 \) the standard symplectic form on \( T^2 \). Since \( [\Omega] \neq 0 \) there exists a constant \( c_0 \neq 0 \) such that \( [\Omega] = c_0[\sigma_0] \). In particular, Moser theorem shows that \((T^2, \Omega) \) is symplectomorphic to \((T^2, c_0\sigma_0) \). Notice that there exists a symplectomorphism from \((T^*T^2, \omega) \) to \((T^2 \times \mathbb{R}^2, \Omega \oplus \omega_0) \) which maps the zero section to \( T^2 \times \{0\} \) for the standard symplectic form \( \omega_0 \) on \( \mathbb{R}^2 \). They together must induce such a symplectomorphism \( \Psi \) from \((T^*T^2, \omega) \) to \((T^2 \times \mathbb{R}^2, (c_0\sigma_0) \oplus \omega_0) \). Now for any \( U_c \) as above there exists a \( r_c > 0 \) such that \( \Psi(U_c) \subset T^2 \times B^2(r_c) \). It is easily checked that \( \Psi \) also induces an injective map \( \Psi_* : \pi_1(U_c) \to \pi_1(T^2 \times B^2(r_c)) \). Hence \( \tilde{C}_{HZ}(U_c, \omega) \leq \tilde{C}_{HZ}(T^2 \times B^2(r_c), (c_0\sigma_0) \oplus \omega_0) \leq \pi r_c^2 \). This leads to the conclusion again. \( \square \)

### 4 The concluding remarks

Our methods can actually be used to deal with a more general question than PMMQ above.

**Definition 4.1.** A smooth, bounded from below function \( H : M \to \mathbb{R} \) is called strong proper if the sublevel \( \{ z \in M \mid H(z) \leq c \} \) is compact for every \( c \in \text{Im}(H) \).

Clearly, \( H_g \) is a strong proper on \( T^*N \). Let \( H : T^*N \to \mathbb{R} \) a strong proper function and

\[ \omega := \omega_{can} + \Omega \]  \hspace{1cm} (4.1)

is a symplectic form on \( T^*N \), where \( \Omega \) is a closed 2-form on \( T^*N \). One may ask the following more general question:

**Question 4.2.** Does the Hamiltonian flow of \( H \) with respect to the symplectic form in (4.1) has a nonconstant periodic orbit on the level \( \{ H = c \} \) for every \( c \in \text{Im}(H) \)?

The proof of Theorem 1.5.A can suitably be modified to show

**Theorem 4.3** Let \( N = M \times L \) be in Theorem 1.5.A. Suppose that a closed two-form \( \Omega \) on \( T^*N \) such that

(i) the 2-form \( \omega := \omega_{can} + \Omega \) is a symplectic form on \( T^*N \) and \( \Omega|_{\pi_2(T^*N)} = 0 \);
(ii) there is a $A \in \text{Symp}(T^* N, \omega_{\text{can}})$ such that $[(A^* \Omega)|_N]$ belongs to the set in (1.2), and $(T^* N, \omega_{\text{can}} + A^* \Omega)$ is symplectomorphic to $(T^* N, \omega_{\text{can}} + \pi_N^* ((A^* \Omega)|_N))$.

Then for every strong proper smooth function $H : T^* N \to \mathbb{R}$ and every $c \in \text{Im}(H)$ it holds that

$$C_H Z (\{ H \leq c \}, \omega) < +\infty.$$ 

Consequently, for almost all $c \in \text{Im}(H)$ the levels $\{ H = c \}$ carries a nonconstant periodic orbit of $X_H$, where $X_H$ is contraction by $i_{X_H} \omega = dH$. Furthermore, if $L$ is simply connected, and an regular value $c_0 \in \text{Im}(H)$ is such that the inclusion $\{ H \leq c_0 \} \hookrightarrow T^* N$ induces an injective homomorphism $\pi_1(\{ H \leq c_0 \}) \to \pi_1(T^* N)$ then for generic $c$ near $c_0$ the levels $\{ H = c \}$ carries a nonconstant periodic orbit of $X_H$ with the contractible projection to $N$.

For other theorems and corollaries in this paper similar results may also be obtained. We omit them.

**Remark 4.4.** One may feel that checking the condition (ii) in Theorem 4.3 is difficult. But under some cases the first claim of it implies the second one. In fact, since $\pi^*_N : T^* N \to N$ induces an isomorphism $\pi^*_N : H^2(N, \mathbb{R}) \to H^2(T^* N, \mathbb{R})$ we have $[A^* \Omega] = [\pi^*_N ((A^* \Omega)|_N)]$, and thus

$$A^* \Omega - \pi^*_N ((A^* \Omega)|_N) = d\alpha$$

for some one-form $\alpha$ on $T^* N$. Using Moser’s technique one can prove that if all two-forms

$$\omega_{\text{can}} + \pi^*_N ((A^* \Omega)|_N) + t d\alpha, \ 0 \leq t \leq 1,$$

are symplectic forms, and $\alpha$ satisfies some conditions( for example, the norm $|\alpha|$ with respect to some complete metric on $T^* N$ is bounded), then the second claim in (ii) may be satisfied.

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