A NOTE ON FRACTIONAL MOMENTS FOR THE ONE-DIMENSIONAL CONTINUUM ANDERSON MODEL

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Abstract. We give a proof of dynamical localization in the form of exponential decay of spatial correlations in the time evolution for the one-dimensional continuum Anderson model via the fractional moments method. This follows via exponential decay of fractional moments of the Green function, which is shown to hold at arbitrary energy and for any single-site distribution with bounded, compactly supported density.

1. Introduction

The fractional moment method (FMM) was initially developed for the discrete Anderson model in [3]. It has recently been extended in [1] and [6] to cover continuum Anderson models, where it was shown that, in any dimension \( d \geq 1 \), exponential decay of fractional moments of the Green function, e.g. (5) below, implies dynamical and spectral localization. In fact, as discussed below, the result on dynamical localization which is obtained via the FMM is stronger than what is obtained by other methods. The fractional moment condition (5) has also been found to be a technically useful tool in other contexts, for example in the proof of Poisson statistics of eigenvalues of the Anderson model in finite volume [18] or vanishing of the d. c. electrical conductivity of an electron gas [2].

The main goal of this note is to fill a gap in the literature, which is to show that the FMM applies to one-dimensional continuum Anderson models. While localization properties of the one-dimensional Anderson model are well understood via other methods, given the mentioned applications it is useful to know that a proof via fractional moments can be given. In dimension \( d = 1 \) localization should hold in the Anderson model at all energies, independent of the disorder strength. To conclude this via the FMM, exponential decay of the fractional moments needs to be verified at all energies. For the discrete Anderson model this was done in the Appendix of [18].

Here we will do this for the continuum one-dimensional Anderson model, which is a random operator in \( L^2(\mathbb{R}) \) of the form

\[
H = H(\omega) = -\frac{d^2}{dx^2} + W + V_\omega.
\]

The background potential, \( W \), is bounded, real-valued and 1-periodic, i.e. \( W(x + 1) = W(x) \). The random potential is given by

\[
V_\omega = \sum_{n \in \mathbb{Z}} \eta_n(\omega) f_n,
\]

where we will assume that the single site potentials \( f_n \) are translates \( f_n(x) = f(x - n) \) of a non-negative and bounded function \( f \). Moreover, we suppose that \( f \) is supported on \([0, 1]\), and that it is strictly positive on a non-trivial subinterval \( J \) of \([0, 1]\), i.e. there exist constants \( C \geq c > 0 \) such that

\[
c \chi_J \leq f \leq C \chi_{[0,1]}.
\]

For the random variables \( \eta_n \), we assume that they are independent and identically distributed. We will also assume that their common distribution \( \mu(A) = \mathbb{P}(\eta_n \in A) \) has a bounded density \( \rho \) with compact support, i.e.

\[
\|\rho\|_\infty < \infty, \quad \text{supp}(\rho) \subset [\eta_{\text{min}}, \eta_{\text{max}}].
\]

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Given any bounded interval $\Lambda$, we will denote by $H_\Lambda = H_\Lambda(\omega)$ the restriction of $H$ to $L^2(\Lambda)$ with Dirichlet boundary conditions. By $G_\Lambda(z) = (H_\Lambda - z)^{-1}$ we denote the resolvent of $H_\Lambda$. We write $\chi_x$ for the characteristic function of the interval $[x, x+1]$. By $\| \cdot \|_2$ we will denote Hilbert-Schmidt norm.

Our main result is

**Theorem 1.1.** For any $E_0 \in \mathbb{R}$ there exists a number $s_0 \in (0,1)$ such that for all $0 < s \leq s_0$ there are $\eta > 0$ and $C < \infty$ such that

$$
E \left( \| \chi_x G_\Lambda(E) \chi_y \|_2^s \right) \leq C e^{-\eta |x-y|},
$$

holds for every interval $\Lambda$ with integer endpoints, all integers $x, y \in \Lambda$ and $E \in (-\infty, E_0]$.

Theorem 1.1 will be proven in Section 3. As preparation we will show in Section 2 that for the continuum Anderson model given by (1) and (2) Furstenberg’s Theorem applies at all energies and thus, in particular, the Lyapunov exponent is positive at all energies. We show this under the weaker assumption that the distribution of the random coupling constants $\eta_n$ has non-discrete support by combining results of [10] and [16].

Theorem 1.1 implies dynamical and spectral localization at all energies:

**Theorem 1.2.** For any $E_0 \in \mathbb{R}$ there exist $\eta > 0$ and $C < \infty$ such that

$$
E(\sup \| \chi_x g(H) P_{E_0}(H) \chi_y \|) \leq C e^{-\mu |x-y|}
$$

for all integers $x$ and $y$. Here the supremum is taken over all Borel measurable functions $g$ which satisfy $|g| \leq 1$ pointwise and $P_{E_0}(H)$ is the spectral projection for $H$ onto $(-\infty, E_0]$. Also, $H$ almost surely has pure point spectrum with exponentially decaying eigenfunctions.

An argument which shows that Theorem 1.2 follows from Theorem 1.1 was provided in [11]. However, to allow single-site potentials of small support as in (3) the proof in [11] needs to be slightly modified. We indicate the changes at the end of Section 3.

The particular choice $g(x) = e^{itx}$, $t \in \mathbb{R}$ arbitrary, shows that (6) is a result on dynamical localization. The exponential decay bound on the right hand side is stronger than what has been obtained with other methods. Note, however, that for the discrete one-dimensional Anderson model the analog of (6) was already obtained in [17] by a method which has not yet been extended to the continuum, however, see [12]. Spectral localization for $H$ is, of course, not new, see e.g. [10] for a more general result. We include it here for completeness and because it was shown in [11] how it follows by an argument using the RAGE theorem from dynamical localization and thus, via Theorem 1.2, is a consequence of (5).

As mentioned above, the discrete analog of our main result is proven in an appendix of [18]. For completeness, we include an alternate proof of this fact in Section 4 where we use methods similar to the ones in our proof of Theorem 1.1. There we will also include a new proof of boundedness of the fractional moments of Green’s function for the discrete Anderson model. For the "off-diagonal case", $x \neq y$ in (5), this slightly streamlines earlier arguments, e.g. [3, 12] by using a change-of-variables argument which was developed for the continuum FMM in [11]. A similar strategy was used in the context of unitary Anderson models in [14].

Our proof of Theorem 1.1 in Section 3 uses Prüfer variables which require to work at real energy $E$. The finite volume resolvent $G_\Lambda(E)$ is almost surely well defined as $H_\Lambda$ has discrete eigenvalues which are strictly monotone in all the random parameters. For some applications and to also have a result for infinite volume it is of interest to be able to extend our main result to complex energy, i.e. to consider energies $E + i\varepsilon$ in Theorem 1.1 and its discrete analog Theorem 1.1 with bounds which are uniform in $\varepsilon > 0$. As discussed in Section 5 this can easily be done for Theorem 1.1. While we expect the same to hold for the continuum, it does not seem to follow with our method of proof.

In order to make our presentation self-contained, we will provide a variety of facts, well-known to those familiar with a-priori solution bounds and the Prüfer formalism, in an Appendix.
2. Furstenberg at All Energies

In this section we consider the continuous one-dimensional Anderson model defined by (1) and (2) under the weaker assumption that the coupling constants have non-discrete distribution, i.e.

\[ \text{supp} \mu \text{ is not discrete.} \]

For fixed \( E \in \mathbb{R} \), let \( T(\eta, E) \) be the transfer matrices of \(-u'' + W u + \eta f u = Eu\) from 0 to 1 and \( G(E) \) the Furstenberg group to energy \( E \), i.e. the closed subgroup of \( SL(2, \mathbb{R}) \) generated by the matrices \( T(\eta, E) \) with \( \eta \) varying in the support of the single site distribution \( \mu \).

The goal of this section is to prove the following result, which is optimal with respect to the use of assumption (2) and thus of some interest by itself.

**Theorem 2.1.** For the continuum one-dimensional Anderson model given by (1), (2) and (7), the Furstenberg group \( G(E) \) is non-compact and strongly irreducible for all \( E \in \mathbb{R} \).

For the definition of strong irreducibility see [3]. By Furstenberg’s Theorem [3], the above result implies that the Lyapunov exponent associated with \( G(E) \) is positive for all energies \( E \in \mathbb{R} \). That \( \mu \) has non-discrete support is crucial here. Examples have been constructed showing that non-trivial but discretely supported single site distributions can lead to a discrete set of critical energies where \( G(E) \) is compact or not strongly irreducible (and the Lyapunov exponent may vanish), see [11] or Section 5 of [9].

Theorem 2.1 follows from applying a slight generalization of the main result in [10], see Theorem 2.2 below, to the methods developed in [10]. For the sake of completeness, we outline this argument.

We begin by stating a generalization of the result in [10]. Let \( Q : \mathbb{R} \rightarrow \mathbb{R} \) be locally integrable and for \( j = 0, 1 \), take \( u_j : \mathbb{R} \rightarrow \mathbb{C} \) to be solutions of

\[ -u''_j + Qu_j = 0, \]

neither of which are identically zero. For any \( V : \mathbb{R} \rightarrow \mathbb{R} \) with \( V \in L^1(\mathbb{R}) \) and support contained in \([0, 1]\), denote by \( u(\lambda) \) the solution of

\[ -u'' + (Q + \lambda V)u = 0 \]

which satisfies \( u(\lambda)(x) = u_0(x) \) for all \( x < 0 \). Here we may consider coupling constants \( \lambda \in \mathbb{C} \). The question of interest in this context is: Given a non-trivial function \( V \), for how many values of \( \lambda \) is it possible that the solution \( u(\lambda) \), which for \( x < 0 \) coincides with \( u_0 \), is proportional to \( u_1 \) for \( x > 1 \)? The case where \( u_0 = u_1 \) is discussed in [10]. Following their arguments, we define the Wronskian

\[ b(\lambda) = W[u_1, u(\lambda)](x) = u_1(x)u'(\lambda)(x) - u_1'(x)u(\lambda)(x) \]

for \( x > 1 \). The \( \lambda \)-set in question is given by the zeros of \( b \).

**Theorem 2.2.** If \( V \) is not identically zero and either

\[ u_0 = u_1 \text{ (and possibly complex-valued)} \]

or

\[ u_0 \text{ and } u_1 \text{ are real-valued}, \]

then the zeros of \( b \) form a discrete set.

In [10] this result is stated and proven for the case \( u_0 = u_1 \). However, for the case of real-valued solutions \( u_0 \) and \( u_1 \), the proof provided in [10] goes through without change if \( u_0 \neq u_1 \). We will use both versions of this result below.

**Proof.** (of Theorem 2.1) Fix \( E_0 \in \mathbb{R} \). Let \( D(E) = \text{Tr}[T(0, E)] \) denote the discriminant of \(-d^2/dx^2 + W \). The first step in our proof demonstrates that, without loss of generality, we may assume both \( 0 \in \text{supp}(\mu) \) and \( D(E_0) \notin \{-2, 0, 2\} \). This is easily seen by adjusting the periodic background \( V_{\text{per}} \). In
fact, let $\eta_0$ be an accumulation point for $\text{supp}(\mu)$. Consider $\tilde{D}(E) = \text{Tr}[T(\eta_0, E)]$, the discriminant of $-\frac{d^2}{dx^2} + \tilde{W}$ where
\begin{equation}
\tilde{W} = W + \eta_0 \sum_{n \in \mathbb{Z}} f(\cdot - n).
\end{equation}
Clearly
\begin{equation}
H_\omega = \tilde{W} + \sum_{n \in \mathbb{Z}} \tilde{\eta}_n(\omega) f(\cdot - n),
\end{equation}
where the random variables $\{\tilde{\eta}_n\}$ have distribution $\tilde{\mu}$ defined by $\tilde{\mu}(M) = \mu(M + \eta_0)$, i.e. $0 \in \text{supp}(\tilde{\mu})$. If $\tilde{D}(E_0) \notin \{-2, 0, 2\}$, then we have completed the first step of this proof. If $\tilde{D}(E_0) \in \{-2, 0, 2\}$, then $E_0$ is an eigenvalue of an operator with quasi-periodic boundary conditions. To see this, define the family of self-adjoint operators
\begin{equation}
H_{\lambda, \theta} = -\frac{d^2}{dx^2} + \tilde{W} + \lambda f \quad \text{on } [0, 1]
\end{equation}
with boundary conditions $u(1) = e^{i\theta} u(0)$ and $u'(1) = e^{i\theta} u'(0)$. It is clear that $E$ is an eigenvalue of $H_{\lambda, \theta}$ if and only if the corresponding discriminant $\text{Tr}[T(\eta_0, \lambda, E)]$ is $2\cos(\theta)$. We conclude that if $\tilde{D}(E_0) = \text{Tr}[T(\eta_0, E_0)] \in \{-2, 0, 2\}$, then $E_0$ is an eigenvalue of $H_{0, \pi}, H_{0, \frac{\pi}{2}}$, or $H_{0, 0}$ respectively. Since $f \geq 0$ and $f \neq 0$, analytic perturbation theory, see e.g. [15], implies that there exists $\delta > 0$ such that for all $\lambda \in (-\delta, \delta) \setminus \{0\}$, $E_0$ is not an eigenvalue of $H_{\lambda, \pi}, H_{\lambda, \frac{\pi}{2}}$, and $H_{\lambda, 0}$. This uses that all the eigenvalues of $H_{\lambda, \theta}$ are analytic and strictly increasing in $\lambda$, the latter being due to the Feynman-Hellmann formula which shows that $[3]$ suffices to get positivity of the $\lambda$-derivative of eigenvalues.

As $\eta_0$ was an accumulation point, there exists $\lambda_1 \in (-\delta, \delta) \setminus \{0\}$ such that $\eta_1 = \eta_0 + \lambda_1 \in \text{supp}(\mu)$. Defining $\tilde{W}$ analogously to (13) with $\eta_0$ replaced by $\eta_1$, we have completed step 1.

Step 2 of this proof demonstrates the validity of Theorem [24] in the event that $D(E_0) \in (-2, 2) \setminus \{0\}$, i.e. $E_0$ is in a band of $-d^2/dx^2 + W$ without being at the “band center”. Let $\phi_\pm$ denote the linearly independent Floquet solutions of $-\phi'' + W\phi = E_0\phi$, see e.g. [10] for details. Denote by $u_{(\eta)}$ the solution of
\begin{equation}
- u'' + (W + \eta f)u = E_0 u
\end{equation}
which satisfies
\begin{equation}
u_{(\eta)}(x) = \begin{cases} 
\phi_+(x) & \text{for } x < 0, \\
a(\eta)\phi_+(x) + b(\eta)\phi_-(x) & \text{for } x > 1.
\end{cases}
\end{equation}
A simple Wronskian argument shows that $a(\eta) \neq 0$ for all $\eta$, and by Theorem [22] (under condition [11]), the set $\{\eta \in \mathbb{C} : b(\eta) = 0\}$ is discrete. Since the support of $\mu$ is not discrete, there exists a $\eta_0 \in \text{supp}(\mu) \setminus \{0\}$ for which $b(\eta_0) \neq 0$. It is shown in [10] that $G(E_0)$ contains a subgroup which is conjugate to the group generated by the matrices
\begin{equation}
Q^{-1} \begin{pmatrix} \rho & 0 \\ 0 & \overline{\rho} \end{pmatrix} Q \quad \text{and} \quad Q^{-1} \begin{pmatrix} a(\eta_0) & b(\eta_0) \\ b(\eta_0) & a(\eta_0) \end{pmatrix} Q \quad \text{where} \quad Q = \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},
\end{equation}
and the numbers $\rho$ and $\overline{\rho}$ are the Floquet multipliers, i.e. the eigenvalues of the transfer matrix $T(0, E_0)$. $D(E_0) \notin (-2, 2) \setminus \{0\}$ means that $\rho = e^{i\omega}$ with $\omega \in (0, \pi) \setminus \{\pi/2\}$. Using this and the explicit form of this group, it was shown to be non-compact and strongly irreducible in [10]. The same readily follows for $G(E_0)$.

Step 3 finishes the proof in the case that $|D(E_0)| > 2$, i.e. $E_0$ is in a gap of $-d^2/dx^2 + W$. In this case, there exist real-valued linearly independent solutions $u_{\pm}$, each not identically zero, of
\begin{equation}
- u'' + W u = E_0 u
\end{equation}
with $u_{\pm}$ in $L^2$ near $\pm\infty$. Similar to above, we denote by $u_{(\eta)}^{\pm}$ the solution of
\begin{equation}
- u'' + (W + \eta f)u = E_0 u
\end{equation}
which satisfies
\begin{equation}
\mathcal{u}(\eta)(x) = \begin{cases} 
\mathcal{u}(x) & \text{for } x < 0, \\
\mathcal{a}(\eta)\mathcal{u}(x) + \mathcal{b}(\eta)\mathcal{u}(x) & \text{for } x > 1.
\end{cases}
\end{equation}

Using Theorem 2.2 (under condition (12)) for each of the four pairs \((u^+, u^-)\), one finds that the set
\begin{equation}
\{ \eta \in \mathbb{C} : a_+(\eta)b_-(\eta)a_-(\eta)b_+(\eta) = 0 \}
\end{equation}
is discrete. Picking \(\eta_0 \in \text{supp}(\mu) \setminus \{0\}\) for which \(a_+(\eta_0)b_-(\eta_0)a_-(\eta_0)b_+(\eta_0) \neq 0\), we will prove that the subgroup generated by \(T(0, E_0)\) and \(T(\eta_0, E_0)\) is non-compact and strongly irreducible repeating arguments from \([10]\).

Since \(|D(E_0)| > 2\), \(T(0, E_0)\) has eigenvalues \(\rho\) and \(\rho^{-1}\) with \(\rho > 1\) or \(\rho < -1\). Denote by
\begin{equation}
v_\pm = \begin{pmatrix}
\mathcal{u}(0) \\
\mathcal{u}'(0)
\end{pmatrix}
\end{equation}
the eigenvectors of \(T(0, E_0)\) corresponding to \(\rho\) and \(\rho^{-1}\), respectively. Clearly, \(w_n = T(0, E_0)^n v_+\) is unbounded, and therefore, the subgroup generated by \(T(0, E_0)\) alone is non-compact. As we have shown that this group is non-compact, to prove that it is also strongly irreducible, we need only show that each direction is mapped onto at least three distinct directions by this group, see e.g. \([8]\). First, suppose \(v\) is not in the direction of \(v_+\) or \(v_-\). Then, the sequence \(w_n = T(0, E_0)^n v\) produces arbitrarily many directions (as \(w_n\) approaches the stable manifold generated by \(v_-\)). If \(v\) is in the direction of \(v_+\) or \(v_-\), then \(T(\eta_0, E_0)v\) is not as \(a_+(\eta_0)b_+(\eta_0)a_-(\eta_0)b_-(\eta_0) \neq 0\). By our previous argument then, \(\tilde{w}_n = T(0, E_0)^n T(\eta_0, E_0)v\) produces arbitrarily many directions. This completes step 3 and the proof of Theorem 2.1. \(\square\)

3. Proof of Theorem 1.1

Non-compactness and strong irreducibility of the Furstenberg group \(G(E)\), if known for all energies in an interval, leads to consequences which go beyond positivity of the Lyapunov exponents. To state the result which we need, denote by \(T(n, k, E) = T_\omega(n, k, E)\) the transfer matrix of \(H\) at energy \(E\) from \(k\) to \(n\), i.e. the \(2 \times 2\)-matrix such that
\begin{equation}
T(n, k, E) \begin{pmatrix}
u(k) \\
u'(k)
\end{pmatrix} = \begin{pmatrix}u(n) \\
u'(n)
\end{pmatrix}
\end{equation}
for all solutions of \(-u'' + (W + V_\omega)u = Eu\).

**Lemma 3.1.** Let \(I \subset \mathbb{R}\) be a compact interval such that \(G(E)\) is non-compact and strongly irreducible for every \(E \in I\). Then there exist \(\alpha_1 > 0\), \(\delta > 0\) and \(n_0 \in \mathbb{N}\) such that for all \(E \in I\), \(n \geq n_0\) and \(x \in \mathbb{R}^2\) normalized,
\begin{equation}
E(\|T(n, 0, E)x\|^{-\delta}) \leq e^{-\alpha_1 n}.
\end{equation}

This is essentially Lemma 5.2 of \([10]\). While the latter is stated in a more concrete setting, the above slightly abstracted version is what one gets from the argument provided in \([10]\) to which we refer for the proof.

Thus, under the assumptions of Theorem 1.1 we conclude from Theorem 2.1 that Lemma 3.1 applies to every compact interval \(I\). To prove Theorem 1.1 it suffices to consider energies \(E \in I := [E_1, E_0]\), where \(E_1\) is a deterministic and strict lower bound of the potential \(W + V_\omega\) (which exists by our assumptions). For energies below \(E_1\) exponential decay of the right hand side of \([5]\) is a deterministic consequence of Combes-Thomas bounds, e.g. \([19]\).

Our main tools in reducing \([5]\) to Lemma 3.1 are the Prüfer amplitudes and phases corresponding to solutions of \(H_\lambda u = Eu\). We introduce these as follows. Write \(\Lambda = [a, b]\) for integers \(a, b\). For any \(E \in \mathbb{R}\), \(c \in [a, b]\) and \(\theta \in \mathbb{R}\) we denote by \(u_c(x, E, \theta)\) the solution of \(-u'' + (W + V_\omega)u = Eu\) which satisfies \(u(c) = \sin \theta\) and \(u'(c) = \cos \theta\). By regarding this solution and its derivative in polar coordinates, we define the Prüfer amplitude, \(R_c(x, E, \theta)\), and the Prüfer phase, \(\phi_c(x, E, \theta)\), by writing
\begin{equation}
u_c(x, E, \theta) = R_c(x, E, \theta) \sin \phi_c(x, E, \theta) \quad \text{and} \quad u'_c(x, E, \theta) = R_c(x, E, \theta) \cos \phi_c(x, E, \theta).
\end{equation}
For fixed $E$, we declare $\phi_c(c, E, \theta) = \theta$ and require continuity of $\phi$ in $x$. In this manner we define uniquely the functions $R_a(x, E, \theta)$ and $\phi_c(x, E, \theta)$ which are jointly continuous in $x$ and $E$.

For the remainder of this section, finite positive constants which can be chosen uniform in the given context may change their value from line to line.

**Proof.** (of Theorem 4.1) We may assume that the integers $x$, $y$ satisfy $x \leq y$ (if $x > y$ use that $\|\chi_x G_A(E) \chi_y\|_2 = \|\chi_x G_A(E) \chi_y\|_*^* = \|\chi_y G_A(E) \chi_x\|_2$). Since $H_A$ satisfies Dirichlet boundary conditions at both $a$ and $b$, the Green’s function can be written in terms of the solutions $u_a = u_a(\cdot, E, 0)$ and $u_b = u_b(\cdot, E, 0)$ if $E$ is not in the spectrum of $H_A$. In this case

$$G_A(s, t; E) = \frac{1}{W(u_a, u_b)} \begin{cases} u_a(s)u_b(t) & \text{if } s \leq t, \\ u_a(t)u_b(s) & \text{if } s > t. \end{cases}$$

where $W(u_a, u_b) = u_a(x,E,0) - u_a(x,E,0)$ is the Wronskian of the solutions $u_a$ and $u_b$. Let us first consider the case $x < y$. As explained in Section 3, a fixed $E$ is almost surely in the resolvent set of $H_A$, and hence, for almost every $\omega$, we have that

$$\|\chi_x G_A(E) \chi_y\|_2^2 = \int_x^{x+1} \int_y^{y+1} \left| \frac{u_a(s)u_b(t)}{W(u_a, u_b)} \right|^2 \, dt \, ds \leq \frac{1}{|W(u_a, u_b)|^2} \int_x^{x+1} \int_y^{y+1} |R_a(s, E, 0)R_b(t, E, 0)|^2 \, dt \, ds \leq \frac{C}{|W(u_a, u_b)|^2} |R_a(x, E, 0)R_b(y, E, 0)|^2.$$

Here (4.2) in Lemma 5.1 in the Appendix was used, where a uniform constant can be chosen since $W + V_\omega - E$ has local $L^1$-bounds which can be chosen uniformly in $\omega$ and $E \in I$. If $x = y$, then the representation (25) leads to two terms in (26), but Lemma 5.1 leads to the same resulting bound. Therefore, we have that

$$\mathbb{E}\left(\|\chi_x G_A(E) \chi_y\|_2^2\right) \leq C \mathbb{E}\left( \frac{R_a(x, E, 0)R_b(y, E, 0)}{|W(u_a, u_b)|^s} \right) \leq C \mathbb{E}\left( \frac{\int_{\eta_{\max}}^{\eta_{\min}} \, d\eta_x}{\int_{\eta_{\min}}^{\eta_{\max}} \, d\eta_x} \right),$$

where $\mathbb{E}$ denotes the expectation with respect to the random variables $\{\eta_x\}_{x \in \mathbb{Z} \setminus \{x\}}$.

By construction, the random variable $\eta_x$ multiplies the single site with support on $[x, x + 1]$, and therefore both $R_a(x, E, 0)$ and $R_b(y, E, 0)$ are independent of $\eta_x$. From this, we conclude that

$$\mathbb{E}\left(\|\chi_x G_A(E) \chi_y\|_2^s\right) \leq C \mathbb{E}\left( \frac{R_b(y, E, 0)}{R_b(y, E, 0)} \int_{\eta_{\min}}^{\eta_{\max}} \frac{\rho(\eta_x)}{|\sin(\phi_b(x, E, 0) - \phi_a(x, E, 0))|^{s/2}} \, d\eta_x \right).$$

The inner integral above may be bounded using Lemma 3.2 which is proven below. Using this result, we find that

$$\mathbb{E}\left(\|\chi_x G_A(E) \chi_y\|_2^s\right) \leq C \mathbb{E}\left( \frac{R_b(y, E, 0)}{R_b(y, E, 0)} \right).$$

It follows from the definition of Prüfer variables that

$$R_b^2(x, E, 0) = R_b^2(y, E, 0) R_b^2(x, E, \phi_b(y, E, 0)),$$

and therefore, the right hand side of (28) can be written in terms of the product of transfer matrices

$$\mathbb{E}\left(\|\chi_x G_A(E) \chi_y\|_2^s\right) = C \mathbb{E}\left( \frac{R_b(y, E, 0)}{R_b(y, E, \phi_b(y, E, 0))} \right) = \left\| T(x, y, E) \left( \begin{array}{c} \sin \phi_b(y, E, 0) \\ \cos \phi_b(y, E, 0) \end{array} \right) \right\|^{-s}. $$
This completes the proof of Theorem 1.1 given Lemma 3.2. We now state and prove this fact.

**Lemma 3.2.** For any bounded interval $I \subset \mathbb{R}$ and $0 < s < 1$, there exists $C < \infty$, such that

$$
\int_{\eta_{\min}}^{\eta_{\max}} \frac{\rho(\eta_x)}{|\sin(\phi_b(x, E, 0) - \phi_a(x, E, 0))|} \sigma_x \, dt_x \leq C
$$

for any integer interval $[a, b]$, any integer $x \in [a, b]$, and $E \in I$.

**Proof.** Observe that the random variable $\phi_b(x, E, 0)$ is determined by the parameters $\{\eta_n\}_{n=1}^{N}$, whereas $\phi_b(x, E, 0)$ depends on $\{\eta_n\}_{n=1}^{N}$. This suggests the change of variables $t(\eta_x) = \phi_b(x, E, 0)$. The result of Lemma 5.4 says in the current context that

$$
t'(\eta_x) = \frac{1}{R_b^2(x, E, 0)} \int_x^{x+1} f_x(t) u_b^2(t, E, 0) \, dt.
$$

Using the condition (30) on the single site potential in combination with Lemmas 5.1 and 5.2 we find constants such that

$$
C_1 R_b^2(x, E, 0) \leq \int_x^{x+1} f_x(t) u_b^2(t, E, 0) \, dt \leq C_2 R_b^2(x, E, 0)
$$

and thus

$$
0 < C_1 \leq t'(\eta_x) \leq C_2 < \infty
$$

uniformly in $\omega$ and $E \in I$. Therefore, we have that

$$
\int_{\eta_{\min}}^{\eta_{\max}} \frac{\rho(\eta_x)}{|\sin(\phi_b(x, E, 0) - \phi_a(x, E, 0))|} \sigma_x \, dt_x \leq C \|\rho\|_{\infty} \int_{t(\eta_{\min})}^{t(\eta_{\max})} \frac{1}{|\sin(t - \phi_a(x, E, 0))|} \, dt.
$$

But by (32) we also have $|t(\eta_{\max}) - t(\eta_{\min})| \leq C$ uniformly in $\omega$ and $E \in I$. The inequality claimed in (30) now follows using (33) and the fact that the resulting integrand has only a finite number of integrable singularities in any bounded interval, independent of the phase shift $\phi_a(x, E, 0)$. \qed

We end this section with some comments on the proof of Theorem 1.3 which follows by a slight adaptation of the proof of Theorem 1.1 in [1]. Essentially, this amounts to avoiding use of the covering condition for the single site potential required in [1] and thus allowing for single site potentials of small support as in (3). To prove (3) for given $E_0 \in \mathbb{R}$, we may again work on the interval $I = [E_1, E_0]$ with $E_1$ as above. As in Section 2 of [1] define, for a finite interval $I$ and integers $x$, $y$,

$$
Y_{\Lambda}(I; x, y) := \sup \{\|\chi_x f(H^\Lambda) \chi_y\| : f \in C_c(I), \|f\|_{\infty} \leq 1\},
$$

where $C_c(I)$ are the continuous functions with compact support inside $I$. Let $E_n$ and $\psi_n$ denote the eigenvalues and corresponding orthonormal eigenfunctions of $H^\Lambda$ and $P_{\psi_n}$ be the orthogonal projector onto $\psi_n$. Thus $f(H^\Lambda) = \sum_{n:E_n \in I} f(E_n) P_{\psi_n}$ and

$$
Y_{\Lambda}(I; x, y) \leq \sum_{n:E_n \in I} \|\chi_x P_{\psi_n} \chi_y\| = \sum_{n:E_n \in I} \|\chi_x \psi_n\| \|\chi_y \psi_n\|
$$

As in (31), using Lemmas 5.1 and 5.2 we have

$$
\|f^1/2 \psi_n\|^2 = \int_y^{y+1} f_y(t) \psi_n^2(t) \, dt \geq C_1 (|\psi_n(y)|^2 + |\psi'_n(y)|^2)
$$
and
\[ \| \chi_y \psi_n \|^2 \leq C_2 (|\psi_n(y)|^2 + |\psi'_n(y)|^2) \]
uniformly in \( \Lambda, n \) and \( \omega \). Thus \( \| \chi_y \psi_n \| \leq C \sqrt[4]{f^y_n} \) and (35) gives \( Y_\Lambda(I;x,y) \leq C_{Q_1}(I;x,y) \), with the eigenfunction correlator
\[ Q_1(I;x,y) := \sum_{n:E_n \in I} \| \chi_x \psi_n \| \sqrt[4]{f^y_n} \psi_n. \]
From here the proof is completed as in [1], where no additional use of the covering condition is made.

4. The discrete case

The one-dimensional discrete Anderson model \( h = h(\omega) \) acts on \( l^2(\mathbb{Z}) \) as
\[ (hn)(n) = -u(n+1) - u(n-1) + \eta_n(\omega) u(n). \]
As before, we assume that the random variables \( (\eta_n) \) are i.i.d. with density \( \rho \) satisfying (1). For \( a, b \in \mathbb{Z}, a < b \), we write \([a,b] := \{a,a+1,\ldots,b\} \), for convenience. The restriction of \( h \) to \( l^2([a,b]) \) is denoted by \( h_{|[a,b]} \), the Green function by \( G_{[a,b]}(x,y;z) \) := \( \langle e_x, (h_{|[a,b]} - z)^{-1} e_y \rangle \).

The following result was first proven by Minami in an appendix of [18]. We include it here to supplement our main result Theorem 1.1 with its discrete analogue and to provide a somewhat different self-contained proof.

**Theorem 4.1.** There exists a number \( s_0 \in (0, 1) \) such that for all \( 0 < s \leq s_0 \), the bound
\[ E \left( \left| G_{[a,b]}(x,y;E) \right|^s \right) \leq C e^{\eta|x-y|}, \]
holds for all \( x, y \in [a,b] \) and \( E \in \mathbb{R} \). Here the numbers \( C > 0 \) and \( \eta > 0 \) depend on \( s \), however, they may be chosen independent of \([a,b]\).

For \( E \) outside the spectrum of \( h_{|[a,b]} \), exponential decay of Green’s function follows from deterministic Combes-Thomas bounds. Thus it will suffice to show (37) for energies \( E \) in, say, \( I = [-3+\eta_{\min},3+\eta_{\max}] \).

We start by establishing a uniform a priori bound on the left hand side of (37). This is well known ever since the ground breaking work [3], but we opt to include a somewhat streamlined proof, using a more recent change of variables idea.

**Lemma 4.2.** Let \( s \in (0, 1) \). There exists a number \( C < \infty \) such that
\[ E \left( \left| G_{[a,b]}(x,y;E) \right|^s \right) \leq C, \]
for all integers \( a < b \) and \( x, y \in [a,b] \) and \( E \in \mathbb{R} \).

**Proof.** For \( x, y \in [a,b], x \neq y \), write \( h = \hat{h} + \eta_x P_x + \eta_y P_y \), where \( P_x = \langle e_x, \cdot \rangle e_x, P_y = \langle e_y, \cdot \rangle e_y. \) Also writing \( P = P_x + P_y \) we get, using Krein’s formula,
\[ G_{[a,b]}(x,y;E) = \left[ A^{-1} + \begin{pmatrix} \eta_x & 0 \\ 0 & \eta_y \end{pmatrix} \right]^{-1} (x,y), \]
with the \( 2 \times 2 \)-matrix \( A = P(\hat{h} - E)^{-1} P \).

We introduce the change of variables \( \alpha = \frac{1}{2}(\eta_x + \eta_y), \beta = \frac{1}{2}(\eta_x - \eta_y) \). With the self adjoint matrices \( A_{\beta} := A^{-1} + \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), the right hand side of (39) becomes \( [A_{\beta} + \alpha I]^{-1}(x,y) \). Therefore,
\[ \int_{\eta_{\min}}^{\eta_{\max}} \int_{\eta_{\min}}^{\eta_{\max}} |G_{[a,b]}(x,y;E)|^s d\mu(\eta_x) d\mu(\eta_y) \leq 2 |\rho|^2 \int_{-(\eta_{\max} - \eta_{\min})/2}^{(\eta_{\max} - \eta_{\min})/2} \int_{\eta_{\min}}^{\eta_{\max}} \| [A_{\beta} + \alpha I]^{-1} \|^s d\alpha d\beta. \]
A general fact, see e.g. Lemma 4.1 of [14], says that there is a constant \( C = C(s, \eta_{\text{max}}, \eta_{\text{min}}) \) such that

\[
\int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \left\| [B + \alpha I]^{-1} \right\|^s \, d\alpha \leq C
\]

for all dissipative \( 2 \times 2 \)-matrices \( B \) (i.e. matrices with \( \text{Im} B \geq 0 \)). In [10] we only need to use this for self adjoint matrices to conclude the required bound for the case \( x \neq y \). The diagonal case \( x = y \) is easier since no change of variable is required and Krein’s formula directly reduces the claim to the elementary analogue of (41) for \( 1 \times 1 \)-matrices, i.e. numbers, see [14] below. \( \square \)

**Proof.** (of Theorem 4.1) Without loss of generality we assume that \( x < y \), using the resolvent identity we see that

\[
G_{[a,b]}(x, y; E) = [1 + G_{[a,b]}(x, x - 1; E)]G_{[x,b]}(x, y; E).
\]

It suffices to prove the exponential decay of \( \text{E} \left( \left| G_{[x,b]}(x, y; E) \right|^s \right) \) for \( s \leq s_1 \). Using Lemma 12 and H"older’s inequality it then follows that (37) holds for \( s \leq s_1/2 \).

We have

\[
G_{[x,b]}(s, t; E) = \frac{1}{W(x, y)} \left\{ \begin{array}{ll}
u_x(s)u_b(t) & \text{if } s \leq t, \\
u_x(t)u_b(s) & \text{if } s > t. \end{array}\right.
\]

Here \( u_x \) and \( u_b \) are the solutions of \(-u(n - 1) - u(n + 1) + \eta_n u(n) = Eu(n)\) with \( u_x(x - 1) = 0, \ u_x(b - 1) = 0, \ u_b(b + 1) = 0 \). The constant Wronskian of \( u_x \) and \( u_b \) is given by

\[
W(x, u_b)(n) = u_x(x + n - 1)u_b(n) - u_x(n)u_b(x + n - 1).
\]

Evaluating the Wronskian at \( n = x \) and denoting by \( \hat{\text{E}} \) the expectation conditioned on \( \eta_x \), we obtain that

\[
\text{E} \left( \left| G_{[x,b]}(x, y; E) \right|^s \right) = \hat{\text{E}} \left( \int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \frac{|u_b(y)|^s}{u_b(x + 1) + (E - \eta_x)u_b(x)} \rho(\eta_x) \, d\eta_x \right).
\]

Now the main task is to show that

\[
\int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \frac{|u_b(y)|^s}{u_b(x + 1) + (E - \eta_x)u_b(x)} \rho(\eta_x) \, d\eta_x \leq C \left\| \left( \begin{array}{c} u_b(y) \\ u_b(y + 1) \\ u_b(x) \\ u_b(x + 1) \end{array} \right) \right\|^s.
\]

Expressed in terms of the discrete transfer matrices \( T(x, y, E) \), the right hand side is equal to \( C\left\| (u_b(y), u_b(y + 1))^t \right\|^s / \left\| T(x, y, E)(u_b(y), u_b(y + 1))^t \right\|^s. \) Thus the required bound follows from (43) and Lemma 5.1 of [7], the discrete analogue of Lemma 3.1.

In order to prove (43), we first note that \( u_b(x), u_b(x + 1) \) as well as \( u_b(y) \) are all independent of \( \eta_x \). With this in mind the proof of (43) is naturally divided into two cases

**Case I:** \( u_b(x) = 0 \), in this case, the left hand side of (43) is simply \( |u_b(y)/u_b(x + 1)|^s \) which is bounded above by \( \left\| (u_b(y), u_b(y + 1))^t \right\|^s / \left\| (u_b(x), u_b(x + 1))^t \right\|^s. \)

**Case II:** If \( u_b(x) \neq 0 \), let \( M = \sup \{|E - \eta| : \eta \in [\eta_{\text{min}}, \eta_{\text{max}}], E \in I\} \). If \( |u_b(x + 1)/u_b(x)| > 2M \), then

\[
\int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \frac{\rho(\eta_x)}{u_b(x + 1) + E - \eta_x} \, d\eta_x \leq 2^s \rho(\infty(\eta_{\text{max}} - \eta_{\text{min}})) \left| \frac{u_b(y)}{u_b(x + 1)} \right|^s
\]

\[
\leq 2^s \rho(\infty(\eta_{\text{max}} - \eta_{\text{min}})) \left( 1 + \frac{1}{4M^2} \right)^{s/2} \left\| \left( \begin{array}{c} u_b(y) \\ u_b(y + 1) \\ u_b(x) \\ u_b(x + 1) \end{array} \right) \right\|^s.
\]

On the other hand if \( |u_b(x + 1)/u_b(x)| \leq 2M \), using that for any \( \beta \in \mathbb{C} \) we have

\[
\int_{\eta_{\text{min}}}^{\eta_{\text{max}}} \left| \frac{1}{\beta - \eta_x} \right|^s \rho(\eta_x) \, d\eta_x \leq C_1(s, \rho),
\]

(44)
we see that
\[
\begin{align*}
|u_b(y)|^s \int_{\eta_{\min}}^{\eta_{\max}} \frac{\rho(\eta)}{|u_b(x) + E - \eta|^s} \, d\eta & \leq C_1(s, \rho) \left(1 + 4M^2\right)^{s/2} \left\| \frac{|u_b(y)|^s}{|u_b(x)|} \right\|_s \\
& \leq C(s, M, \rho) \left\| \begin{pmatrix} u_b(y) \\ u_b(x) \end{pmatrix} \right\|^s.
\end{align*}
\]

We have thus established (42), which ends the proof. \( \square \)

5. Remarks

(i) The proof of Lemma 4.2 works for multi-dimensional discrete Anderson models without any changes.

(ii) With only minor changes the proofs of Lemma 5.2 and Theorem 1.1 extend to complex energy. In particular, this uses that the bound (41) holds uniformly in all dissipative matrices \( B \) (as the matrices \( A_\beta \) are now dissipative) and that (44), the scalar version of (41), holds uniformly in \( \beta \in \mathbb{C} \).

As a consequence, we see that the exponential decay bound (37) holds uniformly in \( E \in \mathbb{C} \).

Working at complex energy our arguments in Section 4 may also be used to establish the analogue of (37) for infinite volume, i.e. to show that
\[
\mathbb{E}(|G(x, y; E + i\varepsilon)|^s) \leq C e^{-\eta|x-y|}
\]
holds uniformly in \( E \in \mathbb{R}, \varepsilon \neq 0 \), where \( G(x, y; z) = \langle e_x, (h-z)^{-1} e_y \rangle \). The only change is that \( u_b \) in (12) is replaced by \( u_{\infty} \), the unique solution (up to a scalar) of \(-u(n-1) - u(n+1) + \eta_n u(n) = (E + i\varepsilon) u(n)\) which is square-summable at \(+\infty\).

(iii) While we expect that Theorem 1.1 extends to complex energy as well, we do not know how to get this with our method of proof. The main problem here is that the Prüfer formalism strongly hinges on working with real-valued solutions. Due to its usefulness in applications, it would be interesting to find a different argument to allow for this extension.

Appendix: Basic facts

In this section, we will collect some basic facts about Prüfer variables and two basic a-priori solution estimates which we use repeatedly throughout the main text. A priori solution estimates like Lemma 5.1 and Lemma 5.2 are standard tools in the theory of Sturm-Liouville operators. Lemma 5.3 as well as its Corollary 5.4 have been frequently used in connection with spectral averaging techniques, e.g. [5]. We provide their proofs merely to make the paper self-contained.

Throughout this appendix, with the exception of the last corollary, the energy parameter \( E \) will be absorbed in the potential term.

Lemma 5.1. For every \( q \in L^1_{\text{loc}}(\mathbb{R}) \), every interval \([c, d]\), and every solution \( u \) of \(-u'' + qu = 0\) on \([c, d]\) one has that
\[
|u(c)|^2 + |u'(c)|^2 \exp \left( -\int_{c}^{d} |1 + q(x)| \, dx \right) \leq |u(d)|^2 + |u'(d)|^2 \leq (|u(c)|^2 + |u'(c)|^2) \exp \left( \int_{c}^{d} |1 + q(x)| \, dx \right).
\]

Proof. Setting \( R(t) := |u(t)|^2 + |u'(t)|^2 \), one easily calculates that
\[
R'(t) = 2\text{Re} \left[ (1 + q(t)) \, u(t)u'(t) \right],
\]
and hence
\[
|R'(t)| \leq |1 + q(t)| \, R(t).
\]
Since \( |u'(c)| \) bounds the derivative of the logarithm of \( R(t) \), the lemma is proven.

\[ \square \]

**Lemma 5.2.** For any positive real numbers \( \ell \) and \( M \) there exists \( C > 0 \) such that

\[ (47) \quad \int_{c}^{c+\ell} |u(t)|^2 \, dt \geq C \left( |u(c)|^2 + |u'(c)|^2 \right) \]

for every \( c \in \mathbb{R} \), every \( L^1_{loc} \)-function \( q \) with \( \int_{c}^{c+\ell} |q(t)| \, dt \leq M \), and any solution \( u \) of \(-u'' + qu = 0\) on \([c, c+\ell]\).

**Proof.** First, we observe that, by rescaling, it is sufficient to prove (47) for real valued solutions with \( |u(c)|^2 + |u'(c)|^2 = 1 \). By Lemma 5.1 there are constants \( 0 < C_1, C_2 < \infty \), depending only on \( \ell \) and \( M \) for which any real-valued solution of \(-u'' + qu = 0\) satisfies

\[ C_1 \leq |u(x)|^2 + |u'(x)|^2 \leq C_2, \]

for all \( x \in [c, c+\ell] \); given the above mentioned normalization. With \( C_3 := (C_1/2)^{1/2} \) and \( C_4 := (2C_2)^{1/2} \), we also have that

\[ (48) \quad C_3 \leq |u(x)| + |u'(x)| \leq C_4. \]

We now claim that for every \( 0 < \alpha < \ell(2 + \ell)^{-1} \) exists an \( x_0(\alpha) = x_0 \in [c, c+\ell] \) for which

\[ |u(x_0)| \geq \alpha C_3. \]

If, for such a fixed value of \( \alpha \), this is not the case, then for all \( x \in [c, c+\ell] \),

\[ |u(x)| < \alpha C_3, \]

and from (48) we may also conclude that

\[ |u'(x)| \geq C_3 - |u(x)| > (1 - \alpha) C_3 > 0. \]

Hence the derivative, \( u' \), is strictly signed. With this we may estimate,

\[ 2\alpha C_3 > |u(c+\ell) - u(c)| = \left| \int_{c}^{c+\ell} u'(x) \, dx \right| \]

\[ = \int_{c}^{c+\ell} |u'(x)| \, dx \]

\[ > \left( 1 - \alpha \right) C_3 \ell. \]

This contradicts the initial assumption on the range of \( \alpha \), and we have proven (49).

The bound (47) now follows as

\[ |u(x) - u(x_0)| \leq \int_{x_0}^{x} |u'(t)| \, dt \leq C_4 |x - x_0|, \]

implies that, in particular, \( |u(x)| \geq \alpha C_3/2 \) for all \( x \in [c, c+\ell] \) for which \( |x - x_0| \leq \alpha C_3/(2C_4) \). \( \square \)

Our remaining results relate to Prüfer variables. In general, for any real potential \( q \in L^1_{loc}(\mathbb{R}) \) and real parameters \( c \) and \( \theta \) let \( u_c \) be the solution of

\[-u'' + qu = 0 \]

with \( u_c(c) = \sin \theta, \ u'_c(c) = \cos \theta \). By regarding this solution and its derivative in polar coordinates, we define the Prüfer amplitude \( R_c(x) \) and the Prüfer phase \( \phi_c(x) \) by writing

\[ (50) \quad u_c(x) = R_c(x) \sin \phi_c(x) \quad \text{and} \quad u'_c(x) = R_c(x) \cos \phi_c(x). \]

For uniqueness of the Prüfer phase we declare \( \phi_c(c) = \theta \) and require continuity of \( \phi_c \) in \( x \). In what follows the initial phase \( \theta \) will be fixed and we thus leave the dependence of \( u_c, R_c \) and \( \phi_c \) implicit in our notation.

In the new variables \( R \) and \( \phi \) the second order equation \(-u'' + qu = 0\) becomes a system of two first order equations, where the equation for \( \phi \) is not coupled with \( R \).
Lemma 5.3. For fixed \( c \) and \( \theta \), one has that

\[
(\ln R_c^2(x))' = (1 + q(x)) \sin(2\phi_c(x)),
\]
and

\[
\phi_c'(x) = 1 - (1 + q(x)) \sin^2(\phi_c(x)).
\]

Proof. It is clear that \( R_c^2 = u^2 + (u')^2 \), and \((51)\) follows from a simple calculation. To see \((52)\), observe the following two equations: \( u' = R_c' \sin(\phi_c) + R_c \cos(\phi_c) \phi_c' \) and \( qu = u'' = R_c' \cos(\phi_c) - R_c \sin(\phi_c) \phi_c' \). Solving for \( \phi_c' \) yields \((52)\). \(\square\)

We have the following formula for the derivative of the Prüfer phase with respect to a coupling constant at a potential.

Lemma 5.4. Let \( W \) and \( V \) be real valued functions in \( L^1_{\text{loc}}(\mathbb{R}) \). For real parameters \( c \), \( \theta \) and \( \lambda \), let \( u_c \) be the solution of

\[
-u'' + Wu + \lambda Vu = 0
\]
normalized so that \( u_c(c) = \sin(\theta) \) and \( u_c'(c) = \cos(\theta) \). Denoting the Prüfer variables of \( u_c \) by \( \phi_c(x,\lambda) \) and \( R_c(x,\lambda) \), indicating their dependence on the coupling constant \( \lambda \), one has that

\[
\frac{\partial}{\partial \lambda} \phi_c(x,\lambda) = -R_c^{-2}(x,\lambda) \int_c^x V(t) u_c^2(t,\lambda) \, dt.
\]

Proof. Using both \((51)\) and \((52)\) from Lemma 5.3 above, one finds that

\[
\frac{\partial^2}{\partial \lambda \partial x} \phi_c(x,\lambda) = -V(x) \sin^2(\phi_c(x,\lambda)) - \frac{\partial}{\partial x} \ln[R_c^2(x,\lambda)] \frac{\partial}{\partial \lambda} \phi_c(x,\lambda),
\]
This implies that

\[
\frac{\partial}{\partial x} \left( R_c^2(x,\lambda) \frac{\partial}{\partial \lambda} \phi_c(x,\lambda) \right) = -V(x) R_c^{-2}(x,\lambda) \sin^2(\phi_c(x,\lambda)) = -V(x) u_c^2(x,\lambda),
\]
for almost every pair \((x,\lambda)\). Since \( \frac{\partial}{\partial \lambda} \phi_c(c,\lambda) = 0 \), \((53)\) follows immediately from \((54)\). \(\square\)

As a special case one finds the energy derivative of the Prüfer phase.

Corollary 5.5. Let \( u \) be the solution of \(-u'' + Wu = Eu \) normalized so that \( u(c) = \sin(\theta) \) and \( u'(c) = \cos(\theta) \), and let \( \phi_c(x,E) \) and \( R_c(x,E) \) be the corresponding Prüfer variables. Then

\[
\frac{\partial}{\partial E} \phi_c(x,E) = R_c^{-2}(x,E) \int_c^x u_c^2(t) \, dt.
\]

Proof. This follows from Lemma 5.4 by setting \( V \) constant to \(-1\). \(\square\)

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