Transport phenomena with chiral fermions in strong magnetic fields

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Abstract
I discuss the electrical conductivity, bulk viscosity, and heavy-quark diffusion coefficient in strong magnetic fields putting emphasis on consequences of the chirality conservation on the perturbative relaxation dynamics.

Keywords: Conductivity, Viscosity, Heavy quarks, Strong magnetic fields, Chirality

1. Introduction
The transport phenomena induced by chiral anomaly and the magnetic field in the relativistic heavy-ion collisions have attracted a number of interests in the last decade (see other contributions to Quark Matter 2018 and, e.g., Ref. [1] for a review). Many aspects of these phenomena can be understood from the special properties of the chiral fermions in the lowest Landau levels (LLLs) which have the (1+1)-dimensional linear dispersion relations. In this contribution, I discuss consequences of such properties reflected in more conventional transport phenomena, i.e., electrical conductivity [2, 3], bulk viscosity [4], and heavy-quark diffusion coefficient [5], when the magnetic field strength is much larger than the temperature scale.

2. Dissipative transport phenomena in strong magnetic fields
2.1. Longitudinal electrical conductivity and bulk viscosity
In the presence of a preferred spatial orientation provided by a magnetic field, there are three and seven components of electrical conductivities and viscosities, respectively [6]. However, the fermion in the LLL can transport the (electric) charge and energy-momentum only along the magnetic field \([\mathbf{B} = (0, 0, B)]\). Consequently, the electric current \(j^\mu\) and the energy-momentum tensor \(T^{\mu\nu}\) have only the temporal and longitudinal components. (More precisely, this is true for \(T^{\mu\nu}(q)\) when the transverse momenta \(q_{xy}\) are sent to zero [4].) Therefore, the quark carriers contribute only to the longitudinal current \(j_z\) and longitudinal pressure \(P_L\), corresponding to the longitudinal conductivity \(\sigma_{zz}\) and bulk viscosity \(\zeta_L\), respectively.
In the linear response regime, the longitudinal transport coefficients \( \sigma_{zz} \) and \( \zeta || \) at \( T \neq 0 \) are given by

\[
\sigma_{zz} = \frac{j_z}{E_z} = N_c \sum_j \frac{|e_j B|}{2\pi} \cdot 2 \int_{-\infty}^{\infty} dp_c \frac{dp_c}{2\pi} \frac{\delta f(p_c)}{E_z},
\]

(1a)

\[
\zeta || = -\frac{1}{3} \frac{\bar{p}||}{\partial_t u_z} = -\frac{N_c}{3} \sum_j \frac{|e_j B|}{2\pi} \cdot 2 \int_{-\infty}^{\infty} dp_c \frac{p_c^2}{2\pi} \frac{\Theta \delta f(p_c)}{ep} \frac{\delta f(p_c)}{\partial_z u_z},
\]

(1b)

where \( N_c \) and \( e_j \) are the number of colors and the electric charge carried by the \( f \)-flavored quark, respectively. The two-dimensional transverse phase space is degenerated with the density of states \( |e_j B|/(2\pi) \), corresponding to the translational invariance for the center coordinates of the cyclotron motions. To obtain the off-equilibrium component of the pressure \( \bar{P}_z \), one needs to subtract the equilibrium component with \( \Theta \beta = (\partial \bar{P}_z/\partial \epsilon)_\beta \), because the pressure decreases even in an adiabatic expansion/compression \([7]\). In the massless limit, the adiabatic expansion rate is given by \( \Theta \beta = 1 \) that is the inverse of the number of spatial dimensions. Clearly, the bulk viscosity vanishes in the massless limit where the scale invariance preserves the equilibrium state at any step of the expansion/compression. Therefore, the bulk viscosity is sensitive to the quark mass that breaks the scale invariance. In the massive case, we have \( \Theta \beta = 1 - 3 m^2/(\pi^2 T^2) \) \([3]\).

The off-equilibrium components \( \delta f(p_c) = f(p_c) - f_{eq}(p_c) \) of the particle distribution functions \( f(p_c) \) will be obtained as solutions of the Boltzmann equations. The relaxation dynamics in the LLL is described by a \((1+1)\)-dimensional Boltzmann equation, since the transverse momentum only serves as a label of the degenerated states. In the presence of an electric field \( E_z \) and an expansion/compression flow \( u_z(z) \), the explicit forms of the Boltzmann equations are, respectively, given by

\[
e_j E_z \frac{\partial f(p_c)}{\partial p_c} = C[f],
\]

(2a)

\[
(\partial_t + \nu_z \partial_z) f(p_c; t, z) = C[f],
\]

(2b)

where \( \nu_z \equiv p_z/ep \) is the velocity in the direction of \( B \). We have taken the steady and homogeneous limits in Eq. (2a). The equilibrium distribution function is given by \( f_{eq}(p_c) = [\exp(\beta ep) + 1]^{-1} \) without the flow velocity and by \( f_{eq}(p_c; t, z) = [\exp(\beta(t)\gamma_u(\epsilon_p - p_z u_z)) + 1]^{-1} \) in the presence of the flow \( u_z \) with \( \gamma_u = (1 - u_z^2)^{-1/2} \) being the gamma factor. In the latter case, the temperature depends on time, since the energy density decreases during the adiabatic expansion/compression of the system \([7]\).

The electrical and momentum currents reach steady states when the (external) driving force and the collisional effects are balanced. To compute the collision term \( C[f] \), we use perturbation theory with respect to the QCD coupling constant on the basis of appropriate resummations in finite temperature and a strong magnetic field. As one can imagine from the occurrence of cyclotron radiation, the leading-order contributions come from the 1-to-2 (2-to-1) processes. On the other hand, the ordinary perturbative expansion, without an external magnetic field, starts from the 2-to-2 processes, since the kinematics of the 1-to-2 processes, i.e., conversions among two fermions and a massless gauge boson, cannot be satisfied. This difference in the kinematics can be understood from the fact that the gluons have the normal \((3+1)\)-dimensional dispersion relations without being subject to the Landau-level discretization, while the quarks have the \((1+1)\)-dimensional ones. Because of this mismatch in the dimensions, the gluon transverse momentum \( |k_\perp|^2 = k_0^2 - k_z^2 \) can be regarded as an “effective mass” in the \((1+1)\)-dimensional kinematics. One may be then convinced that the kinematics of the 1-to-2 processes with a “massive gauge boson” is satisfied \([2, 3]\).

The massless LLL quarks have the linear dispersion relations \( \rho^0 = \pm p_z \) with the upper (lower) sign for the right- (left-) handed chirality. The gluon transverse momentum, or the effective mass, can be written by the fermion momenta \( p, p' \) involved in the 1-to-2 processes as \( |k_\perp|^2 = (\pm (p_z - p'_z))^2 - (p_z - p'_z)^2 = 0 \) (cf., Fig. [1]\(^1\)). This means that the kinematics is satisfied only in the collinear limit \( k \parallel p \parallel p' \), in which the coupling between a physical transverse gluon and the fermions vanishes. Therefore, the collision term is proportional to the square of the current quark mass \( m_q^2 \). This is a consequence of the chirality conservation

\(^1\) The same conclusion can be drawn for all of the 1-to-2 and 2-to-1 processes.
at the perturbative interaction vertex, which forces the velocities \( p_t / p^0 \) and \( p'_t / p'^0 \) to take the same sign. A finite \( m_f \) allows for the chirality mixing.

The Boltzmann equation with the collision term can be solved by linearizing it with respect to the small perturbation. Inserting the solution \( \delta f(p_z) \) into Eqs. (Ia) and (Ib), one obtains the electrical conductivity and the bulk viscosity:

\[
\sigma_{zz} = N_f \sum_i \frac{|e_f| B}{2\pi} \frac{4T}{g^2 C_F m^2_f \ln(T/M)}, \tag{3a}
\]

\[
\zeta_{||} = N_f \sum_i \frac{|e_f| B}{2\pi} \frac{m^2_f}{g^2 C_F \ln(T/M)} \left[ \frac{4}{\pi^2} - \frac{56}{3} \times (0.0304 \ldots) \right], \tag{3b}
\]

where \( C_F = (N_C^2 - 1)/(2N_f) \). The logarithmic factor originates from the collision integral with the ultraviolet and infrared cutoff scales at \( T \) and \( M^2 = \min[m_f^2, g^2/(2\pi) \sum_j |e_j B|/(2\pi)] \), respectively. These transport coefficients are enhanced by a factor of \( 1/m_f^2 \) as a consequence of the chirality conservation when \( m_f \) is small. The bulk viscosity is suppressed by \( m_f^2 \) when \( eB = 0 \) \cite{7}, of which the power is reduced to \( m_f^2 \) as a consequence of the competition between the conformal symmetry and the chirality conservation which govern the behaviors in the massless limit. It is remarkable that the dependences on the current quark mass are important even in the high temperature limit \( T \gg m_f \), as long as \( T^2 \gg eB \).

### 2.2. Heavy-quark diffusion dynamics

I also discuss the heavy-quark diffusion dynamics in the quark-gluon plasma under a strong magnetic field. The heavy quarks are dominantly produced in the initial hard scatterings among the partons in the colliding nuclei, and thus serve as a good probe of the dynamics in the initial and QGP stages. Computing the heavy-quark diffusion coefficient in the presence of a strong magnetic field, I show that the momentum diffusion along the magnetic field is highly suppressed, and the anisotropic diffusion rate gives rise to anisotropic spectra of the open heavy-flavor mesons \cite{13} (see also Ref. \cite{11} for a brief summary).

When the heavy quark is subject to the random kicks by the thermal scatterers, its dynamics is described by the Langevin equations \cite{8}:

\[
\frac{dp_t}{dt} = -\eta_{||} p_z + \xi_z, \quad \frac{dp_\perp}{dt} = -\eta_{\perp} p_\perp + \xi_\perp, \tag{4}
\]

where we have a set of two equations for the directions parallel and perpendicular to the magnetic field. The random forces are assumed to be white noises that satisfy \( \langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t-t') \) and \( \langle \xi_i(t)\xi_j(t') \rangle = \kappa_{\perp}\delta_{ij}(t-t') \) for \( i, j = x, y \). The momentum diffusion coefficients are defined by

\[
\kappa_{||} = \int d^3 q \frac{d\Gamma(q)}{d^3 q} q_z^2, \quad \kappa_{\perp} = \frac{1}{2} \int d^3 q \frac{d\Gamma(q)}{d^3 q} q_\perp^2, \tag{5}
\]

where \( q \) is the momentum transfer from the thermal scatterers to the heavy quark, and the static limit \( (q^0 \to 0) \) is assumed. The leading-order contributions to the momentum transfer rate \( d\Gamma(q)/d^3 q \) come from the one-gluon exchanges with thermal light-quark and gluon scatterers.

\footnote{The gluon contribution is suppressed by a factor of \( 1/|e_f|B \), because of the large phase space volume of the quark scatterers.}

\footnote{The latter is the gluon screening mass from the quark loop which is larger than the gluon-loop contribution when \( eB \gtrsim T^2 \).}
As in the previous section, the linear dispersion relation of the quark scatterers in the LLL gives rise to an important kinematics in the one-gluon exchange \cite{5}. Note that we have $q^0 = \pm(p'_z - p_z)$ with $p, p'$ being the momenta of the quark scatterers (cf., Fig. 1). Again, as a consequence of the chirality conservation, the signs appear only as the overall ones. Therefore, one finds that $q_z = \pm q^0 \to 0$ in the static limit, meaning that the longitudinal momentum transfer is prohibited in the massless limit. The consequence of the chirality conservation suggests a strong anisotropy in the diffusion dynamics.

After straightforward computations, one obtains the finite contributions from the one-gluon exchange diagrams summarized in Table 1 (up to the overall numerical constants). The quark contribution to the transverse component is proportional $eB$, while this factor is replaced by $T^2$ in the isotropic gluon contribution. This difference simply originates from the phase space volumes of the quark and gluon scatterers. Based on these results, one can estimate the anisotropy as

$$\frac{\kappa_\parallel}{\kappa_\perp} = \frac{\kappa_{\text{gluon}}}{\kappa_{\text{quark}} + \kappa_{\text{gluon}}} \sim \frac{T^2}{eB},$$

where $\kappa_{\text{quark}} = \kappa_\parallel$ and $\kappa_{\text{gluon}} = \kappa_\perp = \kappa_{\parallel}$, $\kappa_{\parallel}$, $\kappa_{\perp}$. The diffusion coefficients $\kappa_{\parallel, \perp}$ are related to the drag coefficients $\eta_{\parallel, \perp}$ through the fluctuation-dissipation theorem as $\eta_{\parallel, \perp} = 2M_0 T \kappa_{\parallel, \perp}$. Therefore, the drag force exerting on the heavy quarks is stronger in the transverse direction, which generates a momentum anisotropy $\nu_2$ of the open heavy-flavor mesons in the final state (see Ref. \cite{5} for more discussions).

## 3. Summary

I discussed the relaxation dynamics in strong magnetic fields on the basis of the effective $(1+1)$-dimensional Boltzmann equation. The chirality conservation results in the enhancements of the longitudinal electrical conductivity \cite{2} \cite{3} and bulk viscosity \cite{4} by the inverse factor of the current quark mass. The same results can be obtained from the corresponding diagrammatic calculations \cite{3} \cite{4}. There are related progresses in computations in weaker magnetic fields where the higher Landau levels participate in the relaxation dynamics \cite{9} \cite{10}. I also discussed the heavy-quark diffusion dynamics \cite{5}. The key observation was that the chirality conservation prohibits the longitudinal-momentum exchange between a heavy quark and a massless quark scatterer in the LLL. Consequently, the diffusion coefficient acquires an anisotropy, which in turn gives rise to an origin of the momentum anisotropy of open heavy-flavor mesons.

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