A characterization of involutes and evolutes of a given curve in $\mathbb{E}^n$

Günay ÖZTÜRK

Department of Mathematics, Kocaeli University, Kocaeli, Turkey

Kadri ARSLAN, Betül BULCA

Department of Mathematics, Uludağ University, Bursa, Turkey

Abstract

The orthogonal trajectories of the first tangents of the curve are called the involutes of $x$. The hyperspheres which have higher order contact with a curve $x$ are known osculating hyperspheres of $x$. The centers of osculating hyperspheres form a curve which is called generalized evolute of the given curve $x$ in $n$-dimensional Euclidean space $\mathbb{E}^n$. In the present study, we give a characterization of involute curves of order $k$ (resp. evolute curves) of the given curve $x$ in $n$-dimensional Euclidean space $\mathbb{E}^n$. Further, we obtain some results on these type of curves in $\mathbb{E}^3$ and $\mathbb{E}^4$, respectively.

Keywords: Frenet curve, involutes, evolutes

2010 MSC: 53A04, 53A05

1. Introduction

Let $x = x(t) : I \subset \mathbb{R} \to \mathbb{E}^n$ be a regular curve in $\mathbb{E}^n$, (i.e., $\|x'(t)\| \neq 0$). Then $x$ is called a Frenet curve of osculating order $d$, $(2 \leq d \leq n)$ if $x'(t)$, $x''(t),...,x^{(d)}(t)$ are linearly independent and $x'(t)$, $x''(t),...,x^{(d+1)}(t)$ linearly dependent for all $t$ in $I$ [12]. In this case, $\text{Im}(x)$ lies in an $d$-dimensional Euclidean subspace of $\mathbb{E}^{n+1}$. To each Frenet curve of rank $d$ there can be associated orthonormal $d$-frame $V_1 = \frac{x'(t)}{\|x'(t)\|}, V_2, V_3,..., V_d$ along $x$, the Frenet
\(d\)-frame, and \(d - 1\) functions \(\kappa_1, \kappa_2, \ldots, \kappa_{d-1}: I \to \mathbb{R}\), the Frenet curvature, such that

\[
\begin{bmatrix}
V_1' \\
V_2' \\
V_3' \\
\vdots \\
V_d'
\end{bmatrix} = v \begin{bmatrix}
0 & \kappa_1 & 0 & \cdots & 0 \\
-\kappa_1 & 0 & \kappa_2 & \cdots & 0 \\
0 & -\kappa_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\kappa_{d-1}
\end{bmatrix} \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_d
\end{bmatrix}
\]  

(1)

where, \(v = \|x'(t)\|\) is the speed of the curve \(x\). In fact, to obtain \(V_1, V_2, V_3, \ldots, V_d\) \((2 \leq d \leq n)\) it is sufficient to apply the Gram-Schmidt orthonormalization process to \(x'(t), x''(t), \ldots, x^{(d)}(t)\). Moreover, the functions \(\kappa_1, \kappa_2, \ldots, \kappa_{d-1}\) are easily obtained as by-product during this calculation.

More precisely, \(V_1, V_2, V_3, \ldots, V_d\) and \(\kappa_1, \kappa_2, \ldots, \kappa_{d-1}\) are determined by the following formulas:

\[
E_1(t) : = x'(t) \quad ; V_1 := \frac{E_1(t)}{\|E_1(t)\|},
\]

\[
E_\alpha(t) : = x^{(\alpha)}(t) - \sum_{i=1}^{\alpha-1} <x^{(\alpha)}(t), E_i(t)> \frac{E_i(t)}{\|E_i(t)\|^2}, \quad 2 \leq \alpha \leq n
\]

and

\[
\kappa_\delta(s) := \frac{\|E_{\delta+1}(t)\|}{\|E_\delta(t)\| \|E_1(t)\|},
\]

(3)

respectively, where \(\delta \in \{1, 2, 3, \ldots, d - 1\}\) (see, [2]). For the case \(d = n\), the Frenet curve \(x\) is called a generic curve [12].

The osculating hyperplanes of a generic curve \(x\) at \(t\) is the subspace generated by \(\{V_1, V_2, V_3, \ldots, V_n\}\) that passes through \(x(t)\). The unit vector \(V_n(t)\) is called binormal vector of \(x\) at \(t\). The normal hyperplane of \(x\) at \(t\) is defined to be the one generated by \(\{V_2, V_3, \ldots, V_n\}\) passing through \(x(t)\) [3].

A Frenet curve of rank \(d\) for which the first Frenet curvature \(\kappa_1\) is constant is called a Salkowski curve [10] (or T.C-curve [3]). Further, a Frenet curve of rank \(d\) for which \(\kappa_1, \kappa_2, \ldots, \kappa_{d-1}\) are constant is called (circular) helix or W-curve [3]. Meanwhile, a Frenet curve of rank \(d\) with constant curvature ratios \(\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \ldots, \frac{\kappa_{d-1}}{\kappa_{d-2}}\) is called a ccr-curve (see, [8], [7]). A ccr-curve in \(\mathbb{E}^3\) is known as generalized helix.
Given a generic curve \( x \) in \( \mathbb{E}^4 \), the Frenet 4-frame, \( V_1, V_2, V_3, V_4 \) and the Frenet curvatures \( \kappa_1, \kappa_2, \kappa_3 \) are given by

\[
\begin{align*}
V_1(t) &= \frac{x'(t)}{\|x'(t)\|} \\
V_4(t) &= \frac{x'(t) \wedge x''(t) \wedge x'''(t)}{\|x'(t) \wedge x''(t) \wedge x'''(t)\|} \\
V_3(t) &= \frac{V_4(t) \wedge x'(t)}{\|V_4(t) \wedge x'(t)\|} \\
V_2(t) &= \frac{V_3(t) \wedge V_4(t) \wedge x'(t)}{\|V_3(t) \wedge V_4(t) \wedge x'(t)\|}
\end{align*}
\]

and

\[
\kappa_1(t) = \frac{\langle V_2(t), x''(t) \rangle}{\|x'(s)\|^2}, \quad \kappa_2(t) = \frac{\langle V_3(t), x'''(t) \rangle}{\|x'(t)\|^2 \kappa_1(t)}, \quad \kappa_3(t) = \frac{\langle V_4(t), x''''(t) \rangle}{\|x'(t)\|^4 \kappa_1(t) \kappa_2(t)}
\]

respectively, where \( \wedge \) is the exterior product in \( \mathbb{E}^4 \).

This paper is organized as follows: Section 2 gives some basic concepts of the involute curves of order \( k \) in \( \mathbb{E}^n \). Section 3 explains some geometric properties about the involute curves of order \( k \) in \( \mathbb{E}^3 \), where \( k = 1, 2 \). Section 4 tells about the involute curves of order \( k \) in \( \mathbb{E}^4 \), where \( k = 1, 2, 3 \). Further, these sections provide some properties and results of these type of curves. In the final section we consider generalized evolute curves in \( \mathbb{E}^n \). Moreover, we present some results of generalized evolute curves in \( \mathbb{E}^3 \) and \( \mathbb{E}^4 \), respectively.

2. Involute curves of order \( k \) in \( \mathbb{E}^n \)

Definition 1. Let \( x = x(s) \) be a regular generic curve in \( \mathbb{E}^n \) given with the arclength parameter \( s \) (i.e., \( \|x'(s)\| = 1 \)). Then the curves which are orthogonal to the system of \( k \)-dimensional osculating hyperplanes of \( x \), are called the involutes of order \( k \) (or, \( k^{th} \) involute) of the curve \( x \). For simplicity, we call the involutes of order 1, simply the involutes of the given curve.

In order to find the parametrization of involutes \( \overline{x} \) of order \( k \) of the curve \( x \), we put

\[
\overline{x}(s) = x(s) + \sum_{\alpha=1}^{k} \lambda_\alpha(s)V_\alpha(s), \quad k \leq n - 1
\]
where $\lambda_\alpha$ is a differentiable function and $s$ is the parameter of $\mathbf{x}$ which is not necessarily an arclength parameter. The differentiation of the equation (6) and the Frenet formulae (1) give the following equation

$$
\mathbf{x}'(s) = (1 + \lambda'_1 - \kappa_1 \lambda_2) V_1(s) + \sum_{\alpha=2}^{k-1} (\lambda'_\alpha - \lambda_{\alpha+1} \kappa_{\alpha} + \lambda_{\alpha-1} \kappa_{\alpha-1}) V_\alpha(s) + \lambda_k(s) \kappa_k(s) V_{k+1}(s).
$$

(7)

Furthermore, the involutes $\mathbf{x}$ of order $k$ of the curve $\mathbf{x}$ are determined by

$$
\langle \mathbf{x}'(s), V_j(s) \rangle = 0, 1 \leq j \leq k \leq n - 1.
$$

This condition is satisfied if and only if

$$
1 + \lambda'_1 - \kappa_1 \lambda_2 = 0,
\lambda'_\alpha - \lambda_{\alpha+1} \kappa_{\alpha} + \lambda_{\alpha-1} \kappa_{\alpha-1} = 0,
\lambda'_k + \lambda_{k-1} \kappa_{k-1} = 0,
$$

(8)

where $2 \leq \alpha \leq n - 1$. Consequently, the involutes of order $k$ of a regular generic curve $\mathbf{x}$ are represented by the formulas (8), and when $\lambda_\alpha$ are chosen in this way, $\lambda_k$ does not vanish identically and $V_1(s) = \pm V_{k+1}$ whenever $\lambda_k \neq 0$ [4].

3. Involutes in $\mathbb{E}^3$

In the present section we consider involutes of order 1 and of order 2 of curves in Euclidean 3-space $\mathbb{E}^3$, respectively.

3.1. Involutes of order 1 in $\mathbb{E}^3$

**Proposition 2.** Let $x = x(s)$ be a regular curve in $\mathbb{E}^3$ given with nonzero Frenet curvatures $\kappa_1$ and $\kappa_2$. Then Frenet curvatures $\kappa_1$ and $\kappa_2$ of the involute $\overline{x}$ of the curve $x$ are given by

$$
\kappa_1 = \frac{\sqrt{\kappa_1^2 + \kappa_2^2}}{|\kappa_1| |s - c|}, \quad \kappa_2 = \frac{(\kappa'_1 / \kappa_1)' \kappa_1^2}{(\kappa_1^2 + \kappa_2^2) (c - s)}.
$$

(9)
Proof. Let $\overline{x} = \overline{x}(s)$ be the involute of the curve $x$ in $\mathbb{E}^3$. Then by the use of (7) with (8) we get $1 + \lambda'_1(s) = 0$, and furthermore $\lambda(s) = (c - s)$ for some integral constant $c$. So, we get the following parametrization

$$\overline{x}(s) = x(s) + (c - s)V_1(s). \quad (10)$$

Further, the differentiation of (10) implies that

$$\overline{x}'(s) = \varphi V_2, \quad \overline{x}''(s) = -\varphi \kappa_1 V_1 + \varphi' V_2 + \varphi \kappa_2 V_3,$$
$$\overline{x}'''(s) = -\{(\kappa_1 \varphi)' + \kappa_1 \varphi'\} V_1 + \{\varphi'' - \kappa_1 \varphi - \kappa_2 \varphi'\} V_2 + \{(\kappa_2 \varphi)' + \kappa_2 \varphi'\} V_3. \quad (11)$$

Now, an easy calculation gives

$$\|\overline{x}'(s)\| = |\varphi| = |(c - s)\kappa_1|,$$
$$\|\overline{x}'(s) \times \overline{x}''(s)\| = \varphi^2 \sqrt{\kappa_1^2 + \kappa_2^2},$$
$$\langle \overline{x}'(s) \times \overline{x}''(s), \overline{x}'''(s) \rangle = \varphi^3 (\kappa_1 \kappa_2' - \kappa_2 \kappa_1'). \quad (12)$$

The parameter $s$ is not the arc length parameter of $\overline{x}$, so, as is shown in [1], we have

$$\overline{\kappa}_1 = \frac{\|\overline{x}'(s) \times \overline{x}''(s)\|}{\|\overline{x}'(s)\|^3}, \quad \overline{\kappa}_2 = \frac{\langle \overline{x}'(s) \times \overline{x}''(s), \overline{x}'''(s) \rangle}{\|\overline{x}'(s) \times \overline{x}''(s)\|^2}. \quad (13)$$

Hence, from the relations (11) and (12) we deduce (9). □

By the use of (9) one can get the following result.

**Corollary 3.** If $x = x(s)$ is a cylindrical helix in $\mathbb{E}^3$, then the involute $\overline{x}$ of $x$ is a planar curve.

### 3.2. Involutes of order 2 in $\mathbb{E}^3$

An involute of order 2 of a regular curve $x$ in $\mathbb{E}^3$ has the parametrization

$$\overline{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) \quad (13)$$

where $V_1, V_2$ are tangent and normal vectors of $x$ in $\mathbb{E}^3$ and $\lambda_1, \lambda_2$ are differentiable functions satisfying

$$\lambda'_1(s) = \kappa_1(s)\lambda_2(s) - 1,$$
$$\lambda'_2(s) = -\lambda_1(s)\kappa_1(s). \quad (14)$$

We obtain the following result.
Proposition 4. Let $x = x(s)$ be a regular curve in $\mathbb{E}^3$ given with nonzero Frenet curvatures $\kappa_1$ and $\kappa_2$. Then Frenet curvatures $\bar{\kappa}_1$ and $\bar{\kappa}_2$ of the involute $\bar{x}$ of order 2 of the curve $x$ are given by

$$
\bar{\kappa}_1 = \frac{\text{sgn}(\kappa_2)}{|\lambda_2|}, \quad \bar{\kappa}_2 = \frac{\kappa_2}{\lambda_2}.
$$

(15)

Proof. Let $\bar{x} = \bar{x}(s)$ be the involute of order 2 of the curve $x$ in $\mathbb{E}^3$. Then by the use of (7) with (8) we get

$$
\bar{x}'(s) = \lambda_2(s) \kappa_2(s) V_3(s),
$$

(16)

Further, the differentiation of (16) implies that

$$
\bar{x}''(s) = -\psi(s) \kappa_1(s) \kappa_2(s) V_1(s) - \left\{ (\psi(s) \kappa_2(s))' + \kappa_2(s) \psi'(s) \right\} V_2(s)
$$

$$
+ \left\{ \psi''(s) + \psi(s) \kappa_2^2(s) \right\} V_3(s).
$$

Now, an easy calculation gives

$$
\|ar{x}'(s)\| = |\psi(s)| = |\lambda_2(s) \kappa_2(s)|,
$$

$$
\|ar{x}'(s) \times \bar{x}''(s)\| = \psi(s) \kappa_2(s),
$$

$$
\langle \bar{x}'(s) \times \bar{x}''(s), \bar{x}'''(s) \rangle = \psi(s) \kappa_1(s) \kappa_2^2(s).
$$

(17)

Hence, from the relations (12) and (17) we deduce (15). □

Corollary 5. The involute $\bar{x}$ of order 2 of a generalized helix in $\mathbb{E}^3$ is also a generalized helix in $\mathbb{E}^3$.

Solving the system of differential equations (14) we get the following result.

Corollary 6. Let $x = x(s)$ be a unit speed Salkowski curve in $\mathbb{E}^3$. Then the involute $\bar{x}$ of order 2 of the curve $x$ has the parametrization (13) given with the coefficient functions

$$
\lambda_1(s) = c_1 \sin(\kappa_1 s) + c_2 \cos(\kappa_1 s),
$$

$$
\lambda_2(s) = c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s) - \frac{1}{\kappa_1},
$$

(18)

where $c_1$ and $c_2$ are real constants.
4. Involutes in \( E^4 \)

In the present section we consider involutes of order \( k \), \( 1 \leq k \leq 3 \) of a given curve \( x \) in Euclidean 4-space \( E^4 \).

4.1. Involutes of order 1 in \( E^4 \)

**Proposition 7.** Let \( x = x(s) \) be a regular curve in \( E^4 \) given with the Frenet curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). Then Frenet 4-frame, \( \overline{V}_1, \overline{V}_2, \overline{V}_3 \) and \( \overline{V}_4 \) and Frenet curvatures \( \overline{\kappa}_1, \overline{\kappa}_2 \) and \( \overline{\kappa}_3 \) of the involute \( \overline{x} \) of the curve \( x \) are given by

\[
\begin{align*}
\overline{V}_1(s) &= V_2, \\
\overline{V}_2(s) &= -\kappa_1 V_1 + \kappa_2 V_3 \\
\overline{V}_3(s) &= -\left(\kappa_2 A - \kappa_1 C\right) \kappa_2 V_1 - \left(\kappa_2 A - \kappa_1 C\right) \kappa_1 V_3 + \frac{D \left(\kappa_1^2 + \kappa_2^2\right) V_4}{W \sqrt{\kappa_1^2 + \kappa_2^2}} \\
\overline{V}_4(s) &= \frac{D \kappa_2 V_1 - D \kappa_1 V_3 - \left(\kappa_2 A - \kappa_1 C\right) V_4}{W},
\end{align*}
\]

and

\[
\begin{align*}
\overline{\kappa}_1 &= \frac{\sqrt{\kappa_1^2 + \kappa_2^2}}{|\varphi|}, \quad \varphi := (c - s) \kappa_1, \\
\overline{\kappa}_2 &= \frac{W}{\varphi^2 (\kappa_1^2 + \kappa_2^2)}, \\
\overline{\kappa}_3 &= -\frac{\left(\kappa_2 A - \kappa_1 C\right) \left(\kappa_3 C + D'\right) + D \left(\kappa_2 A' - \kappa_1 C'\right) + D^2 \kappa_2 \kappa_3}{W \varphi^2 \overline{\kappa}_1 \overline{\kappa}_2},
\end{align*}
\]

respectively, where

\[
\begin{align*}
A &= \kappa_1' \varphi + 2 \kappa_1 \varphi' \\
C &= \kappa_2' \varphi + 2 \kappa_2 \varphi' \\
D &= \kappa_2 \kappa_3 \varphi
\end{align*}
\]

and

\[
W = \sqrt{D^2 (\kappa_1^2 + \kappa_2^2) + (\kappa_1 C - \kappa_2 A)^2} = |\varphi| \sqrt{\kappa_2^2 \kappa_3^2 (\kappa_1^2 + \kappa_2^2) + (\kappa_1 \kappa'_2 - \kappa_2 \kappa'_1)^2}.
\]
**Proof.** As in the proof of Proposition 2, the involute \( \overline{x} = \overline{x}(s) \) of the curve \( x \) in \( \mathbb{E}^4 \) has the parametrization
\[
\overline{x}(s) = x(s) + (c - s)V_1(s),
\]
where \( V_1 \) is the unit tangent vector of \( x \).

Further, the differentiation of the position vector \( \overline{x}(s) \) implies that
\[
\begin{align*}
\overline{x}'(s) &= \varphi V_2, \\
\overline{x}''(s) &= -\varphi \kappa_1 V_1 + \varphi' V_2 + \varphi \kappa_2 V_3, \\
\overline{x}'''(s) &= -\{(\kappa_1 \varphi)' + \kappa_1 \varphi'\} V_1 + \{\varphi'' - \kappa_1^2 \varphi - \kappa_2^2 \varphi\} V_2 \\
&\quad + \{(\kappa_2 \varphi)' + \kappa_2 \varphi'\} V_3 + \varphi \kappa_2 \kappa_3 V_4,
\end{align*}
\]
where \( \varphi = (c - s)\kappa_1 \) is a differentiable function. Consequently, substituting
\[
A = \kappa_1^2 \varphi + 2 \kappa_1 \varphi', \\
B = \varphi'' - \kappa_1^2 \varphi - \kappa_2^2 \varphi, \\
C = \kappa_2^2 \varphi + 2 \kappa_2 \varphi', \\
D = \varphi \kappa_2 \kappa_3,
\]
the last vector becomes
\[
\overline{x}''' = -AV_1 + BV_2 + CV_3 + DV_4. \tag{24}
\]
Furthermore, differentiating \( \overline{x}''' \) with respect to \( s \), we get
\[
\overline{x}'''' = \left\{ -\{A' + \kappa_1 B\} V_1 + \{-\kappa_1 A - \kappa_2 C + B'\} V_2 \\
+ \{\kappa_2 B - \kappa_3 D + C''\} V_3 + \{D' + \kappa_3 C\} V_4 \right\}. \tag{25}
\]
Now, by the use of (22), we can compute the vector form \( \overline{x}'(s) \wedge \overline{x}''(s) \wedge \overline{x}'''(s) \) and second principal normal of \( \overline{x} \) as in the following:
\[
\overline{x}'(s) \wedge \overline{x}''(s) \wedge \overline{x}'''(s) = \varphi^2 \left\{ D \kappa_2 V_1 + D \kappa_1 V_3 + (\kappa_1 C - \kappa_2 A) V_4 \right\}
\]
and
\[
\nabla_4(s) = \frac{x'(s) \wedge x''(s) \wedge x'''(s)}{\|x'(s) \wedge x''(s) \wedge x'''(s)\|} = \frac{D \kappa_2 V_1 + D \kappa_1 V_3 - (\kappa_2 A - \kappa_1 C) V_4}{W}
\]
where
\[
W = \sqrt{D^2 (\kappa_1^2 + \kappa_2^2) + (\kappa_2 A - \kappa_1 C)^2}. \tag{27}
\]
Similarly, we can compute the vector form \( \overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s) \) and first principal normal \( \overline{V}_3(s) \) of \( \overline{x} \) as

\[
\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s) = \frac{\varphi^2}{W} \left\{ - (\kappa_2 A - \kappa_1 C) \kappa_2 V_1 - (\kappa_2 A - \kappa_1 C) \kappa_1 V_3 + D (\kappa_1^2 + \kappa_2^2) V_4 \right\}
\]

and

\[
\overline{V}_3(s) = \frac{\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s)}{\left\| \overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s) \right\|} = \frac{-(\kappa_2 A - \kappa_1 C) \kappa_2 V_1 - (\kappa_2 A - \kappa_1 C) \kappa_1 V_3 + D (\kappa_1^2 + \kappa_2^2) V_4}{W \sqrt{\kappa_1^2 + \kappa_2^2}}.
\]

Finally, the vector form \( \overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s) \) and the normal \( \overline{V}_2(s) \) of \( \overline{x} \) becomes

\[
\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s) = \varphi \left\{ D^2 (\kappa_1^2 + \kappa_2^2) - (\kappa_2 A - \kappa_1 C)^2 \right\} (-\kappa_1 V_1 + \kappa_2 V_3)
\]

and

\[
\overline{V}_2(s) = \frac{\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s)}{\left\| \overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s) \right\|} = \frac{-\kappa_1 V_1 + \kappa_2 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}}.
\]

Consequently, an easy calculation gives

\[
\langle \overline{V}_2(s), \overline{x}''(s) \rangle = \varphi \sqrt{\kappa_1^2 + \kappa_2^2}
\]

\[
\langle \overline{V}_3(s), \overline{x}''(s) \rangle = \frac{W}{\sqrt{\kappa_1^2 + \kappa_2^2}}
\]

\[
\langle \overline{V}_4(s), \overline{x}''''(s) \rangle = \frac{-(\kappa_2 A - \kappa_1 C) (\kappa_3 C + D') + D (\kappa_2 A' - \kappa_1 C') + D^2 \kappa_1 \kappa_3}{W}.
\]

Hence, from the relations (30) and (31) we deduce (20). This completes the proof of the proposition. 

For the case \( x \) is a \( W \)-curve one can get the following results.

**Corollary 8.** [11] Let \( \overline{x} \) be an involute of a generic \( x \) curve in \( E^4 \) given with the Frenet curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). If \( x \) is a \( W \)-curve then the Frenet 4-frame, \( \overline{V}_1, \overline{V}_2, \overline{V}_3 \) and \( \overline{V}_4 \) and the Frenet curvatures \( \overline{\kappa}_1, \overline{\kappa}_2 \) and \( \overline{\kappa}_3 \) of the...
involute \( \pi \) of the curve \( x \) are given by

\[
\begin{align*}
V_1(s) &= V_2, \\
V_2(s) &= -\kappa_1 V_1 + \kappa_2 V_3 \\
V_3(s) &= V_4 \\
V_4(s) &= \frac{\kappa_2 V_1 + \kappa_1 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}} 
\end{align*}
\] (31)

and

\[
\begin{align*}
\kappa_1 &= \frac{\sqrt{\kappa_1^2 + \kappa_2^2}}{|\varphi|}, \\
\kappa_2 &= \frac{\kappa_2 \kappa_3}{|\varphi| \sqrt{\kappa_1^2 + \kappa_2^2}} \\
\kappa_3 &= \frac{-\kappa_1 \kappa_3}{|\varphi| \sqrt{\kappa_1^2 + \kappa_2^2}} 
\end{align*}
\] (32)

respectively, where \( \varphi = (c - s) \kappa_1 \).

**Corollary 9.** Let \( \pi \) be an involute of a generic \( x \) curve in \( \mathbb{E}^4 \) given with the Frenet curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). If \( x \) is a \( W \)-curve then \( \pi \) becomes a ccr-curve.

### 4.2. Involutes of order 2 in \( \mathbb{E}^4 \)

An involute of order 2 of a regular curve \( x \) in \( \mathbb{E}^4 \) has the parametrization

\[
\pi(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) 
\] (33)

where \( V_1, V_2 \) are tangent and normal vectors of \( x \) in \( \mathbb{E}^4 \) and \( \lambda_1, \lambda_2 \) are differentiable functions satisfying

\[
\begin{align*}
\lambda_1'(s) &= \kappa_1(s)\lambda_2(s) - 1, \\
\lambda_2'(s) &= -\lambda_1(s)\kappa_1(s). 
\end{align*}
\] (34)

As in the previous subsection we get the following result.
Corollary 10. Let \( x = x(s) \) be a unit speed Salkowski curve in \( E^4 \). Then the involute \( \overline{x} \) of order 2 of the curve \( x \) has the parametrization \((33)\) given with the coefficient functions

\[
\begin{align*}
\lambda_1(s) &= c_1 \sin(\kappa_1 s) + c_2 \cos(\kappa_1 s), \\
\lambda_2(s) &= c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s) - \frac{1}{\kappa_1}.
\end{align*}
\]

where \( c_1 \) and \( c_2 \) are real constants.

We obtain the following result.

Proposition 11. Let \( x = x(s) \) be a regular curve in \( E^4 \) given with nonzero Frenet curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). Then Frenet 4-frame, \( \overline{V}_1, \overline{V}_2, \overline{V}_3 \) and \( \overline{V}_4 \) and Frenet curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) of the involute \( \overline{x} \) of order 2 of a regular curve \( x \) in \( E^4 \) are given by

\[
\begin{align*}
\overline{V}_1(s) &= V_3, \\
\overline{V}_2(s) &= \frac{-\kappa_2 V_2 + \kappa_3 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}}, \\
\overline{V}_3(s) &= \frac{K (\kappa_2^2 + \kappa_3^2) V_1 + (\kappa_2 N - \kappa_3 L) \kappa_3 V_2 + (\kappa_2 N - \kappa_3 L) \kappa_2 V_4}{W \sqrt{\kappa_2^2 + \kappa_3^2}}, \\
\overline{V}_4(s) &= \frac{(\kappa_2 N - \kappa_3 L) V_1 + \kappa_3 K V_2 + \kappa_2 K V_4}{W},
\end{align*}
\]

and

\[
\begin{align*}
\overline{\kappa}_1 &= \frac{\sqrt{\kappa_2^2 + \kappa_3^2}}{|\phi|}; \quad \phi := \lambda_2(s) \kappa_2(s) \\
\overline{\kappa}_2 &= W \phi^2 (\kappa_2^2 + \kappa_3^2), \\
\overline{\kappa}_3 &= \frac{(\kappa_2 N - \kappa_3 L) (\kappa_1 L + K') + (\kappa_2 N' - \kappa_3 L') K + \kappa_1 \kappa_3 K^2}{W \phi^4 \overline{\kappa}_1 \overline{\kappa}_2}
\end{align*}
\]

where

\[
\begin{align*}
K &= \kappa_1 \kappa_2 \phi \\
L &= 2 \kappa_2 \phi' + \kappa_2' \phi \\
N &= 2 \kappa_3 \phi' + \kappa_3' \phi
\end{align*}
\]
and

\[ W = \sqrt{K^2 (\kappa_2^2 + \kappa_3^2) + (\kappa_2 N - \kappa_3 L)^2} \]  \hfill (38)

\[ = |\phi| \sqrt{\kappa_1^2 \kappa_2^2 (\kappa_2^2 + \kappa_3^2) + (2\kappa_2 \kappa_3' - \kappa_3 \kappa_2')^2}. \]

**Proof.** Let \( \overline{x} = \overline{x}(s) \) be the involute of order 2 of the curve \( x \) in \( \mathbb{E}^4 \). Then by the use of (7), we get

\[ \overline{x}'(s) = \phi V_3 \]  \hfill (39)

where \( \phi = \lambda_2(s) \kappa_2(s) \) is a differentiable function. Further, the differentiation of (39) implies that

\[
\overline{x}''(s) = -\phi \kappa_2 V_2 + \phi' V_3 + \phi \kappa_3 V_4, \\
\overline{x}'''(s) = \kappa_1 \kappa_2 V_1 + \{2\kappa_2 \phi' + \kappa_2' \phi\} V_2, \\
\quad + \{\phi'' - \kappa_2^2 \phi - \kappa_3^2 \phi\} V_3 + \{2\kappa_3 \phi' + \kappa_3' \phi\} V_4. \]  \hfill (40)

Consequently, substituting

\[ K = \kappa_1 \kappa_2 \phi, \]
\[ L = 2\kappa_2 \phi' + \kappa_2' \phi, \]
\[ M = \phi'' - \kappa_2^2 \phi - \kappa_3^2 \phi, \]
\[ N = 2\kappa_3 \phi' + \kappa_3' \phi \]

the last vector becomes

\[ \overline{x}''' = K V_1 - L V_2 + M V_3 + N V_4. \]  \hfill (42)

Furthermore, differentiating \( \overline{x}''' \) with respect to \( s \) we get

\[
\overline{x}'''' = \{K' + \kappa_1 L\} V_1 + \{\kappa_1 K - \kappa_2 M - L'\} V_2 \\
\quad + \{M' - \kappa_2 L - \kappa_3 N\} V_3 + \{N' + \kappa_3 M\} V_4 \]  \hfill (43)

Hence, substituting (39)-(43) into (4) and (5), after some calculations as in the previous proposition, we get the result. \( \blacksquare \)

For the case \( x \) is a \( W \)-curve then one can get the following results.

**Corollary 12.** Let \( \overline{x} \) be an involute of order 2 of a generic \( x \) curve in \( \mathbb{E}^4 \) given with the Frenet curvatures \( \overline{\kappa}_1, \overline{\kappa}_2 \) and \( \overline{\kappa}_3 \). If \( x \) is a \( W \)-curve then the
Frenet 4-frame, $\overline{V}_1, \overline{V}_2, \overline{V}_3$ and $\overline{V}_4$ and Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ of the involute $\overline{x}$ of order 2 of a regular curve $x$ in $\mathbb{E}^4$ are given by

\begin{align}
\overline{V}_1(s) &= V_3, \\
\overline{V}_2(s) &= \frac{-\kappa_2 V_2 + \kappa_3 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}}, \\
\overline{V}_3(s) &= V_1, \\
\overline{V}_4(s) &= \frac{\kappa_3 V_2 + \kappa_2 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}}, \\
\end{align}

and

\begin{align}
\overline{\kappa}_1 &= \frac{\sqrt{\kappa_2^2 + \kappa_3^2}}{|\phi|}, \\
\overline{\kappa}_2 &= \frac{\kappa_1 \kappa_2}{|\phi| \sqrt{\kappa_2^2 + \kappa_3^2}}, \\
\overline{\kappa}_3 &= \frac{\kappa_1 \kappa_3}{|\phi| \sqrt{\kappa_2^2 + \kappa_3^2}},
\end{align}

holds, where $\phi(s) = \lambda_2(s) \kappa_2(s)$.

**Corollary 13.** Let $\overline{x}$ be an involute of order 2 of a generic $x$ curve in $\mathbb{E}^4$ given with the Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$. If $x$ is a $W$-curve then $\overline{x}$ becomes a ccr-curve.

**4.3. Involutes of order 3 in $\mathbb{E}^4$**

An involute of order 3 of a regular curve $x$ in $\mathbb{E}^4$ has the parametrization

\[ \overline{x}(s) = x(s) + \lambda_1(s) V_1(s) + \lambda_2(s) V_2(s) + \lambda_3(s) V_3(s) \]

where

\begin{align}
\lambda_1'(s) &= \kappa_1(s) \lambda_2(s) - 1, \\
\lambda_2'(s) &= \lambda_3 \kappa_2 - \lambda_1 \kappa_1, \\
\lambda_3'(s) &= -\lambda_2(s) \kappa_2(s).
\end{align}

By solving the system of differential equations in (47) we get the following result.
Corollary 14. Let \( x = x(s) \) is a unit speed W-curve in \( \mathbb{E}^4 \). Then the involute \( \overline{x} \) of order 3 of the curve \( x \) has the parametrization (46) given with the coefficient functions

\[
\begin{align*}
\lambda_1(s) &= \frac{\kappa_1 (c_2 \sin(\kappa s) - c_3 \cos(\kappa s))}{\kappa} + \frac{c_1 \kappa - \kappa_2^2 s}{\kappa}, \\
\lambda_2(s) &= c_2 \cos(\kappa s) - c_3 \sin(\kappa s) + \frac{\kappa_1}{\kappa}, \\
\lambda_3(s) &= \frac{\kappa_2 (c_2 \sin(\kappa s) - c_3 \cos(\kappa s))}{\kappa} - \frac{c_1 \kappa_1 \kappa - \kappa_1 \kappa_2^2 s}{\kappa \kappa_2},
\end{align*}
\]

where \( \kappa = \kappa_1^2 + \kappa_2^2, \ c_1, \ c_2 \) and \( c_3 \) are real constants.

We obtain the following result.

Proposition 15. Let \( x = x(s) \) be a regular curve in \( \mathbb{E}^4 \) given with nonzero Frenet curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). Then Frenet Frenet 4-frame, \( \overline{V}_1, \overline{V}_2, \overline{V}_3 \) and \( \overline{V}_4 \) and Frenet curvatures \( \overline{\kappa}_1, \overline{\kappa}_2 \) and \( \overline{\kappa}_3 \) of the involute \( \overline{x} \) of order 3 of a regular curve \( x \) in \( \mathbb{E}^4 \) are given by

\[
\begin{align*}
\overline{V}_1(s) &= V_4, \\
\overline{V}_2(s) &= -V_3, \\
\overline{V}_3(s) &= V_2, \\
\overline{V}_4(s) &= V_1,
\end{align*}
\]

and

\[
\begin{align*}
\overline{\kappa}_1 &= \frac{\kappa_3}{|\psi|}, \\
\overline{\kappa}_2 &= \frac{\kappa_2}{|\psi|}, \\
\overline{\kappa}_3 &= -\frac{\kappa_1}{|\psi|},
\end{align*}
\]

where \( \psi(s) = \lambda_3(s) \kappa_3(s) \).

Proof. Let \( \overline{x} = \overline{x}(s) \) be the involute of order 3 of the curve \( x \) in \( \mathbb{E}^4 \). Then by the use of (7) with (8), we get

\[
\overline{x}'(s) = \psi V_4
\]
where $\psi = \lambda_3(s)\kappa_3(s)$ is a differentiable function. Further, the differentiation of (51) implies that
\[
\begin{align*}
\bar{x}''(s) &= -\psi \kappa_3 V_3 + \psi' V_4, \\
\bar{x}'''(s) &= \kappa_2 \kappa_3 \psi V_2 - \{2\kappa'_3 \psi + \kappa'_3 \phi\} V_3 + \{\psi'' - \kappa'_3 \psi\} V_4.
\end{align*}
\]
Consequently, substituting
\[
\begin{align*}
E &= \kappa_2 \kappa_3 \psi \\
F &= 2\kappa'_3 \psi + \kappa'_3 \phi \\
G &= \psi'' - \kappa'_3 \psi
\end{align*}
\]
the last vector becomes
\[
\bar{x}''' = EV_2 - FV_3 + GV_4. \tag{53}
\]
Furthermore, differentiating $\bar{x}'''$ with respect to $s$ we get
\[
\begin{align*}
\bar{x}'''' &= -\kappa_1 EV_1 + \{\kappa_2 F + E'\} V_2 \\
&\quad + \{\kappa_2 E - \kappa_3 G - F'\} V_3 + \{G' - \kappa_3 F\} V_4. \tag{54}
\end{align*}
\]
Hence, substituting (51)-(54) into (4) and (5), after some calculations we get the result. \(\blacksquare\)

**Corollary 16.** The involute $\bar{x}$ of order 3 of a ccr-curve $x$ in $\mathbb{E}^4$ is also a ccr-curve of $\mathbb{E}^4$.

**5. Generalized Evolute Curves in $\mathbb{E}^{m+1}$**

Let $x = x(s)$ be a generic curve in $\mathbb{E}^n$ given with Frenet frame $V_1, V_2, V_3, \ldots, V_n$ and Frenet curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. For simplicity, we can take $n = m + 1$, to construct the Frenet frame $V_1 = T, V_2 = N_1, V_3 = N_2, \ldots, V_n = N_m$ and Frenet curvatures $\kappa_1, \kappa_2, \ldots, \kappa_m$. The centre of the osculating hypersphere of $x$ at a point lies in the hyperplane normal to the $x$ at that point. The curve passing through the centers of the osculating hyperspheres of $x$ defined by
\[
\tilde{x} = x + \sum_{i=1}^{m} c_i N_i, \tag{55}
\]
which is called \textit{generalized evolute} (or focal curve) of $x$, where $c_1, c_2, \ldots, c_m$ are smooth functions of the parameter of the curve $x$. The function $c_i$ is called the $i^{th}$ focal curvature of $\gamma$. Moreover, the function $c_1$ never vanishes and $c_1 = \frac{1}{\kappa_1}$.  

The differentiation of the equation (55) and the Frenet formulae (1) give the following equation

$$
\tilde{x}'(s) = (1 - \kappa_1 c_1) T + (c_1' - \kappa_2 c_2) N_1 + \\
+ \sum_{i=2}^{m-1} (c_{i-1} \kappa_i + c_i' - c_{i+1} \kappa_{i+1}) N_i + (c_{m-1} \kappa_m + c_m') N_m.
$$

(56)

Since, the osculating planes of $\tilde{x}$ are the normal planes of $x$, and the points of $\tilde{x}$ are the center of the osculating sphere of $x$ then the generalized evolutes $\tilde{x}$ of the curve $x$ are determined by

$$
\langle \tilde{x}'(s), T(s) \rangle = \langle \tilde{x}'(s), N_1(s) \rangle = \ldots = \langle \tilde{x}'(s), N_{m-1}(s) \rangle = 0.
$$

(57)

This condition is satisfied if and only if

$$
1 - \kappa_1 c_1 = 0,  \\
c_1' - \kappa_2 c_2 = 0, \\
\vdots \\
c_{i-1} \kappa_i + c_i' - c_{i+1} \kappa_{i+1} = 0, \quad 2 \leq i \leq m - 1.
$$

(58)

hold. So, the focal curvatures of a curve parametrized by arclength $s$ satisfy the following "scalar Frenet equation" for $c_m \neq 0$:

$$
\frac{R_m^2}{2c_m} = c_{m-1} \kappa_m + c_m'.
$$

(59)

where

$$
R_m = \| \tilde{x} - x \| = \sqrt{c_1^2 + c_2^2 + \ldots + c_m^2}
$$

is the radius of the osculating $m$-sphere [13]. Consequently, the generalized evolutes $\tilde{x}$ of the curve $x$ are represented by the formulas (55), and

$$
\tilde{x}'(s) = (c_{m-1} \kappa_m + c_m') N_m.
$$

(60)

If $\tilde{x}'(s) = 0$, then $R_m$ is constant and the curve $x$ is spherical.
Proposition 17. The curvatures of a generic curve \( x = x(s) : I \subset \mathbb{R} \to \mathbb{E}^{m+1} \) parametrized by arc length, may be obtained in terms of the focal curvatures by the formula:

\[
\kappa_i = \frac{c_1 c'_1 + c_2 c'_2 + \ldots + c_{i-1} c'_{i-1}}{c_{i-1} c_i}. \tag{61}
\]

Remark 18. For a generic curve, the functions \( c_i \) or \( c_{i-1} \) can vanish at isolated points. At these points the function \( c_1 c'_1 + c_2 c'_2 + \ldots + c_{i-1} c'_{i-1} \) also vanishes, and the corresponding value of the function \( \kappa_i \) may be obtained by l'Hospital rule. Denote by \( R_m \) the radius of the osculating \( m \)-sphere. Obviously

\[
R_m^2 = c_1^2 + c_2^2 + \ldots + c_m^2. \tag{62}
\]

Theorem 19. Let \( x = x(s) \) be a generic curve in \( \mathbb{E}^{m+1} \) given with Frenet frame \( T, N_1, N_2, \ldots, N_m \) and Frenet curvatures \( \kappa_1, \kappa_2, \ldots, \kappa_m \). Then Frenet frame \( \tilde{T}, \tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_m \) and Frenet curvatures \( \tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_m \) of the generalized evolute \( \tilde{x} \) of \( x \) in \( \mathbb{E}^{m+1} \) are given by

\[
\tilde{T} = \epsilon N_m, \\
\tilde{N}_k = \delta_k N_{m-k}; \quad 1 \leq k \leq m-1 \\
\tilde{N}_m = \pm T
\]

and

\[
\frac{\tilde{\kappa}_1}{|\kappa_m|} = \frac{\tilde{\kappa}_2}{\kappa_{m-1}} = \ldots = \frac{|\tilde{\kappa}_m|}{|\kappa_{m-1} \kappa_m + c'_m|} = \frac{1}{|c_{m-1} \kappa_m + c'_m|} \tag{63}
\]

where \( \epsilon(s) \) is the sign of \( (c_{m-1} \kappa_m + c'_m)(s) \) and \( \delta_k \) the sign of \( (-1)^k \epsilon(s) \kappa_m(s) \).

5.1. Evolutes in \( \mathbb{E}^3 \)

An generalized evolute of a regular curve \( x \) in \( \mathbb{E}^3 \) has the parametrization

\[
\tilde{x}(s) = x(s) + c_1(s) N_1(s) + c_2(s) N_2(s) \tag{64}
\]

where \( N_1 \) and \( N_2 \) are normal vectors of \( x \) in \( \mathbb{E}^3 \) and \( c_1, c_2 \) are focal curvatures satisfying

\[
c_1(s) = \frac{1}{\kappa_1(s)}, \quad c_2(s) = \frac{\rho'(s)}{\kappa_2(s)}. \tag{65}
\]

where \( \rho = c_1 = \frac{1}{\kappa_1} \) is the radius of the curvature of \( x \).

We obtain the following result.
Proposition 20. Let $x = x(s)$ be a regular curve in $\mathbb{E}^3$ given with nonzero Frenet curvatures $\kappa_1$ and $\kappa_2$. Then Frenet curvatures $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ of the evolute $\tilde{x}$ of the curve $x$ are given by

$$
\tilde{\kappa}_1 = \frac{\kappa_2^2}{|\rho\kappa_2^2 + \rho'|}, \quad \tilde{\kappa}_2 = \frac{\kappa_1\kappa_2}{|\rho\kappa_2^2 + \rho'|}.
$$

(66)

where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of $x$.

Proof. As a consequence of (63) we get (64).

Corollary 21. The evolute $\tilde{x}$ of a generalized helix in $\mathbb{E}^3$ is also a generalized helix in $\mathbb{E}^3$.

By the use of (59) with (65) one can get the following result.

Corollary 22. A regular curve with nonzero curvatures $\kappa_1$ and $\kappa_2$ lies in a sphere if and only if

$$
\left(\frac{\rho'}{\kappa_2}\right)' + \rho\kappa_2 = 0
$$

(67)

holds, where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of $x$.

5.2. Evolutes in $\mathbb{E}^4$

An generalized evolute of a generic curve $x$ in $\mathbb{E}^4$ has the parametrization

$$
\tilde{x}(s) = x(s) + c_1(s)N_1(s) + c_2(s)N_2(s) + c_3(s)N_3(s)
$$

(68)

where $N_1$, $N_2$ and $N_3$ are normal vectors of $x$ in $\mathbb{E}^4$ and $c_1$, $c_2$ and $c_3$ are focal curvatures satisfying

$$
c_1(s) = \frac{1}{\kappa_1(s)}, \quad c_2(s) = \frac{\rho'(s)}{\kappa_2(s)}, \quad c_3(s) = \frac{\rho(s)\kappa_2(s) + \left(\frac{\rho'(s)}{\kappa_2(s)}\right)'}{\kappa_3(s)}.
$$

(69)

where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of $x$.

We obtain the following result.
Proposition 23. Let \( x = x(s) \) be a regular curve in \( \mathbb{E}^4 \) given with nonzero Frenet curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). Then Frenet 4-frame, \( \tilde{T}, \tilde{N}_1, \tilde{N}_2 \) and \( \tilde{N}_3 \) and Frenet curvatures \( \tilde{\kappa}_1, \tilde{\kappa}_2 \) and \( \tilde{\kappa}_3 \) of the evolute \( \tilde{x} \) of a regular curve \( x \) in \( \mathbb{E}^4 \) are given by

\[
\begin{align*}
\tilde{T}(s) &= N_3, \\
\tilde{N}_1(s) &= -N_2, \\
\tilde{N}_2(s) &= N_1, \\
\tilde{N}_3(s) &= T,
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\kappa}_1 &= \frac{\kappa_3}{|\psi|}, \\
\tilde{\kappa}_2 &= \frac{\kappa_2}{|\psi|}, \\
\tilde{\kappa}_3 &= -\frac{\kappa_1}{|\psi|}
\end{align*}
\]

where \( \psi(s) = c_2(s)\kappa_3(s) + c_3'(s) \) is a smooth function.

Proof. As a consequence of (62) with (63) we get the result.

Corollary 24. The evolute \( \tilde{x} \) of a ccr-curve \( x \) in \( \mathbb{E}^4 \) is also a ccr-curve of \( \mathbb{E}^4 \).

By the use of (60) with (65) one can get the following result.

Corollary 25. A regular curve with nonzero curvatures \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) lies on a sphere if and only if

\[
\left( \frac{\rho(s)\kappa_2(s) + \left( \frac{\rho'(s)}{\kappa_2(s)} \right)'\kappa_3(s)}{\kappa_3(s)} \right)' + \rho'(s)\frac{\kappa_3(s)}{\kappa_2(s)} = 0
\]

holds, where \( \rho = \frac{1}{\kappa_1} \) is the radius of the curvature.

Proposition 26. A curve \( x = x(s) : I \subset \mathbb{R} \rightarrow \mathbb{E}^4 \) is spherical, i.e., it is contained in a sphere of radius \( R \), if and only if \( x \) can be decomposed as

\[
x(s) = m - \frac{R}{\kappa_1}N_1(s) + \frac{R\kappa_1'}{\kappa_2\kappa_1^2}N_2(s) + \frac{R}{\kappa_3} \left( \frac{\kappa_1'}{\kappa_2\kappa_1^2} \right)'N_3(s).
\]

where \( m \) is the center of the sphere.
References

[1] D. Blaženka and MS. Željka, *Involutes and evolutes in n-dimensional simply isotropic space*, Journal of information and organizational sciences 2(3) (1999), 71-79.

[2] H. Gluck, *Higher curvatures of curves in Euclidean space*, Am. Math. Monthly 73 (1966), 699-704.

[3] O. A. Goncharova, *Ruled surfaces in $E^4$ with constant ratio of the Gaussian curvature and Gaussian torsion*, Journal of Mathematical Physics, Analysis, Geometry 4(3) (2008), 371-379.

[4] G. P. Henderson, *Parallel curves*, Canad. J. Math. 6 (1954), 99-107.

[5] B. Kılıç, K. Arslan and G. Öztürk, *Tangentially cubic curves in Euclidean spaces*, Differential Geometry-Dynamical Systems 10 (2008), 186-196.

[6] F. Klein and S. Lie, *Über diejenigen ebenen kurven welche durch ein geschlossenes system von einfach unendlich vielen vartauschbaren linearen Transformationen in sich übergehen*, Math. Ann. 4 (1871), 50-84.

[7] J. Monterde, *Curves with constant curvature ratios*, Bull. Mexican Math. Soc. Ser. 3A 13(1) (2007), 177-186.

[8] G. Öztürk, K. Arslan and H. H. Hacisalihoglu, *A characterization of ccr-curves in $R^m$*, Proc. Estonian Acad. Sci. 57(4) (2008), 217-224.

[9] M. C. Romero-Fuster and E. Sanabria-Codesal, *Generalized evolutes, vertices and conformal invariants of curves in $R^{n+1}$*, Indag. Mathem., N.S. 10 (1999), 297-305.

[10] E. Salkowski, *Zur transformation von raumkurven*, Math. Ann. 66(4) (1909), 517-557.

[11] M. Turgut and T. A. Ali, *Some characterizations of special curves in the Euclidean space $E^4$*, Acta Univ. Sapientiae, Mathematica 2(1) (2010), 111-122.

[12] R. Uribe-Vargas, *On singularites, “perestroikas” and differential geometry of space curve*, Ens. Math. 50 (2004), 69-101.
[13] R. Uribe-Vargas, *On vertices, focal curvatures and differential geometry of space curves*, Bull Braz. Math. Soc 36 (2005), 285-307.