Splitting of single-particle levels in clusters potentials

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Abstract. In analogy with the Nilsson model, we calculate the splitting of spherical single-particle levels in a deformed field, but for cluster potentials. We study applications to alpha-cluster nuclei with two, three and four alpha particles, in which the deformation corresponds to the relative distance between the alpha particles. The splitting of the single-particle levels is studied for the cases of a dumbbell, equilateral triangle and a regular tetrahedron. The observed patterns may be used to gain insight into how the single-particle levels evolve with deformation.

1. Introduction

Energy level splitting is a very well-known phenomenon in physics, which occurs whenever a set of degenerate levels is split in an external field. One of the best known examples is the Zeeman effect in which rotational states are split in an external magnetic field according to their magnetic substates (Figure 1) [1].

Another example is the Nilsson model of deformed shell model where the single-particle levels are split in a deformed quadrupole field [2]. In this case the energy levels are not only split but also strongly mixed by the quadrupole field. The amount of mixing depends on the strength of the deformed field (Figure 2). Level crossing and repulsion occur due to this deformation. In the case of the Zeeman effect, the levels are split but not mixed, and can therefore be labeled by the same quantum numbers as before the splitting, i.e the angular momentum \( l \) and its projection \( m_l \). In the case of the Nilsson model, the levels are strongly mixed and are labeled by the quantum numbers in the intrinsic or body-fixed frame \([N\lambda JΩ]^*\) where \( Ω \) denotes the...
projection of the single-particle angular momentum on the symmetry axis (z-axis), $\pi$ is the parity, $N$ the principal quantum number, $n_z$ the number of nodes in the wave function in the z-direction and $\Lambda$ the projection of the orbital angular momentum on the symmetry axis.

In our work, we use a deformed field, that of cluster potentials, where the clusters are alpha particles. The deformation is taken to be the relative distance to the center of mass of the whole $k\alpha + x$ nucleon structure, where is $x$ a nucleon. The splitting of the single-particle levels is studied as a function of the relative distance for the cases of a dumbbell, equilateral triangle and a regular tetrahedron, where a symmetry factor is crucial for the level splitting and configuration mixing. Relabeling is also necessary in this case specially for the cases of an equilateral triangle and regular tetrahedron. The main goal for this study is to obtain patterns that gain insight into how the single-particle levels evolve with deformation. That is done by studying the mixing between states, how these are related with the geometrical configuration and what kind of symmetry is present.

2. Cluster Shell Model

We start by reviewing the Cluster Shell Model [3]. In this model nuclei with $Z = N = 2k$ are treated as a cluster of $k$ $\alpha$-particles whose matter and charge density are given by a gaussian form

$$ \rho (\vec{r}) = \left( \frac{\alpha}{\pi} \right)^{\frac{3}{2}} e^{-\alpha (r^2 + \beta^2)} 4\pi \sum_{\lambda \mu} i_\lambda (2\alpha\beta r) Y_{\lambda\mu} (\theta, \phi) \sum_{i=1}^k Y_{\lambda\mu}^* (\theta_i, \phi_i). \quad (1) $$

Here $\vec{r}_i = (\beta, \theta_i, \phi_i)$ where $\beta$ denotes the relative distance of the $\alpha$ particles to the center of mass, and $\theta_i$ and $\phi_i$ are the angles. An important factor is the cluster factor, $\sum_{i=1}^k Y_{\lambda\mu}^* (\theta_i, \phi_i)$, which contains the information about the geometrical configuration of the alpha particles. The potential is obtained by convoluting the density, eq. (1), with a Volkov potential [4] to obtain

$$ V (\vec{r}) = -V_0 \sum_{\lambda \mu} 4\pi e^{-\alpha (r^2 + \beta^2)} i_\lambda (2\alpha\beta r) Y_{\lambda\mu} (\theta, \phi) \sum_{i=1}^k Y_{\lambda\mu}^* (\theta_i, \phi_i). \quad (2) $$
The spin-orbit interaction is taken as

\[ V_{so} (\vec{r}) = V_{0,so} \frac{1}{2} \left[ \frac{1}{r} \frac{\partial V (\vec{r})}{\partial r} (\vec{s} \cdot \vec{l}) + (\vec{s} \cdot \vec{l}) \frac{1}{r} \frac{\partial V (\vec{r})}{\partial r} \right]. \] (3)

Finally, the coulomb potential is found by convoluting the charge density \( Z e^2 k \rho (\vec{r}) \) with Green’s function giving

\[
V_C (\vec{r}) = \frac{Ze^2}{k} \left( \frac{\alpha}{\pi} \right)^{\frac{3}{2}} \sum_{\lambda\mu} \frac{4\pi}{2\lambda + 1} Y_{\lambda\mu} (\theta, \phi) \sum_{i=1}^{k} Y_{\lambda\mu}^* (\theta_i, \phi_i) \times \\
\times \left[ \frac{1}{r^{\lambda+1}} \int_0^r f_\lambda (r') r'^{\lambda+2} dr' + r^\lambda \int_0^\infty f_\lambda (r') \frac{1}{r^{\lambda-1}} dr' \right]. \] (4)

(5)

We then obtain the single-particle energy levels and intrinsic states as a function of \( \beta \) for each configuration \( (k = 2, 3, 4) \) of fig. 3 by solving the single-particle Schrödinger equation

\[ H = \frac{\vec{p}^2}{2m} + V (\vec{r}) + V_{so} (\vec{r}) + V_C (\vec{r}). \] (6)

In the case of neutrons \( V_C (\vec{r}) = 0 \).

3. Results

The Hamiltonian of eq. (6) is diagonalized in the harmonic oscillator basis. The correlation diagrams are shown as a function of \( \beta \) in Figures 4 and 5. The states for the spherical case correspond to \( \beta = 0 \). For the two-body cluster configuration the levels with projection \( \pm \Omega \) are degenerate. As a consequence, a single-particle level \((l, \frac{1}{2})_j\) is split into a series of doublets with \( \Omega = \frac{1}{2}, \frac{3}{2}, \ldots, j \) and parity \( P = (-)^l \). For the triangular configuration neither the angular momentum nor parity is conserved. The spherical single-particle levels are split into a series of doublets characterized by \( E_{2}^{\prime} \) and \( E_{1}^{\prime} \), the double valued representations of \( D_3 \), the rotational subgroup of \( D_{3h} \). For the tetrahedral configuration, the single-particle levels are split into a
Table 1. Resolution of single-particle levels into the double-valued representations of the corresponding point group. Notation from Hamermesh [5].

| $j$     | Dumbbell          | Triangle | Tetrahedron |
|---------|-------------------|----------|-------------|
| 1/2     | $(1/2)^P$         | $E'_2$   | $E'$        |
| 3/2     | $(1/2 + 3/2)^P$   | $E'_1 + E'_2$ | $G'$        |
| 5/2     | $(1/2 + 3/2 + 5/2)^P$ | $E'_1 + 2E'_2$ | $E' + G'$   |
| 7/2     | $(1/2 + 3/2 + 5/2 + 7/2)^P$ | $E'_1 + 3E'_2$ | $2E' + G'$ |

series of doublets and quadruplets characterized by $E'$ and $G'$, respectively, the double valued representations of $T$, the rotational subgroup of $T_d$. The results are summarized in table 1. The first result that we can see by comparing Figure 4 is that the correlation diagrams for protons and neutrons behave in the same manner. This result shows that it does not matter much whether the particle is a neutron or a proton, there is just a small shift in energy.

In order to address the issue of mixing between the levels we study the overlap between intrinsic states of different values of deformation, $\langle \chi_\Omega (\beta) \lvert \chi_\Omega' (\beta') \rangle$. We can then have the overlap with a fixed value of $\beta'$, namely $\beta' = 0$. In that way we can see the mixing between states as we vary $\beta$. This can be seen in Figure 5, where from comparing the graphs of overlap of the triangle and tetrahedron, there is present more mixing in the case of triangle than the case of tetrahedron. A main reason why this occurs is because the density of states of same symmetry is higher in the configuration of triangle, as a result those levels are then closer energetically to each other and can mix more strongly in comparison to the tetrahedron.

Figure 4. Correlation diagrams for the case of dumbbell, obtained from eq. (6) with $V_0 = 32$ MeV, $\alpha = 0.0511$ fm$^{-2}$ and $V_{0,so} = 27.5$ MeV fm$^2$ [3] for a proton (left) and a neutron (right).

For the case of triangle there exist strong repulsion between states, which causes crossings of wave function from one level to another. These cases can be seen in the graph of overlap of the triangle in the region 3.5 to 4.5 fm for the states belonging to the $E'_2$ symmetry. In states
Figure 5. Correlation diagrams (top) and overlaps between states of the same symmetry (bottom) for the case of a triangle (left) and a tetrahedron (right), and a single-particle neutron, obtained from eq. (6) with $V_0 = 32$ MeV, $\alpha = 0.0511$ fm$^{-2}$ and $V_{0,so} = 17$ MeV fm$^2$ for the triangle, and $V_{0,so} = 12.1$ MeV fm$^2$ for the tetrahedron.

belonging to the $E_1'$ that doesn’t happen, and the mixing occurs smoothly with no wave function crossings.

In the case of the tetrahedron, although there occurs a wave function crossing (between 3.5 to 4 fm) this doesn’t happens as much as in the case of triangle. There is strong mixing but states are sufficiently separated so that wave function crossings don’t occur as often as in the case of triangle.

In both cases the difference between states of the same symmetry group arise from how much energetically distanced they are from other states. A simple example are the ground levels, which in comparison to higher levels present much less mixing. That can be seen by comparing...
the curves $\langle \chi_\Omega (\beta) | \chi_\Omega (\beta' = 0) \rangle$ where $\Omega$ can be any of the double-valued representations for triangle and tetrahedron shown in Table 1. The faster those curves decrease the more the state has mixed with others. Again, the mayor difference between the states is seen in the case of the triangle. Also the higher the state of the same group of symmetry the more it mixes. From both correlation diagrams of Figure 5 it can be seen that the single-particle levels asymptotically tend to a certain degeneracy, which corresponds to the case where the alpha particles are too far apart and thus interact as separated alpha particles. This is something that is not obtained in the Nilsson model, where the degeneracy shown is that of an elongated cigar. As this occurs mixing between states decrease. This is shown in the curves of $\langle \chi_\Omega (\beta) | \chi_{\Omega'} (\beta' = 0) \rangle$. Here $\Omega$ and $\Omega'$ are states of the same symmetry, but not the same state. The decrease starts in most cases at 5 fm, hinting that the frontier between the moment where the alphas can still be seen as to be part of the same nucleus and not separate particles is around that value. In the correlation diagrams it is also shown that convergence towards the degeneracy of separate alphas occur after 5 fm.

4. Conclusions
From the obtained patterns it has been illustrated how single-particle levels evolve with deformation and which states interact most and why. From the overlap it can also be thought which interactions could be the predominant ones (quadrupole, octupole, etc). This and the interpretation of the behaviour due to the symmetry will be further analysed. All the information from this study can later be used to extend the Algebraic Cluster Model (ACM)\[6\]\[7\] to odd-cluster nuclei like $^9$Be, $^9$B, $^{13}$C, $^{13}$N, $^{17}$O and $^{17}$F, by developing the Algebraic Cluster-Fermion Model (ACFM)\[8\].

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