Abstract. In this paper, we establish spectral inequalities on measurable sets of positive Lebesgue measure for the Stokes operator, as well as an observability inequalities on space-time measurable sets of positive measure for non-stationary Stokes system. Furthermore, we provide their applications in the theory of shape optimization and time optimal control problems.

Keywords: spectral inequality, observability inequality, Stokes equations, shape optimization problems, time optimal control problem.

Mathematics Subject Classification (2010): 49Q10, 76D07, 76D55, 93B05, 93B07, 93C95.

1. Introduction and main results

Let $T > 0$, and let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded connected open set with a smooth boundary $\partial \Omega$. We will use the notation $Q = \Omega \times (0, T)$, $\Sigma = \partial \Omega \times (0, T)$, and we will denote by $\nu = \nu(x)$ the outward unit normal vector to $\Omega$ at $x \in \partial \Omega$. Throughout the paper spaces of $\mathbb{R}^N$-valued functions, as well as their elements, are represented by boldface letters.

The present paper deals with an observability inequality on measurable sets of positive measure for the Stokes system

\[
\left\{ \begin{array}{ll}
    z_t - \Delta z + \nabla q = 0 & \text{in } Q, \\
    \text{div } z = 0 & \text{in } Q, \\
    z = 0 & \text{on } \Sigma, \\
    z(\cdot, 0) = z_0 & \text{in } \Omega.
\end{array} \right. 
\] (1.1)

System (1.1) is a linearization of the Navier-Stokes system for a homogeneous viscous incompressible fluid (with unit density and unit kinematic viscosity) subject to homogeneous Dirichlet boundary conditions. Here, $z$ is the $\mathbb{R}^N$-valued velocity field and $q$ stands for the scalar pressure.

Our motivation to obtain an observability inequality on measurable sets for the Stokes system (1.1) comes from the well-known fact that observability inequalities are equivalent to controllability properties. In the case we are dealing with, this will be equivalent to the null controllability of system (1.1) with bounded controls acting on measurable sets with positive measure, and will have important applications in shape optimization problems and in the study of the bang-bang property for time and norm optimal control problems for system (1.1) (see Section 3).

Department of Mathematics, Federal University of Pernambuco, CEP 50740-545, Recife, PE, Brazil. E-mail: fchaves@mat.ufpe.br. F. W. Chaves-Silva was supported ERC Project No. 320845: Semi Classical Analysis of Partial Differential Equations, ERC-2012-ADG.

Department of Mathematics, Federal University of Pernambuco, CEP 50740-545, Recife, PE, Brazil. E-mail: diego.souza@mat.ufpe.br. D. A. Souza was supported by the ERC advanced grant 668998 (OCLOC) under the EU’s H2020 research program.

School of Mathematics and Statistics, Wuhan University, 430072 Wuhan, China; Sorbonne Universités, UPMC Univ Paris 06, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005 Paris, France. E-mail: zhangcansx@163.com.
Observability inequalities for system (1.1) from a cylinder $\omega \times (0, T)$, with $\omega \subset \Omega$ being a non-empty open set, have been proved in different ways by several authors in the past few years. For instance, in [11], the observability inequality for the Stokes system is obtained by means of global Carleman inequalities for parabolic equations with zero Dirichlet boundary conditions (see also [7] and [10]). Another proof is given in [12] by means of Carleman inequalities for parabolic equations with non-homogeneous Dirichlet boundary conditions applied to the system satisfied by the vorticity $\text{curl} \, z$. More recently, in [6], a new proof was established based on a spectral inequality for the eigenfunctions of the Stokes operator.

Concerning observability inequalities over general measurable sets in space and time variables, as far as we know, the first result was obtained in [2] for the heat equation in a bounded and locally star-shaped domain, and later extended in [8] and [9] to the case of parabolic systems with time-independent analytic coefficients associated to possibly non self-adjoint elliptic operators and higher order parabolic evolutions with the analytic coefficients depending on space and time variables, when the boundary of the bounded domain in which the equation evolves is global analytic. We also refer the interested reader to [1, 19, 24] for some earlier and closely related results on this subject.

For the Stokes system, the only result we know is the one in [25], which gives an observability inequality from a measurable subset with positive measure in the time variable. In there, the argument is mainly based on the theory of analytic semigroups. In this paper, we extended the result in [25] to the case of observations from sets of positive measure in both time and space variables.

Before presenting our main results, we first introduce the usual spaces in the context of fluid mechanics:

$$
V = \{ y \in H^1_0(\Omega)^N; \text{div} \, y = 0 \},
$$

$$
H = \{ y \in L^2(\Omega)^N; \text{div} \, y = 0, \, y \cdot \nu = 0 \text{ on } \partial \Omega \}.
$$

Throughout the paper, the following notation will be used: $B_R(x_0)$ denotes a ball in $\mathbb{R}^N$ of radius $R > 0$ and with center $x_0 \in \mathbb{R}^N$; $|\omega|$ is the Lebesgue measure of a subset $\omega \subset \Omega$ and $C(\cdots)$ stands for a positive constant depending only on the parameters within the brackets, and it may vary from line to line in the context.

Our first result is a $L^1$-observability inequality from measurable sets with positive measure for system (1.1).

**Theorem 1.1.** Let $B_{4R}(x_0) \subset \Omega$. For any measurable subset $\mathcal{M} \subset B_R(x_0) \times (0, T)$ with positive measure, there exists a positive constant $C_{\text{obs}} = C(N, R, \Omega, \mathcal{M}, T)$ such that the observability estimate

$$
\|z(T, \cdot)\|_H \leq C_{\text{obs}} \int_{\mathcal{M}} |z(x, t)| \, dx dt
$$

(1.2)

holds for all $z_0 \in H$.

**Remark 1.2.** When the observation set is $\mathcal{M} = B_R(x_0) \times (0, T)$, one can see that the observability constant $C_{\text{obs}}$ has the form $Ce^{C/T}$ with $C = C(N, \Omega, R) > 0$. This is in accordance with the very recent result [6, Theorem 1.1].

**Remark 1.3.** The above technical assumption imposed on the measurable set $\mathcal{M}$ is just to simplify the statement of the main result. Without loss of generality, for any measurable set $\mathcal{M} \subset \Omega \times (0, T)$ with positive measure, one can always assume that

$$
\mathcal{M} \subset B_R(x_0) \times (0, T) \quad \text{with} \quad B_{4R}(x_0) \subset \Omega
$$
for some $R > 0$ and $x_0 \in \mathbb{R}^N$. Otherwise, one may choose a new measurable set $\tilde{M} \subset M$ with $|\tilde{M}| \geq c|M|$ for some constant $0 < c < 1$.

The method we shall use to prove Theorem 1.1 relies mainly on the telescoping series method [2] (which is in part inspired by [15] and [22]), the propagation of smallness for real-analytic functions on measurable sets [23] as well as a spectral inequality for Stokes system.

Let $\{e_j\}_{j \geq 1}$ be the sequence of eigenfunctions of the Stokes system

$$
-\Delta e_j + \nabla p_j = \lambda_j e_j \quad \text{in} \quad \Omega,
$$

$$
\text{div} e_j = 0 \quad \text{in} \quad \Omega,
$$

$$
e_j = 0 \quad \text{on} \quad \partial \Omega,
$$

with the sequence of eigenvalues $\{\lambda_j\}_{j \geq 1}$ satisfying

$$
0 < \lambda_1 \leq \lambda_2 \leq \ldots \quad \text{and} \quad \lim_{j \to \infty} \lambda_j = +\infty.
$$

The following inequality is proved in [6].

**Theorem 1.4.** [6, Theorem 3.1] For any non-empty open subset $\Omega \subset \Omega$, there exists a constant $C = C(N, \Omega, \Omega) > 0$ such that

$$
\sum_{\lambda_j \leq \Lambda} a_j^2 \leq \left( \int_\Omega \left| \sum_{\lambda_j \leq \Lambda} a_j e_j(x) \right|^2 \, dx \right)^2 \leq C e^{C \sqrt{\Lambda}} \int_\Omega \left| \sum_{\lambda_j \leq \Lambda} a_j e_j(x) \right|^2 \, dx,
$$

for any sequence of real numbers $\{a_j\}_{j \geq 1} \in \ell^2$ and any positive number $\Lambda$.\(^1\)

Spectral inequality (1.4) allow us to control the low frequencies of the Stokes system with a precise estimate on the cost of controllability with respect to the frequency length which, combined with the decay of solutions of (1.1), implies the null controllability of Stokes system with $L^2$-controls applied to arbitrarily small open sets.

Our second main result is an extension of the spectral inequality (1.4) from open sets to measurable sets of positive measure.

**Theorem 1.5.** Let $B_{4R}(x_0) \subset \Omega$ and let $\omega \subset B_R(x_0)$ be a measurable set with positive measure. Then, there exists a constant $C = C(N, R, \Omega, |\omega|) > 0$ such that

$$
\left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \leq C e^{C \sqrt{\Lambda}} \int_\omega \left| \sum_{\lambda_j \leq \Lambda} a_j e_j(x) \right| \, dx,
$$

for all $\Lambda > 0$ and any sequence of real numbers $\{a_j\}_{j \geq 1} \in \ell^2$.

**Remark 1.6.** Inequality (1.5) leads to a null controllability result for the Stokes system with $L^\infty$-controls (see Theorem 3.5).

\(^1\)Recall that $\ell^2 \triangleq \left\{ \{a_j\}_{j \geq 1} : \sum_{j=1}^{+\infty} a_j^2 < +\infty \right\}$. 


As we will see below, the proof of Theorem 1.5 strongly depends on quantitative estimates of the interior spatial analyticity for finite sums of eigenfunctions of the Stokes system (1.3). As far as we know, for the Navier-Stokes equations, this kind of interior analyticity has been first analyzed in [13] and [14], where the authors consider a nonlinear elliptic system satisfied by the velocity $z$ and the vorticity $\text{curl} z$ and show the interior analyticity for the velocity $z$. However, since the boundary condition for the $\text{curl} z$ is not prescribed, the analyticity up to the boundary cannot be achieved by this method.

In this paper, in order to establish the spectral inequality (1.5), we adapted the arguments in [13] and [14], and [2, Theorem 5], to the low frequencies of the Stokes system.

The paper is organized as follows. In Section 2, we shall present the proofs of Theorems 1.1 and 1.5. Section 3 deals with some applications of main theorems for shape optimization and time optimal control problems of Stokes system. Finally, in Appendix A, we prove some real-analytic estimates for solutions of the Poisson equation.

2. Spectral and Observability Inequalities

2.1. Spectral inequality on measurable sets. This section is devoted to the proof of Theorem 1.5. Compared to the proof of [2, Theorem 5] for the Laplace operator, we here encounter the difficulty due to the pressure in the Stokes system. To circumvent that, we consider the equation satisfied by the $\text{curl}$ of the low frequencies, which is an equation without pressure but with no boundary conditions. This allow us recover and quantify the interior real-analytic estimates based on the $\text{curl}$ operator.

We begin with an estimate of the propagation of smallness for real-analytic functions on measurable sets with positive measure, which plays a core ingredient in the proof of Theorem 1.5.

**Lemma 2.1.** Assume that $f : B_{2R}(x_0) \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$ is real-analytic and verifies

$$|\partial_\alpha f(x)| \leq \frac{M|\alpha|!}{(\rho R)^{\alpha}}, \text{ for } x \in B_{2R}(x_0), \alpha \in \mathbb{N}^N,$$

for some $M > 0$ and $0 < \rho \leq 1$.

For any measurable set $\omega \subset B_R(x_0)$ with positive measure, there are positive constants $C = C(R, N, \rho, |\omega|)$ and $\theta = \theta(R, N, \rho, |\omega|)$, with $\theta \in (0, 1)$, such that

$$\|f\|_{L^\infty(B_R(x_0))} \leq C \left( \int_{\omega} |f(x)| \, dx \right)^\theta M^{1-\theta}. $$

The above-mentioned local observability inequality for real-analytic functions was first established in [23]. The interested reader can also find a simpler proof of Lemma 2.1 in [1, Section 3], and a more general extension in [8, Lemma 2].

**Proof of Theorem 1.5.** For each real number $\Lambda > 0$ and each sequence $\{a_j\}_{j \geq 1} \in \ell^2$, we define

$$u_\Lambda(x) = \sum_{\lambda_j \leq \Lambda} a_j e_j(x), \quad x \in \Omega,$$

and

$$v_\Lambda(x, s) = \sum_{\lambda_j \leq \Lambda} a_j e^{s \sqrt{\lambda_j}} d e_j(x), \quad (x, s) \in \Omega \times (-1, 1),$$
where $d$ denotes the curl operator.\footnote{In fact, $d$ is the differential which maps 1-forms into 2-forms. When a vector field $w$ is identified with a 1-form, then $dw$ can be identified with a $(2N-1)$-dimensional vector.}

Because $\mathbf{v}_A(\cdot,0) = du_A$ and $\text{div}_x u_A = 0$, we have that
\[
\Delta_x u_A(x) = d^* v_A(x,0), \quad x \in \Omega,
\] (2.1)
where $d^*$ is the adjoint of $d$.

Let us now obtain an estimate of the propagation of smallness for $u_A$ on measurable sets with positive measure. According to Lemma 2.1, it is sufficient to quantify the analytic estimates of higher-order derivatives of $u_A$.

Since $v_A(\cdot, \cdot)$ satisfies
\[
-\partial_s^2 v_A(x,s) - \Delta_x v_A(x,s) = 0, \quad (x,s) \in \Omega \times (-1,1),
\]
we have that $d^* v_A$ verifies
\[
-\partial_s^2 d^* v_A(x,s) - \Delta_x d^* v_A(x,s) = 0, \quad (x,s) \in \Omega \times (-1,1)
\]
and, using Lemma A.1 in the appendix with $f \equiv 0$, $d^* v_A$ is real-analytic in $B_{4R}(x_0,0)$ and the following estimate holds
\[
\|\partial_x^\alpha \partial_s^\beta d^* v_A\|_{L^\infty(B_{2R}(x_0,0))} \leq C \frac{|\alpha|!}{(\rho R)^{|\alpha|+|\beta|}} \left( \int_{B_{4R}(x_0,0)} |d^* v_A(x,s)|^2 dx ds \right)^{1/2}, \quad \forall \alpha \in \mathbb{N}^N, \beta \geq 0,
\]
where the positive constants $\rho$ and $C$ only depend on the dimension $N$.

Taking $\beta = 0$ in the previous estimate, we readily obtain
\[
\|\partial_s^\alpha d^* v_A(\cdot,0)\|_{L^\infty(B_{2R}(x_0))} \leq C \frac{|\alpha|!}{(\rho R)^{|\alpha|}} \left( \int_{B_{4R}(x_0,0)} |d^* v_A(x,s)|^2 dx ds \right)^{1/2}, \quad \forall \alpha \in \mathbb{N}^N. \tag{2.2}
\]

To bound the right-hand side in (2.2), we set
\[
\mathbf{w}_A(x,s) = \sum_{\lambda_j \leq \Lambda} a_j e^{\lambda_j s} \mathbf{e}_j(x), \quad (x,s) \in \Omega \times (-1,1)
\]
and then the following estimate holds
\[
\|d^* v_A\|_{L^2(B_{4R}(x_0,0))}^2 \leq C \|\mathbf{w}_A\|_{L^2((-1,1):H^2(\Omega))}^2
\]
\[
\leq C \int_{-1}^1 \|\mathbf{A w}_A(\cdot, s)\|_{H^2}^2 ds,
\]
where we have used the fact that there exists $C = C(N, \Omega) > 0$ such that
\[
\frac{1}{C} \|\mathbf{y}\|_{H^2(\Omega)} \leq \|\mathbf{A y}\|_{H^2(\Omega)} \leq C \|\mathbf{y}\|_{H^2(\Omega)}, \quad \forall \mathbf{y} \in D(\mathbf{A}),
\]
with $\mathbf{A}$ being the Stokes operator\footnote{The Stokes operator $\mathbf{A} : D(\mathbf{A}) \to H$ is defined by $\mathbf{A} = -P \Delta$, with $D(\mathbf{A}) = \{ \mathbf{y} \in V : \mathbf{A y} \in H \}$ and $P : L^2(\Omega) = H \oplus H^\perp \to H$ is the Leray projection.}.
Since \( \{e_j\}_{j \geq 1} \) is an orthonormal basis of \( \mathbf{H} \), the last estimate yields
\[
\|d^* v_\Lambda\|_{L^2(B_{4R}(x_0,0))}^2 \leq C e^{C \sqrt{\Lambda}} \sum_{\lambda_j \leq \Lambda} a_j^2,
\] (2.3)
for some \( C > 0 \).

Therefore, combining (2.2) and (2.3), we have
\[
\|\partial^2_x d^* v_\Lambda(\cdot,0)\|_{L^\infty(B_{2R}(x_0))} \leq C \left( \frac{\|v_\Lambda\|_{L^2(B_{2R}(x_0))}}{(\rho R)^{|\alpha|}} e^{C \sqrt{\Lambda}} \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2}, \quad \forall \alpha \in \mathbb{N}^N,
\] (2.4)
where \( C = C(N, \Omega) \).

Since \( u_\Lambda \) solves the Poisson equation (2.1), we have that \( u_\Lambda \) is real-analytic whenever the exterior force \( d^* v_\Lambda(\cdot,0) \) is real-analytic. Now, thanks to (2.4), we can apply again Lemma A.1 to obtain that
\[
\|\partial^2_x u_\Lambda\|_{L^\infty(B_R(x_0))} \leq (R \rho)^{-|\alpha|-1}|\alpha|! \left( \|u_\Lambda\|_{L^2(B_{2R}(x_0))} + C e^{C \sqrt{\Lambda}} \left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \right), \quad \forall \alpha \in \mathbb{N}^N,
\]
for some constant \( \rho > 0 \).

Noticing that
\[
\|u_\Lambda\|_{L^2(B_{2R}(x_0))}^2 \leq \|u_\Lambda\|_{H}^2 = \sum_{\lambda_j \leq \Lambda} a_j^2,
\]
one can see that
\[
\|\partial^2_x u_\Lambda\|_{L^\infty(B_R(x_0))} \leq \left( \frac{|\alpha|!}{(\rho R)^{|\alpha|}} e^{K \sqrt{\Lambda}} \left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \right), \quad \forall \alpha \in \mathbb{N}^N,
\] (2.5)
where \( \rho \) and \( K \) are positive constants independent of \( \Lambda \).

Using (2.5) and Lemma 2.1, applied to the real-analytic function \( u_\Lambda \), we obtain the estimate
\[
\|u_\Lambda\|_{L^\infty(B_R(x_0))} \leq C \left( \int_{\Omega} |u_\Lambda(x)| \, dx \right)^\theta \left( e^{K \sqrt{\Lambda}} \left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \right)^{1-\theta},
\] (2.6)
for some constants \( C = C(N, R, \Omega, |\omega|) > 0 \) and \( \theta = \theta(N, R, \Omega, |\omega|) \in (0, 1) \).

On the other hand, by the spectral inequality given in Theorem 1.4, there exists \( C = C(\Omega, R, N) \) such that
\[
\left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \leq C e^{C \sqrt{\Lambda}}\|u_\Lambda\|_{L^\infty(B_R(x_0))}.
\]
The above inequality and (2.6) then leads to
\[
\left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \leq C e^{C \sqrt{\Lambda}} \left( \int_{\Omega} |u_\Lambda(x)| \, dx \right)^\theta \left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{(1-\theta)/2},
\]
which give us the desired observability inequality

\[
\left( \sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \leq C e^{C\sqrt{\omega}} \int_\omega |u_\Lambda(x)| \, dx.
\]

\[ \square \]

2.2. Observability inequality on measurable sets in space-time variables. This Section is devoted to the proof of Theorem 1.1.

We begin with an interpolation estimate for the solutions of the Stokes system, which will be estimate a consequence of the spectral inequality given in Theorem 1.5 and the exponential decay of solutions of the Stokes system, and can be seen as a quantitative estimate of the strong uniqueness of solutions. We refer the reader to [2, 8, 25] for closely related results concerning the strong unique continuation property for general parabolic equations.

**Proposition 2.2.** Let \( B_{4R}(x_0) \subset \Omega \) and let \( \omega \subset B_{2R}(x_0) \) be a measurable set with positive measure. Then, there exists \( C = C(\Omega, |\omega|) > 0 \) such that

\[
\|z(\cdot, t)\|_H \leq \left( C e^{C|\omega|} \|z(\cdot, t)\|_{L^1(\omega)} \right)^{1/2} \|z(\cdot, s)\|_H^{1/2}, \quad \forall z_0 \in H,
\]

where \( 0 \leq s < t \leq T \) and \( z \) is the solution of (1.1) associated to \( z_0 \).

**Proof.** It suffices to prove the estimate in the case \( s = 0 \).

For any \( \Lambda > 0 \), we set

\[
H_\Lambda \triangleq \text{span}\{e_j; \lambda_j \leq \Lambda\}.
\]

Given \( z_0 \in H \), the solution \( z \) of (1.1) can be split into \( z = z_\Lambda + z_\perp \), where \( z_\Lambda \) and \( z_\perp \) are the solutions of (1.1) (together with some pressures) associated to \( z_{0,\Lambda} \in H_\Lambda \) and \( z_{0,\perp} \in \overline{H_\perp} \), \( z_0 = z_{0,\Lambda} + z_{0,\perp} \), respectively. Moreover, one has

\[
z_\Lambda(\cdot, t) \in H_\Lambda \quad \text{and} \quad \|z_\perp(\cdot, t)\|_H \leq e^{-\Lambda t} \|z_0\|_H, \tag{2.7}
\]

for every \( t > 0 \).

From (1.5) and (2.7), for each \( t > 0 \) we have

\[
\|z(\cdot, t)\|_H \leq \|z_\Lambda(\cdot, t)\|_H + \|z_\perp(\cdot, t)\|_H
\]

\[
\leq C e^{C\sqrt{\omega}} \|z_\Lambda(\cdot, t)\|_{L^1(\omega)} + e^{-\Lambda t} \|z_0\|_H
\]

\[
\leq C e^{C\sqrt{\omega}} \left( \|z(\cdot, t)\|_{L^1(\omega)} + \|z_\perp(\cdot, t)\|_H \right) + e^{-\Lambda t} \|z_0\|_H
\]

\[
\leq C e^{C\sqrt{\omega}} \left( \|z(\cdot, t)\|_{L^1(\omega)} + e^{-\Lambda t} \|z_0\|_H \right) + e^{-\Lambda t} \|z_0\|_H
\]

\[
\leq C_1 e^{C_1 \sqrt{\omega} - \frac{t\Lambda}{2}} \left( e^{\frac{t\Lambda}{2}} \|z(\cdot, t)\|_{L^1(\omega)} + e^{-\frac{t\Lambda}{2}} \|z_0\|_H \right)
\]

\[
\leq C_2 e^{-\frac{t\Lambda}{2}} \|z(t)\|_{L^1(\omega)}^{1/2} \|z_0\|_H^{1/2},
\]

where in the last inequality we have used that

\[
C_1 \sqrt{\omega} - \frac{t\Lambda}{2} \leq \frac{C_1^2}{2t}, \quad \text{for any} \quad \Lambda > 0
\]

\[
\overline{H_\perp} = \text{span}\{e_j; \lambda_j > \Lambda\}.
\]
and the following lemma:

**Lemma 2.3** ([21]). Let $C_1$, $C_2$ be positive and $M_0$, $M_1$ and $M_2$ be nonnegative. Assume there exist $C_3 > 0$ such that $M_0 \leq C_3 M_1$ and $\delta_0 > 0$ such that

$$M_0 \leq e^{-C_1 \delta} M_1 + e^{-C_2 \delta} M_2$$

for every $\delta \geq \delta_0$. Then, there exits $C_0$ such that

$$M_0 \leq C_0 M_1^{C_2/(C_1+C_2)} M_2^{C_1/(C_1+C_2)}.$$  

\[
\square
\]

For the proof of Theorem 1.1, we will use the following result concerning the property of Lebesgue density point for a measurable set in $\mathbb{R}$.

**Lemma 2.4** ([19], Proposition 2.1). Let $E$ be a measurable set in $(0,T)$ with positive measure and let $l$ be a density point of $E$. Then, for each $\mu > 1$, there is $l_1 = l_1(\mu, E)$ in $(l, T)$ such that the sequence associated to $E$ and $M$ exists

$$|E \cap (l_m, l_m)| \geq \frac{1}{3} (l_m - l_{m+1}), \forall m \geq 1.$$  

(2.8)

**Proof of Theorem 1.1.** For each $t \in (0,T)$, let us define the slice

$$M_t = \{ x \in \Omega : (x,t) \in M \}$$

and

$$E = \left\{ t \in (0,T); |M_t| \geq \frac{|M|}{2T} \right\}.$$  

From Fubini’s Theorem, it follows that $M_t \subset \Omega$ is measurable for a.e. $t \in (0,T)$, $E$ is measurable in $(0,T)$ and

$$|E| \geq \frac{|M|}{2|B_R(x_0)|} \quad \text{and} \quad \chi_E(t) \chi_M(x) \leq \chi_M(x,t), \text{ in } \Omega \times (0,T).$$

For a.e. $t \in E$, we apply Proposition 2.2 to $M_t$ to find a constant $C = C(\Omega, R, |M|/(T|B_R(x_0)|))$ such that

$$\|z(\cdot,t)\|_H \leq \left( C e^{\frac{C}{t-\tau_m}} \|z(\cdot,t)\|_{L^1(M_t)} \right)^{1/2} \|z(\cdot,\cdot)\|_H^{1/2},$$  

(2.9)

for $0 \leq s < t$.

Let $l$ be any density point in $E$. For $\mu > 1$ (to be chosen later), we denote by $\{l_m\}_{m \geq 1}$ the sequence associated to $l$ and $\mu$ as in Lemma 2.4. For each $m \geq 1$, we set

$$\tau_m = l_{m+1} + \frac{(l_m - l_{m+1})}{6}$$

hence,

$$|E \cap (\tau_m, l_m)| = |E \cap (l_m, l_m)| - |E \cap (l_{m+1}, \tau_m)| \geq \frac{(l_m - l_{m+1})}{6}.$$  

Taking $s = l_{m+1}$ in (2.9), we get

$$\|z(\cdot,t)\|_H \leq \left( C e^{\frac{C}{l_{m+1} - \tau_m}} \|z(\cdot,t)\|_{L^1(M_t)} \right)^{1/2} \|z(\cdot,\cdot)\|_H^{1/2}, \text{ for a.e. } t \in E \cap (\tau_m, l_m).$$  

(2.10)
Integrating (2.10) with respect to \( t \) over \( E \cap (t_m, l_m) \), we obtain
\[
\|z(\cdot, l_m)\|_H \leq \left( Ce^{\frac{c}{l_m-l_m+1}} \int_{l_m+1}^{l_m} \chi_E(t) \|z(\cdot, t)\|_{L^1(M_t)} dt \right)^{1/2} \|z(l_{m+1})\|_H^{1/2},
\]
which implies that
\[
\|z(\cdot, l_m)\|_H \leq \epsilon \|z(\cdot, l_{m+1})\|_H + \epsilon^{-1} Ce^{\frac{c}{l_m-l_m+1}} \int_{l_m+1}^{l_m} \chi_E(t) \|z(\cdot, t)\|_{L^1(M_t)} dt,
\]
for any \( \epsilon > 0 \).

Taking \( \epsilon = e^{-\frac{c}{l_m-l_m+1}} \) in the above inequality, we have
\[
e^{-\frac{c}{l_m-l_m+1}} \|z(\cdot, l_m)\|_H - e^{-\frac{c}{l_m-l_m+1}} \|z(\cdot, l_{m+1})\|_H \leq C \int_{l_m+1}^{l_m} \chi_E(t) \|z(\cdot, t)\|_{L^1(M_t)} dt. \tag{2.11}
\]
Finally, choosing \( \mu = \frac{2(C+1)}{2C+1} \), where \( C \) is any constant for which inequality (2.11) holds, we readily obtain
\[
e^{-\frac{c}{l_m-l_m+1}} \|z(\cdot, l_m)\|_H - e^{-\frac{c}{l_m-l_m+1}} \|z(\cdot, l_{m+1})\|_H \leq C \int_{l_m+1}^{l_m} \chi_E(t) \|z(\cdot, t)\|_{L^1(M_t)} dt, \quad \forall m \geq 1,
\]
because \( \mu(l_{m+1} - l_m + 2) = l_m - l_{m+1} \), for all \( m \geq 1 \).

Finally, adding the telescoping series in (2.12) from \( m = 1 \) to \( +\infty \), we get the observability inequality
\[
\|z(\cdot, T)\|_H \leq C \int_{M \cap (\Omega \times [T_1])} |z(x, t)| dx dt,
\]
with some constant \( C = C(N, R, \Omega, M, T) > 0 \). \( \square \)

3. Applications

3.1. Shape optimization problems. As a direct and interesting application of Theorem 1.5, we analyze the following shape optimization problem formulated in [17].

Let \( \{\beta_j^\nu\}_{j \in \mathbb{N}} \) be a sequence of independent real random variables on a probability space \((X, \mathcal{F}, \mathbb{P})\) having mean equal to 0, variance equal to 1, and a super exponential decay (for instance, independent Gaussian or Bernoulli random variables, see [5, Assumption (3.1)] for more details). For every \( \nu \in X \), the solution of (1.1) corresponding to the initial datum
\[
z_0^\nu = \sum_{j \geq 1} \beta_j^\nu a_j e_j, \quad \text{with} \quad \{a_j\}_{j \geq 1} \in \ell^2,
\]
is given by
\[
z^\nu(\cdot, t) = \sum_{j \geq 1} \beta_j^\nu a_j e^{-t\lambda_j} e_j. \tag{3.2}
\]
Given \( L \in (0, 1) \), we define the set of admissible designs
\[
\mathcal{U}_L = \left\{ \chi_\omega \in L^\infty(\Omega; \{0, 1\}) : \omega \subset \Omega \text{ is a measurable subset of measure } |\omega| = L|\Omega| \right\}.
\]
For each $\chi_\omega \in \mathcal{U}_L$, we then define the randomized observability constant by

$$C_{T,\text{rand}}(\chi_\omega) = \inf_{\|z'\|_1 = 1} \mathbb{E} \int_0^T \int_\omega |z'(x,t)|^2 \, dx \, dt,$$

Using (3.2), the properties of random variables $\beta_j$, and the change of variable $b_j = a_j e^{-T\lambda_j}$, we deduce that

$$C_{T,\text{rand}}(\chi_\omega) = \inf_{\sum_{j=1}^\infty |b_j|^2 = 1} \mathbb{E} \int_0^T \int_\omega \left| \sum_{j \geq 1} \beta_j b_j e^{t\lambda_j} e_j(x) \right|^2 \, dx \, dt,$$

where $\mathbb{E}$ is the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$.

From Fubini’s theorem and the independence of the random variables $\{\beta_j\}_{j \in \mathbb{N}}$, a simple computation gives

$$C_{T,\text{rand}}(\chi_\omega) = \inf_{j \geq 1} e^{2T\lambda_j} - \frac{1}{2\lambda_j} \int_\omega |e_j(x)|^2 \, dx.$$

We now consider the optimal design problem of maximizing the randomized observability constant $C_{T,\text{rand}}(\chi_\omega)$ over the set of admissible designs $\mathcal{U}_L$. In other words, we study the problem

$$(P^T) : \sup_{\chi_\omega \in \mathcal{U}_L} C_{T,\text{rand}}(\chi_\omega) = \sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \geq 1} e^{2T\lambda_j} - \frac{1}{2\lambda_j} \int_\omega |e_j(x)|^2 \, dx. \quad (3.3)$$

The optimal shape design problem (3.3) models the best sensor shape and location problem for the control of the Stokes system (1.1).

We have the following result:

**Theorem 3.1.** The problem $(P^T)$ has a unique solution.

**Proof.** We only have to check the following two conditions:

1. If there exists $E \subset \Omega$ of positive Lebesgue measure, an integer $m \in \mathbb{N}^*$, $\beta_1, \ldots, \beta_m \in \mathbb{R}^+$, and $C \geq 0$ such that $\sum_{j=1}^m \beta_j |e_j(x)|^2 = C$ almost everywhere on $E$, then there must hold $C = 0$ and $\beta_1 = \beta_2 = \ldots = \beta_m = 0$.

2. For every $a \in L^\infty(\Omega; [0,1])$ such that $\int_\Omega a(x) \, dx = L|\Omega|$, one has

$$\liminf_{j \to +\infty} \frac{e^{2T\lambda_j} - 1}{2\lambda_j} \int_\Omega a(x)|e_j(x)|^2 \, dx > \frac{e^{2T\lambda_1} - 1}{2\lambda_1}.$$

By the analyticity of the eigenfunctions of Stokes system with homogeneous Dirichlet boundary conditions, it is not difficult to show that the first condition holds.

For the second condition, notice that there exists $\epsilon > 0$ and $E \subset \Omega$ of positive measure such that $a \geq \epsilon \chi_E$ and

$$\int_\Omega a(x)|e_j(x)|^2 \, dx \geq \epsilon \int_E |e_j(x)|^2 \, dx.$$

From Theorem 1.5, we easily see that

$$\liminf_{j \to +\infty} \frac{e^{2T\lambda_j} - 1}{2\lambda_j} \int_\Omega a(x)|e_j(x)|^2 \, dx = +\infty.$$
From [17, Theorem 1], it follows that problem \((P^T)\) has a unique solution.

\[\square\]

**Remark 3.2.** The optimal set given by Theorem 3.1 is open and semi-analytic. This follows from the fact that the eigenfunctions of the Stokes system with homogeneous Dirichlet boundary conditions are analytic.

**Remark 3.3.** A proof of Theorem 3.1 when \(\Omega\) is the unit disk of \(\mathbb{R}^2\) can be found in [17]. However, such proof relies on an explicit knowledge of the eigenfunctions of the Stokes operator, which of course can not be extended to general domains. Notice that to prove Theorem 3.1, in the general case, the key point is to obtain an observability inequalities with observations over measurable sets of positive measure as in Theorem 1.5.

### 3.2. Null controllability for Stokes system with bounded controls.

Let \(\omega\) be a non-empty open subset of \(\Omega\) and consider the following controlled Stokes system

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \nabla p &= v\chi_{\omega} \quad \text{in} \quad Q, \\
\text{div} \ u &= 0 \quad \text{in} \quad Q, \\
u &= 0 \quad \text{on} \quad \Sigma, \\
u(\cdot, 0) &= u_0 \quad \text{in} \quad \Omega.
\end{align*}
\]

(3.4)

It is well known that for any \(T > 0\), \(u_0 \in H\), and \(v \in L^2(\omega \times (0, T))\), there exists exactly one solution \((u, p)\) to the Stokes equations (3.4) with

\[
u \in C^0([0, T]; H) \cap L^2(0, T; V), \quad p \in L^2(0, T; U),
\]

where

\[
U := \left\{ \psi \in H^1(\Omega); \int_{\Omega} \psi(x) \, dx = 0 \right\}.
\]

In the context of the Stokes system (3.4), for \(1 \leq p \leq \infty\), the \(L^p\)-null controllability problem at time \(T\) reads as follows:

*For any \(u_0 \in H\), find a control \(v \in L^p(\omega \times (0, T))\) such that the associated solution to (3.4) satisfies

\[
u(x, T) = 0 \quad \text{in} \quad \Omega.
\]

(3.5)*

The following result is well-known.

**Theorem 3.4.** For any non-empty open subset \(\omega\) of \(\Omega\) and any \(T > 0\), the Stokes system (3.4) is \(L^2\)-null controllable.

For the proof, we refer the reader to [6, 10, 11].

In practice it would be interesting to take the control steering the solution of the Stokes system to rest to be in \(L^\infty(\omega \times (0, T))\). Nevertheless, it is not clear how to construct \(L^\infty(\omega \times (0, T))\) controls from \(L^2(\omega \times (0, T))\) controls. Notice that for the case of the heat equation this is always

\[\text{Here, it is understood that the optimal set is unique up to the set of zero measure. A subset of a real analytic finite-dimensional manifold is said to be semi-analytic if it can be written in terms of equalities and inequalities of real analytic functions.}\]
possible since one can use local regularity results (for more details, see [3]), which is no longer the case for the Stokes system.

From Theorem 1.1 we are able to deduce a null controllability for Stokes system with $L^\infty$-controls. More precisely, we have:

**Theorem 3.5.** For any non-empty open subset $\omega$ of $\Omega$ and any $T > 0$, the Stokes system (3.4) is $L^\infty$-null-controllable.

*Proof.* The proof follows from the duality between observability and controllability and the $L^1$-observability inequality (1.2).

The observability inequality established in Theorem 1.1 allow us to conclude stronger controllability properties for the Stokes system (3.4). In fact it is possible to control the Stokes system with $L^\infty$-controls supported in any measurable set of positive measure:

**Theorem 3.6.** For any $T > 0$ and any measurable set of positive measure $\gamma \subset \Omega \times [0, T]$, the Stokes system (3.4) is $L^\infty$-null controllable with control supported in $\gamma$.

### 3.3. Time optimal control problem for the Stokes system.

Let $| \cdot |_r : \mathbb{R}^N \to [0, \infty)$ be the $r$-euclidean norm in $\mathbb{R}^N$, i.e.,

$$|x|_r = \begin{cases} (|x_1|^r + \ldots + |x_N|^r)^{\frac{1}{r}} & \text{if } r \in [1, \infty), \\ \max\{|x_1|, \ldots, |x_N|\} & \text{if } r = \infty, \end{cases}$$

for every $x \in \mathbb{R}^N$.

For $r \in [1, \infty]$ fixed and any $M > 0$, we consider the set of admissible controls

$$U^M_{ad, r} = \{ v \in L^\infty(\omega \times [0, \infty)) ; |v(x, t)|_r \leq M \text{ a.e. in } \omega \times [0, \infty) \}$$

and for $u_0 \in H$ given, we define the set of reachable states starting from $u_0$:

$$R(u_0, U^M_{ad, r}) = \{ u(\cdot, \tau) ; \tau > 0 \text{ and } u \text{ is the solution of (3.4) with } v \in U^M_{ad, r} \}.$$ 

Thanks to Theorem 3.5, it follows that $0 \in R(u_0, U^M_{ad, r})$, for any $u_0 \in H$.

In this section, we study the following time optimal control problem:

*given* $u_0 \in H$ and $u_f \in R(u_0, U^M_{ad, r})$, find $v^*_r \in U^M_{ad, r}$ such that the corresponding solution $u^*_r$ of (3.4) satisfies

$$u^*_r(\tau^*_r(u_0, u_f)) = u_f,$$

where $\tau^*_r(u_0, u_f)$ is the minimal time needed to steer the initial datum $u_0$ to the target $u_f$ with controls in $U^M_{ad, r}$, i.e.

$$\tau^*_r(u_0, u_f) = \min_{v \in U^M_{ad, r}} \{ \tau ; u(\cdot, \tau) = u_f \}.$$ 

We have the following result:

**Theorem 3.7.** Let $M > 0$ and $r \in [1, \infty]$ be given. For every $u_0 \in H$ and any $u_f \in R(u_0, U^M_{ad, r})$, the time optimal problem (3.7) has at least one solution. Moreover, any optimal control $v^*_r$ satisfies the bang-bang property: $|v^*_r(x, t)|_r = M$ for a.e. $(x, t) \in \omega \times [0, \tau^*_r(u_0, u_f)]$. 

Proof. Since $u_f \in \mathcal{R}(u_0, \mathbb{U}_{M,r}^f)$, there exists a minimizing sequence $(\tau_n, v_n)_{n \geq 1}$ such that $\tau_n \to \tau^*_r(u_0, u_f)$ and $(v_n)_{n \geq 1} \subset \mathbb{U}_{ad}$ has the property that the associated solution $u_n$ to (3.4) satisfies $u_n(\cdot, \tau_n) = u_f$ for all $n \geq 1$. Also, because $(v_n)_{n \geq 1} \subset \mathbb{U}_{ad}^r$, it follows that $(v_n)_{n \geq 1}$ converges weakly-$\star$ to some vector-function $v^* \in \mathbb{U}_{ad}$ in $L^\infty(\omega \times (0, \tau^*_r(u_0, u_f)))$.

Claim: $v^*$ is a solution of the time optimal problem (3.6).

Proof of the Claim. We only have to show that $u^*(\cdot, \tau^*_r(u_0, u_f)) = u_f$, where $u^*$ is the solution of (3.4) associated to $v^*$.

To show this, let $\bar{u}$ be the solution of (3.4) with $v \equiv 0$ and $w = u^* - \bar{u}$, $w_n = u_n - \bar{u}$ solutions of

$$
\begin{align*}
\frac{w_t}{\partial t} - \Delta w + \nabla \pi &= v^* 1_\omega & \text{in} & Q, \\
\text{div }w &= 0 & \text{in} & Q, \\
w &= 0 & \text{on} & \Sigma, \\
w(0) &= 0 & \text{in} & \Omega,
\end{align*}
$$

and

$$
\begin{align*}
\frac{w_n}{\partial t} - \Delta w_n + \nabla \pi_n &= v_n 1_\omega & \text{in} & Q, \\
\text{div }w_n &= 0 & \text{in} & Q, \\
w_n &= 0 & \text{on} & \Sigma, \\
w_n(0) &= 0 & \text{in} & \Omega,
\end{align*}
$$

respectively.

Now, thanks to the continuity in time of $\bar{u}$ and that $\tau_n \to \tau^*_r(u_0, u_f)$, it follows that $\bar{u}(\cdot, \tau_n) \to \bar{u}(\cdot, \tau^*_r(u_0, u_f))$ in $H$. Moreover, it is not difficult to see that

$$
\begin{align*}
\langle w_n(\tau_n) - w_n(\tau^*_r(u_0, u_f)), \varphi \rangle &\to 0 & \forall \varphi \in H, \\
\langle w_n(\tau^*_r(u_0, u_f)), \varphi \rangle &\to \langle w(\tau^*_r(u_0, u_f)), \varphi \rangle & \forall \varphi \in H
\end{align*}
$$

and

$$
\langle w_n(\tau_n), \varphi \rangle \to \langle w(\tau^*_r(u_0, u_f)), \varphi \rangle & \forall \varphi \in H.
$$

Since $u_f = \bar{u}(\cdot, \tau_n) + w_n(\cdot, \tau_n)$, we have that $(u_f, \varphi) = \langle \bar{u}(\cdot, \tau_n) + w_n(\cdot, \tau_n), \varphi \rangle$ for all $\varphi \in H$ and $(u_f, \varphi) = \langle \bar{u}(\cdot, \tau^*_r(u_0, u_f)) + w(\cdot, \tau^*_r(u_0, u_f)), \varphi \rangle = \langle u^*(\cdot, \tau^*_r(u_0, u_f)), \varphi \rangle$, for all $\varphi \in H$. \[\Box\]

Now, let us show that any optimal control $v^* \in \mathbb{U}_{ad}^r$ satisfies the bang-bang property. To do this, we argue by contradiction.

We consider $u^*$ the corresponding state (with some pressure) to (3.4) and suppose that there exist $\epsilon > 0$ and a measurable set of positive measure $\gamma \subset \omega \times (0, \tau^*_r(u_0, u_f))$ such that

$$
|v^*(x, t)|_r < M - \epsilon \quad ((x, t) \in \gamma), \tag{3.8}
$$

Choosing $\delta_0 > 0$ small enough such that

$$
\left\{ \begin{array}{l}
\tau_0 = \tau^*_r(u_0, u_f) - \delta_0 > 0, \\
\text{the set } \Gamma = \{ (x, t) \in \omega \times (0, \tau_0) : (x, t) \in \gamma \} \text{ has positive measure,}
\end{array} \right.
$$

and using the time continuity of $u^*$, there exists $\delta \in (0, \delta_0)$ such that

$$
\|u_0 - u^*(\cdot, \delta)\|_H \leq \frac{\epsilon}{C_{\text{obs}}(\tau_0, \Gamma)}, \tag{3.9}
$$

OBSERVABILITY ESTIMATE AND APPLICATIONS 13
where $C_{obs}(\tau_0, \Gamma)$ is the observability constant given by Theorem 1.1 for the control domain $\Gamma$ at time $\tau_0$.

From Theorem 3.6, there exists a control $v \in L^\infty(\omega \times (0, \tau_0))$ with

$$
\left\{ \begin{array}{l}
supp v \subset \Omega, \\
\text{the associated solution } u \text{ satisfies } u(\cdot, 0) = u_0 - u^*(\cdot, \delta) \text{ and } u(\cdot, \tau_0) = 0,
\end{array} \right.
$$

Thus, from (3.9) we have that

$$\|v\|_{L^\infty(\Omega \times (0, \tau_0))} \leq C_{obs}(\tau_0, \Gamma)\|u_0 - u^*(\delta)\|_H.
$$

Now, let $\tilde{v} \in L^\infty(\omega \times (0, \tau_0))$ be defined by

$$\tilde{v}(x, t) = v^*(x, t + \delta) + v(x, t) \quad (t \in [0, \tau_0]).$$

Noticing that $\tau_0 + \delta \leq \tau^*_0(u_0, u_f)$, using the fact that $\supp v \subset \Gamma$ and estimate (3.8), it follows that $\tilde{v} \in \mathcal{U}_H^{M, r}$.

Finally, setting $\hat{v}(x, t) = u^*(x, t + \delta) + u(x, t)$ and $\tilde{p}(x, t) = p^*(x, t + \delta) + p(x, t)$, we have that

$$\hat{v}(\cdot, 0) = u_0, \quad \tilde{u}(\tau^*_0(u_0, u_f) - \delta) = u_f \quad \text{ and that }
$$

$$\hat{u} - \Delta \hat{u} + \nabla \hat{p} = \tilde{v}1_\omega.
$$

Hence, $\hat{v} \in \mathcal{U}_H^{M, r}$ is a control which steers $u_0$ to $u_f$ at time $\tau^*_0(u_0, u_f) - \delta$. This contradicts the definition of $\tau^*_0(u_0, u_f)$ and the bang-bang property holds. \hfill \square

About the uniqueness of the optimal control for problem (3.7), we have the following result:

**Proposition 3.8.** Let $M > 0$ and $r \in (1, \infty)$. For any $u_0 \in H$ and every $u_f \in \mathcal{R}(u_0, \mathcal{U}_H^{M, r})$, the time optimal control problem (3.6)-(3.7) has a unique solution $v^*_r$ which satisfies a bang-bang property: $|v^*_r(x, t)|_{r} = M$ for a.e. $(x, t) \in \omega \times [0, \tau^*_r(u_0, u_f)]$.

**Proof.** The existence of solution and the bang-bang property is a consequence of Theorem 3.7. We only have to prove the uniqueness of solution. Thus, let $v$ and $h$ be two time optimal controls in $\mathcal{U}_H^{M, r}$. Thanks to the linearity, $w = \frac{1}{2}(v + h)$ is also a time optimal control. From Theorem 3.7, $w$ also satisfies the bang-bang property. Therefore, we have that $|v(x, t)|_{r} = |h(x, t)|_{r} = |w(x, t)|_{r} = M$, a.e. in $\omega \times (0, \tau^*_r(u_0, u_1))$. Now, if $v(x, t) \neq h(x, t)$ in a measurable set of positive measure $D \subset \omega \times (0, \tau^*_r(u_0, u_1))$, then, thanks to the fact that any norm $|\cdot|_r$ for $r \in (1, \infty)$ is uniformly convex in $\mathbb{R}^N$, we have that $|w(x, t)|_r < M$ a.e. in $D \subset \omega \times (0, \tau^*_r(u_0, u_1))$. This contradicts the bang-bang property for $w$. \hfill \square

**APPENDIX A. REAL-ANALYTIC ESTIMATES FOR SOLUTIONS TO THE POISSON EQUATION**

In this appendix we prove the following lemma which was used in the proof of Theorem 1.5.

**Lemma A.1.** Assume that $f$ is an real-analytic function in $B_R(x_0)$ verifying

$$
|\partial_\alpha \alpha f(x)| \leq \frac{M|\alpha|!}{(R\rho_0)^{\alpha}} \quad \text{for all } x \in B_R(x_0) \text{ and } \alpha \in \mathbb{N}^N, \quad \text{(A.1)}
$$

with some positive constants $M$ and $\rho_0$. Let $u \in L^2(B_R(x_0))$ satisfying the Poisson equation

$$
- \Delta u = f \quad \text{in } B_R(x_0). \quad \text{(A.2)}
$$
Then,  is real-analytic in and has the estimate

\[ \| \partial^\alpha_\mathbf{x} u \|_{L^\infty(B_{R/2}(x_0))} \leq \frac{|\alpha|!}{(R\rho)^{|\alpha|+1}} \left( \| u \|_{L^2(B_R(x_0))} + M \right), \quad \text{for all } \alpha \in \mathbb{N}_N, \quad (A.3) \]

where \( \rho \) is a constant depending only on the dimension \( N \) and \( \rho_0 \).

A proof of the lemma A.1 for \( f \equiv 0 \) can be found in [16]. For the sake of completeness, we give a proof for the non-homogeneous case.

**Proof.** By rescaling, it suffices to prove the estimate (A.3) when \( R = 1 \) and \( x_0 = 0 \).

Since \( f \) is real-analytic in \( B_1(0) \), by the interior regularity for solutions of elliptic equations, we have that \( u \) is smooth in \( B_1(0) \). Hence, we have that

\[ -\Delta \partial^\alpha_\mathbf{x} u = \partial^\alpha_\mathbf{x} f \quad \text{for all } \mathbf{x} \in B_1(0), \]

for every \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_N \).

Multiplying the above equation by \( (1 - |\mathbf{x}|^2)^{2(|\alpha|+1)} \partial^\alpha_\mathbf{x} u \) gives

\[- (1 - |\mathbf{x}|^2)^{2(|\alpha|+1)} \partial^\alpha_\mathbf{x} u (\mathbf{x}) \Delta \partial^\alpha_\mathbf{x} u (\mathbf{x}) = (1 - |\mathbf{x}|^2)^{2(|\alpha|+1)} \partial^\alpha_\mathbf{x} u (\mathbf{x}) \partial^\alpha_\mathbf{x} f (\mathbf{x}), \quad \forall \mathbf{x} \in B_1(0), \quad (A.4)\]

and integration by parts gives

\[\begin{align*}
\int_{B_1(0)} (1 - |\mathbf{x}|^2)^{2(|\alpha|+1)} |\nabla \partial^\alpha_\mathbf{x} u|^2 \, d\mathbf{x} &= 4(|\alpha| + 1) \int_{B_1(0)} (1 - |\mathbf{x}|^2)^{2|\alpha|+1} (\nabla \partial^\alpha_\mathbf{x} u \cdot \mathbf{x}) \partial^\alpha_\mathbf{x} u \, d\mathbf{x} \\
&\quad + \int_{B_1(0)} (1 - |\mathbf{x}|^2)^{2|\alpha|+1} |\partial^\alpha_\mathbf{x} u \partial^\alpha_\mathbf{x} f| \, d\mathbf{x}.
\end{align*}\]

Now, thanks to the Young’s inequality, we have the following estimate

\[\begin{align*}
\int_{B_1(0)} (1 - |\mathbf{x}|^2)^{2|\alpha|+1} |\nabla \partial^\alpha_\mathbf{x} u|^2 \, d\mathbf{x} &\leq 16(|\alpha| + 1)^2 + 1 \int_{B_1(0)} (1 - |\mathbf{x}|^2)^{2|\alpha|+1} |\partial^\alpha_\mathbf{x} u|^2 \, d\mathbf{x} \\
&\quad + \int_{B_1(0)} |\partial^\alpha_\mathbf{x} f|^2 \, d\mathbf{x}.
\end{align*}\]

Since \( f \) satisfies (A.1), we get

\[\begin{align*}
\int_{B_1(0)} (1 - |\mathbf{x}|^2)^{2|\alpha|+1} |\nabla \partial^\alpha_\mathbf{x} u|^2 \, d\mathbf{x} &\leq 17(|\alpha| + 1)^2 \int_{B_1(0)} (1 - |\mathbf{x}|^2)^{2|\alpha|} |\partial^\alpha_\mathbf{x} u|^2 \, d\mathbf{x} \\
&\quad + |B_1(0)| \left[ \frac{M|\alpha|!}{\rho_0^{|\alpha|}} \right]^2.
\end{align*}\]

Therefore, we obtain

\[\left\| (1 - |\mathbf{x}|^2)^{|\alpha|+1} \nabla \partial^\alpha_\mathbf{x} u \right\|_{L^2(B_1(0))} \leq 5 \left[ (|\alpha| + 1) \left\| (1 - |\mathbf{x}|^2)^{|\alpha|} \partial^\alpha_\mathbf{x} u \right\|_{L^2(B_1(0))} + \frac{M|\alpha|!}{\rho_0^{|\alpha|}} \right], \quad (A.5)\]

for every \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_N \). In particular, taking \( \alpha = (0, \ldots, 0) \), we deduce the estimate

\[\left\| (1 - |\mathbf{x}|^2) \nabla u \right\|_{L^2(B_1(0))} \leq 5 \left( \| u \|_{L^2(B_1(0))} + M \right).\]

By induction, we have the inequality

\[\left\| (1 - |\mathbf{x}|^2)^{|\alpha|} \partial^\alpha_\mathbf{x} u \right\|_{L^2(B_1(0))} \leq \rho^{-|\alpha|-1}|\alpha|! \left( \| u \|_{L^2(B_1(0))} + M \right), \quad (A.6)\]
for some constant $0 < \rho < \min \{ \rho_0, \frac{1}{6} \}$ and every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$.

It is not difficult to see that estimate (A.6) leads to (A.3). □

Acknowledgements. The authors would like to appreciate Prof. E. Trélat and Prof. G. Lebeau for the stimulating conversations during this work.

REFERENCES

[1] J. Apraiz, L. Escauriaza. Null-control and measurable sets. ESAIM: COCV, 19 (2013), 239–254.
[2] J. Apraiz, L. Escauriaza, G. Wang, C. Zhang. Observability inequalities and measurable sets. J. Eur. Math. Soc., 16 (2014), 2433–2475.
[3] O. Bodart, M. González-Burgos, R. Pérez-García. Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity. Comm. Partial Differential Equations 29 (2004)(7–8), 1017–1050.
[4] F. Boyer, P. Fabrie. Mathematical tools for the study of the incompressible Navier-Stokes equations and related models. Applied Mathematical Sciences, Vol. 183. Springer, Berlin (2013).
[5] N. Burq, N. Tzvetkov. Random data Cauchy theory for supercritical wave equations. I. Local theory. Invent. Math., 173 (2008), 449-475.
[6] F. W. Chaves-Silva, G. Lebeau. Spectral inequality and optimal cost of controllability for the Stokes system. ESAIM: COCV, 22 (2016) 1137-1162.
[7] J. M. Coron, S. Guerrero. Null controllability of the N-dimensional Stokes system with N-1 scalar controls. Journal of Differential Equations, 246 (2009), 2908-2921.
[8] L. Escauriaza, S. Montaner, C. Zhang. Observation from measurable sets for parabolic analytic evolutions and applications. J. Math. Pures Appl. 104 (2015), 837-867.
[9] L. Escauriaza, S. Montaner, C. Zhang. Analyticity of solutions to parabolic evolutions and applications. arXiv1509.04053v1.
[10] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, J. P. Puel. Local exact controllability of the Navier-Stokes system. J. Math. Pures Appl., 83 (2004), 1501-1542.
[11] A. V. Fursikov, O. Yu. Imanuvilov. Controllability of Evolution Equations. Lecture Notes Series 34, Research Institute of Mathematics, Seoul National University, Seoul, 1996.
[12] O. Yu. Imanuvilov, J. P. Puel, M. Yamamoto. Carleman estimates for parabolic equations with nonhomogeneous boundary conditions. Chin. Ann. Math. Ser. B, 30 (2009), 333-378.
[13] C. Kahane. On the spatial analyticity of solutions of the Navier-Stokes equations. Arch. Rational Mech. Anal. 33 (1969), 386-405.
[14] K. Masuda. On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation. Proc. Japan Acad. 43 (1967), 827-832.
[15] L. Miller. A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. AIMS, 14 (2010), 1465-1485.
[16] C. B. Morrey, L. Nirenberg. On the analyticity of the solutions of linear elliptic systems of partial differential equations. Commun. Pur. Appl. Math., X (1957), 271-290.
[17] Y. Privat, E. Trélat, E. Zuazua. Optimal shape and location of sensors for parabolic equations with random initial data. Arch. Ration. Mech. Anal. 216 (2015), 921-981.
[18] Y. Privat, E. Trélat, E. Zuazua, Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains. J. Eur. Math. Soc., 18 (2016), 1043-1111.
[19] K. D. Phung, G. Wang. An observability estimate for parabolic equations from a general measurable set in time and its applications. J. Eur. Math. Soc., 15 (2013), 681–703.
[20] K. D. Phung, G. Wang, X. Zhang. On the existence of time optimal controls for linear evolution equations. Discrete Contin. Dyn. Syst. Ser. B, 8 (2007), 925–941.
[21] L. Robbiano. Fonction de coût et contrôle des solutions des équations hyperboliques, Asympt. Analysis, 10 (1995), 95–115.
[22] T. I. Seidman. How violent are fast controls: III. J. Math. Anal. Appl. 339 (2008), 461-468.
[23] S. Vessella. A continuous dependence result in the analytic continuation problem. Forum Math., 11 (1999), 695–703.
[24] G. Wang. $L^\infty$-null controllability for the heat equation and its consequences for the time optimal control problem. SIAM J. Control Optim., 47 (2008), 1701–1720.

[25] G. Wang, C. Zhang. Observability estimate from measurable sets in time for some evolution equations. To appear in SIAM J. Control Optim.