WHY SHOULD A BIRATIONAL GEOMETER CARE ABOUT BRIDGELAND STABILITY CONDITIONS?

CLAUDIO FONTANARI AND DILETTA MARTINELLI

Abstract. In this survey we borrow from Coskun and Huizenga an example of application of Bridgeland stability conditions to birational geometry and we rephrase it without assuming any previous knowledge about derived categories.

1. Introduction

Before addressing the question in the title, perhaps we first need to justify why on earth the answer should come from two newcomers into the Bridgeland world. Indeed, while approaching the somehow exotic land of derived categories, we deeply felt the urgency of a strong motivation rooted in classical algebraic geometry. Even if it is well known that applications of Bridgeland stability conditions to birational geometry are many and fruitful, it might seem (so it was for us) that in order to appreciate the geometric content of the theory one needs to be already involved in its jungle of technicalities. However, in the end we realized that at least some geometric example could be presented with only a little amount of machinery. Now we would like to humbly share the results of our efforts to get the point, without any claim of originality and exhaustiveness.

Let $X$ be a connected projective scheme over an algebraically closed field $k$ of characteristic zero. If we fix an ample line bundle $\mathcal{O}_X(1)$ on $X$ and a polynomial $P \in \mathbb{Q}[z]$, then according to [9], Theorem 4.3.4, there is a projective scheme whose closed points are in bijection with $S$-equivalence classes of Gieseker semistable sheaves with Hilbert polynomial $P$. In particular, we can consider the moduli space $M = M_{\mathbb{P}^2}(r, c_1, c_2)$ of $S$-equivalence classes of semistable sheaves of rank $r$ and Chern classes $c_1$ and $c_2$ on the projective plane (see for instance [10], Part II).

As explained in the preface to [9], there are good reasons to study moduli spaces of sheaves. In particular, they provide examples of higher dimensional algebraic varieties with a rich and interesting geometry. In fact, in some regions in the classification of higher dimensional varieties the only known examples are moduli spaces of sheaves on a surface. From a birational geometry perspective, many natural questions arise: describe the ample cone, determine the effective cone, run an MMP, and so on.

Quite recently, remarkable progress in the field has been obtained by the application of Bridgeland stability conditions introduced in [5]. In particular, Bayer and Macrì have described the nef cone of the moduli space of Gieseker stable sheaves on a K3 surface in [4] and Coskun and Huizenga have computed the ample cone of Gieseker semistable sheaves on $\mathbb{P}^2$ in [7].

Here we have chosen the description of the ample cone as our guiding example in order to motivate an ideal reader with a background in classical algebraic geometry...
and without any previous knowledge about derived categories. In this spirit, we are going to adopt a slightly unconventional order: indeed, we start with geometric applications, by taking stability conditions as a sort of black box, and we turn to algebraic formalism only at the end. More precisely, in Section 2 we present the beautiful geometry related to Bridgeland stability conditions. Next, in Section 3 we focus on the case of \( \mathbb{P}^2 \) and we outline the procedure to compute the ample cone of \( M \) introduced in [7]. Finally, in Section 4 we collect precise definitions in the case of \( \mathbb{P}^2 \) and specific references for the general case.

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## 2. Geometry of stability conditions

Let \( \mathcal{A} = \text{coh}(X) \) be the abelian category of coherent sheaves on a smooth projective variety \( X \) defined over the complex numbers. Bridgeland’s key idea was to introduce stability conditions not on the abelian category of coherent sheaves, but on the bounded derived category \( D^b(\mathcal{A}) \). As anticipated in the introduction, for now we consider the notion of stability condition as a black box. However, in order to avoid cheating too much, it may be useful to have in mind at least the definition of derived category, which we now recall from [2], Section 3.

Let \( C^b(\mathcal{A}) \) be the category of bounded complexes: objects are complexes
\[
E^\bullet = \ldots \to E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} E^{i+2} \to \ldots
\]
with \( d^{i+1} \circ d^i = 0 \) and \( H^i(E) = 0 \) for all but finitely many \( i \), and morphisms \( f^\bullet : E^\bullet \to F^\bullet \) are morphisms \( f^i : E^i \to F^i \) that commute with the differential. A morphism \( f^\bullet \) is called a quasi-isomorphism if the induced morphism in cohomology \( f_* : H^*(E^\bullet) \to H^*(F^\bullet) \) is an isomorphism for all \( i \).

**Definition 2.1.** The bounded derived category \( D^b(\mathcal{A}) \) of \( \mathcal{A} \) is obtained from \( C^b(\mathcal{A}) \) by inverting quasi-isomorphisms. The morphisms in \( D^b(\mathcal{A}) \) are formal compositions \( f^{-1} \circ g \) where \( f \) is a quasi-isomorphism.

We denote with \( \text{Stab}(X) \) the set of all the Bridgeland stability conditions on \( X \). In [6] Bridgeland proved that \( \text{Stab}(X) \) carries a natural geometric structure (see also [3], Theorem 2.2 and Proposition 2.3): 

**Theorem 2.2.** \( \text{Stab}(X) \) is a finite dimensional complex variety. Moreover, given a fixed numerical invariant \( v \), \( \text{Stab}(X) \) admits a locally finite dimensional walls-chambers decomposition.

Let \( \mathcal{M}_\sigma(v) \) be the stack of \( \sigma \)-semistable objects in \( D^b(\mathcal{A}) \) of fixed numerical invariant \( v \). Bayer and Macrì proved the following crucial result (see [3], Lemma 3.3): 

**Lemma 2.3.** (Positivity Lemma) Let \( \sigma \) be a stability condition and \( v \) a fixed numerical invariant. Then there exists a nef divisor \( \mathcal{L}_\sigma \) on \( \mathcal{M}_\sigma(v) \). Moreover, a curve \( C \) on \( \mathcal{M}_\sigma(v) \) is such that \( \mathcal{L}_\sigma \cdot C = 0 \) if and only if for two general closed points \( c \) and \( c' \) in \( C \), the corresponding objects \( \mathcal{E}_c \) and \( \mathcal{E}'_{c'} \in D^b(\mathcal{A}) \) are \( S \)-equivalent.

## 3. The ample cone of moduli spaces of sheaves on the plane

In this section we focus on the case of \( \mathbb{P}^2 \). In [6] Bridgeland proved that the space of stability conditions \( \text{Stab}(\mathbb{P}^2) \) contains as a subset the upper half plane \( H := \{ (s,t) | s \in \mathbb{R}, t > 0 \} \). In this case, the numerical invariant \( v \) is given by the Chern character \( \xi \).
Theorem 3.1. [1] Let $\xi$ be a fixed Chern character. For each $(s, t) \in H$, there exists a coarse moduli space $M_{s,t}(\xi)$ parametrizing the $\sigma_{s,t}$-semistable objects with fixed Chern character $\xi$.

The geometry of $M_{s,t}(\xi)$ has been recently investigated by Li and Zhao in [12] (see also [11]). Fixing the Chern character $\xi$ implies, via Hirzebruch-Riemann-Roch, that we are also fixing the Euler characteristic and so the Hilbert polynomial. Therefore, we can consider the space $M(\xi)$ of $S$-equivalence classes of Gieseker semistable sheaves with Hilbert polynomial $P_\xi$.

Bridgeland stability conditions have been proven to be useful to study the birational geometry of $M(\xi)$. In particular, the goal of this section is to compute the ample cone of $M(\xi)$, following the approach in [7].

Theorem 3.2. [1] $H \subseteq \text{Stab}(\mathbb{P}^2)$ admits the following walls-chambers decomposition: there is a unique vertical wall and to the left of this wall there is a finite number of distinct nested semicircular walls. If we call the semicircular wall of maximal radius the Gieseker wall and the chamber outside this wall the Gieseker chamber, then for any $(s, t)$ outside the Gieseker wall $M_{s,t}(\xi)$ and $M(\xi)$ are isomorphic.

Now we apply Bayer and Macrì Positivity Lemma to this situation.

Lemma 3.3. For any $(s, t)$ in the Gieseker chamber, there exists a nef divisor $l_{s,t}$ on $M_{s,t}(\xi)$. Moreover, if we choose $(s, t)$ on the Gieseker wall, there exists a curve $C$ such that $l_{s,t} \cdot C = 0$, so $l_{s,t}$ is nef but not ample.

We go back to the problem of computing the ample cone of $M(\xi)$. As shown in Section 18 of [10], the rank of the Néron-Severi of $M(\xi)$ can be only one or two. Moreover, as explained in [7], Proposition 2.4, one of the extremal rays of the ample cone was already known in terms of the Donaldson-Uhlenbeck-Yau compactification. Thanks to Theorem 3.2 and Lemma 3.3, to determine the ample cone of $M(\xi)$ it is enough to compute the Gieseker wall.

4. A FIRST GLIMPSE TO FORMAL DEFINITIONS

In this section we would like to give a gentle introduction to the notion of Bridgeland stability conditions, focusing again on the case of $\mathbb{P}^2$. The goal, more than precision, is to get the reader a little bit familiar with the notions that she will need to learn to enter in the technical core of the topic. For more details, we refer to [2] and [8].

Definition 4.1. A torsion pair in an abelian category $A$ is a pair $(T, F)$ of full additive subcategories of $A$ such that

- $\text{Hom}_A(T, F) = 0$ for all $T \in T$ and $F \in F$;
- Any $E \in A$ fits into a short exact sequence
  $$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0,$$
  where $T \in T$ is called torsion class and $F \in F$ is called torsion-free class.

Definition 4.2. Let $D^b(A)$ be the bounded derived category of $A$. A $t$-structure on $D^b(A)$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full additive subcategories of $D^b(A)$ such that

- $\text{Hom}_{D^b(A)}(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
- Any $D \in D^b(A)$ fits into a triangle
  $$X \rightarrow D \rightarrow Y \rightarrow X[1]$$
  where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
- $\mathcal{X}[1] \subseteq \mathcal{X}$.
The usual notation is $D^{\leq 0} := \mathcal{X}$ and $D^{\geq 1} := \mathcal{Y}$. The heart of a $t$-structure is given by $\mathcal{H} := D^{\leq 0} \cap D^{\geq 0}$. The heart turns out to be an abelian category.

A $t$-structure is called bounded if every object in $D^b(\mathcal{A})$ is contained in $D^{\leq n} \cap D^{\geq -n}$ for $n$ sufficiently large.

**Remark 4.3.** Let $\mathcal{A}$ be an abelian category and let $D^b(\mathcal{A})$ be the bounded derived category of $\mathcal{A}$. Then, the standard bounded $t$-structure is given by

- $D^{\leq 0} = \{ X \in D^b(\mathcal{A}) | H^i(X) = 0 \text{ for } i > 0 \}$;
- $D^{\geq 0} = \{ Y \in D^b(\mathcal{A}) | H^i(Y) = 0 \text{ for } i < 0 \}$.

Now we move on to the definition of Bridgeland stability conditions. We focus on the case of $\mathbb{P}^2$, so that $\mathcal{A} = \text{coh}(\mathbb{P}^2)$ and $D^b(\mathcal{A}) = D^b(\text{coh}(\mathbb{P}^2))$. In this part we mainly refer to [1], Section 5.

**Definition 4.4.** A Bridgeland stability condition on $\mathbb{P}^2$ is a triple $(\mathcal{H}; r, d)$ such that

- $\mathcal{H}$ is the heart of a bounded $t$-structure on $D^b(\text{coh}(\mathbb{P}^2))$.
- $r$ and $d$ are linear maps
  
  \[ r, d : K(D^b(\text{coh}(\mathbb{P}^2))) \to \mathbb{R} \]

  defined on the $K$-group of the derived category, i.e. $r$ and $d$ are additive on triangles. The maps $r$ and $d$ satisfy
  - $r(E) \geq 0$ for each $E \in \mathcal{H}$;
  - If $r(E) = 0$ for some non-zero object $E \in \mathcal{H}$, then $d(E) > 0$.
- All the objects in $\mathcal{H}$ satisfy the property of Harder-Narasimhan.

We now recall the concepts of slope, stability and Harder-Narasimhan filtration.

**Definition 4.5.** The slope of a non-zero object $E \in \mathcal{H}$ with respect to $r, d$ is

\[ \mu(E) = \begin{cases} 
\frac{d(E)}{r(E)} & \text{if } r(E) \neq 0, \\
+\infty & \text{otherwise}.
\end{cases} \]

**Definition 4.6.** An object $E \in \mathcal{H}$ is called stable (resp. semistable) if for any proper non-zero subobject $F \subseteq E$

\[ \mu(F) < \mu(E) \text{ (resp. } \leq ) \]

We recall that an object $E \in \mathcal{H}$ satisfies the property of Harder-Narasimhan if it admits a finite filtration

\[ 0 \subseteq E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E \]

uniquely determined by the fact that $F_i := E_i/E_{i-1}$ is semistable and

\[ \mu(F_1) > \mu(F_2) > \cdots > \mu(F_n). \]

We now want to explicitly construct stability conditions on $\mathbb{P}^2$. We need, therefore, to exhibit a heart and the maps $r$ and $d$ such that they satisfy all the properties of Definition 4.4. The first choice could be the heart of the standard $t$-structure on $D^b(\text{coh}(\mathbb{P}^2))$, see Remark 4.3, with the “ordinary” rank and degree, namely, for $E \in \mathcal{H}$:

- $d(E) := c_1(E) \cdot L$;
- $r(E) := c_0 \cdot L^2$,

where $L$ is the hyperplane class on $\mathbb{P}^2$. Unluckily, these choices do not origin to a Bridgeland stability condition, because if we consider the skyscraper sheaf $\mathcal{C}_p$ we have

\[ r(\mathcal{C}_p) = 0 = d(\mathcal{C}_p). \]
However, the associated slope function $\mu(E) = d(E)/r(E)$ satisfies the following weak property of Harder-Narasimhan: there exists a filtration

$$0 \subseteq E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$$

where $E_0$ is torsion-free and for $i > 0$, $F_i = E_i/E_{i-1}$ are torsion-free semistable sheaves with strictly decreasing slopes $\mu_i := \mu(F_i)$.

Since the standard choice does not work, the idea is now to use the slopes $\mu_i$ to construct a torsion pair on $\text{coh}(\mathbb{P}^2)$ inducing a bounded $t$-structure on $D^b(\text{coh}(\mathbb{P}^2))$.

**Definition 4.7.** Let $s \in \mathbb{R}$ and let $(T_s, F_s)$ be full subcategories of $\text{coh}(\mathbb{P}^2)$ such that

- $T \in T_s$ if $T$ is torsion or if every $\mu_i > s$ in the filtration of Harder-Narasimhan of $T$;
- $F \in F_s$ if $F$ is torsion-free and if every $\mu_i \leq s$ in the filtration of Harder-Narasimhan of $F$.

In this way, for any $s \in \mathbb{R}$, every pair $(T_s, F_s)$ is a torsion pair on $\text{coh}(\mathbb{P}^2)$ that induces the following bounded $t$-structure on $D^b(\text{coh}(\mathbb{P}^2))$:

- $D^s_0 := \{\text{complexes } E|H^{-1}(E) \in F_s \text{ and } H^i(E) = 0 \text{ for } i < -1\}$;
- $D^s_0 := \{\text{complexes } E|H^0(E) \in T_s \text{ and } H^i(E) = 0 \text{ for } i > 0\}$.

The heart of the $t$-structure defined by the torsion-pair is

$$H_s := \{\text{complexes } E|H^{-1}(E) \in F_s, H^0(E) \in T_s \text{ and } H^i(E) = 0 \text{ otherwise}\}.$$

In order to conclude, we just need to introduce the right notion of rank and degree.

**Theorem 4.8.** [6] Let $s \in \mathbb{R}$ and $t > 0$. If we define the functions of rank and degree on $H_s$ in the following way:

- $r_t := t \cdot c_1(E(-s)) \cdot L$;
- $d_t := (t^2/2)c_0(E(-2)) \cdot L^2 + c_2(E(-s))$,

then $(H_s; r_t, d_t)$ is a stability condition on $\mathbb{P}^2$ with slope function $\mu_{s,t} := d_t/r_t$.

Indeed, it turns out that $r_t$ and $d_t$ satisfy all the properties of Definition 4.4 (see [1], remarks following Definition 5.10).

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