Ricci curvature and geodesic flows stability in Riemannian twisted flux tubes

by

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Abstract

Ricci and sectional curvatures of twisted flux tubes in Riemannian manifold are computed to investigate the stability of the tubes. The geodesic equations are used to show that in the case of thick tubes, the curvature of planar (Frenet torsion-free) tubes have the effect of damping the flow speed along the tube. Stability of geodesic flows in the Riemannian twisted thin tubes (almost filaments), against constant radial perturbations is investigated by using the method of negative sectional curvature for unstable flows. No special form of the flow like Beltrami flows is admitted, and the proof is general for the case of thin tubes. It is found that for positive perturbations and angular speed of the flow, instability is achieved, since the sectional Ricci curvature of the twisted tube metric is negative. PACS numbers: 02.40.Hw:Riemannian geometries
I Introduction

The stability of geodesic flows have been recently investigated by Kambe [1] by making use of the technique of Ricci sectional curvature [2], where the negative sectional curvature indicates instability of the flow, while positivity or null indicates stability. In the case of instability the geodesics deviate from the perturbation of the fluid. Following the work of D. Anosov [3] on the perturbation in geodesic flows in three-dimensional Riemannian geometry, in this paper the sectional Riemann curvature of the geodesic flow for a Riemannian flux tube [4, 5], where the axis of the tube flow possesses Frenet curvature and torsion. In the approximation of a thin tube where the radius of the tube is almost null, we show that the flows are unstable, against orthogonal perturbations, which is equivalently due to the negativity of the sectional Ricci sectional curvature. Throughout the paper the elegant coordinate-free language of differential geometry [6] is used. The paper is organized as follows: Section II presents a brief review of Riemannian geometry in the coordinate free language. Section III presents the geodesic flow computation of the Christoffel symbols for the thick flux tube, where we show that the curvature of the tube axihe speed of the flow. Section IV presents the computation of the instability of Riemannian tube flow. Section V presents the conclusions.

II Ricci and sectional Riemann curvatures

In this section we make a brief review of the differential geometry of surfaces in coordinate-free language. The Riemann curvature is defined by

\[ R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \]  

(II.1)

where \(X \in TM\) is the vector representation which is defined on the tangent space \(TM\) to the manifold \(M\). Here \(\nabla_X Y\) represents the covariant derivative given by

\[ \nabla_X Y = (X.\nabla)Y \]  

(II.2)

which for the physicists is intuitive, since we are saying that we are performing derivative along the \(X\) direction. The expression \([X,Y]\) represents the commutator, which on a vector basis
frame $\vec{e}_l$ in this tangent sub-manifold defined by

$$X = X_k \vec{e}_k$$  \hspace{1cm} (II.3)

or in the dual basis $\partial_k$

$$X = X^k \partial_k$$  \hspace{1cm} (II.4)

can be expressed as

$$[X, Y] = (X, Y)^k \partial_k$$  \hspace{1cm} (II.5)

In this same coordinate basis now we are able to write the curvature expression (II.1) as

$$R(X, Y) Z := [R^l_{ijkp} Z^j X^k Y^p] \partial_l$$  \hspace{1cm} (II.6)

where the Einstein summation convention of tensor calculus is used. The expression $R(X, Y) Y$ which we shall compute bellow is called Ricci curvature. The sectional curvature which is very useful in future computations is defined by

$$K(X, Y) := \frac{< R(X, Y) Y, X >}{S(X, Y)}$$  \hspace{1cm} (II.7)

where $S(X, Y)$ is defined by

$$S(X, Y) := ||X||^2 ||Y||^2 - < X, Y >^2$$  \hspace{1cm} (II.8)

where the symbol $<, >$ implies internal product.
III Geodesic equations in Riemannian tube metric

In this section we shall consider the twisted flux tube Riemann metric. The metric $g(X,Y)$ line element can be defined as \[4, 5\]
\[ds^2 = dr^2 + r^2d\theta_R^2 + K^2(s)ds^2\] (III.9)

This line element was used previously by Ricca [4] and the author [5] as a magnetic flux tubes with applications in solar and plasma astrophysics. This is a Riemannian line element
\[ds^2 = g_{ij}dx^i dx^j\] (III.10)

if the tube coordinates are $(r, \theta_R, s)$ [4] where $\theta(s) = \theta_R - \int \tau ds$ where $\tau$ is the Frenet torsion of the tube axis and $K(s)$ is given by
\[K^2(s) = [1 - r\kappa(s)\cos\theta(s)]^2\] (III.11)

Let us now compute the geodesic equations
\[\frac{dv^i}{dt} + \Gamma^i_{jk}v^jv^k = 0\] (III.12)

where $v^s = \frac{ds}{dt}$ and $v^\theta = \frac{d\theta}{dt}$ the Riemann-Christoffel symbols are given by
\[\Gamma^i_{jk} = \frac{1}{2}g^{il}[g_{lj,k} + g_{lk,j} - g_{jk,l}]\] (III.13)

The only nonvanishing components of the Christoffel symbols or Levi-Civita [6] connections of the flux tube are
\[\Gamma^2_{21} = \frac{1}{r}\] (III.14)
\[\Gamma^2_{33} = -\frac{K(s)\kappa\sin\theta}{r}\] (III.15)
\[\Gamma^3_{11} = \frac{r\kappa}{2K}[\tau\sin\theta + \frac{\kappa'}{\kappa}\cos\theta]\] (III.16)
\[\Gamma^3_{33} = K^{-1}K' = -\Gamma^3_{11}\] (III.17)

A simple example can be given by writing the geodesic equation for the untwisted tube, where $v_\theta = 0$, as
\[\ddot{s} + \Gamma^3_{11}[\dot{r}^2 - \dot{s}^2] = 0\] (III.18)
where substitution of the component $\Gamma^{3}_{11}$ above yields

$$dlnv_{s} = -d(ln\kappa)$$ (III.19)

where we use the chain rule of differential calculus, $\frac{dv}{v_s} \frac{dv}{dt} = \frac{dv}{v_s}$. Solution of equation (III.19) is

$$v_s = v_0\kappa^{-1}$$ (III.20)

where $v_0$ is an integration constant. This solution tells us that when the curvature tends to $\infty$ the velocity along the tube axis vanishes. Physically this means that the curvature acts as a damping to the flow, along the tube. In the next section we investigate in some detail the stability of an incompressible or volume preserving flow, using the method of the sign of the Ricci sectional curvature.

**IV Ricci sectional curvature and flow stability**

One of the most important features of the investigation of the stability of flows in the Euclidean manifold $E^3$, is the comprehension of the fact that the covariant derivative in the flow curved manifold is given by the gradient operator in curvilinear coordinates. Thus to compute the Ricci sectional curvature above, we need to make use of the grad operator in the twisted flux tube Riemannian metric given by

$$\nabla = [\partial_r, r^{-1}\partial_\theta, K^{-1}\partial_\phi]$$ (IV.21)

Since the axis of the tube undergoes torsion and curvature, we need some dynamical relations from vector analysis and differential geometry of curves [7] such as the Frenet frame $(\vec{t}, \vec{n}, \vec{b})$ equations

$$\vec{t}' = \kappa \vec{n}$$ (IV.22)

$$\vec{n}' = -\kappa \vec{t} + \tau \vec{b}$$ (IV.23)

$$\vec{b}' = -\tau \vec{n}$$ (IV.24)

and the other frame vectors are

$$\vec{\epsilon}_r = \vec{n}cos\theta + \vec{b}sin\theta$$ (IV.25)
\[ \vec{e}_\theta = -\vec{n} \sin \theta + \vec{b} \cos \theta \]  
(IV.26)

\[ \partial_\theta \vec{e}_\theta = -\vec{n}[(1 + \tau^{-1}\kappa)\sin \theta + \cos \theta] - \vec{b}[(\cos \theta + \sin \theta] \]  
(IV.27)

Let the constant perturbation be given by

\[ X = u_r^1 \vec{e}_r \]  
(IV.28)

The upper index one in this expression refers to the fact that the background original value of \( u_r \) was considered as \( u_r^0 = 0 \) to form the tube. The other variable \( Y \) is given by

\[ Y = u_\theta \vec{e}_\theta + u_s \vec{t} \]  
(IV.29)

Therefore to compute the Ricci tensor step by step we start by the term

\[ \nabla_X Y = u_r^{(1)} \partial_r [u_\theta \vec{e}_\theta + u_s \vec{t}] \]  
(IV.30)

which vanishes since we adopt here the approximation \( u_r^{(1)} \partial_r [u_\theta] \approx 0 \) along with the same relation to the radial partial derivative of \( u_r \). So

\[ \nabla_Y \nabla_X Y \approx 0 \]  
(IV.31)

Now the second term in the Ricci tensor is

\[ \nabla_X \nabla_Y Y = -u_r^{(1)} r^{-2} u_\theta \left( \frac{1 + \tau}{\tau} (\vec{n} \cos \theta + \vec{b} \sin \theta) \right] \]  
(IV.32)

where we have used the approximation of the thin tube where \( K(s) \approx 1 \) and \( r \approx 0 \)

\[ [X,Y] = u_r^{(1)} [r^{-1} u_\theta - \tau u_s] [\vec{e}_r - \tau^{-1} \vec{t}] \]  
(IV.33)

which implies that

\[ \nabla_{[X,Y]} Y = u_r^{(1)} [r^{-1} u_\theta - \tau u_s] [-\tau^{-1} u_\theta \partial_s \vec{e}_\theta + \kappa \vec{n}] \]  
(IV.34)

The Ricci tensor is

\[ R(X,Y)Y = -u_r^{(1)} r^{-1} u_\theta \tau^{-1} \partial_s \vec{e}_\theta \]  
(IV.35)
In the previous computations we have made use of the imcompressibility of the flow

$$\nabla \cdot \vec{u} = 0$$  \hspace{1cm} (IV.36)

which is

$$\partial_s u_\theta = \tau r \kappa u_\theta \approx 0$$  \hspace{1cm} (IV.37)

since \( r \approx 0 \) on the RHS of equation (IV.37). The sectional curvature is thus

$$K(X, Y) = \frac{< R(X, Y)Y, X >}{S(X, Y)} = -\frac{u_\theta [1 + \tau(s)\kappa \cos \theta]}{ru^{(1)}_r [u_\theta^2 + u_s^2]}$$  \hspace{1cm} (IV.38)

when the tube is strongly twisted, \( u_\theta^2 \gg u_s^2 \) thus the sectional Ricci curvature is

$$K(X, Y) = \frac{< R(X, Y)Y, X >}{S(X, Y)} = -\frac{[1 + \tau(s)\kappa \cos \theta]}{ru^{(1)}_r u_\theta}$$  \hspace{1cm} (IV.39)

when the tube, besides is planar or torsion vanishes the last expression reduces to

$$K(X, Y) = \frac{< R(X, Y)Y, X >}{S(X, Y)} = -\frac{1}{ru^{(1)}_r u_\theta}$$  \hspace{1cm} (IV.40)

Note that when both angular velocity and perturbation both keep the same sign, the sectional curvature \( K(X, Y) \) is negative and the flow along the Riemannian flux tube is unstable. There is singularity in this sectional Riemannian curvature in \( r \approx 0 \).

V  Conclusions

An important issue in plasma astrophysics as well as in fluid mechanics is to know when a fluid, charged or not, is unstable or not. In this paper we discuss and present incompressible flows and investigate their stability. Instability is obtained even before the singularity is achieved.
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