GENUS FIELDS OF GLOBAL FIELDS

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Abstract. In this paper we obtain the extended genus field of a global field. First we define the extended genus field of a global function field and we obtain, via class field theory, the description of the extended genus field of an arbitrary global function field. In the last part of the paper we use the techniques for function fields to describe the extended genus field of an arbitrary number field.

1. Introduction

The study of narrow or extended genus fields goes back to C.F. Gauss [8] who introduced the genus concept in the context of quadratic forms. During the first half of the last century, the concept was imported to quadratic number fields. H. Hasse [9] studied genus theory of quadratic number fields by means of class field theory. H.W. Leopoldt [11] generalized the work of H. Hasse by introducing the concept of genus field for a finite abelian extension of the rational field. Leopoldt studied extended genus fields using the arithmetic of abelian fields by means of Dirichlet characters. The first to introduce the concept of genus field and of extended genus field of a nonabelian finite extension of the rational field was A. Fröhlich who defined the concept of genus field of an arbitrary finite extension of \( \mathbb{Q} \) [6, 7]. For a number field \( K \), Fröhlich defined the genus field \( K \) (with respect to the rational field \( \mathbb{Q} \)) as \( K_{\text{ge}} :=KF \) where \( F/\mathbb{Q} \) is the maximum abelian extension such that \( KF/K \) is unramified everywhere. Similarly, the extended genus field is \( K_{\text{gex}} :=KL \) where \( L/\mathbb{Q} \) is the maximum abelian extension such that \( KL/K \) is unramified at the finite primes. Numerous authors have studied genus fields and extended genus fields for finite field extensions \( K/\mathbb{Q} \).

In the case of number fields, the concepts of Hilbert class field and of extended Hilbert class field are defined without any ambiguity. The Hilbert class field \( K_{H} \) and the extended Hilbert class field \( K_{H+} \) of a number field \( K/\mathbb{Q} \) are defined as the maximum abelian unramified extension and the maximum abelian extension unramified at the finite primes of \( K \), respectively. In this way, the concepts of genus field and of extended genus field are defined depending on the concept of the Hilbert class field, and of the extended Hilbert class field respectively. Namely, we have \( K \subseteq K_{\text{ge}} \subseteq K_{H} \) and the Galois group \( \text{Gal}(K_{H}/K) \) is isomorphic to the class group \( Cl_{K} \) of \( K \). The genus field \( K_{\text{ge}} \) corresponds to a subgroup \( G_{K} \) of \( Cl_{K} \) and we have \( \text{Gal}(K_{\text{ge}}/K) \cong Cl_{K}/G_{K} \). The degree \([K_{\text{ge}} : K]\) is called the genus number of \( K \) and \( \text{Gal}(K_{\text{ge}}/K) \) is called the genus group of \( K \). Similarly, \( K \subseteq K_{\text{gex}} \subseteq K_{H+} \) and \( K_{\text{gex}} \) corresponds to a subgroup \( G_{K+} \) of \( \text{Gal}(K_{\text{gex}}/K) \cong Cl_{K+} \).
For global function fields the picture is different due to the fact that there are several concepts of Hilbert class field and of extended Hilbert class field, depending in which aspect you are interested in. The direct definition of the Hilbert class field \( K_H \) of a global function field \( K \) over \( \mathbb{F}_q \) as the maximum unramified abelian extension of \( K \) has the disadvantage of being of infinite degree over \( K \) due to the extensions of constants. In the extensions of constants, every prime is eventually inert, so, if we are interested in a definition of a Hilbert class field of finite degree over the base field, we must impose some condition on the extension of constants. It seems that the first one to consider extended genus fields in the case of function fields was R. Clement in [5], where she considered the case of a cyclic tame extension \( K/\mathbb{F}_q(T) \) of prime degree \( l \) different from the characteristic \( p \) of \( \mathbb{F}_q \). She developed the theory along the lines of the case studied by Hasse in [9]. Later on, S. Bae and J.K. Koo [3] generalized the results of Clement following the development given by Fröhlich. They defined the extended genus field for extensions of an arbitrary global function field \( K \) defining an analogue to the cyclotomic function field extensions of \( \mathbb{F}_q(T) \) given by the Carlitz module.

M. Rosen defined in [15] the Hilbert class field of a global function field \( K \) as the maximum abelian unramified extension of \( K \) such that a fixed nonempty finite set of prime divisors of \( K \) decompose fully. Using this definition of Hilbert class field, G. Peng [14] found the genus field of a cyclic tame extension of prime degree over the rational function field \( k = \mathbb{F}_q(T) \). His method used the analogue for function fields of the Conner–Hurrelbrink exact hexagon in number fields. The wild prime case was presented by S. Hu and Y. Li in [10] where they described explicitly the genus field of an Artin–Schreier extension of the rational function field. In [2, 12, 13] we developed a theory of genus fields using the same concept of Hilbert class field. In those papers, the ideas of Leopoldt using Dirichlet characters were strongly used.

In this paper we are interested in describing, using class field theory, the extended genus field of a finite separable extension of \( k \). B. Anglès and J.-F. Jaulent in [1] established the general theory of extended genus fields of global fields, either function or numeric. We use a concept of extended genus field for function fields different from the one defined by Anglès and Jaulent. With this concept, when we describe the finite abelian extension \( L \) where \( K_{\text{gen}} = KL \), we may write \( L \) as the composition of a sort of \( P \)-components, where \( P \) runs through the finite primes of \( k \). We consider these \( P \)-components \( L_P \) as the composition of \( E_P \), the \( P \)-component of the projection \( E \) of \( L \) in a cyclotomic function field given by the Carlitz module, and a field \( S \) which codifies the behavior of the infinite prime. More precisely, \( S \) codifies the wild ramification and the inertia of the infinite prime of \( k \). To this end, we need to consider the idèle group corresponding to an arbitrary cyclotomic function field. Finally, we describe the field \( S \).

It turns out, that the same approach works for number fields. Indeed, in the number field case, the problem is simpler because, by the Kronecker–Weber theorem, any abelian extension of \( \mathbb{Q} \) is cyclotomic, that is, it is contained in a cyclotomic number field. In the function field case, the maximum abelian extension of \( k \) consists of three components: one cyclotomic, one of constants and one, also cyclotomic, where the infinite prime is totally and wildly ramified and it is the only ramified prime. In the number field case, the “\( p \)-components” can be found explicitly for \( p \geq 3 \) depending only on their degree over \( \mathbb{Q} \). The case \( p = 2 \) does
not depend only on its degree over \(\mathbb{Q}\) since, for \(n \geq 3\), the cyclotomic field \(\mathbb{Q}(\zeta_{2^n})\) is not cyclic. We give a criterion to describe the 2-component of \(K_{\text{get}}\). Finally, we present some results on the behavior of the genus field of a composition. For number fields, a similar result was obtained by M. Bhaskaran in [4] and by X. Zhang [18].

2. Preliminaries and Notations

We denote by \(k = \mathbb{F}_q(T)\) the global rational function filed with field of constants the finite field of \(q\) elements \(\mathbb{F}_q\). Let \(R_T = \mathbb{F}_q[T]\) be the ring of polynomials, that is, the ring of integers of \(k\) with respect to the pole of \(T\), the infinite prime \(p_\infty\). Let \(R_T^* := \{ P \in R_T \mid P \text{ is monic and irreducible} \}\). The elements of \(R_T^*\) are the finite primes of \(k\) and \(p_\infty\) is the infinite prime of \(k\). For \(N \subseteq R_T\), \(\Lambda_N\) denotes the \(N\)-th torsion of the Carlitz module. A finite extension \(F/k\) will be called cyclotomic if there exists \(N \subseteq R_T\) such that \(k \subseteq F \subseteq k(\Lambda_N)\).

Given a cyclotomic function field \(E\), the group of Dirichlet characters \(X\) corresponding to \(E\) is the group \(X\) such that \(X \subseteq (R_T/(N))^* \cong \text{Gal}(k(\Lambda_N)/k) = \text{Hom}((R_T/(N))^*, \mathbb{C}^*)\) and \(E = k(\Lambda_N)^H\) where \(H = \bigcap \chi \in X \ker \chi\). For the basic results on Dirichlet characters, we refer to [17, §12.6].

For a group of Dirichlet characters \(X\), let \(Y = \prod_{P \in R_T} \chi_P\) where \(\chi_P = \chi \mid X\) and \(\chi_P\) is the \(P\)-th component of \(\chi\): \(\chi = \prod_{P \in R_T^*} \chi_P\). If \(E\) is the field corresponding to \(Y\), we define \(E_{\text{get}}\) as the field corresponding to \(Y\). We have that \(E_{\text{get}}\) is the maximum unramified extension at the finite primes of \(E\) contained in a cyclotomic function field. The infinite prime \(p_\infty\) might be ramified in \(E_{\text{get}}/k\) (see [12]).

Let \(L_n = k(\Lambda_{1/T^{n+1}})^{F_q}, n \in \mathbb{N} \cup \{0\}\) where \(F_q \subseteq \left(\mathbb{R}_{1/T}/(1/T^{n+1})\right)^*\), is isomorphic to the inertia group of the prime corresponding to \(T\) in \(k(\Lambda_{1/T^{n+1}})/k\). The prime \(p_\infty\) is the only ramified prime in \(L_n/k\) and it is totally and wildly ramified. For \(m \in \mathbb{N}\), and for any finite extension \(F/k\), \(F_m\) denotes the extension of constants: \(F_m = F\mathbb{F}_q^m\). In particular \(k_m = \mathbb{F}_q^m(T)\).

Given a finite abelian extension \(K/k\), there exist \(n \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}\) and \(N \subseteq R_T\) such that \(K \subseteq L_n k(\Lambda_N)^{k_m} =: n(k(\Lambda_N))_m\) (see [17, Theorem 12.8.31]). We define \(M := L_n k_{m}\). In \(M/k\) no finite prime of \(k\) is ramified.

For any extension \(E/F\) of global fields and for any place \(\mathfrak{P}\) of \(E\) and \(\mathfrak{p} = \mathfrak{P} \cap F\), the ramification index is denoted by \(e_{E/F}(\mathfrak{P}|\mathfrak{p}) = e(\mathfrak{P}|\mathfrak{p})\) and the inertia degree is denoted by \(f_{E/F}(\mathfrak{P}|\mathfrak{p}) = f(\mathfrak{P}|\mathfrak{p})\). When the extension is Galois we denote \(e_{E/F}(\mathfrak{P}|\mathfrak{p}) = e_{E/F}(\mathfrak{P}|\mathfrak{p})\) and \(f_{E/F}(\mathfrak{P}|\mathfrak{p}) = f_{E/F}(\mathfrak{P}|\mathfrak{p})\). In particular for any abelian extension \(E/k\), \(e(\mathfrak{P}|\mathfrak{p})\) and \(f(\mathfrak{P}|\mathfrak{p})\) denote the ramification index and the inertia degree of \(P \in R_T^*\) in \(E/k\) respectively, and we denote by \(e_{\infty}(E/k)\) and \(f_{\infty}(E/k)\) the ramification index and the inertia degree of \(p_\infty\) in \(E/k\). The symbol \(e_{\text{wild}}(E/F)\) denotes the wild ramification part of the infinite primes in \(E/F\). Similarly, \(I_{E/F}(\mathfrak{P}|\mathfrak{p})\) denotes the inertia group and \(D_{E/F}(\mathfrak{P}|\mathfrak{p})\) the decomposition group.

For any finite separable extension \(K/k\) the finite primes of \(K\) are the primes over the primes \(P \in R_T^*\) and the infinite primes of \(K\) are the primes over \(p_\infty\). The Hilbert class field \(K_H\) of \(K\) is the maximum abelian extension of \(K\) unramified at every finite prime of \(K\) and where all the infinite primes of \(K\) are fully decomposed. The genus field \(K_{\text{get}}\) of \(K/k\) is the maximum extension of \(K\) contained in \(K_H\) and
such that it is the composite $K_{ge} = KF$ where $F/k$ is abelian. We choose $F$ the maximum possible. In other words, $F$ is the maximum abelian extension of $k$ contained in $K_H$.

Let $K/k$ be a finite abelian extension. We know that $K_{ge} = KE^H_{ge}$ is the genus field of $K$ where $H$ is the decomposition group of the infinite primes in $KE/K$ and $E := KM \cap k(\Lambda_N)$ (see [2]). We also know that $KE_{ge}/K_{ge}$ and $KE/K$ are extensions of constants.

For a local field $F$ with prime $p$, we denote by $F(p) \cong \mathbb{F}_q$ the residue field of $F$, $U_p(n) = 1 + p^n$ the $n$-th units of $F$, $n \in \mathbb{N} \cup \{0\}$.

Let $\pi = \pi_F = \pi_p$ be a uniformizer element for $p$, that is, $v_p(\pi) = 1$. Then the multiplicative group of $F$ satisfies $F^* \cong \langle \pi \rangle \times U_p \cong \langle \pi \rangle \times \mathbb{F}_q^* \times U_p^{(1)}$ as groups.

### 3. Extended Genus Field of a Global Function Field

Let $K/k$ be a finite abelian extension. Let $n \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}$ and $N \in \mathbb{N}_+$ be such that $K \subseteq n k(\Lambda_N).m$. Let $E = KM \cap k(\Lambda_N)$. Define the extended genus field of $K$ as

$$K_{ge} := KE_{ge}.$$

Note that $K_{ge}/K$ is unramified at the finite primes since $E_{ge}/E$ is unramified at the finite primes, so that $KE_{ge}/KE$ is unramified at the finite primes and we also know that $KE/K$ is unramified at the finite primes ([2]).

![Diagram](image)

Now $KM = EM/E$ is ramified at most at the infinite prime $p_\infty$ and the inertia of $p_\infty$ in the extension $EM = KM/E$ is contained in $M$. Hence $EM/E$ is unramified at the finite primes. The same holds for $KM/K$ and we have $K \subseteq KE \subseteq KM = EM$. In short, $K_{ge}/K$ is unramified at the finite primes. We also have that $K_{ge}/E$ is tamely ramified at $p_\infty$ since $E_{ge}/k$ is tamely ramified at $p_\infty$ so that $KE_{ge}/K$ is tamely ramified at $p_\infty$ and $K_{ge} = KE_{ge}$.

We also have $[E_{ge} : E_{ge}^H] = q - 1$ since $e_\infty(E_{ge}/E) = q - 1$ where in general, for a finite abelian extension $L/F$, $e_\infty(L/F)$ denotes the ramification index of the infinite primes of $F$ in $L$, and $H \subseteq I_\infty(E_{ge}/k)$, where in general $I_\infty(L/F)$ denotes the inertia group of the infinite primes in the Galois extension $L/F$. In other words, the infinite primes of $E_{ge}^H$ are fully ramified in the extension $E_{ge}/E_{ge}^H$. Thus we have

$$[E_{ge} : E_{ge}^H] = e_\infty(E_{ge}/E_{ge}^H)|e_\infty(k(\Lambda_N)/k) = q - 1.$$

Therefore we have that $K_{ge} = KE_{ge}/KE_{ge}^H = K_{ge}$ is unramified at the finite primes, the infinite primes are tamely ramified, and $[K_{ge} : K_{ge}] = q - 1$.

Now let $K/k$ be a finite and separable extension. We define $K_{ge}$ as $KE_{ge}$ where $K_{ge} = KF$, that is, $F$ is the maximum abelian extension of $k$ contained in the Hilbert class field $K_H$ of $K$ (see [2]). Note that $F_{ge} = F$. 
Note that \([K_{\text{ger}} : K_{\text{gt}}] = [F_{\text{ger}} : F_{\text{gt}}]q - 1\) and the only possible ramified primes in \(K_{\text{gt}}/K_{\text{gt}}\) are the infinite primes and they are tamely ramified.

**Definition 3.1.** For a finite separable extension \(K/k\), we define the extended genus field of \(K\) as \(K_{\text{gt}} = KF_{\text{gt}} = KL\) where \(L = F_{\text{gt}}\). We stress that we choose \(F\) to be the maximum abelian extension of \(k\) such that \(K_{\text{gt}} = KF\).

**Remark 3.2.** For a finite prime \(P \in R_T^+\), the tame part of the ramification of \(P\) in \(K_{\text{gt}}/k\) can be obtained in the following way. Let \(d_P = \deg P\) and let \(e_P(L/k) = e_P(0)^{e_P(\ell(w)) = e_{\text{tame}}(P)e_P(\ell(w))\text{ where }\gcd(p, e_P(0)) = 1\text{ and }e_P(\ell(w)) = p^{\alpha_P}\text{ for some integer }\alpha_P \geq 0.\) Since \(L/k\) is abelian, we have \(e_P(0)^{q^{d_P} - 1} = 1\) (see [16, Proposició 10.4.8]).

Consider the extension \(k_{\text{gt}}^{(0)}/k\) where \(P\) is the only finite prime ramified, \(k_{\text{gt}}^{(0)} \subseteq L\) and \(e_P(0)^{[k_{\text{gt}}^{(0)} : k]}\). Note that \([k(\Lambda_P) : k] = q^{d_P} - 1\). Then \(KK_{\text{gt}}^{(0)} \subseteq K_{\text{gt}}\) and \(Kk_{\text{gt}}^{(0)}/K\) is unramified at the finite primes. Thus, by Abhyankar Lemma,

\[
e_P(K[k] = e_P(Kk_{\text{gt}}^{(0)})[k] = \text{lcm}[e_P(K[k]), e_P(k_{\text{gt}}^{(0)})] = \text{lcm}[e_P(K[k]), e_P(0)]
\]

Therefore \(e_P(0)|e_P(K[k])\). Since \(e_P(0)\) is the maximum with this property, it follows that

\[
e_{\text{tame}}(P) = e_P(0) = \gcd(q^{d_P} - 1, e_P(K[k])).
\]

We now obtain \(K_{\text{gt}} = KL\) where \(L\) satisfies \(L_{\text{gt}} = L, L/k\) abelian and \(L\) is the maximum with respect to this property. Let \(L \subseteq n k(\Lambda_n)_m\). If necessary, we may assume \(n, m, N\) are minimum, where \(m \in \mathbb{N}\) is the conductor of constants (see [2]), \(N \in R_T\) and \(\alpha \in \mathbb{N} \cup \{0\}\). In this situation we define the conductor of \(L\) as \((m, N, n)\).

Let \(E := LM \cap k(\Lambda_N)\). Then \(EM = LM\) and \(L_{\text{gt}} = L = E_{\text{gt}}L\) so that \(E_{\text{gt}} \subseteq L\) and \(E = E_{\text{gt}}\). In fact, \(E_{\text{gt}} \subseteq L \subseteq LM = EM\), hence \(E_{\text{gt}}M \subseteq EM\) and from the Galois correspondence, \(E_{\text{gt}} \subseteq E\). Thus \(E_{\text{gt}} = E\).

\[
E = E_{\text{gt}}\quad L\quad LM = EM
\]

Let \(S := L \cap M, S \subseteq M = L_n k_m\).

Let \(X = Y = \prod_{P \in R_T^+} X_P\) be the group of Dirichlet characters associated to \(E_{\text{gt}} = E\). Then if \(E_P\) is the field associated to \(X_P\), with \(P \in R_T^+\), \(E = \prod_{P \in R_T^+} E_P\) where \(E_P = k\) for almost all \(P\) and if \(P_1, \ldots, P_r\) are the finite primes ramified in \(E/k, X_{P_i} \neq \{1\}, E_{P_i} \neq k, E_{P_i} \cap \prod_{j \neq i} E_{P_j} = k, 1 \leq i \leq r\) and \(E = E_{P_1} \cdots E_{P_r}\), \(\text{Gal}(E/k) \cong X = Y = \prod_{P \in R_T^+} X_P = \prod_{P \in R_T^+} \text{Gal}(E_{P_i}/k) \cong \prod_{i=1}^{r} \text{Gal}(E_{P_i}/k)\).

Thus

\[
\text{Gal}(E/k) \cong \prod_{i=1}^{r} \text{Gal}(E_{P_i}/k).
\]
For any nonempty finite subset $A \subseteq R^+_T$, we define $E_A := \prod_{P \in A} E_P$. We may consider $E_P$ as the “$P$-th primary component” of $E$.

\[
E = \prod_{P \in R^+_T} E_P = E_{gs}, \quad L = ES, \quad LM = EM
\]

\[
E_P \quad \quad L_P = E_P S \quad \quad L_P M = E_P M
\]

\[
k \quad \quad S = L \cap M \quad \quad M
\]

We define $L_P := E_P M \cap L$. We have that $E_P \subseteq E \subseteq L$ and $E_P \subseteq E_P M$. Therefore $E_P \subseteq L_P$. From the Galois correspondence, we have

\[
L_P = E_P S.
\]

For any nonempty finite subset $A \subseteq R^+_T$, we let $L_A := E_A M \cap k(\Lambda_N)$. From the Galois correspondence we obtain $L_A = E_A S$ and in particular

\[
L_A = \left( \prod_{P \in A} E_P \right) S = \prod_{P \in A} (E_P S) = \prod_{P \in A} L_P.
\]

We have

**Proposition 3.3.** For any $A, B \in R_T \backslash \{0\}$, let $L_A := E_A M \cap L$, where $E_A := \prod_{P \mid A} E_P$, that is, $E_A = E_A$ and $L_A = L_A$, where $A = \{ P \in R^+_T \mid P \mid A \}$. Then we have

\[
L_{AB} = L_A L_B.
\]

Furthermore, if $\gcd(A, B) = 1$ we have $L_A \cap L_B = S = L \cap M$.

**Proof.** It remains to consider the case $\gcd(A, B) = 1$. We have $E_A = \prod_{P \mid A} E_P$, $E_B = \prod_{P \mid B} E_P$ and $\{ P \in R^+_T \mid P \mid A \} \cap \{ P \in R^+_T \mid P \mid B \} = \emptyset$. Therefore $E_A E_B = k$ and $kL \cap M = L \cap M = S$. The result follows from the Galois correspondence. \qed

Now, for $P \in R^+_T$, $L_P = E_P M \cap L \supseteq M \cap L = S$ and $L_P \neq S \iff P \in \{ P_1, \ldots, P_r \}$. In fact, $L_P = E_P M \cap L \neq S \iff E_P M \neq M \iff E_P \neq k \iff P \in \{ P_1, \ldots, P_r \}$.

Finally, $E = \prod_{P \in R^+_T} E_P = \prod_{i=1}^r E_{P_i}$. Therefore, since $LM = EM$, in particular $L \subseteq EM$. We have

\[
L = EM \cap L = \left( \prod_{i=1}^r E_{P_i} \right) M \cap L = \prod_{i=1}^r (E_{P_i} M \cap L) = \prod_{i=1}^r L_{P_i}
\]

\[
= \prod_{i=1}^r L_{P_i} \cdot \prod_{P \notin \{ P_1, \ldots, P_r \}} S = \prod_{P \in R^+_T} L_P.
\]

Thus

\[
L = \prod_{i=1}^r L_{P_i} = \prod_{P \in R^+_T} L_P.
\]

We have proved
Theorem 3.4. For $A \in R_T$, we define $L_A = E_A M \cap L$. Let $S = L \cap M$. We have

1. For all $A, B \in R_T$, $L_{AB} = L_AL_B$.
2. $L_A \supseteq S$ for all $A \in R_T$ and $L_A = S \iff P_i \mid A$ for all $1 \leq i \leq r$.
3. $L_A \cap L_B = S$ for all $A, B \in R_T$ such that $\gcd(A, B) = 1$.
4. $L = \prod_{P \in R_T^+} L_P = \prod_{i=1}^r L_{P_i}$.

In order to compute $L$ we need to know $S$, that is, the behavior of $p_\infty$, and also each $E_p$ for $P \in R_T^+$. First, we have that if $P \in R_T^+$ is unramified in $K/k$, then $P$ is unramified in $E/k$ and therefore in $L/k$. Indeed, if $P$ were ramified in $L/k$, then we would have

$$e_P(KL|K) = e_P(KL|K)e_P(K|k) = e_P(K|k) = e_P(KL|L)e_P(L|k) > 1$$

so that $e_P(KL|K) > 1$ contrary to the definition of $L$.

\[
\begin{array}{c}
K \\
\downarrow
\end{array} \quad \begin{array}{c}
KL \\
\downarrow
\end{array} \quad \begin{array}{c}
k \\
\downarrow
\end{array} \quad \begin{array}{c}
L
\end{array}
\]

Thus, it suffices to know $E_{P_i}$, $1 \leq i \leq r$ where $P_1, \ldots, P_r$ are the finite primes ramified in $K/k$ and therefore these are the only possible finite primes ramified in $E/k$ and in $L/k$. Now, in $E_P/k$ the only finite prime ramified is $P$ and $p_\infty$ is tamely ramified. Note that the tame ramification index of $p_\infty$ in $E/k$ and in $L/k$ is the same. This is a consequence of Theorem 3.5.

In general we consider an arbitrary global function field $F$. Let $J_F$ be the idèle group of $F$ and let $C_F = J_F/F^\star$ be the idèle class group of $F$. To find $E_p$ for $P \in \{P_1, \ldots, P_r\}$, we must find the idèle subgroup of $J_F$ corresponding to $E_p$. Now, since $E_p$ is cyclotomic and $P$ is the only finite prime ramified, there exists $t \in \mathbb{N}$ such that $E_p \subseteq k(\Lambda_{P^t})$. Therefore, the idèle group corresponding to $E_p$ contains the idèle group corresponding to $k(\Lambda_{P^t})$.

Theorem 3.5. Let $N \in R_T$, $N = P_1^{a_1} \cdots P_r^{a_r}$ with $P_1, \ldots, P_r \in R_T^+$ distinct. Set $R'_T := R_T^+ \setminus \{P_1, \ldots, P_r\}$. Then, the idèle group corresponding to $k(\Lambda_N)$ is

$$\mathcal{X}_N = \prod_{i=1}^r U_{P_i}^{(a_i)} \times \prod_{P \in R'_T} U_P \times [(\pi) \times U_\infty^{(1)}],$$

where $\pi = 1/T$ is a uniformizer for $p_\infty$.

Proof. Let $U' := \prod_{Q \in R_T^+} U_Q \times [(\pi) \times U_\infty^{(1)}]$. We will give an epimorphism

$$\psi_N : U' \longrightarrow \text{Gal}(k(\Lambda_N)/k) =: G_N$$

such that $\ker \psi_N = \mathcal{X}_N$ and hence, $U'/\mathcal{X}_N \cong G_N$.

Let $\bar{\xi} \in U'$. Then $\xi_{P_i} \in U_{P_i} = \{\sum_{j=0}^\infty a_j P_i^j \mid a_j \in R_T/\langle P_i \rangle\}, 1 \leq i \leq r$. Since $k$ is dense in the local field $k_{P_i}$, there exists $Q_i \in R_T$ such that $Q_i \equiv \xi_{P_i} \mod P_i^{a_i}$. By the Chinese Residue Theorem, we have that there exists $C \in R_T$ such that $C \equiv Q_i \mod P_i^{a_i}, 1 \leq i \leq r$ and so $C \equiv \xi_{P_i} \mod P_i^{a_i}, 1 \leq i \leq r$.

Now, if $C_1 \in R_T$ satisfies $C_1 \equiv P_i \mod P_i^{a_i}, 1 \leq i \leq r$, we have that $P_i^{a_i}|C - C_1$ for $1 \leq i \leq r$. It follows that $N|C - C_1$ and thus $C \in R_T$ is unique modulo $N$. On the other hand, $v_{P_i}(\xi_{P_i}) = 0$, so that $P_i \nmid \xi_{P_i}$, and so we obtain that $\gcd(C, N) = 1$. In this way we have that $C \mod N$ defines an element of $G_N = \text{Gal}(k(\Lambda_N)/k)$. 


Given $\sigma \in G_N$, there exists $C \in R_T$ such that $\sigma \lambda_N = \lambda_N^C$, where $\lambda_N$ is a generator of $\Lambda_N$. Let $\vec{\xi} \in U'$ with $\xi_i = C$, $1 \leq i \leq r$ and $\xi_0 = 1$ for all $P \in R_T'$. Therefore $\vec{\xi} \mapsto C \mod N$ and $\psi_N$ es onto. Finally, $\ker \psi_N = \{ \vec{\xi} \in U' \mid \xi_i \equiv 1 \mod P_i^{\alpha_i}, 1 \leq i \leq r \} = X_N$. So we have that $\psi_N$ is an epimorphism and $\ker \psi_N = X_N$.

We will show that $U'/X_N \cong J_k/X_N k^*$. We have the composition

$$U' \xrightarrow{\mu} J_k \xrightarrow{\cdot} J_k/X_N k^*,$$

with $\text{im} \mu = U'X_N k^* / X_N k^*$ and $\ker \mu = U' \cap X_N k^*$.

Now, $X_N \subseteq U'$ so that $X_N \subseteq U' \cap X_N k^*$. Conversely, if $\vec{\xi} \in U' \cap X_N k^*$, the components of $\vec{\xi}$ are given as

$$\xi_i = a \cdot \beta_i, \quad P \in R_T, \quad \beta_i \in X_N, \quad a \in k^*,$$

$$\xi_0 = a \cdot \beta_0, \quad \beta_0 \in (\pi) \times U_1^{(1)}.$$

Since $\xi_i, \beta_i \in U/P$ we have $v_P(\xi_i) = v_P(\beta_i) = 0$ for all $P \in R_T$. It follows that $v_P(a) = 0$ for all $P \in R_T$. Furthermore, since $\deg a = 0$ we have $v_\infty(a) = 0$ and so $a \in \mathbb{F}_q^*$. 

Now $\xi_0, \beta_0 \in (\pi) \times U_1^{(1)} = \ker \phi_\infty$, where $\phi_\infty : k_\infty^* \to \mathbb{F}_q^*$ is the sign function of $k_\infty^*$ defined as $\phi_\infty(\lambda \pi^n u) = \lambda$ where $\lambda \in \mathbb{F}_q^*$, $n \in \mathbb{N}$ and $u \in U_1^{(1)}$. Thus $1 = \phi_\infty(\xi_i) = \phi_\infty(a) \phi_\infty(\beta_0) = \phi_\infty(a)$ and so $a = 1$. It follows that $\vec{\xi} \in X_N$. Therefore $\ker \mu = X_N$ and we obtain a monomorphism $U'/X_N \to J_k/X_N k^*$.

It remains to prove that $\theta$ is surjective. So, we must prove that $J_k = U'X_N k^* = U'k^*$. We have that $U'$ corresponds to the maximum unramified extension at every finite prime. Let $L/k$ be this extension. Since $U_1^{(1)}$ corresponds to the first ramification group, and in this way it corresponds to the wild ramification of $p_\infty$, it follows that in $L/k$ there is at most a ramified prime ($p_\infty$), being tamely ramified and of degree 1. From [16, Proposición 10.4.11], we obtain that $L/k$ is an extension of constants.

Finally, since $1 = \min \{ n \in \mathbb{N} \mid \deg \alpha = n, \alpha \in U' \}$, the field of constants of $L$ is $\mathbb{F}_q$ (see [16, Teorema 17.8.6]) and therefore $L = k$. It follows that $C_k \cong U'$, that is, $J_k/k^* \cong U'$ and thus $J_k = U'k^*$.

\begin{corollary}
With the above notations, we have that for a cyclotomic field $k \subseteq F \subseteq k(\Lambda_N)$, the idèle group corresponding to $F$ is of the form $R_F \times \prod_{Q \in R_T} U_Q \times [(\pi) \times U_1^{(1)}]$ with $R_F$ a group satisfying $\prod_{i=1}^r U_{P_i}^{(\alpha_i)} \subseteq R_F \subseteq \prod_{i=1}^r U_{P_i}$.

\begin{proof}
Let $\Delta$ be the idèle group corresponding to $F$. Thus $\Delta \supseteq X_N$. Now

$$\frac{\prod_{i=1}^r U_{P_i}^{(\alpha_i)}}{\prod_{i=1}^r U_{P_i}} \cong (R_T/(N))^* \cong \text{Gal}(k(\Lambda_N)/k).$$

Therefore $\text{Gal}(k(\Lambda_N)/F) \cong \frac{\theta}{\prod_{i=1}^r U_{P_i}} \cong \text{Gal}(k(\Lambda_N)/k)$ for a group $\Theta \subseteq \prod_{i=1}^r U_{P_i}$.

The group $\Theta$ corresponds to $R_F$. The result follows.
\end{proof}
\end{corollary}
Corollary 3.7. Let $P \in R^+_T$. Then the idèle group corresponding to $E_P$ is of the form
\[
\Delta_P = H_P \times \prod_{Q \neq P} U_Q \times [(\pi) \times U^{(1)}_\infty],
\]
where $U^{(t)}_P \subseteq H_P \subseteq U_P$ for some $t \in \mathbb{N}$.

Proof. Since $E_P$ is cyclotomic and the only finite prime ramified is $P$, there exists $t \in \mathbb{N}$ such that $E_P \subseteq k(\Lambda_P^t)$. The result follows from Corollary 3.6.

For each $P \in R^+_T$, $k_P$ denotes the completion of $k$ at $P$ and $k_\infty$ denotes the completion of $k$ at $p_\infty$. We recall the following result of class field theory.

Theorem 3.8. Let $F$ be a global field and let $R/F$ be the class field extension corresponding to $H$, that is, $H$ is the open subgroup of $C_F$ such that $H = N_{R/F} C_R$ and $\text{Gal}(R/F) \cong C_F/H$. Let $E/F$ be a finite separable extension. Then $ER/E$ is the class field extension corresponding to the subgroup $N^{-1}_{E/F}(H)$ of $C_E$.

\[
\begin{array}{ccc}
E & \xrightarrow{N^{-1}_{E/F}(H)} & ER \\
F & \downarrow & H \\
 & \downarrow & \\
 & R.
\end{array}
\]

Proof. We have that if $E/F$ and $E'/F'$ are two finite abelian extensions of global fields with $F \subseteq F'$ and $E \subseteq E'$ of global fields, and if $\psi_{E/F}$ denotes the Artin map of the extension $E/F$ then we have the following commutative diagram

\[
\begin{array}{ccc}
C_{E'} & \xrightarrow{\psi_{E'/F'}} & \text{Gal}(E'/F') \\
\downarrow N_{F'/F} & & \downarrow \text{rest} \\
C_E & \xrightarrow{\psi_{E/F}} & \text{Gal}(E/F)
\end{array}
\]

where rest denotes the restriction map (see [16, Proposición 17.6.39]).

We apply this result to our situation, that is, we have the commutative diagram

\[
\begin{array}{ccc}
C_E & \xrightarrow{\psi_{ER/E}} & \text{Gal}(ER/E) \\
\downarrow N_{E/F} & & \downarrow \text{rest} \\
C_F & \xrightarrow{\psi_{R/F}} & \text{Gal}(R/F)
\end{array}
\]

Let $\psi_{ER/E} : C_E \rightarrow \text{Gal}(ER/E)$ be the Artin map. The norm group corresponding to $ER/E$ is $\ker \psi_{ER/E}$, that is, $C_E/\ker \psi_{ER/E} \cong \text{Gal}(ER/E)$. Now the restriction map is injective and we have
\[
\text{rest} \circ \psi_{ER/E} = \psi_{R/F} \circ N_{E/F}.
\]
Therefore

\[ \bar{x} \in \ker \psi_{ER/E} \iff \psi_{ER/E}(\bar{x}) = 1 \iff \psi_{ER/E}(\bar{x}) = 1 \iff \psi_{R/F} \circ N_{E/F}(\bar{x}) = 1 \iff N_{E/F}(\bar{x}) \in \ker \psi_{R/F} = H \iff \bar{x} \in N_{E/F}^{-1}(H). \]

\[ \square \]

We apply Theorem 3.8 to the diagram

\[
\begin{array}{c}
K \\
\downarrow \\
KE_P \\
\downarrow \\
E_P
\end{array}
\]

that is, $KE_P$ is the class field of $N_{K/k}^{-1}(\Delta_P)$. Since $E_P$ is maximum in the sense that $P$ is the only finite prime ramified in $E_P/k$ and $KE_P/K$ unramified at every finite prime, we have that $\Delta_P$ satisfies

\[ N_{K/k}^{-1}(\Delta_P) \subseteq \prod_{Q \in R_+^*} \prod_{p \mid Q} U_p \times \prod_{\psi_\infty \mid \mathfrak{p}_\infty} K_{\mathfrak{p}_\infty}^* \subseteq J_K, \]

or, equivalently,

\[ \Delta_P \subseteq N_{K/k} \left( \prod_{Q \in R_+^*} \prod_{p \mid Q} U_p \times \prod_{\psi_\infty \mid \mathfrak{p}_\infty} K_{\mathfrak{p}_\infty}^* \right). \]

Let $\bar{\alpha} \in \prod_{Q \in R_+^*} \prod_{p \mid Q} U_p \times \prod_{\psi_\infty \mid \mathfrak{p}_\infty} K_{\mathfrak{p}_\infty}^* \triangleright \alpha = (\alpha_p)_p$. Then

\[ N_{K/k} \bar{\alpha} = \prod_{Q \in R_+^*} \left( \prod_{p \mid Q} N_{K_{\mathfrak{p}_Q}/kQ}^{\alpha_p} \right) \cdot \prod_{\psi_\infty \mid \mathfrak{p}_\infty} N_{K_{\mathfrak{p}_\infty}/k_{\mathfrak{p}_\infty}}^{\alpha_{\psi_\infty}}. \]

For $Q \neq P$, $Q$ is unramified in $L_P/k$, therefore, for $G | Q$, $K_G/kQ$ is unramified and in particular it is a cyclic extension. Then $N_{K_G/kQ} U_G = U_Q$ (see [16, Teorema 17.2.17]).

For $Q = P$, we have

\[ \prod_{p \mid P} N_{K_{\mathfrak{p}_P}/k_{\mathfrak{p}_P}}^{\alpha_p} = \prod_{j=1}^{m_p} N_{K_{\mathfrak{p}_j}/k_{\mathfrak{p}_j}}^{\alpha_{\mathfrak{p}_j}}, \]

where $\text{con}_{k/K} P = p_1^{c_1} \cdots p_{m_p}^{c_{m_p}}$.

It follows that $\prod_{j=1}^{m_p} N_{K_{\mathfrak{p}_j}/k_{\mathfrak{p}_j}}^{\alpha_{\mathfrak{p}_j}} \in H_P$. In other words, if

\[ S_j := N_{K_{\mathfrak{p}_j}/k_{\mathfrak{p}_j}} U_{\mathfrak{p}_j} \times \prod_{Q \in R_+^*} U_Q \times [\pi \times U_{\psi_\infty}^{(1)}] \subseteq U_P \times \prod_{Q \in R_+^*} U_Q \times [\pi \times U_{\psi_\infty}^{(1)}], \]

we have

\[ \Delta_P = \prod_{j=1}^{m_p} S_j \quad \text{and} \quad H_P = \prod_{j=1}^{m_p} N_{K_{\mathfrak{p}_j}/k_{\mathfrak{p}_j}} U_{\mathfrak{p}_j}. \]

Now, if $S_j$ is the norm group of the field $R_j \subseteq k(\Delta_{P_j})_m$ for some $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ and $c_j \in \mathbb{N}$, then $\prod_{j=1}^{m_p} S_j$ is the norm group of $\cap_{j=1}^{m_p} R_j$. 

It follows that \([C_k : k^*S_j] = [U_P : N_{K_{P_j/kP}} U_P]\) and \(\text{Gal}(R_j/k) \cong C_k/k^*S_j\).

Therefore \([R_j : k] = [C_k : k^*S_j] = [U_P : N_{K_{P_j/kP}} U_P]\). Finally, we have

\[
E_P = \bigcap_{j=1}^{m_P} R_j, \quad [E_P : k] = \left[ \bigcap_{j=1}^{m_P} R_j : k \right] = \left[ U_P : \prod_{j=1}^{m_P} N_{K_{P_j/kP}} U_P \right].
\]

We have proved our main result.

**Theorem 3.9.** Let \(K/k\) be a finite and separable extension, where \(k = \mathbb{F}_q(T)\). With the notations as above, let \(K_{\text{get}} = KL\). Then \(L = \prod_{P \in \mathbb{P}_T} L_P\) where \(L_P = E_P S, S = L \cap M\) and \(k \subseteq E_P \subseteq k(\Lambda_P)\) corresponds to \(\prod_{j=1}^{m_P} N_{K_{P_j/kP}} U_P\). In particular

\[
[E_P : k] = \left[ U_P : \prod_{j=1}^{m_P} N_{K_{P_j/kP}} U_P \right],
\]

where \(\text{con}_{k/K} P = p_1^{e_1} \cdots p_{m_P}^{e_{m_P}}\).

The tamely ramified part of \(L_P/K\) is given by

\[
e^{\text{tame}}(P) = \gcd(e_1, \ldots, e_{m_P}, q^d - 1),
\]

with \(d_P = \text{deg}_k P\). \(\Box\)

### 3.1. The field \(S\)

To study \(S\), recall that for a finite extension \(K/k\), the genus field is \(K_{\text{ge}} = KF\) where \(F/k\) is the maximum abelian extension contained in the Hilbert class field and the extended genus field is \(K_{\text{get}} = KL\), where \(L\) satisfies \(L_{\text{get}} = L\), \(L/k\) is abelian and \(L\) is the maximum with respect to this property. We have \(F_{\text{ge}} = F, L = F_{\text{get}}\) and \(L_{\text{get}} = L\). Let \(L \subseteq \cap k(\Lambda_N)_m\) with \((m, N, n)\) the conductor of \(L\). Then \(M = L_n k_m\) and \(S = L \cap M\).

**Proposition 3.10.** We have that \(L/F\) is totally ramified at the infinite primes, unramified at the finite primes and \([L : F]/q - 1\). In particular, \(L/F\) is tamely ramified.

**Proof.** We have that \(F/k\) is abelian. Let \(F \subseteq \cap k(\Lambda_N)_m\) and \(E = FM \cap k(\Lambda_N)\). Then \(E_{\text{get}} M = F_{\text{get}} M = FM = EM\) (see [2]) and therefore \(E_{\text{get}} = E\).

Since \(e_{\infty}(E_{\text{get}}/E_{\text{ge}}) = q - 1\) and \(e_{\infty}(M/k) = q^n\), it follows that \(e_{\infty}(E_{\text{get}} M/E_{\text{ge}} M) = e_{\infty}(E_{\text{get}}/E_{\text{ge}}) = [E_{\text{get}} : E_{\text{ge}}]\).

Hence, \(e_{\infty}(F_{\text{get}}/F_{\text{ge}}) = e_{\infty}(F_{\text{get}} M/F_{\text{ge}} M) = e_{\infty}(E_{\text{get}} M/E_{\text{ge}} M) = [E_{\text{get}} : E_{\text{ge}}] = [E_{\text{get}} F : E_{\text{ge}} F] = [E_{\text{get}} : E_{\text{ge}}]\). So, the infinite primes are total and tamely ramified in \(L = F_{\text{get}}/F_{\text{ge}} = F\).

On the other hand, \(E_{\text{get}}/E_{\text{ge}}\) is unramified at the finite primes, thus \(E_{\text{get}} F = F_{\text{get}} = L/F = F_{\text{ge}} = E_{\text{ge}} F\) is unramified at the finite primes. \(\Box\)
Proposition 3.11. We have
\[ e_{\infty}^\text{wild}(L|k) = e_{\infty}^\text{wild}(F|k) = e_{\infty}^\text{wild}(S|k) = e_{\infty}(S|k). \]
Furthermore, \( S = L \cap M = F \cap M \).

Proof. We have \( e_{\infty}^\text{wild}(L|k) = e_{\infty}^\text{wild}(L|K) = e_{\infty}^\text{wild}(F|k) = e_{\infty}(F|k) \).
By the definition of \( S \), we have \( e_{\infty}^\text{wild}(S|k) = e_{\infty}(S|k) \) since \( e_{\infty}(S|k)|q^n \). Now, \( L = E_{\text{Gal}}S, E_{\text{Gal}} \cap S = k \) and \( e_{\infty}^\text{wild}(E_{\text{Gal}}|k) = 1 \). Therefore
\[ e_{\infty}^\text{wild}(L|k) = e_{\infty}^\text{wild}(E_{\text{Gal}}S) = e_{\infty}^\text{wild}(S|k) = e_{\infty}(S|k) \]
since \( e_{\infty}^\text{wild}(E_{\text{Gal}}S|k) = 1 \).
We have \( F \cap M \subset L \cap M = S \) and \( F \cap (L \cap M) = F \cap M \).

\[ \begin{array}{c}
S = L \cap M \quad F(L \cap M) \quad L \\
F \cap M \quad F
\end{array} \]

It follows that \( [L \cap M : F \cap M] = [F(L \cap M) : F][F : L]|q - 1 \). We have that \( L/F \) is totally ramified at the infinite primes and therefore \( F(L \cap M)/F \) is also fully ramified at the infinite primes. It follows that \( S = L \cap M/F \cap M \) is fully ramified at the infinite primes (see [16, Corolario 10.4.15]). Thus, \( [S : F \cap M]|q^n \) and \( [S : F \cap M]|q - 1 \) so that \( [S : F \cap M] = 1 \) and \( F \cap M = L \cap M \). \( \square \)

Proposition 3.12. The field of constants of \( S \), of \( L \) and of \( F \) is the same.

Proof. If \( \mathbb{F}_{q^n} \) is the field of constants of \( L \) then \( \mathbb{F}_{q^n} \subset S = L \cap M \) and since \( S \subset F \subset L \), the result follows. \( \square \)

Proposition 3.13. Let \( \text{con}_{K/k} \mathfrak{p}_{\infty} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r} \) and let \( t_i = \deg_K(\mathfrak{P}_i) \). Then the field of constants of \( K_{\text{Gal}} \) is \( \mathbb{F}_{q^n} \) where \( t_0 = \gcd(t_1, \ldots, t_r) \).

Proof. See [13]. \( \square \)

Corollary 3.14. The field of constants of \( S, L \) and \( F \) is \( \mathbb{F}_{q^n} \). \( \square \)

Now we consider a finite abelian extension \( J/k \) such that \( KJ/K \) is unramified and the infinite primes decompose fully. Let \( \mathfrak{P}|\mathfrak{p}_{\infty} \) be a prime divisor of \( KJ, \mathfrak{P} \subset K = \mathfrak{P}_i \) for some \( 1 \leq i \leq r \) and \( \mathfrak{P} \cap J = \mathfrak{Q} \). Taking the completions we have
\[ K_{\mathfrak{P}_i} \supseteq (KJ)_{\mathfrak{Q}} \]
\[ k_{\infty} \supseteq H_{i} \supseteq J_{\mathfrak{Q}} \]
\[ \begin{array}{c}
K_{\mathfrak{P}_i} \supseteq (KJ)_{\mathfrak{Q}} \supseteq k_{\infty} \\
H_{i} \supseteq J_{\mathfrak{Q}}
\end{array} \]
Let \( H_i := N_{J_{\mathfrak{Q}}/k_{\infty}}J_{\mathfrak{Q}}^{\mathfrak{P}_i} \), that is, \( H_i \) is the norm group of \( J_{\mathfrak{Q}} \). Therefore, the norm group corresponding to \( (KJ)_{\mathfrak{Q}} = K_{\mathfrak{P}_i} \) is \( N_{K_{\mathfrak{P}_i}/k_{\infty}}^t(H_i) \) (see Theorem 3.8). Hence
\[ N_{K_{\mathfrak{P}_i}/k_{\infty}}^t(H_i) = K_{\mathfrak{P}_i}^{*} \] That is, \( H_i = N_{K_{\mathfrak{P}_i}/k_{\infty}}(K_{\mathfrak{P}_i}^{*}) \). The maximum global abelian extension \( J/k \) satisfying that \( KJ/K \) is unramified and the infinite primes decompose fully, satisfies, locally at \( \infty \), that its norm gorup is
\[ \prod_{i=1}^{r} H_i = \prod_{i=1}^{r} N_{K_{\mathfrak{P}_i}/k_{\infty}}(K_{\mathfrak{P}_i}^{*}). \]
In this way, if $R/k_\infty$ is the maximum abelian extension with $(KR)_\infty = K_{P_1}$ for some $i$. Thus $R$ corresponds to $\prod_{i=1}^{r} H_i$, that is, $\text{Gal}(R/k_\infty) = k_\infty^* / (\prod_{i=1}^{r} H_i)$ and $[R : k_\infty] = [k_\infty^* : \prod_{i=1}^{r} H_i]$.

Let $[R : k_\infty] = p^a \alpha$ with $\alpha \in \mathbb{N} \cup \{0\}$ and $p \nmid \alpha$. Since $S$ is the maximum abelian extension of $k$ such that the only ramified prime is $p_\infty$, it is fully ramified and $S \subseteq L$, and since $f_\infty(L | S) = 1$, it follows that if $p_\infty$ is the only prime in $S$ dividing $p_\infty$ (recall that the number of primes in $S$ that lie above $p_\infty$ is $h_\infty(S | k) = 1$), then $[S p_\infty : k_\infty] = [S : k] = p^a$. In particular, the norm group corresponding to $S_\infty = S p_\infty$ in $k_\infty$ is the group $S \supseteq \prod_{i=1}^{r} H_i$, which is the minimum such that $[k_\infty^* : S] = p^a$ is a $p$-group.

The conductor $p_\infty^{m_0}$ of $S_\infty$ is such that $n_0$ is the minimum nonnegative integer such that $U_{\infty}^{(n_0)} \subseteq S$. The conductor of constants $m_0$ of $S$, that is, $m_0$ is the minimum natural number such that $S \subseteq k_{m_0} L_{n_0}$ is given as follows (see [2]). Let $t = f_\infty(S | k)$, $d^t = f_\infty(R^t | S)$ where $R^t = S_{m_0} \cap L_{n_0}$ and $d^t = e_\infty(S | F^t)$ where $F^t = S \cap k_{n_0} k(\Lambda_1) = S \cap L_{n_0}$. Therefore

$$m_0 = f_\infty(S | k) e_\infty(S | S \cap L_{n_0}).$$

**Proposition 3.15.** Let $f_\infty(S | k) = t$. Then $F_{q^t}$ is the field of constants of $S$. That is, $t = t_0$.

**Proof.** We have $F_{q^t_0}(T) = k_{t_0} \subseteq S$. Let $m_0, n_0$ be minimum such that $S \subseteq k_{m_0} L_{n_0}$. Then $S \cap k_{m_0} = k_{t_0}$.

We have $S k_{t_0} L_{n_0} = S L_{n_0}$, $k_{t_0} L_{n_0} \subseteq S L_{n_0} \subseteq k_{m_0} L_{n_0}$. Let $S L_{n_0} \cap k_{m_0} = k_{m'}$. From the Galois correspondence we obtain that $k_{m'} k_{t_0} L_{n_0} = k_{m'} L_{n_0} = S L_{n_0} \supseteq S$. It follows that $m' \geq m_0$. Hence $m' = m_0$ and $S L_{n_0} = k_{m_0} L_{n_0}$.

Now $e_\infty(k_{m_0} L_{n_0} | k_{t_0}) = q^n$ and $k_{m_0} L_{n_0} \subseteq k_{m_0} S \subseteq k_{m_0} L_{n_0}$. Then

$$e_\infty(k_{m_0} L_{n_0} | S) = e_\infty(k_{m_0} L_{n_0} L_{k_{m_0}} S) = \frac{q_{m_0}^{n_0}}{[k_{m_0} L_{n_0} S : k_{m_0}]} = \frac{q_{m_0}^{n_0}}{[S : S \cap k_{m_0}]} = \frac{q_{m_0}^{n_0}}{[S : k_{t_0}]}.$$

Thus

$$e_\infty(S | k_{t_0}) = \frac{e_\infty(k_{m_0} L_{n_0} | k_{t_0})}{e_\infty(k_{m_0} L_{n_0} | S)} = \frac{q_{m_0}^{n_0}}{q_{m_0}^{n_0} / [S : k_{t_0}]} = [S : k_{t_0}].$$

It follows that $S/k_{t_0}$ is fully ramified at the infinite prime. In particular, $f_\infty(S | k_{t_0}) = 1$ so that $f_\infty(S | k) = f_\infty(S | k_{t_0}) f_\infty(k_{t_0} | k) = f_\infty(k_{t_0} | k) = t_0$.

We collect the above discussion in the following theorem.
Theorem 3.16. Let \( S = L \cap M \). Let \( \text{con}_{k/K} P_\infty = P_1^{e_1} \cdots P_r^{e_r} \) and let \( t_i = \deg_K(P_i) \), \( 1 \leq i \leq r \). Then the field of constants of \( S \) is \( \mathbb{F}_{q_0} \).

Let \( n_0 \) be the minimum nonnegative integer with \( U^{(n_0)}_\infty \subseteq S \) where \( S \supseteq \prod_{i=1}^r H_i = K \cap K_{P_i} \) and \( S \) is the minimum such that \( [k_\infty : S] = p^\infty \) is a \( p \)-group. Then the conductor of constants of \( S \) is \( m_0 = f_\infty(S|k) e_\infty(S|S \cap L_{n_0}) \) and \( S \) is the local norm group corresponding to \( S \). In particular \( \mathbb{F}_{q_0} \subseteq S \subseteq k_0 L_{n_0} \).

\( \square \)

4. NUMBER FIELDS

The results of Section 3 can be developed in the number field case. In fact, for a number field, the extended genus field is more transparent than in the function field case.

Definition 4.1. Let \( K \) be an arbitrary number field, that is, a finite extension of the rational field \( \mathbb{Q} \). Let \( K_{H^+} \) be the extended or narrow Hilbert class field of \( K \), that is, \( K_{H^+} \) is the maximum abelian extension of \( K \) unramified at every finite prime of \( K \). We define the extended genus field \( K_{gex} \) of \( K \) as the maximum extension of \( K \) contained in \( K_{H^+} \) such that it is of the form \( KL \) with \( L/\mathbb{Q} \) abelian.

Equivalently, if \( L \) is the maximum abelian extension of \( \mathbb{Q} \) contained in \( K_{H^+} \), the extended genus field of \( K \) is \( K_{gex} = KL \).

Again, we stress that we choose \( L \) maximum.

As in the function field case, we have

\[
\text{Proposition 4.2.} \quad \text{Let } K/\mathbb{Q} \text{ be a finite abelian extension and let } X \text{ be the group of Dirichlet characters corresponding to } K. \text{ Then } Y := \prod_{p \text{ prime}} X_p \text{ is the group of Dirichlet characters corresponding to } K_{gex}. \quad \square
\]

In particular, if \( K/\mathbb{Q} \) is any finite extension and \( K_{gex} = KL \), then \( L = L_{gex} \). We want to describe \( K_{gex} \) for a general number field \( K \). Let \( K/\mathbb{Q} \) be a finite extension. Let \( p \) be a prime in \( \mathbb{Q} \) and let

\[
\text{con}_{\mathbb{Q}/K} P = p_1^{e_1} \cdots p_r^{e_r},
\]

that is, \( e_i = e_{K/\mathbb{Q}}(p_i|p) \), \( 1 \leq i \leq r \). Let \( K_{p_1}, \ldots, K_{p_r} \) be the completions of \( K \) at the primes above \( p \). Let \( K_{gex} = KL \) with \( L/\mathbb{Q} \) the maximum abelian extension such that \( K \subseteq K_{gex} \subseteq K_{H^+} \).

\[
\begin{array}{c}
\text{Q} \\
\text{K} \\
\text{K_{gex} = KL} \\
\text{K_{H^+}} \\
\text{L}
\end{array}
\]

Since \( L = L_{gex} \), we let \( L_p \) be the field corresponding to \( X_p \). We have that \( L = \prod_{p \text{ prime}} L_p \) and \( L_p \cap L_q = \mathbb{Q} \) for any primes \( p, q \) such that \( p \neq q \). We have that \( L_p \) is the maximum abelian extension of \( \mathbb{Q} \) with \( p \) the only possible finite prime ramified and such that \( KL_p/K \) is unramified at every finite prime.
Let \( p \) be a fixed prime and let \( L_p \subseteq \mathbb{Q}(\zeta_{p^n}) \). For any \( n \in \mathbb{N} \), the idèle group corresponding to \( \mathbb{Q}(\zeta_n) \) is

\[
\mathcal{X}_n = \prod_{i=1}^{4} U_{p_i}^{(\alpha_i)} \times \prod_{q \text{ prime}} U_q \times \mathbb{R}^+,
\]

where \( n = \prod_{i=1}^{t} p_i^{\alpha_i} \). As in the case of function fields, it follows that the idèle group corresponding to \( L_p \) is of the form

\[
\Delta_p = H_p \times \prod_{q \text{ prime}} U_q \times \mathbb{R}^+,
\]

where \( U_p^{(p^n)} \subseteq H_p \subseteq U_p \).

We have \( e_{K_p, \mathbb{Q}_p} = e_{K, \mathbb{Q}(p, p)} = e \). The extension \( L_p/\mathbb{Q} \) is totally ramified at \( p \) and even we could mix up \( L_p \) with the completion of \( L \) at \( p \). We have, with both meanings of \( L_p \), that \( [L_p : \mathbb{Q}] = [L_p : \mathbb{Q}_p] = e_p(\Delta_p) \).

By Theorem 3.8 we have that the norm group of the abelian extension \( K L_p/K \) is \( N_{K/\mathbb{Q}}(\Delta_p) \). Since \( L_p \) is maximum, we want \( \Delta_p \) to be such that (see [16, Corolario 17.6.47])

\[
N_{K/\mathbb{Q}}^{-1}(\Delta_p) \subseteq \prod_{p \text{ finite}} U_p \times \prod_{p \text{ real}} K_p^* = \prod_{p \text{ finite}} U_p \times \prod_{p \text{ real}} \mathbb{R}^* \subseteq J_K,
\]

or

\[
\Delta_p \subseteq N_{K/\mathbb{Q}} \left( \prod_{p \text{ finite}} U_p \times \prod_{p \text{ real}} \mathbb{R}^* \right).
\]

Let \( \vec{\alpha} \in \prod_{p \text{ finite}} U_p \times \prod_{p \text{ real}} \mathbb{R}^* \subseteq J_K \), \( \vec{\alpha} = (\alpha_p)_p \). Then

\[
N_{K/\mathbb{Q}} \vec{\alpha} = \prod_{q \text{ finite}} \left( \prod_{p | q} N_{K_p/\mathbb{Q}_p} \alpha_p \right) \left( \prod_{p \text{ real}} N_{\mathbb{R}/\mathbb{R}} \alpha_p \right).
\]

As in the case of function fields we obtain that

\[
H_p = \prod_{p | \mathbb{Q}} N_{K_p/\mathbb{Q}_p} U_p \quad \text{ and } \quad \Delta_p = \prod_{p | \mathbb{Q}} N_{K_p/\mathbb{Q}_p} U_p \times \prod_{q \text{ prime}} U_q \times \mathbb{R}^+.
\]

In other words, let

\[
S_i = N_{K_p/\mathbb{Q}_p} U_p \times \prod_{q \text{ prime}} U_q \times \mathbb{R}^+ \subseteq U_p \times \prod_{q \text{ prime}} U_q \times \mathbb{R}^+.
\]

We have

\[
\Delta_p = \prod_{i=1}^{r} S_i.
\]

Now \( S_i \) corresponds to a field \( R_i \subseteq \mathbb{Q}(\zeta_{p^n}) \) and from [16, Teorema 17.6.49] it follows that \( \prod_{i=1}^{r} S_i \) corresponds to \( \bigcap_{i=1}^{r} R_i \). Thus \( L_p = \bigcap_{i=1}^{r} R_i \). Furthermore, since \( R_i \) corresponds to \( S_i \), we have

\[
[C_Q : \mathbb{Q}^* S_i] = [R_i : \mathbb{Q}] \quad \text{and} \quad \text{Gal}(R_i/\mathbb{Q}) \cong C_Q/\mathbb{Q}^* S_i.
\]
Corollary 4.4. With the conditions of Proposition 4.3, we have

\[ [R_i : \mathbb{Q}] = [C_i : \mathbb{Q}^* S_i] = \left[ U_p : N_{K_{p^i}} / \mathbb{Q}_p U_p \right] \]

and

\[ [L_p : \mathbb{Q}] = \left[ \bigcap_{i=1}^{r} R_i : \mathbb{Q} \right] = \left[ U_p : \prod_{i=1}^{r} N_{K_{p^i}} / \mathbb{Q}_p U_p \right]. \]

When \( p \geq 3 \), we have that \( \mathbb{Q}(\zeta_{p^n}) \) is cyclic for every \( m \in \mathbb{N} \), however, when \( p = 2, \mathbb{Q}(\zeta_{2^n}) \) is not cyclic for \( m \geq 3 \). We study the two cases.

Let \( G = \langle \sigma \rangle \cong C_n \) be a finite cyclic group of order \( n \in \mathbb{N} \) and let \( H_i = \langle \sigma^{j_i} \rangle < G \) where \( j_i / n, i = 1, 2 \). Let \( H_1 \cap H_2 = \langle \sigma^a \rangle \) and \( H_1 H_2 = \langle \sigma^b \rangle \) with \( s, t|n \).

We have \( \sigma^a \in H_i, i = 1, 2 \) so that there exist \( a_i \in \mathbb{Z} \) such that \( \sigma^a = \sigma^{j_i a_i}, i = 1, 2 \). Therefore \( t \equiv j_i a_i \mod n, i = 1, 2 \), that is, \( t = j_i a_i + t n, i = 1, 2 \). Hence \( j_i|t, i = 1, 2 \) so \( \text{lcm}(j_1, j_2)|t \). Let \( u = \text{lcm}(j_1, j_2), j_i|u \). Then \( \sigma^u = \sigma^{j_i b_i} \in H_i, i = 1, 2 \). Thus \( \sigma^u \in H_1 \cap H_2 = \langle \sigma^l \rangle \) and \( \sigma^u = \sigma^v \) for some \( c \) and \( u = tc + ln \). It follows that \( t|u = \text{lcm}(j_1, j_2) \). Therefore \( t = u \).

In other words, \( H_1 \cap H_2 = \langle \sigma^{\text{lcm}(j_1, j_2)} \rangle \).

Now, \( H_1 H_2 = \langle \sigma^b \rangle, \frac{n}{s} = [H_1 H_2] = \frac{|H_1||H_2|}{|H_1 \cap H_2|} = \frac{n}{j_1 j_2} \). Therefore \( st = j_1 j_2 \) and \( j_1 j_2 = \gcd(j_1, j_2) \text{lcm}(j_1, j_2) = \gcd(j_1, j_2) t = st \). Hence \( s = \gcd(j_1, j_2) \).

In short, we have

Proposition 4.3. Let \( G = \langle \sigma \rangle \) be a cyclic group of order \( n \) and let \( H_i = \langle \sigma^{j_i} \rangle \) with \( j_i | n, i = 1, 2 \) be two subgroups of \( G \). Then

\[ H_1 \cap H_2 = \langle \sigma^{\text{lcm}(j_1, j_2)} \rangle, \quad H_1 H_2 = \langle \sigma^{\gcd(j_1, j_2)} \rangle. \]

\( \square \)

Corollary 4.4. With the conditions of Proposition 4.3, we have

\[ |H_1 \cap H_2| = \frac{|G|}{\text{lcm}(j_1, j_2)}, \quad |G : H_1 \cap H_2| = \text{lcm}(j_1, j_2) = \text{lcm} \left[ \frac{|G|}{|H_1|}, \frac{|G|}{|H_2|} \right], \quad |H_1 H_2| = \frac{|G|}{\gcd(j_1, j_2)}, \]

\[ |G : H_1 H_2| = \frac{|G|}{|H_1 H_2|} = \gcd(j_1, j_2) = \gcd \left( \left[ G : H_1 \right], \left[ G : H_2 \right] \right). \]

\( \square \)

Corollary 4.5. If \( p > 2 \) is a prime number and \( H_i < \mathbb{Z}_p^* \), \( i = 1, 2 \) are two subgroups of finite index, then \( \left[ \mathbb{Z}_p^* : H_1, H_2 \right] = \gcd \left( \left[ \mathbb{Z}_p^* : H_1 \right], \left[ \mathbb{Z}_p^* : H_2 \right] \right) \).

Proof. We have that \( \mathbb{Z}_p^* \cong C_{p-1} \times \mathbb{Z}_p \) where \( C_{p-1} \) is the cyclic group of order \( p - 1 \). Let \( H_i = H_i' \times p^{n_i} \mathbb{Z}_p \) where \( H_i' \) is the torsion of \( H_i, i = 1, 2 \). Then \( H_1 H_2 = H_1' H_2' \times p^{\min(n_1, n_2)} \mathbb{Z}_p \). Therefore

\[ \left[ \mathbb{Z}_p^* : H_1 H_2 \right] = [C_{p-1} : H_1' H_2'] p^{\min(n_1, n_2)} = \gcd \left( \left[ C_{p-1} : H_1' \right], \left[ C_{p-1} : H_2' \right] \right) p^{\min(n_1, n_2)} = \gcd \left( \left[ \mathbb{Z}_p^* : H_1 \right], \left[ \mathbb{Z}_p^* : H_2 \right] \right). \]

\( \square \)
We apply the previous discussion to the case $p > 2$.

**Proposition 4.6.** If $p > 2$, $\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}$ is a cyclic extension and $L_p/\mathbb{Q}$ is a cyclic extension. For $F_1, F_2$ contained in $\mathbb{Q}(\zeta_{p^m})$, we have

$$[F_1 \cap F_2 : \mathbb{Q}] = \gcd ([F_1 : \mathbb{Q}], [F_2 : \mathbb{Q}]).$$

**Proof.** We consider $F_1 F_2/\mathbb{Q}$ which is cyclic since $\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}$ is a cyclic extension. We have

![Diagram of cyclic extensions]

Let $a = [F_1 \cap F_2 : \mathbb{Q}]$, $b = [F_1 : \mathbb{Q}]$ and $c = [F_2 : \mathbb{Q}]$. We have that $a|b$ and $a|c$ so that $a|\gcd(b, c)$. Now, since $\gcd(b, c)|b$ and $\gcd(b, c)|c$, there exists a unique field $F_0$ satisfying $[F_0 : \mathbb{Q}] = \gcd(b, c)$, $F_0 \subseteq F_1$ and $F_0 \subseteq F_2$. Hence $F_0 \subseteq F_1 \cap F_2$. This implies $\gcd(b, c) = [F_0 : \mathbb{Q}][F_1 \cap F_2 : \mathbb{Q}] = a$. Thus $a = \gcd(b, c)$. □

**Corollary 4.7.** With the conditions of Proposition 4.6, if $p > 2$ and $F_1, \ldots, F_t \subseteq \mathbb{Q}(\zeta_{p^m})$, we have

$$\left[ \bigcap_{i=1}^t F_i : \mathbb{Q} \right] = \gcd_{1 \leq i \leq t} ([F_i : \mathbb{Q}]).$$

**Proof.** Use induction. □

**Remark 4.8.** If $p = 2$, Proposition 4.6 is not longer true. For instance, if $F_1 = \mathbb{Q}(\sqrt{2})$, $F_2 = \mathbb{Q}(i)$, then $[F_1 : \mathbb{Q}] = [F_2 : \mathbb{Q}] = 2$ and since $F_1 \cap F_2 = \mathbb{Q}$, we have $[F_1 \cap F_2 : \mathbb{Q}] = 1$.

**Remark 4.9.** Since

$$\bigcap_{i=1}^r R_i : \mathbb{Q} = [L_p : \mathbb{Q}] = \mathbb{Q}_p^* : \prod_{i=1}^r N_{(R_i)_{p^i}}(R_i)_{p^i}^* = \left[ U_p : \prod_{i=1}^r \left. N_{(R_i)_{p^i}}/\mathbb{Q}_p \right. U_{p^i} \right],$$

for $p > 2$, we have

$$[L_p : \mathbb{Q}] = \gcd_{1 \leq i \leq r} ([R_i : \mathbb{Q}]) = \gcd_{1 \leq i \leq r} \left( [U_p : \prod_{i=1}^r N_{(R_i)_{p^i}}/\mathbb{Q}_p U_{p^i}] \right).$$

The main result on number fields is the following.

**Theorem 4.10.** Let $K/\mathbb{Q}$ be a finite extension. With the above notations, we have $K_{gf} = KL$ where $L = \prod_{p \text{finite}} L_p$ and $L_p$ is a subfield of $\mathbb{Q}(\zeta_{p^m})$ satisfying

$$[L_p : \mathbb{Q}] = \left[ U_p : \prod_{i=1}^r N_{K_{p^i}}/\mathbb{Q}_p U_{p^i} \right],$$

where $\con_{\mathbb{Q}/K} p = p_1^{e_1} \cdots p_r^{e_r}$.

Furthermore, if $p > 2$,

$$[L_p : \mathbb{Q}] = \gcd_{1 \leq i \leq r} \left( [U_p : \left. N_{K_{p^i}}/\mathbb{Q}_p \right. U_{p^i}] \right).$$
$L_p$ is determined by its degree $[L_p : \mathbb{Q}]$ and $L_p$ is the class field of

$$
\prod_{i=1}^{r} \mathbb{N}_{K_p/Q_p} U_p \times \prod_{q \text{ prime}, q \neq p} U_q \times \mathbb{R}^+.
$$

**The tame ramification degree of the extension** $[L_p : \mathbb{Q}]$ is

$$
eq \text{tame} \equiv \gcd(e_1, \ldots, e_r, p - 1).
$$

**Proof.** It remains to find $e^\text{tame}$. Note that necessarily, $p \geq 3$. Let $L_p'$ be the subfield of $L_p$ of degree $b_p$ where $[L_p : \mathbb{Q}] = p^\alpha b_p$ and $\gcd(p, b_p) = 1$.

For any $F \subseteq \mathbb{Q}(\zeta_{p^{m_p}})$ with $[F : \mathbb{Q}] = d$ and $\gcd(p, d) = 1$, $F/\mathbb{Q}$ is tamely ramified. Assume that $KF/K$ is unramified at $p$.

$$
\begin{array}{c}
K \\
\| \\
\mathbb{Q} \\
\| \\
F
\end{array}
$$

By Abhyankar Lemma, if $\mathfrak{p}$ is a prime in $KF$ with $\mathfrak{p} \cap \mathbb{Q} = (p)$, $\mathfrak{p} \cap K = \mathfrak{p}_i$, $\mathbb{Q} = \mathfrak{p} \cap F$, then

$$
eq (\mathfrak{p}_i|p) = \text{lcm}[e(\mathfrak{p}_i|p), e(\mathfrak{Q}|p)] = \text{lcm}[e_i, d].
$$

Therefore $e(\mathfrak{p}_i|p) = \frac{e(\mathfrak{Q}|p)}{e_i}$, that is, $\mathfrak{p}$ is unramified in $KF/K$ if and only if $e(\mathfrak{Q}|p) = e_i$ if and only if $d|e_i$. Hence, $KF/K$ is unramified at every finite prime, if and only if $d|e_i$ for $1 \leq i \leq r$ if and only if $d|\gcd(e_1, \ldots, e_r)$. Since $d|p - 1$, this is equivalent to $d|\gcd(e_1, \ldots, e_r, p - 1)$. It follows that $b_p = \gcd(e_1, \ldots, e_r, p - 1)$. 

**Remark 4.11.** Theorem 4.10 was proved by M. Bhaskaran in [4] and by X. Zhang in [18].

4.1. **Remarks on $L_2$.** For any finite extension $K/\mathbb{Q}$, we have that if $K_{\text{gen}} = KL$ with $L/\mathbb{Q}$ the maximum abelian extension contained in $K_{\text{gen}}$, we have proved that if $L = \prod_{\mathfrak{p} \text{ prime}} L_{\mathfrak{p}}$, then for $p \geq 3$, $L_p$ is completely determined by

$$
[L_p : \mathbb{Q}] = \left[U_p : \prod_{\mathfrak{p}|p} \mathbb{N}_{K_p/Q_p} U_p \right] = \gcd[U_p : \mathbb{N}_{K_p/Q_p} U_p].
$$

This is not so for $p = 2$. We want to study $L_2$.

Let $[L_2 : \mathbb{Q}] = 2^a$, $a \geq 1$. For $a \geq 2$, there are three possible $L_2$, namely

$$
\mathbb{Q}(\zeta_{2a+1}), \quad \mathbb{Q}(\zeta_{2a+2})^+ = \mathbb{Q}(\zeta_{2a+2} + \zeta_{2a+2}^{-1}) \quad \text{and} \quad \mathbb{Q}(\zeta_{2a+2})^- := \mathbb{Q}(\zeta_{2a+2} - \zeta_{2a+2}^{-1}),
$$

see [16, §5.3.1].

If $L_2$ is real, then $L_2 = \mathbb{Q}(\zeta_{2a+2})^+$. If $L_2$ has conductor $2^{a+1}$ then $L_2 = \mathbb{Q}(\zeta_{2a+1})$. In other words, $L_2$ can be determined by means of its conductor and whether it is real or not.
If $K(\zeta_{2n+1})/K$ is unramified, we have $L_2 = \mathbb{Q}(\zeta_{2n+1})$. In any case $\mathbb{Q}(\zeta_{2n+1})^+ \subseteq L_2$, and therefore $K\mathbb{Q}(\zeta_{2n+1})^+/K$ is unramified.

\[ \mathbb{Q}(\zeta_{2n+1})^+ \overset{J}{\longrightarrow} \mathbb{Q}(\zeta_{2n+1}) \]

\[ \mathbb{Q}(\zeta_{2n+1})^- \]

\[ \mathbb{Q}(\zeta_{2n+1})^+ \quad \text{Gal} \left( \mathbb{Q}(\zeta_{2n+1})/\mathbb{Q}(\zeta_{2n+1})^+ \right) \cong C_2 \times C_2 \]

\[ \mathbb{Q}(\zeta_{2n+1}) \]

\[ \mathbb{Q}(\zeta_{4}) \]

We need to determine the group of idèles corresponding to each extension: $L_2 \in \{ \mathbb{Q}(\zeta_{2n+1}), \mathbb{Q}(\zeta_{2n+1})^+, \mathbb{Q}(\zeta_{2n+1})^- \}$.

Recall that for a local field $K$ we have $K^* \cong \mathbb{F}_q^* \times U_p^{(1)} \times (\pi)$, where $\pi$ is a uniformizer element, $v_p(\pi) = 1$, $U_p$ are the units of $K^*$, $U_p^{(1)}$ are the units modulo 1, that is, $U_p^{(1)} = \{ \xi \in U_p \mid \xi - 1 \in (\pi) \} = 1 + \pi \mathcal{O}_K = 1 + p$, and $U_p^{(1)} \times \mathbb{F}_q^* = U_p$ where $\mathbb{F}_p$ is the residue field.

In the particular case of $K = \mathbb{Q}_p$, $q = p$, $\mathbb{F}_p^* \cong C_{p-1} = \mathbb{Z}/(p-1)\mathbb{Z}$ and $U_p = \left\{ \sum_{i=0}^{\infty} a_i p^i \mid a_0 \neq 0, a_i \in \{0, 1, \ldots, p-1\} \text{ for all } i \right\} \cong \mathbb{Z}_p^*$, where $\mathbb{Z}_p$ denotes the ring of $p$–adic integers and $\mathbb{Z}_p^*$ is the multiplicative group of $\mathbb{Z}_p$.

We have

**Proposition 4.12.** (1) If $p > 2$, $\mathbb{Z}_p^* \cong C_{p-1} \times \mathbb{Z}_p$ as groups.

(2) If $p = 2$, $1 + 2\mathbb{Z}_2 \cong \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$ and $1 + 4\mathbb{Z}_2 \cong \mathbb{Z}_2$. In particular,

\[ U_2 = U_2^{(1)} = \mathbb{Z}_2^* \cong 1 + 2\mathbb{Z} \cong \{\pm 1\} \times (1 + 4\mathbb{Z}_2) \cong \{\pm 1\} \times \mathbb{Z}_2. \]

\[ \square \]

We are going to identify complex conjugation $J$ with $-1$ since $J(\zeta_{2n}) = \zeta_{2n}^{-1}$ for all $n$. The non–zero closed subgroups of $U_2 = \mathbb{Z}_2^* \cong \{\pm 1\} \times \mathbb{Z}_2$ are: $\{\pm 1\} \times 2^n\mathbb{Z}_2$, $2^n\mathbb{Z}_2$ and $\{\pm 1\} \cdot 2^n\mathbb{Z}_2$ with $n \in \mathbb{N} \cup \{0\}$.

The quotient groups are respectively

- $\frac{\{\pm 1\} \times \mathbb{Z}_2}{\{\pm 1\} \times 2^n\mathbb{Z}_2} \cong \frac{\mathbb{Z}_2}{2^n\mathbb{Z}_2} \cong C_{2^n}$,
- $\frac{\{\pm 1\} \times \mathbb{Z}_2}{\{\pm 1\} \cdot 2^n\mathbb{Z}_2} \cong \{\pm 1\} \times \frac{\mathbb{Z}_2}{2^n\mathbb{Z}_2} \cong \{\pm 1\} \times C_{2^n}$,
- $\frac{\{\pm 1\} \times \mathbb{Z}_2}{\{\pm 1\} \times 2^n\mathbb{Z}_2} \cong H$.

Let us study $H$. Consider $b := 1 \in \mathbb{Z}_2$. Then $b$ is a topological generator of $\mathbb{Z}_2$. Let $a := -1$ be the unique torsion element of $\mathbb{Z}_2$ of order 2. Let $H$ be the procyclic group with topological generator $ab^{2^n} : H = \langle ab^{2^n} \rangle$ (topological closure). Denote by $\hat{a}$ and $\hat{b}$ the classes of $a$ and $b$ modulo $H$ respectively: $\hat{a} = a \mod H$; $\hat{b} = b \mod H$. 


We have \( \mathcal{G}/\mathcal{H} = \langle \tilde{a}, \tilde{b} \rangle \) where \( \mathcal{G} = \{ \pm 1 \} \times \mathbb{Z}_2 \cong \mathbb{Z}_2^* \). Since \( ab^{2^n} \in \mathcal{H} \), \( \tilde{b}^{2^n} = \tilde{a}^{-1} \mod \mathcal{H} \) and \( \tilde{a}^{-1} = \tilde{a} \mod \mathcal{H} \) (indeed, \( a^{-1} = a = -1 \)). Therefore \( \mathcal{G}/\mathcal{H} = \langle \tilde{b} \rangle \) since \( \tilde{a} = \tilde{b}^{2^n} \in \langle \tilde{b} \rangle \) so that \( \mathcal{G}/\mathcal{H} \) is a cyclic group.

Note that \( b^{2^n} \notin \mathcal{H} \) since otherwise \( a \in \mathcal{H} \) but \( a \) is a torsion element and \( \mathcal{H} \) is torsion free. Therefore \( b^{2^{n+1}} \notin \mathcal{H} \). On the other hand \( b^{2^{n+1}} = b^{2^n} b^{2^n} \equiv ab^{2^n} \mod \mathcal{H} \) so that \( b^{2^{n+1}} \in \mathcal{H} \). It follows that \( o(b) = 2^{n+1} \). Thus \( \mathcal{G}/\mathcal{H} \) is a cyclic group of order \( 2^{n+1} \).

Uniformizing the indexes, we have

- \( \{ \pm 1 \} \times \mathbb{Z}_2 \) \( \mathcal{G}/\mathcal{A}_n \equiv \text{Gal}(Q(\zeta_{2m+2})^+ / Q) \equiv C_{2m} \) since \(-1 \in \mathcal{A}_n\),
- \( \{ \pm 1 \} \times \mathbb{Z}_2 \) \( \mathcal{G}/\mathcal{B}_m \equiv \text{Gal}(Q(\zeta_{2m+1}) / Q) \equiv C_2 \times C_{2m-1} \) since \( \mathcal{G}/\mathcal{B}_n \) is noncyclic,
- \( \{ \pm 1 \} \times \mathbb{Z}_2 \) \( \mathcal{G}/\mathcal{C}_m \equiv \text{Gal}(Q(\zeta_{2m+2}^-) / Q) \equiv C_{2m} \) since it is cyclic and \(-1 \notin \mathcal{C}_n\).

Since \( [L_2 : Q] = 2^m \) and \( [U_2 : \prod_{p|2} U_p] = 2^m \), it follows the following theorem.

**Theorem 4.13.** If \( [L_2 : Q] = 2^m \), then
5. SOME REMARKS ON GENUS FIELDS OF NUMBER FIELDS

Let \(L/\mathbb{Q}\) be a finite Galois extension. Since \(L/\mathbb{Q}\) is normal, \(L\) is either totally real or totally imaginary. Let \(J : \mathbb{C} \to \mathbb{C}\) be the complex conjugation. Since \(J_{|\mathbb{Q}} = \text{Id}_\mathbb{Q}\) and \(L/\mathbb{Q}\) is normal, we have \(J(L) = L = L\). Hence \(J|_L \in G := \text{Gal}(L/\mathbb{Q})\). Furthermore \(J|_L\) has order \(o(J|_L) = 1\) or \(2\). Let \(L^J\) be the fixed field of \(L\) under the action of \(J\). We have \(\text{Gal}(L|L^J) = \langle J|_L \rangle \cong \{1\} \) or \(C_2\), the cyclic group of order 2 and \([L : L^J]|2\). Furthermore, \(L^J \subseteq \mathbb{R}\).

Note that \(L^J\) is neither necessarily normal over \(\mathbb{Q}\) nor totally real. For instance, if \(L = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})\).

Then \(\text{Gal}(L/\mathbb{Q}) = \langle \alpha, \beta \rangle = C_2 \times C_3 \cong S_3\), the symmetric group in 3 elements. \(L\) is totally imaginary and \(L^J = \mathbb{Q}(\sqrt[3]{2})\), the extension \(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}\) is not normal and the 3 embeddings are \(\sqrt[3]{2} \mapsto \begin{cases} \sqrt[3]{2} \\ \sqrt[3]{3} \sqrt[3]{2} \\ \sqrt[3]{\sqrt[3]{2}} \end{cases}\). In other words, with the usual meaning, \(r_1 = 1\) and \(r_2 = 1\).

When \(L/\mathbb{Q}\) is abelian, then \(\langle J|_L \rangle \triangleleft G\), \(L^J/\mathbb{Q}\) is a Galois extension and \(L^J\) is totally real.

In the case of genus fields, we consider \(K/\mathbb{Q}\) a finite extension and let \(K_H\) and \(K_{H^+}\) be the Hilbert class field and the Hilbert extended (narrow) class field of \(K\) respectively. Then the genus field \(K_{ge}\) is the maximum extension such that \(K \subseteq K_{ge} \subseteq K_H\) with \(K_{ge} = KF, F/\mathbb{Q}\) abelian. In particular, \(F = F_{ge}\). The extended or narrow genus field \(K_{ge}\) of \(K\) is the maximum extension such that \(K \subseteq K_{ge} \subseteq K_{H^+}\) with \(K_{ge} = KL\) and \(L/\mathbb{Q}\) is abelian. In particular, \(L_{ge} = L\). Recall that \(L_{ge}\) is the maximum abelian extension of \(\mathbb{Q}\) such that \(L_{ge}/L\) is unramified at every finite prime and \(F_{ge}\) is the maximum abelian extension of \(\mathbb{Q}\) with \(F_{ge}/K\) unramified at every prime.

From the remarks above, it follows that \([K_{ge} : K_{ge}] = 1\) or \(2\) for every finite abelian extension \(K/\mathbb{Q}\). Now, we have \(K_H \subseteq K_{H^+}\) and in fact \(\text{Gal}(K_{H^+}/K_H) \cong C_2\) for some \(r \in \mathbb{N} \cup \{0\}\). In our notation, we have that \(F \subseteq L\) since \(KF/K\) is unramified and \(F/\mathbb{Q}\) is abelian. On the other hand, \(L^J\) is totally real, \(L^J/\mathbb{Q}\) is abelian and \(KL^J/K\) is unramified at every prime. It follows that \(L^J \subseteq F \subseteq L\).
Since \( F = F_{\text{gr}} \) it follows that \([L : F]|2\) and therefore

\[
[K_{\text{gr}} : K_{\text{gr}}] = [KL :KF]|[L : F] = 1 \text{ or } 2.
\]

In short, we have

**Proposition 5.1.** For a finite extension \( K/\mathbb{Q} \), we have \([K_{\text{gr}} : K_{\text{gr}}]|2.\)

Now consider \( K_i/\mathbb{Q}, i = 1, 2 \), two finite extensions and let \( K = K_1K_2 \). We have \( K_i \subseteq K \) for \( i = 1, 2 \). On the other hand, \((K_1)_{\text{gr}}/K_1\) is unramified and abelian, it follows that \( K(K_1)_{\text{gr}}/K_1K_1 = K \) is unramified and abelian. Hence \( K(K_1)_{\text{gr}} \subseteq K_{\text{gr}} \). It follows that \((K_1)_{\text{gr}} \subseteq K_{\text{gr}} \). Similarly \((K_2)_{\text{gr}} \subseteq K_{\text{gr}} \). Therefore \((K_1)_{\text{gr}}(K_2)_{\text{gr}} \subseteq K_{\text{gr}} \).

**Remark 5.2.** Not necessarily \((K_1)_{\text{gr}}(K_2)_{\text{gr}} = K_{\text{gr}} \).

**Example 5.3.** Let \( p, q, p_1, q_1 \) be four odd distinct primes. Let \( K_1 = \mathbb{Q}(\zeta_{pq})^+, K_2 = \mathbb{Q}(\zeta_{p_1q_1})^+. \) Then, using Dirichlet characters, we have that \((K_1)_{\text{gr}} \subseteq \mathbb{Q}(\zeta_{pq})^+ \) and \( \mathbb{Q}(\zeta_{pq})^+/(\zeta_{pq})^+ \) is ramified at \( \infty \), it follows that \((K_1)_{\text{gr}} = K_1 \). Similarly \((K_2)_{\text{gr}} = K_2 \).

Furthermore, since \( p \neq q \) (respectively \( p_1 \neq q_1 \)), \( \mathbb{Q}(\zeta_{pq})^+/(\zeta_{pq})^+ \) is ramified only at \( \infty \), that is, \( \mathbb{Q}(\zeta_{pq})^+/(\zeta_{pq})^+ \) is unramified at every finite prime ([16, Teorema 5.3.2]).

Now \( K_1K_2 = K = \mathbb{Q}(\zeta_{pq})^+\mathbb{Q}(\zeta_{p_1q_1})^+ \subseteq \mathbb{Q}(\zeta_{pqp_1q_1}). \)

\[
\begin{array}{c}
\mathbb{Q}(\zeta_{pq})^+ \quad \mathbb{Q}(\zeta_{pqp_1q_1})^+ \\
\downarrow 2 \quad \downarrow 2 \\
K_1 = \mathbb{Q}(\zeta_{pq})^+ \quad K_1K_2 = K \\
\downarrow \quad \downarrow 2 \\
\mathbb{Q} \quad \mathbb{Q}(\zeta_{p_1q_1})^+ \\
\end{array}
\]

We have that \( \mathbb{Q}(\zeta_{pqp_1q_1})^+ / \mathbb{Q} \) is unramified since \( p \) is unramified in \( \mathbb{Q}(\zeta_{pq})^+/(\zeta_{pq})^+ \) and thus

\[
e_q(\mathbb{Q}(\zeta_{pqp_1q_1})|\mathbb{Q}) = p - 1 = e_q(\mathbb{Q}(\zeta_{pq})^+|\mathbb{Q}) = e_q(K|\mathbb{Q}).
\]

The same holds for \( q, p_1 \) and \( q_1 \). Now, \( \infty \) is ramified in \( \mathbb{Q}(\zeta_{pqp_1q_1})^+/(\zeta_{pqp_1q_1})^+) \).

It follows that \( K_{\text{gr}} = \mathbb{Q}(\zeta_{pqp_1q_1})^+ \) and that \([K_{\text{gr}} : (K_1)_{\text{gr}}(K_2)_{\text{gr}}] = 2 > 1.\)

**Remark 5.4.** For any two finite abelian extensions \( K_i/\mathbb{Q}, i = 1, 2 \) we have \( K_{\text{gr}} = (K_1)_{\text{gr}}(K_2)_{\text{gr}} \) where \( K = K_1K_2 \) (see [2]).

**Theorem 5.5.** Let \( K_i/\mathbb{Q}, i = 1, 2 \) be two finite abelian extensions and let \( K = K_1K_2. \) Then

\[
[K_{\text{gr}} : (K_1)_{\text{gr}}(K_2)_{\text{gr}}]|2.
\]
Proof. In general we consider a finite abelian extension $K/\mathbb{Q}$. Let $L = K_{ge}$. We have $K_{ge} = L^+ K$ (see [2]). Let $K = K_1 K_2$. Then $K_{ge} = (K_1)_{ge}(K_2)_{ge}$. Therefore $L = L_1 L_2$ and $K_{ge} = L^+ K_1 (K_1)_{ge}(K_2)_{ge} = L_1^+ K_1 L_2^+ K_2 = L_1^+ L_2^+ K$. Hence

$$[K_{ge} : (K_1)_{ge}(K_2)_{ge}] = [L^+ K : L_1^+ L_2^+ K] = [L^+ : L_1^+ L_2^+].$$

To prove the result, it suffices to show that for two finite abelian extensions $L_i/\mathbb{Q}, i = 1, 2$, and for $L = L_1 L_2$, we have $[L^+ : L_1^+ L_2^+] = 2$.

In general, we have $L^+ = L \cap \mathbb{Q}(\zeta_n)^+ = L \cap (\mathbb{Q}(\zeta_n))^+$ for $L \subseteq \mathbb{Q}(\zeta_n)$. In particular, if $S := \text{Gal}(\mathbb{Q}(\zeta_n)/L)$, $L^+ = L \cap \mathbb{Q}(\zeta_n)^+ = \mathbb{Q}(\zeta_n)^S \cap \mathbb{Q}(\zeta_n)^I = \mathbb{Q}(\zeta_n)^{SI}$ where $I = \langle J \rangle$ and thus $\text{Gal}(\mathbb{Q}(\zeta_n)/L^+) = SI$.

Let $S_i := \text{Gal}(\mathbb{Q}(\zeta_n)/L_i), i = 1, 2$. Since $L = L_1 L_2$, we have $S = S_1 \cap S_2$. We also have

$$L_1^+ L_2^+ = \mathbb{Q}(\zeta_n)^{S_1} \mathbb{Q}(\zeta_n)^{S_2} = \mathbb{Q}(\zeta_n)^{S_1 \cap S_2 \cap J} \subseteq L^+ = \mathbb{Q}(\zeta_n)^{SI}.$$

Therefore

$$\text{Gal}(L^+/L_1^+ L_2^+) \cong \frac{\text{Gal}(\mathbb{Q}(\zeta_n)/L^+)}{\text{Gal}(\mathbb{Q}(\zeta_n)/L_1^+)} \cong \frac{S_1 \cap S_2 I}{SI} = \frac{S_1 \cap S_2 I}{(S_1 \cap S_2) I}. \quad (5.1)$$

Now

$$|S_1 \cap S_2 I| = \frac{|S_1 I||S_2 I|}{|S_1 S_2 I|} = \frac{|S_1 I||S_2 I|}{|S_1| |S_2 I|} = \frac{|S_1 I||S_2 I|^2}{|S_1 S_2 I| |S_1| |S_2 I|} = \frac{|S_1 S_2 I|^2}{|S_1 S_2 I|}.$$ 

On the other hand $|(S_1 \cap S_2) I| = |SI| = \frac{|S_1 I||S_2 I|}{|S_1 S_2 I|}$. It follows that

$$[S_1 I \cap S_2 I : (S_1 \cap S_2) I] = \frac{|S_1 \cap S_2 I| |S_1 S_2 I| |S_1 \cap I| |S_2 \cap I| |I|}{|S_1 S_2 I| |S_1 \cap I| |S_2 \cap I| |I|} = \frac{|S_1 S_2 I| |S_1 \cap I| |S_2 \cap I| |I|}{|S_1 S_2 I| |S_1 \cap I| |S_2 \cap I| |I|}.$$ 

Now $S \cap I \subseteq S_2 \cap I$. Let $\alpha = [S_2 \cap I : S \cap I] \in \mathbb{N}$. Then

$$[S_1 I \cap S_2 I : (S_1 \cap S_2) I] = \frac{1}{\alpha} \frac{|S_1 S_2 \cap I|}{|S_1 \cap I|} = \frac{1}{\alpha} \frac{|S_1 S_2 I|}{|S_1 S_2 I|} = \frac{1}{\alpha} |S_1 S_2 I| |S_1 I|.$$ 

We have $S_1 S_2 \subseteq S_1 S_2 I$. Let $\beta = [S_1 S_2 I : S_1 S_2] \in \mathbb{N}$. It follows that

$$[S_1 I \cap S_2 I : (S_1 \cap S_2) I] = \frac{1}{\alpha \beta} |S_1 I| = \frac{1}{|S_1 I|} = \frac{1}{|S_1 I|} |I| = \frac{\gamma}{\alpha \beta},$$

with $\gamma = [S_1 \cap I]|I|$. Therefore $[S_1 I \cap S_2 I : (S_1 \cap S_2) I] = \frac{\gamma}{\alpha \beta} \in \mathbb{N}$ and

$$[S_1 I \cap S_2 I : (S_1 \cap S_2) I] = \frac{\gamma}{\alpha \beta} \in \mathbb{N} \text{ and } [S_1 I \cap S_2 I : (S_1 \cap S_2) I] = \frac{\gamma}{\alpha \beta}.$$ 

It follows that

$$[L^+ : L_1^+ L_2^+] = 2 \text{ and } [K_{ge} : (K_1)_{ge}(K_2)_{ge}] = 2.$$ 

$\square$

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