MULTIPLICATIVE ZAGREB INDICES AND COINDICES OF SOME DERIVED GRAPHS

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Abstract. In this note, we obtain the expressions for multiplicative Zagreb indices and coindices of derived graphs such as a line graph, subdivision graph, vertex-semitotal graph, edge-semitotal graph, total graph and paraline graph.

Keywords: multiplicative Zagreb indices and coindices, derived graphs.

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1. INTRODUCTION

In this paper, we are concerned with simple graphs without isolated vertices. Let \( G \) be such a graph with vertex set \( V(G) \), \(|V(G)| = n\), and edge set \( E(G) \), \(|E(G)| = m\). As usual, \( n \) is the order and \( m \) the size of \( G \). The degree of a vertex \( w \in V(G) \) is the number of vertices adjacent to \( w \) and is denoted by \( d_G(w) \). A vertex \( w \in V(G) \) is said to be pendant if \( d_G(w) = 1 \). The degree of an edge \( e = uv \) in \( G \), denoted by \( d_G(e) \), is defined by \( d_G(e) = d_G(u) + d_G(v) - 2 \). We refer to [9] for unexplained terminology and notation.

A graphical invariant is a number related to a graph, in other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also called the topological indices. In 1984, Narumi and Katayama [11] considered the product index as

\[
NK(G) = \prod_{u \in V(G)} d_G(u)
\]

for representing the carbon skeleton of a saturated hydrocarbon, and named it as a simple topological index. Tomović and Gutman renamed this molecular structure
descriptor as the Narumi-Katayama index [15]. In 2010, Todeshine et al. [13,14] proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows:

$$\prod_{G}^{1} = \prod_{u \in V(G)} d_{G}(u)^{2} = [NK(G)]^{2} \quad \text{and} \quad \prod_{uv \in E(G)} d_{G}(u)d_{G}(v).$$

These two graph invariants are called first and second multiplicative Zagreb indices by Gutman [6]. And recently, Eliasi et al. [5] introduced a further multiplicative version of the first Zagreb index as

$$\prod_{G}^{*} = \prod_{uv \in E(G)} [d_{G}(u) + d_{G}(v)].$$

In [18] and [7] the authors called it a multiplicative sum Zagreb index and modified first multiplicative Zagreb index respectively. The second multiplicative Zagreb index for any graph $G$ can also be written as [6]

$$\prod_{G}^{2} = \prod_{u \in V(G)} d_{G}(u)^{d_{G}(u)}.$$

Xu et al. [19] defined the first and second multiplicative Zagreb coindices, respectively, as

$$\prod_{G}^{1} = \prod_{uv \notin E(G)} [d_{G}(u) + d_{G}(v)] \quad \text{and} \quad \prod_{uv \notin E(G)} d_{G}(u)d_{G}(v).$$

The main properties of multiplicative Zagreb indices are summarized in [1, 4, 10, 12, 17, 18, 20].

We introduce the modified second multiplicative Zagreb index as

$$\prod_{G}^{*} = \prod_{uv \in E(G)} [d_{G}(u) + d_{G}(v)]^{[d_{G}(u)+d_{G}(v)].}$$

2. DERIVED GRAPHS

In recent papers [2,3,8], the authors obtained the expressions for Zagreb indices and coindices of derived graphs. This motivates us to find expressions for $\prod_{1}$, $\prod_{2}$, $\prod_{1}^{*}$ and $\prod_{2}^{*}$ of derived graphs.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We are concerned with the following graphs derived from $G$ ([8]):

- **line graph** $L = L(G)$; $V(L) = E(G)$ and the two vertices of $L$ are adjacent if the corresponding edges of $G$ are incident with a common vertex;

- **subdivision graph** $S = S(G)$; $V(S) = V(G) \cup E(G)$ and the vertex of $S$ corresponding to the edge $uv$ of $G$ is inserted in the edge $uv$ of $G$;
- vertex-semitotal graph $T_2 = T_2(G)$; $V(T_2) = V(G) \cup E(G)$ and $E(T_2) = E(S) \cup E(G)$;
- edge-semitotal graph $T_1 = T_1(G)$; $V(T_1) = V(G) \cup E(G)$ and $E(T_1) = E(S) \cup E(L)$;
- total graph $T = T(G)$; $V(T) = V(G) \cup E(G)$ and $E(T) = E(S) \cup E(G) \cup E(L)$;
- paraline graph $PL = PL(G)$ is the line graph of the subdivision graph.

In Figure 1 self-explanatory examples of these derived graphs are depicted.

In [19], Kexiang Xu et al. obtained the expressions for $\Pi_2(G)$ of any connected graph $G$ as

$$\Pi_2(G) = \prod_{u \in V(G)} d_G(u)^{n-1-d_G(u)} \quad \text{and} \quad \Pi_2(G) \Pi_2(G) = \left( \prod_1(G) \right)^{n-1}$$

which are not satisfied for a complete graph. The following lemmas give the correct expressions for $\Pi_2(G)$. 

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**Fig. 1.** Various graphs derived from the graph $G$. The vertices of these derived graphs (except the paraline graph $PL$), corresponding to the vertices of the parent graph $G$, are indicated by circles. The vertices of these graphs corresponding to the edges of the parent graph $G$ are indicated by squares.
Lemma 2.1. For a connected graph $G \neq K_n$, we have
\[
\prod_{2}(G) = \prod_{u \in V(G)} d_G(u)^{n-1-d_G(u)}.
\]

Lemma 2.2. For a connected graph $G \neq K_n$, we have
\[
\prod_{2}(G)\prod_{2}(G) = \left( \prod_{1}(G)^{n-1} \right)^{\frac{1}{2}}.
\]

Next we present the values of multiplicative Zagreb indices and coindices for several classes of graphs.

Example 2.3. Let $P_n$ be the path with $n$ vertices. The pendant vertices have degree 1 and other vertices have degree 2. Hence,

(i) $\prod_{1}(P_n) = 4^{(n-2)}$,  
(ii) $\prod_{2}(P_n) = 4^{(n-2)}$,  
(iii) $\prod_{1}^{*}(P_n) = 9 \cdot 4^{(n-3)}$, $n \geq 3$,  
(iv) $\prod_{1}^{1}(P_n) = 2 \cdot 9^{(n-3)} \cdot 4^{(n-4)!}$, $n \geq 4$,  
(v) $\prod_{2}(P_n) = 2^{(n-2)(n-3)}$, $n \geq 3$,  
(vi) $\prod_{2}^{*}(P_n) = 3^{6} \cdot 4^{4(n-3)}$, $n \geq 3$.

Example 2.4. Consider the cycle $C_n$ with $n$ vertices. Since its every vertex is of degree 2, then

(i) $\prod_{1}(C_n) = 4^{n}$,  
(ii) $\prod_{2}(C_n) = 4^{n}$,  
(iii) $\prod_{1}^{*}(C_n) = 4^{\frac{n(n-3)}{2}}$, $n \geq 4$,  
(iv) $\prod_{1}^{1}(C_n) = 4^{\frac{n(n-3)}{2}}$, $n \geq 4$,  
(v) $\prod_{2}(C_n) = 4^{4^{n}}$,  
(vi) $\prod_{2}^{*}(C_n) = 4^{4^{n}}$.

Example 2.5. Let $K_n$ be the complete graph on $n$ vertices. All vertices of $K_n$ have degree $n - 1$ and so

(i) $\prod_{1}(K_n) = (n-1)^{2n}$, $n \geq 2$,  
(ii) $\prod_{2}(K_n) = (n-1)^{n(n-1)}$, $n \geq 2$,  
(iii) $\prod_{1}^{*}(K_n) = [2(n-1)]^{\frac{n(n-1)}{2}}$, $n \geq 2$,  
(iv) $\prod_{1}^{1}(K_n) = 0$,  
(v) $\prod_{2}(K_n) = 0$,  
(vi) $\prod_{2}^{*}(K_n) = [2(n-1)]^{n(n-1)^2}$, $n \geq 2$.

Example 2.6. Let $K_{r,s}$ be the complete bipartite graph. Then $K_{r,s}$ has $r+s$ vertices and $rs$ edges. Hence,
Multiplicative Zagreb indices and coindices. . . 291

(i) \( \prod_1(K_{r,s}) = r^{2s} \cdot s^{2r} \),
(ii) \( \prod_2(K_{r,s}) = [rs]^{rs} \),
(iii) \( \prod_1(K_{r,s}) = [r + s]^{rs} \),
(iv) \( \prod_1(K_{r,s}) = [2r]^{\frac{s(s-1)}{2}} \cdot [2s]^{\frac{r(r-1)}{2}} \), \( r \neq 1 \) and \( s \neq 1 \),
(v) \( \prod_2(K_{r,s}) = r^{s(s-1)} \cdot s^{r(r-1)} \), \( r \neq 1 \) and \( s \neq 1 \),
(vi) \( \prod_2(K_{r,s}) = [r + s]^{rs(r+s)} \).

Example 2.7. Let \( W_n \) be the wheel on \( n \) vertices. Its central vertex has degree \( n - 1 \) and its other vertices have degree 3. This implies

(i) \( \prod_1(W_n) = (n - 1)^2 \cdot 3^{2(n-1)} \),
(ii) \( \prod_2(W_n) = 3(n - 1)(n-1) \cdot 3^{2(n-1)} \),
(iii) \( \prod_1(W_n) = [n + 2]^{(n-1)} \cdot 6^{(n-1)} \),
(iv) \( \prod_1(W_n) = \frac{6^{(n-1)}}{2^{(n-2)}} \), \( n \geq 5 \),
(v) \( \prod_2(W_n) = 9^{(n-1)\frac{n-1}{2}} \), \( n \geq 5 \),
(vi) \( \prod_2(W_n) = 66^{(n-1)} \cdot [n + 2]^{(n-1)(n+2)} \).

3. RESULTS

Theorem 3.1. Let \( G \) be a graph of order \( n \) and size \( m \). Then

\[ \prod_1(S) = 4^m \prod_1(G). \]

Proof. Note that \( S \) has \( n + m \) vertices.

\[ \prod_1(S) = \prod_{u \in V(S)} d_S(u)^2 = \prod_{u \in V(S) \cap V(G)} d_S(u)^2 \prod_{e \in V(S) \cap E(G)} d_S(e)^2. \]

Note that for \( u \in V(S) \cap V(G) \), \( d_S(u) = d_G(u) \) and for \( e \in V(S) \cap E(G) \), \( d_S(e) = 2 \).

\[ \prod_1(S) = \prod_{u \in V(G)} d_G(u)^2 \prod_{e \in E(G)} 2^2 = 4^m \prod_1(G). \]

Theorem 3.2. Let \( G \) be a graph of order \( n \) and size \( m \). Then

\[ \prod_2(S) = 4^m \prod_2(G). \]

Proof. Since \( S \) has \( n + m \) vertices, then

\[ \prod_2(S) = \prod_{u \in V(S)} d_S(u)^{d_S(u)} = \prod_{u \in V(S) \cap V(G)} d_S(u)^{d_S(u)} \prod_{e \in V(S) \cap E(G)} d_S(e)^{d_S(e)}. \]

Since for \( u \in V(S) \cap V(G) \), \( d_S(u) = d_G(u) \) and for \( e \in V(S) \cap E(G) \), \( d_S(e) = 2 \).

\[ \prod_2(S) = \prod_{u \in V(G)} d_G(u)^{d_G(u)} \prod_{e \in E(G)} 2^2 = 4^m \prod_2(G). \]
Theorem 3.3. Let $G$ be a graph of order $n$ and size $m$. Then

$$\prod_{1}^{*}(S) = \prod_{u \in V(G)} [2 + d_G(u)]^{d_G(u)}.$$ 

Proof. Since $S$ has $n + m$ vertices, then we have

$$\prod_{1}^{*}(S) = \prod_{uv \in E(S)} [d_S(u) + d_S(e)].$$

Since for $u \in V(S) \cap V(G)$, $d_S(u) = d_G(u)$ and for $e \in V(S) \cap E(G)$, $d_S(e) = 2$.

$$\prod_{1}^{*}(S) = \prod_{u \in V(G)} [2 + d_G(u)]^{d_G(u)}.$$ □

Corollary 3.4. Let $G$ be a connected graph of order $n$ and size $m$. Then

$$\prod_{2}^{*}(S) = \frac{2^{m^{2}+nm-3m} \prod_{1}^{1}(G) \prod_{2}^{1}(G)-1}{\prod_{2}^{2}(G)}.$$ 

Proof. From Lemma 2.2 we have

$$\prod_{2}^{2}(S) = \frac{[\prod_{1}^{1}(S)]^{n+m-1}}{\prod_{2}^{2}(S)}.$$ 

From Theorems 3.1 and 3.2 we get the result. □

Theorem 3.5. Let $G$ be a graph of order $n$ and size $m$. Then

$$\prod_{1}^{1}(T_2) = 4^{n+m} \prod_{1}^{1}(G).$$ 

Proof. Note that $T_2$ has $n + m$ vertices.

$$\prod_{1}^{1}(T_2) = \prod_{u \in V(T_2)} d_{T_2}(u)^2 = \prod_{u \in V(T_2) \cap V(G)} d_{T_2}(u)^2 \prod_{e \in V(T_2) \cap E(G)} d_{T_2}(e)^2.$$ 

Note that for $u \in V(T_2) \cap V(G)$, $d_{T_2}(u) = 2d_G(u)$ and for $e \in V(T_2) \cap E(G)$, $d_{T_2}(e) = 2$.

$$\prod_{1}^{1}(T_2) = \prod_{u \in V(G)} [2d_G(u)]^{2} \prod_{e \in E(G)} 2^{2} = 4^{n+m} \prod_{1}^{1}(G).$$ □

Theorem 3.6. Let $G$ be a graph of order $n$ and size $m$. Then

$$\prod_{2}^{2}(T_2) = 64^{m} \prod_{1}^{1}(G) \prod_{2}^{2}(G).$$
Proof. Since $T_2$ has $n + m$ vertices and $3m$ edges, then we have

$$\prod_2(T_2) = \prod_{uv \in E(T_2)} d_{T_2}(u)d_{T_2}(v)$$

$$= \prod_{uv \in E(T_2) \cap E(G)} d_{T_2}(u)d_{T_2}(v) \prod_{ue \in E(T_2) \setminus E(G)} d_{T_2}(u)d_{T_2}(e).$$

Since for $u \in V(T_2) \cap V(G)$, $d_{T_2}(u) = 2d_G(u)$ and for $e \in V(T_2) \cap E(G)$, $d_{T_2}(e) = 2$.

$$\prod_2(T_2) = \prod_{uv \in E(G)} 2d_G(u)2d_G(v) \prod_{ue \in E(T_2) \setminus E(G)} (2)2d_G(u)$$

$$= 4^m \prod_{uv \in E(G)} d_G(u)d_G(v)4^m \prod_{ue \in E(G)} d_G(u)^2$$

$$= 64^m \prod_1(G) \prod_2(G). \quad \square$$

Theorem 3.7. Let $G$ be a graph of order $n$ and size $m$. Then

$$\prod_1^*(T_2) = 8^m \prod_1^*(G) \prod_{u \in V(G)} [1 + d_G(u)]^{d_G(u)}.$$

Proof. Since $T_2$ has $n + m$ vertices and $3m$ edges, then we have

$$\prod_1^*(T_2) = \prod_{uv \in E(T_2)} [d_{T_2}(u) + d_{T_2}(v)]$$

$$= \prod_{uv \in E(T_2) \cap E(G)} [d_{T_2}(u) + d_{T_2}(v)] \prod_{ue \in E(T_2) \setminus E(G)} [d_{T_2}(u) + d_{T_2}(e)].$$

Since for $u \in V(T_2) \cap V(G)$, $d_{T_2}(u) = 2d_G(u)$ and for $e \in V(T_2) \cap E(G)$, $d_{T_2}(e) = 2$.

$$\prod_1^*(T_2) = \prod_{uv \in E(G)} [2d_G(u) + 2d_G(v)] \prod_{ue \in E(T_2) \setminus E(G)} [2 + 2d_G(u)]$$

$$= 2^m \prod_{uv \in E(G)} [d_G(u) + d_G(v)]2^m \prod_{ue \in E(G)} [1 + d_G(u)]^{d_G(u)}$$

$$= 8^m \prod_1^*(G) \prod_{u \in V(G)} [1 + d_G(u)]^{d_G(u)}. \quad \square$$

Corollary 3.8. Let $G$ be a connected graph of order $n$ and size $m$. Then

$$\prod_2(T_2) = \frac{2^{(n+m)^2-n-7m}[\prod_1(G)]^{n+m-3}}{\prod_2(G)}.$$
Proof. From Lemma 2.2 we have
\[ \prod_2(T_2) = \left[ \prod_1(T_2) \right]^{\frac{n+m-1}{2}} / \prod_2(T_2) \]

From Theorems 3.5 and 3.6 we get the result. \qed

Theorem 3.9. Let \( G \) be a graph of order \( n \) and size \( m \). Then
\[ \prod_1(T_1) = \prod_1(G) \left[ \prod_1^*(G) \right]^2. \]

Proof. Note that \( T_1 \) has \( n + m \) vertices.
\[ \prod_1(T_1) = \prod_{u \in V(T_1)} d_{T_1}(u)^2 \]
\[ = \prod_{u \in V(T_1) \cap V(G)} d_{T_1}(u)^2 \prod_{e_i \in V(T_1) \cap E(G)} d_{T_1}(e_i)^2. \]

Note that for \( u \in V(T_1) \cap V(G) \), \( d_{T_1}(u) = d_G(u) \) and for \( e_i \in V(T_1) \cap E(G) \), \( d_{T_1}(e_i) = d_G(u_i) + d_G(v_i) \).
\[ \prod_1(T_1) = \prod_{u \in V(G)} d_G(u)^2 \prod_{u_i, v_i \in E(G)} [d_G(u_i) + d_G(v_i)]^2 \]
\[ = \prod_1(G) \left[ \prod_1^*(G) \right]^2. \] \qed

Theorem 3.10. Let \( G \) be a graph of order \( n \) and size \( m \). Then
\[ \prod_2(T_1) = \prod_2(G) \prod_2^*(G). \]

Proof. Since \( T_1 \) has \( n + m \) vertices, then we have
\[ \prod_2(T_1) = \prod_{u \in V(T_1)} d_{T_1}(u)^{d_{T_1}(u)} \]
\[ = \prod_{u \in V(T_1) \cap V(G)} d_{T_1}(u)^{d_{T_1}(u)} \prod_{e_i \in V(T_1) \cap E(G)} [d_{T_1}(e_i)]^{d_{T_1}(e_i)}. \]

Since for \( u \in V(T_1) \cap V(G) \), \( d_{T_1}(u) = d_G(u) \) and for \( e_i \in V(T_1) \cap E(G) \), \( d_{T_1}(e_i) = d_G(u_i) + d_G(v_i) \).
\[ \prod_2(T_1) = \prod_{u \in V(G)} d_G(u)^{d_G(u)} \prod_{u_i, v_i \in E(G)} [d_G(u_i) + d_G(v_i)]^{d_G(u_i) + d_G(v_i)} \]
\[ = \prod_2(G) \prod_2^*(G). \] \qed
Corollary 3.11. Let $G$ be a connected graph of order $n$ and size $m$. Then
\[
\prod_2(T_1) = \frac{\prod_1(G)^{n+m-1} \prod_1^*(G)^{n+m-1}}{\prod_2(G) \prod_2^*(G)}.
\]

Proof. From Lemma 2.2 we have
\[
\prod_2(T_1) = \frac{[\prod_1(T_1)]^{n+m-1}}{\prod_2(T_1)}.
\]
From Theorems 3.9 and 3.10 we get the result. \hfill \Box

In [16], an incorrect expression for $\prod_1(T)$ was established. The following theorem gives the correct expression for $\prod_1(T)$.

Theorem 3.12. Let $G$ be a graph of order $n$ and size $m$. Then
\[
\prod_1(T) = 4^n \prod_1(G) \left[ \prod_1^*(G) \right]^2.
\]

Proof. Note that $T$ has $n + m$ vertices.
\[
\prod_1(T) = \prod_{u \in V(T)} d_T(u)^2 = \prod_{u \in V(T) \cap V(G)} d_T(u)^2 \prod_{e_i \in V(T) \cap E(G)} d_T(e_i)^2.
\]
Note that for $u \in V(T) \cap V(G)$, $d_T(u) = 2d_G(u)$ and for $e_i \in V(T) \cap E(G)$, $d_T(e_i) = d_G(u_i) + d_G(v_i)$.
\[
\prod_1(T) = \prod_{u \in V(G)} [2d_G(u)]^2 \prod_{u_i,v_i \in E(G)} [d_G(u_i) + d_G(v_i)]^2 = 4^n \prod_1(G) \left[ \prod_1^*(G) \right]^2. \hfill \Box
\]

Theorem 3.13. Let $G$ be a graph of order $n$ and size $m$. Then
\[
\prod_2(T) = 16^m \prod_2^*(G) \left[ \prod_2^*(G) \right]^2.
\]

Proof. Since $T$ has $n + m$ vertices, then we have
\[
\prod_2(T) = \prod_{u \in V(T)} d_T(u)^{d_T(u)} = \prod_{u \in V(T) \cap V(G)} d_T(u)^{d_T(u)} \prod_{e_i \in V(T) \cap E(G)} d_T(e_i)^{d_T(e_i)}.
\]
Note that for $u \in V(T) \cap V(G)$, $d_T(u) = 2d_G(u)$ and for $e_i \in V(T) \cap E(G)$, $d_T(e_i) = d_G(u_i) + d_G(v_i)$.
\[
\prod_2(T) = \prod_{u \in V(G)} [2d_G(u)]^{2d_G(u)} \prod_{u_i,v_i \in E(G)} [d_G(u_i) + d_G(v_i)]^{d_G(u_i) + d_G(v_i)}
\]
\[
= \prod_{u \in V(G)} 2^{2d_G(u)} \left[ d_G(u) \right]^{2d_G(u)} \prod_2^*(G)
\]
\[
= 16^m \prod_2^*(G) \left[ \prod_2^*(G) \right]^2. \hfill \Box
Corollary 3.14. Let \( G \) be a connected graph of order \( n \) and size \( m \). Then
\[
\prod_2(T) = \frac{2^{n(n+m-1)-4m} \prod_1(G)^{n+m-1} \prod_1^*(G)^{n+m-1}}{\prod_2(G) \prod_2^2(G)}.
\]

Proof. From Lemma 2.2 we have
\[
\prod_2(T) = \frac{\prod_1(T)^{n+m-1}}{\prod_2(T)}.
\]

From Theorems 3.12 and 3.13 we get the result. \( \square \)

Theorem 3.15. Let \( G \) be a graph of order \( n \) and size \( m \). Then
\[
\prod_1(PL) = \prod_2(G)^2.
\]

Proof. Note that paraline graph \( PL \) has 2\( m \) vertices, and \( d_G(u) \) of its vertices have the same degree as the vertex \( u \) of the graph \( G \).
\[
\prod_1(PL) = \prod_{u \in V(PL)} d_{PL}(u)^2 = \prod_{u \in V(G)} d_G(u)^{2d_G(u)} = \left[ \prod_2(G) \right]^2. \quad \square
\]

Theorem 3.16. Let \( G \) be a graph of order \( n \) and size \( m \). Then
\[
\prod_2(PL) = \prod_{u \in V(G)} [d_G(u)]^{[d_G(u)]^2}.
\]

Proof. Since \( PL \) has 2\( m \) vertices, then we have
\[
\prod_2(PL) = \prod_{uv \in E(PL)} d_{PL}(u)d_{PL}(v) = \prod_2(G) \prod [d_G(u)][d_G(u)(d_G(u)-1)] = \prod_{u \in V(G)} [d_G(u)]^{[d_G(u)]^2}. \quad \square
\]

Theorem 3.17. Let \( G \) be a graph of order \( n \) and size \( m \). Then
\[
\prod_1^*(PL) = \prod_1^*(G) \prod_{u \in V(G)} [2d_G(u)]^{\frac{d_G(u)(d_G(u)-1)}{2}}.
\]

Proof. Since \( PL \) has 2\( m \) vertices, then we have
\[
\prod_1^*(PL) = \prod_{uv \in E(PL)} [d_{PL}(u) + d_{PL}(v)] = \prod_1^*(G) \prod_{u \in V(G)} [2d_G(u)]^{\frac{d_G(u)(d_G(u)-1)}{2}}. \quad \square
\]

Corollary 3.18. Let \( G \) be a connected graph of order \( n \) and size \( m \). Then
\[
\prod_2(PL) = \frac{[\prod_2(G)]^{2m-1}}{\prod_{u \in V(G)} [d_G(u)]^{[d_G(u)]^2}}.
\]
Proof. From Lemma 2.2 we have
\[ \prod_2^{PL} = \left( \prod_{PL} \right)^{2m-1} \]
From Theorems 3.15 and 3.16 we get the result.

One can easily obtain the expressions for multiplicative Zagreb indices and coindices of line graph \( L \) of graph \( G \) by considering edge degrees of \( G \).

**Theorem 3.19.** Let \( G \) be a graph of order \( n \) and size \( m \). Then

(i) \( \prod_1(L) = \prod_{e \in E(G)} d_G(e)^2 \),
(ii) \( \prod_2(L) = \prod_{e_i \sim e_j} d_G(e_i)d_G(e_j) \),
(iii) \( \prod_1^*(L) = \prod_{e_i \sim e_j} [d_G(e_i) + d_G(e_j)] \),
(iv) \( \prod_1(L) = \prod_{e_i \sim e_j} [d_G(e_i) + d_G(e_j)] \),
(v) \( \prod_2(L) = \prod_{e_i \sim e_j} d_G(e_i)d_G(e_j) \),

where \( e_i \sim e_j \) (resp. \( e_i \sim e_j \)) means that the edges \( e_i \) and \( e_j \) are adjacent (resp. not adjacent) in \( G \).

It remains a task for the future to find the expressions for \( \prod_1^*(T_1) \) and \( \prod_1^*(T) \).

In [19], the total multiplicative sum Zagreb index \( T \prod(G) \) of a graph \( G \) is defined as
\[ T \prod(G) = \prod_{u, v \in V(G)} [d_G(u) + d_G(v)] \].

**Lemma 3.20 ([19]).** For a connected graph \( G \), we have \( \prod_1^*(G)\prod_1(G) = T \prod(G) \).

By Lemma 3.20, one can find the expression for \( \prod_1 \) of derived graphs. But obtaining the expression for \( T \prod \) of derived graphs is a difficult task.

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