On the definition of a unique effective temperature for non-equilibrium critical systems

Pasquale Calabrese\(^1\) and Andrea Gambassi\(^2,3\)

\(^1\)Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, United Kingdom.

\(^2\)Max-Planck-Institut für Metallforschung, Heisenbergstr. 3, D-70569 Stuttgart, Germany.

\(^3\)Institut für Theoretische und Angewandte Physik, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany.

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Abstract

We consider the problem of the definition of an effective temperature via the long-time limit of the fluctuation-dissipation ratio \(X^\infty\) after a quench from the disordered state to the critical point of an \(O(N)\) model with dissipative dynamics. The scaling forms of the response and correlation functions of a generic observable \(O(t)\) are derived from the solutions of the corresponding Renormalization Group equations. We show that within the Gaussian approximation all the local observables have the same \(X^\infty\), allowing for a definition of a unique effective temperature. This is no longer the case when fluctuations are taken into account beyond that approximation, as shown by a computation up to the first order in the \(\epsilon\)-expansion for two quadratic observables. This implies that, contrarily to what often conjectured, a unique effective temperature can not be defined for this class of models.

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I. INTRODUCTION

The non-equilibrium dynamics of physical systems is one of the most challenging problems in statistical mechanics. Equilibrium statistical mechanics has been probably one of the most important achievements during the last century. On the other hand, in nature equilibrium is more an exception rather than a rule. In view of that many efforts are currently aiming at achieving a coherent theoretical picture of non-equilibrium phenomena: Indeed many real systems persist out-of-equilibrium practically forever. One example is naturally provided by slow-relaxing systems, such as structural glasses and spin-glasses, whose equilibration times sometimes exceed any reasonable experimental time scale (in fact they can even evolve on the scale of geological eras). The high degree of complexity of such systems makes the description of their dynamics really awkward, since apparently all the history of a sample has to be known in order to predict its evolution. Conversely, a useful theory should be able to provide a description of the behavior of the system in terms of few and essential effective parameters. The effective temperature, defined on the basis of fluctuation-dissipation relations between correlation and response functions [1], has been proved very fruitful in this direction (at least for mean-field models) and it is currently under intensive experimental investigation [2].

To introduce the concept let us consider the following experiment. Prepare a system (e.g., a glass, a ferromagnet, etc.) in an equilibrium state corresponding to a high temperature \( T_0 \) (where “high” means, here, greater than any critical or glass transition temperature). At time \( t = 0 \), quench the system to some temperature \( T < T_0 \) by taking it into contact with a thermal bath at temperature \( T \), and let it evolve. On a general basis, one expects that the relaxation towards the equilibrium state corresponding to \( T \) is characterized by two different regimes: (A) a transient one with non-equilibrium evolution, for \( t < t_{eq}(T) \), and (B) a stationary regime with equilibrium evolution for \( t > t_{eq}(T) \), where \( t_{eq}(T) \) is some characteristic equilibration time of the system. During (A) the behavior of the system is expected to depend on the specific initial conditions and both time-reversal symmetry and time-translation invariance are broken, while they are recovered in regime (B): The dynamics of fluctuations is given by the “equilibrium” one. We are concerned here with those systems for which the regime (B) is never achieved during experimental times, i.e., for all practical purposes, \( t_{eq} = \infty \). In this case, the standard concepts of equilibrium statistical mechanics do not apply and in particular two-time quantities, such as the response \( R_\mathcal{O}(t, s) = \left. \langle \delta \mathcal{O}(t) \delta h_\mathcal{O}(s) \rangle \right|_{h_\mathcal{O}=0} \) and the correlation functions \( C_\mathcal{O}(t, s) = \langle \mathcal{O}(t) \mathcal{O}(s) \rangle \) (\( \mathcal{O} \) is some observable and \( h_\mathcal{O} \) its conjugated field, we will be more explicit later on) depend separately on \( s \) (usually called the “age” of the system, being the time spent in the phase with \( t_{eq} = \infty \)) and \( t \), even for long times. This behavior is usually referred to as aging [3,4]. Such a useful quantity as the temperature \( T \) of the system is not defined in this genuine non-equilibrium regime. On the other hand one can address the question whether some effective temperature \( T_{eff} \) (in general different from \( T \)) can be still defined and used to understand the physics of the system.

In equilibrium [regime (B)], correlation and response functions depend only on time differences and the fluctuation-dissipation theorem (FDT) states that

\[
TR_\mathcal{O}(t - s) = \partial_s C_\mathcal{O}(t - s),
\]
[here and in the following we assume $t > s$ given that causality implies $R_O(t < s, s) = 0$]

where $T$ is expressed in units $k_B = 1$. Whatever the regime is, one can always define the so-called fluctuation-dissipation ratio (FDR) as [5]

$$X_O(t, s) = \frac{T R_O(t, s)}{\partial_s C_O(t, s)}.$$  \hspace{1cm} (2)

As a consequence of Eq. (1), $X_O(t, s) = 1$ whenever $t > s \gg t_{\text{eq}}(T)$. Some sort of effective temperature [1] in the aging regime [$t_{\text{eq}}(T) = \infty$] can be defined via the long-time limit of $X_O(t, s)$

$$X_O^\infty = \lim_{s \to \infty} \lim_{t \to \infty} X_O(t, s),$$  \hspace{1cm} (3)

through the relation $T_O^{\text{eff}} = T/X_O^\infty$ (that reduces to the thermodynamic temperature of the thermal bath when equilibrium is asymptotically reached). Obviously this definition has to be regarded as formal as long as one is not able to establish a link between $T_{\text{eff}}$ and some thermodynamic properties. Nevertheless, in Ref. [1] it has been shown that $T_{\text{eff}}$ plays the same role as the thermodynamic temperature, in the sense that it determines the direction of heat flows and acts as a criterion for thermalization. Moreover, a thermometer coupled to the observable $O$ measures (on a proper time scale) the temperature $T_O^{\text{eff}}$ [1]. We also mention that it has been argued that $X_O(t, s)$ establishes a bridge between the dynamically inaccessible equilibrium state and the asymptotic dynamics for large times [6]. (See also Refs. [7,8] for a discussion of other properties of $X_O(t, s)$ and $T_O^{\text{eff}}$.)

It has been stressed several times in the literature (see, e.g., Ref. [9]) that the effective temperature can be of interest in order to devise some thermodynamics for the system provided that its value is independent of the observable used to define it. This has been explicitly verified for infinite-range (mean-field) glass models [1]. Beyond these cases, the observable dependence of $T_O^{\text{eff}}$ has been investigated analytically for the trap model [10], for the one-dimensional Ising model [11,12], and for the $d$-dimensional spherical model [13], whereas numerical studies have addressed the problem for supercooled liquids [14] and for the two-dimensional Ising model [11].

In this paper we discuss the problem of the observable independence of the FDR (and consequently of $T_O^{\text{eff}}$) in a simpler (compared to glasses) class of slow-relaxing systems: Critical systems quenched from a high-temperature phase to the critical point and evolving according to a purely dissipative dynamics. In fact, soon after the introduction of the FDR Eq. (2), it was pointed out [15] that also these systems display slow-relaxation (due to $t_{\text{eq}} \sim \xi^z$, where $\xi$ is the correlation length, diverging at the critical point, and $z$ the dynamical critical exponent [16]) and aging. The FDR of the order parameter was then determined for a Gaussian model and for the random walk. Subsequent analytical and numerical calculations on realistic models confirmed and generalized this picture [17–42,11–13] (for a review, see Ref. [43]). For our analysis we can take advantage of the powerful tools of Renormalization Group (RG) and field theory to provide analytical predictions for some FDR’s in an $\epsilon$-expansion, where $\epsilon = 4 - d$ and $d$ is the spatial dimensionality of the system. This kind of study allows us to clarify once and for all whether a unique effective temperature can be defined for such systems in the long-time limit.
The outline of the paper is the following. In Sec. II we introduce the model and the field-theoretical approach to non-equilibrium dynamics. We derive, by means of RG equations, the scaling forms of two quadratic (in the order parameter) observables. In Sec. III we argue that the FDR $X^O_\infty$ (and thus $T^O_{\text{eff}}$) does not depend on the specific observable $O$ within the Gaussian (mean-field) approximation. The effects of the interaction are taken into account in Sec. IV, where we show by an explicit calculation that the FDR’s of two quadratic observables differ already at the lowest order beyond the Gaussian approximation. In Sec. V our results are carefully compared with the numerical and analytic ones available in the literature. In Sec. VI we summarize the results obtained and their implications. In the Appendix we report the details of the computations of the Feynman diagrams.

II. THE MODEL

One among the simplest non-trivial models displaying slow-relaxation and aging is a lattice spin model in $d$ dimensions with $O(N)$ symmetry and short-range interactions evolving according to a purely dissipative dynamics after a quench to the critical point. In the simplest instance its lattice Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle ij \rangle} s_i \cdot s_j,$$

where $s_i$ is a $N$-component spin located at the lattice site $i$, with $s_i^2 = 1$. The sum runs over all pairs $\langle ij \rangle$ of nearest-neighbor lattice sites. A purely dissipative dynamics for this model proceeds by elementary moves that amount to random changes in the direction of the spin $s_i$ (spin-flip sampling). The transition rates can be arbitrarily chosen provided that the detailed-balance condition is satisfied. For analytical studies the most suited are the Glauber ones [44], which allow exact solutions in the one-dimensional case [17–19,11,12].

Despite their simplicity, these models are not exactly solvable (for arbitrary $N$) in physical dimensions $d = 2, 3$, and to obtain information about the non-equilibrium critical dynamics, one has to resort to numerical simulations [20,21,11,29–33].

To investigate analytically the dynamical behavior in physical dimensions, we take advantage of the universality considering the time evolution of a $N$-component field $\varphi(x,t)$ with a purely dissipative dynamics (Model A of Ref. [16]). This is described by the stochastic Langevin equation

$$\partial_t \varphi_i(x,t) = -\Omega \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi_i(x,t)} + \xi_i(x,t),$$

where $\Omega$ is the kinetic coefficient, $\xi(x,t)$ a zero-mean stochastic Gaussian noise with

$$\langle \xi_i(x,t)\xi_j(x',t') \rangle = 2\Omega \delta(x-x')\delta(t-t')\delta_{ij},$$

and $\mathcal{H}[\varphi]$ is the static Hamiltonian. It may be assumed, near the critical point, of the Landau-Ginzburg form

$$\mathcal{H}[\varphi] = \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} r_0 \varphi^2 + \frac{1}{4!} g_0 \varphi^4 \right],$$
where \( r_0 \propto T \) is the temperature parameter, assuming its critical value \( r_{0,c} \) for \( T = T_c \) \( (r_{0,c} = 0) \) within the analytical approach discussed below, and \( g_0 \) is the bare coupling constant of the theory. This coarse-grained continuum dynamics is expected to be in the same universality class as the lattice models with \( O(N) \) symmetry, short-range interactions, and spin-flip dynamics [16].

The equilibrium correlation and response functions can be obtained by means of the field-theoretical action [45,46]

\[
S[\varphi, \tilde{\varphi}] = \int dt \int d^d x \left[ \tilde{\varphi} \frac{\partial \varphi}{\partial t} + \Omega \tilde{\varphi} \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi} - \varphi \Omega \tilde{\varphi} \right].
\]  

Here \( \tilde{\varphi}(x, t) \) is a \( N \)-component auxiliary field, conjugated to the external field \( h \) in such a way that \( \mathcal{H}[\varphi, h] = \mathcal{H}[\varphi] - \int d^d x h \varphi \). As a consequence, the linear response to the field \( h \) of a generic observable \( \mathcal{O} \) is given by

\[
\frac{\delta \langle \mathcal{O} \rangle}{\delta h_i(x, s)} = \Omega \langle \tilde{\varphi}_i(x, s) \mathcal{O} \rangle, \quad i = 1, \ldots, N.
\]  

For this reason \( \tilde{\varphi}(x, t) \) is termed response field.

Within this field-theoretical formalism it is possible to show that the FDT holds for generic \( R_\mathcal{O} \) and \( C_\mathcal{O} \). This has an illuminating derivation in a supersymmetric formulation, where the FDT's are the Ward identities due to the supersymmetry [45].

The effect of a macroscopic initial condition \( \varphi_0(x) = \varphi(x, t = 0) \) may be taken into account by averaging over the initial configuration with a weight \( e^{-H_0[\varphi_0]} \) where [47]

\[
H_0[\varphi_0] = \int d^d x \frac{\tau_0}{2} [\varphi_0(x) - a(x)]^2,
\]

that specifies an initial state \( a(x) \) with Gaussian short-range correlations proportional to \( \tau_0^{-1} \).

Following standard methods [45,46] the response and correlation functions may be obtained by a perturbative expansion of the functional weight \( e^{-(S[\varphi, \tilde{\varphi}]+H_0[\varphi_0])} \) in terms of the coupling constant \( g_0 \). The propagators (Gaussian two-point functions of the fields \( \varphi \) and \( \tilde{\varphi} \)) of the resulting theory are [47]

\[
\langle \tilde{\varphi}_i(q, s) \varphi_j(-q, t) \rangle_0 = \delta_{ij} R_{\mathcal{O}}^0(t, s) = \delta_{ij} \theta(t - s) G(t - s),
\]

\[
\langle \varphi_i(q, s) \varphi_j(-q, t) \rangle_0 = \delta_{ij} C_{\mathcal{O}}^0(t, s) = \frac{\delta_{ij}}{q^2 + r_0} \left[ G(|t - s|) + \left( \frac{r_0 + q^2}{\tau_0} - 1 \right) G(t + s) \right],
\]

where \( \theta(t) \) is the step function \( [\theta(t \leq 0) = 0, \theta(t > 0) = 1] \) and

\[
G(t) = e^{-\Omega(q^2 + r_0)t}.
\]
all the diagrams with loops of response propagators have to be omitted. This ensures that causality holds in the perturbative expansion [47,48,46]. From the technical point of view, the breaking of time-translation invariance does not allow the factorization of connected correlation functions in terms of one-particle irreducible ones as usually done when time-translation invariance holds. As a consequence, as it is the case when dealing with surface critical phenomena [49], all the computation has to be done in terms of connected functions only [47]. Furthermore it has been shown [47] that $\tau_0^{-1}$ is an irrelevant variable for the RG flow affecting only the correction to the leading long-time scaling behavior we are interested in. In view of that we fix it to its fixed-point value $\tau_0^{-1} = 0$ from the very beginning of the calculation.

From scaling arguments [20], and more rigorously from the solution of RG equations [47], it is known that the zero-momentum response and correlation functions of the basic fields satisfy the scaling forms (see Ref. [43] for a review):

$$
R_{q=0}(t, s) = A_R (t - s)^a(t/s)^\theta F_R(s/t),
$$

$$
C_{q=0}(t, s) = A_C s(t - s)^\theta(t/s)^\theta F_C(s/t),
$$

where $a = (2 - \eta - z)/z$, $z$ is the dynamical critical exponent, $\eta$ the anomalous dimension of the fields [45], and $\theta$ the initial-slip exponent [47,48]. We single out explicitly the non-universal amplitudes $A_{R,C}$ by fixing $F_R(0) = 1$. With this normalization $F_{R,C}$ are universal scaling functions. From the previous scaling forms one deduces that

$$
\partial_s C_{q=0}(t, s) = A_{\partial C} (t - s)^a(t/s)^\theta F_{\partial C}(s/t),
$$

where the non-universal amplitude $A_{\partial C}$ has been defined so that $F_{\partial C}(0) = 1$. Accordingly one has $A_{\partial C} = A_C(1 - \theta)$.

Using Eqs. (14) and (16) one finds that

$$
\chi_{q=0}(t, s) = \frac{R_{q=0}(t, s)}{\partial_s C_{q=0}(t, s)} = \frac{A_R F_R(s/t)}{A_{\partial C} F_{\partial C}(s/t)},
$$

is a universal amplitude-ratio (in the sense of Ref. [50]) being the ratio of two quantities [$R_{q=0}(t, s)$ and $\partial_s C_{q=0}(t, s)$] that have the same scaling dimensions. Furthermore it is a function of the ratio $s/t$ only, and not of $s$ and $t$ separately.\(^1\) In lattice simulations (especially in the literature concerning glassy systems [7,8]) response and correlation functions are often measured in the real space $x$, instead of in the momentum space $q$ as done here. From the scaling forms given above one can derive the analogous ones in the real space for $R_x, C_x$, and then define $X_{x=0} \equiv R_{x=0}/\partial_s C_{x=0}$ (originally introduced in Refs. [5,15]) in analogy with $\chi_{q=0}$. In general one expects $X_{x=0} \neq \chi_{q=0}$. Nonetheless it has been argued [25] that the long-time limit of the universal amplitude ratio

$$
X_M^\infty = \lim_{s \to \infty} \lim_{t \to \infty} \frac{R_{q=0}(t, s)}{\partial_s C_{q=0}(t, s)} = \lim_{s/t \to 0} \chi_{q=0}(s/t) = \frac{A_R}{A_{\partial C}} = \frac{A_R}{A_C(1 - \theta)},
$$

\(^1\)This is an important difference compared to mean-field glassy model, where, instead, it turns out that $X_{x=0}$ in the long-time regime can be written as a function of $C(t, s)$ (see, e.g., Refs. [7,8]).
(the subscript \( M \) refers to the fact that \( \varphi_{q=0} \propto M \), the average magnetization) is equal to the same limit of \( X_{x=0} \). This equality was also confirmed by numerical simulations [11] (even if the numerics of Ref. [11] have been questioned [51]).

### A. Scaling forms of composite operators

We now derive scaling forms analogous to Eqs. (14) and (15) for the correlation and response functions of local composite operators, focusing on those of the form \( \varphi^m \). However, the derivation is completely general and can be easily applied to any other operators.

Let us consider an observable \( \mathcal{O} \) (a composite operator, using the field-theoretical terminology) having \( h_{\mathcal{O}} \) as a conjugate field (e.g., \( \mathcal{O} \) is the energy density and \( h_{\mathcal{O}} \) the temperature) and coupling to \( \mathcal{H} \) according to \( \mathcal{H} \rightarrow \mathcal{H} + h_{\mathcal{O}} \mathcal{O} \). As a consequence, the dynamical functional \( S \) changes according to

\[
S \rightarrow S_{\mathcal{O}} = S + \Omega h_{\mathcal{O}} \tilde{\mathcal{O}},
\]

where the associated operator \( \tilde{\mathcal{O}} \) is given by

\[
\tilde{\mathcal{O}} = \int dt d^d x \tilde{\varphi}(x, t) \frac{\delta \mathcal{O}}{\delta \varphi(x, t)}.
\]

The linear response of an observable \( \mathcal{A} \) to a variation in the field \( h_{\mathcal{O}} \) can be expressed as

\[
\left. \frac{\delta \langle \mathcal{A} \rangle_{h_{\mathcal{O}}}}{\delta h_{\mathcal{O}}} \right|_{h_{\mathcal{O}}=0} = \Omega \langle \mathcal{A} \tilde{\mathcal{O}} \rangle,
\]

where \( \langle \cdot \rangle_{h_{\mathcal{O}}} \) stands for the average over the dynamics associated with the dynamical functional in the presence of \( h_{\mathcal{O}} \). This generalizes Eq. (9).

To render finite the correlation functions with insertions of the operator \( \mathcal{O} \), of the form

\[
\langle [\varphi]^n [\tilde{\varphi}]^\tilde{n} [\mathcal{O}]^a [\tilde{\mathcal{O}}]^\tilde{a} \rangle,
\]

one additional renormalization (compared to those necessary when \( \mathcal{O} \) and \( \tilde{\mathcal{O}} \) are not inserted, evaluated in Ref. [46]) is required: \( \mathcal{O}_B = Z_{\mathcal{O}} \mathcal{O}_R \). Here and in the following with the subscript \( B \) we indicate the bare quantities and with \( R \) the renormalized ones, whose correlation functions are finite upon removing the regularization [45]. In the case of operators mixing under renormalization, \( \mathcal{O} \) has to be understood as a vector of suitable operators, while \( Z_{\mathcal{O}} \) will be in general a renormalization matrix [45]. The presence of additive renormalizations does not change the scaling arguments presented below. In view of Eq. (19) one finds that \( \tilde{\mathcal{O}}_B = Z_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}_R \) with

\[
Z_{\tilde{\mathcal{O}}} = (Z_{\tilde{\varphi}}/Z_{\varphi})^{1/2} Z_{\mathcal{O}},
\]

where \( Z_{\varphi} \) and \( Z_{\tilde{\varphi}} \) are the renormalization constants of the fields, defined, as usual, by \( \varphi_B = Z_{\varphi}^{1/2} \varphi_R \) and \( \tilde{\varphi}_B = Z_{\tilde{\varphi}}^{1/2} \tilde{\varphi}_R \). The correlation function (21) can be renormalized according to

\[
\langle [\varphi]^n [\tilde{\varphi}]^\tilde{n} [\mathcal{O}]^a [\tilde{\mathcal{O}}]^\tilde{a} \rangle_B = Z_{\varphi}^{n/2} Z_{\tilde{\varphi}}^{\tilde{n}/2} Z_{\mathcal{O}}^a Z_{\tilde{\mathcal{O}}}^{\tilde{a}} \langle [\varphi]^n [\tilde{\varphi}]^\tilde{n} [\mathcal{O}]^a [\tilde{\mathcal{O}}]^\tilde{a} \rangle_R,
\]

where, on the r.h.s., \( \langle \cdot \rangle_R \) means that all the bare quantities have been replaced by the corresponding renormalized ones. Applying standard techniques it is possible to write the
RG equations by introducing appropriate RG functions [45]. For the theory defined by $S$, in addition to the RG functions of the theory with action $S$, one has to introduce two new functions $\varrho O \equiv \mu \partial_\mu \ln Z_O|_0$ and $\varrho \tilde{O} \equiv \mu \partial_\mu \ln Z_{\tilde{O}}|_0^2$ ($\mu$ is the scale at which the theory has been renormalized [45]). With $|_0$ we indicate that the differentiation has to be done with fixed bare parameters. In view of Eq. (22), $\varrho O$ and $\varrho \tilde{O}$ are related by

$$\varrho \tilde{O} = \varrho O + (\varrho \tilde{\varphi} - \varrho \varphi)/2$$

(where $\varrho \varphi$ and $\varrho \tilde{\varphi}$ are the usual RG functions for the fields). Combining dimensional analysis with the solution of the RG equations one finds that the scaling dimension of the correlation function $\langle [O][\tilde{O}] \rangle_R$ in the $(x, t)$-space is given, at the infrared fixed point of the theory, by

$$\delta(o, \tilde{o}) = o [O]_{\text{scal}} + \tilde{o} [\tilde{O}]_{\text{scal}}.$$  \hspace{1cm} (24)

With $[\cdot]_{\text{scal}}$ we indicate the scaling dimensions. In terms of the fixed-point values of the RG functions and the canonical (engineering) mass dimensions of the operators (denoted by $[\cdot]_{\text{can}}$) they are expressed as:

$$[O]_{\text{scal}} \equiv [O]_{\text{can}} + \varrho O(g^*) ,$$

$$[\tilde{O}]_{\text{scal}} \equiv [\tilde{O}]_{\text{can}} + \varrho \tilde{O}(g^*) ,$$  \hspace{1cm} (25)

where $g^*$ is the fixed-point value of the renormalized coupling constant. In analogy with the anomalous dimensions $\eta = \varrho \varphi(g^*)$ and $\tilde{\eta} = \varrho \tilde{\varphi}(g^*)$ (such that $[\varphi]_{\text{scal}} = (d - 2 + \eta)/2$ and $[\tilde{\varphi}]_{\text{scal}} = (d + 2 + \tilde{\eta})/2$ [46]) one introduces $\eta O$ and $\eta \tilde{O}$:

$$\frac{d - 2 + \eta O}{2} \equiv [O]_{\text{scal}} ,$$

$$\frac{d + 2 + \eta \tilde{O}}{2} \equiv [\tilde{O}]_{\text{scal}} .$$  \hspace{1cm} (26)

(Note that, in contrast to $\eta$ and $\tilde{\eta}$, $\eta O$ and $\eta \tilde{O}$ do not generally vanish in the free theory $g^* = 0$.) It is easy to verify that, in terms of $\eta O$, the critical exponent $\gamma O$ of the susceptibility ($\langle OO \rangle \sim |T - T_c|^{-\gamma O}$) is given by

$$\gamma O = \nu(2 - \eta O) ,$$  \hspace{1cm} (27)

where $\nu$ is the critical exponent of the correlation length. As a consequence of Eq. (19) one has $[\tilde{O}]_{\text{can}} = [O]_{\text{can}} + [\varphi]_{\text{can}} - [\varphi]_{\text{can}} = [O]_{\text{can}} + 2$ (recall that in Model A dynamics $[\varphi]_{\text{can}} = (d - 2)/2$ and $[\tilde{\varphi}]_{\text{can}} = (d + 2)/2$). Using $(\tilde{\eta} - \eta)/2 = z - 2$ (a consequence of the fluctuation-dissipation theorem [45]), we have

$$\frac{\eta \tilde{O} - \eta O}{2} = z - 2$$  \hspace{1cm} (28)

and $[\tilde{O}]_{\text{scal}} = [O]_{\text{scal}} + z$, leading to

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$^2\varrho O$ and $\varrho \tilde{O}$ are the anomalous dimensions of $O$ and $\tilde{O}$, respectively. In the literature they are usually referred to as $\eta O$ or $\gamma O$ [45]. However, in the present case we use $\eta O$ and $\gamma O$ to indicate critical exponents.
\[
\delta(2, 0) = d - 2 + \eta_O,
\]
\[
\delta(1, 1) = d - 2 + \eta_O + z.
\] (29)

Taking into account that \([\text{time}]_{\text{scal}} = -z\), one finds (hereafter we set \(\Omega = 1\))

\[
\langle O(q, 0)O(-q, s) \rangle = (t - s)^{\alpha_{\phi} + 1} F_C(q^* (t - s), s/t),
\]
\[
\langle O(q, 0)\bar{O}(-q, s) \rangle = (t - s)^{\alpha_{\phi}} F_R(q^*(t - s), s/t),
\] (30) (31)

where

\[
a_{\phi} \equiv -\frac{\delta(1, 1) - d}{z} = \frac{2 - \eta_O - z}{z}.
\] (32)

Let us now discuss the effect of the temporal surface. The scaling dimension of \(\tilde{\phi}_0\) has been computed in Ref. [47], i.e., \([\tilde{\phi}_0]_{\text{scal}} = [\tilde{\varphi}]_{\text{scal}} + \eta_0/2\). (In terms of \(\eta_0\) and \(z\) the initial-slip exponent is given by \(\theta = -\eta_0/(2z)\) [47,48].) Consider now the case of the observables \(O^{(m)}(t)\) constructed with \(m\) fields \(\phi\) at time \(t\), i.e. \(O^{(m)}(t) \sim \phi^m(t)\). Eq. (19) implies that \(\tilde{O}^{(m)}(t) \sim \tilde{\phi}(t)\phi^{m-1}(t)\). The observable \(\tilde{O}^{(m)}\), obtained from \(O^{(m)}\) by replacing all the fields \(\phi\) with those \(\tilde{\phi}\), has dimension given by

\[
[\tilde{O}^{(m)}]_{\text{scal}} = [O^{(m)}]_{\text{scal}} + mz.
\] (33)

Moreover, keeping in mind the scaling dimension of \(\tilde{\phi}_0\) one has, for \(t = 0\),

\[
[\tilde{O}^{(m)}(0)]_{\text{scal}} = [\tilde{O}^{(m)}]_{\text{scal}} - m\theta z,
\] (34)

where \(\tilde{O}^{(m)}(0) = \tilde{O}^{(m)}(t = 0)\).

When inserted into correlation functions, \((\partial_t \phi)_{t=0} = 2\Omega \tilde{\phi}_0\), while the insertion of an initial field \(\phi_0\) vanishes [49]. Thus, for the operators \(O^{(m)}\) and \(\tilde{O}^{(m)}\) we expect the following short-distance expansion for \(t \to 0\)

\[
O^{(m)}(t) \sim \rho(t)\tilde{\phi}_0^m + \text{h.o.c.f.},
\]
\[
\tilde{O}^{(m)}(t) \sim \tilde{\rho}(t)\tilde{\phi}_0^m + \text{h.o.c.f.},
\] (35) (36)

where h.o.c.f. stands for higher order composite fields that contribute to this expansion only with subleading terms. As a consequence, the scaling dimensions of \(\rho\) and \(\tilde{\rho}\) are given by

\[
[\rho]_{\text{scal}} = -mz + m\theta z \quad \text{and} \quad [\tilde{\rho}]_{\text{scal}} = -(m - 1)z + m\theta z,
\] (37)

where we used Eqs. (33), (34), and the FDT. Taking into account that \([\text{time}]_{\text{scal}} = -z\), one concludes that, for \(s \to 0\),

\[
O^{(m)}(s) \sim s^{m-\theta} \quad \text{and} \quad \tilde{O}^{(m)}(s) \sim s^{m-1-\theta}.
\] (38)

It is now possible to rewrite the scaling forms (30) and (31) in a way that shows explicitly the behavior of the scaling functions \(F_C\) and \(F_R\) for \(s \to 0\), i.e.,

\[
\langle O^{(m)}(q, t)O^{(m)}(-q, s) \rangle = s(t - s)^{\alpha_{\phi} + (m - 1) + \theta} \hat{F}_C^{\phi}(q^*(t - s), s/t),
\]
\[
\langle O^{(m)}(q, t)\bar{O}^{(m)}(-q, s) \rangle = (t - s)^{\alpha_{\phi} + (m - 1) + \theta} \hat{F}_R^{\phi}(q^*(t - s), s/t),
\] (39) (40)
where now the functions $\hat{F}_C$ and $\hat{F}_R$ are regular for $s \to 0$. Furthermore they are also universal once we fix the normalization for small arguments.

Let us focus on the scaling properties of the correlation and response functions of the observables with $m = 2$. Because of the $O(N)$ symmetry of the underlying theory only two quadratic zero-momentum operators with different scaling dimensions exist, that can be written as

$$E(t) \equiv \int d^dx \sum_{i=1}^N \varphi_i^2(x, t) = \int \frac{d^d q}{(2\pi)^d} \sum_{i=1}^N \varphi_i(q, t) \varphi_i(-q, t), \quad (41)$$

$$T_{ij}(t) \equiv \int d^dx \varphi_i(x, t) \varphi_j(x, t) - \frac{1}{N} \delta_{ij} E(t) = \int \frac{d^d q}{(2\pi)^d} \varphi_i(q, t) \varphi_j(-q, t) - \frac{1}{N} \delta_{ij} E(t). \quad (42)$$

The corresponding response operators are given by

$$\tilde{E}(t) = \sum_{i=1}^N \int \frac{d^d q}{(2\pi)^d} 2\tilde{\varphi}_i(q, t) \tilde{\varphi}_i(-q, t), \quad (43)$$

$$\tilde{T}_{ij}(t) = \int \frac{d^d q}{(2\pi)^d} \left[ \tilde{\varphi}_i(q, t) \tilde{\varphi}_j(-q, t) + \tilde{\varphi}_i(q, t) \tilde{\varphi}_j(-q, t) \right] - \frac{1}{N} \delta_{ij} \tilde{E}(t). \quad (44)$$

In the following we will refer to them, generically, as $\mathcal{O}(t)$. The corresponding scaling functions are

$$C^\mathcal{O}(t, s) \equiv \langle \mathcal{O}(t) \mathcal{O}(s) \rangle = A^\mathcal{C}_C s(t - s)^{a_\mathcal{C}}(t/s)^{1+2\theta} F^\mathcal{C}_C(s/t), \quad (45)$$

$$R^\mathcal{O}(t, s) \equiv \langle \mathcal{O}(t) \tilde{\mathcal{O}}(s) \rangle = A^\mathcal{R}_R (t - s)^{a_\mathcal{O}}(t/s)^{1+2\theta} F^\mathcal{R}_R(s/t), \quad (46)$$

where the non-universal amplitudes $A^\mathcal{O}_{C,R}$ are determined so that $F^\mathcal{O}_{C,R}(0) = 1$.

In terms of these quantities we can write the (zero-momentum) FDR as

$$\chi^\mathcal{O}(t, s) \equiv \frac{\Omega R^\mathcal{O}(t, s)}{\partial_s C^\mathcal{O}(t, s)}, \quad (47)$$

(here we restore the actual value of $\Omega$, currently set to 1) that, as its analogous (17), depends only on the ratio $s/t$ and it is a universal function. In particular its universal long-time limit reads

$$\chi^{\mathcal{O}_\infty} = \lim_{s \to \infty} \lim_{t \to \infty} \chi^\mathcal{O}(t, s) = \frac{1}{2} \frac{A^\mathcal{R}_R}{(1-\theta) A^\mathcal{C}_C}. \quad (48)$$

1. Overview of the known values of the exponents

For the two specific cases of the quadratic operators $E(t)$ and $T_{ij}(t)$, the exponent $a_\mathcal{O}$ can be expressed in terms of more familiar critical exponents. In fact we obtained $a_\mathcal{O} = (2 - \eta_\mathcal{O} - z)/z$, with $\eta_\mathcal{O}$ scaling dimension of the operator $\mathcal{O}$. Using Eq. (27) we can express it in terms of the susceptibility exponent, given by\(^3\) (see, e.g., Ref. [52])

\(^3\)When hyperscaling holds (i.e., for $d \leq 4$) $2 - d\nu$ can be replaced by $\alpha$, the specific heat exponent.
\[ \gamma_\mathcal{O} = \begin{cases} 2 - d\nu, & \mathcal{O} = E, \\ -d\nu + 2\phi, & \mathcal{O} = T, \end{cases} \] (49)

where \( \phi \) is the so-called quadratic crossover exponent [45,52]. The exponents \( \gamma_E \) and \( \gamma_T \) are exactly known in \( d = 2 \) [45]. Even in \( d = 3 \) they are known with very high accuracy, in fact they have been computed up to five loops in the \( \epsilon \)-expansion [53] and six (sometimes seven) loops in fixed dimension \( d = 3 \) [54,52] for generic values of \( N \) (see Ref. [55] for a review). The dynamic critical exponents are currently known with a much less accuracy. In fact \( z \) has been computed up to three loops in the \( \epsilon \)-expansion [56] and up to four in fixed dimensions [57], whereas \( \theta \) is known only up to two loops in the \( \epsilon \)-expansion [47]. For \( d > 4 \) the mean-field results hold: \( a_\mathcal{O} = 1 - d/2 \) (for both \( E \) and \( T \)) and \( \theta = 0 \).

For later convenience, we report here explicitly the \( O(\epsilon) \) expansion of \( a_\mathcal{O}^4 \) (recall that \( z = 2 + O(\epsilon^2) \) [16])

\[ a_\mathcal{O} = \begin{cases} -1 + \frac{4 - N}{2(N + 8)}\epsilon + O(\epsilon^2), & \mathcal{O} = E \\ -1 + \frac{N + 4}{2(N + 8)}\epsilon + O(\epsilon^2), & \mathcal{O} = T \end{cases} \] (50)

and \( \theta \)

\[ \theta = \frac{N + 2}{N + 8} + O(\epsilon^2). \] (51)

The exponents are exactly known in the limit \( N \to \infty \): \( \nu^{-1} = d - 2, z = 2, \theta = 1 - d/4 \) [47], and \( \phi = 2\nu \) in \( d < 4 \), whereas \( \nu = 1/2, z = 2, \theta = 0, \) and \( \phi = 1 \) in \( d > 4 \) (mean-field exponents). Using these values, the scaling forms for the energy in \( d < 4 \) are

\[ C^E(t, s) = A^E_C (t - s)^{d/2 - 3}(t/s)^{1-d/2}F^E_C(s/t), \] (52)
\[ R^E(t, s) = A^E_R (t - s)^{d/2 - 3}(t/s)^{1-d/2}F^E_R(s/t), \] (53)

whereas for \( T \) in \( d < 4 \),

\[ C^T(t, s) = A^T_C (t - s)^{1-d/2}(t/s)^{1-d/2}F^T_C(s/t), \] (54)
\[ R^T(t, s) = A^T_R (t - s)^{1-d/2}(t/s)^{1-d/2}F^T_R(s/t). \] (55)

For \( d > 4 \) the scaling forms for \( E \) and \( T \) are the same:

\[ C^\mathcal{O}(t, s) = A^\mathcal{O}_C (t - s)^{1-d/2}(s/t)F^\mathcal{O}_C(s/t), \] (56)
\[ R^\mathcal{O}(t, s) = A^\mathcal{O}_R (t - s)^{1-d/2}(s/t)F^\mathcal{O}_R(s/t). \] (57)

---

\(^4\)In passing, let us mention that, for \( N = 1 \), \( \phi \) has a non-trivial value, even if the operator \( T_{ij}(t) \) is not defined for the Ising model. This fact has an interpretation in terms of a gas of \( N \)-color interacting loops (see, e.g., Ref. [58]) belonging to the same universality class as the \( O(N) \) model.
III. THE GAUSSIAN APPROXIMATION

For the Gaussian model the response and correlation functions are known exactly, so we can evaluate the FDR (in Ref. [15] this has been done directly in real space). From Eqs. (11), (12), and the definition (17), properly generalized to \( q \neq 0 \), one finds [25]

\[
X^0_q(t, s) = \Omega R^0_q = \frac{1}{1 + e^{-2\Omega(q^2+r_0)s}}. \tag{58}
\]

If the theory is non-critical (\( r_0 \neq 0 \)) the limit of this ratio for \( s \to \infty \) is 1 for all the values of \( q \), in agreement with the idea that in the high-temperature phase all the fluctuating modes have a finite equilibration time, so that equilibrium is recovered and the FDT applies. In the critical theory the limit ratio is again equal to one when \( q \neq 0 \), whereas for \( q = 0 \) one has \( X^0_{q=0}(t, s) = 1/2 \). This shows that the only mode that “does not relax” to the equilibrium is the zero mode in the critical limit. This picture (already presented in Ref. [25]) has been confirmed by a one-loop computation [25].

In the case of the Gaussian theory, with \( g_0 = 0 \) in the Hamiltonian (7), the FDR can be easily computed for a generic observable. In particular we show that the FDR for a set of local one-point observables is always equal to 1/2 in the long-time limit. This allows for a definition of a unique effective temperature in the Gaussian model. We consider operators of the form \( O_{i,n} = \partial_i \varphi^n \). The correlation and response functions of all local operators can be written in terms of those of \( O_{i,n} \).

The critical (i.e., \( r_0 = 0 \)) response and correlation functions of the order parameter Eqs. (11) and (12) are given, in real space, by (their diagrammatic representation is reported in Fig. 1 (b) and (a), respectively)

\[
R_x(t, s) = F_d(t-s, x), \tag{59}
\]

\[
C_x(t, s) = K_d(t-s, x) - K_d(t+s, x), \tag{60}
\]

where (we set \( \Omega = 1 \)) \( F_d(\tau, x) = \int (dq)e^{-q^2\tau}e^{-iq\cdot x} \) and \( K_d(\tau, x) = \int (dq)q^{-2}e^{-q^2\tau}e^{-iq\cdot x} \) (with \( dq = d^dq/(2\pi)^d \)).

Let us explain our argument considering first the operators \( O_{0,n} \). The two-point correlation functions of \( O_{0,n} \) is given by the \( (n-1) \)-loop diagram with the two points connected by \( n \) correlation lines, as shown in Fig. 1 (c).

In the real space its expression is simply given by the product of \( n \) correlators. Thus

\[
C^O_x(t, s) = c_n[C_x(t, s)]^n, \tag{61}
\]

(\( c_n \) is the combinatorial factor associated with the diagram) whose derivative is

\[
\partial_s C^O_x(t, s) = c_n n[C_x(t, s)]^{n-1} \partial_s C_x(t, s). \tag{62}
\]

Analogously the response function is given by the diagram depicted in Fig. 1 (d), obtained from that one contributing to the correlation function (Fig. 1 (c)) by replacing an order-parameter correlator with a response function:

\[
R^O_x(t, s) = c_n n[C_x(t, s)]^{n-1} R_x(t, s), \tag{63}
\]
FIG. 1. Diagrammatic elements: (a) correlation and (b) response functions of the order parameter. (c) Diagrammatic representation for the two-point correlation function $C_x^O(t, s) = \langle O_{0,n}(t, x)O_{0,n}(s, 0) \rangle$ and (d) response function $R_x^O(t, s) = \langle O_{0,n}(t, x)\tilde{O}_{0,n}(s, 0) \rangle$. See the text for further explanations.

where the factor $n$ comes from the fact that the response operator is $\tilde{O}_{0,n} = nO_{0,n-1}\tilde{\phi}$. Note that the combinatorial factor $c_n$ is the same as for the correlation function.

The FDR is given by

$$X_{x}^{O_{0,n}}(t, s) = \frac{R_{x}^{O_{0,n}}(t, s)}{\partial_s C_{x}^{O_{0,n}}(t, s)} = \frac{R_{x}(t, s)}{\partial_s C_{x}(t, s)} = X_{x}^{M}(t, s).$$

Thus we obtain the remarkable result that the FDR of powers of the field in real space is equal, for all times, to the FDR of the field.

Before considering the effects of the derivatives let us consider the previous relation in momentum space. Remembering that the product of two functions after a Fourier transformation becomes a convolution, one has

$$R_{q}^{O}(t, s) = c_n n(C \ast \cdots \ast C \ast R)_q,$$

$$\partial_s C_{q}^{O}(t, s) = c_n n(C \ast \cdots \ast C \ast C')_q,$$

where $\ast$ is the convolution, $C$ and $R$ are in momentum space (with the time dependence understood), $\cdots$ means $n - 1$ times, and $C'_q = \partial_s C_q$. For $q = 0$ the previous relations become

$$R_{q=0}^{O}(t, s) = c_n n \int dp (C \ast \cdots \ast C)_p R_{-p},$$

$$\partial_s C_{q=0}(t, s) = c_n n \int dp (C \ast \cdots \ast C)_p C'_{-p}.$$

Thus

$$\left[X_{q=0}^{O}(t, s)\right]^{-1} = \frac{\int dp (C \ast \cdots \ast C)_p C'_{-p}}{\int dp (C \ast \cdots \ast C)_p R_{-p} X_{-p}^{-1}}.$$

13
i.e., the inverse of $X_q^{0} = 0$ is a weighted average of $X_{-p}^{-1}$, with weight $(C \ast \cdots \ast C)_p R_{-p}$, that in the limit $t \to \infty$, $s \to \infty$ in the proper order, is peaked around $p = 0$, giving $X_q^{s} = \lim_{s \to \infty} \lim_{t \to \infty} X_q^{0}(t, s) = \lim_{s \to \infty} \lim_{t \to \infty} X_{p=0}(t, s) = X_{M}^{s}$. Note that in momentum space, at variance with the relation Eq. (64) holding between FDR in real space, only the long-time limit of the FDR reproduces the FDR for the fields. This is in agreement with some explicit calculations we made for $\varphi^2$ (see below) and $\varphi^3$ FDR’s.$^5$

Let us now take into account the effect of the derivatives. In momentum space they amount simply to a multiplication by $q^i$ and so they change the weight in Eq. (69) by a factor $q^{2i}$ (one $q^i$ for each insertion), not affecting our conclusion on the long-time limit.

Let us note that if instead we introduce some non-local operators, as e.g. $e^{\partial^2 \mathcal{O}}$ (with $\mathcal{O}$ a local operator), the weight in Eq. (69) is no longer exponential in $q^2$ and the long-time limit of FDR may be different from $1/2$.

**IV. TWO-LOOP COMPUTATION**

Here we present the details of the two-loop (i.e., up to $O(\epsilon)$ in the $\epsilon$-expansion) perturbative computation of the correlation and response functions for the zero-momentum energy and the tensor operators given by Eqs. (41) and (42). The one-loop results are the Gaussian ones obtained in the previous section.

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$^5$The Gaussian $\varphi^3$ FDR is obtained from Eqs. (A9) and (B6) of Ref. [26]. The calculation is straightforward, but cumbersome.
The diagrammatic contributions are reported in Fig. 2 for the correlation function \( C^O(t, s) \) and in Fig. 3 for the response function \( R^O(t, s) \). In Table I we list the symmetry factors of the diagrams (depending both on the global topology and on the external legs) and the corresponding color factors (depending on global topology and the index structure of the vertices).

For later convenience let us introduce the function

\[
I_n(t) = \int \frac{d^d q}{(2\pi)^d} \frac{e^{-2q^2 t}}{q^{2n}} = I_n(1)t^{-d/2+n},
\]

where

\[
I_n(1) = N_d \frac{\Gamma(d/2-n)}{2^{n-d/2-n} \Gamma(d/2)},
\]

and \( N_d = 2/[(4\pi)^{d/2}\Gamma(d/2)] \).

Thus we obtain, for the one-loop diagrams\(^6\)

\[\text{FIG. 3. Diagrams contributing to the response function.}\]

\(^6\)Note that without the temporal surface at \( t = 0 \), i.e., using the equilibrium correlator \( C^{(eq)}(t_1 - t_2) \), \( B_C \) would be given only by \( I_2(|t_1 - t_2|) \). For \( d = 4 - \epsilon \), \( I_2 \) has a pole \( [I_2(1) \propto \Gamma(2 - d/2)] \), that is usually subtracted by introducing the well-known additive renormalization for the \( \varphi^2 \) correlation function. In this case, however, the time-boundary term yields two additional factors that exactly cancel the pole of the first term. Accordingly, in contrast to the equilibrium case, no additive renormalization has to be introduced to render finite the energy correlation function. Thus the equilibrium \textit{renormalized} correlation functions for \( q = 0 \) cannot be simply recovered by taking the limit \( t, s \to \infty \) with \( t - s \) fixed, as usually done for field correlation functions.
Using the non-universal constants given in Eqs. (76) and (78) we find that
\[ \langle \mathcal{O} \rangle_{\text{diagrammatic contributions}} = 0 \text{ due to the causality of the response functions).} \]

where here and hereafter we set Ω = 1 to lighten the notation and \( t > s \) (for \( t < s \) the diagrammatic contributions are zero due to the causality of the response functions).

In terms of the two-loop Feynman integrals the response and correlation functions read
\[ \begin{align*}
R_B^O(t, s) &= 4C_1 R_B(t, s) - 4C_{2a} g G_2(t, s) - 2C_{2b} g [L_1(t, s) + L_2(t, s) + L_3(t, s)] + O(g^2), \\
C_B^O(t, s) &= 2C_1 B_C(t, s) - 2C_{2a} g [G_1(t, s) + G_1(s, t)] - 2C_{2b} g [L_4(t, s) + L_4(s, t)] + O(g^2).
\end{align*} \]

The calculations of \( G_i \) and \( L_i \) are quite cumbersome and all the details are reported in the Appendix. Taking into account the expressions therein we find renormalized response and correlation functions that are in agreement with equations (46) and (45), with the exponents given in Eqs. (50) and (51). The non-universal amplitudes and scaling functions are given by (as usual we introduce \( \tilde{g} \equiv N_d \bar{g} \) to simplify the notation)
\[ \begin{align*}
A_R^{E,T} &= \left[ 1 - \frac{\epsilon}{2} + \left( \frac{\epsilon}{2} - \frac{C_{2a}}{C_1} \tilde{g} \right) (\gamma_E + \log 2) + \frac{C_{2b}}{C_1} \tilde{g} \right] N_d C_1, \\
F_R^{E,T}(x) &= 1 + \left( \frac{\epsilon}{2} - \frac{C_{2b}}{C_1} \tilde{g} \right) \left( \frac{\log(1 - x)}{x} \right) + 1 - \log(1 - x), \\
A_C^{E,T} &= \left[ 1 - \frac{\epsilon}{2} + \left( \frac{\epsilon}{2} - \frac{C_{2a}}{C_1} \tilde{g} \right) (\gamma_E + \log 2) + \frac{C_{2b}}{C_1} \tilde{g} \right] N_d C_1, \\
F_C^{E,T}(x) &= \frac{1 - x}{x^2} \left[ - \log(1 - x^2) - \frac{\epsilon}{4} W_1(x) - \frac{C_{2a}}{C_1} \tilde{g} W_2(x) - \frac{C_{2b}}{C_1} \tilde{g} W_3(x) \right],
\end{align*} \]

where the functions \( W_i(x) \) have been defined in Eqs. (A24), (A25), and (A26). The previous equations are valid up to \( O(\bar{g}^2, \epsilon \tilde{g}, \epsilon^2) \).

According to Eq. (48), \( \mathcal{X}^{\infty} \) can be expressed as an amplitude ratio
\[ \mathcal{X}^{\infty} = \frac{A_R^O}{2 A_C^O (1 - \theta)}. \]

Using the non-universal constants given in Eqs. (76) and (78) we find that
\[ \mathcal{X}^{\infty} = \frac{1}{2} \left( 1 - \frac{C_{2b} \tilde{g}^*}{C_1} - \frac{C_{2a}}{C_1} \tilde{g} \right) + O(\epsilon^2), \]
where
\[
\tilde{g}^* = \frac{6}{N+8} \epsilon + O(\epsilon^2),
\] (82)
is the fixed-point value of the coupling constant [45]. Taking into account the combinatorical factors in Table I we finally find:

\[
X_{\infty}^O = \begin{cases} 
\frac{1}{2} \left( 1 - \frac{2N+2}{3N+8} \epsilon \right) + O(\epsilon^2), & O = E, \\
\frac{1}{2} \left( 1 - \frac{1}{12} \frac{3N+16}{N+8} \epsilon \right) + O(\epsilon^2), & O = T.
\end{cases}
\] (83)

Recall that the one-loop FDR for the order parameter (magnetization) is [25]

\[
X_{\infty}^M = \frac{1}{2} \left( 1 - \frac{1}{4} \frac{N+2}{N+8} \epsilon \right) + O(\epsilon^2).
\] (84)

From these results we conclude that the long-time limit of the fluctuation-dissipation ratio depends on the particular observable chosen to compute it.

As a final remark let us comment on the connection between the correlation functions of $\varphi^2$ in Model A dynamics and those of the conserved density $\varepsilon$ in the associated Model C [16]. In equilibrium dynamics it is usually easier to compute $\langle \varphi^2 \varphi^2 \rangle$ in terms of $\langle \varepsilon \varepsilon \rangle$ (see, e.g., Ref. [59]). However, this is no longer possible when studying non-equilibrium dynamics (considered in Ref. [28]) because of the connection between $\varphi$ and $\varepsilon$ does not carry over to this case.

V. COMPARISON WITH OTHER RESULTS

A. The spherical model and the $N \to \infty$ limit

The static critical behavior of the $O(N)$ model in the limit $N \to \infty$ is known to be equivalent to that of the spherical model, defined by the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} \sum_{\langle ij \rangle} (s_i - s_j)^2,
\] (85)

where $s_i$ are real numbers subjected to the constraint $\sum_i s_i^2 = L^d$ ($L$ being the linear dimension of the $d$-dimensional lattice, assumed for simplicity to be hypercubic), and the sum runs over all nearest-neighbor pairs $\langle ij \rangle$. Indeed Stanley proved [60] that the free energies of the two models are exactly the same. From this equality it follows that critical exponents, scaling functions etc. are equal. The same equivalence also holds for equilibrium critical dynamics defined in the spherical model by means of a Langevin equation as Eq. (5)\(^7\). As

\(^7\)This equation has to be properly modified in order to prescribe a dynamics that is compatible with the spherical constraint.
far as we are aware, this equivalence has not yet been carried over to the non-equilibrium critical dynamics we are interested in. Let us recall that the free energies of the models in the presence of a spatial boundary (whose corresponding field theory looks very similar to the non-equilibrium one that we are considering) are well known to be not equal [61].

The spherical model has attracted a lot of attentions, since its essential Gaussian Hamiltonian makes it exactly solvable although the resulting critical behavior is not mean-field like because of the spherical constraint. The FDR of the magnetization was calculated by Godrèche and Luck, who obtained $X^\infty_M = 1 - 2/d$. This result is compatible with the two-loop $\epsilon$-expansion [26] for $N = \infty$ and, moreover, it agrees with the exact result we are going to derive. Recently Sollich [13] has determined several FDR’s of quadratic (in the spin $s_i$) operators that could be compared with our results. He considered the bond energy observable $B_i = \frac{1}{2}(s_i - s_j)^2$, the product observable $P_i = s_is_j$ (with i,j nearest neighbors) both in the real and in the momentum space, and the total energy. The exact results of his analysis are that $X^\infty_P = X^\infty_B = X^\infty_M \neq X^\infty_E$ (the expansion of $X^\infty_E$ close to four dimensions for $d < 4$ nicely agrees with our two-loop $\epsilon$-expansion for $N = \infty$). The scaling forms for these observables have been also derived and they agree with our equations (31) and (30) with $a_B = -1 - d/2$, $a_P = 1 - d/2$, and $a_E = d/2 - 3$ in $d < 4$ (and $a_E = a_P = a_B + 2 = 1 - d/2$ in $d > 4$). All these findings agree with our calculation apart from the fact that the observable $P$ of the spherical model cannot be naively identified with $\varphi^2$ for $N = \infty$. On the other hand $a_B = a_P - 2$ agrees with the naive identification of $B$ as the Laplacian of $P$ in the continuum limit.

We report now the calculation of some FDR’s for the $O(N)$ model directly for $N = \infty$, using the well-known property that for $N = \infty$ the fourth order interaction term can be self-consistently decoupled $g/N(\varphi^2)^2 \rightarrow gC_{x=0}(t)\varphi^2$ (here $g_0$ is replaced by $g/N$) [45]. Taking advantage of this decoupling the exact response function of the theory (for $2 < d < 4$) has been computed [47], finding:

$$R_q(t, s) = \theta(t - s) \left( \frac{t}{s} \right)^{1-d/4} e^{-q^2(t-s)}, \quad (86)$$

whereas, for the correlation function [47]

$$C_q(t, s) = 2 \int_0^\infty dt' R_q(t, t')R_q(s, t') = 2(ts)^{1-d/4}e^{-q^2(t+s)} \int_0^s dt' t'^{d/2-2}e^{2qs}t'. \quad (87)$$

In particular for $q = 0$ the previous expressions become

$$R_{q=0}(t, s) = \theta(t - s) \left( \frac{t}{s} \right)^{1-d/4} \quad \text{and} \quad C_{q=0}(t, s) = \frac{2}{d/2 - 1}s \left( \frac{t}{s} \right)^{1-d/4} . \quad (88)$$

According to Eq. (18) it is straightforward to compute the FDR for the order parameter, finding $X^\infty_M = 1 - 2/d$, the same result as for the spherical model.

Now we consider, in the limit $N \rightarrow \infty$, the scaling functions (45) and (46) of the response and the correlation of the quadratic operators. The explicit computation for $2 < d < 4$ of the scaling functions, the non-universal amplitudes, and, eventually, the FDR of composite operators is not as straightforward as one could have erroneously expected. In fact, if one roughly assumes, on the sole basis of the decoupling in the large-$N$ limit, that the theory
FIG. 4. Chains of bubble diagrams for the (a) correlation and (b) response function of quadratic observables. For the correlation (a), the index $k$ runs between 1 and $n$.

is essentially Gaussian with renormalized two-point functions given by Eqs. (86) and (87), the result $X_\infty^\infty = X_M^\infty$ would follow from the argument we gave for the Gaussian Model, that makes no use of the specific expressions of the response and correlation functions.

Let us consider more closely the case of the energy $E$. For the two-point function $\langle E(t)E(s) \rangle$, a family of diagrams (chains of bubble diagrams, see Fig. 4) that are of order $O(N^0)$ even in the limit $N \to \infty$ exists (although the coupling constant is of order $1/N$, each bubble carries a combinatorical factor $N$). These diagrams are not accounted for by a simple renormalization of the two-point functions of the order parameter. This fact is well known for static observables (see, e.g., the calculation of the structure factors $\langle \varphi^2(q)\varphi^2(-q) \rangle$ up to four loops in Ref. [52]). If and only if the contribution of such diagrams to $X_\infty^\infty$ vanishes (or is cancelled out) one has $X_E^\infty = X_M^\infty$. The results of the previous sections indicate that this is not the case.

Note that if one considers instead of $E$ the operator $T$ then the chains of bubble diagrams are depressed by a combinatorical factor of order $1/N$ and so they do not contribute to the correlation and response functions for $N = \infty$. On this sole basis one concludes that in the limit $N = \infty$, $X_T^\infty = X_M^\infty$ to all order in $\epsilon$ and not only at the first one as we have explicitly obtained (compare Eq. (83) with Eq. (84) considering the limit $N \to \infty$).

This analysis indicates that the observable $P$ in the spherical model has a scaling behavior (and $X_\infty^\infty$) that is the same as that of $T$, contrarily to the naive expectation, suggesting instead $E$.

Finally let us mention that for all the quantities we considered here, the non-equilibrium dynamics of the spherical and $O(\infty)$ models are exactly the same. This fact calls for a more rigorous investigation of a possible correspondence between the two models beyond the case of equilibrium dynamics.
FIG. 5. $X^n_\infty$ for $O = M, E, T$ and $N = 1, 2, 3, \infty$. For $X_M^n$ we report the two-loop result [26]. For $X^n_E$ we report the direct estimate and those obtained by constraining at the lower critical dimensions (with linear constraint for $N = 1, 2, 3, \infty$ and with quadratic one for $N = 2, 3, \infty$). For $X^n_T$ only the direct estimate is reported. For $N = \infty$ we also report as “E exact” the exact result for the spherical model, from Ref. [13].

B. The Ising model

Apart from the spherical model, the FDR of composite operators has been considered so far only for the one- and two-dimensional Ising models, analytically [11,12] and numerically [11].

In one dimension the several FDR’s considered [11,12] turned out to equal $X^n_M = 1/2$, apart from that of the energy: $X^n_E = 0$. This fact has been interpreted in Ref. [11] as an interplay between criticality and coarsening, a peculiarity of those models (such as the one-dimensional Ising model) with $T_c = 0$. Instead, our result indicates that $X^n_E < X^n_M$ is a more general property, rigorously true close to four dimensions ($\epsilon \ll 1$) and perhaps valid up to $\epsilon = 3$.

In two dimensions, the numerical results $X^n_M = 0.340(5) \simeq X^n_E = 0.33(2)$ apparently
indicate the equality of the two FDR’s [11]. Obviously our result calls for a more precise determination of $X_E^\infty = 0.33(2)$ to understand whether this apparent equality is due to the relatively low precision of such a measure or to the fact that $d = 2$ is a peculiar case for some still unknown reasons. In any case, let us stress that $X_E^\infty$ from the $O(\epsilon)$ result for $\epsilon = 2$ gives a value much smaller than $X_M^\infty$. The direct estimate (unconstrained [u]) from the two-loop series in Eq. (83) for $\epsilon = 2$ (giving $X_E^\infty[u] \sim 0.28$) is probably unreliable as it was the case for $X_M^\infty$. ($X_M^\infty[1\text{loop}] \sim 0.42$ [25] whereas $X_M^\infty[2\text{loop}] = 0.30(5)$ [26].) To obtain a more reliable result (without computing the $O(\epsilon^2)$ term, that seems to be a very difficult calculation, requiring the evaluation of three-loop diagrams) one can constrain linearly [l] the $O(\epsilon)$ result to assume the exactly known value for $d = 1$ (i.e., $\epsilon = 3$), as usually done for this kind of expression (see, e.g., Ref. [55]). Assuming a smooth behavior in $\epsilon$ up to $\epsilon = 3$, one can write

$$X_E^\infty[l] = \frac{1}{2} \left( 1 - \frac{\epsilon}{3} \right) \left[ 1 + \frac{\epsilon}{9} + O(\epsilon^2) \right],$$

(89)

that has the same $\epsilon$-expansion as Eq. (83), but it is expected to converge more rapidly to the correct result. From Eq. (89) we get for the two-dimensional Ising model $X_E^\infty[l] \sim 0.20$, that is much lower than the value that has been determined so far $X_M^\infty \simeq 0.33$ [26,29,11,30,32]. However, we stress that a robust field-theoretical prediction for $X_E^\infty$ for the two-dimensional Ising model requires a (difficult) higher-loop computation. For the three-dimensional Ising model we obtain $X_E^\infty[l] \simeq 0.37$, to be compared with the direct estimate $X_E^\infty[u] \simeq 0.39$. Note that, as usual in $d = 3$, the spreading of the different estimates is much smaller, signaling a higher reliability of these predictions. (We recall that, in three dimensions, the field-theoretical estimate for the FDR of the magnetization is $X_M^\infty[2\text{loop}] = 0.429(6)$ [26].)

In Fig. 5 we report Eq. (89) for $0 < \epsilon < 3$ compared with the direct estimate Eq. (83) and with the two-loop FDR of the magnetization [26].

### C. The $O(N)$ model with $N \geq 2$.

Eq. (83) allows us to provide predictions for the $O(N)$ models with arbitrary $N$. So far the non-equilibrium dynamics of models with continuous symmetry ($N > 1$) has been numerically studied only for the $XY$ ($N = 2$) model, both in $d = 2$ [21,31] and $d = 3$ [33]. The value that has been determined in $d = 3$, i.e., $X_M^\infty = 0.43(4)$ [33] is in good agreement with the two-loop field-theoretical prediction $X_M^\infty = 0.416(8)$ [26]. The observable dependence of $X_M^\infty$ has not yet been addressed in these cases.

To obtain an estimate of $X_E^\infty$ for the three-dimensional $O(N)$ model we again constrain the $\epsilon$-expansion at the lower-critical dimension ($d_{l.c.d.} = 2$ in this case). We assume $X_E^\infty(d = 2) = 0$ for $N \geq 2$. This surely holds for $N > 2$, since for $d = 2$ these systems are in the coarsening regime. Such an assumption is instead questionable for $N = 2$, where it is known that a finite-temperature phase transition of topological nature takes place at $T_{KT}$. A linear constraint would lead to

$$X_E^\infty[l] = \frac{1}{2} \left( 1 - \frac{\epsilon}{2} \right) \left[ 1 + \frac{16 - N}{6(N + 8)}\epsilon + O(\epsilon^2) \right].$$

(90)
On the other hand the exact results for the spherical model [13] suggest that the approach to $d = 2$ is quadratic rather than linear, so it is tempting to implement a quadratic constraint also for finite $N$

$$X_E^{\infty}[q] = \frac{1}{2} \left( 1 - \frac{\epsilon}{2} \right)^2 \left[ 1 + \frac{20 + N}{3(N + 8)} \epsilon + O(\epsilon^2) \right].$$

(91)

In Fig. 5, we report Eqs. (90) and (91) for $0 < \epsilon < 2$ and $N = 2, 3, \infty$. It is apparent that for $N = \infty$ the approximation obtained implementing the quadratic constraint reproduces better the result for the spherical model. For the three-dimensional $XY$ model ($N = 2$) we get $X_E^{\infty} \simeq 0.37$ from direct estimate (no constraint, [u]), $X_E^{\infty} \simeq 0.31$ from linear constraint [l], and $X_E^{\infty} \simeq 0.22$ from the quadratic one [q]. For the three-dimensional Heisenberg model ($N = 3$) we find, instead, $X_E^{\infty}[u] \simeq 0.35$, $X_E^{\infty}[l] \simeq 0.30$, and $X_E^{\infty}[q] \simeq 0.21$.

Even the results with constraints are rather scattered, making very difficult to provide robust estimates in $d = 3$. However, we can surely conclude that $X_E^{\infty} < X_M^{\infty}$ for all $2 < d < 4$ and that their difference should be large enough to be observed in Monte Carlo simulations in three dimensions. The analysis of the non-equilibrium behavior within the $\tilde{\epsilon} = d - 2$ expansion [45] may clarify which, between the linear and the quadratic constraint, is the proper one close to $d = 2$, even if the results for the spherical model strongly suggest the latter.

For $X_T^{\infty}$ we note that it is very close to $X_M^{\infty}$, even for $\epsilon \ll 1$ (see Fig. 5), making the numerical detection of such a difference probably very difficult.

VI. CONCLUSIONS

In this paper we considered the problem of the definition of a unique effective temperature via the long-time limit of the fluctuation-dissipation relation for critical systems quenched from a high-temperature phase to the critical point and evolving according to a purely dissipative dynamics. Within the field-theoretical approach to non-equilibrium critical dynamics and by means of appropriate RG equations, we obtained the general scaling forms for the response and correlation functions of a generic local observable $O(t)$, from which it is possible to derive the FDR $X_O(t, s)$.

We found that in the Gaussian approximation all the local operators have $X_O^{\infty} = 1/2$, allowing for a definition of a unique effective temperature. This equality is broken already at the first order in the $\epsilon$ expansion for the quadratic operators we considered (namely the total energy and the tensor, see Eqs. (41) and (42)). Let us point out that our results go further than those obtained for the one-dimensional Ising model [11,12] and for the spherical model [13]. In these cases $X_O^{\infty} = X_M^{\infty}$ for all the observables $O$, except for the total energy. This operator is conjugated to the temperature of the bath but not to the actual one (if any) of the system. On this basis one could doubt that the energy operator is not as suited as others to define the effective temperature, resulting in a different $X^{\infty}$. Nevertheless we find that there is at least one further operator, namely $T_{ij}$, having $X_T^{\infty} \neq X_M^{\infty}, X_E^{\infty}$. This explicitly shows that, at variance with what is often conjectured, a unique effective temperature can not be defined for this class of models.
Our results for \(N \to \infty\) are always in agreement with the recent ones for the spherical model \([13]\), calling for a proof of a possible correspondence (or for a counterexample). They instead disagree (at qualitative level) with the available numerical simulation of the Ising model \([11]\) apparently giving \(X_E^\infty = X_M^\infty\). Probably a more accurate measure of \(X_E^\infty\) is required to detect the difference (if any) between \(X_E^\infty\) and \(X_M^\infty\). We also provided theoretical predictions for the \(O(N)\) model for arbitrary \(N\) in \(2 < d < 4\). It should be possible to check them quantitatively in \(d = 3\), where the \(\epsilon\)-expansion is expected to be more accurate. This calls for numerical simulations or real experiments in three-dimensional systems.

Let us finally comment that it would be interesting to understand how the standard scenario of effective temperature \([1,62,7]\) can be generalized to the case when each sector of a theory has a different \(T_{\text{eff}}\), as our results indicate to be the case for critical systems beyond the mean-field approximation.

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**APPENDIX A: TWO-LOOP FEYNMAN DIAGRAMS**

In this appendix we report all the details of the evaluation of two-loop integrals. Here and in the following we will denote \(t_\prec = \min\{t_1, t_2\}\) and \(t_\succ = \max\{t_1, t_2\}\). For the diagram \(G_1\) one finds

\[
G_1(t_1, t_2) = \int_0^{t_1} dt' B_R(t_1, t') B_{CC}(t', t_2)
= \int_0^{t_1} dt' I_2(t' + t_2)[I_1(t_1 - t') - I_1(t_1)]
+ \int_0^{t_\succ} dt'[I_2(t_2 - t') - 2I_2(t_2)][I_1(t_1 - t') - I_1(t_1)]
+ \theta(t_1 - t_2) \int_{t_2}^{t_1} dt'[I_2(t' - t_2) - 2I_2(t')][I_1(t_1 - t') - I_1(t_1)].
\]  

(A1)

To isolate the dimensional poles of this expression one has to keep in mind that \(I_1(t)\) diverges for \(t \to 0\) and \(d = 4\). These singularities have to be removed by proper subtractions. The remaining part can be expanded in a regular power series of \(\epsilon = 4 - d\). Then one finds that

\[
G_1(t_1, t_2) = I_2(1)I_1(1)t_1 \left\{ \int_0^1 dx \left[ \left( x + \frac{t_2}{t_1} \right)^{\epsilon/2} - \left( 1 + \frac{t_2}{t_1} \right)^{\epsilon/2} \right] \right\} [1 - (1 - x)^{-1+\epsilon/2} - 1]
\]  

23
For the following computations it is useful to introduce the function $W(x)$ defined as

\[
W(x) = -2 + \frac{\pi^2}{2} - 2x + 2 \log(1 - x^2) + \left( x + \frac{1}{x} \right) \log \frac{1 + x}{1 - x} - 2 \log x \log(1 - x) + \log x \log(1 + x) - 2 \log^2(1 - x) - 3 \log^2(1 + x) - 2 \text{Li}_2(1 - x) - \text{Li}_2 \left( \frac{1}{1 + x} \right) + O(x).
\]  

(A4)

whose expansion for small $x$ is $W(x) = -17x^2/6 + O(x^3)$. $W(x)$ enters the expression of $G_1(t, s) + G_1(s, t)$ as

\[
G_1(t, s) + G_1(s, t) = -\frac{N_d}{2} \frac{1}{\epsilon} \log \left( 1 - \frac{s^2}{t^2} \right) - \frac{N_d}{2} (\gamma_E + \log 2 + \log t) \log \left( 1 - \frac{s^2}{t^2} \right) + \frac{N_d^2}{8} W(s/t). 
\]  

(A5)
The expression of the diagram $G_2$ is

$$G_2(t, s) = \int_s^t dt' B_R(t, t')B_R(t', s),$$  \hspace{1cm} (A6)$$

after the proper subtractions one finds

$$G_2(t, s) = I_1^2(1)t^{-1+\epsilon}\left\{ \int_{s/t}^1 dx \left[ (1-x)^{-1+\frac{\epsilon}{2}} - \left(1 - \frac{s}{t}\right)^{-1+\frac{\epsilon}{2}} \right] \left[ (x - \frac{s}{t})^{-1+\frac{\epsilon}{2}} - \left(1 - \frac{s}{t}\right)^{-1+\frac{\epsilon}{2}} \right] \\
+ \left( \frac{4}{\epsilon} - 1 \right) \left(1 - \frac{s}{t}\right)^{-1+\epsilon} - \int_{s/t}^1 dx (1-x)^{-1+\epsilon/2}(x^{-1+\epsilon/2} - 1) \\
- \frac{4}{\epsilon} \left(1 - \frac{s}{t}\right)^{\epsilon/2} + \frac{2}{\epsilon} \left[ 1 - \left(\frac{s}{t}\right)^{\epsilon/2} \right] \right\},$$  \hspace{1cm} (A7)$$

whose expansion is given by

$$G_2(t, s) = \frac{N_d^2}{4} \frac{s}{t t - s} \left\{ \frac{1}{\epsilon} + \gamma_E + \log 2 + \log t + \frac{1}{2} \left(1 + \frac{t}{s}\right) \log \left(1 - \frac{s}{t}\right) \right\} + O(\epsilon).$$  \hspace{1cm} (A8)$$

In order to compute the diagrams $L_i$ one has to determine the tadpole (loop of the correlation function), given by

$$P(t) = \int \frac{d^d q}{(2\pi)^d} C^0(t, t) = -I_1(t),$$  \hspace{1cm} (A9)$$

where the dimensional regularization has been used. The contribution of the tadpole to the two-point correlation function of the order parameter (i.e., $\langle \varphi(q, t_1)\varphi(-q, t_2) \rangle$, a subdiagram of $L_i$), is given by

$$D_{RC}(t_1, t_2; q) = \int_0^{t_1} dt' R^0_q(t_1, t') P(t') C^0_q(t', t_2)$$

$$= -\left[ \int_{t_2}^{t_1} dt' I_1(t') \frac{e^{-q^2(t_1+t_2-2t')}}{q^2} - I_1(1) \frac{t_2^{2-d/2}}{2-d/2} \right]$$

$$-\theta(t_1 - t_2)^2 I_1(1) C^0_q(t_1, t_2) \frac{t_1^{2-d/2} - t_2^{2-d/2}}{2-d/2}. \hspace{1cm} (A10)$$

While the contribution to the response function $\langle \varphi(q, t)\varphi(-q, s) \rangle$ is given by

$$D_{RR}(t, s; q) = \int_s^t dt' R^0_q(t, t') P(t') R^0_q(t', s) = -I_1(1) \frac{t_2^{2-d/2} - s^{2-d/2}}{2-d/2} R_q(t, s).$$  \hspace{1cm} (A11)$$

Using the previous expressions, the diagram $L_1$ can be written as

$$L_1(t, s) = \int \frac{d^d q}{(2\pi)^d} C^0_q(t, s) D_{RR}(t, s; q) = -I_1(1) \frac{t_2^{2-d/2} - s^{2-d/2}}{2-d/2} B_R(t, s)$$

$$= -I_1^2(1) \frac{t_2^{2-d/2} - s^{2-d/2}}{2-d/2} (t-s)^{1-d/2} - t_2^{2-d/2}],$$  \hspace{1cm} (A12)$$
whose expansion is
\[ L_1(t, s) = \frac{N_d^2 s}{16 t} \frac{1}{t - s} \log \frac{s}{t} + O(\epsilon). \] (A13)

Analogously
\[ L_2(t, s) = \int \frac{d^d q}{(2\pi)^d} R_q^0(t, s) D_{RC}(s, t; q) = - \int_0^s dt' I_1(t') I_1(t - t') + I_1(1) I_1(t) \frac{s^{2-d/2}}{2-d/2}, \] (A14)
and subtracting from the integrand its singular behavior for \( t' \to 0 \), we find
\[ L_2(t, s) = - \int_0^s dt' I_1(t') [I_1(t - t') - I_1(t)] = -I_1^2(1)t^{3-d} \int_0^{s/t} dx x^{1-d/2}[(1 - x)^{1-d/2} - 1], \] (A15)
whose expansion is
\[ L_2(t, s) = \frac{N_d^2}{16t} \log \left(1 - \frac{s}{t}\right) + O(\epsilon). \] (A16)

The expression for \( L_3 \) is given by
\[ L_3(t, s) = \int \frac{d^d q}{(2\pi)^d} R_q^0(t, s) D_{RC}(t, s; q) \]
\[ = - \int_0^s dt' I_1(t') I_1(t - t') + I_1(1) I_1(t) \frac{s^{2-d/2}}{2-d/2} - I_1(1) \frac{t^{2-d/2} - s^{2-d/2}}{2-d/2} B_R(t, s), \] (A17)
that, keeping into account Eqs. (A12) and (A14), leads to
\[ L_3(t, s) = L_2(t, s) + L_1(t, s). \] (A18)

The last diagram is \( L_4 \), for which we find
\[ L_4(t_1, t_2) = \int \frac{d^d q}{(2\pi)^d} C_q^0(t_1, t_2) D_{RC}(t_1, t_2; q) \]
\[ = - \left\{ \int_0^{t^{<}_{\epsilon/t^{<}_{\epsilon}}} dt' I_1(t') [I_2(t_> - t') - I_2(t_1 + t_2 - t')] \right. \]
\[ - I_1(1) \frac{t^{2-d/2}_{\epsilon}}{2-d/2} [I_2(t_>) - I_2(t_1 + t_2)] \left. \right\} - \theta(t_1 - t_2) I_1(1) \frac{t^{2-d/2}_{1} - t^{2-d/2}_{2}}{2-d/2} B_C(t_1, t_2). \] (A19)

Subtracting the singular part of the integrands one finds
\[ L_4(t_1, t_2) = -I_1(1) I_2(1) \]
\[ \times \left\{ t^{<}_{\epsilon} \int_0^{t^{<}_{\epsilon/t^{<}_{\epsilon}}} dx x^{1+d/2} \left[ (1 - x)^{d/2} - \left(1 + \frac{t^{<}_{\epsilon}}{t^{>}_{\epsilon}} - x\right)^{d/2} \right] - 1 + \left(1 + \frac{t^{<}_{\epsilon}}{t^{>}_{\epsilon}}\right)^{d/2} \right\} \]
\[ + \theta(t_1 - t_2) \frac{2}{\epsilon} \left( t^{1/2}_{1} - t^{1/2}_{2} \right) \left[ (t_1 - t_2)^{d/2} + (t_1 + t_2)^{d/2} - 2t^{d/2}_{1} \right], \] (A20)
whose expansion is

\[ L_4(t_1, t_2) = \frac{N_d^2}{8} \left[ \text{Li}_2 \left( \frac{t_{<}/t_2}{1 + t_{<}/t_2} \right) - \text{Li}_2 \left( \frac{t_{<}/t_2}{1 + t_{<}/t_2} \right) + \theta(t_1 - t_2) \frac{N_d^2}{8} \log \frac{t_2}{t_1} \log \left( 1 - \frac{t_2^2}{t_1^2} \right) + O(\epsilon) \right]. \]  

(A21)

Inserting the previous expressions into Eqs. (74) and (75) we end up for the response function with

\[ R_B(t, s) = \left( 1 - \frac{C_{2a} \tilde{g}}{C_1} \right) A_{R} \frac{s}{t} \frac{1}{t - s} \times \left[ 1 + \left( \frac{\epsilon}{2} - \frac{C_{2a} \tilde{g}}{C_1} \right) \log(t - s) \right] \left[ 1 + \frac{C_{2b} \tilde{g}}{C_1} \frac{\log t}{s} \right] F_R(s/t) + O(\tilde{g}^2, \epsilon \tilde{g}, \epsilon^2), \]  

(A22)

and for the correlation function

\[ C_B(t, s) = \left( 1 - \frac{C_{2a} \tilde{g}}{C_1} \right) A_{C} \frac{s^2}{t} \frac{1}{t - s} \times \left[ 1 + \left( \frac{\epsilon}{2} - \frac{C_{2a} \tilde{g}}{C_1} \right) \log(t - s) \right] \left[ 1 + \frac{C_{2b} \tilde{g}}{C_1} \frac{\log t}{s} \right] F_C(s/t) + O(\tilde{g}^2, \epsilon \tilde{g}, \epsilon^2), \]  

(A23)

where \( A_R, F_R, A_C, \) and \( F_C \) are the expressions reported in the text [Eqs. (76), (77), (78), and (79)]. To shorten the formulae we introduce the following functions:

\[ W_1(x) = \log^2(1 + x) - \log^2(1 - x) - 2 \log(1 - x) \log(1 + x) + 2 \log(1 - x^2), \]  

(A24)

\[ W_2(x) = W(x) + \log^2(1 + x) + 3 \log^2(1 - x) + 2 \log(1 - x) \log(1 + x) - \frac{5}{6} \log(1 - x^2), \]  

(A25)

\[ W_3(x) = \text{Li}_2 \left( \frac{x}{1 + x} \right) - \text{Li}_2(x) - \log(1 - x^2). \]  

(A26)

The bare expressions (A22) and (A23) have to be renormalized according to [using the fact that \( \mathcal{O}_B = Z_\mathcal{O} \mathcal{O}_R, \mathcal{O}_B = Z_\mathcal{O} \mathcal{O}_R, \mathcal{O}_B = (Z_\mathcal{O}/Z_\phi)^{1/2} \mathcal{O}_R = \mathcal{O}_R + O(\tilde{g}^2, \epsilon \tilde{g}, \epsilon^2), \) and \( Z_\mathcal{O} = (Z_\mathcal{O}/Z_\phi)^{1/2} Z_\mathcal{O} = Z_\mathcal{O} + O(\tilde{g}^2, \tilde{g} \epsilon, \epsilon^2) = (1 - \frac{C_{2a} \tilde{g}}{C_1} \epsilon) + O(\tilde{g}^2, \tilde{g} \epsilon, \epsilon^2) \) [45]]

\[ C_R(t, s) = \left( 1 + \frac{C_{2b} \tilde{g}}{C_1} \right) C_B(t, s) \]

\[ R_R(t, s) = \left( 1 + \frac{C_{2a} \tilde{g}}{C_1} \right) R_B(t, s). \]  

(A27)

Exponentiating the logarithms, we recover the expected scaling forms and exponents with

\[ a_\mathcal{O} + 1 = \frac{\epsilon}{2} - \frac{C_{2a} \tilde{g}}{C_1} = \begin{cases} 
E : & \frac{4 - N}{2(N + 8)} \epsilon + O(\epsilon^2), \\
T : & \frac{N + 4}{2(N + 8)} \epsilon + O(\epsilon^2), 
\end{cases} \]  

(A28)

and

\[ 2\theta = \frac{C_{2b} \tilde{g}}{C_1} = \frac{N + 2}{2(N + 8)} \epsilon + O(\epsilon^2) \]  

for \( E, T \),

(A29)

in agreement with Eqs. (50), (51) and the scaling forms (39) and (40), both for \( E \) and \( T \).
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