Factorization of the Shoenfield-like bounded functional interpretation

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Abstract

We adapt Streicher and Kohlenbach’s proof of the factorization $S = KD$ of the Shoenfield translation $S$ in terms of Krivine’s negative translation $K$ and the Gödel functional interpretation $D$, obtaining a proof of the factorization $U = KB$ of Ferreira’s Shoenfield-like bounded functional interpretation $U$ in terms of $K$ and Ferreira and Oliva’s bounded functional interpretation $B$.

1 Introduction

In 1958, Gödel [5] presented a functional interpretation $D$ of Heyting arithmetic $\text{HA}^\omega$ into itself (actually, into a quantifier-free theory, for foundational reasons). When composed with a negative translation $N$ of Peano arithmetic $\text{PA}^\omega$ into $\text{HA}^\omega$ (Gödel [4]), it results in a two-step functional interpretation $ND$ of $\text{PA}^\omega$ into $\text{HA}^\omega$ [5]. Nine years later, Shoenfield [9] presented a one-step functional interpretation $S$ of $\text{PA}^\omega$ into $\text{HA}^\omega$.

In 2007, Streicher and Kohlenbach [10], and independently Avigad [1], proved the factorization $S = KD$ of $S$ in terms of $D$ and a negative translation $K$ due to Streicher and Reus [11], inspired by Krivine [8].

$$\text{PA}^\omega \overset{K}{\rightarrow} \text{HA}^\omega \overset{D}{\rightarrow} \text{HA}^\omega$$

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In 2005, Ferreira and Oliva [3] presented a functional interpretation \( B \) of Heyting arithmetic with majorizability \( \text{HA}^\omega_2 \) into itself. Like \( D \), when composed with a negative translation \( N \) of Peano arithmetic with majorizability \( \text{PA}^\omega_2 \) into \( \text{HA}^\omega_2 \), it results in a two-step functional interpretation \( NB \) of \( \text{PA}^\omega_2 \) into \( \text{HA}^\omega_2 \) [3]. Two years later, Ferreira [2] presented a one-step functional interpretation \( U \) of \( \text{PA}^\omega_2 \) into \( \text{HA}^\omega_2 \). By adapting Streicher and Kohlenbach’s proof, we obtain the factorization \( U = KB \).

\[
\begin{array}{c}
\text{PA}^\omega_2 \xrightarrow{K} \text{HA}^\omega_2 \\
\text{HA}^\omega_2 \xrightarrow{B} \text{HA}^\omega_2 \\
\text{HA}^\omega_2 \\
\end{array}
\]

2 Framework

**Definition 1** ([3][12]). The Heyting arithmetic \( \text{HA}^\omega \) that we consider is the usual Heyting arithmetic in all finite types, but with a minimal treatment of equality and no extensionality, following Anne Troelstra [12].

The Heyting arithmetic with majorizability \( \text{HA}^\omega_2 \) is obtained from \( \text{HA}^\omega \) by

1. adding new atomic formulas \( t \leq_{\rho} q \) for all finite types \( \rho \) (where \( t \) and \( q \) are terms of type \( \rho \));

2. adding syntactically new bounded quantifications \( \forall x \leq_{\rho} tA \) and \( \exists x \leq_{\rho} tA \) (where \( A \) is a formula and the variable \( x \) does not occur in the term \( t \));

3. adding the axioms

\[
\forall x \leq tA \leftrightarrow \forall x(x \leq t \rightarrow A), \quad \exists x \leq tA \leftrightarrow \exists x(x \leq t \land A)
\]

governing the bounded quantifications;

4. adding the axioms and rule

\[
x \leq_0 y \leftrightarrow x \leq y, \quad x \leq y \rightarrow \forall u \leq v(xu \leq yv \land yu \leq yv),
A_b \land u \leq v \rightarrow tu \leqqv \land qu \leqqv
\]

\[
\frac{A_b \rightarrow t \leq q}{A_b \rightarrow t \leq q}
\]

governing the majorizability symbol \( \leq \) (where \( \leq_0 \) is the usual inequality between terms of type 0, \( A_b \) is a bounded formula, that is, a formula with all quantifications bounded, and in the rule the variables \( u \) and \( v \) do not occur free in the formula \( A_b \) neither in the terms \( t \) and \( q \));

5. extending the induction axiom to the new formulas.

This system is presented in detail in [3].

We will need the following notation.
Notation 2 ([3]). An underlined letter \( t \) means a tuple (possibly empty) of terms \( t_1, \ldots, t_n \). We use the abbreviations
\[
\begin{align*}
\forall \underline{t} A & : \equiv \forall x_1 \cdots \forall x_n A, \\
\exists \underline{t} A & : \equiv \exists x_1 \cdots \exists x_n A,
\end{align*}
\]
\[
\begin{align*}
\forall \underline{t} A & : \equiv \forall x_1 \leq t_1 \cdots \forall x_n \leq t_n A, \\
\exists \underline{t} A & : \equiv \exists x_1 \leq t_1 \cdots \exists x_n \leq t_n A,
\end{align*}
\]
\[
\begin{align*}
\forall \underline{t} A & : \equiv \forall x (x \leq t \rightarrow A), \\
\exists \underline{t} A & : \equiv \exists x (x \leq t \land A), \\
\forall \underline{t} A & : \equiv \forall x \leq t (x \leq x \rightarrow A), \\
\exists \underline{t} A & : \equiv \exists x \leq t (x \leq x \land A).
\end{align*}
\]

We consider two logical principles.

Definition 3. The law of excluded middle for bounded formulas \( \text{B-LEM} \) is the principle
\[
A_b \lor \neg A_b,
\]
where \( A_b \) is a bounded formula.

Definition 4 ([2]). The monotone bounded choice \( \text{B-mAC} \) is the principle
\[
\forall \underline{t} A \equiv \forall x \leq t \forall Y \exists \underline{t} A(x, y) \rightarrow \exists Y \forall \underline{t} A(x, y),
\]
where \( A_b \) is a bounded formula.

3 Negative translation and bounded functional interpretations

For the convenience of the reader, we recall the definitions of \( K \), \( B \) and \( U \).

Definition 5 ([1][8][13][14]). Krivine’s negative translation (extended to arithmetic with majorizability\(^1\)) of a formula \( A \) of \( \text{PA}_\omega^\omega \) based on \( \neg, \lor, \forall \leq, \forall \) is \( A^K : \equiv \neg A_K \), where \( A_K \) is defined by induction on the complexity of formulas.

1. If \( A \) is an atomic formula, then \( A_K : \equiv \neg A \).
2. \( (\neg A)_K : \equiv \neg A_K \).
3. \( (A \lor B)_K : \equiv A_K \land B_K \).
4. \( (\forall x \leq t A)_K : \equiv \exists x \leq t A_K \).
5. \( (\forall x A)_K : \equiv \exists x A_K \).

If we consider \( \land \) a primitive symbol, then:

6. \( (A \land B)_K : \equiv A_K \lor B_K \).

\(^1\)It still holds a soundness theorem \( \text{PA}_\omega^\omega \vdash A \Rightarrow A^K \) and a characterization theorem \( \text{PA}_\omega^\omega \vdash A \leftrightarrow A^K \).

3
Definition 6 ([3]). The bounded functional interpretation \( A^B \) of a formula \( A \) of \( \text{HA}^\omega \) based on \( \bot, \land, \lor, \rightarrow, \forall \exists, \exists \exists, \forall, \exists \) is defined by induction on the complexity of formulas.

1. If \( A \) is an atomic formula, then \( A^B \defeq \exists x y A_B(x, y) \defeq A \), where \( x \) and \( y \) are empty tuples.

If \( A^B \equiv \exists x y A_B(x, y) \) and \( B^B \equiv \exists x' y' B_B(x', y') \), then:

2. \((A \land B)^B \equiv \exists x y, x' y' (A \land B)_B(x, y, x', y') \equiv \exists x, x' y y'[A_B(x, y) \land B_B(x', y')];

3. \((A \lor B)^B \equiv \exists x y, x' y' (A \lor B)_B(x, y, x', y') \equiv \exists x, x' y y'[A_B(x, y) \lor B_B(x', y')];

4. \((A \rightarrow B)^B \equiv \exists x y, x' y' (A \rightarrow B)_B(x, y, x', y') \equiv \exists x', x y y'[A_B(x, y) \rightarrow B_B(x', y')];

5. \((\forall z \leq t A)^B \equiv \exists x y (\forall z \leq t A)_B(x, y) \equiv \exists x y A_B(x, y);

6. \((\exists z \leq t A)^B \equiv \exists x y (\exists z \leq t A)_B(x, y) \equiv \exists x y \exists z \leq t y A_B(x, y);

7. \((\forall z A)^B \equiv \exists x y, w (\forall z A)_B(x, y, w) \equiv \exists x y, w A_B(x, y, w);

8. \((\exists z A)^B \equiv \exists x y, w (\exists z A)_B(x, y, w) \equiv \exists x, w y A_B(x, y).

Remark 7 ([3]). From 1 and 4 we conclude that if \( A^B \equiv \exists x y A_B(x, y) \), then \( (\neg A)^B \equiv \exists x y (\neg A)_B(x, y) \equiv \exists x y \neg A(x, y) \).

Remark 8 ([3]). We can prove by induction on the complexity of formulas that \( A_B(x, y) \) is a bounded formula.

Definition 9 ([2]). The Shoenfield-like bounded functional interpretation \( A^U \) of a formula \( A \) of \( \text{PA}^\omega \) based on \( \neg, \lor, \forall \exists, \forall \) is defined by induction on the complexity of formulas.

1. If \( A \) is an atomic formula, then \( A^U \defeq \exists x y A_U(x, y) \defeq A \), where \( x \) and \( y \) are empty tuples.

If \( A^U \equiv \exists x y A_U(x, y) \) and \( B^U \equiv \exists x' y' B_U(x', y') \), then:

2. \( (\neg A)^U \equiv \exists x y (\neg A)_U(x, y) \equiv \exists x y \neg A_U(x, y);

3. \((A \land B)^U \equiv \exists x y, x' y' (A \land B)_U(x, y, x', y') \equiv \exists x, x' y y'[A_U(x, y) \land B_U(x', y')];

4. \((\forall z \leq t A)^U \equiv \exists x y (\forall z \leq t A)_U(x, y) \equiv \exists x y A_U(x, y).
5. $(\forall zA)^U := \forall w, \exists y (\forall z A)_U(w, x, y) := \forall w, \exists y \forall z \leq w A_U(x, y)$.

If we consider $\land$ a primitive symbol, then:

6. $(A \land B)^U := \forall x, x' \exists y, y'(A \land B)_U(x, x', y, y') := \forall x, x' \exists y, y' [A_U(x, y) \land B_U(x', y')].$

Remark 10 ([2]). We can also prove by induction on the complexity of formulas that $A_U(x, y)$ is a bounded formula.

$U$ is monotone on the second tuple of the variables, in the following sense.

Lemma 11 (monotonicity of $U$ [2]). $\text{HA}^*_2 \vdash \forall x \forall y \forall \tilde{y} \leq y [A_U(x, \tilde{y}) \rightarrow A_U(x, y)].$

4 Factorization

We want to prove $A^U \leftrightarrow (A^K)^B$ by induction on the complexity of formulas. Because it isn’t $A^K$ but $A_K$ that is defined by induction on the complexity of formulas, it would be better to write $A^U \leftrightarrow (-A_K)^B$. If $A^U := \forall x \exists y A_U(x, y)$ and $(A^K)^B := \exists x' \forall y' (A_K)_B(x', y')$, then using B-mAC in the first equivalence and the monotonicity of $U$ in the second equivalence, we have

$$
A^U \equiv \forall x \exists y A_U(x, y)
\leftrightarrow \exists y \forall x \exists y \leq y_x A_U(x, y)
\leftrightarrow \exists y \forall x A_U(x, Y_x),
(1)
\quad (-A_K)^B \equiv \exists x' \forall y' \neg \forall y' \leq y_x A_U(x', y').
(2)
$$

The comparison of formulas (1) and (2) suggests that we first prove $A_U(x, Y_x) \leftrightarrow \neg \forall y \leq Y_x (A_K)_B(x, y)$, or even better, $A_U(x, y) \leftrightarrow \neg \forall \tilde{y} \leq y (A_K)_B(x, \tilde{y})$. Then, by the above argument, we would have $A^U \leftrightarrow (A^K)^B$.

The factorization proof is almost the straightforward adaptation of Streicher and Kohlenbach’s proof but with two tweaks.

1. Instead of proving $A_U(x, y) \leftrightarrow \neg (A_K)_B(x, y)$, along the lines of Streicher and Kohlenbach’s proof, we prove $A_U(x, y) \leftrightarrow \neg \forall \tilde{y} \leq y (A_K)_B(x, \tilde{y})$, where the appearance of the quantification $\forall \tilde{y} \leq y$ is explained by the above argument.

2. In proving $A_U(x, y) \leftrightarrow \neg \forall \tilde{y} \leq y (A_K)_B(x, \tilde{y})$ we need the hypothesis $x \leq x \land y \leq y$ for technical reasons explained in footnotes.

Theorem 12 (factorization $U = KB$). We have

$$
\text{HA}^*_2 + \text{B-LEM} \vdash \forall x, x' [A_U(x, y) \leftrightarrow (A^K)_B(y, x)],
(3)
\text{HA}^*_2 + \text{B-LEM} + \text{B-mAC} \vdash A^U \leftrightarrow (A^K)^B.
(4)
$$
Proof. Step 1. First we prove

$$\text{HA}_0^+ + \text{B-LEM} \vdash \forall x, y[A_U(x, y) \leftrightarrow \neg \forall \tilde{y} \leq y(A_K)_B(x, \tilde{y})]$$  \hspace{1cm} (5)$$

by induction on the complexity of formulas.

Let us consider the case of atomic formulas \(A\). Using \text{B-LEM} in the equivalence, we have

\[
A_U \equiv A
\]

\[
\leftrightarrow \neg A
\]

\[
\equiv \neg (A_K)_B.
\]

Let us now consider the case of negation \(\neg A\). Assume \(Y \subseteq Y\) and \(x \subseteq x\). Using the induction hypothesis in the first equivalence and \text{B-LEM} in the second equivalence, we have

\[
(-A)_U(Y, \hat{x}) \equiv \exists \hat{x} \leq \hat{x} \neg A_U(\hat{x}, Y, \hat{x})
\]

\[
\leftrightarrow \exists \hat{x} \leq \hat{x} \neg \forall \tilde{y} \leq \tilde{y}(A_K)_B(\hat{x}, \tilde{y})
\]

\[
\leftrightarrow \neg \forall \tilde{y} \leq x \neg \forall \tilde{y} \leq \tilde{y}(A_K)_B(\hat{x}, \tilde{y})
\]

\[
\equiv \neg \forall \tilde{y} \leq \hat{x}[(-A)_K]_B(Y, \hat{x}).
\]

Let us now consider the case of disjunction \(A \lor B\). Assume \(x \subseteq x, x' \subseteq x', y \subseteq y,\) and \(y' \subseteq y'\). Using the induction hypothesis in the first equivalence, \text{B-LEM} in the second equivalence, and intuitionistic logic in the third equivalence\(^2\) we have

\[
(A \lor B)_U(x, x', y, y') \equiv A_U(x, y) \lor B_U(x', y')
\]

\[
\leftrightarrow \neg \forall \tilde{y} \leq y(A_K)_B(x, \tilde{y}) \lor \neg \forall \tilde{y} \leq y'(B_K)_B(x', \tilde{y}')
\]

\[
\leftrightarrow \neg [\forall \tilde{y} \leq y(A_K)_B(x, \tilde{y}) \lor \forall \tilde{y} \leq y'(B_K)_B(x', \tilde{y}')]\]

\[
\equiv \neg \forall \tilde{y}, \tilde{y}' \leq y, y'[A \lor B]_K]_B(x, x', \tilde{y}, \tilde{y}').
\]

Let us now consider the case of bounded universal quantification \(\forall z \leq tA\). Assume \(x \subseteq x\) and \(y \subseteq y\). Using the induction hypothesis in the first equivalence and

\[^2\text{The rule for conversion to prenex normal form } \forall u \leq v(C \land D) \rightarrow \forall u \leq v C \land D \text{ (where the variable } u \text{ does not occur free in the formula } D), \text{ despite its innocuous look, does not hold without the hypothesis } v \leq v. \text{ So we need to use the hypothesis } x \leq \hat{x} \land y \leq \hat{y} \text{ in the proof.}\]
intuitionistic logic in the second and third\footnote{Probably the easiest way to prove the third equivalence is to prove}
equivalences, we have
\[(\forall z \leq tA)_U(x, y) \equiv \forall z \leq tA_U(x, y)\]
\[\iff \forall z \leq t \neg \exists \tilde{y} \leq y(A_K)_B(x, \tilde{y})\]
\[\iff \neg \exists z \leq t \tilde{v}y \leq y(A_K)_B(x, \tilde{y})\]
\[\iff \neg \exists y \leq \tilde{y} \exists z \leq t \tilde{v}y \leq y(A_K)_B(x, \tilde{y})\]
\[\equiv \neg \exists y \leq y[(\forall z \leq tA)_K]_B(x, \tilde{y}).\]

Finally, let us consider the case of unbounded universal quantification \(\forall zA\). Assume \(w \leq w, x \leq x,\) and \(y \leq y\). Using the induction hypothesis in the first equivalence and intuitionistic logic in the second and third equivalences, we have
\[(\forall zA)_U(w, x, y) \equiv \forall z \leq wA_U(x, y)\]
\[\iff \forall z \leq w \neg \exists \tilde{y} \leq y(A_K)_B(x, \tilde{y})\]
\[\iff \neg \exists z \leq w \tilde{y} \leq y(A_K)_B(x, \tilde{y})\]
\[\iff \neg \exists \tilde{y} \leq y \exists z \leq w \tilde{y} \leq y(A_K)_B(x, \tilde{y})\]
\[\equiv \neg \exists \tilde{y} \leq y[(\forall zA)_K]_B(w, x, \tilde{y}).\]

In case we consider \(\wedge\) a primitive symbol, let us now see the case of conjunction \(A \wedge B\). Assume \(x \leq x, x' \leq x', y \leq y,\) and \(y' \leq y'\). Using the induction hypothesis in the first equivalence and intuitionistic logic in the second and third equivalences, we have
\[(A \wedge B)_U(x, x', y, y') \equiv A_U(x, y) \wedge B_U(x', y')\]
\[\iff \neg \exist \tilde{y} \leq y(A_K)_B(x, \tilde{y}) \wedge \neg \exist \tilde{y}' \leq y'(B_K)_B(x', \tilde{y}')\]
\[\iff \neg [\neg \exist \tilde{y} \leq y(A_K)_B(x, \tilde{y}) \lor \neg \exist \tilde{y}' \leq y'(B_K)_B(x', \tilde{y}')]\]
\[\iff \neg \exist \tilde{y}, \tilde{y}' \leq y, y'[\exist \tilde{y} \leq \tilde{y}(A_K)_B(x, \tilde{y}) \lor
\neg \exist \tilde{y}' \leq y'(B_K)_B(x', \tilde{y}')]\]
\[\equiv \neg \exist \tilde{y}, \tilde{y}' \leq y, y'[\neg A \wedge B)_K]_B(x, x', \tilde{y}, \tilde{y}').\]

Step 2. Now we prove \(\exists\). Assume \(Y \leq Y\) and \(x \leq x\). Using \(\exists\) in the equivalence, we have
\[A_U(x, Yx) \iff \neg \exist y \leq Yx(A_K)_B(x, y)\]
\[\equiv (\neg A_K)_B(Y, x)\]
\[\equiv (A_K)_B(Y, x).\]
Step 3. Finally, we prove (4). Using $B$-$mAC$ in the first equivalence, the monotonicity of $U$ in the second equivalence and (3) in the third equivalence, we have

$$A^U \equiv \forall x \exists y A_U(x, y)$$

$$\leftrightarrow \exists Y \forall x \exists y \leq Y x A_U(x, y)$$

$$\leftrightarrow \exists Y \forall x A_U(x, Y x)$$

$$\leftrightarrow \exists Y \forall x (A^K)_B(Y x)$$

$$\equiv (A^K)_B.$$

\[\square\]

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