INVERSE-CLOSEDNESS OF THE SUBALGEBRA OF LOCALLY NUCLEAR OPERATORS

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Abstract. Let $X$ be a Banach space and $T$ be a bounded linear operator acting in $l_p(Z^c, X)$, $1 \leq p \leq \infty$. The operator $T$ is called locally nuclear if it can be represented in the form

$$(Tx)_k = \sum_{m \in Z^c} b_{km} x_{k-m}, \quad k \in Z^c,$$

where $b_{km}: X \to X$ are nuclear,

$$\|b_{km}\|_{s_1} \leq \beta_m, \quad k, m \in Z^c,$$

$\|\cdot\|_{s_1}$ is the nuclear norm, $\beta \in l_1(Z^c, \mathbb{C})$ or $\beta \in l_{1,g}(Z^c, \mathbb{C})$, and $g$ is an appropriate weight on $Z^c$. It is established that if $T$ is locally nuclear and the operator $1 + T$ is invertible, then the inverse operator $(1 + T)^{-1}$ has the form $1 + T_1$, where $T_1$ is also locally nuclear. This result is refined for the case of operators acting in $L_p(R^c, \mathbb{C})$.

1. Introduction

A subset $\mathfrak{T}$ of the algebra of all bounded linear operators acting in a Banach space is called a subalgebra if it is closed under addition, multiplication by scalars, and composition. If additionally $\mathfrak{T}$ is closed under inversion, the subalgebra $\mathfrak{T}$ is called full or inverse-closed. If all operators involved in a mathematical problem belong to the same full subalgebra, then the solution of the problem usually also belongs to the same subalgebra. Thus, we obtain some qualitative information about the solution in advance, which can simplify the investigation of the problem. It is clear that the narrower the full subalgebra, the easier and more convenient it is to work with it.

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The main part of applications of operator theory to numerical mathematics [20] is associated with finite-dimensional operators, i.e. having a finite-dimensional image, although, possibly, acting in an infinite-dimensional space. The space $S_1$ of nuclear operators forms a small extension of the space of finite-dimensional operators. Nuclear operators are more convenient at least from the theoretical point of view, because, in contrast to finite-dimensional ones, they form a Banach space.

A bounded linear operator $A: X \to X$, where $X$ is a Banach space, is called [24,33,37,38] nuclear if it can be represented in the form

\begin{equation}
Ax = \sum_{i=1}^{\infty} a_i(x)y_i,
\end{equation}

where $y_i \in X$, $a_i \in X^*$ ($X^*$ is the conjugate of $X$), and $\sum_{i=1}^{\infty} \|a_i\| \cdot \|y_i\| < \infty$. The linear space $S_1 = S_1(X)$ of nuclear operators is Banach with respect to the norm

\begin{equation}
\|A\|_{S_1} = \inf \sum_{i=1}^{\infty} \|a_i\| \cdot \|y_i\|;
\end{equation}

where the infimum is taken over all representations of the operator $A$ in the form (1).

This paper is devoted to two full subalgebras $s_{1,g}$ and $S_{1,g}$. We call operators belonging to $s_{1,g}$ and $S_{1,g}$ locally nuclear. Formally, an operator acting in $L_p(\mathbb{R}^c, \mathbb{C})$ is locally nuclear if its restriction to any compact subset $M \subset \mathbb{R}^c$ is nuclear and its memory decreases at infinity in a special way; and similarly for $l_p(\mathbb{Z}^c, X)$. Our main results (Theorems 7.3 and 8.4) state that locally nuclear operators form full subalgebras. We also show that a locally nuclear operator acting in the space $L_p(\mathbb{R}^c, \mathbb{C})$, $1 \leq p < \infty$, admits an integral representation (Theorem 8.5).

A more detailed definition of a locally nuclear operator is as follows. Let a linear operator $T$ act in $l_p(\mathbb{Z}^c, X)$, $1 \leq p \leq \infty$, where $X$ is a Banach space. We say that the operator $T$ belongs to the class $s_{1,g}$ if it can be represented in the form

\begin{equation}
(Tx)_k = \sum_{m \in \mathbb{Z}^c} b_{km} x_{k-m}, \quad k \in \mathbb{Z}^c,
\end{equation}

where $b_{km}: X \to X$ are nuclear and

\begin{equation}
\|b_{km}\|_{S_1} \leq \beta_m
\end{equation}
for some (see Definition 3.1) $\beta \in l_1(\mathbb{Z}^c, \mathbb{C})$ or $\beta \in l_{1,g}(\mathbb{Z}^c, \mathbb{C})$, where $g$ is an appropriate weight on $\mathbb{Z}^c$. Next, let a linear operator $A$ act in the space $L_p(\mathbb{R}^c, \mathbb{C})$, $1 \leq p \leq \infty$. We represent $\mathbb{R}^c$ as the union of the disjoint sets:

$$
\mathbb{R}^c = \bigcup_{m=(m_1, m_2, \ldots, m_c) \in \mathbb{Z}^c} [m_1, m_1 + 1) \times [m_2, m_2 + 1) \times \cdots \times [m_c, m_c + 1),
$$

and identify the space $L_p(\mathbb{R}^c, \mathbb{C})$ with $l_p(\mathbb{Z}^c, L_p([0,1)^c, \mathbb{C}))$. Let $T$ be the operator acting in $l_p(\mathbb{Z}^c, L_p([0,1)^c, \mathbb{C}))$, which corresponds to $A$ in accordance with this representation. We say that the operator $A$ belongs to the class $S_{1,g}$ if the operator $T$ belongs to the class $s_{1,g}$.

The inverse-closedness of some other classes of operators possessing good properties only locally was investigated in [5,15,25,30–32]. The preservation of the rate of decrease of memory (i.e. an estimate of the kind (4)) while passing to the inverse operator was studied by many authors, see, e.g., [1–4,6,11–13,16,17,19,21–23,27–30,34,35,39–41]. In this paper we combine these two properties.

In Section 2, we clarify terminology, notation, and general facts connected with Banach algebras. In Section 3, we describe the weighted algebra $l_{1,g}(\mathbb{Z}^c, B)$. In Section 4, we recall some facts connecting with convolution dominated operators. We recall the definition and basic properties of nuclear operators in Section 5; in Section 6, we recall the integral representation for nuclear operators acting in $L_p$ (Theorem 6.3) and give an example of nuclear operator (Proposition 6.5). Section 7 is devoted to Theorem 7.3, which is our main result; in Section 8, it is specified to the case of operators acting in $L_p$ (Theorem 8.4).

2. Banach algebras

In this paper, all linear spaces are considered over the field $\mathbb{C}$ of complex numbers.

An algebra [7, Ch. 1, § 1], [26, Ch. 4, § 1.13], [36, Ch. 10, § 10.1] is a linear space $B$ (over the field $\mathbb{C}$ of complex numbers) endowed with a multiplication possessing the properties

$$
A(BC) = (AB)C, \quad \alpha(AB) = (\alpha A)B = A(\alpha B),
$$

$$
(A + B)C = AC + BC, \quad A(B + C) = AB + AC.
$$

If $B$ is a normed space and

$$
\|AB\| \leq \|A\| \cdot \|B\|,
$$

then $B$ is called a normed algebra. If a normed algebra is a complete (i.e. Banach) space, then it is called a Banach algebra.
An algebra $B$ is called \textit{commutative} if $AB = BA$ for all $A, B \in B$.

Let $X$ be a Banach space. We denote by $B(X)$ the Banach algebra of all bounded linear operators acting in $X$. It is the main example of a Banach algebra.

If an algebra $B$ has an element $1 = 1_B$ such that $A1 = 1_A = A$ for all $A \in B$, the element $1$ is called a \textit{unit}. In this case, the algebra $B$ is called a \textit{unital algebra} or an \textit{algebra with unit}. If, in addition, the algebra $B$ is normed (Banach) and $\|1\| = 1$, then $B$ is called a \textit{normed (Banach) unital algebra}.

Let $B$ be a unital algebra and $A \in B$. An element $B \in B$ is called the \textit{inverse} of $A$ if $AB = BA = 1$. The inverse of $A$ is denoted by the symbol $A^{-1}$. If a element $A$ has an inverse, it is called \textit{invertible} (in the algebra $B$).

A subset $R$ of an algebra $B$ is called a \textit{subalgebra} if $R$ is stable under the algebraic operations (addition, scalar multiplication, and multiplication), i.e. $A + B, \lambda A, AB \in R$ for all $A, B \in R$ and $\lambda \in \mathbb{C}$. Obviously, a subalgebra is an algebra itself. It is also evident that the closure of a subalgebra (of a normed algebra) is again a subalgebra. If a subalgebra $R$ of a unital algebra $B$ contains the unit of the algebra $B$, then $R$ is called a \textit{unital subalgebra}.

A unital subalgebra $R$ of a unital algebra $B$ is called \textit{full} \cite[Ch. 1, §1.4]{7} or \textit{inverse-closed} \cite[p. 183]{21} if every $B \in R$ that is invertible in $B$ is also invertible in $R$. This definition is equivalent to the following one: for any $B \in R$, the existence of $B^{-1} \in B$ such that $BB^{-1} = B^{-1}B = 1$ implies that $B^{-1} \in R$.

Let $B$ be a non-unital algebra. We consider the algebra $\tilde{B}$ consisting of all ordered pairs $(\alpha, A)$, where $\alpha \in \mathbb{C}$ and $A \in B$, with the operations

\begin{align*}
(\alpha, A) + (\beta, B) &= (\alpha + \beta, A + B), \\
\lambda(\alpha, A) &= (\lambda \alpha, A), \\
(\alpha, A)(\beta, B) &= (\alpha \beta, \alpha B + \beta A + AB).
\end{align*}

(5)

It is easy to see that $\tilde{B}$ is in fact an algebra and the element $(1, 0)$ is its unit. Clearly, $B$ is isomorphic to the subalgebra of $\tilde{B}$ consisting of elements of the form $(0, A)$. The element $(\alpha, A)$ is usually denoted by $\alpha \mathbf{1} + A$. If $B$ is normed, a norm on $\tilde{B}$ can be defined by the formula

\begin{align*}
\|(\alpha, A)\| &= |\alpha| + \|A\|.
\end{align*}

(6)

Clearly, $\tilde{B}$ is Banach provided that so is $B$. The algebra $\tilde{B}$ is called the algebra derived from $B$ by \textit{adjoining a unit element}. If $B$ is unital, we mean by $\tilde{B}$ the algebra $B$ itself.

In a particular case when the algebra $B$ is a subalgebra of a unital algebra $A$, we identify the algebra $\tilde{B}$ with the subalgebra $\{\alpha \mathbf{1}_A + B \in A : B \in B\}$. 

\textit{Analysis Mathematica} 49, 2023
\( \alpha \in \mathbb{C} \) of the algebra \( \mathbf{A} \). We note that in this case the norm on \( \tilde{\mathbf{B}} \) induced by the imbedding into \( \mathbf{A} \) is equivalent to the norm (6).

Let \( \mathbf{A} \) and \( \mathbf{B} \) be algebras. A mapping \( \varphi : \mathbf{A} \to \mathbf{B} \) is called \cite{7, Ch. 1, § 1} a morphism of algebras if

\[
\varphi(A + B) = \varphi(A) + \varphi(B), \quad \varphi(\alpha A) = \alpha \varphi(A), \quad \varphi(AB) = \varphi(A)\varphi(B)
\]

for all \( A, B \in \mathbf{A} \) and \( \alpha \in \mathbb{C} \). If the algebras \( \mathbf{A} \) and \( \mathbf{B} \) are unital and additionally \( \varphi(1_{\mathbf{A}}) = 1_{\mathbf{B}} \), then \( \varphi \) is called a morphism of unital algebras. If the algebras \( \mathbf{A} \) and \( \mathbf{B} \) are normed (Banach) and the morphism \( \varphi \) is continuous, then \( \varphi \) is called a morphism of normed (Banach) algebras. If \( \varphi^{-1} \) exists and is a morphism of the same type, then \( \varphi \) is called an isomorphism; in this case, \( \mathbf{A} \) and \( \mathbf{B} \) are called isomorphic.

A linear subspace \( \mathbf{J} \) of an algebra \( \mathbf{B} \) is called a (two-sided) ideal if it possesses the property: \( AJ, JA \in \mathbf{J} \) for all \( A \in \mathbf{B} \) and \( J \in \mathbf{J} \). Clearly, each ideal is a subalgebra. For any ideal \( \mathbf{J} \), the quotient space \( \mathbf{B} / \mathbf{J} \) is an algebra. An ideal \( \mathbf{J} \) is called proper if \( \mathbf{J} \neq \{0\} \) and \( \mathbf{J} \neq \mathbf{B} \). It is easy to see that if (\( \mathbf{B} \) is unital and) an ideal \( \mathbf{J} \) contains an invertible element, then \( \mathbf{J} = \mathbf{B} \).

**Proposition 2.1** ([5, Lemma 4.1], see also [30, Theorem 5.3.6]). Let \( \mathbf{B} \) be a unital algebra and \( \mathbf{J} \) be a proper ideal of \( \mathbf{B} \). Then the algebra \( \tilde{\mathbf{J}} \) (obtained from \( \mathbf{J} \) by adjoining a unit element of \( \mathbf{B} \)) is a full subalgebra of \( \mathbf{B} \).

**Proof.** Let an element \( T = \alpha 1 + J \), where \( \alpha \in \mathbb{C} \) and \( J \in \mathbf{J} \), be invertible in \( \mathbf{B} \). We consider the quotient algebra \cite[11.4]{36} \( \mathbf{B} / \mathbf{J} \). Clearly, the natural projection of \( T \) into \( \mathbf{B} / \mathbf{J} \) coincides with \( \alpha 1_{\mathbf{B} / \mathbf{J}} \). Therefore the natural projection of \( T^{-1} \) into \( \mathbf{B} / \mathbf{J} \) coincides with \( \frac{1}{\alpha} 1_{\mathbf{B} / \mathbf{J}} \) (in particular this implies that \( \alpha \neq 0 \)), which means that \( T^{-1} = \frac{1}{\alpha} 1 + J_1 \) with \( J_1 \in \mathbf{J} \). \( \square \)

### 3. Algebra \( l_{1,g}(\mathbb{Z}^c) \)

In this section, we recall the definition of the weighted algebra \( l_{1,g}(\mathbb{Z}^c) \). We denote by \( \mathbb{Z}^c, c \in \mathbb{N} \), the Cartesian product of \( c \) copies of the group \( \mathbb{Z} \) of integers.

**Definition 3.1.** A weight on the group \( \mathbb{Z}^c \) is a function \( g : \mathbb{Z}^c \to (0, +\infty) \). We always assume that the weight \( g \) on \( \mathbb{Z}^c \) possesses the following properties:

(a) \( g(0) = 1 \),
(b) \( g(m + n) \leq g(m)g(n) \),
(c) \( g(-n) = g(n) \),
(d) \( g(n) \geq 1, \)
(e0) for any \( t \in \mathbb{Z}^c, \) one has \( \lim_{n \to \infty} \frac{\ln g(nt)}{n} = 0, \)
(e1) for any \( t \in \mathbb{Z}^c, \) one has \( \lim_{n \to \infty} \sqrt{g(nt)} = 1. \)

Clearly, (d) is a consequence of (b) and (c).

**Proposition 3.2.** Properties (e0) and (e1) are equivalent.

**Proof.** Indeed,
\[
\lim_{n \to \infty} \sqrt{g(nt)} = 1 \iff \lim_{n \to \infty} \ln \sqrt{g(nt)} = \ln 1 \iff \lim_{n \to \infty} \frac{\ln g(nt)}{n} = 0.
\]

**Example 3.3** ([21, Example 5.21], [23]). Let \( 0 \leq b < 1, a \geq 0 \) and \( s, t \geq 0. \) Let \( \| \cdot \| \) denote the restriction of any norm on \( \mathbb{R}^c \) to \( \mathbb{Z}^c. \) Then the functions
\[
g(n) = 1, \quad g(n) = (1 + |n|)^s, \quad g(n) = e^{a|n|^b}(1 + |n|)^s, \\
g(n) = e^{a|n|^b}(1 + |n|)^s \ln(e + |n|)
\]
satisfy assumptions (a)–(e) from Definition 3.1. It is clear that in this list, each example is a special case of the next one.

Let \( g \) be a weight on \( \mathbb{Z}^c, \) and let \( B \) be a Banach algebra. The space \( l_{1,g} = l_{1,g}(\mathbb{Z}^c, B) \) is the set of all families \( a = \{a_m \in B: m \in \mathbb{Z}^c\} \) such that
\[
\|a\| = \|a\|_{l_{1,g}} = \sum_{m \in \mathbb{Z}^c} g(m)\|a_m\| < \infty.
\]

We endow \( l_{1,g} \) with the coordinate-wise operations of addition and multiplication by scalars; clearly, \( l_{1,g} \) becomes a linear space. If \( g(m) = 1, m \in \mathbb{Z}^c, \) the space \( l_{1,g} = l_{1,g}(\mathbb{Z}^c, B) \) coincides with the ordinary space \( l_1 = l_1(\mathbb{Z}^c, B). \) Clearly, assumption (d) from Definition 3.1 implies that \( l_{1,g} \subseteq l_1. \)

**Proposition 3.4** (see, e.g., [21, p. 196, Lemma 5.22]). Let assumptions (a) and (b) be fulfilled. Then the space \( l_{1,g} = l_{1,g}(\mathbb{Z}^c, B) \) is a Banach algebra with respect to the operation of convolution
\[
(a * b)_k = \sum_{m \in \mathbb{Z}^c} a_m b_{k-m}
\]
taken as multiplication. If \( B \) is unital, so is \( l_{1,g} \); the unit of the algebra \( l_{1,g} \) is the family \( \delta = \{\delta_k : k \in \mathbb{Z}^c\} \) defined by the formula
\[
\delta_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases} \quad k \in \mathbb{Z}^c.
\]

The algebra \( l_{1,g}(\mathbb{Z}^c, \mathbb{C}) \) is commutative.
4. Convolution dominated operators

Convolution dominated operators were investigated by many authors [2–4,6,11–13,16,19,21–23,27–30,34,35,39–41]. A self-contained detailed exposition with a discussion of motivation and history can be found, e.g., in [21]. In this section we recall the main facts.

Let $X$ be a Banach space. For $1 \leq p < \infty$, we denote by $l_p = l_p(Z_c, X)$ the Banach space consisting of all families \( \{x_k \in X : k \in Z_c\} \) such that \( \sum_{k \in Z_c} \|x_k\|^p < \infty \). We endow the space $l_p = l_p(Z_c, X)$ with the usual norm

\[
\|x\| = \|x\|_{l_p} = \left( \sum_{k \in Z_c} \|x_k\|^p \right)^{1/p}.
\]

We denote by $l_\infty = l_\infty(Z_c, X)$ the Banach space consisting of all bounded families \( \{x_k \in X : k \in Z_c\} \) with the norm \( \|x\| = \|x\|_{l_\infty} = \sup_{k \in Z_c} \|x_k\| \).

We call a matrix (indexed by elements of the set $Z_c$ with values in $B(X)$) any family \( \{a_{kl} \in B(X) : k, l \in Z_c\} \). We say that an operator \( T \in B(l_p(Z_c, X)) \), \( 1 \leq p \leq \infty \), is generated by a matrix \( \{a_{kl} \in B(X) : k, l \in Z_c\} \) if

\[
(Tx)_k = \sum_{l \in Z_c} a_{kl} x_l, \quad k \in Z_c,
\]

for all \( x \in l_p \), where the series converges in norm. Obviously, not every matrix generates a (bounded) operator. It is less obvious, that not every (bounded) operator is generated by some matrix, see a counterexample in [30, Example 1.6.4]. For our aims, it is convenient to change the enumeration of matrix elements. Namely, we make the substitution \( l = k - m \) and introduce the notation $b_{km} = a_{k,k−m}$. Now we have

\[
(Tx)_k = \sum_{m \in Z_c} b_{km} x_{k−m}, \quad k \in Z_c.
\]

**Definition 4.1.** Let $g$ be a weight on $Z_c$ satisfying assumptions (a)–(e) from Definition 3.1. We denote by $s_{1,g} = s_{1,g}(Z_c, B(X))$ the set of all operators $T \in B(l_p(Z_c, X))$, \( 1 \leq p \leq \infty \), of the form (9), where the family of coefficients \( \{b_{km} \in B(X) : k, m \in Z_c\} \) satisfies the estimate

\[
\|b_{km}\|_{B(X)} \leq \beta_m
\]

for some $\beta \in l_1(Z_c, \mathbb{C})$. In other words, $s_{1,g} = s_{1,g}(Z_c, B(X))$ consists of operators generated by matrices satisfying estimate (10) with $\beta \in l_1(Z_c, \mathbb{C})$. In the case $g(n) = 1$, $n \in Z_c$, we use the brief symbol $s_1$ instead of $s_{1,g}$.
If estimate (10) holds, they say that operator (9) is dominated by the convolution operator

\[(Bx)_k = \sum_{m \in \mathbb{Z}^c} \beta_m x_{k-m}, \quad k \in \mathbb{Z}^c.\]

**Proposition 4.2** (see, e.g., [30, Theorem 4.4.8]). Let \(1 \leq p \leq \infty\) and estimate (10) be fulfilled with \(\beta \in l_{1,g}(\mathbb{Z}^c, \mathbb{C})\). Then series (9) converges absolutely for all \(x \in l_p(\mathbb{Z}^c, X)\) and defines a linear operator \(T \in \mathcal{B}(l_p(\mathbb{Z}^c, X))\). Furthermore,

\[\|T\|_{\mathcal{B}(l_p)} \leq \sum_{m \in \mathbb{Z}^c} \beta_m = \|\beta\|_{l_1} \leq \|\beta\|_{l_{1,g}}.\]

**Proposition 4.3** (see, e.g., [21, Lemma 5.28], cf. also the proof of Proposition 7.2). The set \(s_{1,g} = s_{1,g}(\mathbb{Z}^c, \mathcal{B}(X))\) forms a subalgebra in the algebra \(\mathcal{B}(l_p(\mathbb{Z}^c, X)), 1 \leq p \leq \infty\).

**Theorem 4.4** ([28], [29, Theorem 2.2.7], [30, Ch. 5, § 5.2, Theorem 5.2.6]). The subalgebra \(s_1(\mathbb{Z}^c, \mathcal{B}(X))\) is full in the algebra \(\mathcal{B}(l_p(\mathbb{Z}^c, X))\) for all \(1 \leq p \leq \infty\).

**Theorem 4.5** ([2–4]). Let assumptions (a)–(e) from Definition 3.1 be fulfilled. Then the subalgebra \(s_{1,g}(\mathbb{Z}^c, \mathcal{B}(X))\) is full in the algebra \(\mathcal{B}(l_p(\mathbb{Z}^c, X))\) for all \(1 \leq p \leq \infty\).

### 5. Nuclear operators

Let \(X\) be a Banach space and \(X^*\) be its conjugate. An operator \(A \in \mathcal{B}(X)\) is called nuclear [24,33,37,38] if it can be represented in the form

\[(11) \quad Ax = \sum_{i=1}^{\infty} a_i(x)y_i,\]

where \(y_i \in X, a_i \in X^*, \) and \(\sum_{i=1}^{\infty} \|a_i\| \cdot \|y_i\| < \infty\). It is usually written briefly as

\[A = \sum_{i=1}^{\infty} a_i \otimes y_i.\]

We denote the set of all nuclear operators \(A \in \mathcal{B}(X)\) by the symbol \(\mathfrak{S}_1(X)\). We set

\[(12) \quad \|A\|_{\mathfrak{S}_1} = \inf \sum_{i=1}^{\infty} \|a_i\| \cdot \|y_i\|,\]

Analysis Mathematica 49, 2023
where the infimum is taken over all representations of the operator $A$ in the form (11). Clearly,

\begin{equation}
\|A\|_{B(X)} \leq \|A\|_{\mathcal{E}_1}, \quad A \in \mathcal{S}_1(X). \tag{13}
\end{equation}

It is interesting to note that the natural mapping from $X^* \otimes_{\pi} X$ to $\mathcal{S}_1(X)$ is not injective in general [9, p. 34].

**Proposition 5.1** ([33, 6.3.2]). *The set $\mathcal{S}_1(X)$ is an ideal in $B(X)$. Moreover,

\begin{align*}
\|JA\|_{\mathcal{S}_1(X)}, \quad \|AJ\|_{\mathcal{S}_1(X)} &\leq \|J\|_{\mathcal{S}_1(X)} \|A\|_{B(X)}, \quad J \in \mathcal{S}_1(X), A \in B(X). \\
\text{The ideal } &\mathcal{S}_1(X) \text{ is complete with respect to the norm (12).}
\end{align*}

**Proposition 5.2** ([33, 6.3.1 and 1.11.2]). *Any nuclear operator is compact.*

**Corollary 5.3.** *If the space $X$ is infinite-dimensional, then the ideal $\mathcal{S}_1(X)$ is proper.*

**Proof.** The fact [14, Ch. IV § 3, Theorem 5] that the closed unit ball is not compact in any infinite-dimensional Banach space implies that a compact operator can not be invertible. Now the assertion follows from Proposition 5.2. □

We denote by $\widetilde{\mathcal{S}_1}(X)$ the ideal $\mathcal{S}_1(X)$ with an adjoint unit. We realize $\widetilde{\mathcal{S}_1}(X)$ as a subalgebra of $B(X)$.

**Theorem 5.4.** *Let $X$ be a Banach space. Then $\widetilde{\mathcal{S}_1}(X)$ is a full subalgebra of the algebra $B(X)$.**

**Proof.** The proof follows from Propositions 5.1 and 2.1. □

In some Banach spaces, necessary and sufficient conditions for the nuclearity of an operator are known.

**Proposition 5.5** ([33, 6.3.6]). *Let $d \in \mathbb{N}$ and $l_1 = l_1(\mathbb{Z}^d, \mathbb{C})$ be the space of all summable families $y = \{y_i \in \mathbb{C} : i \in \mathbb{Z}^d\}$. An operator $A \in B(l_1(\mathbb{Z}^d, \mathbb{C}))$ is nuclear if and only if it can be represented in the matrix form

\begin{equation}
(Ay)_i = \sum_{j \in \mathbb{Z}^d} a_{ij} y_j,
\end{equation}

with

\begin{equation}
\sum_{i \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}^d} |a_{ij}| < \infty.
\end{equation}

In this case

\begin{align*}
\|A\|_{\mathcal{S}_1(l_1)} &\equiv \sum_{i \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}^d} |a_{ij}|.
\end{align*}
6. Nuclear operators in $L_p$

In this section we recall an integral representation for nuclear operators in the space $L_p$ (Theorem 6.3).

Let $E \subseteq \mathbb{R}^c$ be a measurable subset. We denote the integral of a summable function $x: E \to \mathbb{C}$ with respect to the Lebesgue measure by the symbol $\int_E x(t) \, dt$.

We denote by $L_p = L_p(E, \mathbb{C})$, $1 \leq p < \infty$, the space of all measurable functions $u: E \to \mathbb{C}$ bounded in the semi-norm

$$||u|| = ||u||_{L_p} = \left( \int_E |u(t)|^p \, dt \right)^{1/p},$$

and we denote by $L_\infty = L_\infty(E, \mathbb{C})$ the space of all measurable essentially bounded functions $u: E \to \mathbb{C}$ with the semi-norm

$$||u|| = ||u||_{L_\infty} = \operatorname{ess sup}_{t \in E} |u(t)|.$$

Sometimes it is convenient to admit that functions $u \in L_p$ may be undefined on a negligible (or measure zero) set. We denote by $L_p = L_p(E)$, $1 \leq p \leq \infty$, the Banach space of all classes of functions $u \in L_p$ with the identification almost everywhere. Usually they do not distinguish between $L_p$ and $L_p$. For more details, see [8].

Numbers $p, q \in [1, +\infty]$ connected by the relation $\frac{1}{p} + \frac{1}{q} = 1$ are called conjugate exponents (see e.g. [8, Ch. IV, § 6.4]).

**Proposition 6.1 ([8, Ch. 4, § 6, Corollary 4 and Proposition 3]).** Let $p, q \in [1, +\infty]$ be conjugate exponents. Then for any functions $x \in L_p(E, \mathbb{C})$ and $y \in L_q(E, \mathbb{C})$, one has $xy \in L(1, \mathbb{C})$ and

$$\left| \int_E x(t)y(t) \, dt \right| \leq ||x||_{L_p} \cdot ||y||_{L_q}.$$

Moreover,

$$||x||_{L_p} = \sup \left\{ \left| \int_E x(t)y(t) \, dt \right| : ||y||_{L_q} \leq 1 \right\},$$

$$||y||_{L_q} = \sup \left\{ \left| \int_E x(t)y(t) \, dt \right| : ||x||_{L_p} \leq 1 \right\}.$$

**Proposition 6.2.** Let $x_i \in L_p(E, \mathbb{C})$, $1 \leq p \leq \infty$, and $\sum_{i=1}^{\infty} ||x_i||_{L_p} < \infty$. Then for almost all $t \in E$, the series $\sum_{i=1}^{\infty} x_i(t)$ converges absolutely; we denote the sum of the series $\sum_{i=1}^{\infty} x_i(t)$ by $x(t)$; thus, we obtain a function $x: E \to \mathbb{C}$ defined almost everywhere. It is claimed that $x \in L_p(E, \mathbb{C})$ and the series $\sum_{i=1}^{\infty} x_i(t)$ converges to $x$ in the $L_p$-norm.
Proof. The case $1 \leq p < \infty$ is carried through in [8, Ch. 4, § 3, Proposition 6], the case $p = \infty$ is actually analyzed in [8, Ch. 4, § 6]. We describe it in more detail. So, let $p = \infty$. We denote by $F_i$ the set of all points $t$ such that $|x_i(t)| < 2\|x_i\|_{L_\infty}$. By the definition of the space $L_\infty$, the sets $F_i$ have negligible complements. Therefore the set $\bigcap_{i=1}^{\infty} F_i$ has a negligible complement, too. Obviously, the series $\sum_{i=1}^{\infty} x_i(t)$ converges absolutely for all $t \in \bigcap_{i=1}^{\infty} F_i$. □

**Theorem 6.3** ([10, Lemma 2.1 and Theorem 2.2]). Let $p, q \in [1, +\infty]$ be conjugate exponents with $p < \infty$. Let $E \subseteq \mathbb{R}^c$ be a summable set, with measure $M < \infty$. Let an operator $A \in \mathcal{S}_1(L_p(E, \mathbb{C}))$ be represented in the form

$$A = \sum_{i=1}^{\infty} a_i \otimes y_i,$$

where $y_i \in L_p(E, \mathbb{C})$ and $a_i \in L_q(E, \mathbb{C})$, with

$$\sum_{i=1}^{\infty} \|a_i\|_{L_q} \|y_i\|_{L_p} < \infty.$$

Then the series

$$k(t, s) = \sum_{i=1}^{\infty} a_i(s)y_i(t)$$

converges absolutely in the norm of $L_1$ (and consequently, by Proposition 6.2, converges almost everywhere on $E \times E$) to a function $k \in L_1(E \times E, \mathbb{C})$. Moreover,

$$\|k\|_{L_1} \leq M \sum_{i=1}^{\infty} \|a_i\|_{L_q} \|y_i\|_{L_p},$$

and for all $x \in L_p$ for almost all $t \in E$ one has

$$(Ax)(t) = \int_{E} k(t, s)x(s) \, ds.$$

**Lemma 6.4.** Let $p, q \in [1, +\infty]$ be conjugate exponents, with $p < \infty$; let $0 \leq \alpha < \beta \leq 1$; let $a_t \in L_q([0, 1, \mathbb{C}), t \in [\alpha, \beta]$; let the function $t \mapsto a_t$ be measurable [8] with respect to the norm of $L_q([0, 1, \mathbb{C})$. Then the operator

$$(Ax)(t) = \chi_{[\alpha, \beta]}(t) \int_{0}^{1} a_t(s)x(s) \, ds, \quad t \in [0, 1),$$

Analysis Mathematica 49, 2023
where
\[
\chi_{[\alpha, \beta)}(t) = \begin{cases} 
1 & \text{for } t \in [\alpha, \beta), \\
0 & \text{for } t \notin [\alpha, \beta)
\end{cases}
\]

is the characteristic function of \([\alpha, \beta)\), acts in \(L_p([0,1), \mathbb{C})\) and
\[
\|A\|_{L_p([0,1), \mathbb{C})} \leq M(\beta - \alpha)^{1/p},
\]
where
\[
M = \sup_{t \in [\alpha, \beta)} \|a_t\|_{L_q([0,1))} < \infty.
\]

**Proof.** Let \(x \in L_p([0,1), \mathbb{C})\) with \(\|x\|_{L_p([0,1))} = 1\). Then, by Proposition 6.1,
\[
|(Ax)(t)| \leq \chi_{[\alpha, \beta)}(t) \left| \int_0^1 a_t(s)x(s) \, ds \right| \leq \chi_{[\alpha, \beta)}(t) \|a_t\|_{L_q([0,1))} \leq \chi_{[\alpha, \beta)}(t) M.
\]

Obviously, the function \(Ax\) is measurable. Thus, \(A\) acts continuously from \(L_p([0,1), \mathbb{C})\) to \(L_\infty([0,1), \mathbb{C}) \subset L_p([0,1), \mathbb{C})\).

From the above, it is clear that
\[
\|Ax\|_{L_p([0,1))} \leq \sqrt[p]{\int_\alpha^\beta M^p \chi_{[\alpha, \beta)}^p(t) \, dt} = M(\beta - \alpha)^{1/p}. \quad \square
\]

The following proposition gives an example of nuclear operator in \(L_p\).

**Proposition 6.5.** Let \(p, q \in [1, +\infty)\) be conjugate exponents, with \(p < \infty\). Let \(n: [0,1) \times [0,1) \to \mathbb{C}\) be a measurable function; let \(n(t, \cdot) \in L_q([0,1), \mathbb{C})\) for all \(t \in [0,1)\); and let the function \(t \mapsto n(t, \cdot)\) satisfy the Lipschitz condition
\[
\|n(t_1, \cdot) - n(t_2, \cdot)\|_{L_q([0,1))} \leq K|t_1 - t_2|, \quad t_1, t_2 \in [0,1),
\]
for some constant \(K < \infty\). Then the formula
\[
(Tx)(t) = \int_0^1 n(t, s)x(s) \, ds
\]
defines a nuclear operator \(T: L_p([0,1), \mathbb{C}) \to L_p([0,1), \mathbb{C})\) with
\[
\|T\|_{\mathcal{B}} \leq \|n(1/2, \cdot)\|_{L_q([0,1))} + \frac{K}{2} \left(2^{1/p} - 1\right).
\]

**Proof.** By Lemma 6.4, \(T\) continuously acts from \(L_p([0,1), \mathbb{C})\) to \(L_p([0,1), \mathbb{C})\).
We set
\[(T_0x)(t) = \chi_{[0,1)}(t) \int_0^1 n(1/2, s)x(s) \, ds, \quad t \in [0, 1).\]
Clearly, \(T_0 = n(1/2, \cdot) \otimes \chi_{[0,1)}\). We consider the operator
\[(T - T_0)x(t) = \chi_{[0,1)}(t) \int_0^1 (n(t, s) - n(1/2, s))x(s) \, ds, \quad t \in [0, 1).\]
By Lemma 6.4, \(\|T - T_0\| \leq K/2\). Next, we consider the operators
\[
\begin{align*}
(T_{1,1}x)(t) &= \chi_{[0,1/2)}(t) \int_0^1 (n(1/4, s) - n(1/2, s))x(s) \, ds, \quad t \in [0, 1), \\
(T_{1,2}x)(t) &= \chi_{[1/2, 1)}(t) \int_0^1 (n(3/4, s) - n(1/2, s))x(s) \, ds, \quad t \in [0, 1).
\end{align*}
\]
Clearly,
\[
\begin{align*}
T_{1,1} &= (n(1/4, \cdot) - n(1/2, \cdot)) \otimes \chi_{[0,1/2)}, \\
T_{1,2} &= (n(3/4, \cdot) - n(1/2, \cdot)) \otimes \chi_{[1/2, 1)}.
\end{align*}
\]
By Lemma 6.4,
\[
\begin{align*}
\|T_{1,1}\| &\leq \|n(1/4, \cdot) - n(1/2, \cdot)\|_{L_q[0,1)} \left(\frac{1}{2}\right)^{1/p} \leq \frac{K}{4} \left(\frac{1}{2}\right)^{1/p}, \\
\|T_{1,2}\| &\leq \|n(3/4, \cdot) - n(1/2, \cdot)\|_{L_q[0,1)} \left(\frac{1}{2}\right)^{1/p} \leq \frac{K}{4} \left(\frac{1}{2}\right)^{1/p}.
\end{align*}
\]
We observe that
\[
\begin{align*}
(T - T_0 - T_{1,1} - T_{1,2})x(t) &= \chi_{[0,1)}(t) \int_0^1 (n(t, s) - n(1/2, s))x(s) \, ds \\
&\quad - \chi_{[0,1/2)}(t) \int_0^1 (n(1/4, s) - n(1/2, s))x(s) \, ds \\
&\quad - \chi_{[1/2, 1)}(t) \int_0^1 (n(3/4, s) - n(1/2, s))x(s) \, ds \\
&\quad = \chi_{[0,1/2)}(t) \int_0^1 (n(t, s) - n(1/4, s))x(s) \, ds \\
&\quad + \chi_{[1/2, 1)}(t) \int_0^1 (n(t, s) - n(3/4, s))x(s) \, ds, \quad t \in [0, 1).
\end{align*}
\]
By Lemma 6.4,
\[ \|T - T_0 - T_{1,1} - T_{1,2}\| \leq \frac{K}{4} \left( \frac{1}{2} \right)^{1/p} + \frac{K}{4} \left( \frac{1}{2} \right)^{1/p} = \frac{K}{2} \left( \frac{1}{2} \right)^{1/p}. \]

In a similar way, we consider the operators

\[ (T_{2,1}x)(t) = \chi_{[0,1/2^2]}(t) \int_0^1 (n(1/2^3, s) - n(1/4, s))x(s) \, ds, \quad t \in [0, 1), \]

\[ (T_{2,2}x)(t) = \chi_{[1/2^2, 2/2^2]}(t) \int_0^1 (n(3/2^3, s) - n(1/4, s))x(s) \, ds, \quad t \in [0, 1), \]

\[ (T_{2,3}x)(t) = \chi_{[2/2^2, 3/2^2]}(t) \int_0^1 (n(5/2^3, s) - n(3/4, s))x(s) \, ds, \quad t \in [0, 1), \]

\[ (T_{2,4}x)(t) = \chi_{[3/2^2, 1]}(t) \int_0^1 (n(7/2^3, s) - n(3/4, s))x(s) \, ds, \quad t \in [0, 1). \]

Clearly,

\[ T_{2,1} = (n(1/2^3, \cdot) - n(1/4, \cdot)) \otimes \chi_{[0,1/2^2]}, \quad \ldots, \]

\[ T_{2,4} = (n(7/2^3, \cdot) - n(3/4, \cdot)) \otimes \chi_{[3/2^2, 1)}. \]

By Lemma 6.4,

\[ \|T_{2,1}\| \leq \|n(1/2^3, \cdot) - n(1/4, \cdot)\|_{L_q[0,1]} \left( \frac{1}{2^2} \right)^{1/p} \leq \frac{K}{23} \left( \frac{1}{2^2} \right)^{1/p}, \quad \ldots, \]

\[ \|T_{2,4}\| \leq \|n(7/2^3, \cdot) - n(3/4, \cdot)\|_{L_q[0,1]} \left( \frac{1}{2^2} \right)^{1/p} \leq \frac{K}{23} \left( \frac{1}{2^2} \right)^{1/p}. \]

We observe that

\[
\begin{align*}
((T - T_0 - T_{1,1} - T_{1,2} - T_{2,1} - T_{2,2} - T_{2,3} - T_{2,4})x)(t) \\
= \chi_{[0,1/2^2]}(t) \int_0^1 (n(t, s) - n(1/2^3, s))x(s) \, ds \\
+ \chi_{[1/2^2, 2/2^2]}(t) \int_0^1 (n(t, s) - n(3/2^3, s))x(s) \, ds \\
+ \chi_{[2/2^2, 3/2^2]}(t) \int_0^1 (n(t, s) - n(5/2^3, s))x(s) \, ds \\
+ \chi_{[3/2^2, 1]}(t) \int_0^1 (n(t, s) - n(7/2^3, s))x(s) \, ds, \quad t \in [0, 1). 
\end{align*}
\]
By Lemma 6.4,
\[
\|T - T_0 - T_{1,1} - T_{1,2} - T_{2,1} - T_{2,2} - T_{2,3} - T_{2,4}\| \\
\leq 2^2 \left( \frac{K}{2^3} \left( \frac{1}{2^2} \right)^{1/p} \right) = \frac{K}{2} \left( \frac{1}{2^2} \right)^{1/p},
\]
and so on. It is seen that \( T_0 + \sum_{k=1}^{\infty} \sum_{i=1}^{2^k} T_{k,i} \) converges to \( T \) in norm and
\[
\left\| T_0 + \sum_{k=1}^{\infty} \sum_{i=1}^{2^k} T_{k,i} \right\| \leq \|n(1/2, \cdot)\|_{L_q[0,1]} + 2^2 \frac{K}{4} \left( \frac{1}{2} \right)^{1/p} + 2^2 \frac{K}{2^3} \left( \frac{1}{2^2} \right)^{1/p} + \ldots
\]
\[
= \|n(1/2, \cdot)\|_{L_q[0,1]} + \frac{K}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i/p}} = \|n(1/2, \cdot)\|_{L_q[0,1]} + \frac{K}{2} \frac{1}{2^{1/p} - 1},
\]
which means that \( T \in \mathcal{S}_1 \). \( \square \)

7. Locally nuclear operators in \( l_p \)

**Definition 7.1.** Let \( g \) be a weight on \( \mathbb{Z}^c \) satisfying assumptions (a)–(e) from Definition 3.1. Let \( X \) be a Banach space. We denote by \( \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(X)) \) the set of all operators \( T \in \mathcal{B}(l_p(\mathbb{Z}^c, X)) \) of the form
\[
(Tx)_k = \sum_{m \in \mathbb{Z}^c} b_{km} x_{k-m}, \quad k \in \mathbb{Z}^c,
\]
where the family of coefficients \( \{b_{km} \in \mathcal{B}(X) : k, m \in \mathbb{Z}^c\} \) satisfies the estimate
\[
\|b_{km}\|_{\mathcal{S}_1(X)} \leq \beta_m
\]
for some \( \beta \in l_{1,g}(\mathbb{Z}^c, \mathbb{C}) \), cf. Definition 4.1. By (13) and Proposition 4.2, the operator \( T \) indeed acts in \( l_p(\mathbb{Z}^c, X) \). We call operators \( T \in \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(X)) \) locally nuclear.

**Proposition 7.2.** The subalgebra \( \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(X)) \) of locally nuclear operators forms an ideal in the algebra \( \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{B}(X)) \). If \( X \) is infinite dimensional, the ideal \( \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(X)) \) is proper.

**Proof.** Clearly, \( \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(X)) \) is a subalgebra of \( \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{B}(X)) \). Thus, it remains to show that \( K \in \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(X)) \) and \( T \in \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{B}(X)) \) imply \( KT, TK \in \mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(X)) \).
So, let $K \in s_{1,g}(\mathbb{Z}^c, \mathcal{G}_1(X))$ and $T \in s_{1,g}(\mathbb{Z}^c, \mathcal{B}(X))$. These mean that $K$ and $T$ admit the representations

$$(Kx)_k = \sum_{m \in \mathbb{Z}^c} a_{km}x_{k-m}, \quad (Tx)_k = \sum_{l \in \mathbb{Z}^c} b_{kl}x_{k-l}, \quad k \in \mathbb{Z}^c,$$

where

$$\|a_{km}\|_{\mathcal{G}_1(X)} \leq \alpha_m, \quad \|b_{kl}\|_{\mathcal{B}(X)} \leq \beta_l,$$

with $\alpha, \beta \in l_{1,g}(\mathbb{Z}^c, \mathbb{C})$. By the definition of the product of operators, for any $x \in l_p(\mathbb{Z}^c, X)$, we have

$$(KTx)_k = \sum_{m \in \mathbb{Z}^c} a_{km}(Tx)_{k-m} = \sum_{m \in \mathbb{Z}^c} a_{km} \sum_{l \in \mathbb{Z}^c} b_{k-m,l}x_{k-m-l}, \quad k \in \mathbb{Z}^c.$$

Since $l_p(\mathbb{Z}^c, X) \subseteq l_{\infty}(\mathbb{Z}^c, X)$, the family $\{x_i : i \in \mathbb{Z}^c\}$ is bounded. Therefore the latter (double) series converges absolutely (for a fixed $k$). Consequently, any rearrangement of the series converges to the same sum.

We make the change $l = r - m$ in the internal sum,

$$(KTx)_k = \sum_{m \in \mathbb{Z}^c} a_{km} \sum_{r \in \mathbb{Z}^c} b_{k-m,r-m}x_{k-r}, \quad k \in \mathbb{Z}^c.$$  

and interchange the order of summation:

$$(15) \quad (KTx)_k = \sum_{r \in \mathbb{Z}^c} \left( \sum_{m \in \mathbb{Z}^c} a_{km} b_{k-m,r-m} \right) x_{k-r}, \quad k \in \mathbb{Z}^c.$$

By the estimate from Proposition 5.1, we have

$$\|a_{km}b_{k-m,r-m}\|_{\mathcal{G}_1(X)} \leq \|a_{km}\|_{\mathcal{G}_1(X)} \cdot \|b_{k-m,r-m}\|_{\mathcal{B}(X)} \leq \alpha_m \beta_{r-m}.$$

Therefore (see Proposition 3.4),

$$\sum_{m \in \mathbb{Z}^c} \|a_{km}b_{k-m,r-m}\|_{\mathcal{G}_1(X)} \leq \sum_{m \in \mathbb{Z}^c} \alpha_m \beta_{r-m} = (\alpha \ast \beta)_r.$$  

This estimate shows that the series $\sum_{m \in \mathbb{Z}^c} a_{km} b_{k-m,r-m}$ converges absolutely in the norm $\|\cdot\|_{\mathcal{G}_1}$. By the completeness of the ideal $\mathcal{G}_1(X)$ (Proposition 5.1), this implies that the sum $\sum_{m \in \mathbb{Z}^c} a_{km} b_{k-m,r-m}$ belongs to $\mathcal{G}_1(X)$ and

$$\left\| \sum_{m \in \mathbb{Z}^c} a_{km} b_{k-m,r-m} \right\|_{\mathcal{G}_1(X)} \leq (\alpha \ast \beta)_r.$$

By Proposition 3.4, $\alpha \ast \beta \in l_{1,g}(\mathbb{Z}^c, \mathbb{C})$. Hence it follows from formula (15) that $KT \in s_{1,g}(\mathbb{Z}^c, \mathcal{G}_1(X))$.

Similarly, one verifies that $TK \in s_{1,g}(\mathbb{Z}^c, \mathcal{G}_1(X))$.  

*Analysis Mathematica 49, 2023*
Finally, we prove that the ideal $s_{1,g}(Z^c, G_1(X))$ is proper provided $X$ is infinite-dimensional. Indeed, if $\dim X = \infty$, then, by Corollary 5.3, the identity operator $1_X : X \to X$ is not nuclear. Therefore if a matrix of an operator $T$ defined by formula (14) contains at least one identity operator $b_{km}$ (all other elements of the matrix may be zero), then the operator $T$ cannot be locally nuclear. □

We denote by $s_{1,g}(Z^c, G_1(X))$ the subalgebra derived from the ideal $s_{1,g}(Z^c, G_1(X))$ by adjoining the unit element of the algebra $B(l_p(Z^c, X))$ if $X$ is infinite-dimensional; and we denote by $\tilde{s}_{1,g}(Z^c, G_1(X))$ the algebra $s_{1,g}(Z^c, G_1(X))$ itself provided that $X$ is finite-dimensional.

The following theorem is the main result of this paper.

**Theorem 7.3.** The subalgebra $\tilde{s}_{1,g}(Z^c, G_1(X))$ is full in the algebra $B(l_p(Z^c, X))$ for all $1 \leq p \leq \infty$.

**Proof.** If $X$ is finite dimensional, then, because of the equality $B(X) = G_1(X)$, the assertion of the theorem coincides with that of Theorem 4.5. Therefore, without loss of generality, we assume that $X$ is infinite-dimensional.

As a first step, we observe that the subalgebra $\tilde{s}_{1,g}(Z^c, G_1(X))$ is full in the algebra $s_{1,g}(Z^c, B(X))$. This follows from Proposition 2.1 since, by Proposition 7.2, the subalgebra $s_{1,g}(Z^c, G_1(X))$ forms a proper ideal in the algebra $s_{1,g}(Z^c, B(X))$.

To complete the proof, we recall that the subalgebra $s_{1,g}(Z^c, B(X))$ is full in the algebra $B(l_p(Z^c, X))$ by Theorem 4.5. □

**Example 7.4.** Let $c, d \in \mathbb{N}$. We consider the space $l_1(Z^c, l_1(Z^d, C))$. We note that the space $l_1(Z^c, l_1(Z^d, C))$ consists of sequences $x = \{x_k \in l_1(Z^d, C) : k \in Z^c\}$, where each $x_k$ is a sequence $(x_{ki})_{i \in C}$, $i \in Z^d$, itself. We use the brief notation $x_{k,i}$ instead of $(x_{ki})_i$. Obviously,

$$
\|x\|_{l_1(Z^c, l_1(Z^d, C))} = \sum_{k \in Z^c} \|x_k\|_{l_1(Z^d, C)} = \sum_{k \in Z^c} \sum_{i \in Z^d} |x_{k,i}|.
$$

Thus, $l_1(Z^c, l_1(Z^d, C))$ is naturally isometrically isomorphic to $l_1(Z^{c+d}, C)$.

In this example, we show that an operator $T \in B(l_1(Z^c, l_1(Z^d, C)))$ belongs to the class $s_{1,g}(Z^c, G_1(l_1(Z^d, C)))$ if and only if it can be represented in the form

$$(Tx)_{k,i} = \sum_{m \in Z^c} \sum_{j \in Z^d} a_{kmij} x_{k-m,j}, \quad k \in Z^c, \quad i \in Z^d,$$

$Analysis Mathematica 49, 2023$
where $a_{kmij} \in \mathbb{C}$ satisfy the estimate
\[
\sum_{i \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}^d} |a_{kmij}| \leq \beta_{m,k} \quad m, k \in \mathbb{Z}^c,
\]
for some $\beta \in l_{1,g}(\mathbb{Z}^c, \mathbb{C})$. Indeed, by Definition 7.1, $T \in s_{1,g}(\mathbb{Z}^c, \mathcal{S}_1(l_1(\mathbb{Z}^d, \mathbb{C})))$ has the form (14) with $\|b_{km}\|_{\mathcal{S}_1(l_1(\mathbb{Z}^d, \mathbb{C}))} \leq \beta_{m,k}$, and, by Proposition 5.5, each $b_{km} \in \mathcal{S}_1(l_1(\mathbb{Z}^d, \mathbb{C}))$ has the form
\[
(b_{km} y)_i = \sum_{j \in \mathbb{Z}^d} a_{kmij} y_j
\]
with $\|b_{km}\|_{\mathcal{S}_1(l_1(\mathbb{Z}^d, \mathbb{C}))} = \sum_{i \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}^d} |a_{kmij}|$.

8. Locally nuclear operators in $L_p$

We represent the set $\mathbb{R}^c$ as the union (of disjoint subsets)
\[
\mathbb{R}^c = \bigcup_{m \in \mathbb{Z}^c} [0, 1)^c + m,
\]
where $m = (m_1, m_2, \ldots, m_c)$ and
\[
[0, 1)^c + m = [m_1, m_1 + 1) \times [m_2, m_2 + 1) \times \cdots \times [m_c, m_c + 1).
\]

**Proposition 8.1.** The following properties hold.

(a) A set $E \subseteq \mathbb{R}^c$ is measurable if and only if its intersection with each of the sets $[0, 1)^c + m$, $m \in \mathbb{Z}^c$, is summable.

(b) A set $N \subseteq \mathbb{R}^c$ is negligible if and only if its intersection with each of the sets $[0, 1)^c + m$, $m \in \mathbb{Z}^c$, is negligible.

(c) A function $x: \mathbb{R}^c \to \mathbb{C}$ is measurable if and only if its restriction to each of the sets $[0, 1)^c + m$, $m \in \mathbb{Z}^c$, is measurable.

(d) A function $x: \mathbb{R}^c \to \mathbb{C}$ is negligible if and only if its restriction to each of the sets $[0, 1)^c + m$, $m \in \mathbb{Z}^c$, is negligible.

**Proof.** The proof is straightforward. □

**Proposition 8.2 ([18], [30, 1.6.3]).** Let $1 \leq p \leq \infty$. Then the correspondence $\varphi: L_p(\mathbb{R}^c, \mathbb{C}) \to l_p(\mathbb{Z}^c, L_p([0, 1)^c, \mathbb{C}))$ defined by the rule $\varphi(x) = \{x_m\}$, where
\[
x_m(t) = x(t + m), \quad t \in [0, 1)^c,
\]
preserves the norm. We claim that (after identifying of equivalent functions) the correspondence $\varphi$ correctly defines an isometric isomorphism
\[
\varphi: L_p(\mathbb{R}^c, \mathbb{C}) \to l_p(\mathbb{Z}^c, L_p([0, 1)^c, \mathbb{C}))
\]
(which we denote by the same symbol $\varphi$).
Proof. We consider the correspondence
\[ \varphi: \mathcal{L}_p(\mathbb{R}^e, \mathbb{C}) \to l_p(\mathbb{Z}^e, \mathcal{L}_p([0,1]^c, \mathbb{C})) \]
defined by the rule \( \varphi(x) = \{x_m\} \), where
\[ x_m(t) = x(t + m), \quad t \in [0,1]^c. \]
Clearly, for any \( x \in \mathcal{L}_p(\mathbb{R}^e, \mathbb{C}) \), the sequence \( \{x_m\} \) consists of measurable functions and
\[ \|x\|_{L_p} = \left\| \left\{ \|x_m\|_{L_p} \right\} \right\|_{l_p} \]
or, in more detail,
\[ \|x\|_{L_p} = \sqrt[p]{\int_{\mathbb{R}^e} |x(t)|^p \, dt} = \sqrt[p]{\sum_{m \in \mathbb{Z}^e} \int_{[0,1]^c} |x_m(t)|^p \, dt}, \quad p < \infty, \]
\[ \|x\|_{L_\infty} = \text{ess sup}_{t \in \mathbb{R}^e} |x(t)| = \sup_{m \in \mathbb{Z}^e} \text{ess sup}_{t \in [0,1]^c} |x_m(t)|, \quad p = \infty. \]
Thus, \( \varphi \) actually acts from \( \mathcal{L}_p(\mathbb{R}^e, \mathbb{C}) \) to \( l_p(\mathbb{Z}^e, \mathcal{L}_p([0,1]^c, \mathbb{C})) \) and preserves the norm.

The linearity of \( \varphi \) is evident. The preservation of the norm implies that \( \varphi \) is injective.

Let \( \{x_m\} \in l_p(\mathbb{Z}^e, \mathcal{L}_p([0,1]^c, \mathbb{C})) \). Obviously, the sequence \( \{x_m\} \) is the inverse image of the function
\[ x(t) = x_m(t - m), \quad t \in [0,1]^c + m. \]
Thus, \( \varphi \) is surjective and
\[ (\varphi^{-1}\{x_m\})(t) = x_k(t - k), \quad t \in [0,1]^c + k. \]

By Proposition 8.1(d), a measurable function \( x \) is negligible if and only if all members of the sequence \( \varphi(x) = \{x_m\} \) are negligible functions. Therefore, \( \varphi \) generates an isomorphic isomorphism \( \varphi: \mathcal{L}_p(\mathbb{R}^e, \mathbb{C}) \to l_p(\mathbb{Z}^e, \mathcal{L}_p([0,1]^c, \mathbb{C})). \)

Definition 8.3. Since the spaces \( \mathcal{L}_p(\mathbb{R}^e, \mathbb{C}) \) and \( l_p(\mathbb{Z}^e, \mathcal{L}_p([0,1]^c, \mathbb{C})) \) are isomorphic, the algebras \( \mathcal{B}(\mathcal{L}_p(\mathbb{R}^e, \mathbb{C})) \) and \( \mathcal{B}(l_p(\mathbb{Z}^e, \mathcal{L}_p([0,1]^c, \mathbb{C}))) \) are isomorphic as well. We denote by \( \mathcal{S}_{1,g}(\mathbb{R}^e, \mathcal{G}_1) = \mathcal{S}_{1,g}(\mathbb{R}^e, \mathcal{G}_1(\mathcal{L}_p([0,1]^c, \mathbb{C}))), \)
\[ 1 \leq p \leq \infty, \] the set of all operators \( A \in \mathcal{B}(\mathcal{L}_p(\mathbb{R}^e, \mathbb{C})) \) that correspond to operators of the class \( \mathcal{S}_{1,g}(\mathbb{Z}^e, \mathcal{G}_1(\mathcal{L}_p([0,1]^c, \mathbb{C}))) \) according to the isomorphism \( \varphi \) described in Proposition 8.2. More precisely, an operator \( A \) belongs to the
class $\mathcal{S}_{1,g}(\mathbb{R}^c, \mathcal{G}_1)$ if and only if the operator $T = \varphi A \varphi^{-1}$, which renders commutative the diagram

$$
\begin{array}{ccc}
L_p & \xrightarrow{\varphi} & l_p \\
\downarrow A & & \downarrow T \\
L_p & \xrightarrow{\varphi} & l_p
\end{array}
$$

belongs to the class $\mathcal{s}_{1,g}(\mathbb{Z}^c, \mathcal{G}_1(L_p([0,1]^c, \mathbb{C})))$. We call operators belonging to the class $\mathcal{S}_{1,g}(\mathbb{R}^c, \mathcal{G}_1)$ locally nuclear as well.

We denote by $\widetilde{\mathcal{S}}_{1,g}(\mathbb{R}^c, \mathcal{G}_1)$ the subalgebra derived from $\mathcal{S}_{1,g}(\mathbb{R}^c, \mathcal{G}_1)$ by adjoining the unit element of the algebra $\mathcal{B}(L_p(\mathbb{R}^c, X))$.

The following theorem is the most interesting special case of Theorem 7.3.

**Theorem 8.4.** The subalgebra $\widetilde{\mathcal{S}}_{1,g}(\mathbb{R}^c, \mathcal{G}_1)$ is full in the algebra $\mathcal{B}(L_p(\mathbb{R}^c, \mathbb{C}))$ for all $1 \leq p \leq \infty$.

**Proof.** The proof follows from isomorphism (16) and Theorem 7.3. □

The following theorem shows that operators of the class $\mathcal{S}_{1,g}(\mathbb{R}^c, \mathcal{G}_1)$ possess integral representation.

**Theorem 8.5.** Let $1 \leq p < \infty$. Then, for each operator $A \in \mathcal{S}_{1,g}(\mathbb{R}^c, \mathcal{G}_1)$, there exists a measurable function $n: \mathbb{R}^c \times \mathbb{R}^c \to \mathbb{C}$ such that for any $x \in L_p(\mathbb{R}^c, \mathbb{C})$ at almost all points $t \in \mathbb{R}^c$ (the following integral exists and)

$$(Ax)(t) = \int_{\mathbb{R}^c} n(t, s) x(s) \, ds.$$
where \( b_{km} \in \mathcal{S}_1(L_p([0, 1]^c, \mathbb{C})) \), and the series converges absolutely, because

\[
\|b_{km}\| \leq \|b_{km}\|_{\mathcal{S}_1} \leq \beta_m.
\]

By Proposition 6.2, for almost all \( t \in [0, 1]^c \), we have

\[
((Tz)_k)(t) = \sum_{m \in \mathbb{Z}^c} (b_{km}x_{k-m})(t), \quad k \in \mathbb{Z}^c.
\]

Applying to the sequence \( Tz = \{(Tz)_i : i \in \mathbb{Z}^c\} \) the isomorphism \( \varphi^{-1} \), we arrive at the function \( Ax \). According to the previous formula,

\[
(Ax)(t) = (\varphi^{-1}(Tz))(t) = ((Tz)_k)(t - k)
\]

\[
= \sum_{m \in \mathbb{Z}^c} (b_{km}x_{k-m})(t - k), \quad t \in [0, 1]^c + k.
\]

We consider the blocks \( b_{km} \in \mathcal{B}(L_p([0, 1]^c, \mathbb{C})) \), \( k, m \in \mathbb{Z}^c \), which constitute the matrix of the operator \( T \). By Theorem 6.3, for each of \( b_{km} \) there exists a measurable function \( n_{km} : [0, 1]^c \times [0, 1]^c \to \mathbb{C} \) such that for any \( u \in L_p([0, 1]^c, \mathbb{C}) \) at almost all \( t \in [0, 1]^c \) (the following integral exists and)

\[
(b_{km}u)(t) = \int_{[0, 1]^c} n_{km}(t, s)u(s) \, ds, \quad t \in [0, 1]^c.
\]

In particular, we have (almost everywhere)

\[
(b_{km}x_{k-m})(t - k) = \int_{[0, 1]^c} n_{km}(t - k, s)x_{k-m}(s) \, ds, \quad t \in [0, 1]^c + k,
\]

or

\[
(b_{km}x_{k-m})(t - k) = \int_{[0, 1]^c + k-m} n_{km}(t - k, \sigma - k + m)x_{k-m}(\sigma - k + m) \, d\sigma
\]

\[
= \int_{[0, 1]^c + k-m} n_{km}(t - k, \sigma - k + m)x(\sigma) \, d\sigma, \quad t \in [0, 1]^c + k.
\]

Hence, for almost all \( t \in [0, 1]^c + k \) (which implies that for almost all \( t \in \mathbb{R}^c \))

\[
(Ax)(t) = \sum_{m \in \mathbb{Z}^c} (b_{km}x_{k-m})(t - k)
\]

\[
= \sum_{m \in \mathbb{Z}^c} \int_{[0, 1]^c + k-m} n_{km}(t - k, s - k + m)x(s) \, ds = \int_{\mathbb{R}^c} n(t, s)x(s) \, ds,
\]

Analysis Mathematica 49, 2023
where
\begin{equation}
\tag{17}
n(t, s) = n_{km}(t - k, s - k + m), \quad t \in [0, 1)^c + k, \ s \in [0, 1)^c + k - m,
\end{equation}
or
\begin{equation}
n(t, s) = n_{k,k-l}(t - k, s - l), \quad t \in [0, 1)^c + k, \ s \in [0, 1)^c + l.
\end{equation}

**Example 8.6.** We show that the operator
\[(Ax)(t) = \int_{\mathbb{R}} e^{-|t-s|} x(s) \, ds,
\]
acting in $L_p(\mathbb{R}, \mathbb{C})$ for $1 < p < \infty$ belongs to $S_1(\mathbb{R}, S_1)$.

From the proof of Theorem 8.5 it is seen that the operator $T = \varphi A \varphi^{-1}$ has the form (14) with
\[(b_{km} u) = \int_{[0,1]} n_{km}(t, s) u(s) \, ds
\]
with (see (17))
\[n_{km}(t, s) = n(t + k, s - k + m) = e^{-|t-s+m|}, \quad t \in [0, 1), \ s \in [0, 1).
\]

In order to make use of Proposition 6.5, we establish an estimate of $\|n_{km}(\tau, \cdot) - n_{km}(t, \cdot)\|_{L_q[0,1]}$. We have
\[\|n_{km}(\tau, \cdot) - n_{km}(t, \cdot)\|_{L_q[0,1]}^q = \int_{[0,1]} |n_{km}(\tau, s) - n_{km}(t, s)|^q \, ds
\]
\[= \int_{[0,1]} \left| e^{-|\tau-s+m|} - e^{-|t-s+m|} \right|^q \, ds = \int_{[0,1]} \left| \int_0^1 \frac{\partial}{\partial \lambda} e^{-|t+\lambda(\tau-t)-s+m|} \, d\lambda \right|^q \, ds
\]
\[= \int_{[0,1]} \left| \int_0^1 e^{-|t+\lambda(\tau-t)-s+m|} \cdot \frac{(t-\tau)(t + \lambda(\tau-t) - s + m)}{|t + \lambda(\tau-t) - s + m|} \, d\lambda \right|^q \, ds
\]
\[\leq \int_{[0,1]} \left| \int_0^1 e^{-|t+\lambda(\tau-t)-s+m|} \cdot |t - \tau| \, d\lambda \right|^q \, ds
\]
\[\leq |t - \tau|^q \int_{[0,1]} \left| \int_0^1 e^{-|t+\lambda(\tau-t)-s+m|} \, d\lambda \right|^q \, ds
\]
(since $t, \tau, s \in [0, 1)$, we have
\[t + \lambda(\tau - t) = (1 - \lambda)t + \lambda \tau \in [0, 1) \text{ and } t + \lambda(\tau - t) - s \in (-1, 1);
\]
therefore \(|t + \lambda(\tau - t)| \leq 1\) and \(|t + \lambda(\tau - t) - s + m| \geq |m| - 1\), and thus we have

\[
\leq |\tau - t|^q \int_{[0,1]} \left| \int_0^1 e^{-|m|+1} d\lambda \right|^q ds
= |\tau - t|^q \int_{[0,1]} (e^{-|m|+1})^q ds = |\tau - t|^q (e^{-|m|+1})^q.
\]

Consequently,

\[
\|n_{km}(\tau, \cdot) - n_{km}(t, \cdot)\|_{L_q[0,1]} \leq |\tau - t| e^{-|m|}.
\]

In a similar way, we have

\[
\|n_{km}(1/2, \cdot)\|_{L_q[0,1]} = \sqrt{\int_{[0,1]} |n_{km}(1/2, s)|^q ds} = \sqrt{\int_{[0,1]} (e^{-1/2-s+m})^q ds}
\leq \sqrt{\int_{[0,1]} (e^{-|m|+1/2})^q ds} = \sqrt{(e^{-|m|+1/2})^q} = e^{-|m|+1/2} = e^{1/2} e^{-|m|}.
\]

Finally, by Proposition 6.5, we have

\[
\|b_{km}\|_{S_1} \leq e^{1/2} e^{-|m|} + \frac{e \cdot e^{-|m|}}{2} \frac{1}{2^{1/p} - 1}.
\]

We consider the family

\[
\beta_m = e^{1/2} e^{-|m|} + \frac{e \cdot e^{-|m|}}{2} \frac{1}{2^{1/p} - 1}, \quad m \in \mathbb{Z}.
\]

Evidently, \(\beta \in l_1\). (Actually, \(\beta \in l_{1,g}\) for any \(g\) from Example 3.3.) Thus, \(A \in S_1(\mathbb{R}, \mathcal{S}_1)\).

References

[1] I. Asekritova, Yu. Karlovich, and N. Kruglyak, One-sided invertibility of discrete operators and their applications, *Aequationes Math.*, 92 (2018), 39–73.

[2] A. G. Baskakov, Wiener’s theorem and asymptotic estimates for elements of inverse matrices. *Funktional. Anal. i Prilozhen.*, 24 (1990), 64–65 (in Russian); English translation in *Funct. Anal. Appl.*, 24 (1990) 222–224.

[3] A. G. Baskakov, Asymptotic estimates for elements of matrices of inverse operators, and harmonic analysis, *Sibirsk. Mat. Zh.*, 38 (1997), 14–28 (in Russian); English translation in *Siberian Math. J.*, 38 (1997), 10–22.

[4] A. G. Baskakov, Representation theory for Banach algebras, Abelian groups, and semi-groups in the spectral analysis of linear operators, *J. Math. Sci.*, 137 (2006), 4885–5036.
[5] I. Beltiță and D. Beltiță, Inverse-closed algebras of integral operators on locally compact groups, *Ann. Henri Poincaré*, 16 (2015), 1283–1306.

[6] I. A. Blatov and A. A. Terteryan, Estimates for the elements of inverse matrices and incomplete block factorization methods based on matrix sweep, *Zh. Vychisl. Mat. i Mat. Fiz.*, 32 (1992) 1683–1696 (in Russian); English translation in *Comput. Math. Math. Phys.*, 32 (1992), 1509–1522.

[7] N. Bourbaki, *Éléments de mathématique. Fascicule XXXII. Théories spectrales. Chapitre I: Algèbres normées. Chapitre II: Groupes localement compacts commutatifs*, Actualités Scientifiques et Industrielles, No. 1332. Hermann, Paris, 1967.

[8] N. Bourbaki, *Integration. I. Chapters 1–6*, translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian, *Elements of Mathematics (Berlin)*, Springer-Verlag (Berlin, 2004); Translated from the 1959, 1965 and 1967 French originals by Sterling K. Berberian.

[9] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co. (Amsterdam–London–New York–Tokyo, 1993).

[10] J. Delgado, A trace formula for nuclear operators on $L^p$, in: *Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations*, B.-W. Schulze and M.W. Wong, eds., Operator Theory: Advances and Applications, vol. 205, Birkhäuser Verlag AG (Basel–Boston–Berlin, 2010), pp. 181–193.

[11] S. Demko, Inverses of band matrices and local convergence of spline projections. *SIAM J. Numer. Anal.*, 14 (1977), 616–619.

[12] S. Demko, Spectral bounds for $\|A^{-1}\|_\infty$, *J. Approx. Theory*, 48 (1986), 207–212.

[13] S. Demko, W. F. Moss, and Ph. W. Smith, Decay rates for inverses of band matrices, *Math. Comp.*, 43 (1984), 491–499.

[14] N. Dunford and J. T. Schwartz, *Linear operators. Part I. General theory*, Wiley Classics Library, John Wiley & Sons, Inc. (New York, 1988).

[15] B. Farrell and Th. Strohmer, Inverse-closedness of a Banach algebra of integral operators on the Heisenberg group, *J. Operator Theory*, 64 (2010), 189–205.

[16] G. Fendler, K. Gröchenig, and M. Leinert, Convolution-dominated operators on discrete groups, *Integral Equations Operator Theory*, 61 (2008), 493–509.

[17] G. Fernández-Torres and Yu. Karlovich, Two-sided and one-sided invertibility of Wiener-type functional operators with a shift and slowly oscillating data, *Banach J. Math. Anal.*, 11 (2017), 554–590.

[18] J. J. F. Fournier and J. Stewart, Amalgams of $L^p$ and $l^q$, *Bull. Amer. Math. Soc. (N.S.)*, 13 (1985), 1–21.

[19] I. Gohberg, M. A. Kaashoek, and H. J. Woerdeman, The band method for positive and strictly contractive extension problems: an alternative version and new applications, *Integral Equations Operator Theory*, 12 (1989) 343–382.

[20] G. H. Golub and Ch. F. Van Loan, *Matrix Computations*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press (Baltimore, MD, 2013).

[21] K. Gröchenig, Wiener’s lemma: theme and variations. An introduction to spectral invariance, in: *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser (Boston–Basel–Berlin, 2010), Ch. 5, pp. 175–244.

[22] K. Gröchenig and A. Klotz, Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices, *Constr. Approx.*, 32 (2010), 429–466.

[23] K. Gröchenig and M. Leinert, Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices, *Trans. Amer. Math. Soc.*, 358 (2006), 2695–2711.
[24] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., no. 16, American Mathematical Society (Providence, RI, 1966).

[25] E. Yu. Guseva and V. G. Kurbatov, Inverse-closedness of subalgebras of integral operators with almost periodic kernels, *Complex Anal. Oper. Theory*, 14 (2020), Paper No. 4, 23 pp.

[26] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, American Mathematical Society Colloquium Publications, vol. 31, Amer. Math. Soc. (Providence, RI, 1957).

[27] S. Jaffard, Propriétés des matrices “bien localisées” près de leur diagonale et quelques applications, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7 (1990), 461–476.

[28] V. G. Kurbatov, Algebras of difference and integral operators, *Funktional. Anal. i Prilozhen.*, 24 (1990), 87–88 (in Russian); English translation in *Funct. Anal. Appl.*, 24 (1990), 156–158.

[29] V. G. Kurbatov, *Linear Differential-difference Equations*, Voronezh State University (Voronezh, 1990) (in Russian).

[30] V. G. Kurbatov, *Functional Differential Operators and Equations*, Mathematics and its Applications, vol. 473, Kluwer Academic Publishers (Dordrecht, 1999).

[31] V. G. Kurbatov, Some algebras of operators majorized by a convolution, *Funct. Differ. Equ.*, 8 (2001), 323–333.

[32] V. G. Kurbatov and V. I. Kuznetsova, Inverse-closedness of the set of integral operators with $L_1$-continuously varying kernels, *J. Math. Anal. Appl.*, 436 (2016), 322–338.

[33] A. Pietsch, *Operator Ideals*, Mathematische Monographien [Mathematical Monographs], vol. 16, VEB Deutscher Verlag der Wissenschaften (Berlin, 1978).

[34] V. S. Rabinovich, S. Roch, and B. Silbermann, Fredholm theory and finite section method for band-dominated operators, *Integral Equations Operator Theory*, 30 (1998), 452–495.

[35] V. S. Rabinovich, S. Roch, and B. Silbermann, *Limit Operators and Their Applications in Operator Theory*, Operator Theory: Advances and Applications, vol. 150, Birkhäuser Verlag (Basel, 2004).

[36] W. Rudin, *Functional Analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co. (New York–Düsseldorf–Johannesburg, 1973).

[37] A. F. Ruston, On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach space, *Proc. London Math. Soc. (2)*, 53 (1951), 109–124.

[38] A. F. Ruston, Direct products of Banach spaces and linear functional equations, *Proc. London Math. Soc. (3)*, 1 (1951), 327–384.

[39] J. Sjöstrand, Wiener type algebras of pseudodifferential operators, *Séminaire sur les Équations aux Dérivées Partielles*, 1994–1995, Exp. No. IV, École Polytech. (Palaiseau, 1995), 21 pp.

[40] Q. Sun, Wiener’s lemma for infinite matrices with polynomial off-diagonal decay, *C. R. Math. Acad. Sci. Paris*, 340 (2005), 567–570.

[41] Q. Sun, Wiener’s lemma for infinite matrices. II, *Constr. Approx.*, 34 (2011), 209–235.