Functional relations from the Yang-Baxter algebra:
Eigenvalues of the $XXZ$ model with non-diagonal twisted and open boundary conditions

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**Abstract**

In this work we consider a functional method in the theory of exactly solvable models based on the Yang-Baxter algebra. Using this method we derive the eigenvalues of the $XXZ$ model with non-diagonal twisted and open boundary conditions for general values of the anisotropy and boundary parameters.

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1 Introduction

Functional equations methods appeared in the theory of exactly solvable lattice models intimately connected with Baxter’s commuting transfer matrix method [1]. In the early seventies Baxter introduced in his pioneer work [1] the concept of $Q$-operators and $T - Q$ relations determining the eigenvalues of the transfer matrix of the corresponding vertex model.

The complex calculations involved in Baxter’s construction of $Q$-operators seems to have restricted its use and other functional methods, such as the Reshetikhin’s analytical Bethe ansatz, were employed instead to obtain the spectrum of transfer matrices related with quantum Kac-Moody algebras [2]. However we remark here the recent progresses in the construction of $Q$-operators by employing quantum algebras representation theoretic methods [3, 4].

On the other hand, the advent of the algebraic Bethe ansatz in the late seventies provided a systematic approach to find the eigenvalues and eigenvectors of transfer matrices of integrable vertex models. This method, originally proposed by Takhtadzhan and Faddeev [5], is based on the existence of a pseudovacuum or reference state and appropriate commutation rules provenient from the Yang-Baxter algebra, which is a common algebraic structure associated with integrable vertex models.

Although the algebraic Bethe ansatz method is a powerful tool exhibiting a rich mathematical structure, its implementation for models that do not possess a trivial reference state is still an obstacle to be overcome. The aim of this paper is to show that the Yang-Baxter algebra can be explored in order to generate functional relations determining the spectrum of transfer matrices. In order to illustrate that we consider the $XXZ$ model with non-diagonal twists and open boundaries for general values of the anisotropy and boundary parameters. Besides the relevance of studying models with general boundary conditions in the context of statistical mechanics [6], the $XXZ$ model with non-diagonal twists and open boundaries are both included in a class of models where a trivial reference state is absent.

This paper is organized as follows. In section 2 we describe the $XXZ$ model with general toroidal boundary conditions. In particular, we discuss the case of non-diagonal twists and we derive functional relations for its eigenvalues making use of the Yang-Baxter algebra. In section 3 we approach the eigenvalue problem for the $XXZ$ model with non-
diagonal open boundaries using the algebraic-functional method devised in the section 2. Concluding remarks are discussed in the section 4 and in the appendices A-D we give some extra results and technical details.

2 The XXZ model with non-diagonal twisted boundary conditions

The advent of the Quantum Inverse Scattering Method \cite{5,7} was an important stage in the development of the theory of exactly solvable quantum systems. This method unveiled a deep connection between solutions of the Yang-Baxter equation, quantum integrable systems and exactly solvable lattice models of statistical mechanics in two dimensions \cite{5}. In statistical mechanics an important role is played by vertex models whose respective transfer matrix is constructed from local Boltzmann weights contained in a operator $L_{Aj}$. Let $V$ be a finite dimensional linear space, the integrability of the vertex model is achieved when the operator valued function $L : \mathbb{C} \rightarrow \text{End}(V \otimes V)$ is a solution of the Yang-Baxter equation, namely

$$L_{12}(\lambda - \mu)L_{13}(\lambda)L_{23}(\mu) = L_{23}(\mu)L_{13}(\lambda)L_{12}(\lambda - \mu),$$

defined in the space $V_1 \otimes V_2 \otimes V_3$. Here we use the standard notation $L_{ij} \in \text{End}(V_i \otimes V_j)$. The complex valued operator $L_{Aj}(\lambda)$ can be viewed as a matrix in the space of states $\mathcal{A}$ denoting for instance the horizontal degrees of freedom of a square lattice, while its matrix elements are operators acting non-trivially in the $j$-th position of $\bigotimes_{i=1}^{L} V_i$. In its turn the space $V_j$ represents the space of states of the vertical degrees of freedom at the $j$-th site of a chain of length $L$.

The transfer matrix of the corresponding vertex model can be written in terms of the monodromy matrix $T_A(\lambda)$ defined by the following ordered product

$$T_A(\lambda) = L_{AL}(\lambda)L_{AL-1}(\lambda) \ldots L_{A1}(\lambda).$$

As a consequence of the Yang-Baxter equation, the monodromy matrix satisfies the following quadratic relation

$$R(\lambda - \mu) T_A(\lambda) \otimes T_A(\mu) = T_A(\mu) \otimes T_A(\lambda) R(\lambda - \mu),$$

$$(3)$$
usually denominated Yang-Baxter algebra. The $R$-matrix appearing in (3) follows from the solution of the Yang-Baxter equation through the relation $R(\lambda) = PL(\lambda)$ where $P$ denotes the usual permutation operator. The above $R$-matrix plays the role of structure constant for the Yang-Baxter algebra and it consist of an invertible complex valued matrix acting on the tensor product $A \otimes A$.

The invariance of the Yang-Baxter algebra plays an important role in the description of integrable spin chains with general toroidal boundary conditions. One can easily verify the invariance of (3) under the transformation $T_A(\lambda) \rightarrow G_A T_A(\lambda)$ provided that the $c$-number matrix $G_A$ is a symmetry of the $R$-matrix, i.e.

$$[R(\lambda), G_A \otimes G_A] = 0.$$  \hfill (4)

Consequently we can define the operator

$$T(\lambda) = \text{Tr}_A [G_A T_A(\lambda)]$$  \hfill (5)

which constitutes an one parameter family of commuting transfer matrices, i.e. $[T(\lambda), T(\mu)] = 0$.

Now restricting ourselves to the $XXZ$ model, $V_i \equiv C^2$ and the corresponding $L$-operator is that of the anisotropic six vertex model

$$L(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix},$$  \hfill (6)

whose Boltzmann weights are given by $a(\lambda) = \sinh(\lambda + \gamma), b(\lambda) = \sinh(\lambda)$ and $c(\lambda) = \sinh(\gamma)$.

In the Ref. \cite{9, 10} the authors discuss the possible classes of twist matrices $G_A$ compatible with the $R$-matrix associated with (6). These twist matrices turn out to be

(i) $G_A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ \hfill (ii) $G_A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$

where $\alpha$ and $\beta$ are arbitrary complex parameters. The above matrices $G_A$ are non-singular for $\alpha, \beta \neq 0$ and the $XXZ$ model with generalized toroidal boundary conditions is obtained as the logarithmic derivative of the transfer matrix (5) at the point $\lambda = 0$ \cite{9, 10}.

The understanding of physical properties of vertex models demands the exact diagonalization of their respective transfer matrices, which can provide us information about the
free energy behaviour and the nature of the elementary excitations. When the twist matrix of type (i) is considered, a trivial reference state is available and the eigenvalues of the corresponding transfer matrix can be obtained through the algebraic Bethe ansatz \cite{5,7}. By way of contrast, the twist matrix of type (ii) breaks the $U(1)$ symmetry of the system leaving only a $Z_2$ invariance.

Although a trivial reference state in the framework of the algebraic Bethe ansatz is no longer available when the matrix $G_A$ of type (ii) is considered, the eigenvalues of the corresponding transfer matrix were obtained in \cite{9} by means of the Baxter’s $T−Q$ method. It is worthwhile to remark here that the isotropic case associated with the $XXX$ spin chain admits any $2 \times 2$ twist matrix and interesting enough the algebraic Bethe ansatz solution can be obtained by exploring the $GL(2)$ symmetry \cite{11}.

From the historical point of view, the $T−Q$ method was introduced in Baxter’s remarkable works on the eight vertex model \cite{1} and more recently it has found applications in many areas such as the study of integrable systems \cite{12}, conformal field theory \cite{3,13}, correlations functions \cite{14} and efficient description of finite temperature properties \cite{15}. Motivated by the ideas of Baxter’s $T−Q$ method \cite{1,8} and the algebraic Bethe ansatz \cite{5,7}, in what follows we shall demonstrate how we can use the Yang-Baxter algebra to obtain functional relations determining the spectrum of the transfer matrix $T(\lambda)$.

Let us consider the eigenvalue problem for the transfer matrix (5),

$$T(\lambda) |\psi\rangle = \Lambda(\lambda) |\psi\rangle,$$  \hspace{1cm} (8)

taking into account the type (ii) $G_A$ matrix given in (7). As shown in the appendix A, we can set $\alpha = \beta = 1$ without loss of generality for the purposes of this paper. Now considering the definition (2), the monodromy matrix $T_A(\lambda)$ consist of a $2 \times 2$ matrix whose elements are operators that we denote

$$T_A(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$  \hspace{1cm} (9)

Therefore, the transfer matrix (5) reads

$$T(\lambda) = B(\lambda) + C(\lambda).$$  \hspace{1cm} (10)

In contrast to the case when $G_A$ is diagonal \cite{16}, the state $|0\rangle$ defined as

$$|0\rangle = \bigotimes_{j=1}^{L} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$  \hspace{1cm} (11)
is not an eigenstate of the transfer matrix \( T_{A}(\lambda) \). Nevertheless, the state \(|0\rangle\) is still of great utility. Considering the definition (2) together with (6), the elements of \( T_{A}(\lambda) \) satisfy the following relations

\[
A(\lambda) |0\rangle = a(\lambda)^{L} |0\rangle \quad \quad D(\lambda) |0\rangle = b(\lambda)^{L} |0\rangle
\]

\[
B(\lambda) |0\rangle = \dagger \quad \quad C(\lambda) |0\rangle = 0,
\]

where the symbol \( \dagger \) stands for a non-null value. Notice the relations (12) imply in

\[
T(\lambda) |0\rangle = B(\lambda) |0\rangle \tag{13}
\]

and

\[
T(\lambda)B(\lambda) |0\rangle = B(\lambda)B(\lambda) |0\rangle + C(\lambda)B(\lambda) |0\rangle \tag{14}
\]

Now we have reached a point of fundamental importance. The term \( C(\lambda)B(\lambda) |0\rangle \) present in the right hand side of (14) can be evaluated with the help of the Yang-Baxter algebra (3). In order to show that we collect the following commutation rule among the ones encoded in the relation (3),

\[
C(\lambda)B(\mu) = B(\mu)C(\lambda) + \frac{c(\lambda - \mu)}{b(\lambda - \mu)} [A(\mu)D(\lambda) - A(\lambda)D(\mu)]. \tag{15}
\]

The commutation rule (15), together with the relations (12), results in the following identity

\[
C(\lambda)B(\mu) |0\rangle = \frac{c(\lambda - \mu)}{b(\lambda - \mu)} \left[ a(\mu)^{L}b(\lambda)^{L} - a(\lambda)^{L}b(\mu)^{L} \right] |0\rangle. \tag{16}
\]

Therefore, as we are interested in \( C(\lambda)B(\lambda) |0\rangle \), the next step is to consider the limit \( \mu \to \lambda \) in the above relation which can be evaluated using L’Hopital’s rule. Thus we are left with

\[
C(\lambda)B(\lambda) |0\rangle = M(\lambda) |0\rangle \tag{17}
\]

where \( M(\lambda) = Lc(\lambda)^{2}a(\lambda)^{L-1}b(\lambda)^{L-1} \).

At this stage we have already gathered the basic ingredients to obtain functional relations determining the eigenvalues of \( T(\lambda) \). Operating with the dual eigenvector \( \langle \psi | \) on the left side of Eqs. (13) and (14) we are left with the relations

\[
\Lambda(\lambda)F_{0} = F_{1}(\lambda) \tag{18}
\]

\[
\Lambda(\lambda)F_{1}(\lambda) = F_{2}(\lambda) + M(\lambda)F_{0}, \tag{19}
\]
where \( F_0 = \langle \psi | 0 \rangle \), \( F_1(\lambda) = \langle \psi | B(\lambda) | 0 \rangle \) and \( F_2(\lambda) = \langle \psi | B(\lambda)B(\lambda) | 0 \rangle \).

Here we remark one of the roles played by integrability in this approach. Since the transfer matrix \( T(\lambda) \) belongs to a commuting family, the eigenvectors \( | \psi \rangle \) are independent of the spectral parameter \( \lambda \). Thus the dependence of the functions \( F_i(\lambda) \) with \( \lambda \) is solely determined by the operator \( B(\lambda) \). In the appendix C we have computed that dependence making use of Eqs. (2), (6) and (9).

From the Eq. (18) we can see that the functions \( \Lambda(\lambda) \) and \( F_1(\lambda) \) differ only by the constant factor \( F_0 \), thus they possess the same zeroes. Now we can eliminate \( \Lambda(\lambda) \) from the Eq. (18) and substitute the result into the Eq. (19) which yields the following relation involving only the functions \( F_i(\lambda) \),

\[
F_1(\lambda)^2 = F_2(\lambda)F_0 + M(\lambda)F_0^2. \tag{20}
\]

Considering the Eqs. (2), (6) and (9), in the appendix C we have demonstrated that the functions \( F_1(\lambda) \) and \( F_2(\lambda) \) can be written as

\[
F_1(\lambda) = F_1(0) \prod_{i=1}^{L-1} \frac{\sinh(\lambda^{(1)}_i - \lambda)}{\sinh(\lambda^{(1)}_i)} \quad \text{and} \quad F_2(\lambda) = F_2(0) \prod_{i=1}^{2(L-1)} \frac{\sinh(\lambda^{(2)}_i - \lambda)}{\sinh(\lambda^{(2)}_i)}, \tag{21}
\]

where \( \lambda^{(i)}_j \) denote the zeroes of the function \( F_i(\lambda) \). Now a closer look in the Eqs. (18) and (19) reveals that we can obtain the eigenvalue \( \Lambda(\lambda) \) by determining the variables \( \lambda^{(i)}_j \) together with the ratios \( \frac{F_i(0)}{F_0} \).

In order to obtain analogues of Bethe ansatz equations determining the variables \( \lambda^{(i)}_j \), we first consider the Eq. (20) at the points \( \lambda = \lambda^{(1)}_j \). Then we find

\[
F_2(\lambda^{(1)}_j) + M(\lambda^{(1)}_j)F_0 = 0. \tag{22}
\]

Next we consider the points \( \lambda = \lambda^{(2)}_j \) in Eq. (20) which yields

\[
F_1(\lambda^{(2)}_j)^2 - M(\lambda^{(2)}_j)F_0^2 = 0. \tag{23}
\]

Furthermore, we observe by setting \( \lambda = 0 \) in the Eqs. (18) and (19) that the ratios \( \frac{F_i(0)}{F_0} \) can be written in terms of \( \Lambda(0) \) which can be easily evaluated. Therefore we are left with the following expression for the ratios \( \frac{F_i(0)}{F_0} \),

\[
\frac{F_1(0)}{F_0} = \Lambda(0), \quad \frac{F_2(0)}{F_0} = \Lambda(0)^2. \tag{24}
\]
In order to avoid an overcrowded section we have performed the diagonalization of $T(0)$ in the appendix B. The eigenvalue $\Lambda(0)$ turns out to be

$$\Lambda(0) = \sinh(\gamma) L e^{\frac{2\pi r}{L}} \quad r = 0, \ldots, 2L - 1. \quad (25)$$

Gathering our results so far, from the Eqs. (18, 22, 23, 24) we obtain the following expression for the transfer matrix eigenvalues $\Lambda(\lambda)$,

$$\Lambda(\lambda) = \sinh(\gamma) L e^{\frac{2\pi r}{L}} \frac{\prod_{i=1}^{L-1} \sinh(\lambda^{(1)}_i - \lambda)}{\sinh(\lambda^{(1)}_i)} \quad (26)$$

recalling that $r = 0, \ldots, 2L - 1$ and provided that the variables $\lambda^{(i)}_j$ satisfy the following system of non-linear algebraic equations

$$\left[ \frac{\sinh(\lambda^{(1)}_j + \gamma)}{\sinh(\gamma)} \frac{\sinh(\lambda^{(1)}_j)}{\sinh(\gamma)} \right]^{L-1} = -e^{\frac{2\pi r}{L}} \frac{2(L-1)}{\prod_{i=1}^{L-1} \sinh(\lambda^{(2)}_i - \lambda^{(1)}_i)} \quad j = 1, \ldots, L - 1 \quad (27)$$

$$\left[ \frac{\sinh(\lambda^{(2)}_j + \gamma)}{\sinh(\gamma)} \frac{\sinh(\lambda^{(2)}_j)}{\sinh(\gamma)} \right]^{L-1} = \frac{e^{\frac{2\pi r}{L}}}{L} \prod_{i=1}^{L-1} \left[ \frac{\sinh(\lambda^{(1)}_i - \lambda^{(2)}_j)}{\sinh(\lambda^{(1)}_i)} \right]^2 \quad j = 1, \ldots, 2(L - 1) \quad (28)$$

Now we shall examine some aspects of the Eqs. (27) and (28) taking into account the crossing properties discussed in the appendix D. Considering the Eq. (D.9), it follows that

$$\Lambda^t(\lambda) = (-1)^{L+1} \Lambda(-\lambda - \gamma) \quad (29)$$

where $\Lambda^t(\lambda)$ denotes the eigenvalue of the transposed transfer matrix $T^t(\lambda)$. Although the relation (29) does not imply in $\Lambda(\lambda) = (-1)^{L+1} \Lambda(-\lambda - \gamma)$, we have verified the existence of eigenvalues satisfying that relation through the direct diagonalization of $T(\lambda)$ for small chain length $L$.

Let us suppose the relation $\Lambda(\lambda) = (-1)^{L+1} \Lambda(-\lambda - \gamma)$ holds for some eigenvalue. Hence, from the Eq. (18) we can conclude that the function $F_1(\lambda)$ also satisfy

$$F_1(\lambda) = (-1)^{L+1} F_1(-\lambda - \gamma). \quad (30)$$

Since the function $M(\lambda)$ enjoys the property $M(\lambda) = M(-\lambda - \gamma)$, from the Eq. (19) we also find that

$$F_2(\lambda) = F_2(-\lambda - \gamma). \quad (31)$$
Assuming that the relations (30) and (31) hold they have remarkable implications concerning the solutions of Eqs. (27) and (28). Let us analyze first the odd $L$ case. Thus $L - 1$ is an even number and the Eq. (30) implies, for instance, in the following relation for the variables $\lambda^{(1)}_{j}$,

$$\lambda^{(1)}_{i} = -\gamma - \lambda^{(1)}_{i} - \gamma \quad i = 1, \ldots, \frac{L - 1}{2}.$$  

(32)

Thus altogether we have only $\frac{L - 1}{2}$ independent variables $\lambda^{(1)}_{j}$.

Now we turn our attention to the $L$ even case. In that case the Eq. (30) implies for instance in the relation

$$\lambda^{(1)}_{i} = -\lambda^{(1)}_{i} - \gamma \quad i = 1, \ldots, \frac{L - 2}{2},$$

(33)

while the remaining root $\lambda^{(1)}_{L - 1}$ is fixed at the value $-\frac{\gamma}{2}$.

The above analysis can also be carried out for the roots $\lambda^{(2)}_{j}$ taking into account the Eq. (31). Since $2(L - 1)$ is an even number, the variables $\lambda^{(2)}_{j}$ must be related for instance by

$$\lambda^{(2)}_{i} = -\lambda^{(2)}_{i} - \gamma \quad i = 1, \ldots, L - 1.$$  

(34)

These implications of the crossing symmetry have been verified by a numerical analysis of the Eqs. (27) and (28) for small values of $L$. By comparing the eigenvalues given by the Eq. (26) with the direct diagonalization of $T(\lambda)$, we have also verified that the Eqs. (26,27,28) indeed describes a complete spectrum.

We close this section by remarking the dependence with $\frac{1}{L}$ in the right hand side of Eqs. (27) and (28). As far as we know, this kind of dependence with the chain length had not appeared previously in the context of Bethe ansatz, which may be of relevance for the description of the thermodynamical limit $L \to \infty$.

### 3 The $XXZ$ model with non-diagonal open boundaries

In Sklyanin’s pioneer work [17], the author has been able to generalize the Quantum Inverse Scattering Method to accomodate integrable spin chains with open boundaries. The Yang-Baxter equation is still the corner stone of Sklyanin’s approach and it turns out that the $XXZ$ model with general open boundary conditions can be obtained from the following
double-row transfer matrix
\[ t(u) = \text{Tr}_A \left[ K_A^+(u) T_A(u) K_A^-(u) T_A(u) \right], \]
where \( T_A(u) = L_{AL}(u) L_{AL-1}(u) \ldots L_{A1}(u) \) and \( T_A(u) = L_{A1}(u) L_{A2}(u) \ldots L_{AL}(u) \) are the standard monodromy matrices that generate the corresponding closed spin chain with \( L \) sites. The integrability at the boundaries is governed by the matrices \( K_A^-(u) \) and \( K_A^+(u) \), each one describing the reflection at one of the ends of an open chain.

Moreover, the boundary conditions compatible with the bulk integrability are constrained by the so-called reflection equations, which for \( K_A^-(u) \) reads
\[ \mathcal{L}_{21}(u-v) K_A^-(u) \mathcal{L}_{12}(u+v) K_A^-(v) = K_A^-(v) \mathcal{L}_{21}(u+v) K_A^-(u) \mathcal{L}_{12}(u-v), \]
while a similar equation should also hold for the matrix \( K_A^+(u) \).

Turning our attention to the \( XXZ \) model, the bulk hamiltonian is described by the \( \mathcal{L} \)-operator (6) and the most general boundary matrices satisfying the reflection equations are of the form
\[ K_A^-(u) = \begin{pmatrix} k_{11}^-(u) & k_{12}^-(u) \\ k_{21}^-(u) & k_{22}^-(u) \end{pmatrix}, \quad K_A^+(u) = \begin{pmatrix} k_{11}^+(u) & k_{12}^+(u) \\ k_{21}^+(u) & k_{22}^+(u) \end{pmatrix}, \]
whose matrix elements are given by
\[
\begin{align*}
k_{11}^-(u) &= \sinh(h_1^- + u) & k_{12}^-(u) &= \sinh(2u) \\
k_{21}^-(u) &= h_3^- \sinh(2u) & k_{22}^-(u) &= \sinh(h_1^- - u) \\
k_{11}^+(u) &= \sinh(h_1^+ - u - \gamma) & k_{12}^+(u) &= \sinh(-2u - 2\gamma) \\
k_{21}^+(u) &= h_3^+ \sinh(-2u - 2\gamma) & k_{22}^+(u) &= \sinh(h_1^+ + u + \gamma).
\end{align*}
\]
The above \( K \)-matrices possess altogether six free boundary parameters \( \{h_i^\pm\} \) and we find the \( XXZ \) model hamiltonian with open boundaries,
\[
\mathcal{H} = \sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(\gamma) \sigma_i^z \sigma_{i+1}^z \\
+ \frac{\sinh(\gamma)}{\sinh(h_1^-)} \left[ (h_2^- + h_3^-) \sigma_i^x + i(h_2^- - h_3^-) \sigma_i^y + \cosh(h_1^-) \sigma_i^z \right] \\
- \frac{\sinh(\gamma)}{\sinh(h_1^+)} \left[ (h_2^+ + h_3^+) \sigma_L^x + i(h_2^+ - h_3^+) \sigma_L^y + \cosh(h_1^+) \sigma_L^z \right],
\]
related to the former by
\[ t'(0) = 2 \sinh(h_1^-) \sinh(h_1^+) \sinh(\gamma)^{2L-1} \left[ \cosh(\gamma) \mathcal{H} + L \cosh(\gamma)^2 + \sinh(\gamma)^2 \right], \]
where $\sigma_i^x$, $\sigma_i^y$, and $\sigma_i^z$ denote the usual Pauli matrices acting on the $i$-th site.

For general values of the boundary parameters $h_i^\pm$ the double-row transfer matrix (35) does not exhibit $U(1)$ invariance, which makes the application of the algebraic Bethe ansatz quite complex due to the lack of a trivial reference state. By imposing $h_2^\pm = h_3^\pm = 0$ the matrices $K_\pm(\lambda)$ (37) become diagonal and the $U(1)$ symmetry exhibited by the bulk of the system is restored. In that case the spectrum of (35) was obtained in [17] through the algebraic Bethe ansatz. However, several progresses concerning non-diagonal open boundaries have been reported in the literature over the past years when the boundary parameters or the bulk anisotropy satisfy a given constraint [18]-[31].

In what follows we shall explore the Yang-Baxter algebra to generate functional relations determining the complete spectrum of the double-row transfer matrix (35) for general values of the bulk anisotropy $\gamma$ and boundary parameters $h_i^\pm$. In order to tackle the eigenvalue problem

$$t(u)\ket{\Phi} = \Delta(u)\ket{\Phi} \tag{41}$$

for the double-row transfer matrix (35), we first notice the $L$-operator (6) satisfies the property $L_{12}(u)L_{12}(-u) = a(u)a(-u)\text{Id} \otimes \text{Id}$, where $\text{Id}$ denotes the $2 \times 2$ identity matrix. Therefore, the monodromy matrices $T_A(u)$ and $\bar{T}_A(u)$ are related by $\bar{T}_A(u) = [a(u)a(-u)]^L T_A^{-1}(-u)$ and we find the following relation,

$$\left[\bar{T}_A(u) \otimes \text{Id}\right] R(2u) \left[T_A(u) \otimes \text{Id}\right] = \left[\text{Id} \otimes T_A(u)\right] R(2u) \left[\text{Id} \otimes \bar{T}_A(u)\right], \tag{42}$$

with a straightforward manipulation of the Yang-Baxter algebra (3). Analogously to Eq. (9), we denote the matrix elements of $\bar{T}_A(u)$ by

$$\bar{T}_A(u) = \begin{pmatrix} \bar{A}(u) & \bar{B}(u) \\ \bar{C}(u) & \bar{D}(u) \end{pmatrix}. \tag{43}$$

In this way the double-row transfer matrix (35) reads

$$t(u) = k_{11}^+(u)U_1(u) + k_{22}^+(u)U_2(u) + k_{12}^+(u)U_4(u) + k_{21}^+(u)U_3(u) \tag{44}$$

where

$$U_1(u) = k_{11}^-(u)A(u)\bar{A}(u) + k_{12}^-(u)A(u)\bar{C}(u) + k_{21}^-(u)B(u)\bar{A}(u) + k_{22}^-(u)B(u)\bar{C}(u)$$

$$U_2(u) = k_{11}^-(u)C(u)\bar{B}(u) + k_{12}^-(u)C(u)\bar{D}(u) + k_{21}^-(u)D(u)\bar{B}(u) + k_{22}^-(u)D(u)\bar{D}(u)$$
Next we consider the action of \( t(u) \) on the state \(|0\rangle\) defined in (41), which will require the following relations

\[
\begin{align*}
\tilde{A}(u) |0\rangle &= a(u) |0\rangle & \tilde{D}(u) |0\rangle &= b(u) |0\rangle \\
\tilde{B}(u) |0\rangle &= \dagger & \tilde{C}(u) |0\rangle &= 0
\end{align*}
\]

obtained from the structure of the monodromy matrix \( \tilde{T}_A(u) \), and the relations (12) from the previous section.

Then, in order to evaluate \( t(u) |0\rangle \), we shall make use of the commutation rules between the elements of \( T_A(u) \) and \( \tilde{T}_A(u) \) enclosed in the relation (12). In particular, we shall use the following ones,

\[
\begin{align*}
A(u)\tilde{B}(u) &= \frac{b(2u)}{a(2u)} \tilde{B}(u)A(u) - \frac{c(2u)}{a(2u)} B(u)\tilde{D}(u) \\
D(u)\tilde{B}(u) &= \frac{a(2u)}{b(2u)} B(u)D(u) + \frac{c(2u)}{b(2u)} \tilde{A}(u)B(u) \\
\tilde{A}(u)B(u) &= \frac{b(2u)}{a(2u)} B(u)\tilde{A}(u) - \frac{c(2u)}{a(2u)} B(u)D(u) \\
C(u)\tilde{B}(u) &= \frac{c(2u)}{a(2u)} \left[ \tilde{A}(u)A(u) - D(u)\tilde{D}(u) \right] + \tilde{B}(u)C(u) \\
B(u)\tilde{B}(u) &= \tilde{B}(u)B(u).
\end{align*}
\]

Considering the Eqs. (41), (45), (12) and (46) together with the relations (47), a straightforward calculation leave us with

\[
t(u) |0\rangle = Y_0(u) |0\rangle + Y_1(u)B(u) |0\rangle + \tilde{Y}_1(u)\tilde{B}(u) |0\rangle + Y_2(u)B(u)\tilde{B}(u) |0\rangle,
\]

where the functions \( Y_0(u) \), \( Y_1(u) \), \( \tilde{Y}_1(u) \) and \( Y_2(u) \) are given by

\[
\begin{align*}
Y_0(u) &= a^{2L} (u) \sinh(h_1^+ - u) \sinh(h_1^- + u) \frac{\sinh(2u + 2\gamma)}{\sinh(2u + \gamma)} \\
&\quad - b^{2L} (u) \sinh(u + \gamma + h_1^+) \sinh(u + \gamma - h_1^-) \frac{\sinh(2u)}{\sinh(2u + \gamma)} \\
&\quad - a^{L} (u) b^{L} (u) \left( h_3^+ h_3^- + h_3^- h_3^+ \right) \sinh(2u) \sinh(2u + 2\gamma) \\
Y_1(u) &= a^{L} (u) h_3^- \sinh(h_1^+ - u) \sinh(2u) \frac{\sinh(2u + 2\gamma)}{\sinh(2u + \gamma)}
\end{align*}
\]
\[ + b^L(u)h_{\bar{3}}\sinh(u + \gamma - h_{\bar{1}})\sinh(2u)\frac{\sinh(2u + 2\gamma)}{\sinh(2u + \gamma)} \]

From the results of the appendix D we also have the crossing invariance of the double-row
functions \(Y_i(u)\), given in \((D.8)\), imply in the crossing invariance of the function \(f(u)\),

\[ Y_1(u) = -a^L(u)h_{\bar{3}}\sinh(h_{\bar{1}} + u)\sinh(2u)\frac{\sinh(2u + 2\gamma)}{\sinh(2u + \gamma)} \]

\[ Y_2(u) = -h_{\bar{3}}^2h_{\bar{1}}\sinh(2u)\sinh(2u + 2\gamma). \quad (49) \]

We can now follow the scheme devised in the previous section and operating with the dual
eigenvector \(\langle \Phi \rangle\) on the left side of Eq. \((48)\) we obtain

\[ \Delta(u)f_0 = Y_0(u)f_0 + Y_1(u)f_1(u) + \bar{Y}_1(u)f_1(u) + Y_2(u)f_2(u), \]

\[ \text{where } f_0 = \langle \Phi | 0 \rangle, \; f_1(u) = \langle \Phi | B(u) | 0 \rangle, \; \bar{f}_1(u) = \langle \Phi | \bar{B}(u) | 0 \rangle \text{ and } f_2(u) = \langle \Phi | B(u)\bar{B}(u) | 0 \rangle. \]

Before we proceed it is important to examine first the Eq. \((50)\) under the light of the
results obtained in the appendix D. Indeed, the crossing symmetry enables us to establish
the relation \(B(u) = (-1)^{L+1}\bar{B}(-u - \gamma)\) \((D.8)\) which implies in

\[ \bar{f}_1(u) = (-1)^{L+1}f_1(-u - \gamma). \]

Furthermore, the commutation rule \([B(u), \bar{B}(u)] = 0\) given in \((47)\) together with the relation
\((D.8)\), imply in the crossing invariance of the function \(f_2(u)\), i.e.

\[ f_2(u) = f_2(-u - \gamma). \]

From the results of the appendix D we also have the crossing invariance of the double-row
transfer matrix \(t(u) = t(-u - \gamma)\), which implies in

\[ \Delta(u) = \Delta(-u - \gamma). \]

In their turn the functions \(Y_i(u)\) given in \((49)\) fulfill the relations \(Y_0(u) = Y_0(-u - \gamma), \quad \bar{Y}_1(u) = (-1)^{L+1}Y_1(-u - \gamma)\) and \(Y_2(u) = Y_2(-u - \gamma)\). Taking into account the above properties, one can easily verify the crossing invariance of the Eq. \((50)\), which simplifies to the following functional relation,

\[ \Delta(u)f_0 = Y_0(u)f_0 + Y_1(u)f_1(u) + Y_1(-u - \gamma)f_1(-u - \gamma) + Y_2(u)f_2(u). \quad (54) \]

Considering now the results from the appendix C, the function \(f_1(u)\) \((C.13)\) can be
written as

\[ f_1(u) = f_1(0)\prod_{i=1}^{L-1}\frac{\sinh(u^{(1)}_i - u)}{\sinh(u^{(1)}_i)}. \quad (55) \]
On the other hand, the functions $f_2(u)$ and $\Delta(u)$ given by (C.14) and (C.16) will receive extra simplifications due to the relations (52) and (53). Thus they can be written as

$$f_2(u) = g_2 \sinh(u) \sinh(u + \gamma) \prod_{i=1}^{L-2} \frac{\sinh(u_i^{(2)} - u) \sinh(u_i^{(2)} + \gamma + u)}{\sinh(u_i^{(2)}) \sinh(u_i^{(2)} + \gamma)}$$  \hspace{1cm} (56)$$

$$\Delta(u) = \Delta(0) \prod_{i=1}^{L+2} \frac{\sinh(u_i^{(0)} - u) \sinh(u_i^{(0)} + \gamma + u)}{\sinh(u_i^{(0)}) \sinh(u_i^{(0)} + \gamma)}.$$  \hspace{1cm} (57)$$

The initial condition $\Delta(0)$ has been determined in the appendix B through a direct analysis of $t(0)$. Then the next step is to determine the variables $u_j^{(i)}$. In order to do that we consider the Eq. (54) at the points $u = u_j^{(0)}$, $u = u_j^{(1)}$ and $u = u_j^{(2)}$. By doing so we find the following system of algebraic equations,

$$Y_0(u_j^{(0)}) f_0 + Y_1(u_j^{(0)}) f_1(u_j^{(0)}) + Y_1(-u_j^{(0)} - \gamma) f_1(-u_j^{(0)} - \gamma) + Y_2(u_j^{(0)}) f_2(u_j^{(0)}) = 0$$  \hspace{1cm} (58)$$

$$[Y_0(u_j^{(1)}) - \Delta(u_j^{(1)})] f_0 + Y_1(u_j^{(1)} - u_j^{(0)} - \gamma) f_1(-u_j^{(1)} - \gamma) + Y_2(u_j^{(1)}) f_2(u_j^{(1)}) = 0$$  \hspace{1cm} (59)$$

$$[Y_0(u_j^{(2)}) - \Delta(u_j^{(2)})] f_0 + Y_1(u_j^{(2)}) f_1(u_j^{(2)}) + Y_1(u_j^{(2)} - u_j^{(0)} - \gamma) f_1(-u_j^{(2)} - \gamma) = 0.$$  \hspace{1cm} (60)$$

A direct inspection of the Eqs. (58-60) shows that the ratios $\frac{f_1}{f_0}$ and $\frac{f_2}{f_0}$ are also required. In contrast to the case of twisted boundary conditions considered in the previous section, where $M(0) = 0$, the point $u = 0$ of Eq. (54) is not very enlightening. However, in order to determine the ratios $\frac{f_1}{f_0}$ and $\frac{f_2}{f_0}$ we consider, for instance, the non-trivial points $u = u_1$ and $u = u_2$ such that $Y_1(u_1) = Y_1(u_2) = 0$. Then we are left with the following equations

$$[Y_0(u_i) - \Delta(u_i)] f_0 + Y_1(-u_i - \gamma) f_1(-u_i - \gamma) + Y_2(u_i) f_2(u_i) = 0$$  \hspace{1cm} (61)$$

which can be solved for the required ratios.

Now our results can be gathered and we are left with the following expression for the eigenvalues $\Delta(u)$,

$$\Delta(u) = 2 \cosh(\gamma) \sinh(h_1^+) \sinh(h_1^-) \sinh(\gamma) \prod_{i=1}^{L+2} \frac{\sinh(u_i^{(0)} - u) \sinh(u_i^{(0)} + \gamma + u)}{\sinh(u_i^{(0)}) \sinh(u_i^{(0)} + \gamma)},$$  \hspace{1cm} (62)$$

provided that the variables $u_1$, $u_2$ and $u_j^{(i)}$ satisfy the following system of non-linear algebraic equations,

$$\left[ \frac{\sinh(u_i + \frac{\gamma}{2})}{\sinh(u_i - \frac{\gamma}{2})} \right]^{L} \frac{h_3^- \sinh(u_i - \frac{\gamma}{2} - h_i^+)}{h_3^+ \sinh(u_i + \frac{\gamma}{2} - h_i^-)} = 1$$  \hspace{1cm} i = 1, 2 \hspace{1cm} (63)$$

\textsuperscript{1}We have performed the shifts $u_i \rightarrow u_i - \frac{\gamma}{2}$.}
\[ Y_0(u_j^{(0)}) = \]
\[- \left[ Y_1(u_j^{(0)}) \prod_{i=1}^{L-1} \frac{\sinh(u_i^{(1)} - u_j^{(0)})}{\sinh(u_i^{(1)})} + Y_1(-u_j^{(0)} - \gamma) \prod_{i=1}^{L-1} \frac{\sinh(u_i^{(1)} + \gamma + u_j^{(0)})}{\sinh(u_i^{(1)})} \right] \frac{\det(H_1)}{\det(H_0)}
\]
\[-Y_2(u_j^{(0)}) \sinh(u_j^{(0)}) \sinh(u_j^{(0)} + \gamma) \prod_{i=1}^{L-2} \frac{\sinh(u_i^{(2)} - u_j^{(0)})}{\sinh(u_i^{(2)})} \frac{\sinh(u_i^{(2)} + \gamma + u_j^{(0)})}{\sinh(u_i^{(2)} + \gamma)} \frac{\det(H_2)}{\det(H_0)} \]
\[ j = 1, \ldots, L + 2 \]

\[ (64) \]

\[ \left[ 2 \cosh(\gamma) \sinh(h_1^+) \sinh(h_1^-) \sinh(\gamma)^{2L} \prod_{i=1}^{L+2} \frac{\sinh(u_i^{(0)} - u_j^{(1)})}{\sinh(u_i^{(0)})} \frac{\sinh(u_i^{(0)} + \gamma + u_j^{(1)})}{\sinh(u_i^{(0)} + \gamma)} - Y_0(u_j^{(1)}) \right] = \]

\[ +Y_2(u_j^{(1)}) \sinh(u_j^{(1)}) \sinh(u_j^{(1)} + \gamma) \prod_{i=1}^{L-2} \frac{\sinh(u_i^{(2)} - u_j^{(1)})}{\sinh(u_i^{(2)})} \frac{\sinh(u_i^{(2)} + \gamma + u_j^{(1)})}{\sinh(u_i^{(2)} + \gamma)} \frac{\det(H_2)}{\det(H_0)} \]

\[ +Y_1(-u_j^{(1)} - \gamma) \prod_{i=1}^{L-1} \frac{\sinh(u_i^{(1)} + \gamma + u_j^{(1)})}{\sinh(u_i^{(1)})} \frac{\det(H_1)}{\det(H_0)} \]
\[ j = 1, \ldots, L - 1 \]

\[ (65) \]

\[ \left[ 2 \cosh(\gamma) \sinh(h_1^+) \sinh(h_1^-) \sinh(\gamma)^{2L} \prod_{i=1}^{L+2} \frac{\sinh(u_i^{(0)} - u_j^{(2)})}{\sinh(u_i^{(0)})} \frac{\sinh(u_i^{(0)} + \gamma + u_j^{(2)})}{\sinh(u_i^{(0)} + \gamma)} - Y_0(u_j^{(2)}) \right] = \]

\[ Y_1(u_j^{(2)}) \prod_{i=1}^{L-1} \frac{\sinh(u_i^{(1)} - u_j^{(2)})}{\sinh(u_i^{(1)})} + Y_1(-u_j^{(2)} - \gamma) \prod_{i=1}^{L-1} \frac{\sinh(u_i^{(1)} + \gamma + u_j^{(2)})}{\sinh(u_i^{(1)})} \frac{\det(H_1)}{\det(H_0)} \]
\[ j = 1, \ldots, L - 2 \]

\[ (66) \]

where \( H_i \) are \( 2 \times 2 \) matrices resulting from the solution of the system of equations for the ratios \( \frac{L_0^{(0)}}{j_0} \) and \( \frac{L_0^{(2)}}{j_0} \). These matrices turn out to be

\[ H_0 = \begin{pmatrix} \phi_1(u_1) & \phi_2(u_1) \\ \phi_1(u_2) & \phi_2(u_2) \end{pmatrix}, \quad H_1 = \begin{pmatrix} \phi_0(u_1) & \phi_2(u_1) \\ \phi_0(u_2) & \phi_2(u_2) \end{pmatrix}, \quad H_2 = \begin{pmatrix} \phi_1(u_1) & \phi_0(u_1) \\ \phi_1(u_2) & \phi_0(u_2) \end{pmatrix} \]

\[ (67) \]

where

\[ \phi_0(u) = 2 \cosh(\gamma) \sinh(h_1^+) \sinh(h_1^-) \sinh(\gamma)^{2L} \prod_{i=1}^{L+2} \frac{\sinh(u_i^{(0)} + \frac{\gamma}{2} - u)}{\sinh(u_i^{(0)})} \frac{\sinh(u_i^{(0)} + \frac{\gamma}{2} + u)}{\sinh(u_i^{(0)} + \gamma)} - Y_0(u - \frac{\gamma}{2}) \]

\[ 14 \]
\[ \phi_1(u) = Y_1(-u - \frac{\gamma}{2}) \prod_{i=1}^{L-1} \frac{\sinh(u_i^{(1)} + \frac{\gamma}{2} + u)}{\sinh(u_i^{(1)})} \]

\[ \phi_2(u) = Y_2(u - \frac{\gamma}{2}) \sinh(u - \frac{\gamma}{2}) \sinh(u + \frac{\gamma}{2}) \prod_{i=1}^{L-2} \frac{\sinh(u_i^{(2)} + \frac{\gamma}{2} - u)}{\sinh(u_i^{(2)})} \]

\[ \frac{\sinh(u_i^{(2)} + \frac{\gamma}{2} + u)}{\sinh(u_i^{(2)})}, \]  

(68)

Taking into account the Eq. (40) we are left with the following expression for the eigenergies \( E \) of the hamiltonian (39),

\[ E = - \sinh^2(\gamma) \sum_{i=1}^{L+2} \frac{1}{\sinh(u_i^{(0)}) \sinh(u_i^{(0)} + \gamma)} - L \cosh(\gamma) - \frac{\sinh(\gamma)^2}{\cosh(\gamma)}, \]

(69)

given in terms of the roots \( u_i^{(0)} \). We end this section by remarking that numerical checks performed for \( L = 2, 3, 4 \) show that the spectrum generated by the Eqs. (62)-(66) is complete.

### 4 Concluding Remarks

In this paper we have proposed a functional method in the theory of exactly solvable models based on the Yang-Baxter algebra. Using this method we were able to derive the eigenvalues of the \( XXZ \) model with non-diagonal twists and open boundaries for general values of the bulk anisotropy and boundary parameters. Our solution is presented in terms of analogues of Bethe ansatz equations whose variables involved are precisely the roots of the transfer matrix eigenvalues and roots of auxiliary functions defined in terms of the monodromy matrix elements.

Concerning the \( XXZ \) model with non-diagonal twists discussed in the section 2, we stress the unusual dependence of the Eqs. (27) and (28) with the chain length \( L \). We hope the computation of physical properties in the thermodynamical limit \( L \to \infty \), such as the interfacial tension obtained in [9], will be benefited by the use of Eqs. (27, 28). As shown in [34], we also remark that our solution corresponds to the eigenvalues of Baxter’s eight vertex model with elliptic modulus \( \kappa = 1 \) and a particular choice of diagonal twist matrix.

Regarding the case of non-diagonal open boundaries described in the section 3, we remark the solution recently presented for general values of the anisotropy and boundary parameters obtained from the representation theory of \( q \)-Onsager algebra [35] and the multiple reference state structure found in [32, 33]. An interesting problem would be unveiling the
connection between these approaches. Since our method is based on the Yang-Baxter algebra, which is a common algebraic structure underlying integrable vertex models, we expect that our approach can be applied for other models with general open boundary conditions based on $q$-deformed Lie algebras [36].

Finally, we observe that our solutions involve more than one kind of variable resembling the so-called nested Bethe ansatz equations, which are typical of models with higher rank symmetry. We remark that a similar result had been reported previously in the literature obtained from generalized $T – Q$ relations [27].

5 Acknowledgements

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[36] R. Malara and A. Lima-Santos, J. Stat. Mech.: Theor. Exp., (2006) P09013.
Appendix A: The condition $\alpha = \beta = 1$

Let $\tilde{T}(\lambda)$ denote the transfer matrix

$$\tilde{T}(\lambda) = \text{Tr}_A \left[ \tilde{G}_A \mathcal{L}_{AL}(\lambda) \mathcal{L}_{AL-1}(\lambda) \ldots \mathcal{L}_{A1}(\lambda) \right]$$

(A.1)

where $\mathcal{L}_{Aj}(\lambda)$ is the $\mathcal{L}$-operator given in (6) and $\tilde{G}_A$ is the following twist matrix

$$\tilde{G}_A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.$$  

(A.2)

Now considering the matrix $\mathcal{M} = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}$ one can verify that $\mathcal{M}_A^{-1} \tilde{G}_A \mathcal{M}_A = \sqrt{\alpha \beta} G_A$

where $G_A$ is given by

$$G_A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$  

(A.3)

which corresponds to $\tilde{G}_A$ with $\alpha = \beta = 1$. By noticing that $\mathcal{M}_j^{-1} \mathcal{M}_A^{-1} \mathcal{L}_{Aj}(\lambda) \mathcal{M}_A \mathcal{M}_j = \mathcal{L}_{Aj}(\lambda)$, we then find the relation

$$\prod_{j=1}^L \mathcal{M}_j^{-1} \tilde{T}(\lambda) \prod_{i=j}^L \mathcal{M}_j = \sqrt{\alpha \beta} T(\lambda),$$

(A.4)

where $T(\lambda)$ is the transfer matrix (10) considered in the section 2. Thus the eigenvalues $\hat{\Lambda}(\lambda)$ of the transfer matrix $\tilde{T}(\lambda)$ are related to the ones of $T(\lambda)$ by

$$\hat{\Lambda}(\lambda) = \sqrt{\alpha \beta} \Lambda(\lambda).$$

(A.5)

Appendix B: The Eigenvalues $\Lambda(0)$ and $\Delta(0)$

In this appendix we consider the diagonalization of the transfer matrices $T(0)$ and $t(0)$ associated with the $XXZ$ model with twisted and open boundary conditions respectively.

- Twisted Boundary Conditions:

The $\mathcal{L}$-operator (5) consist of a regular solution of the Yang-Baxter equation, i.e. $\mathcal{L}(0) = \sinh(\gamma) P$, where $P$ denotes the permutation operator. In this way the transfer matrix (10) at the point $\lambda = 0$ can be written as $T(0) = \sinh(\gamma)^L \hat{O}$ where the operator $\hat{O}$ is given by

$$\hat{O} = \text{Tr}_A \left[ G_A P_{AL} P_{AL-1} \ldots P_{A1} \right].$$

(B.1)
Using the permutator algebra, \( P^2 = \text{Id} \otimes \text{Id} \) and \( P_{Aj} P_{Ai} = P_{Ai} P_{ij} \), we can readily obtain the following expression for \( \hat{O} \),
\[
\hat{O} = G_1 P_{1L} \ldots P_{13} P_{12},
\]
and its \( n \) times product
\[
\hat{O}^n = G_1 G_2 \ldots G_n \prod_{j=1}^{n} P_{jL} P_{jL-1} \ldots P_{jn+1}.
\]
From the Eq. (B.3) we have
\[
\hat{O}^L = G_1 G_2 \ldots G_L,
\]
which implies, with \( G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), in \( \hat{O}^{2L} = I \) where \( I \) is the identity matrix. Thus the eigenvalues \( O \) of the operator \( \hat{O} \) are given by
\[
O = e^{i\pi r} \quad r = 0, 1, \ldots, 2L - 1
\]
leaving us with the following expression for \( \Lambda(0) \),
\[
\Lambda(0) = \sinh(\gamma)^L e^{i\pi r} \quad r = 0, 1, \ldots, 2L - 1.
\]

**Open Boundary Conditions:**

The matrix \( K_A(u) \) presented in Eqs. (37) and (38) consist of a regular solution of the reflection equation, i.e. \( K_A(0) = \sinh(h^-_1) \text{Id} \). Together with the regularity property of the \( L \)-operator (6) and the permutator property \( P^2 = \text{Id} \otimes \text{Id} \), the double-row transfer matrix (35) at the point \( u = 0 \) is given by
\[
t(0) = \sinh(\gamma)^{2L} \sinh(h^-_1) \text{Tr}_A \left[ K_A(0)^L \right] I.
\]
Therefore, we can see that the transfer matrix \( t(0) \) is proportional to the identity matrix \( I \) with the following eigenvalue
\[
\Delta(0) = 2 \cosh(\gamma) \sinh(h^+_1) \sinh(h^-_1) \sinh(\gamma)^{2L}.
\]

**Appendix C: The functions \( F_i(\lambda) \) and \( f_i(\lambda) \)**

In this appendix we derive the form of the functions \( F_i(\lambda) \) and \( f_i(\lambda) \) used in the sections 2 and 3, considering the algebraic properties of the monodromy matrices \( T_A(\lambda) \) and \( \tilde{T}_A(\lambda) \).
In order to determine the dependence of the functions \( F_i(\lambda) \) with the spectral parameter \( \lambda \), let us first introduce the following notation for the monodromy matrix \( T_A^{(L)}(\lambda) \)

\[
T_A^{(L)}(\lambda) = \begin{pmatrix}
A_L(\lambda) & B_L(\lambda) \\
C_L(\lambda) & D_L(\lambda)
\end{pmatrix},
\]

(C.1)
differing from the one used in (9) by the index \( L \) inserted to emphasize we are considering the ordered product of \( L \) matrices \( L_A(\lambda) \). For instance, the matrices \( L_A(\lambda) \) are given by

\[
L_A(\lambda) = \begin{pmatrix}
\alpha_j(\lambda) & \beta_j(\lambda) \\
\gamma_j(\lambda) & \delta_j(\lambda)
\end{pmatrix}
\]

(C.2)

where

\[
\alpha_j(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ 0 & b(\lambda) \end{pmatrix}_j, \quad \beta_j(\lambda) = \begin{pmatrix} 0 & 0 \\ c(\lambda) & 0 \end{pmatrix}_j,
\]

\[
\gamma_j(\lambda) = \begin{pmatrix} 0 & c(\lambda) \\ 0 & 0 \end{pmatrix}_j, \quad \delta_j(\lambda) = \begin{pmatrix} b(\lambda) & 0 \\ 0 & a(\lambda) \end{pmatrix}_j
\]

(C.3)

are matrices acting non-trivially in the \( j \)-th site of the chain.

Moreover, the monodromy matrix \( T_A^{(L)}(\lambda) \) defined by Eq. (2) satisfies the recurrence relation

\[
T_A^{(L+1)}(\lambda) = L_{A_L+1}(\lambda)T_A^{(L)}(\lambda),
\]

(C.4)
corresponding to the following relations for its matrix elements

\[
A_{L+1}(\lambda) = \alpha_{L+1}(\lambda)A_L(\lambda) + \beta_{L+1}(\lambda)C_L(\lambda)
\]

\[
B_{L+1}(\lambda) = \alpha_{L+1}(\lambda)B_L(\lambda) + \beta_{L+1}(\lambda)D_L(\lambda)
\]

\[
C_{L+1}(\lambda) = \gamma_{L+1}(\lambda)A_L(\lambda) + \delta_{L+1}(\lambda)C_L(\lambda)
\]

\[
D_{L+1}(\lambda) = \gamma_{L+1}(\lambda)B_L(\lambda) + \delta_{L+1}(\lambda)D_L(\lambda).
\]

(C.5)

At this stage, it is important to keep in mind that the spectral parameter \( \lambda \) enters in the monodromy matrix only through the Boltzmann weights \( a(\lambda) \) and \( b(\lambda) \), since \( c(\lambda) \) is \( \lambda \) independent (6). For the forthcoming discussion, it is convenient to introduce the weight \( w(\lambda) \) only to denote the Boltzmann weights possessing dependence with \( \lambda \). Thus, \( w(\lambda) \) can represent both \( a(\lambda) \) and \( b(\lambda) \). The reason for the introduction of \( w(\lambda) \) will become clear in what follows. From the Eqs. (2), (C.2) and (C.3) it is clear that \( B_1(\lambda) \) and \( C_1(\lambda) \) are polynomials of degree 0 in \( w \), while \( A_1(\lambda) \) and \( D_1(\lambda) \) are polynomials of degree 1 in \( w \). These
where the constant \( p \) can be written as \( a(\lambda) = \frac{1}{2} (x q - x^{-1} q^{-1}) \) and \( b(\lambda) = \frac{1}{2} (x - x^{-1}) \). Thus \( w(\lambda) \) is of the form \( w(\lambda) \sim x + x^{-1} \), and expanding its powers we find

\[
B(\lambda) \sim x^{L-1} + \ldots + x^{-(L-1)} = x^{-(L-1)} \left[ c_{L-1}(x^2)^{L-1} + c_L(x^2)^L + \ldots + c_0 \right]. \tag{C.6}
\]

Now considering the functions \( F_1(\lambda) = \langle \psi | B(\lambda) | 0 \rangle \) and \( F_2(\lambda) = \langle \psi | B(\lambda) B(\lambda) | 0 \rangle \) used in section 2, and the fact that \( |\psi\rangle \) is \( \lambda \) independent, we can conclude that

\[
F_1(\lambda) = x^{-(L-1)} P_1(x), \quad F_2(\lambda) = x^{-2(L-1)} P_2(x), \tag{C.7}
\]

where \( P_1(x) \) and \( P_2(x) \) are polynomials in the variable \( x^2 \) of degrees \( L - 1 \) and \( 2(L - 1) \) respectively. Therefore, the polynomials \( P_1(x) \) and \( P_2(x) \) can be written as

\[
P_1(x) = p_1 \prod_{i=1}^{L-1} \left[ x^2 - (x_i^{(1)})^2 \right],
\]

\[
P_2(x) = p_2 \prod_{i=1}^{2(L-1)} \left[ x^2 - (x_i^{(2)})^2 \right] \tag{C.8}
\]

where \( p_j \) are constants and \( x_i^{(j)} \) denote the zeroes of the polynomial \( P_j(x) \). Using the variables \( x_i^{(j)} = e^{\lambda_i^{(j)}} \) we then have

\[
F_1(\lambda) = F_1(0) \prod_{i=1}^{L-1} \frac{\sinh(\lambda_i^{(1)} - \lambda)}{\sinh(\lambda_i^{(1)})},
\]

\[
F_2(\lambda) = F_2(0) \prod_{i=1}^{2(L-1)} \frac{\sinh(\lambda_i^{(2)} - \lambda)}{\sinh(\lambda_i^{(2)})} \tag{C.9}
\]

where the constant \( p_j \) is absorbed by \( F_j(0) \).

Let us now consider the monodromy matrix \( \mathcal{T}_A^{(L)}(\lambda) = \mathcal{L}_{A1}(\lambda) \mathcal{L}_{A2}(\lambda) \ldots \mathcal{L}_{AL}(\lambda) \), with matrix elements

\[
\mathcal{T}_A^{(L)}(\lambda) = \begin{pmatrix} \bar{A}_L(\lambda) & \bar{B}_L(\lambda) \\ \bar{C}_L(\lambda) & \bar{D}_L(\lambda) \end{pmatrix}. \tag{C.10}
\]

appearing in the construction of the double-row transfer matrix \([35]\). Compared to the notation previously used \([43]\), we have also included the index \( L \) stressing we are considering the
ordered product of $L$ matrices $\mathcal{L}_{\mathcal{A}j}(\lambda)$. The monodromy matrix $\tilde{\mathcal{T}}_{\mathcal{A}}^{(L)}(\lambda)$ obeys the recurrence formula

$$\tilde{\mathcal{T}}_{\mathcal{A}}^{(L+1)}(\lambda) = \tilde{\mathcal{T}}_{\mathcal{A}}^{(L)}(\lambda)\mathcal{L}_{\mathcal{A}L+1}(\lambda), \quad (C.11)$$

corresponding to the following recursion relations for the matrix elements

$$\begin{align*}
\tilde{A}_{L+1}(\lambda) &= \tilde{A}_L(\lambda)\alpha_{L+1}(\lambda) + \tilde{B}_L(\lambda)\gamma_{L+1}(\lambda) \\
\tilde{B}_{L+1}(\lambda) &= \tilde{A}_L(\lambda)\beta_{L+1}(\lambda) + \tilde{B}_L(\lambda)\delta_{L+1}(\lambda) \\
\tilde{C}_{L+1}(\lambda) &= \tilde{C}_L(\lambda)\alpha_{L+1}(\lambda) + \tilde{D}_L(\lambda)\gamma_{L+1}(\lambda) \\
\tilde{D}_{L+1}(\lambda) &= \tilde{C}_L(\lambda)\beta_{L+1}(\lambda) + \tilde{D}_L(\lambda)\delta_{L+1}(\lambda).
\end{align*} \quad (C.12)$$

The same arguments previously used for the elements of $\mathcal{T}_{\mathcal{A}}^{(L)}(\lambda)$ can now be considered for the elements of $\tilde{\mathcal{T}}_{\mathcal{A}}^{(L)}(\lambda)$. Thus we obtain that the operators $\tilde{B}_L(\lambda)$ and $\tilde{C}_L(\lambda)$ are polynomials of degree $L-1$ in $w$ while $\tilde{A}_L(\lambda)$ and $\tilde{D}_L(\lambda)$ are of degree $L$. Hence the function $f_1(u) = \langle \Phi | B(u) | 0 \rangle$ required in the section 3 can be written as

$$f_1(u) = f_1(0) \prod_{i=1}^{L-1} \frac{\sinh(u^{(1)}_i - u)}{\sinh(u^{(1)}_i)} \cdot (C.13)$$

On the other hand the function $f_2(u) = \langle \Phi | B(u)\tilde{B}(u) | 0 \rangle$ requires an extra analysis. Noticing that $\alpha_j(0) = \frac{1}{\epsilon(0)}\gamma_j(0)\beta_j(0)$ and $\delta_j(0) = \frac{1}{\epsilon(0)}\beta_j(0)\gamma_j(0)$ together with $\beta_j^2(\lambda) = \gamma_j^2(\lambda) = 0$, one can easily see from the Eqs. (C.5) and (C.12) that $B(0)\tilde{B}(0) = 0$. This same analysis can be performed for the point $\lambda = -\gamma$ yielding $B(-\gamma)\tilde{B}(-\gamma) = 0$. Thus the function $f_2(u)$ can be written as

$$f_2(u) = g_2 \sinh(u) \sinh(u + \gamma) \prod_{i=1}^{2(L-2)} \frac{\sinh(u^{(2)}_i - u)}{\sinh(u^{(2)}_i)} \cdot (C.14)$$

In the section 3 we also require the function $\Delta(u)$ defined as

$$\Delta(u) = \frac{\langle \Phi | t(u) | \Phi \rangle}{\langle \Phi | \Phi \rangle} \cdot (C.15)$$

The above results together with the Eqs. (44), (45), and (38) show that the double-row transfer matrix consist of a polynomial of degree $2(L+2)$ in $w$. Thus the eigenvalues $\Delta(u)$ are of the form

$$\Delta(u) = \Delta(0) \prod_{i=1}^{2(L+2)} \frac{\sinh(u^{(0)}_i - u)}{\sinh(u^{(0)}_i)} \cdot (C.16)$$

where $u^{(0)}_i$ denote the zeroes of $\Delta(u)$. 

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Appendix D: Crossing symmetry

In this appendix we deduce some important properties used in the sections 2 and 3 which are consequence of the crossing symmetry.

Besides the Yang-Baxter equation, the \( \mathcal{L} \)-operator satisfies other important relations, namely

\[
\text{Temporal invariance: } \quad \mathcal{L}_{12}^{t_1 t_2}(\lambda) = \mathcal{L}_{12}(\lambda) \\
\text{Crossing symmetry: } \quad \mathcal{L}_{12}(\lambda) = -V_1 \mathcal{L}_{12}^{t_2}(\lambda - \gamma)V_1^{-1} \tag{D.1}
\]

where \( V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( t_i \) stands for the transposition in the space with index \( i \).

In order to establish a relation between the matrices \( \mathcal{T}_A(\lambda) \) and \( \bar{\mathcal{T}}_A(\lambda) \), we first recall the definitions

\[
\mathcal{T}_A(\lambda) = \mathcal{L}_{AL}(\lambda)\mathcal{L}_{AL-1}(\lambda) \ldots \mathcal{L}_{A1}(\lambda) \quad \tag{D.2}
\]

\[
\bar{\mathcal{T}}_A(\lambda) = \mathcal{L}_{A1}(\lambda) \ldots \mathcal{L}_{AL-1}(\lambda)\mathcal{L}_{AL}(\lambda), \quad \tag{D.3}
\]

and observe that \( \mathcal{L}_{12}^{t_2}(\lambda) = \mathcal{L}_{12}(\lambda) \) which follows from the temporal invariance \( \text{(D.1)} \). Next we consider the transposition in the auxiliary space \( \mathcal{A} \),

\[
\mathcal{T}_{\mathcal{A}}^{t_i}(\lambda) = \mathcal{L}_{\mathcal{A}1}^{t_i}(\lambda) \ldots \mathcal{L}_{\mathcal{A}L}^{t_i}(\lambda) = \mathcal{L}_{\mathcal{A}1}^{t_i}(\lambda) \ldots \mathcal{L}_{\mathcal{A}L}^{t_i}(\lambda), \quad \tag{D.4}
\]

which allows us to use the crossing symmetry \( \text{(D.1)} \) to obtain

\[
\mathcal{T}_{\mathcal{A}}^{t_i}(\lambda) = (-1)^L V_A^{-1} \mathcal{L}_{A1}(\lambda - \gamma)V_A \ldots V_A^{-1} \mathcal{L}_{AL}(\lambda - \gamma)V_A \\
= (-1)^L V_A^{-1} \bar{\mathcal{T}}_A(-\lambda - \gamma)V_A. \quad \tag{D.5}
\]

Now we also consider the transposition \( t = t_1 \ldots t_L \). Then we find

\[
\mathcal{T}_{\mathcal{A}}^{t_a t}(\lambda) = [\mathcal{L}_{AL}(\lambda) \ldots \mathcal{L}_{A1}(\lambda)]^{t_a t_1 \ldots t_L} \\
= \mathcal{L}_{\mathcal{A}1}^{t_a t_1}(\lambda) \ldots \mathcal{L}_{\mathcal{A}L}^{t_a t_L}(\lambda) \\
= \mathcal{L}_{A1}(\lambda) \ldots \mathcal{L}_{AL}(\lambda), \quad \tag{D.6}
\]

which leads immediately to the relation

\[
\mathcal{T}_{\mathcal{A}}^{t_a t}(\lambda) = \bar{\mathcal{T}}_A(\lambda). \quad \tag{D.7}
\]
In terms of the matrix elements, the relations (D.5) and (D.7) read,

\[
\begin{align*}
A(\lambda) &= (-1)^L \tilde{D}(-\lambda - \gamma) \\
B(\lambda) &= (-1)^{L+1} \tilde{B}(-\lambda - \gamma) \\
A^t(\lambda) &= \tilde{A}(\lambda) \\
B^t(\lambda) &= \tilde{C}(\lambda)
\end{align*}
\]

\[C(\lambda) = (-1)^{L+1} \tilde{C}(-\lambda - \gamma) \quad D(\lambda) = (-1)^L \tilde{A}(-\lambda - \gamma) \quad C^t(\lambda) = \tilde{B}(\lambda) \quad D^t(\lambda) = \tilde{D}(\lambda). \tag{D.8} \]

A straightforward calculation taking into account the relations (D.8) reveals that the twisted transfer matrix (10) satisfies the relation

\[T^t(\lambda) = (-1)^{L+1} T(-\lambda - \gamma), \tag{D.9} \]

while we have the following identity for the double-row transfer matrix (35)

\[t(u) = t(-u - \gamma). \tag{D.10} \]