Barycentric Subdivision of Cayley Graphs With Constant Edge Metric Dimension

ALI N. A. KOAM\(^1\) AND ALI AHMAD\(^2\)

\(^1\)Department of Mathematics, College of Science, Jazan University, New Campus, Jazan 2097, Saudi Arabia
\(^2\)College of Computer Science and Information Technology, Jazan University, Jazan 2097, Saudi Arabia

Corresponding author: Ali Ahmad (ahmadsms@gmail.com)

ABSTRACT A motion of a robot in space is represented by a graph. A robot change its position from point to point and its position can be determined itself by distinct labelled landmarks points. The problem is to determine the minimum number of landmarks to find the unique position of the robot, this phenomena is known as metric dimension. Motivated by this a new modification was introduced by Kelenc. In this paper, we computed the edge metric dimension of barycentric subdivision of Cayley graphs \(\text{Cay}(\mathbb{Z}_\alpha \oplus \mathbb{Z}_\beta)\), for every \(\alpha \geq 6, \beta \geq 2\) and an observation is made that it has constant edge metric dimension and only three carefully chosen vertices can appropriately suffice to resolve all the edges of barycentric subdivision of Cayley graphs \(\text{Cay}(\mathbb{Z}_\alpha \oplus \mathbb{Z}_\beta)\).

INDEX TERMS Metric dimension, edge metric dimension, resolving set, barycentric subdivision, Cayley graph.

I. INTRODUCTION Slater [1] and Harary el at [2] independently introduced the concept of metric dimension. Metric dimension has contributed in different real world applications like navigation of robots [3], wireless communications and sensor networks [4], pattern recognition and image processing [5], and above all it has most applications in chemistry that are discussed in [6]–[8]. A lot of interesting work has been done on metric dimension for several types of graphs that can be seen in [9]–[19].

Let \(G = (V, E)\) be a simple and connected graph. Let \(d(\alpha, \beta)\) denotes the distance between two vertices \(\alpha, \beta \in V(G)\). Let \(\alpha \in V(G), e = \beta \gamma \in E(G)\), then \(d(\alpha, e) = \min(d(\alpha, \beta), d(\alpha, \gamma))\). A vertex \(\alpha\) distinguish two vertices \(\beta, \gamma\), if \(d(\alpha, \beta) \neq d(\alpha, \gamma)\), similarly a vertex \(\alpha\) distinguish two edges \(e, e'\), if \(d(\alpha, e) \neq d(\alpha, e')\). A set \(R \subseteq V(G)\) be an edge metric generator for \(G\) if \(\exists\) some \(\alpha_1 \in R\) for every two edges \(e, e' \in E(G)\) such that \(d(\alpha_1, e) \neq d(\alpha_1, e')\). Let \(R = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}\) be an ordered set of vertices of \(G\). The representation of an edge \(e \in E(G)\) with respect to \(R\) is \(d(e, \alpha_1), d(e, \alpha_2), d(e, \alpha_3), \ldots, d(e, \alpha_t))\).

It is denoted \(r(e|R)\). The minimum cardinality of \(R\) is called the edge metric dimension and is denoted by \(\text{edim}(G)\) [20].

It is observed that in some graphs metric generator and edge metric generator are same. That is why it is misunderstood that any edge metric generator \(R\) is same as standard metric generator. But the matter of fact is that there are a few families of graphs in which such coincidence happen. Kelenc et al. [20] explained a detailed comparison between the edge metric generator and standard metric generator. In this paper, they also showed the \(\text{edim}(G)\) is a NP-hard problem and determined that the \(\text{edim}(G)\) of grid graph is 2. Kelenc et al. [20] proved that the \(\text{edim}(W_{1,n})\) of wheel graph \(W_{1,n}\) is \(n\) for \(n = 3, 4\) and \(n - 1\) for \(n \geq 5\). They also determined the edge metric dimension of path, cycle, complete graph, complete bipartite, Fan graphs, cartesian product of cycles and bounds for some families of graphs.

An operation to split an edge into two edges by inserting a new vertex into the interior of an edge is called subdivision an edge. If this operation is applied on a sequence on edges of graph then it is called subdividing a graph \(G\) and resulting graph is known as subdivision of the graph \(G\). The subdivision is used to convert a pseudograph graph into a simple graph. If subdivision operation is performed on all edge of the graph \(G\), then this subdivision is called the barycentric subdivision of \(G\). Gross and Yellen [22] proved the results that the barycentric subdivision of any graph is a simple and bipartite graph. In this paper, we discussed the edge metric dimension of barycentric subdivision of Cayley graphs \(\text{Cay}(\mathbb{Z}_\alpha \oplus \mathbb{Z}_\beta)\).
II. THE EDGE METRIC DIMENSION OF BARYCENTRIC SUBDIVISION OF CAYLEY GRAPHS

Let $G$ be a semigroup, and $K \neq \emptyset \subseteq G$. The Cayley graph $\Gamma = \text{Cay}(G, K)$ is defined as:

- $V(\Gamma) \cong G$
- edge set $E(\Gamma)$ consists of those pairs $(x, y)$ with $x \in G$ such that $kx = y$ for some $k \in K$.

The Cayley graph $\Gamma$ of a group $G$ is symmetric or undirected if and only if $K = K^{-1}$.

The Cayley graphs $\Gamma = \text{Cay}(\mathbb{Z}_a \oplus \mathbb{Z}_b), a \geq 3, b \geq 2,$ is a graph which can be obtained as the cartesian product of a path with $b$ vertices with a cycle on $a$ vertices. The vertex set and edge set of $\Gamma$ defined as: $V(\Gamma) = \{(a, b) : 1 \leq i \leq a, 1 \leq j \leq b\}$ and $E(\Gamma) = \{(a, b)(a_{i+1}, b) : 1 \leq i \leq a, 1 \leq j \leq b\} \cup \{(a, b)(a, b_{j+1}) : 1 \leq i \leq a, 1 \leq j \leq b - 1\}$. We have $|V(\Gamma)| = 2ab$, $|E(\Gamma)| = (2b - 1)a$, where $|V(\Gamma)|, |E(\Gamma)|$ denote the number of vertices, edges of the Cayley graphs $\Gamma$, respectively.

Javaid et al [23] determined the metric dimension of Cayley graphs $\text{Cay}(\mathbb{Z}_a \oplus \mathbb{Z}_b)$ for all $a \geq 7$ and $K = \{\pm 1, \pm 3\}$ and Caceres et al [13] studied the metric dimension $\Gamma$ for $\beta = 2$.

The barycentric subdivision graph $BS(\Gamma)$ can be obtained by adding a new vertex $(c_i, d_j)$ between $(a_i, b_j)$ and $(a_{i+1}, b_j)$ and adding a new vertex $(u_i, v_j)$ between $(a_i, b_j)$ and $(a_{i+1}, b_j)$. Clearly, $BS(\Gamma)$ has $3a\beta - a$ vertices and $4a\beta - 2a$ edges.

In the next theorem, we proved that the metric dimension of the barycentric subdivision $BS(\Gamma)$ of is constant and only three vertices appropriately chosen suffice to resolve all the vertices of the $BS(\Gamma)$. Selection of appropriate basis vertices (also refereed as landmarks in [3]) is core of the problem. For our purpose, we call the sets of points as $H_1 = \{(a_1, b_1) : 1 \leq i \leq a, 1 \leq j \leq b\}$, $H_2 = \{(c_i, d_j) : 1 \leq i \leq a, 1 \leq j \leq b\}$ and $H_3 = \{(u_i, v_j) : 1 \leq i \leq a, 1 \leq j \leq b - 1\}$.

**Theorem 1:** Let $BS(\Gamma)$ be the barycentric subdivision of Cayley graphs $\Gamma$; then $\text{edim}(BS(\Gamma)) = 3$ for every $a \geq 6$, $b \geq 2$.

**Proof 1:** The above equality can be proved using double inequalities as following:

**Case 1.** When $a$ is even.

Let $R = \{(a_1, b_1), (a_{\frac{a}{2}} + 1, b_1), (a_{\frac{a}{4}} + 1, b_1)\} \subset V(BS(\Gamma))$, we show that $R$ is a resolving set for $BS(\Gamma)$ in this case. For representations of any edge of $E(BS(\Gamma))$ with respect to $R$.

Representations for the edges of $BS(\Gamma)$ are:

For $1 \leq j \leq b$

- $r((a_1, b_1)(c_{i, d})|R) = (2i + 2j - 4, a + 2j - 2i, 2i + 2j - 2), \quad \text{for } 1 \leq i \leq \frac{a}{2}, 1 - 1$.
- $r((a_1, b_1)(c_{i, d})|R) = (a - 2, 1, \alpha - 1), \quad \text{for } i = \frac{a}{2}.$
- $r((a_1, b_1)(c_{i, d})|R) = (2a - 2i + 1, 2i - a - 2, 2a - 2), \quad \text{for } \frac{a}{2} + 1 \leq i \leq a - 1.$
- $r((a_1, b_1)(c_{i, d})|R) = (1, \alpha - 2, 0), \quad \text{for } i = \alpha.$

For $1 \leq i \leq \frac{a}{2}$

- $r((a_1, b_1)(a_{i+1}, b_1)|R) = (2i + 2j - 3, a - 2i + 2j - 2, 2i + 2j - 1), \quad \text{for } 1 \leq i \leq \frac{a}{2} - 1.$
- $r((a_1, b_1)(a_{i+1}, b_1)|R) = (a + 2j - 3, 2j - 2, a + 2j - 4), \quad \text{for } i = \frac{a}{2}.$
- $r((a_1, b_1)(a_{i+1}, b_1)|R) = (2a - 2i + 2j - 2, 2i + 2j - a - 3, 2a + 2j - 2i - 4), \quad \text{for } \frac{a}{2} + 1 \leq i \leq a - 1.$
- $r((a_1, b_1)(a_{i+1}, b_1)|R) = (2j - 2, a + 2j - 3, 2j - 1), \quad \text{for } i = \alpha.$
When both vertices belong to the set \( H_2 = \{(c_i, d_j) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta \} \) and on the same level. Without loss of generality, we can suppose that one resolving vertex is \((c_1, d_j)\). Suppose that the second resolving vertex is \((c_2, d_j)\), \((2 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta)\). Then for \(2 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta - 1\), we have \( r((a_1, b_j)(u_1, v_1))((c_1, d_j), (c_2, d_j)) = r((a_1, b_j)(c_2, d_j))((c_1, d_j), (c_2, d_j)) = (1, 2k - 1) \), and for \(2 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta\), we have \( r((a_1, b_j)(u_1, v_1 - 1))((c_1, d_j), (c_2, d_j)) = r((a_1, b_j)(u_1, v_1))((c_1, d_j), (c_{\beta + 1, d_j})) = (0, \alpha - 1) \), a contradiction.

When both vertices belong to the set \( H_2 = \{(c_i, d_j) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta \} \) and on the different level. Without loss of generality, we can suppose that one resolving vertex is \((c_1, d_j)\). Suppose that the second resolving vertex is \((c_2, d_j)\), \((1 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta)\). Then for \(k = 1, 1 \leq j \leq \beta, r((a_1, b_1)(c_2, d_j))((c_1, d_j), (c_2, d_j)) = (0, 2(s - j) + 1)\), and for \(k = 2, 1 \leq j \leq \beta, r((a_1, b_1)(c_2, d_j))((c_1, d_j), (c_2, d_j)) = (2(s - j) + 1, 2(s - j))\), and for \(3 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta\), we have \( r((a_1, b_1)(c_2, d_j))((c_1, d_j), (c_2, d_j)) = (2(s - j) + 1, 2k - 4)\), a contradiction.

When both vertices belong to the set \( H_3 = \{(u_i, v_j) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta - 1 \} \) and on the different level. Without loss of generality, we can suppose that one resolving vertex is \((u_1, v_j)\). Suppose that the second resolving vertex is \((u_2, v_j)\), \((1 \leq k \leq \frac{\alpha}{2} + 1, 1 \leq j \leq \beta - 1)\). Then for \(2 \leq k \leq \frac{\alpha}{2} + 1, 1 \leq j \leq \beta - 1\), we have \( r((u_1, v_j)(u_1, v_{j+1}))((u_1, v_j), (u_2, v_j)) = r((a_1, b_j)(u_1, v_j))((u_1, v_j), (u_2, v_j)) = (0, 2k - 1)\), a contradiction.

(2) When both vertices do not belong to the same set. The subcases are as follows:

- When one vertex belong to the set \( H_1 = \{(a_i, b_j) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta \} \) and the second vertex belong to the set \( H_2 = \{(c_i, d_j) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta \} \). There are two possibilities.

(i) When both vertices are at the same level. Without loss of generality, we can suppose that one resolving vertex is \((a_1, b_j) \subset H_1\). Suppose that the second resolving vertex is \((c_k, d_j) \subset H_2\), \((1 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta)\). Then for \(1 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta - 1\), we have \( r((a_1, b_j)(c_k, d_j))((a_1, b_j), (c_k, d_j)) = r((a_1, b_j)(a_1, b_j))((a_1, b_j), (c_k, d_j)) = (0, 2k - 1)\), and for \(1 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta\), we have \( r((a_1, b_j)(c_k, d_j))((a_1, b_j), (c_{\beta + 1, d_j})) = r((a_1, b_j)(a_1, b_j))((a_1, b_j), (c_{\beta + 1, d_j})) = (0, 2k - 1)\). For \(k = \frac{\beta}{2} + 1, 1 \leq j \leq \beta - 1\), we have \( r((a_1, b_j)(c_k, d_j))((a_1, b_j), (c_{\beta + 1, d_j})) = r((a_1, b_j)(a_1, b_j))((a_1, b_j), (c_{\beta + 1, d_j})) = (0, \alpha - 1)\), a contradiction.

(ii) When both vertices are on the different level. Without loss of generality, we can suppose that one resolving vertex is \((a_1, b_j) \subset H_1\). Suppose that the second resolving vertex is \((c_k, d_j) \subset H_2\), \((1 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta - 1)\). Then for \(k = 1, 1 \leq j \leq \beta, r((a_1, b_1)(c_1, d_j))((a_1, b_1), (c_1, d_j)) = r((a_1, b_1)(a_1, b_1))((a_1, b_1), (c_1, d_j)) = (0, 2s - 2j + 1)\), and for \(1 \leq k \leq \frac{\beta}{2} + 1, 1 \leq j \leq \beta\), we have \( r((a_1, b_1)(c_1, d_j))((a_1, b_1), (c_1, d_j)) = (0, 2s - 2j - 2)\), and for \(k = \frac{\beta}{2} + 1, 1 \leq j \leq \beta\), we have \( r((a_1, b_1)(c_1, d_j))((a_1, b_1), (c_{\beta + 1, d_j})) = r((a_1, b_1)(a_1, b_1))((a_1, b_1), (c_{\beta + 1, d_j})) = (0, \beta - 2s - 2j - 1)\), a contradiction.

- When one vertex belong to the set \( H_1 = \{(a_i, b_j) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta \} \) and the second vertex belong to the set \( H_3 = \{(u_i, v) : 1 \leq i \leq \alpha, 1 \leq j \leq \beta - 1\} \). Without loss of generality, we can suppose that one resolving vertex is \((u_k, v) \subset H_3\), \((1 \leq k \leq \frac{\alpha}{2} + 1, 1 \leq j \leq \beta - 1)\). Due to the symmetry of the graph the behavior of the inner most cycle and the outer most cycle is same. Therefore, it is sufficient to discuss the representation of the vertices for \(1 \leq k \leq \frac{\alpha}{2} + 1, 1 \leq j \leq \beta - 1\). Then for \(k = 1, 1 \leq j \leq \beta - 1, r((a_1, b_1)(c_1, d_j))((a_1, b_1), (u_1, v_j)) = r((a_1, b_1)(c_1, d_j))((a_1, b_1), (u_1, v_j)) = (0, 2s - 2j + 1)\). For \(2 \leq k \leq \frac{\alpha}{2} + 1, 1 \leq j \leq \beta - 1, 1 \leq s \leq \beta, r((u_1, v_j)(u_{k - 1}, v_j))((u_1, v_j), (u_{k - 1}, v_j)) = (0, 2k - 1)\), and for \(k = \frac{\alpha}{2} + 1, 1 \leq j \leq \beta - 1, 1 \leq s \leq \beta\), we have \( r((u_1, v_j)(u_{k - 1}, v_j))((u_1, v_j), (u_{k - 1}, v_j)) = (0, \alpha)\), a contradiction.
A. N. A. Koam, A. Ahmad: Barycentric Subdivision of Cayley Graphs With Constant Edge Metric Dimension

For $1 \leq k \leq \frac{\alpha}{2} + 1$, we have $r((a_{i+1}, b_{j+1})(c_{i}, d_{j}, (u_{j}, v_{j}))) = r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = (2i + 2j - 1)$, for $1 \leq i \leq \frac{\alpha}{2} - 1$. For $k = \frac{\alpha}{2} + 1$, we have $r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = (1, 2k - 1)$, and for $k = \frac{\alpha}{2} + 1$, we have $r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = (2, i - 1)$. For $k = 2, 1 \leq j < s \leq \beta - 1$, we have $r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = (2s - 2j + 1, 1)$, for $3 \leq k \leq \frac{\alpha}{2} + 1, 1 \leq j < s \leq \beta - 1$, and $r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = r((a_{i}, b_{j})(c_{i}, d_{j}, (u_{j}, v_{j}))) = (1, 2k - 2s - 2j - 4), a contradiction.

Hence from above it follows that there is no resolving set with two vertices for $V(BS(\Gamma))$ implying that $edim(BS(\Gamma)) \neq 2$ in this case. Therefore, $edim(BS(\Gamma)) = 3$.

Case 2. When $\alpha$ is odd.

Let $R = \{a_{1}, b_{1}, a_{2}, b_{2}, a_{3}, b_{3}\} \subseteq V(BS(\Gamma))$, we show that $R$ is a resolving set for $BS(\Gamma)$ in this case. For this we give representations of any edge of $E(BS(\Gamma))$ with respect to $R$.

Representations for the edges of $BS(\Gamma)$ are:

For $1 \leq j \leq \beta$

$r((a_{i}, b_{j})(c_{i}, d_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $1 \leq i \leq \frac{\alpha}{2} - 1$.

$r((a_{i}, b_{j})(c_{i}, d_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $i = \frac{\alpha}{2}$.

$r((a_{i}, b_{j})(c_{i}, d_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $i = \frac{\alpha}{2} + 1 \leq i \leq \alpha - 1$.

$r((a_{i}, b_{j})(c_{i}, d_{j}))[R] = (1, \alpha - 1, 0)$, for $i = \alpha$.

$r((c_{i}, d_{j})(a_{i+1}, b_{j}))[R] = (2i + 2j - 3, \alpha - 2i + 2j - 3, 2i + 2j - 1)$, for $1 \leq i \leq \frac{\alpha}{2} - 2$.

$r((c_{i}, d_{j})(a_{i+1}, b_{j}))[R] = (2i + 2j - 3, \alpha - 2i + 2j - 3, 2i + 2j - 1)$, for $i = \frac{\alpha}{2}$.

$r((c_{i}, d_{j})(a_{i+1}, b_{j}))[R] = (2i + 2j - 3, \alpha - 2i + 2j - 3, 2i + 2j - 1)$, for $i = \frac{\alpha}{2} + 1 \leq i \leq \alpha - 1$.

$r((c_{i}, d_{j})(a_{i+1}, b_{j}))[R] = (2i + 2j - 3, \alpha - 2i + 2j - 3, 2i + 2j - 1)$, for $i = \alpha$.

$r((c_{i}, d_{j})(a_{i+1}, b_{j}))[R] = (2i + 2j - 3, \alpha - 2i + 2j - 3, 2i + 2j - 1)$, for $i = \alpha$.

For $1 \leq j \leq \beta - 1$

$r((a_{i}, b_{j})(u_{i}, v_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $1 \leq i \leq \frac{\alpha}{2} - 1$.

$r((a_{i}, b_{j})(u_{i}, v_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $i = \frac{\alpha}{2}$.

$r((a_{i}, b_{j})(u_{i}, v_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $i = \frac{\alpha}{2} + 1 \leq i \leq \alpha - 1$.

$r((a_{i}, b_{j})(u_{i}, v_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $i = \alpha$.

$r((a_{i}, b_{j})(u_{i}, v_{j}))[R] = (2i + 2j - 4, \alpha - 2i + 2j - 2, 2i + 2j - 2)$, for $i = \alpha$.

Again there are no two vertices having the same representations which implies that $edim(BS(\Gamma)) \leq 3$, and suppose that $edim(BS(\Gamma)) = 2$, then there are the same possibilities as in case (1) and contradictions can be deduced analogously. This implies that $edim(BS(\Gamma)) = 3$ in this case, which completes the proof.

III. CONCLUSION

With the help of previously known results in this article this can be concluded that it is not compulsory that the metric dimension of all graphs is equal to the edge metric dimension of graphs. We prove that these subdivisions of Cayley graphs have constant edge metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$.

REFERENCES

[1] P. J. Slater, “Leaves of trees,” in Proc. 6th Southeastern Conf. Combinatorics, Graph Theory, and Comput., 1975, pp. 549–559.

[2] F. Harary and R. A. Melter, “On the metric dimension of a graph,” Ars. Combin., vol. 2, pp. 191–195, May 1976.

[3] S. Khuller, B. Raghavachari, and A. Rosenfeld, “Landmarks in graphs,” Discrete Appl. Math., vol. 70, no. 3, pp. 217–229, Oct. 1996.

[4] Z. Beerlova, F. Eberhardt, T. Erlebach, A. Hall, M. Hoffmann, M. Hinkal, and L. Ram, “Network discovery and verification,” IEEE J. Sel. Areas Commun., vol. 24, no. 12, pp. 2168–2181, Dec. 2006.

[5] R. A. Melter and I. Tomescu, “Metric bases in digital geometry,” Comput. Vis., Graph., Image Process., vol. 25, no. 1, pp. 113–121, Jan. 1984.

[6] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, “Resolvability in graphs and the metric dimension of a graph,” Discrete Appl. Math., vol. 105, p. 11, Aug. 2003.

[7] G. Chartrand, C. Poisson, and P. Zhang, “Resolvability and the upper dimension of graphs,” Comput. Math. with Appl., vol. 39, no. 12, pp. 19–28, Jun. 2000.

[8] M. Johnson, “Structure-activity maps for visualizing the graph variables arising in drug design,” J. Biopharmaceutical Statist., vol. 3, no. 2, pp. 203–236, Jan. 1993.

[9] A. Ahmad, M. Imran, O. Al-Mushtay, and S. A. H. Bakher, “On the metric dimension of barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_m)$,” Miskolc Math. Notes, vol. 16, no. 2, pp. 637–646, 2015.

[10] A. Ahmad, M. Baca, and S. Sultan, “Minimal doubly resolving sets of Necklace graphs,” Mathematical, vol. 20, no. 2, pp. 123–129, 2018.

[11] R. F. Bailey and P. J. Cameron, “Base size, metric dimension and other invariants of groups and graphs,” Bull. London Math. Soc., vol. 43, no. 2, pp. 209–242, Apr. 2011.

[12] R. F. Bailey and M. Geheger, “On the metric dimension of Grassmann graphs,” Discret. Math. Theor. Comput. Sci., vol. 13, no. 4, pp. 97–104, 2011.

[13] J. Cerez, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, and C. Seara, “Wood, On the metric dimension of Cartesian products of graphs,” SIAM J. Discrete Math., vol. 21, pp. 423–441, May 2007.
[14] M. Imran, S. A. U. H. Bokhary, and A. Ahmad, “On classes of regular graphs with constant metric dimension,” Acta Math. Sci., vol. 33, no. 1, pp. 187–206, Jan. 2013.

[15] M. Imran, A. Q. Baig, and A. Ahmad, “Families of plane graphs with constant metric dimension,” Utilitas Math., vol. 88, pp. 43–57, May 2012.

[16] M. A. Malik and M. Sarwar, “On the metric dimension of two families of convex polytopes,” Afrika Matematika, vol. 27, nos. 1–2, pp. 229–238, Mar. 2016.

[17] M. Imran and H. M. A. Siddiqui, “Computing the metric dimension of convex polytopes generated by wheel related graphs,” Acta Math. Hungarica, vol. 149, no. 1, pp. 10–30, Jun. 2016.

[18] J. Kratica, V. Kovačević-Vujčić, M. Čangalović, and M. Stojanović, “Minimal doubly resolving sets and the strong metric dimension of some convex polytopes,” Appl. Math. Comput., vol. 218, no. 19, pp. 9790–9801, Jun. 2012.

[19] T. Vetrík and A. Ahmad, “Computing the metric dimension of the categorical product of some graphs,” Int. J. Comput. Math., vol. 94, no. 2, pp. 363–371, Feb. 2017.

[20] A. Kelenc, N. Tratnik, and I. G. Yero, “Uniquely identifying the edges of a graph: The edge metric dimension,” Discrete Appl. Math., vol. 251, pp. 204–220, Dec. 2018.

[21] A. Kelenc, D. Kuziak, A. Taranenko, and I. G. Yero, “On the mixed metric dimension of graphs,” Appl. Math. Comput., vol. 314, pp. 429–438, May 2017.

[22] J. L. Gross and J. Yellen, Graph Theory and its Applications. New York, NY, USA: Chapman & Hall, 2006.

[23] I. Javaid, M. N. Azhar, and M. Salman, “Metric dimension and determining number of Cayley graphs,” World Appl. Sci. J., vol. 18, no. 12, pp. 1800–1812, 2012.

**ALI N. A. KOAM** received the B.Sc. degree in mathematics from King Abdulaziz University, Jeddah, Saudi Arabia, in 2007, and the Ph.D. degree in mathematics from Leicester University, Leicester, U.K., in 2016. He is currently the Vice Dean of academic development of quality with Jazan University. His research interest is homological algebra, associative algebra, logical algebra, decision-making, and fuzzy set theory. He has authored several journal articles on these topics.

**ALI AHMAD** received the M.Sc. degree in mathematics from Punjab University, Lahore, Pakistan, in 2000, the M.Phil. degree in mathematics from Bahauddin Zakariya University, Multan, Pakistan, in 2005, and the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan, in 2010. His research interest are graph labeling, metric dimension, minimal doubly resolving sets, distances in graphs, and topological indices of graphs. He has authored more than 70 journal articles on these topics. He has been awarded twice Outstanding Performance Award during his Ph.D. research work in 2008 and 2009. He is an Editorial Board Member of two foreign scientific journals.

* * *