Abstract

We discuss the quantum equivalence, to all orders of perturbation theory, between the Yang-Mills theory and its first order formulation through a second rank antisymmetric tensor field. Moreover, the introduction of an additional nonphysical vector field allows us to interpret the Yang-Mills theory as a kind of perturbation of the topological BF model.
1 Introduction

Among the various hypothesis proposed to explain the quark confinement, the description of the QCD vacuum as a dual magnetic superconductor is rather appealing [1, 2, 3]. Particularly noticeable is the formulation given by t’Hooft [4] who attempted to describe the vacuum of QCD by making use of an electric and a magnetic order parameter. The expectation values of these order parameters can exhibit a perimeter or an area law, these different behaviours labelling the various phases of QCD. However, in spite of the many efforts devoted to put this description on firm mathematical basis, such an achievement has not yet been satisfactorily accomplished. In order to improve such a situation, some of the present authors have studied an alternative first order formulation of the non-Abelian Yang-Mills (YM) gauge theory [5, 6]. This formulation (called from now on BF-YM) makes use of an anti-symmetric tensor $B_{\mu\nu}$ and of an additional vector field $\eta_\mu$, yielding the following classical action

$$I_{BF-YM} = -\int \text{Tr}[B \wedge F + g^2(B - D\eta) \wedge *(B - D\eta)],$$

(1.1)

where $F = dA + A \wedge A$ is the YM field strength and $(D\eta = d\eta + [A, \eta])$ the covariant derivative. All the fields are Lie algebra valued, the generators $T^a$ of the corresponding gauge group G being chosen to be antihermitians and normalized in the fundamental representation as $\text{Tr}(T^a T^b) = -\delta^{ab}/2$. The Hodge dual of a p-form in $D$ dimension is defined as $* = \varepsilon^{\mu_1 \ldots \mu_D}/(D-p)!$, and the exterior product between a $p$ form $\omega$ and a $q$ form $\xi$ is given by

$$\omega \wedge \xi = \frac{1}{p!q!} \varepsilon^{\mu_1 \ldots \mu_{p+q}} \omega_{\mu_1 \ldots \mu_p} \xi^{\mu_{p+1} \ldots \mu_{p+q}} d^{p+q}x.$$

(1.2)
It is very easy to see that if one eliminates the antisymmetric field $B$ from the action (1.1) using the equations of motion

$$\frac{1}{2} D_\nu \ast B^{\mu\nu} = g^2 [\eta_\sigma, B^{\mu\sigma} - D^{[\mu} \eta^{\nu]},$$

$$\frac{1}{2} \ast F^{\mu\nu} = -g^2 (B^{\mu\nu} - D^{[\mu} \eta^{\nu]},$$

$$D_\rho B^{\rho\nu} = (D^2 g^{\mu\nu} - D^\mu D^\nu) \eta_\mu,$$

one recovers the usual form of the YM action. Moreover, as discussed in [5], the tensor field $B$ allows to obtain an explicit expression for the magnetic order parameter whose commutation relation with the electric order parameter gives the correct result and whose expectation value turns out to obey the desired perimeter law [5]. Furthermore, once acting on physical states, this magnetic operator gives a singular gauge transformation, as it should be. Having clarified the role of the tensor field $B$, let us now spend a few words about the additional vector field $\eta_\mu$ present in the expression (1.1) [17]. In order to motivate its introduction, let us first observe that in the limit of zero coupling constant the action (1.1) reduces to the topological $BF$ action [7] which, in addition to the gauge invariance, is known to possess a further local tensorial invariance whose origin is deeply related to the topological character of the $BF$ system. Of course, by adding to the pure $BF$ only the term $B \wedge \ast B$ one always recovers the usual YM action, but the topological tensor symmetry is lost. However, as one can easily understand, the introduction of the vector field $\eta_\mu$ provides a simple way to compensate the breaking induced by the term $B \wedge \ast B$, restoring thus the topological tensor invariance. In other words the action (1.1), although classically equivalent to the YM theory, preserves all the symmetries of the topological $BF$ model, giving us the interesting possibility of looking at the pure YM as a perturbation of a topological model [12]. Let also remark that, as we shall see in the next section, the transformation law of the vector field $\eta_\mu$ is simply given by a shift, meaning that all the components of $\eta_\mu$ are nonphysical. It is worth to notice that such a similar vector field has been recently used [8] in order to implement an alternative Higgs mechanism in which the YM gauge fields acquire a mass
through the breaking of a topological symmetry. This is another rather attractive aspect related to the action (1.1). Let us focus, for the time being, on the aim of this letter, i.e. on the study of the quantization and the renormalizability of (1.1) as well as of the proof of its quantum equivalence with the ordinary YM theory, this being a necessary first consistency check supporting the usefulness of the action (1.1). In particular, we shall be able to give a complete algebraic proof of the quantum equivalence of (1.1) with YM based on BRST cohomological arguments. We emphasize here that such an algebraic proof extends to all orders of perturbation theory and does not rely on the existence of a regularization preserving the symmetries, being particularly adapted to the present case due to the presence of the Levi-Civita tensor. Finally, let us mention that the study of the quantum equivalence has been recently discussed in three dimensions by [10], and in four dimensions, using different techniques, by [12]. The work is organized as follows. In Sect.2 we analyse the symmetry content of the action (1.1) and we establish the classical Slavnov-Taylor identity. Sect.3 is devoted to the study of the quantum aspects.

2 Quantization of BF-YM theory

The action (1.1) is easily seen to be left invariant by the following transformations

\begin{align*}
\delta A_\mu &= \delta_G A_\mu + \delta_T A_\mu + \delta' A_\mu, \\
\delta B_{\mu\nu} &= \delta_G B_{\mu\nu} + \delta_T B_{\mu\nu} + \delta' B_{\mu\nu}, \\
\delta \eta_\mu &= \delta_G \eta + \delta_T \eta_\mu + \delta' \eta_\mu,
\end{align*}

(2.1)

where \(\delta_G\) and \(\delta_T\) denote respectively the generators of the gauge and of the tensorial topological invariance defined by

\begin{align*}
\delta_G A_\mu &= D_\mu \theta, \\
\delta_G B_{\mu\nu} &= [B_{\mu\nu}, \theta], \\
\delta_G \eta_\mu &= [\eta_\mu, \theta],
\end{align*}

(2.2)
and

\[ \begin{align*}
\delta_T A_\mu &= 0, \\
\delta_T B_{\mu\nu} &= D_{[\mu\epsilon_{\nu}]}, \\
\delta_T \eta_\mu &= \epsilon_\mu.
\end{align*} \tag{2.3} \]

In particular, from the eqs. (2.3) one can see that the topological tensor transformation of the vector field \( \eta_\mu \) is given by a shift, meaning that all the components of \( \eta_\mu \) are nonphysical. The third generator \( \delta' \) appearing in eqs. (2.1) is associated to a further local invariance whose transformations are given by

\[ \begin{align*}
\delta'_A A_\mu &= 0, \\
\delta'_B B_{\mu\nu} &= [F_{\mu\nu}, \sigma], \\
\delta'_\eta_\mu &= D_\mu \sigma.
\end{align*} \tag{2.4} \]

We remark that the symmetry (2.1) is reducible since \( \delta_T \) and \( \delta' \) are not independent, as it can be seen by choosing \( \epsilon_\mu = D_\mu \sigma \). In order to gauge fix the local invariance (2.1) of the action (1.1) we adopt the linear gauge conditions

\[ \begin{align*}
\partial^\mu A_\mu &= 0, \\
\partial^\mu B_{\mu\nu} &= 0, \\
\partial^\mu \eta_\mu &= 0.
\end{align*} \tag{2.5} \]

Following the BRST formalism \[13\], the gauge fixing action in a Landau type gauge is then given by

\[ I_{gf} = \int d^4 x \text{Tr} \left[ \bar{c} \partial^\mu A_\mu + \bar{\psi} \partial^\mu B_{\mu\nu} + \bar{\rho} \partial^\mu \eta_\mu + (\partial^\mu \bar{\psi}_\mu) u + \bar{\phi} \partial^\mu \psi_\mu \right], \tag{2.6} \]

where \((c, \bar{c}, h_A), (\psi, \bar{\psi}, h_B), (\rho, \bar{\rho}, h_\eta)\) are respectively the ghost, the antighost and the lagrangian multiplier for \( \delta_G, \delta_T, \delta' \); \((\phi, \bar{\phi}, h_\psi)\) the ghost, the antighost and the lagrangian multiplier for the zero modes of the topological symmetry \( \delta_T \), and \((u, h_\bar{\psi})\) a pair of fields which takes into account a further degeneracy associated with \( \bar{\psi} \). The dimensions and the ghost numbers of all the fields are summarized in Table 1.

| Fields | A | B | \( \eta \) | c | \( \bar{c} \) | \( \psi \) | \( \bar{\psi} \) | \( h_A \) | \( h_B \) |
|--------|---|---|---------|---|--------|-------|--------|--------|--------|
| dimension | 1 | 2 | 1 | 0 | 2 | 1 | 1 | 2 | 1 |
| ghost # | 0 | 0 | 0 | 1 | -1 | -1 | 0 | 0 | 0 |

| Fields | \( \phi \) | \( \bar{\phi} \) | \( h_\psi \) | \( \rho \) | \( \bar{\rho} \) | \( h_\eta \) | u | \( h_\bar{\psi} \) |
|--------|--------|--------|--------|---|--------|-------|---|--------|
| dimension | 0 | 2 | 2 | 0 | 2 | 2 | 2 | 2 |
| ghost # | 2 | -2 | -1 | 1 | -1 | 0 | 0 | 1 |
The BRST transformations of the fields are

\[ sA_\mu = -D_\mu c, \]
\[ sB_{\mu \nu} = -[B_{\mu \nu}, c] + D_{[\mu} \psi_{\nu]} + [F_{\mu \nu}, \rho], \]
\[ s\eta_\mu = -[\eta_\mu, c] + \psi_\mu + D_\mu \rho, \]
\[ sc = \frac{1}{2} [c, c], \quad s\bar{c} = h_A, \]
\[ s\psi_\mu = [\psi_\mu, c] + D_\mu \phi, \quad s\bar{\psi}_\mu = h_B, \]
\[ sh_A = 0, \quad sh_B = 0, \quad sh_\psi = 0, \quad sh_\eta = 0, \]
\[ s\phi = -[\phi, c], \quad s\bar{\phi} = h_\psi, \]
\[ s\rho = [\rho, c] - \phi, \quad s\bar{\rho} = h_\eta, \]
\[ su = h_\bar{\psi}, \quad sh_\bar{\psi} = 0, \]

where the parenthesis \([\cdot, \cdot]\) denotes the graded commutator. It is easy to see that \([s, s] = 0\) on all the fields, i.e. the operator \(s\) is nilpotent off-shell. In order to write down a Slavnov-Taylor identity corresponding to the transformations \((2.7)\) we introduce a set of external sources \((A^*_\mu, B^*_{\mu \nu}, \eta^*_\mu, \psi^*_\mu, \rho^*, c^*, \phi^*)\) coupled to the non-linear variations of eqs.\((2.7)\)

\[
I_{\text{ext}} = \text{Tr} \int d^4 x \left( -A^*_\mu D^{\mu} c + \frac{1}{2} B^*_{\mu \nu} (D^{[\mu} \psi^{\nu]} - [B_{\mu \nu}, c] + [F_{\mu \nu}, \rho]) \right.
\]
\[ + \eta^*_\mu (\psi^\mu - [\eta^\mu, c] + D^\mu \rho) + \psi^*_\mu (D^\mu \phi + [\psi^\mu, c]) \]
\[ + \phi^*(-\phi + [\rho, c]) + \frac{1}{2} c^*[c, c] - \phi^* [\phi, c] \bigg). \]

Therefore, the complete action

\[
\Sigma = I_{BF-YM} + I_{gf} + I_{\text{ext}}, \]

satisfies the Slavnov-Taylor identity

\[
\mathcal{S}(\Sigma) = 0, \]

where

\[
\mathcal{S}(\Sigma) = \text{Tr} \int d^4 x \left( \frac{\delta \Sigma}{\delta A_\mu} \frac{\delta \Sigma}{\delta A^*_\mu} + \frac{1}{2} \frac{\delta \Sigma}{\delta B_{\mu \nu}^*} \frac{\delta \Sigma}{\delta B^{*\mu \nu}} + \frac{\delta \Sigma}{\delta \eta_\mu} \frac{\delta \Sigma}{\delta \eta^*_\mu} + \frac{\delta \Sigma}{\delta \psi_\mu} \frac{\delta \Sigma}{\delta \psi^*_\mu} + \frac{\delta \Sigma}{\delta \rho} \frac{\delta \Sigma}{\delta \phi^*} \right.
\]
\[ + \frac{\delta \Sigma}{\delta c} \frac{\delta \Sigma}{\delta \bar{c}^*} + \frac{\delta \Sigma}{\delta \phi} \frac{\delta \Sigma}{\delta \bar{\phi}^*} + h_A \frac{\delta \Sigma}{\delta \bar{\psi}} + h_B \frac{\delta \Sigma}{\delta \bar{\psi}^*} + h_\psi \frac{\delta \Sigma}{\delta \bar{\phi}} + h_\eta \frac{\delta \Sigma}{\delta \bar{\rho}} \bigg). \]
In addition to the Slavnov-Taylor identity \((2.10)\), the complete action \(\Sigma\) turns out to be characterized by the following additional constraints:

- the Landau gauge fixing conditions
  \[
  \frac{\delta \Sigma}{\delta h_A} = \partial^\mu A_\mu, \quad \frac{\delta \Sigma}{\delta h_B^\nu} = \partial^\mu B_{\mu\nu} - \partial_\nu u, \\
  \frac{\delta \Sigma}{\delta h_\eta} = \partial^\mu \eta_\mu, \quad \frac{\delta \Sigma}{\delta h_\psi} = \partial^\mu \psi_\mu, \\
  \frac{\delta \Sigma}{\delta h_\bar{\psi}} = \partial^\mu \bar{\psi}_\mu, \quad \frac{\delta \Sigma}{\delta u} = \partial^\mu h_B^\mu; 
  \]
  \((2.12)\)

- the antighost equations following from the Slavnov-Taylor identity \((2.10)\) and the gauge conditions \((2.12)\)
  \[
  \frac{\delta \Sigma}{\delta \bar{c}} + \partial^\mu \frac{\delta \Sigma}{\delta A_\mu^*} = 0, \quad \frac{\delta \Sigma}{\delta \bar{\psi}_\nu} + \partial^\mu \frac{\delta \Sigma}{\delta B_{\mu\nu}^*} = \partial_\nu h_\bar{\psi}, \\
  \frac{\delta \Sigma}{\delta \bar{\rho}} + \partial^\mu \frac{\delta \Sigma}{\delta \eta_\mu^*} = 0, \quad \frac{\delta \Sigma}{\delta \bar{\phi}} - \partial^\mu \frac{\delta \Sigma}{\delta \psi_\mu^*} = 0; 
  \]
  \((2.13)\)

- the ghost equation, usually valid in the Landau gauge \([14]\)
  \[
  \int d^4x \left( \frac{\delta \Sigma}{\delta \bar{c}} + \left[ \frac{\delta \Sigma}{\delta h_A}, \bar{c} \right] + \left[ \bar{\psi}_\nu^*, \frac{\delta \Sigma}{\delta h_B^\nu} \right] + \left[ u, \frac{\delta \Sigma}{\delta h_\eta} \right] + \left[ \bar{\rho}, \frac{\delta \Sigma}{\delta h_\psi} \right] + \left[ \bar{\phi}, \frac{\delta \Sigma}{\delta h_\bar{\psi}} \right] \right) = \Delta^\epsilon_{cl}, \quad (2.14) 
  \]
  where \(\Delta^\epsilon_{cl}\) is a linear classical breaking given by
  \[
  \Delta^\epsilon_{cl} = \int d^4x \ Tr \left( -[A_\mu^*, A^\mu] + \frac{1}{2}[B_{\mu\nu}^*, B^{\mu\nu}] + [\eta_\mu^*, \eta^\mu] - [\psi_\mu^*, \psi^\mu] - [\rho^*, \rho] + [\phi^*, \phi] - [c^*, c] \right); 
  \]
  \((2.15)\)

- the Ward identity for the rigid gauge invariance stemming from the Slavnov-Taylor identity \((2.10)\) and the ghost equation \((2.14)\)
  \[
  \mathcal{H}_{rig} \Sigma = \sum_{\text{all fields}} \Phi \int d^4x \left[ \Phi, \frac{\delta \Sigma}{\delta \Phi} \right] = 0; \quad (2.16) 
  \]

- the linearly broken Ward identity corresponding to the ghost \(\phi\), typically of a topological \(BF\) system \([13]\)
  \[
  \int d^4x \left( \frac{\delta \Sigma}{\delta \phi} - \left[ \bar{\phi}, \frac{\delta \Sigma}{\delta h_A} \right] \right) = \Delta^\phi_{cl}, \quad (2.17) 
  \]
where $\Delta^\phi_{cl}$ is given by

$$\Delta^\phi_{cl} = \int d^4x \text{Tr}\left([\psi^*_\mu, A^\mu] - \rho^* + [\phi^*, c]\right); \quad (2.18)$$

- the Ward identity following from the $\phi$ ghost equation (2.17) and the Slavnov-Taylor identity (2.10)

$$\int d^4x \left(\frac{\delta \Sigma}{\delta \rho} + [A^\mu, \frac{\delta \Sigma}{\delta \psi_\mu}] + [c, \frac{\delta \Sigma}{\delta \phi}] - [\psi^*_\mu, \frac{\delta \Sigma}{\delta A^*_\mu}] + [\phi^*, \frac{\delta \Sigma}{\delta c^*}] + [\bar{\phi}, \frac{\delta \Sigma}{\delta c}] - [h_\psi, \frac{\delta \Sigma}{\delta h_A}]\right) = 0. \quad (2.19)$$

Notice, finally, that the breaking terms in the left hand side of the equations (2.14) and (2.17), being linear in the quantum fields, are classical breakings, i.e. they are present only at the classical level and will not get renormalized by the radiative corrections [14].

### 3 Renormalization and algebraic equivalence with YM theory

We face now the problem of the quantum extension of the Slavnov-Taylor identity (2.10) and of the equivalence with the YM theory. By following standard arguments [14], all the constraints (2.12)–(2.19) derived in the previous section can be shown to be renormalizable. They can therefore be assumed to hold for the quantum vertex functional

$$\Gamma = \Sigma + O(h). \quad (3.1)$$

In particular, the gauge conditions (2.12) imply that the higher order contributions to the vertex functional $\Gamma$ are independent from the lagrangian multipliers and that, due to the equations (2.13), the antighosts ($\bar{c}, \bar{\psi}, \bar{\rho}, \bar{\phi}$) enter only through the combinations

$$\hat{A}^*_\mu = A^*_\mu + \partial_\mu \bar{c},$$
\[ \hat{B}^*_{\mu\nu} = B^*_{\mu\nu} + \partial_{[\mu} \bar{\psi}_{\nu]}, \]
\[ \hat{\eta}^*_{\mu} = \eta^*_{\mu} + \partial_{\mu} \bar{\rho}, \]
\[ \hat{\psi}^*_{\mu} = \psi^*_{\mu} - \partial_{\mu} \bar{\phi}. \]  \hspace{1cm} (3.2)

Introducing then the reduced action \( \hat{\Sigma} \) \[ \hat{\Sigma} = I_{BF-YM} + \int d^4x \text{Tr} \left( -\hat{A}^*_\mu D^\mu c + \frac{1}{2} \hat{B}^*_{\mu\nu} (D^\mu \psi^\nu - [B^\mu\nu, c] + [F^\mu\nu, \rho]) \right) 
+ \hat{\eta}^*_{\mu} (\psi^\mu - [\eta^\mu, c] + D^\mu \rho) + \hat{\psi}^*_{\mu} (D^\mu \phi + [\psi^\mu, c])
+ \rho^* (-\phi + [\rho, c]) + \frac{1}{2} c^*[c, c] - \phi^*[\phi, c], \]  \hspace{1cm} (3.3)

the Slavnov-Taylor identity (2.10) takes the following simpler form
\[ B_{\hat{\Sigma}} \hat{\Sigma} = 0. \]  \hspace{1cm} (3.4)

where the operator \( B_{\hat{\Sigma}} \) is defined as
\[ B_{\hat{\Sigma}} = \int d^4x \text{Tr} \left( \frac{\delta \hat{\Sigma}}{\delta A^\mu} \frac{\delta}{\delta A^\mu} + \frac{\delta \hat{\Sigma}}{\delta \bar{\eta}^*_{\mu}} \frac{\delta}{\delta \bar{\eta}^*_{\mu}} + \frac{\delta \hat{\Sigma}}{\delta \eta^\mu} \frac{\delta}{\delta \eta^\mu} + \frac{1}{2} \frac{\delta \hat{\Sigma}}{\delta B^\mu_{\nu}} \frac{\delta}{\delta B^*_{\mu\nu}} 
+ \frac{\delta \hat{\Sigma}}{\delta \psi^\mu} \frac{\delta}{\delta \psi^\mu} + \frac{\delta \hat{\Sigma}}{\delta \bar{\psi}^*_{\mu}} \frac{\delta}{\delta \bar{\psi}^*_{\mu}} + \frac{\delta \hat{\Sigma}}{\delta \bar{\phi}^*} \frac{\delta}{\delta \bar{\phi}^*} + \frac{\delta \hat{\Sigma}}{\delta \phi} \frac{\delta}{\delta \phi} 
+ \frac{\delta \hat{\Sigma}}{\delta \bar{c}} \frac{\delta}{\delta \bar{c}} + \frac{\delta \hat{\Sigma}}{\delta c} \frac{\delta}{\delta c} \right), \]  \hspace{1cm} (3.5)

with \[ B_{\hat{\Sigma}} B_{\hat{\Sigma}} = 0. \]  \hspace{1cm} (3.6)

Its action on the fields and on the sources is given by
\[ B_{\hat{\Sigma}} A^\mu = -D^\mu c, \]
\[ B_{\hat{\Sigma}} B^\mu_{\nu} = -[B^\mu_{\nu}, c] + D^\mu \psi^\nu + [F^\mu_{\nu}, \rho], \]
\[ B_{\hat{\Sigma}} \eta^\mu = -[\eta^\mu, c] + \psi^\mu + D^\mu \rho, \]
\[ B_{\hat{\Sigma}} c = \frac{1}{2} [c, c], \]
\[ B_{\hat{\Sigma}} \psi^\mu = [\psi^\mu, c] + D^\mu \phi, \]
\[ B_{\hat{\Sigma}} \phi = -[\phi, c]. \]
As it is well known, both the anomalies and the invariant counterterms can be characterized as nontrivial cohomology classes of the operator $\mathcal{B}_\Sigma$, i.e. they are solution of the consistency condition

$$\mathcal{B}_\Sigma \Delta = 0,$$

(3.8)

$\Delta$ being a local integrated polynomial of canonical dimension 4 and ghost number 0 for the counterterms and 1 for the anomalies. In order to compute the cohomology of $\mathcal{B}_\Sigma$ we begin by analysing the cohomology of the operator $\mathcal{B}_\Sigma^{(0)}$, corresponding to the linearized approximation of eqs.(3.7), i.e.

$$\mathcal{B}_\Sigma^{(0)} A_\mu = -\partial_\mu c,$$

$$\mathcal{B}_\Sigma^{(0)} B_{\mu\nu} = \partial_{[\mu} \psi_{\nu]},$$

$$\mathcal{B}_\Sigma^{(0)} \eta_\mu = \psi_\mu + \partial_\mu \rho,$$

$$\mathcal{B}_\Sigma^{(0)} \psi_\mu = \partial_\mu \phi,$$

$$\mathcal{B}_\Sigma^{(0)} \phi = 0,$$

$$\mathcal{B}_\Sigma^{(0)} c = 0,$$

$$\mathcal{B}_\Sigma^{(0)} \rho = -\phi,$$

(3.7)
\begin{align}
B^{(0)}_{\Sigma} A^*_\mu &= \frac{1}{2} \varepsilon_{\mu\rho\sigma} \partial^\rho B^{\rho\sigma}, \\
B^{(0)}_{\Sigma} \hat{B}^*_\mu &= \frac{1}{2} \varepsilon_{\mu\rho\sigma} \partial^{[\rho} A^{\sigma]} + 2g^2 (B_{\mu\nu} - \partial_{[\mu} \eta_{\nu]}), \\
B^{(0)}_{\Sigma} \hat{\eta}^*_\mu &= -2g^2 (\partial^\nu B_{\mu\nu} + \partial^2 \eta_{\mu} - \partial^\nu \partial_{\mu} \eta_{\nu}), \\
B^{(0)}_{\Sigma} \hat{\psi}^*_\mu &= -\partial^\nu \hat{B}^*_{\mu\nu} - \hat{\eta}^*_\mu, \\
B^{(0)}_{\Sigma} \phi^* &= -\rho^* - \partial^\mu \hat{\psi}^*_\mu, \\
B^{(0)}_{\Sigma} c^* &= -\partial^\mu \hat{A}^*_\mu, \\
B^{(0)}_{\Sigma} \rho^* &= \partial^\mu \hat{\eta}^*_\mu. 
\end{align}

(3.9)

with

\[ B^{(0)}_{\Sigma} B^{(0)}_{\Sigma} = 0. \tag{3.10} \]

The reason for looking at the operator \( B^{(0)}_{\Sigma} \) relies on a very general theorem on BRST cohomology stating that the cohomology of the operator \( B_{\Sigma} \) is isomorphic to a subspace of the cohomology of its linearized approximation \( B^{(0)}_{\Sigma} \). Making now the following linear change of variables

\begin{align}
B_{\mu\nu} \rightarrow \tau_{\mu\nu} &= B_{\mu\nu} - \partial_{[\mu} \eta_{\nu]} + \frac{1}{4g^2} \varepsilon_{\mu\rho\sigma} \partial^{[\rho} A^{\sigma]} , \\
\psi_\mu \rightarrow \varphi_\mu &= \psi_\mu + \partial_\mu \rho, \\
\hat{A}^*_\mu \rightarrow \hat{\omega}^*_\mu &= \hat{A}^*_\mu - \frac{1}{4g^2} \varepsilon_{\mu\rho\sigma} \partial^\rho \hat{B}^{*\rho\sigma}, \\
\hat{\eta}^*_\mu \rightarrow \hat{\lambda}^*_\mu &= \hat{\eta}^*_\mu + \partial^\nu \hat{B}^*_{\mu\nu}, \\
\rho^* \rightarrow \xi^* &= \rho^* + \partial^\mu \hat{\psi}^*_\mu. 
\end{align}

(3.11)

the other fields and sources remaining unchanged, the action of \( B^{(0)}_{\Sigma} \) can be written as

\begin{align}
B^{(0)}_{\Sigma} A_\mu &= -\partial_\mu c, \\
B^{(0)}_{\Sigma} c &= 0, \\
B^{(0)}_{\Sigma} c^* &= -\partial^\mu \hat{\omega}^*_\mu, \\
B^{(0)}_{\Sigma} \hat{\omega}^*_\mu &= -\frac{1}{2g^2} \partial^\nu \partial_{[\mu} A_{\nu]}.
\end{align}

(3.12)
and

\[ B_\Sigma^{(0)} \hat{\tau} = 2g^2 \tau, \quad B_\Sigma^{(0)} \tau = 0, \]
\[ B_\Sigma^{(0)} \varphi = \varphi, \quad B_\Sigma^{(0)} \varphi = 0, \]
\[ B_\Sigma^{(0)} \rho = -\rho, \quad B_\Sigma^{(0)} \rho = 0, \]
\[ B_\Sigma^{(0)} \hat{\lambda}^* = -\hat{\lambda}^*, \quad B_\Sigma^{(0)} \hat{\lambda}^* = 0, \]
\[ B_\Sigma^{(0)} \phi^* = -\phi^*, \quad B_\Sigma^{(0)} \phi^* = 0. \] (3.13)

From eqs. (3.13) it is apparent that the variables \((\tau, \hat{\tau}^*), (\varphi, \eta), (\rho, \rho), (\hat{\lambda}^*, \hat{\psi}^*), (\xi^*, \phi^*)\) are grouped in BRST doublets \([14]\) and therefore cannot contribute to the cohomology of \(B_\Sigma^{(0)}\). We are left thus only with the fields and sources appearing in the equations (3.12). However, the latters are easily recognized to be nothing but the linearized transformations characterizing the cohomology of the pure YM theory \([14]\). This means that we can take as the representatives of the cohomology classes of the operator \(B_\Sigma\) those of the pure YM.

Therefore, the only nontrivial elements (of canonical dimension 4 and ghost number 1 and 0) of the cohomology of \(B_\Sigma\) are given by

- the usual nonabelian gauge anomaly

\[ A = \varepsilon_{\mu \nu \rho \sigma} \int d^4 x \ c_a \partial^\mu \left( d^{abc} \partial^\nu A^\rho_A^c + \frac{D_{abcd}}{12} A^\rho_A^b A^\sigma_A^c A^e_A^d \right), \] (3.14)

where \(d_{abc}\) is the totally symmetric invariant tensor of rank 3 defined by

\[ d_{abc} = \frac{1}{2} Tr(T_a \{T_b, T_c\}), \] (3.15)

and

\[ D_{abcd} = d^{\alpha}_{ab} f_{\nu cd} + d^{\alpha}_{ac} f_{\nu db} + d^{\alpha}_{ad} f_{\nu bc}. \] (3.16)

where \(f_{abc}\) are the structure constants of the gauge group. Since in our model all the fields are in the adjoint representation, the anomaly coefficient automatically vanishes to one loop order. The Adler-Bardeen theorem guar-
anties thus that the gauge anomaly is definitively absent to all orders of perturbation theory.

- the most general invariant counterterm can be written as

\[ \Sigma^c = -\frac{Z_g}{4g^2} \int d^4x \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + B\hat{\Sigma}\Delta^{-1}, \]  

(3.17)

where \( Z_g \) is an arbitrary free parameter and \( \Delta^{-1} \) is some local integrated polynomial with ghosts number -1 and dimension 4.

To a first look the counterterm (3.17) does not seem to have the form of the action \( I_{BF-YM} \) of eq.(1.1). However, it is very easy to check that expression (3.17) can be rewritten in the same form of the original action \( I_{BF-YM} \), the difference being an irrelevant trivial cocycle, i.e.

\[ \frac{1}{4g^2} \int d^4x \text{Tr}(F_{\mu\nu}F^{\mu\nu}) = 2I_{BF-YM} \]

\[ + \frac{1}{4g^2} B\Sigma \int d^4x \text{Tr}\left( B^{*\mu\nu} \left[ \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} + 2g^2 (B_{\mu\nu} - D_{[\mu}\eta_{\nu]}^\mu) \right] \right). \]  

(3.18)

From eqs.(3.17) and (3.18) it then follows that there is only one physical renormalization, associated to the parameter \( Z_g \), which gives a non-vanishing \( \beta_g \) for the coupling constant \( g \). As expected, the numerical value of the 1-loop contribution to the \( \beta_g \) function is the same of the standard YM [11].

Let us summarize our results. We have found that the BF-YM theory can be characterized in terms of the BRST cohomology of the pure YM theory. Moreover, the only non trivial counterterm (3.17) can be equivalently rewritten in terms of the classical BF-YM action, yielding a renormalization of the gauge coupling \( g \).

This shows the algebraic equivalence between the model described by the classical action \( I_{BF-YM} \) and the standard Yang-Mills theory.

These same conclusions are drawn in Ref.[16] which appeared few hours before this paper was ready to be sent to the publisher.
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