SEMI-NAIVE DIMENSIONAL RENORMALIZATION

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ABSTRACT

We propose a treatment of $\gamma^5$ in dimensional regularization which is based on an algebraically consistent extension of the Breitenlohner-Maison-'t Hooft-Veltman (BMHV) scheme; we define the corresponding minimal renormalization scheme and show its equivalence with a non-minimal BMHV scheme.

The restoration of the chiral Ward identities requires the introduction of considerably fewer finite counterterms than in the BMHV scheme. This scheme is the same as the minimal naive dimensional renormalization in the case of diagrams not involving fermionic traces with an odd number of $\gamma^5$, but unlike the latter it is a consistent scheme.

As a simple example we apply our minimal subtraction scheme to the Yukawa model at two loops in presence of external gauge fields.

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Introduction

Dimensional regularization \([1, 2]\) is the most powerful scheme for making perturbative computations in quantum field theory. It is particularly effective in absence of chiral symmetries, since minimal subtraction \([3]\) preserves the vectorial gauge symmetries.

In \([1]\) it has been shown that defining \(\gamma^5\) as in four dimensions the axial anomaly is reproduced. In \([4]\) it has been proven that this definition of \(\gamma^5\) leads to a consistent regularization scheme (BMHV) at all orders in perturbation theory.

The restoration of the chiral Ward identities in the BMHV scheme has been studied in \([5]\).

In the BMHV scheme many of the computational advantages of the dimensional regularization scheme are lost, since the restoration of the chiral Ward identities requires the introduction of non-invariant counterterms for practically any diagram containing \(\gamma^5\).

Due to this difficulty, comparatively few quantities have been computed in this scheme. Among them, there are some one-loop processes in the standard model \([6]\), two-loop anomalous dimensions of four-fermion operators in QCD \([7]\), the three-loop anomalous dimension of axial currents in QCD \([8]\). The Adler-Bardeen non-renormalization theorem \([9]\) has been verified at two loops in this scheme in \([10]\).

Systematic computations of the bare action in the BMHV scheme have been made in non-abelian gauge theories with chiral fermions \([11]\), where the finite one-loop counterterms necessary to restore the Slavnov-Taylor identities have been computed; in the simplest Yukawa model at two loop in the minimal scheme \([12]\); in a general class of Yukawa models at two loops, where the finite two-loop counterterms necessary to restore the rigid Ward identities have been computed, together with some of those necessary to recover the local Ward identities in presence of external gauge fields \([13]\).

Most of the computations in chiral theories are made in some version of the naive dimensional regularization scheme (NDR), first considered in \([14]\) and \([15]\); other proposals of NDR have been made in \([16, 17]\). The common feature of all these versions of NDR is that in fermionic loops with an even number of \(\gamma^5\)’s, these are considered to be anticommuting with all the gamma matrices, thus avoiding the spurious anomalies present in the BMHV scheme. However all these regularization schemes have some algebraic inconsistency. To avoid these algebraic inconsistencies it has been suggested that the cyclic property of the trace be upheld \([18]\), but in this case bosonic symmetry is not respected, and no consistency proof for these renormalization schemes is available.
Reviews on various proposals for dimensional regularization in chiral theories can be found in [19, 20].

In this paper we introduce a consistent regularization scheme, in which to the usual $\gamma_\mu$ and $\gamma^5$ formal objects of the BMHV scheme two new $\gamma^5$-like objects $\eta_1$ and $\eta_2$ are added; the former is not anticommuting with all the gamma matrices, as $\gamma^5$, while the latter is anticommuting; they are defined in such a way that traces with an even number of $\eta_1$ or an odd number of $\eta_2$ vanish.

We will call this scheme semi-naive dimensional regularization (SNDR); due to its algebraic consistency in $d = 4 - \epsilon$ dimension, algebraic manipulations on the bare quantities can be done unambiguously.

We define minimal subtraction (MS) in this scheme in such a way that, after taking the limit $\epsilon \to 0$ on the minimally subtracted Green functions, there exists a homomorphism from these abstract subtracted Green functions to the usual renormalized Green functions, which preserves the algebraic properties on convergent diagrams, like the skeleton expansion.

We prove the equivalence of the MS-SNDR scheme with a non-minimal BMHV scheme.

The proof of the consistency of the MS-SNDR scheme is the main result of this paper. We do not deal with the practical problem of restoring the Ward identities, apart from an explicit example and some general considerations, showing that in this scheme there are considerably fewer spurious anomalies than in the MS-BMHV scheme.

As an example of the practical convenience of our scheme with respect to the BMHV scheme, we treat in some detail the two-loop renormalization of the Yukawa model in presence of external gauge currents, which we have worked out in the BMHV scheme in [13]; while in the latter scheme minimal subtraction does not preserve the (anomalous) chiral Ward identities, so that finite counterterms must be added to restore them for every relevant term, in the SNDR scheme, after performing minimal subtraction only two finite two-loop counterterms are needed to preserve the anomalous Ward identities, with the anomaly appearing in the Bardeen form [21]; the Adler-Bardeen theorem [9] is verified.

We discuss briefly the case of chiral gauge theories and we compare SNDR with previously proposed NDR schemes.

1 Semi-naive dimensional regularization and minimal subtraction
1.1 Breitenlohner-Maison-’t Hooft-Veltman scheme

We recall how gamma matrices are treated in the ’t Hooft and Veltman \[1\] dimensional regularization scheme as elaborated by Breitenlohner and Maison (BMHV) \[4\]. We use Euclidean space conventions.

In the BMHV scheme one considers the Lorentz covariants $I$, $\delta_{\mu\nu}$, $\gamma_{\mu}$, $p_{\mu}$, etc. as formal objects, satisfying the usual tensorial rules. $\delta_{\mu\nu}$ is the Kronecker delta in $d = 4 - \epsilon$ dimensions; a formal rule for summed indices is given:

$$\delta_{\mu\nu}\delta_{\nu\rho} = \delta_{\mu\rho} ; \quad \delta_{\mu\nu}p_{\nu} = p_{\mu} ; \quad \delta_{\mu\mu} = d$$

(1)

The Lorentz indices of these formal covariants do not take a specific value.

The gamma ‘matrices’ $\gamma_{\mu}$ satisfy the relation

$$\{\gamma_{\mu}, \gamma_{\nu}\} = -2\delta_{\mu\nu}I$$

(2)

where $I$ is the identity,

$$I^2 = I ; \quad I\gamma_{\mu} = \gamma_{\mu}I = \gamma_{\mu}$$

(3)

The trace is cyclic and satisfies

$$tr \, I = 4$$

(4)

In the BMHV scheme additional ‘$(d - 4)$-dimensional’ or ‘evanescent’ tensors $\hat{\delta}_{\mu\nu}$, $\hat{p}_{\mu}$ and $\hat{\gamma}_{\mu}$ are introduced; the Kronecker delta in the $(d - 4)$-dimensional space is $\hat{\delta}_{\mu\nu}$, satisfying

$$\hat{\delta}_{\mu\nu}\delta_{\nu\rho} = \hat{\delta}_{\mu\nu}\hat{\delta}_{\nu\rho} = \hat{\delta}_{\mu\rho} ; \quad \hat{\delta}_{\mu\mu} = -\epsilon$$

$$\hat{p}_{\mu} \equiv \hat{\delta}_{\mu\nu}p_{\nu} ; \quad \hat{\gamma}_{\mu} \equiv \hat{\delta}_{\mu\nu}\gamma_{\nu}$$

(5)

The Kronecker delta in four dimensions in $\tilde{\delta}_{\mu\nu}$, satisfying

$$\tilde{\delta}_{\mu\nu} \equiv \delta_{\mu\nu} - \hat{\delta}_{\mu\nu} ; \quad \tilde{p}_{\mu} \equiv \tilde{\delta}_{\mu\nu}p_{\nu} ; \quad \tilde{\gamma}_{\mu} \equiv \tilde{\delta}_{\mu\nu}\gamma_{\nu}$$

(6)

The Levi-Civita antisymmetric tensor has no evanescent component:

$$\hat{\delta}_{\mu\nu}\epsilon_{\nu\rho\sigma\tau} = 0$$

(7)

and satisfies

$$\epsilon_{\mu_{1}\mu_{2}\mu_{3}\mu_{4}}\epsilon_{\nu_{1}\nu_{2}\nu_{3}\nu_{4}} = \sum_{\pi \in S_{4}} sign(\pi)\Pi_{i=1}^{4} \tilde{\delta}_{\mu_{i}\nu_{\pi(i)}},$$

(8)
The ‘matrix’ $\gamma^5$ is defined by
\[
\gamma^5 = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma
\]  
which implies
\[
\{\gamma^5, \gamma^\mu\} = 2\gamma^\mu \gamma^5 \quad (10)
\]
\[
\gamma^5 \gamma^5 = I ; \quad [\gamma^5, \gamma_\mu] = 0 \quad (11)
\]

After having performed the Dirac algebra with these rules, reduced the number of Levi-Civita tensors, eliminated traces, reduced products of gamma matrices to the corresponding antisymmetric products and eliminated summed indices, the BMHV elements are reduced to normal form. The Feynman graphs are expressed in terms of meromorphic functions of $\epsilon$, which are the coefficients of these normal form terms.

The minimal subtraction on a $l$-loop $1PI$ Feynman graph renormalized up to order $l - 1$ consists in subtracting local terms, polynomial in momenta and masses, which are the poles of the Laurent series in $\epsilon$ of the meromorphic functions defining such a graph, both for hatted and non-hatted tensor structures in normal form.

The bare action is a formal object, from which the Feynman graphs are constructed following the usual rules. In the minimal subtraction scheme the bare action $l$-loop counterterms are the poles of the $l$-loop $1PI$ Green functions, computed using the Feynman rules derived from the $(l - 1)$-loop bare action.

After making the minimal subtraction, the limit for $\epsilon \to 0$ is taken in the entire functions which are the coefficients of the normal form terms, and all the hatted tensors are set to zero; finally the formal sums, traces and the normal form covariants are identified with the usual four-dimensional operations and objects respectively.

In Euclidean space the reflection symmetry takes the place of hermiticity\textsuperscript{22}. Reflection symmetry is an antilinear involution, with
\[
\Theta \psi(x) = \bar{\psi}(x') \gamma_1 \quad ; \quad \Theta \bar{\psi}(x) = \gamma_1 \psi(x')
\]  
\[
\Theta \gamma_\mu = -\gamma_\mu \quad ; \quad \Theta \gamma^5 = \gamma^5
\]  
where $x^1 = -x^1, \; x'_\mu = x^\mu$ for $\mu \neq 1$. $\delta_{\mu\nu}, \epsilon_{\mu\nu\rho\sigma}$ and $\epsilon$ are invariant under reflection symmetry.

Considering a multiplet of fermions, the following fermionic bilinears are reflection symmetric:
\[
\int \bar{\psi}(H_1 \gamma_\mu + H_2 \gamma_\mu + H_3 \gamma^5 \gamma_\mu + iH_4 \gamma^5 \gamma_\mu) \partial_\mu \psi + \bar{\psi}(iH_5 A + H_6 \gamma^5 B) \psi
\]
\[
+ \bar{\psi}(iH_7 \gamma_\mu V_\mu + iH_8 \gamma^5 \gamma_\mu A_\mu + iH_9 \gamma_\mu V_\mu + H_{10} \gamma^5 \gamma_\mu A_\mu) \psi
\]  
\[
(14)
\]
where the scalar \( A \), the pseudoscalar \( B \), the vector \( V_\mu \) and the pseudovector \( A_\mu \) are real fields, with

\[
\Theta A(x) = A(x') \quad ; \quad \Theta B(x) = B(x') \quad ; \quad \Theta V_\mu(x) = V'_\mu(x') \quad ; \quad \Theta A_\mu(x) = A'_\mu(x')
\]

and \( H_i \) are hermitian matrices commuting with the gamma matrices.

A non-minimal subtraction consists in performing, after the minimal subtraction, the subtraction of finite local counterterms; to avoid ambiguities, one must characterize which are the normal form terms on which the finite subtractions are made.

In the case of a reflection symmetric theory, a convenient choice of fermionic bilinears finite counterterms are the non-hatted terms in (14). The advantage of this choice is that in the case of non-chiral theories the \( d \)-dimensional Lorentz symmetry of the bare action is preserved.
1.2 Extension of the BMHV algebra

Add to the BMHV Dirac algebra the objects $\eta$ and $\eta_1$ satisfying the following defining relations, assumed to be valid for arbitrary $\epsilon$:

\[
\{\eta, \gamma_\mu\} = 2\hat{\gamma}_\mu \eta_1 \\
\eta^2 = I \\
I\eta = \eta \\
[\eta_1, \hat{\gamma}_\mu] = 0 \\
[\eta_1, \eta] = 0 \\
tr \eta \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = 4 \epsilon_\mu\nu\rho\sigma \\
tr \eta_1^2 = 0
\]  

(15)  

(16)  

(17)  

(18)  

(19)  

(20)  

(21)

We assume that the trace has the property (4) and that it is cyclic on this enlarged algebra. Under reflection symmetry, $\Theta \eta = \eta$ and $\Theta \eta_1 = \eta_1$.

Define

\[
\eta_2 \equiv \eta - \eta_1
\]

(22)

Consider first the case $\epsilon \neq 0$.

Let us prove that

\[
\eta I = \eta \\
\{\eta_1, \gamma_\mu\} = 2\hat{\gamma}_\mu \eta_1 \\
\{\eta_2, \gamma_\mu\} = 0 \\
\eta_1 I = I \eta_1 = \eta_1 \\
\eta_1 \eta_2 = \eta_2 \eta_1 = 0 \\
\eta_1^2 + \eta_2^2 = I; \quad \eta_1^3 = \eta_1; \quad \eta_2^3 = \eta_2
\]

(23)  

(24)  

(25)  

(26)  

(27)  

(28)

Eq. (23) is obvious. From (15) one gets

\[
\epsilon \eta_1 = \frac{1}{2} \hat{\gamma}_\mu \{\eta, \gamma_\mu\} = \frac{1}{2} \epsilon \eta + \frac{1}{2} \hat{\gamma}_\mu \eta \hat{\gamma}_\mu
\]

(29)

Using (15), (18), (22) and (23) one gets the relations (24), (25) and (26). From (15), (24) and (25) one gets easily

\[
[\eta_1^2, \gamma_\mu] = [\eta_2^2, \gamma_\mu] = 0 \\
\gamma_\mu \eta_1 \eta_2 \gamma_\mu = (d - 8) \eta_1 \eta_2 \\
[\eta_1, \eta_2] = 0
\]

(30)
From (16), (22) and (30) one has

\[-d = \gamma_{\mu} \eta_1^2 \gamma_{\mu} = \gamma_{\mu} (\eta_1^2 + \eta_2^2 + \{\eta_1, \eta_2\}) \gamma_{\mu} =
\] 
\[-d(\eta_1^2 + \eta_2^2) + (d - 8)\{\eta_1, \eta_2\}\]

so that

\[\epsilon\{\eta_1, \eta_2\} = 0 \quad (31)\]

Since \(\epsilon \neq 0\), we get \(\{\eta_1, \eta_2\} = 0\), which together with (14) implies (27).

Finally (28), which follows trivially from (16) and (27), says that \(\eta_1^2\) and \(\eta_2^2\) are orthogonal projectors commuting with all the elements of this algebra.

Let us prove the following relations for the trace:

\[tr \eta_1 = tr \eta_2 = 0 \quad (32)\]
\[tr \eta_1^2 \gamma_{\mu_1} \cdots \gamma_{\mu_r} = 0 \quad (33)\]
\[tr \eta_2^2 \gamma_{\mu_1} \cdots \gamma_{\mu_r} = tr \gamma_{\mu_1} \cdots \gamma_{\mu_r} \quad (34)\]
\[tr \eta_2 \gamma_{\mu_1} \cdots \gamma_{\mu_r} = 0 \quad (35)\]
\[tr \eta_1 \gamma_{\mu_1} \cdots \gamma_{\mu_r} = tr \gamma_{\mu_1} \cdots \gamma_{\mu_r} \quad (36)\]

for arbitrary choice of indices \(\mu_1, \ldots, \mu_r, r \geq 1\). One has

\[-d tr \eta_1 = tr \eta_1 \gamma_{\mu} \gamma_{\mu} = tr \gamma_{\mu} \eta_1 \gamma_{\mu} = (8 - d) tr \eta_1\]
\[-d tr \eta_2 = tr \eta_2 \gamma_{\mu} \gamma_{\mu} = tr \gamma_{\mu} \eta_2 \gamma_{\mu} = d tr \eta_2 \quad (37)\]

from which (32) follows. To prove the remaining relations (33-36), observe that if for all \(\mu\)

\[X_{\pm} \gamma_{\mu} = \pm \gamma_{\mu} X_{\pm} \quad (38)\]

then, for indices \(\mu_1, \ldots, \mu_r\) all different

\[-d tr X_{\pm} \gamma_{\mu_1} \cdots \gamma_{\mu_r} = tr X_{\pm} \gamma_{\mu_1} \cdots \gamma_{\mu_r} \gamma_{\mu} \gamma_{\mu} =
\] 
\[\pm tr X_{\pm} \gamma_{\mu_1} \cdots \gamma_{\mu_r} \gamma_{\mu} = \pm (-1)^r(2r - d) tr X_{\pm} \gamma_{\mu_1} \cdots \gamma_{\mu_r} \quad (39)\]

which for non-integer \(d\) has solution only for

\[tr X_{\pm} \gamma_{\mu_1} \cdots \gamma_{\mu_r} = 0 \quad (40)\]

Using this observation and the fact that \(\eta_1^2, \eta_1 \gamma^5\) and \(\eta_2^2\) commute with all the gamma matrices, while \(\eta_2\) anticommutes with all of them, one gets relations (33-36) for indices \(\mu_1, \ldots, \mu_r\) all different. In particular one gets

\[tr \eta_2 \gamma^5 = 0 \quad (41)\]
The general case for (33-36) follows from this particular case since, simplifying gamma matrices with equal indices, one is reduced to the relations

\[ tr \eta_1^2 = tr \eta_2 = 0 \quad ; \quad tr \eta_2^2 = 4 \quad ; \quad tr \eta_1 \gamma^5 = 4 \]  

which follow trivially from the relations (16-21) and from (32), (41).

Using (32-36) all the traces can be computed, observing that the traces with more than two \( \eta_1 \) or two \( \eta_2 \) can be reduced to the cases (32-36) using (15-28).

Observe that in \( d \neq 4 \) dimensions this algebra of covariants does not include \( \epsilon_{\mu_1...\mu_d} \), as might be thought due to the presence of \( \eta_2 \), which is a \( \gamma^5 \)-like object anticommuting with all gamma matrices. In fact \( tr \eta_2 \gamma_{\mu_1}...\gamma_{\mu_d} = 0 \) due to (35).

In the case \( \epsilon = 0 \), \( \gamma_\mu \) vanishes and \( \eta_1 \) is an independent generator, since eq.(29) becomes trivial; in that case, add the defining relations

\[ \{ \eta_1, \gamma_\mu \} = 0 \]  
\[ \eta \eta_1 = \eta_1^2 \]  
\[ I \eta_1 = \eta_1 I = \eta_1 \]  

which we obtained from (15-19) in the case \( \epsilon \neq 0 \).

In the case \( \epsilon = 0 \), after identifying the formal sums (e.g. \( p_\mu \gamma_\mu \)) as usual Einstein convention sums, there is a homomorphism \( \phi \) between the subalgebra generated by \( \gamma_\mu = \gamma_\mu \) and \( \eta \), with the trace \( tr \), and the usual four-dimensional Dirac algebra generated by \( \gamma'_\mu \), with unit matrix \( I' \) and with the usual trace \( tr' \): This homomorphism is uniquely determined by mapping the generators,

\[ \phi(\gamma_\mu) = \gamma'_\mu \quad ; \quad \phi(\eta) = \gamma^5' \equiv \frac{1}{4!} \epsilon_{\mu \nu \rho \sigma} \gamma'_\mu \gamma'_\nu \gamma'_\rho \gamma'_\sigma \]  

and by the relations defining the homomorphism

\[ \phi(I) = I' \]  
\[ \phi(X_1 X_2) = \phi(X_1) \phi(X_2) \quad ; \quad \phi(a X_1 + b X_2) = a \phi(X_1) + b \phi(X_2) \]  
\[ tr' X = tr' \phi(X) \]  

for \( X, X_1 \) and \( X_2 \) belonging to this subalgebra; \( a \) and \( b \) are complex numbers. \( p_\mu, \delta_{\mu \nu} \) and \( \epsilon_{\mu \nu \rho \sigma} \) are mapped into the corresponding usual tensors in four dimensions.
To verify that this is a homomorphism with the properties (47) it is sufficient to verify them on the relations (4,15,16,17,20) defining the subalgebra for \( \epsilon = 0 \).

The kernel of this homomorphism is obtained projecting this subalgebra with \((I - \eta \gamma^5)/2\), which commutes with all the elements of the subalgebra.

\( \gamma_\mu (I + \eta \gamma^5)/2 \) generates the orthogonal subalgebra, establishing an isomorphism between this subalgebra and the usual Dirac algebra.

This homomorphism cannot be extended to \( \eta_1 \), since

\[
4 = tr (\gamma^5 - \eta) \eta_1 \neq tr' (\gamma^{5'} - \gamma^{5'}) \phi(\eta_1) = 0
\]

for any value of \( \phi(\eta_1) \).

Furthermore it cannot be extended to the case \( \epsilon \neq 0 \); for instance

\[
8\epsilon \, \delta_{\mu\nu} = tr (\gamma^5 - \eta) \gamma_\mu \gamma_\rho \eta_\rho \gamma_\nu \neq tr' (\gamma^{5'} - \gamma^{5'}) \phi(\gamma_\mu \gamma_\rho \eta_\rho \gamma_\nu) = 0 \quad (48)
\]

for any value of \( \phi(\gamma_\mu) \).

In order to check the consistency of the defining relations (15-21), let us give an explicit representation of this extension of the Dirac algebra for \( d = 2n \) even dimensions. Denote by \( \gamma'_\mu \) the usual \( 2^{d/2} \)-dimensional gamma matrices; define further \( \gamma^{5'} \equiv \gamma'_1 \gamma'_2 \gamma'_3 \gamma'_4 \) and \( \eta' \equiv i^{(d-1)d/2} \gamma'_1 ... \gamma'_d \) (the \( \gamma_5 \) in the right-hand-side of the latter expression is \( \gamma_\mu \) for \( \mu = 5 \) and should not be confused with the pseudoscalar object called \( \gamma^5 \) defined in the former expression).

The extended algebra elements are given by the following tensor products:

\[
I \equiv I' \times diag(1,1,1,1) \\
\gamma_\mu \equiv \gamma'_\mu \times diag(1,1,1,1) \\
\gamma^5 \equiv \gamma^{5'} \times diag(1,1,1,1) \quad (49) \\
\eta_1 \equiv \gamma^{5'} \times diag(1,-1,0,0) \\
\eta_2 \equiv \eta' \times diag(0,0,1,-1) \\
tr X = 2^{1-\frac{d}{2}} Sp[(I' \times diag(1,-1,1,1))X]
\]

where \( Sp \) is the ordinary trace on \( 2^{d/2+2} \times 2^{d/2+2} \) matrices, while \( tr \) is a not a matrix trace, but it is a cyclic multilinear operator on the space generated by the gamma matrices, \( \eta \) and \( \eta_1 \).

In this representation all the defining relations for the extended Dirac algebra hold. All the properties which we derived above hold in this representation, including (33), which however does not follow from (33) in the case \( r = 2n \).
1.3 Semi-naive dimensional regularization

In the semi-naive dimensional regularization (SNDR) the Feynman amplitudes are expressed in terms of meromorphic functions of the complex parameter \( d = 4 - \epsilon \); in the case of fermionic amplitudes, there is a meromorphic function for each element of the generalized gamma matrix algebra described above, written in normal form (NF).

A basis for the gamma matrix normal form quantities belonging to the generalized Dirac algebra consists in

\[
\{ \Gamma_A, \eta \Gamma_A, \eta \Gamma_A, \eta \Gamma_A \} \quad (50)
\]

\[
\{ \Gamma_A \} = \{ I, \gamma_\mu, \gamma_\mu \gamma_\nu, \ldots, \gamma_\mu \gamma_\nu \ldots \} \quad (51)
\]

where [...] indicates antisymmetrization.

It can be decomposed in the following way:

\[
NF = NF_0 + NF_1 + \hat{NF} \quad (52)
\]

\[
NF_0 = \{ \bar{\Gamma}_A, \eta \bar{\Gamma}_A \} \quad (53)
\]

\[
NF_1 = \{ \eta \bar{\Gamma}_A, \eta \eta \bar{\Gamma}_A \} \quad (54)
\]

\[
\{ \bar{\Gamma}_A \} = \{ I, \tilde{\gamma}_\mu, \tilde{\gamma}_\mu \tilde{\gamma}_\nu, \ldots, \tilde{\gamma}_\mu \tilde{\gamma}_\nu \} \quad (55)
\]

\( \hat{NF} \) consists of the remaining NF terms, which have at least a \( \tilde{\gamma}_\mu \).

Denote by \( A_0 \) and \( A_1 \) the subalgebras generated by \( NF_0 \) and \( NF_1 \) respectively, on the complex numbers (constant in \( \epsilon \)).

Denote by \( \hat{A} \) the subalgebra generated by \( \hat{NF} \), multiplied by an entire function in \( \epsilon \), and by any NF term multiplied by an entire function with a zero in \( \epsilon = 0 \). The objects in \( \hat{A} \) are evanescent.

One has

\[
A_0 A_0 = A_0 \quad A_0 A_1 = A_1 A_0 = A_1 A_1 = A_1 \\
\hat{A} A_0 = A_0 \hat{A} = \hat{A} A_1 = A_1 \hat{A} = \hat{A} \quad (56)
\]

Denote by \( T_0 \) the algebra on constants in \( \epsilon \), whose generators are the normal form terms in \( NF_0 \), the momenta \( \bar{p}_\mu \), \( \delta_{\mu\nu}, \epsilon_{\mu\nu\rho\sigma} \) and the normal forms obtained from these taking tensor products of these (e.g. \( \tilde{\delta}_{\mu\nu}, I, \bar{p}_\mu I \times I, \tilde{\delta}_{\mu\nu} I \times I, \tilde{\gamma}_\mu \times \eta \) are normal forms in \( T_0 \); these tensor products are introduced to cover the case of Feynman graphs with more than one open fermionic line).

Denote by \( T_1 \) the algebra on constants in \( \epsilon \), whose generators are the normal form terms in \( NF_1 \) and the normal forms obtained taking the tensor product of at least one \( NF_1 \) term and \( T_0 \) normal forms.

Finally denote by \( \hat{T} \) the space formed by normal forms with an hatted quantity, on constants in \( \epsilon \), and by any normal form multiplied by an entire
function with a zero in $\epsilon = 0$, and by the elements obtained making a tensor product of at least one of these elements and those previously defined in $T_0$ and $T_1$. This is the algebra of evanescent quantities. One has

$$T_0 T_0 = T_0; \quad T_0 T_1 = T_1 T_0 = T_1 T_1 = T_1$$

$$\hat{T}_0 = T_0 \hat{T} = \hat{T} T_1 = T_1 \hat{T} = \hat{T}$$

The homomorphism $\phi$ defined in the previous subsection extends canonically to $T_0$, but cannot be extended to $T_1$ or to $\hat{T}$.

After the subtraction procedure, a renormalized but still regularized 1PI Feynman graph $G_{nren}$, written in normal form, belongs to the abstract algebra $T_0 + T_1 + \hat{T}$. The subscript $n$ indicates the Lorentz and internal indices of the Feynman graph.

The algebraic consistence of the extended BMHV algebra guarantees that it is equivalent to compute $G_{nren}$ directly in terms of convergent integrals, or to compute separately the divergent integrals for $G_{nunren}$ and its counterdiagram graphs, and then sum them; the latter procedure is much easier.

$G_{nren}$ must be evaluated as a four-dimensional quantity $G_{nren}$, expressed in terms of the usual Dirac algebra. This is done in two steps:

i) set $\epsilon$ and the hatted objects to zero; denote by $G_{\epsilon=0}\text{ren}$ the resulting quantity, belonging to $T_0 + T_1$;

ii) Map the abstract objects in the generalized BMHV algebra to the usual Dirac algebra quantities:

$$\phi(G_{\epsilon=0}\text{ren}) = G_{n\text{ren}}$$

where $G_{n\text{ren}}$ is the renormalized 1PI graph after the regulator has been removed. The map $\phi$ must be the trace-preserving homomorphism previously defined.

It is necessary that this map be a trace-preserving homomorphism to preserve the algebraic properties on convergent diagrams, like making a skeleton expansion of a convergent 1PI Green function; these expansions involve making products of relevant 1PI vertices and dressed propagators, sums over Lorentz indices and Dirac traces, which are preserved by a homomorphism preserving the traces. It is clear that this homomorphism is unique. Since it cannot be defined on $T_1$, it follows that a consistent renormalization scheme in the SNDR scheme must be such that the $T_1$ terms are absent from $G_{\epsilon=0}\text{ren}$ at all orders in loops (and hence from the $\epsilon^0$ term of the Laurent series for $G_{n\text{ren}}$, which therefore must belong to $T_0 + \hat{T}$, while $G_{\epsilon=0}\text{ren}$ must belong to $T_0$).

This is a non-trivial requirement, since the $T_1$-dependent counterterms can only be local, while non-local terms are produced in perturbation theory;
for instance in a unsubtracted two-loop fermionic two-point function (without its one-loop counterdiagrams) there are generally both non-local poles (this is true also in the BMHV case) and non-local finite $A_1$ terms; the point is that these non-local terms are canceled by the one-loop counterdiagrams. In the next subsection we will show that this non-trivial property holds at all orders in perturbation theory.

The second step is trivial in the BMHV scheme, since the BMHV Dirac algebra for $\epsilon = 0$ is isomorphic with the usual Dirac algebra in four dimensions. This is not so in the SNDR Dirac algebra for $\epsilon = 0$.

We will call a quantity, belonging to the extended Dirac algebra, to be weakly evanescent if it belongs to the kernel of the homomorphism $\phi$, modulo evanescent terms. The space of weakly evanescent quantities is called $W$.

The kernel of $\phi$ on the domain $A_0$ is $\frac{1-\eta\gamma^5}{2}A_0$. The subspace $\frac{1+\eta\gamma^5}{2}A_0$ is isomorphic to the usual Dirac algebra. Clearly

$$WA_0 = WW = W$$ (59)

In the general tensor product case

$$WT_0 = WW = W$$ (60)

Summarizing: the regularized quantities (bare action, $G_{\epsilon\text{ren}}$) live in a space with four times as many Dirac algebra elements as in BMHV; $G_{\epsilon=0\text{ren}}$ lives in a space with twice as many Dirac algebra elements as in BMHV; $G_{\text{ren}}$ differs from $G_{\epsilon=0\text{ren}}$ by evanescent quantities; $G_{\text{ren}}$ is isomorphic to $G_{\epsilon=0\text{ren}}$ modulo weakly evanescent quantities; projecting each $A_0$ component of $G_{\epsilon=0\text{ren}}$ to $\frac{1+\eta\gamma^5}{2}A_0$ one gets an object isomorphic to $G_{\text{ren}}$.

1.4 Minimal subtraction in SNDR

The renormalization by minimal subtraction (MS) consists in a minimal subtraction procedure and in the evaluation in four dimensions of the formal objects defined in dimensional regularization.

We define it recursively starting from $l = 1$ loops.

Denote by $G_{\epsilon}(l)(k)$ a $l$-loop $1PI$ Feynman graph with momenta $k$, minimally renormalized according to our MS rules up to $l' < l$ loops but yet unrenormalized at $l$ loops. Denote by $\Gamma_{\epsilon}(l)(k)$ the corresponding Green function, which is the sum of various $G_{\epsilon}(l)(k)$.

We define the minimal subtraction procedure for the SNDR in the following way.

The minimal subtraction on $G_{\epsilon}(l)(k)$ consists in subtracting all the local terms, polynomial in momenta and masses, which are:
i) the poles of the Laurent expansion of the meromorphic functions defining such a graph, both for hatted and non-hatted tensor structures in normal form;

ii) the $T_1$ terms

The resulting renormalized but yet regularized 1PI Feynman graph $G_n^{(l)\text{ren}}$ is evaluated in four dimensions following the procedure described in the previous subsection.

The $l$-loop bare action counterterms consist in the local terms subtracted from the 1PI $l$-loop Green functions $\Gamma_n^{(l)}(k)$ according to these rules.

In a renormalizable theory without insertions of composite operators of dimension larger than five, the only $T_1$ counterterms are $A_1$ terms (there are no relevant four-fermion terms).

As in MS-BMHV, we have to show that $G_n^{(l)\text{ren}}$ thus defined has no non-local poles; furthermore we have to show that it has no non-local $T_1$ term.

Consider $G_n'\epsilon^{(l)}(k)$ and $\Gamma_n'\epsilon^{(l)}(k)$ to be the corresponding quantity in the BMHV scheme.

Suppose by induction hypothesis that in the BMHV scheme finite local counterterms can be chosen for the renormalized (but still regularized) relevant 1PI Green functions $\Gamma_{n'}^{\epsilon\text{ren}(l')}$ up to $l' = l - 1$ loops such that $\Gamma_{n'}^{\epsilon\text{ren}(l')}$ is weakly evanescent, so that this difference vanishes after evaluation in four dimensions (in general the MS-SNDR scheme is equivalent to a non-minimal BMHV scheme).

Let us show that the same can be done at $l$ loops.

Let $d_n$ be the superficial degree of divergence of $G_n^{(l)}(k)$. For $r > d_n$, $\partial_k^r G_n^{(l)}(k)$ and $\partial_k^r G_n'^{\epsilon(l)}(k)$ admit a skeleton expansion.

$\partial_k^r$ denotes $r$ derivatives with respect to the external momenta $k$, with indices not saturated neither among themselves nor with the Lorentz indices included in the subscript $n$.

The difference between the renormalized and still regularized Green functions in the two schemes at $l' < l$ loops is weakly evanescent by induction hypothesis, so that using the skeleton expansion property and (57.50) it follows that for $r > d_n$ also

$$\partial_k^r G_n^{(l)}(k) - \partial_k^r G_n'^{\epsilon(l)}(k) \in W$$

(61)

This equation implies that $G_n^{(l)}(k) - G_n'^{\epsilon(l)}(k)$ is a polynomial of degree $d_n$ in the momenta plus a weakly evanescent quantity. The polynomial part depends on the poles in $\epsilon$ (of degree less or equal to $l$) and on the normal forms, including the $T_1$ terms. An important point in this kind of arguments [23] is that $\partial_k^r$ commutes with the operations of extracting the poles or the $\epsilon^0$ terms.
of the Laurent series on which the meromorphic functions are expanded, and that it commutes also with the operation of projecting the $T_0$ elements to the kernel of $\phi$ and with the extraction of the $T_1$ terms.

Making the minimal subtraction on $G^{\epsilon(l)}_n(k)$ using our rules and making an appropriate non-minimal subtraction in the BMHV sense on $G^{\epsilon(l)}_n(k)$, the polynomial part of $G^{\epsilon(l)}_n(k) - G'^{\epsilon(l)}_n(k)$ can be subtracted and the difference between the renormalized $1PI$ Feynman graphs, yet regularized, is weakly evanescent.

We have therefore proven by induction that the MS-SNDR scheme is equivalent to a (generally non-minimal) BMHV scheme at all orders in perturbation theory.

The proof of polynomiality in masses of the minimal subtraction counterterms proceeds in the same way [24] and holds also for the $T_1$ counterterms.

Let us show from another point of view that step (ii) is necessary to define our minimal subtraction scheme; consider a different definition of minimal subtraction for SNDR, in which only poles are subtracted, and in which the evaluation procedure in four dimensions includes the rule of setting $\eta_1$ to zero and identifying $\eta$ with $\gamma^5$ (this map is not a trace-preserving homomorphism, as we saw in Subsection 1.2). Let us repeat the above argument, in which now $G^{\epsilon(l)}_n$ refers to this modified scheme, while $G'^{\epsilon(l)}_n$ has the same interpretation as above. In this case the renormalization parts $\Gamma^{\text{cren}(l')}_n$ of the skeleton expansion of $\partial^l_{k'} G^{\epsilon(l)}_n$ differ from the corresponding renormalization parts in the BMHV scheme by $T_1$ terms and by weakly evanescent terms. Performing a trace involving a weakly evanescent renormalization part and a $T_1$ contribution in another renormalization part of the skeleton expansion, one gets in general a finite non-local $T_0$ contribution, which is absent in $\partial^l_{k'} G'^{\epsilon(l)}_n(k)$. In such a case it is not possible to choose $l$-loop local counterterms to get the same renormalized Green functions in the two schemes, up to weakly evanescent terms or local $T_1$ terms. Therefore one cannot go from the new scheme to the BMHV scheme with a renormalization group transformation. This is not possible, so that the proposed scheme is wrong.

The necessity of the subtraction of the $T_1$ terms to have a consistent renormalization scheme makes our subtraction scheme minimal; the polynomiality in the masses of these $T_1$ terms supports the notion that MS-SNDR is really an MS scheme.

A characteristic of the usual MS scheme is that to compute the $l$-loop counterterms one has to compute the $O(1/\epsilon)$ part of $l$-loop Feynman integrals, the $O(1)$ part of $(l-1)$-loop Feynman integrals, the $O(\epsilon)$ part of $(l-2)$-loop Feynman integrals, and so on. In MS-SNDR the same happens. In fact the $T_1$ $l$-loop terms in $\Gamma^{\epsilon(l)}_n(k)$ arise from a counterterm graph with a counterterm
containing a $NF_1$ factor ($T_1$ terms are absent at tree level), from a graph with vertices containing only $NF_0$ factors or from a graph containing at least a vertex with a $NF$ factor.

In the first case the Feynman integrals are clearly of $(l-1)$-loop order; in the second case a $NF_1$ term is produced, in the reduction of the graph to normal form, by anticommuting $\eta$ with $\gamma_\mu$, generating a $\hat{\gamma}_\mu \eta_1$ factor; to get a $T_1$ term this $\hat{\gamma}_\mu$ must be contracted with another $\hat{\gamma}_\mu$, producing a $\epsilon$, so that only the pole part of the $l$-loop Feynman integral is involved in producing $T_1$; the same is true in the third case.

Actually the subtraction of the $l$-loop $T_1$ terms in $\Gamma^{(l)}(k)$ can be done algebraically, without doing any analytic computation; in fact one can subtract algebraically all the $NF_1$ terms (both poles and finite parts) from $\Gamma^{(l)}(k)$, subtracting subsequently the remaining poles. (In practice this algebraic subtraction can be done simply by treating $\eta$ in the open fermionic lines as anticommuting with all the gamma matrices and by setting $\eta_1$ to zero on the same lines). The analytic computation of the $T_1$ $l$-loop counterterms matters only for computing $\Gamma^{(l+1)ren}(k)$.

On the contrary, the subtraction of the $l$-loop poles from $\Gamma^{(l)}(k)$ requires some analytic computation, so that from the analytic point of view the $T_1$ counterterms require little effort with respect to the pole counterterms.

Let us discuss briefly finite renormalization, which are needed to recover the chiral symmetry or more in general to renormalize the Green functions in a non-minimal way, e.g. at a momentum subtraction point.

They belong to $T_0$, with the possible addition of $\hat{T}$ terms to have counterterms respecting $d$-dimensional Lorentz symmetry in the non-chiral cases; the discussion is analogous to the one made in the first subsection for the non-minimal BMHV scheme.

If the tree-level action is reflection symmetric, the bare action in MS-SNDR is also reflection symmetric.

In a non-minimal SNDR scheme, reflection symmetry is preserved by choosing reflection symmetric $T_0$ finite renormalization terms.

Obviously non-minimal $T_1$ counterterms are forbidden.
2 Applications

After having shown that the MS-SNDR is a consistent renormalization scheme, let us show that it is convenient from a computational point of view.

In this scheme the renormalized Green functions are the same as in MS-NDR as long as fermionic loops with an odd number of $\gamma^5$ does not occur. In these cases the computations are exactly the same in the two schemes and the chiral Ward identities are satisfied.

In the remaining cases, in MS-SNDR the traces with odd number of $\eta$ or $\eta_1$ enter in the game; in these cases one must check explicitly the chiral Ward identities; if they are broken, they can be recovered as usual by adding local finite counterterms to the MS counterterms, order by order in perturbation theory. These are the cases in which NDR becomes ambiguous or inconsistent.

In the case of external vector gauge fields, the vectorial Ward identities are preserved in the MS-SNDR as well as in the MS-BMHV scheme; in fact, since the vectorial gauge transformations do not involve neither the dimension, nor $\eta$ or $\eta_1$, the operations of extracting the poles and the $T^1$ terms commute with gauge transformations, so that the bare action is vector gauge invariant. The extension of this argument to gauge theories will be discussed later.

2.1 Yukawa models

In the Yukawa models it is possible to use NDR with anticommuting $\gamma^5$ without finding inconsistencies; MS-NDR preserves the rigid chiral symmetries. However coupling these models to external gauge fields and considering the corresponding Ward identities, NDR is not anymore a consistent regularization scheme, due to the presence of the chiral anomalies.

In the MS-BMHV scheme the chiral Ward identities are broken by a large number of spurious terms, which must be subtracted by introducing non-invariant local finite counterterms in the bare action. This has been done at one-loop and partly at two loops in [13], choosing renormalization conditions which, in the cases in which fermionic loops with an odd number of $\gamma^5$ do not occur, give the same renormalized Green functions as in MS-NDR.

In SNDR the tree-level bare action is

$$S^{(0)} = \int \frac{1}{2}(D_\mu \phi_i)^2 + \frac{1}{2}\bar{\psi}\gamma_\mu D_\mu \psi - \frac{1}{2}(D_\mu \bar{\psi})\gamma_\mu \psi +$$

$$i\bar{\psi} y_i \phi_i + \frac{1}{4!} h_{ijkl} \phi_i \phi_j \phi_k \phi_l$$

We have set the dimensional regularization scale $\mu$ to one.
We use the same notation as in [13], with the difference that in the tree-level action and in the gauge transformations on the fermions we replace $\gamma^5$ with $\eta$; the difference between the two cases being weakly evanescent, this is an equivalent starting point for making perturbation theory. We restrict our attention to the massless Yukawa theories. Furthermore we do not require that $\hat{A}_\mu^a = 0$, which was chosen in [13] to reduce the number of counterterms in the BMHV bare action.

The fermions have internal indices which are not indicated ($\psi_I$). The matrices $S_i, P_i$ are hermitian.

We will consider a group $G$ which is not necessarily semi-simple.

The fermions belong to a (reducible) representation $t^a$; under an infinitesimal transformation

$$\delta \psi = i\epsilon^a t^a \psi$$

(64)

We will consider chiral representations

$$t^a = t^a_R P_R + t^a_L P_L = t^a_s I + t^a_p \eta ; \quad \bar{t}^a = t^a_R P_R + t^a_L P_L = t^a_s I - t^a_p \eta$$

$$t^a_R = t^a_s + t^a_p ; \quad t^a_L = t^a_s - t^a_p$$

$$P_R = \frac{I + \eta}{2} ; \quad P_L = \frac{I - \eta}{2}$$

(65)

where $t^a_R$ and $t^a_L$ belong in general to different representations.

The Dirac conjugate fermion transforms as

$$\delta \bar{\psi} = -i\epsilon^a \bar{\psi} \bar{t}^a$$

(66)

The scalars belong to a real (reducible) representation $(\theta^a)_{ij}$; under an infinitesimal transformation

$$\delta \phi_i = i\epsilon^a \theta^a_{ij} \phi_j$$

(67)

The covariant derivatives are

$$D_\mu \psi = (I \partial_\mu + iA_\mu^a t^a) \psi$$

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} - iA_\mu^a \bar{\psi} \bar{t}^a$$

$$D_\mu \phi_i = (\partial_\mu \delta_{ij} + iA_\mu^a \theta^a_{ij}) \phi_j$$

(68)

The Yukawa coupling $\bar{\psi} y^i \psi \phi_i$, with

$$y_i = S_i I + iP_i \eta = Y^i R + Y^{i\dagger} P_L$$

$$Y_i = S_i + iP_i$$

(69)
is invariant under these transformations provided
\[ y^j \theta^a_{ji} + y^i t^a_i - \bar{t}^a y^i = 0 \quad (70) \]
or equivalently
\[ Y^j \theta^a_{ji} + Y^i t^a_i - t^a_L Y^i = 0 \quad ; \quad Y^j \theta^a_{ji} + Y^i t^a_i - t^a_R Y^i = 0 \]

The tree-level action is reflection symmetric.

We use a modified subtraction scheme \[26\], in which the bare constants are chosen of the form
\[ c_A = \sum_{l \geq 1} h^l N^l_A c_A^{(l)}(\epsilon) \]
\[ N_d = (4\pi)^{\epsilon/2-2} \Gamma(1 + \frac{\epsilon}{2}) ; \quad c_A^{(l)}(\epsilon) = \sum_{r \geq 0} c_A^{(l)}(\epsilon) \]

For all counterterms which do not involve \( NF_1 \), in minimal subtraction \( c_A^{(l)}(\epsilon) \) has no \( \epsilon^0 \) term.

The considerations made in the first section are easily generalized to the modified subtraction schemes.

Let us discuss the MS-SNDR renormalization at one and two loops, comparing it with the MS-NDR computations in the MS-NDR scheme \[25\] and in the non-minimal BMHV scheme \[13\].

Consider first the scalar Green functions; since it is not possible to produce renormalization parts involving \( \epsilon_{\mu\nu\rho\sigma} \), these Green functions have only fermionic loops with an even number of \( \eta \)'s, which behave in these traces as the anticommuting objects \( \eta_2 \). Therefore in this case our MS scheme is the same as the MS-NDR scheme; only poles are subtracted, respecting the chiral Ward identities. All the spurious terms present in BMHV \[13\] are absent.

The same is true for the \( \epsilon_{\mu\nu\rho\sigma} \)-independent parts of the Green functions involving external gauge fields with or without external scalar fields. There is none of the spurious terms present in BMHV, with or without \( \hat{A}_\mu^a \) (the former can be avoided in the case of external gauge fields setting \( \hat{A}_\mu^a = 0 \), as in \[13\], but they must be included when the gauge fields are promoted to quantum fields, leading to many other spurious terms in BMHV, besides those computed in \[13\]). Let us now consider the relevant fermionic Green functions. At one loop, multiplying them by \( \eta_2^2 \), they become identical to those in NDR (after the replacement \( \gamma^5 \rightarrow \eta_2 \)), and are minimal subtracted as in that case; multiplying them by \( \eta_1^2 \), they become identical to those in BMHV (after the replacement \( \gamma^5 \rightarrow \eta_1 \)).
Expressing the sum of the two sectors in the base of NF terms involving \( \eta \) and \( \eta_1 \), MS consists in subtracting, besides the poles, the finite NF terms, that is the \( A_1 \) terms. Projecting the MS subtracted graphs with \( \eta_1^2 \), one has MS-NDR; projecting the MS subtracted graphs with \( \eta_1^2 \), one has BMHV graphs renormalized in a non-minimal way (in the BMHV sense), with a finite subtraction such that in four dimensions the renormalized graphs are the same as in MS-NDR. This is precisely the renormalization choice made in [13].

To renormalize the fermionic Green functions at two loops in our scheme it is sufficient to multiply them by \( \eta_1^2 \) and perform MS-NDR. The terms \( S_{NF_1}^{(2)} \) are useful only to compute renormalized Green functions at three or more loops, so we will not compute them here; let us only remark that all contributions to \( S_{NF_1}^{(2)} \) apart from those with a fermionic loop subdiagram can be read out from the corresponding spurious terms in [13], replacing \( \gamma^5 \) with \( \eta_1 \) and projecting them with \( \eta_1^2 \). The difference in the remaining diagrams is that in [13] the fermionic loop contains \( \gamma^5 \), not \( \eta_2 \), leading to some difference in the coefficients of those counterterms, but to the same renormalized Green functions.

The last renormalization parts to be considered are those involving \( \varepsilon_{\mu\nu\rho\sigma} \), which are those with fermionic loops involving an odd number of \( \eta \)'s, which behave as the non-naive \( \gamma^5 \)-objects \( \eta_1 \). At one loop the relevant graphs are the \( \varepsilon_{\mu\nu\rho\sigma} \)-dependent parts of the three-vector and four-vector graphs, which are the same as in BMHV, since \( \eta_1 \) acts as \( \gamma^5 \) in these cases. In our MS scheme, there are no spurious terms in the bare action, corresponding to the fact that the anomaly appears in the Bardeen form [21], preserving vectorial gauge invariance (for a discussion, see [13]).

At two loops, the \( \varepsilon_{\mu\nu\rho\sigma} \)-dependent part of the three-vector graphs involves again \( \eta_1 \); the fermionic counterterms are therefore those obtained projecting the one-loop fermionic counterterms with \( \eta_1^2 \), that is are the same as in BMHV in [13]. Therefore the two-loop anomaly terms cancel as in [13], in agreement with the Adler-Bardeen theorem [9].

The complete bare action in MS-SNDR scheme is

\[
S = \int \frac{1}{2} c_{ij} D_\mu \phi_i D_\mu \phi_j + \frac{1}{2} \bar{\psi} \gamma_\mu (P_R c_\psi + P_L \bar{c}_\psi) D_\mu \psi - \frac{1}{2} (D_\mu \bar{\psi})(P_L c_\psi + P_R \bar{c}_\psi) \gamma_\mu \psi + i \bar{\psi} c_i \psi \phi_i + \frac{1}{4!} c_{ijkl} \phi_i \phi_j \phi_k \phi_l + \frac{1}{4} c_{ab} F_\mu^a F_\mu^b + S_{NF_1} + \Delta S[A^a_\mu] \tag{72}
\]
where

\[
\bar{c}_\psi \equiv \bar{c}_\psi(Y_i, Y_i^\dagger) \equiv c_\psi(Y_i^\dagger, Y_i)
\]

\[
c_\psi \equiv c_\psi(Y_i, Y_i^\dagger)
\]

(73)

are hermitian due to reflection symmetry and \(\Delta S[A_{\mu}^a]\) is the non-naive part of the pure gauge part of the bare action. It is due to Feynman graphs containing one or more fermionic traces with odd number of \(\eta\) or \(\eta_1\), giving each a Levi-Civita tensor.

At one loop one has

\[
c_{ij}^{(1)} = -\frac{4}{\epsilon} Y_{ij}
\]

\[
c_\psi^{(1)} = -\frac{1}{\epsilon} Y_i^\dagger Y_i
\]

\[
c_i^{(1)} = \frac{2}{\epsilon} y_j y_i^\dagger y_j
\]

\[
c_{ijkl}^{(1)} = -\frac{48}{\epsilon} Y_{ijkl} + \frac{3}{\epsilon} h_{ij}^{rs} h_{kl}^{rs}
\]

(74)

and

\[
S_{NF1}^{(1)} = \int \left[ -i \bar{\psi} \eta_1 \gamma_\mu P_i y_i D_\mu \psi + \frac{i}{\epsilon} (D_\mu \bar{\psi}) \eta_1 y_i P_i \gamma_\mu \psi + \bar{\psi} \eta y_j P_j y_j \psi \phi_i - i \bar{\psi} \eta y_i t^a_i y_i^\dagger \gamma_\mu \psi A^a_\mu \right]
\]

\[
\Delta S^{(1)}[A_{\mu}^a] = 0
\]

(75)

where \(Y_{i_1 i_2 \ldots i_{2n-1} i_{2n}}\), \(S_2(F)\) and \(S_2(S)\) are defined as in [13], as well as the group covariant \(K_2^{ab}\) in the next formula.

We checked explicitly that \(S_{NF1}\) is invariant under vectorial gauge transformations, i.e. under transformations satisfying the constraint \(e^a t^a_p = 0\) (see (65)), as expected by the argument given at the beginning of this section.
At two loops one has \[25, 13\]

\[
c_{ij}^{(2)} = \frac{1}{12\epsilon}h_{ikmn}h_{jkmn} + \left(-\frac{8}{\epsilon^2} + \frac{2}{\epsilon}\right)Y_{i}^{jk} + \left(-\frac{4}{\epsilon^2} + \frac{3}{\epsilon}\right)Y_{ij}^{kk}
\]

\[
c_{\psi}^{(2)} = \left(-\frac{1}{8\epsilon} + \frac{1}{2\epsilon}\right)Y_{j}^{j}Y_{i}^{i}Y_{j}^{i}Y_{i}^{j} - \frac{2}{\epsilon^2}Y_{j}^{j}Y_{j}^{i}Y_{i}^{i} + \left(-\frac{2}{\epsilon^2} + \frac{3}{2\epsilon}\right)Y_{ij}^{i}Y_{j}^{j}Y_{i}^{j}Y_{j}^{i}
\]

\[
c_{i}^{(2)} = \left(\frac{4}{\epsilon^2} - \frac{2}{\epsilon}\right)Y_{jk}y_{k}^{j}y_{i}^{i}y_{j}^{j} + \left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\right)y_{k}y_{j}^{j}y_{i}^{i}y_{j}^{j} + \left(\frac{1}{\epsilon^2} - \frac{1}{2\epsilon}\right)y_{k}y_{j}^{j}y_{i}^{i}y_{j}^{j} + \left(\frac{1}{\epsilon^2} - \frac{1}{2\epsilon}\right)y_{k}y_{j}^{j}y_{i}^{i}y_{j}^{j} + \left(\frac{1}{\epsilon^2} - \frac{1}{2\epsilon}\right)y_{k}y_{j}^{j}y_{i}^{i}y_{j}^{j} + \frac{1}{\epsilon}h_{ijkl}y_{k}^{i}y_{l}^{j} + \frac{1}{\epsilon}h_{ijkl}y_{k}^{i}y_{l}^{j}
\]

\[
c_{ijkl}^{(2)} = \left(-\frac{192}{\epsilon^2} + \frac{96}{\epsilon}\right)Y_{nijkl} + \frac{48}{\epsilon}Y_{nijk} + \frac{48}{\epsilon}Y_{njk} + \frac{96}{\epsilon^2}Y_{nij} + \frac{96}{\epsilon}Y_{nijkl} + \frac{48}{\epsilon}Y_{nij} + \frac{12}{\epsilon^2}Y_{ijkl}
\]

\[
\frac{3}{\epsilon^2}h_{mpq}h_{pql} + \left(\frac{6}{\epsilon^2} - \frac{3}{\epsilon}\right)h_{mpq}h_{pql}
\]

\[
c_{ab}^{(2)} = -\frac{2}{\epsilon}K_{2}^{ab}
\]

(76)

where in the expression for \(c_{ijkl}^{(2)}\) the symmetrization in the indices \(i, j, k, l\) is understood.

We didn’t compute \(\Delta S^{(2)}[A_{a}^{a}]\), which might be different from zero; in that case, it must be an \(\epsilon_{\mu\nu\rho\sigma}\) term, and it is the same as in BMHV scheme used in [13]. Observe that \(\Delta S^{(2)}[A_{a}^{a}]\) does not have to do with the anomaly, since it does not involve the totally symmetric tensor \(d^{abc}\).

We didn’t compute \(S_{NF}^{(2)}\), which, as explained in the previous section, is not necessary to compute the two-loop renormalized Green functions.

In absence of the external gauge fields MS-SNDR preserves the rigid chiral Ward identities at all loops. In fact it is not possible to produce the relevant terms with \(\epsilon_{\mu\nu\rho\sigma}\), which are at the origin of the breaking of the symmetries.

In presence of the external gauge fields the chiral Ward identities might be broken by the \(\epsilon_{\mu\nu\rho\sigma}\) contributions, so that a finite counterterm \(\Delta S^{(2)}[A_{a}^{a}]^{non\ min}\) might be necessary to recover the chiral Ward identities; at more than two loops \(\Delta S^{(l)}[A_{a}^{a}]\) can contain also parity even terms, coming from an even number of Levi-Civita tensors.

Let us finally discuss what would have happened if, as discussed at the end of the previous section, we had not subtracted the \(A_{1}\) terms.

In the computation of the anomaly, the difference from the correctly MS-subtracted case and the one now discussed consists of the counterterm graph
in Figure 1, in which there is the insertion of the $A_1$ counterterm in (75), leading to a non-local finite contribution, which conspires to the absence of two-loop corrections to the anomaly; apart from non satisfying the Adler-Bardeen theorem, the suggested theory would have the following inconsistency: in the one-loop approximation it has the same renormalized Green functions as in a consistent renormalization scheme, the non-minimal BMHV scheme used in [13]; from general renormalization theory arguments it follows that the renormalized Green functions in two different renormalization schemes can be connected by a renormalization group transformation; being the one-loop relevant terms equal, it follows that the marginal Green functions can differ at two loops only by a local term; however we just found a case in which this difference is non-local; this is not possible, so that the proposed scheme is inconsistent, in agreement with the arguments in the previous subsection.

2.2 Gauge theories

In the Yukawa theories only the pure gauge Green functions do not satisfy the chiral Ward identities in the MS-SNDR scheme; the remaining Green functions satisfy them at all orders in perturbation theory. The vector Ward identities are respected in this scheme.

In chiral gauge theories no Green function satisfies the chiral Ward identity in this scheme (at order high enough in loops); however SNDR breaks the chiral symmetries in a gentler way than BMHV.

As long as graphs involving fermionic loops with an odd number of $\gamma^5$-like objects are absent, the chiral Ward identities will be satisfied. In general, we can expect that the level at which MS is not sufficient in SNDR is the same at which NDR reveals its inconsistencies. Apart from the graphs mentioned in the previous section, it is reasonable to expect that other superficially divergent graphs containing the subgraph in Figure 2(a) (giving non-vanishing contributions to the two-loop anomaly [13]), like the one in Figure 2(b) need
to be subtracted in a non-minimal way.

In presence of composite operators the need of introducing a finite renormalization appears already at low loop order; for instance coupling an external gauge current to the theory, the minimally subtracted triangle graph involving it and two chiral gauge fields gives the anomaly in the Bardeen form which does not respect the chiral Ward identities so that a finite counterterm must be added to restore them.

Apart from this issue, the MS-SNDR scheme for a chiral gauge theory differs qualitatively from that in Yukawa theories for the appearance of $\gamma^5$ itself (not $\eta$ or $\eta_1$) in the bare action. It is only at this point that the bare action starts living in the extended algebra four times bigger than the BMHV algebra, as can be seen observing that, for instance, in absence of counterterms with $\gamma^5$ in a chiral gauge theory the relevant bare vectorial part of the vertices contains $\gamma_\mu$ and $\eta\gamma_\mu$, but not $\eta_1\gamma^5\gamma_\mu$ and $\eta\gamma^5\gamma_\mu$.

For example consider the graph in Figure 2(c), contributing to the vertex for an external gauge field (it is chosen to be external to avoid anomaly cancellation); in MS it generates counterterms involving $\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma/\epsilon = \gamma^5\gamma_\mu/\epsilon$, while at lower loop level only vertices involving $\eta$ and $\eta_1$ appeared in the bare action. Similar counterterms can be expected in the graph in Figure 2(b); this counterterm cannot be replaced by $\eta\gamma_\mu/\epsilon$, since the difference is not weakly evanescent (see the example in eq. (18)).

As far as vector gauge invariance is concerned, let us argue that MS-SNDR preserves it in the case of a non-abelian vector gauge theory with chiral Yukawa couplings (i.e. the gauged version of the Yukawa model in the previous subsection, with $t^a_p = 0$).

The bare action is BRST-invariant at tree level. Suppose that it is BRST-invariant at $(l-1)$ loops; then the $l$-loop functional generator of the 1PI Green functions renormalized up to $(l-1)$ loops, called $\Gamma^{(l)}$, satisfies the
Slavnov-Taylor identities, which in the Zinn-Justin form \( [27] \) read

\[
S^{(0)} \ast \Gamma^{\epsilon(l)} + \Gamma^{\epsilon(l)} \ast S^{(0)} = - \sum_{m=1}^{l-1} \Gamma^{\epsilon(m)} \ast \Gamma^{\epsilon(l-m)} \tag{77}
\]

In MS-SNDR \( \Gamma^{\epsilon(l')} \), for \( l' < l \), is finite and belongs to \( T_0 \) for \( \epsilon = 0 \); due to \([27]\), for \( \epsilon = 0 \) the right-hand-side of (77) belongs also to \( T_0 \); it follows that the \( T_1 \) local term of \( \Gamma^{\epsilon(l)} \) is BRST invariant, as well as its poles; therefore the bare action is BRST invariant also at \( l \) loops.

### 2.3 Comparison with other NDR schemes

The NDR prescriptions proposed in the past have, as regularization schemes, some inconsistency, so that they require some care in using them, usually on a diagram-by-diagram basis, while in a true regularization scheme one should be able to get unambiguous results by using the bare action and the Feynman rules, whatever computational routine is followed.

Furthermore no proof has been given, in any of these schemes, that the renormalized Green functions are equal, modulo finite local renormalization terms, to those in some consistent scheme, e.g. the BMHV scheme, at all orders in perturbation theory.

The issue of preserving the Ward identities comes only after settling these points, and it is solved by adding local finite terms to the bare action. In absence of a consistent regularization system it is not true that checking the Ward identities on the relevant vertices one is guaranteed that the theory is renormalized consistently. In fact consider a set of ‘renormalized’ Green functions which differs from any consistent set of renormalized Green functions by non-local quantities. Then check the Ward identities on the relevant terms defined at a subtraction point. If there are no anomalies, one can add local finite counterterms to restore the Ward identities on the relevant terms. Clearly the new set of Green functions is still inconsistent, since non-local \( l \)-loop terms cannot be canceled by local \( l \)-loop counterterms.

\(^2\) i.e. such that there is no renormalization group transformation transforming one set of Green functions in the other; one can construct easily similar pathological cases by choosing, within a consistent regularization system, two different one-loop sets of renormalization conditions, say \( A \) and \( B \), and by computing a subset of the two-loop Green functions in system \( A \), another subset in system \( B \); the resulting mixing of Green functions differs by non-local terms from the set of Green functions computed within a single scheme. Working without a consistent regularization system without great care it is likely that a similarly inconsistent system is produced at some point.
Figure 3:

The dangers of using algebraically inconsistent regularization systems have been reviewed in [19], where it is emphasized the fact that the renormalization conditions are compatible with the Ward identities does not guarantee the validity of the quantum action principle, in absence of a true regularization scheme.

Let us consider briefly some of the NDR schemes discussed in the literature.

To avoid confusion with previously defined objects, call \( \Gamma^5 \) the algebraically undefined \( \gamma^5 \)-like object; \( \gamma^5 \) is defined in (9).

In [15] \( \Gamma^5 \) is chosen to be anticommuting in open fermionic lines and in loops with an even number of \( \Gamma^5 \), while in the loops with an odd number of \( \Gamma^5 \) the trace rules are given modulo some evanescent term, to be fixed by the Ward identities. The authors emphasize that this is not a true regularization scheme, but that it is a convenient prescription for one-loop computations.

In [17] \( \Gamma^5 \) is chosen to be anticommuting in open fermionic lines and in loops with an even number of \( \Gamma^5 \), while in the loops with an odd number of \( \Gamma^5 \) one reduces the traces with an odd number of fermions, before evaluating them, to the case of one \( \Gamma^5 \) only, by using anticommuting \( \Gamma^5 \); this rule does not preserve the Bose symmetry.

In [18] a non-cyclic definition of the trace is used, together with a totally anticommuting \( \Gamma^5 \). Bose symmetry is broken in this scheme; to avoid this problem one can add the rule of choosing an appropriate reading point in the graphs.

In [28] it is suggested the use of \( \gamma^5 \) together with a trace rule which, in the case of even number of \( \gamma^5 \), acts as if the \( \gamma^5 \) were naive, while in the case of odd number of \( \gamma^5 \) it acts as the BMHV trace. In this case there are ambiguities in the treatment of objects like \( tr(\{\gamma^5,\gamma_\mu\}\{\gamma^5,\gamma_\mu\})\gamma_\nu\gamma_\rho \), which is zero or not depending on the fact that \( (\{\gamma^5,\gamma_\mu\}\{\gamma^5,\gamma_\mu\}) \) is given or not its value \( 4\epsilon \). For instance using this trace rule the renormalization of the two-loop graph in Figure 3 requires some care due to this ambiguity.

Trueman [29] has used the trick of renormalizing the axial vector-fermion-fermion vertex as the corresponding vector-fermion-fermion vertex multiplied by \( \gamma^5 \), as a way of restoring the Ward identities and for treating \( \gamma^5 \) as if
it were anticommuting, while keeping the BMHV rules. Unlike the other prescriptions reviewed in this subsection, this one is consistent; however, apart from the fact that this trick has been defined only for certain cases, it has the drawback of using the BMHV scheme, with all its spurious anomalies (i.e. finite counterterms which must be added to restore the Ward identities); for instance in the case of the Yukawa theory in [13] a trick similar to the one proposed by Trueman has been used, with the result that the number of spurious anomalies is conspicuous. Trueman’s trick works well as long as one is able to make computations without resorting to the full form of the bare action, implicitly defined; for a three-loop application of this trick see [8].

Let us finally describe the MS-SNDR as a NDR prescription.

Γ₅ is chosen to be anticommuting in open fermionic lines and in loops with an even number of Γ₅, while in the loops with an odd number of Γ₅ it is considered to be equal to γ₅.

Apart from subtracting the poles, the MS-SNDR requires a finite subtraction in the fermionic subdiagrams which occur in the fermionic traces with an odd number of Γ₅; this finite subtraction is chosen such that the corresponding fermionic renormalization part has the same value, apart from evanescent terms, as the fermionic renormalization part occurring in an open fermionic line, or in a closed fermionic line with an even number of Γ₅ (minimal subtraction of the T₁ terms).

In MS-SNDR the BMHV γ₅ does not appear in the tree-level action, but can be produced in divergent graphs, by application of the Levi-Civita tensor on the gamma matrices. The resulting counterterms must not be confused with those with Γ₅; in particular the counterterms (Γ₅ − γ₅)/ε cannot be neglected ((η − γ₅)/ε is not weakly evanescent).

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