Angular distributions of nonlinear Thomson scattering in combined fields with a general elliptical polarization laser and a background magnetic field

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Abstract

Nonlinear Thomson scattering of an electron motion in a combined fields is investigated. The fields are constituted by an intense elliptically polarized laser field and a strong background magnetic field. It is found that the angular distributions of scattering spectra with respect to the azimuthal angle exhibits the symmetry for the laser intensity, the magnetic resonance parameter, the initial axial momentum of electron as well as the order of harmonics in emission spectra. Meanwhile the distribution of polar angle approaches the direction of laser propagation more and more with parameters increase. The optimal angle for maximum radiated power per unit of solid, the corresponding photon number and its brightness can be obtained, which implies that the high quality XUV or/and X-ray light can be generated by the studied scheme when the suitable parameters are choosen.

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I. INTRODUCTION

Since the first ruby laser was invented in the world, the laser intensity is continuously increased and widely applied to many research fields. In particular, the laser intensity has a further great increasing by the chirped pulse amplification technique. In the past decades, more and more research are focused on the ultra-short ultra-intense lasers interacting with matter. Among them the nonlinear Thomson scattering (NTS) has become an important research area due to its great value and application prospect to generate bright and ultrashort X-ray source [2][3] or/and γ-ray pulse [10][12]. Furthermore, the attosecond pulse [13][14][30], higher order of harmonics [15] and the optical vortices [16] can also be generated by NTS.

It is well known that obvious difference of spatial radiation features when parameters of the incident laser pulse are changing, such as polarization state, initial phase, beam waist and pulse width. For linearly polarization laser field, Chen et al. [17] experimentally measured the angular radiation profiles of the second and third harmonics. In 2009, the spatial characteristics of the Thomson scattering were theoretically studied in a linearly polarized laser field by Lan et al., which shows that the spatial distributions depends sensitively on the initial phase [18]. Vais and Yu studied the influence of the beam waist on spatial characteristics [19][22]. Zhang found the four-fold and two-fold rotational symmetry pattern for the space radiation characteristics under different laser intensities [20] and Li discovered the bifoliate radiation pattern (BRP) [21]. Ruijter theoretically and numerically showed the dependency of the spectrum on the intensity of the laser and the carrier envelope phase [23]. For circularly polarized laser field [24][31], the research indicated that the initial phase has a great influence on the spatial distribution [24][26][34]. Wu et al. found that the radiation was significantly affected by beam waist radius [28]. With the increasing of laser intensity, the peak radiation power increases, polar angle decreases and the direction of radiation approaches the direction of laser propagation [27][29][32][33]. He et al. indicated the different effects by varying the initial phase for the circularly and linearly polarized laser fields in the combination of an intense laser field and a strong uniform magnetic field [24]. For elliptically polarized laser field, Shi et al. researched the spatial radiation distribution with varied incident pulse widths [35]. Fruhling et al. reported experimental results of the radiated polarization states by elliptically polarized light [36].

In recent years, the properties of Thomson back-scattering spectra are explored concretely
in combined magnetic and laser fields \[22, 23\] and the influence of the relevant parameters such as the laser intensity, the initial electron’s axial momentum are found \[37–40\]. On the other hand, many studies on spatial radiation characteristics of NTS are limited only for the single linear or/and circular polarized laser field. To our knowledge, the investigation of complete spatial radiation characteristics of single electron interacting with the combination of an intense elliptically polarized laser field and a strong magnetic field are missed, therefore, it is worth to study since some new features maybe useful to get the high quality x-ray source by the NTS in this situation.

In this paper, we study the NTS in this combined fields by an analytical treatment and numerical calculation. We get the angular distribution with respect to azimuthal angle and polar angle. And the relationship between radiation characteristics and parameters is revealed such as the laser intensity, the magnetic resonance parameter, the electron’s initial axial momentum as well as the order of harmonics. The results show that NTS in the combined fields with an elliptically polarized laser field and a strong magnetic field can produce high quality XUV or/and X-ray light. In the optimal observation angles, the number of photons radiated by electron is approaching \(10^{11} - 10^{14}\) and its photons brightness can reach the magnitude \(10^{17}\) (photons / s mm\(^2\) mrad\(^2\) 0.1%BW).

The paper is organized as follows. In Sec.II, the equations of the momentum and trajectory of the electron are obtained in combined elliptically polarized laser and magnetic field. The radiation power per unit of solid angle is given in emission direction. In Sec.III, the radiation distribution with respect to azimuthal angle and polar angle is obtained. The features of the angular distributions of NTS are presented with different parameters. The largest radiated power, the number of photons and photons brightness are given in optimum observation direction. Finally the main conclusions and some discussions are given briefly in Sec. IV.

II. BASIC EQUATIONS

Let us consider the spatial angular distributions of NTS that the electron (with mass \(m\) and charge \(-e\)) moves in the simultaneous laser and magnetic field. As shown in Fig. 1, we assume that the amplitude of external magnetic field is \(B_0\), while the laser field is a left-hand elliptically polarized plane wave with vector potential amplitude \(A_0\), frequency \(\omega_0\)
FIG. 1: (colour online) A schematic illustration of physical model.

and ellipticity α. The laser propagation direction and the magnetic direction are both along the +z direction. By denoting the phase of the laser field as $\eta = \omega_0 t - k \cdot r$, where $k$ and $r$ are the laser wave vector and electron displacement vector respectively. The combined fields can be expressed by the total vector potential:

$$A = \frac{A_0}{\sqrt{1 + \alpha^2}} \left[ -\sin \eta \mathbf{i} + \alpha \cos \eta \mathbf{j} \right] + B_0 x \mathbf{j}. \quad (1)$$

From Eq. (1), the corresponding electric field $E$ and magnetic field $B$ can be obtained

$$E = -\frac{1}{c} \frac{\partial A}{\partial t}, \quad (2)$$

$$B = \nabla \times A. \quad (3)$$

Then, the dynamics of electron will be studied according to the momentum-energy evolving equations:

$$\frac{dp}{dt} = -e(E + \beta \times B), \quad (4)$$

and

$$d(\gamma mc^2)/dt = -ec\beta \cdot E, \quad (5)$$

where $p$ is the electron’s momentum, $\beta$ is the electron’s velocity, $\gamma = (1 - \beta^2)^{-1/2}$ is the electron relativistic factor. Note that in order to simplify the calculations, we have normalized time by $1/\omega_0$, distance by $1/k_0$, velocity by $c$, momentum by $mc$, the electric field by $eE_0/m\omega_0 c$, and the magnetic field by $eB_0/m\omega_0 c$. 
By substituting expression of fields into the equations of motion of the electron, then they are written as the following

\[ \frac{d^2p_x}{d\eta^2} + \omega_b^2 p_x = (\omega_b \alpha + 1)a \sin \eta, \quad (6) \]

\[ \frac{d^2p_y}{d\eta^2} + \omega_b^2 p_y = -(\alpha + \omega_b)a \cos \eta, \quad (7) \]

where \( a = eA_0/\sqrt{1 + \alpha^2 mc^2} \) is the normalized vector potential amplitude, \( \omega_b = b/\varsigma \) is the cyclotron frequency of the electron motion in the combined field and \( b \) is the magnetic field strength normalized by \( eB_0/m\omega_0c \). We assume that the phase \( \eta \) has an initial value \( \eta_{in} = -z_{in} \) at \( t = 0 \). Since an electron moves in a constant amplitude elliptically polarized laser field, the constant of the motion is readily obtained as \( \varsigma = \gamma - p_z = \gamma_0 - p_{z0} = \sqrt{1 + p_{z0}^2} - p_{z0} \) according to Eq.(4) and Eq.(5) where \( p_{z0} = \frac{1}{2\varsigma} - \frac{\varsigma}{2} \) is the initial momentum in the \( z \) direction.

By solving Eqs.(6) and (7), the momentum can be acquired, and the electron trajectories via \( dr/d\eta = p/\varsigma \) can be also easily obtained as

\[ p_x = na \left\{ \varepsilon_1 \sin \eta - \varepsilon_1 \cos \omega_b (\eta - \eta_{in}) \right\} \sin \eta_{in} - \varepsilon_2 \sin \left[ \omega_b (\eta - \eta_{in}) \right] \cos \eta_{in} \right\}, \quad (8) \]

\[ p_y = na \left\{ -\varepsilon_2 \cos \eta - \varepsilon_2 \cos \omega_b (\eta - \eta_{in}) \right\} \cos \eta_{in} - \varepsilon_1 \sin \left[ \omega_b (\eta - \eta_{in}) \right] \sin \eta_{in} \right\}, \quad (9) \]

\[ p_z = \frac{n^2 a^2}{2\varsigma} \left\{ \varepsilon_1^2 \left( \sin^2 \eta + \sin^2 \eta_{in} \right) + \varepsilon_2^2 \left( \cos^2 \eta + \cos^2 \eta_{in} \right) \right. \\
-2 \cos \omega_b (\eta - \eta_{in}) \left[ \varepsilon_1^2 \sin \eta \sin \eta_{in} + \varepsilon_2^2 \cos \eta \cos \eta_{in} \right] \left. -2\varepsilon_1 \varepsilon_2 \sin \omega_b (\eta - \eta_{in}) \sin (\eta - \eta_{in}) \right\} + \frac{1}{2\varsigma} - \frac{\varsigma}{2} \right\}, \quad (10) \]

and

\[ x(\eta) = \frac{na}{\varsigma} \left\{ -\varepsilon_1 \cos \eta - \frac{\varepsilon_1}{\omega_b} \sin \omega_b (\eta - \eta_{in}) \right\} \sin \eta_{in} \\
+ \frac{\varepsilon_2}{\omega_b} \cos \omega_b (\eta - \eta_{in}) \right\} \cos \eta_{in} + \left( \varepsilon_1 - \frac{\varepsilon_2}{\omega_b} \right) \cos \eta_{in} \right\}, \quad (11) \]

\[ y(\eta) = \frac{na}{\varsigma} \left\{ -\varepsilon_2 \sin \eta + \frac{\varepsilon_1}{\omega_b} \cos \omega_b (\eta - \eta_{in}) \right\} \sin \eta_{in} \\
+ \frac{\varepsilon_2}{\omega_b} \sin \omega_b (\eta - \eta_{in}) \right\} \cos \eta_{in} + \left( \varepsilon_2 - \frac{\varepsilon_1}{\omega_b} \right) \sin \eta_{in} \right\}, \quad (12) \]
\[ z(\eta) = (na)^2 \left\{ -\frac{\varepsilon_1^2}{2} \left[ \sin \frac{2\eta}{4} - \left( \frac{\sin^2 \eta_{in} + 1}{2} \right)(\eta - \eta_{in}) \right] \\
+ \frac{\sin 2\eta}{4} \left[ \cos^2 \eta_{in} + \frac{1}{2} \right] (\eta - \eta_{in}) \right\} \\
+ \frac{1}{2} (\varepsilon_1^2 - \varepsilon_2^2) \left\{ \sin \left[ \frac{(\omega_b - 1)(\eta - (\omega_b + 1)\eta_{in})}{\omega_b - 1} \right] \right. \\
- \frac{1}{2} (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_1 \varepsilon_2) \sin \left[ \frac{(\omega_b - 1)(\eta - \eta_{in})}{\omega_b - 1} \right] \right\} \\
- \frac{1}{2} (\varepsilon_1^2 + \varepsilon_2^2) - \varepsilon_1 \varepsilon_2 \sin \left[ \frac{(\omega_b + 1)(\eta - \eta_{in})}{\omega_b + 1} \right] \right\} \\
\left. + \left( \frac{1}{2} \varepsilon_2^2 - \frac{1}{2} \right) (\eta - \eta_{in}) \right\}, \quad (13) \]

where \( n \) is the magnetic resonance parameter defined as \( n = 1/(\omega_b - 1) \), refer to Ref. [11]. So, the strength of external magnetic field is easy to get

\[ B_0 = \left( 1 + \frac{1}{n} \right) \left( \sqrt{1 + p_{z0}^2} - p_{z0} \right) m\omega_0 c / e \approx [(1 + 1/n)(\sqrt{1 + p_{z0}^2} - p_{z0})/\lambda[\mu m]] \times 100 \text{MG}, \]

which is obviously related to the resonance parameter \( n \) and the initial axial momentum \( p_{z0} \). We have \( B_0 \approx (1 + 1/n) \times 100 \text{MG} \) when \( p_{z0} \ll 1 \) and \( B_0 \approx [(1 + 1/n)/2p_{z0}] \times 100 \text{MG} \) when \( p_{z0} \gg 1 \).

The angular distributions of the emission power detected far away from the electron in the direction \( \mathbf{n} \) (not to be confused with the magnetic resonance parameter \( n \)) can be calculated as [42]:

\[ \frac{d^2 I}{d\Omega d\omega} = \frac{e^2 \omega^2}{4\pi^2 c} |\mathbf{n} \times [\mathbf{n} \times \mathbf{F}(\omega)]|^2 , \quad (14) \]

with a function of

\[ \mathbf{F}(\omega) = \frac{1}{\xi} \int_{-\infty}^{+\infty} d\eta p(\eta) \exp \{ i\omega [\mathbf{n} \cdot \mathbf{r}(\eta) + z(\eta)] \}, \quad (15) \]

where the detected direction is denoted as \( \mathbf{n} = \sin \theta \cos \varphi \mathbf{i} + \sin \theta \sin \varphi \mathbf{j} + \cos \theta \mathbf{k} \). Note that the normalized \( \omega = \omega / \omega_0 \) is used for the radiation frequency.

Obviously the electron’s motion is periodic with \( T = 2\pi n \), thus, we can simplify Eq.(15) by an infinite series of delta functions to calculate the emission power along any random direction [37]. Similarly, there is a dimensionless fundamental frequency \( \omega_1 = 2\pi / (T - \mathbf{n} \cdot \mathbf{r}_0 + z_0) \). It is noted that the dimensionless vector \( \mathbf{F}(\omega) \) can be expanded as an infinite series of delta functions [43] at the harmonics of the fundamental frequency as

\[ \mathbf{F}(\omega) = \frac{1}{\xi} \sum_{m=-\infty}^{+\infty} \mathbf{F}_m \delta (\omega - m\omega_1) , \quad (16) \]
so that the $m^{th}$ amplitude is

$$F_m = \omega_1 \int_{\eta_n}^{\eta_n+T} d\eta p(\eta) \exp \{im2\pi h(\eta)\}, \quad (17)$$

with a function of

$$h(\eta) = \frac{\eta - \mathbf{n} \cdot \mathbf{r}(\eta) + z(\eta)}{T - \mathbf{n} \cdot \mathbf{r}_0 + z_0}, \quad (18)$$

where $\mathbf{r}_0 = (0, 0, z_0) = \left(0, 0, T \left[\frac{\epsilon_1^2 \epsilon_2^2}{2\gamma^2} \left(\epsilon_1^2 \left(\sin^2 \eta_{in} + \frac{1}{2}\right) + \epsilon_2^2 \left(\cos^2 \eta_{in} + \frac{1}{2}\right)\right) + \left(\frac{1}{2\gamma} - \frac{1}{2}\right)\right]\right)$ is the drift displacement vector of the electron during one period.

Finally, we can obtain the radiation spectra in units of erg/s per unit solid angle by integrate the frequency $\omega$, in particular, the $m^{th}$ harmonic is

$$\frac{d^2 I_m}{d\Omega dt} = \frac{e^2 \omega_0^2}{4\pi^2 c \gamma^2} (m\omega_1)^2 |\mathbf{n} \times [\mathbf{n} \times \mathbf{F}_m]|^2, \quad (19)$$

where

$$|\mathbf{n} \times [\mathbf{n} \times \mathbf{F}_m]|^2 = |\mathbf{n} \cdot \mathbf{F}_m \mathbf{F}_m - \mathbf{F}_m \mathbf{F}_m\mathbf{F}_m|^2. \quad (20)$$

Thus, based on Eq. (8)-Eq. (20), the angular distributions of NTS when an electron moves in the laser and magnetic fields can be calculated numerically. For simplicity, the parameters are chosen to make the radiation reaction effect (RRE) negligible and the radiated power per unit of solid angle is normalized by $e^2 \omega_0^2/4\pi^2 c$.

### III. NUMERICAL RESULTS AND ANALYSIS

In general, the final spectra of NTS via explicit analytical expression in combined fields with a general elliptical polarization is hard to get. Insteading of it, however, we can use Romberg integral algorithm to calculate $d^2 I_m/d\Omega dt$ and get the spatial distribution features of the NTS numerically. And through the numerical results, the maximum radiated power per unit of solid angle, the corresponding number of photons and photons brightness can be obtained in optimum observation direction. By the way, for a convenience and simplicity, meanwhile without the loosing of the generality, the ellipticity $\alpha = 0.5$, the wavelength $\lambda = 1\mu m$ for laser field and initial phase $\eta_{in} = 0$ for electron are fixed in our study in the following.

It is emphasized that the spatial distribution features of the NTS are researched from three perspectives. One is angular distributions with respect to the azimuthal

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angle $\varphi$, when the polar angle $\theta = \pi/2$ is fixed. The second is angular distributions with respect to the polar angle $\theta$, when the azimuthal angle $\varphi = 0$ is fixed. Thirdly, we look into the small angle spatial distribution by the contour plotting. It is very useful to write some important quantities in three cases like of, in the first case, the detected direction $\mathbf{n} = (\cos \varphi, \sin \varphi, 0)$, the fundamental frequency $\omega_1 = 2\pi/(2\pi n + z_0)$, and $h(\eta) = [\eta - \cos \varphi x(\eta) - \sin \varphi y(\eta) + z(\eta)]/[2\pi n + z_0]$. Similarly, in the second case, they are $\mathbf{n} = (\sin \theta, 0, \cos \theta)$, $\omega_1 = 2\pi/[2\pi n + (1 - \cos \theta)z_0]$ and $h(\eta) = [\eta - \sin \theta x(\eta) + (1 - \cos \theta)z(\eta)]/[2\pi n + (1 - \cos \theta)z_0]$. However, in the third case, the azimuthal angle $\varphi$ and polar angle $\theta$ are no longer special angles. Since the detected direction is $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, the fundamental frequency $\omega_1 = 2\pi/[2\pi n + (1 - \cos \theta)z_0]$, and $h(\eta) = [\eta - \sin \theta \cos \varphi x(\eta) - \sin \theta \sin \varphi y(\eta) + (1 - \cos \theta)z(\eta)]/[2\pi n + (1 - \cos \theta)z_0]$ are used.

Before starting our study for elliptical laser field, in order to check our validity of algorithm, we use Romberg algorithm to calculate the radiation spectrum of Thomson backscattering in combined field with circularly polarized laser. The comparison of theoretical and numerical solutions is shown in Fig.2, we can clearly see that they are in perfect agreement. From the Fig.2 we have checked that the numerical solution is in agreement with the theoretical solution in the Ref.[37] of harmonic order $1 \sim 400$. Moreover, by the way, we correct a trivial typo of Ref.[37], which should be written as

$$d^2 I_m/d\Omega dt = \frac{e^2 \omega_n^2}{4\pi^2 c^2} (m\omega_1)^2 (\pi \omega_1 n^2 a)^2 \left[ A^2_{(m,n)} + B^2_{(m,n)} \right].$$

A. Angular distributions with respect to the azimuthal angle $\varphi$ when $\theta = \pi/2$

Now let us concentrate on the spatial distribution characteristics of NTS in the combined laser and magnetic fields when $\theta = \pi/2$. The angular distributions of the emission power with respect to the azimuthal angle $\varphi$ are investigated. From Ref.[37], it is known that the Thomson scattering spectrum is consisted of a series of harmonics. Here, $m$ represents the order of the harmonics. The azimuthal angle distributions of the emitted power for various parameters are shown in Fig.3(a),(b),(c),(d). The relevant parameters are chosen as (a) $a = 2, n = 5, p_{z0} = 0, m = 500, 1000, 1500, 2000$; (b) $m = 1000, n = 5, p_{z0} = 0, a = 1, 1.5, 2, 3$; (c) $m = 1000, a = 2, p_{z0} = 0, n = 3, 4, 5, 6$; (d) $m = 1000, a = 2, n = 5, p_{z0} = 0, 0.1, 0.25, 0.5$. 

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FIG. 2: (colour online) A check and comparison of the theoretical solution with the numerical solution for the backscattering spectra of the $m^{th}$ harmonic when $\alpha = 1$, $n = 5$, $a = 2$, $p_{z0} = 1$ and $\eta_{in} = 0$.

From Fig.3, the shape of spatial distribution of the radiation power is demonstrated clearly. They are all symmetric with respect to $y$ axis whatever the laser intensity, the order of harmonics, the resonance parameter and the initial axial momentum are. Most of them show the BRP, but it is different under some parameters. In Fig.3(a), it presents an unifoliate radiation pattern (URP) when $m = 500$. In Fig.3(b), it presents a polyphyllous radiation pattern when $a = 1$, the shape can turn into URP when $a = 3$. In Fig.3(c), it show the URP when $n = 6$. In Fig.3(d), the shape is uniformly distributed in the $xy$ plane when $p_{z0} = 0.5$, this rule is the same as Ref. [27] as the initial axial momentum increases. If electron has large initial momentum, the radiation is approximately uniformly distributed with respect to the azimuthal angle $\varphi$. Meanwhile, the strongest radiation power mainly distributes between $\varphi = \pi$ and $\varphi = 2\pi$. We focus on the difference of the variation tendency of $d^2 I/d\Omega dt$ and $d^2 I_{max}/d\Omega dt$, Fig.3(a) and Fig.3(d) show the angular distributions of the
emission power with respect to the azimuthal angle $\varphi$ for different harmonic orders $m^\text{th}$ and different initial axial momentum $p_{z0}$, respectively. Their overall trend of $d^2 I/d\Omega dt$ decrease with the increasing of parameters $m$ or $p_{z0}$. Fig. 3(b) and Fig. 3(c) show the angular distributions of the emission with respect to the azimuthal angle $\varphi$ for different laser intensity $a$ and various resonance parameters $n$, respectively. Their overall trend of $d^2 I/d\Omega dt$ first increased and then decreased with the increasing of parameters $a$ or $n$. They shows the nonlinear variation and the complexity of spatial radiation of the NTS. For the $d^2 I_{\max}/d\Omega dt$, they are all decrease with the increase of parameters $m$ or $p_{z0}$, the corresponding azimuthal angle $\varphi$ is getting closer and closer to $\varphi = 240^\circ$ within a given range of parameters. For
Fig. 3(b) and Fig. 3(c), we can obviously see that their variation trend of $d^2I_{max}/d\Omega dt$ first increased and then decreased with the increase of parameters $a$ or $n$. For Fig. 3(b), the corresponding azimuthal angle $\varphi$ of maximum radiation power varies in the range from $\varphi = 240^\circ$ to $\varphi = 270^\circ$. For Fig. 3(c), the angle $\varphi$ of maximum radiation power is closing to the orientation of $\varphi = 270^\circ$, which varies in the range from $\varphi = 210^\circ$ to $\varphi = 270^\circ$. It is worth noting that we can get the same maximum radiation power in two different directions due to the symmetry.

In general, the radiation angular distributions with respect to the azimuthal angle $\varphi$ show the symmetry. The symmetry of radiation distributions will not be broken while the parameters are changed, the strongest radiation power mainly distributes between $\varphi = \pi$ and $\varphi = 2\pi$. The study shows that the overall trend of $d^2I/d\Omega dt$ first increased and then decreased with the increasing of parameters $a$ or $n$. So we can obtain maximal radiation power and corresponding azimuthal angle by choosing suitable parameters in $xy$ plane which is perpendicular to the direction of laser propagation.

**B. Angular distributions with respect to the polar angle $\theta$ when $\varphi = 0$**

For this part, the angular distributions of NTS in the combined laser and magnetic fields with respect to the polar angle $\theta$ are investigated. The results are shown in Fig. 4 for different orders of harmonic radiation $m$, laser intensity $a$, resonance parameter $n$ and initial axial momentum $p_{z0}$. The relevant parameters are chosen as (a) $a = 1$, $n = 4$, $p_{z0} = 0$, $m = 500, 1000, 1500, 2000$ and (b) $m = 1000$, $n = 4$, $p_{z0} = 0$, $a = 1.5, 2, 3, 4$; (c) $m = 1000$, $a = 1$, $p_{z0} = 0$, $n = 3, 5, 10, 40$; (d) $m = 1000$, $a = 1$, $n = 4$, $p_{z0} = 1, 1.5, 2, 5$.

We know that the polar angle ranges from $0^\circ$ to $180^\circ$, and it is apparent that the spatial distribution of the radiation is mainly in the neighbour of forward direction. From Fig. 4(a), we can see that the maximum radiation power values are $124.4456, 81.3008, 60.5982, 26.7768$, the corresponding polar angles are $17.8^\circ, 18.8^\circ, 19.2^\circ, 18.8^\circ$, respectively. The maximum of the radiation energy emitted per unit solid angle $d\Omega$ per unit time $dt$ decreases when $m$ increases from 500 to 2000. Meanwhile, the corresponding polar angle varies within a small range. In addition, the corresponding polar angle range is relatively wide and the range of polar angle is not significantly decreased with the increasing of $m$. The influence of laser intensity on angular distribution of radiation power is shown in Fig. 4(b), the radiation power
increases and the polar angle decreases with the increasing of laser intensity $a$. The maximum radiation power values are 443.0468, 922.1881, 2198.4269, 3866.7586 and the corresponding polar angles are $12.4^\circ$, $9.4^\circ$, $5.9^\circ$, $4.7^\circ$, respectively. The larger the laser intensity $a$ is, the closer the radiation distribution is to the laser propagation direction and the narrower the corresponding polar angle range becomes. Moreover, we will take the 1000th harmonic radiation and laser intensity $a = 1$ to study the dependencies on other parameters in the following.

Upon the dependence of angular distribution of the radiation power on the resonance parameter $n$ and the initial axial momentum $p_{z0}$ of the electron are also researched. In Fig.4(c), the maximum radiation power values are 12.7063, 205.1229, 785.0971, 2007.8381 and the corresponding polar angles are $24.9^\circ$, $16.1^\circ$, $7.5^\circ$, $1.6^\circ$, respectively. From Fig.4(d), we can see that the maximum radiation power values are 552.7177, 1043.4371, 1736.1382, 8853.4894 and the corresponding polar angles are $7.6^\circ$, $5.9^\circ$, $4.6^\circ$, $2^\circ$, respectively. For Fig.4(c)-(d), it is found that the radiation power increases and the polar angle decreases with the increasing of resonance parameter $n$ or initial axial momentum $p_{z0}$. The corresponding polar angle $\theta$ of the radiation power gradually closes to the laser propagation direction. In Fig.4(c), the larger resonance parameter $n$ is, the narrower the corresponding polar angle range become. However, the corresponding polar angle range of high radiation power are always narrow from Fig.4(d). Specially, the radiation distribution is well collimated along the laser-propagation direction (+$z$) when $p_{z0} = 5$.

The radiation’s frequency, the corresponding radiation energy and photon number per unit solid angle per unit time for several sets of parameters are shown in Table I. The influence of the laser intensity, the resonance parameter and the initial axial momentum on their values is considered. The radiation’s frequency $\omega/\omega_0 = 1.650 \times 10^2$ and polar angle $\theta_{max} = 19.1^\circ$ can be obtained, the maximum radiation energy per unit solid angle per unit time is about 106.79 (normalized by $e^2\omega_0^2/(4\pi^2c)$) and the number of radiated photons is about 0.6470 which is normalized by $e^2\omega_0^2/(4\pi^2c)\omega_0 = e^2\omega_0/4\pi^2c$ \approx $4 \times 10^{11}$ when $m = 1000$, $a = 1$, $n = 4$, $p_{z0} = 0$. Therefore, $10^{11}$ photons can be emitted per unit solid angle per second in this situation. From Table I, one see that the $(d^2I/d\Omega dt)_{max}$ and $N_{\text{photon}}/d\Omega dt$ are both increasing obviously as laser intensity $a$ increases. For the fundamental frequency, it is affected greatly by changing the resonance parameter $n$, see the 2nd line and other lines as comparison. This situation can also be explained by using the analytical expression of
FIG. 4: (colour online) Spatial distributions of the NTS with respect to the polar angle $\theta$, for the different $m$, $a$, $n$ and $p_{z0}$, are shown in (a), (b), (c) and (d), respectively. (a) $a = 1$, $n = 4$, $p_{z0} = 0$; $m = 500, 1000, 1500, 2000$; (b) $m = 1000$, $n = 4$, $p_{z0} = 0$, $a = 1.5, 2, 3, 4$; (c) $m = 1000$, $a = 1$, $p_{z0} = 0$, $n = 3, 5, 10, 40$; (d) $m = 1000$, $a = 1$, $n = 4$, $p_{z0} = 1, 1.5, 2, 5$.

TABLE I: The frequency $\omega$ (normalized by $\omega_0$), corresponding radiation energy (normalized by $e^2\omega_0^2/4\pi^2c$) and photon number (normalized by $e^2\omega_0/4\pi^2ch$) per unit solid angle per unit time for several sets of parameters. The results show the influences of the laser intensity, the resonance parameter and the initial axial momentum on $\theta_{\text{max}}$, $\omega_1$, $(d^2I/d\Omega dt)_{\text{max}}$ and $N_{\text{photon}}/d\Omega dt$ when $m = 1000$, respectively.

| $m$  | $a$ | $n$ | $p_{z0}$ | $\theta_{\text{max}}$ (°) | $\omega_1$ | $\omega = m\omega_1$ | $(d^2I/d\Omega dt)_{\text{max}}$ | $N_{\text{photon}}/d\Omega dt$ |
|------|-----|-----|----------|-----------------|----------|-----------------|-----------------|-----------------|
| 1000 | 1   | 4   | 0        | 19.1            | 0.1650   | $1.650 \times 10^2$ | $1.0679 \times 10^2$ | 0.6470           |
| 1000 | 1   | 5   | 0        | 16.1            | 0.1275   | $1.275 \times 10^2$ | $2.0512 \times 10^2$ | 1.6089           |
| 1000 | 1   | 4   | 1        | 7.6             | 0.1667   | $1.667 \times 10^2$ | $5.5272 \times 10^2$ | 3.3156           |
| 1000 | 2   | 4   | 0        | 9.4             | 0.1665   | $1.665 \times 10^2$ | $9.2219 \times 10^2$ | 5.5404           |
\( \omega_1 \). Finally, the polar angle of the maximal radiation power is seriously influenced by \( p_{z0} \), see the 3rd line and other lines as comparison. In fact the polar angle when \( p_{z0} = 1 \) is rather smaller than that when \( p_{z0} = 0 \) even if the other parameters are fixed, see the 1st and 3rd lines in Table I.

To sum up, the angular distributions with respect to the polar angle \( \theta \) is mainly distributed in the laser propagation direction. Under the condition that the radiation reaction effect (RRE) can be ignored, the radiation power increases as the laser intensity \( a \), the initial axial momentum \( p_{z0} \) and the resonance parameters \( n \) increase, and it increases most obviously as \( a \) increases. In particular, if the electron moves fast enough along the \(+z\) axis initially, the radiation can be focused on the forward. It is possible that the radiation can be collimated in the forward direction if an appropriate set of parameters are chosen.

C. The optimum observation direction

From results and discussion mentioned above, we can know that the radiation power with respect to the polar angle \( \theta \) is mainly distributed in the forward direction when \( \varphi = 0 \). The maximum radiation power \( (d^2I/d\Omega dt)_{\text{max}} \), the corresponding polar angle \( \theta_{\text{max}} \) and photon number can be obtained. In this subsection, the optimum observation direction are investigated, which is expressed by the polar angle \( \theta_{\text{max}} \) and the azimuthal angle \( \varphi_{\text{max}} \). The photon number and radiated power are retrieved, the photons brightness is also obtained.

From Fig.5, the relevant parameters are chosen as (a) \( m = 1000, a = 1, n = 5, p_{z0} = 0 \); (b) \( m = 1000, a = 4, n = 4, p_{z0} = 0 \); (c) \( m = 1000, a = 1, n = 40, p_{z0} = 0 \); (d) \( m = 1000, a = 1, n = 10, p_{z0} = 10 \).

For elliptically polarized laser field, we have \( a = \sqrt{I_0[W/cm^2] \lambda[\mu m] / (1.38 \times 1.25)} \). So, we can estimate the laser intensity as \( I_0 = [(1.38 \times 1.25)a^2/\lambda[\mu m]^2] \times 10^{18} W/cm^2 \approx 1.7 \times 10^{18} W/cm^2 \) when \( a = 1, \lambda = 1 \mu m \). Thus, it is worth noting that the RRE can be ignored when the laser field is weak. Otherwise, the radiation of the electron is very complex and the RRE should be taken into account. Based on the approximation of the external magnetic field \( B_0 \) in Sec II, \( B_0 \approx (1 + 1/10) \times 100MG = 110MG \) when \( n = 10, p_{z0} = 0 \). However, the external magnetic field can be reduced to \( B_0 \approx [(1 + 1/10)/2 \times 20] \times 100MG = 5.5MG \) if the the initial axial momentum \( p_{z0} = 10 \) when the resonance parameter keeps still as \( n = 10 \). It indicates that by increasing the initial axial momentum \( p_{z0} \) of the electron, the
FIG. 5: (colour online) Contour of the spatial distributions of the NTS with respect to the polar angle $\theta$ and the azimuthal angle $\varphi$. (a) $m = 1000$, $a = 1$, $n = 5$, $p_z = 0$; (b) $m = 1000$, $a = 4$, $n = 4$, $p_z = 0$; (c) $m = 1000$, $a = 1$, $n = 40$, $p_z = 0$; (d) $m = 1000$, $a = 1$, $n = 10$, $p_z = 10$.

The intensity of the magnetic field can be reduced by nearly 20 times, which can be achieved in the experiments now. From Fig. 5, the position and size of bright spots change with the change of parameters are discovered first. For the azimuthal angle $\varphi$, the bright spots of Fig. 5(c) are concentrated in the range $180^\circ$ to $360^\circ$ and other figures are concentrated in the range $0^\circ$ to $180^\circ$. This angular distribution is different from that $xy$ plane which is perpendicular to laser direction. Furthermore, they are all symmetric with respect to $y$ axis and the results are consistent with previous studies. For the polar angle $\theta$, they are all in the forward direction. Since the values in the other ranges are small, we focus only on the forward part. It can be see from the pictures, the radiation scattered by electron can be well collimated in the forward ($\theta_{max} \leq 5$ degrees) as long as the laser intensity and magnetic field strength are satisfied.
TABLE II: The frequency $\omega$ (normalized by $\omega_0$), corresponding radiation energy (normalized by $e^2\omega_0^2/4\pi^2 c$) and photon number (normalized by $e^2\omega_0/4\pi^2 ch$) per unit solid angle per unit time in the optimum observation direction. They are compared with the previous results for the azimuthal angle $\phi = 0$.

| m   | a  | n  | $p_{z0}$ | $\theta_{max}$ (°) | $\varphi$ (°) | $\omega_1$ | $\omega = m\omega_1$ | $(d^2I/d\Omega dt)_{max}$ | $N_{\text{photon}}/d\Omega dt$ |
|-----|----|----|----------|-------------------|-------------|-----------|-------------------|----------------------------|----------------------------|
| 1000 | 1  | 5  | 0        | 16.1              | 0           | 0.1275    | 1.275 x 10^2      | 2.0512 x 10^2               | 1.6089 x 10^6               |
| 1000 | 4  | 4  | 0        | 4.7               | 0           | 0.1664    | 1.664 x 10^2      | 3.8667 x 10^3               | 2.3244 x 10^1               |
| 1000 | 1  | 40 | 0        | 1.6               | 0           | 0.0185    | 1.850 x 10^1      | 2.0078 x 10^3               | 1.0861 x 10^2               |
| 1000 | 1  | 10 | 10       | 0.5               | 0           | 0.0531    | 5.312 x 10^1      | 5.3319 x 10^4               | 1.0038 x 10^3               |

Comparisons of our results with previous study and a brief data summary of them are listed in Table II. We can find that $(d^2I/d\Omega dt)_{max}$ and $N_{\text{photon}}/d\Omega dt$ have increased in value, but not in magnitude. So, it can get the optimum observation direction and higher radiation intensity and photon counts. We can clearly see that the azimuthal angle $\varphi_{max}$ corresponding to the maximum radiation power is not zero, the corresponding polar angle $\theta_{max}$ is also changed. Moreover, the data in Table II indicate that $10^{11} - 10^{14}$ photons can be emitted by the electron per second per unit solid angle. The brightness of the beam at the harmonics we adopt is good enough to apply in experiments, and the strong radiation of the NTS can reach the frequency range of soft X-ray with photon energy about hundreds of eV.

Finally, the photons brightness of the optimum observation direction of the nonlinear Thomson scattering in combined fields with a general elliptical polarization can be estimated as follows. It can be defined as the phase space density of the photon flux, which is given
by (refer to Ref. [3] and Ref. [44])

\[ B \left[ \text{photons} / \text{s mm}^2 \text{ mrad}^2 \right] \approx 1.40 \times 10^9 N_e \frac{\lambda_0 [\mu m]}{r_0^4 [\mu m]} \left( \frac{\Delta \omega}{\omega} \right) \frac{d^2 I}{\omega^2 d\Omega dt}. \]  

(21)

where \( N_e = n_e \sigma_0 L_p \) is the electron number for a plasma interacting with the laser pulse, \( n_e \) is the electron density, \( \sigma_0 = \pi r_0^2 / 2 \) is the laser cross section and \( L_p = 2 \pi r_0^2 / \lambda_0 \) is the laser-electron interaction distance, \( r_0 \) is the laser spot size. Here, we remember that the laser wavelength \( \lambda_0 = 1 \mu m \), the bandwidth (BW) \( \Delta \omega / \omega \approx 0.1\% \), the electron density \( n_e = 10^{20} \text{ cm}^{-3} \) and it is neglected for the space-charge potential aroused from the high-density plasmas. Now we see that the photon brightness of the high-order harmonics \( m = 1000 \) of the optimum observation direction in Table II are about \( 2.46 \times 10^{13}, 3.08 \times 10^{14}, 1.10 \times 10^{16}, 9.62 \times 10^{16} \) (photons / s mm\(^2\) mrad\(^2\) 0.1\%BW), respectively. Obviously the maximum photon brightness of the high-order harmonics is larger than that of the low order harmonics in Ref. [44].

IV. CONCLUSION AND DISCUSSION

In a summary, in this study, the angular distributions of NTS are researched in detail by an electron moving in combined elliptically polarized laser and magnetic fields. Firstly, the angular distribution with respect to the azimuthal angle \( \varphi \) when the polar angle \( \theta = 90^\circ \) presents interesting shape which is strictly symmetric. They are symmetric about the \( y \) axis whatever the laser intensity, the order of harmonics, the resonance parameter, and the initial axial momentum are. And the radiation power is stronger when the azimuthal angle is in the range of \( 180^\circ \sim 360^\circ \). Furthermore, the radiated power per solid angle does not decrease monotonously with the increase of laser intensity or the resonance parameter, the maximum radiation power can be obtained by choosing appropriate parameters. However, it decreases appreciably and gradually uniformly distributed in the \( xy \) plane as the initial momentum increases. Nextly, the angular distribution of radiation with respect to the polar angle \( \theta \) when the azimuthal angle \( \varphi = 0^\circ \) is well collimated in the forward direction. The larger the resonance parameter \( n \), the laser intensity \( a \) and the initial axial momentum \( p_{z0} \) are, the closer the radiation distributes to the laser-propagation direction and the larger the maximum radiation power will be.

Accordingly, it is possible that the radiation can be concentrated in the forward direction
if an appropriate set of parameters is chosen. We find that the angular distribution with respect to the azimuthal angle $\varphi$ when the polar angle $\theta = 90^\circ$ and the angular distribution with respect to the polar angle $\theta$ when the azimuthal angle $\varphi = 0^\circ$ is different. Finally, the best angle of observation $\theta_{\text{max}}$ and $\varphi_{\text{max}}$ can be accurately figured out and the high frequency part of the radiation can reach the range of XUV and X-ray. In the best observation angles, the number of photons radiated by electron is approaching $10^{11} - 10^{14}$ and its photons brightness can reach the magnitude $10^{17}$ (photons / s mm$^2$ mrad$^2$ 0.1%BW). Photons of different frequencies can be radiated along different directions by the electron of NTS.

While the fixed ellipticity is used for the second-stage numerical study, we believe that the similar treatment can be extended to case of an arbitrary elliptical polarization laser field since the analytical expressions of momenta and displacement of electrons in the combined fields are got in the first-stage theoretical description. Meanwhile, the conclusions about azimuthal angle symmetry and polar angle dependence of radiation spectrum which are mentioned above in this paper are general. Certainly the dependence of studied problem on the ellipticity is worthy to study for the sake of completeness, which beyond the scope of this paper. Importantly, for practical applications, we think it is particularly important to choose the parameter to produce the photons with specified frequencies that we need. It can be used as a reference to help the experiment researcher to obtain high quality high intensity radiation at the optimal spatial distribution.

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[1] P. Maine, D. Strickland, P. Bado, M. Pessot, G. Mourou, IEEE J. Quantum Electron 24, 398 (1988).
[2] P. Sprangle, E. Esarey, A. Fisher, J. Appl. Phys. 72, 5032 (1992).
[3] E. Esarey, S. K. Ride, P. Sprangle, Phys. Rev. E 48, 3003 (1993).
[4] K. J. Kim, S. Chattopadhay, C. V. Shank, Nucl. Instrum. Methods Phys. Res. Sect. A 341, 351 (1994).
[5] H. Kotaki, M. Kando, H. Dewa, et al., Nucl. Instrum. Methods Phys. Res. Sect. A 455, 166 (2000).
[6] W. J. Brown, F V. Hartemann, Phys. Rev. ST AB 7, 060703 (2004).
[7] S. G. Anderson, C. P. J. Barty, S. M. Betts, W. J. Brown, J. K. Crane, Appl. Phys. B: Lasers Opt, 78, 891 (2004).
[8] P. Tomassini, A. Giuliani, D. Giuliani, L. A. Gizzi, Appl. Phys. B: Lasers Opt, 80, 419 (2005).
[9] H. Schwoerer, B. Liesfeld, H. P. Schlenvoigt, K. U. Anthon, R. Sauerbrey, Phys. Rev. Lett. 96, 014802 (2006).
[10] Y. Taira, T. Hayakawa, M. Katoh, Sci. Rep. 7, 5018 (2017).
[11] Y. Taira, M. Katoh, Phys. Rev. A 98, 052130 (2018).
[12] Y. Y. Chen, K. Z. Hatsagortsyan, C. H. Keitel Matter Radiat. Extrem. 4, 024401 (2019).
[13] P. F. Lan, P. X. Lu, W. Cao, Phys. Plasmas 13, 013106 (2006).
[14] P. F. Lan, P. X. Lu, W. Cao, X. L. Wang, Phys. Rev. E 72, 066501 (2005).
[15] p. Venkat, A R. Holkundkar, Phys. Rev. Accel. Beams 22, 084401 (2019).
[16] Y. Taira, M. Katoh, ApJ 860, 45 (2018).
[17] S. Chen, D. Umstadter Nature 396, 653 (1998).
[18] P. F. Lan, P. X. Lu, W. Cao, Plasma Sci. Technol. 9, 143 (2007).
[19] O. E. Vais, S. G. Bochkarev, V. Y. Bychenkov, Plasma Phys. Rep. 42, 818 (2016).
[20] Q. Y. Zhang, Y. W. Tian, Optik 137, 262 (2017).
[21] K. Li, L. X. Li, Q. Shu, Y. W. Tian, Y. Y. Shi, Z. X. Zhang, Optik 183, 813 (2019).
[22] P. H. Yu, H. N. Liu, Z. Y. Gu, Y. W. Tian, Laser Phys. 30, 045301 (2020).
[23] M. Ruijter, V. Petrillo, T C. Teter, M. Valialshchikov, S. Rykovonov, Crystals 11, 528 (2021).
[24] X. K. He, B. Shuai, X. C. Ge, R. X. Li, Z. Z. Xu, Phys. Rev. E 68, 056501 (2003).
[25] M. Boca, A. Oprea, Phys. Scr. 83, 055404 (2011).
[26] Y. W. Tian, D. X. Yue, E. Feng, M. Zhao, J. Tian, Q. Y. Zhang, Opt Quant Electron 45, 1125 (2013).
[27] L. Zhao, Z. J. Chen, H. B. Sang, B. S. Xie, Chinese Phys. Lett. 36, 074101 (2019).
[28] Y. Wu, Y. H. Liu, D. Liu, Y. W. Tian, Laser Phys. 30, 115301 (2020).
[29] Y. Q. Wang, C. Wang, K. Li, L. X. Li, Y. W. Tian, Laser Phys. Lett. 18, 015303 (2021).
[30] Y. Q. Wang, C. Wang, Q. Y. Zhou, L. X. Li, Y. W. Tian, Laser Phys. 31, 015301 (2021).
[31] K. Liu, T. Yu, D. Zou, X. R. Xu, Y. Yin, F. Q. Shao, Eur. Phys. J. D 74, 7 (2020).
[32] Z. J. Chen, H. Qin, X. Chen, Y. W. Tian, Laser Phys. 31, 075401 (2021).
[33] Y. Q. Wang, C. Wang, K. Li, L. X. Li, Y. W. Tian, Opt Quant Electron 53, 229 (2021).
[34] Y. X. Shi, Z. Xu, J. Y. Wang, Z. H. Huang, H. Liu, Y. W. Tian, Laser Phys. 32, 015401 (2022).
[35] Y. X. Shi, J. Y. Wang, B. G. Wu, Z. Xu, H. Liu, Y. W. Tian, Laser Phys. 31, 105401 (2021).
[36] C. Fruhling, J. Wang, D. Umstadter, C. Schulzke, M. Romero, M. Ware, J. Peatross, Phys. Rev. A 104, 053519 (2021).
[37] Y. J. Fu, C. Jiang, C. Lv, F. Wan, H. B. Sang, B. S. Xie, Phys. Rev. A 94, 052102 (2016).
[38] C. Jiang, H. Z. Xie, H. B. Sang, B. S. Xie, EPL 117, 44002 (2017).
[39] J. L. Zhu, B. S. Xie, EPL 126, 34001 (2019).
[40] Z. J. Chen, L. Zhao, C. Jiang, H. B. Sang, B. S. Xie, EPL 125, 64002 (2019).
[41] J. Li, B. S. Xie, H. B. Sang, X. R. Hong, S. Zhang, M. Y. Yu, Appl. Phys. Lett. 95, 161105 (2009).
[42] J. D. Jackson, Classical Electrodynamics, Wiley, New York (1975).
[43] F. He, Y. Y. Lau, D. P. Umstadter, T. Strickler, Phys. Plasmas 9, 4325 (2002).
[44] X. R. Hong, Y. N. Li, D. Wei, R. A. Tang, J. A. Sun, W. S. Duan, Phys. Plasmas 29, 043102 (2022).