Numerical differentiation on the Bakhvalov mesh in the presence of an exponential boundary layer

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Abstract. The question of the application of formulas of the numerical differentiation of functions in the presence of the exponential boundary layer is investigated. The problem is that the application of classical formulas, which are based on the differentiation of the Lagrange polynomial on the uniform mesh in this case leads to significant errors. It is proposed to study the formulas for derivatives on the Bakhvalov mesh, which is widely used in the construction of difference schemes for singularly perturbed problems. It is proved that applying of classical difference formulas for derivatives on a Bakhvalov mesh have error estimate that is uniform with respect to a small parameter. The results of numerical experiments are presented.

1. Introduction
Various convective-diffusion processes with prevailing convection are modeled, basing on singularly perturbed problems. The use of classical difference schemes on a uniform mesh for the numerical solution of such problems leads to significant errors if a small parameter is commensurate with the mesh step [1], [2]. To achieve uniform convergence of difference schemes, the mesh dense in the boundary layer is widely used [1], [3]. The development of formulas of numerical differentiation for functions having large gradients in the boundary layers is of interest. The application of classical numerical differentiation formulas based on Lagrange polynomials to functions with large gradients in the boundary layer can lead to significant errors. It was shown in [4]. To ensure that the error of the formula for numerical differentiation does not depend on large gradients of the function in boundary layer, in [4, 5, 6] an approach based on fitting the formula to singular component responsible for large gradients of the function in the boundary layer.

In [7], it was proved that numerical differentiation formulas based on Lagrange polynomials can be applied to functions with large gradients on Shishkin mesh [1], since the error estimate is uniform in a small parameter.

A review of approaches for constructing numerical differentiation formulas for functions with large gradients is given in [8]. The work [9] shows how the numerical differentiation formula that is exact on a singular component can be constructed based on differentiation of an exponential spline. Then the derivative is found as a continuously differentiable function.

In this paper, we show that the classical difference formulas for calculation of the first and second derivatives can be successfully applied on the Bakhvalov mesh [3].

So, we assume that for the function $u(x)$ for a given $m$ the following decomposition holds

$$u(x) = p(x) + \Phi(x), \quad x \in [0, 1],$$

(1)
where

\[ |p^{(j)}(x)| \leq C_1, \quad |\Phi^{(j)}(x)| \leq \frac{C_1}{\varepsilon^j} e^{-\alpha x / \varepsilon}, \quad 0 \leq j \leq m, \tag{2} \]

functions \( p(x) \) and \( \Phi(x) \) are not explicitly defined, \( \alpha > 0, \varepsilon > 0 \). Coefficient \( \alpha \) is separated from zero, parameter \( \varepsilon \) may be close to zero. According to (2), the regular component \( p(x) \) has derivatives, bounded to a given order, and derivatives of the singular component \( \Phi(x) \) are not uniformly bounded with respect to the parameter \( \varepsilon \in (0, 1] \).

According to [1, 10], for a given \( m \) one can construct the decomposition (1) with constraints (2) for the solution of a singularly perturbed boundary value problem:

\[ \varepsilon u''(x) + a_1(x)u'(x) - a_2(x)u(x) = f(x), \quad u(0) = A, \quad u(1) = B, \tag{3} \]

where \( a_1(x) \geq \alpha > 0, \quad a_2(x) \geq 0, \quad \varepsilon > 0, \quad \) functions \( a_1(x), a_2(x), f(x) \) are smooth enough. For small \( \varepsilon \) the solution of the problem (3) has the region of large gradients at the boundary \( x = 0 \).

Throughout the paper, by \( C \) and \( C_j \) we mean positive constants independent of the parameter \( \varepsilon \) and the number of grid nodes \( N \). We will restrict various values to one constant \( C_j \) if it is clear from the text.

2. The first derivative calculating error

Let \( \Omega^h \) be the mesh on the interval \([0, 1]\):

\[ \Omega^h = \{ x_n, n = 0, 1, \ldots, N, \quad x_n = x_{n-1} + h_n, \quad x_0 = 0, \quad x_N = 1 \}. \]

Let \( L_k(u, x) \) be a Lagrange polynomial for the function \( u(x) \) with \( k \) interpolation nodes.

Consider the classic formula for derivative calculating,

\[ u'(x) \approx L'_2(u, x) = \frac{u_n - u_{n-1}}{h}, \quad x \in [x_{n-1}, x_n]. \tag{4} \]

Let us dwell on the case of a uniform mesh with step \( h \). Consider a function of the form (1), \( u(x) = e^{-x/\varepsilon} \) on the interval \([0, 1]\). Then for \( \varepsilon = h \)

\[ \varepsilon \left| \frac{u(h) - u(0)}{h} - u'(0) \right| = e^{-1}. \]

Thus, in the case of a uniform grid, the relative error of the formula (4) is not \( \varepsilon \)-uniform, for \( \varepsilon = h \) the order of accuracy error is \( O(1) \). Accuracy does not increase with decreasing \( h \) if \( \varepsilon = h \).

In the case of the Shishkin mesh in [7] \( \varepsilon \)-uniform error estimation:

\[ \varepsilon \left| \frac{u_n - u_{n-1}}{h} - u'(x) \right| \leq C \frac{\ln N}{N}, \quad x \in [x_{n-1}, x_n], \quad 1 \leq n \leq N. \]

was obtained. Let’s move on to error estimation formula (4) on the Bakhvalov mesh [3].

Define the Bakhvalov mesh \( \Omega^h \) with nodes \( x_n = g(n/N), \quad n = 0, 1, \ldots, N \), where the function \( g(t), \quad t \in [0, 1] \) is defined as follows:

\[ g(t) = -\frac{r \varepsilon}{\alpha} \ln \left[ 1 - 2(1 - \varepsilon) t \right], \quad 0 \leq t \leq \frac{1}{2}, \quad \varepsilon \leq e^{-1}, \tag{5} \]

\[ g(t) = \sigma + (2t - 1)(1 - \sigma), \quad 1/2 \leq t \leq 1. \tag{6} \]

where \( r \) is positive and integer number.
Set the boundary layer region $[0, \sigma]$ in accordance with (5),

$$\sigma = g(1/2) = -\frac{r_\varepsilon}{\alpha} \ln \varepsilon. \quad (7)$$

Define the Bakhvalov mesh $\Omega^h$ in accordance with the relations (5)–(7) under the condition $\varepsilon \leq e^{-1}$. For $\varepsilon > e^{-1}$ we set the uniform mesh $\Omega^h$.

Given (5), we obtain

$$x_n = -\frac{r_\varepsilon}{\alpha} \ln \left[1 - 2(1 - \varepsilon)n/N\right], \ n = 0, 1, \ldots, \frac{N}{2}. \quad (8)$$

Consequently,

$$h_n = \frac{r_\varepsilon}{\alpha} \ln \left[1 + \frac{2(1 - \varepsilon)/N}{1 - 2(1 - \varepsilon)n/N}\right], \ n = 1, 2, \ldots, \frac{N}{2}. \quad (9)$$

It is easy to verify that the sequence of steps $h_n, \ n = 1, 2, \ldots, N/2$ is strictly increasing. From (9) follows

$$h_{N/2} = \frac{r_\varepsilon}{\alpha} \ln \left[1 + \frac{2(1 - \varepsilon)}{N\varepsilon}\right].$$

Therefore, for some constant $C_2$ the estimate is true:

$$h_n \leq \frac{C_2}{N}, \ n = 1, 2, \ldots, N. \quad (10)$$

**Lemma 1** Let the function $u(x)$ have the representation (1) for $m \geq 2$, in (5) $r \geq 2$. Then for some constant $C$ error estimates are valid

$$\varepsilon |L_2'(u, x) - u'(x)| \leq \frac{C}{N}, \ x \in [x_{n-1}, x_n], \ n \leq \frac{N}{2}, \quad (11)$$

$$|L_2(u, x) - u'(x)| \leq \frac{C}{N}, \ x \in [x_{n-1}, x_n], \ n > \frac{N}{2}. \quad (12)$$

**Proof.** We use the well-known error estimate

$$|L_2'(u, x) - u'(x)| \leq \int_{x_{n-1}}^{x_n} |u''(s)| \, ds. \quad (13)$$

Let us dwell on the case $n \leq N/2$. Given (2) and (8), from (13) we get

$$\varepsilon |L_2'(\Phi, x) - \Phi'(x)| \leq C \left[e^{-\alpha x_{n-1}/\varepsilon} - e^{-\alpha x_n/\varepsilon}\right] = C(b_n^{\varepsilon} - a_n^{\varepsilon}) \leq Cb_n(b_n - a_n) \leq \frac{C}{N},$$

where

$$b_n^{\varepsilon} = e^{-\alpha x_{n-1}/\varepsilon}, \ a_n^{\varepsilon} = e^{-\alpha x_n/\varepsilon}, \ 1 > b_n > a_n > 0.$$

So, for some constant $C$

$$\varepsilon |L_2'(\Phi, x) - \Phi'(x)| \leq \frac{C}{N}, \ x \in [x_{n-1}, x_n], \ 1 \leq n \leq \frac{N}{2}. \quad (14)$$

Considering (2), (10), (13), for some constant $C$ we get

$$|L_2'(p, x) - p'(x)| \leq \frac{C}{N}, \ x \in [x_{n-1}, x_n], \ 1 \leq n \leq N. \quad (15)$$

From (14), (15) it follows (11).

We dwell on the case $n > N/2$. For $x \geq \sigma$ according to (2), (7) the estimation $|\Phi'(x)| \leq C_1$ holds. Then according to (1), (2), (13) the estimation (12) is valid. The lemma is proved.

So, in case of Bakhvalov mesh in the region of boundary layer, the relative error estimate (11) is true for the formula (4), and outside of the boundary layer region the absolute error estimate (12) is true.
3. The error in the second derivative calculating

Let \( L_3(u, x) \) be Lagrange polynomial, interpolating the function \( u(x) \) with interpolation nodes \( x_{n-1}, x_n, x_{n+1} \) on the interval \([x_{n-1}, x_{n+1}]\). Then the difference formula for the second derivative on this interval has the form

\[ u''(x) \approx L_3''(u, x) = \frac{2u_{n-1}}{h_n(h_n + h_{n+1})} - \frac{2u_n}{h_n h_{n+1}} + \frac{2u_{n+1}}{h_{n+1}(h_n + h_{n+1})}, \quad x \in [x_{n-1}, x_{n+1}]. \] (16)

**Lemma 2** Let the function \( u(x) \) have the representation (1) for \( m \geq 3 \), in (5) \( r \geq 3 \) and \( N \) is a multiple of four. Then for some constant \( C \) error estimates hold

\[ \varepsilon^2 |L_3''(u, x) - u''(x)| \leq \frac{C}{N}, \quad x \in [x_{n-1}, x_{n+1}], \quad n < \frac{N}{2}, \] (17)

and

\[ \varepsilon^2 |L_3''(u, x) - u''(x)| \leq \frac{C}{N}, \quad x \in [x_{n-1}, x_{n+1}], \quad n > \frac{N}{2}. \] (18)

**Proof.** If \( N \) is a multiple of four, then each interval \([x_{n-1}, x_{n+1}]\) completely in the boundary layer region \([0, \sigma]\) or from this region. Given (16), easy to get

\[ \varepsilon^2 \left(L_3''(u, x) - u''(x)\right) = \frac{\varepsilon^2}{h_n h_{n+1}(h_n + h_{n+1})} \times \]

\[ \left[2h_{n+1}u_{n-1} - 2(h_n + h_{n+1})u_n + 2h_n u_{n+1} - h_n h_{n+1}(h_n + h_{n+1})u''_n\right] + \varepsilon^2(u''_n - u''(x)). \] (19)

Now we use the Taylor series expansion with the remainder term in the integral form

\[ u(x) = u(x_0) + (x - x_0)u'(x_0) + \frac{1}{2}(x - x_0)^2u''(x_0) + \frac{1}{2} \int_{x_0}^{x} (x - s)^2u''(s) \, ds \]

and get

\[ u_{n-1} = u_n - h_n u'_n + \frac{1}{2} h_n^2 u''_n + \frac{1}{2} \int_{x_{n-1}}^{x_n} (x_n - s)^2u''(s) \, ds, \] (20)

\[ u_{n+1} = u_n + h_{n+1} u'_n + \frac{1}{2} h_{n+1}^2 u''_n + \frac{1}{2} \int_{x_{n+1}}^{x_n} (x_n - s)^2u''(s) \, ds. \] (21)

Substituting (20), (21) into (19), we get

\[ \varepsilon^2 \left(L_3''(u, x) - u''(x)\right) = \frac{\varepsilon^2}{h_n h_{n+1}(h_n + h_{n+1})} \times \]

\[ \left[h_{n+1} \int_{x_{n-1}}^{x_n} (x_n - s)^2u''(s) \, ds - h_n \int_{x_n}^{x_{n+1}} (x_n - s)^2u''(s) \, ds\right] + \varepsilon^2 \int_{x}^{x_n} u''(s) \, ds. \] (22)

From (22) it follows

\[ \varepsilon^2 |L_3''(u, x) - u''(x)| \leq 2\varepsilon^2 \int_{x_{n-1}}^{x_{n+1}} |u''(s)| \, ds. \] (23)
For \( m = 3 \) in decomposition (1) the derivative \( p'''(x) \) is bounded, therefore, in accordance with (10), (23) for some constant \( C \)

\[
\left| L_3''(p, x) - p''(x) \right| \leq \frac{C}{N}, \ x \in [x_{n-1}, x_{n+1}], \ n = 1, 3, \ldots, N - 1. \quad (24)
\]

Let us dwell on the estimation of the error on the component \( \Phi(x) \).

Consider the case \( n < N/2 \). Given estimates (2), (23), we get

\[
\varepsilon^2 \left| L_3''(\Phi, x) - \Phi''(x) \right| \leq \frac{2C_1}{\varepsilon} \int_{x_{n-1}}^{x_{n+1}} e^{-\alpha s/\varepsilon} ds = \frac{2C_1}{\alpha} \left[ e^{-\alpha x_{n-1}/\varepsilon} - e^{-\alpha x_{n+1}/\varepsilon} \right]. \quad (25)
\]

It is easy to show that for \( 1 \geq b > a > 0 \) the inequality is true:

\[
b^r - a^r < b^r - 1(b-a)r. \]

Then from (25) for some constant \( C_3 \) we get

\[
\varepsilon^2 \left| L_3''(\Phi, x) - \Phi''(x) \right| \leq \frac{C_3}{N}, \ x \in [x_{n-1}, x_{n+1}], \ n = 1, 3, \ldots, \frac{N}{2} - 1. \quad (26)
\]

Consider the case \( n > N/2 \). Then \( x_{n-1} \geq \sigma \) and, taking into account (2), (7), from (23) we get

\[
\left| L_3''(\Phi, x) - \Phi''(x) \right| \leq \frac{C}{N}, \ x \in [x_{n-1}, x_{n+1}], n = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N - 1. \quad (27)
\]

From the estimates (24), (26), (27) we get estimates (17), (18). Lemma is proved.

4. Results of numerical experiments

Let us numerically compare the accuracy of the formula (16) for second derivative of the function of the form (1) using the uniform mesh, Shishkin mesh and Bakhvalov mesh.

Consider the Shishkin mesh [1]. We define the grid steps on the basis of ratios:

\[
\sigma = \min\left\{ \frac{1}{2}, \frac{3\varepsilon}{\alpha \ln N} \right\}, \ h_n = \frac{2\sigma}{N}, \ n \leq \frac{N}{2}; \ h_n = \frac{2(1-\sigma)}{N}, \ n > \frac{N}{2}. \quad (28)
\]

According to [7], in the case of a function of the form (1) and Shishkin mesh (28) for some constant \( C \)

\[
\varepsilon^2 |u''(x) - L_3''(u, x)| \leq C \frac{\ln N}{N}, \ x \in [0, 1]. \quad (29)
\]

Comparing the estimates (17)-(18) and (29) we get that in the case of the Bakhvalov mesh, the error in the second derivative calculating is smaller compared to the Shishkin mesh. Let us confirm this with the results of computational experiments.

We define a function of the form (1)

\[
u(x) = \cos \frac{\pi x}{2} + e^{-x/\varepsilon}, \ x \in [0, 1], \ \varepsilon \in (0, 1).
\]

Then in (1) \( \Phi(x) = e^{-x/\varepsilon} \).
Tables 1-3 show the relative error \( \Delta_{N,\varepsilon} \) and computed order of accuracy

\[
M_{N,\varepsilon} = \log_2 \frac{\Delta_{N,\varepsilon}}{\Delta_{2N,\varepsilon}}
\]
in cases of uniform mesh, Shishkin mesh and Bakhvalov mesh, where

\[
\Delta_{N,\varepsilon} = \varepsilon^2 \max_{n,j} |L_3'(u, \bar{x}_{n,j}) - u''(\bar{x}_{n,j})|,
\]

where \( \bar{x}_{n,j} \) are nodes of a finer mesh formed by dividing each interval \([x_{n-1}, x_n]\) of the original mesh by 10 equal sub-intervals.

In tables \( \varepsilon - m \) stands for \( 10^{-m} \). From table 1 it follows that the uniform mesh using is unacceptable, for \( \varepsilon = 1/N \) the error does not decrease with the increasing of \( N \). The results of table 2 are consistent with the error estimate (29) in the case of Shishkin mesh. In accordance with table 3, in the case of the Bakhvalov mesh for all relations between \( \varepsilon \) and \( N \) the first order of accuracy corresponding to the estimate (17) is confirmed.

On the meshes of Bakhvalov and Shishkin, the error stabilizes with the decreasing of parameter \( \varepsilon \).

**Table 1.** The error in second derivative calculating on the uniform grid

| \( \varepsilon \) | \( N \) |
|---|---|
| 32 | 64 | 128 | 256 | 512 | 1024 |
| 1 | 1.74e-1 | 8.74e-2 | 4.38e-2 | 2.19e-2 | 1.10e-2 | 5.48e-3 |
| \( 16^{-1} \) | 4.19e-1 | 2.86e-1 | 1.68e-1 | 9.17e-2 | 4.79e-2 | 2.45e-2 |
| \( 32^{-1} \) | 4.83e-1 | 4.19e-1 | 2.86e-1 | 1.68e-1 | 9.17e-2 | 4.79e-2 |
| \( 64^{-1} \) | 3.89e-1 | 4.83e-1 | 2.86e-1 | 1.68e-1 | 9.17e-2 |
| \( 128^{-1} \) | 1.86e-1 | 3.89e-1 | 4.83e-1 | 2.86e-1 | 1.68e-1 |
| \( 256^{-1} \) | 3.69e-2 | 1.86e-1 | 3.89e-1 | 4.83e-1 | 2.86e-1 |
| \( 512^{-1} \) | 9.77e-4 | 3.69e-2 | 1.86e-1 | 3.89e-1 | 4.83e-1 | 4.9e-1 |

**Table 2.** The error and calculated accuracy order for the second derivative computing on the Shishkin mesh

| \( \varepsilon \) | \( N \) |
|---|---|
| 32 | 64 | 128 | 256 | 512 | 1024 |
| 1 | 1.74e-1 | 8.74e-2 | 4.38e-2 | 2.19e-2 | 1.10e-2 | 5.48e-3 |
| \( 16^{-1} \) | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| \( 32^{-1} \) | 3.50e-1 | 2.58e-1 | 1.68e-1 | 9.17e-2 | 4.79e-2 | 2.45e-2 |
| \( 64^{-1} \) | 0.44 | 0.62 | 0.88 | 0.94 | 0.97 | 0.98 |
| \( 128^{-1} \) | 0.44 | 0.57 | 0.67 | 0.74 | 0.79 | 0.83 |
| \( 256^{-1} \) | 0.44 | 0.57 | 0.67 | 0.74 | 0.79 | 0.83 |
| \( 512^{-1} \) | 0.44 | 0.57 | 0.67 | 0.74 | 0.79 | 0.83 |

Now we consider the formula for calculating the first derivative by the function values in three mesh nodes on an arbitrary mesh interval \([x_{n-1}, x_{n+1}]\):

\[
u' (x) \approx L_3' (u, x) = u_{n-1} \frac{2x - x_n - x_{n+1}}{h_n (h_n + h_{n+1})} - u_n \frac{2x - x_{n-1} - x_{n+1}}{h_n h_{n+1}} + u_{n+1} \frac{2x - x_n - x_{n-1}}{h_{n+1} (h_n + h_{n+1})}.
\]
Table 3. The error and calculated accuracy order for the second derivative computing on the Bahvalov mesh

| ε    | 32  | 64  | 128 | 256 | 512 | 1024 |
|------|-----|-----|-----|-----|-----|------|
| 1    | 1.74e-1 | 8.74e-2 | 4.38e-2 | 2.19e-2 | 1.10e-2 | 5.48e-3 |
| 16^{-1} | 2.38e-1 | 1.30e-1 | 6.76e-2 | 3.45e-2 | 1.74e-2 | 8.75e-3 |
| 32^{-1} | 2.48e-1 | 1.36e-1 | 7.08e-2 | 3.62e-2 | 1.83e-2 | 9.18e-3 |
| 64^{-1} | 2.49e-1 | 1.37e-1 | 7.14e-2 | 3.64e-2 | 1.84e-2 | 9.25e-3 |

Table 4. The error for the first derivative computing by three nodes on the uniform mesh

| ε    | 32  | 64  | 128 | 256 | 512 | 1024 |
|------|-----|-----|-----|-----|-----|------|
| 1    | 2.25e-3 | 5.68e-4 | 1.42e-4 | 3.57e-5 | 8.92e-6 | 2.23e-6 |
| 16^{-1} | 6.67e-2 | 2.56e-2 | 8.14e-3 | 2.30e-3 | 6.12e-4 | 1.58e-4 |
| 32^{-1} | 1.26e-1 | 6.67e-2 | 2.56e-2 | 8.14e-3 | 2.30e-3 | 6.12e-4 |
| 64^{-1} | 1.32e-1 | 1.26e-1 | 6.67e-2 | 2.56e-2 | 8.14e-3 | 2.30e-3 |
| 128^{-1} | 1.04e-1 | 1.32e-1 | 6.67e-2 | 2.56e-2 | 8.14e-3 | 2.30e-3 |
| 256^{-1} | 6.74e-2 | 1.04e-1 | 1.32e-1 | 6.67e-2 | 2.56e-2 | 6.12e-4 |
| 512^{-1} | 3.90e-2 | 6.71e-2 | 1.04e-1 | 1.32e-1 | 6.12e-1 | 6.67e-2 |

Table 5. The error and calculated accuracy order for the first derivative computing by three nodes on the Shishkin mesh

| ε    | 32  | 64  | 128 | 256 | 512 | 1024 |
|------|-----|-----|-----|-----|-----|------|
| 1    | 2.25e-3 | 5.68e-4 | 1.42e-4 | 3.57e-5 | 8.92e-6 | 2.23e-6 |
| 16^{-1} | 4.10e-2 | 2.05e-2 | 8.14e-3 | 2.30e-3 | 6.12e-4 | 1.58e-4 |
| 32^{-1} | 4.10e-2 | 2.05e-2 | 8.71e-3 | 3.30e-3 | 1.15e-3 | 3.77e-4 |
| 64^{-1} | 4.10e-2 | 2.05e-2 | 8.71e-3 | 3.30e-3 | 1.15e-3 | 3.77e-4 |

The relative error in the first derivative calculating according to the formula:

$$\Delta_{N,\varepsilon} = \varepsilon \max_{n,j} \left| L_3'(u, \tilde{x}_{n,j}) - u'((\tilde{x}_{n,j})\right|$$

is shown in tables 4-6 on the uniform mesh, Shishkin mesh and Bahvalov mesh, by analogy with the case of the second derivative calculating.
Table 6. The error and calculated accuracy order for the first derivative computing by three nodes on the Bakhvalov mesh

| $\varepsilon$ | 32   | 64   | 128  | 256  | 512  | 1024 |
|---------------|------|------|------|------|------|------|
| 1             | 2.25e-3 | 5.68e-4 | 1.42e-4 | 3.57e-5 | 8.92e-6 | 2.23e-6 |
| 1.99          | 2.00  | 2.00  | 2.00  | 2.00  | 2.00  |
| 16$^{-1}$     | 1.81e-2 | 4.84e-3 | 1.25e-3 | 3.17e-4 | 7.99e-5 | 2.00e-5 |
| 1.90          | 1.98  | 1.99  | 1.99  | 2.00  |
| 32$^{-1}$     | 1.92e-2 | 5.16e-3 | 1.33e-3 | 3.38e-4 | 8.53e-5 | 2.14e-5 |
| 1.90          | 1.98  | 1.99  | 2.00  |
| 64$^{-1}$     | 1.98e-2 | 5.32e-3 | 1.38e-3 | 3.49e-4 | 8.80e-5 | 2.21e-5 |
| 1.90          | 1.98  | 1.99  | 2.00  |
| 128$^{-1}$    | 2.01e-2 | 5.40e-3 | 1.40e-3 | 3.55e-4 | 8.94e-5 | 2.24e-5 |
| 1.89          | 1.95  | 1.98  | 1.99  | 2.00  |
| 256$^{-1}$    | 2.02e-2 | 5.44e-3 | 1.41e-3 | 3.58e-4 | 9.01e-5 | 2.26e-5 |
| 1.89          | 1.95  | 1.98  | 1.99  | 2.00  |

Calculation results show the unacceptability of the uniform mesh using. Shishkin mesh calculation results are consistent with the errors estimate [7]

$$\varepsilon|L_3'(u, x) - u'(x)| \leq C \left( \frac{\ln N}{N} \right)^2, \ x \in [0, 1].$$

In the case of the Bakhvalov mesh, the following error estimate is confirmed

$$\varepsilon|L_3'(u, x) - u'(x)| \leq \frac{C}{N^2}, \ x \in [0, 1].$$

5. Conclusion

Difference formulas for calculating the first and the second derivatives of functions of one variable in the presence of exponential boundary layer are studied. It is proved that the relative error is uniform in a small parameter, i.e. independent of large gradients of the function in the boundary layer, when difference formulas for calculating the first and second derivatives are applied on the Bakhvalov mesh. The results of computational experiments are consistent with obtained error estimates.

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References
[1] Shishkin G I 1992 Discrete Approximations of Singularly Perturbed Elliptic and Parabolic Equations (Ekaterinburg: UB RAS)
[2] Miller J J H, O’Riordan E and Shishkin G I 2012 Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions (Singapore: World Scientific)
[3] Bakhvalov N S 1969 The optimization of methods of solving boundary value problems with a boundary layer USSR Comput. Math. Math. Phys. 9 139-166
[4] Zadorin A I and Zadorin N A 2010 Spline interpolation on a uniform grid for functions with a boundary-layer component Comput. Math. Math. Phys. 50 211-223
[5] Zadorin A I and Zadorin N A 2012 Interpolation formula for functions with a boundary layer component and its application to derivatives calculation Sib. Electron. Math. Rep. 9 445-455
[6] Il’in V P and Zadorin A I 2019 Adaptive formulas of numerical differentiation of functions with large gradients
   *J. Phys.: Conf. Ser.* **1260** 042003

[7] Zadorin A I 2018 Analysis of Numerical Differentiation Formulas in a Boundary Layer on a Shishkin Grid
   *Numerical Analysis and Applications* **11** 193-203

[8] Blatov I A and Zadorin A I 2019 Approaches to the calculation of derivatives of functions with large gradients
   in the boundary layer under the values at the grid nodes *J. Phys.: Conf. Ser.* **1158** 022029

[9] Blatov I A, Zadorin A I and Kitaeva E V 2018 An application of the exponential spline for the approximation
   of a function and its derivatives in the presence of a boundary layer *J. Phys.: Conf. Ser.* **1050** 012012

[10] Linss T 2001 The Necessity of Shishkin Decompositions *Appl. Math. Lett.* **14** 891-896