Remarks on Nöther charges and black holes entropy

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Abstract. We criticize and generalize some properties of Nöther charges presented in a paper by V. Iyer and R. M. Wald and their application to entropy of black holes. The first law of black holes thermodynamics is proven for any gauge-natural field theory. As an application charged Kerr-Newman solutions are considered. As a further example we consider a (1 + 2) black hole solution.

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1. Introduction

An expression for entropy of black holes has recently been proposed in a paper by Iyer and Wald (see [1] and references quoted therein). This paper relates variations of entropy of stationary black holes (then generalized to the non-stationary case) to variations of a suitable conserved quantity.

This idea relies on a well known and by now classical analogy between thermodynamic laws and the evolution law for the horizon area of black holes. However, we were intrigued by the relation established in [1] between entropy and Nöther charges especially in view of a number of unclear questions, which we hope to clarify here. In an appendix we summarize the main results presented in [1], suitably restated with the notation we shall use in the next Sections (see also [2] and [3]).

First of all, we claim that problems about the definition of integrals (i.e. on the choice of representatives for superpotentials) have been only partially faced in [1], where the solutions proposed are insufficiently motivated. We also remark that the results of [1] were achieved in a somewhat “twisted” way. In fact, field equations, as well as the expression for their solutions, were there used also when there was no need of doing so. Moreover, a particular class of Lagrangians has been chosen with the aim of applying the framework to General Relativity; the representatives of this class of Lagrangians depend on fields and their derivatives through involved expressions which make results difficult to be achieved. Finally, most of the properties of Nöther charges used more or less explicitely in [1] have been in fact known for long time in various frameworks, some of which much simpler and more general than the framework introduced in [1] (see, for example, [3], [4], [5], [6], [7], [8] and references quoted therein).

For these reasons, in addition to the interest for applications not only in Relativity but also in much more general frameworks of current physical interest, we believe that the property (10.9) in the appendix is worth being proved in a much more general framework using all the simple but powerful modern mathematical tools which are naturally involved in it (and this will also enlight all problems in the definition of integrals). In particular our attitude towards field theories brings us to use the geometric language of fiber bundles in which variational calculus is naturally formulated.

This framework allows us in fact to clearly distinguish between quantities and properties which hold only along classical solutions (i.e. on-shell) from quantities and properties which hold on general configurations (i.e. off-shell). The reasons for pursuing this distinction are not only mathematical, but they are physical reasons mainly. In fact it is generally accepted that in quantum field theory (QFT) physical contributions may come from configurations which are not classical solutions. Thus, at least at QFT level, two quantities which
differ off-shell may have very different physical meaning even though they are classically identical (i.e. they coincide on-shell).

Moreover, the needless use of fields equations may break down properties which hold off-shell (e.g. see below for strong conservation laws) or hide properties which instead hold only on-shell. As an example, we can remark that global superpotentials, which always exist in natural theories as General Relativity, show that conserved currents are not only closed forms on-shell, but they are also exact forms on-shell (regardless of the topology of space-time $M$). Well, this is a relatively simple result to be proved working with off-shell Nöther currents (also known as momentum maps; see [9]), but it is relatively difficult to produce a potential if one works with differential forms on space-time instead of using forms on (suitable prolongations of) the configuration bundle. (Of course expressions for on-shell currents do not change, thus they were and thus remain exact forms; however, it often happens that just local potentials are built or, even worse, global potentials are produced but they are claimed to exist just locally.)

The requirement that the superpotentials $U(L, \xi, \sigma)$ are the pull-backs of the same bundle theoretic quantity $U(L, \xi)$, which does not depend on the section, also enlights a number of definition problems. In fact $U(L, \xi)$ is global, so that integrals on boundaries are well defined; canonical ways to construct $U(L, \xi)$ are provided in the correct framework, so that also integrals on closed domains are defined; finally our approach ensures regularity conditions on the way $U(L, \xi, \sigma)$ depends on the section $\sigma$, which make variations of entropy meaningful.

2. Natural and gauge-natural theories

Let then $C = (C, M, \pi, F)$ be the configuration bundle of a field theory. $C$ is a bundle over a (space-time) manifold $M$ which is assumed to be orientable, paracompact and of dimension $n$. A Lagrangian of order $k$ is a morphism

$$L : J^kC \to A_n(M)$$

(2.1)

from the Lagrangian phase bundle $J^kC$ (namely, the $k$-order jet prolongation of the configuration bundle, i.e. the bundle where fields live together with their partial derivatives up to order $k$ included) into the bundle $A_n(M)$ of $n$-forms over $M$.

One can define the bundle morphism $\delta L : J^kC \to V^*(J^kC) \otimes A_n(M)$, where $V^*(\cdot)$ denotes the dual of the vertical bundle, where vertical refers to the relevant bundle projection (i.e. it denotes the kernel of the relevant tangent map, or equivalently it refers to bundle-vectors which are tangent to fibers).
The (global) morphism $\delta L$ is defined by

$$< \delta L \circ j^k \sigma \mid j^k X > = \frac{d}{dt} \left( L \circ j^k \hat{\Psi}_t \circ j^k \sigma \right) \bigg|_{t=0}$$  \hspace{1cm} (2.2)

where $X$ is any vertical vector field on $C$, $\hat{\Psi}_t$ is its flow and we denoted by $\langle \cdot \mid \cdot >$ the standard pairing between the dual vertical bundle $V^*(C)$ (or some prolongation $V^*(J^kC)$ of it) and the vertical bundle $V(C)$ (or $V(J^kC)$, respectively).

It has been shown (see [3] and [10]) that for each Lagrangian $L$ the Calculus of Variations induces a unique (global) morphism, called the Euler-Lagrange morphism

$$\mathcal{E}(L) = J^{2k}C \rightarrow V^*(C) \otimes A_n(M)$$  \hspace{1cm} (2.3)

together with a family of (global) morphisms (which depend on the Lagrangian and possibly on a connection $\gamma$ on $M$) called Poincaré-Cartan morphisms

$$\mathcal{F}(L, \gamma) = J^{2k-1}C \rightarrow V^*(J^{k-1}C) \otimes A_{n-1}(M)$$  \hspace{1cm} (2.4)

The Euler-Lagrange morphism and the Poincaré-Cartan morphisms are in fact defined so that the so-called first-variation formula holds for any vertical vector field $X$ on $C$:

$$< \delta L \mid j^k X > = < \mathcal{E}(L) \mid X > + \text{Div} < \mathcal{F}(L, \gamma) \mid j^{k-1} X >$$  \hspace{1cm} (2.5)

where the formal divergence operator on forms is defined by

$$\text{Div}(f) \circ j^{k+1} \sigma = d(f \circ j^k \sigma), \hspace{1cm} f : J^kC \rightarrow A(M)$$  \hspace{1cm} (2.6)

$d(\cdot)$ being the exterior differential operator on forms and $A(M) \equiv \oplus_k A_k(M)$ denotes the bundle of forms over $M$.

We remark that the first-variation formula (2.5) is written on the bundle $C$ and it encompasses the whole class of first-variation formulae that can be written on the base $M$ by pull-back along sections of the configuration bundle $C$ (not necessarily solutions of field equations, i.e. of the Euler-Lagrange equations $\mathcal{E}(L) \circ j^{2k} \sigma = 0$). We will show that performing calculations in this off-shell fashion turns out to be important and certainly it clarifies relations among intrinsic quantities that can be hidden by direct evaluation along solutions.

The Poincaré-Cartan morphisms are uniquely defined for $k = 0, 1$; for $k = 2$ (which is the case of interest for all metric theories of gravitation) they are not unique but still there is a canonical choice which is independent on the connection $\gamma$. For $k \geq 3$ also the possibility of this canonical choice is lost and
one has to work with the whole class of Poincaré-Cartan morphisms (see [11], [12], [13] and [14]). Fortunately, in all physically relevant cases we have \( k \leq 2 \) and therefore we shall speak of the Poincaré–Cartan morphism (and form).

If we consider a projectable vector field \( \hat{\Xi} \) over \( C \), i.e. a vector field such that its projection \( \xi = \xi(\hat{\Xi}) = \pi_* (\hat{\Xi}) \) is a well defined vector field over \( M \), we say that \( \hat{\Xi} \) is an infinitesimal symmetry of \( L \) iff

\[
\mathcal{L}_\xi (L \circ j^k \sigma) = < \delta L \circ j^k \sigma | j^k \mathcal{L}_{\hat{\Xi}} \sigma >
\]  

(2.7)

where \( j^k \sigma \) denotes the \( k \)-order jet prolongation of any section \( \sigma \) and the Lie-derivative of a section \( \sigma \) with respect to \( \hat{\Xi} \) is defined by the following prescription:

\[
\mathcal{L}_{\hat{\Xi}} \sigma = T \sigma (\xi) - \hat{\Xi} \circ \sigma
\]  

(2.8)

If the configuration bundle \( C \) is a natural bundle, i.e. it is associated to the \( s \)-frame bundle \( L^s(M) \) for some \( s \geq 1 \) (or, equivalently, its transition functions depend functionally on local changes of coordinates in the base \( M \) together with their partial derivatives up to order at most \( s \) ), we can restrict our attention to projectable vector fields \( \hat{\xi} \) on \( C \) which are natural lifts of vector fields \( \xi \) on the base \( M \). If all such vector fields \( \hat{\xi} \) are infinitesimal symmetries for the Lagrangian, we say that \( L \) is a natural Lagrangian (see [15], [16] and references quoted therein).

A field theory is said to be a natural theory if and only if both its configuration bundle and its Lagrangian are natural. An important example of natural theories is provided by the so-called metric theories of gravitation, where the configuration bundle is chosen to be the bundle \( C = \text{Lor}(M) \) of Lorentzian metrics on \( M \), which is a natural bundle associated to \( L(M) \) \((s = 1)\); the prototype of metric theories is of course General Relativity. The covariance principle of General Relativity just asserts that the Lagrangian is natural and the Hilbert-Einstein Lagrangian is a particular second-order natural Lagrangian \((k = 2)\).

However, not all physically relevant field theories are natural theories. As an example, in Yang-Mills theories the gauge potentials are described by principal connections on a \( SU(m) \)-principal bundle \( \mathcal{P} = (P, M, \pi, SU(m)) \). In general the bundle \( \text{Con}(\mathcal{P}) = J^1 P/SU(m) \) of principal connections on \( \mathcal{P} \) is not a natural bundle, so that Yang-Mills theories cannot be natural. Moreover, in Yang-Mills theories pure gauge symmetries are not the lift of diffeomorphisms on the base \( M \) (they are all vertical and thence project on the identity).

Let now \( G \) be any Lie group and \( \mathcal{P} = (P, M, p, G) \) be a \( G \)-principal bundle, called the structure bundle; we say that a bundle \( C \) is a gauge-natural bundle of order \((r, s)\) associated to \( \mathcal{P} \) (see [17], [18] and references quoted therein).
therein) if it is associated to the principal bundle $J^r P \times_M L^s(M)$ (with $s \geq r$), where $\times_M$ denotes the fiber product over $M$. A canonical action of the automorphisms of the structure bundle is defined on gauge-natural bundles (and it completely characterizes them). By means of this action, one can uniquely define for any infinitesimal generator $\Xi$ of automorphisms on $\mathcal{P}$ a projectable vector field $\hat{\Xi}$ on the gauge-natural bundle $\mathcal{C}$ associated to the structure bundle. We then say that the Lagrangian $L$ is gauge-natural if all such vector fields $\hat{\Xi}$ are infinitesimal symmetries of $L$.

We say that a field theory is a gauge-natural theory if both its configuration bundle and its Lagrangian are gauge-natural. Gauge-natural theories encompass, in a single and coherent framework, all fields theories which are of interest in modern theoretical physics, since they include in a unifying scheme all natural theories, gauge theories, Bosonic and Fermionic matter as well as their mutual interactions. In order to achieve a canonical treatment of conserved quantities (as shown in [18], [19], [20] and briefly recalled below) we also require that two principal connections $(\gamma, \omega)$ on $L(M)$ and $\mathcal{P}$, respectively, can be built out of the dynamical fields. This latter requirement amounts to build two morphisms (which, by an abuse of language, will still be denoted by $(\gamma, \omega)$) from (some jet prolongation of) the configuration bundle $\mathcal{C}$ to the bundles of linear connections $\text{Con}(L(M))$ and gauge connections $\text{Con}(\mathcal{P})$, respectively. Using these two morphisms, whenever we fix a section $\sigma$ of $\mathcal{C}$, we induce two connections on $L(M)$ and $\mathcal{P}$, which are thence called dynamical connections. We stress that in higher order gauge-natural theories one can always choose the representative of the Poincaré-Cartan morphism induced by the dynamical connection on $M$, so that no ambiguity occurs.

As we said, most of the physically relevant field theories admit a formulation in the gauge-natural framework. Clearly, natural theories are particular gauge-natural theories when $\mathcal{P}$ is a $G$-principal bundle with respect to the trivial group $G = \{e\}$ and the configuration bundle is a gauge-natural bundle of order $(0, s)$ associated to $\mathcal{P}$. Analogously, pure gauge theories are gauge-natural provided one suitably reformulates them on a gauge-natural bundle associated to $J^1 P \times_M L(M)$ (see [21]).

3. Conserved quantities in gauge-natural theories

Both in natural and gauge-natural theories conserved quantities are canonically defined, because symmetries are encoded in the definition of the theory. The aim of this Section is to extend to gauge-natural theories some properties of Nöther charges that are known to hold in natural theories (see [1], [3], [22], [23], [24] and [25]).
Let us consider the infinitesimal generator $\Xi$ of a 1-parameter subgroup of automorphisms $\Phi_t$ of the structure bundle $\mathcal{P}$. As we said above, $\Xi$ induces a projectable vector field $\hat{\Xi}$ on the configuration bundle $\mathcal{C}$; $\hat{\Xi}$ is the infinitesimal generator of the flow $\hat{\Phi}_t$ induced on $\mathcal{C}$ by the flow $\Phi_t$ by means of the action of the group $\text{Aut}(\mathcal{P})$ of automorphisms of $\mathcal{P}$ on the configuration bundle $\mathcal{C}$ itself.

Since in gauge-natural theories fields are not natural objects, Lie-derivatives of fields with respect to generic vector fields on $\mathcal{M}$ are meaningless; however we can define Lie-derivatives of fields with respect to infinitesimal generators of automorphisms on the structure bundle $\mathcal{P}$ as follows:

$$\mathcal{L}_\Xi \sigma := \mathcal{L}_{\hat{\Xi}} \sigma = T\sigma(\xi) - \hat{\Xi} \circ \sigma, \quad \xi = p_*(\Xi) = \pi_*(\hat{\Xi}) \quad (3.1)$$

The Lagrangian is by definition $\text{Aut}(\mathcal{P})$-covariant, i.e. it satisfies

$$\mathcal{L}_\xi (L \circ j^k \sigma) = < \delta L \circ j^k \sigma \mid j^k \mathcal{L}_\Xi \sigma > \quad (3.2)$$

The first-variation formula (2.5) and the gauge covariance condition (3.2) alone provide a complete treatment of conserved quantities in higher order field theories. In fact, these two formulae provide an intrinsic version of Nöther theorem by simply noticing that the Lie derivative of a section $\sigma$ is vertical. Hence one can apply the first-variation formula

$$\mathcal{L}_\xi (L \circ j^k \sigma) = < \delta L \circ j^k \sigma \mid j^k \mathcal{L}_\Xi \sigma > =$$

$$= < \mathcal{E}(L) \circ j^{2k} \sigma \mid \mathcal{L}_\Xi \sigma > + \text{Div} < \mathcal{F}(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} \mathcal{L}_\Xi \sigma > \quad (3.3)$$

which, by expanding the l.h.s. Lie-derivative, allows to define the following currents:

$$\mathcal{E}(L, \Xi) = < \mathcal{F}(L, \gamma) \mid j^{k-1} \Xi_{\mathcal{L}} > - i_\xi L$$

$$\mathcal{W}(L, \Xi) = - < \mathcal{E}(L) \mid \Xi_{\mathcal{L}} > \quad (3.4)$$

Here $\Xi_{\mathcal{L}} : \mathcal{J}^1 \mathcal{C} \rightarrow \mathcal{V} \mathcal{C}$ is the global bundle morphism intrinsically defined by

$$(j^1 \sigma)^* \Xi_{\mathcal{L}} = \mathcal{L}_\Xi \sigma \quad (3.5)$$

where $\mathcal{L}_\Xi \sigma$ is the Lie derivative of a section $\sigma$ defined by (3.1).

Because of (3.3), these two currents obey a conservation law

$$\text{Div} \mathcal{E}(L, \Xi) = \mathcal{W}(L, \Xi) \quad (3.6)$$

Let us then denote by $\mathcal{W}(L, \Xi, \sigma)$ and $\mathcal{E}(L, \Xi, \sigma)$ the pull-back of $\mathcal{W}(L, \Xi)$ and $\mathcal{E}(L, \Xi)$, respectively, along a section $\sigma$ of $\mathcal{C}$. Since $\mathcal{W}(L, \Xi, \sigma)$ vanishes
because of (3.4) whenever \( \sigma \) is a solution of field equations \( E(L) \circ j^{2k} \sigma = 0 \), \( E(L, \Xi, \sigma) \) is conserved on-shell. In other words, one can pull-back eq. (3.6) along solutions producing a whole class of currents (one for each section) which are closed forms on \( M \). We remark that eq. (3.6) is again written on \( C \) (and not on \( M \)), a fact which will turn out to be important below when discussing superpotentials.

The currents we defined, namely \( E(L, \Xi) \) and \( W(L, \Xi) \), are linear with respect to the (infinite) jet prolongation \( j \Xi \) of the infinitesimal generator of automorphisms \( \Xi = \xi^\mu \partial_\mu + \xi^A \rho_A \) (here \( \rho_A \) is a local basis for right invariant vertical vector fields on \( \mathcal{P} \)). As a consequence, if the configuration bundle is a gauge-natural bundle of order \((r, s)\), Lie derivatives of sections can be written as linear combinations of symmetrised covariant derivatives with respect to the dynamical connections \((\gamma, \omega)\)

\[
\xi^\mu, \nabla_{\rho_1} \xi^\mu, \ldots, \nabla_{\rho_1 \cdots \rho_s} \xi^\mu, \xi^A, \nabla_{\rho_1} \xi^A, \ldots, \nabla_{\rho_1 \cdots \rho_r} \xi^A
\]  

(3.7)

and, consequently, currents can be written as

\[
E(L, \Xi) = \left[ T^\lambda_\mu \xi^\mu + T^\lambda_\rho_1 \nabla_{\rho_1} \xi^\mu + \ldots + T^\lambda_\rho_1 \cdots \rho_{k+s-1} \nabla_{\rho_1 \cdots \rho_{k+s-1}} \xi^\mu \right. \\
+ T^\lambda_A \xi^A + T^\lambda_\rho_1 \nabla_{\rho_1} \xi^A + \ldots + T^\lambda_\rho_1 \cdots \rho_{k+r-1} \nabla_{\rho_1 \cdots \rho_{k+r-1}} \xi^A \right] ds
\]  

(3.8)

\[
W(L, \Xi) = \left[ W^\mu_\mu \xi^\mu + W^\mu_\rho_1 \nabla_{\rho_1} \xi^\mu + \ldots + W^\mu_\rho_1 \cdots \rho_s \nabla_{\rho_1 \cdots \rho_s} \xi^\mu \right. \\
+ W_A^\rho_1 \xi^A + W_A^\rho_1 \nabla_{\rho_1} \xi^A + \ldots + W_A^{\rho_1 \cdots \rho_r} \nabla_{\rho_1 \cdots \rho_r} \xi^A \right] ds
\]  

(3.9)

where \( ds = dx^1 \wedge \ldots \wedge dx^n \) is the local volume form on \( M \), while \( ds_\mu = i_{\partial_\mu} \cdot ds \) are the local generators of \((n-1)\)-forms. The coefficients \((T^\lambda_\mu, T^\lambda_\rho_1, \ldots, T^\lambda_\rho_1 \cdots \rho_{k+s-1}, T^\lambda_A, T^\lambda_\rho_1, \ldots, T^\lambda_\rho_1 \cdots \rho_{k+r-1})\) as well as \((W^\mu_\mu, W^\mu_\rho_1, \ldots, W^\mu_{\rho_1 \cdots \rho_s}, W_A^\rho_1, \ldots, W_A^{\rho_1 \cdots \rho_r})\) are tensor densities with respect to automorphisms of the structure bundle.

Whenever we have such a linear combination we can perform covariant integration by parts to obtain for the same quantity an equivalent linear expansion whose coefficients are all symmetric with respect to upper indices, while the integrated terms are all pushed into a formal divergence (see [25]). For example, we can recast the current \( W(L, \Xi) \) as

\[
W(L, \Xi) = \mathcal{B}(L, \Xi) + \text{Div} \tilde{E}(L, \Xi)
\]  

(3.10)
The current $\tilde{E}(L, \Xi)$ vanishes identically on shell because of (3.4) and (3.9) and $B(L, \Xi)$ turns out to be identically vanishing along any section (i.e. off-shell). This holds because of $\text{Div}^2 = 0$ and since $\text{Div} B(L, \Xi) = 0$ for arbitrary $\Xi$ implies $B(L, \Xi) = 0$ when $B(L, \Xi)$ is symmetric with respect to upper indices. The quantity $\tilde{E}(L, \Xi)$ is usually called the reduced current, and the identities

$$B(L, \Xi) = 0$$

are called generalized Bianchi identities.

The same kind of covariant integration by parts (which by the way is a well defined, canonical and global operation in the bundle framework) is performed on the current $E(L, \Xi)$ and we obtain

$$E(L, \Xi) = \tilde{E}(L, \Xi) + \text{Div} U(L, \Xi)$$

where $\tilde{E}(L, \Xi)$ is the reduced current defined in (3.10). Again we shall denote by $\tilde{E}(L, \Xi, \sigma)$ and $U(L, \Xi, \sigma)$ the pull-back along a section $\sigma$ of $\tilde{E}(L, \Xi)$ and $U(L, \Xi)$, respectively; $U(L, \Xi)$ is called a superpotential. Obviously, because of $\text{Div}^2 = 0$, superpotentials are not unique but rather defined modulo formal divergences. Once again both the reduced current and superpotentials are calculated on the bundle $C$ and can be pulled-back on the base $M$ along any section. Reduced currents and superpotentials are easily shown to be global, thanks to their tensorial character; if they are pulled-back along a solution $\sigma$ the reduced current vanishes and the current $E(L, \Xi, \sigma)$ is not only a closed form, but it is also exact on-shell (see [25]).

In other words, while the current $E(L, \Xi)$ is conserved just along solutions, the quantity $E(L, \Xi) - \tilde{E}(L, \Xi) = \text{Div} U(L, \Xi)$ is conserved along any section of $C$. We express this different behaviour by saying that $E(L, \Xi)$ is weakly conserved while $\text{Div} U(L, \Xi)$ is said to be strongly conserved.

As it is well known, natural theories as well as gauge-natural theories always allow (global) superpotentials (see [20] and [25]). Thus one can say that in this kind of theories conserved currents are exact forms on-shell and a representative for their potentials is explicitly produced. This is a further result which can be completely hidden by performing calculations on the base $M$.

Let us finally consider a region $D \subset M$ of spacetime, i.e. a compact $(n-1)$-submanifold with a boundary $\partial D \subset D \subset M$ which is a compact $(n-2)$-submanifold; let us define the conserved quantity along a section $\sigma$ as

$$Q_D(L, \Xi, \sigma) = \int_D E(L, \Xi, \sigma) = \int_D \tilde{E}(L, \Xi, \sigma) + \int_{\partial D} U(L, \Xi, \sigma)$$

(3.13)
If $\sigma$ is a solution of field equations, the reduced current $\tilde{\mathcal{E}}(L, \Xi, \sigma)$ vanishes and $Q_D(L, \Xi, \sigma)$ is just given by the integral of the superpotential on the boundary of the region $D$. In natural theories, the vector field $\Xi$ is usually taken to be the natural lift $\hat{\xi}$ of a vector field $\xi$ over the base $M$. In this case one simply writes $U(L, \xi)$ instead of $U(L, \hat{\xi})$ and so on, for notational convenience.

4. Intrinsic ADM formalism

One of the standard approaches to conserved quantities in an Hamiltonian setting for General Relativity is due to Arnowitt, Deser and Misner (ADM). This method (see [26]) is motivated by the need of selecting a preferred conserved quantity (the mass) which in the Hamiltonian formalism determines the evolution of fields. This is achieved by a $(n+1)$ decomposition of space-time; this decomposition is not unique and it destroys covariance of the theory, which has to be restored at the end. A covariant ADM formalism has been later developed (see e.g. [2] and [27]) which does not require explicitly a $(n+1)$ decomposition.

In this framework, let us consider a current $\mathcal{E}(L, \Xi)$ and a vertical vector field $X$ on $\mathcal{C}$; we calculate the variation of the current along $X$ and find:

\[
\delta_X \mathcal{E}(L, \Xi) = \delta_X (< \mathcal{I}(L, \gamma) \mid j^{k-1} \Xi > - i\xi L) =
\]
\[
= \delta_X < \mathcal{I}(L, \gamma) \mid j^{k-1} \Xi > - i\xi < \delta L \mid j^k X > =
\]
\[
= \delta_X < \mathcal{I}(L, \gamma) \mid j^{k-1} \Xi > - i\xi < \mathcal{E}(L) \mid X > +
\]
\[
- i\xi \text{Div} < \mathcal{I}(L, \gamma) \mid j^{k-1} X >=
\]
\[
= \delta_X < \mathcal{I}(L, \gamma) \mid j^{k-1} \Xi > - \mathcal{L}_\xi < \mathcal{I}(L, \gamma) \mid j^{k-1} X > +
\]
\[
- i\xi < \mathcal{E}(L) \mid X > + \text{Div} i\xi < \mathcal{I}(L, \gamma) \mid j^{k-1} X >
\]

Integrating along a solution $\sigma$ we have then:

\[
\delta_X Q_D(L, \Xi, \sigma) = \int_D \delta_X \mathcal{E}(L, \Xi, \sigma) =
\]
\[
= \int_D \delta_X < \mathcal{I}(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} \mathcal{L}_\Xi \sigma > +
\]
\[
- \int_D \mathcal{L}_\xi < \mathcal{I}(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X > +
\]
\[
- \int_D i\xi < \mathcal{E}(L) \circ j^{2k} \sigma \mid X > + \int_{\partial D} i\xi < \mathcal{I}(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X >
\]
The third integral on the r.h.s. vanishes because \( \sigma \) is a solution. Following \([2]\), let us now suppose that boundary conditions for \( X \) can be chosen so that a global \((n-2)\)-form over \( M \) \( B(L, \xi, \sigma) = \frac{1}{2} B^{\mu\nu} ds_{\mu\nu} \) exists and the following holds:

\[
\delta_X B(L, \xi, \sigma) \bigg|_{\partial D} = i_\xi < F(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X > \bigg|_{\partial D} \tag{4.3}
\]

where, as usual, \( \xi \) is the projection of \( \Xi \) onto \( M \).

As shown in \([2]\) and \([27]\), this is certainly the case of General Relativity. Then we can redefine the conserved quantity (3.13) by subtracting a boundary term

\[
\hat{Q}_D(L, \Xi, \sigma) = \int_D E(L, \Xi, \sigma) - \int_{\partial D} B(L, \xi, \sigma) = \int_D \tilde{E}(L, \Xi, \sigma) + \int_{\partial D} [U(L, \Xi, \sigma) - B(L, \xi, \sigma)]
\]

If \( \sigma \) is a solution then the reduced current vanishes and \( \hat{Q}_D(L, \Xi, \sigma) \) is a purely "boundary" quantity.

Since in asymptotically simple space-times the quantity \( B(L, \xi, \sigma) \) has to be integrated on spatial infinity, it has to be a global and therefore covariant \((n-2)\)-form (otherwise the integral itself would depend on coordinates and may not be defined if \( B(L, \xi, \sigma) \) is not defined on the whole of spatial infinity). Of course it may happen that, in some particular situation or for some special solution, some weaker recipe works; but, if this is the case, then it cannot be claimed that these extend to \textit{general} recipes. Furthermore by using the non-uniqueness of \( U(L, \Xi) \) and \( B(L, \xi) \) one suitably imposes further conditions able to ensure that the modified conserved quantities (4.4) vanish on some particular field configuration, so to fix the closed-form ambiguity mentioned above.

In General Relativity we have the Lagrangian \( L_\mu = r \sqrt{g} \) \( ds \) (\( r \) is the scalar curvature of the Levi-Civita connection \( \gamma \) of the dynamical metric field \( g \)). The Poincaré-Cartan morphism is (recall that \( k = 2 \))

\[
< F(L, \gamma) \mid j^1 X > = f^{\lambda\mu}_\rho \nabla_\mu X^\rho\sigma, \quad f^{\lambda\mu}_\rho = \sqrt{g} [g^{\lambda\mu} g_{\rho\sigma} - \delta_\rho^{(\lambda} \delta_\sigma^{\mu)}] \tag{4.5}
\]

where \( X^\rho\sigma = \delta g^{\rho\sigma} \) is a vertical vector of \( \text{Lor}(M) \).

If we choose the standard boundary conditions \( X|_{\partial D} = 0 \), we can set

\[
\tilde{B} = -\sqrt{g} g^{\alpha\beta} \xi^{[\lambda} u^{\mu]}_{\alpha\beta} \) ds_{\lambda\mu}, \quad u^{\mu}_{\alpha\beta} = \gamma^{\mu}_{\alpha\beta} \gamma^\epsilon_{(\alpha} \delta^\mu_{\beta)} \tag{4.6}
\]
Unfortunately this choice of $\tilde{\mathbf{B}}$ is not covariant, i.e. $\tilde{\mathbf{B}}$ it is not a $(n-2)$-form $(n = \dim(M))$. However, if one chooses a background connection $\Gamma$, a covariant boundary term exists and it is defined as follows:

$$
\mathbf{B}(L, \xi, \sigma) = - \sqrt{|g|} g^{\alpha\beta} \xi^{[\lambda} u_{\alpha\beta}^{\mu]} ds_{\lambda\mu}
$$

$$
u_{\alpha\beta}^{\mu} = \nu_{\alpha\beta}^{\mu} - U_{\alpha\beta}^{\mu}, \quad U_{\alpha\beta}^{\mu} = \Gamma_{\alpha\beta}^{\mu} - \Gamma_{e(\alpha\beta)}^{\mu} \tag{4.7}
$$

We stress here that some background connection $\Gamma$ is always chosen in current literature, though usually for asymptotically flat solutions this is not evident since one generally fixes it to be the Levi-Civita connection of Minkowski metric, which in suitable coordinates is even vanishing. In fact, a background field has to be chosen to provide covariance. If a background connection is not chosen, this simply means that one is in fact fixing some particular class of coordinate systems and implicitly a connection which is zero in those coordinates.

As discussed in [2] and [27], the conserved quantity $\hat{Q}_D(L, \Xi, \sigma)$ defined by (4.4) has to be interpreted as the relative conserved quantity with respect to the background connection $\Gamma$. According to this interpretation, $\Gamma$ is considered unaffected by deformations $X$ while it is dragged by symmetry generators $\Xi$. Of course the background connection $\Gamma$ is not actually a physical field but rather a sort of “gauge fixing”; in other words, it has to be regarded as a mere parameter (i.e. we are interested, for instance, in the mass of the field $g$ alone, and not in the total mass of $g$ and $\Gamma$ together). The interested reader may find discussed examples in [2], [27], [28], whereby a comparison among various background choices and different prescriptions is also provided. Application to the conserved quantities in spherically symmetric spacetimes can be found in [29].

If the spacetime is asymptotically flat (according to one of the standard definitions), one can choose $\Gamma$ to be the Levi-Civita connection of Minkowski space and in this case one can recover the standard ADM formalism.

In any case, these techniques apply to a much more general situation than asymptotically flat spaces. If one performs now a $(n+1)$ decomposition (see [2] and [27] for further details) in the general case, one finds the boundary corrections known as Regge-Teitelboim terms (see [30]) plus some further boundary corrections (which of course vanish in the asymptotically flat case).

### 5. First order General Relativity

As we said in the previous Section, ADM conserved quantities are defined by choosing a background connection $\Gamma$. The same ingredients are needed to correct the so-called anomalous factors in Komar superpotentials (see [31]) and to give a covariant first-order formulation of General Relativity (see [3], [12]).
We shall show that this mechanism is equivalent to intrinsic ADM formalism as described above and then to standard ADM formalism when it applies.

As we have already done in the previous Sections, let us denote systematically by capital letters the background quantities and by lower case letters the corresponding dynamical quantities, i.e.

\[
\begin{align*}
    g_{\mu\nu} & \quad \text{metric} \\
    \gamma^\alpha_{\beta\nu} & \quad \text{connections} \\
    R^\alpha_{\beta\mu\nu} & \quad \text{Riemann tensors} \\
    R_{\mu\nu} & \quad \text{Ricci tensors} \\
    R & \quad \text{scalar curvatures}
\end{align*}
\]

\[
\begin{align*}
    u^\mu_{\alpha\beta} = \gamma^\mu_{\alpha\beta} - \gamma^\epsilon_{(\alpha} \delta^\mu_{\beta)} & \quad, \\
    U^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \Gamma^\epsilon_{(\alpha} \delta^\mu_{\beta)}
\end{align*}
\]

and let us define the relative quantities by:

\[
\begin{align*}
    q^\mu_{\alpha\beta} &= \gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\alpha\beta} , \\
    w^\mu_{\alpha\beta} &= u^\mu_{\alpha\beta} - U^\mu_{\alpha\beta}
\end{align*}
\]

Since the Hilbert-Einstein Lagrangian is of the second order, field equations should be expected to be of fourth order, while they are instead of the second order as if the Lagrangian were of the first order only. As is well known, this depends on the fact that second order derivatives can be hidden in a divergence which does not affect field equations:

\[
L_H = \mathcal{L} \, ds = r \sqrt{g} \, ds = \left[ p^{\alpha\beta} (\gamma^\rho_{\alpha\sigma} \gamma^\sigma_{\rho\beta} - \gamma^\sigma_{\sigma\rho} \gamma^\rho_{\alpha\beta}) + d_\sigma (p^{\alpha\beta} u^\sigma_{\alpha\beta}) \right] \, ds
\]

where we have set \( p^{\alpha\beta} = \sqrt{g} g^{\alpha\beta} = \partial \mathcal{L} / \partial R_{\alpha\beta} \).

Unfortunately, this has only a local meaning since this splitting of \( L_H \) is not covariant. Again, if we use a background connection \( \Gamma \) as shown in [28] we can achieve a covariant splitting \( L_H = L_1 + \text{Div} (p^{\alpha\beta} w^\sigma_{\alpha\beta} \, ds_\sigma) \), where

\[
L_1 = \left[ p^{\alpha\beta} (\gamma^\rho_{\alpha\sigma} \gamma^\sigma_{\rho\beta} - \gamma^\sigma_{\sigma\rho} \gamma^\rho_{\alpha\beta}) + d_\sigma (p^{\alpha\beta} U^\sigma_{\alpha\beta}) \right] \, ds = \left[ R \sqrt{g} + p^{\alpha\beta} (q^\rho_{\alpha\sigma} q^\sigma_{\rho\beta} - q^\sigma_{\sigma\rho} q^\rho_{\alpha\beta}) \right] \, ds
\]

The first expression shows that the background \( \Gamma \) has no dynamics (so there is no restriction to be fulfilled when we fix it); the second expression shows that \( L_1 \) is covariant (the quantities \( q^\rho_{\alpha\sigma} \) being tensors on \( M \)).
Now we can calculate conserved currents for both $L_H$ and $L_1$, finding:

\[ \mathcal{E}_H = \mathcal{E}(L_H, \xi) = [p^{\alpha \beta} \mathcal{L}_\xi u^\lambda_{\alpha \beta} - L_H \xi^\lambda] \, ds_\lambda = \tilde{\mathcal{E}}_H + \text{Div} \, \mathcal{U}_H \]
\[ \mathcal{E}_1 = \mathcal{E}(L_1, \xi) = [-w^\lambda_{\alpha \beta} \mathcal{L}_\xi p^{\alpha \beta} - L_1 \xi^\lambda] \, ds_\lambda = \tilde{\mathcal{E}}_1 + \text{Div} \, \mathcal{U}_1 \]

where we have set

\[ \tilde{\mathcal{E}}_H = 2\sqrt{g} \left( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} \right) g^{\mu \lambda} \xi^\nu \, ds_\lambda, \quad \mathcal{U}_H = \nabla_{\alpha} \xi^{[\lambda} p^{\sigma]_{\alpha \beta} \, ds_{\lambda \sigma} \]
\[ \tilde{\mathcal{E}}_1 = \mathcal{E}_H + p^{\alpha \beta} \mathcal{L}_\xi U^\lambda_{\alpha \beta} \, ds_\lambda, \quad \mathcal{U}_1 = \mathcal{U}_H + \xi^{[\lambda} w_{\alpha \beta}^{\sigma]} p^{\alpha \beta} \, ds_{\lambda \sigma} \]

The superpotential $\mathcal{U}_H$ is a generalization of the Komar potential (see expression (10.3) in the appendix) which was originally written for timelike Killing vectors (see [22]).

The background fixing produces an additional boundary term in conserved quantities, namely

\[ Q_1 = Q_H + \int_{\partial D} \xi^{[\lambda} w_{\alpha \beta}^{\sigma]} p^{\alpha \beta} \, ds_{\lambda \sigma} \]

which is exactly the same contribution to conserved quantities (4.4) due to the boundary correction $B(L, \xi, \sigma)$ given in ADM framework (see eq. (4.7)). Consequently, both methods correct the anomalous factor problem endemic to the Komar potential. It has in fact been explicitly shown (see [7] and [28]) that they provide the correct values for both angular momentum and mass for Schwarzschild, Kerr and Kerr-Newman solutions.

These two methods are by no means general. The first method (ADM) uses space infinity and thence it does not apply to compact solutions, like e.g. some cosmological solutions; an explicit recipe to build boundary corrections is then provided just for standard General Relativity, though the method may as well apply to generalized metric theories, provided one gives a way of building the boundary correction and exhibits a correct interpretation. The second method has definitively stuck to standard General Relativity.

6. Variation of conserved quantities

Let us now consider a gauge-natural field theory together with an infinitesimal generator $\Xi$ of automorphisms of the structure bundle $\mathcal{P}$; let $X$ be a vertical vector field on the configuration bundle $\mathcal{C}$. Thence one can define the variation of the current $\mathcal{E}(L, \Xi)$ along $X$ as given by (4.1). \[ \delta_X \mathcal{E}(L, \Xi) = \delta_X \tilde{\mathcal{E}}(L, \Xi) + \text{Div} \, \delta_X \mathcal{U}(L, \Xi) \]

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and comparing eqs. (4.1) and (6.1) we obtain

\[
\text{Div}(\delta_X U(L, \Xi) - i_\xi < IF(L, \gamma) \mid j^{k-1}X>) =
\]
\[
= \delta_X < IF(L, \gamma) \mid j^{k-1}\Xi \xi > - \mathcal{L}_\xi < IF(L, \gamma) \mid j^{k-1}X> + (6.2)
\]
\[
- \delta_X \mathcal{E}(L, \Xi) - i_\xi < E(L) \mid X>
\]

Eq. (6.2) holds off-shell; if it is pulled-back along a solution \( \sigma \) then the r.h.s. term \( i_\xi < E(L) \circ j^{2k}\sigma \mid X > \) vanishes. The following two lemmas also hold.

**Lemma (6.3):** If \( \mathcal{L}_\Xi \sigma = 0 \) then

\[
\delta_X < IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}\mathcal{L}_\Xi \sigma > - \mathcal{L}_\xi < IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}X> = 0.
\]

Outline of proof: Since both \( \delta_X \) and \( \mathcal{L}_\xi \) are derivatives the following holds

\[
\delta_X < IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}\mathcal{L}_\Xi \sigma > +
\]
\[
- \mathcal{L}_\xi < IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}X>
\]
\[
= < \delta_X IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}\mathcal{L}_\Xi \sigma > +
\]
\[
+ < IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}\delta_X \mathcal{L}_\Xi \sigma > +
\]
\[
- < \mathcal{L}_\Xi IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}X> +
\]
\[
- < IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}\mathcal{L}_\Xi X>
\]

Because of \( \delta_X \mathcal{L}_\Xi \sigma = \mathcal{L}_\Xi X \) the second and the fourth term on the r.h.s. cancel each other, while \( < \delta_X IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}\mathcal{L}_\Xi \sigma > \) vanishes by the hypothesis \( \mathcal{L}_\Xi \sigma = 0 \) and \( < \mathcal{L}_\Xi IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^{k-1}X> = 0 \) since \( L \) is a gauge-natural Lagrangian.

Let us consider the morphism \( \delta E(L) : J^{2k}C \to V^*(J^{2k}C) \otimes V^*(C) \otimes A_n(M) \). We say that \( X \) is a solution of the **linearized field equations** if

\[
\delta E(L) (j^{2k}X, Y) = 0 \quad \text{for any vertical field } Y \text{ on } C \quad (6.5)
\]

**Lemma (6.6):** If \( \sigma \) is a solution of field equations and \( X \) is a solution of linearized field equations then \( \delta_X \mathcal{E}(L, \Xi, \sigma) = 0 \).
Proof: We have
\[
\delta_X \mathcal{W}(L, \Xi) \overset{(3.4)}{=} -\delta_X <\mathcal{E}(L) \mid \Xi_L > = \\
= -< \delta_X \mathcal{E}(L) \mid \Xi_L > - < \mathcal{E}(L) \mid \delta_X \Xi_L > = \\
= -\delta \mathcal{E}(L) (j^{2k} X, \Xi_L) - < \mathcal{E}(L) \mid \delta_X \Xi_L > 
\] (6.7)
The last term \(< \mathcal{E}(L) \mid \delta_X \Xi_L > \) vanishes on-shell.

We recall that integrating by parts \( \mathcal{W}(L, \Xi) \) (see (3.10)) and using Bianchi identities (3.11), one obtains a pure divergence, i.e. \( \mathcal{W}(L, \Xi) = \text{Div} \tilde{\mathcal{E}}(L, \Xi) \); this easily proves that
\[
0 = \delta \mathcal{E}(L) (j^{2k} X, \delta_X \Xi_L \sigma) = \delta_X \mathcal{W}(L, \Xi, \sigma) = \text{Div} \delta_X \tilde{\mathcal{E}}(L, \Xi, \sigma) 
\] (6.8)
This implies that \( \delta_X \tilde{\mathcal{E}}(L, \Xi, \sigma) = 0 \), again because of the symmetry of the coefficients. Thence also Lemma (6.6) is proved. □

Applying these two lemmas to (6.2), the following theorem is then proved:

**Theorem (6.9):** If \( \sigma \) is a solution of field equations, \( \mathcal{L} \Xi \sigma = 0 \) and \( X \) is a solution of linearized field equations (6.5) then
\[
\text{Div} (\delta_X \mathcal{U}(L, \Xi, \sigma) - i_\xi < \mathcal{F}(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X >) = 0 
\] (6.10)

This theorem generalizes to all gauge-natural theories the result that Iyer and Wald proved in [1] for a particular class of natural theories and which has been used in [1] and [32] to define black holes entropy and to prove the first principle of thermodynamics.

As noticed by Iyer and Wald in [1], the objects involved in the theory of conserved quantities, e.g. the superpotential, may be ambiguous. In fact, one can add a total divergence to the Lagrangian without actually changing field equations but changing the superpotential itself. As we said in Section 5, this is the key property to cure the anomalous factor problem.

Adding a total divergence \( \text{Div} \theta \) to the Lagrangian \( L \), we induce in fact an extra term \( i_\xi \theta \) in the superpotential \( \mathcal{U}(L, \Xi) \). Nevertheless, in the framework developed in [1] this latter term does not cause any ambiguity because \( \xi \) vanishes on the bifurcation surface \( \Sigma \) (see eq. (10.7) in the appendix). We stress that, in our framework, the relevant quantity (see (10.14) in the appendix) is instead
\[
\delta_X \mathcal{U}(L, \Xi) - i_\xi < \mathcal{F}(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X > 
\] (6.11)
which again is not affected by any ambiguity since the extra term \( \delta_X (i_\xi \theta) \) in the variation of the superpotential is exactly cancelled by the contribution \( i_\xi \delta_X (\theta) \) due to the second term in (6.11).
7. Kerr-Newman black holes

We now apply the result of the previous Section to treat the cases of Kerr-Newman solutions. In some way this is a toy example, since, even though electromagnetic field is involved (and hence a gauge-natural theory should in principal be needed), the electromagnetic potential $A_\mu$ is at the same time a (local) 1-form on $M$ as well as a $U(1)$-connection (see e.g. [33] for more details on the naturality of Maxwell theory). Thus if one gives up the description of electric charges (and (pure) gauge transformations) the Einstein-Maxwell system can be treated as a natural theory and can be then described by standard results for natural theories. If one wants to analyze truly gauge-natural theories one should cope with SU($n$) Yang-Mills theories, but in these cases some problems about the interpretation of exact solutions arise; in particular there may be problems about the correct conserved quantities to be produced.

Let us then consider a gravity-electromagnetic (GEM) system. Its configuration bundle is

$$ C = \text{Lor}(M) \times_M \text{Con}(\mathcal{P}) $$

(7.1)

where $\mathcal{P}$ is the trivial $U(1)$-bundle over the maximal extension of the Kerr-Newman spacetime (see, e.g. [34] and [35]). Local fibered coordinates on $C$ are $(x^\mu, g_{\mu\nu}, A_\mu)$.

The (GEM) system is described by the following Lagrangian

$$ L = \mathcal{L}_H \, ds + \mathcal{L}_{\text{EM}} \, ds = \frac{\sqrt{g}}{16\pi} \left[ r_{\mu\nu} g^{\mu\nu} - F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \right] ds $$

(7.2)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature of the Maxwell connection $A_\mu$.

The bundle $C$ is a gauge-natural bundle of order $(1,1)$ associated to the structure bundle $\mathcal{P}$ and $L$ is a gauge-natural Lagrangian. The dynamical connections are the Levi-Civita connection of $g_{\mu\nu}$ and the Maxwell connection $A_\mu$.

Let us consider a Kerr-Newman solution of the (GEM) system. Ingoing Kerr-Schild coordinates $(t, r, \theta, \phi)$ on $M$ can be choosen so that the solution reads as

$$ g = \eta + \rho^{-2}(2mr - e^2) \left[ dt + dr - a \sin^2 \theta d\phi \right]^2 $$

$$ A = -er \rho^{-2} \left[ dt + dr - a \sin^2 \theta d\phi \right] $$

(7.3)

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, $m^2 \geq e^2 + a^2$, and we have set

$$ \eta = -dt^2 + \left[ dr - a \sin^2 \theta \, d\phi \right]^2 + \rho^2 \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right] $$

(7.4)
The metric in (7.3) is singular in $\rho^2 = 0$ and has two horizons $r_{\pm} = m \pm \sqrt{m^2 - a^2 - e^2}$. The singularity in $\rho^2 = 0$ is a true one, as one can see, e.g., by computing the invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (see [34]).

We remark that, when $m = 0$ and $e = 0$, $g$ reduces to $\eta$ which is the flat Kerr-Schild metric. The Levi-Civita connection of $\eta$ is chosen as a background $\Gamma$. As a background for the gauge-field we shall choose the vanishing gauge potential $A_{\mu}^{(B)} = 0$.

The vector field $\Xi = \xi^\mu \partial_\mu + \zeta \hat{\rho}$, where $\hat{\rho}$ is the right invariant generator of the Lie algebra $i \mathfrak{R}$ of $U(1)$ on the structure bundle $P$, induces an infinitesimal symmetry on $\mathcal{C}$. The induced superpotential is of the following form:

$$ U(L, \Xi) = \left[ \nabla_\beta \xi^{[\alpha} \eta^{\sigma]}_{\beta\beta} \right] \text{d}s_{\alpha\sigma} - \left[ (4\pi)^{-1} F^{\alpha\sigma}_{\mu} A_{\mu} \xi^\mu \right] \sqrt{g} \text{d}s_{\alpha\sigma}$$

(7.5)

The first contribution is the generalized Komar potential, the second comes from the electromagnetic field and the third comes from gauge invariance. The boundary correction (4.7) is given by

$$ B(L, \xi) = -\xi^{[\alpha} \eta^{\sigma]}_{\mu\nu} \eta^{\mu\nu} \text{d}s_{\alpha\sigma}$$

(7.6)

and the corrected superpotential is defined as follows:

$$ U_1(L, \Xi) = U(L, \Xi) - B(L, \xi)$$

(7.7)

This is the correct superpotential since $\mathcal{L}_\xi \Gamma = 0$ and $\mathcal{L}_\xi A_{\mu}^{(B)} = 0$ (if this were not the case, the reduced current would not vanish on shell because we would be changing also the background with respect to which we are computing “conserved quantities”).

If we consider $\partial_t$ (which is a Killing vector for both backgrounds since $\mathcal{L}_\xi U_{\mu\nu} = 0$ and $\mathcal{L}_\xi A_{\mu}^{(B)} = 0$) integrating on spatial spheres $S^2_r = \{r = \text{constant} > r_+\}$ we have the mass

$$ \mathcal{E} = \int_{S^2_r} U_1(L, \partial_t, \sigma) = m + \frac{e^2 r}{2(a^2 + r^2)}$$

(7.8)

As long as $r$ goes to infinity, we get $\mathcal{E} = m$, as expected.
If we consider $-\partial_\phi$ (which is again a Killing vector for both backgrounds) integrating on spatial spheres $S^2_r$ we have the angular momentum

$$\mathcal{J} = -\int_{S^2_r} \mathcal{U}_1(L, \partial_\phi, \sigma) = ma \tag{7.9}$$

If we consider $\Xi = \hat{b} \rho$ with $b$ constant (which is again a Killing vector for the background since $g^{\mu\nu}$ is not affected by pure gauge transformations and $A^{(a)}$ is invariant because $b$ is constant) integrating on spatial spheres $S^2_r$ we have the electric charge

$$Q = \int_{S^2_r} \mathcal{U}_1(L, \hat{b} \rho, \sigma) = -be \tag{7.10}$$

Let us now consider the vector field $\Xi = \partial_t + \Omega_H \partial_\phi + \hat{b} \rho$ (where $\Omega_H = a/(a^2 + r_+^2)$ and $b = em/(2mr_+ - e^2)$). It is a Killing vector for the backgrounds, it leaves the solution invariant (i.e. $\mathcal{L}_\Xi \sigma = 0$); thus, if $X$ is a solution of the linearized field equations (i.e. it is “tangent” to the space of solutions) we have (see (10.10))

$$\frac{\kappa}{2\pi} \delta_X S = \int_{S^2_r} \left[ \delta_X \mathcal{U}(L, \Xi, \sigma) - i_\xi < IF(L, \gamma) | j^1X > \right] =$$

$$= \int_{\infty} \delta_X \left[ \mathcal{U}(L, \Xi, \sigma) - \mathcal{B}(L, \xi, \sigma) \right] =$$

$$= \delta_X E - \Omega_H \delta_X J - b \delta_X Q = (1 - a\Omega)\delta m - m\Omega \delta a - b\delta e \tag{7.11}$$

Setting now for the surface gravity

$$\kappa = \frac{\sqrt{m^2 - a^2 - e^2}}{2mr_+ - e^2} \tag{7.12}$$

and integrating (7.11) one gets

$$S = 2\pi mr_+ \tag{7.13}$$

Using then the expressions for $(r_+, \Omega, b, \kappa)$ one can expand variations with respect to $(\delta m, \delta a, \delta e)$ and verify directly that (7.13) satisfies (7.11). We remark that in these coordinates (as well as in any other usual coordinates but the Kruskal-like ones) the contribution from the term $i_\xi < IF(L, \gamma) | j^1X >$ never vanishes on any sphere for any $r$. This is because $\Sigma$, though belonging to the maximal extension of the Kerr-Newman solution, does not intersect the domain
of the coordinates used. For this reason we believe that the definition (10.14) in appendix for the variation of entropy turns out to be much more useful than its simplification (10.7) in the appendix. Moreover further hypotheses (as well as more calculations) are required to perform such a simplification.

8. Another example

As observed in [36] Einstein equations with (negative) cosmological constant allow a \((1 + 2)\)-dimensional black hole solution. Let us consider the Lagrangian

\[
L = \alpha \left( r + \frac{2}{l^2} \right) \sqrt{g} \, ds \tag{8.1}
\]

Considering its variation one finds

\[
< E(L) | X > = \alpha \sqrt{g} \left[ r_{\mu \nu} - \left( \frac{1}{2} R + \frac{1}{l^2} \right) g_{\mu \nu} \right] X^{\mu \nu} \, ds
\]

\[
< F(L, \gamma) | j^1 X > = \alpha \sqrt{g} \left[ g^{\lambda \sigma} \gamma_{\alpha \beta} - \delta^{\lambda}_{(\alpha} \delta^{\sigma}_{\beta)} \right] \nabla_\sigma X^{\alpha \beta} \, ds_\lambda
\]

\[
\tilde{E}(L, \xi) = 2\alpha \sqrt{g} \left[ r_\nu^\lambda - \left( \frac{1}{2} R + \frac{1}{l^2} \right) \delta^\lambda_\nu \right] \xi^\nu \, ds_\lambda
\]

\[
U(L, \xi) = \alpha \sqrt{g} \nabla^{[\beta} \xi^{\alpha]} \, ds_{\alpha \beta}
\]

This model allows the following black hole solution

\[
g = -N^2 \, dt^2 + N^{-2} dr^2 + r^2 (N_\phi dt + d\phi)^2
\]

where we have set

\[
N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad M = \frac{r_+^2 + r_-^2}{l^2}
\]

\[
N_\phi = -\frac{J}{2r^2}, \quad J = \frac{2r_+ r_-}{l}
\]

Once again, if one computes the entropy as shown in appendix using (10.14) (choosing any \(\Sigma'\) with \(r = \) constant in the coordinate used) in place of (10.7), one easily gets:

\[
\delta S = 4\pi^2 \delta r_+ \quad \Rightarrow \quad S = 4\pi^2 r_+ \quad \left( \kappa = \frac{r_+^2 - r_-^2}{r_+ l^2} \right)
\]
which is exactly the entropy expected (see [37]).

If the contribution of the term \( i_\xi \langle I F(L, \gamma) \mid j^1 X \rangle \) is considered alone it can be easily checked that this contribution cannot be discarded on any \( S_r \).

9. Conclusions

We have generalized the definition of entropy given in [1]. Our formalism applies not only to a general-covariant metric theory, but to any gauge-natural theory.

In particular this, together with the Legendre equivalence of \( R, R^2 \) and \( R_{\mu\nu}R^{\mu\nu} \) theories proved in [38], [39] and [40] explains some recent results on black holes entropy in non linear theories of gravitation by Maeda et al. ([41]), which will form the subject of further investigation. Moreover, our formalism can be fruitfully applied to black hole entropy in string theory, which is a hot subject in current literature (see, e.g. [42]). Also this will form the subject of further investigation.

The definition for the variation of the entropy is

\[
\delta_X S = \frac{1}{T} \int_{\Sigma'} [\delta_X U(L, \Xi, \sigma) - i_\xi \langle IF(L, \gamma) \circ j^{2k-1}\sigma \mid j^1 X \rangle] \tag{9.1}
\]

where \( \Xi = \partial_t + \Omega \partial_\phi + b^A \rho_A \) is a Killing vector for the solution \( \sigma \) and \((\Omega, b^A)\) are chosen to provide the expected form of the first principle of thermodynamics:

\[
\delta_X E = T \delta_X S + \Omega \delta_X J + b^A \delta_X Q_A \tag{9.2}
\]

The domain of integration is than any \((n-2)\)-surface \( \Sigma' \), such that \( \infty - \Sigma' \) is a homological border (e.g. it does not include singularities).

This prescription for entropy may be regarded exactly as an algorithm to define a quantity \( S \) such that (9.2) holds.

We remark that the coefficients \((T, \Omega, b^A)\) has to be provided by some other argument. In some case, under some severe hypotheses, the fully general prescription (9.1) reduces to the simplified prescription given in [1].

10. Appendix

This appendix is devoted to summarize and recast the partial results presented in [1] into the general theory we presented above. All the results reproduce exactly those of [1] and are just restated in our notation; we stress that some imprecise statements (especially on locality of potentials and some of their ambiguity) are to be compared with their correct counterparts in our framework.
As is well known, if one considers a Lagrangian \( L : J^k B \to A_n(M) \), then its variation defines two morphisms \(<I E(L) | X >\) and \(<IF(L, \gamma) | j^{k-1}X >\) such that the first variation formula holds:

\[
< \delta L | j^{k}X >= < I E(L) | X > + \text{Div} < IF(L, \gamma) | j^{k-1}X >
\]  

(10.1)

Then \( IE \circ j^{2k} \sigma = 0 \) are field equations, while \(<IF(L, \gamma) | j^{k-1}X >\) is a \((n-1)\)-form depending on fields and (linearly) on their deformation. Though the quantity \(<IF(L, \gamma) | j^{k-1}X >\) does not play any role in field equations, it is known to be tightly related to conserved quantities. In fact, Nöther theorem claims that if an infinitesimal Lagrangian symmetry \( \xi \) is considered, then one can define a conserved current \( \mathcal{E}(L, \xi, \sigma) = (j^{2k} \sigma)^* [<IF(L, \gamma) | j^{k-1}L \xi \sigma > - i_\xi L] \), where \( L \xi \sigma \) denotes Lie derivative of fields. The current \( \mathcal{E}(L, \xi, \sigma) \) is a closed \((n-1)\)-form on spacetime \( M \) (\( \dim(M) = n \)). The current \( \mathcal{E}(L, \xi, \sigma) \) depends on the Lagrangian, on the symmetry \( \xi \) considered and on a solution \( \sigma \) of field equations. Since the currents \( \mathcal{E}(L, \xi, \sigma) \) are closed \((n-1)\)-forms, they allow potentials which are \((n-2)\)-forms \( \mathcal{U}(L, \xi, \sigma) \) on \( M \) locally constructed from fields and \( \xi \) (see [43], [44]) such that

\[
d \mathcal{U}(L, \xi, \sigma) = \mathcal{E}(L, \xi, \sigma)
\]

(10.2)

Conserved quantities \( Q_D(L, \xi, \sigma) \) are defined integrating the conserved current \( \mathcal{E}(L, \xi, \sigma) \) on a regular domain \( D \subset M \). More generally one can consider the integrals of the potential \( \mathcal{U}(L, \xi, \sigma) \) on a closed \((n-2)\)-submanifold \( C \), which may not be the boundary of a regular domain \( D \subset M \) (here boundary is used in the homological sense).

Of course, the potentials \( \mathcal{U}(L, \xi, \sigma) \) are not uniquely defined, since also \( \mathcal{U}'(L, \xi, \sigma) = \mathcal{U}(L, \xi, \sigma) + \alpha(\sigma) \) is a potential provided \( d\alpha(\sigma) = 0 \). [Authors’ note: if physical sense has to be given to integrals of the potential \( \mathcal{U}(L, \xi, \sigma) \) on a closed region \( C \) which are not boundaries, then a canonical way of choosing a representative \( \mathcal{U}(L, \xi, \sigma) \) has to be provided, the integral depending on the form \( \alpha(\sigma) \). Of course if we change representatives then the integral values change.]

Specializing to standard General Relativity, a superpotential exists having the form

\[
\mathcal{U}_{\text{Kom}}(L, \xi, \sigma) = \sqrt{g} \nabla^\mu \xi^\nu ds_{\nu\mu}
\]

(10.3)

It was given by Komar (see [22]) for a timelike Killing vector \( \xi \) and then generalized to an arbitrary vector field. Here \( ds_{\mu\nu} \) denotes the standard local basis for \((n-2)\)-forms.

Let \( \sigma \) be an asymptotically flat solution of standard General Relativity, \( t \) a Killing vector which is an asymptotic time translation and \( \lambda \) a Killing vector
which is an asymptotic spatial rotation; if one considers the conserved quantities associated to them integrating the Komar superpotential on \textit{space infinity} (which is not a boundary because of singularities), $Q_\infty(L, \lambda, \sigma)$ produces the angular momentum but $Q_\infty(L, t, \sigma)$ gives just one half of the expected mass. This is commonly known as the \textit{anomalous factor problem} (see [31]) which can be \textit{cured} considering a suitable boundary term $B_{ADM}(L, \xi, \sigma)$ such that

\[
\delta X B_{ADM}(L, \xi, \sigma)\bigg|_\infty = i \xi < J^F(L, \gamma) \mid j^1 X > \bigg|_\infty
\]  

(10.4)

Then one can correct the definition of conserved quantities in the following way

\[
\mathcal{M}_{ADM} = \int_\infty [U_{Kom}(L, t, \sigma) - B_{ADM}(L, t, \sigma)]
\]

\[
\mathcal{J}_{ADM} = -\int_\infty [U_{Kom}(L, \lambda, \sigma) - B_{ADM}(L, \lambda, \sigma)]
\]

(10.5)

which both give the expected values since the correction on angular momentum vanishes. The recipe to define $B_{ADM}(L, \xi, \sigma)$ was given by Arnowitt, Deser and Misner (ADM) using a $(n+1)$ decomposition and it is generally known as \textit{ADM formalism}.

In [1] a general covariant metric theory is considered; it is described by a Lagrangian of the form

\[
L = \mathcal{L}(g_{\mu\nu}, R_{\alpha\beta\mu\nu}, \nabla_{\lambda} R_{\alpha\beta\mu\nu}, \ldots, \nabla_{(\lambda_1 \ldots \lambda_m)} R_{\alpha\beta\mu\nu}), \, ds
\]  

(10.6)

That is simply a particular class of natural (and thence gauge-natural) Lagrangians.

A stationary, asymptotically flat solution $\sigma$ is considered and assumed to have a bifurcate Killing horizon (see [32], [45]). Let us consider a Killing vector $\xi = t + \Omega_\parallel \lambda$ which vanishes on the bifurcation $(n-2)$-surface $\Sigma$ which is the only \textit{internal boundary}. To correct anomalous factor, a boundary term $\tilde{B}$ analogous to (4.6) is considered. [Authors’ note: the boundary correction $\tilde{B}$ in [1] is explicitly not required to be covariant so that it is definitely local and coordinate dependent! The \textit{corrected} conserved quantities are thence undefined, being the integrand \textit{a priori} not even defined on the domain of integration. Even assuming that $\tilde{B}$ is defined on the whole of space infinity, still the integral is coordinate dependent, against the very basic prescriptions of General Relativity. Alternatively, this quantities are, at best, related to some preferred (and undefined) class of coordinate systems. Of course these problems are avoided when introducing the background connection $\Gamma$ as shown in Section 4. We stress that strictly speaking $\Gamma$ has not to be regard as a background field, but
as a parameter; thus it does not violate prescriptions of General Relativity. As we already explained, the conserved quantities one defines using $\Gamma$ have to be interpreted as *relative* conserved quantities with respect to $\Gamma$.

Then the entropy of this stationary black hole is implicitly defined by

$$\frac{\kappa}{2\pi} \delta_X S = \int_{\Sigma} \delta_X U(L, \xi, \sigma)$$

(10.7)

where $X = \delta g$ is a *deformation*, i.e. a vertical vector field, and where $\kappa$ is the *surface gravity* defined by

$$\kappa^2 = -\frac{1}{2} \nabla^a \xi^b \nabla_a \xi_b$$

(10.8)

and it can be shown to be constant on $\Sigma$.

Of course $\delta S$ is not well defined since $\Sigma$ is not a boundary so that one has to explain why (or at least how) a particular representative for $U(L, \xi, \sigma)$ is chosen. [Authors’ note: of course Komar superpotential is a possible choice but it is a superpotential for *standard* General Relativity with the *standard* Hilbert-Einstein Lagrangian; here $U(L, \xi, \sigma)$ is a superpotential of the general covariant metric theory taken under consideration. In [1] an algorithm is provided for building such a representative, but it is there also claimed that it is *one* of many possible algorithms, none of which seems to be the *best* to be chosen.] Anyway, if we want variations of entropy to be defined we also have to ensure some regularity condition on the way the representative $U(L, \xi, \sigma)$ for the superpotential depends on the section $\sigma$.

Provided that all these *details* can be fixed, now first evolution law of black holes has to be derived; it establishes a relation between variations of the entropy, the mass and angular momentum. This is of course tricky because entropy is computed integrating on $\Sigma$ while mass and angular momentum are computed on spatial infinity. Anyway, one can use the following property which holds if $\mathcal{L}_\xi \sigma = 0$, $\sigma$ is a solution of field equations and $X$ is a [global] solution of the linearized field equations:

$$d[\delta_X U(L, \xi, \sigma) - i_\xi < I F(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X >] = 0$$

(10.9)

Eq. (10.9) integrated on $M$ gives

$$\int_{\infty - \Sigma} [\delta_X U(L, \xi, \sigma) - i_\xi < I F(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X >] = 0$$

(10.10)
Now $\xi = t + \Omega \lambda$ and $i_\xi < \mathcal{F}(L, \gamma) | j^{k-1}X >$ (which is linear in $\xi$) vanish on $\Sigma$; thus it is easy to show that:

$$\frac{\kappa}{2\pi} \delta X S = \int_{\Sigma} \delta X U(L, \xi, \sigma) =$$

$$= \int_{\Sigma} \left[ \delta X U(L, \xi, \sigma) - i_\xi < \mathcal{F}(L, \gamma) \circ j^{2k-1}X | j^{k-1}X > \right] =$$

$$= \int_{\Sigma} \left[ \delta X U(L, \xi, \sigma) - i_\xi < \mathcal{F}(L, \gamma) \circ j^{2k-1}X | j^{k-1}X > \right] +$$

$$- \int_{\Sigma} \left[ \delta X U(L, \xi, \sigma) - i_\xi < \mathcal{F}(L, \gamma) \circ j^{2k-1}X | j^{k-1}X > \right] =$$

$$= \int_\infty \left[ \delta X U(L, \xi, \sigma) - B(L, \xi, \sigma) \right] = \delta X \mathcal{M}_{ADM} - \Omega \delta X \mathcal{J}_{ADM}$$

$$\Rightarrow \delta X \mathcal{M}_{ADM} = \frac{\kappa}{2\pi} \delta X S + \Omega \delta X \mathcal{J}_{ADM}$$

which is in fact the first law of thermodynamics. Again $\mathcal{M}_{ADM}$ and $\mathcal{J}_{ADM}$ are a generalization of ADM conserved quantities (10.5) which reproduce the standard ones in standard General Relativity when $U(L, \xi, \sigma) = U_{Kon}(L, \xi, \sigma)$ and $B(L, \xi, \sigma) = B_{ADM}(L, \xi, \sigma)$. We stress that it is essential for this result that the term $i_\xi < \mathcal{F}(L, \gamma) | j^{k-1}X >$ vanishes on $\Sigma$. Under these hypotheses, eq. (10.7) provides an expression for the variation of the entropy. If one wants to have an expression for the entropy itself, one has to look for a new quantity $\tilde{U}(L, \xi, \sigma)$ satisfying the variational equation

$$\delta X U(L, \xi, \sigma) = \kappa \delta X \tilde{U}(L, \xi, \sigma)$$

(10.12)

so that one has

$$S = 2\pi \int_{\Sigma} \tilde{U}(L, \xi, \sigma)$$

(10.13)

We stress that the vanishing of $i_\xi < \mathcal{F}(L, \gamma) | j^{k-1}X >$ on $\Sigma$ is completely useless to derive (10.11), if one defines the variation of the entropy, instead of using (10.7), by means of the following

$$\frac{\kappa}{2\pi} \delta X S = \int_{\Sigma} \left[ \delta X U(L, \xi, \sigma) - i_\xi < \mathcal{F}(L, \gamma) \circ j^{2k-1}X | j^{k-1}X > \right]$$

(10.14)
Accordingly, the evaluation on $\Sigma$ is a further step needed to simplify expression (10.14). In fact, the integral

$$\int_{\Sigma} \left[ \delta_X U(L, \xi, \sigma) - i_\xi < \mathcal{F}(L, \gamma) \circ j^{2k-1} \sigma \mid j^{k-1} X > \right]$$

is completely unaffected by deformations of the region $\Sigma'$ because of (10.9).

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