Abstract—The sequence reconstruction problem, introduced by Levenshtein in 2001, considers a communication channel in which the receiver obtains a noisy version of a codeword transmitted by the sender. The problem is to determine the number of channel outputs required to uniquely reconstruct the transmitted codeword. Levenshtein showed that the problem remains open for the case where the channel involves insertion errors only and the codebook comprises of all binary words. Recently, in [13]–[15], the authors study the problem of code design under such scenarios. Specifically, they fix the number of channel outputs available to the receiver and design codes that allow the receiver to uniquely reconstruct the transmitted codeword.

In this work, we revisit the open problem posed by Levenshtein (and later by Gabrys and Yaakobi) and provide an asymptotic solution for all values of \( t \geq 2 \). Specifically, let \( 1 \leq \ell \leq t \) and we transmit codewords from an \((\ell-1)\)-deletion-correcting code over a \( t \)-deletion channel. In this work, we first show that the number of channels required for unique reconstruction is upper bounded by the quantity \((2^t)n^{t-\ell}\) (see Theorem 4). Subsequently, we provide a matching lower bound. That is, we construct a pair of codewords \( X \) and \( Y \) with Levenshtein distance at least \( \ell \) and show that the number of channels required to disambiguate \( X \) or \( Y \) is at least \((2^t)n^{t-\ell}\). This therefore implies that our estimate is asymptotically exact (see Theorem 5). Furthermore, in the special case where \( \ell = t \), we determine that \( N(n, \ell, t) = \binom{n}{t} \).

Before we formally state our contributions, we briefly remark on our proof techniques. The main difficulty lies with the proof of the upper bound. While our arguments bear certain similarities to that of Levenshtein and Gabrys and Yaakobi, an analysis that mimics these works is too tedious. Instead, we turn to techniques in subsequence combinatorics. Of particular interest is [16], where Elzinga et al. developed certain recursion rules and used dynamic programming to provide quadratic-time algorithms to enumerate certain subsequence problems. In our paper, we modify these recursion rules to provide an inductive proof of the upper bound in Section III. We believe that the recursion rules developed in this work will provide insights for other problems related to sequence reconstruction.

I. INTRODUCTION

The sequence reconstruction problem, introduced by Levenshtein in 2001, considers a communication channel in which the receiver obtains a noisy version of a codeword transmitted by the sender. The problem is to determine the number of channel outputs required to uniquely reconstruct the transmitted codeword. Levenshtein studied the sequence reconstruction problem for the \((t-1)\)-deletion channel. For the case where \( \ell \) is the set of all binary sequences, Levenshtein determined the minimum number of channel outputs required for unique reconstruction. However, when \( \ell \) is a \((t-1)\)-deletion-correcting code where \( \ell \geq 1 \), little results are known. Only recently, Gabrys and Yaakobi solved the sequence reconstruction problem for the \((t-1)\)-deletion channel when \( \ell \) is a single-deletion-correcting code and in the paper, they noted that the problem remains open for the case where \( \ell \) is an \((t-1)\)-deletion correcting codes for \( \ell \geq 2 \).

However, little progress was made on this open problem. Nevertheless, there was a slew of related results. The sequence reconstruction problem was solved in the following instances:

- when the channel involves insertions only and the codebook is any \( e \)-insertion error-correcting code;
- when the channel involves combinations of single substitution and single insertion error and the codebook comprises of all binary words [12].

Recently, in [13]–[15], the authors study the problem of code design under such scenarios. Specifically, they fix the number of channel outputs available to the receiver and design codes that allow the receiver to uniquely reconstruct the transmitted codeword.

In this work, we revisit the open problem posed by Levenshtein (and later by Gabrys and Yaakobi) and provide an asymptotic solution for all values of \( \ell \geq 2 \). Specifically, let \( 1 \leq \ell \leq t \) and we transmit codewords from an \((\ell-1)\)-deletion-correcting code over a \( t \)-deletion channel. In this work, we first show that the number of channels required for unique reconstruction is upper bounded by the quantity \((2^t)n^{t-\ell}\) (see Theorem 4). Subsequently, we provide a matching lower bound. That is, we construct a pair of codewords \( X \) and \( Y \) with Levenshtein distance at least \( \ell \) and show that the number of channels required to disambiguate \( X \) or \( Y \) is at least \((2^t)n^{t-\ell}\). This therefore implies that our estimate is asymptotically exact (see Theorem 5). Furthermore, in the special case where \( \ell = t \), we determine that \( N(n, \ell, t) = \binom{n}{t} \).

Before we formally state our contributions, we briefly remark on our proof techniques. The main difficulty lies with the proof of the upper bound. While our arguments bear certain similarities to that of Levenshtein and Gabrys and Yaakobi, an analysis that mimics these works is too tedious. Instead, we turn to techniques in subsequence combinatorics. Of particular interest is [16], where Elzinga et al. developed certain recursion rules and used dynamic programming to provide quadratic-time algorithms to enumerate certain subsequence problems. In our paper, we modify these recursion rules to provide an inductive proof of the upper bound in Section III. We believe that the recursion rules developed in this work will provide insights for other problems related to sequence reconstruction.

II. PRELIMINARIES

Let \( \Sigma \) denote the binary alphabet \( \{0, 1\} \). We use \( \Sigma^n \) to denote the set of all length-\( n \) binary sequences.

Let \( x \in \Sigma^n \). Then the deletion ball of radius \( t \) centered at \( x \) is defined to be the set of all length-(\( n-t \)) subsequences of \( x \) and this ball is denoted by \( D_t(x) \). Given two binary sequences \( x \) and \( y \) with \( |y| = |x| + k \), we are interested in the intersection of their deletion balls and we use \( D(x, y, t, t+k) \) to denote the set \( D_t(x) \cap D_{t+k}(y) \). Furthermore, we define their Levenshtein distance to be \( d_L(x, y) \triangleq \min \{ t \geq 0 : D(x, y, t, t+k) \neq \emptyset \} \). Equivalently, if \( d_L(x, y) = \ell \), we have that

\[
D(x, y, \ell, \ell+k) \neq \emptyset \text{ and } D(x, y, \ell-1, \ell+k-1) = \emptyset.
\]

Hence, a codebook \( C \) is an \((\ell-1)\)-deletion-correcting code if \( d_L(x, y) \geq \ell \) for all distinct \( x, y \in C \).
We now formally define our problem statement. For \(1 \leq \ell \leq t < n\), the task of sequence reconstruction problem for deletion channels is to determine the following quantity.

\[
N(n, \ell, t) \triangleq \max\left\{ |\mathcal{D}(x, y, t, k)| : x, y \in \Sigma^n, d_L(x, y) \geq \ell \right\}.
\]

(1)

Suppose we have an \((\ell - 1)\)-deletion correcting code \(\mathcal{C}\) of length \(n\). If a codeword from \(\mathcal{C}\) is transmitted over a \(t\)-deletion channel, Levenshtein showed that \(N(n, \ell, t) + 1\) distinct channel outputs are sufficient to allow unique reconstruction of the transmitted word \(\mathcal{C}\).

For \(\ell \in \{1, 2\}\), the exact values \(N(n, \ell, t)\) have been determined in [1], [10]. To state these results, we require the maximum size of a \(t\)-deletion ball. Specifically, for \(0 \leq t < n\), we use \(D(n, t)\) to denote the quantity \(\max\{ |\mathcal{D}(x) : x \in \Sigma^n\} \). We know from [18] that

\[
D(n, t) = \sum_{i=0}^{t} \binom{n-t}{i} = \frac{1}{t!} n^t - O(n^{t-1}),
\]

(2)

and the maximum is achieved when the \(x\) is alternating. For convenience, we extend the domain of \((1)\) to include \(\ell = 0\) and hence, \(N(n, 0, t)\) is given by \(D(n, t)\).

When \(\ell = 1\), we have the following landmark result of Levenshtein.

**Theorem 1** (Levenshtein [1]). For \(1 \leq t < n\),

\[
N(n, 1, t) = 2D(n - 2, t - 1) = \frac{2}{(t-1)!} n^{t-1} - O(n^{t-2}).
\]

(3)

The result for \(\ell = 2\) is obtained more than a decade later by Gabrys and Yaakobi.

**Theorem 2** (Gabrys and Yaakobi [10]). For \(2 \leq t < n\) and \(n \geq 8\),

\[
N(n, 2, t) = 2D(n - 4, t - 2) + 2D(n - 5, t - 2) + 2D(n - 7, t - 2) + D(n - 6, t - 3) + D(n - 7, t - 3) = \frac{6}{(t - 2)!} n^{t-2} - O(n^{t-3}).
\]

(4)

**A. Our Contributions**

In this paper, we provide asymptotically exact estimates of \(N(n, \ell, t)\) for all values of \(0 \leq \ell \leq t\). Specifically, we establish the following theorem.

**Theorem 3** (Main Theorem). For \(0 \leq \ell \leq t < n\), we have that

\[
N(n, \ell, t) = \left(\frac{2\ell}{t}\right) \left(\frac{2\ell}{(t-\ell)!}\right) n^{t-\ell} - O(n^{t-1}).
\]

(5)

**Remark 1.**

- Observe that when \(\ell \in \{0, 1, 2\}\), the main theorem recovers the asymptotic estimates of [2], [3], and [4], respectively.
- In [3], [4], [4], [4], we use the usual big \(O\) notation where the asymptotics is measured in terms of \(n\). That is, \(f(n) = O(g(n))\) means that \(\limsup_{n \to \infty} f(n) / g(n)\) is bounded by some constant \(C\). Also, we emphasize that whenever the big \(O\) notation is used, both \(f(n)\) and \(g(n)\) are positive functions. Hence, for example, from [2], we have that \(D(n, t)\) is at most \(n^t / t!\).

Here, we outline the proof for the main theorem. First, we demonstrate an upper bound for \(N(n, \ell, t)\) in Section III. In particular, we study a general version of the quantity \(N(n, \ell, t)\) where the transmitted sequences are of different lengths. Specifically, we consider two binary sequences \(x, y\) with \(|y| = |x| + k\) and \(k \geq 0\), set \(\Delta(x, y, t, k)\) to be the size of the intersection \(\mathcal{D}(x, y, t, k)\). Section III is dedicated to an induction proof of the following theorem.

**Theorem 4** (Upper bound). Let \(0 \leq \ell \leq t \leq n\). Suppose that \(x \in \Sigma^n\) and \(y \in \Sigma^{n+k}\) with \(k \geq 0\). If \(d_L(x, y) \geq \ell\), then we have

\[
\Delta(x, y, t, k) \leq \frac{k^{2\ell}}{(t-\ell)!} n^{t-\ell}.
\]

(6)

Next, in Section IV, we provide a matching lower bound. Specifically, we demonstrate the following proposition.

**Proposition 5** (Lower Bound). Fix \(\ell > 0\). For \(n \geq 4\ell - 2\), there exists two sequences \(X, Y \in \Sigma^n\) such that \(d_L(X, Y) \geq \ell\) and

\[
\Delta(X, Y, t, t) \geq \left(\frac{2\ell}{t}\right) D(n - 4\ell + 2, t - \ell) = \frac{2^{\ell}}{(t-\ell)!} n^{t-\ell} - O(n^{t-1}), \text{ for all fixed } t \geq \ell.
\]

Theorem 5 now follows from Theorem 4 and Proposition 5.

Observe that when we set \(\ell = 1\) and \(k = 0\) in \(6\), we have that \(N(n, t, t) \leq \left(\frac{2\ell}{t}\right)\). On the other hand, for \(n \geq 4\ell - 2\), it follows from \(7\) that \(N(n, t, t) \geq \left(\frac{2\ell}{t}\right)\). Therefore, we have determined the exact value of \(N(n, t, t)\).

**Corollary 6.** Set \(\ell = t\). For \(n \geq 4t - 2\), we have that

\[
N(n, \ell, t) = \left(\frac{2\ell}{t}\right) = \Theta(1) \text{ for fixed values of } t.
\]

Next, we make a conjecture on the exact value of \(N(n, \ell, t)\).

**Conjecture.** For \(1 \leq \ell \leq t < n\) and sufficiently large \(n\), we have that

\[
N(n, \ell, t) = N(n - 1, \ell, t) + N(n - 2, \ell, t - 1).
\]

(8)

Finally, in the spirit of Levenshtein’s work [1], [9], we provide a polynomial-time reconstruction method in Section V for the special case where \(\ell = t\).

**Proposition 7.** Let \(\mathcal{C}\) be an \((t - 1)\)-deletion-correcting code of length \(n\) for some \(2 \leq t < n\). Suppose further that \(\mathcal{C}\) has an \((t - 1)\)-deletion-correcting decoder that runs in \(T(n)\) time. If we transmit \(x \in \mathcal{C}\) over a \(t\)-deletion channel and obtain \(M \triangleq N(n, t, t) + 1 = \left(\frac{2\ell}{t}\right) + 1\) distinct outputs, then we can determine \(x\) in time \(O(T(n) + M n)\) time.

So, in particular, when \(t = 2\), the classic Varshamov-Tenengolts (VT) codes are single-deletion-correcting codes equipped with a linear-time decoder [17]. Furthermore, [4] states that \(N(n, 2, 2) + 1 = 7\) and so, we can uniquely reconstruct a codeword from any distinct seven reads. Then Proposition 7 implies that this reconstruction can be done in linear time.

### III. Upper Bound

In this section, we prove Theorem 4 using induction. For convenience, we rewrite the upper bound in the following form that is amenable to an inductive analysis.
Theorem 8. Let $S(\ell, t, k)$ denotes the statement:
For all $n \geq t$, we have that
$$\Delta(x, y, t + k) \begin{cases} \leq (\frac{k + 2t}{\ell - t})^{n-t}, & \text{if } t \geq \ell, \\ = 0, & \text{if } t < \ell. \end{cases}$$
Then $S(\ell, t, k)$ is true for all $\ell, t, k \geq 0$.

Note that in Theorem 8 we extend the domain to include the cases where $t < \ell$. We justify this in the following lemma where we demonstrate $S(\ell, t, k)$ for certain bases cases.

Lemma 9 (Base Cases). The following instances are true.
(i) $S(0, t, k)$ is true for all $t, k \geq 0$.
(ii) $S(\ell, t, k)$ is true for all $0 \leq t < \ell$.

Proof. For (i), we have $\ell = 0$. Since $D(x, y, t, t + k) \subseteq D_t(x)$, we apply (2) to have $\Delta(x, y, t + k) \leq n!t!$. For (ii), since $t < \ell$, it follows from the definition of Levenshtein distance that $D(x, y, t, t + k) = \varnothing$. In other words, $\Delta(x, y, t + k) = 0$. \(\square\)

Hence, it remains to demonstrate the induction step. To this end, we define a total order on the set of triples $\{(\ell, t, k) : \ell, t, k \geq 0\}$. Specifically, we use $\prec$ to denote the usual lexicographic order on the triples. That is, $(\ell, t, k) \prec (\ell_0, t_0, k_0)$ means one of the following:
- $\ell < \ell_0$, or
- $\ell = \ell_0$ and $t < t_0$, or
- $\ell = \ell_0$, $t = t_0$ and $k < k_0$.
It is well-known that the lexicographic order defines a total order on the set of triples. Hence, in any nonempty subset of triples, there is always a smallest triple with respect to $\prec$.

Now, we are ready to state the induction step.

Lemma 10 (Induction Step). Suppose that $0 < \ell_0 \leq t_0$ and $k_0 \geq 0$. If $S(\ell, t, k)$ is true for $(0, 0, 0) \prec (\ell_0, t_0, k_0)$, then $S(\ell_0, t_0, k_0)$ is true.

As the proof of Lemma 10 is fairly technical, we defer the detailed arguments to Subsection 11.8. In what follows, we assume that the lemma is true and complete the induction proof of Theorem 8.

Proof of Theorem 8. Suppose otherwise that $S(\ell, t, k)$ fails to hold for some triple $(\ell_0, t_0, k_0)$ with respect to the order $\prec$. Since the triple is smallest, we have that $S(\ell, t, k)$ is true for all $(\ell, t, k) \prec (\ell_0, t_0, k_0)$. Furthermore, Lemma 9 implies that $\ell_0 > 0$ and $t \geq 0$.

Therefore, the conditions of Lemma 10 are met and so, $S(\ell_0, t_0, k_0)$ must be true, contradicting our assumption. \(\square\)

For the rest of this section, we prove the induction step, and we adopt for the following convention. For $x \in \Sigma^n$, we write $x$ as $x_1x_2 \cdots x_k$. In other words, for $1 \leq i \leq n$, the $i$th bit of $x$ is denoted by $x_i$. Furthermore, the $\ell$th prefix of $x$ is denoted by $x^\ell$. That is, $x^\ell = x_1x_2 \cdots x_\ell$. Similarly, for $k \geq 0$, we consider a binary sequence $y \in \Sigma^{n+k}$ and let $y = y_1y_2 \cdots y_{n+k}$.

A. Recursion Rules

Our induction relies on two recursion rules, Lemmas 12 and 13. To state the recursion rules, we use the following notation used extensively in [10]. Given a bit $a \in \Sigma$ and a set $X$ of nonempty binary sequences, we use $X_a$ to denote the set of sequences in $X$ that end with $a$. Also, we use $X \circ a$ to denote the set of sequences obtained by appending $a$ to all sequences in $X$. Hence, $|X \circ a| = |X|$ while $|X_a| \leq |X|$.

Then the following result is folklore.

Lemma 11. Given $x \in \Sigma^n$ and $a \in \Sigma$, let $i$ be the largest index integer such that $x_i = a$, then we have that $(D_t(x))_a = D_{t-i}(x^{i}) \circ a$. Therefore, the deletion ball centered at $x$ can be recursively computed using the rule:
$$D_t(x) = \left(D_t\left(x^{(n-1)}\circ x_n\right) \cup D_{t-1}(x^{(n-1)})\right) \circ a$$

Our first recursion rule provides a lower bound on the Levenshtein distance and is simple modification of the usual recursion rules used in dynamic programming (see for example, [16]).

Lemma 12. Suppose that $x \in \Sigma^n$ and $y \in \Sigma^{n+k}$ with $k \geq 0$ and $d_L(x, y) \geq \ell$.
- When $x_n = y_{n+k}$, $d_L(x^{(n-1)}, y^{(n+k-1)}) \geq \ell$.
- When $x_n \neq y_{n+k}$ and $k = 0$, $d_L(x^{(n-1)}, y) \geq \ell - 1$, $d_L(x, y^{(n+k-1)}) \geq \ell - 1$.
- When $x_n \neq y_{n+k}$ and $k > 0$, $d_L(x^{(n-1)}, y) \geq \ell - 1$, $d_L(x, y^{(n+k-1)}) \geq \ell$. \(9\)

Proof. If $x_n = y_{n+k}$, we claim that $D(x^{(n-1)}, y^{(n+k-1)}), \ell - 1, \ell + k - 1) = \varnothing$. Since otherwise that $z$ belongs to $D(x^{(n-1)}, y^{(n+k-1)}), \ell - 1, \ell + k - 1)$, then we have $z \circ x_n = z \circ y_{n+k} \in D(x, y, \ell - 1, \ell + k - 1)$, which is a contradiction.

If $x_n \neq y_{n+k}$, we claim $D(x^{(n-1)}, y, \ell - 2, \ell + k - 1) = \varnothing$. Again, suppose otherwise that $z \in D(x^{(n-1)}, y, \ell - 2, \ell + k - 1)$. Then we have $z \in D(x, y, \ell - 1, \ell + k - 1)$ because $D_{t-2}(x^{(n-1)}) \subseteq D_{t-1}(x)$.

- When $k = 0$, we have $D(x, y^{(n+k-1)}, \ell - 1, \ell + k - 2) = \varnothing$ by symmetry.
- When $k > 0$, we claim $D(x, y^{(n+k-1)}, \ell - 1, \ell + k - 2) = \varnothing$. Again, suppose otherwise that $z \in D(x, y^{(n+k-1)}, \ell - 1, \ell + k - 2)$. Then $z \in D(x, y, \ell - 1, \ell + k - 1)$ because $D_{t+k-2}(y^{(n+k-1)}) \subseteq D_{t+k-1}(y)$. This is a contradiction. \(\square\)

The next recursion rule is crucial to our inductive proof. In particular, we show that we can bound the size of $D(x, y, t, t + k)$ using the corresponding values for the prefixes of $x$ and $y$.

Lemma 13. Let $x \in \Sigma^n$, $y \in \Sigma^{n+k}$. Then the following are true.
- If $x_n = y_{n+k}$, then
$$\Delta(x, y, t + k) \leq \Delta(x^{(n-1)}, y^{(n+k-1)}, t + k)$$
- $+ \Delta(x^{(n-1)}, y^{(n+k-1)}, t - 1, t + k - 1)$.
- If $x_n \neq y_{n+k}$, then
$$\Delta(x, y, t + k) \leq \Delta(x, y^{(n+k-1)}, t, t + k - 1)$$
- $+ \Delta(x^{(n-1)}, y, t - 1, t + k).$

Proof. In both cases, we apply Lemma 11.
When \( x_n = y_{n+k} \),
\[
\Delta(x, y, t, t + k) = |D(x, y, t, t + k)_{x_n}| + |D(x, y, t, t + k)_{y_n}|
\leq \Delta(x^{(n-1)}, y, t, t + k) + \Delta(x^{(n-1)}, y^{(n+k-1)}, t, t + k - 1).
\]

When \( x_n \neq y_{n+k} \),
\[
\Delta(x, y, t, t + k) = |D(x, y, t, t + k)_{x_n}| + |D(x, y, t, t + k)_{y_n}|
\leq \Delta(x, y^{(n+k-1)}, t, t + k - 1) + \Delta(x^{(n-1)}, y, t, t + k - 1).
\]

\[
\Box
\]

\hspace{1em}

**B. Proof of Induction Step**

Finally, we prove Lemma 10. Specifically, we suppose that \( 0 < \ell_0 \leq t_0 \) and \( k_0 \geq 0 \) and assume \( S(\ell, t, k) \) is true for \( (t, k) < (t_0, t_0, k_0) \). Our aim is to show that \( S(\ell_0, t_0, k_0) \) is true. In other words, we show that (9) is true for all \( n \geq t_0 \) and we do so by induction on \( n \).

Suppose that \( n = t_0 \). Then the set \( D(x, y, t_0, t_0 + k_0) \) is a singleton set that comprises the empty string. So, we have:
\[
\Delta(x, y, t_0, t_0 + k_0) = 1 \leq \frac{(2\ell_0)}{(t_0 - \ell_0)!} (n_0 + 1)^{t_0 - \ell_0}.
\]
The last inequality holds because \((t_0 + 1)^{t_0 - \ell_0} \geq (t_0 - \ell_0)! (n_0 + 1)^{t_0 - \ell_0} \geq (t_0 - \ell_0)! \).

Next, we assume that (9) is true for all \( n \leq n_0 \). We will prove that, for \( x \in \Sigma^{n_0 + 1} \) and \( y \in \Sigma^{n_0 + k_0 + 1} \) with \( d_L(x, y) \geq \ell_0 \), we have:
\[
\Delta(x, y, t_0, t_0 + k_0) \leq \frac{(2\ell_0)}{(t_0 - \ell_0)!} (n_0 + 1)^{t_0 - \ell_0}.
\]

We have the following two cases.

(I) When \( x_{n_0+1} = y_{n_0+k_0+1} \), it follows from (9) that
\[
d_L(x^{(n_0)}, y^{(n_0+k_0)}) \geq \ell_0.
\]

When \( t_0 > \ell_0 \), Lemma 13 implies that
\[
\Delta(x, y, t_0, t_0 + k_0) \leq \Delta(x^{(n_0)}, y^{(n_0+k_0)}, t_0, t_0 + k_0) + \Delta(x^{(n_0)}, y^{(n_0+k_0)}, t_0 - 1, t_0 + k_0 - 1)
\leq \frac{(k_0 + 2\ell_0)}{(t_0 - \ell_0)!} (n_0 + 1)^{t_0 - \ell_0} + \frac{(k_0 + 2\ell_0)}{(t_0 - \ell_0 - 1)!} (n_0 + 1)^{t_0 - \ell_0 - 1}
\leq \frac{(k_0 + 2\ell_0)}{(t_0 - \ell_0)!} (n_0 + 1)^{t_0 - \ell_0},
\]
as desired. The last inequality follows from the fact that \((n + 1)^{t_0 - \ell_0} = \sum_{i=0}^{n_0} (i + 1)^{t_0 - \ell_0} i^n i! \).

On the other hand, if \( t_0 = \ell_0 \), we have \( \Delta(x^{(n_0)}, y^{(n_0+k_0)}, t_0 - 1, t_0 + k_0 - 1) = 0 \). It is then not difficult to proceed as above and show that \( \Delta(x, y, t_0, t_0 + k_0) \leq \frac{(k_0 + 2\ell_0)}{(t_0)!} \).

(II) Suppose that \( x_{n_0+1} \neq y_{n_0+k_0+1} \). When \( k_0 = 0 \), Lemma 10 implies that \( d_L(x, y^{(n_0)}) \geq \ell_0 - 1 \) and \( d_L(x^{(n_0)}, y) \geq \ell_0 - 1 \). Again, we apply Lemma 13 to have
\[
\Delta(x, y, t_0 + 1, t_0 + 1) \leq \Delta(x, y^{(n_0)}, t_0 + 1, t_0) + \Delta(x^{(n_0)}, y, t_0, t_0 + 1)
\leq \frac{2(2\ell_0 - 1)}{(t_0 - \ell_0)!} n_0^{t_0 - \ell_0} + \frac{(2\ell_0 - 1)}{(t_0 - \ell_0 - 1)!} n_0^{t_0 - \ell_0 - 1}
\leq \frac{2(2\ell_0 - 1)}{(t_0 - \ell_0)!} n_0^{t_0 - \ell_0} \leq \frac{2(2\ell_0 - 1)}{(t_0 - \ell_0)!} (n_0 + 1)^{t_0 - \ell_0 - 1}.
\]

For the last inequality, observe that \( 2(2\ell_0 - 1) = (2\ell_0) \).

On the other hand, when \( k_0 > 0 \), (11) implies that \( d_L(x, y^{(n_0+k_0)}) \geq \ell_0 \) and \( d_L(x^{(n_0)}, y) \geq \ell_0 - 1 \). Again, applying Lemma 13, we have that
\[
\Delta(x, y, t_0 + 1, t_0 + k_0)
\leq \Delta(x, y^{(n_0+k_0)}, t_0 + 1, t_0 + k_0 - 1) + \Delta(x^{(n_0)}, y, t_0 - 1, t_0 + k_0)
\leq \frac{(k_0 + 2\ell_0 - 1)}{(t_0 - \ell_0)!} n_0^{t_0 - \ell_0} + \frac{(k_0 + 2\ell_0 - 1)}{(t_0 - \ell_0 - 1)!} n_0^{t_0 - \ell_0 - 1}
\leq \frac{(k_0 + 2\ell_0 - 1)}{(t_0 - \ell_0)!} (n_0 + 1)^{t_0 - \ell_0}.
\]

This completes the induction proof.

**IV. LOWER BOUND**

In this section, we prove Proposition 5. Specifically, for \( n \geq 4\ell - 2 \) we explicitly construct two length-\( n \) sequences \( X \) and \( Y \) such that the intersection of their \( t \)-deletion balls has size at least the quantity defined by (7).

To this end, for \( \ell > 0 \), we consider the following two sequences \( A_{\ell} \) and \( B_{\ell} \) of length \( 4\ell - 2 \).
\[
A_{\ell} \triangleq (11010)^{\ell-1}10
B_{\ell} \triangleq (01101)^{\ell-1}101.
\]

Here, \( x^\ell \) denotes the concatenation of \( \ell \) copies of \( x \). Now, using \( A_{\ell} \) and \( B_{\ell} \), we construct the desired sequences \( X \) and \( Y \). Specifically, let \( Z \) be an alternating sequence of length \( n - 4\ell + 2 \) that starts with one and we set
\[
X \triangleq A_{\ell}Z \quad \text{and} \quad Y \triangleq B_{\ell}Z.
\]

Now, to show that \( X \) and \( Y \) satisfy the conditions of Proposition 5, we first consider \( A_{\ell} \) and \( B_{\ell} \) only, and exhibit the required properties.

Now, we show that \( A_{\ell} \) and \( B_{\ell} \) have the required Levenshtein distance. To this end, we consider the number of runs in both \( A_{\ell} \) and \( B_{\ell} \). Formally, for \( x \in \Sigma^n \), a run of \( x \) refers to contiguous repetition of the same bit and we denote the number of runs in \( x \) with \( R(x) \). The following lemma bounds the changes to \( R(x) \) when we delete bits from \( x \).

**Lemma 14.** Let \( x \in \Sigma^n \) and suppose that we delete the bit \( x_i \) from \( x \) to obtain \( x' \). Then
- If \( i = 1 \) or \( i = n \), then \( R(x) - 1 \leq R(x') \leq R(x) \).
- If \( 1 < i < n \), then \( R(x) - 2 \leq R(x') \leq R(x) \).

We are ready to show that \( A_{\ell} \) and \( B_{\ell} \) are far apart in terms of Levenshtein distance.
Lemma 15. For all $\ell > 0$, we have $d_L(A_\ell, B_\ell) \geq \ell$.

Proof. First, we count the number of runs in each sequence. Clearly, $R(A_\ell) = 4\ell - 2$ and $R(B_\ell) = 2\ell$.

It remains to show that $D(A_\ell, B_\ell, 0 - 1, \ell - 1) = \emptyset$. Suppose that we delete $\ell - 1$ bits from $A_\ell$ and $B_\ell$ to obtain $A'$ and $B'$, respectively. We consider two cases.

If we delete the first bit of $A_\ell$, then by Lemma\[14\]
\[
R(A') \geq 4\ell - 2 - 1 - 2(\ell - 2) = 2\ell + 1 > R(B_\ell) \geq R(B').
\]
Since the number of runs of $A'$ is strictly greater than that of $B'$, we have $A' \neq B'$.

On the other hand, suppose that we do not delete the first bit of $A_\ell$. Then, as before, using Lemma\[14\] we have that $R(A') \geq 2\ell$. However, since the first bit of $A_\ell$ and $B_\ell$ differ, we need to delete the first bit of $B_\ell$ to obtain a common subsequence. Thus, we assume that $R(B') < R(B_\ell)$. Again, we have that $R(A') > R(B')$ and so, $A' \neq B'$.

Next, we show that the Levenshtein distance of $A_\ell$ and $B_\ell$ is indeed $\ell$. Furthermore, the intersection of the corresponding deletion balls is of size at least $\left(\begin{array}{c} \ell \end{array}\right)$.

Lemma 16. For all $\ell > 0$, we have that $D(A_\ell, B_\ell, t, t) \geq \left(\begin{array}{c} \ell \end{array}\right)$.

Proof. We partition the indices of $\{1, 2, \ldots, 4\ell - 2\}$ into pairs $\{1, 2\}, \{3, 4\}, \ldots, \{4\ell - 1, 4\ell - 2\}$ and split $A_\ell$ and $B_\ell$ according to these pairs. For convenience, we call a 01-pair and a 00-pair on these index pairs a 01-block and an 00-block, respectively. Hence, $A_\ell$ comprises $2\ell - 1$ copies of 00-blocks, while $B_\ell$ comprises $\ell$ 01-blocks, and $(\ell - 1)$ 10-separators. We consider the index set $I = \{1 \leq i \leq 4\ell - 2 : i \equiv 1, 2 \text{ (mod 4)}\}$. Notice that the ith bits of $A_\ell$ and $B_\ell$ differ if and only if i belongs to $I$. Furthermore, $I$ corresponds to the $A_\ell$-blocks in $B_\ell$ and so, $|I| = 2\ell$.

We consider the collection of subsequences of $B$ obtained by deleting bits from the 01-blocks of $B_\ell$. In other words,

$E \triangleq \{e \in D_\ell(B_\ell) : \text{we delete } \ell \text{ bits whose indices belong to } I\}$. Clearly, $E \subset D_\ell(B_\ell)$. In what follows, we show that $|E| = \left(\begin{array}{c} \ell \end{array}\right)$ and $E \subset D_\ell(A_\ell)$.

First, let $J$ and $J'$ be two $\ell$-subsets of $I$. Let $e$ and $e'$ be the resulting subsequences of $B_\ell$ obtained by deleting the indices in $J$ and $J'$, respectively. We claim that $e \neq e'$. Indeed, let us consider the smallest block where $J$ and $J'$ differ. That is, $i^* = \min\{1 \leq i \leq \ell : \{4i - 3, 4i - 2\} \cap J \neq \{4i - 3, 4i - 2\} \cap J'\}$. Then, up to the $i^*$-th separator, both $e$ and $e'$ coincide. Since $J$ and $J'$ differ, $e$ and $e'$ differ for the next two bits and so, $e \neq e'$ (see Example\[2\]i)). So, there are $\left(\begin{array}{c} \ell \end{array}\right)$ choices for $J$, we have that $|E| = \left(\begin{array}{c} \ell \end{array}\right)$.

Next, let $e \in E$. We claim that $e \in D_\ell(A_\ell)$. To do so, we insert $\ell$ bits into $e$ to obtain $A_\ell$. Now, recall that only bits in the 01-blocks of $B_\ell$ are deleted. To obtain $A_\ell$, we insert zero, one, or two bits according to the number of deletions in each 01-block. Specifically, we adopt the following rule.

(a) If no bits of a 01-block are deleted, we insert two bits to obtain two additional 10-separators.
(b) If only one bit of a 01-block is deleted, we insert one bit to obtain an additional 10-separator.
(c) If the entire 01-block is deleted, we do not insert any bit.

Now, let the number of 01-blocks with zero, one, and two deletions be $\alpha, \beta$ and $\gamma$, respectively. Because there are $\ell$ 01-blocks deletions, we have $\alpha + \beta + \gamma = \ell$. Also, since the number of deletions is $\ell$, we have that $\beta + 2\gamma = \ell$.

Therefore, the number of 10-separators created is $2\alpha + \beta = 2(\alpha + \beta + \gamma) - (\beta + 2\gamma) = \ell$ and so, $\alpha + \beta + 2\gamma = \ell$. Together with the remaining $(\ell - 1)$ 10-separators in $e$, we have $(2\ell - 1)$ 10-separators which is $A_\ell$ (see Example\[2\]ii)).

Example 2. Let $\ell = 4$. Then

$A_4 = 101000100101010$, 
$B_4 = 0100010010100101$.

Here, we underline the 01-separators and $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14\}$.

(a) We choose four out of the eight indices in $I$ to delete from $B_4$. Consider $J = \{1, 2, 5, 9\}$ and $J' = \{1, 2, 5, 10\}$. Then the resulting subsequences are

$e = 101000100101$, 
$e' = 10100110100101$.

Then we follow the steps in the proof of Lemma\[16\] to find that $i^* = 3$ Indeed, up to the second separator, $e$ and $e'$ share the same prefix $10110$, and the next two bits of $e$ and $e'$ differ.

(b) We insert four bits into both $e$ and $e'$ according to the theorem of Lemma\[16\] and obtain $A_4$. Indeed, $e, e' \in D_4(A_4)$.

$A_4 = 101000100100100101010100$, 
$A_4 = 101000100100100101010100$.

Here, the inserted bits are highlighted in blue.

Proof of Proposition\[5\] Since $X$ and $Y$ contains $A_\ell$ and $B_\ell$, respectively, we have that $d_L(X, Y) \geq d_L(A_\ell, B_\ell) = \ell$.

Next, we provide a lower bound for $d(X, Y, t)$. Recall the definition of $E$ given in the proof of Lemma\[16\] and we consider the following set of length-$(n - t)$ sequences.

$T = \{ef : e \in E, D_{n-t}(Z)\}$.

Since $E \subset D_\ell(A_\ell, B_\ell, \ell, \ell)$, we have that $|T| \geq \left(\begin{array}{c} \ell \end{array}\right)|D_{n-t}(Z)| = \left(\begin{array}{c} \ell \end{array}\right)D(n - 4\ell + 2, t - \ell)$, as required.

V. EFFICIENT RECONSTRUCTION FROM CHANNEL OUTPUTS

Throughout this section, let $t \geq 2$ and $\mathcal{C}$ be an $(\ell - 1)$-deletion-correcting code of length $n$ equipped with a $(\ell - 1)$-deletion-correcting decoder DEC. Specifically, DEC is a map from $\Sigma^{n-t+1}$ to $\mathcal{C} \cup \{\text{FAILURE}\}$ such that

$$\text{DEC}(y) = \begin{cases} x, & \text{if } y \in D_{n-t}(x) \text{ and } x \in \mathcal{C}, \\ \text{FAILURE}, & \text{if } y \notin \bigcup_{x \in \mathcal{C}} D_{n-t}(x). \end{cases}$$

In this section, we propose a simple and efficient algorithm that makes use of DEC to recover a transmitted codeword from their noisy outputs. Specifically, we prove a general version of Proposition\[7\] where we consider a subcode $\mathcal{T}$ of $\mathcal{C}$. 
Proposition 17. Suppose that $\mathcal{C}$ is a $(t-1)$-deletion-correcting code equipped with a decoder $\text{DEC}$ that runs in $O(n)$ time.

Consider a subcode $\mathcal{X}$ with the property that $D(x, x', t, t) < M$ for all pairwise distinct $x, x' \in \mathcal{X}$.

If we are given $M$ distinct outputs $y_1, y_2, \ldots, y_M \in D_t(x)$, then we are able to reconstruct $x$ in $O(T(n) + M)$ time.

We provide a high level description of our reconstruction algorithm. First, we pick any two noisy outputs, say, $y_1$ and $y_2$. Next, we compute two possible candidates $x$ and $x'$. Finally, we use the remaining $M - 2$ outputs to eliminate the incorrect candidate. The following lemma justifies the first step.

Lemma 18. Let $X \subseteq \Sigma^n$ and $Y, Z \in D_t(X)$. Suppose that $U$ is the longest common suffix of $Y$ and $Z$. Without loss of generality, assume that $Y = B0U$ and $Z = C1U$. Then either $B01U \in D_{t-1}(X)$ or $C10U \in D_{t-1}(X)$.

Proof. Let $X = AV$ where $V$ is the shortest suffix of $X$ that contains $U$. Let $k = |V| - |U|$. Then we have $D_k(X)U = D_{k-t}(A) \circ U$. In other words, $B0$ and $C1$ are in $D_{k-t}(A)$, or specifically, $B0 \in D_{k-t}(A)_0$ and $C1 \in D_{k-t}(A)_1$. Now there are two cases to consider:

(i) If $A = A'$, then $D_{k-t}(A) = D_{k-t}(A') = D_{k-t-1}(A')_1$. This means $C1 \in D_{k-t-1}(A')$. Hence, $C10 \in D_{k-t-1}(A)$. Since $D_{k-t-1}(A) \circ U = D_{t-1}(X)U$, we have that $C10U \in D_{t-1}(X)$.

(ii) Similarly, if $A = A'$, then $D_{k-t}(A)_0 = D_{k-t-1}(A')_0 \subseteq D_{k-t-1}(A')$. This means $B0 \in D_{k-t-1}(A')$. Hence, $B01 \in D_{k-t-1}(A)$. Since $D_{k-t-1}(A) \circ U = D_{t-1}(X)U$, we have that $B01U \in D_{t-1}(X)$.

We are now ready to present our algorithm for Proposition 17. Recall that $\mathcal{X}$ is a subcode of a $(t-1)$-deletion-correcting code with the property that $D(x, x', t, t) < M$ for all pairs of codewords $x, x' \in \mathcal{X}$.

INPUT: $y_1, y_2, \ldots, y_M \in D_t(x)$ for some $x \in \mathcal{X}$.

OUTPUT: $x \in \mathcal{X}$

(1) We pick two outputs, $y_1, y_2$, and set $U$ to be the longest common suffix of $y_1$ and $y_2$. Without loss of generality, assume that $y_1 = a10U$ and $y_2 = a21U$.

(2) By Lemma 18 we have two possible scenarios.

- If $a101U \in D_{t-1}(X)$, then we can recover $x$ by using the $(t-1)$-deletion-correcting decoder. Specifically, we set $x_1 \leftarrow \text{DEC}(a101U)$.

- Similarly, if $a210U \in D_{t-1}(X)$, we can also recover $x$ using $\text{DEC}$. So, we set $x_2 \leftarrow \text{DEC}(y_110U)$.

(3) Finally, to distinguish between the two scenarios, we use the remaining outputs $y_3, y_4, \ldots, y_M$. Specifically, if $\{y_3, y_4, \ldots, y_M\} \subseteq D_t(x_1)$, we return the codeword $x_1$. Otherwise, we return $x_2$.

To complete the proof of Proposition 17 we analyse the running time. Clearly, Steps 1 and 2 can be completed in $O(Mn)$ time and $2T(n)$ time respectively. For Step 3, we need to determine if $y_1$ is a subsequence of $x_i$ for $3 \leq j \leq M$ and $i \in \{1, 2\}$. Since each verification can be completed in $O(n)$ time, Step 3 can be completed in $O(Mn)$ time and the proposition follows.

To conclude this section, we discuss the implication of Proposition 17 for the case where $t = 2$, that is, the channel that introduces two deletions.

Consider a VTL code $\mathcal{C}$ of length $n$. As mentioned earlier, $\mathcal{C}$ is a single-deletion-correcting code that is equipped with a linear-time decoder. In [10] (or Theorem 2), Gabrys and Yaakobi showed that we can uniquely reconstruct any codeword of $\mathcal{C}$ using seven distinct reads. Later, Chrisnata and Kiah considered a subcode $\mathcal{X}$ of $\mathcal{C}$ with 2log2n + O(log2 log2 n) redundant bits and showed that any codeword of $\mathcal{C}$ can be reconstructed using five distinct reads [15]. In both cases, naïvely, we can reconstruct the transmitted word in quadratic time by trying all possibilities for the missing two positions. However, if we apply the algorithms in Propositions 7 and 17 we are able to recover the transmitted word in linear time.

REFERENCES

[1] V. I. Levenshtein, “Efficient reconstruction of sequences,” IEEE Trans. on Information Theory, 47(1), pp. 2–22, 2001.

[2] G. M. Church, Y. Gao, and S. Kosuri. “Next-generation digital information storage in DNA,” Science, 337(6102):1628–1628, 2012.

[3] N. Goldman, P. Bertone, S. Chen, C. Dessensouz, E. M. LeProust, B. B. Bip, and B. Birney. “Towards practical, high-capacity, low-maintenance information storage in synthesized DNA,” Nature, 494:77–80, 2013.

[4] S. Yazdi, H. M. Kiah, E. R. Garcia, J. Ma, H. Zhao, and O. Milenkovic. DNA-based storage: Trends and methods. IEEE Trans. Molecular, Biolo
cial Multi-Scale Commum., 1(3):230–248, 2015.

[5] L. Organick, S. Ang, Y.-J. Chen, R. Lopez, S. Yekhanin, K. Makarychev, M. Razz, G. Kanath, P. Gopalak, B. Nguyen, C. Takahashi, S. Newman, H.-Y. Parker, C. Rashtchian, K. Stewart, G. Gupta, R. Carlson, J. Mulligan, D. Carmean, G. Seelig, L. Ceze, and K. Strauss. “Random access in large-scale DNA data storage.” Nature Biotechnology, 36(3), 242, 2018.

[6] A. Lenz, P. H. Siegel, A. Wachter-Zeh, and E. Yaakobi. “Coding over sets for DNA storage.” IEEE Trans. Info. Theory, 66(4), 2331–2351.

[7] S. S. Parkin, M. Hayashi, and L. Thomas, “Magnetic domain-wall racetrack memory,” Science, vol. 320, pp. 190–194, 2008.

[8] Y. M. Chee, H. M. Kiah, A. Vardy, E. Yaakobi, and V. K. Vu. “Coding for racetrack memories,” IEEE Trans. on Information Theory, 64(11), pp. 7094–7112, 2018.

[9] V. I. Levenshtein, “Efficient Reconstruction of Sequences from Their Subsequences or Supersequences,” Journal of Combinatorial Theory, Series A, 93, pp. 310–332, 2001.

[10] R. Gabrys, and E. Yaakobi. “Sequence reconstruction over the deletion channel,” IEEE Trans. on Information Theory, 64(4), pp.2924–2931, 2018.

[11] F. Sala, R. Gabrys, C. Schoeny, and L. Dolecek, “Exact reconstruction from insertions in synchronization codes,” IEEE Trans. Info. Theory, vol. 63, no. 4, pp. 2428–2445, Apr. 2017.

[12] M. Abu-Sini, and E. Yaakobi, “Levenshtein’s Reconstruction Problem Under Insertions, Deletions, and Substitutions”. IEEE Trans. Info. Theory, vol. 67, no. 11, pp. 7132–7158, Nov. 2021.

[13] K. Cai, H. M. Kiah, T. T. Nguyen and E. Yaakobi, “Coding for Sequence Reconstruction for Single Edits,” IEEE Trans. Info. Theory, 2021, doi: 10.1109/TIT.2021.3122798.

[14] J. Chrisnata, H. M. Kiah, E. Yaakobi. “Optimal Reconstruction Codes for Deletion Channels,” In Proc. IEEE Int. Symp. Info. Theory and its Applications (ISITA), pp. 279–283, 2020. (arXiv preprint arXiv:2004.06032)

[15] J. Chrisnata, and H. M. Kiah, “Correcting Two Deletions with More Reads,” presented at the IEEE Int. Symp. Info. Theory (ISIT), Melbourne, Australia, Jul. 2021.

[16] Elzinga, Cees, Sven Rahmann, and Hui Wang. “Algorithms for subsequence combinatorics.” Theoretical Computer Science, vol. 409, no. 3, pp. 394–404, 2008.

[17] V. I. Levenshtein, “Binary codes capable of correcting deletions, insertions, and reversals,” in Soviet physics doklady, vol. 10, no. 8, pp. 707–710, 1966.

[18] L. Calabi. “On the Computation of Levenshtein’s Distances,” TN-9-0030, Parkhe Math. Labs., Inc., Carlisle, MA, 1967.