Invariance Principles for Tempered Fractionally Integrated Processes

Farzad Sabzikar\(^1\) and Donatas Surgailis\(^2\)

September 24, 2018

\(^1\)Iowa State University and \(^2\)Vilnius University

Abstract

We discuss invariance principles for autoregressive tempered fractionally integrated moving averages in \(\alpha\)-stable \((1 < \alpha \leq 2)\) i.i.d. innovations and related tempered linear processes with vanishing tempering parameter \(\lambda \sim \lambda_*/N\). We show that the limit of the partial sums process takes a different form in the weakly tempered \((\lambda_* = 0)\), strongly tempered \((\lambda_* = \infty)\), and moderately tempered \((0 < \lambda_* < \infty)\) cases. These results are used to derive the limit distribution of the OLS estimate of AR(1) unit root with weakly, strongly, and moderately tempered moving average errors.

Keywords: invariance principle; tempered linear process; autoregressive fractionally integrated moving average; tempered fractional stable/Brownian motion; tempered fractional unit root distribution;

1 Introduction

The present paper discusses partial sums limits and invariance principles for tempered moving averages

\[
X_{d,\lambda}(t) = \sum_{k=0}^{\infty} e^{-\lambda k} b_d(k)\zeta(t-k), \quad t \in \mathbb{Z}
\]  

(1.1)

in i.i.d. innovation process \(\{\zeta(t)\}\) with coefficients \(b_d(k)\) regularly varying at infinity as \(k^{d-1}\), viz.

\[
b_d(k) \sim \frac{c_d}{\Gamma(d)} k^{d-1}, \quad k \to \infty, \quad c_d \neq 0, \quad d \neq 0
\]  

(1.2)

where \(d \in \mathbb{R}\) is a real number, \(d \neq -1, -2, \ldots\) and \(\lambda > 0\) is tempering parameter. In addition to (1.2) we assume that

\[
\sum_{k=0}^{\infty} k^j b_d(k) = 0, \quad 0 \leq j \leq [-d], \quad -\infty < d < 0
\]  

(1.3)

\[
\sum_{k=0}^{\infty} |b_d(k)| < \infty, \quad \sum_{k=0}^{\infty} b_d(k) \neq 0, \quad d = 0
\]  

(1.4)
An important example of such processes is the two-parametric class ARTFIMA\((0, d, \lambda, 0)\) of tempered fractionally integrated processes, generalizing the well-known ARFIMA\((0, d, 0)\) class, written as

\[
X_{d,\lambda}(t) = (1 - e^{-\lambda B})^{-d} \zeta(t) = \sum_{k=0}^{\infty} e^{-\lambda k} \omega_{-d}(k) \zeta(t - k), \quad t \in \mathbb{Z} \tag{1.5}
\]

with coefficients given by power expansion \((1 - e^{-\lambda z})^{-d} = \sum_{k=0}^{\infty} e^{-\lambda k} \omega_{-d}(k) z^k, |z| < 1\), where \(B x(t) = x(t - 1)\) is the backward shift. Due to the presence of the exponential tempering factor \(e^{-\lambda k}\) the series in (1.1) and (1.5) absolutely converges a.s. under general assumptions on the innovations, and defines a strictly stationary process. On the other hand, for \(\lambda = 0\) the corresponding stationary processes in (1.1) and (1.5) exist under additional conditions on the parameter \(d\). See Granger and Joyeux [12], Hosking [13], Brockwell and Davis [5], Kokoszka and Taqqu [15]. We also note (see e.g. [10], Ch. 3.2) that the (untempered) linear process \(X_{d,0}\) of (1.1) with coefficients satisfying (1.2) for \(0 < d < 1/2\) is said long memory, while (1.2) and (1.3) for \(-1/2 < d < 0\) is termed negative memory and (1.4) short memory, respectively, parameter \(d\) usually referred to as memory parameter.

The model in (1.5) appeared in Giraitis et al. [8], which noted that for small \(\lambda > 0\), \(X_{d,\lambda}\) has a covariance function which resembles the covariance function of a long memory model for arbitrary large number of lags but eventually decays exponentially fast. [8] termed such behavior ‘semi long-memory’ and noted that it may have empirical relevance for modelling of financial returns. Giraitis et al. [9] propose the semi-long memory ARCH(\(\infty\)) model as a contiguous alternative to (pure) hyperbolic and exponential decay which are often very hard to distinguish between in a finite sample. On the other side, Meerschaert et al. [20] effectively apply ARTFIMA\((0, d, \lambda, 0)\) in (1.5) for modeling of turbulence in the Great Lakes region.

The present paper obtains limiting behavior of tempered linear processes in (1.1) with small tempering parameter \(\lambda = \lambda_N \to 0\) tending to zero together with the sample size. The important statistic is the partial sums process

\[
S_{N}^{d,\lambda}(t) := \sum_{k=1}^{[Nt]} X_{d,\lambda}(k), \quad t \in [0, 1] \tag{1.6}
\]

of \(X_{d,\lambda}\) in (1.1) with i.i.d. innovations \(\{\zeta(t)\}\) in the domain of attraction of \(\alpha\)-stable law, \(1 < \alpha \leq 2\). Functional limit theorems for the partial sums process play a crucial role in the R/S analysis, unit root testing, change-point analysis and many other time series inferences. See Lo [17], Phillips [21], Giraitis et al. [9], Lavancier et al. [16] and the references therein.

We prove that the limit behavior of (1.6) essentially depends on how fast \(\lambda = \lambda_N\) tends to 0. Assume that there exists the limit

\[
\lim_{N \to \infty} N \lambda_N = \lambda_* \in [0, \infty]. \tag{1.7}
\]
Depending on the value of $\lambda_*$, the process $X_{d,\lambda N}$ will be called strongly tempered if $\lambda_* = \infty$, weakly tempered if $\lambda_* = 0$, and moderately tempered if $0 < \lambda_* < \infty$. While the behavior of $S_N^{d,\lambda N}$ in the strongly and weakly tempered cases is typical for short memory and long memory processes, respectively, the moderately tempered decay $\lambda_N \sim \lambda_* / N, \lambda_* \in (0, \infty)$ leads to tempered fractional stable motion of second kind (TFSM II) $Z_{H,\alpha,\lambda}^H, H = d + 1/\alpha > 0$ defined as a stochastic integral

$$Z_{H,\alpha,\lambda}^H(t) := \int_{\mathbb{R}} h_{H,\alpha,\lambda}(t; y) M_\alpha(y), \quad t \in \mathbb{R}$$

with respect to $\alpha$-stable Lévy process $M_\alpha$ with integrand

$$h_{H,\alpha,\lambda}(t; y) := (t - y)^{H - 1/\alpha} e^{-\lambda(t-y)^+} - (-y)^{H - 1/\alpha} e^{-\lambda(-y)^+} + \lambda \int_0^t (s - y)^{H - 1/\alpha} e^{-\lambda(s-y)^+} s, \quad y \in \mathbb{R}.$$

TFSM II and its Gaussian counterpart tempered fractional Brownian motion of second kind (TFBM II) were recently introduced in Sabzikar and Surgailis [22], the above processes being closely related to the tempered fractional stable motion (TFSM) and the tempered fractional Brownian motion (TFBM) defined in Meerschaert and Sabzikar [19] and Meerschaert and Sabzikar [18], respectively. As shown in [22], TFSM and TFSM II are different processes, especially striking are their differences as $t \to \infty$.

As an application of our invariance principles we obtain the limit distribution of the OLS estimator $\hat{\beta}_N$ of the slope parameter in AR(1) model with tempered ARTFIMA($0, d, \lambda_N, 0$) errors and small tempering parameter $\lambda_N \to 0$ satisfying (1.7), under the null (unit root) hypothesis $\beta = 1$. In the case of (untempered) ARFIMA($0, d, 0$) error process with finite variance and standardized i.i.d. innovations, Sowell [24] proved that the distribution of the normalized statistic $N^{1/(1+2d)}(\hat{\beta}_N - 1)$ tends to the so-called fractional unit root distribution written in terms of fractional Brownian motion with parameter $H = d + \frac{1}{2}$. Sowell’s [24] result extends the classical unit root distribution for weakly dependent errors in Phillips [21] to fractionally integrated error process, yielding drastically different limits for $0 < d < 1/2, d = 0$ and $-1/2 < d < 0$.

It turns out that in the case of ARTFIMA($0, d, \lambda_N, 0$) error process with $\lambda_N \sim \lambda_* / N$, the limit distribution of $\hat{\beta}_N$ depends on $\lambda_* \in [0, \infty]$ and $d$. Roughly speaking (see Theorem 5.2 for precise formulation), in the moderately tempered case $0 < \lambda_* < \infty$ the limit distribution of $\hat{\beta}_N$ writes similarly to Sowell [24] with FBM $B_H$ replaced by TFBM II $B_{H,\lambda_*}^H$, and the convergence holds for all $-1/2 < d < \infty$ in contrast to [24] which is limited to $|d| < 1/2$. Under strong tempering $\lambda_N / N \to \lambda_* = \infty$, the limit distribution of $\hat{\beta}_N$ is written in terms of standard Brownian motion but takes a different form in the cases $d > 0, d = 0$ and $d < 0, d \neq N_-$; moreover, except for the i.i.d. case $d = 0$, this limit is different from Sowell’s limit in [24] and also from the unit root distribution in Dickey and Fuller [6] and Phillips [21].
The paper is organized as follows. Section 2 introduces ARTFIMA($p,d,\lambda,q$) class and provides basic properties of these processes. In Section 3 we define TFSM II/TFBM II. Section 4 contains the main results of the paper (invariance principles). Section 5 discusses the application to unit root testing. The proofs of the main results are relegated to Section 6.

In what follows, $C$ denotes generic constants which may be different at different locations. We write $d \to d$, $d \overset{d}{\to}$, $f \overset{d}{\to}$, $f \overset{d}{=} f$ for the weak convergence and equality of distributions and finite-dimensional distributions. $\mathbb{N}_\pm := \{\pm 1, \pm 2, \ldots \}$, $\mathbb{R}_+ := (0, \infty)$, $(x)_\pm := \max(\pm x, 0), x \in \mathbb{R}$, $f := \int_{\mathbb{R}}$. $LP(\mathbb{R}) (p \geq 1)$ denotes the Banach space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ with finite norm $\|f\|_p = (\int |f(x)|^p x^1/p$.

## 2 Tempered fractionally integrated process

In this section, we define ARTFIMA($p,d,\lambda,q$) process and discuss its basic properties. Let $\Phi(z) = 1 - \sum_{i=1}^p \phi_i z^i$ and $\Theta(z) = 1 + \sum_{i=1}^q \theta_i z^i$ be polynomials with real coefficients of degree $p,q \geq 0$, such that $\Phi(z)$ does not vanish on $\{z \in \mathbb{C}, |z| \leq 1\}$ and $\Phi(z)$ and $\Theta(z)$ have no common zeros. Let $d \in \mathbb{R} \setminus \mathbb{N}_+$. Consider Taylor’s expansion

$$\frac{\Theta(z)}{\Phi(z)}(1-z)^d = \sum_{k=0}^{\infty} a_d(k) z^k, \quad |z| < 1. \quad (2.1)$$

Note that

$$a_d(k) = \sum_{s=0}^{k} \omega_d(k) \psi(k-s), \quad k \geq 0, \quad (2.2)$$

where $\omega_d(k) = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(d)}$, and $\psi(j)$ are the coefficients of the power series $\sum_{j=0}^{\infty} \psi(j) z^j = \Theta(z)/\Phi(z), |z| \leq 1$. We use the fact (see Kokoszka and Taqqu (15), Lemma 3.1) that for any $d \in \mathbb{R} \setminus \mathbb{N}_-\n$

$$\omega_{-d}(k) = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)} = \Gamma(d)^{-1} k^{d-1} (1 + O(1/k)), \quad k \to \infty. \quad (2.3)$$

### Proposition 2.1

Let $\Theta(1) \neq 0$. For any $d \in \mathbb{R} \setminus \mathbb{N}_-, \quad \text{the coefficients} \quad a_{-d}(k), k \geq 0 \quad \text{satisfy conditions (1.2)-(1.3). In particular, for any} \quad d \in \mathbb{R} \setminus \{0,-1,-2,\ldots\} \quad a_{-d}(k) \sim \frac{\Theta(1)}{\Phi(1)\Gamma(d)} k^{d-1} \left(1 + O\left(\frac{1}{k}\right)\right), \quad k \to \infty. \quad (2.4)$

### Proof

Let us prove (2.4). By definition, $a_{-d}(k) = \sum_{j=0}^{k} \psi(j) \omega_{-d}(k-j), k \geq 0$, see (2.2). It is well-known that $|\psi(j)| \leq Ce^{-cj}$ for some constants $C,c > 0$, see (15), proof of Lemma 3.2). Note $\Theta(1)/\Phi(1) = \sum_{j=0}^{\infty} \psi(j) \neq 0$. We have

$$|a_{-d}(k) - (\Theta(1)/\Phi(1)) \omega_{-d}(k)|$$

$$= \left| \sum_{j=0}^{k} \psi(j) (\omega_{-d}(k-j) - \omega_{-d}(k)) - \omega_{-d}(k) \sum_{j>k} \psi(j) \right| \leq \sum_{i=1}^{3} \ell_{k,i},$$

where $\ell_{k,i}$ are constants.
where
\[
\ell_{k,1} := \sum_{0 \leq j \leq k^{1/4}} |\psi(j)||\omega_{-d}(k-j) - \omega_{-d}(k)|,
\]
\[
\ell_{k,2} := \sum_{k^{1/4} < j \leq k} |\psi(j)|(|\omega_{-d}(k-j)| + |\omega_{-d}(k)|),
\]
and \(\ell_{k,3} := |\omega_{-d}(k)| \sum_{j > k} |\psi(j)| \leq CKd^{-1} \sum_{j > k} e^{-cj} = o(kd^{-2})\) since \(\psi(j)\) decay exponentially. Similarly, \(\ell_{k,2} \leq Ce^{-ck^{1/4}k} \max_{1 \leq j \leq k} |\omega_{-d}(j)| = o(kd^{-2})\). Using (2.3) we obtain \(|\ell_{k,1}| \leq C(\ell'_{k,1} + \ell'_{k,2})\), where \(\ell''_{k,2} := k^{d-2} \sum_{j \geq 0} |\psi(j)| = O(kd^{-2})\) and
\[
\ell'_{k,1} := \sum_{0 \leq j \leq k^{1/4}} |\psi(j)|(k^{d-1} - (k-j)^{d-1}) = k^{d-1} \sum_{0 \leq j \leq k^{1/2}} |\psi(j)|(1 - (1-j/k)^{d-1})
\]
proving \(|a_{-d}(k) - (\Theta(1)\Phi(1))\omega_{-d}(k)| = O(kd^{-2})\) and hence (2.4) in view of (2.3). Thus, \(a_{-d}(k), d \neq 0\) satisfy (1.2). Condition (1.4) is obvious from \(a_0(k) = \psi(k)\) and properties of \(\psi(k)\) stated above.

It remains to prove (1.3). Let \(j < -d < j + 1\) for some \(j = 0, 1, \ldots\). Then since \(\Psi(z) := \Theta(z)/\Phi(z)\) is analytic on \(\{z \in \mathbb{C}, |z| \leq 1+\delta\}\) with \(|\psi(z)| \leq CKd^{-1}\), see (2.4), the function \(\sum_{k=0}^\infty a_{-d}(k)z^k = \Psi(z)(1-z)^{-d}\) is \(j\) times differentiable on \(\{|z| \leq 1\}\) and \(\frac{\partial^i}{\partial z^i} \Psi(z)(1-z)^{-d}\big|_{z=1} = \sum_{r=0}^i \binom{i}{r} \frac{\partial^r \psi(z)}{\partial z^r}\big|_{z=1} = 0, 0 \leq i \leq j\). Hence, \(0 = \sum_{k=1}^\infty k(k-1) \cdots (k-i+1)a_{-d}(k) = \sum_{k=1}^\infty k(k-1) \cdots (k-i+1)a_{-d}(k)\) for any \(0 \leq i \leq j\), proving (1.3) and the proposition, too. \(\square\)

**Definition 2.2** Let the autoregressive polynomials \(\Theta(z), \Phi(z)\) of degree \(p, q\) satisfy the above conditions, and \(d \in \mathbb{R} \setminus \mathbb{N}, \lambda \geq 0\). Moreover, let \(\zeta = \{\zeta(t), t \in \mathbb{Z}\}\) be a stationary process with \(\text{E}|\zeta(0)| < \infty\). By ARTFIMA\((p, d, \lambda, q)\) process with innovation process \(\zeta\) we mean a stationary moving-average process \(X_{p,d,\lambda,q} = \{X_{p,d,\lambda,q}(t), t \in \mathbb{Z}\}\) defined by
\[
X_{p,d,\lambda,q}(t) = \sum_{k=0}^\infty e^{-\lambda k}a_{-d}(k)\zeta(t-k), \quad t \in \mathbb{Z}
\]
where the series converges in \(L_1\).

**Remark 2.3** (i) For \(\lambda = 0\) and \(|d| < 1/2\) and i.i.d. innovations with zero mean and unit variance, ARTFIMA\((p, d, 0, q)\) process \(X_{p,d,0,q}\) coincides with ARFIMA\((p, d, q)\) process, see e.g. Brockwell and Davis [5]. Particularly, \(X_{d,0} = X_{0,d,0,0}\) in (2.5) is a stationary solution of the AR(\(\infty\)) equation
\[
(1 - B)^d X_{d,0}(t) = \sum_{j=0}^\infty \omega_d(j)X_{d,0}(t-j) = \zeta(t)
\]
and the series in (2.5) and (2.6) converge in \(L_2\), meaning that \(X_{d,0}\) is invertible.
(ii) For \( \lambda = 0 \) and zero mean i.i.d. \( \alpha \)-stable innovations, \( 1 < \alpha < 2 \), the definition of \( X_{p,d,0,q} \) in Definition 2.2 agrees with the definition of ARFIMA(\( p, d, q \)) process in [15], who showed that the series in (2.5) converges a.s. and in \( L_1 \) for \( d < 1 - \frac{1}{\alpha} \).

**Proposition 2.4** Let \( \Theta(z), \Phi(z) \) satisfy the above conditions and \( \lambda > 0 \). Then the series in (2.5) converges in \( L_1 \) for any \( d \in \mathbb{R} \setminus \mathbb{N}_- \) hence ARTFIMA(\( p, d, \lambda, q \)) process \( X_{p,d,\lambda,q} \) in Definition 2.2 is well-defined for arbitrary (stationary) innovation process \( \zeta \) with finite mean. Moreover, if \( |\Theta(z)| > 0, |z| \leq 1 \) then \( X_{p,d,\lambda,q} \) is invertible:

\[
\zeta(t) = \sum_{k=0}^{\infty} e^{-\lambda k} \tilde{a}_d(k) X_{p,d,\lambda,q}(t-k),
\]

where

\[
\sum_{k=0}^{\infty} \tilde{a}_d(k) z^k = \frac{\Phi(z)}{\Theta(z)}(1-z)^d, \quad |z| < 1
\]

and the series in (2.7) converges in \( L_1 \).

**Proof.** The convergence in \( L_1 \) of the series in (2.5) follows from (2.4). To show invertibility of these series, note that by Proposition 2.1 the coefficients in (2.8) satisfy the bound \( |\tilde{a}_d(k)| \leq C k^{-d-1}, k \geq 1 \) for \( d \in \mathbb{R} \setminus \mathbb{N}_+ \), and an exponential bound \( |\tilde{a}_d(k)| \leq e^{-ck}, k \geq 1, c > 0 \) for \( d \in \mathbb{N}_+ \).

Hence, \( e^{-\lambda k}|\tilde{a}_d(k)| \leq e^{-\lambda k} k^{-d-1} \), for any \( d \in \mathbb{R} \) implying the convergence in \( L_1 \) of the series in (2.7). Finally, equality in (2.7) follows from identity 1 = \( \frac{\Phi(z)}{\Theta(z)}(1-z)^d \), \( \frac{\Theta(z)}{\Phi(z)}(1-z)^{-d} \), \( |z| < 1 \).

**Proposition 2.5** describes some second order properties of ARTFIMA(\( p, d, \lambda, q \)) with standardized innovations.

**Proposition 2.5** Let \( X_{p,d,\lambda,q} \) be ARTFIMA(\( p, d, \lambda, q \)) process in (2.5), \( d \in \mathbb{R} \setminus \mathbb{N}_-, \lambda > 0 \), with standardized i.i.d. innovations \( \{\zeta(t), t \in \mathbb{Z}\}, \mathbb{E}\zeta(0) = 0, \mathbb{E}\zeta^2(0) = 1 \). Then \( \mathbb{E}X_{p,d,\lambda,q}(t) = 0, \mathbb{E}X^2_{p,d,\lambda,q}(t) = \sum_{k=0}^{\infty} e^{-2\lambda k} a_d^2(k) < \infty \) and

(i) The spectral density of \( X_{p,d,\lambda,q} \) is given by

\[
h(x) = \frac{1}{2\pi} \left| \frac{\Theta(1+ix)}{\Phi(1+ix)} \right|^2 (1 - 2e^{-\lambda} \cos x + e^{-2\lambda})^{-d}, \quad -\pi \leq x \leq \pi.
\]

(ii) The covariance function of \( X_{0,d,\lambda,0} \) is given by

\[
\gamma_d,\lambda(k) = \mathbb{E}X_{0,d,\lambda,0}(0)X_{0,d,\lambda,0}(k) = \frac{e^{-\lambda k} \Gamma(d+k)}{\Gamma(d+1)} 2F_1(d, k + d; k + 1; e^{-2\lambda}),
\]

where \( 2F_1(a, b; c; z) \) is the Gauss hypergeometric function (see e.g. [11]). Moreover,

\[
\sum_{k \in \mathbb{Z}} |\gamma_d,\lambda(k)| < \infty, \quad \sum_{k \in \mathbb{Z}} \gamma_d,\lambda(k) = (1 - e^{-\lambda})^{-2d}
\]

and

\[
\gamma_d,\lambda(k) \sim Ak^{d-1}e^{-\lambda k}, \quad k \to \infty, \quad \text{where} \quad A = (1 - e^{-2\lambda})^{-d} \Gamma(d)^{-1}.
\]
\textbf{Proof.} (i) From the transfer function \((\Theta(e^{-ix})/\Phi(e^{-ix}))(1-e^{-ix-\lambda})^{-d}\) of the filter in (2.5) we have that \(h(x) = \frac{1}{2\pi} |\Theta(e^{-ix})/\Phi(e^{-ix})|^2 |1-e^{-ix-\lambda}|^{-2\gamma}\), where \(|1-e^{-ix-\lambda}|^2 = 1-2e^{-\lambda}\cos(x) + e^{-2\lambda}\), proving part (i). (ii) \((2.9)\) follows from \(\gamma_{d,\lambda}(k) = \int_{\pi}^{\pi} \cos(kx)h(x)\) and (11), Eq. 9.112. The first relation in (2.10) follows from \(\sum_{j \in \mathbb{Z}} |e^{-\lambda j}\omega_d(j)| < \infty\), see (2.3), and the second one from \(\sum_{k \in \mathbb{Z}} \gamma_{d,\lambda}(k) = 2\pi h(0) = (1-e^{-\lambda})^{-2\gamma}\). Finally, (2.11) is proved in ([8], (4.15)). \(\square\)

3 Tempered fractional Brownian and stable motions of second kind

This section contains the definition of TFBM II/TFSM II and some of its properties from Sabzikar and Surgailis [22]. The reader is referred to the aforementioned paper for further properties of these processes including relation to tempered fractional calculus, relation between TFBM II/TFSM II and TFBM/TFSM, dependence properties of the increment process (tempered fractional Brownian/stable noise), local and global asymptotic self-similarity.

For \(1 < \alpha \leq 2\), let \(M_\alpha = \{M_\alpha(t), t \in \mathbb{R}\}\) be an \(\alpha\)-stable Lévy process with stationary independent increments and characteristic function

\[ \mathbb{E}e^{i\theta M_\alpha(t)} = e^{-\sigma^\alpha|\theta|^\alpha t \left(1-i\beta\tan(\pi\alpha/2)\text{sign}(\theta)\right)}, \quad \theta \in \mathbb{R}, \] \hspace{1cm} (3.1)

where \(\sigma > 0\) and \(\beta \in [-1,1]\) are the scale and skewness parameters, respectively. For \(\alpha = 2\), \(M_2(t) = \sqrt{2\sigma} B(t)\), where \(B\) is a standard Brownian motion with variance \(\mathbb{E}B^2(t) = t\). Stochastic integral \(I_\alpha(f) \equiv \int f(x)M_\alpha(x)\) is defined for any \(f \in L^\alpha(\mathbb{R})\) as \(\alpha\)-stable random variable with characteristic function

\[ \mathbb{E}e^{i\theta I_\alpha(f)} = \exp\{-\sigma^\alpha|\theta|^\alpha \int |f(x)|^\alpha \left(1-i\beta\tan(\pi\alpha/2)\text{sign}(\theta f(x))\right) dx\}, \quad \theta \in \mathbb{R}. \] \hspace{1cm} (3.2)

see e.g. [23] Chapter 3.

For \(t \in \mathbb{R}, H > 0, 1 < \alpha \leq 2, \lambda \geq 0\) consider the function \(y \mapsto h_{H,\alpha,\lambda}(t; y) : \mathbb{R} \rightarrow \mathbb{R}\) given in (1.9). Note \(h_{H,\alpha,\lambda}(t; \cdot) \in L^\alpha(\mathbb{R})\) for any \(t \in \mathbb{R}, \lambda > 0, 1 < \alpha \leq 2, H > 0\) and also for \(\lambda = 0, 1 < \alpha \leq 2, H \in (0,1)\). We will use the following integral representation of (1.9) (see [22]).

For \(H > \frac{1}{\alpha}\):

\[ h_{H,\alpha,\lambda}(t; y) = (H - \frac{1}{\alpha}) \int_0^t (s-y)^{H-\frac{1}{\alpha}-1} e^{-\lambda(s-y)+s} \] \hspace{1cm} (3.3)

For \(0 < H < \frac{1}{\alpha}\):

\[ h_{H,\alpha,\lambda}(t; y) = (H - \frac{1}{\alpha}) \begin{cases} \int_0^t (s-y)^{H-\frac{1}{\alpha}-1} e^{-\lambda(s-y)+s}, & y < 0, \\ -\int_{-\infty}^t (s-y)^{H-\frac{1}{\alpha}-1} e^{-\lambda(s-y)+s} + \lambda^{\frac{1}{\alpha}} \Gamma(H - \frac{1}{\alpha}), & y \geq 0. \end{cases} \] \hspace{1cm} (3.4)
**Definition 3.1** Let $M_\alpha$ be $\alpha$-stable Lévy process in (3.1), $1 < \alpha \leq 2$ and $H > 0$, $\lambda > 0$. Define, for $t \in \mathbb{R}$, the stochastic integral

$$Z^{H}_{H,\alpha,\lambda}(t) := \int h_{H,\alpha,\lambda}(t;y)M_\alpha(y). \quad (3.5)$$

The process $Z^{H}_{H,\alpha,\lambda} = \{Z^{H}_{H,\alpha,\lambda}(t), t \in \mathbb{R}\}$ will be called *tempered fractional stable motion of second kind* (TFSM II). A particular case of (3.5) corresponding to $\alpha = 2$

$$B^{H}_{H,\lambda}(t) := \frac{1}{\Gamma(H + \frac{1}{2})} \int h_{H,2,\lambda}(t;y)B(y) \quad (3.6)$$

will be called *tempered fractional Brownian motion of second kind* (TFBM II).

The next proposition states some basic properties of TFSM II $Z^{H}_{H,\alpha,\lambda}$.

**Proposition 3.2** (22)

(i) $Z^{H}_{H,\alpha,\lambda}$ in (3.5) is well-defined for any $t \in \mathbb{R}$ and $1 < \alpha \leq 2, H > 0, \lambda > 0$, as a stochastic integral in (3.2).

(ii) $Z^{H}_{H,\alpha,\lambda}$ in (3.5) has stationary increments and $\alpha$-stable finite-dimensional distributions. Moreover, it satisfies the following scaling property: $\{Z^{H}_{H,\alpha,\lambda}(bt), t \in \mathbb{R}\} \overset{fd}{=} \{b^{H}Z^{H}_{H,\alpha,b\lambda}(t), t \in \mathbb{R}\}, \forall b > 0$.

(iii) $Z^{H}_{H,\alpha,\lambda}$ in (3.5) has a.s. continuous paths if either $\alpha = 2, H > 0$, or $1 < \alpha < 2, H > 1/\alpha$ hold.

**4 Invariance principles**

In this section, we discuss invariance principles and the convergence of (normalized) partial sums process

$$S^{d,\lambda}_{N}(t) := \sum_{k=1}^{[Nt]} X_{d,\lambda}(k), \quad t \in [0, 1] \quad (4.1)$$

of tempered linear process $X_{d,\lambda}$ in (1.1) with i.i.d. innovations belonging to the domain of attraction of $\alpha$-stable law, $1 < \alpha \leq 2$ (see below). We shall assume that the tempering parameter $\lambda \equiv \lambda_{N} \rightarrow 0$ as $N \rightarrow \infty$ and following limit exists:

$$\lim_{N \rightarrow \infty} N\lambda_{N} = \lambda_{*} \in [0, \infty]. \quad (4.2)$$

Recall that $X_{d,\lambda_{N}}$ is said *strongly tempered* if $\lambda_{*} = \infty$, *weakly tempered* if $\lambda_{*} = 0$, and *moderately tempered* if $0 < \lambda_{*} < \infty$. We show that the limits of the partial sums process in (4.1) exist under condition (4.2) and depend on $\lambda_{*}, d, \alpha$; moreover, in all cases these limits belong to the class of TFSM II processes defined in Section 3.
Definition 4.1 Write $\zeta \in D(\alpha)$, $1 < \alpha \leq 2$ if

(i) $\alpha = 2$ and $\mathbb{E}\zeta = 0$, $\sigma^2 := \mathbb{E}\zeta^2 < \infty$, or

(ii) $1 < \alpha < 2$ and there exist some constants $c_1, c_2 \geq 0, c_1 + c_2 > 0$ such that $\lim_{x \to \infty} x^\alpha \mathbb{P}(\zeta > x) = c_1$ and $\lim_{x \to -\infty} |x|^\alpha \mathbb{P}(\zeta \leq x) = c_2$; moreover, $\mathbb{E}\zeta = 0$.

Condition $\zeta \in D(\alpha)$ implies that r.v. $\zeta$ belongs to the domain of normal attraction of an $\alpha$-stable law. In other words, if $\zeta(i), i \in \mathbb{Z}$ are i.i.d. copies of $\zeta$ then

$$N^{-1/\alpha} \sum_{i=1}^{[N]} \zeta(i) \xrightarrow{fdd} M_\alpha(t), \quad N \to \infty,$$

(4.3)

where $M_\alpha$ is an $\alpha$-stable Lévy process in (3.1) with $\sigma, \beta$ determined by $c_1, c_2$, see ([7], pp. 574-581).

We shall use the following criterion for convergence of weighted sums in i.i.d. r.v.s. See ([10], Prop. 14.3.2), [2], [14].

Proposition 4.2 Let $1 < \alpha \leq 2$ and $Q(g_N) = \sum_{t \in \mathbb{Z}} g_N(t) \zeta(t)$ be a linear form in i.i.d. r.v.s $\zeta(t) \in D(\alpha)$ with real coefficients $g_N(t), t \in \mathbb{Z}$. Assume that there exists $p \in [1, \alpha)$ if $\alpha < 2$, $p = 2$ if $\alpha = 2$ and a function $g \in L^p(\mathbb{R})$ such that the functions

$$\bar{g}_N(x) := N^{1/\alpha} g_N([xN]), \quad x \in \mathbb{R}$$

(4.4)

satisfy

$$\|\bar{g}_N - g\|_p \to 0 \quad (N \to \infty).$$

(4.5)

Then $Q(g_N) \xrightarrow{d} \int g(x) M_\alpha(x)$, where $M_\alpha$ is as in (4.3) and (3.1).

Write $\xrightarrow{D[0,1]}$ for weak convergence of random processes in the Skorohod space $D[0,1]$ equipped with $J_1$-topology, see [4].

In Theorem 4.3 below, $X_{d,\lambda_N}$ is a tempered linear process of (1.1) with i.i.d. innovations $\zeta(t) \in D(\alpha), 1 < \alpha \leq 2$, coefficients $b_d(k), k \geq 0, d \in \mathbb{R} \setminus \mathbb{N}_-$ satisfying (1.2)-(1.4), and tempering parameter $0 < \lambda_N \to 0 (N \to \infty)$ satisfying (4.2). W.l.g., we shall assume that the asymptotic constant $c_d$ in (1.2)-(1.4) equals 1: $c_d = 1 \forall d \in \mathbb{R} \setminus \mathbb{N}_-$.

Theorem 4.3 (i) (Strongly tempered process.) Let $\lambda_* = \infty$ and $d \in \mathbb{R} \setminus \mathbb{N}_-$. Then

$$N^{-1/\lambda_*} \lambda_*^{d/\lambda_*} S_N^{d,\lambda_N}(t) \xrightarrow{fdd} M_\alpha(t),$$

(4.6)

where $M_\alpha$ is $\alpha$-stable Lévy process in (4.3). Moreover, if $\alpha = 2$ and $\mathbb{E}|\zeta(0)|^p < \infty$ for some $p > 2$ then

$$N^{-1/\lambda_*} \lambda_*^{d/\lambda_*} S_N^{d,\lambda_N}(t) \xrightarrow{D[0,1]} \sigma B(t),$$

(4.7)
where \( B \) is a standard Brownian motion and \( \sigma > 0 \) some constant.

(ii) (Weakly tempered process.) Let \( \lambda_* = 0 \) and \( H = d + \frac{1}{\alpha} \in (0, 1) \). Then

\[
N^{-H} S_N^{d,\lambda_N}(t) \xrightarrow{fdd} \Gamma(d+1)^{-1} Z_{H,0,0}(t), \tag{4.8}
\]

where \( Z_{H,0,0} \) is a linear fractional \( \alpha \)-stable motion, see Definition 3.1. Particularly, for \( \alpha = 2 \), \( Z_{H,2,0} \) is a multiple of FBM.

Moreover, if either \( 1 < \alpha \leq 2, 1/\alpha < H < 1 \), or \( \alpha = 2, 0 < H < 1/2 \) and \( \mathbb{E}|\zeta(0)|^p < \infty \) \((\exists p > 1/H)\) hold, then \( \xrightarrow{fdd} \) in (4.8) can be replaced by \( \xrightarrow{D[0,1]} \).

(iii) (Moderately tempered process.) Let \( \lambda_* \in (0, \infty) \) and \( H = d + \frac{1}{\alpha} > 0 \). Then

\[
N^{-H} S_N^{d,\lambda_N}(t) \xrightarrow{fdd} \Gamma(d+1)^{-1} Z_{H,\lambda_*}(t), \tag{4.9}
\]

where \( Z_{H,\lambda_*} \) is a TFSM II as defined in Definition 3.1.

Moreover, if either \( 1 < \alpha \leq 2, 1/\alpha < H \), or \( \alpha = 2, 0 < H < 1/2 \) and \( \mathbb{E}|\zeta(0)|^p < \infty \) \((\exists p > 1/H)\) hold, then \( \xrightarrow{fdd} \) in (4.9) can be replaced by \( \xrightarrow{D[0,1]} \).

Remark 4.4 Note that for \( \lambda_N = \lambda_*/N \) the normalization in (4.6) becomes \( N^{-(\frac{1}{\alpha} + d)} \lambda_*^d \) where the exponent \( \frac{1}{\alpha} + d = H \) is the same as in (4.8) and (4.9).

Remark 4.5 The functional convergence in (4.6), case \( 1 < \alpha < 2 \) (the case of discontinuous limit process) is open and apparently does not hold in the usual \( J_1 \)-topology, see [3]. In the case of (4.8) and (4.9) and \( 1 < \alpha < 2, 0 < H < \frac{1}{\alpha} \), functional convergence cannot hold in principle since the limit processes do not belong to \( D[0,1] \).

5 Tempered fractional unit root distribution

A fundamental problem of time series is testing for the unit root \( \beta = 1 \) in the AR(1) model

\[
Y(t) = \beta Y(t-1) + X(t), \quad t = 1, 2, \ldots, N, \quad Y(0) = 0 \tag{5.1}
\]

with stationary error process \( X = \{X(t), t \in \mathbb{Z}\} \). The classical approach to the unit root testing is based on the limit distribution of the OLS estimator \( \hat{\beta}_N \)

\[
\hat{\beta}_N = \frac{\sum_{t=1}^N Y(t)Y(t-1)}{\sum_{t=1}^N Y^2(t-1)}. \tag{5.2}
\]

The limit theory for \( \hat{\beta}_N \) in the case of weakly dependent errors \( X \) was developed in Phillips [21]. We note that [21] makes an extensive use of invariance principle for the error process. Sowell [24] obtained the limit distribution of \( \hat{\beta}_N \) in the case of strongly dependent ARFIMA(0,d,0) error.
process with finite variance and standardized i.i.d. innovations. [24] proved that the distribution of the normalized statistic $N^{1/(1+2d)}(\hat{\beta}_N - 1)$ tends to that of the ratio

$$\frac{1}{2 \int_0^1 B_H^2(s)^{\frac{1}{2}}} \begin{cases} B_H^2(1), & 0 < d < 1/2, \\ B^2(1) - 1, & d = 0, \\ -H\Gamma(H + \frac{1}{2})/\Gamma(\frac{3}{2} - H), & -1/2 < d < 0, \end{cases}$$

(5.3)

where $H = d + \frac{1}{2}$ and $B_H$ is a FBM with parameter $H \in (0, 1)$, $B = B_{1/2}$ being a standard Brownian motion.

In this section we extend Sowell’s [24] result to ARTFIMA$(0, d, \lambda_N, 0)$ error process with small tempering parameter $\lambda_N \sim \lambda_\ast/N \to 0$ as in (4.2). Although Theorem 5.2 can be generalized to more general tempered processes with finite variance as in Theorem 4.3, our choice of ARTFIMA$(0, d, \lambda_N, 0)$ as the error process is motivated by better comparison to [24]. As noted in Section 1, the degree of tempering has a strong effect on the limit distribution of $\hat{\beta}_N$ and leads to a new two-parameter family of tempered fractional unit root distributions. Following [24], we decompose

$$\hat{\beta}_N - 1 = \hat{A}_N - \hat{B}_N,$$

(5.4)

where

$$\hat{A}_N := \frac{Y^2(N)}{2 \sum_{t=1}^N Y^2(t - 1)}, \quad \hat{B}_N := \frac{\sum_{t=1}^N X^2(t)}{2 \sum_{t=1}^N Y^2(t - 1)}.$$  

(5.5)

Under the unit root hypothesis $\beta = 1$ we have $Y(t) = \sum_{i=1}^t X(i) = S_N(t/N)$, where $S_N(x) := \sum_{t=1}^{[Nx]} X(t), x \in [0, 1]$ is the partial sums process. Particularly, the statistics in (5.5) can be rewritten as

$$\hat{A}_N = \frac{S_N^2(1)}{2N \int_0^1 S_N^2(s)^{\frac{1}{2}}}, \quad \hat{B}_N = \frac{\sum_{t=1}^N X^2(t)}{2N \int_0^1 S_N^2(s)^{\frac{1}{2}}}.$$  

(5.6)

For ARTFIMA error process $X_{d,\lambda_N}$, the behavior of $S_N^2(1)$ and $\int_0^1 S_N^2(s)^{\frac{1}{2}}$ can be derived from Theorem 4.3. The behavior of $\sum_{t=1}^N X^2(t)$ is established in the following proposition.

**Proposition 5.1** Let $X_{d,\lambda_N}$ be an ARTFIMA$(0, d, \lambda_N, 0)$ process in (2.3) with i.i.d. innovations $\{\zeta(t)\}, E\zeta(0) = 0, E\zeta^2(0) = 1$, fractional parameter $d \in \mathbb{R} \setminus \mathbb{N}_-$ and tempering parameter $\lambda_N \to 0$. Moreover, let $E|\zeta(0)|^p < \infty$ ($\exists p > 2$). Then

$$\frac{1}{N} \sum_{t=1}^N X^2_{d,\lambda_N}(t) \xrightarrow{p} \frac{\Gamma(1 - 2d)}{\Gamma^2(1 - d)}, \quad d < 1/2,$$

(5.7)

$$\frac{\lambda^{2d-1}}{N} \sum_{t=1}^N X^2_{d,\lambda_N}(t) \xrightarrow{p} \frac{\Gamma(d - 1/2)}{2\sqrt{\pi}\Gamma(d)}, \quad d > 1/2,$$

(5.8)

$$\frac{1}{N|\log \lambda_N|} \sum_{t=1}^N X^2_{d,\lambda_N}(t) \xrightarrow{p} \frac{1}{\pi}, \quad d = 1/2.$$  

(5.9)
The main result of this section is the following theorem.

**Theorem 5.2** Consider the AR(1) model in (5.1) with \( \beta = 1 \) and ARTFIMA \((0,d,\lambda_N,0)\) error process \( X = \{X_{d,\lambda_N}(t)\} \) in (2.5) with i.i.d. innovations \( \{\zeta(t), t \in \mathbb{Z}\}, E\zeta(0) = 0, E\zeta^2(0) = 1, E|\zeta(0)|^p < \infty \) \((3p > 2 \vee 1/(d+1/2))\), fractional parameter \( d \in \mathbb{R} \setminus \mathbb{N}_- \) and tempering parameter \( \lambda_N > 0 \) satisfying (4.2).

(i) (Strongly tempered errors.) Let \( \lambda_\ast = \infty, d \in \mathbb{R} \setminus \mathbb{N}_- \). Then

\[
\min(1, \lambda_N^{-2d}) N(\hat{\beta}_N - 1) \quad \overset{d}{\longrightarrow} \quad \frac{1}{2 \int_0^1 B^2(s) s} \begin{cases} 
B^2(1), & d > 0, \\
B^2(1) - 1, & d = 0, \\
-\Gamma(1 - 2d)/\Gamma(1 - d)^2, & d < 0,
\end{cases}
\]

where \( B \) is a standard Brownian motion.

(ii) (Weakly tempered errors.) Let \( \lambda_\ast = 0 \) and \( H = d + \frac{1}{2} \in (0,1) \). Then

\[
N^{1 \wedge (1+2d)}(\hat{\beta}_N - 1) \quad \overset{d}{\longrightarrow} \quad \frac{1}{2 \int_0^1 B_H^2(s) s} \begin{cases} 
B_H^2(1), & \frac{1}{2} < H < 1, \\
B^2(1) - 1, & H = \frac{1}{2}, \\
-\Gamma(H + \frac{1}{2})/\Gamma(\frac{3}{2} - H), & 0 < H < \frac{1}{2},
\end{cases}
\]

where \( B_H \) is a FBM with variance \( E B_H^2(t) = t^{2H} \), \( B = B_{1/2} \).

(iii) (Moderately tempered errors.) Let \( 0 < \lambda_\ast < \infty \) and \( H = d + \frac{1}{2} > 0 \). Then

\[
N^{1 \wedge (1+2d)}(\hat{\beta}_N - 1) \quad \overset{d}{\longrightarrow} \quad \frac{1}{2 \int_0^1 (B_{H,\lambda_\ast}^H(s))^2} \begin{cases} 
(B_{H,\lambda_\ast}^H(1))^2, & H > \frac{1}{2}, \\
(B_{H,\lambda_\ast}^H(1))^2 - 1, & H = \frac{1}{2}, \\
-\Gamma(2(1 - H))/\Gamma(\frac{3}{2} - H)^2, & 0 < H < \frac{1}{2},
\end{cases}
\]

where \( B_{H,\lambda}^H \) is a TFBM II given by (3.6).

**Remark 5.3** The limit (5.10) in the weakly tempered case coincides with Sowell’s limit (5.3). Since \( B_{H,0}^H \overset{\text{fdd}}{=} C_1 B_H \), where \( C_1 = \Gamma(1 - H)/2^{2H} \sqrt{\pi} \) \((H + 1/2)\sqrt{\pi} \), see [22], the r.v. on the r.h.s. of (5.11) for \( \lambda_\ast = 0, 0 < H < 1 \) also coincides with (5.3), however the convergence (5.11) holds for any \( H > 0 \) in contrast to \( H \in (0,1) \) in (5.10).

**Proof of Theorem 5.2** From Theorem 4.3 and Proposition 5.1 we obtain the joint convergence of

\[
\left(a_N (S_{N,\lambda_\ast}^{d,\lambda_N}(1))^2, a_N \int_0^1 (S_{N,\lambda_\ast}^{d,\lambda_N}(s))^2 s, b_N \sum_{t=1}^N X_{d,\lambda_N}(t)\right)
\]

where \( a_N \to 0, b_N \to 0 \) are normalizations defined in these theorems and depending on \( d \) and \( \lambda_\ast \), in each case (i)-(iii) of Theorem 5.2. Then the statement of Theorem 5.2 follows from (5.12), the continuous mapping theorem and the representation of \( \hat{\beta}_N - 1 \) in (5.4)-(5.6) through the corresponding quantities in (5.12). \( \square \)
6 Proofs of Theorem 4.3 and Proposition 5.1

Proof of Theorem 4.3 (i) We restrict the proof of finite-dimensional convergence in (4.6) to one-dimensional convergence at \( t > 0 \) since the general case follows similarly. We use Proposition 4.2 accordingly, write \( N^{-d} S_N \lambda_N^d(t) = Q(g_N(t, \cdot)) = \sum_{i \in \mathbb{Z}} g_N(t; i) \zeta(i) \), where \( g_N(t; i) := N^{-1/\alpha} \lambda_N^d \sum_{k=1}^{[Nt]} \alpha \lambda_N^d \lambda_N^d \zeta(k-i) b_d(k-i) \). It suffices to prove (4.5) for suitable \( p \) and \( g(t; x) := 1_{[0,t]}(x) \). We have

\[
\tilde{g}_N(t; x) = \lambda_N^d \sum_{k=1}^{[Nt]} e^{-\lambda_N^d[Nx]} b_d(k - [Nx]).
\]

Let us prove the point-wise convergence:

\[
\tilde{g}_N(t; x) \to g(t; x) = 1_{[0,t]}(x), \quad \forall \ x \neq 0, t.
\]

Let us prove that conditions (1.2)-(1.4) imply that

\[
G_N := \lambda_N^d \sum_{k=0}^{\infty} e^{-\lambda_N^d k} b_d(k) \to 1 \quad (N \to \infty).
\]

First, let \( d > 0 \). Then since \( \sum_{k=0}^{n} b_d(k) \sim (1/d_{\Gamma}(d))n^d, n \to \infty \) according to (1.2), then by applying the Tauberian theorem for power series (Feller [7], Ch. 13, § 5, Thm. 5) we have \( \sum_{k=0}^{\infty} b_d(k) e^{-\lambda_N^d k} \sim (1 - e^{-\lambda_N^d})^{-d}, N \to \infty \), proving (6.3) for \( d > 0 \). Next, let \( d = 0 \). Then in view of (1.4) the dominated convergence theorem applies yielding \( \sum_{k=0}^{\infty} e^{-\lambda_N^d k} b_0(k) \to \sum_{k=0}^{\infty} b_0(k) = 1 \) and (6.3) follows again.

Next, let \(-1 < d < 0\). Then \( \tilde{b}_d(k) := \sum_{i=k}^{\infty} b_d(i) \sim (-1/d_{\Gamma}(d))k^{d-1}, k \to \infty, \tilde{d} := d + 1 \in (0, 1) \), \( \tilde{b}_d(0) = 0 \) and \( \sum_{k=0}^{\infty} e^{-\lambda_N^d k} b_d(k) = -e^{-\lambda} \sum_{k=1}^{\infty} e^{-\lambda_N^d k} b_d(k) \) using summation by parts. Then the aforementioned Tauberian theorem implies \( \sum_{k=0}^{\infty} e^{-\lambda_N^d k} b_d(k) \sim (1 - e^{-\lambda})^{1-\tilde{d}} = (1 - e^{-\lambda_N^d})^{-d} \) proving (6.3) for \(-1 < d < 0\). In the general case \(-j < d < -j + 1, j = 1, 2, \ldots \) relation (6.3) follows similarly using summation by parts \( j \) times.

Let \( 0 < x < t \) first. Then \( \tilde{g}_N(t; x) = G_N - \tilde{g}_N^*(t; x) \), where

\[
\tilde{g}_N^*(t; x) := \lambda_N^d \sum_{k=[Nx]}^{[Nt]-[Nx]} b_d(k)e^{-\lambda_N^d k}.
\]

Using (1.2) for \( d \neq 0 \) we obtain

\[
|\tilde{g}_N^*(t; x)| \leq C(N\lambda_N)^d N^{-1} \sum_{k=[Nx]}^{[Nt]-[Nx]} e^{-\lambda_N^d(k/N)}(k/N)^{d-1}
\]

\[
\leq C(N\lambda_N)^d \int_{t-x}^{\infty} e^{-\lambda_N^d y} y^{d-1} dy
\]

\[
= C \int_{(t-x)(N\lambda_N)}^{\infty} e^{-z} z^{d-1} dz \to 0
\]

(6.4)
since $N\lambda_N \to \infty$. A similar result for $d = 0$ follows directly from (1.4). In view of (6.3), this proves (6.2) for $0 < x < t$. Next, let $x < 0$. Then similarly as above

$$\bar{g}_N(t; x) \leq C(N\lambda_N)^d N^{-d} \sum_{k > [Nx]} e^{-(N\lambda_N)(k/N)(k/N)^{d-1}}$$

$$\leq C(N\lambda_N)^d \int_{|x|}^{\infty} e^{-N\lambda_N y} y^{d-1} y$$

$$= C \int_{|x|/(N\lambda_N)}^{\infty} e^{-z} z^{d-1} z \to 0,$$  \hspace{1cm} (6.5)

proving (6.2). Note also that $|\bar{g}_N(t; x)| \leq C\lambda_N^d \sum_{k=0}^{\infty} |b_d(k)| \leq C\lambda_N^d \leq C$ for $d < 0$ and $|\bar{g}_N(t; x)| \leq C(N\lambda_N)^d \int_0^{\infty} e^{-N\lambda_N y} y^{d-1} y \leq C\int_0^{\infty} e^{-z} z^{d-1} z \leq C$ for $d > 0$, implying that $|\bar{g}_N(t; x)|$ is bounded uniformly in $x \in \mathbb{R}, N \geq 1$; moreover, according to (6.5) $|\bar{g}_N(t; x)| \leq C e^{-c|x|}, x < -2$ decays exponentially with $x \to -\infty$ with some $c' > 0$ uniformly in $N \geq 1$. This proves (4.5) and hence (4.6).

Consider the functional convergence in (4.7). This follows from the tightness criterion

$$N^{-p/2} \lambda_N^{d} \mathbb{E}[S_{N}^{d,\lambda_N}(t) - S_{N}^{d,\lambda_N}(s)]^p \leq C|L_N(t) - L_N(s)|^{p/2}, \quad \forall \ 0 \leq s < t \leq 1,$$ \hspace{1cm} (6.6)

where $L_N(t) := [Nt]/N$, see [1], also (10), Lemma 4.4.1). By Rosenthal's inequality (see e.g. [10], Proposition 4.4.3), $\mathbb{E}[S_{N}^{d,\lambda_N}(t) - S_{N}^{d,\lambda_N}(s)]^p \leq C\mathbb{E}^{p/2} |S_{N}^{d,\lambda_N}(t) - S_{N}^{d,\lambda_N}(s)|^2$ where $\mathbb{E}|S_{N}^{d,\lambda_N}(t) - S_{N}^{d,\lambda_N}(s)|^2 = N\lambda_N^{-2d} \int |\bar{g}_N(t; x) - \bar{g}_N(s; x)|^2 x \leq CN\lambda_N^{-2d}|L_N(t) - L_N(s)|$ follows similarly as above, proving (6.6) and part (i), too.

(ii) Relation (4.8) is well-known with $S_{N}^{d,\lambda_N}(t)$ replaced by $S_{N}^{d,0}(t)$, see e.g. [2], also (10), Cor. 4.4.1), so that it suffices to prove

$$R_N(t) := S_{N}^{d,\lambda_N}(t) - S_{N}^{d,0}(t) = o_p(NH).$$  \hspace{1cm} (6.7)

With Proposition 4.2 in mind, (6.7) follows from $\|\bar{g}_N^0(t; \cdot)\|_p \to 0$, where

$$|\bar{g}_N^0(t; x)| := N^{-d} \left[ \sum_{k=1}^{\lfloor Nt \rfloor} b_d(k - [Nx]) \left( 1 - e^{-\lambda_N(k-[Nx])} \right) \right]$$

$$\leq C(N\lambda_N) \frac{1}{N^{d+1}} \sum_{k=1}^{\lfloor Nt \rfloor} (k - [Nx])^d \leq C(N\lambda_N) \to 0$$

uniformly in $x \in \mathbb{R}$, where we used (1.2), inequality $1 - e^{-x} \leq x (x \geq 0)$ and the fact that $H \in (0, 1), 1 < \alpha \leq 2$ imply $-1 < d < 1 - \frac{1}{\alpha}$. For $x < -1$ a similar argument leads to

$$|\bar{g}_N^0(t; x)| \leq CN^{-d} \sum_{k=1}^{\lfloor Nt \rfloor} (k - [Nx])^{d-1} \leq CN^{-d} (\lfloor Nt \rfloor - [Nx])^d \leq C((-x)^d)$$  \hspace{1cm} (6.8)

$$\leq C((-x)^d) \leq C(-x)^{d-1}$$
implying the dominating bound $|\tilde{g}_N^0(t; x)| \leq C/(1+|x|)^{1-d} =: \bar{g}(x)$ where $\|\bar{g}\|_p < \infty$ for $1 \leq p < \alpha$ sufficiently close to $\alpha$ due to condition $d < 1 - \frac{1}{\alpha}$. This proves $\|\tilde{g}_N^0(t; \cdot)\|_p \to 0$, hence (6.7) and (4.8), too.

To prove the tightness part of (ii), we use a similar criterion as in (6.6), viz.,

$$N^{-pH} \mathbb{E}|S_{N}^{d, \lambda N}(t) - S_{N}^{d, \lambda N}(s)|^{p} \leq C|L_{N}(t) - L_{N}(s)|^{q}, \quad \forall \ 0 \leq s < t \leq 1,$$

with $L_{N}(t) = [Nt]/N$ and suitable $p,q > 1$. Let first $\frac{1}{\alpha} < H < 1$, or $0 < d < 1 - \frac{1}{\alpha}$. Let $\tilde{g}_{N}(t; x) = N^{-d} \sum_{k=1}^{[Nt]} b_{d}(k - [Nt]) \ e^{-\lambda N(k - [Nt])}$. Then for $0 < s < t$

$$|\tilde{g}_{N}(t; x) - \tilde{g}_{N}(s; x)| \leq N^{-d} \sum_{k=\lfloor Ns \rfloor + 1}^{\lfloor Nt \rfloor} |b_{d}(k - [Nt])| \leq CN^{-d} \sum_{k=\lfloor Ns \rfloor + 1}^{\lfloor Nt \rfloor} (k - [Nt])^{d-1}$$

and therefore for $1 \leq p < \alpha$ sufficiently close to $\alpha$

$$N^{-pH} \mathbb{E}|S_{N}^{d, \lambda N}(t) - S_{N}^{d, \lambda N}(s)|^{p} \leq C\|\tilde{g}_{N}(t; \cdot) - \tilde{g}_{N}(s; \cdot)\|_{p}^{p}$$

Next, let $\alpha = 2$ and $0 < H < \frac{1}{2}$, or $-1/2 < d < 0$. Then (6.9) holds for $S_{N}^{d,0}$ instead of $S_{N}^{d, \lambda N}$ with $q = pH > 1$, see ([10], proof of Prop. 4.4.4). Hence, it suffices to prove a similar bound for $R_{N}(t)$ in (6.7). By Rosenthal’s inequality (see the proof (6.6)) $\mathbb{E}|R_{N}(t) - R_{N}(s)|^{p} \leq C\mathbb{E}^{p/2}|R_{N}(t) - R_{N}(s)|^{2}$ and hence $N^{-pH} \mathbb{E}|R_{N}(t) - R_{N}(s)|^{p} \leq C\|\tilde{g}_{N}^{0}(t; \cdot) - \tilde{g}_{N}^{0}(s; \cdot)\|_{2}^{p}$, where

$$|\tilde{g}_{N}^{0}(t; x) - \tilde{g}_{N}^{0}(s; x)| \leq CN^{-d} \sum_{k=\lfloor Ns \rfloor + 1}^{\lfloor Nt \rfloor} (k - [Nt])^{d-1} (1 - e^{-\lambda N(k - [Nt])})$$

similarly as in (6.10). Hence $N^{-pH} \mathbb{E}|R_{N}(t) - R_{N}(s)|^{p} \leq C\|\tilde{g}_{N}^{0}(t; \cdot) - \tilde{g}_{N}^{0}(s; \cdot)\|_{2}^{p} \leq C(L_{N}(t) - L_{N}(s))^{1+p(1+d)}$ follows, proving (6.7) and part (ii), too.

(iii) Similarly as in the proof of (ii), let us prove $\|\tilde{g}_{N}(t; \cdot) - g(t; \cdot)\|_{p} \to 0$, where

$$\tilde{g}_{N}(t; x) := \frac{1}{N^{d}} \sum_{k=1}^{[Nt]} b_{d}(k - [Nt]) \ e^{-\lambda N(k - [Nt])}, \quad g(t; x) := \frac{1}{1^{(1+d)}} \tilde{h}_{H,\alpha, \lambda_{*}}(t; x),$$

15
see the definition of $h_{H,\alpha,\lambda}$ in \textbf{(1.3)}. First, let $d > 0$ or $H > \frac{1}{\alpha}$. Then using \textbf{(1.2)} we obtain the point-wise convergence
\[
\tilde{g}_N(t; x) = \frac{1}{NT(d)} \sum_{k \leq [Nx]} \left( \frac{k}{N} - \frac{[Nx]}{N} \right)^{-1} \left( 1 + \epsilon_{N1}(k, x) \right) e^{-\lambda_s \left( \frac{k}{N} - \frac{[Nx]}{N} \right) (1 + \epsilon_{N2})}
\]
\[
\rightarrow \frac{1}{\Gamma(d)} \int_0^t (y-x)^{d-1} e^{-\lambda_s (y-x)} dy = \frac{1}{\Gamma(1+d)} h_{H,\alpha,\lambda_s}(t; x), \quad \forall x \neq 0, t, \quad \text{(6.14)}
\]
see \textbf{(3.3)}, where
\[
\epsilon_{N1}(k, x) := \Gamma(d) \left( k - [Nx] \right)^{-d-1} b_d(k - [Nx]) - 1 \rightarrow 0, \quad \epsilon_{N2} := (N\lambda_N/\lambda_s) - 1 \rightarrow 0
\]
as $N \rightarrow \infty, k - [Nx] \rightarrow \infty$ and $|\epsilon_{N1}(k, x)| + |\epsilon_{N2}| < C$ is bounded uniformly in $N, k, x$.
Therefore \textbf{(6.14)} holds by the dominated convergence theorem. We also have from \textbf{(6.13)} that
\[
|\tilde{g}_N(t; x)| \leq C N^{-d} \sum_{k=1}^{[Nx]} (k - [Nx])^{d-1} e^{-\left(\lambda_s/2\right)(k-[Nx])} \leq C \int_0^t (s-x)^{d-1} e^{-\left(\lambda_s/2\right)(s-x)} + g = C h_{d+1/\alpha,\lambda_s/2}(t; x) =: \tilde{g}(x)
\]
is dominated by an integrable function, see \textbf{(3.3)}, with $\|\tilde{g}\|_p < \infty$.
This proves \textbf{(1.9)} for $d > 0$.

Next, let $-\frac{1}{\alpha} < d < 0$. Decompose $\tilde{g}_N(t; x)$ in \textbf{(6.13)} as
\[
\tilde{g}_N(t; x) = \tilde{g}_{N1}(t; x) - \tilde{g}_{N2}(t; x),
\]
where
\[
\tilde{g}_{N1}(t; x) := N^{-d} \sum_{k=1}^{[Nx]} b_d(k - [Nx]) e^{-\lambda_s (k-[Nx])},
\]
\[
\tilde{g}_{N2}(t; x) := N^{-d} \sum_{[Nx]+1}^{\infty} b_d(k - [Nx]) e^{-\lambda_s (k-[Nx])}.
\]
Let $0 < x < t$. First we have
\[
\tilde{g}_{N1}(t; x) = \sum_{j=0}^{[Nt]} \frac{b_d(j) e^{-\lambda_s j}}{\lambda_s^d} (N\lambda_N)^{-d} \rightarrow \lambda_s^{-d}
\]
since the last ratio tends to 1 as $N \rightarrow \infty$, see \textbf{(6.3)}. We also have
\[
\tilde{g}_{N2}(t; x) = \frac{1}{NT(d)} \sum_{k \leq [Nx]} \left( \frac{k}{N} - \frac{[Nx]}{N} \right)^{-1} \left( 1 + \epsilon_{N1}(k, x) \right) e^{-\lambda_s \left( \frac{k}{N} - \frac{[Nx]}{N} \right) (1 + \epsilon_{N2})}
\]
\[
\rightarrow \frac{1}{\Gamma(d)} \int_0^t (s-x)^{d-1} e^{-\lambda_s (s-x)} dy = \lambda_s^{-d} - \frac{1}{\Gamma(1+d)} h_{H,\alpha,\lambda_s}(t; x), \quad \text{(6.14)}
\]
see \textbf{(3.4)}, similarly to \textbf{(6.14)}. This proves the point-wise convergence $\tilde{g}_N(t; x) \rightarrow g(t; x) = \Gamma(1 + d)^{-1} h_{H,\alpha,\lambda_s}(t; x)$ for $0 < x < t$ and the proof for $x < 0$ is similar. Then $\|\tilde{g}_N(t; \cdot) - g(t; \cdot)\|_p \rightarrow 0$ or \textbf{(4.9)} for $-\frac{1}{\alpha} < d < 0$ follows similarly as in the case $d > 0$ above.

Consider the proof of tightness in (iii). We use the same criterion \textbf{(6.5)} as in part (ii). Let first $d > 0$. Then for $|x| \leq 1$ the bound in \textbf{(6.10)} and hence $\int_0^1 |\tilde{g}_N(t; x) - \tilde{g}_N(s; x)|^p dx \leq C (L_N(t) - L_N(s))^{1+pd}$ follows as in \textbf{(6.11)}. On the other hand, for $x < -1$ we have
\[
|\tilde{g}_N(t; x) - \tilde{g}_N(s; x)| \leq C e^{-\left(\lambda_s/2\right)|x|} \int_{[Nx]/N}^{[Nt]/N} \chi = C e^{-\left(\lambda_s/2\right)|x|} (L_N(t) - L_N(s))
\]
implying $\int_{-\infty}^{-1} \mid \tilde{g}_N(t; x) - \tilde{g}_N(s; x) \mid dx \leq C \mid L_N(t) - L_N(s) \mid^p$. Consequently, (6.9) for $d > 0$ holds with $q = (1 + pd) \land p > 1$. Finally, (6.9) for $\alpha = 2, -1/2 < d < 0$ follows as in case (ii) since (6.12) holds in the case $\lambda_N = O(1/N)$ as well. This ends the proof of Theorem 4.3. \hfill \Box

**Proof of Proposition 5.7** (i) By stationarity, $N^{-1} \mathbb{E} \sum_{t=1}^{N} X_{d, \lambda N}^2(t) = \mathbb{E} X_{d, \lambda N}^2(0)$. Let us first prove the convergence of expectations:

$$
\mathbb{E} X_{d, \lambda N}^2(0) \rightarrow \frac{\Gamma(1 - 2d)}{\Gamma^2(1 - d)}, \quad d < 1/2,
$$

$$
\lambda_{N}^{2d-1} \mathbb{E} X_{d, \lambda N}^2(0) \rightarrow \frac{\Gamma(d - 1/2)}{2\sqrt{\pi} \Gamma(d)}, \quad d > 1/2,
$$

$$
\mid \log \lambda_{N} \mid^{-1} \mathbb{E} X_{d, \lambda N}^2(0) \rightarrow \frac{1}{\pi}, \quad d = 1/2,
$$

as $N \rightarrow \infty$. Since $\mathbb{E} X_{d, \lambda N}^2(0) = \sum_{k=0}^{\infty} e^{-2\lambda_N k} \omega_{-d}(k)$, with $\omega_{-d}(k) \equiv \omega_{-d}(k)$ defined in (2.3), and

$$
\sum_{k=0}^{n} \omega_{-d}(k) \sim \begin{cases} 
(1/2d - 1)\Gamma^2(d)n^{2d-1}, & d > 1/2, \\
(1/\Gamma^2(1/2)) \log(n), & d = 1/2, \\
\sum_{k=0}^{\infty} \omega_{-d}(k) = \Gamma(1 - 2d)/\Gamma^2(1 - d), & d < 1/2
\end{cases}
$$

as $n \rightarrow \infty$, the convergences in (6.15)-(6.17) follows from the Tauberian theorem in [7] used in the proof of Theorem 4.3 (i) above.

With (6.15)-(6.17) in mind, (5.7) - (5.9) follow from

$$
Q_N = N^{-1} \sum_{t=1}^{N} \left[ X_{d, \lambda N}^2(t) - \mathbb{E} X_{d, \lambda N}^2(t) \right] = \begin{cases} 
o_p(1), & d < 1/2, \\
 o_p(\lambda_{N}^{1-2d}), & d > 1/2, \\
 o_p(\mid \log \lambda_{N} \mid), & d = 1/2.
\end{cases}
$$

Let $\omega_{-d, \lambda}(k) := \omega_{-d}(k)e^{-\lambda k}$. We have $Q_N = Q_{N1} + Q_{N2}$, where

$$
Q_{N1} = N^{-1} \sum_{s \leq N} (\xi(s) - \mathbb{E} \xi(s)) \sum_{t=1 \land s}^{N} \omega_{-d, \lambda N}(t - s),
$$

$$
Q_{N2} = N^{-1} \sum_{s_2 < s_1 \leq N} \xi(s_1) \sum_{t=1 \land s_1}^{N} \omega_{-d, \lambda N}(t - s_1)\omega_{-d, \lambda N}(t - s_2) \xi(s_2).
$$

Note $Q_{Ni}, i = 1, 2$ are sums of martingale differences. We shall use the well-known moment inequality for sums of martingale differences:

$$
\mathbb{E} \left[ \mid \sum_{i \geq 1} \xi_i \mid^\alpha \right] \leq 2 \sum_{i \geq 1} \mathbb{E} \mid \xi_i \mid^\alpha
$$

see e.g. [10], Prop. 2.5.2, which is valid for any $1 \leq \alpha \leq 2$ and any sequence $\{\xi_i, i \geq 1\}$ with $\mathbb{E} \mid \xi_i \mid^\alpha < \infty, \mathbb{E} \xi_i \xi_j, 1 \leq j < i = 0, i \geq 1$. 17
First, let $d > 1/2$. Using $E|\zeta(0)|^p < \infty, 2 < p < 4$ and (6.19) with $\alpha = p/2$ we obtain

\[ E|Q_{N1}|^{p/2} \leq CN^{-p/2} \sum_{s \leq N} \left| \sum_{t=1}^N \omega_{-d,\lambda_N}(t-s) \right|^{p/2} \]

\[ \leq CN^{-p/2} \sum_{s \leq N} \left( \sum_{t=1}^N (t-s)^{2(d-1)} e^{-2\lambda_N(t-s)} \right)^{p/2} \]

\[ \leq CN^{-p/2} \left\{ \int_0^t \left( \int_0^N (t+s)^{2(d-1)} e^{-2\lambda_N(t+s)} s \right)^{p/2} + N \left( \int_0^N t^{2(d-1)} e^{-2\lambda_N t} s \right)^{p/2} \right\} \]

\[ \equiv CN^{-p/2} \{ I_{N1} + I_{N2} \}. \]

Here, $I_{N2} \leq N \left( \int_0^t \left( \int_0^N s^{2(d-1)} e^{-2\lambda_N s} \right)^{p/2} = CN^{p/2}(1-2d) \right)$ and $I_{N1} \leq CN^{-p/2} \int_0^t s^{(d-1)p} e^{-\lambda_N s} s \leq CN^{p/2} \lambda_N^{(1-d)p-1}$. Therefore,

\[ E|Q_{N1}|^{p/2} \leq C(\lambda_N^{(1-d)p-1} + N^{1-p/2} \lambda_N^{p/2}(1-2d)). \]  

(6.20)

Next,

\[ E|Q_{N2}|^2 \leq CN^{-2} \sum_{s_2 < s_1 \leq N} E \left( \sum_{t=1}^N \omega_{-d,\lambda_N}(t-s_1) \omega_{-d,\lambda_N}(t-s_2) \zeta(s_2) \right)^2 \]

\[ \leq CN^{-2} \sum_{s_2 < s_1 \leq N} \sum_{t=1}^N \omega_{-d,\lambda_N}(t-s_1) \omega_{-d,\lambda_N}(t-s_2) \]

\[ \leq CN^{-2} \sum_{t=1}^N \left( \sum_{s \leq t} \omega_{-d,\lambda_N}(t-s) \right)^2 \]

\[ \leq CN^{-2} \int_0^t \left( \int_{-\infty}^0 (t-s)^{2(d-1)} e^{-2\lambda_N(t-s)} s \right)^2 \]

\[ = CN^{-1} \left( \int_0^t s^{2(d-1)} e^{-2\lambda_N s} s \right)^2 \leq CN^{-1} \lambda_N^{2(1-2d)}. \]  

(6.21)

Since $(1-d)p - 1 > (p/2)(1-2d)$ and $1 - (p/2) < 0$, (6.20) and (6.21) prove (6.18) for $d > 1/2$.

Next, let $d < 1/2$. Then similarly as above we obtain

\[ E|Q_{N1}|^{p/2} \leq CN^{-p/2} \sum_{s \leq N} \left( \sum_{t=1}^\infty \left( \int_{-\infty}^t (t-s)^{2(d-1)} s \right)^{p/2} \right) \]

\[ = CN^{-p/2} \left\{ N \left( \sum_{t=1}^N t^{2(d-1)} \right)^{p/2} + \sum_{s \geq N} \left( \sum_{t=1}^N (s+t)^{2(d-1)} \right)^{p/2} \right\} \]

\[ \leq CN^{1-p/2} + CN^{-p/2} \sum_{s \geq N} (Ns^{2(d-1)})^{p/2} \leq CN^{1-p/2} \]  

(6.22)

and

\[ E|Q_{N2}|^2 \leq CN^{-2} \sum_{s_2 < s_1 \leq N} \sum_{t=1}^N \omega_{-d,\lambda_N}(t-s_1) \omega_{-d,\lambda_N}(t-s_2) \]

\[ \leq CN^{-2} \sum_{t=1}^N \left( \sum_{s < t} (t-s)^{2(d-1)} \right)^2 \leq CN^{-1}. \]  

(6.23)
\[ (6.22) \text{ and } (6.23) \text{ prove } (6.18) \text{ for } d < 1/2. \]

Finally, let \( d = 1/2 \). Then since \( p > 2 \)

\[
\mathbb{E}|Q_{N1}|^{p/2} \leq CN^{-p/2} \left( \sum_{s \leq N} \left( \sum_{t=1}^{\infty} (t-s)_+^{-1} \right)^{p/2} \right)
\]

\[ = CN^{-p/2} \left\{ N \left( \sum_{t=1}^{N} t^{-1} \right)^{p/2} + \sum_{s \geq N} \left( \sum_{t=1}^{N} (s+t)^{-1} \right)^{p/2} \right\} \]

\[ \leq CN^{1-p/2} (N \log N)^{p/2} + CN^{-p/2} \sum_{s \geq N} (Ns^{-1})^{p/2} \]

\[ \leq CN^{1-p/2} (N \log N)^{p/2} = o(1) \quad (6.24) \]

while

\[
\mathbb{E}|Q_{N2}|^2 \leq CN^{-2} \sum_{t=1}^{N} \left( \sum_{s < t} (t-s)^{-1} \exp^{-\lambda N(t-s)} \right)^2 \leq CN^{-1} (\log \lambda N)^2. \quad (6.25)
\]

\[ (6.24) \text{ and } (6.25) \text{ prove } (6.18) \text{ for } d = 1/2. \text{ Proposition 5.1 is proved.} \]

\[ \square \]

References

[1] Abramowitz, M., Stegun, I.: Handbook of mathematical functions, ninth edition, Dover, New York (1965).

[2] Astrauskas, A. (1983). Limit theorems for sums of linearly generated random variables. *Lithuanian J. Math.* 23 127–134.

[3] Balan, R., Jakubowski, A. and Louhichi, S. (2016). Functional convergence of linear processes with heavy-tailed innovations. *J. Theoret. Probab.* 29 491–526.

[4] Billingsley, P. (1968). *Convergence of Probability Measures*. New York: Wiley.

[5] Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods*, 2nd ed.. New York: Springer.

[6] Dickey, D. and Fuller, W. (1979). Distribution of the estimators for autoregressive time series with a unit root. *JASA* 74 427–431.

[7] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*, vol. 2. New York: Wiley.

[8] Giraitis, L., Kokoszka, P. and Leipus, R. (2000). Stationary ARCH models: dependence structure and central limit theorem. *Econometric Theory* 16 3–22.

[9] Giraitis, L., Kokoszka, P., Leipus, R. and Teyssiére, G. (2003). Rescaled variance and related tests for long memory in volatility and levels. *J. Econometrics* 112 265–294.

[10] Giraitis, L., Koul, H.L. and Surgailis, D. (2012). *Large Sample Inference for Long Memory Processes*. London: Imperial College Press.

[11] Gradshteyn, I.S. and Ryzhik, I.M. (2000). *Tables of Integrals and Products*. 6th edition. New York: Academic Press.

[12] Granger, C.W.J. and Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *J. Time Series Anal.* 1 15–29.
[13] Hosking, J.R.M. (1981). Fractional differencing. *Biometrika* **68** 165–176.

[14] Kasahara, Y. and Maejima, M. (1988). Weighted sums of i.i.d. random variables attracted to integrals of stable processes. *Probab. Theory Relat. Fields* **78** 75–96.

[15] Kokoszka, P.S. and Taqqu, M.S. (1995). Fractional ARIMA with stable innovations. *Stochastic Process. Appl.* **60** 19–47.

[16] Lavancier, F., Leipus, R., Philippe, A. and Surgailis, D. (2013). Detection of non-constant long memory parameter. *Econometric Theory* **29** 1009–1056.

[17] Lo, A. (1991). Long-term memory in stock market prices. *Econometrica* **59** 1279–1313.

[18] Meerschaert, M.M. and Sabzikar, F. (2013). Tempered fractional Brownian motion. *Statist. Probab. Lett.* **83** 2269–2275.

[19] Meerschaert, M.M. and Sabzikar, F. (2016). Tempered fractional stable motion. *J. Theoret. Probab.* **29** 681–706.

[20] Meerschaert, M.M., Sabzikar, F., Phanikumar, M.S. and Zeleke, A. (2014). Tempered fractional time series model for turbulence in geophysical flows. *J. Stat. Mech. Theory Exp.* **2014** P09023.

[21] Phillips, P.C.B. (1987). Time series regression with a unit root. *Econometrica* **55** 277–301.

[22] Sabzikar, F. and Surgailis, D. (2017). Tempered fractional Brownian and stable motions of second kind. Preprint. Available on [http://arxiv.org/abs/1702.07258](http://arxiv.org/abs/1702.07258)

[23] Samorodnitsky, G. and Taqqu, M.S. (1996). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Boca Raton etc: Chapman and Hall.

[24] Sowell, F. (1990). The fractional unit root distribution. *Econometrica* **58** 495–505.