Graphic-analytical calculation method of straight centrally compressed bars on stability

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Abstract. In designing bars a significant drawback of employing the method for calculating slenderness using the coefficient of reducing allowable stress lies in the need to use the successive approximation method requiring a considerable amount of calculation. This paper suggests an alternative grapho-analytical method of calculations devoid of this drawback. The essence of the method proposed consists in analytically receiving the dependence between the coefficient of reducing the allowable stress and the slenderness and in graphically determining their real values as the coordinates of the point of intersection of the graph of the dependence received and the graph built using the reference values.

1. Introduction
Anyone who has taken the course on materials resistance would be familiar with the graphic representation of the dependence between normal stress $\sigma_{kr}$, corresponding to the critical force which, if exceeded, would result in losing the stability of the original shape of the homogeneous straight centrally compressed bar, from its slenderness $\lambda$ (Figure 1).

![Figure 1. Dependence between critical stress and slenderness of the bar.](image)

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Curve I represents the calculated dependence \( \sigma_{iy} = \frac{\pi^2 E}{l^2} \), where \( \lambda = \frac{\beta l}{i} \) for bar slenderness; \( i = \sqrt{\frac{J_{\text{min}}}{A}} \) for the minimum radius of inertia; \( J_{\text{min}} \) is the minimum axial moment of inertia of the cross section; \( A \) is the cross-section area; \( \beta \) is the length coefficient depending on the mode of bar fixation and \( l \) is the length of the bar.

As it follows from the nature of curve I, with the reduction in the bar slenderness, the stress \( \sigma_{iy} \) rises indefinitely, which must not be the case in reality. When the stress reaches its limit value \( \sigma_L \) (the yield stress under compression \( \sigma_{T,C} \) for plastic materials or the stress limit under compression \( \sigma_{B,C} \) for fragile materials) the bar loses its load-bearing capacity, thus the limiting straightforward II is introduced to cut off the upper section of curve I. When plasto-elastic deformations occur, the calculations for stability have to be made using the Yasinsky formula: \( \sigma_{iy} = a - b\lambda + cl\lambda^2 \), where \( a, b, c \) are the coefficients depending on the material of the bar.

Thus taking that \( \lambda < \lambda_0 \), i.e. when the bar’s slenderness is low, it has to be calculated for durability; with the medium slenderness of the bar, when \( \lambda_0 < \lambda < \lambda_{\text{lim}} \), its stability has to be calculated using the Yasinsky formula; when \( \lambda > \lambda_{\text{lim}} \) the calculations have to be made using the Euler formula (bars with high slenderness).

2. Formulating the task

There exists the method for calculating centrally compressed bars for slenderness using the coefficient for the allowable stress reduction. The allowable stress equals \( [\sigma_c] = \frac{\sigma_L}{n_c} \) (where \( n_c \) is a coefficient of resistance). The allowed stress on stability is determined using the \( [\sigma]_{\text{y}} = \frac{\sigma_{iy}}{n_y} \) formula. The ratio of allowed stresses may be formulated as: \( \frac{[\sigma]_c}{[\sigma]_{\text{y}}} = \frac{\sigma_{iy} \cdot n_c}{n_y \cdot \sigma_L} \), or \( [\sigma]_{\text{y}} = \phi \cdot [\sigma]_c \), where \( \phi = \frac{\sigma_{iy} \cdot n_c}{n_y \cdot \sigma_L} \) is a coefficient of resistance of allowed stresses.

The idea for the method of calculating bar stability consists in the fact that for low slenderness bars, the allowable compression stress equals \( [\sigma]_c \), whereas with the increase in slenderness it has to be reduced gradually multiplying it by the \( \phi \) coefficient whose value is below 1. The values of the \( \phi \) coefficient depending on slenderness \( \lambda \) are available in books of reference for different materials.

Conventionally, depending on the task set there exist checking calculations and designing calculations for stability. In case of checking calculations, there usually arises no difficulty in using coefficient \( \phi \). Stemming from geometrical dimensions of the bar under calculation and the conditions of its fixation, the minimum value of the inertia radius is determined, as well as slenderness and allowable stress on stability. Then the stresses occurring in cross-sections of the bar as a result of compression force \( F \left( \sigma = \frac{F}{A} \right) \) are compared with the allowable stress for stability and the conclusion is drawn.

In designing calculations some uncertainty occurs due to the fact that the calculations of the cross-section area stem from \( \sigma = \frac{F}{\phi A} \leq [\sigma]_c \), i.e. \( A \geq \frac{F}{\phi [\sigma]_c} \), therefore, the value of coefficient \( \phi \) remains unknown. This results in the need to employ the method of successive approximations [1], [2]. Taking
the initial value of $\varphi_1 = 0.05...0.6$, we calculate the required area of cross-section $A_1$. Knowing the type of the section, we determine the minimum radius of inertia $i_1$ and the bar slenderness $\lambda_1$. Using the value of $\lambda_1$ we determine the respective value of the coefficient of allowable stress reduction in $\varphi'$. In case $\varphi_1$ and $\varphi_1$ significantly diverge, the calculations are repeated at the second stage taking the value of the second coefficient $\varphi_2 = \frac{\varphi_1 + \varphi'}{2}$. New values of area $A_2$ are calculated, as well as those for the minimum inertia radius $i_2$, slenderness $\lambda_2$ and the respective value of coefficient $\varphi'_2$. In case $\varphi'_2$ and $\varphi_2$ significantly diverge, the calculations are repeated taking the value of $\varphi_3 = \frac{\varphi_2 + \varphi'_2}{2}$.

This procedure may be repeated several times, which allows to gradually approach the real value of the cross-section area of the bar.

Thus, the need to employ the step-by-step method requiring a large number of calculations is a drawback inherent in the application of the method for calculating the stability using the coefficient for reducing the main allowable unit stress in designing. It is necessary to formulate a method devoid of this drawback, i.e. the one that does not require numerous iterations.

3. Solution

The grapho-analytical method of calculating the stability of centrally compressed bars is devoid of the drawback inherent in the method of calculations involving the coefficient of reducing the allowable main stress. Its essence consists in the analytical obtaining the $\varphi(\lambda)$ dependence between the coefficient $\varphi$ and slenderness $\lambda$, and the graphical determining of the real values of $\varphi$ and $\lambda$, as the coordinates of the intersection of the graph of the dependence received and the graph built using the values of $\varphi$ and $\lambda$ taken from the reference books. In particular, Table 1 [3] features the respective parameters for steel $\sigma_{F.C.} = 400\, MPa$ [3].

| $\lambda$ | 0   | 10  | 20  | 30  | 40  | 50  | 60  | 70  | 80  | 90  | 100 | 110 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\varphi$ | 1.000 | 0.982 | 0.949 | 0.905 | 0.854 | 0.796 | 0.721 | 0.623 | 0.532 | 0.447 | 0.369 | 0.306 |
| $\lambda$ | 120 | 130 | 140 | 150 | 160 | 170 | 180 | 190 | 200 | 210 | 220 |
| $\varphi$ | 0.260 | 0.223 | 0.195 | 0.171 | 0.152 | 0.136 | 0.123 | 0.111 | 0.101 | 0.093 | 0.086 |

As an example, let us consider a fix-ended straight bar with the square cross-section $b\times b$ centrally loaded by the applied compression force $F$ (Figure 2). Here $A = b^2$; $J_{\min} = \frac{b^4}{12}$; $i = \sqrt{\frac{J_{\min}}{A}} = \frac{b}{\sqrt{12}}$; $\lambda = \frac{\beta l}{i} = \sqrt{\frac{12}{\lambda} \beta l}$. As $\sqrt{A} = b$, consequently $\lambda = \frac{\sqrt{12} \beta l}{\sqrt{A}}$. Thus $A = \frac{12 \beta^2 l^2}{\lambda^2}$. On the other hand, $A \geq \frac{F}{\varphi|\sigma|_{c}}$. The comparison the latter two expressions allows us to conclude that the minimum value for the cross-section area of the bar will be determined using the formula: $\varphi = \frac{F \lambda^2}{12 \beta^2 l^2 |\sigma|_{c}}$.

We proceed by making the graph showing the change in $\varphi$ depending on $\lambda$, for example, on the basis of Table 1 and the graph of the latter dependence and taking $l = 1\, m$, $\beta = 2$ (rigid restraint of one of the bar ends), $F = 500\, kN$. We obtain the values of $\varphi$ and $\lambda$, as the coordinates of the intersection
point of these two graphs. Figure 3 features the corresponding graphs. The value of \( \lambda \approx 109 \) identified in Figure 3 is further used in the formula \( A = \frac{12B^2l^2}{\lambda^2} \). Taking into account that for a square \( b = \sqrt{A} \), we obtain the value of \( b = 64 \text{ mm} \).

![Figure 2. Bar design scheme.](image)

![Figure 3. Determining the coefficient of reducing allowed stress for a square section.](image)

Figure 4 features the outcomes of the calculations for the stability of the respective bar when fixed using the Nastran software (the bar is depicted as deformed under the compressive force of 700 kN, with the scale reflecting full shift), figure 5 helps to visualize the process using the graph of moving \( \delta \) of the unfixed end of the bar perpendicular to its original axis in its undeformed state received as a result of the step-by-step calculations involving Nastran. It is seen that with the value of the compressing force at approximately 500 kN a huge increase in the bending occurs. Thus the quantitative solution confirms that the grapho-analytical method use is justified.

![Figure 4. Outcomes of calculations using the Nastran software.](image)
Now let us consider a similar bar having a round cross section which diameter equals \(d\). Here \(A = \frac{\pi d^2}{4}\); \(J_{\text{min}} = \frac{\pi d^4}{64}\); \(i = \sqrt{\frac{J_{\text{min}}}{A}} = \sqrt{\frac{\pi d^4 \cdot 4}{64 \pi d^2}} = \frac{d}{4} \); \(\lambda = \frac{\beta l}{i} = \frac{4\beta l}{d}\). As \(d = 2\sqrt{\frac{A}{\pi}}, \lambda = \frac{4\beta l}{2\sqrt{A/\pi}}\). It follows that \(A = \frac{4\pi \beta^2 l^2}{\lambda^2}\). On the other hand, \(A \geq \frac{F}{\varphi \sigma_{\text{lc}}}\). By comparing the last two expressions we conclude that the minimum value of the cross-section area of the bar will be determined using the formula: \(\varphi = \frac{F \lambda^2}{4\pi \beta^2 l^2 \sigma_{\text{lc}}}\).

We proceed by building the graph reflecting the change in \(\varphi\) depending on \(\lambda\), for example, on the basis of Table 1, and the graph of the ultimate dependence taking \(l = 1\, \text{m}, \beta = 2\) (rigid restraint of one of the bar ends), \(F = 500\, \text{kN}\). We obtain the values of \(\varphi\) and \(\lambda\) as coordinates of the intersection point of these two graphs. The respective graphs are shown in Figure 6. The value of \(\lambda \approx 111\) calculated on the basis of the data featured in Figure 6 is then used in the \(A = \frac{4\pi \beta^2 l^2}{\lambda^2}\) formula. Taking into account that for a circle \(d = 2\sqrt{\frac{A}{\pi}}\), we obtain the value of \(b = 72\, \text{mm}\).

![Figure 5](image5.png)

**Figure 5.** Dependence between the moving of the free end of the bar and the compressing force.

![Figure 6](image6.png)

**Figure 6.** Determining the coefficient of reducing allowed stress for a round section.

### 4. Conclusions
The method described allows to determine the dimensions of cross-sections of centrally compressed straight bars rather fast and easily while calculating their stability. The method may be used for bars whose cross-sections allows to obtain the analytical dependency between area \(A\) and axial moment of inertia \(J_{\text{min}},\) – for a circle, a square, a triangle, a rectangle with a given ratio of sides, etc., i.e. for simple types of sections. This limits its use in case of calculating for bars having more complex sections, which is a drawback.

With the view to facilitating the process of using the method suggested, it is possible to obtain the ultimate analytical dependencies between coefficients \(\varphi\) and slenderness \(\lambda\) and those between areas \(A\) and \(\lambda\). Then, resorting to the Excel software, it is possible to achieve automation in rather a simple calculation process. This calculation method does not require any sophisticated systems, which may also be considered its advantage.
5. References

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