Black Hole solutions in a cosmological spacetime background

Metin Arık and Yorgo Şenikoğlu
Department of Physics, Boğaziçi University, Bebek, Istanbul, Turkey
(Dated: January 18, 2014)

We propose and analyze a new metric that has two conformal factors a(t) and b(t) that combine the expansion of the universe and its effects on the spatial and temporal part of the Schwarzschild metric in isotropic coordinates. We present the solutions, their descriptions and we comment on their shortcomings. In the spatially flat case of an expanding universe, we derive from the proposed metric the special solutions of the field equations for the dust approximation and the McVittie metric. We show that the presence of a black hole does not modify the a(t)α r^1/3 law for dust and \( H = \text{constant} \) for dark energy.

PACS numbers: 04.20.Cv, 04.20.Jb, 04.70.-s

I. INTRODUCTION

In Einstein’s theory of general relativity, two exact solutions were well known and studied throughout the years. One is the Schwarzschild solution that describes the gravitational field outside a spherical non-rotating mass, without charge, and the cosmological constant set to zero. This was practical to model spacetime outside a star, a planet or a black hole. In Schwarzschild coordinates (on a spherically symmetric space-time) the line element for the Schwarzschild metric has the form

\[
ds^2 = (1 - \frac{2Gm}{r})dt^2 - (1 - \frac{2Gm}{r})^{-1}dr^2 - r^2 d\Omega^2
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \).

Another formulation commonly used of this spacetime is in the isotropic coordinates. The metric takes the form

\[
ds^2 = \left(1 - \frac{Gm}{2r} \right)^2 dt^2 - \left(1 + \frac{Gm}{2r} \right)^4 \left(dr^2 + r^2 d\Omega^2 \right).
\]

The issue of the Schwarzschild metric is that it ignores the fact the universe is expanding in the background in which the mass is present.

Another exact solution of the Einstein Field Equations is the Friedmann Robertson Walker Lemaître (FRWL) metric \( [1],[2] \), with the assumption that space is homogeneous and isotropic, and the spatial part of the metric may depend on time.

For a spatially flat universe, the FRWL metric is

\[
ds^2 = dt^2 - a(t)^2 \left(dr^2 + r^2 d\Omega^2 \right).
\]

The non-vanishing components of the Einstein tensor in the orthonormal basis are

\[
G_{00} = 3 \frac{\ddot{a}}{a^2}, \quad G_{11} = G_{22} = G_{33} = -2 \frac{\dot{a}}{a} - \frac{\ddot{a}}{a^2}.
\]

Concerning the impacts in an expanding universe, authors V.Faraoni \( [3] \) and A.Jacques \( [4] \) worked on the cosmological effects of expansion on local systems. They explored the local attraction in a gravitationally bound system and analysed the solution of general relativity representing a black hole embedded in a special cosmological background. H.Arakida \( [5] \) presented the dominant effects due to cosmological expansion, and the use of a time dependent spacetime model.

With a time dependent model McVittie \( [6],[7] \) incorporated two solutions to depict a spherically symmetric metric that describes a point mass embedded in an expanding spatially-flat universe.

The McVittie metric is

\[
ds^2 = \left(1 - \frac{Gm}{2ra(t)} \right)^2 dt^2 - a(t)^2 \left(1 + \frac{Gm}{2ra(t)} \right)^4 \left(dr^2 + r^2 d\Omega^2 \right).
\]

The non-vanishing components of the Einstein tensor in the orthonormal basis are

\[
G_{00} = 3 \frac{\ddot{a}}{a^2}, \quad G_{11} = G_{22} = G_{33} = - \frac{1}{\left(1 - \frac{Gm}{2ra(t)} \right)^2} \left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \left( \frac{Gm}{2ra(t)} \right)^2 \left(2 \frac{\ddot{a}}{a} - 5 \frac{\dot{a}^2}{a^2} \right) \right).
\]

K.Lake and M.Abdelqader \( [8] \) studied the McVittie solution that contains a black hole in an expanding universe, worked on specific solutions that asymptote ΩCDM cosmology and A.M. da Silva, M.Fontanini and D.C.Guariento \( [9] \), the main characteristics of the McVittie solution for different choices of the scale factor. R.Nandra, A.N.Lasenby and M.P.Hobson \( [10] \)}
present a tetrad based procedure to solve Einstein’s field equations to derive metrics describing a point mass in an expanding universe and B.C.Nolan\[11\] proves the existence of solutions representing more general spherical objects embedded in a Robertson Walker universe.

This leads us to a very naive question; why should the conformal factor (scale factor) a(t) influence the same way the temporal part and spatial part of the metric. Can a more general expression of the metric be presented? The motivation that drove us also to this, was the question whether or not the Schwarzschild Radius was increasing in an expanding universe. Thus we wanted to see if a conformal factor can influence with a time dependency, the radius of this black hole and therefore the dynamics that it creates with time. In the following section we will propose more generalized solutions of the Field Equations that lead to interesting findings.

II. GENERAL FORMULATION

Let us write the metric with two conformal factors a(t), the scale factor and b(t), a conformal factor.

$$ds^2 = \frac{(1 - \frac{Gm(t)}{2r})^2}{(1 + \frac{Gm(t)}{2r})^2} dt^2 - a(t)^2 \left(1 + \frac{Gmb(t)}{2r}\right)^4 (dr^2 + r^2 d\Omega^2).$$

The calculations lead us to the following non-vanishing components of the Einstein tensor in the orthonormal basis

$$G_{00} = \frac{3}{a^2} \left(1 + \frac{Gm}{2r}\right)^2 \left(1 + \frac{Gmb}{2r}\right)^2 (b \dot{a} + b \dot{a} + b a) \frac{Gm}{a^2} ;$$

$$G_{01} = \frac{2Gm}{r^2 \left(1 - \frac{Gm}{2r}\right)^2 \left(1 + \frac{Gmb}{2r}\right)^2} \frac{(b \dot{a} + b \dot{a})}{a^2} ;$$

$$G_{11} = G_{22} = G_{33} \frac{1}{\left(1 - \frac{Gm}{2r}\right)^2 \left(1 + \frac{Gmb}{2r}\right)^2} \left(2 \ddot{a} + \ddot{a}\right)$$

$$+ \frac{1}{a^2} \left(\frac{Gm}{2r}\right)^2 (b \dot{a}^2 + 16a \dot{a}^2 + 2a \dot{a} b + 4a^2 \dot{b})$$

$$+ \frac{1}{a^2} \left(\frac{Gm}{2r}\right)^2 (-b^2 a^2 + 4ab \dot{b} - 2ab \dot{a} a + 16a^2 b^2)$$

$$+ \frac{1}{a^2} \left(\frac{Gm}{2r}\right)^2 (-b^3 a^2 - 12a^2 \dot{a} b - 2ab \dot{a} a - 8a^2 b \dot{b} - 4a^2 b^2 \dot{b}) .$$

III. DISCUSSIONS OF SOLUTIONS

A. Dust approximation

Matter dominated Universe is modeled by dust approximation. The matter is approximated as stationary dust particles which produce no pressure.

For p = 0,

$$G_{11} = G_{22} = G_{33} = 0.$$

Equating (12) to zero, we obtain a system of differential equations.

$$2 \dot{a} + \frac{\dot{a}^2}{a^2} = 0,$$

$$(b \dot{a}^2 + 16a \dot{a} b + 2a \dot{b} + 4a^2 \dot{b}) = 0,$$

$$(-b^2 a^2 + 4ab \dot{b} - 2ab \dot{a} a + 16a^2 b^2) = 0,$$

$$(-b^3 a^2 - 12a^2 \dot{a} b - 2ab \dot{a} a - 8a^2 b \dot{b} - 4a^2 b^2 \dot{b}) = 0.$$}

The unique solution that satisfies this system of equations is

$$a(t) = ct^{2/3} \text{ and } b(t) = 1$$

where c is a constant. We have found here, from our generalized metric (9), the well known behaviour of the scale factor a(t).

In addition for

$$a(t) = ct^{2/3} \text{ and } b(t) = 1$$

we have

$$G_{00} = \frac{4}{3r^2} \left(1 + \frac{Gm}{2r}\right)^2 ;$$

$$G_{01} = \frac{4}{3r^2 \left(1 - \frac{Gm}{2r}\right)^2 \left(1 + \frac{Gmb}{2r}\right)^2} \frac{c^5}{3r}. \quad (14)$$

We note here that the behaviour of $G_{01}$ is as $\frac{1}{r^2}$, since $G_{01}$ is a momentum component, its positivity shows that matter falls outward. This may be a fact that supports the proposition that the universe was created from a black hole.

From this we note that the energy due to dust calculated for any finite volume including the black hole is infinite. In his latest talk [12], S.W.Hawking states that "black holes are not black", this can be a manifestation of what we have stated previously.

Another fact that we need to state is that, as we propose it, our event horizon is not static because of b(t). Our result for dust solution is b(t)=1 , which implies a static event horizon. This has been pointed also by A.Davidson and S.Rubin [13]: an evolving universe can host locally a static event horizon.

B. The case for a diagonal Einstein Tensor

For $G_{01}$=0, from equation (11) we obtain simply

$$b(t) = \frac{1}{a(t)}. \quad (15)$$
Replacing the value of \( b(t) \) into (10) and (12), we get the following equations

\[
G_{00} = 3 \frac{\dot{a}^2}{a^2} , \tag{16}
\]

\[
G_{11} = G_{22} = G_{33} = \frac{1}{(1 - \frac{Gm}{2ra(t)})} \left( \frac{2 \ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \left( \frac{Gm}{2ra(t)} \right) \left( \frac{2 \ddot{a}}{a} - \frac{3 \dot{a}^2}{a^2} \right) \right) . \tag{17}
\]

We notice that this is the solution produced again by our generalized metric (9). They are identical to equations (7) and (8) of the McVittie metric.

Let us introduce the Hubble parameter:

\[
H = \frac{\dot{a}}{a} . \tag{18}
\]

Consequently the equations (16) and (17) become

\[
G_{00} = 3H^2 , \quad G_{11} = G_{22} = G_{33} = \frac{1}{(1 - \frac{Gm}{2ra(t)})} \left( 2H + 3H^2 + \frac{Gm}{2ra(t)}(2H - 3H^2) \right) . \tag{19}
\]

For

\[
H = \frac{\dot{a}}{a} = \text{constant} , \tag{20}
\]

we have: \( G_{00} = 3H^2 \) and

\[
G_{11} = G_{22} = G_{33} = -3H^2 \quad \text{which is independent of } r .
\]

Equivalently, it can be shown that if \( G_{11}, G_{22}, G_{33} \) are independent of \( r \), \( H \) is constant.

We find that for \( b(t) = \frac{1}{a(t)} \) and \( H = \text{constant} \)

\[
G_{00} = -G_{11} = -G_{22} = -G_{33} = 3H^2 \quad \text{which is constant. This is the vacuum (dark) energy for dark energy dominated universe.}
\]

IV. CONCLUSION

We have investigated the effects of conformal factors \( a(t) \) and \( b(t) \) in the background of a homogeneous and isotropic expanding universe. Consequently, field equations were presented for various cases to illustrate our point of view. From a more generalized metric that we have proposed in the beginning of this paper, we have shown that, depending on given conditions, we can obtain the characteristics of \( a(t) \) and \( b(t) \) for the matter dominated universe \((p = 0, \text{ matter dominated dust approximation})\) and the vacuum (dark) energy dominated universe \((p = -\rho)\) with the McVittie metric.

Concerning the radius of the black hole, we need to make some remarks. In our terms, the radius is \( r = 2Gmb(t) \). For the expanding universe, with the dust approximation, \( b(t) = 1 \) we note that the radius is \( a(t)r = 2Gma(t) \). This means that the black hole radius increases with the expansion. In the case that \( b(t) = \frac{1}{a(t)} \), the McVittie case, we have \( a(t)r = 2Gm = \text{constant} \); the black hole radius is constant with the expansion.

V. ACKNOWLEDGEMENTS

Y. Şenikoğlu and M. Arık thank Prof. A. Davidson for his insights and helpful conversations, and the referee, whose suggestions remarkably improved the presentation of the paper.

[1] A.A.Friedmann. On the curvature of space. Z.Phys., 10:377, 1922.
[2] G.Lemaître. Expansion of the universe. Mon.Not.R.Astron.Soc., 91:483, 1931.
[3] V.Faraoni. An analysis of the Sultana-Dyer cosmological black hole solution of the Einstein equations. Phys.Rev.D, 80:044013, 2009.
[4] A.Jacques V.Faraoni. Cosmological expansion and local physics. Phys.Rev.D, 76:063510, 2007.
[5] H.Araikawa. Time delay in Robertson McVittie spacetime. New Astronomy, 14:264–268, 2008.
[6] G.C.McVittie. The mass-particle in an expanding universe. Mon.Not.R.Astron.Soc., 93:325–329, 1933.
[7] G.C.McVittie. General Relativity and Cosmology. University of Illinois Press, 2nd edition, 1965.
[8] M.Abdelgader K.Lake. A Schwarzschild - de Sitter black and white hole embedded in an asymptotically LambdaCDM cosmology. Phys.Rev.D, 84:044045, 2011.
[9] D.C.Guarento A.M. da Silva, M.Fontanini. How the expansion of the universe determines the causal structure of mcvittie spacetimes. Phys.Rev.D, 87:064030, 2013.
[10] M.P.Hobson R.Nandra, A.N.Lasenby. The effect of a massive object on an expanding universe. Mon.Not.R.Astron.Soc., 4:2931–2944, 2012.
[11] B.C.Nolan. A point mass in an isotropic universe. Phys.Rev.D, 58:064006, 1998.
[12] S.W.Hawking. Information preservation and weather forecasting for black holes. arXiv, 2014. hep-th 1401.5761.
[13] S.Rubin A.Davidson. Can an evolving universe host a static event horizon. Phys.Rev.D, 86:104061, 2012.