SPECTRUM AND EIGENFUNCTIONS OF THE LATTICE HYPERBOLIC RUIJSENAARS-SCHNEIDER SYSTEM WITH EXPONENTIAL MORSE TERM

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Abstract. We place the hyperbolic quantum Ruijsenaars-Schneider system with an exponential Morse term on a lattice and diagonalize the resulting n-particle model by means of multivariate continuous dual $q$-Hahn polynomials that arise as a parameter reduction of the Macdonald-Koornwinder polynomials. This allows to compute the n-particle scattering operator, to identify the bispectral dual system, and to confirm the quantum integrability in a Hilbert space set-up.

1. Introduction

It is well-known that the hyperbolic Calogero-Moser n-particle system on the line can be placed in an exponential Morse potential without spoiling the integrability [A, LW]. An extension of Manin’s Painlevé-Calogero correspondence links the particle model in question to a multicomponent Painlevé III equation [H]. Just as for the conventional Calogero-Moser system without Morse potential, the integrability is preserved upon quantization and the corresponding spectral problem gives rise to a rich theory of remarkable novel hypergeometric functions in several variables [IM, O, H, HL].

An integrable Ruijsenaars-Schneider type ($q$-)deformation [RS, H1] of the hyperbolic Calogero-Moser system with Morse potential was introduced in [S] and in a more general form in [D2, Sec. II.B]. Recently, it was pointed out that particle systems of this kind can be recovered from the Heisenberg double of SU($n,n$) via Hamiltonian reduction [M]. In the present work we address the eigenvalue problem for a quantization of the latter hyperbolic Ruijsenaars-Schneider system with Morse term. Specifically, it is shown that the eigenfunctions are given by multivariate continuous dual $q$-Hahn polynomials that arise as a parameter reduction of the Macdonald-Koornwinder polynomials [K, M2]. As immediate by-products, one reads-off the n-particle scattering operator and the commuting quantum integrals of a bispectral dual system [DG, G].

The material is organized as follows. In Section 2 we place the hyperbolic Ruijsenaars-Schneider system with Morse term from [D2] on a lattice. The diagonalization of the resulting quantum model in terms of multivariate continuous dual $q$-Hahn polynomials is carried out in Section 3. In Sections 4 and 5 the n-particle
scattering operator and the bispectral dual integrable system are exhibited. Finally, the quantum integrability of both the hyperbolic Ruijsenaars-Schneider system with Morse term on the lattice and its bispectral dual system are addressed in Section 6.

2. Hyperbolic Ruijsenaars-Schneider system with Morse term

The hyperbolic quantum Ruijsenaars-Schneider system on the lattice was briefly introduced in [R2, Sec. 3C2] and studied in detail from the point of view of its scattering behavior in [R4] (see also [D4, Sec. 6] for a further generalization in terms of root systems). In this section we formulate a corresponding lattice version of the hyperbolic quantum Ruijsenaars-Schneider system with Morse term introduced in [D2, Sec. II.B].

2.1. Hamiltonian. The Hamiltonian of our \( n \)-particle model is given by the formal difference operator [D2, Eqs. (2.25), (2.26)]:

\[
H := \sum_{j=1}^{n} \left( w_+(x_j) \left( \prod_{1 \leq k \leq n, k \neq j} \frac{t^{-1} - q^{x_j-x_k}}{1 - q^{x_j-x_k}} \right) (T_j - 1) \right. \\
+ \left. w_-(x_j) \left( \prod_{1 \leq k \leq n, k \neq j} \frac{t - q^{x_j-x_k}}{1 - q^{x_j-x_k}} \right) (T_j^{-1} - 1) \right),
\]

(2.1)

where

\[
w_+(x) := \sqrt{\frac{qt_0 t_3}{t_1 t_2}} (1 - t_1 q^x)(1 - t_2 q^x), \quad w_-(x) := \sqrt{\frac{t_1 t_2}{qt_0 t_3}} (1 - t_0 q^x)(1 - t_3 q^x),
\]

and \( T_j (j = 1, \ldots, n) \) acts on functions \( f : \mathbb{R}^n \to \mathbb{C} \) by a unit translation of the \( j \)th position variable

\[
(T_j f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n).
\]

Here \( q \) denotes a real-valued scale parameter, \( t \) plays the role of the coupling parameter for the Ruijsenaars-Schneider inter-particle interaction, and the parameters \( t_r (r = 0, \ldots, 3) \) are coupling parameters governing the exponential Morse interaction. Upon setting \( t_0 = \epsilon t^{n-1} q^{-1} \) and \( t_r = \epsilon \) for \( r = 1, 2, 3 \), one has that \( w_\pm(x_j) \to \epsilon^{(n-1)/2} \) when \( \epsilon \to 0 \). We thus recover in this limit the Hamiltonian of the hyperbolic quantum Ruijsenaars-Schneider system given in terms of Ruijsenaars-Macdonald difference operators [R1, M1]. By a translation of the center-of-mass of the form \( q^{x_j} \to cq^{x_j} (j = 1, \ldots, n) \) for some suitable constant \( c \), it is possible to normalize one of the \( t_r \)-parameters to unit value; from now on it will therefore always be assumed that \( t_3 \equiv 1 \) unless explicitly stated otherwise.

2.2. Restriction to lattice functions. Let \( \rho + \Lambda := \{ \rho + \lambda \mid \lambda \in \Lambda \} \), where \( \Lambda \) denotes the cone of integer partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with weakly decreasingly ordered parts \( \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \), and \( \rho = (\rho_1, \ldots, \rho_n) \) with

\[
\rho_j = (n - j) \log(t) \quad (j = 1, \ldots, n).
\]

(2.2)
The action of $H$ \((2.1)\) (with $t_3 = 1$) preserves the space of lattice functions $f : \rho + \Lambda \to \mathbb{C}$:

$$
(Hf)(\rho + \lambda) = \sum_{1 \leq j \leq n} v^+_j(\lambda) (f(\rho + \lambda + e_j) - f(\rho + \lambda)) + \sum_{1 \leq j \leq n} v^-_j(\lambda) (f(\rho + \lambda - e_j) - f(\rho + \lambda)),
$$

where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$ and

$$
v^+_j(\lambda) = \sqrt{\frac{qt_0}{t_1t_2}} (1 - t_1t^{- n-j} q^\lambda_j)(1 - t_2t^{- n-j} q^\lambda_j) \prod_{\substack{1 \leq k \leq n \\lambda \neq k \neq j}} \frac{t^{-1} - t^{-k-j} q^\lambda_j - \lambda_k}{1 - t^{-k-j} q^\lambda_j - \lambda_k},
$$

$$
v^-_j(\lambda) = \sqrt{\frac{t_1t_2}{qt_0}} (1 - t_0t^{- n-j} q^\lambda_j)(1 - t^n t^{- n-j} q^\lambda_j) \prod_{\substack{1 \leq k \leq n \\lambda \neq k \neq j}} \frac{t - t^{-k-j} q^\lambda_j - \lambda_k}{1 - t^{-k-j} q^\lambda_j - \lambda_k}.
$$

Indeed, given $\lambda \in \Lambda$, one has that $v^+_j(\lambda) = 0$ if $\lambda + e_j \not\in \Lambda$ due to a zero stemming from the factor $t^{-1} - t^{-1} q^\lambda_{j-1} - \lambda_j$ when $\lambda_{j-1} = \lambda_j$ and one has that $v^-_j(\lambda) = 0$ if $\lambda - e_j \not\in \Lambda$ due to a zero stemming from either the factor $t - t q^\lambda_j - \lambda_{j+1}$ when $\lambda_j = \lambda_{j+1}$ or from the factor $(1 - q^\lambda_n)$ when $\lambda_n = 0$.

### 3. Spectrum and eigenfunctions

Ruijsenaars’ starting point in \([R4]\) is the fact that the hyperbolic quantum Ruijsenaars-Schneider system on the lattice is diagonalized by the celebrated Macdonald polynomials \([M1]\) Ch.VI. In this section we show that in the presence of the Morse interaction the role of the Macdonald eigenpolynomials is taken over by multivariate continuous dual $q$-Hahn eigenpolynomials that arise as a parameter reduction of the Macdonald-Koornwinder polynomials \([K][M2]\).

#### 3.1. Multivariate continuous dual $q$-Hahn polynomials

Continuous dual $q$-Hahn polynomials are a special limiting case of the Askey-Wilson polynomials in which one of the four Askey-Wilson parameters is set to vanish \([KLS]\) Ch. 14.3]. The corresponding reduction of the Macdonald-Koornwinder multivariate Askey-Wilson polynomials \([K][M2]\) is governed by a weight function of the form

$$
\tilde{\Delta}(\xi) := \frac{1}{(2\pi)^n} \prod_{1 \leq j \leq n} \left| \frac{e^{2\imath t_0 e^{\imath \xi_j}}}{\prod_{0 \leq r < 2(\imath t, e^{\imath \xi_j})}} \right|^2 \prod_{1 \leq j < k \leq n} \left| \frac{e^{\imath (\xi_j + \xi_k)}, e^{\imath (\xi_j - \xi_k)}}{\left( \xi_j e^{\imath (\xi_j + \xi_k)}, e^{\imath (\xi_j - \xi_k)} \right)} \right|^2
$$

supported on the alcove

$$
\Lambda := \{(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \mid \pi > \xi_1 > \xi_2 > \cdots > \xi_n > 0\},
$$

where $(x)_m := \prod_{r=0}^{m-1}(1 - xq^r)$ and $(x_1, \ldots, x_l)_m := (x_1)_m \cdots (x_l)_m$ refer to the $q$-Pochhammer symbols, and it is assumed that

$$
q, t \in (0, 1) \quad \text{and} \quad \hat{t}_r \in (-1, 1) \setminus \{0\} \quad (r = 0, 1, 2).
$$
More specifically, the multivariate continuous dual $q$-Hahn polynomials $P_{\lambda}(\xi),$ $\lambda \in \Lambda$ are defined as the trigonometric polynomials of the form

$$P_{\lambda}(\xi) = \sum_{\mu \leq \lambda} c_{\lambda,\mu} m_{\mu}(\xi) \quad (c_{\lambda,\mu} \in \mathbb{C})$$ (3.4a)

such that

$$c_{\lambda,\lambda} = \prod_{1 \leq j \leq n} \frac{t^{j\lambda_j} t^{(n-j)\lambda_j}}{(t_0 t_1 t^{n-j}, t_0 t_2 t^{n-j})_{\lambda_j}} \prod_{1 \leq j < k \leq n} \frac{(t^{k-j})_{\lambda_j - \lambda_k}}{(t^{1+k-j})_{\lambda_j - \lambda_k}}$$ (3.4b)

and

$$\int_{\mathbb{R}} P_{\lambda}(\xi) \overline{P_{\mu}(\xi)} \Delta(\xi) d\xi = 0 \text{ if } \mu < \lambda.$$ (3.4c)

Here we have employed the dominance partial order

$$\forall \mu, \lambda \in \Lambda : \quad \mu \leq \lambda \text{ iff } \sum_{1 \leq j < k} \mu_j \leq \sum_{1 \leq j < k} \lambda_j \quad \text{for} \quad k = 1, \ldots, n,$$ (3.5)

and the symmetric monomials

$$m_{\lambda}(\xi) := \sum_{\nu \in W_{\lambda}} e^{i(\nu_1 \xi_1 + \cdots + \nu_n \xi_n)}, \quad \lambda \in \Lambda,$$ (3.6)

associated with the hyperoctahedral group $W = S_n \ltimes \{1,-1\}^n$ of signed permutations.

The present choice of the leading coefficient $c_{\lambda,\lambda}$ in Eq. (3.4b) normalizes the polynomials in question such that $P_{\lambda}(i\hat{\rho}) = 1,$ where $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_n)$ is given by $\hat{\rho}_j = (n-j) \log(t) + \log(t_0),$ $j = 1, \ldots, n$ (cf. [D3 Sec. 6], [M2 Ch. 5.3]). With this normalization, the orthogonality relations obtained as the degeneration of those for the Macdonald-Koornwinder polynomials [K Sec. 5], [D3 Sec. 7], [M2 Ch. 5.3] read:

$$\int_{\mathbb{R}} P_{\lambda}(\xi) \overline{P_{\mu}(\xi)} \Delta(\xi) d\xi = \begin{cases} \Delta_{\lambda}^{-1} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}, \end{cases}$$ (3.7a)

where

$$\Delta_{\lambda} := \Delta_0 \prod_{1 \leq j \leq n} \frac{(t_0 t_1 t^{n-j}, t_0 t_2 t^{n-j})_{\lambda_j}}{(t_0 t_1 t^{n-j}, t_0 t_2 t^{n-j})_{\lambda_j}} \left( 1 - \frac{t^{k-j} q^\lambda_j - \lambda_k}{1 - t^{k-j}} \right)_{\lambda_j - \lambda_k}$$ (3.7b)

and

$$\Delta_0 := \prod_{1 \leq j \leq n} \left( \frac{(q,t)}{(t)_\infty} \right) \prod_{0 \leq r<s \leq 2} \left( \hat{t}_r \hat{t}_s t^{n-j} \right).$$ (3.7c)

### 3.2. Diagonalization

Let $l^2(\rho + \Lambda, \Delta)$ denote the Hilbert space of lattice functions $f : \rho + \Lambda \to \mathbb{C}$ determined by the inner product

$$\langle f, g \rangle_{\Delta} := \sum_{\lambda \in \Lambda} f(\rho + \lambda) \overline{g(\rho + \lambda)} \Delta_{\lambda} \quad (f, g \in l^2(\rho + \Lambda, \Delta)).$$ (3.8)
with ρ and Δλ as in Eqs. (2.2) and (3.7a)–(3.7c), and let \( L^2(\mathbb{A}, \Delta(\xi)d\xi) \) be the Hilbert space of functions \( \hat{f} : \mathbb{A} \to \mathbb{C} \) determined by the inner product
\[
\langle \hat{f}, \hat{g} \rangle_{\Delta} := \int_{\mathbb{A}} \hat{f}(\xi) \overline{\hat{g}(\xi)} \Delta(\xi) d\xi \quad (\hat{f}, \hat{g} \in L^2(\mathbb{A}, \Delta(\xi)d\xi)),
\]
with \( \Delta \) taken from Eq. (3.1). We denote by \( \psi_{\xi} : \rho + \Lambda \to \mathbb{C} \) the lattice wave function given by
\[
\psi_{\xi}(\rho + \lambda) := P_{\lambda}(\xi) \quad (\xi \in \mathbb{A}, \lambda \in \Lambda).
\]
Then the orthogonality relations in Eqs. (3.7a)–(3.7c) imply that the associated function given by
\[
\rho \text{ with } (3.10)
\]
constitutes a Hilbert space isomorphism with an inversion formula given by
\[
\langle F^{-1} \hat{f}(\rho + \lambda) \rangle_{\Delta} = \int_{\mathbb{A}} \hat{f}(\xi) \overline{\psi_{\xi}(\rho + \lambda)} \Delta(\xi) d\xi \quad (f \in L^2(\rho + \Lambda, \Delta d\xi)).
\]

**Theorem 1.** Let \( \hat{E} \) denote the bounded real multiplication operator acting on \( \hat{f} \in L^2(\mathbb{A}, \Delta d\xi) \) by \( \hat{E} \hat{f}(\xi) := \hat{E}(\xi) \hat{f}(\xi) \) with
\[
\hat{E}(\xi) := \sum_{1 \leq j \leq n} \left(2 \cos(\xi_j) - t^{n-j} \hat{t}_0 - t^{j-n} \hat{t}_1\right).
\]

For
\[
t_0 = q^{-1} \hat{t}_1 \hat{t}_2, \quad t_1 = \hat{t}_0 \hat{t}_2, \quad t_2 = \hat{t}_0 \hat{t}_1
\]
with \( q, t \) and \( \hat{t}_r \) in the parameter domain (6.3), the hyperbolic lattice Ruijsenaars-Schneider Hamiltonian with Morse interaction \( H \) (2.2) constitutes a bounded self-adjoint operator in the Hilbert space \( L^2(\rho + \Lambda, \Delta) \) diagonalized by the Fourier transform \( F \) (3.11a), (3.11b):
\[
H = F^{-1} \circ \hat{E} \circ F.
\]

**Proof.** It suffices to verify that the Fourier kernel \( \psi_{\xi} (3.10) \) satisfies the eigenvalue equation \( H \psi_{\xi} = \hat{E}(\xi) \psi_{\xi} \), or more explicitly that:
\[
\sum_{1 \leq j \leq n} \frac{\psi_j^{+} (\lambda)}{\lambda + e_j - \xi} (\psi_{\xi}(\rho + \lambda + e_j) - \psi_{\xi}(\rho + \lambda)) + \sum_{1 \leq j \leq n} \frac{\psi_j^{-} (\lambda)}{\lambda - e_j - \xi} (\psi_{\xi}(\rho + \lambda - e_j) - \psi_{\xi}(\rho + \lambda)) = \hat{E}(\xi) \psi_{\xi}(\rho + \lambda).
\]
This eigenvalue equation amounts to the continuous dual \( q \)-Hahn reduction of the Pieri recurrence formula for the Macdonald-Koornwinder polynomials corresponding to Eqs. (6.4), (6.5) and Section 6.1 of [102].

It is immediate from Theorem 1 that the hyperbolic lattice Ruijsenaars-Schneider Hamiltonian with Morse interaction \( H \) (2.2) has purely absolutely continuous spectrum in \( L^2(\rho + \Lambda, \Delta) \), with the wave functions \( \psi_{\xi} \), \( \xi \in \mathbb{A} \) in Eq. (3.11a), constituting an orthogonal basis of (generalized) eigenfunctions.
Remark 2. For $\hat{t}_2 \to 0$ the lattice Hamiltonian $H$ (3.12a) becomes of the form

$$H = \sum_{j=1}^{n} \left( \hat{t}_0^{-1}(1 - \hat{t}_0 \hat{t}_1 q^x_j)(\prod_{1 \leq k \leq n \atop k \neq j} \frac{t - q^{x_j - x_k}}{1 - q^{x_j - x_k}}) T_j \right)$$  

(3.13a)

$$+ \hat{t}_0(1 - q^{x_j})(\prod_{1 \leq k \leq n \atop k \neq j} \frac{t - q^{x_j - x_k}}{1 - q^{x_j - x_k}}) T_j^{-1} + (\hat{t}_0 + \hat{t}_1)q^x_j) - \varepsilon_0,$$

with $x = \rho + \lambda$ and

$$\varepsilon_0 := \sum_{j=1}^{n} (\hat{t}_0 t^{n-j} + \hat{t}_0^{-1} t^{n-j}).$$  

(3.13b)

Indeed, this readily follows from Eqs. (2.1), (3.12b) with the aid of the elementary polynomial identity (cf. Example 2. (a) of [M1, Ch. VI.3])

$$\sum_{j=1}^{n} (1 + z_j) \prod_{1 \leq k \leq n \atop k \neq j} \frac{t - z_j/z_k}{1 - z_j/z_k} = \sum_{j=1}^{n} (z_j + t^{n-j}).$$

4. SCATTERING

In this section we rely on results from [D4], permitting to describe briefly how the $n$-particle scattering operator for the hyperbolic quantum Ruijsenaars-Schneider system on the lattice computed by Ruijsenaars [R4] gets modified due to the presence of the external Morse interactions. Specifically, the scattering process of the present model with Morse terms turns out to be governed by an $n$-particle scattering matrix $S(\xi)$ that factorizes in two-particle and one-particle matrices:

$$S(\xi) := \prod_{1 \leq \ell < k \leq n} s(\xi_j - \xi_k) s(\xi_j + \xi_k) \prod_{1 \leq j \leq n} s_0(\xi_j),$$

(4.1a)

with

$$s(x) := \frac{(q^x e^{ix}, e^{-ix})_\infty}{(q^{eix}, e^{ix})_\infty} \quad \text{and} \quad s_0(x) := \frac{(q^{2ix} e^{-ix}, e^{ix})_\infty}{(q^{eix} e^{-ix}, e^{ix})_\infty},$$

(4.1b)

which compares to Ruijsenaars’ scattering matrix $\prod_{1 \leq j < k \leq n} s(\xi_j - \xi_k)$ for the corresponding model without Morse interactions [D4].

To substantiate further some additional notation is needed. Let us denote by $\mathcal{H}_0$ the self-adjoint discrete Laplacian in $\ell^2(\Lambda)$ of the form

$$(\mathcal{H}_0 f)(\lambda) := \sum_{1 \leq j \leq n \atop \lambda + e_j \in \Lambda} f(\lambda + e_j) + \sum_{1 \leq j \leq n \atop \lambda - e_j \in \Lambda} f(\lambda - e_j) \quad (f \in \ell^2(\Lambda)),$$

and let

$${\Delta} := \Delta^{1/2}(H + \varepsilon_0) \Delta^{-1/2},$$

(4.2)

with $H$ and $\varepsilon_0$ taken from (2.3) and (3.13b), respectively. Here the operator $\Delta^{1/2} : \ell^2(\rho + \Lambda, \Delta) \to \ell^2(\Lambda)$ refers to the Hilbert space isomorphism

$$(\Delta^{1/2} f)(\lambda) := \Delta^{1/2} f(\rho + \lambda) \quad (f \in \ell^2(\rho + \Lambda, \Delta))$$

(4.3)
(with $\Delta^{-1/2} := (\Delta^{1/2})^{-1}$). Then (by Theorem 1)

$$\mathcal{H} = \mathcal{F}^{-1}(\hat{E} + \epsilon_0)\mathcal{F},$$

with $\mathcal{F} := \hat{\Delta}^{1/2}F\Delta^{-1/2}$, (4.4)

where $\hat{\Delta}^{1/2} : L^2(\mathbb{A}, \Delta d\xi) \to L^2(\mathbb{A})$ denotes the Hilbert space isomorphism

$$\left(\hat{\Delta}^{1/2}\hat{f}\right)(\xi) := \hat{\Delta}^{1/2}(\xi)\hat{f}(\xi) \quad (\hat{f} \in L^2(\mathbb{A}, \Delta d\xi))$$

(4.5)

(and $\hat{E}$ is now regarded as a self-adjoint bounded multiplication operator in $L^2(\mathbb{A})$). Furthermore, one has that

$$\mathcal{H}_0 = \mathcal{F}_0^{-1}(\hat{E} + \epsilon_0)\mathcal{F}_0,$$

where $\mathcal{F}_0 : \ell^2(\Lambda) \to L^2(\mathbb{A})$ denotes the Fourier isomorphism recovered from $\mathcal{F}$ in the limit $q, t \to 0$, $\hat{t}_r \to 0$ ($r = 0, 1, 2$). Specifically, this amounts to the Fourier transform

$$\left(\mathcal{F}_0\hat{f}\right)(\xi) = \sum_{\lambda \in \Lambda} f(\lambda)\chi_{\lambda}(\lambda)$$

(6.6a)

with the inversion formula

$$\left(\mathcal{F}_0^{-1}\hat{f}\right)(\lambda) = \int_{\hat{\Lambda}} \hat{f}(\xi)\chi_{\lambda}(\lambda) d\xi$$

(6.6b)

($\hat{f} \in L^2(\mathbb{A})$) associated with the anti-invariant Fourier kernel

$$\chi_{\lambda}(\lambda) := \frac{1}{(2\pi)^{n/2} \pi^n} \sum_{w \in W} \text{sign}(w)e^{i\left(w(\rho_0 + \lambda), \xi\right)},$$

where $\text{sign}(w) = \epsilon_1 \cdots \epsilon_n \text{sign}(\sigma)$ for $w = (\sigma, \epsilon) \in W = S_n \times \{1, -1\}^n$ and $\rho_0 = (n, n - 1, \ldots, 2, 1)$.

Let $C_0(\mathbb{A}_{\text{reg}})$ be the dense subspace of $L^2(\mathbb{A})$ consisting of smooth test functions with compact support in the open dense subset $\mathbb{A}_{\text{reg}} \subset \mathbb{A}$ on which the components of the gradient

$$\nabla\hat{E}(\xi) = (-2\sin(\xi_1), \ldots, -2\sin(\xi_n)),$$

$\xi \in \mathbb{A}$

do not vanish and are all distinct in absolute value. We define the following unitary multiplication operator $\hat{S} : L^2(\mathbb{A}, d\xi) \to L^2(\mathbb{A}, d\xi)$ via its restriction to $C_0(\mathbb{A}_{\text{reg}})$:

$$\left(\hat{S}\hat{f}\right)(\xi) := \hat{S}(w_\xi)\hat{f}(\xi) \quad (\hat{f} \in C_0(\mathbb{A}_{\text{reg}}),$$

(4.7)

where $w_\xi \in W$ for $\xi \in \mathbb{A}_{\text{reg}}$ is the signed permutation such that the components of $w_\xi \nabla\hat{E}(\xi)$ are all positive and reordered from large to small.

Theorem 4.2 and Corollary 4.3 of Ref. [D4] now provide explicit formulas for the wave operators and scattering operator comparing the large-times asymptotics of the interacting particle dynamics $e^{it\mathcal{H}}$ relative to the Laplacian’s reference dynamics $e^{i\mathcal{H}_0}$ as a continuous dual $q$-Hahn reduction of [D4] Thm. 6.7.

**Theorem 3 (Wave and Scattering Operators).** The operator limits

$$\Omega^\pm := s - \lim_{t \to \pm \infty} e^{it\mathcal{H}} e^{-it\mathcal{H}_0}$$

converge in the strong $\ell^2(\Lambda)$-norm topology and the corresponding wave operators $\Omega^\pm$ intertwining the interacting dynamics $e^{it\mathcal{H}}$ with the discrete Laplacian’s dynamics $e^{i\mathcal{H}_0}$ are given by unitary operators in $\ell^2(\Lambda)$ of the form

$$\Omega^\pm = \mathcal{F}^{-1} \circ \hat{S}^{\mp 1/2} \circ \mathcal{F}_0,$$
where the branches of the square roots are to be chosen such that
\[
s(x)^{1/2} = \frac{(qe^{ix})_\infty}{(qe^{ix})_\infty} \left(\frac{te^{ix}}{te^{ix}}\right)_\infty \quad \text{and} \quad s_0(x)^{1/2} = \frac{(qe^{-2ix})_\infty}{(qe^{-2ix})_\infty} \prod_{0 \leq r \leq 2} \left(\frac{\dot{t}_r e^{ix}}{t_r e^{ix}}\right)_\infty.
\]
The scattering operator relating the large-times asymptotics of \(e^{i\hat{H}_\mathbb{R}}\) for \(t \to -\infty\) and \(t \to +\infty\) is thus given by the unitary operator
\[
S := (\Omega^+)^{-1} \Omega^- = \mathcal{F}_0^{-1} \circ \hat{S} \circ \mathcal{F}_0.
\]

5. Bispectral dual system

The bispectral dual in the sense of Duistermaat and Grünbaum \cite{DG, G} of the hyperbolic quantum Ruijsenaars-Schneider system on the lattice is given by the trigonometric Ruijsenaars-Macdonald \(q\)-difference operators \cite{R1, M1}. This bispectral duality is a quantum manifestation of the duality between the classical Ruijsenaars-Schneider systems with hyperbolic/trigonometric dependence on the position/momentum variables and vice versa \cite{R3}, which (at the classical level) states that the respective action-angle transforms linearizing the two systems under consideration are inverses of each other. As a degeneration of the Macdonald-Koornwinder \(q\)-difference operator \cite{K} Eq. (5.4)], we immediately arrive at a bispectral dual Hamiltonian for our hyperbolic quantum Ruijsenaars-Schneider system with Morse term.

Indeed, the continuous dual \(q\)-Hahn reduction of the \(q\)-difference equation satisfied by the Macdonald-Koornwinder polynomials \cite{K} Thm. 5.4] reads
\[
\hat{H} P_\lambda = E_\lambda P_\lambda \quad \text{with} \quad E_\lambda = \sum_{j=1}^n \tilde{v}^{-1}(q^{-\lambda_j} - 1) \quad (\lambda \in \Lambda), \tag{5.1a}
\]
where
\[
\hat{H} = \sum_{j=1}^n \left(\hat{v}_j(\xi)\hat{T}_{j,q} - 1 + \hat{v}_j(-\xi)\hat{T}_{j,q}^{-1} - 1\right), \tag{5.1b}
\]
and
\[
\hat{v}_j(\xi) = \frac{\prod_{0 \leq r \leq 2} (1 - \dot{t}_r e^{i\xi_j})}{(1 - e^{2i\xi_j})(1 - qe^{2i\xi_j})} \prod_{1 \leq k \leq n, k \neq j} \frac{1 - te^{i(\xi_j + \xi_k)} - 1 - te^{i(\xi_j - \xi_k)}}{1 - e^{i(\xi_j + \xi_k)} - 1 - e^{i(\xi_j - \xi_k)}}. \tag{5.1c}
\]
Here \(\hat{T}_{j,q}\) acts on trigonometric (Laurent) polynomials \(\hat{p}(e^{i\xi_1}, \ldots, e^{i\xi_n})\) by a \(q\)-shift of the \(j\)th variable:
\[
(\hat{T}_{j,q}\hat{p})(e^{i\xi_1}, \ldots, e^{i\xi_n}) := \hat{p}(e^{i\xi_1}, \ldots, e^{i\xi_{j-1}}, qe^{i\xi_j}, e^{i\xi_{j+1}}, \ldots, e^{i\xi_n}).
\]
In other words, the bispectral dual Hamiltonian \(\hat{H}\) \cite{5.1b, 5.1c} constitutes a nonnegative unbounded self-adjoint operator with purely discrete spectrum in \(L^2(\hat{\Omega}, d\xi)\) that is diagonalized by the (inverse) Fourier transform \(F\) \cite{5.1a}, \cite{3.11b}:
\[
\hat{H} = F \circ E \circ F^{-1}, \tag{5.2}
\]
where \(E\) denotes the self-adjoint multiplication operator in \(\ell^2(\rho + \Lambda, \Delta)\) of the form \((Ef)(\rho + \lambda) := E_\lambda f(\rho + \lambda)\) (for \(\lambda \in \Lambda\) and \(f \in \ell^2(\rho + \Lambda, \Delta)\) with \(\langle Ef, Ef \rangle_\Delta < \infty\).
6. Quantum integrability

In this final section we provide explicit formulas for a complete system of commuting quantum integrals for the hyperbolic quantum Ruijsenaars-Schneider Hamiltonian with Morse term on the lattice $H$ (2.3) and for its bispectral dual Hamiltonian $H$ (5.1e), (5.1f). This confirms the quantum integrability of both Hamiltonians in the present Hilbert space set-up.

6.1. Hamiltonian. The quantum integrals for the hyperbolic Ruijsenaars-Schneider Hamiltonian with Morse term are given by commuting difference operators $H_1, \ldots, H_n$ that are defined via their action on $f \in \ell^2(\rho + \Lambda, \Delta)$ (cf. [12] Eqs. (2.20)–(2.23)):

$$(H_t f)(\rho + \lambda) := \sum_{J_+, J_- \subset \{1, \ldots, n\}} U_{J_+ \cap J_- - |J_+| - |J_-|} (\lambda) V_{J_+, J_-}(\lambda) f(\rho + \lambda + e_{J_+} - e_{J_-})$$

$$(\lambda \in \Lambda, l = 1, \ldots, n),$$

where $e_J := \sum_{j \in J} e_j$ for $J \subset \{1, \ldots, n\}$, $J^c := \{1, \ldots, n\} \setminus J$ and

$$V_{J_+, J_-}(\lambda) = t^{-\frac{1}{2}|J_+|(|J_+|-1)+\frac{1}{2}|J_-|(|J_-|-1)} \times \prod_{j \in J_+} \sqrt{\frac{qt_0}{t_1 t_2}} (1 - t_1 t^{n-j} q^{\lambda_j})(1 - t_2 t^{n-j} q^{\lambda_j})$$

$$\times \prod_{j \in J_-} \sqrt{\frac{t_1 t_2}{qt_0}} (1 - t_0 t^{n-j} q^{\lambda_j})(1 - t^{n-j} q^{\lambda_j})$$

$$\times \prod_{j \in J_+, k \in J_-} \left(1 - t^{1+k-j} q^{\lambda_j} - \lambda_k \right) \left(1 - t^{k-j} q^{\lambda_j} - \lambda_k + 1 \right) \prod_{j \in J_-, k \in J_+} \left(1 - t^{k-j} q^{\lambda_j} - \lambda_k \right) \prod_{j \in J_+} \left(1 - t^{k-j} q^{\lambda_j} - \lambda_k \right),$$

$$U_{K,\rho}(\lambda) = (-1)^p \times \sum_{I_+ \cup I_- \subset K \atop I_+ \cap I_- = \emptyset, |I_+| + |I_-| = p} \left( \prod_{j \in I_+} \sqrt{\frac{qt_0}{t_1 t_2}} (1 - t_1 t^{n-j} q^{\lambda_j})(1 - t_2 t^{n-j} q^{\lambda_j}) \right)$$

$$\times \prod_{j \in I_-} \sqrt{\frac{t_1 t_2}{qt_0}} (1 - t_0 t^{n-j} q^{\lambda_j})(1 - t^{n-j} q^{\lambda_j})$$

$$\times \prod_{j \in I_+, k \in I_-} \left(1 - t^{1+k-j} q^{\lambda_j} - \lambda_k \right) \left(1 - t^{k-j} q^{\lambda_j} - \lambda_k + 1 \right) \prod_{j \in I_-, k \in I_+} \left(1 - t^{k-j} q^{\lambda_j} - \lambda_k \right) \prod_{j \in I_+} \left(1 - t^{k-j} q^{\lambda_j} - \lambda_k \right).$$
For \( l = 1 \) the action of \( H_l \) (6.1) is seen to reduce to that of \( H \) (2.3). The diagonalization in Theorem 4 generalizes to these higher commuting quantum integrals as follows.

**Theorem 4.** For parameters of the form as in Theorem 4 the difference operators \( H_1, \ldots, H_n \) (6.1) constitute bounded commuting self-adjoint operators in the Hilbert space \( \ell^2(\rho + \Lambda, \Delta) \) that are simultaneously diagonalized by the Fourier transform \( F \) (3.11a), (3.11b):

\[
H_l = F^{-1} \circ \hat{E}_l \circ F \quad (l = 1, \ldots, n),
\]

where \( \hat{E}_l \) denotes the bounded real multiplication operator acting on \( \hat{f} \in L^2(\Lambda, \Delta d\xi) \) by \( (\hat{E}_l \hat{f})(\xi) := \hat{E}_l(\xi) \hat{f}(\xi) \) with

\[
\hat{E}_l(\xi) := \sum_{1 \leq j_1 < \cdots < j_l \leq n} (2 \cos(\xi_{j_1}) - t^{j_1-1}i_0 - t^{-(j_1-1)}i_0^{-1}) \cdots (2 \cos(\xi_{j_l}) - t^{j_l-1}i_0 - t^{-(j_l-1)}i_0^{-1}).
\]

**Proof.** The eigenvalue equation \( H_l \psi_\xi = \hat{E}_l(\xi) \psi_\xi \) reads explicitly

\[
\sum_{J+J' \subseteq \{1, \ldots, n\}} U_{J_+, J_-; J_{+}-J_-} \psi_\xi(\rho + \lambda + e_{J_+} - e_{J_-})
= \hat{E}_l(\xi) \psi_\xi(\rho + \lambda).
\]

This eigenvalue identity corresponds to the continuous dual q-Hahn reduction of the Pieri recurrence formula for the Macdonald-Koornwinder polynomials in [D3 Thm. 6.1], where we have expressed the eigenvalues \( \hat{E}_l(\xi) \) in a compact form stemming from [KNS Eq. (5.1)] (cf. also [DE Sec. 2.2]). \( \square \)

6.2. **Bispectral dual Hamiltonian.** The continuous dual q-Hahn reduction of the system of higher q-difference equations for the Macdonald-Koornwinder polynomials in [D3 Sec. 5.1] reads

\[
\hat{H}_l P_\lambda = E_{\lambda,l} P_\lambda \quad (\lambda \in \Lambda, l = 1, \ldots, n),
\]

where

\[
E_{\lambda,l} := t^{-l(l-1)/2} \sum_{1 \leq j_1 < \cdots < j_l \leq n} (t^{j_1-1}q^{\lambda_{j_1}} - t^{n-j_1}) \cdots (t^{j_l-1}q^{\lambda_{j_l}} - t^{n+1-l-j_l})
\]

(cf. [KNS Eq. (5.1)]), and

\[
\hat{H}_l := \sum_{J \subseteq \{1, \ldots, n\}}, 0 \leq |J| \leq l e_j \in \{1, \ldots, n\}, j \in J \hat{V}_{e_j} \hat{T}_{e_j, \rho},
\]

with \( \hat{T}_{e_j, \rho} := \prod_{j \in J} \hat{T}_{e_j}^{e_j} \) and

\[
\hat{V}_{e_j} = \prod_{j \in J} \frac{1}{1 - te^{(e_j \xi_j + \xi_k)}} \prod_{j \in J, k \in J} \frac{1 - te^{(e_j \xi_j + \xi_k)}}{1 - e^{(e_j \xi_j + \xi_k)}} \prod_{j \in J, k \in J} \frac{1 - te^{(e_j \xi_j + e_j \xi_k)}}{1 - e^{(e_j \xi_j + e_j \xi_k)}} \times \prod_{j, k \in J, j < k} \frac{1 - te^{(e_j \xi_j + e_j \xi_k)}}{1 - e^{(e_j \xi_j + e_j \xi_k)}}.
\]
\[ \hat{U}_{K,p} = (-1)^p \sum_{I \subset K, |I| = p} \prod_{l \in I} \prod_{j \in I} \frac{1 - te^{i(\epsilon_j + \xi_k)}}{1 - e^{i(\epsilon_j + \xi_k)}} \left( (1 - te^{i\epsilon_l \xi_j}) (1 - qe^{i\epsilon_l \xi_j}) \right) \]

For \( l = 1 \), this reproduces the continuous dual \( q \)-Hahn reduction of the Macdonald-Koornwinder \( q \)-difference equation in Eqs. (5.1a–5.1c).

The \( q \)-difference operators \( \hat{H}_1, \ldots, \hat{H}_n \) extend the bispectral dual Hamiltonian \( \hat{H} \) (5.1a–5.1c) into a complete system of commuting quantum integrals that are simultaneously diagonalized by the multivariate continuous dual \( q \)-Hahn polynomials.

**Theorem 5.** For parameter values in the domain (3.3), the \( q \)-difference operators \( \hat{H}_1, \ldots, \hat{H}_n \) constitute nonnegative unbounded self-adjoint operators with purely discrete spectra in \( L^2(\Delta, \Delta d\xi) \) that are simultaneously diagonalized by the (inverse) Fourier transform \( F \) (3.11a), (3.11b):

\[ \hat{H}_l = F \circ E_l \circ F^{-1}, \quad l = 1, \ldots, n, \quad (6.4) \]

where \( E_l \) denotes the self-adjoint multiplication operator in \( \ell^2(\rho + \Lambda, \Delta) \) given by \( (E_l f) (\rho + \lambda) := E_{\lambda,l} f (\rho + \lambda) \) (on the domain of \( f \) in \( \ell^2(\rho + \Lambda, \Delta) \)) such that \( \langle E_l f, E_l f \rangle_\Delta < \infty \).

Notice in this connection that although the domain of the unbounded operator \( \hat{H}_l \) in \( L^2(\Delta, \Delta d\xi) \) depends on \( l \), the resolvent operators \( (\hat{H}_1 - z_1)^{-1}, \ldots, (\hat{H}_n - z_n)^{-1} \) (with \( z_1, \ldots, z_n \in \mathbb{C} \setminus [0, +\infty) \)) commute as bounded operators on \( L^2(\Delta, \Delta d\xi) \), and the \( q \)-difference operators \( \hat{H}_1, \ldots, \hat{H}_n \) moreover commute themselves on the joint polynomial eigenbasis \( P_\lambda \), \( \lambda \in \Lambda \).

**Remark 6.** To infer that the eigenvalues \( E_{\lambda,l} \) (6.3b) are nonnegative—thus indeed giving rise to a nonnegative operator \( \hat{H}_l \) in Theorem 5—it is helpful to note that these can be rewritten as (cf. [DX Sec. 5.1])

\[ E_{\lambda,l} = t^{-l(l-1)/2} E_{l,n}(q^{-\lambda_1}, q^{-\lambda_2}, \ldots, q^{-\lambda_n}; t^{l-1}, t^l, \ldots, t^{n-1}) \]

with

\[ E_{l,n}(z_1, \ldots, z_n; y_1, \ldots, y_n) := \sum_{0 \leq k \leq l} (-1)^{l+k} e_k(z_1, \ldots, z_n) h_{l-k}(y_1, \ldots, y_n). \]

Here \( e_k(z_1, \ldots, z_n) \) and \( h_k(y_1, \ldots, y_n) \) refer to the elementary and the complete symmetric functions of degree \( k \) (cf. [M1 Ch. I.2]), with the convention that \( e_0 = h_0 = 1 \). The nonnegativity of the eigenvalues now readily follows inductively in the particle number \( n \) by means of the recurrence (cf. [D1 Lem. B.2])

\begin{align*}
E_{l,n}(q^{-\lambda_1}, & q^{-\lambda_2}, \ldots, q^{-\lambda_n}; t^{l-1}, t^l, \ldots, t^{n-1}) = \\
(q^{-\lambda_1} - t^{l-1}) E_{l-1,n-1}(q^{-\lambda_2}, & q^{-\lambda_3}, \ldots, q^{-\lambda_n}; t^{l-1}, \ldots, t^{n-1}) + E_{l,n-1}(q^{-\lambda_2}, \ldots, q^{-\lambda_n}; t^{l-1}, \ldots, t^{n-1})
\end{align*}
and the homogeneity
\[ E_{l,n}(t z_1, \ldots, t z_n; ty_1, \ldots, ty_n) = t^l E_{l,n}(z_1, \ldots, z_n; y_1, \ldots, y_n). \]

**Remark 7.** The hyperbolic Ruijsenaars-Schneider Hamiltonian with Morse term \( (2.1) \) can be retrieved as a limit of the Macdonald-Koornwinder \( q \)-difference operator \( [D2] \). In this limit the center-of-mass is sent to infinity, which causes the hyperoctahedral symmetry of the Macdonald-Koornwinder operator to be broken: while the permutation-symmetry still persists the parity-symmetry is no longer present. Indeed, the limit in question restores the translational-invariance of the interparticle pair-interactions enjoyed by the original Ruijsenaars-Schneider model and gives moreover rise to additional Morse terms that are not parity-invariant. It turns out that most of our results above can in fact be lifted to the Macdonald-Koornwinder level, even though such a generalization is presumably somewhat less relevant from a physical point of view. Specifically, the scattering of the corresponding quantum lattice model associated with the full six-parameter family of Macdonald-Koornwinder polynomials was briefly discussed in \([D1]\) Sec. 6.4], its commuting quantum integrals can be read-off from the Pieri formulas for the Macdonald-Koornwinder polynomials in \([D3]\) Thm. 6.1], and the pertinent bispectral dual Hamiltonian and its commuting quantum integrals are given by the Macdonald-Koornwinder \( q \)-difference operator \( [K] \) and its higher-order commuting \( q \)-difference operators \( [D3]\) Thm. 5.1].

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