A Study of the S-Generalized Gauss Hypergeometric Function and Its Associated Integral Transforms

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Abstract The aim of the present paper is to further investigate the S-generalized Gauss hypergeometric function which was recently introduced by Srivastava et al. [8]. In the course of our study, we first present an integral representation, the Mellin transform and a complex integral representation of the S-generalized Gauss hypergeometric function. Next, we introduce a new integral transform whose kernel is the S-generalized Gauss hypergeometric function and point out its three special cases which are also believed to be new. We specify that the well-known Gauss hypergeometric function transform follows as a simple special case of our integral transforms. Finally, we establish an inversion formula for the integral transform which we have introduced in this investigation.

Keywords: S-Generalized Gauss hypergeometric function, Integral representation, Complex integral representation, Mellin transform, Integral transform, Inversion formula

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1. Introduction and Definitions

The S-generalized Gauss hypergeometric function:

\[
F_p^{(a, b, \tau, \mu)}(a, b; c; z)\]

was introduced and investigated by Srivastava et al. [[8], p. 350, Eq. (1.12)]. It is represented in the following manner:

\[
f_p^{(a, b, \tau, \mu)}(a, b; c; z) = \sum_{n=0}^{\infty} \left( a_n \right) \frac{B_n^{(a, b, \tau, \mu)}(b+n, c-b) z^n}{n!} \quad (|z|<1)
\]

(1.1)

\[
\left\{ \begin{array}{ll}
\Re(p) \geq 0; \\
\min \{ \Re(\alpha), \Re(\beta), \Re(\tau) \} > 0; \\
\Re(c) > \Re(b) > 0
\end{array} \right.
\]

in terms of the classical Beta function \(B(\lambda, \mu)\) and the S-generalized Beta function \(B_p^{(a, b, \tau, \mu)}(x, y)\), which was also defined by Srivastava et al. [[8], p. 350, Eq. (1.13)] as follows:

\[
B_p^{(a, b, \tau, \mu)}(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} \left( \frac{p}{1-t^\tau} \right)^\mu dt
\]

(1.2)

\[
\left\{ \begin{array}{ll}
\Re(p) \geq 0; \\
\min \{ \Re(x), \Re(y), \Re(\alpha), \Re(\beta) \} > 0; \\
\min \{ \Re(\tau), \Re(\mu) \} > 0
\end{array} \right.
\]

and \((\lambda)_n\) denotes the Pochhammer symbol defined (for \(\lambda \in \mathbb{C}\)) by (see [11], p. 2 and pp. 4-6); see also [10], p. 2):

\[
(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}
\]

provided that the Gamma quotient exists (see, for details, [13], p. 16 et seq.) and [15], p. 22 et seq.).

For \(\tau = \mu\), the S-generalized Gauss hypergeometric function defined by (1.1) reduces to the following generalized Gauss hypergeometric function \(F_p^{(a, b, \tau)}(a, b; c; z)\) studied earlier by Parmar [7], p.44):

\[
F_p^{(a, b, \tau)}(a, b; c; z) = \sum_{n=0}^{\infty} \left( a_n \right) \frac{B_n^{(a, b, \tau)}(b+n, c-b) z^n}{n!} \quad (|z|<1)
\]

(1.4)

\[
\left\{ \begin{array}{ll}
\Re(p) \geq 0; \\
\min \{ \Re(\alpha), \Re(\beta), \Re(\tau) \} > 0; \\
\Re(c) > \Re(b) > 0
\end{array} \right.
\]

which, in the further special case when \(\tau = 1\), reduces to the following extension of the generalized Gauss hypergeometric function (see, e.g., [16], p. 4606, Section 3; see also [15], p. 39]).
Differentiation and the Mellin Transform of the S-Generalized Gauss Hypergeometric Function

Theorem 1. Suppose that
\[ \mathcal{R}(p) \geq 0; \arg(1-z) < \pi, \]
\[ \min \{ \mathcal{R}(\tau), \mathcal{R}(\mu), \mathcal{R}(b+\tau a), \mathcal{R}(c-b+\mu a) \} > 0, \]
and \( \mathcal{R}(c) > \mathcal{R}(b) > 0. \)

Then the following integral representation holds true:
\[
F^{(\alpha, \beta, \tau, \mu)}_p (a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 z^{-\frac{1}{\tau}} F_1 \left( \alpha; \beta; -\frac{p}{t^\tau (1-t)^\mu} \right) dt, \quad \left( |z| < 1, \frac{1}{\tau} > 1 \right) \quad (2.1)
\]
where the S-generalized Gauss hypergeometric function 
\[ F^{(\alpha, \beta, \tau, \mu)}_p (a, b; c; z) \]
is given by (1.1).

Proof. Using Eq. (1.1) on the left-hand side of (2.1), we find that
\[
F^{(\alpha, \beta, \tau, \mu)}_p (a, b; c; z) = \sum_{n=0}^\infty \frac{b_p (b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \mathcal{R}(p) \geq 0; \mathcal{R}(c) > \mathcal{R}(b) > 0.
\]

In the present paper, we propose to further investigate the S-generalized Gauss hypergeometric function defined by (1.1). We first derive an integral representation, the Mellin transform of (2.1), we obtain
\[
F^{(\alpha, \beta, \tau, \mu)}_p (a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 \frac{z^{-\frac{1}{\tau}} F_1 \left( \alpha; \beta; -\frac{p}{t^\tau (1-t)^\mu} \right) dt}{\tau^\tau (1-t)^\mu}.
\]

which proves Theorem 1.

2.2. The Mellin Transform of the S-Generalized Gauss Hypergeometric Function

As usual, the Mellin transform of a function \( f(t) \) is defined by (see, for example, [[2], p. 340, Eq. (8.2.5)])
\[
\mathcal{M} \left[ f(t) \right](s) = \int_0^\infty t^{s-1} f(t) dt \quad (\mathcal{R}(s) > 0). \quad (2.2)
\]
provided that the improper integral exists.

Theorem 2. If
\[
\mathcal{R}(p) \geq 0, \min \left\{ \mathcal{R}(\tau), \mathcal{R}(\mu), \mathcal{R}(b+\tau a), \mathcal{R}(c-b+\mu a) \right\} > 0,
\]
and \( \mathcal{R}(c) > \mathcal{R}(b) > 0. \)

then
\[
\mathcal{M} \left[ F^{(\alpha, \beta, \tau, \mu)}_p (a, b; c; t) \right](s) = (-1)^s \frac{B(s, a-s) B^{(\alpha, \beta, \tau, \mu)}_p (b-s, c-b)}{B(b, c-b)}. \quad (2.3)
\]

Proof. In order to prove the assertion (2.3), by taking the Mellin transform of (2.1), we obtain
\[
\Delta(s) := \left[ \int_0^\infty z^{s-1} \left( \frac{1}{B(b, c-b)} \int_0^1 \frac{z^{-\frac{1}{\tau}} F_1 \left( \alpha; \beta; -\frac{p}{t^\tau (1-t)^\mu} \right) dt}{\tau^\tau (1-t)^\mu} \right) dz \right].
\]
Upon interchanging the order of the $t$- and the $z$-integrals (which is permissible under the conditions stated), if we evaluate the resulting $z$-integral first, we get

$$
\Delta(s) = \frac{1}{B(b,c-b)} \int_0^1 \left[ t^{b-1} (1-t)^{c-b-1} \right] \left[ \frac{t^{1-	au} - \frac{1}{t} \sqrt{1-t}}{\Gamma(-\tau) \Gamma(a-s)} \right] dt.
$$

Now, with the help of (1.2), we get the desired result (2.3) after a little simplification.

### 2.3. A Complex Integral Representation of the S-Generalized Gauss Hypergeometric Function

If we take the inverse Mellin transform of (2.3), we easily arrive at the following complex integral transform defined by (2.5):  

$$
F_p^{(a,b;\tau,\mu)}(a,b;c;z): 
$$

$$
\phi(z) = \int_{0}^{\infty} F_p^{(a,b;\tau,\mu)}(a,b;c;z) f(z) dz,
$$

where $\Lambda$ denotes the class of functions for which

$$
f(z) = \begin{cases} 
O(z^c) & (z \to 0) \\
O(z^{-\infty}) & (|z| \to \infty),
\end{cases}
$$

provided that the existence conditions in (1.1) for the S-generalized Gauss hypergeometric function

$$
F_p^{(a,b;\tau,\mu)}(a,b;c;z)
$$

are satisfied and

$$
\Re(\zeta) > -1
$$

and

$$
\Re(w_2) > 0 \text{ or } \Re(w_2) = 0 \text{ and } \Re(w_1 - a + 1) < 0.
$$

### 2.4. The S-Generalized Gauss Hypergeometric Function Transform

We define the S-generalized Gauss hypergeometric transform by the following equation (see also a recent work [14] dealing with several new families of integral transforms):

$$
\mathfrak{S}[f(z);s] = \phi(s) := \int_{0}^{\infty} F_p^{(a,b;\tau,\mu)}(a,b;c;z) f(z) dz,
\tag{2.5}
$$

where $\mathfrak{S}$ denotes the class of functions for which

$$
\Re\left(\frac{F_p^{(a,b;\tau,\mu)}(a,b;c;z)}{f(z)}\right) > -1
$$

and

$$
\Re\left(w_2\right) > 0 \text{ or } \Re\left(w_2\right) = 0 \text{ and } \Re\left(w_1 - a + 1\right) < 0.
\tag{2.7}
$$

### 2.5. Special Cases

In this section, we give three special cases of our integral transform defined by (2.5).

#### 2.5.1. Generalize Gauss Hypergeometric Function Transform

If we put $\tau = \mu$ in (2.5), the transform in (2.5) reduces to the generalized Gauss hypergeometric function transform given by

$$
\phi(s) = \int_{0}^{\infty} F_p^{(a,\beta;\tau,\mu)}(a,b;c;z) f(z) dz.
\tag{2.8}
$$

#### 2.5.2. Extension of the Generalized Gauss Hypergeometric Function Transform

By taking $\tau = \mu = 1$ in (2.8), we get the following extension of the generalized Gauss hypergeometric function transform:

$$
\phi(s) = \int_{0}^{\infty} F_p^{(a,\beta)}(a,b;c;z) f(z) dz.
\tag{2.9}
$$

Moreover, if we take $\alpha = \beta$ in (2.9), it reduces to the extended Gauss hypergeometric function transform given below:

$$
\phi(s) = \int_{0}^{\infty} F_p^{(a,\beta)}(a,b;c;z) f(z) dz.
\tag{2.10}
$$

if we set $p = 0$ in the integral transforms defined by (2.8), (2.9) and (2.10), we easily get the Gauss hypergeometric transform (see, for details, [12]).

### 2.6. Inversion Formula for the S-Generalized Gauss Hypergeometric Function Transform

**Theorem 3.** If $y^{-1} f(y) \in L(0,\infty)$, the function $f(y)$ is of bounded variation in the neighborhood of the point $y = z$, and

$$
\phi(s) = \mathfrak{S}[f(z);s] = \int_{0}^{\infty} F_p^{(a,\beta;\tau,\mu)}(a,b;c;z) f(z) dz,
\tag{2.11}
$$

then

$$
\frac{1}{2} \left\{ f(t+0) + f(t-0) \right\} = \int_{0}^{\infty} \left[ -\text{sign} \left( \frac{1}{z} \right) \right] \frac{(-1)^{-1} B(b,c-b)}{B(b,c-b)} z^{-\kappa} \Omega(\kappa) dz,
\tag{2.12}
$$

where

$$
\Omega(\kappa) := \int_{0}^{\infty} s^{-\kappa} \phi(s) ds,
\tag{2.13}
$$

provided that existence conditions for the S-generalized Gauss hypergeometric function $F_p^{(a,\beta;\tau,\mu)}(a,b;c;z)$ given by (1.1) are satisfied, the S-generalized Gauss hypergeometric function transform of $f(z)$ exists, and

$$
\Re(1-\kappa) > 0 \text{ and } \Re(1-a-\kappa) < 0.
\tag{2.14}
$$

**Proof.** In order to prove the inversion formula (2.12), we substitute the value of $\phi(s)$ from (2.11) into the right-hand side of (2.13). We thus find that...
Upon interchanging the order of the \( z \)- and the \( s \)-integrals in (2.14) (which is permissible under the given conditions), if we evaluate the \( s \)-integral by using (2.3), we obtain

\[
\Omega(\kappa) = \int_0^\infty s^{-\kappa} \varphi(s) ds = \int_0^\infty s^{-\kappa} \left( \int_0^\infty F_p^{(\alpha, \beta; \gamma, \delta)} (a, b; c; s) f(z) dz \right) ds. \tag{2.14}
\]

Finally, by applying the Mellin Inversion Formula to the above integral (2.15), we get the desired result (2.12) after a little simplification.

3. Concluding Remarks and Observations

In our present investigation, we have further studied the \( S \)-generalized Gauss hypergeometric function:

\[
F_p^{(\alpha, \beta; \gamma, \delta)} (a, b; c; z),
\]

which was recently introduced by Srivastava et al. [8]. In the course of our study, we have presented an integral representation, the Mellin transform and a complex integral representation of the \( S \)-generalized Gauss hypergeometric function. We have also introduced a new integral transform whose kernel is the \( S \)-generalized Gauss hypergeometric function and pointed out its three special cases which are also believed to be new. Furthermore, we have specified that the well-known Gauss hypergeometric function transform follows as a simple special case of our integral transforms. Finally, we have established an inversion formula for the integral transform which we have introduced in this investigation.

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