Note
Forbidding a set difference of size 1
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How large can a family $A \subseteq \mathcal{P}[n]$ be if it does not contain $A, B$ with $|A \setminus B| = 1$? Our aim in this paper is to show that any such family has size at most $2 + o(1) \left(\frac{n}{\lfloor n/2 \rfloor}\right)$. This is tight up to a multiplicative constant of 2. We also obtain similar results for families $A \subseteq \mathcal{P}[n]$ with $|A \setminus B| \neq k$, showing that they satisfy $|A| \leq C_k \left(\frac{n}{\lfloor n/2 \rfloor}\right)$, where $C_k$ is a constant depending only on $k$.

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1. Introduction

A family $A \subseteq \mathcal{P}[n] = \mathcal{P}(\{1, \ldots, n\})$ is said to be a Sperner family or antichain if $A \not\subseteq B$ for all distinct $A, B \in A$. Sperner’s theorem [9], one of the earliest results in extremal combinatorics, states that every Sperner family $A \subseteq \mathcal{P}[n]$ satisfies

$$|A| \leq \left(\frac{n}{\lfloor n/2 \rfloor}\right).$$

(1)

(We remark that this paper is self-contained; for background on Sperner’s theorem and related results see [2].)

Kalai [5] noted that the Sperner condition can be rephrased as follows: $A$ does not contain two sets $A$ and $B$ such that, in the unique subcube of $\mathcal{P}[n]$ spanned by $A$ and $B$, $A$ is the bottom point and $B$ is the top point. He asked: what happens if we forbid $A$ and $B$ to be at a different position in this subcube? In particular, he asked how large $A \subseteq \mathcal{P}[n]$ can be if we forbid $A$ and $B$ to be at a ‘fixed ratio’ $p : q$ in this subcube. That is, we forbid $A$ to be $p/(p + q)$ of the way up this subcube and $B$ to be $q/(p + q)$ of the way up this subcube. Equivalently, $q|A \setminus B| \neq p|B \setminus A|$ for all distinct $A, B \in A$. Note that the Sperner condition corresponds to taking $p = 0$ and $q = 1$. In [8], we gave an asymptotically tight answer for all ratios $p : q$, showing that one cannot improve on the ‘obvious’ example, namely the $q - p$ middle layers of $\mathcal{P}[n]$.

\textbf{Theorem 1.1} ([8]). Let $p, q$ be coprime natural numbers with $q \geq p$. Suppose $A \subseteq \mathcal{P}[n]$ does not contain distinct $A, B$ with $q|A \setminus B| = p|B \setminus A|$. Then

$$|A| \leq (q - p + o(1)) \left(\frac{n}{\lfloor n/2 \rfloor}\right).$$

(2)

Up to the $o(1)$ term, this is best possible. Indeed, the proof of Theorem 1.1 in [8] also gives the exact maximum size of such $A$ for infinitely many values of $n$.

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Another natural question considered in [8] asks how large a family \( A \subset \mathcal{P}[n] \) can be, if instead of forbidding a fixed ratio, we forbid a 'fixed distance' in these subcubes. For example, how large can \( A \subset \mathcal{P}[n] \) be if \( A \) is not at distance 1 from the bottom of the subcube spanned with \( B \) for all \( A, B \in A \)? Equivalently, \(|A \setminus B| \neq 1 \) for all \( A, B \in A \). Here the following family \( A^* \) provides a lower bound: let \( A^* \) consist of all sets \( A \) of size \( \lfloor n/2 \rfloor \) such that \( \sum_{i \in A} i \equiv r \pmod{n} \) where \( r \in \{0, \ldots, n-1\} \) is chosen to maximise \(|A^*|\). Such a choice of \( r \) gives \( |A^*| \geq \left( \frac{n}{n/2} \right) \). Note that if we had \(|A \setminus B| = 1 \) for some \( A, B \in A^* \), since \(|A| = |B|\), we would also have \(|B \setminus A| = 1\) – letting \( A \setminus B = \{i\} \) and \( B \setminus A = \{j\} \) we then have \( i-j \equiv 0 \pmod{n} \) giving \( i = j \), a contradiction.

In [8], we showed that any such family \( A \subset \mathcal{P}[n] \) satisfies \(|A| \leq \frac{\sqrt{2n}}{n} = O\left( \frac{1}{n^{1/2}} \left( \frac{n}{n/2} \right) \right) \) for some absolute constant \( C \). We conjectured that the family \( A^* \) constructed in the previous paragraph is asymptotically maximal (Conjecture 5 of [8]). In Section 2, we prove that this is true up to a factor of 2.

Theorem 1.2. Suppose that \( A \subset \mathcal{P}[n] \) is a family of sets with \(|A \setminus B| \neq 1 \) for all \( A, B \in A \). Then \(|A| \leq \frac{2+o(1)}{n} \left( \frac{n}{n/2} \right) \).

One could also ask what happens if we forbid a fixed set difference of size \( k \), instead of 1 (where we think of \( k \) as fixed and \( n \) as varying). This turns out to be harder. In [8] we noted that the following family \( A_k^* \subset \mathcal{P}[n] \) gives a lower bound of \( \frac{1}{n^{k/2}} \left( \frac{n}{n/2} \right)^k \) : supposing \( n \) is prime, let \( A_k^* \) consist of all sets \( A \) of size \( \lfloor n/2 \rfloor \) which satisfy \( \sum_{i \in A} i^d \equiv 0 \pmod{n} \) for all \( 1 \leq d \leq k \).

Theorem 1.3. Let \( k \in \mathbb{N} \). Suppose that \( A \subset \mathcal{P}[n] \) with \(|A \setminus B| \neq k \) for all \( A, B \in \mathcal{P}[n] \). Then \(|A| \leq \frac{\sqrt{k}}{n^k} \left( \frac{n}{n/2} \right) \), where \( C_k \) is a constant depending only on \( k \).

Our notation is standard. We write \([n]\) for \( \{1, \ldots, n\} \), and \([a, b]\) for the interval \( \{a, \ldots, b\} \). For a set \( X \), we write \( \mathcal{P}(X) \) for the power set of \( X \) and \( X^{(k)} \) for the collection of all \( k \)-sets in \( X \). We often suppress integer-part signs.

2. Proof of Theorem 1.2

Our proof of Theorem 1.2 uses Katona’s averaging method (see [6]), but modified in a key way. Ideally here, as in the proof of Spencer’s theorem or Theorem 1.1, we would find configurations of sets covering \( \mathcal{P}[n] \) so that each configuration has at most \( C/n^{3/2} \) proportion of its elements in \( A \), for any family \( A \) satisfying \(|A \setminus B| \neq 1 \) for \( A, B \in A \). Then, provided that these configurations cover \( \mathcal{P}[n] \) uniformly, we could count incidences between elements of \( A \) and these configurations to get an upper bound on \(|A|\).

However, we do not see how to find such configurations. Instead our approach is as follows. Rather than insisting that each of the sets in our configuration is well-behaved (in the sense above), we will only require that most of them have at most \( C/n^{3/2} \) proportion of their elements in \( A \). It turns out that this can be achieved, and that it is good enough for our purposes.

Proof. We will prove the proposition under the assumption that \( n \) is even—this can easily be removed. To begin with, remove all elements in \( A \) of size smaller than \( n/2 - n^{2/3} \) or larger than \( n/2 + n^{2/3} \). By Chernoff’s inequality (see Appendix A of [1]), we have removed at most \( o\left( \frac{\sqrt{n}}{n/2} \right) \) \( \frac{n}{n/2} \) sets. Let \( B \) denote the remaining sets in \( A \). It suffices to show that \(|B| \leq \frac{2+o(1)}{n} \left( \frac{n}{n/2} \right) \).

We write \( I = [1, n/2 + n^{2/3}] \) and \( J = [n/2 + n^{2/3} + 1, n] \) so that \( |n| = I \cup J \). Let us choose a permutation \( \sigma \in S_n \) uniformly at random. Given this choice of \( \sigma \), for all \( i \in I, j \in J \) let \( C_{i,j} = \{\sigma(1), \ldots, \sigma(i), \sigma(j)\} \). Let \( C = \bigcup_{i,j} C_{i,j} \).

Now, for any choice of \( \sigma \in S_n \), at most one of the partial chains of \( C \) can contain an element of \( B \). Indeed, suppose both \( C_{i,j_1} = C_{i,j} \cup \{\sigma(j_1)\} \) and \( C_{i,j_2} = C_{i,j} \cup \{\sigma(j_2)\} \) lie in \( A \) for distinct \( j_1, j_2 \in J \). Since \( C_i \) and \( C_i \) are elements of a chain, without loss of generality we may assume \( C_{i,j} \subset C_{i,j} \). But then \( (C_i \cup \{\sigma(j_1)\}) \cap (C_i \cup \{\sigma(j_2)\}) = \{\sigma(j_1)\} \), which contradicts \(|A \setminus B| \neq 1 \) for all \( A, B \in \mathcal{P}[n] \).

Note that the above bound alone does not guarantee the upper bound on \(|A|\) stated in the theorem, since a fixed partial chain \( C_i \) may contain many elements of \( A \). We now show that this cannot happen too often.

For \( i \in I \) and \( j \in J \), let \( X_{i,j} \) denote the random variable given by

\[
X_{i,j} = \begin{cases} 
1 & \text{if } C_{i,j} \subset B \text{ and } C_{i,j} \not\subset B \text{ for } k < i; \\
0 & \text{otherwise.}
\end{cases}
\]

From the previous paragraph, we have

\[
\sum_{i,j} X_{i,j} \leq 1 \quad (3)
\]

where both here and below the sum is taken over all \( i \in I \) and \( j \in J \). Taking expectations on both sides of (3) this gives

\[
\sum_{i,j} \mathbb{E}(X_{i,j}) \leq 1 \quad (4)
\]
Rearranging we have
\[ \sum_{i,j} \mathbb{E}(X_{i,j}) = \sum_{i,j} \sum_{B \in \mathcal{B}} \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \not\in \mathcal{B} \text{ for } k < i). \] (5)

We now bound \( \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \not\in \mathcal{B} \text{ for } k < i) \) for sets \( B \in \mathcal{B} \). Note that we can only have \( C_{i,j} = B \) if \( |B| = i + 1 \). Furthermore, for such \( B \), since \( C_{i,j} \) is equally likely to be any subset of \([n]\) of size \( i + 1 \), we have \( \mathbb{P}(C_{i,j} = B) = 1/\binom{n}{i+1} \). We will show that for all such \( B \)
\[ \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \not\in \mathcal{B} \text{ for } k < i) = (1 - o(1)) \mathbb{P}(C_{i,j} = B). \] (6)
To see this, note that given any set \( D \subset [n] \), there is at most one element \( d \in D \) such that \( D - d \in \mathcal{B} \). Indeed, \(|(D - d') \setminus (D - d)| = 1 \) for any distinct choices of \( d, d' \in D \). Recalling that \( C_{k,j} = C_{j} - \{\sigma(k + 1), \ldots, \sigma(i)\} \) for all \( k < i \) and that \( \sigma(k + 1) \) is chosen uniformly at random from the \( k + 1 \) elements of \( C_{k+1,j} - \{\sigma(j)\} \), we see that for \( k + 1 \geq n/2 - n^{2/3} \) we have
\[ \mathbb{P}(C_{k,j} \not\in \mathcal{B}|C_{k+1,j}, \ldots, C_{i,j}) \geq \left(1 - \frac{1}{k+1}\right) \geq \left(1 - \frac{1}{n/2 - n^{2/3}}\right). \] (7)
Also, since \( \mathcal{B} \) contains no sets of size less than \( n/2 - n^{2/3} \), for \( k + 1 < n/2 - n^{2/3} \) we have
\[ \mathbb{P}(C_{k,j} \not\in \mathcal{B}|C_{k+1,j}, \ldots, C_{i,j}) = 1. \] (8)
But now by repeatedly applying (7) and (8) we get that for any \( B \) of size \( i + 1 \in [n/2 - n^{2/3}, n/2 + n^{2/3}] \) we have
\[ \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \not\in \mathcal{B} \text{ for } k < i) \geq \left(1 - \frac{1}{n/2 - n^{2/3}}\right)^{(i-n/2-n^{2/3})} \mathbb{P}(C_{i,j} = B) \geq \left(1 - \frac{1}{n/2 - n^{2/3}}\right)^{2n^{2/3}} \mathbb{P}(C_{i,j} = B) = (1 - o(1)) \mathbb{P}(C_{i,j} = B). \]

Now combining (6) with (4) and (5) we obtain
\[
1 \geq \sum_{i,j} \mathbb{E}(X_{i,j}) \\
= \sum_{i,j} \sum_{B \in \mathcal{B}} \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \not\in \mathcal{B} \text{ for } k < i) \\
= \sum_{i,j} \sum_{B \in \mathcal{B}^{(i+1)}} (1 - o(1)) \mathbb{P}(C_{i,j} = B) \\
= (1 - o(1)) \sum_{i,j} \frac{|\mathcal{B}^{(i+1)}|}{\binom{n}{i+1}} \\
= (1 - o(1)) |\mathcal{B}| \sum_{i} \frac{|\mathcal{B}^{(i+1)}|}{\binom{n}{i+1}}.
\]
Since \(|\mathcal{B}| = n/2 - n^{2/3}\), this shows that
\[ \frac{2 + o(1)}{n} \geq \sum_{i} \frac{|\mathcal{B}^{(i+1)}|}{\binom{n}{i+1}} \\
giving |\mathcal{B}| \leq \frac{2 + o(1)}{n} \binom{n}{n/2}, \text{as required.} \]

3. Proof of Theorem 1.3

The proof of Theorem 1.3 will use the following result of Frankl and Füredi [4].

**Theorem 3.1 (Frankl–Füredi).** Let \( r, k \in \mathbb{N} \) with \( 0 \leq k < r \). Suppose that \( \mathcal{A} \subset [n]^k \) with \( |A \cap B| \neq k \) for all \( A, B \in \mathcal{A} \). Then
\[ |A| \leq d_r n^{\max(k,r-k-1)} \] where \( d_r \) is a constant depending only on \( r \).

We will also make use of the Erdős–Ko–Rado theorem [3].

**Theorem 3.2 (Erdős–Ko–Rado).** Suppose that \( k \in \mathbb{N} \) and that \( 2k \leq n \). Then any family \( \mathcal{A} \subset [n]^k \) with \( A \cap B \neq \emptyset \) for all \( A, B \in \mathcal{A} \) satisfies
\[ |\mathcal{A}| \leq \binom{n-1}{k-1}. \]
We are now ready for the proof of the main result. Given a set $U \subset [n]$ and a permutation $\sigma \in S_n$, below we write $\sigma(U) = \{\sigma(u) : u \in U\}$.

**Proof of Theorem 1.3.** We will assume for convenience that $n$ is a multiple of 3k—this assumption can easily be removed. To begin, remove all elements in $A$ of size smaller than $n/2 - n^{2/3}$ or larger than $n/2 + n^{2/3}$. By Chernoff's inequality (see Appendix A of [1]), we have removed at most $o\left(\frac{1}{e^3}\left(\frac{n}{n/2}\right)^n\right)$ sets. Let $\mathcal{B}$ denote the remaining sets in $A$. For each $l \in [0, k - 1]$, let $\mathcal{B}_l = \{B \in \mathcal{B} : |B| \equiv l \text{ (mod } k)\}$.

To prove the theorem it suffices to prove that for all $l \in [0, k - 1]$ we have $|\mathcal{B}_l| \leq \frac{c'}{\pi^k}\left(\frac{n}{n/2}\right)^n$, where $c' = C(k) > 0$. We will show this when $l = 0$ as the other cases are similar.

Let $l = [1, n/3]$ and $J = [n/3 + 1, n]$ so that $|\mathcal{B}_l| = |J| \cup J$. Let us choose a permutation $\sigma \in S_n$ uniformly at random. Given this choice of $\sigma$, for all $i \in [n/3k]$, let $C_{i, S} = \sigma((1, \ldots, ik)) \cup \sigma(S)$.

Let $C_S = \{C_{i, S} : i \in [n/3k]\}$ and call these sets ‘partial chains’. We write $D = \left\{S \in \binom{J}{n/3} : C_S \cap \mathcal{B}_0 \neq \emptyset\right\} \subset \binom{J}{n/3}$.

We claim that for any choice of $\sigma \in S_n$, we have

$$|D| \leq \frac{d_{2k}(12k^2)^k}{n^k} \left(\frac{|J|}{n/3}\right)^2, \tag{9}$$

where $d_{2k}$ is as in Theorem 3.1. Indeed otherwise, by averaging, there exists $T \in J^{(n/3 - 2k)}$ for which the family $D_T = \left\{U \in (J \setminus T)^{(2k)} : U \cup T \in D\right\} \subset (J \setminus T)^{(2k)}$ satisfies $|D_T| > \frac{d_{2k}(12k^2)^k}{n^k} \left(\frac{|J|}{2k}\right)^2$. This gives that

$$|D_T| > \frac{d_{2k}(12k^2)^k}{n^k} \left(\frac{|J \setminus T|}{2k}\right)^2 \geq \frac{d_{2k}(12k^2)^k}{n^k} \frac{|J \setminus T|^{2k}}{(2k)^{2k}} = \frac{d_{2k}|J \setminus T|^2k}{(n/3)^k} \geq \frac{d_{2k}|J \setminus T|^k}{k},$$

since $|J \setminus T| = n/3 + 2k \geq n/3$. However, applying Theorem 3.1 to $D_T$ with $r = 2k$ we find $U, U' \in D_T$ with $|U \cup U'| = k$. This then gives $C_{i, U \cup T}, C_{i', U' \cup T} \in \mathcal{B}_0$ for some $i, i' \in [n]$. Without loss of generality, we have $i \leq i'$. But then, as $\sigma((1, \ldots, ik)) \subset \sigma((1, \ldots, i'))$, we have

$$|C_{i, U \cup T} \setminus C_{i', U' \cup T}| = |\sigma(U) \setminus \sigma(U')| = |U \setminus U'| = |U| - |U \cup U'| = 2k - k = k.$$}

However $|A \setminus B| \neq k$ for all $A, B \in \mathcal{B}_0$. This contradicts that $(9)$ must hold.

Now the bound $(9)$ shows that for any choice of $\sigma \in S_n$, at most $c_1/n^k$ proportion of the sets $C_S$ can contain elements of $\mathcal{B}_0$. Note however that any of these partial chains may still contain many elements from $\mathcal{B}_0$. As in the proof of Theorem 1.2, we now show that this cannot happen too often.

For $i \in [n/3k]$ and $S \in J^{(n/3)}$, let $X_{i, S}$ denote the random variable given by

$$X_{i, S} = \begin{cases} 1 & \text{if } C_{i, S} \in \mathcal{B}_0 \text{ and } C_{i', S} \not\in \mathcal{B}_0 \text{ for all } i' < i; \\ 0 & \text{otherwise}. \end{cases}$$

From the previous paragraph, we have

$$\sum_{i, S} X_{i, S} \leq \frac{d_{2k}(12k^2)^k}{n^k} \left(\frac{|J|}{n/3}\right)^2 \tag{10}$$

where both here and below the sum is taken over all $i \in [n/3k]$ and $S \in J^{(n/3)}$. Taking expectations on both sides of $(3)$ this gives

$$\sum_{i, S} \mathbb{E}(X_{i, S}) \leq \frac{d_{2k}(12k^2)^k}{n^k} \left(\frac{|J|}{n/3}\right)^2. \tag{11}$$

Rearranging we have

$$\sum_{i, S} \mathbb{E}(X_{i, S}) = \sum_{i, S} \sum_{B \in \mathcal{B}_0} \mathbb{P}(C_{i, S} = B \text{ and } C_{i', S} \not\in \mathcal{B}_0 \text{ for all } i' < i). \tag{12}$$
We now bound $P(C_{i,S} = B$ and $C'_{r,S} \notin B_0$ for $i' < i$) for sets $B \in B_0$. Note that we can only have $C_{i,S} = B$ if $|B| = ik + n/3$. Furthermore, for such $B$, since $C_{i,S}$ is equally likely to be any subset of $[n]$ of size $ik + n/3$, we have $P(C_{i,S} = B) = 1/\binom{n}{ik+n/3}$. We will prove that for all such $B$

$$P(C_{i,S} = B$ and $C_{r,S} \notin B_0$ for $i' < i$) = (1 - o(1))P(C_{i,S} = B).

To see this, note that given any set $D \subset [n]$ and two sets $E_1, E_2 \in D^{(k)}$ for which $D \setminus E_1, D \setminus E_2 \in B_0$, we must have $E_1 \cap E_2 \neq \emptyset$—otherwise $|D \setminus E_1 \setminus (D \setminus E_2)| = k$. Therefore, for $|D| \geq 2k$, by Theorem 3.2, there are at most $\binom{|D| - 1}{k - 1} = \frac{|D|!}{k!(|D| - k)!}$ choices of $E \in D^{(k)}$ with $D \setminus E \in B_0$. Recalling that $C_{r,S} = C_{i,S} - [\sigma(i'k+1), \ldots, \sigma((i'+1)k)]$ for all $i' < i$ and that $[\sigma(i'k+1), \ldots, \sigma((i'+1)k)]$ is chosen uniformly at random among all $k$-sets in $[\sigma(1), \ldots, \sigma((i'+1)k)]$, we see that for $(i'+1)k+n/3 \geq (n/2-n^2/3)$ we have

$$P(C_{r,S} \notin B_0 | C_{i+1,S}, \ldots, C_{i,S}) \geq \left(1 - \frac{k}{(i'+1)k}\right) \geq \left(1 - \frac{k}{n/6 - n^2/3}\right).$$

Also, since $B_0$ contains no sets of size less than $n/2 - n^2/3$, for $(i'+1)k+n/3 < (n/2 - n^2/3)$ we have

$$P(C_{r,S} \notin B_0 | C_{i+1,S}, \ldots, C_{i,S}) = 1.$$

But now by repeatedly applying (14) and (15), we get that for any $B$ of size $ik + n/3 \in [n/2 - n^2/3, n/2 + n^2/3]$ we have

$$P(C_{i,S} = B$ and $C_{r,S} \notin B_0$ for $i' < i$) \geq \left(1 - \frac{k}{n/6 - n^2/3}\right)^{2n^2/3/k} P(C_{i,S} = B)$$

$$\geq \left(1 - \frac{k}{n/6 - n^2/3}\right)^{2n^2/3/k} P(C_{i,S} = B)$$

$$= (1 - o(1))P(C_{i,S} = B).$$

Now combining (13) with (11) and (12) we obtain

$$\frac{d_{2k}(12k^2)^k}{n^k} \left(\frac{|U|}{n/3}\right) \geq \sum_{i,S} \sum_{B \in B_0} P(C_{i,S} = B$ and $C_{r,S} \notin B_0$ for $i' < i$$)

$$= \sum_{i,S} \sum_{B \in B_0} (1 - o(1))P(C_{i,S} = B)$$

$$= (1 - o(1)) \sum_{i,S} \frac{|B_0^{(r)(n/3)}|}{\binom{n}{ik+n/3}}$$

$$= (1 - o(1)) \left(\frac{|U|}{n/3}\right) \sum_{j \in [n]} \frac{|B_0^{(j)}|}{\binom{n}{j}}.$$

But this shows that

$$\frac{d_{2k}(12k^2)^k}{n^k} \geq \sum_{j \in [n]} \frac{|B_0^{(j)}|}{\binom{n}{j}}$$

giving $|B_0| \leq \frac{d_{2k}(12k^2)^k}{n^k} \binom{n}{n/2}$, as required. \hfill \Box

4. Concluding remarks

It would be very interesting to determine the true answer in Theorem 1.2, i.e., to remove the factor of 2. This is related to the well-known problem of finding the maximum size of a set system in which no two members are at Hamming distance 2, where there is also a 'gap' of a multiplicative constant 2. Indeed, our proof of Theorem 1.2 can be modified to show that the answers to these two questions are asymptotically equal. See Katona [7] for background on this problem.

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References

[1] N. Alon, J. Spencer, The Probabilistic Method, third ed., Wiley, 2008.
[2] B. Bollobás, Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability, first ed., Cambridge University Press, 1986.
[3] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 12 (1961) 313–320.
[4] P. Frankl, Z. Füredi, Forbidding just one intersection, J. Combin. Theory Ser. A 39 (1985) 160–176.
[5] G. Kalai, Personal communication, 2010.
[6] G.O.H. Katona, A simple proof of the Erdős–Chao Ko–Rado theorem, J. Combin. Theory Ser. B 13 (1973) 83–84.
[7] G.O.H. Katona, Forbidden inclusion patterns in the families of subsets (introducing a method), in: E. Győri, G.O.H. Katona, L. Lovász (Eds.), Horizons of Combinatorics, in: Bolyai Society Mathematical Studies, vol. 17, Springer-Verlag, 2008, pp. 119–140.
[8] I. Leader, E. Long, Tilted Sperner families, Discrete Appl. Math. 163 (2) (2014) 194–198.
[9] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928) 544–548.