PERMUTATION CODE EQUIVALENCE IS NOT HARDER THAN GRAPH ISOMORPHISM WHEN HULLS ARE TRIVIAL

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ABSTRACT. The paper deals with the problem of deciding if two finite-dimensional linear subspaces over an arbitrary field are identical up to a permutation of the coordinates. This problem is referred to as the permutation code equivalence. We show that given access to a subroutine that decides if two weighted undirected graphs are isomorphic, one may deterministically decide the permutation code equivalence provided that the underlying vector spaces intersect trivially with their orthogonal complement with respect to an arbitrary inner product. Such a class of vector spaces is usually called linear codes with trivial hulls. The reduction is efficient because it essentially boils down to computing the inverse of a square matrix of order the length of the involved codes. Experimental results obtained with randomly drawn binary codes having trivial hulls show that permutation code equivalence can be decided in a few minutes for lengths up to 50,000.

1. INTRODUCTION

This paper deals with the problem of deciding whether two finite dimensional linear subspaces of $\mathbb{F}^n$, where $\mathbb{F}$ is a field (not necessarily finite) are identical up to a permutation of the coordinates. This problem which is referred to as the permutation code equivalence is a special case of the isomorphism problem that asks, for a given metric, whether there exists a linear isometry between two linear subspaces. These problems are usually encountered in coding theory where the ambient space $\mathbb{F}^n$ is endowed with the Hamming distance. The isomorphism problem is of great importance in the classification of codes because isomorphic codes display identical characteristics such as length, dimension, minimum distance, weight distribution and correction capabilities. The permutation code equivalence problem was used in cryptography by McEliece in [McE78] as a tool for improving the security of public key encryption schemes. Although the security of the McEliece scheme is not directly related to the permutation equivalence problem, there exist some encryption schemes that can be fully cryptanalyzed if an attacker efficiently solves the equivalence problem (see for instance [MS07, BCD+16]). Consequently a better assessment of the computational difficulty of this problem will permit to more accurately evaluate the security of code-based cryptographic primitives.

Its computational difficulty was studied in [PR97] where it was shown that if the permutation equivalence problem over the binary field is NP-complete, then the polynomial hierarchy collapses. Hence this problem is unlikely to be NP-complete. They also provided a polynomial-time reduction from the graph isomorphism to the permutation equivalence problem. However, in light of the recent breakthrough showing that graph isomorphism might not be a hard problem [Bab15], this reduction becomes less interesting. A worst-case upper-bound was given in [BCGQ11] showing that the permutation equivalence problem can be solved with $(2 + o(1))^n$ operations.

Another very important question connected to the isomorphism problem consists in characterizing the group of isometries that leaves globally invariant a given linear space. This group is called the automorphism group. The set of isometries between two equivalent linear subspaces is then a coset of the automorphism group. In particular the number of solutions (when they exist) to the isomorphism problem is exactly the size of the automorphism group. Since permutations are always isometries for any distance over $\mathbb{F}^n$, the set of solutions...
permutations that globally stabilize a linear space is called the permutation group. Leon [Leo82] introduced a method for fully determining the permutation group of a linear code over a finite field \( \mathbb{F}_q \). The algorithm requires the computation of all codewords of a given weight. The time complexity is therefore \( n q^{O(k)} \) for \( k \)-dimensional vector subspaces assuming that the time complexity of field operations are constant. The techniques developed in Leon’s algorithm can then be used to solve the permutation equivalence problem for linear codes of small dimension over small fields. This algorithm is for instance implemented in Magma software [BCP97]. Later Sendrier [Sen00] considered the hull of codes which is the intersection of a vector subspace with its orthogonal complement with respect to the Euclidean inner product.

The underlying motivation is that (permutation) equivalent linear subspaces have permutation equivalent hulls. Although the permutation problem between hulls may have more solutions, and moreover inequivalent codes may have equivalent hulls, the practical advantage is that hulls tend to have very small dimensions [Sen97]. Actually hulls are very likely to be reduced to \( \{0\} \) which paradoxically does not help in solving the problem. Consequently Sendrier’s approach is interesting when the hull is very small but not trivial, and its permutation group is ideally trivial. The technique developed in [Sen00] first punctures at each position the code to be tested, and then computes for each punctured code the weight enumerator of its hull. All these weight enumerators provide a compact “signature” with the property that permutation equivalent codes have equivalent signatures. The permutation equivalence problem is therefore reduced to testing whether two signatures are equivalent. The time complexity of [Sen00] is at least exponential in the dimension of the hull. Furthermore, in order to take into consideration the cases where there may exist many solutions, not to mention the fact that two inequivalent codes may have equivalent signatures, several successive puncturing operations have to be applied, which complicates the analysis of the algorithm. However it is claimed (see for example [SS13]) that its empirical time complexity is \( O \left( n^2 + 2^h n^2 \log n \right) \) where \( h \) is the dimension of the hull, which means that the algorithm becomes polynomial when restricted to codes with hulls of bounded dimension.

Recently a new algebraic approach was introduced in [ST17] in order to solve the permutation code equivalence. It builds a quadratic polynomial system whose binary solutions are permutations. It was then shown that Gröbner basis techniques fully solve the code equivalence but the cost becomes rapidly prohibitive as the length of the codes increases. Several improvements are then proposed enabling to efficiently solve the permutation equivalence when the codes have small hulls or are defined over large fields.

These previous works tend to prove that the hardness of the permutation code equivalence is not well understood. From a practical point of view, there is a prevalent belief that this problem is easy (see for example [Sen02]) when dealing with codes having very small hulls, as it is the case for random linear codes. On the other hand, no rigorous theoretical bound supporting these concrete observations is known, especially there is no proof showing that permutation code equivalence is polynomial in time for codes with trivial hulls. Furthermore several works are dedicated to design practical algorithms [Leo82, Sen00, Bou07, Feu09] but none seems to be able to decide efficiently if two codes of rate say one-half and of length greater than 200 are equivalent or not. The situation is getting worse when the fields are large, not to mention fields of characteristic 0, because all these methods require small finite fields.

**Our contributions.** We show that given access to a subroutine that decides if two weighted undirected graphs are isomorphic, we may deterministically decide the permutation code equivalence over an arbitrary field \( \mathbb{F} \) assuming that the hulls of the involved vector spaces are trivial. Our reduction given in Theorem 5 is very efficient since it essentially boils down to computing the inverse of a square matrix of order \( n \) where \( n \) is the length of the codes considered. We exploit the fact that the direct sum of a vector subspace \( U \subset \mathbb{F}^n \) and its orthogonal \( U^\perp \) is equal to the full vector space \( \mathbb{F}^n = U \oplus U^\perp \) when \( U \cap U^\perp = \{0\} \). It is then possible to decompose each vector \( e_i \) of the canonical basis \( \{e_i : i \in \{1, n\}\} \) as \( e_i = \sigma_U(e_i) + \sigma_{U^\perp}(e_i) \) where \( \sigma_U(e_i) \) (resp. \( \sigma_{U^\perp}(e_i) \)) is the projection of \( e_i \) on \( U \) (resp. \( U^\perp \)). We then define the square matrix \( \Sigma_U \) whose rows are formed by the projections \( \sigma_U(e_i) \). In the same way, we define \( \Sigma_{U^\perp} \) which enables to
write $\Sigma_U + \Sigma_{U^\perp} = I_n$. We prove that $\Sigma_U$ and $\Sigma_{U^\perp}$ are symmetric and for any permutation matrix $X$ it holds that $X^T (\Sigma_U + \Sigma_{U^\perp}) X = X^T \Sigma_U X + X^T \Sigma_{U^\perp} X$. By the uniqueness of the decomposition, it entails that $\Sigma_{U^\perp} X = X^T \Sigma_U X$ and $\Sigma_{U^\perp} X = X^T \Sigma_{U^\perp} X$. These relations express the property that graphs represented by $\Sigma_U X$ and $\Sigma_U$ viewed as adjacency matrices are isomorphic.

The advantage of this reduction is twofold. From a practical point of view, the permutation equivalence would benefit from the (numerous) efficient existing tools that solve the problem of isomorphism between graphs. Our experimentations with binary codes of length up to 20,000 show that we can decide if two codes are equivalent in less than two seconds. Our computations were performed on Intel Core i7, 2.5 GHz, 16 GB ram with Magma software which relies on version 2.6r7 of the nautily and Traces packages. The other advantage is that, in light of the recent result obtained in [Bab15], our reduction validates in a sense the belief that codes with trivial hulls form a class of easy instances for the permutation code equivalence problem.

Finally, we propose two solutions to treat the cases of codes having a non-trivial intersection with their orthogonal complement. The first one applies more particularly to codes defined over an extension field $\mathbb{F}_{p^m}$ where $m > 1$ and $p$ is a prime. We equip the space $\mathbb{F}_{p^m}^n$ with an Hermitian inner product by means of an automorphism of $\mathbb{F}_{p^m}$. This enables us to adopt a different perspective because a code may not have a non-trivial hull for one inner product but has a trivial hull with respect to another one. With little effort, we are able to generalize our reduction to any Hermitian inner product. This generalization permits to encompass all but an exponentially small fraction of linear codes. The remaining cases are then essentially codes defined over a prime field $\mathbb{F}_p$. By shortening sufficiently enough it is possible to obtain codes with trivial hulls. Given access to an oracle that decides the graph isomorphism, we then propose an algorithm whose time complexity is $O(h \pi^{\omega+1} \text{Gl}(n))$ where $\omega$ is the exponent of matrix multiplication ($2 < \omega < 3$) and $\text{Gl}(n)$ is the time complexity for deciding if two weighted graphs with $n$ vertices and weights in $\mathbb{F}$ are isomorphic.

The rest of the paper is organized as follows. Section 2 introduces notation and classical notions on algebra and graph theory. It also introduces the permutation code equivalence problem. Section 3 is dedicated to the presentation of a deterministic reduction from the permutation code equivalence to graph isomorphism. Section 4 is concerned with methods that deal with codes having non-trivial hulls.

2. Preliminaries

2.1. Notation. Throughout the paper $\mathbb{F}$ is an arbitrary field. For any $m$ and $n$ in $\mathbb{Z}$ the notation $\llbracket m, n \rrbracket$ stands for the set of integers $i$ such that $m \leq i \leq n$. The (Euclidean) inner product between $x$ and $y$ from $\mathbb{F}^n$ is $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$. $\mathfrak{S}_n$ is the Symmetric group on $\llbracket 1, n \rrbracket$. The action of $\mathfrak{S}_n$ over $\mathbb{F}^n$ is defined for any $\pi$ in $\mathfrak{S}_n$ and $u$ in $\mathbb{F}^n$ by $u^\pi = (u_{\pi(1)}, \ldots, u_{\pi(n)})$. We extend this notation to any subset $U \subseteq \mathbb{F}^n$, namely $U^\pi = \{u^\pi : u \in U\}$. Finally, we will rely on the fact that the computation of the inverse of an $n \times n$ matrix over a field $\mathbb{F}$ can be performed in $O(n^\omega)$ operations assuming that the operations in $\mathbb{F}$ are constant where $\omega$ is the exponent of matrix multiplication ($2 < \omega < 3$). The best theoretical bound is currently $\omega < 2.373$.

2.2. Permutation code equivalence. A linear code $U$ of length $n$ and dimension $k$ over a field $\mathbb{F}$ is a $k$-dimensional vector subspace of $\mathbb{F}^n$. Any $k \times n$ matrix whose rows form a basis is called a generator matrix. The orthogonal complement of a code $U \subseteq \mathbb{F}^n$ is the linear space $U^\perp$ containing all vectors $z$ from $\mathbb{F}^n$ such that for all $u \in U$, we have $\langle u, z \rangle = 0$. We always have $\dim U^\perp = n - \dim U$, and any generator matrix of $U^\perp$ is called a parity check matrix of $U$. Throughout the paper we use the convention that $G_U$ and $H_U$ represent respectively a generator matrix and a parity check matrix of a linear code $U$. The Hull of a code
When $U \cap U^\perp = \{0\}$ we say that $U$ has a trivial hull or is a linear code with a complementary dual. It means that both $G_U G_U^T$ and $H_U H_U^T$ are invertible. It is also equivalent to say that $\mathbb{F}^n = U \oplus U^\perp$, that is to say $H_U U \cap U^\perp$ is invertible with

$$H_U^{-1} = \left[ G_U^T \left( G_U G_U^T \right)^{-1} H_U^T \left( H_U H_U^T \right)^{-1} \right].$$

(1)

Two linear codes $A$ and $B$ of length $n$ are permutation equivalent if there exists $\pi$ in $\mathcal{S}_n$ such that $B = A^\pi$. We shall use the notation $B \sim_\pi A$. This is also equivalently denoted by $B = AX \triangleq \{aX : a \in A\}$ where $X$ is a matrix representing the permutation $\pi$.

**Definition 1.** Given two linear codes $A$ and $B$ of length $n$, the permutation code equivalence problem asks if there exists a permutation $\pi$ in $\mathcal{S}_n$ such that $B = A^\pi$.

**Remark 2.** Two linear codes $A$ and $B$ are permutation equivalent if and only if $G_B = SG_A X$ where $S$ is an invertible matrix and $X$ is a permutation matrix. Furthermore it is not difficult to see that $B \sim_\pi A$ if and only if $B^\perp \sim_\pi A^\perp$.

### 2.3. Graph theory

A weighted (or edge-labeled) graph $G = (V, E)$ is composed of a finite set of vertices $V$, a set of edges $E \subseteq V \times V$, a set of weights $W$, and a function which assigns a weight $w(i, j)$ to each edge $(i, j) \in E$. We will always assume that $V = [1, n]$. The graph $G$ is undirected if for each $(i, j)$ in $E$, we also have $(j, i)$ in $E$ and $w(i, j) = w(j, i)$. The adjacency matrix $A(G) = [a_{i,j}]$ of a graph $G$ is an $n \times n$ matrix such that $a_{i,j} = w(i, j)$ when $(i, j) \in E$ and $a_{i,j} = 0$ otherwise. In particular $A(G)$ is symmetric when $G$ is undirected. The graph isomorphism (GI) problem is given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $n$ vertices, determine whether $G_2$ can be obtained from $G_1$ by permutation of its vertices while keeping both adjacent vertices and weight edges. It is equivalent to determine if there exists a permutation $\pi$ in $\mathcal{S}_n$ such that $(\pi(i), \pi(j))$ in $E_2$ if and only if $(i, j)$ in $E_1$ and $w(\pi(i), \pi(j)) = w(i, j)$.

We shall write in that case $G_2 \sim_\pi G_1$. This also means that there exists an $n \times n$ permutation matrix $X$ such that

$$A(G_2) = X^T A(G_1) X.$$  

(2)

### 3. A Reduction to Weighted Graph Isomorphism

Let us assume that $U \subseteq \mathbb{F}^n$ is a code with a trivial hull, and consequently we have $\mathbb{F}^n = U \oplus U^\perp$. Let $\sigma_U : \mathbb{F}^n \rightarrow U$ and $\sigma_U^\perp : \mathbb{F}^n \rightarrow U^\perp$ be the projections associated to $U$ and $U^\perp$ respectively. Any vector $v$ in $\mathbb{F}^n$ can be uniquely written as $v = \sigma_U(v) + \sigma_U^\perp(v)$. The linear maps $\sigma_U$ and $\sigma_U^\perp$ satisfy the following relations: $\sigma_U + \sigma_U^\perp = \text{id}$, $\sigma_U^2 = \sigma_U$, $\sigma_U^\perp = \sigma_U^\perp$ and $\sigma_U \circ \sigma_U = \sigma_U$ and $\sigma_U^\perp \circ \sigma_U^\perp = \text{id}$. We also have that $\text{Im}(\sigma_U) = U$ and $\text{Im}(\sigma_U^\perp) = U^\perp$. In order to characterize matrices defining $\sigma_U$ and $\sigma_U^\perp$, we introduce the following $n \times n$ matrices

$$\begin{cases} 
\Sigma_U & \triangleq G_U^T \left( G_U G_U^T \right)^{-1} G_U, \\
\Sigma_U^\perp & \triangleq H_U^T \left( H_U H_U^T \right)^{-1} H_U.
\end{cases}$$

(3)

**Proposition 3.** Let $U$ be a linear code with trivial hull then it holds that

$$\forall v \in \mathbb{F}^n, \quad \sigma_U(v) = v \Sigma_U \quad \text{and} \quad \sigma_U^\perp(v) = v \Sigma_U^\perp.$$  

(4)
WE present two approaches to extend the reduction given in Theorem 6 to codes with non-trivial hulls. Theorem 6.

Let \( v \) which means that \( \sigma_U(v) \) is equivalent to finding \( x_U \) from \( \mathbb{F}^k \) and \( x_{U\perp} \) from \( \mathbb{F}^{n-k} \) such that \( \sigma_U(v) = x_UG_U \) and \( \sigma_{U\perp}(v) = x_{U\perp}H_U \), which means that \( v = (x_U, x_{U\perp})H_U \). By assumption \( H_{U\cap U\perp} \) is invertible and by using (1) we obtain then \( x_U = v G^T_U \) \( (G_UG_U^T)^{-1} \) and \( x_{U\perp} = v H^T_U \) \( (H_UH_U^T)^{-1} \).

Note that \( \Sigma_U \) only depends on the code \( U \) and not on a particular generator matrix of \( U \) since for any invertible matrix \( S \) it holds that
\[
(SG_U)^T (SG_U (SG_U)^T)^{-1} (SG_U) = \Sigma_U
\]
Moreover \( \Sigma_U \) (and \( \Sigma_{U\perp} \)) satisfies the relations \( \text{Im} (\Sigma_U) = U \) (and consequently \( \text{rank}\Sigma_U = \dim U \)), \( \Sigma^2_U = \Sigma_U, \Sigma_U + \Sigma_{U\perp} = I_n \) and \( \Sigma_U \Sigma_{U\perp} = 0 \) and \( \Sigma_U = \Sigma_U^T \). Finally because \( \Sigma_U \) is symmetric, we interpret it as an adjacency matrix of a weighted undirected graph.

Definition 4. Let \( U \subset \mathbb{F}^n \) be a linear code with a trivial hull. The weighted graph \( \mathcal{G}_U \) associated to \( U \) is the graph defined by the adjacency matrix \( \Sigma_U \).

Theorem 5. Let \( A \) and \( B \) be two linear codes of length \( n \) over a field \( \mathbb{F} \) having both a trivial hull. Then \( B \sim_{\pi} A \) for some permutation \( \pi \) in \( S_n \) if and only if \( \mathcal{G}_B \sim_{\pi} \mathcal{G}_A \).

Proof. Let assume that there exists a permutation matrix such that \( B = AX \). Since \( B \) has a trivial hull the square matrix \( \Sigma_B \) exist and one has
\[
\Sigma_B = (G_A X)^T \left( G_A X (G_A X)^T \right)^{-1} (G_A X)
\]
\[
= X^T G_A^T \left( G_A X X^T G_A^T \right)^{-1} G_A X
\]
Because \( XX^T = I_n \) we have \( \Sigma_B = X^T \Sigma_A X \). Consequently \( \mathcal{G}_B \) is isomorphic to \( \mathcal{G}_A \) by means of \( X \). Reciprocally let us assume that there exists a permutation matrix \( X \) such that it holds
\[
\left\{
\begin{aligned}
\Sigma_B &= X^T \Sigma_A X, \\
\Sigma_{B\perp} &= X^T \Sigma_{A\perp} X.
\end{aligned}
\right.
\]
By definition of \( \Sigma_A \) and \( \Sigma_B \), we know that \( \text{Im} (\Sigma_A) = A \) and \( \text{Im} (\Sigma_B) = B \). From (6) we also have \( \text{Im} (\Sigma_B) = \text{Im} (\Sigma_A) \) \( X \), which proves that \( B \) and \( A \) are permutation equivalent.

Note that because \( B\perp \) and \( A\perp \) are permutation equivalent through the same permutation \( X \), we also have that \( \mathcal{G}_{B\perp} \) and \( \mathcal{G}_{A\perp} \) are isomorphic. We are now able to state the main result of the paper.

Theorem 6. There is a deterministic reduction running in \( O(n^2) \) time from deciding if two codes with trivial hulls over a field \( \mathbb{F} \) and length \( n \) are permutation equivalent to deciding whether two weighted undirected graphs having \( n \) vertices and weights in \( \mathbb{F} \) are isomorphic.

Proof. The reduction comes from Theorem 5 which in turn bowls down to computing \( \Sigma_A \) and \( \Sigma_B \). This can be done in \( O(n^2) \) operations.

Remark 7. We gathered in Table 1 the time to solve the permutation equivalence for binary linear codes of length \( n \) and dimension \( n/2 \) with trivial hulls.

4. CODES WITH NON-TRIVIAL HULLS

We present two approaches to extend the reduction given in Theorem 5 to codes with non trivial hulls. The first approach applies to codes defined over a field \( \mathbb{F} = \mathbb{F}^m \) with \( m > 1 \) and \( p \) is a prime number as long as they have a trivial hull with respect to a Hermitian product. The second approach deals with the remaining cases, in particular codes defined over a prime field \( \mathbb{F}_p \). In the sequel we explain in more details these two methods.
Table 1. Time in seconds to solve the permutation equivalence for randomly drawn binary codes with a trivial hull of length \( n \) and dimension \( n/2 \). Results were obtained with Magma on MacBookPro, Intel Core i7, 2.5 GHz, 16GB ram (version 2.6r7 nauty and Traces packages).

| \( n \) | \( \Sigma_A \) | \( \text{Gl}(n) \) |
|---|---|---|
| 1,000 | 0.06 | 0.01 |
| 5,000 | 1.6 | 0.11 |
| 10,000 | 6.5 | 0.5 |
| 20,000 | 31.1 | 1.8 |
| 30,000 | 81.1 | 4.1 |
| 40,000 | 153 | 7.5 |
| 50,000 | 283 | 11.9 |

4.1. Hermitian inner product. In this part we consider codes that are defined over \( \mathbb{F} = \mathbb{F}_p^m \) with \( m > 1 \) and \( p \) is a prime number. The idea is to consider a Hermitian inner product \( \langle x, y \rangle_\theta \triangleq \sum_{i=1}^n x_i \theta(y_i) \) where \( \theta \) is an automorphism of \( \mathbb{F}_p^m \). We recall that such automorphisms are of the form \( z \mapsto z^e \) where \( e \) lies in \([0, m - 1]\). From now on, we consider an automorphism \( \theta \). We define in the same manner the orthogonal \( U^\perp_\theta \) of a set \( U \subset \mathbb{F}_p^n \) as the set of vectors that are orthogonal to \( U \) with respect to \( \langle \cdot, \cdot \rangle_\theta \). Furthermore for any permutation \( \pi \in \Sigma_n \) and for any \( x, y \) in \( \mathbb{F}_p^n \), we have

\[
\langle x^\pi, y^\pi \rangle_\theta = \langle x, y \rangle_\theta,
\]

Hence \( A \sim_\pi B \) if and only if \( A^\perp_\theta \sim_\pi B^\perp_\theta \) for any linear codes \( A \subset \mathbb{F}_p^n \) and \( B \subset \mathbb{F}_p^n \). Following (3) we define \( \Sigma_U^{(\theta)} \) for a trivial-hull linear code \( U \subset \mathbb{F}_p^n \) with respect to the hermitian inner product as

\[
\Sigma_U^{(\theta)} \triangleq \theta(G_U^T) \left( G_U \theta(G_U^T) \right)^{-1} G_U,
\]

with the convention that \( \theta(Z) \triangleq [\theta(z_{i,j})] \) for any matrix \( Z = [z_{i,j}] \). It is not difficult to see that \( \Sigma_U^{(\theta)} \) does not depend on \( G_U \) and it is possible to define a graph \( G_U^{(\theta)} \) with respect to \( \Sigma_U^{(\theta)} \) as in Definition 4. Furthermore since for any matrix \( X \) with binary entries we have \( \theta(X) = X \), we are able to generalize Theorem 3.

**Proposition 8.** Let \( A \) and \( B \) be two linear codes of length \( n \) over a field \( \mathbb{F}_p^m \) having both a trivial hull with respect to the hermitian product \( \langle \cdot, \cdot \rangle_\theta \) where \( \theta \) is any automorphism of \( \mathbb{F}_p^m \). Then \( B \sim_\pi A \) for some permutation \( \pi \) in \( \Sigma_n \) if and only if \( G_B^{(\theta)} \sim_\pi G_A^{(\theta)} \).

4.2. General case. The result obtained in Proposition 8 permits us to solve the permutation code equivalence in more cases, especially when \( m \) is very large. The remaining unsolved cases are therefore codes defined over a prime field \( \mathbb{F}_p \), and those defined over \( \mathbb{F}_p^m \) with \( m > 1 \) that have non trivial hulls with respect to any hermitian inner product over \( \mathbb{F}_p^m \). We develop a general approach based on testing the equivalence between shortened codes. We recall that the shortened code of \( U \) over \( \mathcal{I} \subset [1,n] \) is defined as

\[
\mathcal{S}_\mathcal{I}(U) \triangleq \left\{ u \in U : \forall i \in \mathcal{I}, \ u_i = 0 \right\}. \quad (7)
\]

Moreover, a set \( \mathcal{I} \subset [1,n] \) is an information set for a linear code \( U \) if there exists a generator matrix such that its restriction to the columns that belong to \( \mathcal{I} \) is the identity matrix. We have then the following property.

**Proposition 9.** If \( \mathcal{I} \subset [1,n] \) is an information set for \( U \cap U^\perp \) then \( \mathcal{S}_\mathcal{I}(U) \) has a trivial hull.
We now describe a method for testing if two linear codes $A$ and $B$ with this issue is to observe that for all $\ell \in [1, n] \setminus I$ the codes $S_{\pi(\ell)} \cup I \setminus \{\ell\} \setminus \{\pi(i)\} (U)$ and $S_{\pi(\ell)} \cup I \setminus \{\pi(i)\} (U^\pi)$ are equivalent for each $i \in I$. Note that the recourse to an integer $\ell \notin I$ is necessary because our test requires to have codes shortened on sets of size $\dim U \cap U^\perp$.

We now describe a method for testing if two linear codes $A$ and $B$ of length $n$ both having hulls of dimension $h$ are equivalent or not. Firstly we fix a set $I \subset [1, n]$ and an integer $\ell \in [1, n] \setminus I$ such that $S_I (A)$ has a trivial hull. Next, we search for a set $J \subset [1, n]$ of cardinality $h$ such that $S_J (B)$ has a trivial hull, an integer $\ell' \in [1, n] \setminus J$ and a permutation $\gamma : I \rightarrow J$ such that the following holds

1. $S_I (A)$ and $S_J (B)$ are equivalent which can be decided in $O((n-h)^w \text{Gl}(n-h))$ operations.

2. $S_I (A)$ punctured at $\ell$ is equivalent to $S_J (B)$ punctured at $\ell'$. This step can be done with at most $O((n-1-h)^w \text{Gl}(n-1-h))$ operations.

3. For $i \in I$, $S_{\{\ell\} \cup I \setminus \{i\}} (A)$ and $S_{\{\ell'\} \cup J \setminus \{\gamma(i)\}} (B)$ are equivalent. This step requires $O((n-h)^w \text{Gl}(n-h))$ operations for each $i$.

When $A$ and $B$ are equivalent, this procedure will necessarily find the quantities $J$, $\ell'$ and $\gamma$, unlike the case where $A$ and $B$ are inequivalent codes. The (worst-case) cost of this approach is therefore $O\left(\binom{n}{h} h! (n-h)^{\omega+1} (h+2) \text{Gl}(n-h)\right)$ operations, which asymptotically gives $O\left(h^n \omega^{h+1} \text{Gl}(n)\right)$.

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