AN UPPER BOUND FOR THE HEIGHT FOR REGULAR AFFINE AUTOMORPHISMS OF $\mathbb{A}^n$

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Abstract. In [6], Kawaguchi proved a lower bound for height of $h(f(P))$ when $f$ is a regular affine automorphism of $\mathbb{A}^2$, and he conjectured that a similar estimate is also true for regular affine automorphisms of $\mathbb{A}^n$ for $n \geq 3$. In this paper we prove Kawaguchi’s conjecture. This implies that Kawaguchi’s theory of canonical heights for regular affine automorphisms of projective space is true in all dimensions.

1. Introduction

Let $\zeta$ be a rational map on $\mathbb{P}^n$ and $Z(\zeta)$ be the indeterminacy locus of $\zeta$. We will say that a family of rational maps $\{\zeta_i\}$ is jointly regular if $\bigcap Z(\zeta_i)$ is empty. Silverman [14] proved the following result for jointly regular maps.

Theorem 1.1. If $\zeta_1, \cdots, \zeta_m : \mathbb{P}^n \rightarrow \mathbb{P}^n$ is a family of jointly regular maps, then

$$\sum_{i=1}^{m} \frac{1}{d_i} h(\zeta_i(P)) \geq h(P) - C,$$

where $d_i$ be the degree of $\zeta_i$, and the constant $C$ is independent of the point $P$.

In special cases, we can improve the bound. Suppose that

$$f : \mathbb{A}^n \rightarrow \mathbb{A}^n$$

is an affine automorphism with inverse function $f^{-1}$. Since $\mathbb{A}^n$ is a dense subset of $\mathbb{P}^n$, we can find $\phi_0$ and $\psi_0$, rational functions that extend $f$ and $f^{-1}$,

$$\mathbb{P}^n \xleftarrow{\psi_0} \mathbb{P}^n \xrightarrow{\phi_0} \mathbb{P}^n \quad \Downarrow \quad \Downarrow$$

$$\mathbb{A}^n \xleftarrow{f^{-1}} \mathbb{A}^n \xrightarrow{f} \mathbb{A}^n$$

We say $f$ is a regular affine automorphism if $\{\phi_0, \psi_0\}$ is jointly regular. Kawaguchi showed in [6] that Silverman’s result can be improved when we have a regular affine automorphism of $\mathbb{A}^2$.

Theorem 1.2. Let $f$ be a regular affine automorphism on $\mathbb{A}^2$. Then,

$$\frac{1}{\deg f} h(f(P)) + \frac{1}{\deg f^{-1}} h(f^{-1}(P)) > \left(1 + \frac{1}{\deg f \deg f^{-1}}\right) h(P) + C$$
Kawaguchi conjectured that Theorem 1.2 is true for regular affine automorphisms of $\mathbb{A}^n$ for all $n \geq 2$. In this paper we prove Kawaguchi’s conjecture. From now on, we will let $H$ be the hyperplane at infinity, $f$ and $f^{-1}$ an affine automorphism and its inverse, and $\phi_0$ and $\psi_0$ morphisms that are meromorphic extensions of $f$ and $f^{-1}$. By $Z(\zeta)$ we will mean the indeterminacy locus of the rational map $\zeta$.

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2. Resolving Indeterminacy and the Essential Divisor of $\phi$

As a standard application of blowing up subschemes, we know that for any rational map $f : \mathbb{P}^n \to \mathbb{P}^n$, there is a blowup $V$ of $\mathbb{P}^n$ and a birational morphism $\pi : V \to \mathbb{P}^n$ such that $f \circ \pi$ is a morphism. In general, not all birational morphisms are decomposed into monoidal transformations (blowing up along closed subvarieties). Fortunately, we have Hironaka’s theorem on resolution of indeterminacy.

Theorem 2.1. Let $\zeta : V \to W$ be a rational map between smooth proper varieties. Then there is a sequence of varieties $V_i$ such that:

1. $V_i$ is a blowup of $V_{i-1}$ along a smooth irreducible subvariety.
2. The meromorphic extension $\zeta_r$ of $\zeta$ on $V_r$ is a morphism.

\begin{center}
\begin{tikzpicture}
  \node (V) {$V_r$};
  \node (V0) [below left of=V] {$V$};
  \node (W) [above right of=V] {$W$};
  \node (E) [below right of=V] {$E$};
  \draw[->] (V) to node [above] {$\pi$} (W);
  \draw[->] (V0) to node [above] {$\phi_0$} (V);
  \draw[->] (V) to node [left] {$\zeta_r$} (E);
  \draw[->] (V0) to node [left] {$\zeta$} (E);
\end{tikzpicture}
\end{center}

Proof. [4, Main Theorem II, Corollary 3] \qed

Definition 2.2. We call a birational map $\pi : X \to Y$ a monoidal transformation if $X$ is a blowup of $Y$ whose center is a subvariety.

Definition 2.3. Let $\pi : W \to V$ be a birational morphism with center the scheme whose ideal sheaf is $\mathfrak{I}$, and let $D$ be an irreducible divisor on $V$. We define the proper transformation of $D$ to be $\pi^{-1}(D \cap U)$, where $U = V \setminus Z(\mathfrak{I})$, and where $Z(\mathfrak{I})$ is the underlying subvariety that is the zero set of the ideal $\mathfrak{I}$.

We now assume that $V_0, \ldots, V_k$ is a sequence of varieties as in Theorem 2.1 which resolves the indeterminacy of $\phi_0$. We let $V = V_k$, we write $\pi_V : V \to \mathbb{P}^n$ for the birational morphism that is a composition of monoidal transformations, and we let $\phi = \phi_0 \circ \pi_V : V \to \mathbb{P}^n$ be the morphism extending $\phi_0$. We prove a lemma describing the Picard group of $V$.

Lemma 2.4. With notation as above, let $H_V \in \text{Div}(V)$ be the proper transform in $V$ of $H$, and for each $1 \leq i \leq k$, let $E_i \in \text{Div}(V)$ be the proper transform in $V$ of the exceptional divisor of the monoidal transformation $V_i \to V_{i-1}$. Then

$$\text{Pic}(V) = \mathbb{Z}H_V \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \cdots \oplus \mathbb{Z}E_k.$$ 

In other words, $\text{Pic}(V)$ is a free abelian group of rank $k + 1$ generated by $H_V$ and $E_1, \ldots, E_k$. 

Proof. Let $\pi : \tilde{X} \to X$ be the blowup of a smooth variety along a smooth subvariety and let $E$ denotes the exceptional divisor of the blowup. Then it is well known that $\text{Pic}(\tilde{X}) = \pi^* \text{Pic}(X) \oplus \mathbb{Z}E$; see for example [3, II.8.Ex 5]. Applying this fact repeatedly, starting from $V_0 = \mathbb{P}^n$ and $\text{Pic}(V_0) = \mathbb{Z}H$, an easy induction gives the desired result. □

Similarly, for $f^{-1}$, we let $W = W_j$ be a variety which resolves the indeterminacy of $\psi_0$, we let $\pi_W : W \to \mathbb{P}^n$ be a birational morphism that is a composition of monoidal transformations, we write $\psi = \psi_0 \circ \pi_W : W \to \mathbb{P}^n$ for the morphism extending $\psi$, and we let

$$ \text{Pic}(W) = \mathbb{Z}H_W \oplus \mathbb{Z}F_1 \oplus \mathbb{Z}F_2 \oplus \cdots \oplus \mathbb{Z}F_l, $$

where $H_W \in \text{Div}(W)$ is the proper transform in $W$ of $H$, and for each $1 \leq j \leq l$, let $F_j \in \text{Div}(W)$ be the proper transform in $W$ of the exceptional divisor of the monoidal transformation $W_j \to W_{j-1}$.

This is summarized in the following commutative diagrams.

Here are some lemmas which will help us to define the essential divisor and to prove Kawaguchi’s conjecture in section 3.

Lemma 2.5.

$$ \phi_0(H \setminus Z(\phi_0)) \subset Z(\psi_0). $$

Proof. Let

$$ \phi_0 = (X_0^d, F_1, \cdots, F_n), \quad \psi_0 = (X_0^c, G_1, \cdots, G_n). $$

Then using the fact that $\phi_0$ and $\psi_0$ are inverse maps, we have

$$ \psi_0 \circ \phi_0 = \left( X_0^{de}, G_1 X_0^{d}, F_1, \cdots, F_n, \cdots \right) = \left( X_0^{de}, X_0^{de-1} X_1, \cdots, X_0^{de-1} X_n \right). $$

Now let $P = [0, x_1, \cdots, x_n] \in H \setminus Z(\phi_0)$. Then,

$$ \psi_0 (\phi_0(P)) = (0, G_1(\phi_0(P)), \cdots) = \left( 0^{de}, 0^{de-1} X_1, \cdots, 0^{de-1} X_n \right), $$

and hence $\phi_0(P) \in Z(\psi_0)$. □

Lemma 2.6.

$$ \phi_0^* \phi_0^* H = H. $$

Proof. For this proof, we let $H$, $H_V$ and $E_i$ be specific closed subvarieties of codimension 1, not linear equivalence classes. Let $C_i$ are the image of $E_i$ by $\phi$. Since $\phi$ is an automorphism on

$$ \mathbb{A}^n \cong V \setminus \left( H_V \cup \left( \bigcup_{i=1}^{k} E_i \right) \right) \cong \mathbb{P}^n \setminus H, $$

the $C_i$ are algebraic subsets of $H$. So we can choose a hyperplane $H'$ which does not contain any of the $C_i$. Let $H'^{\psi}$ be the preimage of $H'$ by $\phi$. Then, $H'^{\psi}$ does not contain any of $E_i$. It also
means that the \( H^i \cap E \) always has codimension larger than 1, since \( E \) is irreducible. And the codimension of \( H^i \cap H_V \) is also larger than 1. So,

\[
\phi \left( H^i \cap \left( H_V \cup \bigcup_{i=1}^{k} E_i \right) \right).
\]

is a closed cycle of codimension larger than 1, since \( \phi_* \) is a graded group homomorphism. Therefore,

\[
\phi_* (H^i) = [K(H^i) : K(\phi(H^i))] \cdot \phi(H^i)
\]

Moreover, since \( \phi \) is one-to-one outside of \( H_V \) and \( E_i \),

\[
[K(H^i) : K(\phi(H^i))] = 1 \quad \text{and} \quad \phi \left( H^i \cap \left( H_V \cup \bigcup_{i=1}^{k} E_i \right) \right) = H' \setminus H.
\]

Therefore, \( \phi_* \phi^* H' = H' \), and hence \( \phi_* \phi^* H = \phi_* \phi^* H' = H' = H. \)

Lemma 2.7.

\[ \phi(H_V) \subset H \quad \text{and} \quad \phi(E_i) \subset H. \]

Proof. Suppose that there is a point \( Q \in H_V \cup (\bigcup_i E_i) \) such that \( \phi(Q) \in \mathbb{A}^n = \mathbb{P}^n \setminus H \). Then, since \( Z(\psi_0) \subset H, \phi(Q) \notin Z(\psi_0) \), and hence \( \psi_0(\phi(Q)) \in \mathbb{A}^n \).

We now consider some rational maps. Let \( \phi : V \dashrightarrow W \) and \( \psi : W \dashrightarrow V \) be rational maps that extend \( \phi \) and \( \psi \), and let \( \zeta \) be the composition of \( \psi \) and \( \phi \).

It is clear that \( \psi_0 \circ \phi = \pi_V \circ \zeta \) where they are defined. Further, \( \zeta : V \dashrightarrow V \) is a rational map which satisfies

\[
(\pi_V \circ \zeta \circ \pi_V^{-1}) \big|_{\mathbb{A}^n} = f^{-1} \circ f = id_{\mathbb{A}^n}.
\]

But we have a trivial extended morphism \( id_V : V \rightarrow V \) which also satisfies

\[
(\pi_V \circ id_V \circ \pi_V^{-1}) \big|_{\mathbb{A}^n} = id_{\mathbb{A}^n},
\]

so that \( \zeta = id_V \). Therefore,

\[
\psi_0 \circ \phi(Q) = \pi_V \circ \zeta(Q) = \pi_V(Q) \in H,
\]

which contradicts the assumption. \( \square \)
Lemma 2.8. Write \( \phi^*H \) in terms of the basis for \( \text{Pic}(V) \), say

\[
\phi^*H = b_0H_V + \sum_{i=1}^{k} b_iE_i.
\]

Then \( b_i > 0 \) for all \( i \).

Proof. Let \( u \) be a uniformizer for the hyperplane \( H \) at infinity,

\[
u(x_0, \ldots, x_n) = x_0.
\]

Then by definition,

\[
\phi^*H = \text{ord}_{H_V}(u \circ \phi) \cdot H_V + \sum \text{ord}_{E_i}(u \circ \phi) \cdot E_i.
\]

Furthermore, \( u = 0 \) on \( H \) because it is a uniformizer of \( H \), while \( \phi(E_i) \subset H \) and \( \phi(H_V) \subset H \) by Lemma 2.7. Therefore, \( u \circ \phi = 0 \) on \( H_V \) and on \( E_i \), and hence \( \text{ord}_{H_V}(u \circ \phi) \geq 1 \) and \( \text{ord}_{E_i}(u \circ \phi) \geq 1 \). \( \square \)

Theorem 2.9. Write

\[
\phi^*H = b_0H_V + \sum_{i=1}^{k} b_iE_i
\]

as in Lemma 2.8. Then there exists a unique index \( t \neq 0 \) such that

\[
b_t = 1, \quad \phi_*E_t = H \quad \text{and} \quad \phi_*E_j = 0 \quad \text{for all} \quad j \neq t.
\]

Proof. It is clear that \( \phi_*H_V \) and \( \phi_*E_i \) are nonnegative multiples of \( H \), since \( \text{Pic}(\mathbb{P}^n) = \mathbb{Z} \). So we may write \( \phi_*E_i = s_iH \) with \( s_i \geq 0 \). For any irreducible divisor \( D \), the definition of \( \phi_*D \) is

\[
[K(D) : K(\phi(D))] \cdot \phi(D).
\]

From this definition, we get

\[
(1) \quad \phi_*H_V = 0,
\]

because \( \phi_0(H \setminus Z(\phi_0)) \subset Z(\psi_0) \) from Lemma 2.5, and \( Z(\psi_0) \) has codimension greater than 1. Furthermore, since \( \phi_*\phi^*H = H \) from Lemma 2.6, we have

\[
H = \phi_*\phi^*H = \phi_*\left( aH_V + \sum_{i=1}^{k} b_iE_i \right) = \sum_{i=1}^{k} b_i\phi_*E_i = \sum_{i=1}^{k} b_i s_i H.
\]

Lemma 2.8 says that \( b_j > 0 \) for all \( j \), and also \( s_i \geq 0 \) for all \( i \). Therefore there is exactly one \( t \) satisfying \( s_t b_t = 1 \), and \( s_j b_j = 0 \) for all \( j \neq t \). Then the fact that every \( b_j > 0 \) implies that \( s_j = 0 \) for all \( j \neq t \). Finally, since \( s_t \) and \( b_t \) are non-negative integers, the equality \( s_t b_t = 1 \) implies that \( s_t = b_t = 1 \). \( \square \)

Definition 2.10. Let \( E_t \) be the unique exceptional divisor satisfying \( \phi_*E_t = H \) as described in Theorem 2.9. We call \( E_t \) the essential exceptional divisor of \( \phi \).
3. Proof of Kawaguchi’s Conjecture

**Theorem 3.1.** Let \( f \) be a regular affine automorphism of \( \mathbb{A}^n \). Then there is a constant \( C = C(f) \) such that for all \( P \in \mathbb{A}^n \),

\[
\frac{1}{\deg f} h(f(P)) + \frac{1}{\deg f^{-1}} h(f^{-1}(P)) \geq \left(1 + \frac{1}{\deg f \deg f^{-1}}\right) h(P) + C.
\]

We recall an important definition.

**Definition 3.2.** Let \( D \in \text{Pic}(X) \). We say that \( D \) is numerically effective if

\[
D^r \cdot Y \geq 0
\]

for all \( r \geq 1 \) and all integral subschemes \( Y \subset X \) of dimension \( r \).

**Proposition 3.3.** Let \( \zeta : X \to X' \) be a morphism of projective varieties, and let \( D \in \text{Pic}(X') \) be numerically effective. Then \( \zeta^* D \) is also numerically effective.

**Proof.** This is a standard result, see for example [9, Example 1.4.4(i)], but for the readers conveience, we provide the short proof. From the projection property of intersection,

\[
(\zeta^* D)^r \cdot Y' = \zeta^* D \cdot ( (\zeta^* D)^{r-1} \cdot Y') = \zeta^* [D \cdot \zeta_* \{(\zeta^* D)^{r-1} \cdot Y'] = \cdots = D^r \cdot \zeta_* Y'.
\]

Since \( \dim(\zeta_* Y') = r \) and \( D \) is numerically effective, this last intersection is nonnegative. \( \Box \)

An important ingredient in the proof of Theorem 3.1 is a result that describes the pullback of the pushforward of a numerically effective divisor.

**Lemma 3.4.** Let \( \rho : \tilde{X} \to X \) be a birational morphism and let \( Z \) be an divisor on \( \tilde{X} \) that is both effective and numerically effective. Then \( \rho^* \rho_* Z \geq Z \).

**Proof.** Let \( Z \) be an effective and numerically effective divisor, and let

\[
\text{Pic}(V) = \langle D_1, \cdots, D_r \rangle
\]

be a generating set for \( \text{Pic}(V) \), where the \( D_i \) are effective divisors of \( V \). Then we can find a generating set

\[
\text{Pic}(W) = \langle D_1^\#, \cdots, D_r^\#, F_1, \cdots, F_s \rangle,
\]

where \( D_i^\# \) is the proper transformation of \( D_i \), and where the \( F_i \) are exceptional divisors of the birational morphism \( \rho \). We write the pullback of \( D_i \) as

\[
\rho^* D_i = D_i^\# + \sum_j m_{ij} F_j.
\]

This allows us to write the given divisor \( Z \) as

\[
Z = \sum_i a_i D_i^\# + \sum_j b_j F_j
\]

\[
= \sum_i a_i \left( \rho^* D_i + \sum_j m_{ij} F_j \right) + \sum_j b_j F_j
\]

\[
= \sum_i a_i \rho^* D_i + \sum_j f_j F_j,
\]
where each $f_j$ is some expression involving the $b_j$ and $m_{ij}$. Further, and this is a key point, the fact that $Z$ is numerically effective implies that $f_j \leq 0$ for all $j$. This follows from [7, Lemma 2.19].

We now compute
\[
\rho^* \rho_* Z = \sum_i a_i \rho^* \rho_* D_i + \sum_j f_j \rho^* \rho_* F_j
\]
\[
= \sum_i a_i \rho^* D_i \quad \text{since } \rho_* \rho^* D_i = D_i \text{ and } \rho_* F_j = 0.
\]
Hence
\[
\rho^* \rho_* Z - Z = -\sum_j f_j F_j = \sum_j (-f_j) F_j \geq 0
\]
is effective. □

We recall that we have the following diagram of maps,
\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & P^n \\
\downarrow \pi_V & & \downarrow \phi_0 \\
\mathbb{P}^n & \rightarrow & \mathbb{P}^n
\end{array}
\]

For notational convenience, we let
\[
d = \deg(f) \quad \text{and} \quad d' = \deg(f^{-1}).
\]

**Lemma 3.5.** Let $E_t$ be the essential divisor of $\phi$. Then the pull-backs of $H$ by $\pi_V$ and $\phi$ have the form
\[
\pi_V^* H = H_V + d'E_t + \sum_{i \neq t} a_i E_i, \quad \phi^* H = dH_V + E_t + \sum_{i \neq t} b_i E_i,
\]
where the coefficients of the non-essential exceptional divisors satisfy
\[
d'b_i \geq a_i \geq 0 \quad \text{for all } i \neq t.
\]

**Proof of Lemma 3.5.** Let
\[
\pi_V^* H = a_0 H_V + \sum_{i=1}^k a_i E_i \quad \text{and} \quad \phi^* H = b_0 H_V + \sum_{i=1}^k b_i E_i,
\]
Clearly $\pi_V^* \pi_V^* H = H$ by Lemma 2.6 and $\pi_V^* E_t = 0$ because $\pi_V(E_t) \subset Z(\phi_0)$. So,
\[
H = \pi_V^* \pi_V^* H = \pi_V^* \left( a_0 H_V + \sum_{i=1}^k a_i E_i \right) = \pi_V^* (a_0 H_V) = a_0 H,
\]
and hence $a_0 = 1$. And we have $b_t = 1$ by the definition of the essential exceptional divisor. Next we compute $b_0$ and $a_t$. Letting $u$ be a uniformizer at $H$, we have by definition
\[
\phi^* H = \ord_{H_V} (u \circ \phi) \cdot H_V + \sum_{i=1}^k \ord_{E_i} (u \circ \phi) \cdot E_i.
\]
Taking the push-forward of $\phi^* H$ by $\pi_V$, we get
\[
\pi_V^* \phi^* H = \ord_{H_V} (u \circ \phi) \cdot \pi_V^* H_V + \sum_{i=1}^k \ord_{E_i} (u \circ \phi) \cdot \pi_V^* E_i = \ord_{H_V} (u \circ \phi) \cdot H_V,
\]
since we showed
\[ \pi_{V*}H_V = H \quad \text{and} \quad \pi_{V*}E_i = 0. \]
Furthermore, because \( \phi = \phi_0 \circ \pi_V \) and \( \phi_0 = [x^d_0, f_1, \ldots, f_n] \), we have
\[ \ord_{H_V}(u \circ \phi) = \ord_H(u \circ \phi \circ \pi_V^{-1}) = \ord_H(u \circ \phi_0) = d. \]
On the other hand,
\[ \pi_{V*}\phi^*H = \pi_{V*} \left( b_0H_V + \sum_{i=1}^{k} b_iE_i \right) = b_0\pi_{V*}(H_V) + \sum_{i=1}^{k} b_i\pi_{V*}(E_i) = b_0H, \]
and hence \( b_0 = d. \)
Similarly,
\[ \pi_V^*H = \ord_{H_V}(u \circ \pi_V) \cdot H_V + \sum_{i=1}^{k} \ord_{E_i}(u \circ \pi_V) \cdot E_i, \]
and
\[ \phi_*\pi_V^*H = \ord_{H_V}(u \circ \pi_V) \cdot \phi_*H_V + \sum_{i=1}^{k} \ord_{E_i}(u \circ \pi_V) \cdot \phi_*E_i = \ord_{E_i}(u \circ \phi) \cdot H, \]
since \( \phi_*H_V = 0, \quad \phi_*E_t = H \quad \text{and} \quad \phi_*E_j = 0 \quad \text{for all} \quad j \neq t. \)
Furthermore, because
\[ \pi_V = \psi_0 \circ \phi \quad \ord_{E_i}(u \circ \phi) = b_V = 1, \quad \text{and} \quad \phi_0 = [x^d_0, g_1, \ldots, g_n], \]
we have
\[ \ord_{E_i}(u \circ \pi_V) = \ord_{E_i}(u \circ \psi_0 \circ \phi) = \ord_H(u \circ \psi_0) = d'. \]
On the other hand,
\[ \phi_*\pi_V^*H = \phi_* \left( a_0H_V + \sum_{i=1}^{k} a_iE_i \right) = a_0\phi_*H_V + \sum_{i=1}^{k} a_i\phi_*E_i = a_tH, \]
and hence \( a_t = d'. \)

It is clear that all of the \( a_i \) and \( b_i \) are non-negative, since \( H \) is effective and \( H_V \) and the \( E_i \) are the divisors whose support is contained in \( \pi_V^{-1}(H) \) and \( \phi^{-1}(H) \). It remains to prove the inequality \( d'b_i \geq a_i \). We apply Lemma 3.4 to the divisor \( \pi_V^*H \), which is both effective and numerically effective, and to the map \( \phi_V \). Lemma 3.4 tells us that
\[ \phi_V^*\phi_{V*}(\pi_V^*(H)) \geq \pi_V^*(H). \]
We also have
\[ \phi_{V*}\pi_V^*(H) = \phi_{V*} \left( H_V + d'E_i + \sum_{i \neq t} a_iE_i \right) \]
from above,
\[ = d'H \quad \text{from Theorem 2.9} \]
(Note that \( \phi_*E_i = 0 \) for \( i \neq t \) be the definition of the essential divisor, and we have already seen in (1) that \( \phi_{V*}H_V = 0. \)) Hence
\[ \phi_V^*(d'H) = \phi_V^*(\phi_{V*}\pi_V^*(H)) \geq \pi_V^*(H). \]
Using the expressions for $\phi_V^*(H)$ and $\pi_V^+(H)$ from above, this inequality gives

$$d' \left( dH_V + E_t + \sum_{i \neq t} a_i E_i \right) \geq H_V + d'E_t + \sum_{i \neq t} b_i E_i.$$ 

A little bit of algebra yields

$$(dd' - 1)H_V + \sum_{i \neq t} (d'a_i - b_i)E_i \geq 0.$$ 

Then the following lemma shows that $d'a_i \geq b_i$ for all $i \neq t$. □

Lemma 3.6. The divisor

$$c_0 H_V + \sum_{i \neq t} c_i E_i \in \text{Div}(V)$$

is linearly equivalent to an effective divisor if and only if $c_i \geq 0$ for all $i \neq t$.

Proof. One direction is obvious. For the other, suppose that

$$c_0 H_V + \sum_{i \neq t} c_i E_i \sim n_0 H_V + \sum_i n_i E_i + \sum_j m_j D_j,$$

where $n_i \geq 0$ and $m_j \geq 0$, and where the $D_j$ are irreducible divisors distinct from $H_V$ and the $E_i$. Note that the $D_j$ have the property that $\pi_s D_j$ has nontrivial intersection with $\mathbb{A}^n$, since $H_V$ is the proper transform of $H = \mathbb{P}^n \setminus \mathbb{A}^n$ and the $E_i$ are the exceptional divisors of the blowup $\pi : V \rightarrow \mathbb{P}^n$. Then the fact that $\phi_0$ is an automorphism of $\mathbb{A}^n$ implies that $\phi_*(D_j) \neq 0$. Hence there are positive integers $k_j$ such that $\phi_*(D_j) \sim k_j H$.

We know from before that $\phi_* H_V = 0$ and $\phi_* E_i = 0$ for $i \neq t$, and also $\phi_* E_t = H$. Therefore

$$0 = \phi_* \left( c_0 H_V + \sum_{i \neq t} c_i E_i \right) = n_t \phi_* E_t + \sum_j m_j \phi_* D_j \sim n_t H + \sum_j m_j k_j H.$$ 

Here $n_t \geq 0$, $m_j \geq 0$, and $k_j > 0$, and $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$, so we conclude that $n_t = 0$ and $m_j = 0$ for all $j$.

This proves that

$$c_0 H_V + \sum_{i \neq t} c_i E_i \sim n_0 H_V + \sum_{i \neq t} n_i E_i.$$ 

Lemma 2.4 says that $H_V$ and the $E_i$ are linearly independent in $\text{Pic}(V)$, so $c_i = n_i \geq 0$ for all $i$. □

Proof of Theorem 2.1. To ease notation, we let $E_V = E_t$ be the essential divisor, and we define

$$M_V = \sum_{i \neq t} a_i E_i \quad \text{and} \quad I_V = \sum_{i \neq t} b_i E_i.$$ 

Then Lemma 3.4 can be rewritten as

$$\pi_V^* H = H_V + d'E_V + M_V, \quad \phi^* H = dH_V + E_V + I_V,$$

where $M_V, I_V \geq 0$ and $d'I_V \geq M_V$. Further, the supports of $I_V$ and $M_V$ do not contain the support of $H_V$ or $E_V$. 

Now do the same with a blowup that resolves the indeterminacy of \( \psi_0 \),
\[
\begin{array}{ccc}
\psi & \downarrow \pi_W & \phi_W \\
\mathbb{P}^n & \psi_0 - & \mathbb{P}^n
\end{array}
\]

Since \( \psi_0 \) is the extension of the affine automorphism \( f^{-1} \) to \( \mathbb{P}^n \), we have using similar notation,
\[
\pi_W^* H = H_W + d'F_W + N_W, \quad \psi^* H = dH_W + E_W + J_W,
\]
with \( J_W, N_W \geq 0 \) and \( dJ_W \geq N_W \). Here \( F_W \) is the essential divisor of \( \psi \), and \( N_W \) and \( J_W \) are \( \mathbb{A}^n \)-effective divisors whose supports do not contain the support of \( H_W \) or \( F_W \).

Now, consider a blowup \( U \) of \( \mathbb{P}^n \) that resolves both \( \phi_0 \) and \( \psi_0 \). The assumption that \( f \) is regular implies that the centers of the blowups \( V \) and \( W \) are disjoint, so \( U \) is a blowup of \( \mathbb{P}^n \) whose center is the scheme-theoretic sum of the centers of the blowups \( V \) and \( W \). We have the following diagram:

\[
\begin{array}{ccc}
\psi & \downarrow \pi_W & \phi_W \\
\mathbb{P}^n & \psi_0 - & \mathbb{P}^n
\end{array}
\]

Let \( H_U \) be the proper transformation of \( H \) by \( \pi_U : U \to \mathbb{P}^n \), and let \( E, M, I, F, N, J \) be the proper transformations of \( E_V, M_V, I_V, F_W, N_W, J_W \), respectively. Then we have
\[
\begin{align*}
\pi_U^* H &= H_U + d'E + dF + M + N, \\
\phi_U^* H &= d(H_U + dF + N) + E + I, \\
\psi_U^* H &= d'(H_U + d'E + M) + F + J.
\end{align*}
\]

Finally, we compute the divisor
\[
D = \frac{1}{d} \phi_U^* H + \frac{1}{d'} \psi_U^* H - \left( 1 + \frac{1}{dd'} \right) \pi_U^* H
\]
\[
= (H_U + dF + N) + \frac{1}{d}(E + I) + (H_U + d'E + M) + \frac{1}{d'}(F + J)
\]
\[
- \left( 1 + \frac{1}{dd'} \right) (H_U + d'E + dF + M + N)
\]
\[
= \left( 1 - \frac{1}{dd'} \right) H_U + \frac{1}{dd'} (dJ - N + d'I - M).
\]

The fact that \( d'I > M \) and \( dJ > N \) implies that \( D \) is effective, which completes the proof of Theorem 3.1. \( \square \)

4. **Canonical Height Functions for Regular Affine Automorphism**

In Sections 4 and 5 of [6], Kawaguchi constructed canonical heights for regular affine automorphisms under the assumption that his height lower bound conjecture was true. Since we have proved his conjecture, his construction of canonical heights, as described in the following theorem, is now valid in all dimensions.
Theorem 4.1. Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be an affine automorphism of degree $d$ and $f^{-1}$ be its inverse map of degree $d'$. Define

$$\hat{h}_+(P) = \limsup \frac{1}{d} h(f^n(P)), \quad \hat{h}_-(P) = \limsup \frac{1}{d'} h(f^{-1})^n(P)$$

and

$$\hat{h}(P) = \hat{h}_+(P) - \hat{h}_-(P)$$

Then,

$$\frac{1}{d} \hat{h}(f(P)) + \frac{1}{d'} \hat{h}(f^{-1}(P)) = \left(1 + \frac{1}{dd'}\right) \hat{h}(P)$$

and $\hat{h}(P) = 0$ if and only if $P$ is $f$-periodic.

5. An Example

An example of a Hénon map on $\mathbb{A}^n$, which is the regular affine automorphism, is

$$f(x_1, \cdots, x_n) = (x_2, x_3 + x_2^2, x_4 + x_2^3, \cdots, x_n + x_{n-1}^2, x_1 + x_n^2).$$

On $\mathbb{A}^2$, properties of Hénon maps are well known, but less is known for Hénon maps on $\mathbb{A}^n$. In this section we describe what happens for a Hénon map of degree 2 on $\mathbb{A}^3$. Since the details of the proof are somewhat lengthy, we just give a sketch and refer the reader to [11] for details.

Let $f$ be a Hénon map defined as

$$(x, y, z) = (y, z + y^2, x + z^2).$$

It’s an affine automorphism with inverse

$$f^{-1} = (z - (y - x^2), x, y - x^2)$$

and meromorphic extensions

$$\phi_0([x, y, z, w]) = [yw, zw + y^2, xw + z^2, w^2],$$

$$\psi_0([x, y, z, w]) = [zw^3 - (yw - x^2), xw^3, yw^3 - x^2w^2, w^4].$$

The map $f$ is regular, since the indeterminacy loci are $Z(\phi_0) = \{[1, 0, 0, 0]\}$ and $Z(\psi_0) = \{[0, y, z, 0]\}$.

After a number of blow-ups along subvarieties of dimension zero and one, and using intersection theory to compute relevant coefficients, we find that the resolution of $f$ gives a map $\phi_V : V \to \mathbb{P}^3$ with

$$\pi_V^* H = H + E_1 + 2E_2 + 2E_3 + 4E_4 + 4E_5,$$

$$\phi_V^* H = 2H + E_1 + 2E_2 + E_3 + 2E_4 + E_5.$$ 

Further, $E_5$ is the essential exceptional divisor. Similarly, resolving $f^{-1}$ gives a map $\psi_W : W \to \mathbb{P}^3$ with

$$\pi_W^* (H) = H + F_1 + 2F_2 + 2F_3 + 2F_4 + F_5 + 2F_6 + 2F_7,$$

$$\psi_W^* (H) = 4H + 2F_1 + 4F_2 + 2F_3 + F_4 + F_5 + 2F_6 + F_7.$$ 

Combining the two resolutions to form a single variety that resolves both $f$ and $f^{-1}$ yields

$$\pi^* H = H + 4E_5 + 2F_7 + (E_1 + 2E_2 + 2E_3 + 4E_4) + (F_1 + 2F_2 + 2F_3 + F_5 + 2F_6 + 2F_7),$$

$$\phi^* H = 2(H + 2F_7 + (F_1 + 2F_2 + 2F_3 + F_5 + 2F_6 + 2F_7)) + E_5 + (E_1 + 2E_2 + E_3 + 2E_4),$$

$$\psi^* H = 4(H + 4E_5 + (E_1 + 2E_2 + 2E_3 + 4E_4)) + F_4 + (2F_1 + 4F_2 + 2F_3 + F_5 + 2F_6 + F_7),$$
and hence
\[
\frac{1}{2} \phi^* H + \frac{1}{4} \psi^* H - \left(1 + \frac{1}{8}\right) \pi^* H
= \frac{7}{8} H + \frac{3}{8} E_1 + \frac{3}{4} E_2 + \frac{1}{2} E_3 + \frac{1}{2} E_4 + 0 E_5 + \frac{3}{8} F_1 + \frac{3}{4} F_2 + \frac{1}{4} F_3 + 0 F_4 + \frac{1}{8} F_5 + \frac{1}{4} F_6 + 0 F_7
\geq 0.
\]

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