LINEAR PULLBACK COMPONENTS OF THE SPACE OF CODIMENSION ONE FOLIATIONS

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Abstract. The space of holomorphic foliations of codimension one and degree \( d \geq 2 \) in \( \mathbb{P}^n \) (\( n \geq 3 \)) has an irreducible component whose general element can be written as a pullback \( F^*F \), where \( F \) is a general foliation of degree \( d \) in \( \mathbb{P}^2 \) and \( \mathbb{F} : \mathbb{P}^n \rightarrow \mathbb{P}^2 \) is a general rational linear map. We give a polynomial formula for the degrees of such components.

Introduction

Codimension one holomorphic foliations in \( \mathbb{P}^n \) are defined by nonzero integrable twisted 1−forms, i.e., \( \omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2)) \) satisfying
\[
\omega \wedge d\omega = 0
\]
Since \( \omega, \lambda \omega \) yield the same foliation for any \( \lambda \in \mathbb{C}^* \), the space of such foliations is in fact a closed subscheme \( \mathbb{F}(d,n) \) of the projective space \( \mathbb{P}(H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2))) \) defined by the equations derived from (1).

Explicitly, these equations are as follows. Any element \( \omega \) of \( H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+2)) \) can be written \( \omega = A_0dZ_0 + \cdots + A_n dZ_n \), where \( A_i \) are homogeneous polynomials of degree \( d+1 \) in the variables \( Z_0, \ldots, Z_n \), such that \( A_0Z_0 + \cdots + A_nZ_n = 0 \). The integrability condition (1) imposes relations arising from
\[
A_i\left( \frac{\partial A_k}{\partial Z_j} - \frac{\partial A_j}{\partial Z_k} \right) + A_j\left( \frac{\partial A_i}{\partial Z_k} - \frac{\partial A_k}{\partial Z_i} \right) + A_k\left( \frac{\partial A_j}{\partial Z_i} - \frac{\partial A_i}{\partial Z_j} \right) = 0.
\]
with \( 0 \leq i < j < k \leq n \). These equations are quadratic in the coefficients of the polynomials \( A_i \). For \( n = 2 \) the space of foliations is a projective space: the integrability condition is automatically satisfied.

The geometry of the space of codimension one foliations in \( \mathbb{P}^n \) for \( n \geq 3 \) is a rich field of research. In particular the problem of describing the irreducible components of these spaces has received many contributions, cf. [2], [4], [5], [12] and references therein just to quote a few.

In [11] Jouanolou describes the irreducible components for the space of foliations of degrees \( d = 0, 1 \). We have that \( \mathbb{F}(0,n) \) is naturally isomorphic to the grassmanian of subspaces of codimension 2 in \( \mathbb{P}^n \). The space of foliations of degree \( d = 1 \) has two irreducible components.

For foliations of degree 2 in \( \mathbb{P}^n \), \( n \geq 3 \) Cerveau and Lins Neto [2] have shown that there are just 6 components. We recall some known components of \( \mathbb{F}(d,n) \) pinpointing those for which the degree has been found.

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1. Pull-back of projective 1-forms.

1.1. Projective 1-forms. The main reference for this material is [11]. A projective 1-form of degree $d$ in $\mathbb{P}^n$ is a global section of $\Omega^1_{\mathbb{P}^n}(d+2)$, for some $d \geq 0$.

We denote by $S_d$ the space $H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d)) = \text{Sym}_d(\mathbb{C}^{n+1})^\vee$ of homogeneous polynomials of degree $d$ in the variables $Z_0, \ldots, Z_n$. We write $\partial_i = \partial/\partial Z_i$, thought of as a vector field basis for $\mathbb{C}^{n+1}$. The dual basis will also be written as $dZ_0, \ldots, dZ_n$.
whenever we think of differential forms. Twisting the Euler exact sequence (cf. [10, Thm 8.13, p. 176]): we get
\[
0 \to \Omega_{\mathbb{P}^n}(d+2) \to \mathcal{O}_{\mathbb{P}^n}(d+1) \otimes S_1 \to \mathcal{O}_{\mathbb{P}^n}(d+2) \to 0.
\]

Taking global sections we find the exact sequence
\[
(2) \quad 0 \to V_d^p := H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1(d+2)) \to S_{d+1} \otimes S_1 \xrightarrow{i_{R_n}} S_{d+2} \to 0
\]
where \(i_{R_n}(\sum A_i dZ_i) = \sum A_i Z_i\) is the contraction by the radial vector field. Thus a 1-form \(\omega \in V_d^p\) can be written in homogeneous coordinates as
\[
\omega = A_0 dZ_0 + \cdots + A_n dZ_n
\]
where the \(A_i\)'s are homogeneous polynomials of degree \(d+1\) satisfying
\[
A_0 Z_0 + \cdots + A_n Z_n = 0.
\]

1.2. Linear Pullback. Let \(F : \mathbb{P}^n \dashrightarrow \mathbb{P}^2\) be a linear projection, i.e., \(F = [F_0 : F_1 : F_2]\), with \(F_i\) linearly independent homogeneous polynomials of degree 1. Pick \(\omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(d+2))\). Write \(\omega = B_0 dX_0 + B_1 dX_1 + B_2 dX_2\) where \(B_i \in S_{d+1}(\mathbb{R}^2)\). The pullback \(F^*(\omega)\) is the 1-form
\[
(3) \quad F^*(\omega) = F^* B_0 dF_0 + F^* B_1 dF_1 + F^* B_2 dF_2.
\]
A simple application of Euler relation shows that \(i_{R_n}(F^*(\omega)) = F^*(i_{R_2}(\omega)) = 0\).

On the other hand, as any projective 1-form in \(\mathbb{P}^2\) is integrable we have \(0 = F^*(\omega \wedge d\omega) = F^*(\omega) \wedge dF^*(\omega)\) i.e., \(F^*(\omega)\) is a projective integrable 1-form of degree \(d\),
\[
F^*(\omega) \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1((d+2)))\).
\]

1.3. Remark. For a fixed map \(F\) as above we obtain injective linear maps
\[
H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(d+2)) \longrightarrow H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1(d+2)),
\]
\[
\mathbb{F}(d,2) \longrightarrow \mathbb{F}(d,n).
\]

1.4. Parameter space for rational maps. Let \(F = [F_0 : F_1 : F_2]\) be a rational map as above, with \(F_i \in S_1\). Note that if we change the basis of the linear system \((F_0, F_1, F_2)\) then we obtain projectively equivalent pullbacks. So the natural parameter space for rational linear maps is the Grassmannian of dimension 3 subspaces of the space \(S_1\) of forms of degree 1,
\[
(4) \quad G := G(3, S_1).
\]

2. Linear Pullback Component

We show next that the locus in \(\mathbb{P}(V_d^p)\) (cf. 2) corresponding to codimension one foliations obtained by linear pullbacks of foliations in \(\mathbb{P}^2\) is the birational image of a natural projective bundle over the Grassmannian \(G(3, S_1)\).

2.1. Proposition. Notation as in (2) and (4), let
\[
\mathcal{V}_d := \{(F, \mu) \in G \times V_d^p \mid \mu = F^* \omega \text{ for some } \omega \in V_d^p\}.
\]

Then
(i) \(\mathcal{V}_d\) is a vector subbundle of \(G \times V_d^p\) of rank \((d+1)(d+3)\).

Let \(q_2 : \mathbb{P}(V_d) \subset G \times \mathbb{P}(V_d^p) \to \mathbb{P}(V_d^p)\) be the projection and set
\[
\mathcal{Y} := q_2(\mathbb{P}(V_d)), \quad g := \dim G = 3(n - 2).
\]

Then
(ii) the dimension of $Y$ is $g + (d + 1)(d + 3) - 1$ and
(iii) the degree of $Y$ is given by the Segre class $s_g(Y_d)$.

**Proof.** Consider the tautological exact sequence of vector bundles over $G$, (cf. 4)

\[ 0 \to \mathcal{T} \to G \times S_1 \to Q \to 0 \]
where $\mathcal{T}$ is of rank 3 with fiber $T_F = \langle F_0, F_1, F_2 \rangle, F_i \in S_1$. We obtain a natural rational map,

\[ G \times \mathbb{P}^n \xrightarrow{T} \mathbb{P}(\mathcal{T}^\vee) \]

We interpret the map $T$ as the family of $F_i$’s as in 1.2 and $\mathbb{P}(\mathcal{T}^\vee)$ as the family of $\mathbb{P}^2$’s obtained as the fibers of $\psi$. The relative cotangent bundle of $\psi$ fits into the relative Euler exact sequence (cf. \[6, B.5.8, p. 435\]):

\[ 0 \to \Omega^1_{\psi}(1) \to \psi^* T \to \mathcal{O}_{T^\vee}(1) \to 0. \]

Twisting by $\mathcal{O}_{T^\vee}(d + 1)$ we find

\[ 0 \to \Omega^1_{\psi}(d + 2) \to \psi^* T \to \mathcal{O}_{T^\vee}(d + 2) \to 0. \]

Taking direct image yields the following exact sequence over $G$

\[ 0 \to \psi_* \Omega^1_{\psi}(d + 2) \to \text{Sym}_{d+1} \mathcal{T} \otimes \mathcal{T} \to \text{Sym}_{d+2}(\mathcal{T}) \to 0. \]

Define

\[ V_d := \psi_* \Omega^1_{\psi}(d + 2). \]

The fiber of $V_d$ over each $F \in G$, is

\[ H^0(\mathbb{P}(T_F^\vee), \Omega^1_{\mathbb{P}(T_F^\vee)}(d + 2)) \]

i.e., the space of 1–forms defining foliations of degree $d$ in the varying $\mathbb{P}^2 \cong \mathbb{P}(T_F^\vee)$.

On the other hand, we obtain from (5) injective maps of vector bundles over $G$,

\[ \iota_1 : \text{Sym}_{d+1} \mathcal{T} \otimes \mathcal{T} \to S_{d+1} \otimes S_1 \]

and

\[ \iota_2 : \text{Sym}_{d+2} \mathcal{T} \to S_{d+2}. \]

These two maps fit into the following diagram of exact sequences

\[ \begin{array}{ccc}
\ker(a) = V_d & \xrightarrow{j} & \ker(b) = V_d^n \\
0 & \xrightarrow{a} & \text{Sym}_{d+1} \mathcal{T} \otimes \mathcal{T} \xrightarrow{\iota_1} S_{d+1} \otimes S_1 \\
0 & \xrightarrow{b} & \text{Sym}_{d+2} \mathcal{T} \xrightarrow{\iota_2} S_{d+2}. \\
\end{array} \]

In this way we obtain an injective map of vector bundles over $G$,

\[ j : V_d \to G \times V_d^n. \]

The vector subbundle $\mathcal{V}_d \subset G \times V_d^n$ is as stated in (i).
Let \( q_1 : \mathbb{P}(V_d) \rightarrow \mathbb{G} \) and \( q_2 : \mathbb{P}(V_d) \rightarrow \mathbb{P}(V^*_d) \) be the maps induced by projection:

\[
\begin{array}{c}
\mathbb{P}(V_d) \\
\mathbb{G} \\
\mathbb{Y} \subset \mathbb{P}(V^*_d)
\end{array}
\]

\( q_1 \quad q_2 \)

We prove in Lemma 2.3 below that \( q_2 : \mathbb{P}(V_d) \rightarrow \mathbb{P}(V^*_d) \) is generically injective. Set \( u := \dim \mathbb{Y} = \dim \mathbb{P}(V_d) \). Write \( H \) for the hyperplane class of \( \mathbb{P}(V^*_d) \). We have \( q_2^*H = c_1O_{V_d}(1) =: h \). Using [6, §3.1, p. 47, Prop.4.4, p. 83 and Ex. 8.3.14, p. 143], we may compute

\[
\deg \mathbb{Y} = \int_{\mathbb{P}(V^*_d)} H^n \cap \mathbb{Y} = \int_{\mathbb{P}(V_d)} h^u = \int_{\mathbb{G}} q_1(h^u) = \int_{\mathbb{G}} s(h(V_d)).
\]

\( \square \)

2.2. Remark. From sequence (6) we obtain \( s(V_d) = s(\text{Sym}_{d+1}(T \otimes T)c_1(\text{Sym}_{d+2}(T))) \).

2.3. Lemma. For general \( (F, \omega) \in \mathbb{G} \times H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(d + 2)) \), \( F^*(\omega) \) determines uniquely \( \omega \) and \( F \).

Proof. With notation as in 1.2, let \( I(F) = Z(F_0, F_1, F_2) \subset \mathbb{P}^n \) denote the indeterminacy locus of \( F \). For generic \( (F, \omega) \), the singular set of \( F^*(\omega) \) consists of linear components of codimension two of the form \( F^{-1}(q) \) where \( q \in \text{Sing}(\omega) \). These linear components intersect in \( I(F) \). Indeed, we can suppose \( q_1 = [0 : 0 : 1] \) and \( q_2 = [0 : 1 : 0] \), so \( F^{-1}(q_1) = Z(F_0, F_1) \) and \( F^{-1}(q_2) = Z(F_0, F_2) \). From \( I(F) \) we retrieve the linear system \( (F_0, F_1, F_2) \) i.e., the point \( F \in \mathbb{G} \).

On the other hand, consider the blow-up \( \pi : B \rightarrow \mathbb{P}^n \) of \( \mathbb{P}^n \) in \( I(F) \). We proceed to show that it is possible to recover the 1-form \( \omega \) from the strict transform \( \omega^* \) of \( \omega \). Indeed, recall \( B = \{(p, [x_0 : x_1 : x_2]) \in \mathbb{P}^n \times \mathbb{P}^2 \mid x_iF_j(p) - x_jF_i(p) = 0 \} \). Therefore over \( U := \{x_0 \neq 0 \} \) we have \( B_U = \{(p, [1 : t : s]) \mid F_1(p) = tF_0(p); F_2(p) = sF_0(p) \} \) and the equation of the exceptional divisor is \( F_0 = 0 \). In this chart we have \( dF_1 = F_0 dt + t dF_0, dF_2 = F_0 ds + sdF_0 \), therefore

\[
\pi^*F^*(\omega) = B_0(F_0, tF_0, sF_0) dF_0 + B_1(F_0, tF_0, sF_0) dF_1 + B_2(F_0, tF_0, sF_0) dF_2 =
\]

\[
F_0^{d+1}[B_0(1, t, s) dF_0 + B_1(1, t, s)(F_0 dt + t dF_0) + B_2(1, t, s)(F_0 ds + sdF_0)] =
\]

\[
F_0^{d+1}[(B_0(1, t, s) + tB_1(1, t, s) + sB_2(1, t, s))dF_0 + B_1(1, t, s)F_0 dt + B_2(1, t, s)F_0 ds]
\]

Recalling \( \omega \) is a projective 1-form, so \( B_0(1, t, s) + tB_1(1, t, s) + sB_2(1, t, s) = 0 \), we obtain

\[
\pi^*F^*(\omega) = F_0^{d+2}[B_1(1, t, s)dt + B_2(1, t, s)ds].
\]

Therefore the strict transform of \( F^*\omega \) is \( \bar{\omega} = B_1(1, t, s)dt + B_2(1, t, s)ds \). Ditto for the other local charts of the blowup. This shows that we may recover \( \omega \) from \( \bar{\omega} \).

\( \square \)

2.4. Corollary. The component \( \text{LPB}(d, n) \subset \mathcal{F}(d, n) \) is rational.
3. Computation of the degree of LPB(d,n)

3.1. Proposition. Notation as in 2.1, \( \deg(LPB(d,n)) \) is a polynomial in \( d \) of degree \( 3g = 9(n-2) \).

We need some preliminary results.

3.2. Lemma. Let \( E \) be a vector bundle of rank \( r \) on a variety \( X \). The \( k \)-Segre class

\[ s_k(\text{Sym}_d(E)) = \sum_{|\lambda|=k} p_\lambda(d) c_\lambda(E) \]

where \( p_\lambda(d) \) is a polynomial in \( d \) of degree \( \leq rk \), and there exists \( p_\lambda \) of degree \( rk \). Here the sum runs over the partitions of \( k \), and if \( \lambda = (\lambda_1, \ldots, \lambda_l) \) is a partition of \( k \), \( c_\lambda(E) := c_{\lambda_1}(E) \cdots c_{\lambda_l}(E) \).

Proof. First we prove the following assertion relating the Chern characters to the Segre classes of \( \text{Sym}_d(E) \).

3.3. Claim A(k): Assume the Chern character

\[ ch_j(\text{Sym}_d(E^\vee)) = \sum_{|\mu|=j} q_\mu(d)c_\mu(E) \]

where \( q_\mu(d) \) are polynomials in \( d \) of degree \( \leq r+j-1 \) for all \( j \geq 0 \). Then

\[ s_k(\text{Sym}_d(E)) = \sum_{|\lambda|=k} p_\lambda(d) c_\lambda(E) \]

where \( p_\lambda(d) \) are polynomials in \( d \) of degree \( \leq rk \). Moreover, if \( ch_1(\text{Sym}_d(E)) = q_1(d)c_1(E) \) and \( \deg(q_1) = r \), then \( s_k(\text{Sym}_d(E)) \) has a coefficient of degree \( rk \).

For a vector bundle \( F \) of rank \( r \), let \( x_1, \ldots, x_r \) be the Chern roots of \( F^\vee \). Then for all \( k \geq 0 \), the Segre class

\[ s_k(F) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq r} x_{i_1} \cdots x_{i_k} \]

is the \( k \)-complete symmetric function (cf. [7, p. 28]).

On the other hand the complete symmetric functions can be expressed in term of the power sum symmetric functions, \( p_k = \sum_i x_i^k \). We borrow from ([13, p. 25]) the explicit relations:

\[ s_k(F) = \sum_{|\lambda|=k} w_\lambda ch_\lambda(F^\vee) \]

where \( \lambda = (\lambda_1, \ldots, \lambda_l) \) is a partition of \( k \). Following the notation in [13] we write \( \lambda = (1^{m_1}, 2^{m_2}, \ldots) \) where \( m_i := \# \{ j \mid \lambda_j = i \} \) and \( w_\lambda = \prod_{i \geq 1} \frac{\lambda_i!}{i^{m_i} m_i!} \).

Write for short \( ch_j = ch_j(\text{Sym}_d(F^\vee)) \). Whenever the coefficients of \( ch_j \) are polynomials in \( d \) of degree \( m \) we will write \( \deg(ch_j) = m \), by abuse of notation.

We observe that \( \deg(ch_\lambda) = \deg(ch_{\lambda_1} \cdots ch_{\lambda_l}) \leq l(r-1)+k \leq rk \) and the equality holds if and only if \( l = k \), i.e. if \( \lambda = (1, \ldots, 1) \) in which case the coefficient of \( ch_k^k \) is \( \frac{1}{k!} \). In other words

\[ s_k(\text{Sym}_d(E)) = \frac{1}{k!} ch_k^k + \text{l.o.t} \]

Hence \( s_k(\text{Sym}_d(E)) \) is a linear combination of monomials in the Chern classes of \( E \) whose coefficients are polynomials in \( d \) of degree \( \leq rk \).

Next we prove the following claim by induction on \( r = \text{rank}(E) \) and on \( k \):
3.4. Claim $P(k)$: For $k \geq 0$, $ch_k(\text{Sym}_d(E)) = \sum_{|\mu|=k} q_\mu(d)c_\mu(E)$, where $q_\mu(d)$ are polynomials in $d$ of degree $\leq r + k - 1$. Moreover, $ch_1(\text{Sym}_d(E)) = q_1(d)c_1(E)$ where $q_1(d)$ is a polynomial of degree $r$.

For $r = 1$ we have $ch_k(\text{Sym}_d(E)) = \frac{1}{k!}d^kc_1(E)^k$.

Suppose that $P(k)$ is true for vector bundles of rank $r - 1$. Let $\pi : E \to X$ be a vector bundle of rank $\text{rank}(E) = r$ and $p : \mathbb{P}(E) \to X$ the induced projective bundle. For $k = 0$ we have $ch_0(\text{Sym}_d(E)) = \text{rank}(\text{Sym}_d(E)) = (d+r-1)$, a polynomial in $d$ of degree $r - 1$. Suppose that $k \geq 1$, and that $P(s)$ holds for $s < k$.

Over $\mathbb{P}(E)$ we have the tautological exact sequence:

$$(10) \quad 0 \to \mathcal{O}_E(-1) \to p^*E \to Q \to 0.$$

It induces the following exact sequence for $d \geq 1$

$0 \to \mathcal{O}_E(-1) \otimes \text{Sym}_{d-1}(p^*E) \to \text{Sym}_d(p^*E) \to \text{Sym}_d(Q) \to 0.$

Hence we may write the relation for the Chern characters

$$(11) \quad ch(\text{Sym}_d(p^*E)) = ch(\mathcal{O}_E(-1) \otimes \text{Sym}_{d-1}(p^*E)) + ch(\text{Sym}_d(Q)).$$

On the other hand, $ch(\mathcal{O}_E(-1) \otimes \text{Sym}_{d-1}(p^*E)) = ch(\mathcal{O}_E(-1))ch(\text{Sym}_{d-1}(p^*E))$. So each graded part satisfies

$$(12) \quad ch_k(\mathcal{O}_E(-1) \otimes \text{Sym}_{d-1}(p^*E)) = \sum_{i=0}^{k} \frac{\alpha_i}{i!}ch_{k-i}(\text{Sym}_{d-1}(p^*E))$$

where $\alpha := c_1(\mathcal{O}_E(-1))$. It follows from (11) and (12) that

$$ch_k(\text{Sym}_d(p^*E)) = \sum_{i=0}^{k} \frac{\alpha_i}{i!}ch_{k-i}(\text{Sym}_{d-1}(p^*E)) + ch_k(\text{Sym}_d(Q)).$$

Hence

$$(13) \quad ch_k(\text{Sym}_d(p^*E)) - ch_k(\text{Sym}_{d-1}(p^*E)) = \sum_{i=1}^{k} \frac{i}{i!}ch_{k-i}(\text{Sym}_{d-1}(p^*E)) + ch_k(\text{Sym}_d(Q)).$$

Observe that the right hand side of (13) involves:

- $ch_k(\text{Sym}_d(Q))$, which by induction, since $\text{rank}(Q) = r - 1$, is of the form $\sum_{|\mu|=k} q_\mu(d)c_\mu(Q)$ where $q_\mu(d)$ are polynomials in $d$ of degree $\leq r - 1 + k - 1 = r + k - 2$. Moreover, by (10) $c_r(Q) = c_r(p^*E) - \alpha c_{r-1}(p^*E)$. Thus $ch_k(\text{Sym}_d(Q))$ is a linear combination of monomials in the Chern classes of $p^*E$ and in $\alpha$ whose coefficients are polynomials in $d$ of degree $\leq r + k - 2$.

- $ch_s(\text{Sym}_{d-1}(p^*E))$ with $s < k$ which, by induction on $k$ is of the form $\sum_{|\mu|=k} q_\mu(d)c_\mu(p^*(E))$ where $q_\mu(d)$ are polynomials in $d$ of degree $\leq r + s - 1$; the maximal degree appearing is $r + k - 2$ (coming from the coefficients in $ach_{k-1}(\text{Sym}_{d-1}(p^*E))$).

Recall that the pullback $p^* : A_*(X) \to A_*(\mathbb{P}(E))$ is a monomorphism with left inverse $\beta \mapsto p_*(c_1(\mathcal{O}_E(1))^{r-1} \cap (\beta))$ ( [6, p. 49]). Applying this inverse to (13) we conclude that

$$ch_k(\text{Sym}_d(E)) - ch_k(\text{Sym}_{d-1}(E))$$

is a linear combination of monomials in the Chern classes of $E$ whose coefficients are polynomials in $d$ of degree $\leq r + k - 2$, and this implies that the coefficients in $ch_k(\text{Sym}_d(E))$ are polynomials in $d$ of degree $\leq r + k - 1$. 


Observe that for $k = 1$ we obtain that $\chi_1(\text{Sym}_d(E)) - \chi_1(\text{Sym}_{d-1}(E)) = (\frac{d+r-2}{r-1})c_1(E)$, a polynomial of degree $r - 1$. So $\chi_1(\text{Sym}_d(E)) = q_1(d)c_1(E)$ where $q_1(d)$ is polynomial of degree $r$.

Using A(k) 3.3, we deduce that $s_k(\text{Sym}_d(E))$ is a linear combination of monomials in the Chern classes of $E$ whose coefficients are polynomials in $d$ of degree $\leq \text{rank}(E)k$ and there exists a coefficient of degree $\text{rank}(E)k$.

□

Next we prove Proposition 3.1.

Proof. From the exact sequence

$$0 \to V_d \to \text{Sym}_{d+1} T \otimes T \to \text{Sym}_{d+2}(T) \to 0$$

we obtain the relation for Chern characters

$$\chi(V_d^\vee) = \chi(\text{Sym}_{d+1} T^\vee \otimes T^\vee) - \chi(\text{Sym}_{d+2}(T^\vee)).$$

Thus for each $k \geq 1$ we have

$$\chi_k(V_d^\vee) = \sum_{i=0}^{k} \chi_{k-i}(\text{Sym}_{d+1} T^\vee)\chi_i(T^\vee) - \chi_k(\text{Sym}_{d+2}(T^\vee)).$$

By assertion P(k) (3.4) $\chi_k(\text{Sym}_d(T^\vee))$ is a polynomial in $d$ of degree $\leq k + 2$. Therefore $\chi_k(V_d^\vee)$ is a polynomial in $d$ of degree $\leq k + 2$. Moreover,

$$\chi_1(\text{Sym}_d(T^\vee)) = q_1(d)c_1(T^\vee)$$

where $q_1(d)$ is a polynomial of degree 3. Therefore

$$\chi_1(V_d^\vee) = \chi_1(\text{Sym}_{d+1} T^\vee)\chi_0(T^\vee) + \chi_0(\text{Sym}_{d+1} T^\vee)\chi_1(T^\vee) - \chi_1(\text{Sym}_{d+2}(T^\vee)) =$$

$$= (3q_1(d + 1) + \binom{d+3}{2} - q_1(d + 2))c_1(T^\vee)$$

is a polynomial in $d$ of degree 3.

As in the proof of assertion A(k) (3.3), we conclude that $s_k(V_d)$ is polynomial in $d$ of degree $3k$.

□

3.5. Some formulas. To get explicit formulas for $s_d(V_d)$ for any fixed $d, n$ we use sequence (6), and Macaulay2 [9]. We find (cf. script §5 below)

$$\deg(\text{LB}(d, 3)) = \frac{29}{25}(d + 4)(d^2 + 6d + 11)(d^2 + 2d + 3).$$

$$\deg(\text{LB}(d, 4)) = \frac{1}{8308880(d + 4)!}[8d^{12} + 192d^{11} + 2176d^{10} + 15360d^9 + 75090d^8 + 267552d^7 + 711859d^6 + 1423716d^5 + 2119892d^4 + 2279136d^3 + 1662291d^2 + 730188d + 125388)(2 + d)].$$

4. Higher degree pullback components

As stated in the Introduction, the set of foliations obtained by pullback of foliations in $\mathbb{P}^2$ by rational maps of degree $m$ also form an irreducible component $PB(m, k, n)$ of $F(d, n)$, $d := (k + 2)m - 2$.

As in the linear case, a natural parameter space of rational maps $F : \mathbb{P}^n \to \mathbb{P}^2$ of degree $m$ is the grassmannian $G(3, S_m)$. Mimicking 2.1, we can construct a fiber bundle

$$V_{m, k} := \{(F, \mu) \in G(3, S_m) \times V_k^m | \mu = F^*\omega \text{ for some } \omega \in V_k^2\}.$$
However, for $m \geq 2$, the map $j : \mathcal{V}_{m,k} \to V^n_d$ (analogous to the map in (8)) is injective only over the open subset consisting of dominant maps. So its image is not a subbundle of $V^n_d$. In fact, for a non-dominant map $F \in G(3,S_m)$, $j_F(\omega) = F^* (\omega) = 0$ for all $\omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1 (k + 2))$ defining a foliation that leaves invariant the closure of $\text{Im}(F)$.

We could in principle find the locus $Z \subset G(3,S_m)$ where the rank of $\text{Im}(j)$ drops and then blow-up $G(3,S_m)$ along $Z$. Doing this we expect to build a subbundle $\tilde{\mathcal{V}}_k$ of $V^n_d$ over the blow-up that coincides with $\text{Im}(j)$ over the open set of dominant maps. The projection of $\mathbb{P}(\tilde{\mathcal{V}}_k)$ to $\mathbb{P}(V^n_d)$ is the space of foliations obtained as pullback by dominant maps. To make it work, we’d need to know how to get our hands on the Segre classes of $Z$. We hope to report on this elsewhere.

5. Scripts

5.1. Scripts for Macaulay2. In order to compute $\deg(\text{LPB}(d, n))$ for any given value of $n \geq 3$, just set $N = n$ at the beginning of the script below. It can be fed into http://habanero.math.cornell.edu:3690/.

loadPackage "Schubert2"
N=3 --plug-in 3,4...
pt= base d
-- set d to be a free parameter in the ‘‘intersection ring’’
--of the base variety
G=flagBundle({3,N-2}, pt)
-- Grassmannian of 3-planes in N+1-space,
(S,Q)=G.Bundles
-- names the sub and quotient bundles on G
A=symmetricPower(d+2, S)
B=symmetricPower(d+1, S)*S
integral(chern(A-B))

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