Amalgamated Free Products, Unstable Homotopy Invariance, and the Homology of $SL_2(Z[t])$

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Abstract We prove that if $R$ is a domain with many units, then the natural inclusion $E_2(R) \to E_2(R[t])$ induces an isomorphism in integral homology. This is a consequence of the existence of an amalgamated free product decomposition of $E_2(R[t])$. We also use this decomposition to study the homology of $E_2(Z[t])$ and show that a great deal of the homology of $E_2(Z[t])$ maps nontrivially into the homology of $SL_2(Z[t])$. As a consequence, we show that the latter is not finitely generated in all positive degrees.

Key words amalgamated free product – group homology – homotopy invariant presheaf – elementary matrices

Introduction

The fundamental theorem of algebraic $K$-theory asserts that if $A$ is a regular ring, then there is a natural isomorphism $K_i(A[t]) \cong K_i(A)$ for all $i \geq 0$. More generally, a presheaf $\mathcal{F}$ on the category of schemes over a base $S$ is \textit{homotopy invariant} if $\mathcal{F}(X \times A^1) \cong \mathcal{F}(X)$ for all schemes $X$. The utility of homotopy invariant presheaves is well-documented (see \textit{e.g.}, [10,11,12]).

The existence of homotopy invariance in $K$-theory suggests that one search for unstable analogues. In a previous work [4], the author showed that such a statement is true for infinite fields $k$: the natural map $G(k) \to G(k[t])$ induces an isomorphism in integral homology for

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G = SL_n, GL_n, PGL_n. Since homotopy invariance in K-theory holds for all regular rings, one might hope that the above isomorphism holds for rings other than infinite fields.

For a ring A, denote by E_n(A) the subgroup of GL_n(A) generated by the elementary matrices. In many cases, the group E_n(A) coincides with SL_n(A) (e.g., A local or Euclidean), but this need not always be so. In this paper, we study the homology of the group E_2(R[t]).

Our first result is the following.

**Theorem A** If R is an integral domain with many units, then the inclusion E_2(R) → E_2(R[t]) induces an isomorphism

\[ H_{\bullet}(E_2(R), \mathbb{Z}) \cong H_{\bullet}(E_2(R[t]), \mathbb{Z}). \]

The definition of a ring with many units will be recalled below. The principal example of such a ring is a local ring with infinite residue field (or an algebra over such a ring).

Let k be an infinite field and denote by Sch/k the category of smooth affine schemes over k. In this setting, Theorem A admits the following geometric interpretation.

**Corollary** The presheaves \( \mathcal{H}_i = H_i(E_n(-), \mathbb{Z}) \) are homotopy invariant on Sch/k.

One would hope that a similar statement holds for \( H_i(E_n(-), \mathbb{Z}) \) for \( n > 2 \), but so far we have been unable to prove it. What is clear, as the following example of Weibel [13] shows, is that if Theorem A is to admit a generalization to the case \( n > 2 \), then the hypotheses on the ring R will have to be strengthened.

**Example** Consider the local ring of the node at the singularity—

\[ R = (k[x, y]/(y^2 - x^3 - x^2))(x, y). \]

This is a one-dimensional noetherian local domain, and, if k is infinite, is a ring with many units. However, if \( H_\bullet(E_n(R), \mathbb{Z}) \cong H_\bullet(E_n(R[t]), \mathbb{Z}) \) for all \( n \), then in particular \( H_2(E(R), \mathbb{Z}) \cong H_2(E(R[t]), \mathbb{Z}) \); i.e., \( K_2(R) \cong K_2(R[t]) \). But in [13] it is shown that \( K_2(R) \neq K_2(R[t]) \).

Thus, it seems that one must assume that R is at least regular, and it is not at all clear how to use this property.

There is another reason to study the homology presheaves \( \mathcal{H}_i = H_i(E_n(-), \mathbb{Z}) \). If one could show that the \( \mathcal{H}_i \) are homotopy invariant and admit transfer maps (i.e., for a finite flat morphism \( R \to S \) there is a map \( \mathcal{H}_i(S) \to \mathcal{H}_i(R) \) satisfying the usual properties), then one could prove the Friedlander–Milnor conjecture [2] for \( SL_n \): If G is a reductive group scheme over an algebraically closed field k and if
$p \neq \text{char } k$, then

$$H^\bullet_{et}(BG_k, \mathbb{Z}/p) \cong H^\bullet(BG(k), \mathbb{Z}/p).$$

Here, $BG_k$ is the simplicial classifying scheme of $G_k$ and $BG(k)$ is the classifying space (= simplicial set) of the discrete group $G(k)$ of $k$-rational points. Unfortunately, the $H_i$ do not appear to admit transfers. However, it would still be interesting to know if the $H_i$ are homotopy invariant.

Theorem A is a consequence of the existence of an amalgamated free product decomposition

$$E_2(R[t]) \cong E_2(R) \ast_{B(R)} B(t)$$

where, for a ring $A$, $B(A)$ denotes the upper triangular subgroup. This follows from Nagao's decomposition of $SL_2(k[t])$ ($k$ a field) and some straightforward group-theoretic considerations. In particular, for $R = \mathbb{Z}$ we have $E_2(\mathbb{Z}[t]) \cong SL_2(\mathbb{Z}) \ast_{B(\mathbb{Z})} B(\mathbb{Z}[t])$ (note that $E_2(\mathbb{Z}) = SL_2(\mathbb{Z})$). This allows us to make partial computations of the homology of $SL_2(\mathbb{Z}[t])$. Grunewald, Mennicke, and Vaserstein [3] have shown that $SL_2(\mathbb{Z}[t])$ has free quotients $F$ of countable rank and that the projection $SL_2(\mathbb{Z}[t]) \rightarrow F$ may be taken so that all unipotent elements (and hence all elements of $E_2(\mathbb{Z}[t])$) lie in the kernel. This was generalized and improved by Krstić and McCool [5], who showed that the same is true for any domain $R$ which is not a field. This implies that $H_\bullet(SL_2(\mathbb{Z}[t]), \mathbb{Z})$ contains the homology of a countably generated free group as a direct summand. Note, however, that this only shows that $H_1(SL_2(\mathbb{Z}[t]), \mathbb{Z})$ has infinite rank.

The amalgamated free product decomposition for $E_2(\mathbb{Z}[t])$ allows us to compute its homology. We show that $H_i(E_2(\mathbb{Z}[t]), \mathbb{Z})$ contains a countably generated free summand for $i \geq 1$. It is not clear, a priori, that any homology classes map nontrivially into $H_\bullet(SL_2(\mathbb{Z}[t]), \mathbb{Z})$. We show that this is indeed the case.

**Theorem B** For each $i \geq 0$, the map

$$H_i(E_2(\mathbb{Z}[t]), \mathbb{Z}) \rightarrow H_i(SL_2(\mathbb{Z}[t]), \mathbb{Z})$$

is nontrivial.

**Corollary** For each $i \geq 1$, the group $H_i(SL_2(\mathbb{Z}[t]), \mathbb{Z})$ is not finitely generated.

In the case $i = 1$ we see that $H_1(E_2(\mathbb{Z}[t]))$ maps nontrivially into $H_1(SL_2(\mathbb{Z}[t]))$. Since all elementary matrices map to 1 in the free quotient $F$, the image of $H_1(E_2(\mathbb{Z}[t]))$ intersects the copy of
$H_1(F) \subset H_1(SL_2(\mathbb{Z}[t]))$ trivially. Hence we have found a nontrivial, nonfinitely generated summand orthogonal to $H_1(F)$.

Theorem B is proved by considering the (surjective!) homomorphisms $\varphi_p : E_2(\mathbb{Z}[t]) \to SL_2(\mathbb{F}_p[t])$ obtained by reducing polynomials modulo $p$. We show that classes in $H_i(E_2(\mathbb{Z}[t]), \mathbb{Z})$ map nontrivially into $H_i(SL_2(\mathbb{F}_p[t]), \mathbb{Z})$ under $\varphi_p^*$. The result follows since the map $\varphi_p$ factors through $SL_2(\mathbb{Z}[t])$.

The paper is organized as follows. In Section 1 we deduce the amalgamated free product decomposition for $E_2(R[t])$. In Section 2 we prove Theorem A. In Section 3 we discuss the homology of $E_2(\mathbb{Z}[t])$. In Section 4 we study the homology of $SL_2(\mathbb{F}_p[t])$. Section 5 contains the proof of Theorem B. Finally, Section 6 deals with other homology classes in $H_*(SL_2(\mathbb{Z}[t]), \mathbb{Z})$.

Notation If $A$ is a ring (always assumed commutative with unit), then $A^\times$ denotes the multiplicative group of units. If $G$ is an abelian group, then $nG$ denotes the subgroup of $G$ annihilated by $n$. If no coefficient module is specified, homology groups are to be interpreted as integral homology.

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1 Amalgamated Free Products

Recall the following theorem of Nagao [6]. Let $k$ be a field. Then we have an amalgamated free product decomposition

$$SL_2(k[t]) \cong SL_2(k) *_{B(k)} B(k[t])$$

where for a ring $A$, $B(A)$ denotes the subgroup of upper triangular matrices. We will make use of the following result of Serre ([8], Prop. 3, p. 6).

Proposition 1.1 Suppose that $G$ is an amalgamated free product, $G = G_1 *_{A} G_2$. Let $H_i \subseteq G_i$ be subgroups such that $H_1 \cap A = B = H_2 \cap A$. Denote by $H$ the subgroup of $G$ generated by the $H_i$. Then the evident homomorphism

$$H_1 *_{B} H_2 \longrightarrow H$$

is an isomorphism.  \(\Box\)
Let $R$ be an integral domain with field of fractions $Q$ and consider the subgroups $E_2(R) \subseteq E_2(Q) (= SL_2(Q))$ and $B(R[t]) \subseteq B(Q[t])$. Observe that $E_2(R) \cap B(Q) = B(R) = B(R[t]) \cap B(Q)$.

**Corollary 1.2** There is a decomposition

$$E_2(R[t]) \cong E_2(R) *_{B(R)} B(R[t]).$$

**Proof** It is clear that the group $H$ generated by $E_2(R)$ and $B(R[t])$ is contained in $E_2(R[t])$. The reverse inclusion follows since $E_2(R)$ contains the permutation matrix $\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$ and hence $H$ contains all upper and lower triangular matrices. \qed

## 2 Rings with Many Units

In this section, we shall prove Theorem A.

**Definition 2.1** A ring $A$ is an $S(n)$-ring if there exist $a_1, \ldots, a_n \in A^\times$ such that the sum of any nonempty subfamily of the $\{a_i\}$ is a unit. If $A$ is an $S(n)$-ring for every $n$, then we say that $A$ has many units.

Examples of rings with many units include local rings with infinite residue fields and algebras over such rings. However, a ring having infinitely many units need not be a ring with many units. An example is the local ring $\mathbb{Z}(p)$. In fact, $\mathbb{Z}(p)$ is an $S(p^1)$-ring, but not an $S(p)$-ring.

Rings with many units satisfy the following property (see [7], Thm. 1.10).

**Proposition 2.2** If $R$ has many units, then the inclusion $B(R) \to B(R[t])$ induces an isomorphism in integral homology. \qed

**Theorem 2.3** Suppose $R$ is an integral domain with many units. Then the inclusion $E_2(R) \to E_2(R[t])$ induces an isomorphism

$$H_\bullet(E_2(R), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(E_2(R[t]), \mathbb{Z}).$$

**Proof** The amalgamated free product decomposition yields a Mayer–Vietoris sequence for computing $H_\bullet(E_2(R[t]), \mathbb{Z})$. Since the evident map $H_\bullet(B(R), \mathbb{Z}) \to H_\bullet(B(R[t]), \mathbb{Z})$ is split injective for any domain $R$, the long exact sequence breaks up into short exact sequences

$$0 \to H_i(B(R)) \to H_i(B(R[t])) \oplus H_i(E_2(R)) \to H_i(E_2(R[t])) \to 0$$
(this is valid for any domain $R$). By Proposition 2.2, $H_i(B(R)) \cong H_i(B(R[t]))$ if $R$ has many units so that the map

$$H_i(E_2(R), \mathbb{Z}) \to H_i(E_2(R[t]), \mathbb{Z})$$

is an isomorphism. \( \square \)

Remark 2.4 Evidently, this approach will not work for computing $H_\bullet(E_n(R[t]))$ for $n \geq 3$ since no such amalgamated free product decomposition exists.

3 The Homology of $E_2(\mathbb{Z}[t])$

We now shift gears and study the homology of $E_2(\mathbb{Z}[t])$. Theorem 2.3 does not apply in this case since $\mathbb{Z}$ does not have many units. It is possible to calculate $H_\bullet(E_2(\mathbb{Z}[t]), \mathbb{Z})$ explicitly, but the final answer is a bit complicated. For this reason, we will compute only a part of the integral homology in detail. We also compute the homology of $E_2(\mathbb{Z}[t])$ with $F_p$-coefficients.

For each $i$ we have a short exact sequence

$$0 \to H_i(B(\mathbb{Z})) \to H_i(B(\mathbb{Z}[t])) \oplus H_i(SL_2(\mathbb{Z})) \to H_i(E_2(\mathbb{Z}[t])) \to 0.$$

Observe that $B(\mathbb{Z}[t]) = B(\mathbb{Z}) \times t\mathbb{Z}[t]$ so that for any principal ideal domain $k$ we have the Künneth exact sequence

$$0 \to \bigoplus_{l+m=i} H_l(B(\mathbb{Z}), k) \otimes H_m(t\mathbb{Z}[t], k) \to H_i(B(\mathbb{Z}[t]), k) \to \bigoplus_{l+m=i-1} \operatorname{Tor}_1^k(H_l(B(\mathbb{Z}), k), H_m(t\mathbb{Z}[t], k)) \to 0.$$

In particular, if $k$ is a field we have

$$H_\bullet(B(\mathbb{Z}[t]), k) \cong H_\bullet(B(\mathbb{Z}), k) \otimes H_\bullet(t\mathbb{Z}[t], k).$$

Suppose $k$ is the finite field $F_p$. Then we have the following result.

**Proposition 3.1** If $p \geq 3$, then for $R = \mathbb{Z}, \mathbb{Z}[t]$,

$$H_\bullet(B(R), F_p) = H_\bullet(R, F_p) = \bigwedge_{F_p}^\bullet (R \otimes F_p).$$

If $p = 2$, then

$$H_\bullet(B(\mathbb{Z}[t]), F_2) = (H_\bullet(\mathbb{Z}/2, F_2) \otimes H_\bullet(\mathbb{Z}, F_2)) \otimes \bigwedge_{F_2}^\bullet tF_2[t].$$
If $p \geq 3$, the homology of $E_2(\mathbb{Z}[t])$ is

$$H_i(E_2(\mathbb{Z}[t]), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & i = 0 \\ t\mathbb{F}_p[t] \oplus \mathbb{Z}/12 \otimes \mathbb{F}_p & i = 1 \\ \wedge_i \mathbb{F}_p[t] \oplus H_i(SL_2(\mathbb{Z}), \mathbb{F}_p) & i \geq 2. \end{cases}$$

In particular, if $p \geq 5$, then for $i \geq 2$

$$H_i(E_2(\mathbb{Z}[t]), \mathbb{F}_p) = \wedge_i \mathbb{F}_p[t].$$

Proof Consider the short exact sequence

$$0 \to H_i(B(\mathbb{Z}), \mathbb{F}_p) \to H_i(B(\mathbb{Z}[t]), \mathbb{F}_p) \oplus H_i(SL_2(\mathbb{Z}), \mathbb{F}_p) \to H_i(E_2(\mathbb{Z}[t]), \mathbb{F}_p) \to 0.$$
The integral homology is similarly complicated and we leave it to the interested reader to write it down. What is important for our purposes is the following.

**Proposition 3.4** For each $i$, $H_i(E_2(Z[t]), Z)$ contains $\wedge_i tZ[t]$ as a direct summand. Moreover, the element $t^{l_1} \wedge \cdots \wedge t^{l_i}$ maps to the element $t^{l_1} \wedge \cdots \wedge t^{l_i}$ in $\wedge_i F_p tP_p[t] \subseteq H_i(E_2(Z[t]), F_p)$ under the map induced by reducing coefficients modulo $p$.

**Proof** Arguing as in the proof of Proposition 3.3 we see that the group $H_i(B(Z[t]), Z)/H_i(B(Z), Z)$ contains a copy of the tensor product $H_0(B(Z), Z) \otimes H_i(tZ[t], Z)$. Since $tZ[t]$ is a torsion-free abelian group, its integral homology is simply an exterior algebra; i.e., $H_i(tZ[t], Z) = \wedge_i Z[t]$. The second assertion follows by considering the homomorphism of Mayer–Vietoris sequences induced by reducing coefficients modulo $p$. \qed

4 The Homology of $SL_2(F_p[t])$

In this section we compute the homology of the group $SL_2(F_p[t])$. As before, the amalgamated free product decomposition of $SL_2(F_p[t])$ yields a collection of short exact sequences

$$0 \to H_i(B(R)) \to H_i(B(F_p[t])) \oplus H_i(SL_2(F_p[t])) \to H_i(SL_2(F_p[t])) \to 0.$$ 

Observe that if $\ell$ is a prime distinct from $p$, then the natural map

$$H_i(B(F_p), F_\ell) \to H_i(B(F_p[t]), F_\ell)$$

is an isomorphism for all $i$. This follows by considering the Hochschild-Serre spectral sequence associated to the extension

$$0 \to R \to B(R) \to F_p^\times \to 1 \quad (4.1)$$

for $R = F_p, F_p[t]$. Since $H_q(R, F_\ell) = 0$ for $q > 0$, the claimed isomorphism follows. As a consequence, we see that for all $i$,

$$H_i(SL_2(F_p[t]), F_\ell) = H_i(SL_2(F_p), F_\ell)$$

for $\ell \neq p$. (This was proved, in greater generality, by C. Soulé [9].)

If we consider integral or $F_p$ coefficients, however, the situation is much different.

**Lemma 4.1** For $R = F_p, F_p[t]$, we have

$$H_i(B(R), F_p) = H_0(F_p^\times, H_i(R, F_p)).$$
Proof Consider the Hochschild–Serre spectral sequence associated to the extension (4.1); it has $E^2$-term

$$E^2_{r,s} = H_r(F_p^\times, H_s(R, F_p)).$$

Since $|F_p^\times| = p - 1$ is invertible in $F_p$, we see that $E^2_{r,s} = 0$ for $r > 0$ (see [1], p. 84). □

The action of $\alpha \in F_p^\times$ on $R$ is $\alpha : x \mapsto \alpha^2 x$. Thus, if $p = 2$ or $3$, we have the following.

Corollary 4.2 If $p = 2$ or $3$, then for all $i \geq 0$,

$$H_i(B(R), F_p) = H_i(R, F_p).$$

Proof In this case, $(F_2^\times)^2 = \{1\} = (F_3^\times)^2$ so that $F_p^\times$ acts trivially on $H_i(R, F_p)$. □

Corollary 4.3 If $p = 2$ or $3$, then for all $i \geq 0$,

$$H_i(SL_2(F_p[t]), F_p) = H_i(SL_2(F_p), F_p) \oplus H_i(B(F_p[t]), F_p)/H_i(B(F_p), F_p).$$

In particular, $H_i(SL_2(F_p[t]), F_p)$ contains a copy of $\bigwedge^i_p tF_p[t]$ as a direct summand.

Proof If $p = 2$ or $3$, then $B(F_p[t]) = B(F_p) \times tF_p[t]$. Hence the group $H_i(B(F_p[t]), F_p)/H_i(B(F_p), F_p)$ consists of

$$\bigoplus_{l+m=i, l<i} H_l(F_p, F_p) \otimes H_m(tF_p[t], F_p).$$

This contains $H_i(tF_p[t], F_p)$ as a summand and this, in turn, contains $\bigwedge^i_p tF_p[t]$ as a summand. □

The calculation of the integral homology is obviously more complicated, even for $p = 2, 3$. However, we make the following observation.

Proposition 4.4 If $p = 2$ or $3$, then for all $i \geq 0$, $H_i(SL_2(F_p[t]), Z)$ contains a copy of $\bigwedge^i_Z tF_p[t]$. Furthermore, the element $t^{l_1} \cdots t^{l_i} \in \bigwedge^i_Z tF_p[t]$ maps to $t^{l_1} \wedge \cdots \wedge t^{l_i} \in \bigwedge^i_F tF_p[t] \subseteq H_i(SL_2(F_p[t]), F_p)$ under the map induced by reducing coefficients modulo $p$.

Proof For any abelian group $A$, the map $\bigwedge^i_Z A \to H_*(A, Z)$ is injective [1], p. 123. It follows that $H_i(B(F_p[t]), Z)$ contains a copy of $H_0(B(F_p), Z) \otimes H_i(tF_p[t], Z)$, which, in turn, contains a copy of $\bigwedge^i_Z tF_p[t]$. Thus, the group $H_i(SL_2(F_p[t]), Z)$ does also. The second assertion follows from the naturality of the map induced in homology by the coefficient homomorphism $Z \to F_p$. □
The situation for primes greater than 3 is made more difficult by the fact that $(F_p^\times)^2 \neq \{1\}$ for $p \geq 5$. Thus, the calculation of $H_*(B(F_p))$ and $H_*(B(F_p[t]))$ is more involved. Since considering the primes 2 and 3 is sufficient for our purposes here, we leave the case $p \geq 5$ to the interested reader.

5 The Homology of $SL_2(Z[t])$

In this section we prove Theorem B (which was stated in the introduction). We first demonstrate the following result.

Proposition 5.1 If $p = 2$ or 3, then the natural map

$$H_i(E_2(Z[t]), Z) \to H_i(SL_2(F_p[t]), Z)$$

induced by $\varphi_p : E_2(Z[t]) \to SL_2(F_p[t])$ maps the element $t^{l_1} \land \cdots \land t^{l_i} \in \land^i_j tZ[t]$ to $t^{l_1} \land \cdots \land t^{l_i} \in \land^i_j tF_p[t]$. The same is true with $F_p$-coefficients.

Proof Consider the commutative diagram

$$\begin{array}{c}
H_i(B(Z)) \to H_i(B(Z[t])) \oplus H_i(SL_2(Z)) \to H_i(E_2(Z[t])) \\
\downarrow \quad \downarrow \quad \downarrow \\
H_i(B(F_p)) \to H_i(B(F_p[t])) \oplus H_i(SL_2(F_p)) \to H_i(SL_2(F_p[t])).
\end{array}$$

Since $t^j \in tZ[t]$ maps to $t^j \in tF_p[t]$, the claim is clear. The second assertion follows similarly. $\square$

Consider the inclusion $j : E_2(Z[t]) \to SL_2(Z[t])$ and the induced map $j_*$ on homology. For each $i$-tuple $\underline{l} = (l_1, \ldots, l_i)$, $l_1 < \cdots < l_i$, denote by $x_{\underline{l}}$ the homology class $j_*(t^{l_1} \land \cdots \land t^{l_i}) \in H_i(SL_2(Z[t]), Z)$.

Theorem 5.2 For each $i$-tuple $\underline{l}$, the class $x_{\underline{l}}$ is nontrivial. Moreover, if $\underline{m}$ is another $i$-tuple distinct from $\underline{l}$, then $x_{\underline{l}}$ and $x_{\underline{m}}$ are distinct. Thus, the group $H_i(SL_2(Z[t]), Z)$ is not finitely generated for all $i \geq 1$.

Proof Let $p = 2$ or 3. Consider the commutative diagram

$$\begin{array}{c}
E_2(Z[t]) \xrightarrow{j} SL_2(Z[t]) \\
\varphi_p \downarrow \quad \downarrow \pi_p \\
SL_2(F_p[t])
\end{array}$$

and the induced commutative diagram

$$\begin{array}{c}
H_i(E_2(Z[t]), Z) \xrightarrow{j_*} H_i(SL_2(Z[t]), Z) \\
\varphi_{p*} \downarrow \quad \downarrow \pi_{p*} \\
H_i(SL_2(F_p[t]), Z)
\end{array}$$
in homology. Since \( \varphi_{p^k}(t^i \wedge \cdots \wedge t^i) = t^i \wedge \cdots \wedge t^i \), we see that 
\[ \pi_{p^k}(t^i \wedge \cdots \wedge t^i) \neq 0 \] 
and hence \( x_\ell \) is nontrivial. Moreover, since \( \varphi_{p^k}(t^i \wedge \cdots \wedge t^i) \neq \varphi_{p^k}(t^{m_1} \wedge \cdots \wedge t^{m_i}) \), we see that \( x_\ell \neq x_m \) provided \( \ell \neq m \). \hfill \Box

**Corollary 5.3** For each \( i \)-tuple \( \ell \), the element \( x_\ell \) has either infinite order or order divisible by 6.

**Proof** Note that \( \varphi_{p^k}(t^i \wedge \cdots \wedge t^i) \) is an element of order \( p \) in the group \( H_i(SL_2(F_p[t]), \mathbb{Z}) \) (since this group is all \( p \)-torsion). It follows that \( x_\ell \) has order divisible by the least common multiple of 2 and 3 (if it is not infinite). \hfill \Box

**Remark 5.4** We can glean further information by reducing modulo primes \( p \geq 5 \). By Lemma 4.1,

\[ H_i(B(F_p[t]), F_p) = H_0(F_p^\times, H_i(F_p[t], F_p)) \]

where \( \alpha \in F_p^\times \) acts on \( F_p[t] \) via \( \alpha : x \mapsto \alpha^2 x \). Now, \( H_i(F_p[t], F_p) \) contains \( \Lambda_{F_p}^i t F_p[t] \) as a direct summand and if \( i \) is a multiple of \( (p - 1)/2 \), then, by Fermat’s Little Theorem, \( F_p^\times \) acts trivially on \( \Lambda_{F_p}^i t F_p[t] \). This implies that for \( i = n(p - 1)/2 \), \( H_i(SL_2(F_p[t]), F_p) \) contains a copy of \( \Lambda_{F_p}^i t F_p[t] \). Hence, if \( \ell \) is an \( i \)-tuple, \( i = n(p - 1)/2 \), then \( x_\ell \) has order divisible by \( 2 \cdot 3 \cdot p \). For example, the \( x_\ell \) in \( H_{2n} \) have order divisible by \( 2 \cdot 3 \cdot 5 = 30 \), those in \( H_{3n} \) have order divisible by \( 2 \cdot 3 \cdot 7 = 42 \), etc. Given this, it seems reasonable to conjecture that the \( x_\ell \) have infinite order.

**6 Other Classes in** \( H_\bullet(SL_2(Z[t]), Z) \)

In [5], Krstić and McCool exhibit a basis of a free retract \( F \) of the quotient \( SL_2(Z[t])/U_2(Z[t]) \) (here, \( U_2(Z[t]) \) denotes the subgroup generated by unipotent matrices). It consists of elements \( h_{p,k} \) where \( p \geq 2 \) and \( k \geq 1 \). The elements \( h_{p,k} \) are defined as

\[ h_{p,k} = \begin{pmatrix} 1 + pt^k & t^{3k} \\ p^3 & 1 - pt^k + p^2t^{2k} \end{pmatrix}. \]

Thus, the elements \( \overline{h}_{p,k} \in H_1(SL_2(Z[t]), Z) \) form the basis of a free direct summand; denote this summand by \( Z\{\overline{h}_{p,k}\} \). Then we may write

\[ H_1(SL_2(Z[t]), Z) = H_1(SL_2(Z), Z) \oplus Z\{\overline{h}_{p,k}\} \oplus X, \]
where \( X \) is some abelian group.

Since \( E_2(\mathbb{Z}[t]) \subset U_2(\mathbb{Z}[t]) \), the elements

\[
x_k = \begin{pmatrix} 1 & t^k \\ 0 & 1 \end{pmatrix}
\]

map to 1 in \( F \). Thus, the elements \( \overline{x}_k \) map into the summand \( X \subset H_1(SL_2(\mathbb{Z}[t]), \mathbb{Z}) \). Hence we see that \( X \) is also not finitely generated.

Consider the homomorphism \( \pi_p : SL_2(\mathbb{Z}[t]) \rightarrow SL_2(\mathbb{F}_p[t]) \). Then we have the following.

**Proposition 6.1** The kernel of the map

\[
\pi_{ps} : H_1(SL_2(\mathbb{Z}[t]), \mathbb{Z}) \rightarrow H_1(SL_2(\mathbb{F}_p[t]), \mathbb{Z})
\]

is not finitely generated.

**Proof** If \( p \geq 5 \), then \( H_1(SL_2(\mathbb{F}_p[t]), \mathbb{Z}) = H_1(SL_2(\mathbb{F}_p), \mathbb{Z}) = 0 \) so that \( \pi_{ps} = 0 \). If \( p = 2 \) or 3 then it is not clear that any summand lies in the kernel. But note that \( \pi_p(x_k) = \pi_p(h_{p,k}) \) so that \( \pi_{ps}(\overline{x}_k - \overline{h}_{p,k}) = 0 \) for all \( k \geq 1 \). \( \square \)

Consider the elements \( g_{p,k} \) defined by

\[
g_{p,k} = \begin{pmatrix} 1 & -t^k \\ -p & 1 + pt^k \end{pmatrix}.
\]

Note that each \( g_{p,k} \) is not unipotent; i.e., the matrix

\[
n_{p,k} = \begin{pmatrix} 0 & -t^k \\ -p & pt^k \end{pmatrix}
\]

has infinite order. However, we do have the following.

**Lemma 6.2** For each \( k, l \), \( g_{p,k} \equiv g_{p,l} \) modulo \( U_2(\mathbb{Z}[t]) \).

**Proof** An easy calculation shows that

\[
g_{p,k}^{-1} g_{p,l} = \begin{pmatrix} 1 & t^k - t^l \\ 0 & 1 \end{pmatrix}.
\]

The result follows. \( \square \)

Thus, the set of all \( g_{p,k} \) (for a fixed \( p \)) lies in a single coset of \( U_2(\mathbb{Z}[t]) \). Denote by \( \overline{g}_p \) the induced (nontrivial) element of \( \mathbb{Z}\{\overline{h}_{p,k}\} \).

**Proposition 6.3** If \( p = 2 \) or 3, then \( \overline{g}_p + \overline{x}_k \) lies in the kernel of \( \pi_{ps} \) for all \( k \geq 1 \).
Proof Note that \( \pi_p(g_{p,k}) = x_k^{-1} \). It follows then that \( \pi_p^*(\gamma_p) = -t^k \in \bigwedge^1_{\mathbb{Z}} t\mathbb{F}_p[t] \subset H_1(SL_2(\mathbb{F}_p[t], \mathbb{Z})). \) Since \( \pi_p^*(\pi_k) = t^k \), the result follows.

Note that we also have \( \gamma_p + \gamma_{p,k} \in \ker(\pi_p) \) since \( \gamma_p + \pi_k \) and \( \gamma_{p,k} - \pi_k \) lie in the kernel.

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