Semidistributive Laurent Series Rings
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Abstract. If $A$ is a ring with automorphism $\varphi$ and the skew Laurent series ring $A((x, \varphi))$ is a right semidistributive semilocal ring then $A$ is a right semidistributive right Artinian ring. The Laurent series ring $A((x))$ is a right semidistributive semilocal ring if and only if $A$ is a right semidistributive right Artinian ring.

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1 Introduction

All rings are assumed to be associative and with non-zero identity element; all modules are unitary and, unless otherwise specified, all modules are right modules. The words of type an ”Artinian ring” mean ”a right and left Artinian ring”.

1.1. Semidistributive and serial modules and rings. A module $M$ is said to be distributive if $X \cap (Y + Z) = X \cap Y + X \cap Z$ for any three its submodules $X, Y, Z$. A module is said to be uniserial if any two its submodules are comparable with respect to inclusion. It is clear that any uniserial module is distributive. The ring $\mathbb{Z}$ of integers is a distributive non-uniserial $\mathbb{Z}$-module. Direct sums of distributive (resp. uniserial) modules are called semidistributive (resp. serial) modules.

1.2. The Laurent series rings $A((x, \varphi))$ and their modules. If $A$ is a ring with automorphism $\varphi$, then $A((x, \varphi))$ denotes the skew Laurent series ring with coefficient ring $A$; this ring is formed by all series $f = \sum_{i=k}^{+\infty} f_i x^i$, where $x$ is a variable, $k$ is an integer (maybe, negative), and all the coefficients $f_i$ are contained in the ring $A$. In the ring $A((x, \varphi))$, addition is naturally defined and multiplication is defined with regard to the relation $xa = \varphi(a)x$ (for all
elements \( a \in A \). For \( \varphi = 1_A \), we obtain the ordinary Laurent series ring \( A((x)) \).

For every right \( A \)-module \( M \), we denote by \( M((x, \varphi)) \) the set of all formal series \( \sum_{i=0}^{\infty} m_i x^i \), where \( m_i \in M \), \( t \in \mathbb{Z} \), and either \( m_t \neq 0 \) or \( m_i = 0 \) for all \( i \). The set \( M((x, \varphi)) \) is a natural right \( A((x, \varphi)) \)-module, where module addition is defined naturally and multiplication by elements of \( A((x, \varphi)) \) is defined by the relation

\[
(\sum_{i=0}^{\infty} m_i x^i)(\sum_{j=0}^{\infty} a_j x^j) = \sum_{k=0}^{\infty} \left( \sum_{i+j=k} m_i \varphi^i(a_j) \right) x^k.
\]

1.3. Remark. In [11, Theorem 13.7], it is proved that \( A((x, \varphi)) \) is a right distributive semilocal ring if and only if \( A \) is a finite direct product of right uniserial right Artinian rings \( A_i \) and \( \varphi(A_i) = A_i \) for all \( i \). In [10], it is proved that the ring \( A((x, \varphi)) \) is right serial of and only if \( A \) is a right serial right Artinian ring. In the both cases, the ring \( A((x, \varphi)) \) is a right Artinian ring.

In connection to Remark 1.3, we prove Theorem 1.4 which is the main result of the given paper.

1.4. Theorem. Let \( A \) be a ring with automorphism \( \varphi \).

1. If \( A((x, \varphi)) \) is a right semidistributive semilocal ring, then \( A \) is a right semidistributive right Artinian ring and \( A((x, \varphi)) \) is a right Artinian ring.

2. Assume that \( \varphi(e) = e \) for every local idempotent \( e \in A \). Then \( A((x, \varphi)) \) is a right semidistributive semilocal ring if and only if \( A \) is a right semidistributive right Artinian ring. In this case, \( A((x, \varphi)) \) is a right Artinian ring.

3. \( A((x)) \) is a right semidistributive semilocal ring if and only if \( A \) is a right semidistributive right Artinian ring. In this case, \( A((x)) \) is a right Artinian ring.

In connection to Theorem 1.4, we give Remark 1.5 and Remark 1.6.

1.5. Remark. Let \( F \) be a field and let \( A \) be the 5-dimensional \( F \)-algebra generated by all \( 3 \times 3 \) matrices of the form

\[
\begin{pmatrix}
f_{11} & f_{12} & f_{13} \\
0 & f_{22} & 0 \\
0 & 0 & f_{33}
\end{pmatrix},
\]

where \( f_{ij} \in F \).

It is directly verified that \( A \) is a right semidistributive, left serial, Artinian ring which is not right serial. Therefore, \( A((x, \varphi)) \) is not a right serial ring. Thus, it follows from Theorem 1.4 that \( A((x)) \) is a right semidistributive ring which is not right serial.

1.6. Remark. If \( A \) is a field, the the formal power series ring \( A[[x]] \) is a right distributive local ring which is not right Artinian.
We present some necessary notation and definitions.

Let $A$ be a ring. We denote by $J(A)$ the Jacobson radical of $A$. A ring $A$ is said to be **semilocal** if $A/J(A)$ is a semisimple Artinian ring. A ring $A$ is said to be **local** if $A/J(A)$ is a division ring.

A module $M$ is said to be **finite-dimensional** if $M$ does not contain an infinite direct sum of non-zero submodules. A module $M$ is said to be **quotient finite-dimensional** if all factor modules of the module $M$ are finite-dimensional.

## 2 Proof of Theorem 1.4

### 2.1. Remark.
In [3, Corollary 5.7], it is proved that any finite direct sum of quotient finite-dimensional modules is a quotient finite-dimensional module; also see [2, Corollary 5.24].

### 2.2. Lemma.
Let $R$ be a ring which has only a finite number of non-isomorphic simple modules.

1. If $M$ is a distributive right $R$-module, then $M$ is finite-dimensional.
2. Every distributive right $R$-module is quotient finite-dimensional.
3. If the ring $R$ is right semidistributive, then every cyclic right $R$-module is quotient finite-dimensional.

**Proof.**

1. Assume the contrary. Then $M$ contains a submodule $X = \bigoplus_{i=1}^{\infty}X_i$, where $X_i$ is a non-zero-cyclic module, $i = 1, 2, \ldots$. For each $i$, the module $X_i$ has a submodule $Y_i$ such that $X_i/Y_i$ is simple. We denote by $Y$ the submodule $\bigoplus_{i=1}^{\infty}Y_i$ of $X$. The distributive module $M/Y$ contains the submodule $X/Y$ which is isomorphic to the infinite direct sum $\bigoplus_{i=1}^{\infty}(X_i/Y_i)$ of simple modules $X_i/Y_i$. Since the ring $A$ has only a finite number of non-isomorphic simple modules, $X_i/Y_i \cong X_j/Y_j$ for some $i \neq j$. Then $X_i/Y_i \oplus X_j/Y_j$ is a distributive module which is the direct sum of isomorphic simple modules. This is impossible, by [6].

2. Since all homomorphic image of distributive modules are distributive, the assertion follows from 1.

3. The assertion follows from 2 and Remark 2.1. \hfill \Box

### 2.3. Lemma.
If $R$ is a right semidistributive semilocal ring, then every cyclic right $R$-module is quotient finite-dimensional.

Since any semilocal ring has only a finite number of non-isomorphic simple modules, Lemma 2.3 follows from Lemma 2.2(3).
2.4. Lemma. Let $A$ be a ring with automorphism $\varphi$ and $R = A((x, \varphi))$ the skew Laurent series ring.

1. The ring $A$ is right Artinian if and only if the ring $R$ is right Artinian.
2. If every cyclic right $R$-module is quotient finite-dimensional, then the rings $R$ and $A$ are right Noetherian.
3. If every cyclic right $R$-module is quotient finite-dimensional, then the Jacobson radical $J(R)$ of $R$ is nilpotent.
4. If $R$ is a semilocal ring and every cyclic right $R$-module is quotient finite-dimensional, then the rings $R$ and $A$ are right Artinian.
5. If $R$ is a right semidistributive semilocal ring, then the rings $R$ and $A$ are right Artinian.
6. $A$ is a right Artinian right uniserial ring if and only if $R$ is a right Artinian right uniserial ring, if and only if $R$ is a right uniserial ring.
7. If $A$ is a right Artinian right uniserial ring and $M = mA$ is a cyclic right $A$-module, then the right $R$-module $mR$ of skew Laurent series is a cyclic uniserial Artinian module.

Proof. 1. The assertion is proved in [11, Proposition 9.2].
2. The assertion is proved in [11, Proposition 13.5].
3. By 2, the ring $R$ is right Noetherian. In this case, the Jacobson radical $J(R)$ is nilpotent, by [8, Theorem 1(1)].
4. By 2 and 3, $R$ is a right Noetherian ring with nilpotent Jacobson radical. In addition, $R$ is a semilocal ring, by assumption. It is directly verified that any right Noetherian semilocal ring with nilpotent Jacobson radical is right Artinian. Since the ring $R$ is right Artinian, the ring $A$ is right Artinian, by 1.
5. By Lemma 2.3, every cyclic $R$-module is quotient finite-dimensional. Thus the assertion follows from 4.
6. The assertion is proved in [11, Proposition 12.4].
7. By 6 $R$ is a right Artinian right uniserial ring. Therefore, $mR \cong R_R/S$, where $S$ is a right ideal of $R$. Therefore, $mR$ is a cyclic uniserial Artinian right $R$-module. □

2.5. Local idempotents and modules, semiperfect and local rings. A ring is said to be local if all its non-invertible elements are contained in the Jacobson radical of the ring. For a ring $A$, a right $A$-module $M$ is said to be local if $M$ is a cyclic module and its quotient module modulo its
Jacobson radical is simple. For a ring $A$, a non-zero idempotent $e \in A$ is said to be local if $eAe$ is a local ring (equivalently, $eA$ is a local module). A ring $A$ is said to be semiperfect if for its identity element $1_A$, there is a decomposition $1_A = e_1 + \cdots + e_n$ into a sum of some orthogonal local idempotents $e_1, \ldots, e_n \in A$; this decomposition is called a local decomposition for the ring $A$.

In the following familiar assertions 1-4, we fix a semiperfect ring $A$ with local decomposition $1_A = e_1 + \cdots + e_n$.

1. If $1_A = f_1 + \cdots + f_m$ is one more local decomposition for $A$, then $m = n$ and there is a permutation $\tau$ of the set $\{1, \ldots, n\}$ such that the ring $e_iAe_i$ is isomorphic to the ring $f_{\tau(i)}Af_{\tau(i)}$ and there is an isomorphism of right $A$-modules $e_iA \cong f_{\tau(i)}A$.

2. If $e$ is a non-zero idempotent of $A$, then there is a non-empty subset $K$ of $\{1, \ldots, n\}$ such that $eA \cong \oplus_{k \in K}e_kA$.

3. A right $A$-module $M$ is distributive if and only if $Me_i$ is a uniserial right $e_iAe_i$-module, for each $e_i$.

4. The ring $A$ is right semidistributive if and only if $e_jAe_i$ is a uniserial right $e_iAe_i$-module, for each $e_i$ and $e_j$.

5. The ring $A$ is right semidistributive if and only if the right $A$-module $e_iA$ is distributive for each $i$, if and only if for any local decomposition $1_A = f_1 + \cdots + f_m$, the right $A$-module $f_iA$ is distributive for each $i$, if and only if for any local decomposition $1_A = f_1 + \cdots + f_m$, the right $f_iAf_j$-module $f_jAf_i$ is uniserial for each $i$.

**Proof.** 1, 2. The assertions are well known; e.g., see [1] Section 27] or [7] Section 6.3].

3. The assertion is proved in [3] Lemma 4].

4. The assertion follows from 3.

5. The assertion follows from 1 and 4. □

2.6. Lemma. Let $A$ be a ring, $\varphi$ be an automorphism of $A$ such that $\varphi(e) = e$ for every idempotent $e \in A$, and let $R = A((x, \varphi))$ be the skew Laurent series ring.

1. For any non-zero idempotent $e \in A$ and each right $A$-module $M$, the skew Laurent series ring $(eAe)((x, \varphi))$ is naturally isomorphic to the ring $eRe$ and the right $(eAe)((x, \varphi))$-module $(Me)((x, \varphi))$ of skew Laurent series can be naturally identified with the right $eRe$-module $(Me)((x, \varphi))$. 


2. If \( e \) is a non-zero idempotent of the ring \( A \) such that \( eAe \) is a right uniserial right Artinian ring, then the ring \( eRe \) is a right uniserial right Artinian ring and \( e \) is a local idempotent of \( R \).

3. If \( A \) is a right semidistributive right Artinian ring, then \( R \) is a right semidistributive right Artinian ring.

**Proof.** 1. The assertion is directly verified.

2. By 2.4(6), the skew Laurent series ring \((eAe)((x, \varphi))\) is a right uniserial right Artinian ring. By 1, the ring \( eRe \) is a right uniserial right Artinian ring. Therefore, \( e \) is a local idempotent of \( R \).

3. Since \( A \) is a right Artinian ring, it follows from Lemma 2.4(1) that \( R \) is a right Artinian ring. In particular, the ring \( A \) is semiperfect and its identity element \( 1_A \) has a decomposition \( 1_A = e_1 + \cdots + e_n \) into a sum of some orthogonal local idempotents \( e_1, \ldots, e_n \in A \). Since \( A \) is a right semidistributive semiperfect ring, it follows from 2.5(4) that \( e_jAe_i \) is a uniserial right \( e_iAe_i \)-module, for each \( e_i \) and \( e_j \). Since \( A \) is a right Artinian ring, it is directly verified that each ring \( e_iAe_i \) is right Artinian. By 2, each \( e_i \) is a local idempotent of \( R \). By 1 and Lemma 2.4(6), the skew Laurent series ring \((e_iAe_i)((x, \varphi))\) is naturally isomorphic to the ring \( e_iRe_i \) and is right uniserial. Since \( e_iAe_i \) is a right uniserial right Artinian ring, all cyclic right \( e_iRe_i \)-modules are uniserial Artinian right modules, by Lemma 2.4(7). By 1, the right \((e_iAe_i)((x, \varphi))\)-module \((e_jAe_i)((x, \varphi))\) of skew Laurent series can be naturally identified with the right \( e_iRe_i \)-module \((e_jAe_i)((x, \varphi))\). Since \( 1_R = 1_A = e_1 + \cdots + e_n \) is the sum of orthogonal local idempotents \( e_1, \ldots, e_n \in R \), it follows from 2.5(4) that \( R \) is a right semidistributive ring. \( \square \)

**2.7. The completion of the proof of Theorem 1.4.** Let \( R = A((x, \varphi)) \).

1. Let \( R \) be a right semidistributive semilocal ring. By Lemma 2.4(5), the rings \( R \) and \( A \) are right Artinian.

Let \( \{e_1, \ldots, e_n\} \) be a complete set of local orthogonal idempotents of the right Artinian right semidistributive ring \( A \). By 2.5(4), each of the rings \( e_iAe_i \) are right Artinian right uniserial rings. By Lemma 2.4(6), each of the rings \( e_iRe_i \) are right Artinian right uniserial rings. In particular, each of the rings \( e_iRe_i \) are right Artinian right uniserial rings and \( \{e_1, \ldots, e_n\} \) is a complete set of local orthogonal idempotents of the right Artinian ring \( R \).

By applying 2.5(5) to the right semidistributive right Artinian ring \( R \), we obtain that for all \( i \) and \( j \), the right \( e_iRe_i \)-module \( e_jRe_i \) is uniserial. We fix \( i \) and \( j \). By 2.5(5), it is sufficient to prove the right \( e_iAe_i \)-module \( e_jAe_i \) is
uniserial.

Let $e_jae_i, e_jbe_i \in e_jAe_i$ and $e_jae_i \notin e_jbe_iA$. If $e_jae_i \in e_jbe_iR$, then $e_jae_i = e_jbe_if$ for some $f \in R$. Then $e_jae_i = e_jbe_if_0$ for the constant term $f_0$ of $f$ and $e_jae_i \in e_jbe_iA$; this is a contradiction. Since the right $e_iRe_i$-module $e_jRe_i$ is uniserial, we have $e_jbe_i \in e_jae_iR$ and $e_jbe_i = e_jae_ig$ for some $g \in R$. Then $e_jbe_i = e_jae_ig_0$ for the constant term $g_0$ of $g$ and $e_jbe_i \in e_jae_iA$. Therefore, the right $e_iRe_i$-module $e_jAe_i$ is uniserial. By 2.5(5), $A$ is a right semidistributive right Artinian ring.

2. The assertion follows from 1 and Lemma 2.6(2).

3. Let $A$ be a right semidistributive right Artinian ring. By Lemma 2.6(3), $R$ is a right semidistributive right Artinian ring.

Let $R$ be a right semidistributive right Artinian ring. By 1, $A$ is a right semidistributive right Artinian ring. □

3 Open Questions

Let $A$ be a ring with automorphism $\varphi$.

3.1. Let $A$ be a right semidistributive right Artinian ring. Is it true that $A((x, \varphi))$ is a right semidistributive ring?

3.2. Let $R = A((x, \varphi))$ be a regular ring, i.e., $r \in rRr$ for each $r \in R$. Is it true that the ring $R$ is Artinian? This is true if the automorphism $\varphi$ is of finite order; see [5, Theorem 1].

3.3. Let $A$ be a ring such that the ring $A((x, \varphi))$ is semilocal. Is it true that $A$ is semiperfect and the Jacobson radical of $A$ is nil? This is true if $\varphi = 1_A$; see [12].

3.4. When is the ring $A((x, \varphi))$ semilocal?

3.5. When is the ring $A((x, \varphi))$ right distributive?

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