The reduction theorem
for relatively maximal subgroups *

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Abstract

Let $\mathcal{X}$ be a class of finite groups closed under taking subgroups, homomorphic images and extensions. It is known that if $A$ is a normal subgroup of a finite group $G$ then the image of an $\mathcal{X}$-maximal subgroup $H$ of $G$ in $G/A$ is not, in general, $\mathcal{X}$-maximal in $G/A$. We say that the reduction $\mathcal{X}$-theorem holds for a finite group $A$ if, for every finite group $G$ that is an extension of $A$ (i.e. contains $A$ as a normal subgroup), the number of conjugacy classes of $\mathcal{X}$-maximal subgroups in $G$ and $G/A$ is the same. The reduction $\mathcal{X}$-theorem for $A$ implies that $HA/A$ is $\mathcal{X}$-maximal in $G/A$ for every extension $G$ of $A$ and every $\mathcal{X}$-maximal subgroup $H$ of $G$. In this paper, we prove that the reduction $\mathcal{X}$-theorem holds for $A$ if and only if all $\mathcal{X}$-maximal subgroups are conjugate in $A$ and classify the finite groups with this property in terms of composition factors.

Key words: complete class, $\mathcal{X}$-maximal subgroup, $\mathcal{X}$-submaximal subgroup, finite simple group.

To the 110th anniversary of the birth of Helmut Wielandt.

1 Introduction

1.1 Relatively maximal subgroups and reduction theorems

In this paper, we consider only finite groups. $G$ always denotes a finite group.

Since its inception in the papers by É. Galois [12] and C. Jordan [28, 29], group theory has had the following as one of its central problems: given a group $G$ and a class

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\( \mathfrak{X} \) of groups (the class of solvable groups, for example), determine the \( \mathfrak{X} \)-subgroups of \( G \) (i.e. the subgroups belonging to \( \mathfrak{X} \)). If \( \mathfrak{X} \) has good properties resembling those of solvable groups then to solve the general problem it suffices to know the so-called maximal \( \mathfrak{X} \)-subgroups (or \( \mathfrak{X} \)-maximal subgroups), i.e. the maximal by inclusion \( \mathfrak{X} \)-subgroups. Following H. Wielandt [56, 58], a nonempty class \( \mathfrak{X} \) is said to be complete (‘vollständig’) if subgroups and homomorphic images of an \( \mathfrak{X} \)-group and extensions of an \( \mathfrak{X} \)-group by an \( \mathfrak{X} \)-group are always \( \mathfrak{X} \)-groups. Examples of complete classes are

- \( \mathfrak{G} \), the class of all finite groups;
- \( \mathfrak{S} \), the class of all finite solvable groups.
- \( \mathfrak{S}_\pi \), the class of all \( \pi \)-groups for a set \( \pi \) of primes (i.e. the groups \( G \) such that every prime divisor of \( |G| \) belongs to \( \pi \));
- \( \mathfrak{S}_\pi \), the class of all solvable \( \pi \)-groups for a set \( \pi \) of primes.

In fact, the latter two cases are extremal for every complete \( \mathfrak{X} \): if we denote by \( \pi = \pi(G) \) the union of the sets \( \pi(G) \) of prime divisors of \( |G| \), where \( G \) runs over \( \mathfrak{X} \), then

\[
\mathfrak{S}_\pi \subseteq \mathfrak{X} \subseteq \mathfrak{G}.
\]

We will henceforth assume that \( \mathfrak{X} \) is a fixed complete class. We denote by \( m_X(G) \) the set of \( \mathfrak{X} \)-maximal subgroups of \( G \).

Following an approach practiced in group theory, it is natural to try to reduce the problem of determining \( \mathfrak{X} \)-maximal subgroups of \( G \) to the sections of a normal or subnormal series

\[
G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = 1,
\]

i.e. a series such that, for each \( i = 1, \ldots, n \), \( G_i \) is normal in \( G \) or \( G_{i-1} \), respectively. In this case, instead of a subgroup \( H \) of \( G \), the projection

\[
H^i = (H \cap G_{i-1})G_i/G_i
\]

in each section \( G^i = G_{i-1}/G_i \) is considered. It is easy to see that \( H \in \mathfrak{X} \) if and only if \( H^i \in \mathfrak{X} \) for all \( i = 1, \ldots, n \). Moreover, if \( H^i \in m_X(G^i) \) for all \( i = 1, \ldots, n \), then \( H \in m_X(G) \) [57 (2.5) and (2.6)]. In order to apply inductive arguments to studying \( \mathfrak{X} \)-maximal subgroups, it is natural to ask: is every projection \( H^i \) an \( \mathfrak{X} \)-maximal subgroup of \( G^i \) if \( H \) is \( \mathfrak{X} \)-maximal in \( G \)? This question is equivalent to the following.

(*) Assume that \( A \) is a normal subgroup and \( H \) is an \( \mathfrak{X} \)-maximal subgroup of \( G \). Is it true that \( HA/A \in m_X(G/A) \) and \( H \cap A \in m_X(A) \)?

Complete classes \( \mathfrak{X} \) satisfying (*) are known. The class \( \mathfrak{G}_p \) of all \( p \)-groups for a prime \( p \) is one of them: the \( \mathfrak{G}_p \)-maximal subgroup are exactly the Sylow \( p \)-subgroups by the Sylow theorem, and, for any Sylow \( p \)-subgroup \( H \) of \( G \), if \( A \) is normal in \( G \) then \( H \cap A \) is a Sylow \( p \)-subgroup of \( A \), as well as \( HA/A \) is a Sylow \( p \)-subgroup of \( G/A \). More generally, if the index of an \( \mathfrak{X} \)-subgroup \( H \) of \( G \) is not divisible by the primes in \( \pi = \pi(G) \) then \( H \in m_X(G) \) and, moreover, \( HA/A \in m_X(G/A) \) and \( H \cap A \in m_X(A) \) for each normal subgroup \( A \) of \( G \).

\(^1\)To distinguish \( \mathfrak{X} \)-maximal subgroups for a class \( \mathfrak{X} \) from maximal (among proper) ones, Wielandt [57] suggested using the term “relatively maximal” (w.r.t. a class \( \mathfrak{X} \)).

\(^2\)Recall that a group \( G \) is an extension of a group \( A \) by a group \( B \) if there is an epimorphism \( G \to B \) with kernel isomorphic to \( A \).
In this case, $H$ is a so-called $\pi$-Hall subgroup, i.e. a $\pi$-subgroup whose index is divisible by no primes from $\pi$. But, in general, the answer to question (*) is negative. We are concerned with this question for $H \cap A$ in detail in [13]. Here we consider the first part of this question concerning with the behavior of $\mathfrak{X}$-maximal subgroups under epimorphisms.

Wielandt notices [56, 14.2], [58, 4.3] that, if there exists a group $L$ with at least two conjugacy classes of $\mathfrak{X}$-maximal subgroups for a given complete class $\mathfrak{X}$, then there are no restrictions for an $\mathfrak{X}$-subgroup of an arbitrary group $G_0$ to coincide with the the image of an $\mathfrak{X}$-maximal subgroup under an appropriate epimorphism onto $G_0$:

- every (not only $\mathfrak{X}$-maximal) $\mathfrak{X}$-subgroup of $G_0$ is the image of an $\mathfrak{X}$-maximal subgroup of the regular wreath product $G = L \wr G_0$ under the natural epimorphism $G \to G_0$.

However, Wielandt [56, Sections 12 and 15] shows that, under some restrictions on the kernel of a homomorphism $\phi$ from a group $G$, there is a natural bijection between the conjugacy classes of $\mathfrak{X}$-maximal subgroups of $G$ and those of $G^\phi$.

Denote by $k_{\mathfrak{X}}(G)$ the number of conjugacy classes of $\mathfrak{X}$-maximal subgroups of $G$.

**Definition 1** We say that the $\mathfrak{X}$-Reduktionssatz (the reduction $\mathfrak{X}$-theorem) holds for a group $A$, if $k_{\mathfrak{X}}(G) = k_{\mathfrak{X}}(G/N)$ for every group $G$ containing a normal subgroup $N$ isomorphic to $A$.

It is not hard to show that if $\phi : G \to G_0$ is an epimorphism then, for every $K \in m_{\mathfrak{X}}(G_0)$, there exists $H \in m_{\mathfrak{X}}(G)$ such that $K = H^\phi$. As a consequence,

- $k_{\mathfrak{X}}(G_0) \leq k_{\mathfrak{X}}(G)$ and
- $k_{\mathfrak{X}}(G_0) = k_{\mathfrak{X}}(G) \implies m_{\mathfrak{X}}(G_0) = \{ H^\phi \mid H \in m_{\mathfrak{X}}(G) \}$.

It means that the $\mathfrak{X}$-Reduktionssatz for $A$ is equivalent to the following statement:

- If $G$ contains a normal subgroup $N$ isomorphic to $A$ then $H \mapsto H N / N$ maps $m_{\mathfrak{X}}(G)$ onto $m_{\mathfrak{X}}(G/N)$ and induces the natural bijection between the conjugacy classes of $\mathfrak{X}$-maximal subgroups of $G$ and $G/N$.

If a group $G$ has a normal subgroup $N$ satisfying the $\mathfrak{X}$-Reduktionssatz then we can study $\mathfrak{X}$-maximal subgroups in $G/N$ instead of those in $G$. Thus, in order to determine $\mathfrak{X}$-maximal subgroups of finite groups up to conjugation, it would be useful to know all groups with the $\mathfrak{X}$-Reduktionssatz. The program of studying such groups was outlined by Wielandt in his lectures in Tübingen in 1963-64 [56, Chapter III]. The goal of this paper is to classify all groups with the $\mathfrak{X}$-Reduktionssatz for any complete $\mathfrak{X}$. As a matter of fact, this result is a realization of Wielandt’s program.

Wielandt shows [56, 12.9] that the $\mathfrak{X}$-Reduktionssatz holds for the so-called $\mathfrak{X}$-separable (‘$\mathfrak{X}$-reihig’) groups, i.e. for groups possessing a subnormal series whose sections either are $\mathfrak{X}$-groups or have no nontrivial $\mathfrak{X}$-subgroups. In particular, the $\mathfrak{X}$-Reduktionssatz holds for the solvable groups.

Notice that if the $\mathfrak{X}$-Reduktionssatz holds for a group $G$ then $k_{\mathfrak{X}}(G) = k_{\mathfrak{X}}(G/G) = 1$, which means that all $\mathfrak{X}$-maximal subgroups are conjugate in $G$. Does the converse statement hold, i.e. does the $\mathfrak{X}$-Reduktionssatz hold for $G$ if the $\mathfrak{X}$-maximal subgroup of $G$

\[\text{The existence of a group with at least two conjugacy classes of } \mathfrak{X}\text{-maximal subgroups for a complete } \mathfrak{X}\text{ is equivalent to the fact that } \mathfrak{X} \text{ differs from each of the classes } \mathfrak{S}, \mathfrak{S}_g, \text{ and } \mathfrak{S}_p, \text{ where } p \text{ is prime, see } [39].\]
are conjugate? In another form, this question, as well as the problem of classification of groups with the \(X\)-Reduktionssatz, was asked by Wielandt in \([56]\), see 1.2.

The following statement answers this question and completely determines the groups satisfying the \(X\)-Reduktionssatz in terms of their composition factors:\(^4\)

**Theorem 1** (mod CFSG) *For a finite group \(G\), the following statements are equivalent:*

(i) the \(X\)-Reduktionssatz hold for \(G\);

(ii) the \(X\)-maximal subgroups of \(G\) are conjugate;

(iii) for every composition factor \(S\) of \(G\), either \(S \in \mathcal{X}\) or the pair \((S, \mathcal{X})\) satisfies one of Conditions I–VII in Appendix.

Theorem 1 can be considered as the most complete among other reduction theorems (see \([56, \text{sections 12 and 15}]\), for example) and includes them as particular cases. Our proof is based on Wielandt’s ideas and implements his strategy suggested in \([56]\). To prove Theorem 1 we have to solve some problems posed by Wielandt, see 1.2.

Other ingredients of the proof are connected with Hall subgroups and the so-called \(\mathcal{P}_\pi\)-groups, i.e. groups such that the complete analog of the Sylow theorem holds for their \(\pi\)-subgroups. It is easy to show that if the \(X\)-maximal subgroups are conjugate in \(G\) then every \(X\)-maximal subgroup is a \(\pi\)-Hall subgroup for \(\pi = \pi(X)\) and \(G\) is like a \(\mathcal{P}_\pi\)-group. Hall subgroups and \(\mathcal{P}_\pi\)-groups are well-studied \([24, 33–38, 41, 42, 48, 50, 51]\). Theorem 1 is an application of these results, see 1.3.

**1.2 Wielandt’s strategy and \(X\)-submaximal subgroups**

In order to prove the equivalence of (i) and (ii) in Theorem 1 we have to prove Wielandt’s conjecture that the conjugacy of the \(X\)-maximal subgroups implies the conjugacy of the so-called \(X\)-submaximal subgroups introduced in \([56, \text{Definition 15.1}]\) and \([58, \text{p. 170}]\). We discuss the concept of an \(X\)-submaximal subgroup generalizing that of an \(X\)- maximal subgroup below. Briefly, Wielandt proved \([56, 15.4]\) that, for every group, the conjugacy of the \(X\)-submaximal subgroups implies the \(X\)-Reduktionssatz. As we have seen, this implies the conjugacy of the \(X\)-maximal subgroups:

\[
\begin{bmatrix}
\text{the conjugacy of the } X\text{-submaximal subgroups} \\
\Rightarrow \\
\text{the } X\text{-Reduktionssatz}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\text{the conjugacy of the } X\text{-maximal subgroups}
\end{bmatrix}
\]

The problem of the equivalence of (i) and (ii) in Theorem 1 may be extended to

**Problem 1** (H. Wielandt, \([56, \text{offene Frage zu 15.4}]\)) Does the conjugacy of the \(X\)-maximal subgroups of a group \(G\) imply the conjugacy of the \(X\)-submaximal subgroups of \(G\)?

\[
\begin{bmatrix}
\text{the conjugacy of the } X\text{-submaximal subgroups} \\
\Leftrightarrow \\
\text{the conjugacy of the } X\text{-maximal subgroups}
\end{bmatrix}
\]

\(^4\)Recall that the composition factors of a finite group are the sections if its composition series, i.e. a subnormal series whose sections are simple. According to the Jordan–Hölder theorem, the multiset of composition factors of a group \(G\) is an invariant of \(G\) up to isomorphisms and does not depend on the composition series.
This problem is central in our paper and we solve it in the positive.

Wielandt's definitions of $X$-submaximal subgroups given in [56, Definition 15.1] and [58, p. 170] are different and even non-equivalent [40, Section 2]. Although we solve the problems presented in the lectures [56], the working definition for us is the broader definition of an $X$-maximal subgroup given in the talk [58]. Following [40], the $X$-submaximal subgroups in the sense of [56, Definition 15.1] are called strongly $X$-submaximal.

**Definition 2** A subgroup $H$ of a group $G$ is called strongly $X$-submaximal or $X$-submaximal in the sense of [56] (respectively, $X$-submaximal or $X$-submaximal in the sense of [58]), if there is an embedding $\phi : G \hookrightarrow G^*$ in a group $G^*$ such that $G^\phi$ is normal (respectively, subnormal, i.e. a member of a subnormal series) in $G^*$ and $H^\phi = K \cap G^\phi$ for an $X$-maximal subgroup $K$ of $G^*$.

We denote the set of strongly $X$-submaximal subgroups of $G$ by $\text{sm}_X^*(G)$ and the set of $X$-submaximal subgroups of $G$ by $\text{sm}_X(G)$. It is clear that

$$\text{m}_X(G) \subseteq \text{sm}_X^*(G) \subseteq \text{sm}_X(G).$$

Thus, if we prove that the conjugacy of the members of $\text{m}_X(G)$ always implies that of $\text{sm}_X(G)$, then we solve Problem 1 to be understood in either sense.

Let us assume we have already done that. Then, applying induction and the $X$-Reduktionssatz, one can show that the $X$-maximal (and, consequently, the $X$-submaximal) subgroups are conjugate in $G$ if they are conjugate in every composition factor of $G$, cf. [59, p. 57, Satz 1(a)]. Using his result [54], Wielandt notices that, if $G$ itself [56, 15.6] or even every composition factor of $G$ [56, 15.4] possesses a nilpotent $\pi$-Hall subgroup for $\pi = \pi(X)$, then $G$ has exactly one conjugacy class of (strongly) $X$-submaximal subgroups.

In view of this fact, he asks [56, offene Frage, p. 37 (643)], [59, p. 57, Frage 2]: for a simple group $G \notin \mathcal{X}$, does the conjugacy of the $X$-submaximal subgroups imply the existence of a nilpotent $\pi$-Hall subgroup for $\pi = \pi(X)$? A posteriori, with the results of this paper in mind, it is easy to give a counterexample: for $\mathcal{X} = \mathcal{S}_{3,7}$, the $X$-maximal (and $X$-submaximal) subgroups of $\text{PSL}_2(7)$ are conjugate, while $\text{PSL}_2(7)$ has no nilpotent $\{3,7\}$-Hall subgroups. However, in the natural way, Wielandt’s question [56, offene Frage, p. 37 (643)] gives rise to the following

**Problem 2** In what finite simple groups are all $X$-(sub)maximal subgroups conjugate?

A complete solution to Problem 2 turns out to be a by-product of our approach to Problem 1 outlined in 1.3 and the previous results on $D_\pi$-groups.

### 1.3 $D_X$-groups and $D_\pi$-groups

We need the concept of a $D_X$-group introduced in [21] which is natural in the context of our article.

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5For the role of $X$-maximal subgroups in Wielandt’s program presented in [58], see 1.3.

6In Wielandt’s lectures, the word “simple” is missing from the wording of this question. But it is clear from the context that it is a misprint. This follows from the meaning of the statement [56, 15.4"], p. 37 (643)] which immediately precedes the formulation of [56, offene Frage, p. 37 (643)]. The problem of the existence of a simple group possessing exactly one conjugacy class of $X$-submaximal subgroups, but containing no nilpotent $\pi$-Hall subgroup, is written down in Wielandt’s scientific diary on the last page of the main part [59, p. 57, Frage 2 (entry dated August 20, 1982)]. Examples of nonsimple groups without nilpotent $\pi$-Hall subgroups but with a unique conjugacy class of $X$-submaximal subgroups were known to Wielandt, see [56, 12.9].
Definition 3 A finite group $G$ is called a $\mathcal{D}_X$-group (we also say that $G$ belongs to $\mathcal{D}_X$ or $G$ satisfies $\mathcal{D}_X$ and write $G \in \mathcal{D}_X$), if $k_X(G) = 1$, i.e. the $X$-maximal subgroups of $G$ are conjugate.

Obviously, Problem 1 can be rewritten in the form: Is it true that an $X$-submaximal subgroup of a $\mathcal{D}_X$-group is always $X$-maximal? Also, it can be equivalently reformulated as follows, see [21, Theorem 2]:

Problem 3 Is an extension of a $\mathcal{D}_X$-group by a $\mathcal{D}_X$-group always a $\mathcal{D}_X$-group? In other words, if $N \trianglelefteq G$ and $N, G/N \in \mathcal{D}_X$, is it true that $G \in \mathcal{D}_X$?

If $X = \mathfrak{S}_\pi$ is the class of $\pi$-groups, we write $\mathcal{D}_\pi$ instead of $\mathcal{D}_X$. The statement $G \in \mathcal{D}_\pi$ is equivalent to the complete analog of the Sylow theorem for $\pi$-subgroup of $G$ which means the existence and conjugacy of $\pi$-Hall subgroups and the fact that every $\pi$-subgroup is contained in a $\pi$-Hall subgroup. The notation $\mathcal{D}_\pi$ was introduced by P. Hall [25]. The particular case of Problem 3 for $X = \mathfrak{S}_\pi$ was first formulated in the one-hour talk by Wielandt at the 13th International Congress of Mathematicians in Edinburgh in 1958 [54]. Problem 3 in this case for $X = \mathfrak{S}_\pi$ is solved in the affirmative using the classification of finite simple groups (see [51, Theorem 6.6]). We are going to use this result to solve Problems 1 and 3 in general situation. Moreover, for every $\pi$, the simple $\mathcal{D}_\pi$-groups are classified [51, Theorem 6.9]. This classification allows us to solve Problem 2.

The possibility to reduce the case of an arbitrary complete class $X$ to the well-studied case $X = \mathfrak{S}_\pi$ and thereby to solve Problems 1, 2, and 3 and to obtain Theorem 1 is ensured by the following theorem which is the main result of the paper.

Theorem 2 (mod CFSG) Let $X$ be a complete class of finite groups, let $\pi = \pi(X)$, and let $G$ be a finite simple group. Then

$$G \in \mathcal{D}_X \text{ if and only if either } G \in X \text{ or } \pi(G) \not\subseteq \pi \text{ and } G \in \mathcal{D}_\pi.$$  
In particular, if $G \in \mathcal{D}_X$ then $G \in \mathcal{D}_\pi$.

Since $G \in \mathcal{D}_\pi$ if $G$ has a nilpotent $\pi$-Hall subgroup [54], Theorem 2 can be considered as a weakening of the original Wielandt’s conjecture that the conjugacy of the $X$-submaximal subgroups of a simple group $G \notin X$ implies the existence of a nilpotent $\pi$-Hall subgroup [56, offene Frage, p. 37 (643)]. Moreover, it follows from [24, Lemma 4] that if $G$ is a simple $\mathcal{D}_\pi$-group and $\pi(G) \not\subseteq \pi$ then either $2 \not\in \pi \cap \pi(G)$ or $3 \not\in \pi \cap \pi(G)$ and the description of $\pi$-Hall subgroups of simple groups [51, Appendix 1] implies that $G$ possesses a solvable $\pi$-Hall subgroup $H$ of Fitting height at most 2 (i.e. $H$ is an extension of a nilpotent group by nilpotent one).

Briefly, the proof of Theorem 2 is based on the following simple consequence of the Sylow theorem: in a $\mathcal{D}_X$-group, every $X$-maximal subgroup is a $\pi$-Hall subgroup for $\pi = \pi(X)$. We prove the main part ‘only if’ of the theorem considering case by case the simple groups $G$ with a Hall subgroup $H$ (the Hall subgroups of simple groups are known, see [51, Appendix 1]). We show that if $G \notin \mathcal{D}_\pi$ where $\pi = \pi(H)$ is the set of prime divisors of $|H|$, then $G$ contains a subgroup $U$ such that $U$ is not conjugate to a subgroup of $H$, while every composition factor of $U$ is isomorphic to a section (i.e. a quotient of a subgroup) of $H$. It means that if $H \in \mathfrak{m}_\pi(G)$ then $U \notin X$, while there are no $X$-maximal subgroups

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7 Moreover, this case of Problem 3 is mentioned in the surveys [8, 10, 55] and in the textbooks [18, 19, 45, 47], and was also included by L. Shemetkov into the “Kourovka Notebook” [52, Problem 3.62].
of $G$ containing $U$ and conjugate to $H$. In particular, $G \not\in \mathcal{D}_\mathcal{X}$. This approach explains the rather large size of this article.

In [21], Problems 1 and 3 are reduced to simple groups. Using [21, Theorems 1 and 2] and the mentioned above results on $\mathcal{D}_\pi$-groups, we obtain from Theorem 2 the following statements which solve, respectively, the equivalent Problems 3 and 1.

**Corollary 1** (mod CFSG) Let $\mathcal{X}$ be a complete class. Assume that $N$ is a normal subgroup of a finite group $G$. Then $G \in \mathcal{D}_\mathcal{X}$ if and only if $N \in \mathcal{D}_\mathcal{X}$ and $G/N \in \mathcal{D}_\mathcal{X}$.

**Corollary 2** (mod CFSG) Let $\mathcal{X}$ be a complete class. Then the conjugacy of the $\mathcal{X}$-maximal subgroups of a finite group is equivalent to the conjugacy of the $\mathcal{X}$-submaximal subgroups.

**Corollary 3** (mod CFSG) Let $\mathcal{X}$ be a complete class of finite groups. Then, for a finite simple group $S$, the following statements are equivalent:

(i) All $\mathcal{X}$-submaximal subgroups are conjugate;

(ii) All $\mathcal{X}$-maximal subgroups are conjugate (i.e. $S \in \mathcal{D}_\mathcal{X}$);

(iii) Either $S \in \mathcal{X}$, or the pair $(S, \mathcal{X})$ satisfies one of Conditions I–VII in Appendix.

Conditions I–VII are arithmetic and given in terms of natural parameters of simple groups. These conditions appear in [35] [38] as necessary and sufficient for a simple group $S$ to satisfy $\mathcal{D}_\pi$, see also [51 Theorem 6.9 and Appendix 2]. Note that Condition I here differs from Condition I in [37,51]. In these articles, Condition I includes the case $\pi(S) \subseteq \pi$.

Theorem 1 immediately follows from Corollaries 1, 3 and Wielandt’s theorem [50, 15.4] which states that the conjugacy of $\mathcal{X}$-submaximal subgroups implies the $\mathcal{X}$-Reduktionssatz.

For $\pi = \pi(\mathcal{X})$, the classes $\mathcal{D}_\mathcal{X}$ and $\mathcal{D}_\pi$ are closely related. Although it is easy to understand that they are different in general (the groups in $\mathcal{D}_\mathcal{X} \setminus \mathcal{X}$ are $\mathcal{D}_\pi$-groups but are not $\mathcal{D}_\mathcal{X}$-groups), it follows by induction from Theorem 2 and Corollary 1 that

$$\mathcal{D}_\mathcal{X} \subseteq \mathcal{D}_\pi.$$ 

This observation allows us to establish rather unobvious properties of $\mathcal{X}$-subgroups in the groups with conjugate $\mathcal{X}$-maximal subgroups. So, in these groups, every $\pi$-subgroup belongs to $\mathcal{X}$. Applying the results of [31,49,50] we conclude that every subgroup $K$ of $G \in \mathcal{D}_\mathcal{X}$ which contains an $\mathcal{X}$-maximal subgroup of $G$ is itself a $\mathcal{D}_\mathcal{X}$-group. In particular, an $\mathcal{X}$-maximal subgroup in $K$ is always $\mathcal{X}$-maximal in $G$.

### 1.4 Remarks and open problems

1° It follows from Corollary 1 that the $\mathcal{D}_\mathcal{X}$-groups, like solvable or nilpotent groups, form a Fitting class. Recall that a Fitting class is a class of finite groups $\mathcal{F}$ such that

- $N \in \mathcal{F}$ if $N \triangleleft G$ and $G \in \mathcal{F}$ and
- $G \in \mathcal{F}$ if $G = MN$ where $M, N \triangleleft G$ and $M, N \in \mathcal{F}$.
For such $\mathfrak{F}$, every group $G$ has the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$, i.e., the largest normal $\mathfrak{F}$-subgroup.

- Let $D$ be the $\mathcal{D}_x$-radical of a group $G$. Then $D$ coincides with the subgroup generated by all (sub)normal subgroups $U$ of $G$ such that every composition factor of $U$ either is an $\mathfrak{X}$-group or satisfies one of Conditions I–VII.

By Theorem 1 there is a bijection between the conjugacy classes of $\mathfrak{X}$-maximal subgroups in $G$ and $G/D$, and we can study the members of $m_{\mathfrak{X}}(G/D)$ instead of $m_{\mathfrak{X}}(G)$.

Considering the $\mathcal{D}_x$-radical $D$ specifically from the point of view of reduction theorems, we can say that $D$ is an absolute radical of $G$ in the sense that $D$ is defined by its inner structure and does not depend on the way of embedding in $G$. Is there a subgroup of a group $G$ which is the largest among the normal subgroups $N$ such that $k_{\mathfrak{X}}(G) = k_{\mathfrak{X}}(G/N)$? Equivalently, is it true that if $M$ and $N$ are normal subgroups of a finite group $G$ then $k_{\mathfrak{X}}(G) = k_{\mathfrak{X}}(G/M) = k_{\mathfrak{X}}(G/N)$ always implies $k_{\mathfrak{X}}(G) = k_{\mathfrak{X}}(G/MN)$? Obviously, an affirmative answer to this question would follow from the validity of the following conjecture: if $N$ is a normal subgroup of a group $G$ then $k_{\mathfrak{X}}(G) = k_{\mathfrak{X}}(G/N)$ always implies $k_{\mathfrak{X}}(N) = 1$, i.e., $N \in \mathcal{D}_x$. The authors do not know any counterexamples to this conjecture.

2° The concept of an $\mathfrak{X}$-submaximal subgroup is useful not only in connection with reduction theorems. In his plenary talk in 1979, Wielandt suggested a general program on studying $\mathfrak{X}$-maximal subgroups of finite groups, and $\mathfrak{X}$-submaximal subgroups are a cornerstone of this program. We have mentioned that the image under an epimorphism of an $\mathfrak{X}$-maximal subgroup is not an $\mathfrak{X}$-maximal subgroup of the image. Also,

- the intersection of an $\mathfrak{X}$-maximal subgroup $H$ with a normal subgroup $N$ of $G$ is not an $\mathfrak{X}$-maximal in $N$, in general. For example, a Sylow 2-subgroup $H$ of $G = PGL_2(7)$ is $\mathfrak{X}$-maximal in $G$ for $\mathfrak{X} = \mathfrak{S}_{_{2,3}}$ but $H \cap N \notin m_{\mathfrak{X}}(N)$ for $N = PSL_2(7)$, because $H \cap N$ is a Sylow 2-subgroup of $N$ and is contained in a subgroup of order 24 [17 Example 2, p. 170].

In contrast to $\mathfrak{X}$-maximal subgroups, $\mathfrak{X}$-submaximal ones have the following evident inductive property:

- If $H \in \text{sm}_{\mathfrak{X}}(G)$ and $N$ is a (sub)normal subgroup of $G$ then $H \cap N \in \text{sm}_{\mathfrak{X}}(N)$.

Notice that not every $\mathfrak{X}$-subgroup of a group is $\mathfrak{X}$-submaximal. An obstruction here is the Wielandt–Hartley theorem. The strong version of this theorem announced in [58, 5.4(a)] and proven in [40, Theorem 2] states:

- If $H \in \text{sm}_{\mathfrak{X}}(G)$ then $N_G(H)/H$ contains no non-trivial $\mathfrak{X}$-subgroups.

For the case where $H$ is strongly $\mathfrak{X}$-submaximal, this theorem was proven in [56, 13.2] and [26, Lemmas 2 and 3]. It is due to the Wielandt–Hartley theorem and the above-mentioned inductive property that the concept of an $\mathfrak{X}$-submaximal subgroup becomes useful and efficient. For example, it helps to easily see that the $\mathfrak{X}$-maximal subgroups are determined uniquely up to conjugacy by their projections on the sections of a subnormal series [58, 5.4(c)], [40, Corollary 1]. It was shown in [20] that the knowledge of $\mathfrak{X}$-submaximal subgroups in simple groups for a given class $\mathfrak{X}$ would make it possible to inductively construct the $\mathfrak{X}$-maximal subgroups in an arbitrary finite group and, consequently, would make great progress in solving the general problem of determining the $\mathfrak{X}$-maximal subgroups. Thus, the central problem in Wielandt’s program and in the topic
related to the search of \( \mathfrak{X} \)-maximal subgroups for a given complete class \( \mathfrak{X} \) is the description of \( \mathfrak{X} \)-submaximal subgroups in simple groups which is equivalent to the description of \( \mathfrak{X} \)-maximal subgroups in the automorphism groups of simple groups \([58, 5.3]\). As a part of this problem pointed out by Wielandt himself \([58, \text{Frage g}]\), the classification of \( \mathfrak{X} \)-submaximal subgroups in minimal non-solvable groups is completed in \([22]\).

3° In light of Theorem 1 and the importance of the \( \mathfrak{X} \)-submaximal subgroups, it is natural to ask about the validity of reduction theorems for these subgroups themselves. Wielandt thought that such theorems hold, which can be seen from his latest diary entries. For example, \([59, \text{p. 57, Satz 1}]\) states for \( \mathfrak{X} = \mathfrak{D}_\pi \): if a normal subgroup \( N \) of a group \( G \) is such that the \( \mathfrak{X} \)-submaximal subgroups are conjugate in every composition factor of \( N \), then there is a natural one-to-one correspondence between the \( G \)- and \( \overline{G} \)-conjugacy classes of \( m_\mathfrak{X}(G) \) and \( m_\mathfrak{X}(\overline{G}) \), where \( \overline{G} = G/N \), as well as between the classes of \( \text{sm}_\mathfrak{X}(G) \) and \( \text{sm}_\mathfrak{X}(\overline{G}) \). However, the ‘as well as’ part is incorrect even in the case when \( N \) is a \( \pi \)-separable group. Counterexamples are given in \([43, 44]\). The number of conjugacy classes of \( \mathfrak{X} \)-submaximal subgroups in \( G \) can be both strictly greater and strictly smaller than in \( G/N \). The following statement is a consequence of our results. It can be considered as a weak analog of Theorem 1 for \( \mathfrak{X} \)-submaximal subgroups.

**Corollary 4** (mod CFSG) Let \( \mathfrak{X} \) be a complete class of finite groups, let \( \mathfrak{F} \) be a Fitting class such that \( \mathfrak{F} \subseteq \mathfrak{D}_\mathfrak{X} \), and let \( G \) be a finite group. Denote by \( \pi : G \to G/G_\mathfrak{F} \) the canonical epimorphism. If \( H \in \text{sm}_\mathfrak{X}(G) \) then \( \pi(H) \in \text{sm}_\mathfrak{X}(\overline{G}) \). In particular, if \( k \) and \( \overline{k} \) are the numbers of conjugacy classes of \( \mathfrak{X} \)-submaximal subgroups in \( G \) and \( \overline{G} \), respectively, then \( \overline{k} \geq k \).

The classes of nilpotent groups, solvable groups, \( \mathfrak{X} \)-groups, groups without non-trivial \( \mathfrak{X} \)-subgroups, \( \mathfrak{X} \)-separable groups, \( \mathfrak{D}_\mathfrak{X} \)-groups are examples of Fitting classes \( \mathfrak{F} \) satisfying the assumption of Corollary 4. In some sense, \( \mathfrak{X} \)-submaximal subgroups can be studied modulo the \( \mathfrak{F} \)-radical for such \( \mathfrak{F} \). Of course, Corollary 4 does not look as impressive as Theorem 1. Examples from \([43, 44]\) show that we can not replace the \( \mathfrak{F} \)-radical \( G_\mathfrak{F} \) in the assumption of Corollary 4 with an arbitrary normal \( \mathfrak{D}_\mathfrak{X} \)-subgroup \( N \), because the image in \( G/N \) of an \( \mathfrak{X} \)-submaximal subgroup of \( G \) is not \( \mathfrak{X} \)-submaximal in \( G/N \) in general. However, we do not know of any such examples where \( N \) is characteristic. Notice that even under the assumption of Corollary 4, \( G/G_\mathfrak{F} \) may possesses an \( \mathfrak{X} \)-submaximal subgroup which is not the image of any \( \mathfrak{X} \)-submaximal subgroup of \( G \) and the inequality in Corollary 4 may be strict.

2 Preliminaries

2.1 Notation

According to \([11, 7, 30]\), we use the following notation.

\( \varepsilon \) and \( \eta \) always denote either +1 or −1 and the sign of this number. Sometimes (in the notation of orthogonal groups of odd dimension) \( \eta \) can be used as an empty symbol.

\( n \) denotes the cyclic group of order \( n \), where \( n \) is a positive integer.

\( A^n \) denotes the direct product of \( n \) copies of \( A \). In particular,
$p^n$ denotes the elementary abelian group of order $p^n$, where $p$ is a prime.

Sym($\Omega$) denotes the symmetric group on $\Omega$.

Sym$_n$ is the symmetric group of degree $n$, i.e. Sym$_n$ = Sym($\Omega$), where $\Omega$ = \{1, 2, \ldots, n\}.

Alt$_n$ denotes the alternating group of degree $n$.

GL$_n$(q) or GL$_n^+$ (q) denotes the general linear group of degree $n$ over a field of order $q$.

SL$_n$(q) or SL$_n^+$ (q) denotes the special linear group of degree $n$ over a field of order $q$.

PSL$_n$(q) or PSL$_n^+$ (q) denotes the projective special linear group of degree $n$ over a field of order $q$.

PGL$_n$(q) or PGL$_n^+$ (q) denotes the projective general linear group of degree $n$ over a field of order $q$.

GU$_n$(q) or GL$_n^-$(q) denotes the general unitary group of degree $n$ over a field of order $q$.

SU$_n$(q) or SL$_n^-$(q) denotes the special unitary group of degree $n$ over a field of order $q$.

PSU$_n$(q) or PSL$_n^-$(q) denotes the projective special unitary group of degree $n$ over a field of order $q$.

PGU$_n$(q) or PGL$_n^-$(q) denotes the projective general unitary group of degree $n$ over a field of order $q$.

SO$_n^\varepsilon$(q) is the orthogonal group of degree $n$ over a field of order $q$, where $\varepsilon$ $\in$ \{+1, -1\} for $n$ even and $\varepsilon$ is an empty symbol for $n$ odd.

$\Omega_n^\varepsilon(q)$ is the derived subgroup of SO$_n^\varepsilon(q)$.

$P\Omega_n^\varepsilon(q)$ is the reduction of $\Omega_n^\varepsilon(q)$ modulo scalars.

Sp$_n$(q) denotes the symplectic group of even degree $n$ over a field of order $q$.

PSp$_n$(q) denotes the projective symplectic group of even degree $n$ over a field of order $q$.

$r^{1+2n}$ denotes an extra special group of order $r^{1+2n}$, where $r$ is a prime.

$A : B$ means a split extension of a group $A$ by a group $B$ ($A$ is normal).

$A : B$ means a non-split extension of a group $A$ by a group $B$ ($A$ is normal).

$A : B$ means an arbitrary (split or non-split extension) of a group $A$ by a group $B$ ($A$ is normal).

$A^{m+n}$ means $A^m : A^n$.

PG, for a linear group $G$, means the reduction of $G$ modulo scalars.

$\mathfrak{X}$ is a complete class of groups.
$\mathfrak{S}$ is a class of all solvable groups.

$\mathfrak{D}_\mathcal{X}$ is a class of groups with all maximal $\mathcal{X}$-subgroups conjugate.

$\pi$ is a set of primes.

$\mathfrak{S}_\pi$ is a class of all solvable $\pi$-groups

$\mathfrak{D}_\pi$ is a class of groups with all maximal $\pi$-subgroups conjugate, i.e. $\mathfrak{D}_\pi = \mathfrak{D}_{\mathfrak{S}_\pi}$.

$\mathfrak{E}_\pi$ is a class of groups possessing $\pi$-Hall subgroups, i.e. $G \in \mathfrak{E}_\pi$, if $\text{Hall}_\pi(G)$ is nonempty.

$G_{\mathcal{X}}$ means the $\mathcal{X}$-radical of $G$, i.e. the subgroup generated by all normal $\mathcal{X}$-subgroups of $G$. In particular,

$G_{\mathfrak{S}_\pi}$ means the solvable radical of $G$.

$O_\pi(G)$ means the $\pi$-radical of $G$, the subgroup generated by all normal $\pi$-subgroups of $G$, i.e. $O_\pi(G) = G_{\mathfrak{S}_\pi}$.

$\mu(G)$ denotes the degree of the minimal faithful permutation representation of a finite group $G$, i.e. the smallest $n$ such that $G$ is isomorphic to a subgroup of $\text{Sym}_n$.

$\mathcal{X}$-Hall subgroup, is a subgroup $H$ of $G$ such that $H$ is an $\mathcal{X}$-subgroup and a $\pi(\mathcal{X})$-Hall subgroup.

$\text{Hall}_\mathcal{X}(G)$ is the set of all $\mathcal{X}$-Hall subgroups of $G$, i.e. $\text{Hall}_\mathcal{X}(G) = \mathcal{X} \cap \text{Hall}_\pi(\mathcal{X})(G)$.

### 2.2 Known properties of $\pi$-Hall subgroups, $\mathfrak{D}_\pi$- and $\mathfrak{D}_\mathcal{X}$-groups

**Lemma 2.1** \[25, Lemma 1\] Let $N$ be a normal subgroup and let $H$ be a $\pi$-Hall subgroup of a group $G$. Then $H \cap N \in \text{Hall}_\pi(N)$ and $HN/N \in \text{Hall}_\pi(G/N)$.

**Lemma 2.2** \[15, Theorem A\] If $2 \not\in \pi$ and $G \in \mathfrak{E}_\pi$, then every two $\pi$-Hall subgroups of $G$ are conjugate.

**Lemma 2.3** \[11, Theorem 7.7\], \[51, Theorem 6.6\] Let $N$ be a normal subgroup of $G$. Then $G \in \mathfrak{D}_\pi$ if and only if $N \in \mathfrak{D}_\pi$ and $G/N \in \mathfrak{D}_\pi$.

**Lemma 2.4** \[37, Theorem 3\] Let $\pi$ be a set of primes and $S$ a simple group. Then $S \in \mathfrak{D}_\pi$ if and only if either $S$ is a $\pi$-group or $(S, \mathcal{X})$ for $\mathcal{X} = \mathfrak{S}_\pi$ satisfies one of Conditions I–VII.

**Lemma 2.5** \[21, Proposition 1\] Let $\mathcal{X}$ be a complete class, $\pi = \pi(\mathcal{X})$, and $G \in \mathfrak{D}_\mathcal{X}$. Then $m_{\mathcal{X}}(G) = \text{Hall}_\mathcal{X}(G) \subseteq \text{Hall}_\pi(G)$. In particular, $G \in \mathfrak{E}_\pi$.

**Lemma 2.6** \[21, Theorem 1\] If $G \in \mathfrak{D}_\mathcal{X}$ and $N \trianglelefteq G$ then $N \in \mathfrak{D}_\mathcal{X}$ and $G/N \in \mathfrak{D}_\mathcal{X}$.

**Lemma 2.7** \[21, Theorem 2\] For complete $\mathcal{X}$, the following statements are equivalent.

1. The elements of $\text{sm}_{\mathcal{X}}(G)$ are conjugate in any $G \in \mathfrak{D}_\mathcal{X}$.

2. $\text{sm}_{\mathcal{X}}(G) = m_{\mathcal{X}}(G)$ for any $G \in \mathfrak{D}_\mathcal{X}$.  

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(3) \( \mathcal{D}_X \) is closed under taking extensions.

(4) \( \text{Aut} S \in \mathcal{D}_X \) for every simple \( S \in \mathcal{D}_X \).

(5) The elements of \( \text{sm}_X(S) \) are conjugate in any simple \( S \in \mathcal{D}_X \).

**Lemma 2.8** [56, 12.9] Let \( N \) be a normal \( X \)-separable subgroup of \( G \). Then the map given by the rule \( M \mapsto MN/N \) is a surjection between sets \( m_X(G) \) and \( m_X(G/N) \). Moreover, this map induces a bijection between the sets of conjugacy classes of \( X \)-maximal subgroups of \( G \) and \( G/N \). In particular, \( G \in \mathcal{D}_X \) if and only if \( G/N \in \mathcal{D}_X \).

**Lemma 2.9** [42, Lemma 2.1(e)] Let \( N \) be a normal subgroup of \( G \) and \( \pi(G/N) \subseteq \pi \). Assume \( N \) contains a \( \pi \)-Hall subgroup \( H_0 \). Then the following statements are equivalent.

1. There is \( H \in \text{Hall}_\pi(G) \) such that \( H_0 = H \cap N \).
2. For every \( g \in G \) there exists \( x \in N \) such that \( Hg_0 = Hx_0 \).

**Lemma 2.10** (The Wielandt–Hartley theorem, strong form) [58, 5.4(a)], [10, Theorem 2] Let \( K \) be an \( X \)-submaximal subgroup of a finite group \( G \). Then \( N_G(K)/K \) is a \( \pi(X)' \)-group.

### 2.3 Arithmetic Lemmas

For an odd integer \( q \), denote by \( \varepsilon(q) \) the number \( \varepsilon = \pm 1 \) such that \( q \equiv \varepsilon \) (mod 4).

If \( r \) is an odd prime and \( k \) is an integer not divisible by \( r \), then \( e(k, r) \) is the smallest positive integer \( e \) with \( k^e \equiv 1 \) (mod \( r \)). So, \( e(k, r) \) is the multiplicative order of \( k \) modulo \( r \).

For a natural number \( e \) set

\[
e^* = \begin{cases} 2e & \text{if } e \equiv 1 \pmod{2}, \\ e & \text{if } e \equiv 0 \pmod{4}, \\ e/2 & \text{if } e \equiv 2 \pmod{4}. \end{cases}
\]

It follows from the definition that if \( e \) divides an even number \( n \) then \( e^* \) divides \( n \) again. Moreover, \( e^{**} = e \) for every \( e \).

For a real \( x \), the integer part \([x]\) of \( x \) is defined as a unique integer \( k \) such that

\[
k \leq x < k + 1.
\]

The following lemma is evident.

**Lemma 2.11** If \( m \) is a positive integer and \( x \) is a real then

\[
[x]/m = [x/m].
\]

The next result may be found in [52].

**Lemma 2.12** ([52], [17, Lemmas 2.4 and 2.5]) Let \( r \) be an odd prime, \( k \) an integer not divisible by \( r \), and \( m \) a positive integer. Denote \( e(k, r) \) by \( e \).

Then the following identities hold:

\[
(k^m - 1)_r = \begin{cases} (k^e - 1)_r(m/e)_r & \text{if } e \text{ divides } m, \\ 1 & \text{if } e \text{ does not divide } m; \end{cases}
\]

\[
(k^m - (-1)^m)_r = \begin{cases} (k^{e^*} - (-1)^{e^*})_r(m/e^*)_r & \text{if } e^* \text{ divides } m, \\ 1 & \text{if } e^* \text{ does not divide } m. \end{cases}
\]
Lemma 2.13 Let $q > 1$ and $n$ be positive integers, let $r$ be an odd prime such that $(q, r) = 1$, and let $e = e(r, q)$. Then the following statements hold:

(1) $(n!)_r = r^\alpha$, where $\alpha = \sum_{i=1}^{\infty} \lfloor n/r^i \rfloor$;

(2) $\prod_{i=1}^{n}(q^i - 1)_r = (q^e - 1)_{[n/e]}([n/e]!)_r$;

(3) $\prod_{i=1}^{m}(k^i - (-1)^i)_r = (k^{e^*} - (-1)^{e^*})^r_{[m/e^*]}([m/e^*]!)_r$;

(4) $\prod_{i=1}^{n}(q^i - 1)_r = (n!)_r$ if and only if $e = r - 1$, $(q^{r-1} - 1)_r = r$ and $[n/r] = [n/(r-1)]$.

(5) $\prod_{i=1}^{m}(q^i - (-1)^i)_r = (n!)_r$ if and only if $e^* = r - 1$, $(q^{(r-1)^*} - (-1)^{(r-1)^*})_r = r$ and $[n/r] = [n/(r-1)]$.

Proof. The statement (1) is well-known (see, for example, [48, Lemma 2]). Statements (2) and (3) follow from Lemma 2.12.

Now we prove (4). Let $A = \prod_{i=1}^{n}(q^i - 1)_r$. Then by (2) and in view of the Little Fermat Theorem,

$$A = (q^e - 1)_{[n/e]}([n/e]!)_r \geq r^{[n/e]}([n/e]!)_r \geq r^{[n/(r-1)]}([n/(r-1)]!)_r \geq r^{[n/r]}([n/r]!)_r = r^\beta,$$

where by (1) and, in view of Lemma 2.11 for $x = n/r$ and $m = r^i$, we have

$$\beta = [n/r] + \sum_{i=1}^{\infty} \lfloor n/r^i \rfloor = [n/r] + \sum_{i=1}^{\infty} \lfloor n/r^{i+1} \rfloor = \sum_{i=1}^{\infty} \lfloor n/r^i \rfloor = \log_r(n!)_r.$$

Therefore, $A \geq (n!)_r$. Moreover, this inequality becomes an equality if and only if all inequalities in (2.1) are equalities, i.e. if and only if

$$r - 1 = e, \quad (q^{r-1})_r = r, \quad \text{and} \quad [n/(r-1)] = [n/r].$$

This implies (4).

Now we prove (5). Let $A' = \prod_{i=1}^{m}(q^i - (-1)^i)_r$. Since $r$ is odd in view of the Little Fermat Theorem $e$ divides the even number $r - 1$. Consequently, $e^*$ also divides $r - 1$ and, by (3),

$$A' = (q^{e^*} - (-1)^{e^*})^{r}_{[n/e^*]}([n/e^*]!)_r \geq r^{[n/e^*]}([n/e^*]!)_r \geq r^{[n/(r-1)]}([n/(r-1)]!)_r \geq r^{[n/r]}([n/r]!)_r = r^\beta,$$

where $\beta$ is as above.

Therefore, $A' \geq (n!)_r$. Again this inequality becomes an equality if and only if all the inequalities in (2.2) are equalities, i.e. if and only if one of the equalities

$$r - 1 = e^*, \quad (q^{e^*} - (-1)^{e^*})_r = r, \quad \text{and} \quad [n/(r-1)] = [n/r].$$

This implies (v).
2.4 On Hall subgroups of finite simple groups

Lemma 2.14 [23, Theorem A4 and the notices after it], [48, Main result], [51, Theorem 8.1] Suppose that \( n \geq 5 \) and \( \pi \) is a set of primes with \( |\pi \cap \pi(n!)| > 1 \) and \( \pi(n!) \not\subseteq \pi \). Then

(1) The complete list of possibilities for \( \text{Sym}_n \) containing a \( \pi \)-Hall subgroup \( H \) is given in Table 1.

(2) \( M \in \text{Hall}_\pi(\text{Alt}_n) \) if and only if \( M = H \cap \text{Alt}_n \) for some \( H \in \text{Hall}_\pi(\text{Sym}_n) \).

In particular, every proper nonsolvable \( \pi \)-Hall subgroup of a symmetric group of degree \( n \) is isomorphic to a symmetric group of degree \( n \) or \( n - 1 \) and has a unique nonabelian composition factor isomorphic to an alternating group of the same degree.

Table 1:

| \( n \) | \( \pi \) | \( H \in \text{Hall}_\pi(\text{Sym}_n) \) |
|---|---|---|
| prime | \( \pi((n-1)!)) \) | \( \text{Sym}_{n-1} \) |
| 7 | \( \{2, 3\} \) | \( \text{Sym}_3 \times \text{Sym}_4 \) |
| 8 | \( \{2, 3\} \) | \( \text{Sym}_4 \triangleleft \text{Sym}_2 \) |

Lemma 2.15 [21, Proposition 3] Let \( \pi = \pi(\mathcal{X}) \). Then for \( G = \text{Alt}_n \) the following conditions are equivalent.

(1) \( G \in \mathcal{D}_\mathcal{X} \).

(2) \( G \in \mathcal{D}_\mathcal{X} \cap \mathcal{D}_\pi \).

(3) either \( |\pi \cap \pi(G)| \leq 1 \) or \( G \in \mathcal{X} \).

(4) All submaximal \( \mathcal{X} \)-subgroups are conjugate in \( G \).

Lemma 2.16 [35, Theorem 4.1], [51, Theorem 8.2] Let \( G \) be either one of the 26 sporadic groups or a Tits group, \( \pi \) be such that \( 2 \in \pi \), \( \pi(G) \not\subseteq \pi \), and \( |\pi \cap \pi(G)| > 1 \), and \( H \) be a \( \pi \)-Hall subgroup of \( G \). Then the corresponding intersections \( \pi \cap \pi(G) \) and the structure of \( H \) are indicated in Table 2.

Table 2:

| \( G \) | \( \pi \cap \pi(G) \) | Structure of \( H \) |
|---|---|---|
| \( M_{11} \) | \( \{2, 3\} \) | \( 3^2 : Q_8 \cdot 2 \) |
| & \( \{2, 3, 5\} \) | \( \text{Alt}_6 \cdot 2 \) |
| \( M_{22} \) | \( \{2, 3, 5\} \) | \( 2^4 : \text{Alt}_6 \) |
| \( M_{23} \) | \( \{2, 3\} \) | \( 2^4 : (3 \times A_4) : 2 \) |
| & \( \{2, 3, 5\} \) | \( 2^4 : \text{Alt}_6 \) |
| & \( \{2, 3, 5\} \) | \( 2^4 : (3 \times \text{Alt}_5) : 2 \) |
| & \( \{2, 3, 5, 7\} \) | \( L_3(4) : 2_2 \) |
| & \( \{2, 3, 5, 7\} \) | \( 2^4 : \text{Alt}_7 \) |
| & \( \{2, 3, 5, 7, 11\} \) | \( M_{22} \) |
| \( M_{24} \) | \( \{2, 3, 5\} \) | \( 2^6 : 3 \cdot \text{Sym}_6 \) |
Lemma 2.17 [12, Lemma 3.1], [51, Lemma 8.10] Let $\pi$ be a set of primes with $2, 3 \in \pi$. Assume that $G \cong \mathrm{SL}_2(q) \cong \mathrm{SL}_2^\pi(q) \cong \mathrm{Sp}_2(q)$, where $q$ is a power of an odd prime $p \notin \pi$, and $\varepsilon = \varepsilon(q)$. Recall that for a subgroup $A$ of $G$ we denote by $PA$ the reduction modulo scalars. Then the following statements hold.

(A) If $G \in \mathcal{E}_\pi$ and $H \in \mathrm{Hall}_\pi(G)$, then one of the following statements holds.

(a) $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $PH$ is a $\pi$-Hall subgroup in the dihedral subgroup $D_{q-\varepsilon}$ of order $q - \varepsilon$ of $PG$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(b) $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{(2, 3)} = 24$, $PH \cong \mathrm{Alt}_4$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(c) $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{(2, 3)} = 48$, $PH \cong \mathrm{Sym}_4$. There exist exactly two classes of conjugate subgroups of this type, and $\mathrm{PGL}_2^\pi(q)$ interchanges these classes.

(d) $\pi \cap \pi(G) = \{2, 3, 5\}$, $(q^2 - 1)_{(2, 3, 5)} = 120$, $PH \cong \mathrm{Alt}_5$. There exist exactly two classes of conjugate subgroups of this type, and $\mathrm{PGL}_2^\pi(q)$ interchanges these classes.

(B) Conversely, if $\pi$ and $(q^2 - 1)_\pi$ satisfy one of statements (a)–(d), then $G \in \mathcal{E}_\pi$.

(C) Every $\pi$-Hall subgroup of $PG$ can be obtained as $PH$ for some $H \in \mathrm{Hall}_\pi(G)$. Conversely, if $PH \in \mathrm{Hall}_\pi(PG)$ and $H$ is a full preimage of $PH$ in $G$, then $H \in \mathrm{Hall}_\pi(G)$.

Lemma 2.18 [12, Lemma 3.2], [51, Corollary 8.11] Let $G = \mathrm{GL}_2^\pi(q)$, $PG = G/Z(G) = \mathrm{PGL}_2^\pi(q)$, where $q$ is a power of a prime $p$, and $\varepsilon = \varepsilon(q)$. Let $\pi$ be a set of primes such that $2, 3 \in \pi$ and $p \notin \pi$. A subgroup $H$ of $G$ is a $\pi$-Hall subgroup if and only if $H \cap \mathrm{SL}_2(q)$ is a $\pi$-Hall subgroup of $\mathrm{SL}_2(q)$, $|H : H \cap \mathrm{SL}_2(q)|_\pi = (q - \eta)_\pi$, and either statement (a), or statement (b) of Lemma 2.17 holds. More precisely, one of the following statements holds.

(a) $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, where $\varepsilon = \varepsilon(q)$, $PH$ is a $\pi$-Hall subgroup in the dihedral group $D_{2(q-\varepsilon)}$ of order $2(q-\varepsilon)$ of $PG$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(b) $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{(2, 3)} = 24$, $PH \cong \mathrm{Sym}_4$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

Lemma 2.19 [14, Theorem 3.2], [16, Theorem 3.1], [34, Theorem 1.2], [51, Theorems 8.3–8.7] Let $\pi$ be a set of primes and $G$ a group of Lie type over the field $\mathbb{F}_q$ of characteristic $p \in \pi$. Assume, $G \in \mathcal{E}_\pi$ and $H \in \mathrm{Hall}_\pi(G)$. Then one of the following statements holds.

(1) $H = G$. 

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$J_1$ & $\{2, 3\}$ & $2 \times \mathrm{Alt}_4$
\hline
$\{2, 3, 5\}$ & $2 \times \mathrm{Alt}_5$
\hline
$\{2, 3, 7\}$ & $2^3 : 7 : 3$
\hline
$\{2, 7\}$ & $2^3 : 7$
\hline
$J_4$ & $\{2, 3, 5\}$ & $2^{11} : (2^6 : 3 : \mathrm{Sym}_6)$
\hline
\end{tabular}
\end{table}
Lemma 2.20 [42, Lemma 4.3], [51, Theorem 8.12] Assume $G = SL_n^v(q)$ is a special linear or unitary group with the base field $\mathbb{F}_q$ of characteristic $p$ and $n \geq 2$. Let $\pi$ be a set of primes such that $2, 3 \in \pi$ and $p \not\in \pi$. Then the following statements hold.

(A) Suppose $G \in \mathcal{E}_\pi$, and $H$ is a $\pi$-Hall subgroup of $G$. Then for $G$, $H$, and $\pi$ one of the following statements holds.

(a) $n = 2$ and one of the statements (a)–(d) of Lemma 2.17 holds.

(b) $n = 4$, $(n, q - 1) = 1$, $s = 2$, $n_1, n_2 \in \{1, n - 1\}$;

(c) $n = 5$, $(2 \cdot 3, q - 1) = 1$, $s = 2$, $n_1 = n_2 = 2$;

(d) $n = 5$, $(2 \cdot 3 \cdot 5, q - 1) = 1$, $s = 3$, $n_1, n_2, n_3 \in \{1, 2\}$;

(e) $n = 7$, $(5 \cdot 7, q - 1) = 1$, $(3, q + 1) = 1$, $s = 2$, $n_1, n_2 \in \{3, 4\}$;

(f) $n = 8$, $(2 \cdot 5 \cdot 7, q - 1) = 1$, $(3, q + 1) = 1$, $s = 2$, $n_1 = n_2 = 4$;

(g) $n = 11$, $(2 \cdot 3 \cdot 7 \cdot 11, q - 1) = 1$, $(5, q + 1) = 1$, $s = 2$, $n_1, n_2 \in \{5, 6\}$.

Pic. 1. Dynkin diagram of the root system of $D_l(q)$.
(b) either $q \equiv \eta \pmod{12}$, or $n = 3$ and $q \equiv \eta \pmod{4}$, $\text{Sym}_n$ satisfies $\mathcal{E}_\pi$, $\pi \cap \pi(G) \subseteq \pi(q-\eta) \cup \pi(n!)$, and if $r \in (\pi \cap \pi(n!)) \setminus \pi(q-\eta)$, then $|G|_r = |\text{Sym}_n|_r$; $H$ is included in

$$M = L \cap G \cong \mathbb{Z}^{n-1} \cdot \text{Sym}_n,$$

where $L = Z \cdot \text{Sym}_n \leq \text{GL}_n^\eta(q)$ and $Z = \text{GL}_1^\eta(q)$ is a cyclic group of order $q - \eta$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(c) $n = 2m + k$, where $k \in \{0, 1\}$, $m \geq 1$, $q \equiv -\eta \pmod{3}$, $\pi \cap \pi(G) \subseteq \pi(q^2 - 1)$, the groups $\text{Sym}_m$ and $\text{GL}_2^\eta(q)$ satisfy $\mathcal{E}_\eta$, and

$$M = L \cap G \cong (\text{GL}_2^\eta(q) \circ \cdots \circ \text{GL}_2^\eta(q)) \cdot \text{Sym}_m \circ Z,$$

$m$ times

where $L = \text{GL}_2^\eta(q) \cdot \text{Sym}_m \times Z \leq \text{GL}_n(q)$ and $Z$ is a cyclic group of order $q - \eta$ for $k = 1$ and $Z$ is trivial for $k = 0$. The subgroup $H$ acting by conjugation on the set of factors in the central product

$$\underbrace{\text{GL}_2^\eta(q) \circ \cdots \circ \text{GL}_2^\eta(q)}_{m \text{ times}}$$

has at most two orbits. The intersection of $H$ with each factor $\text{GL}_2^\eta(q)$ in (2.3) is a $\pi$-Hall subgroup in $\text{GL}_2^\eta(q)$. The intersections with the factors from the same orbit all satisfy the same statement (a) or (b) of Lemma 2.18. Two $\pi$-Hall subgroups of $M$ are conjugate in $G$ if and only if they are conjugate in $M$.

Moreover $M$ possesses one, two, or four classes of conjugate $\pi$-Hall subgroups, while all subgroups $M$ are conjugate in $G$.

(d) $n = 4$, $\pi \cap \pi(G) = \{2, 3, 5\}$, $q \equiv 5\eta \pmod{8}$, $(q + \eta)_3 = 3$, $(q^2 + 1)_5 = 5$, and $H \cong 4 \cdot 2^3 \cdot \text{Alt}_6$. In this case, $G$ possesses exactly two classes of conjugate $\pi$-Hall subgroups of this type and $\text{GL}_4^\eta(q)$ interchanges these classes.

(e) $n = 11$, $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{(2,3)} = 24$, $q \equiv -\eta \pmod{3}$, $q \equiv \eta \pmod{4}$, $H$ is included in a subgroup $M = L \cap G$, where $L$ is a subgroup of $G$ of type $((\text{GL}_2^\eta(q) \cdot \text{Sym}_4) \downarrow (\text{GL}_4^\eta(q) \cdot \text{Sym}_3))$, and

$$H = (((Z \circ 2 \cdot \text{Sym}_4) \cdot \text{Sym}_4) \times (Z \cdot \text{Sym}_4)) \cap G,$$

where $Z$ is a Sylow 2-subgroup of a cyclic group of order $q - \eta$. All $\pi$-Hall subgroups of this type are conjugate in $G$. 

Pic. 2. Dynkin diagram of the root system of $^2D_4(q)$. 

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The Dynkin diagram of the root system of $^2D_4(q)$ is shown in the figure. The root system is described by the graph, where each node represents a root and the connections between nodes represent the relationships between the roots.

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(B) Conversely, if the conditions on $\pi$, $n$, $\eta$, and $q$ in one of statements (a)–(e) are satisfied, then $G \in \mathcal{E}_\pi$.

**Lemma 2.21** [12] Lemma 4.4, [51] Theorem 8.13 Let $G = \text{Sp}_{2n}(q)$ be a symplectic group over a field $\mathbb{F}_q$ of characteristic $p$. Assume that $\pi$ is a set of primes such that $2, 3 \in \pi$ and $p \not\in \pi$. Then the following statements hold.

(A) Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then both $\text{Sym}_n$ and $\text{SL}_2(q)$ satisfy $\mathcal{E}_\pi$ and $\pi \cap \pi(G) \subseteq \pi(q^2 - 1)$. Moreover, $H$ is a $\pi$-Hall subgroup of

$$M = \text{Sp}_2(q) \wr \text{Sym}_n \cong \left( \frac{\text{SL}_2(q) \times \cdots \times \text{SL}_2(q)}{\text{Sym}_n} \right) : \text{Sym}_n \leq G. \quad (2.4)$$

(B) Conversely, if both $\text{Sym}_n$ and $\text{SL}_2(q)$ satisfy $\mathcal{E}_\pi$ and $\pi \cap \pi(G) \subseteq \pi(q^2 - 1)$, then $M \in \mathcal{E}_\pi$ and every $\pi$-Hall subgroup $H$ of $M$ is a $\pi$-Hall subgroup of $G$.

(C) Two $\pi$-Hall subgroups of $M$ are conjugate in $G$ if and only if they are conjugate in $M$.

**Lemma 2.22** [12] Lemma 6.7, [51] Theorem 8.14 Assume that $G = \Omega^\eta_q(n)$, $\eta \in \{+,-,o\}$, $q$ is a power of a prime $p$, $n \geq 7$, $\varepsilon = \varepsilon(q)$. Let $\pi$ be a set of primes such that $2, 3 \in \pi$, $p \not\in \pi$. Then the following statements hold.

(A) If $G$ possesses a $\pi$-Hall subgroup $H$, then one of the following statements holds.

(a) $n = 2m + 1$, $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $\pi \equiv \varepsilon \pmod{12}$, $\text{Sym}_m \in \mathcal{E}_\pi$, and $H$ is a $\pi$-Hall subgroup in $M = \left( \text{O}^m_2(q) \wr \text{Sym}_m \times \text{O}_1(q) \right) \cap G$. All $\pi$-Hall subgroup of this type are conjugate.

(b) $n = 2m$, $\eta = \varepsilon^m$, $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $\pi \equiv \varepsilon \pmod{12}$, $\text{Sym}_m \in \mathcal{E}_\pi$, and $H$ is a $\pi$-Hall subgroup in $M = \left( \text{O}^m_2(q) \wr \text{Sym}_m \right) \cap G$. All $\pi$-Hall subgroup of this type are conjugate.

(c) $n = 2m$, $\eta = -\varepsilon^m$, $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $\pi \equiv \varepsilon \pmod{12}$, $\text{Sym}_{m-1} \in \mathcal{E}_\pi$, and $H$ is a $\pi$-Hall subgroup of $M = \left( \text{O}^m_2(q) \wr \text{Sym}_{m-1} \times \text{O}^m_2(q) \right) \cap G$. All $\pi$-Hall subgroup of this type are conjugate.

(d) $n = 11$, $\pi \cap \pi(G) = \{2, 3\}$, $\pi \equiv \varepsilon \pmod{12}$, $(q^2 - 1)_{\pi} = 24$, and $H$ is a $\pi$-Hall subgroup of $M = \left( \text{O}^2_2(q) \wr \text{Sym}_4 \times \text{O}_1(q) \right) \cap G$. All $\pi$-Hall subgroup of this type are conjugate.

(e) $n = 12$, $\eta = -\pi \cap \pi(G) = \{2, 3\}$, $\pi \equiv \varepsilon \pmod{12}$, $(q^2 - 1)_{\pi} = 24$, and $H$ is a $\pi$-Hall subgroup of $M = \left( \text{O}^2_2(q) \wr \text{Sym}_4 \times \text{O}_1(q) \right) \cap G$. There exist precisely two classes of conjugate subgroups of this type in $G$, and the automorphism of order 2 induced by the group of similarities of the natural module interchanges these classes.

(f) $n = 7$, $\pi \cap \pi(G) = \{2, 3, 5, 7\}$, $|G|_\pi = 2^9 \cdot 3^4 \cdot 5 \cdot 7$, and $H \cong \Omega^2_7(2)$. There exist precisely two classes of conjugate subgroups of this type in $G$, and $\text{SO}_7(q)$ interchanges these classes.

(g) $n = 8$, $\eta = +\pi \cap \pi(G) = \{2, 3, 5, 7\}$, $|G|_\pi = 2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$, and $H \cong 2 \cdot \Omega^5_8(2)$. There exist precisely four classes of conjugate subgroups of this type in $G$. The subgroup of $\text{Out}(G)$ generated by diagonal and graph automorphisms is
isomorphic to Sym$_4$ and acts on the set of these classes as Sym$_4$ in its natural permutation representation, and every diagonal automorphism acts without fixed points.

(h) $n = 9$, $\pi \cap \pi(G) = \{2, 3, 5, 7\}$, $|G|_\pi = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, and $H \cong 2.O^+_8(2).2$. There exist precisely two classes of conjugate subgroups of this type in $G$, and SO$_9(q)$ interchanges these classes.

(B) Conversely, if one of the statements (a)–(h) holds, then $G$ possesses a $\pi$-Hall subgroup with the given structure.

**Lemma 2.23** [2 Lemmas 7.1–7.6], [51 Theorem 8.13] Assume that

$$G \in \{E_6^0(q), E_7(q), E_8(q), F_4(q), G_2(q),^3D_4(q)\},$$

where $q$ is a power of a prime $p$. Let $\varepsilon = \varepsilon(q)$. Let $\pi$ be a set of primes such that $2, 3 \in \pi$, $p \notin \pi$. Then $G$ contains a $\pi$-Hall subgroup $H$ if and only if one of the following statements hold:

(a) $G$ is a group in Table 3 and the values for the untwisted Lie rank $l$ of $G$, $\delta$ and the structure of the Weyl group $W$ are given in the Table 3: if $G = E_6^0(q)$ then $\eta = \varepsilon$; $\pi(W) \subseteq \pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $H$ is a $\pi$-Hall subgroup of a group $T . W$, where $T$ is a maximal torus of order $(q - \varepsilon)^l/\delta$. All $\pi$-Hall subgroups of this type are conjugate in $G$;

(b) $G = ^3D_4(q)$, $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$ and $H$ is a $\pi$-Hall subgroup in $T . W(G_2)$, where $T$ is a maximal torus of order $(q - \varepsilon)(q^3 - \varepsilon)$. All $\pi$-Hall subgroups of this type are conjugate in $G$;

(c) $G = E_6^\varepsilon(q)$, $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$ and $H$ is a $\pi$-Hall subgroup in $T . W(F_4)$, where $T$ is a maximal torus of order $(q^2 - 1)^2(q - \varepsilon)^3/(3, q + \varepsilon)$, $24$, $q^4 + q^2 + 1) = 7$, $H \cong G_2(2)$, and all $\pi$-Hall subgroups of this type are conjugate in $G$.

| $G$       | $l$     | $\delta$ | $W$                              | $|W|$     |
|-----------|---------|-----------|----------------------------------|----------|
| $E_6^0(q)$| 6       | $(3, q - \eta)$ | $W(E_6) \cong Sp_4(3)$            | $2^7.3^4.5$ |
| $E_7(q)$  | 7       | 2         | $W(E_7) \cong 2 \times P\Omega_7(2)$ | $2^{10}.3^4.5.7$ |
| $E_8(q)$  | 8       | 1         | $W(E_8) \cong 2 \cdot P\Omega_8^+(2)$ | $2^{14}.3^5.5^2.7$ |
| $F_4(q)$  | 4       | 1         | $W(F_4)$                         | $2^7.3^2$ |
| $G_2(q)$  | 2       | 1         | $W(G_2)$                         | $2^3$    |

Table 3: Weyl groups of exceptional root systems

**Lemma 2.24** [16 Theorem 3.1] Let $G$ be a group of Lie type with base field $\mathbb{F}_q$ of some characteristic $p$. Assume that $\pi$ is such that $2, p \in \pi$, and $3 \notin \pi$. Suppose $G \in E_\pi$ and $H \in \text{Hall}_\pi(G)$. Then $p = 2$ and one of the following statements holds.

(1) $\pi \cap \pi(G) \subseteq \pi(q - 1) \cup \{2\}$, a Sylow 2-subgroup $P$ of $H$ is normal in $H$ and $H/P$ is Abelian.
(2) $p = 2$, $G \cong B_2(2^{2n+1})$ and $\pi(G) \subseteq \pi$.

**Lemma 2.25** [21, Lemma 4] Let $G$ be a nonabelian simple group. Then $G \in \mathcal{D}_{(2,3)}$ if and only if $G$ is a Suzuki group $2B_2(q)$. In this case every $\pi$-subgroup of $G$ is 2-group.

**Lemma 2.26** [11, Lemma 5.1 and Theorem 5.2] Let $G$ be a group of Lie type over a field of characteristic $p$. Assume that $\pi$ is such that $3, p \notin \pi$ and $2 \in \pi$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then either $H$ possesses a normal abelian 2'-Hall subgroup or $G \cong 2G_2(3^{2n+1})$ and $\pi \cap \pi(G) = \{2, 7\}$.

**Lemma 2.27** Let $G$ be a simple nonabelian group. Assume that $\pi$ is such that $\pi(G) \not\subseteq \pi$, $|\pi \cap \pi(G)| > 1$, $2 \in \pi$ and $3 \notin \pi$. Suppose $G \in \mathcal{E}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then the following statements hold.

1. $H$ is solvable.

2. If every solvable $\pi$-subgroup of $G$ is conjugate to a subgroup of $H$, then $G$ is a group of Lie type over a field of characteristic $p \notin \pi$ and $G \in \mathcal{D}_\pi$.

**Proof.** Statement (1) is proved in [21, Lemma 10].

Prove (2). Consider all possibilities for $G$, according to the classification of finite simple groups (see [3, Theorem 0.1.1]).

**Case 1:** $G \cong \text{Alt}_n$, $n \geqslant 5$. This case is impossible by Lemma 2.14.

**Case 2:** $G$ is either a sporadic group or a Tits group. By Lemma 2.16 it follows that $G \cong J_1$ and $\pi \cap \pi(G) = \{2, 7\}$. Now, by the Burnside theorem [9, Ch. I, 2], every $\pi$-subgroup of $G$ is solvable and is conjugate to a subgroup of $H$, that is $G \in \mathcal{D}_\pi$. It contradicts Lemma 2.4.

**Case 3:** $G$ is a group of Lie type over a field $\mathbb{F}_q$ of characteristic $p \in \pi$. By Lemma 2.24 $p = 2$, $H$ is solvable and a Sylow 2-subgroup $P$ of $H$ is normal in $H$. Moreover, $\pi \cap \pi(G) \subseteq \pi(q - 1) \cup \{2\}$. Condition $|\pi \cap \pi(G)| > 1$ implies that $q > 2$. It is known that $G$ has a subgroup which is a homomorphic image of $\text{SL}_2(q) = \text{PSL}_2(q)$. Since $\text{PSL}_2(q)$ is simple, we assume that $\text{PSL}_2(q) \leqslant G$. Take $r \in \pi \cap \pi(q - 1)$. Then $\text{PSL}_2(q)$ contains a dihedral subgroup $U$ of order $2r$ and $U$ has no normal Sylow 2-subgroups. Hence, $U$ is not conjugate to any subgroup of $H$.

**Case 4:** $G$ is a group of Lie type over a field of characteristic $p \notin \pi$. Lemma 2.26 implies that either $H$ possesses a normal abelian 2'-Hall subgroup or $G \cong 2G_2(3^{2n+1})$, $\pi \cap \pi(G) = \{2, 7\}$. In the last case every $\pi$-subgroup of $G$ is solvable by the Burnside theorem [9, Ch. I, 2] and $G \in \mathcal{D}_\pi$. Suppose, $H$ possesses a normal abelian 2'-Hall subgroup. It is sufficient to prove that every $\pi$-subgroup of $G$ is solvable. Suppose $U$ is a nonsolvable subgroup of $G$. Then $U/U_\pi \neq 1$ and every minimal subnormal subgroup of $U/U_\pi$ is nonabelian simple and is isomorphic to a Suzuki group $2B_2(2^{2m+1})$ in view of condition $3 \notin \pi$ and the Thompson-Glauberman theorem [13, Chapter II, Corollary 7.3]. Take in $U$ the full preimage $V$ of a Borel subgroup of $2B_2(2^{2m+1})$. Then $V$ is solvable and is conjugate to a subgroup of $H$. In particular, $V$ and a Borel subgroup $V/U_\pi$ of $2B_2(2^{2m+1})$ possesses a normal 2'-Hall subgroup, but this is not true.

**Lemma 2.28** Let $G \in \mathcal{D}_\pi$ be a nonabelian simple group. Then either $G$ is a $\pi$-group or every $\pi$-Hall subgroup of $G$ is solvable. In particular, if $G$ is not a $\pi$-group then $G \in \mathcal{D}_\pi$ for every $\pi \subseteq \pi$.

**Proof.** Lemma follows from Lemmas 2.25 and 2.27 and the solvability of groups of odd order [11].
2.5 Degrees of minimal faithful permutation representation

In the following Lemma we collect some statements about minimal degrees of faithful permutation representations of some groups.

**Lemma 2.29** The following statements hold.

1. If $H \leq G$ then $\mu(H) \leq \mu(G)$.

2. [27] Theorem 2] Let $G$ be a finite group. Let $\mathfrak{L}$ be a complete class of finite groups. Let $N$ be the $\mathfrak{L}$-radical of $G$, that is the maximal normal $\mathfrak{L}$-subgroup of $G$. Then $\mu(G) \geq \mu(G/N)$.

3. [10] Theorem 3.1] If $G = L_1 \times L_2 \times \cdots \times L_r$ and $L_1, L_2, \ldots, L_r$ are simple then $\mu(G) = \mu(L_1) + \mu(L_2) + \cdots + \mu(L_r)$.

4. If $G$ is simple then $\mu(G)$ is equal to the minimum of indices of maximal subgroups in $G$.

5. $\mu(\text{Sym}_n) = \mu(\text{Alt}_n) = n$.

2.6 Some subgroups of quasisimple and almost simple groups

**Lemma 2.30** [4, Tables 8.1 and 8.2] Assume that $q^2 \equiv 1 \pmod{5}$ and $q$ is a power of an odd prime. Then $\text{SL}_2(q)$ contains a subgroup isomorphic to $\text{SL}_2(5)$ and $\text{PSL}_2(q)$ contains a subgroup isomorphic to $\text{PSL}_2(5) \cong \text{Alt}_5$.

**Lemma 2.31** [4, Tables 8.8 and 8.10] Assume that $q \equiv \eta \pmod{4}$, where $q$ is a power of an odd prime and $\eta = \pm 1$. Then $\text{SL}_4^\eta(q)$ contains a subgroup isomorphic to $4 \circ 2^{1+4}.\text{Alt}_6$.

**Lemma 2.32** [35, Lemma 1.24] Let $l$ be an odd prime, $q > 2$ be a power of a prime. Assume $G = \langle D, x \rangle$, where $D$ is the group of all diagonal matrices of determinant 1, and

$$x = \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots & 0 & 1 \\
1 & 0 & 1 & 0 & \cdots & 0 & 1 \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{pmatrix} \in \text{SL}_r(q).$$

Then $|G| = (q - 1)^{r-1}r$ and $G$ is absolutely irreducible.

**Lemma 2.33** Let $q$ be a power of an odd prime. Let pair $(G^*, m)$, where $G^*$ is a quasisimple group and $m$ is an positive integer, appear in the following list:

1. $(G^*, m) = (\text{SL}_n^\eta(q), [n/2]), n > 2, \eta = \pm$;
2. $(G^*, m) = (\text{Sp}_n(q), n/2), n > 2$ is even;
3. $(G^*, m) = (\Omega_n(q), 2[n/4]), n > 5$ is odd;
4. $(G^*, m) = (\Omega^n_n(q), 2[n/4]), n > 6$ is even;
(5) \((G^*, m) = (\Omega_n^f(q), 2([n - 1]/4)), n > 6\) is even;

(6) \((G^*, m) = (E_6^0(q), 4), G^*\) is a quotient of the universal group by a central subgroup;

(7) \((G^*, m) = (E_7(q), 7), G^*\) is a quotient of the universal group by a central subgroup;

(8) \((G^*, m) = (E_8(q), 8)\).

Then \(G^*\) contains a collection \(\Delta\) of subgroups such that

(a) every member of \(\Delta\) is isomorphic to \(SL_2(q)\),

(b) if \(K^*, L^* \in \Delta\) are distinct then \([K^*, L^*] = 1\), and

(c) \(|\Delta| = m\).

**Proof.** Lemma follows from Aschbacher’s theory of fundamental subgroups. Recall that, if \(G^*\) is a group from the lemma, then \(K^*\) is a fundamental subgroup, if \(K^*\) is conjugate to a subgroup generated by a long root subgroup \(U\) and its opposite \(U\). Every fundamental subgroup of \(G^*\) is isomorphic to \(SL_2(q)\). Fix a Sylow 2-subgroup \(S^*\) of \(G^*\) and consider \(\Delta = \text{Fun}(S^*)\) consisting of all fundamental subgroups \(L^*\) such that \(S^* \cap L^*\) is a Sylow 2-subgroup of \(L^*\). It follows from [1] 6.2 that distinct elements of \(\Delta\) elementwise commute and follows from [2] Theorem 2 that \(|\Delta| = m\). \(\square\)

**Lemma 2.34** Let \(G = \text{Sym}_n\), where \(n \geq 5\), and \(G\) contains a subgroup \(H\) having \(\text{Alt}_m\) as a homomorphic image for some \(m \in \{n - 1, n\}\). Then \(H \cong \text{Alt}_m\).

**Proof.** Suppose, \(m = n\). In this case \(|G : H| \leq 2\) and \(H \in \{\text{Sym}_n, \text{Alt}_n\}\). Since \(\text{Sym}_n\) has no \(\text{Alt}_n\) as a homomorphic image, we have \(H = \text{Alt}_n\).

Suppose, \(m = n - 1\). First of all, note the following well-known fact: every subgroup \(H_0\) of \(G\) of index \(n\) is isomorphic to \(\text{Sym}_{n-1}\). Indeed, let \(K\) be the kernel of the action of \(G\) by right multiplication on the set \(\Omega\) of right cosets of \(H\) in \(G\). Then \(K \leq H_0\) and \(|G : K| \geq |G : H_0| = n > 2\). Since \(G\) has a unique minimal normal subgroup \(\text{Alt}_n\) and its index equals 2, we have \(K = 1\). Therefore, \(G\) is embedded in \(\text{Sym}(\Omega) \cong \text{Sym}_n\) and \(G \cong \text{Sym}(\Omega)\). Since \(H_0\) is a point stabilizer in \(G\), we have \(H_0 \cong \text{Sym}_{n-1}\).

Now let \(L\) be the kernel of an epimorphism \(H \to \text{Alt}_{n-1}\). We need to show that \(L = 1\). If not, then

\(|H| = |L||\text{Alt}_{n-1}| \geq 2|\text{Alt}_{n-1}| = (n - 1)!\) and \(|G : H| \leq n|\),

Since \(G\) has no proper subgroups of index less than \(n\), except \(\text{Alt}_n\), we have either \(H = G = \text{Sym}_n\), or \(H \cong \text{Sym}_{n-1}\). But \(\text{Alt}_{n-1}\) is not a homomorphic image of any of the groups \(S_n\) or \(\text{Sym}_{n-1}\). \(\square\)

**Lemma 2.35** Let \(G = \text{Sym}_n\). Then \(H = N_G(H)\) if \(H \leq G\) and \(|G : H|\) is odd.

**Proof.** Let \(S\) be a Sylow 2-subgroup of \(H\). Then \(S\) is a Sylow 2-subgroup of \(G\) and \(S = N_G(S)\) by [3] Lemma 4. So, \(S = N_G(S) \leq H\), and \(H = N_G(H)\) by the Frattini Argument. \(\square\)

**Lemma 2.36** Let \(H\) be a \(\pi\)-Hall subgroup of \(G = L \wr \text{Sym}_n\). Denote by \(L_1 \times \ldots \times L_n\) the base of the wreath product. Assume that \(L_i\) possesses a \(\pi\)-Hall subgroup that is isomorphic to \(H \cap L_i\) and is not conjugate with \(H \cap L_i\) in \(L_i\). Then \(G\) possesses a \(\pi\)-Hall subgroup \(K\) such that \(H\) and \(K\) have the same composition factors and are not conjugate in \(G\).
PROOF. We can assume for the simplicity that \( i = 1 \). We set \( A = L_1 \times \ldots \times L_n \) and \( H_1 = H \cap L_1 \) and denote by \( K_1 \) a subgroup of \( L_1 \) that is isomorphic to \( H_1 \) but is not conjugate to \( H_1 \). Note that \( G \) acts on the set \( \Omega = \{ L_1, \ldots, L_n \} \) via conjugation and \( A \) is the kernel of this action. Moreover, it follows from the definition of a wreath product that \( N_G(L_1) = C_G(L_1)L_1 \).

Renumbering \( \{ L_1, \ldots, L_n \} \), if necessary, we may choose a right transversal \( h_1 = 1, \ldots, h_m \) of \( N_H(L_1) \) in \( H \) so that \( L_1^{h_i} = L_i \). Then \( L_1^{h_i} \neq L_1^{h_j} \) if \( i \neq j \). In particular, \( m \leq n \). So \( \{ L_1, \ldots, L_m \} \) is an orbit of \( H \) on \( \Omega \). Thus both \( \Delta = \{ L_1, \ldots, L_m \} \) and \( \Gamma = \{ L_{m+1}, \ldots, L_n \} \) are \( H \)-invariant. Set

\[
K_i = \begin{cases} 
K_1^{h_i} & \text{for } i = 1, \ldots, m \\
H \cap L_i & \text{for } i = m + 1, \ldots, n 
\end{cases}
\]

and \( K_0 = \langle K_i \mid i = 1, \ldots, n \rangle = K_1 \times \ldots \times K_n \). By construction, \( K_0 \leq A \) and \( K_0 \cong H \cap A \in \text{Hall}_\pi(A) \).

We claim that for every \( h \in H \) there exists \( a \in A \) such that \( K_0^h = K_0^a \).

Indeed, take \( h \in H \). Then there exists \( \sigma \in \text{Sym}_n \) such that \( L_1^h = L_i^\sigma \) for \( i = 1, \ldots, n \). Since \( \Delta \) and \( \Gamma \) are both \( H \)-invariant, we obtain that \( i^\sigma \in \{ 1, \ldots, m \} \) for \( i = 1, \ldots, m \) and \( i^\sigma \in \{ m + 1, \ldots, n \} \) for \( i = m + 1, \ldots, n \).

Take \( i \leq m \). Then \( h_i h = x h_i \) for some \( x \in N_H(L_1) \). In this case

\[
K_i^h = K_1^{h_i} = K_1^{xh_i}. 
\]

Since \( x \in N_H(L_1) \leq N_G(L_1) = C_G(L_1)L_1 \) and \( K_1 \leq L_1 \), \( K_i^h = K_i^a \) for some \( b \in L_1 \). Set \( b_i = b_i^{h_i} \in L_i \). Then we have

\[
K_i^h = K_1^{xh_i} = K_1^{bh_i} = (K_1^{h_i})^{b^h} = K_i^b. 
\]

Thus, we see that there are \( b_1 \in L_1, \ldots, b_m \in L_m \) such that

\[
K_i^h = K_i^{b_i}, 
\]

for every \( i \leq m \).

Let \( a = b_1 \ldots b_m \). We show that \( K_0^h = K_0^a \). Indeed, we have seen that \( K_i^h = K_i^{b_i} = K_i^a \) if \( i \leq m \). If \( i > m \) then

\[
K_i^h = H \cap L_i^h = H \cap L_i^\sigma = K_i^a, 
\]

since \( a \) centralizes \( K_j \) for all \( j > m \). Hence,

\[
K_0^h = \langle K_i^h \mid i = 1, \ldots, n \rangle = \langle K_i^a \mid i = 1, \ldots, n \rangle = K_0^a. 
\]

Now Lemma 2.30 implies that there is \( K \in \text{Hall}_\pi(HA) \subseteq \text{Hall}_\pi(G) \) such that \( K_0 = K \cap A \).

The groups \( H \) and \( K \) have the same composition factors, since \( K/K \cap A \cong KA/A = HA/A \cong H/H \cap A \) and \( K \cap A = K_0 \cong H \cap A \).

Suppose, \( K = H^g \) for some \( g \in G \). Then the image of \( g \) in \( G/A \) normalizes \( HA/A = KA/A \). Note that \( 2 \in \pi \) in view of Lemma 2.22. Therefore, the index of \( HA/A \) in \( G/A \cong \text{Sym}_n \) is odd. Lemma 2.35 implies that \( g \in H/A \), and so we may assume that \( g \in A \). Thus \( K \cap A = H^g \cap A \), i.e. \( K_0 = K \cap A = K_1 \times \ldots \times K_n \) and \( H \cap A = H_1 \times \ldots \times H_n \) are conjugate in \( A = L_1 \times \ldots \times L_n \). In particular, \( K_1 \) and \( H_1 \) are conjugate in \( L_1 \), a contradiction. \( \square \)

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3 Proof of Theorem 2

Theorem 2 says that, for a finite simple group $G$, the following statements are equivalent:

(1) $G \in \mathcal{D}_X$ and

(2) either $G \in X$ or $\pi(G) \nsubseteq \pi$ and $G \in D_\pi$.

(2) $\Rightarrow$ (1). Obviously $X \subseteq D_X$. So we need to prove that if $\pi(G) \nsubseteq \pi$ and $G \in D_\pi$ then $G \in D_X$. Lemma 2.28 implies that every $\pi$-Hall subgroup (hence every $\pi$-subgroup of $G$, since $G \in D_\pi$) is solvable, thus it belongs to $X$. On the other hand, every $X$-subgroup of $G$ is a $\pi$-subgroup and so is contained in a $\pi$-Hall subgroup (we again use $G \in D_\pi$ here). Therefore, $m_X(G) = \text{Hall}_\pi(G)$. Hence every two $X$-maximal subgroups of $G$ are conjugate, i.e. $G \in D_X$.

(1) $\Rightarrow$ (2). This implication is much harder to prove. Its proof requires case by case consideration and we organize it in a series of steps, and divide it in subsections.

3.1 Proof of the implication (1) $\Rightarrow$ (2): general remarks

Assume that $G \in D_X$ and $G \notin X$. We need to show that $G \in D_\pi$.

Lemma 2.3 implies that

(i) $m_X(G) = X \cap \text{Hall}_\pi(G) = \text{Hall}_X(G)$. In particular, $G \in \mathcal{E}_\pi$ and all elements of

Hall$_X(G)$ are conjugate.

Suppose by contradiction that $G \notin D_\pi$. Then

(ii) There exists a $\pi$-subgroup of $G$ which does not belong to $X$.

Otherwise the $\pi$-subgroups of $G$ are exactly the $X$-subgroups, thus the $\pi$-maximal subgroups of $G$ are conjugate, i.e. $G \in D_\pi$.

The inclusion $\mathcal{G}_\pi \subseteq X$ and (ii) immediately imply

(iii) There exists a non-solvable $\pi$-subgroup in $G$.

The solvability of primary and biprimary groups [9, Ch. I, 2] and (iii) implies

(iv) $|\pi \cap \pi(G)| > 2$.

The Feit–Thompson theorem [11] implies

(v) $2 \in \pi \cap \pi(G)$.

Moreover, it follows from (v) and Lemma 2.27 that

(vi) $3 \in \pi \cap \pi(G)$.

Now we prove that

(vii) $G$ has no solvable $\pi$-Hall subgroups.

Indeed, if $G$ has a solvable $\pi$-Hall subgroup $H$, then $H \in X \cap \text{Hall}_\pi(G) = m_X(G)$. In view of (v), (vi) and the Hall theorem, $H$ contains a $\{2,3\}$-Hall subgroup $H_0$ and $H_0 \in \text{Hall}_{\{2,3\}}(G)$. Take an arbitrary $\{2,3\}$-subgroup $U$ in $G$. Since $U$ is solvable and in view of (v) and (vi) we have $U \in X$. Now $G \in D_X$ implies that $U$ is conjugate to a subgroup of $H$. Moreover, the solvability of $H$ means that $U$ is conjugate to a subgroup of $H_0$ by the Hall theorem. Hence $G \in D_{\{2,3\}}$, a contradiction with Lemma 2.25.

Now we exclude all possibilities for $G$, considering finite simple groups case by case, according to the Classification of the finite simple groups.
3.2 Alternating groups

The following statement follows from Lemma 2.15

(viii) $G$ is not isomorphic to an alternating group.

3.3 Sporadic groups and Tits group

Now, exclude any possibilities for $G$ to be a sporadic group.

(ix) $G$ is not isomorphic to the Mathieu group $M_{11}$.

Suppose, $G = M_{11}$. According to Lemma 2.16 and Table 2 and in view of (v)-(vii) it is sufficient to consider the situation $\pi \cap \pi(G) = \{2, 3, 5\}$ and a Hall $\mathcal{X}$-subgroup $H$ of $G$ is $M_{10} = \text{Alt}_6 \cdot 2$. Take a $\{2, 3\}$-Hall subgroup $U$ of $G$ (this group appears in Table 2). Since $U$ is a solvable $\pi$-group, we have $U \in \mathcal{X}$. Now $G \in \mathcal{X}$ implies that $U$ is conjugate to a subgroup of $H$. But this means that $H$ and its unique nonabelian composition factor $\text{Alt}_6$ satisfy $\delta_{\{2,3\}}$. A contradiction with Lemma 2.14.

(x) $G$ is not isomorphic to the Mathieu group $M_{22}$.

According to Lemma 2.16 and Table 2 if $G = M_{22}$ then an $\mathcal{X}$-Hall subgroup $H$ of $G$ is isomorphic to $2^4 : \text{Alt}_6$. But $G$ contains a maximal subgroup $U \cong 2^4 : \text{Sym}_5$ which is an $\mathcal{X}$-group and is not isomorphic to a subgroup of $H$.

(xi) $G$ is not isomorphic to the Mathieu group $M_{23}$.

Suppose, $G = M_{23}$ and $H \in \text{Hall}_{\mathcal{X}}(G)$. Lemma 2.16 and (vii) imply that one of the following cases holds.

(a) $\pi \cap \pi(G) = \{2, 3, 5\}$ and $H \cong 2^4 : \text{Alt}_6$;
(b) $\pi \cap \pi(G) = \{2, 3, 5\}$ and $H \cong 2^4 : (3 \times \text{Alt}_5)$;
(c) $\pi \cap \pi(G) = \{2, 3, 5, 7\}$ and $H \cong \text{PSL}_3(4) : 2$;
(d) $\pi \cap \pi(G) = \{2, 3, 5, 7\}$ and $H \cong 2^4 : \text{Alt}_7$;
(e) $\pi \cap \pi(G) = \{2, 3, 5, 7, 11\}$ and $H \cong M_{22}$.

In Case (a), we consider an $\mathcal{X}$-subgroup $U \cong 2^4 : (3 \times \text{Alt}_5)$ which is a $\pi$-Hall subgroup (and appears in Case (b)) and is not isomorphic to $H$.

Suppose, Case (b) holds. In $G$, consider a $\{2, 3\}$-subgroup $U \cong 3^2 : Q_8$, a Frobenius group which is contained in $\text{PSL}_3(4)$, see [7]. Suppose, $U$ is a subgroup of $H$. Let

$$\varphi : H \to H/O_2(H)$$

be the natural epimorphism. Since $U$ has no non-trivial normal 2-subgroups, we have

$$U \cong \overline{U} \leq \overline{H} \cong 3 \times \text{Alt}_5.$$ 

Now $|\overline{U}|_3 = |\overline{H}|_3 = 3^2$, i.e. $\overline{U}$ contains a Sylow 3-subgroup of $\overline{H}$ and the cyclic subgroup $O_3(H)$ of order 3 must be a normal subgroup in $\overline{U}$. But $U \cong \overline{U}$ has no normal subgroups of order 3. A contradiction.

We exclude Cases (c), (d) and (e), since the subgroup $H$ does not contain elements of order 15 in these cases while $M_{23}$ has a cyclic subgroup $U$ of order 15 and $U \in \mathcal{S}_\mathcal{X} \subseteq \mathcal{X}$. 

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(xii) $G$ is not isomorphic to the Mathieu group $M_{24}$.

If $G = M_{24}$ then an X-Hall subgroup $H$ is isomorphic to $2^6 : 3 \cdot \text{Sym}_6$. Consider an $X$ subgroup $U \cong 2^4 : \text{Alt}_6$ which is included in a maximal subgroup $M = M_{23}$ of $G$ and is a $\{2, 3, 5\}$-Hall subgroup of $M$. Since $G \in \mathcal{X}$, without loss of generality, we can assume that $U \leq H$. Now $U$ contains a subgroup $U_0 \cong \text{Alt}_6$ and, clearly, $U_0 \cap O_2(H) = 1$. Let

$$\pi : H \to H/O_2(H)$$

be the natural epimorphism. We have

$$\text{Alt}_6 \cong U_0 \cong \overline{U}_0 \leq \overline{H} \cong 3 \cdot \text{Sym}_6.$$ 

But this means that

$$\text{Alt}_6 \cong \overline{U}_0 = \overline{U}_0 \leq \overline{H} \cong 3 \cdot \text{Alt}_6$$

and we have a contradiction.

(xiii) $G$ is not isomorphic to the Janko group $J_1$.

Suppose, $G = J_1$ and $H \in \text{Hall}_X(G)$. It follows from Lemma 2.16, Table 2 and (vii) that $H \cong 2 \times \text{Alt}_5$. Clearly, $H$ contains no elements of order 15. But $G$ has a cyclic subgroup $U$ of order 15 (see [7]) and $U \in \mathcal{S}_X \subseteq \mathcal{X}$.

(xiv) $G$ is not isomorphic to the Janko group $J_4$.

Suppose, $G = J_1$ and $H \in \text{Hall}_X(G)$. Lemma 2.16 and Table 2 imply that

$$H \cong 2^{11} : 2^6 : 3 \cdot \text{Sym}_6.$$ 

We exclude this possibility arguing exactly as in (xii), because $G$ contains a subgroup isomorphic to $M_{24}$.

(xv) $G$ is not isomorphic to any sporadic group or a Tits group.

This statement follows from (v), Lemma 2.16 and (ix)-(xiv).

3.4 Groups of Lie type of characteristic in $p \in \pi(\mathcal{X})$

Now, according to Lemma 2.19 we exclude the possibilities for $G$ to be isomorphic to a group of Lie type whose characteristic belongs to $\pi$.

(xvi) If $G$ is a group of Lie type, then $G$ has no $\pi$-Hall subgroups contained in a Borel subgroup.

Since every Borel subgroup of $G$ is solvable, (xvi) follows from (vii).

(xvii) $G$ is not isomorphic to $D_l(q)$, where $q$ is a power of some $p \in \pi$. 

Suppose, $G \cong D_l(q)$ and the numeration of the roots in a fundamental root system $\Pi$ of $G$ is chosen as in the Dynkin diagram on Pic. 1. It follows from Lemma 2.19 that $q$ is a power of 2, $l$ is a Fermat prime (in particular, $l \geq 5$), and $(l, q - 1) = 1$. Moreover, if $H \in \text{Hall}_2(G)$, then $H$ is conjugate to the canonic parabolic maximal subgroup corresponding to the set $\Pi \setminus \{r_1\}$ of fundamental roots. This parabolic subgroup has a composition factor isomorphic to $D_{l-1}(q)$. Since $\mathfrak{X}$ is a complete class, we obtain that

$$D_l(q) \in \mathfrak{X} \quad \text{for} \quad i \leq l - 1, \quad \text{and} \quad A_1(q) \in \mathfrak{X}.$$ 

Moreover, $\pi(q - 1) \subseteq \pi$. Consider the canonic parabolic maximal subgroup $P_J$ of $G$, corresponding to the set $J = \Pi \setminus \{r_2\}$. Above remarks and the completeness of $\mathfrak{X}$ under extensions implies that $P_J \in \mathfrak{X}$: the nonabelian composition factors of $P$ are isomorphic to $D_{l-2}(q)$ and, possibly, $A_1(q)$, while the orders of abelian composition factors belong to $\pi(q - 1) \cup \{2\} \subseteq \pi$. But the maximality of $P$ means that $P_J$ is not conjugate to any subgroup of $H$, a contradiction with $G \in \mathfrak{D}_\mathfrak{X}$.

(xviii) $G$ is not isomorphic to $^2D_l(q)$, where $q$ is a power of some $p \in \pi$.

Suppose, $G \cong ^2D_l(q)$ and the numeration of the roots in a fundamental root system $\Pi^1$ of $G$ is chosen as in the Dynkin diagram on Pic. 2. It follows from Lemma 2.19 that $q$ is a power of 2, $l - 1$ is a Mersenne prime, and $(l - 1, q - 1) = 1$. Take $H \in \text{Hall}_2(G)$. Then $H$ is conjugate to the canonic parabolic maximal subgroup corresponding to the set $\Pi^1 \setminus \{r_1^1\}$ of fundamental roots. This parabolic subgroup has a composition factor isomorphic to $^2D_{l-1}(q)$ if $l > 4$ or isomorphic to $^2A_3(q)$ if $l = 4$. Consider the canonic parabolic maximal subgroup $P_J$ of $G$ which corresponds to the set $J = \Pi^1 \setminus \{r_2^1\}$ of fundamental roots. Arguing as in (xviii), we see that $P_J \in \mathfrak{X}$ and $P$ is not conjugate to any subgroup of $H$, a contradiction with $G \in \mathfrak{D}_\mathfrak{X}$.

(xix) $G$ is not isomorphic to $A_{l-1}(q) \cong \text{PSL}_l(q)$, where $q$ is a power of some $p \in \pi$.

Suppose, $G = \text{PSL}_n(q)$, where $q$ is a power of some $p \in \pi$, and let $G^* = \text{SL}_n(q)$. Lemma 2.3 implies that $G \in \mathfrak{D}_\mathfrak{X}$ if and only if $G^* \in \mathfrak{D}_\mathfrak{X}$. Thus, $G^* \in \mathfrak{D}_\mathfrak{X}$ and, moreover, it follows from (vi) and Lemma 2.3 that there are no solvable $\pi$-Hall subgroups in $G^*$.

Identify $G^*$ with $\text{SL}(V)$, where $V = \mathbb{F}_q^n$ is the natural $n$-dimensional module for $G^*$. Let $H^* \in \text{Hall}_\mathfrak{X}(G^*)$. By Lemma 2.19, $H^*$ is the stabilizer in $G^*$ of a series

$$0 = V_0 < V_1 < \cdots < V_s = V$$

of subspaces such that $\dim V_i/V_{i-1} = n_i$, $i = 1, 2, \ldots, s$, and one of the following conditions holds:

(a) $n$ is a prime, $s = 2, n_1, n_2 \in \{1, n - 1\}$;
(b) $n = 4, s = 2, n_1 = n_2 = 2$; moreover, $q = 2^{2t+1}$;
(c) $n = 5, s = 2, n_1, n_2 \in \{2, 3\}$;
(d) $n = 5, s = 3, n_1, n_2, n_3 \in \{1, 2\}$;
(e) $n = 7, s = 2, n_1, n_2 \in \{3, 4\}$;
(f) $n = 8, s = 2, n_1 = n_2 = 4$; moreover, $q = 2^t$;

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(g) $n = 11$, $s = 2$, $n_1, n_2 \in \{5, 6\}$.

In cases (a), (c), (e), and (g), $H^*$ is the stabilizer of a subspace of some dimension $m \neq n - m$ and the stabilizer $K^*$ of a subspace of dimension $n - m$ is isomorphic to $H^*$ (in particular, $K^* \in \mathfrak{X}$) but is not conjugate to $H^*$. It contradicts $G^* \in \mathcal{D}_\mathfrak{X}$.

If case (d) holds, then there are exactly three conjugacy classes of $\pi$-Hall subgroups with the same composition factors and $H^*$ belongs to one of them. Thus, case (d) is impossible for $G^* \in \mathcal{D}_\mathfrak{X}$.

Now consider cases (b) and (f). In these cases $n = 4$ and $n = 8$, respectively. Moreover, if $q = 2$ then case (b) holds and $G = \text{PSL}_4(2) \cong \text{Alt}_5 \notin \mathcal{D}_\mathfrak{X}$ in view of (viii). Therefore, we assume that $q > 2$ if $n = 4$. Define $r = n - 1 = 3$ in (b) and $r = n - 1 = 7$ in (f). It is easy to check that $r \in \pi$ in both cases. Consider the subgroup $U^*$ of $G^*$, consisting of all matrices of type

$$
\begin{pmatrix}
a & \\
1 & 
\end{pmatrix},
$$

where $a \in \langle D, x \rangle \subseteq \text{SL}_r(q)$, $D$ is the group of all diagonal matrices in $\text{SL}_r(q)$ and

$$
x = \begin{pmatrix}
0 & 1 & & \\
0 & 1 & & \\
& & \ddots & \\
1 & 0 & & 0
\end{pmatrix} \in \text{SL}_r(q).
$$

By Lemma 2.32 it follows that there is a subspace $W$ of $V$ of dimension $r$, such that $U^*$ acts irreducibly on $W$. Clearly, $U^*$ cannot stabilize any subspace of dimension $n/2 = 2$ in case (b) or $n/2 = 4$ in case (f). Therefore, $U^*$ is not conjugate to any subgroup of $H^*$.

By Lemma 2.32 $U^*$ is a solvable $\pi$-group, so $U^* \in \mathfrak{X}$, a contradiction with $G^* \in \mathcal{D}_\mathfrak{X}$.

(xx) $G$ is not isomorphic to any group of Lie type of characteristic $p \in \pi$.

This statement follows from (xvi)–(xix) and Lemma 2.19.

### 3.5 Classical groups of characteristic $p \notin \pi(\mathfrak{X})$

In view of (xx), $G$ is a group of Lie type over a field of an order $q$ and characteristic $p \notin \pi$. In particular, $p \neq 2, 3$.

We start with the smallest case $G = \text{PSL}_2(q)$.

(xxii) $G$ is not isomorphic to $\text{PSL}_2(q)$.

Suppose $G = \text{PSL}_2(q)$, and denote $G^* = \text{SL}_2(q)$. Then $G^* \in \mathcal{D}_\mathfrak{X}$ and $G^*$ has no solvable $\pi$-Hall subgroups by (vii) and Lemmas 2.1 and 2.8, so statement (d) of Lemma 2.17 holds. Therefore if $H^*$ is an $\mathfrak{X}$-Hall subgroup of $G^*$ then the image of $H^*$ in $G^*/Z(G^*) \cong G$ is isomorphic to $\text{Alt}_5$. But in this case there are exactly two conjugacy classes of $\mathfrak{X}$-Hall subgroup in $G$. It contradicts $G \in \mathcal{D}_\mathfrak{X}$.

Now we show that $G$ is not isomorphic to a classical group. First we consider the most transparent case of symplectic groups. Similar, but more complicated, arguments appear in the consideration of the other types of classical groups: linear, unitary and orthogonal.

(xxii) $G$ is not isomorphic to $\text{PSp}_{2n}(q)$.
Suppose $G = \text{PSp}_{2n}(q)$ and denote $G^* = \text{Sp}_{2n}(q)$. By (vii) and Lemma 2.38, we have $G^* \in \mathcal{D}_X$ and $G^*$ has no solvable $\pi$-Hall subgroups. Consider $H^* \in \text{Hall}_X(G^*)$. We claim that

- $\pi \cap \pi(G^*) \subseteq \pi(q^2 - 1)$;
- $H^*$ is included in a subgroup $M^* \cong \text{SL}_2(q) \wr \text{Sym}_n$,

we denote by $B^*$ the base of this wreath product;

- $H^*/(H^* \cap B^*)$ is isomorphic to a $\pi$-Hall subgroup of $\text{Sym}_n$;
- $H^* \cap B^*$ is solvable.

First two items can be found in Lemma 2.21, the third item follows by Lemma 2.1. The last item follows by Lemma 2.36, since if $H^* \cap B^*$ is nonsolvable, then, for some component $L^* = \text{SL}_2(q)$, $H^* \cap L^*$ is a nonsolvable $\pi$-Hall subgroup of $L^*$. Now Lemma 2.17 implies that $L^*$ possesses $\pi$-Hall subgroup that is isomorphic and nonconjugate to $H^* \cap L^*$. Finally, Lemma 2.36 implies that $M^*$ possesses a $\pi$-Hall subgroup that is nonconjugate to $H^*$ but have the same composition factors. Lemma 2.21(C) implies that $G^*$ possesses nonconjugate $X$-Hall subgroups, a contradiction with $G^* \in \mathcal{D}_X$.

The nonsolvability of $H^*$ and Lemma 2.14 imply that $H^*/(H^* \cap B^*)$ is isomorphic to a symmetric group of degree $n$ or $n - 1$ and this degree is at least 5. In particular, $5 \in \pi \cap \pi(G^*)$, $\text{Alt}_5 \in X$, and 5 divides $q^2 - 1$. Moreover, $H^* \cap B^*$ coincides with the solvable radical $H^*_S$ of $H^*$.

Lemma 2.33 implies that $G^*$ possesses a collection $\Delta$ of subgroups isomorphic to $\text{SL}_2(q)$ such that $|\Delta| = n$ and $[K^*, L^*] = 1$ for every $K^*, L^* \in \Delta$ and $K^* \neq L^*$. Since 5 divides $q^2 - 1$, Lemma 2.30 implies that $\text{SL}_2(q)$ possesses a subgroup isomorphic to $\text{SL}_2(5)$. For every $K^* \in \Delta$ fix some $U(K^*) \leq K^*$ such that $U(K^*) \cong \text{SL}_2(5)$. Set

$$U^* = \langle U(K^*) \mid K^* \in \Delta \rangle.$$ 

It follows from the definition that

$$U^*/U^*_S \cong \text{Alt}_5 \times \cdots \times \text{Alt}_5$$

and $U^*_S$ is a 2-group. Thus, $U^* \in X$.

We show that $U^*$ is not conjugate to a subgroup of $H^*$, and this contradicts $G^* \in \mathcal{D}_X$. Indeed, if $U^*$ is conjugate to a subgroup of $H^*$, then we can assume that $U^* \leq H^*$. Denote by $R^* = H^* \cap B^*$ the solvable radical of $H^*$ and let

$$\eta : H^* \to H^*/R^* = \overline{H}^*$$

be the natural epimorphism. We have seen above that $\overline{H}^*$ is isomorphic to a subgroup of $\text{Sym}_n$. Therefore,

$$\mu(\overline{U}^*) \subseteq \mu(\overline{H}^*) \leq n.$$ 

On the other hand, $U^*/U^*_S \cong U^*/U^*_S$ and Lemma 2.29 implies that

$$\mu(U^*) = \mu(U^*/U^*_S) = 5n > n.$$ 

It contradicts the previous inequality.

Thus, (xxii) is proved.
Suppose \( G = \text{PSL}_n^\eta(q) \) and denote \( G^* = \text{SL}_n^\eta(q) \). By (vii) and Lemma 2.18 we have \( G^* \in \mathcal{D}_X \) and \( G^* \) has no solvable \( \pi \)-Hall subgroups. Let \( H^* \in \text{Hall}_X(G^*) \). Consider all possibilities for \( H^* \) given in statements (a)–(c) of Lemma 2.20.

In case (a) \( n \) is equal to 2, and this case is excluded in view of (xxiii).

In case (d) \( H^* \) is isomorphic to \( 4 \cdot 2^4 \cdot \text{Alt}_6 \). In this case \( G^* \) has two conjugacy classes of \( \pi \)-Hall subgroups isomorphic \( 4 \cdot 2^4 \cdot \text{Alt}_6 \). So if \( H^* \) satisfies (d), then this contradicts \( G^* \in \mathcal{D}_X \).

In case (e) we have \( \pi \cap \pi(G^*) = \{2, 3\} \). So \( H^* \) is solvable and this case is excluded in view of (vii).

Thus, one of the following statements holds.

(b) \( q \equiv \eta \pmod{4} \), \( \text{Sym}_n \) satisfies \( \mathcal{E}_\pi \), \( \pi \cap \pi(G) \subseteq \pi(q - \eta) \cup \pi(n!) \), and if \( r \in (\pi \cap \pi(n!)) \setminus \pi(q - \eta) \), then \( |G^*|_r = |\text{Sym}_n|_r \). In this case \( H^* \) is included in

\[
M^* = L^* \cap G^* \cong (q - \eta)^{n-1} \cdot \text{Sym}_n,
\]

where \( L^* = \text{GL}_1^\eta(q) \triangleleft \text{Sym}_n \leq \text{GL}_n^\eta(q) \).

(c) \( n = 2m + k \), where \( k \in \{0, 1\} \), \( m \geq 1 \), \( q \equiv -\eta \pmod{3} \), \( \pi \cap \pi(G) \subseteq \pi(q^2 - 1) \), the groups \( \text{Sym}_m \) and \( \text{GL}_2^\eta(q) \) satisfy \( \mathcal{E}_\pi \). In this case \( H^* \) is contained in

\[
M^* = L^* \cap G^* \cong (\text{GL}_2^\eta(q) \circ \cdots \circ \text{GL}_2^\eta(q)) \cdot \text{Sym}_m \circ Z,
\]

where \( L^* = \text{GL}_2^\eta(q) \triangleleft \text{Sym}_m \times Z \leq \text{GL}_n(q) \) and \( Z \) is a cyclic group of order \( q - \eta \) for \( k = 1 \), and \( Z \) is trivial for \( k = 0 \). The intersection of \( H^* \) with each factor \( \text{GL}_2^\eta(q) \) is a \( \pi \)-Hall subgroup in \( \text{GL}_2^\eta(q) \).

By Lemma 2.18 \( \pi \)-Hall subgroups of \( \text{GL}_2^\eta(q) \) are solvable. Since \( H^* \) is nonsolvable, every nonabelian composition factor of \( H^* \) is a composition factor of a \( \pi \)-Hall subgroup of a symmetric group of degree at most \( n \). In both cases (b) and (c), it follows from Lemma 2.14 that

- every nonabelian composition factor of \( H^* \) is isomorphic to an alternating group; in particular
- \( 5 \in \pi \cap \pi(G^*) \), \( \text{Alt}_5 \in \mathcal{X} \) and \( n \geq 5 \); and
- \( H^*/H^*_\mathcal{X} \) is isomorphic to a subgroup of \( \text{Sym}_n \).

Now we consider two cases: 5 divides \( q^2 - 1 \) and 5 does not divide \( q^2 - 1 \).

Suppose, 5 divides \( q^2 - 1 \). In this case we argue similarly the case of symplectic groups above. Lemma 2.33 implies that \( G^* \) possesses a collection \( \Delta \) of subgroups isomorphic to \( \text{SL}_2(q) \) such that \( |\Delta| = [n/2] \) and \( [K^*, L^*] = 1 \) for every \( K^*, L^* \in \Delta \) and \( K^* \neq L^* \). Since 5 divides \( q^2 - 1 \), Lemma 2.33 implies that \( \text{SL}_2(q) \) possesses a subgroup isomorphic to \( \text{SL}_2(5) \). For every \( K^* \in \Delta \) fix some \( U(K^*) \leq K^* \) such that \( U(K^*) \cong \text{SL}_2(5) \). Set

\[
U^* = \langle U(K^*) \mid K^* \in \Delta \rangle.
\]
It follows from the definition that
\[
U^*/U^*_\Phi \cong \begin{array}{c}
\text{Alt}_5 \times \cdots \times \text{Alt}_5 \\
\text{[n/2] times}
\end{array}
\]
and \(U^*_\Phi\) is a 2-group. Thus, \(U^* \in X\).

We claim that \(U^*\) is not conjugate to a subgroup of \(H^*\) and this contradicts \(G^* \in D_X\). Indeed, if \(U^*\) is conjugate to a subgroup of \(H^*\), then we can assume that \(U^* \leq H^*\). Denote by \(R^*\) the solvable radical of \(H^*\) and let
\[
\eta: H^* \to H^*/R^* = \overline{H}^*
\]
be the natural epimorphism. It follows from above and from Lemma 2.14 that \(\overline{H}^*\) is isomorphic to a symmetric group of degree at most \(n\). Therefore,
\[
\mu(\overline{U}^*) \leq n.
\]

On the other hand, Lemma 2.29 implies that
\[
\mu(\overline{U}^*) \geq \mu(U^*/U^*_\Phi) = 5[n/2] > n.
\]
It contradicts the previous inequality. Hence 5 does not divide \(q^2 - 1\).

Suppose, 5 does not divide \(q^2 - 1\). It means that case (b) holds (in particular, the solvable radical \(H^*_\Phi\) of \(H^*\) is abelian) and \(|G^*|_5 = |\text{Sym}_n|_5\). We have
\[
|G^*|_5 = \prod_{i=1}^{n}(q^i - \eta^i)_5 \quad \text{and} \quad |\text{Sym}_n|_5 = (n!)_5.
\]

Lemma 2.13 implies that \([n/4] = [n/5]\). Since \(n \geq 5\), this means
\[
n \in \{5, 6, 7, 10, 11, 15\}.
\]

Assume that \(n \in \{5, 6, 7\}\) first. Since Sym \(n \in D_\pi\) and a \(\pi\)-Hall subgroup of Sym \(n\) belongs to \(X\), it follows from Lemma 2.14 that \(\text{Alt}_5 \in X\). Moreover, if \(n = 6\) or \(n = 7\), then \(\text{Alt}_6 \in X\).

The group \(G^*\) has a subgroup isomorphic to \(\text{SL}_n^2(q)\). Moreover, (b) implies that \(q \equiv \eta \pmod{4}\) and by Lemma 2.31 \(G^*\) has a subgroup
\[
W^* \cong 4 \circ 2^{1+4} \cdot \text{Alt}_6.
\]

Define \(U^* \leq G^*\) in the following way. If \(n = 6, 7\), then \(U^* = W^*\). If \(n = 5\), then \(W^*/W^*_\Phi \cong \text{Alt}_6\) contains a subgroup isomorphic to \(\text{Alt}_5\), and we set \(U^*\) to be equal to its full preimage in \(W^*\). By construction \(U^* \in X\).

We claim that \(U^*\) is not conjugate to any subgroup of \(H^*\) and this contradicts \(G^* \in D_X\). Indeed, if \(U^* \leq H^*\) and \(R^* = H^*_\Phi\) is the solvable radical of \(H^*\), then \(U^*/(U^* \cap R^*)\) is isomorphic to a subgroup of \(H^*/R^* \leq \text{Sym}_n\). We have that \(U^*/U^*_\Phi \cong \text{Alt}_m\) for some \(m \in \{n, n - 1\}\). Since \(U^*/U^*_\Phi\) is a homomorphic image of \(U^*/(U^* \cap R^*)\), it follows by Lemma 2.34 that \(U^*/(U^* \cap R^*) \cong \text{Alt}_m\). Therefore, \(U^*_\Phi = U^* \cap R^*\). This is impossible, since \(R^*\) is abelian, while \(U^*_\Phi \cong 4 \circ 2^{1+4}\) contains an extra special 2-subgroup of order \(2^5\).

Assume finally that \(n \in \{10, 11, 15\}\). Lemma 2.13 implies that \(\text{Alt}_{10} \in X\). Therefore \(\text{Alt}_6 \in X\). It is clear that \(G^*\) has a subgroup
\[
\text{SL}_4^n(q) \circ \text{SL}_4^n(q) \text{ if } n \in \{10, 11\}, \text{ and } \text{SL}_3^n(q) \circ \text{SL}_3^n(q) \circ \text{SL}_3^n(q) \text{ if } n = 15.
\]
By Lemma 2.31 and in view of \( q \equiv \eta \pmod{4} \), we can find a subgroup \( U^* \) in \( G^* \) such that \( U^*_\Sigma = O_2(U^*) \) and

\[
U^*/U^*_\Sigma \cong \begin{cases} 
\operatorname{Alt}_6 \times \operatorname{Alt}_6, & \text{if } n \in \{10, 11\}, \\
\operatorname{Alt}_6 \times \operatorname{Alt}_6 \times \operatorname{Alt}_6, & \text{if } n = 15.
\end{cases}
\]

Clearly, \( U^* \in \mathfrak{X} \). But \( U^* \) is not conjugate to a subgroup of \( H^* \). Indeed, if \( U^* \leqslant H^* \), then \( U^*/(U^* \cap R^*) \) is isomorphic to a subgroup of \( \operatorname{Sym}_n \), where \( R^* = H^*_\Sigma \). Therefore, by Lemma 2.29 we have

\[
n \geq \mu(U^*/(U^* \cap R^*)) \geq \mu(U^*/U^*_\Sigma) = \begin{cases} 
\mu(\operatorname{Alt}_6 \times \operatorname{Alt}_6) = 12, & \text{if } n \in \{10, 11\}, \\
\mu(\operatorname{Alt}_6 \times \operatorname{Alt}_6 \times \operatorname{Alt}_6) = 18, & \text{if } n = 15,
\end{cases}
\]

a contradiction.

Thus, (xxiii) is proven.

(b) \( \mu(\operatorname{Alt}_6 \times \operatorname{Alt}_6) = 12 \), \( n \in \{10, 11\} \).

(c) \( \mu(\operatorname{Alt}_6 \times \operatorname{Alt}_6 \times \operatorname{Alt}_6) = 18 \), \( n = 15 \).

\( G \) is not isomorphic to \( \mathcal{P} \Omega^m_2(q) \), \( \eta \in \{+, -, \circ\} \).

Suppose \( G = \mathcal{P} \Omega^m_2(q) \), \( n \geq 7 \) and denote \( G^* = \Omega^m_2(q) \). By (vii) and Lemma 2.8 we have \( G^* \in \mathcal{R}_\mathfrak{X} \), and \( G^* \) has no solvable \( \pi \)-Hall subgroups and has exactly one class of \( \mathfrak{X} \)-Hall subgroups. Let \( H^* \in \operatorname{Hall}_\mathfrak{X}(G^*) \). Consider all possibilities for \( H^* \) given in statements (a)–(h) of Lemma 2.22.

In cases (d) and (e) we have \( \pi \cap \pi(G^*) = \{2, 3\} \), and we exclude these cases in view of (vii) and the solvability of \( \{2, 3\} \)-groups.

We exclude cases (f), (g) and (h), since in all these cases there are at least two conjugacy classes of \( \mathfrak{X} \)-Hall subgroups of \( G^* \) isomorphic to \( H^* \).

Thus, one of the following statements holds.

(a) \( n = 2m + 1 \), \( \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon) \), \( q \equiv \varepsilon \pmod{12} \), \( \operatorname{Sym}_m \in \mathcal{E}_\pi \), and \( H^* \) is a \( \pi \)-Hall subgroup in

\[
M^* = \left( O_2^\varepsilon(q) \wr \operatorname{Sym}_m \times O_1(q) \right) \cap G^*.
\]

(b) \( n = 2m \), \( \eta = \varepsilon^m \), \( \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon) \), \( q \equiv \varepsilon \pmod{12} \), \( \operatorname{Sym}_m \in \mathcal{E}_\pi \), and \( H \) is a \( \pi \)-Hall subgroup in

\[
M^* = \left( O_2^\varepsilon(q) \wr \operatorname{Sym}_m \right) \cap G^*.
\]

(c) \( n = 2m \), \( \eta = -\varepsilon^m \), \( \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon) \), \( q \equiv \varepsilon \pmod{12} \), \( \operatorname{Sym}_{m-1} \in \mathcal{E}_\pi \), and \( H^* \) is a \( \pi \)-Hall subgroup of

\[
M^* = \left( O_2^\varepsilon(q) \wr \operatorname{Sym}_{m-1} \times O_2^{-\varepsilon}(q) \right) \cap G^*.
\]

Here \( \varepsilon = \pm 1 \) and \( q - \varepsilon \) is divisible by 4.

Groups \( O_2^+(q) \) and \( O_2^-(q) \) are solvable. As in the proofs of (xxii) and (xxiii), we see that the symmetric group of degree \( m \), in cases (a) and (b), and of degree \( m - 1 \) in case (c) has nonsolvable \( \mathfrak{X} \)-Hall subgroup which is isomorphic to a symmetric group. Moreover, this \( \mathfrak{X} \)-Hall subgroup is isomorphic to \( H^*/H^*_\Sigma \). Thus,

- \( 5 \in \pi \cap \pi(G^*) \subseteq \pi(q - \varepsilon) \subseteq \pi(q^2 - 1) \),
- \( \operatorname{Alt}_5 \in \mathfrak{X} \), and
• $H^*/H^*_\Phi$ is isomorphic to a subgroup of $\text{Sym}_m$ in cases (a) and (b) and of $\text{Sym}_{m-1}$ in case (c). Therefore,

$$\mu(H^*/H^*_\Phi) \leq m = \lceil n/2 \rceil.$$  

Lemma 2.33 implies that $G^*$ possesses a collection $\Delta$ of subgroups isomorphic to $\text{SL}_2(q)$ such that $|\Delta| = k \geq 2((n-1)/4)$ and $[K^*, L^*] = 1$ for every $K^*, L^* \in \Delta$ and $K^* \neq L^*$. Since $5$ divides $q^2 - 1$, Lemma 2.30 implies that $\text{SL}_2(q)$ possesses a subgroup isomorphic to $\text{SL}_2(5)$. For every $K^* \in \Delta$ fix some $U(K^*) \leq K^*$ such that $U(K^*) \cong \text{SL}_2(5)$. Set $U^* = \langle U(K^*) \mid K^* \in \Delta \rangle$.

It follows from the definition that

$$U^*/U^*_\Phi \cong \text{Alt}_5 \times \cdots \times \text{Alt}_5 \underbrace{\times \cdots \times \text{Alt}_5}_{k \text{ times}}$$

and $U^*_\Phi$ is a 2-group. Thus, $U^* \in \mathfrak{X}$.

We show that $U^*$ is not conjugate to a subgroup of $H^*$. Otherwise we can assume that $U^* \leq H^*$. Let $R^* = H^*_\Phi$ and let

$$\pi: H^* \to H^*/R^* = \overline{H^*}$$

be the natural epimorphism. Therefore,

$$\mu(\overline{U^*}) \leq \mu(\overline{H^*}) \leq \lceil n/2 \rceil.$$  

On the other hand, since $n \geq 7$ and in view of Lemma 2.29, we have

$$\mu(\overline{U^*}) \geq \mu(U^*/U^*_\Phi) = 5k \geq 10 \left\lceil \frac{n-1}{4} \right\rceil \geq 10 \left( \frac{n-4}{4} \right) = \frac{5(n-4)}{2} > \frac{n+1}{2} \geq \left\lceil \frac{n}{2} \right\rceil,$$

a contradiction.

$(xxv)$ $G$ is not isomorphic to a classical group.

This statement follows from $(xx)$ if characteristic of a group belongs to $\pi$ and from $(xxii)$–$(xxiv)$ in over cases.

### 3.6 Exceptional groups of Lie type of characteristic $p \notin \pi(\mathfrak{X})$

$(xxv)$ $G$ is not isomorphic to one of groups $^2\text{B}_2(2^{2m+1})$, $^2\text{G}_2(3^{2m+1})$, and $^2\text{F}_4(2^{2m+1})$.

This statement follows from $(v)$, $(vi)$ and $(xx)$.

$(xxvi)$ $G$ is not isomorphic to $G_2(q)$.

Suppose that $G = G_2(q)$ and $H \in \text{Hall}_\mathfrak{X}(G)$. By Lemma 2.23, either $H$ is solvable, which contradicts $(vii)$, or statement $(d)$ of Lemma 2.23 holds:

$(d)$ $G = G_2(q)$, $\pi \cap \pi(G) = \{2, 3, 7\}$, $(q^2 - 1)_{(2,3,7)} = 24$, $(q^4 + q^2 + 1)_7 = 7$, and $H \cong G_2(2)$.  

33
By [1], $H' \cong \text{PSU}_3(3)$ has a maximal subgroup isomorphic to $\text{SL}_3(2) \cong \text{PSL}_2(7)$ and every maximal subgroup of $H'$ not isomorphic to $\text{SL}_3(2)$ is solvable. This implies that $\text{SL}_3(2) \in \mathcal{X}$ and $H$ has no subgroups isomorphic to $2^3 \cdot \text{SL}_3(2)$, which belongs to $\mathcal{X}$. On the other hand, it follows from [6] Table 1 that $G$ has a subgroups isomorphic to $2^3 \cdot \text{SL}_3(2)$.

(xxvii) $G$ is not isomorphic to one of groups $3^3D_4(q)$ and $F_4(q)$.

By Lemma 2.23 every Hall $\mathcal{X}$-subgroup of $3^3D_4(q)$ and $F_4(q)$ is solvable, which contradicts (vii), if $G \in \{3^3D_4(q), F_4(q)\}$.

(xxviii) $G$ is not isomorphic to one of groups $E_6(q)$ and $2E_6(q)$.

Suppose $G = E_6^q(q)$, $\eta = \pm$ and $H \in \text{Hall}_X(G)$. Since $H$ is not solvable, statement (c) of Lemma 2.23 does not hold, and we have case (a) of this Lemma for $E_6^q(q)$:

- $4$ divides $q - \eta$, $\{2, 3, 5\} \subseteq \pi \cap \pi(G) \subseteq \pi(q - \eta)$, $H$ is a $\pi$-Hall subgroup of a group $T$. $\text{Sp}_4(3)$, where $T$ is a maximal torus of order $(q - \eta)^6/3$.

Note that $\text{Sp}_4(3)$ is a $\pi$-group. This implies that $\text{Sp}_4(3)$ is a homomorphic image of $H$ and $\text{Sp}_4(3) \in \mathcal{X}$. Furthermore, $H/H_\mathcal{X} \cong \text{Sp}_4(3)$. By information in [7], $\text{Sp}_4(3)$ has a subgroup isomorphic to $\text{Alt}_5$. Therefore, $\text{Alt}_5 \in \mathcal{X}$.

Lemma 2.33 implies that $G$ possesses a collection $\Delta$ of subgroups isomorphic to $\text{SL}_2(q)$ such that $|\Delta| = 4$ and $[K, L] = 1$ for every $K, L \in \Delta$ and $K \neq L$. Since $5$ divides $q^2 - 1$, Lemma 2.33 implies that $\text{SL}_2(q)$ possesses a subgroup isomorphic to $\text{SL}_2(5)$. For every $K \in \Delta$ fix some $U(K) \leq K$ such that $U(K) \cong \text{SL}_2(5)$. Set

$$U = \langle U(K) \mid K \in \Delta \rangle.$$ 

It follows from the definition that

$$U/U_\mathcal{X} \cong \text{Alt}_5 \times \text{Alt}_5 \times \text{Alt}_5 \times \text{Alt}_5$$

and $U_\mathcal{X}$ is a 2-group. Thus, $U \in \mathcal{X}$. Suppose that $U$ is conjugate to a subgroup of $H$. Then $H/H_\mathcal{X} \cong \text{Sp}_4(3)$ contains a subgroup for which $U/U_\mathcal{X}$ is a homomorphic image. But

$$|H/H_\mathcal{X}|_5 = |\text{Sp}_4(3)|_5 = 5^4 = |\text{Alt}_5|_5^4 = |U/U_\mathcal{X}|_5,$$

and this is impossible.

(xxix) $G$ is not isomorphic to $E_7(q)$.

Suppose $G = E_7(q)$ and $H \in \text{Hall}_X(G)$. By Lemma 2.23 we have:

- $\{2, 3, 5, 7\} \subseteq \pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, where $\varepsilon = \pm 1$ is such that $4$ divides $q - \varepsilon$, $H$ is a $\pi$-Hall subgroup of a group $T$. $(2 \times P\Omega_7(2))$, where $T$ is a maximal torus of order $(q - \eta)^7/2$.

This implies that $P\Omega_7(2) \in \mathcal{X}$ and $H/H_\mathcal{X} \cong P\Omega_7(2)$. In $P\Omega_7(2)$ there is a maximal subgroup $\Omega_n^t(2) \cong \text{Sym}_n$. Therefore, $\text{Alt}_5 \in \mathcal{X}$.

Now we argue as in (xxviii). Lemma 2.33 implies that $G$ possesses a collection $\Delta$ of subgroups isomorphic to $\text{SL}_2(q)$ such that $|\Delta| = 7$ and $[K, L] = 1$ for every $K, L \in \Delta$ and $K \neq L$. Since $5$ divides $q^2 - 1$, Lemma 2.33 implies that $\text{SL}_2(q)$ possesses a subgroup
isomorphic to $\text{SL}_2(5)$. For every $K \in \Delta$ fix some $U(K) \leq K$ such that $U(K) \cong \text{SL}_2(5)$. Set

$$U = \langle U(K) \mid K \in \Delta \rangle.$$ 

It follows from the definition that

$$U/U_{\Theta} \cong \underbrace{\text{Alt}_5 \times \cdots \times \text{Alt}_5}_{7 \text{ times}}$$

and $U_{\Theta}$ is a 2-group. Thus, $U \in \mathfrak{X}$. We claim that $U$ is not conjugate to a subgroup of $H$. It is sufficient to show that $|U/U_{\Theta}|_5 > |H/H_{\Theta}|_5$. Indeed,

$$|H/H_{\Theta}|_5 = |\text{P}\Omega^+_8(2)|_5 = 5 \cdot 5^7 = |\text{Alt}_5|_5^8 = |U/U_{\Theta}|_5,$$

a contradiction with $G \in \mathcal{D}_X$.

$(xxx)$ $G$ is not isomorphic to $E_8(q)$.

Suppose $G = E_7(q)$ and $H \in \text{Hall}_X(G)$. By Lemma 2.23 we have:

- $\{2, 3, 5, 7\} \subseteq \pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, where $\varepsilon = \pm 1$ is such that 4 divides $q - \varepsilon$, $H$ is a $\pi$-Hall subgroup of a group $T \cdot 2 \cdot \text{P}\Omega^+_8(2) \cdot 2$, where $T$ is a maximal torus of order $(q - \eta)^8$.

This implies that $\text{P}\Omega^+_8(2) \in \mathfrak{X}$ and $H/H_{\Theta} \cong \text{P}\Omega^+_8(2) \cdot 2$. In $\text{P}\Omega^+_8(2)$ there is a maximal subgroup $\Theta^7(2)$. Therefore, $\text{Alt}_5 \in \mathfrak{X}$.

Now we argue as in $(xxviii)$. Lemma 2.33 implies that $G$ possesses a collection $\Delta$ of subgroups isomorphic to $\text{SL}_2(q)$ such that $|\Delta| = 8$ and $[K, L] = 1$ for every $K, L \in \Delta$ and $K \neq L$. Since 5 divides $q^2 - 1$, Lemma 2.30 implies that $\text{SL}_2(q)$ possesses a subgroup isomorphic to $\text{SL}_2(5)$. For every $K \in \Delta$ fix some $U(K) \leq K$ such that $U(K) \cong \text{SL}_2(5)$. Set

$$U = \langle U(K) \mid K \in \Delta \rangle.$$ 

It follows from the definition that

$$U/U_{\Theta} \cong \underbrace{\text{Alt}_5 \times \cdots \times \text{Alt}_5}_{8 \text{ times}}$$

and $U_{\Theta}$ is a 2-group. Thus, $U \in \mathfrak{X}$. We claim that $U$ is not conjugate to a subgroup of $H$. It is sufficient to show that $|U/U_{\Theta}|_5 > |H/H_{\Theta}|_5$. Indeed,

$$|H/H_{\Theta}|_5 = |\text{P}\Omega^+_8(2) \cdot 2|_5 = 5^2 \cdot 5^8 = |\text{Alt}_5|_5^8 = |U/U_{\Theta}|_5,$$

a contradiction with $G \in \mathcal{D}_X$.

$(xxxi)$ $G$ is not isomorphic to any exceptional group of Lie type.

This statement follows from $(xxvi)-(xxx)$.

### 3.7 Final proof of the implication $(1) \Rightarrow (2)$

$(xxxii)$ $G$ does not exist.

Indeed, according to the classification of finite simple groups [3, Theorem 0.1.1], in $(viii)$, $(xiv)$, $(xx)$, $(xxv)$, and $(xxxi)$ we have exclude for $G$ all possibilities to be a finite simple group.

Theorem 2 is proven. □
4 Proofs of Corollaries and Theorem 1

4.1 Proofs of Corollaries 1 and 2

In view of Lemma 2.6, in order to prove Corollary 1 it is sufficient to prove that any extension of a $\mathcal{D}_X$-group by a $\mathcal{D}_X$-group is a $\mathcal{D}_X$-group. Now by Lemma 2.7 to prove Corollaries 1 and 2 it is sufficient to show that if $G \in \mathcal{D}_X$ is a simple group then $\hat{G} = \text{Aut}(G) \in \mathcal{D}_X$.

By Theorem 2 we need to consider two cases: $G \in \mathcal{X}$ and $G \in \mathcal{D}_X \cap \mathcal{D}_\pi$, where $\pi = \pi(\mathcal{X})$, and in the last case $G$ is not a $\pi$-group.

In the first case, since $\text{Aut}(G)/\text{Inn}(G)$ is solvable and $\text{Inn}(G) \cong G \in \mathcal{X}$, we conclude that $\hat{G}$ is $\mathcal{X}$-separable and $\hat{G} \in \mathcal{D}_X$ follows from Lemma 2.8.

In the last case, every $\pi$-Hall subgroup of $G$ is solvable by Lemma 2.28. Consequently, the $\mathcal{X}$-subgroups of $G$ are exactly the solvable $\pi$-subgroups. Since $\text{Aut}(G)/\text{Inn}(G)$ is solvable, the same statement holds for the $\mathcal{X}$-subgroups of $\hat{G} = \text{Aut}(G)$. In particular, $m_\mathcal{X}(G) = \text{Hall}_\pi(G)$. Now Lemma 2.3 implies that $G \in \mathcal{D}_\pi$. Hence the elements of $m_\mathcal{X}(\hat{G}) = \text{Hall}_\pi(\hat{G})$ are conjugate and $\hat{G} \in \mathcal{D}_X$. □

4.2 Proof of Corollary 3

Corollary 1 means that $G \in \mathcal{D}_X$ if and only if every composition factor $S$ of $G$ is a $\mathcal{D}_X$-group. Now Corollary 3 immediately follows from Theorem 2 and Lemma 2.4. □

4.3 Proof of Theorem 1

Theorem 1 follows from Corollaries 1–3 and [56, 15.4]. □

4.4 Proof of Corollary 4

We need the following two lemmas which are consequences of Theorem 1:

Lemma 4.1 Let $H \in m_\mathcal{X}(G)$ for a group $G$. If $A$ is a normal $\mathcal{D}_X$-subgroup of $G$ and $\pi: G \to G/A$ is the canonical epimorphism then $N_{\mathcal{X}}(H) = N_G(H)$.

Proof. We clearly have $N_{\mathcal{X}}(H) \leq N_G(H)$. Suppose $x \in N_G(HU)$ (equivalently, $\pi \in N_{\mathcal{X}}(H)$). Then $H^xA = HA$. The $\mathcal{X}$-Reduktionssatz for $A$ implies that there is $a \in A$ such that $H^a = H^a$, i.e. $xa^{-1} \in N_G(H)$, $x \in N_G(H)A$, and $\pi \in N_G(H)$. □

Lemma 4.2 Let $G \trianglelefteq G^*$ and let $H = K \cap G$ for some $K \in m_\mathcal{X}(G^*)$. Suppose that $A \in \mathcal{D}_X$ is a normal subgroup of $G^*$ and $\pi: G^* \to G^*/A$ is the canonical epimorphism. Then $\overline{H} = \overline{K} \cap \overline{G}$.

Proof. Clearly, $\overline{H} = \overline{K} \cap G \leq \overline{K} \cap \overline{G}$. Suppose that $\overline{H} < \overline{K} \cap \overline{G}$. Since $G \trianglelefteq G^*$, we have $H \trianglelefteq K$, $\overline{H} \trianglelefteq \overline{K}$, and $\overline{H} \trianglelefteq \overline{K} \cap \overline{G}$.

Since $\overline{H} < \overline{K} \cap \overline{G}$, the index of $\overline{H}$ in $N_{\mathcal{X}}(\overline{H})$ is divisible by a prime $p \in \pi(\mathcal{X})$. By Lemma 4.1 we have $N_{\mathcal{X}}(\overline{H}) = N_G(\overline{H})$. Now

$$p \mid |N_{\mathcal{X}}(\overline{H}) : H| = \left| \frac{N_G(H)A}{HA} \right| = \frac{|N_G(H)|}{|N_A(H)|} \cdot \frac{|H|}{|H \cap A|} = \frac{|N_G(H) : H|}{|N_A(H) : (H \cap A)|}.$$
Therefore, \( p \mid |N_G(H)/H| \) contrary to Lemma 2.10.

Now we prove Corollary \( 3 \) Denote by \( \phi \) the canonical epimorphism \( G \to G/N \). We need to show that \( H^\phi \in \sm_X(G^\phi) \). We can assume that there exists a group \( G^* \) such that \( G \trianglelefteq G^* \) and \( H = G \cap K \) for some \( K \in \m_X(G^*) \). Denote by \( A \) the normal closure \( \langle N^x \mid x \in G^* \rangle \) of \( N \) in \( G^* \). As \( N^x \trianglelefteq G^x \trianglelefteq G^* \) for every \( x \in G^* \) and \( \m \) is a Fitting class, we conclude that \( A \in \m \). Moreover, \( G \cap A \trianglelefteq A \). Consequently, \( G \cap A \in \m \). Since \( G \cap A \trianglelefteq G \), we have
\[
N \cap A \trianglelefteq G = N.
\]
Therefore, \( G \cap A = N \). Consider the restriction \( \tau : G \to G^*/A \) of the canonical epimorphism \( G^* \to G^*/A \). Then the kernel \( G \cap A \) of \( \tau \) coincides with \( N \). By the homomorphism theorem, there exists an injective homomorphism \( \psi : G^\phi = G/A \to G^*/A \) such that the following diagram is commutative:
\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & G^*/A \\
\phi \downarrow & & \downarrow \psi \\
G^\phi & & \\
\end{array}
\]
Then \( G^\phi \circ \psi = G^\phi / A \trianglelefteq G^*/A \). Moreover, by Lemma 1.2 we have
\[
H^\phi \circ \psi = H^\tau = HA/A = (G \cap K)A/A = (GA/A) \cap (KA/A) = G^\phi \cap (KA/A),
\]
where \( KA/A \in m_X(G^*/A) \) by Theorem 1. Thus \( H^\phi \in \sm_X(G^\phi) \) by the definition of an \( X \)-submaximal subgroup.

In order to prove the inequality \( k \geq k \), it is sufficient to prove that if \( H, H_1 \in \sm_X(G) \) and \( HN/N \) and \( H_1N/N \) are conjugate in \( G/N \), then \( H \) and \( H_1 \) are conjugate in \( G \). Without loss of generality, we can assume that \( HN/N = H_1N/N \) and \( HN = H_1N \). By Corollary 1 we have \( G_0 := HN \in \mathcal{D}_X \). Moreover,
\[
H \cap N \in \sm_X(N) = m_X(N) = \text{Hall}_X(N)
\]
by Corollary 2 and Lemma 2.5. Therefore, \( H \in \text{Hall}_X(G_0) = m_X(G_0) \). Similarly, \( H_1 \in m_X(G_0) \). Hence, \( H \) and \( H_1 \) are conjugate in \( G_0 \).

\[\square\]

Appendix

In Conditions I–VII below, \( S \) is a finite simple group, \( X \) is a complete class of finite groups, and \( \pi = \pi(X) \).

**Condition I.** We say that \((S, X)\) satisfies Condition I if \( S \in X \) or \(|\pi \cap \pi(S)| \leq 1 \).

**Condition II.** We say that \((S, X)\) satisfies Condition II if one of the following cases holds:

1. \( S \cong M_{11} \) and \( \pi \cap \pi(S) = \{5, 11\} \);
2. \( S \cong M_{12} \) and \( \pi \cap \pi(S) = \{5, 11\} \);
3. \( S \cong M_{22} \) and \( \pi \cap \pi(S) = \{5, 11\} \);
4. \( S \cong M_{23} \) and \( \pi \cap \pi(S) \) coincide with one of the following sets \( \{5, 11\} \) and \( \{11, 23\} \);

\[\text{For a Fitting class } \mathfrak{F}, \text{ if } N_1, \ldots, N_s \text{ are subnormal } \mathfrak{F}\text{-subgroups of } G \text{ then } \langle N_1, \ldots, N_s \rangle \in \mathfrak{F} \text{ [2][II(2.8)].}\]
(5) $S \cong M_{24}$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{5, 11\}$ and $\{11, 23\}$;

(6) $S \cong J_4$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{3, 5\}$, $\{3, 7\}$, $\{3, 19\}$, and $\{5, 11\}$;

(7) $S \cong J_4$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{5, 7\}$, $\{5, 11\}$, $\{5, 31\}$, $\{7, 29\}$, and $\{7, 43\}$;

(8) $S \cong O'N$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{5, 11\}$ and $\{5, 31\}$;

(9) $S \cong L_9$ and $\pi \cap \pi(S) = \{11, 67\}$;

(10) $S \cong Ru$ and $\pi \cap \pi(S) = \{7, 29\}$;

(11) $S \cong Co_1$ and $\pi \cap \pi(S) = \{11, 23\}$;

(12) $S \cong Co_2$ and $\pi \cap \pi(S) = \{11, 23\}$;

(13) $S \cong Co_3$ and $\pi \cap \pi(S) = \{11, 23\}$;

(14) $S \cong M(23)$ and $\pi \cap \pi(S) = \{11, 23\}$;

(15) $S \cong M(24)'$ and $\pi \cap \pi(S) = \{11, 23\}$;

(16) $S \cong B$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{11, 23\}$ and $\{23, 47\}$;

(17) $S \cong M$ and $\pi \cap \pi(S)$ coincide with one of the following sets $\{23, 47\}$ and $\{29, 59\}$.

**Condition III.** Let $S$ be isomorphic to a group of Lie type over the field $\mathbb{F}_q$ of characteristic $p \in \pi$ and let $\tau = (\pi \cap \pi(S)) \setminus \{p\}$. We say that $(S, \mathfrak{X})$ satisfies Condition III if $\tau \subseteq \pi(q-1)$ and every prime in $\pi$ does not divide the order of the Weyl group of $S$.

In order to formulate Conditions IV and V, we need the following notation. If $r$ is an odd prime and $q$ is an integer not divisible by $r$, then $e(q, r)$ is the smallest positive integer $e$ with $q^e \equiv 1 \pmod{r}$.

**Condition IV.** Let $S$ be isomorphic to a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$. Let $2, p \notin \pi$. Denote by $r$ the minimum in $\pi \cap \pi(S)$ and put $\tau = (\pi \cap \pi(S)) \setminus \{r\}$ and $a = e(q, r)$. We say that $(S, \mathfrak{X})$ satisfies Condition IV if there exists $t \in \tau$ with $b = e(q, t) \neq a$ and one of the following statements holds.

(1) $S \cong A_{n-1}(q)$, $a = r - 1$, $b = r$, $(q^{r-1} - 1)_r = r$, $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right]$, and $e(q, s) = b$ for every $s \in \tau$;

(2) $S \cong A_{n-1}(q)$, $a = r - 1$, $b = r$, $(q^{r-1} - 1)_r = r$, $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right] + 1$, $n \equiv -1 \pmod{r}$, $e(q, s) = b$ for every $s \in \tau$;

(3) $S \cong 2A_{n-1}(q)$, $r \equiv 1 \pmod{4}$, $a = r - 1$, $b = 2r$, $(q^{r-1} - 1)_r = r$, $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right]$ and $e(q, s) = b$ for every $s \in \tau$;

(4) $S \cong 2A_{n-1}(q)$, $r \equiv 3 \pmod{4}$, $a = \frac{r-1}{2}$, $b = 2r$, $(q^{r-1} - 1)_r = r$, $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right]$ and $e(q, s) = b$ for every $s \in \tau$;

(5) $S \cong 2A_{n-1}(q)$, $r \equiv 1 \pmod{4}$, $a = r - 1$, $b = 2r$, $(q^{r-1} - 1)_r = r$, $\left[\frac{n}{r-1}\right] = \left[\frac{n}{r}\right] + 1$, $n \equiv -1 \pmod{r}$, and $e(q, s) = b$ for every $s \in \tau$;
(6) $S \cong 2A_{n-1}(q)$, $r \equiv 3 \pmod{4}$, $a = \frac{n-1}{2}$, $b = 2r$, $(q^{r-1} - 1)_r = r$, $\left[\frac{n}{r} \right] = \left[\frac{n}{s} \right] + 1$, $n \equiv -1 \pmod{r}$ and $e(q, s) = b$ for every $s \in \tau$;

(7) $S \cong 2D_n(q)$, $a \equiv 1 \pmod{2}$, $n = b = 2a$ and for every $s \in \tau$ either $e(q, s) = a$ or $e(q, s) = b$;

(8) $S \cong 2D_n(q)$, $b \equiv 1 \pmod{2}$, $n = a = 2b$ and for every $s \in \tau$ either $e(q, s) = a$ or $e(q, s) = b$.

**Condition V.** Let $S$ be isomorphic to a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p$. Suppose, $2, p \notin \pi$. Let $r$ be the minimum in $\pi \cap \pi(S)$, let $\tau = (\pi \cap \pi(S)) \setminus \{r\}$, and let $e = e(q, r)$. We say that $(S, \pi)$ satisfies Condition V if $e(q, t) = c$ for every $t \in \tau$ and one of the following statements holds.

1. $S \cong A_{n-1}(q)$ and $n < cs$ for every $s \in \tau$;
2. $S \cong 2A_{n-1}(q)$, $c \equiv 0 \pmod{4}$ and $n < cs$ for every $s \in \tau$;
3. $S \cong 2A_{n-1}(q)$, $c \equiv 2 \pmod{4}$ and $2n < cs$ for every $s \in \tau$;
4. $S \cong 2A_{n-1}(q)$, $c \equiv 1 \pmod{2}$ and $n < 2cs$ for every $s \in \tau$;
5. $S$ is isomorphic to one of the groups $B_n(q)$, $C_n(q)$, or $2D_n(q)$, $c$ is odd and $2n < cs$ for every $s \in \tau$;
6. $S$ is isomorphic to one of the groups $B_n(q)$, $C_n(q)$, or $D_n(q)$, $c$ is even and $n < cs$ for every $s \in \tau$;
7. $S \cong D_n(q)$, $c$ is even and $2n \leq cs$ for every $s \in \tau$;
8. $S \cong 2D_n(q)$, $c$ is odd and $n < cs$ for every $s \in \tau$;
9. $S \cong 3D_4(q)$;
10. $S \cong E_6(q)$, and if $r = 3$ and $c = 1$ then $5, 13 \notin \tau$;
11. $S \cong 2E_6(q)$, and if $r = 3$ and $c = 2$ then $5, 13 \notin \tau$;
12. $S \cong E_7(q)$, if $r = 3$ and $c \in \{1, 2\}$ then $5, 7, 13 \notin \tau$, and if $r = 5$ and $c \in \{1, 2\}$ then $7 \notin \tau$;
13. $S \cong E_8(q)$, if $r = 3$ and $c \in \{1, 2\}$ then $5, 7, 13 \notin \tau$, and if $r = 5$ and $c \in \{1, 2\}$ then $7, 31 \notin \tau$;
14. $S \cong G_2(q)$;
15. $S \cong F_4(q)$, and if $r = 3$ and $c = 1$ then $13 \notin \tau$.

**Condition VI.** We say that $(S, \pi)$ satisfies Condition VI if one of the following statements holds.

1. $S$ is isomorphic to $2B_2(2^{2m+1})$ and $\pi \cap \pi(S)$ is contained in one of the sets

$$\pi(2^{2m+1} - 1), \, \pi(2^{2m+1} \pm 2^{m+1} + 1);$$
\(S\) is isomorphic to \(2G_2(3^{2m+1})\) and \(\pi \cap \pi(S)\) is contained in one of the sets
\[
\pi(3^{2m+1} - 1) \setminus \{2\}, \quad \pi(3^{2m+1} + 3^{m+1} + 1) \setminus \{2\};
\]
\(\pi(2^{2(2m+1)} + 1), \quad \pi(2^{2m+1} + 2^{m+1} + 1),
\]
\(\pi(2^{2(2m+1)} + 2^{3m+2} + 2^{m+1} - 1), \quad \pi(2^{2(2m+1)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1).\)

**Condition VII.** Let \(S\) be isomorphic to a group of Lie type with the base field \(\mathbb{F}_q\) of characteristic \(p\). Suppose that \(2 \in \pi\) and \(3, p \notin \pi\). Put \(\tau = (\pi \cap \pi(S)) \setminus \{2\}\) and \(\varphi = \{t \in \tau \mid t\) is a Fermat number\}. We say that \((S, \mathfrak{X})\) satisfies Condition VII if \(\tau \subseteq \pi(q - \varepsilon)\), where the number \(\varepsilon = \pm 1\) is such that 4 divides \(q - \varepsilon\), and one of the following statements holds.

1. \(S\) is isomorphic to either \(A_n(q)\) or \(2A_n(q)\), \(s > n\) for every \(s \in \tau\), and \(t > n + 1\) for every \(t \in \varphi\);
2. \(S \cong B_n(q)\), and \(s > 2n + 1\) for every \(s \in \tau\);
3. \(S \cong C_n(q)\), \(s > n\) for every \(s \in \tau\), and \(t > 2n + 1\) for every \(t \in \varphi\);
4. \(S\) is isomorphic to either \(D_n(q)\) or \(2D_n(q)\), and \(s > 2n\) for every \(s \in \tau\);
5. \(S\) is isomorphic to either \(G_2(q)\) or \(2G_2(q)\), and \(7 \notin \tau\);
6. \(S \cong F_4(q)\) and \(5, 7 \notin \tau\);
7. \(S\) is isomorphic to either \(E_6(q)\) or \(2E_6(q)\), and \(5, 7 \notin \tau\);
8. \(S \cong E_7(q)\) and \(5, 7, 11 \notin \tau\);
9. \(S \cong E_8(q)\) and \(5, 7, 11, 13 \notin \tau\);
10. \(S \cong 3D_4(q)\) and \(7 \notin \tau\).

**References**

[1] M. Aschbacher, Characterization of Chevalley groups over fields of odd order, Ann. Math., 106 (1977), 353–468.

[2] M. Aschbacher, On finite groups of Lie type and odd characteristic, J. Algebra, 66:2 (1980), 400–424.

[3] M. Aschbacher, R. Lyons, S. D. Smith, R. Solomon, The classification of finite simple groups. Groups of characteristic 2 type. Mathematical Surveys and Monographs, 172. American Mathematical Society, Providence, RI, 2011. xii+347 pp.

[4] J. N. Bray, D. F. Holt, C. M. Roney-Dougal, The Maximal Subgroups of the Low-Dimensional Finite Classical Groups. Cambridge: Cambridge Univ. Press, 2013. 438 p.

[5] R. Carter, P. Fong, The Sylow 2-subgroups of the finite classical groups, J. Algebra, 1:1 (1964), 139–151.
[6] A. M. Cohen, M. W. Liebeck, J. Saxl, G. M. Seitz, The local maximal subgroups of exceptional groups of Lie type, finite and algebraic, Proc. London Math. Soc. Ser. III, 64 (1992), N1, 21–48.

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of Finite Groups. Oxford: Clarendon Press, 1985. 252 p.

[8] S. A. Chunikhin, L. A. Shemetkov, Finite groups, J. Soviet Math. 1:3 (1973), 291–332.

[9] K. Doerk, T. Hawkes, Finite Soluble Groups, Berlin, New York, Walter de Gruyter, 1992.

[10] D. Easdown, C. E. Praeger, On minimal faithful permutation representations of finite groups, Bulletin Australian Mathematical Society, 38(1988), 207–220.

[11] W. Feit, J. G. Thompson, Solvability of groups of odd order, Pacif. J. Math., 13 (1963), N 3, 775–1029.

[12] É. Galois, Mémoire sur les conditions de résolubilité des équations par radicaux, J. Math. Pures Appl. (Liouville) 11 (1846), 417–433.

[13] G. Glauberman, Factorization in local subgroups of finite groups. Conf. Ser. Math., 33, Amer. Math. Soc., Providence, RI, 1976.

[14] F. Gross, On a conjecture of Philip Hall, Proc. Lond. Math. Soc. (3), 52:3 (1986), 464–494.

[15] F. Gross, Conjugacy of odd order Hall subgroups, Bull. Lond. Math. Soc., 19:4 (1987), 311–319.

[16] F. Gross, Hall subgroups of order not divisible by 3, Rocky Mount. J. Math., 23:2 (1993), 569–591.

[17] F. Gross, Odd order Hall subgroups of the classical linear groups, Math. Z., 220:3 (1995), 317–336.

[18] W. Guo, The Theory of Classes of Groups, Beijing, New York, Kluwer Acad. Publ., 2006.

[19] W. Guo, Structure Theory of Canonical Classes of Finite Groups, Berlin, Springer, 2015.

[20] W. Guo, D. O. Revin, On maximal and submaximal X-subgroups, Algebra and logic, 56:1 (2018), 9–28.

[21] W. Guo, D. O. Revin, Conjugacy of maximal and submaximal X-subgroups, Algebra and Logic, 57:3 (2018), 169–181.

[22] W. Guo, D. O. Revin, Classification and properties of the π-submaximal subgroups in minimal nonsolvable groups, Bulletin of Mathematical Sciences, 8:2 (2018), 325–351.

[23] W. Guo, D. O. Revin, Pronormality and submaximal X-subgroups in finite groups, Communications in Mathematics and Statistics, 6:3 (2018), 289–317.

[24] W. Guo, D. O. Revin, E. P. Vdovin, Confirmation for Wielandt’s conjecture, J. Algebra, 434 (2015), 193–206.

[25] P. Hall, Theorems like Sylow’s, Proc. London Math. Soc., 6:22 (1956), 286–304.

[26] B. Hartley, A theorem of Sylow type for a finite groups, Math. Z., 122:4 (1971), 223–226.

[27] D. F. Holt, Representing quotients of permutation groups, Quarterly Journal of Mathematics, 48, No. 2 (1997), 347–350.
[28] C. M. Jordan, Commentaire sur le Mémoire de Galois. Comptes rendus 60 (1865), 770–774.
[29] C. M. Jordan, Traité des substitutions et des équations algébriques, Paris: Gauthier-Villars, 1870.
[30] P. B. Kleidman, M. Liebeck, The subgroup structure of the finite classical groups. Cambridge: Cambridge Univ. Press, 1990. 303 p.
[31] N. Ch. Manzaeva, Heritability of the property $\mathcal{D}_{\pi}$ by overgroups of $\pi$-Hall subgroups in the case where $2 \in \pi$, Algebra and Logic, 53:1 (2014), 17–28.
[32] V. D. Mazurov and E. I. Khukhro (eds.), The Kourovka notebook. Unsolved problems in group theory, 17th ed., Institute of Mathematics, Siberian Branch of RAS, Novosibirsk 2010.
[33] V. D. Mazurov, D. O. Revin, On the Hall $D_{\pi}$-property for finite groups, Siberian Math. J., 38:1 (1997), 106–113.
[34] D. O. Revin, Hall $\pi$-subgroups of finite Chevalley groups whose characteristic belongs to $\pi$, Siberian Advances in Mathematics, 1999, 9:2, 25–71.
[35] D. O. Revin, The $D_{\pi}$-property in a class of finite Groups, Algebra and Logic, 41:3 (2002), 187–206.
[36] D. O. Revin, The $D_{\pi}$ property of finite groups in the case $2 \notin \pi$, Proceedings of the Steklov Institute of Mathematics, 257:suppl.1 (2007), S164–S180.
[37] D. O. Revin, The $D_{\pi}$-property in finite simple groups, Algebra and Logic, 47:3 (2008), 210–227.
[38] D. O. Revin, The $D_{\pi}$-property of linear and unitary groups, Siberian Math. J., 49:2 (2008), 353–361.
[39] D. O. Revin, submaximal and epimaximal $\mathcal{X}$-subgroups, Algebra and Logic, 58:6 (2019), 475–479.
[40] D. O. Revin, S. V. Skresanov, A. V. Vasilev, The Wielandt–Hartley theorem for submaximal $\mathcal{X}$-subgroups, Monatshefte für Mathematik, 193:1 (2020), 143–155.
[41] D. O. Revin, E. P. Vdovin, Hall subgroups of finite groups, Contemporary Mathematics, 402 (2006), 229–265.
[42] D. O. Revin, E. P. Vdovin, On the number of classes of conjugate Hall subgroups in finite simple groups, J. Algebra, 324:12 (2010), 3614–3652.
[43] D. O. Revin, A. V. Zavarintsine, On the behavior of $\pi$-submaximal subgroups under homomorphisms, Communications in Algebra, 48:2 (2020), 702–707.
[44] D. O. Revin, A. V. Zavarintsine, The behavior of $\pi$-submaximal subgroups under homomorphisms with $\pi$-separable kernels, Siberian Electronic Mathematical Reports, 17 (2020), accepted, see arXiv:2006.09752.
[45] L. A. Shemetkov, Formations of Finite Groups (in Russian), Nauka, Moscow (1978).
[46] L. A. Shemetkov, Two directions in the development of the theory of non-simple finite groups, Russian Math. Surveys 30:2 (1975), 185–206.
[47] M. Suzuki, Group Theory II, Springer-Verlag, New York–Berlin–Heidelberg–Tokyo, 1986.
[48] J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc., 74 (1968), 383–437.

[49] E. P. Vdovin, N. Ch. Manzaeva, D. O. Revin, On the heritability of the property $D_\pi$ by subgroup, Proceedings of the Steklov Institute of Mathematics, 279:Suppl. 1 (2012), 130–138.

[50] E. P. Vdovin, N. Ch. Manzaeva, D. O. Revin, On the heritability the Sylow $\pi$-theorem by subgroups, Sb. Math., 211:3 (2020), 309–335.

[51] E. P. Vdovin, D. O. Revin, Theorems of Sylow type, Russian Math. Surveys, 66:5 (2011), 829–870.

[52] A. J. Weir, Sylow $p$-subgroups of the classical groups over finite fields with characteristic prime to $p$, Proceedings of the American Mathematical Society 6:4 (1955), 529–533.

[53] H. Wielandt, Zum Satz von Sylow, Math. Z., 60 (1954), N4. 407–408.

[54] H. Wielandt, Entwicklungslinien in der Strukturtheorie der endlichen Gruppen, Proc. Intern. Congress Math., Edinburg, 1958. London: Cambridge Univ. Press, 1960, 268–278.

[55] H. Wielandt, Arithmetische Struktur und Normalstruktur endlicher Gruppen, Conv. Internaz. di Teoria dei Gruppi Finiti e Applicazioni (Firenze 1960), Edizioni Cremonese, Roma 1960, 56–65.

[56] H. Wielandt Zum Satz von Sylow, Vorlesung an der Universität Tübingen im Wintersemester 1963/64. Helmut Wielandt: Mathematical Works, Vol. 1, Group theory (ed. B. Huppert and H. Schneider, de Gruyter, Berlin, 1994), 607–655.

[57] H. Wielandt, On the structure of composite groups, Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra, August 1965, pp. 379-388. Gordon and Breach Science Publishers, Inc., New York 1967.

[58] H. Wielandt, Zusammengesetzte Gruppen: Hölzer Programm heute, The Santa Cruz conf. on finite groups, Santa Cruz, 1979. Proc. Sympos. Pure Math., 37, Providence RI: Amer. Math. Soc., 1980, 161–173.

[59] H. Wielandt, Tagebücher, D17, (Mathematical Diary XVII), 1980, available in photocopied and transcribed form at https://www3.math.tu-berlin.de/numerik/Wielandt/index_en.html