The Generalized Multiplicative Gradient Method
and Its Convergence Rate Analysis

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Abstract

Multiplicative gradient method is a classical and effective method for solving the positron emission tomography (PET) problem. In this work, we propose a generalization of this method on a broad class of problems, which includes the PET problem as a special case. We show that this generalized method converges with rate $O(1/k)$.

1 Introduction

The multiplicative gradient (MG) method is a simple optimization algorithm for solving a class of concave differentiable maximization problems, where in each iteration $t$, one obtains the next iterate $x^{t+1}$ by (element-wise) multiplying the current iterate $x^t$ with the gradient of the objective function $f$ at $x^t$, namely $\nabla f(x^t)$. Mathematically, this can be written as

$$x^{t+1}_i := x^t_i \nabla_i f(x^t), \quad \forall i \in [n],$$

(MG)

where $n$ is the dimension of the decision variable, and $x_i$ denotes the $i$-th entry of $x$ (and the same for $\nabla_i f(x^t)$). This method was originally developed in the information theory community to compute the channel capacity and rate distortion function (see e.g., [1–3]), which involves solving the following problem:

$$\max_{x \in \Delta_n} \left\{ f(x) := \sum_{j=1}^m p_j \ln(a_j^\top x) \right\}.$$  (1)

Specifically, in (1), $\Delta_n := \{x \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^n x_i = 1\}$ denotes the unit simplex in $\mathbb{R}^n$, $p_j > 0$ for all $j \in [m]$, $a_j \in \mathbb{R}^n_+$, for all $j \in [m]$ and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n]\}$, namely the nonnegative orthant in $\mathbb{R}^n$. Without loss of generality, we may assume that $\sum_{j=1}^m p_j = 1$. Furthermore, we assume that $a_j \neq 0$ for all $j \in [m]$ so that $\text{dom } f \cap \Delta_n \neq \emptyset$ and hence (1) has an optimal solution.

Later on, it was noticed that the problem in (1), despite having a simple form, has wide applications that goes beyond information theory. Therefore, the MG method appears in several diverse areas, including the log-optimal investment [4], the positron emission tomography (PET) in medical imaging [5], and the mixture model estimation in statistics [6]. Indeed, under appropriate statistical models, the MG method amounts to the expectation-maximization (EM) algorithm for performing maximum likelihood (ML) estimation (see e.g., [6]), and consequently, it is commonly referred to as EM algorithm in the literature.

For a fairly long time, the MG method was perceived as a specialized method for solving (1), and the convergence rate of this method was not well-understood. Until very recently, some progress has been made towards applying this method to some other problems as well as understanding its convergence rate. In particular, Zhao [7] showed that the MG method...
applied to solving (1) enjoys a $O(1/k)$ convergence rate, both in the ergodic and non-ergodic senses. In [8], Cohen et al. applied the MG method to solving the D-optimal design problem (see e.g., [9]), and they showed the $O(1/k)$ ergodic convergence rate of this method. In addition, Lin et al. [10] applied a variant of the MG method to solving the quantum state tomography (QST) problem (see e.g., [11]), and similar to [8], they showed the $O(1/k)$ ergodic convergence rate of this method. Given the recent progress, one may naturally ask the following questions:

i) What is the essential problem structure that drives the success of the MG method?

ii) Is there a general problem class that (MG) works well?

iii) What is the interaction between the complexity of (MG) and the problem structure?

In this work, we make progress in answering the questions above. Specifically, we identify a broad class of problems that includes the PET problem as a special case, and propose a generalization of the MG method that works for this class of problems. The feasible set in this problem class is defined via the notion of symmetric cones. Using the framework of Euclidean Jordan algebra (EJA), we show that the generalized MG method converges with rate $O(1/k)$.

1.1 A general problem class

We first present a general class of optimization problems where the (generalized) MG method can be applied. This problem reads:

$$\max \ F(x) := f(Ax)$$
$$\text{s.t. } x \in C := \{x \in K_1 : \langle e, x \rangle = 1\}. \quad (P)$$

We first describe the constraint set in (P). Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be a finite-dimensional real inner-product space. We let $K_1 \subseteq \mathbb{V}$ be a symmetric cone, which is a cone that is both self-dual and homogeneous. We say that $K_1$ is self-dual if $K_1 = K_1^\ast := \{z \in \mathbb{V} : \langle z, x \rangle \geq 0, \ \forall x \in K_1\}$, i.e., the dual cone of $K_1$. By homogeneity, we mean that for any $x, y \in \text{int} K_1$, there always exists a linear automorphism on $K_1$, denoted by $T$, such that $Tx = y$. The notion of symmetric cone encompasses several useful instances, such as the nonnegative orthant $\mathbb{R}^n_+$, the second order cone $Q^{n+1} := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geq \|x\|_2\}$ and the cone of $n \times n$ real symmetric positive semi-definite (PSD) matrices $\mathbb{S}^n_+$. Informally, the point $e \in \text{int} K_1$ is the “center” of $K_1$, for example, $e = 1_n := (1, 1, \ldots, 1)$ if $K_1 = \mathbb{R}^n_+$ and $e = I_n$ (the $n \times n$ identity matrix) if $K_1 = \mathbb{S}^n_+$. The formal definition of $e$ will be given in Section 2.1, under the framework of EJA. Lastly, note that when $K_1 = \mathbb{R}^n_+$ and $e = 1_n$, $C$ becomes the standard unit simplex, and for this reason, $C$ is often referred to as the “generalized unit simplex” in the literature (see e.g., [12, 13]).

Next, we describe the objective function in (P). We let $A : K_1 \rightarrow K_2$ be a linear operator, where $K_2 \subseteq \mathbb{W}$ is a regular cone in a finite-dimensional vector space $\mathbb{W}$. (We call a cone regular if it is closed, convex, pointed, and has nonempty interior.) Denote $A^* : K_2^\ast \rightarrow K_1$ as the adjoint of $A$. We require both $A : \text{int} K_1 \rightarrow \text{int} K_2$ and $A^* : \text{int} K_2^\ast \rightarrow \text{int} K_1$. Let $f : K_2 \rightarrow \mathbb{R} \cup \{-\infty\}$ be a concave function that is three-times differentiable and 1-logarithmically-homogeneous (1-LH) on $\text{int} K_2$, namely

$$f(ty) = f(y) + \ln t, \quad \forall t > 0, \ \forall y \in \text{int} K_2. \quad (2)$$

Consequently, we see that

i) $F : K_1 \rightarrow \mathbb{R} \cup \{-\infty\}$ is a concave function that is three-times differentiable and 1-LH on $\text{int} K_1$,

ii) $\langle \nabla f(y), y \rangle = 1$ for all $y \in \text{int} K_2$ (see e.g., [14, Proposition 2.3.4]),

iii) $\nabla f(y) \in \text{int} K_2^\ast$ for all $y \in \text{int} K_2$, and hence $\nabla F(x) = A^* \nabla f(Ax) \in \text{int} K_1$ for all $x \in \text{int} K_1$.

Note that ergodic and non-ergodic convergence rates refer to the convergence rates of the averaged iterate and last iterate produced by an algorithm, respectively.
Note that $F$ need not belong to the class of self-concordant functions. Instead, we require $F$ to fall under the class of gradient log-convex functions, whose definition will be given in Section 3.4.

At this point, the problem in (P) may seem somewhat abstract, and one may naturally wonder the usefulness of this problem class. Therefore, let us provide a few applications below.

1.2 Applications

1.2.1 Positron emission tomography (PET)

PET is a medical imaging technique that measures the metabolic activities of human tissues and organs. The mathematical model behind this process is described as follows. Suppose that an emission object (e.g., a human organ) has been discretized into $n$ voxels. The number of events emitted by voxel $i$ ($i \in [n]$) is a Poisson random variable $X_i \sim \text{Pois}(x_i)$ with unknown mean $x_i \geq 0$, and $\{X_i\}_{i=1}^n$ are independent. We also have a scanner array with $m$ bins. Each event emitted by voxel $i$ has a known probability $p_{ij}$ of being detected by bin $j$ ($j \in [m]$), and $\sum_{j=1}^m p_{ij} = 1$. Let $Y_j$ denote the total number of events detected by bin $j$, whereby $E[Y_j] := y_j := \sum_{i=1}^n p_{ij} x_i$. By Poisson thinning and superposition, it follows that $\{Y_j\}_{j=1}^m$ are independent random variables and $Y_j \sim \text{Pois}(y_j)$ for all $j \in [m]$. We seek to perform maximum-likelihood (ML) estimation of the unknown means $\{x_i\}_{i=1}^n$ based on observations $\{Y_j\}_{j=1}^m$ of the random variables $\{Y_j\}_{j=1}^m$. From the model above, we easily see that the ML estimation problem can be written as

$$\max_{x \geq 0} \left[ l(x) := - \sum_{i=1}^n x_i + \sum_{j=1}^m Y_j \ln \left( \sum_{i=1}^n p_{ij} x_i \right) \right]. \quad (3)$$

Using first-order optimality conditions, and by re-scaling both the function $l$ and the variable $x$, we see that (3) can be equivalently written as follows:

$$\max_z F(z) := \left( \sum_{j=1}^m Y_j \right)^{-1} \sum_{j=1}^m Y_j \ln \left( \sum_{i=1}^n p_{ij} z_i \right)$$

$$\text{s.t. } z \in \Delta_n := \{ z \in \mathbb{R}_+^n : \sum_{i=1}^n z_i = 1 \}, \quad (4)$$

where $\Delta_n$ is called the unit simplex in $\mathbb{R}^n$.

Note that the problem (4) appears in many other applications as well, including computing the rate distortion function in information theory [3], maximum likelihood estimation for mixture models in statistics [6] and the log-optimal investment problem [4].

To fit (4) into the problem class (P), we can take $K_1 = \mathbb{R}_+^n$, $K_2 = \mathbb{R}_m^+$, $e = (1, 1, \ldots, 1) \in \mathbb{R}^n$, $A : z \mapsto Pz$, where $P := (p_{ij})_{i,j\in[m],i\in[n]} \in \mathbb{R}_m^{m \times n}$ has non-zero rows (and non-zero columns since $\sum_{j=1}^m p_{ij} = 1$ for all $i \in [n]$) and $f : w \mapsto \left( \sum_{j=1}^m Y_j \right)^{-1} \sum_{j=1}^m Y_j \ln w_j$ for $w \in \mathbb{R}_m^+$, where $\mathbb{R}_m^+$ denotes the set of positive real numbers.

1.2.2 $D$-optimal design

Given $m$ points $a^1, \ldots, a^m \in \mathbb{R}^n$ whose affine hull is $\mathbb{R}^n$, the $D$-optimal design problem reads:

$$\max_x F(x) := n^{-1} \ln \det \left( \sum_{i=1}^m x_i a_i a_i^\top \right) \quad \text{s.t. } \ x \in \Delta_m. \quad (5)$$

In the domain of statistics, the $D$-optimal design problem corresponds to maximizing the determinant of the Fisher information matrix $E(aa^\top)$; see [15, 16], as well as the exposition in [17]. In computational geometry, $D$-optimal design arises as a Lagrangian dual problem of the minimum volume covering ellipsoid (MVCE) problem, which dates back at least 70 years to [18] (see [19] for a modern treatment). Indeed, the problem (5) is useful in a variety of other areas, for example, computational statistics [20] and data mining [21].

To fit (5) into the problem class (P), we can take $K_1 = \mathbb{R}_m^n$, $K_2 = \mathbb{S}_m^+$, $e = (1, 1, \ldots, 1) \in \mathbb{R}^m$, $A : x \mapsto \sum_{i=1}^m x_i a_i a_i^\top$, and $f : Y \mapsto -n^{-1} \ln \det Y$ for $Y \in \mathbb{S}_m^+$, where $\mathbb{S}_m^+$ denotes the cone of $n \times n$ real symmetric positive-definite matrices.
1.2.3 Quantum state tomography (QST)

Consider the problem of state reconstruction of a quantum system, where the unknown quantum state is described by a matrix \( X^* \in \mathbb{H}_n^+ \), where \( \mathbb{H}_n^+ \) denotes the cone of \( n \times n \) complex Hermitian PSD matrices [11]. For normalization purpose, we let \( \text{tr}(X^*) = 1 \), where \( \text{tr}(\cdot) \) denotes the trace of a square matrix. To estimate \( X^* \), we prepare \( N \) particles in the state \( X^* \), and independently observe the output of each particle in one of the \( m \) channels in the measuring equipment. In particular, for \( j \in [m] \), the probability \( p_j \) of observing a particle in the \( j \)-th channel is given by \( p_j := \langle X^*, a_j a_j^H \rangle \) for some known \( a_j \in \mathbb{C}^n \), where \( (\cdot)^H \) denotes the conjugate transpose operation. We let \( \sum_{j=1}^m a_j a_j^H = I_n \) so that \( \sum_{j=1}^m p_j = 1 \). For a particular experiment, let \( n_j \) denote the number of times that the output of a particle is observed in the \( j \)-th channel, so that \( \sum_{j=1}^m n_j = N \). Using multinomial probability, the log-likelihood of the outcome of this experiment is given by \( N^{-1} \sum_{j=1}^m n_j \ln(\langle X^*, a_j a_j^H \rangle) \), up to some constants. Therefore, to find the ML estimate of \( X^* \), one solves the following optimization problem:

\[
\max_X \ F(X) := N^{-1} \sum_{j=1}^m n_j \ln(\langle X, a_j a_j^H \rangle) \quad \text{s. t.} \quad X \in \mathbb{H}_n^+, \ \text{tr}(X) = (I_n, X) = 1.
\]  

(6)

To fit (6) into the problem class (P), we can take \( K_1 = \mathbb{H}_n^+ \) (which is a symmetric cone; see Section 2.2 for details), \( K_2 = \mathbb{R}_n^m \), \( e = I_n \), \( A : X \mapsto (\langle X, a_j a_j^H \rangle)_{j=1}^m \), and \( f : w \mapsto \sum_{j=1}^m n_j \ln w_j \) for \( w \in \mathbb{R}_n^m \).

1.2.4 Semidefinite relaxation of Boolean quadratic problem

The Boolean quadratic program (BQP)

\[
q^* : = \max_{x \in \{\pm 1\}^n} \ x^T A x
\]

is a classical problem in discrete optimization, where without loss of generality we may assume that the \( n \times n \) matrix \( A \) is symmetric and positive definite, namely \( A \in \mathbb{S}_n^+ \). Despite being NP-hard, Nesterov [22] showed that the semidefinite relaxation

\[
s^* : = \min_y \ \sum_{i=1}^n y_i \quad \text{s. t.} \quad \text{diag}(y) \succeq A
\]  

(7)

provides a \((2/\pi)\)-approximation of the BQP, namely \((2/\pi)s^* \leq q^* \leq s^*\), and later in [23, Lemma 6] it was shown that (7) can be equivalently written in the following form:

\[
\max_X \ \ 2 \ln(\sum_{i=1}^n \langle X, a_i a_i^T \rangle^{1/2}) \quad \text{s. t.} \quad X \in \mathbb{S}_n^+, \ \text{tr}(X) = (I_n, X) = 1,
\]  

(8)

where \( a_i \in \mathbb{R}^n \) is the \( i \)-th column of \( A^{1/2} \) for \( i \in [n] \).

To fit (7) into the problem class (P), we can take \( K_1 = \mathbb{S}_n^+ \), \( K_2 = \mathbb{R}_n^m \), \( e = I_n \), \( A : X \mapsto ((\langle X, a_i a_i^T \rangle)_{i=1}^m \), and \( f : w \mapsto 2 \ln(\sum_{i=1}^n \sqrt{w_i}) \) for \( w \in \mathbb{R}_n^m \setminus \{0\} \). In particular, note that \( f \) is not a self-concordant function.

2 Background

Throughout this work, all the algebras and vector spaces that appear are over reals.

2.1 Euclidean Jordan algebra (EJA)

**Jordan algebra.** Let \( (\mathbb{V}, \circ) \) be a (finite-dimensional) Jordan algebra, where \( \mathbb{V} \) is a finite-dimensional vector space and \( \circ : \mathbb{V} \times \mathbb{V} \to \mathbb{V} \) is a bilinear binary operation on \( \mathbb{V} \) such that for all \( x, y \in \mathbb{V} \), we have \( x \circ y \in \mathbb{V} \). In addition, for all \( x, y \in \mathbb{V} \), the binary operation \( \circ \) satisfies the following two properties:

i) \( x \circ y = y \circ x \),

ii) \( x^2 \circ (x \circ y) = x \circ (x^2 \circ y) \).
We call any bilinear binary operation $\circ$ that satisfies these two properties a Jordan product on $\mathbb{V}$. For notational convenience, we will sometimes use $\mathbb{V}$ to denote $(\mathbb{V}, \circ)$, if the operation $\circ$ is clear from context. We let $\mathbb{V}$ possess an identity element, namely an element $e \in \mathbb{V}$ such that $x \circ e = e \circ x = x$ for all $x \in \mathbb{V}$. We can define the power of $x \in \mathbb{V}$ recursively as

$$
x^0 := e, \quad x^k := x \circ x^{k-1}, \quad \forall k \geq 1.
$$

Let $\deg(x)$ denote the degree of $x \in \mathbb{V}$, which is defined to be the smallest positive integer $k$ such that $\{e, x, \ldots, x^k\}$ is linearly dependent. Then we define the rank of $\mathbb{V}$ to be the largest degree of all elements in $\mathbb{V}$, i.e., $\text{rank}(\mathbb{V}) := \max\{\deg(x) : x \in \mathbb{V}\}$. Denote $r := \text{rank}(\mathbb{V})$, and so for all $x \in \mathbb{V}$, the set $\{e, x, \ldots, x^r\}$ is linearly dependent, whereby there exist $\{a_i(x)\}_{i=1}^r \subseteq \mathbb{R}$ such that

$$
x^r - a_1(x)x^{r-1} + a_2(x)x^{r-2} + \cdots + (-1)^r a_r(x)e = 0. \quad (10)
$$

(Note that $0$ denotes the zero element in $\mathbb{V}$.) From [24, Proposition II.2.1], we know that for each $i \in [r]$, $a_i$ is a homogeneous polynomial of degree $i$. We call the monic polynomial

$$
\chi_x(\lambda) := \lambda^r - a_1(x)\lambda^{r-1} + a_2(x)\lambda^{r-2} + \cdots + (-1)^r a_r(x) \quad (\lambda \in \mathbb{R})
$$

the characteristic polynomial of $x$. Define the $r$ eigenvalues of $x$ (counting multiplicities) to be the roots of $\chi_x$, which we denote by $\{\lambda_i(x)\}_{i=1}^r$, and define

$$
\text{tr}(x) := \sum_{i=1}^r \lambda_i(x) = a_1(x) \quad \text{and} \quad \det(x) := \prod_{i=1}^r \lambda_i(x) = a_r(x). \quad (12)
$$

Note that $\text{tr}(\cdot)$ is a linear function on $\mathbb{V}$.

**EJA.** We call the Jordan algebra $\mathbb{V}$ (with identity element $e$) Euclidean if there exists an inner product $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ on $\mathbb{V}$ that satisfies

$$
\langle x, y \circ z \rangle = \langle x \circ y, z \rangle, \quad \forall x, y, z \in \mathbb{V}. \quad (13)
$$

From [24, Proposition III.1.5], we know that $\mathbb{V}$ is Euclidean if and only if the symmetric bilinear form $(x, y) \mapsto \text{tr}(x \circ y)$ is positive definite on $\mathbb{V}$, i.e., it is an inner product on $\mathbb{V}$. As such, throughout this work we fix $\langle x, y \rangle := \text{tr}(x \circ y)$, and hence $\langle e, x \rangle = \text{tr}(x)$ for $x \in \mathbb{V}$. Let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$.

**Spectral decomposition.** A crucial property of the EJA $\mathbb{V}$ is that any $x \in \mathbb{V}$ admits a spectral decomposition, that is $x = \sum_{i=1}^r \lambda_i(x)q_i$, where $\{\lambda_i(x)\}_{i=1}^r \subseteq \mathbb{R}$ are the eigenvalues of $x$ and $\{q_i\}_{i=1}^r \subseteq \mathbb{V}$ is a complete system of primitive orthogonal idempotents, which satisfy the following three properties:

i) (Idempotency and primitiveness) $q_i^2 = q_i$ and $\|q_i\| = 1$, $\forall i \in [r]$,

ii) (Orthogonality) $q_i \circ q_j = 0$, $\forall i \neq j$, $i, j \in [r]$,

iii) (Completeness) $\sum_{i=1}^r q_i = e$.

We call $\{q_i\}_{i=1}^r$ a Jordan frame in $\mathbb{V}$. (For notational brevity, we omit showing the dependence of $\{q_i\}_{i=1}^r$ on $x$.) A simple consequence is that

$$
\langle q_i, q_j \rangle = \langle q_i^2, q_j \rangle = \langle q_i, q_i \circ q_j \rangle = 0, \quad \forall i \neq j, \quad i, j \in [r]. \quad (14)
$$

In addition, for convenience, denote the set of primitive idempotents in $\mathbb{V}$ by $\Omega(\mathbb{V})$, i.e.,

$$
\Omega(\mathbb{V}) := \{u \in \mathbb{V} : u^2 = u, \|u\| = 1\} \subseteq \mathbb{C}.
$$

Based on spectral decomposition, given any uni-variate function $f : \mathcal{D}_f \to \mathbb{R}$, where $\mathcal{D}_f \subseteq \mathbb{R}$ denotes the natural domain of $f$, we can define

$$
f(x) := \sum_{i=1}^r f(\lambda_i(x))q_i, \quad (15)
$$

for any $x \in \mathbb{V}$ such that $\{\lambda_i(x)\}_{i=1}^r \subseteq \mathcal{D}_f$. Common examples of $f$ include $\ln(\cdot)$, $\exp(\cdot)$ and $(\cdot)^\alpha$ with $\alpha \in \mathbb{R}$. In particular, if $x$ has no zero eigenvalues, we call $x$ invertible and define it inverse
For convenience, denote the set of invertible elements in \( V \) by \( \mathcal{I}(V) \). In addition, we easily see two simple calculus rules: given another uni-variate function \( g : D_g \to \mathbb{R} \), where \( D_g \subseteq \mathbb{R} \) denotes the natural domain of \( g \), we have
\[
f(x) + g(x) = (f + g)(x), \quad f \circ g(x) = (fg)(x),
\]
for any \( x \in V \) such that \( \{\lambda_i(x)\}_{i=1}^r \subseteq D_f \cap D_g \). In particular, we have \( x^\alpha \circ x^\beta = x^{\alpha+\beta} \) for any \( \alpha, \beta \in \mathbb{R} \) such that both \( x^\alpha \) and \( x^\beta \) exist.

**Linear and quadratic representations.** For any \( x \in V \), define its linear representation \( L(x) : V \to V \) as a linear operator on \( V \) such that \( L(x)y := x \circ y \) for all \( y \in V \). Note that \( L(\cdot) \) is linear due to the bilinearity of \( \circ \). Then, we define the **quadratic representation** of \( x \) as
\[
P(x) := 2L(x)^2 - L(x^2).
\]
(17)

Note that both \( L(x) \) and \( P(x) \) are self-adjoint w.r.t. the inner product \( \langle \cdot, \cdot \rangle \). The operator \( P(\cdot) \) will play a pivotal role in our analysis, and so we list a few of its useful properties below:

i) \( P(x) \) is invertible if and only if \( x \in \mathcal{I}(V) \), and \( P(x)^{-1} = P(x^{-1}) \) for \( x \in \mathcal{I}(V) \),

ii) \( P(x^\alpha) = P(x)^\alpha \) for any \( \alpha \in \mathbb{R} \) such that \( x^\alpha \) exists

iii) \( P(x^\alpha x^\beta) = x^{2\alpha+\beta} \) for any \( \alpha, \beta \in \mathbb{R} \) such that both \( x^\alpha \) and \( x^\beta \) exist,

iv) \( (P(x)y)^{-1} = P(x^{-1}y^{-1}) = P(x^{-1})y^{-1} \), if both \( x, y \in \mathcal{I}(V) \)

v) \( P(\alpha x) = \alpha^2 P(x) \) for all \( \alpha \in \mathbb{R} \),

vi) \( P(P(x)y) = P(x)P(y)P(x) \).

### 2.2 Symmetric cones

Let \( V \) be a (finite-dimensional) vector space equipped with inner product \( \langle \cdot, \cdot \rangle \), and \( \mathcal{K} \) be a symmetric cone in \( V \). From [24, Theorem III.3.1], we know that there exists a binary operation \( \circ : V \times V \to V \) such that when equipped with it, \( V \) becomes an EJA and \( \mathcal{K} \) becomes the **cone of squares** of \( V \), namely \( \mathcal{K} = \{x^2 : x \in V\} \). This is called the **Jordan algebraic characterization** of symmetric cones. (Note that the converse is also true: indeed, one can easily show that the cone of squares of any EJA is symmetric; see e.g., [25, Section 2.5].) Such a characterization provides a convenient way to study symmetric cones under the EJA framework, and in the following we will provide a few useful properties of cone of squares.

Let \( \mathcal{K} \) be the cone of squares in the EJA \( V \). From the spectral decomposition, we see that \( \mathcal{K} = \{x \in V : \lambda_i(x) \geq 0, \ \forall i \in [r]\} \), namely \( \mathcal{K} \) consists of all the elements in \( V \) with nonnegative eigenvalues. Similarly, we have \( \text{int} \mathcal{K} = \{x \in V : \lambda_i(x) > 0, \ \forall i \in [r]\} = \{x^2 : x \in \mathcal{I}(V)\} \). We denote the partial orders induced by \( \mathcal{K} \) and \( \text{int} \mathcal{K} \) by \( \succeq_{\mathcal{K}} \) and \( \succ_{\mathcal{K}} \), respectively. On the other hand, we can characterize the dual cone of \( \mathcal{K} \), i.e., \( \mathcal{K}^* = \{x \in V : \langle L(x)z, z \rangle \geq 0, \ \forall z \in V\} \), namely \( \mathcal{K}^* \) consists of all the elements in \( V \) such that \( L(x) \) is a PSD linear operator. Since \( \mathcal{K} \) is self-dual, we can also characterize \( \mathcal{K} \) in this way. Based on this characterization, we have that for any \( x, y \succeq_{\mathcal{K}} 0 \),
\[
\langle x, y \rangle = \sum_{i=1}^n \lambda_i(x)\langle y, q_i \rangle = \sum_{i=1}^n \lambda_i(x)\langle y, q_i^2 \rangle = \sum_{i=1}^n \lambda_i(x)\langle L(y)q_i, q_i \rangle \geq 0,
\]
(18)

In particular, we have that if \( y \succeq_{\mathcal{K}} x \) and \( z \succeq_{\mathcal{K}} 0 \), then \( \langle y, z \rangle \geq \langle x, z \rangle \).

Next, we introduce some properties of \( P(\cdot) \) on \( \mathcal{K} \). If \( x \in \mathcal{I}(V) \), then we can easily show that \( P(x) \) is a (linear) automorphism on both \( \mathcal{K} \) and \( \text{int} \mathcal{K} \), namely \( P(x)\mathcal{K} = \mathcal{K} \) and \( P(x)\text{int} \mathcal{K} = \text{int} \mathcal{K} \).

As a result, if \( x \in \mathcal{I}(V) \), then
\[
y \succeq_{\mathcal{K}} z \iff P(x)y \succeq_{\mathcal{K}} P(x)z, \quad y \succ_{\mathcal{K}} z \iff P(x)y \succ_{\mathcal{K}} P(x)z, \quad \forall y, z \in V,
\]
(19)

namely the (invertible) linear operator \( P(x) \) preserves the partial orders \( \succeq_{\mathcal{K}} \) and \( \succ_{\mathcal{K}} \) and vice versa. Indeed, for any \( x, y \in \text{int} \mathcal{K} \), we have \( P(w)x = y \), where \( w = P(x^{-1/2})(P(x^{1/2}y)^{1/2}) \in \text{int} \mathcal{K} \) is the scaling point for the pair \( (x, y) \). Lastly, note that for all \( x \in V \), we always have \( P(x)\mathcal{K} \subseteq \mathcal{K} \) and therefore, if \( y \succeq_{\mathcal{K}} 0 \), then \( P(x)y \succeq_{\mathcal{K}} P(x)z \).
2.3 Simple EJAs and primitive symmetric cones

Given a Jordan algebra \((V, \circ)\), its Jordan sub-algebra is given by \((V', \circ)\), where \(V'\) is a linear subspace of \(V\) that is closed under \(\circ\), namely for any \(x, y \in V'\), we have \(x \circ y \in V'\). We call a \(V, \circ\) simple if it only has trivial Jordan sub-algebras, namely \(\{0\}, \circ\) and \((V, \circ)\), and \(V \neq \{0\}\). We say that a \((V, \circ)\) is a direct sum of \(n\) (Euclidean) Jordan sub-algebras \(\{(V_i, \circ)\}_{i=1}^n\), written as \(V = \bigoplus_{i=1}^n V_i\), if i) the linear space \(V\) is a direct sum of the linear spaces \(\{V_i\}_{i=1}^n\) and ii) \(V_i \circ V_j := \{u_i \circ u_j : u_i \in V_i, u_j \in V_j\} = \{0\}\), for all \(i, j \in [n]\) such that \(i \neq j\).

From [24, Proposition III.4.4], we know that any EJA can be written as a direct sum of simple EJAs, and from [26] (see also [24, Chapter 5]), we know that any simple EJA is isomorphic to one of the following five EJAs (let \(n\) be a positive integer):

1. The Jordan spin algebra \(\mathbb{L}^{n+1}\): \(V = \mathbb{R}^{n+1}\). For any \(x := (x_0, x_1) \in \mathbb{R} \times \mathbb{R}^n\) and \(y := (y_0, y_1) \in \mathbb{R} \times \mathbb{R}^n\), we have \(x \circ y := (x^T y, x_0 y_1 + y_0 x_1)\) and the identity element \(e := (1, 0) \in \mathbb{R} \times \mathbb{R}^n\).

2. \(\text{Herm}(n, \mathbb{N})\): \(V = \mathbb{S}^n\), namely the space of the \(n \times n\) real symmetric matrices. For any \(X, Y \in \mathbb{S}^n\), we have \(X \circ Y := (X Y + Y X)/2\) and the identity element \(e := I_n\).

3. \(\text{Herm}(n, \mathbb{C})\): \(V\) is the space of \(n \times n\) complex Hermitian matrices. The binary operation \(\circ\) and identity element are defined in the same way as \(\text{Herm}(n, \mathbb{R})\).

4. \(\text{Herm}(n, \mathbb{Q})\): \(V\) is the space of \(n \times n\) quaternion \((\mathbb{Q})\) Hermitian matrices. The binary operation \(\circ\) and identity element are defined in the same way as \(\text{Herm}(n, \mathbb{R})\).

5. \(\text{Herm}(3, \mathbb{O})\): \(V\) is the space of \(3 \times 3\) octonion \((\mathbb{O})\) Hermitian matrices. The binary operation \(\circ\) and identity element are defined in the same way as \(\text{Herm}(n, \mathbb{R})\).

In particular, we see that \((\mathbb{R}^n, \circ)\) with \(\circ : (x, y) \mapsto (x_0 y_1, \ldots, x_n y_n)\) is a EJA, since \(\mathbb{R} = \text{Herm}(1, \mathbb{R})\). (Perhaps somewhat subtly, the last three cases involve matrices with non-real entries, however, from [24, Chapter 5], we know that all of these EJAs are isomorphic to some EJAs over reals, and so it suffices to consider real EJAs in this work, as stated at the beginning of Section 2.)

For later reference, the spectral decomposition in each of the EJAs \(\mathbb{R}^n\), \(\mathbb{L}^{n+1}\) and \(\text{Herm}(n, \mathbb{R})\) is provided below (recall that \(r = \text{rank}(V)\)):

1. \(V = \mathbb{R}^n\) (\(r = n\)): for any \(x \in \mathbb{R}^n\), \(\lambda_i(x) = x_i\) and \(q_i = e_i\) for \(i \in [n]\), where \(e_i\) is the \(i\)-th column of \(I_n\).

2. \(V = \mathbb{L}^{n+1}\) (\(r = 2\)): for any \(x = (x_0, x_1) \in \mathbb{R} \times \mathbb{R}^n\), \(\lambda_1(x) = x_0 - \|x_1\|_2\), \(q_1 = (1/2)(1, -u)\), \(\lambda_2(x) = x_0 + \|x_1\|_2\), and \(q_1 = (1/2)(1, u)\), where \(u = x_1/\|x_1\|_2\) if \(x_1 \neq 0\) and any vector with unit \(\ell_2\)-norm otherwise.

3. \(V = \text{Herm}(n, \mathbb{R})\) (\(r = n\)): the spectral decomposition coincides with the one in linear algebra.

We call a symmetric cone primitive if it is the cone of squares of a simple EJA. The classification of simple EJAs above imply that there are only five primitive symmetric cones:

1. the second-order cone \(\mathbb{Q}^{n+1}\) (associated with \(\mathbb{L}^{n+1}\)),
2. the cone of \(n \times n\) Hermitian PSD matrices over \(\mathbb{A}\) (associated with \(\text{Herm}(n, \mathbb{B})\)), where \(\mathbb{B} = \mathbb{R}, \mathbb{C}\) or \(\mathbb{Q}\),
3. the cone of \(3 \times 3\) Hermitian PSD matrices over octonions (associated with \(\text{Herm}(3, \mathbb{O})\)).

Since any EJA can be decomposed into the direct sum of simple EJAs, one can easily show that any symmetric cone can be written as a Cartesian product of primitive symmetric cones. In particular, since \(\mathbb{R}_+\) is the cone of squares in \(\text{Herm}(1, \mathbb{R})\), we see that \(\mathbb{R}_+^n\) is the the cone of squares in \(\mathbb{R}^n\), and hence is symmetric.
2.4 Transforming an optimization problem to (P)

Based on the background above, we illustrate two cases where the Problem (P) can be transformed from other problems. First, consider the case where in (P), the objective function remains the same and the constraint set becomes \( C_a := \{ x \in K_1 : \langle a, x \rangle = 1 \} \) for some \( a \in \text{int} K_1 \). Let \((V, \circ)\) be the EJA such that its cone of squares is \( K_1 \), and \( P(\cdot) \) be the quadratic representation in \( V \). Define \( w := P(a^{-1/2})a^{1/2} \in \text{int} K_1 \) so that \( P(w) = e \). Then by a change of variable \( z := P(w^{-1})x \), we can rewrite the problem \( \max_{x \in C_a} f(Ax) \) as \( \max_{x \in C} f(Bz) \), where \( B := AP(w) \). Since \( P(w) \) is a self-adjoint automorphism on both \( K_1 \) and \( \text{int} K_1 \), we easily see that both \( B \) and \( B^* = P(w)A^* \) satisfy all the requirements in Section 1.1. As the second case, suppose now that the constraint set is given by \( D(C) \), where \( D \) is a linear operator (and the objective function is unchanged). In this case, we can rewrite this problem as \( \max_{z \in \mathbb{R}^q} f(Ez) \), where \( E := AD \). If \( E \) (and \( E^* \)) satisfies the requirements in Section 1.1, then this problem also falls in the class (P). As a concrete example, consider the PET problem in (4), where the constraint set is given by a polytope that has \( n \) non-zero vertices in \( \mathbb{R}^q_+ \) (for some \( q \geq 2 \)).

3 A generalized MG method and its convergence analysis

3.1 A generalized MG method

We generalize the MG method in (MG) to solve the general class of problems in (P). This method starts with any point \( x^0 \in \text{ri} C := \mathcal{H} \cap \text{int} K_1 \), where \( \mathcal{H} := \{ x \in V : \text{tr}(x) = 1 \} \), and in each iteration \( t \geq 0 \), it consists of two steps:

\[
\begin{align*}
\hat{x}^{t+1} &:= \exp \left( \ln(x^t) + \ln(\nabla F(x^t)) \right), \\
x^{t+1} &:= \hat{x}^{t+1} / \text{tr}(\hat{x}^{t+1}).
\end{align*}
\]

To see how (MG) generalized (MG), note that from Section 2.3, when \( V = \mathbb{R}^n \) and \( K_1 = \mathbb{R}^n_+ \), the spectral decomposition of any \( x \in \mathbb{R}^n \) is \( x = \sum_{i=1}^n x_i e_i \), and hence both \( \exp(\cdot) \) and \( \ln(\cdot) \) are interpreted elementwise. Therefore, we have \( \hat{x}^{t+1} = \hat{x}^{t+1}_i = \hat{x}_i = 1 \). Hence in (MG), we have \( x^{t+1} = \hat{x}^{t+1} \) and (MG) is recovered. However, note that in general, the algorithm in (MG) cannot be simplified, since for some \( x, y \in V \), we may not have \( \exp(x + y) = \exp(x) \circ \exp(y) \). In fact, we may not even have \( \exp(x) \circ \exp(y) \succ_K 0 \) (but note that \( \exp(x + y) \succ_K 0 \) for all \( x, y \in V \)).

As an example, consider \( V = \text{Herm}(2, \mathbb{R}) \) and \( K_1 = \mathbb{S}^2_+ \), and choose \( X, Y \in V \) such that

\[
W := \exp(X) = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 2 & 2 \end{bmatrix} \succ_{K_1} 0, \quad R := \exp(Y) = \begin{bmatrix} 1 & 1 \\ 1 & 13 \end{bmatrix} \succ_{K_1} 0 \quad \text{and} \quad W \circ R = \begin{bmatrix} 1 & -5 \\ -5 & 25 \end{bmatrix}.
\]

Since \( \det(W \circ R) = 0 \), we see that \( W \circ R \not\succ_{K_1} 0 \). In addition, the \( \exp(\ln(\cdot) + \ln(\cdot)) \) structure of (MG) is inspired from the MG-type algorithm in [10] for solving the specific QST problem (cf. Section 1.2.3), in the context of which \( \exp(\cdot) \) and \( \ln(\cdot) \) denote the matrix exponential and logarithm, respectively.

At this point, one may wonder if (MG) is even well-defined, and let us provide some explanations on this. Suppose that \( x^t \in \text{ri} C \) for some \( t \geq 1 \). Recall that \( \text{ri} C = \mathcal{H} \cap \text{int} K_1 \).

First, note that \( \ln(\nabla F(x^t)) \) is well-defined, since \( \nabla F : \text{int} K_1 \to \text{int} K_1 \) (cf. Section 1.1) and hence \( \nabla F(x^t) \in \text{int} K_1 \). Second, the definition of \( \exp(\cdot) \) implies that \( \hat{x}^{t+1} \in \text{int} K_1 \) and hence \( \text{tr}(\hat{x}^{t+1}) > 0 \). Consequently, we see that \( x^{t+1} \in \text{int} K_1 \) and \( \text{tr}(x^{t+1}) = 1 \), and hence \( x^{t+1} \in \text{int} K_1 \cap \mathcal{H} = \text{ri} C \). Since \( x^0 \in \text{ri} C \), we see that sequence of iterates \( \{ x_t \}_{t \geq 0} \subseteq \text{ri} C \).

3.2 An important conic inequality

From (MG), we can show the following conic inequality in the “log-domain”, which is crucial in the analysis of (MG). In the rest of this paper, let \( n := \text{rank}(V) \).
Proposition 1. For all \( t \geq 0 \), we have \( \text{tr}(\dot{x}^{t+1}) \leq 1 \) and hence
\[
\ln(x^{t+1}) \succeq_{\mathcal{K}_1} \ln(x^t) + \ln(\nabla F(x^t)).
\tag{20}
\]

Again, the inequality (20) is trivial (and actually holds with equality) in the special case \( \mathbb{V} = \mathbb{R}^n \), where (GMG) becomes (MG). However, in the general case, the proof of this inequality is non-trivial, and makes use the following two lemmas.

Lemma 1 (Generalized Golden-Thompson inequality [27, Theorem 5.1]). For any EJA \( \mathbb{V} \) and any \( x, y \in \mathbb{V} \), we have
\[
\text{tr}(\exp(x + y)) \leq \text{tr}(\exp(x) \circ \exp(y)).
\tag{21}
\]

Lemma 1 is a generalization of the Golden-Thompson inequality in matrix theory [28, 29], i.e., when \( \mathbb{V} = \text{Herm}(n, \mathbb{R}) \) or \( \text{Herm}(n, \mathbb{C}) \). It indicates that although the relationship between \( \exp(x + y) \) and \( \exp(x) \circ \exp(y) \) can be quite complicated in general, the traces of these two quantities are related via a simple inequality.

Lemma 2 (Operator monotonicity of \( \ln(\cdot) \)). For any \( x, y \in \mathbb{V} \) such that \( y \succeq x > 0 \), we have \( y^{-1} \preceq x^{-1} \) and \( \ln(y) \succeq \ln(x) \), where the partial orders \( \succeq \) and \( \preceq \) are defined w.r.t. \( \mathcal{K}_1 \).

Proof. Define \( d := y - x \geq 0 \) so that \( y = x + d \). Since \( x + \mathcal{K} \subseteq \text{int} \mathcal{K} \), we have \( y \in \text{int} \mathcal{K} \), or \( y \succeq 0 \). We first show that \( y^{-1} \preceq x^{-1} \). Indeed, we have
\[
y \succeq x \iff z := P(x^{-1/2})y \succeq P(x^{-1/2})x = e
\]
\[
\iff e = P(z^{-1/2})z \preceq P(z^{-1/2})e = z^{-1}
\]
\[
\iff x^{-1} = P(x^{-1/2})e \preceq P(x^{-1/2})z^{-1},
\]
where all the steps follow from (19). Since by definition, \( z^{-1} = P(x^{-1/2})y^{-1} \), the last step becomes \( x^{-1} \succeq P(x^{-1/2})P(x^{-1/2})y^{-1} = y^{-1} \). It only remains to show \( \ln(y) \succeq \ln(x) \).

Let \( p_\tau(\lambda ) := -(\tau + \lambda )^{-1} \) for \( \tau \geq 0 \) and \( \lambda > 0 \), and we first show that \( p_\tau(y) \succeq p_\tau(x) \) for all \( \tau \geq 0 \). Indeed, based on the spectral decomposition \( x = \sum_{i=1}^n \lambda_i(x)q_i \), we have
\[
p_\tau(x) = -\sum_{i=1}^n (\tau + \lambda_i(x))^{-1}q_i = -\left( \sum_{i=1}^n (\tau + \lambda_i(x))q_i \right)^{-1} = -(x + \tau e)^{-1}.
\]
Similarly, we have \( p_\tau(y) = -(y + \tau e)^{-1} \). For all \( \tau \geq 0 \), since \( y + \tau e \succeq x + \tau e > 0 \), by the results above, we have \( (y + \tau e)^{-1} \succeq (x + \tau e)^{-1} \), which is equivalent to \( p_\tau(y) \succeq p_\tau(x) \). Next, we notice that the uni-variate function \( \ln : (0, +\infty) \to \mathbb{R} \) can be expressed as a (Riemann) integral:
\[
\ln(\lambda ) = \ln \left( \frac{\lambda + 1}{\tau + \lambda} \right)_{\tau=0}^{\tau=\infty} = \int_0^{\infty} \frac{1}{\tau + 1} - \frac{1}{\tau + \lambda} \, d\tau = \int_0^{\infty} \frac{1}{\tau + 1} + p_\tau(\lambda ) \, d\tau, \quad \forall \lambda > 0,
\]
and therefore,
\[
\ln(x) = \sum_{i=1}^n \int_0^{\infty} \frac{1}{\tau + 1} + p_\tau(\lambda_i(x)) \, d\tau \quad q_i = \int_0^{\infty} \frac{1}{\tau + 1} \sum_{i=1}^n q_i + \sum_{i=1}^n p_\tau(\lambda_i(x))q_i \, d\tau
\]
\[
= \int_0^{\infty} e + p_\tau(x) \, d\tau. \tag{22}
\]
Therefore, we have
\[
\ln(y) = \int_0^{\infty} \frac{1}{\tau + 1} e + p_\tau(y) \, d\tau \succeq \int_0^{\infty} \frac{1}{\tau + 1} e + p_\tau(x) \, d\tau = \ln(x). \tag{23}
\]

Proof of Proposition 1. From Lemma 1, we have
\[
\text{tr}(\dot{x}^{t+1}) = \text{tr} \left( \exp \left( \ln(x^t) + \ln(\nabla F(x^t)) \right) \right) \preceq \text{tr} \left( \exp \left( \ln(x^t) \right) \circ \exp \left( \ln(\nabla F(x^t)) \right) \right)
\]
\[
= \text{tr} \left( x^t \circ \nabla F(x^t) \right) = \langle \nabla F(x^t), x^t \rangle = 1,
\]
where the last step follows from that \( F \) is 1-LH (on \( \text{int} \mathcal{K}_1 \)). Since \( \text{tr}(\dot{x}^{t+1}) \leq 1 \) and \( \dot{x}^{t+1} \succeq_{\mathcal{K}_1} 0 \), we have
\[
x^{t+1} \succeq_{\mathcal{K}_1} \dot{x}^{t+1} = \exp \left( \ln(x^t) + \ln(\nabla F(x^t)) \right). \tag{23}
\]
Now, take \( \ln(\cdot) \) on both sides of (23) and apply Lemma 2, we arrive at (20). \qed
3.3 A growth bound of $F$

For convenience, let us denote $X^*$ as the set of optimal solutions of (P), which is nonempty due to the compactness of the constraint set $C$, and define the sub-optimality gap of (P) as

$$
\delta(x) := F(x^*) - F(x), \quad \forall x \in X, \forall x^* \in X^*.
$$

(24)

In this section, we first derive a growth bound of $F$ due to its 1-logarithmic-homogeneity (on $\text{int } K$), which plays an important role in the analysis of (GMG).

**Proposition 2.** For any $x \in \text{ri } C$ and any $x^* \in X^*$, we have $\delta(x) \leq \ln((\nabla F(x), x^*))$.

Indeed, this is an immediate consequence of the following lemma that applies to any LH function, which may be of independent interest.

**Lemma 3.** Let $K$ be a proper cone with polar cone $K^\circ := -K^\circ$. If $f : \text{int } K \to \mathbb{R}$ is a convex and differentiable function such that $-f$ is $\theta$-logarithmically-homogeneous for some $\theta > 0$ and $\nabla f : \text{int } K \to K^\circ$, then for any $x, z \in \text{int } K$, we have

$$
f(x) - f(z) \leq \theta \ln(-\langle \nabla f(x), z \rangle).\tag{25}
$$

**Proof.** Define the Fenchel conjugate of $f$ as

$$
f^*(s) := \sup_{x \in \text{int } K} \langle s, x \rangle - f(x), \quad \forall s \in K^\circ.\tag{26}
$$

(Indeed, since $f$ is $\theta$-LH, one can easily show that $f^*(s) < +\infty$ if and only if $s \in K^\circ$.) Using Fenchel’s inequality (see e.g., [30, pp. 105]), for any $z \in \text{int } K$ and $s \in K^\circ$, we have $\langle s, z \rangle < 0$ and

$$
f(z) + f^*(s + \theta \ln(-\langle s, z \rangle)) \geq \langle s, z \rangle + \theta \ln(-\langle s, z \rangle) \geq \min_{\tau > 0} -\tau + \theta \ln \tau = \theta \ln \theta.
$$

Therefore, for any $x \in \text{int } K$, by letting $s = \nabla f(x) \in K^\circ$, we have

$$
\theta \ln(-\langle \nabla f(x), z \rangle) \geq -\theta - f(z) - f^*(\nabla f(x)) = \langle \nabla f(x), x \rangle - f(z) - f^*(\nabla f(x)) = f(x) - f(z),
$$

where in the second step we use $\langle \nabla f(x), x \rangle = -\theta$ since $f$ is $\theta$-LH, and in the third step we use Fenchel’s inequality (see e.g., [30, pp. 105]). This completes the proof. \qed

In Lemma 3, if we set $f = -F$, $\theta = 1$ and $z = x^*$, then we obtain Proposition 2. Next, based on Proposition 2 and the structure of the generalized unit simplex $C$, we derive a growth bound of $F$ that is independent of the optimal solutions $X^*$.

**Proposition 3.** For any $x \in \text{ri } C$, we have

$$
\delta(x) \leq \max_{u \in C} \ln(-\langle \nabla F(x), u \rangle) = \ln(\lambda_{\max}(\nabla F(x))) = \max_{u \in C} \langle \ln(\nabla F(x)), u \rangle.\tag{27}
$$

Proposition 3 is an immediate consequence of the following lemma (and the monotonicity of $\ln(\cdot)$), which studies the maximization of a linear function over $C$.

**Lemma 4.** For any $b \in \mathbb{V}$ with spectral decomposition $b = \sum_{i=1}^r \lambda_i(b) q_i$, where $\lambda_1(b) \geq \ldots \geq \lambda_n(b)$, we have $\lambda_{\max}(b) := \lambda_1(b) = \langle b, q_1 \rangle = \max_{x \in C} \langle b, x \rangle$.

**Proof.** Define $c_i := \langle x, q_i \rangle$ for all $i \in [n]$. Since $x \in C$ and the Jordan frame $\{q_i\}_{i=1}^n \subseteq C$, we have $c_i \geq 0$ for all $i \in [n]$ and $\sum_{i=1}^n c_i = \langle e, x \rangle = 1$. Therefore, we have

$$
\max_{x \in C} \langle b, x \rangle = \max_{c \in \Delta_n} \sum_{i=1}^n \lambda_i(b) c_i = \lambda_1(b),\tag{28}
$$

where $\Delta_n := \{ c \in \mathbb{R}_+^n : \sum_{i=1}^n c_i = 1 \}$ is the standard unit simplex. In addition, the definition of the Jordan frame clearly suggests that $\langle b, q_1 \rangle = \lambda_1(b)$. This completes the proof. \qed
3.4 Convergence analysis of (GMG)

Before our analysis, we first formally define the class of gradient log-convex functions.

**Definition 1** (Gradient Log-convex Functions). Let $F$ be the objective function in (P) so that $\nabla F : \text{int } K_1 \to \text{int } K_1$. We call $F$ gradient log-convex if the following holds:

$$\lambda \ln(\nabla F(x)) + (1 - \lambda) \ln(\nabla F(y)) \succeq_{K_1} \ln(\nabla F(\lambda x + (1 - \lambda)y)), \quad \forall x, y \in \text{ri } C, \ \forall \lambda \in [0, 1].$$

**Remark 1.** Since one can easily show that $z \succeq_{K_1} 0$ if and only if $\langle z, u \rangle \geq 0$ for any $u \in \Omega(V)$ (which consists of all the primitive idempotents in $V$), we see that Assumption 1 is equivalent to the following statement: for any $u \in \Omega(V)$, the real-valued function

$$\phi_u : x \mapsto \langle \ln(\nabla F(x)), u \rangle$$

is convex on $\text{ri } C$. Although this assumption seems somewhat technical and restrictive, as we will show in Section 4, it actually holds for a wide class of functions. Note that for the particular QST problem (cf. Section 1.2.3), Lin et al. [10] have shown that this assumption to be satisfied, and made use of this assumption to analyze their MG-type algorithm.

We now present our main convergence theorem.

**Theorem 1.** For any $T \geq 1$, define the averaged iterate $\bar{x}^T := (1/T) \sum_{t=0}^{T-1} x^t$. Then we have

$$\delta(\bar{x}^T) \leq \frac{\ln \lambda_{\min}^{-1}(x^0)}{T}. \quad (30)$$

**Proof.** Let $u \in C$ be any primitive idempotent in $V$, and from (20), we have for all $t \geq 0$,

$$\langle \ln(x^{t+1}), u \rangle \geq \langle \ln(x^t), u \rangle + \langle \ln(\nabla F(x^t)), u \rangle,$$

and hence for all $T \geq 1$,

$$\langle \ln(x^T), u \rangle \geq \langle \ln(x^0), u \rangle + T \langle \ln(\nabla F(x^T)), u \rangle \geq \langle \ln(x^0), u \rangle + T \langle \ln(\nabla F(\bar{x}^T)), u \rangle,$$

where the last step follows from Assumption 1. Next, we claim that for all $y \in \text{ri } C$, $\langle \ln(y), u \rangle \leq 0$. To see this, let $y := \sum_{i=1}^n \lambda_i(y)q_i$ be the spectral decomposition of $y$, and hence

$$\langle \ln(y), u \rangle = \sum_{i=1}^n \ln(\lambda_i(y))\langle q_i, u \rangle \leq \sum_{i=1}^n (\lambda_i(y) - 1)\langle q_i, u \rangle$$

$$\leq \sum_{i=1}^n \lambda_i(y)\langle q_i, u \rangle - \langle e, u \rangle$$

$$\leq \sum_{i=1}^n \lambda_i(y) - 1$$

$$\leq \text{tr}(y) - 1 = 0,$$

where in (a) follows from $\langle q_i, u \rangle \geq 0$ (since both $q_i, u \in K_1$) and $\ln(x) \leq x - 1$ for $x > 0$, (b) follows from $\sum_{i=1}^n q_i = e$ and (c) follows from $\lambda_i(y) > 0$ for $i \in [n]$ (since $y \in \text{int } K_1$), $\langle q_i, u \rangle \leq \|q_i\|\|u\| = 1$ and $\langle e, u \rangle = \langle e, u^2 \rangle = \langle u, u \rangle = \|u\|^2 = 1$. Therefore, since $\bar{x}^T \in \text{ri } C$, we have $\langle \ln(\bar{x}^T), u \rangle \leq 0$ in (32) and hence

$$\langle \ln(\nabla F(\bar{x}^T)), u \rangle \leq \frac{-\langle \ln(x^0), u \rangle}{T} = \frac{\langle \ln((x^0)^{-1}), u \rangle}{T}. \quad (33)$$

Now, taking maximum of both sides over $u \in C$, and we have

$$\delta(\bar{x}^T) \leq \max_{u \in C} \langle \ln(\nabla F(\bar{x}^T)), u \rangle \leq \frac{\max_{u \in C} \langle \ln((x^0)^{-1}), u \rangle}{T} \leq \frac{\ln \lambda_{\max}((x^0)^{-1})}{T} = \frac{\ln \lambda_{\min}^{-1}(x^0)}{T},$$

where in both (a) and (b) we use Proposition 3. This complete the proof. □

**Remark 2.** Note that $\lambda_{\min}(x^0)$ characterizes the distance of $x^0$ to $\text{bd } K_1$ in the direction of $-e$. Since $x^0 \in \text{ri } C$, the optimal choice of $x^0$ should be $x^0 = (1/n)e$, and the convergence rate becomes $\ln(n)/T$. 

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4 Problem instances satisfying Assumption 1

In this section we provide several problem instances that satisfy Assumption 1. We first introduce the notion of associatively generated EJAs and a Cauchy-Schwartz-type (CS-type) inequality on these algebras. This inequality will play an important role in our theory.

4.1 Representable EJAs and a CS-type inequality

An algebra \((\mathbb{A}, \cdot)\) is a real associative algebra if \(\mathbb{A}\) is a finite-dimensional real vector space that is closed under the binary operation \(\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}\). For convenience, we will omit \(\cdot\) and simply write \(x \cdot y\) as \(xy\) for \(x, y \in \mathbb{A}\). Our description mainly follows from [31, Section 2], wherein \(\mathbb{A}\) is always assumed to have an identity element \(e\) such that \(xe = ex = x\), and there exists a linear automorphism on \(\mathbb{A}\), called conjugation and denoted by \((\cdot)’\), such that \((xy)’ = x’ y’\) for any \(x, y \in \mathbb{A}\). We call \(x \in \mathbb{A}\) symmetric if \(x’ = x\), and denote the subspace of symmetric elements in \(\mathbb{A}\) by \(S\), namely \(S := \{ x \in \mathbb{A} : x’ = x \}\). We define a simple Euclidean Jordan associative (EJA) system (over reals) as a triple \((\mathbb{A}, S, V)\) such that

i) \(\mathbb{A}\) is an associative algebra as described above such that \(\text{tr}(xx’) \geq 0\) for all \(x \in \mathbb{A}\),

ii) \((S, \circ)\) is a EJA, where the binary operation \(\circ : (x, y) \mapsto (xy + yx)/2\) for all \(x, y \in S\),

iii) \((V, \circ)\) is a simple Euclidean Jordan sub-algebra of \(S\).

Similar to the decomposition of EJA (cf. Section 2.3), any EAJ system \((\mathbb{A}, S, V)\) can be written as a finite direct sum of simple EAJ systems \(\{ (\mathbb{A}_i, S_i, V_i) \}_{i=1}^{m}\) (for some \(m \geq 1\)), namely \((\mathbb{A}, S, V) = \left( \bigoplus _{i=1}^{m} \mathbb{A}_i, \bigoplus _{i=1}^{m} S_i, \bigoplus _{i=1}^{m} V_i \right)\).

We call a EJA representable if it can be generated through a EAJ system as described above. Note that among all the simple EJAs listed in Section 2.3, only the first four are representable, namely \(\mathbb{L}^{n+1}\) and \(\text{Herm}(n, \mathbb{B})\), where \(\mathbb{B} = \mathbb{R}, \mathbb{C}\) or \(\mathbb{Q}\) (see [31, Section 4] for details). As such, any EJA that can be written as the direct sum of the first four simple EJAs is representable. Also, we call the cone of squares of any representable EJA as a representable symmetric cone.

The importance of representable EJAs can be seen from the following CS-type inequality, which is pivotal in proving many problem instances satisfying Assumption 1. Due to its technicality, we defer the proof of this inequality to Section 5.

**Proposition 4** (CS-type inequality for representable EJAs). Let \(V\) be a representable EJA and \(K\) be its cone of squares. In addition, let \(v_i \in K\) for \(i \in [n]\) and \(\alpha_i, \beta_i \in \mathbb{R}\) for \(i \in [n]\), such that \(\sum _{i=1}^{n} \alpha_i^2 v_i \in \text{int} K\). Then we have

\[
\sum _{i=1}^{n} \beta_i^2 v_i \succeq_K P(\sum _{i=1}^{n} \alpha_i \beta_i v_i)(\sum _{i=1}^{n} \alpha_i^2 v_i)^{-1}. \tag{34}
\]

**Remark 3.** Note that if \(K = \mathbb{R}_+\), then (39) becomes the “usual” elementary CS inequality:

\[
(\sum _{i=1}^{n} \beta_i^2 v_i)(\sum _{i=1}^{n} \alpha_i^2 v_i)^{-1} \geq \sum _{i=1}^{n} \alpha_i \beta_i v_i^2,
\]

where \(v_i \geq 0\), \(\forall i \in [n]\).

Next, we illustrate several problem instances that satisfy Assumption 1, and as remarked in Remark 1, we shall show that on these instances, the function \(\phi_u\) is convex on \(ri C\) for any \(u \in \Omega(V)\). In the following, we present these instances according to different types of \(K_1\) and \(K_2\).

4.2 First instance: \(K_1 = \mathbb{R}^n_+\) and \(K_2\) is any symmetric cone

Since \(K_2\) is symmetric, from the Jordan algebraic characterization in Section 2.2, we can associate \(K_2\) with a EJA \((\mathbb{W}, \circ)\) such that \(K_2\) is the cone of squares in \(\mathbb{W}\). Let \(r := \text{rank}(\mathbb{W})\). Also, due to the concavity of \(\ln(\cdot)\), for any \(v \in \text{int} K_1\), we have

\[
\langle \ln(v), u \rangle \leq \ln \langle v, u \rangle, \quad u \in \Omega(V). \tag{35}
\]
However, note that when \( K_1 = \mathbb{R}^n_+ \) (and hence \( V = \mathbb{R}^n \)), we have \( \Omega(V) := \{e_1, \ldots, e_n\} \) and (35) holds with equality. This fact leads to a simpler form of \( \phi_u \), namely

\[
\phi_u(x) = \langle \ln(\nabla F(x)), u \rangle = \ln(\langle \nabla F(x), u \rangle) = \ln(\langle A^* \nabla f(Ax), u \rangle) = \ln(\langle f(Ax), Au \rangle).
\] (36)

Since \( Ax \in \text{int}K_2 \) for all \( x \in rIC \) and \( Au \in K_2 \) for all \( u \in \Omega(V) \), to show the convexity of \( \phi_u \) on \( rIC \), we can indeed show a stronger statement: that is, the function

\[
\psi_w : y \mapsto \ln(\langle \nabla f(y), w \rangle)
\] (37)
is convex on \( \text{int}K_2 \), for any \( w \in K_2 \). There are (potentially) several choices of \( f \) to make \( \psi_w \) convex on \( \text{int}K_2 \) (for any \( w \in K_2 \)). Indeed, an obvious one would be \( f(y) = \ln(\langle c, y \rangle) \) for some \( c \in \text{int}K_2 \). However, this choice would make (P) trivial to solve. A less obvious choice is the (negative) rank-normalized log-determinant barrier for \( K_2 \), as shown in the proposition below.

**Proposition 5.** If \( f(y) = r^{-1} \ln \det(y) \) for \( y \in \text{int}K_2 \), then \( \psi_w \) is convex on \( \text{int}K_2 \) for any \( w \in K_2 \).

**Proof.** It suffices to consider \( r = 1 \). From standard results (see e.g., [25, Proposition 2.6.1]), we know that \( \nabla f(y) = y^{-1} \), and for convenience, let us define

\[
\zeta_w(y) := \langle \nabla f(y), w \rangle = \langle y^{-1}, w \rangle, \quad \forall y \in \text{int}K_2,
\]

so \( \psi_w = \ln \zeta_w \) on \( \text{int}K_2 \). Fix any \( y \in \text{int}K_2 \) and any \( h \in \mathbb{W} \), and define the uni-variate function \( \xi_w(\alpha) := \zeta_w(y + \alpha h) = \langle (y + \alpha h)^{-1}, w \rangle \) with \( \text{dom} \xi_w := \{ \alpha \in \mathbb{R} : y + \alpha h \in \text{int}K_2 \} \). Note that

\[
\xi_w(\alpha) = \langle (P(y^{-1/2})(e + \alpha P(y^{-1/2})h))^{-1}, w \rangle = \langle P(y^{-1/2})(e + \alpha P(y^{-1/2})h)^{-1}, w \rangle.
\]

Write the spectral decomposition of \( P(y^{-1/2})h = \sum_{i=1}^r \lambda_i q_i \) and using the self-adjoint property of \( P(y^{-1/2}) \), we have

\[
\xi_w(\alpha) = \langle \sum_{i=1}^r (1 + \alpha \lambda_i)^{-1} q_i, P(y^{-1/2})w \rangle = \sum_{i=1}^r (1 + \alpha \lambda_i)^{-1} c_i,
\] (38)

where \( c_i := \langle q_i, P(y^{-1/2})w \rangle \geq 0, \forall i \in [r] \), and the nonnegativity of \( \{c_i\}_{i=1}^r \) follows from the fact that \( P(y^{-1/2}) \) is an automorphism on \( K_2 \). From (38), we easily see that

\[
\xi_w(0) = \sum_{i=1}^r c_i, \quad \xi_w'(0) = -\sum_{i=1}^r c_i \lambda_i, \quad \xi_w''(0) = 2 \sum_{i=1}^r c_i \lambda_i^2,
\]

and consequently,

\[
\xi_w''(0)\xi_w(0) - \xi_w'(0)^2 \overset{(a)}{=} \left( \sum_{i=1}^r c_i \lambda_i^2 \right) \left( \sum_{i=1}^r c_i \right) - \left( \sum_{i=1}^r c_i \lambda_i \right)^2 \overset{(b)}{\geq} 0,
\] (39)

where (a) follows from \( c_i \geq 0 \) for all \( i \in [r] \) and (b) follows from Cauchy-Schwartz inequality. Now, define \( \varphi_w(\alpha) := \psi_w(y + \alpha h) = \ln \xi_w(\alpha) \) with \( \text{dom} \varphi_w = \text{dom} \xi_w \), and hence

\[
\langle \nabla^2 \psi_w(y)h, h \rangle = \varphi_w''(0) = \frac{\xi_w''(0)\xi_w(0) - \xi_w'(0)^2}{\xi_w(0)^2} \geq 0,
\]

where the last step follows from (39). This completes the proof. \( \square \)

Note that if \( K_2 \) is a Cartesian product of \( s \) symmetric cones (for some \( s \geq 2 \)), then we can choose \( f \) to be the convex combination of the \( s \) rank-normalized log-determinant barriers, each for one component symmetric cone. This, together with Proposition 5, covers the first two applications in Section 1.2, namely PET and D-optimal design.
4.3 Second case: $\mathcal{K}_1 = \mathbb{R}^n_+$ and $\mathcal{K}_2$ is the intersection of symmetric cones

Let us consider a setting related to the first case, that is, $\mathcal{K}_2$ is the intersection of $s$ symmetric cones $\{K^s\}_{s=1}^n$ that lie within the same linear space $\mathcal{W}$. By equipping $\mathcal{W}$ with $s$ different Jordan products $\{\circ_s : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}\}_{s=1}^n$, we effectively create $s$ Jordan algebras $\{\mathcal{W}_s, \circ_s\}_{s=1}^n$. We assume that each $(\mathcal{W}, \circ_s)$ to be Euclidean, and let $\mathcal{K}^s$ be the cone of squares in $(\mathcal{W}, \circ_s)$. A representative example of $\mathcal{K}_2$ would be the doubly nonnegative (matrix) cone, where $\mathcal{W} = \mathbb{S}^n$, namely the the space of $n \times n$ real symmetric matrices, $s = 2$, $\mathcal{K}_1 = \mathbb{S}^n_+$ and $\mathcal{K}_2$ is the cone of nonnegative matrices (i.e., matrices with nonnegative entries). For any $X, Y \in \mathbb{S}^n$, if we let $X \circ_1 Y := (XY + YX)/2$ and $X \circ_2 Y := X \odot Y$ (where $\odot$ denotes entrywise product), then one can easily verify the $\mathcal{K}_i$ is indeed the cone of squares in $(\mathcal{W}, \circ_i)$, for $i = 1, 2$.

Denote the rank of $(\mathcal{W}, \circ_i)$ by $r_i \geq 1$ and its determinant by $\det_i : \mathcal{W} \rightarrow \mathbb{R}$. Let

$$f(y) := \sum_{i=1}^s \beta_ir_i^{-1} \ln \det_i(y), \quad \forall y \in \text{int} \mathcal{K}_2 = \bigcap_{i=1}^s \text{int} \mathcal{K}^i, \quad (40)$$

where $\beta_i > 0$ for all $i \in [s]$ and $\sum_{i=1}^s \beta_i = 1$. For $i \in [s]$ and $y \in \text{int} \mathcal{K}_2$, denote the inverse of $y$ w.r.t. $(\mathcal{W}, \circ_i)$ by $g_i(y) \in \text{int} \mathcal{K}^i$, and so $g_i(y) = \nabla \ln \det_i(y)$. As a result,

$$\nabla f(y) = \sum_{i=1}^s \beta_ir_i^{-1} g_i(y) \in \sum_{i=1}^s \text{int} \mathcal{K}^i \subseteq \text{int} \left( \sum_{i=1}^s \mathcal{K}^i \right)$$

and since $\mathcal{K}^s = \left( \bigcap_{i=1}^s \mathcal{K}^i \right)^* = \bigcap_{i=1}^s \mathcal{K}^i$, we have $\nabla f(y) \in \text{int} \mathcal{K}^s$, which satisfies our assumption in Section 1.1. Next, let us show that this choice of $f$ makes $\psi_w$ (defined in (37)) convex on $\text{int} \mathcal{K}_2$ for any $w \in \mathcal{K}_2$.

**Proposition 6.** With the choice of $f$ in (40), $\psi_w$ is convex on $\text{int} \mathcal{K}_2$ for any $w \in \mathcal{K}_2$.

**Proof.** The proof leverages the proof of Proposition 5. Indeed, fix any $y \in \text{int} \mathcal{K}_2$ and any $h \in \mathcal{W}$, and define the uni-variate function $\xi^s_w(\alpha) := (g_i(y + ah), w)$ with $\text{dom} \xi^s_w := \{\alpha \in \mathbb{R} : y + ah \in \text{int} \mathcal{K}_2\}$ for $i \in [s]$. The proof of Proposition 5 suggests that both $\xi^s_w(0), (\xi^s_w)'(0) \geq 0$ and

$$(\xi^s_w)'(0) \xi^s_w(0) \geq (\xi^s_w)'(0)^2, \quad \forall i \in [s]. \quad (41)$$

Using CS-inequality and (41), we have

$$\left( \sum_{i=1}^s \beta_i r_i^{-1} (\xi^s_w)'(0) \right) \left( \sum_{i=1}^s \beta_i r_i^{-1} \xi^s_w(0) \right) \geq \left( \sum_{i=1}^s \beta_i r_i^{-1} (\xi^s_w)'(0)^{1/2} \xi^s_w(0)^{1/2} \right)^2 \geq \left( \sum_{i=1}^s \beta_i r_i^{-1} (\xi^s_w)'(0) \right)^2 \geq \left( \sum_{i=1}^s \beta_i r_i^{-1} (\xi^s_w)'(0) \right)^2. \quad (42)$$

Define $\varphi_w(\alpha) := \psi_w(y + ah) = \ln(\sum_{i=1}^s \beta_i r_i^{-1} \xi^s_w(\alpha))$ with $\text{dom} \varphi_w = \bigcap_{i=1}^s \text{dom} \xi^s_w$, and hence

$$\langle \nabla^2 \psi_w(y) h, h \rangle = \varphi_w''(0) = \frac{\sum_{i=1}^s \beta_i r_i^{-1} (\xi^s_w)'(0)^{1/2}}{\sum_{i=1}^s \beta_i r_i^{-1} (\xi^s_w)(0)^{1/2}} \geq 0,$$

where the last step follows from (42). This completes the proof.

4.4 Third case: $\mathcal{K}_1$ is any representable symmetric cone and $\mathcal{K}_2 = \mathbb{R}^n_+$

Next, let us consider the case where $\mathcal{K}_1$ is any representable symmetric cone (and hence $\mathcal{V}$ is a representable EJA) and $\mathcal{K}_2 = \mathbb{R}^n_+$. Note that we unlike the case where $\mathcal{K}_1 = \mathbb{R}^n_+$, we cannot write $\phi_u$ in the simpler form as in (36), and consequently, the proof of the convexity of $\phi_u$ on $\text{ri} \mathcal{C}$ (for any $u \in \Omega(\mathcal{V})$) is more complicated. Nevertheless, in the following, we will show that this is true for at least two choices of $f$ (defined on $\text{int} \mathcal{K}_2 = \mathbb{R}^m_+$). The first one is the standard (normalized) log-barrier of $\mathbb{R}^n_+$, namely

$$f(y) = \sum_{j=1}^m w_j \ln y_j, \quad (\forall y \in \mathbb{R}^m_++), \quad \text{where } \sum_{j=1}^m w_j = 1, \ w_j > 0, \ \forall j \in [m]. \quad (43)$$
The second one is the logarithm of the $p$-pseudo-norm (where $0 < p < 1$), namely
\[
  f(y) = \ln \|y\|_p := p^{-1} \ln(\sum_{j=1}^m y_j^p) \quad (\forall \, y \in \mathbb{R}_+^m).
\] (44)
In addition, from Section 1.1, since $A^* : \mathbb{R}_+^m \to K_1$, without loss of generality, we can let
\[
  A^* : y \mapsto \sum_{j=1}^m v_j y_j \quad \text{for some } \{v_j\}_{j=1}^m \subseteq K_1 \quad (\text{where } y \in \mathbb{R}_+^m),
\]
and \{\(v_j\)\}_{j=1}^m are chosen such that $A^* : \mathbb{R}_+^m \to \text{int} K_1$. Throughout this section, let $\preceq$ and $\succeq$ denote the partial orders induced by $K_1$.

Given any function $g : \text{int} K_1 \to \mathbb{V}$, let us define its directional derivative at any $x \in \text{int} K_1$ along any $d \in \mathbb{V}$ as
\[
  Dg(x)[d] := \lim_{\alpha \downarrow 0} \frac{g(x + \alpha d) - g(x)}{\alpha} \in \mathbb{V},
\]
and similarly, define $D^2g(x)[d, d'] := \lim_{\alpha \downarrow 0} \alpha^{-1}(Dg(x + \alpha d')[d] - Dg(x)[d]) \in \mathbb{V}$ for any $d' \in \mathbb{V}$. Our proof will make use of the following two important lemmas.

**Lemma 5.** Let $x \in \text{int} K_1$. Then for any $d \in \mathbb{V}$, we have
\[
  Dx^{-1}[d] = -P(x^{-1})d,
\]
\[
  D^2x^{-1}[d, d] = 2P(x^{-1/2})(P(x^{-1/2})d)^2.
\]
Consequently, we have
\[
  D\ln(x)[d] = \int_0^{+\infty} P(x + \tau e)^{-1}d\tau \quad (45)
\]
\[
  D^2\ln(x)[d, d] = -2\int_0^{+\infty} P(x + \tau e)^{-1}P(d)(x + \tau e)^{-1}d\tau. \quad (46)
\]

**Proof.** Define $\xi_{x, d}(\alpha) := (x + \alpha d)^{-1}$ with $\text{dom} \xi_{x, d} = \{\alpha \in \mathbb{R} : x + \alpha d \in \text{int} K_1\}$. Let $v := P(x^{-1/2})d$ with spectral decomposition $\sum_{i=1}^n \mu_i p_i$, and we have
\[
  \xi_{x, d}(\alpha) = (P(x^{1/2})(e + \alpha v))^{-1} = P(x^{-1/2})(e + \alpha v)^{-1} = P(x^{-1/2})\sum_{i=1}^n (1 + \alpha \mu_i)^{-1} p_i.
\]
As a result,
\[
  Dx^{-1}[d] = \xi_{x, d}'(0) = -P(x^{-1/2})\sum_{i=1}^n (1 + \alpha \mu_i)^{-2}\mu_i p_i|_{\alpha=0} = -P(x^{-1/2})\sum_{i=1}^n \mu_i p_i = -P(x^{-1/2})v = -P(x^{-1})d,
\]
\[
  D^2x^{-1}[d, d] = \xi_{x, d}''(0) = 2P(x^{-1/2})\sum_{i=1}^n \mu_i^2 p_i = 2P(x^{-1/2})v^2 = 2P(x^{-1/2})(P(x^{-1/2})d)^2.
\]
Now, define $\zeta(a) := \ln(x + ad)$ with $\text{dom} \zeta = \{a \in \mathbb{R} : x + ad \in \text{int} K_1\}$. From (22), we have
\[
  \zeta(a) = \int_0^{+\infty} (\tau + 1)^{-1} e - (x + \tau e + ad)^{-1} d\tau = \int_0^{+\infty} (\tau + 1)^{-1} e - \xi_{x + \tau e, d}(a) d\tau
\]
and hence
\[
  D\ln(x)[d] = \zeta'(0) = \int_0^{+\infty} -\xi_{x + \tau e, d}'(0) d\tau = \int_0^{+\infty} P(x + \tau e)^{-1}d\tau
\]
\[
  D^2\ln(x)[d, d] = \zeta''(0) = \int_0^{+\infty} -\xi_{x + \tau e, d}''(0) d\tau = -2\int_0^{+\infty} P(x + \tau e)^{-1/2}(P((x + \tau e)^{-1/2})d)^2 d\tau
\]
\[
  \overset{(a)}{=} -2\int_0^{+\infty} P(x + \tau e)^{-1/2}P(x + \tau e)^{-1/2}P(d)(x + \tau e)^{-1} d\tau
\]
\[
  = -2\int_0^{+\infty} P(x + \tau e)^{-1}P(d)(x + \tau e)^{-1} d\tau,
\]
where in (a) we use $(P(x)y)^2 = P(P(x)y)e = P(x)P(y)e = P(x)P(y)x^2$ for all $x, y \in \mathbb{V}$. \(\square\)
Lemma 6. Let $\pi : \mathcal{I} \to \text{int} \mathcal{K}_1$ be a uni-variate twice differentiable function, where $\mathcal{I} \subseteq \mathbb{R}$ is an interval that contains 0. Define $\omega(\alpha) := \ln \pi(\alpha)$ for $\alpha \in \mathcal{I}$, where the function $\ln : \text{int} \mathcal{K}_1 \to \mathbb{V}$. Then for any $u \in \mathcal{K}_1$, we have

$$\langle \omega''(0), u \rangle = 2 \int_{0}^{+\infty} \langle B(\tau), w(\tau) \rangle \, d\tau \geq 2 \int_{0}^{+\infty} \langle B(0), w(\tau) \rangle \, d\tau,$$  

(47)

where for all $\tau \geq 0$,

$$B(\tau) := (1/2)\pi''(0) - P(\pi'(0))(\pi(0) + \tau e)^{-1} \quad \text{and} \quad w(\tau) := P(\pi(0) + \tau e)^{-1} u \in \mathcal{K}_1.$$  

Proof. Using chain rule, we clearly have

$$\omega'(0) = D \ln(\pi(0))[\pi'(0)],$$  

(49)

$$\omega''(0) = D \ln(\pi(0))[\pi'(0)] + D^2 \ln(\pi(0))[\pi'(0), \pi'(0)].$$  

(50)

Consequently, using (45) and (46) in Lemma 5, we have

$$\langle \omega''(0), u \rangle = \int_{0}^{+\infty} \langle P(\pi(0) + \tau e)^{-1}\pi''(0) - 2P(\pi(0) + \tau e)^{-1}P(\pi'(0))(\pi(0) + \tau e)^{-1}, u \rangle \, d\tau,$$

$$= \int_{0}^{+\infty} \langle P(\pi(0) + \tau e)^{-1}(\pi''(0) - 2P(\pi'(0))(\pi(0) + \tau e)^{-1}), u \rangle \, d\tau,$$

$$= \int_{0}^{+\infty} \langle \pi''(0) - 2P(\pi'(0))(\pi(0) + \tau e)^{-1}, P(\pi(0) + \tau e)^{-1}u \rangle \, d\tau,$$

$$= \int_{0}^{+\infty} 2 \langle B(\tau), w(\tau) \rangle \, d\tau.$$

This shows the equality in (47). To show the inequality in (47), it suffices to show $B(\tau) \geq B(0)$ for all $\tau \geq 0$. Indeed, since $\pi(0) + \tau e \geq \pi(0) > 0$, from Lemma 2, we have $(\pi(0) + \tau e)^{-1} \leq \pi(0)^{-1}$. Hence, from the last part of Section 2.2, we have $P(\pi'(0))(\pi(0) + \tau e)^{-1} \leq P(\pi'(0))\pi(0)^{-1}$, and this implies $B(\tau) \geq B(0)$, for all $\tau \geq 0$. \qed

Proposition 7. With the choice of $f$ in either (43) or (44), the function $\phi_u$ is convex on $\text{int} \mathcal{K}_1$ for any $u \in \mathcal{K}_1$.

Proof. Fix any $u \in \mathcal{K}_1$. For convenience, let us define $g(y) := \ln(A^* \nabla f(y)) = \ln(\sum_{j=1}^{m} \nabla_j f(y)v_j)$ for any $y \in \mathbb{R}^{m}_{++}$, and so $\phi_u(x) := \langle g(Ax), u \rangle$ for $x \in \text{int} \mathcal{K}_1$. Consequently,

$$D^2 \phi_u(x)[d,d] = \langle D^2 g(Ax)[A\pi, A\pi], u \rangle, \quad \forall d \in \mathbb{V}.$$  

To show $D^2 \phi_u(x)[d,d] \geq 0$ for all $x \in \text{int} \mathcal{K}_1$ and $d \in \mathbb{V}$, it suffices to show

$$\langle D^2 g(y)[h,h], u \rangle \geq 0, \quad \forall y \in \mathbb{R}^{m}_{++}, \forall h \in \mathbb{R}^{m}.$$  

(51)

To this end, define $\pi(\alpha) := \sum_{j=1}^{m} \nabla_j f(y + \alpha h)v_j$ and $\omega(\alpha) := \ln \pi(\alpha) = g(y + \alpha h)$, with $\text{dom} \omega = \text{dom} \pi = \{\alpha \in \mathbb{R} : y + \alpha h \in \mathbb{R}^{m}_{++}\}$. Note that we have $\langle D^2 g(y)[h,h], u \rangle = \langle \omega''(0), u \rangle$ and

$$\pi(0) = \sum_{j=1}^{m} \nabla_j f(y)v_j, \quad \pi'(0) = \sum_{j=1}^{m} D^2 f(y)[e_j,h]v_j \quad \text{and} \quad \pi''(0) = \sum_{j=1}^{m} D^3 f(y)[e_j,h,h]v_j.$$  

From Lemma 6, to show $\langle \omega''(0), u \rangle \geq 0$, it suffices to show $B(0) \geq 0$ (which then implies that $B(0), w(\tau) \rangle \geq 0$ for all $\tau \geq 0$ in (47)), where

$$B(0) := (1/2)\pi''(0) - P(\pi'(0))\pi(0)^{-1}$$

$$= (1/2) \sum_{j=1}^{m} D^3 f(y)[e_j,h,h]v_j - P\left( \sum_{j=1}^{m} D^2 f(y)[e_j,h]v_j \right) \left( \sum_{j=1}^{m} \nabla_j f(y)v_j \right)^{-1}.$$  

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Let us consider the first case, namely \( f(y) = \sum_{j=1}^{m} w_j \ln y_j \) as in (43). We have
\[
\nabla_j f(y) = w_j y_j^{-1}, \quad D^2 f(y)[e_j, h] = -w_j y_j^{-2} h_j \quad \text{and} \quad D^3 f(y)[e_j, h, h] = 2w_j y_j^{-3} h_j^2
\]
for all \( j \in [m] \) and hence
\[
(1/2)B(0) = \sum_{j=1}^{m} w_j y_j^{-3} h_j^2 v_j - P(\sum_{j=1}^{m} w_j y_j^{-2} h_j v_j)(\sum_{j=1}^{m} w_j y_j^{-1} v_j)^{-1} = \sum_{j=1}^{m} b_j^2 - P(\sum_{j=1}^{m} a_j \circ b_j)(\sum_{j=1}^{m} a_j^2)^{-1} \geq 0,
\]
where \( a_j := \frac{w_j^{1/2} - y_j^{-1/2} v_j^{1/2}}{y_j^{1/2} - y_j^{-1/2} v_j^{1/2}} \) and \( b_j := \frac{w_j^{1/2} - 3/2 h_j v_j^{1/2}}{y_j^{1/2} - y_j^{-1/2} v_j^{1/2}} \) for \( j \in [m] \), and the last step follows from the CS-type inequality in Proposition 4.

Now, let us consider the second case, namely \( f(y) = p^{-1} \ln(\sum_{j=1}^{m} y_j^p) \) as in (44), so \( \nabla_j f(y) = y_j^{p-1}/(\sum_{j=1}^{m} y_j^p) \) for \( j \in [m] \). Define \( \tilde{g} : y \mapsto \ln(\sum_{j=1}^{m} y_j^{p-1} v_j) \) for \( y \in \mathbb{R}_{++}^m \), and hence
\[
g(y) = \ln \left( \frac{\sum_{j=1}^{m} y_j^{p-1} v_j}{\sum_{j=1}^{m} y_j^p} \right) = \tilde{g}(y) - \ln \left( \sum_{j=1}^{m} y_j^p \right) e = \tilde{g}(y) - pf(y)e.
\]
Since \( f \) is concave on \( \mathbb{R}_{++}^m \), we have \( D^2 g(y)[h, h] \succeq D^2 \tilde{g}(y)[h, h] \) for all \( y \in \mathbb{R}_{++}^m \) and \( h \in \mathbb{R}^m \), and hence to show (51), it suffices to show
\[
(1/2) \sum_{j=1}^{m} D^3 \tilde{f}(y)[e_j, h, h] v_j - P(\sum_{j=1}^{m} D^2 \tilde{f}(y)[e_j, h] v_j)(\sum_{j=1}^{m} \nabla_j \tilde{f}(y) v_j)^{-1} \geq 0,
\]
where for all \( j \in [m] \),
\[
\nabla_j \tilde{f}(y) = y_j^{p-1}, \quad D^2 \tilde{f}(y)[e_j, h] = (p-1)y_j^{p-2} h_j, \quad D^3 \tilde{f}(y)[e_j, h, h] = (1-p)(2-p)y_j^{p-3} h_j^2.
\]
To show (53), note that using (54), the left-hand side of (53) can be written as
\[
(1-p)(1-p/2) \sum_{j=1}^{m} y_j^{p-3} h_j^2 v_j - (1-p)^2 P(\sum_{j=1}^{m} y_j^{p-2} h_j v_j)(\sum_{j=1}^{m} y_j^{p-1} v_j)^{-1} \\
\geq (1-p)^2 \left[ \sum_{j=1}^{m} y_j^{p-3} h_j^2 v_j - P(\sum_{j=1}^{m} y_j^{p-2} h_j v_j)(\sum_{j=1}^{m} y_j^{p-1} v_j)^{-1} \right] \\
\geq (1-p)^2 \left[ \sum_{j=1}^{m} b_j^2 - P(\sum_{j=1}^{m} a_j \circ b_j)(\sum_{j=1}^{m} a_j^2)^{-1} \right] \geq 0,
\]
where \( a_j := y_j^{(p-1)/2} v_j^{1/2} \) and \( b_j := y_j^{(p-3)/2} h_j v_j^{1/2} \) for \( j \in [m] \), and the last step follows from the CS-type inequality in Proposition 4. This completes the proof.

Note that Proposition 7 covers the last two applications in Section 1.2, namely QST and the semidefinite relaxation of BQP.

### 5 Proof of Proposition 4

The proof of Proposition 4 leverages the following two lemmas. Throughout this section, let us denote the EAJ system that generates \( V \) by \((A, S, V)\), and denote the cone of squares in \( S \) by \( K(S) \).

**Lemma 7** ([31, Lemma 3]). For any \( x \in A \), we have \( xx' \succeq_{K(S)} 0 \).

**Lemma 8.** We have \( V \cap K(S) = K \).

**Proof.** By definition, clearly \( K \subset V \) and \( K \subseteq K(S) \). Thus it suffices to show the \( \subseteq \) direction. Let \( x \in V \cap K(S) \), so that \( x = z^2 \) for some \( z \in S \). Since \( x \in V \), we have \( z = x^{1/2} \in V \) and \( x \in K \).
To show Proposition 4, let us first define \( a_i := \alpha_i v_i^{1/2} \) and \( b_i := \beta_i v_i^{1/2} \) for \( i \in [n] \) (note that \( a_i, b_i \in \mathbb{V} \subseteq \mathbb{S} \) for \( i \in [n] \)), and let \( c \in \mathbb{A} \). From Lemma 7, we have
\[
\sum_{i=1}^{n} (b_i - ca_i) (b_i - ca_i)’ \succeq_{\mathcal{K}(\mathbb{S})} 0.
\]
On the other hand, by the symmetry of \( b_i \) and \( a_i \), we have
\[
\sum_{i=1}^{n} (b_i - ca_i) (b_i - ca_i)’ = \sum_{i=1}^{n} (b_i - ca_i) (b_i - a_i c’)
= \sum_{i=1}^{n} b_i^2 - (\sum_{i=1}^{n} b_i a_i) c’ - c(\sum_{i=1}^{n} a_i b_i) + c(\sum_{i=1}^{n} a_i^2) c’.
\]
(Note that since \( x \cdot x = x \circ x \) for all \( x \in \mathbb{S} \), \( b_i^2 \) is defined w.r.t. both binary operations \( \cdot \) and \( \circ \).)
By choosing \( c = (\sum_{i=1}^{n} b_i a_i) (\sum_{i=1}^{n} a_i^2)^{-1} \) so that \( c’ = (\sum_{i=1}^{n} a_i^2)^{-1} (\sum_{i=1}^{n} a_i b_i) \), we have
\[
0 \preceq_{\mathcal{K}(\mathbb{S})} \sum_{i=1}^{n} (b_i - ca_i) (b_i - ca_i)’ = \sum_{i=1}^{n} b_i^2 - (\sum_{i=1}^{n} b_i a_i)^2 (\sum_{i=1}^{n} a_i^2)^{-1} (\sum_{i=1}^{n} a_i b_i)
= \sum_{i=1}^{n} b_i^2 - (\sum_{i=1}^{n} a_i^2 b_i v_i) (\sum_{i=1}^{n} a_i^2 v_i)^{-1} (\sum_{i=1}^{n} a_i^2 b_i v_i)^{-1}
\overset{(a)}{=} \sum_{i=1}^{n} b_i^2 v_i - P(\sum_{i=1}^{n} a_i b_i v_i) (\sum_{i=1}^{n} a_i^2 v_i)^{-1} \in \mathbb{V},
\]
where in (a) we use the fact that \( P(x) y = x y x \) for all \( x, y \in \mathbb{S} \), which can be easily derived from the definition \( x \circ y = (x + y)/2 \). Thus, we have
\[
\sum_{i=1}^{n} b_i^2 v_i - P(\sum_{i=1}^{n} a_i b_i v_i) (\sum_{i=1}^{n} a_i^2 v_i)^{-1} \in \mathbb{V} \cap \mathcal{K}(\mathbb{S}) = \mathcal{K},
\]
where the last equality follows from Lemma 8. This completes the proof.

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