Varieties of aperiodic monoids with central idempotents whose subvariety lattice is distributive

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Abstract
We completely classify all varieties of aperiodic monoids with central idempotents whose subvariety lattice is distributive.

Keywords Monoid · Aperiodic monoid · Monoid with central idempotents · Variety · Subvariety lattice · Distributive lattice

Mathematics Subject Classification 20M07

1 Introduction and summary

A variety $V$ is distributive if its lattice $\mathcal{L}(V)$ of subvarieties is distributive.

There was genuine interest to investigate distributive varieties of groups. At the turn of the 1960s and 1970s, there were a lot of articles on this topic (see Cossey [2], Kovács and Newman [17] and Roman’kov [25], for instance). However, over time, the activity began to fade. The general problem of describing distributive varieties of groups turned out to be highly infeasible. Here it suffices to refer to the result of Kozhevnikov [18], which implies that there exist uncountably many group varieties whose subvariety lattice is isomorphic to the 3-element chain.

The problem of describing (in term of identities) distributive varieties of rings was raised by Bokut’ in 1976 in [3, Problem 19]. This problem remains open so far. Quoting from a paper [29] by Volkov on this topic, ‘There is extensive literature on the subject...”

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so that even the mere list of relevant publications is far too long to be placed here. Roughly speaking, one may characterize the current stage of investigations as a period of searching for a border separating varieties with distributive and non-distributive subvariety lattice’.

In 1979, Shevrin [28, Problem 2.60a] posed the problem of classifying all distributive varieties of semigroups. This problem includes the problem of identifying all distributive varieties of periodic groups. In view of the above-mentioned result by Kozhevnikov [18], the last problem seems to be extremely difficult. So, it is natural to speak about the classification modulo group varieties here. In the early 1990s, in a series of papers, Volkov described distributive varieties of semigroups in a very wide partial case, resulting in an almost complete description modulo group varieties (see Shevrin at al. [27, Section 11] for more details).

The present article is concerned with the distributive varieties of monoids, i.e., semigroups with an identity element. Even though monoids are very similar to semigroups, the story turns out to be very different and difficult. Distributive varieties of monoids have not been systematically examined earlier, although non-trivial examples of such varieties have long been known. We mean the variety of all commutative monoids (Head [11]) and the variety of all idempotent monoids (Wismath [30]). Since the middle of the 2000s, papers began to appear with some other non-trivial examples of distributive varieties of monoids (see Gusev [4, 5], Gusev and Sapir [8], Gusev and Vernikov [9, 10], Jackson [12], Jackson and Lee [14], Lee [19, 21, 23], Zhang and Luo [31]).

The present paper is the first attempt at a systematic study of distributive varieties of monoids. As in the semigroup case, in view of the result by Kozhevnikov [18], the general problem of classifying distributive varieties of monoids seems to be extremely difficult because it includes the problem of identifying all distributive varieties of periodic groups. Thus, it is natural to begin with the study of monoid varieties with the mentioned property within the class of monoids that do not contain non-trivial subgroups. Such monoids are called aperiodic. Nevertheless, experience suggests that even the problem of classifying distributive varieties of aperiodic monoids remains quite difficult. So, it is natural at first to try solving this problem within some subclass of the class of all aperiodic monoids.

The class $A_{cen}$ of aperiodic monoids with central idempotents is a natural candidate. This class is quite wide. It includes, in particular, all nilpotent monoids, that is, monoids obtained from nilsemigroups by adjoining a new identity element. Subvarieties of $A_{cen}$ have been intensively studied for two last decades. This class is rich in examples of varieties interesting from specific points of view (see Gusev [4], Gusev and Lee [6], Jackson [12, 13], Jackson and Lee [14], Jackson and Zhang [16]). Besides that, one managed to completely describe subvarieties of $A_{cen}$ with some natural and important properties: hereditary finitely based varieties (Lee [20]), almost Cross varieties (Lee [22]) and inherently non-finitely generated varieties (Lee [23]). The main result of the present paper naturally fits into this series of results. Namely, we completely classify distributive subvarieties of $A_{cen}$.

To formulate the main result of the article, we need some definitions and notation. Let $\mathcal{X}$ be a countably infinite set called an alphabet. As usual, we denote by $\mathcal{X}^*$ the free monoid over the alphabet $\mathcal{X}$; elements of $\mathcal{X}^*$ are called words, while elements of
are said to be letters. Words unlike letters are written in bold. An identity is written as $u \approx v$, where $u, v \in \mathcal{X}^*$; it is non-trivial if $u \neq v$.

As usual, $\mathbb{N}$ denote the set of all natural numbers. For any $n \in \mathbb{N}$, we denote by $S_n$ the full symmetric group on the set $\{1, 2, \ldots, n\}$. For convenience, we put $S_0 = S_1$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $n, m, k \in \mathbb{N}_0$, $\rho \in S_{n+m}$ and $\tau \in S_{n+m+k}$, we define the words:

\[
\begin{align*}
\mathbf{a}_{n,m}[\rho] &= \left( \prod_{i=1}^{n} z_i t_i \right) x \left( \prod_{i=1}^{n+m} z_i \rho \right) x \left( \prod_{i=n+1}^{n+m} t_i z_i \right), \\
\mathbf{a'}_{n,m}[\rho] &= \left( \prod_{i=1}^{n} z_i t_i \right) x^2 \left( \prod_{i=1}^{n+m} z_i \rho \right) \left( \prod_{i=n+1}^{n+m} t_i z_i \right), \\
\mathbf{c}_{n,m,k}[\tau] &= \left( \prod_{i=1}^{n} z_i t_i \right) x y t \left( \prod_{i=n+1}^{n+m} z_i t_i \right) x \left( \prod_{i=1}^{n+m+k} z_i \tau \right) y \left( \prod_{i=n+m+1}^{n+m+k} t_i z_i \right), \\
\mathbf{c'}_{n,m,k}[\tau] &= \left( \prod_{i=1}^{n} z_i t_i \right) y x t \left( \prod_{i=n+1}^{n+m} z_i t_i \right) x \left( \prod_{i=1}^{n+m+k} z_i \tau \right) y \left( \prod_{i=n+m+1}^{n+m+k} t_i z_i \right).
\end{align*}
\]

We denote by $\mathbf{d}_{n,m,k}[\tau]$ and $\mathbf{d'}_{n,m,k}[\tau]$ the words obtained from the words $\mathbf{c}_{n,m,k}[\tau]$ and $\mathbf{c'}_{n,m,k}[\tau]$, respectively, when reading the last words from right to left. We fix notation for the following two identities:

\[
\begin{align*}
\alpha : \ xytytx \approx xytxy & \quad \text{and} \quad \beta : \ xzytxy \approx xzyxy.
\end{align*}
\]

Let $\var$ denote the monoid variety given by a set $\Sigma$ of identities. We fix notation for the following monoid varieties:

\[
\begin{align*}
\mathbf{P}_n &= \var \left\{ x^n \approx x^{n+1}, \ xy \approx yx, \mathbf{a}_{k,\ell}[\rho] \approx \mathbf{a'}_{k,\ell}[\rho], \mathbf{c}_{k,\ell,m}[\tau] \approx \mathbf{c'}_{k,\ell,m}[\tau], \mathbf{d}_{k,\ell,m}[\tau] \approx \mathbf{d'}_{k,\ell,m}[\tau] \right\}, \\
\mathbf{Q}_n &= \var \left\{ x^n \approx x^{n+1}, \ xy \approx yx, \ x^2 y \approx xyx \right\}, \\
\mathbf{R}_n &= \var \left\{ x^n \approx x^{n+1}, \ xy \approx yx, \ x^2 y \approx xyx, \ \alpha, \ \beta \right\},
\end{align*}
\]

where $n \in \mathbb{N}$. By $\mathbf{V}^\delta$, we denote the monoid variety dual to the variety $\mathbf{V}$ (in other words, $\mathbf{V}^\delta$ consists of monoids dual to members of $\mathbf{V}$).

Our main result is the following

**Theorem 1.1** A subvariety of $\mathbf{A}_{cen}$ is distributive if and only if it is contained in one of the varieties $\mathbf{P}_n$, $\mathbf{Q}_n$, $\mathbf{R}_n$, or $\mathbf{R}_n^\delta$ for some $n \in \mathbb{N}$.

Notice that the proof Theorem 1.1 implies that set of all distributive subvarieties of $\mathbf{A}_{cen}$ is countably infinite (see Remark 1 at the end of Sect. 5).

The article consists of five sections. Some background results are first given in Sect. 2. Section 3 contains several examples of non-distributive varieties of monoids. In Sect. 4, we prove a number of auxiliary assertions. Results from Sects. 3 and 4 will then be used in Sect. 5 to prove Theorem 1.1.
2 Preliminaries

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to the monograph of Burris and Sankappanavar [1] for more information.

2.1 Deduction

An identity \( u \approx v \) is directly deducible from an identity \( s \approx t \) if there exist some words \( a, b \in \mathcal{X}^* \) and substitution \( \phi : \mathcal{X} \to \mathcal{X}^* \) such that \{\( u, v \)\} = \{\( a\phi(s)b, a\phi(t)b \)\}. A non-trivial identity \( u \approx v \) is deducible from a set \( \Sigma \) of identities if there exists some finite sequence \( u = w_0, w_1, \ldots, w_m = v \) of words such that each identity \( w_i \approx w_{i+1} \) is directly deducible from some identity in \( \Sigma \).

The following assertion is a specialization for monoids of a well-known universal-algebraic fact (see Burris and Sankappanavar [1, Theorem II.14.19], for instance).

**Proposition 2.1** Let \( V \) be the variety defined by some set \( \Sigma \) of identities. Then \( V \) satisfies an identity \( u \approx v \) if and only if \( u \approx v \) is deducible from \( \Sigma \). \( \square \)

2.2 Factor monoids

For any set of words \( W \), the factor monoid of \( W \), denoted by \( M(W) \), is the monoid that consists of all factors of the words of \( W \) and a zero element 0, with multiplication \( \cdot \) given by

\[
u \cdot v = \begin{cases} uv & \text{if } uv \text{ is a factor of some word in } W, \\ 0 & \text{otherwise}; \end{cases}
\]

the empty word, more conveniently written as 1, is the identity element of \( M(W) \). This construction was used by Perkins [24] for exhibiting the first example of a non-finitely based finite semigroup. Since the beginning of the millennium, such monoids were consistently and systematically studied in the many articles (see Jackson [12], Jackson and Sapir [15], Sapir [26], for instance).

A word \( w \) is an isoterm for a variety \( V \) if \( V \) violates any non-trivial identity of the form \( w \approx w' \). Equivalently, \( w \) is an isoterm for \( V \) if and only if the identities satisfied by \( V \) cannot be used to convert \( w \) into a different word.

Given any word \( w \), let \( M(W) \) denote the variety generated by the factor monoid \( M(W) \). One advantage in working with factor monoids is the relative ease of checking if a variety \( M(W) \) is contained in some given variety.

**Lemma 2.2** (Jackson [12, Lemma 3.3]) Let \( V \) be a monoid variety and \( W \) be a set of words. Then \( M(W) \) lies in \( V \) if and only if each word in \( W \) is an isoterm for \( V \). \( \square \)

For brevity, if \( w_1, w_2, \ldots, w_k \) are words, then we write \( M(w_1, w_2, \ldots, w_k) \) [respectively, \( M((w_1, w_2, \ldots, w_k)) \)] rather than \( M((w_1, w_2, \ldots, w_k)) \) [respectively, \( M((w_1, w_2, \ldots, w_k)) \)].
2.3 Subvarieties of $A_{cen}$

The following fact is well known and can be easily verified.

**Lemma 2.3** Any subvariety of $A_{cen}$ satisfies the identities

\[ x^n \approx x^{n+1} \quad \text{and} \quad x^n y \approx yx^n \quad (2.1) \]

for some $n \in \mathbb{N}$. \hfill \square

Actually, the following fact is well known. We provide its proof for the sake of completeness.

**Lemma 2.4** Let $V$ be a subvariety of $A_{cen}$. If $M(xy) \notin V$, then $V$ is commutative.

**Proof** If $x$ is an isoterms for $V$, then $V$ satisfies $xy \approx yx$ by Lemma 2.2. If $x$ is not an isoterms for $V$, then $V$ is completely regular, i.e., $V$ consists of unions of groups by [9, Lemma 2.4 and Corollary 2.6]. It is well known that every completely regular variety of aperiodic monoids consists of idempotent monoids. In view of Lemma 2.3, $V$ satisfies the identity $x^n y \approx yx^n$ for some $n \geq 1$. It remains to notice that every idempotent monoid satisfying this identity is commutative. \hfill \square

**Corollary 2.5** Let $X$ and $Y$ be subvarieties of $A_{cen}$. If $X \land Y$ is commutative, then either $X$ or $Y$ is commutative.

**Proof** If both $X$ and $Y$ are non-commutative, then $M(xy) \in X \land Y$ by Lemma 2.4. To complete the proof, it remains to notice that $M(xy)$ and so $X \land Y$ are non-commutative. \hfill \square

2.4 Some known results

The following fact is obvious.

**Lemma 2.6** Let $V$ be a monoid variety and $n \in \mathbb{N}$. Then $x^n$ is not an isoterms for $V$ if and only if $V$ satisfies the identity $x^n \approx x^m$ for some $m > n$. \hfill \square

**Lemma 2.7** (Gusev and Vernikov [10, Lemma 2.10]) Let $X$ and $Y$ be aperiodic monoid varieties. If $X \land Y$ satisfies the identity $x^n \approx x^m$ for some $n, m \in \mathbb{N}$, then this identity holds in either $X$ or $Y$. \hfill \square

**Proposition 2.8** (Head [11]) Each commutative monoid variety can be defined by the identities $xy \approx yx$ and $x^n \approx x^m$ for some $n, m \in \mathbb{N}$. \hfill \square

The content of a word $w$, that is, the set of all letters occurring in $w$ is denoted by $\text{con}(w)$. For a word $w$ and a letter $x$, let $\text{occ}_x(w)$ denote the number of occurrences of $x$ in $w$. A letter $x$ is called simple [multiple] in a word $w$ if $\text{occ}_x(w) = 1$ [respectively, $\text{occ}_x(w) > 1$]. The set of all simple [multiple] letters of a word $w$ is denoted by $\text{sim}(w)$ [respectively, $\text{mul}(w)$]. Let $w$ be a word and $\text{sim}(w) = \{t_1, t_2, \ldots, t_m\}$. We
will assume without loss of generality that \( w(t_1, t_2, \ldots, t_m) = t_1 t_2 \cdots t_m \). Then \( w = w_0 t_1 w_1 \cdots t_m w_m \) for some words \( w_0, w_1, \ldots, w_m \). The words \( w_0, w_1, \ldots, w_m \) are called blocks of the word \( w \). The representation of the word \( w \) as a product of alternating simple in \( w \) letters and blocks is called a decomposition of the word \( w \).

**Lemma 2.9** (Gusev and Vernikov [10, Lemma 2.17]) Let \( u \approx v \) be an identity of \( M(\text{xy}) \). Suppose that \( u_0 \prod_{i=1}^{m} (t_i u_i) \) is the decomposition of the word \( u \). Then \( \text{con}(u) = \text{con}(v) \) and the decomposition of the word \( v \) has the form \( v_0 \prod_{i=1}^{m} (t_i v_i) \).

A non-empty word \( w \) is called linear if \( \text{occ}_x(w) \leq 1 \) for each letter \( x \). Let \( u \) and \( v \) be words and \( u_0 \prod_{i=1}^{m} (t_i u_i) \) and \( v_0 \prod_{i=1}^{m} (t_i v_i) \) be decompositions of \( u \) and \( v \), respectively. A letter \( x \) is called linear-balanced in the identity \( u \approx v \) if \( x \) is multiple in \( u \) and \( \text{occ}_x(u_i) = \text{occ}_x(v_i) \leq 1 \) for all \( i = 0, 1, \ldots, m \); the identity \( u \approx v \) is called linear-balanced if any letter \( x \in \text{mul}(u) \cup \text{mul}(v) \) is linear-balanced in this identity.

**Lemma 2.10** (Gusev and Vernikov [10, Lemma 3.1]) Let \( V \) be a monoid variety such that the word \( \prod_{i=1}^{k} x t_i x \) is an isoterm for \( V \), \( u \) be a word such that all its blocks are linear words and \( \text{occ}_x(u) \leq k + 1 \) for every letter \( x \). Then every identity of the form \( u \approx v \) that holds in the variety \( V \) is linear-balanced.

The subvariety of a variety \( V \) defined by a set \( \Sigma \) of identities is denoted by \( V/\Sigma \).

**Lemma 2.11** (Gusev and Vernikov [9, Lemma 4.9]) Let \( V \) be a monoid variety with \( M(\text{xyx}) \in V \).

(i) If \( M(\text{xyzyx}) \notin V \), then \( V \) satisfies the identity \( \alpha \).

(ii) If \( M(\text{zxytx}) \notin V \), then \( V \) satisfies the identity \( \beta \).

The subvariety of a variety \( V \) defined by a set \( \Sigma \) of identities is denoted by \( V/\Sigma \).

**Lemma 2.12** (Gusev and Vernikov [10, Lemma 2.19]) Let \( V \) be a variety. Suppose that there is a set \( \Sigma \) of identities such that:

(i) if \( U \subseteq V \), then \( U = V\Phi \) for some subset \( \Phi \) of \( \Sigma \);

(ii) if \( U, U' \subseteq V \) and \( U \land U' \) satisfies an identity \( \sigma \in \Sigma \), then \( \sigma \) holds in either \( U \) or \( U' \).

Then the the lattice \( \mathcal{L}(V) \) is distributive.

3 Certain varieties with non-distributive subvariety lattice

3.1 The variety \( M(x^2 y, yx^2) \)

Let \( T \) and \( SL \) denote the variety of trivial monoids and the variety of all semilattice monoids, respectively.

**Proposition 3.1** (Gusev and Lee [7, Proposition 3.1]) The lattice \( \mathcal{L}(M(x^2 y, yx^2)) \) is given in Fig. 1. In particular, this lattice is modular but not distributive.
3.2 The variety $M(xzxyty) \lor N$

Let

$$N = \var{ x^2 \approx x^3, \ x^2y \approx yx^2, \ xyxzx \approx x^2yz, \ \alpha, \ \beta }.$$  

If $w$ is a word and $X \subseteq \con(w)$, then we denote by $w(X)$ the word obtained from $w$ by deleting all letters except letters from $X$. If $X = \{x_1, x_2, \ldots, x_k\}$, then we write $w(x_1, x_2, \ldots, x_k)$ rather than $w([x_1, x_2, \ldots, x_k])$.

**Proposition 3.2** The lattice $\mathcal{L}(M(xzxyty) \lor N)$ is not modular.

**Proof** Suppose that $M(xzxyty) \lor N$ satisfies an identity $xysxzytz \approx v$ for some $v \in \mathfrak{F}^*$. In view of Lemma 2.2, $v(x, z, s, t) = xsztz$ and $v(y, z, s, t) = ysztz$. Then $v = v' sztz$, where $v' \in \{xy, yx\}$. Since $N$ violates

$$xyzxy \approx yxzxy, \quad (3.1)$$

we have $v' \neq yx$. Therefore, $v = xysxzytz$. We see that $xysxzytz$ is an isoterm for the variety $M(xzxyty) \lor N$. Let $V = M(xysxzytz)$. Then

$$(M(xzxyty) \lor N) \land V = V$$

by Lemma 2.2. It is routine to check that (3.1) holds in $M(xysxzytz)$. Then $N \land V$ satisfies

$$xysxzytz \beta xysxzytz \approx yysxzytz \approx yysxzytz.$$
Clearly, $xysxzytz \approx yxsxzytz$ is satisfied by $M(xzyt)$ as well. Since the sub-monoid of $M(xysxzytz)$ generated by $\{x, z, ys, yt, 1\}$ is isomorphic to $M(xzyt)$ and so $M(xzyt) \subset V$, we have

$$M(xzyt) \vee (N \wedge V) \subset (M(xzyt) \vee N) \wedge V = V.$$ 

It follows that the lattice $\mathcal{L}(M(xzyt) \vee N)$ is not modular. \hfill \square

### 3.3 Varieties of the form $M(a_{n,m}[\rho])$

For any $n, m \in \mathbb{N}_0$ and $\rho \in S_{n+m}$, we put $\hat{a}_{n,m}[\rho] = (a_{n,m}[\rho])_X$. Let

$$\hat{N}_0^2 = \{(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid |n - m| \leq 1\}.$$ 

For $n, m \in \mathbb{N}_0$, a permutation $\rho$ from $S_{n+m}$ is a $(n, m)$-permutation if, for all $i = 1, 2, \ldots, n + m - 1$, one of the following holds:

- $1 \leq i \rho \leq n$ and $n < (i + 1) \rho \leq n + m$;
- $1 \leq (i + 1) \rho \leq n$ and $n < i \rho \leq n + m$.

Evidently, if $\rho$ is a $(n, m)$-permutation, then $(n, m) \in \hat{N}_0^2$. The set of all $(n, m)$-permutations is denoted by $S_{n,m}$. If $w$ is a word and $X \subseteq \text{con}(w)$, then we denote by $w_X$ the word obtained from $w$ by deleting all letters from $X$. If $X = \{x\}$, then we write $w_x$ rather than $w_{\{x\}}$.

The proof of the following lemma is quite analogous to the proof of Lemma 4.10 in [9] and so we omit it.

**Lemma 3.3** For any $(n, m) \in \hat{N}_0^2$ and $\rho \in S_{n,m}$, the word $\hat{a}_{n,m}[\rho]$ is an isoterm for the variety $M(xzyt)$. \hfill \square

For any non-empty word $w$ of length $\ell$, $0 \leq k \leq \ell$ and $0 \leq m \leq \ell - k$, let $w[k; m]$ denote a factor of $w$ of length $m$ directly succeeding the prefix of $w$ of length $k$. The expression $i w x$ means the $i$th occurrence of a letter $x$ in a word $w$. If the $i$th occurrence of $x$ precedes the $j$th occurrence of $y$ in a word $w$, then we write $(i w x) < (j w y)$. For any monoid variety $V$, let $\text{FIC}(V)$ denote the fully invariant congruence on $X^*$ corresponding to $V$.

**Proposition 3.4** The lattice $\mathcal{L}(M(a_{n,m}[\rho]))$ is not distributive for any $(n, m) \in \hat{N}_0^2$ and $\rho \in S_{n,m}$.

**Proof** There are four possibilities:

- $n = m$ and $1 \leq 1 \rho \leq n$;
- $n = m$ and $n + 1 \leq 1 \rho \leq 2n$;
- $n = m + 1$ and so $1 \leq 1 \rho \leq n$;
- $n = m - 1$ and so $n + 1 \leq 1 \rho \leq n + m$. 

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We will consider only the first possibility because the other ones are considered quite analogous. In this case, \(1 \leq i \theta \leq n\) and \(n + 1 \leq (i + 1) \theta \leq 2n\) for any \(i = 1, 3, \ldots, 2n - 1\).

For \(k \in \mathbb{N}\) and an arbitrary permutation \(\theta \in S_{k,k}\) with \(1 \leq i \theta \leq k\) and \(k + 1 \leq (i + 1) \theta \leq 2k\) for any \(i = 1, 3, \ldots, 2k - 1\), we define the permutation \(\theta' \in S_{k+2,k+2}\) as follows:

- \(10' = 1 \theta + 1\) and \(20' = 2 \theta + 3\);
- \(30' = k + 2\) and \(40' = 2k + 4\);
- \(i \theta' = (i - 2) \theta + 1\) for any \(i = 5, 7, \ldots, 2k - 1\);
- \(i \theta' = (i - 2) \theta + 3\) for any \(i = 6, 8, \ldots, 2k\);
- \((2k + 1) \theta' = 1\) and \((2k + 2) \theta' = k + 3\);
- \((2k + 3) \theta' = (2k - 1) \theta + 1\) and \((2k + 4) \theta' = (2k) \theta + 3\).

Let \(a_{k+2,k+2}[\theta'] \approx a\) be an identity of \(M(a_{k,k}[\theta])\). It is easy to see that a non-trivial identity of the form \(xyzxyt \approx w\) implies a non-trivial identity of the form \(a_{k,k}[\theta] \approx w'\). This observation and Lemma 2.2 imply that \(xyzxyt \approx w\) is an isomer for \(M(a_{k,k}[\theta])\). Then \(a_{x} = a_{k+2,k+2}[\theta']\) by Lemma 3.3. Since \(a_{k+2,k+2}[\theta'](X)\) coincides (up to renaming of letters) with \(a_{k,k}[\theta]\) for

\[
X = \{x\} \cup \{z_i, t_i \mid 2 \leq i \leq n + 1\} \cup \{z_i, t_i \mid k + 4 \leq i \leq 2k + 3\},
\]

Lemma 2.2 implies that \(a(X) = a_{k,k}[\theta](X)\). In particular, \(\text{occ}_x(a) = 2\). We notice also that the first and the second occurrences of \(x\) in \(a\) lie in the same block because \(xyzxyt\) and so \(zx\) are isomers for \(M(a_{k,k}[\theta])\). Therefore, \((1a_{l}t_{n+2}) < (1a_{x})\) and \((2a_{x}) < (1a_{l}n+3)\). It follows that \(a = a_{k+2,k+2}[\theta']\). We see that the word \(a_{k+2,k+2}[\theta']\) is an isomer for \(M(a_{k,k}[\theta])\). Then \(M(a_{k+2,k+2}[\theta']) \subseteq M(a_{k,k}[\theta])\) by Lemma 2.2.

Since it suffices to verify that some subvariety of \(M(a_{n,n}[\rho])\) has a non-distributive subvariety lattice, we may further assume without any loss that \(\rho = \xi'\) for some \(\xi \in S_{n-2,n-2}\). In particular,

\[
2 \leq 1 \rho, (2n - 1) \rho \leq n - 1 \text{ and } n + 2 \leq 2 \rho, (2n) \rho \leq 2n - 1. \tag{3.2}
\]

For any \(0 \leq s \leq t \leq 6n\), put \(v_{s,t} = p \cdot q[s; s] \cdot x \cdot q[s; t - s] \cdot x \cdot q[t; 6n - t] \cdot r\), where

\[
p = \left( \prod_{i=1}^{1} z_{i}'' \right) \left( \prod_{i=1}^{(n-1)} z'_{i} t_{i} \right) \left( \prod_{i=1}^{n} z_{i} t_{i} \right) \left( \prod_{i=(2n-1)}^{n} z''_{i} t_{i} \right) \left( \prod_{i=1}^{n} z_{i} t_{i} \right),
\]

\[
q = z_{1} \rho z_{2} \rho \cdot \left( \prod_{i=1}^{2n} z_{i}'' \right) \left( \prod_{i=3}^{2n} z_{i} \right) \left( \prod_{i=1}^{2n} z''_{i} \right) : z(2n-1) \rho z(2n) \rho,
\]

\[
r = \left( \prod_{i=1}^{2} t_{i}'' \right) \left( \prod_{i=1}^{(2n) - 1} t_{i} z_{i}'' \right) \left( \prod_{i=1}^{2n} t_{i} z_{i}'' \right) \left( \prod_{i=1}^{2n} t_{i} z_{i}'' \right) \left( \prod_{i=1}^{2n} t_{i} z_{i}'' \right).
\]

Evidently, \(v_{0,6n}\) coincides (up to renaming of letters) with \(a_{3n,3n}[\tau]\) for some \(\tau \in S_{3n,3n}\). Then arguments similar to ones from the second paragraph of this proof imply that the word \(v_{0,6n}\) is an isomer for \(M(a_{n,n}[\rho])\).
Let \( v_{1,6n} \approx v \) be an identity of \( M(a_{n,n}[\rho]) \). Arguments similar to ones from the second paragraph of this proof imply that \( v_x = pqr \). Note that if

\[
X = \{x, z_{(2n-1)\rho}, t_{(2n-1)\rho}, z_{(2n)\rho}, t_{(2n)\rho}\} \cup \{z_i', t_i', | 1 \leq i \leq 2n - 2\},
\]

then \( v_{1,6n}(X) \) coincides (up to renaming of letters) with \( a_{n,n}[\rho] \). Then \( v(X) = v_{1,6n}(X) \) by Lemma 2.2. In particular, \( \text{occ}_x(v) = 2 \). Arguments similar to ones from the second paragraph of this proof imply that \( (11t'_{n+1}) < (1v x) < (2v x) < (1v x') \). It follows that \( v \) is either \( v_{0,6n} \) or \( v_{1,6n} \) or \( v_{2,6n} \). Since \( v_{0,6n} \) is an isoterms for \( M(a_{n,n}[\rho]) \), the word \( v \) cannot coincide with \( v_{0,6n} \). Therefore, \( v \in \{v_{1,6n}, v_{2,6n}\} \). Finally, it is routine to check that the identity \( v_{1,6n} \approx v_{2,6n} \) holds in \( M(a_{n,n}[\rho]) \). We see that the set \( \{v_{0,6n-1}, v_{0,6n-2}\} \) is a FIC\( (M(a_{n,n}[\rho])) \)-class. By similar arguments we can show that the set \( \{v_{0,6n-1}, v_{0,6n-2}\} \) is a FIC\( (M(a_{n,n}[\rho])) \)-class.

Let

\[
X = M(a_{n,n}[\rho]) \land \text{var}[v_{0,6n} \approx v_{2,6n}] \quad \text{and} \quad Y = M(a_{n,n}[\rho]) \land \text{var}[v_{0,6n} \approx v_{0,6n-2}]
\]

and \( A = \{v_{0,6n}, v_{1,6n}, v_{2,6n}\} \). Consider an identity \( u \approx u' \) of \( X \) with \( u \in A \). We are going to show that \( u' \in A \). In view of Proposition 2.1, we may assume without loss of generality that either \( u \approx u' \) holds in \( M(a_{n,n}[\rho]) \) or \( u \approx u' \) is directly deducible from \( v_{0,6n} \approx v_{2,6n} \). In view of the above, \( A \) is a union of two FIC\( (M(a_{n,n}[\rho])) \)-classes. Therefore, it remains to consider the case when \( u \approx u' \) is directly deducible from \( v_{0,6n} \approx v_{2,6n} \), i.e., there exist some words \( a, b \in X^* \) and substitution \( \phi: X \to X^* \) such that \( \{u, u'\} = \{a\phi(v_{0,6n})b, a\phi(v_{2,6n})b\} \).

If \( \phi(x) = 1 \), then \( \phi(v_{0,6n}) = \phi(v_{2,6n}) \) and so \( u = u' \), whence \( u' \in A \). It remains to consider the case when \( \phi(x) \neq 1 \). Clearly, \( t_i, t_i', t_i'' \notin \text{con}(\phi(x)) \) for any \( i = 1, 2, \ldots, 2n \) because \( t_i, t_i', t_i'' \in \text{sim}(u) \), while \( x \in \text{mul}(v_{0,6n}) = \text{mul}(v_{2,6n}) \). We note also that the letters of the form \( z_i, z_i' \) or \( z_i'' \) do not occur in \( \phi(x) \) as well because the first and the second occurrences of these letters in \( u \) lie in different blocks, while the first and the second occurrences of \( x \) in both \( v_{0,6n} \) and \( v_{2,6n} \) lie in the same block. Therefore, \( \text{con}(\phi(x)) = \{x\} \). Since \( \text{occ}_x(u) = \text{occ}_x(v_{0,6n}) = \text{occ}_x(v_{2,6n}) = 2 \), we have \( \phi(x) = x \). It follows that \( x \notin \text{con}(\phi(z_i, z_i', z_i'')) \) for any \( i = 1, 2, \ldots, 2n \). Clearly, \( t_i, t_i', t_i'' \notin \text{con}(\phi(z_j, z_j', z_j'')) \) for any \( i, j = 1, 2, \ldots, n \) because \( t_i, t_i', t_i'' \in \text{sim}(u) \), while \( z_j, z_j', z_j'' \notin \text{mul}(v_{0,6n}) = \text{mul}(v_{2,6n}) \). We note also that \( \phi(z_i), \phi(z_i') \) and \( \phi(z_i'') \) are either empty words or letters for any \( i = 1, 2, \ldots, n \) because any factor of \( u \) of length greater than 1 occurs only once in \( u \). This implies that

\[
\{\phi(z_i), \phi(z_i'), \phi(z_i'') | 1 \leq i \leq 2n\} \subseteq \{1\} \cup \{z_i, z_i', z_i'' | 1 \leq i \leq 2n\}. \tag{3.3}
\]

In view of the above, if \( u = a\phi(v_{2,6n})b \), then \( u' = a\phi(v_{0,6n})b \in A \). So, it remains to consider the case when \( u = a\phi(v_{0,6n})b \) and \( u' = a\phi(v_{2,6n})b \). Since \( u = v_{\ell,2n} \) for some \( \ell \in \{0, 1, 2\} \), we have \( \phi(q) = q[\ell; 6n - \ell] \).

If \( \ell = 0 \), then, since \( \phi(z_{1\rho}) \) and \( \phi(z_{2\rho}) \) are either empty words or letters by (3.3), we have \( u' \in A \).
Suppose that \( \ell = 1 \). Then \( \phi(z_{1\rho}) = 1 \) because \( \phi(z_{1\rho}) \) cannot coincide with \( z_{2\rho} \). If \( \phi(z_{2\rho}) = 1 \), then \( \phi(v_{0,6n}) = \phi(v_{2,6n}) \) and so \( u = u' \), whence \( u' \in A \). If \( \phi(z_{2\rho}) \neq 1 \), then \( \phi(z_{2\rho}) = z_{2\rho} \) by (3.3). It follows that \( u' = v_{2,6n} \) and so \( u' \in A \).

Suppose now that \( \ell = 2 \). Then

\[
\{z_{i_1}, z'_{i_2}, z''_{i_3} \mid 1 \leq i_1, i_2, i_3 \leq n\} \setminus \{z_{1\rho}\}
\]

is a subset of

\[
\{\phi(z_{i_1}), \phi(z'_{i_2}), \phi(z''_{i_3}) \mid 1 \leq i_1, i_2, i_3 \leq n\}
\]

by (3.3) and, moreover, the sets (3.4) and \( \{\phi(t_{i_1}), \phi(t'_{i_2}), \phi(t''_{i_3}) \mid 1 \leq i \leq n\} \) are disjoint. If \( \phi(z_{1\rho}) = 1 \), then \( \phi(z_{2\rho}) = 1 \) because \( \phi(z_{2\rho}) \) cannot coincide with \( z'_{1\rho} \). In this case, \( \phi(v_{0,6n}) = \phi(v_{2,6n}) \) and so \( u = u' \), whence \( u' \in A \). So, we may assume that \( \phi(z_{1\rho}) \neq 1 \). Then \( \phi(z_{1\rho}) = z'_{1\rho} \) by (3.3). Two cases are possible:

(a) \( 1 \rho < (2n - 1) \rho \);  
(b) \( (2n - 1) \rho < 1 \rho \).

Suppose that (a) is true. Then the set

\[
\{z_{i_1}, z'_{i_2}, z''_{i_3} \mid 1 \leq i_1 \leq n, 1 \rho + 1 \leq i_2, i_3 \leq n\} \setminus \{z_{1\rho}\}
\]

must be a subset of

\[
\{\phi(z_{i_1}), \phi(z'_{i_2}), \phi(z''_{i_3}) \mid 1 \rho + 1 \leq i_1, i_3 \leq n, (2n - 1) \rho + 1 \leq i_2 \leq n\}.
\]

But this is impossible because the first of these sets contains \( (n - 1) + 2 \cdot (n - 1 \rho) \) elements, the second one has at most \( (n - (2n - 1) \rho) + 2 \cdot (n - 1 \rho) \) elements and \( 1 < (2n - 1) \rho \) by (3.2).

Suppose now that (b) is true. Then the set

\[
\{z_{i_1}, z'_{i_2}, z''_{i_3} \mid 1 \leq i_1 \leq n, 1 \leq i_2, i_3 \leq 1 \rho\} \setminus \{z_{1\rho}\}
\]

must be a subset of

\[
\{\phi(z_{i_1}), \phi(z'_{i_2}), \phi(z''_{i_3}) \mid 1 \leq i_1, i_3 \leq 1 \rho, 1 \leq i_2 \leq (2n - 1) \rho\}.
\]

But this is also impossible because the first of these sets contains \( (n - 1) + 2 \cdot (1 \rho) \) elements, the second one has at most \( (2n - 1 \rho) + 2 \cdot (1 \rho) \) elements and \( (2n - 1 \rho) < 1 \rho < n \) by (3.2).

We see that \( u' \in A \) in either case. This means that \( A \) forms a FIC(\( X \))-class. By similar arguments we can show that \( \{v_{0,6n}, v_{0,6n-1}, v_{0,6n-2}\} \) is a FIC(\( Y \))-class. It follows that the word \( v_{0,6n} \) is an isoterm for \( X \lor Y \). Put \( Z = M(v_{0,6n}) = M(a_{3n,3n}[\pi]) \).

According to Lemma 2.2, \( Z \subseteq X \lor Y \) and, therefore, \( (X \lor Y) \land Z = Z \). It is routine to check that \( Z \) satisfies \( a_{k,k}[\pi] \approx a'_{k,k}[\pi] \) for any \( k = 1, 2, \ldots, 3n - 1 \) and any \( \pi \in S_{k,k} \).

Then the identity \( v_{0,6n} \approx v_{0,0} \) is satisfied by \( X \lor Z \) because \( v_{0,6n} \approx v_{2,6n} \) holds in \( X \),
Proposition 3.5
and, for any \( i \in \mathbb{Z} \),
that holds in 

Let \( \theta \in S_{p+1} \) as follows: \( q \theta = r \), for any \( i = 1, 2, \ldots, q \),
and, for any \( i = q + 1, q + 2, \ldots, p + 1 \),

Put \( k = n + m + 1 \) and \( \pi = (\rho_{1, k})_{2k+1, 2k+1} \). Let \( c_{n,m,k+2}[\pi] \approx c \) be an identity that holds in \( M(c_{n,m,k+2}[\rho]) \). Clearly, the word \( x y x \) is an isoterm for \( M(c_{n,m,k}[\rho]) \). Then Lemma 2.10 implies that

where \( c' \) and \( c'' \) are linear words with

Let \( 1 \leq p \leq 2k \). Evidently, \( x y z x t y \) is an isoterm for \( M(c_{n,m,k}[\rho]) \). Then if \( n + m < p \pi, (p + 1) \pi \), then \( (1e^* z_{p \pi}) < (1e^* z_{(p+1) \pi}) \). Further, since \( k > 0 \), one can show that a non-trivial identity of the form \( x y z x t y \approx \psi \) implies some non-trivial identity of the form \( c_{n,m,k}[\rho] \approx \psi \), whence \( x y z x t y \approx \psi \) is an isoterm for \( M(c_{n,m,k}[\rho]) \). Hence if \( p \pi \leq n + m < (p + 1) \pi \) or \( (p + 1) \pi \leq n + m < p \pi \), then \( (1e^* z_{p \pi}) < (1e^* z_{(p+1) \pi}) \). The case when \( p \pi, (p + 1) \pi \leq n + m \) is impossible by definition of \( \pi \) and the fact that \( \rho \in S_{n+m,n+m+1} \). We see that \( (1e^* z_{p \pi}) < (1e^* z_{(p+1) \pi}) \) in either case. By a similar
argument we can show that \((1e^v x) < (1e^v z_1 \pi)\) and \((1e^v z_{(2k+1)} \pi) < (1e^v y)\). Hence \(c'' = xz_1 \pi z_{2 \pi} \cdots z_{(2k+1)} \pi y\).

Further, by definition, \((c_{n,m,k+2}[\pi])_X\) coincides (up to renaming of letters) with \(c_{n,m,k}[\rho]\) for \(X = \{z_1 \pi, z_{(2k+1)} \pi, t_1 \pi, t_{(2k+1)} \pi\}\). Then Lemma 2.2 implies that \(c_X = (c_{n,m,k+2}[\pi])_X\). In particular, \(c' = xy\). Thus, \(c = c_{n,m,k+2}[\pi]\). We see that \(c_{n,m,k+2}[\pi]\) is an isoterm for \(M(c_{n,m,k}[\rho])\). By similar arguments we can show that if \(\tau = (\rho_1 2k)_{2k+1,k}\), then \(c_{n,m,k+2}[\tau]\) is an isoterm for \(M(c_{n,m,k}[\rho])\). Then \(M(c_{n,m,k+2}[\pi])\) and \(M(c_{n,m,k+2}[\tau])\) lie in \(M(c_{n,m,k}[\rho])\). It is routine to check that the monoids \(M(c_{n,m,k+2}[\pi])\) and \(M(c_{n,m,k+2}[\tau])\) satisfy the identities \(c_{n,m,k+2}[\tau] \approx c_{n,m,k+2}[\pi]\) and \(c_{n,m,k+2}[\pi] \approx c_{n,m,k+2}[\pi]\), respectively. So, we have proved that the variety \(M(c_{n,m,k}[\rho])\) contains two incomparable subvarieties \(M(c_{n,m,k+2}[\pi])\) and \(M(c_{n,m,k+2}[\tau])\). In view of this fact and Lemma 5.1 in [15], it suffices to show that the lattice \(L(M(c_{n,m,k+2}[\pi]), c_{n,m,k+2}[\tau]))\) is not modular.

For any \(\xi, \eta \in S_2\), we define the word:

\[
v_{\xi, \eta} = p a_1 b_1 x_1 \xi x_2 \eta y_1 y_2 b_2 a_2 q r s,
\]

where

\[
p = \left( \prod_{i=1}^{n} z_i t_i \right) \left( \prod_{i=1}^{n} z'_i t'_i \right) \left( \prod_{i=1}^{n} z''_i t''_i \right),
\]

\[
q = t \left( \prod_{i=n+1}^{k-1} z_i t_i \right) \left( \prod_{i=n+1}^{k-1} z'_i t'_i \right) \left( \prod_{i=n+1}^{k-1} z''_i t''_i \right),
\]

\[
r = x_1 z'_1 \pi a_1 \left( \prod_{i=2}^{2k} z'_{i, \pi} \right) b_2 z_{(2k+1) \pi} x_2 \left( \prod_{i=1}^{2k+1} z_i \pi \right) y_1 z''_{1, \pi} b_1 \left( \prod_{i=2}^{2k} z''_{i, \pi} \right) a_2 z''_{(2k+1) \pi} y_2,
\]

\[
s = \left( \prod_{i=k}^{2k+1} t_i z_i \right) \left( \prod_{i=k}^{2k+1} t'_i z'_i \right) \left( \prod_{i=k}^{2k+1} t''_i z''_i \right).
\]

Let \(\varepsilon\) denote the trivial permutation from \(S_2\). We need the following two auxiliary facts.

\section*{Lemma 3.6}

The set \(\{v_{\xi, \eta} \mid \xi, \eta \in S_2\}\) forms a FIC(M(c_{n,m,k+2}[\tau]))-class.

\section*{Proof of Lemma 3.6}

It is routine to check that \(M(c_{n,m,k+2}[\tau])\) satisfies \(v_{\xi_1, \eta_1} \approx v_{\xi_2, \eta_2}\) for any \(\xi_1, \eta_1, \xi_2, \eta_2 \in S_2\). Let now \(v'_{\varepsilon, \varepsilon} \approx v\) be an identity of \(M(c_{n,m,k+2}[\tau])\). Clearly, \(x y x\) is an isoterm for \(M(c_{n,m,k+2}[\tau])\). Then Lemma 2.10 implies that \(v = p v' q r s\), where \(v'\) and \(r'\) are linear words with

\[
\text{con}(v') = \{a_1, a_2, b_1, b_2, x_1, x_2, y_1, y_2\}\] and \(\text{con}(r) = \text{con}(r')\).

Further, arguments similar to ones from the second paragraph of the proof of Proposition 3.5 imply that all the letters occur in \(r'\) in the same order as in \(r\) and so \(r' = r\).
Since \((v_{\epsilon,\epsilon})_n\) coincides (up to renaming of letters) with \(c_{n,m,k+2}[\tau]\) for

\[X = \{a_1, b_1, t \} \cup \{z_i, t_i \mid 1 \leq i \leq 2k + 1\},\]

Lemma 2.2 implies that \((1_a a_1) < (1_v b_1)\). By a similar argument we can show that

\[(1_v b_1) < (1_v x_1), (1_v b_1) < (1_v x_2), (1_v x_1) < (1_v y_1),\]
\[(1_v x_1) < (1_v y_2), (1_v x_2) < (1_v y_1), (1_v y_2) < (1_v y_1), (1_v y_2) < (1_v b_2), (1_v b_2) < (1_v a_2).\]

It follows that \(v = v_{\xi,\eta}\) for some \(\xi, \eta \in S_2\). Therefore, the set \(\{v_{\xi,\eta} \mid \xi, \eta \in S_2\}\) forms a FIC(\(M(c_{n,m,k+2}[\tau])\))-class. \(\Box\)

**Lemma 3.7** Let \(\xi_1, \eta_1, \xi_2, \eta_2 \in S_2\). If a non-trivial identity \(v_{\epsilon,\epsilon} \approx v\) is directly deducible from the identity \(v_{\xi_1,\eta_1} \approx v_{\xi_2,\eta_2}\), then \(\{v_{\epsilon,\epsilon}, v\} = \{v_{\xi_1,\eta_1}, v_{\xi_2,\eta_2}\}\).

**Proof of Lemma 3.7** Since \(v_{\epsilon,\epsilon} \approx v\) is directly deducible from \(v_{\xi_1,\eta_1} \approx v_{\xi_2,\eta_2}\), there are words \(a, b \in \xi^*\) and an endomorphism \(\phi\) of \(\xi^*\) such that \(v_{\epsilon,\epsilon} = a\phi(v_{\xi_1,\eta_1})b\) and \(v = a\phi(v_{\xi_2,\eta_2})b\). Further, Lemma 3.6 implies that \(v = v_{\xi,\eta}\) for some \(\xi, \eta \in S_2\). Since the identity \(v_{\epsilon,\epsilon} \approx v\) is non-trivial, \((\xi, \eta) \neq (\epsilon, \epsilon)\). By symmetry, we may assume that \(\xi \neq \epsilon\). Then \((1_v x_2) < (1_v x_1)\). This is only possible when one of the following holds:

- \(\xi_1 \neq \xi_2, x_1 \in \text{con}(\phi(x_{1\xi_1}))\) and \(x_2 \in \text{con}(\phi(x_{2\xi_2}))\);
- \(\eta_1 \neq \eta_2, x_1 \in \text{con}(\phi(y_{1\eta_1}))\) and \(x_2 \in \text{con}(\phi(y_{2\eta_2}))\).

We note that every factor of length \(> 1\) of the word \(v_{\epsilon,\epsilon}\) has exactly one occurrence in this word. It follows that

- \((*)\) \(\phi(c)\) is either the empty word or a letter for any \(c \in \text{mul}(v_{\xi_1,\eta_1})\).

In view of this fact, one of the following holds:

- (a) \(\xi_1 \neq \xi_2, \phi(x_{1\xi_1}) = \phi(x_{2\xi_2}) = x_1\) and \(\phi(x_{2\xi_1}) = x_2\);
- (b) \(\eta_1 \neq \eta_2, \phi(y_{1\eta_1}) = \phi(y_{2\eta_2}) = x_1\) and \(\phi(y_{2\eta_1}) = \phi(y_{2\eta_2}) = x_2\).

Suppose that (a) holds. Since \((2v_{\epsilon,\epsilon} x_1) < (2v_{\epsilon,\epsilon} x_2)\), we have \((2v_{\xi_1,\eta_1} x_{1\xi_1}) < (2v_{\xi_1,\eta_1} x_{2\xi_2})\). This implies that \(1\xi_1 = 1\) and \(2\xi_1 = 2\), whence \(\xi_1 = \epsilon\). Then

\[\phi(z'_{1\pi} a_1 \prod_{i=2}^{2k} z'_{i\pi}) b_2 z'_{(2k+1)\pi} = z'_{1\pi} a_1 \prod_{i=2}^{2k} z'_{i\pi} b_2 z'_{(2k+1)\pi}.\]

It follows from \((*)\) that \(\phi(a_1) = a_1, \phi(b_2) = b_2\) and \(\phi(z'_{i\pi}) = z'_{i\pi}\) for any \(i = 1, 2, \ldots, 2k + 1\). Then

\[\phi(b_1 x_{1\xi_1}, x_{2\xi_1}, y_{1\eta_1}, y_{2\eta_1}) = b_1 x_1 x_2 y_1 y_2.\]

Now we apply \((*)\) again and obtain that \(\phi(b_1) = b_1\) and \(\phi(y_i) = y_{i\eta_1}\) for any \(i = 1, 2\). Since \((2v_{\epsilon,\epsilon} y_1) < (2v_{\epsilon,\epsilon} y_2)\), we have \((2v_{\xi_1,\eta_1} y_{1\eta_1}) < (2v_{\xi_1,\eta_1} y_{2\eta_1})\). This
implies that \(1\eta_1 = 1\) and \(2\eta_1 = 2\), whence \(\eta_1 = \varepsilon\). Then \(v = v_{\xi_2,\eta_2}\) and so \(\{v_\varepsilon,\varepsilon; v\} = \{v_{\xi_1,\eta_1}, v_{\xi_2,\eta_2}\}\).

Suppose that \((b)\) holds. Since \((2v_\varepsilon,\varepsilon) < (2v_\varepsilon,\varepsilon)\), we have \((2v_{\xi_1,\eta_1}, y_{1\eta_1}) < (2v_{\xi_1,\eta_1}, y_{2\eta_1})\). This implies that \(1\eta_1 = 1\) and \(2\eta_1 = 2\), whence \(\eta_1 = \varepsilon\). Then

\[
\phi\left(z''_{1\pi}b_1\left(\prod_{i=2}^{2k} z''_{i\pi}\right)a_2z''_{(2k+1)\pi}\right) = z''_{1\pi}a_1\left(\prod_{i=2}^{2k} z''_{i\pi}\right)b_2z''_{(2k+1)\pi}.
\]

It follows from \((*)\) that \(\phi(b_1) = a_1\) and \(\phi(a_2) = b_2\). Then

\[
\phi(x_{1\xi_1}x_{2\xi_1}y_{1\eta_1}y_{2\eta_1}b_2) = b_1x_1x_2y_1y_2.
\]

Now we apply \((*)\) again and obtain that \(\phi(y_{2\eta_1}) = y_1\), which contradicts \((b)\). Therefore, \((b)\) is impossible.

Lemma 3.7 is proved. \(\square\)

One can return to the proof of Proposition 3.5. Let

\[
X = M(c_{n,m+k+2}[\pi], c_{n,m,k+2}[\tau]) \wedge \mathrm{var}\{v_1 \approx v_2, v_3 \approx v_4\},
\]

\[
Y = M(c_{n,m,k+2}[\pi], c_{n,m,k+2}[\tau]) \wedge \mathrm{var}\{v_2 \approx v_4\},
\]

\[
Z = M(c_{n,m,k+2}[\pi], c_{n,m,k+2}[\tau]) \wedge \mathrm{var}\{v_1 \approx v_3, v_2 \approx v_4\}.
\]

Consider an identity \(u \approx u'\) of \(X\) with \(u \in \{v_1, v_2\}\). We are going to show that \(u' \in \{v_1, v_2\}\). In view of Proposition 2.1, we may assume without loss of generality that either \(u \approx u'\) holds in \(M(c_{n,m,k+2}[\pi], c_{n,m,k+2}[\tau])\) or \(u \approx u'\) is directly deducible from \(v_1 \approx v_2\) or \(v_3 \approx v_4\). According to Lemma 3.7, \(u \approx u'\) cannot be directly deducible from \(v_3 \approx v_4\) and if \(u \approx u'\) is directly deducible from \(v_1 \approx v_2\), then \(\{u, u'\} \subseteq \{v_1, v_2\}\). Therefore, it remains to consider the case when \(u \approx u'\) is satisfied by \(M(c_{n,m,k+2}[\pi], c_{n,m,k+2}[\tau])\). It follows from Lemma 3.6 that \(u' \in \{v_1, v_2, v_3, v_4\}\).

If \(u' \in \{v_3, v_4\}\), then the identity \(u(X) \approx u'(X)\) coincides (up to renaming of letters) with the identity \(c_{n,m,k+2}[\pi] \approx c'_{n,m,k+2}[\pi]\), where

\[
X = \{y_1, y_2, t\} \cup \{z''_{i\pi}, t''_{i\pi} | 1 \leq i \leq 2k + 1\}.
\]

But this is impossible because the variety \(M(c_{n,m,k+2}[\pi], c_{n,m,k+2}[\tau])\) violates this identity by Lemma 2.2. We see that \(u' \in \{v_1, v_2\}\) in either case and so the set \(\{v_1, v_2\}\) forms a FIC\((X)\)-class. By similar arguments we can show that the set \(\{v_1, v_3\}\) forms a FIC\((Z)\)-class and the word \(v_1\) is an isoterms for \(Y\). This implies that \(v_1\) is an isoterm for \((X \lor Z) \wedge Y\). Clearly, both \(X \wedge Y\) and \(Z\) satisfy the identity \(v_1 \approx v_3\). Therefore, \(v_1\) is not an isoterm for \((X \wedge Y) \lor Z\). Since \(Z \subseteq Y\), we have

\[
(X \wedge Y) \lor Z \subset (X \vee Z) \wedge Y.
\]

It follows that the lattice \(\mathcal{L}(M(c_{n,m,n+m+1}[\rho]))\) is not modular. \(\square\)
The proof of the following statement is very similar to the proof of Proposition 3.5 and so we omit it.

**Proposition 3.8** The lattice \( \Sigma(\mathcal{M}(c_{n,m,0}[\rho])) \) is not modular and so not distributive for any \( m, m \in \mathbb{N}_0 \) and \( \rho \in S_{n+m} \).

### 4 Auxiliary results

#### 4.1 Identities formed by words with one multiple letter

Let

\[ A = \text{var}\{x^2 y \approx y x^2\}. \]

**Lemma 4.1** Let \( r, e_0, f_0, e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0 \). Then the identity

\[ x^{e_0} \left( \prod_{i=1}^{r} t_i x^{e_i} \right) \approx x^{f_0} \left( \prod_{i=1}^{r} t_i x^{f_i} \right), \tag{4.1} \]

is equivalent within \( A \) to some identity

\[ x^{e_0'} \left( \prod_{i=1}^{r} t_i x^{e_i'} \right) \approx x^{f_0'} \left( \prod_{i=1}^{r} t_i x^{f_i'} \right) \]

with \( e_1', f_1', e_2', f_2', \ldots, e_r', f_r' \leq 1 \).

**Proof** Let \( p_i \) and \( q_i \) be the greatest even numbers such that \( p_i \leq e_i \) and \( q_i \leq f_i \) for any \( i = 1, 2, \ldots, r \). Then the required conclusion follows from the fact that the identities

\[ x^{e_0} \left( \prod_{i=1}^{r} t_i x^{e_i} \right) \approx x^{e_0+\sum_{i=1}^{r} p_i} \left( \prod_{i=1}^{r} t_i x^{e_i-p_i} \right) \]

\[ \approx x^{f_0+\sum_{i=1}^{r} q_i} \left( \prod_{i=1}^{r} t_i x^{f_i-q_i} \right) \]

are consequences of \( x^2 y \approx y x^2 \).

For any \( n, m \in \mathbb{N}_0 \), we fix notation for the following identity:

\[ \delta_{n,m} : x^m t_1 x t_2 x \cdots t_n x \approx x^{m+n} t_1 t_2 \cdots t_n. \]

**Lemma 4.2** Let \( V \) be an aperiodic monoid subvariety of \( A \), \( 0 \leq e_1, f_1, \ldots, e_n, f_r \leq 1 \) and \( e_0, f_0 \in \mathbb{N}_0 \). If the identity (4.1) is non-trivial, then

\[ V \{ (4.1) \} = V \{ x^e \approx x^f, \delta_{e-e_0,e_0}, \delta_{f-f_0,f_0} \}. \]
where $e = \sum_{i=0}^{r} e_i$ and $f = \sum_{i=0}^{r} f_i$.

**Proof** The identity (4.1) follows from $\{x^e \approx x^f, \delta_{e-e_0, e_0}, \delta_{f-f_0, f_0}\}$ because

$$x^{e_0} \left( \prod_{i=1}^{r} t_i x^{e_i} \right) \sum_{i=1}^{r} t_i x^{e_i} \approx x^{f_0} \left( \prod_{i=1}^{r} t_i x^{f_i} \right).$$

Evidently, $x^e \approx x^f$ is a consequence of (4.1). So, it remains to show that $\mathbb{V}(\{4.1\})$ satisfies $\delta_{e-e_0, e_0}$ and $\delta_{f-f_0, f_0}$. Two cases are possible.

**Case 1:** $e_i = f_i = 1$ for any $i = 1, 2, \ldots, r$. Then $e = e_0 = f = f_0$. Since the identity (4.1) is non-trivial, by symmetry, we may assume that $e_0 = f_0 + g$ for some $g \in \mathbb{N}$. In view of Lemma 2.3, $\mathbb{V}$ satisfies the identity $x^s \approx x^{s+1}$ for some $s \in \mathbb{N}$. Then the identities

$$(4.1) \quad \sum_{i=1}^{r} t_i x^{e_i} = x^{e_0} + \sum_{i=1}^{r} t_i x^{f_i},$$

and so $\delta_{r, e_0}$ are satisfied by $\mathbb{V}(\{4.1\})$. It remains to notice that $\delta_{r, f_0}$ is consequence of $\delta_{r, e_0}$.

**Case 2:** $(e_j, f_j) \neq (1, 1)$ for some $j \in \{1, 2, \ldots, r\}$. We use induction on $r$.

**Induction base:** $r = 1$. We may assume without any loss that (4.1) coincides with either $x^{e_0} t_1 \approx x^{f_0} t_1$ or $x^{e_0} t_1 \approx x^{f_0} t_1 x$. If (4.1) coincides with $x^{e_0} t_1 \approx x^{f_0} t_1$, then $e = e_0$ and $f = f_0$. In this case, the identities $\delta_{e-e_0, e_0}$ and $\delta_{f-f_0, f_0}$ are trivial and so follow from (4.1). If (4.1) is equal to $x^{e_0} t_1 \approx x^{f_0} t_1 x$, then $\delta_{e-e_0, e_0}$ is trivial, while $\delta_{f-f_0, f_0}$ follows from (4.1) because

$$(4.1) \quad x^{f_0} t_1 x \approx x^{e_0} t_1 = x^{e_0} t_1 \approx x^{f_0} t_1.$$

We see that (4.1) implies $\delta_{e-e_0, e_0}$ and $\delta_{f-f_0, f_0}$ in either case.

**Induction step:** $r > 1$. For any $\ell = 1, 2, \ldots, r$, we put

$$p_{\ell} = \left( \prod_{i=1}^{r} t_i x^{e_i} \right)_{t_{\ell}} \quad \text{and} \quad q_{\ell} = \left( \prod_{i=1}^{r} t_i x^{f_i} \right)_{t_{\ell}}.$$

Suppose that $e_k + e_{k+1}, f_k + f_{k+1} \leq 1$ for some $k \in \{1, 2, \ldots, r-1\}$. Consider the identity $x^{e_0} p_{k+1} \approx x^{f_0} q_{k+1}$. If $e_i = f_i = e_k + e_{k+1} = f_k + f_{k+1} = 1$ for any $i = 1, 2, \ldots, k-1, k+2, \ldots, r$, then $\mathbb{V}(x^{e_0} p_{k+1} \approx x^{f_0} q_{k+1})$ satisfies the identities $\delta_{e-e_0-1, e_0}$ and $\delta_{f-f_0-1, f_0}$ by Case 1 and so the identities $\delta_{e-e_0, e_0}$ and $\delta_{f-f_0, f_0}$. If either $(e_k + e_{k+1}, f_k + f_{k+1}) \neq (1, 1)$ or $(e_q, f_q) \neq (1, 1)$ for some $q \in \{1, 2, \ldots, k-1, k+2, \ldots, r\}$, then $\delta_{e-e_0, e_0}$ and $\delta_{f-f_0, f_0}$ hold in $\mathbb{V}(x^{e_0} p_{k+1} \approx x^{f_0} q_{k+1})$ by the induction assumption. Since $x^{e_0} p_{k+1} \approx x^{f_0} q_{k+1}$ is consequence of (4.1), the identities $\delta_{e-e_0, e_0}$ and $\delta_{f-f_0, f_0}$ are satisfied by $\mathbb{V}(\{4.1\})$. So, we may further assume that $e_i + e_{i+1} > 1$.

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or \( f_i + f_{i+1} > 1 \) for any \( i = 1, 2, \ldots, r - 1 \). In particular, \( (e_i, f_i) \neq (0, 0) \) for any \( i = 1, 2, \ldots, r \).

Then \( (e_j, f_j) \in \{(0, 1), (1, 0)\} \). We may assume without any loss that \( (e_j, f_j) = (0, 1) \) and \( j \neq r \). Then \( f_{j+1} = 1 \). Consider the identity \( x^{e_0} p_{j+1} \approx x^{f_0} q_{j+1} \). Clearly, \( x^2 y \approx yx^2 \) implies \( x^{f_0} q_{j+1} \approx x^{f_0+2} q_{j+1}' \), where

\[
q_{j+1}' = \left( \prod_{i=1}^{j-1} t_i x^{f_i} \right) \cdot t_j \cdot \left( \prod_{i=j+2}^r t_i x^{f_i} \right).
\]

By the induction assumption, \( \mathsf{V}\{x^{e_0} p_{j+1} \approx x^{f_0+2} q_{j+1}' \} \) and so \( \mathsf{V}\{(4.1)\} \) satisfy \( \delta_{e-e_0, e_0} \) and \( \delta_{f-(f_0+2), f_0+2} \). Then \( \delta_{f-f_0, f_0} \) holds in \( \mathsf{V}\{(4.1)\} \) because

\[
x^{f_0} \left( \prod_{i=1}^{r} t_i x^{f_i} \right) \approx x^{e_0} \left( \prod_{i=1}^{r} t_i x^{e_i} \right) \approx x^{e-e_0} x^{e_0} x^{\ell} \approx x^{f} \left( \prod_{i=1}^{r} t_i \right).
\]

Lemma 4.2 is proved. \( \square \)

**Corollary 4.3** Let \( X \) and \( Y \) be aperiodic monoid subvarieties of \( A \). If the variety \( X \land Y \) satisfies the identity \( \delta_{n, m} \) for some \( n, m \in \mathbb{N}_0 \), then this identity holds in either \( X \) or \( Y \).

**Proof** If \( M(xy) \notin X \), then \( X \) is commutative by Lemma 2.4. Then \( X \) satisfies the identity \( \delta_{n, m} \) because it is a consequence of the commutative law. By a similar argument we can show that if \( M(xy) \notin Y \), then \( \delta_{n, m} \) holds in \( Y \). So, we may further assume that \( M(xy) \in X \land Y \).

It follows from Proposition 2.1 that there is a sequence of pairwise distinct words \( w_1, w_2, \ldots, w_\ell \) such that \( w_1 = x^{m_1} t_1 x t_2 \cdots t_n x \), \( w_\ell = x^{m_\ell} t_1 t_2 \cdots t_n \) and \( w_i \approx w_{i+1} \) holds in either \( X \) or \( Y \) for any \( i = 1, 2, \ldots, \ell - 1 \). By symmetry, we may assume that \( w_1 \approx w_2 \) is satisfied by \( X \). Since \( M(xy) \in X \), Lemma 2.9 implies that \( w_2 = x^{e_0} \prod_{i=1}^{n} (t_i x^{e_i}) \) for some \( e_0, e_1, \ldots, e_n \in \mathbb{N}_0 \). In view of Lemma 4.1, we may assume that \( e_1, e_2, \ldots, e_n \leq 1 \). Now Lemma 4.2 applies and we conclude that \( X \) satisfies \( \delta_{n, m} \). \( \square \)

### 4.2 Identities formed by words with two multiple letters

**Lemma 4.4** Let \( X \) and \( Y \) be aperiodic monoid subvarieties of \( A \). If \( X \land Y \) satisfies the identity

\[
p \ x y \ q \approx p \ y x \ q, \tag{4.2}
\]

where

\[
p = a_1 t_1 \cdots a_k t_k \quad \text{and} \quad q = t_{k+1} a_{k+1} \cdots t_{k+\ell} a_{k+\ell} \quad \text{for some} \ k, \ell \in \mathbb{N}_0 \quad \text{and} \ a_1, a_2, \ldots, a_{k+\ell} \quad \text{are letters such that} \ \{a_1, a_2, \ldots, a_{k+\ell}\} = \{x, y\}. \tag{4.3}
\]
then this identity is true in either $X$ or $Y$.

**Proof** If $M(xy) \notin X$, then $X$ is commutative by Lemma 2.4 and, therefore, satisfies (4.2). By a similar argument we can show that $M(xy) \notin Y$, then $Y$ satisfies (4.2).

So, we may further assume that $M(xy) \in X \land Y$.

Put $u = px y q$. We note that sim$(u) = \{t_1, t_2, \ldots, t_{k+\ell}\}$. If the word $u_y$ is not an isoterms for $X$, then $X$ satisfies a non-trivial identity of the form $u_y \approx u'$. Lemma 2.9 implies that $u' = x^{g_0}(\prod_{i=1}^{k+\ell} t_i x^{g_i})$ for some $f_0, f_1, \ldots, f_{k+\ell} \in \mathbb{N}_0$. In view of Lemma 4.1, we may assume that $f_1, f_2, \ldots, f_{k+\ell} \leq 1$. Then we can apply Lemma 4.2 with the conclusion that $X$ satisfies the identity $δ_{occ}(u) − 1, 1$. This identity together with $x^2 y \approx yx^2$ implies $u \approx x^{occ}(u) u_x \approx p x y q$, and we are done. Analogous considerations show that if the word $u_x$ is not an isoterms for $X$ or $Y$ or the word $u_y$ is not an isoterms for $Y$, then the identity (4.2) is true in either $X$ or $Y$.

So, it remains to consider the case both the words $u_x$ and $u_y$ are isoterms for $X$ and $Y$. Proposition 2.1 implies that one of the varieties $X$ or $Y$, say $X$, satisfies a non-trivial identity of the form $u \approx v$. In view of Lemma 2.10, the identity $u \approx v$ is linear-balanced. This means that sim$(v) = \{t_1, t_2, \ldots, t_{k+\ell}\}$ and blocks of the word $v$ (in order of their appearance from left to right) are $a_1, a_2, \ldots, a_k, w, a_{k+1}, \ldots, a_{k+\ell}$, where $w \in \{xy, yx\}$. Since the identity $u \approx v$ is non-trivial, $w = yx$, whence $v = px y q$.

**Lemma 4.5** Let $X$ and $Y$ be aperiodic monoid subvarieties of $A(\alpha, \beta)$. If $X \land Y$ satisfies the identity

$$x^{e_0} y^{f_0} \left( \prod_{i=1}^{r} t_i x^{e_i} y^{f_i} \right) \approx y^{f_0} x^{e_0} \left( \prod_{i=1}^{r} t_i x^{e_i} y^{f_i} \right),$$

(4.4)

with $r \in \mathbb{N}_0$, $e_0, f_0 \in \mathbb{N}$, $e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0$, $\sum_{i=0}^{r} e_i \geq 2$ and $\sum_{i=0}^{r} f_i \geq 2$, then this identity is true in either $X$ or $Y$.

**Proof** If $e_i > 1$ or $f_i > 1$ for some $i \in \{0, 1, \ldots, r\}$, then it is routine to check that (4.4) is a consequence of $\{x^2 y \approx yx^2, \beta\}$. So, we may further assume that $e_0 = f_0 = 1$ and $e_1, f_1, e_2, f_2, \ldots, e_r, f_r \leq 1$.

If $M(xy) \notin X$, then $X$ is commutative by Lemma 2.4 and, therefore, satisfies (4.4). By a similar argument we can show that $M(xy) \notin Y$, then $Y$ satisfies (4.4). So, we may further assume that $M(xy) \in X \land Y$. Let $u$ denote the the left-hand side of the identity (4.4). We note that sim$(u) = \{t_1, t_2, \ldots, t_r\}$. If the word $u_x$ is not an isoterms for $X$, then $X$ satisfies a non-trivial identity of the form $u_y \approx u'$. Lemma 2.9 implies that $u' = x^{g_0}(\prod_{i=1}^{r} t_i x^{g_i})$ for some $g_0, g_1, \ldots, g_r \in \mathbb{N}_0$. In view of Lemma 4.1, we may assume that $g_1, g_2, \ldots, g_r \leq 1$. Then we can apply Lemma 4.2 with the conclusion that $X$ satisfies the identity $δ_{e-1, 1}$, where $e = \sum_{i=0}^{r} e_i$. This identity together with $\{x^2 y \approx yx^2, \beta\}$ implies

$$u^{δ_{e-1, 1}} x^e u_x = x^e y^{f_0} \left( \prod_{i=1}^{r} t_i y^{f_i} \right) \{x^2 y \approx yx^2, \beta\} \approx y^{f_0} x^e \left( \prod_{i=1}^{r} t_i y^{f_i} \right).$$
Analogous considerations show that if the word \( u_x \) is not an isoterm for \( X \) or \( Y \) or the word \( u_y \) is not an isoterm for \( Y \) or \( X \), then the identity (4.4) is true in either \( X \) or \( Y \).

So, it remains to consider the case when both the words \( u_x \) and \( u_y \) are isotersms for \( X \) and \( Y \). Since (4.4) is satisfied by in \( X \land Y \), Proposition 2.1 implies that there is a sequence of pairwise distinct words \( w_1, w_2, \ldots, w_\ell \) such that \( w_1 = u, w_\ell \) is the right-hand side of (4.4) and \( w_i \approx w_{i+1} \) holds in either \( X \) or \( Y \). Then there exists \( j \in \{1, 2, \ldots, \ell - 1\} \) such that \((1w_jx) < (1w_jy) \) but \((1w_{j+1}y) > (1w_{j+1}x) \). Since \( u_x \) and \( u_y \) are isotersms for \( X \land Y \), Lemma 2.10 implies that \( w_j = xy(\prod_{i=1}^r t_i x^{e_i} y^{f_i}) \) and \( w_{j+1} = yx(\prod_{i=1}^r t_i b_i) \),

where \( a_i, b_i \in \{x^{e_i} y^{f_i}, y^{f_i} x^{e_i}\} \) for any \( i = 1, 2, \ldots, r \). We may assume without any loss that \( w_j \approx w_{j+1} \) holds in \( X \). Then \( X \) satisfies (4.4) because

\[
xy(\prod_{i=1}^r t_i x^{e_i} y^{f_i}) \approx yx(\prod_{i=1}^r t_i b_i) \approx x y(\prod_{i=1}^r t_i a_i) \approx y x(\prod_{i=1}^r t_i x^{e_i} y^{f_i}).
\]

Lemma 4.5 is proved.

4.3 Identities of the form \( a_{n,m}[\rho] \approx a'_{n,m}[\rho] \)

Let

\[
A' = A[a_{n,m}[\rho] \approx a'_{n,m}[\rho] \mid (n, m) \in \mathbb{N}_0^2 \text{ and } \rho \in S_{n,m}].
\]

Lemma 4.6 (Gusev and Vernikov [10, Lemma 3.8]) Let \( V \) be a subvariety of \( A' \). If \( w = pxqxr \) and \( \text{con}(q) \subseteq \text{mul}(w) \), then \( V \) satisfies the identity \( w \approx px^2qr \). □

Lemma 4.7 Let \( V \) be a subvariety of \( A \) satisfying the identities (3.1) and

\[
xyzxy \approx xzyyx,
\]

(4.5)

\((n, m) \in \mathbb{N}_0^2 \text{ and } \rho \in S_{n,m} \). Suppose that \( V \) satisfies a non-trivial identity \( a_{n,m}[\rho] \approx a \) for some \( a \in X^* \) with \( a_x = \hat{a}_{n,m}[\rho] \). Suppose also that one of the following holds:

(i) \( x^2 \) is a factor of \( a \);
(ii) \( \text{occ}_x(a) > 2 \).

Then \( V \) satisfies the identity \( a_{n,m}[\rho] \approx a'_{n,m}[\rho] \).
Proof. Clearly, 
\[ a = \left( \prod_{i=1}^{n} x^{e_i} z_i x^{f_i} t_i \right) x^{g_0} \left( \prod_{i=1}^{n+m} z_{i,j} x^{g_i} \right) \left( \prod_{i=n+1}^{n+m} t_i x^{e_i} z_i x^{f_i} \right) \]
for some \( g_0, e_1, f_1, g_1, \ldots, e_{n+m}, f_{n+m}, g_{n+m} \in \mathbb{N}_0 \). Then \( V \) satisfies 
\[ \left( \prod_{i=1}^{n} t_i \right) x^2 \left( \prod_{i=n+1}^{n+m} t_i \right) \approx \left( \prod_{i=1}^{n} x^{e_i+f_i} z_i t_i \right) x^{\sum_{i=0}^{n+m} g_i} \left( \prod_{i=n+1}^{n+m} t_i x^{e_i+f_i} z_i \right). \]  
(4.6)

(i) If \( x^2 \) is a factor of \( a \), then \( V \) satisfies the identities 
\[ a \approx a'_{n,m}[\rho], \]
and we are done.

(ii) In view of Part (i), we may further assume that \( x^2 \) is not a factor of \( a \).

First, we consider the case when \( n + m = 1 \). Let \( \epsilon \) denote the trivial permutation from \( S_1 \). If \( n = 1 \) and \( m = 0 \), then \( a_{n,m}[\rho] \approx a \) is nothing but \( a_{1,0}[\epsilon] = z_1 t_1 x z_1 x \approx x^{e_1} z_1 x^{f_1} t_1 x^{g_0} z_1 x^{g_1} \). Since \( x^2 \) is not a factor of \( a \) and \( \text{occ}_x(a) > 2 \), at least three of the numbers \( e_1, f_1, g_0, g_1 \) are equal to 1. Then \( V \) satisfies 
\[ a_{1,0}[\epsilon] \approx a = x^{e_1} z_1 x^{f_1} t_1 x^{g_0} z_1 x^{g_1} (3.1, (4.5)) \approx x^{e_1+f_1} z_1 t_1 x^{g_0+g_1} z_1 \]  
(4.6)  
\[ \approx z_1 t_1 x^2 z_1 = a'_{1,0}[\epsilon], \]
and we are done. By a similar argument we can show that if \( n = 0 \) and \( m = 1 \), then \( a_{0,1}[\rho] \approx a'_{0,1}[\epsilon] \) holds in \( V \). So, we may further assume that \( n, m \geq 1 \).

If every block of \( a \) contains at most one occurrence of \( x \), then \( a \approx a'_{n,m}[\rho] \) is a consequence of (4.6), whence \( V \) satisfies \( a_{n,m}[\rho] \approx a'_{n,m}[\rho] \). So, it remains to consider the case when \( x \) is multiple in some block of \( a \).

Suppose that \( e_j + f_j > 1 \) for some \( j \in \{1, 2, \ldots, n+m\} \). Then \( e_j = f_j = 1 \). We may assume without loss of generality that \( j \leq n \). Clearly, \( x^2 \) is a factor of \( a(x, z_{n+1}, t_{n+1}) \) and \( a_{n,m}[\rho](x, z_{n+1}, t_{n+1}) \) coincides (up to renaming of letters) with \( a_{0,1}[\epsilon] \). Then Part (i) implies that \( V \) satisfies \( a_{0,1}[\epsilon] \approx a'_{0,1}[\epsilon] \). Evidently,
follows from $a_{0,1}[\varepsilon] \approx a_{0,1}'[\varepsilon]$. Now Part (i) applies again and we conclude that $a_{n,m}[\rho] \approx a_{n,m}'[\rho]$ is satisfied by $V$.

Suppose now that $e_i + f_i \leq 1$ for any $i = 1, 2, \ldots, n + m$. Then $\sum_{i=0}^{n+m} g_i \geq 2$. In this case, there are $1 \leq s \leq r \leq n + m$ such that $xz_{s}z_{(s+1)} \cdots z_{r}x$ is a factor of $a$. Let

$$X = \{x, t_1, t_2, \ldots, t_{n+m}, z_{s}, z_{(s+1)}, \ldots, z_{r}\}.$$

It is easy to see that the identity $a_{n,m}[\rho](X) \approx a(X)$ implies the identity

$$a \approx \left( \prod_{i=1}^{n} e_i t_i \right) \left( \prod_{i=1}^{s-1} x^{g_{i-1}} z_{i,\rho} \right) x^{\sum_{i=0}^{r-1} g_i} \left( \prod_{i=s}^{n+m} z_{i,\rho} \right) x^{\sum_{i=r}^{n+m} g_i} \left( \prod_{i=t+1}^{n+m} z_{i,\rho} x^{g_i} \right) \left( \prod_{i=n+1}^{n+m} e_i t_i \right),$$

(4.7)

where

$$e_i = \begin{cases} x^{2e_i} z_i x^{2f_i} & \text{if } s \leq i \rho \leq r; \\ x^{2e_i+f_i} z_i x^{f_i} & \text{if } e_i = 1 \text{ and } i \rho < s \text{ or } r < i \rho; \\ x^{e_i} z_i x^{e_i+2f_i} & \text{if } e_i = 0 \text{ and } i \rho < s \text{ or } r < i \rho. \end{cases}$$

Evidently, if $e_j = 1$ or $f_j = 1$ for some $1 \leq j \leq n + m$, then $x^2$ is a factor of the left hand-side of (4.7). If $e_i = f_i = 0$ for any $i = 1, 2, \ldots, n + m$, then either $\sum_{i=0}^{s-1} g_i > 1$ or $\sum_{i=r}^{n+m} g_i > 1$ because $\text{occ}_x(a) > 2$ and, therefore, $x^2$ is a factor of the left hand-side of (4.7) as well. We see that the left hand-side of (4.7) contains the factor $x^2$ in either case. According to Part (i), the identity $a_{n,m}[\rho] \approx a_{n,m}'[\rho]$ holds in $V$.

Lemma 4.7 is proved.  

For any $n, m \in \mathbb{N}_0$, $\rho \in S_{n+m}$ and $0 \leq p \leq q \leq n + m$, we put

$$a_{n,m}^{p,q}[\rho] = \left( \prod_{i=1}^{n} z_{i,\rho} \right) \left( \prod_{i=1}^{p} z_{i,\rho} \right) x \left( \prod_{i=p+1}^{q} z_{i,\rho} \right) x \left( \prod_{i=q+1}^{n+m} z_{i,\rho} \right) \left( \prod_{i=n+1}^{n+m} t_i z_i \right).$$

Lemma 4.8 Let $V$ be a monoid variety such that $M(\varepsilon)$ \subseteq $V \subseteq A((3.1), \ (4.5))$. If, for any $(n, m) \in \hat{\mathbb{N}}_0^2$ and $\rho \in S_{n,m}$, the monoid $M(a_{n,m}[\rho])$ does not lie in $V$, then $V \subseteq A'$.

Proof If $M(xzxyt) \notin V$, then $V$ satisfies $\beta$ by Lemma 2.11. Then the same arguments as in the proof of Lemma 3.11 in [10] imply that $V$ satisfies $a_{n,m}[\rho] \approx a_{n,m}'[\rho]$ for any $(n, m) \in \hat{\mathbb{N}}_0^2$ and $\rho \in S_{n,m}$. So, we may assume that $M(xzxyt) \in V$.

Since $M(a_0[\varepsilon]) \notin V$, it follows from Lemma 2.2 that $V$ satisfies a non-trivial identity $a_{0}[\varepsilon] \approx a$ for some $a \in X^*$. Lemma 3.3 implies that $a_{x} = a_{1,0}[\varepsilon]$. If $x^2$ is a factor of $a$ or $\text{occ}_x(a) > 2$, then $V$ satisfies the identity $a_{1,0}[\varepsilon] \approx a_{1,0}'[\varepsilon]$ by
Lemma 4.7. If \( x^2 \) is not a factor of \( a \) and \( \text{occ}_x(a) \leq 2 \), then, since \( xyx \) is an isoterms for \( V \), the identity \( a_{1,0}[ε] ≈ a \) must coincide (up to renaming of letters) with

\[
ztxzx ≈ xzxtz. \tag{4.8}
\]

By a similar argument one can show that the assumption that \( M(a_{0,1}[ε]) \notin V \) implies that \( V \) satisfies one of the identities \( a_{0,1}[ε] ≈ a_{1,0}'[ε] \) or \( \tag{4.8} \).

Let \( τ \in S_2 = S_{1,1}. \) Since \( M(a_{1,1}[τ]) \notin V \), Lemma 2.2 implies that \( V \) satisfies a non-trivial identity \( a_{1,1}[τ] ≈ a \) for some \( a \in X^* \). It follows from Lemma 3.3 that \( a_x = \hat{a}_{1,1}[τ] \). If \( x^2 \) is a factor of \( a \) or \( \text{occ}_x(a) > 2 \), then \( a_{1,1}[τ] ≈ a_{1,1}'[τ] \) holds in \( V \) by Lemma 4.7. Suppose now that \( \text{occ}_x(a) \leq 2 \) and \( x^2 \) is not a factor of \( a \). Since \( xyx \) and so \( x \) are isoterms for \( V \), we have \( \text{occ}_x(a) = 2 \). Assume that some occurrence of \( x \) lies between the first occurrences of \( t_1 \) and \( t_2 \) in \( a \). Since \( x^2 \) is not a factor of \( a \) and the identity \( a_{1,1}[τ] \approx a \) is non-trivial, we may assume without any loss that \( a = a_{1,2} \).

Hence \( (a_{1,1}[τ])(t_2z,t_2x,t_1) ≈ (a_{1,2} τ \{ t_2z,t_2x,t_1 \}) \) and \( x^2y \approx yx^2 \) imply either \( a_{1,0}[ε] ≈ a_{1,0}'[ε] \) or \( a_{0,1}[ε] = a_{0,1}'[ε] \). Then both \( a_{0,1}[ε] = a_{0,1}'[ε] \) and \( a_{1,0}[ε] = a_{1,0}'[ε] \) hold in \( V \) because

\[
A[a_{1,0}[ε] ≈ a_{1,0}'[ε]]. \tag{4.8} = A[a_{0,1}[ε] ≈ a_{0,1}'[ε]]. \tag{4.8}
\]

Then \( V \) satisfies \( a_{1,1}[τ] ≈ a_{1,1}'[τ] \) because \( a = a_{1,2} \approx a_{1,1}'[τ] \) is a consequence of

\[
\{ a_{1,0}[ε] ≈ a_{1,0}'[ε], a_{0,1}[ε] ≈ a_{0,1}'[ε], x^2y ≈ yx^2 \}.
\]

Assume now that there are no occurrences of \( x \) between the first occurrences of \( t_1 \) and \( t_2 \) in \( a \). Then we may assume without any loss that some occurrence of \( x \) precedes the first occurrence of \( t_1 \) in \( a \). Since \( xyx \) is an isoterms for \( V \) and \( x^2 \) is not a factor of \( a \), we have \( a = xz_1tx_1z_1tx_1z_2t_2z_2t_2z_2x \). Recall that either \( \tag{4.8} \) or \( a_{0,1}[ε] = a_{0,1}'[ε] \) holds in \( V \). If \( \tag{4.8} \) holds in \( V \), then \( V \) satisfies a non-trivial identity \( a ≈ a_{1,1} \) for some \( 0 ≤ ℓ_1 ≤ ℓ_2 ≤ 2 \). In this case, the same arguments as in the above show that \( a_{1,1}[τ] ≈ a_{1,1}'[τ] \) holds in \( V \). If \( a_{0,1}[ε] = a_{0,1}'[ε] \) holds in \( V \), then \( V \) satisfies \( a_{0,1}[ε] = a_{0,1}'[ε] = a_{1,0}[ε] = a_{1,0}'[ε] \) because they are consequences of \( a_{1,1}[τ] = a_{1,1}'[τ] \).

We have proved that there exists a number \( r \) such that \( V \) satisfies the identity \( a_{r_1,r_2}[ρ] = a_{r_1,r_2}'[ρ] \) for all \( (r_1,r_2) \in \mathbb{N}_0^2 \) and \( ρ \in S_{r_1,r_2} \) with \( r_1 + r_2 ≤ r \) (for instance, \( r = 1 \)). We are going to verify that an arbitrary \( r \) possesses this property. Arguing by contradiction, we suppose that the mentioned claim is true for \( r = 1, 2, \ldots, k - 1 \) but is false for \( r = k \). Then \( V \) violates \( a_{n,m}[ρ] = a_{n,m}'[ρ] \) for some \( (n,m) = \mathbb{N}_0^2 \) and \( ρ \in S_{n,m} \) such that \( n + m = k \). According to Lemma 2.2, \( V \) satisfies a non-trivial identity \( a_{n,m}[ρ] = a \). In view of Lemma 3.3, \( a_x = \hat{a}_{n,m}[ρ] \). Then \( \text{occ}_x(a) \leq 2 \) and \( x^2 \) is not a factor of \( a \) by Lemma 4.7. It follows from the fact that \( x \) is an isoterms for \( V \) that \( \text{occ}_x(a) = 2 \).
Suppose that some occurrence of \( x \) lies between the first occurrences of \( t_n \) and \( t_{n+1} \) in \( a \). Since \( x^2 \) is not a factor of \( a \) and the identity \( a_{n,m}[\rho] \approx a \) is non-trivial, we may assume without any loss that \( a = a_{n,m}^{p,q}[\rho] \) for some \( 0 < p < q \leq n + m \). Let

\[
X = \{ z_1\rho, t_1\rho, z_2\rho, t_2\rho, \ldots, z_p\rho, t_p\rho, z(q+1)\rho, t(q+1)\rho, z(q+2)\rho, t(q+2)\rho, \ldots, z_k\rho, t_k\rho \},
\]

Clearly, \((a_{n,m}[\rho])X\) coincides (up to renaming of letters) with \( a_{c,d}[\pi] \) for some \((c, d) \in \mathbb{N}^2_0 \) and \( \pi \in S_{c,d} \) such that \( c + d = q - p \). Since \( a_{c,d}[\pi] \approx a_{c,d}'[\pi] \) holds in the variety \( V \), this variety must satisfy \( a = a_{n,m}^{p,q}[\rho] \approx a_{n,m}^{p,q}[\rho] \approx a_{n,m}^{p,q}[\rho] \) contradicting the choice of \( n, m \) and \( \rho \).

Suppose now that there are no occurrences of \( x \) between the first occurrences of \( t_n \) and \( t_{n+1} \) in \( a \). Then we may assume without any loss that some occurrence of \( x \) precedes the first occurrence of \( t_n \) in \( a \). Since \( xyx \) is an isoterm for \( V \) and \( x^2 \) is not a factor of \( a \), we have

\[
a = \left( \prod_{i=1}^{j-1} z_i t_i \right) \cdot (xz_j xt_j) \cdot \left( \prod_{i=j+1}^{n} z_i t_i \right) \cdot \left( \prod_{i=1}^{n+m} z_i \right) \cdot \left( \prod_{i=n+1}^{n+m} t_i z_i \right)
\]

for some \( j \in \{1, 2, \ldots, n\} \). Taking into account the fact that the identities \( \epsilon_1 \approx \epsilon_1' \) and \( x^2y \approx yx^2 \) hold in \( V \), we obtain that \( V \) satisfies \( a = a_{n,m}^{p,q}[\rho] \approx a_{n,m}^{p,q}[\rho] \approx a_{n,m}^{p,q}[\rho] \) contradicting the choice of \( n, m \) and \( \rho \) again.

### 4.4 Identities of the form \( c_{n,m,k}[\rho] \approx c_{n,m,k}'[\rho] \)

**Lemma 4.9** Let \( V \) be a variety such that \( M(xyxy) \subseteq V \) and \( N \not\subseteq V \). Suppose that \( V \) does not contain the monoids \( M(c_{n,m,0}[\pi]) \) and \( M(c_{n,m,n+m+1}[\pi]) \) for all \( n, m \in \mathbb{N}_0 \), \( \tau \in S_{n+m} \) and \( \pi \in S_{n+m,n+m+1} \). Then \( V \) satisfies the identity \( c_{n,m,k}[\rho] \approx c_{n,m,k}[\rho] \) for any \( n, m, k \in \mathbb{N}_0 \) and \( \rho \in S_{n+m+k} \).

**Proof** It is routine to check that, for any \( p, q, r \in \mathbb{N}_0 \) and \( \rho \in S_{p+q+r} \), one can construct \( \pi \in S_{n+m,n+m+1} \) for some \( n, m \in \mathbb{N}_0 \) such that \( c_{p,q,r}[\rho] \approx c_{p,q,r}[\rho] \) is a consequence of

\[
c_{n,m,n+m+1}[\pi] \approx c_{n,m,n+m+1}[\pi]. \tag{4.9}
\]

In view of this fact, it suffices to show that \( V \) satisfies the identity (4.9) for any \( n, m \in \mathbb{N}_0 \) and \( \pi \in S_{n+m,n+m+1} \).

Let us fix \( n, m \in \mathbb{N}_0 \) and \( \pi \in S_{n+m,n+m+1} \). Put \( k = n + m \). Suppose that \( xyzxyt \) is an isoterm for \( V \). Then it is easy to see that if \( c_{n,m,k+1}[\pi] \approx c \) is an identity of \( V \), then \( c \in \{ c_{n,m,k+1}[\pi], c_{n,m,k+1}[\pi] \} \). Since \( M(c_{n,m,k+1}[\pi]) \notin V \), this fact and Lemma 2.2

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imply that (4.9) holds in V. Thus, we may further assume that \(xzxyty\) is not an isoterm for V. Then V satisfies \(\beta\) by Lemma 2.11(ii). It follows that V satisfies the identities

\[
c_{n,m,k+1}[\pi] \approx \left(\prod_{i=1}^{n} z_{i}t_{i}\right)xyt\left(\prod_{i=1}^{k} z_{i}t_{i}\right)x\left(\prod_{i=1}^{2k+1} z_{i}^{\pi}\right)Z_{1}\left(\prod_{i=1}^{2k+1} z_{i}^{\pi}\right)Z_{2}\left(\prod_{i=1}^{2k+1} t_{i}z_{i}\right),
\]

\[c'_{n,m,k+1}[\pi] \approx \left(\prod_{i=1}^{n} z_{i}t_{i}\right)yx\left(\prod_{i=1}^{k} z_{i}t_{i}\right)x\left(\prod_{i=1}^{2k+1} z_{i}^{\pi}\right)Z_{1}\left(\prod_{i=1}^{2k+1} z_{i}^{\pi}\right)Z_{2}\left(\prod_{i=1}^{2k+1} t_{i}z_{i}\right),
\]

(4.10)

where \(Z_{1} = \{z_{i} \mid k + 1 \leq i \leq 2k + 1\}\) and \(Z_{2} = \{z_{i} \mid 1 \leq i \leq k\}\). Evidently, there exists \(\tau \in S_{k}\) such that

\[c_{n,m,k}[\pi](X) = c_{n,m,0}[\tau] \quad \text{and} \quad c'_{n,m,k}[\pi](X) = c'_{n,m,0}[\tau],
\]

where \(X = \{x, y, z_{i}, t_{i} \mid 1 \leq i \leq k\}\). Since \(M(c_{n,m,0}[\tau]) \notin V\), Lemma 3.14 in [10] implies the V satisfies \(c_{n,m,0}[\tau] \approx c'_{n,m,0}[\tau]\). Then the identity (4.9) holds in V because it is a consequence of \(c_{n,m,0}[\tau] \approx c'_{n,m,0}[\tau]\) and (4.10). \(\square\)

5 Proof of Theorem 1.1

Necessity. Let V be a distributive subvariety of \(A_{cen}\). In view of Lemma 2.3, any subvariety of \(A_{cen}\) satisfies the identities

\[x^{n} \approx x^{n+1} \quad \text{and} \quad x^{n}y \approx yx^{n} \quad (5.1)
\]

for some \(n \in \mathbb{N}\). Let \(n\) be the least number such that V satisfies (5.1). If \(n = 1\), then V \(\subseteq \text{SL}_{1} = P_{1}\), and we are done. So, we may further assume that \(n > 1\). Two cases are possible.

Case 1: \(M(xyxy) \subseteq V\). If \(n = 2\), then V satisfies the identity \(x^{2}y \approx yx^{2}\) because this identity is nothing but the second identity in (2.1). Let now \(n > 2\). In view of Proposition 3.1, \(M(x^{2}y) \notin V\). Then Lemma 2.2 implies that V satisfies a non-trivial identity \(x^{2}y \approx v\). According to Lemma 2.6, \(x^{2}\) is an isoterm for V, whence \(\text{con}(v) = \{x, y\}\), \(\text{occ}_{x}(v) = 2\) and \(\text{occ}_{y}(v) = 1\). This is only possible when \(v = yx^{2}\) because \(M(xyxy) \subseteq V\) and the identity \(x^{2}y \approx v\) is non-trivial. We see that V satisfies \(x^{2}y \approx yx^{2}\) in either case. Two subcases are possible.

Subcase 1.1: \(N \subseteq V\) or \(N^{\delta} \subseteq V\). By symmetry, we may assume that \(N \subseteq V\). It is shown in [4, Theorem 1.1] that the lattice \(\mathcal{L}(M(xyxy) \vee N)\) is not modular. Therefore, \(M(xyxy) \notin V\). Then V satisfies \(\alpha\) by Lemma 2.11(i). In view of Lemma 3.2, \(M(xytxy) \notin V\). Then Lemma 2.11(ii) implies that V satisfies \(\beta\). Hence V \(\subseteq \text{R}_{n}\).

Subcase 1.2: \(N, N^{\delta} \not\subseteq V\). According to Propositions 3.5 and 3.8 and the statements dual to them, the variety does not contain the monoids

\[M(c_{n,m,n+m+1}[\pi]), M(d_{n,m,n+m+1}[\pi]), M(c_{n,m,0}[\tau]) \quad \text{and} \quad M(d_{n,m,0}[\tau])
\]
for all $n, m \in \mathbb{N}_0, \pi \in S_{n+m,n+m+1}$ and $\tau \in S_{n+m}$. Then Lemma 4.9 and the dual to it imply that $V$ satisfies the identities

$$c_{n,m,k}[\rho] \approx c'_{n,m,k}[\rho] \quad \text{and} \quad d_{n,m,k}[\rho] \approx d'_{n,m,k}[\rho]$$

for any $n, m, k \in \mathbb{N}_0$ and $\rho \in S_{n+m+k}$.

Further, $M(a_{n,m}[\rho]) \notin V$ for any $(n, m) \in \mathbb{N}_0^2$ and $\rho \in S_{n,m}$ by Proposition 3.4. Then Lemma 4.8 implies that $V$ satisfies $a_{n,m}[\rho] \approx a'_{n,m}[\rho]$ for any $(n, m) \in \mathbb{N}_0^2$ and $\rho \in S_{n,m}$. It follows from the proof of Lemma 4.4 in [9] that $a_{n,m}[\rho] \approx a'_{n,m}[\rho]$ holds in $V$ for any $n, m \in \mathbb{N}_0$ and $\rho \in S_{n+m}$. Hence $V \subseteq P_n$.

Case 2: $M(xyx) \not\in V$. According to Lemma 2.2, $V$ satisfies a non-trivial identity $xyx \approx v$. Since $n > 1$, Lemma 2.6 implies that $x$ is an isorm for $V$. Then $v = x^s y x^t$, where either $s \geq 2$ or $t \geq 2$. By symmetry, we may assume that $s \geq 2$. If $n = 2$, then $V$ satisfies $x^2y \approx xyx$ because

$$xyx \approx x^s y x^t \approx x \approx x^2 y x^t \approx x^2 y \approx x^2 y \approx x^2 y = x^2 y.$$  

If $n > 2$, then $s = 2$ and $t = 0$ by Lemma 2.6. We see that $x^2y \approx xyx$ is satisfied by $V$ in either case, whence $V \subseteq Q_n$.

Sufficiency. By symmetry, it suffices to show that the varieties $P_n, Q_n$ and $R_n$ are distributive. It is shown in [10, Corollary 5.8] that the lattice $\mathcal{L}(Q_n)$ is distributive. So, it remains to prove that the lattices $\mathcal{L}(P_n)$ and $\mathcal{L}(R_n)$ are distributive.

Distributivity of $\mathcal{L}(P_n)$. We are going to deduce the required fact from Lemma 2.12 with $V = P_n$ and $\Sigma = \Phi$, where $\Phi$ consists of the identity $xy \approx yx$, all identities of the form $x^k \approx x^\ell$ with $k, \ell \in \mathbb{N}$, all identities of the form $\delta_{k,\ell} \approx \delta_{k,\ell}$ with $k, \ell \in \mathbb{N}$ and all identities of the form (4.2) such that the equalities (4.3) hold. In view of Lemmas 2.7 and 4.4 and Corollaries 2.5 and 4.3, to do this, it remains to prove only that each subvariety of $P_n$ may be given within $P_n$ by some subset of $\Phi$.

Let $w' \approx v'$ be an arbitrary identity. It suffices to verify that $P_n[w' \approx v'] = P_n \Gamma$ for some $\Gamma \subseteq \Phi$. If $P_n[w' \approx v']$ is commutative, then it can be defined by some subset of $\{x^k \approx x^\ell, xy \approx yx \mid k, \ell \in \mathbb{N}\}$ by Proposition 2.8. Let us now consider the case when $P_n[w' \approx v']$ is not commutative. Then $M(xy) \in P_n[w' \approx v']$ by Lemma 2.4. Let $w_0 \prod_{i=1}^m (t_i w'_i)$ be the decomposition of $w'$. Lemma 2.9 implies that the decomposition of $v'$ has the form $v'_0 \prod_{i=1}^m (t_i v'_i)$. According to Lemma 4.6, the identities satisfied by $P_n$ can be used to convert the words $w'$ and $v'$ into some words $w$ and $v$, respectively, such that the following hold:

- the decompositions of $w$ and $v$ are of the form $w_0 \prod_{i=1}^m (t_i w_i)$ and $v_0 \prod_{i=1}^m (t_i v_i)$, respectively;
- $\text{occ}_x(w_i) = \text{occ}_x(w'_i)$ and $\text{occ}_x(v_i) = \text{occ}_x(v'_i)$ for any $x \in \mathcal{X}$ and $i = 0, 1, \ldots, m$;
- $x^{\text{occ}_x(w_i)}$ and $x^{\text{occ}_x(v_i)}$ are factors of $w_i$ and $v_i$, respectively, for any $x \in \text{con}(w_i) = \text{con}(v_i)$ and $i = 0, 1, \ldots, m$.  

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In view of this fact, it suffices to show that the identity \( \mathbf{w} \approx \mathbf{v} \) is equivalent within \( \mathbf{P}_n \) to some subset of \( \Phi \). Clearly, the set

\[
\{ \mathbf{w}(x, t_1, t_2, \ldots, t_m) \approx \mathbf{v}(x, t_1, t_2, \ldots, t_m) \mid x \in \text{mul}(\mathbf{w}) = \text{mul}(\mathbf{v}) \}
\]

(5.2)

of identities can be used to convert the word \( \mathbf{w} \) into some word \( \mathbf{u} \) such that the following hold:

- the decomposition \( \mathbf{u} \) has the form \( \mathbf{u}_0 \prod_{i=1}^{m} (t_i \mathbf{u}_i) \);
- \( \text{occ}_x(\mathbf{u}_i) = \text{occ}_x(\mathbf{v}_i) \) for any \( x \in \mathcal{X} \) and \( i = 1, 2, \ldots, m \);
- \( x^{\text{occ}_x(\mathbf{u}_i)} \) is a factor of \( \mathbf{u}_i \) for any \( x \in \text{con}(\mathbf{u}_i) \) and \( i = 1, 2, \ldots, m \).

Evidently, every identity from (5.2) is of the form (4.1). Then it follows from Lemmas 4.1 and 4.2 that the set (5.2) is equivalent modulo \( x^2y \approx xy^2 \) to some identities of the form \( \delta_{k,e} \) and \( x^k \approx x^e \). In view of this fact, it remains to show that the identity \( \mathbf{u} \approx \mathbf{v} \) is equivalent within \( \mathbf{P}_n \) to a subset of \( \Phi \).

We call an identity \( \mathbf{c} \approx \mathbf{d} \) 1-invertible if \( \mathbf{c} = e'xyx'' \) and \( \mathbf{d} = e'yxyx'' \) for some words \( e', x', x'' \) and letters \( x, y \in \text{con}(e'e'') \). Let \( k > 1 \). An identity \( \mathbf{c} \approx \mathbf{d} \) is called \( k \)-invertible if there is a sequence of words \( \mathbf{c} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_k = \mathbf{d} \) such that the identity \( \mathbf{w}_i \approx \mathbf{w}_{i+1} \) is 1-invertible for each \( i = 0, 1, \ldots, k-1 \) and \( k \) is the least number with such a property. For convenience, we will call the trivial identity 0-invertible.

Notice that the identity \( \mathbf{u} \approx \mathbf{v} \) is \( r \)-invertible for some \( r \in \mathbb{N}_0 \) because \( \text{occ}_x(\mathbf{u}_i) = \text{occ}_x(\mathbf{v}_i) \) for any \( x \in \mathcal{X} \) and \( i = 1, 2, \ldots, m \). We will use induction by \( r \).

**Induction base.** If \( r = 0 \), then \( \mathbf{u} = \mathbf{v} \), whence \( \mathbf{P}_n[\mathbf{u} \approx \mathbf{v}] = \mathbf{P}_n[\emptyset] \).

**Induction step.** Let \( r > 0 \). Obviously, \( \mathbf{u}_j \neq \mathbf{v}_j \) for some \( j \in \{0, 1, \ldots, m\} \). Then there are letters \( x \) and \( y \) such that \( \mathbf{u}_j = b'y^q x^p b'' \) for some \( b', b'' \in \mathcal{X}^* \) and the word \( x^p \) precedes the word \( y^q \) in the block \( \mathbf{v}_j \), where \( p = \text{occ}_x(\mathbf{u}_j) = \text{occ}_x(\mathbf{v}_j) \) and \( q = \text{occ}_y(\mathbf{u}_j) = \text{occ}_y(\mathbf{v}_j) \). We denote by \( \hat{\mathbf{u}} \) the word that is obtained from \( \mathbf{u} \) by swapping of the words \( x^p \) and \( y^q \) in the block \( \mathbf{u}_j \).

Suppose that \( \text{occ}_x(\mathbf{u}_c) > 1 \) for some \( c \in \{0, 1, \ldots, m\} \). We may assume without loss of generality that \( j < c \). If \( c \neq j \), then \( \mathbf{u}_c = f'x^\text{occ}_x(\mathbf{u}_c)f'' \) for some \( f', f'' \in \mathcal{X}^* \) and \( \mathbf{P}_n \) satisfies the identities

\[
\begin{align*}
\mathbf{u} & \overset{x^2y \approx xy^2}{\approx} \mathbf{u}_0 \cdot \left( \prod_{i=1}^{j-1} t_i \mathbf{u}_i \right) \cdot (t_j b' x^2 y^q x^p b'') \\
& \quad \cdot \left( \prod_{i=j+1}^{c-1} t_i \mathbf{u}_i \right) \cdot (t_c f' x^\text{occ}_x(\mathbf{u}_c)-2 f'') \cdot \left( \prod_{i=c+1}^{m} t_i \mathbf{u}_i \right) \\
\text{Lemma 4.6} & \approx \mathbf{u}_0 \cdot \left( \prod_{i=1}^{j-1} t_i \mathbf{u}_i \right) \cdot (t_j b' x^{2+p} y^q b'') \\
& \quad \cdot \left( \prod_{i=j+1}^{c-1} t_i \mathbf{u}_i \right) \cdot (t_c f' x^\text{occ}_x(\mathbf{u}_c)-2 f'') \cdot \left( \prod_{i=c+1}^{m} t_i \mathbf{u}_i \right)
\end{align*}
\]
If $c = j$, then $P_n$ satisfies the identities

$$u \approx \left( \prod_{i=1}^{j-1} t_i u_i \right) \cdot (t_j b' x^p y^q b'') \cdot \left( \prod_{i=j+1}^{m} t_i u_i \right) = \hat{u}.$$  

Lemma 4.6

$$u \approx \left( \prod_{i=1}^{j-1} t_i u_i \right) \cdot (t_j b' x^p y^q b'') \cdot \left( \prod_{i=j+1}^{m} t_i u_i \right) = \hat{u}.$$  

We see that $\hat{u} \approx v$ is satisfied by $P_n$. The identity $\hat{u} \approx v$ is $(r - pq)$-invertible. By the induction assumption, $P_n[\hat{u} \approx v] = P_n \Gamma$ for some $\Gamma \subseteq \Phi$. Then $P_n[u \approx v] = P_n \Gamma$, and we are done. By similar arguments we can show that if $\text{occ}_y(u_d) = \text{occ}_y(v_d) > 1$ for some $d \in \{0, 1, \ldots, m\}$, then $P_n[u \approx v] = P_n \Gamma$ for some $\Gamma \subseteq \Phi$. Thus, we may further assume that $\text{occ}_x(u_i), \text{occ}_y(u_i) \leq 1$ for any $i = 0, 1, \ldots, m$. In particular, $p = q = 1$.

Suppose that $x, y \in \text{con}(u_a) = \text{con}(v_a)$ for some $a \neq j$. Then Lemma 3.12 in [10] or the statement dual to it implies that $P_n$ satisfies the identity $u \approx \hat{u}$. The identity $\hat{u} \approx v$ is $(r - 1)$-invertible. By the induction assumption, $P_n[\hat{u} \approx v] = P_n \Gamma$ for some $\Gamma \subseteq \Phi$. Then $P_n[u \approx v] = P_n \Gamma$, and we are done. Thus, we may further assume that at most one of the letters $x$ and $y$ occurs in $u_i$ for any $i \neq j$.

Let $s$ be the least number such that $\text{con}(u_s) \cap \{x, y\} \neq \emptyset$. Put

$$T_{x,y} = \{t_j \mid s < j \leq m, \text{con}(u_j) \cap \{x, y\} \neq \emptyset\}.$$  

(5.3)

In view of the above, the identity

$$u([x, y] \cup T_{x,y}) \approx v([x, y] \cup T_{x,y})$$  

(5.4)

must be linear-balanced. Then, since at most one of the letters $x$ and $y$ occurs in $u_i$ for any $i \neq j$, the identity (5.4) must coincide (up to renaming of letters) with an identity of the form (4.2) such that the equalities (4.3) hold. It is clear that $u \approx \hat{u}$ and the identity $\hat{u} \approx v$ is $(r - 1)$-invertible. By the induction assumption, $P_n[u \approx v] = P_n \Gamma$ for some $\Gamma \subseteq \Phi$. Then $P_n[u \approx v] = P_n \Gamma$. In view of the above, this implies that $P_n[w' \approx v']$ can be given within $P_n$ by some subset of $\Phi$, and we are done.

**Distributivity of $\mathcal{L}(R_n)$.** We are going to deduce the required fact from Lemma 2.12 with $V = R_n$ and $\Sigma = \Phi$, where $\Phi$ consists of the identity $xy \approx yx$, all identities of the form $x^k \approx x^\ell$ with $k, \ell \in \mathbb{N}$, all identities of the form $\delta_{k,\ell}$ with $k, \ell \in \mathbb{N}_0$ and all identities of the form (4.4) with $r \in \mathbb{N}_0$, $e_0, f_0 \in \mathbb{N}$, $e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0$, $\sum e_i + \sum f_i = r$. For any $x, y \in \Phi$ and $\mathcal{A} \subseteq \Phi$, let $\text{d}(x, y; \mathcal{A})$ denote the smallest integer $n$ such that $x \approx y$ if and only if $\mathcal{A} \subseteq \Gamma$. Then $\text{d}(x, y; \Phi) = 0$, and $\text{d}(x, y; \Phi) = \infty$ if and only if $xy \not\approx yx$.
\[ \sum_{i=0}^{r} e_i \geq 2 \quad \text{and} \quad \sum_{i=0}^{r} f_i \geq 2. \]

In view of Lemmas 2.7 and 4.5 and Corollaries 2.5 and 4.3, to do this, it remains to prove only that each subvariety of \( R_n \) may be given within \( R_n \) by some subset of \( \Phi \).

Let \( u \approx v \) be an arbitrary identity. It suffices to verify that \( R_n \{ u \approx v \} = R_n \Gamma \) for some \( \Gamma \subseteq \Phi \). If the variety \( R_n \{ u \approx v \} \) is commutative, then it can be defined by a subset of \( \{ x^k \approx x^\ell, \; xy \approx yx \mid k, \ell \in \mathbb{N} \} \) by Proposition 2.8. So, it remains to consider the case when \( R_n \{ u \approx v \} \) is not commutative. In view of Proposition 4.1 in [20] and the inclusion \( R_n \subseteq \text{var} \{ \alpha, \beta \} \), we may assume that one of the following two statements holds:

(a) the identity \( u \approx v \) coincides with an identity of the form (4.1) with \( r, e_0, f_0, e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0; \)

(b) the identity \( u \approx v \) coincides with an identity of the form (4.4) with \( r \in \mathbb{N}_0, e_0, f_0 \in \mathbb{N}, e_1, f_1, \ldots, e_r, f_r \in \mathbb{N}_0, \sum_{i=0}^{r} e_i \geq 2 \) and \( \sum_{i=0}^{r} f_i \geq 2. \)

Notice that if the claim (b) holds, then the identity \( u \approx v \) lies in \( \Phi \), and we are done. Suppose now that the claim (a) holds. Lemma 4.1 and the inclusion \( R_n \subseteq A \) allow us to assume that \( e_1, f_1, e_2, f_2, \ldots, e_r, f_r \leq 1 \). In view of Lemma 4.2, the identity (4.1) is equivalent within \( R_n \) to the set \( \{ x^e \approx x^f, \; \delta e - e_0,0, \; \delta f - f_0,0 \} \), where \( e = \sum_{i=0}^{r} e_i \) and \( f = \sum_{i=0}^{r} f_i \). We see that the identity \( u \approx v \) is equivalent within \( R_n \) to some subset of \( \Phi \) in either case.

Theorem 1.1 is proved. \( \square \)

**Remark 1** Analysing the proof of Theorem 1.1, one can notice that if \( X \in \{ P_n, Q_n, R_n \} \), then each subvariety of \( X \) may be given within \( X \) by a finite set of identities. Consequently, set of all distributive subvarieties of \( A_{\text{cen}} \) is countably infinite.

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