Some Topological Invariants of Generalized Möbius Ladder

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Abstract. The Hosoya polynomial of a graph $G$ was introduced by H. Hosoya in 1988 as a counting polynomial, which actually counts the number of distances of paths of different lengths in $G$. The most interesting application of the Hosoya polynomial is that almost all distance-based graph invariants, which are used to predict physical, chemical and pharmacological properties of organic molecules, can be recovered from it. In this article we give the general closed form of the Hosoya polynomial of the generalized Möbius ladder $M(m, n)$ for arbitrary $m$ and for $n = 3$. Moreover, we recover Wiener, hyper Wiener, Tratch-Stankevitch-Zefirov, and Harary indices from it.

Subject Classification (2010). 05C12, 05C31

Keywords. Hosoya polynomial, Generalized Möbius ladder, Topological indices

1. Introduction

The Hosoya polynomial of a graph was introduced by H. Hosoya in 1988 as a counting polynomial; it actually counts the number of distances of paths of different lengths in the graph \cite{Hosoya1988}.

Hosoya polynomial is very well studied. In 1993, Gutman introduced Hosoya polynomial for a vertex of a graph \cite{Gutman1993}, which is related with Hosoya polynomial of the graph. The most interesting application of the Hosoya polynomial is that almost all distance-based graph invariants, which are used to predict physical, chemical and pharmacological properties of organic molecules, can be recovered from it.

Hosoya polynomial has been computed for several classes of graphs. In 2002 Diudea computed the Hosoya polynomial of several classes of toroidal nets
and recovered their Wiener indices \([15]\). In 2011 Ali found the Hosoya polynomial of concatenated pentagonal rings \([1]\). In 2012 Kishori gave a recursive method for calculating the Hosoya polynomial of Hanoi graphs, and computed some of their distance-based invariants \([12]\). In 2013 Farahi computed the Hosoya polynomial of polycyclic aromatic hydrocarbons \([6]\).

There are some useful topological indices that are related to Hosoya polynomial as Wiener, hyper Wiener, Tratch-Stankevitch-Zefirov (TSZ), and Harary indices. The Wiener index was introduced by Harry Wiener in 1947 and was used to correlate with boiling points of alkanes \([20]\). Later it was observed that the Wiener index can be used to determine a number of physico-chemical properties of alkanes as heats of formation, heats of vaporization, molar volumes, and molar refractions \([7]\). Moreover, it can be used to correlate those physico-chemical properties which depend on the volume-surface ratio of molecules and to Gas-chromatographic retention data for series of structurally related molecules. Another topological index whose mathematical properties are relatively well investigated is the hyper-Wiener index and was introduced by Randic in 1993 \([17]\). It is also used to predict physico-chemical properties of organic compounds, particularly to pharmacology, agriculture, and environment protection \([4]\); for more details, see also \([18, 14, 13, 5, 9]\).

In 1993 Plavsic et al. introduced a new topological index, known as Harary index, to characterize chemical graphs \([16]\). Tratch, Stankevitch, and Zefirov introduced Tratch-Stankevitch-Zefirov (TSZ) index as expanded Wiener index in 1990 \([19]\).

This article is organized as follows: Section 2 covers the basic definitions as of graph, distance, Hosoya polynomial, topological index, and generalized Möbius ladder; Section 3 contains the general closed forms of the Hosoya polynomial of generalized Möbius ladder for arbitrary \(m\) and \(n = 3\); Section 4 covers the topological indices of this ladder.

2. Basic Definitions

A graph \(G\) is a pair \((V, E)\), where \(V\) is the set of vertices and \(E\) is the set of edges. A path from a vertex \(v\) to a vertex \(w\) in a graph \(G\) is a sequence of vertices and edges that starts from \(v\) and stops at \(w\). The number of edges in a path is the length of that path. A graph is said to be connected if there is a path between any two of its vertices. The distance \(d(u, v)\) between two vertices \(u, v\) of a connected graph \(G\) is the length of a shortest path between them. The diameter of \(G\), denoted by \(d(G)\), is the longest distance in \(G\).
A molecular graph is a representation of a chemical compound in terms of graph theory. Specifically, molecular graph is a graph whose vertices correspond to (carbon) atoms of the compound and whose edges correspond to chemical bonds. For instance, Figure 2 represents the molecular graph of 1-bromopropyne ($CH_3 - C \equiv C - Br$).

**Definition 2.1.** The Hosoya polynomial in a variable $x$ of a molecular graph $G = (V, E)$ is defined as

$$H(G, x) = \sum_{\{v, u\} \in V} x^{d(u, v)} = \sum_{k=1}^{d(G)} d(G, k)x^k,$$

where $d(G, k)$ is the number of pairs of vertices of $G$ laying at distance $k$ from each other.

A function $I$ which assigns to every connected graph $G$ a unique number $I(G)$ is called a graph invariant. Instead of the function $I$ it is custom to say the number $I(G)$ as the invariant. An invariant of a molecular graph which can be used to determine structure-property or structure-activity correlation is called the topological index. A topological index is said to be distance-based if it depends on paths in the graph.

The following are definitions of some those distance-based indices that have connections with the Hosoya polynomial.

Let $u, v$ be arbitrary vertices of a connected graph $G = (V, E)$, and let $d(v, G)$ is the sum of distances of $v$ with all vertices of $G$. The Wiener index $W(G)$ of the graph $G$ is defined as

$$W(G) = \sum_{v < u; u, v \in V} d(v, u) = \frac{1}{2} \sum_{v \in V} d(v, G).$$

The Wiener index and the Hosoya polynomial are related by the equation

$$W(G) = \frac{d}{dx} H(G, x)|_{x=1}.$$

The hyper-Wiener index $WW(G)$ of a graph $G$ is defined as

$$WW(G) = \sum_{v < u; u, v \in V} d(v, u) = \frac{1}{2} \sum d(v, u)^2 + \frac{1}{2} \sum d(v, u).$$

The hyper-Wiener index and the Hosoya polynomial are related by the equation

$$WW(G) = \frac{1}{2} \frac{d^2}{dx^2} xH(G, x)|_{x=1}.$$
The Harary index $Ha(G)$ of a graph $G$ is defined as

$$Ha(G) = \sum_{i<j} \frac{1}{d(u_i, v_j)^2},$$

and is related to Hosoya polynomial by

$$Ha(G) = \int_0^1 \frac{H(G, x)}{x} dx.$$

The Tratch-Stankevitch-Zefirov index is also related to the Hosoya polynomial under the relation

$$TSZ(G) = \frac{1}{3!} \frac{d^3}{dx^3} x^2 H(G, x)|_{x=1}.$$

**Definition 2.2.** [10] Consider the Cartesian product $P_m \times P_n$ of paths $P_m$ and $P_n$ with vertices $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, u_n$, respectively. Take a $180^\circ$ twist and identify the vertices $(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_n)$ with the vertices $(u_m, v_n), (u_m, v_{n-1}), \ldots, (u_m, v_1)$, respectively, and identify the edge $((u_1, i), (u_1, i + 1))$ with the edge $((u_m, v_{n+1-i}), (u_m, v_{n-i}))$, where $1 \leq i \leq n - 1$. What we receive is the generalized Möbius ladder $M_{m,n}$.

You can see $M_{7,3}$ in the following figure.

![Figure 3: The grid form of the generalized Möbius ladder $M_{7,3}$](image)

The original form of $M_{7,3}$ is:

![Figure 4: The generalized Möbius ladder $M_{7,3}$](image)

### 3. Main Results

In this section we give the closed form of the Hosoya polynomial of the generalized Möbius ladder $M_{m,n}$ for $n = 3$. Actually, we give two results, one for even $m$ and other for odd $m$. 
Theorem 3.1. The Hosoya polynomial of the $M_{m,3}$, when $m \geq 6$ is even, is

$$H(M_{m,3}) = \sum_{k=1}^{m/2} c_k x^k,$$

where

$$c_k = \begin{cases} 5(m-1), & k = 1 \\ 8(m-1), & k = 2 \\ 9(m-1), & 2 < k < \frac{m}{2} \\ 8(m-1), & k = \frac{m}{2}. \end{cases}$$

Proof. In order to find the $k$th coefficient of the Hosoya polynomial we first give the upper triangular entries of the $3(m-1) \times 3(m-1)$ distance matrix $D = (d_{ij})$ corresponding to $M_{m,3}$ and then find the sum of all the entries $d_{ij} = k, j > i$, to find $c_k$.

Since $D$ is symmetric, we only need to know its upper-triangular part. For this we use only $m - 1$ matrices $B_i, 0 \leq q \leq m - 2$, each having order $3 \times 3$. Each $B_q$ appears $m - q - 1$ times on the $q$th secondary diagonal; by the 0th secondary diagonal we mean the main diagonal. These matrices are

$$B_0 = \begin{pmatrix} 0 & 1 & 2 \\ . & 0 & 1 \\ . & . & 0 \end{pmatrix}, \quad B_q = \begin{pmatrix} q-1 & q & q+1 \\ q & q-1 & q \\ q+1 & q & q-1 \end{pmatrix}, \quad B_{\frac{m}{2}} = \begin{pmatrix} m+2 & m+2 & \frac{m}{2} \\ \frac{m+2}{2} & \frac{m+2}{2} & \frac{m}{2} \\ \frac{m}{2} & \frac{m}{2} & \frac{m}{2} \end{pmatrix}, \quad \text{and} \quad B_q = \begin{pmatrix} m-q+2 & m-q+1 & m-q \\ m-q+1 & m-q & m-q+1 \\ m-q & m-q+1 & m-q+2 \end{pmatrix},$$

for $q = 1, 2, \ldots, \frac{m}{2}$.

We now give the coefficients $c_k$ of the polynomial. Depending on the special behavior, we give $c_1$, $c_2$, and $c_{\frac{m}{2}}$ one-by-one, and give all the remaining in a single general form:

$c_1$: The entry 1 appears only in $B_0$, $B_1$, and $B_{m-2}$. Since 1 appears in each $B_0$ twice and there are $m-1$ $B_0$s in $D$, the number of 1s in all $B_0$s is $2(m-1)$. Since 1 appears in each $B_1$ thrice and there are $m-2$ $B_1$s in $D$, the total number of 1s in $B_1$s is $3(m-2)$. Since 1 appears in each $B_{m-2}$ thrice and there is only one $B_{m-2}$ in $D$, the number of 1s in $B_{m-2}$ is 3. Hence $c_1 = 2(m-1) + 3(m-2) + 3 = 5(m-1)$.

$c_2$: The entry 2 appears in $B_0$, $B_1$, $B_2$, $B_{m-3}$, and $B_{m-2}$. Since 2 appears in each $B_0$ once and there are $m-1$ $B_0$s in $D$, the number of 2s in all $B_0$s is $m-1$. Since 2 appears in each $B_1$ four times and there are $m-2$ $B_1$s in $D$, the number of 2s in all $B_1$s is $4(m-2)$. Since 2 appears in each $B_2$ thrice and there are $m-3$ $B_2$s in $D$, the number of 2s in all $B_2$s is $3(m-3)$. Since 2 appears in each $B_{m-3}$ thrice and there are 2 $B_{m-3}s$ in $D$, the number of 2s in all $B_{m-3}s$ is 6. Since 2 appears in each $B_{m-2}$ four times and there is 1 $B_{m-2}s$ in $D$, the number of 2s in $B_{m-2}s$ is 4. Hence,
\( c_2 = (m - 1) + 4(m - 2) + 3(m - 3) + 2(3) + 4 = 8(m - 1). \)

\( c_{\frac{m}{2}}: \) The entry \( \frac{m}{2} \) appears 2 times in \( B_{\frac{m-4}{2}}, \) 2 times in \( B_{\frac{m+2}{2}}, \) 6 times in \( B_{\frac{m-2}{2}}, \) and 6 times in \( B_{\frac{m}{2}}. \) Since the matrices \( B_{\frac{m-4}{2}}, B_{\frac{m+2}{2}}, B_{\frac{m-2}{2}}, \) and \( B_{\frac{m}{2}} \) appear respectively \( \frac{m+2}{2}, \frac{m-4}{2}, \frac{m}{2}, \) and \( \frac{m-2}{2} \) times in \( D, \) we have \( c_{\frac{m}{2}} = 2\left(\frac{m+2}{2}\right) + 2\left(\frac{m-4}{2}\right) + 6\left(\frac{m}{2}\right) + 6\left(\frac{m-2}{2}\right) = 8(m - 1). \)

\( c_k, 3 \leq k \leq \frac{m-2}{2}: \) The entry \( k \) appears 2 times in \( B_{k-1}, \) 2 times in \( B_{m-k+2}, \) 4 times in \( B_k, \) 4 times in \( B_{m-k+1}, \) 3 times in \( B_{k+1}, \) and 3 times in \( B_{m-k}. \) The matrices \( B_{k-1}, B_{m-k+2}, B_k, B_{m-k+1}, B_{k+1}, \) and \( B_{m-k} \) appear respectively \( m - k + 1, k - 2, m - k, k - 1, m - k - 1, \) and \( k \) times in \( D. \) Hence, \( c_k = 2(m-k+1)+2(k-2)+4(m-k)+4(k-1)+3(m-k-1)+3k = 9(m-1), \) and we are done.

For better understanding, let us have a look at the example:

**Example.** Consider \( M_{10,3}: \)

![Figure 5: The grid form of M_{10,3}](image)

The block matrices are:

\[
B_0 = \begin{pmatrix} 0 & 1 & 2 \\ . & 0 & 1 \\ . & . & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}, \\
B_3 = \begin{pmatrix} 3 & 4 & 5 \\ 4 & 3 & 4 \\ 5 & 4 & 3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 4 & 5 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 4 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 5 & 5 & 4 \\ 5 & 4 & 5 \\ 4 & 5 & 5 \end{pmatrix}, \\
B_6 = \begin{pmatrix} 5 & 4 & 3 \\ 4 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}, \quad B_7 = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}, \quad B_8 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}.
\]

The distance matrix \( D \) in the form of these block matrices is

\[
D = \begin{pmatrix}
B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \\
. & B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \\
. & . & B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
. & . & . & B_0 & B_1 & B_2 & B_3 & B_4 & B_5 \\
. & . & . & . & B_0 & B_1 & B_2 & B_3 & B_4 \\
. & . & . & . & . & B_0 & B_1 & B_2 & B_3 \\
. & . & . & . & . & . & B_0 & B_1 & B_2 \\
. & . & . & . & . & . & . & B_0 & B_1 \\
. & . & . & . & . & . & . & . & B_0
\end{pmatrix}.
\]
It is now evident from $D$ that
\[
H(M_{10,3}) = c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5
= 5(m - 1)x^1 + 8(m - 1)x^2 + 9(m - 1)x^3 + 9(m - 1)x^4 + 8(m - 1)x^5
= 45x^1 + 72x^2 + 81x^3 + 81x^4 + 72x^5.
\]

For $m = 4$ we receive the special case:

**Proposition 3.2.** $H(M_{4,3}) = 15x + 21x^2$.

**Proof.** Since the distance matrix $D$ corresponding to $M_{4,3}$ is symmetric, we give only its upper-triangular part:

\[
D = \begin{pmatrix}
0 & 1 & 2 & 1 & 2 & 2 & 2 & 1 \\
0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\
0 & 2 & 2 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\
0 & 2 & 2 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\
0 & 2 & 2 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\
\end{pmatrix}
\]

It immediately follows that $c_1 = 15$ and $c_2 = 21$.

For odd $m$ we have the result:

**Theorem 3.3.** The Hosoya polynomial of $M_{m,3}$, when $m \geq 7$ is odd, is
\[
H(M_{m,3}) = \sum_{k=1}^{\frac{m+1}{2}} c_k x^k,
\]
where
\[
c_k = \begin{cases}
5(m - 1), & k = 1 \\
8(m - 1), & k = 2 \\
9(m - 1), & 2 < k < \frac{m-1}{2} \\
17\left(\frac{m-1}{2}\right), & k = \frac{m-1}{2} \\
4(m - 1), & k = \frac{m+1}{2}.
\end{cases}
\]

**Proof.** The block matrices, which were explained in Theorem 3.1 of the distance matrix of $M_{m,3}$, when $m \geq 7$ is odd, are:

\[
B_0 = \begin{pmatrix}
0 & 1 & 2 \\
. & 0 & 1 \\
. & . & 0
\end{pmatrix},
B_q = \begin{pmatrix}
q - 1 & q & q + 1 \\
q & q - 1 & q \\
q + 1 & q & -1
\end{pmatrix}
\]

for $q = 1, 2, \ldots, \frac{m-3}{2}$, $B_{\frac{m-1}{2}} = \begin{pmatrix}
\frac{m-3}{2} & \frac{m-1}{2} & \frac{m-3}{2} \\
\frac{m-1}{2} & \frac{m-3}{2} & \frac{m-1}{2} \\
\frac{m-3}{2} & \frac{m-1}{2} & \frac{m-3}{2}
\end{pmatrix}$,

and
\[
B_q = \begin{pmatrix}
m - q + 2 & m - q + 1 & m - q \\
m - q + 1 & m - q & m - q + 1 \\
m - q & m - q + 1 & m - q + 2
\end{pmatrix}
\]

for $q = \frac{m+1}{2}, \frac{m+3}{2}, \ldots, m - 1$.

The proofs $c_1$, $c_2$, and $c_k$, $2 < k < \frac{m-1}{2}$, are similar to the proofs given in Theorem 3.1.
We need to find $c_{m-1}$ and $c_{m+1}$. The entry $\frac{m-1}{2}$ appears 5 times in $B_{m-1}$, 4 times in $B_{m+1}$, 4 times in $B_{m-1}$, 2 times in $B_{m-5}$, and 2 times in $B_{m+3}$. Since $B_{m-1}, B_{m+1}, B_{m-3}, B_{m-5},$ and $B_{m+3}$ appear respectively $\frac{m-1}{2}, \frac{m-3}{2}, \frac{m+1}{2}, \frac{m+3}{2},$ and $\frac{m-5}{2}$. Hence, $c_{m-1} = 5\left(\frac{m-1}{2}\right) + 4\left(\frac{m-3}{2}\right) + 4\left(\frac{m+1}{2}\right) + 2\left(\frac{m+3}{2}\right) + 2\left(\frac{m-5}{2}\right) = \frac{17}{2}(m - 1)$.

Finally, we go for $c_{m+1}$. The entry $\frac{m+1}{2}$ appears 4 times in $B_{m+1}$, 2 times in $B_{m+1}$, and 2 times in $B_{m+3}$. Since $B_{m-1}, B_{m+1},$ and $B_{m+3}$ repeat respectively $\frac{m-1}{2}, \frac{m-3}{2},$ and $\frac{m+1}{2}$ times in $D$. Therefore, $c_{m+1} = 4\left(\frac{m-1}{2}\right) + 2\left(\frac{m-3}{2}\right) + 2\left(\frac{m+1}{2}\right) = 4(m - 1)$. This completes the proof. □

Here, again, we receive a special case:

**Proposition 3.4.** $H(M_{5,3}, x) = 20x + 30x^2 + 16x^3$.

**Proof.** Here $D$ is:

$$D = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 \\ 0 & 3 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 2 & \end{pmatrix}$$

It is now evident that $c_1 = 20$, $c_2 = 30$, and $c_3 = 16$. □

4. **Topological Indices**

In this section we give the distance-based topological indices, Wiener, hyper Wiener, Harary, and Tracht-Stankevitch-Zefirovof, of $M_{m,3}$ for both even and odd $m$.

**Proposition 4.1.** For even $m$ we get

1. $W(M_{m,3}) = \frac{1}{8}[9m^3 + 5m^2 - 62m + 48]$
2. $WW(M_{m,3}) = \frac{3}{16}m^4 + \frac{13}{16}m^3 + \frac{4}{4}m^2 - \frac{33}{4}m + 7$
3. $H_a(M_{m,3}) = 9m + 7 - \frac{16}{m} + 9(m - 1) \sum_{i=3}^{m-2} \frac{1}{i}$
4. $TSZ(M_{m,3}) = \frac{1}{16}m^4 + \frac{43}{24}m^2 + \frac{31}{4}m^3 + \frac{19}{3} - \frac{53}{6}m$

**Proof.** We prove these relations one by one using Theorem 3.1 and the relations given in Introduction section:
1.

\[
W(M_{m,3}) = \frac{d}{dx} [H(M_{m,3}, x)]_{x=1} \\
= \frac{d}{dx} [5(m-1)x + 8(m-1)x^2 + 8(m-1)x^{m \over 2} \\
+ \sum_{i=3}^{m-2} 9(m-1)x^i]_{x=1} \\
= 21m - 21 + 4m(m-1) + 9(m-1) \sum_{i=3}^{m-2} i \\
= \frac{1}{8} [9m^3 + 5m^2 - 62m + 48]
\]

2.

\[
WW(M_{m,3}) = \frac{1}{2} \frac{d^2}{dx^2} [xH(M_{m,3}, x)]_{x=1} \\
= \frac{1}{2} \frac{d^2}{dx^2} [5(m-1)x^2 + 8(m-1)x^3 + 8(m-1)x^{m \over 2} + 1} \\
+ \sum_{i=3}^{m-2} 9(m-1)x^{i+1}]_{x=1} \\
= \frac{1}{2} [10(m-1) + 48(m-1) + 4m(m-1)(m+2) \\
+ 9(m-1) \sum_{i=3}^{m-2} ii + 1] \\
= \frac{3}{16} m^4 + \frac{13}{16} m^3 + \frac{1}{4} m^2 - \frac{33}{4} m + 7
\]

3.

\[
H_a(M_{m,3}) = \int_0^1 \frac{H(M_{m,3}, x)}{x} dx \\
= \int_0^1 \frac{5(m-1)x + 8(m-1)x^2 + 8(m-1)x^{m \over 2} + \sum_{i=3}^{m-2} 9(m-1)x^i}{x} dx \\
= 9m + 7 - \frac{16}{m} + 9(m-1) \sum_{i=3}^{m-2} \frac{1}{i}
\]
4.

\[ TSZ(M_{m,3}) = \frac{1}{3!} \frac{d^3}{dx^3} [x^2 H(M_{m,3})]_{x=1} \]

\[ = \frac{1}{3!} \frac{d^3}{dx^3} [5(m-1)x^3 + 8(m-1)x^4 + \sum_{i=3}^{m-2} 9(m-1)x^{i+2} \]

\[ + 8(m-1)x^{m+2}]_{x=1} \]

\[ = \frac{1}{3!} [30(m-1) + 96(m-1) \]

\[ + \sum_{i=3}^{m-2} i(i+1)(i+2)9(m-1)] + 2(m+2)(m+4)(m-1) \]

\[ = \frac{1}{16} m^4 + \frac{43}{24} m^2 + \frac{31}{48} m^3 + \frac{19}{3} - \frac{53}{6} m \]

□

Proposition 4.2. For odd \( m \) we get

1. \( W(M_{m,3}) = \frac{1}{8} [9m^3 + 5m^2 - 53m + 39] \)
2. \( WW(M_{m,3}) = \frac{3}{16} m^4 + \frac{13}{16} m^3 + \frac{13}{16} m^2 - \frac{125}{16} m + 6 \)
3. \( H_a(M_{m,3}) = \frac{m(9m+25)}{m+1} + 9(m-1) \)

\[ \sum_{i=3}^{m-2} \frac{1}{i} \]

4. \( TSZ(M_{m,3}) = \frac{1}{16} m^4 + \frac{31}{48} m^3 + \frac{95}{48} m^2 - \frac{133}{16} m + \frac{45}{8} \)

Proof. The proof is similar to the proof of Proposition 4.1 □

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