On the Minimum Number of Colors for Links: Change of Behavior at p=11

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Abstract

This article concerns the minimum number of colors it takes to assemble a non-trivial coloring modulo a prime \( p \), i.e. a non-trivial \( p \)-coloring. Let us use the term \( p \)-Minimal Sufficient Set of Colors for a set with the minimum number of colors, which realizes such a coloring.

The work developed so far by several authors shows that for each prime \( p < 11 \) there is a \( p \)-Minimal Sufficient Set of Colors which depends on the prime \( p \) and not on the link at issue. Namely for each such prime \( p \) there is a unique positive integer \( m_p \) (the minimum number of colors modulo \( p \)), and a specific \( p \)-Minimal Sufficient Set of Colors, say \( \{c_1, c_2, \ldots, c_{m_p}\} \), such that, given any link admitting non-trivial \( p \)-colorings, there is a diagram of the link equipped with a non-trivial \( p \)-coloring using colors \( c_1, \ldots, c_{m_p} \).

In this article we show that this is no longer true for \( p = 11 \).

We also address obstructions to minimizing the number of colors and use this to calculate the minimum number of colors for a set of knots of determinant 11 and 13. Finally we propose directions for further work.

Keywords: links, colorings, equivalence classes of colorings, sufficient sets of colors, minimal sufficient sets of colors.

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1 Introduction

We begin by recalling the definition of a (Fox) coloring and how the existence of non-trivial colorings depends on the given modulus and on the determinant of the link.

Consider a link along with one of its diagrams. Regard the arcs of the diagram as algebraic variables and at each crossing read off the coloring condition: twice the “over-arc” minus the “under-arcs” equals zero, see Figure 1. Consider the system of linear homogeneous equations that we so associate to the link

\[
\begin{align*}
\begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} & \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} \\
\end{align*}
\]

\( a \) \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} \quad b \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} \quad c \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} \quad a - 2b - c
\]

Figure 1: The coloring condition (in an equivalent form), \( c = 2b - a \), relating the arcs that meet at a crossing.
Definition 1.1. (Minimum Number of Colors) Let $p$ be a positive integer, $L$ a link admitting non-trivial $p$-colorings, and $D$ a diagram of $L$. Let $c_{p,D}$ stand for the minimum number of colors it takes to assemble a diagram. Its solutions are the Fox colorings. There are always trivial solutions i.e., the so-called trivial colorings; each of these is obtained by assigning the same color (integer) to every arc. The existence of non-trivial colorings i.e., solutions which involve more than one color, requires further considerations.

Let us call the matrix of the coefficients of the aforementioned system of linear homogeneous equations, the coloring matrix of the diagram. When a Reidemeister move is performed on this diagram, a new diagram is obtained and with it a new system of linear homogeneous equations by associating with each crossing of the new diagram a coloring condition. It turns out that the coloring matrix of the new diagram is related to the coloring matrix of the old diagram via standard elementary transformations of matrices over the integers. These are the following.

- Multiplication of a row (column) by $-1$.
- Addition to one row (column) of integer linear combinations of other rows (columns).
- Insertion (deletion) of a row and column made up of 0's except for a 1 at the diagonal entry.
- Permutations of rows (columns).

Since different diagrams of the same link are related by a finite sequence of Reidemeister moves, then the equivalence class of the coloring matrix with respect to the elementary transformations of matrices is a topological invariant of this link. Furthermore, any coloring matrix is a matrix over the integers: along each row there is one 2, two $-1$'s and the rest is 0's. When we add each and every row but the last to the last row we obtain a column of zeros. It follows that the determinant of the coloring matrix is zero. Also because the coloring matrix is a matrix over the integers, there is an outstanding representative of its equivalence class, the Smith Normal Form (SNF). A matrix in SNF has the following features. It is diagonal. Down the diagonal suppose we have $d_1, d_2, \ldots, d_k, 0, 0, \ldots, 0$. Then $d_i | d_{i+1}$ and the multi-set \{\{d_1, d_2, \ldots, d_k, 0, 0, \ldots, 0\}\} is an invariant of the equivalence class of matrices. The $d_i$'s are called the invariant factors of this equivalence class. Since the coloring matrix has zero determinant and the determinant is an invariant over an equivalence class of matrices, then the SNF of our coloring matrix has at least one 0 along the diagonal. This 0 stands for the trivial colorings: one variable can assume any value but all the others are forced to assume specific values, if there are no more 0's along the diagonal. Going back to the original coloring matrix setting, this corresponds to all variables assuming the same value i.e., the trivial colorings. Again for a coloring matrix, the product of all the elements down the diagonal of its SNF except for that 0 is called the determinant of the link, notation $\det L$ for link $L$. It is known that for knots i.e., for 1-component links, the determinant of the knot is an odd integer (\[\mathbb{Z}\]). So in order to force non-trivial colorings, at least for knots, we have to work on a setting where at least one $d_i$ along the diagonal of the SNF becomes zero. This occurs if we work over the integers modulo a factor $m$ of one of the invariant factors, the $d_i$'s. Then at least one other variable is free to take on any value and so there are non-trivial colorings over the integers modulo $m$. Note that over a modulo which does not divide any of the $d_i$'s there are only trivial colorings. We end here this brief exposition of (Fox) colorings.

Remark Split links (which have null determinant) are non-trivially colored with two colors in any modulus. In general links of null determinant admit colorings in any modulus and we believe they deserve an independent study which we plan on doing in a separate article. For this reason all links referred to in this article are links of non-null determinant.

It follows that if a link admits non-trivial $p$-colorings for prime $p$, then $p$ divides one of the invariant factors of the equivalence class of the coloring matrix, thus it divides the determinant of the link. Now that we have a method for checking when a link $L$ admits non-trivial $p$-colorings, for a given prime $p$, the next topic is to know what is the minimum number of colors it takes to assemble a non-trivial $p$-coloring of $L$. This notion was first introduced in \[\mathbb{Z}\] and is the main concern of the current article. We remark that the moduli with respect to which we are considering the colorings in this article will always be prime (except where noted). This is due to a result stating that the minimum number of colors obtained for a link on a given modulus is also necessarily obtained on a certain prime divisor of this modulus (Lemma 1.6 in \[\mathbb{Z}\]).
a non-trivial $p$-coloring on $D$. We denote

$$\text{mincol}_p L := \min\{c_{p,D} \mid D \text{ is a diagram of } L\}$$

and call it the minimum number of colors for $L$, modulo $p$.

We remark that this is clearly a link invariant. We also remark how challenging it is to compute this invariant. In fact, by the definition, we have to consider each diagram of the link at issue in order to compute its minimum number of colors. But there are infinitely many diagrams for each link which makes this calculation an impossible task when working directly from the definition. This difficulty has been circumvented in the following way, up to prime modulus 7.

**Theorem 1.1.** (A, T, A, T, S) Let $L$ be a link with $\det L \neq 0$.

1. $\text{mincol}_2 L = 2$ iff $2 \mid \det L$;
2. $\text{mincol}_3 L = 3$ iff $3 \mid \det L$;
3. $\text{mincol}_5 L = 4$ iff $5 \mid \det L$;
4. $\text{mincol}_7 L = 4$ iff $7 \mid \det L$.

**Proof.** (sketch of) Since $\text{mincol}_m L > 1$ is equivalent to stating that $L$ admits non-trivial $m$-colorings and since for prime $m$ this is equivalent to $m$ dividing $\det L$ this takes care of the proof in one direction. Now for the converse.

1. If $2 \mid \det L$ then $L$ admits non-trivial 2-colorings and clearly $\text{mincol}_2 L = 2$.
   Furthermore if $\text{mincol}_m L = 2$, with colors $a \neq b$, then at a crossing where the colors meet the coloring condition reads $2(b - a) = 0$ which implies that $2 \mid m$ (to be used below in this proof).

2. If $3 \mid \det L$ then $L$ admits non-trivial 3-colorings. As we just saw if only two distinct colors meet at a crossing then the modulus is divisible by 2 so $\text{mincol}_3 L = 3$.
   Furthermore if $\text{mincol}_m L = 3$, then at a certain crossing the three distinct colors meet; the over-arc of this crossing has to become an under-arc at another crossing where again the three distinct colors meet for otherwise this over-arc would be part of a split component of the link, conflicting with $\det L \neq 0$. Conjuring the coloring conditions associated to these two crossings we obtain $3 \mid m$ (to be used below in this proof).

3. Since $2 \mid 5$ and $3 \mid 5$ then $\text{mincol}_5 L \geq 4$. On the other hand, it has been proved that any non-trivial 5-coloring can be accomplished with colors 1, 2, 3, 4 (12). Thus $\text{mincol}_5 L = 4$.

4. Finally, since $2 \mid 7$ and $3 \mid 7$, then $\text{mincol}_7 L \geq 4$. On the other hand, it has been proved that any non-trivial 5-coloring can be accomplished with colors 0, 1, 2, 4 (9). Thus $\text{mincol}_7 L = 4$.

This ends this sketch of proof; a more detailed analysis with stronger conclusions is found in [8].

We now come to the main issue of this article. Let us introduce some terminology in order to simplify statements.

**Definition 1.2.** (Sufficient Set of Colors; Minimal Sufficient Set of Colors) Let $m$ be a positive integer and let $L$ be a link admitting non-trivial $m$-colorings.

- A $m$-**Sufficient Set of Colors** (for $L$) is a set of integers mod $m$ such that a non-trivial $m$-coloring can be realized on a diagram of $L$ with colors from this set.
- A $m$-**Minimal Sufficient Set of Colors** (for $L$) is a $m$-Sufficient Set of Colors (for $L$) whose cardinality is $\text{mincol}_m L$.

**Definition 1.3.** (Common Minimal Sufficient Set of Colors) Suppose $m$ is a positive integer such that for any two links, $L$ and $L'$, admitting non-trivial $m$-colorings, $\text{mincol}_m L = \text{mincol}_m L'$.

In these circumstances, a Common $m$-**Minimal Sufficient Set of Colors** is a set of $\text{mincol}_m L$ integers mod $m$ such that for any link $L''$ admitting non-trivial $m$-colorings, such a coloring can be realized on a diagram of $L''$ with colors from this set.
We will drop the \( m \)- from these designations when it is clear from context which modulus we are working on.

**Corollary 1.1.** *(Corollary to Theorem 1.1)* We keep the hypothesis of Theorem 1.1.

For each prime \( p \leq 11 \), there is a Common \( p \)-Minimal Sufficient Set of Colors.

**Proof.** We refer to cases 1. - 4. in the statement of the Theorem 1.1 in each case indicating which set can be a Common \( p \)-Minimal Sufficient Set of Colors, leaning on the proof of the Theorem.

1. \( \{0,1,\} \), 2. \( \{0,1,2,\} \), 3. \( \{1,2,3,4,\} \), 4. \( \{0,1,2,4,\} \).

There is no dependence on the link at issue in each of these cases thereby concluding the proof. \( \square \)

The main result of this article is the following.

**Theorem 1.2.** There is no Common 11-Minimal Sufficient Set of Colors.

The proof of Theorem 1.2 is found in Section 3. We also prove that certain sets of colors cannot be Sufficient Sets of Colors. This is helpful when choosing the next color to eliminate in the process of lowering the number of colors of a non-trivial coloring.

**Theorem 1.3.** Let \( k \) be a positive integer and \( L \) a link with non-null determinant, admitting non-trivial \((2k + 1)\)-colorings.

If \( S \subseteq \{0,1,2,\cdots, k\}, \mod 2k + 1 \), then \( S \) is not a \((2k + 1)\)-Sufficient Set of Colors for \( L \).

The proof of Theorem 1.3 is found in Section 2 along with a proof of Corollary 2.1 and of Theorem 2.1 which concern also obstructions to minimizing the sets of colors. These results are applied in Sections 3 and 4 to obtain the minimum numbers of colors of certain knots of prime determinant 11 and 13 and Minimal Sufficient Sets of Colors for these knots. This is summarized in Tables 1.1 and 1.2. In Section

| \( L \) | \( \text{mincol}_{11} L \) | 11-Minimal Sufficient Set of Colors for \( L \) |
|---|---|---|
| 6_2 * | 5 | \( \{0,2,3,4,8\} \) (Fig. 2) |
| 7_2 | 5 | \( \{0,3,4,5,6\} \) (Fig. 3) |
| 10_125 * | 5 | \( \{0,1,5,8,10\} \) (Figs. 5, 15 and 16) |
| 10_152 * | 5 | \( \{0,2,3,4,8\} \) (Fig. 7) |
| \( T(2,11) \) | 5 | \( \{0,1,2,3,6\} \) (Fig. 3) |

Table 1.1: \( \text{mincol}_{11} L \) for the knots of prime determinant 11 from Rolfsen’s tables ([10]) and for \( T(2,11) \) ([5]). Knots with a * after their designation admit the same 11-Minimal Sufficient Set of Colors; the same for knots without * (see Sections 3 and 4). A knot with a * and a knot without a * do not admit the same 11-Minimal Sufficient Set of Colors (see Section 3).

| \( L \) | \( \text{mincol}_{13} L \) | 13-Minimal Sufficient Set of Colors for \( L \) |
|---|---|---|
| 6_3 * | 5 | \( \{0,1,2,6,11\} \) (Fig. 8) |
| 7_3 * | 5 | \( \{0,2,3,4,9\} \) (Figs. 9 and 10) |
| 8_1 | \( \leq 6 \) | \( \subseteq \{0,1,2,3,4,7\} \) ? (Fig. 13 and 14) |
| 9_{43} | \( \leq 6 \) | \( \subseteq \{0,1,3,6,8,12\} \) ? (Figs. 15 through 19) |
| 10_{154} * | 5 | \( \{0,3,4,5,10\} \) (Figs. 11 and 12) |
| \( T(2,13) \) | \( \leq 6 \) | \( \subseteq \{0,1,2,3,4,7\} \) ? (Fig. 3, Figs. 4, 5, and 6) |

Table 1.2: \( \text{mincol}_{13} L \) for the knots of prime determinant 13 from Rolfsen’s tables ([10]) and for \( T(2,13) \) ([5]). Knots with a * after their designation admit the same 13-Minimal Sufficient Set of Colors (see Section 4). For each of the other knots either the \( \text{mincol}_{13} L \) is 6 and the 13-Minimal Sufficient Set of Colors is the set right before the ?; or the \( \text{mincol}_{13} L \) is 5 and the 13-Minimal Sufficient Set of Colors may bear no relationship to the set right before the ? (see Section 4).

We raise questions concerning the possible new behavior of the minimum number of colors for larger primes and we present a procedure for reducing the number of colors of a non-trivial coloring.
A boxed integer at the top left of a Figure indicates the module with respect to which the colorings in the Figure are being considered.

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2 Obstructions to Minimizing the Set of Colors of a Coloring

Proof. (of Theorem 1.3)

Assume to the contrary and suppose there is a diagram of $L$ endowed with a non-trivial $(2k + 1)$-coloring whose sequence of representatives of the distinct colors is $(c_i) \subseteq \{0, 1, 2, \ldots, k\}$. Then

\[0 \leq 2c_{i_1} \leq 2k\quad \text{and} \quad 0 \leq c_{i_2} + c_{i_3} \leq 2k\quad \text{for any } c_{i_1}, c_{i_2}, c_{i_3} \in \{0, 1, 2, \ldots, k\}.

Let us then consider the triples $(c_{i_1}, c_{i_2}, c_{i_3})$ for which the coloring condition holds at a crossing of the diagram i.e. for which

\[2c_{i_1} = c_{i_2} + c_{i_3} \mod 2k + 1.

Since each side of these equalities takes on a value between 0 and $2k$ then each of these equalities also holds over the integers.

This further implies that these coloring conditions hold for any prime $p > 2k + 1$ which means that the link $L$ has non-trivial colorings modulo infinitely many primes. But this conflicts with the standing hypothesis that $\det L \neq 0$, thus completing the proof.

We will now obtain a Corollary for Theorem 1.3 (Corollary 2.1), after Definition 2.1, and some facts, Propositions 2.1 and 2.2.

Definition 2.1. Let $m$ be a positive integer. A permutation, $f$, of the integers mod $m$ i.e., of the set $\mathbb{Z}/m\mathbb{Z}$, is called an $(m)$-coloring automorphism if for any two integers $a$ and $b$, the coloring condition is preserved i.e.,

\[f(a \ast b) = f(a) \ast f(b) \mod m\]

where $a \ast b := 2b - a$.

Proposition 2.1. Coloring automorphisms preserve colorings.

Specifically, let $D$ be a link diagram with arcs $a_1, \ldots, a_n$ as we travel along the diagram. Let $f$ be an $m$-coloring automorphism.

If $D$ admits an $m$-such that for each $i$, arc $a_i$ receives color $c_i$, then the assignment arc $a_i$ receives color $f(c_i)$, for each $i$, is also an $m$-coloring of $D$.

Proof. We keep the notation and terminology of the statement with the following addition. We let $a_{i_j}, \in \{a_1, \ldots, a_n\}$ denote the over-arc at the crossing where arcs $a_i$ and $a_{i+1}$ meet.

Since the assignment arc $a_i$ receives color $c_i$, for each $i$, is an $m$-coloring, then $c_{i+1} = 2c_{i_j} - c_i$ (mod $m$) for each $i$. Thus

\[f(c_{i+1}) = f(2c_{i_j} - c_i) = 2f(c_{i_j}) - f(c_i) \mod m\]

for each $i$ i.e., the assignment arc $a_i$ receives color $f(c_i)$ for each $i$ is also an $m$-coloring of $D$. This completes the proof.

Proposition 2.2. (\[2\]) Given a positive integer $m$, an $m$-coloring automorphism is of the form:

\[f_{\lambda, \mu}(x) = \lambda x + \mu\]

where $\lambda$ is any element from the set of units of the integers mod $m$ and $\mu$ is any integer mod $m$. Furthermore, the set of $m$-coloring automorphisms equipped with composition of functions for binary operation constitutes a group.

Proof. The proof can be found in \[2\].
Corollary 2.1 (Corollary to Theorem 1.3). Let $k$ be a positive integer. Let $f$ be a $(2k+1)$-coloring automorphism. Let $L$ be a link admitting non-trivial $(2k+1)$-colorings.

A set $S \subseteq \{f(0), f(1), \ldots, f(k)\}$ cannot be a $(2k+1)$-Sufficient Set of Colors for $L$.

Proof. Assume to the contrary and suppose there is a non-trivial $(2k+1)$-coloring consisting of colors from the set:

\[ \{f(0), f(1), f(2), \cdots, f(k)\}. \]

Let $g$ be the inverse permutation to $f$. Then colors from the set

\[ \{g(f(0)), g(f(1)), g(f(2)), \cdots, g(f(k))\} = \{0, 1, 2, \cdots, k\} \]

also give rise to a non-trivial $(2k+1)$-coloring which contradicts Theorem 1.3. The proof is complete. \qed

2.1 Further obstructions: formalizing an argument of Kauffman (5) and of Saito (11).

The relevance of Theorem 1.3 and Corollary 2.1 is the following. A possible way of reducing the number of colors is to start from a diagram of the link under study endowed with a non-trivial $m$-coloring and via Reidemeister moves followed by local reassignment of colors, obtain a new diagram of the same link endowed with a non-trivial $m$-coloring using a proper subset of the set of colors used by the former coloring. Theorem 1.3 and Corollary 2.1 may help recognizing which colors of the former coloring may be eliminated. These results indicate which colors cannot be removed; the remaining ones are the candidates to colors to be eliminated.

Theorem 2.1 provides us with a subsequent test on these candidates.

Theorem 2.1. Let $k$ be a positive integer. Let $L$ be a link of non-null determinant and admitting non-trivial $(2k+1)$-colorings.

Suppose $S = \{c_1, \ldots, c_n\}$ is a $(2k+1)$-Sufficient Set of Colors for $L$. (WLOG we take the $c_i$’s from $\{0,1,2,\ldots,2k\}$.) For each $c_i$ ($i = 1, 2, \ldots, n$), let $S_i$ be the set of unordered pairs $\{c_1, c_2\}$ (with $c_1 \neq c_2$ from $S$) such that $2c_i = c_1^2 + c_2^2 \mod 2k+1$.

1. There is at least one $i$ such that the expression $2c_i = c_1^2 + c_2^2$ does not make sense over the integers.

2. If there is $i \in \{1, 2, \ldots, n\}$ such that for all $j \in \{1, 2, \ldots, n\}$, $c_i$ does not belong to any of the unordered pairs in $S_j$, then $n > \text{mincol}_p(L)$ and $S \setminus \{c_i\}$ is also a $(2k+1)$-Sufficient Set of Colors.

Proof. 1. If for all $i$’s each of the expressions $2c_i = c_1^2 + c_2^2$ makes sense without considering it mod $2k+1$, then we have colorings whose coloring conditions hold modulo any prime. But this conflicts with the det $L \neq 0$ hypothesis which implies that det $L$ admits only a finite number of prime divisors.

2. Since $c_i$ cannot be the color of an under-arc at a polychromatic crossing, then this implies the existence of a split component of $L$ colored with $c_i$ alone, which conflicts with the hypothesis.

The proof is complete. \qed

3 Proof of Theorem 1.2

We begin with some background material. It was proved in [3] that given a link diagram admitting non-trivial $p$-colorings, its set of non-trivial $p$-colorings admits an equivalence relation according to the following definition.

Definition 3.1. (3) Let $p$ be a prime. Let $L$ be a link admitting non-trivial $p$-colorings and $D$ be one of its diagrams. Let $C$ and $C'$ be non-trivial $p$-colorings of $D$ i.e., $C$ and $C'$ are mappings from the arcs of $D$ to $\mathbb{Z}/p\mathbb{Z}$ such that if $a$, $b$, and $c$ are the arcs as in Figure 4 then $2C(b) - C(a) = C(c)$ and $2C'(b) - C'(a) = C'(c)$ (note that here we let $a,b,$ and $c$ stand for the arcs). Then $C$ is related to $C'$ if, by definition, there is a $p$-coloring automorphism, $f$, such that,

$$C'(a) = f(C(a))$$

for any arc $a$ of the diagram $D$. 

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It was further proved in [3] that this equivalence relation complies with Reidemeister moves followed by reassignment of colors. This is to say that if two non-trivial $p$-colorings are related on a diagram $D$ and if this diagram undergoes a Reidemeister move, then the unique assignment of colors to new arcs and deletion of colors from deleted arcs produces $p$-colorings in the new diagram from the $p$-colorings on diagram $D$ such that the two $p$-colorings on the new diagram are also related. In this way the following Theorem is obtained.

**Theorem 3.1.** ([3]) The equivalence relation presented in Definition 3 is a topological invariant. In particular, the number of equivalence classes is a topological invariant. For a given prime $p$ and for a link $L$ admitting non-trivial $p$-colorings, the number of equivalence classes is 

$$\frac{p^{n-1} - 1}{p - 1}$$

where $n$ is the number of 0’s, modulo $p$, in the diagonal of the SNF of any coloring matrix of $L$.

Hence, if a link has prime determinant, $p$, then there is only one equivalence class of non-trivial $p$-colorings for this link. Thus, any two non-trivial coloring of this link realized over the same diagram are related (by a $p$-coloring automorphism). Now suppose $D_0$ is a diagram of this link, $L$, of prime determinant $p$, which is equipped with a $p$-coloring using $\text{mincol}_p L$ colors. Then any other non-trivial $p$-coloring on $D_0$ is related to the former $p$-coloring by a $p$-coloring automorphism. Moreover, both these $p$-colorings use $\text{mincol}_p L$ colors. If for each prime $p$ there is a common $p$-Minimal Sufficient Set of Colors then the following is true. For any two links, $L$ and $L'$, of prime determinant $p$, with $\text{mincol}_p L = \text{mincol}_p L'$, and such that colors $c_1, \ldots, c_n, (c'_1, \ldots, c'_m$, respect.) realize a non-trivial $p$-coloring of $L (L'$, respect.), there is a $p$-coloring automorphism, $f$, such that $c'_i = f(c_i)$, for all $i = 1, \ldots, m$. Otherwise, there would not be a common set of colors with respect to which both links admit a non-trivial $p$-coloring using $m_p$ colors i.e., there would not be a Common $p$-Minimal Sufficient Set of Colors. We are now ready to prove Theorem 1.2.

**Proof.** (of Theorem 1.2) The strategy is the following. We consider two specific knots of prime determinant 11, say $K$, and $K'$. We show that $\text{mincol}_{11} K = 5 = \text{mincol}_{11} K'$ (which is the minimum number of colors since for prime modulus $p > 7$ the least number of colors is greater then 4, see [11] and [8]) and exhibit 11-Minimal Sufficient Sets of Colors for both $K$ and $K'$. Finally, we prove that there is no 11-coloring automorphism that maps one of these 11-Minimal Sufficient Sets of Colors onto the other. According to the preceding discussion this proves that there is no Common 11-Minimal Sufficient Set of Colors. That knots $K$ and $K'$ with the indicated features exist is the content of Proposition 3.1 below. This completes the proof of Theorem 1.2.

**Proposition 3.1.** Knots $6_2$ and $7_2$ are such that $\det 6_2 = 11 = \det 7_2$ and $\text{mincol}_{11} 6_2 = 5 = \text{mincol}_{11} 7_2$. But an 11-Minimal Sufficient Set of Colors for $6_2$ can never be an 11-Minimal Sufficient Set of Colors for $7_2$ (and conversely).

**Proof.** The determinant of both $6_2$ and $7_2$ is 11 (see [10]).

In Figures 2 and 3 we show that both $6_2$ and $7_2$ admit non-trivial 11-colorings using only 5 colors. This is their least number of colors modulo 11 since for prime $p > 7$ the least number of colors needed to assemble a non-trivial $p$-coloring has to be greater than 4 ([8] [11]). Note the relevance of Theorem 1.3 and of Corollary 2.1 in deciding which colors to eliminate. For example, 8 could not have been eliminated in Figure 2.

In Table 3.1 we show that there is no 11-coloring automorphism that maps the 5 colors used by $6_2$ ($\{0, 2, 3, 4, 8\}$, see Figure 2) to the 5 colors used by $7_2$ ($\{0, 3, 4, 5, 6\}$, see Figure 3). We reason as follows. If there was such an automorphism, call it $f$, then $f(0)$ and $f(2)$ would be distinct elements of $\{0, 3, 4, 5, 6\}$. We then explore the different possibilities for these values $f(0)$ and $f(2)$ can take on and check that none of them gives rise to an automorphism that maps $\{0, 2, 3, 4, 8\}$ onto $\{0, 3, 4, 5, 6\}$. For instance, in the first row of Table 3.1 we assume $f(0) = 0, f(2) = 3$. Since $f(x) = \lambda x + \mu$ then $0 = 0\lambda + \mu, 3 = 2\lambda + \mu$ so $\lambda = 7, \mu = 0$. Then $f(3) = 3 \cdot 7 + 0 = 21 = 11 \cdot 10$ (which does not belong to $\{0, 3, 4, 5, 6\}$ and this is indicated by the X), $f(4) = 4 \cdot 7 + 0 = 28 = 11 \cdot 2$, and $f(8) = 8 \cdot 7 + 0 = 56 = 11 \cdot 5$ (which does not belong to $\{0, 3, 4, 5, 6\}$ and this is indicated by the X). The calculations leading to the expressions in other rows of Table 3.1 are performed analogously. In each row we see that there is always at least one element of $\{0, 3, 4, 5, 6\}$ which does not belong to $f(\{0, 2, 3, 4, 8\})$. This completes the proof.
Figure 2: The knot $6_2$ whose determinant is 11. On the left-hand side, a diagram with minimum number of crossings equipped with a non-trivial 11-coloring; the dotted lines indicate the move that will take to the diagram on the right-hand side. On the right-hand side a diagram equipped with a non-trivial 11-coloring using a minimum number of colors modulo 11: $\{0, 2, 3, 4, 8\}$.

Figure 3: The knot $7_2$ whose determinant is 11. On the left-hand side, a diagram with minimum number of crossings equipped with a non-trivial 11-coloring; the dotted lines indicate the move that will take to the diagram on the right-hand side. On the right-hand side a diagram equipped with a non-trivial 11-coloring using a minimum number of colors modulo 11: $\{0, 3, 4, 5, 6\}$.

We remark that $f(\{0, 1, 2, 3, 6\}) = \{0, 3, 4, 5, 6\}$ with $f(x) = 11 - x + 6$ which allows to state in the caption to Table 1.1 that $7_2$ and $T(2, 11)$ have the same 11-Minimal Sufficient Set of Colors.
### 4 Removing Colors From Non-Trivial 11-Colorings (Knots 10_{125} and 10_{152}) and From Non-Trivial 13-Colorings (Knots 6_{3}, 7_{3}, 8_{1}, 9_{43}, and 10_{154}).

In this Section we apply again the obstruction results of Section 2: Theorem 1.3 and Corollary 2.1 and Theorem 2.1. Among other things we obtain the following results.

\[ 5 = \text{mincol}_{11}10_{125} = \text{mincol}_{11}10_{152} \quad (11 = \det 10_{125} = \det 10_{152}). \]

\[ 5 = \text{mincol}_{13}6_{3} = \text{mincol}_{13}7_{3} = \text{mincol}_{13}10_{154} \quad (13 = \det 6_{3} = \det 10_{125} = \det 10_{152}). \]

\[ 6 \geq \text{mincol}_{13}8_{1}, \quad 6 \geq \text{mincol}_{13}9_{43} \quad (13 = \det 6_{3} = \det 10_{125} = \det 10_{152}). \]

**Figure 4:** The knot 10_{125} whose determinant is 11. The dotted lines indicate the move that will take to the next diagram.
Figure 5: The knot $10_{125}$ whose determinant is 11. Continued from Figure 4. The regions boxed with dotted lines are treated in Figure 6.

Figure 6: The knot $10_{125}$ whose determinant is 11, conclusion. The circle around the 3 tells us it still has to be removed; this is done below in this Figure. With this we accomplish coloring $10_{125}$ mod 11 with 5 colors, $\{0, 1, 5, 8, 10\}$. We refrain from incorporating the adjustments in this figure in Figure 5 in order not to over-burden things. We note that the 11-coloring automorphism $f(x) = 8x$ maps $\{0, 1, 2, 4, 7\}$ onto $\{0, 1, 5, 8, 10\}$.

Figure 7: The knot $10_{152}$ whose determinant is 11. The dotted lines indicate the move that will take to the next diagram. A non-trivial 11-coloring of $10_{152}$ is obtained with colors $\{0, 2, 3, 4, 8\}$. The 11-coloring automorphism $f(x) = 2x$ maps the set $\{0, 1, 2, 4, 7\}$ to the set $\{0, 2, 3, 4, 8\}$.
Figure 8: The knot 6₃ whose determinant is 13. On the left-hand side, a diagram with minimum number of crossings equipped with a non-trivial 13-coloring; the dotted lines indicate the move that will take to the diagram on the right-hand side. On the right-hand side a diagram equipped with a non-trivial 13-coloring using a minimum number of colors modulo 13: \{0, 1, 2, 6, 11\}.

Figure 9: The knot 7₃ whose determinant is 13. On the left-hand side, a diagram with minimum number of crossings equipped with a non-trivial 13-coloring; the dotted lines indicate the move that will take to the next diagram.

Figure 10: The knot 7₃ whose determinant is 13, conclusion. A minimum number of 5 colors is needed to obtain a non-trivial 13-coloring of 7₃ here with colors \{0, 2, 3, 4, 9\}. Also, the 13-coloring automorphism \(f(x) = 6x + 3\) maps the set of colors \{0, 1, 2, 6, 11\} to the set \{0, 2, 3, 4, 9\}.

4.1 Further discussion on the 13-Sufficient Sets of Colors \{0, 1, 3, 6, 8, 12\} for knot \(9_{43}\) and \{0, 1, 2, 3, 4, 7\} for knots \(8_1\) and \(T(2, 13)\).

Let us consider \(8_1\). In Figure 14 we show it can be colored modulo 13 with 6 colors, \{0, 1, 2, 3, 4, 7\}. This was obtained by experimenting. It is conceivable that \(8_1\) may be colored with 5 colors. In this case and starting from the set \{0, 1, 2, 3, 4, 7\}, which color could be removed? We begin by applying Theorem 1.3 and/or Corollary 2.1. Color 7 could not be removed for otherwise the remaining set would clearly be in conflict with Theorem 1.3 since all the remaining colors are less than 6. Color 0 cannot be removed also since otherwise upon subtraction by 1, a set of colors each of which is less than 6 is obtained. Analogously, colors 2 and 3 cannot be simultaneously removed from the original set for otherwise multiplication of the remaining colors by 4 (modulo 13) yields a set each of whose colors are less than 6. Thus, a potential
Figure 11: The knot 10\textsubscript{154} whose determinant is 13. A minimum number of 5 colors is needed to obtain a non-trivial 13-coloring of 10\textsubscript{154}. In Figure 12 we show how to remove color 2 thus showing 10\textsubscript{154} can be colored with a minimum of 5 colors mod 13, \{0, 3, 4, 5, 10\}.

Figure 12: The knot 10\textsubscript{154} whose determinant is 13, conclusion. The circle around the 7 tells us it still has to be removed; this is done below in this figure. With this we accomplish coloring 10\textsubscript{154} mod 13 with 5 colors, \{0, 3, 4, 5, 10\}. We refrain from incorporating the adjustments in this figure in Figure 11 in order not to over-burden things. Also, the 13-coloring automorphism \(f(x) = 5x\) maps the set of colors \{0, 1, 2, 6, 11\} to the set \{0, 3, 4, 5, 10\}.

Figure 13: The knot 8\textsubscript{1} whose determinant is 13. We believe that at this point no more comments are necessary.

candidate set of 5 colors obtained from \{0, 1, 2, 3, 4, 7\} has to have colors 0, 7 and at least one of 2 or 3. The possibilities are then

\[
\{0, 2, 3, 4, 7\}, \quad \{0, 1, 2, 3, 7\}, \quad \{0, 1, 3, 4, 7\}, \quad \{0, 1, 2, 4, 7\}.
\]

Now we apply Theorem 2.1. Let us first consider the set \{0, 2, 3, 4, 7\} and let us see which polychromatic crossings it supports modulo 13. With this set of colors there is no polychromatic crossing with 0 on its over-arc; if 2 is the color of an over-arc at a crossing then the under-arcs can only be colored with 0 and
Figure 14: The knot 8\_1 whose determinant is 13, conclusion. We managed to eliminate colors until we were left with the 6 colors of the set \{0, 1, 2, 3, 4, 7\}. Modulo 13 the minimum number of colors cannot be less than 5.

Figure 15: The knot 9\_43 whose determinant is 13. We believe that at this point no more comments are necessary.

Figure 16: The knot 9\_43 whose determinant is 13. The boxed regions involve color 5; there are three distinct cases. In Figures 17, 18, and 19 we show how to eliminate color 5 from each of them.

4; if 3 is the color of an over-arc at a crossing then the under-arcs can only be colored with 2 and 4; there is also no polychromatic crossing with 4 or with 7 on its over-arc, with this set of colors. Thus each of the coloring conditions supported by this set holds over the integers, which conflicts with Theorem 2.1. Hence this set cannot be a 13-Minimal Sufficient Set of Colors for 8\_1.

Let us now consider the set \{0, 1, 2, 3, 7\}. It does not support polychromatic crossings with 0 or 3 on the over-arc; if 1 is the color of an over-arc at a crossing then the under-arcs bear colors 0 and 2; if 2
is the color of an over-arc at a crossing then the under-arcs bear colors 1 and 3; if 7 is the color of an
over-arc at a crossing then the under-arcs bear colors 0 and 1. Since 7 is never a color of an under-arc at
a polichromatics crossing, there is a conflict with Theorem 2.1. Hence this set cannot be a 13-Minimal
Sufficient Set of Colors for $8_1$.

For the set $\{0, 1, 3, 4, 7\}$ we see that 3 and 4 are never the colors of under-arcs at polichromatic
crossings. Hence this set cannot be a 13-Minimal Sufficient Set of Colors for $8_1$.

Finally an analogous analysis on the set $\{0, 1, 2, 4, 7\}$ does not show any obstruction for it to be a
13-Minimal Sufficient Set of Colors for $8_1$. We were not able however to obtain a non-trivial 13-coloring
of $8_1$ by eliminating color 3 from the diagram in Figure 14. Furthermore, since $f(x) = 4x + 11 \text{ mod } 13
is such that $f(\{0, 1, 2, 4, 7\}) = \{0, 1, 2, 6, 11\}$, then should $\{0, 1, 2, 4, 7\}$ be a 13-Minimal Sufficient Set of Colors for $8_1$ then $8_1$ and $6_3$ (and 7_3 and 10_154) share the same 13-Minimal Sufficient Set of Colors.

An analogous analysis on the 13-Sufficient Set of Colors for $9_{43}$, $\{0, 1, 3, 6, 8, 12\}$, shows that $\{0, 1, 3, 8, 12\}$,
$\{0, 1, 3, 6, 12\}$, and $\{0, 3, 6, 8, 12\}$ are candidates to 13-Minimal Sufficient Sets of Colors for knot $9_{43}$, after
application of Theorem 1.3 and Corollary 2.1. Further screening using Theorem 2.1 still confirms these
as candidates to 13-Minimal Sufficient Sets of Colors for knot $9_{43}$. We were not able however to further

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Figure 17: The knot $9_{43}$ whose determinant is 13. Taking care of the 5 in one of the boxed regions in
Figure 16; first case.

Figure 18: The knot $9_{43}$ whose determinant is 13. Taking care of the 5 in one of the boxed regions in
Figure 16; second case.

Figure 19: The knot $9_{43}$ whose determinant is 13. Taking care of the 5 in one of the boxed regions in
Figure 16; third case. We refrain from including these treatments in the diagram of Figure 16 in order
not to overburden it. This concludes our attempt at minimizing the number of colors for a non-trivial
13-coloring of $9_{43}$: we did it with 6 colors, $\{0, 1, 3, 6, 8, 12\}$. 

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Figure 17: The knot 9

Figure 18: The knot 9

Figure 19: The knot 9
eliminate colors from \{0, 1, 3, 6, 8, 12\} on non-trivial 13-colorings of 9_{43}. Furthermore, since

\begin{align*}
  f(\{0, 1, 3, 6, 8, 12\}) &= \{0, 1, 2, 6, 11\} \quad \text{with } f(x) = 12x + 1 \mod 13, \\
  f(\{0, 1, 3, 6, 12\}) &= \{0, 1, 2, 6, 11\} \quad \text{with } f(x) = 4x + 2 \mod 13, \\
  f(\{0, 3, 6, 8, 12\}) &= \{0, 1, 2, 6, 11\} \quad \text{with } f(x) = 10x + 11 \mod 13,
\end{align*}

then if any of these subsets is a 13-Minimal Sufficient Set of Colors for 9_{43}, then 9_{43} and 6_3 (and 7_3 and 10_{154}) share the same 13-Minimal Sufficient Set of Colors.

5 Directions for Future Work

We distinguish three main topics to be looked into: 1. Common Minimal Sufficient Set of Colors; 2. Minimum number of colors; 3. Procedure for reducing the number of colors. We suggest the domain of applicability be links of prime determinants or at most links with only one equivalence class of colorings for the prime \( p \) at issue (see statement of Theorem 3.1).

1. (Common Minimal Sufficient Set of Colors)

Let \( p \) be a prime. As we have shown in Theorem 1.2 there is a change of behavior of the minimum number of colors at \( p = 11 \). For each prime \( p < 11 \) there is a Common \( p \)-Minimal Sufficient Set of Colors for any link admitting non-trivial \( p \)-colorings. At \( p = 11 \) no such phenomenon occurs.

Should this scenario persist for larger primes i.e., the lack of Common \( p \)-Minimal Sufficient Set of Colors for some/all primes \( p > 11 \), we believe the following objects are worth looking into. In any case they already are worth looking into for \( p = 11 \).

**Definition 5.1.** (\( p \)-Normal Set of Colors) Let \( p \) be a positive integer. We define \( p \)-Normal Set of Colors to be a \( p \)-Sufficient Set of Colors which is minimal with respect to the following property. For any link \( L \) admitting non-trivial \( p \)-colorings there is a diagram of \( L \) which can be equipped with a non-trivial \( p \)-coloring using the colors from this set alone.

How many \( p \)-Normal Sets of Colors are there per \( p \)? If there is more than one, are their cardinalities equal?

**Definition 5.2.** (\( p \)-Super Normal Set of Colors) Let \( p \) be a positive integer. We define \( p \)-Super Normal Set of Colors to be a \( p \)-Sufficient Set of Colors which is minimal with respect to the following property. For any link \( L \) admitting non-trivial \( p \)-colorings there is a diagram of \( L \) which can be equipped with a non-trivial \( p \)-coloring using \( \mincol_p L \) colors from this set alone.

How many \( p \)-Super Normal Sets of Colors are there per \( p \)? If there is more than one, are their cardinalities equal? Can a \( p \)-Super Normal Set of Colors be chosen from the set of \( p \)-Normal Sets of Colors?

2. (Minimum number of colors)

Given a prime \( p = 2k + 1 \), if there are links \( L \) and \( L' \) such that \( \mincol_p L \neq \mincol_p L' \), what is the range of the function \( \mincol_p(\ldots) \)? Is there a link \( L \) such that \( \mincol_p L > k \)?

3. (Procedure for reducing the number of colors)

We used an example to explain this procedure in Subsection 4.1. The basic idea is, given a prime \( p \), and given a diagram \( D \), endowed with a non-trivial \( p \)-coloring let \( S = \{c_1, \ldots, c_n\} \mod p \) be the set of colors used in the coloring. Use Theorem 1.3 and Corollary 2.1 to list the candidates to \( p \)-Sufficient Sets of Colors of the form \( S \setminus \{c_i\} \). Use Theorem 2.1 to further screen this list. Use the list just obtained to guide in the choice of the next color to be eliminated from the coloring.

We plan to look into these issues in the near future.
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