ON STEADY STATES OF VAN DER WAALS FORCE DRIVEN
THIN FILM EQUATIONS

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Abstract. Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \) be a bounded smooth domain and \( \alpha > 1 \). We are interested in the singular elliptic equation

\[
\Delta h = \frac{1}{\alpha} h^{-\alpha} - p \quad \text{in } \Omega
\]

with Neumann boundary conditions. In this paper, we gave a complete description of all continuous radially symmetric solutions. In particular, we constructed nontrivial smooth solutions as well as rupture solutions. Here a continuous solution is said to be a rupture solution if its zero set is nonempty. When \( N = 2 \) and \( \alpha = 3 \), the equation has been used to model steady states of van der Waals force driven thin films of viscous fluids. We also considered the physical problem when total volume of the fluid is prescribed.

1. Introduction

The equation

\[
ht = \nabla \cdot (h^3 \nabla p)
\]

has been used to model the dynamics of van der Waals force driven thin films of viscous fluids\cite{27}\cite{28}\cite{29}\cite{30}. Here \( h \) is the thickness of the thin film and the pressure

\[
p = \frac{1}{3} h^{-3} - \Delta h,
\]

is a sum of contributions from disjoining pressure due to attractive van der Waals force and a linearized curvature term corresponding to surface tension effects. Hence, (1.1) becomes

\[
ht = -\nabla \cdot (h^{-1} \nabla h) - \nabla \cdot \left( h^3 \nabla \Delta h \right),
\]

which is a special case of the generalized thin film equation

\[
ht = -\nabla \cdot (h^m \nabla h) - \nabla \cdot (h^n \nabla \Delta h)
\]

where the exponents \( m, n \) represent the powers in the destabilizing second-order and the stabilizing fourth-order diffusive terms, respectively. This class of equations occurs in connection with many physical models involving fluid interfaces\cite{28}\cite{24}. For example, when \( n = 1 \) and \( m = 1 \), it describes a gravity driven Hele-Shaw cell\cite{11}\cite{24}\cite{13}\cite{23}; for \( n = m = 3 \) it describes fluid droplets hanging from a ceiling\cite{15}; and for \( n = 0 \) and \( m = 1 \), it is a modified Kuramoto-Sivashinsky equation which describes solidification of a hyper-cooled melt\cite{3}\cite{7}.

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decades, these models have also been the focus of rigorous and extensive mathematical analysis\[2\] [3] [8] [10] [12] [16] [19] [20] [21] [22] [26].

As in the van der Waals force case, when $n - m \neq 1$, letting
\begin{equation}
(1.5)
p = -\frac{1}{m - n + 1} h^{m-n+1} - \Delta h,
\end{equation}
we can rewrite (1.4) as
\begin{equation}
h_t = \nabla \cdot (h^n \nabla p).
\end{equation}

Now we consider viscous fluids in a cylindrical container whose bottom is represented by $\Omega$, a bounded smooth domain in $\mathbb{R}^2$. Since there is no flux across the boundary, we have the Neumann boundary condition
\begin{equation}
(1.6) \quad \frac{\partial p}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{equation}
We also ignore the wetting or nonwetting effect, and assume that the fluid surface is perpendicular to the boundary of the container, i.e.,
\begin{equation}
(1.7) \quad \frac{\partial h}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{equation}
Whenever $m - n \neq -1$ or $-2$, we can associate (1.4) with energy
\begin{equation}
E(h) = \int_{\Omega} \left( \frac{1}{2} |\nabla h|^2 - \frac{1}{(m-n+1)(m-n+2)} h^{m-n+2} \right),
\end{equation}
and formally, using (1.6), (1.7), we have
\begin{equation}
\frac{d}{dt} E(h) = \int_{\Omega} \left( \Delta hh_t - \frac{1}{m-n+1} h^{m-n+1} h_t \right)
= \int_{\Omega} p \nabla \cdot (h^n \nabla p) = - \int_{\Omega} h^n |\nabla p|^2.
\end{equation}
Hence, for a thin film fluid at rest, $p$ has to be a constant, and $h$ satisfies (1.6).

Therefore, letting $\alpha = -(m-n+1)$, we are led to the elliptic problem
\begin{equation}
(1.8) \quad \left\{ \begin{array}{ll}
\Delta h = \frac{1}{\alpha} h^{-\alpha} - p & \text{ in } \Omega, \\
\frac{\partial h}{\partial \nu} = 0 & \text{ on } \partial \Omega,
\end{array} \right.
\end{equation}
where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$ and $p$ is a constant.

When $N = 1$, this equation has been studied by R. Laugesen and M. Pugh in [20] where they produced positive, smooth steady states for all $\alpha$ and touchdown steady states for $\alpha < 1$. In [5], A. L. Bertozzi, G. Grün and T. P. Witelski considered (1.8) with additional Born repulsion term which leads to the elliptic equation
\begin{equation}
(1.9) \quad \left\{ \begin{array}{ll}
\Delta h = \frac{1}{\alpha} h^{-\alpha} \left( 1 - \left( \frac{\varepsilon}{h} \right)^\beta \right) - p & \text{ in } \Omega, \\
\frac{\partial h}{\partial \nu} = 0 & \text{ on } \partial \Omega,
\end{array} \right.
\end{equation}
where $\beta$ is a positive constant. When $\varepsilon > 0$, the associated energy to (1.9) is bounded from below which makes a variational approach possible and enables them to show the existence of an energy minimizer in any dimensions. It seems difficult to extend this approach to the limiting case $\varepsilon = 0$.

The goal of this paper is to understand radial solutions of (1.8) when $N \geq 2$ and $\alpha > 1$ which we will assume throughout this paper. In particular, when $N = 2$ and $\alpha = 3$, we come to the van der Waals force driven thin films in the physically realistic dimension. When $\alpha > 1$, except the limited discussions in [18], there
seems no established elliptic theory for (1.8), and hence it is the mathematically
more interesting case. We remark that energy method can be applied to yield
nontrivial solutions to (1.8) when $\alpha < 1$. On the other hand, the behavior of radial
solutions is quite different when $\alpha > 1$. For example, we will show that for
$N \geq 2$, the radial solutions will never vanish away from the origin which contrasts
with the $\alpha < 1$ case where touchdown steady states can be shown to exist in any
dimensions.

Due to the singular nature of (1.8), we need to be careful in discussing "solu-
tions" to (1.8). We say $h$ is a continuous solution of (1.8) in $\Omega$, if
$h \not\equiv 0$ and is a nonnegative continuous function in $\Omega$ satisfying the equation in (1.8) in the open
set \( \{ x \in \Omega : h(x) > 0 \} \). The rupture set of $h$,
$$\Sigma = \{ x \in \Omega : h(x) = 0 \},$$
corresponds to "dry spots" in the thin film, which is of great significance in the
coatings industry where nonuniformities are very undesirable. Standard elliptic
theory implies that $h$ is smooth and hence a classical solution of (1.8) in $\Omega$
\( \setminus \Sigma \). An interesting Hausdorff dimension estimate of $\Sigma$ can be found in [18] where it is shown
that any finite energy solution satisfies $H^\mu (\Sigma) = 0$ where $\mu = N - 2 + \frac{4}{\alpha + 1}$. For
van der Waals force driven thin film, we have $N = 2$ and $\alpha = 3$, hence $H^1 (\Sigma) = 0$,
i.e., the thin film with finite energy can’t have one dimensional rupture set.

For any $p > 0$, let
\[
(1.10) \quad \xi = (\alpha p)^{-\frac{1}{\alpha}}
\]
then $h \equiv \xi$ is always a solution to (1.6). The natural question is whether it is the
only solution. For the radially symmetric case, after a simple scaling, the uniqueness
theorem in [13] implies:

**Proposition 1.1.** Let $N \geq 2$ and $\alpha > 1$. For any given $R > 0$, there exists a
constant $p_0$, such that for any $p \leq p_0$, $h \equiv \xi$ is the only radial solution of (1.8) in $B_R (0)$.

When $p$ is large, nontrivial solutions do exist. In fact, we have

**Theorem 1.2.** Let $N \geq 2$ and $\alpha > 1$. For any given $R > 0$, there exists a
nondecreasing sequence of $\{ p_k \}$, such that for any $p > p_k$, (1.8) has at least $k$
nontrivial smooth radial solutions in $B_R (0)$.

Theorem 1.2 is an application of Theorem 1.3 below which gives a complete
description of all nontrivial smooth radial solutions.

**Theorem 1.3.** Let $N \geq 2$ and $\alpha > 1$. For any given $p > 0$, and for any $\eta > 0,
\eta \neq \xi$, there exists an increasing sequence $\{ r_k^{p, \eta} \}_{k=1}^\infty$ with
$$r_1^{p, \eta} \geq \max \left\{ \left( \frac{2N \alpha (\eta - \xi)}{\xi^{\alpha - \eta} - \eta^{\alpha - \xi}} \right)^{\frac{1}{2+\alpha}}, \left( \frac{p_0}{p} \right)^{\frac{1}{2+\alpha}} \right\}$$
where $p_0$ is the constant in Proposition 1.1 and
\[
(1.11) \quad \lim_{k \to \infty} (r_{k+1}^{p, \eta} - r_k^{p, \eta}) = \pi (\alpha p)^{-\frac{1+\alpha}{2+\alpha}},
\]
such that for each $r_k^{p, \eta}$, there exists a unique smooth radial solution of (1.8) in $B_{r_k^{p, \eta}} (0)$ satisfying $h(0) = \eta$. 

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We say a continuous solution to (1.8) is a rupture solution if \( \Sigma \) is not empty. It will be shown that for radial solutions, rupture can only occur at the origin. (See Corollary 2.2 below.) Our main result in this paper is as follows.

**Theorem 1.4.** Let \( N \geq 2 \) and \( \alpha > 1 \). For any given \( p > 0 \), there exists an increasing sequence \( \{ r_k^{p,0} \}_{k=1}^\infty \) with

\[
(1.12) \quad r_k^{p,0} \geq \left( \frac{p_0}{p} \right)^{\frac{1+\alpha}{2\alpha}},
\]

and

\[
(1.13) \quad \lim_{k \to \infty} \left( r_{k+1}^{p,0} - r_k^{p,0} \right) = \pi \left( \frac{1+\alpha}{2\alpha} \right),
\]

where \( p_0 \) is the constant in Proposition 1.1 such that for each \( r_k^{p,0} \), there exists a unique radial rupture solution to (1.8) in \( B_{r_k^{p,0}}(0) \). Furthermore, if \( R \neq r_k^{p,0} \) for any \( k \), then there is no radial rupture solution to (1.8) in \( B_R(0) \).

**Remark 1.5.** The rupture solutions constructed in Theorem 1.4 above are weak solutions to (1.8) in the distributional sense. (See Remark 4.3 below.) We also remark that when \( N = 1 \) and \( \alpha > 1 \), there is no radial rupture solutions [20].

For any given \( p > 0 \), there exists a nontrivial radial solutions to (1.8) in \( B_R(0) \) if and only if \( r_k^{p,\eta} = R \) holds for some \( \eta \geq 0 \), \( \eta \neq \xi \) and for some integer \( k \geq 1 \).

In physical experiments, usually the total volume of the fluid is known, i.e., the average film thickness

\[
\bar{h} = \frac{1}{|\Omega|} \int_{\Omega} h(x) \, dx
\]

given while the pressure \( p \) is an unknown constant. Hence, given \( \bar{h} > 0 \), we need to find function \( h \) and constant \( p \), such that

\[
(1.14) \quad \begin{cases}
\Delta h = \frac{1}{\alpha} h^{-\alpha} - p & \text{in } \Omega, \\
\frac{1}{|\Omega|} \int_{\Omega} h(x) \, dx = \bar{h} \\
\left. \frac{\partial h}{\partial \nu} \right|_{\partial \Omega} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

When \( \Omega = B_1(0) \), all radial solutions of (1.14) can be obtained by scaling from solutions in Theorems 1.3 and 1.4. We will discuss such scaling in Section 5. In particular, we will show

**Theorem 1.6.** Let \( N \geq 2 \), \( \alpha > 1 \) and \( \Omega = B_1(0) \subset \mathbb{R}^N \). There exists a sequence of thickness \( \bar{h}_1, \bar{h}_2, \cdots \) satisfying

\[
\lim_{k \to \infty} \sqrt{k \pi} \bar{h}_k = 1,
\]

such that for any \( k \), (1.14) with \( \bar{h} = \bar{h}_k \) has a radial rupture solution which, viewed as a function in \( r \), has exactly \( k - 1 \) critical points in \( (0,1) \). Furthermore, if \( \bar{h} \neq \bar{h}_k \) for any \( k \), then (1.14) has no radial rupture solution.

When \( \Omega = B_1(0) \), Proposition 1.1 implies that nontrivial solutions to (1.14) must satisfy \( p > p_0 \). Since

\[
p = \frac{1}{|\Omega|} \int_{\Omega} \frac{1}{\alpha} h^{-\alpha},
\]

we may ask the existence of a critical average film thickness \( \bar{h}_0 \) so that there is no nontrivial solutions to (1.14) whenever \( h \geq \bar{h}_0 \). Numerical analysis suggests that
\( \bar{h}_0 \) does not exist. However, we are unable to provide an analytical proof. Such a proof could be possible if we have better understanding of \( \bar{h} (p, \eta, k) \) which is defined in section 5. Moreover, the detailed property of \( \bar{h} (p, \eta, k) \) could also provide us a statement similar to Theorem 1.6 for the smooth radial solutions.

The paper is organized in the following way: In Section 2, we show that any radial solution can be extended to a global solution which is oscillating around \( \xi \). In Section 3 and Section 4, we discuss smooth radial solutions and rupture solutions respectively and Theorems 1.3 and 1.4 are proved. In Section 5, we use scaling argument to prove Theorems 1.2 and 1.6.

2. Preliminaries

Recall that given \( \alpha > 1 \) and \( p > 0 \), \( h \in C^0 (B_R (0)) \) is said to be a continuous solution of

\[
\triangle h = \frac{1}{\alpha} h^{-\alpha} - p
\]

in \( B_R (0) \) if \( h \geq 0 \) and it satisfies (2.1) in the open set \( \{ x \in B_R (0) : h (x) > 0 \} \).

Let \( h \) be a radially symmetric solution to (2.1), we can view \( h \) as a continuous function defined on \( [0, R) \) satisfying

\[
h'' + \frac{N-1}{r} h' + f(h) = 0
\]

in the set \( S^+ = \{ r \in (0, R) : h(r) > 0 \} \). Here

\[
f(h) = -\frac{1}{\alpha} h^{-\alpha} + p
\]

is monotone increasing and its antiderivative

\[
F(h) = \frac{1}{\alpha (\alpha - 1)} h^{1-\alpha} + ph
\]

is convex in \( (0, \infty) \). Let

\[
\xi = (\alpha p)^{-\frac{1}{\alpha}},
\]

then \( F' (\xi) = f (\xi) = 0 \), and \( F \) achieves its absolute minimum at \( \xi \). Furthermore, \( F(h) \rightarrow \infty \) as \( h \rightarrow 0^+ \) or \( h \rightarrow \infty \).

For each \( r \in S^+ \), letting

\[
e_1 (r) = \frac{1}{2} (h' (r))^2 + F(h(r)), \]

\[
e_2 (r) = \frac{1}{2} \left( r^{N-1} h' (r) \right)^2 + r^{2(N-1)} F(h(r)) = r^{2(N-1)} e_1 (r),
\]

we have

\[
dr [e_1 (r)] = -\frac{N-1}{r} (h' (r))^2 \leq 0,
\]

and

\[
dr [e_2 (r)] = 2 (N-1) F (h (r)) r^{2N-3} \geq 0.
\]

The monotonicity of \( e_1 \) and \( e_2 \) will be used to obtain a priori bounds.
Lemma 2.1. Let $0 < r_1 < r_2$ be such that $(r_1, r_2) \subset S^+$. Given $\bar{r} \in (r_1, r_2)$, we have, for any $r \in (r_1, r_2)$,

$$e_1 (r) \leq \left( \frac{\bar{r}}{r} \right)^{2(N-1)} e_1 (\bar{r}).$$

Furthermore,

$$c_1 \leq h (r) \leq c_2$$

where $c_1, c_2$ are two positive constants depending on $\alpha, p, N, r_1, \bar{r}, h (\bar{r})$ and $h' (\bar{r})$, and are independent of $r_2$.

Proof. Since $e_2 (r)$ is monotone increasing, we have, for any $r \in (r_1, \bar{r}]$,

$$e_1 (r) = r^{2(1-N)} e_2 (r) \leq \bar{r}^{2(1-N)} e_2 (\bar{r}) = \left( \frac{\bar{r}}{r} \right)^{2(N-1)} e_1 (\bar{r}).$$

On the other hand, since $e_1 (r)$ is monotone decreasing, we have, for any $r \in [\bar{r}, r_2)$,

$$e_1 (r) \leq e_1 (\bar{r}).$$

Combining (2.7) and (2.8), we obtain (2.6). Now for any $r \in (r_1, r_2)$,

$$F (h (r)) \leq e_1 (r) \leq \left( \frac{\bar{r}}{r_1} \right)^{2(N-1)} e_1 (\bar{r}),$$

so (2.6) follows from the fact that $F (h) \to \infty$ as $h \to 0^+$ or $h \to \infty$. \hfill $\Box$

Corollary 2.2. $h$ can not have rupture away from the origin, i.e., $S^+ = (0, R)$. Furthermore, $h$ can be uniquely extended to a positive smooth solution of (2.2) in $(0, \infty)$.

Proof. Since $S^+$ is open, it is a union of open intervals of the form $(r_1, r_2)$ with $r_1, r_2 \notin S^+$. Given any such interval, if $r_1 > 0$, Lemma 2.1 implies

$$\liminf_{r \to r_1^+} h (r) > 0,$$

and since $h$ is continuous, we conclude $h (r_1) > 0$, which contradicts the assumption $r_1 \notin S^+$. Similarly, we can get a contradiction if $r_2 < R$. Hence, $S^+ = (0, R)$. Extending $h$ to a maximal interval of existence $(0, R^*)$. If $R^* < \infty$, applying Lemma 2.1 again, we have for some positive constants $c_1, c_2$

$$c_1 \leq h (r) \leq c_2$$

for any $r \in (R/2, R^*)$, so the solution can be extended beyond $R^*$. Hence, $R^* = \infty$. \hfill $\Box$

Now, redefine $S^+ = \{ r > 0 : h (r) > 0 \}$, we observe that $S^+ = (0, \infty)$ and Lemma 2.1 still holds. In particular, (2.6) holds for all $r_1 < r < \infty$. In the remaining part of this section, we shall show that $h$ oscillates around $\xi$ near $r = \infty$.

We will need Sturm’s Separation Theorem.

Lemma 2.3. [17] Let $q (t)$ be a real-valued continuous function such that

$$0 < m \leq q (t) \leq M.$$

Given $t_2 > t_1 > 0$, if $u = u (t)$ is a nontrivial solution of

$$u'' + q (t) u = 0$$

then $u (t)$ is a nontrivial solution of

$$u'' + q (t) u = 0$$

in $(0, \infty)$ and

$$\limsup_{r \to \infty} \frac{u (r)}{r} < \infty, \quad \liminf_{r \to \infty} \frac{u (r)}{r} > 0.$$
satisfying $u(t) > 0$ on $(t_1, t_2)$, then

$$t_2 - t_1 \leq \frac{\pi}{\sqrt{m}}.$$ 

And if in addition $u(t_1) = u(t_2) = 0$, then

$$t_2 - t_1 \geq \frac{\pi}{\sqrt{M}}.$$ 

**Lemma 2.4.** For any $r_0 > 0$, there exists $r_1 > r_0$ such that $h'(r_1) = 0$.

**Proof.** Suppose this is false, then we have either $h'(r) > 0$ for all $r \in (r_0, \infty)$ or $h'(r) < 0$ for all $r \in (r_0, \infty)$. Hence, $h$ is strictly monotone increasing or decreasing on $(r_0, \infty)$. From Lemma 2.1 and the observation above, it follows that $h$ is also bounded at $\infty$. So we can assume

$$\lim_{r \to \infty} h(r) = \zeta$$

for some $\zeta > 0$. For any $r > r_0$, integrating (2.2) from $r_0$ to $r$, we obtain

$$h'(r) - h'(r_0) + \int_{r_0}^{r} \frac{N - 1}{s} h'(s) \, ds + \int_{r_0}^{r} f(h(s)) \, ds = 0. \tag{2.9}$$

Since

$$\frac{1}{2} (h'(r))^2 \leq e_1(r) \leq e_1(r_0),$$

$h'(r)$ is bounded in $[r_0, \infty)$. Now, as $h'(r)$ does not change sign,

$$\left| \int_{r_0}^{r} \frac{N - 1}{s} h'(s) \, ds \right| \leq \frac{N - 1}{r_0} \left| \int_{r_0}^{r} h'(s) \, ds \right| = \frac{N - 1}{r_0} |h(r) - h(r_0)|$$

which is also bounded in $[r_0, \infty)$. Hence identity (2.9) implies that

$$\int_{r_0}^{r} f(h(s)) \, ds$$

is bounded in $[r_0, \infty)$. Since $\lim_{r \to \infty} h(r) = \zeta$, we have

$$\lim_{r \to \infty} f(h(r)) = f(\zeta)$$

which must be 0. Thus $\zeta = \xi$ and

$$\lim_{r \to \infty} h(r) = \xi.$$ 

Now let

$$v(r) = r^{\frac{N-1}{2}} (h(r) - \xi),$$

then

$$v_{rr} + \frac{(N - 1)(3 - N)}{4r^2} v + r^{\frac{N-1}{2}} f(h) = 0. \tag{2.10}$$

Since $f(\xi) = 0$, we can rewrite (2.10) as

$$v_{rr} + B(r) v = 0, \tag{2.11}$$

where

$$B(r) = \frac{f(h) - f(\xi)}{h - \xi} + \frac{(N - 1)(3 - N)}{4r^2}.$$ 

Now $\lim_{r \to \infty} h(r) = \xi$ implies

$$\lim_{r \to \infty} B(r) = f'(\xi) > 0.$$
By Lemma 2.3,
\[ v(r) = r^{N-1}(h(r) - \xi) \]
will be oscillating around 0 as \( r \to \infty \), which contradicts the assumption that \( h(r) \to \xi \) in a strictly monotonically manner.

\[ \square \]

Next, we have

**Lemma 2.5.** Let \( h'(r_0) = 0 \) for some \( r_0 \geq 0 \).

(i). If \( h(r_0) = \xi \), then \( h(r) \equiv \xi \).

(ii). If \( h(r_0) > \xi \), then there exists \( r_1 > r_0 \), such that \( h'(r) < 0 \) on \( (r_0, r_1) \), \( h'(r_1) = 0 \), and \( F(h(r_1)) < F(h(r_0)) \).

(iii). If \( 0 < h(r_0) < \xi \), then there exists \( r_1 > r_0 \), such that \( h'(r) > 0 \) on \( (r_0, r_1) \), \( h'(r_1) = 0 \), and \( F(h(r_1)) < F(h(r_0)) \).

**Proof.**

(i). This is the standard ODE uniqueness result.

(ii). Since \( h(r_0) > \xi \), we have \( f(h(r_0)) > 0 \). Now
\[ \left(r^{N-1}h'\right)' = -r^{N-1}f(h), \]
implies that \( r^{N-1}h' \) is strictly monotone decreasing in \( (r_0, r_0 + \delta) \) for some \( \delta > 0 \). Hence we have \( h'(r) < 0 \) on \( (r_0, r_0 + \delta) \). Applying Lemma 2.4, there exists \( r_1 > r_0 \), such that \( h'(r_1) = 0 \), and we also have \( h'(r) < 0 \) on \( (r_0, r_1) \) if we choose the smallest such \( r_1 \). If \( h(r_1) > \xi \), we would have \( r^{N-1}h' \) is strictly decreasing near \( r_1 \), hence \( h'(r_1) < 0 \), which gives a contradiction. And if \( h(r_1) = \xi \), then \( h \equiv \xi \), which contradicts the hypothesis \( h(r_0) > \xi \). Hence we have \( h(r_1) < \xi \). Finally, \( F(h(r_1)) < F(h(r_0)) \) follows from (2.3).

(iii). Similar to the proof of part (ii). \[ \square \]

Let \( h \) be a nontrivial global solution of (2.2), starting with \( r_1 > 0 \) such that \( h'(r_1) = 0 \). The existence of \( r_1 \) is guaranteed by Lemma 2.4. Without loss of generality, we assume \( h(r_1) < \xi \). For \( k = 1, 2, \ldots \), we define through Lemma 2.5,
\[ r_{2k} = \sup \{ r > r_{2k-1} : h'(s) > 0 \text{ for all } s \in (r_{2k-1}, r) \}, \]
\[ r_{2k+1} = \sup \{ r > r_{2k} : h'(s) < 0 \text{ for all } s \in (r_{2k}, r) \}. \]

**Lemma 2.6.**
\[ \lim_{k \to \infty} r_k = \infty. \]

**Proof.** If it is not true, then we have
\[ \lim_{k \to \infty} r_k = r^* \]
for some \( r^* > 0 \). Since \( h \) is smooth, we have
\[ h(r^*) = \xi, \quad h'(r^*) = 0, \]
hence Lemma 2.4 implies \( h \equiv \xi \), which is a contradiction. \[ \square \]

Next, we show that the lengths of oscillating intervals \( r_{k+1} - r_k \) are bounded.

**Lemma 2.7.** There exists positive constants \( C_1 \) and \( C_2 \) such that
\[ C_1 \leq (r_{k+1} - r_k) \leq C_2 \]
for any \( k = 1, 2, 3, \ldots \).
Proof. Since
\[ \lim_{k \to \infty} r_k = \infty, \]
we only need to prove the lemma when \( r_k \) is sufficiently large. Differentiating (2.2), we have
\[ \frac{N - 1}{r} h'' - \frac{N - 1}{r^2} h' + h^{-\alpha-1} h' = 0. \]
Let
\[ w(r) = r^{\frac{N-1}{2}} h'(r), \]
then \( w \) satisfies
\[ w'' + \left( h^{-\alpha-1} - \frac{N^2 - 1}{4r^2} \right) w = 0. \]
Since \( h \) is bounded away from both zero and infinity when \( r \to \infty \) by (2.5), we have for some \( R > 0 \) such that for any \( r > R \),
\[ c_1 \leq h^{-\alpha-1}(r) - \frac{N^2 - 1}{4r^2} \leq c_2 \]
for some positive constants \( c_1, c_2 \). Since \( r_k, k = 1, 2, \ldots \), are zeros of \( w \), the conclusion follows from Lemma 2.8. \( \square \)

Finally, we have

**Lemma 2.8.**
\[ \lim_{r \to \infty} h(r) = \xi. \]

**Proof.** Starting with \( r_1 > 0 \) such that \( h'(r_1) = 0, h(r_1) < \xi \), we define \( r_k \) as above. Since \( F(h(r_k)) \) is monotone decreasing in \( k \), and \( h(r_{2k}) > \xi, h(r_{2k-1}) < \xi \), the property of function \( F \) implies \( h(r_{2k}) \) is monotone decreasing and \( h(r_{2k-1}) \) is monotone increasing. Hence we have the limits
\[ \eta_1 = \lim_{k \to \infty} h(r_{2k}) \geq \xi \geq \eta_2 = \lim_{k \to \infty} h(r_{2k-1}). \]
Now
\[ \eta_1 - \eta_2 \leq |h(r_{k+1}) - h(r_k)| = \int_{r_k}^{r_{k+1}} |h'(r)| \, dr \]
\[ \leq (r_{k+1} - r_k)^{\frac{1}{2}} \left( \int_{r_k}^{r_{k+1}} |h'|^2 \, dr \right)^{\frac{1}{2}} \leq \sqrt{C_2} \left( \int_{r_k}^{r_{k+1}} |h'|^2 \, dr \right)^{\frac{1}{2}}, \]
which implies
\[ (2.13) \quad \int_{r_k}^{r_{k+1}} \frac{|h'|^2}{r} \, dr \geq \frac{1}{r_{k+1}} \int_{r_k}^{r_{k+1}} |h'|^2 \geq \frac{(\eta_1 - \eta_2)^2}{C_2 r_{k+1}} \geq \frac{(\eta_1 - \eta_2)^2}{2C_2^2} \int_{r_k}^{r_{k+1}} \frac{1}{r} \, dr \]
when \( k \) is sufficiently large. In the last inequality, we used \( r_{k+1} \leq 2r_k \) when \( k \) is large. From (2.3), we have for any \( r > r_1 \),
\[ \int_{r_1}^{r} \frac{|h'|^2}{r} = \frac{1}{N - 1} [e_1(r_1) - e_1(r)] \leq \frac{e_1(r_1)}{N - 1}. \]
Therefore \( \frac{|h'|^2}{r} \) is integrable at \( \infty \). Since \( \frac{1}{r} \) is not integrable at \( \infty \), (2.13) implies that \( \eta_1 = \eta_2 = \xi \) and
\[ \lim_{r \to \infty} h(r) = \xi. \]
\( \square \)
Corollary 2.9.  

\[(2.14) \lim_{k \to \infty} (r_{k+1} - r_k) = \pi (\alpha p)^{-\frac{1}{2\alpha}}.\]

Proof. In equation (2.12), we now have  

\[\lim_{r \to \infty} \left( h - \frac{N^2 - 1}{4r^2} \right) = \xi^{-\alpha - 1} = (\alpha p)^{\frac{1}{2\alpha}}.\]

Hence (2.14) follows from Lemma 2.3. \(\square\)

3. Nontrivial Smooth Radially Symmetric Solutions

This section is devoted to the proof of Theorem 1.3. Given \(\eta > 0\), we consider (2.2) with the initial values  

\[h(0) = \eta > 0, h_r(0) = 0.\]

The local existence and uniqueness of a smooth solution is standard since \(f\) is smooth when \(h\) is bounded away from zero. And such solution is actually a global solution from Corollary 2.2. For any \(\eta \neq \xi\), without loss of generality, we assume \(\eta > \xi\). Since \((r^{N-1}h')' = -r^{N-1}f(h) = -r^{N-1} \left( -\frac{1}{\alpha} h^{-\alpha} + p \right)\), we have \((r^{N-1}h')' < 0\) in \((0, \delta)\) for some small \(\delta > 0\). From \(h'(0) = 0\), we conclude \(h'(r) < 0\) in \((0, \delta)\). Then we can define  

\[r_1 = \min \{ r > 0 : h'(r) = 0 \}.\]

The existence of \(r_1 > 0\) is guaranteed by Lemma 2.4 with \(r_0 = \frac{\delta}{2}\). From the analysis in the previous section, \(h\) will be oscillating around \(\xi\), and all critical points of \(h\) can be listed as \(r_1 < r_2 < r_3 < \cdots\), with  

\[C_1 \leq r_{k+1} - r_k \leq C_2,\]

and  

\[\lim_{k \to \infty} (r_{k+1} - r_k) = \pi (\alpha p)^{-\frac{1}{2\alpha}}.\]

Hence for any \(k \geq 1\), \(h\) is a nontrivial smooth solution of  

\[\Delta h = \frac{1}{\alpha} h^{-\alpha} - p \quad \text{in} \quad B_{r_k}(0),\]

\[\frac{\partial h}{\partial r} = 0 \quad \text{on} \quad \partial B_{r_k}(0).\]

And all nontrivial smooth radial solutions of (1.14), when \(\Omega\) is a ball, can be obtained this way. More precisely, let \(\Omega = B_R(0)\) for a given \(R > 0\), then (1.14) has a nontrivial smooth radial solution if and only if \(R = r_k^{p,\eta}\) for some \(\eta > 0, \eta \neq \xi\) and for some \(k \geq 1\), here we write \(r_k = r_k^{p,\eta}\) to recognize its dependence on \(p\) and \(\eta\).

Now we recall the uniqueness result of M. Del Pino and G. Hernandez [13].

Proposition 3.1. Given \(\alpha > 1\), there exists \(d_0 > 0\), such that  

\[\begin{cases} -d\Delta u + u^{-\alpha} = 1 & \text{in} \quad B_1(0), \\ \frac{\partial u}{\partial r} = 0 & \text{on} \quad \partial B_1(0) \end{cases}\]

has no nontrivial radial solution whenever \(d \geq d_0\).
It is easy to verify
\[ \tilde{h}(x) = (\alpha p)^{\frac{1}{\alpha}} h(r_1 x) \]
satisfies (3.1) with
\[ d = \frac{\alpha}{(\alpha p)^{\frac{\alpha+1}{\alpha}} r_1^\frac{\alpha+1}{\alpha}}. \]
Hence Proposition 3.1 implies
\[ r_1 > \left( \frac{\alpha}{(\alpha p)^{\frac{\alpha+1}{\alpha}} d_0} \right)^{\frac{1}{2}} \equiv \left( \frac{p_0}{\alpha} \right)^{\frac{\alpha+1}{\alpha}}, \]
here
\[ p_0 = \left( \frac{1}{\alpha \frac{1}{\alpha} d_0} \right)^{\frac{\alpha+1}{\alpha}} = \alpha^{-\frac{1}{\alpha+1}} d_0^{-\frac{\alpha+1}{\alpha}}. \]

In general, \( r_k \) depends on both \( p \) and \( \eta \). We refer to Corollary 5.2 for the scaling of \( r_k \) when \( p, \eta \) changes.

**Lemma 3.2.** For any \( \eta > 0, \eta \neq \xi \), we have
\[ r_1(\eta) \geq \sqrt{\frac{2N\alpha (\eta - \xi)}{\xi^{-\alpha} - \eta^{-\alpha}}}. \]

In particular,
\[ \lim_{\eta \to \infty} r_1(\eta) = \infty. \]

**Proof.** First we assume \( \eta > \xi \). From the definition of \( r_1(\eta) \) and Lemma 2.5 we have \( h'(r_1) = 0, h(r_1) < \xi \) and for any \( r \in (0, r_1), 0 < h(r) < \eta, h'(r) < 0 \). Now
\[ (r^{N-1}h')' = -r^{N-1} \left( \frac{\xi^{-\alpha} - h^{-\alpha}}{\alpha} \right) \geq -r^{N-1} \frac{\xi^{-\alpha} - \eta^{-\alpha}}{\alpha}. \]
Integrating from 0 to \( r \), we have
\[ r^{N-1}h'(r) \geq - \frac{r^N}{N} \frac{\xi^{-\alpha} - \eta^{-\alpha}}{\alpha}, \]
i.e.
\[ h'(r) \geq - \frac{r}{N\alpha} (\xi^{-\alpha} - \eta^{-\alpha}). \]
Integrating again from 0 to \( r_1 \), we have
\[ h(r_1) - h(0) \geq - \frac{r_1^2}{2N\alpha} (\xi^{-\alpha} - \eta^{-\alpha}), \]
therefore
\[ r_1(\eta) \geq \sqrt{\frac{2N\alpha (h(0) - h(r_1))}{\xi^{-\alpha} - \eta^{-\alpha}}} \geq \sqrt{\frac{2N\alpha (\eta - \xi)}{\xi^{-\alpha} - \eta^{-\alpha}}}. \]
The bound when \( \eta < \xi \) can be proved similarly. \( \Box \)
4. Rupture Solutions

In this section, we will consider radial solutions to (2.1) which are not smooth and prove Theorem 1.4. From Corollary 2.2, we need to consider $h \in C^0([0, \infty))$ such that $h(0) = 0$ and $h$ satisfies (2.2) in $(0, \infty)$.

First, we check the growth rate of $h$ near the origin.

Lemma 4.1. Let $h$ be a radially symmetric rupture solution, then for any $\delta > 0$, there exists positive constant $c_1$ such that

$$h(r) \geq c_1 r^{\frac{2}{\alpha+1}}$$

holds for any $r \in [0, \delta]$.

Proof. Since $h$ is positive and smooth away from the origin, we only need to prove the bound for small $\delta$. Let $\delta > 0$ be sufficiently small so that

$$\frac{1}{\alpha} h^{-\alpha}(r) - p \geq \frac{1}{2\alpha} h^{-\alpha}(r)$$

holds for any $r \in (0, \delta]$. Now

(4.1) \( (r^{N-1} h')' = r^{N-1} \left( \frac{h^{-\alpha}}{\alpha} - p \right) \geq \frac{1}{2\alpha} h^{-\alpha} r^{N-1} \)

implies \( r^{N-1} h' \) is monotone increasing in $(0, \delta]$. Since $h(0) = 0$ and $h(r)$ is positive away from the origin, there exists a sequence $r_i \to 0$ such that $h'(r_i) > 0$. Hence, $h'(r) > 0$ for any $r \in (0, \delta]$. Integrating (4.1) from $\varepsilon$ to $r$, and using the fact that $h$ is increasing, we have

$$r^{N-1} h'(r) - \varepsilon^{N-1} h'(\varepsilon) \geq \frac{1}{2N\alpha} h^{-\alpha}(r) \left( r^N - \varepsilon^N \right).$$

Letting $\varepsilon \to 0$, we have

$$r^{N-1} h'(r) \geq \frac{1}{2N\alpha} h^{-\alpha}(r) r^N.$$

Hence for any $r \in (0, \delta]$,

$$\frac{d}{dr} h^{\alpha+1}(r) \geq \frac{\alpha + 1}{2N\alpha} r^\alpha.$$

Integrating from $0$ to $r$, we have

$$h^{\alpha+1}(r) \geq \frac{\alpha + 1}{4N\alpha} r^2,$$

i.e. for any $r \in (0, \delta]$,

$$h(r) \geq \left( \frac{\alpha + 1}{4N\alpha} \right)^{\frac{1}{\alpha+1}} r^{\frac{2}{\alpha+1}}.$$

$\Box$

Lemma 4.2. Let $h$ be a radially symmetric rupture solution, then we have for some positive constant $c_2$,

$$h(r) \leq c_2 r^{\frac{2}{\alpha+1}}$$

for any $r \in [0, \infty)$.
Proof. Since $h$ is uniformly bounded at $\infty$, we only need to prove the inequality near the origin. First we claim

\begin{equation}
\lim_{r \to 0^+} r^{N-1} h'(r) = 0.
\end{equation}

From

\[ (r^{N-1} h')' = r^{N-1} \left( \frac{h^{-\alpha}}{\alpha} - p \right), \]

it follows that $r^{N-1} h'$ is monotone increasing near the origin. Thus, if (4.2) is false, we would have
\[ r^{N-1} h'(r) \geq c > 0 \]

near the origin, hence
\[ h'(r) \geq cr^{1-N}. \]

Since $r^{1-N}$ is not integrable near zero, the above inequality contradicts the fact that $h$ is continuous.

Given $\delta > 0$, for any $r \in (0, \delta)$,
\[ (r^{N-1} h')' = r^{N-1} \left( \frac{h^{-\alpha}}{\alpha} - p \right) \leq \frac{h^{-\alpha}}{\alpha} r^{N-1} \leq \frac{c_1^{-\alpha}}{\alpha} r^{N-1 - \frac{2\alpha}{\alpha + 1}} \]

by Lemma 4.1. Integrating from $\varepsilon$ to $r$, we obtain
\[ r^{N-1} h'(r) - \varepsilon^{N-1} h'(\varepsilon) \leq \frac{1}{\alpha} \left( N - \frac{2\alpha}{\alpha + 1} \right) c_1^{-\alpha} \left( r^{N-1 - \frac{2\alpha}{\alpha + 1}} - \varepsilon^{N-1 - \frac{2\alpha}{\alpha + 1}} \right) \]

Letting $\varepsilon \to 0$, we have
\[ r^{N-1} h'(r) \leq \frac{1}{\alpha} \left( N - \frac{2\alpha}{\alpha + 1} \right) c_1^{-\alpha} r^{N-1 - \frac{2\alpha}{\alpha + 1}}, \]

i.e.,
\[ h'(r) \leq \frac{1}{\alpha} \left( N - \frac{2\alpha}{\alpha + 1} \right) c_1^{-\alpha} r^{1 - \frac{2\alpha}{\alpha + 1}}. \]

Integrating from 0 to $r$, we have, for any $r \in (0, \delta)$,
\[ h(r) \leq \frac{\alpha + 1}{2\alpha} \left( N - \frac{2\alpha}{\alpha + 1} \right) c_1^{-\alpha} r^{\frac{\alpha}{\alpha + 1}}. \]

\[ \square \]

Lemmas 4.1 and 4.2 imply that $h(r)$ is of order $r^{\frac{\alpha}{\alpha + 1}}$ near the origin. Now we write

\begin{equation}
(4.3) \quad h = c^* \varphi(r) r^{\frac{\alpha}{\alpha + 1}},
\end{equation}

where
\[ c^* = \left[ \frac{2\alpha}{\alpha + 1} \left( N - 2 + \frac{2}{\alpha + 1} \right) \right]^{-\frac{1}{\alpha + 1}}. \]

Observe that $h = c^* r^{\frac{\alpha}{\alpha + 1}}$ is a solution of
\[ \Delta h - \frac{1}{\alpha} h^{-\alpha} = 0 \]
in \((0, \infty)\). Direct calculation yields

\[ \phi'' + (A + 1) \frac{\phi'}{r} + \frac{g(\phi)}{r^2} + Cr^{-\frac{2}{\alpha+1}} = 0 \]

where

\[ A = N - 2 + \frac{4}{\alpha + 1} > 0, \quad C = \frac{p}{c^*} > 0 \]

and

\[ g(\phi) = \frac{2}{\alpha + 1} \left( N - 2 + \frac{2}{\alpha + 1} \right) (\phi - \phi^{-\alpha}) \]

Lemmas 4.1 and 4.2 imply that

\[ 0 < \liminf_{r \to 0^+} \phi(r) \leq \limsup_{r \to 0^+} \phi(r) < \infty. \]

On the other hand, let \( \phi \) be a positive solution of (4.4) satisfying (4.5). Then \( h \) defined by (4.3) is a rupture solution.

Locally, there exists at least one solution of (4.4) with initial values

\[ \phi(0) = 1, \phi'(0) = 0. \]

To see this, we rewrite the equation as

\[ \phi'' + (A + 1) \frac{\phi'}{r} + \frac{g'(1)(\phi - 1)}{r^2} + \frac{\tilde{g}(\phi)}{r^2} + Cr^{-\frac{2}{\alpha+1}} = 0 \]

Denoting

\[ \psi = \phi - 1, \]

we have

\[ \psi'' + (A + 1) \frac{\psi'}{r} + \frac{g'(1) \psi}{r^2} + \frac{\tilde{g}(\psi)}{r^2} + Cr^{-\frac{2}{\alpha+1}} = 0, \]

where

\[ \tilde{g}(\psi) = g(\psi + 1) - g'(1) \psi \]
satisfies \( \tilde{g}(0) = 0, \tilde{g}'(0) = 0 \). Now let \( a_1, a_2 \) be two numbers satisfying

\[ a_1 + a_2 = A = N - 2 + \frac{4}{\alpha + 1}, \quad a_1a_2 = g'(1) = 2 \left( N - 2 + \frac{2}{\alpha + 1} \right), \]

then the real parts of \( a_1, a_2 \) are both positive and it is easy to verify that

\[ \psi'' + (A + 1) \frac{\psi'}{r} + \frac{g'(1) \psi}{r^2} = r^{-a_2-1} (r^{a_2-a_1-1} (r^{a_1})_r)'. \]

Hence, we have

\[
\psi = -r^{-a_1} \int_0^r \left\{ s^{a_1-a_2-1} \int_0^s t^{a_2+1} \left( \frac{\tilde{g}(\psi(t))}{t^2} + Ct^{-\frac{2}{\alpha+1}} \right) dt \right\} ds
\]

\[
= -\frac{C}{(a_1 + \frac{2a}{\alpha+1}) (a_2 + \frac{2a}{\alpha+1})} r^{-a_2-1} \int_0^r \left\{ s^{a_1-a_2-1} \int_0^s t^{a_2+1} \tilde{g}(\psi(t)) dt \right\} ds.
\]

Let

\[
L\psi = -\frac{C}{(a_1 + \frac{2a}{\alpha+1}) (a_2 + \frac{2a}{\alpha+1})} r^{\frac{2a}{\alpha+1}} -r^{-a_1} \int_0^r \left\{ s^{a_1-a_2-1} \int_0^s t^{a_2+1} \tilde{g}(\psi(t)) dt \right\} ds,
\]

then for \( \delta \) sufficiently small, \( L \) is a contraction mapping from

\[ X = \{ \psi \in C([0, \delta]) : |\psi(r)| \leq \delta \text{ for any } r \in [0, \delta] \} \]
into itself. Here $L$ is a real mapping even though $a_1, a_2$ could be complex numbers.

Let $\psi$ be the unique fixed point of $L$ in $X$, then $\varphi = 1 + \psi$ is a solution to (4.4) satisfying (4.6).

Let $\varphi$ be the local solution of (4.4) we just constructed, then $h$ defined by (4.3) is continuous with $h (0) = 0$ and satisfies (2.2) in $\mathbb{R}$. Such solution can be uniquely extended to a solution in $(0, \infty)$ which converges to $\xi$ by Lemma 2.8. Thus we have constructed a global rupture solution.

Remark 4.3. From the bounds in Lemmas 4.1 and 4.2, it is easy to see that the rupture solution we constructed is actually a weak solution of (2.1) in $\mathbb{R}^N$, $N \geq 2$.

More precisely, we have $h \in W^{2,P}_{\text{loc}} (\mathbb{R}^N)$, $h^{-\alpha} \in L^P_{\text{loc}} (\mathbb{R}^N)$ for any $1 \leq P < \frac{\alpha + 1}{2\alpha} N$, and

$$
\int_{\mathbb{R}^N} h \Delta \phi = \int_{\mathbb{R}^N} \left( \frac{1}{\alpha} h^{-\alpha} - p \right) \phi 
$$

holds for any $\phi \in C_c^\infty (\mathbb{R}^N)$.

From the proof of Lemma 4.1 we have $h' > 0$ near the origin, so we can define $r_1 = \min \{ r > 0 : h'(r) = 0 \}$, the existence of $r_1$ is guaranteed by Lemma 2.4. Furthermore, as in the smooth solution case, we have a sequence $\{r_k\}_{k=1}^\infty$ such that for each $k$, $h$ is a rupture solution of

$$
\begin{cases}
\Delta h = \frac{1}{\alpha} h^{-\alpha} - p & \text{in } B_{r_k} (0), \\
\frac{\partial h}{\partial \nu} = 0 & \text{on } \partial B_{r_k} (0).
\end{cases}
$$

In the remaining part of this section, we will show that the rupture solution to (2.2) is actually unique.

In (4.4), with $r = e^{-t}$, $\phi (t) = \varphi (r)$, direct calculation yields

$$
(4.7) \quad \phi_{tt} - A\phi_t + g (\phi) + Ce^{-\frac{2p}{\alpha + 1} t} = 0
$$
on $(-\infty, \infty)$.

Lemma 4.4. There exists a unique global solution to (4.7) satisfying

$$
(4.8) \quad 0 < \liminf_{t \to \infty} \phi (t) \leq \limsup_{t \to \infty} \phi (t) < \infty.
$$

Noticing that $r \to 0^+$ is equivalent to $t \to \infty$, the uniqueness of rupture solution follows from Lemma 4.4. Before proving Lemma 4.4 we first study the behavior of $\phi$ at $\infty$.

We write

$$
G (\phi) = \frac{2}{\alpha + 1} \left( N - 2 + \frac{2}{\alpha + 1} \right) \left( \frac{\phi^2}{2} + \frac{\phi^{1-\alpha}}{\alpha - 1} \right),
$$
hence,

$$
G' (\phi) = g (\phi).
$$

Multiplying equation (4.7) with $\phi_t$, we have

$$
(4.9) \quad \frac{d}{dt} \left( \frac{\phi_t^2}{2} + G (\phi) \right) = A \phi_t^2 - C e^{-\frac{2p}{\alpha + 1} t} \phi_t \geq A \frac{\phi_t^2}{2} - C e^{-\frac{4p}{\alpha + 1} t}.
$$
Lemma 4.5. Let \( \phi \) be a global solution to (4.7) satisfying (4.8), then
\[
-\infty < \liminf_{t \to \infty} \phi_t(t) \leq \limsup_{t \to \infty} \phi_t(t) < \infty
\]
and
\[
\int_0^\infty \phi_t^2(s) < \infty.
\]
Furthermore, the limit
\[
\lim_{t \to \infty} \left( \frac{\phi_t^2}{2} + G(\phi) \right)
\]
exists and is finite.

Proof. If \( \phi_t \) is unbounded at \( \infty \), then \( \frac{\phi_t^2}{2} + G(\phi) \) will be unbounded, so there exists a sequence \( \{t_k\}_{k=1}^{\infty} \) with \( \lim_{k \to \infty} t_k = \infty \), such that
\[
\lim_{k \to \infty} \left( \frac{\phi_{t_k}^2}{2} + G(\phi(t_k)) \right) = \infty.
\]
For any \( t > t_k \), integrating
\[
\frac{d}{dt} \left( \frac{\phi_t^2}{2} + G(\phi) \right) \geq -\frac{C^2}{2A} e^{-\frac{4\alpha}{\alpha+1} t},
\]
from \( t_k \) to \( t \), we have
\[
\frac{\phi_t^2}{2} + G(\phi(t)) \geq \frac{\phi_{t_k}^2}{2} + G(\phi(t_k)) - \int_{t_k}^t \frac{C^2}{2A} e^{-\frac{4\alpha}{\alpha+1} s} ds
\]
\[
\geq \frac{\phi_{t_k}^2}{2} + G(\phi(t_k)) - \frac{(\alpha+1) C^2}{8\alpha A}.
\]
Hence,
\[
\lim_{t \to \infty} \left( \frac{\phi_t^2}{2} + G(\phi) \right) = \infty.
\]
From (4.9), \( G(\phi) \) is bounded at \( \infty \), so we deduce
\[
\lim_{t \to \infty} \frac{\phi_t^2}{2} = \infty,
\]
which is impossible for bounded \( \phi \). The \( L^2(0, \infty) \) bound of \( \phi_t \) follows from (4.9) and the fact that \( \frac{\phi_t^2}{2} + G(\phi) \) is bounded at \( \infty \). Finally, since the right hand side of
\[
\frac{d}{dt} \left( \frac{\phi_t^2}{2} + G(\phi) \right) = A\phi_t^2 - Ce^{-\frac{4\alpha}{\alpha+1} t} \phi_t
\]
is absolutely integrable at \( \infty \), we have for any \( t_0 \),
\[
\lim_{t \to \infty} \left[ \frac{\phi_t^2}{2} + G(\phi) \right] = \frac{\phi_{t_0}^2}{2} + G(\phi(t_0)) + \int_{t_0}^\infty \left( A\phi_t^2 - Ce^{-\frac{4\alpha}{\alpha+1} t} \phi_t \right)
\]
which is finite. \( \square \)

Lemma 4.6. If
\[
\lim_{t \to \infty} \phi = \varsigma,
\]
then
\[
\varsigma = 1.
\]
Proof. If $\varsigma \neq 1$, we first assume $\varsigma < 1$, then for some small $\delta > 0$,

$$
\phi_{tt} - A\phi_t = -g(\phi) - Ce^{-\frac{\alpha}{\alpha + 1}t} > \delta,
$$

for any $t \geq T_0$, where $T_0$ is a sufficiently large constant. Hence

$$
\left(e^{-At}\phi_t\right)_t > \delta e^{-At}
$$

for any $t \geq T_0$. Now since $\phi_t^2$ is integrable, we can choose $T_1 > T_0$ with $|\phi_t(T_1)|$ sufficiently small. For any $t > T_1$, integrating (4.10) from $T_1$ to $t$, we have

$$
\phi_t(t) > \left[ \frac{\delta}{A} \left( e^{-AT_1} - e^{-At} \right) + e^{-AT_1} \phi_t(T_1) \right] e^{At} > \frac{\delta}{2A} e^{-AT_1} e^{At}
$$

when $t$ is sufficiently large, which contradicts the boundedness of $\phi_t$ at $\infty$. The case $\varsigma > 1$ can be treated in the same manner. \qed

Lemma 4.7.

$$
\lim_{t \to \infty} G(\phi(t)) = G(1),
$$

and hence

$$
\lim_{t \to \infty} \phi(t) = 1.
$$

Proof. Since $G(1) = \min G(\phi)$, if

$$
\lim_{t \to \infty} \left( \frac{\phi^2}{2} + G(\phi) \right) = G(1),
$$

then $\lim_{t \to \infty} G(\phi)$ exists and equals $G(1)$, as desired. We proceed by contradiction and assume that

$$
\lim_{t \to \infty} \left( \frac{\phi^2}{2} + G(\phi) \right) = L > G(1).
$$

We claim

$$
\liminf_{t \to \infty} G(\phi) < L.
$$

Otherwise,

$$
\lim_{t \to \infty} G(\phi) = L,
$$

which implies

$$
\lim_{t \to \infty} \phi = \varsigma
$$

for some $\varsigma$ with $G(\varsigma) = L$, a contradiction to Lemma 4.0. Hence, there exists a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \to \infty$ and

$$
G(1) \leq G(\phi(t_k)) < L - \delta
$$

for some $\delta > 0$. Now we consider

$$
\delta_k = \sup \left\{ s > t_k : \text{For any } t \in (t_k, s), \ G(\phi(t)) < L - \frac{\delta}{2} \right\}.
$$

Observe that $\delta_k$ is finite. Otherwise $\phi^2_t > \delta_k$ for any $t$ sufficiently large, and then $\phi$ is monotone with derivative bounded away from zero, hence it will be unbounded, which gives a contradiction. Since $G(\phi(s_k)) = L - \frac{\delta}{2}$, we must have

$$
|\phi(t_k) - \phi(s_k)| > \delta_1,
$$
where $\delta_1 > 0$ is a constant depending on $\delta$, $L$ and $G$. Since $\phi_t$ is bounded, we have

$$|s_k - t_k| > \delta_2 \equiv \frac{\delta_1}{\|\phi_t\|_{L^\infty}} > 0.$$

However, for $t_k$ so large that

$$\frac{\phi_t^2}{2} + G(\phi) > L - \frac{\delta}{4}$$

on $(t_k, \infty)$, we have

$$\phi_t^2(t) > \frac{\delta}{2}$$

on $(t_k, s_k)$. Hence

$$\int_{t_k}^{s_k} \phi_t^2 > \frac{\delta}{2} \delta_2.$$

On the other hand, Lemma 4.5 says

$$\lim_{k \to \infty} \int_{t_k}^{\infty} \phi_t^2 = 0,$$

which is a contradiction. $\square$

Now we are ready to prove Lemma 4.4

**Proof of Lemma 4.4.** Let $\phi$ and $\tilde{\phi}$ be two global solutions of (4.7) satisfying (4.8). Letting $\psi = \phi - \tilde{\phi}$, we have

$$\psi_{tt} - A\psi_t + B(t) \psi = 0.$$

Here

$$A = N - 2 + \frac{4}{\alpha + 1},$$

and

$$B(t) = \frac{2}{\alpha + 1} \left( N - 2 + \frac{2}{\alpha + 1} \right) \left( 1 + \frac{\tilde{\phi}^{-\alpha} - \phi^{-\alpha}}{\phi^{-\alpha}} \right).$$

Since

$$\lim_{t \to \infty} \tilde{\phi}(t) = \lim_{t \to \infty} \phi(t) = 1,$$

we have

$$\lim_{t \to \infty} B(t) = B_0 = 2 \left( N - 2 + \frac{2}{\alpha + 1} \right) > 0.$$

It is easy to check that for any $\lambda$ such that

$$\lambda^2 - A\lambda + B_0 = 0,$$

we have Re $\lambda > 0$. Since $\psi$ is bounded at $\infty$, Lemma 4.8 below with $\lambda_0 = 0$ implies $\psi \equiv 0$. $\square$

The following result seems standard and should be well-known. A proof is included here for the convenience of the reader.
Lemma 4.8. Let \( u \) satisfy the linear equation
\begin{equation}
(4.11) \quad u_{tt} - Au_t + B(t)u = 0.
\end{equation}
Here \( A \) is a constant, and \( B(t) \) is a continuous function such that
\[ \lim_{t \to \infty} B(t) = B_0. \]
Let \( \lambda_1, \lambda_2 \) be solutions of
\[ \lambda^2 - A\lambda + B_0 = 0. \]
Suppose that there exists a constant \( \lambda_0 \) satisfying
\[ \lambda_0 < \lambda_m = \min (\Re \lambda_1, \Re \lambda_2) \]
such that, for some positive constants \( T \) and \( c \),
\[ |u(t)| \leq ce^{\lambda_0 t} \]
holds for any \( t \geq T \), then \( u \equiv 0 \).

Proof. Let \( u \) be any function satisfying (4.11). For any \( \lambda \in (\lambda_0, \lambda_m) \), let \( v = e^{-\lambda t}u \). It is easy to check
\begin{equation}
(4.12) \quad v_{tt} - (A - 2\lambda) v_t + (B(t) - A\lambda + \lambda^2) v = 0.
\end{equation}
Since \( \lambda < \lambda_m \), we have
\[ A - 2\lambda = \Re \lambda_1 + \Re \lambda_2 - 2\lambda \geq 2\lambda_m - 2\lambda > 0 \]
and
\[ B_0 - A\lambda + \lambda^2 > 0. \]
Multiplying (4.12) with \( v_t \), we obtain
\[ \frac{d}{dt} (v_t^2 + (B_0 - A\lambda + \lambda^2) v^2) = 2 (A - 2\lambda) v_t^2 + 2 (B_0 - B(t)) vv_t. \]
For any \( \varepsilon_1 > 0 \) since
\[ \lim_{t \to \infty} B(t) = B_0, \]
there exists \( T_1 > 0 \), such that
\[ |(B_0 - B(t)) vv_t| \leq \varepsilon_1 \left( v_t^2 + (B_0 - A\lambda + \lambda^2) v^2 \right) \]
holds for any \( t \geq T_1 \). Hence for any \( t \geq T_1 \), we have
\[ \frac{d}{dt} (v_t^2 + (B_0 - A\lambda + \lambda^2) v^2) \leq 2 (A - 2\lambda + \varepsilon_1) \left( v_t^2 + (B_0 - A\lambda + \lambda^2) v^2 \right). \]
Gronwall’s inequality then implies that for any \( t \geq T_1 \),
\begin{equation}
(4.13) \quad v_t^2 + (B_0 - A\lambda + \lambda^2) v^2 \leq c_{\varepsilon_1} e^{2(A-2\lambda+\varepsilon_1)t}
\end{equation}
where
\[ c_{\varepsilon_1} = \left[ v_t^2 (T_1) + (B_0 - A\lambda + \lambda^2) v^2 (T_1) \right] e^{-2(A-2\lambda+\varepsilon_1)T_1}. \]
Now let \( u_1 \) be the solution of (4.11) in the Lemma such that
\[ |u_1(t)| \leq ce^{\lambda_0 t}. \]
holds for any \( t \geq T \). Then for any \( \lambda \in (\lambda_0, \lambda_m) \) and for any \( \varepsilon \in (0, A - 2\lambda) \), \( v_1 = e^{-\lambda t} u_1 \) satisfies
\[
\frac{d}{dt} \left( v_1^2, t \right) + (B_0 - A\lambda + \lambda^2) v_1^2 \geq 2 (A - 2\lambda) v_1^2 + 2 (B_0 - B(t)) v_1 v_1, t \\
\geq 2 (A - 2\lambda - \varepsilon) v_1^2 - v_1^2
\]
for any \( t \geq T_\epsilon \) if we choose \( T_\epsilon \geq T \) sufficiently large. Since \( A - 2\lambda - \varepsilon > 0 \) and
\[(4.14)\]
holds for any \( t \geq T \), we have, by similar arguments as in the proof of Lemma 4.5,
\[
|v_1 (t)| \leq ce^{-(\lambda - \lambda_0)t}
\]
holds for any \( t \geq T_\epsilon \), for some positive constant \( C \). Hence for any \( \varepsilon_2 \in (0, \lambda_m - \lambda_0) \), we have
\[
|u_{1,t} (t)| \leq C_2 e^{(\lambda_0 + \varepsilon_2)t}
\]
holds for any \( t \geq T_\epsilon \) where \( C_2 \) is a large constant. Now for fixed \( \lambda \in (\lambda_0, \lambda_m) \), we have
\[(4.15)\]
for any \( t \geq T_\epsilon \) where \( C_3 \) is a large constant. If \( u_1 \) is a nontrivial solution to (4.11), then \( v_1 \) is a nontrivial solution to (4.12). Let \( v_2 \) be another solution of (4.12) which is linearly independent of \( v_1 \). Then (4.13) holds for \( v_2 \). Combining with (4.14) and (4.16), we have for any \( t \geq \max \{ T_1, T_\epsilon \} \),
\[
W (t) = v_1 v_2, t - v_2 v_1, t
\]
satisfies
\[(4.16)\]
we have
\[
W' (t) = (A - 2\lambda) W (t),
\]
Choosing \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough so that
\[
\lambda_0 - \lambda + \varepsilon_1 + \varepsilon_2 < 0,
\]
we conclude from (4.10) and (4.17) that \( W (t) \equiv 0 \) which contradicts to the assumption that \( v_1, v_2 \) are two linearly independent solutions. Hence \( v_1 \equiv 0 \) and \( u_1 \equiv 0 \).
Scaling of solutions

In this section, we will use a scaling argument to prove Theorems 1.2 and 1.6. Let \( h^{p,\eta} \) be the unique solution to (2.2) satisfying \( h(0) = \eta \neq (ap)^{-\frac{1}{\alpha}} \). When \( \eta = 0 \), \( h^{p,0} \) is the unique rupture solution. Let \( r_k^{p,\eta}, k = 1, 2, \ldots \), be the increasing sequence of positive critical points of \( h^{p,\eta} \). Then

\[
\begin{align*}
      h^{p,\eta,k}(x) &= \left( r_k^{p,\eta} \right)^{-\frac{2}{1+\alpha}} h^{p,\eta}(r_k^{p,\eta}|x|)
\end{align*}
\]

satisfies

\[
\begin{align*}
      \Delta h &= \frac{1}{h^{\frac{1}{\alpha}+(ap)^{\frac{1}{\alpha}\eta}}} (ap)^{\frac{1}{\alpha}\eta} \frac{1}{x} x \quad \text{in } B_1(0), \quad \frac{\partial h}{\partial \nu} = 0 \quad \text{on } \partial B_1(0) \quad \text{with} \quad p^{p,\eta,k} = p \left( r_k^{p,\eta} \right)^{\frac{2}{1+\alpha}}.
\end{align*}
\]

Let \( \bar{h}(p,\eta,k) = \frac{1}{|B_1(0)|} \int_{B_1(0)} h^{p,\eta,k}(x) \, dx = \frac{\left( r_k^{p,\eta} \right)^{-\frac{1}{1+\alpha}}}{|B_{r_k^{p,\eta}}(0)|} \int_{B_{r_k^{p,\eta}}(0)} h^{p,\eta}(x) \, dx. \) Then \( h^{p,\eta,k}(x) \) is a solution to (1.14) with \( \bar{h} = \bar{h}(p,\eta,k). \)

**Lemma 5.1.** For any \( p > 0 \) and \( \eta \geq 0, \eta \neq (ap)^{-\frac{1}{\alpha}}, \)

\[
h^{p,\eta}(x) = (ap)^{-\frac{1}{\alpha}} h^{\frac{1}{\alpha}+(ap)^{\frac{1}{\alpha}\eta}} \left( (ap)^{\frac{1}{\alpha}\eta} x \right).
\]

**Proof.** Let \( f(x) = (ap)^{-\frac{1}{\alpha}} h^{\frac{1}{\alpha}+(ap)^{\frac{1}{\alpha}\eta}} \left( (ap)^{\frac{1}{\alpha}\eta} x \right), \) we have \( f(0) = \eta \) and

\[
\Delta f(x) = \alpha p \left( \frac{1}{h^{\frac{1}{\alpha}+(ap)^{\frac{1}{\alpha}\eta}}} \right)^{-\alpha} - \frac{1}{\alpha} \frac{1}{f^{\alpha}} - p.
\]

So the lemma follows from the uniqueness of the radial solution. \( \square \)

**Corollary 5.2.** For each \( k, \)

\[
r_k^{p,\eta} = (ap)^{-\frac{1+\alpha}{\alpha}} r_k^{\frac{1}{\alpha}+(ap)^{\frac{1}{\alpha}\eta}} , \quad h^{p,\eta,k} = h^{\frac{1}{\alpha}+(ap)^{\frac{1}{\alpha}\eta},k} , \quad p^{p,\eta,k} = p^{\frac{1}{\alpha}+(ap)^{\frac{1}{\alpha}\eta},k}
\]

and

\[
\bar{h}(p,\eta,k) = \bar{h} \left( \frac{1}{\alpha}, (ap)^{\frac{1}{\alpha}\eta}, k \right)
\]

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Lemma 2.5 implies that for fixed \( \eta > 1, r_k^{\frac{1}{\alpha},\eta}, k = 1, 2, \ldots, \) are well separated. Hence, we can apply standard ODE theory to conclude that for each \( k = 1, 2, \ldots, r_k^{\frac{1}{\alpha},\eta}, \) viewed as a function of \( \eta, \) is continuous in \((1, \infty)\).
From Lemma 3.2 and noticing $r_k^{\frac{1}{\alpha}, \eta}$ is monotone increasing in $k$, we have for each $k = 1, 2, \cdots$, 
\[
\lim_{\eta \to \infty} r_k^{\frac{1}{\alpha}, \eta} = \infty.
\]
Let 
\[
R_k = \inf_{\eta > 1} r_k^{\frac{1}{\alpha}, \eta},
\]
therefore $R_k$ is monotone nondecreasing in $k$. Furthermore, the interval $(R_k, \infty)$ is contained in the range of $r_k^{\frac{1}{\alpha}, \eta}$. Given $R > 0$, let 
\[
p_k = \frac{1}{\alpha} \left( \frac{R}{R_k} \right)^{\frac{2}{\alpha + 1}},
\]
then $p_k$ is monotone nondecreasing in $k$. For any $p > p_k$, and for any $1 \leq i \leq k$, let $\eta_i \in (R_i, \infty)$ be such that 
\[
\eta_i = \left( \frac{p_k^{\frac{1}{\alpha}, \eta_i}}{R_k} \right)^{\frac{1}{1 + \alpha}}.
\]
Then we have 
\[
\bar{h}^{p, (\alpha p)^{-\frac{1}{\alpha}}, \eta_i} = (\alpha p)^{-\frac{1}{2 + \alpha}} r_k^{\frac{1}{\alpha}, \eta_i} = R,
\]
i.e., $h^{p, (\alpha p)^{-\frac{1}{\alpha}}, \eta_i}$ is a nontrivial smooth radial solution to (1.18) in $B_R(0)$. Since $h^{p, (\alpha p)^{-\frac{1}{\alpha}}, \eta_i}$, viewed as a function in $r$, has exactly $i - 1$ critical points in $(0, R)$, we have found $k$ distinctive solutions as desired.

**Remark 5.3.** For fixed $k \geq 1$, numerical computation suggests that $r_k^{\frac{1}{\alpha}, \eta}$ is not monotone increasing for $\eta \in [0, 1) \cup (1, \infty)$. Hence, given $p > p_0$, we may have two different solutions with the same number of critical points in $(0, R)$.

From Corollary 5.2 to get a solution to (1.14) through scaling, we can fix either $p$ or $\eta$. Without loss of generality, we assume $p = \frac{1}{\alpha}$, and $\eta \neq 1, \eta \geq 0$.

**Proof of Theorem 1.6.** All radial solutions can be obtained by scaling. So (1.14) has a rupture solution only when 
\[
\bar{h} = \bar{h}_k \equiv \bar{h} \left( \frac{1}{\alpha}, 0, k \right).
\]
Let 
\[
r_k \equiv r_k^{\frac{1}{\alpha}, 0},
\]
then 
\[
\bar{h}_k = \frac{(r_k)^{-\frac{1}{\alpha}}}{|B_{r_k}(0)|} \int_{B_{r_k}(0)} h^{r_k^{\frac{1}{\alpha}, 0}}(x) \, dx.
\]
Since 
\[
\lim_{r \to \infty} h^{r_k^{\frac{1}{\alpha}, 0}}(r) = 1,
\]
we have 
\[
\lim_{k \to \infty} \frac{1}{|B_{r_k}(0)|} \int_{B_{r_k}(0)} h^{r_k^{\frac{1}{\alpha}, 0}}(x) \, dx = 1.
\]
Hence the conclusion that 
\[
\lim_{k \to \infty} \sqrt{k \pi} h_k = 1
\]
follows from Corollary 2.9. \qed
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