Analytical theory of enhanced logarithmic Rayleigh scattering in amorphous solids

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The damping or attenuation coefficient of sound waves in solids due to impurities scales with the wavenumber to the fourth power, also known as Rayleigh scattering. In amorphous solids, Rayleigh scattering may be enhanced by a logarithmic factor although computer simulations offer conflicting conclusions regarding this enhancement and its microscopic origin. We present a tensorial replica field-theoretic derivation based on heterogeneous elasticity, which suggests that long-range (power-law) spatial correlations of elastic constants might be the possible origin of the logarithmic enhancement to Rayleigh scattering of phonons in amorphous solids. Conversely, internal stresses, or stress correlations, are not necessary, within the considered framework, to give rise to the logarithmic correction, although we cannot exclude that they might provide alternative routes to this effect.

I. INTRODUCTION

Amorphous solids exhibit anomalous thermal and vibrational properties at low temperature. Thanks to improved scattering experiments, as well as numerical simulations, recent years have witnessed huge achievements in our understanding of glassy materials. One interesting property, as reported in [1], is that long-wavelength phonons are more damped in glasses than in ordinary crystalline solids, with an attenuation or damping coefficient that scales with wavenumber $k$ as $\sim -k^{d+1}\ln k$ (in dimension $d$), with a logarithmic enhancement compared with the well-known Rayleigh scattering law $\sim k^{d+1}$, whose validity has never been questioned in the last fifty years of studies of sound attenuation in amorphous materials [2].

To be more specific, a compilation of many experiments with X-ray and light scattering demonstrates that the wavenumber dependence of the longitudinal sound attenuation coefficient, $\Gamma_L (k)$, is in general divided into three regimes [3–10]: (1) $\Gamma_L (k) \sim k^2$ for low $k$; (2) $\Gamma_L (k) \sim k^4$ for an intermediate $k$ regime; and (3) $\Gamma_L (k) \sim k^2$ for large $k$. The latter regime can be ascribed to thermal and anharmonic effects, since experiments are usually performed at finite temperature. On the other hand, most computer studies address sound attenuation problem at zero temperature in order to remove anharmonic effects. Regardless of system size, a recent numerical study of 2D systems reveals that the logarithmic correction to the cubic scaling, $\Gamma_\lambda (k) \sim -k^3 \ln k$ ($\lambda = L, T$ stands for longitudinal and transverse) emerges in the boson peak (BP) regime, while it disappears as the wavenumber approaches the continuum limit, where $\Gamma_\lambda (k) \sim k^3$ is recovered [11]. Authors in [1] even revisited data in experimental systems to confirm the damping coefficient indeed corresponds to the enhanced $-k^{d+1}\ln k$ law.

To rationalise the observed logarithmic correction to the Rayleigh law, one interpretation is to invoke the existence of correlated inhomogeneities of the elastic constants within the framework of heterogeneous elasticity (HE), yet neither quantitative nor qualitative arguments have been presented. Also, the possible relation between the logarithmic correction to the Rayleigh law and the long-range nature of elastic modulus has been questioned in [11]. From pure theoretical perspective, DeGiuli argued that the long-range stress correlations are responsible for this anomalous scattering phenomenon by applying ”Edwards field theory” to granular matter at long length scales [12, 13].

In this paper, by applying tensorial replica field theory to athermal amorphous systems with power-law decay in elastic constant correlations, we attempt to reveal the origin of the enhanced phonon damping, especially where the logarithmic enhancement is prompted. The analytical theory shows that the logarithmic enhancement is possibly due to the long-range power-law correlations of elastic constants [14–17]. We will consider systems with similar elastic property as in [11]. However, such systems usually have coupled internal longitudinal and transverse propagators, the explicit form of damping is thus not as clearly defined [13]. We then generalise our theory to other systems with explicit dependence of damping on wavenumber, where purely longitudinal (transverse) waves propagate through the solid, where we find the logarithmic enhancement in the same form $\sim -k^{d+1}\ln k$ as in the simulations of Ref. [1].

II. FORMALISM

Throughout this paper, we focus on 2D systems. All results can be generalized to 3D case. In elastic media, deformations of a generic material point (or a particle)
are expressed in terms of microscopic displacements \( \mathbf{u} \) defined as the current position of the particle at time \( t \), \( \mathbf{r}(\mathbf{r}, t) \) minus its initially position located at \( \mathbf{r} \), i.e., \( \mathbf{u} = \mathbf{r}(\mathbf{r}, t) - \mathbf{r} \). In the absence of body forces and assuming spatially uniform density \( \rho \), the Lagrangian form of the elastic wave equation can be written as \[ \rho \frac{\partial^2 u^\alpha(\mathbf{r})}{\partial t^2} = \frac{\partial}{\partial r^\beta} \left[ S^{\alpha\beta\kappa\chi}(\mathbf{r}) \frac{\partial u^\kappa(\mathbf{r})}{\partial r^\chi} \right] \] (1)

with \[ S^{\alpha\beta\kappa\chi}(\mathbf{r}) = C^{\alpha\beta\kappa\chi}(\mathbf{r}) + \delta^{\alpha\beta} \sigma^{\kappa\chi}(\mathbf{r}) \] (2)

where \( C^{\alpha\beta\kappa\chi} \) and \( \sigma^{\kappa\chi} \) are the elastic constants and the Cauchy stress in the reference configuration respectively. Greek subscripts refer to Cartesian coordinates and \( \delta^{\alpha\beta} \) denotes the Kronecker delta. We note that, with the pair interaction \( V_{ij} \), \( C^{\alpha\beta\kappa\chi}_{ij} = h_{ij} r_{ij} \delta^{\alpha\beta} n_i n_j \) where \( r_{ij} \) is the interatomic distance, \( \mathbf{u}_{ij} \) is the unit vector pointing from \( i \) to \( j \) and \( h_{ij} = V''_{ij}(r_{ij}) r_{ij}^2 - V'_{ij}(r_{ij}) r_{ij} \). Prime denotes the derivative with respect to distance. The second term on RHS in Eq. (2) involves the pair contributions to the internal stress field and hence carries long-range spatial correlations due to stress. Without loss of generality, we assume it is much smaller in amplitude than \( C^{\alpha\beta\kappa\chi}_{ij} \) and can be essentially ignored. Writing \( \mathbf{u}_{ij} = (\cos \theta_{ij}, \sin \theta_{ij}) \), the elastic constants appear to be of form \( C^{\alpha\beta\kappa\chi}_{ij} = h_{ij} \cos^2 \theta_{ij} \sin^4 \theta_{ij}, n = 0, ..., 4 \). There are, hence, five local constants for each pair, they are \[ C^{1}_{ij} = h_{ij}, \]
\[ C^{2}_{ij} = h_{ij} \cos(2\theta_{ij}), \]
\[ C^{3}_{ij} = h_{ij} \sin(2\theta_{ij}), \]
\[ C^{4}_{ij} = h_{ij} \cos(4\theta_{ij}), \]
\[ C^{5}_{ij} = h_{ij} \sin(4\theta_{ij}) \]

The contributions of each pair to the Lamé constants are \( \mu_{ij} = (1/8)(C^{1}_{ij} - C^{3}_{ij}) \) and \( \lambda_{ij} = (1/8)(C^{1}_{ij} + C^{3}_{ij}) \). We are able to express effective elastic constants \( S^{\alpha\beta\kappa\chi}_{ij} \approx C^{\alpha\beta\kappa\chi}_{ij} \) in terms of these five local constants:

\[ C^{xxx}_{ij} = \frac{C^{1}_{ij}}{8} + \frac{C^{2}_{ij}}{2} + \frac{3C^{4}_{ij}}{8} + \frac{C^{5}_{ij}}{8}, \]
\[ C^{xyy}_{ij} = \frac{C^{1}_{ij}}{8} + \frac{C^{2}_{ij}}{2} + \frac{3C^{4}_{ij}}{8}, \]
\[ C^{yyx}_{ij} = \frac{C^{1}_{ij}}{8} + \frac{C^{2}_{ij}}{2} + \frac{3C^{4}_{ij}}{8}, \]
\[ C^{yyy}_{ij} = \frac{C^{1}_{ij}}{8} - \frac{C^{2}_{ij}}{2} + \frac{3C^{4}_{ij}}{8}. \]

Then, Eq. (1) becomes

\[ \rho \frac{\partial^2 u^x(\mathbf{r})}{\partial t^2} = \frac{\partial}{\partial r^y} \left[ \left( \frac{C^{1}_{ij}}{8} + \frac{C^{2}_{ij}}{2} + \frac{3C^{4}_{ij}}{8} \right) \frac{\partial u^x}{\partial r^x} + \left( \frac{C^{5}_{ij}}{8} + \frac{C^{3}_{ij}}{4} \right) \frac{\partial u^y}{\partial r^y} \right] + \frac{\partial}{\partial r^x} \left[ \left( \frac{C^{1}_{ij}}{8} + \frac{C^{2}_{ij}}{2} + \frac{3C^{4}_{ij}}{8} \right) \frac{\partial u^x}{\partial r^y} + \left( \frac{C^{5}_{ij}}{8} + \frac{C^{3}_{ij}}{4} \right) \frac{\partial u^y}{\partial r^x} \right] \]

A. Toy model with vanishing \( C^1, C^2, C^4, C^5 \)

To probe the simplest possible scenario of long-range correlations in the elastic constants, we assume \( C^1(\mathbf{r}), C^2(\mathbf{r}), C^4(\mathbf{r}), C^5(\mathbf{r}) = 0 \) while \( C^3(\mathbf{r}) = C(\mathbf{r}) = \rho C_0 + \rho \tilde{C}(\mathbf{r}) \) is expressed in terms of its mean value plus a Gaussian random part, i.e. \( \tilde{C}(\mathbf{r}) = 0 \) and \( C(\mathbf{r}) = \frac{1}{r^2} \tilde{C}(\mathbf{r}) \) for some constants \( \gamma, \mu(\mathbf{r}) \) may depend on \( \mathbf{r} \) and scales as \( \sim \sigma(\chi^2) \) when \( r \to \infty \). In other words, only the effect of non-vanishing \( C^3 \) is considered, whose spatial autocorrelation scales as \( 1/r^2 \).

Equation (5) then reduces to

\[ C(\mathbf{r})C(\mathbf{r} + \mathbf{r}') = B(\mathbf{r}) = \frac{C^2}{r^2} \]
\[
\rho \frac{\partial^2 u^x(\mathbf{r})}{\partial t^2} = \frac{1}{4} \left[ \frac{\partial C(\mathbf{r})}{\partial r^x} \frac{\partial u^x}{\partial r^y} + \frac{\partial C(\mathbf{r})}{\partial r^y} \frac{\partial u^x}{\partial r^x} + 2C(\mathbf{r}) \frac{\partial^2 u^x}{\partial r^x \partial r^y} + \frac{\partial C(\mathbf{r})}{\partial r^y} \frac{\partial u^y}{\partial r^x} + \frac{\partial C(\mathbf{r})}{\partial r^x} \frac{\partial u^y}{\partial r^y} + C(\mathbf{r}) \left( \frac{\partial^2 u^y}{\partial r^x \partial r^y} + \frac{\partial^2 u^x}{\partial r^y \partial r^y} \right) \right]
\]
\[
= \frac{1}{4} \sum_{\alpha \neq \beta} \left[ \nabla_\alpha \nabla_\beta \right] u^x + \frac{1}{4} \sum_{\alpha} \left[ \nabla_\alpha \nabla_\alpha \right] u^y
\]
\[
\rho \frac{\partial^2 u^y(\mathbf{r})}{\partial t^2} = \frac{1}{4} \left[ \frac{\partial C(\mathbf{r})}{\partial r^x} \frac{\partial u^y}{\partial r^y} + \frac{\partial C(\mathbf{r})}{\partial r^y} \frac{\partial u^y}{\partial r^x} + 2C(\mathbf{r}) \frac{\partial^2 u^y}{\partial r^x \partial r^y} + \frac{\partial C(\mathbf{r})}{\partial r^y} \frac{\partial u^x}{\partial r^y} + \frac{\partial C(\mathbf{r})}{\partial r^x} \frac{\partial u^x}{\partial r^y} + C(\mathbf{r}) \left( \frac{\partial^2 u^x}{\partial r^y \partial r^y} + \frac{\partial^2 u^y}{\partial r^x \partial r^y} \right) \right]
\]
\[
= \frac{1}{4} \sum_{\alpha} \left[ \nabla_\alpha \nabla_\alpha \right] u^x + \frac{1}{4} \sum_{\alpha \neq \beta} \left[ \nabla_\alpha \nabla_\beta \right] u^y.
\]

In frequency space, upon letting \( z = \omega + i0 \), the equation of motion of the frequency-dependent displacement vector \( \mathbf{u}(z, \mathbf{r}) \) is (we have dropped the ring)

\[
A(z)u(\mathbf{r}, z) = 0,
\]
with

\[
A^{xx} = A^{yy} = -\rho z^2 - \frac{1}{4} \sum_{\alpha \neq \beta} \left[ \nabla_\alpha \nabla_\beta \right],
\]

\[
A^{xy} = A^{yx} = \frac{1}{4} \sum_{\alpha} \left[ \nabla_\alpha \nabla_\alpha \right].
\]  

The fluctuation of \( C(\mathbf{r}) \) may be implemented by the probability distribution for its fluctuating part,

\[
P[\dot{C}(\mathbf{r})] = P_0 \exp \left[ -\frac{1}{2} \int d^2 r d' r' B^{-1}(\mathbf{r} - \mathbf{r}') \dot{C}(\mathbf{r}') \right]
\]

where \( B^{-1} \) is the inverse of \( B(\mathbf{r} - \mathbf{r}') \) such that

\[
\int d^2 p B(\mathbf{r} - \mathbf{p}) B^{-1}(\mathbf{r} - \mathbf{q}) = \delta(\mathbf{r} - \mathbf{q}),
\]

while \( P_0 \) is a normalization factor. The Lagrangian is expressed as (scaled by \( \rho \)),

\[
L = \frac{1}{2} \int d^2 r u^T A u = u^x (A_{xx} u^x + A_{xy} u^y) + u^y (A_{yx} u^x + A_{yy} u^y)
\]

\[
= \frac{1}{2} \int d^2 r \left\{ -z^2 u^x \cdot u - \frac{1}{4} \sum_{\alpha \neq \beta} u^x \left[ \nabla_\alpha \nabla_\beta u^x \right] - \frac{1}{4} \sum_{\alpha} u^x \left[ \nabla_\alpha \nabla_\alpha u^x \right] - \frac{1}{4} \sum_{\alpha \neq \beta} u^y \left[ \nabla_\alpha \nabla_\beta u^y \right] - \frac{1}{4} \sum_{\alpha} u^y \left[ \nabla_\alpha \nabla_\alpha u^y \right] \right\}
\]

\[
= \frac{1}{2} \int d^2 r \left\{ -z^2 u^x \cdot u - \frac{1}{4} \sum_{\alpha \neq \beta} \left[ \nabla_\alpha \left[ u^x \nabla_\beta u^x \right] - \left( \nabla_\alpha u^x \right) \left( \nabla_\beta u^x \right) \right] - \frac{1}{4} \sum_{\alpha} \left[ \nabla_\alpha \left[ u^x \nabla_\alpha u^x \right] - C(\nabla_\alpha u^x) (\nabla_\alpha u^x) \right] - x \leftrightarrow y \right\}
\]

\[
= \frac{1}{2} \int d^2 r \left\{ -z^2 u^x \cdot u - \frac{1}{4} \nabla_x \left[ u^x \nabla_y u^x + u^y \nabla_x u^y + u^y \nabla_x u^y + u^x \nabla_x u^y \right] - \frac{1}{4} \nabla_y \left[ x \leftrightarrow y \right] \right\}
\]

\[
+ \frac{1}{2} C(\nabla_x u^x)(\nabla_y u^x) + (\nabla_x u^y)(\nabla_y u^y) + (\nabla_x u^y)(\nabla_y u^y) + (\nabla_y u^x)(\nabla_y u^x)) \right\}
\]

\[
= \frac{1}{2} \int d^2 r \left\{ -z^2 u^x \cdot u - \frac{1}{4} \nabla_x \left[ u^x \nabla_y u^x + u^y \nabla_x u^y + u^y \nabla_x u^x + u^x \nabla_x u^y \right] + \frac{1}{2} C(\nabla \cdot u)(\nabla_x u^y + \nabla_y u^x) \right\}
\]

\[
= \frac{1}{2} \int d^2 r \left\{ -z^2 u^x \cdot u + \frac{1}{2} C(\nabla \cdot u)(\nabla_x u^y + \nabla_y u^x) \right\}
\]
\[ \langle Z(0) \rangle \approx \lim_{n \to 0} \int \mathcal{D}[\mathbf{u}_n(x)] \mathcal{D}[\mathcal{C}(x)] P_0 \times \exp \left[ -\frac{1}{2} \sum_{a=1}^{n} \int d^2 r \left\{ -z^2 \mathbf{u}_a(x)^2 + \frac{C(r)}{2} \left( \nabla \cdot \mathbf{u}_a(x) \right) \left( \nabla_x u_a^y + \nabla_y u_a^x \right) \right\} - \frac{1}{2} \int d^2 r d^2 r' \mathcal{C}(x) B^{-1}(x-x') \mathcal{C}(x') \right] \]

\[
\approx \lim_{n \to 0} \int \mathcal{D}[\mathbf{u}_n(x)] \mathcal{D}[\mathcal{C}(x)] \exp \left[ -\frac{1}{2} \sum_{a=1}^{n} \int d^2 r \left\{ -z^2 \mathbf{u}_a(x)^2 + \frac{1}{2} C_0 \left( \nabla \cdot \mathbf{u}_a(x) \right) \left( \nabla_x u_a^y + \nabla_y u_a^x \right) \right\} \right.
\]

\[
\left. + \frac{1}{16} \sum_{a,b=1}^{n} \int d^2 r d^2 r' \left( \nabla \cdot \mathbf{u}_a(x) \right) \left( \nabla_x u_a^y(x) + \nabla_y u_a^x(x) \right) B(x-x') \left( \nabla \cdot \mathbf{u}_b(x') \right) \left( \nabla_x u_b^y(x') + \nabla_y u_b^x(x') \right) \right]\]

\[
(11)
\]

where \( a = 1, ..., n \) is a replica index (same as \( b \)), and the \( n \to 0 \) limit eliminates the determinant factor. By means of a Hubbard-Stratonovich transformation, we introduce

\[
\langle Z(0) \rangle \approx \lim_{n \to 0} \int \mathcal{D}[\mathbf{u}_n(x)] \mathcal{D}[\Lambda_{ab}^{\alpha \kappa \chi \beta}(x,x',z)] P_0 \]

\[
\times \exp \left\{ -\frac{1}{2} \sum_{a=1}^{n} \int d^2 r \left\{ -z^2 \mathbf{u}_a(x)^2 - \frac{C_0}{4} \left( \sum_{\alpha \neq \beta} \left( u_a^{x \alpha} \nabla_\alpha u_a^{x \beta} + u_a^{y \alpha} \nabla_\alpha u_a^{y \beta} + \sum_{\alpha} (u_a^{x \alpha} \nabla_\alpha u_a^{y \beta} + u_a^{y \alpha} \nabla_\alpha u_a^{x \beta}) \right) \right\} \right\}
\]

\[
\left. - \frac{1}{4} \sum_{a,b=1}^{n} \sum_{\alpha \kappa \chi \beta=x,y} \int d^2 r d^2 r' \left[ \Lambda_{ab}^{\alpha \kappa \chi \beta}(x,x',z) B^{-1}(x-x') \left( \alpha \kappa \chi \beta \sum_{\kappa \chi'} \Lambda_{ab}^{\alpha \kappa \chi \beta}(x,x',z) - u_a^{x \alpha}(x) \nabla_\kappa \Lambda_{ab}^{\alpha \kappa \chi \beta}(x,x',z) \nabla_\chi u_b^{y \beta}(x') \right) \right] \right\}
\]

\[
(12)
\]

where \( \Lambda_{ab}^{\alpha \kappa \chi \beta} = 0 \) if \( \alpha = \beta, \kappa = \chi \) or \( \alpha \neq \beta, \kappa \neq \chi \).

The way \( \Lambda_{ab}^{\alpha \kappa \chi \beta} \) is introduced is to make Eq. (12) consistent with Eq. (7). \( \Lambda_0 \) is a normalization constant. The generating function including source \( J_{ab}(x,x') \) is
\[ \langle Z(J) \rangle = \lim_{n \to 0} \int \mathcal{D}[\mathbf{u}_n(r)] \mathcal{D}[\Lambda_{ab}^\alpha \chi^\beta (\mathbf{r}, \mathbf{r}')] N_0 \]
\[ \times \exp \left\{ -\frac{1}{2} \sum_{a=1}^{n} \int d^2 r \left[ -Z^2 \mathbf{u}_n(r)^2 - \frac{C_0}{4} \left( \sum_{a \neq \beta} (u_a^x \nabla_\alpha \nabla_\beta u_a^x + u_a^y \nabla_\alpha \nabla_\beta u_a^y) + \sum_\alpha (u_a^x \nabla_\alpha u_a^x + u_a^y \nabla_\alpha u_a^y) \right) \right] \right\} \]
\[ \times \exp \left\{ -\frac{1}{4} \sum_{a,b=1}^{n} \sum_{\alpha \chi = x,y} \int d^2 r d^2 r' \sum_{\alpha' \chi'} \Lambda_{ab}^{\alpha \chi \beta} B^{-1}(\mathbf{r} - \mathbf{r}') \Lambda_{a'b'}^{\alpha' \chi' \beta} - u_a^x(r) \nabla_\alpha \Lambda_{ab}^{\alpha \chi \beta} \nabla_\chi u_b^y(\mathbf{r}') + 4 J_{ab}^{\alpha \chi \beta} \right\} \]
\[ \times \exp \left\{ -\frac{1}{2} \sum_{a,b=1}^{n} \sum_{\alpha \chi = x,y} \int d^2 r d^2 r' \left[ \mathbf{u}_n(r) A_{ab}^{\alpha \chi \beta} \mathbf{u}_n(r') \right] + \frac{1}{2} \sum_{\alpha' \chi'} \Lambda_{ab}^{\alpha \chi \beta} B^{-1}(\mathbf{r} - \mathbf{r}') \Lambda_{a'b'}^{\alpha' \chi' \beta} + 2 J_{ab}^{\alpha \chi \beta} \right\} \]  
(13a)

where

\[ A_{ab}(\Lambda) \equiv \delta(\mathbf{r} - \mathbf{r}') \left( -Z^2 - \frac{C_0}{4} \sum_{\alpha \neq \beta} \nabla_\alpha \nabla_\beta \right) - \frac{C_0}{4} \sum_\alpha \nabla_\alpha \nabla_\alpha \right) - \frac{1}{4} \sum_{\kappa \chi} \left( \nabla_\kappa A_{ab}^{\kappa \chi \chi} (\mathbf{r}, \mathbf{r}', z) \nabla_\chi \Lambda_{ab}^{\alpha \chi \beta} (\mathbf{r}, \mathbf{r}', z) \nabla_\chi \right). \]  
(13b)

By evaluating derivatives of \( \langle Z(J) \rangle \) with respect to \( J_{ab}^{\alpha \chi \beta} \) at \( J_{ab}^{\alpha \chi \beta} = 0 \), we are able to find the averaged Green’s function of \( \Lambda_{ab}^{\alpha \chi \beta} \). Physically, \( \langle \Lambda_{ab}^{\alpha \beta} (\mathbf{r}, \mathbf{r}') \rangle \) measures the average response of the \( \alpha \) component of the displacement field at \( \mathbf{r} \) to an impulse in the \( \beta \)th component at \( \mathbf{r}' \).

Integrating \( \mathcal{D}[\mathbf{u}] \) out in Eq. (12), we obtain a field theory involving only the \( \Lambda \) field:

\[ \langle Z(0) \rangle \propto \lim_{n \to 0} \int \mathcal{D}[\Lambda] \exp \left\{ -\frac{1}{2} \sum_{a,b=1}^{n} \sum_{\alpha \beta \kappa \chi} \left( \ln \det A_{ab}^{\alpha \chi \beta} \right) + \frac{1}{2} \sum_{\alpha' \chi'} \int d^2 r d^2 r' \Lambda_{ab}^{\alpha \chi \beta} B^{-1}(\mathbf{r} - \mathbf{r}') \Lambda_{a'b'}^{\alpha' \chi' \beta} \right\} . \]  
(14)

Expanding in \( \hat{\Lambda} \), \( A_{ab}(\Lambda) \) is written as

\[ A_{ab}(\Lambda) = A_{ab}(\Lambda_0) - \frac{1}{4} \sum_{\kappa \chi} \left( \nabla_\kappa \hat{A}^{\kappa \chi \chi} (\mathbf{r}, \mathbf{r}', z) \nabla_\chi \Lambda_{ab}^{\alpha \chi \beta} (\mathbf{r}, \mathbf{r}', z) \nabla_\chi \right) \equiv A_{ab}(\Lambda_0) + \hat{A}_{ab}(\hat{\Lambda}). \]  
(16)

Making use of the identity

\[ \ln \det(A(\Lambda_0) + \hat{A}(\hat{\Lambda})) = \ln \det(A(\Lambda_0)) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr}(A^{-1}{\hat{A}} \cdots A^{-1}{\hat{A}}) \]  
(17)

The corresponding saddle-point equations can be solved with a replica-diagonal field \( \sum_{\kappa \chi} \nabla_\kappa A_{ab}^{\alpha \chi \beta} (\mathbf{r}, \mathbf{r}', z) \nabla_\chi = \)
Taking diagonal saddle points need only be taken into account at the stage of renormalization. Then we have 
\[ \ln \det(A_0) = \text{Tr} \ln(A_0) \] and \( \langle \Sigma^{\alpha \beta} \rangle \) can be determined by taking
\[ \frac{\delta}{\delta \Sigma^{\alpha \beta}} \left( \text{Tr} \ln(A(\Sigma^{\alpha \beta})) + \frac{1}{2} \int d^2 r d^2 r' \Sigma^{\alpha \beta} B^{-1}(\mathbf{r} - \mathbf{r}') \Sigma^{\alpha \beta} \right) = 0 \] (18)
which yields
\[ \langle \Sigma^{\alpha \beta}_0 \rangle = \frac{1}{4} \sum_{\kappa \chi} \nabla_{\kappa} B(\mathbf{r} - \mathbf{r}') \nabla_{\beta} (G_0(r^\alpha, r'^\chi, z)) \epsilon^{\alpha \kappa \chi \beta} \] (19a)
\[ G_0(r^x, r'^x, z) = \left[ -z^2 - \frac{(C_0 + \Sigma_{0 x})}{4} \left( \nabla_x \nabla_y + \nabla_y \nabla_x \right) \right]^{-1} \] (19b)
\[ G_0(r^y, r'^y, z) = \left[ -z^2 - \frac{(C_0 + \Sigma_{0 y})}{4} \left( \nabla_x \nabla_y + \nabla_y \nabla_x \right) \right]^{-1} \] (19c)
\[ G_0(r^x, r'^y) = \left[ \frac{(C_0 + \Sigma_{x y})}{4} \left( \nabla_x \nabla_y + \nabla_y \nabla_x \right) \right]^{-1} \] (19d)
\[ G_0(r^y, r'^x) = \left[ \frac{(C_0 + \Sigma_{x y})}{4} \left( \nabla_x \nabla_y + \nabla_y \nabla_x \right) \right]^{-1} \] (19e)

where \( \epsilon^{\alpha \kappa \chi \beta} = 0 \) if \( \alpha = \beta, \kappa = \chi \) or \( \alpha \neq \beta, \kappa \neq \chi \). Translational invariance holds after taking the ensemble average, hence the CPA Green’s function \( G_0 \) depends only on the difference between two points in space.

### B. Theory with non-zero \( C^1, C^2, C^4, C^5 \)

We weaken our condition on the other elastic constants by letting \( C^1, C^2, C^4 \) and \( C^5 \) be all non-zero constants. The propagator \( A \) in \( A(z)u(\mathbf{r}, z) = 0 \) takes the form (scaled with \( \rho \))
\[ A^{\alpha \beta} = -z^2 \delta^{\alpha \beta} - \sum_{\kappa \chi} C^{\alpha \kappa \chi \beta} \nabla_{\kappa} \nabla_{\chi} - \frac{1}{4} \sum_{\kappa \chi} [\nabla_{\kappa} (\tilde{C} \nabla_{\chi})] \epsilon^{\alpha \kappa \chi \beta} \] (20)
where \( C^{\alpha \kappa \chi \beta} \) corresponds to the \( r \)-independent part of elastic constants \( C^i, i = 1, 2, 3, 4, 5 \). The explicit form is not important and we do not provide it here. The Lagrangian becomes
\[ L = \frac{1}{2} \int d^2 r \left\{ -z^2 u^2 + \frac{1}{2} \tilde{C} (\nabla \cdot u)(\nabla u^y + \nabla y u^x) \right. \]
\[ \left. - \sum_{\alpha \beta \kappa \chi} u^{\alpha} C^{\alpha \kappa \chi \beta} \nabla_{\kappa} \nabla_{\chi} u^{\beta} \right\} \] (21)
and \( \langle Z(J) \rangle \) is

\[ \langle Z(J) \rangle = \lim_{n \to 0} \int D[u^{\alpha}](\mathbf{r}) D[A^{\alpha \kappa \chi \beta}(\mathbf{r}, \mathbf{r}', z)] \Lambda_0 \]
\[ \times \exp \left\{ -\frac{1}{2} \sum_{\alpha, \beta = 1}^{n} \sum_{\alpha \kappa \beta = x, y} \int d^2 r d^2 r' \left[ u^{\alpha}_{\alpha}(\mathbf{r}) A_{\alpha \beta}(\alpha \kappa \beta_{x, y}) u^{\beta}_{\beta}(\mathbf{r}') + \frac{1}{2} \sum_{\kappa' \chi'} A^{\alpha \kappa \chi \beta}(\mathbf{r} - \mathbf{r}') A^{\alpha \kappa' \chi' \beta}(\mathbf{r} - \mathbf{r}') + 2 J_{\alpha \beta} A_{\alpha \beta} \right] \right\} \] (22a)

with
\[ A_{\alpha \beta}^{\alpha \kappa \chi \beta}(\mathbf{r}) \equiv \delta(\mathbf{r} - \mathbf{r}') \left( -z^2 \delta^{\alpha \beta} - \sum_{\kappa \chi} C^{\alpha \kappa \chi \beta} \nabla_{\kappa} \nabla_{\chi} \right) - \frac{1}{4} \sum_{\kappa \chi} \nabla_{\kappa} A_{\alpha \beta}^{\alpha \kappa \chi \beta}(\mathbf{r}, \mathbf{r}', z) \nabla_{\chi} \] (22b)

Again, letting \( \sum_{\kappa \chi} \nabla_{\kappa} A_{\alpha \beta}^{\alpha \kappa \chi \beta} \nabla_{\chi} = \sum_{\kappa \chi} \nabla_{\kappa} \sum_{\beta \alpha} \delta_{ab} \nabla_{\chi} \) and finding the saddle point of \( \text{Tr} \ln A(\Sigma^{\alpha \beta}) + \frac{1}{2} \int d^2 r d^2 r' \Sigma^{\alpha \beta} B(\mathbf{r} - \mathbf{r}') \Sigma^{\alpha \beta} \), we obtain the self-consistent equations for the self-energy:
\begin{align}
\langle \Sigma_0^{\alpha\beta} \rangle &= \frac{1}{4} \sum_{\kappa \chi} \nabla_\kappa B(\mathbf{r} - \mathbf{r}^\prime) \nabla_\chi (G_0(r^\alpha, r^\beta, z)) \epsilon^{\alpha \kappa \chi \beta} \\
G_0(r^x, r^x, z) &= \left[ -z^2 - \left( \frac{C^4}{8} + \frac{C^2}{2} + \frac{3C^1}{8} \right) \nabla_y \nabla_y - \left( \frac{C^1}{8} - \frac{C^4}{8} \right) \nabla_y \nabla_y - \left( \frac{C^5}{8} + \frac{C_0 + \Sigma_0^{xy}}{4} \right) \left( \nabla_y \nabla_y + \nabla_x \nabla_x \right) \right]^{-1} \\
G_0(r^y, r^y, z) &= \left[ -z^2 - \left( \frac{C^4}{8} - \frac{C^2}{2} + \frac{3C^1}{8} \right) \nabla_y \nabla_y - \left( \frac{C^1}{8} - \frac{C^4}{8} \right) \nabla_y \nabla_y - \left( \frac{C^5}{8} + \frac{C_0 + \Sigma_0^{yy}}{4} \right) \left( \nabla_y \nabla_y + \nabla_x \nabla_x \right) \right]^{-1} \\
G_0(r^x, r^y, z) &= \left[ -\left( \frac{C^1}{8} - \frac{C^4}{8} \right) \left( \nabla_y \nabla_y + \nabla_x \nabla_x \right) - \left( \frac{C_0 + \Sigma_0^{xy}}{4} \right) \left( \nabla_y \nabla_y + \nabla_x \nabla_x \right) \right]^{-1} \\
G_0(r^y, r^x, z) &= \left[ -\left( \frac{C^1}{8} - \frac{C^4}{8} \right) \left( \nabla_y \nabla_y + \nabla_x \nabla_x \right) - \left( \frac{C_0 + \Sigma_0^{yx}}{4} \right) \left( \nabla_y \nabla_y + \nabla_x \nabla_x \right) \right]^{-1} \\
\end{align}

Defining the Fourier transform as
\[ \Sigma(k, z) = \int d^2(\mathbf{r} - \mathbf{r}^\prime) e^{ik(\mathbf{r} - \mathbf{r}^\prime)} \Sigma(\mathbf{r} - \mathbf{r}^\prime, z), \tag{24} \]

\begin{align}
\langle \Sigma_0^{\alpha\beta} \rangle &= \frac{1}{4} \sum_{\kappa \chi} \epsilon^{\alpha \kappa \chi \beta} k_\kappa k_\chi \int d^2q \hat{B}(k - q) \langle G_0(q) \rangle \\
\hat{B}(k) &= \int d^2re^{ik\cdot r} B(r) \\
G_0(k^x, k^x, z) &= \left[ -z^2 - \left( \frac{C^4}{8} + \frac{C^2}{2} + \frac{3C^1}{8} \right) k^x k^x - \left( \frac{C^1}{8} - \frac{C^4}{8} \right) k^y k^y - \left( \frac{C^5}{4} + \frac{C_0 + \Sigma_0^{xy}}{2} \right) k^x k^x \right]^{-1} \\
G_0(k^y, k^y, z) &= \left[ -z^2 - \left( \frac{C^4}{8} - \frac{C^2}{2} + \frac{3C^1}{8} \right) k^y k^y - \left( \frac{C^1}{8} - \frac{C^4}{8} \right) k^x k^x - \left( \frac{C^5}{4} + \frac{C_0 + \Sigma_0^{yy}}{2} \right) k^x k^x \right]^{-1} \\
G_0(k^x, k^y) &= \left[ -\left( \frac{C^1}{4} - \frac{C^4}{4} \right) k^x k^y - \left( \frac{C_0 + \Sigma_0^{xy}}{4} \right) k^2 - \frac{C^5}{8} (k^x k^x - k^y k^y) \right]^{-1} \\
G_0(k^y, k^x) &= \left[ -\left( \frac{C^1}{4} - \frac{C^4}{4} \right) k^x k^y - \left( \frac{C_0 + \Sigma_0^{yx}}{4} \right) k^2 - \frac{C^5}{8} (k^x k^x - k^y k^y) \right]^{-1} \\
\end{align}

which must be solved self-consistently since the self-energy of the Green’s function involves the full propagator itself. In the weak scattering limit, approximate solutions are possible because \( \text{Im} \Sigma(k, z) \) is small compared with \( C_0 \) and also the imaginary part of the propagator takes the form of a \( \delta \)-function, \( \text{Im} \langle G_0(q) \rangle \propto -\delta(\omega^2 - q^2) \) upon averaging over all possible directions of dummy variable \( q \) and upon re-scaling redundant constants. The correlation function in Eq. (25a) may be evaluated for acoustic dispersion relation \( |\mathbf{q}| = \omega \) and pulled out of the integral via introducing an angular average of the Fourier transform of the correlation function over the directions of the unit vector \( \hat{q} \).

\[ \langle \hat{B}(k - \omega \hat{q}) \rangle_{\hat{q}} \equiv \frac{\int d\Omega_q \hat{B}(k - \omega \hat{q})}{\int d\Omega_q} \]

Since the correlation function takes the form of power-law decay, we calculate the angular average as \( \langle B \rangle \equiv \langle \hat{B}(k - \omega \hat{q}) \rangle_{k=\omega} \) to be

\[ \langle B \rangle \propto \int d\Omega_q d^2r e^{i\omega(k - \hat{q}) \cdot \mathbf{r}} B(r) \]

\[ \propto \int_0^\infty dr J_0^2(\omega r) B(r) r \]

\[ \propto I_0(\omega a) K_0(\omega a) \tag{27} \]
where $J_0, I_0$ and $K_0$ are (modified) Bessel functions. The detailed derivation is outlined in Appendix A. When $\omega \to 0$, we note that $I_0(\omega a) \sim 1$ and $K_0(\omega a) \sim -\ln(\omega a)$. The imaginary part of the propagator after integration over $q$ takes the form of density of states,

$$-\int d^2 q \text{Im}(G_0(q)) \sim N(\omega^2).$$

Combining results (25a), (27) and (28), we get, when linear dispersion $k = \omega$ holds and $\omega$ is small, the averaged self-energy (susceptibility) as:

$$\text{Im} \Sigma_0(k) \sim -k^2 \ln k.$$  \hspace{1cm} (29)

A similar result was obtained by John and Stephen \cite{15} in a different context of Anderson localization where a scalar model with power-law correlation in the spatially varying atomic masses was considered. To our knowledge the one presented here is the first derivation of this effect in the context of phonon propagation in elastic media, thus accounting for the full tensorial nature of the problem.

We note that there is no purely longitudinal and transverse wave with respect to direction of $k$. This is different from the cases considered in \cite{14,16}. However, cross terms (25e) and (25f) essentially contribute nothing to the density of states. One might define a more general relation between damping and self-energy over different directions. Hence Eq. (29) demonstrates that the self-energy of the phonon Green’s function, which is closely related to the phonon damping coefficient, does indeed exhibit a logarithmic enhancement correction to the Rayleigh law as a result of power-law spatial correlations in at least the elastic constant $C^3$. Hence this result holds for materials that are described within the heterogeneous elasticity framework.

C. Theory with non-zero $C^1, C^4, C^5$ and long-range correlations in $C^2$ and $C^3$

In addition to letting $C^3(l) \equiv \rho C_3 + \rho \tilde{C}_3(l)$, we further require $C^2(l) \equiv \rho C_2 + \rho C_2(l)$ with $C_{2,3}(l) = 0$ and $\tilde{C}_{2,3}(l)C_{2,3}(l + l') = B_{2,3}(l') = \frac{\gamma^2_{3,2}}{r_{2,3}^{\gamma_{3,2}}}$. In this case, the configurational average is due to spatial fluctuations of both $C^2$ and $C^3$ and is hence given by

$$P[\tilde{C}(l)] \propto \exp \left[ -\frac{1}{2} \sum_{i=2,3} \int d^2 r d^2 r' \tilde{C}_i(l) B_i^{-1}(l - l') \tilde{C}_i(l') \right].$$

(30)

To implement the same formalism as above, we now introduce two effective fields to get the CPA for the one-particle Green’s function. Eqs. (20) and (22b) become

$$A^{\alpha \beta} = -z^2 \delta_{\alpha \beta} - \sum_{\kappa \chi} C^{\alpha \kappa \chi \beta} \nabla_\kappa \nabla_\chi - \frac{1}{4} \sum_{\kappa \chi} [\nabla_\kappa (\tilde{C}_2 \nabla_\chi)] C^{\alpha \kappa \chi \beta} + \frac{1}{2} \nabla_\alpha \tilde{C}_2 \nabla_\beta \delta_{\alpha \beta} \eta_{\alpha \beta}$$

(31)

$$A^{\alpha \chi \beta}_{ab} (A_2, A_3) \equiv \delta(l - l') \left( -z^2 \delta_{\alpha \beta} - \sum_{\kappa \chi} C^{\alpha \kappa \chi \beta} \nabla_\kappa \nabla_\chi \right) - \frac{1}{4} \sum_{\kappa \chi} \nabla_\kappa A^{\alpha \chi \beta}_{ab,3} (l - l', z) \nabla_\chi + \frac{1}{2} \nabla_\alpha A^{\alpha \beta}_{ab,2} \nabla_\beta \delta_{\alpha \beta} \eta_{\alpha \beta}$$

(32)

where $\eta_{xx} = 1$, $\eta_{yy} = -1$. The self-consistent equations take the following form
\[
\begin{align*}
\langle \Sigma_{0,2}^{\alpha\beta} \rangle &= \frac{1}{2} \nabla_\alpha B_2(\mathbf{r} - x) \nabla_\beta (G_0(r^\alpha, r^\beta, z)) \eta_{\alpha\beta} \\
\langle \Sigma_{0,3}^{\alpha\beta} \rangle &= \frac{1}{4} \sum_{\kappa} \nabla_\kappa B_3(\mathbf{r} - x) \nabla_\chi (G_0(r^\alpha, r^\beta, z)) \epsilon^{\alpha\kappa\chi\beta} \\
G_0(r^x, r^y, z) &= \left[ -z^2 - \left( \frac{C_4}{8} + \frac{C_2 + \Sigma_{0,2}^{xy}}{2} + 3C_1 \right) \nabla_y \nabla_x - \left( \frac{C_4}{8} - \frac{C_4}{8} \right) \nabla_y \nabla_y - \left( \frac{C_5}{8} + \frac{(C_3 + \Sigma_{0,3}^{xy})}{4} \right) \nabla_y \nabla_y \right]^{-1} \\
G_0(r^x, r^y, z) &= \left[ -z^2 - \left( \frac{C_4}{8} - \frac{C_2 - \Sigma_{0,2}^{yy}}{2} + 3C_1 \right) \nabla_y \nabla_y - \left( \frac{C_4}{8} - \frac{C_4}{8} \right) \nabla_y \nabla_x - \left( \frac{C_5}{8} + \frac{(C_3 + \Sigma_{0,3}^{yy})}{4} \right) \nabla_y \nabla_y \right]^{-1} \\
G_0(r^x, r^z, z) &= \left[ -z^2 - \left( \frac{C_4}{8} - \frac{C_2 - \Sigma_{0,2}^{xy}}{2} + 3C_1 \right) \nabla_y \nabla_y - \left( \frac{C_4}{8} - \frac{C_4}{8} \right) \nabla_y \nabla_x - \left( \frac{C_5}{8} + \frac{(C_3 + \Sigma_{0,3}^{xy})}{4} \right) \nabla_y \nabla_y \right]^{-1}
\end{align*}
\]

Using the same arguments as in the last section, we can easily verify that \( \text{Im} \Sigma_{0,2} \sim -k^2 \ln k \) and \( \text{Im} \Sigma_{0,3} \sim -k^2 \ln k \) at small \( k = \omega \). Hence, the logarithmic enhancement to Rayleigh scattering law remains confirmed also in the case of power-law spatial correlations in two elastic constants, \( C^2 \) and \( C^3 \).

### III. The Case of Purely Longitudinal Waves

The main finding we want to show is that the presence of \( 1/r^{d-1} \) correlations in elasticity cause the logarithmic enhancement of scattering. However, the cases we have considered above do not present purely longitudinal/transverse wave components and hence it is not easy to directly link the averaged response \( \Sigma \) to the damping coefficient of longitudinal and transverse phonons. However, we notice that, in \[10\], if the local sound velocity has the same decaying behavior as \( C^3(z) \), then we would deduce that the damping is \( \Gamma(k) \sim -k^3 \ln k \). To see it more clearly, we consider longitudinal waves in a random medium with an elastic constant

\[
\lambda + \frac{2\mu}{m_0} = c^2(z) = c_0^2 + \Delta(z).
\]

Here, \( \lambda \) and \( \mu \) are the two Lame’s constants (note that, in the previous sections, we assumed that the Lame’s constants do not vary in space, in agreement with \[1\]). \( c(z) \) is the local sound velocity and \( c_0 \) is the spatially averaged one, which means \( \Delta = 0 \). We also let \( \Delta(z) \DeltaMZ = B(z) \). The equation of motion in frequency space for the matrix of Green’s functions \( G_{ij}(x, x', z) \) of the waves is

\[
\sum_{l=1}^{2} (z^2 \delta_{ii} + \nabla_i \tilde{K}(x) \nabla_i) G_{ij}(x, x', z) = -\delta_{jj} \delta(x - x').
\]

Upon replacing spatial fluctuation by effective field \( \Sigma(x, x', z) \), one can show that

\[
\text{Im} \Sigma_L(k, z) \propto k^2 \langle \tilde{B}(k, \omega) \rangle \int d^2 q \text{Im} G_L(q, z)
\]

\[
\sim -\omega^2 \ln \omega = -k^2 \ln k
\]

with \( k = \omega \). Since the dynamical structure factor \( S_L(k, z) \) for longitudinal excitations is given by \[20\]

\[
S_L(k, \omega) = \frac{\text{Im} \langle \Sigma_L(k, z) \rangle}{\pi \left[ n(\omega) + 1 \right]}
\]

\[
= \frac{\left[ n(\omega) + 1 \right] 2k^4 \text{Im} \Sigma(\omega)}{\pi \left\{ k^2 c_L(\omega)^2 - \omega^2 \right\}^2 + 4k^4 \text{Im} \Sigma(\omega)^2}
\]

\[
= \frac{1}{\pi} \left[ n(\omega) + 1 \right] \frac{k^2}{2 \omega} \frac{k^2 \text{Im} \Sigma(\omega) / \omega}{\left[ k^2 c_L(\omega)^2 - \omega^2 \right] + \left[ k^2 \text{Im} \Sigma(\omega) / \omega \right]^2}
\]

where the phonon damping coefficient is given by

\[
\Gamma(\omega) = \frac{2}{c_L(\omega)^2} \omega \text{Im} \Sigma(\omega) \sim -\omega^2 \ln \omega \sim -k^2 \ln k,
\]

where \( c_L \) is the longitudinal speed of sound, and the linear dispersion relation of \( k \) and \( \omega \) is implied at small \( \omega \). Generally, in \( d \)-dimensional space, we would expect the damping \( \Gamma(k) \sim -k^{d+1} \ln k \). Thus, if a suitable form for the dynamical structure factor \( S(k, \omega) \) in the general case of entangled longitudinal and transverse components could be found, then the damping coefficient would be proportional to the imaginary part of \( \Sigma(k, z) \) (as the result of linear response theory), and then Eq. (37) would hold also for the cases discussed in the previous sections.
IV. CONCLUSION

We have theoretically shown, in two dimensions, that long-range elastic correlations cause a logarithmic enhancement to Rayleigh scattering of phonons in amorphous systems where internal stresses are absent. In particular, when longitudinal and transverse waves can be disentangled, the phonon damping coefficient is found to exhibit a logarithmic enhancement factor in front of the Rayleigh law in the low frequency (wavenumber) regime. Recent work \cite{22} on jammed harmonic sphere packings showed that the Rayleigh scattering law without the logarithmic correction is observed in the low wavevector limit in those systems, however the authors showed that in the systems they studied there were no long-range correlations in the elastic moduli.

We also note that many numerical simulations addressing this problem so far only extract affine elastic constants, while the nonaffine contribution to elasticity \cite{13,21} might be important in some systems and should be examined in detail in future work. Moreover, Ref. \cite{22} argued on the basis of numerical data on computer glasses that Rayleigh scaling is expected over, Ref. \cite{22} argued on the basis of numerical data on computer glasses that Rayleigh scaling is expected over, Ref. \cite{22} argued on the basis of numerical data on computer glasses that Rayleigh scaling is expected over, Ref. \cite{22} argued on the basis of numerical data on computer glasses that Rayleigh scaling is expected over, Ref. \cite{22} argued on the basis of numerical data on computer glasses that Rayleigh scaling is expected over. Furthermore, our analysis is restricted to the athermal limit. At finite temperature, elastic correlators would receive additional effects from anharmonicity \cite{23,24} and other thermal effects. We expect this problem to be important also for plasticity and yielding, which could be the object of future work.

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Appendix A: Steps in the derivation of Eq. (27)

We aim to show the equality of second line in Eq. (27) for averaged elastic correlation function $B(r)$. The Bessel function of the first kind in integral representation is defined as

$$J_n(x) = \int_0^\pi \cos(n\theta - x \sin \theta) d\theta. \quad (A1)$$
When \( n = 0 \), we have

\[
J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos(x \sin \theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin \theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \theta} d\theta. \tag{A2}
\]

We write \( \langle B \rangle \) as

\[
\langle B \rangle \propto \int d\Omega d^2 r e^{i\omega (\hat{k} - \hat{q}) \cdot \mathbf{r}} B(r)
\]

\[
= \int_{r=0}^{\infty} \int_{\theta=-\pi}^{\pi} \int_{\tau=-\pi}^{\tau} r B(r) e^{i\omega r (\cos \theta - \cos(\phi - \theta))} dr d\theta d\phi \tag{A3}
\]

where \( \hat{k} \) is aligned with the x-axis, forming an angle \( \phi \) and an angle \( \theta \) with \( \hat{q} \) and with \( \mathbf{r} \), respectively. We thus can write

\[
\int_{\theta=-\pi}^{\pi} \int_{\tau=-\pi}^{\tau} e^{i\omega r (\cos \theta - \cos(\phi - \theta))} d\theta d\phi
\]

\[
= \int_{\theta=-\pi}^{\pi} \int_{\tau=-\tau}^{\tau} e^{i\omega r (\cos \theta - \cos(\phi - \theta))} d\theta d\tau
\]

\[
= \int_{\theta=-\pi}^{\pi} \int_{\tau=-\pi}^{\tau} e^{i\omega r (\cos \theta - \cos(\phi - \theta))} d\theta d\tau
\]

\[
= \int_{\theta=-\pi}^{\pi} e^{i\omega r \cos \theta} d\theta \int_{\tau=-\pi}^{\tau} e^{-i\omega r \cos \tau} d\tau
\]

\[
= \int_{\theta=-\pi}^{\pi} e^{i\omega r \cos \theta} d\theta \int_{\tau=-\pi}^{\tau} e^{-i\omega r \cos \tau} d\tau
\]

\[
\propto J_0^2(\omega r) \tag{A4}
\]

where we have used the periodicity of trigonometric functions.