Vanishing theorems of the basic harmonic forms on a complete foliated Riemannian manifold

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Abstract. It is well-known that on a compact foliated Riemannian manifold $(M, \mathcal{F})$ with some transversal curvature conditions, there are no nontrivial basic harmonic $r$-forms ($0 < r < q = \text{codim}\mathcal{F}$)-forms (M. Min-Oo et al., J. Reine Angew. Math. 415 (1991) [9]). In this paper, we extend the above facts to a complete foliated Riemannian manifold.

1 Introduction

Let $(M, g, \mathcal{F})$ be a foliated Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric $g$ with respect to $\mathcal{F}$. A foliated Riemannian manifold is a Riemannian manifold with a Riemannian foliation, i.e., a foliation on a smooth manifold such that the normal bundle is endowed with a metric whose Lie derivative is zero along leaf directions (see [13]). A Riemannian metric on $M$ is bundle-like if the leaves of the foliation $\mathcal{F}$ are locally equidistant, that is, the metric $g$ on $M$ induces a holonomy invariant transverse metric on the normal bundle $Q = TM/T\mathcal{F}$, where $T\mathcal{F}$ is the tangent bundle of $\mathcal{F}$. Every Riemannian foliation admits bundle-like metrics. Many researchers have studied basic forms and the basic Laplacian on foliated Riemannian manifolds. Basic forms are locally forms on the space of leaves; that is, forms $\phi$ satisfying $i(X)\phi = i(X)d\phi = 0$ for all $X \in T\mathcal{F}$. Basic forms are preserved by the exterior derivative and are used to define basic de-Rham cohomology groups $H^*_B(\mathcal{F})$. The basic Laplacian $\Delta_B$ for a given bundle-like metric is a version of the Laplace operator that preserves the basic forms. It is well-known [6,14] that on a closed oriented manifold $M$ with a transversally oriented Riemannian foliation $\mathcal{F}$, $H^*_B(\mathcal{F}) \cong \mathcal{H}^*_B(\mathcal{F})$, where $\mathcal{H}^*_B(\mathcal{F}) = \ker\Delta_B$ is finite dimensional. And so $\chi_B(\mathcal{F}) = \sum_{r=0}^{q} (-1)^r \dim \mathcal{H}^*_B(\mathcal{F})$, where $\chi_B(\mathcal{F})$ is the basic Euler characteristic [2]. In 1991, M. Min-Oo et al. [9] proved that on a closed foliated Riemannian manifold $M$, if the transversal curvature operator of

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If $F$ is positive definite, then $H_B^r(F) = 0$ $(0 < r < q)$, that is, any basic harmonic $r$-form is trivial.

In this paper, we study the basic $r$-forms on a complete foliated Riemannian manifold.

**Main Theorem.** Let $(M, g, F)$ be a complete foliated Riemannian manifold and all leaves be compact. Assume that the mean curvature form is bounded and coclosed.

1. If the transversal Ricci curvature of $F$ is positive-definite, then any $L^2$-basic harmonic $1$-forms $\phi$ with $\phi \in S_B$ are trivial.
2. If the curvature endomorphism of $F$ is positive-definite, then any $L^2$-basic harmonic $r$-forms $\phi$ with $\phi \in S_B$ are trivial.

Here $S_B$ is the Sobolev space of basic forms whose derivative belong to $L^2 \Omega^r_B(F)$.

Note that in 1980, H. Kitahara [8] proved that under the same condition of the transversal Ricci curvature, there are no nontrivial basic $\Delta_T$-harmonic $1$-forms with finite global norms. Here $\Delta_T$ is a different operator to the basic Laplacian $\Delta_B$. If $F$ is minimal, then $\Delta_T = \Delta_B$.

## 2 Preliminaries

Let $(M, g, F)$ be a $(p + q)$-dimensional complete foliated Riemannian manifold with a foliation $F$ of codimension $q$ and a bundle-like metric $g$ with respect to $F$. Let $TM$ be the tangent bundle of $M$, $TF$ its integrable subbundle given by $F$, and $Q = TM/TF$ the corresponding normal bundle of $F$. Then we have an exact sequence of vector bundles

$$0 \rightarrow TF \rightarrow TM \overset{\pi}{\rightarrow} Q \rightarrow 0,$$

where $\pi : TM \rightarrow Q$ is a projection and $\sigma : Q \rightarrow TF^\perp$ is a bundle map satisfying $\pi \circ \sigma = id$. Let $g_Q$ be the holonomy invariant metric on $Q$ induced by $g$, i.e., $\theta(X)g_Q = 0$ for any vector field $X \in TF$, where $\theta(X)$ is the transverse Lie derivative [5]. Let $R^Q$ and $\text{Ric}^Q$ be the transversal curvature tensor and transversal Ricci operator of $F$ with respect to the transversal Levi-Civita connection $\nabla^Q \equiv \nabla$ in $Q$ [14], respectively. A differential form $\phi \in \Omega^r(M)$ is basic if $i(X)\phi = 0$ and $i(X)d\phi = 0$ for all $X \in TF$. In a distinguished chart $(x_1, \cdots, x_p; y_1, \cdots, y_q)$ of
\( \mathcal{F} \), a basic \( r \)-form \( \phi \) is expressed by

\[
\phi = \sum_{a_1 < \cdots < a_r} \phi_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},
\]

where the functions \( \phi_{a_1 \cdots a_r} \) are independent of \( x \). Let \( \Omega^r_B(\mathcal{F}) \) be the set of all basic \( r \)-forms on \( M \). Then \( \Omega^r(M) = \Omega^r_B(\mathcal{F}) \oplus \Omega^r_B(\mathcal{F})^\perp \) [1]. Now, we recall the star operator \( \ast : \Omega^r_B(\mathcal{F}) \rightarrow \Omega^{r-1}_B(\mathcal{F}) \) given by [6,12]

\[
\ast \phi = (-1)^{p(q-r)} (\phi \wedge \chi_\mathcal{F}), \quad \forall \phi \in \Omega^r_B(\mathcal{F}),
\]

where \( \chi_\mathcal{F} \) is the characteristic form of \( \mathcal{F} \) and \( \ast \) is the Hodge star operator associated to \( g \). For any basic forms \( \phi, \psi \in \Omega^r_B(\mathcal{F}) \), it is well-known [12] that \( \phi \wedge \ast \psi = \psi \wedge \ast \phi \) and \( \ast^2 \phi = (-1)^{r(q-r)} \phi \). The operator \( d_B \) is the restriction of \( d \) to the basic forms, i.e., \( d_B = d|_{\Omega^r_B(\mathcal{F})} \). Let \( d_T = d_B - \kappa_B \wedge \) and \( \delta_T = (-1)^{q(r+1)+1} \ast d_B \ast \), where \( \kappa_B \) is the basic part of the mean curvature form \( \kappa \) of \( \mathcal{F} \) [1]. Note that \( \kappa_B \) is closed, i.e., \( d\kappa_B = 0 \) [10,14]. The operator \( \delta_B : \Omega^r_B(\mathcal{F}) \rightarrow \Omega^{r-1}_B(\mathcal{F}) \) is defined by

\[
\delta_B \phi = (-1)^{q(r+1)+1} \ast d_T \ast \phi = \delta_T \phi + i(\kappa_B^\ast) \phi,
\]

where \( (\cdot)^\ast \) is the \( g_Q \)-dual vector field of \( (\cdot) \). Generally, \( \delta_B \) is not a restriction of \( \delta \) on \( \Omega^r_B(\mathcal{F}) \), i.e., \( \delta_B \neq \delta|_{\Omega^r_B(\mathcal{F})} \), where \( \delta \) is the formal adjoint of \( d \). But \( \delta_B \omega = \delta \phi \) for any basic 1-form \( \phi \). Let \( \Delta_B = d_B \delta_B + \delta_B d_B \) be a basic Laplacian. Then \( \Delta^M|_{\Omega^q_B(\mathcal{F})} = \Delta_B \) [9], where \( \Delta^M \) is the Laplacian on \( M \). Let \( \{E_a\}_{a=1,\cdots,q} \) be a local orthonormal basic frame of \( Q \) and \( \theta^a \) a \( g_Q \)-dual 1-form to \( E_a \). We define \( \nabla^*_B \nabla_B : \Omega^r_B(\mathcal{F}) \rightarrow \Omega^r_B(\mathcal{F}) \) by

\[
\nabla^*_B \nabla_B = -\sum_a \nabla^2_{E_a} + \nabla_{\kappa_B^\ast}, \quad (2.4)
\]

where \( \nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y} \) for any \( X, Y \in TM \) and \( \nabla^M \) is the Levi-Civita connection with respect to \( g \). Then the generalized Weitzenböck type formula on \( \Omega^q_B(\mathcal{F}) \) is given by [4]

\[
\Delta_B \phi = \nabla^*_B \nabla_B \phi + F(\phi) + A_{\kappa_B^\ast} \phi \quad (2.5)
\]

for any \( \phi \in \Omega^q_B(\mathcal{F}) \), where \( F = \sum_{a,b=1}^q \theta^a \wedge i(E_b) R^Q(E_b, E_a) \) and

\[
A_Y \phi = \theta(Y) \phi - \nabla_Y \phi. \quad (2.6)
\]
In particular, for a 1-form $\phi$, $F(\phi)^2 = \text{Ric}^Q(\phi^2)$ and $A_Y s = -\nabla_{\sigma(s)} \pi(Y)$. Let $\Omega^*_{B,o}(\mathcal{F})$ be the space of basic forms with compact supports.

Let $\nu$ be the transversal volume form, i.e., $\ast \nu = \chi_{\mathcal{F}}$. The pointwise inner product $\langle \cdot, \cdot \rangle$ on $\Omega^r_B(\mathcal{F})$ is given by

$$\langle \phi, \psi \rangle_{\nu} = \phi \wedge \bar{\psi}$$

(2.7)

for any basic forms $\phi, \psi \in \Omega^r_B(\mathcal{F})$. And the global inner product $\ll \cdot, \cdot \gg$ on $\Omega^r_B(\mathcal{F})$ is defined by

$$\ll \phi, \psi \gg = \int_M \langle \phi, \psi \rangle_{\mu_M}$$

(2.8)

for any $\phi, \psi \in \Omega^r_B(\mathcal{F})$, one of which has compact support, where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form with respect to $g$. It is well-known [4] that $\ll \nabla_{\text{tr}}^* \nabla_{\text{tr}} \phi, \psi \gg = \ll \nabla_{\text{tr}} \phi, \nabla_{\text{tr}} \psi \gg$ for any $\phi, \psi \in \Omega^r_{B,0}(\mathcal{F})$ and

$$\ll d_B \phi, \psi \gg = \ll \phi, \delta_B \psi \gg$$

(2.9)

for any $\phi \in \Omega^r_{B,o}(\mathcal{F}), \psi \in \Omega^{r+1}_{B,o}(\mathcal{F})$. The basic form $\phi$ is said to be $L^2$-basic form if $\phi$ has finite global norm, i.e., $\|\phi\|^2 < \infty$. Let $\mathcal{H}^r_{B,2}(\mathcal{F})$ be the space of $L^2$-basic harmonic forms, i.e.,

$$\mathcal{H}^r_{B,2}(\mathcal{F}) = \{ \phi \in L^2 \Omega^r_B(\mathcal{F}) \mid d_B \phi = \delta_B \phi = 0 \}. \quad (2.10)$$

Generally, the space $\mathcal{H}^r_{B,2}(\mathcal{F})$ can have infinite dimension. And if the dimension of $\mathcal{H}^r_{B,2}(\mathcal{F})$ is finite, then it depends on the bundle-like metric. Trivially, if $M$ is compact, then $\mathcal{H}^r_{B,2}(\mathcal{F}) \cong H^r_B(\mathcal{F})$. So we study the vanishing properties of the $L^2$-basic harmonic spaces on a complete foliated Riemannian manifold.

Remark 2.1 (1) The operator $\delta_T$ is the formal adjoint of $d_B$ with respect to the global norm $\langle \cdot, \cdot \rangle$, which is given by

$$\langle \phi, \psi \rangle = \int_M \langle \phi, \psi \rangle_{\nu} dx_1 \wedge \cdots \wedge dx_p.$$  (2.11)

Let $\Delta_T = d_B \delta_T + \delta_T d_B$ be a Laplacian. If the foliation is minimal, then $\delta_T = \delta_B$. So $\Delta_B = \Delta_T$.

(2) In 1980, H. Kitahara [8] proved that if the transversal Ricci curvature is nonnegative and positive at some point, then there are no nontrivial $L^2$-basic $\Delta_T$-harmonic 1-forms.
3 Vanishing theorem

Let \((M, g, \mathcal{F})\) be a complete foliated Riemannian manifold with a foliation \(\mathcal{F}\) of codimension \(q\) and a bundle-like metric \(g\) with respect to \(\mathcal{F}\). Assume that all leaves of \(\mathcal{F}\) are compact. Now, we consider a smooth function \(\mu\) on \(\mathbb{R}\) satisfying

\[
\begin{align*}
(i) \quad & 0 \leq \mu(t) \leq 1 \text{ on } \mathbb{R}, \\
(ii) \quad & \mu(t) = 1 \text{ for } t \leq 1, \\
(iii) \quad & \mu(t) = 0 \text{ for } t \geq 2.
\end{align*}
\]

Let \(x_0\) be a point in \(M\). For each point \(y \in M\), we denote by \(\rho(y)\) the distance between leaves through \(x_0\) and \(y\). For any real number \(l > 0\), we define a Lipschitz continuous function \(\omega_l\) on \(M\) by

\[
\omega_l(y) = \mu(\rho(y)/l).
\]

Trivially, \(\omega_l\) is a basic function. Let \(B(l) = \{ y \in M | \rho(y) \leq l \}\) for \(l > 0\). Then \(\omega_l\) satisfies the following properties:

\[
\begin{align*}
0 \leq \omega_l(y) & \leq 1 \quad \text{for any } y \in M, \\
\text{supp } \omega_l & \subset B(2l), \\
\omega_l(y) & = 1 \quad \text{for any } y \in B(l), \\
\lim_{l \to \infty} \omega_l(y) & = 1 \quad \text{for any } y \in B(l), \\
|d_B \omega_l| & \leq \frac{C}{l} \quad \text{almost everywhere on } M,
\end{align*}
\]

where \(C\) is a positive constant independent of \(l\). Hence \(\omega_l \psi\) has compact support for any basic form \(\psi \in \Omega_B^r(\mathcal{F})\) and \(\omega_l \psi \to \psi\) (strongly) when \(l \to \infty\).

**Lemma 3.1** For any \(\phi \in \Omega_B^r(\mathcal{F})\), there exists a number \(A\) depending only on \(\mu\), such that

\[
\begin{align*}
\|d_B \omega_l \wedge \phi\|^2_{B(2l)} & \leq \frac{qA^2}{l^2} \|\phi\|^2_{B(2l)}, \\
\|d_B \omega_l \wedge \bar{\phi}\|^2_{B(2l)} & \leq \frac{qA^2}{l^2} \|\phi\|^2_{B(2l)}, \\
\|d_B \omega_l \otimes \phi\|^2_{B(2l)} & \leq \frac{qA^2}{l^2} \|\phi\|^2_{B(2l)},
\end{align*}
\]

where \(\|\phi\|^2_{B(2l)} = \ll \phi, \phi \gg_{B(2l)} = \int_{B(2l)} \langle \phi, \phi \rangle \mu_M\).

**Proposition 3.2** For any \(L^2\)-basic form \(\psi\), if \(\Delta_B \psi = 0\), then \(d_B \psi = 0\) and \(\delta_B \psi = 0\).
**Proof.** Let \( \psi \) be a \( L^2 \)-basic form. Then we have

\[
\ll \Delta_B \psi, \omega_l^2 \psi \gg_{B(2l)} = \ll d_B \psi, d_B (\omega_l^2 \psi) \gg_{B(2l)} + \ll \delta_B (\omega_l^2 \psi), \delta_B (\omega_l^2 \psi) \gg_{B(2l)}. \tag{3.1}
\]

By a direct calculation, we have

\[
d_B (\omega_l^2 \psi) = \omega_l^2 d_B \psi + 2 \omega_l d_B \omega_l \wedge \psi, \tag{3.2}
\]

\[
\delta_B (\omega_l^2 \psi) = \omega_l^2 \delta_B \psi + (-1)^{q(r+1)+1} \bar{x} (2 \omega_l d_B \omega_l \wedge \bar{x} \psi). \tag{3.3}
\]

From (3.1), (3.2) and (3.3), if \( \Delta_B \psi = 0 \), then

\[
\| \omega_l d_B \psi \|^2_{B(2l)} + \| \omega_l \delta_B \psi \|^2_{B(2l)} = -2 \ll \omega_l d_B \psi, d_B \omega_l \wedge \psi \gg_{B(2l)} + 2 (-1)^{q(r+1)} \ll \omega_l \delta_B \psi, \bar{x} (d_B \omega_l \wedge \bar{x} \psi) \gg_{B(2l)}.
\]

Hence by the the Schwartz’s inequality and Lemma 3.1, we have

\[
\| \omega_l d_B \psi \|^2_{B(2l)} + \| \omega_l \delta_B \psi \|^2_{B(2l)} \leq \epsilon_1 \| \omega_l d_B \psi \|^2_{B(2l)} + \epsilon_2 \| \omega_l \delta_B \psi \|^2_{B(2l)} + \frac{B_1}{l} \| \psi \|^2_{B(2l)}
\]

for some positive real numbers \( \epsilon_1, \epsilon_2 \) and \( B_1 \). Therefore, we have

\[
\| \omega_l d_B \psi \|^2_{B(2l)} + \| \omega_l \delta_B \psi \|^2_{B(2l)} \leq \frac{B_2}{l} \| \psi \|^2_{B(2l)}
\]

for some positive real number \( B_2 \). Since \( \psi \) is the \( L^2 \)-basic form, letting \( l \to \infty \), \( d_B \psi = \delta_B \psi = 0 \). \( \square \)

**Remark 3.3** In 1979, H. Kitahara \([7]\) proved the corresponding result with the Laplacian \( \Delta_T \). Namely, on a complete foliated manifold, if \( \Delta_T \phi = 0 \), then \( d_B \phi = \delta_T \phi = 0 \).

Now we prove the vanishing theorem of the \( L^2 \)-basic harmonic form on a complete foliated Riemannian manifold. First of all, we prepare some lemmas.

**Lemma 3.4** Let \((M,g,F)\) be a complete foliated Riemannian manifold whose leaves are compact. Suppose that \( \kappa_B \) is bounded and coclosed. Then for any \( L^2 \)-basic harmonic form \( \phi \),

\[
\limsup_{l \to \infty} \ll A_{\kappa_B} \phi, \omega_l^2 \phi \gg_{B(2)} = 0. \tag{3.4}
\]
Proof. Let $\phi$ be a $L^2$-basic harmonic form. Since $\theta(X)\phi = d_Bi(X)\phi$, from (2.6) we have

$$A_{\kappa_B^2} \phi, \omega_1^2 \phi \gg B(2l) \Leftrightarrow d_Bi(\kappa_B^2)\phi, \omega_1^2 \phi \gg B(2l) - \nabla_{\kappa_B^2} \phi, \omega_1^2 \phi \gg B(2l) \quad (3.5)$$

Since $\delta_B \phi = 0$, from (3.3) and Lemma 3.1, we have

$$\left| \left| d_Bi(\kappa_B^2)\phi, \omega_1^2 \phi \gg B(2l) \right| \right| = 2\left| \left| \omega_l i(\kappa_B^2)\phi, \ast (d_B \omega_l \wedge \ast \phi) \gg B(2l) \right| \right| \leq \epsilon_3 \left| \omega_l i(\kappa_B^2)\phi \right|^2 + \frac{1}{\epsilon_3} \left( d_B \omega_l \wedge \ast \phi \right)^2 \leq \epsilon_3 \left| \omega_l i(\kappa_B^2)\phi \right|^2 + \frac{B_3}{l^2} \| \phi \|^2_{B(2l)}$$

for some positive real numbers $\epsilon_3$ and $B_3$. By using $|i(\kappa_B^2)\phi|^2 + |\kappa_B \wedge \phi|^2 = |\kappa_B|^2|\phi|^2$, we have

$$\left| \left| d_Bi(\kappa_B^2)\phi, \omega_1^2 \phi \gg B(2l) \right| \right| \leq \epsilon_3 \max(|\kappa_B|^2) \left| \omega_l \phi \right|^2 + \frac{B_3}{l^2} \| \phi \|^2_{B(2l)} \quad (3.6)$$

On the other hand, since $\delta_B \kappa_B = 0$, by a direct calculation, we have

$$\nabla_{\kappa_B^2} \phi, \omega_1^2 \phi \gg B(2l) = \frac{1}{2} d_B(|\omega_l|^2) \kappa_B \gg B(2l) - \omega_l \phi, \kappa_B^2(\omega_l) \phi \gg B(2l)$$

Hence by the Schwartz inequality, we have

$$\left| \left| \nabla_{\kappa_B^2} \phi, \omega_1^2 \phi \gg B(2l) \right| \right| = \left| \left| \omega_l \phi, \kappa_B^2(\omega_l) \phi \gg B(2l) \right| \right| \leq \epsilon_4 \left| \omega_l \phi \right|^2 + \frac{B_4}{l^2} \max(|\kappa_B|^2) \| \phi \|^2_{B(2l)} \quad (3.7)$$

for a positive real numbers $\epsilon_4$ and $B_4$. From (3.6) and (3.7), by letting $l \to \infty$, we have

$$\limsup_{l \to \infty} \left| \left| d_Bi(\kappa_B^2)\phi, \omega_1^2 \phi \gg B(2l) \right| \right| \leq \epsilon_3 \max(|\kappa_B|^2) \| \phi \|^2,$nabla_{\kappa_B^2} \phi, \omega_1^2 \phi \gg B(2l) \right| \right| \leq \epsilon_4 \| \phi \|^2.$$}

Since $\epsilon_3$ and $\epsilon_4$ are arbitrary positive numbers, we have

$$\limsup_{l \to \infty} \left| \left| d_Bi(\kappa_B^2)\phi, \omega_1^2 \phi \gg B(2l) \right| \right| = 0, \quad (3.8)$$

$$\limsup_{l \to \infty} \left| \left| \nabla_{\kappa_B^2} \phi, \omega_1^2 \phi \gg B(2l) \right| \right| = 0. \quad (3.9)$$

Hence from (3.5), (3.8) and (3.9), the proof is completed. □
Theorem 3.5 Let \((M, g, F)\) be as in Lemma 3.4. Suppose that \(\kappa_B\) is bounded and coclosed. If the curvature endomorphism \(F\) of \(F\) is positive-definite, then any \(L^2\)-basic harmonic \(r\)-forms \(\phi\) with \(\phi \in S_B\) are trivial, i.e., \(\mathcal{H}_{B,2}^r(F) = \{0\}\).

Proof. Let \(\phi\) be a \(L^2\)-basic harmonic \(r\)-form. From (2.5) and Proposition 3.2, we have
\[
\langle \nabla^*_{tr} \nabla_{tr} \phi, \omega^2 \phi \rangle + \langle F(\phi), \omega^2 \phi \rangle + \langle A_{\kappa_B} \phi, \omega^2 \phi \rangle = 0. \tag{3.10}
\]
On the other hand, a direct calculation gives
\[
\ll \nabla^*_{tr} \nabla_{tr} \phi, \omega^2 \phi \gg_{B(2l)} = \ll \nabla_{tr} \phi, 2\omega_l d_B \omega_l \otimes \phi \gg_{B(2l)} + \|\omega_l \nabla_{tr} \phi\|^2_{B(2l)}. \tag{3.11}
\]
From Lemma 3.1, we have
\[
| \ll \nabla_{tr} \phi, 2\omega_l d_B \omega_l \otimes \phi \gg_{B(2l)} | \leq \epsilon_5 \|\omega_l \nabla_{tr} \phi\|^2_{B(2l)} + \frac{B_5}{l^2} \|\phi\|^2_{B(2l)}
\]
for some positive constants \(\epsilon_5\) and \(B_5\). Hence by letting \(l \to \infty\), we have
\[
\limsup_{l \to \infty} \ll \nabla_{tr} \phi, 2\omega_l d_B \omega_l \otimes \phi \gg_{B(2l)} \leq \epsilon_5 \|\nabla_{tr} \phi\|^2
\]
for arbitrary \(\phi \in S_B\) (i.e., \(\|\nabla_{tr} \phi\|^2 < \infty\)), we have
\[
\limsup_{l \to \infty} \ll \nabla_{tr} \phi, 2\omega_l d_B \omega_l \otimes \phi \gg_{B(2l)} = 0. \tag{3.12}
\]
Hence from (3.11), (3.12) and Lemma 3.4, we have
\[
\|\nabla_{tr} \phi\|^2 + \limsup_{l \to \infty} \ll F(\phi), \omega^2 \phi \gg_{B(2l)} = 0, \tag{3.13}
\]
which complete the proof. \(\square\)

Since \(F(\phi^\sharp) = \text{Ric}^Q(\phi^\sharp)\) for any basic 1-form \(\phi\), we have the following corollary.

Corollary 3.6 Let \((M, g, F)\) be as in Lemma 3.4. Suppose that \(\kappa_B\) is bounded and coclosed. If the transversal Ricci curvature \(\text{Ric}^Q\) is positive-definite, then any \(L^2\)-basic harmonic 1-forms \(\phi\) with \(\phi \in S_B\) are trivial, \(\mathcal{H}_{B,2}^1(F) = \{0\}\).

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