An Improved Limited Tolerance Dominance Relation in Incomplete Ordered Decision Systems

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Abstract. A method for analyzing incomplete ordered decision system (IODS) is proposed. First, we analyze the limitations of existing generalized dominance relations. On this basis, we define an improved limited tolerance dominance relation, and study its properties. Consider that the monotonicity of boundary region of IODS is not established, the attribute reduction based on the approximate quality is defined. The significance of attributes based on the approximate quality is defined, and a heuristic attribute reduction algorithm with time complexity $O(|U|^3|C|)$ is proposed. Finally, the validity of this method is verified by an example.

Introduction

To analyze the ordered decision systems (ODS), a dominance rough set approach (DRSA) is proposed in [1]. The method has been successfully applied to multicriteria decision problems [2,3]. At present, DRSA has been extended to handle various types of ODS [4-6]. To analyze the inconsistencies caused by errors, the variable precision DRSA [7] and variable consistency DRSA [8] were proposed. Especially, to handle incomplete ODS (IODS), the dominance relations (DR) is generalized, and some generalized DRs have been defined, such as tolerance DR [9], limited tolerance DR (LTDR) [10], $\lambda$-degree LTDR [11], similarity DR [12], characteristic DR [13] and valued DR [14]. In [5], the properties of the extensions of DRSA were studied.

But, the existing generalized DRs have some limitations in dealing with IODS. In tolerance DR, two objects that uncertainly satisfy dominance relation on all attributes may be considered as indiscernible [9]. In similarity DR, two objects that certainly satisfy dominance relation on many attributes may be separated on all attributes. The characteristic DR is a combination of similarity DR and tolerance DR, and it can not avoid their limitations. In valued DR, the prior probability distribution of attribute values in IODS need to be known, and it is difficult to choose the appropriate threshold for different IODSs. Besides, there are three different LTDRs proposed by Hu et al. [10,11] and Luo et al.[16] respectively. In this paper, based on the analysis of their limitations, an improved LTDR denoted by ILTDR is proposed, which can better handle incomplete information in IODS.

Preliminaries

Definition 1 [1]. An ODS is $S=<U, C \cup D, V, f>$, here $U$ is the objects set, $C \cap D = \emptyset$, $C$ is the condition criteria set, $D$ is the decision criteria set, $V=\bigcup_{a \in C \cup D} V_a$ and $V_a$ is the range of $a$, and $f$ is a function such that $f(x,a) \in V_a$ for $a \in C \cup D$ and $x \in U$.

In an ODS, $D$ partitions $U$ into $CL=\{Cl_1, Cl_2, \ldots, Cl_n\}$. The upward union is $Cl^x_r=\bigcup_{s \geq r} Cl_s$, and the downward union is $Cl^x_r=\bigcup_{s \leq r} Cl_s$, where $r=1,2, \ldots,n$ [1].

In addition, if there are some unknown conditional attribute values in ODS, the ODS is called IODS, otherwise a complete ODS (Cods). For simplicity, $a(x)$ represents the value of $x$ on $a$, and all the missing conditional attribute values in IODS are denoted as $\ast^\ast$.

Definition 2 [1]. Let $S$ be Cods, the DR with respect to $A \subseteq C$ is $D_A=\{ <x,y> \in U^2 | \forall a \in A (a(x) \geq a(y)) \}$, here $a(x) \geq a(y)$ implies that $x$ is at least as good as $y$ on $a$. 

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Definition 3. Let $S$ be an IODS and $A \subseteq C$, four different types of generalized dominance relations in [9,10,11,16] are respectively defined as:

$$TD_A=\{<x,y> \in U^2 \mid \forall a \in A( a(x) \geq a(y) \lor a(x)=\ast \lor a(y)=\ast ) \},$$
(1)

$$LD_A^I=\{<x,y> \in U^2 \mid <x,y> \in TD_A \land M_A(x,y) \neq \emptyset \} ,$$
(2)

$$LD_A^II=\{<x,y> \in U^2 \mid <x,y> \in TD_A \land |M_A(x,y)|/|C| \geq \lambda \},$$
(3)

$$LD_A^III=\{<x,y> \in U^2 \mid \forall a \in A( a(x) \geq a(y) \lor (a(x)=\max V_a \land a(y)=\ast ) \lor (a(x)=\ast \land a(y)=\min V_a)) \} \cup I_U.$$  
(4)

here $I_U=\{<x,x> \mid x \in U\}$, $M_A(x,y)=\{a \in A | a(x) \neq \ast \land a(y) \neq \ast \}$, $\lambda \in [0,1]$, $\min V_a$ and $\max V_a$ respectively represent the worst value and the best value in $V_a$.

Obviously, $LD_A^I$ and $LD_A^II$ are the improvements of $TD_A$, and $LD_A^III$ is an improvement of $LD_A^I$ by the threshold $\lambda$. Both $TD_A$ and $LD_A^I$ are special cases of $LD_A^II$, that is, $LD_A^II=TD_A$ for $\lambda=0$, and $LD_A^III=LD_A^II$ for $\lambda=1/|C| [11]$. But, it is difficult to choose the appropriate threshold for a given IODS.

In order to analyze $LD_A^I$, $LD_A^II$ and $LD_A^III$, an example is given as follows.

Example 1. Table 1 shows an IODS, which is the comments on 12 papers. Here, $U=\{p_i | i=1,2,\ldots,12\}$, $C=\{a_1,a_2,a_3\}$ and $D=\{d\}$. $V_{a_1}=V_{a_2}=V_{a_3}=\{0-poor,1-fair,2-good,3-excellent\}$ and $V_{d}=\{1-reject,2-major revision,3-minor revision,4-accept\}$.

| Papers | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ | $p_{11}$ | $p_{12}$ |
|--------|------|------|------|------|------|------|------|------|------|---------|---------|---------|
| Originality $a_1$ | | | | | | | | | | | | |
| Presentation $a_2$ | | | | | | | | | | | | |
| Technical soundness $a_3$ | | | | | | | | | | | | |
| Overall evaluation $d$ | 3 | 2 | 3 | 2 | 2 | 1 | 3 | 2 | 4 | 3 | 3 | 1 |

From table 1, we can see the following results.

- For $p_1$ and $p_2$, there are $<p_1,p_2>$, $<p_2,p_1> \in TD_C$. But they uncertainly satisfy dominance relation on $\forall a \in C$, and the probability of satisfying dominance relation on $C$ is little. $p_2$ and $p_12$ also have this problem. Therefore, $TD_A$ is too loose to describe the order relationship between objects.

- For $p_2$ and $p_3$, there are $<p_2,p_3> \notin LD_C$, and $<p_2,p_3> \notin LD_C^II$ for $M_C(p_2,p_3)=\emptyset$. But $a_1(p_3)\geq a_1(p_2)$ for $a_1(p_3)=3=\max V_{a_1}$ and $a_1(p_2)=\ast$. So, the $M_A(x,y)$ in (2) and (3) is not suitable. The $LD_A^I$ and $LD_A^II$ do not consider the case of the maximum and minimum attribute values in the attribute domain.

- For $p_3$ and $p_4$, although $p_3$ definitely dominates $p_4$ on $a_1$ and $a_3$, there is $<p_3,p_4> \notin LD_C^III$ for $a_2(p_3)=\ast$ and $a_2(p_4)=1 \land \min V_{a_2}$. This problem also exist for $p_{10}$ and $p_{11}$, $p_{10}$ and $p_{12}$. The $LD_C^III$ considers the case of the maximum and minimum attribute values. But, its definition constraints are too strict.

An ILTDR in IODS

Definition 4. Let $S$ be an IODS and $A \subseteq C$, an ILTDR is defined as

$$ILD_A=\{<x,y> \in U^2 \mid \forall a \in A( a(x) \geq a(y) \lor a(x)=\ast \lor a(y)=\ast ) \lor IM_A(x,y) \neq \emptyset \} \cup I_U,$$

here $IM_A(x,y)=\{a \in A | a(x)=\max V_a \lor a(y)=\ast \lor (a(x)=\ast \land a(y)=\min V_a) \}$, $I_U=\{<x,x> \mid x \in U\}$.

From table 1, we have the following results:

- There are $<p_1,p_2>$, $<p_2,p_{12}>$, $<p_2,p_3> \in TD_C$, but $<p_1,p_2>$, $<p_2,p_{12}>$, $<p_2,p_3> \notin ILD_C$. This indicates that the ILTDR overcomes the limitations of tolerance DR.

- There are $<p_3,p_2> \notin LD_C^I$ and $<p_3,p_2> \notin LD_C^II$, but $<p_3,p_2> \in ILD_C$, for $IM_C(p_3,p_2)=\{a_1\}$. There is $<p_3,p_2> \notin LD_C^III$, but $<p_3,p_2> \in ILD_C$. These indicate that the ILTDR is an improvement of I, II and III LTDR.

Further, the lower approximations and upper approximations of $C_{l}$ and $C_{r}$, $r=1,2, \ldots, n$, are
\[ \text{ILD}_A(C_l^r) = \{x \in U \mid \text{ILD}_A(x) \subseteq C_l^r\}, \text{ILD}_A(C_l^r) = \{x \in U \mid \text{ILD}_A(x) \cap C_l^r \neq \emptyset\} = \bigcup_{x \in C_l^r} \text{ILD}_A(x), \] (6)

\[ \text{ILD}_A(C_l^r) = \{x \in U \mid \text{ILD}_A(x) \subseteq C_l^r\}, \text{ILD}_A(C_l^r) = \{x \in U \mid \text{ILD}_A(x) \cap C_l^r \neq \emptyset\} = \bigcup_{x \in C_l^r} \text{ILD}_A(x), \] (7)

where \( \text{ILD}_A(x) = \{z \in U \mid x \succ z \in \text{ILD}_A\} \) and \( \text{ILD}_A(x) = \{z \in U \mid z \succ x \in \text{ILD}_A\} \).

The boundary and positive regions of \( Y \in \{C_l^r, C_l^\leq, C_l^\geq\}, r=1,2, \ldots, n \), are defined as

\[ B_{n,ld}(Y) = \overline{\text{ILD}_A(Y) - \text{ILD}_A(Y)} \), \text{Pos}_{ld}(Y) = \text{ILD}_A(Y). \] (8)

The approximation quality (or sorting quality) of the partition \( CL \) is

\[ \gamma_{ld}(CL) = \frac{U - (\bigcup_{r=1}^n B_{n,ld}(C_l^r))) \bigcup (\bigcup_{r=1}^n B_{n,ld}(C_l^r))] \} \}} / |U|. \] (9)

This ratio \( \gamma_{ld}(CL) \in [0,1] \) shows the relationship between all properly classified objects and all objects in \( U \). If \( \gamma_{ld}(CL) < 1 \), then \( S \) is inconsistent, otherwise consistent.

Considering an IODS \( S, x \in U \) and \( A, B \subseteq C \) such that \( A \subset B \). According to definitions (5)-(7), we can know \( LD_B \subseteq LD_B \subseteq LD_B \subseteq TD_B \) and \( LB_B \subseteq LB_B \subseteq LB_B \), but are not always \( ILD_B = \cap_{b \in B} LD_B \), \( ILD_B = \cap_{b \in B} (LD_B(b^\#)) \), \( ILD_B \subseteq ILD_A \) and \( ILD_B(b^\#) \subseteq ILD_A(b^\#) \), where \#\{+, -\}. So the monotonicity of the ILTDR and the dominating (dominated) sets are not valid. Further, the monotonicity of approximations and boundary regions of \( C_l^\Delta \) (here \( \Delta \in \{\geq, \leq\} \) and \( r=1,2, \ldots, n \) is not valid, i.e., \( \text{ILD}_A(C_l^\Delta) \not\subseteq \text{ILD}_B(C_l^\Delta) \), \( \text{ILD}_B(C_l^\Delta) \not\subseteq \text{ILD}_A(C_l^\Delta) \) and \( B_{n,ld}(C_l^\Delta) \not\subseteq B_{n,ld}(C_l^\Delta) \).

**Proposition 1.** Let \( S \) be an IODS, for any \( A \subset C \), we have the following results:

- Let \( \Delta \in \{\geq, \leq\} \) and \( r=1,2, \ldots, n \), \( \text{ILD}_A(C_l^r) \subseteq C_l^\Delta \subseteq \overline{\text{ILD}_A(C_l^r)} \). (10)

- For \( r=1,2, \ldots, n-1 \), \( \text{ILD}_A(C_l^{r+1}) = U - \overline{\text{ILD}_A(C_l^r)} \), \( \overline{\text{ILD}_A(C_l^{r+1})} = U - \text{ILD}_A(C_l^r) \). (11)

- For \( r=1,2, \ldots, n-1 \), \( B_{n,ld}(C_l^r) = B_{n,ld}(C_l^r) \), \( B_{n,ld}(C_l^{r+1}) = B_{n,ld}(C_l^r) \). (12)

- For \( r=1,2, \ldots, n-1 \) such that \( r \succ r \),

\[ \text{ILD}_A(C_l^r) \subseteq \text{ILD}_A(C_l^{r+1}), \text{ILD}_A(C_l^r) \subseteq \text{ILD}_A(C_l^{r+1}), \text{ILD}_A(C_l^r) \supseteq \text{ILD}_A(C_l^{r+1}), \text{ILD}_A(C_l^r) \supseteq \text{ILD}_A(C_l^{r+1}). \] (13)

- For \( r=1,2, \ldots, n \) and \( \Delta \in \{\geq, \leq\} \),

\[ \text{ILD}_A(C_l^r) \cap C_l^\Delta = \text{ILD}_A(C_l^r) \cap \text{ILD}_A(C_l^r), \text{ILD}_A(C_l^r) \cap C_l^\Delta = \overline{\text{ILD}_A(C_l^r)} \cap \text{ILD}_A(C_l^r). \] (14)

**Proof.**

- Obviously, (10) is true for the reflexivity of \( ILD_A \) and definitions (6)-(7).

- For \( \forall x \in U \), if \( \text{ILD}_A(x) \subset C_l^{1-} \) \( (r=1,2, \ldots, n-1) \), then \( x \in \text{ILD}_A(C_l^{1-}) \). Otherwise, there is \( \text{ILD}_A(x) \subset C_l^{1-} \iff \text{ILD}_A(x) \cap (U - C_l^{1-}) \neq \emptyset \iff \text{ILD}_A(x) \cap C_l^{1-} \neq \emptyset \). This implies \( x \in \overline{\text{ILD}_A(C_l^{1-})} \).

So, we have \( \text{ILD}_A(C_l^{1-}) = U - \overline{\text{ILD}_A(C_l^{1-})} \). Similarly, \( \overline{\text{ILD}_A(C_l^{1-})} = U - \text{ILD}_A(C_l^{1-}) \) can be obtained. So, (11) is true.

- Because \( C_l^{\geq} = U - C_l^{\leq} \), there is \( B_{n,ld}(C_l^{\geq}) = B_{n,ld}(C_l^{\leq}) \). According to (11), the equalities \( B_{n,ld}(C_l^{1-}) = \overline{\text{ILD}_A(C_l^{1-})} - \text{ILD}_A(C_l^{1-}) = [U - \text{ILD}_A(C_l^{1-})] - [U - \overline{\text{ILD}_A(C_l^{1-})}] = B_{n,ld}(C_l^{1-}) \) imply that (12) is true.

- For \( \forall r, \in \{1,2, \ldots, n \} \) such that \( r \succ r \), there is \( C_l^{\geq} \subseteq C_l^{\leq} \). So, \( \text{ILD}_A(x) \subseteq C_l^{\geq} \) and \( \text{ILD}_A(x) \cap C_l^{\leq} \neq \emptyset \) mean, respectively, \( \text{ILD}_A(x) \subseteq C_l^{\leq} \) and \( \text{ILD}_A(x) \cap C_l^{\leq} \neq \emptyset \) for any \( x \in U \). Due to (6), there are \( \overline{\text{ILD}_A(C_l^{r})} \subseteq \text{ILD}_A(C_l^{r}) \) and \( \overline{\text{ILD}_A(C_l^{r})} \subseteq \text{ILD}_A(C_l^{r}) \). Similarly, the remaining two can be proved. So, (13) is true.
Due to (13), (14) is true.

In fact, (11) is an approximate complementarity, which indicates that if $x$ definitely belongs to $Cl_{r+1}$ or better, it cannot belong to $Cl_r$ or worse. (12) is the identity of boundary region, which indicates that if $x$ is uncertain about $Cl_{r+1}$, it is also uncertain about $Cl_r$. (13) is the monotonicity of the approximations about class labels, which indicates that the upper and lower approximations of objects set increase monotonically with its increase.

**Proposition 2.** Let $S$ be an IODS, for any $A \subseteq C$, there are

$$ILD_A (Cl_r^z) = \bigcap_{r=1}^n Pos_{ILD_A} (Cl_r^z) = \bigcap_{r=1}^n Pos_{ILD_A} (Cl_r^z) = ILD_A (Cl_r^z) \cap \bigcup_{r=1}^n Bn_{ILD_A} (Cl_r^z) = \bigcup_{r=1}^n Bn_{ILD_A} (Cl_r^z). \quad (15)$$

**Proof.** Due to (8), (12) and (13), (15) is obviously true.

According to (15), the definition (9) of approximation quality for $CL$ can be modified to

$$\gamma_{ILD,A}(CL) = 1 - \frac{|\bigcup_{r=1}^n Bn_{ILD_A} (Cl_r^z)|}{|U|}. \quad (16)$$

Rough approximation is the basic operation of knowledge acquisition. The algorithm for calculating rough approximation induced by ILTDR in IODS is shown in Algorithm 1. Because the number of $Cl_r^z (r=1,2,\ldots,n)$ is much smaller than $|U|$, it’s easy to know that the time complexity of Algorithm 1 is $O(|U|^2)$.

**Algorithm 1.** Calculating rough approximations induced by ILTDR

**Input:** an IODS $S$ and $A \subseteq C$.

**Output:** the boundary regions and approximations of $Cl_r^z$ and $Cl_r^z$, $r=1,2,\ldots,n$.

1: Let $ILD_A(Y) = \emptyset$, $ILD_A(Y) = \emptyset$, $Bn_{ILD_A}(Y) = \emptyset$, here $Y \in \{Cl_r^z, Cl_r^z\}$ and $r=1,2,\ldots,n$;
2: Calculate the dominating (dominated) sets $ILD_A^+(y)$ ($ILD_A^-(y)$) for each $y \in U$;
3: for each $Cl_r^z (r=1,2,\ldots,n)$ do
4: \hspace{1em} for each $y \in U$ do
5: \hspace{2em} if $ILD_A^+(y) \subset Cl_r^z$ then $ILD_A(Cl_r^z) = ILD_A(Cl_r^z) \cup \{y\}$;
6: \hspace{2em} if $y \in Cl_r^z$ then $ILD_A(Cl_r^z) = ILD_A(Cl_r^z) \cup ILD_A^-(y)$;
7: end for
8: \hspace{1em} $Bn_{ILD_A}(Cl_r^z) = ILD_A(Cl_r^z) - ILD_A(Cl_r^z)$;
9: \hspace{1em} if $r=1$ then $ILD_A(Cl_r^z) = ILD_A(Cl_r^z)$, \hspace{1em} $ILD_A(Cl_r^z) = ILD_A(Cl_r^z)$, \hspace{1em} $Bn_{ILD_A}(Cl_r^z) = Bn_{ILD_A}(Cl_r^z)$;
10: \hspace{1em} if $r>1$ then $ILD_A(Cl_{r-1}) = U - ILD_A(Cl_{r-1})$, \hspace{1em} $ILD_A(Cl_{r-1}) = U - ILD_A(Cl_{r-1})$, \hspace{1em} $Bn_{ILD_A}(Cl_{r-1}) = Bn_{ILD_A}(Cl_{r-1})$;
11: end for
12: return $ILD_A(Y)$, $ILD_A(Y)$, $Bn_{ILD_A}(Y)$, where $Y \in \{Cl_r^z, Cl_r^z\}$ and $r=1,2,\ldots,n$;

**Example 2.** From Table 1, $CL = \{Cl_1, Cl_2, Cl_3, Cl_4\}$: $Cl_1 = \{p_6p_{12}\}$, $Cl_2 = \{p_2p_4p_5p_8\}$, $Cl_3 = \{p_1p_3p_7, p_{10}p_{11}\}$, $Cl_4 = \{p_9\}$. Thus, $Cl_1^z = U$, $Cl_2^z = U - \{p_6p_{12}\}$, $Cl_3^z = \{p_1p_3p_7, p_{10}p_{11}\}$, $Cl_4^z = \{p_9\}$; $Cl_1^z = \{p_6, p_{12}\}$, $Cl_2^z = \{p_2p_4p_5p_8p_{12}\}$, $Cl_3^z = U - \{p_9\}$, $Cl_4^z = U$. By Algorithm 1, we have the following results:

$$ILD_A(Cl_1^z) = U, \hspace{1em} ILD_A(Cl_1^z) = \{p_1, p_3, p_7, p_{10}, p_{11}\}, \hspace{1em} ILD_A(Cl_1^z) = \{p_6, p_{12}\};$$

$$ILD_A(Cl_2^z) = U - \{p_6\}, \hspace{1em} ILD_A(Cl_2^z) = \{p_2p_4p_5p_8\};$$

$$Bn_{ILD_A}(Cl_1^z) = \emptyset, \hspace{1em} Bn_{ILD_A}(Cl_1^z) = \{p_1, p_3, p_7, p_{10}, p_{11}\}, \hspace{1em} Bn_{ILD_A}(Cl_1^z) = \{p_6, p_{12}\};$$

$$ILD_A(Cl_3^z) = U - \{p_9\}, \hspace{1em} ILD_A(Cl_3^z) = \{p_1, p_3, p_7, p_{10}, p_{11}\};$$

$$Bn_{ILD_A}(Cl_3^z) = \{p_1, p_3, p_7, p_{10}, p_{11}\}, \hspace{1em} Bn_{ILD_A}(Cl_3^z) = \{p_6, p_{12}\};$$

$$ILD_A(Cl_4^z) = \emptyset, \hspace{1em} ILD_A(Cl_4^z) = \{p_9\};$$

**Attribute Reduction Based on ILTDR in IODS**

Under the ILTDR in IODS, the monotonicity of boundary region of $Cl_r^z (r=1,2,\ldots,n)$ is not established. Therefore, the monotonicity of approximate quality for CL is not valid. In other words,
\( \gamma_{\text{ILD}}(CL) \leq \gamma_{\text{ILD}}(CL) \) does not necessarily hold for \( A \subseteq C \). At this time, the attribute reduction method which keeps the approximate quality unchanged is no longer applicable. So, under the ILTDR in IODS, the definition of attribute reduction that does not reduce the approximate quality of partition CL can be shown in Definition 5.

**Definition 5.** \( S \) is an IODS and \( A \subseteq C \). If \( \gamma_{\text{ILD}}(CL) \geq \gamma_{\text{ILD}}(CL) \) and \( \gamma_{\text{ILD}}(CL) < \gamma_{\text{ILD}}(CL) \) for any \( B \subseteq A \), \( A \) is an attribute reduction of \( S \).

**Definition 6.** Let \( S \) be an IODS and \( A \subseteq C \). Due to (16), for any \( a \in A \), the significance of \( a \) with respect to \( CL \) is defined as
\[
\text{sig}(a, A, CL) = \gamma_{\text{ILD}}(CL) - \gamma_{\text{ILD}}(CL) = (|\bigcup_{i=1}^{m} B_{\text{ILD}}(CL_i^a)| - |\bigcup_{i=1}^{m} B_{\text{ILD}}(CL_i)|)/|U|.
\]

The \( \text{sig}(a, A, CL) \in [-1, 1] \) describes the effect of \( a \in A \) on the approximation quality of partition \( CL \). On this basis, a heuristic attribute reduction algorithm of IODS is proposed in Algorithm 2, which does not reduce the approximation quality of \( CL \) with respect to \( C \).

**Algorithm 2** Heuristic attribute reduction algorithm of IODS

**Input:** an IODS \( S \).

**Output:** the attribute reduction \( A \) of \( S \).

1. Let \( A = C \), \( A^* = \emptyset \).
2. Compute \( \gamma_{\text{ILD}}(CL) = 1 - |\bigcup_{i=1}^{m} B_{\text{ILD}}(CL_i)|/|U| \);
3. for each \( a \in A \) do
4. \( \text{sig}(a, A, CL) = \gamma_{\text{ILD}}(CL) - \gamma_{\text{ILD}}(CL) \);
5. if \( \text{sig}(a, A, CL) \geq 0 \) then \( A^* = A^* \cup \{a\} \);
6. end for
7. if \( A^* \neq \emptyset \) then \( \text{sig}(a^*, A, CL) = \max \{\text{sig}(a, A, CL) | a \in A^* \} \), \( A = A^* \), go to Step 3;
8. return \( A \).

The Algorithm 2 can be analyzed as follows.

- In Step 2, the calculation of \( \gamma_{\text{ILD}}(CL) \) is \( O(|U|^2(|C|+1)) \) according to Algorithm 1.
- In Steps 3-6, \( |A| \) boundary regions are calculated and the each one requires \( O(|U|^2(|A|-1)) \) according to Algorithm 1. Because \(|A| \leq |C|\), its calculation is \( O(|U|^2|C|^3) \).
- In Step 7, there are \(|C|-i\) candidate attributes in the \( i \)th loop, and the calculation of each loop is \( O(|U|^2(|C|-i)^2) \) according to Steps 3-6. The amount of calculation is \( \sum_{i=1}^{m} (|C|-i)^2|U|^2 = \sum_{i=1}^{m} (|C|^2-2|C|+i^2)|U|^2 < |U|^2|C|^3 \). Here \( m < |C| \). So the calculation of Step 7 is \( O(|U|^2|C|^3) \).

Therefore, in the worst case, the time complexity of Algorithm 2 is \( O(|U|^2|C|^3) \).

**Example 3.** Use Algorithm 2 to find the attribute reduction of IOD in Table 1. First, we take \( A = C \), \( A^* = \emptyset \). The approximation quality \( \gamma_{\text{ILD}}(CL) \) of partition \( CL \) is 0.17. Second, the significance of each attribute in \( A = \{a_1, a_2, a_3\} \) are \( \text{sig}(a_1, A, CL) = \text{sig}(a_2, A, CL) = 0 \), \( \text{sig}(a_3, A, CL) = -0.08 \). Thus there is \( A^* = \{a_1, a_2\} \). Next, here we select the attribute \( a_1 \) from \( A^* \), \( A = \{a_2, a_3\} \), and \( \gamma_{\text{ILD}}(CL) = 0.17 \). And then, the significance of each attribute in \( A = \{a_2, a_3\} \) are \( \text{sig}(a_2, A, CL) = 0 \) and \( \text{sig}(a_3, A, CL) = 0.25 \). There is \( A^* = \{a_2, a_3\} \). \( a_3 \) is selected from \( A^* \), and \( A = \{a_2\} \). So, the attribute reduction is \( A = \{a_2\} \).

**Conclusions**

How to deal with data missing is a key problem in knowledge acquisition. The basic problem of knowledge acquisition in rough set theory is attribute reduction. This paper introduced an ILTDR, which is an extension the DRSA in IODS. This approach improves the limitations of the existing limited tolerance dominance relations. After analyzing the properties of rough approximation under the ILTDR, it is known that the monotonicity of positive region in IODS is no longer valid. Based on this, a heuristic attribute reduction algorithm of IODS is designed, which does not decrease the approximate quality. Examples show that this method can effectively deal with IODS. In future, we will study the decision rule extraction method based on the ILTDR.
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