THE DERIVED CATEGORY OF A LOCALLY COMPLETE INTERSECTION RING

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Abstract. In this paper, we answer a question of Dwyer, Greenlees, and Iyengar by proving a local ring $R$ is a complete intersection if and only if every complex of $R$-modules with finitely generated homology is proxy small. Moreover, we establish that a commutative noetherian ring $R$ is locally a complete intersection if and only if every complex of $R$-modules with finitely generated homology is virtually small.

1. Introduction

The relation of the structure of a commutative noetherian ring $R$ and that of its category of modules has long been a major topic of study in commutative algebra. More recently, it has been extended to studying the relations between the structure of $R$ and that of its derived category $\mathcal{D}(R)$. Working in this setting allows one to use ideas from algebraic topology and triangulated categories to gain insight into properties of $R$.

Basic information on $\mathcal{D}(R)$ is contained in its full subcategory consisting of complexes with finitely generated homology, denoted $\mathcal{D}^{f}(R)$. A complex of $R$-modules is said to be perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules. The following homotopical characterization of regular rings is well known: a commutative noetherian ring $R$ is regular if and only if every object of $\mathcal{D}^{f}(R)$ is a perfect complex.

In many respects, the local rings that are closest to being regular are complete intersections. We characterize of complete intersections in terms of how each object of $\mathcal{D}^{f}(R)$ relates to the perfect complexes. Moreover, this yields a homotopical characterization of a locally complete intersection ring. Following [10] and [11], we say that a complex of $R$-modules $M$ finitely builds a complex of $R$-modules $N$ provided that $N$ can be obtained by taking finitely many cones and retracts starting from $M$. More precisely, $M$ finitely builds $N$ provided $N$ is in $\text{Thick}_{\mathcal{D}(R)} M$ (see Section 2.5). The main result of the paper is the following:

Theorem. A commutative noetherian ring $R$ is locally a complete intersection if and only if every nontrivial object of $\mathcal{D}^{f}(R)$ finitely builds a nontrivial perfect complex.

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2. Preliminaries

2.1. Differential Graded Algebra. Fix a commutative noetherian ring $Q$. Let $A = \{A_i\}_{i \in \mathbb{Z}}$ denote a DG $Q$-algebra. We only consider left DG $A$-modules.

2.1.1. Let $M$ and $N$ be DG $A$-modules. We say that $\varphi : M \to N$ is a morphism of DG $A$-modules provided $\varphi$ is a morphism of the underlying complexes of $Q$-modules such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$. We write $\varphi : M \cong N$ when $\varphi$ is a quasi-isomorphism.

2.1.2. Let $M$ be a DG $A$-module. The differential of $M$ is denoted by $\partial M$. For each $i \in \mathbb{Z}$, $\Sigma^i M$ is the DG $A$-module given by $(\Sigma^i M)_n := M_{n-i}$, $\partial \Sigma^i M := (-1)^i \partial M$, and $a \cdot m := (-1)^{|a||i|} am$.

2.1.3. A DG $A$-module $P$ is semiprojective if for every morphism of DG $A$-modules $\alpha : P \to N$ and each surjective quasi-isomorphism of DG $A$-modules $\gamma : M \to N$ there exists a unique up to homotopy morphism of DG $A$-modules $\beta : P \to M$ such that $\alpha = \gamma \beta$.

2.1.4. A semiprojective resolution of a DG $A$-module $M$ is a surjective quasi-isomorphism of DG $A$-modules $\epsilon : P \to M$ where $P$ is a semiprojective DG $A$-module. Semiprojective resolutions exist and any two semiprojective resolutions of $M$ are unique up to homotopy equivalence [12, 6.6].

2.1.5. For DG $A$-modules $M$ and $N$, define

$$\text{Ext}_A^*(M, N) := H(\text{Hom}_A(P, N))$$

where $P$ is a semiprojective resolution of $M$ over $A$. Since any two semiprojective resolutions of $M$ are homotopy equivalent, $\text{Ext}_A^*(M, N)$ is independent of choice of $M$. An element $[\alpha]$ of $\text{Ext}_A^*(M, N)$ is the class of a morphism of DG $A$-modules $\alpha : P \to \Sigma^{|\alpha|} N$.

Moreover, given $[\alpha]$ and $[\beta]$ in $\text{Ext}_A^*(M, N)$, then $[\alpha] = [\beta]$ if and only if $\alpha$ and $\beta$ are homotopic morphisms of DG $A$-modules.

2.1.6. Let $D(A)$ denote the derived category of $A$ (see [15] for an explicit construction). Recall that $D(A)$, equipped with $\Sigma$, is a triangulated category. Define $D^f(A)$ to be the full subcategory of $D(A)$ consisting of all $M$ such that $H(M)$ is a finitely generated graded module over $H(A)$. We use $\cong$ to denote isomorphisms in $D(A)$ and reserve $\simeq$ for isomorphisms of DG $A$-modules.
2.2. Koszul Complexes. Fix a commutative noetherian ring $Q$. Let $f = f_1, \ldots, f_n$ be a list of elements in $Q$. Define $\text{Kos}^Q(f)$ to be the DG $Q$-algebra with $\text{Kos}^Q(f)$ the exterior algebra on a free $Q$-module with basis $\xi_1, \ldots, \xi_n$ of homological degree $1$, and differential $\partial \xi_i = f_i$. We write $\text{Kos}^Q(f) = Q\langle \xi_1, \ldots, \xi_n | \partial \xi_i = f_i \rangle$.

2.2.1. Let $f' = f'_1, \ldots, f'_m$ be in $Q$. Assume there exists $a_{ij} \in Q$ such that $f_i = \sum_{j=1}^m a_{ij} f'_j$.

There exists a unique morphism of DG $Q$-algebras $\text{Kos}^Q(f) \to \text{Kos}^Q(f')$ satisfying $\xi_i \mapsto \sum_{j=1}^m a_{ij} \xi'_j$.

Therefore, $\text{Kos}^Q(f')$ is a DG $\text{Kos}^Q(f)$-module where the action is given by $\xi_i \cdot e' = \sum_{j=1}^m a_{ij} \xi'_j e'$ for all $e' \in E'$.

2.2.2. Assume that $(Q, n, k)$ is a commutative noetherian local ring. Define $K^Q$ to be the Koszul complex on some minimal generating set for $n$. Then $K^Q$ is unique up to DG $Q$-algebra isomorphism.

2.3. Map on Ext. Let $Q$ be a commutative noetherian ring. Fix a morphism of DG $Q$-algebras $\varphi : A' \to A$. Let $M$ and $N$ be DG $A$-modules, $\epsilon : P \to M$ be a semiprojective resolution of $M$ over $A$, and $\epsilon' : P' \to M$ a semiprojective resolution of $M$ over $A'$. There exists a unique up to homotopy morphism of DG $A'$-modules $\alpha : P' \to P$ such that $\epsilon' = \alpha \epsilon$. Define $\text{Hom}_{A'}(\alpha, N)$ to be the composition

\[
\text{Hom}_A(P, N) \xrightarrow{\text{Hom}_{A'}(P, N)} \text{Hom}_{A'}(P, N) \xrightarrow{\text{Hom}_{A'}(\alpha, N)} \text{Hom}_{A'}(P', N).
\]

This induces a map in cohomology

\[
\text{Ext}^*_A(M, N) : \text{Ext}^*_A(M, N) \to \text{Ext}^*_A(M, N)
\]

given by $\text{Ext}^*_A(M, N) = H(\text{Hom}_{A'}(\alpha, N))$; it is independent of choice of $\alpha$, $P$, and $P'$.

2.3.1. Let $\varphi : A' \to A$ be a morphism of DG $Q$-algebras and let $M$ and $N$ be DG $A$-modules. If $\varphi$ is a quasi-isomorphism, then $\text{Ext}^*_A(M, N)$ is an isomorphism [12, 6.10].

In the following theorem, the theory of DG $\Gamma$-algebras is used. See [2, Section 6] or [13, Chapter 1] as a reference for definitions and notation.

**Theorem 2.3.2.** Assume $(Q, n, k)$ is a regular local ring. Let $R = Q/I$ where $I$ is minimally generated by $f = f_1, \ldots, f_n \in n^2$. Let $E$ be the Koszul complex on $f$ over $Q$. Let $\varphi : E \to R$ denote the augmentation map. The canonical map

\[
\text{Ext}^*_E(k,k) : \text{Ext}^*_R(k,k) \to \text{Ext}^*_E(k,k)
\]

is surjective.
Proof. Write $E = Q(\xi_1, \ldots, \xi_n | \partial \xi_i = f_i)$. For an element $a \in Q$, let $\pi$ denote the image of $a$ in $R$. Let $s_1, \ldots, s_e$ be a minimal generating set for $n$. Let $X = \{x_1, \ldots, x_e\}$ be a set of exterior variables of homological degree 1 and $Y = \{y_1, \ldots, y_n\}$ a set of divided power variables of homological degree 2. By [2, 7.2.10], the morphism of DG $\Gamma$-algebras $\varphi : E \to R$ extends to a morphism of DG $\Gamma$-algebras

$$\varphi(X) : E(X|\partial x_i = s_i) \to R(X|\partial x_i = \pi_i)$$

such that $\varphi(X)(x_i) = x_i$ for each $1 \leq i \leq e$.

Since $f_i \in n^2$, there exists $a_{ij} \in n$ such that

$$f_i = \sum_{j=1}^e a_{ij} s_j.$$

For each $1 \leq i \leq n$, we have degree 1 cycles

$$z_i := \sum_{j=1}^n a_{ij} x_j - \xi_i \quad \text{and} \quad \pi_i := \sum_{j=1}^n a_{ij} x_j$$

in $E(X)$ and $R(X)$, respectively, where $\varphi(X)(z_i) = \pi_i$. Applying [2, 7.2.10] yields a morphism of DG $\Gamma$-algebras

$$\varphi(X, Y) : E(X,Y|\partial y_i = z_i) \to R(X,Y|\partial y_i = \pi_i)$$

extending $\varphi(X)$ such that $\varphi(X, Y)(y_i) = y_i$ for each $1 \leq i \leq n$.

By [2, 6.3.2], $E(X, Y)$ is an acyclic closure of $k$ over $E$. In particular, $E(X, Y)$ is a semiprojective resolution of $k$ over $E$. Next, $\pi_1, \ldots, \pi_n$ is a minimal generating set for the maximal ideal of $R$. Also, since $f_1, \ldots, f_n$ minimally generates $I$, it follows that $[\pi_1], \ldots, [\pi_n]$ is a minimal generating set for $H_1(R(X))$ (see [18, Theorem 4] or [13, 1.5.4]). Thus, $R(X, Y)$ is the second step in forming an acyclic closure of $k$ over $R$. Let $\iota : R(X, Y) \to R(X, Y, V)$ denote the inclusion of DG $\Gamma$-algebras where $R(X, Y, V)$ is an acyclic closure of $k$ over $R$ and $V$ consists of $\Gamma$-variables of homological degree at least 3. Define $\alpha : E(X, Y) \to R(X, Y, V)$ to be the morphism of DG $\Gamma$-algebras given by $\alpha := \iota \circ \varphi(X, Y)$.

The following is a commutative diagram of $\Gamma$-algebras

$$
\begin{array}{ccc}
E(X, Y) \otimes_E k & \alpha \otimes k & R(X, Y, V) \otimes_R k \\
\cong & \cong & \\
\downarrow & \downarrow & \\
k<X,Y> & \subseteq & k<X,Y,V>
\end{array}
$$

Therefore, $\alpha \otimes k$ is an injective morphism of $\Gamma$-algebras. In particular, $\alpha \otimes k$ is injective as a map of graded $k$-vector spaces. Also, the following is a commutative diagram of graded $k$-vector spaces

$$
\begin{array}{ccc}
\Hom_k(R(X, Y, V) \otimes_R k, k) & (\alpha \otimes k)^* & \Hom_k(E(X, Y) \otimes_E k, k) \\
\cong & & \cong \\
\downarrow & \downarrow & \\
\Hom_R(R(X, Y, V), k) & \Hom_{\varphi}(\alpha, k) & \Hom_{E}(E(X, Y), k)
\end{array}
$$

Since $\alpha \otimes k$ is injective, $(\alpha \otimes k)^*$ is surjective. Thus, $\Hom_{\varphi}(\alpha, k)$ is surjective. Moreover, $\Hom_{\varphi}(E(X, Y), k)$ and $\Hom_R(R(X, Y, V), k)$ have trivial differential (see [2, 6.3.4]). Thus, $\Ext^*_\varphi(k, k) = \Hom_{\varphi}(\alpha, k)$, and so $\Ext^*_\varphi(k, k)$ is surjective. \qed
2.4. Support of a Complex of Modules. Let $R$ be a commutative noetherian ring and $\text{Spec } R$ denote the set of prime ideals of $R$. For a complex of $R$-modules $M$, define the support of $M$ to be
\[ \text{Supp}_R M := \{ p \in \text{Spec } R : M_p \neq 0 \}. \]

2.4.1. Let $M$ be in $D^f(R)$ and let $x$ generate an ideal $I$ of $R$. It follows from Nakayama’s lemma that $x$ generates a maximal ideal $m$ of $R$ with $m \in \text{Supp}_R M$, then
\[ \text{Supp}_R (M \otimes_R \text{Kos}^R(x)) = \text{Supp}_R M \cap \text{Supp}_R (R/I). \]

In particular, if $x$ generates a maximal ideal $m$ of $R$ with $m \in \text{Supp}_R M$, then
\[ \text{Supp}_R (M \otimes_R \text{Kos}^R(x)) = \{ m \}. \]

**Lemma 2.4.2.** Let $n$ be a nonzero integer and let $M$ be in $D^f(R)$. If $\alpha : M \to \Sigma^n M$ is a morphism in $D(R)$, then
\[ \text{Supp}_R M = \text{Supp}_R (\text{cone}(\alpha)). \]

**Proof.** Let $C := \text{cone}(\alpha)$. We have an exact triangle
\[ M \to \Sigma^n M \to C \to \]

in $D(R)$. For each $p \in \text{Spec } R$, there is an exact triangle
\[ M_p \to \Sigma^n M_p \to C_p \to \]

in $D(R_p)$. It follows that $\text{Supp}_R C \subseteq \text{Supp}_R M$.

If $p \notin \text{Supp}_R C$, then $M_p \cong \Sigma^n M_p$ in $D(R_p)$. Since $M_p \cong \Sigma^n M_p$, $M_p$ is in $D^f(R_p)$, and $n \neq 0$, it follows that $M_p \cong 0$. Thus, $p \notin \text{Supp}_R M$. \(\square\)

2.5. Thick Subcategories. Let $\mathcal{T}$ denote a triangulated category. A full subcategory $\mathcal{T}'$ of $\mathcal{T}$ is called thick if it is closed under suspension, has the two out of three property on exact triangles, and is closed under direct summands. For an object $X$ of $\mathcal{T}$, define the thick closure of $X$ in $\mathcal{T}$, denoted $\text{Thick}_\mathcal{T} X$, to be the intersection of all thick subcategories of $\mathcal{T}$ containing $X$. Since an intersection of thick subcategories is a thick subcategory, $\text{Thick}_\mathcal{T} X$ is the smallest thick subcategory of $\mathcal{T}$ containing $X$. See [5, Section 2] for an inductive construction of $\text{Thick}_\mathcal{T} X$ and a discussion of the related concept of levels. If $Y$ is an object of $\text{Thick}_\mathcal{T} X$, then we say that $X$ finitely builds $Y$.

2.5.1. Let $R$ be a commutative ring. Recall that a complex of $R$-modules $M$ is perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules. By [11, 3.7], $\text{Thick}_{D(R)} R$ consists exactly of the perfect complexes.

2.5.2. Let $R$ be a commutative ring and let $m$ be a maximal ideal of $R$. By [11, 3.10], $\text{Thick}_{D(R)} (R/m)$ consists of all objects $M$ of $D^f(R)$ such that $\text{Supp}_R M = \{ m \}$.

2.5.3. Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories with right adjoint exact functor $G$. Let $\epsilon : FG \to \text{id}_{\mathcal{T}'}$ and $\eta : \text{id}_{\mathcal{T}} \to GF$ be the co-unit and unit transformations.

The full subcategory of $\mathcal{T}$ consisting of all objects $X$ such that the natural map $\eta_X : X \to GF(X)$ is an isomorphism is a thick subcategory of $\mathcal{T}$. For each $X$ in $\mathcal{T}$, the composition
\[ F(X) \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{\epsilon_{F(X)}} F(X) \]
is an isomorphism. Therefore, if \( \eta_X \) is an isomorphism in \( T \) then \( \epsilon_F(X) \) is an isomorphism in \( T' \) and \( F \) induces an equivalence of categories

\[
\text{Thick}_T X \xrightarrow{\sim} \text{Thick}_{T'} F(X).
\]

**Lemma 2.5.4.** Let \( \varphi : R \to S \) be flat morphism of commutative rings. Suppose \( M \) is in \( D(R) \) and the natural map \( M \to M \otimes_R S \) is an isomorphism in \( D(R) \). Then the functor \( - \otimes_R S : D(R) \to D(S) \) induces an equivalence of categories

\[
\text{Thick}_{D(R)} M \xrightarrow{\sim} \text{Thick}_{D(S)} (M \otimes_R S).
\]

In particular, for each \( N \) in \( \text{Thick}_{D(R)} M \) the natural map \( N \to N \otimes_R S \) is an isomorphism in \( D(R) \).

**Proof.** The restriction of scalar functor \( G : D(S) \to D(R) \) is a right adjoint to \( - \otimes_R S : D(R) \to D(S) \). By assumption, the natural map

\[
M \to G(M \otimes_R S)
\]

is an isomorphism in \( D(R) \). Hence, (2.5.3) completes the proof. \( \square \)

2.6. **Support of Cohomology Graded Modules.** Let \( \mathcal{A} = \{A^i\}_{i \geq 0} \) be a cohomologically graded, commutative noetherian ring. Recall that \( \text{Proj} \mathcal{A} \) denotes the set of homogeneous prime ideals of \( \mathcal{A} \) not containing \( \mathcal{A}^{>0} := \{A^i\}_{i>0} \). For homogeneous elements \( a_1, \ldots, a_m \in \mathcal{A} \) define

\[
\mathcal{V}(a_1, \ldots, a_m) = \{ p \in \text{Proj} \mathcal{A} : a_i \in p \text{ for each } i \}.
\]

For a (cohomologically) graded \( \mathcal{A} \)-module \( X \), set

\[
\text{Supp}_\mathcal{A}^+ X := \{ p \in \text{Proj} \mathcal{A} : X_p \neq 0 \}.
\]

The following properties of (cohomologically) graded \( \mathcal{A} \)-modules follow easily from the definition of support; see [6, 2.2]

**Proposition 2.6.1.** Let \( \mathcal{A} = \{A^i\}_{i \geq 0} \) be a cohomologically graded, commutative noetherian ring.

1. Let \( X \) be a graded \( \mathcal{A} \)-module and \( n \in \mathbb{Z} \). Then \( \text{Supp}_\mathcal{A}^+ X = \text{Supp}_\mathcal{A}^+(\Sigma^n X) \).

2. Given an exact sequence of graded \( \mathcal{A} \)-modules \( 0 \to X' \to X \to X'' \to 0 \) then

\[
\text{Supp}_\mathcal{A}^+ X = \text{Supp}_\mathcal{A}^+ X' \cup \text{Supp}_\mathcal{A}^+ X''.
\]

3. If \( X \) is a finitely generated graded \( \mathcal{A} \)-module, then \( \text{Supp}_\mathcal{A}^+ X = \emptyset \) if and only if \( X^{>0} = 0 \).

3. **Cohomology Operators and Support Varieties**

3.1. **Fixed Notation.** Throughout this section, let \( Q \) be a commutative noetherian ring. When \( Q \) is local, we will let \( n \) denote its maximal ideal and \( k \) its residue field.

Let \( I \) be an ideal of \( Q \) and fix a generating set \( f = f_1, \ldots, f_n \) for \( I \). Set \( R := Q/I \) and \( E := Q[\xi_1, \ldots, \xi_n] / \partial_\xi_i = f_i \). The augmentation map \( E \to R \) is a map of DG \( Q \)-algebras. Hence, we consider DG \( R \)-modules as DG \( E \)-modules via restriction of scalars along \( E \to R \).

Let \( S := Q[\chi_1, \ldots, \chi_n] \) be a graded polynomial ring where each \( \chi_i \) has cohomological degree 2. When \( Q \) is local, set

\[
\mathcal{A} := S \otimes_Q k = k[\chi_1, \ldots, \chi_n].
\]
Define $\Gamma$ to be the graded $Q$-linear dual of $S$, i.e., $\Gamma$ is the graded $Q$-module with

$$
\Gamma_i := \text{Hom}_Q(S^i, Q).
$$

Let $\{y^{(H)}\}_{H \in \mathbb{N}^n}$ be the $Q$-basis of $\Gamma$ dual to $\{\chi^H := \chi_1^{h_1} \cdots \chi_n^{h_n}\}_{H \in \mathbb{N}^n}$ the standard $Q$-basis of $S$. Then $\Gamma$ is a graded $S$-module via the action

$$
\chi_i \cdot y^{(H)} := \begin{cases} 
y^{(h_1, \ldots, h_{i-1}, h_i, h_{i+1}, \ldots, h_n)} & h_i \geq 1 \\
0 & h_i = 0
\end{cases}
$$

3.2. Cohomology Operators. Let $M$ be a DG $E$-module. A semiprojective resolution $\epsilon : P \xrightarrow{\sim} M$ over $Q$ such that $P$ has the structure of a DG $E$-module and $\epsilon$ is a morphism of DG $E$-modules is called a Koszul resolution of $M$. A semiprojective resolution of $M$ over $E$ is a Koszul resolution of $M$, and hence Koszul resolutions exist.

Let $\epsilon : P \xrightarrow{\sim} M$ be a Koszul resolution of $M$. Define $U_E(P)$ to be the DG $E$-module with

$$
U_E(P)^\mathbb{Z} \cong (E \otimes_Q \Gamma \otimes_Q P)^\mathbb{Z}
$$

and differential given by the formula

$$
\partial = \partial^E \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \partial^P + \sum_{i=1}^n (1 \otimes \chi_i \otimes \lambda_i - \lambda_i \otimes \chi_i \otimes 1)
$$

where $\lambda_i$ denotes left multiplication by $\xi_i$. By [4, 2.4], $U_E(P) \to M$ is a semiprojective resolution over $E$ where the augmentation map is given by

$$
a \otimes y^{(H)} \otimes x \mapsto \begin{cases} 
\alpha(x) & |H| = 0 \\
0 & |H| > 1
\end{cases}
$$

Notice that $U_E(P)$ has a DG $S$-module structure where $S$ acts on $U_E(P)$ via its action on $\Gamma$. For a DG $E$-module $N$, $\text{Hom}_E(U_E(P), N)$ is a DG $S$-module and hence,

$$
\text{Ext}^*_E(M, N) = \text{H}(\text{Hom}_E(U_E(P), N))
$$

is a graded module over $S$.

Remark 3.2.1. Let $M$ and $M'$ be DG $E$-modules and assume that $\alpha : M \to M'$ is a morphism of DG $E$-modules. Let $F$ and $F'$ be semiprojective resolutions of $M$ and $M'$ over $E$, respectively. Since $F$ is semiprojective over $E$, there exists a morphism of DG $E$-modules $\hat{\alpha} : F \to F'$ lifting $\alpha$ that is unique up to homotopy. Moreover, $\hat{\alpha}$ induces a morphism of DG $E$-modules $1 \otimes 1 \otimes \hat{\alpha} : U_E(F) \to U_E(F')$ that is $S$-linear and unique up to homotopy.

In particular, if $F$ and $F'$ are both semiprojective resolutions of a DG $E$-module $M$, then there exists a DG $E$-module homotopy equivalence $U_E(F) \to U_E(F')$ that is $S$-linear and unique up to homotopy. Thus, the $S$-module structures of $\text{H}(\text{Hom}_E(U_E(F), N))$ and $\text{H}(\text{Hom}_E(U_E(F'), N))$ coincide when $F$ and $F'$ are both semiprojective resolutions of $M$ over $E$.

Proposition 3.2.2. Let $M$ and $N$ be in $D(E)$. Then the $S$-module structure on $\text{Ext}^*_E(M, N)$ is independent of choice of Koszul resolution for $M$. Moreover, the $S$-module action on $\text{Ext}^*_E(M, N)$ is functorial in $M$ and given an exact triangle $M' \to M \to M'' \to$ in $D(E)$, there exists an exact sequence of graded $S$-modules

$$
\Sigma^{-1} \text{Ext}^*_E(M', N) \to \text{Ext}^*_E(M'', N) \to \text{Ext}^*_E(M, N) \to \text{Ext}^*_E(M', N).
$$
Proof. Let $P$ be a Koszul resolution of $M$ and $F$ a semiprojective resolution of $M$ over $E$. There exists a morphism of DG $E$-modules $\tilde{\alpha} : F \to P$ lifting the identity on $M$ which is unique up to homotopy. This induces a DG $E$-module homotopy equivalence $1 \otimes 1 \otimes \tilde{\alpha} : U_E(F) \to U_E(P)$ that is $S$-linear and unique up to homotopy. Thus, $F$ and $P$ determine the same $S$-module structure on $\text{Ext}^*_E(M, N)$. From Remark 3.2.1, it follows that the $S$-module structure on $\text{Ext}^*_E(M, N)$ is independent of choice of Koszul resolution for $M$.

Moreover, by Remark 3.2.1 the $S$-module structure on $\text{Ext}^*_E(M, N)$ is functorial in $M$. Thus, $\text{Ext}^*_E(-, N)$ sends exact triangles in $D(E)$ to exact sequences of graded $S$-modules. \hfill \Box

3.2.3. Assume that $(Q, n, k)$ is a local ring and recall that $A = S \otimes_Q k$. Let $M$ be in $D(E)$. The $S$-action on $\text{Ext}^*_E(M, k)$ factors through $S \to A$, and hence, $\text{Ext}^*_E(M, k)$ is a graded $A$-module. Therefore, by Proposition 3.2.2, for any exact triangle $M' \to M \to M'' \to$ in $D(E)$, we get an exact sequence of graded $A$-modules

$$\Sigma^{-1} \text{Ext}^*_E(M', k) \to \text{Ext}^*_E(M'', k) \to \text{Ext}^*_E(M, k) \to \text{Ext}^*_E(M', k).$$

Lemma 3.2.4. Assume that $(Q, n, k)$ is a local ring and $M$ is in $D(E)$. For any $x \in n$, there exists an exact sequence of graded $A$-modules

$$0 \to \Sigma^{-1} \text{Ext}^*_E(M, k) \to \text{Ext}^*_E(M \otimes_Q \text{Kos}^Q(x), k) \to \text{Ext}^*_E(M, k) \to 0.$$

Proof. By (3.2.3), applying $\text{Ext}^*_E(-, k)$ to the exact triangle $M \to M \to M \otimes_Q \text{Kos}^Q(x) \to$ in $D(E)$ gives us an exact sequences of graded $A$-modules

$$\Sigma^{-1} \text{Ext}^*_E(M, k) \to \text{Ext}^*_E(M \otimes_Q \text{Kos}^Q(x), k) \to \text{Ext}^*_E(M, k) \to \text{Ext}^*_E(M, k).$$

Since $x$ is in $n$, we obtain the desired result. \hfill \Box

Proposition 3.2.5. Assume that $(Q, n, k)$ is a regular local ring. For each $M$ in $D^f(E)$, $\text{Ext}^*_E(M, k)$ is a finitely generated graded $A$-module.

Proof. As $H(M)$ is finitely generated over $Q$ and $Q$ is regular, there exists a Koszul resolution $P \xrightarrow{\sim} M$ such that $P$ is a bounded complex of finitely generated free $Q$-modules (see [4, 2.1]). Also, we have an isomorphism of graded $A$-modules

$$\text{Hom}_E(U_E(P), k) \cong A \otimes_k \text{Hom}_Q(P, k)^2.$$ Thus, $\text{Hom}_E(U_E(P), k)$ is a noetherian graded $A$-module. As $A$ is a noetherian graded ring and $\text{Ext}^*_E(M, k)$ is a graded subquotient of $\text{Hom}_E(U_E(P), k)$, it follows that $\text{Ext}^*_E(M, k)$ is a noetherian graded $A$-module. \hfill \Box

Remark 3.2.6. Suppose the local ring $(Q, n, k)$ is regular. By (2.2.1), $K^Q$ is a DG $E$-module. Assume that $I \subseteq n^2$. Left multiplication by $\xi_i$ on $K^Q$ is zero modulo $n$. Thus, we have an isomorphism of DG $A$-modules

$$\text{Hom}_E(U_E(K^Q), k) \cong A \otimes_k \text{Hom}_Q(K^Q, k).$$

where both DG $A$-modules have trivial differential (see (2.2.1)). Therefore, there is an isomorphism of graded $A$-modules

$$\text{Ext}^*_E(k, k) \cong A \otimes_k \text{Hom}_Q(K^Q, k).$$

In particular,

$$\text{Supp}^+_A(\text{Ext}^*_E(k, k)) = \text{Proj} A.$$
3.3. Support Varieties. For the rest of the section, further assume that \((Q, n, k)\) is a regular local ring, \(f\) minimally generates \(I\), and \(I \subseteq n\). Recall that

\[ A = S \otimes_Q k = k[\chi_1, \ldots, \chi_n]. \]

By Proposition 3.2.5, \(\text{Ext}^*_{E}(M, k)\) is a finitely generated graded \(A\)-module for each \(M\) in \(\mathcal{D}^f(E)\). This leads to the following definition which recovers the support varieties of Avramov in \([3]\) in the case that \(f\) is a \(Q\)-regular sequence. The varieties, defined below, are investigated and further developed in \([17]\).

**Definition 3.3.1.** Let \(M\) be in \(\mathcal{D}^f(E)\). Define the support variety of \(M\) over \(E\) to be

\[ V_E(M) := \text{Supp}_A^+(\text{Ext}^*_{E}(M, k)). \]

**Theorem 3.3.2.** With the assumptions above, the following hold.

1. Let \(M\) and \(N\) be in \(\mathcal{D}^f(E)\). If \(N\) is in \(\text{Thick}_{\mathcal{D}(E)} M\), then \(V_E(N) \subseteq V_E(M)\).
2. For any \(M\) in \(\mathcal{D}^f(E)\), \(V_E(M) = V_E(M \otimes_Q K^Q)\).
3. \(f\) is a \(Q\)-regular sequence if and only if \(V_E(R) = \emptyset\).

**Proof.** Using (3.2.3) and Proposition 2.6.1, it follows that the full subcategory of \(\mathcal{D}^f(E)\) consisting of objects \(L\) such that \(V_E(L) \subseteq V_E(M)\) is a thick subcategory of \(\mathcal{D}^f(E)\). Therefore, (1) holds.

Iteratively applying Lemma 3.2.4 and Proposition 2.6.1(2), establishes (2).

For (3), first assume that \(f\) is a \(Q\)-regular sequence. Hence, the augmentation map \(E \to R\) is a quasi-isomorphism. Therefore, (2.3.1) yields an isomorphism

\[ \text{Ext}^*_{E}(R, k) \cong \text{Ext}^*_{R}(R, k) = k. \]

Thus, \(V_E(R) = \text{Supp}_R^+ k = \emptyset\).

Conversely, assume that \(V_E(R) = \emptyset\). Hence, by Proposition 3.2.5 and Proposition 2.6.1(3),

\[ \text{Ext}_{E}^{\geq 0}(R, k) = 0. \]

Next, let \(g = g_1, \ldots, g_n\) be a minimal generating set for \(I\) such that \(g' = g_1, \ldots, g_c\) is a maximal \(Q\)-regular sequence in \(I\) for some \(c \leq g\). Set \(\overline{Q} := Q/\langle g' \rangle\), \(\overline{g}\) to be the image of \(g_{c+1}, \ldots, g_n\) in \(\overline{Q}\), and \(\overline{E} := \text{Kos}^Q(\overline{g})\). Since \(g'\) is a \(Q\)-regular sequence, we have a quasi-isomorphism of DG \(Q\)-algebras \(E \xrightarrow{\sim} \overline{E}\). Hence, (2.3.1) yields an isomorphism of graded \(k\)-vector spaces

\[ \text{Ext}^*_{E}(R, k) \cong \text{Ext}^*_{E}(R, k). \]

In particular, \(\text{Ext}^*_{E}^{\geq 0}(R, k) = 0\) by (1). Therefore, \(pd_{\overline{Q}} R < \infty\) (c.f. \([7, B.10]\)). Since \(R = \overline{Q} / I\overline{Q}\) where \(I\overline{Q}\) contains no \(Q\)-regular element, it follows that \(I\overline{Q} = 0\) (see \([9, 1.4.7]\)). Thus, \(g = g'\), that is, \(I\) is generated by a \(Q\)-regular sequence. Therefore, by \([9, 1.6.19]\), \(f\) is \(Q\)-regular sequence. \(\square\)

**Remark 3.3.3.** In \([17]\), a different argument is used to establish Theorem 3.3.2(c). In fact, the following is shown: \(f\) is a \(Q\)-regular sequence if and only if \(V_E(M) = \emptyset\) for some nonzero finitely generated \(R\)-module \(M\).

**Theorem 3.3.4.** Assume \((Q, n, k)\) is a regular local ring. Let \(R = Q/I\) where \(I\) is minimally generated by \(f = f_1, \ldots, f_n \in n^2\). Let \(E\) be the Koszul complex on
Let $f$ over $Q$ and set $A = k[\chi_1, \ldots, \chi_n]$. For each homogeneous element $g \in A$, there exists a complex of $R$-modules $C(g)$ in $\text{Thick}_{D(R)} k$ such that 

$$V_E(C(g)) = V(g).$$

**Proof.** As $Q$ is regular, the Koszul complex $K^Q$ is a free resolution of $k$ over $Q$. Moreover, (2.2.1) says that $K^Q$ is a Koszul resolution of $k$. By (3.2), there exists a semiprojective resolution $\epsilon : U \to k$ over $E$ where $U := U_E(K^Q)$. Let $d$ denote the degree of $g$. Define 

$$\tilde{C}(g) := \text{cone}(U \xrightarrow{g} \Sigma^dU).$$

The same proof of [6, 3.10] and Remark 3.2.6 yields 

(2) $V_E(\tilde{C}(g)) = V(g)$.

Fix a projective resolution $\delta : P \to k$ over $R$. Since $U$ is a semiprojective DG $E$-module there exists a morphism of DG $E$-modules $\alpha : U \to P$ such that $\delta \alpha = \epsilon$. Note that $\alpha$ is a quasi-isomorphism.

By Theorem 2.3.2 and (2.1.5), there exists a morphism of complexes of $R$-modules $\gamma : P \to \Sigma^d k$ such that 

(3) $U \xrightarrow{g} \Sigma^dU \xrightarrow{\gamma} \Sigma^d k$

is a diagram of DG $E$-modules that commutes up to homotopy. Define 

$$C(g) := \text{cone}(\gamma).$$

Since $P \simeq k$ and $\gamma$ is a morphism of complexes of $R$-modules, it follows that $C(g)$ is in $\text{Thick}_{D(R)} k$. Also, as $\alpha$ are $\Sigma^d \epsilon$ quasi-isomorphisms and (3) commutes up to homotopy, we get an isomorphism 

$$C(g) \simeq \tilde{C}(g)$$

in $D(E)$. Therefore, Equation (2) yields 

$$V_E(C(g)) = V_E(\tilde{C}(g)) = V(g).$$

□

4. **Virtually Small Complexes**

Let $R$ be a commutative noetherian ring. A complex of $R$-modules $M$ is **virtually small** if $M \simeq 0$ or there exists a nontrivial object $P$ in $\text{Thick}_{D(R)} M \cap \text{Thick}_{D(R)} R$. If in addition $P$ can be chosen with $\text{Supp}_R M = \text{Supp}_R P$, we say $M$ is **proxy small**. These notions were introduced by Dwyer, Greenlees, and Iyengar in [10] and [11], where the authors apply methods from commutative algebra to homotopy theory and vice versa.

**Remark 4.1.** In [10] and [11], the objects of $\text{Thick}_{D(R)} R$ are called the **small objects** of $D(R)$. With this terminology, the nontrivial virtually small objects of $D(R)$ are the complexes that finitely build a nontrivial small object.

4.2. A nontrivial object $M$ of $D^f(R)$ is virtually small if and only if there exists a maximal ideal $m = (x)$ of $R$ such that $\text{Kos}^R(x)$ is in $\text{Thick}_{D(R)} M$. In particular, if $R$ is local, a nontrivial complex $M$ in $D^f(R)$ is virtually small if and only if $K^R$ is in $\text{Thick}_{D(R)} M$. This was observed in [11, 4.5], and is a consequence of a theorem of M. Hopkins [14] and Neeman [16].


As a matter of notation, let $\mathcal{VS}(R)$ to be the full subcategory of $\mathcal{D}^f(R)$ consisting of all virtually small complexes. In the following lemma, the argument for “(1) implies (2)” is abstracted from the proof of [11, 9.4].

**Lemma 4.3.** Let $R$ be a commutative noetherian ring. The following are equivalent:

1. Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$ is a subcategory of $\mathcal{VS}(R)$ for each maximal ideal $\mathfrak{m}$ of $R$.
2. $\mathcal{D}^f(R) = \mathcal{VS}(R)$.
3. $\mathcal{VS}(R)$ is a thick subcategory of $\mathcal{D}(R)$.

**Proof.** (1) $\implies$ (2): Let $M$ be a nontrivial object of $\mathcal{D}^f(R)$. Since $M$ is nontrivial, there exists a maximal ideal $\mathfrak{m}$ in $\text{Supp}_R M$. Let $x$ generate $\mathfrak{m}$ and set

$$N := M \otimes_R \text{Kos}^R(x).$$

By (2.4.1), $\text{Supp}_R N = \{\mathfrak{m}\}$ and hence, $N$ is in Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$ (see (2.5.2)). By assumption, there exists a nontrivial object $P$ in Thick$_{\mathcal{D}(R)}(R) \cap$Thick$_{\mathcal{D}(R)}(R)$. Finally, since $N$ is in Thick$_{\mathcal{D}(R)}(R)$, Thick$_{\mathcal{D}(R)}(R)$ is a subcategory of Thick$_{\mathcal{D}(R)}(R)$. Thus, $P$ is in Thick$_{\mathcal{D}(R)}(R)$. That is, $M$ is virtually small.

(2) $\implies$ (3): Whenever $R$ is noetherian, $\mathcal{D}^f(R)$ is a thick subcategory of $\mathcal{D}(R)$.

(3) $\implies$ (1): By (2.5.2), $\text{Kos}^R(x)$ is in Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$. Thus, $R/\mathfrak{m}$ is in $\mathcal{VS}(R)$. Since $\mathcal{VS}(R)$ is a thick subcategory of $\mathcal{D}(R)$, it follows that Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$ is contained in $\mathcal{VS}(R)$. $\square$

**Lemma 4.4.** Let $\varphi : R \to S$ be a flat morphism of commutative noetherian rings. Suppose $\mathfrak{m}$ is a maximal ideal of $R$ such that $\mathfrak{m}S$ is a maximal ideal of $S$ and the canonical map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Then Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$ is a subcategory of $\mathcal{VS}(R)$ if and only if Thick$_{\mathcal{D}(S)}(S/\mathfrak{m}S)$ is a subcategory of $\mathcal{VS}(S)$.

**Proof.** Set $K := \text{Kos}^R(x)$ where $x$ generates $\mathfrak{m}$. Let $x'$ denote the image of $x$ under $\varphi$ and set $K' := \text{Kos}^S(x')$. Hence, we have an isomorphism of DG $S$-algebras $K' \cong K \otimes_R S$.

Assume Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$ is a subcategory of $\mathcal{VS}(R)$. Let $N$ be a nontrivial object of Thick$_{\mathcal{D}(S)}(S/\mathfrak{m}S)$. By Lemma 2.5.4, there exists a nontrivial complex $M$ in Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$ such that $M \otimes_R S \cong N$ in $\mathcal{D}(S)$. By assumption and (4.2), $K$ is in Thick$_{\mathcal{D}(S)}(S/\mathfrak{m}S)$. Hence, $K \otimes_R S$ is in Thick$_{\mathcal{D}(R)}(M \otimes_R S)$ and $N \cong M \otimes_R S$. Since $K' \cong K \otimes_R S$ and $N \cong M \otimes_R S$, we conclude that $K'$ is in Thick$_{\mathcal{D}(S)}(S/\mathfrak{m}S)$. Thus, $N$ is in $\mathcal{VS}(S)$.

Let $M$ be a nontrivial object of Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$. Thus, $M \otimes_R S$ is a nontrivial object of Thick$_{\mathcal{D}(S)}(S/\mathfrak{m}S)$. By assumption and (4.2), $K'$ is in Thick$_{\mathcal{D}(S)}(M \otimes_R S)$. Therefore,

$$K' \in \text{Thick}_{\mathcal{D}(R)}(M \otimes_R S).$$

Since the natural map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism and $K$ and $M$ are in Thick$_{\mathcal{D}(R)}(R/\mathfrak{m})$, by applying Lemma 2.5.4 we obtain the following isomorphisms in $\mathcal{D}(R)$

$$K \xrightarrow{\cong} K \otimes_R S \cong K' \text{ and } M \xrightarrow{\cong} M \otimes_R S.$$ 

These isomorphisms and (4) imply that $K$ is in Thick$_{\mathcal{D}(R)}(R)$. That is, $M$ is in $\mathcal{VS}(R)$. $\square$

**Proposition 4.5.** Let $R$ be a commutative noetherian ring.
(1) Then $D^f(R) = VS(R)$ if and only if $D^f(R_m) = VS(R_m)$ for every maximal ideal $m$ of $R$.

(2) In addition, assume $(R, m, k)$ is local and let $\hat{R}$ denote its $m$-adic completion. Then $D^f(R) = VS(R)$ if and only if $D^f(\hat{R}) = VS(\hat{R})$.

Proof. By Lemma 4.3, $D^f(R) = VS(R)$ if and only if $\text{Thick}_{D(R)}(R/m)$ is a subcategory of $VS(R)$ for each maximal ideal of $m$ of $R$. By Lemma 4.4, the latter holds if and only if $\text{Thick}_{D(R_m)}(\kappa(m))$ is a subcategory of $VS(R_m)$ for each maximal ideal $m$ of $R$ where $\kappa(m) = R_m/mR_m$. Equivalently, $D^f(R_m) = VS(R_m)$ for each maximal ideal $m$ of $R$ by Lemma 4.3. Thus, (1) holds.

Next, Lemma 4.4 yields that $\text{Thick}_{D(\hat{R})}k$ is a subcategory of $VS(R)$ if and only if $\text{Thick}_{D(\hat{R})}k$ is a subcategory of $VS(\hat{R})$. Applying Lemma 4.3, finishes the proof of (2). □

5. The Main Results

Let $(R, m)$ be a commutative noetherian local ring and let $\hat{R}$ denote its $m$-adic completion. The local ring $R$ is said to be a complete intersection provided

$\hat{R} \cong Q/(f_1, \ldots, f_c)$

where $Q$ is a regular local ring and $f_1, \ldots, f_c$ is a $Q$-regular sequence. In [11, 9.4], the following was established: if $R$ is a complete intersection every object of $D^f(R)$ virtually small. If in addition $R$ is a quotient of a regular local ring, every object of $D^f(R)$ is proxy small. Moreover, the authors posed the following question:

Question 5.1. [11, 9.4] If every object of $D^f(R)$ is virtually small, is $R$ a complete intersection?

Theorem 5.2, below, answers Question 5.1 in the affirmative. Much of the work in establishing “(1) implies (3)” is done in the proof of a theorem of Bergh [8, 3.2]. The theory of support varieties developed in Section 3.3 is the key ingredient used to prove “(2) implies (1).”

Theorem 5.2. Let $R$ be a commutative noetherian local ring. The following are equivalent.

(1) $R$ is a complete intersection.
(2) Every object of $D^f(R)$ is virtually small.
(3) Every object of $D^f(R)$ is proxy small.

Proof. (1) $\implies$ (3): Let $M$ be in $D^f(R)$. In the proof of [8, 3.2], it is shown there exist positive integers $n_1, \ldots, n_t$ and exact triangles in $D(R)$

$M \to \Sigma^{n_1}M \to M(1) \to$
$M(1) \to \Sigma^{n_2}M(1) \to M(2) \to$
$\vdots \quad \vdots \quad \vdots \quad \vdots$
$M(t-1) \to \Sigma^{n_t}M(t-1) \to M(t) \to$

such that $M(t)$ is in $\text{Thick}_{D(R)} R$. Also, it is clear that $M(t)$ is in $\text{Thick}_{D(R)} M$. Since each $n_i \neq 0$, Lemma 2.4.2 yields

$\text{Supp}_R M = \text{Supp}_R(M(1)) = \ldots = \text{Supp}_R(M(t))$. 

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Thus, $M$ is proxy small.

(3) $\Rightarrow$ (2): Clear from the definitions.

(2) $\Rightarrow$ (1): By Proposition 4.5(2), we may assume that $R$ is complete. Write $R = Q/I$ where $(Q, n, k)$ is a regular local ring. Assume $I$ is minimally generated by $f = f_1, \ldots, f_n \in n^2$ and let $E$ be the Koszul complex on $f$.

Fix $1 \leq i \leq n$. By Theorem 3.3.4, there exists $C(i)$ in $\text{Thick}_{D(R)} k$ with

$$V_E(C(i)) = V(\chi_i).$$

By assumption, each $C(i)$ is virtually small. Therefore, (4.2) implies that $K^R$ is in $\text{Thick}_{D(R)} C(i)$. Hence,

$$V_E(K^R) \subseteq V_E(C(i)) = V(\chi_i),$$

by Theorem 3.3.2(1). Applying Theorem 3.3.2(2) with $M = R$ yields

$$V_E(R) = V_E(K^R),$$

and hence, $V_E(R) \subseteq V(\chi_i)$. Therefore,

$$V_E(R) \subseteq V(\chi_1) \cap \ldots \cap V(\chi_n).$$

That is, $V_E(R) = \emptyset$ and so by Theorem 3.3.2(3), $f$ is a $Q$-regular sequence. Thus, $R$ is a complete intersection. □

This structural characterization of a complete intersection’s derived category yields the following corollary which was first established by Avramov in [1].

**Corollary 5.3.** Assume a commutative noetherian local ring $R$ is a complete intersection. For any $p \in \text{Spec } R$, $R_p$ is a complete intersection.

**Proof.** For any $p \in \text{Spec } R$, the functor $- \otimes_R R_p : D^f(R) \to D^f(R_p)$ is essentially surjective. Also, the property of proxy smallness localizes. These observations and Theorem 5.2 complete the proof. □

Let $R$ be a commutative noetherian ring. We say that $R$ is **locally a complete intersection** if $R_p$ is a complete intersection for each $p \in \text{Spec } R$. By Corollary 5.3, $R$ is locally a complete intersection if and only if $R_m$ is a complete intersection for every maximal ideal $m$ of $R$. We obtain the following homotopical characterization of rings that are locally complete intersections.

**Theorem 5.4.** A commutative noetherian ring $R$ is locally a complete intersection if and only if every object of $D^f(R)$ is virtually small.

**Proof.** As remarked above, $R$ is locally a complete intersection if and only if $R_m$ is a complete intersection for each maximal ideal $m$ of $R$. By Theorem 5.2, the latter holds if and only if $D^f(R_m) = \text{VS}(R_m)$ for each maximal ideal $m$ of $R$. Equivalently, $D^f(R) = \text{VS}(R)$ by Proposition 4.5(1). □

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