On the thin-shell limit of branes in the presence of Gauss-Bonnet interactions

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In this paper we study thick-shell braneworld models in the presence of a Gauss-Bonnet term. We discuss the peculiarities of the attainment of the thin-shell limit in this case and compare them with the same situation in Einstein gravity. We describe the two simplest families of thick-brane models (parametrized by the shell thickness) one can think of. In the thin-shell limit, one family is characterized by the constancy of its internal density profile (a simple structure for the matter sector) and the other by the constancy of its internal curvature scalar (a simple structure for the geometric sector). We find that these two families are actually equivalent in Einstein gravity and that the presence of the Gauss-Bonnet term breaks this equivalence. In the second case, a shell will always keep some non-trivial internal structure, either on the matter or on the geometric sectors, even in the thin-shell limit.

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I. INTRODUCTION

In the light of the Randall-Sundrum (RS) braneworld paradigm, the cosmological evolution of the universe would acquire important modifications at early times with respect to the standard lore provided by the Friedmann equation of General Relativity (an exhaustive study of the phase space of the new cosmological models can be found in [3]). The new Friedmann-like equation has now contributions that are quadratic in the density, dark-radiation contributions and possibly even other types of dark contributions (coming for example from a fundamental electromagnetic field in the bulk).

This same paradigm, with the characteristic presence of extra dimensions, naturally suggests that we should look for further modifications to the generalized Friedmann equation caused by the existence of a Gauss-Bonnet (GB) term in the field equations. In five dimensions, the most general geometric field equations of the form $G_{AB} = 0$, with $G_{AB}$ a symmetric and conserved tensor constructed by using the metric and its first and second derivatives only, is a linear combination of the metric tensor itself, the Einstein tensor and the Lanczos tensor (or GB term). From this point of view, the value of the three coefficients of this combination, basically, the cosmological constant, Newton’s constant and the $\alpha$ constant (to be defined later) respectively, are parameters to be determined experimentally. The GB term has particular relevance when considering the string-inspired nature of the RS proposal. In string theory, the GB term appears as the first higher curvature correction to Einstein gravity. Possible consequences of the GB interactions on cosmological inflation have already been considered in [15, 16]. Other analysis concerning different aspects of GB corrections to braneworld physics can be found in [17]–[36] (this is not intended to be an exhaustive list).

In trying to generalize the thin-shell cosmological models to incorporate the effects of a GB term it has been some controversy. First, it was claimed that the GB equations were not well defined in the distributional sense required by the thin-shell formalism. It would be necessary to study the microphysics to solve the ambiguities that would arise. However, later, in references [18]–[27], it was assumed that the particular structure of the GB Lagrangian (linear in second order derivatives) posed no problems in the distributional limit (at least in models with $\mathbb{Z}_2$ symmetry). Remarkably, the results obtained in those papers were not consistent with each other. The generalized Friedmann equation found in papers [25]–[27] (quadratic on the brane density $\rho_b$) is different from that found in papers [18]–[24] (with a complicated dependence on $\rho_b$ coming from obtaining the real root of a cubic equation). More recently, there has been an argument [37, 38] based on the form of the surface term in GB (see [39]) that strongly support this later complicated $\rho_b$ dependence (see also [40]).

In this paper, we will study the cosmological behaviour of shells (or branes) that are thin but still of a finite thickness $T$. In this way we want to shed some light on how the zero thickness is attained in the presence of GB interactions. This limit has been studied for Einstein gravity in [41]. Thick shells in the context of GB interactions have been already studied in [42] and [43], but with a focus on different aspects than those in this paper. The conclusion of our analysis here is twofold. On the one side, our results show that the generalized Friedmann equation in [18]–[24] can

1 We are thinking on the standard RS scenario, in which there is only one additional extra dimension; for 7 or more extra dimensions one will have to consider more general Lovelock terms [3].
be found by using a completely general procedure, in which the energy density of the brane in the thin-limit is related to the averaged density. Instead, the equation in \[25\]–\[27\] can only be found for specific geometric configurations and with a procedure in which the energy density of the brane in the thin-limit comes from the value of the boundary density in the thick-brane model. On the other side, we argue that the information lost when treating a real thin shell as infinitely thin is in a sense larger in Einstein-Gauss-Bonnet gravity than in the analogous situation in standard General Relativity.

Let us explain further this last point. From a physical point of view, in the process of passing from the notion of function to that of distribution one loses information. Many different series of functions define the same limiting distribution. For example, the series of function to that of distribution one loses information. Many different series of functions define the same limiting Dirac’s delta distribution. The distribution only takes into account the total conserved area delimited by the series of functions. The gravitational field equations relate geometry with matter content. If we take the matter content to have some distributional character, the geometry will acquire also a distributional character. When analyzing the thin-limit of branes in Einstein gravity, by constructing series or families of solutions parametrized by their thickness, we observe that the blowing up parts of the series of functions that describe the density-of-matter profile transfer directly to the same kind of blowing up parts in the description of its associated geometry. Very simple density profiles [like the previous function \(f_T(y)\)] are associated with very simple geometric profiles, and vice-versa. However, when considering Einstein-Gauss-Bonnet gravity this does not happen. The blowing up parts of the series describing the matter density and the geometry are inequivalent. A simple density profile does not correspond to a very simple geometric profile and vice-versa; on the contrary, we observe that they have some sort of complementary behaviour. This result leads us to argue that the distributional description of the cosmological evolution of a brane in Einstein-Gauss-Bonnet gravity is hiding important aspects of the microphysics, not present when dealing with pure Einstein gravity. Also, we find that for simple models of the geometry, one can make compatible the two seemingly distinct generalized Friedmann equations found in the literature.

The paper is organized as follows. In the next Section we build thick-shell models for static branes (direct generalization of the RS model \[1\]). This provides a simple situation in which the main ideas of the paper can already be seen. For clarity, we will separate the Einstein and Einstein-Gauss-Bonnet cases. Section \[11\] will deal with the dynamical cosmological case. Finally, we will make a brief summary of the results found in Section \[14\]. General formulae for the construction of the field equations are given in the Appendix.

**II. STATIC THICK SHELLS IN EINSTEIN AND GAUSS-BONNET**

**A. Einstein**

To fix ideas and notation let us first describe the simple case in which we have a static thick brane in an anti-de Sitter (adS) bulk. We take an ansatz for the metric of the form\(^2\)

\[
ds^2 = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 ,
\]

where \(\eta_{\mu\nu}\) is the four-dimensional Minkowski metric. Comparing with the formulas given in Appendix \[A\] [in particular equation \[A1\]] this means taking \(a(t,y) = n(t,y) = \exp(-2A(y))\), and \(b(t,y) = 1\). The energy-momentum tensor has the form

\[
\kappa_5^2 T_{AB} = \rho u_A u_B + p_L h_{AB} + p_T n_A n_B ,
\]

where \(u_A = (-e^{-2A}, 0, 0)\), \(n_A = (0, 0, 1)\) and \(h_{AB} = g_{AB} + u_A u_B - n_A n_B\) with \(g_{AB}\) the 5-dimensional bulk metric. Here, \(\kappa_5^2\) denotes the bulk gravitational coupling constant and \(\rho, p_L\) and \(p_T\) represent respectively the energy density, the longitudinal pressure and the transverse pressure, and are taken to depend only on \(y\). The Einstein equations

\(^2\) Five-, four-, and three-dimensional indices are written using upper-case Latin, Greek, and lower-case Latin letters respectively.
\[ G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2 T_{AB} \]

with a negative cosmological constant, \( \Lambda_5 \equiv -6/\ell^2 \), result on the following independent equations for the metric function \( A(y) \):

\[
3A'' - 6A'^2 = \rho - \frac{6}{\ell^2},
\]

\[
6A'^2 = p_T + \frac{6}{\ell^2},
\]

\[
p_L = -\rho.
\]

For convenience, we will hide the \( \kappa_5^2 \) dependence inside the matter magnitudes, \( \rho = \kappa_5^2 \rho_{\text{true}} \), etc. We also consider a \( \mathbb{Z}_2 \)-symmetric geometry around \( y = 0 \). The brane extend in thickness from \( y = -T/2 \) to \( y = +T/2 \). Outside this region \( \rho = p_T = 0 \), so we have a purely \( \text{AdS} \) spacetime: \( A(y) = -y/\ell + b \) for \( y \in (-\infty, -T/2) \) and \( A(y) = y/\ell + b \) for \( y \in (T/2, +\infty) \) where \( b \) is an irrelevant constant. The junction conditions at \( y = -T/2, +T/2 \) [see Eqs. (A6) and (A7)] tell us that

\[
A(-T^-/2) = A(-T^+/2), \quad A(T^-/2) = A(T^+/2),
\]

\[
A'(-T^-/2) = A'(-T^+/2), \quad A'(T^-/2) = A'(T^+/2).
\]

From here and using (A8) we deduce that the transversal pressure is zero at the brane boundaries \( p_T(-T/2) = p_T(T/2) = 0 \). Since we are imposing a \( \mathbb{Z}_2 \)-symmetry with \( y = 0 \) as fixed point, hereafter we will only specify the value of the different functions in the interval \((-T/2, 0)\).

The function \( A' \) is odd and therefore interpolates from \( A'(-T/2) = -1/\ell \) to \( A'(0) = 0 \). If in addition we ask for the null-energy condition \( \rho + p_T = 3A'' \geq 0 \) to be satisfied everywhere inside the brane, then \( p_T \) has to be a negative and monotonically decreasing function from \( p_T(-T/2) = 0 \) to \( p_T(0) = -6/\ell^2 \). This condition will turn to be fundamental in defining a thin-shell limit.

By isolating \( A'' \) from equations (A4) and (A5) we can relate the total bending of the geometry on passing through the brane with its total \( \rho + p_T \)

\[
\frac{6}{\ell} = 3A''_{T/2} - T/2 = \int_{-T/2}^{T/2} (\rho + p_T) \, dy.
\]

At this stage of generality, one can create different one-parameter families of thick-brane versions of the Randall-Sundrum thin brane geometry, by parameterizing each member of a given family by its thickness \( T \). The only requirement needed to do this is that the value of the previous integral must be kept fixed independently of the thickness of the particular thick-brane geometry. Thus, each particular family can be seen as a regularization of Dirac’s delta distribution.

We can realize that, provided the condition \( \rho + p_T \geq 0 \) is satisfied, there exists a constant \( C \), independent of the thickness \( T \), such that \( p_T < C \), that is, the profile for \( p_T \) is bounded and will not blow up in the thin-shell limit. Therefore, in the limit in which the thickness of the branes goes to zero, \( T \to 0 \), the integral of \( p_T \) goes to zero with the thickness. (Strictly speaking, the thin-shell limit is reached when \( T/\ell \to 0 \) but throughout this paper we are going to maintain \( \ell \) constant.) Instead, the profile of \( \rho \) has to develop arbitrarily large values in order to fulfil

\[
\frac{6}{\ell} = \lim_{T \to 0} \int_{-T/2}^{T/2} \rho \, dy.
\]

In the thin-shell limit, we can think of Einstein’s equations as providing a relation between the characteristics of the density profile and the shape of the internal geometry. A very complicated density profile will have associated a very complicated function \( A(y) \). Physically we can argue that when a shell becomes very thin one does not care about its internal structure and, therefore, one tries to describe it in the most simple terms. But what is it exactly the meaning of simple? Here we will adopt two different definitions of simple: The first one is to consider that the internal density is distributed homogeneously throughout the shell when the shell becomes very thin. The second case is to consider that the profile for \( A' \) is such that it interpolates from \( A'(-T/2) = -1/\ell \) to \( A'(0) = 0 \) through a straight line, or what is the same, that the internal profile of \( A'' \) is constant. Again, we require this for very thin shells. This geometric prescription is equivalent to asking for a constant internal scalar curvature, since \( R = 8A'' - 20A'^2 \) and for every thin shell the term \( A'^2 \) is negligible with respect to the constant \( A'' \) term. Hereafter, we will use indistinctly the names straight interpolation or constant curvature for these models. In building arbitrarily thin braneworld models, one needs that the profiles for the internal density \( \rho \) and the internal \( A'' \) acquire arbitrarily high values (they will become distributions in the limit of strictly zero thickness). In the first of the two simple models described, the simplicity applies to the blowing up parts of the matter content; in the second, the simplicity applies to the blowing up parts of the geometry. From the physical point of view advocated in the introduction, these simple profiles are those that do not involve losing information in the process of taking the limit of strictly zero thickness.

Let us analyze both cases independently.
1. Constant density profile

Let us first define for convenience $z \equiv y/T$ as a scale invariant coordinate inside the brane. Then, mathematically, the idea that the density profile, which we will assume to be analytic inside the brane for simplicity, becomes constant in the thin-shell limit can be expressed as follows:

$$\rho(z) = \sum_n \beta_n(T) z^{2n},$$  \hspace{1cm} (11)

where

$$\lim_{T \to 0} T \beta_n(T) \to 0, \quad \forall n \neq 0; \quad \lim_{T \to 0} T \beta_0(T) \to \rho_b : \text{constant}.$$  \hspace{1cm} (12)

For these density profiles, the Einstein equations in the thin-shell limit tell us that

$$3A'' = \beta_0(T) - \frac{6}{T^2} + 6A'^2.$$  \hspace{1cm} (13)

From here we get the profile for $A'$:

$$A' = \sqrt{\frac{\beta_0(T)}{6} - \frac{1}{T^2}} \tan \left(2\sqrt{\frac{\beta_0(T)}{6} - \frac{1}{T^2}} y\right).$$  \hspace{1cm} (14)

Notice that this expression only makes sense for $\beta_0(T) > \frac{6}{T^2}$, but this is just the regime we are interested in. We have to impose now the boundary condition $A'(T/2) = 1/\ell$ to the previous expression.

$$1/\ell = \sqrt{\frac{\beta_0(T)}{6} - \frac{1}{T^2}} \tan \left(\sqrt{\frac{\beta_0(T)}{6} - \frac{1}{T^2}} T\right).$$  \hspace{1cm} (15)

In this manner, we have implicitly determined the form of the function $\beta_0(T)$. In the limit in which $T \to 0$ with $T\beta_0(T) \to \rho_b$, we find the following relation

$$\frac{6}{\ell} = \rho_b.$$  \hspace{1cm} (16)

This condition is just what we expected from the average condition.

2. Straight interpolation

In this case, the mathematical idea that in thin-shell limit the profile for $A'$ corresponds to a straight interpolation, can be formulated as

$$A''(z) = \sum_n \gamma_n(T) z^{2n},$$  \hspace{1cm} (17)

with

$$\lim_{T \to 0} T \gamma_n(T) \to 0, \quad \forall n \neq 0; \quad \lim_{T \to 0} T \gamma_0(T) \to \frac{2}{\ell}.$$  \hspace{1cm} (18)

For these geometries we find that the associated profiles for $p_T$ and $\rho$ in the thin-shell limit have the following form

$$p_T = -\frac{6}{T^2} \left(1 - 4z^2\right) + \omega_1(T, z),$$  \hspace{1cm} (19)

$$\rho = \frac{6}{T} + \frac{6}{T^2} \left(1 - 4z^2\right) + \omega_2(T, z),$$  \hspace{1cm} (20)

where here and along this paper $\omega_n(T, z)$ will denote functions that vanish in the limit $T \to 0$. Now, from this density profile we can see that

$$\lim_{T \to 0} \int_{-T/2}^{T/2} \rho dy = \frac{6}{\ell},$$  \hspace{1cm} (21)
as we expected. Moreover, we can see that the boundary value of the density satisfies $T \rho \big|_{T/2} \to 6/\ell$ in the thin shell limit, which is the same condition satisfied by the averaged density, $T \langle \rho \rangle \to 6/\ell$.

An additional interesting observation for what follows is the following: The sets of profiles that yield constant density in the thin-shell limit (11) and straight interpolation for the geometric profile (17) coincide. Therefore, in the thin-shell limit one can assume at the same time a constant internal structure for the density and a straight-interpolation for the geometry.

B. Einstein-Gauss-Bonnet

Let us move now to the analysis of the same ideas in the presence of the Gauss-Bonnet term. The Einstein-Gauss-Bonnet Lagrangian is

$$S = \frac{1}{2\kappa^4_5} \int dx^5 \sqrt{-g} [R - 2\Lambda_5 + \alpha L_{GB}], \quad (22)$$

with

$$L_{GB} = R^2 - 4R^{AB}R_{AB} + R^{ABCD}R_{ABCD}. \quad (23)$$

Now, the field equations deduced from this Lagrangian are

$$G_{AB} + \alpha H_{AB} = -\Lambda_5 g_{AB} + \kappa^2_5 T_{AB}, \quad (24)$$

where $H_{AB}$ is the Lanczos tensor [10]:

$$H_{AB} = 2R_{ACDE}R^C_B R^{D} - 4R_{ACBD}R^{CD} - 4R_{AC}R^C_B + 2RR_{AB} - \frac{1}{2}g_{AB}L_{GB}. \quad (25)$$

For the ansatz (2) we obtain [compare with equations (A18-A20) in Appendix A]

$$3A''(1 - 4\alpha A'^2) - 6A'^2(1 - 2\alpha A'^2) = \rho - \frac{6}{\ell^2}, \quad (26)$$

$$6A'^2(1 - 2\alpha A'^2) = p_T + \frac{6}{\ell^2}, \quad (27)$$

$$p_L = -\rho. \quad (28)$$

The junction conditions for the geometry are the same as before [8], implying again the vanishing of the transversal pressure at the boundaries, $p_T = 0$.

In the outside region the solution is a pure adS spacetime but with a modified length scale

$$\frac{1}{\ell} \equiv \sqrt{\frac{1}{4\alpha} \left(1 - \sqrt{1 - \frac{8\alpha}{\ell^2}}\right)}. \quad (29)$$

Now, isolating $A''$ from (26) and (27) we can relate the total bending of the geometry on passing through the brane with the integral of $\rho + p_T$

$$\frac{6}{\ell} \left(1 - \frac{4}{3} \frac{\alpha}{\ell^2}\right) = (3A' - 4\alpha A^3)|_{-T/2}^{T/2} = \int_{-T/2}^{T/2} (\rho + p_T) dy. \quad (30)$$

Again, if the condition $\rho + p_T \geq 0$ is fulfilled throughout the brane we will have that in the thin shell limit

$$\frac{6}{\ell} \left(1 - \frac{4}{3} \frac{\alpha}{\ell^2}\right) = \lim_{T \to 0} \int_{-T/2}^{T/2} \rho dy. \quad (31)$$

At this point we can pursue this analysis in the two simple cases of constant density profile and straight interpolation.
Following the same steps as before for a constant density profile \([11, 12]\), the equation that one has to solve in the thin-shell limit is

\[
3A''(1 - 4\alpha A'^2) = \beta_0(T) - \frac{6}{T^2} + 6A'^2(1 - 2\alpha A'^2).
\] (32)

Introducing the notation \(B \equiv A'\) we reduce this equation to the following integral

\[
y = \frac{1}{4\alpha} \int_0^B \frac{(4\alpha B^2 - 1) dB}{B^2 - \frac{1}{2\alpha}B^2 - \frac{1}{2\alpha} \left( \frac{\beta_0(T)}{6} - \frac{1}{T^2} \right)}.
\] (33)

The result of performing the integration is

\[
y = \frac{1}{2} \left[ \frac{1}{\sqrt{-R_-}} \tan^{-1} \left( \frac{B}{\sqrt{-R_-}} \right) - \frac{1}{\sqrt{R_+}} \tanh^{-1} \left( \frac{B}{\sqrt{R_+}} \right) \right],
\] (43)

where \(R_{\pm}\) are

\[
R_{\pm} = \frac{1}{4\alpha} \left[ 1 \pm \sqrt{1 + 8\alpha \left( \frac{\beta_0(T)}{6} - \frac{1}{T^2} \right)} \right].
\] (44)

Again, by imposing the boundary condition

\[
T \beta_0(T) \to \lim_{T \to 0} \frac{6}{T^2} \left( 1 - \frac{4\alpha}{3} \right),
\] (37)

in agreement with condition \([31]\).

Using this same asymptotic expansion, we can see that, in the thin-shell limit, the profile for \(A'(y)\) satisfies

\[
A'(y) - \frac{4\alpha}{3} A'(y)^3 = \frac{1}{3} \beta_0(T) y.
\] (38)

Recursively, one can create a Taylor expansion for \(A'(y)\). The first two terms are

\[
A'(y) = \frac{1}{3} \beta_0(T) y + \frac{4\alpha}{81} \beta_0(T)^3 y^3 + \mathcal{O}(y^5) = \frac{1}{3} T \beta_0(T) z + \frac{4\alpha}{81} T^3 \beta_0(T)^3 z^3 + \mathcal{O}(z^5).
\] (39)

By differentiating this expression we find

\[
A''(y) = \frac{1}{3} \beta_0(T) + \frac{4\alpha}{27} T^2 \beta_0(T)^3 z^2 + \mathcal{O}(z^4).
\] (40)

Now, contrarily to what happens in Einstein theory, this profile does not correspond to the set considered in the straight interpolation before (see Fig. 1). By looking at \([17]\) we can identify

\[
\gamma_0(T) \equiv \frac{1}{3} \beta_0(T), \quad \gamma_1(T) \equiv \frac{4\alpha}{27} T^2 \beta_0(T)^3.
\] (41)

Then, we can see that

\[
\lim_{T \to 0} T \gamma_0(T) = \frac{2}{T} \left( 1 - \frac{4\alpha}{3} \right) \neq \frac{2}{T}, \quad \lim_{T \to 0} T \gamma_1(T) = \frac{32\alpha}{T^3} \left( 1 - \frac{4\alpha}{3} \right)^3 \neq 0.
\] (42)

The coefficients \(\gamma_n\) do not satisfy the conditions in \([13]\). Therefore, unlike the energy density, the scalar of curvature does not have a constant profile.
2. **Straight interpolation**

As in the Einstein case, the straight interpolation profile for $A''$ corresponds to

$$A''(z) = \sum_n \gamma_n(T) z^n,$$  

with

$$\lim_{T \to 0} T \gamma_n(T) \rightarrow 0, \quad \forall n \neq 0; \quad \lim_{T \to 0} T \gamma_0(T) \rightarrow \frac{2}{\ell},$$

(44)

From here we can deduce the associated profiles for $p_T$ and $\rho$ by substituting on (26) and (27).

In the limit $T \to 0$, the dominant part in the density profile is

$$\rho = 3\gamma_0(T)(1 - 4\alpha\gamma_0(T)^2 T^2 z^2).$$

(45)

Identifying

$$\beta_0(T) \equiv 3\gamma_0(T), \quad \beta_1(T) \equiv -12\alpha T^2 \gamma_0(T)^3,$$

(46)

we find that

$$\lim_{T \to 0} T \beta_0(T) = \frac{6}{\ell}, \quad \lim_{T \to 0} T \beta_1(T) \neq 0.$$  

(47)

Therefore, even in the thin shell limit a straight interpolation in the geometry does not correspond to a constant density profile (see Fig.2). In the presence of a Gauss-Bonnet term it is not compatible to ascribe to have a simple description for the interior density profile and for the geometric warp factor at the same time. In the limit of strictly zero thickness (distributional limit) one will unavoidably lose some information on the combined matter-geometry system.

To finish this section let us make an additional observation. From expressions (43) and (44), we can see that

$$A'' = \frac{2}{T} A'|_{T/2} + \nu(T, z) \quad \text{with} \quad \lim_{T \to 0} T \nu(T, z) = 0.$$  

(48)

Using this property in (26,27)

$$\rho + p_T = 3A''(1 - 4\alpha A^2) = \frac{6}{T} A'(T/2)(1 - 4\alpha A^2)$$

(49)
We can see that contrarily to what happens in the Einstein case, this condition is different from the averaged condition

\[ T\rho|_{T/2} = 6A'(1 - 4\alpha A')|_{T/2}. \]  

We can see that contrarily to what happens in the Einstein case, this condition is different from the averaged condition. Evaluating at \( y = T/2 \) we find

\[ T\rho|_{T/2} = 6A'(1 - 4\alpha A')|_{T/2}. \]  

Therefore, the averaged density and the boundary density are different, and this is independent of the brane thickness. For this simple model, in thin-shell limit one can define two different internal density parameters characterizing the thin brane. One represents the total averaged internal density and can be defined as

\[ \rho_{av} \equiv \lim_{T \to 0} T\langle\rho\rangle. \]  

The other represents an internal density parameter calculated by extrapolating to the interior the value of the density on the boundary. This density can be defined as

\[ \rho_{bv} \equiv \lim_{T \to 0} T\rho|_{T/2}. \]  

The junction conditions for a thin shell given in [18], corresponds to the averaged condition or and therefore relate the total bending of the geometry in passing through the brane with its total averaged density. Instead, the particular condition analyzed for the boundary value of \( \rho|_{T/2} \) yields in the thin-shell limit the junction condition in [25]. This condition is only considering information about the boundary value of the density and not about its average.

As a summary, what these analysis suggest is that in the presence of the Gauss-Bonnet term we cannot forget the interior structure of the brane, by modelling it by a simple model, even in the thin-shell limit. We will see again this feature in the next section on the cosmological dynamics of thick shells.

**III. DYNAMIC THICK SHELLS IN EINSTEIN AND GAUSS-BONNET**

We are now going to study cosmological thick branes. To that end we will use the class of spacetime metrics given in [A1], which contain a Friedmann-Robertson-Walker (FRW) cosmological model in every hypersurface \( \{y = \text{const.}\} \), with a matter content described by an energy-momentum tensor of the form [8]. We consider the additional
assumption of a static fifth dimension: \( \dot{b} = 0 \). We can rescale the coordinate \( y \) in such a way that \( b = 1 \). Then, the line element (A1) becomes

\[
ds^2 = -n^2(t,y)dt^2 + a^2(t,y)h_{ij}dx^i dx^j + dy^2,
\]

In Appendix A we show that the \( \{ty\} \)-component of the Einstein-Gauss-Bonnet field equations, for the case with a well-defined limit in Einstein gravity, leads to the equation (A13). In our case it implies the following relation:

\[
n(t,y) = \xi(t) \dot{a}(t,y).
\]

In this situation the rest of field equations can be written in the form given in (A18,A19,A20). In our case they become

\[
\left[ a^4 \left( \Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} \right) \right] = \frac{1}{6} (a^4)' \rho,
\]

where now \( \Phi \) is given by

\[
\Phi = \frac{\dot{a}^2}{n^2 a^2} + \frac{k}{a^2} - \frac{a'^2}{a^2} = H^2 + \frac{k}{a^2} - \frac{a'^2}{a^2},
\]

where we have identified the first term with the square of the Hubble function associated with each \( y = \text{const.} \) slide,

\[
H(t,y) = \frac{\dot{a}}{na}.
\]

With the assumption \( \dot{b} = 0 \), the field equation (57) leads to a conservation equation for matter of the same form as in the FRW models [see Eq.(A10)]:

\[
\dot{\rho} = -3 \frac{\dot{a}^2}{a} \left( \rho + p_L \right).
\]

In the same way we did in the static scenario, we consider here the situation in which there is a \( \mathbb{Z}_2 \) symmetry and a fixed proper thickness \( T \) for the brane. Then one has to solve separately the equations for the bulk (\( \vert y \vert > T/2 \)) and the equations for the thick brane (\( \vert y \vert < T/2 \)). The first step has already been done and the result is (25):

\[
\Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} = \frac{M}{a^4}, \quad \text{for } \vert y \vert > \frac{T}{2},
\]

where \( M \) is a constant that can be identified with the mass of a black hole present in the bulk. Once the solution inside the thick brane has been found one has to impose the junction conditions (A6,A7) at \( y = \pm T/2 \).

The first thing we can deduce from the junction conditions is that the quantity \( \Phi \) is continuous across the two boundaries \( y = \pm T/2 \). But in general, its transversal derivative, \( \Phi' \), will be discontinuous. Then, using equation (63) it follows that the transversal pressure has to be zero on the boundary, \( p_T(t,\pm T/2) = 0 \). At the same time, from (63) we deduce that we must always have:

\[
\left. a^4 \left( \Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} \right) \right|_{y = \pm T/2} = M.
\]

3 The coupling constant \( \alpha \) used here is one half the one used in [25].
On the other hand, using again the relation \[ \Phi = -2a^2 - \frac{\Phi}{a}, \] we find that
\[ H' = -\frac{a'}{a} H \implies \Phi' = -2 \frac{a'}{a} \left( \Phi + \frac{\Phi}{a} \right), \]
and then, expanding \[ \Phi', \] we get
\[ (1 + 4\alpha \Phi) \frac{a''}{a} = \Phi - \frac{2}{\ell^2} - \frac{1}{3} \rho. \]

In the limit \( T \to 0, \) the profiles for the density \( \rho \) and for \( a'' \) blow up, therefore, these dominant terms in expression \[ (66) \]
have to be equated. This results in the following equation
\[ (1 + 4\alpha \Phi) \left( \frac{a'}{a} \right)' = -\frac{1}{3} \rho. \]

In what follows we consider the analysis of the Einstein and Einstein-Gauss-Bonnet theories separately.

A. Einstein

In Einstein gravity it is not difficult to write down an equation describing the dynamics of every layer in the interior of a thick shell. To that end we will take \( \alpha = 0 \) in the equations above. By integrating \[ (64) \] over the interval \((-T/2, y_*)\) and using \[ (63, 66) \] we arrive at
\[ \left( H^2 + \frac{k}{a^2} + \frac{1}{\ell^2} \right) = \left( \frac{\left( \frac{\rho'}{\rho} \right)}{a} \right)^2 \left( \frac{\rho}{a^4} \right) + \frac{M}{a^2} + \frac{1}{6a^4} \int_{-T/2}^{y_*} (\rho^2) dy = \frac{1}{36} \left( \int_{-y_*}^{y_*} \rho dy \right)^2 + \frac{M}{a^2} + \frac{1}{6a^4} \int_{-T/2}^{y_*} (a^4) \rho dy. \]

A particular layer of matter inside the shell, located at \( y = y_* \), can be seen as separating an internal spacetime from a piece of external spacetime. From the previous equation we can see that the cosmological evolution of each layer \( y = y_* \) in the thick shell depends on the balance between the integrated density beyond the layer (external spacetime) and a weighted contribution of the integrated density in the internal spacetime. Therefore, the dynamics of each shell layer will be influenced by the particular characteristics of the internal density profile inside the shell. However, by looking at this same equation we can see that the dynamics of the boundary layer \( y_* = -T/2 \) is only influenced by the total integrated density throughout the shell:
\[ \left( H^2 + \frac{k}{a^2} + \frac{1}{\ell^2} \right) \bigg|_{-T/2} = \frac{1}{36} \left( \int_{-T/2}^{T/2} \rho dy \right)^2 = \frac{1}{36} (T(\rho))^2 = \frac{1}{36} \rho_{av}^2 \]

This is the modified Friedmann equation for the cosmological evolution of the brane \[ (2) \].

In the same manner as we proceeded with static shells, let us analyze the case in which the density profile tends to a (time-dependent) constant in the thin-shell limit:
\[ \rho = \beta_0(T, t) + \omega_3(T, t, z), \quad \lim_{T \to 0} T\beta_0(T, t) = \rho_{av}(t), \quad \lim_{T \to 0} T\omega_3(T, t, z) = 0. \]

When \( \alpha = 0, \) equation \[ (66) \] tells us that if the density profile depends only on \( t \) in the thin-shell limit, then, in this same limit, the blowing up part of the geometry \( (a'/a)' \) is also constant through the brane interior, describing what we called before a straight interpolation. A simple density profile amounts to a simple and equivalent geometric profile and vice-versa. In this same case but including the Gauss-Bonnet term, \( \alpha \neq 0, \) the geometrical factor \( (a'/a)' \) will exhibit a non trivial profile in \( y, \) even in the thin-shell limit. We will see this fact in more detail in the next subsection.

Now, in the case in which \( \rho \) depends only on time, equation \[ (67) \] reads
\[ \left( H^2 + \frac{k}{a^2} + \frac{1}{\ell^2} \right) \bigg|_{y_*} = \left( \frac{\rho'}{\rho} \right)^2 \bigg|_{y_*} + \frac{M}{a^4(y_*)} + \frac{1}{6} \rho(t) \left( 1 + \frac{a^2}{a^4} \right) \bigg|_{y_*}. \]

From equation \[ (66) \] we deduce that
\[ \frac{a'}{a} \bigg|_{y_*} = \frac{1}{6} \rho_{av} - \frac{1}{3} \int_{-T/2}^{y_*} \rho dy = \frac{1}{3} \rho_{av} z_*, \]
and integrating we obtain

\[ a(t, y) = a_0(t) \exp \left( -\frac{1}{6} \rho_{av}(t) T z^2 \right), \]  

(72)

(remember that \( z \equiv y/T \)). Therefore, in the lowest order in \( T \) we have an equation for the internal geometry of the following form

\[ H_0^2 + \frac{k}{a_0^2} + \frac{1}{\ell^2} = \frac{1}{9} \rho_{av}^2 z^2 + \frac{1}{36} \rho_{av}^2 (1 - 4z^2) + \frac{M}{a_0^2} = \frac{1}{36} \rho_{av}^2 + \frac{M}{a_0^2}, \]  

(73)

which is exactly the standard braneworld generalized Friedmann equation \[2\].

\[ \text{B. Einstein-Gauss-Bonnet} \]

In the general Einstein-Gauss-Bonnet case, equation \[66\] can be written as

\[ \left[ 1 + 4\alpha \left( H^2 + \frac{k}{a^2} \right) \right] \left( \frac{a'}{a} \right)' - 4\alpha \left( \frac{a'}{a} \right)^2 \left( \frac{a'}{a} \right)' = \frac{-1}{3} \rho. \]  

(74)

Then, integrating between \(-T/2\) and \(T/2\) yields

\[ 2 \left[ 1 + 4\alpha \left( H^2 + \frac{k}{a^2} \right) \right] \left( \frac{a'}{a} \right) \bigg|_{T/2} - \frac{8\alpha}{3} \left( \frac{a'}{a} \right)^3 \bigg|_{T/2} = -\frac{1}{3} \langle \rho \rangle T = -\frac{1}{3} \rho_{av}. \]  

(75)

The boundary equation \[83\] can be written as

\[ \left( H^2 + \frac{k}{a^2} \right) \bigg|_{T/2} - \left( \frac{a'}{a} \right)^2 \bigg|_{T/2} + 2 \alpha \left[ \left( H^2 + \frac{k}{a^2} \right)^2 - \left( \frac{a'}{a} \right)^2 \right]_{T/2} - \frac{M}{a^4(T/2)} + \frac{1}{\ell^2} = 0, \]  

(76)

This is a quadratic equation for \( (a'/a)^2 \) with solutions

\[ \left( \frac{a'}{a} \right)^2 \bigg|_{T/2} = \frac{1}{4\alpha} \left[ 1 + 4\alpha \left( H^2 + \frac{k}{a^2} \right) \right]_{T/2} \pm \sqrt{1 + \frac{8\alpha}{\ell^2} - \frac{8\alpha M}{a^4}}. \]  

(77)

From these two roots we will take only the minus sign as it is the only one with a well defined limit when \( \alpha \) tends to zero. Now, by squaring \[65\] and substituting the previous root we arrive to a cubic equation for \( H^2 + k/a^2 \) first found in \[18\]. This cubic equation has a real root that can be expressed as \[43\]

\[ H^2 + \frac{k}{a^2} = \frac{1}{8\alpha} \left[ (\sqrt{\lambda^2 + \zeta^3 + \lambda})^{2/3} + (\sqrt{\lambda^2 + \zeta^3 - \lambda})^{2/3} - 2 \right], \]  

(78)

where

\[ \lambda \equiv \sqrt{\frac{\alpha}{2}} \rho_{av}, \quad \zeta \equiv \sqrt{1 + 8\alpha V(a)} \equiv \sqrt{1 + \frac{8\alpha}{\ell^2} - \frac{8\alpha M}{a^4}}. \]  

(79)

In addition to this equation we need the conservation equation

\[ \hat{\rho} = 3H(\rho + p_L), \]  

(80)

which is valid for each section \( y = y_\ast \), and in particular, for the boundary, \( y = T/2 \). This equation can be averaged to give

\[ T(\hat{\rho}) = 3(HT(\rho + p_L)) = 3H_{T/2}(T(\rho) + T(p_L)) \pm \mathcal{O}(T), \]  

(81)

or written in another way

\[ \hat{\rho}_{av} = 3H_{T/2}(\rho_{av} + p_{avL}). \]  

(82)
This happens because

\[ H(t, y) \rightarrow H_0(t, y_0) + \mathcal{O}(T) \]  

(83)

for whatever \( y_0 \in [-T/2, T/2] \), which we have taken as \( y_0 = T/2 \) for convenience.

Let us analyze now the simple case of a constant density profile. For consistency with the \( T \rightarrow 0 \) case, we know that

\[ a(t, y) = a_0(t)[1 + T\tilde{a}(t, z)] + \mathcal{O}(T^2), \]  

(84)

and therefore, from (56)

\[ n(t, y) = \xi(t) [a_0 (1 + T\tilde{a}(t, z))]'. \]  

(85)

Then, in the same limit equation (60) reads

\[ \tilde{a}_{zz} = -\frac{1}{3} \frac{\beta_0(T, t)T}{[1 + 4\alpha \left( H_0^2 + \frac{k}{a_0^2} - \tilde{a}_z^2 \right)]} \]  

(86)

(Here the subscript \( , z \) denotes differentiation with respect to \( z \).) A necessary condition to have a straight interpolation for the geometry is that \( \tilde{a}(t, z) = b(t)Z(z) \). To check whether or not a simple density profile corresponds to a straight geometrical profile we can therefore try to solve the previous equation by separation of variables. It is not difficult to see that in order to find a solution with a well defined Einstein limit we need that

\[ b(t) = \mu, \quad \mu^{-1} \beta_0(T, t)T = \mu^{-1} \beta_0(T)T = \rho_{av} : \text{constant}, \quad H_0^2 + \frac{k}{a_0^2} = \Lambda_4, \]  

(87)

where \( \mu \) is a constant that can be absorbed into the function \( Z(z) \), so we will take it to be \( \mu = 1 \). In this way we will recover the anti-de Sitter and de Sitter solutions for the brane (depending on the sign of the effective four-dimensional cosmological constant). To find the specific \( y \) profile we have to solve

\[ Z_{zz} = -\frac{1}{3} \frac{\rho_{av}}{[1 + 4\alpha \left( \Lambda_4 - Z_z^2 \right)]}. \]  

(88)

This equation can be integrated to get

\[ (1 + 4\alpha\Lambda_4)Z_{zz} - \frac{4\alpha}{3} Z_{zz} = -\frac{1}{3} \rho_{av} z. \]  

(89)

For our proposes the specific solution of this cubic equation is not important. What we want to point out is that the solution does not correspond to a straight interpolation as it happened in the Einstein case. So, in general, simple solutions for the matter profile lead to non-trivial profiles for the scalar curvature even in the thin-shell limit.

Let us see now what happens when we take a simple model for the geometry, the straight interpolation model:

\[ a(t, z) = a_0(t) - \frac{1}{2} b(t) z^2 T. \]  

(90)

From equation (74) we can deduce the density profile in this situation

\[ \lim_{T \to 0} T\rho = 3 \left[ 1 + 4\alpha \left( H_0^2 + \frac{k}{a_0^2} \right) \left( \frac{b}{a_0} \right)^3 - 12\alpha \left( \frac{b}{a_0} \right)^3 z^2. \]  

(91)

As in the static case, even for very small thickness the density profile has now a non-trivial structure. We can observe that

\[ \left. \frac{a'}{a} \right|_{T/2} = -\frac{1}{2} \frac{b}{a_0} + \mathcal{O}(T), \quad \left( \frac{a'}{a} \right)' = \frac{2}{T} \left[ \frac{a'}{a} \right]_{T/2} + \mathcal{O}(T) \].  

(92)

The second relation and equation (45) coincide in the thin-shell limit. Therefore, evaluating (74) on \( y = T/2 \) we obtain

\[ \left. \left[ 1 + 4\alpha \left( H^2 + \frac{k}{a^2} \right) \right] \left( \frac{a'}{a} \right)' \right|_{T/2} - 4\alpha \left( \frac{a'}{a} \right)^3 \right|_{T/2} = -\frac{1}{6} T \rho \big|_{T/2} = -\frac{1}{6} \rho_{av}. \]  

(93)
Now, following the same steps that we followed previously but using this condition instead of (75) we arrive at a cosmological generalized Friedmann equation \[25\] different from that in \[18\] in its form and in the fact that it depends on the quantity associated with the boundary value of the energy density, \(\rho_{\text{bv}}\), instead of the value associated with the average of the energy density, \(\rho_{\text{av}}\). Remarkably, the cubic equation that results from combining the last equation \[93\] with the boundary condition \[76\] becomes in this case linear. That is, the coefficients of the terms quadratic and cubic in \(H^2 + k/a^2\) vanish \[25\]. The modified Friedmann equation found in this case is:

\[
H^2 + \frac{k}{a^2} = \frac{1}{1 + \frac{8\alpha}{a} - \frac{8\alpha M}{a^4}} \left(1 - \frac{2}{1 + \frac{8\alpha}{a} - \frac{8\alpha M}{a^4}}\right) - \frac{1}{4\alpha} \left(\sqrt{1 - \frac{8\alpha}{a} - \frac{8\alpha M}{a^4}} - 1\right).
\] (94)

In contrast with the modified Friedmann equation \(25\), which was obtained by using a completely general procedure, in order to obtain this equation we had to use a procedure which required to consider an extra assumption, namely equation \(92\), and hence, it will not work for profiles of the metric function \(a(t, z)\) that do not satisfy these requirements or equivalent ones. On the other hand, by looking at the developments here presented, we can conclude that the different results found in the literature for the dynamics of a distributional shell have their origin in the additional internal richness introduced in the brane by the presence of the GB term.

### IV. SUMMARY

We have analyzed and compared how the thin shell limit of static and cosmological braneworld models is attained in Einstein and Einstein-Gauss-Bonnet gravitational theories. We have seen that the generalized Friedmann equation proposed in \[18\] is always valid and relates the dynamical behaviour of the shell’s boundary with its total internal density (obtained by integrating transversally the density profile). Instead, the generalized Friedmann equation proposed in \[25\] relates the dynamical behaviour of the shell’s boundary with the boundary value of the density within the brane. This equation is not always valid, only for specific geometrical configurations.

Einstein equations in these models transfer the blowing up contributions of the thin-shell internal density profile to the structure of the internal geometry in a faithful way. If we don’t know the internal structure of the shell we can always model it in simple terms by assuming an (almost) constant density profile and an (almost) constant internal curvature. However, the GB term makes incompatible to have both magnitudes (almost) constant. If the density is (almost) constant, then the curvature is not, and vice-versa. Therefore, we can say that the particular structure of the Einstein-Gauss-Bonnet theory introduces important microphysical features on the matter-geometry configurations beyond those in Einstein gravity, that are hidden in the distributional limit.

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### APPENDIX A: THE 5D METRIC AND ITS ASSOCIATED GEOMETRICAL QUANTITIES AND FIELD EQUATIONS

In this appendix we present the main geometrical quantities and field equations associated with the following 5D metric

\[
ds^2 = g_{AB} dx^A dx^B = -n^2(t, y) dt^2 + a^2(t, y) h_{ij}(x^k) dx^i dx^j + b^2(t, y) dy^2,
\] (A1)

where \(h_{ij}\) is the metric of the three-dimensional maximally symmetric surfaces \(\{t = \text{const.}, y = \text{const.}\}\), whose spatial curvature is parametrized by \(k = -1, 0, 1\). A particular representation of \(h_{ij}\) is

\[
h_{ij} dx^i dx^j = \frac{1}{\left(1 + \frac{k}{a^2}\right)^2} \left(dr^2 + r^2 d\Omega_2^2\right),
\] (A2)

being \(d\Omega_2^2\) the metric of the 2-sphere. The metric \(A1\) contains as particular cases the metrics used along this paper.
The non-zero components of the Einstein tensor $G_{AB}$ corresponding to this line element are given by ($\dot{Q} = \partial_t Q$, $Q' = \partial_y Q$):

$$G_{tt} = 3 \left\{ n^2 \Phi + \frac{\dot{a} \dot{b}}{ab} - n^2 \frac{a''}{a} \left[ \frac{a' b'}{a b} - \frac{a'' b''}{a b} \right] \right\},$$

$$G_{ty} = 3 \left\{ \frac{\dot{a} n'}{a n} + \frac{\dot{b} \dot{a'}}{a b} - \dot{a}' \right\},$$

$$G_{ij} = \frac{a^2}{b^2} h_{ij} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + 2 \frac{n'}{n} \right) - \frac{b'}{b} \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) + 2 \frac{a''}{a} + \frac{n''}{n} \right\},$$

$$- \frac{a^2}{n^2} h_{ij} \left\{ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \dot{\frac{n}{n}} \right) - \frac{b}{b} \left( \frac{\dot{n}}{n} - 2 \frac{\dot{a}}{a} \right) + 2 \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right\} - k h_{ij},$$

$$G_{yy} = 3 \left\{ -b^2 \Phi + \frac{a' n'}{a n} - \frac{b^2}{n^2} \left[ \frac{\dot{a}}{a} - \dot{\frac{n}{n}} \right] \right\},$$

where

$$\Phi(t, y) = \frac{1}{n^2} \frac{\dot{a}^2}{a^2} - \frac{1}{b^2} \frac{a'^2}{a^2} + \frac{k}{a^2}.$$  \hfill (A4)

Apart from the metric and the Einstein tensor, the field equations in Einstein-Gauss-Bonnet gravity contain a term quadratic in the curvature, namely $H_{AB}$ [see Eq. (25)]. The non-zero components of this tensor can be written as follows

$$H_{tt} = 6 \Phi \left[ \frac{\dot{a} \dot{b}}{ab} + \frac{n^2}{b^2} \left( \frac{a' b'}{a b} - \frac{a''}{a} \right) \right],$$

$$H_{ty} = 6 \Phi \left[ \frac{\dot{a} n'}{a n} + \frac{\dot{b} \dot{a'}}{a b} - \dot{a}' \right],$$

$$H_{ij} = 2a^2 h_{ij} \left\{ \Phi \left[ \frac{1}{n^2} \left( \frac{\dot{n} \dot{b}}{n b} - \frac{\dot{b}}{b} \right) - \frac{1}{b^2} \left( \frac{n' \dot{b}'}{n b} - \frac{n''}{n} \right) \right] \right\}$$

$$+ \frac{2}{a^2 b n} \left[ \frac{\dot{a}^2 \dot{b} n}{n^2} + \frac{a'^2 b' n'}{b^2} + \frac{\dot{a} \dot{a}' n}{b^2 n^2} \left( b' \dot{n} - \dot{b} n' \right) \right]$$

$$- 2 \left[ \frac{1}{n^2} \frac{\dot{a}}{a} \left( \frac{1}{n^2} \frac{\dot{b}}{b} + \frac{1}{b^2} \frac{a'}{a} \right) - \frac{1}{a^2} \left( \frac{1}{n^2} \frac{\dot{n}}{n} + \frac{1}{b^2} \frac{a'}{a} \right) \right]$$

$$+ \frac{2}{b^2 n^2} \left[ \frac{\dot{a} a''}{a} - \frac{\dot{a}^2 n^2}{a^2 n^2} - \frac{a'^2 b^2}{a^2 b^2} - \frac{\dot{a}'}{a} \left( \frac{\dot{a}}{a} - 2 \frac{\dot{n}}{n} - 2 \frac{\dot{b}}{b} \right) \right] \right\},$$

$$H_{yy} = 6 \Phi \left[ \frac{a' n'}{a n} + \frac{b^2}{n^2} \left( \frac{\dot{a}}{a} - \dot{\frac{n}{n}} \right) \right].$$  \hfill (A5)

In this paper we consider the situation in which a thick brane is embedded in the five-dimensional spacetime described by (A1), whose boundaries are located at $y = \text{const.}$ hypersurfaces. Let us consider the usual junction conditions at a hypersurface $\Sigma_{y_c} \equiv \{ p \in V_5 | y(p) = y_c \}$, that is, the continuity of the induced metric, $q_{AB} = g_{AB} - n_A n_B$ and the extrinsic curvature, $K_{AB} = -q_{C(A} q_{B)} \nabla_C n_D$, of $\Sigma_{y_c}$:

$$n(t, y_c^+) = n(t, y_c^-), \quad a(t, y_c^+) = a(t, y_c^-),$$  \hfill (A6)

$$\frac{n'(t, y_c^+)}{b(t, y_c^+)} = \frac{n'(t, y_c^-)}{b(t, y_c^-)}, \quad \frac{a'(t, y_c^+)}{b(t, y_c^+)} = \frac{a'(t, y_c^-)}{b(t, y_c^-)}. \hfill (A7)$$

Now, let us assume a matter content described by an energy-momentum tensor of the form

$$\kappa_5^2 T_{AB} = \rho u_A u_B + p_L h_{AB} + p_T n_A n_B,$$  \hfill (A8)
\( u_A = (-n(t, y), 0, 0), \quad h_{AB} = g_{AB} + u_A u_B - n_A n_B, \quad n_A = (0, 0, b(t, y)) \),

where \( \rho, p_L \), and \( p_T \) denote, respectively, the energy density and the longitudinal and transverse pressures with respect to the observers \( u^A \). They are functions of \( t \) and \( y \). The energy-momentum conservation equations, \( \nabla_A T^{AB} = 0 \), reduce to the following two equations:

\[
\dot{\rho} = -\frac{b}{b} (\rho + p_T) - 3\frac{\dot{a}}{a} (\rho + p_L), \tag{A10}
\]

\[
p_T' = -3\frac{a'}{a} (p_T - p_L) - \frac{n'}{n} (\rho + p_T). \tag{A11}
\]

In this situation, the \( \{ty\} \)-component of the field equations for the metric \( A1 \) in Einstein-Gauss-Bonnet gravity [Eq. (24)] has the form

\[
(1 + 4\alpha \Phi) \left( \frac{\dot{a} n'}{a n} + \frac{a' b}{a b} - \frac{\dot{a}'}{a} \right) = 0. \tag{A12}
\]

If we discard the possibility \( 1 + 4\alpha \Phi = 0 \) by restricting ourselves to models with a well-defined limit in Einstein gravity \( (\alpha \rightarrow 0) \), we have that the metric functions must satisfy the following relation

\[
\frac{\dot{a}'}{a} = \frac{n'}{n} \dot{a} + \frac{b}{b} a'. \tag{A13}
\]

Using this consequence of the \( \{ty\} \)-component, we can rewrite the rest of components of \( G_{AB} \) and \( H_{AB} \) as

\[
G_{tt} = \frac{3n^2}{2a^3 a'} (a^4 \Phi)', \quad G_{yy} = -\frac{3b^2}{2a^3 a'} (a^4 \Phi)', \tag{A14}
\]

\[
G_{ij} = \frac{1}{2a a'} h_{ij} \left\{ \frac{b a'}{b a} (a^4 \Phi)' + \frac{n'}{n} \frac{a'}{a} (a^4 \Phi)' - (a^4 \Phi)' \right\}, \tag{A15}
\]

\[
H_{tt} = \frac{3n^2}{2a^3 a'} (a^4 \Phi^2)', \quad H_{yy} = -\frac{3b^2}{2a^3 a'} (a^4 \Phi^2)', \tag{A16}
\]

\[
H_{ij} = \frac{1}{2a a'} h_{ij} \left\{ \frac{b a'}{b a} (a^4 \Phi^2)' + \frac{n'}{n} \frac{a'}{a} (a^4 \Phi^2)' - (a^4 \Phi^2)' \right\}. \tag{A17}
\]

Then, the field equations (24) for the metric \( A1 \) are equivalent to equation (A13) and the following three equations

\[
\left[ a^4 \left( \Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} \right) \right]' = \frac{1}{6} (a^4)' \rho, \tag{A18}
\]

\[
\frac{b a'}{b a} \left[ a^4 \left( \Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} \right) \right] + \frac{n'}{n} \frac{a'}{a} \left[ a^4 \left( \Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} \right) \right]' - \left[ a^4 \left( \Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} \right) \right]' = 2a a' a^2 p_L, \tag{A19}
\]

\[
\left[ a^4 \left( \Phi + 2\alpha \Phi^2 + \frac{1}{\ell^2} \right) \right]' = -\frac{1}{6} (a^4)' p_T. \tag{A20}
\]

Introducing (A18) and (A20) into (A10) we get the conservation equation (A10).

[1] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” Phys. Rev. Lett. 83, 3370 (1999) [arXiv:hep-ph/9905221]: “An alternative to compactification,” Phys. Rev. Lett. 83, 4690 (1999) [arXiv:hep-th/9906064].
[36] J. E. Lidsey, S. Nojiri and S. D. Odintsov, “Braneworld cosmology in (anti)-de Sitter Einstein-Gauss-Bonnet-Maxwell gravity,” JHEP 0206, 026 (2002) [arXiv:hep-th/0202198].

[37] S. C. Davis, “Generalised Israel junction conditions for a Gauss-Bonnet brane world,” Phys. Rev. D 67, 024030 (2003) [arXiv:hep-th/0208205].

[38] E. Gravanis and S. Willison, “Israel conditions for the Gauss-Bonnet theory and the Friedmann equation on the brane universe,” Phys. Lett. B 562, 118 (2003) [arXiv:hep-th/0209076].

[39] R. C. Myers, “Higher Derivative Gravity, Surface Terms And String Theory,” Phys. Rev. D 36, 392 (1987).

[40] N. Deruelle and J. Madore, “On the quasi-linearity of the Einstein- Gauss-Bonnet gravity field equations,” arXiv:gr-qc/0305004.

[41] P. Mounaix and D. Langlois, “Cosmological equations for a thick brane,” Phys. Rev. D 65, 103523 (2002) arXiv:gr-qc/0202089.

[42] O. Corradini and Z. Kakushadze, “Localized gravity and higher curvature terms,” Phys. Lett. B 494, 302 (2000) arXiv:hep-th/0009022.

[43] M. Giovannini, “Thick branes and Gauss-Bonnet self-interactions,” Phys. Rev. D 64, 124004 (2001) arXiv:hep-th/0107233.

[44] J. P. Gregory and A. Padilla, “Braneworld holography in Gauss-Bonnet gravity,” arXiv:hep-th/0304250.