ON AN OPERATOR PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. Let \( \mathcal{P}_n \) denote the space of all complex polynomials \( P(z) = \sum_{j=0}^{n} a_j z^j \) of degree \( n \) and \( \mathcal{B}_n \) a family of operators that maps \( \mathcal{P}_n \) into itself. In this paper, we consider a problem of investigating the dependence of

\[
\left| \mathcal{B}[P(Rz)] - \alpha \mathcal{B}[P(rz)] + \beta \left\{ \left( \frac{R+k}{k+r} \right)^n - |\alpha| \right\} \mathcal{B}[P(rz)] \right|
\]

on the maximum and minimum modulus of \(|P(z)|\) on \(|z| = k\) for arbitrary real or complex numbers \( \alpha, \beta \in \mathbb{C} \) with \(|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k \) and establish certain sharp operator preserving inequalities between polynomials, from which a variety of interesting results follows as special cases.

Keywords: Polynomials; Inequalities in the complex domain; \( \mathcal{B}_n \)-operator.

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1. Introduction

Let \( \mathcal{P}_n \) denote the space of all complex polynomials \( P(z) = \sum_{j=0}^{n} a_j z^j \) of degree \( n \).

A famous result known as Bernstein’s inequality (for reference, see [8, p.531], [10, p.508] or [11]) states that if \( P \in \mathcal{P}_n \), then

\[
Max_{|z|=1} |P'(z)| \leq nMax_{|z|=1} |P(z)|,
\]

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whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have

\begin{equation}
\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|, \quad R \geq 1.
\end{equation}

(2)

(for reference, see [7, p.442] or [8, vol.I, p.137]).

If we restrict ourselves to the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, then inequalities (1) and (2) can be respectively replaced by

\begin{equation}
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|,
\end{equation}

(3)

and

\begin{equation}
\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \quad R \geq 1.
\end{equation}

(4)

Inequality (3) was conjectured by Erdös and later verified by Lax [5], whereas inequality (4) is due to Ankey and Ravilin [1]. Aziz and Dawood [2] further improved inequalities (3) and (4) under the same hypothesis and proved that,

\begin{equation}
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\},
\end{equation}

(5)

\begin{equation}
\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \frac{R^n - 1}{2} \min_{|z|=1} |P(z)|, \quad R \geq 1.
\end{equation}

(6)

As a compact generalization of Inequalities (1) and (2), Aziz and Rather [3] have shown that if $P \in \mathcal{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > 1$ and $|z| \geq 1$,

\begin{equation}
\left| P(Rz) - \alpha P(z) + \beta \left\{ \left( \frac{R + 1}{2} \right)^n - |\alpha| \right\} P(z) \right|
\leq |z|^n \left| R^n - \alpha + \beta \left\{ \left( \frac{R + 1}{2} \right)^n - |\alpha| \right\} \right| \max_{|z|=1} |P(z)|.
\end{equation}

(7)

The result is sharp and equality in (7) holds for the polynomial $P(z) = az^n, a \neq 0$.

As a corresponding compact generalization of Inequalities (3) and (4), they [3] have also shown that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with
\(|\alpha| \leq 1, |\beta| \leq 1, R > 1 \text{ and } |z| \geq 1,\)

\[
\left| P(Rz) - \alpha P(z) + \beta \left\{ \left( \frac{R + 1}{2} \right)^n - |\alpha| \right\} P(z) \right|
\leq \frac{1}{2} \left[ \left| R^n - \alpha + \beta \left\{ \left( \frac{R + 1}{2} \right)^n - |\alpha| \right\} \right| |z|^n \right. \\
+ \left. \left| 1 - \alpha + \beta \left\{ \left( \frac{R + 1}{2} \right)^n - |\alpha| \right\} \right| \right] \max_{|z|=1} |P(z)|.
\]

(8)

The result is best possible and equality in (8) holds for \(P(z) = az^n + b, |a| = |b|\).

Q. I. Rahman [9] (see also Rahman and Schmeisser [10, p. 538]) introduced a class \(\mathcal{B}_n\) of operators \(B\) that carries a polynomial \(P \in \mathcal{P}_n\) into

(9)  \[ B[P(z)] = \lambda_0 P(z) + \lambda_1 \left( \frac{nz}{2} \right) \frac{P'(z)}{1!} + \lambda_2 \left( \frac{nz}{2} \right)^2 \frac{P''(z)}{2!}, \]

where \(\lambda_0, \lambda_1\) and \(\lambda_2\) are such that all the zeros of

(10)  \[ U(z) = \lambda_0 + n\lambda_1 z + \frac{n(n-1)}{2} \lambda_2 z^2 \]

lie in half plane \(|z| \leq |z - n/2|\).

As a generalization of the inequalities (1) and (3), Q. I. Rahman [9, inequalities 5.2 and 5.3] proved that if \(P \in \mathcal{P}_n\), then

(11)  \[ |B[P(z)]| \leq |B[z^n]| \max_{|z|=1} |P(z)|, \text{ for } |z| \geq 1, \]

and if \(P \in \mathcal{P}_n, P(z) \neq 0 \text{ in } |z| < 1\), then

(12)  \[ |B[P(z)]| \leq \frac{1}{2} (|B[z^n]| + |\lambda_0|) \max_{|z|=1} |P(z)|, \text{ for } |z| \geq 1, \]

where \(B \in \mathcal{B}_n\).

1. Preliminaries

For the proof of our results, we need the following Lemmas.
Lemma 1.1. If \( P \in \mathcal{P}_n \) and \( P(z) \) have all its zeros in \( |z| \leq k \) where \( k \geq 0 \), then for every \( R \geq r \), \( Rr \geq k^2 \) and \( |z| = 1 \), we have

\[
|P(Rz)| \geq \left( \frac{R + k}{r + k} \right)^n |P(rz)|.
\]

The above is due to Aziz and Zargar [4]. The next lemma follows from Corollary 18.3 of [6, p. 86].

Lemma 1.2. If \( P \in \mathcal{P}_n \) and \( P(z) \) has all zeros in \( |z| \leq k \), where \( k > 0 \) then all the zeros of \( B[P(z)] \) also lie in \( |z| \leq k \).

Lemma 1.3. If \( P \in \mathcal{P}_n \) and \( P(z) \) have no zero in \( |z| < k \), where \( k > 0 \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \), \( R > r \geq k \) and \( |z| \geq 1 \),

\[
\left| B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)] \right| \leq k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] \right|,
\]

where \( Q(z) = z^n P(1/z) \) and

\[
\Phi_k(R, r, \alpha, \beta) = \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} - \alpha.
\]

Proof. By hypothesis, the polynomial \( P(z) \) does not vanish in \( |z| < k \). Therefore, all the zeros of polynomial \( Q(z/k^2) \) lie in \( |z| < k \). As

\[
|k^n Q(z/k^2)| = |P(z)| \text{ for } |z| = k,
\]

applying Theorem 2.1 to \( P(z) \) with \( F(z) \) replaced by \( k^n Q(z/k^2) \), we get for arbitrary real or complex numbers \( \alpha, \beta \) with \( |\alpha| \leq 1 \), \( |\beta| \leq 1 \), \( R > r \geq k \) and \( |z| \geq 1 \),

\[
\left| B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)] \right| \leq k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] \right|,
\]

This proves Lemma 1.3.

\[\square\]
Lemma 1.4. If \( P \in \mathcal{P}_n \) and \( Q(z) = z^n P(1/z) \) then for \( \alpha, \beta \in \mathbb{C} \), with \( |\alpha| \leq 1, |\beta| \leq 1, R > r \geq k, k \leq 1 \) and \( |z| \geq 1 \),

\[
\left| B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)] \right| + k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] \right| \leq \left\{ |\lambda_0| \left| 1 + \Phi_k(R, r, \alpha, \beta) \right| + \frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| \right\} M_{\text{ax}} |P(z)|,
\]

where \( \Phi_k(R, r, \alpha, \beta) \) is given as (14).

Proof. Let \( M = \text{Max}_{|z|=k} |P(z)| \), then by Rouche’s theorem, the polynomial \( F(z) = P(z) - \mu M \) does not vanish in \( |z| < k \) for every \( \mu \in \mathbb{C} \) with \( |\mu| > 1 \). Applying Lemma 1.3 to polynomial \( F(z) \), we get for \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( |z| \geq 1 \),

\[
\left| B[F(Rz)] + \Phi_k(R, r, \alpha, \beta)B[F(rz)] \right| \leq k^n \left| B[H(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[H(rz/k^2)] \right|,
\]

where \( H(z) = z^n F(1/z) = Q(z) - \overline{\mu} M z^n \). Replacing \( F(z) \) by \( P(z) - \mu M \) and \( H(z) \) by \( Q(z) - \overline{\mu} M z^n \), we have for \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( |z| \geq 1 \),

\[
\left| B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)] - \mu \lambda_0 \left( 1 + \Phi_k(R, r, \alpha, \beta) \right) M \right| \leq k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] \right|,
\]

\[
- \frac{\overline{\mu}}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) MB[z^n]
\]

(16)

where \( Q(z) = z^n P(1/z) \).

Now choosing argument of \( \mu \) in the right hand side of inequality (16) such that

\[
k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] \right| = \frac{\overline{\mu}}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) MB[z^n],
\]

which is possible by applying Corollary 2.3 to polynomial \( Q(z/k^2) \), and using the fact \( \text{Max}_{|z|=k} |Q(z/k^2)| = M/k^n \), we get for \( |\alpha| \leq 1, |\beta| \leq 1 \) and \( |z| \geq 1 \),

\[
\left| B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)] \right| - |\mu \lambda_0| \left( 1 + \Phi_k(R, r, \alpha, \beta) \right) M \leq \frac{\overline{\mu}}{k^n} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) |B[z^n]| M - k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] \right|
\]
Equivalently for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,
\[
|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| + k^n |B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)]|
\leq |\mu| \left( |\lambda_0| + 1 + \Phi_k(R, r, \alpha, \beta) + \frac{1}{k^n} |\frac{R^n + r^n\Phi_k(R, r, \alpha, \beta)}{|z^n|} \right) M
\]
Letting $|\mu| \to 1$, we get the conclusion of Lemma 1.4 and this completes proof of Lemma 1.4.

2. Main results

**Theorem 2.1.** If $F \in \mathcal{P}_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$ where $k > 0$ and $P(z)$ is a polynomial of degree at most $n$ such that

\[
|P(z)| \leq |F(z)|\quad \text{for } |z| = k,
\]
then for $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k$ and $|z| \geq 1$,

\[
|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| \leq |B[F(Rz)] + \Phi_k(R, r, \alpha, \beta)B[F(rz)]|,
\]
where

\[
\Phi_k(R, r, \alpha, \beta) = \beta \left\{ \left( \frac{R + k}{k + r} \right)^n - |\alpha| \right\} - \alpha.
\]
The result is best possible and the equality holds for the polynomial $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$.

**Proof of Theorem 2.1.** Since polynomial $F(z)$ of degree $n$ has all its zeros in $|z| \leq k$ and $P(z)$ is a polynomial of degree at most $n$ such that

\[
|P(z)| \leq |F(z)|\quad \text{for } |z| = k,
\]
therefore, if $F(z)$ has a zero of multiplicity $s$ at $z = ke^{i\theta_0}$, $0 \leq \theta_0 < 2\pi$, then $P(z)$ has a zero of multiplicity at least $s$ at $z = ke^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then inequality (17) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum
modulus principle, it follows that

$$|P(z)| < |F(z)| \text{ for } |z| > k.$$  

Suppose $F(z)$ has $m$ zeros on $|z| = k$ where $0 \leq m < n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree $m$ whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < k$. This implies with the help of inequality (19) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Again, from inequality (19), we have

$$|P_1(z)| \leq |F_2(z)| \text{ for } |z| = k$$

where $F_2(z) \neq 0$ for $|z| = k$. Therefore for every real or complex number $\lambda$ with $|\lambda| > 1$, a direct application of Rouche’s theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < k$ hence the polynomial

$$G(z) = F_1(z) (P_1(z) - \lambda F_2(z)) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$G(z) = (z - te^{i\delta})H(z)$$

where $t < k$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq k$. Applying Lemma 1.1 to the polynomial $H(z)$, we obtain for every $R > r \geq k$ and
0 ≤ θ < 2π,

\[ |G(Re^{iθ})| = |Re^{iθ} - te^{iδ}| |H(Re^{iθ})| \]

\[ ≥ |Re^{iθ} - te^{iδ}| \left( \frac{R + k}{k + r} \right)^{n-1} |H(re^{iθ})|, \]

\[ = \left( \frac{R + k}{k + r} \right)^{n-1} \left| \frac{Re^{iθ} - te^{iδ}}{re^{iθ} - te^{iδ}} \right| |(re^{iθ} - te^{iδ})H(re^{iθ})|, \]

\[ ≥ \left( \frac{R + k}{k + r} \right)^{n-1} \left( \frac{R + r}{r + t} \right) |G(re^{iθ})|. \]

This implies for \( R > r ≥ k \) and \( 0 ≤ θ < 2π \),

(20) \[ \left( \frac{r + t}{R + t} \right) |G(Re^{iθ})| ≥ \left( \frac{R + k}{k + r} \right)^{n-1} |G(re^{iθ})|. \]

Since \( R > r ≥ k \) so that \( G(Re^{iθ}) \neq 0 \) for \( 0 ≤ θ < 2π \) and \( \frac{r + k}{k + r} > \frac{r + t}{R + t} \), from inequality (20), we obtain

(21) \[ |G(Re^{iθ})| > \left( \frac{R + k}{k + r} \right)^{n} |G(re^{iθ})|, \quad R > r ≥ k \text{ and } 0 ≤ θ < 2π. \]

Equivalently,

\[ |G(Rz)| > \left( \frac{R + k}{k + r} \right)^{n} |G(rz)| \]

for \( |z| = 1 \) and \( R > r ≥ k \). Hence for every real or complex number \( α \) with \( |α| ≤ 1 \) and \( R > r ≥ k \), we have

(22) \[ |G(Rz) - αG(rz)| ≥ |G(Rz)| - |α| |G(rz)| \]

\[ > \left\{ \left( \frac{R + k}{k + r} \right)^{n} - |α| \right\} |G(rz)|, \text{ for } |z| = 1. \]

Also, inequality (21) can be written in the form

(23) \[ |G(re^{iθ})| < \left( \frac{k + r}{R + k} \right)^{n} |G(Re^{iθ})| \]

for every \( R > r ≥ k \) and \( 0 ≤ θ < 2π \). Since \( G(Re^{iθ}) \neq 0 \) and \( \left( \frac{k + r}{R + k} \right)^{n} < 1 \), from inequality (23), we obtain for \( 0 ≤ θ < 2π \) and \( R > r ≥ k \),

\[ |G(re^{iθ})| < |G(Re^{iθ})|. \]

That is,

\[ |G(rz)| < |G(Rz)| \text{ for } |z| = 1. \]
Since all the zeros of \( G(Rz) \) lie in \(|z| \leq (k/R) < 1\), a direct application of Rouche’s theorem shows that the polynomial \( G(Rz) - \alpha G(rz) \) has all its zeros in \(|z| < 1\) for every real or complex number \( \alpha \) with \(|\alpha| \leq 1\). Applying Rouche’s theorem again, it follows from (22) that for arbitrary real or complex numbers \( \alpha, \beta \) with \(|\alpha| \leq 1, |\beta| \leq 1\) and \( R > r \geq k\), all the zeros of the polynomial

\[
T(z) = G(Rz) - \alpha G(rz) + \beta \left\{ \left(\frac{R + k}{k + r}\right)^n - |\alpha| \right\} G(rz)
\]

lie in \(|z| < 1\).

Applying Lemma 1.3 to the polynomial \( T(z) \) and noting that \( B \) is a linear operator, it follows that all the zeros of polynomial

\[
B[T(z)] = \left[ B[P(Rz)] - \alpha B[P(rz)] + \beta \left\{ \left(\frac{R + k}{k + r}\right)^n - |\alpha| \right\} B[P(rz)] \right]
\]

\[
- \lambda \left[ B[F(Rz)] - \alpha B[F(rz)] + \beta \left\{ \left(\frac{R + k}{k + r}\right)^n - |\alpha| \right\} [F(rz)] \right]
\]

lie in \(|z| < 1\). This implies

\[
(24) \quad |B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| \leq |B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]|,
\]

for \(|z| \geq 1\) and \( R > r \geq k\). If inequality (24) is not true, then there a point \( z = z_0 \) with \( |z_0| \geq 1 \) such that

\[
\{|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]\}_{z=z_0} \geq \{|B[F(Rz)] + \Phi_k(R, r, \alpha, \beta)B[F(rz)]\}_{z=z_0},
\]

But all the zeros of \( F(Rz) \) lie in \(|z| < (k/R) < 1\), therefore, it follows (as in case of \( G(z) \)) that all the zeros of \( F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R + k}{k + r}\right)^n - |\alpha| \right\} F(rz) \) lie in \(|z| < 1\). Hence, by Lemma 1.3,

\[
\{B[F(Rz)] + \Phi_k(R, r, \alpha, \beta)B[F(rz)]\}_{z=z_0} \neq 0
\]

with \(|z_0| \geq 1\). We take

\[
\lambda = \frac{\{B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]\}_{z=z_0}}{\{B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]\}_{z=z_0}}.
\]
then $\lambda$ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of $\lambda$, we obtain $\{B[T(z)]\}_{z=z_0} = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $B[T(z)]$ lie in $|z| < 1$. Thus (24) holds for $|\alpha| \leq 1$, $|\beta| \leq 1$, $|z| \geq 1$, and $R > r > k$.

For $\alpha = 0$ in Theorem 2.1, we obtain the following result.

**Corollary 2.2.** If $F \in P_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$, where $k > 0$ and $P(z)$ is a polynomial of degree at most $n$ such that

$$|P(z)| \leq |F(z)| \text{ for } |z| = k,$$

then for $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$|B[P(Rz)] + \beta \left(\frac{R+k}{k+r}\right)^n B[P(rz)]| \leq |B[F(Rz)] + \beta \left(\frac{R+k}{k+r}\right)^n B[F(rz)]|.$$  

(25)

The result is sharp, and the equality holds for the polynomial $P(z) = e^{i\gamma} F(z)$ where $\gamma \in \mathbb{R}$.

If we choose $F(z) = z^n M/k^n$, where $M = \text{Max}_{|z|=k} |P(z)|$ in Theorem 2.1, we get the following result.

**Corollary 2.3.** If $P \in P_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$ and $|z| = 1$,

$$|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]|$$

(26)

$$\leq \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| |B[z^n]| \text{Max}_{|z|=k} |P(z)|,$$

where $\Phi_k(R, r, \alpha, \beta)$ is given by (18). The result is best possible and equality in (26) holds for $P(z) = az^n$, $a \neq 0$.

Next, we take $P(z) = z^n m/k^n$, where $m = \text{Min}_{|z|=k} |P(z)|$ in Theorem 2.1, we get the following result.
Corollary 2.4. If $F \in \mathcal{P}_n$ and $F(z)$ have all its zeros in the disk $|z| \leq k$, where $k > 0$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$

\[
\min_{|z|=1} \left| B[F(Rz)] + \Phi_k(R, r, \alpha, \beta)B[F(rz)] \right| \geq \frac{|B[z^n]|}{k^n} \left| R^n + r^n \Phi_k(R, r, \alpha, \beta) \right| \min_{|z|=k} |P(z)|,
\]

(27)

where $\Phi_k(R, r, \alpha, \beta)$ is given by (18). The result is Sharp.

If we take $\beta = 0$ in (26), we get the following result.

Corollary 2.5. If $P \in \mathcal{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq k > 0$ and $|z| \geq 1$,

\[
|B[P(Rz)] - \alpha B[P(rz)]| \leq \frac{1}{k^n} |R^n - \alpha r^n| |B[z^n]| \max_{|z|=k} |P(z)|,
\]

(28)

The result is best possible as shown by $P(z) = az^n, a \neq 0$.

For polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < k$, we establish the following result which leads to the compact generalization of inequalities (3),(4),(8) and (12).

Theorem 2.6. If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$ and $|z| \geq 1$,

\[
|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| \leq \frac{1}{2} \left[ \frac{|B[z^n]|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| \right. + |1 + \Phi_k(R, r, \alpha, \beta)| |\lambda_0| \left. \right] \max_{|z|=k} |P(z)|
\]

(29)

where $\Phi_k(R, r, \alpha, \beta)$ is given by (18).

Proof of Theorem 2.6. Since $P(z)$ does not vanish in $|z| < k$, $k \leq 1$, by Lemma 1.3, we have for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

\[
|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| \leq k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] \right|
\]

(30)

where $Q(z) = z^n P(1/z)$. Inequality (30) in conjunction with Lemma 1.4 gives for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \), \( R > r \geq k \) and \( |z| \geq 1 \),

\[
2|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| \\
\leq |B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| + k^n |B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)]| \\
\leq \left\{ |\lambda_0| + |\Phi_k(R, r, \alpha, \beta)| + \left| \frac{|B[z^n]|}{k^n} |R^n + r^n\Phi_k(R, r, \alpha, \beta)| \right| \right\} \max_{|z|=k} |P(z)| .
\]

This completes the proof of Theorem 2.6.

\( \square \)

We finally prove the following result, which is the refinement of Theorem 2.6.

**Theorem 2.7.** If \( P \in \mathcal{P}_n \) and \( P(z) \) does not vanish in the disk \( |z| < k \), where \( k \leq 1 \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \), \( R > r \geq k > 0 \) and \( |z| = 1 \),

\[
\left| B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)] \right| \\
\leq \frac{1}{2} \left\{ \left| \frac{B[z^n]}{k^n} |R^n + r^n\Phi_k(R, r, \alpha, \beta)| + |1 + \Phi_k(R, r, \alpha, \beta)| |\lambda_0| \right\} \max_{|z|=k} |P(z)| \\
- \left\{ \left| \frac{B[z^n]}{k^n} |R^n + r^n\Phi_k(R, r, \alpha, \beta)| - |1 + \Phi_k(R, r, \alpha, \beta)| |\lambda_0| \right\} \min_{|z|=k} |P(z)| \right\} ,
\]

where \( \Phi_k(R, r, \alpha, \beta) \) is given by (18).

**Proof of Theorem 2.7.** Let \( m = \min_{|z|=k} |P(z)| \). If \( P(z) \) has a zero on \( |z| = k \), then the result follows from Theorem 2.6. We assume that \( P(z) \) has all its zeros in \( |z| > k \) where \( k \leq 1 \) so that \( m > 0 \). Now for every \( \delta \) with \( |\delta| < 1 \), it follows by Rouche’s theorem \( h(z) = P(z) - \delta m \) does not vanish in \( |z| < k \). Applying Lemma 1.3 to the polynomial \( h(z) \), we get for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \leq 1, |\beta| \leq 1 \), \( R > r \geq k \) and \( |z| \geq 1 \)

\[
|B[h(Rz)] + \Phi_k(R, r, \alpha, \beta)B[h(rz)]| \leq k^n \left| B[q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[q(rz/k^2)] \right| ,
\]
where \( q(z) = z^n h(1/z) = z^n P(1/z) - \delta m z^n \). Equivalently,

\[
|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)] - \delta \lambda_0 (1 + \Phi_k(R, r, \alpha, \beta)) m | \\
\leq k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)] \\
- \frac{\delta}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) m B[z^n] \right|
\]

(32)

where \( Q(z) = z^n P(1/z) \). Since all the zeros of \( Q(z/k^2) \) lie in \(|z| \leq k\), \( k \leq 1 \) by Corollary 2.4 applied to \( Q(z/k^2) \), we have for \( R > 1 \) and \(|z| = 1\),

\[
|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)]| \\
\geq \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| B[z^n]| \min Q(z/k^2)_{|z|=k} \\
= \frac{1}{k^{2n}} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| B[z^n]|m.
\]

(33)

Now, choosing the argument of \( \delta \) on the right hand side of inequality (32) such that

\[
k^n \left| B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)] - \frac{\delta}{k^{2n}} (R^n + r^n \Phi_k(R, r, \alpha, \beta)) m B[z^n] \right|
\]

\[
= k^n |B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)]| - \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| B[z^n]|m.
\]

for \(|z| = 1\), which is possible by inequality (33). We get for \(|z| = 1\),

\[
|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]| - |\delta| |\lambda_0||1 + \Phi_k(R, r, \alpha, \beta)| m \\
\leq k^n |B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)]| \\
- \frac{|\delta|}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| B[z^n]|m.
\]

(34)

Equivalently for \(|z| = 1, R > r \geq k\), we have

\[
|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]| - k^n |B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)]| \\
\leq |\delta| \left\{ |\lambda_0||1 + \Phi_k(R, r, \alpha, \beta)| - \frac{1}{k^n} |R^n + r^n \Phi_k(R, r, \alpha, \beta)| B[z^n]| \right\} m.
\]

(35)
Letting $|\delta| \to 1$ in inequality (35), we obtain for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R \geq r \geq k$ and $|z| = 1$,

$$|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]| - k^n|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)]|$$

(36) \quad \leq \begin{cases} |\lambda_0||1 + \Phi_k(R, r, \alpha, \beta)| - \frac{1}{k^n}|R^n + r^n\Phi_k(R, r, \alpha, \beta)||B[z^n]| \end{cases} m.

Inequality (36) in conjunction with Lemma 1.4 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > 1$ and $|z| = 1$,

$$2|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]|$$

\quad \leq \begin{cases} |\lambda_0||1 + \Phi_k(R, r, \alpha, \beta)| + \frac{1}{k^n}|R^n + r^n\Phi_k(R, r, \alpha, \beta)||B[z^n]| \end{cases} \text{Max}|P(z)|

\quad + \begin{cases} |\lambda_0||1 + \Phi_k(R, r, \alpha, \beta)| - \frac{1}{k^n}|R^n + r^n\Phi_k(R, r, \alpha, \beta)||B[z^n]| \end{cases} \text{Min}|P(z)|.

which is equivalent to inequality (31) and thus completes the proof of theorem 2.7.

\[ \square \]

If we take $\alpha = 0$, we get the following.

**Corollary 2.8.** If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \leq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq k$ and $|z| = 1$,

$$\left|B[P(Rz)] + \beta \left( \frac{R + k}{k + r} \right)^n B[P(rz)] \right|$$

\quad \leq \begin{cases} \left|B[z^n]\right| \left\{ \frac{R}{k^n} + r^n\beta \left( \frac{R + k}{k + 1} \right)^n \right\} + \left|1 + \beta \left( \frac{R + k}{k + 1} \right)^n \right| |\lambda_0| \end{cases} \text{Max}|B[P(z)]|$$

\quad - \begin{cases} \left|B[z^n]\right| \left\{ \frac{R}{k^n} + r^n\beta \left( \frac{R + k}{k + 1} \right)^n \right\} - \left|1 + \beta \left( \frac{R + k}{k + 1} \right)^n \right| |\lambda_0| \end{cases} \text{Min}|B[P(z)]|.

(37)

For $\beta = 0$, Theorem 2.6 reduces to the following result.
Corollary 2.9. If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \leq 1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq k$ and $|z| = 1$,

$$|B[P(Rz)] - \alpha B[P(z)]| \leq \frac{1}{2} \left[ \left\{ \frac{|B[z^n]|}{k^n} |R^n - \alpha r^n| + |1 - \alpha| |\lambda_0| \right\} \underset{|z|=k}{\text{Max}} |P(z)| - \left\{ \frac{|B[z^n]|}{k^n} |R^n - \alpha r^n| - |1 - \alpha| |\lambda_0| \right\} \underset{|z|=k}{\text{Min}} |P(z)| \right].$$

(38)

The result is sharp and extremal polynomial is $P(z) = az^n + b, |a| = |b| \neq 0$.

References

[1] N. C. ANKENY and T. J. RIVLIN, On a theorem of S. Bernstein, Pacific J. Math., 5 (1955), 849 - 852.

[2] A. AZIZ and Q. M. DAWOOD, Inequalities for a polynomial and its derivatives, J. Approx. Theory 54 (1998), 306 -311.

[3] A. AZIZ and N. A. RATHER, Some compact generalization of Bernstein-type inequalities for polynomials, Math. Inequal. Appl., 7(3) (2004), 393 - 403.

[4] A. AZIZ and B. A. ZARGAR, Inequalities for a polynomial and its derivatives, Math. Inequal. Appl., 1 (1998), 263-270.

[5] P. D. LAX, Proof of a conjecture of P. Erdös on the derivative of a polynomial, Bull. Amer. Math. Soc., 50 (1944), 509-513.

[6] M. MARDEN, Geometry of Polynomials, Math. Survves No. 3, Amer. Math. Soc., Providence, R I (1966).

[7] G. V. MILOVANOVIC, D. S. MITRINOVIC and TH. M. RASSIAS, Topics in Polynomials: Extremal Properties, Inequalities, Zeros, World scientific Publishing Co., Singapore, (1994).

[8] G. PÓLYA and G. SZEGÖ, Aufgaben und lehrsätze aus der Analysis, Springer-Verlag, Berlin (1925).

[9] Q. I. RAHMAN, Functions of exponential type, Trans. Amer. Soc., 135(1969), 295 C 309

[10] Q. I. RAHMAN and G. SCHMESSIER, Analytic theory of polynomials, Claredon Press, Oxford, 2002.

[11] A. C. SCHAFFER, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, Bull. Amer. Math. Soc., 47(1941), 565-579.