Calderón-Zygmund estimates for stochastic elliptic systems on bounded Lipschitz domains

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November 10, 2022

Abstract
Concerned with elliptic operator with stationary random coefficients of integrable correlations and bounded Lipschitz domains, arising from stochastic homogenization theory, this paper is mainly devoted to studying Calderón-Zygmund estimates. As an application, we obtain the homogenization error in the sense of oscillation and fluctuation, and these results are optimal up to a quantity $O(\ln(1/\epsilon))$, which is coming from the quantified sublinearity of correctors in dimension two and less smoothness of the boundary of the domain.

The main scheme relies on (weighted) annealed Calderón-Zygmund estimates, which was recently developed by Josien and Otto \cite{27} via a robust non-perturbative argument, independent of the quenched Calderón-Zygmund estimate that originally developed by Armstrong and Daniel \cite{1} and Gloria, Neukamm and Otto \cite{19}.

Enlightened by Duerinckx and Otto’s job \cite{9}, the main innovation of the present work is to apply Shen’s real arguments (of weighted version) to study elliptic systems with Dirichlet or Neumann boundary conditions. We start from a qualitative description of homogenization theorem in a local way (without using boundary correctors), and find a new form of minimal radius, which proved to be a suitable key to open quantitative homogenization on the boundary value problems, if adopting Gloria-Neukamm-Otto’s strategy that originally inspired by Naddaf and Spencer’s pioneering work.

Key words: Homogenization error; Calderón-Zygmund estimates; oscillation; fluctuation; Lipschitz domains.

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1 Introduction

1.1 Motivation and main results

Quantitative stochastic homogenization theory has been extensively studied in the last decade, and the crucial progress was made by overcoming the missing Poincaré estimate on probability space to upgrade a quantified sublinearity of correctors. Roughly speaking, there are two approaches to reach this purpose. The one based upon quantified ergodicity in terms of finite range assumption was developed by S. Armstrong and C. Smart [4] and culminated with the monograph [2]; The other one rooted in concentration inequalities (which quantifies ergodicity) has been well studied by A. Gloria, S. Neukamm and F. Otto [17, 18, 19, 20]. Considering the authors’ limited knowledge and the length of the article, we refer the reader to [2, 3, 19, 27] and references therein for more details.

Nevertheless, the second approach seldom touches the boundary value problems (except of periodic boundary condition). To the authors’ best knowledge, with the help of boundary correctors originally developed by M. Avellaneda and F. Lin [5] for quantitative periodic homogenization theory, J. Fischer and C. Raithel [14] derived large-scale regularity theory for random elliptic operators on the half-space, and recently regarding a region with a corner in dimension two, M. Josien, C. Raithel and M. Schäffner [28] obtained large-scale regularity theory by a non-standard expansion argument. Instead, without introducing boundary correctors, the present work is managing to extend Gloria-Neukamm-Otto’s methods [19, 20] to study boundary value problems involving non-smooth domains.

We need to acknowledge that our results will be expected by some experts. However, the potential value of this paper is to tread Gloria-Neukamm-Otto’s results [19, 20] on correctors as the “input”, and systematically investigate the impact of lower regularity of the boundary on the relevant conclusions by virtue of Shen’s real arguments[35] together with the new definition of minimal radius which originally defined in [19] (also similar to “random scale” $\mathcal{X}_s$ given in [2]). On the other hand, the study of boundary value problems on non-smooth domains is a classical branch in PDEs and fruitful research results have been obtained (see for example [21, 24]). The current job manage to establish a reasonable link between it and quantitative stochastic homogenization.

Precisely, we are interested in a family of elliptic operators in divergence form\textsuperscript{1}

$$\mathcal{L}_\varepsilon := -\nabla \cdot a^\varepsilon \nabla = -\nabla \cdot a(\cdot/\varepsilon) \nabla, \quad \varepsilon > 0,$$

\textsuperscript{1}For the ease of the statement, we adopt scalar notation and language throughout the paper.
where \( a \) satisfies \( \lambda \)-uniformly elliptic conditions, i.e., for some \( \lambda \in (0, 1) \) there holds

\[
\lambda |\xi|^2 \leq \xi \cdot a(x)\xi \leq \lambda^{-1}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d.
\]

One can introduce the configuration space that is the set of coefficient fields satisfying (1.1), equipped with a probability measure on it, which is also called as an ensemble and denote the expectation by \( \langle \cdot \rangle \). This ensemble is assumed to be stationary, i.e., for all shift vectors \( z \in \mathbb{R}^d \), \( a(\cdot + z) \) and \( a(\cdot) \) have the same law under \( \langle \cdot \rangle \). In order to get a quantitative theory, we also assume that the ensemble \( \langle \cdot \rangle \) satisfies the spectral gap condition: for any random variable, which is a functional of \( a \), there exists \( \lambda_1 \geq 1 \) such that

\[
\langle (F - \langle F \rangle)^2 \rangle \lesssim_{\lambda_1} \int_{\mathbb{R}^d} \left( \int_{B_1(x)} \frac{\partial F}{\partial a} \right)^2 dx, \tag{1.2}
\]

where the symbol “\( \lesssim_{\lambda_1, \lambda_2, \ldots, \lambda_n} \)” reads “\( \leq C \) for a constant \( C \) depending only on the tuple \( (\lambda, \lambda_1, \ldots, \lambda_n) \) of previously parameters”, and the definition of the functional derivative of \( F \) with respect to \( a \) can be found in (1.15) (or see [27, pp.15]). Moreover, for obtaining pointwise estimates we also assume that there exists \( \sigma_0 \in (0, 1) \) such that for any \( 0 < \sigma < \sigma_0 \) and \( 1 \leq \gamma < \infty \) there holds

\[
\langle \|a\|_{C^{0,\sigma}(B_1)}^\gamma \rangle \lesssim_{\lambda_2} 1. \tag{1.3}
\]

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \), and \( R_0 > 0 \) represents the diameter of \( \Omega \). Throughout the paper, let \( B_r(x) \) denote the open ball centered at \( x \) of radius \( r \) and we usually omit the center point of \( B_r(x) \) without confusions. We say \( \Omega \) is Lipschitz (resp. \( C^1 \)) if there exist \( r_0 > 0, M_0, \{z_k : k = 1, 2, \ldots, N_0\} \subset \partial \Omega \) such that \( \partial \Omega \subset \bigcup_k B_{r_0}(z_k) \), and for each \( k \), there exist a Lipschitz (resp. \( C^1 \)) function \( \psi_k \) in \( \mathbb{R}^{d-1} \) and a coordinate system in \( \mathbb{R}^d \) obtained from the standard Euclidean coordinate system by translation and rotation so that \( B_{r_0}(0) \cap \Omega = B_{r_0}(0) \cap \{(x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi_k(x')\} \), where \( \psi_k \) satisfies \( \psi_k(0) = 0 \) and \( \|\nabla \psi_k\|_{L^\infty(\mathbb{R}^{d-1})} \leq M_0 \).

We now consider the following Dirichlet (or Neumann) boundary value problems:

\[
(D_\varepsilon) \begin{cases} 
-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = \nabla \cdot f & \text{in } \Omega; \\
u_\varepsilon = 0 & \text{on } \partial \Omega.
\end{cases}
\quad \text{or} \quad
(N_\varepsilon) \begin{cases} 
-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = \nabla \cdot f & \text{in } \Omega; \\
n \cdot a^\varepsilon \nabla u_\varepsilon = -n \cdot f & \text{on } \partial \Omega,
\end{cases} \tag{1.4}
\]

where \( n \) is the outward unit normal vector to \( \partial \Omega \). Based upon the sublinearity of correctors (see [39, Theorem 2]), it follows from Tartar’s test function method that for \( \langle \cdot \rangle \)-a.e. (admissible) coefficient \( a \), the flux quantity \( a^\varepsilon \nabla u_\varepsilon \) weakly converges to the effective one denoted by \( \bar{a} \nabla \bar{u} \), where \( \bar{a}e_i := \langle a(\nabla \phi_i + e_i) \rangle \) with \( e_i \) being the canonical basis of \( \mathbb{R}^d \), and the so-called corrector \( \phi_i \) satisfies \( \nabla \cdot a(\nabla \phi_i + e_i) = 0 \) in \( \mathbb{R}^d \) (see also Lemma 2.1). Moreover, there holds the following strong convergence:

\[
\lim_{\varepsilon \to 0} \int_\Omega \langle |u_\varepsilon - \bar{u}|^2 \rangle = 0; \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_\Omega \langle |\nabla u_\varepsilon - (\nabla \phi_i + e_i)\partial_i \bar{u}|^2 \rangle = 0 \tag{1.5}
\]

(see for example [39, Theorem 3]), where \( \partial_i \bar{u} \) represents partial derivative of \( \bar{u} \) with respect to \( e_i \)-direction, and \( \bar{u} \) satisfies the effective equations \( -\nabla \cdot \bar{a} \nabla \bar{u} = \nabla \cdot f \) in \( \Omega \) with the related Dirichlet (or Neumann) boundary conditions, respectively. We mention that the effective equations are totally deterministic, which has shown the potential merit of random homogenization theory, i.e., the possible reduction in numerical complexity.

In fact, the real implementation of the numerical algorithm requires more quantitative theory (see for example [18, 8]), and one of the interesting topic is to quantify the homogenization error (1.5). In
this regard, not only the oscillating property of the error of two-scale expansions but also the fluctuation estimate has been established in the present work. In order to achieve this application-oriented goal, as an important tool, we plan to develop some Calderón-Zygmund estimates, and the main results are as follows.

**Theorem 1.1** (Calderón-Zygmund estimates). Let \( \Omega \subset \mathbb{R}^d \) with \( d \geq 2 \) be a bounded Lipschitz domain and \( \varepsilon \in (0, 1] \). Suppose that \( \langle \cdot \rangle \) is stationary and satisfies spectral gap condition (1.2), and the (admissible) coefficient additionally satisfies (1.3) with the symmetry condition \( a = a^* \). Let \( u_{\varepsilon} \) be associated with \( f \) by the elliptic systems (1.4). Then, there exists a stationary random field \( \chi_* \) with a Lipschitz continuity, satisfying \( \langle \chi_*^\beta \rangle \lesssim \lambda, \lambda_1, \lambda_2, d, \beta \) 1 for any \( \beta < \infty \), such that the quenched Calderón-Zygmund estimate

\[
\int_\Omega \left( \int_{D_{\varepsilon, \varepsilon}(x)} |\nabla u_{\varepsilon}|^2 \right)^{\frac{p}{2}} \lesssim_{\lambda, d, M_0, p} \int_\Omega \left( \int_{D_{\varepsilon, \varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}},
\]

(1.6)

holds for any \( p > 1 \) satisfying \( \left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{2d} + \theta \) with \( 0 < \theta < 1 \), where we have the notation \( D_{e, e}(x) := \Omega \cap B_{e, e}(x) \) with \( B_{e, e}(x) := B_{e, e}(x/\varepsilon) \), and the average integral \( \int_E f \) is given by \( \frac{1}{|E|}\int_E f(x)dx \) with \( |E| \) being the Euclidean volume of \( E \). Also, for any \( \frac{1}{q} - \frac{1}{2} \leq \frac{1}{2d} + \theta \) and \( \bar{p} > p \), we have the annealed Calderón-Zygmund estimate

\[
\int_\Omega \left( \int_{D_{\varepsilon, \varepsilon}(x)} |\nabla u_{\varepsilon}|^2 \right)^{\frac{p}{2}} \lesssim_{\lambda, d, M_0, p, q} \int_\Omega \left( \int_{D_{\varepsilon, \varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}},
\]

(1.7)

where \( D_{\varepsilon}(x) := B_{\varepsilon}(x) \cap \Omega \). Particularly, if the equations (1.4) are scalar, the estimates (1.6) and (1.7) then hold for (4/3) \( - \theta < p, q < 4 + \theta \) if \( d = 2 \); and for (3/2) \( - \theta < p, q < 3 + \theta \) if \( d \geq 3 \). Moreover, if \( \Omega \) is a regular-SKT (or \( C^1 \)) domain\(^2\) (in such the case the symmetry condition \( a = a^* \) is not needed), then the estimates (1.6) and (1.7) hold for any \( 1 < p, q < \infty \) with \( \bar{p} > p \); For any \( \omega \in A_q^3 \), we also obtain weighted annealed Calderón-Zygmund estimate

\[
\left( \int_\Omega \left( \int_{D_{\varepsilon, \varepsilon}(x)} |\nabla u_{\varepsilon}|^2 \right)^{\frac{p}{2}} \omega \right)^{\frac{1}{p}} \lesssim_{\lambda, d, M_0, p, q, \omega, A_q} \left( \int_\Omega \left( \int_{D_{\varepsilon, \varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \omega \right)^{\frac{1}{p}}.
\]

(1.8)

In particular, there exists \( \omega_\sigma \in A_q \) such that

\[
\left( \int_\Omega \left( \int_{D_{\varepsilon}(x)} |\nabla u_{\varepsilon}|^2 \right)^{\frac{p}{2}} \omega_\sigma^{\pm 1} \right)^{\frac{1}{p}} \lesssim_{\lambda, d, M_0, p, q} \left( \int_\Omega \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \omega_\sigma^{\pm 1} \right)^{\frac{1}{p}}.
\]

(1.9)

where the multiplicative constant never depends on \( \varepsilon \) and \( R_0 \).

In the case of Dirichlet boundary conditions for \( \partial \Omega \in C^{1, \alpha} \) with \( \alpha \in (0, 1] \), the result analogy to (1.6) was established by S. Armstrong, T. Kuusi and J.-C. Mourrat [2] under a finite range assumption, and the estimate similar to (1.7) was recently stated by M. Duerinckx [10]. If \( \Omega = \mathbb{R}^d \), for some weight \( \omega \in A_q \) the result analogy to (1.8) had been given by A. Gloria, S. Neukamm and F. Otto in their early version of [19]. In terms of Reifenberg flat domains\(^3\), the Calderón-Zygmund estimates have been

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\(^2\)“SKT” is an abbreviation of Semmes, Kenig and Toro, and the regular-SKT is also known as chord-arc domains with vanishing constant and we refer the reader to [24, Definition 4.8] for the details. Roughly speaking, this kind of domain can be regarded as a Lipschitz domain with very small Lipschitz constant (i.e., \( M_0 \) is very small), which is also closely related to the Reifenberg flat boundary with vanishing constant (see [24, pp.2691]).

\(^3\)The notation \( A_q \) with \( 1 \leq q \leq \infty \) is known as a Muckenhoupt’s weight class (see for example [11, Chapter 7]), and the Muckenhoupt characteristic constant of \( \omega \) is denoted by \( \omega[A_q] \).
well studied by S. Byun, L. Wang \cite{6} for deterministic elliptic operators with small BMO coefficients, originally motivated by L. Caffarelli and I. Peral’s work \cite{7}, while regarding general Lipschitz domains, Z. Shen \cite{30} has developed a real method to simplify the related estimates to a reverse Hölder inequality, which is also the main philosophy behind Theorem 1.1. In view of D. Jerison and C. Kenig’s work \cite{26}, the range of $p$ which the estimate (1.6) holds for is optimal in the scalar case, and sharp in the case of systems for $d = 2, 3$, that have already been recognized in \cite{30, 15} for periodic homogenization. The estimate (1.8) is new even for smooth domains, which is also sharp in the sense of Muckenhoupt’s weight class. When the large-scale average (1.8) is changed into the small-scale one in (1.9), except for the loss in the random index of the right-hand side, the weighted estimate (1.9) is not valid for all of the related weight functions.

Although the small-scale regularity on coefficients was imposed, the estimates (1.7) and (1.9) can not be easily enhanced by pointwise form (on integrand function) because of the randomness involved in the multiplicative “constant”. The possible way to overcome this difficulty is appealing to the robust non-perturbative argument developed in \cite{19, 27}, and we will address it in a separate work. Nevertheless, the assumption (1.3) which leads to pointwise estimates of correctors (see Lemma 2.1) still simplifies the demonstrations on the higher order moment estimates of the new introduced minimal radius $\chi_*$. We also mention that the ensemble $\langle \cdot \rangle$ satisfying the assumptions (1.1), (1.2) and (1.3) can be found in literature \cite[Section 3.2]{27}, where the stationary centered Gaussian fields with integrable covariance function was introduced. Finally, since we avoid imposing boundary correctors in the boundary estimates, our methods are completely applicable to different types of boundary conditions.

As an application of the above theorem, we have the following results.

**Theorem 1.2** (homogenization errors). Let $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ be a bounded Lipschitz domain and $\varepsilon \in (0, 1]$. Suppose that $\langle \cdot \rangle$ is stationary and satisfies the spectral gap condition (1.2), and the (admissible) coefficient additionally satisfies (1.3) with the symmetry condition $a = a^*$. Let $u_\varepsilon, u_0 \in H^1_0(\Omega)$ be associated with $f \in C^1_0(\Omega; \mathbb{R}^d)$ by

$$(D_\varepsilon) \begin{cases} -\nabla \cdot a^* \nabla u_\varepsilon = \nabla \cdot f & \text{in } \Omega; \\ u_\varepsilon = 0 & \text{on } \partial \Omega; \end{cases} \quad (D_0) \begin{cases} -\nabla \cdot \tilde{a} \nabla \tilde{u} = \nabla \cdot f & \text{in } \Omega; \\ \tilde{u} = 0 & \text{on } \partial \Omega. \end{cases}$$

Then, there exists $p_1 : = \frac{2d}{d-1} + \theta$ with $\theta > 0$ depending only on $\Omega$, such that for all $p < p_1$ we have the strong norm estimate

$$\left\langle \left( \int_\Omega |u_\varepsilon - \tilde{u}|^2 \right)^{\frac{p}{2}} \right\rangle \lesssim \varepsilon \mu_d(R_0/\varepsilon) \ln(R_0/\varepsilon) \left( \int_\Omega |\nabla \cdot f|^2 \right)^{\frac{p}{2}},$$

where $\mu_d$ defined in (2.3) has been used to quantify sublinearity of correctors. Moreover, if $\Omega$ is a bounded regular-SKT (or $C^1$) domain (in such the case the symmetry condition $a = a^*$ is not required). Then, the estimate (1.11) holds for any $1 < p < \infty$. For any $h \in C^\infty_0(\Omega; \mathbb{R}^d)$ and some $\varphi_i \in H^1_0(\Omega)$ with $i = 1, \cdots, d$, define the random variable as

$$H^\varepsilon := \int_\Omega h \cdot (a^\varepsilon - \tilde{a}) (\nabla u_\varepsilon - \nabla \tilde{u} - \nabla \phi_\varepsilon^\varepsilon \varphi_i).$$

Then, there also holds the following weak norm estimate

$$\varepsilon^{-\frac{d}{2}} \left\langle \left( H^\varepsilon - \langle H^\varepsilon \rangle \right)^{2p} \right\rangle^{\frac{1}{2p}} \lesssim \varepsilon \mu_d(R_0/\varepsilon) \ln^{\frac{1}{2p}}(R_0/\varepsilon) \left( \int_\Omega |\nabla h|^{2s} \right)^{\frac{1}{2s'}} \left( \int_\Omega |R_0 \nabla f|^{2s'} \right)^{\frac{1}{2s'}}$$

for all $p < \infty$, where $s, s' > 1$ with $1/s + 1/s' = 1$, and the multiplicative constant is independent of $\varepsilon$. 

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In the case of periodic homogenization, the non-smoothness of the boundary will lead to the loss of $O(\ln(R_0/\varepsilon))$ in the convergence rate, first observed by C. Kenig, F. Lin and Z. Shen [23]. However, their method relies on Rellich-type estimates established for $L_{\varepsilon}$, so the Hölder continuity of coefficients at small-scales is inevitably required at least. Inspired by T. Suslina’s job [36], the second author of this paper removed the smoothness assumption of coefficients to develop the result similar to the estimate (1.11). In the random setting, M. Josien, C. Raithel and M. Schäffner [28] recently discovered the same phenomenon for a region with a corner in a much stranger norm, while the stated estimate (1.11) is new established for more general class of non-smooth domains by a different argument.

As we have already mentioned previously, the other source of the loss of $O(\ln(R_0/\varepsilon))$ in (1.11) and (1.13) is also related to the quantified sublinearity of the corrector in dimension two (see Lemma 2.1), which originally recognized by A. Gloria and F. Otto [17] for discrete elliptic differential operator with independently and identically distributed random coefficients. In the case of $\partial \Omega \in C^{1,1}$ or convex domains, the result analogy to (1.11) with a better random integrability was established by S. Armstrong, T. Kuusi and J.-C. Mourrat [2] under a finite range assumption. Regarding stationary random coefficients with slowly decaying correlations in [20], A. Gloria, S. Neukamm and F. Otto obtained a sharp homogenization error in the sense of quenched norm on $\mathbb{R}^d$. In terms of weak norm estimate, the result analogy to (1.13) has been shown in [27] without considering boundary layers, and we additionally employ weighted annealed Calderón-Zygmund estimate (1.9) to get the almost sharp result. We also mention that the weighted annealed estimate similar to (1.9) can be applied to the fluctuation estimates of second-order correctors (see [8, Proposition 3]), and weighted quenched Calderón-Zygmund estimate was used to study the optimal error estimates for periodic homogenization problems on perforated domains (see [37]).

Finally, we mention that the starting point in the proofs of Theorems 1.1 and 1.2 is the error of the following two-scale expansions:

$$w_{\varepsilon} := u_{\varepsilon} - \bar{u} - \varepsilon \phi_i \varphi_i,$$

(1.14)

and $\varphi_i \in H^1_0(\Omega)$ will be fixed according to the concrete environment. In general, we will impose a cut-off function to make it satisfy homogenous boundary conditions. Therefore, the region will be divided into “layer” and “co-layer” parts, and we reduced the boundary layer problem to control the gradient of the effective solution on the layer part and second order derivatives on the co-layer one, respectively. That is where the nontangential maximal function played a role, and some classical results of boundary value problems on non-smooth domains entered.

1.2 Organization of the paper

In Section 2. Based upon the quantitative properties of extended correctors (see Lemma 2.1), by introducing the new minimal radius $\chi_*$ in Lemma 2.2 we show the $H$-convergence in a local way, and $\frac{1}{L}$-Lipschitz continuity of $\chi_*$ with a local boundedness property has been shown in Remarks 2.4, 2.5, respectively. Shen’s real method of the weight version, as the basic tool, has been stated in Lemma 2.9, which more or less released us from repeatedly using the real argument developed in [30]. Involved in the new defined minimal radius $\chi_*$, the property of smoothing operator and some geometry property of integrals were introduced in Lemmas 2.6 and 2.10, respectively.

In Section 3. All the Calderón-Zygmund estimates stated in Theorem 1.1 will be discussed separately in two subsections, which are known as quenched estimates and annealed ones, respectively. Roughly speaking, with the help of Lemma 2.9, our approach is still starting from quenched estimates and then transfer to annealed ones. In Subsection 3.1, the quenched estimates have been summarized in Proposition 3.1. In order to overcome the difficulties caused by the non-smoothness of the boundary, we
first make use of the interior quenched $W^{1,p}$ estimates of the correctors, whose proof essentially belongs to F. Otto (see Lemma 3.3). After that, we establish the reverse Hölder inequality in Lemma 3.4, which will play an important role not only in the quenched estimates, but also in the annealed ones in our demonstration. We show the proof of Proposition 3.1 with details, the novelty of which can be easily recognized compared to the methods directly appealing to large-scale Lipschitz estimates. In Subsection 3.2, the annealed estimates have been shown in Proposition 3.6. Because the integrable index of spatial variables was determined by the regularity of the boundaries, weighted Calderón-Zygmund estimates are unlikely to hold for all the members in the Muckenhoupt’s weight class (otherwise there will be a contradiction by extrapolation). Therefore, we first establish the weighted estimates for the regular-SKT (or $C^1$) domains (see Lemma 3.8). Furthermore, we employ some convexity and inequalities, as well as higher-order moment estimates of minimal radius $\chi_*$, to reduce the large-scale averages to small-scale ones with the loss of the stochastic integrability at the right-hand side (see Lemma 3.10). Obviously, not all of the weights can be equally interchangeable with average integrals. In the end, we point out that we only need to modify our proof in a few places and the same estimates hold true for the Neumann boundary value problems, which has been included in several remarks.

In Section 4. Both the homogenization errors stated in Theorem 1.2 will be addressed separately in two subsections. In Subsection 4.1, taking Dirichlet boundary value problems as an example, we discuss the homogenization error in the strong norm. In essence, it still inherits the idea of using duality and weight functions to overcome the loss of the convergence rate, caused by boundary layers. However, the present model involves randomness, and we need to use annealed Calderón-Zygmund estimates even if we are working under the energy framework in terms of spatial variables. Although we introduce the distance function as a weight to accelerate the convergence rate, here we do not rely on weighted annealed Calderón-Zygmund estimates. Besides, it is not hard to observe that the non-smooth boundary condition will affect the stochastic integrability of the homogenization error. In Subsection 4.2, modulo the homogenization commutator (see for example [27, pp. 29]), we discuss the estimate of the homogenization error in the weak norm for the regular-SKT (or $C^1$) domains, where we employ the weighted annealed Calderón-Zygmund estimates to accelerate the convergence rate and it will not appear if the model does not produce any boundary layer. Besides, to coincide with the framework of sensitive estimates, a rescaling argument still played a crucial role in the whole proof.

In Section 5. This section, as an appendix, contains two lemmas. One is an improved Meyer estimates by using a convexity argument; The other one is related to the reverse Hölder’s inequality for the elliptic operator (with constant coefficients) under different boundary conditions. Issues related to them are actually of independent interest.

### 1.3 Notation

1. Notation for estimates.
   
   (a) $\lesssim$ and $\gtrsim$ stand for $\leq$ and $\geq$ up to a multiplicative constant, which may depend on some given parameters in the paper, but never on $\varepsilon$. If we emphasize the dependence of the multiplicative constant on some parameters, we will give it in subscript form like $\lesssim_{\lambda_1,\ldots,\lambda_n}$. In addition, we will also use superscripts like $\lesssim^{(2.1)}$ to indicate the formula or estimate referenced. We write $\sim$ when both $\lesssim$ and $\gtrsim$ hold.

   (b) We use $\gg$ instead of $\gtrsim$ to indicate that the multiplicative constant is much larger than 1 (but still finite). Therefore, we have the notation $\ll$.

2. Notation for derivatives.
(a) Spatial derivatives: $\nabla v = (\partial_1 v, \ldots, \partial_d v)$ is the gradient of $v$, where $\partial_i v = \partial_v/\partial x_i$ denotes the $i^{th}$ derivative of $v$. $\nabla^2 v$ denotes the Hessian matrix of $v$; $\nabla \cdot v = \sum_{i=1}^d \partial_i v_i$ denotes the divergence of $v$, where $v = (v_1, \ldots, v_d)$ is a vector-valued function.

(b) Functional (or vertical) derivative: the random tensor field $\frac{\partial F}{\partial a}$ (depending on $(a, x)$) is the functional derivative of $F$ with respect to a defined by

$$\lim_{\varepsilon \to 0} \frac{F(a + \varepsilon \delta a) - F(a)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\partial F(a)}{\partial a_{ij}(x)}(\delta a)_{ij}(x) dx.$$  \hspace{1cm} (1.15)

3. Geometric notation.

(a) $d \geq 2$ is the dimension, and $R_0$ represents the diameter of $\Omega$.

(b) Let $B_r(x)$ denote the open ball centered at $x$ of radius $r$. For any ball $B \subset \mathbb{R}^d$, let $x_B$ and $r_B$ represent the center and radius of $B$, respectively. Let $D_{*,\varepsilon}(x) := \Omega \cap B_{*,\varepsilon}(x)$ with $B_{*,\varepsilon}(x) := B_{2\chi_*(x/\varepsilon)}(x)$, where $\chi_*$ is the minimum radius given in Lemma 2.2 or Corollary 2.8.

(c) $D_r(x) := B_r(x) \cap \Omega$ and $\Delta_r(x) := B_r(x) \cap \partial \Omega$ with $x$ near to $\partial \Omega$. We usually omit the center point of $B_r(x), D_r(x), \Delta_r(x)$ without confusions.

4. Notation for functions.

(a) The function $I_E$ is the characteristic function of the region $E$.

(b) We denote $F(\cdot/\varepsilon)$ by $F^\varepsilon(\cdot)$ for the simplicity, and $(f)_r = \int_{B_r} f = \frac{1}{|B_r|} \int_{B_r} f(x) dx$.

Finally, we mention that: (1) when we say that the multiplicative constant depends on the character of the domain, it means that the constant relies on $M_0$; (2) the Einstein’s summation convention for repeated indices is used throughout.

2 Preliminaries

Lemma 2.1 (extended correctors [19, 29]). Let $d \geq 1$. Assume that $\langle \cdot \rangle$ satisfies stationary and ergodic\footnote{In terms of “ergodic”, we refer the reader to [19, pp.105] or [29, pp.222-225] for the definition.} conditions. Then, there exist two random fields $\{\phi_i\}_{i=1, \ldots, d}$ and $\{\sigma_{ijk}\}_{i,j,k=1, \ldots, d}$ with the following properties: the gradient fields $\nabla \phi_i$ and $\nabla \sigma_{ijk}$ are stationary in the sense of $\nabla \phi_i(a; x + z) = \nabla \phi_i(a; \cdot + z); x)$ for any shift vector $z \in \mathbb{R}^d$ (and likewise for $\nabla \sigma_{ijk}$). Let $(\phi, \sigma)$ be the extended correctors and satisfy

$$\begin{align*}
\nabla \cdot a(\nabla \phi_i + e_i) &= 0; \\
\nabla \cdot \sigma_i &= q_i := a(\nabla \phi_i + e_i) - \bar{a}e_i; \\
-\Delta \sigma_{ijk} &= \partial_j q_{ik} - \partial_k q_{ij},
\end{align*}$$  \hspace{1cm} (2.1)

in the distributional sense on $\mathbb{R}^d$ with $\bar{a}e_i := \langle a(\nabla \phi_i + e_i) \rangle$ and $(\nabla \cdot \sigma_i)_j := \sum_{k=1}^d \partial_k \sigma_{ijk}$. Also, the field $\sigma$ is skew-symmetric in its last indices, that is $\sigma_{ijk} = -\sigma_{ikj}$. Moreover, if $\langle \cdot \rangle$ additionally satisfies the spectral gap condition (1.2), as well as (1.3). For any $p \in [1, \infty)$, there holds

$$\langle |\nabla(\phi, \sigma)|^p \rangle \lesssim_{\lambda, \lambda_1, \lambda_2, d, p} 1.$$ \hspace{1cm} (2.2)
and

\[ \left( \frac{1}{R} \right)^{\frac{1}{p'}} \left( \int_{B_{2R}} |(\phi, \sigma) - (\phi, \sigma)|^{2p} \right)^{\frac{1}{p}} \leq \theta \quad \text{and} \quad \left( \frac{1}{R} \right)^{\frac{1}{p'}} \left( \int_{B_{2R}} |u|^2 \right)^{\frac{1}{p}} \leq \theta \quad \forall R \geq \chi_* , \quad (2.4) \]

Proof. See for example [27, Proposition 4.1] coupled with [19, Lemma 1]. \( \square \)

**Lemma 2.2** (qualitative theory). Let \( \Omega \) be a bounded Lipschitz domain and \( \varepsilon \in (0,1] \). Assume that the ensemble \( \langle \cdot \rangle \) is stationary and satisfies the spectral gap condition (1.2), as well as (1.3). Let \( p, \gamma \in (1,\infty) \) and \( p', \gamma' \) be the associated conjugate index satisfying \( 0 < p' - 1 < 1 \) and \( 0 < \gamma' - 1 < 1 \). Then, for any \( \epsilon > 0 \), there exist \( \theta \in (0,1) \), depending on \( \lambda, \gamma, p, M, \epsilon \), and the stationary random field \( \chi_* \), satisfying \( \langle \chi_* \rangle \lesssim \lambda_1 \lambda_2 a d, \beta^1 \) for any \( \beta < \infty \), such that

\[ \left( \frac{1}{R} \right)^{\frac{1}{p'}} \left( \int_{B_{2R}} |(\phi, \sigma) - (\phi, \sigma)|^{2p} \right)^{\frac{1}{p}} \leq \theta \quad \text{and} \quad \left( \frac{1}{R} \right)^{\frac{1}{p'}} \left( \int_{B_{2R}} |u|^2 \right)^{\frac{1}{p}} \leq \theta \quad \forall R \geq \chi_* , \quad (2.4) \]

and for any solution \( u_\varepsilon \) of \( \nabla \cdot a^2 \nabla u_\varepsilon = 0 \) in \( D_2 \) with \( u_\varepsilon = 0 \) on \( \Delta_1 \), there exists \( \bar{u}_r \in H^1(D_{2r}) \) satisfying \( \nabla \cdot \bar{a} \nabla \bar{u}_r = 0 \) in \( D_{2r} \) with \( \bar{u}_r = u_\varepsilon \) on \( \partial D_{2r} \), such that

\[ \int_{D_r} |\nabla u_\varepsilon - (\varepsilon_i + (\nabla \phi_i)\varepsilon_i) \partial_i \bar{u}_r|^2 \leq \epsilon \int_{D_{2r}} |\nabla u_\varepsilon|^2 , \quad (2.5) \]

and

\[ \int_{D_r} |u_\varepsilon - \bar{u}_r|^2 \leq \epsilon \int_{D_{2r}} |u_\varepsilon|^2 \quad (2.6) \]

hold for any \( \varepsilon \chi_* \leq r \leq 1 \).

**Remark 2.3.** If the (admissible) coefficients satisfy the symmetry condition \( a = a^* \), then we can choose \( \gamma = 1 \) in (2.4). Concerned with the Neumann boundary problem, we only modify the equations which \( \bar{u}_r \) satisfies by the new ones: \( \nabla \cdot \bar{a} \nabla \bar{u}_r = 0 \) in \( D_{2r} \) with \( n \cdot \bar{a} \nabla \bar{u}_r = n \cdot a^2 \nabla u_\varepsilon \) on \( \partial D_{2r} \), where \( n \) is the outward unit normal vector associated with \( \partial \Omega \). Then the proof on the Neumann boundary conditions follows from the same arguments as those given for the Dirichlet one.

Proof. By rescaling it is fine to assume \( r = 1 \). We start from denoting the “layer” region of \( D_2 \) with the width \( r \) by \( O_\varepsilon(D_2) := \{ x \in D_2 : \text{dist}(x, \partial D_2) \leq r \} \). Without confusion, \( O_\varepsilon(D_2) \) is simply written as \( O_\varepsilon \) throughout the proof. Let \( \eta_* \in C_0^1(D_2) \) be a cut-off function satisfying \( \eta_* = 1 \) on \( D_2 \setminus O_{2r}, \eta_* = 0 \) outside \( O_\varepsilon \) and \( |\nabla \eta_*| \lesssim 1/\varepsilon \) on \( O_{2r} \setminus O_\varepsilon \). Therefore, we can construct the equations: \( \nabla \cdot \bar{a} \nabla \bar{u}_1 = 0 \) in \( D_2 \) with \( \bar{u}_1 = u_\varepsilon \) on \( \partial D_2 \). By choosing \( \phi_i = \eta_* \partial_i \bar{u}_1 \) in (1.14), and employing the equations (2.1) together with the skew-symmetry of \( \sigma_i \), we can derive

\[ -\nabla \cdot a^2 \nabla w_\varepsilon = \nabla \cdot \left[ \varepsilon (a^2 \phi_i - \sigma_i) \nabla \phi_i + (a^2 - \bar{a}) (\nabla \bar{u}_1 - \varphi) \right] \quad \text{in} \quad D_2 , \quad (2.7) \]

with \( w_\varepsilon = 0 \) on \( \partial D_2 \), and the details may be found in [27, pp.8-10]. Then the energy estimate together with the definition of \( \eta_* \) leads to

\[ \int_{D_2} |\nabla w_\varepsilon|^2 \lesssim \lambda_d \int_{D_2} |(\phi_i, \sigma_i)^2| |\nabla (\eta_* \partial_i \bar{u}_1)|^2 + \int_{O_{2r}} |\nabla \bar{u}_1|^2 \]

\[ \lesssim \int_{O_{2r}} |(\phi_i, \sigma_i)^2| |\partial_i \bar{u}_1|^2 + \int_{D_2 \setminus O_{2r}} |(\phi_i, \sigma_i)^2| |\nabla \partial_i \bar{u}_1|^2 + \varepsilon \beta \left( \int_{D_2} |\nabla \bar{u}_1|^{2p} \right)^{\frac{1}{p}}, \quad (2.8) \]
and the argument is therefore reduced to showing the estimates on \( I_1, I_2 \) and \( I_3 \) above.

We start from the term \( I_1 \) (i.e., the layer part estimate), and it follows from Hölder’s inequality that

\[
I_1 \leq \left( \int_{B_2} |(\phi^\varepsilon, \sigma^\varepsilon)|^{2p'} \right)^{\frac{1}{2p'}} \left( \int_{O_{2\varepsilon}} |\nabla \bar{u}\bar{1}|^{2p'} \right)^{\frac{1}{2p'}} \leq \varepsilon \frac{1}{\gamma p'} \left( \int_{B_2} |(\phi^\varepsilon, \sigma^\varepsilon)|^{2p} \right)^{\frac{1}{2p}} \left( \int_{D_2} |\nabla \bar{u}\bar{1}|^{2p'\gamma' \gamma''} \right)^{\frac{1}{2p'}},
\]

where we mention that \( p', \gamma' \) are the conjugate index of \( p, \gamma \), satisfying \( 0 < p' - 1 \ll 1 \) and \( 0 < \gamma' - 1 \ll 1 \), respectively. By using \( W^{1,q} \) estimates (see for example [16] with \( 0 < q - 2 \ll 1 \) and Lemma 5.1, we have

\[
\left( \int_{D_2} |\nabla \bar{u}\bar{1}|^{2p'\gamma'} \right)^{\frac{1}{2p'}} \lesssim \left( \int_{D_2} |\nabla u_{\varepsilon}|^{2p'\gamma''} \right)^{\frac{1}{2p'}} \lesssim (5.1) \int_{D_4} |\nabla u_{\varepsilon}|^2.
\]

This together with (2.9) gives us

\[
I_1 \lesssim \varepsilon \frac{1}{\gamma p'} \left( \int_{B_{2/\varepsilon}} |(\phi, \sigma)|^{2p} \right)^{\frac{1}{p}} \int_{D_4} |\nabla u_{\varepsilon}|^2.
\]

We now continue to study the term \( I_2 \) (i.e., the co-layer part estimate), and similarly begin from Hölder’s inequality

\[
I_2 \leq \left( \int_{B_2} |\varepsilon (\phi^\varepsilon, \sigma^\varepsilon)|^{2p} \right)^{\frac{1}{p}} \left( \int_{D_2 \setminus O_{\varepsilon}} |\nabla \partial_i \bar{u}_1|^{2p'} \right)^{\frac{1}{p'}},
\]

and then applying for the interior Lipschitz estimate, Hölder’ inequality and the geometry properties of integral in orders, we have

\[
\int_{D_2 \setminus O_{\varepsilon}} |\nabla^2 \bar{u}_1|^{2p'} \lesssim \int_{D_2 \setminus O_{\varepsilon}} \frac{1}{\delta(x)^{2p'}} \int_{B(x, \delta(x)/4)} |\nabla \bar{u}_1|^{2p'}
\]

\[
\lesssim \varepsilon \frac{1}{p'} \left( \int_{D_2 \setminus O_{\varepsilon}} \int_{B(x, \delta(x)/4)} |\nabla \bar{u}_1|^{2p'\gamma'} \right)^{\frac{1}{p'}} \lesssim \varepsilon \frac{1}{p'} \left( \int_{D_2} |\nabla \bar{u}_1|^{2p'\gamma''} \right)^{\frac{1}{p'}},
\]

where \( \delta(x) := \text{dist}(x, \partial D_2) \). Plugging this back into the estimate (2.12) leads to

\[
I_2 \lesssim \varepsilon \frac{1}{p'} \left( \int_{B_{2/\varepsilon}} |(\phi^\varepsilon, \sigma^\varepsilon)|^{2p} \right)^{\frac{1}{p}} \left( \int_{D_2} |\nabla \bar{u}_1|^{2p'\gamma'} \right)^{\frac{1}{p'}} \lesssim (2.10) \varepsilon \frac{1}{p'} \left( \int_{B_{2/\varepsilon}} |(\phi, \sigma)|^{2p} \right)^{\frac{1}{p}} \int_{D_4} |\nabla u_{\varepsilon}|^2.
\]

Concerned with the last term \( I_3 \), it follows from the estimate (2.10) that

\[
I_3 \lesssim \varepsilon \frac{1}{p} \int_{D_4} |\nabla u_{\varepsilon}|^2.
\]

As a result, plugging (2.11), (2.14) and (2.15) back into (2.8), we immediately obtain that

\[
\int_{D_2} |\nabla w_{\varepsilon}|^2 \lesssim \left\{ \varepsilon \frac{1}{p'} \left( \int_{B_{2/\varepsilon}} |(\phi, \sigma)|^{2p} \right)^{\frac{1}{p}} + \varepsilon \frac{1}{p} \right\} \int_{D_4} |\nabla u_{\varepsilon}|^2.
\]

In view of (1.14), we have

\[
\int_{D_1} |\nabla u_{\varepsilon} - (e_i + (\nabla \phi_i)^\varepsilon) \partial_i \bar{u}_1|^2 \lesssim \int_{D_2} |\nabla w_{\varepsilon}|^2 + \int_{D_2 \setminus O_{\varepsilon}} |\varepsilon \phi_i \nabla \partial_i \bar{u}_1|^2 + \int_{O_{\varepsilon}} |(\nabla \phi_i)^\varepsilon \partial_i \bar{u}_1|^2.
\]
Moreover, there holds
\[
\int_{\mathcal{O}_\varepsilon} \left| \left( \nabla \phi_1 \right)^\varepsilon \partial_i \bar{u}_1 \right|^2 \lesssim^{(2.10)} \varepsilon^{-\frac{1}{\varepsilon^{p}}} \left( \int_{B_{2r}/\varepsilon} \left| \nabla \phi_1 \right|^{2p} \right)^{\frac{1}{p}} \int_{D_4} \left| \nabla u_\varepsilon \right|^2,
\]
and plugging this and the estimate (2.14) back into (2.17), we obtain
\[
\int_{D_1} \left| \nabla u_\varepsilon - (e_i + (\nabla \phi_1)\varepsilon) \partial_i \bar{u}_1 \right|^2 \lesssim \left\{ \varepsilon^{-\frac{1}{\varepsilon^{p}}} \left( \int_{B_{2\varepsilon}/\varepsilon} \left| (\phi, \sigma, \nabla \phi) \right|^{2p} \right)^{\frac{1}{p}} + \varepsilon^{\frac{1}{p}} \right\} \int_{D_4} \left| \nabla u_\varepsilon \right|^2. \tag{2.18}
\]
By triangle inequality, Poincaré’s inequality, and Caccioppoli’s inequality we arrive at
\[
\int_{D_1} \left| u_\varepsilon - \bar{u}_1 \right|^2 \leq \int_{D_2} |w_\varepsilon|^2 + \int_{D_2} \left| \varepsilon \phi_1 \sigma \varepsilon \partial_i \bar{u}_1 \right|^2 \lesssim \left\{ \varepsilon^{-\frac{1}{\varepsilon^{p}}} \left( \int_{B_{2\varepsilon}/\varepsilon} \left| (\phi, \sigma) \right|^{2p} \right)^{\frac{1}{p}} + \varepsilon^{\frac{1}{p}} \right\} \int_{D_4} \left| \nabla u_\varepsilon \right|^2. \tag{2.19}
\]

If replacing the corrector \( \phi_i \) in the error term (1.14) with \( \phi_i - \int_{B_r} \phi_i \) (denoted by \( \bar{\phi}_i \), and likewise for \( \sigma \)), and rescaling back, then the stated estimates (2.18) can be rewritten as
\[
\int_{D_r} \left| \nabla u_\varepsilon - (e_i + (\nabla \phi_1)\varepsilon) \partial_i \bar{u}_r \right|^2 \lesssim \left\{ \varepsilon^{-\frac{1}{\varepsilon^{p}}} \left( \int_{B_{2\varepsilon}/\varepsilon} \left| (\bar{\phi}, \bar{\sigma}, \nabla \phi) \right|^{2p} \right)^{\frac{1}{p}} \right\} \int_{D_4} \left| \nabla u_\varepsilon \right|^2.
\]
In this regard, for any \( \theta > 0 \) (to be determined later) we can define the stationary field \( \chi_\varepsilon \) (we call it minimal radius, which plays the same role like \( r_\varepsilon \) defined in [19]) as follows:
\[
\chi_\varepsilon(0) := \inf \left\{ l > 0 : \left( \frac{1}{R} \right)^{\frac{1}{\varepsilon^{p}}} \left[ \int_{B_{2R}} \left| (\phi, \sigma) - (\phi, \sigma)_{2R} \right|^{2p} \right]^\frac{1}{p} + \left( \int_{B_{2R}} \left| \nabla \phi \right|^{2p} \right)^\frac{1}{p} \leq \theta \right\} \quad \forall R \geq l \quad \theta \geq \varepsilon^{\chi_\varepsilon}, \tag{2.20}
\]
(we usually omit the center point, denoted by \( \chi_\varepsilon \)). By the definition of \( \chi_\varepsilon \), one can straightforwardly derive that
\[
\int_{D_r} \left| \nabla u_\varepsilon - (e_i + (\nabla \phi_1)\varepsilon) \partial_i \bar{u}_r \right|^2 \lesssim \theta \int_{D_4} \left| \nabla u_\varepsilon \right|^2 \leq \varepsilon \int_{D_4} \left| \nabla u_\varepsilon \right|^2 \quad \forall r \geq \varepsilon^{\chi_\varepsilon},
\]
which proved the stated result (2.5), and we prefer \( \theta = \varepsilon/C(\lambda, d, p, \gamma, M_0) \). By the same token, we have proved
\[
\int_{D_r} \left| u_\varepsilon - \bar{u}_r \right|^2 \lesssim \theta \int_{D_4} \left| u_\varepsilon \right|^2 \leq \varepsilon \int_{D_4} \left| u_\varepsilon \right|^2 \quad \forall r \geq \varepsilon^{\chi_\varepsilon}.
\]

Finally, by a dyadic decomposition of scales, the \( p \)-th moment of \( \chi_\varepsilon \) follows from (2.2) and (2.3). \( \Box \)

**Remark 2.4.** In [19], the authors gave a way to make \( r_\varepsilon \) be \( \frac{1}{8} \)-Lipschitz continuous. In terms of \( \chi_\varepsilon \), introduced in (2.20), one can similarly construct \( \frac{1}{L} \)-Lipschitz continuous random fields, where \( L := \max\{R_0, d - 1, 8\} \). Let \( \theta_0 \) be such that (2.5) holds for any \( r \geq \varepsilon^{\chi_\varepsilon}(\theta_0) \), and also for \( r \geq \varepsilon^{\chi_\varepsilon}(\theta) \) with \( 0 < \theta < \theta_0 \). We can define \( \chi_\varepsilon \) to be the largest function with Lipschitz constant \( L \) less than \( \chi_\varepsilon(\theta) \) and \( \theta \) will be chosen later, i.e.,
\[
\chi_\varepsilon := \inf_{y \in \mathbb{R}^d} \left( \chi_\varepsilon(y; \theta) + |y|/L \right).
\]
By setting \( \theta_1 := 2(L + 1)^{\frac{d}{2} + 1} \theta_0 \), it concludes that \( \chi_\varepsilon \) is \( \frac{1}{L} \)-Lipschitz continuous satisfying \( \chi_\varepsilon(\theta_0) \leq \chi_\varepsilon \leq \chi_\varepsilon(\theta_1) \). For the ease of the statement, we still use the original notation to represent the minimal radius with \( \frac{1}{L} \)-Lipschitz continuity.
Remark 2.5. By $\frac{1}{\varepsilon}$-Lipschitz continuity of $\chi_*$ and $\langle \chi_*^\beta \rangle \lesssim 1$ with $\beta < \infty$, it is not hard to see that for any $x \in \mathbb{R}^d$, there holds
\[
\left\langle \left( \sup_{y \in B_1(x)} |\chi_*(y)| \right)^\beta \right\rangle \lesssim 1, \tag{2.21}
\]
where the multiplicative constant is independent of $x$. Moreover, for any $\kappa > 0$, $\beta_1 < \infty$ and $x_0 \in \mathbb{R}^d$, let $X_R(x_0) := R^{-\kappa} \sup_{y \in B_R(x_0)} |\chi_*(y)|$, where $R \geq 1$. Then, it follows from a covering argument, subadditivity of $\langle \cdot \rangle$ and Markov’s inequality that
\[
\langle |X_R(x_0)| > \rho \rangle \lesssim \sum_{i=1}^{R^d} \left\langle \sup_{y \in B_i(x_i)} |\chi_*(y)| > R^\kappa \rho \right\rangle \lesssim \frac{\langle \sup_{y \in B_i(x_i)} |\chi_*(y)| \rangle^\beta}{\rho^\beta R_{\kappa \beta}} \lesssim (2.21) \rho^{-\beta} R^{-d-\kappa \beta} \lesssim \rho^{-\beta},
\]
p provided $\beta \geq d/\kappa$. By choosing $\beta > \max\{d/\kappa, \beta_1\}$, there uniformly holds
\[
\langle |X_R(x_0)|^\beta \rangle \lesssim 1 + \int_1^\infty \rho^{\beta_1-1-\beta} \rho d \rho \lesssim 1, \tag{2.22}
\]
with respective to $x_0 \in \mathbb{R}^d$. One can further derive that
\[
\sup_{y \in B_R(x_0)} |\chi_*(y)| \leq X_R(x_0) R^\kappa. \tag{2.23}
\]

Lemma 2.6 (random shift). Let $0 < \varepsilon \leq 1$ and $1 \leq p < \infty$, and fix $\zeta \in C_0^\infty(B_{1/2}(0))$ with $\int_{\mathbb{R}^d} \zeta = 1$, and $\zeta_\varepsilon(x) := r^{-d} \zeta(x/r)$. Given $\chi_*$ as in Lemma 2.2 with $\frac{1}{\varepsilon}$-Lipschitz continuity stated in Remark 2.4 and for any fixed $x \in \mathbb{R}^d$, define the smoothing operator as
\[
S_{\varepsilon,\zeta}(f)(x) := \int_{\mathbb{R}^d} \zeta_{\chi_*(x/\varepsilon)}(y-x) f(y)dy. \tag{2.24}
\]
Let $f \in W^{1,p}(\mathbb{R}^d)$ with $1 \leq p < \infty$, then we obtain
\[
\int_{\mathbb{R}^d} |f - S_{\varepsilon,\zeta}(f)|^p \leq \int_{\mathbb{R}^d} \chi_\varepsilon^p(\cdot/\varepsilon) |\nabla f|^p. \tag{2.25}
\]
Proof. Although $\chi_*$ involves the randomness, the proof is totally deterministic like that given for periodic homogenization (see [33]), inspired by Steklov smoothing operator originally applied to homogenization problem by V. Zhikov and S. Pastukhova [40]. We provide a proof for the sake of the completeness.

We start from changing a variable, Hölder’s inequality and Fubini’s theorem that
\[
\int_{\mathbb{R}^d} |f(x) - S_{\varepsilon,\zeta}(f)(x)\rangle^p dx = \int_{\mathbb{R}^d} \left| \int_{|z| \leq 1} \zeta(z) \langle f(x) - f(x - \varepsilon \chi_*(x/\varepsilon)z)\rangle dz \right|^p dx \lesssim \int_{|z| \leq 1} \zeta(z) \int_{\mathbb{R}^d} |f(x) - f(x - \varepsilon \chi_*(x/\varepsilon)z)|^p dx dz. \tag{2.26}
\]
For any $x \in \mathbb{R}^d$ and $|z| \leq 1$, we obtain that
\[
|f(x) - f(x - \varepsilon \chi_*(x/\varepsilon)z)|^p \lesssim \varepsilon^p \int_0^1 \langle \chi_*(x/\varepsilon)\rangle^p |\nabla f(x - t \varepsilon \chi_*(x/\varepsilon)z)|^p dt. \tag{2.27}
\]
Let $\tilde{x} := x - t \varepsilon \chi_*(x/\varepsilon)z$, and we have $|\tilde{x} - x| \leq \varepsilon \chi_*(x/\varepsilon)$. In terms of $\frac{1}{\varepsilon}$-Lipschitz continuity of $\chi_*$, there holds $\chi_*(\tilde{x}/\varepsilon) \sim \chi_*(x/\varepsilon)$. Moreover, we have $d\tilde{x} = \frac{\partial \tilde{x}}{\partial x} dx$ (where the Jacobian matrix $\frac{\partial \tilde{x}}{\partial x}$ is
Corollary 2.8. Let \( \phi \) be introduced as in Lemma 2.2, and we also assume the same conditions as in Lemma 2.2. Let \( \varphi_i := S_{s, \varepsilon}(\eta \partial_t \tilde{u}_r) \) in (1.14), where the definition of \( S_{s, \varepsilon} \) is given in Lemma 2.6 and \( \eta \) is a cut-off function similar to that given in (2.26). Thus, for any \( r > 0 \), there exist \( \theta \in (0, 1) \), depending on \( \lambda, d, \gamma, p, M_0, \varepsilon \), and the stationary random field \( \tilde{\chi}_s \), satisfying \( \langle \tilde{\chi}_s \rangle \leq 1 \) for any \( \beta < \infty \), such that
\[
\int_{D_r} |\nabla u_\varepsilon(r) - (\epsilon_i + (\nabla \phi_i)^{\varepsilon})\varphi_i|^2 \leq \varepsilon \int_{D_{4\varepsilon}} |\nabla u_\varepsilon|^2, \tag{2.30}
\]
holds for any \( \varepsilon \tilde{\chi}_s \leq r \leq 1 \).

Proof. By the definition of \( \varphi_i \), it follows from the estimate (2.28) that
\[
\int_{D_r} |\nabla u_\varepsilon(r) - (\epsilon_i + (\nabla \phi_i)^{\varepsilon})\varphi_i|^2 \leq \int_{D_r} |\nabla u_\varepsilon - (\epsilon_i + (\nabla \phi_i)^{\varepsilon})\eta \partial_t \tilde{u}_r + \int_{D_r} |(e_i + (\nabla \phi_i)^{\varepsilon})\eta \partial_t \tilde{u}_r - \varphi_i|^2
\]
\[
\leq \left\{ \left( \frac{\varepsilon}{r} \right)^{\frac{1}{p'}} \left( \int_{B_{2r/\varepsilon}} |(\tilde{\varphi}, \tilde{\sigma})|^{2p} \right) \right\}^{\frac{1}{p'}} \int_{D_{4\varepsilon}} |\nabla u_\varepsilon|^2 + \int_{D_r} |(e_i + (\nabla \phi_i)^{\varepsilon})\eta \partial_t \tilde{u}_r - \varphi_i|^2, \tag{2.31}
\]
where \((\tilde{\varphi}, \tilde{\sigma})\) is taken the same notation from the proof of Lemma 2.6. We continue to estimate the last term above by Hölder’s inequality, Lemma 2.6 and Remark 2.5 as follows:
\[
\int_{D_r} |(e_i + (\nabla \phi_i)^{\varepsilon})\eta \partial_t \tilde{u}_r - \varphi_i|^2 \leq \left( \int_{D_r} |(e_i + (\nabla \phi_i)^{\varepsilon})|^{2p} \right)^{\frac{1}{2p}} \left( \int_{D_r} |(\eta \partial_t \tilde{u}_r - \varphi_i)|^{2p'} \right)^{\frac{1}{2p'}}
\]
\[
\lesssim \varepsilon^{2} \left\{ 1 + \int_{B_{2r}} |\nabla \phi_i|^{2p} \right\}^{\frac{1}{2p}} \left\{ \int_{D_r} |\nabla \eta \nabla \tilde{u} + \eta \nabla^2 \tilde{u}|^{2p'} |\chi_s(\cdot/\varepsilon)|^{2p'} \right\}^{\frac{1}{2p'}} \tag{2.32}
\]
\[
\lesssim \left( \frac{\varepsilon^{2}}{r} \right)^{\frac{1}{2p}-\kappa} \left\{ 1 + \int_{B_{2r}} |\nabla \phi_i|^{2p} \right\}^{\frac{1}{2p}} \int_{D_{4\varepsilon}} |\nabla u_\varepsilon|^2,
\]
where we also employ the layer and co-layer type estimates (2.10), (2.13) in the last step. Then, plugging the estimate (2.32) back into (2.31) we have

$$\int_{D_{\epsilon}} |\nabla u_\epsilon - (e_i + (\nabla \phi)\xi)|^2 \lesssim X_\epsilon \left(\frac{\xi}{\eta}\right)^{\nu} \left\{ \left( \int_{B_{2\epsilon}/\epsilon} |(\phi, \sigma, \nabla \phi)|^{2p} \right)^{\frac{1}{p}} + 1 \right\} \int_{D_{\epsilon}} |\nabla u_\epsilon|^2,$$

where one can prefer \(0 < \kappa < \frac{1}{\nu p'}\) in the above estimate. As a result, similar to the definition of \(\chi_*\) given in Lemma 2.2, we can modify the minimal radius as

$$\bar{\chi}_*(0) := \inf \left\{ l > 0 : X_R \left(\frac{1}{R}\right)^{\nu} \left[ \left( \int_{B_{2R}} |(\phi, \sigma) - (\phi, \sigma)_{2R}|^{2p} \right)^{\frac{1}{p}} + \left( \int_{B_{2R}} |\nabla \phi|^{2p} \right)^{\frac{1}{p}} + 1 \right] \leq \theta; \forall R \geq l \right\}.$$

By definition, it is not hard to observe that \(\bar{\chi}_* \geq \chi_*\). Also, the \(\beta\)-th moment of \(\bar{\chi}_*\), follows from (2.2), (2.3) and (2.22).

**Lemma 2.9** (Shen’s lemma of weighted version). Let \(0 < p_0 < p < p_1\), \(\omega \in A^p_{p_0}\) weight and \(\Omega\) be a bounded Lipschitz domain, \(F \in L^{p_0}(\Omega)\) and \(f \in L^{p_1}(\Omega)\). Let \(B_0 := B_{r_0}(x_0)\), where \(x_0 \in \partial \Omega\) and \(0 < r_0 < c_0\text{diam}(\Omega)\). Suppose that for each ball \(B\) with the property that \(|B| \leq c_1|B_0|\) and either \(4B \subset B_0 \cap \Omega\) or \(B\) is centered on \(\partial \Omega \cap B_0\), there exist two measurable functions \(F_B\) and \(R_B\) on \(\Omega \cap 2B\), such that \(|F| \leq |F_B| + |R_B|\) on \(\Omega \cap 2B\), and

\[
\left( \int_{2B \cap \Omega} |R_B|^{p_1} \right)^{\frac{1}{p_1}} \leq N_1 \left\{ \left( \int_{4B \cap \Omega} |F|^{p_0} \right)^{\frac{1}{p_0}} + \sup_{4B_1 \supseteq 4B} \left( \int_{B \cap \Omega} |f|^{p_0} \right)^{\frac{1}{p_0}} \right\} \left( \int_{\Omega} \omega \right)^{1/p_0},
\]

\[
\left( \int_{2B \cap \Omega} |F|^{p_0} \right)^{\frac{1}{p_0}} \leq N_2 \sup_{4B_1 \supseteq 4B} \left( \int_{B \cap \Omega} |f|^{p_0} \right)^{\frac{1}{p_0}} + \theta \left( \int_{4B_1 \cap \Omega} |F|^{p_0} \right)^{\frac{1}{p_0}},
\]

where \(N_1, N_2 > 0\), \(\eta > 0\) and \(0 < c_0, c_1 < 1\). Then there exists \(\theta_0 > 0\), depending on \(d, p_0, p_1, p, N_1, N_2\), the \(A_{p/p_0}\) constant of \(\omega\) and the Lipschitz character of \(\Omega\), with the property that if \(0 \leq \theta \leq \theta_0\), then

\[
\left( \int_{4B_0 \cap \Omega} |F|^{p_0} \right)^{\frac{1}{p_0}} \leq C \left\{ \left( \int_{4B_0 \cap \Omega} |F|^{p_0} \right)^{\frac{1}{p_0}} \left( \int_{B_0} \omega \right)^{1/p} + \left( \int_{4B_0 \cap \Omega} |f|^{p_0} \right)^{\frac{1}{p_0}} \right\},
\]

where \(C\) depends only on \(N_1, N_2, c_0, p, q, \) the \(A_{p/p_0}\) constant of \(\omega\) and the Lipschitz character of \(\Omega\).

**Proof.** See [35, pp.6-8].

**Lemma 2.10** (primary geometry on integrals). Let \(\chi_*\) be given as in Lemma 2.2 or Corollary 2.8 with \(\frac{1}{4}\)-Lipschitz continuity, and \(\varepsilon \in (0, 1]\). Let \(f \in L^1_{\text{loc}}(\mathbb{R}^d)\), and \(\Omega \subset \mathbb{R}^d\) be a bounded Lipschitz domain. Then there hold the following inequalities:

- For all \(B_r \subset \mathbb{R}^d\) and \(x_0 \in B_r \cap \Omega\) with \(r < \frac{4}{5}\chi_*(x_0/\varepsilon)\), we have

  \[
  \left( \int_{B_{s\varepsilon}(x_0) \cap \Omega} |f| \right)^{\frac{1}{s}} \lesssim \int_{B_{5r} \cap \Omega} \left( \int_{B_{s\varepsilon}(x_0) \cap \Omega} |f| \right)^{\frac{1}{s}} dx,
  \]

  where \(s > 0\) and \(B_{s\varepsilon}(x) := B_{\varepsilon\chi_*(x/\varepsilon)}(x)\).

- For all balls \(B_r \subset \mathbb{R}^d\) with \(r \geq \frac{4}{3}\chi_*(0)\), we have

  \[
  \int_{B_{r} \cap \Omega} |f| \lesssim \int_{B_{2r} \cap \Omega} \int_{B_{s\varepsilon}(x) \cap \Omega} |f| \lesssim \int_{B_{7r} \cap \Omega} |f|,
  \]

  (2.36)
where the up to constant depends only on $d$ and the character of $\Omega$.

Proof. The proof is standard and we provide a proof for the sake of the completeness, while in the case of $\Omega = \mathbb{R}^d$ we refer the reader to [9, Lemma 6.5] and [37, Lemma 6.2] for a deterministic case. Throughout the proof, the key ingredient is $\frac{1}{\varepsilon}$-Lipschitz continuity of $\chi_*$, and we first show the estimate (2.35).

We first show the estimate (2.35) for the case of $s \geq 1$. For any $x \in B_r \cap \Omega$, noting that $B_{\varepsilon \chi_*(x_0/\varepsilon) + 2r}(x) \cap \Omega$, we have

$$\left( \int_{B_{\varepsilon \chi_*(x_0)} \cap \Omega} |f| \right)^{\frac{1}{2}} \lesssim \int_{B_r \cap \Omega} \left( \sum_{i=1}^{N} \int_{B_{\varepsilon \chi_*(x_i/\varepsilon) - r}(x + x_i) \cap \Omega} |f| \right)^{\frac{1}{2}}.$$

The other fact is that there exist a finite integer $N > 0$, depending only on $d$, and a family of points $\{z_i\}_{i=1}^{N} \subset B_{4r}$, such that $B_{\varepsilon \chi_*(x_0/\varepsilon) + 2r} \subset \bigcup_{i=1}^{N} B_{\varepsilon \chi_*(x_i/\varepsilon) - r}(z_i)$. Therefore, by noting $z_i + B_r \subset B_{5r}$ and $\varepsilon \chi_*(x_0/\varepsilon) - r \leq \varepsilon \chi_*(x/\varepsilon) \leq 2 \varepsilon \chi_*(x_0/\varepsilon)$ holding for any $x \in B_{5r}$ (due to $\frac{1}{\varepsilon}$-Lipschitz continuity of $\chi_*$), we can derive that

$$\left( \int_{B_{\varepsilon \chi_*(x_0)} \cap \Omega} |f| \right)^{\frac{1}{2}} \lesssim \sum_{i=1}^{N} \int_{B_r \cap \Omega} \left( \int_{B_{\varepsilon \chi_*(x_i/\varepsilon) - r}(x + x_i) \cap \Omega} |f| \right)^{\frac{1}{2}} \lesssim \int_{B_{5r} \cap \Omega} \left( \int_{B_{\varepsilon \chi_*(x)} \cap \Omega} |f| \right)^{\frac{1}{2}}.$$ (2.37)

Concerned with the case $0 < s < 1$, we appeal to the special case $s = 1$ proved above and the desired result follows from Hölder’s inequality, i.e.,

$$\left( \int_{B_{\varepsilon \chi_*(x_0)} \cap \Omega} |f| \right)^{\frac{1}{2}} \lesssim \left( \int_{B_{\varepsilon \chi_*(x_0)} \cap \Omega} |f| \right)^{\frac{1}{2}} \lesssim \left( \int_{B_{\varepsilon \chi_*(x)} \cap \Omega} |f| \right)^{\frac{1}{2}}.$$

This coupled with (2.37) gives us the whole proof of (2.35).

Then, we turn to the estimate (2.36). For any $|x - y| \leq \varepsilon \chi_*(x/\varepsilon)$, it follows from $\frac{1}{\varepsilon}$-Lipschitz continuity of $\chi_*$ that $\varepsilon \chi_*(x/\varepsilon) \sim \varepsilon \chi_*(y/\varepsilon)$. Then we have

$$\int_{B_{2r} \cap \Omega} \int_{B_{\varepsilon \chi_*(x)} \cap \Omega} |f| \, dx \sim \frac{1}{|B_{2r} \cap \Omega|} \int_{\mathbb{R}^d} |f(y)| \, \frac{|B_{2r} \cap \Omega \cap B_{\varepsilon \chi_*(y)}|}{|B_{\varepsilon \chi_*(y)} \cap \Omega|} \, dy.$$ (2.38)

On the one hand, for all $y \in B_r \cap \Omega$ we have $\frac{1}{5} B_{\varepsilon \chi_*(y)} \subset B_{r + \frac{1}{5} \varepsilon \chi_*(y/\varepsilon)} \subset B_{2r}$ (due to $\frac{1}{\varepsilon}$-Lipschitz continuity of $\chi_*$), and this together with (2.38) leads to

$$\int_{B_{2r} \cap \Omega} \int_{B_{\varepsilon \chi_*(x)} \cap \Omega} |f| \, dx \geq \frac{C_d}{|B_{2r}|} \int_{\mathbb{R}^d} |f(y)| \, \frac{|B_{2r} \cap \Omega \cap B_{\varepsilon \chi_*(y)}|}{|B_{\varepsilon \chi_*(y)} \cap \Omega|} \, dy \gtrsim \int_{B_r \cap \Omega} |f(y)| \, dy.$$ (2.39)

On the other hand, we collect $y \in \mathbb{R}^d$ to be such that $B_{2r} \cap B_{\varepsilon \chi_*(y)} \neq \emptyset$, which allows us to have the concrete range of $y$ by a simple computation: $|y| \leq 2r + \varepsilon \chi_*(y/\varepsilon) \leq 2r + \varepsilon \chi_*(0) + \frac{|y|}{r} \Rightarrow |y| \leq 7r$. In this regard, from (2.38) we find

$$\int_{B_{2r} \cap \Omega} \int_{B_{\varepsilon \chi_*(x)} \cap \Omega} |f| \, dx \leq \frac{C_d}{|B_{2r}|} \int_{\mathbb{R}^d} |f(y)| \, \frac{|B_{2r} \cap \Omega \cap B_{\varepsilon \chi_*(y)}|}{|B_{\varepsilon \chi_*(y)} \cap \Omega|} \, dy \lesssim \int_{B_r \cap \Omega} |f(y)| \, dy.$$ (2.40)

Therefore, combining the estimates (2.39) and (2.40) leads to the stated estimate (2.36), and this ends the proof.
3 Calderón-Zygmund estimates

3.1 Quenched estimates

**Proposition 3.1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $d \geq 2$ and $\varepsilon \in (0, 1]$. Suppose that $\langle \cdot \rangle$ is stationary, satisfying the spectral gap condition (1.2), and the (admissible) coefficient additionally satisfies (1.3) and the symmetry condition $a = a^*$. Let $u_\varepsilon$ and $f$ be associated with the equations.

\[
\begin{cases}
    \mathcal{L}_\varepsilon(u_\varepsilon) = \nabla \cdot f & \text{in } \Omega; \\
    u_\varepsilon = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3.1)

Then, there exists a stationary random field $\chi_*$ given as in Corollary 2.8 with $1 \over L$-Lipschitz continuity (where the Lipschitz constant is given as in Remark 2.4) such that

\[
\int_\Omega \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{q}{2}} \lesssim_{\lambda,d,M_0,p} \int_\Omega \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{q}{2}},
\]

(3.2)

holds for any $p > 1$ satisfying $|1 \over p - 1 | \leq 1 \over 2d + \theta$ with $0 < \theta \ll 1$. Also, for some $\omega \in A_1$ we have

\[
\int_\Omega \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{q}{2}} \omega \lesssim_{\lambda,d,M_0,p} \int_\Omega \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{q}{2}} \omega,
\]

(3.3)

where we recall the notation $B_{\varepsilon}(x) := B_{\varepsilon}(x/\varepsilon)(x)$, and we set $D_{\varepsilon}(x) := \Omega \cap B_{\varepsilon}(x)$. Moreover, if $\Omega$ is a regular-SKT (or $C^1$) domain (in such the case the symmetry condition $a = a^*$ is not needed), then the estimate (3.2) holds for any $1 < p < \infty$; For any $\omega \in A_p$ with $1 < p < \infty$, we have the following weighted estimate

\[
\int_\Omega \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{q}{2}} \omega \lesssim_{\lambda,d,M_0,[\omega],A_p} \int_\Omega \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{q}{2}} \omega.
\]

(3.4)

**Remark 3.2.** The same results are also valid for the elliptic systems with Neumann boundary conditions as long as we modify Lemma 3.4 below accordingly. Besides, in terms of the scalar case of the equations (3.1), the range of $p$ satisfies $4/3 - \theta < p < 4 + \theta$ if $d = 2$; and for $(3/2) - \theta < p < 3 + \theta$ if $d \geq 3$, which is coming from Lemma 3.4. Since except of the range of $p$ it follows from the same proof as that given for systems, we don’t reproduce it any more.

**Lemma 3.3** (interior $W^{1,p}$ estimates). Let $2 \leq q < \infty$ and $\varepsilon \in (0, 1]$. Suppose that $\langle \cdot \rangle$ is stationary and satisfies the spectral gap condition (1.2). Let $u_\varepsilon$ and $f$ be associated by $-\nabla \cdot a^\varepsilon \nabla u_\varepsilon = \nabla \cdot f$ in $\mathbb{R}^d$. Then, there exists a stationary random fields $\chi_*$ as given in Lemma 2.2 with $1 \over L$-Lipschitz continuity such that there holds

\[
\int_{B_R} \left( \int_{B_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{q}{2}} \lesssim \left( \int_{B_{2R}} \int_{B_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{q}{2}} + \int_{B_{2R}} \left( \int_{B_{\varepsilon}(x)} |f|^2 \right)^{\frac{q}{2}},
\]

(3.5)

for any $R > 0$, where the up to constant depends only on $d$, $\lambda$ and $q$.

**Proof.** Based upon large-scale Lipschitz estimates, the original proof is due to [1] and [19], while the present work starts from $H$-convergence of local version thanks to the new introduced minimal radius $\chi_*$ in Lemma 2.2. The main idea of the proof is due to Prof. Otto’s lecture in Toulouse 2019, and we modified it for our purpose. The proof is divided into three steps. By rescaling arguments, it suffices to consider the case $R = 1$. 

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Step 1. Decomposition of the solutions and reduction. For any ball $20B \subset B_1$, let $w_\varepsilon \in H_0^1(10B; \mathbb{R}^m)$ and $v_\varepsilon := u_\varepsilon - w_\varepsilon$ satisfy the following equations:

$$- \nabla \cdot a^\varepsilon \nabla w_\varepsilon = \nabla \cdot f_{10B}, \quad \text{and} \quad \nabla \cdot a^\varepsilon \nabla v_\varepsilon = 0 \text{ in } 10B. \quad (3.6)$$

Then we consider the approximating function $\nabla \cdot a \nabla \bar{v} = 0$ in $9B$ with $\bar{v} = v_\varepsilon$ on $\partial(9B)$. For the ease of the statement, we first introduce the following notation:

$$U(x) := \left( \int_{B_{x,\varepsilon}} |\nabla u_\varepsilon|^2 \right)^{1/2}; \quad F(x) := \left( \int_{B_{x,\varepsilon}} |f|^2 \right)^{1/2};$$

$$W_B(x) := \left( \int_{B_{x,\varepsilon}} |(e_i + (\nabla \phi_i)^\varepsilon) \partial_i \bar{v}|^2 \right)^{1/2}; \quad V_B(x) := \left( \int_{B_{x,\varepsilon}} |\nabla u_\varepsilon - (e_i + (\nabla \phi_i)^\varepsilon) \partial_i \bar{v}|^2 \right)^{1/2}; \quad (3.7)$$

$$V_B^{(1)}(x) := \left( \int_{B_{x,\varepsilon}} |\nabla v_\varepsilon - (e_i + (\nabla \phi_i)^\varepsilon) \partial_i \bar{v}|^2 \right)^{1/2}; \quad V_B^{(2)}(x) := \left( \int_{B_{x,\varepsilon}} |\nabla w_\varepsilon|^2 \right)^{1/2},$$

where $B_{x,\varepsilon}(x) := B_{\varepsilon \chi_*(x/\varepsilon)}(x)$.

Then, one can immediately observe the following relationships:

$$U \leq W_B + V_B; \quad \text{and} \quad V_B \leq V_B^{(1)} + V_B^{(2)}. \quad (3.8)$$

By Lemma 2.9 (it is fine to choose $\omega$ be a constant therein), the desired estimate (3.5) will immediately follow from the following two estimates:

$$\sup_{\frac{1}{2}B} W_B \lesssim \left( \int_{20B} U^2 + F^2 \right)^{1/2}; \quad (3.9)$$

and (for $0 < \varepsilon \ll 1$, there holds)

$$\int_{\frac{1}{2}B} V_B^2 \lesssim \int_{20B} (F^2 + \varepsilon U^2). \quad (3.10)$$

Let $x_B$ and $r_B$ represent the center and radius of $B$, respectively. The proof on the above estimates should be divided into two cases: (1) $0 < r_B < \frac{2}{\varepsilon} \chi_*(x_B/\varepsilon)$; (2) $r_B \geq \frac{2}{\varepsilon} \chi_*(x_B/\varepsilon)$. The first case is simple, and with the help of (2.35) we merely choose $W_B = U$ and $V_B = 0$ to show the estimates (3.9) and (3.10), respectively, in such the case. Therefore, we will put more effort in proving the second case.

Step 2. Show the estimate (3.9) under the case of $r_B \geq \frac{2}{\varepsilon} \chi_*(x_B/\varepsilon)$. For any $x \in \frac{1}{2}B$, recalling the notation $W_B(x)$ in (3.7), and noting that $B_{x,\varepsilon}(x) \subset 5B$ for all $x \in \frac{1}{2}B$ (due to $\frac{1}{2}$-Lipschitz continuity of $\chi_*$), we have

$$\left( \int_{B_{x,\varepsilon}(x)} |(e_i + (\nabla \phi_i)^\varepsilon) \partial_i \bar{v}|^2 \right)^{1/2} \lesssim \left( \int_{B_{x,\varepsilon}(x)} |e_i + (\nabla \phi)^\varepsilon|^2 \right)^{1/2} \sup_{5B} |\nabla \bar{v}(y)|$$

$$\lesssim \left\{ \left( \int_{B_{\chi_*(x/\varepsilon)}(x/\varepsilon)} |\nabla \phi|^2 \right)^{1/2} + 1 \right\} \left( \int_{6B} |\nabla \bar{v}(y)|^2 \right)^{1/2}.$$
Step 3. Show the estimate (3.10) under the case of \( r_B \geq \frac{\varepsilon}{4} \chi_*(x_B/\varepsilon) \). In view of the second inequality of (3.8) and Lemmas 2.2 and 2.10, we have
\[
\int_{\frac{r}{2} B} V_B^2 \lesssim (2.36) \int_B |\nabla v_\varepsilon - (e_i + (\nabla \phi_i)^\varepsilon) \partial_i \tilde{u}|^2 + \int_B |\nabla w_\varepsilon|^2 \\
\lesssim (2.5) \varepsilon \int_{10B} |\nabla v_\varepsilon|^2 + \int_{10B} |f|^2 \lesssim (2.36) \int_{20B} (F^2 + \varepsilon U^2),
\]
where we also employ the energy estimate for the first equation of (3.6) in the last two inequalities. This ends the whole proof.

\[\square\]

Lemma 3.4 (quenched reverse Hölder’s inequality). Let \( \Omega \) be a bounded Lipschitz domain. Let \( 0 < p_0 < q < p_1 \) and \( p_1 = \frac{2d}{d-1} + \theta \) with \( \theta > 0 \) depending only on \( \Omega \). Assume the same conditions as in Proposition 3.1 and \( 0 < \varepsilon \leq 1 \) and \( R > 0 \). Let \( u_\varepsilon \) be the solution of
\[
\begin{aligned}
-\nabla \cdot a^\varepsilon \nabla u_\varepsilon &= 0 \quad \text{in } B_{2R} \cap \Omega; \\
u_\varepsilon &= 0 \quad \text{on } B_{2R} \cap \partial \Omega.
\end{aligned}
\tag{3.12}
\]
Then, there exists a stationary random field \( \chi_* \) given as in Corollary 2.8 with \( \frac{1}{a} \)-Lipschitz continuity such that
\[
\left( \int_{B_{rB} \cap \Omega} \left( \int_{D_{\varepsilon} \cap (x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_0}{2}} \right) \lesssim_{\lambda,d,p,q} \left( \int_{D_{\varepsilon} \cap (x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_0}{2}}.
\tag{3.13}
\]
Proof. The main idea is similar to that given in Lemma 3.3. The proof is divided into four steps. Throughout the proof, we assume that \( 0 < p_0 \leq 2 \) and \( 1 < p_1 < \infty \) are given as in Lemma 3.4 above.

Step 1. Approximating the solutions and reduction. For any ball \( B \), it owns the property that \( r_B \leq (1/100)R \) and either \( x_B \in B_{2R} \cap \partial \Omega \) or \( 10B \subset B_{4R} \cap \Omega \), where \( x_B \) and \( r_B \) represent the center and radius of \( B \), respectively. We now consider the approximating equation: \( \nabla \cdot a \nabla \tilde{u} = 0 \) in \( 10B \cap \Omega \) with \( \tilde{u} = u_\varepsilon \) on \( \partial (10B \cap \Omega) \), and introduce the following notation:
\[
\begin{aligned}
U(x) := \left( \int_{D_{\varepsilon} \cap (x)} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}}; \\
D_{\varepsilon} \cap (x) \subseteq B_{2R} \cap \Omega = B_{\varepsilon \chi_*(x/\varepsilon)}(x) \cap \Omega;
\end{aligned}
\]
\[
\begin{aligned}
W_B(x) := \left( \int_{D_{\varepsilon} \cap (x)} |(e_i + (\nabla \phi_i)^\varepsilon \varphi_i|^2 \right)^{\frac{1}{2}}; \\
V_B(x) := \left( \int_{D_{\varepsilon} \cap (x)} |\nabla u_\varepsilon - (e_i + (\nabla \phi_i)^\varepsilon \varphi_i|^2 \right)^{\frac{1}{2}};
\end{aligned}
\]
where \( \varphi_i := S_{\varepsilon}(\eta_i \partial_i \tilde{u}) \) (see Lemma 2.6 for the definition of \( S_{\varepsilon} \)), and we mention that the random field \( \chi_* \) is from Corollary 2.8. Then, one can immediately observe the relationships: \( U \leq W_B + V_B \).

By Lemma 2.9 (simply let \( \omega \) be a constant therein), the desired estimate (3.13) will immediately follow from the following two estimates:
\[
\left( \int_{B \cap \Omega} W_B^{p_1} \right)^{\frac{1}{p_1}} \lesssim \left( \int_{20B \cap \Omega} U^{p_0} \right)^{\frac{1}{p_0}},
\tag{3.14}
\]
and (for \( 0 < \varepsilon \ll 1 \), there holds)
\[
\int_B V_B^{p_0} \lesssim \varepsilon \int_{20B \cap \Omega} U^{p_0}.
\tag{3.15}
\]

\[\text{If } \Omega \text{ is a regular-SKT domain, one may take } p_1 = \infty. \text{ If the equations (3.12) are scalar, let } p_1 = 3 \text{ as } d \geq 3; \text{ and } p_1 = 4 \text{ as } d = 2. \text{ In essence, the value of } p_1 \text{ is determined by Lemma 5.3.} \]
The proof on the above estimates should be divided into two cases: (1) \(0 < r_B < \frac{\varepsilon}{4} \chi_*(x_B/\varepsilon)\); (2) \(r_B \geq \frac{\varepsilon}{4} \chi_*(x_B/\varepsilon)\). The first case is simple, and we will discuss the estimates (3.14) and (3.15) in Step 2, while we will put more effort in proving them for the second case.

**Step 2.** Establish the estimates (3.14) and (3.15) in the case of \(0 < r_B < \frac{\varepsilon}{4} \chi_*(x_B/\varepsilon)\). We merely choose \(W_B = U\) and \(V_B = 0\), and the estimate (3.15) is trivial in such the case. To see the estimate (3.14), on account of \(\frac{1}{T}\)-Lipschitz continuity of \(\chi_*\) and Lemma 2.10, for any \(x_0 \in \frac{1}{2}B \cap \Omega\), we have \(0 < r_B < \frac{\varepsilon}{3} \chi_*(x_0/\varepsilon)\) and

\[
\left( \int_{D_{\varepsilon}(x_0)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_1}{2}} \lesssim \left( \int_{10B \cap \Omega} \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^\frac{p_1}{2} \right)^{\frac{1}{p_1}},
\]

which further implies that for any \(p_1 \geq p_0\),

\[
\left( \int_{D_{\varepsilon}(x_0)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_1}{2}} \lesssim \left( \int_{10B \cap \Omega} \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^\frac{p_0}{2} \right)^{\frac{1}{p_0}}.
\]

Integrating both sides above with respective to \(x_0 \in \frac{1}{2}B \cap \Omega\), we arrive at

\[
\left( \int_{\frac{1}{2}B \cap \Omega} \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_1}{2}} \right)^{\frac{1}{p_1}} \lesssim \left( \int_{10B \cap \Omega} \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^\frac{p_0}{2} \right)^{\frac{1}{p_0}},
\]

which offers us the stated estimate (3.14).

**Step 3.** Show the estimate (3.9) under the case of \(r_B \geq \frac{\varepsilon}{4} \chi_*(x_B/\varepsilon)\). Let \(\alpha'\) be the conjugate index of \(\alpha\) satisfying \(0 < \alpha' - 1 < 1\). According to the definition of \(S_{*,\varepsilon}\) in (2.24), it follows from Hölder’s inequality, the Fubini theorem that

\[
\int_{D_{\varepsilon}(x)} |(e_i + (\nabla \phi_i)^\varepsilon)S_{*,\varepsilon}(\eta_\varepsilon \partial u)|^2 \lesssim \int_{D_{\varepsilon}(x)} |(e_i + (\nabla \phi_i)^\varepsilon)|^2 \int_{\mathbb{R}^d} \zeta_{\varepsilon \chi_*(z/\varepsilon)}(z-y)|\eta_\varepsilon \partial \bar{u}(y)|^2 dy dz
\]

\[
\lesssim \int_{2D_{\varepsilon}(x)} |\partial \bar{u}(y)|^2 \int_{\mathbb{R}^d} \zeta_{\varepsilon \chi_*(y/\varepsilon)}(y-z)|(e_i + (\nabla \phi_i)^\varepsilon)(z)|^2 dz dy,
\]

where we also analyze the compact support under the convolution in the second inequality, and it relies on two facts (1): \(\chi_*(y/\varepsilon) \sim \chi_*(z/\varepsilon)\) provided \(|z-y| \leq \varepsilon \chi_*(z/\varepsilon)\); (2): For any \(z \in D_{\varepsilon}(x)\) and any \(y \in \mathbb{R}^d\) satisfying \(|y-z| \leq \varepsilon \chi_*(z/\varepsilon)\) we can infer that \(|y-x| \leq 2\varepsilon \chi_*(x/\varepsilon)\). We can further derive that

\[
\int_{D_{\varepsilon}(x)} |(e_i + (\nabla \phi_i)^\varepsilon)S_{*,\varepsilon}(\eta_\varepsilon \partial u)|^2 \lesssim \int_{2D_{\varepsilon}(x)} |\partial \bar{u}|^2 \int_{B_{\varepsilon}(y)} |(e_i + (\nabla \phi_i)^\varepsilon)|^2 dy.
\]

Thus, by using Hölder’s inequality we have

\[
\left( \int_{\frac{1}{2}B \cap \Omega} W_{B_{\varepsilon}}^p \right)^{\frac{1}{p_1}} \lesssim \left( \int_{\frac{1}{2}B \cap \Omega} \left( \int_{2D_{\varepsilon}(x)} |\partial \bar{u}|^{2\alpha'} \right)^{\frac{p_1}{p_0}} \left( \int_{2D_{\varepsilon}(x)} \left( \int_{B_{\varepsilon}(y)} |(\nabla \phi_i)^\varepsilon + e_i|^2 \right)^{\alpha} \right)^{\frac{p_1}{p_0}} \right)^{\frac{1}{p_1}}.
\]  

(3.17)

In terms of the right-hand side of (3.17), we claim that

\[
\sup_{x \in \frac{1}{2}B \cap \Omega} \left( \int_{2D_{\varepsilon}(x)} \left( \int_{B_{\varepsilon}(y)} |(\nabla \phi_i)^\varepsilon + e_i|^2 \right)^{\alpha} \right)^{\frac{p_1}{2\alpha}} \lesssim 1;
\]  

(3.18a)
\[
\left( \int_{B^c \cap \Omega} \left( \int_{2D_{*,\epsilon}(x)} |\nabla \bar{u}|^{2\alpha'} \right)^{\frac{p_1}{2\alpha'}} \right)^{\frac{1}{p_1}} \lesssim \left( \int_{10B^c \cap \Omega} |\nabla u_\epsilon|^{p_0} \right)^{\frac{1}{p_0}}. \tag{3.18b}
\]

Admitting them for a while, plugging them back into (3.17) and using Lemma 2.10 we derive that
\[
\left( \int_{B^c \cap \Omega} W_B^{p_1} \right)^{\frac{1}{p_1}} \lesssim (3.17), (3.18a), (3.18b) \left( \int_{10B^c \cap \Omega} |\nabla u_\epsilon|^{p_0} \right)^{\frac{1}{p_0}} \lesssim (2.36) \left( \int_{20B^c \cap \Omega} U^{p_0} \right)^{\frac{1}{p_0}},
\]

which is the stated estimate (3.14) in such the case.

We now turn to address the estimate (3.18a). It follows from Lemma 3.3 that
\[
\left( \int_{2D_{*,\epsilon}(x)} \left( \int_{B_{*,\epsilon}(y)} |(\nabla \phi_i)^\epsilon + e_i|^2 \right)^{\frac{\alpha}{p_1}} \right)^{\frac{1}{\alpha}} \lesssim \left( \int_{4B_{*,\epsilon}(x)} \int_{B_{*,\epsilon}(y)} |(\nabla \phi_i)^\epsilon + e_i|^2 \right)^{\frac{\alpha}{p_1}} + 1 \lesssim 1,
\]

where we also employ the same computation as those given in (3.11) for the last inequality. We proceed to show the estimate (3.18b). By using Hölder’s inequality, Lemmas 2.10, 5.3, energy estimates and Lemma 5.1 in the order, we have
\[
\left( \int_{B^c \cap \Omega} \left( \int_{2D_{*,\epsilon}(x)} |\nabla \bar{u}|^{2\alpha'} \right)^{\frac{p_1}{2\alpha'}} \right)^{\frac{1}{p_1}} \lesssim (2.36) \left( \int_{B^c \cap \Omega} |\nabla \bar{u}|^{p_1} \right)^{\frac{1}{p_1}} \lesssim (5.4) \left( \int_{4B^c \cap \Omega} |\nabla \bar{u}|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{9B^c \cap \Omega} |\nabla u_\epsilon|^2 \right)^{\frac{1}{2}} \lesssim (5.1) \int_{10B^c \cap \Omega} |\nabla u_\epsilon|.
\]

**Step 4.** Show the estimate (3.15) under the case of \( r_B \geq \frac{x}{x} \chi_*(x_B/\epsilon) \). It suffices to consider the case \( 0 < p_0 \leq 2 \). By using Hölder’s inequality and Corollary 2.8 and Lemma 2.10 in the order, there holds
\[
\int_{B^c \cap \Omega} V_B^{p_0} \lesssim \left( \int_{B^c \cap \Omega} |\nabla u_\epsilon - (e_i + (\nabla \phi_i)^\epsilon)\phi_i|^2 \right)^{\frac{p_0}{2}} \lesssim e^{p_0} \int_{12B^c \cap \Omega} |\nabla u_\epsilon|^{p_0} \lesssim e^{p_0} \int_{20B^c \cap \Omega} U^{p_0}.
\]

We have the stated estimate (3.15), and this ends the whole proof. \( \square \)

**Proof of Proposition 3.1.** Basically, we should prove quenched Calderón-Zygmund estimates for general Lipschitz domains (in such the case we take \( p_1 = \frac{2d}{d+1} + \theta \) with \( 0 < \theta < 1 \) and \( \omega = 1 \) in the later computations) and regular-SKT (or \( C^1 \)) domains (we choose \( 2 < p_1 < \infty \) to be arbitrarily fixed), separately. Let \( 0 < p_0 - 1 < 1 \).

**Step 1.** Decomposition of the solutions and outline the proof of (3.4). Arbitrarily fix a ball \( B_0 \) centered on \( \partial \Omega \) with radius \( r_{B_0} > 1 \). For any ball \( B \), it owns the property that \( r_B \leq (1/100)r_{B_0} \), and either \( x_B \in 2B_0 \cap \partial \Omega \) or \( 2B \subset 4B_0 \cap \Omega \), where \( x_B \) and \( r_B \) represent the center and radius of \( B \), respectively. Let \( w_\epsilon \in H^1_0(12B \cap \Omega) \) and \( v_\epsilon := u_\epsilon - w_\epsilon \) satisfy the following equations:
\[
- \nabla \cdot a^\epsilon \nabla w_\epsilon = \nabla \cdot fI_{12B}, \quad \text{and} \quad \nabla \cdot a^\epsilon \nabla v_\epsilon = 0 \quad \text{in} \quad 12B \cap \Omega.
\]

Then we consider the approximating function \( \nabla \cdot \bar{u} \nabla \bar{v} = 0 \) in \( 9B \cap \Omega \) with \( \bar{v} = v_\epsilon \) on \( \partial(9B \cap \Omega) \). For the ease of the statement, we first introduce the following notation: (Recall that \( D_{*,\epsilon}(x) := B_{*,\epsilon}(x) \cap \Omega = B_{\epsilon \chi_*(x/\epsilon)}(x) \cap \Omega \))
\[
U(x) := \left( \int_{D_{*,\epsilon}(x)} |\nabla u_\epsilon|^2 \right)^{\frac{1}{2}}; \quad F(x) := \left( \int_{D_{*,\epsilon}(x)} |f|^2 \right)^{\frac{1}{2}};
\]
\[
W_B(x) := \left( \int_{D_{*,\epsilon}(x)} |(e_i + (\nabla \phi_i)^\epsilon)\phi_i|^2 \right)^{\frac{1}{2}}; \quad V_B(x) := \left( \int_{D_{*,\epsilon}(x)} |\nabla u_\epsilon - (e_i + (\nabla \phi_i)^\epsilon)\phi_i|^2 \right)^{\frac{1}{2}};
\]
\[
V^{(1)}_B(x) := \left( \int_{D_{*,\epsilon}(x)} |\nabla \bar{v} - (e_i + (\nabla \phi_i)^\epsilon)\phi_i|^2 \right)^{\frac{1}{2}}; \quad V^{(2)}_B(x) := \left( \int_{D_{*,\epsilon}(x)} |\nabla u_\epsilon|^2 \right)^{\frac{1}{2}}.
\]
where \( \varphi_i = S_{*,\varepsilon}(\eta_i \partial_i \tilde{v}) \) similar to that introduced in Lemma 3.4.

Then, one can observe the following relationships: \( U \leq W_B + V_B \) and \( V_B \leq V_B^{(1)} + V_B^{(2)} \) (recalling (3.8)). Once we established the following two estimates:

\[
\left( \int_{\frac{1}{2}B \cap \Omega} W_B^{p_1} \right)^{\frac{1}{p_1}} \lesssim \left( \int_{20B \cap \Omega} U^{p_0} + F^{p_0} \right)^{\frac{1}{p_0}} \left( \int_{B} \omega \right)^{\frac{1}{p_1}}, \tag{3.19}
\]

and

\[
\int_{\frac{1}{2}B \cap \Omega} V_B^{p_0} \lesssim \int_{20B \cap \Omega} (F^{p_0} + \epsilon U^{p_0}). \tag{3.20}
\]

By Lemma 2.9, for any \( p_0 < p < p_1 \) we have

\[
\left( \int_{\frac{1}{2}B_0 \cap \Omega} U^{p} \omega \right)^{\frac{1}{p}} \lesssim_{(2.34)} \left\{ \left( \int_{2B_0 \cap \Omega} U^{p_0} \right)^{\frac{1}{p_0}} \left( \int_{B_0} \omega \right)^{\frac{1}{p_0}} + \left( \int_{2B_0 \cap \Omega} |F|^p \omega \right)^{\frac{1}{p}} \right\}. \tag{3.21}
\]

By a covering argument (we will show the details in the next step) it is not hard to get

\[
\int_{\Omega} U^{p} \omega \lesssim_{(2.34)} \left\{ \left( \frac{\Omega}{|B_0|} \right)^{p-1} \left( \int_{\Omega} U^{p_0} \right)^{\frac{1}{p_0}} \left( \int_{B_0} \omega \right)^{\frac{1}{p_0}} + \int_{\Omega} |F|^p \omega \right\}. \tag{3.22}
\]

In particular, by taking \( \omega = 1 \) in (3.21) we simply have

\[
\int_{\frac{1}{2}B_0 \cap \Omega} U^{p} \lesssim_{(2.34)} \left( \int_{2B_0 \cap \Omega} U^{p_0} \right)^{\frac{1}{p_0}} + \int_{2B_0 \cap \Omega} |F|^p, \tag{3.23}
\]

which actually implies the desired estimate (3.2).

Admit the desired estimate (3.2) for a while. Recall 0 < \( p_0 - 1 \ll 1 \), and by using Hölder’s inequality and the property of \( A_{p/p_0} \) class (due to the reverse property it is known that \( \omega \in A_p \) implies \( \omega \in A_{p/p_0} \) provided 0 < \( p_0 - 1 \ll 1 \), and see for example [11, pp.137]), we can derive that

\[
\left( \int_{\Omega} |U|^{p_0} \right)^{\frac{1}{p_0}} \lesssim_{(3.2)} \left( \int_{\Omega} |F|^{p_0} \right)^{\frac{1}{p_0}} \lesssim_{(3.2)} \left( \int_{\Omega} |F|^{p_0} \right)^{\frac{1}{p_0}} \left( \int_{\Omega} \omega^{-\frac{p_0}{p-p_0}} \right)^{\frac{p-p_0}{p}} \lesssim_{(3.2)} \left( \int_{\Omega} |F|^{p_0} \right)^{\frac{1}{p}} \left( \int_{\Omega} \omega \right)^{-\frac{1}{p}}.
\]

Plugging this back into the right-hand side of (3.22), we have established the desired estimate (3.4).

Similar to those given for Lemma 3.3, the proof on the estimates (3.19) and (3.20) should be divided into two cases: (1) 0 < \( r_B < \frac{\varepsilon}{4} \chi_\ast(x_B/\varepsilon) \); (2) \( r_B \geq \frac{\varepsilon}{4} \chi_\ast(x_B/\varepsilon) \). We plan to address the first case in Step 3, while the second case will be discussed in Steps 4a, 4b and 5.

**Step 2.** Show the estimate (3.2) by a covering argument and duality. There exist \( N > 1 \) and a family of balls, denoted by \( \{B_r(x_i)\}_{i=1}^N \), such that \( \Omega \subset \bigcup_{i=1}^N B_r(x_i) \cap \Omega \). Therefore, we have

\[
\int_{\Omega} U^{p} \leq \sum_i \int_{B_r(x_i) \cap \Omega} U^{p} \lesssim_{(3.23)} r_0^{d(1-p)} \sum_i \left( \int_{B_r(x_i) \cap \Omega} U^{p_0} \right)^{\frac{1}{p_0}} + \sum_i \int_{B_r(x_i) \cap \Omega} F^{p} \nonumber \lesssim_{(3.24)} r_0^{d(1-p)} \left( \int_{\Omega} U^{p_0} \right)^{\frac{1}{p_0}} + \int_{\Omega} F^{p}. \]

Then we turn to the special case \( p = 2 \). Let \( \tilde{u}_\varepsilon \) be a \( H^1(\mathbb{R}^d) \)-extension of \( u_\varepsilon \), satisfying \( \tilde{u}_\varepsilon = u_\varepsilon \) in \( \Omega \) and \( \|
abla \tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^d)} \lesssim \|
abla u_\varepsilon\|_{L^2(\Omega)} \). On the one hand, by using the extension theorem, Fubini’s theorem and energy estimate in the order, we arrive at

\[
\int_{\Omega} U^{2} \leq \int_{\mathbb{R}^d} \int_{B_r(x)} |\nabla \tilde{u}_\varepsilon|^2 = \int_{\mathbb{R}^d} |\nabla \tilde{u}_\varepsilon|^2 \lesssim \int_{\Omega} |\nabla u_\varepsilon|^2 \lesssim \int_{\Omega} |f|^2. \tag{3.25}
\]
On the other hand, for each point \( x \in \Omega \) there exists a ball \( B \) with radius \( r_B > \frac{\varepsilon}{1} \chi_*(x_B/\varepsilon) \). Collecting all these balls provides us an open cover of \( \Omega \), and there must exist a finite sub-cover, denoted by \( \Omega \subset \bigcup_i B_i \).

On account of Lemma 2.10, we proceed to compute the right-hand side above as follows:

\[
\int_\Omega |f|^2 \leq \sum_i |B_i| \int_{B_i \cap \Omega} |f|^2 \lesssim \sum_i |B_i| \int_{2B_i \cap \Omega} \int_{D_{*,\varepsilon}(x)} |f|^2 \lesssim \int_\Omega \int_{D_{*,\varepsilon}(x)} |f|^2 = \int_\Omega F^2, \tag{3.26}
\]

where each point of \( \Omega \) is covered at most finite number of times by different balls. Thus, combining the estimates (3.25) and (3.26) we have

\[
\int_\Omega U^2 \lesssim \int_\Omega F^2. \tag{3.27}
\]

Hence, for any \( 2 < p < p_1 \), plugging the estimate (3.27) back into (3.24), we derive that

\[
\int_\Omega U^p \lesssim \left( \int_\Omega U^2 \right)^{\frac{p}{2}} + \int_\Omega F^p \lesssim \int_\Omega F^p.
\]

As a result, the case of \( p_1' < p < 2 \) follows from the duality argument and we left the details to the reader. Note that \( p_1 \) is given as the same as in Lemma 3.4.

**Step 3.** Establish the estimates (3.19) and (3.20) in the case of \( 0 < r_B < \frac{\varepsilon}{1} \chi_*(x_B/\varepsilon) \). We adopt the same strategy as that given in Lemma 3.4. We merely choose \( W_B = U \) and \( V_B = 0 \), and we focus on the estimate (3.19). Similar to that given for (3.16), for any \( \alpha \geq 1 \), we have

\[
\left( \int_{\frac{1}{2}B \cap \Omega} \left( \int_{D_{*,\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_1}{2}} \omega(x) dx \right)^{\frac{1}{p_1}} \leq \left( \int_{\frac{1}{2}B \cap \Omega} \left( \int_{D_{*,\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{\beta_1 p_1}{2}} \right)^{\frac{1}{\beta_1} p_1} \left( \int_{\frac{1}{2}B \cap \Omega} \omega^{\beta_1} \right)^{\frac{1}{\beta_1} p_1} \tag{3.29}
\]

Combining the estimates (3.28) and (3.29) we obtain

\[
\left( \int_{\frac{1}{2}B \cap \Omega} \left( \int_{D_{*,\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_1}{2}} \omega(x) dx \right)^{\frac{1}{p_1}} \lesssim \left( \int_{10B \cap \Omega} \left( \int_{D_{*,\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_1}{2}} \right)^{\frac{1}{p_1}} \left( \int_{B} \omega \right)^{\frac{1}{p_1}},
\]

which in fact proves the stated estimate (3.19).

**Step 4a.** Show the estimate (3.19) in the case of \( r_B \geq \frac{\varepsilon}{1} \chi_*(x_B/\varepsilon) \) for the general Lipschitz domain. Let \( \alpha' \) be the conjugate index of \( \alpha \) satisfying \( 0 < \alpha' - 1 \ll 1 \). As a preparation, let \( p_1 \in (2, \infty) \) and some \( \omega \in A_1 \) be given such that

\[
\left( \int_{\frac{1}{2}B \cap \Omega} |\nabla \bar{v}|^{p_1} \omega \right)^{\frac{1}{p_1}} \lesssim \left( \int_{\frac{1}{2}B \cap \Omega} |\nabla \bar{v}|^2 \right)^{\frac{1}{2}} \left( \int_{B} \omega \right)^{\frac{1}{p_1}} \tag{3.30}
\]
Then, by Fubini's theorem and the definition of $A_1$ weight (see [11, pp.134]), we can further derive that

\[
\left(\int_{\frac{1}{2}B \cap \Omega} W_{B}^{p_1} \omega \right)^{\frac{1}{p_1}} \lesssim \left(\int_{\frac{1}{2}B \cap \Omega} |\nabla \overline{v}|^2 \omega \right)^{\frac{1}{2}} \left(\int_{B} \omega \right)^{\frac{1}{2}} \lesssim \left(\int_{\frac{1}{2}B \cap \Omega} |\nabla \bar{v}|^2 \omega \right)^{\frac{1}{2}} \left(\int_{B} \omega \right)^{\frac{1}{2}}
\]

\[
\lesssim \left(\int_{10B \cap \Omega} |\nabla \bar{v}|^{p_0} \right)^{\frac{1}{p_0}} \left(\int_{B} \omega \right)^{\frac{1}{p_1}} \lesssim \left(\int_{12B \cap \Omega} U^{p_0} + V_{B}^{(2)p_0} \right)^{\frac{1}{p_0}} \left(\int_{B} \omega \right)^{\frac{1}{p_1}} \tag{3.32}
\]

We now choose $p_1' < p_0 \leq 2$, and it follows that

\[
\left(\int_{\frac{1}{2}B \cap \Omega} W_{B}^{p_1} \omega \right)^{\frac{1}{p_1}} \lesssim d \left(\int_{\frac{1}{2}B \cap \Omega} |\nabla \bar{v}|^2 \right)^{\frac{1}{2}} \left(\int_{B} \omega \right)^{\frac{1}{2}} \lesssim \left(\int_{B \cap \Omega} |\nabla \bar{v}|^2 \omega \right)^{\frac{1}{2}} \left(\int_{B} \omega \right)^{\frac{1}{2}}
\]

\[
\lesssim \left(\int_{10B \cap \Omega} |\nabla \bar{v}|^{p_0} \right)^{\frac{1}{p_0}} \left(\int_{B} \omega \right)^{\frac{1}{p_1}} \lesssim \left(\int_{12B \cap \Omega} U^{p_0} + V_{B}^{(2)p_0} \right)^{\frac{1}{p_0}} \left(\int_{B} \omega \right)^{\frac{1}{p_1}} \tag{3.33}
\]

This consequently leads to the stated estimate (3.19) for the general Lipschitz domain.

**Step 4b.** Show the estimate (3.19) in the case of $r_B \geq \frac{1}{4} \chi_s(x_B/\varepsilon)$ for the regular-SKT domain. Let $p_1 \in (2, \infty)$ and $1 < p_0 \leq 2$ be arbitrarily fixed. We start from the estimate (3.31), and by Hölder’s inequality, the reverse Hölder’s inequality of $A_p$-weight function (see for example [11, Theorem 7.4]) and Lemmas 5.1 and 5.3, we obtain

\[
\left(\int_{\frac{1}{2}B \cap \Omega} W_{B}^{p_1} \omega \right)^{\frac{1}{p_1}} \lesssim \left(\int_{\frac{1}{2}B \cap \Omega} |\nabla \bar{v}|^{p_1 \beta} \right)^{\frac{1}{p_1 \beta}} \left(\int_{B} \omega^{\beta} \right)^{\frac{1}{p_1 \beta}} \lesssim \left(\int_{10B \cap \Omega} |\nabla \bar{v}|^{p_0} \right)^{\frac{1}{p_0}} \left(\int_{B} \omega \right)^{\frac{1}{p_1 \beta}} \lesssim \left(\int_{12B \cap \Omega} U^{p_0} + V_{B}^{(2)p_0} \right)^{\frac{1}{p_0}} \left(\int_{B} \omega \right)^{\frac{1}{p_1 \beta}} \tag{3.34}
\]

This consequently leads to the stated estimate (3.19) for the regular-SKT (or $C^1$) domain.

**Step 5.** Show the estimate (3.20) in the case of $r_B \geq \frac{1}{4} \chi_s(x_B/\varepsilon)$. The argument is similar to that given for (3.15). In view of $V_B \leq V_{B}^{(1)} + V_{B}^{(2)}$, we first have

\[
\int_{\frac{1}{2}B \cap \Omega} V_{B}^{p_0} \leq \int_{\frac{1}{2}B \cap \Omega} \left(\int_{D_{s,\varepsilon}(x)} |\nabla \bar{v}_\varepsilon - (e_i + (\nabla \phi_i)\varepsilon)\varphi_1|^{2} \right)^{\frac{p_0}{2}} + \int_{\frac{1}{2}B \cap \Omega} \left(\int_{D_{s,\varepsilon}(x)} |\nabla \bar{v}_\varepsilon|^{2} \right)^{\frac{p_0}{2}} \tag{3.35}
\]

\[
\lesssim \left(\int_{\frac{1}{2}B \cap \Omega} |\nabla \bar{v}_\varepsilon - (e_i + (\nabla \phi_i)\varepsilon)\varphi_1|^{2} \right)^{\frac{p_0}{2}} + \int_{\frac{1}{2}B \cap \Omega} \left(\int_{D_{s,\varepsilon}(x)} |f_\varepsilon|^{2} \right)^{\frac{p_0}{2}}. \tag{3.36}
\]
Therefore, it suffices to address the first term in the right-hand side. With the help of Corollary 2.8 together with Lemma 5.1, there holds

\[
\left( \int_{B^{c}\cap\Omega} |\nabla v_\varepsilon - (e_i + (\nabla \phi_i)\varepsilon)\varphi_i|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon \int_{4B^{c}\cap\Omega} |\nabla v_\varepsilon|
\]

and then a similar computation as that given for (3.33) leads to

\[
\left( \int_{B^{c}\cap\Omega} |\nabla v_\varepsilon - (e_i + (\nabla \phi_i)\varepsilon)\varphi_i|^2 \right)^{\frac{p}{p_0}} \lesssim \varepsilon \int_{20B^{c}\cap\Omega} U^{p_0} + F^{p_0}.
\]

Plugging this back into (3.34), we have derived the stated estimate (3.20).

**Step 6.** Show the details on the estimate (3.30). Here we choose \( p_1 = 2 + \theta \) with \( 0 < \theta \ll 1 \) and \( \omega(x) := [\text{dist}(x, \partial\Omega)]^\sigma \) with \( \sigma \in (-1,0) \). In this regard, it concludes that \( \omega \in A_1 \). In terms of the equation that \( \bar{v} \) satisfies, by decomposing the region \( B \cap \Omega \) into “layer” part and “co-layer” part, and then applying nontangential maximal function estimates to the layer part\(^6\), we obtain

\[
\int_{B^{c}\cap\Omega} |\nabla \bar{v}|^{p_1} \omega \lesssim r^\sigma \left( \int_{B\cap\partial\Omega} |(\nabla \bar{v})^*|^{p_1} dS + \int_{B^{c}\cap\Omega} |\nabla \bar{v}|^{p_1} \right)
\]

\[
\lesssim r^\sigma \int_{B^{c}\cap\Omega} |\nabla \bar{v}|^{p_1} \lesssim \left( \int_{2B^{c}\cap\Omega} |\nabla \bar{v}|^2 \right)^{\frac{p_0}{p}} \left( \int_B \omega \right),
\]

which implies the stated estimate (3.30). This ends the whole proof. \( \square \)

**Remark 3.5.** Due to the estimate (3.30) proved in Step 6 above, one can derive reverse Hölder’s inequality of weighted version on general Lipschitz domains as follows. Let \( p_1 = 2 + \theta \) with \( 0 < \theta \ll 1 \) and \( 0 < p_0 < q < p_1 \). Assume \( u_\varepsilon \) is the solution of (3.12). Then, for some weight \( \omega \in A_1 \) given in Step 6 above, we have

\[
\left( \int_{B^{c}\cap\Omega} \left( \int_{D_{s,\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{q}{q-1}} \omega \right)^{\frac{q}{q}} \lesssim \left( \int_{B} \omega \right)^{\frac{p_0}{p}} \left( \int_{B^{c}\cap\Omega} \left( \int_{D_{s,\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p_0}{q}} \right)^{\frac{q}{p}}.
\]

It can be derived by modifying Step 2,3 in the proof of Lemma 3.4 and the details can be found in Step 3, 4a in the proof of Proposition 3.1, respectively.

### 3.2 Annealed estimates

**Proposition 3.6.** Let \( \Omega \) be a general Lipschitz domain and \( q_1 := \frac{2d}{d-1} + \theta \) with \( 0 < \theta \ll 1 \) and \( \varepsilon \in (0,1] \). Suppose that \( \langle \cdot \rangle \) is stationary, satisfying the spectral gap condition (1.2), and the (admissible) coefficient additionally satisfies (1.3) and the symmetry condition \( a \equiv a^* \). Let \( u_\varepsilon \) be a weak solution to (3.1). Then, there exists a stationary random field \( \chi_\varepsilon \) given as in Corollary 2.8 with \( \frac{1}{2} \)-Lipschitz continuity such that

\[
\int_{\Omega} \left( \left( \int_{D_{s,\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}} \lesssim \int_{\Omega} \left( \left( \int_{D_{s,\varepsilon}(x)} |f|^2 \right)^{\frac{q}{q-1}} \right)^{\frac{q}{q}},
\]

\( ^6 \)In this regard, we refer the reader to the proof of (4.46) for the details.
holds for any $q'_1 < p, q < q_1$ (where $q'_1$ is the conjugate index of $q_1$), and for any $p < \bar{p} < \infty$ we have
\[
\int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \leq \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p}.
\] (3.37)
Moreover, if $\Omega$ is a bounded regular-SKT (or $C^1$) domain (in such the case the symmetry condition $a = a^*$ is not needed), and $1 < p, q < \infty$. Then, for any $\omega \in A_q$, we have
\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \omega \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \omega \right)^{\frac{1}{q}},
\] (3.38)
where up to constant is deterministic. Also, for any $p < \bar{p} < \infty$, there holds
\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \int_{B_{\varepsilon}(x)} \omega \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \int_{B_{\varepsilon}(x)} \omega \right)^{\frac{1}{q}}.
\] (3.39)
In particular, we have
\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \right)^{\frac{1}{q}},
\] (3.40)
and by taking $\omega_{\sigma}(x) := [\text{dist}(x, \partial \Omega_0)]^{q-1}$ with $\Omega_0 \supset \Omega$ satisfying $\partial \Omega_0 \in C^2$ and $\text{dist}(\cdot, \partial \Omega_0) = O(\varepsilon)$ on $\partial \Omega$, we obtain
\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \omega_{\sigma}^{\frac{1}{q}} \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \omega_{\sigma}^{\frac{1}{q}} \right)^{\frac{1}{q}}.
\] (3.41)
\textbf{Remark 3.7.} The same results are also true for the elliptic systems with Neumann boundary conditions since the annealed estimate is based upon the related quenched one, and the arguments from scales $\chi_*$ getting down to 1 are independent of PDEs (see Lemma 3.10). Similar to Proposition 3.1, in the scalar case of the equations (3.1), take $q_1 = 4$ if $d = 2$; and $q_1 = 3$ if $d \geq 3$.

\textbf{Lemma 3.8.} Let $\Omega$ be a bounded regular-SKT (or $C^1$) domain, and $1 < p, q < \infty$. Assume the same conditions as in Proposition 3.6 in such the case. Let $u_\varepsilon$ and $f$ be associated with the equation (3.1). Then, there exists a stationary random field $\chi_*$ given as in Corollary 2.8 with $\frac{1}{2}$-Lipschitz continuity such that for any $\omega \in A_q$, we have
\[
\int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \omega \leq \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \omega.
\] (3.42)
In particular, there holds
\[
\int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p} \leq \int_{\Omega} \left\langle \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}} \right\rangle^\frac{q}{p}.
\] (3.43)
\textbf{Proof.} Based upon Proposition 3.1, the idea is to repeat using Shen’s real arguments, which was first employed in [9]. We modify the original proof to fulfill our purpose, and the whole proof will be completed by two parts: \textit{Part I}(the first three steps). We in fact proved the estimate (3.42) for all weight $\omega \in A_1$, which will be done by the first three steps. \textit{Part II} (the last three steps). Based upon
the derived result (3.43) in Part I, reusing the same real arguments leads to the desired estimate (3.42) for all weight $\omega \in A_q$.

**Step 1.** Reduction and outline the proof of Part I. Let $p, q_1 \in (1, \infty)$, we need to prove the estimate (3.42) for the case $p \leq q < q_1$, and then the case $q < p$ follows from a duality argument. Similar as in the proof of Proposition 3.1, we start from decomposition of the solutions (3.1). For any ball $2B \cap \Omega \subset \Omega$, let $w_\varepsilon, v_\varepsilon \in H_0^1(\Omega)$ satisfy the following equations:

$$-\nabla \cdot a^\varepsilon \nabla w_\varepsilon = \nabla \cdot f I_{2B \cap \Omega}, \quad \text{and} \quad \nabla \cdot a^\varepsilon \nabla v_\varepsilon = \nabla \cdot f I_{\Omega \cap 2B} \quad \text{in} \quad \Omega. \tag{3.44}$$

For the ease of the statement, we introduce the following notations:

$$U(x) := \left( \int_{D_{x, \varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{1 \over 2}; \quad F(x) := \left( \int_{D_{x, \varepsilon}(x)} |f|^2 \right)^{1 \over 2},$$

and

$$W_B(x) := \left( \int_{D_{x, \varepsilon}(x)} |\nabla w_\varepsilon|^2 \right)^{1 \over 2}; \quad V_B(x) := \left( \int_{D_{x, \varepsilon}(x)} |\nabla v_\varepsilon|^2 \right)^{1 \over 2}.$$

Thus we have $U \leq W_B + V_B$ due to the relationship $u_\varepsilon = w_\varepsilon + v_\varepsilon$ in $\Omega$. According to the preconditions sated in Lemma 2.9, we have to establish the following two estimates:

$$\int_{\frac{1}{2} B \cap \Omega} W_B^p \leq \int_{10 B \cap \Omega} F^p; \tag{3.45a}$$

$$\left( \int_{\frac{1}{2} B \cap \Omega} V_B^{q_1} \omega \right)^{\frac{p}{q_1}} \leq \left( \int_{4 B \cap \Omega} (U^p + F^p) \right)^{\frac{p}{q}}. \tag{3.45b}$$

Then, with the help of the estimate (3.2), one can derive the stated estimate (3.42) under the case $p \leq q < \infty$ for all weight $\omega \in A_1$, since $q/p \geq 1$ can be arbitrary. Although we merely derived the estimate (3.42) within $A_1$ for now, taking the weight $\omega = 1$ will simply offer us the desired estimate (3.43), which will benefit in the second part of the proof. Therefore, the remainder of the proof is mainly to demonstrate the estimates (3.45a) and (3.45b).

**Step 2.** Show the estimate (3.45a). It is reduced to show

$$\int_{\frac{1}{2} B \cap \Omega} \left( \int_{D_{x, \varepsilon}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{p}{2}} \lesssim \int_{10 B \cap \Omega} \left( \int_{D_{x, \varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}}. \tag{3.46}$$

The main work is to analyze the size of support set of $\int_{D_{x, \varepsilon}(x)} f I_{2B \cap \Omega}$, and we discuss it by two cases: (1) $r_B \geq \frac{\varepsilon}{4} \chi_\varepsilon(x_B/\varepsilon)$; (2) $r_B < \frac{\varepsilon}{4} \chi_\varepsilon(x_B/\varepsilon)$. We start from the first case, and notice that if $\int_{D_{x, \varepsilon}(x)} f I_{2B} \neq 0$, then there must hold $|x - x_B| - \varepsilon \chi_\varepsilon(x_B/\varepsilon) < 2r_B$ due to the geometry fact. In this respect, we conclude that

$$|x - x_B| \leq 2r_B + \varepsilon \chi_\varepsilon(x_B/\varepsilon) \leq 2r_B + \varepsilon \chi_\varepsilon(x_B/\varepsilon) + |x - x_B|/\L \Rightarrow |x - x_B| \leq 7r_B. \tag{3.47}$$

Hence, with the help of Proposition 3.1 we have

$$\int_{B \cap \Omega} \left( \int_{D_{x, \varepsilon}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{p}{2}} \lesssim \frac{1}{|B|} \int_{\Omega} \left( \int_{D_{x, \varepsilon}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{p}{2}} \lesssim (3.2) \frac{1}{|B|} \int_{\Omega} \left( \int_{D_{x, \varepsilon}(x)} |f I_{2B \cap \Omega}|^2 \right)^{\frac{p}{2}} \lesssim (3.47) \int_{7B \cap \Omega} \left( \int_{D_{x, \varepsilon}(x)} |f|^2 \right)^{\frac{p}{2}}. \tag{3.48}$$
Now, we proceed to talk about the case (2) \( r_B < \frac{\varepsilon}{4} \chi_\ast(x_B/\varepsilon) \). For any \( x_0 \in \frac{1}{2}B \cap \Omega \), by noting \( 2B \subset B_{\ast,c}(x_0) \) and using energy estimates we have

\[
\int_{B_{\ast,c}(x_0)} |\nabla w_\varepsilon|^2 \lesssim \int_{B_{\ast,c}(x_0)} |f|^2. \tag{3.49}
\]

Then appealing to Lemma 2.10 (by noting \( r_B < (\frac{2L}{2L-1})^{\frac{2}{2}} \chi_\ast(x_0) \)), it follows that

\[
\left( \int_{B_{\ast,c}(x_0)} |\nabla w_\varepsilon|^2 \right)^{\frac{p}{2}} \lesssim \left( \int_{B_{\ast,c}(x_0)} |f|^2 \right)^{\frac{p}{2}} \lesssim \int_{10B \cap \Omega} \left( \int_{B_{\ast,c}(x)} |f|^2 \right)^{\frac{p}{2}}.
\]

Integrating both sides above with respect to \( x_0 \in \frac{1}{2}B \cap \Omega \) yields the estimate (3.46) in such the case. This together with (3.48) completes the whole argument of the estimate (3.46), and finally taking expectation on the both sides of (3.46) offers us the stated estimate (3.45a).

**Step 3.** Establish the estimate (3.45b). Let \( \gamma, \gamma' \) be such that \( 1/\gamma + 1/\gamma' = 1 \) with \( 0 < \gamma' - 1 \ll 1 \). By using Minkowski’s inequality, Hölder’s inequality and Lemma 3.4 coupled with reverse Hölder’s inequality of the weight (see [11, Theorem 7.4]) in the order, we obtain

\[
\left( \int_{\frac{1}{2}B \cap \Omega} V_{B,\varepsilon}^{q_1} \omega \right)^{\frac{p}{q_1}} = \left( \int_{\frac{1}{2}D} \left( \int_{B_{\ast,c}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{q_1}{p}} \omega \right)^{\frac{p}{q_1}} \leq \left( \int_{\frac{1}{2}B \cap \Omega} \left( \int_{B_{\ast,c}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{q_1}{q}} \right)^{\frac{p}{q_1}} \left( \int_{\frac{1}{2}B \cap \Omega} \omega^{\frac{q_1}{q}} \right)^{\frac{p}{q_1}} \lesssim (3.13) \int_{2B \setminus \Omega} \left( \int_{B_{\ast,c}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{p}{q_1}} \left( \int_{B} \omega \right)^{\frac{p}{q_1}},
\]

which together with the relationship \( u_\varepsilon = w_\varepsilon + v_\varepsilon \) in \( \Omega \) and the estimate (3.46) leads to the desired estimate (3.45b). This ends the first part of the proof.

**Step 4.** Show the estimate (3.42). Let \( 0 < q_0 - 1 \ll 1 \), and \( q_1 \in (1, \infty) \) is as before. On account of Lemma 2.9, it suffices to establish

\[
\int_{\frac{1}{2}B \cap \Omega} W_{\varepsilon}^{q_0} \leq \int_{10B \setminus \Omega} F^{q_0}; \tag{3.51a}
\]

\[
\left( \int_{\frac{1}{2}B \cap \Omega} V_{B,\varepsilon}^{q_1} \omega \right)^{\frac{q}{q_1}} \leq \int_{2B \setminus \Omega} \left( \int_{B_{\ast,c}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{q_1}{q}} \left( \int_{B} \omega \right)^{\frac{q_1}{q}}. \tag{3.51b}
\]

**Step 5.** Show the estimate (3.51a), which is parallel to Step 2. For the case \( r_B \geq \frac{\varepsilon}{4} \chi_\ast(x_B/\varepsilon) \), with the help of the estimate (3.43) and the geometry fact (3.47), we have

\[
\int_{B \cap \Omega} \left( \int_{B_{\ast,c}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{q_1}{p}} \leq \frac{1}{|B|} \int_{\Omega} \left( \int_{B_{\ast,c}(x)} |\nabla w_\varepsilon|^2 \right)^{\frac{q_1}{p}} \lesssim (3.43) \frac{1}{|B|} \int_{\Omega} \left( \int_{B_{\ast,c}(x)} |f|^{2B \setminus \Omega}\right)^{\frac{q_1}{p}} \lesssim \int_{B \setminus \Omega} F^{q_0}.
\]

Then we turn to the case \( r_B < \frac{\varepsilon}{4} \chi_\ast(x_B/\varepsilon) \). For any \( x_0 \in \frac{1}{2}B \cap \Omega \), in view of the estimate (3.49), we have

\[
\left( \int_{B_{\ast,c}(x_0)} |\nabla w_\varepsilon|^2 \right)^{\frac{q_1}{p}} \lesssim \left( \int_{B_{\ast,c}(x_0)} |f|^2 \right)^{\frac{q_1}{p}}.
\]
and integrate both sides above with respect to $x_0 \in \frac{1}{2}B \cap \Omega$. This result coupled with (3.52) leads to the estimate (3.51a).

**Step 6.** Show the estimate (3.51b), which is parallel to Step 3. For the same reason, one can simply replace the index $p$ in (3.50) with $q_0$ to obtain

\[
\left( \int_2 2B \mid \nabla v_\varepsilon \mid^2 \right)_{q_1}^{q_0} \lesssim 2 \int_2 2B \mid \nabla v_\varepsilon \mid^2 \leq \int_2 2B \mid \nabla v_\varepsilon \mid^2 \leq \int_2 2B \mid \nabla v_\varepsilon \mid^2,
\]

and this ends the whole proof.

**Remark 3.9.** Let the range of $q_1$ be given as in Proposition 3.6, and the proof of the estimate (3.36) can be similarly given by Steps 1, 2, 3 in the proof of Lemma 3.8, and we omit the details.

**Lemma 3.10.** Let $\chi_\varepsilon$ be given as in Lemma 2.2 or Corollary 2.8 with $\frac{1}{L}$-Lipschitz continuity. Assume that $F \in C_0^\infty(\Omega; L^q_\varepsilon)$ and $1 \leq p \leq q < \infty$ and $1 < s < \infty$. Then, for any $p_1, p_2 > p$, there hold

\[
\left( \int_\Omega (\int_{D_s(x)} |F|^s) \frac{1}{s} \omega(x) dx \right)^{\frac{1}{s}} \lesssim \left( \int_\Omega (\int_{D_s(x)} |F|^s) \frac{1}{s} \omega(x) dx \right)^{\frac{1}{s}}; \tag{3.53}
\]

and

\[
\left( \int_\Omega (\int_{D_s(x)} |F|^s) \frac{1}{s} \omega(x) dx \right)^{\frac{1}{s}} \lesssim \left( \int_\Omega (\int_{D_s(x)} |F|^s) \frac{1}{s} \omega(x) dx \right)^{\frac{1}{s}}; \tag{3.54}
\]

for any $\omega \in A_q$, where we recall the notation $D_{s,\varepsilon}(x) := B_{\varepsilon \chi_\varepsilon(x)}(x) \cap \Omega$.

**Proof.** Arguments for (3.53). The main idea is as the same as that given in [9], and we provide a proof with details for the reader’s convenience. The basic idea is to use the properties of conditional expectation to decompose the random scale into dyadic balls, and by a polynomial decay (growth) estimate for each determined scale, we finally use the loss of random index to trade the boundedness. By rescaling arguments, it suffices to show

\[
\int_{\Omega \varepsilon} (\int_{D_s(y)} |\tilde{F}|^s) \frac{1}{s} \tilde{\omega}(y) dy \lesssim \int_{\Omega \varepsilon} (\int_{D_s(y)} |\tilde{F}|^s) \frac{1}{s} \tilde{\omega}(y) dy, \tag{3.55}
\]

where $x = \varepsilon y \in \Omega$, and we use the following notations throughout the proof: $\tilde{F}(y) = F(\varepsilon y) = F(x)$ and $\tilde{\omega} = \omega(\varepsilon \cdot)$, as well as $D_s(y) := B_{\chi_\varepsilon(y)}(y) \cap (\Omega \varepsilon)$ and $D_R(y) := B_R(y) \cap (\Omega \varepsilon)$.

**Step 1.** We start from establishing the following deterministic estimate, i.e., for any $y \in \Omega \varepsilon$ and $R \geq 1$, there holds

\[
\int_{\Omega \varepsilon} (\int_{D_R(y)} |\tilde{F}|^s) \frac{1}{s} \tilde{\omega} dy \lesssim \int_{\Omega \varepsilon} (\int_{D_R(y)} |\tilde{F}|^s) \frac{1}{s} \tilde{\omega} dy, \tag{3.56}
\]

for any $1 \leq p \leq q < \infty$, where $(\frac{q}{p} - \frac{2}{s})_+ := \max\{0, \frac{q}{p} - \frac{2}{s}\}$. It is reduced to show

\[
(\int_{D_R(y)} |\tilde{F}|^s)^\frac{1}{s} \lesssim \int_{D_R(y)} (\int_{D_R(z)} |\tilde{F}|^s) \frac{1}{p}, \tag{3.57}
\]
since by using Fubini’s theorem, i.e., for any $R > 0$

$$\int_{\Omega/\varepsilon} \left( \int_{D_R(x)} f \right) g(x) dx = \int_{\Omega/\varepsilon} \left( \int_{D_R(y)} g \right) f(y) dy,$$

(3.58)

(where we mention that $\text{supp}(f) \subset \Omega/\varepsilon$), and the property of $A_p$ class (see [11, pp.134]), i.e.,

$$\omega(Q)\left(\frac{|S|}{|Q|}\right)^p \leq C_0\omega(S), \quad \forall S \subset Q.$$  (3.59)

it follows from the estimate (3.57) that

$$\int_{\Omega/\varepsilon} \left( \frac{\langle \int_{D_R(y)} |\tilde{F}|^s \rangle^p}{\omega} \right)^{\frac{q}{p}} dy \approx \int_{\Omega/\varepsilon} \left( \frac{\langle \int_{D_R(y)} |\tilde{F}|^s \rangle^p}{\omega} \right)^{\frac{q}{p}} dy$$

$$= \int_{\Omega/\varepsilon} \left( \frac{\langle \int_{D_R(y)} |\tilde{F}|^s \rangle^p}{\omega} \right)^{\frac{q}{p}} dy$$

$$\approx \int_{\Omega/\varepsilon} \left( \frac{\langle \int_{D_R(y)} |\tilde{F}|^s \rangle^p}{\omega} \right)^{\frac{q}{p}} dy$$

Step 2. Show the estimate (3.57). It suffices to establish

$$\left( \int_{D_R(y)} |\tilde{F}|^s \right)^{\frac{p}{q}} \approx \int_{D_R(y)} \left( \int_{D_1(z)} |\tilde{F}|^s \right)^{\frac{p}{q}} dy,$$

(3.60)

since taking $\langle \cdot \rangle^{\frac{p}{q}}$ on the both sides above we can immediately derive (3.57). The estimate (3.60) will be demonstrated by two cases:

Case 1. For any $p \geq s$, it follows from Lemma 2.10 and Hölder’s inequality:

$$\left( \int_{D_R(y)} |\tilde{F}|^s \right)^{\frac{p}{q}} \approx \int_{D_R(y)} \left( \int_{D_1(z)} |\tilde{F}|^s \right)^{\frac{p}{q}} dy \leq \int_{D_R(y)} \left( \int_{D_1(z)} |\tilde{F}|^s \right)^{\frac{p}{q}} dy.$$

(3.61)

Case 2. For any $p \leq s$, we employ a covering argument and Lemma 2.10 to have

$$\left( \int_{D_R(y)} |\tilde{F}|^s \right)^{\frac{p}{q}} \approx \left( \sum_i \int_{D_1(z_i)} |\tilde{F}|^s \right)^{\frac{p}{q}} \approx \sum_i \int_{D_1(z_i)} |\tilde{F}|^s \approx \int_{D_R(y)} \left( \int_{D_1(z)} |\tilde{F}|^s \right)^{\frac{p}{q}} dy,$$

which coupled with (3.61) leads to the stated estimate (3.60).

Step 3. Show the estimate (3.55). For the ease of the statement, let $d\mu(y) := \omega dy$. On account of the property of conditional expectations, we have

$$\int_{\Omega/\varepsilon} \left( \frac{\langle \int_{D_s(y)} |\tilde{F}|^s \rangle^p}{\omega} \right)^{\frac{q}{p}} d\mu(y) \leq \int_{\Omega/\varepsilon} \left( \sum_{n=1}^{2^{n-1}} I_{(2^{n-1} \leq \chi_s(y) < 2^{n+1}-1)} \left( \int_{D_{2^n}(y)} |\tilde{F}|^s \right)^{\frac{p}{q}} d\mu(y) \right)^{\frac{q}{p}}$$

$$\leq \int_{\Omega/\varepsilon} \left( \sum_{n=1}^{\infty} 2^{-nd s} \chi_s(y)^{d s} I_{(2^{n-1} \leq \chi_s(y) < 2^{n+1}-1)} \left( \int_{D_{2^n}(y)} |\tilde{F}|^s \right)^{\frac{p}{q}} d\mu(y) \right)^{\frac{q}{p}}$$

$$\leq \int_{\Omega/\varepsilon} \left( \sum_{n=1}^{\infty} 2^{-nd s} \left( \int_{D_{2^n}(y)} |\tilde{F}|^s \right)^{\frac{p}{q}} \right)^{\frac{q}{p}} d\mu(y)$$
By Holder’s inequality (we split $nd\delta = nd\delta(\frac{p}{q} + \frac{2-p}{q})$), the last line above will be controlled by

$$\int_{\Omega/\varepsilon} \left( \sum_{n=1}^{\infty} 2^{-nd\delta} \left\langle f_n \tilde{\mu} \right\rangle \left\langle f_n \right\rangle \right)^{\frac{p}{2}} d\mu \lesssim \delta^{1-p} \sum_{n=1}^{\infty} 2^{-nd\delta} \int_{\Omega/\varepsilon} \left\langle f_n \right\rangle^{\frac{p}{2}} d\mu$$

$$\lesssim \delta^{1-p} \sum_{n=1}^{\infty} 2^{nd\delta(\frac{p}{q}-1)} \int_{\Omega/\varepsilon} \left\langle \int_{D_{2n+1}(y)} I_{\{2^n-1 \leq \tilde{\chi}_s(x) < 2^{n+1}-1\}} |\tilde{F}|^s \right\rangle^{\frac{p}{2}} d\mu(y)$$

$$\lesssim (3.56) \delta^{1-p} \sum_{n=1}^{\infty} (2^{nd\delta} (\frac{p}{q} - \frac{1}{p}) + \delta (\frac{1}{p} - \frac{1}{q}) + 1 - \frac{1}{q}) \int_{\Omega/\varepsilon} \left\langle \int_{D_1(y)} I_{\{2^n-1 \leq r_s(x) < 2^{n+1}-1\}} |\tilde{F}|^s \right\rangle^{\frac{p}{2}} f_{B_1(y)} \tilde{\omega} dy.$$

Then, by using Markov’s inequality, one can choose $N_0 \gg 1$ such that

$$\left\langle I_{\tilde{\chi}_s \geq \tilde{\chi}_0} \right\rangle \lesssim 2^{-n} \frac{(\tilde{\chi}_s \geq \tilde{\chi}_0)}{pp} \lesssim 2^{-n} N_0 \left(\frac{(\tilde{\chi}_s \geq \tilde{\chi}_0)}{pp} \right),$$

where $\mathbb{P}$ is the probability measure associated with the ensemble $\langle \cdot \rangle$, and $dq[(\frac{1}{p} - \frac{1}{s})_+ + \delta (\frac{1}{p} - \frac{1}{q}) + 1 - \frac{1}{q}] - N_0 \left(\frac{(\tilde{\chi}_s \geq \tilde{\chi}_0)}{pp} \right) < 0$. This consequently gives

$$\int_{\Omega/\varepsilon} \left\langle \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} \tilde{\omega}(y) \right\rangle^{\frac{p}{2}} dy \lesssim \int_{\Omega/\varepsilon} \left\langle \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} f_{B_1(y)} \tilde{\omega} dy \right\rangle,$$

which implies (3.53).

**Arguments for (3.54).** The idea is similar to that given for the estimate (3.53). For any $1 \leq p \leq q < \infty$, we need first establish

$$\left( \int_{\Omega/\varepsilon} \left\langle \left( \int_{D_{2R}(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} \tilde{\omega}(y) \right\rangle^{\frac{p}{2}} dy \right)^{\frac{1}{2}} \lesssim R^{-d(\frac{1}{p} - \frac{1}{q}) + \frac{d\delta}{p} + \frac{d\delta}{q}} \left( \int_{\Omega/\varepsilon} \left\langle \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} f_{B_1(y)} \tilde{\omega} dy \right\rangle \right)^{\frac{1}{2}},$$

where the index $\delta_0 > 0$ is very small, coming from reverse Hölder property of $A_p$-weight. In fact, the proof the deterministic part is harder than its counterpart (3.56), and we divided into three cases to complete the argument.

**Case 1.** For $p \leq q \leq s$, we show the estimate

$$\int_{\Omega/\varepsilon} \left\langle \left( \int_{D_{2R}(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} \tilde{\omega}(y) \right\rangle^{\frac{p}{2}} dy \lesssim R^{-\delta_0 d} \int_{\Omega/\varepsilon} \left\langle \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} f_{B_1(y)} \tilde{\omega} dy \right\rangle.$$  

We start from introducing the reverse Hölder’s property of $A_p$ class, i.e., there exists $\delta_0 > 0$ such that

$$\omega(S) \leq C_0 \left( \frac{|S|}{Q} \right)^{\delta_0} \omega(Q), \quad \forall S \subset Q \quad (3.64)$$

(see for example [11, Corollary 7.6]). By using the reverse Hölder’s property above and Fubini’s theorem, we have

$$\int_{\Omega/\varepsilon} \left\langle \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} f_{D_1(y)} \tilde{\omega} dy \right\rangle \lesssim (3.64) \int_{\Omega/\varepsilon} \left\langle \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{2}} f_{D_1(y)} \tilde{\omega} dy \right\rangle$$

$$= (3.58) R^{-\delta_0 d} \int_{\Omega/\varepsilon} \int_{D_{2R}(y)} \left\langle \left( \int_{D_1(z)} |\tilde{F}|^s \right)^{\frac{p}{2}} \tilde{\omega} dy \right\rangle.$$

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Recalling the notation \( d\mu(y) = \omega(y)dy \), and then employing Hölder’s inequality, Minkowski’s inequality and Lemma 2.10 in the order, we obtain

\[
\int_{\Omega/\varepsilon} \left( \int_{D_R(y)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} d\mu(y) \leq \int_{\Omega/\varepsilon} \left( \int_{D_R(y)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} d\mu(y) \\
\leq \int_{\Omega/\varepsilon} \left( \int_{D_R(y)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} d\mu(y) \\
\lesssim_{(2.36)} \int_{\Omega/\varepsilon} \left( \int_{D_R(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} d\mu(y).
\]  

(3.66)

Combining the estimates (3.65) and (3.66), we have the stated estimate (3.63).

Case 2. For \( p \leq s \leq q \), we show the estimate

\[
\int_{\Omega/\varepsilon} \left( \int_{D_2R(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \omega dy \gtrsim R^{-\frac{da}{s} + d + \delta_0 d} \int_{\Omega/\varepsilon} \left( \int_{D_1(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \tilde{\omega} dy.
\]

(3.67)

For any \( y \in \Omega/\varepsilon \), applying for Lemma 2.10, Minkowski’s inequality in the order, we first arrive at

\[
\langle \left( \int_{D_2R(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \rangle^{\frac{q}{p}} \geq \left( \int_{D_1(y)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} \geq \left( \int_{D_1(y)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} =: I.
\]

Then, by covering the region \( D_R(y) \) by a family of balls (or half-balls) denoted by \( \{ D_1(y_i) \} \), and using Hölder’s inequality of \( l^p \)-version and Hölder’s inequality of \( L^p \)-version in the order, we obtain

\[
R^{-\frac{da}{s}} \sum \left( \int_{D_1(y_i)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} \geq \sum \left( \int_{D_1(y_i)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} \approx \int_{D_R(y)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \tilde{\omega} dy.
\]

(3.69)

Combining the inequalities (3.68) and (3.69) and integrating both sides with respect to \( y \in \Omega/\varepsilon \) in terms of \( \omega dx \) (denoted by \( d\mu \)), we obtain

\[
\int_{\Omega/\varepsilon} \left( \int_{D_2R(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} d\mu(y) \lesssim R^{-\frac{da}{s} + d + \delta_0 d} \int_{\Omega/\varepsilon} \left( \int_{D_1(y)} \left( \int_{D_1(z)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} \tilde{\omega} dy.
\]

which completes the proof of (3.67).

Case 3. For \( s \leq p \leq q \), we will establish

\[
\int_{\Omega/\varepsilon} \left( \int_{D_2R(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \omega dy \gtrsim R^{-\frac{da}{s} + d + \delta_0 d} \int_{\Omega/\varepsilon} \left( \int_{D_1(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \tilde{\omega} dy.
\]

(3.70)

The argument is similar to that given for Case 2. For any \( y \in \Omega/\varepsilon \), a routine decomposition leads to

\[
\langle \left( \int_{D_2R(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \rangle^{\frac{q}{p}} \lesssim \left( \sum_{y \in \Omega/\varepsilon} \left( \int_{D_1(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}} \gtrsim R^{-\frac{da}{s}} \left( \sum_{y \in \Omega/\varepsilon} \left( \int_{D_1(y)} \left| \tilde{F}' \right|^s \right)^{\frac{p}{s}} \right)^{\frac{q}{p}}.
\]
By using twice Hölder’s inequality of $l^p$, one can further derive that
\[
\left< \left( \int_{D_{2r}(y)} |\tilde{F}|^s \right)^{\frac{p}{s}} \right>^\frac{s}{p} \geq R^{-\frac{dn}{d-1}} \left( \sum_i \left< \left( \int_{D_1(y_i)} |\tilde{F}|^s \right)^{\frac{p}{s}} \right> \right)^\frac{s}{p} \geq R^{-\frac{dn}{d-1}} \sum_i \left< \left( \int_{D_1(y_i)} |\tilde{F}|^s \right)^{\frac{p}{s}} \right> \approx R^{-\frac{dn}{d-1} + d} \int_{D_{r}(y)} \left< \left( \int_{D_1(z)} |\tilde{F}|^s \right)^{\frac{p}{s}} \right>^\frac{s}{p}.
\]
Integrating both sides above with respect to $y \in \Omega/\varepsilon$ in terms of $d\mu$, and then appealing to the following estimate:
\[
\int_{\Omega/\varepsilon} \int_{D_r(y)} \left< \left( \int_{D_1(z)} |\tilde{F}|^s \right)^{\frac{p}{s}} \right>^\frac{s}{p} d\mu(y) = \int_{\Omega/\varepsilon} \int_{D_1(z)} \left< \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{s}} \right>^\frac{s}{p} d\mu(y) \geq \int_{\Omega/\varepsilon} \int_{D_1(z)} \left< \left( \int_{D_1(y)} |\tilde{F}|^s \right)^{\frac{p}{s}} \right>^\frac{s}{p} d\mu(y),
\]
we can immediately reach the stated estimate (3.70).

As a result, combining the estimates (3.63), (3.67), (3.70), we have the stated estimate (3.62). The remainder of the proof for (3.54) is analogy to those computations shown in Step 3, and we don’t reproduce it here. □

**Proof of Proposition 3.6.** The estimates (3.36) and (3.37) follows from Remark 3.9, while the estimate (3.38) comes from (3.42) in Lemma 3.8. Based upon Lemma 3.10, combining the estimates (3.38), (3.53) and (3.54) leads to (3.39). Taking $\omega = 1$ in (3.39), we have the estimate (3.40). Moreover, let $\Omega_0 \subseteq \Omega$ satisfy $\partial \Omega_0 \in C^2$ and $\text{dist}(\cdot, \partial \Omega_0) = O(\varepsilon)$ on $\partial \Omega$. By setting $\omega_\sigma(x) := |\text{dist}(x, \partial \Omega_0)|^q$, it is known that $\omega_\sigma \in A_q$, and $\int_{B_r(x)} \omega_\sigma \sim \omega_\sigma(x)$ for any $x \in \Omega$. This together with (3.39) provides us with (3.41) for $\omega_\sigma$, and $\omega_\sigma^{-1}$ follows from a duality argument. □

**Proof of Theorem 1.1.** The results of Theorem 1.1 follows from Propositions 3.1 and 3.6. □

## 4 Homogenization errors

### 4.1 Strong norms

**Proposition 4.1** (optimal errors). Let $\Omega$ be a bounded Lipschitz domain and $\varepsilon \in (0,1]$. Suppose that $\langle \cdot \rangle$ is stationary, satisfying the spectral gap condition (1.2), and the (admissible) coefficient additionally satisfies (1.3) and the symmetry condition $a = a^*$ (not necessary for some special Lipschitz domains). Let $u_\varepsilon, u_0 \in H^1_0(\Omega)$ be associated with $f \in L^2(\Omega)$ and $b \in H^1(\partial \Omega)$ by

\[
\left\{ \begin{array}{ll}
-\nabla \cdot a^* \nabla u_\varepsilon = f & \text{in } \Omega; \\
u_\varepsilon = b & \text{on } \partial \Omega;
\end{array} \right. \quad \left\{ \begin{array}{ll}
-\nabla \cdot \bar{a} \nabla \bar{u} = f & \text{in } \Omega; \\
\bar{u} = b & \text{on } \partial \Omega.
\end{array} \right.
\]

Then, there exists $p_1 := \frac{2d}{d-1} + \theta$ with $\theta > 0$ depending only on $\Omega$, such that for all $p < p_1$ we have

\[
\left< \left( \int_\Omega |u_\varepsilon - \bar{u}|^2 \right)^{\frac{p}{2}} \right>^{\frac{1}{p}} \lesssim \mu_d(R_0/\varepsilon) \varepsilon \ln(R_0/\varepsilon) \left\{ \|f\|_{L^2(\Omega)} + \|b\|_{H^1(\partial \Omega)} \right\}.
\]
Lemma 4.2 (random cancellation). Let $\Omega \subset \mathbb{R}^d$ be a subset with $d \geq 2$ (not necessary to be bounded). For any $\beta < \infty$, assume that the random field $\varpi$ satisfies
\[
\left\langle \left( \int_{B_1(x)} |\varpi|^2 \right)^{\frac{d}{2}} \right\rangle^{\frac{1}{\beta}} \leq c_0 \mu_d(x) \tag{4.3}
\]
Then, for any $2 \leq p, q < \infty$, $f \in C_0^\infty(\Omega_\varepsilon)$ where $\Omega_\varepsilon := \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \varepsilon \}$, we have
\[
\int_{\Omega} \left\langle \left( \int_{D_\varepsilon} |\varpi\xi S_\varepsilon(f)|^2 \right)^{\frac{d}{2}} \right\rangle^{\frac{1}{p}} \lesssim_{d, c_0} \int_{\Omega} |f|^q \mu_d'(\cdot / \varepsilon); \tag{4.4a}
\]
\[
\int_{\Omega} \left\langle \left( \int_{D_\varepsilon(x)} |\varpi\xi S_\varepsilon(f)|^2 \right)^{\frac{d}{2}} \delta^{\pm 1}(x) dx \right\rangle \lesssim_{d, c_0} \int_{\Omega} |f|^2 \mu_d^2(\cdot / \varepsilon) \delta^{\pm 1}, \tag{4.4b}
\]
and for some local integrable function $g$ involving randomness such that
\[
\left\langle \left| \int_{\Omega} \text{g}_\varepsilon S_\varepsilon(f) \right|^p \right\rangle^{\frac{1}{p}} \lesssim_{d, c_0} \int_{\Omega} \left\langle S_\varepsilon(|g|^2)^{\frac{d}{2}} \right\rangle^{\frac{1}{\bar{p}}} |f| \mu_d(\cdot / \varepsilon). \tag{4.5}
\]
holds for $\bar{p} > p$, where $S_\varepsilon$ is a smoothing operator$^7$.

Proof. The proof is straightforward, and we provide a proof for the sake of completeness. We start from
\[
\int_{D_\varepsilon(x)} |\varpi\xi S_\varepsilon(f)|^2 \leq \int_{D_\varepsilon(x)} |\varpi\xi|^2 S_\varepsilon(|f|^2) \leq \int_{D_\varepsilon(x)} S_\varepsilon(|\varpi\xi|^2) |f|^2, \tag{4.6}
\]
and by the geometry property of integrals we further derive that
\[
\int_{\Omega} \left\langle \left( \int_{D_\varepsilon(x)} |\varpi\xi S_\varepsilon(f)|^2 \right)^{\frac{d}{2}} \right\rangle^{\frac{1}{p}} \leq \left( \int_{\Omega} \left\langle \left( \int_{D_\varepsilon(x)} S_\varepsilon(|\varpi\xi|^2) |f|^2 \right)^{\frac{d}{2}} \right\rangle \right)^{\frac{1}{p}} \tag{4.7}
\]
\[
\leq \int_{\Omega} \int_{D_\varepsilon(x)} |f(z)|^2 \left\langle \left( \int_{B_\varepsilon(z)} |\varpi\xi|^2 \right)^{\frac{d}{2}} \right\rangle^{\frac{1}{p}} \delta^{\pm 1}(x) dx \leq \int_{\Omega} |f(x)|^2 \left\langle \left( \int_{B_\varepsilon(x)} |\varpi\xi|^2 \right)^{\frac{d}{2}} \right\rangle^{\frac{1}{p}} \int_{D_\varepsilon(x)} \delta^{\pm 1}(x) dx \tag{4.8}
\]
To present the proof of (4.4b), by Fubini’s theorem we merely modify (4.7) by the following computation
\[
\int_{\Omega} \int_{D_\varepsilon(x)} |f(z)|^2 \left\langle \left( \int_{B_\varepsilon(z)} |\varpi\xi|^2 \right)^{\frac{d}{2}} \right\rangle^{\frac{1}{p}} \delta^{\pm 1}(x) dx \leq \int_{\Omega} |f(x)|^2 \left\langle \left( \int_{B_\varepsilon(x)} |\varpi\xi|^2 \right)^{\frac{d}{2}} \right\rangle^{\frac{1}{p}} \int_{D_\varepsilon(x)} \delta^{\pm 1}(x) dx \tag{4.8}
\]
We now proceed to show the estimate (4.5). Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and Minkowski’s inequality and H"older’s inequality lead to
\[
\left\langle \left| \int_{\Omega} \text{g}_\varepsilon S_\varepsilon(f) \right|^p \right\rangle^{\frac{1}{p}} \leq \int_{\Omega} \left\langle |S_\varepsilon(|g\varpi\xi|)|^p \right\rangle^{\frac{1}{p}} |f|
\]
\[
\leq \int_{\Omega} \left\langle \left( \int_{D_\varepsilon(x)} |g|^2 \right)^{\frac{d}{2p_1}} \right\rangle^{\frac{1}{p_1}} \left\langle \left( \int_{B_\varepsilon(x)} |\varpi\xi|^2 \right)^{\frac{d}{2p_2}} \right\rangle^{\frac{1}{p_2}} |f| \lesssim_{d, c_0} \int_{\Omega} \left\langle \left( \int_{D_\varepsilon(x)} |g|^2 \right)^{\frac{d}{p_1}} \right\rangle^{\frac{1}{p_1}} |f| \mu_d(1/\varepsilon),
\]
which finally implies the stated estimate (4.5). This ends the proof. \[\Box\]

\[\tag{4.8}\]

$^7$The definition of $S_\varepsilon$ can be found in (2.24) by setting $\chi_* \equiv 1$ therein.
Lemma 4.3 (errors in $H^1$-norm). Assume the same conditions as in Proposition 4.1. Let $u_\varepsilon, u_0 \in H^1_0(\Omega)$ be associated with $f \in L^2(\Omega)$ and $b \in H^1(\partial \Omega)$ by the equations (4.1), respectively. Let $\eta \in C^1_0$ be a cut-off function satisfying (4.9), and we consider the following error term

$$z_\varepsilon = u_\varepsilon - \bar{u} - \varepsilon \phi_i \varphi_i \quad \text{with} \quad \varphi_i = S_\varepsilon(\eta \partial_i \bar{u}).$$

Then, there exists $p_1 := \frac{2d + 1}{d - 1} + \theta$ with $\theta > 0$ depending only on $\Omega$, such that for all $p < p_1$ we have

$$\left( \int_\Omega \left( \int_{D_\varepsilon(x)} |\nabla z_\varepsilon|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \lesssim \mu_{d}(R_0/\varepsilon) \varepsilon^{\frac{1}{2}} \left\{ ||f||_{L^2(\Omega)} + ||b||_{H^1(\partial \Omega)} \right\},$$

where the up to constant is independent of $\varepsilon$.

Proof. By using the antisymmetry of flux corrector and the equations (4.1), the routine computation leads to

$$- \nabla \cdot a^\varepsilon \nabla z_\varepsilon = \nabla \cdot \left[ \varepsilon(a^\varepsilon \phi_i^\varepsilon \sigma_i^\varepsilon) \nabla \varphi_i - (a^\varepsilon - \bar{a})(\nabla u_\varepsilon - \varphi) \right] \quad \text{in} \quad \Omega,$$

with $z_\varepsilon = 0$ on $\partial \Omega$, where $\varphi = (\varphi_1, \cdots, \varphi_d)$. Therefore, with the help of Proposition 3.6, we have

$$\int_\Omega \left( \int_{D_\varepsilon(x)} |\nabla z_\varepsilon|^2 \right)^{\frac{p}{2}} \lesssim (3.37) \text{or} (3.40) \int_\Omega \left( \int_{D_\varepsilon(x)} |\varepsilon(\phi_i^\varepsilon, \sigma_i^\varepsilon) \nabla \varphi_i|^2 \right)^{\frac{p}{2}} + \int_\Omega \left( \int_{D_\varepsilon(x)} |\nabla u - \varphi|^2 \right)^{\frac{p}{2}} := I_1 + I_2,$$

where $\tilde{p} > p$. On the random part $I_1$, appealing to Lemma 4.2, we have

$$I_1 = \int_\Omega \left( \int_{D_\varepsilon(x)} |\varepsilon(\phi_i^\varepsilon, \sigma_i^\varepsilon) \nabla S_\varepsilon(\eta \partial \bar{u})|^2 \right)^{\frac{p}{2}} \lesssim (4.4a) \varepsilon^2 \int_\Omega |\nabla(\eta \partial \bar{u})|^2 \mu_{d}^2(\cdot/\varepsilon).$$

On the deterministic part $I_2$, we have

$$I_2 = \int_\Omega \left( \int_{D_\varepsilon(x)} |\nabla \bar{u} - \varphi|^2 \right)^{\frac{p}{2}} \leq \int_\Omega |\nabla \bar{u} - \varphi|^2 \leq \int_{\Omega \setminus \Sigma_\varepsilon} |\nabla \bar{u}|^2 + \int_{\Sigma_\varepsilon} |\eta \nabla \bar{u} - S_\varepsilon(\eta \nabla \bar{u})|^2$$

$$\lesssim \int_{\Omega \setminus \Sigma_\varepsilon} |\nabla \bar{u}|^2 + \varepsilon^2 \int_{\Sigma_\varepsilon} |\nabla^2 \bar{u}|^2.$$

Consequently, combining the above estimates we have

$$\int_\Omega \left( \int_{D_\varepsilon(x)} |\nabla z_\varepsilon|^2 \right)^{\frac{p}{2}} \lesssim (4.13), (4.14), (4.15) \mu_{d}^2(R_0/\varepsilon) \int_{\Omega \setminus \Sigma_\varepsilon} |\nabla \bar{u}|^2 + \varepsilon^2 \mu_{d}^2(R_0/\varepsilon) \int_{\Sigma_\varepsilon} |\nabla^2 \bar{u}|^2.$$
In this regard, the problem is reduce to the following “layer” and “co-layer” type estimates (which had been introduced in [38], inspired by Shen’s work [33]. For the ease of the statement, it is fine to assume $\|f\|_{L^2(\Omega)} + \|b\|_{H^1(\partial\Omega)} = 1$):

\[
\begin{align*}
\|\nabla \tilde{u}\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\frac{1}{2}} \|\nabla \tilde{u}\delta \frac{1}{2}\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|\nabla^2 \tilde{u}\|_{L^2(\Omega, \partial \Omega_\varepsilon)} & \lesssim \varepsilon^{-\frac{1}{2}}; \\
\|\nabla \tilde{u}\delta \frac{1}{2}\|_{L^2(\Omega, \partial \Omega_\varepsilon)} + \|\nabla^2 \tilde{u}\delta \frac{1}{2}\|_{L^2(\Omega, \partial \Omega_\varepsilon)} & \lesssim (R_0/\varepsilon).
\end{align*}
\] (4.17)

Plugging the estimates (4.17) back into (4.16)

\[
\int_{\Omega} \left( \int_{D_{\varepsilon}(x)} |\nabla \zeta_{\varepsilon}|^2 \right)^{\frac{2}{p}} \lesssim (4.17) \mu_d(R_0/\varepsilon) \varepsilon,
\]

which offers us the desired estimate (4.11).

**Proof of Proposition 4.1.** The main idea is similar to that given in [38], and with the help of Lemma 4.2 the proof is shorter. We will finish the whole argument by three steps.

**Step 1.** Construct the auxiliary equations and simplify the problem. Given arbitrary $g \in C_0^\infty(\Omega)$, we consider the following the adjoint problems:

\[
\begin{align*}
- \nabla \cdot a^\varepsilon \nabla v_\varepsilon &= g & \text{in} & \Omega; \\
\frac{\partial v_\varepsilon}{\partial \mathbf{n}} &= 0 & \text{on} & \partial \Omega; \\
v_\varepsilon &= 0 & \text{on} & \partial \Omega.
\end{align*}
\] (4.18)

We denoted the related homogenization error term by $\tilde{z}_\varepsilon := v_\varepsilon - \bar{v} - \varepsilon \phi_j^\varepsilon S^j(\tilde{\eta}\partial_j \bar{v})$, and have

\[
\nabla v_\varepsilon &= \nabla \tilde{z}_\varepsilon + \nabla \bar{v} + \nabla \phi_j^\varepsilon S^j(\tilde{\eta}\partial_j \bar{v}) + \varepsilon \phi_j^\varepsilon \nabla S^j(\tilde{\eta}\partial_j \bar{v}),
\] (4.19)

where $\tilde{\eta}$ is the cut-off function defined in (4.9). Moreover, recalling the error term (4.10), it follows from the equations (4.12) and (4.18) that

\[
\int_{\Omega} z_\varepsilon g = - \int_{\Omega} \nabla v_\varepsilon \cdot \varepsilon (a^\varepsilon \phi_i^\varepsilon - \sigma_i^\varepsilon) \nabla \varphi_i - \int_{\Omega} \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a})(\nabla \bar{u} - \varphi).
\]

By the linearity of the equations, one may assume $\|g\|_{L^2(\Omega)} = 1$ and $\|f\|_{L^2(\Omega)} + \|b\|_{H^1(\partial \Omega)} = 1$. Thus, the desired estimate (4.2) immediately follows from these two estimates below, the duality argument and triangle inequality: For any $p < \infty$, (it suffices to show the case $p > 2$ while the case $p < 2$ follows trivially.)

\[
\begin{align*}
\left\langle \int_{\Omega} \nabla v_\varepsilon \cdot \varepsilon (a^\varepsilon \phi_i^\varepsilon - \sigma_i^\varepsilon) \nabla \varphi_i \right\rangle^{\frac{1}{2}} & \lesssim \mu_d(R_0/\varepsilon) \varepsilon \ln(R_0/\varepsilon). \quad (4.20a) \\
\left\langle \int_{\Omega} \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a})(\nabla \bar{u} - \varphi) \right\rangle^{\frac{1}{2}} & \lesssim \varepsilon \ln(R_0/\varepsilon). \quad (4.20b)
\end{align*}
\]

**Step 2.** Arguments for (4.20a). Set $p_1 > p$ throughout the proof. The main idea is to accelerate the convergence rate by appealing to the constructed error term (4.19). Therefore, plugging the formula (4.19) into the left-hand side of (4.20a), we have four terms in hand. Basically, we mainly employ the random cancellation results given in Lemma 4.2. Together with the properties of cut-off functions
\( \eta, \tilde{\eta} \) defined in (4.9), we reduce the error estimate to the “layer” and “co-layer” type estimates (4.17). Replacing \( \nabla v_\varepsilon \) with \( \nabla \bar{v} \) in the left-hand side of (4.20a), by using Lemmas 4.2 and 4.3, we first obtain

\[
\left\langle \int_\Omega \nabla \bar{v} \cdot (a^\varepsilon \phi^\varepsilon_t - \sigma^\varepsilon_t) \nabla \varphi_t \right\rangle^{\frac{p}{2}} \lesssim \varepsilon \int_\Omega S_\varepsilon (|\nabla \bar{v}|^2) \frac{\varepsilon}{\mu_d(\cdot/\varepsilon)} \int_\Omega \mu_d(\cdot/\varepsilon) \mu_d(\cdot/\varepsilon) \varepsilon.
\]

Replacing \( \nabla v_\varepsilon \) with \( \nabla \bar{v} \) in the left-hand side of (4.20a), and by noting that the distance function \( \delta \) can exchange with the smoothing operator (see [38, Lemma 3.2]), we then have

\[
\left\langle \int_\Omega \nabla \bar{v} \cdot (a^\varepsilon \phi^\varepsilon_t - \sigma^\varepsilon_t) \nabla \varphi_t \right\rangle^{\frac{p}{2}} \lesssim \varepsilon \int_\Omega S_\varepsilon (|\nabla \bar{v}|^2) \frac{\varepsilon}{\mu_d(\cdot/\varepsilon)} \int_\Omega \mu_d(\cdot/\varepsilon) \delta^{-1}(\varepsilon) \varepsilon.
\]

Replacing \( \nabla v_\varepsilon \) with \( \nabla \phi^\varepsilon_j S_\varepsilon(\tilde{\eta} \partial_j \bar{v}) \) in the left-hand side of (4.20a), by repeatedly using of Lemma 4.2, we arrive at

\[
\left\langle \int_\Omega \nabla \phi^\varepsilon_j S_\varepsilon(\tilde{\eta} \partial_j \bar{v}) \cdot (a^\varepsilon \phi^\varepsilon_t - \sigma^\varepsilon_t) \nabla \varphi_t \right\rangle^{\frac{p}{2}} \lesssim \varepsilon \int_\Omega \mu_d(\cdot/\varepsilon) \varepsilon \ln(R_0/\varepsilon).
\]

Replacing \( \nabla v_\varepsilon \) with \( \varepsilon \phi^\varepsilon_j S_\varepsilon(\nabla (\tilde{\eta} \partial_j \bar{v})) \) in the left-hand side of (4.20a), by the same token, we derive that

\[
\left\langle \int_\Omega \varepsilon \phi^\varepsilon_j S_\varepsilon(\nabla (\tilde{\eta} \partial_j \bar{v})) \cdot (a^\varepsilon \phi^\varepsilon_t - \sigma^\varepsilon_t) \nabla \varphi_t \right\rangle^{\frac{p}{2}} \lesssim \varepsilon \int_\Omega \mu_d(\cdot/\varepsilon) \varepsilon \ln(R_0/\varepsilon).
\]

**Step 3.** Arguments for (4.20b). With the help of the cut-off function \( \eta \), we can first decompose the left-hand side of (4.20b) into “layer” and “co-layer” parts:

\[
\int_\Omega \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a}) (\nabla \bar{u} - \varphi) = \int_\Omega \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a}) \nabla \bar{u}(1 - \eta) + \int_\Omega \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a}) (\eta \nabla \bar{u} - S_\varepsilon(\eta \nabla \bar{u})).
\]

We start from addressing layer parts. By noting that \( \text{supp}(\tilde{\eta}) \cap \text{supp}(\nabla \eta) = \emptyset \), when we plug the formula (4.19) into the first term in the right-hand side above, from Hölder’s inequality, it follows that

\[
\int_\Omega \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a}) \nabla \bar{u}(1 - \eta) = \int_\Omega \nabla \bar{v} \cdot (a^\varepsilon - \bar{a}) \nabla \bar{u}(1 - \eta) \lesssim \left( \int_\Omega |\nabla z_\varepsilon|^2 \right)^{\frac{1}{2}} + \left( \int_{O_{4\varepsilon}} |\nabla \bar{v}|^2 \right)^{\frac{1}{2}} \left( \int_{O_{4\varepsilon}} |\nabla \bar{u}|^2 \right)^{\frac{1}{2}}.
\]

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Only the term containing \( z_\varepsilon \) involves randomness, and appealing to geometry property of integral and Lemma 4.3 we have

\[
\left\langle \int_\Omega \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a}) \nabla \bar{u} (1 - \eta) \right\rangle^\frac{1}{p} \leq \frac{1}{p} \left( \int_\Omega \nabla z_\varepsilon (1 - \eta) \right)^\frac{1}{p} \leq \frac{1}{p} \left( \int_\Omega \nabla \bar{u} (1 - \eta) \right)^\frac{1}{p} \leq (4.11), (4.17) \mu_d(R_0 / \varepsilon) \varepsilon. \tag{4.24}
\]

We now turn to dealing the second term in the right-hand side of (4.23) (i.e., the co-layer part). By plugging the formula (4.19) into it we have

\[
\int_\Omega \nabla v_\varepsilon \cdot (a^\varepsilon - \bar{a}) (\eta \nabla \bar{u} - S_\varepsilon (\eta \nabla \bar{u})) = \int_\Omega (\nabla z_\varepsilon + \nabla \bar{v} + \nabla \phi_j^\varepsilon S_\varepsilon (\tilde{\eta} \partial_j \bar{v}) + \varepsilon \phi_j^\varepsilon \nabla S_\varepsilon (\tilde{\eta} \partial_j \bar{v})) \cdot (a^\varepsilon - \bar{a}) (\eta \nabla \bar{u} - S_\varepsilon (\eta \nabla \bar{u})) =: I_1 + I_2 + I_3 + I_4. \tag{4.25}
\]

Then, following a similar computation we can obtain the related estimates term by term, and the only difference here is that we will employ the following estimates (see [33, Lemma 2.2] and [38, Lemma 3.3]):

\[
\int_{\Omega \setminus O_{2\varepsilon}} |f - S_\varepsilon(f)|^2 \lesssim \varepsilon^2 \int_{\Omega \setminus O_{2\varepsilon}} |\nabla f|^2; \tag{4.26a}
\]

\[
\int_{\Omega \setminus O_{2\varepsilon}} |f - S_\varepsilon(f)|^2 \delta \lesssim \varepsilon^2 \int_{\Omega \setminus O_{2\varepsilon}} |\nabla f|^2 \delta. \tag{4.26b}
\]

A routine computation leads to

\[
\left\langle I_1^p \right\rangle^\frac{1}{p} \lesssim \frac{1}{p} \left( \int_\Omega \nabla |\nabla \bar{u} - S_\varepsilon (\eta \nabla \bar{u})| \right)^\frac{1}{p} \lesssim (4.26a) \varepsilon \left( \int_\Omega |\nabla \eta \nabla \bar{u}|^2 \right)^\frac{1}{p} \left( \int_\Omega \nabla z_\varepsilon (1 - \eta) \right)^\frac{1}{p} \lesssim (4.17), (4.11) \mu_d(R_0 / \varepsilon) \varepsilon. \tag{4.27}
\]

\[
\left\langle I_2^p \right\rangle^\frac{1}{p} \lesssim \left( \int_{\text{supp}(\eta)} |\nabla \bar{v}|^2 \frac{dx}{\delta(x)} \right)^\frac{1}{p} \left( \int_\Omega |\eta \nabla \bar{u} - S_\varepsilon (\eta \nabla \bar{u})|^2 \delta(x) dx \right)^\frac{1}{p} \lesssim (4.26b), (4.17) \ln(R_0 / \varepsilon) \varepsilon. \tag{4.28}
\]

In terms of \( I_3 \), by noting the geometry property of integrals and commutation between distance function and average operators (i.e., \( \delta \sim S_\varepsilon(\delta) \) on \( \Omega \setminus O_{2\varepsilon} \) whose proof can be found in [38, Lemma 3.1]), we first arrive at

\[
I_3 \lesssim \frac{1}{p} \left( \int_\Omega |\nabla \eta \nabla \bar{u}|^2 \delta(x) dx \right)^\frac{1}{p} \left( \int_\Omega \nabla \phi_j^\varepsilon S_\varepsilon (\tilde{\eta} \partial_j \bar{v}) |^2 \delta^{-1}(x) dx \right)^\frac{1}{p},
\]

which, together with Minkowski’s inequality, then offers us

\[
\left\langle I_3^p \right\rangle^\frac{1}{p} \lesssim (4.4b) \varepsilon \left( \int_\Omega |\nabla \eta \nabla \bar{u}|^2 \delta(x) dx \right)^\frac{1}{p} \left( \int_\Omega |\tilde{\eta} \nabla \bar{u}|^2 \delta^{-1}(x) dx \right)^\frac{1}{p} \lesssim (4.17) \ln(R_0 / \varepsilon) \varepsilon. \tag{4.28}
\]

In terms of \( I_4 \), we begin from

\[
I_4 \lesssim (4.26a) \varepsilon \left( \int_\Omega |\nabla \eta \nabla \bar{u}|^2 \right)^\frac{1}{p} \left( \int_\Omega \nabla \phi_j^\varepsilon S_\varepsilon (\tilde{\eta} \partial_j \bar{v}) |^2 \right)^\frac{1}{p},
\]
which leads to
\[
\langle |I_4|^p \rangle \lesssim \varepsilon^2 \left( \int_\Omega |\nabla (\eta \nabla \tilde{u})|^2 \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla (\hat{\eta} \partial_j \tilde{v})|^2 \mu^2_{\alpha}(\cdot) / \varepsilon \right)^{\frac{1}{2}} \lesssim \mu_d(R_0 / \varepsilon) \varepsilon. \tag{4.29}
\]

Combining the above estimates (4.25), (4.27), (4.28) and (4.29), we have
\[
\langle | \int_\Omega \nabla v_{\varepsilon} \cdot (a^\varepsilon - \bar{a})(\eta - S_\varepsilon(\eta \nabla \tilde{u})) \rangle \rangle \lesssim \mu_d(R_0 / \varepsilon) \varepsilon,
\]
and this together with (4.24) finally gives the desired estimate (4.20b). This ends the whole proof. \qed

4.2 Weak norms

Proposition 4.4. Let \( \Omega \) be a bounded regular-SKT (or \( C^1 \)) domain and \( \varepsilon \in (0,1] \). Suppose that \( \langle \cdot \rangle \) is stationary, satisfying the spectral gap condition (1.2), as well as (1.3). Let \( u_\varepsilon, u_0 \in H_0^1(\Omega) \) be associated with \( f \in C_0^1(\Omega; \mathbb{R}^d) \) by
\[
\begin{aligned}
-\nabla \cdot a^\varepsilon \nabla u_\varepsilon &= \nabla \cdot f \quad \text{in } \Omega; \\
u_\varepsilon &= 0 \quad \text{on } \partial \Omega; \\
-\nabla \cdot \bar{a} \nabla \tilde{u} &= \nabla \cdot f \quad \text{in } \Omega; \\
\tilde{u} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\tag{4.30}
\]
For any \( h \in C_0^\infty(\Omega; \mathbb{R}^d) \), the random variable \( H^\varepsilon \) is defined as in (1.12). Then for all \( p < \infty \) we have
\[
\varepsilon^{-\frac{d}{2}} \langle (\langle H^\varepsilon - \langle H^\varepsilon \rangle \rangle)^{2p} \rangle \lesssim \varepsilon \mu_d(R_0 / \varepsilon) \ln \frac{\mu_d(R_0 / \varepsilon)}{\varepsilon} \left( \int_\Omega |\nabla h|^2 \right)^{\frac{1}{2p}} \left( \int_\Omega |R_0 \nabla f|^{2s} \right)^{\frac{1}{2s}},
\tag{4.31}
\]
where \( s, s' > 1 \) with \( 1/s' + 1/s = 1 \).

Proof. The main idea is appealing to sensitive arguments, which is similar to that given as in [27, Proposition 6.1], while the present contribution is to establish fluctuation property of the two-scale expansion error with a bounded domain considered. Let \( s, s' > 1 \) satisfy \( 1/s + 1/s' = 1 \), and the proof is divided into four steps.

Step 1. Representation of the functional derivative of \( H^\varepsilon \) and reduction. Let \( \hat{h}, \hat{f}, \tilde{v} \) represent the different rescale way of \( h, f, v \), as follows:
\[
\hat{h} := \varepsilon^d h(\varepsilon \cdot); \quad \hat{f} := f(\varepsilon \cdot); \quad \tilde{v} := \frac{1}{\varepsilon} v(\varepsilon \cdot).
\tag{4.32}
\]
Thus, in view of the equations (4.30) we can rewrite it as into
\[
\begin{aligned}
-\nabla \cdot a \nabla \hat{u} &= \nabla \cdot \hat{f} \quad \text{in } \Omega / \varepsilon; \\
\hat{u} &= 0 \quad \text{on } \partial \Omega / \varepsilon; \\
-\nabla \cdot \bar{a} \nabla \tilde{u} &= \nabla \cdot \tilde{f} \quad \text{in } \Omega / \varepsilon; \\
\tilde{u} &= 0 \quad \text{on } \partial \Omega / \varepsilon.
\end{aligned}
\]
We now fix \( \varphi_i = \eta \partial_i \tilde{u} \), where \( \eta \in C_0^1(\Omega) \) is a cut-off function satisfying (4.9), and it follows from the fact \( \nabla \tilde{v} = \nabla v \) that
\[
\hat{\varphi}_i = \hat{\eta} \partial_i \hat{u}; \quad \tilde{w} = \hat{u} - \tilde{u} - \phi_i \hat{\varphi}_i; \quad \text{and} \quad \nabla \tilde{w} = \nabla w_\varepsilon.
\tag{4.33}
\]
Therefore, the error of two-scale expansions satisfies
\[
-\nabla \cdot a \nabla \tilde{w} = \nabla \cdot [(a \phi_i - \sigma_i) \nabla \hat{\varphi}_i + (a - \bar{a})(1 - \hat{\eta}) \nabla \tilde{u}] \quad \text{in } \Omega / \varepsilon,
\]
with \( \tilde{w} = 0 \) on \( \partial \Omega / \varepsilon \). We now denote the random variable by

\[
H_\varepsilon := \int_{\Omega / \varepsilon} \tilde{h} \cdot (a - \bar{a}) \left( \nabla \bar{u} - \nabla \bar{u} - \nabla \phi_i \hat{\phi}_i \right).
\]

and we in fact have \( H_\varepsilon = H^e \) by changing variable in (1.12). The advantage of the expression of \( H_\varepsilon \) is that we can directly appeal to the \( \varepsilon \)-version spectral gap inequality\(^8\)

\[
\left( \langle H_\varepsilon - \langle H_\varepsilon \rangle \rangle^{2p} \right)^{\frac{1}{p}} \leq \left( \int_{\mathbb{R}^d} \left( \int_{B_1(y)} \left| \frac{\partial H_\varepsilon}{\partial a} \right|^2 dy \right)^p \right)^{\frac{1}{p}} \tag{4.34}
\]

to obtain the desired estimate (4.31), whenever we have the concrete expression of \( \frac{\partial H_\varepsilon}{\partial a} \).

To do so, we construct auxiliary equations:

\[
\begin{aligned}
- \nabla \cdot a^* \nabla \tilde{v}^* &= \nabla \cdot (a^* \phi_j^* - \sigma_j^*) \nabla \tilde{h}_j & \text{in } \Omega / \varepsilon; \\
v^* &= 0 & \text{on } \partial \Omega / \varepsilon;
\end{aligned}
\tag{4.35}
\]

and

\[
- \nabla \cdot a^* \nabla z_j^* = \nabla \cdot (a^* \phi_i^* - \sigma_i^*) \nabla (\tilde{h}_i \hat{\varphi}_j) & \text{in } \mathbb{R}^d.
\]

Then, a routine computation leads to

\[
\frac{\partial H_\varepsilon}{\partial a} = \tilde{h}_j (e_j + \nabla \phi_j^* \otimes (\nabla \tilde{w} + \phi_i \nabla \hat{\phi}_i)) + \nabla \phi_j^* \tilde{h}_j \otimes (\nabla \bar{u} - \bar{\varphi}) \\
- \left[ \nabla z_j^* + \phi_i^* \nabla (\tilde{h}_i \hat{\phi}_j) \right] \otimes (\nabla \phi_j + e_j) + (\nabla \tilde{v}^* + \phi_j^* \nabla \tilde{h}_j) \otimes \nabla \bar{u},
\tag{4.36}
\]

and the details may be found in [27, pp.31-32] and left to the reader. To complete the whole proof, we need to establish the following estimates:

\[
\left\langle \left( \int_{\mathbb{R}^d} \left( \int_{B_1(y)} \left| \tilde{h}_j (e_j + \nabla \phi_j^* \otimes (\nabla \tilde{w} + \phi_i \nabla \hat{\phi}_i)) \right|^2 dy \right)^p \right)^{\frac{1}{p}} \right\rangle \leq \varepsilon^{d+2} \mu_2^2 (R_0 / \varepsilon) \ln^{\frac{1}{2}} (R_0 / \varepsilon) (\int_{\Omega} |\nabla \tilde{h}|^{2s} dx)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla f|^{2s'} dx \right)^{\frac{1}{s'}}, \tag{4.37a}
\]

\[
\left\langle \left( \int_{\Omega / \varepsilon} \left( \int_{B_1(y)} \left| \nabla \phi_j^* \tilde{h}_j \otimes (\nabla \bar{u} - \bar{\varphi}) \right|^2 dy \right)^p \right)^{\frac{1}{p}} \right\rangle \leq \varepsilon^{d+2} \left( \int_{\Omega} |\nabla \tilde{h}|^{2s} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla f|^{2s'} dx \right)^{\frac{1}{s'}}, \tag{4.37b}
\]

where the constant is independent of \( \varepsilon \), and

\[
\left\langle \left( \int_{\Omega} \left( \int_{B_1(x)} \left| (\varepsilon^{-\frac{d}{2}} \nabla \tilde{v}^* + \varepsilon^{1+\frac{d}{2}} \phi_j^* \nabla \tilde{h}_j) \otimes \nabla u_\varepsilon \right|^2 dx \right)^p \right)^{\frac{1}{p}} \right\rangle \leq \varepsilon^{d+2} \mu_d^2 (R_0 / \varepsilon) (\int_{\Omega} |\nabla \tilde{h}|^{2s} dx)^{\frac{1}{2}} \left( \int_{\Omega} |f|^{2s'} dx \right)^{\frac{1}{s'}}, \tag{4.38a}
\]

\(^{8}\)It can be derived from the spectral condition (1.2) see for example [27, pp.17-18].
\[
\left\langle \left( \int_{\mathbb{R}^{2}} \left( \int_{B_{1}(y)} \left( (\nabla z_{j}^{*} + \phi_{j}^{*}(h_{j}\phi_{j}^{*})) \otimes (\nabla \phi_{j} + e_{j}) \right)^{2} dy \right) \right)^{\frac{1}{p}} \right\rangle^{q} \leq \varepsilon^{d+2} \mu_{2}(R_{0}/\varepsilon) \left( \int_{\Omega} |\nabla h|^{2s} \right)^{\frac{1}{2}} + \ln^{2} (R_{0}/\varepsilon) \left( \int_{\Omega} |R_{0} \nabla f|^{2s} \right)^{\frac{1}{2}}. \tag{4.38b} \]

Admitting them for a while, collecting the estimates (4.34), (4.36), (4.37a), (4.37b), (4.38a) and (4.38b) leads to the desired estimate (4.31).

**Step 2.** As a preparation, we need to show the following layer and co-layer type estimates:

\[
\begin{align*}
\int_{\Omega_{s}} |\nabla \bar{u}|^{2s'} &\leq \varepsilon R_{0}^{2s'-1} \int_{\Omega} |\nabla f|^{2s'}, \tag{4.39a} \\
\int_{\Omega_{s} \setminus \Omega_{c}} |\nabla^{2} \bar{u}|^{2s'}(x) dx &\leq \ln(\varepsilon R_{0}^{-1}) R_{0}^{2s'-1} \int_{\Omega} |\nabla f|^{2s'}. \tag{4.39b}
\end{align*}
\]

The original idea of proof is similar to that given in [33], and we provide a proof for the reader’s convenience. Due to the non-smoothness of the boundary, we first divide the effective equation of (4.30) into

\[
\begin{align*}
(1) &\quad - \nabla \cdot \bar{a} \nabla \bar{u}^{(1)} = f_{0} \quad \text{in} \quad \mathbb{R}^{d}, \\
(2) &\quad \nabla \cdot \bar{a} \nabla \bar{u}^{(2)} = 0 \quad \text{in} \quad \Omega; \quad \bar{u}^{(2)} = -\bar{u}^{(1)} \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \(f_{0}\) is the zero-extension of \(\nabla \cdot f\) to the whole space \(\mathbb{R}^{d}\). It is not hard to see that \(\bar{u} = \bar{u}^{(1)} + \bar{u}^{(2)}\) on \(\overline{\Omega}\), and our later analysis is based upon this decomposition.

We first address the estimate (4.39a), and

\[
\int_{\Omega_{c}} |\nabla \bar{u}|^{2s'} \leq \int_{\Omega_{c}} |\nabla \bar{u}^{(1)}|^{2s'} + \int_{\Omega_{c}} |\nabla \bar{u}^{(2)}|^{2s'}. \tag{4.41}
\]

For any \(t \in [0, 2\varepsilon]\), one may define \(S_{t} := \Gamma_{t}(\partial \Omega)\) and \(\Gamma_{t}(x') := x' - tn(x')\) for a.e. \(x' \in \partial \Omega\) and \(n\) is the outward unit normal vector associated with \(\partial \Omega\). There exists a vector field \(\rho \in C^{1}(\Omega; \mathbb{R}^{d})\) such that \(\rho \cdot n \geq c_{0}\) on \(S_{t}\) for each \(t \in [0, 2\varepsilon]\), as well as \(|\nabla \rho| \lesssim 1/R_{0}\), where \(c_{0} > 0\). It follows the divergence theorem and Hölder’s inequality,

\[
\int_{S_{t}} |\nabla \bar{u}^{(1)}|^{2s'} dS \leq \frac{1}{c_{0}} \int_{S_{t}} n \cdot \rho |\nabla \bar{u}^{(1)}|^{2s'} dS \lesssim \frac{1}{R_{0}} \int_{\Omega} |\nabla \bar{u}^{(1)}|^{2s'} + \int_{\Omega} |\nabla \bar{u}^{(1)}|^{2s'-1} |\nabla^{2} \bar{u}^{(1)}| \lesssim \left( \int_{\Omega} |\nabla^{2} \bar{u}^{(1)}|^{\frac{2s'd}{2s'd-1}} \right)^{\frac{2s'd-1}{2s'd}} + \left( \int_{\Omega} |\nabla^{2} \bar{u}^{(1)}|^{\frac{2s'd}{2s'd-1}} \right)^{\frac{2s'd-1}{2s'd}}. \tag{4.42}
\]

Then, by boundedness of Riesz potential and singular integrals (see for example [16, Chapter 7]), we have

\[
\left( \int_{\mathbb{R}^{d}} |\nabla \bar{u}^{(1)}|^{\frac{2s'd}{d}} \right)^{\frac{d}{2s'+d-1}} \lesssim \left( \int_{\Omega} |\nabla f|^{\frac{2s'd}{2s'+d-1}} \right)^{\frac{2s'+d-1}{d}}; \tag{4.43a}
\]

\[
\int_{\mathbb{R}^{d}} |\nabla^{2} \bar{u}^{(1)}|^{q} \lesssim q \int_{\Omega} |\nabla f|^{q} \quad \forall \; 1 < q < \infty. \tag{4.43b}
\]

(Here we merely employ the case \(q = \frac{2s'd}{2s'+d-1}\).) Plugging the estimates (4.43a) and (4.43b) back into (4.42), there holds

\[
\int_{S_{t}} |\nabla \bar{u}^{(1)}|^{2s'} dS \lesssim \left( \int_{\Omega} |\nabla f|^{\frac{2s'd}{2s'+d-1}} \right)^{\frac{2s'+d-1}{d}} \leq CR_{0}^{2s'-1} \int_{\Omega} |\nabla f|^{2s'}. \tag{4.44}
\]
for any $t \in [0, 2\varepsilon]$. By co-area formula, we have that

$$
\int_{O_\varepsilon} |\nabla \bar{u}^{(1)}|^{2s'} \lesssim \varepsilon \max_{t \in [0,2\varepsilon]} \int_{\partial S_t} |\nabla \bar{u}^{(1)}|^{2s'} dS \lesssim \varepsilon R_0^{2s'-1} \int_{\Omega} |\nabla f|^{2s'}.
$$

(4.45)

Moreover, by the definition of non-tangential maximal function (see for example [34, pp.6]), as well as the related estimates (see [31, Theorem 1.4] for general Lipschitz domains as $0 < s' - 1 < 1$; [24, Theorem 7.2] for a regular-SKT domain; and [12, Theorem 2.4] for a $C^1$ domain), we have

$$
\int_{O_\varepsilon} |\nabla \bar{u}^{(2)}|^{2s'} \lesssim \varepsilon \int_{\partial \Omega} |(\nabla \bar{u}^{(2)})^*|^{2s'} dS \lesssim \varepsilon R_0^{2s'-1} \int_{\Omega} |\nabla f|^{2s'},
$$

(4.46)

where $\nabla_{\text{tan}}$ is tangential derivative with respect to $\partial \Omega$. As a result, combining the estimates (4.41), (4.45) and (4.46) leads to the desired estimate (4.39a).

We now turn to the estimate (4.39b). To do so, we first have

$$
\int_{\Omega \setminus O_\varepsilon} |\nabla^2 \bar{u}^{(1)}|^{2s'} \lesssim R_0^{2s'-1} \int_{\mathbb{R}^d} |\nabla^2 \bar{u}^{(1)}|^{2s'} \lesssim (4.43b) R_0^{2s'-1} \int_{\Omega} |\nabla f|^{2s'}.
$$

(4.47)

Then, by using the interior Lipschitz estimates for the solution of the equations (2) in (4.40), i.e.,

$$
|\nabla^2 \bar{u}^{(2)}(x)|^{2s'} \lesssim \frac{1}{[\delta(x)]^{2s'}} \int_{B_{\delta(x)/2}(x)} |\nabla \bar{u}^{(2)}|^{2s'},
$$

we obtain

$$
\int_{\Omega \setminus O_\varepsilon} |\nabla^2 \bar{u}^{(2)}|^{2s'} \lesssim \int_{\Omega \setminus O_\varepsilon} \int_{B_{\delta(x)/4}(x)} |\nabla \bar{u}^{(2)}|^{2s'} \lesssim \int_{\partial \Omega} |(\nabla \bar{u}^{(2)})^*|^{2s'} + \frac{1}{R_0} \int_{\Omega} |\nabla \bar{u}^{(1)}|^{2s'}
$$

(4.48)

$$
\lesssim (4.46),(4.43a),(4.44) \ln(R_0/\varepsilon) R_0^{2s'-1} \int_{\Omega} |\nabla f|^{2s'}.
$$

By the same token, combining the estimates (4.47) and (4.48) offers us the desired estimate (4.39b).

**Step 3.** Show the estimates (4.37a) and (4.37b). By changing variables, we start from

$$
\left\langle \left( \int_{\mathbb{R}^d} \left( \int_{B_1(y)} \left| \tilde{h}_j (e_j + \nabla \phi_j^* \otimes (\nabla \tilde{w} + \phi_i \nabla \tilde{\phi}_i) \right| \right)^2 dy \right)^p \right\rangle^{\frac{1}{p}}
$$

$$
= \varepsilon^d \left\langle \left( \int_{\mathbb{R}^d} \left( \int_{B_{\varepsilon}(x)} \left| h_j (e_j + \nabla \phi_j^{\varepsilon*} \otimes \left( \nabla w_\varepsilon + \varepsilon \phi_i \nabla \phi_j \right) \right| \right)^2 dx \right)^p \right\rangle^{\frac{1}{p}}.
$$

(4.49)

---

9We refer the reader to [34, pp.213-214] for the concrete definition and properties of the tangential derivative.
Then employing Hölder’s inequality, Minkowski’s inequality and Lemma 2.1 in the order, we derive that

\[
\left\langle \left( \int_{\mathbb{R}^d} \left( \int_{B_{c}(x)} |h_j(e_j + \nabla \phi^e_j) \otimes F_e| \right)^2 \, dx \right)^{\frac{1}{p'}} \rightangle^{\frac{1}{p}} \\
\leq \int_{\mathbb{R}^d} \left\langle \left( \int_{D_{c}(x)} |(e_j + \nabla \phi^e_j)^{2s'}|^{\frac{1}{p'}} \right)^{\frac{1}{p}} \left\langle \left( \int_{D_{c}(x)} |F_e|^2 \right)^{\frac{1}{p'}} \right\rangle^{\frac{1}{p'}} \left( \int_{D_{c}(x)} |h_j|^{2s} \right)^{\frac{1}{p'}} \, dx \\
\lesssim (\ref{2.2}) \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |F_e|^2 \right)^{\frac{1}{p'}} \right\rangle^{\frac{1}{p'}} \delta^{2s'-1}_\sigma(x) \, dx \int_{\Omega} \left( \int_{D_{c}(x)} |h_j|^{2s} \delta^{2s-2s}(x) \, dx \right)^{\frac{1}{p'}}
\]

(4.50)

where \( \delta_\sigma(x) := \text{dist}(x, \partial \Omega_0) \) and \( \Omega_0 \supseteq \Omega \) satisfying \( \partial \Omega_0 \in C^2 \) and \( \text{dist}(\cdot, \partial \Omega_0) = O(\varepsilon) \) on \( \partial \Omega \) (similar to that given in Proposition 3.6). On the one hand, from Fubini’s theorem and Hardy’s inequality, it follows that

\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |h|^{2s} \delta^{4s-2s}(x) \, dx \right)^{\frac{1}{p'}} \right\rangle^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \lesssim \left( \int_{\Omega} |h|^{2s} \int_{B_{c}(x)} \delta^{4s-2s}(x) \, dx \right)^{\frac{1}{p'}} \lesssim \left( \int_{\Omega} |\nabla h|^{2s} \delta^{4s}(x) \, dx \right)^{\frac{1}{p'}} \lesssim R_0 \left( \int_{\Omega} |\nabla h|^{2s} \right)^{\frac{1}{p'}}.
\]

(4.51)

On the other hand, appealing to Proposition 3.6 we have

\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |\nabla w_{\varepsilon}|^{2s'} \right)^{\frac{1}{p'}} \delta^{2s'-1}(x) \, dx \right\rangle^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \lesssim \left( \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |\varepsilon(\phi^e_i, \sigma^e_i) \nabla \varphi_i|^{2s'} \right)^{\frac{1}{p'}} \delta^{2s'-1}(x) \, dx \right\rangle^{\frac{1}{p'}} \right)^{\frac{1}{p'}} + R_\varepsilon,
\]

where \( R_\varepsilon := \left( \int_{O_\varepsilon} |\nabla \bar{u}|^{2s'} \delta^{2s'-1}(x) \right)^{\frac{1}{p'}}. \) This further implies

\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |F_e|^2 \right)^{\frac{1}{p'}} \delta^{2s'-1}(x) \, dx \right\rangle^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \lesssim \left( \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |\varepsilon(\phi^e_i, \sigma^e_i) \nabla \varphi_i|^{2s'} \right)^{\frac{1}{p'}} \delta^{2s'-1}(x) \, dx \right\rangle^{\frac{1}{p'}} \right)^{\frac{1}{p'}} + \varepsilon^2 R_0 \frac{1}{p'} \left( \int_{\Omega} |\nabla f|^{2s'} \right)^{\frac{1}{p'}}.
\]

(4.52)

From Hölder’s inequality, Minkowski’ inequality and Lemma 2.1, the right-hand side of (4.52) can be controlled by

\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |\varepsilon(\phi^e_i, \sigma^e_i)|^{2s'} \right)^{\frac{1}{p'}} \delta^{2s'-1}(x) \, dx \right\rangle^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \lesssim (\ref{2.3}) \varepsilon^2 \mu_d^2(R_0/\varepsilon) \delta^{2s'-1}(x) \, dx
\]

\[
\lesssim \varepsilon^2 \mu_d^2(R_0/\varepsilon) \left( \int_{\Omega} \left( \int_{\Omega} |\nabla \bar{u}|^{2s'} \delta^{2s'-1}(x) \, dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \lesssim \varepsilon^2 \mu_d^2(R_0/\varepsilon) \varepsilon^{-1} \left( \int_{O_\varepsilon} |\nabla \bar{u}|^{2s'} + \int_{\Omega \setminus O_\varepsilon} |\nabla^2 \bar{u}|^{2s'} \delta^{2s'-1}(x) \, dx \right)^{\frac{1}{p'}}
\]

where we notice that \( \delta \sim \delta_\sigma \) on \( \Omega \setminus O_\varepsilon \). Then, plugging this back into (4.52) and then using the layer and co-layer type estimates (4.39a) and (4.39b) we derive that

\[
\left( \int_{\Omega} \left\langle \left( \int_{D_{c}(x)} |F_e|^2 \right)^{\frac{1}{p'}} \delta^{2s'-1}(x) \, dx \right\rangle^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \lesssim R_0^{-\frac{1}{p'}} \varepsilon^2 \mu_d^2(R_0/\varepsilon) \ln \frac{1}{\delta}(R_0/\varepsilon) \left( \int_{\Omega} |\nabla f|^{2s'} \right)^{\frac{1}{p'}}.
\]
and this combining with the estimates (4.51), (4.50) and (4.49) gives the stated estimate (4.37a).

We now turn to the estimate (4.37b). In view of the notation (4.32) and (4.33) we have

\[
\left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| \nabla \phi_{j}^{s} h_{j} \otimes (1 - \eta_{e}) \nabla \bar{u} \right| \right)^{2} dx \right)^{\frac{1}{p}} = \varepsilon^{\text{nd}} \left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| (\nabla \phi_{j}^{s} h_{j}) \otimes (1 - \eta_{e}) \nabla \bar{u} \right| \right)^{2} dx \right)^{\frac{1}{p}},
\]

and it follows from Hölder’s inequality, Minkowski’s inequality and Lemma 2.1 that

\[
\left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| (\nabla \phi_{j}^{s} h_{j}) \otimes (1 - \eta_{e}) \nabla \bar{u} \right| \right)^{2} dx \right)^{\frac{1}{p}} \leq \int_{\Omega} \int_{D_{e}} \left( \int_{\Omega} \left| (\nabla \phi_{j}^{s} h_{j}) \otimes (1 - \eta_{e}) \nabla \bar{u} \right| \right)^{\frac{1}{p}} dx \right)^{\frac{1}{p}} \leq \varepsilon^{2} \left( \int_{\Omega} \left| h \right|^{2s} dx \right)^{\frac{1}{p}} \left( \int_{\Omega} \left| R_{0} \nabla f \right|^{2s'} dx \right)^{\frac{1}{p}}.
\]

By using Hardy’s inequality and the layer type estimate (4.39a), the right-hand side above is given by

\[
I_{0} \lesssim R_{0}^{-\frac{1}{p} - \frac{1}{2s'}} \varepsilon^{2} \left( \int_{\Omega} \left| h \right|^{2s} dx \right)^{\frac{1}{p}} \left( \int_{\Omega} \left| R_{0} \nabla f \right|^{2s'} dx \right)^{\frac{1}{p}},
\]

and this implies the desired estimate (4.37b).

**Step 4.** Show the estimate (4.38a) and (4.38b). We start from establishing (4.38a). By a rescaling argument, the auxiliary equations (4.35) can be rewrite as

\[
\begin{cases}
-\nabla \cdot a^{s} \nabla v_{\varepsilon} = \varepsilon^{d+1} \nabla \cdot (a^{s} \phi_{j}^{s} - \sigma_{j}^{s}) \nabla h_{j} & \text{in } \Omega; \\
v_{\varepsilon} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

and we also have

\[
\left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| \nabla \phi_{j}^{s} h_{j} \otimes (1 - \eta_{e}) \nabla \bar{u} \right| \right)^{2} dx \right)^{\frac{1}{p}} \leq \varepsilon^{2} \left( \int_{\Omega} \left| h \right|^{2s} dx \right)^{\frac{1}{p}} \left( \int_{\Omega} \left| R_{0} \nabla f \right|^{2s'} dx \right)^{\frac{1}{p}}.
\]

Its right-hand side above can be further controlled by

\[
\varepsilon^{-d} \left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| \nabla v_{\varepsilon}^{s} \right|^{2} dx \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} + \varepsilon^{2+d} \left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| \phi_{j}^{s} \nabla h_{j} \otimes \nabla u_{\varepsilon}^{s} \right|^{2} dx \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}.
\]

We start from dealing with $I_{1}$. By using Hölder’s inequality and Minkowski’s inequality,

\[
I_{1} \leq \int_{\Omega} \left( \int_{D_{e}(x)} \left| \nabla v_{\varepsilon}^{s} \right|^{sp} dx \right)^{\frac{1}{p}} \left( \int_{D_{e}(x)} \left| \nabla u_{\varepsilon}^{s} \right|^{s'p'} dx \right)^{\frac{1}{p}} \leq \left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| \nabla v_{\varepsilon}^{s} \right|^{sp} dx \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \left( \int_{\Omega} \left( \int_{D_{e}(x)} \left| \nabla u_{\varepsilon}^{s} \right|^{s'p'} dx \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}.
\]
In view of the equations (4.54), mainly applying Proposition 3.6 and Lemma 2.1 to the right-hand side of (4.56), we obtain

\[
\left( \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{s/p} \right)^{\frac{1}{s}} dx \right)^\frac{1}{\varepsilon} \lesssim (4.30) \varepsilon^{2d+2} \left( \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |(\phi_j^{s,\varepsilon}, \sigma_j^{s,\varepsilon}) \nabla h_j|^2 \right)^{s/p} \right)^{\frac{1}{s}} dx \right)^\frac{1}{\varepsilon} \\
\lesssim \varepsilon^{2d+2} \left( \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |(\phi_j^{s,\varepsilon}, \sigma_j^{s,\varepsilon})|^{2s/p} \nabla h_j|^2 \right)^{s/p} \right)^{\frac{1}{s}} dx \right)^\frac{1}{\varepsilon} \\
\lesssim (2.3) \varepsilon^{2d+2} \mu_\varepsilon^2 (R_0/\varepsilon) \left( \int_\Omega |\nabla h|^{2s} dx \right)^\frac{1}{s},
\]

and

\[
\left( \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{s/p} \right)^{\frac{1}{s}} dx \right)^\frac{1}{\varepsilon} \lesssim (4.30) \left( \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |f|^2 \right)^{s/p} \right)^{\frac{1}{s}} dx \right)^\frac{1}{\varepsilon} \lesssim \left( \int_\Omega |f|^{2s} dx \right)^{\frac{1}{s}} \tag{4.58}
\]

Combining the estimates (4.56), (4.57) and (4.58) we have

\[
\varepsilon^{-d} I_1 \lesssim \varepsilon^{d+2} \mu_\varepsilon^2 (R_0/\varepsilon) \left( \int_\Omega |\nabla h|^{2s} \right)^\frac{1}{s} \left( \int_\Omega |f|^{2s} \right)^\frac{1}{s}. \tag{4.59}
\]

We now proceed to handle $I_2$, a similar computation as given for $I_1$ leads to

\[
I_2 \leq \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |\phi_j^{s,\varepsilon} \nabla h_j|^2 \right)^{s/p} \right)^{\frac{1}{s}} \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{s/p} dx \\
\leq \left( \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |\phi_j^{s,\varepsilon} \nabla h_j|^2 \right)^{s/p} \right)^{\frac{1}{s}} dx \right) \left( \int_\Omega \left( \left( \int_{D_{\varepsilon}(x)} |\nabla u_\varepsilon|^2 \right)^{s/p} \right)^{\frac{1}{s}} dx \right)^\frac{1}{s} \\
\lesssim (4.58) \mu_\varepsilon^2 (R_0/\varepsilon) \left( \int_\Omega |\nabla h|^{2s} \right)^{\frac{1}{s}} \left( \int_\Omega |f|^{2s} \right)^{\frac{1}{s}}.
\]

As a result, plugging this back into (4.55) and together with the estimate (4.59) we can derive the stated estimate (4.38a).

Finally, we turn to the estimate (4.38b). From Hölder’s inequality, Minkowski’s inequality Lemma 2.1 and triangle inequality, it follows that

\[
I_3 := \left( \left( \int_{\mathbb{R}^d} \left( \int_{B_{1}(y)} \left| (\nabla z_j^* + \phi_j^* \nabla (\hat{h}_j \hat{\varphi}_j)) \otimes (\nabla \phi_j + e_j) \right|^{2s/p} dy \right)^{\frac{1}{s}} \right)^{\frac{1}{p}} \\
\leq \int_{\mathbb{R}^d} \left( \int_{B_{1}(y)} \left| (\nabla z_j^* + \phi_j^* \nabla (\hat{h}_j \hat{\varphi}_j)) \right|^{2s/p} \right)^{\frac{1}{s}} dy + \int_{\mathbb{R}^d} \left( \int_{B_{1}(y)} \left| \phi_j^* \nabla (\hat{h}_j \hat{\varphi}_j) \right|^{2s/p} \right)^{\frac{1}{s}} dy.
\]

By annealed Calderón-Zygmund estimate (see [9, Theorem 6.1]), for each $j = 1, \ldots, d$, we have

\[
\int_{\mathbb{R}^d} \left( \int_{B_{1}(y)} \left| \nabla z_j^* \right|^{2s/p} \right)^{\frac{1}{s}} dy \leq \int_{\mathbb{R}^d} \left( \int_{B_{1}(y)} \left| (\phi_j^* \nabla (\hat{h}_j \hat{\varphi}_j)) \right|^{2s/p} \right)^{\frac{1}{s}} dy.
\]

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This together with (4.60) provides us with
\[
I_3 \lesssim \int_{\mathbb{R}^d} \left( \int_{B_1(y)} \left| (\phi_i^*, \sigma_i^*) \nabla (h_i \dot{\varphi}) \right|^2 \right)^{\frac{p}{2}} dy = \varepsilon^{d+2} \int_{\mathbb{R}^d} \left( \int_{B_{\varepsilon}^c(x)} \left| (\phi_i^x, \sigma_i^x) \nabla (h_i \varphi) \right|^2 \right)^{\frac{p}{2}} dx
\]
\[
\lesssim \varepsilon^{d+2} \mu_d^2(R_0/\varepsilon) \left\{ \left( \int_{\Omega} |\nabla h|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\varphi|^{2s'} \right)^{\frac{1}{2s'}} + \left( \int_{\Omega} |h|^{2s} \delta^{\frac{s}{2} - 2s} \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \varphi|^{2s'} \delta^{2s' - 1} \right)^{\frac{1}{2}} \right\}.
\]
Applying $W^{1,p}$ estimates (with $0 < p - 2 \ll 1$), the estimates (4.53) and (4.39b) to the right-hand side above, we further derive that
\[
I_3 \lesssim \varepsilon^{d+2} \mu_d^2(R_0/\varepsilon) \left\{ \left( \int_{\Omega} |\nabla h|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |f|^{2s'} \right)^{\frac{1}{2}} + \ln \left( \int_{\Omega} |\nabla h|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |R_0 \nabla f|^{2s'} \right)^{\frac{1}{2}} \right\},
\]
which has already proved the desired estimate (4.38b), and this ends the whole proof. \hfill \square

**Proof of Theorem 1.2.** The results of Theorem 1.2 follows from Propositions 4.1 and 4.4. \hfill \square

## 5 Appendix

**Lemma 5.1** (improved Meyer’s inequality). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and $\varepsilon \in (0, 1]$ and $R > 0$. Suppose that the (admissible) coefficient satisfies (1.1), and $u_\varepsilon$ is the solution of (3.12). Then, for any $0 < p - 2 \ll 1$ and $p_0 > 0$, there holds
\[
\left( \int_{B_R \cap \Omega} |\nabla u_\varepsilon|^p \right)^{\frac{1}{p}} \lesssim_{\lambda, p, p_0} \left( \int_{B_{2R} \cap \Omega} |\nabla u_\varepsilon|^{p_0} \right)^{\frac{1}{p_0}}.
\]
(5.1)

**Proof.** Based upon the uniform ellipticity conditions (1.1), we start from Caccioppoli’s inequality, i.e.,
\[
\left( \int_{B_R \cap \Omega} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} \lesssim_{\lambda, d} \frac{1}{R} \left( \int_{B_{2R} \cap \Omega} |u_\varepsilon|^2 \right)^{\frac{1}{2}},
\]
(5.2)
and we omit the details (see for example [16, Chapter 4]). Then, the Meyer’s estimate comes from reverse Hölder’s inequality (see for example [16, Theorem 6.38]), i.e., there exists $0 < \theta \ll 1$ depending only on $\Omega$ such that for any $2 < p < 2 + \theta$ we have
\[
\left( \int_{B_R \cap \Omega} |\nabla u_\varepsilon|^p \right)^{\frac{1}{p}} \lesssim_{\lambda, d, p} \left( \int_{B_{2R} \cap \Omega} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}}.
\]
(5.3)
Moreover, with the help of a covering argument we can derive that for any $0 < s < t < 1$ there holds
\[
\left( \int_{D_s} |\nabla u_\varepsilon|^p \right)^{\frac{1}{p}} \lesssim (t-s)^{d(\frac{1}{p} - \frac{1}{2})} \left( \int_{D_t} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}}.
\]
Thus, it follows from a convexity argument ([13, pp.173]) that for any $p_0 > 0$ we have
\[
\left( \int_{D_{t_0}} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{D_{t_0}} |\nabla u_\varepsilon|^{p_0} \right)^{\frac{1}{p_0}}
\]
where $t_0 \in (0, 1)$. This together with (5.3) will lead to the stated estimate (5.1) after a rescaling argument. \hfill \square
Remark 5.2. If concerning the Neumann boundary conditions, one can similarly derive the estimate (5.1). To see this, the right-hand side of (5.2) is merely replaced by $\frac{1}{R} \inf_{c \in \mathbb{R}} \left( \int_{B_2 R} |u - c|^2 \right)^{\frac{1}{2}}$, and the rest of the proof is the same.

Lemma 5.3. Let $\Omega$ be a bounded regular-SKT (or $C^1$) domain. Assume $u$ is a solution of $\nabla \cdot \bar{a} \nabla v = 0$ in $B_2 R \cap \Omega$ with $v = 0$ on $B_2 R \cap \partial \Omega$. Then, for any $2 < p < \infty$, there holds

$$\left( \int_{B_2 R \cap \Omega} |\nabla v|^p \right)^{\frac{1}{p}} \lesssim \left( \int_{B_2 R \cap \Omega} |\nabla v|^2 \right)^{\frac{1}{2}}, \quad (5.4)$$

where the up to constant relies on $\lambda, d, p$ and the character of $\Omega$. Moreover, if $\Omega$ is a general Lipschitz domain, the estimate (5.4) holds for $2 < p < \frac{2d}{d-1} + \theta$ with $0 < \theta \ll 1$. In particular, if the equation is scalar, the estimate (5.4) holds for $2 < p < 3 + \theta$ if $d \geq 3$; and $2 < p < 4 + \theta$ if $d = 2$.

Proof. If $\Omega$ is a $C^1$ domain, the desired estimate (5.4) will immediately follows from interior Lipschitz estimates coupled with boundary Hölder’s estimate via the geometry property of integrals. We focus on the case of the bounded regular-SKT domain. In a nutshell, our method is a bootstrap method, which contains two ingredients: (1) the related nontangential maximal function estimates, i.e.,

$$\| (\nabla v)^* \|_{L^p(\partial D_R)} \lesssim_p \| \nabla \tan v \|_{L^p(\partial D_R)} \quad \forall p \in (1, \infty) \quad (5.5)$$

(see [24, Theorem 7.2]); (2) a powerful embedding inequality associated with nontangential maximal functions, i.e.,

$$\| \nabla v \|_{L^p(D_R)} \lesssim_{p,d} \| (\nabla v)^* \|_{L^q(\partial D_R)}; \quad p = \frac{qd}{(d-1)} \quad (5.6)$$

(see for example [22, Remark 9.3]).

Let $t \in [1, 2]$, and it follows that

$$\| \nabla v \|_{L^p(D_1)} \lesssim_{(5.6)} \| (\nabla v)^* \|_{L^q(\partial D_1)} \lesssim_{(5.5)} \| \nabla \tan v \|_{L^q(\partial D_1 \Delta_1)}.$$  

By co-area formula, we further derive that

$$\int_1^2 \| \nabla v \|^q_{L^p(D_t)} dt \lesssim \int_1^2 \| \nabla v \|^q_{L^q(\partial D_1 \Delta_1)} dt \lesssim \| \nabla v \|^q_{L^q(D_2)}; \quad (5.7)$$

On the other hand, there exists $t_0 \in [1, 2]$ such that

$$\| \nabla v \|^q_{L^p(D_{t_0})} \leq 2 \int_1^{t_0} \| \nabla v \|^q_{L^p(D_t)} dt. \quad (5.8)$$

Thus, combining the estimates (5.7) and (5.8) we obtain

$$\| \nabla v \|_{L^p(D_{t_0})} \lesssim \| \nabla v \|_{L^p(D_{t_0})} \lesssim \| \nabla v \|_{L^q(D_2)}.$$  

(5.9)

Then, by the relationship $p = \frac{qd}{d-1}$ we can infer the iteration formula $p_k = p_{k-1} \kappa_d = \cdots = p_0 \kappa_d^k$ with $p_0 := 2$ and $\kappa_d := \frac{d}{d-1}$. In view of (5.9), we have $\| \nabla v \|_{L^{p_k}(D_{t_1})} \lesssim \| \nabla v \|_{L^2(D_{2k})}$, which together with covering and rescaling arguments leads to

$$\| \nabla v \|_{L^{pk}(D_R)} \leq C_k R^d \left( \frac{1}{pk} - \frac{1}{2} \right)^{\frac{1}{2}} \| \nabla v \|_{L^2(D_{2k})}, \quad \text{and} \quad \lim_{k \to \infty} C_k = \infty.$$

This offers us the stated estimate (5.4) by mention that the dependence of the multiplicative constant on $k$ is essentially the dependence on $p$. For a general Lipschitz domain, we refer the reader to the remark below. \qed

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**Remark 5.4.** In terms of Neumann boundary conditions, the estimate (5.5) should be replaced by\[
\| \nabla (\nabla u)^* \|_{L^p(\partial D_R)} \lesssim_p \| \partial v / \partial \nu \|_{L^p(\partial D_R)} \quad \text{(see [24, Theorem 7.3])},
\]
and then following the same arguments we can analogically establish (5.4) for Neumann boundary problems.

**Remark 5.5.** If Ω is a general Lipschitz domain, the estimate (5.5) is not true, and the upper bound of 1 < p < 2 + θ with 0 < θ ≪ 1 is optimal (see [31] and references therein for more details). In this regard, the proceed of “bootstrap” given in the proof of Lemma 5.3 can not happen. In the scalar case, the result has been shown in [30, Theorem B] together with [26, Theorem 1.2, Proposition 1.4].

**Acknowledgements.** The first author is supported by China Postdoctoral Science Foundation (Grant No.2022M710228). The second author deeply appreciated Prof. Felix Otto for his instruction and encourages when he held a post-doctoral position in the Max Planck Institute for Mathematics in the Sciences (in Leipzig). The second author would like to thank the Department of Mathematics in Peking University for providing him with excellent office condition during his visit in spring 2022. The second author was supported by the Young Scientists Fund of the National Natural Science Foundation of China (Grant No. 11901262); the Fundamental Research Funds for the Central Universities (Grant No.lzujbky-2021-51);

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