MONOGENIC PURE CUBICS

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Abstract. Let $k \geq 2$ be a square-free integer. We prove that the number of square-free integers $m \in [1, N]$ such that $(k, m) = 1$ and $\mathbb{Q}(\sqrt[3]{k^2m})$ is monogenic is $\gg N^{1/3}$ and $\ll N/\log N)^{1/3-\epsilon}$ for any $\epsilon > 0$. Assuming ABC, the upper bound can be improved to $O(N^{1/3}+\epsilon)$. Let $F$ be the finite field of order $q$ with $(q, 3) = 1$ and let $g(t) \in F[t]$ be non-constant square-free. We prove unconditionally the analogous result that the number of square-free $h(t) \in F[t]$ such that $\deg(h) \leq N$, $(g, h) = 1$ and $F(t, \sqrt[3]{g^2h})$ is monogenic is $\gg q^{N/3}$ and $\ll N^2q^{N/3}$.

1. Introduction

A number field $K$ is called monogenic if its ring of integers $\mathcal{O}_K$ is $\mathbb{Z}[\theta]$ for some $\theta \in \mathcal{O}_K$. Number fields that are fundamental to the development of algebraic number theory such as quadratic and cyclotomic fields are all monogenic. Certain questions about monogenic number fields (as well as monogenic orders) are closely related to the so called discriminant form equations which have been studied extensively by Evertse, Györy, and other authors. The readers are referred to [EG15] [EG16] [Ngu17] [BN18] [Ga19] and the references there for many interesting results including those over positive characteristic fields.

A pure cubic field is a number field of the form $\mathbb{Q}(\sqrt[3]{n})$ where $n > 1$ is cube-free. In a certain sense, pure cubic fields are the “next” family of number fields to investigate after quadratic fields especially from the computational point of view (for example, see [WCS80] [WDS83] [SS99] of which the third paper treats the function field analogue of pure cubic fields). While every quadratic field is monogenic, many pure cubics are not and the goal of this paper is to study the density of monogenic number fields of degree $n$ for any $n > 1$. The questions considered in this paper are quite different in nature since we restrict to the 1-parameter family $\mathbb{Q}(\sqrt[3]{n})$ as well as the family of polynomials $X^3 - n$ none of which have square-free discriminant.

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To see why the above question has a negative answer, we start with the following [Mar18, p. 35–36]:

**Theorem 1.1.** (Dedekind). Let $n > 1$ be a cube-free integer, let $\alpha = \sqrt[3]{n}$, and write $n = k^2m$ where $k$ and $m$ are square-free positive integers. We have the following:

- If $n \equiv \pm 1 \mod 9$ then $\{1, \alpha, \alpha^2/k\}$ is an integral basis of $K$.
- If $n \equiv 1 \mod 9$ then $\{1, \alpha, (k^2 \pm k^2\alpha + \alpha^2)/(3k)\}$ is an integral basis of $K$.

An immediate consequence is that $\mathbb{Q}(\sqrt[3]{n})$ is monogenic when $n > 1$ is square-free and $n \not\equiv \pm 1 \mod 9$. In fact this is the only case when we have a positive density result. For the remaining cases (i.e. $k > 1$ or $n \equiv \pm 1 \mod 9$), the conclusion is in stark contrast with the above. As a side note, a recent paper of Gassert, Smith, and Stange [GSS19] considers the 1-parameter family of quartic fields given by $X^4 - 6X^2 - \alpha X - 3$ and shows that a positive density of them are monogenic.

Throughout this paper, for each square-free positive integer $k$, let:

$S_k = \{\text{square-free } m > 0: (m, k) = 1, k^2m \not\equiv \pm 1 \mod 9, \mathbb{Q}(\sqrt[3]{k^2m}) \text{ is monogenic}\}$, and if $(k, 3) = 1$ let

$T_k = \{\text{square-free } m > 0: (m, k) = 1, k^2m \equiv \pm 1 \mod 9, \mathbb{Q}(\sqrt[3]{k^2m}) \text{ is monogenic}\}$.

From now on, whenever $T_k$ is mentioned, we tacitly assume the condition that $(k, 3) = 1$. Our main results for pure cubic number fields are the following:

**Theorem 1.2.** For every $\epsilon > 0$ and square-free integer $k \geq 2$, we have $N^{1/3} \ll_k \mathbb{Q}(\sqrt[3]{k^2m}) |S_k \cap [1, N]| \ll_{k, \epsilon} N/(\log N)^{1/3-\epsilon}$ as $N \to \infty$. For every square-free $k \geq 1$, we have $N^{1/3} \ll_k \mathbb{Q}(\sqrt[3]{k^2m}) |T_k \cap [1, N]| \ll_{k, \epsilon} N/(\log N)^{1/3-\epsilon}$ as $N \to \infty$. Consequently, the sets $S_k$ for $k \geq 2$ and the sets $T_k$ for $k \geq 1$ have zero density.

A table of monogenic pure cubic fields with discriminant up to $12 \cdot 10^6$ has been computed by Gaál-Szabó [GS10, Gaa19] and it is noted in [Gaa19, p. 111] that “the frequency of monogenic fields is decreasing”. Our zero density result illustrates this observation. Further investigations and computations involving integral bases and monogenicity of higher degree pure number fields have been done by Gaál-Remete [GR17].

Assuming ABC, we can arrive at the much stronger upper bound:

**Theorem 1.3.** Assume that the ABC Conjecture holds. Let $\epsilon > 0$ and let $k$ be a square-free positive integer. We have $|T_k \cap [1, N]| = O_{\epsilon,k}(N^{(1/3)+\epsilon})$. And if $k \geq 2$, we have $|S_k \cap [1, N]| = O_{\epsilon,k}(N^{(1/3)+\epsilon})$.

**Remark 1.4.** From the lower bound in Theorem 1.2, we have that the exponent $1/3$ in Theorem 1.3 is best possible. It is not clear if we can replace $N^{(1/3)+\epsilon}$ by some $N^{1/3} f(N)$ where $f(N)$ is dominated by $N^{\epsilon}$ for any $\epsilon$.

**Remark 1.5.** In principle, we can break $T_k$ into $T_k^+ = \{m \in T_k : k^2m \equiv 1 \mod 9\}$ and $T_k^- = \{m \in T_k : k^2m \equiv -1 \mod 9\}$. When choosing the $\pm$ signs appropriately, all results and arguments for $T_k$ remain valid for each individual $T_k^+$ and $T_k^-$. We now consider the function field setting. For the rest of this section, let $F$ be a finite field of order $q$ and characteristic $p \neq 3$. A polynomial $f(t) \in F[t]$ is called square-free (respectively cube-free) if it is not divisable by the square (respectively cube) of a non-constant element of $F[t]$. Every cube-free $f(t)$ can be
written uniquely as \( f(t) = g(t)^2 h(t) \) in which \( g(t), h(t) \in F[t] \) are square-free and \( g(t) \) is monic. We have the analogue of Dedekind’s theorem for \( F[t] \):

**Theorem 1.6** (function field Dedekind). Let \( f(t) \in F[t] \) be cube-free, let \( \alpha = \sqrt[3]{t} \), \( K = F(t, \alpha) \), and let \( \mathcal{O}_K \) be the integral closure of \( F[t] \) in \( K \). Express \( f(t) = g(t)^2 h(t) \) as above. Then \( \{1, \alpha, \alpha^2 / g\} \) is a basis of \( \mathcal{O}_K \) over \( F[t] \).

**Proof.** The proof is a straightforward adaptation of steps in the proof of Theorem [11] given in [18] p. 35–36

As before, \( K = F(t, \sqrt[3]{t}) \) is called monogenic if \( \mathcal{O}_K = F[t, \theta] \) for some \( \theta \in \mathcal{O}_K \). For each monic square-free \( g(t) \in F[t] \), let

\[
\mathcal{U}_g = \{ \text{square-free } h \in F[t] : (g, h) = 1, F(t, \sqrt[3]{g^2}h) \text{ is monogenic} \}.
\]

For each positive integer \( N \), let \( F[t]_{\leq N} \) denote the set of polynomials of degree at most \( N \). It is easy to show that there are \( q^{N+1} - q^N \) square-free polynomials in \( F[t]_{\leq N} \). Therefore, if we define the density of a subset \( A \) of \( F[t] \) to be

\[
\lim_{N \to \infty} \frac{|A \cap F[t]_{\leq N}|}{|F[t]_{\leq N}|}
\]

(assuming the limit exists), then the set \( \mathcal{U}_1 \) has density \( 1 - 1/q \). As before, this is in stark contrast to the case \( \deg(g) > 0 \):

**Theorem 1.7.** Let \( g \) be a non-constant monic square-free polynomial in \( F[t] \). We have

\[
q^{N/3} \ll |\mathcal{U}_g \cap F[t]_{\leq N}| \ll N^2 q^{N/3}
\]

as \( N \to \infty \) where the implied constants depend only on \( F \) and \( g \).

**Remark 1.8.** In Theorem [17] an analogous upper bound to the number field setting would be \( q^{(1/3+\epsilon)N} \). The bound \( N^2 q^{N/3} \) obtained here is much stronger; this is a typical phenomenon thanks to the uniformity of various results over function fields.

We end this section with a brief discussion on the methods of the proofs. As mentioned above, it is well-known that monogenicity is equivalent to the fact that a certain discriminant form equation has a solution in \( \mathbb{Z} \) (or \( F[t] \) if we are in the function field case). For the questions involving pure cubic fields considered here, we end up with an equation of the form \( aX^3 + bY^3 = c \) where \( a \) and \( c \) are fixed and \( b \) varies so that the equation has a solution \((X, Y)\). There are several methods to study those Thue equations \([11, 10, 19]\) and we can effectively bound the number of solutions or the size of a possible solution. However, the question considered here is somewhat different: we are estimating how many \( b \) for which we have at least one solution. The unconditional upper bound \( N/(\log(N))^{1/3} \) in the number field case follows from a sieving argument together with a simple instance of the Chebotarev density theorem. The much stronger bound \( N^{1/3 + \epsilon} \) in the number field case as well as the bound \( N^2 q^{N/3} \) in the function field case follow from the use of ABC together with several combinatorial arguments that might be of independent interest.

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2. The number field case

We start with the following:

**Proposition 2.1.** Let \( k \) and \( m \) be square-free positive integers. We have:

(a) \( m \in S_k \) if and only if \( k^2m \not\equiv \pm 1 \mod 9 \) and the equation \( kX^3 - mY^3 = 1 \) has a solution \( X, Y \in \mathbb{Z} \).

(b) \( m \in T_k \) if and only if \( k^2m \equiv \pm 1 \mod 9 \) and the equation \( kX^3 - mY^3 = 9 \) has a solution \( X, Y \in \mathbb{Z} \).

**Proof.** For (a), suppose \( k^2m \not\equiv \pm 1 \mod 9 \), let \( \alpha = \sqrt[3]{k^2m} \), \( K = \mathbb{Q}(\alpha) \), and consider the integral basis \( \{1, \alpha, \alpha^2/k\} \). To find \( \theta \in \mathcal{O}_K \) such that \( \mathcal{O}_K = \mathbb{Z}[\theta] \), it suffices to consider \( \theta \) of the form \( \theta = u\alpha + v(\alpha^2/k) \) with \( u, v \in \mathbb{Z} \). Then we have:

\[
\theta^2 = 2uvkm + v^2m\alpha + u^2\alpha^2.
\]

We represent \((1, \theta, \theta^2)\) in terms of the given integral basis and the corresponding matrix has determinant \( kv^3 - mv^2 \). Therefore \( \mathcal{O}_K = \mathbb{Z}[\theta] \) if and only if the equation \( kX^3 - mY^3 = 1 \) has a solution \( X, Y \in \mathbb{Z} \).

The proof of part (b) is similar with some tedious algebraic expressions as follows.

Suppose \( k^2m \equiv \pm 1 \mod 9 \), let \( \alpha = \sqrt[3]{k^2m} \), \( K = \mathbb{Q}(\alpha) \), and consider the integral basis \( \{1, \alpha, \beta = (k^2 \pm k^2\alpha + \alpha^2)/(3k)\} \). As before, consider \( \theta = u\alpha + v\beta \) with \( u, v \in \mathbb{Z} \). Then depending on whether \( k^2m \equiv \pm 1 \mod 9 \), we have:

\[
\theta^2 = u^2\alpha^2 + \frac{2uv}{3k}(k^2\alpha \pm k^2\alpha^2 + \alpha^3) + \frac{v^2}{3k}(k^2 \pm k^2\alpha + \alpha^2)^2
\]

\[
= \frac{2uvkm}{3} + \frac{v^2k^2}{9} \pm \frac{2v^2k^2m}{9} + \left(\frac{2uv}{3} + \frac{v^2m}{9} \pm \frac{2v^2k^2}{9}\right)\alpha
\]

\[
= c_1 + c_2\alpha + c_3\beta
\]

where \( c_3 = 3k \left(\frac{u^2}{3} \pm \frac{v^2k^2}{9} + \frac{2v^2}{9}\right) \), \( c_2 = \frac{2uv}{3} + \frac{v^2m}{9} \pm k^2u^2 - \frac{2uvk^3}{3} \pm \frac{v^2k^4}{9} \), and the precise value of \( c_1 \) is not needed for our purpose. We represent \((1, \theta, \theta^2)\) in terms of the given integral basis and the corresponding matrix has determinant:

\[
3ku^3 \pm 3k^2u^2v + k^3uv^2 - \frac{m + k^4}{9}v^3 = \frac{1}{9}(k(3u \pm kv)^3 - mv^3).
\]

Therefore if \( \mathcal{O}_K = \mathbb{Z}[\theta] \) then the equation \( kX^3 - mY^3 = 9 \) has a solution \( X, Y \in \mathbb{Z} \). Conversely, if \((X_0, Y_0)\) is a solution, we can choose \( v = Y_0 \) and \( u = (X_0 \mp kY_0)/3 \) and we need to explain why \( u \in \mathbb{Z} \). From \( k^2m \equiv \pm 1 \mod 9 \), we have \( m \equiv \pm k^4 \mod 9 \). Using this and the equation \( kX_0^3 - mY_0^3 = 9 \), we have \( X_0^3 \equiv \pm k^3Y_0^3 \mod 9 \). Hence \( X_0 \equiv \pm kY_0 \mod 3 \). □

The following establish the upper bounds in Theorem 1.2:

**Proposition 2.2.** Let \( a \) and \( b \) be positive integers such that \( b/a \) is not the cube of a rational number. As \( N \to \infty \), the number of integers \( m \in [1, N] \) such that the equation \( aX^3 - mY^3 = b \) has an integer solution is \( O_{a,b}(N/(\log N)^{1/3}) \).
Proof. Let \( L = \mathbb{Q}(\sqrt[3]{b/a}) \) and let \( L' \) be its Galois closure. Let \( S \) be the set of primes \( p \not| 3ab \) such that \( b/a \) is not a cube mod \( p \). This means \( p \) remains a prime in \( L \) and \( pO_L \) splits completely in \( L' \); in other words the Frobenius of \( p \) with respect to \( L'/\mathbb{Q} \) is the conjugacy class of the 2 elements of order 3. The Chebotarev density theorem gives that \( S \) has Dirichlet as well as natural density 1/3. Put \( s(x) = |S \cap [1, x]| \) so that \( s(x) = \frac{\pi(x)}{3} + o(\pi(x)) \); put \( r(x) = s(x) - \frac{\pi(x)}{3} \). Then partial summation gives:

\[
\sum_{p \in S, p \leq x} \frac{1}{p} = \int_{2}^{x} \frac{ds(t)}{t} = \frac{1}{3} \int_{2}^{x} \frac{d\pi(t)}{t} + \int_{2}^{x} \frac{dr(t)}{t} \sim \frac{1}{3} \log \log x \text{ as } x \to \infty
\]

thanks to the Prime Number Theorem and the fact that \( r(t) = o(\pi(t)) \). This implies

\[
(1) \quad \prod_{p \in S, p \leq x} \left( 1 - \frac{1}{p} \right) = (\log x)^{-1/3} e^{o(\log \log x)}.
\]

Now observe that if \( m \in [1, N] \) is divisible by some \( p \in S \) then the equation \( aX^3 - mY^3 = b \) cannot have an integer solution since \( b/a \) is not a cube mod \( p \). By sieving \([MV06\) Chapter 3.2], the number of \( m \in [1, N] \) such that \( p \not| m \) for all \( p \in S \) is \( O \left( \prod_{p \in S, p \leq N} \left( 1 - \frac{1}{p} \right) N \right) \) and we use (1) to finish the proof.

**Proof of Theorem 1.2.** The upper bound in Theorem 1.2 follows from Propositions [23] and [22]. For the lower bound, first we consider \( S_k \) and the equation \( kX^3 - mY^3 = 1 \). We can always take \( m = kX_0^3 - 1 \) for \( X_0 \in [1, (N/k)^{1/3}] \) so that the above equation has a solution \((X_0, 1)\). We need that \( k^2m \not\equiv 0 \mod 9 \) and \( m \) is square-free for a positive proportion of such \( X_0 \). A direct calculation shows that regardless of the possibility of \( k \mod 9 \), we can always find \( r \in \{0, \ldots, 8\} \) such that \( k^2(kr^3 - 1) \not\equiv 0 \mod 9 \). We now choose \( X_0 \) of the form \( X_0 = 9t + r \) for \( t \in [1, cN^{1/3}] \) where \( c \) is a positive constant depending only on \( k \). By classical results of Hooley \([Hoo67\] \([Hoo68\) (also see \([Gra98\) for a more general result assuming ABC), the irreducible cubic polynomial \( f(t) = k(9t + r)^3 - 1 \in \mathbb{Z}[t] \) admits square-free values for at least \( c'cN^{1/3} \) many \( t \) where \( c' > 0 \) depends only on \( k \) and \( r \). The proof of \( N^{1/3} \ll_k |T_k \cap [1, N]| \) is completely similar.

We will obtain the stronger upper bound \( O(N^{1/3 + \epsilon}) \) assuming the ABC Conjecture:

**Conjecture 2.3.** Let \( \epsilon > 0 \), then there exists a positive constant \( C \) depending only on \( \epsilon \) such that the following holds. For all relatively prime integers \( a, b, c \in \mathbb{Z} \) with \( a + b = c \), we have:

\[
\max \{|a|, |b|, |c|\} \leq C \left( \prod_{\text{prime } p \mid abc} p \right)^{1+\epsilon}
\]

Theorem 1.3 follows from Proposition 2.1 and the following:

**Proposition 2.4.** Assume Conjecture 2.3. Let \( a \) and \( b \) be positive integers such that \( b/a \) is not the cube of a rational number and let \( \epsilon > 0 \). The number of integers
m such that $|m| \leq N$ and the equation $aX^3 - mY^3 = b$ has an integer solution $(X, Y)$ is $O_{a, b, \epsilon}(N^{(1/3)+\epsilon})$.

Proof. Let $\delta$ be a small positive number depending on $\epsilon$ that will be specified later. The implicit constants in this proof depends only on $a$, $b$, and $\delta$. Except for the finitely many $m$ for which $b/m$ is the cube of an integer, any $(m, X_0, Y_0)$ such that $aX_0^3 - mY_0^3 = b$, $|m| \leq N$, and $X_0, Y_0 \in \mathbb{Z}$ satisfies $mX_0Y_0 \neq 0$. An immediate consequence of ABC gives:

$$\max\{|X_0^3|, |mY_0^3|\} \ll |mX_0Y_0|^{1+\delta}.$$  

From $aX_0^3 - mY_0^3 = b$, we get $|Y_0| \ll |m^{-1/3}X_0|$. Combining with the above, we get: $|X_0|^3 \ll |m^{2/3}X_0^3|^{1+\delta}$. Put $\delta' = \frac{2(1+\delta)}{3(1-2\delta)} - \frac{2}{3}$ so that we have:

$$|X_0| \ll m^{(2/3)+\delta'} \text{ and } |Y_0| \ll m^{(1/3)+\delta'}.$$  

Therefore, in order to estimate the number of $m$, we estimate the number of pairs $(X_0, Y_0)$ with $X_0 = O(N^{(2/3)+\delta'})$ and $Y_0 = O(N^{(1/3)+\delta'})$ such that $\frac{aX_0^3 - b}{Y_0^3}$ is an integer in $[-N, N]$. Fix such a $Y_0$, we have the obvious bound $|X_0| \ll N^{1/3}|Y_0|$ and we now study the congruence $aX_0^3 \equiv b \mod Y_0^3$. Let $p$ be a prime divisor of $Y_0$ and let $d > 0$ such that $p^d \mid Y_0$. If $p \nmid ab$, the equation $aX_0^3 \equiv b \mod p^{3d}$ has at most 3 solutions in $\mathbb{Z}/p^{3d}\mathbb{Z}$ thanks to the structure of $(\mathbb{Z}/p^{3d}\mathbb{Z})^*$. If $p \mid ab$ and $3d > \max(v_p(a), v_p(b))$, for the above congruence equation to have a solution, we must have that $v_p(b) - v_p(a)$ is a positive integer divisible by 3 and any solution must have the form $p^{v_p(b) - v_p(a)/3}x$ where $x$ satisfies $x^3 \equiv u \mod p^{3d-(v_p(b) - v_p(a))}$ and $u$ is given by $\frac{b}{a} = p^{v_p(b) - v_p(a)/3}u$. Again, there are at most 3 solutions in this case. In conclusion, there are $O(3^{\omega(Y_0)})$ many solutions in $\mathbb{Z}/Y_0^3\mathbb{Z}$ of the equation $aX_0^3 \equiv b \mod Y_0^3$; here $\omega(n)$ denotes the number of distinct prime factors of $n$.

Overall, the number of pairs $(X_0, Y_0)$ is at most:

$$\sum_{Y_0 = O(N^{(1/3)+\delta'})} O\left(3^{\omega(Y_0)} \left(\frac{N^{1/3}|Y_0|}{|Y_0^3|} + 1\right)\right).$$

This is $O(N^{((1/3)+\delta')(1+\delta')})$ since $3^{\omega(Y_0)}$ is dominated by $|Y_0|^{\delta'}$. Now choosing $\delta$ sufficiently small so that $((1/3) + \delta')(1 + \delta') < (1/3) + \epsilon$ and we get the desired conclusion.  

3. THE FUNCTION FIELD CASE

Throughout this section, let $F$ be a finite field of order $q$ and characteristic $p \neq 3$. A polynomial $f(t) \in F[t]$ is called square-free (respectively cube-free) if it is not divisible by the square (respectively cube) of a non-constant polynomial in $F[t]$. Every cube-free $f(t)$ can be written uniquely as $f(t) = g(t)^2h(t)$ in which $g(t), h(t) \in F[t]$ are square-free and $g(t)$ is monic. We have the function field analogue of Proposition 2.1 whose proof is completely similar:

**Proposition 3.1.** Let $f, g, h \in F[t]$ be as above. Then $F(t, \sqrt[3]{f})$ is monogenic if and only if there exists $X, Y \in F[t]$ such that $gX^3 - hY^3 \in F^*$. 

In function fields, the Mason-Stothers theorem plays a similar role to ABC:
Theorem 3.2 (Mason-Stothers). Let $E$ be a field and let $A, B, C \in E[t]$ be relatively prime polynomials with $A + B = C$. Suppose that at least one of the derivatives $A', B', C'$ is non-zero then

$$\max\{\deg(A), \deg(B), \deg(C)\} \leq r(ABC) - 1$$

where $r(ABC)$ denotes the number of distinct roots of $ABC$ in $\bar{E}$.

In order to guarantee the condition on derivatives in the above theorem, we need:

Lemma 3.3. Let $g(t), h(t) \in F[t]$ be non-constant square-free polynomials. Suppose there exist $X, Y \in F[t]$ such that $gX^3 - hY^3 \in F^*$. Then there exist $X_1, Y_1 \in F[t]$ such that $gX_1^3 - hY_1^3 \in F^*$ and at least one of the derivatives $(gX_1^3)'$ and $(hY_1^3)'$ is non-zero.

Proof. Write $g = g_1 \cdots g_u$ and $h = h_1 \cdots h_v$ where the $g_i$’s and $h_j$’s are irreducible over $F$. Let $n$ be the largest non-negative integer such that both $gX^3$ and $hY^3$ are $p^n$-th power of some element of $F[t]$. Write $gX^3 = X^{p^n}$ and $hY^3 = Y^{p^n}$, we have that $X - Y \in F^*$ and at least one of the derivatives $X'$ and $Y'$ is non-zero.

Since $p \neq 3$, from $gX^3 = X^{p^n}$ and $hY^3 = Y^{p^n}$ we can express $X$ and $Y$ as:

$$\tilde{X} = g_1^{b_1} \cdots g_u^{b_u} X_0^3 \quad \text{and} \quad \tilde{Y} = h_1^{c_1} \cdots h_v^{c_v} Y_0^3$$

where the $b_i$’s and $c_j$’s are positive integer, $\gcd(X_0, g_1 \cdots g_u) = \gcd(Y_0, h_1 \cdots h_v) = 1$, and $b_ip^n - 1 \equiv c_j p^n - 1 \equiv 0 \mod 3$ for $1 \leq i \leq u$ and $1 \leq j \leq v$.

Hence the $b_i$’s and $c_j$’s have the same non-zero congruence mod 3. Depending on whether they are 1 mod 3 or respectively 2 mod 3, we can write

$$\tilde{X} = gX_1^3 \quad \text{and} \quad \tilde{Y} = hY_1^3$$

or respectively

$$\tilde{X} = g^2 X_1^3 \quad \text{and} \quad \tilde{Y} = h^2 Y_1^3.$$
Now we choose the smallest $k$ such that $N \leq d_k$. This implies that $\omega(P)$ is at most the number of monic irreducible polynomials of degree at most $k$:

$$\omega(P) \leq \sum_{n=1}^{k} \frac{q^n + O(q^{n/2})}{n} = O(q^k/k).$$

From the above formula for $d_k$ and the choice of $k$, we have that $q^k \ll N$ and $k \ll \log N$; this finishes the proof. \hfill \Box

It turns out that we will need an estimate for $\sum 3^{\omega(P)}$ where $\deg(P) \leq N$. Using the above bound $N/\log N$ for each individual $\omega(P)$ would yield $O(q^{N+O(N/\log N)})$ for the above sum which would not be good enough for our purpose. Instead, we have:

**Lemma 3.5.** \[ \sum_{\deg(P) \leq N} 3^{\omega(P)} = O(N^2 q^N). \]

**Proof.** Let $s_N$ be $\sum 3^{\omega(P)}$ where $P$ ranges over all monic polynomials of degree equal to $N$. It suffices to show $s_N = O(N^2 q^N)$. The generating series $\sum_n s_n T^n$ has the Euler product:

$$\prod_P \frac{1 + 3T^{\deg(P)} + 3T^{2\deg(P)} + \ldots}{1 - T^{\deg(P)}},$$

where $P$ ranges over all the monic irreducible polynomials over $F$. The denominator is simply the zeta-function $1/(1 - qT)$ while the coefficients of the numerator are bounded above by the coefficients of $\prod_P (1 + 3\deg(P) + 3^2\deg(P) + \ldots)^2 = \frac{1}{(1 - qT)^2}$.

Therefore the $s_N$’s are bounded above by the coefficients of $1/(1 - qT)^3$ and this finishes the proof. \hfill \Box

**Proof of Theorem 1.7.** For the lower bound, we simply study the equation $gX^3 - hY^3 = 1$. Either by adapting the arguments in \cite{Hoo67, Hoo68, Gra98} or using a general result of Poonen \cite[Theorem 3.4]{Poo03} which is valid for a multivariable polynomial, we have that for a positive proportion of the polynomials $X \in F[t]$ with $\deg(X) \leq (N - \deg(g))/3$, we have that $gX^3 - 1$ is squarefree; we now simply take $Y = 1$ and $h = gX^3 - 1$ for those $X$’s. This proves the lower bound.

For the upper bound, we prove that for an arbitrary $\alpha \in F^*$, there are $O(q^{N/3})$ many $h$ of degree at most $N$ such that the equation $gX^3 - hY^3 = \alpha$ has a solution $X,Y \in F[t]$; since $\deg(g) > 0$ we must have that $Y \neq 0$. By Lemma 3.3 we may assume that at least one of the derivatives $(gX^3)'$ and $(hY^3)'$ is non-zero. The Mason-Stothers theorem yields:

$$\deg(h) + 3\deg(Y) \leq \deg(g) + \deg(X) + \deg(h) + \deg(Y) - 1,$$

and

$$\deg(g) + 3\deg(X) \leq \deg(g) + \deg(X) + \deg(h) + \deg(Y) - 1.$$ 

The first inequality gives $\deg(Y) \leq \deg(X)/2 + O(1)$ then we use this and the second inequality to obtain $\deg(X) \leq \frac{2}{3} \deg(h) + O(1)$. Then it follows that $\deg(Y) \leq \frac{N}{3} + O(1)$

We now count the number of pairs $(X,Y)$ with $Y \neq 0$ such that $\deg(Y) \leq \frac{N}{3} + O(1)$ and $\frac{2X^3 - \alpha}{Y^3}$ is a polynomial in $F[t]$ of degree at most $N$. Hence $\deg(X) \leq \deg(Y) + (N/3) - \deg(g)$. Arguing as before, for each prime power factor $P^n$
of $Y$, the congruence equation $gX^3 - \alpha = 0 \mod P^{3n}$ has at most 3 solutions $\mod P^{3n}$. Therefore by the Chinese Remainder Theorem, the congruence equation $gX^3 - \alpha = 0 \mod Y^3$ has at most $3^{\omega(Y)}$ solutions $\mod Y^3$. Therefore once $Y$ is fixed, there are at most $3^{\omega(Y)} (q^{\deg(Y)} + (N/3 - \deg(g) - 3\deg(Y)+1) + 1)$ possibilities for $X$. Hence the number of pairs $(X, Y)$ is at most:

$$\sum_{k=0}^{(N/3)+O(1)} \sum_{Y: \deg(Y)=k} 3^{\omega(Y)} (q^{(N/3) - 2\deg(Y)} + 1) \ll \sum_{k=2} \sum_{Y: \deg(Y)=k} (q-1)q^{k(3\log k)}q^{(N/3) - 2k} + \sum_{k=0}^{(N/3)+O(1)} \sum_{Y: \deg(Y)=k} 3^{\omega(Y)}.$$ 

The first term is $O(q^{N/3})$ since $\sum_{k=0}^{\infty} 3^{O(k/\log k)}q^{-k} < \infty$ while the second term is $O(N^2q^{N/3})$ thanks to the previous lemma and this finishes the proof. \quad \square

4. Further Questions

Thanks to the lower bounds in our results, we know that the “main terms” $N^{1/3}$ and $q^{N/3}$ in the upper bounds are optimal. However, it seems possible that the “extra factors” $N^c$ in the number field case and $N^2$ in the function field case can be improved. This motivates:

**Question 4.1.**

(a) In Theorem 1.3, can one replace the bound $O(N^{1/3+\epsilon})$ by $O(N^{1/3-f(N)})$ where $f(N)$ is dominated by any $N^c$?

(b) In Theorem 1.7, can one improve the bound $O(N^2q^{N/3})$? Could this upper bound even be $O(q^{N/3})$?

(c) In the number field case, can one obtain an unconditional power-saving bound $O(N^c)$ where $c < 1$?

References

[BN18] J. P. Bell and K. D. Nguyen, *Some finiteness results on monogenic orders in positive characteristic*, Int. Math. Res. Not. **2018** (2018), 1601–1637.

[BSW] M. Bhargava, A. Shankar, and X. Wang, *Squarefree values of polynomial discriminants I*, arXiv:1611.09806.

[EG15] J.-H. Evertse and K. Győry, *Unit Equations in Diophantine Number Theory*, Cambridge Studies in Advanced Mathematics, vol. 146, Cambridge University Press, Cambridge, 2015.

[EG16] Discriminant Equations in Diophantine Number Theory, New Mathematical Monographs, vol. 32, Cambridge University Press, Cambridge, 2016.

[Gaal19] I. Gaál, *Diophantine Equations and Power Integral Bases: Theory and Algorithms*, second ed., Birkhäuser, Cham, Switzerland, 2019.

[GR17] I. Gaál and L. Remete, *Integral bases and monogenicity of pure fields*, J. Number Theory **173** (2017), 129–146.

[Gra98] A. Granville, *ABC allows us to count squarefrees*, Int. Math. Res. Not. IMRN (1998), no. 19, 991–1009.

[GS10] I. Gaál and T. Szabó, *A note on the minimal indices of pure cubic fields*, JP J. Algebra Number Theory Appl. **19** (2010), 129–139.

[GSS19] T. A. Gassert, H. Smith, and K. Stange, *A family of monogenic $S_4$ quartic fields arising from elliptic curves*, J. Number Theory **197** (2019), 361–382.
[Hoo67] C. Hooley, *On the power free values of polynomials*, Mathematika 14 (1967), 21–26.

[Hoo68] ______, *On the square-free values of cubic polynomials*, J. reine angew. Math. 229 (1968), 147–154.

[Mar18] D. Marcus, *Number fields*, second ed., Universitext, Springer Nature, Cham, Switzerland, 2018.

[MV06] H. Montgomery and R. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2006.

[Ngu17] K. D. Nguyen, *On modules of integral elements over finitely generated domains*, Trans. Amer. Math. Soc. 369 (2017), 3047–3066.

[Poo03] B. Poonen, *Squarefree values of multivariable polynomials*, Duke Math. J. 118 (2003), 353–373.

[SS99] R. Scheidler and A. Stein, *Voronoi’s algorithm in purely cubic congruence function fields of unit rank 1*, Math. Comp. 69 (1999), 1245–1266.

[WCS80] H. C. Williams, G. Cormack, and E. Seah, *Calculation of the regulator of a pure cubic field*, Math. Comp. 34 (1980), 567–611.

[WDS83] H. C. Williams, G. W. Dueck, and B. K. Schmidt, *A rapid method of evaluating the regulator and class number of a pure cubic field*, Math. Comp. 41 (1983), 235–286.

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