Computation of critical exponent $\eta$ at $O(1/N_f^2)$ in quantum electrodynamics in arbitrary dimensions.

J.A. Gracey,
Department of Applied Mathematics and Theoretical Physics,
University of Liverpool,
P.O. Box 147,
Liverpool,
L69 3BX,
United Kingdom.

Abstract. We present a detailed evaluation of $\eta$, the critical exponent corresponding to the electron anomalous dimension, at $O(1/N_f^2)$ in a large flavour expansion of QED in arbitrary dimensions in the Landau gauge. The method involves solving the skeleton Dyson equations with dressed propagators in the critical region of the theory. Various techniques to compute massless two loop Feynman diagrams, which are of independent interest, are also given.
1 Introduction.

The important properties of a renormalizable quantum field theory are contained within the renormalization group equation from which one can, for instance, determine how Green’s functions depend on the renormalization scale. Central to this equation are the $\beta$-function and $\gamma(g)$, the anomalous dimension of the basic fields. Ordinarily one computes these within conventional perturbation theory, in a particular renormalization scheme, and hence determines the properties of the Green’s functions to the same order. One difficulty with perturbation theory is that computations at successive higher orders become exceedingly tedious due in part to the increase in number of graphs to be considered and the complexity of the integrals which appear. Thus it is not possible to determine the higher order contributions to a renormalization group analysis with ease. One approach to alleviate this difficulty is to examine models in an approximation different from conventional perturbation theory, such as the large $N$ expansion. In this approach the quantity $gN$, where $g$ is the perturbative coupling constant, is held fixed as $N \to \infty$ so that one remains in the perturbative régime of the model. Theories which admit such an expansion are those with an internal symmetry and included in this class is quantum electrodynamics, (QED), with $N_f$ flavours of electrons which we will examine in detail in this paper. From the graphical point of view the large $N$ expansion is a reordering of perturbation theory such that chains of bubble graphs are summed first. As in conventional perturbation theory one can renormalize Green’s functions and extract the pole structure from which one can deduce the large $N$ approximation to $\beta(g)$ and $\gamma(g)$. However, one will also run into the same difficulties as perturbation theory, such as the appearance of intractable integrals, which will occur at next to leading order.

This problem is overcome by an alternative approach developed in [1, 2] for the $O(N)$ bosonic $\sigma$ model, which involves examining the theory at the $d$-dimensional critical point, defined as the non-trivial zero of the $\beta$-function, where the theory is finite and massless. Moreover, the fields also obey asymptotic scaling, [1], where the propagator will take the simple conformal structure $1/(x^2)^\alpha$ in coordinate space, with $\alpha$ its critical exponent. In particular, one examines the skeleton Dyson equations of the theory, which are valid at the critical point, and derives a critical point consistency equation which can be solved within the large $N$ expansion for the anomalous dimension, [1, 2]. Since the anomalous dimension exponent, $\eta$, of, say, a bosonic field is related to $\gamma(g)$ via $\eta = (d - 2) + \gamma(g_c)$, where $g_c$ is the
critical coupling, then one can deduce the coefficients of $\gamma(g)$ to all orders in perturbation theory within the particular large $N$ approximation made. This and similar scaling relations emerge because the renormalization group equation takes a simplified form at criticality. Further, the absence of a mass for the basic fields means that one can quite straightforwardly probe the model beyond the leading order of the conventional large $N$ approach.

Recently this technique was applied to QED with $N_f$ flavours of electrons in [3]. There $\eta$ was computed at leading order in the Landau gauge and so it is the purpose of this paper to present a detailed evaluation of the $O(1/N_f^2)$ corrections to the result of [3] using some of the techniques introduced in earlier works, [1, 2, 4, 5], as well as developing others for the specific case in hand. Since the calculation of $\eta$ is in arbitrary dimensions, we will therefore not only provide additional coefficients of $\gamma(g)$ in four dimensional perturbation theory in the $\overline{\text{MS}}$ scheme but also $O(1/N_f^2)$ corrections for the three dimensional model, which is perturbatively super-renormalizable and currently of interest in various problems. As far as we are aware the only previous $O(1/N_f^2)$ calculation in QED was carried out in [6], where $\eta$ was computed in the Feynman gauge but precisely in three dimensions. Unlike the arbitrary dimension analytic result we give here, whose derivation is predominantly algebraic, the calculation of [3] was partly carried out numerically. Moreover, the techniques developed here for four dimensional QED, will be very important when other more physically consistent theories like QCD are solved within this critical point large $N_f$ formalism. Although QED is not a consistent theory in isolation due to the occurrence of the Landau pole at large values of the coupling this does not prevent us from accessing the perturbative region of QED in large $N_f$ as then $g$ is small.

Earlier large $N_f$ analysis of QED was carried out in [7, 8] where the pole structure of the electron self energy and photon electron vertex were determined by explicitly carrying out the large $N_f$ bubble sum in the Landau gauge. The $O(1/N_f)$ corrections to the $\beta$-function were determined as well as the renormalization group function corresponding to the dependence of the renormalized mass with the renormalization scale both in $\overline{\text{MS}}, [7, 8]$, though the wave function renormalization was not studied. To go beyond this leading order by explicitly computing the next to leading order corrections to the bubble sum would be very involved and thus it is appropriate to follow the more efficient and elegant methods of [1, 2] to compute $O(1/N_f^2)$ corrections since it turns out that there are only two 2-loop corrections to consider. Finally, we refer the interested reader to previous $O(1/N^2)$ calculations in other models, ie [1, 2, 4, 5, 9], since they will very much serve as
a basis for the calculation presented here. For instance, since QED involves fermions we will use several results from the much more straightforward $O(1/N^2)$ calculation of $\eta$ in the $O(N)$ Gross Neveu model, [4], which is also fermionic. We note that the large $N$ exponents which have been derived in these other models have all been shown to be in agreement with the appropriate renormalization group functions to as many orders as they have been calculated within explicit perturbation theory using dimensional regularization in the $\overline{MS}$ scheme.

The paper is organised as follows. In section 2, we introduce the necessary notation and formalism we will use and review the leading order analysis of [3], but in coordinate space here. The formal corrections to the leading order consistency equation for $\eta$ are discussed in section 3, where we also deduce the renormalization required to analyse the theory in the critical region. Section 4 deals with some of the calculational techniques required and difficulties encountered, in explicitly computing the $O(1/N^2_f)$ correction graphs. We discuss their application to the photon self energy in section 5, as well as providing some detail on the calculation of the component bosonic graphs which arise and further techniques for their evaluation. A similar discussion for the corrections to the electron self energy are given in section 6, whilst we provide the main result of our efforts in section 7, where $\eta$ is deduced at $O(1/N^2)$.

Finally, we provide various appendices which either list useful identities for general massless two loop Feynman diagrams or contain a library of the more involved basic two loop component graphs which occurred in our analysis.

## 2 Preliminaries.

We begin by introducing the formalism we will use in our analysis as well as recalling the important features of QED we require. To fix notation and conventions we will calculate with the (massless) QED lagrangian,

$$L = i \bar{\psi}^i \slashed{\partial} \psi^i + A_\mu \bar{\psi}^i \gamma^\mu \psi^i - \frac{(F_{\mu\nu})^2}{4e^2} - \frac{(\partial_\mu A^\mu)^2}{2be^2}$$

(2.1)

where unlike the usual formulation we have rescaled the coupling constant, $e$, into the definition of the kinetic term of the $U(1)$ gauge field $A_\mu$ so that the dimensional analysis of each term in (2.1) is completely analogous to the earlier critical point treatment of other models, [1, 3]. Also, we have set $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the field $\psi^i$, $1 \leq i \leq N_f$, corresponds to the electron with
The parameter, $b$, of (2.1) denotes the conventional parameter which appears in a covariant gauge fixing term.

First, we recall the renormalization group functions, $\beta(g)$ and $\gamma(g)$, of QED to the perturbative orders in which they are known in $d = 4 - 2\epsilon$ dimensions $\epsilon < 0$ small, where we set $g = (e/2\pi)^2$. Recently, $\beta(g)$ has been computed to four loops in \(\overline{\text{MS}}\) in $[10]$ and is

\[
\beta(g) = -2\epsilon g + \frac{2N_f}{3}g^2 + \frac{N_f}{2}g^3 - \frac{N_f(22N_f + 9)}{144}g^4 \\
- \frac{N_f}{64}\left[ \frac{616N_f^2}{243} + \left( \frac{416\zeta(3)}{9} - \frac{380}{27} \right)N_f + 23 \right]g^5 + O(g^6) \quad (2.2)
\]

where we use the conventions of $[3, 11]$ and $g$ is the dimensionless coupling constant in $d$ dimensions. It is important to note that (2.2) is what is determined as the $\beta$-function in $(4 - 2\epsilon)$-dimensions in $\overline{\text{MS}}$ using dimensional regularization prior to setting $\epsilon = 0$ and that the coefficients are $d$-independent.

(If they were $d$-dependent then one would not be using $\overline{\text{MS}}$.) It is easy to see that when $d < 4$ (2.2) indicates the existence of a non-trivial zero, $g_c$, of $\beta(g)$ which corresponds to a phase transition. To deduce the perturbative coefficients of the $(4 - 2\epsilon)$-dimensional functions of the renormalization group equation from the large $N_f$ exponents we compute later, the location of the critical point is given from (2.2) by

\[
g_c = \frac{3\epsilon}{N_f} - \frac{27\epsilon^2}{4N_f^2} + \frac{99\epsilon^3}{16N_f^2} + \frac{77\epsilon^4}{16N_f^2} + O\left(\epsilon^5; \frac{1}{N_f^3}\right) \quad (2.3)
\]

In (2.2), the coefficients of the large $N_f$ term at each order in $g$ agree with the explicit large $N_f$ $\overline{\text{MS}}$ renormalization of the $\beta$-function of QED carried out in $[12]$ using dimensional regularization in $d = 4 - 2\epsilon$ dimensions. Further, the critical exponent $1/\nu = -\beta'(g_c)$, which in effect carries information on the perturbative $\beta$-function has recently been calculated using the formalism of $[1, 3]$ by examining (2.1) at the $d$-dimensional critical point, $[3]$, and is in exact agreement with the results of $[12, 10]$. The electron anomalous dimension, $\gamma(g)$, is also known to several orders by restricting the QCD results of $[14, 13]$ to the abelian case and is

\[
\gamma(g) = \frac{b}{2}g - \frac{(4N_f + 3)}{16}g^2 + \frac{(40N_f^2 + 54N_f + 27)}{576}g^3 + O(g^4) \quad (2.4)
\]

in a general covariant gauge. In fact it has been proved in $[16]$ that the only $b$-dependence which appears in the electron anomalous dimension is
at one loop and therefore the coefficients of the two loop and higher terms of (2.4) are gauge independent. Our remarks concerning the lack of $d$-dependence of the coefficients of the powers of $g$ in (2.2) hold equally for this function since it was also computed in $(4-2\epsilon)$-dimensions using dimensional regularization with $\overline{\text{MS}}$. In an earlier work, [3], we computed $\eta = \gamma(g_c)$ in the Landau gauge, $b = 0$, and deduced the $O(1/N_f)$ terms of $\gamma(g)$ in the large $N_f$ expansion to compare with (2.4). These were in total agreement with the coefficients appearing in (2.4) to $O(g^3)$ and subsequently we deduced all the higher order $O(1/N_f)$ coefficients. So, for example, in this gauge, [3],

$$\gamma(g) = -\frac{(4N_f + 3)}{16}g^2 + \frac{(40N_f^2 + 54N_f + 27)}{576}g^3 + \left(\frac{35N_f^3}{1296} + b_1 N_f^2 + c_1 N_f + d_1\right)g^4 + O(g^5)$$

(2.5)

where $b_1$, $c_1$ and $d_1$ are unknown. By calculating $\eta_2$, we will be able to determine the next to leading order coefficients of $\gamma(g)$. Essentially, expanding $\eta$ at a particular order in $1/N_f$ in powers of $\epsilon = 2 - \frac{d}{2}$, the coefficients of $\epsilon^n$ in $\eta$ are related to the $n$th order coefficient of $\gamma(g)$ since $g_c \sim 3\epsilon/N_f$ from (2.3). Moreover, since the first non-trivial terms of the expansion of both $\eta$ at $O(1/N_f)$ and $O(1/N_f^2)$ in powers of $\epsilon$ begin at $O(\epsilon^2)$ they will produce information to all orders on the gauge independent part of the renormalization group function. So in this sense we will regard $\eta$ as gauge independent.

Having recalled the perturbative structure of the field theory in $d = 4 - 2\epsilon$ dimensions and the equivalence of the leading order $1/N_f$ exponents already computed with the technique of [1, 2] for QED we now introduce the formalism of the method we use. First, we note the consequences for the Green’s functions as a result of the existence of a non-trivial fixed point in $d \neq 4$ dimensions. From a statistical physics point of view, near a critical point physical quantities obey simple power law behaviour where the power or critical exponent depends purely on the dimension of spacetime and the parameters corresponding to any internal symmetry by the universality principle. (See, for example, [17].) From the continuous field theory point of view, one is dealing with fields which are conformal at criticality and thus do not involve any mass. Therefore, we take the following coordinate space forms for the asymptotic scaling forms of the propagators of the fields of (2.1), which are consistent with Lorentz symmetry, near criticality.
as, \[3, 4, 18\],

\[
\psi(x) \sim \frac{A[f]}{(x^2)^\alpha} \\
A_{\mu\nu}(x) \sim \frac{B}{(x^2)^\beta} \left[ \eta_{\mu\nu} + \frac{2\beta}{(2\mu - 2\beta - 1)} \frac{x_{\mu} x_{\nu}}{x^2} \right]
\]

(2.6)
as \(x^2 \to 0\). The point is that in this limit, or equivalently \(k^2 \to \infty\), this structure is the dominant part of the propagator which governs the renormalization and is therefore the part which is relevant for deducing the anomalous parts of the critical exponents \(\alpha\) and \(\beta\) which are equivalent to the critical renormalization group functions. We have chosen to work in the Landau gauge for the following reason. Perturbatively, if one used any (covariant) gauge, other than the Landau gauge, then the gauge parameter gets renormalized and thus the gauge changes. Therefore, since the large \(N_f\) expansion is a reordering of perturbation theory such that chains of bubbles are summed first, it is thus important that one works in a gauge, ie the Landau gauge, which is unaffected by renormalization effects, \[8\]. Whilst \(A_{\mu\nu}(x)\) does not appear to take the usual form for the Landau gauge it is easy to transform (2.6) to momentum space using a Fourier transform to observe that its structure will be proportional to \(P_{\mu\nu}(k) = (\eta_{\mu\nu} - k_\mu k_\nu/k^2)\) where \(k\) is the momentum conjugate to \(x\). For completeness and to fix conventions, we note that the Fourier transform we use is \[1\],

\[
\frac{1}{(x^2)^\alpha} = \frac{a(\alpha)}{2^{2\alpha-\mu}} \int \frac{e^{ikx}}{(k^2)^{\mu-\alpha}}
\]

(2.7)
where we have set \(a(\alpha) = \Gamma(\mu - \alpha)/\Gamma(\alpha)\) and the dimension of spacetime, \(d\), to be \(d = 2\mu\). For the most part of this paper, we will work in coordinate space though it is straightforward to map from one space to the other via (2.7) since the fields are massless. The quantities \(A\) and \(B\) in (2.6) are the amplitudes of each field and are independent of \(x\) whilst \(\alpha\) and \(\beta\) are the exponents of the respective fields. They are related to the exponents we will calculate, via, \[3\],

\[
\alpha = \mu + \frac{1}{2}\eta \quad , \quad \beta = 1 - \eta - \chi
\]

(2.8)
where \(\eta\) is the electron anomalous dimension and \(\chi\) is the anomalous dimension of the electron photon vertex of (2.1). Both the latter quantities depend only on \(\mu\) and \(N_f\) and are \(O(1/N_f)\) within the large \(N_f\) expansion and will be calculated to \(O(1/N_f^2)\) and \(O(1/N_f)\) respectively in this paper.
As a preliminary to introducing the formal equations which will be solved to give an expression for $\eta_2$ we review briefly the leading order analysis. Whilst this was done initially in [3], that calculation was carried out in momentum space. Although the same results will be obtained in coordinate space as in momentum space we will mostly work throughout this paper in coordinate space. The exponent $\eta$ is obtained by examining the skeleton Dyson equations with dressed propagators at the critical point $g_c$, (2.3). As the fields obey asymptotic scaling there then one can replace the propagators constituting the Dyson equations with (2.6) which will therefore result in equations involving $\alpha$ and $\beta$ which can be solved. The Dyson equations which we consider here are illustrated in figs. 1 and 2 having been truncated at $O(1/N^2)$. (Ordinarily at $O(1/N^2)$ in this approach one has in addition several three loop graphs but these vanish in QED due to Furry’s theorem.)

The quantities $\psi^{-1}$ and $A^{-1}_{\mu\nu}$ correspond to the inverse propagators and their asymptotic scaling forms are obtained from (2.6) by first transforming them to momentum space, using (2.7) and its derivatives, and then inverting them via $G^{-1}G = 1$, before transforming back to coordinate space, [1, 2]. As in other critical point analyses of models involving a gauge field, [18], the asymptotic form of its inverse propagator is determined by restricting the inversion to be on the transverse subspace of momentum space, since only the transverse piece is physically relevant, [19]. Following this procedure,

$$
\psi^{-1}(x) \sim \frac{r(\alpha - 1)\delta}{A(x^2)^{2\mu - \alpha + 1}}
$$

$$
A^{-1}_{\mu\nu}(x) \sim \frac{m(\beta)}{B(x^2)^{2\beta - \mu}} \left[ \eta_{\mu\nu} + \frac{2(2\mu - \beta)(x_{\mu}x_{\nu})}{(2\beta - 2\mu - 1)x^2} \right]
$$

as $x^2 \to 0$ where

$$
r(\alpha) = \frac{a(\alpha - \mu)}{\pi^{2\mu}(\mu - \alpha)a(\alpha)} , \quad m(\beta) = \frac{[4(\mu - \beta)^2 - 1]a(\beta - \mu)}{4\pi^{2\mu}(\mu - \beta)^2a(\beta)}
$$

and (2.9) was first given in [4].

Thus with (2.6), (2.9) and (2.10) and, for the moment, retaining only the leading one loop graphs of figs. 1 and 2 the Dyson equations at criticality are equivalent to

$$
0 = r(\alpha - 1) + \frac{2(2\mu - 1)(\beta - \mu + 1)z}{(2\mu - 2\beta - 1)}
$$

(2.12)
for the electron where we have factored off the common piece \( \frac{x}{x^2} \). Also,

\[
0 = \frac{m(\beta)}{(x^2)^{2\mu - \beta}} \left[ \eta_{\mu\nu} + \frac{2(2\mu - \beta)}{(2\beta - 2\mu - 1)} \frac{x_\mu x_\nu}{x^2} \right] - \frac{4zN_f}{(x^2)^{2\alpha - 1}} \left( \eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \right)
\]

(2.13)

for the photon and we have set \( z = A^2 B \). As the transverse part of the gauge field in momentum space is the only physically meaningful part of (2.13), \[19\], we project this out by first transforming to momentum space, multiply the resulting equation by the projection operator \( P_{\mu\nu}(k) \), before mapping back to coordinate space. Equivalently one can make the following replacement for the longitudinal components of (2.13) in \( x \)-space, valid for all exponents \( \alpha \), which corresponds to this operation, ie

\[
\frac{x_\mu x_\nu}{(x^2)^{\alpha}} \rightarrow \frac{\eta_{\mu\nu}}{2(\alpha - 1)(x^2)^{\alpha - 1}}
\]

(2.14)

Thus the relevant piece of (2.13) is

\[
0 = \frac{(\mu - \beta)(\beta - \mu + 1)m(\beta)}{[4(\mu - \beta)^2 - 1]} + \frac{4zN_f}{(2\mu - 1)(2\alpha - 1)}
\]

(2.15)

In writing down (2.15) we note that the powers of \( x^2 \), which cancel in the leading order analysis of (2.12), are cancelled after projecting out the relevant piece of (2.13).

We now have two equations, (2.12) and (2.15), involving two unknowns, \( z \) and \( \eta \), and so eliminating \( z \) between both the consistency equation for \( \eta \) emerges, ie

\[
0 = \frac{(\mu - \beta)(\beta - \mu + 1)m(\beta)}{[4(\mu - \beta)^2 - 1]} + \frac{2N_f(\alpha - 1)r(\alpha - 1)}{(2\mu - 1)(2\alpha - 1)}
\]

(2.16)

which can be solved with \( \alpha = \mu + \frac{1}{2} \eta \) and \( \beta = 1 \) to give

\[
\eta_1(\mu) = - \frac{(2\mu - 1)(2 - \mu)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)}
\]

(2.17)

where \( \eta = \sum_{i=1}^{\infty} \eta_i/N_i \) and hence from either (2.12) or (2.15),

\[
z_1 = \frac{(2\mu - 3)\Gamma(\mu + 1)\Gamma(\mu)\eta_1}{4\pi^2\mu(2\mu - 1)(\mu - 2)}
\]

(2.18)

which is required later. Although \( z_1 = 0 \) in three dimensions this does not imply that the formalism breaks down in this dimension. First, there are
corrections to \( z \) which ensure \( A \neq 0 \) and \( B \neq 0 \) and if one were to compute \( z_1 \) in arbitrary covariant gauge then the factor becomes \( (2\mu - 3 + b) \), [17]. Second, in the determination of the final results for exponents it turns out that the appearance of such \( (2\mu - 3) \) factors always arise with a factor \( (2\mu - 3)^{-1} \) and therefore there is no difficulty in obtaining non-zero and non-singular results when the final \( d \)-dimensional results are restricted to \( d = 3 \). Indeed when (2.17) is evaluated in three dimensions it agrees with the wave function renormalization constant calculated in an explicit three dimensional large \( N_f \) renormalization of QED, [20], which is another strong check on our analysis in addition to the already stated consistency of the \( \epsilon \)-expansion of (2.17) with the leading order \( O(g^3) \) terms of (2.14). Whilst (2.17) was obtained in a similar fashion in momentum space the coordinate space approach provides the starting point for computing the corrections to \( \eta_1 \). For instance, one can expand (2.16) to the next order in \( 1/N_f \) but this, of course, neglects the higher order graphs which have to be included.

3 Corrections to consistency equations.

In this section, we derive the formal corrections to the consistency equations to determine \( \eta_2 \) by including the higher order corrections to the skeleton Dyson equations with dressed propagators. For QED there are only two such correction graphs which are given in figs. 1 and 2. First, we formally denote the values of the two loop integrals by \( \Sigma \) for the electron self-energy and \( \Pi_{\mu\nu} = \eta_{\mu\nu}\Pi + x_\mu x_\nu \Xi/x^2 \) for the photon self energy in coordinate space. We will discuss their explicit evaluation in subsequent sections but note that by value of the graph we mean the result one would obtain by computing the integrals with unit amplitudes and symmetry factors (such as minus signs for fermion loops etc) excluded.

Thus, the graphs of figs. 1 and 2 are equivalent to

\[
0 = r(\alpha - 1) + (x^2)^{\chi} f(\beta) z + (x^2)^2 \chi z^2 \Sigma \tag{3.1}
\]

and

\[
0 = \frac{m(\beta)}{(x^2)^{2\mu - \beta}} \left[ \eta_{\mu\nu} + \frac{2(2\mu - \beta)}{(2\beta - 2\mu - 1)} \frac{x_\mu x_\nu}{x^2} \right] - \frac{4zN_f}{(x^2)^{2\alpha - 1}} \left( \eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \right)
- \frac{z^2N_f}{(x^2)^{4\alpha + \beta - 2\mu - 2}} \left[ \eta_{\mu\nu}\Pi + \frac{x_\mu x_\nu \Xi}{x^2} \right] \tag{3.2}
\]

where \( f(\beta) = 2(2\mu - 1)(\beta - \mu + 1)/(2\mu - 2\beta - 1) \). Unlike at leading order we cannot cancel off the powers of \( x^2 \) since now \( \chi = O(1/N_f) \) which will give
Further, from the naive computation of $\Sigma$, $\Pi$ and $\Xi$ it turns out that they are in fact infinite due to divergences which arise from vertex subgraphs when $\chi = 0$. This situation, indeed, is completely analogous to that at $O(1/N^2)$ in other models, [1, 2, 4, 5]. As a first step in treating these infinities we introduce a regulator, $\Delta$, by shifting the exponent of the gauge field, $\beta$, by $\beta \rightarrow \beta - \Delta$, where $\Delta$ is an infinitesimal quantity playing much the same role as $\epsilon = (4 - d)/2$ does in dimensional regularization of four dimensional perturbative calculations. With the introduction of $\Delta$ we formally define the following quantities

$$
\Sigma = \frac{K}{\Delta} + \Sigma' , \quad \Pi = \frac{P}{\Delta} + \Pi' , \quad \Xi = \frac{X}{\Delta} + \Xi'
$$

(3.3)

where the prime, $'$, denotes the completely finite parts of $\Sigma$, $\Pi$ and $\Xi$ with respect to $\Delta$ and both the residues, $K$, $P$ and $X$, and finite pieces are purely functions of $\mu$, $\alpha$ and $\beta$. The poles with respect to $\Delta$ of (3.3) which will therefore appear in (3.1) and (3.2) are removed by the vertex counterterm which is available at each vertex in the one loop graphs of figs. 1 and 2. Thus denoting this counterterm by $u$, which can be expanded about $u = 1$ in the large $N_f$ expansion, as

$$
u = 1 + \frac{u_1}{N_f} + O\left(\frac{1}{N_f^2}\right)
$$

(3.4)

the regularized Dyson equations therefore become,

$$
n = r(\alpha - 1) + zu^2(x^2)^{\chi + \Delta} f(\beta - \Delta) + z^2(x^2)^{2\chi + 2\Delta} \left(\frac{K}{\Delta} + \Sigma'\right)
$$

(3.5)

and

$$
n = \frac{m(\beta - \Delta)}{(x^2)^{2\mu - \beta + \Delta}} \left[\eta_{\mu\nu} + \frac{2(\beta - 2\mu - 1 - 2\Delta)}{(2\beta - 2\mu - 1 - 2\Delta)} x_{\mu} x_{\nu} x^2\right]
$$

$$
- \frac{4zu^2N_f}{(x^2)^{2\alpha - 1}} \left(\eta_{\mu\nu} - \frac{2x_{\mu} x_{\nu}}{x^2}\right)
$$

$$
- \frac{z^2N_f}{(x^2)^{4\alpha + \beta - 2\mu - 2 - \Delta}} \left[\frac{1}{\Delta} \left(P\eta_{\mu\nu} + X x_{\mu} x_{\nu} x^2\right) + \Pi'\eta_{\mu\nu} + \Xi' x_{\mu} x_{\nu} x^2\right]
$$

(3.6)

With (3.4) and expanding each term of the electron equation, (3.5), to $O(1/N^2_f)$ and the finite parts in $\Delta$, then the divergent terms at $O(1/N^2_f)$ with respect to $\Delta$ are

$$
\frac{4u_1 z_1 (2\mu - 1)(\beta - \mu + 1)}{(2\mu - 2\beta - 1)} + \frac{Kz^2}{\Delta}
$$

(3.7)
Setting (3.7) to zero gives a finite consistency equation for the electron so that the $\Delta \to 0$ limit can be achieved without difficulties. Of course this choice corresponds to a minimal scheme. If one were to absorb finite parts into $u_1$ then this would alter only the values of the amplitudes but not the exponents. If one were to absorb finite parts into $u_1$ then this would alter only the values of the amplitudes but not the exponents. The resulting finite equation, however, contains $\ln x^2$ type terms which would otherwise spoil the analysis at criticality when $x^2 \to 0$. To avoid this difficulty, $\chi$, which has yet to be determined is defined in such a way that the $\ln x^2$ terms are absent. So with

$$\chi_1 = -\frac{(2\mu - 2\beta - 1)Kz_1}{2(2\mu - 1)(\beta - \mu + 1)}$$

the finite Dyson equation at $O(1/N_f^2)$ in the critical region becomes

$$0 = r(\alpha - 1) + \frac{2z(2\mu - 1)(\beta - \mu + 1)}{(2\mu - 2\beta - 1)} + z^2 \Sigma' + \frac{Kz^2}{(2\mu - 2\beta - 1)(\beta - \mu + 1)}$$

Unlike in other $O(1/N^2)$ analyses, where only the finite parts of the higher order corrections contributed to $\eta_2$, here, at least formally, the residue $K$ appears in (3.9). This is due to the fact that in the non-regularized equation there is a non-zero function of $\beta$ multiplying the term involving the counterterm, $u$. Thus when this function is expanded in powers of $\Delta$, a finite term remains in the $\Delta \to 0$ limit, when the linear term of $f(\beta - \Delta)$ multiplies the counterterm $u_1$, which involves the residue $K$.

The treatment of the photon self energy corrections are somewhat similar to those of the electron, though as at leading order we consider only that part of (3.6) which corresponds to the transverse part in momentum space since this is the physically important piece. Thus, using (2.14) we consider

$$0 = \frac{2(\mu - \beta + \Delta)m(\beta - \Delta)}{(2\mu - 2\beta + 1 + 2\Delta)N_f} - \frac{8zu^2(\alpha - 1)(x^2)^{\alpha + \Delta}}{(2\alpha - 1)}$$

$$- z^2(x^2)^{2\alpha + 2\Delta} \left[ \Pi + \frac{\Xi}{2(4\alpha + \beta - 2\mu - 2 - \Delta)} \right]$$

where we note that the same counterterm as (3.5), of course, arises. Again isolating the divergent terms with respect to $\Delta$ at $O(1/N_f^2)$ they are absorbed by setting

$$u_1 = -\frac{(2\alpha - 1)z_1}{16(\alpha - 1)} \left[ P + \frac{X}{2(4\alpha + \beta - 2\mu - 2)} \right]$$
which does not appear to be equivalent to that obtained from (3.7). For the moment, we note that when the explicit values for $P$, $X$ and $K$ are determined in a later section, we will observe that (3.7) and (3.11) are in agreement. Further, $\ln x^2$ terms are removed by again defining $\chi_1$ appropriately. In this case, we set

$$
\chi_1 = -\frac{(2\alpha - 1)z_1}{8(\alpha - 1)} \left[ P + \frac{X}{2(4\alpha + \beta - 2\mu - 2)} \right] \quad (3.12)
$$

which will, of course, also agree with (3.8) giving us at least one check on the explicit evaluation of $\Sigma$ and $\Pi_{\mu\nu}$. Consequently, the finite Dyson equation for the photon at criticality is

$$
0 = \frac{2(\mu - \beta)m(\beta)}{(2\mu - 2\beta + 1)N_f} - \frac{8z(\alpha - 1)}{(2\alpha - 1)} - z^2 \left[ \Pi' + \frac{\Xi'}{2(2\alpha - 1)} + \frac{X}{2(2\alpha - 1)^2} \right] \quad (3.13)
$$

where, like (3.9), the residue of the two loop correction, $X$, will also give a contribution. It arises in a different way to the appearance of $K$ in (3.9), through the expansion in powers of $\Delta$ of the coefficient which multiplies $\Xi$ after restriction to the transverse piece.

Whilst we have formally derived the finite corrections to (2.12) and (2.15) we require the explicit values of $\Sigma$, $\Pi$ and $\Xi$, and hence $\chi_1$, which is needed to evaluate the functions of $\beta$ in (3.9) and (3.13) to $O(1/N_f^2)$ in order to obtain the formal consistency equation satisfied by $\eta_2$.

## 4 Computational tools for computing $\Sigma$ and $\Pi_{\mu\nu}$.

Before detailing the explicit calculation of the two loop corrections we will now review and develop the necessary techniques which will be required. First, we recall that in solving the bosonic and supersymmetric $O(N)$ $\sigma$ models at $O(1/N^2)$ in $\mathbb{1}$ $\mathbb{2}$ $\mathbb{3}$ $\mathbb{4}$ $\mathbb{5}$, extensive use was made of the technique known as uniqueness which was first introduced in $\mathbb{21}$ and subsequently used and extended in various forms in $\mathbb{22}$-$\mathbb{24}$. It is applicable to models which involve a 3-vertex, where the exponents of the propagators forming the vertex are initially arbitrary. In endeavouring to compute such a 3-vertex in coordinate space it turns out that the calculation cannot be completed in closed form unless the sum of the exponents are restricted to be the dimension of spacetime, $2\mu$, for a purely bosonic vertex with no derivative
couplings, [23]. When this uniqueness criterion is satisfied then the integral
can be completed to yield the product of various propagators multiplied
by a factor dependent on the exponents of the initial vertex. We have
summarized this rule in fig. 3 where the Greek letter beside each line denotes
the exponent of that propagator and we have written the product of the
three resulting propagators graphically as a triangle. Also, \( \nu(\alpha_1, \alpha_2, \alpha_3) = \pi^\mu \prod_{i=1}^{3} a(\alpha_i) \). Further, if \( \alpha + \beta + \gamma = 2\mu + n \) for any positive integer \( n \) then this vertex is also integrable, yielding a product of triangle graphs. We
have outlined this rule in detail since in the \( \sigma \) models its 3-vertex exponents
indeed sum to the uniqueness value, [2, 5], which meant that the technique
could be applied there. More recently, a similar rule was developed for the
vertex of the Gross Neveu model, where the basic uniqueness condition due
to the presence of fermions becomes \( 2\mu + 1 \), [18].

In the model we are concerned with here the relevant 3-vertex contains
a gauge field interacting with two fermionic fields and therefore we need
to develop the analogous uniqueness integration value for this vertex. It is
illustrated graphically in fig. 4 in coordinate space, where the indices \( \mu \) and
\( \nu \) refer to those which appear in (2.6) and the numerator of the integral is
\( (\gamma - \hat{z}) \gamma^\mu (\hat{z} - \hat{y}) \). We use the convention that a fermion propagating from
\( x \) to \( y \) has a factor \( (\hat{x} - \hat{y}) \). Repeating the analysis described to derive the
result of fig. 3 yields a more involved expression. After introducing Feynman
parameters for each propagator and completing two of the four integrations
one obtains a sum of various integrals each involving one hypergeometric
function. Essentially, the uniqueness condition emerges by choosing the
individual arguments of this function in such a way that it is equivalent to
a simple algebraic function, after which the integral can be computed and
the result of fig. 3 obtained for the purely bosonic vertex. In the case of
fig. 4, it turns out that the minimal uniqueness condition for this vertex is
\( \alpha_1 + \alpha_2 + \alpha_3 = 2\mu + 2 \), for the Landau gauge, which results in a sum of
triangle graphs analogous to the right side of fig. 3 but with bosonic and
fermionic propagators. We do not give the full graphical expression here
for the following simple reason. If one now examines the electron photon
vertex, then the value of the vertex, when the leading order exponents are
substituted, is \( 2\alpha + \beta = 2\mu + 1 \), which is one step from the uniqueness
value unfortunately. This means that unlike the models of [2, 4, 5], we do
not have a direct integration rule with which to perform our calculations.

To circumvent this difficulty a different approach will be required which
we now outline. The ideas we use, however, derive from similar difficulties in
other models. For instance, in the Gross Neveu model, [4], it was noted that
one can always rewrite a graph involving fermions in terms of its constituent purely bosonic graphs, by explicitly computing the trace over $\gamma$-matrices. A consequence of this is that some of the exponents of the resulting graphs are reduced by an integer so that the vertices become equal to the (bosonic) uniqueness value and therefore integrable from fig. 3. Whilst it was possible to introduce methods in the Gross Neveu model, [4], where taking the trace is not necessary, it is an important observation for the present case. The application of this to QED will, in principle, be similar though to derive the constituent bosonic graphs involves taking traces over a larger number of $\gamma$-matrices. We will mention further simplifying aspects later but note that in our manipulations we will make use only of the algebra (A.1) which is, of course, valid in arbitrary dimensions.

Secondly, one technique which is used frequently in computing massless Feynman integrals is that of integration by parts, [2]. Indeed a recursion relation was developed for a two loop integral in [2] which was primarily required for changing the value of the vertices by $\pm1$, to obtain two loop graphs which had either integrable triangles or vertices. Equally we can apply this technique to the gauge vertex of fig. 4 where the gauge field propagator is

$$\frac{1}{(z^2)^{\alpha_1}} \left[ \eta_{\mu\nu} + \frac{2\alpha_1}{(2\mu - 2\alpha_1 - 1) z^2} z_{\mu} z_{\nu} \right]$$  \hspace{1cm} (4.1)

and its numerator was noted earlier. In the corresponding integral we choose to rewrite part of the second term of (4.1) as

$$\frac{z_{\mu}}{(z^2)^{\alpha_1+1}} = -\frac{1}{2\alpha_1} \frac{\partial}{\partial z^\mu} \left( \frac{1}{(z^2)^{\alpha_1}} \right)$$  \hspace{1cm} (4.2)

so that when one integrates by parts, factors from differentiating the denominators will combine with the numerator to reduce the number of $\gamma$-matrices present by two in some terms. Thus carrying this out and rearranging, one obtains the result illustrated graphically in fig. 5 for arbitrary values of the exponents. Therefore, one can rewrite the 3-vertex involving a photon in Landau gauge in terms of three other graphs which have a simpler structure. For instance, the first graph is proportional to what one would obtain if the second term of (4.1) was ignored. In the remaining two 3-vertices the long dashed vertical line, with an index at its external end, joining the point of integration, $z$, is to be understood as the propagator $z_{\mu}/(z^2)^{\alpha_1}$, and we also note they involve only one $\gamma$-matrix. The usefulness of this rule will become apparent later but we remark that using it in a two loop graph will, for
instance, alter the structure of the propagator joining to the external ends of the original 3-vertex, in the case of the second and third terms of the right side of fig. 5. Therefore, two further rules are required, which are illustrated in figs. 6 and 7, and each is established by an integration by parts similar to (4.2). We note that the numerator of the graph of the left side of fig. 6 is $(y - \hat{r})(\hat{r} - \hat{s})$, whilst that of fig. 7 is $(y - \hat{s})\gamma_{\mu}$.

5 Computation of $\Pi_{\mu\nu}$.

In this section, we outline the calculation of $\Pi_{\mu\nu}(x)$ and concentrate on the main features. As discussed previously, since the method of uniqueness cannot be used directly, we have to develop an algorithm which will give rise to integrals to which one can apply this method.

First, the detailed graphical expression for $\Pi_{\mu\nu}(x)$ is given in fig. 8, where the Greek indices at the vertices (internal and external) correspond to the Lorentz index of the corresponding uncontracted $\gamma$-matrix. Therefore, this two loop integral involves a trace over eight $\gamma$-matrices. As a first step in reducing this number, we use the integration by parts rules given in the previous section. Applying fig. 5 first yields three graphs, one of which is the first graph of fig. 9, with the remainder treated using fig. 6 at the opposite vertex. After a suitable rearrangement they correspond to the final three graphs of fig. 9 where one has, of course, to include the regularization $\Delta$. We have given only the graphical form of these graphs since the simple factors associated with each can be readily deduced from figs. 5 and 6. The final graph is multiplied by the factor,

$$\text{tr}(\not{x}\gamma_{\mu}\not{x}\gamma_{\nu}) = 4[2x^\mu x^\nu - x^2 \eta^{\mu\nu}]$$

For simplicity, we denote the contribution from the first graph of fig. 9, with correct factors by $I$ and that from the others by $II$, treating each separately.

To illustrate the type of calculations required once the integral has been broken up into its constituents, we concentrate for the moment on the final graph of fig. 9, ie $\langle \alpha, \alpha, \alpha, \alpha, \alpha, \beta - 1 - \Delta \rangle$, where we have defined $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$ to be the general bosonic two loop integral of fig. 10. To compute $\langle \alpha, \alpha, \alpha, \alpha, \beta - 1 - \Delta \rangle$, we use the methods of subtractions developed in [2] and note that in the absence of the regulator, $\Delta$, the graph is integrable using the uniqueness rule of fig. 3, since $2\alpha + (\beta - 1) = 2\mu$, as $\alpha = \mu$ and $\beta = 1$, at this approximation in large $N_f$. Carrying out the naive integration at $\Delta = 0$, will yield a result proportional to $\Gamma(0)$, which
is infinite. However, when the theory is regularized by $\Delta$, one loses uniqueness at each vertex and cannot naively apply fig. 3. Instead, recalling that only the pole and finite pieces of $\Pi_{\mu\nu}$, with respect to $\Delta$, are required to deduce $\eta_2$, one need only consider the combination $(A - B) + B$, where $A = \langle \alpha, \alpha, \alpha, \alpha, \beta - 1 - \Delta \rangle$, and $B$ is such that in the presence of $\Delta$ it is calculable but has the same divergence structure as $A$. Thus, $(A - B)$ is finite and can be deduced for $\Delta = 0$, [2]. A suitable choice for $B$ is

$$\langle \alpha, \alpha, 0, \beta - 1 - \Delta \rangle \ + \ \langle \alpha, 0, \alpha, \beta - 1 - \Delta \rangle \quad (5.2)$$

which corresponds to a sequence of integrable chains of propagators for $\Delta \neq 0$. The combination $(A - B)$ is obtained by completing one integration with $\Delta = 0$ first, before shifting each exponent of the propagators of the resulting chain by a temporary regulator, $\delta$, so that poles in $1/\delta$ can be seen to cancel before setting $\delta$ to zero, [2]. Thus, one obtains

$$\langle \alpha, \alpha, \alpha, \alpha, \beta - 1 - \Delta \rangle = \frac{2\pi^2 a^2(\alpha)a(\beta - 1)}{\Gamma(\mu)} \left[ 1 + \Delta \left( B(\beta) - B(\alpha - 1) \right) + \frac{1}{(\beta - 1)} + \frac{(\beta - 1)}{2(\alpha - 1)(\mu - \beta)} \right] \quad (5.3)$$

where $B(x) = \psi(\mu - x) + \psi(x)$, and $\psi(x)$ is the logarithmic derivative of the $\Gamma$-function, without specifying the leading order values of $\alpha$ and $\beta$ as $\mu$ and 1, respectively. This last remark is actually crucial, since naively setting $\alpha = \mu$ and $\beta = 1$ in (5.3) one immediately discovers that the formal expression appears to be infinite which is due to the fact that the value $\alpha = \mu$ is the anti-uniqueness value of a bosonic field, [2]. In reality this problem is not significant since there are cancelling infinities which arise from other graphs comprising $\Pi_{\mu\nu}$, so that the sum of all the constituent pieces is indeed finite upon setting $\alpha = \mu$ and $\beta = 1$ in the final expression. Indeed this cancellation will provide us with a stringent check on our analysis and was also a feature of the much simpler Gross Neveu model, [4], if one rewrites the $O(1/N^2)$ corrections there in terms of its bosonic components.

For the remainder of the calculation one essentially breaks up the original integral into similar bosonic two loop graphs and computes them using this method of subtractions though some are not as straightforward as the one outlined above. For instance, the first two graphs of fig. 11 can be treated directly, except that in the second there is only one subtraction to consider, since the divergence arises in the left vertex subgraph. Whilst the vertices of the third graph of fig. 11 are unique, one cannot apply the
subtraction procedure directly, since the infinity structure of the naive subtraction $\langle \alpha, 0, \alpha - 1, \alpha, \beta - \Delta \rangle$ does not match that of the original. Instead an alternative technique is required which involves rewriting the integral in terms of other graphs, which can be computed by subtractions. In \cite{2}, recursion relations for graphs of the type of fig. 10 were developed by integration by parts which have the effect that the exponents of the lines comprising the new integrals are adjusted by $\pm 1$ relative to the original. Whilst we were unable to make use of that rule here it is clear that a similar rule will be needed.

Such an alternative was given in \cite{24, 25}, being derived in the latter by considering the uniqueness rule for a bosonic 3-vertex when its value is $2\mu + 1$ and is illustrated in fig. 12. Applying it to the upper internal vertex of the graph of fig. 10, for example, one obtains the recursion relation \cite{25},

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle = \frac{\alpha_5(\mu - \alpha_5 - 1)}{(\alpha_1 - 1)(\alpha_2 - 1)x^2} \langle \alpha_1 - 1, \alpha_2 - 1, \alpha_3, \alpha_4, \alpha_5 + 1 \rangle$$

$$+ \frac{(\alpha_1 + \alpha_2 - \mu - 1)}{(\alpha_1 - 1)} \langle \alpha_1 - 1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$+ \frac{(\alpha_1 + \alpha_2 - \mu - 1)}{(\alpha_2 - 1)} \langle \alpha_1, \alpha_2 - 1, \alpha_3, \alpha_4, \alpha_5 \rangle \quad (5.4)$$

Further, one can derive more relations by either applying fig. 12 to other vertices or by first making use of the possible general transformations given in the table of \cite{2}, then applying the rule before undoing the initial transformation by applying its inverse. In appendix B, we have listed the other recursion relations we required for this calculation, but note that we believe this list is not exhaustive. Returning to the third graph of fig. 11 and applying (B.10), results in the following three integrals, $\langle \alpha, \alpha, \alpha, \alpha, \beta - \Delta \rangle$, $\langle \alpha, \alpha - 1, \alpha, \alpha - 1, \beta - \Delta \rangle$ and $\langle \alpha - 1, \alpha, \alpha - 1, \alpha, \beta - \Delta \rangle$, where the final two are equivalent under a change of integration variable and each is calculable by the subtraction procedure.

In the first part of appendix C, we have listed the basic library of two loop building block integrals we required to compute $\Pi_{\mu\nu}$, each expanded to its finite part with respect to $\Delta$. However, we have not given the values of those integrals which can be derived directly by subtractions, only those which used the recursion relations given in appendix B since they are more tedious to deduce. Where possible, we have calculated several of these integrals in independent ways, using different recursion relations and so are confident that the expressions are in fact correct.
One basic integral, \( \langle \alpha, \alpha, \alpha, \alpha, \beta - 2 - \Delta \rangle \), deserves special attention, since it does not have any uniqueness values at \( \Delta = 0 \) and also cannot be related to any other basic integral for reasons which will become apparent. We compute it by first applying the transformation \( \leftarrow \), in the notation of \[2\], and then use the recursion relation (B.12), which results in \( \langle \alpha, \mu - \alpha + 1, \mu - \alpha, \alpha, \mu - 1 - \Delta \rangle \) and \( \langle \alpha - 1, \mu - \alpha + 1, \mu - \alpha, \alpha, \mu - 1 - \Delta \rangle \). Applying \( \leftarrow \) to the former yields \( \langle \alpha, \mu - \alpha + 1, \mu - \alpha, \alpha, \mu - 1 - \Delta \rangle \) and \( \langle \alpha - 1, \mu - \alpha + 1, \mu - \alpha, \alpha, \mu - 1 - \Delta \rangle \).

One then applies (B.8) appropriately to the other graph before undoing the initial \( \leftarrow \) transformation. One of the resulting two graphs can be treated as in an earlier discussion whilst the other is \( \langle \alpha - 1, \alpha - 1, \alpha, \alpha - 1, \beta - \Delta \rangle \).

We were unable to evaluate this graph directly for arbitrary \( \alpha \) and \( \beta \), though it is in fact finite, both with respect to \( \Delta \) and setting \( \alpha = \mu \) and \( \beta = 1 \), and had therefore to leave it unevaluated in our analysis until we set \( \alpha = \mu \) and \( \beta = 1 \) after all contributions to \( \Pi_{\mu \nu} \) had been summed. After this substitution we can relate it to a known integral \( ChT(1, 1) \), in the notation of \[2\], by applying the transformation \( \rightarrow \). Thus,

\[
\langle \mu - 1, \mu - 1, \mu - 1, \mu - 1, 1 \rangle = \frac{a^3(\mu - 1)}{a(2\mu - 3)} ChT(1, 1) 
\] (5.5)

and \( ChT(1, 1) = 3\pi^{2\mu} a(2\mu - 2)\Gamma(\mu - 1)[\psi'(\mu - 1) - \psi'(1)] \). Adding all the pieces, we have

\[
\langle \alpha, \alpha, \alpha, \alpha, \beta - 2 - \Delta \rangle = \frac{2\pi^{2\mu}(\mu - 1)^2 a^2(\alpha) a(\beta - 1)}{(\alpha - 1)^2(\beta - 1)\Gamma(\mu)} + \frac{2(\beta + 1 - \mu)(\beta - \mu)(2\alpha - 3)}{(\alpha - 1)^3} \times \langle \alpha - 1, \alpha - 1, \alpha - 1, \alpha - 1, \beta - \Delta \rangle
\] (5.6)

Several other basic building block integrals also involved \( \langle \alpha - 1, \alpha - 1, \alpha - 1, \alpha - 1, \beta - \Delta \rangle \), which have therefore been left unevaluated in the list in appendix C.

We conclude the section by returning to the graphs of fig. 9 and briefly mention the key points required to compute the first three aside from those already discussed. For the remaining graphs of \( II \), one occurs in the evaluation of \( I \), whilst the other can be treated by a subtraction, even though it involves one fermion propagator. (The rules for integrating chains of propagators involving fermions were given in \[4\], for example.)
Finally, to treat \( I \) we have used the result
\[
\gamma^\rho \gamma^\mu \gamma^\nu \gamma^\sigma \gamma_\rho = 4 \eta^\mu^\sigma \gamma^\nu - 2 \gamma^\mu \gamma^\sigma \gamma^\nu \\
- 2 \gamma^\nu \gamma^\mu \gamma^\sigma - 2(\mu - 1) \gamma^\mu \gamma^\nu \gamma^\sigma \tag{5.7}
\]
which is valid for arbitrary dimensions and has been derived solely from (A.1). It turns out that the three terms of (5.7) with three \( \gamma \)-matrices give graphs, which although involve fermions, can be computed by the subtraction procedure without having to explicitly project out the \( \eta^{\mu\nu} \) and \( x^\mu x^\nu \) components. For instance, the graph involving only the three \( \gamma \)-matrices of the last term of (5.7) is
\[
\frac{8\pi^2 a^2 (\alpha - 1) a(\beta) Q_{\mu\nu}}{\Delta(\alpha - 1)^2 \mu \Gamma(\mu - 1)} \left[ 1 + \Delta \left( B(\beta) - B(\alpha - 1) - \frac{1}{(\alpha - 1)} \right) \right] \tag{5.8}
\]
where \( Q_{\mu\nu}(x) = \eta_{\mu\nu} - 2 x^\mu x^\nu / x^2 \). For the graph corresponding to the first term of (5.7), we had to treat its transverse and longitudinal components separately but in doing so each piece involved at most four \( \gamma \)-matrices, and therefore we employed (A.5).

Thus having given an extensive discussion of how to compute \( \Pi_{\mu\nu} \) we note that adding all contributions, \( I \) is, at \( \alpha = \mu \) and \( \beta = 1 \),
\[
- \frac{32\pi^2 a^2 (\alpha - 1) a(\beta) Q_{\mu\nu}}{\Delta(\alpha - 1)^2 \mu \Gamma(\mu - 1)} \left[ \frac{1}{\mu} + \Delta \left( \frac{3}{2(\mu - 1)^3} - \frac{2}{\mu(2\mu - 3)} - \frac{3\hat{\Theta}(\mu)}{2(\mu - 1)} \right) \right] \tag{5.9}
\]
where \( \hat{\Theta}(\mu) = \psi'(\mu - 1) - \psi'(1) \). Whilst we find
\[
\frac{16\pi^2 Q_{\mu\nu}}{(2\mu - 3) \Gamma(\mu) \Delta} \left[ 1 - \frac{2\Delta}{(2\mu - 3)} \right] \tag{5.10}
\]
for \( II \). Thus summing (5.8) and (5.9) we obtain
\[
\Pi_{\mu\nu} = \frac{16\pi^2 a^2 (\alpha - 1) a(\beta) Q_{\mu\nu}}{\Delta(\alpha - 1)^2 \mu \Gamma(\mu - 1)} \left[ \frac{1}{\mu} + \Delta \left( \frac{3}{2(\mu - 1)^3} - \frac{2}{\mu(2\mu - 3)} - \frac{3\hat{\Theta}(\mu)}{2(\mu - 1)} \right) \right] \tag{5.11}
\]
As mentioned earlier we have used the stringent check that the result for \( \Pi_{\mu\nu} \) must not be singular at \( \alpha = \mu \) and \( \beta = 1 \) when adding all the contributions from all the pieces. Also the tensor structure is the same as at one loop and if one were to transform (5.11) in the context of (3.6) to momentum space the transverse projector, \( P_{\mu\nu}(k) \), will emerge so that the result is gauge invariant. This provides another useful check on the calculation since the individual constituent graphs do not each have the necessary \( Q_{\mu\nu} \) structure.
6 Computation of $\Sigma$.

In the previous section, we outlined in detail the methods required to determine $\Pi_{\mu\nu}$. The algorithm discussed there can also be used to calculate $\Sigma$ though there are several differences which we mention here. First, in $\Sigma$ of fig. 1 we have to compute a graph involving two gauge fields. Again the first step is to break the graph up by using the integration by parts rule on one of the gauge fields which will result in three graphs, denoted by $I$, $II$ and $III$, respectively corresponding to the three terms of fig. 5. To proceed further one applies either the rule of fig. 5 again or that of fig. 7.

It turns out that the resulting nine graphs are much more straightforward to analyse in terms of the basic bosonic two loop building block integrals than for $\Pi_{\mu\nu}$ and in particular there is no graph analogous to $\langle \alpha, \alpha, \alpha, \alpha, \beta - 2 - \Delta \rangle$. However, as the graph $\Sigma$ will be proportional to $\not{q}$ overall, where we use the convention that the left external vertex of fig. 1 is the origin, the result of any manipulations will yield graphs with at most three $\gamma$-matrices using the results (A.2)-(A.4). To proceed further, one can either, if possible, use fermion subtractions similar to the bosonic subtractions of section 5, which were introduced in [4], or, if, for example, the naive subtraction is insufficient to match the divergence structure of the particular graph then it is best to take the explicit trace by first multiplying the graph by $\not{q}$. This will yield a sum of purely bosonic graphs which can be treated individually. We have given the library of essential integrals in the latter half of appendix C where we have again listed only those expressions which were derived from recursion relations. The two loop graphs which are directly obtained by the same type of subtractions discussed in the previous section are not given since they are easy to deduce.

As an example we discuss the computation of the graph obtained after applying the integration by parts rule of fig. 5 to each of the internal vertices, which is equivalent to that of fig. 1, but where the second term of (4.1) is absent. For concreteness, the numerator of this integral is

$$\gamma^\nu(-\hat{y})\gamma^\sigma(\hat{y} - \hat{x})\gamma^\mu(\hat{x} - \hat{y})\gamma_\sigma$$  \hfill (6.1)

where $y$ is the location of the top vertex of integration and $z$ the lower which involves seven $\gamma$-matrices. However, this number can be reduced by using (A.4) several times and gives after a suitable rearrangement

$$4(\mu - 1)[2\hat{y}[(x - y)^2 - (y - z)^2 - (x - z)^2] + (\mu - 4)\hat{y}(\hat{y} - \hat{x})(\hat{x} - \hat{y})]$$  \hfill (6.2)
where we have made a change of variables to obtain the factor 2 in the first term. Whilst the final term involves three fermion propagators one observes that its form is equivalent to the fermion self energy of the Gross Neveu model, which was calculated in [4]. If we introduce the notation \( \langle \tilde{\alpha}_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle \) to mean that the propagator with exponent \( \alpha_1 \) corresponds to a fermion then \( \langle \tilde{\alpha}, \beta - \Delta, \alpha, \beta - \Delta, \alpha - 1 \rangle \) can be calculated directly by using the subtraction \( \langle \tilde{\alpha}, \beta - \Delta, \alpha, 0, \alpha - 1 \rangle \) and the result, to the finite part in \( \Delta \), is

\[
\frac{\pi^{2\mu} a^2(\alpha - 1)a(\beta)}{\Delta \Gamma(\mu)(\mu - \alpha)(\alpha - 1)} + O(\Delta) \tag{6.3}
\]

Again, one observes that this expression is infinite at \( \alpha = \mu \) and \( \beta = 1 \) but in the overall sum for \( \Sigma \), such infinities will in fact cancel. For the remaining graphs, \( \langle \tilde{\alpha}_1, \beta - \Delta, \alpha - 1, \beta - \Delta, \alpha \rangle \) and \( \langle \tilde{\alpha}_1, \beta - 1 - \Delta, \alpha, \beta - \Delta, \alpha \rangle \), one has to take a trace for each to obtain

\[
\frac{2\pi^{2\mu} a(\alpha - 1)a(\alpha)a(\beta)}{\Delta \Gamma(\mu)} \left[ 1 + \Delta \left( B(\beta) - B(\alpha - 1) - \frac{1}{\mu} + \frac{(\beta - 1)}{4\mu(\alpha - 1)} \right) \right] \tag{6.4}
\]

for the former and

\[
\frac{\pi^{2\mu} a(\alpha)a(\alpha - 1)a(\beta)}{2\Delta \Gamma(\mu)} \left[ \left( 6 - \frac{(2\mu + 1)(\beta - 1)}{\mu(\alpha - 1)} \right) 
+ \Delta \left( \left( 2 - \frac{(\beta - 1)}{\mu(\alpha - 1)} \right) [B(\beta) - B(\alpha - 1)] + \frac{\alpha(\beta - 1)}{(\alpha - 1)^2} - \frac{4}{\mu} 
- \frac{(\beta - 1)(\mu - 1)}{\mu(\alpha - 1)} - \frac{(\beta - 1)(\mu^2 - 3\mu + 1)}{\mu^2(\alpha - 1)} \right) \right] \tag{6.5}
\]

for the latter. The computation of the remaining graphs is also similar.

Finally, we close this section by giving the results of the three component graphs which comprise \( \Sigma \), as an aid to the interested reader, having checked that each is finite after the substitution of \( \alpha = \mu \) and \( \beta = 1 \). Thus

\[
\Sigma_I = -\frac{16\pi^{2\mu}(2\mu - 1)(\mu - 1)^2}{\Delta(2\mu - 3)^2 \Gamma^2(\mu)} \left[ \mu - 2 + \Delta \left( \frac{3\mu - 4}{2\mu} - \frac{4(\mu - 2)}{(2\mu - 3)^2} \right) \right] \tag{6.6}
\]

\[
\Sigma_{II} = -\frac{8\pi^{2\mu}(2\mu - 1)}{\Delta(2\mu - 3)^2 \Gamma^2(\mu)} \left[ \mu - 2 + \Delta \left( \frac{2}{\mu} - \frac{4(\mu - 2)}{(2\mu - 3)^2} \right) \right] \tag{6.7}
\]

\[
\Sigma_{III} = -\frac{4\pi^{2\mu}(2\mu - 1)(\mu - 2)}{(2\mu - 3)^2 \Gamma^2(\mu)} \tag{6.8}
\]
We note that one of the constituent graphs of $\Sigma_{II}$ involves the relatively large expression (C.17) which is deduced from a recursion relation containing (C.15) and (C.16). Hence,

$$
\Sigma = \frac{4\pi^{2\mu}(2\mu - 1)}{\mu \Gamma^2(\mu)(2\mu - 3)^2 \Delta} \left[ 2(2\mu - 1)(\mu - 2)^2 \right.
\quad + \left. \Delta \left( (2\mu - 5)\mu + \frac{4(\mu - 1)^2(\mu - 2)}{\mu} - \frac{8(2\mu - 1)(\mu - 2)^2}{(2\mu - 3)} \right) \right] \quad (6.9)
$$

We conclude our discussion of the two loop calculations by remarking that in computing both $\Pi_{\mu \nu}$ and $\Sigma$ we made extensive use of the relation $2\alpha + \beta = 2\mu + 1$ between the exponents $\alpha$ and $\beta$, which is valid at the order we are calculating, to simplify substantial amounts of tedious algebra.

### 7 Derivation of $\eta_2$.

Having discussed the derivation of the two loop corrections we now return to the formalism developed in section 3 and derive the consistency equation for $\eta_2$. From (5.11) and (6.9), we first of all check that the values one obtains for $\chi_1$ in both renormalizations agree. Thus with

$$
K = \frac{8\pi^{2\mu}(2\mu - 1)^2(\mu - 2)^2}{(2\mu - 3)^2 \Gamma(\mu) \Gamma(\mu + 1)} \quad , \quad P = \frac{16\pi^{2\mu}(2\mu - 1)(2 - \mu)}{(2\mu - 3) \Gamma(\mu) \Gamma(\mu + 1)} \quad (7.1)
$$

then

$$
\chi_1 = - \eta_1 \quad (7.2)
$$

from either (3.8) or (3.12), and, as was noted in [26], this corresponds to the QED Ward identity. Likewise, the expressions for the vertex renormalization constant, $u$, both agree. With this value for $\chi_1$, then $\beta = 1 + O(1/N_f^2)$, so that we can now write down the formal consistency equation for $\eta_2$, which is given by eliminating $z_2$ from (3.9) and (3.13), as

$$
\frac{2\eta_2}{\eta_1^2} = \frac{1}{\mu - (2\mu - 1)(\mu - 1)}
\quad + \frac{(2\mu - 3)^2 \Gamma(\mu) \Gamma(\mu + 1)}{4\pi^{2\mu}(2\mu - 1)^2(\mu - 2)^2} \left( \Sigma' - \frac{K}{(2\mu - 3)(\mu - 2)} \right)
\quad + \frac{(2\mu - 3) \Gamma(\mu) \Gamma(\mu + 1)}{16\pi^{2\mu}(\mu - 1)(\mu - 2)} \left( \Pi' + \frac{\Xi'}{2(\mu - 1)} + \frac{X}{2(2\mu - 1)^2} \right) \quad (7.3)
$$
The explicit expressions for $\Pi'$ and $\Sigma'$ can be read off from (5.11) and (6.9), respectively, and with (7.1) some straightforward algebra therefore leads to

$$
\eta_2 = \eta_1^2 \left[ \frac{3\mu(\mu - 1)\hat{\Theta}(\mu)}{(2\mu - 1)(\mu - 2)} + \frac{3}{2\mu} \frac{1}{(\mu - 1)} \right. \\
- \left. \frac{1}{3(\mu - 2)^2} \frac{28}{9(2\mu - 1)} \frac{35}{18(\mu - 2)} \right] ^{(7.4)}
$$

which is the main result of this paper, and is an arbitrary dimension expression for the $O(1/N_f^2)$ part of the electron anomalous dimension in the Landau gauge and as was noted earlier encodes information on the gauge independent part of the 4-dimensional renormalization group function, $\gamma(g)$.

Aside from the internal consistency checks on the explicit derivation of the values of the two loop integrals already discussed, we have checked that the final result for $\eta_2$ agrees with the three loop $\overline{\text{MS}}$ anomalous dimension, $\gamma(g_c)$, expanded near four dimensions. It is worth noting that this three loop result is in fact another very stringent check on our result since we have only evaluated two 2-loop graphs. Moreover, we can now derive the coefficients of the higher order terms of $\gamma(g)$ at $O(1/N_f^2)$, which have not been given previously. For example, from (2.5), with (2.3) and (7.4), we find

$$
\gamma(g) = - \frac{[4N_f + 3]}{16} g^2 + \frac{[40N_f^2 + 54N_f + 27]}{576} g^3 \\
+ \left[ \frac{35N_f^3}{1296} + N_f^2 \left( \frac{1}{27} - \frac{\zeta(3)}{4} \right) + c_1 N_f + d_1 \right] g^4 + O(g^5) \quad (7.5)
$$

where the unknown constants $c_1$ and $d_1$ can only be deduced from $\eta$ at $O(1/N_f^3)$ and $O(1/N_f^4)$ respectively. The next to leading order coefficients with respect to $N_f$, at each subsequent perturbative order, are deduced by first extracting all the $O(1/N_f^2)$ corrections to $g_c$, which are contained within the large $N_f$ $\beta$-function of (3). Comparing the numerical structure of (7.5) with the four loop $\beta$-function of (2.2), we note that they are similar in that the transcendental number, $\zeta(3)$, appears at fourth order but not at leading order in $N_f$.

As (7.4) is valid in arbitrary dimensions, we can evaluate it in three dimensions and find

$$
\eta = - \frac{8}{3\pi^2 N_f} + \frac{16(32 - 3\pi^2)}{9\pi^4 N_f^2} + O\left( \frac{1}{N_f^2} \right) \quad (7.6)
$$
where the $O(1/N_f)$ term agrees with the calculation of [21]. From (7.6), we can obtain estimates for the anomalous dimension for various values of $N_f$. For instance, at $N_f = 3$, $\eta = -0.09$ and at $N_f = 4$, $\eta = -0.06$.

8 Discussion.

We conclude with various observations. First, we have given an analytic expression for the electron anomalous dimension at $O(1/N_f^2)$ by solving the appropriate Dyson equations at criticality within the large $N_f$ expansion, algebraically. Indeed, the information contained in the $d$-dimensional expression, (7.4), relates to the perturbative coefficients of the gauge independent part of the electron anomalous dimension. For completeness, we note that this model is now solved at leading order in large $N_f$, since the critical exponents, or equivalently, the appropriate renormalization group functions, relating to the $\beta$-function and the electron mass anomalous dimension, $\gamma_m(g)$, have been given elsewhere, [8], and in the notation of this paper, they are

$$\gamma_m(g_c) = -\frac{2\eta_1}{(\mu - 2)N_f} \tag{8.1}$$

$$\lambda = (\mu - 2) - \frac{(2\mu - 3)(\mu - 3)}{N_f} \eta_1 \tag{8.2}$$

where $2\lambda = -\beta'(g_c)$. So, for example, in three dimensions there is no $O(1/N_f)$ correction to the $\beta$-function which is gauge independent and this is consistent with the ultraviolet superrenormalizability of that theory. Indeed the four dimensional perturbative information, which is encoded within (7.4), (8.1) and (8.2), will provide useful checks for future explicit perturbative calculations. Second, with the techniques given in this paper, it ought now to be possible to go beyond the $O(1/N_f)$ expressions in (8.1) and (8.2), by extending our critical Dyson equation approach. For instance, the $O(1/N^2)$ corrections to the $\beta$-function of the bosonic and supersymmetric $O(N)\sigma$ models are known, [23, 4]. Indeed, the method of [2, 3] is such that one has always to first compute the exponent $\eta$, which we have done here, before attempting to determine $\lambda$ at the same order in large $N$ since, for example, one needs to know $z_2$ which is determined from (3.9) and (3.13) once $\eta_2$ is available. Finally, we remark that the methods which have been developed here for computing anomalous dimensions in an abelian gauge theory,
will be applicable in solving similar models, such as the bosonic $CP(N)$ $\sigma$ model, beyond the $O(1/N)$ exponents which are presently known, \[27\].

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A Conventions.

In this appendix, we briefly list our conventions and several results involving \(\gamma\)-matrices which we required. Although we work in arbitrary dimensions we take the trace convention to be \(\text{tr}1 = 4\), and to manipulate \(\gamma\)-matrices we made use only of

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (A.1)
\]

working in Euclidean space throughout. From (A.1) we derive,

\[
\gamma^\sigma \gamma^\mu \gamma^\nu = -2(\mu - 1)\gamma^\mu \quad (A.2)
\]

\[
\gamma^\sigma \gamma^\mu \gamma^\nu \gamma^\rho = 4\eta^{\mu\nu} \gamma^\rho + 2(\mu - 2)\gamma^\mu \gamma^\nu \quad (A.3)
\]

\[
\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho = 4\eta^{\mu\sigma} \gamma^\nu \gamma^\rho - 2\eta^{\mu\nu} \gamma^{\sigma\rho} - 2\gamma^\nu \gamma^\sigma \gamma^\rho - 2(\mu - 1)\gamma^\mu \gamma^\nu \gamma^\sigma \quad (A.4)
\]

Although we were able to manipulate the two loop integrals to reduce the number of \(\gamma\)-matrices involved, we needed

\[
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho) = 4[\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho} + \eta^{\nu\rho} \eta^{\mu\sigma}] \quad (A.5)
\]

to complete the calculations of \(\Pi_{\mu\nu}\) and \(\Sigma\).

B Summary of recursion relations.

To compute various basic bosonic two loop building block integrals making up \(\Sigma\) and \(\Pi_{\mu\nu}\), we required various recursion relations, which we list in this appendix. Whilst this list is perhaps not complete, given the large number of transformations one can make on the basic two loop integral of fig. 10, they were all that we required for our purposes. To compactify our expressions a little, we define a new five argument quantity similar to \(\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle\), but involving square brackets. In the following it will denote the two loop graph obtained from the general graph of fig. 10, but with the respective exponents adjusted by the arguments of the expression involving square brackets. For instance, \([0, 0, 0, 0, 1]\) means \(\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 + 1 \rangle\) etc. Also, when a square bracket appears with a subscript \(\pm\) it will correspond to multiplying the overall expression by a factor \((x^2)^{\pm 1}\), so that each term has the correct dimensions. We also use the notation of \([\ref{2}]\) and define

\[
\alpha_1 + \alpha_2 + \alpha_5 = s_1 \quad , \quad \alpha_3 + \alpha_4 + \alpha_5 = s_2
\]

\[
\alpha_1 + \alpha_4 + \alpha_5 = t_1 \quad , \quad \alpha_2 + \alpha_3 + \alpha_5 = t_2
\]

\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = d \quad (B.1)
\]
From the rule of fig. 12, we have derived the following recursion relations, which we believe have not been given before, and emphasise that the relations are completely general and applicable to any graph of the form of fig. 10, and not solely for the specific problem dealt with in this paper. So, we have,

\[
\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle = \frac{(3\mu - d)(d - 2\mu - 1)}{(\alpha_2 - 1)(\alpha_3 - 1)} [0, -1, -1, 0, 1]_+ \\
+ \frac{(\alpha_2 + \alpha_3 - \mu - 1)}{(\alpha_2 - 1)} [0, -1, 0, 0, 1] \\
+ \frac{(\alpha_2 + \alpha_3 - \mu - 1)}{(\alpha_3 - 1)} [0, 0, -1, 0, 1] \quad (B.2)
\]

\[
= \frac{\alpha_1(\mu - \alpha_1 - 1)}{(\alpha_2 - 1)(\alpha_5 - 1)} [1, -1, 1, -1, 0]_+ \\
+ \frac{(\alpha_2 + \alpha_5 - \mu - 1)}{(\alpha_2 - 1)} [0, -1, 1, 0, 0] \\
+ \frac{(\alpha_2 + \alpha_5 - \mu - 1)}{(\alpha_5 - 1)} [0, 0, 1, 0, -1] \quad (B.3)
\]

\[
= \frac{\alpha_5(\mu - \alpha_5 - 1)}{(\alpha_1 - 1)(\alpha_2 - 1)} [-1, -1, 0, 0, 1]_+ \\
+ \frac{(\alpha_1 + \alpha_2 - \mu - 1)}{(\alpha_2 - 1)} [0, -1, 0, 0, 0]_+ \\
+ \frac{(\alpha_1 + \alpha_2 - \mu - 1)}{(\alpha_1 - 1)} [-1, 0, 0, 0, 0]_+ \quad (B.4)
\]

\[
= \frac{\alpha_5(\mu - \alpha_2 - \alpha_5 - 1)}{(2\mu - s_1 - 1)(s_1 - \mu)} [0, 0, 0, -1, 1] \\
+ \frac{\alpha_2(\mu - \alpha_2 - \alpha_5 - 1)}{(2\mu - s_1 - 1)(s_1 - \mu)} [0, 1, 0, 0, 0]_+ \\
+ \frac{\alpha_2\alpha_5}{(2\mu - s_1 - 1)(s_1 - \mu)} [-1, 1, -1, 0, 1] \quad (B.5)
\]

\[
= \frac{\alpha_3(2\mu - s_1)(s_1 - \mu - 1)}{(2\mu - s_2 - 1)(s_2 - \mu)(\alpha_1 - 1)} [-1, 0, 1, 0, 0] \\
+ \frac{\alpha_5(\alpha_1 - \alpha_3 - 1)(\mu - \alpha_5 - 1)}{(s_2 - \mu)(2\mu - 1 - s_2)(\alpha_1 - 1)} [-1, 0, 0, 0, 1] \\
+ \frac{\alpha_3(\alpha_1 - \alpha_3 - 1)}{(s_2 - \mu)(2\mu - s_2 - 1)} [0, 0, 1, 0, 0]_+ \quad (B.6)
\]
\[
\begin{align*}
\langle \alpha_d, \alpha \rangle & = \frac{\alpha_3 (t_2 - \mu) (\mu - \alpha_3 - 1)}{(d - 2\mu)(3\mu - d - 1)(\alpha_5 - 1)} [0, 1, 1, 0, -1]_- \\
& \quad + \frac{\alpha_4 (\mu - \alpha_2 - \alpha_3 - 1)(\mu - \alpha_4 - 1)}{(d - 2\mu)(3\mu - d - 1)(\alpha_5 - 1)} [0, 1, 0, 1, -1]_- \\
& \quad + \frac{(t_2 - \mu)(\mu - \alpha_2 - \alpha_3 - 1)}{(d - 2\mu)(3\mu - d - 1)} \alpha_2 \alpha_3 [0, 1, 0, 0, 0]_- \quad \text{(B.7)}
\end{align*}
\]

Further, we record the recursion relation derived using integration by parts which we also required in evaluating $\Pi_{\mu \nu}$, which is, \cite{2, 22}:

\[
\begin{align*}
\frac{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle}{(d + t_1 - 4\mu)^{-1}} & = \alpha_2 ([0, 1, 0, 0, 0]_- - [-1, 1, 0, 0, 0]) \\
& \quad + \alpha_3 ([0, 0, 1, 0, 0]_- - [0, 0, 1, -1, 0]) \quad \text{(B.12)}
\end{align*}
\]
C  Basic two loop integrals.

In this appendix, we give a list of the basic bosonic two loop integrals required to compute \( \Sigma, \Pi \) and \( \Xi \), expanded to the finite part with respect to \( \Delta \). As discussed in sect. 5 these fall into classes i.e. those which are computed directly using the method of subtractions and those which are not but which are determined by using recursion relations. As the former set are easy to establish, we list only those of the second class which we require. First, we consider the basic graphs for \( \Pi \) and \( \Xi \), using the notation of fig. 10. We note that the integral \( \langle \alpha, \alpha, \alpha, \alpha, \alpha \rangle \) has already been discussed earlier.

\[
\langle \alpha, \alpha - 1, \alpha - 1, \alpha, \beta \rangle = - \frac{2\pi^2 \mu (\mu - 1) a(\alpha) a(\alpha - 1) a(\beta)}{\Gamma (\mu + 1)} \tag{C. 1}
\]

\[
\langle \alpha - 1, \alpha, \alpha, \alpha, \beta \rangle = \frac{2\pi^2 \mu a(\alpha) a(\alpha - 1) a(\beta)}{\Gamma (\mu)} \left[ \frac{1}{\Delta} + B(\beta) \right] - B(\alpha - 1) - \frac{(\beta - 2)}{(\beta - 1)} - \frac{(\mu + \beta - 1)}{2\mu (\alpha - 1)} \tag{C. 2}
\]

\[
\langle \alpha - 1, \alpha - 1, \alpha, \alpha, \beta \rangle = \frac{2\pi^2 \mu a(\alpha - 1) a(\alpha) a(\beta)}{\Gamma (\mu)} \left[ \frac{1}{\Delta} + B(\beta) \right] - B(\alpha - 1) - \frac{1}{\alpha - 1} + \frac{2}{\mu - 1} \tag{C. 3}
\]

\[
\langle \alpha, \alpha - 1, \alpha - 1, \alpha, \beta - 1 \rangle = \frac{\pi^2 \mu a(\beta - 1)}{a^2 (\mu - \alpha) \Gamma (\mu)} \left[ \frac{1}{\Delta} + B(\beta) - B(\alpha - 1) \right] + \frac{1}{\mu - 1} \left[ (\beta + 1 - \mu)(2\alpha - 3) \right] \tag{C. 4}
\]

\[
\langle \alpha - 1, \alpha, \alpha - 1, \alpha, \beta - 1 \rangle = \frac{\pi^2 \mu (\mu - 1) a^2(\alpha) a(\beta)}{\Gamma (\mu)} + \frac{(\mu - 1 - \beta)(2\alpha - 3)}{(\alpha - 1)^2} \times \langle \alpha - 1, \alpha - 1, \alpha - 1, \alpha - 1, \beta \rangle \tag{C. 5}
\]
\[ \langle \alpha, \alpha - 2, \alpha, \beta - \Delta \rangle = \frac{\pi^{2 \mu} a^2(\alpha) a(\beta - 1)}{\Gamma(\mu + 1)} \left[ \frac{1}{\Delta} + B(\beta) - B(\alpha - 1) \right. \\
+ \frac{\mu^2 + \mu - 1}{\mu(\mu - 1)} \left. - \frac{3}{2} \frac{(\alpha - 1)^2}{(\mu - \beta)} \right. \\
+ \frac{\alpha}{\beta - 1} \left. \frac{\alpha(\alpha - 2)(\beta - 1)}{2(\alpha - 1)(\mu - \beta)} \right] \] (C.6)

\[ \langle \alpha, \alpha - 1, \alpha - 2, \alpha, \beta - \Delta \rangle = \frac{\pi^{2 \mu} (2 - \mu) a(\alpha) a(\alpha - 1) a(\beta)}{\Gamma(\mu + 1)} \left[ \frac{1}{\Delta} + B(\beta) \right. \\
- B(\alpha - 1) + \frac{\mu^2 + \mu - 1}{\mu(\mu - 1)} \left. + \frac{\mu - 1}{\mu - 2} \right. \\
- \frac{(\beta - 1)}{(\mu - 2)(\alpha - 1)} \] (C.7)

Next, the basic integrals for \( \Sigma \) are,

\[ \langle \alpha, \beta - 1 - \Delta, \alpha - 1, \beta - \Delta, \alpha \rangle = \frac{\pi^{2 \mu} a(\alpha - 1) a(\alpha) a(\beta)(\mu - 1)}{\Gamma(\mu + 1)} \left[ \frac{\beta - 1}{\alpha - 1} - 3 \right] \] (C.8)

\[ \langle \alpha - 1, \beta - \Delta, \alpha, \beta - \Delta, \alpha \rangle = \frac{\pi^{2 \mu} a(\alpha - 1) a(\alpha) a(\beta)}{\Gamma(\mu)} \left[ \frac{2}{\Delta} + B(\beta) - 3 \right. \\
- B(\alpha - 1) + \frac{\beta - 1}{\alpha - 1} - \frac{1}{\mu} \left. \right] \] (C.9)

\[ \langle \alpha, \beta - \Delta, \alpha, \beta - 1 - \Delta, \alpha \rangle = \frac{\pi^{2 \mu} a^2(\alpha) a(\beta - 1)}{\Gamma(\mu)} \left[ \frac{2}{\Delta} + B(\beta) - B(\alpha - 1) \right. \\
+ \frac{1}{\beta - 1} \left. + \frac{\alpha(\beta - 1)}{2(\alpha - 1)(\mu - \beta)} \right. \\
- \left. \frac{(3\mu - 1)(\alpha - 1)}{2\mu(\mu - \beta)} - \frac{2}{\mu} \right] \] (C.10)

\[ \langle \alpha - 1, \beta + 1 - \Delta, \alpha, \beta - \Delta, \alpha - 1 \rangle = \frac{\pi^{2 \mu} a^2(\alpha - 1) a(\beta + 1)}{(\mu - \alpha) \mu \Gamma(\mu - 1)} \left[ \mu - \alpha \right. \\
+ \left. 1 - \frac{\beta(\mu - \beta + \alpha - 2)}{(\alpha - 1)} \right] \] (C.11)
\[
\langle \alpha - 1, \beta - 1 - \Delta, \alpha, \beta - \Delta, \alpha \rangle = \frac{\pi^{2\mu} a(\alpha - 1)a(\alpha)a(\beta)}{\Delta \Gamma(\mu)} \left[ \frac{3 - (\beta - 1)}{\alpha - 1} \right] + \Delta \left( B(\beta) - B(\alpha - 1) + \frac{1}{\mu - 1} \right) \quad \text{(C.12)}
\]

\[
\langle \alpha, \beta - 2 - \Delta, \alpha - 1, \beta - \Delta, \alpha \rangle = \frac{\pi^{2\mu}(2 - \mu)a(\alpha - 1)a(\alpha)a(\beta)}{\Gamma(\mu + 1)} \times \left[ \frac{1}{\Delta} + B(\beta) - B(\alpha - 1) \right.
\]
\[
+ \left. \frac{\mu^2 + \mu - 1}{\mu(\mu - 1)} + \frac{2(\mu - \beta)}{\alpha - 1} \right] \quad \text{(C.13)}
\]

\[
\langle \alpha, \beta - 2 - \Delta, \alpha, \beta - \Delta, \alpha \rangle = \frac{2\pi^{2\mu} a(\alpha - 1)a(\alpha)a(\beta)}{\Gamma(\mu)} \left[ \frac{1}{\Delta} + B(\beta) - B(\alpha - 1) \right.
\]
\[
+ \left. \frac{4(\mu - \beta)}{\alpha - 1} - \frac{1}{\alpha - 1} \right]
\]
\[
+ \left( 1 - \frac{(\beta - 1)(\mu - 1)}{\mu(\alpha - 1)} \right) \left( \frac{1}{\Delta} + B(\beta) \right) 
\]
\[
+ \left[ \frac{\mu^2 + \frac{\mu - 1}{\mu(\mu - 1)}}{\alpha - 1} \right] \right] \quad \text{(C.14)}
\]

\[
\langle \alpha + 1, \beta - 2 - \Delta, \alpha, \beta - 1 - \Delta, \alpha \rangle = \frac{\pi^{2\mu} a^2(\alpha)(\beta - 1)(\beta - 1)}{2(\mu - \alpha - 1)\Gamma(\mu + 1)} \left[ \frac{(\beta - 2)(\beta - 3)}{2\alpha} + \mu - \beta + 1 \right.
\]
\[
- \left. \frac{4(\alpha - 1)(\beta - 2)}{\beta - 1} \right]\right] \quad \text{(C.15)}
\]

\[
\langle \alpha + 1, \beta - 1 - \Delta, \alpha, \beta - \Delta, \alpha - 1 \rangle \right] = \frac{\pi^{2\mu} a(\alpha)(\alpha - 1)a(\beta - 1)}{(\mu - \alpha - 1)\Gamma(\mu + 1)} \times \left[ \frac{\mu - \alpha + 1}{\beta - 1} - \frac{(\mu - 1)}{\alpha} \right] \quad \text{(C.16)}
\]
\[
\langle \alpha + 1, \beta - 1 - \Delta, \alpha, \beta - 1 - \Delta, \alpha \rangle \\
= \frac{\pi^{2\mu} a(\alpha) a(\alpha - 1) a(\beta - 1)}{\Gamma(\mu)(\alpha - 1)(\beta - 1)} \left[ \frac{4}{\beta - 1} + \frac{2}{\alpha} + \frac{2}{\mu - \alpha - 1} \right] \\
+ 2(B(\beta) - B(\alpha - 1)) + \frac{(\beta - 1)}{(\mu - \beta)(\alpha - 1)} \\
+ \frac{4(2\alpha + 2\beta - 3)(\alpha - \mu - 1)(\alpha - 1)}{\mu(\mu - 1)(\mu - \alpha - 1)} \left( \frac{\beta - 3}{2\alpha} + \frac{\mu - \beta + 1}{\beta - 2} \right) \\
- \frac{2(\alpha + \beta - 1)(\alpha - \mu - 1)(\beta - 2)}{\mu(\mu - 1)(\mu - \alpha - 1)} \left( \frac{\alpha}{\beta - 1} - \frac{(\mu - 1)}{(\mu - \alpha + 1)} \right) \right] \quad (C.17)
\]
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Figure Captions.

Fig. 1. Skeleton Dyson equation for the electron.

Fig. 2. Skeleton Dyson equation for the photon.

Fig. 3. Uniqueness rule for a bosonic vertex.

Fig. 4. Electron photon vertex.

Fig. 5. Integration by parts rule for gauge vertex.

Fig. 6. Additional integration by parts rule.

Fig. 7. Further integration by parts rule for gauge vertex.

Fig. 8. Graphical representation of $\Pi_{\mu\nu}(x)$.

Fig. 9. Photon self energy after integrating by parts.

Fig. 10. Basic two loop self energy graph.

Fig. 11. Various basic bosonic graphs contributing to $\Pi_{\mu\nu}$.

Fig. 12. Basic rule for recursion relations.
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