G-graphs and special representations
for binary dihedral groups in GL(2, ℂ)

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Abstract

We give the classification of all possible G-graphs for any small binary dihedral subgroup G in GL(2, ℂ) and use this classification to give the combinatorial description of the special representations of G in terms of its maximal cyclic normal subgroup.

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1 Introduction

The G-graphs were introduced by Nakamura [Nak01] to construct a crepant resolution of the quotient ℂ^3/G for abelian subgroups G ⊂ SL(3, ℂ). This resolution was the moduli space G-Hilb of G-clusters introduced by Ito and Nakamura [IN96], which among other properties, has been proved to be the minimal resolution of ℂ^2/G for small finite subgroups G ⊂ GL(2, ℂ) in [Ish02] and a “distinguished” crepant resolution for subgroups G ⊂ SL(3, ℂ) in [BKR01].

Since then, G-graphs have been proved to be a useful tool to describe G-Hilb mainly in toric cases (for instance [Kid01], [Ito02] for G ⊂ GL(2, ℂ) and [CR02] for G ⊂ GL(3, ℂ)). The first attempt to extend the notion of G-graph to calculate G-Hilb for non-abelian subgroups was made by Leng in [Len02] for binary dihedral subgroups G ⊂ SL(2, ℂ) and some binary trihedral subgroups G ⊂ SL(3, ℂ) (see also [Seb05]). In this paper we calculate every possible G-graph for small binary dihedral subgroups G ⊂ GL(2, ℂ).

The key fact in the construction of these G-graphs is to consider the action of a dihedral group G on ℂ^2 as the cyclic action by its maximal abelian subgroup A = < 1/2(1, a) > of index two, followed by a dihedral involution. This interpretation allows us to construct G-graphs from the union of two symmetric
cyclic $A$-graphs, that we call $qG$-graphs, from whom we construct the corresponding $G$-graphs in a unique way according to the representation theory of $G$. In Theorem 4.17 we classify the $G$-graphs into types $A$, $B$, $C$ and $D$.

Given any $G$-graph $\Gamma$ there exist a corresponding affine open set $U_\Gamma$ in $G\mathrm{Hilb}(\mathbb{C}^2)$ which consists of all $G$-clusters $Z$ such that $O_Z$ admits $\Gamma$ for basis as a vector space over $\mathbb{C}$. In Theorem 5.2 and the following Corollary we prove that the number of distinct $G$-graphs gives us a covering of $G\mathrm{Hilb}(\mathbb{C}^2)$ with the minimum number of open sets. The explicit equations of these open sets are given in [NdC09] using the moduli space of representations of the McKay quiver.

In the last part of the paper we apply the classification of $G$-graphs to give the list of special representations of any small binary dihedral group $G \subset \text{GL}(2, \mathbb{C})$. Using the explicit description of the ideals corresponding to the $G$-graphs that we give in Theorem 4.17 we are able to give a 1-parameter family of ideals connecting any $G$-cluster at the exceptional divisor $E$ in $G\mathrm{Hilb}(\mathbb{C}^2)$. Thanks to the special McKay correspondence every exceptional curve appearing in the minimal resolution correspond one-to-one to the special irreducible representations, which now we have expressed in terms of $G$-graphs. This minimal resolution is $G\mathrm{Hilb}(\mathbb{C}^2)$ by Ishii’s Theorem (see [Ish02] and Theorem 6.1), which allows us to give in Theorem 6.2 the explicit description of the special representations of a small binary dihedral group $G = \text{BD}_{2n}(a)$ in terms of the continued fraction $\frac{2a}{n}$. We must note that the classification of the special representations was discovered independently by Wemyss and Iyama in [IW08] in a more general setting using the homological definition of special representation and a counting argument on a non-commutative ring. It is worth to mention that this counting argument do not give a general proof of which 1-dimensional representations are special in the dihedral case, while the $G$-graphs give a method to get round this problem.

The paper is distributed as follows: In the preliminary Section 2 we give the notations and definitions we use for the rest of the paper. In Section 3 we present the definition of dihedral group $G$ that we consider as well as their representation theory and the resolution of the quotient $\mathbb{C}^2/G$, and Section 4 is dedicated to the calculation of the $G$-graphs for these dihedral groups and their classification into the types $A$, $B$, $C$ and $D$. In Section 6 we prove that any $G$-cluster $Z$ admits as basis for $O_Z$ a $G$-graph of type $A$, $B$, $C$ or $D$, thus obtaining an open cover for $G\mathrm{Hilb}(\mathbb{C}^2)$. The last Section is devoted to the special representations and their description in terms of the continued fraction $\frac{2a}{n}$.

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2 Preliminaries

We start by describing the $G$-invariant Hilbert scheme $G\mathrm{Hilb}$ which motivates the definition of $G$-graph.

**Definition 2.1.** Let $G \subset \text{GL}(n, \mathbb{C})$ be a finite subgroup. A $G$-cluster is a $G$-invariant zero dimensional subscheme $Z \subset \mathbb{C}^n$, defined by an ideal $I_Z \subset \mathbb{C}[x_1, \ldots, x_n]$, such that $O_Z = \mathbb{C}[x_1, \ldots, x_n]/I_Z \cong \mathbb{C}G$ the regular representation as $\mathbb{C}G$-modules. The $G$-Hilbert scheme $G\mathrm{Hilb}(\mathbb{C}^n)$ is the moduli space parametrizing $G$-clusters.

Recall that the regular representation $\mathbb{C}G$ is a direct sum of irreducible representations $\rho_i$, where every irreducible $\rho_i$ appears $(\dim \rho_i)$ times in the sum. That is

$$\mathbb{C}G = \bigoplus_{\rho_i \in \text{Irr } G} (\rho_i)^{\dim \rho_i}$$

Therefore if $I_Z$ is an ideal defining a $G$-cluster, or a point in $G\mathrm{Hilb}(\mathbb{C}^n)$, then the vector space $O_Z$ has in its basis $(\dim \rho_i)$ elements in each irreducible representation $\rho_i$ (see Section 3.2). To describe distinguished basis with this property for these coordinate rings, it is convenient to use the notion of $G$-graph.
Definition 2.2. Let $G \subset \text{GL}(n, \mathbb{C})$ be a finite subgroup. A $G$-graph is a subset $\Gamma \subset \mathbb{C}[x_1, \ldots, x_n]$ satisfying the following two conditions:

1. It contains $(\dim \rho)$ number of elements in each irreducible representation $\rho$.
2. If a monomial $x_1^{\mu_1} \cdots x_n^{\mu_n}$ is a summand of a polynomial $P \in \Gamma$, then for every $0 \leq \mu_j \leq \lambda_j$ the monomial $x_1^{\mu_1} \cdots x_n^{\mu_n}$ must be a summand of some polynomial $Q_{\mu_1,\ldots,\mu_n} \in \Gamma$.

Note that given any ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, we can choose a basis for the vector space $\mathbb{C}[x_1, \ldots, x_n]/I$ which is a $G$-graph. This choice will never be unique.

From now on we restrict ourselves to the surface case, i.e. we consider finite subgroups $G \subset \text{GL}(2, \mathbb{C})$.

Representation of a $G$-graph

When the group $G$ is abelian there are $|G|$ irreducible representations, every irreducible representation $\rho$ of $G$ is 1-dimensional, and every monomial in $\mathbb{C}[x, y]$ belongs to some representation. Therefore the $G$-graphs for abelian groups are represented in the lattice of monomials $M$ by the monomials contained in $\Gamma$. For example, the following picture represents the $\frac{1}{6}(1, 5)$-graph $\Gamma = \{1, x, y, x^2, x^3, x^4\}$:

![Diagram](image)

As the figure suggests, the representation of $\Gamma$ consists of all monomials in $\mathbb{C}[x, y]$ which do not belong to the ideal $I = (x^5, xy, y^2)$, so we may also say that the $G$-graph $\Gamma$ is defined by the ideal $I$.

When $G$ is non-abelian, irreducible representations may not contain monomials, and the elements in $G$-graphs consist of sums of monomials. In this case, the representation of a $G$-graph is also drawn on the lattice of monomials $M$, but now using the following rule: a monomial $x^iy^j$ is contained in the representation of a $G$-graph $\Gamma$ if it is contained as a summand in some polynomial $P \in \Gamma$. For example, consider the binary dihedral group $D_4 \subset \text{SL}(2, \mathbb{C})$ and the $D_4$-graph

$$\Gamma = \{1, x^4 - y^4, x^2 + y^2, x^3 - y^2, (x, y), (y^3, -x^3)\}$$

Note that it has one element in each 1-dimensional representation and two elements in the 2-dimensional representation $V$ (see Table 1). In this case, $\Gamma$ is represented by the following figure

![Diagram](image)

where the basis elements $x^2 + y^2 \in \rho_2$ and $x^2 - y^2 \in \rho_1$ are represented by $x^2$ and $y^2$ respectively. We also say that $\Gamma$ is defined by the ideal $I = (xy, x^4 + y^4)$. Counting the number of monomials in the figure we have that $|\Gamma| = 9 > |D_4| = 8$, but notice that $x^4 - y^4$ belongs to the basis of $\mathbb{C}[x, y]/I$, so we cannot exclude $x^4$ and $y^4$. On the other hand we have the relation $x^4 + y^4 = 0$, i.e. $x^4 = -y^4$ and both monomials count as one in the basis. We say that $x^4$ and $y^4$ are "twins".

Example 2.3. The cyclic case. Let $A = \left\langle \frac{1}{2}(1, a) \right\rangle := \left\{(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{smallmatrix}) \mid |\varepsilon| = 1 \text{ primitive} \right\}$ be the cyclic group in $\text{GL}(2, \mathbb{C})$ with $(k, a) = 1$. The quotient singularity is toric and the minimal resolution is $Y = A\text{-Hilb}(\mathbb{C}^2)$.

The $A$-graphs are completely determined by the positive nonzero lattice points $e_i$ on the boundary of the Newton polygon of the lattice $L := \mathbb{Z}^2 + \frac{1}{2}(1, a) \cdot \mathbb{Z}$. Indeed, let $\frac{b}{a} = [b_1, \ldots, b_m]$, then there are $m$ exceptional curves $E_1, \ldots, E_m$ in $A\text{-Hilb}(\mathbb{C}^2)$, each $E_i \cong \mathbb{P}^1$ with selfintersections $-b_1, \ldots, -b_m$ respectively.
covered by \( m + 1 \) affine open sets.

Let \( e_i = (r, s) \) and \( e_{i+1} = (u, v) \) be two consecutive boundary lattice points of \( L \), then the corresponding open set in \( Y \) is of the form \( Y_i \cong \mathbb{C}(\xi_i, \eta_i) \), where \( \xi_i = x^y / y^x \) and \( \eta_i = y^y / x^x \), and \( Y = Y_0 \cup \ldots \cup Y_m \). Therefore every point \((\xi_i, \eta_i) \in Y_i \) corresponds to the \( A \)-cluster \( Z_{\xi_i, \eta_i} \) defined by the ideal

\[
I_{\xi_i, \eta_i} = (x^y - \xi_i^r y^x, y^y - \eta_i^s x^x, x^{s-r} y^{u-r} - \xi_i \eta_i)
\]

The \( A \)-graph \( \Gamma \) is obtained by setting \( \xi_i = \eta_i = 0 \), giving the following “stair” shape:

\[
\begin{array}{ccc}
\vdots & & \\
1 & x \hdots & \\
y & & y^x \\
u & & v
\end{array}
\]

For any \((\xi_i, \eta_i) \in \mathbb{C}^2 \), modulo the ideal \( I_{Z_{\xi_i, \eta_i}} \) we have that any monomial in \( \mathbb{C}[x, y] \) can be written in terms of elements in \( \Gamma \). In other words, the vector space \( \mathbb{C}[x, y] / I_{Z_{\xi_i, \eta_i}} \) has \( \Gamma \) as basis. Note also that \( k = su - rv \), so the number of elements agrees with the order of the group. Thus \( \mathbb{C}^2_{\xi_i, \eta_i} \) is an open set in \( A\text{-Hilb}(\mathbb{C}^2) \).

**Example:** Consider the group \( A = \langle 1, 12 \rangle \). We have \( \mathbb{D} = [2, 4, 2] \) and therefore the resolution of singularities \( Y = A\text{-Hilb}(\mathbb{C}^2) \to \mathbb{C}^2/G \) is of the form \( Y = Y_0 \cup Y_1 \cup Y_2 \cup Y_3 \), where each \( Y_i \cong \mathbb{C}^2(\xi_i, \eta_i) \), \( i = 0, \ldots, 3 \). See Figure [I]

![Figure 1: Resolution of singularities Y of the cyclic singularity of type \( \frac{1}{12}(1, 7) \).](image)

The corresponding \( A \)-clusters for these affine pieces are defined by the following ideals:

\[
I_{\xi_0, \eta_0} = \left( \frac{x^{12} - \xi_0}{y - \eta_0 x^7} \right) \quad I_{\xi_1, \eta_1} = \left( \frac{x^7 - \xi_1 y}{y^2 - \eta_1 x^2} \right) \quad I_{\xi_2, \eta_2} = \left( \frac{x^2 - \xi_2 y^2}{y^7 - \eta_2 x} \right) \quad I_{\xi_3, \eta_3} = \left( x - \xi_3 y^7 \right)
\]

so each of the complex planes \( Y_i \) parametrise an open set in \( A\text{-Hilb}(\mathbb{C}^2) \). To see which \( A \)-graph corresponds to each of the open sets, it is useful to look at the origin in each of the \( Y_i \), for \( i = 0, 1, 2, 3 \), i.e. consider \( \xi_i = \eta_i = 0 \). Then we obtain the \( A \)-graphs \( \Gamma_i \) for \( i = 0, \ldots, 3 \) shown in Figure [II].
In other words, BD classified by Brieskorn in \([\text{Bri68}]\) as follows:

\[ \beta \]

Dihedral groups in \(\text{GL}(2, \mathbb{C})\)

We consider the following representation of binary dihedral subgroups in \(\text{GL}(2, \mathbb{C})\) in terms of their action on the complex plane \(\mathbb{C}^2\):

\[ \text{BD}_{2n}(a) = \langle \alpha = \left( \begin{array}{cc} \varepsilon & 0 \\ 0 & \varepsilon^a \end{array} \right), \beta = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) : \varepsilon^{2n} = 1 \text{ primitive, } (2n, a) = 1, a^2 \equiv 1 \pmod{2n} \rangle \]

In other words, \(\text{BD}_{2n}(a)\) is the group of order \(4n\) generated by the cyclic group \(\frac{1}{2n}(1, a)\) and the dihedral symmetry \(\beta\) which interchanges the coordinates \(x\) and \(y\). The subgroup \(A := \langle \alpha \rangle \leq G\) is a choice of maximal cyclic subgroup of \(G\), which is of index \(2\) (note that \(\beta^2 \in A\)). The condition \(a^2 \equiv 1 \pmod{2n}\) is equivalent to the classical dihedral condition of \(\alpha \beta = \beta \alpha^a\), and it creates a lot of symmetry in \(A\).

An element \(g \in G\) is a \textit{quasireflection} if it fixes a hyperplane, and a group \(G\) is called \textit{small} if it does not contain any quasireflection. A theorem of Chevalley, Shephard and Todd [ST54] states that if \(H \subset \text{GL}(n, \mathbb{C})\) is generated by reflections then \(\mathbb{C}^n/H \cong \mathbb{C}^n\), which traditionally reduces the study of these quotients to small groups. Since the groups \(\text{BD}_{2n}(a)\) may contain quasireflections, we present in the next proposition a criteria for a binary dihedral group \(\text{BD}_{2n}(a)\) to be small.

**Proposition 3.1.** \(\text{BD}_{2n}(a)\) is small \(\iff\) \(\gcd(a + 1, 2n) \mid n\)

**Proof.** The elements of \(G\) are of the form \(\alpha^i\) and \(\alpha^i \beta\) for \(i = 0, \ldots, n - 1\). Since \((2n, a) = 1\) the subgroup \(A\) is small, so quasireflections can only occur among the elements of the form \(\alpha^i \beta\) (except for \(i = n\)), and

\[ \alpha^i \beta \text{ is a quasireflection} \iff \det(\alpha^i \beta - I) = 0 \]

\[ \iff \begin{vmatrix} -1 & \varepsilon^i \\ -\varepsilon^a & -1 \end{vmatrix} = 1 + \varepsilon^{(a+1)i} = 0 \]

\[ \iff (a + 1)i \equiv n \pmod{2n} \]

Therefore, \(G\) has no quasireflections if and only do not exist any solutions to the equation \((a + 1)x \equiv n \pmod{2n}\). As a linear congruence, it only has solutions if the \(\gcd(a + 1, 2n)\) divides \(n\). Indeed, if \(g = \gcd(a + 1, 2n)\) and \(u\) is a solution, then \((a + 1)u \equiv n \pmod{g}\) and so \((a + 1)u \equiv n \equiv 0 \pmod{g}\), which only can happen if \(g | n\).

**Remark 3.2. Brieskorn classification.** Small binary dihedral groups in \(\text{GL}(2, \mathbb{C})\) were originally classified by Brieskorn in [Bri68] as follows:

\[ D_{N,q} := \begin{cases} \langle \psi_{2q}, \tau, \phi_{2k} \rangle, & \text{if } k := N - q \equiv 1 \pmod{2} \\ \langle \psi_{2q}, \tau \circ \phi_{4k} \rangle, & \text{if } k \equiv 0 \pmod{2} \end{cases} \]
where

\[ \psi_r = \begin{pmatrix} \varepsilon_r & 0 \\ 0 & \varepsilon_r^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \phi_r = \begin{pmatrix} \varepsilon_r & 0 \\ 0 & \varepsilon_r \end{pmatrix}, \]

with \( \varepsilon_r = \exp \frac{2\pi i}{2n} \) and \( |DN,q| = 4kq \).

The groups \( BD_{2n}(a) \) which are small correspond to the case \( k \) odd, where \( q = \frac{2n}{(a+1,2n)} \) and \( n = 4k \). The case \( k \) even is obtained by taking \( \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) which we denote by \( BD_{2n}(a,q) \) (See [NdC08], §3 for more details).

For simplicity, in this paper we only treat the groups \( BD_{2n}(a) \) but we want to emphasize that the methods used here apply to the groups \( BD_{2n}(a,q) \) as well.

### 3.1 Resolution of dihedral singularities

Let \( G = BD_{2n}(a) \) be a small binary dihedral group with cyclic maximal subgroup \( A = \langle \alpha \rangle \leq G \). Let us now look at the geometric construction of the resolution of a dihedral singularity \( \mathbb{C}^2/G \).

Consider first the action of \( A \) on \( \mathbb{C}^2 \). The quotient affine variety \( Y = \mathbb{C}^2/A \) is the toric quotient singularity of type \( \frac{1}{2n}(1,a) \), with an isolated singular point at the origin. The resolution of singularities \( Y = A - \text{Hilb}(\mathbb{C}^2) \to \mathbb{X} \) is given by the Hirzebruch–Jung continued fraction \( \frac{2n}{a} = [b_1, \ldots, b_k] \) as follows: \( Y = Y_0 \cup \ldots \cup Y_k \), where each \( Y_i \cong \mathbb{C}^2 \) and the exceptional divisor on \( Y \) is \( E = \bigcup_{i=1}^k E_i \) with \( E_i \cong \mathbb{P}^1 \), and \( E_i^2 = -b_i \).

**Lemma 3.3.** If \( a^2 \equiv 1 \pmod{2n} \) then the continued fraction \( \frac{2n}{a} \) is symmetric with respect to the middle term, i.e.

\[ \frac{2n}{a} = [b_1, b_2, \ldots, b_{m-1}, b_m, b_{m-1}, \ldots, b_2, b_1] \]

**Proof.** Let \( L \) be the lattice of weights and \( M \) the dual lattice of monomials, and consider the continued fractions \( \frac{2n}{2n-a} = [a_1, \ldots, a_s] \) and \( \frac{2n}{a} = [b_1, \ldots, b_t] \). If a monomial \( x^iy^j \) is \( A \)-invariant then \( i + aj \equiv 0 \pmod{2n} \), and by the assumption \( a^2 \equiv 1 \pmod{2n} \), \( x^iy^j \) is also invariant. Therefore, the continued fraction \( \frac{2n}{2n-a} \) is symmetric, i.e. \( a_i = a_{r-i} \). Indeed, let \( u_{i-1} = x^{k}y^{l'} \), \( u_i = x^{k}y^{l'} \), \( u_{i+1} = x^{k'}y^{l''} \) be three consecutive invariant monomials and \( u_{r-(i+1)} = x^{k''}y^{l''} \), \( u_{r-i} = x^{k''}y^{l''} \) their symmetric partners. Since \( u_{k-i}u_{k+1} = u_k^{a_k} \) we have

\[ k' + k'' = a_i k = a_{r-i} k \quad \text{and} \quad l' + l'' = a_i l = a_{r-i} l \]

and then \( a_i = a_{r-i} \) for all \( i \). The symmetry of \( \frac{2n}{2n-a} \) implies the symmetry of \( \frac{2n}{a} \) and then \( b_i = b_{r-i} \). \( \square \)

At the moment we have only considered the cyclic part of the dihedral group \( G \). To complete its action on \( \mathbb{C}^2 \) we need to act on \( Y = A - \text{Hilb}(\mathbb{C}^2) \) with \( G/A \cong \mathbb{Z}/2\mathbb{Z} \), which is generated by \( \beta \). Notice that the symmetry in the continued fraction \( \frac{2n}{a} \) implies that the coordinates along the exceptional curves \( Y_i \) in the resolution \( Y \to \mathbb{C}^2/A \) are also symmetric, i.e. \( \beta \) identifies the affine subsets \( Y_i \cong \mathbb{C}^2_{(x^i)y^j/(y^i,x^j)} \) with \( Y_{r-i} \cong \mathbb{C}^2_{(x^{i'}y^{l''}/y^{l''}x^{i''})} \). Furthermore, the rational exceptional curves in \( E \) covered by these affine patches are also identified.

In the case when the number of rational curves in \( Y \) is even, i.e. \( l \) is even, there exists a middle affine set \( Y_m \) covering the intersection of the two middle rational curves \( E_{m-1} \) and \( E_m \). The \( \mathbb{Z}/2 \)-action of \( \beta \) identifies \( Y_m \) with itself, fixing the point \( E_{m-1} \cap E_m \). In fact, this fixed point will not be singular in the quotient since we have the following lemma.

**Lemma 3.4.** Suppose that the continued fraction \( \frac{2n}{a} \) has an even number of elements. Then \( G \) has quasireflections.

**Proof.** Let \( \{Y_i\}_{i=1}^{r} \) be the open affine covering of \( A - \text{Hilb}(\mathbb{C}^2) \) with middle open set \( Y_m \cong \mathbb{C}^2_{\lambda,\mu} \), where \( \lambda = x^u/y^v \) and \( \mu = y^u/x^v \). The action of \( \beta \) on this open set is

\[
\begin{align*}
\lambda = x^u/y^v &\mapsto (-1)^v/x^v = (-1)^u\mu \\
\mu = y^u/x^v &\mapsto (-1)^ux^u/y^v = (-1)^u\lambda
\end{align*}
\]
Hence the action is given by the matrix $M = \begin{pmatrix} 0 & (-1)^n \\ (-1)^n & 0 \end{pmatrix}$ with $\det M = -1$. In other words, $M$ is a reflection and the group is not small. 

From now on we suppose that $\frac{2n}{m}$ has odd number of elements, i.e. the exceptional divisor $E \subset Y$ has an odd number of irreducible components $E_i$. There exists a middle rational curve $E_m \cong \mathbb{P}^1$ with coordinate ratio $(x^q : y^q)$ covered by $Y_{m-1}$ and $Y_m$, where $q = \frac{2n}{(a-1, 2n)}$.

Then $\beta$ identifies $Y_{m-1}$ with $Y_m$ and it is an involution on $E_m$, having two fixed points on $E_m$ which are singular on $\tilde{Y} := Y/\langle \beta \rangle$ depending on whether or not there are quasireflections. If there are quasireflections, the resolution will have a type $A$ Dynkin diagram. If there are no quasireflections, $\tilde{Y}$ has 2 singular $A_1$ points, and blowing them up we obtain the Dynkin diagram of type $D$ that we were looking for.

**Remark 3.5.** For a small finite subgroup $G \subset \text{GL}(2, \mathbb{C})$, by a result of Ishii (Ish02) $G\text{Hilb}(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$. By the uniqueness of minimal models in dimension 2, the previous construction can be expressed in the following diagram:

$$
\mathbb{C}^2 \curvearrowright A \leq G \subset \text{GL}(2, \mathbb{C})
$$

$$
G/A \curvearrowright A\text{Hilb}(\mathbb{C}^2) \longrightarrow \mathbb{C}^2/A
$$

$$
G\text{Hilb}(\mathbb{C}^2) \longrightarrow \tilde{Y}
$$

In other words, we have that $G\text{Hilb}(\mathbb{C}^2) \cong (G/A)\text{Hilb}(A\text{Hilb}(\mathbb{C}^2))$.

The next proposition describes de Dynkin diagram of the exceptional divisor in $\text{BD}_{2n}(a)\text{Hilb}(\mathbb{C}^2)$.

**Proposition 3.6.** Let $\text{BD}_{2n}(a)$ be a small binary dihedral group and $E$ be the exceptional divisor in $\text{BD}_{2n}(a)\text{Hilb}(\mathbb{C}^2)$, the minimal resolution of $\mathbb{C}^2/\text{BD}_{2n}(a)$. Then $E$ has the following Dynkin diagram of type $D$:

\[
\begin{array}{c}
-2 \\
-2 \\
-2 \\
\end{array}
\]

$$
-b_1 \quad -b_2 \quad -b_{m-1} \quad -\frac{b_m+2}{2}
$$

where $\frac{2n}{m} = [b_1, \ldots, b_{m-1}, b_m, b_{m-1}, \ldots, b_1]$ and $b_m$ is even.

**Proof.** Let $X = A\text{Hilb}(\mathbb{C}^2)$ and $\pi: X \to X/\langle \beta \rangle$ be the quotient map, and denote by $E_i$, $i = 1, \ldots, r$ the exceptional curves in $X$ where $E_i^2 = E_{r-i}^2 = -b_i$. Denote also by $E_j$ the exceptional curves in $X/\langle \beta \rangle$ for $j = 1, \ldots, m-1$.

Since the group is small, the number of elements in the continued fraction $\frac{2n}{m}$ is odd. Then we have a middle rational curve $E_m$ in the exceptional locus of $A\text{Hilb}(\mathbb{C}^2)$, covered by the affine charts $Y_m \cong \mathbb{C}^2_{(\lambda, \mu)}$ and $Y_{m+1} \cong \mathbb{C}^2_{(\lambda', \mu')}$, where $\lambda = x^i/y^i$, $\mu = y^q/x^q$, $\lambda = x^q/y^q$ and $\mu = y^q/x^q$. Using the fact that $(i, j)$ and $(q, q)$ are consecutive points in the Newton polygon of $\frac{1}{2n}(1, a)$, we know that $i + j = b_m q$, and the action of $\beta$ in this case is

\[
\begin{align*}
\lambda &\mapsto (-1)^i y^i x^j = (-1)^i \lambda \mu^{b_m} \\
\mu &\mapsto (-1)^q x^q y^q = (-1)^q 1/\mu
\end{align*}
\]

First note that since $i$ and $j$ have the same parity and $i + j = b_m q$, we have that $b_m$ and $q$ cannot be both odd. In addition, $j$ and $q$ cannot be both even. Indeed, let

$$
(0, 2n), (1, a), \ldots, (r, s), (u, v), \ldots, (j, i), (q, q), (i, j), \ldots, (a, 1), (2n, 0)
$$

7
the sequence of points in the boundary of the Newton polygon for \( \frac{1}{2m}(1,a) \). Then \( q,j \) (and therefore \( i \)) even implies \( u = b_{m-1}j - q \) is even, \( r = b_{m-2}u - j \) is even, and by induction 1 is even, a contradiction.

Hence, the only possibilities for fixed locus of the action of \( \beta \) on \( A\text{-Hilb}(\mathbb{C}^2) \) are shown in Table 2.

In the case when the group \( BD_{2n}(a) \) is small, we see that \( b_m \) is even and \( X/\langle \beta \rangle \) has two singular \( A_1 \) points along \( E_m \equiv \mathbb{P}^1 \) and \( E_m = -b_m/2 \). Let \( f : Y \to X/\langle \beta \rangle \) the resolution of these to \( A_1 \) singularities denoting by \( C_1, C_2 \) the corresponding rational curves in \( Y \), and by \( E'_m \) the strict transform of \( E_m \). Then

\[
-b_m/2 = (E'_m + C_1/2 + C_2/2)^2 = (E'_m)^2 + C_1^2/4 + C_2^2/4 + E'_m C_1 + E'_m C_2 = (E'_m)^2 + 1
\]

so that \( (E'_m)^2 = -(b_m + 2)/2 \). \( \square \)

**Remark 3.7.** If \( a = 1 \) the group \( BD_{2n}(1) \) is Abelian, and by Proposition 3.1 it is small if and only if \( n \) is odd. Therefore, \( Y \) has a type \( A \) Dynkin diagram of the form

\[
-2 \quad -(n+1) \quad -2
\]

\[\bullet \quad \bullet \quad \bullet \quad \bullet \]

**Remark 3.8.** If we consider the non small group \( BD_{12}(5) \), the quotient variety \( X/\langle \beta \rangle \) is non singular and the exceptional divisor is of the form

\[-3 \quad -1
\]

\[\bullet \quad \bullet \]

To obtain the minimal resolution we need to contract the \(-1\)-curve, obtaining a single \( \mathbb{P}^1 \) with selfintersection \(-2 \). To see this more explicitly, the group of its quasireflections is generated by

\[
H = \langle \alpha \beta, \alpha^3 \beta, \alpha^5 \beta, \alpha^7 \beta, \alpha^9 \beta, \alpha^{11} \beta \rangle
\]

which has order 12. The quotient is \( \mathbb{C}^2_{x,y}/H \cong \mathbb{C}^2_{u,v} \) where \( u = x^6 - y^6 \) and \( v = xy \). Now the action of \( \alpha \) in the new coordinates is \( (u,v) \mapsto (\varepsilon^6 u, \varepsilon^6 v) \). In other words, the action is \( \frac{1}{12}(6,6) \cong \frac{1}{2}(1,1) \), and the exceptional divisor in the minimal resolution of \( \mathbb{C}^2/BD_{12}(5) \) consists of just one \( \mathbb{P}^1 \) with selfintersection \(-2 \).

**3.2 Representation theory for \( BD_{2n}(a) \) groups**

Let \( G = BD_{2n}(a) \subset \text{GL}(2, \mathbb{C}) \) be a small binary dihedral group. As an abstract group it has the following presentation:

\[
BD_{2n}(a) = \langle \alpha, \beta \mid \alpha^{2n} = \beta^4 = 1, \alpha^a = \beta^2, \alpha \beta = \beta \alpha^a \rangle
\]

Therefore any representation \( \rho : G \to \text{GL}(n, \mathbb{C}) \) of \( G \) must also satisfy \( \rho(\alpha)^{2n} = \rho(\beta)^4 = 1, \rho(\alpha)^a = \rho(\beta)^2 \) and \( \rho(\alpha) \rho(\beta) = \rho(\beta) \rho(\alpha)^a \). Then, if we denote by \( \rho_j^{\pm} \) the 1-dimensional irreducible representations, we have

\[
\rho_j^{+}(\alpha) = \varepsilon^j, \quad \rho_j^{+}(\beta) = i^n \\
\rho_j^{-}(\alpha) = \varepsilon^j, \quad \rho_j^{-}(\beta) = -i^n
\]
where \( \varepsilon \) is a \( 2n \)-th primitive root of unity, and for the irreducible 2-dimensional representations \( V_\tau \) we get

\[
V_\tau(\alpha) = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^ar \end{pmatrix}, \quad V_\tau(\beta) = \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}
\]

By the definition of the groups BD\(_{2n}(a)\) in terms of their action on \( \mathbb{C}^2 \) (see Section 3) the natural representation is \( V_1 \).

The group \( G \) acts on the complex plane \( \mathbb{C}^2_{x,y} \), so it will also act on the polynomial ring \( \mathbb{C}[x,y] \) breaking it into different eigenspaces.

**Definition 3.9.** We say that a polynomial \( f \in \mathbb{C}[x,y] \) is \( \rho \)-invariant, or \( f \) is semi-invariant with respect to \( \rho \), if

\[
f(g \cdot P) = \rho(g)f(P) \quad \text{for all } g \in G, \ P \in \mathbb{C}^2_{x,y}
\]

(compare [Muk03], Definition 6.10). We denote by \( S_\rho := \{ f \in \mathbb{C}[x,y] : f \in \rho \} \) the \( \mathbb{C}[x,y] \)-module of \( \rho \)-invariants, and we say that a polynomial belongs to a representation \( \rho \) if it belongs to the corresponding module \( S_\rho \).

In the case of BD\(_{2n}(a)\) groups we have that:

\[
f(x,y) \in \rho_i^k \iff \alpha(f) = \varepsilon^j f \text{ and } \beta(f) = \varepsilon^nf \quad (2)
\]

\[
(f,\beta(f)) \in V_r \iff \alpha(f,\beta(f)) = (\varepsilon^rf,\varepsilon^ar\beta(f)) \quad (3)
\]

where we are making the abuse of notation \( \alpha(f(x,y)) = f(\alpha(x),\alpha(y)) = f(\varepsilon x,\varepsilon^a y) \) and \( \beta(f(x,y)) = f(\beta(x),\beta(y)) = f(y,-x) \) (see [NdC08] §3.4.2 for more details).

To calculate the number of one and two dimensional irreducible representations, notice that the number of 1-dimensional representations coincide with the number of scalar diagonal elements in \( \frac{1}{2n}(1,a) \). Since there exists \( k \in \mathbb{N} \) such that \( n = kq \), the number of 1-dimensional representations is \( 4k \). If we call \( d \) the number of 2-dimensional irreducible representations, by the well-known formula

\[
|G| = \sum_{\rho \in \text{Irr} G} (\dim \rho)^2
\]

where \( \text{Irr} G \) is the set of irreducible representations of \( G \), we have \( 4n = 4k + 4d \). This implies that the number of 2-dimensional elements is \( d = n - k \).

**Example 3.10.** In Table 3 we present the irreducible representations for the group BD\(_{12}(7)\) together with some elements belonging to the corresponding modules \( S_\rho \).

## 4 \( G \)-graphs for BD\(_{2n}(a)\) groups

Let \( G = \text{BD}_{2n}(a) \) be a small binary dihedral group and let \( A = \langle \frac{1}{2n}(1,a) \rangle \) be the maximal normal cyclic subgroup of \( G \). As we have seen in Section 3.1, the resolution \( Y \rightarrow \mathbb{C}^2/G \) is obtained by acting with \( \beta \) on \( A\text{-Hilb}(\mathbb{C}^2) \). The \( G \)-graphs are constructed in the same way by translating the action of \( \beta \) into the \( A \)-graphs. Recall that \( \beta : (x,y) \mapsto (y,-x) \) interchanges the coordinates \( x \) and \( y \). The symmetry along the coordinates of the exceptional divisor in \( A\text{-Hilb}(\mathbb{C}^2) = \bigcup_{i \in \mathbb{N}} Y_i \) implies that \( \beta \) identifies \( Y_i \) with \( Y_{2m-i} \), as well as identifying the corresponding \( A \)-graphs contained in those affine pieces. The union of these two symmetric \( A \)-graph is what we call a \( qG \)-graph.

### 4.1 \( qG \)-graphs

Let \( Z_i \) be an \( A \)-cluster in \( Y_i \) and \( \beta(Z_i) \) its image under \( \beta \) in \( Y_{2m-i} \), with ideals \( I_{Z_i} \) and \( I_{\beta(Z_i)} \) respectively. Denote also by \( \tilde{Z} \) the point in the quotient \( \tilde{X} := A\text{-Hilb}(\mathbb{C}^2)/\langle \beta \rangle \) corresponding to the orbit \( \{ Z_i, \beta(Z_i) \} \).
we have that \(G\tilde{\gamma}\) where \(\tilde{\gamma}\) is a \(G\)-invariant graph symmetric with respect to the diagonal. But, notice that \(\tilde{\gamma}\) it is not a \(G\)-graph. Indeed, \(\gamma_i\) and \(\beta(\gamma_i)\) have an overlap (their common basis elements), so the number of elements

| \(\alpha\) | \(\beta\) | \(S_\rho\) |
|---|---|---|
| \(\rho_0^+\) | 1 | 1, \(x^{12}+y^{12}, x^5y-xy^5, x^6y^6, x^8y^4+x^4y^8\) |
| \(\rho_0^-\) | 1 | \(-1, x^{12}-y^{12}, x^5y+xy^5, x^4y^4\) |
| \(\rho_1^+\) | \(\epsilon^2\) | 1, \(x^2+y^2, x^2y-xy^2\) |
| \(\rho_1^-\) | \(\epsilon^2\) | \(-1, x^2-y^2, x'y+xy^2\) |
| \(\rho_2^+\) | \(\epsilon^4\) | 1, \(x^4+y^4, x^3y-xy^3, x^2y^2\) |
| \(\rho_2^-\) | \(\epsilon^4\) | \(-1, x^4-y^4, x^3y+yxy^3, x^5y^5\) |
| \(\rho_3^+\) | \(-1\) | 1, \(x^6+y^6, x^{11}y-xy^{11}, x^{10}y^2+x^{10}y^2\) |
| \(\rho_3^-\) | \(-1\) | \(-1, x^6-y^6, x^{11}y+yxy^{11}, x^{10}y^2-x^{10}y^2\) |
| \(\rho_4^+\) | \(\epsilon^8\) | 1, \(x^8+y^8, x^6y^2+x^2y^6, x^4y^4\) |
| \(\rho_4^-\) | \(\epsilon^8\) | \(-1, x^8-y^8, x^6y^2-x^2y^6, xy\) |
| \(\rho_5^+\) | \(\epsilon^{10}\) | 1, \(x^{10}+y^{10}, x^3y-xy^3\) |
| \(\rho_5^-\) | \(\epsilon^{10}\) | \(-1, x^{10}-y^{10}, x^3y+xy^3\) |
| \(V_1\) | \(\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\) | \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) | \((x, y), (y^7, -x^7), (x^6y, -xy^6), (x^2y^5, -x^5y^2)\) |
| \(V_2\) | \(\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\) | \(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\) | \((x^3, y^3), (y^9, -x^9), (xy^2, x^2y), (x^6y, -xy^6)\) |
| \(V_3\) | \(\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\) | \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) | \((x^5, y^5), (y^{11}, -x^{11}), (xy^4, x^4y), (x^{10}y, -xy^{10})\) |

Table 3: Some semi-invariant elements in each \(S_\rho\) for \(BD_{12}(7)\)

(see Figure 3). Suppose that \(Z_i\) and \(\beta(Z_i)\) are not one of the two points in \(A-Hilb(\mathbb{C}^2)\) fixed by \(\beta\). In this case, \(\tilde{Z}\) is not one of the singular \(A_1\) points in the quotient and, since \(G-Hilb(\mathbb{C}^2)\) is the minimal resolution, \(\tilde{Z}\) corresponds one-to-one with a \(G\)-cluster \(Z\).

As clusters in \(\mathbb{C}^2\), we have that \(Z \supset Z_i \cup \beta(Z_i)\), or equivalently \(I_Z \subset I_{Z_i} \cap I_{\beta(Z_i)}\). In terms of graphs, if we denote by \(\Gamma, \Gamma_i\) and \(\beta(\Gamma_i)\) the graphs corresponding to the ideals \(I_Z, I_{Z_i}\) and \(I_{\beta(Z_i)}\) respectively, we have that

\[\Gamma \supset \Gamma_i \cup \beta(\Gamma_i) = \Gamma_i\]

where \(\Gamma_i\) is a \(G\)-invariant graph symmetric with respect to the diagonal. But, notice that \(\Gamma_i\) it is not a \(G\)-graph. Indeed, \(\Gamma_i\) and \(\beta(\Gamma_i)\) have an overlap (their common basis elements), so the number of elements
of $\tilde{\Gamma}$ is always smaller than $|G| = 2 \cdot |A|$. We call these new graphs “quasi $G$-graphs” ($qG$-graphs), and there are two types of them, $A$ and $B$, which correspond to the two possible shapes for $\Gamma_i \cup \beta(\Gamma_i)$.

In the case when $Z_i = \beta(Z_i)$ is fixed by $\beta$, we have that $\Gamma_i = \beta(\Gamma_i)$ so the $qG$-graph is all overlap, making the extension to $G$-graphs in this case not unique. Indeed, for each of the two $\beta$-fixed points there is a projective line of $G$-clusters corresponding to the exceptional curves of the blow-up. The $G$-graphs at these $G$-clusters are called type $C$ and type $D$, and they do not come from $A$-graphs.

Remark 4.1. Since every cyclic $A$-graph is given by two consecutive points $\{e_i, e_{i+1}\}$ in the boundary of the Newton polygon of $L$, every $qG$-graph is given by a consecutive pair $\{e_i, e_{i+1}\}$ together with its symmetric pair with respect to the diagonal. Therefore, in order to calculate all possible $qG$-graphs, we need only consider half of the lattice points. More precisely, we just have to look at the $qG$-graphs coming from consecutive points of the list: $e_0 = (0, 2n)$, $e_1 = (1, a)$, ..., $e_m = (q, q)$.

Example 4.2. Continuing with the previous example, the action of $\beta$ glues together $\Gamma_0$ with $\Gamma_3$ and $\Gamma_1$ with $\Gamma_2$, obtaining the $qG$-graphs $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$. See Figure 4.

![Figure 4: $qG$-graphs for the group BD$_{2n}(a)$.](image)

Now we look at how two symmetric $A$-graphs can merge together into a $qG$-graph. The following proposition shows that there are only two types of gluing.

**Proposition 4.3.** Let $G = \text{BD}_{2n}(a) \subset \text{GL}(2, \mathbb{C})$ a small binary dihedral group. Let $\Gamma$ be the $A$-graph defined by the consecutive Newton polygon points $e_i = (r, s)$ and $e_{i+1} = (u, v)$. Then

- $\tilde{\Gamma}$ is of type $A$ if $s - v > u$
- $\tilde{\Gamma}$ is of type $B$ if $s - v = u - r$

Their shape is shown in the following diagram:

![Type A and Type B](image)

The colored area represents the overlap between the $A$-graphs.
Proof. Let $e_{i+1} = (t, w)$ the next point in the boundary of the Newton polygon. Since any $qG$-graph \( \tilde{\Gamma} \) is the union of two symmetric $A$-graphs with respect to the diagonal, the shape of the $qG$-graph will depend on the relation between $s - v$ and $u$.

Suppose that $s - v \leq u$. This means that $s - v = u - k$ for some $0 \leq k < u$. Then since $w = b_{i+1}v - s$ we have that

\[
k = u + v - s = u + (1 - b_{i+1})v + w\]

which implies $b_{i+1} = 2$ (otherwise $k < u - 2v + w$, and since $v \geq u$ and $v > w$, then $k < 0$ a contradiction). Thus $k = u - v + w$, and applying the same argument to $w$ we obtain that $b_{i+2} = b_{i+2} = \ldots = b_m = 2$, that is, \[\frac{2n}{n} = [b_0, \ldots, b_i, 2, \ldots, 2, b_i, \ldots, b_0].\]

Finally, the chain of 2s in the middle of the continued fraction \[\frac{2n}{n}\] gives us the value of $k$: Let $e_{m-1} = (c, d)$, $e_m = (q, q)$, $e_{m+1} = (d, c)$ be the three middle Newton polygon points. Then proceeding with the previous argument we get

\[
k = u - d + q
= u - (2q - c) + q = u - q + c
\]

which gives the $qG$-graph of type $B$. \hfill \Box

From the previous proof, we can deduce that the distribution of the $qG$-graphs of type $A$ and $B$ in the exceptional locus depend on the number of 2s in the middle of the continued fraction $\frac{2n}{n}$. For example, the continued fraction $[\ldots, b, 2, 2, 2, 2, b, \ldots]$ with $b \neq 2$, gives three $qG$-graphs of type $B$.

Let \( \tilde{\Gamma}_0, \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{m-1} \) be the sequence of $qG$-graphs for a given group \( G = BD_{2n}(a) \), then if $b_i = 2$ for $k \leq i \leq m$ and $b_{k-1} \neq 2$, we have that $\tilde{\Gamma}_0, \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{k-1}$ are of type $A$ and $\tilde{\Gamma}_k, \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{m-1}$ are of type $B$. As a consequence we get the following corollary.

**Corollary 4.4.** There are no type $B$ $qG$-graphs if and only if the middle entry $b_m$ in the continued fraction $\frac{2n}{n}\] is different from 2.

In the case when $a = 2n - 1$, i.e. $BD_{2n}(2n - 1) \subset SL(2, \mathbb{C})$, the coefficients of the continued fraction $\frac{2n}{2n-1}$ are all 2, and every $qG$-graph is of type $B$.

### 4.2 From qG-graphs to G-graphs

In this section we will construct the $G$-graph corresponding to a given $qG$-graph. Every element of the $qG$-graph is obviously in the $G$-graph, and the number of elements that we have to add to the $qG$-graph is exactly the number of elements in the overlap $M$. We find these new members by looking at the representations of $G$ that are contained in $M$.

Let $I$ be an ideal defining a $G$-cluster and $\Gamma_I$ its graph. The elements of $\Gamma_I$ form a basis for the vector space $\mathbb{C}[x, y]/I$, and every irreducible representation $\rho$ has $(\dim \rho)$ number of elements in $\Gamma_I$. Suppose for simplicity that $\rho$ is a 1-dimensional irreducible representation of $G$. Then $\rho$ can be considered as a 1-dimensional vector space with basis $g$, and whenever an element $f$ in $\rho$ is not basis of $\rho$ we have a relation of the form $f = ag$ for some $a \in \mathbb{C}$. In particular $f - ag \in I$.

In order to easily describe the $G$-graph we can always take $a = 0$, so that $f = 0$ in $\mathbb{C}[x, y]/I$. More precisely, the choice of taking all constants to be equal to zero for a given $G$-cluster corresponds to looking at the origin of the affine open set defined by the $G$-graph. The set of these $G$-clusters located at the origins describes the 0-dimensional strata of $G$-Hilb$(\mathbb{C}^2)$.

As we have seen in Section 3.2 instead of having monomials in every irreducible representation we can have sums of monomials. This creates the new phenomenon of "twin" elements in the $G$-graphs which does not appear in the Abelian case. For example, let $f = x^i y^j - x^j y^i$ with $i > j$ be an element...
in some 1-dimensional representation $\rho$. If $f$ is not the basis for $\rho$, then $x^iy^j = x^jy^i$ and both monomials become the same element in $C[x, y]/I$, that is, they are twins. Moreover, multiplying the above equality by $x^py^q$ for any positive integers $p$ and $q$, we have $x^{i+p}y^{j+q} = x^{j+p}y^{i+q}$, which means that once we have a pair of symmetric twin elements we get a pair of symmetric “twin regions”. Thus, in order to count the number of basis elements when this phenomenon occurs, a pair of twin elements will count as a single basis element.

### 4.2.1 G-graphs of type $A$

Suppose that $\tilde{\Gamma}$ is a $qG$-graph of type $A$ and let $e_i = (r, s)$ and $e_{i+1} = (u, v)$ be the corresponding lattice points in the Newton polygon. Then $\tilde{\Gamma} = \Gamma_I \cup \Gamma_{\beta(I)}$ where $I_i = (x^s, y^u, x^{s-v}y^{u-r})$ and $\beta(I_i) = (x^u, y^s, x^{u-r}y^{s-v})$ define the $\frac{1}{q}(1, a)$-graphs. In addition, we have the inequalities $r < u \leq v < s$ and $u < s - v$, coming from the lattice $L$ and the type $A$ condition respectively.

The general shape for a type $A qG$-graph is the following:

$$
\begin{align*}
\text{If we call } M = \Gamma_I \cap \Gamma_{\beta(I)} \text{ the overlap, clearly } \#\tilde{\Gamma} = 2 \cdot \#\Gamma_I - \#M < 2n. \text{ Then, in order for } \tilde{\Gamma} \text{ to be a } G \text{-graph it needs to be extended by } \#M = u^2 \text{ elements, and these extra elements belong to the irreducible representations appearing in } M.
\end{align*}
$$

Let us denote by $\Gamma$ (or by $\Gamma_A(r, s; u, v)$ when a reference to the lattice points is necessary) the $G$-graph corresponding to $\tilde{\Gamma}$ that we are looking for, and by $I$ its defining ideal. The following lemmas show that we just need to look at some key representations.

**Lemma 4.5.** The polynomial $x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}$ is $G$-invariant, i.e. it belongs to $I$.

**Proof.** We need to show that $x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}$ is invariant under the action of $\alpha$ and $\beta$. We know that $ar \equiv s$ and $au \equiv v \pmod{2n}$, then

$$
\begin{align*}
\alpha(x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}) &= x^{s-v} + au - ar)x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r} + as - av)x^{u-r}y^{s-v} \\
&= x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}
\end{align*}
$$

and since $s - v$ and $u - r$ must have the same parity, we also obtain that

$$
\begin{align*}
\beta(x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}) &= (-1)^{u-r}x^{u-r}y^{s-v} + (-1)^{u-r+s-v}x^{s-v}y^{u-r} \\
&= x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}.
\end{align*}
$$

**Lemma 4.6.** The monomial $x^uy^u$ is not in the $G$-graph $\Gamma$.
Lemma 4.7. The polynomial \( x^u u^v \notin I \). Notice that \( x^u u^v \) and \( x^u+v + (-1)^u y^{u+v} \) are in the same 1-dimensional representation \( \rho_{u+v} \), so \( x^u+v + (-1)^u y^{u+v} \in I \) and it forms a twin region. Observe also that the following elements belong to the same 2-dimensional representation:

\[
(x^u y^{u+1}, (-1)^u x^{u+1} y^u), (y^{u+v+1}, -x^{u+v+1}), (x^l, y^l) \in V_I
\]

for some \( 0 < l < u+v+1 \). Since the monomials \( y^{u+v+1}, x^{u+v+1}, x^l, y^l \in \Gamma \) they must belong to \( \Gamma \) so the pair \( (x^u y^{u+1}, (-1)^u x^{u+1} y^u) \notin \Gamma \), hence \( x^u y^{u+1}, x^{u+1} y^u \in I \).

This last condition implies that on the diagonal, the \( qG \)-graph can only be extended with the element \( x^u y^u \). We claim the and along the sides it cannot be extended enough to obtain a \( G \)-graph. Indeed, by the previous lemma we know that \( x^{s-v} y^{u-r} + (-1)^{u-r} x^{u-r} y^{s-v} \in I \) so it forms a pair of twins. Also \( x^r (x^{s-v} y^{u-r} + (-1)^{u-r} x^{u-r} y^{s-v}) = x^{s+r-v} y^{u-r} + (-1)^{u-r} x^{u} y^{s-v} \in I \). Now, the type \( A \) condition \( u < s - v \) implies that \( x^u y^{s-v} \notin I \), and therefore \( x^{s+r-v} y^{u-r} \in I \) (same for \( x^{u-r} y^{s+r-v} \)). Thus the twin regions have \( r^2 \) elements each.

In the same way, the element \( x^{u+v} + (-1)^u y^{u+v} \) forms another twin region, and since \( u + u < s \), their size is at most \( (u-r)^2 - (u-r) \). So, the \( qG \)-graph is extended with at most \( r^2 + (u-r)^2 - (u-r) + 1 < u^2 = \#M \) monomials, and this is a contradiction.

As a consequence we have that \( x^{s-v} y^{u-r} + (-1)^{u-r} x^{u-r} y^{s-v} \) creates a twin region \( T_1 \) of size \( r^2 \). The following lemma shows that there exists another polynomial creating a second twin region.

Lemma 4.7. The polynomial \( x^{r+s} + (-1)^r y^{r+s} \in I \) and creates a twin region \( T_2 \) of size \( (u-r)^2 \).

Proof. It is easy to see that \( x^r y^r \) and \( x^{r+s} + (-1)^r y^{r+s} \) are in the same representation \( \rho_{r+s} \). But \( x^r y^r \) belongs to the \( qG \)-graph so it must be in \( \Gamma \), and therefore \( x^{r+s} + (-1)^r y^{r+s} \notin I \).

Now combining the previous lemmas we see that \( x^{r+s}, y^{r+s} \in I \) and the size of the twin region \( T_2 \) is equal to \( (u-r)^2 \). 

Hence, the extension to a \( G \)-graph from any type \( A \) \( qG \)-graph has two twin regions \( T_1 \) and \( T_2 \), except for the first \( qG \)-graph \( T_0 \) given by \( e_0 = (0, 2n) \) and \( e_1 = (1, a) \) where only \( T_2 \) appears \( (r = 0 \text{ in this case}) \). Figure 5 represents the shape of the \( G \)-graph we were looking for.

Proposition 4.8. The ideal \( I_A = (x^u u^v, x^{s-v} y^{u-r} + (-1)^{u-r} x^{u-r} y^{s-v}, x^{r+s} + (-1)^r y^{r+s}) \) defines a \( G \)-graph \( \Gamma_A(r, s; u, v) \) of type \( A \).

![Figure 5: Extension from a qG-graph to a G-graph of type A by the elements in the overlap.](image)
Proof. The previous lemmas show that the $qG$-graph only expands along its sides, and it does not grow further than the twin regions $T_1$ and $T_2$. Then, the number of elements added is
\[
\#T_1 + \#T_2 + \#I + \#II = r^2 + (r - u)^2 + 2r(u - r) = u^2
\]
exactly the number of elements in the overlap.

We still have to prove that $\Gamma$ contains 1 polynomial for each 1-dimensional irreducible representation $\rho_k$ and 2 pairs of polynomials for each 2-dimensional irreducible representation $V_l$. Since the $qG$-graph is made from two cyclic graphs symmetric with respect to the diagonal, every representation not appearing in the overlap will appear twice, as required. Therefore we need only check that the representations of the new extended blocks correspond exactly to the ones in the overlap.

Every representation contained in regions I and II is 2-dimensional, and we have one basis element coming from the overlap $M$ and a second from the extended region.

In the twin regions $T_1$ and $T_2$ it looks like we have doubled the basis elements, but the presence of the twin relations gives the correct number. We show in diagram below the configuration of the representations contained in the twin region $T_1$. The case of $T_2$ is analogous.

\[\text{Diagram}\]

The elements on the diagonals (marked by dots) are in 1-dimensional representations. The rest (marked by letters) are in 2-dimensional representation pairs, that is, a monomial $x^iy^j$ and its symmetric with respect to the diagonal $x^jy^i$, will form an element in some 2-dimensional representation (we omit the sign here). Elements with the same letter are in the same representation.

Since twin symmetric regions count as one, we can see how the representations of the overlap are fully represented in the extension. Note also that in these twin regions, there are three pairs of elements in the graph for each 2-dimensional representation when we should have only two. For this, observe that the twin relations
\[
x^{s-r}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v} = 0 \quad \text{and} \quad x^{r+s} + (-1)^r y^{r+s} = 0
\]
identify two of these pairs, and we have a $G$-graph.

Therefore, for every 2-dimensional representation contained in a twin region we have one basis element coming from the overlap region and a second basis element coming from a combination of elements in the outer twin regions. \[\square\]

4.2.2 $G$-graphs of type $B$

Suppose that the $qG$-graph $\tilde{\Gamma}$ defined by $e_i = (r,s)$ and $e_{i+1} = (u,v)$ is of type B, that is, we can define
\[
m := s - v = u - r
\]
As before, we will denote by $\Gamma$, or $\Gamma_B(r,s;u,v)$, the $G$-graph corresponding to $\tilde{\Gamma}$.

The general shape for a type B $qG$-graph is the following:
In addition, we suppose that $x^uy^m$ and $x^m y^n$ are not in $\Gamma$, that is,

$$x^y y^m = x^m y^n = 0$$

Note that $\alpha(x^uy^m) = \varepsilon^{u+a} - x^uy^m = \varepsilon^{u+r} - x^uy^m = \varepsilon^x y^m$, which implies that the pair $(x^uy^m, (-1)^m y^m)$ belongs to the 2-dimensional representation $V_r$.

**Remark 4.9.** This assumption characterises the $G$-graph. If we consider them as part of the basis, i.e. $x^uy^m, x^m y^n \in \Gamma$, we will be dealing with the next type B $G$-graph in the exceptional locus of the resolution $Y = G$-Hilb$(\mathbb{C}^2)$.

**Lemma 4.10.** Let $G = BD_{2n}(a)$ and let $\Gamma = \Gamma_B(r, s; u, v)$ be a $qG$-graph of type B. Then the monomial $x^m y^n$ is $(\alpha)$-invariant with $m$, odd, and $x^2m y^n$ is $G$-invariant.

**Proof.** We have that $\alpha(x^m y^n) = \varepsilon^{s-v} + \alpha(a - r)x^m y^n = \varepsilon^{s-v} + x^m y^n = x^m y^n$, so it is $(\alpha)$-invariant.

By definition we know that $2m = s - r$, and since the last $qG$-graph is defined by the lattice points $(r, s)$ and $(q, q)$, we also know that $2n = q(r - s)$. Then $n = qm$, and since $G = BD_{2n}(a)$ we conclude that $m$ is odd. Therefore, $\beta(x^m y^n) = -x^m y^n$ so the $G$-invariant monomial is $x^2m y^n$.

Note also that $m$ remains the same for every $qG$-graph of type B (if it exists). Indeed, if we consider the next type B $qG$-graph $\Gamma_B(u, v; t, w)$, we have that $w = 2v - s$ so that $s - v = m$.

Now we split up the type B graphs into two different cases as follows: type B.1 will be the case when $u < 2m$, and type B.2 the case when $u \geq 2m$. See Figure 6.

![Figure 6: G-graphs of type B.1 and B.2 according to the size of the overlap.](image)

**Remark 4.11.** Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{l-1}$ be the sequence of $qG$-graphs for a given group $G = BD_{2n}(a)$. Let $\frac{2a}{a} = [b_1, \ldots, b_i, \ldots, b_l]$ and suppose that $b_i = 2$ for $k \leq i \leq l - 1$ and $b_{k-1} \neq 2$, for some $0 \leq k \leq l - 1$.

By Remark ?? the $qG$-graphs $\Gamma_i$ are of type $A$ for $i \leq k - 1$, and of type $B$ for $i \leq k \leq l - 1$. We claim that $\Gamma_k$ is always of type $B.1$, while $\Gamma_i$ for $k < i \leq l - 1$ is of type $B.2$.

Indeed, let $\Gamma_{k-1} = \Gamma_A(p, q; r, s)$ a $G$-graph of type $A$ and $\Gamma_{k} = \Gamma_B(r, s; u, v)$ a $G$-graph of type $B$. Then we can define $m = s - v = u - r$, and since the corresponding $b_i$ in the continued fraction is 2 ($\Gamma_k$ is of type $B$) we have $q = 2s - v$. Now, since $\Gamma_{k-1}$ is of type $A$ we have

$$r < q - s = (2s - v) - s = s - v = m$$

and then $u = m + r < 2m$, so $\Gamma_k$ is of type $B.1$.

Suppose now that $k < l - 1$ so that exists $\Gamma_{k+1} = \Gamma_B(u, v; w, t)$ a $qG$-graph of type $B$. Then $w = 2u - r$, and if we call $m' = w - u$, then

$$2m' = 2(w - u) = 2w - 2u = 2w - (w + r) = w - r$$

Therefore $w \geq 2m'$, i.e. a type $B.2$ $qG$-graph. By induction the rest of the $qG$-graphs are also of type $B.2$. 

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**Type B.1**

In this case $u < 2m$. The description of the $G$-graphs of type $B.1$ is given by the following proposition:

**Proposition 4.12.** The ideal $I_{B.1} = (x^{r+s} + (-1)^r y^{r+s}, x^{m+s}y^{m-r}, (-1)^{m-r}x^{m-r}y^{m+s}, x^u y^m, x^m y^u)$ defines a $G$-graph $\Gamma_{B.1}(r, s; u, v)$ of type $B.1$.

Proof. Since $u < 2m$ we know that $r < u$. Then $x^r y^r \in \Gamma$ is a basis element and we have the following relation:

$$x^{r+s} + (-1)^r y^{r+s} = 0$$

which creates a twin region. Also, using the condition $x^{u} y^{m} = x^{m} y^{u} = 0$ and multiplying the previous relation by $x^{m}$ and $y^{m}$ we get

$$x^{s+u} = y^{s+u} = 0$$

Looking now at the representations $\rho_0^+$ and $\rho_0^-$, we see that $x^{m+s}y^{m-r} + (-1)^{m-r}x^{m-r}y^{m+s} = 0$ (since 1 is always basic) and $x^{m+s}y^{m-r} - (-1)^{m-r}x^{m-r}y^{m+s} = 0$ (otherwise $x^m y^m = 0$ and the $G$-graph would not have enough elements), which implies

$$x^{m+s}y^{m-r} = x^{m-r}y^{m+s} = 0$$

All these relations are enough to determine a $G$-graph.

To finish we need to check that in $\Gamma$ we have the correct number of elements in each representation. As in the type $A$ case, this is true since the representations involved in the overlap are exactly those for the elements we have just added. In Figure 7 we show the $G$-graph and the underlying relation between the representations of $G$ for the type $B.1$ case.

**Type B.2**

Now we have that $u \geq 2m$ and $x^u y^m = x^m y^u = 0$.

**Proposition 4.13.** The ideal $I_{B.2} = (x^{2m} y^m, x^{s+m}, y^{s+m}, x^u y^m, x^m y^u)$ defines a $G$-graph $\Gamma_{B.2}(r, s; u, v)$ of type $B.2$.

Proof. First observe that $x^r y^m$ and $x^m y^r$ are elements of the basis. Indeed, the pairs

$$(x^{r-m}, y^{r-m}), (y^{s+m}, (-1)^{s+m} x^{s+m}), (x^r y^m, -x^m y^r) \in V_{2r-u}$$
The monomials $x^{r-m}$ and $y^{r-m}$ must be in $\Gamma$ since they belong to the $qG$-graph. If $x^r y^m = x^m y^r = 0$ then $y^{s+m}, x^{s+m} \in \Gamma$, which is the case of the previous type $B G$-graph. Therefore we can assume that

$$x^{s+m} = y^{s+m} = 0$$

From this, together with the rest of assumptions and the $G$-invariant relation $x^{2m} y^{2m} = 0$, we obtain the result. In the Figure 8 we show the resulting $G$-graph and the relation with the representations in the overlap.

### 4.2.3 Remaining $G$-graphs: types C and D

From any $qG$-Graph we are able to find uniquely its corresponding $G$-graph by adding some suitable basis elements. This procedure will give almost all possible $G$-graphs. Indeed, the involution of the middle rational curve to itself by $\beta$ gives two isolated fixed points, and therefore two more rational curves in the exceptional locus for the resolution of $\mathbb{C}^2/G$. This part of the exceptional locus cannot be recovered with $qG$-graphs, i.e. from the toric information of $A$-Hilb($\mathbb{C}^2$).

In this section we construct the $G$-graphs which correspond to the neighbourhood of the two exceptional curves coming from the involution of the middle curve (the discontinuous area in the diagram below).

They will be called type $C$ and type $D$, and each of them have two cases, $C^+$ and $C^-$ ($D^+$ and $D^-$ respectively).

Let $e_m = (r, s)$ and $e_{m+1} = (q, q)$ be the lattice points giving the last $qG$-graph $\Gamma_{m-1}$, and define

$$m_1 := s - q \quad \text{and} \quad m_2 := q - r$$
Note that if the last $qG$-graph is of type $B$ then $m_1 = m_2$, and if it is of type $A$ then $m_1 > m_2$. The new
$G$-graphs depend on a choice of basis of the following two 1-dimensional representations of $G$:
\begin{align*}
x^q + (-i)^q y^q, & \quad x^q y^{m_2} + (-1)^i i^q x^{m_2} y^s \in \rho_q^+ \\
x^q - (-i)^q y^q, & \quad x^q y^{m_2} - (-1)^i i^q x^{m_2} y^s \in \rho_q^-
\end{align*}
which correspond to the two curves we want to cover (see Section 6).

Note that the polynomials $x^q + (-i)^q y^q$ and $x^q - (-i)^q y^q$ cannot be both out of the basis of $\mathbb{C}[x, y]/I$ at the same time. Indeed, if $x^q + (-i)^q y^q, x^q - (-i)^q y^q \in I$ then also $x^q, y^q \in I$, which is a contradiction since they are elements of the last $qG$-graph.

**Remark 4.14.** The monomial $x^s-r y^{s-r}$ is $G$-invariant, and therefore is in $I$. Indeed,
\[ \alpha(x^s-r y^{s-r}) = \beta(x^s-r y^{s-r}) = x^s-r y^{s-r} \]
since $r$ and $s$ have the same parity, and $ar \equiv s$ and $as \equiv r \pmod{2n}$. Similarly, the pairs of monomials
\[ x^{s-r+iq} y^{s-r-iq} + (-1)^iq x^{s-r-iq} y^{s-r+iq} \]
are also $G$-invariant for every $i$.

Now we give the description of these $G$-graphs starting with the type $D$ case first, followed by the type $C$.

**$G$-graphs of type $D$**

Suppose that $x^q + (-i)^q y^q \in I$. Then $x^q y^{m_2} + (-1)^i i^q x^{m_2} y^s$ must be in the $G$-graph $\Gamma$, and the basis elements are $x^q - (-i)^q y^q$ and $x^q y^{m_2} + (-1)^i i^q x^{m_2} y^s$. Similarly, we can choose $x^q - (-i)^q y^q \in I$, which now implies $x^q + (-i)^q y^q$ and $x^q y^{m_2} - (-1)^i i^q x^{m_2} y^s$ are in the basis. The first case corresponds to type $D^+$ and the second to type $D^-$.

The assumption $x^q + (-i)^q y^q = 0$ (or analogously, $x^q - (-i)^q y^q = 0$ for the Case $D^-$) identifies the monomials $x^q$ with $y^q$ as twin elements in $\mathbb{C}[x, y]/I$, and together with $x^s-r y^{s-r} = 0$ characterise completely the shape of the $G$-graph $\Gamma$. This gives the following Proposition:

**Proposition 4.15.** The ideal $I_D^+ = (x^q+(-i)^q y^q, x^s-r y^{s-r})$ defines a $G$-graph of type $D^+$, and similarly, the ideal $I_D^- = (x^q - (-i)^q y^q, x^s-r y^{s-r})$ defines a $G$-graph of type $D^-$.

![Figure 9: $G$-graph of type D. The monomials $x^q$ and $y^q$ are identified.](image-url)
Proof. We check that the number of basis elements is correct: by Remark 4.14 we have that \(x^{s-r+iq}y^{s-r-iq}+(-1)^iqx^{s-r-iq}y^{s-r+iq} \in \rho_0^+\) for all \(i\), i.e. they belong to \(I\). Also, \(x^{s-r+iq}y^{s-r-iq}+(-1)^iqx^{s-r-iq}y^{s-r+iq} \in \rho_0^-\) belong to \(I\) because now \(x^{(s-r)/2}y^{(s-r)/2}\) is in the basis of \(\rho_0\). Hence \(x^{s-r+iq}y^{s-r-iq}, x^{s-r-iq}y^{s-r+iq} \in I\), and the \(G\)-graph has a “stair” shape with stairs of height \(q\) as shows Figure 4.\!

The condition \(x^q + (-1)^iqy^q \in I\) gives the identification \(y^q = -(-1)^iqy^q\), so every monomial \(x^iy^j\) in \(\Gamma\) with \(i \geq q\) can be written in terms of monomials \(x^ky^l\) with \(k < q\). So, if we define \(k, t\) such that \(s - r = kq + t\), we have that that the number of basis elements after the twin identifications are

\[
t(s - r + (k + 1)q) + (q - t)(s - r + kq) = q(s - r + t + qk) = 2q(s - r)
\]

But, since the points \((r, s)\) and \((q, q)\) define an \(A\)-cluster, we have that \(qs - qr = 2n\), and the number above is equal to \(4n = |G|\). \(\square\)

**G-graphs of type C**

Now \(x^q + (-i)^qy^q\) and \(x^q - (-i)^qy^q\) are basis elements for \(\rho_q^+\) and \(\rho_q^-\), i.e. \(x^sy^{m_2} \pm (-1)^iqx^{m_2}y^s \in I\), which implies that \(x^sy^{m_2}, x^{m_2}y^s \in I\).

The following pairs belong to the same 2-dimensional representation (omitting the corresponding signs)

\[
(x^r, y^r),(y^s, (-1)^s x^s), (x^q, y^m, (-1)^{m_1}x^{m_1}y^q) \in V_r
\]

and all monomials in them must belong to the basis (otherwise the \(G\)-graph would have less than \(|G|\) elements). Therefore there must be an identification between pairs of the same degree, that is, we take as basic elements \((x^r, y^r)\) and a linear combination of \((y^s, (-1)^s x^s)\) and \((x^q, y^m, (-1)^{m_1}x^{m_1}y^q)\). We have two possibilities:

**Case** \(C^+: y^{m_1}(x^q + (-i)^qy^q), x^{m_1}(x^q + (-i)^qy^q) \in I\) and the elements in the basis are:

\[
(x^r, y^r), (y^{m_1}(x^q + (-i)^qy^q), x^{m_1}(x^q + (-i)^qy^q)).
\]

**Case** \(C^-: y^{m_1}(x^q - (-i)^qy^q), x^{m_1}(x^q - (-i)^qy^q) \in I\) and the elements in the basis are:

\[
(x^r, y^r), (y^{m_1}(x^q - (-i)^qy^q), x^{m_1}(x^q + (-i)^qy^q)).
\]

Note also that the irreducible representation \(\rho_{2q}^\pm\) contain the elements

\[
x^{2q} + (-1)^qy^{2q} \quad \text{and} \quad x^sy^q,
\]

and since the \(i\)-dimensional representation, we need only one element to be the basis for \(\rho_{2q}^\pm\). On the other hand, both of them must belong to \(\Gamma\), otherwise there will be not \(|G|\) elements in \(\Gamma\). This implies that we have to take a combination of them as our basis for \(\rho_{2q}^\pm\). We will take

\[
(x^s + (-i)^qy^q)^2 = x^{2q} + (-1)^qy^{2q} + 2(-1)^qx^qy^q \in I \quad \text{for type} \ C^+, \quad (4)
\]

\[
(x^q - (-i)^qy^q)^2 = x^{2q} + (-1)^qy^{2q} - 2(-1)^qx^qy^q \in I \quad \text{for type} \ C^-.
\]

Note that in both cases, if the last \(q\) \(G\)-graph \(\Gamma_t\) is of type \(B\) then this last equation is redundant, so it is only needed in the type \(A\) case. Indeed, substituting the value of \(s = 2q - r\) into the equations for case \(C^+\) we get

\[
y^{2q-r} + x^qy^{q-r}, x^{2q-r} + (-1)^qx^qy^q \in I
\]

Now multiplying \((4)\) by \((-1)^qy^r\) and \((5)\) by \(x^r\), and adding them together we get \(x^{2q} + (-1)^qy^{2q} + 2(-1)^qx^qy^q\) as desired.

In the case when the last \(G\)-graph \(\Gamma_t\) is of type \(A\), we do need the equation \(x^{2q} + (-1)^qy^{2q} + 2(-1)^qx^qy^q\) together with the \(G\)-invariant \(x^{m_1}y^{m_2} + (-1)^{m_2}x^{m_2}y^{m_1}\). In this case \(x^{s-r}y^{s-r}\) can be obtained using the rest of identities. In other words,

\[
x^{s-r}y^{s-r} \in (x^{2q} + (-1)^qy^{2q} + 2(-1)^qx^qy^q, x^sy^{m_2} + (-1)^qx^{m_2}y^s, x^{m_1}y^{m_2} + (-1)^{m_2}x^{m_2}y^{m_1})
\]

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Indeed, using the second and third generators of the ideal we have that
\[ x^{s-r}y^{s-r}z = (-i)^q x^{2(s-q)} y^{2(q-r)} = \begin{cases} -(-1)^{q-r}(i)^q x^{r-s}y^{r-s} & \text{if } s < 2(q-r) \\
-(-1)^{q-r}(i)^q x^{r-s}y^{r-s} & \text{if } s > 2(q-r) \end{cases} \]

Since \( s < 2(q-r) \) we can multiply the first generator by \( x^{s-r}y^{s-r}z \) getting
\[ (-1)^{q-r}x^{r-s}y^{r-s} - (-1)^{r-s}y^{s-r} + 2x^{s-r}y^{s-r} \]

The only possibility for this to be identically zero is only if \( q \) and \( r \) are both even at the same time. But this is impossible because it would imply that every boundary lattice point in the lattice \( L \) is even, a contradiction. Therefore, the sum above is not identically zero, which implies that \( x^{s-r}y^{s-r}z = 0 \).

![Figure 10: G-graph of type C when \( \Gamma_l \) is (a) of type B.1, and (b) of type B.2.](image)

Now we have the following proposition

**Proposition 4.16.** Let \( \Gamma_l \) be the last \( qG \)-graph. Then the ideal
\[
I_{C+} = (\begin{cases} (x^q + (-i)^q y) x^s y^m + (-1)^{r-s} y^{s-r} & if \Gamma_l \text{ is of type A, or} \\
(y^m(x^q + (-i)^q y)) x^s y^m + (-1)^{r-s} y^{s-r} & if \Gamma_l \text{ is of type B, defines a } G \text{-graph of type } C^+ \text{ (and similarly for } C^- \text{ changing the corresponding signs).} \end{cases}
\]

The shape of the \( G \)-cluster is the same in both cases, and as in the type \( D \) case, it has a stair shape. In this case, the conditions \( x^s y^m, x^m y^s \in I \) make the stair smaller. Figure 10 represents the type \( B \) situation, i.e. when the last \( qG \)-graph is of type \( B \) (the type \( A \) case is similar).

We collect in the following theorem the results of all the cases explained previously, obtaining the general result.

**Theorem 4.17.** Let \( (r, s; u, v) \) be the \( G \)-graph corresponding to the two consecutive lattice points \( e_i = (r, s) \) and \( e_{i+1} = (u, v) \) of the Newton polygon. Then we have the following possibilities:

- **If** \( u < s - v \) **then** \( \Gamma \) **is of type A** and it is defined by the ideal
  \[
  I_A = (x^u y^v, x^{u-v} y^{s-r} + (-1)^{u-r} x^{u-r} y^{s-v}, x^{r+s} + (-1)^r y^{r+s})
  \]

- **If** \( u - v = m \) **and** \( u < 2m \) **then** \( \Gamma \) **is of type B.1** and it is defined by the ideal
  \[
  I_{B.1} = (x^{r+s} + (-1)^r y^{r+s}, x^{m+s} y^{m-r} + (-1)^{m-r} x^{m-r} y^{m+s}, x^u y^v, x^m y^m)
  \]

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b) If \( u \geq 2m \) then \( \Gamma \) is of type B.2 and it is defined by the ideal

\[
I_{B.2} = (x^{2m} y^{2m}, x^{s+m}, y^{s+m}, x^u y^m, x^m y^u)
\]

- In addition, when \( u = v = q := \frac{2n}{(n-1,2n)} \) we have four types of \( G \)-graphs: types \( C^+, C^-, D^+ \) and \( D^- \). The \( G \)-graphs of types \( D^\pm \) are defined by the ideals:

\[
I_{D^\pm} = (x^q \pm (-i)^q y^q, x^{s-r} y^{s-r})
\]

For \( G \)-graphs of types \( C^\pm \) we have two cases:

a) If \( 2q < s \), and we call \( m_1 := s - q \) and \( m_2 := q - r \), they are defined by the ideals

\[
I_{C^\pm_A} = ((x^q \pm (-i)^q y^q)^2, x^y y^{m_2} \pm (-1)^i q x^{m_2} y^s, x^{m_1} y^{m_2} \pm (-1)^{m_2} x^{m_2} y^{m_1})
\]

b) If \( 2q = r + s \) they are defined by the ideals

\[
I_{C^\pm_B} = (y^m (x^q \pm (-i)^q y^q), x^m (x^q \pm (-i)^q y^q), x^{s-r} y^{s-r}, x^s y^m, x^m y^s)
\]

**Example 4.18.** Consider the group \( \text{BD}_{42}(13) \). We have that \( \frac{42}{13} = [4, 2, 2, 2, 4] \) and the lattice points in the Newton polygon that we need to consider are \( e_0 = (0, 42) \), \( e_1 = (1, 13) \), \( e_2 = (4, 10) \) and \( e_3 = (7, 7) \). Therefore we have 7 distinguished \( \text{BD}_{42}(13) \)-graphs: \( e_0, e_1 \) gives a type \( A \), \( e_1, e_2 \) gives a type \( B.1 \) and \( e_2, e_3 \) gives the types \( B.2, C^+, C^-, D^+ \) and \( D^- \). They are shown in Figure 11 together with their corresponding ideals.

![Figure 11: The G-graphs for the group BD_{42}(13)](image-url)
5 Walking along the exceptional divisor

In this section we prove that at every G-cluster in $G$-Hilb($\mathbb{C}^2$) can be expressed by an ideal with $G$-graph of type either $A$, $B$, $C$ or $D$. We start giving a one-parameter family of ideals which connects any two consecutive $G$-graphs, obtaining every $G$-cluster at the exceptional divisor $E$ of $G$-Hilb($\mathbb{C}^2$) by passing through all $G$-graphs for a given $BD_{2n}(a)$ group.

**Theorem 5.1.** Let $G = BD_{2n}(a)$ and let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = C^+, C^-, D^+, D^-$ be the sequence of $G$-graphs with $I_{\Gamma_0}, I_{\Gamma_1}, \ldots, I_{\Gamma_m}, I_{C^+}, I_{C^-}, I_{D^+}, I_{D^-}$ the corresponding defining ideals. Then,

1. For any two consecutive $G$-graphs, $\Gamma_i$ and $\Gamma_{i+1}$, there exists a family of ideals $J_{(\xi_i : \eta_i)}$ with $(\xi_i : \eta_i) \in \mathbb{P}^1$ such that
   - $J_{(0:1)} = I_{\Gamma_i}$, $J_{(1:0)} = I_{\Gamma_{i+1}}$ and
   - $J_{(\xi_i : \eta_i)}$ defines a $G$-cluster.

2. There exists a family of ideals $J_{(\gamma_+ : \delta_+)}$ (respectively $J_{(\gamma_- : \delta_-)}$) with $(\gamma_+, \delta_+) \in \mathbb{P}^1$ such that
   - $J_{(0:1)} = I_{C^+}$, $J_{(1:0)} = I_{D^+}$ and
   - $J_{(\gamma_+ : \delta_+)}$ defines a $G$-cluster.
   (Similarly for $J_{(\gamma_- : \delta_-)}$).

3. There exists a family of ideals $J_{(\tau : \mu)}$ with $(\tau, \mu) \in \mathbb{P}^1$ such that
   - $J_{(0:1)} = I_{C^+}$, $J_{(1:0)} = I_{C^-}$, $J_{(1:1)} = I_{\Gamma_i}$ and
   - $J_{(\tau : \mu)}$ defines a $G$-cluster.

**Proof.** The proof goes through case by case analysis. For any two consecutive $G$-graphs $\Gamma_1$ and $\Gamma_2$ with defining ideals $I_{\Gamma_1}$ and $I_{\Gamma_2}$, we give a family of ideals parametrised by a $\mathbb{P}^1$ joining the two ideals, and prove that every ideal of this family defines a $G$-cluster.

- $A(r,s;u,v) \to A(u,v;t,w)$: Suppose that we have two consecutive $G$-graphs of type $A$, $\Gamma_1 = \Gamma_A(r,s;u,v)$ and $\Gamma_2 = \Gamma_A(u,v;t,w)$, and let $P_1$ and $P_2$ the corresponding clusters in $G$-Hilb($\mathbb{C}^2$) defined by them. Then we have the conditions $r < u < t < w < s$, $t = b_i u - r$ and $w = b_i v - s$ from the lattice $L$, and $u < s - v$, $t < v - w$ from the type $A$ assumption.

We claim that the family of ideals $I_{(a:b)}$ given by the polynomials

\[
R_1 := ax^n y^u - b(x^{u+v} + (-1)^u y^{u+v}) \\
R_2 := x^r y^s + (-1)^r y^{r+s} \\
R_3 := x^{s-r} y^{u-r} + (-1)^{s-r} x^{u-r} y^{s-v} \\
R_4 := x^t y^l \\
R_5 := x^{v-w} y^{l-u} + (-1)^{l-u} x^{l-u} y^{v-w}
\]

define a 1-parameter family of $G$-clusters parametrised by a $\mathbb{P}^1$ with coordinates $a$ and $b$, such that the cluster $P_1$ is defined by $I_{(1:0)}$ and $P_2$ is defined by $I_{(0:1)}$.

Indeed, let $P = (0 : 1)$, i.e. $a = 0$ and $R_1 = x^{u+v} + (-1)^u y^{u+v} = 0$. We need to prove that the ideal is generated by $R_1, R_2$ and $R_5$. Let us start proving that $R_3$ is a combination of them. First notice that $u + v \leq s - v$. Otherwise $s = b_i v - w < u + 2v$, which can only happen when $b_i = 2$. In that case $u - r = t - u$ and $s - v = v - w$ which implies that the equations $R_3$ and $R_5$ are the same. Thus, we can multiply $R_3$ by $x^{s-v-(u+v)} y^{u-r} + (-1)^r x^{u-r} y^{s-v-(u+v)}$ to get

\[
R_3 + (-1)^u (x^{s-v-(u+v)} y^{2u+v-r} + (-1)^u r x^{2u+v-r} y^{s-v-(u+v)}) = 0
\]
So, if the inequalities \( t \leq 2u + v - r \) and \( t \leq s - v - (u + v) \) as satisfied, we can use the equation \( x^i y^j = 0 \) to conclude that the generator \( R_3 \) is redundant. The first inequality is true since \( t < t + w < u + v < u + v + (u - r) \). For the second inequality, suppose that is false, i.e. suppose that \( t > s - 2v - w \). Then \( s = b_i v - w < t + 2v + u \), i.e. \( (b_i - 2)v < t + u + w \) since \( t + w \) and \( u \) are both less than \( v \). Hence \( b_i = 2 \) or \( 3 \). As before, if \( b_i = 2 \) the equations \( R_3 \) and \( R_5 \) are the same, and if \( b_i = 3 \) then \( v < t + u + w \) or equivalently \( u + v < t + w \), which is impossible since \( t < v - w \) by the type \( A \) condition.

For \( R_2 \), note that multiplying \( R_1 \) by \((x^{(r+s)-(u+v)} + (-1)^v y^{(r+s)-(u+v)})\) we get

\[
R_2 + (-1)^u(x^{(r+s)-(u+v)}y^{u+v} + (-1)^v x^{u+v} y^{(r+s)-(u+v)}) = 0
\]

But \( t < v < u + v \) and by the previous case \( t < s - v - (u + v) < (r + s) - (u + v) \), so the second term is divisible by \( x^i y^j \) and therefore the equation \( R_2 = x^{r+s} + (-1)^r y^{r+s} \) is also not needed to define \( P_1 \).

If \( P = (0 : 1) \), i.e. \( b = 0 \), we have that \( R_1 = x^u y^u = 0 \) and we need to check that in this case the generators \( R_4 \) and \( R_5 \) are redundant. The inequalities \( u < t \) imply that \( R_4 \) is divisible by \( R_1 \). For \( R_5 \), note that we always have \( u + v < w \) and \( u + v < t - u \) except when \( b_i = 2 \), in which case \( R_5 = R_3 \) are the same. Thus, \( R_5 \) is also divisible by \( R_1 \).

Now suppose that \( P \) is neither \( P_1 \) nor \( P_2 \), i.e. \( a, b \neq 0 \). We now prove that the ideal defines a \( G \)-cluster, or in other words, it admits as basis the \( G \)-graph \( \Gamma = A(r, s; u, v) \):

Since \( t < s - v \) (otherwise \( s = b_i v - w < t + v \) and \((b_i - 1)v < t + w < v \) which contradicts \( b_i \geq 2 \)), we have that

\[
x^{t-(u+r)}R_3 = x^{r+s-(u+v)+t}y^{u-r} + (-1)^u r x^{t} y^{s-v} = 0
\]

\[
y^{t-(u-r)}R_3 = x^{s-v} y^{t} + (-1)^r u x^{r+s-v} y^{(r+s)-(u+v)} = 0,
\]

which implies that

\[
x^{r+s-(u+v)+t}y^{u-v} = x^{u-v} y^{(r+s)-(u+v)+t} = 0
\]

On the other hand we have that

\[
x^{u+i} y^u = x^i x^u y^u = x^i R_1 = \frac{b}{a} (x^{u+v+i} + (-1)^u x^i y^{u+v})
\]

and similarly for \( y^i R_1 \). Note that we are interested only for \( i \in \{1, \ldots, (r+s) - (u + v)\} \), so that the elements above lie in the boundary of the \( G \)-graph we are aiming for. In this case, \( u + v + i \leq s \) so that \( x^{u+v+i} \) is always in \( \Gamma_1 \), thus we also need to check that the monomial \( x^i y^{u+v} \) can be written in terms of elements in \( \Gamma_1 \).

If \( i = u \) then

\[
x^{2u} y^u = \frac{b}{a} (x^{2u+v} + (-1)^u x^u y^{u+v})
\]

\[
= \frac{b}{a} (x^{2u+v} + (-1)^u b/a(x^{u+v} y^u + (-1)^u y^{u+2v}))
\]

but \( t < v < u + v \), so using \( R_4 \) we write \( x^{2u} y^u \) (and similarly \( x^{u} y^{2u} \)) as a combination of elements in \( \Gamma_1 \). Also, since \( t < u + v \), by equation \( 7 \) we have that \( x^{u+v+i} y^{u} = \frac{b}{a} (x^{u+v+i} y^{u+v}) \), so we are reduced to checking the values of \( i \) in the interval \((u, t)\).

Note that if \( u + v \geq (s - v + r) + t - u \), using equation \( 7 \) we can conclude that the monomials \( x^i y^{u+v} \) are either 0 or an element in \( \Gamma_1 \) for all \( i \). So suppose that \( u + v < (s - v + r) + t - u \). Then, for any \( i \in \{1, t - u\} \), monomials of the form \( x^{2u+v} y^u \) are a combination of elements in \( \Gamma_1 \) if and only if \( x^i y^{u+2v} \) also are. Again, if \( u + 2v \geq (s - v + r) + t - u \) all these monomials will be a combination of elements in \( \Gamma_1 \). Otherwise, iterating the same procedure, we will end up with some \( j \in \mathbb{N} \) such that \( u + jv \geq (s - v + r) + t - u \) and we are done.
• $A(r,s;u,v)\rightarrow B1(u,v;t,w)$: In this case the family of $G$-clusters is parametrised by the $\mathbb{P}^1_{(b,d)}$, and they are defined by the ideal generated by the following polynomials

\[
\begin{align*}
R_1 &:= cx^y y^t - d(x^u + (-1)^v y^v) & R_4 &:= x^t y^m \\
R_2 &:= x^{r+s} + (-1)^v y^s & R_5 &:= x^m y^t \\
R_3 &:= x^{s-r} y^r - (-1)^u x^r y^r & R_6 &:= x^{m+v} y^{m+u} + (-1)^m x^m y^{m+v}
\end{align*}
\]

where the point $P_1 = (1 : 0)$ is defined by a $G$-graph of type $A$, and the point $P_2 = (0 : 1)$ by a $G$-graph of type $B1$.

Let $d = 0$. We want to prove that the ideal is generated by $R_1$, $R_2$ and $R_3$. We have that $R_1 = x^u y^u = 0$ and $u \leq m$ (Otherwise $m = t - u < u$, so $t = b_i u - r < 2u$ or equivalently $(b_i - 2) u < r$. This implies that $b_i = 2$, but this would mean that $P_1$ is defined by a $G$-graph of type $B$, a contradiction). Thus, $R_4$ and $R_5$ are divisible by $R_1$. For $R_6$, note that $u < v < 2v - w = m + v$ and $u \leq m - u$ except when $b_i = 2$ or $3$. As before the case $b_i = 3$ is impossible, and when $b_i = 3$ we have that $R_3 = R_6$.

In the case when $c = 0$ we want to show that $R_2$ and $R_3$ redundant. Since $m = t - u < t$, with the same calculation as before we obtain $R_2$. Similarly, since $u + v < s - v$ (again unless $b_i = 2$) we can use $R_1 = x^{u+v} + (-1)^u y^{u+v}$ to obtain $R_3$.

• $B1(r,s;u,v)\rightarrow B2(u,v;t,w)$: In this case the family of ideals parametrised by $\mathbb{P}^1_{(e,f)}$ is given by the following generators:

\[
\begin{align*}
R_1 &:= ex^y y^t - fy^s & R_5 &:= x^f y^m \\
R_2 &:= ex^m y^u + (-1)^v f x^s & R_6 &:= x^m y^t \\
R_3 &:= x^{r+s} + (-1)^v y^s & R_7 &:= x^{2m} y^{2m} \\
R_4 &:= x^{m+s} y^{m-r} + (-1)^{m-r} x^{m-r} y^{m+s}
\end{align*}
\]

If $f = 0$ the ideal defines a $G$-graph of type $B1$, i.e. the family is generated by $R_1$, $R_2$, $R_3$ and $R_4$. Indeed, since $x^m y^u = x^u y^m = 0$ and $u < t$ we have that $R_5 = R_6 = 0$. And since $P_1 = (0 : 1)$ is of type $B1$ we know that $u < 2m$ which implies that $R_7$ also vanishes.

If $e = 0$ then $x^s = y^s = 0$, and consequently $x^{r+s} = y^{r+s} = x^{m+s} y^{m-r} = x^{m-r} y^{m+s} = 0$. Thus the ideal at $P_2 = (1 : 0)$ is defined by $R_1$, $R_2$, $R_4$, $R_5$ and $R_6$ as desired.

For the rest of the points $(e : f)$ with $e, f \neq 0$ we have that $x^s = (-1)^v e / f x^m y^u$ and $y^s = e / f x^m y^m$. Therefore $x^s y^i = (-1)^v e / f x^m y^{u+i}$ for $1 \leq i < m$, where $x^m y^{u+i} \in \Gamma_{B,2}$ (and similarly with $x^i y^j$ for $1 \leq i < m$). This two equations, together with $R_3$, $R_6$ and $R_7$ allow us to take the $G$-graph corresponding to $P_2$ the basis for every point in the rational curve parametrised by $c$ and $d$.

• $B2(r,s;u,v)\rightarrow B2(u,v;t,w)$: This case is very similar to the previous one. The difference is that now all the equations except $R_7$ involve 2-dimensional representations. The family of ideals is given by the following generators:

\[
\begin{align*}
R_1 &:= gx^y y^t - hy^s & R_5 &:= x^f y^m \\
R_2 &:= gx^m y^u + (-1)^v h x^s & R_6 &:= x^m y^t \\
R_3 &:= y^{s+m} & R_7 &:= x^{2m} y^{2m} \\
R_4 &:= x^{s+m}
\end{align*}
\]

which are parametrised by a $\mathbb{P}^1$ with coordinates $(g : h)$.

From now on suppose that the last $G$-graph is $\Gamma_1 = \Gamma(r,s;q,q)$. Depending on the type of $\Gamma_1$ we have a different $G$-graph of type $C$, so we will denote by a subindex, $A$ or $B$, whether $\Gamma_1$ is of type $A$ or $B$.

• $C_A(r,s;q)\rightarrow C_A(r,s;q,q)$: In this case the family of ideals

\[
\begin{align*}
R_1 &:= j \cdot (x^q + (-1)^q y^q)^2 - f \cdot (x^q - (-1)^q y^q)^2 & R_4 &:= x^{m_1} y^{m_2} + (-1)^{m_2} x^{m_2} y^{m_1} \\
R_2 &:= x^q y^{m_2} + (-1)^q q x^{m_2} y^q & R_5 &:= x^{r+s} + (-1)^r y^{r+s} \\
R_3 &:= x^q y^{m_2} - (-1)^q q x^{m_2} y^q
\end{align*}
\]
Theorem 5.2. Let $G$ and $r, s$ curves are covered by the ideals $G$ basis. As a corollary of the previous theorem we have that the can always chose a basis for $C$ clusters in $C$ graphs. 

Proof. By construction, every point in $D$ graph is of type $A_B$ and let $R$ be small and let $R_1$ we have that 

\begin{align*}
R_1 &= (j_+ - j_+)(x^2y + (-1)^{y2y}) + 2(-i)^{y2y}(j_+ + j_+)x^iy^y \\

\end{align*}

Therefore, at the point $Q = (1:1)$ the ideal is generated by $R_4$, $R_5$ and $x^y y^y$, i.e. it is defined by the last $G$-graph $\Gamma_1(r, s; q, q)$ of type $A$.

- $C_B^+(r, s; q, q) \rightarrow C_B^-(r, s; q, q)$ If the last $G$-graph is of type $B_1$ then the family is given by 

\begin{align*}
R_1 &= k_- (y^s + x^y y^m) - k_+ (y^s - x^y y^m) \\
R_2 &= k_- ((-1)^s x^s - x^m y^q) - k_+ ((-1)^s x^s + x^m y^q) \\
R_3 &= x^{2m} y^{2m} \\
R_4 &= x^y m^+ + (-1)^{y2m} y^m \\
R_5 &= x^m y^s - (-1)^{y2m} x^m y^s \\
R_6 &= x^{r+s} + (-1)^{y2m} y^s
\end{align*}

If the last $G$-graph is of type $B_2$, we have to replace the equation $R_6$ involving the 1-dimensional representation $\rho_{r+s}$ by the relations $R_6^+ := y^{s+m}$ and $R_6^- := x^{s+m}$, which now involve monomials in the 2-dimensional representation $V_{r-s}$.

- $C_A^+(r, s; q, q) \rightarrow D^+(r, s; q, q)$ We have the family in this case, is given by 

\begin{align*}
R_1 &= z_+ (x^y \pm (-1)^{y2y}) - w_+ (x^y y^m \pm (-1)^{y2m} m^2 y^m) \\
R_2 &= x^{s-r} y^{s-r} \\
R_3 &= x^{m_1 y^m_2 + (-1)^{m_2 m_2} x^m y^m} \\
R_4 &= (x^y \pm (-1)^{y2y})^2
\end{align*}

When $w_+ = 0$ the ideal is generated by $R_1$ and $R_2$, i.e. it has a type $B$ basis, and when $z_+ = 0$ we obtain the $G$-graph of type $C_A$ defined by $R_1$, $R_3$ and $R_4$.

- $C_B^+ \rightarrow D^+$ This case is similar to the previous case changing the generator $R_3$ for the two equations belonging to the corresponding 2-dimensional representation.

Theorem 5.2. Let $G = BD_{2n}(a)$ be small and let $P \in G$-Hilb($\mathbb{C}^2$) be defined by the ideal $I$. Then we can always chose a basis for $\mathbb{C}[x, y]/I$ from one of the following list:

| $\Gamma_A$, $\Gamma_B$, $\Gamma_C^+$, $\Gamma_C^-$, $\Gamma_D^+$, $\Gamma_D^-$ |
|---|

Proof. By construction, every point in $G$-Hilb($\mathbb{C}^2$) away from the “horns” corresponds to a pair of $A$-clusters in $A$-Hilb($\mathbb{C}^2$). Therefore, we can chose for these $A$-clusters the symmetric $A$-graphs $\tilde{\Gamma}(r, s; u, v)$ and $\tilde{\Gamma}(v; u; s; r)$ for some $(r, s)$ and $(u, v)$ boundary lattice points in the Newton polygon for $\frac{1}{2n}(1, a)$, and we can take $\Gamma(r, s; u, v)$ to be the $G$-graph (of type $A$ or $B$) for our $G$-cluster.

For the clusters in the exceptional “horns” $E^+$ and $E^-$, we know by Theorem 5.1 that these exceptional curves are covered by the ideals $J_{(\gamma_+, \beta_+)}$ and $J_{(\gamma_-, \beta_-)}$, which correspond to $G$-graphs of type $C^\pm$ and $D^\pm$, and we are done.

Let $U_\Gamma$ the open set in $G$-Hilb($\mathbb{C}^2$) which consists of all $G$-clusters $Z$ such that $O_Z$ admits $\Gamma$ as its basis. As a corollary of the previous theorem we have that the $G$-graphs for a $BD_{2n}(a)$ group gives us the open set for the covering of $G$-Hilb($\mathbb{C}^2$) that we are looking for.

Corollary 5.3. Let $G = BD_{2n}(a)$ a small binary dihedral group and let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1}, \Gamma_{C^+}, \Gamma_{C^-}, \Gamma_{D^+}, \Gamma_{D^-}$ the list of $G$-graphs. Then 

\begin{align*}
U_{\Gamma_0}, U_{\Gamma_1}, \ldots, U_{\Gamma_{m-1}}, U_{\Gamma_{C^+}}, U_{\Gamma_{C^-}}, U_{\Gamma_{D^+}}, U_{\Gamma_{D^-}}
\end{align*}

form an open cover of $G$-Hilb($\mathbb{C}^2$).
Remark 5.4. By deforming the $G$-graph $\Gamma$, located at the origin, we can calculate the explicit equation of the open set $U_{\Gamma}$, as it is done in [Len02] for binary dihedral subgroups in $\text{SL}(2, \mathbb{C})$. Extending the groups to $\text{GL}(2, \mathbb{C})$ increases the amount of choices for generators of these ideals, which makes this approach much harder in practice. Fortunately, one can associate to any $G$-graph an open set of the moduli space $\mathcal{M}_0(Q, R)$ of $\theta$-stable quiver representations of the bound McKay quiver $(Q, R)$, which coincides with $G$-$\text{Hilb}(\mathbb{C}^2)$, and where the calculation of the open set turns out to be much easier. This is the content of [NdC09].

6 Special representations

For a finite small subgroup $G \subset \text{GL}(2, \mathbb{C})$, the McKay correspondence states that there is a one-to-one correspondence between exceptional divisors $E_i$ in the minimal resolution of $\mathbb{C}^2/G$ and the special irreducible representations $\rho_i$ of $G$. In [Ish02] Ishii proves that the minimal resolution is in fact $G$-$\text{Hilb}(\mathbb{C}^2)$.

Theorem 6.1 ([Ish02], §7.1). Let $G \subset \text{GL}(2, \mathbb{C})$ be small and denote by $I_y$ the ideal corresponding to $y \in G$-$\text{Hilb}(\mathbb{C}^2)$ and by $m$ the maximal ideal of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin $0$. If $y$ is in the exceptional locus, then we have an isomorphism

$$I_y/mI_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i, \text{ and } y \notin E_j \text{ for } j \neq i, \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j, \end{cases}$$

as representations of $G$, where $\rho_i$ is the special representation associated with the irreducible exceptional curve $E_i$.

In other words, for any point in the exceptional divisor of $G$-$\text{Hilb}(\mathbb{C}^2)$, only the trivial and the special representations corresponding to the curves which the point lies on are involved in the ideal defining the $G$-cluster. In our case, the explicit description of these ideals is the following:

Proposition 6.2. Let $G = \text{BD}_{2n}(a)$ be small and $y \in G$-$\text{Hilb}(\mathbb{C}^2)$ be a point in the exceptional locus. Denote by $I_y$ the ideal defining $y$ and by $\gamma_y = \Gamma(r, s, u, v)$ the corresponding $G$-graph. Then

$$I_y/mI_y \cong \begin{cases} \rho_{r+s}^{(-1)^r} \oplus \rho_{u+v}^{(-1)^u} \oplus \rho_0^+ & \text{if } \gamma_y \text{ is of type A}, \\ \rho_{r+s}^{(-1)^r} \oplus V_r \oplus \rho_0^+ & \text{if } \gamma_y \text{ is of type B.1}, \\ V_{2r-u} \oplus V_r \oplus \rho_0^+ & \text{if } \gamma_y \text{ is of type B.2}, \\ \rho_{r}^{(-1)^r} \oplus \rho_0^+ & \text{if } \gamma_y \text{ is of type } C^\pm \text{ and } \Gamma_{m-1} \text{ is of type A}, \\ V_r \oplus \rho_0^+ \oplus \rho_0^+ & \text{if } \gamma_y \text{ is of type } C^\pm \text{ and } \Gamma_{m-1} \text{ is of type B}, \\ \rho_0^+ & \text{if } \gamma_y \text{ is of type } D^\pm, \end{cases}$$

where $m$ is of the maximal ideal of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin 0.

Proof. Reformulating Theorem 4.1.7 in the language of Theorem 6.1 we see that the representations involved in the open set of each of the ideals are the ones presented above. Now by Theorem 5.1 we can find a rational curve $E_i$ connecting any two consecutive $G$-graphs, which must agree with the exceptional divisor of $G$-$\text{Hilb}(\mathbb{C}^2)$.

As an immediate consequence of both 6.1 and 6.2 we obtain the special representations for any group $\text{BD}_{2n}(a)$ in terms of the continued fraction $\frac{2n}{a}$, which we list in the following theorem.

Theorem 6.3. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1}$ the sequence of $qG$-graphs given by $e_0 = (0, 2n), e_1 = (1, a), e_2 = (c_1, d_1), \ldots, e_{m-1} = (c_{m-2}, d_{m-2}), e_m = (q, q)$, where

$$\Gamma_0, \ldots, \Gamma_i \text{ are of type A}$$
$$\Gamma_{i+1} \text{ is of type B.1}$$
$$\Gamma_{i+2}, \ldots, \Gamma_{m-1} \text{ are of type B.2}$$

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Then, the special representations are
\[ \rho_{1+a}, \rho_{c_1+d_1}, \rho_{c_2+d_2}, \ldots, \rho_{c_{i+1}+d_{i+1}} \] from type A,
\[ V_{c_i} \] from type B.1,
\[ V_{c_{i+1}}, \ldots, V_{c_{m-2}} \] form type B.2 and
\[ \rho_{\frac{q}{q}}^+, \rho_{\frac{q}{q}}^- \] from types C and D

Remark 6.4. We want to note that the same result holds for groups of the form BD\(_{2n}(a, q)\) since the G-graphs are not affected by the change in the generator \(\beta\). In other words, Theorem 6.3 is valid also for any small binary dihedral subgroup \(G \subset \text{GL}(2, \mathbb{C})\) with maximal cyclic subgroup \(A = \langle \frac{1}{2n}(1, a) \rangle\).

There is also a relation between G-graphs of types A and B, and the dimension of the corresponding irreducible special representations. Let \(E = \bigcup E_i \subset \text{BD}_{2n}(a)\)-Hilb(\(\mathbb{C}^2\)) be the exceptional divisor. By [WB98], the dimension of the special representations \(\rho_1\) corresponding to \(E_i\) is equal to the coefficient of \(E_i\) in the fundamental cycle \(Z_{\text{fund}}\) (the smallest effective divisor such that \(Z_{\text{fund}} \cdot E_i \leq 0\)). Let \(-2, -2, -a_m, \ldots, -a_2, -a_1\) be the selfintersections along the minimal resolution of \(G/\text{fund}q\) and \(\rho_{\frac{q}{q}}\) corresponding to the “horns” of the Dynkin diagram.

Corollary 6.5. The special irreducible representations of BD\(_{2n}(a)\) are all 1-dimensional if and only if the middle entry \(b_m\) in the continued fraction \(\frac{a}{q} = [b_1, \ldots, b_m, \ldots, b_1]\) is different from 2.

Proof. Let \(Z_{\text{fund}} = \sum c_i E_i\) for \(i = 1, \ldots, m + 2\) and \(c_i \geq 0\). Then we have to find the minimum integers \(a_i\) such that

\[
\begin{align*}
-a_1c_1 + c_2 &\leq 0 \\
c_1 - a_2c_2 + c_3 &\leq 0 \\
c_2 - a_3c_3 + c_4 &\leq 0 \\
& \vdots \\
c_{m-2} - a_{m-1}c_{m-1} + c_m &\leq 0 \\
c_{m-1} - a_mc_m + c_{m+1} + c_{m+2} &\leq 0 \\
c_m - 2c_{m+1} &\leq 0 \\
c_m - 2c_{m+2} &\leq 0.
\end{align*}
\]

First note that if \(c_i = 0\) for some \(i\), then \(c_i = 0\) for all \(i\). Also, the value of \(c_i\) corresponds to the dimension of an irreducible representation, so it is at most 2. Thus \(1 \leq c_i \leq 2\).

By minimality \(c_n + c_{n+2} = 1\) if and only if \(a_n > 2\). The same argument shows that if \(a_i > 2\) for some \(i\) then we can take \(c_j = 1\) for \(j \leq i\). Therefore, if \(a_n > 2\) we have that \(c_j = 1\) for all \(j\), so every special representation is 1-dimensional.

By Theorem 6.3 we know that \(a_n = \frac{a_{m+2}}{2}\), so \(a_2 = 2\) if and only if \(b_m = 2\) and the result is proved.

Thus we have a one-to-one correspondence between \(qG\)-graphs of type A and 1-dimimensional special representations (except the two corresponding to the “horns” which are also 1-dimensional and they are covered by the \(G\)-graphs of type C and D), and another correspondence between \(qG\)-graphs of type B and 2-dimensional special representations.

References

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