LIE ALGEBRAS OF CONSERVATION LAWS OF
VARIATIONAL ORDINARY DIFFERENTIAL EQUATIONS

EMANUELE FIORANI AND ANDREA SPIRO

Abstract. We establish a new version of the first Noether Theorem, according to which the (equivalence classes of) first integrals of given Euler-Lagrange equations in one independent variable are in exact one-to-one correspondence with the (equivalence classes of) vector fields satisfying two simple geometric conditions, namely they simultaneously preserve the holonomy distribution of the jets space and the action from which the Euler-Lagrange equations are derived.

1. Introduction

The first Noether Theorem is surely one of the most celebrated and widely studied results on conservation laws: see, for instance, [9, 10, 5, 12] and references therein. As far as we know, the strongest and most general version of this theorem has been given by Olver in [10, 11]. There, in a very clear and precise way, Olver shows that there exists an exact one-to-one correspondence between the family of (equivalence classes of) conservation laws for given Euler-Lagrange equations on sections of a bundle \( \pi : E \to M \), and the collection of (equivalence classes of) some special vector fields, called \textit{generalized infinitesimal symmetries}, defined on the bundle \( \pi^\infty : J^\infty(E) \to M \) of the infinite jets of sections of \( E \).

We now recall that any jet bundle of finite order \( \pi^k : J^k(E) \to M \) is completely determined, up to local equivalences, by the pair \((N, D)\), formed by:

- the total manifold \( N := J^k(E) \) of the jet bundle;
- a special distribution \( D \subset TN \), called \textit{canonical differential system} or \textit{holonomy distribution} (14, 15, 13).

Indeed, by a result by Yamaguchi, the pair \((N, D)\) characterizes the bundle \( \pi^k : J^k(E) \to M \) in the following sense: if \((N', D')\) is another pair, formed by a manifold \( N' \) of \( \dim N' = \dim N \) and a non-integrable distribution \( D' \subset TN' \) on \( N' \), satisfying an appropriate set of conditions, then there

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\end{itemize}
exists a local diffeomorphism between $N'$ and $N = J^k(E)$, which maps $\mathcal{D}'$ into $\mathcal{D}$ and allows to consider locally $N'$ as a jet bundle of order $k$ ([15], Thm. 2.4').

It is therefore natural to expect that Olver’s correspondence between conservation laws and generalized infinitesimal symmetries might admit an equivalent formulation in terms of vector fields on the jet bundle satisfying the following simple conditions: their local flows preserve

a) the holonomy distribution $\mathcal{D}$ and

b) the action $\mathcal{I}$, from which the Euler-Lagrange equations are derived.

Such alternative formulation of Noether-Olver’s correspondence is actually possible.

In this paper, we prove it for Euler-Lagrange equations in one independent variable. The proof for the general case of equations in several independent variables will appear in a forthcoming paper ([3]; see also [2]).

Let us call *infinitesimal symmetries for the action* $\mathcal{I}$, or shortly $\mathcal{I}$-symmetries, the vector fields of a jet bundle $J^k(E)$, satisfying conditions (a) and (b). Our result indicates that the correspondence between $\mathcal{I}$-symmetries and conserved quantities (better to say, *constants of motion*, depending on derivatives up to a fixed finite order, possibly higher than the order of the system), is an almost perfect analogue of the well-known bijection between first integrals of a time-independent Hamiltonian system and the Hamiltonian vector fields that preserve the Hamiltonian function $H$ (see e.g. [6], §5.5).

However, this analogy breaks down in the following crucial aspect. First of all, we stress the fact that the above correspondence is established for *any* system of Euler-Lagrange equations, derived by some variational principle. In particular, it equally applies to both Lagrangian and Hamiltonian settings. Hence, one can explicitly apply our construction to determine the $\mathcal{I}$-symmetries associated with the first integrals of a time-independent Hamiltonian system that depend just on phase space coordinates (to distinguish them from all other constants of motion, we call them *first integrals of elementary type*). Comparing them with the Hamiltonian vector fields associated with such first integrals, one can realize the following somehow unexpected fact: the $\mathcal{I}$-symmetries and Hamiltonian vector fields are different objects, even though there exists a very natural bijection between them. There is however a very simple reason behind such a difference: an Hamiltonian vector field corresponds to a first integral of elementary type (determined up to a constant) by means of a contraction with the canonical symplectic 2-form of the phase space; an $\mathcal{I}$-symmetry corresponds to a first integral of the same kind by means of a contraction with the Poincaré-Cartan 1-form of the Hamiltonian system (see [13], §4.3 for details).

On the basis of this fact, our alternative presentation of the correspondence between conservation laws and $\mathcal{I}$-symmetries can be considered as
the natural generalization of the correspondence between first integrals of elementary type and infinitesimal symmetries of a Poincaré-Cartan 1-form, and not of the canonical symplectic form.

In addition, the explicit details of our proof show the following facts:

1) For any $k \geq 0$ and for any action $I$ on curves $\gamma : I \subset \mathbb{R} \to E$, determined by a Lagrangian which depends on the $k'$-th order jets of such curves with $k' \leq \left[ \frac{k}{2} \right] - 1$, there exists at least one 1-form, which is a natural analogue of the Poincaré-Cartan 1-forms of Hamiltonian systems (we call it 1-form of Poincaré-Cartan type).

2) For a generic action $I$, there exist several (not just one!) associated 1-forms of Poincaré-Cartan type and the explicit correspondence between $I$-symmetries and constants of motion does depend on the choice of one such 1-form. It is only the associated map between equivalence classes of $I$-symmetries and of conservation laws, which is independent of this choice.

3) For any fixed $u \in J^k(E)$, the collection $g^I$ of germs at $u$ of $I$-symmetries has a natural structure of an infinite-dimensional Lie algebra, determined by the usual Lie brackets between vector fields. However, in general, the Lie algebra structure of $g^I$ does not induce a natural Lie algebra structure on the space $\text{ConstMot}$ of germs of (locally defined) constants of motion. One can impose a corresponding natural Lie algebra structure on certain subspaces of $\text{ConstMot}$ only if special restrictions are considered, as for instance if one consider only Hamiltonian systems and first integrals of elementary type. Nonetheless, there always exists a natural linear representation of $g^I$ on $\text{ConstMot}$, which makes $\text{ConstMot}$ a $g$-module (see §3.3 below, for details).

We observe that our construction of the correspondence between $I$-symmetries and conservation laws makes use only of classical operators of Differential Geometry, like e.g. exterior differentials, Lie derivatives etc., and it has been designed to admit simple and direct generalizations to Euler-Lagrange equations on supermanifolds. We plan to undertake this task in a future paper.

As a conclusive remark, we remind that Noether theorems have a long story, clearly exposed in Kosmann-Schwarzbach’s book [5] and summarized also in Olver’s review [12]. In [5], p. 143-144, the author stresses the clarity and completeness of Olver’s presentation in [10] and suggests further investigations towards other kinds of geometrical approaches to Noether theorems (see, for instance [4]). In our opinion, the results of this paper may be considered as a contribution in this direction.

The paper is structured as follows. In §2, we introduce the definition of the holonomic distribution of a jet bundle $J^k(E)$, associated with a bundle $\pi : E \to \mathbb{R}$ with 1-dimensional basis, and of variational equivalences between
$p$-forms on $J^k(E)$. The interest for such equivalence relations is motivated by the following facts:

1. a Hamiltonian or Lagrangian action $I$ on curves $\gamma : I \subset \mathbb{R} \rightarrow E$, depending on the $k$-order jets of these curves, can be always defined as the integral of a 1-form of $J^k(E)$ along the traces in $J^k(E)$ of the curves of jets $t \mapsto j^k(\gamma)|_t$;

2. two 1-forms on $J^k(E)$ determine the same action $I$ if and only if they are variationally equivalent;

3. the Euler-Lagrange equations, which characterize the stationary curves for $I$, are given by the components of a special 2-form, which is variationally equivalent to the exterior differentials of the (variationally equivalent) 1-forms that determine $I$.

In §3, we introduce the notion of infinitesimal symmetries of an action $I$ and prove the advertised correspondence between (equivalence classes of) such infinitesimal symmetries and (equivalence classes of) constants of motion for the Euler-Lagrange equations of $I$. In §4, we determine the infinitesimal symmetries of the action, associated with a (time-independent) Hamiltonian system, and compare them with the Hamiltonian vector fields, associated with first integrals of elementary type. Finally, using Darboux Theorem, we get our final result, Theorem 4.4, which generalizes a previous theorem by Mukunda (8).

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2. Geometrization of Euler-Lagrange equations of one independent variable

2.1. Notational remarks. In this paper we are concerned with the systems of ordinary differential equations for curves $\gamma : I \subset \mathbb{R} \rightarrow M$ on an $n$-dimensional manifold $M$, which are Euler-Lagrange equations determined by some variational principle.

Main examples of such equations are given by the differential systems occurring in Lagrangian and Hamiltonian mechanics. In these cases, the manifold $M$ plays the role of the configuration space or phase space of the considered physical system. The parameter $t \in I \subset \mathbb{R}$ of the curve has to be considered as the time coordinate.

In our discussion, the 1-dimensional manifold $\mathbb{R}$ is constantly considered with a fixed orientation, namely the one determined by the trivial coordinate system $\text{Id}_\mathbb{R} = (t) : \mathbb{R} \rightarrow \mathbb{R}$. The globally defined 1-form $dt$ is referred to as standard volume form of $\mathbb{R}$.

It is immediate to realise that any (smooth) parameterized curve $\gamma : I \subset \mathbb{R} \rightarrow M$ is uniquely associated with the corresponding (local) section of
the trivial bundle $\pi : E = \mathbb{R} \times M \to \mathbb{R}$

$$\widetilde{\gamma}_t := (t, \gamma_t) .$$

So, with no loss of generality, in place of parameterized curves in $M$, all results of this paper are expressed in terms of local (smooth) sections of the trivial bundle $(E = M \times \mathbb{R}, \mathbb{R}, \pi)$.

Consider an integer $k \geq 1$. Given a local section $\gamma : I \to E = \mathbb{R} \times M$, we use the notation $j^k_t(\gamma)$ for the $k$-th order jet of $\gamma$ at $t \in I$. The space of $k$-jets of local sections of the bundle $(E, \mathbb{R}, \pi)$ is denoted by $J^k(E)$.

For any $1 \leq \ell \leq k$, we indicate by $\pi^k_\ell : J^k(E) \to J^\ell(E)$, $\pi^k_\ell(j^k_t(\gamma)) := j^\ell_t(\gamma)$.

We also consider the natural projections $\pi^k_0 : J^k(E) \to E$ and $\pi^k_{-1} : J^k(E) \to \mathbb{R}$, defined by

$$\pi^k_0(j^k_t(\gamma)) := \gamma_t , \quad \pi^k_{-1}(j^k_t(\gamma)) := t .$$

Given a section $\gamma : I \subset \mathbb{R} \to E$, we call lift of $\gamma$ to the $k$-th order the associated curve of jets

$$\gamma^{(k)} : I \subset \mathbb{R} \to J^k(E) , \quad \gamma^{(k)}(t) := j^k_t(\gamma) .$$

For a given system of coordinates $\xi = (y^i) : U \subset M \to \mathbb{R}^n$ on $U \subset M$, the coordinates on $E$, defined by

$$\tilde{\xi} : I \times U \subset E \to \mathbb{R}^{n+1} , \quad \tilde{\xi}(t, x) = (t, y^1(x), \dots, y^n(x)) ,$$

are called associated with $\xi = (y^i)$. In general, any set of coordinates on $E$ of this form is called set of adapted coordinates.

Given a set of adapted coordinates $\tilde{\xi} = (t, y^i)$ on $I \times U \subset E$, we may consider the naturally associated coordinates

$$\tilde{\xi}^{(k)} = (t, y^i, y^i_{(1)}, \dots, y^i_{(k)}) : (\pi^k_0)^{-1}(U \times I) \subset J^k(E) \to \mathbb{R}^{n(k+1)+1} ,$$

which sends a given $k$-th order jet $u = j^k_t(\gamma)$ into the $N$-tuple, with $N = n(k + 1) + 1$, defined by

$$\left(t, y^i, y^i_{(1)}, \dots, y^i_{(k)}\right)(u) := \left(t, \gamma_t, \frac{d\gamma_i}{ds}\bigg|_{s=t}, \dots, \frac{d^k\gamma_i}{ds^k}\bigg|_{s=t}\right) .$$

We call such coordinates a set of adapted coordinates of $J^k(E)$.
2.2. Holonomic distributions and variational classes.

**Definition 2.1.** The *holonomic distribution* of $J^k(E)$ is the distribution $\mathcal{D} \subset TJ^k(E)$, which is defined at any $u \in J^k(E)$ by

$$
\mathcal{D}_u = \text{Span} \left\{ v \in T_uJ^kE : v = \frac{d\gamma(k)}{ds} \bigg|_{s=t} \right\}
$$

for some $\gamma : I \rightarrow E$ such that $j_t^k(\gamma) = u$.

The vectors in $\mathcal{D}$ and the vector fields with values in $\mathcal{D}$ are called *holonomic*.

Consider a system of adapted coordinates $(t, y^i, y^i_1, \ldots, y^i_k)$. If $\gamma$ is a section such that $u = j_t^k(\gamma)$, the components of $v = \frac{d\gamma(k)}{ds} \bigg|_{s=t}$ along the direction of $\frac{\partial}{\partial t} \bigg|_u$ and $\frac{\partial}{\partial y^i(a)} \bigg|_u$, $1 \leq a \leq k-1$, are completely fixed by the coordinates $(y^i_1, \ldots, y^i_k)$. The other components are not determined by the coordinates of $u$ and may vary arbitrarily. From these observation, a basis for the subspace $\mathcal{D}_u \subset T_uJ^k(E)$ is given by the vectors

$$
\left. \frac{d}{dt} \right|_{u=(t,y^i_0)} := \left. \frac{\partial}{\partial t} \right|_u + \sum_{a=0}^{k-1} y^i(a+1) \left. \frac{\partial}{\partial y^i(a)} \right|_u \quad \text{and} \quad \left. \frac{\partial}{\partial y^i(k)} \right|_u , \quad 1 \leq i \leq n ,
$$

and the vector fields $\left. \frac{d}{dt}, \frac{\partial}{\partial y^i(k)} \right|$ is a collection of local generators for $\mathcal{D}$.

**Definition 2.2.** A locally defined $p$-form $\lambda$ of $J^k(E)$, $p \geq 1$, is called *holonomic* if $\iota_X \lambda = 0$ for any holonomic vector field $X$. A local 0-form (i.e., a $C^\infty$ function on an open set) is called *holonomic* if it vanishes identically.

If $\alpha, \alpha'$ are $p$-forms on the same open subset $U \subset J^k(E)$, they are called *variationally equivalent* if

$$
\alpha - \alpha' = \lambda + d\mu
$$

for some holonomic $p$-form $\lambda$ and some holonomic $(p-1)$-form $\mu$.

By previous remarks, given a set of adapted coordinates $\xi^k := (t, y^i_0)$, the holonomic 1-forms are exactly those that are linear combination of the 1-forms (here, $y^i_0 := y^i$)

$$
\omega^i_{(a)} := dy^i_{(a)} - y^i_{(a+1)}dt , \quad a = 0, \ldots, k-1 ,
$$

at all points.

Note also that if $\mu$ is holonomic, its differential $d\mu$ might be non-holonomic. For instance, the 1-forms $\omega^i_{(a)}, a \leq k-2$, are holonomic, but
their differentials are of the form \( d\omega_i^{(a)} = dy_i^{(a+1)} \land dt = \omega_i^{(a+1)} \land dt \) and are not holonomic. Indeed,
\[
\frac{d}{dt} d\omega_i^{(a)} = -\omega_i^{(a+1)} \neq 0.
\]
The relation of variational equivalence is an equivalence relation between \( p \)-forms defined on the same open subset \( U \subset J^k(E) \). If \( \alpha \) is a \( p \)-form on \( U \), we call \( \alpha \) the collection \([\alpha]\) of all \( p \)-forms that are variationally equivalent to \( \alpha \).

The main motivation for considering the notion of variational classes is discussed in the next section.

2.3. Actions, Lagrangians and Poincaré-Cartan forms. As mentioned in the Introduction, we are concerned with conservation laws for Hamiltonian systems as well as for any set of Euler-Lagrange equations on sections \( \gamma : I \rightarrow E = M \times \mathbb{R} \), originating from functionals of the form
\[
I_L(\gamma) = \int_a^b L(\gamma^{(k)}) dt.
\]
(2.3)
Here \( L : J^k(E) \rightarrow \mathbb{R} \) denotes a \( k \)-th order Lagrangian, that is a \( C^\infty \) real function on \( J^k(E) \).

For such purposes, it is very convenient to consider the following notion.

**Definition 2.3.** Let \([\alpha]\) be the variational class of a 1-form \( \alpha \) on \( J^k(E) \). We call **action associated with** \([\alpha]\) the functional
\[
\mathcal{I}_{[\alpha]} : \{ \text{local sections } \gamma : I \subset \mathbb{R} \rightarrow E \} \rightarrow \mathbb{R},
\]
\[
\mathcal{I}_{[\alpha]}(\gamma) := \int_{\gamma^{(k)}(I)} \alpha.
\]
(2.4)
(in this formula, \( \int_{\gamma^{(k)}(I)} \alpha \) indicates the integral of \( \alpha \) along the 1-dimensional submanifold \( \gamma^{(k)}(I) \subset J^k(E) \)).

Here is a sequence of remarks that motivate this definition.

(1) The functional \( \mathcal{I}_{[\alpha]} \) is well defined. Namely, if \( \alpha, \alpha' \) are such that \([\alpha'] = [\alpha]\) then \( \alpha' = \alpha + \lambda \) for some holonomic \( \lambda \) and
\[
\int_{\gamma^{(k)}(I)} \alpha' = \int_{\gamma^{(k)}(I)} \alpha + \int_{\gamma^{(k)}(I)} \lambda = \int_{\gamma^{(k)}(I)} \alpha,
\]
since \( \lambda \) is 0 on all vectors tangent to \( \gamma^{(k)}(I) \), because they are holonomic.
For any Lagrangian $L : J^k(E) \to \mathbb{R}$, the 1-form $\alpha_L := L \cdot (\pi_k^*)^*dt$ is such that, for any section $\gamma$,

$$I_{[\alpha_L]}(\gamma) = \int_{\gamma^{(k)}(I)} \alpha_L = \int_a^b L(\gamma^{(k)})dt = I_L(\gamma).$$

This shows that (2.3) coincides with the action associated with $[\alpha_L]$.

Let $\alpha$ be a 1-form on $J^k(E)$ and denote by $\tilde{\alpha} = (\pi_k^{k+1})^*(\alpha)$ the pull-back of $\alpha$ on the jet space $J^{k+1}(E)$. Let also $\mathcal{W} \subset J^{k+1}(E)$ be an open subset admitting a set of adapted coordinates $\tilde{\xi}^{(k)} = (t, y^i, y^{j(\gamma)})$. The collection of 1-forms $(dt, \omega^i(a), dy^{j(k+1)})$ is a coframe field on $\mathcal{W}$ and any 1-form is a linear combination of such 1-forms at any point. Since

$$\frac{\partial}{\partial y^i(a)} \tilde{\alpha} = \alpha \left( \pi_k^{k+1} \left( \frac{\partial}{\partial y^{j(k)}} \right) \right) = 0 \quad \text{for all } 1 \leq i \leq n,$$

it follows that $\tilde{\alpha}|_\mathcal{W}$ has trivial components along the 1-forms like $dy^{j(k+1)}$. It is therefore of the form

$$\tilde{\alpha}|_\mathcal{W} = Ldt + \sum_{a=0}^k \alpha_i(a)\omega^i(a),$$

for some smooth real functions $L$, $\alpha_i(a)$ on $\mathcal{W}$. Since $\sum_{a=0}^k \alpha_i(a)\omega^i(a)$ is holonomic and $dt$ coincides with the pull-back of the standard volume form $dt$ of $\mathbb{R}$, we conclude that $[\tilde{\alpha}|_\mathcal{W}] = [\alpha_L]$ and the values of the functional $I_{[\alpha]}$ on sections of $\mathcal{W}$ are given by

$$I_{[\alpha]}(\gamma) = \int_{\gamma^{(k+1)}(I)} \tilde{\alpha} = \int_a^b L(\gamma^{(k+1)})dt = I_L(\gamma).$$

This means that, locally, $I_{[\alpha]}$ can be always identified with a functional of the form $I_L$, given by an appropriate $(k+1)$-th order Lagrangian $L$.

Let $M = T^*\mathbb{R}^N$ be the phase space of a classical mechanical system and $H : T^*\mathbb{R}^N \to \mathbb{R}$ the Hamiltonian, which determines the dynamics of the system. As it is well known, the Hamilton equations $\dot{q}^i = \frac{\partial H}{\partial p^i}$, $\dot{p}_j = -\frac{\partial H}{\partial q^j}$ are the Euler-Lagrange equations that arise from a variational principle on the action

$$I(\gamma) = \int_{\gamma(I)} p_i dq^i - Hdt$$

on sections of $\pi : (T^*\mathbb{R}^N) \times \mathbb{R} \to \mathbb{R}$. The 1-form $\alpha_H = p_i dq^i - Hdt$ is usually called Poincaré-Cartan form. Note that (2.5) coincides with the action $I_{[\alpha_H]}$ associated with the variational class $[\alpha_H]$. 


By these observations, it is clear that the actions determined by variational classes constitute a set that naturally includes and extends the class of all actions in Lagrangian and Hamiltonian mechanics. With the purpose of dealing with both kinds of such actions on the same footing, from now on our discussion is done in the general terms of variational classes and associated actions.

We conclude with a very convenient definition.

**Definition 2.4.** A $p$-form $\tilde{\beta}$ on (an open subset of) $J^k(E)$ is said of order $r$ for some $0 \leq r \leq k$ if there exists a $p$-form $\beta$ on (an open subset of) $J^r(E)$ such that $\tilde{\beta} = (\pi_k^r)^* \beta$.

Using this definition, by a pull-back, a $p$-form $\alpha$ on $J^r(E)$ can be considered as a $p$-form of order $r$ on a jet space $J^k(E)$ for any $k \geq r$.

### 2.4. Variational Principles and Euler-Lagrange equations

We now want to introduce a definition of variational principles for actions given by variational classes, which directly implies the usual Euler-Lagrange equations in Lagrange or Hamiltonian settings. For this, we first need to consider the following generalized definition of variation with fixed boundary.

Let $\gamma : I \to E = M \times \mathbb{R}$ be a local section and $[a, b] \subset I$ a closed subinterval of its domain $I$. We call smooth variation of $\gamma$ with fixed boundary up to order $k$ any smooth map $F : [a, b] \times [-\varepsilon, \varepsilon] \subset \mathbb{R}^2 \to M$, such that:

a) all maps $\gamma^{(s)} := F(\cdot, s) : [a, b] \to E$, $s \in [-\varepsilon, \varepsilon]$, admit $C^\infty$ extensions $\gamma^{(s)} : I^2 \to E$ on intervals $I^2 \supset [a, b]$, which are sections of $E$;

b) $\gamma^{(0)} = \gamma$;

c) the $k$-th order jet curves $(\gamma^{(s)})^{(k)} : [a, b] \to J^k(E)$ are such that

$$(\gamma^{(s)})^{(k)}(a) = \gamma^{(k)}(a), \quad (\gamma^{(s)})^{(k)}(b) = \gamma^{(k)}(b)$$

for all $s \in [-\varepsilon, \varepsilon]$.

**Definition 2.5.** Let $\gamma : I \subset \mathbb{R} \to E = M \times \mathbb{R}$ be a section and $\mathcal{I}_{[\alpha]}$ the action determined by a 1-form $\alpha$ of order $r$ in $J^k(E)$. We say that $\gamma$ satisfies the variational principle determined by $\mathcal{I}_{[\alpha]}$ if

$$\frac{d\mathcal{I}_{[\alpha]}(\gamma^{(s)})}{ds} \bigg|_{s=0} = 0, \quad \gamma^{(s)} := F(\cdot, s),$$

for all smooth variations $F$ with fixed boundaries up to order $r$ of the restrictions $\gamma|_{[a, b]}$ on all closed subintervals $[a, b] \subset I$.

Condition (2.6) clearly depends only on the first order jet in the variable $s$ of the variation $F$. Indeed it is equivalent to a condition which involve some special vector fields, which we now introduce.
Let $\gamma : I \to E = M \times \mathbb{R}$ be a section and

$$W : \gamma^{(k)}([a,b]) \to T^{k}(E)|_{\gamma^{(k)}([a,b])}$$

a vector field, which is defined only at the points of $\gamma^{(k)}([a,b]), \ [a,b] \subset I$. We say that $W$ is a $k$-th order variational field if there exists a smooth variation $F : [a,b] \times [-\varepsilon, \varepsilon] \to E$ of $\gamma$ with fixed boundary up to order $k$, such that

$$W = F^{(k)}( \frac{\partial}{\partial s} \big|_{(x,0)} ) , \quad (2.7)$$

where $F^{(k)}$ is the map

$$F^{(k)} : [a,b] \times [-\varepsilon, \varepsilon] \to J^{k}(E) , \quad F^{(k)}(t, s) = (\gamma^{(s)})^{(k)}(t) = J^{k}_{t}(F(\cdot, s)) .$$

**Proposition 2.6.** A section $\gamma : I \to E$ satisfies the variational principle determined by $I[\alpha]$ if and only if for any closed subinterval $[a, b] \subset I$ and any $k$-th order variational field $W$ at the points of $\gamma^{(k)}([a,b]),$

$$\int_{\gamma^{(k)}([a,b])} W d\alpha = 0 . \quad (2.8)$$

**Proof.** Let $J = [a, b] \subset I$ and denote by $F$ a smooth variation with fixed boundaries up to order $r$ of $\gamma|_{J}$. We also indicate by $W$ the variational field along $\gamma|_{[a,b]}$, which is determined by $F$ by means of (2.7). By Stokes Theorem and the conditions satisfied by $F$ at the points $(a, s)$ and $(b, s),$

$$\frac{dI_{[a]}(\gamma^{(s)})}{ds}\bigg|_{s=0} = \lim_{h \to 0} \frac{1}{h} \left( \int_{(\gamma^{(h)})^{(k)}(J)} \alpha - \int_{(\gamma^{(0)})^{(k)}(J)} \alpha \right) =$$

$$= \lim_{h \to 0} \frac{1}{h} \left( \int_{F(J \times [0,h])} d\alpha \right) = \int_{\gamma^{(k)}([a,b])} W d\alpha .$$

From this, the claim follows. \(\square\)

At a first glance, condition (2.8) looks difficult to be handled, because it involves the notion of variational vector fields, which are objects that might be hard to characterize in terms of explicit differential equations.

On the other hand, we observe that (2.8) is satisfied if and only if

$$\int_{\gamma^{(k)}([a,b])} W \beta = 0 \text{ for any } \beta \in [d\alpha].$$

Indeed, if $\beta = d\alpha + \lambda + d\mu$, for some holonomic $\lambda$ and $\mu$,\n
$$\int_{\gamma^{(k)}([a,b])} W d\alpha \lambda \text{ is holonomic} = \int_{\gamma^{(k)}([a,b])} W \beta - \int_{\gamma^{(k)}([a,b])} W d\mu \text{ Stokes Thm.}$$

$$= \int_{\gamma^{(k)}([a,b])} W \beta - \mu(W)|_{\gamma^{(k)}(b)} + \mu(W)|_{\gamma^{(k)}(a)} \text{ fixed boundary} = \int_{\gamma^{(k)}([a,b])} W \beta .$$

(2.9)
By this fact, it turns out that it is very convenient to consider the following kind of 2-forms, which, as we will shortly see, lead naturally to the Euler-Lagrange equations of the considered variational principle.

**Definition 2.7.** A 2-form $\sigma$ on $J^k(E)$, $k \geq 1$ is called source form if
\[ \iota_V \sigma = 0 \] (2.10)
for any vector field $V$ such that $(\pi^k_0)_*(V) = 0$.

If $\xi^{(k)} = (t, y^i, y^i_{(1)}, \ldots y^i_{(k)})$ is a system of adapted coordinates on an open subset $U \subset J^k(E)$, condition (2.10) is equivalent to the equations
\[ \sigma \left( \frac{\partial}{\partial y^i_{(a)}}, \cdot \right) = 0, \quad 1 \leq a \leq k, \] (2.11)
to be satisfied at all points of $U$. It follows that $\sigma$ satisfies (2.10) if and only if it is of the form
\[ \sigma = \sigma_j dy^j \wedge dt + \sum_{k<\ell} \sigma_{k\ell} dy^k \wedge dy^\ell = \]
\[ = \left( \bar{\sigma}_j + \sum_{k<\ell} (\bar{\sigma}_{k\ell} y^i_{(1)} - \bar{\sigma}_{k\ell} y^i_{(1)}) \right) \omega^i_{(0)} \wedge dt + \sum_{k<\ell} \bar{\sigma}_{k\ell} \omega^i_{(0)} \wedge \omega^i_{(0)} \]
for some smooth functions $\bar{\sigma}_j$ and $\bar{\sigma}_{k\ell}$.

Coming back to (2.8) and (2.4), by [13] Prop. A.2, if $\alpha$ is a 1-form, which is locally variationally equivalent to a 1-form $L dt$ of order $r$, and it is considered (through a pull-pack) as a 1-form on $J^k(E)$, with $k \geq 2r$, the class $[d\alpha]$ on $J^k(E)$ contains exactly one source form $\sigma \in [d\alpha]$, which has the expression
\[ \sigma = \alpha \omega^i_{(0)} \wedge dt \] (2.12)
in any set of adapted coordinates.

For reader’s convenience, we show the existence of a source form as above in the simple case, in which $\alpha$ is defined on an open set $U \subset J^k(E)$, $k \geq 2$, endowed with adapted coordinates $\xi^{(k)} = (t, y^i_{(a)})$, and it is already of the form $\alpha = L dt$ for some Lagrangian $L$ of order 1. In this case,
\[ d\alpha = \frac{\partial L}{\partial y^i} dy^i \wedge dt + \frac{\partial L}{\partial y^i_{(1)}} dy^i_{(1)} \wedge dt = \frac{\partial L}{\partial y^i} \omega^i_{(0)} \wedge dt + \frac{\partial L}{\partial y^i_{(1)}} d\omega^i_{(0)} = \]
\[ = \frac{\partial L}{\partial y^i} \omega^i_{(0)} \wedge dt + dt \left( \frac{\partial L}{\partial y^i_{(1)}} \omega^i_{(0)} \right) - dt \left( \frac{\partial L}{\partial y^i_{(1)}} \omega^i_{(0)} \right) \wedge \omega^i_{(0)} = \]
\[ = \left( \frac{\partial L}{\partial y^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^i_{(1)}} \right) \right) \omega^i_{(0)} \wedge dt + dt \left( \frac{\partial L}{\partial y^i_{(1)}} \omega^i_{(0)} \right) - \]
\[ - \left( \frac{\partial^2 L}{\partial y^i \partial y^i_{(1)}} \right) \omega^i_{(0)} \wedge \omega^i_{(0)} - \left( \frac{\partial^2 L}{\partial y^i_{(1)} \partial y^i_{(1)}} \right) \omega^i_{(1)} \wedge \omega^i_{(0)} . \]
Since \( \mu = \frac{\partial L}{\partial y^i} \omega^i_{(0)} \) and \( \lambda_{(a)} := -\left( \frac{\partial^2 L}{\partial y^j(a) y^i_{(1)}} \right) \omega^j_{(a)} \wedge \omega^i_{(0)} \), \( a = 0, 1 \), are holonomic, we see that the variational class \([d\alpha] \) contains the source form
\[
\sigma = \sigma_i \omega^i_{(0)} \wedge dt \quad \text{with} \quad \sigma_i = \frac{\partial L}{\partial y^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^i(a)} \right).
\]
(2.13)

We are now able to prove that the sections which satisfy a variational principle, are exactly the solutions of an appropriate system of Euler-Lagrange equations, as expected.

**Theorem 2.8.** Assume that \( \alpha \) is a 1-form on a jet space \( J^k(E) \), which is (locally) variationally equivalent to some form of order \( r \) of the kind \( L dt \) for some \( r \leq \frac{k}{2} \). Let also \( \sigma \) be a source form in \([d\alpha] \). A section \( \gamma : I \rightarrow E \) satisfies the variational principle of \( I[\alpha] \) if and only if
\[
\dot{\gamma}^{(k)}(t) \sigma = 0 \quad \text{for any} \quad t \in I.
\]
(2.14)

**Proof.** First of all, we observe that if \( \sigma \) and \( \sigma' \) are source forms in the same variational class \([d\alpha] \), i.e., such that \( \sigma - \sigma' = \lambda + d\mu \) for some holonomic \( \lambda \) and \( \mu \), then \( d\mu \) is holonomic and the whole difference \( \sigma - \sigma' \) is holonomic. In fact, if \( d\mu \neq 0 \) and not holonomic, in some set of adapted coordinates \( d\mu \) is necessarily of the form
\[
d\mu = \sum_{1 \leq a} \mu^a \frac{d}{dt} y^i(a) \wedge dt
\]
for some non-trivial functions \( \mu^a \). But this would contradict the fact that \( \sigma \) and \( \sigma' \) are both source forms, hence both satisfying (2.11). Due to this and the fact that, for any section \( \gamma \), the tangent vectors \( \dot{\gamma}^{(k)} \) are in \( \mathcal{D} \), we get that \( \dot{\gamma}^{(k)}(t) \sigma = \dot{\gamma}^{(k)}(t) \sigma' \).

By this remark, with no loss of generality, from now on we may assume that \( \sigma \) is the unique source form of \([d\alpha] \) described in (2.12). By (2.4) and Proposition 2.6 \( \gamma \) satisfies the variational principle if and only if
\[
\int_{\gamma^{(k)}(a,b)} \varepsilon W \sigma = 0
\]
(2.15)

for any closed subinterval \([a, b] \subset I \) and any \( k \)-th order variational field \( W \). If we consider \([a, b] \) so small so that \( \gamma^{(k)}(a,b) \) is included in the domain of a system of adapted coordinates \( \xi^{(k)} = (t, y^i(a)) \), we have that \( W \) and \( \varepsilon W \sigma \) are of the form
\[
W = W^i \frac{\partial}{\partial y^i} + \sum_{a=1}^k W^i(a) \frac{\partial}{\partial y^i(a)}, \quad \varepsilon W \sigma = \varepsilon (\sigma_i \omega^i_{(0)} \wedge dt) = (W^i \sigma_i) dt.
\]
(2.16)

We now observe that for any choice of functions \( f^i : \gamma^{(k)}(a,b) \rightarrow \mathbb{R} \) that vanish identically on neighborhoods of \( a \) and \( b \), one can construct a smooth
variation $F$ with fixed boundary up to order $k$, whose associated variational field $W$ satisfies

$$W^i|_{\gamma(t)} = f^i|_{\gamma(t)}$$

at any $t \in [a, b]$. This fact together with (2.16) implies that (2.15) is satisfied for all subintervals $[a, b]$ and all variational fields $W$ if and only if the functions $\sigma_i|_{\gamma(t)}$ are identically vanishing. Since

$$t_{\gamma(t)}(\gamma(t)) = \left(\sigma_i|_{\gamma(t)(t)}\right) \left(\omega^i(\gamma(t)) dt - \omega^i(0)\right) = -\sigma_i|_{\gamma(t)(t)} \omega^i(0)$$

the claim follows.

By previous remarks and the proof of Theorem 2.8, using a set of adapted coordinates, the equation (2.14) is equivalent to the system

$$\sigma_i(\gamma(t)) = 0, \quad 1 \leq i \leq n,$$

where the $\sigma_i$'s are the components of the unique source form $\sigma \in [d\alpha]$ described in (2.12). By (2.13), when $\alpha$ is of the form $\alpha = L dt$ for some Lagrangian $L$ of first order, the equations (2.17) are the Euler-Lagrange equations

$$\left(\frac{\partial L}{\partial y^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial y_i^{(1)}}\right)\right)_{\gamma(t)} = 0.$$

The reader can directly check that (2.17) coincide with the Euler-Lagrange equations of a Lagrangian $L$ also in the cases in which $L$ is of order higher than one.

3. A geometric proof of Noether Theorem for variational systems of O.D.E.'s

3.1. Conservation laws and symmetries of variational o.d.e.'s. Let $\alpha$ be a 1-form on $J^k(E)$ and $f : U \subset J^k(E) \rightarrow \mathbb{R}$ a smooth function, defined on an open subset of $J^k(E)$.

**Definition 3.1.** The function $f$ is said constant of motion for the variational principle of $I_{[\alpha]}$ if for any section $\gamma : I \rightarrow E$ that satisfies the variational principle,

$$\frac{d(f \circ \gamma(t))}{dt} \bigg|_t = 0 \quad \text{for any } t \in I.$$

As we will shortly see, the (first) Noether Theorem establishes a natural correspondence between symmetries of $I_{[\alpha]}$ and conservation laws. Indeed, such correspondence appears to be a bijection, provided that the objects that are called symmetries are specified in an appropriate way. To this purpose, the following definition is crucial.

**Definition 3.2.** Let $X$ be a vector field and $\alpha$ a 1-form on $J^k(E)$.
a) $X$ is called \textit{infinitesimal symmetry of} $\mathcal{D}$ (shortly, $\mathcal{D}$-symmetry) if, for any holonomic vector field $Y$, the Lie derivative $L_XY$ is also a holonomic vector field.

b) $X$ is called \textit{infinitesimal symmetry for} $\mathcal{I}_{[\alpha]}$ if it is a $\mathcal{D}$-symmetry and $L_X\alpha$ is holonomic for some (and hence for all) $\alpha \in [\alpha]$.

\textbf{Remark 3.3.} As a direct consequence of definitions, $X$ is a $\mathcal{D}$-symmetry if and only if the local flow $\Phi^X_t$ of $X$ around any jet $u \in J^k(E)$, maps the holonomic distribution $\mathcal{D}$ into itself. This implies the following two crucial facts:

a) A vector field $X$ on some open subset $U \subset J^k(E)$ generates a 1-parameter family of diffeomorphisms that transform $k$-lifts $\gamma^{(k)}$ of sections into curves that are also lifts of sections if and only if it is a $\mathcal{D}$-symmetry.

This is the main reason of interest for $\mathcal{D}$-symmetries.

b) If $X$ is a $\mathcal{D}$-symmetry and $\lambda$ is a holonomic $p$-form, also the $p$-forms $\Phi^X_t \ast \lambda$, $t \in ]-\varepsilon, \varepsilon[ \subset \mathbb{R}$, and the Lie derivative $L_X \lambda$, are holonomic. From this, it follows that if $\alpha$ and $\alpha'$ are variationally equivalent (i.e. $\alpha - \alpha' = \lambda + d\mu$, with $\lambda, \mu$ holonomic), then $L_X \alpha$ is holonomic if and only if $L_X \alpha'$ is holonomic.

This explains why the definition of infinitesimal symmetry for $\mathcal{I}_{[\alpha]}$ depends on the variational class $[\alpha]$ and not on the choice of the 1-form $\alpha$ in that class.

If we denote by $\mathcal{S}^{(k)}$ the class of $k$-th order lifts of sections of $\pi : E \to \mathbb{R}$ and we consider $\mathcal{I}_{[\alpha]}$ as an operator $\mathcal{I}_{[\alpha]} : \mathcal{S}^{(k)} \to \mathbb{R}$ with domain $\mathcal{S}^{(k)}$, (a) and (b) lead to the following interpretation of the notions in Definition 3.2:

- The local flows of $\mathcal{D}$-symmetries can be considered as 1-parameter groups of local transformations of $\mathcal{S}^{(k)}$;
- The local flows of infinitesimal symmetries of $\mathcal{I}_{[\alpha]}$ can be considered as 1-parameter groups of local transformations of $\mathcal{S}^{(k)}$, with orbits along which the functional $\mathcal{I}_{[\alpha]}$ is constant.

In the next proposition, we show that the $\mathcal{D}$-symmetries and the infinitesimal symmetries for an action $\mathcal{I}_{[\alpha]}$ coincide with the vector fields that satisfy an appropriate system of partial differential equations.

\textbf{Proposition 3.4.} Let $X$ and $\alpha$ be a vector field and a 1-form on $J^k(E)$, respectively, and $\hat{\xi}^{(k)} = (t, y_i^{(k)})$ a system of adapted coordinates on $U \subset J^k(E)$. Then:
1) $X|_U$ is a $\mathcal{D}$-symmetry if and only if it satisfies the following system of p.d.e.'s

$$
\begin{align*}
\omega_i^j ((\mathcal{L}_X \frac{d}{dt}) &= 0 , \quad 0 \leq a \leq k - 1 , \quad 1 \leq i \leq n , \\
\omega_i^j ((\mathcal{L}_X \frac{\partial}{\partial y_{(k)}}) &= 0 , \quad 0 \leq a \leq k - 1 , \quad 1 \leq i, j \leq n .
\end{align*}
$$

(3.1)

2) $X|_U$ is an infinitesimal symmetry for $\mathcal{T}_{\alpha}$ (considered as functional on sections in $U$) if and only if it satisfies (3.1) and the equations

$$
\begin{align*}
(\mathcal{L}_X \alpha) (\frac{d}{dt}) &= 0 , \\
(\mathcal{L}_X \alpha) (\frac{\partial}{\partial y_{(k)}}) &= 0 .
\end{align*}
$$

(3.2)

Proof. We recall that $\mathcal{D}|_U$ is generated by the vector fields $\frac{d}{dt}$ and $\frac{\partial}{\partial y_{(k)}}$, $1 \leq j \leq n$. This implies that a vector field takes values in $\mathcal{D}|_U$ if and only if it is in the intersection of the kernels of the 1-forms $\omega_i^j (a)$, $0 \leq a \leq k - 1$.

From these two facts, it follows that $X|_D$ is a $\mathcal{D}$-symmetry if and only if (1) holds and that $\mathcal{L}_X \alpha$ is holonomic (i.e. it vanishes on all holonomic vector fields) if and only if (2) is satisfied. $\square$

We conclude with an explicit description of $\mathcal{D}$-symmetries in adapted coordinates. In the next statement, $\xi^{(k)} = (t, y^i, y_{(a)}^i)$ is a fixed system of adapted coordinates on an open subset $U \subset J^k(E)$. Moreover, for any smooth map $v = (v^0, v^1, \ldots, v^n) : U \subset J^k(E) \rightarrow \mathbb{R}^{n+1}$ we adopt the notation $X_v$ to indicate the vector field on $U$ defined by

$$
X_v := v^0 \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial y^i} + \sum_{a=1}^k v_{(a)}^i \frac{\partial}{\partial y_{(a)}^i}
$$

(3.3)

where

$$
v_{(a)}^i := \frac{d}{dt} a (v^i - y_{(1)}^i v^0) + y_{(a+1)}^i v^0 .
$$

(3.4)

(in this formula, we assume $y_{(k+1)}^i = 0$). Notice that, by (3.3) and (3.4), we may also write $X_v$ as

$$
X_v := v^0 \frac{d}{dt} + (v^i - y_{(1)}^i v^0) \frac{\partial}{\partial y^i} + \sum_{a=1}^k \frac{d}{dt} a (v^i - y_{(1)}^i v^0) \frac{\partial}{\partial y_{(a)}^i}
$$

(3.5)

Proposition 3.5. If $\dim M \geq 2$, a vector field $X$ on $U$ is a $\mathcal{D}$-symmetry if and only if $X = X_v$ for some $v = (v^i)$ such that $\frac{\partial v}{\partial y_{(k)}^i} = 0$ for all $1 \leq i \leq n$. 

Proof. By Proposition 3.4 a vector field
\[ X = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial y^i} + X^i_{(1)} \frac{\partial}{\partial y^i_{(1)}} + \ldots + X^i_{(k)} \frac{\partial}{\partial y^i_{(k)}} \]
onumber
on \mathcal{U} is a \mathcal{D}\text{-symmetry if and only if it satisfies the equations
\[ \omega^i_{(a)} \left( \mathcal{L}_X \frac{d}{dt} \right) = 0, \quad \omega^i_{(a)} \left( \mathcal{L}_X \frac{\partial}{\partial y^j_{(k)}} \right) = 0 \quad (3.6) \]
for any \( 0 \leq a \leq k - 1 \). We recall that
\[ \mathcal{L}_X \frac{d}{dt} = -\frac{dX^0}{dt} \frac{\partial}{\partial t} + \sum_{a=0}^{k-1} \left( X^i_{(a+1)} - \frac{dX^i_{(a)}}{dt} \right) \frac{\partial}{\partial y^i_{(a)}} - \frac{dX^i_{(k)}}{dt} \frac{\partial}{\partial y^i_{(k)}}. \]
Hence, the first set of equations in (3.6) means that, for any \( 0 \leq a \leq k - 1 \),
\[ 0 = \omega^i_{(a)} \left( \mathcal{L}_X \frac{d}{dt} \right) = X^i_{(a+1)} - \frac{dX^i_{(a)}}{dt} + y^i_{(a+1)} \frac{dX^0}{dt}. \quad (3.7) \]
This shows that all components \( X^i_{(a)} \), \( a \geq 1 \), are uniquely determined by the components \( X^i \) and, by induction, one can check that \( X \) is as in (3.3).

In order to conclude, it suffices to show that the other equations in (3.6) are equivalent to
\[ \frac{\partial X^0}{\partial y^j_{(k)}} = \frac{\partial X^i}{\partial y^j_{(k)}} = \frac{\partial X^i_{(a)}}{\partial y^j_{(k)}} = 0 \quad (3.8) \]
for any \( 0 \leq a \leq k - 1 \), so that \( \frac{\partial \mathbf{v}^\ell}{\partial y^j_{(k)}} = \frac{\partial X^\ell}{\partial y^j_{(k)}} = 0 \). Indeed, denoting by \( z^A \) an arbitrary coordinate amongst \((t, y^i, y^i_{(a)})\), one has that \( \mathcal{L}_X \frac{\partial}{\partial y^j_{(k)}} = -\frac{\partial X^A}{\partial y^j_{(k)}} \frac{\partial}{\partial z^A} \).

This means that the second set of equations in (3.6) is equivalent to
\[ \frac{\partial X^i_{(a)}}{\partial y^j_{(k)}} = y^i_{(a+1)} \frac{\partial X^0}{\partial y^j_{(k)}} \quad (3.9) \]
for some \( i \neq j \). Now, setting \( a = k - 1 \) and taking the derivative of (3.9) w.r.t. \( y^i_{(k)} \) for some \( i \neq j \), we get
\[ \frac{\partial^2 X^i_{(k-1)}}{\partial y^j_{(k)} \partial y^i_{(k)}} = \frac{\partial X^0}{\partial y^j_{(k)}} + y^i_{(k)} \frac{\partial^2 X^0}{\partial y^j_{(k)} \partial y^i_{(k)}}. \]
On the other hand, considering equation (3.9) with \( j = i \) and taking the derivative w.r.t. \( y^i_{(k)} \) we have
\[ \frac{\partial^2 X^i_{(k-1)}}{\partial y^j_{(k)} \partial y^i_{(k)}} = y^i_{(k)} \frac{\partial^2 X^0}{\partial y^j_{(k)} \partial y^i_{(k)}}. \]
Taking the difference, we obtain \( \frac{\partial X^a}{\partial y^{(k)}} = 0 \). Inserting this in (3.9), equalities (3.8) follow. □

We have now all the ingredients for the two parts of the Noether Theorem, which are stated and proved in the next section.

3.2. Noether Theorem.

**Definition 3.6.** Let \([\alpha]\) be a variational class of 1-forms on \( J^k(E) \), determined by a 1-form \( \alpha \), which is locally variationally equivalent to 1-forms \( Ldt \) for some \( r \leq \frac{k}{2} \). A 1-form \( \alpha_o \in [\alpha] \) is called of Poincaré-Cartan type if \( d\alpha_o \) is a source form modulo a holonomic 2-form.

The main example of such kind of 1-forms is given by the Poincaré-Cartan form \( \alpha_o = p_i dq^i - H dt \) discussed in (4) of §2.3. In fact

\[
d\alpha_o = \frac{\partial H}{\partial p_j} dp_j \wedge dt + \frac{\partial H}{\partial q^k} dq^k \wedge dt + dp_i \wedge dq^i,
\]

which is a source form on any jet space \( J^k(E), k \geq 1 \), of the trivial bundle \( \pi: E = T^*\mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \).

Note that if \( \alpha \) is a 1-form on \( J^k(E) \), satisfying the assumptions of (3.6), then for any \( u \in J^k(E) \) there exists a neighborhood \( U \) of \( u \) such that the variational class \([\alpha|_U]\) contains a 1-form of Poincaré-Cartan type. This can be directly seen as follows: consider a neighbourhood \( U \) admitting a system of adapted coordinates, and let \( \sigma \in [d\alpha|_U] \) be the source form described in (2.12). Then \( \sigma = d\alpha|_U + d\mu + \lambda = d(\alpha|_U + \mu) + \lambda \), for some holonomic \( \mu \) and \( \lambda \), and \( \alpha_o = \alpha|_U + \mu \) is a 1-form of Poincaré-Cartan type in the variational class \([\alpha|_U]\).

We also remark that, replacing \( J^k(E) \) by a jet space of higher order, one may safely assume that the variational class \([\alpha|_U]\) contains at least one 1-form of Poincaré-Cartan type of order \( r \leq k - 1 \). We will shortly see that such harmless assumption is often quite convenient.

The notion of 1-forms of Poincaré-Cartan type leads to the following useful characterisation of infinitesimal symmetries of a given action. As in Proposition 3.5, we consider as fixed a system of adapted coordinates \( \hat{\xi}^{(k)} = (t, y^i, y^i_a) \) on an open subset \( U \subset J^k(E) \) and for any \( \mathbb{R}^{n+1} \)-valued smooth map \( v = (v^i) \) on \( U \), we denote by \( X_v \) the associated vector field defined in (3.3).

**Proposition 3.7.** Assume that \( \dim M \geq 2 \) and let \( \alpha_o \) be a 1-form of Poincaré-Cartan type in \([\alpha]\) of order \( r \leq k - 1 \) and \( X_v \) a \( D \)-symmetry
on \( \mathcal{U} \) associated with \( \mathbf{v} = (\mathbf{v}^i) \). Then \( X_\mathbf{v} \) is an infinitesimal symmetry for \( I_{[\alpha]} \) if and only if it satisfies the linear differential equation

\[
\frac{d}{dt} \left( \alpha_o(X_\mathbf{v}) \right) = \sigma \left( \frac{d}{dt} X_\mathbf{v} \right),
\]

where \( \sigma \) is any source form of the variational class \( [\alpha|_\mathcal{U}] \).

**Proof.** Let \( \lambda \) be the holonomic 2-form defined by \( \lambda = d\alpha_o - \sigma \). By Proposition 3.3 (2) and the fact that \( \frac{\partial}{\partial y^j(k)} \lambda = \frac{\partial}{\partial y^j(k)} \sigma = 0 \), \( X_\mathbf{v} \) is an infinitesimal symmetry for \( I_{[\alpha]} \) if and only if

\[
\begin{align*}
\mathcal{L}_{X_\mathbf{v}} \alpha_o \left( \frac{d}{dt} \right) &= \frac{d}{dt} (\alpha_o(X_\mathbf{v})) + \sigma(X_\mathbf{v}, \frac{d}{dt}) = 0, \\
\mathcal{L}_{X_\mathbf{v}} \alpha_o \left( \frac{\partial}{\partial y^j(k)} \right) &= \frac{\partial}{\partial y^j(k)} (\alpha_o(X_\mathbf{v})) + \sigma(X_\mathbf{v}, \frac{\partial}{\partial y^j(k)}) = 0.
\end{align*}
\]

Since \( \frac{\partial X^A_\mathbf{v}}{\partial y^j(k)} = 0 \) for all components \( X^A_\mathbf{v} \) of \( X_\mathbf{v} \) (Proposition 3.5) and \( \alpha_o \) is of order \( r \leq k - 1 \), the second equality is trivially satisfied for any \( 1 \leq i \leq k \).

By the first equation in (3.11), the claim follows.

We can now state and prove the Noether Theorem in its two parts, direct and inverse.

**Theorem 3.8 (Noether Theorem – first part).** Let \( [\alpha] \) be a variational class of 1-forms on \( J^k(E) \) and assume that \( \alpha_o \) is a 1-form of Poincaré-Cartan type in \( [\alpha] \).

If a vector field \( X \) on \( \mathcal{U} \subset J^k(E) \) is an infinitesimal symmetry for \( I_{[\alpha]} \) (considered as functional on sections in \( \mathcal{U} \)), then

\[
f^{(X)} := \iota_X \alpha_o : \mathcal{U} \to \mathbb{R}
\]

is a constant of motion for the variational principle of \( I_{[\alpha]} \).

**Proof.** By definition of 1-forms of Poincaré-Cartan type, \( d\alpha_o = \sigma + \lambda \), where \( \sigma \) is a source form in \( [d\alpha_o] \) and \( \lambda \) is a holonomic 2-form. It follows that, for any section \( \gamma : I \to E \)

\[
\frac{df^{(X)} \circ \gamma(k)}{dt} = d(\iota_X \alpha_o) \left( \frac{d\gamma(k)}{dt} \right) =
\]

\[
= \mathcal{L}_{X} \alpha_o \left( \frac{d\gamma(k)}{dt} \right) - d\alpha_o \left( X, \frac{d\gamma(k)}{dt} \right) \quad \mathcal{L}_{X} \alpha_o \text{ is holonomic} 
\]

\[
= -\sigma \left( X, \frac{d\gamma(k)}{dt} \right).
\]
Since $\sigma$ is a source form of \([d\alpha_o]\), by Theorem [2.8] if $\gamma$ is a solution of the variational principle of $I_o$, we have $\frac{df(X)\sigma(k)}{dt} = -\sigma(X, \frac{d\gamma(k)}{dt}) = 0$. \(\square\)

Now, in order to state and prove the inverse of this result, we need to consider a new notion.

Let $[\alpha]$ be a variational class of 1-forms of $J^k(E)$ and assume that $\sigma = \sigma_i\omega_i^{(0)} \wedge dt$ is a source form of the kind (2.12) on some open subset $\mathcal{W} \subset J^k(E)$. Assume also that $\sigma$ is of order $r_o \leq k - 1$ and consider the differentials $d\sigma_i$ of the components $\sigma_i$ of $\sigma$. By the assumption on the order of $\sigma$, such differentials are equal to

$$d\sigma_i = \frac{\partial \sigma_i}{\partial t} dt + \sum_{a=0}^{k-1} \frac{\partial \sigma_i}{\partial y_j^{(a)}} dy_j^{(a)}.$$  

Due to this, for any $k$-th order lift $\gamma^{(k)} : I \rightarrow \mathcal{W}$ of a section $\gamma$ of $E$, we have

$$d \left( \sigma_i(\gamma^{(k)}(t)) \right) = d\sigma_i \left( \dot{\gamma}^{(k)}(t) \right) = \left. \frac{d\sigma_i}{dt} \right|_{\gamma^{(k)}(t)}.$$  

Hence a lifted section $\gamma^{(k)}$ corresponds to a solution of the Euler-Lagrange equations

$$\sigma_i(\gamma_t^{(k)}) = 0, \quad 1 \leq i \leq n,$$  

if and only if it is a solution of the system of partial differential equations

$$\sigma_i(\gamma_t^{(k)}) = \frac{d\sigma_i}{dt}(\gamma_t^{(k)}) = 0, \quad 1 \leq i \leq n.$$  

The system (3.13) is usually called first prolongation of (3.12). We stress the fact if the functions (i.e., 0-forms) $\sigma_i$ which define the Euler-Lagrange equations are of order $r_o$, the functions that define the first prolongation (3.13) are 0-forms of order $r_o + 1 \leq k$.

Consider now the integer $p_o := k - r_o$. Iterating the above argument, we can directly prove that the system (3.12) is equivalent to

$$\sigma_i(\gamma_t^{(k)}) = \frac{d\sigma_i}{dt}(\gamma_t^{(k)}) = \ldots = \left( \frac{d}{dt} \right)^{p_o} (\sigma_i)(\gamma_t^{(k)}) = 0, \quad 1 \leq i \leq n.$$  

We call it full prolongation of (3.12) on the $k$-order jet space $J^k(E)$.

Note that the order of the collection of functions appearing in a full prolongation is generically not less than $k$.

**Definition 3.9.** Let $F_\sigma : \mathcal{W} \subset J^k(E) \rightarrow \mathbb{R}^{n-(p_o+1)}$ be the smooth function

$$F_\sigma := \left( \sigma_i, \frac{d}{dt}(\sigma_j), \left( \frac{d}{dt} \right)^2 (\sigma_\ell), \ldots, \left( \frac{d}{dt} \right)^{p_o}(\sigma_m) \right)$$  

(3.15)
and set \( Z_\sigma := \{ u \in W : F_\sigma(u) = 0 \} \subset J^k(E) \). The system of Euler-Lagrange equations (3.12) is called regular in \( J^k(E) \) if the map \( F_\sigma \) is a submersion at all points of \( Z_\sigma \).

We may now state the second part of Noether Theorem.

**Theorem 3.10 (Noether Theorem – second part).** Assume that \( \dim M \geq 2 \) and let \( \alpha \in [\alpha] \) be a 1-form of Poincaré-Cartan type of order \( r \leq k - 1 \) in a variational class \([\alpha]\) of 1-forms on \( J^k(E) \). Assume also that there exists an open subset \( W \subset J^k(E) \), where the following non-degeneracy conditions are satisfied:

- a) there exists a source form \( \sigma = \sigma_i \omega_{(0)}^i \wedge dt \) of order \( r_o \leq k - 1 \) on \( W \) of the kind (2.12), which determines a system of Euler-Lagrange equations \( \sigma_i = 0 \), which is regular in \( J^k(E) \);
- b) \( \alpha_o \left( \frac{d}{dt} \right)|_u \neq 0 \) at all \( u \)’s in \( W \).

If \( f : W \to \mathbb{R} \) is a constant of motion of order \( k - 1 \) for the variational principle of \( I[\alpha] \), then there exist

1) a neighborhood \( U \) of \( Z_\sigma = \{ u \in W : F_\sigma(u) = 0 \} \), where \( F_\sigma \) is defined in (3.15);
2) an infinitesimal symmetry \( X(f) \) for \( I[\alpha] \) on \( U \);
3) a \( p_o \)-tuple of constants of motion \( (g^{(1)}, \ldots, g^{(p_o)}) \), \( p_o = k - r_o - 1 \), on \( U \), vanishing at all points \( \gamma(k)(t) \) of all lifts of the solutions of the variational principle

such that

\[
\iota_{X(f)} \alpha = f|_U + g^{(1)} + \ldots + g^{(p_o)}. \tag{3.16}
\]

**Proof.** Consider a system of adapted coordinates \( \tilde{\xi}^{(k)} = (t, y^i, y_i^{(a)}) \) and let \( \sigma = \sigma_i \omega_{(0)}^i \wedge dt \) on \( W \) be a source form satisfying the non-degeneracy condition (a). By Propositions 3.5 and 3.7 we need to show that there exists a neighbourhood \( U \) of \( Z_\sigma \), a smooth \( \mathbb{R}^{n+1} \)-valued map \( \mathbf{v} = (v^0, v^i) : U \to \mathbb{R}^{n+1} \) and \( p_o \) constants of motion \( g^{(\ell)} \) on \( U \), vanishing on lifts \( \gamma(k)(t) \) of solutions, such that the vector field \( X_\mathbf{v} \) satisfies the system of linear equations

\[
\alpha(X_\mathbf{v}) = f + \sum_{\ell=1}^{p_o} g^{(\ell)}, \quad (\iota_{\frac{d}{dt}} \sigma)(X_\mathbf{v}) = \frac{df}{dt} + \sum_{\ell=1}^{p_o} \frac{dg^{(\ell)}}{dt}. \tag{3.17}
\]

If we express \( \alpha \) and \( \sigma \) as sums of the form

\[
\alpha = \alpha_0 dt + \sum_{0 \leq a \leq k-1} \alpha_{(a)}^i \omega_i^{(a)} \quad \text{and} \quad \sigma = \sum_{1 \leq i \leq n} \sigma_i \omega_{(0)}^i \wedge dt,
\]
equations (3.17) become

\[
\begin{align*}
    v^0 a_0 &= -\sum_{i=1}^{n} \left( v^i - y^i_{(1)} v^0 \right) a_i^{(0)} - \sum_{1 \leq a \leq k-1} \sum_{1 \leq i \leq n} \frac{d^a}{dt^a} \left( v^i - y^i_{(1)} v^0 \right) a_i^{(a)} + \\
    &\quad + f + \sum_{\ell=1}^{p_o} g^{(\ell)}, \\
\end{align*}
\]

(3.18)

We claim that the function \( \frac{df}{dt} : W \rightarrow \mathbb{R} \) vanishes identically on \( Z_\sigma \). Indeed, since \( Z_\sigma = \{ F_\sigma = 0 \} \) is equal to the collection of the jets of the \((k\text{-th order})\) solutions to the variational principle, for any \( u \in Z_\sigma \),

\[
    \left. \frac{d}{dt} \right|_u \gamma^{(k)} |_{t=0} \in \text{Span} \left\{ \left. \frac{\partial}{\partial y^i_{(k)}} \right|_u \right\},
\]

where we denoted by \( \gamma^{(k)} \) the \( k\text{-th order lift} \) of a solution with \( u = \gamma^{(k)}(t_o) \). Since \( f \) is a constant of motion and it is of order \( k-1 \), we get

\[
    df \left( \left. \frac{d}{dt} \right|_u \right)\gamma^{(k)}(t_o) = 0,
\]

which proves the claim.

From this, the fact that \( F_\sigma : \mathcal{W} \rightarrow \mathbb{R}^{n(p_o+1)} \) is a submersion at any \( u \in Z_\sigma \) and standard properties of submanifolds (see e.g., [7], Lemma 2.1 and [10], Prop. 2.10), there exists a neighborhood \( \mathcal{U} \subset \mathcal{W} \) and \( n \cdot (p_o + 1) \) smooth functions \( \tilde{v}^{i}_{(\ell)} \), \( 1 \leq j \leq n, 0 \leq \ell \leq p_o \), on \( \mathcal{U} \) (not uniquely determined!), such that

\[
    -\left. \frac{df}{dt} \right|_u = \sum_{i=1}^{n} \tilde{v}^{i}_{(0)} \sigma_i + \sum_{i=1}^{n} \tilde{v}^{i}_{(1)} \frac{d\sigma_i}{dt} + \ldots + \sum_{i=1}^{n} \tilde{v}^{i}_{(p_o)} \left( \frac{d}{dt} \right)^{p_o-1} (\sigma_i). \quad (3.19)
\]

Let \( g^{(1)} : \mathcal{U} \rightarrow \mathbb{R} \) be the smooth function defined by

\[
    g^{(1)} := \sum_{i=1}^{n} \tilde{v}^{i}_{(p_o)} \left( \frac{d}{dt} \right)^{p_o-1} (\sigma_i). \quad (3.20)
\]

This function vanishes identically on the jets of the solutions (it is pointwise equal to a linear combination components of the map \( F_\sigma \)) and it is therefore a constant of motion. Furthermore,

\[
    \sum_{i=1}^{n} \tilde{v}^{i}_{(p_o)} \left( \frac{d}{dt} \right)^{p_o} (\sigma_i) = \frac{dg^{(1)}}{dt} - \sum_{i=1}^{n} \frac{d}{dt} (\tilde{v}^{i}_{(p_o)}) \cdot \left( \frac{d}{dt} \right)^{p_o-1} (\sigma_i),
\]
so that (3.19) can be re-written in the form

\[- \frac{df}{dt} - \frac{dg^{(1)}}{dt} = \sum_{a=0}^{p_0-1} \left( \sum_{i=1}^{n} \hat{v}^{(a)}_i \left( \frac{d}{dt} \right)^a (\sigma_i) \right) + \sum_{i=1}^{n} \hat{v}^{(p_0-1)}_i \left( \frac{d}{dt} \right)^{p_0-1} (\sigma_i) \]  

(3.21)

where we set

\[ \tilde{v}^{(p_0-1)}_i := \hat{v}^{(p_0-1)}_i - \frac{d}{dt} (\hat{v}^{(p_0-1)}_i) . \]

Iterating this line of arguments, we conclude that (3.19) is equivalent to an equality of the form

\[- \frac{df}{dt} - \frac{dg^{(1)}}{dt} - \ldots - \frac{dg^{(p_0)}}{dt} = \sum_{i=1}^{n} \tilde{v}^i (\sigma_i) , \]  

(3.22)

for some appropriate smooth functions \( \tilde{v}^i, g^{(\ell)} : U \to \mathbb{R} \), where the \( g^{(\ell)} \) are constants of motion that vanish identically on the jets of the solutions of the variational principle.

Since \( \alpha_0 = \alpha \left( \frac{d}{dt} \right) \) is nowhere vanishing on \( \mathcal{W} \), we may consider the function

\[ v^0 := - \sum_{i=1}^{n} \tilde{v}^i (\alpha_i^0) - \sum_{1 \leq a \leq k-1}^{1 \leq i \leq n} \frac{d^a \tilde{v}^i}{dt^a} (\alpha_i^0) + \frac{f + \sum_{p_0} g^{(p_0)}}{\alpha_0} \]

and the corresponding \((n+1)\)-tuple of functions on \( U \subset \mathcal{W} \)

\[ v := \left( v^0, v^1 = \tilde{v}^1 + y_1^1 v^0, \ldots, v^n = \tilde{v}^n + y_n^1 v^0 \right) . \]

By construction, \( v \) satisfies (3.18) and \( X(f) := X_v \) is an infinitesimal symmetry satisfying (3.16).

3.3. Correspondence between infinitesimal symmetries and constants of motion. Let \([\alpha]\) be a variational class on \( J^k(E) \), which is locally determined by a 1-form \( Ldt \) of order \( r \) with \( 2r \leq k \), and assume that \( X \) is an infinitesimal symmetry \( X \) for the action \( I_{[\alpha]} \) on some open subset \( U \subset J^k(E) \).

By the proof of the first part of the Noether Theorem, if \( \alpha_0, \alpha'_0 \) are distinct 1-forms of Poincaré-Cartan type in \([\alpha]\) and \( f(X) \) and \( g(X) \) are the constants of motion associated with \( X \) via \( \alpha_0 \) and \( \alpha'_0 \), i.e.,

\[ f(X) := i_X \alpha_0 , \quad g(X) := i_X \alpha'_0 , \]

the difference \( h = f(X) - g(X) \) is a constant of motion with the property that, for any \( k \)-lift \( \gamma^{(k)} \) of a section of \( E \) (here, \( \sigma \) is a source form of \([d\alpha_0]\))

\[ \frac{dh \circ \gamma^{(k)}}{dt} = \sigma \left( X, \frac{d\gamma^{(k)}}{dt} \right) - \sigma \left( X, \frac{d\gamma^{(k)}}{dt} \right) = 0 . \]

It is therefore convenient to consider the following definition.
Definition 3.11. An infinitesimal symmetry $X$ for the action $\mathcal{I}[\alpha]$ on $U \subset J^k(E)$ is called

1) $\alpha_o$-trivial if the constant of motion $f^{(X)} := i_X \alpha_o$, determined by $\alpha_o \in [\alpha]$ of Poincaré-Cartan type, is the zero function $f^{(X)} = 0$;

2) trivial if for any $u \in U$ there exists at least one $\alpha_o \in [\alpha]$ of Poincaré-Cartan type on a neighbourhood $U' \subset U$ of $u$ such that $f^{(X)} := i_X \alpha_o$ is constant on any $k$-th order lift $\gamma^{(k)}$ of a section $\gamma$ of $E$.

By previous observations, the property of being trivial does not depend on the choice of the 1-form $\alpha_o$ of Poincaré-Cartan type and it is equivalent to the condition

$$\sigma \left( X, \frac{d}{dt} \right) = 0 ,$$

where $\sigma$ is an arbitrary source form $[d\alpha_o]$.

Take now a fixed variational class $[\alpha]$ on $J^k(E)$, with the usual assumption that $\alpha \simeq Ldt$ for some 1-form $Ldt$ of order $r$ with $2r \leq k$, and let $\alpha_o \in [\alpha]$ be of Poincaré-Cartan type of order $r' \leq k - 1$ on some open set $U \subset J^k(E)$. Fix $u_o \in U$ and consider the following classes of germs at $u_o$ (here, given a vector field $X$ or a function $f$, we denote by $\underline{X}$ and $\underline{f}$, respectively, their germs at $u_o$):

$$\Sigma := \{ \text{germs at } u_o \text{ of infinitesimal symmetries of } \mathcal{I}[\alpha] \}$$

$$\mathcal{Triv}^{(\alpha_o)} := \{ \underline{X} \in \Sigma : X \text{ is an } \alpha_o\text{-trivial symmetry} \}$$

$$\mathcal{Triv} := \{ \underline{X} \in \Sigma : X \text{ is a trivial symmetry} \}$$

$$\mathcal{ConstMot} := \{ \text{germs at } u_o \text{ of constants of motion for } \mathcal{I}[\alpha] \}$$

$$\mathcal{Null} := \{ \underline{f} \in \mathcal{ConstMot} : f \text{ is identically equal to } 0 \text{ at points of solutions} \}$$

$$\mathcal{Const} := \{ \underline{f} \in \mathcal{ConstMot} : f \text{ is constant along any section } \gamma^{(k)} \}$$

All such classes of germs have natural structures of vector spaces. The space $\Sigma$ is also endowed with a natural Lie algebra structure, given by the usual Lie brackets between vector fields.

Using the above notation, when $\dim M \geq 2$ and the non-degeneracy conditions (a) and (b) of Theorem 3.10 are satisfied, the two parts of Noether Theorem can be restated saying that for any given choice of a 1-form $\alpha_o \in [\alpha]$ of Poincaré-Cartan type of order $r' \leq k - 1$, there exists a natural surjective linear map

$$\varphi^{(\alpha_o)} : \Sigma \longrightarrow \mathcal{ConstMot}/\mathcal{Null} ,$$

(3.23)
From the definition of the map $\varphi^{(\alpha)}$, one has that $\ker \varphi^{(\alpha)} = \text{Triv}^{(\alpha)}$ and the above homomorphism induces an isomorphism of vector spaces

$$
\iota^{(\alpha)} : \Sigma / \text{Triv}^{(\alpha)} \xrightarrow{\sim} \text{ConstMot} / \text{Null}.
$$

This isomorphism does depend on the choice of $\alpha$. However, if one considers the quotients of the vector spaces $\Sigma$ and $\text{ConstMot}$ by the subspaces $\text{Triv}$ and $\text{Null} + \text{Const}$, respectively, the surjective map (3.23) establishes a vector space isomorphism

$$
\iota : \Sigma / \text{Triv} \xrightarrow{\sim} \text{ConstMot} / (\text{Null} + \text{Const}),
$$

which is now independent on the choice of $\alpha$.

A priori, there is no reason for $\text{Triv}^{(\alpha)}$ or $\text{Triv}$ to be ideals of the Lie algebra $\Sigma$. Due to this, the quotients $\Sigma / \text{Triv}^{(\alpha)}$ and $\Sigma / \text{Triv}$ cannot be expected to have a natural Lie algebra structure.

However, something can be said on this regard, provided that we consider the following restricted class of infinitesimal symmetries.

**Definition 3.12.** Given $\alpha \in [\alpha]$ of Poincaré-Cartan type and with the above conditions satisfied, an infinitesimal symmetry $X$ for $I^{[\alpha]}$ is called $\alpha$-symmetry if $L_X \alpha = 0$.

Denote by $\Sigma^{(\alpha)} \subset \Sigma$ the subalgebra of the germs at $u_0$ of $\alpha$-symmetries. We claim that the Lie brackets between vector fields induce a linear action of $\Sigma^{(\alpha)}$ on $\text{Triv}^{(\alpha)}$. Indeed, if $X \in \Sigma^{(\alpha)}$ and $Y \in \text{Triv}^{(\alpha)}$

$$
\iota[X,Y]^{(\alpha)} = d(\alpha(Y))(X) - L_X \alpha(Y) = d(\alpha(Y))(X) \equiv 0
$$

showing that the germ $[X,Y]$ is in $\text{Triv}^{(\alpha)}$. Hence, the map

$$
\tilde{\text{ad}} : \Sigma^{(\alpha)} \to \text{Hom} \left( \Sigma / \text{Triv}^{(\alpha)}, \Sigma / \text{Triv}^{(\alpha)} \right),
$$

$$
\tilde{\text{ad}}_X(Z) \mod \text{Triv}^{(\alpha)} := [X,Z] \mod \text{Triv}^{(\alpha)}. \quad (3.24)
$$

is well-defined and is a linear representation. Composing with the isomorphism $\iota^{(\alpha)}$, we get the following linear map for any $X \in \Sigma^{(\alpha)}$:

$$
\rho(X) : \text{ConstMot} \to \text{ConstMot},
$$

$$
\rho(X)(f) := \iota[X,Z(f)]^{(\alpha)} = X \iota(Z(f))^{(\alpha)} = X(f),
$$

where $Z(f)$ is any germ in $\Sigma^{(\alpha)}$ that is mapped onto $f$ by $\varphi^{(\alpha)}$. By construction, the map $\rho$ determines a linear representation of $\Sigma^{(\alpha)}$ and we have the following:

**Proposition 3.13.** Given $u_0 \in J^k(E)$ and $\alpha \in [\alpha]$ of Poincaré-Cartan type and satisfying the hypothesis of Theorem 3.10, the map

$$
\rho : \Sigma^{(\alpha)} \to \text{Hom} \left( \text{ConstMot} / \text{Null}, \text{ConstMot} / \text{Null} \right),
$$

$$
\rho(X)([f]_{\text{Null}}) := X(f) \mod \text{Null}, \quad (3.25)
$$
is a linear representation of the space of (germs of) $\alpha_o$-symmetries $\Sigma^{(\alpha_o)}$ on the quotient space of (germs of) constants of motion $\text{ConstMot}/\text{Null}$.

Remark 3.14. A similar argument can be used to show the existence of a natural linear representation of $\Sigma^{(\alpha_o)}$ also on the quotient space $\text{ConstMot}/(\text{Null} + \text{Const})$.

4. Infinitesimal symmetries and Hamiltonian vector fields in Hamiltonian mechanics

4.1. Notational issues. From now on, we assume that the configuration space $M$ is a cotangent bundle $M = T^*N$ of an $n$-dimensional manifold $N$.

We denote by $\tilde{\pi} : T^*N \to N$ the canonical projection of $T^*N$ and for any system of coordinates $\eta = (q^1, \ldots, q^n) : U \subset N \to \mathbb{R}^n$ of $N$, we call associated coordinates on $T^*N$ the map $\tilde{\xi}_{\eta} : \tilde{\pi}^{-1}(U) \subset T^*N \to \mathbb{R}^{2n}$, which associates to any 1-form $\beta = p_i dq^i \mid_x \in T^*_x N$, $\eta(x) = (q^1, \ldots, q^n)$, the coordinates

$$\beta = p_i dq^i \mid_x \tilde{\xi}_{\eta} = (q^1, \ldots, q^n, p_1, \ldots, p_n).$$

In the following, we consider only this kind of coordinates on $T^*N$ and the systems of adapted coordinates on $J^k(E), E = T^*N \times \mathbb{R}$, are assumed associated with such coordinates and of the form

$$\tilde{\xi}^{(k)} = (t, q^i, p^j, q^i_{(1)}, p^j_{(1)}, \ldots, q^i_{(k)}, p^j_{(k)}) : U \subset J^k(E) \to \mathbb{R}^{2n(k+1)+1}.$$

The components of a vector field $X$ on $U \subset J^k(E)$ along the coordinate vector fields $\frac{\partial}{\partial \theta^{(a)}}$ (resp. $\frac{\partial}{\partial q^i_{(a)}}$) are denoted by $X^i_{(a)}$ (resp. $X^i_{j(a)}$), that is

$$X = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + X^i_{(1)} \frac{\partial}{\partial q^i_{(1)}} + X^i_{j(1)} \frac{\partial}{\partial p^j_{(1)}} + \ldots.$$

The holonomic 1-forms (2.2) are now denoted by

$$\omega^i_{(a)} := dq^i - q^i_{(a)} dt, \quad \omega^i_{j(a)} := dp^i - p^i_{j(a)} dt.$$

We finally denote by $\vartheta$ and $\Omega$ the tautological 1-form and canonical symplectic 2-form, respectively, of $T^*N$. We recall that they are defined by $\vartheta|_{\beta} := \beta(\tilde{\pi}_*(\cdot))$ and $\Omega = d\vartheta$ and that, in coordinates $\xi_{\eta} = (q^i, p^j)$, they are given by the well-known expressions

$$\vartheta = p_i dq^i, \quad \Omega = d\vartheta = dp_i \wedge dq^i.$$
4.2. Infinitesimal symmetries of Hamiltonian actions. According to the standard terminology of Hamiltonian mechanics, a \textit{(time independent) Hamiltonian} is a smooth real function $H : \mathcal{U} \subset M = T^*N \longrightarrow \mathbb{R}$ defined on some open subset of $T^*N$.

For a given Hamiltonian $H$, let us consider the following definition.

\textbf{Definition 4.1.} The Poincaré-Cartan 1-form of $H$ is the 1-form $\alpha^H$ on the bundle $\pi : E = \mathcal{U} \times \mathbb{R} \subset T^*N \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\alpha^H := \vartheta - Hdt$$

( in coordinates, $\alpha^H := p_i dq^i - Hdt$).

For any $k \geq 1$, the Hamiltonian action of $H$ on $J^k(E)$ is the action $I[\alpha^H]$, defined by the variational class on $J^k(E)$ of the (pull-back on $J^k(E)$ of) $\alpha^H$.

If $\alpha^H$ is considered as a 1-form of $J^1(E)$, we may see that it is (locally) variationally equivalent to the 1-form $\alpha^H - p_i \omega^i = (H - p_i q^i_{(1)})dt$. This means that the action $I[\alpha^H]$ is (locally) determined by the Lagrangian $L = H - p_i q^i_{(1)}$, which is clearly of order 1. Furthermore,

$$d\alpha^H = \Omega - dH \wedge dt = dp_i \wedge dq^i - dH \wedge dt,$$

showing that $d\alpha^H$ is a source form, hence that $\alpha^H$ is of Poincaré-Cartan type.

These observations show that:

1) Theorems 3.8 and 3.10 can be used for $I[\alpha^H]$ whenever $\alpha^H$ is considered on a jet space $J^k(E)$ with $k \geq 2$.
2) If $\alpha^H$ is taken as a 1-form on $J^2(E)$ and we consider adapted coordinates $(t, q^i_{(a)}, p^j_{(a)})_{a=0,1,2}$, the source form $\sigma$ in the variational class $[d\alpha^H]$ of the kind (2.12) is

$$\sigma = \left( \frac{\partial H}{\partial q^i} + p^i_{(1)} \right) \omega^i_{(0)} \wedge dt + \left( \frac{\partial H}{\partial p^i} - q^i_{(1)} \right) \omega^i_{(0)} \wedge dt. \tag{4.1}$$

3) The system given by the full prolongation of the Euler-Lagrange equations, determined by (4.1), is

$$\begin{align*}
\sigma_i = \frac{\partial H}{\partial q^i} + p^i_{(1)} &= 0, \\
\bar{\sigma}^i = \frac{\partial H}{\partial p^i} - q^i_{(1)} &= 0, \\
\frac{d\sigma_i}{dt} = \frac{\partial^2 H}{\partial q^i q^j} q^j_{(1)} + \frac{\partial^2 H}{\partial q^i p^j} p^j_{(1)} + p_{(2)i} &= 0, \\
\frac{d\bar{\sigma}^i}{dt} = \frac{\partial^2 H}{\partial p^i q^j} q^j_{(1)} + \frac{\partial^2 H}{\partial p^i p^j} p^j_{(1)} - q^i_{(2)} &= 0. \tag{4.2}
\end{align*}$$

4) $\alpha^H$ satisfies condition (a) of Theorem 3.10.
5) \( \alpha^H \) satisfies also condition (b) of Theorem 3.10, provided that it is restricted to the open subset \( \mathcal{W} = \{(H - p_iq^{(1)}_i)(u_o) \neq 0\} \).

Due to this and Proposition 3.7, given a \((2n+1)\)-tuple \( v = (v^0, v^i, v^j) \) of smooth functions on a subset \( \mathcal{W} \subset J^2(E) \), the \( \mathcal{D} \)-symmetry

\[
X_v := v^0 \frac{d}{dt} + v^i \frac{\partial}{\partial q^i} + v_i \frac{\partial}{\partial p_i} + \sum_{a=1}^2 \frac{d^a}{dt^a} \left( v^{(a)}_i - q^{(1)}_i v^0 \right) \frac{\partial}{\partial q^{(a)}_i} + \sum_{a=1}^2 \frac{d^a}{dt^a} \left( v_i - p^{(1)}_i v^0 \right) \frac{\partial}{\partial p^{(a)}_i}
\]

is an infinitesimal symmetry for \( I_{[\alpha]} \) if and only if \( v \) satisfies the equation

\[
p^{(1)}_i v^i + p_i \frac{d v^i}{dt} - \frac{dH}{dt} v^0 - H \frac{dv^0}{dt} = - \left( \frac{\partial H}{\partial q^i} + p^{(1)}_i \right) (v^i - q^{(1)}_i v^0) - \left( \frac{\partial H}{\partial p_i} - q^{(1)}_i \right) (v_i - p^{(1)}_i v^0). \tag{4.3}
\]

In addition, by Theorem 3.10, given a constant of motion \( f \) on \( \mathcal{W} \subset J^2(E) \), of order less than or equal to 1, we may locally determine a \((2n+1)\)-tuple \( v \), corresponding to an infinitesimal symmetry \( X_v \) for \( I_{[\alpha]} \) and such that

\[
r_{X_v} \alpha^H = f + g,
\]

where \( g \) is a constant of motion that vanishes identically along the solutions of the variational principle. By the proof of Theorem 3.10, the constant of motion \( g \) (identically vanishing on solutions) and the infinitesimal symmetry \( X_v \) are determined by the following steps:

**Step 1.** Find smooth functions \( (\tilde{v}^i, \tilde{v}_j, \tilde{v}^{(1)}_j, \tilde{v}^{(1)}_{j(1)}) \) such that

\[
- \frac{df}{dt} = \tilde{v}^i \left( \frac{\partial H}{\partial q^i} + p^{(1)}_i \right) + \tilde{v}_i \left( \frac{\partial H}{\partial p_i} - q^{(1)}_i \right) + \tilde{v}^{(1)}_j \left( \frac{\partial^2 H}{\partial q^i \partial q^j} q^{(1)}_i + \frac{\partial^2 H}{\partial q^i \partial p_j} p^{(1)}_j + \frac{\partial^2 H}{\partial p_i \partial q^j} q^{(1)}_j + \frac{\partial^2 H}{\partial p_i \partial p_j} p^{(1)}_j - q^{(2)}_j \right).
\]

(Note that such functions do exist, but are not uniquely determined by \( f \).)

**Step 2.** Determine the constant of motion \( g \) by the formula

\[
g := \tilde{v}^{(1)}_j \left( \frac{\partial H}{\partial q^i} + p^{(1)}_i \right) + \tilde{v}^{(1)}_{j(1)} \left( \frac{\partial H}{\partial p_i} - q^{(1)}_i \right).
\]
Then the infinitesimal symmetry $X_{\nu}$ is determined by the $(2n + 1)$-tuple $v$

\[
\begin{align*}
\nu^0 &:= \frac{1}{p_iq_i(1)} - H \left( - \sum_{i=1}^{n} \left( \tilde{\nu}^i - \frac{\partial f}{\partial t} \right) p_i + f + g \right), \\
\nu^i &:= \tilde{\nu}^i - \frac{\partial f_i(1)}{\partial t} + q_i^0 \nu^0, \\
v_i &:= \tilde{\nu}_j - \frac{\partial f_j(1)}{\partial t} + p_j(1) \nu^0 \quad (4.4)
\end{align*}
\]

and it is therefore equal to

\[
X_{\nu} = \nu^0 \frac{d}{dt} + \left( \tilde{\nu}^i - \frac{\partial f_i(1)}{\partial t} \right) \frac{\partial}{\partial q^i} + \left( \tilde{\nu}_j - \frac{\partial f_j(1)}{\partial t} \right) \frac{\partial}{\partial p_j} + \sum_{a=1}^{2} \frac{d^a}{dt^a} \left( \tilde{\nu}^i - \frac{\partial f_i(1)}{\partial t} \right) \frac{\partial}{\partial q^i_{(a)}} + \sum_{a=1}^{2} \frac{d^a}{dt^a} \left( \tilde{\nu}_j - \frac{\partial f_j(1)}{\partial t} \right) \frac{\partial}{\partial p_j_{(a)}} \quad (4.5)
\]

4.3. Infinitesimal symmetries and $H$-symplectic symmetries. Consider now a constant of motion $f$ on $J^2(E)$ for the action $\mathcal{I}_{[\alpha]}$ of order 0 and time-independent, i.e. which is the pull-back on $J^2(E)$ of a function $\tilde{f}: \mathcal{U} \subset T^*M \rightarrow \mathbb{R}$, with $d\tilde{f} \left( \frac{\partial}{\partial t} \right) = 0$. We call any such constant of motion first integral of elementary type.

Now, recall that the system of equations that are satisfied by the solutions of the variational principle of $\mathcal{I}_{[\alpha]}$ are (4.2). It is then clear that, for any $x_o \simeq (t_o, q^i_{o, p_{oj}}) \in T^*M$, there exists a jet $u_0 \simeq (t_o, q^i_{o, p_{oj}}, q^i_{(1)o}, p_{j(1)o}) \in J^2(E)$, satisfying the full prolongation of Euler-Lagrange equations and a solution $\gamma: I \subset \mathbb{R} \rightarrow E$, such that $\gamma(t_o) = u_o$. The coordinates $q_{(1)o}, p_{j(1)o}$ of such point $u_o$ are equal to

\[
q_{(1)o} = \frac{\partial H}{\partial p_i} \bigg|_{(q^i_{o, p_{oj}})}, \quad p_{j(1)o} = \frac{\partial H}{\partial q^i} \bigg|_{(q^i_{o, p_{oj}})}.
\]

Since the first integral $f$ depends only on the coordinates of $T^*M$, it follows that

\[
0 = df|_{u_o} \left( \gamma_{t_o}^{(2)} \right) = \frac{\partial f}{\partial q^i} \bigg|_{(q^i_{o, p_{oj}})} \gamma_{(1)}(t_o) + \frac{\partial f}{\partial p_i} \bigg|_{(q^i_{o, p_{oj}})} \gamma_{(1)}(t_o) = \frac{\partial H}{\partial q^i} \bigg|_{(q^i_{o, p_{oj}})} - \frac{\partial f}{\partial p_i} \bigg|_{(q^i_{o, p_{oj}})} \frac{\partial H}{\partial q^j} \bigg|_{(q^j_{o, p_{oj}})} + d^2f|_{(q^i_{o, p_{oj}}, p_{j(1)o})} \quad (4.6)
\]

for any $(q^i_{o, p_{oj}}) \in T^*M$. By arbitrariness of $(q^i_{o, p_{oj}})$, it follows that

\[
- \frac{df}{dt} = \frac{\partial f}{\partial q^i} q^i_{(1)} + \frac{\partial f}{\partial p_j} p_{j(1)} = \frac{\partial f}{\partial q^i} q^i_{(1)} + \frac{\partial f}{\partial p_j} \left( p_{j(1)} + \frac{\partial H}{\partial q^i} \right) .
\]
This shows that Step 1 of previous section can be easily solved by setting
\[ \hat{v}_i = \frac{\partial f}{\partial p_i}, \quad \hat{v}_j = -\frac{\partial f}{\partial q_j}, \quad v_i^{(1)} = v_j^{(1)} = 0. \]
From this, following Step 2, we get that the (vanishing along solutions) constant of motion \( g \) is identically vanishing and that the infinitesimal symmetry \( X(f) \) associated with \( f \) is
\[ X(f) = \frac{1}{p_i q_j^{(1)}} - H \left( -\left( \frac{\partial f}{\partial p_i} \right) p_i + f \right) \frac{d}{dt} + \left( \frac{\partial f}{\partial p_j} \right) \frac{\partial}{\partial q^i} - \left( \frac{\partial f}{\partial q_i} \right) \frac{\partial}{\partial p_j} + \]
\[ + \sum_{a=1}^2 \frac{d^a}{dt^a} \left( \frac{\partial f}{\partial q^i} \right) \frac{\partial}{\partial q_i^{(a)}} - \sum_{a=1}^2 \frac{d^a}{dt^a} \left( \frac{\partial f}{\partial q_j^{(a)}} \right) \frac{\partial}{\partial p_j^{(a)}}. \] (4.7)
Consider now the natural immersion
\[ \iota : E = T^*M \times \mathbb{R} \rightarrow J^2(E), \quad \iota(\beta, t) = j^2_t(\tilde{\beta}), \]
where \( \tilde{\beta} : \mathbb{R} \rightarrow E \) denotes the constant section \( \tilde{\beta}(t) \equiv \beta \), and the projection
\[ \hat{\pi} : E = T^*M \times \mathbb{R} \rightarrow T^*M. \]
For any infinitesimal symmetry [4.3], consider the associated vector field \( Y(f) \) on (an open subset of) \( T^*M \), defined by
\[ Y(f) := \hat{\pi}_* \left( X(f) \mid_{\iota(E)} \right) = \left( \frac{\partial f}{\partial p_i} \right) \frac{\partial}{\partial q^i} - \left( \frac{\partial f}{\partial q_i} \right) \frac{\partial}{\partial p_j}. \] (4.8)
Notice that:

i) The vector field \( Y(f) \) is the Hamiltonian vector field associated with \( f \), i.e. the unique vector field on \( T^*M \) that satisfies the equation
\[ t_{Y(f)} \Omega = df. \] (4.9)
Note also that, by [4.3], it satisfies the condition
\[ t_{Y(f)} dH = 0. \] (4.10)

ii) Conversely, if \( Y(f) \) is a Hamiltonian vector field on \( \mathcal{U} \subset T^*M \), associated with a function \( f \) on \( \mathcal{U} \) and satisfying [4.11], one can directly check that \( f : \mathcal{U} \rightarrow \mathbb{R} \) is a first integral of elementary type.

iii) In any open subset \( \mathcal{W} \subset J^2(E) \), where a system of adapted coordinates \( \hat{\xi}^{(2)} = (t, q^i, p_j, q_i^{(1)}, p_j^{(1)}, q_i^{(2)}, p_j^{(2)}) \) are defined, given a Hamiltonian vector field as in (ii), there is a unique infinitesimal symmetry \( X(f) \) for the variational principle of \( I_{[\alpha, H]} \) of the form [4.3] and such that \( Y(f) := \hat{\pi}_* \left( X(f) \mid_{\iota(E)} \right) \).

iv) Given \( u = \iota(\beta_0) \in \iota(E) \subset J^2(E) \), the correspondence \( X \mapsto \hat{\pi}_* \left( X \mid_{\iota(E)} \right) \) determines an isomorphism between the Lie algebra of germs at \( \beta_0 \) of the infinitesimal symmetries as in [4.3] and the Lie
algebra of germs at $\beta_0$ of the Hamiltonian vector fields on $T^*M$, which satisfy (4.10).

These facts can be nicely summarized using the following notion.

**Definition 4.2.** The vector fields $Y$ on $\mathcal{U} \subset T^*M$ that satisfy the equations

\[ \mathcal{L}_Y \Omega = 0, \quad \iota_Y dH = 0 \]  

are called $H$-symplectic symmetries of $T^*M$.

Fix now a point $u \in \mathcal{U} \subset T^*M$ and consider the following classes of germs at $u$:

- $\mathfrak{sp}^H := \{ \text{germs at } u \text{ of } H\text{-symplectic symmetries} \}$,
- $\mathfrak{I}^{\text{elem}} := \{ \text{germs at } u \text{ of first integrals for } \mathcal{I}_{[u,H]} \text{ of elementary type} \}$.

By the above discussion, the correspondence between infinitesimal symmetries and constants of motion, given by Noether Theorems, determines the isomorphism of vector spaces

\[ \varphi : \mathfrak{sp}^H \rightarrow \mathfrak{I}^{\text{elem}}/\mathbb{R}, \]  

where, for any $Y \in \mathfrak{sp}^H$, the corresponding equivalence class $\varphi(Y) \in \mathfrak{I}^{\text{elem}}/\mathbb{R}$ is determined by the $f$ (determined up to an additive constant) such that

\[ \iota_Y \Omega_{|u} = df_{|u}. \]

Since $\mathfrak{sp}^H$ has a natural structure of Lie algebra, the vector space isomorphism $\varphi$ induces a natural Lie algebra structure on $\mathfrak{I}^{\text{elem}}/\mathbb{R}$.

We remark that the Lie brackets of the induced Lie algebra structure are the usual Poisson brackets of the symplectic manifold $(T^*M, \Omega)$.

4.4. The infinite-dimensional Lie algebra $\mathfrak{sp}^H$. Let $\Omega_o$ be the standard symplectic form of $\mathbb{R}^{2n}$, i.e.

\[ \Omega_o = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \ldots + dx^{2n-1} \wedge dx^n, \]  

and denote by $\mathfrak{sp}_\infty(2n, \mathbb{R})$ the Lie subalgebra of $\mathfrak{sp}_\infty(2n, \mathbb{R})$, determined by the vector fields, commuting with $\frac{\partial}{\partial x^1}$. We recall that $\mathfrak{sp}_\infty(2n, \mathbb{R})$ is the infinite-dimensional Lie algebra of the germs at 0 of vector fields of $\mathbb{R}^{2n}$, which preserve $\Omega_o$. Consequently, the (infinite-dimensional) Lie algebra $\mathfrak{sp}_\infty(2n, \mathbb{R})$ is made of all germs of vector fields $X$ satisfying the pair of conditions

\[ \mathcal{L}_X \Omega_o = 0 \quad \text{and} \quad \mathcal{L}_{\frac{\partial}{\partial x^1}} X = 0. \]  

Note that the second condition in (4.14) is equivalent to require that

\[ dx^2(X) = \mathcal{L}_X dx^2 = \mathcal{L}_X (1, \frac{\partial}{\partial x^1}, \Omega_o) = 0. \]
One can directly check that $X \in \mathfrak{sp}^{(1)}_{\infty}(2n, \mathbb{R})$ if and only if $X$ is of the form

$$X = h \frac{\partial}{\partial x^1} + \bar{X} \quad \text{with} \quad \bar{X} = \sum_{j=3}^{2n} \bar{X}_j \frac{\partial}{\partial x^j}$$

where $h$ and $\bar{X}^i$ are functions that satisfy the equations

$$\frac{\partial h}{\partial x^1} = \frac{\partial \bar{X}^j}{\partial x^1} = 0, \quad \mathcal{L}_X \Omega_o' = -dh \wedge dx^2,$$

where $\Omega_o'$ denotes the standard symplectic form of $\mathbb{R}^{2n-2} = \{ x \in \mathbb{R}^{2n} : x^1 = x^2 = 0 \}$.

This means that

$$\mathfrak{sp}^{(1)}_{\infty}(2n, \mathbb{R}) \supset \mathfrak{sp}_{\infty}(2n-2, \mathbb{R}) \times \mathbb{R}.$$

Let $H : \mathcal{U} \subset T^*M \rightarrow \mathbb{R}$ be a time-independent Hamiltonian. An element $\beta \in \mathcal{U}$ is called point of non-degeneracy for $H$ if $dH|_\beta \neq 0$.

**Proposition 4.3.** For any Hamiltonian $H$ and any point of non-degeneracy $u$ for $H$, the Lie algebra $\mathfrak{sp}^{H_0}$ is isomorphic to the infinite dimensional Lie algebra $\mathfrak{sp}^{(1)}_{\infty}(2n, \mathbb{R})$.

**Proof.** By the proof of Darboux Theorem (see e.g. [1]), since $dH|_u \neq 0$, there exists a system of coordinates around $u$, in which $\Omega$ assumes the same expression of the standard symplectic form $\Omega_o$ and the function $H$ is equal to $H = x^2$. From this, the conclusion follows. \qed

From the above proposition, around points of non-degeneracy, all Lie algebras of (germs of) first integrals of elementary type of all Hamiltonians are infinite-dimensional and mutually isomorphic. The same clearly occurs for any subalgebra $\mathfrak{g}$ of such Lie algebras and gives rise to the following phenomenon (see also [8] for a constructive proof of this property for some special Lie algebras).

**Theorem 4.4.** Assume that for a given Hamiltonian $H$ there exists a collection of first integrals of elementary type, which (by means of Poisson brackets) constitutes a (finite or infinite) dimensional Lie algebra $\mathfrak{g} \subset \mathfrak{sp}^H$ at a point of non-degeneracy $u$.

Then the same occurs for any other Hamiltonian $H'$ in the following sense: around any point of non-degeneracy of $H'$, there exists a collection of (locally defined!) first integrals of elementary type for $H'$, which constitutes a Lie algebra $\mathfrak{g}' \subset \mathfrak{sp}^{H'}$ that is isomorphic to $\mathfrak{g}$. 
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Emanuele Fiorani and Andrea Spiro, Scuola di Scienze e Tecnologie, Università di Camerino, Via Madonna delle Carceri 9, I-62032 Camerino (Macerata), ITALY

E-mail address: emanuele.fiorani@unicam.it
E-mail address: andrea.spiro@unicam.it