THE SYNERGY BETWEEN NUMERICAL AND
PERTURBATIVE APPROACHES TO BLACK HOLES

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Abstract. I describe approaches to the study of black hole spacetimes via numerical relativity. After a brief review of the basic formalisms and techniques used in numerical black hole simulations, I discuss a series of calculations from axisymmetry to full 3D that can be seen as stepping stones to simulations of the full 3D coalescence of two black holes. In particular, I emphasize the interplay between perturbation theory and numerical simulation that build both confidence in present results and tools to aid and to interpret results of future simulations of black hole coalescence.

1. Introduction
When I started graduate school, I began a thesis project in perturbation theory of spherical spacetimes. I still remember well how my advisor, Vincent Moncrief, an expert in perturbation theory, advised me to study a paper by Vishveshwara on black hole perturbations [1]. “That’s the best place to find the perturbation formalism”, he told me. At that time, of course, I had no idea how important this subject would continue to be years later. I was very lucky to become grounded in this subject at an “early age”, and I knew it would provide important insight into problems that were intractable in numerical relativity at that time. However I did not appreciate that even as numerical relativity would become more and more mature, harnessing hundreds of processors in parallel to solve ever larger problems, perturbation theory would continue to play such an important role in so many ways. In fact, its role in numerical relativity has become even more important in recent years, as I describe below. Vishu’s work in
this area influenced me in ways that I appreciate even more as my own research moves into large scale numerical simulation.

2. Numerical Evolutions of Black Holes

The numerical evolution of black holes is very difficult, as one must simultaneously deal with singularities inside them, follow the highly non-linear regime near the horizons, and also calculate the linear regime in the radiation zone where the waves represent a very small perturbation on the background spacetime metric. In axisymmetry this has been achieved, for example, for stellar collapse [2], rotating collisionless matter [3], distorted vacuum black holes with rotation [4] and without [5], and for equal mass colliding black holes [6, 7], but with difficulty. These 2D evolutions can be carried out to roughly $t = 100M$, where $M$ is the ADM mass of the spacetime, but beyond this point large gradients related to singularity avoiding slicings usually cause the codes to become very inaccurate and crash. This is one of the fundamental problems associated with black hole evolutions: if one uses the gauge freedom in the Einstein equations to bend time slices up and around the singularities, one ends up with pathological behavior in metric functions describing the warped slices that eventually leads to numerical instabilities.

In 3D the problems are even more severe with this traditional, singularity avoiding time slicing approach. To simulate the coalescence of two black holes in 3D, evolutions of time scales $t \approx 10^2 - 10^3 M$ will be required. Traditional approaches can presently carry evolutions only to about $t = 50M$. However, in spite of these difficulties, great progress is being made on several fronts. Alternative approaches to standard numerical evolution of black holes, such as apparent horizon boundary conditions and characteristic evolution, promise much longer evolutions. Apparent horizon conditions cut away the causally disconnected region interior to the black hole horizon, allowing better behaved slicings. These have been well developed in 1D, spherically symmetric studies [8, 9, 10, 11, 12, 13] and full 3D evolutions. Characteristic evolution with ingoing null slices have very recently been successful in evolving 3D single black holes for essentially unlimited times, even with distortions away from spherical or axisymmetry [14]. These alternate approaches look promising, but will take time to develop into general approaches to the two black hole coalescence problem, and I will not have space to cover them here. Instead, I focus on how traditional approaches, aided by perturbative studies, are providing insight into the dynamics of distorted and colliding black holes. When the alternate techniques for black hole evolutions mature, they too will be aided by perturbative studies to both verify and interpret numerical simulations.
3. Interplay of Perturbative and Numerical Black Hole Studies

While numerical relativity is making steady progress in its ability to simulate the dynamics of black holes, an equally important and exciting development of the last few years has been the application of perturbation theory to aid in the verification and interpretation numerical simulations. As I discuss below, there are various ways that perturbation theory is used in conjunction with numerical relativity, from the extraction of waveforms from fully non-linear numerical simulations, to full evolutions of black hole initial data sets, treated as perturbations about the Schwarzschild black hole background.

In this section I present a series of ideas and calculations that lead to what I call a “Ladder of Credibility”. Problems need to be studied in sequence from easier to more complicated, leading ultimately to the 3D spiraling coalescence problem. I will discuss the foundations of black hole perturbation theory, and show how it can be used both to test and to interpret results of full numerical simulations, beginning with axisymmetric distorted black holes, moving on to axisymmetric black hole collisions, and building to full 3D simulations.

3.1. PERTURBATION THEORY

In this section I give a very brief overview of the theory of perturbations of the Schwarzschild spacetime. This is a topic which is very rich and has a long history, and to which Vishu has contributed immensely. A more detailed discussion can be found in [15]. In the next section, I will discuss how we apply the theory to study black hole data sets.

One begins the analysis by writing the full metric as a sum of the Schwarzschild metric $g_{\alpha\beta}$ and a small perturbation $h_{\alpha\beta}$:

$$g_{\alpha\beta} = g^\circ_{\alpha\beta} + h_{\alpha\beta}. \quad (1)$$

One then plugs this expression into the vacuum Einstein equations to get equations for the perturbation tensor $h_{\alpha\beta}$. One can separate off the angular part of the solution by expanding $h_{\alpha\beta}$ in spherical tensor harmonics, as was originally done by Regge and Wheeler [16]. For each $\ell, m$ mode, one gets separate equations for the perturbed metric functions, which we now denote by $h^{(\ell m)}_{\alpha\beta}$.

There are two independent expansions: one, known as even parity, which does not introduce any rotational motion in the hole, and one, known as odd parity, which does. We will concentrate here on even parity perturbations, as these are the ones which will be relevant for studying the data-sets.
discussed here. The odd parity perturbations produce equations which are very similar, and both can be considered in the general case.

We also note here that this treatment is presently restricted to perturbations of Schwarzschild black holes. For the more general rotating case, one would like to use the Teukolsky formalism describing perturbations of Kerr. This is much more complicated, and has not yet been applied to numerical black hole simulations of the kind discussed in this paper. This is an important research topic that needs attention soon!

I have discussed here only the linearized theory of black hole perturbations, but this can also be extended to higher order. This has recently been accomplished by Gleiser, Nicasio, Price, and Pullin [17], who worked out equations describing perturbations of Schwarzschild to second order in an expansion parameter. Hence the metric is written as

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + \epsilon h_{\alpha\beta}^{(1)} + \epsilon^2 h_{\alpha\beta}^{(2)}.$$  \hspace{1cm} (2)

and the Einstein equations are expanded to second order in $\epsilon$. As we will see below, this second order formalism is very useful in black hole studies [18].

When dealing with perturbations in relativity, one must be careful about interpreting the various metric components $h_{\alpha\beta}$ in terms of physics. Under a coordinate transformation of the form $x^\mu \rightarrow x^\mu + \delta x^\mu$, the metric coefficients will transform as well. One can use this gauge freedom to eliminate certain metric functions to simplify the corresponding equations for the perturbations. Another, more powerful approach, developed first by Moncrief, is to consider linear combinations of the $h_{\alpha\beta}$ and their derivatives that are actually invariant under the gauge transformation above. In either case, the analysis leads to a single wave equation for the perturbations of the black hole:

$$\frac{\partial^2 \psi^{(\ell m)}}{\partial r^*^2} - \frac{\partial^2 \psi^{(\ell m)}}{\partial t^2} + V^{(\ell)}(r) \psi^{(\ell m)} = 0,$$  \hspace{1cm} (3)

where the potential function $V^{(\ell)}(r)$ is given by

$$V^{(\ell)}(r) = \left(1 - \frac{2M}{r}\right) \times \left\{ \frac{1}{\Lambda^2} \left[ \frac{72M^3}{r^5} - \frac{12M}{r^3} (\ell - 1)(\ell + 2) \left(1 - \frac{3M}{r}\right) \right] + \frac{\ell(\ell - 1)(\ell + 2)(\ell + 1)}{r^2\Lambda} \right\},$$  \hspace{1cm} (4)

where $r$ is the standard Schwarzschild radial coordinate, and $r^*$ is the so-called tortoise coordinate, given by $r^* = r + 2M \ln (r/2M - 1)$, and $\Lambda$ is a function of $r$ described below. For all $\ell$, the potential function is positive.
and has a peak near $r = 3M$. This equation was first derived by Zerilli [19]. Regge and Wheeler found an analogous equation for the odd parity perturbations, which are much simpler than the even parity perturbations, 13 years earlier in 1957 [16]. Note that the potential depends on $\ell$, but is independent of $m$. This remarkable equation (along with its odd-parity counterpart, known as the Regge-Wheeler equation) completely describes gravitational perturbations of a Schwarzschild black hole. It continues to be studied by many researchers now 40 years after this perturbation program was begun in 1957.

When carried out to second order in the perturbation parameter, the first order equations are of course the same, but they are accompanied by an equation representing the second order correction. Remarkably, the second order equation has exactly the same form as the first order equation, with source terms which are non-linear in the first order perturbation quantities. As I mention below, the second order perturbation theory has been used in application to the head-on collision of two black holes to extend the range of validity of the first order treatment. But perhaps its most important use is in providing a check on the validity of the first order perturbation treatment. When second order corrections are small compared to first order results, one expects the first order results to be reliable.

Using the first order equation, it has been shown that Schwarzschild is stable to perturbations, and has characteristic oscillation frequencies known as quasinormal modes [20]. These modes are solutions to the Zerilli equation as given in Eq. (3) which are completely ingoing at the horizon ($r^* = -\infty$) and completely outgoing at infinity ($r^* = \infty$). For each $\ell$-mode, independent of $m$, there is a fundamental frequency, and overtones. These are very important results! One expects that a black hole, when perturbed in an arbitrary way, will oscillate at these quasinormal frequencies. This will give definite signals to look for with gravitational wave observatories.

These quasinormal frequencies are complex, meaning they have an oscillatory and a damping part (not growing—black holes are stable!), so the oscillations die away as the waves carry energy away from the system. The frequencies depend only on the mass, spin, and charge of the black hole.

There are numerous ways in which this perturbation theory has become essential in numerical black hole simulations, and the rest of this paper will concentrate on this subject. First of all, the fact that perturbation theory reveals that black holes have quasinormal mode oscillations raises expectations about the evolution of distorted black holes: they should, at least in the linear regime, oscillate at these frequencies which should be seen in fully non-linear numerical simulations. But are they still seen in highly non-linear interactions, e.g., in the collision of two black holes? Secondly, as we will see, this perturbation theory provides a method by which to sep-
arate out the Schwarzschild background from the wave degrees of freedom, which can be used to find waves in numerical simulations. Finally, as the perturbations are governed by their own evolution equation, this equation should be useful to actually evolve some classes of black hole initial data, as long as they represent slightly perturbed black holes, and this can be used as an important check of fully non-linear numerical codes. Certainly, during the late stages of black hole coalescence, the system will settle down to a slightly perturbed black hole, and numerical codes had better be able to accurately compute waves from such systems if they are to be used to help researchers find signals in actual data collected by gravitational wave observatories.

3.2. WAVEFORM EXTRACTION

In this section I show how to take this perturbation theory and apply it in a practical way to numerical black hole simulations. One considers now the numerically generated metric $g_{\alpha\beta,\text{num}}$ to be the sum of a spherically symmetric part and a perturbation: $g_{\alpha\beta,\text{num}} = g_{\alpha\beta} + h_{\alpha\beta}$, where the perturbation $h_{\alpha\beta}$ is expanded in tensor spherical harmonics as before. To compute the elements of $h_{\alpha\beta}$ in a numerical simulation, one integrates the numerically evolved metric components $g_{\alpha\beta,\text{num}}$ against appropriate spherical harmonics over a coordinate 2–sphere surrounding the black hole. The orthogonality of the $Y_{\ell m}$’s allows one to “project” the contributions of the general wave signal into individual modes, as explained below. The resulting functions can then be combined in a gauge-invariant way, following the prescription given by Moncrief [21], leading directly to the Zerilli function. This procedure was originally developed by Abrahams [22] and developed further by various groups.

As mentioned above, we assume the general metric can be decomposed into its spherical and non-spherical parts. We assume here that the background is Schwarzschild, written in Schwarzschild coordinates, but the treatment of a more general spherical background in an arbitrary coordinate system is possible [23]. The nonspherical perturbation tensor $h_{\mu\nu}$ for even-parity perturbations can be written

\begin{align*}
    h_{tt} &= -SH_0^{(\ell m)} Y_{\ell m} \\
    h_{tr} &= H_1^{(\ell m)} Y_{\ell m} \\
    h_{t\theta} &= h_0^{(\ell m)} Y_{\ell m,\theta} \\
    h_{t\phi} &= h_0^{(\ell m)} Y_{\ell m,\phi} \\
    h_{rr} &= S^{-1}H_2^{(\ell m)} Y_{\ell m}
\end{align*}
where \( S = (1 - 2M/r) \) comes from the Schwarzschild background, and \( r \) is the standard Schwarzschild radial coordinate. I have taken these expressions directly from Vishu’s 1970 paper [1] where I first learned them.

Each \( \ell m \)-mode of \( h_{\mu\nu} \) can then be obtained numerically by projecting the full metric against the appropriate \( Y_{\ell m} \). So, for example,

\[
H_2^{(\ell m)} = (1 - 2M/r) \int g_{rr,\text{num}} Y_{\ell m} d\Omega. \tag{15}
\]

More complex expressions can be developed for all metric functions, and on a more general background, as detailed in [24, 25]. These functions can be computed numerically, given a numerically generated spacetime, as described below.

Once the perturbation functions \( H_2, K, G, \text{etc.} \) are obtained, one still needs to create the special combination that obeys the Zerilli equation. Moncrief showed that the Zerilli function that obeys this wave equation is gauge invariant in the sense discussed above, and can be constructed from the Regge-Wheeler variables as follows:

\[
\psi^{(\ell m)} = \sqrt{2(\ell - 1)(\ell + 2)} \frac{4r S^2 k_2^{(\ell m)}}{\ell(\ell + 1)} + \ell(\ell + 1) r k_1^{(\ell m)}, \tag{16}
\]

where

\[
\Lambda \equiv \ell(\ell + 1) - 2 + \frac{6M}{r} \tag{17}
\]

\[
k_1^{(\ell m)} \equiv K^{(\ell m)} + 2S r G_{r}^{(\ell m)} - 2S \frac{S}{r} h_{1}^{(\ell m)} \tag{18}
\]

\[
k_2^{(\ell m)} \equiv \frac{H_{2}^{(\ell m)}}{2S} - \frac{1}{2\sqrt{S}} \frac{\partial}{\partial r} \left( \frac{r K^{(\ell m)}}{\sqrt{S}} \right) \tag{19}
\]

\[
S \equiv 1 - \frac{2M}{r}. \tag{20}
\]

In order to compute the Regge-Wheeler perturbation functions \( h_1, H_2, G, \text{and} K \), one needs the spherical metric functions on some 2-sphere. We get these in 3D by interpolating the Cartesian metric functions onto a
surface of constant coordinate radius, and computing the spherical metric functions from these using the standard transformation. These are straightforward but messy calculations which are covered in detail in Refs. [24, 25].

In summary, in this section I have outlined a practical approach to the use of perturbation theory as a tool to construct a gauge invariant measure of the gravitational radiation in a numerically generated black hole spacetime. There are various ways in which this information can be used, which is the subject of the following sections.

3.3. APPLICATIONS

In the sections above, I showed how to extract the gauge invariant Zerilli function at a given radius on some time slice of the numerical spacetime. In the following sections I show several ways in which this information can be used, including (a) evolving a numerically (or analytically) generated initial data set with perturbation theory and (b) extracting waveforms from a fully non-linear evolution, possibly from the same data set. This perturbative approach has also recently been successful in providing outer boundary conditions for a 3D numerical simulation [26].

3.3.1. Axisymmetric Distorted Black Holes

We begin with the study of axisymmetric single black holes that have been distorted by the presence of an adjustable torus of non-linear gravitational waves which surround it. The amplitude and shape of the initial wave can be specified by hand, as described below, and can create very highly distorted black holes. Such initial data sets, and their evolutions in axisymmetry, have been studied extensively, as described in Refs. [5, 27, 28]. For our purposes, we consider them as convenient initial data that create a distorted black hole that mimics the merger, just after coalescence, of two black holes colliding in axisymmetry [7].

Following [27], we write the 3–metric in the form originally used by Brill [29]:

$$dτ^2 = \tilde{ψ}^4 \left( e^{2q} \left( dη^2 + dθ^2 \right) + \sin^2 θ dφ^2 \right),$$

(21)

where $η$ is a radial coordinate related to the standard Schwarzschild isotropic radius $\bar{r}$ by $\bar{r} = e^η$. (We have set the scale parameter $m$ in [27] to be 2 in this paper.) We choose our initial slice to be time symmetric, so that the extrinsic curvature vanishes. Thus, given a choice for the “Brill wave” function $q$, the Hamiltonian constraint leads to an elliptic equation for the conformal factor $\tilde{ψ}$. The function $q$ represents the gravitational wave surrounding the black hole, and is chosen to be

$$q(η, θ, φ) = a \sin^n θ \left( e^{-\left( \frac{η+b}{2a} \right)^2} + e^{-\left( \frac{η-b}{2a} \right)^2} \right) \left( 1 + c \cos^2 φ \right).$$

(22)
Thus, an initial data set is characterized by the parameters \((a, b, w, n, c)\), where, roughly speaking, \(a\) is the amplitude of the Brill wave, \(b\) is its radial location, \(w\) its width, and \(n\) and \(c\) control its angular structure. A study of full 3D initial data and their evolutions are discussed elsewhere \([30, 31, 32]\).

If the amplitude \(a\) vanishes, the undistorted Schwarzschild solution results, leading to

\[
\tilde{\psi} = 2 \cosh \left( \frac{\eta}{2} \right),
\]

which puts the metric (21) in the standard Schwarzschild isotropic form.

**Linear Evolution** We now consider the evolution of these distorted black holes. If the amplitude is low enough, this should represent a small perturbation on a Schwarzschild black hole, and hence the system should be amenable to a perturbative treatment. The idea is to actually compute \(\psi(r, t = 0)\), by extracting the Zerilli function at every radial grid point on the initial data slice, and use the Zerilli evolution equation to actually evolve the system as a perturbation. This will allow us to compare the waveform at some radius \(\psi_{\text{lin}}(r_0, t)\) obtained in this way, to that obtained with a well-tested 2D axisymmetric code that performs full non-linear evolutions. This will serve as a test of the initial Zerilli function being given to the linear code, and of the linear evolution code itself. It will also help us determine for which Brill wave amplitudes this procedure breaks down. Beyond a certain point, perturbation theory will fail and non-linear effects will become important.

Let us first consider the data set \((a, b, w, n, c) = (0.05, 1, 1, 4, 0)\), in the notation above. In this case the Brill wave is initially far from the black hole, and will propagate in, hitting it and exciting the normal mode oscillations. In Figure 1 we show the \(\ell = 2\) and \(\ell = 4\) Zerilli functions as a function of time, at a radius of \(r = 15M\). Data are shown from both the linear and 2D non-linear codes. We see that for both functions, the linear and non-linear results line up nicely until about \(t = 50M\), when a phase shift starts to be noticeable. This phase shift and widening of the wave at late times is known from previous studies of numerical simulations of distorted black hole spacetimes in axisymmetry \([5]\).

As second example, let us look at \((a, b, w, n, c) = (0.05, 0, 1, 4, 0)\). In this case the Brill wave is initially right on the throat. In Figure 2 we show the \(\ell = 2\) and the \(\ell = 4\) waveforms as a function of time extracted at a radius of \(r = 15M\). Again, data from both the linear and 2D non-linear codes are shown. The data line up well until about \(t = 50M\), when phase errors again show up.

This is the first step in our ladder of credibility. We have shown that these data sets provide an important testbed for a numerical black hole
Figure 1. We show the (a) $\ell = 2$ and (b) $\ell = 4$ Zerilli functions as a function of time, extracted during linear and 2D non-linear evolutions of the data set $(a, b, w, n, c) = (0.05, 1, 1, 4, 0)$. The data were extracted at a radius of $r = 15M$.

evolution. First, they confirm that these data sets can be treated as linear perturbations on a Schwarzschild background, since both linear and fully non-linear evolutions agree. Second, they remarkably confirm the results of a complex, non-linear evolution code which evolves a black hole (in maximal slicing). This gives great confidence in the ability of this code to treat black holes and extract waveforms, even in the more highly distorted cases where perturbation theory breaks down (but waveform extraction will not necessarily break down, at least far from the hole). We will use this technique in various ways below.

3.3.2. Axysymmetric Black Hole Collisions

We now turn to another application of this basic idea of evolving dynamic black hole spacetimes with perturbation theory, but this time we consider two black holes colliding head on. It might seem to be impossible to treat colliding black holes perturbatively, but there are two limits in which perturbation theory has been shown to be incredibly successful. First, if the two
holes are so close together initially that they have actually already merged into one, they might be considered as a single perturbed Schwarzschild hole (the so-called “close limit”). Using similar ideas to those discussed above, Price and Pullin and others [33, 34, 35, 36, 37] used this technique to produce waveforms for colliding black holes in the Misner [38] and Brill and Lindquist [39] black hole initial data. These initial data sets for multiple black holes are actually known analytically. The original paper of Price and Pullin [33] is what spurred on so much interest in these many applications of perturbation theory as a check on numerical relativity. Second, when the holes are very far apart, one can consider one black hole as a test particle falling into the other. Then one rescales the answer obtained by formally allowing the “test particle” to be a black hole with the same mass as the one it is falling into [6, 7, 34].

The details of this success has provided insights into the nature of collisions of holes, and should also apply to many systems of dynamical black holes. The waveforms and energies agree remarkably well with numerical
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simulations. Moreover, second order perturbation theory \[37\] spectacularly improved the agreement between the close limit and full numerical results for even larger distances between the holes, although ultimately beyond a certain limit the approximation is simply inappropriate and breaks down.

The success of these techniques suggests, among other things, that these are very powerful methods that can be used hand-in-hand with fully nonlinear numerical evolutions, and can be applied in a variety of black hole spacetimes where one might naively think they would not work. For these reasons, many researchers are continuing to apply these techniques to the axisymmetric case with more and more complicated black hole spacetimes (e.g., the collision of boosted black holes \[36\], or counter-rotating, spinning black holes, colloquially known as the “cosmic screw”\). Furthermore, these techniques will become even more essential in 3D, where we cannot achieve resolution as high as we can with 2D codes.

Finally, this is yet another rung on the “ladder of credibility”: we now have not only slightly perturbed Schwarzschild spacetimes to consider, but also a series of highly nontrivial colliding black hole spacetimes that are now well understood in axisymmetry due to the nice interplay between perturbative and fully numerical treatments of the same problems. These then provide excellent testbeds for 3D simulations, which we turn to next.

3.3.3. 3D Testbeds

Armed with robust and well understood axisymmetric black hole codes, we now consider the 3D evolution of axisymmetric distorted black hole initial data. These same axisymmetric initial data sets can be ported into a 3D code in cartesian coordinates, evolved in 3D, and the results can be compared to those obtained with the 2D, axisymmetric code discussed above. The 3D code used to evolve these black hole data sets is described in Refs. \[10, 40\], and the simulations described here are major simulations on very large supercomputers: they require about 12 GBytes of memory and take more than 24 hours on a 128 processor SGI Origin 2000 computer.

In the first of these simulations, we study the evolution of the distorted single black hole initial data set \((a, b, w, n, c) = (0.5, 0, 1, 2, 0)\). As the azimuthal parameter \(c\) is zero, this is axisymmetric and can also be evolved in 2D. In Figure 3a we show the result of the 3D evolution, focusing on the \(\ell = 2\) Zerilli function extracted at a radius \(r = 8.7M\) as a function of time. Superimposed on this plot is the same function computed during the evolution of the same initial data set with a 2D code, based on the one described in detail in \[5, 28\]. The agreement of the two plots over the first peak is a strong affirmation of the 3D evolution code and extraction routine. It is important to note that the 2D results were computed with a different slicing (maximal), different coordinate system, and a different spatial gauge.
Yet the physical results obtained by these two different numerical codes, as measured by the waveforms, are remarkably similar (as one would hope). A full evolution with the 2D code to \( t = 100M \), by which time the hole has settled down to Schwarzschild, shows that the energy emitted in this mode at that time is about \( 4 \times 10^{-3} M \). This result shows that now it is possible in full 3D numerical relativity, in cartesian coordinates, to study the evolution and waveforms emitted from highly distorted black holes, even when the final waves leaving the system carry a small amount of energy.

In Fig. 3b we show the \( \ell = 4 \) Zerilli function extracted at the same radius, computed during evolutions with 2D and 3D codes. This waveform is more difficult to extract, because it has a higher frequency in both its angular and radial dependence, and it has a much lower amplitude: the energy emitted in this mode is about three orders of magnitude smaller than the energy emitted in the \( \ell = 2 \) mode, yet it can still be accurately evolved and extracted. This is quite a remarkable result, and bodes well for the ability of numerical relativity codes ultimately to compute accurate waveforms that will be of great use in interpreting data collected by gravitational wave detectors. (However, as I point out below, there is a quite a long way to go before the general 3D coalescence can be studied!)

These results have been reported in much more detail in [30, 40].

3.3.4. True 3D Distorted Black Holes

We now turn to radiation extraction in true 3D black hole evolutions. This is of major importance for the connection between numerical relativity and gravitational wave astronomy. Gravitational wave detectors such as LIGO, VIRGO, and GEO will measure these waves directly, and may depend on numerical relativity to provide templates to both extract the signals from the experimental data and to interpret the results.

In the sections above, I showed by comparison to 2D results that a 3D code is able to accurately simulate distorted black holes. Armed with these tests, we now consider evolutions of initial data sets which are non-axisymmetric distorted black holes, \( i.e. \), data sets which have non-vanishing azimuthal parameter \( c \). We also consider the evolution of the same distorted black hole data sets via linearized theory, as we did with the 2D results presented above. The techniques are the same, although the details in a full 3D treatment are more complicated. Please refer to Refs. [24, 25] for more details.

The initial data set \( (a = -0.1, b = 0, c = 0.5, w = 1, n = 4) \) was evolved with the 3D numerical relativity Cartesian code described above, with \( 300^3 \) grid zones points in each coordinate direction. In Figure 4 we show the \( \ell = m = 2 \) Zerilli function computed during the evolution, comparing both the full non-linear theory with the linearized treatment. This is the first
Figure 3. We show the (a) $\ell = 2$ and (b) $\ell = 4$ Zerilli functions vs. time, extracted during 2D and 3D evolutions of the data set $(a, b, v, n, c) = (0.5, 0, 1, 2, 0)$. The functions were extracted at a radius of $8.7M$. The 2D data were obtained with $202 \times 54$ grid points, giving a resolution of $\Delta \eta = \Delta \theta = 0.03$. The 3D data were obtained using $300^3$ grid points and a resolution of $\Delta x = 0.0816M$.

Time a 3D non-linear numerical relativity code has been used to compute waveforms from fully 3D distorted black hole (but also see recent results using a characteristic formulation [14]). From the figure it is clear that the 3D results agree well with the perturbative treatment, even though the energy carried by these waves is very small.

Nonaxisymmetric modes are extracted not just because they can be, but because they should be quite important for gravitational wave observatories. It turns out that the $\ell = 2, m = 2$ is considered to be one of the most promising black hole modes to be seen by gravitational wave detector. In realistic black hole coalescence, the final hole is expected to have a large amount of angular momentum, possibly near the Kerr limit $a = 1$. This particular mode is one of the least damped (much less damped than for Schwarzschild, as seen here), and is also expected to be strongly excited [41]. Therefore, it is important to begin exhaustively testing the code's
Figure 4. We show waveforms for the $\ell = m = 2$ Zerilli function extracted from the linear and non-linear evolution codes for a fully 3D, nonaxisymmetric distorted black hole. The dotted line shows the linear evolution, evolving only the Zerilli equation, and the solid line shows the non-linear evolution in full 3D cartesian coordinates, using a massively parallel supercomputer.

ability to generate and cleanly extract such nonaxisymmetric modes, even in the case studied here without rotation.

Many more modes can be extracted, including $\ell = 4, m$ modes, and details and analysis can be found in [30, 42, 24, 25]. Comparisons between the perturbative and non-linear results reveal that waveforms can be accurately extracted in the linear regime, and that even at very low distortion amplitudes non-linear effects appear. The comparisons with perturbation theory are essential in understanding these effects, and will continue to be for some years to come.

4. Summary

I have given a brief overview of work on evolutions of distorted black holes and black hole collisions over the last decade, from 2D axisymmetric studies to recent 3D studies. At each stage along the way, perturbation theory has turned out to be an essential ingredient in the program. Our understanding of the 2D collision of two black holes has been aided immensely from perturbation theory, and in the last year our ability to simulate true 3D distorted black holes, which model the late stages of 3D binary black hole coalescence, has matured considerably.

Unfortunately, we still have a very long way to go! Although one can now do 3D evolutions of distorted black holes, and accurately extract very low amplitude waves, the calculations one can presently do are actually very limited. With present techniques, the evolutions can only be carried out for a fraction of the time required to simulate the 3D orbiting coalescence.
Most of what has been described here has been with certain symmetries, or with a single black hole in full 3D. At the present time, I am only aware of one attempt to study the collision of two black holes in 3D without any symmetries, which was recently reported by Brügmann [43]. However, this calculation is treated as a feasibility study, without detailed waveform extraction at this point, and again the evolution times are quite limited (less than those reported here.)

Many new techniques are under development to extend the simulations to the time scales required for true binary black hole coalescence, such as apparent horizon boundary conditions [8, 13], hyperbolic systems [44, 45], and characteristic evolution [14]. Many of these (or perhaps all!) may be needed to handle the general, long term evolution of coalescing black holes. Each of these techniques may introduce numerical artifacts, even if at very low amplitude, to which the waveforms may be very sensitive. As new methods are developed and applied to numerical black hole simulations, they can now be tested on evolutions such as those presented here to ensure that the waveforms are accurately represented in the data.

In closing, I want to emphasize that the kind of work that Vishu helped to pioneer 30 years ago is still at the forefront of numerical simulations aided by the world’s most powerful computers, and it seems clear that perturbation theory will continue to play an essential role in both the verification and physical understanding of large scale numerical simulations. There is still much work to do in this area, both on the theoretical and numerical areas. I expect it will continue to excite researchers for years to come!

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