PSEUDO-SYMMETRY ON UNIT TANGENT SPHERE BUNDLES

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Abstract. In this paper, we study the pseudo-symmetry of unit tangent sphere bundle. We prove that if the unit tangent sphere bundle $T_1M$ with standard contact metric structure over a locally symmetric $M^n$, $n \geq 3$ is pseudo-symmetric, then $M$ is of constant curvature.

1. Introduction

It is interesting to study the geometric interplay between a given Riemannian manifold $(M, g)$ and its unit tangent sphere bundle $T_1M$. Many authors have studied the standard contact metric structure of unit tangent sphere bundle. For example, D. E. Blair ([2]) proved that the unit tangent sphere bundle of a Riemannian manifold is locally symmetric if and only if the base manifold is of constant curvature 0 or 2-dimensional and of constant curvature 1. This says that local symmetry gives too strong restriction for the unit tangent sphere bundle. In this context, E. Boeckx and G. Calvaruso ([5]) studied the so-called semi-symmetry on $T_1M$ as a natural generalization of local symmetry ([5]). A Riemannian manifold $(\bar{M}, \bar{g})$ is said to be semi-symmetric if its curvature tensor $\bar{R}$ satisfies the condition

$$\bar{R}(\bar{X}, \bar{Y}) \cdot \bar{R} = 0$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$, where $\bar{R}(\bar{X}, \bar{Y})$ acts as a derivation on $\bar{R}$. They obtained that if the unit tangent sphere bundle of a Riemannian manifold is semi-symmetric, then it is already locally symmetric.

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As a generalization of semi-symmetry, we may consider the notion of pseudo-symmetry which was introduced by R. Deszcz ([9]). A Riemannian manifold \((\bar{M}, \bar{g})\) is said to be pseudo-symmetric if there exists a function \(L\) such that
\[
\bar{R}(\bar{X}, \bar{Y}) \cdot \bar{R} = L\{(\bar{X} \wedge \bar{Y}) \cdot \bar{R}\}
\]
for any vector fields \(\bar{X}\) and \(\bar{Y}\) on \(\bar{M}\). Here \(\bar{X} \wedge \bar{Y}\) is the endomorphism field defined by \((\bar{X} \wedge \bar{Y}) \bar{Z} = \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{Z}, \bar{X})\bar{Y}\). In particular, a pseudo-symmetric space is said to be constant type if \(L\) is constant. Then a semi-symmetric space is a pseudo-symmetric space of constant type (with \(L = 0\)).

In [8], the first author and J. Inoguchi proved that the unit tangent sphere bundle over a 2-dimensional Riemannian manifold \(M\) is pseudo-symmetric if and only if \(M\) is of constant curvature. Then the following question will naturally arise. “Can we extend the above result to higher dimensional cases?”

In the present paper, we obtained the partial answer about this question. The main theorem is the following.

**Main Theorem.** Let \((M, g)\) be an \(n(\geq 3)\)-dimensional locally symmetric space and \(T_1M\) be the unit tangent sphere bundle with standard contact metric structure over \(M\). If \(T_1M\) is pseudo-symmetric, then \(M\) is of constant curvature.

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class \(C^\infty\). We start by collecting some fundamental materials of contact metric geometry. We refer to [1] for further details. A \((2n + 1)\)-dimensional manifold \(M^{2n+1}\) is said to be a contact manifold if it admits a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) everywhere. Given a contact form \(\eta\), we have a unique vector field \(\xi\), the characteristic vector field, satisfying \(\eta(\xi) = 1\) and \(d\eta(\xi, \bar{X}) = 0\) for any vector field \(\bar{X}\) on \(M\). It is well-known that there exists a Riemannian metric \(\bar{g}\) on \(\bar{M}\) and a \((1, 1)\)-tensor field \(\phi\) such that

\[
(1) \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi,
\]

where \(\bar{X}\) and \(\bar{Y}\) are vector fields on \(\bar{M}\). From (1) it follows that

\[
(2) \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi \bar{X}, \phi \bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}).
\]
A Riemannian manifold $\bar{M}$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (1) is said to be a contact metric manifold and is denoted by $\bar{M} = (\bar{M}; \eta, \bar{g}, \phi, \xi)$. Given a contact metric manifold $\bar{M}$, we define the structural operator $h$ by $h = \frac{1}{2} \mathcal{L}_{\xi} \phi$, where $\mathcal{L}$ denotes Lie differentiation. Then we may observe that $h$ is symmetric and satisfies

$$h \xi = 0 \quad \text{and} \quad h \phi = -\phi h,$$

(3) where $\nabla$ is the Levi-Civita connection. From (3) and (4) we see that each trajectory of $\xi$ is a geodesic. We denote by $R$ the Riemannian curvature tensor defined by

$$\bar{R}([\bar{X}, \bar{Y}], \bar{Z}) = \nabla_{\bar{Y}}(\nabla_{\bar{X}} \bar{Z}) - \nabla_{\nabla_{\bar{X}} \bar{Y}} \bar{Z} - \nabla_{[\bar{X}, \bar{Y}]} \bar{Z}$$

for all vector fields $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ on $\bar{M}$. Along a trajectory of $\xi$, the Jacobi operator $\ell = \bar{R}(\cdot, \xi) \xi$ is a symmetric $(1,1)$-tensor field. We call it the characteristic Jacobi operator. A contact metric manifold for which $\xi$ is Killing is called a $K$-contact manifold. It is easy to see that a contact metric manifold is $K$-contact if and only if $h = 0$ or, equivalently, $\ell = I - \eta \otimes \xi$.

3. The unit tangent sphere bundle

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [10], [11], [13]). We only briefly review some notations and definitions. Let $M = (M, g)$ be an $n$-dimensional Riemannian manifold and let $TM$ denote its tangent bundle with the projection $\pi: TM \to M$, $\pi(p, u) = p$. For a vector field $X$ on $M$, its vertical lift $X^v$ on $TM$ is the vector field defined by $X^v \omega = \omega (X) \circ \pi$, where $\omega$ is a 1-form on $M$. For the Levi Civita connection $\nabla$ on $M$, the horizontal lift $X^h$ of $X$ is defined by $X^h \omega = \nabla_X \omega$. The tangent bundle $TM$ can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called Sasaki metric, depending only on the Riemannian metric $g$ on $M$. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields $X$ and $Y$ on $M$. Also, $TM$ admits an almost complex structure tensor $J$ defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure $J$.

The unit tangent sphere bundle $\bar{\pi}: T_1 M \to M$ is a hypersurface of $TM$ given by $g_p(u, u) = 1$. Note that $\bar{\pi} = \pi \circ i$, where $i$ is the immersion.
of $T_1 M$ into $T M$. A unit normal vector field $N = u^v$ to $T_1 M$ is given by the vertical lift of $u$ for $(p, u)$. The horizontal lift of a vector is tangent to $T_1 M$, but the vertical lift of a vector is not tangent to $T_1 M$ in general. So, we define the tangential lift of $X$ to $(p, u) \in T_1 M$ by

$$X^t_{(p,u)} = (X - g(X, u)u)^v.$$ 

Clearly, the tangent space $T^t_{(p,u)} T_1 M$ is spanned by vectors of the form $X^h$ and $X^t$, where $X \in T_p M$.

We now define the standard contact metric structure of unit tangent sphere bundle $T_1 M$ over a Riemannian manifold $(M, g)$. The metric $g'$ on $T_1 M$ is induced from the Sasaki metric $\tilde{g}$ on $T M$. Using the almost complex structure $J$ on $T M$, we define a unit vector field $\xi'$, a 1-form $\eta'$ and a $(1,1)$-tensor field $\phi'$ on $T_1 M$ by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$ 

Since $g'(\tilde{X}, \phi'\tilde{Y}) = 2d\eta'(\tilde{X}, \tilde{Y})$, $(\eta', g', \phi', \xi')$ is not a contact metric structure. If we rescale this structure by

$$\xi = 2\xi', \quad \eta = \frac{1}{2} \eta', \quad \phi = \phi', \quad \tilde{g} = \frac{1}{4} g',$$

we get the standard contact metric structure $(\eta, \tilde{g}, \phi, \xi)$. The tensors $\xi$ and $\phi$ are explicitly given by

$$(5) \quad \xi = 2u^h, \quad \phi^X^t = -X^h + \frac{1}{2} g(X, u)\xi, \quad \phi^X^h = X^t$$

where $X$ and $Y$ are vector fields on $M$. From now on, we consider $T_1 M = (T_1 M, \eta, \tilde{g}, \phi, \xi)$ with the standard contact metric structure. Then we have the following formulas (cf. [2], [3], [4], [7], [12]).

The Levi-Civita connection $\nabla$ of $T_1 M$ is described by

$$\begin{align*}
\nabla_{X^t} Y^t &= -g(Y, u)X^t, \\
\nabla_{X^h} Y^h &= \frac{1}{2} (R(u, X)Y)^h, \\
\nabla_{X^t} Y^h &= (\nabla_X Y)^t + \frac{1}{2} (R(u, Y)X)^h, \\
\nabla_{X^h} Y^t &= (\nabla_X Y)^h - \frac{1}{2} (R(X, Y)u)^t
\end{align*}$$

(6)

for all vector fields $X$ and $Y$ on $M$. 
Also the Riemann curvature tensor \( \bar{R} \) of \( T_1M \) is given by

\[ \bar{R}(X^t, Y^t)Z^t = -(g(X, Z) - g(X, u)g(Z, u))Y^t \]
\[ + (g(Y, Z) - g(Y, u)g(Z, u))X^t, \]
\[ \bar{R}(X^t, Y^t)Z^h = \left\{ R(X - g(X, u)u, Y - g(Y, u)u)Z \right\}^h \]
\[ + \frac{1}{4} \left\{ [R(u, X), R(u, Y)]Z \right\}^h, \]
\[ \bar{R}(X^h, Y^t)Z^t = -\frac{1}{2} \left\{ R(Y - g(Y, u)u, Z - g(Z, u)u)X \right\}^h \]
\[ - \frac{1}{4} \left\{ R(u, Y)R(u, Z)X \right\}^h, \]
\[ \bar{R}(X^h, Y^h)Z^h = \frac{1}{4} \left\{ R(X, Z)(Y - g(Y, u)u) \right\}^t - \frac{1}{4} \left\{ R(X, R(u, Y)Z)u \right\}^t \]
\[ + \frac{1}{2} \left\{ (\nabla_X R)(u, Y)Z \right\}^h, \]
\[ \bar{R}(X^h, Y^h)Z^t = \left\{ R(X, Y)(Z - g(Z, u)u) \right\}^t \]
\[ + \frac{1}{4} \left\{ R(Y, R(u, Z)X)u - R(X, R(u, Z)Y)u \right\}^t \]
\[ + \frac{1}{2} \left\{ (\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X \right\}^h, \]
\[ \bar{R}(X^h, Y^h)Z^h = (R(X, Y)Z)^h + \frac{1}{2} \left\{ R(u, R(X, Y)u)Z \right\}^h \]
\[ - \frac{1}{4} \left\{ R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y \right\}^h \]
\[ + \frac{1}{2} \left\{ (\nabla_X R)(X, Y)u \right\}^t \]

for all vector fields \( X, Y \) and \( Z \) on \( M \). Using the formulae (7), we get

\[ \ell X^t = (R_u^2 X)^t + 2(R_u^t X)^h, \]
\[ \ell X^h = 4(R_u X)^h - 3(R_u^2 X)^h + 2(R_u^t X)^t \]

where \( R_u = R(\cdot, u)u \), \( R_u^t = (\nabla_u R)(\cdot, u)u \) and \( R_u^2 = R(R(\cdot, u)u, u)u \).

4. Proof of Main Theorem

Suppose that \( T_1M \) is pseudo-symmetric. Then we have

\[ \bar{R}(X, Y) \cdot \bar{R}(Z, \bar{V})W = L((X \wedge Y) \cdot \bar{R})(Z, \bar{V})\bar{W} \]
for any vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{W}$ and any smooth function $L$ on $T_{1}M$.

We put $\bar{Y} = \bar{V} = \bar{W} = \bar{\xi}$. Then from (9) we have the following equation:

$$
\bar{R}(\tilde{X}, \bar{\xi})\ell\tilde{Z} - \ell(\bar{R}(\tilde{X}, \bar{\xi})\tilde{Z}) - \bar{R}(\tilde{Z}, \ell\tilde{X})\bar{\xi} - \bar{R}(\bar{\xi}, \ell\tilde{X}) = L\{2\eta(\tilde{X})\ell\tilde{Z} - \eta(\tilde{Z})\ell\tilde{X} - \bar{g}(\tilde{X}, \tilde{Z})\bar{\xi} - \bar{R}(\tilde{Z}, \tilde{X})\tilde{\xi} - \bar{R}(\bar{\xi}, \ell\tilde{X})\tilde{X}\}.
$$

(10)

Setting $\tilde{X} = X^{t}$, $\tilde{Z} = Z^{t}$ in (10), and applying a Riemmanian metric $\bar{g}$ with respect to $Y^{t}$ on both sides, then we have

$$
g(R(X, R^{2}_{u}Z)u, Y) + 4g(X, u)g(R^{3}_{u}Z, Y) + \frac{1}{2}g(R(X, u)R^{3}_{u}Z, Y)
$$

$$
-2g((\nabla_{u}R)(u, X)R^{2}_{u}Z, Y) - 4g(R(X, Z)u, R_{u}Y) + 3g(R(X, Z)u, R^{2}_{u}Y)
$$

$$
+4g(Z, u)g(R^{2}_{u}X, Y) - 6g(Z, u)g(R^{3}_{u}X, Y) - 4g(X, u)g(R^{2}_{u}Z, Y)
$$

$$
+2g(R(u, X)R_{u}Z, R_{u}Y) - \frac{3}{2}g(R(u, X)R_{u}Z, R^{2}_{u}Y) - 3g(Z, R^{2}_{u}X)u, Y)
$$

$$
- g(R(Z, u)R^{2}_{u}X, Y) + \frac{1}{2}g(R(R^{3}_{u}X, u)R_{u}Z, Y) + 2g((\nabla_{R^{2}_{u}X}R)(u, Z)u, Y)
$$

$$
+2g((\nabla_{u}R)(u, Z)R^{2}_{u}X, Y)
$$

$$
= L\{-\frac{1}{2}g(Y, u)g(X, R^{2}_{u}Z) - 3g(R(Z, X)u, Y) + 3g(X, u)g(R_{u}Z, Y)
$$

$$
-3g(Z, u)g(R_{u}X, Y) + \frac{1}{2}g(R(X, u)R_{u}Z, Y) - g(R(Z, u)R_{u}X, Y)\}.
$$

Setting $\tilde{X} = X^{h}$, $\tilde{Z} = Z^{h}$ in (10), and applying a Riemmanian metric $\bar{g}$ with respect to $Y^{t}$ on both sides, then we have

$$
2g(R(X, u)R^{2}_{u}Z, Y) - 2g(Y, u)g(R(X, u)R^{2}_{u}Z, u) + \frac{1}{2}g(R(R^{2}_{u}Z, u)X, R_{u}Y)
$$

$$
- \frac{1}{2}g(R(R^{3}_{u}Z, u)X, Y) + 2g((\nabla_{R^{2}_{u}X}R)(X, u)u, Y) - 2g(R(X, u)Z, R^{2}_{u}Y)
$$

$$
+ 2g(Z, u)g(R^{3}_{u}X, Y) - \frac{1}{2}g(R(Z, u)X, R^{3}_{u}Y) - 2g((\nabla_{X}R)(u, Z)u, R_{u}Y)
$$

$$
+ 2g((\nabla_{u}R)(u, Z)X, R^{2}_{u}Y) - 2g(R(R_{u}Z, R_{u}X)u, Y) + \frac{3}{2}g((R_{u}Z, R^{2}_{u}X)u, Y)
$$

$$
- 2g(R(Z, u)R_{u}X, R_{u}Y) + \frac{3}{2}g(R(Z, u)R^{2}_{u}X, R_{u}Y) + \frac{1}{2}g(R(R_{u}Z, X)u, R^{2}_{u}Y)
$$

$$
= L\{g(X, u)g(R^{2}_{u}Z, Y) - \frac{1}{2}g(R(R_{u}Z, X)u, Y) - \frac{1}{2}g(R(Z, u)X, R_{u}Y)\}.
$$

(12)
Setting $\bar{X} = X^h$, $\bar{Z} = Z^h$ in (10), and applying a Riemannian metric $\bar{g}$ with respect to $Y^h$ on both sides, then we have

$$8g(R(X, u)R_u X, Y) + 4g(R(R_u X)R_u X, Y) - 2g(R(R^2_u X, u)X, Y)$$

$$+ 2g(R(R_u X, Z)u, R_u Y) - 6g(R(X, u)R^2_u X, Y) - 3g(R(X, u)R^2_u Z, Y)$$

$$+ 32g(R(R^3_u X, u)X, Y) - \frac{3}{2}g(R(R^3_u X, u)u, R_u Y) + 2g(\nabla_X R)(u, R_u Z)u, Y)$$

$$- 2g((\nabla_u R)(u, R^2_u Z)X, Y) - 8g(R(X, u)Z, R_u Y) + 6g(R(X, u)Z, R^2_u Y)$$

$$- 4g(R(u, R_u X)Z, R_u Y) + 3g(R(u, R_u X)Z, R^2_u Y) + 2g(R(R_u Z, u)X, R_u Y)$$

$$- 32g(R(R_u Z, u)X, R^2_u Y) + 2g(R(X, Z)u, R^2_u Y) - \frac{3}{2}g(R(X, Z)u, R^2_u Y)$$

$$- 2g((\nabla_Z R)(u, u)R^2_u Y) - 8g(R(Z, R_u X)u, Y) - 6g(R(R_u X, Z)u, R_u Y)$$

$$- 6g(R(R_u Z, R_u X, Y) + 6g(R(Z, R^2_u X)u, Y) + \frac{9}{2}g(R(R^2_u Z, Z)u, R_u Y)$$

$$+ \frac{9}{2}g(R(u, R_u Z)R^2_u Z, Y) - 4g((\nabla_R R)(u, R_u X)u, Y) - 8g(R(Z, u)R_u X, Y)$$

$$+ 6g(R(Z, u)R^2_u Z)Y + 2g((\nabla_u R)(u, R^2_u X)Z, Y)$$

$$= L\{4g(X, u)g(R_u Z, Y) - 3g(X, u)g(R^2_u Z, Y) - 2g(Z, u)g(R_u Z, Y)$$

$$+ \frac{3}{2}g(Z, u)g(R^2_u Z, Y) - 2g(Y, u)g(X, R_u Z) + \frac{3}{2}g(Y, u)g(X, R^2_u Z)$$

$$- 2g(R(X, Z)u, Y) + \frac{3}{2}g(R(Z, X)u, R_u Y) - \frac{3}{2}g(R(u, R_u Z)X, Y)$$

$$- 2g(R(Z, u)X, Y)\}.$$  

Now, we assume that the base manifold $M$ is an $n(\geq 3)$-dimensional locally symmetric space and let $\{u, e_1, e_2, \cdots, e_{n-1}\}$ be an orthonormal basis of $M$ such that $R_u e_i = \lambda_i e_i$ and $R_u e_j = \lambda_j e_j$ for $i, j = 1, 2, \cdots, n - 1$. Putting $X = Z = e_j$, $Y = e_i$ in (11), then we get

$$3\lambda^2_i \lambda_j - 4\lambda_i \lambda_j + \lambda_i \lambda_j g(R(e_i, e_j)u, e_j) = 0.$$  

And putting $X = Y = e_j$, $Z = e_i$ in (12), then we obtain

$$(\lambda^2_i + 4\lambda^2_j - 4\lambda_i \lambda_j + 4\lambda_i \lambda_j - \lambda_i \lambda_j)g(R(e_i, e_j)u, e_j) = 0.$$  

Similarly, putting $X = Y = e_j$, $Z = e_i$ in (13), then we obtain

$$(9\lambda^2_i \lambda_j - 12\lambda^2_j - 12\lambda_i \lambda_j + 16\lambda_i + 3\lambda_j - 4L)g(R(e_i, e_j)u, e_j) = 0.$$  

Here, we suppose that $g(R(e_i, e_j)u, e_j) \neq 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j \in \{1, 2, \cdots, n - 1\}$. Then from (14) $\sim$ (16), we obtain the following equations:

$$3\lambda^2_i \lambda_j - 4\lambda_i \lambda_j + \lambda_i \lambda_j L = 0,$$
From (17), we have \( \lambda_j = 0 \) or \( L = -3\lambda_i^2 + 4\lambda_i \). Then we consider the following three cases:

(I) Let \( \lambda_j = 0 \) and \( L \neq -3\lambda_i^2 + 4\lambda_i \). Then from (19), we obtain \( L = -3\lambda_i^2 + 4\lambda_i \). But, by the assumption, this is a contradiction.

(II) Let \( \lambda_j = 0 \) and \( L = -3\lambda_i^2 + 4\lambda_i \). Then from (18), we have \( \lambda_i(\lambda_i^2 + 4\lambda_i - L) = 0 \). Under the assumption of pseudo-symmetry with \( L \neq 0 \), it follows from \( L = -3\lambda_i^2 + 4\lambda_i \) that \( \lambda_i \neq 0 \). So we have

\[
L = \lambda_i^2 + 4\lambda_i.
\]

From the assumption of case (II) and (20), we get \( \lambda_i = 0 \), which is a contradiction to \( L \neq 0 \).

(III) Let \( \lambda_j \neq 0 \) and \( L = -3\lambda_i^2 + 4\lambda_i \). Then from (18), we have \( \lambda_i^3 - \lambda_i\lambda_j^2 - \lambda_j^2 + \lambda_i\lambda_j = 0 \), that is,

\[
(\lambda_i - \lambda_j)(\lambda_i^2 + \lambda_i\lambda_j + \lambda_j) = 0.
\]

Since \( \lambda_i \neq \lambda_j \), we obtain the equation

\[
\lambda_i^2 + \lambda_i\lambda_j + \lambda_j = 0.
\]

This time, we put \( X = Z = e_i, Y = e_j \) in (11). Then we get

\[
(3\lambda_i\lambda_j^2 - 4\lambda_i\lambda_j + \lambda_iL)g(R(e_j, e_i)u, e_i) = 0.
\]

Put \( X = Y = e_i, Z = e_j \) in (12) to obtain

\[
(\lambda_j^3 + 4\lambda_i^2 - 4\lambda_i^2\lambda_j + 4\lambda_i\lambda_j - \lambda_jL)g(R(e_j, e_i)u, e_i) = 0.
\]

Put \( X = Y = e_i, Z = e_j \) in (13) to obtain

\[
(9\lambda_i\lambda_j^2 - 12\lambda_i^2 - 12\lambda_i\lambda_j + 16\lambda_j + 3\lambda_iL - 4L)g(R(e_j, e_i)u, e_i) = 0.
\]

From (22) ~ (24), we have the following equations:

\[
3\lambda_i\lambda_j^2 - 4\lambda_i\lambda_j + \lambda_iL = 0,
\]

\[
\lambda_j^3 + 4\lambda_i^2 - 4\lambda_i^2\lambda_j + 4\lambda_i\lambda_j - \lambda_jL = 0,
\]

\[
9\lambda_i\lambda_j^2 - 12\lambda_i^2 - 12\lambda_i\lambda_j + 16\lambda_j + 3\lambda_iL - 4L = 0.
\]

From (25), we have \( \lambda_i = 0 \) or \( L = -3\lambda_j^2 + 4\lambda_j \). Similarly, we consider the three cases, and then we see that all the cases yield contradiction. For example, in the case \( \lambda_i \neq 0 \) and \( L = -3\lambda_i^2 + 4\lambda_j \), from (26) we obtain

\[
\lambda_i^2 + \lambda_i\lambda_j + \lambda_i = 0.
\]
Since $\lambda_i \neq \lambda_j$, from (21) and (28), we have
\begin{equation}
(29) \quad \lambda_i + \lambda_j - 1 = 0.
\end{equation}
Also, from $L = -3\lambda_i^2 + 4\lambda_i$ and $L = -3\lambda_j^2 + 4\lambda_j$, we have
\begin{equation}
(30) \quad 3\lambda_i + 3\lambda_j - 4 = 0.
\end{equation}
But, we see that $\lambda_i$ and $\lambda_j$ satisfying (29) and (30) do not exist. After all, we find that $M$ should satisfy $g(R(e_i,e_j)u,e_j) = 0$ or $\lambda_i = \lambda_j$. If $\lambda_i = \lambda_j$, the Jacobi operator $R_u$ for an arbitrary $u \in T_1M$ has only one eigenvalue. Thus $M$ is of constant curvature. And if $M$ satisfies $g(R(e_i,e_j)u,e_j) = 0$, $M$ must have constant sectional curvature when $\text{dim } M \geq 3$, by Cartan’s theorem ([6]). Therefore, we have completed the proof of Main Theorem.

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