ON WEYL-REDUCIBLE CONFORMAL MANIFOLDS
AND LCK STRUCTURES

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Abstract. A recent result of M. Kourganoff states that if $D$ is a closed, reducible, non-flat Weyl connection on a compact conformal manifold $M$, then the universal cover of $M$, endowed with the metric whose Levi-Civita covariant derivative is the pull-back of $D$, is isometric to $\mathbb{R}^q \times N$ for some irreducible, incomplete Riemannian manifold $N$. Moreover, he characterized the case where the dimension of $N$ is 2 by showing that $M$ is then a mapping torus of some Anosov diffeomorphism of the torus $\mathbb{T}^{q+1}$. We show that in this case one necessarily has $q = 1$ or $q = 2$.

1. WEYL-REDUCIBLE MANIFOLDS

Let $(M, c)$ be a compact conformal manifold. A Weyl structure on $M$ is a torsion-free linear connection $D$ preserving the conformal structure $c$, in the sense that for every Riemannian metric $g \in c$, $D_X g = \theta_g(X)g$ for some 1-form $\theta_g$ on $M$ called the Lee form of $D$ with respect to $g$. If $g' := e^f g$ is another metric in the conformal class, then

$$\theta_{g'} = \theta_g + df.$$ 

The Weyl structure $D$ is called closed if $\theta_g$ is closed for one (and thus all) metrics $g \in c$ and exact if $\theta_g$ is exact for all $g \in c$. From the above formula we see that if $D$ is exact, so that $\theta_g = df$ for some $g \in c$, then $\theta_{e^{-f} g} = 0$, thus $D$ is the Levi-Civita connection of the metric $e^{-f} g \in c$.

The manifold $(M, c, D)$ is called Weyl-reducible if the Weyl structure $D$ is reducible and non-flat.

Based on some evidence given by the Gallot theorem on Riemannian cones [4], it was conjectured in [2] that every closed, non-exact Weyl structure on a compact conformal manifold is either irreducible or flat. Matveev and Nikolayevsky [7] constructed a counterexample to the general conjecture, but later on Kourganoff proved that a weaker form of this conjecture holds:
Theorem 1. (cf. [6, Thm. 1.5]). A closed non-exact Weyl structure \( D \) on a compact conformal manifold \( M \), is either flat or irreducible, or the universal cover \( \tilde{M} \) of \( M \) together with the Riemannian metric \( g_D \) whose Levi-Civita connection is \( D \), is the Riemannian product of a complete flat space \( \mathbb{R}^q \) and an incomplete Riemannian manifold \( (N, g_N) \) with irreducible holonomy:

\[
(\tilde{M}, g_D) = \mathbb{R}^q \times (N, g_N).
\]

In [6, Example 1.6] (see also [7]), examples of closed reducible Weyl structures on compact manifolds are constructed using a linear map \( A \in \text{SL}_{q+1}(\mathbb{Z}) \), such that:

1. there exists an \( A \)-invariant decomposition \( \mathbb{R}^{q+1} = E^s \oplus E^u \) with \( \dim(E^u) = 1 \) and \( A|_{E^u} = \lambda^q \text{Id}_{E^u} \) for some real number \( \lambda > 1 \);
2. there exists a positive definite symmetric bilinear form \( b \) on \( E^s \), such that \( \lambda A|_{E^s} \) is orthogonal with respect to \( b \).

Then \( A \) induces a diffeomorphism (also denoted by \( A \)) of the torus \( \mathbb{T}^{q+1} \), whose mapping torus \( M_A := \mathbb{T}^{q+1} \times (0, \infty)/(x, t) \sim (Ax, 1/\lambda t) \), carries a reducible non-flat closed Weyl structure \( D_\phi \) obtained by projecting to \( M_A \) the Levi-Civita connection of the metric on \( \mathbb{T}^{q+1} \times (0, \infty) \) given by:

\[
g_\phi := dx_1^2 + \cdots + dx_q^2 + \varphi(t)dx_{q+1}^2 + dt^2,
\]

where \( x_1, \ldots, x_{q+1} \) are the local coordinates with respect to an orthonormal basis \( (e_1, \ldots, e_{q+1}) \) with \( e_1, \ldots, e_q \in E^s \), \( e_{q+1} \in E^u \), and \( \varphi: (0, +\infty) \rightarrow (0, +\infty) \) is any smooth function satisfying \( \varphi(\lambda t) = \lambda^{2q+2}\varphi(t) \) for every \( t \in (0, +\infty) \).

Moreover, Kourganoff proved that these are, up to diffeomorphism, the only examples of Weyl-reducible manifolds when the incomplete factor \( N \) is 2-dimensional:

Theorem 2. [6, Theorem 1.7] Assume that \( D \) is a closed non-exact Weyl structure \( D \) on a compact conformal manifold \( M \) which is neither flat nor irreducible. If the irreducible manifold \( N \) given by Theorem 1 is 2-dimensional, then \((M, D)\) is isomorphic to one of the Riemannian manifolds \((M_A, D_\phi)\).

It turns out, however, that matrices \( A \in \text{SL}_{q+1}(\mathbb{Z}) \) satisfying the conditions (1) and (2) above, only exist for \( q = 1 \) or \( q = 2 \). This is the object of the next section.
2. A number-theoretical result

**Proposition 3.** Let $q \in \mathbb{N}^*$ and $A \in \text{SL}_{q+1}(\mathbb{Z})$, such that there is a direct sum decomposition $\mathbb{R}^{q+1} = E^s \oplus E^u$ invariant by $A$, with $\dim(E^u) = 1$. If there exists a positive definite symmetric bilinear form $b$ on $E^s$ and a real number $\lambda > 1$, such that $\lambda A|_{E^s}$ is orthogonal with respect to $b$, then $q \in \{1, 2\}$.

**Proof.** Let $C$ be a symmetric positive definite matrix, such that $b = \langle C^2, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product. Then the following equivalence holds:

$$\lambda A|_{E^s} \in O(E^s, b) \iff C \cdot (\lambda A|_{E^s}) \cdot C^{-1} \in O(q).$$

In particular, each eigenvalue of $\text{Spec}(\lambda A|_{E^s})$ has modulus 1 and the characteristic polynomial of $A$ denoted by $\mu_A$ is given by:

$$\mu_A(X) = (X - \lambda^q) \prod_{j=1}^{q} \left(X - \frac{z_j}{\lambda}\right),$$

where $z_j$ are complex numbers with $|z_j| = 1$ for all $j \in \{1, \ldots, q\}$, and $\prod_{j=1}^{q} z_j = 1$. Note that $\mu_A$ is irreducible in $\mathbb{Z}[X]$, since if it were a product of two non-constant polynomials with integer coefficients, one of them would have all roots of modulus less than 1, which is impossible. We distinguish the following two cases:

**Case 1.** If $q = 2p$ is even, denoting $\mu_A(X) = \sum_{j=0}^{2p+1} a_j X^j$ with $a_j \in \mathbb{Z}$ and $a_{2p+1} = 1$, $a_0 = -1$, we get

$$\lambda^{2p} + \frac{1}{\lambda} \sum_{j=1}^{2p} z_j = -a_{2p}, \quad \lambda^{-2p} + \lambda \sum_{j=1}^{2p} \frac{1}{z_j} = a_1.$$

This shows that the sum $s := \sum_{j=1}^{2p} z_j$ is real, and since $|z_j| = 1$ for all $j \in \{1, \ldots, 2p\}$, $s$ is also equal to $\sum_{j=1}^{2p} \frac{1}{z_j}$. Eliminating $s$ from the two equations above, yields

$$\lambda^{4p+2} + a_{2p} \lambda^{2p+2} + a_1 \lambda^{2p} - 1 = 0.$$

Consequently, $\lambda^2$ is root of the polynomial

$$Q(X) := X^{2p+1} + a_{2p} X^{p+1} + a_1 X^p - 1.$$

Denote by $r_1, \ldots, r_{2p}$ the other complex roots of $Q$. Newton’s relations show that there exists a monic polynomial $\tilde{Q} \in \mathbb{Z}[X]$ whose roots are $\lambda^2 r_1, \ldots, \lambda^2 r_{2p}$. The monic polynomials $\mu_A$ and $\tilde{Q} \in \mathbb{Z}[X]$ have both degree $2p + 1$ and $\lambda^{2p}$ is a common root. Since $\mu_A$ is irreducible, they
must coincide, so up to a permutation, one can assume that \( r_j = \frac{z_j}{\lambda} \) for all \( j \in \{1, \ldots, 2p\} \). This shows that \( \lambda^\frac{1}{p} r_j \) are complex numbers of modulus one for all \( j \in \{1, \ldots, 2p\} \).

If \( p \geq 2 \), the coefficients of \( X^{2p} \) and \( X \) in the polynomial \( Q \) vanish, so

\[
\lambda^2 + \sum_{j=1}^{2p} r_j = 0 = \frac{1}{\lambda^2} + \sum_{j=1}^{2p} \frac{1}{r_j}.
\]

Thus \( \sum_{j=1}^{2p} r_j = -\lambda^2 \) and as \( |\lambda^\frac{1}{p} r_j| = 1 \) for all \( j \),

\[
-\lambda^{-2} = \sum_{j=1}^{2p} \frac{1}{r_j} = \lambda^\frac{1}{p} \sum_{j=1}^{2p} r_j = -\lambda^\frac{1}{p} \lambda^2.
\]

This contradicts the fact that \( \lambda > 1 \), showing that \( p = 1 \) and therefore \( q = 2 \) (see also [1, Lemma 3.5]).

**Case 2.** If \( q \) is odd, then \( \mu_A \) has at least one further real root, so either \( \frac{1}{\lambda} \) or \( -\frac{1}{\lambda} \) is a root of \( \mu_A \). Up to reordering the subscripts one thus has \( z_1 = \pm 1 \). Assume that \( z_1 = 1 \) (the argument for \( z_1 = -1 \) is the same). The monic polynomial \( P \in \mathbb{Z}[X] \) defined by \( P(X) := X^{q+1} \mu_A(\frac{1}{X}) \) satisfies \( P(0) = 1 \), and its roots are \( \{\lambda^{-q}, \lambda, \frac{\lambda}{z_2}, \ldots, \frac{\lambda}{z_q}\} \).

By Newton’s identities again, there exists a monic polynomial \( \tilde{P} \in \mathbb{Z}[X] \) with \( \tilde{P}(0) = 1 \), whose roots are \( \{\lambda^{-q}, \lambda^q, (\frac{\lambda}{z_2})^q, \ldots, (\frac{\lambda}{z_q})^q\} \).

Since the monic polynomials \( \mu_A \) and \( \tilde{P} \in \mathbb{Z}[X] \) (of same degree) have \( \lambda^q \) as common root, and \( \mu_A \) is irreducible, they must coincide. In particular \( \lambda^{-q^2} \) is a root of \( \mu_A \). On the other hand every root of \( \mu_A \) has complex modulus equal to either \( \lambda^q \) or \( \frac{1}{\lambda} \). Since \( \lambda > 1 \), we obtain \( q = 1 \).

**Remark 4.** As pointed out by V. Vuletescu, for odd \( q \), Proposition 3 also follows from a more general result of Ferguson [3], whose proof, however, is rather involved.

### 3. Applications

Our main application concerns locally conformally Kähler manifolds. Recall that a Hermitian manifold \((M, g, J)\) of complex dimension \( n \geq 2 \) is called **locally conformally Kähler** (in short, lcK) if around every point in \( M \) the metric \( g \) can be conformally rescaled to a Kähler metric. This condition is equivalent to the existence of a closed 1-form \( \theta \), such that \( d\Omega = \theta \wedge \Omega \).
where $\Omega := g(J\cdot, \cdot)$ denotes the fundamental 2-form. Let now $\widetilde{M}$ be the universal cover of an lcK manifold $(M, J, g, \theta)$, endowed with the pull-back lcK structure $(\widetilde{J}, \widetilde{g}, \widetilde{\theta})$. Since $\widetilde{M}$ is simply connected, $\widetilde{\theta}$ is exact, i.e., $\widetilde{\theta} = d\varphi$, and by the above considerations, the metric $g^K := e^{-\varphi}\widetilde{g}$ is Kähler.

The group $\pi_1(M)$ acts on $(\widetilde{M}, \widetilde{J}, g^K)$ by holomorphic homotheties. Furthermore, we assume that the lcK structure is strict, in the sense that $\pi_1(M)$ is not a subgroup of the isometry group of $(\widetilde{M}, g^K)$. In particular, the Levi-Civita connection of the Kähler metric $g^K$ projects to a closed, non-exact Weyl structure on $M$, called the standard Weyl structure. Its Lee form with respect to $g$ is exactly $\theta$.

Due to the fact that the real dimension of an lcK manifold is even, applying Proposition 3 to the special case of a compact strict lcK manifold whose standard Weyl structure is reducible, we obtain the following:

**Proposition 5.** Let $M$ be a compact Weyl-reducible strict lcK manifold. If the irreducible factor $N$ in the splitting of the universal cover $(\widetilde{M}, g^K)$ as a Riemannian product $\mathbb{R}^q \times N$ given by Theorem 7 is 2-dimensional, then $q = 2$ and thus $M$ is an Inoue surface $S^0$, cf. [5].

Let us remark that if in Proposition 5 we drop the assumption on the dimension of the irreducible factor, then there are many more examples of Weyl-reducible lcK structures. They are obtained on lcK manifolds constructed by Oeljeklaus and Toma [9], for every integer $s \geq 1$, on certain compact quotients $M_\Gamma$ of $\mathbb{C} \times \mathbb{H}^s$, where $\mathbb{H}$ denotes the upper complex half-plane, $\Gamma$ are certain groups whose action on $\mathbb{C} \times \mathbb{H}^s$ is cocompact and properly discontinuous (for the precise definition of $\Gamma$ and its action see [9]). We will briefly review them here.

In order to define the lcK structure on the quotient $M_\Gamma$, Oeljeklaus and Toma consider the function

$$F : \mathbb{C} \times \mathbb{H}^s \to \mathbb{R}, \quad F(z, z_1, \ldots, z_s) := |z|^2 + \frac{1}{y_1 \cdots y_s},$$

with $z_k = x_k + iy_k$ and claim that it is a global Kähler potential on $\mathbb{C} \times \mathbb{H}^s$ (note a small sign error in [9]). To check this, we introduce

$$u : \mathbb{H}^s \to \mathbb{R}, \quad u(z_1, \ldots, z_s) := \frac{1}{y_1 \cdots y_s} = \frac{(2i)^s}{\prod_{j=1}^s (z_j - \bar{z}_j)},$$

and compute

$$\bar{\partial}u = u \sum_{j=1}^s \frac{d\bar{z}_j}{z_j - \bar{z}_j}, \quad \partial u = -u \sum_{j=1}^s \frac{dz_j}{z_j - \bar{z}_j}.$$
\[ \partial \bar{\partial} u = \partial u \wedge \sum_{j=1}^{s} \frac{dz_j}{z_j - \bar{z}_j} - u \sum_{j=1}^{s} \frac{dz_j \wedge d\bar{z}_j}{(z_j - \bar{z}_j)^2} \]

\[ = -u \sum_{j,k=1}^{s} \frac{1 + \delta_{jk}}{(z_j - \bar{z}_j)(z_k - \bar{z}_k)} dz_j \wedge d\bar{z}_k, \]

whence

\[ (2) \quad \partial \bar{\partial} u = \frac{u}{4} \sum_{j,k=1}^{s} \frac{1 + \delta_{jk}}{y_j y_k} dz_j \wedge d\bar{z}_k. \]

This shows that \( i \partial \bar{\partial} u \) is the fundamental 2-form of a Kähler metric \( h \) on \( \mathbb{H}^s \) whose coefficients are \( h_{j\bar{k}} = \frac{u}{4} \frac{1 + \delta_{jk}}{y_j y_k} \).

**Proposition 6.** The Kähler metric on \( \mathbb{H}^s \) with Kähler potential \( u \) is irreducible.

**Proof.** The matrix \((h_{j\bar{k}})\) can be written as the product of 3 matrices

\[
(h_{j\bar{k}}) = \frac{u}{4} \begin{pmatrix}
\frac{1}{y_1} & 0 & \ldots & 0 \\
0 & \frac{1}{y_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{y_s}
\end{pmatrix}
\begin{pmatrix}
2 & 1 & \ldots & 1 \\
1 & 2 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 2
\end{pmatrix}
\begin{pmatrix}
\frac{1}{y_1} & 0 & \ldots & 0 \\
0 & \frac{1}{y_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{y_s}
\end{pmatrix},
\]

so its determinant equals

\[
\det(h_{j\bar{k}}) = \left(\frac{u}{4}\right)^s (s + 1) \frac{1}{(y_1 \ldots y_s)^2} = \frac{(s + 1)u^{s+2}}{4^s}.
\]

The usual formula for the Ricci form \( \rho \) of \( h \) (cf. e.g. [8, Eq. (12.6)]) together with (1) and (2) gives

\[
\rho = -i \partial \bar{\partial} \ln(\det(h_{j\bar{k}})) = -i(s + 2) \partial \bar{\partial} \ln(u) = -i(s + 2) \partial \left( \frac{1}{u} \partial \bar{\partial} u \right)
\]

\[
= -i(s + 2) \left( \frac{1}{u} \partial \bar{\partial} u - \frac{1}{u^2} \partial u \wedge \bar{\partial} u \right)
\]

\[
= -i(s + 2) \sum_{j,k=1}^{s} \frac{2 + \delta_{jk}}{y_j y_k} dz_j \wedge d\bar{z}_k.
\]

This shows that the Ricci tensor of \( h \) is negative definite on \( \mathbb{H}^s \), so \( h \) is irreducible. \( \square \)

As a consequence of Proposition 6, the Kähler metric on \( \mathbb{C} \times \mathbb{H}^s \) with fundamental 2-form \( \Omega = i \partial \bar{\partial} F = i dz \wedge d\bar{z} + i \partial \bar{\partial} u \) is the product of the flat metric on \( \mathbb{C} \) with an irreducible Kähler metric on \( \mathbb{H}^s \). Therefore, the
induced lcK structure on the compact quotient $M_\Gamma$ is Weyl-reducible, and the irreducible factor of the universal cover given by Theorem 1 is exactly $N = \mathbb{H}^s$, so it has dimension $2s$.

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