Review Article

On the study of flow between unsteady squeezing rotating discs with cross diffusion effects under the influence of variable magnetic field

Rehan Ali Shah, Aamir Khan∗, Muhammad Shuaib

Abstract

The aim of this article is to provide an analytical and numerical investigation to the viscous fluid flow, heat and mass transfer under the influence of a variable magnetic field. The governing system of partial differential equations are transformed by means of similarity transformations to a system of ordinary differential equations which are solved by Homotopy Analysis Method (HAM) and BVP4c. The effects of involved physical parameters are illustrated for the velocity components, magnetic field components, heat and mass transfers. Authentication of HAM results for various involved physical parameters are supported by comparison with numerical results obtained by BVP4c. It is observed that increasing distance between discs increase pressure on lower disc and torque on upper disc. It is also observed that increase in axial component of magnetic field increase fluid’s axial velocity and increase in magnetic Reynold’s number decrease magnetic flux it lower disc. Heat flux from lower to upper disc is increased by increase in Dufour number.

Keywords: Applied mathematics
1. Introduction

Squeeze flow is a type of flow in which a fluid is compressed between two parallel discs approaching each other. The unsteady squeeze flow between two discs in motion is regarded as one of the most important research topics due to its scientific and engineering applications such as compression and injection moulding, blood flow due to expansion and contraction of vessels, movable pistons in engines, hydraulic brakes, lubrication and material processing, cooling towers etc.

Von Karman [1] was the first to introduce the famous transformations and reduced the Navier–Stokes equations to a fourth order ordinary differential equations which were solved by approximate integral method. Lance et al. [2] used shooting techniques and obtained the numerical solutions of Navier–Stokes equations by using the similarity transformations of Von Karman [1]. Holodniok [3] solved the same problem using finite difference method and Newton’s iteration for obtaining solutions at higher Reynolds number. Mustafa et al. [4] studied heat and mass transfer characteristics in a viscous fluid which is squeezed between parallel plates. They found that the magnitude of local Nusselt number is an increasing function of Prandtl and Eckert number.

To understand the magneto-hydrodynamics (MHD) flow it is strongly related to the study of physical effects which take place in MHD. According to Lenz’s law motion of a conductor into a magnetic field, electric current is induced in the conductor and creates its own magnetic field. When currents are induced by motion of a conducting fluid through a magnetic field a Lorentz force acts on the fluid and modifies its motion. In MHD, the motion modifies the field and vice versa. This makes the theory highly non-linear [5, 6]. Hughes et al. [7] studied squeezed flow of an electrically conducting fluid between two discs in the presence of a magnetic field. It has been shown that the load capacity of the normal force which the fluid exerts on the upper disc is dependent on the magnetohydrodynamics interactions in the fluid. Verma [8] studied the squeeze film lubrication of a magnetic fluid between two approaching surfaces in the presence of an externally applied magnetic field. He assumed that the applied magnetic field $M$ has components of the form:

$$M_x = M(x) \cos \theta, \quad M_y = M(x) \sin \theta, \quad M_z = 0 \quad \text{where} \quad \theta = \theta(x, y).$$

The Navier–Stokes equation with applied magnetic field is also solved by Siddiqui et al. [9]. They reduced the governing equations to a fourth order coupled ordinary differential equations. Approximate solutions are obtained up to first order by assuming that the flow is symmetric at $y = 0$ and meets no slip conditions at the upper plate. Kuzma et al. [10] studied MHD squeeze films both theoretically and experimentally taking into account the fluid inertia effects and buoyant forces. Excellent agreement has been obtained between theory and experiment. Krieger
et al. [11] studied the MHD lubrication flow between parallel stationary disks in an axial magnetic field. He also obtained excellent agreement between theory and experimental results until the transition to turbulent flow occurred. Sheikholislami [26, 27] studied the effect of magnetic field on nanofluid flow and concluded that nanoparticles can change the thermal behavior of nanofluids. Effect of magnetic field on fluid flow is also studied by [22, 23, 24, 25]. Acharya et al. (2016) studied the squeezing flow of two types of nanofluids such as Cu water and Cu-kerosene between parallel plates in the presence of variable magnetic field. The non-linear differential equations are solved numerically by RK-4 method with shooting technique and analytically using differential transformation method (DTM). Saidi et al. (2016) investigated the unsteady three dimensional nanofluid flow, heat and mass transfer in a rotating system in the presence of an externally applied magnetic field. Muhyyud-Din (2015) studied the MHD flow of a viscous incompressible fluid between parallel discs. The governing system of equations are solved by using variational iteration method (VIM). Hayat et al. (2016) studied the squeezing flow of nanofluid between two parallel plates in the presence of magnetic field. They neglected the induced magnetic field for small magnetic Reynolds number and concluded that temperature of fluid is decreasing for large values of squeezing parameter. Nabhani et al. (2016) numerically investigated the squeezing film between two porous circular discs in the presence of an externally applied magnetic field. An implicit finite difference scheme is used to discretized the governing unsteady nonlinear equation and solve the new system of equations by Gauss–Seidal method. The radiation effect on squeezing flow between parallel discs is studied by Muhyyud-Din et al. (2016). Homotopy Analysis Method is employed to obtained the expression for velocity and temperature profiles. They have concluded opposite behavior of the velocity profile for suction and injection of fluid for all involved parameters. Ganji et al. (2015) investigated the heat transfer of nanofluid flow between parallel plates in presence of variable magnetic field. Using homotopy perturbation method they concluded that Nusselt number has direct relationship with Brownian motion parameter.

Existing information on the topic witnessed that the flow between unsteady squeezing rotating discs with cross diffusion effects under the influence of variable magnetic field in polar coordinates has never been reported and is the very first study in the literature. The present paper is time dependent squeezed flow between two parallel discs which at time $t = 0^+$ are spaced at a distance $d(t) = D(1 - \delta t)^{0.5}$, where $D$ and $\delta^{-1}$ denotes representative length and time respectively [12, 13, 14]. The lower disc situated at $z = 0$ is fixed/stationary and porous while upper disc at $z = d(t)$ is movable and rotating with angular velocity proportional to $\Omega(1 - \delta t)^{-1}$, where $\Omega$ represents the angular velocity with dimension $t^{-1}$ [12, 14, 15]. The fluid is also under the influence of an external applied magnetic field $M$, which will give rise to an induced magnetic field $B(r, z, t)$ with components $B_r$, $B_\theta$ and $B_z$ between discs. The model problem has been solved by HAM proposed by Liao [16] and by
In the following sections, the problem is formulated, analyzed and discussed through graphs and tables.

2. Calculation

An axisymmetric flow of an incompressible viscous fluid is considered between two parallel discs separated at a distance of \( d(t) = D(1-\delta t)^{0.5} \), where \( D \) is representative length equivalent to the disc separation at \( t = 0 \). The upper disc is rotating with an angular velocity \( \Omega \) and is moving towards or away from the fixed lower disc. Also the upper disc is under the influence of an external applied magnetic field \( M \) with radial, tangential and axial components;

\[
M_r = \alpha r M_1 (1-\delta t)^{-1}, \quad M_\theta = r M_2 (1-\delta t)^{-1}, \quad M_z = -\alpha M_1 (1-\delta t)^{-0.5},
\]

where \( M_1 = \frac{N_o}{\mu_2} \) and \( M_2 = \frac{M_o}{\mu_1} \) such that \( N_o \) and \( M_o \) are dimensional quantities used to make \( M_r, M_\theta \) and \( M_z \) dimensionless and \( \mu_1, \mu_2 \) the magnetic permeabilities of the media outside and inside the two discs respectively. If \( \mu_o \) is the permeability of free space then for liquid metal \( \mu_o = \mu_2 \) [11]. On the lower disc \( M_\theta \) and \( M_z \) are assumed to be zero [18]. The external applied magnetic field \( M \) will produce an induced magnetic field \( B(r, z, t) \) with components \( B_r, B_\theta, B_z \). The cylindrical coordinates \( (r, \theta, z) \) with the origin fixed at the center of lower disc is used. The upper and lower discs are maintained at constant temperature \( T_u \) and \( T_l \) respectively.

The unsteady incompressible and axisymmetric equations of continuity, momentum, magnetic field, energy and transport are [19, 20, 21, 22],

\[
\nabla \cdot U = 0, \quad (1)
\]
\[
\nabla \cdot B = 0, \quad (2)
\]
\[ \rho \left[ \frac{\partial U}{\partial t} + (U \cdot \nabla)U \right] + \nabla p - \mu \nabla^2 U - \frac{1}{\mu_2} \left[ (\nabla \times B) \times B \right] = 0, \quad (3) \]
\[ \frac{\partial B}{\partial t} - \nabla \times (V \times B) - \frac{1}{\sigma \mu_2} \nabla^2 B = 0, \quad (4) \]
\[ \rho(V \cdot \nabla)T - \frac{1}{C_p} \nabla(kV)T - \frac{1}{C_p} \nabla q_r - \frac{Dk_T}{C_s C_p} \nabla \cdot (\nabla C) = 0, \quad (5) \]
\[ \rho(V \cdot \nabla)C - \nabla \cdot (D \nabla)C - \frac{Dk_T}{C_s T_m} \nabla \cdot (\nabla T) = 0. \quad (6) \]

**Boundary conditions**

The boundary conditions for the squeezing flow under consideration is given by

\[ u_r = 0, \quad u_\theta = \Omega r, \quad u_z = \frac{d(d(t))}{dt}, \quad B_r = ar M_2 (1 - \delta t)^{-1}, \quad B_\theta = r M_1 (1 - \delta t)^{-1}, \]
\[ B_z = -\alpha M_2 (1 - \delta t)^{-0.5} \]
\[ T = T_w, \quad C = C_w \quad \text{at} \quad z = d(t), \]
\[ u_r = u_\theta = B_\theta = B_z = 0, \quad u_z = -w_0 (1 - \delta t)^{-0.5}, \quad T = T_w, \quad C = C_w \quad \text{at} \quad z = 0. \quad (7) \]

**Velocity and magnetic field:**

\[ V = V(u_r, u_\theta, u_z), \quad u_r = (r, \theta, z), \quad u_\theta = (r, \theta, z), \quad u_z = (r, \theta, z) \]
\[ B_r = (r, z, t), \quad B_\theta = (r, z, t), \quad B_z = (r, z, t) \quad (8) \]

where \( u_r, u_\theta \) and \( u_z \) are the velocity components along radial, tangential and axial directions respectively, \( \rho \) is the pressure, \( T \) is temperature, \( C \) is concentration, \( \alpha \) is thermal diffusivity, \( \rho \) is fluid density, \( D \) is diffusion coefficient, \( T_m \) is mean fluid temperature, \( T_i \) and \( C_i \) denotes the temperature and concentration at the lower disc while \( T_u \) and \( C_u \) are temperature and concentration at upper disc respectively, \( v \) is kinematic viscosity, \( \sigma \) is electrical conductivity, \( c_p \) is specific heat at constant pressure, \( \kappa \) is conductivity, \( \kappa_T \) is thermal diffusion ratio, \( T_m \) is mean fluid temperature, \( \sigma_r \) is Stefan–Boltzmann constant, \( \kappa_r \) is the mean absorption co-efficient and \( q_r \) is the radiative heat flux such that \( \nabla q_r = \frac{16\sigma T_u^4}{3\kappa_r} \frac{\partial T^2}{\partial z^2} \).

To reduce the partial differential Eqs. (1), (2), (3), (4), (5) and (6) to a system of ordinary differential equations, we use the following similarity transformations [17].

\[ u_r = 0.5 \delta r (1 - \delta t)^{-1} f^\prime(\eta), \quad u_\theta = \Omega r (1 - \delta t)^{-1} g(\eta), \]
\[ u_z = -\delta D (1 - \delta t)^{-0.5} f(\eta) \]
\[ B_r = 0.5 \delta r D^{-1} (1 - \delta t)^{-1} m'(\eta), \quad B_\theta = r N_0 (1 - \delta t)^{-0.5} n(\eta), \]
\[ B_z = -\delta M_2 (1 - \delta t)^{-0.5} m(\eta), \]
\[ \theta = \frac{T - T_u}{T_i - T_u}, \quad \phi = \frac{C - C_u}{C_i - C_u}, \quad \eta = z D^{-1} (1 - \delta t)^{-0.5} \quad (9) \]
The Equation of continuity is identically satisfied and the Momentum, Magnetic field, Energy and Transport equations take the following form

\[
S_q^2 \frac{d^4 f}{d\eta^4} - S_q^3 \left[ \eta \frac{d^3 f}{d\eta^3} + 3 \frac{d^2 f}{d\eta^2} - 2 f \frac{d^3 f}{d\eta^3} + 2 N_z^2 R_{em} \left( \eta \frac{d^2 m}{d\eta^2} + m \frac{d m}{d\eta} + 2 m^2 \frac{d^2 f}{d\eta^2} \right) - 2 m f \frac{d^2 m}{d\eta^2} \right] + 2 N_z^2 \left( g \frac{d g}{d\eta} - N_\theta^2 \eta \frac{d n}{d\eta} \right) = 0
\]  
(10)

\[
\frac{d^2 g}{d\eta^2} - S_q \left( 2g + \eta \frac{d g}{d\eta} + 2g \frac{d f}{d\eta} - 2 f \frac{d g}{d\eta} \right) + 2 N_z N_\theta \left( m \frac{d n}{d\eta} - n \frac{d m}{d\eta} \right) = 0
\]  
(11)

\[
\frac{d^2 m}{d\eta^2} - R_{em} \left( m + \eta \frac{d m}{d\eta} - 2 f \frac{d m}{d\eta} + 2 m \frac{d f}{d\eta} \right) = 0
\]  
(12)

\[
N_\theta \frac{d^2 n}{d\eta^2} - N_\theta R_{em} \left( 2n + \eta \frac{d n}{d\eta} - 2 f \frac{d n}{d\eta} \right) - 2 N_z m \frac{d g}{d\eta} = 0
\]  
(13)

\[
(3 R_d + 4) \frac{d^2 \theta}{d\eta^2} + 3 R_d D_u \frac{d^2 \phi}{d\eta^2} - 6 R_d P_s S_f \frac{d \theta}{d\eta} = 0
\]  
(14)

\[
\frac{d^2 \phi}{d\eta^2} + S_s S_f \frac{d^2 \theta}{d\eta^2} + 2 S_s S_f \frac{d \phi}{d\eta} = 0
\]  
(15)

and the boundary conditions are reduced to

\[
f(0) = A, \quad f'(0) = 0, \quad g(0) = 0, \quad \theta(0) = 1, \quad \phi(0) = 1, \quad m(0) = 0, \quad n(0) = 0
\]

\[
f(1) = 0.5, \quad f'(1) = 0, \quad g(1) = \Omega, \quad \theta(1) = 0, \quad \phi(1) = 0, \quad m(1) = 1, \quad n(1) = 1
\]  
(16)

where \( S_q = \frac{\delta D^2}{2\nu} \) is the squeeze Reynolds number, \( N_r = \frac{\Omega D^2}{\nu} \) the rotational Reynolds number, \( B_t = \sigma \mu_2 \nu \) the Bachelor number, \( R_{em} = S_q B_t \) the magnetic Reynolds number, \( N_z = \frac{M_s}{D_s \sqrt{\mu_2 \rho}} \) the strength of magnetic field in z direction, \( N_\theta = \frac{N_\theta}{\Omega v \sqrt{\mu_2 \rho}} \) the strength of magnetic field in \( \theta \) direction, \( A = \frac{\nu D}{\delta D} \) the suction/injection parameter, \( P_s = \frac{\nu}{\delta} \) the Prandtl number, \( S_s = \frac{v}{D} \) the Schmidt number, \( R_d = \frac{\kappa_s \kappa}{4 \sigma T_k} \) the radiation parameter, \( S_o = \frac{D(T_l - T_u)}{v T_k (C_l - C_u)} \) the Soret number and \( D_u = \frac{D(G_l - C_l)}{C_l v \nu (T_l - T_u)} \) is the Dufour number.

### 3. Analysis

The analytic method HAM is used to solve system of Eqs. (9), (10), (11), (12), (13), (14) and (15). Due to HAM, the functions \( f(\eta), g(\eta), m(\eta), n(\eta), \theta(\eta) \) and \( \phi(\eta) \) can be expressed, by a set of base functions \( \eta^c, c \geq 0 \) as:

\[
f_m(\eta) = \sum_{\zeta=0}^{\infty} a_{\zeta} \eta^\zeta, \quad (17)
\]

\[
g_m(\eta) = \sum_{\zeta=0}^{\infty} b_{\zeta} \eta^\zeta, \quad (18)
\]
\[ m_m(\eta) = \sum_{\zeta=0}^{\infty} c_\zeta \eta^\zeta, \quad (19) \]
\[ n_m(\eta) = \sum_{\zeta=0}^{\infty} d_\zeta \eta^\zeta, \quad (20) \]
\[ \theta_m(\eta) = \sum_{\zeta=0}^{\infty} e_\zeta \eta^\zeta, \quad (21) \]
\[ \phi_m(\eta) = \sum_{\zeta=0}^{\infty} f_\zeta \eta^\zeta, \quad (22) \]
where \( a_\zeta, b_\zeta, c_\zeta, d_\zeta, e_\zeta \) and \( f_\zeta \) are the constant coefficients to be determined. Initial approximations are chosen as follows:
\[ f_0(\eta) = (2A - 1)\eta^3 - \frac{3}{2}(2A - 1)\eta^2 + A, \quad (23) \]
\[ g_0(\eta) = \Omega \eta, \quad (24) \]
\[ m_0(\eta) = \eta, \quad (25) \]
\[ n_0(\eta) = \eta, \quad (26) \]
\[ \theta_0(\eta) = 1 - \eta, \quad (27) \]
\[ \phi_0(\eta) = 1 - \eta. \quad (28) \]
The auxiliary operators are chosen as
\[ \ell_f = \frac{\partial^4}{\partial \eta^4}, \quad \ell_g = \frac{\partial^2}{\partial \eta^2}, \quad \ell_m = \frac{\partial^2}{\partial \eta^2}, \quad \ell_n = \frac{\partial^2}{\partial \eta^2}, \quad \ell_\theta = \frac{\partial^2}{\partial \eta^2}, \quad \ell_\phi = \frac{\partial^2}{\partial \eta^2}, \quad (29) \]
with the following properties
\[ \ell_f(\zeta_1 \eta^3 + \zeta_2 \eta^2 + \zeta_3 \eta + \zeta_4) = 0, \quad (30) \]
\[ \ell_g(\zeta_5 \eta + \zeta_6) = 0, \quad (31) \]
\[ \ell_m(\zeta_7 \eta + \zeta_8) = 0, \quad (32) \]
\[ \ell_n(\zeta_9 \eta + \zeta_{10}) = 0, \quad (33) \]
\[ \ell_\theta(\zeta_{11} \eta + \zeta_{12}) = 0, \quad (34) \]
\[ \ell_\phi(\zeta_{13} \eta + \zeta_{14}) = 0, \quad (35) \]
where \( \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9, \zeta_{10}, \zeta_{11}, \zeta_{12}, \zeta_{13} \) and \( \zeta_{14} \) are arbitrary constants.

The Zeroth order deformation problems can be obtained as:
\[ (1; \beta) \ell_f [\tilde{f}(\eta; \beta) - f_0(\eta)] = q \hbar f N_f [\tilde{f}(\eta; \beta), \tilde{g}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)], \quad (36) \]
\[ (1; \beta) \ell_g [\tilde{g}(\eta; \beta) - g_0(\eta)] = q \hbar g N_g [\tilde{f}(\eta; \beta), \tilde{g}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)], \quad (37) \]
\[ (1; \beta) \ell_m [\tilde{m}(\eta; \beta) - m_0(\eta)] = q \hbar m N_m [\tilde{f}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)], \quad (38) \]
\[ (1; \beta) \ell_n [\tilde{n}(\eta; \beta) - n_0(\eta)] = q \hbar n N_n [\tilde{f}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)], \quad (39) \]
\[ (1; \beta) \ell_\theta [\tilde{\theta}(\eta; \beta) - \theta_0(\eta)] = q \hbar \theta N_\theta [\tilde{f}(\eta; \beta), \tilde{\theta}(\eta; \beta), \tilde{\phi}(\eta; \beta)]. \quad (40) \]
\[(1; \beta) \mathcal{L}_q [\tilde{\Phi}(\eta; \beta) - \Phi_0(\eta)] = q \mathcal{H}_q N_\phi \{ \tilde{f}(\eta; \beta), \tilde{\theta}(\eta; q), \tilde{\Phi}(\eta; \beta) \}. \tag{41}\]

The nonlinear operators of Eqs. (9), (10), (11), (12), (13) and (14) are defined as

\[
N_f[\tilde{f}(\eta; \beta), \tilde{g}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)] = S_q \left( \frac{\partial^4 \tilde{f}(\eta; \beta)}{\partial \eta^4} - S_q \left( \frac{\partial^3 \tilde{f}(\eta; \beta)}{\partial \eta^3} + 3 \frac{\partial^2 \tilde{f}(\eta; \beta)}{\partial \eta^2} - 2 \tilde{f}(\eta; \beta) \frac{\partial^3 \tilde{f}(\eta; \beta)}{\partial \eta^3} \right) \right.
\nonumber
\left. + 2N_z R_{em} \left( \eta \tilde{m}(\eta; \beta) \frac{\partial^2 \tilde{m}(\eta; \beta)}{\partial \eta^2} + \tilde{m}(\eta; \beta) \frac{\partial \tilde{m}(\eta; \beta)}{\partial \eta} \right) \right]
\nonumber
\left. + 2m^2(\eta; \beta) \frac{\partial^2 \tilde{f}(\eta; \beta)}{\partial \eta^2} - 2m(\eta; \beta) \tilde{f}(\eta; \beta) \frac{\partial^2 \tilde{m}(\eta; \beta)}{\partial \eta^2} \right) \right]
\left. + 2N_z \left( \tilde{g}(\eta; \beta) \frac{\partial \tilde{g}(\eta; \beta)}{\partial \eta} - N_\phi \tilde{n}(\eta; \beta) \frac{\partial \tilde{n}(\eta; \beta)}{\partial \eta} \right), \tag{42}\right]
\]

\[
N_g[\tilde{f}(\eta; \beta), \tilde{g}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)] = \frac{\partial^2 \tilde{g}(\eta; \beta)}{\partial \eta^2} - S_q \left[ 2 \tilde{g}(\eta; \beta) + \eta \frac{\partial \tilde{g}(\eta; \beta)}{\partial \eta} + 2 \tilde{g}(\eta; \beta) \frac{\partial \tilde{f}(\eta; \beta)}{\partial \eta} \right]
\nonumber
\left. - 2 \tilde{f}(\eta; \beta) \frac{\partial \tilde{g}(\eta; \beta)}{\partial \eta} + 2N_z \tilde{N}_\beta \left( \eta \tilde{m}(\eta; \beta) \frac{\partial \tilde{m}(\eta; \beta)}{\partial \eta} + \tilde{m}(\eta; \beta) \frac{\partial \tilde{m}(\eta; \beta)}{\partial \eta} \right) \right], \tag{43}\]

\[
N_m[\tilde{f}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)] = \frac{\partial^2 \tilde{m}(\eta; \beta)}{\partial \eta^2} - R_{em} \left( \tilde{m}(\eta; \beta) + \eta \frac{\partial \tilde{m}(\eta; \beta)}{\partial \eta} - 2 \tilde{f}(\eta; \beta) \frac{\partial \tilde{m}(\eta; \beta)}{\partial \eta} \right)
\nonumber
\left. + 2 \tilde{m}(\eta; \beta) \frac{\partial \tilde{f}(\eta; \beta)}{\partial \eta} \right), \tag{44}\]

\[
N_n[\tilde{f}(\eta; \beta), \tilde{m}(\eta; \beta), \tilde{n}(\eta; \beta)] = N_\beta \frac{\partial^2 \tilde{n}(\eta; \beta)}{\partial \eta^2} - N_\beta R_{em} \left( 2 \tilde{n}(\eta; \beta) + \eta \frac{\partial \tilde{n}(\eta; \beta)}{\partial \eta} - 2 \tilde{f}(\eta; \beta) \frac{\partial \tilde{n}(\eta; \beta)}{\partial \eta} \right)
\nonumber
\left. - 2N_z \tilde{m}(\eta; \beta) \frac{\partial \tilde{g}(\eta; \beta)}{\partial \eta} \right), \tag{45}\]

\[
N_\beta[\tilde{f}(\eta; \beta), \tilde{\theta}(\eta; \beta), \tilde{\Phi}(\eta; \beta)] = \left( 3R_d + 4 \right) \frac{\partial^2 \tilde{\theta}(\eta; \beta)}{\partial \eta^2} + 3R_d S_c \frac{\partial^2 \tilde{\Phi}(\eta; \beta)}{\partial \eta^2} - 6R_d P_s S_q \tilde{f}(\eta; \beta) \frac{\partial \tilde{\Phi}(\eta; \beta)}{\partial \eta}, \tag{46}\right]
\]

\[
N_\phi[\tilde{f}(\eta; \beta), \tilde{\theta}(\eta; \beta), \tilde{\Phi}(\eta; \beta)] = \frac{\partial^2 \tilde{\Phi}(\eta; \beta)}{\partial \eta^2} + S_c S_o \frac{\partial^2 \tilde{\Phi}(\eta; \beta)}{\partial \eta^2} + 2S_c S_q \tilde{f}(\eta; \beta) \frac{\partial \tilde{\Phi}(\eta; \beta)}{\partial \eta}, \tag{47}\]

where \( \beta \) is an embedding parameter, \( h_f, h_g, h_m, h_n, h_\theta \), and \( h_\Phi \) are the nonzero auxiliary parameter and \( N_f, N_g, N_m, N_n, N_\theta \) and \( N_\phi \) are the nonlinear parameters.
For $\beta = 0$ and 1, we have

\begin{align*}
\tilde{f}(\eta, 0) &= f_0(\eta), & \tilde{f}(\eta, 1) &= f(\eta), \\
\tilde{g}(\eta, 0) &= g_0(\eta), & \tilde{g}(\eta, 1) &= g(\eta), \\
\tilde{m}(\eta, 0) &= m_0(\eta), & \tilde{m}(\eta, 1) &= m(\eta), \\
\tilde{n}(\eta, 0) &= n_0(\eta), & \tilde{n}(\eta, 1) &= n(\eta), \\
\tilde{\theta}(\eta, 0) &= \theta_0(\eta), & \tilde{\theta}(\eta, 1) &= \theta(\eta), \\
\tilde{\phi}(\eta, 0) &= \phi_0(\eta), & \tilde{\phi}(\eta, 1) &= \phi(\eta),
\end{align*}

(48)

so we can say that as $\beta$ varies from 0 to 1, $\tilde{f}(\eta, 0), \tilde{g}(\eta, 0), \tilde{m}(\eta, 0), \tilde{n}(\eta, 0), \tilde{\theta}(\eta, 0), \tilde{\phi}(\eta, 0)$ varies from initial guesses $f_0(\eta), g_0(\eta), m_0(\eta), n_0(\eta), \theta_0(\eta)$ and $\phi_0(\eta)$ to exact solution $f(\eta), g(\eta), m(\eta), n(\eta), \theta(\eta)$ and $\phi(\eta)$ respectively.

Taylor’s series expansion of these functions yields:

\begin{align*}
f(\eta; \beta) &= f_0(\eta) + \sum_{m=1}^{\infty} \beta^m f_m(\eta), \\
g(\eta; \beta) &= g_0(\eta) + \sum_{m=1}^{\infty} \beta^m g_m(\eta), \\
m(\eta; \beta) &= m_0(\eta) + \sum_{m=1}^{\infty} \beta^m m_m(\eta), \\
n(\eta; \beta) &= n_0(\eta) + \sum_{m=1}^{\infty} \beta^m n_m(\eta), \\
\theta(\eta; \beta) &= \theta_0(\eta) + \sum_{m=1}^{\infty} \beta^m \theta_m(\eta), \\
\phi(\eta; \beta) &= \phi_0(\eta) + \sum_{m=1}^{\infty} \beta^m \phi_m(\eta),
\end{align*}

(49-54)

\begin{align*}
f_m(\eta) &= \frac{1}{m!} \frac{\partial^m f(\eta; \beta)}{\partial \eta^m} \bigg|_{\beta=0}, \quad g_m(\eta) = \frac{1}{m!} \frac{\partial^m g(\eta; \beta)}{\partial \eta^m} \bigg|_{\beta=0}, \\
m_m(\eta) &= \frac{1}{m!} \frac{\partial^m m(\eta; \beta)}{\partial \eta^m} \bigg|_{\beta=0}, \\
n_m(\eta) &= \frac{1}{m!} \frac{\partial^m n(\eta; \beta)}{\partial \eta^m} \bigg|_{\beta=0}, \\
\theta_m(\eta) &= \frac{1}{m!} \frac{\partial^m \theta(\eta; \beta)}{\partial \eta^m} \bigg|_{\beta=0}, \\
\phi_m(\eta) &= \frac{1}{m!} \frac{\partial^m \phi(\eta; \beta)}{\partial \eta^m} \bigg|_{\beta=0},
\end{align*}

(55)

it should be noted that the convergence of above series strongly depends upon $h_f, h_g, h_m, h_n, h_\theta$ and $h_\phi$.

Assuming that these nonzero auxiliary parameters are chosen so that Eqs. (35), (36), (37), (38), (39) and (40) converges at $\beta = 1$. Therefore one can obtain

\begin{align*}
f(\eta) &= f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \\
g(\eta) &= g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta),
\end{align*}

(56-57)
\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \quad (58) \]
\[ \phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \quad (59) \]

Differentiating the deformation Eqs. (35), (36), (37), (38), (39) and (40) \(m\)-times with respect to \(\beta\) and putting \(\beta = 0\), we have

\[ \frac{\partial}{\partial \eta} [f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_{f,m}(\eta), \quad (60) \]
\[ \frac{\partial}{\partial \eta} [g_m(\eta) - \chi_m g_{m-1}(\eta)] = h_g R_{g,m}(\eta), \quad (61) \]
\[ \frac{\partial}{\partial \eta} [m_m(\eta) - \chi_m m_{m-1}(\eta)] = h_m R_{m,m}(\eta), \quad (62) \]
\[ \frac{\partial}{\partial \eta} [n_m(\eta) - \chi_m n_{m-1}(\eta)] = h_n R_{n,m}(\eta), \quad (63) \]
\[ \frac{\partial}{\partial \eta} [\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R_{\theta,m}(\eta), \quad (64) \]
\[ \frac{\partial}{\partial \eta} [\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = h_\phi R_{\phi,m}(\eta), \quad (65) \]

subject to the boundary conditions

\[ f_m(0) = A, \quad f_m'(0) = 0, \quad g_m(0) = 0, \quad m_m(0) = 0, \quad n_m(0) = 0, \quad (66) \]
\[ \theta_m(0) = 0, \quad \phi_m(0) = 1, \quad \theta_m'(0) = 0, \quad \phi_m'(0) = 0, \]

where

\[ R_{f,m}(\eta) = S_q f_m'''(\eta) - S_q^2 \left[ 3 f_m''(\eta) + \eta f_m'''(\eta) - 2 \sum_{j=0}^{m-1} f_j(\eta) f_{m-j-1}'''' \right. \]
\[ + 2 N_2^2 R_{m,m} \sum_{j=0}^{m-1} m_j(\eta) \left( \eta m_{m-j-1}' + m_{m-j-1}' + 2 m_{m-j-1} f_{m-j-1}'' \right) \]
\[ - 2 f_{m-j-1} m_{m-j-1}''' \right] \]
\[ + 2 N_2^2 \left( \sum_{j=0}^{m-1} g_j(\eta) g_{m-j-1}'(\eta) - N_2^2 \sum_{j=0}^{m-1} n_j(\eta) n_{m-j-1}'(\eta) \right) \]
\[ - 8 \Omega g_m''(\eta) - M^2 f_m''(\eta), \quad (67) \]

\[ R_{g,m}(\eta) = g_m''(\eta) - S_q \left[ \eta g_m'(\eta) + 2 g_m(\eta) f_m'(\eta) \right. \]
\[ - 2 \sum_{j=0}^{m-1} f_j(\eta) g_{m-j-1}'(\eta) \]
\[ + 2 N_2 N_\theta \left( \sum_{j=0}^{m-1} m_j(\eta) n_{m-j-1}'(\eta) - \sum_{j=0}^{m-1} n_j(\eta) m_{m-j-1}'(\eta) \right) \] \]
\[ - 2 \sum_{j=0}^{m-1} f_j(\eta) g_{m-j-1}'(\eta) \]
\[ + 2 N_2 N_\theta \left( \sum_{j=0}^{m-1} m_j(\eta) n_{m-j-1}'(\eta) - \sum_{j=0}^{m-1} n_j(\eta) m_{m-j-1}'(\eta) \right), \quad (68) \]
Finally, and

\[ f_m(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta h_f R_{f,m}(z) d\eta d\eta d\eta d\eta + \chi_m f_{m-1} + \zeta_1 \eta^3 + \zeta_2 \eta^2 + \zeta_3 \eta + \zeta_4, \]

\[ g_m(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta h_g R_{g,m}(z) d\eta d\eta d\eta d\eta + \chi_m g_{m-1} + \zeta_5 \eta + \zeta_6, \]

\[ m_m(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta h_m R_{m,m}(z) d\eta d\eta d\eta d\eta + \chi_m m_{m-1} + \zeta_7 \eta + \zeta_8, \]

\[ n_m(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta h_n R_{n,m}(z) d\eta d\eta d\eta d\eta + \chi_m n_{m-1} + \zeta_9 \eta + \zeta_9, \]

\[ \theta_m(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta h_\theta R_{\theta,m}(z) d\eta d\eta d\eta d\eta + \chi_m \theta_{m-1} + \zeta_{11} \eta + \zeta_{12}, \]

\[ \phi_m(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta h_\phi R_{\phi,m}(z) d\eta d\eta d\eta d\eta + \chi_m \phi_{m-1} + \zeta_{13} \eta + \zeta_{14}, \]

and so the exact solution \( f(\eta), g(\eta), m(\eta), n(\eta), \theta(\eta) \) and \( \phi(\eta) \) becomes

\[ f(\eta) \approx \sum_{n=0}^m f_n(\eta), \quad g(\eta) \approx \sum_{n=0}^m g_n(\eta), \quad m(\eta) \approx \sum_{n=0}^m m_n(\eta), \]

\[ n(\eta) \approx \sum_{n=0}^m n_n(\eta), \quad \theta(\eta) \approx \sum_{n=0}^m \theta_n(\eta), \quad \phi(\eta) \approx \sum_{n=0}^m \phi_n(\eta). \]
3.1. Optimal convergence control parameters

It must be remarked that the series solutions (59), (60), (61), (62), (63) and (64) contain the nonzero auxiliary parameters \( h_f, h_g, h_m, h_n, h_\theta \) and \( h_\phi \) which determine the convergence region and also rate of the homotopy series solutions. To obtain the optimal values of \( h_f, h_g, h_m, h_n, h_\theta \) and \( h_\phi \) here the so called average residual error defined by Liao [16] were used as:

\[
\varepsilon_m^f = \frac{1}{\zeta + 1} \sum_{j=0}^{\zeta} \left[ N_f \left( \sum_{i=0}^{m} f_\eta, \sum_{i=0}^{m} g_\eta, \sum_{i=0}^{m} m_\eta, \sum_{i=0}^{m} n_\eta \right) \right]_{n=j\Delta n}^2 \, d\eta, \tag{80}
\]

\[
\varepsilon_m^g = \frac{1}{\zeta + 1} \sum_{j=0}^{\zeta} \left[ N_g \left( \sum_{i=0}^{m} f_\eta, \sum_{i=0}^{m} g_\eta, \sum_{i=0}^{m} m_\eta, \sum_{i=0}^{m} n_\eta \right) \right]_{n=j\Delta n}^2 \, d\eta, \tag{81}
\]

\[
\varepsilon_m^m = \frac{1}{\zeta + 1} \sum_{j=0}^{\zeta} \left[ N_m \left( \sum_{i=0}^{m} f_\eta, \sum_{i=0}^{m} m_\eta, \sum_{i=0}^{m} n_\eta \right) \right]_{n=j\Delta n}^2 \, d\eta, \tag{82}
\]

\[
\varepsilon_m^n = \frac{1}{\zeta + 1} \sum_{j=0}^{\zeta} \left[ N_n \left( \sum_{i=0}^{m} f_\eta, \sum_{i=0}^{m} m_\eta, \sum_{i=0}^{m} n_\eta \right) \right]_{n=j\Delta n}^2 \, d\eta, \tag{83}
\]

\[
\varepsilon_m^\theta = \frac{1}{\zeta + 1} \sum_{j=0}^{\zeta} \left[ N_\theta \left( \sum_{i=0}^{m} f_\eta, \sum_{i=0}^{m} m_\eta, \sum_{i=0}^{m} n_\eta \right) \right]_{n=j\Delta n}^2 \, d\eta, \tag{84}
\]

\[
\varepsilon_m^\phi = \frac{1}{\zeta + 1} \sum_{j=0}^{\zeta} \left[ N_\phi \left( \sum_{i=0}^{m} f_\eta, \sum_{i=0}^{m} m_\eta, \sum_{i=0}^{m} n_\eta \right) \right]_{n=j\Delta n}^2 \, d\eta. \tag{85}
\]

Due to Liao [16]

\[
\varepsilon_m^i = \varepsilon_m^f + \varepsilon_m^g + \varepsilon_m^m + \varepsilon_m^n + \varepsilon_m^\theta + \varepsilon_m^\phi,
\]

where \( \varepsilon_m^i \) is the total squared residual error. Total average squared residual error is minimized by employing Mathematica package BVPh 2.0 [16].

4. Discussion

4.1. The torques exerted on the discs

According to Hamza et al. [17] the frictional moment or torque exerted by fluid on the upper disc is given by

\[
\tau_u = 2\pi\mu \int_0^c r^2 \left( \frac{\partial u_\theta}{\partial z} \right)_{z=d} \, dr,
\]

but \( u_\theta = \Omega r(1 - \delta t)^{-1}g(\eta) \) so the above equation becomes

\[
\tau_u = \frac{\pi\mu\Omega c^4}{2D(1 - \delta t)^{1.5}} g'(1),
\]
or \( \tau^*_u = g'(1) \) where \( \tau^*_u = \frac{2D(1 - \delta t)^{1.5}}{\pi \mu \Omega c^4} \). (87)

For the lower disc the corresponding result is

\( \tau^*_l = g'(0) \), (88)

which are the dimensionless exerted torques of the fluid on upper and lower discs.

### 4.2. The pressure or the normal force of fluid on upper disk

According to Hamza et al. [17], the pressure or the normal force which the fluid exerts on the upper disc is given as:

\[
F = 2\pi \left[ \int_0^a rP(r, 1, t) dr - \int_0^a rP^+(r, 1, t) dr \right],
\]

where \( P^+(r, 1, t) \) in for conditions on the side of the disc and \( p(r, 1, t) \) denotes the pressure at the edge of the disc at time \( t \) [19]. Let us assume that \( \frac{\partial P^+(r, 1, t)}{\partial r} = 0 \) and using Eq. (9), we get

\[
\frac{\partial p}{\partial r} = \frac{\rho r}{(1 - \delta t)^2} \left[ \frac{\delta \nu}{2D^2} \frac{\partial f}{\partial \eta} + \Omega^2 g^2(\eta) - \frac{\delta^2}{4} \left( \frac{\partial f}{\partial \eta} + \eta \frac{\partial^2 f}{\partial \eta^2} + \frac{\partial f}{\partial \eta} - 2f(\eta) \frac{\partial^2 f}{\partial \eta^2} \right) \right.
\]

\[\left. - \frac{2}{\rho \mu_2} \left( \frac{\delta^2 M_o^2}{4D^2} m(\eta) \frac{\partial^2 m}{\partial \eta^2} + N_o^2 n^2(\eta) \right) \right], \tag{90}
\]

where \( Y(t, \eta) = \frac{1}{\rho} \frac{\partial p}{\partial r} \). Using Eqs. (90) and (91) in Eq. (89), we have

\[
F = \frac{\pi \rho \delta^2 a^4}{16(1 - \delta t)^2} \left[ 2N_c^2 \frac{\partial^2 m(1)}{\partial \eta^2} - S_q^{-1} \frac{\partial^2 f(1)}{\partial \eta^2} - \left( \frac{N_c}{S_q} \right)^2 \left( \Omega^2 - 2N_q^2 \right) \right],
\]

\[
F_{\text{Pres}} = \frac{2N_c^2 \frac{\partial^2 m(1)}{\partial \eta^2} - S_q^{-1} \frac{\partial^2 f(1)}{\partial \eta^2} - \left( \frac{N_c}{S_q} \right)^2 \left( \Omega^2 - 2N_q^2 \right) \}, \tag{92}
\]

where \( F_{\text{Pres}} = \frac{16(1 - \delta t)^2}{\pi \rho \delta^2 a^4} \). Which is the dimensionless pressure on the upper disc. The positive or negative numerical values of \( F_{\text{Pres}} \) will be according the force acting by the fluid on the upper disc is in the positive or negative direction of the z-axis respectively.
Table 1. Total residual error for different order of approximations taking fixed values of $A = 2$, $P_r = 1$, $N_\theta = 0.1$, $S_c = 0.5$, $\Omega = 0.1$, $D_u = 0.1$, $R_{em} = 0.01$, $S_e = 0.01$, $S_q = 0.01$, $N_r = 0.05$, $N_z = 0.01$ and $R_d = 0.5$.

| $m$ | $e^f_m$ | $e^g_m$ | $e^m_m$ | $e^n_m$ | $e^0_m$ | $e^g_m$ |
|-----|---------|---------|---------|---------|---------|---------|
| 2   | 1.07301 \times 10^{-17} | 4.42065 \times 10^{-14} | 2.98217 \times 10^{-13} | 6.86906 \times 10^{-13} | 2.83487 \times 10^{-15} | 5.37582 \times 10^{-15} |
| 4   | 2.40529 \times 10^{-27} | 1.90871 \times 10^{-23} | 5.56441 \times 10^{-23} | 5.08355 \times 10^{-23} | 1.27111 \times 10^{-18} | 1.85421 \times 10^{-18} |
| 8   | 1.20203 \times 10^{-29} | 1.82379 \times 10^{-36} | 1.28962 \times 10^{-36} | 1.34294 \times 10^{-36} | 1.57018 \times 10^{-25} | 2.29037 \times 10^{-25} |
| 12  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.24513 \times 10^{-34} | 1.34294 \times 10^{-36} | 1.98572 \times 10^{-32} | 2.83025 \times 10^{-32} |
| 16  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.21864 \times 10^{-34} | 1.34294 \times 10^{-36} | 8.78104 \times 10^{-35} | 1.16154 \times 10^{-38} |
| 20  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.22061 \times 10^{-34} | 1.34294 \times 10^{-36} | 8.78104 \times 10^{-35} | 4.57892 \times 10^{-39} |
| 24  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.22061 \times 10^{-34} | 1.34294 \times 10^{-36} | 8.78104 \times 10^{-35} | 4.57892 \times 10^{-39} |
| 28  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.22061 \times 10^{-34} | 1.34294 \times 10^{-36} | 8.78104 \times 10^{-35} | 4.57892 \times 10^{-39} |
| 32  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.22061 \times 10^{-34} | 1.34294 \times 10^{-36} | 8.78104 \times 10^{-35} | 4.57892 \times 10^{-39} |
| 36  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.22061 \times 10^{-34} | 1.34294 \times 10^{-36} | 8.78104 \times 10^{-35} | 4.57892 \times 10^{-39} |
| 40  | 1.20203 \times 10^{-29} | 1.83755 \times 10^{-36} | 1.22061 \times 10^{-34} | 1.34294 \times 10^{-36} | 8.78104 \times 10^{-35} | 4.57892 \times 10^{-39} |

Table 2. Computations for $f(\eta)$, $m(\eta)$, $\theta(\eta)$, $\phi(\eta)$ with $A = 2$, $D_u = 0.1$, $\Omega = 0.1$, $P_r = 1$, $S_c = 0.01$, $R_d = 0.5$, $S_0 = 0.1$, $N_r = 0.05$, $S_q = 0.001$, $N_z = 0.1$, $N_\theta = 0.1$ and various values of $\eta$.

| $\eta$ | $f(\eta)$ | $g(\eta)$ | $m(\eta)$ | $\theta(\eta)$ | $\phi(\eta)$ |
|--------|------------|------------|------------|----------------|------------|
| 0.1001 | 1.9579     | 0.1000     | 0.1017     | 0.8999         | 0.8998     |
| 0.2002 | 1.8437     | 0.0200     | 0.2030     | 0.7998         | 0.7997     |
| 0.3003 | 1.6754     | 0.0300     | 0.3040     | 0.6997         | 0.6996     |
| 0.4004 | 1.4711     | 0.0400     | 0.4045     | 0.5996         | 0.5995     |
| 0.5005 | 1.2488     | 0.0500     | 0.5047     | 0.4995         | 0.4994     |
| 0.6006 | 1.0266     | 0.0600     | 0.6045     | 0.3994         | 0.3993     |
| 0.7007 | 0.8226     | 0.0700     | 0.7039     | 0.2993         | 0.2992     |
| 0.8008 | 0.6548     | 0.0800     | 0.8031     | 0.1992         | 0.1991     |
| 0.9009 | 0.5412     | 0.0900     | 0.9020     | 0.0991         | 0.0990     |
| 0.9999 | 0.5000     | 0.0999     | 0.9999     | 0.0001         | 0.0001     |

4.3. Error analysis

The problem under consideration is solved by mathematica package BVPh 2.0 for a maximum residual error $10^{-40}$. Analysis are carried out using 40th-order approximations. Error analysis performed in Figure 1 and tabulated results given in Tables 1, 2, 3, 4, 5, 6, 7 and 8 are provided to support the authentification of results for different involved physical parameters.

Figure 1 illustrates the maximum average residual error at different orders of approximation for $f(\eta)$, $g(\eta)$, $m(\eta)$, $n(\eta)$, $\theta(\eta)$ and $\phi(\eta)$. It is clear from subfigures that error is almost continuously reduced up to 8th-order of approximation. Table 1 and Figure 2 presents the total residual error for different order of approximations taking fixed values of $A = 2$, $P_r = 1$, $N_\theta = 0.1$, $S_c = 0.5$, $\Omega = 0.1$, $D_u = 0.1$, $R_{em} = 0.01$, $S_e = 0.01$, $S_q = 0.01$, $N_r = 0.05$, $N_z = 0.01$ and $R_d = 0.5$. Table 2 shows comparison of HAM results to numerical values of $f(\eta)$, $g(\eta)$, $m(\eta)$, $n(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ with $A = 2$, $D_u = 0.1$, $\Omega = 0.1$, $P_r = 1$, $S_c = R_e = 0.01$, $R_d = 0.5$, $S_0 = 0.1$, $N_r = 0.05$, $S_q = 0.001$, $N_z = 0.1$, $N_\theta = 0.1$ and various values of $\eta$. Table 3
Table 3. Convergence of HAM solution for different orders of approximation for \( f''(0), -g'(0), -m'(0), -\theta'(0) \) and \(-\phi'(0)\) when \( A = 2, P_r = 1, N_p = 0.1, S_q = 0.01, S_r = 0.5, \Omega = 0.1, D_u = 0.1, R_{cm} = 0.01, S_c = 0.01, S_t = 0.01, N_r = 0.05, N_z = 0.01 \) and \( R_d = 0.5 \).

| \( m \) | \( f''(0) \) | \( -g'(0) \) | \( -m'(0) \) | \( -\theta'(0) \) | \( -\phi'(0) \) |
|-------|----------|----------|----------|----------|----------|
| 1     | -9.0417 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 5     | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 10    | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 15    | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 20    | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 25    | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 30    | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 35    | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |
| 40    | -9.0419 | -0.1016 | -1.0182  | -0.2731  | -1.1358  |

Table 4. Optimal values of convergence control parameters versus different orders of approximation with fixed values of \( A = 2, \Omega = D_u = N_p = 0.1, P_r = 1, S_c = R_{cm} = S_q = N_z = 0.01, S_r = R_d = 0.5, N_r = 0.05 \).

| Order | \( h_f \) | \( h_g \) | \( h_m \) | \( h_o \) | \( h_o \) | \( r_{me} \) |
|-------|----------|----------|----------|----------|----------|-------------|
| 2     | -1.00189 | -0.991363 | -1.00334 | -10.1448 | -0.273176 | -1.1358276 |
| 3     | -1.00055 | -9.995376 | -1.00183 | -10.9938 | -0.272869 | -1.13548   |
| 4     | -1.01000 | -1.024640 | -0.97735 | -10.2198 | -0.272675 | -1.00149   |
| 5     | -1.02940 | -1.056300 | -1.03446 | -10.4458 | -0.281715 | -1.17377   |
| 6     | -1.02988 | -1.086410 | -1.07524 | -10.6428 | -0.281541 | -1.21301   |

Table 5. Computations for \( f''(0), -g'(0), -m'(0), -\theta'(0) \) and \(-\phi'(0)\) with \( A = 2, D_u = R_d = 0.5, \Omega = S_o = P_r = N_z = N_p = 0.1, S_c = R_{cm} = 0.01, N_r = 0.05 \) and various values of \( S_q \).

| \( S_q \) | HAM \( f''(0), -g'(0), -m'(0), -\theta'(0), -\phi'(0) \) | BVP4c \( f''(0), -g'(0), -m'(0), -\theta'(0), -\phi'(0) \) |
|----------|-------------------------------------------------|--------------------------------------------------|
| 0.1      | -9.4289 -0.1177 -1.0182 0.9955 1.0015 -9.4289 -0.1177 -1.0183 0.9956 1.0016 |
| 0.2      | -9.8800 -0.1381 -1.0182 0.9911 1.0031 -9.8800 -0.1381 -1.0182 0.9912 1.0031 |
| 0.3      | -10.3532 -0.1614 -1.0181 0.9868 1.0046 -10.3532 -0.1615 -1.0181 0.9869 1.0046 |
| 0.4      | -10.8484 -0.1881 -1.0181 0.9825 1.0061 -10.8584 -0.1881 -1.0181 0.9826 1.0061 |
| 0.5      | -10.3654 -0.2182 -1.0180 0.9783 1.0077 -11.3655 -0.2183 -1.0180 0.9784 1.0077 |

Table 6. Computations for \( f''(0), -g'(0), -m'(0), -\theta'(0) \) and \(-\phi'(0)\) with \( A = 2, P_r = S_c = 1, S_c = \Omega = N_z = N_p = 0.1, N_r = 0.05, S_q = 0.01, R_d = 0.5, R_{cm} = 0.01 \) and various values of \( D_u \).

| \( D_u \) | HAM \( f''(0), -g'(0), -m'(0), -\theta'(0), -\phi'(0) \) | BVP4c \( f''(0), -g'(0), -m'(0), -\theta'(0), -\phi'(0) \) |
|----------|-------------------------------------------------|--------------------------------------------------|
| 0.1      | -9.0419 -0.0107 -1.0183 0.9953 1.0161 -9.0419 -0.0107 -1.0183 0.9953 1.0161 |
| 0.2      | -9.0419 -0.0107 -1.0183 0.9949 1.0161 -9.0419 -0.0107 -1.0183 0.9949 1.0161 |
| 0.5      | -9.0419 -0.0107 -1.0183 0.9936 1.0162 -9.0419 -0.0107 -1.0183 0.9936 1.0162 |
| 0.9      | -9.0419 -0.0107 -1.0183 0.9917 1.0164 -9.0419 -0.0107 -1.0183 0.9917 1.0164 |
| 1.3      | -9.0419 -0.0107 -1.0183 0.9898 1.0166 -9.0419 -0.0107 -1.0183 0.9898 1.0166 |
| 1.7      | -9.0419 -0.0107 -1.0183 0.9868 1.0167 -9.0419 -0.0107 -1.0183 0.9868 1.0167 |

presents optimal values of convergence control parameters as well as the maximum values of total average squared residual error versus different order of approximation. Here, it is noticed that the solution obtained from momentum, energy and transport equations converges to exact solution as we increase the order of approximation.
Table 7. Computations for \( f''(0) \), \( g'(0) \), \( m'(0) \), \( \theta'(0) \) and \( \phi'(0) \) with \( A = 2 \), \( D_u = R_d = 0.5 \), \( \Omega = P_e = N_z = N_q = S_q = 0.1 \), \( S_z = 0.01 \), \( S_q = 0.001 \), \( N_z = 0.05 \) and various values of \( R_{em} \).

| \( R_{em} \) | HAM | BVP4c |
| --- | --- | --- |
| \( f''(0) \) | \( -g'(0) \) | \( -m'(0) \) | \( -\theta'(0) \) | \( -\phi'(0) \) | \( f''(0) \) | \( -g'(0) \) | \( -m'(0) \) | \( -\theta'(0) \) | \( -\phi'(0) \) |
| 0.6 | -9.0041 | -0.1001 | -2.9004 | 0.9999 | 1.0000 | -9.0041 | -0.1002 | -2.9004 | 1.0000 | 1.0000 |
| 0.7 | -9.0042 | -0.1001 | -3.4443 | 0.9999 | 1.0000 | -9.0042 | -0.1002 | -3.4443 | 1.0000 | 1.0000 |
| 0.8 | -9.0042 | -0.1001 | -4.0792 | 0.9999 | 1.0000 | -9.0042 | -0.1002 | -4.0793 | 1.0000 | 1.0000 |
| 0.9 | -9.0042 | -0.0999 | -4.8215 | 0.9999 | 1.0000 | -9.0042 | -0.1002 | -4.8220 | 1.0000 | 1.0000 |
| 1.0 | -9.0043 | -0.1002 | -5.6842 | 0.9999 | 1.0000 | -9.0042 | -0.1002 | -5.6862 | 1.0000 | 1.0000 |
| 1.1 | -9.0043 | -0.1002 | -6.6814 | 0.9999 | 1.0000 | -9.0042 | -0.1002 | -6.6881 | 1.0000 | 1.0000 |

Table 8. Fluid’s pressure and torques for fixed values of \( N_z = 0.2 \), \( P_e = 1 \), \( R_d = 1 \), \( N_q = 2 \), \( D_u = 2 \), \( S_z = 2 \), \( R_{em} = 1 \), \( S_q = 0.1 \), \( N_z = 0.05 \), \( A = 2 \), \( \Omega = 0.5 \) and various values of \( S_q' \).

| \( S_q' \) | HAM | BVP4c |
| --- | --- | --- |
| \( F_{pres} \) | \( g'(1) \) | \( g'(0) \) | \( F_{pres} \) | \( g'(1) \) | \( g'(0) \) |
| 0.1 | -175.089 | 0.5028 | 0.5713 | -175.089 | 0.5028 | 0.5713 |
| 0.3 | -56.5676 | 0.5075 | 0.7394 | -56.5676 | 0.5075 | 0.7394 |
| 0.5 | -32.9558 | 0.5136 | 0.9460 | -32.9558 | 0.5136 | 0.9460 |
| 0.7 | -22.9836 | 0.5212 | 1.1950 | -22.9836 | 0.5212 | 1.1950 |
| 1.0 | -15.6742 | 0.5143 | 1.6996 | -15.6742 | 0.5143 | 1.6996 |

Figure 2. Error profile of (a) \( f(\eta) \), (b) \( g(\eta) \), (c) \( m(\eta) \), (d) \( n(\eta) \), (e) \( \theta(\eta) \) and (f) \( \phi(\eta) \) for fixed values of \( S, A, D_u, P_e, M, S_z, S_q, R \) and \( \Omega \).

More justification of our accurate solution is supported with the help of Table 4 for numerical values of \( f''''(0) \), \( g'(0) \), \( m'(0) \), \( n'(0) \), \( \theta'(0) \) and \( \phi'(0) \). Here, it is clear that the solutions are almost converge at 5th-order of approximations. Tables 5, 6, 7 and 8 are made to compare the numerical values of HAM and BVP4c for different values of \( S_q, D_u, R_{em} \) and fixed values of remaining involved parameters.

5. Results

Influence of different flow parameters involved in the nonlinear ordinary differential Eqs. (10), (11), (12), (13), (14) and (15) subject to the boundary conditions given in
The magnetic squeeze flow components Reynolds number velocity is shown in Figures 3, Eq. (16) are discussed in this section. Influence of these flow parameters are shown both graphically and numerically for the velocity components \( f'(\eta), g(\eta), f(\eta) \), magnetic field components \( m(\eta), n(\eta) \), temperature variation \( \theta(\eta) \) and mass transport variation \( \phi(\eta) \).

The effect of flow parameters such as squeeze Reynolds number \( S_q \), rotational Reynolds number \( N_r \), magnetic Reynolds number \( R_{em} \), magnetic field strength components \( N_\theta, N_z \), suction/injection parameter \( A \), radiation parameter \( R_d \), Prandtl number \( P_r \), Schmidt number \( S_c \), Soret number \( S_q \) and Dufour number \( D_q \) are shown for both suction \((A > 0)\) and injection \((A < 0)\). Here it is important to mention that the positive values of \( S_q \) means that upper disc is moving away from lower disc while negative values shows that upper disc is moving towards stationary lower disc. It is also clear that large or small values of \( S_q \) may be regarded as rapid or slow vertical velocities of upper disc respectively or the increase of distance between two discs.

Figures 3, 4, 5, 6, 7, 8, and 9 are plotted to investigate the unsteady squeezing flow with cross diffusion and magnetic effect between two parallel discs. Effect of squeeze Reynolds number \((S_q)\) for suction \((A = 1 > 0)\) both fixed values of \( P_r = D_u = S_c = 1, S_o = R_d = N_z = N_\theta = 0.5, R_{em} = 0.1, N_r = 3, \Omega = 0.01 \) is shown in Figure 3(a–d). Increasing the vertical velocities \((S_q = 0.1, 1, 2, 3)\) of upper disc away from lower disc is decreasing the radial velocity \( f'(\eta) \) as fluid is moving in azimuthal direction (towards upper disc). This decrease in radial velocity is seen near the center of fluid domain, however as fluid passes central region it start increasing as shown in Figure 3(a). Figure 3(b) shows effect of \( S_q \) on azimuthal velocity component \( g(\eta) \). It is clear that increasing the distance between discs will
Figure 4. Profile of (a) $f(\eta)$, (b) $f'(\eta)$, (c) $g(\eta)$ and (d) $m(\eta)$ for different values of squeezing number $-S_q$ and fixed values of $P_r, R_d, A, D_u, S_x, S_y, N_2, N_r, N_p, \Omega$ and $R_{cm}$.

Figure 5. Profile of (a) $f(\eta)$, (b) $f'(\eta)$, (c) $g(\eta)$, (d) $m(\eta)$, (e) $n(\eta)$ and (f) $\theta(\eta)$ for different values of rotation parameter $\Omega$ and fixed values of $P_r, R_d, A, D_u, S_x, S_y, N_2, N_r, N_p, S_q$ and $R_{cm}$.
Figure 6. Profile of (a) $\theta(\eta)$ and (b) $\phi(\eta)$ for different values of Prandtl number $Pr$ and fixed values of $\omega, R_d, A, D_u, S_c, S_o, N_z, N_r, N_\theta, S_q$ and $R_{em}$.

Figure 7. Profile of (a) $f(\eta)$, (b) $f'(\eta)$, (c) $g(\eta)$, (d) $m(\eta)$, (e) $\theta(\eta)$ and (f) $\phi(\eta)$ for different values of $R_{em}$ and fixed values of $Pr, R_d, A, D_u, S_c, S_o, N_z, N_r, \Omega$ and $S_q$.

allow fluid to move in azimuthal direction which will increase velocity as shown in Figure 3(b), however after central region $g(\eta)$ have opposite behavior. It is also noted that increasing the fluid between the discs region increase temperature due to the fraction of inflow with lower disc. The same but opposite behavior is seen for $\phi(\eta)$ in Figure 3(d). Figure 4(a–d) depicts the effect of the squeeze rate of discs on axial, radial velocity components, heat and mass transfer rate. It is observed from Figure 4(a) that squeezing upper disc toward lower is decreasing axial velocity due
Figure 8. Profile of (a) $\theta(\eta)$ and (b) $\phi(\eta)$ for different values of Soret number $S_o$ and fixed values of $P_r$, $R_d$, $A$, $D_u$, $S_c$, $S_q$, $N_r$, $N_\theta$, $\Omega$ and $R_{em}$.

Figure 9. Profile of (a) $f(\eta)$, (b) $\theta(\eta)$, (c) $m(\eta)$ and (d) $n(\eta)$ for different values of Soret number $N_z$ and fixed values of $P_r$, $R_d$, $A$, $D_u$, $S_c$, $S_q$, $S_o$, $N_r$, $N_\theta$, $\Omega$ and $R_{em}$.

to the fact that fluid is pushed toward lower disc and so it need to move in radial direction i.e. radial velocity needs to increasing but here it is decreasing due to fluid injection ($A = -1 < 0$) from lower porous disc. This effect is seen near central region of domain and so after central region it start increasing as injection effect is dominated by squeezing rate. Similarly Figure 4(c–d) shows that fluid injection from lower discs decreasing both heat and mass transfer rate.

Figure 5 is made to show influence of rotation parameter $\Omega$ over $f(\eta)$, $f'(\eta)$, $g(\eta)$, $m(\eta)$, $n(\eta)$ and $\theta(\eta)$ with fixed values of $D_u = A = R_{em} = P_r = S_o = S_q = 1$, $S_c = N_r = N_\theta = 5$, $N_z = 0.8$, $R_d = 10$. It is clear that increase in rate of rotation of upper disc will push the fluid to move in azimuthal direction which will decrease axial and radial velocity component $f(\eta)$, $f'(\eta)$ respectively and increase azimuthal velocity $g(\eta)$ as shown in Figure 5(d–f) that increase on $\Omega$ decrease the effect of magnetic field components in axial and azimuthal direction. Figures 6 and 8 are made to plot
Table 9. Fluid’s pressure and torques for fixed values of \( N_z = 0.2, P_r = 1, R_d = 5, S_q = 1, D_u = 0.1, S_a = 0.5, R_{em} = 3, S_e = 0.1, N_y = 1, A = 2, \Omega = 1 \) and various values of \( N_\theta \).

| \( N_\theta \) | HAM \( F_{pres} \) | \( g'(1) \) | \( g'(0) \) | BVP4c \( F_{pres} \) | \( g'(1) \) | \( g'(0) \) |
|---|---|---|---|---|---|---|
| 0.1 | -16.9193 | 0.0663 | 4.3532 | -16.9193 | 0.0663 | 4.3532 |
| 0.5 | -16.6554 | -0.0932 | 4.4119 | -16.6554 | -0.0932 | 4.4119 |
| 2  | -12.2329 | -0.8771 | 5.0162 | -12.2329 | -0.8771 | 5.0162 |
| 5  | 13.0811  | -2.5151 | 6.0753 | 13.0811  | -2.5151 | 6.0753 |
| 10 | 101.406  | -5.7279 | 8.1201 | 101.406  | -5.7279 | 8.1201 |

The effect of Prandtl number \( P_r \) and Soret number \( S_s \) on \( \theta(\eta) \) and \( \phi(\eta) \). It is shown that increase in \( P_r \) or \( S_s \) increase \( \theta(\eta) \) and decrease \( \phi(\eta) \). Maximum increase and decrease is seen in middle of fluid domain. Figure 7 shows effect of magnetic Reynolds number \( R_{em} \). Magnetic Reynolds number \( R_{em} = S_q Bt \) represents the ratio of fluid flux to the magnetic diffusivity. Increase in magnetic Reynolds number means that either the fluid flux is increasing or magnetic diffusivity decreasing. It is seen from subfigures that increase in \( R_{em} \) decrease both axial and azimuthal velocities while radial velocity is increasing after central region. Also this phenomena decreasing temperature and azimuthal component of induced magnetic field component \( m(\eta) \). Figure 11 represent the H-curves of \( f''(0), g'(0), m'(0), n'(0), \theta'(0) \) and \( \phi'(0) \).

Table 4 is made to compare the numerical values of \( f(\eta), g(\eta), m(\eta), n(\eta), \theta(\eta) \) and \( \phi(\eta) \) obtained through BVPPh 2.0 and BVP4c. It is noticed that both solutions have excellent agreement for different values of \( \eta \). Tables 5, 6 and 7 shows comparison of \( f''(0), -g(0), -m(0), -n(0), -\theta(0) \) and \( -\phi(0) \) through HAM and BVP4c for different values of \( S_q, D_u, R_{em} \) and fixed values of remaining involved parameters. Table 5 indicates that increasing the distance between discs decreasing the heat flux, radial and tangential skin friction while mass flux increasing. Table 6 is made to show effect of Dufour number. It is clear from this table that \( D_u \) has no effect on \( f''(0), -g'(0) \) and \( -m'(0) \) while it increasing both heat and mass flux. Effect of \( R_{em} \) is shown in Table 7 where \( -m'(0) \) decreases and \( -g'(0), -\theta'(0) \) and \( -\phi'(0) \) have no influence of \( R_{em} \). The variation of load or pressure and the torque that the fluid exerts on discs are shown in Tables 8 and 9. It is clear from Table 8 that increasing distance between increasing load along-z-axis. Also torque on lower disc is always positive and increasing where on the other hand torque is decreasing near the upper disc. Table 9 shows effect of \( N_\theta \) on load and torques. It is clear from table that load and torque on lower disc is increasing with increase in \( N_\theta \) where torque on upper disc is decreasing. These results agree with the theoretical and experimental results obtained by Hughes et al. [7]. Figure 9 shows effect of the axial component \( N_z \) of the magnetic field on \( f(\eta), \theta(\eta), m(\eta), n(\eta) \). As we increase \( N_z \), the axial component of fluid velocity start increasing. Hence, \( N_z \) can be used to increase the axial velocity and temperature of the fluid as shown in Figure 9(a–b). Also \( m(\eta) \) increases as \( N_z \) increases, which is an expected result as the normal component of the induced
Figure 10. 3D Profile view of (a) $f(\eta)$, (b) $g(\eta)$, (c) $m(\eta)$, (d) $n(\eta)$, (e) $\theta(\eta)$ and (f) $\phi(\eta)$ for fixed values of $S_q = A = 2$, $\Omega = R_m = 1$, $D_u = 0.1$, $P_r = R_d = 0.5$, $N_z = N_\theta = 4$, $N_r = 0.05$. 

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Figure 11. H-curves of $f''(0), g'(0), m'(0), n'(0), \theta'(0)$ and $\phi'(0)$.

Figure 12. Effect of axial magnetic field component on $m(\eta)$ and $n(\eta)$ of previous research paper.

magnetic field must increase with the increase of the normal component of external applied magnetic field. This phenomena is decreasing azimuthal component $n(\eta)$ (Figure 12). These results agree with the results obtained by Hamza [12] and S. Elshekh [19]. Figure 10 represents the 3D view of $f(\eta), g(\eta), m(\eta), n(\eta), \theta(\eta), \phi(\eta)$ with fixed values of involved parameters.

6. Conclusion

In this paper, Navier–Stokes equations along with variable magnetic field and heat/mass transfers are taken into account for the squeezing flow of viscous fluid between parallel discs. HAM is used to determine the series solution of fluid velocity components, magnetic field components, mass and temperature distribution. Main upshots of this paper are presented as below:

- It is concluded that magnetic field can be utilize to increase the axial fluid velocity.
• Increasing the velocity of upper disc away from lower is decreasing flow movement in radial direction while moving upper disc toward lower is increasing radial movement of fluid flow.
• Increase in rate of rotations of upper disc increase fluid flow in tangential direction and decrease velocities in remaining directions along with magnetic field components. Also this phenomena increase fluid temperature.
• Change in Dufour number has opposite behavior on heat and mass flux.
• Increase in magnetic Reynold’s number decrease magnetic flux at lower disc.
• It is found that numerical solution obtained by BVP4c are in excellent agreement with analytical solution obtained by HAM.
• It is observed from Table 8 that increase in distance between discs increase pressure on lower disc. The same table shows that this phenomena increase torque on upper disc and decrease it on lower disc.

Declarations

Author contribution statement

Rehan Ali Shah, Aamir Khan, Muhammad Shuaib: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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