Arbitrage from a Bayesian’s Perspective*

Ayan Bhattacharya†

Abstract

This paper builds a model of interactive belief hierarchies to derive the conditions under which judging an arbitrage opportunity requires Bayesian market participants to exercise their higher order beliefs. Methodologically, the approach of the paper is to take the standard asset pricing setup that gives rise to arbitrage and transform it into a Bayesian decision problem faced by a representative market agent. As a Bayesian, such an agent must carry a complete recursion of priors over the uncertainty about future asset payouts, the strategies employed by other market participants that are aggregated in the price, other market participants’ beliefs about the agent’s strategy, other market participants beliefs about what the agent believes their strategies to be, and so on ad infinitum. Defining this infinite recursion of priors — the belief hierarchy so to speak — along with how they update gives the Bayesian decision problem equivalent to the standard asset pricing formulation of the question. In this setting, any update to the belief hierarchy of an agent is one of two kinds: a change in belief about the asset payouts, or a change in belief about the strategies and beliefs employed by other market participants. The main results of the paper show that an arbitrage trade corresponds to special updates of the second kind. When an agent anticipates market participant responses will be generated using \( k \) levels of the belief hierarchy but finds that the actual asset prices are supported by \( k + 1 \) or higher levels, there is an arbitrage opportunity. It is shown that the presence of arbitrage depends on the degree of optimality of the belief hierarchies employed by market agents and responsiveness of the price aggregation mechanism, and is closely related to market tatonnement. The paper connects the foundations of finance to the foundations of game theory by identifying a bridge from market arbitrage to market participant belief hierarchies.

Keywords: Arbitrage opportunity, Beliefs in asset price, Level-k reasoning, Higher-order belief, Bayesian belief hierarchy, Fundamental theorem of asset pricing.

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†Email: ayan.bhattacharya@gmail.com. Affiliation: University of Chicago – Booth School of Business, and Arrow Markets. My thanks to Stefan Nagel for helpful comments.
1 Introduction

This paper connects the foundations of finance to the foundations of game theory. Specifically, it builds a model of interactive belief hierarchies to derive the theoretical conditions under which judging an arbitrage opportunity requires Bayesian market participants to exercise their higher order beliefs. A no-arbitrage argument is at the heart of many of the modern developments in asset pricing theory and Dybvig and Ross (1987) term the equivalence results linking the various characterizations of no-arbitrage the “fundamental theorem of asset pricing.” Likewise, the “hierarchy of beliefs” technique proposed in Harsányi (1967) for Bayesian players, that evolved into the epistemic approach to the field, underpins many of the important developments in game theory. A rigorous bridge from market arbitrage to market participant belief hierarchies, therefore, offers a potential path to transferring strategic concerns of agents that underlie game theoretic modeling to asset pricing theory.

Methodologically, the approach of the paper is to take the standard asset pricing setup that gives rise to arbitrage and transform it into a Bayesian decision problem faced by a representative market agent. The traditional definition of an arbitrage portfolio is given in terms of a set of conditions on the ex-ante prices of assets relative to their ex-post payouts. Nonetheless, the ex-ante prices arise as a result of individual optimization of some sort or another, and market aggregation, so the price formation process embeds an implicit strategic interaction among the participants. When taking this interaction into account, the uncertainty facing an agent in the market is no longer just about the ex-post payouts from assets, but also about the strategies chosen by other market participants that get aggregated into the price. An agent must carry a prior over this uncertainty as a Bayesian. That is not all, though. The agent is also uncertain about what other market participants believe her strategy to be, about what other market participants believe about what she believes their strategies to be, and so on ad infinitum. As a Bayesian, an agent must carry a prior over each of these layers of uncertainty. Defining this infinite recursion of priors — the belief hierarchy so to speak — along with how they update gives the Bayesian decision problem equivalent to the standard asset pricing formulation of the question.

A fundamental insight in epistemic game theory is that if the basic uncertainty spaces are suitably restricted, one may define a universal domain of uncertainty for an agent that encompasses all the recursive layers of uncertainty arising in an interaction. Defining a prior over this universal domain is then equivalent to defining the infinite recursion of
priors, as long as the agents’ beliefs obey certain minimal consistency requirements. We adopt this epistemic approach in the paper. Any update to the belief hierarchy of a market agent could then be one of two kinds: a change in belief about the asset payoffs, or a change in belief about the strategies and beliefs employed by other market participants. In this paper we switch off the first type of update by assuming symmetric information about ex-post payouts and focus exclusively on updates of the the second kind. That is to say, an agent starts with a prior belief hierarchy about the market environment she is facing, and after seeing the actual asset prices generated as a result of the choices made by market participants, revises her belief hierarchy if she has to. The main results of the paper show that an arbitrage trade corresponds to a subset of revisions of this type.

An arbitrage opportunity is not an equilibrium phenomenon, so the market aggregation mapping that gives rise to such an opportunity cannot be the standard individual-optimization/market-clearing pricing function that is used to study equilibrium. Nevertheless, if the mapping that aggregates individual choices into a market price is known to a market participant, she may in-principle back out a set of choices for other market participants that are consistent with the market price she observes. In other words, there is a set of market participant belief hierarchies that support an asset price which can be backed out by a market agent who observes the asset price and knows the aggregation mapping. An arbitrage opportunity arises when an agent finds that the belief hierarchies in actual use are in some sense “more optimal” than what she had anticipated them to be.

Optimality in our setup is defined in terms of the number of levels of the belief hierarchy that are taken into account by agents to generate undominated responses — responses that cannot be unequivocally improved. When an agent anticipates that other market participants will be generating their responses using only $k$ levels of their belief hierarchy but subsequently discovers — when backing out actual choices from the price — that they are using $k + 1$ or higher levels, there is an arbitrage opportunity. For example, the agent might anticipate that the rest of the market will use undominated responses that account for two levels of their respective hierarchies, and choose her initial strategy to be an optimal response to this anticipation. For this agent to discover an arbitrage opportunity in the market subsequently, it is necessary that the rest of the market actually use undominated responses that account for at least three levels of their hierarchy.

Whether or not an arbitrage trade exists depends both on the degree of optimality of the anticipated belief hierarchies and on the extent to which the market aggregation mapping allows an agent to uncover the actual belief hierarchies supporting an observed
price. It is the interplay between these two factors that gives the study of arbitrage with belief hierarchies its distinct flavor. For example, if the market aggregation is a constant mapping (i.e., prices stay unchanged no matter what choices market participants make) there is no arbitrage opportunity no matter what belief hierarchy agents employ. This is because an agent simply cannot discern any feature of the actual belief hierarchies that are supporting the observed prices. So, no matter what belief hierarchy she anticipates, after observing the prices the agent is unable to make the case that she would do unequivocally better if she used a different hierarchy. On the other hand, if the market aggregation mapping is one-to-one (i.e., observed prices allow agents to perfectly distinguish the belief hierarchies actually used) then no-arbitrage implies that agents are optimizing over the entire infinite hierarchy. This is because a competitive game ensues among the market participants: every agent wants to optimize one level more than the rest of the market so as to not leave any money on the table, in effect pushing the orders of optimization to infinity for everyone. Such a competitive process would not go all the way to infinity if an agent could not distinguish her counterparty’s belief hierarchy with sufficient precision from the prices.

The analysis in the paper highlights the importance of higher order reasoning in markets. It is not enough if an agent reasons optimally about just fundamentals or beliefs of other market participants; to leave no money on the table, the agent must also reason optimally about beliefs about beliefs of other market participants, about beliefs about beliefs about beliefs of other market participants, and so on, as far as needed under the market aggregation mapping. Yet common experience seems to suggest that traders in the real world don’t go very far up the orders when reasoning deliberately about their environment. The resolution to this apparent incongruity lies in recognizing that a market tatonnement process — in which traders are simply responding to immediate market circumstances — can deliver the same outcome as a higher order reasoning process. At each step a trader could be reasoning deliberately just one-step ahead, yet stacking up a series of such “one-step aheads” in a tatonnement sequence leads to market outcomes that are indistinguishable from higher order reasoning.

**Related Literature:** The need to study how agents forecast the forecast of others has motivated financial economics ever since Keynes (1936) introduced his influential metaphor of markets as a beauty contest. Beginning with the seminal work of Townsend (1978; 1983), Phelps (1983) and Sargent (1991), there is a large literature at the intersection of finance and macroeconomics that looks at models where agents have heterogeneous ex-
pectations about the future realizations of economic variables, and thus have to forecast the forecast of others. In asset pricing theory, papers like Biais and Bossaerts (1993), Allen, Morris and Shin (2006), Banerjee, Kaniel and Kremer (2009) and Makarov and Rytchkov (2012) build on this metaphor to create rational (or near rational) expectations equilibrium models of asset prices that rely on heterogeneous beliefs and differential information among agents. While related to this literature, the main point of departure of the present study is that neither heterogeneous expectations about fundamentals nor asymmetric information play any role in the model. In fact, it is assumed throughout that the physical probability measure describing the ex-post payoff uncertainty is known to all the market participants.

On the epistemic side, a number of papers have explored both the belief-about-belief based foundation as well as reasoning based foundation of rational expectations equilibrium in economies with asymmetric information. In the reasoning based “eductive” approach pioneered by Guesnerie (see Guesnerie 1992, 2002 for important papers; and Guesnerie 2005 and Desgranges 2014 for surveys of this literature) the equilibrium solutions are based on the assumption that each agent reasons about the reasoning of other agents in line with common knowledge of rationality and the model. To microfound the adaptive learning that is necessary in reasoning based models, Adam and Marcet (2011) provide a decision theoretic framework based on agents who are “internally rational” but may not be “externally rational”. On the other hand, the belief-about-belief based approach that hews more closely to the traditional epistemic foundation for solution concepts relies on various alternative notions of common knowledge of market clearing, player beliefs and rationality to justify rational expectation outcomes (see MacAllister 1990, Morris 1995, Dutta and Morris 1997, Ben-Porath and Heifetz 2011). At a broad level, the approach in the present paper is similar to the one taken in this literature. However, unlike this body of work, there is no asymmetric information in the present study. Further, arbitrage is not an equilibrium phenomenon, so the rational expectations equilibrium framework is of limited use in understanding the genesis of arbitrage trade.

The rigorous study of arbitrage was initiated in Ross (1976, 1978) and Harrison and Kreps (1979), and a number of alternative formulations for such trade were proposed in the ensuing years. Dybvig and Ross (1987) termed the equivalence of the various alternative conditions that led to (no) arbitrage the fundamental theorem of asset pricing. One way to interpret the results in this paper is that they add to this list of equivalent conditions — in this case, using the Bayesian belief hierarchies of market agents. Another way to think
about the results is that they highlight the fundamental significance of level-k reasoning (Stahl and Wilson 1994, Stahl and Wilson 1995, Nagel 1995) and the recursion of priors for asset pricing and arbitrage theory. More references to the literature are interspersed throughout the text.

**Organization of the paper:** The main results of the paper are in Section 5 and the preceding sections build the tools that are necessary to derive them. Section 2 describes the classical no-arbitrage condition that is the starting point of the study. Section 3 then gradually transforms the classical definition into a form that is more explicit about the strategic concerns of a representative Bayesian market agent, with Section 3.1 providing the model of the market that such an agent uses and Section 3.2 defining arbitrage in the model. Section 4 describes the construction of the recursive hierarchy of beliefs that are the building blocks for asset pricing under strategic uncertainty, with Section 4.1 outlining the basic hierarchy that is used in epistemic game theory and Section 4.2 defining the specific hierarchy that we use in our analysis. Section 5 contains the main results of the paper linking traditional descriptions of arbitrage to belief hierarchy based descriptions. Section 5.1 contains the central theorems of the paper characterizing arbitrage and no-arbitrage in terms of belief hierarchies, and emphasizes the importance of responsiveness of the market aggregation mapping for arbitrage. Section 5.2 highlights the close analogy between tatonnement and reasoning-based price adjustment processes. Section 6 contains a discussion on ways to empirically measure the orders of belief used by market participants.

2 A First Definition of Arbitrage

The textbook definition of an arbitrage opportunity (for example, Back 2017) is a portfolio that requires no investment to set up, is guaranteed to not lose money, and generates positive income almost surely. More specifically, consider a financial market with a vector of $d$ assets. One of the assets in the market is risk-free, in the sense that it always pays out a fixed amount ex-post. The payoffs from the other assets are stochastic. To model the fundamental uncertainty in the market, let us fix a measurable space $(S, \mathcal{S})$. The state space $S$ is assumed to be compact and metric, and the elements of this space are labeled *states of nature*. The ex-post asset payouts are given to be non-negative, bounded random
variables\(^1\)
\[
\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^d).
\] (1)

The assets are priced in the market ex-ante (i.e., before the uncertainty is realized), and let us denote the market prices by
\[
q = (q^1, \ldots, q^d) \in \mathbb{R}^d.
\] (2)

A portfolio is a vector
\[
\theta = (\theta^1, \ldots, \theta^d) \in \mathbb{R}^d,
\] (3)
where \(\theta_i\) represents the number of units of the \(i^{th}\) asset in the collection. The ex-ante cost of creating the portfolio \(\theta\) is \(\theta \cdot q\), and the portfolio pays off \(\theta \cdot \tilde{x}\) ex-post depending on the realized state of nature \(s \in S\).

**Definition 1.** (Arbitrage opportunity) A portfolio \(\theta \in \mathbb{R}^d\) is called an arbitrage opportunity with respect to a probability measure \(P\) on the measurable space \((S, \mathcal{S})\) if
\[
\theta \cdot q \leq 0, \text{ but } \theta \cdot \tilde{x} \geq 0 \text{ P-a.s. and } P[\theta \cdot \tilde{x} > 0] > 0.
\] (4)

Notice that the probability measure \(P\) is relevant in so far as it fixes the null sets of the measurable space. One could alternatively formulate the definition using the state space directly (a portfolio that loses money under no event and makes money under some events) but such a definition would come with the assumption that at least some events that make money for the portfolio have a non-zero probability.

Trading models based on no-arbitrage usually take \(\tilde{x}\) and \(q\) as model primitives and impose conditions that rule out arbitrage opportunities in the market (for example, a positive stochastic discount factor, or a risk-neutral measure condition). Our journey in the sequel will be in the opposite direction. We start with the no-arbitrage condition and try to obtain “deeper” foundations for the concept in terms of the Bayesian beliefs of a market participant.

\(^1\)To facilitate easy identification, random variables in the paper are distinguished by a tilde above the variable.
3 Transforming the Arbitrage Definition

In this section, we set about transforming the definition of arbitrage provided in the previous section to arrive at an interpretation that is more amenable to strategic analysis (this interpretation in contained in Proposition 1). We take the viewpoint of a particular market participant — agent $i$ (she) — who is reasoning about the market, and try to decipher the meaning of arbitrage from her perspective (the terms market participant and agent are used interchangeably in the sequel). The market under focus need not be in equilibrium, so the analysis relies on deriving the value that our agent attaches to her strategies given an arbitrary market stochastic discount factor (SDF) — without explicitly specifying an equilibrium price formation process. In a Bayesian universe, SDFs and strategies must themselves arise as a result of beliefs held by agents, and in subsequent sections we extend the analysis by connecting it explicitly to underlying belief hierarchies.

The classical definition of arbitrage keeps silent on how the arbitrage opportunity is actually exploited in the market. The implicit assumption is that if prices satisfy the condition in Definition 1, then at least one market participant will notice this and exploit the profit-making opportunity to improve her utility. Let us restate the definition of arbitrage to make this point explicit.

**Definition 2.** (Tradeable arbitrage opportunity) There is a tradeable arbitrage opportunity in the market with respect to a probability measure $\mathbb{P}$ if and only if there is a market participant $i \in I$ who can trade a portfolio $\theta_i \in \mathbb{R}^d$ with the property

$$\theta_i \cdot q \leq 0, \quad \text{but} \quad \theta_i \cdot \tilde{x} \geq 0 \text{ P-a.s. and } \mathbb{P}[\theta_i \cdot \tilde{x} > 0] > 0. \quad (5)$$

We use the term tradeable arbitrage opportunity to distinguish opportunities that are exploitable. Unless some market participant can actually trade an arbitrage opportunity that is said to exist, its presence is contestable at best. The limits to arbitrage literature (starting with Shleifer and Vishny 1997)\(^2\) shows that a wedge may exist between Definitions 1 and 2, and in such cases it is the latter definition that takes precedence. In the sequel, we shall focus exclusively on tradeable arbitrage opportunities, and when there is no chance of confusion we shall drop the prefix “tradeable” and refer to these as just

\(^2\)Gromb and Vyanos (2010) provide a survey of the limits to arbitrage literature.
arbitrage opportunities.

Notice that condition (5) is somewhat ambiguous about the price impact of arbitrage orders. One way to interpret the definition is that it makes no allowance for any price impact: the price vector $q$ is fixed no matter what quantities are traded (call this, the restricted interpretation). In reality, however, trading almost always entails a price impact. If we would like to take such impact into account, an alternative way to interpret the arbitrage definition is that the price vector $q$ in condition (5) is portfolio specific. In other words, if the arbitrage portfolio were $\theta'$ instead of $\theta$, the ex ante prices would be $q'$ instead of $q$. In our analysis, we will favor this alternative interpretation of arbitrage. To highlight the portfolio specific nature of prices, let us use the notation $q_\theta$ to denote the ex-ante prices when the traded arbitrage portfolio is $\theta$.

**Definition 3.** (Tradeable arbitrage opportunity with portfolio specific price) There is a tradeable arbitrage opportunity in the market with respect to a probability measure $P$ if and only if there is a market participant $i \in I$ who can trade a portfolio $\theta_i \in \mathbb{R}^d$ with the property

$$\theta_i \cdot q_{\theta_i} \leq 0, \text{ but } \theta_i . \xi \geq 0 \text{ P-a.s. and } P[\theta_i . \xi > 0] > 0. \quad (6)$$

The only difference between Definitions 2 and 3 is the use of $q_{\theta_i}$ for prices instead of $q$. The ex-ante prices are now transaction specific: for each portfolio that an agent proposes, the market generates a customized price. Though the change in notation is minor, Definition 3 provides a much more general view of arbitrage. Different orders may affect the prices differently, and market participants have to take the order specificity of price into account before judging an arbitrage opportunity. In other words, there is no “global” ex-ante price vector for the assets to begin with, but each arbitrage portfolio $\theta$ generates its own specific ex-ante price vector $q_\theta$. Of course, one could always recover the restricted interpretation from this general interpretation by assuming $q_\theta = q$ for all $\theta$.

Another related point worth noting is that the original arbitrage definition purports to look at the arbitrage trade in complete isolation — the price vector $q$ in (2) is simply taken as exogenously given. Nevertheless, an arbitrage opportunity is in itself a trading opportunity, so any definition of arbitrage implicitly posits a price formation process. For

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3An entire sub-area of Finance, the field of Market Microstructure, is dedicated to understanding the nuances of price impact. See Foucault et al. (2013) for a survey.
the original arbitrage definition, this is a market process where prices stay unchanged no matter what portfolios are traded, if we go by the restricted interpretation above. If this were indeed the case, however, prices would never correct to become arbitrage-free. The source of such conundrums lies in the lack of an explicit model of a market that is not in equilibrium.

An arbitrage opportunity is not an equilibrium phenomenon, so any agent exploiting such an opportunity has to build, in her mind, a model of a market that is independent of the specifics of an equilibrium notion. Constructing such a model is the focus of the next subsection.

3.1 A Model of the Market, Not Necessarily in Equilibrium, from a Participant’s Perspective

How should agent \( i \) reason about the market and asset prices? To make this question more concrete, we need a model of the market from \( i \)’s perspective. A first version of the model that we employ (this version does not lay out the belief hierarchies explicitly) is provided in equation (9) below, and in the intervening paragraphs we focus on delineating the primitives of this model.

A maintained assumption throughout is that agent \( i \) is Bayesian and she is aware that the market is populated by a finite set \( I \) of Bayesian agents. Agent \( i \) faces the fundamental uncertainty described by the measurable space \((S, S)\) and has access to the vector of \( d \) tradeable assets with measurable payoffs \( \tilde{x} \) as described in Section 2, to navigate the uncertainty. Importantly, the market that agent \( i \) reasons about need not be in equilibrium.

Agent \( i \) selects a strategy from a compact metric space \( A_i \) in order to express her personal preference for the ex-ante market prices. Each strategy \( a_i \in A_i \) is a vector representing quantities of the various assets (i.e., a portfolio) to be bought — or sold, sell being a negative buy — in a certain sequence.

There is a mapping \( \tilde{f} : (a_i, a_{-i}) \mapsto \tilde{m} \), where \( \tilde{m} \) is a bounded random variable on \((S, S)\) representing the SDF chosen by the market after aggregating the individual market participant choices \((a_i)_{i\in I}^{4,5} \). For all the SDFs under consideration, we shall assume that the risk-free asset has a positive gross return, i.e., \( \int_S \tilde{m} dP > 0 \). \( \tilde{f} \) is known to all market partic-

\(^4\)We are implicitly assuming the “law of one price” holds in the market. This assumption is not strictly required for the results, but making it simplifies the presentation considerably.

\(^5\)We use the standard notation \( a_{-i} \in A_{-i} = \prod_{j \neq i} A_j \).
ipants, and the ex-ante price of the assets prevailing in the market for the transaction can be backed out from the market SDF $\tilde{m}$ and physical probability measure $P$ in the standard manner: $q^n = \int_S \tilde{m} \tilde{x}^n dP, \ n \in \{1, \ldots, d\}$. An equilibrium outcome would impose constraints on the form of $\tilde{f}$, but since arbitrage is not an equilibrium phenomenon, we shall not concern ourselves here with defining any special restrictions (like market clearing) for the mapping.

Every strategy that agent $i$ may choose is associated with a stochastic net gain. When the agent selects $a_i$ as her strategy, her ex-ante cost is $a_i \cdot q_{a_i}$. If $\tilde{f}(a_i, a_{-i}) = \tilde{m}_{a_i}$, this cost works out to be $a_i \cdot \int_S \tilde{m}_{a_i} \tilde{x} dP$. To move this cost from the ex-ante to ex-post, the said amount needs to be invested in a risk free asset. Since we are using portfolio specific pricing, let us designate the market SDF that results from investing exclusively in the risk-free asset by $\tilde{m}_{rf}$, so that an investment of $-a_i \cdot q_{a_i}$ exclusively into the risk-free asset invokes the SDF $\tilde{m}_{rf}^{-a_i \cdot q_{a_i}}$. Thus, the cost to the agent, ex-post, of choosing strategy $a_i$ is $\frac{a_i \cdot \int_S \tilde{m}_{a_i} \tilde{x} dP}{\int_S \tilde{m}_{rf}^{-a_i \cdot q_{a_i}} dP}$. The agent’s stochastic payout from the strategy is $a_i \cdot \tilde{x}$. Therefore, the stochastic net gain to the agent from selecting strategy $a_i$ is given by the random vector

$$\tilde{g}(a_i, a_{-i}) = a_i \cdot ( \tilde{x} - \frac{\int_S \tilde{m}_{a_i} \tilde{x} dP}{\int_S \tilde{m}_{rf}^{-a_i \cdot q_{a_i}} dP} ).$$

(7)

An agent’s utility depends on the ex-ante prices and ex-post payoffs from the assets, along with her holding of the assets. Since the market SDF depends on the strategies of all market participants, we have $U_i : \prod_{j \in I} A_j \times S \rightarrow \mathbb{R}$ for agent $i$’s utility. Agent $i$ thus reasons that her utility depends on the choices of the other agents in the market (which collectively determine the market SDF) and this necessitates strategic thinking on her part. The strategic thinking entails belief hierarchies which we take up in subsequent sections.

The value that agent $i$ derives from a particular contingency $\psi \in S$ in the probability space $(S, \mathcal{S}, P)$ is determined by her selected strategy’s net gain in that contingency. Ceteris paribus, higher her gain from a strategy in a contingency, greater the utility she attaches to it, and vice-versa. That is to say,

$$g(a_i', a_{-i}, \psi) > g(a_i'', a_{-i}, \psi) \iff U_i(a_i', a_{-i}, \psi) > U_i(a_i'', a_{-i}, \psi).$$

(8)

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6 A loss would be a negative gain in our terminology.
Using the primitives described above, from agent $i$’s perspective the market may be represented by the tuple

\[ M_i = ((S, S), \tilde{x}, I, (A_j)_{j \in I}, \tilde{f}, U_i). \]  

(9)

Given $M_i$, the agent makes her choice $a_i \in A_i$ and is told the aggregate market selection in the form of an SDF $\tilde{m}$. Notice that beyond the fact that there are $I$ market participants who are making selections from their respective sets $A_j$, agent $i$ is not presumed to know anything about other market agents.

One can recover the regular “price-quantity” version of the market by defining $M = (M_i)_{i \in I}$, and using the market SDF with $\tilde{f}$ to obtain prices, and the strategies $a_j, j \in I$, to obtain the quantities traded. An equilibrium notion imposes its own restrictions on the elements of $M$ (for example, market clearing and utility maximization in the Walrasian equilibrium), but it is not necessary to impose these restrictions for an analysis of non-equilibrium phenomena like arbitrage. $M_i$ in (9) is not linked to any specific equilibrium notion or solution concept: it is just a description of the market from agent $i$’s perspective.

3.2 Arbitrage in $M_i$

We would like to extend the definition of arbitrage to the setup of $M_i$, and this is accomplished in Proposition 1 below. Let $a_i$ denote agent $i$’s initial choice of strategy. Recall that the mapping $\tilde{f}$ aggregates all market participant strategies into the SDF $\tilde{m}$. Since $\tilde{f}$ could be many-one, the inverse mapping $\tilde{f}^{-1}$ is a correspondence. So, if agent $i$ sees that the market SDF is $\tilde{m}$ when she has chosen $a_i$, she can back out a set of strategies for other market participants $A_{-i}(\tilde{m}, a_i) \subseteq \prod_{j \neq i} A_j$ which satisfies the condition

\[ a_i \times A_{-i}(\tilde{m}, a_i) = \tilde{f}^{-1}(\tilde{m}). \]  

(10)

That is to say, $A_{-i}(\tilde{m}, a_i)$ denotes the set of strategies that might have been used by other market participants according to agent $i$, given she chose $a_i$ and observed the market SDF $\tilde{m}$.

Agent $i$ has a tradeable arbitrage opportunity if she can find an alternative strategy that increases her utility weakly in every plausible state of nature — and strictly in at least some states — given the strategies of other market participants she has backed out from asset prices. The utility based characterization of arbitrage in Proposition 1 serves as
a useful stepping stone on the road towards more general descriptions of arbitrage based on belief hierarchies.

**Proposition 1.** There is a tradeable arbitrage opportunity in the market with respect to a probability measure \( P \) if and only if there is a market participant \( i \in I \) who may change her strategy from \( a_i \) to \( a_i^* \), such that

\[
\bar{U}_i(a_i^*, a_{-i}) \geq \bar{U}_i(a_i, a_{-i}) \quad \text{P-a.s. and } \quad P[\bar{U}_i(a_i^*, a_{-i}) > \bar{U}_i(a_i, a_{-i})] > 0,
\]

(11)

for all \( a_{-i} \in A_{-i}(\bar{m}, a_i) \).

**Proof.** In the Appendix.

The definition of arbitrage takes on a particularly simple form in (11). There is an arbitrage opportunity if and only if a market participant can revise her strategy to generate weakly higher utility in every state of nature and strictly higher utility in some states. In a concrete sense, therefore, an arbitrage opportunity implies that there is a market participant \( i \) who is responding to the aggregate strategy of other participants with a strategy that can be improved. The aggregate strategy of all participants in the market is embodied in the ex-ante prices, and this gives the equivalence between (11) and earlier definitions of arbitrage. The formulation of arbitrage in Proposition 1 is handy for us because it rests on strategic foundations (the market SDF depends on the choice of all market participants) despite bypassing the intricacies of equilibrium formation.

Having characterized an arbitrage opportunity in terms of agent \( i \)'s strategies and utility, we can now approach the question in terms of “dominated” and “undominated” responses. Agent \( i \)'s strategy, \( a_i \), is said to be a dominated response to \( A_{-i}(\bar{m}, a_i) \) if there is an \( a_i^* \in A_i \) that satisfies condition (11) in the Proposition above. We shall label such strategies dominated-wrtp responses, where the acronym w.r.t.p. stands for “with respect to price”. The label comes from the fact that when agent \( i \) employs such a strategy, she can improve her response given the resulting prices.

**Definition 4.** (Dominated/Undominated-wrtp response) Given her current strategy \( a_i \in A_i \) and other market participant strategies \( a_{-i} \in A_{-i}(\bar{m}, a_i) \), agent \( i \) is said to be using a dominated-wrtp response if she can use another strategy \( a_i^* \neq a_i \) such that \( a_i^* \) and \( a_i \) satisfy condition (11) in Proposition 1. It given by the set

\[
D_i^{\text{wrtp}} = \{ a_i \in A_i : \text{Given } a_{-i} \in A_{-i}(\bar{m}, a_i), \text{ there exists } a_i^* \in A_i \text{ s.t.}
\]

\[
\bar{U}_i(a_i^*, a_{-i}) > \bar{U}_i(a_i, a_{-i}) \quad \text{P-a.s. and } \quad P[\bar{U}_i(a_i^*, a_{-i}) > \bar{U}_i(a_i, a_{-i})] > 0.
\]
\[ \tilde{U}_i(a^*_i, a_{-i}) \geq \hat{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\tilde{U}_i(a^*_i, a_{-i}) > \hat{U}_i(a_i, a_{-i})] > 0 \}. \] (12)

A strategy that is not dominated-wrt \( p \) is termed an undominated-wrt \( p \) response. It is given by the set
\[ UD_i^{wrt} = \{ a_i \in A_i : a_i \not\in D_i^{wrt} \}. \]

The set \( A_{-i}(\tilde{m}, a_i) \subseteq \prod_{j \neq i} A_j \) plays an important role in our analysis. Though we’ve labeled the responses in Definition 4 dominated with respect to price, in reality these are dominated with respect to the market participant strategies that support the price; i.e., the set \( A_{-i}(\tilde{m}, a_i) \). In these terms, Proposition 1 says that there is a tradeable arbitrage opportunity in the market only when a market participant \( i \in I \) is using a dominated-wrt \( p \) response.

**Corollary 1.** (Proposition 1). There is a tradeable arbitrage opportunity in the market with respect to a probability measure \( P \) if and only if there is a market participant \( i \in I \) who is using a dominated-wrt \( p \) response.

We could, in fact, define subsets of the strategy space \( \prod_{j \neq i} A_j \) using many different criteria. As we will see when we define agent \( i \)’s hierarchy of beliefs in the next section, at each level of such a hierarchy there is a different subset of \( \prod_{j \neq i} A_j \) that agent \( i \) could deem plausible for other market participants, and each case generates a different set of dominated responses. To decide whether or not the new dominated response sets coincide with the agent’s dominated-wrt \( p \) response, we will have to compare \( A_{-i}(\tilde{m}, a_i) \) with the subsets of \( \prod_{j \neq i} A_j \) generated by the belief hierarchy. A tradeable arbitrage opportunity is equivalent to a dominated-wrt \( p \) response, so this creates a way to bridge the classical approach to the study of arbitrage with a belief hierarchy based approach.

## 4 A Bayesian’s Belief Hierarchy

In this section, we continue to reason from the standpoint of a particular market participant, but instead of SDF as primitive, we introduce the belief hierarchies supporting the SDF as the basis of our model. A takeaway is that the space of uncertainty that agent \( i \) actually faces in the market is much larger than \( S \) — the domain of fundamental uncertainty — that has been been the traditional focus in asset pricing. This is because agent \( i \) is
uncertain not just about fundamentals, but also about the choices made by other market participants, about what other market participants believe her choice to be, about what other market participants believe about what she believes their choices to be, and so on ad infinitum. Despite the seemingly limitless size, such uncertainty spaces are well-studied mathematical objects, and an agent navigates them by assigning plausibility to certain subsets of belief hierarchies over others. In this section, we define such a plausible hierarchy of beliefs, \( W^k_i \), that plays a special role in the analysis of arbitrage opportunities. We characterize some of the salient properties of \( W^k_i \) and define the notion of a dominated \( k^{th} \) order response for this hierarchy. In Section 5, we shall use the hierarchy \( W^k_i \) to provide a belief based foundation for an arbitrage opportunity.

We continue to use the market model described by condition (9) for agent \( i \). However, the focus in this Section will be primarily on how agent \( i \) reasons about the choice of strategy of other market participants, \( j \neq i \), from \( (A_j)_{j \neq 1} \) before she has herself chosen a strategy \( a_i \) and received the aggregate market choice \( \tilde{m} \). Thus, we shall be discussing the reasoning process employed by the agent in the absence of any signal from the market. Recall that agent \( i \)’s utility depends on \( \tilde{x} \) and \( \tilde{f} \) through the relation in (8). For this Section, we shall abstract away from this dependence and assume directly that agent \( i \) knows the utility mapping \( U_i : \Pi_{j \in I} A_j \times \mathcal{S} \to \mathbb{R} \). Effectively, therefore, we shall be using only the reduced form

\[
\mathcal{M}_i = ((S, \mathcal{S}), I, (A_j)_{j \in I}, U_i),
\]

of the original market model \( \mathcal{M}_i \) for agent \( i \).

### 4.1 Belief Hierarchies and Canonical Homeomorphism

This subsection describes the construction of belief hierarchies for agent \( i \). While not common in finance, the construction below is standard in the epistemic game theoretic literature (see Dekel and Siniscalchi 2015). A reader who is familiar with that literature may skip ahead to the next subsection after browsing through our notation.

How should a Bayesian market participant reason about her situation? To start with, agent \( i \) is uncertain about the state of nature \( S \). However, all our definitions of arbitrage presume the physical probability measure \( P \) is known to market participants, so agent \( i \)’s prior belief over \( (S, \mathcal{S}) \) is predetermined. We label \( b^0_i = P \) agent \( i \)’s zeroth order belief. \( b^0_i \) is a member of the singleton set \( B^0_i \), the set of permitted zeroth order beliefs for agent \( i \).
Agent $i$ is also uncertain about the strategies that are chosen by other market participants. $S$ and $(A_j)_{j \neq i}$ together determine the state space for agent $i$’s layer-0 uncertainty

$$Y^0_i = S \times \prod_{j \neq i} A_j$$

and, as a Bayesian, she must have a prior belief on this space. $Y^0_i$ is a compact metric space since $S$ and $A_j$ are compact metric spaces, and we let $\Delta(Y^0_i)$ denote the set of probability measures on the Borel $\sigma$-field of $Y^0_i$ endowed with the topology of weak convergence. Then $B^1_i = \Delta(Y^0_i)$ is again a compact metric space. Agent $i$’s first order belief $b^1_i$ is a member of the set $B^1_i$.

Agent $i$ realizes that as Bayesians, all agents in the market carry a first order belief in their head, but she is uncertain about the rest of the market’s first order beliefs. Her second order belief is a prior over the first order beliefs of other agents in the market and her own layer-0 uncertainty. Iterating such arguments, we get the state space for agent $i$’s layer-$k$ uncertainty

$$Y^k_i = Y^{k-1}_i \times \prod_{j \neq i} B^k_j.$$ (15)

The agent’s $(k + 1)^{th}$ order belief $b^{k+1}_i$ is a member of the set $B^{k+1}_i = \Delta(Y^k_i)$.

Notice that given such a hierarchy of beliefs, an agent may compute the probability of an event in multiple ways. For instance, both $b^1_i$ and $\text{marg}_{Y^0_i} b^2_i$ give agent $i$’s beliefs on $S \times \prod_{j \neq i} A_j$. Belief hierarchies are termed coherent when they lead to the same probability for events, no matter how the probability is calculated; i.e., when for all $k > 1$,

$$\text{marg}_{Y^{k-1}_i} b^{k+1}_i = b^k_i.$$ (16)

As is standard, we shall assume that not only does agent $i$ have coherent beliefs, but also every other agent in the set $I$ has coherent beliefs, and this fact is common knowledge in the market. We shall use the label consistent to denote beliefs that reflect coherence and
common knowledge of coherence of beliefs in the market. Coherence and consistency of beliefs are important properties of a belief hierarchy, and a maintained assumption throughout the sequel shall be that the beliefs under consideration satisfy these two conditions.

Agent $i$’s belief hierarchy $b_i$ is, therefore, a point in the space

$$B_i = \{(b_i^0, b_i^1, b_i^2, \ldots) \in \prod_{k \geq 0} B_i^k : \text{(17)}\}$$

The set $B_i$ in (17) is a compact and metric subset of $\prod_{k \geq 0} B_i^k$ under the product topology, and foundational work in epistemic game theory (Armbruster & Böge 1979, Boge and Eisele 1979, Mertens and Zamir 1985, Brandenburger and Dekel 1993, Heifetz 1993, among others) has shown that the sets $B_i$ and $\Delta(Y_i^0 \times \prod_{j \neq i} B_j)$ are homeomorphic. Hence, these two sets are of the “same size” and agent $i$ does not need to consider any further priors. Further, the Daniell-Kolmogorov existence theorem and its extensions (see Rogers and Williams 2000, Chapter 3) guarantee that the homeomorphism

$$\phi_i : B_i \rightarrow \Delta(Y_i^0 \times \prod_{j \neq i} B_j)$$

is canonical, with the property that for $b_i \in B_i$, $\text{marg}_{Y_i^{k-1}}[\phi_i(b_i)] = b_i^k$.

In sum, the state space for the total domain of uncertainty faced by agent $i$ in the market is $\Omega = S \times \prod_{j \neq i} A_j \times \prod_{j \neq i} B_j$. This is a much larger space than the domain of fundamental uncertainty, $S$, that is normally used in standard asset pricing theories. Yet, only by accounting for the space of belief hierarchies, $B_i$ in equation (17), does one exhaust all the uncertainty faced by agent $i$. Since finite ordinals are enough to characterize the hierarchies, standard mathematical induction suffices to characterize their attributes. In the next subsection we define a subset of the consistent belief hierarchies that is pertinent to our characterization of arbitrage, and derive some of its properties via induction.

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7To define consistent hierarchies, let $H_i$ denote the set of coherent belief hierarchies for agent $i$. A consistent belief hierarchy for agent $i$ is the set

$$B_i = \{(b_i^0, b_i^1, b_i^2, \ldots) \in H_i : \text{marg}_{Y_i^{k-1}}[\phi_i(b_i)] = b_i^k \forall k \geq 1, \forall j \neq i \text{ and } (b_j^0, b_j^1, b_j^2, \ldots) \in H_j\}.$$
4.2 Dominated and Undominated Responses and Belief Hierarchy Set $W^k_i$

Consistent belief hierarchies, in themselves, are too broad to be useful. Consequently, one imposes further natural restrictions on such hierarchies depending on the application at hand. This is the motivation behind epistemic solution techniques like rationalizability (Bernheim 1984, Pearce 1984) or rationality and common knowledge of rationality (Brandenburger and Dekel 1987, Tan and Werlang 1988). For our application — investigating the link with arbitrage — it is simpler to use a notion of optimization that is a tad bit different from conventional rationality. In conventional rationality under uncertainty, rational agents are deemed to choose strategies that maximize their expected subjective utility. In our case, optimizing agents shall be deemed to choose strategies that give them a weakly higher utility in every state of nature and strictly higher utility in at least some states. This leads to the belief hierarchy set $W^k_i$.

In this section, we provide a microfoundation for the belief hierarchy set $W^k_i$. We describe this hierarchy as the subset of consistent belief hierarchies for agent $i$ that imposes two additional restrictions: (i) the agent believes that other participants do not use dominated responses, (ii) the agent believes that other market participants believe that she does not use dominated responses. In order that her belief about the belief of other market participants be valid, the agent uses a belief hierarchy in $W^k_i$ only when she does not actually use a dominated response. The definition of dominated and undominated response sets use our notion of optimizing agents instead of conventional rationality. Finally, we derive a few key properties of this hierarchy that are useful for the subsequent analysis. Table 1 on the following page gives an intuitive summary of the main characteristics of the belief hierarchy set $W^k_i$, and dominated and undominated responses on this hierarchy. We use the term “unequivocally improve utility” in the table as a shorthand for utility that weakly increases in every state of nature in the support of $P$, and strictly increases in at least some states. Readers primarily interested in the arbitrage side of the story may skip ahead to Section 5 after perusing the table if they so prefer, and come back to this section for details as needed.

Recall that each belief $b^k_i$ in agent $i$’s hierarchy is a probability measure that has a support — the states to which the measure assigns non-zero probability. The probability

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8 The support of a probability measure $Q$ on a measurable space $(B, B)$, denoted by supp$(Q)$, is the smallest closed subset $\bar{B}$ of $B$ such that $Q(\bar{B}) = 1$. 

Table 1: Dominated and Undominated responses, and Belief hierarchy set $W^k_i$

| Condition | Short explanation of the condition |
|-----------|-----------------------------------|
| $a_i \in D^1_i$ | Agent $i$’s strategy $a_i$ is in the dominated first order response set $D^1_i$ if she can unequivocally improve her utility by choosing an alternative response, no matter what strategies other market participants use. Her strategy is in the undominated first order response set $UD^1_i$ if it is not a member of the set $D^1_i$. |
| $a_i \in UD^1_i$ | |
| $a_i \in D^k_i, k > 1$ | Agent $i$’s strategy $a_i$ is in the dominated $k^{th}$ order response set $D^k_i$ if it is a member of her undominated $(k - 1)^{th}$ order response set, and she can unequivocally improve her utility by choosing an alternative response when her beliefs about other market participants’ strategies come from a belief hierarchy $b_i \in W^k_i$. |
| $a_i \in UD^k_i, k > 1$ | Agent $i$’s strategy $a_i$ is in the undominated $k^{th}$ order response set $UD^k_i$ if it is a member of her undominated $(k - 1)^{th}$ order response set, and she cannot unequivocally improve her utility by choosing an alternative response when her beliefs about other market participants’ strategies come from a belief hierarchy $b_i \in W^k_i$. |
| $b_i \in W^k_i, k > 1$ | Agent $i$ is using a belief hierarchy $b_i$ in the set $W^k_i$ means: |
| | • She believes that other agents in the market are using strategies in their undominated $(k - 1)^{th}$ order response sets |
| | • She believes that other agents in the market believe that she is using a strategy in her undominated $(k - 1)^{th}$ order response set |
| | Agent $i$ uses a belief hierarchy $b_i \in W^k_i$ only if her own strategy is a member of her undominated $(k - 1)^{th}$ order response set. |

measure induced by the entire hierarchy of beliefs is given by $\phi_i(b_i)$, and since the canonical homeomorphism $\phi_i$ “preserves beliefs,” agent $i$ can recover her $k^{th}$ order belief from $\phi_i$ by taking the appropriate marginal; i.e., $\text{marg}_{X^{k-1}}[\phi_i(b_i)] = b^k_i$.9

By a dominated response in the context of a belief hierarchy, we mean, roughly, a strategy of agent $i$ that can be improved no matter what strategies other market participants follow and which state of nature realizes ex-post, given they are in the support of agent $i$’s beliefs. Dominated responses with respect to belief hierarchies are closely related to dominated-wrt responses and arbitrage opportunities, and we shall establish the connection rigorously in subsequent sections. For now, to give a precise meaning to a dominated response in the context of belief hierarchies, we need to provide an inductive definition for the notion.

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9$\text{marg}_X[Q]$ is the marginal of probability measure $Q$ on set $X$. $\text{supp} \text{marg}_X[Q]$ is the support of the marginal of $Q$ on $X$. 

19
Agent $i$’s strategy is dominated with respect to her zeroth order belief if she has an alternative strategy that can increase her utility weakly in every state of nature in the support of the probability measure $b_i^0 = P$, no matter what strategies the other market participants select, and further, in at least some states of nature the alternative increases her utility strictly. Such strategies are labeled dominated first order responses. An undominated response is a strategy that is not dominated. Thus, an undominated first order response by agent $i$ to her zeroth order belief cannot be weakly improved upon in every state of nature that agent $i$ presumes possible as well as strictly improved upon in some states, for every strategy that other market participants can use.

**Definition 5.** (Dominated/Undominated first order response) Given a probability measure $P$ over states of nature, agent $i \in I$ is said to be using a *dominated first order response* if her strategy lies in the set

$$D_1^i = \{ a_i \in A_i : \text{For all } a_{-i} \in \prod_{j \neq i} A_j, \text{ there exists } a_i^* \in A_i \text{ s.t. } \tilde{U}_i(a_i^*, a_{-i}) \geq \tilde{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\tilde{U}_i(a_i^*, a_{-i}) > \tilde{U}_i(a_i, a_{-i})] > 0 \}. \quad (19)$$

A strategy that is not a dominated response to agent $i$’s zeroth order belief is termed an *undominated first order response*. It is given by the set

$$UD_1^i = \{ a_i \in A_i : a_i \notin D_1^i \}. \quad (20)$$

Since dominated first order responses can be unequivocally improved, it would seem reasonable to suppose that agent $i$ should not believe that other market participants shall employ such strategies. In other words, agent $i$ should employ a belief hierarchy in

$$V_1^2 = \{ b_i \in B_i : \text{ marg } [\phi_i(b_i)] [a_j \in D_1^j] = 0 \text{ for all } j \neq i \}, \quad (21)$$

since this set excludes beliefs in $B_i$ that give a non-zero probability weight to the use of dominated strategies in $D_1^j$ by other participants. By the same token, agent $i$ must also anticipate that other agents would not believe that she has employed a strategy in $D_1^i$. In this case, the agent should employ a belief hierarchy in

$$W_1^2 = \{ b_i \in V_1^2 : b_i^1 \in \text{ supp marg } [\phi_i(b_i)] \implies \text{ marg } [b_i^1] [a_i \in D_1^i] = 0 \text{ for all } j \neq i \}, \quad (22)$$
since this set excludes the beliefs in $V^2_i$ that permit agent $i$ to believe that other market participants believe that $i$ has used a dominated strategy in $D^1_i$. It also seems reasonable to suppose that if agent $i$ is indeed using a belief hierarchy in $W^2_i$, then she should have selected her strategy from $UD^1_i$. That is,

$$b_i \in W^2_i \implies a_i \in UD^1_i.$$  \hspace{1cm} (23)

Condition (23) ensures that agent $i$’s belief about other market participants’ beliefs (about $i$’s strategy) are valid. One would like to rule out invalid presumptions from $i$’s belief hierarchy because utility gains supported by invalid beliefs would not actually fructify for the agent.

The procedure can be extended iteratively to provide a general inductive definition for $k^{th}$ order belief hierarchy sets. Let $UD^k_i \subseteq UD^{k-1}_i \subseteq \cdots \subseteq UD^2_i \subseteq UD^1_i$ be a series of subsets of $UD^1_i$, $i \in I$. We shall microfound these subsets as higher order undominated responses shortly, but for now let us just take these to be nested subsets of $UD^1_i$. Let $D^2_i = UD^1_i \setminus UD^2_i$, and more generally $D^k_i = UD^{k-1}_i \setminus UD^k_i$. We can define the belief hierarchy set $W^k_i$ almost exactly as we defined $W^2_i$ in the foregoing paragraph. In the first stage define the set

$$V^k_i = \{b_i \in W^{k-1}_i : \text{marg}_{A_j} [\phi_i(b_i)] [a_j \in D^{k-1}_j] = 0 \text{ for all } j \neq i\},$$ \hspace{1cm} (24)

which excludes the beliefs in $W^{k-1}_i$ that give a non-zero probability weight to the use of dominated strategies in $D^{k-1}_{j \neq i}$ by other participants. Next, define $W^k_i$ as the subset of $V^k_i$ with the following property

$$W^k_i = \{b_i \in V^k_i : b^{k-1}_j \in \text{supp marg}_{b_i} [\phi_i(b_i)] \implies \text{marg}_{A_i} [b^{k-1}_j] [a_i \in D^{k-1}_j] = 0 \text{ for all } j \neq i\}.$$ \hspace{1cm} (25)

As with $W^1_i$, the set $W^k_i$ excludes the beliefs in $V^k_i$ that permit agent $i$ to believe that other market participants believe that $i$ has used a dominated strategy in $D^{k-1}_{i \neq j}$. Finally, if agent $i$ is indeed using a belief hierarchy in $W^k_i$, then she should have actually selected her strategy from $UD^{k-1}_i$, to ensure agent $i$’s belief about other market participants’ beliefs (about $i$’s strategy) are valid. That is,

$$b_i \in W^k_i \implies a_i \in UD^{k-1}_i.$$ \hspace{1cm} (26)
Conditions (24) – (26) characterize the belief hierarchy set $W^k_i$. Notice that this is an inductive definition — the definition of $W^k_i$ depends on the definition of $W^{k-1}_i$ ... depends on the definition of $W^2_i$. The set $W^2_i$ is a subset of $B_i$, the set of consistent belief hierarchies, and we adopt the convention $W^1_i = B_i$.

The definition of $W^k_i$ also rests on the definitions of dominated and undominated sets. We have already defined $D^1_i$ and $UD^1_i$ in Definition 5. We now proceed to define the higher order dominated and undominated sets. Intuitively, a dominated $k^{th}$ order response for agent $i$ is dominated with respect to her $(k-1)^{th}$ order belief about other market participant strategies and states of nature. That is to say, she has an alternative response that can increase her utility weakly in every state of nature in the support of the probability measure $P$ — given the strategies of the other market participants are chosen from the support of the marginal of $b^k_{i-1}$ on $\prod_{j \neq i} A_j$ — and further, in at least some states of nature the alternative increases her utility strictly. The belief hierarchies used for dominated and undominated $k^{th}$ order responses come from the set $W^k_i$.

**Definition 6.** (Dominated/Undominated $k^{th}$ order response on $W^k_i$) Given belief hierarchy $b_i \in W^k_i$ for agent $i$, she is said to be using a dominated $k^{th}$ order response if her strategy is a member of the set

$$D^k_i = \{a_i \in UD^{k-1}_i : \text{for all } a_{-i} \in \text{supp marg } \phi_i(b_i), \text{ there exists } a^*_i \in A_i \text{ s.t. } \tilde{U}_i(a^*_i, a_{-i}) \geq \tilde{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\tilde{U}_i(a^*_i, a_{-i}) > \tilde{U}_i(a_i, a_{-i})] > 0\}, \quad (27)$$

A strategy that is not a dominated response to agent $i$’s $k^{th}$ order belief is termed an undominated $k^{th}$ order response. It is given by the set

$$UD^k_i = \{a_i \in UD^{k-1}_i : a_i \notin D^k_i\}. \quad (28)$$

Recall that the set $UD^1_i$ is a subset of $A_i$, and we adopt the convention $UD^0_i = A_i$. Figure 1 on the following page provides a pictorial representation of dominated and undominated responses. Notice that the undominated responses form a nested hierarchy, which is to say that for $k_1 \leq k_2$, we have $UD^{k_1}_i \supseteq UD^{k_2}_i$. Further, the union of the dominated and undominated response sets at any level of the hierarchy is equal to the undominated response set at the preceding level of the hierarchy, i.e. $D^k_i \cup UD^k_i = UD^{k-1}_i$.

A salient point worth emphasizing is that the use of belief hierarchy sets $W^k_i$ for $k \geq 2$,
or dominated and undominated $k^{th}$ order responses for $k \geq 2$, entails higher order reasoning. An agent engages in higher order reasoning when she is reasoning about the reasoning of other market participants (… about the reasoning of other market participants). Recall that $W_i^2$ imposes a restriction on the set $B_i^2$, the set of second order beliefs of agent $i$. Now, agent $i$’s second order belief is a probability measure on $Y_i^1 = Y_i^0 \times \prod_{j \neq i} B_j^1$, and since $B_j^1$ is the set of first order beliefs of agent $j \neq i$, $B_i^2$ encapsulates agent $i$’s reasoning about the reasoning of $j$. This is second order reasoning. We use the terms reasoning and belief interchangeably since it is traditionally assumed that an agent forms a belief about an object after she has reasoned about it; in other words, a higher order belief is the outcome of an agent’s higher reasoning about the belief. Thus, higher the value of $k$ in $W_i^k$, higher the order of reasoning that agent $i$ has employed to arrive at her response.

We justified the definition of belief hierarchy set $W_i^k$ and corresponding higher order response sets using the criteria of optimality and naturalness: it seems unnatural that any agent would choose a suboptimal, dominated response deliberately, or presume that other agents would choose such responses, especially when the order $k$ is not very high. One could, however, decide on other criteria of naturalness to delineate alternative belief hierarchies and their corresponding higher order response sets. Therefore, we would like to pin down a few key properties of $W_i^k$ that become essential in developing the link with arbitrage. We shall work exclusively with $W_i^k$ in the rest of the paper but any belief hierarchy set that satisfies analogous properties should give us similar results.

The first of these properties is that there is a whittling down of the set of belief hierar-
chies that agent $i$ attributes to agent $j$ as $i$ uses belief hierarchy sets of higher and higher orders. That is to say, the belief hierarchies of agent $j$, $j \neq i$, embedded within $b_i \in W_i^k$ form a nested series of subsets as the order $k$ in $W_i^k$ goes up.

**Proposition 2.** Denote $b_j(W_i^k) = \{b_j \in B_j : b_j \in \text{supp } \text{marg}_{b_i}[\phi_i(b_i)], b_i \in W_i^k\}$ for $j \neq i$. Then

$$B_j \supseteq b_j(W_i^2) \supseteq \cdots \supseteq b_j(W_i^{k-1}) \supseteq b_j(W_i^k) \supseteq \cdots$$  \hspace{1cm} (29)

That is, a belief hierarchy in $W_i^k$ embeds a set of belief hierarchies for agent $j$, given by $b_j(W_i^k)$, that is a subset of the set of hierarchies for $j$ embedded in $W_i^{k-1}$, given by $b_j(W_i^{k-1})$.

*Proof.* In the Appendix. $\Box$

Proposition 2 describes how the belief hierarchy sets that $i$ attributes to other market participants shrinks as she climbs up her own hierarchy. The next proposition describes how the space of strategies that $i$ considers viable for other market participants shrinks as she uses belief hierarchies of higher and higher orders. Just like the belief hierarchies of agent $j$ embedded within $b_i$, the space of strategies for $j$ that $i$ believes plausible, too, form a nested series of subsets as the order $k$ in $W_i^k$ goes up.

**Proposition 3.** Denote $A_j(W_i^k) = \{a_j \in A_j : a_j \in \text{supp } \text{marg}_{A_j}[\phi_i(b_i)], b_i \in W_i^k\}$. Then

$$A_j \supseteq A_j(W_i^2) \supseteq \cdots \supseteq A_j(W_i^{k-1}) \supseteq A_j(W_i^k) \cdots$$  \hspace{1cm} (30)

That is, a belief hierarchy in $W_i^k$ embeds a set of strategies for agent $j$, given by $A_j(W_i^k)$, that is a subset of the set of strategies for $j$ embedded in $W_i^{k-1}$, given by $A_j(W_i^{k-1})$.

*Proof.* In the Appendix. $\Box$

The set $A_j(W_i^k)$ represents the set of strategies that agent $i$ deems plausible for agent $j$ when she uses a belief hierarchy in the set $W_i^k$, and the definition of $W_i^k$ implies that $A_j(W_i^k)$ is the set of undominated responses for $j, \mathcal{U}D_j^{k-1}$. Thus, intuitively, Proposition 3 is another expression of the nestedness of undominated responses. Taken in conjunction, Propositions 2 and 3 indicate that as agent $i$ climbs higher and higher up the order $k$ in $W_i^k$, she believes all market participants use strategies in correspondingly high order undominated response sets by responding optimally to their respective belief hierarchies. Since $i \in I$ is a generic agent in the market, when $k$ is unbounded this is in essence an affirmation of rationality and common knowledge of rationality in our setting.
For the sequel, we will assume that equations (21) – (26) that determine the belief hierarchies in $W_i^k$ and Definitions (5) and (6) that determine dominated and undominated responses on the hierarchies in $W_i^k$ characterize our agent $i$. Their properties are summarized in Table 1 on page 19. We shall at times refer to the set of belief hierarchies in $W_i^k$ as belief hierarchies of order $k$. As noted earlier, as the order $k$ in $W_i^k$ goes up, so does the order of reasoning used by agent $i$.

The discussion in this section focused primarily on outlining the appropriate belief hierarchy for a Bayesian agent in the market. It turns out that there is an intimate connection between an arbitrage opportunity and optimal behavior using a belief hierarchy. This is the subject of the next section.

### 5 Arbitrage and Higher Order Beliefs

Having laid out the requisite background in the previous sections, we are now ready to forge the link between arbitrage and market participant belief hierarchies that we’ve been building towards. Specifically, we see that an arbitrage opportunity results only when an agent underestimates the degree to which market participants are optimizing: when choosing her belief hierarchy the agent presumes that market participants will employ undominated responses of order $k$, but on seeing the actual asset prices she infers that they have used undominated responses of a higher order (Theorems 1 and 2).

Crucially, determining the presence of an arbitrage opportunity (or absence) requires the invocation of higher order beliefs of market participants, and no-arbitrage implies that all market participants are reasoning about other market participants’ strategic choices up to a sufficiently high order (Theorem 3).

Precisely how many orders of belief need to be invoked depends on how well the market aggregation mapping $\tilde{f}$ separates among different sets of market participant choices. This is because the arbitrage trade relies on a disparity between the initial presumption of the market agent and what she finds in actuality when she backs out the real choices made by market participants, using $\tilde{f}$ and the asset prices. If the aggregation mapping $\tilde{f}$ is not sufficiently responsive, an agent cannot distinguish the actual choices employed by market participants well, which means that she need not have reasoned too far up her hierarchy, initially, to have avoided arbitrage (Propositions 4 and 5).

An important question raised by this analysis is the extent to which participants have
to *deliberately* reason in market settings to reach a state of no-arbitrage. Common experience seems to suggest that real-world traders don’t go very far up their belief hierarchies when reasoning deliberately; yet arbitrage opportunities are hard to come by in actual markets. Section 5.2 tackles this question by showing that a market tatonnement process — in which agents are simply responding to immediate market circumstances — delivers the same outcomes as the hierarchy-based reasoning process (Corollary 2 and Proposition 6). In other words, the higher order reasoning that we impute to market agents need not be a completely deliberate process. At each stage, market agents could be reasoning just one-step ahead — yet stacking up a series of such “one-step-aheads” in a tatonnement sequence would lead to market outcomes that are indistinguishable from a setting where agents use higher order reasoning.

### 5.1 Link Between Arbitrage and Belief Hierarchy

This subsection establishes the formal link between arbitrage and belief hierarchies. As before, we shall primarily reason from the perspective of a particular market participant, agent $i$, though we shall make the assumption that all the market agents are symmetric in how they approach the problem of selecting their respective strategies. Specifically, the sequence of steps that any agent $i \in I$ follows will be assumed to be:

- *Step-1*: Agent $i \in I$ reasons about the market $\mathcal{M}_i$ (equations 9 and 13) using a belief hierarchy in the set $W^k_i$ defined by equations (21)–(26);

- *Step-2*: She then makes her choice $a_i \in A_i$, and is told the market’s aggregate selection in the form of an SDF $\tilde{m}$.

We shall term Steps 1 and 2 the *eductive sequence* in the market.\(^{10}\) We have analyzed Step-1 separately in Section 4, and Step-2 separately in Section 3, and we now intend to bring the analyses in the two sections together.

As a preliminary, in order to link Steps 1 and 2 in the eductive sequence, we shall assume that any market participant’s choice of strategy in Step-2 is undominated with respect to the belief hierarchy she employs in Step-1.

**Assumption 1.** Each market participant $i \in I$ uses a belief hierarchy in $W^k_i$, $k \geq 0$, in Step-1 of the eductive sequence, and their choice of strategy in Step-2 is an undominated $k^{th}$ order response from $UD^k_i$.

---

\(^{10}\)The term “eductive” was introduced in Binmore (1987) to distinguish play that arises from agents reasoning about the reasoning of other agents in a game.
Assumption 1 simply says that agent $i$ judges a certain subset of strategies plausible for other market participants in Step-1 of the eductive sequence, and in Step-2 chooses her strategy to be an undominated response to that subset. Since a belief hierarchy in $W^k_i$ attributes plausibility to the subset $\prod_{j \neq i} UD_{j}^{k-1} \subseteq \prod_{j \neq i} A_j$ of strategies for other market participants, another way to state Assumption 1 is that it mandates that agent $i$ choose an undominated response to the strategies in $\prod_{j \neq i} UD_{j}^{k-1}$ in Step-2.

Intuitively, $\prod_{j \neq i} UD_{j}^{k-1}$ represents agent $i$’s initial assessment of market participant strategies before she has had a chance to interact with the market. That is to say, when reasoning with a belief hierarchy in $W^k_i$ in Step-1 of the eductive sequence, agent $i$ assigns plausibility to the subset of strategies $\prod_{j \neq i} A_j(W^k_i) = \prod_{j \neq i} UD_{j}^{k-1}$ for other market participants. Subsequently, in Step-2, from the market SDF $\tilde{m}$ and aggregation mapping $\tilde{f}$, she can back out the set of strategies $A_{-i}(\tilde{m}, a_i)$ that have been actually used by other market participants. $A_{-i}(\tilde{m}, a_i)$ thus represents agent $i$’s assessment of the set of strategies that were actually used by the other market participants in Step-1. In simple terms, arbitrage arises when the agent’s actual finding differs from her initial assessment. At a technical level, it is the set-theoretic relationship between $\prod_{j \neq i} UD_{j}^{k-1}$ and $A_{-i}(\tilde{m}, a_i)$ that determines the nuances of the link between belief hierarchy and arbitrage, and we explore the connection in detail in the propositions that follow.

While we shall be working exclusively with the belief hierarchy set $W^k_i$ in this section, most of the results that follow hold more generally for any set of belief hierarchies that satisfy conditions analogous to the ones in Propositions 2 and 3. Essentially, what we need is that the space of strategies that agent $i$ deems plausible for $j \neq i$ form a series of subsets as agent $i$ increases the order of her belief hierarchy set. If this happens, choosing an optimal strategy using level $k$ automatically implies the strategy is optimal for levels $0 \leq n < k$. Most of the proofs below employ some version of this argument.

Our first theorem provides a necessary condition for arbitrage. It says that a tradeable arbitrage opportunity entails at least one market agent misanticipating the responses of other market participants. How might such misanticipation arise? Recall that undominated response sets form a nested hierarchy, i.e. for $k_1 \leq k_2$, we have $\prod_{j \neq i} UD_{j}^{k_1} \supseteq \prod_{j \neq i} UD_{j}^{k_2}$. So, if agent $i$ uses a belief hierarchy in $W^k_i$ to generate her own response, anticipating other market participants to select their responses from their respective undominated $(k - 1)^{th}$ order response sets, the anticipation can go wrong only if other market

---

11The set $A_j(W^k_i)$ was defined in Proposition 3.
participants actually use responses in higher order undominated sets. That is to say, the agent initially underestimates how far up their hierarchy other market participants are optimizing (i.e., choosing undominated responses) and this renders her own initial response suboptimal, creating an opportunity for a riskless profitable trade when she gets to know the actual responses of the market participants.

**Theorem 1.** There is a tradeable arbitrage opportunity in the market with respect to a probability measure $P$ only if there is a market participant $i \in I$ who uses a belief hierarchy in $W_i^k$ in Step-1 of the eductive sequence and finds $A_{-i}(\tilde{m}, a_i) \subset \prod_{j \neq i} UD_j^{k-1}$ in Step-2.

**Proof.** From Proposition 1, a tradeable arbitrage opportunity means we are given that there is an agent $i \in I$ who may change her strategy from $a_i$ to $a_i^*$ to obtain

$$\tilde{U}_i(a_i^*, a_{-i}) \geq \tilde{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\tilde{U}_i(a_i^*, a_{-i}) > \tilde{U}_i(a_i, a_{-i})] > 0,$$

for any $a_{-i} \in A_{-i}(\tilde{m}, a_i)$. We will use a proof by contradiction to obtain the result. Suppose $A_{-i}(\tilde{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k-1}$. Recall that for undominated response sets, $UD_i^{m1} \supseteq UD_i^{m2}$ when $m1 \leq m2$ (see Definition 6), i.e. the undominated response sets form a nested sequence as one increases the order. By Assumption 1, $a_i \in UD_i^k$, which means that for any strategy $a_{-i}$ that other market participants choose in the set $\prod_{j \neq i} UD_j^{k-1}$, agent $i$ cannot unequivocally improve her utility\textsuperscript{12} by changing her strategy. Therefore, when $A_{-i}(\tilde{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k-1}$, agent $i$ cannot unequivocally improve her utility by changing her strategy if she is already choosing her strategy from the set $UD_i^k$. This means there is no $a_i^*$ satisfying condition (31) above. Therefore, we have the result.

\[ \square \]

Assumption 1 links the agent’s belief hierarchy to her undominated response, so an equivalent way to state Proposition 1 is that there is a tradeable arbitrage opportunity in the market only if there is a market participant $i \in I$ who uses a response in $UD_i^k$ but finds $A_{-i}(\tilde{m}, a_i) \subset \prod_{j \neq i} UD_j^{k-1}$.

Is the condition in Theorem 1 also sufficient for arbitrage? Not quite. To see why, notice that a belief hierarchy in $W_i^k$ — which requires that agent $i$ choose a strategy in the undominated set $UD_i^k$ under Assumption 1 — doesn’t completely pin down agent $i$’s

\textsuperscript{12}Recall that we the term “unequivocally improve utility” is a short-form for condition (31), i.e. a change of strategy from $a_i$ to $a_i^*$ for agent $i$ leading to

$$\tilde{U}_i(a_i^*, a_{-i}) \geq \tilde{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\tilde{U}_i(a_i^*, a_{-i}) > \tilde{U}_i(a_i, a_{-i})] > 0 \text{ for any } a_{-i} \in A_{-i}(\tilde{m}, a_i).$$
strategy set. Since undominated responses form nested subsets, agent $i$ could very well be choosing a strategy in the set $\cap_{k \geq 0} UD_i^k$ when the only restriction is that she select in $UD_i^k$. If this is the case, however, she has no tradeable arbitrage opportunity no matter what responses other market participants select. This is because agent $i$ has already chosen from the best possible response set she can find — a response in $\cap_{k \geq 0} W_i^k$ — and there is no way for her to improve it any further. Intuitively, she has exhausted all the infinite orders of her hierarchy and, therefore, her response cannot be made any better. We need to exclude this case from our specification of undominated responses to derive a sufficient condition. In other words, we need to specify a set $UD_i^k \setminus UD_i^{k+n}$, $n$ finite, for agent $i$'s strategy, in addition to specifying the belief hierarchy set $W_i^k$. Since $UD_i^k \setminus UD_i^{k+n} = \bigcup_{l=0}^n UD_i^{k+l} \setminus UD_i^{k+l+1}$, we can as well work with the set $UD_i^k \setminus UD_i^{k+1}$ without loss of generality, and this is what we do.

The condition $A_{-i}(\tilde{m}, a_i) \subset \prod_{j \neq i} UD_j^{k-1}$ in Theorem 1, too, needs a slight modification to generate sufficiency. If agent $i$ selects an undominated response to the set $\prod_{j \neq i} UD_j^{k-1}$, by the nestedness of undominated sets, the response is automatically undominated for the sets $\prod_{j \neq i} UD_j^{k-2}$, $\prod_{j \neq i} UD_j^{k-3}$, ..., $\prod_{j \neq i} UD_j^0$. This is the rationale behind the requirement that $A_{-i}(\tilde{m}, a_i)$ be a strict subset of $\prod_{j \neq i} UD_j^{k-1}$ when there is an arbitrage opportunity. However, $A_{-i}(\tilde{m}, a_i) \subset \prod_{j \neq i} UD_j^{k-1}$ is a characterization based on specifying the regions where arbitrage cannot exist, it does not say anything about the regions where the arbitrage opportunity does exist. For sufficiency, we need to recognize that $UD_i^k \setminus UD_i^{k+1} = D_i^{k+1}$ (see Figure 1 on page 23), which is a dominated response to the set $\prod_{j \neq i} UD_j^k$. Now, a dominated response to a set implies the response is also dominated with respect to every subset of the said set. Therefore, whenever $A_{-i}(\tilde{m}, a_i)$ belongs in a subset of $\prod_{j \neq i} UD_j^k$ the agent has an arbitrage opportunity. These sufficient conditions are summarized in Theorem 2.

**Theorem 2.** If there is a market participant $i \in I$ who uses a belief hierarchy in $W_i^k$ in Step-1 of the eductive sequence, and selects an undominated response in the set $UD_i^k \setminus UD_i^{k+1}$ to find $A_{-i}(\tilde{m}, a_i) \subset \prod_{j \neq i} UD_j^k$ in Step-2, there is a tradeable arbitrage opportunity in the market with respect to the probability measure $P$.

**Proof.** From condition (28) in Definition 6, $UD_i^k \setminus UD_i^{k+1} = D_i^{k+1}$. The definition of $D_i^{k+1}$ implies that agent $i$ may change her strategy from $a_i$ to $a_i^*$ to obtain

$$\tilde{U}_i(a_i^*, a_{-i}) \geq \tilde{U}_i(a_i, a_{-i}) \text{ a.s. and } P[\tilde{U}_i(a_i^*, a_{-i}) > \tilde{U}_i(a_i, a_{-i})] > 0,$$

(32)
for any $a_{-i} \in \prod_{j \neq i} \mathcal{U}D_k^j$. Next, as noted before, undominated response sets have the property $\mathcal{U}D_m^{i+1} \supseteq \mathcal{U}D_m^i$ when $m_1 \leq m_2$, i.e. the undominated response sets form a nested sequence as one increases the order. Thus, $A_{-i}(\tilde{m}, a_i) \subseteq \prod_{j \neq i} \mathcal{U}D_k^j$ means condition (32) above holds for any $a_{-i} \in A_{-i}(\tilde{m}, a_i)$. From Proposition 1, this implies that there is a tradeable arbitrage opportunity in the market with respect to the probability measure $P$, since agent $i$ may change her strategy to $a_i^*$. Therefore, we have the result.

In sum, Theorems 1 and 2 say that a tradeable arbitrage opportunity in the market is equivalent to there being at least one agent in the market who has not exhausted her entire order hierarchy when choosing her response, and who finds (when she sees the actual asset prices) that other market participants are going further up their respective hierarchies in selecting responses than she had anticipated.

The contrapositive of the above theorems gives us a characterization of no-arbitrage in this setup. As we’ve already discussed in the context of Theorem 2, if market participants are exhausting their entire order hierarchy when choosing their response (i.e., choosing in the set $\bigcap_{n \geq 0} \mathcal{U}D_n^i$), there is no way for them to improve their responses any further. This is one of the avenues through which no-arbitrage may be achieved in markets. The second avenue is through agents choosing an undominated response with order just high enough that they cannot improve on it even after observing the actual asset prices. It is this second avenue for no-arbitrage that makes the market environment distinct from traditional game-theoretic environments (more on this below). Theorem 3 shows that these are the only two avenues for no-arbitrage in markets. Like the characterizations in the fundamental theorems of asset pricing (Dybvig and Ross 1987), the theorem generates an alternative definition for arbitrage-free markets.

**Theorem 3.**

i. There is no tradeable arbitrage opportunity in the market if every market participant $i \in I$ either chooses a response in the set $\mathcal{U}D_k^i \setminus \mathcal{U}D_{i+1}^k$ and finds $A_{-i}(\tilde{m}, a_i) \supseteq \prod_{j \neq i} \mathcal{U}D_j^{k_i-1}$, or chooses a response in the set $\bigcap_{n \geq 0} \mathcal{U}D_n^i$.

ii. There is no tradeable arbitrage opportunity in the market only if every market participant $i \in I$ either chooses a response in the set $\mathcal{U}D_k^i \setminus \mathcal{U}D_{i+1}^k$ and finds $A_{-i}(\tilde{m}, a_i) \supseteq \prod_{j \neq i} \mathcal{U}D_j^{k_i}$, or chooses a response in the set $\bigcap_{n \geq 0} \mathcal{U}D_n^i$.

**Proof.** Part-1a. If every market participant $i \in I$ chooses a response in the set $\mathcal{U}D_k^i \setminus \mathcal{U}D_{i+1}^k$ and finds $A_{-i}(\tilde{m}, a_i) \supseteq \prod_{j \neq i} \mathcal{U}D_j^{k_i-1}$ there is no tradeable arbitrage opportunity:
The most important ingredient is the belief hierarchy, and no-arbitrage requires that a purely belief-based phenomenon, without any explicit reference to asset fundamentals. If a market participant \(i \in I\) chooses a response in the set \(\mathcal{UD}^{k_i}_i \setminus \mathcal{UD}^{k_i+1}_i\), we have that \(a_i \in \mathcal{UD}^{k_i}_i\). Thus, \(a_i\) is an undominated response for any \(a_{-i} \in \prod_{j \neq i} \mathcal{UD}^{k_j}_{j-1}\). Further, since \(A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} \mathcal{UD}^{k_j}_{j-1}\), \(a_i\) is undominated for at least some \(a_{-i} \in A_{-i}(\bar{m}, a_i)\). Therefore, there is no \(a_i^*\) such that \(i\) may change her strategy from \(a_i\) to \(a_i^*\) to obtain

\[
\hat{U}_i(a_i^*, a_{-i}) \geq \hat{U}_i(a_i, a_{-i}) \text{ \(\mathbf{P}\)-a.s. and } \mathbf{P}[\hat{U}_i(a_i^*, a_{-i}) > \hat{U}_i(a_i, a_{-i})] > 0,
\]

for every \(a_{-i} \in A_{-i}(\bar{m}, a_i)\). If this condition is true for all market participants, the contrapositive of Corollary 1 implies there is no tradeable arbitrage opportunity in the market.

Part 1b. If every market participant chooses a response in the set \(\bigcap_{n \geq 0} \mathcal{UD}^n_i\), there is no tradeable arbitrage opportunity:

If a market participant \(i \in I\) chooses a response in the set \(\bigcap_{n \geq 0} \mathcal{UD}^n_i\), she is in effect using a belief hierarchy in \(\bigcap_{k \geq 0} \mathcal{W}^k_i\), and has thus exhausted all the orders of the belief hierarchy set. Thus, there is no \(a_i^*\) such that \(i\) may change her strategy from \(a_i\) to \(a_i^*\) to obtain

\[
\hat{U}_i(a_i^*, a_{-i}) \geq \hat{U}_i(a_i, a_{-i}) \text{ \(\mathbf{P}\)-a.s. and } \mathbf{P}[\hat{U}_i(a_i^*, a_{-i}) > \hat{U}_i(a_i, a_{-i})] > 0,
\]

no matter what \(a_{-i}\) the other market participants select. In particular, this holds for \(a_{-i} \in A_{-i}(\bar{m}, a_i)\). If no participant can choose an \(a_i^*\) satisfying the condition, the contrapositive of Corollary 1 implies there is no tradeable arbitrage opportunity in the market.

Part 2. There is no tradeable arbitrage opportunity in the market only if every market participant \(i \in I\) either chooses a response in the set \(\mathcal{UD}^k_i \setminus \mathcal{UD}^{k+1}_i\) to find \(A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} \mathcal{UD}^{k_j}_{j-1}\), or chooses a response in the set \(\bigcap_{n \geq 0} \mathcal{UD}^n_i\):

We will employ a proof by contradiction for this part. We work with the contrapositive of the statement. Suppose there is a market participant \(i \in I\) who neither chooses a response in the set \(\bigcap_{n \geq 0} \mathcal{UD}^n_i\) nor chooses a response in the set \(\mathcal{UD}^k_i \setminus \mathcal{UD}^{k+1}_i\) to find \(A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} \mathcal{UD}^{k_j}_{j-1}\). In this case, \(i\) will have chosen a response in the set \(\mathcal{UD}^k_i \setminus \mathcal{UD}^{k+1}_i\) to find \(A_{-i}(\bar{m}, a_i) \subseteq \prod_{j \neq i} \mathcal{UD}^{k_j}_{j-1}\). By Theorem 2 we then have that there is a tradeable arbitrage opportunity in the market. Thus, a contradiction.

\[\Box\]

Taken together, Theorems 1–3 illustrate that arbitrage in markets could be described as a purely belief-based phenomenon, without any explicit reference to asset fundamentals. The most important ingredient is the belief hierarchy, and no-arbitrage requires that mar-

\[\text{\footnote{Recall that } \neg(P \rightarrow Q)\text{, the negation of an implication, is } P \land \neg Q\text{. In our case } P\text{ is “chooses a response in the set } \mathcal{UD}^k_i \setminus \mathcal{UD}^{k+1}_i\text{” and } Q\text{ is “finds } A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} \mathcal{UD}^{k_j}_{j-1}\text{”.}}\]

31
ket participants optimize till a sufficiently high order. In other words, checking the optimality of first order beliefs is not enough to ensure no-arbitrage in markets, and higher order beliefs can play a pivotal role. If a market is arbitrage-free, it is quite likely that agents are engaging in higher order reasoning using their belief hierarchies.\textsuperscript{14} Such higher order reasoning need not always be deliberate, as we highlight in the next subsection, and could also follow from a mechanical process of tatonnement. Nevertheless, higher order beliefs and higher order reasoning are an essential component of arbitrage-free markets and likely play a much more fundamental role in asset pricing than traditionally envisaged for them.

At this point, it may be worthwhile to note the distinctions between no-arbitrage and epistemic game-theoretic solution concepts like rationalizability (Bernheim 1984, Pearce 1984) or rationality and common knowledge of rationality (Brandenburger and Dekel 1987, Tan and Werlang 1988). First, of course, there is the difference in the optimization criteria discussed in Section 4.2. More importantly, the epistemic notions mandate that agents must exercise all the infinite orders of their hierarchy, i.e. the agents have to select from (the equivalent of) $\bigcap_{n \geq 0} UD^n_i$, and there is no provision for taking in feedback from the game using a set like $A_{-i}(\hat{m}, a_i)$. In case of no-arbitrage, on the other hand, all agents $i \in I$ selecting from $\bigcap_{n \geq 0} UD^n_i$ is only one avenue to achieve the outcome. No-arbitrage may equally well be achieved through agents choosing an undominated response with order just high enough that they cannot improve on it even after observing the actual asset prices.

Whether or not $\prod_{j \neq i} UD_{j}^{k_i-1}$ is a subset of $A_{-i}(\hat{m}, a_i)$ depends on how well the aggregation mapping $\hat{f}$ distinguishes among sets of market participant choices. For example, if the stochastic discount factor $\hat{m}$ stays unchanged no matter what strategies $(a_i)_{i \in I}$ market participants choose, we have $A_{-i}(\hat{m}, a_i) = \prod_{j \neq i} A_j$. This means that $A_{-i}(\hat{m}, a_i) \supseteq \prod_{j \neq i} UD_{j}^{k_i-1}$ no matter which order value $k_i \geq 1$ is used.\textsuperscript{15} That is to say, the no-arbitrage condition places no restriction, whatsoever, on the belief hierarchies that may prevail in the market in this case. In such a market, there is no possibility of arbitrage regardless of which belief hierarchies are used because market participants cannot distinguish at all the beliefs that support the observed prices.

\textsuperscript{14}As discussed in Section 4.2, an agent engages in higher order reasoning when she is reasoning about the reasoning \ldots about the reasoning of other market participants, and a higher order belief is the outcome of an agent’s higher reasoning.

\textsuperscript{15}Recall that $UD^n_j = A_j$ by convention, so that for $k = 1$ we have $\prod_{j \neq i} A_j = \prod_{j \neq i} UD_{j}^{k_i-1}$.
On the other hand, if \( \tilde{m} \) changes for every new combination of market participant choices (i.e., \( \tilde{f} \) is one-to-one), then Proposition 4 shows that no-arbitrage implies that all market participants are exercising their entire infinite hierarchy of beliefs.

**Proposition 4.** When \( \tilde{f} \) is one-to-one, there is no tradeable arbitrage opportunity in the market if and only if every market participant \( i \in I \) chooses a response in the set \( \bigcap_{n \geq 0} UD_i^n \).

**Proof.** In the Appendix.

In other words, if the aggregation mapping \( \tilde{f} \) allows market participants to distinguish the beliefs supporting asset prices really well, then the participants must reason about rather high order beliefs if they have to achieve a state of no-arbitrage. Proposition 4 encapsulates the extreme end of this story — with \( \tilde{f} \) one-to-one, agents must choose in the set \( \bigcap_{n \geq 0} UD_i^n \). In fact, we can derive a comparative statics result based on the responsiveness of the aggregation mapping that highlights this narrative more generally.

Recall from (10) that \( a_i \times A_{\sim i}(\tilde{m}, a_i) = \tilde{f}^{-1}(\tilde{m}) \). The notation \( A_{\sim i}(\tilde{m}, a_i) \) presumes that the aggregation mapping \( \tilde{f} \) is fixed. When considering comparative statics involving the market aggregation mapping, it helps to use the notation \( A_{\sim i}^{f_2}(\tilde{m}, a_i) \) to indicate that the inversion is undertaken with respect to the mapping \( \tilde{f} \). We say that market aggregation mapping \( \tilde{f}_2 \) is more responsive than market aggregation mapping \( \tilde{f}_1 \) if

\[
A_{\sim i}^{f_2}(\tilde{m}, a_i) \subseteq A_{\sim i}^{f_1}(\tilde{m}, a_i) \quad \forall i \in I.
\]

In intuitive terms, \( \tilde{f}_2 \) allows agents to distinguish among market participant choices better than \( \tilde{f}_1 \). If this is the case, the proposition below shows that agents must be using a belief hierarchy set with higher order under \( \tilde{f}_2 \), than under \( \tilde{f}_1 \), to attain no-arbitrage.

**Proposition 5.** If \( \tilde{f}_2 \) is more responsive than \( \tilde{f}_1 \) — and \( k_{i_2}, k_{i_1} \) are the minimum orders for which \( A_{\sim i}^{f_2}(\tilde{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k_{i_2}} \) and \( A_{\sim i}^{f_1}(\tilde{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k_{i_1}} \), \( i \in I \) — then

\[
k_{i_2} \geq k_{i_1},
\]

and the minimum order of the undominated response set from which a market participant selects, given no-arbitrage, increases weakly under \( \tilde{f}_2 \) (ceteris paribus).

**Proof.** In the Appendix.
In order to get the intuitive gist of Propositions 4 and 5, it helps to go over a somewhat heuristic analogy. Imagine a trader in the market deciding how many orders of reasoning to use. In order to leave no money on the table, the trader knows he has to reason one step ahead of the market (more accurately, ensure $\prod_{j \neq i} UD_j \subseteq A^{-f}_{-1}(\hat{m}, a_i)$). Since the objective is to stay one step ahead, how many steps ahead the trader actually reasons depends on his perception of the number of steps the market reasons. After he observes the market prices, this perception of the trader is captured in a variable like $A^{-f}_{-1}(\hat{m}, a_i)$. When the inverse aggregation mapping is completely precise ($f$ is one-one), so that the trader knows the number of steps used by the market exactly, there ensues a competitive game — if our trader is one step ahead of the market, then the market is one step behind and plays catch-up in order to leave no money on the table, and vice-versa — that pushes all traders to reason ahead an unbounded number of steps. On the other hand, if the inverse aggregation mapping conveys a rather vague description of the number of steps ($f$ is not very responsive), then each trader can convince himself that he is leaving no money on the table despite not reasoning very many steps ahead. This happens because arbitrage uses a rather demanding interpretation for “money on the table” in case of vague descriptions: only when no money is lost in every scenario that can be conceived under the vague description, and some money gets made in at least some scenarios, do we have an arbitrage. As the descriptions get more and more vague, the number of scenarios that may be conceived under the descriptions multiply, rendering it ever more likely for traders to uncover a scenario where they can lose money — despite not reasoning too many steps ahead.

In narrative above, a very important role is played by the reasoning faculty of agents. Is it possible for markets to be arbitrage-free when agents do not possess such powers of reason? This is the question we tackle in the next subsection.

### 5.2 Tatonnement versus Reasoning

In the previous subsection we analyzed no-arbitrage through the lens of a market participant’s reasoning. We could also proceed with a similar analysis using the lens of market tatonnement. Corollary 1 told us that a necessary and sufficient condition for a tradeable arbitrage opportunity is the presence of a market participant who uses a dominated-wrtwp response. Another way to state the corollary would be as follows.

**Corollary 2.** (Proposition 1). *There is no tradeable arbitrage opportunity in the market with*
respect to a probability measure \( P \) if and only if there is no market participant \( i \in I \) who is using a dominated-wrt\( p \) response.

Corollaries 1 and 2 make no reference to the reasoning process (if any) employed by market participants, and in this subsection we make no assumptions on how market agents actually undertake their reasoning (they could very well be not reasoning at all, at least deliberately). Our sole assumption here will be that agents trade away any immediate arbitrage opportunity that becomes available to them. The main takeaway is that this single assumption about trading behavior results in a market adjustment process that is, for all purposes, equivalent to market participants reasoning about progressively higher orders of their belief hierarchy. Such an adjustment process continues until a no-arbitrage condition like Corollary 2 (or Theorem 3, when employing belief hierarchies) is met. In other words, even when agents are not deliberately using higher order beliefs, they end up behaving as if they do, as long as they trade away arbitrage opportunities.

We use the term tatonnement to describe the adjustment process by which a market with arbitrage opportunities transforms into a market with no-arbitrage. More precisely, market tatonnement describes the following algorithm:

- **Step-1:** Each market participant \( i \in I \) selects a strategy \( a_i \in A_i \) and then gets to know the market SDF \( \tilde{m}_1 = \tilde{f}((a_i)_{i \in I}) \). If no market participant finds that she is using a dominated-wrt\( p \) response, the process concludes. Otherwise, the process moves to Step-\( n \), with \( n = 2 \).

- **Step-\( n \):** A market participant \( i \in I \) who finds that she has used a strategy in her dominated-wrt\( p \) response set \( D_{i}^{\text{wrt} p} \), given \( a_i, \tilde{m}_{n-1} \) and \( \tilde{f} \), selects an alternative strategy \( a_i^\ast \) from her undominated-wrt\( p \) response set \( UD_{i}^{\text{wrt} p} \). This gives rise to a new market SDF \( \tilde{m}_n \). After these adjustments, if no market participant finds that she is using a dominated-wrt\( p \) response, the process concludes. Otherwise, a counter \( l \) is first set to \( l = n \), and the process moves back to Step-\( n \), but with \( n = l + 1 \).

Thus, the tatonnement process concludes when the condition in Corollary 2 is satisfied and the market is arbitrage-free.

At the outset, let us note that a market where the tatonnement process has concluded is equivalent (in terms of outcome) to a market in which every market participant \( i \in I \) either chooses a response in the set \( UD_{i}^{k_i} \setminus UD_{i}^{k_i+1} \) and finds \( A_{-i}(\tilde{m}, a_i) \supset \prod_{j \neq i} UD_{j}^{k_j} \), or chooses a response in the set \( \bigcap_{n \geq 0} UD_{i}^{n} \). This follows from the fact that both Corollary 2
and Theorem 3 are equivalent descriptions of an arbitrage-free market. An outside analyst who has access to only the final outcome of the market adjustment process (i.e. the final asset prices and final quantities traded by market participants) has no way of guessing whether the tatonnement algorithm, or the belief hierarchy based reasoning procedure in Section 5.1, was used in arriving at the outcome.

The proposition below shows that a similar outcome equivalence holds for the most part at every step of the tatonnement algorithm.

**Proposition 6.** A market participant \( i \in I \) changing her strategy from \( a_i \in D_i^{wtp} \) initially, to \( a_i^* \in UD_i^{wtp} \) subsequently, corresponds to the following change,

1. Initially, the participant uses a belief hierarchy in \( W_i^k \) for the eductive sequence and finds \( A_{-i}(\tilde{m}, a_i) \subset \prod_{j \neq i} UD_{j}^{k-1} \).
2. Subsequently, the participant either uses a belief hierarchy in \( W_i^{k+\alpha} \) for the eductive sequence and finds \( A_{-i}(\tilde{m}, a_i) \supset \prod_{j \neq i} UD_{j}^{k+\alpha} \), or uses a belief hierarchy in \( \bigcap_{\beta > \alpha} W_i^{k+\beta} \), for some integer \( \alpha \geq 0 \).

**Proof.** In the Appendix. 

Proposition 6 implies that each step of the tatonnement process increases the order of the belief hierarchy set for some agent \( i \in I \) by the quantity \( \alpha \) (i.e., \( W_i^k \) to \( W_i^{k+\alpha} \)). The value of \( \alpha \) may be zero because of the gap between necessary and sufficient conditions for arbitrage,\(^{16}\) but \( \alpha \) is always non-negative. In other words, every round of tatonnement (weakly) increases the order of the belief hierarchy of a market participant who trades away the arbitrage opportunity in that round. Thus, as more and more rounds of the tatonnement process pile on, it gets increasingly likely that agents in the market are all exercising their higher order reasoning — assuming the market is arbitrage-free. This climb higher and higher up the belief hierarchy is especially striking because it does not need the agents to “deliberately” engage in higher order reasoning. In each round of tatonnement, the agents are simply trading away an immediate arbitrage opportunity. Nevertheless, the cumulative effect of repeated rounds of tatonnement is as if the agents are undertaking higher order reasoning.

To see the content of Proposition 6 more intuitively, it helps to note that one may

\(^{16}\)More discussion on the \( \alpha = 0 \) scenario follows below, see footnote 17.
Figure 2: **Tattonnement and reasoning.** Tattonnement adjustments in the market can be represented using participant belief hierarchies and reasoning.

partition any individual market participant’s strategy space as follows:

\[
A_i = \bigcup_{n \geq 0} D^n_i \cup \left( \bigcap_{l \geq 0} UD^l_i \right), \quad i \in I. \tag{37}
\]

So, in case of a two-participant market (say agents \(i\) and \(j\)), we may represent the strategy space of market participants using the chess-like board in Figure 2. Every strategy pair that the agents decide on can be represented as a point on this board. For example, suppose agent \(i\) initially chooses a point in \(D^4_i = UD^3_i \setminus UD^4_i\), while agent \(j\) initially chooses a point in \(D^6_j = UD^5_j \setminus UD^6_j\). This pair of strategies is the red cross on the left in the figure. Suppose the responsiveness of the market aggregation mapping is such that \(i\) cannot distinguish \(j\)'s responses in the horizontally hatched region, and \(j\) cannot distinguish \(i\)'s responses in the vertically hatched region. Then, \(i\)'s initial choice is dominated-wrt to while \(j\)'s initial choice is not. Alternatively, using the reasoning based narrative, we have that agent \(i\) chose a belief hierarchy in \(W^3_i\) (i.e., \(k = 3\)) initially, then found out that \(A_{-i}(\tilde{m}, a_i)\) lay inside the set \(UD^4_j\), which implies \(A_{-i}(\tilde{m}, a_i) \subset UD^{k-1}_j\) (with \(k = 3\)). Thus, \(i\)'s status
satisfies the initial condition in (i) in Proposition 6. By the same token, agent \( j \) chose a strategy in \( W^5_j \) (i.e., \( k = 5 \)) and found out that \( A_j(\tilde{m}, a_j) \) was a member of \( UD^3_i \), which implies \( A_j(\tilde{m}, a_j) \not\subset UD^{k-1}_i \) (with \( k = 5 \)). Thus \( j \)'s status does not satisfy the initial condition in Proposition 6. After realizing that she has a tradeable arbitrage opportunity, suppose agent \( i \) selects a new response in \( D^7_i = UD^6_i \setminus UD^7_i \). The pair of strategies of the agents is now the blue cross on the right in the figure. Now, agent \( i \)'s strategy is undominated-wrtp. In the reasoning based narrative, she now uses a belief hierarchy in \( W^6_i \) (i.e., \( k = 6 \)). \( A_i(\tilde{m}, a_i) \) is still in the set \( UD^4_j \) from \( i \)'s perspective since \( j \)'s response is unchanged from earlier, and therefore \( A_i(\tilde{m}, a_i) \supset UD^k_j \) (with \( k = 6 \)). Agent \( i \)'s status now satisfies the subsequent condition in (ii) in Proposition 6. \( i \)'s belief hierarchy set changed from \( W^3_i \) to \( W^6_i \), so \( \alpha \) in this case is 3.

The object of Proposition 6 and the example is to highlight the point that an outside analyst can build an alternative narrative for most tatonnement adjustments in the market using the technique of belief hierarchies and higher order reasoning, even if market participants are not deliberately engaging in the reasoning. This is because any tatonnement adjustment process traces out a trajectory in the \( I \)-dimensional space \( A_1 \times A_2 \times \cdots \times A_I \), through the grids formed by the partitions of \( A_i, i \in I \) (in the figure, the squares formed by overlapping \( D^k_i \) and \( D^k_j \)).

The only tatonnement adjustments that an analyst cannot capture using belief hierarchies are the ones that happen within the grids. When the grids formed by the partition of strategy spaces described in (37), based on the belief hierarchy set \( \bigcap_{k \geq 0} W^k_i \), are too coarse to capture the move from dominated-wrtp to undominated-wrtp responses, the reasoning based narrative is no longer sufficient to describe the tatonnement adjustments.\(^{17}\) In such a case, though, the analyst can move to a different belief hierarchy set (not based on \( W_i \) that gives finer partitions of \( A_i, i \in I \). As mentioned before, there is nothing sacrosanct about the set \( W^k_i \), and we can undertake similar analysis with other hierarchy sets that satisfy the requisite properties in Section 4.2.

6 Discussion

Understanding arbitrage or the lack of it, in markets, has been a central preoccupation of finance, and this paper provides a new characterization of arbitrage using belief hier-

\(^{17}\) This corresponds to the scenario \( \alpha = 0 \) in Proposition 6, in which case both the initial and subsequent condition in the proposition have the participant using a belief hierarchy in the same set \( W^k_i \).
archies of Bayesian market participants. It is shown that an arbitrage opportunity exists only when a market participant underestimates the order of belief hierarchies actually used for reasoning in the market. The arbitrage trade depends both on the responsiveness of the market aggregation mapping and the order of the belief hierarchies employed by market agents, and the more responsive the aggregation mapping, the higher the order of reasoning needed from agents to avoid arbitrage. In the Bayesian belief hierarchy based approach to markets, therefore, no-arbitrage seems to go almost hand-in-hand with higher order beliefs of participants. Can we measure from empirical data the extent to which higher order beliefs get used by market participants?

There seem to be multiple ways to do this using our analysis. What we have termed responsiveness of the market aggregation mapping is not very different, empirically, from the price impact function — the degree to which asset prices move in response to a trade. In case there is no price impact at all, no matter what trade is undertaken, the aggregation mapping is likely to be unresponsive. As we discussed earlier, this means market participants do not have to employ higher order beliefs to achieve a state of no-arbitrage. On the other hand, when the price impact of trades is high, the aggregation mapping is likely to be more responsive and the use of higher order beliefs becomes necessary for no-arbitrage. Therefore, the magnitude of the price impact function can serve as a proxy for the extent to which higher order beliefs are in use in a market, assuming the asset prices provide no arbitrage opportunities. Of course, such an assertion comes with a number of caveats. Price impact often has a component attributed to asymmetric information, and since our model has no explicit role for information asymmetry, this component would need to be carefully decoupled. Further, the precise nature of the relationship between price impact and responsiveness of market aggregation mapping would need more careful consideration before taking such a methodology to the data. With the responsiveness of the aggregation mapping, what we are after is how well agents can distinguish among market choices that support an observed price. The size of the trade, which is the main ingredient for measuring traditional price impact, is only one dimension along which market choices may be distinguished. In this sense, price impact is a necessary condition for market aggregation responsiveness when its value is low, and sufficient condition for market aggregation responsiveness when its value is high. How might one enrich the traditional measurement of price impact so that we have a measure that is closer to a necessary and sufficient condition? Empirical methodology questions like these would need to be carefully addressed if we have to establish a credible procedure for determining
higher order beliefs from the data through price impact functions.

Another avenue to measure the extent of higher order beliefs in markets could be through surveys. There is a growing literature that uses survey data to decipher belief formation for asset prices (see Adam and Nagel (2022) for a recent survey) but the emphasis in this area has been mostly on first order beliefs.\footnote{In the empirical macroeconomics literature, researchers have recently started to work with higher order expectations data. Notably, Coibion et al. (2021) provides an analysis of survey data that estimates higher order macroeconomic expectations of firm managers.} How might we gauge through surveys if higher order beliefs are in use? The analysis in this paper suggests that higher the order of beliefs over which a market participant optimizes, smaller the set of strategies she considers feasible for other market participants. So, for instance, periodically surveying hedge fund managers about how they think Reddit-based retail traders might trade, or surveying retail traders about how institutional investors might trade, and so on, can provide a way to decipher the time variation of higher order beliefs in the market. If a category of traders attributes a wide variety of strategies to another category, it is likely that the order of beliefs at play is low. On the other hand, if the attributed strategies belong to a narrow set, it is quite likely that higher order beliefs are being used. Another possibility could be to include tatonnement-styled questions in the surveys that query traders on their strategies in a series of hypothetical scenarios. Responses to such scenario-based questions could then be used to gauge the degree to which traders stand ready to engage in the back and forth needed for a market adjustment process. Needless to say, these are all very rough outlines, and a lot more work needs to be undertaken to establish such methodologies in practice.

From an applied perspective, one of the main takeaways from the paper is that even the most basic of concepts in finance — the notion of arbitrage — is inextricably linked to higher order beliefs. There is increasing recognition in the finance community that asset prices may move despite the lack of commensurate movement in fundamentals, and a key to disentangling such puzzling market movements likely lies in higher order beliefs. Exploring the role of Bayesian belief hierarchies in more detail in the context of financial markets should thus be a promising agenda of research.
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Appendix: Omitted Proofs

Proof. (Propostion 1).

Part-1: There is a tradeable arbitrage opportunity in the market implies the condition in the Proposition holds.

From Definition 3, there is a tradeable arbitrage opportunity if and only if agent \( i \) can trade a portfolio \( \theta_i \in \mathbb{R}^d \) with the property

\[ \theta_i \cdot q_{\theta_i} \leq 0, \text{ but } \theta_i \cdot \bar{x} \geq 0 \text{ P-a.s. and } \mathbb{P}[\theta_i \cdot \bar{x} > 0] > 0. \]  

(38)

Let us use the superscript \( rf \) to label the risk-free asset; i.e., the risk-free asset’s ex-ante price is \( q_{\theta_i}^{rf} \), its payout is \( x^{rf} \), and the portfolio weight on risk-free is \( \theta_i^{rf} \). Let us use \( q_{\theta_i}^{-rf} \), \( \bar{x}^{-rf} \) and \( \theta_i^{-rf} \) to designate the corresponding variables for the set of risky assets. Since \( \theta_i \cdot q_{\theta_i} \leq 0 \), and \( \int \bar{m} d\mathbb{P} > 0 \) is positive for all \( \bar{m} \) under consideration, we have

\[ 0 \geq \frac{\theta_i q_{\theta_i}}{\int \bar{m} d\mathbb{P}} = \frac{\theta_i^{rf} q_{\theta_i}^{rf}}{\int \bar{m} d\mathbb{P}} + \frac{\theta_i^{-rf} q_{\theta_i}^{-rf}}{\int \bar{m} d\mathbb{P}}. \]

This implies

\[ \theta_i^{-rf} \cdot \bar{x}^{rf} - \frac{\theta_i^{-rf} q_{\theta_i}^{-rf}}{\int \bar{m} d\mathbb{P}} \geq \theta_i^{rf} \cdot \bar{x}^{rf} + \frac{\theta_i^{rf} q_{\theta_i}^{rf}}{\int \bar{m} d\mathbb{P}} = \theta_i \cdot \bar{x} \]  

(39)

for all \( \bar{m} \). Since \( \theta_i \cdot \bar{x} \) is \( \mathbb{P} \)-a.s. non-negative and strictly positive with non-vanishing probability from (38), the same must be true of \( \theta_i^{-rf} \cdot \bar{x}^{rf} - \frac{\theta_i^{-rf} q_{\theta_i}^{-rf}}{\int \bar{m} d\mathbb{P}} \). As for the risk-free asset, we have \( \theta_i^{rf} \cdot x^{rf} - \frac{\theta_i^{rf} q_{\theta_i}^{rf}}{\int \bar{m} d\mathbb{P}} = 0 \).

Thus, from (7), we have that the gains from trading the portfolio \( \theta_i \) satisfy
\[ \bar{g}(\theta_i, a_{-i}) \geq 0 \text{ P-a.s. and } P[\bar{g}(\theta_i, a_{-i}) > 0] > 0. \]  

We are given that \( a_i \) is agent \( i \)'s original strategy. Define her new strategy \( a_i^* \) as the composition operation applied to her original strategy and the arbitrage trade. That is to say, the original strategy followed by the arbitrage trade is her new strategy, so that

\[ \bar{g}(a_i, a_{-i}) + \bar{g}(\theta_i, a_{-i}) = \bar{g}(a_i^*, a_{-i}). \]  

(41)

Since \( \bar{g}(\theta_i, a_{-i}) \) satisfies the condition in (40), we get

\[ \bar{g}(a_i^*, a_{-i}) - \bar{g}(a_i, a_{-i}) \geq 0 \text{ P-a.s. and } P[\bar{g}(a_i^*, a_{-i}) - \bar{g}(a_i, a_{-i}) > 0] > 0. \]

(42)

Using the equivalence in (8) we therefore obtain the requisite relation among the utilities listed in (11).

**Part-2: The condition in the Proposition holds implies there is a tradeable arbitrage opportunity in the market.**

In this case we are given that \( i \) may change her strategy from \( a_i \) to \( a_i^* \) to obtain

\[ \bar{U}_i(a_i^*, a_{-i}) \geq \bar{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\bar{U}_i(a_i^*, a_{-i}) > \bar{U}_i(a_i, a_{-i})] > 0. \]

(43)

Using the equivalence in (8), we get

\[ \bar{g}(a_i^*, a_{-i}) - \bar{g}(a_i, a_{-i}) \geq 0 \text{ P-a.s. and } P[\bar{g}(a_i^*, a_{-i}) - \bar{g}(a_i, a_{-i}) > 0] > 0. \]

(44)

Construct the following portfolio: (i) Go long the portfolio \( a_i^* \) by investing \( a_i^* \cdot q_{a_i^*} \) (ii) Go short the portfolio \( a_i \) by investing \( -a_i \cdot q_{a_i} \), (iii) Fund the investments by trading \( -a_i^* \cdot q_{a_i^*} \) in the risk-free asset initially, followed by another trade of \( +a_i \cdot q_{a_i} \) in the risk free asset. Here, \( q_{a_i^*} = \int_S \bar{f}(a_i^*, a_{-i}) \tilde{x} \text{dP} \), \( q_{a_i} = \int_S \bar{f}(a_i, a_{-i}) \tilde{x} \text{dP} \).

We claim that this is a tradeable arbitrage portfolio. Indeed, the ex-ante investment in constructing the portfolio is zero. The stochastic payout from (i) and (ii) is

\[ a_i^* \tilde{x} - a_i \tilde{x}, \]

and from (iii) is

\[ \frac{-a_i^* \cdot q_{a_i^*}}{\int_S \bar{m}_i\bar{f}_i a_i^* \text{dP}} + \frac{a_i \cdot q_{a_i}}{\int_S \bar{m}_i\bar{f}_i a_i \text{dP}}. \]

(46)

Summing (45) and (46), we get that the total stochastic payout is \( \bar{g}(a_i^*, a_{-i}) - \bar{g}(a_i, a_{-i}) \), which is \( P \)-a.s non-negative and strictly positive with nonvanishing probability, by (44). Therefore, from (6), this is an
Proof. (Proposition 2). The result follows from the Axiom of specification in Set theory which states that: To every set $\mathcal{A}$ and to every condition $S(x)$ there corresponds a set $\mathcal{B} \subseteq \mathcal{A}$ whose elements are exactly those elements $x$ of $\mathcal{A}$ for which $S(x)$ holds.\(^{19}\)

From equation (22), $W^2_i$ imposes the additional condition $\text{marg}_{A_i}[b_i][a_i \in D_1^i] = 0$ on members of the set $B_j$. Therefore, by the axiom of specification $B_j \supseteq b_j(W^2_i)$. Similarly, from equation (25), $W^k_i$ imposes the additional condition $\text{marg}_{A_i}[b_i^{-1}][a_i \in D_{i-1}^k] = 0$ on members of $b_j(W^{k-1}_i)$. So, by the axiom of specification $b_j(W^{k-1}_i) \supseteq b_j(W^k_i)$. Thus, we have the relation in the statement of the proposition.

Proof. (Proposition 3). One way to prove this proposition is to follow the strategy in the proof of Proposition 2 and use the Axiom of specification.

A more direct way is to observe that equation (24) imposes the restriction that any belief hierarchy $b_i$ in $V_i^k$ and $W_i^k$ satisfies $\text{marg}_{A_i}[\phi_i(b_i)][a_i \in D_{i-1}^k] = 0$. Next, since $b_i \in W_i^k$ only if $b_i \in W_i^{k-1}$, we get, additionally, $\text{marg}_{A_i}[\phi_i(b_i)][a_i \in D_{i-2}^k] = 0$. Iterating this argument repeatedly, we obtain

$$\text{marg}_{A_i}[\phi_i(b_i)][a_j \in D_{i-n}^k] = 0 \text{ for all } k - 1 \leq n \leq 1. \tag{47}$$

Therefore, $\text{supp} \text{marg}_{A_i}[\phi_i(b_i)] = A_j \setminus \bigcup_{n=1}^{k-1} D_{i-n}^k = U D_{j}^{k-1}$, which implies $A_j(W_k^j) = U D_{j}^{k-1}$. Finally, by the definition of undominated response sets in equations (28) and (20), we obtain

$$A_j \supseteq U D^1_i \supseteq \cdots \supseteq U D^{k-1}_j \supseteq U D^k_j \supseteq \cdots \tag{48}$$

Thus, we have the relation in the statement of the proposition.

Proof. (Proposition 4). We have shown sufficiency in part-1b in the proof of Theorem 3. To establish necessity, we need to rule out the possibility that $k$ is finite. When $k$ is finite, there are two scenarios that may result. In both the scenarios, we show below, there is an arbitrage opportunity. This implies $k$ cannot be finite, i.e. all market participants choose a response in $\bigcap_{k \geq 0} U D_i^k$.

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\(^{19}\)See Halmos (1960), Chapter 2.
Thus, analogous arguments for cases where only a subset of the market participants were using finite orders.

Agent $i$ for any $a_i$ used by other market participants. So, $A_i \subseteq \prod_i I$ and the definition of $D_i^{k+1}$ implies that agent $i, i \in I$, may change her strategy from $a_i$ to $a_i^*$ to obtain

$$\hat{U}_i(a_i, a_{-i}) \geq \hat{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\hat{U}_i(a_i^*, a_{-i}) > \hat{U}_i(a_i, a_{-i})] > 0, \quad (49)$$

for any $a_{-i} \in \prod_{j \neq i} U_i^{k,j}$. Since $\hat{f}$ is one-to-one, the agents can accurately discern the strategies used by other market participants. So, $A_i \subseteq \prod_i I$. Theorem 2, then, implies that each agent $i \in I$ has a tradeable arbitrage opportunity.

Scenario 2. Different market participants use different finite orders $k_i$ for their belief hierarchy and choose responses in the set $U_i^{k_i} \cap U_i^{k+1}, i \in I$.

In this case, the participant that chooses $k_{\text{min}} = \min \{k_i : i \in I\}$ — call her $i_{\text{min}}$ — has a tradeable arbitrage opportunity. This is because, similar to Case 1 above, $U_i^{k_{\text{min}}} \cap U_i^{k_{\text{min}}+1} = D_i^{k_{\text{min}}+1}$, and the definition of $D_i^{k_{\text{min}}+1}$ implies that agent $i_{\text{min}}$, may change her strategy from $a_i$ to $a_i^*$ to obtain

$$\hat{U}_i(a_i^*, a_{-i}) \geq \hat{U}_i(a_i, a_{-i}) \text{ P-a.s. and } P[\hat{U}_i(a_i^*, a_{-i}) > \hat{U}_i(a_i, a_{-i})] > 0, \quad (50)$$

for any $a_{-i} \in \prod_{j \neq i} U_i^{k_{\text{min}}}$. Since $\hat{f}$ is one-to-one, the agents can accurately discern the strategies used by other market participants. So, $A_{i_{\text{min}}} \subseteq \prod_{j \neq i} U_i^{k_{\text{min}}}$. Theorem 2, then, implies that agent $i_{\text{min}}$ has a tradeable arbitrage opportunity.

Scenarios 1 and 2 were described with all market participants using finite orders, but we could make analogous arguments for cases where only a subset of the market participants were using finite orders. Thus, $k$ cannot be finite for any market participant, given there is no tradeable arbitrage opportunity.

Proof. (Proposition 5). From the definition of responsiveness, $A_{i_{\text{min}}}^f(\bar{m}, a_i) \subseteq A_{i_{\text{min}}}^f(\bar{m}, a_i)$ when $\bar{f}$ is more responsive than $\bar{f}_1$, for $i \in I$. So, if $A_{i_{\text{min}}}^f(\bar{m}, a_i) \supseteq \prod_{j \neq i} U_i^{k_{\text{min}}}$, then we must also have $A_{i_{\text{min}}}^f(\bar{m}, a_i) \supseteq \prod_{j \neq i} U_i^{k_{\text{min}}}$.

As noted before, from Definition 6, we have that undominated response sets are nested as one increases the order. Therefore, if $k_{i_1}$ is the minimum order for which $A_{i_{\text{min}}}^f(\bar{m}, a_i) \supseteq \prod_{j \neq i} U_i^{k_{i_1}}$, we must have $k_{i_1} \leq k_{i_2}$.

From Theorem 3-(ii), under no-arbitrage, the order of the undominated response under $\bar{f}_1$ needs to be at least $k_{i_1}$ given $A_{i_{\text{min}}}^{f_1}(\bar{m}, a_i) \supseteq \prod_{j \neq i} U_i^{k_{i_1}}$, while under $\bar{f}_2$ it needs to be at least $k_{i_2}$ given...
Theorem 1, we then get the condition that agent \( a_i \) uses a belief hierarchy in \( W_i^k \) for the eductive sequence and finds

\[
A_{-i}(\bar{m}, a_i) \subseteq \prod_{j \neq i} UD_j^{k-1}, \quad i \in I.
\]

Thus, the minimum order of the undominated response set from which the market participant selects, given no-arbitrage, increases weakly under \( (\text{Proposition 6}) \). Since we are focusing exclusively on agent \( i \)'s strategies here, we will assume without loss of generality that no other market participant has a tradeable arbitrage opportunity.\(^{20}\)

\( a_i \in D_i^{\text{wtp}} \) implies, from Corollary 1, that agent \( i \) has a tradeable arbitrage opportunity. From Theorem 1, we then get the condition that agent \( i \) uses a belief hierarchy in \( W_i^k \) for the eductive sequence and finds

\[
A_{-i}(\bar{m}, a_i) \subseteq \prod_{j \neq i} UD_j^{k-1}.
\]  

(51)

This gives the correspondence of \( a_i \in D_i^{\text{wtp}} \) with (i) in the statement of the Proposition.

Next, \( a_i \in UD_i^{\text{wtp}} \) implies, from Corollary 2, that agent \( i \) does not have a tradeable arbitrage opportunity. From Theorem 3, we then get the condition that agent \( i \) either chooses a response in the set \( UD_i^{k_i} \setminus UD_i^{k_i+1} \) and finds \( A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k_i} \), or chooses a response in the set \( \cap_{n \geq 0} UD_i^n \).

We show below that the first scenario is equivalent to the agent using a belief hierarchy in \( W_i^{k+\alpha} \) for the eductive sequence and finding \( A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k+\alpha} \), and the second scenario is equivalent to the agent using a belief hierarchy in \( \cap_{\beta > \alpha} W_i^{k+\beta} \). This gives the correspondence of \( a_i \in UD_i^{\text{wtp}} \) with (ii) in the statement of the Proposition. Let us look at each scenario in turn:

**Scenario 1.** Agent \( i \) chooses a response in the set \( UD_i^{k_i} \setminus UD_i^{k_i+1} \) and finds \( A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k_i} \).

If the agent chooses a response in the set \( UD_i^{k_i} \) she is using a belief hierarchy in \( W_i^{k_i} \) for the purpose of the eductive sequence. Thus, to establish the requisite equivalence, we need to show \( k_i = k + \alpha \), where the value of \( k \) is fixed by the initial belief hierarchy \( W_i^k \) in (i) of the statement of the Proposition. As noted before, from Definition 6 we have that undominated response sets satisfy the property \( UD_i^{m_1} \supseteq UD_i^{m_2} \) when \( m_1 \leq m_2 \), i.e. the undominated response sets form a nested sequence as one climbs up the orders. This implies that if \( UD_i^{m_1} \supseteq UD_i^{m_2} \), then \( m_1 < m_2 \). Therefore, from \( A_{-i}(\bar{m}, a_i) \subseteq \prod_{j \neq i} UD_j^{k_i-1} \) in (51), and \( A_{-i}(\bar{m}, a_i) \supseteq \prod_{j \neq i} UD_j^{k_i} \) in the present scenario, we get that

\[
k_i > k - 1.
\]

(52)

This implies \( k_i = k + \alpha \), for some integer \( \alpha \geq 0 \), as needed.

**Scenario 2.** Agent \( i \) chooses a response in the set \( \cap_{n \geq 0} UD_i^n \):

An undominated \( k^{th} \) order response \( UD_i^k \) is defined with respect to the belief hierarchy \( W_i^k \) (Definition 6), so in this case, given the response is in the set \( \cap_{n \geq 0} UD_i^n \), \( i \) employs a belief hierarchy in \( \cap_{n \geq 0} W_i^n \).

\(^{20}\)It is easy to see that each proposition above on the existence of a tradeable arbitrage opportunity in the market can be restated as a proposition about the existence of a tradeable arbitrage opportunity for agent \( i \).
Next, we have that the set $\cap_{n \geq 0}^{W_i^n} = \cap_{\beta > \alpha}^{W_i^{k+\beta}}$, for any given $\alpha \geq 0$, because the belief hierarchy sets $W_i^n$ are nested, by definition (see equations (21) – (26)). That is to say, $W_i^0 \supseteq W_i^1 \supseteq W_i^2 \supseteq \ldots$. Therefore, the agent using a belief hierarchy in $\cap_{\beta > \alpha}^{W_i^{k+\beta}}$ is equivalent to her choosing a response in the set $\cap_{n \geq 0}^{UD^n_i}$.

□