Geometrical Theory of Separation of Variables, 
a review of recent developments

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Abstract

The Separation of Variables theory for the Hamilton-Jacobi equation is ‘by definition’ related to the use of special kinds of coordinates, for example Jacobi coordinates on the ellipsoid\(^1\) or Stäckel systems\(^2\) in the Euclidean space. However, it is possible and useful to develop this theory in a coordinate-independent way: this is the Geometrical Theory of Separation of Variables. It involves geometrical objects (like special submanifolds and foliations) as well as special vector and tensor fields like Killing vectors and Killing two-tensors (i.e. isometries of order one and two), and their conformal extensions; quadratic first integrals are associated with the Killing two-tensors. In the recent years Separable Systems provide mathematical structures studied from different points of view. We present here a short review of some of these structures and of their applications with particular consideration to the underlying geometry. Algebraic relations among Killing

\(^1\)C. G. Jacobi, Vorlesungen über Dynamik, Gesammelte Werke, Berlin (1884)  
\(^2\)P. Stäckel, Math. Ann. 42 537 (1893)
tensors, quadratic first integrals or their associated second order differential operators and some aspects of approximation with separable systems are considered. This paper has been presented as a poster at Dynamics Days Europe 2008, Delft 25-29 August 2008.

Introduction

There are two levels of the Geometrical Theory of Separation of Variables (GTSOV): (i) general or non-orthogonal separation (i.e., not necessarily orthogonal), (ii) orthogonal separation. A geometrical characterization (in terms of foliations and Killing vectors and tensors) of the general separation has been proposed by Benenti\(^3\). Among the applications of such a characterization we find a coordinate-independent proof of a theorem of Kalnins and Miller\(^4\): on a Riemannian manifold with constant curvature the SOV always occurs in orthogonal coordinates. A second application is a finer classification of the orthogonal separable systems. Eisenhart’s theorem on Stäckel systems\(^5\) provides a first result on the geometrical characterization of the Orthogonal-SOV (OSOV). However, the necessary and sufficient conditions written in the Eisenhart statements are redundant. Minimal conditions have been proposed by Benenti. According to one of the various possibilities, OSOV of the geodesic flow on a Riemannian manifold occurs if and only if there exists a Killing tensor \(K\) with simple eigenvalues and normal eigenvectors. Such a tensor has been called a characteristic tensor. Furthermore, if a potential energy \(V\) is present, it must satisfy the characteristic equation \(dK \, dV = 0\). With respect to Cartesian coordinates on an Euclidean space this equation reduces to the so-called Bertrand-Darboux equation. The existence of a characteristic KT implies the existence of other \(n-1\) linearly independent KT with common eigenvectors \(K_i\) which are associated with \(n-1\) quadratic first integrals of \(H\) by \(H_i = \frac{1}{2}K^{jl}_i p_j p_l + V_i\) with \(dV_i = K_i \, dV\). In many cases (for example when the Ricci tensor is null) it is possible to build quantum symmetry operators for the Schrödinger equation associated with \(H\) by putting \(\hat{H}_i = \nabla_j (K^{jl}_i \nabla_l) + V_i\) and obtaining multiplicative separation for the Schrödinger equation in the same coordinates as for the

\(^{3}\)S. Benenti, J.Math.Phys. 38 6578 (1997)

\(^{4}\)E. Kalnins and W. Miller Jr, SIAM J. Math. Anal. 11 1011 (1980)

\(^{5}\)L. P. Eisenhart, Riemannian geometry, Princeton University Press (1949)
classical system. In the following, some few topics of GTSOV are sketched, together with some of their possible applications.

1 Benenti systems

Among the orthogonal separable systems we find the special class of the so-called Benenti systems (or L-systems) for which the separation is characterized by a special conformal Killing tensor \( L \) called Benenti tensor (or L-tensor). For these systems a complete set of quadratic first integrals can be constructed by a pure algebraic and coordinate independent method starting from \( L \). The geodesic flow on an asymmetric ellipsoid (Jacobi) is an example of a Benenti system. Several recent papers have shown that Benenti systems have a very rich structure closely related to other fields of research. An L-tensor is a conformal KT of order (1,1) which is torsionless with (real) simple eigenvalues. Let \( L \) a symmetric 2-tensor, then the tensors \( K_a, a = 0, \ldots, n-1 \), defined by

\[
K_0 = I, \quad K_a = \frac{1}{a}tr(K_{a-1}L)I - K_{a-1}L, \quad a > 1,
\]

are \( n \) independent KT with common normal eigenvectors iif \( L \) is a L-tensor. The KT's \( K_a \) determine an orthogonal separable system, a Stäckel system called Benenti system. Not all the Stäckel systems are Benenti systems. The first integrals are obtained from the KT as exposed in the Introduction; a recursive formula similar to the previous one can produce the potentials in each of the first integrals.

1.1 Separable coordinates for triangular Newton equations

(see [8]) Two dynamical systems on the same configuration manifold \( Q \) are equivalent if their motions on \( Q \) locally coincide as unparametrized curves; i.e. up to time transformations of the kind \( dt = \mu(q)dt' \). For natural systems: a dynamical system admits a Lagrangian equivalent system iif the

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6S.Benenti, C.Chanu and G. Rastelli, J.Math. Phys. 43 5183 (2002)
7S.Benenti, Acta Applicanda Mathematica 87 33 (2005)
8K.Marciniak and S. Rauch-Wojciechowski, Studies in Applied Mathematics 118 45 (2007)
force $F$ is such that $F = -A^{-1} \nabla V$, where $A$ is the cofactor of a special conformal Killing tensor $J$: $A = (det J) J^{-1}$. A special conformal KT with simple eigenvalues is a L tensor. An L tensor is a special conformal KT. A cofactor pair system is a cofactor system in two distinct ways. A cofactor pair system such that $\bar{J}$ has real simple eigenvalues is equivalent to an L system generated by $\bar{J}$. The triangular Newton equations

$$\ddot{q}_i = M_i(q_1, \ldots, q_i), \quad i = 1 \ldots n$$

are not in general a Lagrangian system, but if $M = -(\text{cof } G)^{-1} \nabla k$, where $G$ is the matrix associated with the characteristic KT of elliptic coordinates in the Euclidean space, it is equivalent to a Lagrangian one. In this case in fact $G$ is an L tensor, then a special conformal KT and generates a Benenti system. The system equivalent to the TNE is then Hamiltonian and admits SOV in the orthogonal coordinates of the Benenti system.

Example:

Let $(x_1, x_2)$ Cartesian coordinates, let the triangular system be $\ddot{x}_1 = -4x_1$, $\ddot{x}_2 = 6x_1^2 - 4x_2$; this is a cofactor system with $G = \begin{pmatrix} 1 & -x_1 \\ -x_1 & -2x_2 \end{pmatrix}$.

The separable coordinates (not a Stöckel system of the Euclidean plane) are $u_1 = x_1$ and $u_2 = \frac{1}{2} x_1^2 + x_2$ and their associated quadratic first integrals provide the quadrature of the system.

1.2 Separation curves, Stackel systems and soliton systems

(see [9]) A set of $n$ relations of the form $\phi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0 \quad a_i \in \mathbb{R}$, such that $(\lambda_i, \mu_i)$ are canonical coordinates and $\det(\partial \phi_i/\partial a_j) \neq 0$ are called separation relations, when they are all of the same form are called separation curve. If the separation curve is of the form

$$H_1 \lambda^{n-1} + H_2 \lambda^{n-2} + \ldots + H_n = \frac{1}{2} \lambda^m \mu^2 + \lambda^k$$

where the $H_i$ are the polynomials $H_i = \frac{1}{2} \mu^T K_i G^{(m)} \mu + V_i^{(k)}$ with $G^{(m)} = L^m G^{(0)}$, $G^{(0)} = \text{diag} (1/\Delta_1, \ldots, 1/\Delta_n)$, $\Delta_i = \prod_{j \neq i}(\lambda_i - \lambda_j)$ and where $L = \text{diag} (\lambda_1, \ldots, \lambda_n)$ is a conformal KT with respect to the metrics $G^{(m)}$. The

\[9\] M.Blaszak and K. Marciniak, JMP 47 032904 (2006)
tensors $K_i$ are given by $K_{i+1} = L K_i + q_i I$, $K_1 = I$, $K_{n+1} = 0$ where the $q_i(\lambda)$ are coefficients of the characteristic polynomial of $L$. The potentials $V^k_i$ are obtained by the recursion $V^k_i = V^{k-1}_{i+1} - q_i V^{k-1}_1$. Therefore, the $H_i$ form a Benenti system generated by the L-tensor $L$ with separable coordinates $\lambda_i$. The $q_i$ are new coordinates on the configuration manifold and for the Benenti systems of above a hierarchy of integrable dispersionless equations, called Killing dispersionless system, whose solutions are called in general solitons, is obtained from the equation $\frac{d}{dt} q_j = K_i(q) \frac{d}{dx} q$ where $t_i$ is the evolution parameter (time) of the Hamiltonian flow of $H_i$ and $x = t_1$.

Example:

\[ H_1 \lambda + H_2 = \frac{1}{2} \lambda \mu^2 + \lambda^4 \]

be the separation curve, the diagonalized Hamiltonians (Benenti system) are

\[ H_1 = \frac{1}{2(\lambda_1 - \lambda_2)}(\lambda_1 \mu^2_1 - \lambda_2 \mu^2_2 + 2(\lambda_1^2 - \lambda_2^2)), \]
\[ H_2 = \frac{\lambda_1 \lambda_2}{2(\lambda_2 - \lambda_1)}(\mu_1^2 - \mu_2^2 + 2(\lambda_1^3 - \lambda_2^3)). \]

\[ q_1 = \lambda_1 + \lambda_2, \quad q_2 = -4(\lambda_1 \lambda_2) \]

where $(\lambda_i, \mu_i)$ are canonical Parabolic coordinates. In Cartesian canonical coordinates $(q_i, p_i)$ they are

\[ H_1 = \frac{1}{2}(p_1^2 + p_2^2) + q_1^3 + \frac{1}{2} q_1 q_2^2, \quad H_2 = \frac{1}{2}(q_2 p_1 p_2 - q_1 p_2^2 + \frac{1}{8} q_2^4 + \frac{1}{2} q_1^2 q_2^2), \]

$H_1$ is the Hamiltonian of a separable case of the Henon-Heiles system. The integrable dispersionless PDE are

\[ q_{1,t} = \frac{1}{2} q_2 q_{2,x}, \quad q_{2,t} = \frac{1}{2} q_2 q_{1,x} - q_1 q_{2,x} \]

that, after differentiation and elimination of $q_2$ yield the KdV soliton equation $q_{1,t} + \frac{1}{2} q_{1,xxx} + 3q_1 q_{1,x} = 0$ together with one differential consequence of it.

2 Superintegrability

Classical Hamiltonian systems in $n$ dimensional manifolds admit at most $2n - 1$ functionally independent (f.i.) first integrals. They are Liouville integrable if $n$ f.i. first integrals are in involution. Integrable systems with more than $n$ f.i. first integrals are called superintegrable. Well known examples are the harmonic oscillator and the Kepler-Coulomb system. Because the integral curves of the systems stay in the intersection of the level hypersurfaces of the first integrals, the occurrence of the maximum number of functionally independent first integrals, i.e $2n - 1$, implies that the integral curves of the
system can be determined by algebraic methods. Hamiltonian systems separable in multiple coordinate systems admits often more than \( n \) f.i. quadratic first integrals and they are said to be quadratically superintegrable. The Kepler-Coulomb system is an example of quadratically maximally superintegrable system. By searching among multiseparable systems many instances of superintegrability have been found recently and their classification is in progress.\(^{10}\) \(^{11}\) \(^{12}\) \(^{13}\) \(^{14}\)

2.1 Superintegrable 3-body systems on the line

(see\(^{15}\)) By using Cylindrical coordinates \((r, \psi, z)\), with rotational axis \(z\), and by indicating with \((p_r, p_\psi, p_z)\) their conjugate momenta, let us consider the natural Hamiltonian with potential

\[
V = \frac{F(\psi)}{r^2}.
\]

The potential \(V\) is separable, and therefore Liouville integrable, in Cylindrical, Spherical, Parabolic, Ellipsoidal Prolate and Oblate coordinate systems. The five quadratic first integrals associated with the multiseparability take the form

\[
\begin{align*}
H &= \frac{1}{2} (p_r^2 + \frac{1}{r^2} p_\psi^2 + p_z^2) + \frac{F(\psi)}{r^2}, \\
H_1 &= \frac{1}{2} p_\psi^2 + F(\psi), \\
H_2 &= \frac{1}{2} p_z^2, \\
H_3 &= \frac{1}{2} \left((rp_z - zp_r)^2 + \left(1 + \frac{z^2}{r^2}\right) p_\psi^2\right) + \left(1 + \frac{z^2}{r^2}\right) F(\psi), \\
H_4 &= \frac{1}{2} \left(zp_r^2 + \frac{z^2}{r^2} p_\psi^2 - rp_z p_r\right) + \frac{z}{r^2} F(\psi).
\end{align*}
\]

Only four of them are f.i. The first three integrals allow the separation of the system in cylindrical coordinates. Let \(x^i, i = 1 \ldots 3\) the positions of three points on a straight line, \((x^i)\) can be interpreted as Cartesian coordinates of a single point in the space. It is possible to show that all interactions among the points of the form

\[
V = \sum_i \frac{1}{X_i^2} F_i \left(\frac{X_{i+1}}{X_i}, \frac{X_{i+2}}{X_i}\right) \quad i = 1, \ldots, 3 \text{ (mod 3)},
\]

\(^{10}\) E. G. Kalnins, G. Williams, W. Miller Jr and G. S. Pogosyan, J. Math. Phys. 40 708 (1999)  
\(^{11}\) S. Gravel and P. Winternitz, J. Math. Phys. 43 5902 (2002)  
\(^{12}\) C. Daskaloyannis and K. Ypsilantis, J. Math. Phys. 47 042904 (2006)  
\(^{13}\) S. Benenti, C. Chanu and G. Rastelli, J. Math. Phys. 41, 4654 (2000)  
\(^{14}\) P. Tempesta et al. eds: Superintegrability in Classical and Quantum Systems, CRM proceedings & Lecture Notes 37 Amer. Math. Soc. (2004)  
\(^{15}\) C. Chanu, L. Degiovanni and G. Rastelli: Superintegrable three body systems on the line JMP 49, 112901 (2008)
are superintegrable with 4 f.i. first integrals, where $F_i$ are arbitrary functions of two variables and $X_i = x^i - x^{i+1}, \ i = 1, 2, 3 \ (\text{mod } 3)$, $r \cos \psi = \frac{1}{\sqrt{2}}(x^1 - x^2)$, $r \sin \psi = \frac{1}{\sqrt{6}}(x^1 + x^2 - 2x^3)$, $z = \frac{1}{\sqrt{3}}(x^1 + x^2 + x^3)$. Indeed, it can be proved that such $V$ are in the form (1), then, admit all the first-integrals $H_i$ four of which are always functionally independent.

Examples:

The Calogero system

$$V_C = \frac{k_1}{(x^1 - x^2)^2} + \frac{k_2}{(x^2 - x^3)^2} + \frac{k_3}{(x^3 - x^1)^2} = \sum_{i=1}^{3} \frac{k_i}{X_i^2}, \ k_i \in \mathbb{R}.$$  

The Wolfes system

$$V_W = \frac{k_1}{(x^1 + x^3 - 2x^2)^2} + \frac{k_2}{(x^2 + x^1 - 2x^3)^2} + \frac{k_3}{(x^3 + x^2 - 2x^1)^2}$$

$$= \sum_{i=1}^{3} \frac{1}{X_i^2} \left( \frac{X_{i+1}}{X_i} - \frac{X_{i+2}}{X_i} \right)^{-2}, \ i = 1, \ldots, 3 \ (\text{mod } 3).$$

And a new one

$$V = \sum_{i=1}^{3} \frac{k_i}{X_i^2 + X_{i+1}^2} = \sum_{i=1}^{3} \frac{k_i}{X_{i+2}^2} \left( \frac{X_i^2}{X_{i+2}^2} + \frac{X_{i+1}^2}{X_{i+2}^2} \right)^{-1} \ (\text{mod } 3).$$

### 3 Stäckel approximation

Several attempts have been made to use the Stäckel systems as suitable integrable Hamiltonians to perturb in order to approximate given physical systems. For example \cite{16} and \cite{17} for the dynamics of stars in elliptical galaxies. There, for given separable systems and by using the separable coordinates themselves, the possible separable potentials (i.e. satisfying the condition $d(K dV) = 0$) are analyzed with the purpose to find the separable dynamics closest to the gravitational potentials of elliptical galaxies. The best fitting integrable Hamiltonian is then determined by using perturbative methods.

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\cite{16} J. Binney, S. Tremaine, *Galactic Dynamics*, Princeton University press (1994)

\cite{17} P. T. De Zeeuw, D. Lynden-Bell, MNRAS 215 713 (1985)
3.1 Decomposition of a potential into integrable and perturbative terms

Given a natural (non integrable) Hamiltonian $H$ we search for the separable system $(H_i), i = 1 \ldots n$ representing "the closest", among all separable systems, approximation to the dynamics determined by $H$. Due to the structure of the separable systems, the quantities $m_i = \{H, L_i\} = dL_i/dt$ are linear homogeneous polynomials in the momenta. Then, the scalars $\mu_i = g_{ij}m^j_i m^l_i$ provide informations on the time-variation of the functions $L_i$ along the integral curves of $H$. It is remarkable that $d\mu_i = dK_i dV$. We can conjecture that the minimization of all these quantities, which are coordinate-independent, among all the Stäckel systems in some neighborhood of the configuration manifold determines the separable system which provides the best approximation of the given (nonintegrable) Hamiltonian dynamics in the same neighborhood.

Example:

Let us consider the Quadrupole field in two dimensions: $H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{G}{r} + \frac{D}{r^3}\left(3\frac{x^2}{r^2} - 1\right)$ where $G, D$ are real constants. By applying the criteria of above we determine the separable Hamiltonian system associated to elliptic coordinates with foci of cartesian coordinates, if $D/C > 0$, $x^2 = 2D/G, y = 0$ and with potential $W_e = \frac{2}{v^2 - u^2} (Gv - 4\frac{D}{v})$, where $(u, v)$ are the same elliptic coordinates. The dynamics of this system can be compared with the original one and with the system separable in Polar coordinates $^{18}$.

4 Et cetera

Several advances in GTSOV cannot take place in this short presentation. It follows a short list just to mention them and to provide some references. Conformal separation$^{19,20}$, Stäckel transformation$^{21,22}$, Classification

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\textsuperscript{24}J. F. Carinena, M. F. Rañada, M. Santander and T. Sanz-Gil, J. Nonl. Math. Phys. \textbf{12} 230 (2005)