A new approach
for the vertical part of the contour
in thermal field theories

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Abstract

A lot of work has been devoted in the past to understand the role of vertical branch of the time path in thermal field theories, and in particular to see how to deal with it in the real-time formalism.

Unlike what is commonly believed, I emphasize on the fact that the vertical part of the path contributes to real-time Green’s functions, and I prove that this contribution is taken into account simply by the substitution $n(\omega_k) \rightarrow n(|k_o|)$ in the real time Feynman rules. This new proof is based on very simple algebraic properties of the contour integration.

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1 Introduction

When deriving the matrix Feynman rules of the closed time path (CTP in the following) formalism, an intriguing problem is to understand how one can reach a 2 components matrix formulation from the path represented on figure 1. Indeed, it is widely accepted that each component of the matrix formalism corresponds to one of the two horizontal branches $C_1$ and $C_2$ of the path $\bar{1} \bar{2} \bar{4} \bar{3} \bar{1} \bar{3}$. In this picture, there is no room for the vertical part $C_v$. Therefore, most of the derivations of the matrix formalism found in the literature got rid in some way of the vertical part of the path. Usually, one invokes the limits $t_I \to -\infty$ and $t_F \to +\infty$, in conjunction with an ad hoc choice of the asymptotic properties of the source $j(x)$ coupled to the field in the generating functional.

This derivation seems highly artificial, as one can judge by the numerous attempts to find other, less ad hoc, justifications. Among them, the most “revolutionary” approach was that of Niegawa who rejected the hypothesis according to which the vertical part of the time path does not contribute to Green’s functions in the real time formalism. Instead, he argued that it does contribute in some cases, and this contribution can be taken into account by the so called “$n(|k^0|)$ prescription”. Although correct in his statement, his proof is controversial since he still makes use of the artificial limits $t_I \to -\infty$ and $t_F \to +\infty$.

In a previous paper, I attempted to show this result without using these limits at all. Indeed, I started by showing that (i) the vertical part of the path contributes in general, (ii) the time integrations involved in the calculation of Feynman diagrams in time coordinates give a result which is totally independent of the time $t_I$ and $t_F$. As a consequence, all the arguments based upon specific limits for $t_I$ and $t_F$ were suspect since nothing nontrivial could occur when taking these limits. After that, I showed in the case of self-energy insertions between propagators that the contribution from the vertical part is precisely the term that corresponds to the difference between the $n(|k^0|)$ prescription and the $n(\omega_k)$ prescription. This proof was quite intricate, mainly because it involved dealing with delicate products of distributions, and seemed also to leave open the possibility for the vertical part to contribute in many other cases.

After that, the intricacies of the products of distributions were elegantly avoided by Le Bellac and Mabilat, who introduced a regularization of the propagators, which had the main property of preserving the Kubo Martin
Schwinger (KMS in the following) boundary conditions, as well as the holomorphy of the propagators. In their proof, the necessity of using \(|k^0|\) as the argument of statistical factors appeared quite naturally, but in a way which was not obviously related to the vertical part of the path.

My purpose in the present paper is to present an alternative proof of this result, in a way which avoids all the intricacies of the multiplication of distributions, while being more complete than [10] since all the situations in which the vertical part of the path can contribute are clearly identified. The part of the proof dealing with self-energy insertions, which was nontrivial in [10], is now quite straightforward thanks to the use of simple algebraic properties of the contour integration.

The structure of this paper is as follows. In section 2, I start by recalling the origin of the vertical part of the path, and its precise role for the consistent perturbative expansion of a theory of quantum fields in thermal equilibrium. Then, in section 3, I explain why performing the Fourier transform to go from the time variable to the energy variable is more complicated at finite temperature than it is at zero temperature. In this section, I also derive the matrix formalism in a naive (and incorrect) way, assuming first for the sake of simplicity that the vertical part of the path does not contribute.

Section 4 is devoted to a detailed study of the circumstances in which the vertical part of the path contributes. It is shown that it can contribute only in two simple cases: vacuum diagrams and self-energy insertions.

In section 5, I study the effect of the vertical part in the case of self-energy insertions and show how the matrix Feynman rules must be modified to generate properly the contribution of the vertical part. This proof starts by the almost trivial case of repeated concatenation of free propagators, which is then generalized to the case of general self-energy insertions by simple algebraic arguments. Finally, the last section is devoted to concluding remarks. Some technical details are relegated into two appendices.

2 Origin of the vertical part

In [10], I derived the perturbative expansion of a thermal field theory in time coordinates by using the canonical approach in order to make more explicit the role of the vertical part of the path. I will just summarize here the main points of this derivation. When doing this perturbative expansion, the main difference with respect to the zero temperature situation is related to the fact that the

\footnote{Other justifications of the matrix formalism used regularizations as well, like [3, 9]. But, in these papers, the regularization scheme had the effect to break KMS and to make the contribution of the vertical part artificially vanish. Then, one introduces by hand the “n(|k^0|) prescription” in order to reinforce KMS, and it happens that this prescription is precisely what was needed to take into account the contribution of the vertical part, as we shall see later. A consistent justification of the matrix formalism should never need to “reinforce KMS” since it should never use intermediate steps that break KMS.}

\footnote{The necessity of modified Feynman rules in order to calculate vacuum diagrams in the real time formalism is known for a long time. Justifications can be found in [3, 14, 16].}
parameter of the expansion (the coupling constant of the theory) appears not
only in the dynamics of the fields via their evolution equation, but also in the
averaging procedure itself via the density operator \( e^{-\beta H} \) \( (\beta \equiv 1/T, \ k_B = 1) \).
Indeed, the Hamiltonian \( H \) contains the coupling constant. One then sees easily
that the two horizontal branches are necessary in order to expand in powers of
the coupling constant the time evolution of the fields. But in order to have a
consistent perturbative expansion (in particular to preserve thermal equilibrium
order by order in the coupling constant), one needs also to expand in powers of
the coupling constant the density operator itself. This is done easily thanks to
the following formula \[ 1 \]

\[ e^{-\beta H} = e^{-\beta H_0} T_c \exp i \int_{C_v \times \mathbb{R}^3} \mathcal{L}_\text{in}(\phi_\text{in}(x)) \ d^4 x , \]

where \( H_0 \) is the free part of the Hamiltonian, \( C_v \) is a path in the complex time
plane going from \( t_I \) to \( t_I - i\beta \), \( \mathcal{L}_\text{in} \) is the interaction part of the Lagrangian
density, and \( \phi_\text{in} \) is the field in the interaction picture (i.e. a free field). With
this formula, it is now obvious that the perturbative expansion of the density
operator itself made possible by the addition of the vertical part \( C_v \) to the
previous two horizontal branches.

The physical meaning of the vertical part \( C_v \) is now quite clear: this piece of
the contour is needed because the interaction modifies the equilibrium density
operator. Therefore, it is likely that this vertical part is crucial for the consis-
tency of the perturbative expansion, and that arguments suggesting that it can
simply be dropped are wrong. \[ 4 \]

3 From time to energy -
First approach to the RTF

3.1 From time to energy
At this stage, we have definite Feynman rules to calculate perturbatively a
Green’s function in time coordinates: at each vertex, one must integrate over
time along the whole path \( C \equiv C_1 \cup C_2 \cup C_v \). As at zero temperature, the problem
is that these Feynman rules are not very convenient for practical calculations.
One usually prefers to work in the Fourier space with the conjugate variables
\((k^0, k)\). Since in the sector of spatial variables, everything is similar to the
zero temperature case, going from position to 3-momentum is trivial and works

\[ ^3 \text{The reason why such a formula is possible is related to the analogy between the canonical}
\text{density operator and an evolution operator. The role of the vertical part of the path seems}
\text{to have remained unnoticed by particle physicists, who usually derived the perturbative ex-
\text{pansion of thermal field theories by functional methods based on the Feynman-Kac formula}
\text{[3].}} \]

\[ ^4 \text{We are now in a position to understand why enforcing by hand KMS and taking into}
\text{account the vertical part can be related: without the vertical part, the perturbative expansion}
\text{would be inconsistent because the density operator would not be expanded in powers of the}
\text{coupling constant. In other words, statistical equilibrium, i.e. KMS, would be broken.} \]}
exactly in the same way as at $T = 0$ (in the following, I assume that the transformation $x \rightarrow k$ has already been performed, and I do not write explicitly the spatial variables).

Problems arise when one tries to go from time to energy. Indeed, the property behind the usefulness of the Fourier transform is the relation existing between the Fourier transform (FT in the following) and the convolution product. More precisely, given two 2-point functions $f(x_1^0, x_2^0)$ and $g(x_1^0, x_2^0)$, one expects the FT to satisfy the identity

$$\text{FT}(f * g)(k_1^0, k_2^0) = \left[\text{FT}(f)(k_1^0, k_2^0)\right]\left[\text{FT}(g)(k_1^0, k_2^0)\right].$$  \hspace{1cm} (2)

The problem comes from the fact that the relevant convolution product at finite temperature is defined by an integration along the path $C$ instead of the real axis $\mathbb{R}$:

$$f(x_1^0, x_2^0) \equiv \int_C dy^0 f(x_1^0, y^0)g(y^0, x_2^0).$$  \hspace{1cm} (3)

Obviously, the usual definition of the Fourier transform cannot accommodate the relations of Eq. (3) and Eq. (2). This definition should be modified in order to make these relations compatible. A first solution that I will not develop here is provided by the so called imaginary time formalism, which can be seen in this context as a work-around for the above problem. More precisely, one makes use of the $-i\beta$-periodicity properties of thermal Green’s functions in order to expand them in Fourier series, the Fourier modes (called Matsubara frequencies in this context) being imaginary since the period is imaginary.

### 3.2 Naive approach to the RTF

Another solution is provided by the matrix formulation (often called real time formalism when the context makes obvious the fact that we are in the Fourier space). For each $n$-point function, one defines $2^n$ distinct Fourier transforms labelled by $n$ superscripts $a_i = 1$ or $2$, via the relations

$$G^{(a_1, \cdots, a_n)}(k_1, \cdots, k_n) \equiv \prod_{i=1}^{n} \int_{C_{a_i} \times \mathbb{R}^4} d^4 x_i \, e^{i k_i \cdot x_i} \left[ G(x_1, \cdots, x_n) \right].$$  \hspace{1cm} (4)

Then, one would like to have Feynman rules enabling a direct calculation of these new Green’s functions, without going through the stage of the function in time coordinates. As a first approach, let us first assume that the vertical part $C_v$ does not contribute to the calculation of the function $G(x_1, \cdots, x_n)$. This hypothesis has been at the basis of most of the attempts to derive the matrix formalism, and the focus has mainly been on findings arguments to

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5I am implicitly assuming that $C_{1,2}$ are extended from $-\infty$ to $+\infty$ in this definition of the Fourier transforms, in order to make them as close as possible to the usual one. Nevertheless, it should be emphasized that this limit has no effect as far as the contribution of the vertical part is concerned, since the integrand $G(x_1, \cdots, x_n)$ is totally independent upon the times $t_i$ and $t_F$ (see appendix A). Therefore, we still have to find how to deal with the contribution of $C_v$ in this matrix formalism.
justify it. In the present paper, I will first derive the Feynman rules of the real
time formalism in situations where the vertical part does not contributes. If we
assume that the vertical part of the path does not contribute in the convolution
\( P(x_1, x_2) \equiv (F * G)(x_1, x_2) \), then we have obviously in terms of the previously
defined Fourier transforms:
\[
P^{ab} = F^{a1} G^{1b} - F^{a2} G^{2b} = (F \tau_3 G)^{ab},
\]  
where the Pauli matrix \( \tau_3 \equiv \text{Diag}(1, -1) \) deals with the minus sign associated
to type 2 indices. This relation can be seen as a particular form of Eq. (4),
the product of the right hand side being a matrix product. Therefore, we see
that transforming 2-point functions into \( 2 \times 2 \) matrices enables to generalize the
usual relationship between the Fourier transform and the convolution product
to the thermal case.

Going on along this line, we would of course obtain the standard matrix
formulation for the real time formalism in Fourier space. Nevertheless, this
justification is valid only for situations in which the vertical branch does not
contribute. At this point, the standard way has been to try to get rid of the
vertical part. Instead of that, I will determine precisely the situations in which
it contributes, and show that its contribution can be included in the matrix
formalism by a minor modification of its Feynman rules.

4 Diagrams in which
the vertical part contributes

4.1 Example

A simple example showing that the vertical part can contribute to the result of
a path integration is provided by the convolution of two bare propagators. Such
a calculation would appear for instance in the insertion of a mass term. This
calculation has been done explicitly in [10] and shows the following features:
(i) The vertical part is mandatory in order to have a result invariant under
time translation.
(ii) The vertical part enables to get rid of the \( t_i \) dependence that would show
up in the result if one where using only \( C_1 \cup C_2 \) (the validity of this result is
quite general, see appendix A).
(iii) A \( t_i\)-independent, invariant under time translation, contribution of the
vertical part of the path is left in the result.

The existence of such an explicit example definitively rules out the justifica-
tions based on the initial hypothesis that the vertical part does not contribute.
4.2 Generic contour integration

It is convenient to work with the mixed coordinates \((x^0, k)\) in which the bare propagator has the following explicit expression:

\[
G_o(x^0, y^0; k) = \frac{1}{2\omega_k} \sum_{s=\pm} G^E_{o,s}(x^0, y^0),
\]

with

\[
G^E_{o,s}(x^0, y^0) \equiv e^{-isE(y^0-x^0)} \left[ \theta_c(s(y^0-x^0)) + n_B(E) \right]
\]

and

\[
\omega_k \equiv \sqrt{k^2 + m^2}, \quad n_B(E) \equiv \frac{1}{e^{\beta E} - 1}.
\]

Because of the structure of this bare propagator, it is a priori obvious that every time integration can be reduced to integrals of the following type:

\[
I_C(\Sigma) \equiv \int_C dx^0 e^{-i\Sigma x^0} f(x^0, \Sigma).
\]

In the above integral, \(\Sigma\) is a linear combination of the on-shell energies \(\omega_k\) corresponding to the various legs (internal as well as external) of the diagram, with coefficients 0, 1 or \(-1\), while the function \(f(\cdot)\) is a product of factors like \(\theta_c(\pm(x^0-x^0_i)) + n_B(\omega_k, E)\). This function is therefore piece-wise constant along the path \(C\). Moreover, the KMS boundary condition is such that the integrand takes equal values at both ends of the path:

\[
e^{-i\Sigma t_i} f(t_i, \Sigma) = e^{-i\Sigma(t_i - i\beta)} f(t_i - i\beta, \Sigma).
\]

I want now to show that the object \(I_C(\Sigma)\) receives a contribution of the vertical part \(C_v\) if and only if \(\Sigma = 0\). Let us first assume that \(\Sigma \neq 0\). Therefore, an integration by parts gives immediately:

\[
I_C(\Sigma) = \frac{1}{i\Sigma} \int_C dx^0 e^{-i\Sigma x^0} \frac{\partial f(x^0, \Sigma)}{\partial x^0}.
\]

Then, since the function \(f(\cdot)\) is piece-wise constant, its derivative is a discrete sum of Dirac’s distributions \(\delta_c(\cdot)\). Therefore, the generic structure of the above integral is a sum like

\[
I_C(\Sigma) = \frac{1}{i\Sigma} \sum_i c_i e^{-i\Sigma x^0_i},
\]

where the \(c_i\) are coefficients we don’t need to make more explicit (\(c_i\) is the value at the point \(x^0 = x^0_i\) of the coefficient in front of \(\delta(x^0 - x^0_i)\) in \(\partial f/\partial x^0\)) and the \(x^0_i\) are the times at which the value of \(f(x^0, \Sigma)\) changes. Now, in order to see if there is in the above result a contribution which is specific to the vertical

\[\text{In situations where fermions with chemical potential are present in the theory, this result remains true because the fermions always come in pairs at vertices and because charges are conserved at each vertex.}\]
part of the path, let us calculate the same Feynman diagram using only $C_1 \cup C_2$. This means that $x_0^\alpha$, as well as all the $x_i^\alpha$ are now restricted to the horizontal branches of the time path. For the integral $I(\Sigma)$, the result would be the same

$$I_{C_1 \cup C_2}(\Sigma) = \frac{1}{i \Sigma} \sum_{\{i | x_i^\alpha \in C_1 \cup C_2\}} c_i e^{-i \Sigma x_i^\alpha}. \quad (13)$$

But, by definition of the calculation based on only $C_1 \cup C_2$, all the other times $x_0^\alpha$ are also on $C_1 \cup C_2$, so that the “restricted” sum contains in fact all the terms of the full sum, with the same coefficients $c_i$. Therefore:

$$\text{if } \Sigma \neq 0, \quad I_C(\Sigma) = I_{C_1 \cup C_2}(\Sigma), \quad (14)$$

and there is no contribution specific to $C_v$ in this case. In other words, all the contour integrals give the same result whether they appear in the calculation with the full path or in the calculation with only the horizontal branches, if $\Sigma \neq 0$.

Let us now consider the case where $\Sigma = 0$. The integration by parts gives now

$$I_C(\Sigma) = -i \beta f(t_f, 0) - \int_C dx^\alpha \frac{\partial f(x^\alpha, 0)}{\partial x^\alpha}. \quad (15)$$

By the same arguments as before, we can show that there is no contribution specific to the vertical part in the second term. But now the factor $-i \beta$ in the first term comes from the difference $t_f - (t_f - i \beta)$ of the two extremities of the time path. Therefore, this term would vanish if we were dropping the vertical part. From that, we conclude that this first term is a contribution from the vertical part.

4.3 Localization of the contribution of $C_v$

The condition $\Sigma = 0$ necessary to have a contribution of the vertical part is a constraint on the 3-momenta (both internal and external) of the diagram. But not all the situations where $\Sigma = 0$ lead to a contribution of the vertical part at the very end of the calculation. Indeed, since the function $I_C(\Sigma)$ is continuous at $\Sigma = 0$, we won’t have a contribution of $C_v$ at the end if the condition $\Sigma = 0$ defines a sub-manifold of zero measure in the space accessible to 3-momenta (taking into account the constraints provided by 3-momentum conservation). This is in fact the generic case.

There are only two distinct situations in which the condition $\Sigma = 0$ does not reduce the accessible space more than the 3-momentum conservation does. The first of these two cases correspond to vacuum diagrams (diagrams without external legs) for which the last time integration has always $\Sigma = 0$ because of the invariance under time translation (because a function of a single time must

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\footnote{We see that the contribution of the vertical part is not a continuous function of $\Sigma$. Nevertheless, the total contribution is a continuous function of $\Sigma$.}
be a constant if invariance under time translation holds). The fact that the last

time integration plays a particular role in such a diagram is at the origin of the

specific Feynman rules for vacuum diagrams: (i) the last time integration just
gives an extra factor \( -i\beta \), and (ii) one of the vertices must be kept fixed to type
1 or type 2. A justification of these additional rules is given in [10] and won’t
be reproduced in the present paper.

The second situation in which a contribution of the vertical part is left at the
end of the calculation is encountered for self-energy insertions between propa-
gators. Indeed, in that case, the frequency \( \Sigma \) can be the difference \( \omega_{k_1} - \omega_{k_2} \)
of the incoming and outgoing on-shell energies while 3-momentum conserva-
tion imposes \( k_1 = k_2 \), i.e. \( \Sigma = 0 \). This is the situation I will study in detail in
the next section. It is worth noticing that compared to [10], the insertion of
self-energies is shown to be the only situation in which the vertical part con-
tributes.\(^8\)

5 Effect of the vertical part
on the RTF Feynman rules

5.1 Basic example

I now study in detail the case of self-energy insertions which is the only one in
which the self-energy contributes, besides vacuum diagrams, in order to show
that the contribution of \( C_v \) is automatically included by the matrix formalism
provided that one uses \( |k^o| \) for the argument of statistical weights. The general
philosophy of the proof is to start from a Green’s function expressed in time
coordinates, for which we have unambiguous Feynman rules. Then we have to
Fourier transform it in order to obtain the corresponding matrix. Finally, we
must deduce from the result the Feynman rules in Fourier space that would
have given the same function. I will start by the trivial example of repeated
mass insertions. But contrary to [10] where this example was only used as an
illustration for the more general case of self-energy insertions, this example is
in the present paper at the very heart of the proof. Indeed, I show in the
next paragraph that the most general case can be reduced to the trivial one by
making use of simple algebraic properties of the contour integration.

The object we are interested in is the propagator obtained after the resum-

mation of an additional mass term \( -i\mu^2 \):

\[
G(x^o_1, x^o_2) \equiv \sum_{n=0}^{+\infty} (-i\mu^2)^n (G_o \ast \cdots \ast G_o)(x^o_1, x^o_2),
\]

where the convolution product appearing at order \( n \) in the sum contains \( n+1 \)
factors. Of course, the result of this sum is well known without the need of

\(^8\)In [10], I identified the condition \( \Sigma = 0 \) as the necessary condition to have a contribution
of the vertical part in \( I_C(\Sigma) \), but didn’t realize that this condition is relevant only if it defines
a sub-manifold of strictly positive measure.
performing the calculation

\[
G(x_1^o, x_2^o) = G_o(x_1^o, x_2^o)|_{m^2 \to m^2 + \mu_2^2},
\]

where the notation \( m^2 \to m^2 + \mu^2 \) means that each occurrence of \( m^2 \) in \( G_o \) is replaced by \( m^2 + \mu^2 \). Since the Fourier transform given by Eq. (16) does not involve the mass, the above result for the resummed propagator also holds for its Fourier transform. Therefore, we have to find out the Feynman rules that would give the matrix propagator in which the mass squared is translated by an amount equal to \( \mu^2 \). Let us now do the same resummation in the matrix formalism by making use of Eq. (5), in order to determine how it should be modified in order to reach the expected result Eq. (18). To that effect, it is convenient to factorize the free matrix propagator as follows

\[
G_o(k) = U(k) \begin{pmatrix} \Delta(k) & 0 \\ 0 & \Delta^*(k) \end{pmatrix} U(k),
\]

where \( \Delta(k) \equiv i \mathbb{P}/(k^2 - m^2) + \pi \delta(k^2 - m^2) \) is the usual Feynman propagator, and \( U(k) \) is a matrix containing the statistical factors:

\[
U(k) = \begin{pmatrix} \sqrt{1 + n_B} & (\theta(k^0) + n_B)/\sqrt{1 + n_B} \\ \theta(k^0) + n_B)/\sqrt{1 + n_B} & \sqrt{1 + n_B} \end{pmatrix}.
\]

At this stage, it seems that we still have the choice \(|k^o|\) or \( \omega_k \) for the arguments of the statistical weights. In the matrix formalism, the resummation is performed by

\[
G(k) = \sum_{n=0}^{+\infty} (-i\mu^2)^n G_o(k)[\tau_3 G_o(k)]^n
\]

\[
= U(k) \left[ \sum_{n=0}^{+\infty} (-i\mu^2)^n D_o(k)[\tau_3 D_o(k)]^n \right] U(k)
\]

\[
= U(k) \left[ D_o(k)|_{m^2 \to m^2 + \mu^2} \right] U(k),
\]

where I denote \( D_o(k) \equiv \text{Diag}(\Delta(k), \Delta^*(k)) \). In order to do the sum, I have used the algebraic relation \( U\tau_3 U = \tau_3 \). It is now obvious that if we want to have the relation \( G(k) = G_o(k)|_{m^2 \to m^2 + \mu^2} \), we need \( U(k) = U(k)|_{m^2 \to m^2 + \mu^2} \), which means that the matrix \( U(k) \) should be independent of \( m^2 \). The only way to achieve that is to use \(|k^o|\) as the argument of \( n_B \) in \( U(k) \).

Therefore, we have justified in the case of this simple example the fact that the prescription \( n_B(|k^o|) \) should be used in the RTF Feynman rules in order to

\[
\text{For the term of order } n \text{ in the infinite sum, this result implies:}
\]

\[
G_o \ast \cdots \ast G_o = \frac{1}{n!} \left[ \frac{\partial}{\partial m^2} \right]^n G_o,
\]

a relation known as the mass derivative formula [16].
get the correct result. Moreover, since we have seen in section 2 that the vertical part enables to take into account the interaction (here the term \(-i\mu^2\)) in the density operator, i.e. in the statistical factors, we can conclude that choosing the right argument for the statistical functions reintroduces the contribution of the vertical part in the result.

5.2 Repeated self-energy insertions

I want now to generalize the previous result concerning the \(n_B(|k^o|)\) prescription to the general case of self-energy insertions, illustrated on figure 2.

![Figure 2: Generic configuration giving a contribution of the vertical part.](image)

This situation seems more complicated at first sight since we don’t know \textit{a priori} the result. The calculation to be performed in time coordinates is

\[
G(x_1^o, x_2^o) \equiv \sum_{n=0}^{+\infty} (G_o * \Pi * \cdots * \Pi * G_o)(x_1^o, x_2^o),
\]

where the term of order \(n\) in the right hand side contains \(n\) factors \(\Pi\) and \(n + 1\) factors \(G_o\). This is where the properties of the contour convolution discussed in appendix 3 are quite helpful. Indeed, if we use now the commutativity of this product of convolution (which holds here since all the convoluted objects satisfy KMS), we can rewrite

\[
G(x_1^o, x_2^o) = \sum_{n=0}^{+\infty} ((G_o * \cdots * G_o) * (\Pi * \cdots * \Pi))(x_1^o, x_2^o).
\]

Now that the free propagators \(G_o\) are grouped together, we have reduced the problem to the previous one. Indeed, we know that we don’t have any contribution of the vertical part in the convolution of two objects if at least one of them is one particle irreducible, which is the case of \(\Pi\). Therefore, \(C_v\) does not contribute in the product \(\Pi * \cdots * \Pi\), and we can obtain its Fourier transform with the Feynman rules (Eq. 3) established under the hypothesis that \(C_v\) does not contribute. The only problem related to \(C_v\) comes from the product \(G_o * \cdots * G_o\) which has already been considered in the previous paragraph. Its Fourier transform is obtained by the Feynman rules with the \(n_B(|k^o|)\) prescription. Therefore, the Fourier transform is given by

\[
G(k) = \sum_{n=0}^{+\infty} G_o(k)[\tau_3G_o(k)]^n\Pi(k)[\tau_3\Pi(k)]^{n-1},
\]
in which one should use the \( n_p(|k^n|) \) prescription for the \( n + 1 \) \( G_o \)'s. At this stage, it is trivial to put the various factors back into a more natural order to get:

\[
G(k) = \sum_{n=0}^{+\infty} G_o(k)[\tau_3 \Pi(k) \tau_3 G_o(k)]^n .
\] (25)

This trick based on the commutativity of the contour convolution enabled us to reduce the general case to the simpler one treated in the previous paragraph, and to see again that the argument of the statistical weights for the propagators along the chain must be \( |k^n| \).

6 Concluding remarks

In this paper, I have given a new, quite compact, justification for the matrix formalism for the RTF in Fourier space. The focus has been on a correct treatment of the vertical part of the time path. In particular, no use is made of the limits \( t_I \to -\infty \) and \( t_F \to +\infty \), since KMS implies a total independence of the Green’s functions with respect to these parameters. The justification is made in three steps: (i) identify the diagrams in which the vertical part contributes (ii) show in the case of mass insertions that the contribution of \( C_v \) is included by the \( n(|k^n|) \) prescription and (iii) use simple properties of the contour convolution to reduce the general case to the previous one.

The present justification is complementary to that of Le Bellac and Mabilat, since it provides a better control on which are the topologies receiving a contribution from the vertical part, while in [11, 12] all the topologies appear on the same footing. Compared to that of [10], this proof is more complete since the situations in which the vertical part contributes are clearly delimited, and the end of the proof is considerably simplified by making use of the commutativity of the contour convolution.

I would also like to emphasize again on the physics encoded in the vertical part of the path. Indeed, since we know that the role of the vertical part in the perturbative expansion is to extract the dependence upon the coupling constant contained in the density operator, it was obvious right from the beginning that its effects on the Feynman rules could only affect the statistical factors.

To end this paper, it is worth making a comment on the Keldysh formalism [17] used in out-of-equilibrium situations. This formalism is based on a time path which does not contain the vertical part \( C_v \). Indeed, there is no need for it here since the initial density operator is not related to the Hamiltonian and therefore does not contain the coupling constant. All the properties of the equilibrium Green’s functions that are related to the presence of the vertical part are lost: out-of-equilibrium Green’s functions depend explicitly on the initial time \( t_I \), and

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\(^{10}\)This is possible because we can write \( G_o(k) = UD_o U \) and \( \Pi(k) = UP U \) with \( D_o \) and \( P \) diagonal matrices, and because \( U \) satisfies \( U\tau_3 U = \tau_3 \). This merely says that the commutativity of the contour convolution is transported in the matrix formalism.

\(^{11}\)For the other propagators, the prescription for the statistical factors is indifferent.
are not invariant under time translation. For this reason, going to Fourier space is also much less straightforward.

A Path independence of contour integrations

For the purpose of discussing the effect of the vertical part of the path in the real time formalism, we need first to recall some basic properties of the contour integration.

The most noticeable property of this integration is that it gives a result which is independent of the initial time \( t_I \) used to define the path \([10]\). This property is in fact a quite direct consequence of the KMS relations satisfied by propagators appearing in the perturbative expansion, and was to be expected given the physical meaning of thermal equilibrium.

Indeed, we can write any Green’s function \( G(x_1^0, \cdots, x_n^0) \) calculated perturbatively as:

\[
G(x_1^0, \cdots, x_n^0) = \prod_{i=1}^V \int_C dy_i^0 \ g(x_1^0, \cdots, x_n^0|y_1^i, \cdots, y_{\nu_i}^i),
\]

where \( V \) is the total number of vertices in the diagram, and the \( y_i^0 \) the inner times. Now, if we consider a function

\[
a(y^0) = \theta_c(y^0 - y^0_+^+)a^+(y^0) + \theta_c(y^0_+ - y^0_-)a^-(y^0)
\]

on the path \( C \), with holomorphic functions \( a^\pm(y^0) \), and then calculate the integral

\[
A \equiv \int_C dy^0 a(y^0),
\]

we have the following two properties:

(i) \( A \) depends only on the extremities of the path, and on the other times \( y_{\pm}^0 \), but not on its precise shape.

(ii) we have \( dA/dt_I = a(t_i - i\beta) - a(t_i) \).

Looking now at the structure of the bare propagators (see Eq. \([11]\)), we see that the integrand \( g \) satisfies the conditions of the previous lemma with the additional property of taking the same value at both extremities of the path for each inner variable \( y_i^0 \). Applying therefore (ii), we conclude that the function \( G(x_1^0, \cdots, x_n^0) \) is independent upon \( t_i \). Using then the possibility to deform the path (i), we can change \( t_F \) without changing the result of the integrals. The function \( G(x_1^0, \cdots, x_n^0) \) is therefore also independent of \( t_F \). Finally, the only dependence of \( G(x_1^0, \cdots, x_n^0) \) upon the path comes through the external times \( x_i^0 \) which are supposed to be on the path.

B Properties of the contour convolution

In order to deal simply with self-energy insertions, it is convenient to discuss first a few properties of the contour convolution defined by Eq. \([3]\).
The first obvious property is that the result is independent of both $t_I$ and $t_F$ provided that the two functions one is convoluting satisfy the KMS relations and correspond to two particles of the same nature.\footnote{This limitation is not important in practice since convoluting a bosonic function with a fermionic one would be totally meaningless.}

In order to simplify the study of this operation for functions satisfying KMS, the first step is to write two-point functions by means of their spectral representation:\footnote{It is possible to group the two terms in this sum in order to obtain a single term containing the full free propagator. This splitting is natural here since $G^E_{o,s}$ is the smallest part of the free propagator that still satisfies KMS. Any property that is a consequence of KMS can be obtained by limiting the study to this very simple piece.}

$$F(x_1^0, x_2^0) = \sum_{s = \pm} \int_0^{+\infty} dE f_s(E) G^E_{o,s}(x_1^0, x_2^0), \quad (29)$$

where the $G^E_{o,s}$ are the building blocks of the free propagator given by Eq. (7).

If one uses this spectral representation, it is sufficient to limit the study of the operation $\ast$ to its action on simple objects like $G^E_{o,s}$.

An elementary integration based on Eq. (11) gives immediately:

$$G^A_{o,\epsilon} \ast G^B_{o,\eta} = \frac{1}{i(\epsilon A - \eta B)} \left[ \epsilon G^B_{o,\eta} - \eta G^A_{o,\epsilon} \right]. \quad (30)$$

We notice that the result is unchanged if we permute the two objects we are convoluting. This property is transported to general two-point functions through their spectral representation: the contour convolution is commutative.

Iterating the above relation, we obtain:

$$\left( G^A_{o,\epsilon} \ast G^B_{o,\eta} \right) \ast G^C_{o,\mu} = \frac{\eta}{i(\epsilon A - \eta B)} \frac{\mu}{i(\epsilon A - \mu C)} G^A_{o,\epsilon} + \frac{\epsilon}{i(\eta B - \epsilon A)} \frac{\mu}{i(\eta B - \mu C)} G^B_{o,\eta} + \frac{\epsilon}{i(\mu C - \epsilon A)} \frac{\eta}{i(\mu C - \eta B)} G^C_{o,\mu}. \quad (31)$$

The remarkable property of this result is its symmetry under any permutation of the three objects one is convoluting. Again, this is trivially extended to any triplet of two-point functions: the contour convolution is associative.

To conclude this appendix, one can say that as far as functions satisfying KMS are concerned, the contour convolution possesses the same basic properties as the ordinary convolution product.

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