3-degenerate induced subgraph of a planar graph

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Abstract

A graph $G$ is $d$-degenerate if every non-null subgraph of $G$ has a vertex of
degree at most $d$. We prove that every $n$-vertex planar graph has a 3-degenerate
induced subgraph of order at least $3n/4$.

Keywords: planar graph; graph degeneracy.

1 Introduction

Graphs in this paper are simple, having no loops and no parallel edges. For a graph
$G = (V, E)$, the neighbourhood of $x \in V$ is denoted by $N(x) = N_G(x)$, the degree of
$x$ is denoted by $d(x) = d_G(x)$, and the minimum degree of $G$ is denoted by $\delta(G)$. Let
$\Pi = \Pi(G)$ be the set of total orderings of $V$. For $L \in \Pi$, we orient each edge $vw \in E$
as $(v, w)$ if $w <_L v$ to form a directed graph $G_L$. We denote the out-neighbourhood,
also called the back-neighbourhood, of $x$ by $N_G^L(x)$, the out-degree, or back-degree, of $x$
bys $d_G^L(x)$. We write $\delta^+ (G_L)$ and $\Delta^+ (G_L)$ to denote the minimum out-degree and the

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maximum out-degree, respectively, of $G_L$. We define $|G| := |V|$, called the order of $G$, and $\|G\| := |E|$.

An ordering $L \in \Pi(G)$ is $d$-degenerate if $\Delta^+(G_L) \leq d$. A graph $G$ is $d$-degenerate if some $L \in \Pi(G)$ is $d$-degenerate. The degeneracy of $G$ is $\min_{L \in \Pi(G)} \Delta^+(G_L)$. It is well known that the degeneracy of $G$ is equal to $\max_{H \subseteq G} \delta(H)$.

Alon, Kahn, and Seymour [4] initiated the study of maximum $d$-degenerate induced subgraphs in a general graph and proposed the problem on planar graphs. We study maximum $d$-degenerate induced subgraphs of planar graphs. For a non-negative integer $d$ and a graph $G$, let

$$\alpha_d(G) = \max\{|S| : S \subseteq V(G), \ G[S] \text{ is } d\text{-degenerate}\} \text{ and }$$

$$\bar{\alpha}_d = \inf\{\alpha_d(G)/|V(G)| : G \text{ is a non-null planar graph}\}.$$

Let us review known bounds for $\bar{\alpha}_d$. Suppose that $G = (V, E)$ is a planar graph. For $d \geq 5$, trivially we have $\bar{\alpha}_d = 1$ because planar graphs are 5-degenerate.

For $d = 0$, a 0-degenerate graph has no edges and therefore $\alpha_0(G)$ is the size of a maximum independent set of $G$. By the Four Colour Theorem, $G$ has an independent set $I$ with $|I| \geq |V(G)|/4$. Both $K_4$ and $C_8^2$ witness that $\bar{\alpha}_0 \leq 1/4$, so $\bar{\alpha}_0 = 1/4$. In 1968, Erdős (see [5]) asked whether this bound could be proved without the Four Colour Theorem. This question still remains open. In 1976, Albertson [2] showed that $\bar{\alpha}_0 \geq 2/9$ independently of the Four Colour Theorem. This bound was improved to $\bar{\alpha}_0 \geq 3/13$ independently of the Four Colour Theorem by Cranston and Rabern in 2016 [8].

For $d = 1$, a 1-degenerate graph is a forest. Since $K_4$ has no induced forest of order greater than 2, we have $\bar{\alpha}_1 \leq 1/2$. Albertson and Berman [3] and Akiyama and Watanabe [1] independently conjectured that $\bar{\alpha}_1 = 1/2$. In other words, every planar graph has an induced forest containing at least half of its vertices. This conjecture received much attention in the past 40 years; however, it remains largely open. Borodin [7] proved that the vertex set of a planar graph can be partitioned into five classes such that the subgraph induced by the union of any two classes is a forest. Taking the two largest classes yields an induced forest of order at least $2|V(G)|/5$. So $\bar{\alpha}_1 \geq 2/5$. This remains the best known lower bound on $\bar{\alpha}_1$. On the other hand, the conjecture of Albertson and Berman, Akiyama and Watanabe was verified for some subfamilies of planar graphs. For example, $C_3$-free, $C_5$-free, or $C_6$-free planar graphs were shown in [20] [11] to be 3-degenerate, and a greedy algorithm shows that the vertex set of a 3-degenerate graph can be partitioned into two parts, each inducing a forest. Hence $C_3$-free, $C_5$-free, or $C_6$-free planar graphs satisfy the conjecture. Moreover, Raspaud and Wang [16] showed that $C_4$-free planar graphs can be partitioned into two induced forests, thus satisfying the conjecture. In fact, many of these graphs have larger induced forests. Le [14] showed that if a planar graph $G$ is $C_3$-free, then it has an induced forest with at least $5|V(G)|/9$ vertices; Kelly and Liu [12] proved that if in addition $G$ is $C_4$-free, then $G$ has an induced forest with at least $2|V(G)|/3$ vertices.

Now let us move on to the case that $d = 2$. The octahedron has 6 vertices and is 4-regular, so a 2-degenerate induced subgraph has at most 4 vertices. Thus $\bar{\alpha}_2 \leq 2/3$. 


We conjecture that equality holds. Currently, we only have a more or less trivial lower bound: \( \bar{\alpha}_2 \geq 1/2 \), which follows from the fact that \( G \) is 5-degenerate, and hence we can greedily 2-colour \( G \) in an ordering that witnesses its degeneracy so that no vertex has three out-neighbours of the same colour, i.e., each colour class induces a 2-degenerate subgraph. Dvořák and Kelly [10] showed that if a planar graph \( G \) is \( C_3 \)-free, then it has a 2-degenerate induced subgraph containing at least \( 4|V(G)|/5 \) vertices.

For \( d = 4 \), the icosahedron has 12 vertices and is 5-regular, so a 4-degenerate induced subgraph has at most 11 vertices. Thus \( \bar{\alpha}_4 \leq 11/12 \). Again we conjecture that equality holds. The best known lower bound is \( \bar{\alpha}_4 \geq 8/9 \), which was obtained by Lukoťka, Mazák and Zhu [15].

In this paper, we study 3-degenerate induced subgraphs of planar graphs. Both the octahedron \( C_6^2 \) and the icosahedron witness that \( \bar{\alpha}_3 \leq 5/6 \). Here is our main theorem.

**Theorem 1.1.** Every \( n \)-vertex planar graph has a 3-degenerate induced subgraph of order at least \( 3n/4 \).

We conjecture that the upper bounds for \( \bar{\alpha}_d \) mentioned above are tight. We remark that it is possible to obtain infinitely many 3-connected tight examples for each \( d \) by gluing together many copies of the tight example discussed above.

**Conjecture 1.1.** \( \bar{\alpha}_2 = 2/3, \bar{\alpha}_3 = 5/6, \) and \( \bar{\alpha}_4 = 11/12 \).

The problem of colouring the vertices of a planar graph \( G \) so that colour classes induce certain degenerate subgraphs has been studied in many papers. Borodin [7] proved that every planar graph \( G \) is acyclically 5-colourable, meaning that \( V(G) \) can be coloured in 5 colours so that a subgraph of \( G \) induced by each colour class is 0-degenerate and a subgraph of \( G \) induced by the union of any two colour classes is 1-degenerate. As a strengthening of this result, Borodin [6] conjectured that every planar graph has degenerate chromatic number at most 5, which means that the vertices of any planar graph \( G \) can be coloured in 5 colours so that for each \( i \in \{1,2,3,4\} \), a subgraph of \( G \) induced by the union of any \( i \) colour classes is \( (i-1) \)-degenerate. This conjecture remains open, but it was proved in [13] that the list degenerate chromatic number of a graph is bounded by its 2-colouring number, and it was proved in [9] that the 2-colouring number of every planar graph is at most 8. As consequences of the above conjecture, Borodin posed two other weaker conjectures: (1) Every planar graph has a vertex partition into two sets such that one induces a 2-degenerate graph and the other induces a forest. (2) Every planar graph has a vertex partition into an independent set and a set inducing a 3-degenerate graph. Thomassen confirmed these conjectures in [18] and [19].

This paper is organized as follows. In Section 2 we will present our notation. In Section 3 we will formulate a stronger theorem that allows us to apply induction. This will involve identifying numerous obstructions to a more direct proof. In Section 4 we will organize our proof by contradiction around the notion of an *extreme counterexample*. In Sections 5–7, we will develop properties of extreme counterexamples that eventually lead to a contradiction in Section 8.
2 Notation

For sets $X$ and $Y$, define $Z = X \cup Y$ to mean $Z = X \cup Y$ and $X \cap Y = \emptyset$. Let $G = (V, E)$ be a graph with $v, x, y \in V$ and $X, Y \subseteq V$. Then $\|v, X\|$ is the number of edges incident with $v$ and a vertex in $X$ and $\|X, Y\| = \sum_{v \in X} \|v, Y\|$. When $X$ and $Y$ are disjoint, $\|X, Y\|$ is the number of edges $xy$ with $x \in X$ and $y \in Y$. In general, edges in $X \cap Y$ are counted twice by $\|X, Y\|$. Let $N(X) = \bigcup_{x \in X} N(x) - X$.

We write $H \subseteq G$ to indicate that $H$ is a subgraph of $G$. The subgraph of $G$ induced by a vertex set $A$ is denoted by $G[A]$. The path $P$ with $V(P) = \{v_1, \ldots, v_n\}$ and $E(P) = \{v_1v_2, \ldots, v_{n-1}v_n\}$ is denoted by $v_1 \cdots v_n$. Similarly the cycle $C = P + v_nv_1$ is denoted by $v_1 \cdots v_nv_1$.

Now let $G$ be a simple connected plane graph. The boundary of the infinite face is denoted by $B = B(G)$ and $V(B(G))$ is denoted by $B = B(G)$. Then $B$ is a subgraph of the outerplanar graph $G[B]$. For a cycle $C$ in $G$, let $\text{int}_{G}[C]$ denote the subgraph of $G$ obtained by removing every exterior vertices and edges and let $\text{ext}_{G}[C]$ be the subgraph of $G$ obtained by removing all interior vertices and edges. Usually the graph $G$ is clear from the text, and we write $\text{int}[C]$ and $\text{ext}[C]$ for $\text{int}_{G}[C]$ and $\text{ext}_{G}[C]$. Let $\text{int}(C) = \text{int}[C] - V(C)$ and $\text{ext}(C) = \text{ext}[C] - V(C)$. Let $N^\circ(x) = N(x) - B$ and $N^\circ(X) = N(X) - B$.

For $L \in \Pi$, the up-set of $x$ in $L$ is defined as $U_L(x) = \{y \in V : y >_L x\}$ and the down-set of $x$ in $L$ is defined as $D_L(x) = \{y \in V : y <_L x\}$. Note that for each $L \in \Pi$, $y <_L x$ means that $y \leq_L x$ and $y \neq x$. For two sets $X$ and $Y$, we say $X \leq_L Y$ if $x \leq_L y$ for all $x \in X$, $y \in Y$.

3 Main result

In this section we phrase a stronger, more technical version of Theorem [17] that is more amenable to induction. This is roughly analogous to the proof of the 5-Choosability Theorem by Thomassen [17].

If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[A]$ for a set $A$ of vertices, then we would like to join two 3-degenerate subgraphs obtained from $G_1$ and $G_2$ by induction to form a 3-degenerate subgraph of $G$. The problem is that vertices from $A$ may have neighbours in both subgraphs. Dealing with this motivates the following definitions.

Let $A \subseteq V(G)$. A subgraph $H$ of $G$ is $(k, A)$-degenerate if there exists an ordering $L \in \Pi(G)$ such that $A \leq_L V - A$ and $d_H'(v) \leq k$ for every vertex $v \in V(H) - A$. Equivalently, every subgraph $H'$ of $H$ with $V(H') - A \neq \emptyset$ has a vertex $v \in V(H') - A$ such that $d_{H'}(v) \leq k$. A subset $Y$ of $V$ is $A$-good if $G[Y]$ is $(3, A)$-degenerate. We say a subgraph $H$ is $A$-good if $V(H)$ is $A$-good. Thus if $A = \emptyset$ then $G$ is $A$-good if and only if $G$ is 3-degenerate. Let

$$f(G; A) = \max\{|Y| : Y \subseteq V(G) \text{ is } A\text{-good}\}.$$ 

Since $\emptyset$ is $A$-good, $f(G; A)$ is well defined.
For an induced subgraph $H$ of $G$ and a set $Y$ of vertices of $H$, we say $Y$ is \emph{collectable} in $H$ if the vertices of $Y$ can be ordered as $y_1, y_2, \ldots, y_k$ such that for each $i \in \{1, 2, \ldots, k\}$, either $y_i \notin A$ and $d_{H-\{y_1, y_2, \ldots, y_{i-1}\}}(y_i) \leq 3$ or $V(H)-\{y_1, y_2, \ldots, y_{i-1}\} \subseteq A$.

In order to build an $A$-good subset, we typically apply a sequence of operations of deleting and collecting. Deleting $X \subset V$ means replacing $G$ with $G-X$. An ordering witnessing that $Y$ is collectable is called a \emph{collection} order. For disjoint subsets $V_1, \ldots, V_s$ of $V$, if $V_i$ is collectable in $G-\bigcup_{j=1}^{s-1} V_j$ for each $i = 1, 2, \ldots, s$, then collecting $V_1, \ldots, V_s$ means first putting $V_1$ at the end of $L$ in a collection order for $V_1$, then putting $V_2$ at the end of $L-V_1$ in a collection order for $V_2$ in $G-V_1$, etc. Note that if $Y$ is a collectable set in $G$ and $V-Y$ is $A$-good, then $V$ is $A$-good.

\textbf{Definition 3.1.} A path $v_1v_2 \ldots v_\ell$ of a plane graph $G$ is \emph{admissible} if $\ell > 0$ and it is a path in $B(G)$ such that for each $1 < i < \ell$, $G-v_i$ has no path from $v_{i-1}$ to $v_{i+1}$.

A path of length 0 has only 1 vertex in its vertex set.

\textbf{Definition 3.2.} A set $A$ of vertices of a plane graph $G$ is \emph{usable} in $G$ if for each component $G'$ of $G$, $A \cap V(G')$ is the empty set or the vertex set of an admissible path of $G'$.

\textbf{Lemma 3.1.} Let $G$ be a plane graph and let $A$ be a usable set in $G$. Then for each vertex $v$ of $G$, $|N_G(v) \cap A| \leq 2$.

\textit{Proof.} This is clear from the definition of an admissible path. \hfill \Box

\textbf{Observation 3.1.} If $G$ is outerplanar and $A$ is a usable set in $G$, then $G$ is $(2, A)$-degenerate.

Observation 3.1 motivates the expectation that plane graphs with large boundaries have large 3-degenerate induced subgraphs. Roughly, we intend to prove that $f(G; A) \leq 3|V(G)|/4 + |B|/4$. This formulation provides a potential function for measuring progress as we collect and delete vertices. For example, deleting a boundary vertex with at least four interior neighbours provides a smaller graph whose potential is at least as large. Some of the bonus $|B|/4$ is needed for dealing with chords. But this does not quite work; $C_6^2$ is a counterexample, and there are infinitely many more. The rest of this section is devoted to formulating a more refined potential function.

A set $Z$ of vertices is said to be \emph{exposed} if $Z \subseteq B$. We say that a vertex $z$ is \emph{exposed} if $\{z\}$ is exposed. We say that deleting $Y$ and collecting $X$ \emph{exposes} $Z$ if $Z \subseteq B(G-Y-X)-B$.

\textbf{Definition 3.3.} Let $Q = \{Q_1, Q_2, Q_2^+, Q_3, Q_4, Q_4^+, Q_4^{++}\}$ be the set of plane graphs shown in Figure 4. For a plane graph $G$, a cycle $C$ of $G$ is \emph{special} if $G_C := \text{int}_G[C]$ is isomorphic to a plane graph in $Q$, where $C$ corresponds to the boundary. In this case, $G_C$ is also \emph{special}. 

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For a special cycle $C$ of a plane graph $G$, we define

\[ T_C := \text{int}_G(C), \] which is isomorphic to $K_3$,  
\[ X_C := \{ v \in V(C) : \text{there is a facial cycle } D \text{ such that} \]
\[ v \in V(C) \cap V(D) \text{ and } |V(T_C) \cap V(D)| = 2 \}, \]
\[ V_C := V(G_C), Y_C := X_C \cup V(T_C), \text{ and } \overline{Y}_C := V_C - Y_C = V(C) - X_C. \]

Then $V(C) = X_C \cup \overline{Y}_C$.

**Observation 3.2.** Let $A$ be a usable set in a plane graph $G$. Let $C = v_1 \ldots v_kv_1$ be a special cycle of $G$. If $G_C$ is (not only isomorphic but also equal to a plane graph) in $Q$, then the following hold.

(a) $T_C = xyzx$ with $N_G(x) = \{y, z, v_2, v_3\}$ and $N_G(y) = \{x, z, v_1, v_2\}$.
(b) $X_C = \{v_1, v_2, v_3\}$ if $G_C \neq Q_3$ and $X_C = \{v_1, v_2, v_3, v_4\}$ if $G_C = Q_3$.
(c) Deleting any vertex in $X_C \cap B$ exposes two vertices of $T_C$.
(d) For each vertex $v \in X_C$, $V(T_C)$ is collectable in $G - v$, except that if $G_C = Q_4^{++}$ and $v = v_2$ then only $\{x, y\}$ is collectable in $G - v$.
(e) If $\overline{Y}_C \neq \emptyset$ then there is a facial cycle $C^*$ containing $\overline{Y}_C \cup \{v\}$ for some $v \in V(T_C)$.
   Moreover, $v = z$ is unique, and if $|\overline{Y}_C| = 2$, then $C^*$ is unique.
(f) $T_C$ has at least two vertices $v$ such that $d_G(v) = 4$.

Note that vertices on $C$ may have neighbours in $\text{ext}(C)$ or maybe contained in $A$. Thus we may not be able to collect vertices of $C$.

A special cycle $C$ is called exposed if $X_C \subseteq B(G)$. A special cycle packing of $G$ is a set of exposed special cycles $\{C_1, \ldots, C_m\}$ such that $Y_{C_i} \cap Y_{C_j} = \emptyset$ for all $i \neq j$. Let

Figure 1: Plane graphs in $Q$ defining special subgraphs $G_C$ where $C$ corresponds to the boundary cycle. Solid black vertices denote vertices in $X_C$. 
τ(G) be the maximum cardinality of a special cycle packing and
\[ \partial(G) = \frac{3}{4}|V(G)| + \frac{1}{4}(|B| - \tau(G)). \]
We say that a special cycle packing of G is \textit{optimal} if its cardinality is equal to \( \tau(G) \).

\textbf{Theorem 3.2.} For all plane graphs G and usable sets \( A \subseteq B(G) \),
\[ f(G; A) \geq \partial(G). \quad (3.1) \]

Clearly \(|B| - \tau(G) \geq 2\) for any plane graph G with at least 2 vertices. This is trivial if \( \tau(G) = 0 \). If \( \tau(G) = k \), then each of the \( k \) exposed cycles in the maximum cardinality special cycle packing of G has at least 3 vertices in B and therefore \(|B| - \tau(G) \geq 2k \geq 2\).

The following consequence of Theorem 3.2 is the main result of this paper.

\textbf{Corollary 3.3.} Every \( n \)-vertex planar graph G (with \( n \geq 2 \)) has an induced 3-degenerate subgraph H with \(|V(H)| \geq (3n + 2)/4\).

\section{Setup of the proof}

Suppose Theorem 3.2 is not true. Among all counterexamples, choose \((G; A)\) so that
(i) \(|V(G)|\) is minimum,
(ii) subject to (i), \(|A|\) is maximum, and
(iii) subject to (i) and (ii), \(|E(G)|\) is maximum.
We say that such a counterexample is \textit{extreme}.

If \( A' \not\subseteq V(G') \), then we may abbreviate \((G'; A' \cap V(G'))\) by \((G'; A')\), but still (ii) refers to \(|A' \cap V(G')|\). We shall derive a sequence of properties of \((G; A)\) that leads to a contradiction. Trivially \(|V(G)| > 2\), G is connected (if G is the disjoint union of \( G_1 \) and \( G_2 \), then \( f(G; A) = f(G_1; A) + f(G_2; A) \) and \( \partial(G) = \partial(G_1) + \partial(G_2) \)).

\textbf{Lemma 4.1.} Let G be a plane graph and \( X \) be a subset of \( V(G) \). If \( A \) is usable in \( G \), then \( A - X \) is usable in \( G - X \).

\textbf{Proof.} We may assume that G is connected and \( X = \{v\} \). If \( v \notin A \), then it is trivial. Let \( P = v_0v_1 \cdots v_k \) be the admissible path in G such that \( A = V(P) \). If \( v = v_0 \) or \( v = v_k \), then again it is trivial. If \( v = v_i \) for some \( 0 < i < k \), then by the definition of admissible paths, \( G - v_i \) is disconnected, and \( v_{i-1} \) and \( v_{i+1} \) are in distinct components. Thus again \( A - \{v\} \) is usable in \( G - v \). \( \square \)

Suppose \( Y \) is a nonempty subset of \( V(G) \) and \( G[Y] \) is connected. Let \( C \) be an exposed special cycle of \( G' = G - Y \). Then \( C \) satisfies one of the following conditions.
(a) \( C \) is an exposed special cycle of \( G \).
(b) \( C \) is a non-exposed special cycle of \( G \); in this case \( X_C \cap (B(G') - B) \neq \emptyset \).
(c) \( C \) is not a special cycle of \( G \); in this case \( Y \subseteq \text{int}_G(C) \), and so \( Y \cap B = \emptyset \).
Lemma 4.2. Let $Y$ be a nonempty subset of $V(G)$ such that $G[Y]$ is connected. Let $G' = G - Y$. If $C$, $C'$ are distinct exposed type-c special cycles of $G'$, then $Y_C \cap Y_{C'} \neq \emptyset$.

Proof. Let $C$, $C'$ be distinct exposed type-c special cycles of $G'$. Since $B \cap Y = \emptyset$ and $G[Y]$ is connected, there exists a facial cycle $D$ of $G'$ such that $\text{int}_C(D) = G[Y]$. Then $D$ is a facial cycle of both $G'_C$ and $G'_{C'}$. Arguing by contradiction, suppose $Y_C \cap Y_{C'} = \emptyset$.

Since $V(D) \subseteq V(G'_C) = Y_C \cup \bar{Y}_C$ and $V(D) \subseteq V(G'_{C'}) = Y_{C'} \cup \bar{Y}_{C'}$, we have

$$V(D) \subseteq V(G'_C) \cap V(G'_{C'}) \subseteq \bar{Y}_C \cup \bar{Y}_{C'}.$$

By symmetry we may assume that $|\bar{Y}_C \cap V(D)| \geq |\bar{Y}_{C'} \cap V(D)|$. Using $|\bar{Y}_C|, |\bar{Y}_{C'}| \leq 2$, we deduce that

$$3 \leq |V(D)| \leq |V(G'_C) \cap V(G'_{C'})| \leq 4, \quad \bar{Y}_C \subseteq V(D), \quad |\bar{Y}_C| = 2, \quad \text{and} \quad \bar{Y}_{C'} \cap V(D) \neq \emptyset.$$

We will show that $H$ is isomorphic to $H_1$ or $H_2$ in Figure 2. Since $|\bar{Y}_C| = 2$, by Observation 3.2(e), $D$ is the unique facial cycle in $G'_C$ such that there is a vertex $\hat{z} \in V(T_C)$ with $\bar{Y}_C \cup \{\hat{z}\} \subseteq V(D)$. As $\hat{z} \in V(D)$ and $\hat{z} \in Y_C$, we have $\hat{z} \in \bar{Y}_{C'}$. Since $\bar{Y}_{C'} \neq \emptyset$, again by Observation 3.2(e), there is a unique vertex $\bar{z} \in V(T_{C'})$ such that $\bar{Y}_{C'} \cup \{\bar{z}\}$ is contained in a facial cycle of $G'_{C'}$. Then $\bar{z} \in \bar{Y}_{C'} \cap V(D)$.

First we show that $|V(G'_C) \cap V(G'_{C'})| = 4$. Assume to the contrary that $|V(G'_C) \cap V(G'_{C'})| = 3$. Since $V(D) \subseteq V(G'_C) \cap V(G'_{C'})$, we conclude that $|V(D)| = 3$. Then $\bar{z}\bar{z}$ is an edge, and the two inner faces of $G'$ incident with $\bar{z}\bar{z}$ are contained in $V(G'_C) \cap V(G'_{C'})$.

Since the intersection of any two inner faces of $G'_C$ has at most 2 vertices, we have $|V(G'_C) \cap V(G'_{C'})| \geq 4$, a contradiction.

As $V(G'_C) \cap V(G'_{C'}) = \bar{Y}_C \cup \bar{Y}_{C'}$, we conclude that $|\bar{Y}_{C'}| = 2$ and $|V(C)| = 5 = |V(C')|.$

Let $Q \in Q$ be the plane graph isomorphic to $G'_C$. By inspection of Figure 1, $G'_{C'}$ is isomorphic to $Q$. We may assume that $G'_C = Q$ by relabelling vertices. Let $u \mapsto u'$ be an isomorphism from $G'_C$ to $G'_{C'}$. Using uniqueness from Observation 3.2(e), $z = \bar{z}$, $z' = \bar{z}$, $\bar{Y}_C = \{v_4, v_5\}$ and $\bar{Y}_{C'} = \{v'_4, v'_5\}$. To prove our claim let us divide our analysis into two cases, resulting either in $H_1$ or $H_2$.

- If $|V(D)| = 4$, then $Q = Q_4^{+}$ and $V(G'_C) \cap V(G'_{C'}) = V(D) = \{v_4, v_5, v_1, z\}$. Since $v_4, v_5 \in \bar{Y}_C$, we have $v_1, z \in \bar{Y}_{C'}$. Then $v'_4 = v_1, v'_5 = z$, and $v_5 = z'$. As $X_C$ and $X_{C'}$ are exposed in $G'$, the cycle $v_1v_2v_3v'_4v'_3v_2v_3v_1$ is in $G'[B(G')]$ and so $H = H_1$ in Figure 2(a).

- If $|V(D)| = 3$, then $Q = Q_4^{++}$, $V(D) = \{z, v_4, v_5\}$. By symmetry, we may assume that $z' = v_5$. Then $V(G'_C) \cap V(G'_{C'}) = \{z, v_4, v_5, v_1\}$, as $C'$ contains all common
neighbors of $z$ and $z'$ in $G'$, which is a property of $Q_{41}^+$. Since $v_4, v_5 \in \overline{Y}_C$ and $V(G'_C) \cap V(G''_C) \subseteq \overline{Y}_C \cup \overline{Y}_{C'}$, we deduce that $v_1, z \in \overline{Y}_{C'}$. By symmetry in $G''_C$, we may assume that $z = v'_5$ and $v_1 = v'_4$. As $X_C$ and $X_{C'}$ are exposed in $G'$, the cycle $v_1v_2v_3v'_1v'_2v'_3v_1$ is in $G''[B(G')]$. So $H = H_1 + zz' = H_2$ in Figure 2(b).

Notice that in both cases, $v_4 = v'_1 \in B(G')$ and $v_4 \in V(D)$. Set $Y' = \{v_4, x', y'\}$ and $G'' = G - Y'$. As $V(\text{int}_G(D)) = Y$, in $G - v_4$, we can collect both $x'$ and $y'$ and at least one vertex of $Y$ is exposed. Thus $B(G'') - B$ contains $z, z'$ and $(B(G'') - B) \cap Y \neq \emptyset$. So $|B(G'') - B| \geq 3$.

Let $\mathcal{P}$ be an optimal special cycle packing of $G''$, and put

$$\mathcal{P}_0 = \{C^* \in \mathcal{P} : C^* \text{ is a non-exposed special cycle of } G\}.$$

Consider $C^* \in \mathcal{P}_0$. As $v_4 = v'_1 \in B \cap Y'$, there is no exposed type-c special cycle in $G''$. Thus $C^*$ is type-b, and so $X_{C^*} \cap (B(G'') - B) \neq \emptyset$. Let $w \in X_{C^*} \cap (B(G'') - B)$. Since $T_{C^*}$ is connected, has a neighbour of $w$, and has no vertex from $B(G'')$, we have $V(T_{C^*}) \subseteq Y$ and $X_{C^*} \subseteq (B(G'') - B) \cup \{v_1\}$.

As $\mathcal{P}_0$ is a packing, $3|\mathcal{P}_0| \leq |B(G'') - B| + 1$. This implies that $|\mathcal{P}_0| \leq |B(G'') - B| - 2$, because $|B(G'') - B| \geq 3$. We now deduce that

$$|\mathcal{P}_0| \leq |B(G'') - B| - 2 = |B(G'')| - (|B| - 1) - 2 = |B(G'')| - |B| - 1.$$ 

Therefore

$$\tau(G) \geq \tau(G'') - |\mathcal{P}_0| \geq \tau(G'') - |B(G'')| + |B| + 1.$$ 

Hence, using $V(G) = V(G'') \cup Y'$,

$$\partial(G) = \frac{3}{4}|V(G)| + \frac{1}{4}(|B| - \tau(G)) \leq \frac{3}{4}(|V(G'')| + 3) + \frac{1}{4}(|B(G'')| - \tau(G'') - 1) = \partial(G'') + 2.$$ 

Now, as we have already collected $x', y'$, we have

$$f(G; A) \geq f(G''; A) + 2 \geq \partial(G'') + 2 \geq \partial(G).$$
This contradicts the assumption that $G$ is a counterexample.

\begin{lemma}
Let $Y$ be a nonempty subset of $V(G)$ such that $G[Y]$ is connected and let $G' = G - Y$. Then
\[ \partial(G) \leq \partial(G') + \frac{3|Y|}{4} + \frac{|B - B(G')| + \delta(Y)}{4}. \]
Moreover, if $G$ has an exposed special cycle $C$ such that $Y_C \cap Y \neq \emptyset$ and $Y_C \cap Y_{C'} = \emptyset$ for any other exposed special cycle $C'$ of $G$, then
\[ \partial(G) \leq \partial(G') + 3\frac{|Y|}{4} + \frac{|B - B(G')| + \delta(Y) - 1}{4}. \]
\end{lemma}

\begin{proof}
An optimal special cycle packing of $G'$ has at most $|B(G') - B|$ type-b cycles by definition and has $\delta(Y)$ type-c cycles by Lemma 4.2. We can remove such cycles from the special cycle packing of $G$ to obtain a special cycle packing of $G$. So
\[ \tau(G) \geq \tau(G') - |B(G') - B| - \delta(Y) = \tau(G') - |B(G')| + |B| - |B - B(G')| - \delta(Y). \]
Plugging this into the definition of $\partial(G)$, we obtain
\[ \partial(G) \leq \partial(G') + \frac{3|Y|}{4} + \frac{|B - B(G')| + \delta(Y)}{4}. \]
If $G$ has an exposed special cycle $C$ such that $C$ is not a special cycle of $G'$ and $Y_C$ is disjoint from $Y_{C'}$ for any other exposed special cycle $C'$ of $G$, then we can add cycle $C$ to the special cycle packing of $G$ obtained above. So
\[ \tau(G) \geq \tau(G') - |B(G') - B| - \delta(Y) + 1 = \tau(G') - |B(G')| + |B| - |B - B(G')| - \delta(Y) + 1. \]
Plugging this into the definition of $\partial(G)$, we obtain
\[ \partial(G) \leq \partial(G') + 3\frac{|Y|}{4} + \frac{|B - B(G')| + \delta(Y) - 1}{4}. \]
\end{proof}

\begin{lemma}
Every vertex $v \in V - A$ satisfies $d(v) \geq 4$.
\end{lemma}

\begin{proof}
Suppose that $d(v) \leq 3$. Apply Lemma 4.3 with $Y = \{v\}$. Let $G' = G - Y$. Note that if $v$ is a boundary vertex, then $\delta(Y) = 0$. So $|B - B(G')| + \delta(Y) \leq 1$. Therefore
\[ \partial(G) \leq \partial(G') + \frac{3}{4} + \frac{1}{4}. \]
By the minimality of $(G; A)$, $f(G'; A) \geq \partial(G')$. Therefore $f(G; A) = f(G'; A) + 1 \geq \partial(G)$, a contradiction.
\end{proof}

\begin{lemma}
There are no disjoint nonempty subsets $X, Y$ of $V(G)$ such that $Y$ is a set of $4|X|$ interior vertices of $G$, $G[X \cup Y]$ is connected, and $Y$ is collectable in $G - X$.
\end{lemma}
Proof. Suppose that there exist disjoint nonempty sets $X, Y \subseteq V(G)$ such that $Y$ is a subset of $4|X|$ interior vertices of $G$, $G[X \cup Y]$ is connected, and $Y$ is collectable in $G - X$. Let $G' = G - (X \cup Y)$. We apply Lemma 4.3. Since $|B - B(G')| + \delta(X \cup Y) \leq |X|$, we have $\partial(G) \leq \partial(G') + \frac{2}{3}(|X| + |Y|) + \frac{1}{4}|X| = \partial(G') + 4|X|$. As $G$ is extreme, $f(G'; A) \geq \partial(G')$. Hence $f(G; A) \geq f(G'; A) + |Y| = f(G'; A) + 4|X| \geq \partial(G') + 4|X| \geq \partial(G)$, a contradiction. \hfill\(\Box\)

Lemma 4.6. For any two distinct special cycles $C_1, C_2$ of $G$, $Y_{C_1} \cap Y_{C_2} = \emptyset$.

Proof. Assume to the contrary that $C_1, C_2$ are two special cycles of $G$ with $Y_{C_1} \cap Y_{C_2} \neq \emptyset$. Observe that for each $i = 1, 2$, $V(T_{C_i})$ has two vertices of degree 4 and one vertex of degree 4, 5, or 6 in $G$.

If $T_{C_1}$ and $T_{C_2}$ share an edge, say $T_{C_1} = xyz$ and $T_{C_2} = xyz'$, then one of $x, y, z$, has degree 4. Since $G$ is simple, $z \neq z'$. Let $v$ be the other neighbor of $x$. By inspecting all graphs in $Q$, we deduce that each of $z, z'$ is either adjacent to $v$ or has degree at most 5 in $G$. So in $G - v$, the set $\{x, y, z, z'\}$ is collectable, contrary to Lemma 4.5.

Assume $T_{C_1}$ and $T_{C_2}$ have a common vertex, say $T_{C_1} = xyz$ and $T_{C_2} = xyz'$. If none of $y, z, y', z'$ have degree 6, then we can delete $x$ and collect $y, z, y', z'$, contrary to Lemma 4.5. So we may assume that $d_G(y) = 6$ and hence $d_G(x) = d_G(z) = 4$ and all the faces incident to $x$ are triangles because $G_{C_1}$ is isomorphic to $Q_4^4$. Thus we may assume $yy', zz' \in E(G)$. By deleting $y$, we can collect $x, z, z'$, and $y'$, again contrary to Lemma 4.5. (We collect $y'$ ahead of $z'$ if $d_G(z') = 6$ and collect $z'$ ahead of $y'$ otherwise.) Thus $V(T_{C_1}) \cap V(T_{C_2}) = \emptyset$.

If $X_{C_1} \cap V(T_{C_2}) \neq \emptyset$, then for a vertex $v$ of maximum degree in $V(T_{C_2})$, after deleting $v$, we can collect the other two vertices of $T_{C_2}$ and two vertices of $T_{C_1}$, contrary to Lemma 4.5. So $X_{C_1} \cap V(T_{C_2}) = \emptyset$ and by symmetry, $X_{C_2} \cap V(T_{C_1}) = \emptyset$.

If $X_{C_1} \cap X_{C_2}$ contains a vertex $v$, then by deleting $v$, we can collect two vertices from each of $T_{C_1}$ and $T_{C_2}$, again contrary to Lemma 4.5 because $V(T_{C_1}) \cap V(T_{C_2}) = \emptyset$. \hfill\(\Box\)

Lemma 4.7. If $C$ is a special cycle of $G$, then there is a vertex $u \in X_C$ such that $V(T_C)$ is collectable in $G - u$ and $G' = G - (V(T_C) \cup \{u\})$ has no type-c special cycle.

Proof. Suppose the lemma fails for some special cycle $C$ of $G$ with $|E(C)| = k$. Then $G_C$ is isomorphic to a graph $Q \in Q$. We may assume $G_C = Q$. Then $V(T_C)$ is collectable in $G - v_1$. Put $Y = V(T_C) \cup \{v_1\}$ and $G' = G - Y$. Since $G'$ has a type-c special cycle $C'$, $G'_{C'}$ has a facial cycle $C''$ with $Y = V(int(C''))$.

Then $C''$ consists of the subpath $C - v_1$ from $v_2$ to $v_k$ of length $k - 2$ and a path $P$ from $v_k$ to $v_2$ in $G'$. As $G'_{C'}$ is special, $3 \leq |E(C'')| \leq 5$. So $|E(P)| \leq 5 - (k - 2) \leq 4$. Now $N_G(v_1) \subseteq V(P) \cup \{y, z\}$, so $d_G(v_1) \leq |E(P)| + 3 \leq 10 - k \leq 7$. If $d_G(v_1) \leq 6$, then after deleting $v_2$ we can collect $Y$: use the order $x, y, v_1, z$ if $d_G(v_1) \leq 5$; else $d_G(v_1) = 6$ and $k \leq 4$, so use the order $x, y, z, v_1$. This contradicts Lemma 4.5. Thus $d_G(v_1) = 7$. So $k = 3$, $|E(P)| = 4$, $|E(C'')| = 5$, $G_C = Q_1$, and $v_1$ is adjacent to all vertices of $P$.

Setting $u = v_3$, and using symmetry between $v_1$ and $v_3$, we see that $v_3$ is also an interior vertex with $d_G(v_3) = 7$.
Let $P = v_3u_1u_2u_3v_2$. Now $G'_{C'}$ is isomorphic to $Q_4$ since $C''$ is a facial 5-cycle. Assume $u \mapsto u'$ is an isomorphism from $Q_4$ to $G'_{C'}$. Then $C'' = z'v'_3v'_4v'_5v'_1z'$.

If there is $w \in \{v_2, v_3\} \cap \{v'_1, v'_3\}$ then after deleting $w$ we can collect $\{x, y, z, x', y', z'\}$, contrary to Lemma 4.5. Else $\{v'_1, v'_3\} = \{u_1, u_3\}$ and therefore $z' = u_2$, see Figure 3. After deleting $\{v_2, u_1\}$, we can collect $\{x, y, z, v_3, v_1, z', x', y'\}$, contrary to Lemma 4.5, as both $v_1$ and $v_3$ are interior vertices.

\[ \square \]

**Lemma 4.8.** $G$ has no special cycle.

**Proof.** Assume to the contrary that $C$ is a special cycle of $G$. By Lemma 4.7 there is a vertex $u \in X_C$ such that $V(T_C)$ is collectable in $G - u$ and $G' = G - (V(T_C) \cup \{u\})$ has no type-c special cycle. Observe that $f(G; A) \geq f(G'; A) + 3$. So it suffices to show that $\partial(G) \leq \partial(G') + 3$. Since $G'$ has no type-c special cycles, every exposed special cycle of $G'$ is a special cycle of $G$.

If $u \notin B$, then $B(G') = B$ and so $B - B(G') = \emptyset$. As $\delta(V(T_C) \cup \{u\}) = 0$, we deduce from Lemma 4.3 that $\partial(G) \leq \partial(G') + \frac{3}{4} \cdot 4$.

Thus we may assume that $u \in B$ and so $|B - B(G')| = 1$. If $C$ is exposed in $G$, then by Lemmas 4.3 and 4.6, $\partial(G) \leq \partial(G') + \frac{3}{4} \cdot 4 + \frac{1}{4}$. If $C$ is not exposed in $G$, then $X_C$ has some interior vertex $v$. Since $v$ is adjacent to a vertex of $T_C$, $v$ is exposed in $G'$. By Lemma 4.6 $v \notin X_{C'}$ for every exposed special cycle $C'$ of $G'$, because $C'$ is a special cycle of $G$. Therefore, in an optimal special cycle packing of $G'$, at most $|B(G') - B| - 1$ of the cycles are not exposed in $G$. So,

\[
\tau(G) \geq \tau(G') - (|B(G') - B| - 1) = \tau(G') - (|B(G')| - |B| + 1) + 1.
\]

Thus

\[ \partial(G) = \frac{3}{4}|V(G)| + \frac{1}{4}(|B| - \tau(G)) \leq \frac{3}{4}(|V(G')| + 4) + \frac{1}{4}(|B| + (-\tau(G') + |B(G')| - |B|)) = \partial(G') + 3. \]
Lemma 4.9. Let s be an integer. Let X and Y be disjoint subsets of V(G) such that Y is collectable in G−X. If \(|B(G − (X ∪ Y))| ≥ |B(G)| + s\), G[X ∪ Y] is connected, and \((X ∪ Y) ∩ B(G) ≠ \emptyset\), then \(s + |Y| < 3|X|\).

Proof. Let \(G′ = G − (X ∪ Y)\). Since \((X ∪ Y) ∩ B(G) ≠ \emptyset\) and G[X ∪ Y] is connected, any special cycle of \(G′\) is also a special cycle of G. So \(τ(G′) = 0\) and \(\partial(G) ≤ \partial(G′) + \frac{3}{4}|X ∪ Y| − \frac{s}{4}\). As \(X ∪ Y ≠ \emptyset\) and \((G; A)\) is extreme, \(f(G′; A) ≥ \partial(G′)\). Thus

\[ f(G′; A) + |Y| ≤ f(G; A) < \partial(G) ≤ \partial(G′) + \frac{3}{4}|X ∪ Y| − \frac{s}{4} ≤ f(G′; A) + \frac{3}{4}|X ∪ Y| − \frac{s}{4}. \]

This implies that \(s + |Y| < 3|X|\). \(\Box\)

Lemma 4.10. G is 2-connected and |A| = 2.

Proof. Suppose G is not 2-connected. If \(|V(G)| ≤ 3\), then G is (3, A)-degenerate, so \(f(G; A) = \partial(G)\) and we are done. Else \(|V(G)| > 3\). As G is connected, it has a cut-vertex x. Let \(G_1, G_2\) be subgraphs of G such that \(G = G_1 ∪ G_2\), \(V(G_1) ∩ V(G_2) = \{x\}\), and \(|V(G_1) ∩ A| ≤ |V(G_2) ∩ A|\). Observe that if \(x ∉ A\), then \(A ∩ V(G_1) = \emptyset\) by the choice of G_1 because A is usable in G.

Let \(A_1 = V(G_1) ∩ A\) if \(x ∉ A\) and \(A_1 = \{x\}\) otherwise. Let \(A_2 = V(G_2) ∩ A\). Note that for each \(i = 1, 2\), \(A_i\) is usable in \(G_i\). For \(i = 1, 2\), let \(X_i\) be a maximum \(A_i\)-good set in \(G_i\).

Let \(X := (X_1 ∪ X_2 − \{x\}) ∪ (X_1 ∩ X_2)\). We claim that X is \(A\)-good in G. If \(x ∈ A\), then collect \(X_1 − A, X_2 − A, A ∩ X\). If \(x ∉ A\) and \(x ∈ X_1 ∩ X_2\), then collect \(X_1 − \{x\}, X_2\). If \(x ∉ A\) and \(x ∉ X_1 ∩ X_2\), then collect \(X_1 − \{x\}, X_2 − \{x\}\). This proves the claim that X is \(A\)-good in G.

As \((G; A)\) is extreme, \(f(G_i; A_i) ≥ \partial(G_i)\) for \(i = 1, 2\).

If \(x ∈ B\) then \(B(G_i) = B(G) ∩ V(G_i)\) for \(i = 1, 2\). Note that any special cycle of \(G_i\) is a special cycle of G and so \(τ(G_i) = 0\) for \(i = 1, 2\) by Lemma 4.8 and hence \(\partial(G) = \partial(G_1) + \partial(G_2) − 1\).

If \(x ∉ B\), then we may assume \(V(G_1) ∩ B(G) = \emptyset\). Hence \(B(G) = B(G_2)\). Since only one inner face of \(G_2\) contains vertices of \(G_1\), \(τ(G_2) ≤ 1\) by Lemma 4.8. Note that

\[ \partial(G) = \partial(G_1) + \partial(G_2) − \frac{3}{4} + \frac{1}{4}τ(G_2) − \frac{1}{4}(|B(G_1)| − τ(G_1)). \]

Since \(τ(G_1) ≤ |B(G_1)| − 2\), we have \(\partial(G) ≤ \partial(G_1) + \partial(G_2) − 1\).

In both cases, we have the contradiction:

\[ f(G; A) ≥ |X_1| + |X_2| − 1 = f(G_1; A_1) + f(G_2; A_2) − 1 ≥ \partial(G_1) + \partial(G_2) − 1 ≥ \partial(G). \]

Thus G is 2-connected, and hence |A| ≤ 2. As \((G; A)\) is extreme, we have |A| = 2. \(\Box\)

In the following, set \(A = \{a, a′\}\).

Lemma 4.11. The boundary cycle B has no chord.
Proof. Assume \( B \) has a chord \( e := xy \). Let \( P_1, P_2 \) be the two paths from \( x \) to \( y \) in \( B \) such that \( A \subseteq V(P_1) \). Since \( e \) is a chord, both \( P_1 \) and \( P_2 \) have length at least two.

Set \( G_1 = \text{int}[P_1 + e] \) and \( G_2 = \text{int}[P_2 + e] \). As \( \tau(G) = 0 \) by Lemma 4.3, we know that \( \tau(G_1) = \tau(G_2) = 0 \). Hence \( \partial(G) = \partial(G_1) + \partial(G_2) - 2 \). We may assume that \( A \subseteq V(G_2) \). Let \( A_1 = \{x, y\} \) and \( A_2 = A \).

For \( i = 1, 2 \), let \( X_i \) be a maximum \( A_i \)-good set in \( G_i \). Then \( X = (X_1 \cup X_2 - \{x, y\}) \cup (X_1 \cap X_2) \) is an \( A \)-good set in \( G \): collect \( X_1 - \{x, y\}, (X_2 - \{x, y\}) \cup (X_1 \cap X_2) \). Thus

\[
f(G; A) \geq f(G_1; A_1) + f(G_2; A_2) - 2 \geq \partial(G_1) + \partial(G_2) - 2 = \partial(G),
\]

contrary to the choice of \( G \).

\( \square \)

Lemma 4.12. \( G \) is a near plane triangulation.

Proof. By Lemma 4.10, every face boundary of \( G \) is a cycle of \( G \). Assume to the contrary that \( G \) has an interior face \( F \) which is not a triangle. Then \( V(F) \) has a pair of vertices non-adjacent in \( G \) because \( G \) is a plane graph. Let \( e \notin E(G) \) be an edge drawn on \( F \) joining them. Then \( G' = G + e \) is a plane graph with \( B(G') = B(G) \). As \( G \) is extreme, \( G' \) is not a counterexample. As \( f(G'; A) \leq f(G; A) \), we conclude \( \tau(G') > \tau(G) \), and hence \( G' \) has an exposed special cycle \( C \) and \( e \) is an edge of \( G' \). By (d) of Observation 3.2, there is a vertex \( v \in X_C \) such that after deleting \( v \), we can collect all the three vertices of \( T_C \). In \( G - (V(T_C) \cup \{v\}), \) all vertices in \( V(G) - (V(T_C) \cup \{v\}) \) are exposed. By Lemma 4.9, none of these vertices can be an interior vertex of \( G \), because otherwise \( B(G - (V(T_C) \cup \{v\})) \geq |B(G)| \). So all these vertices are boundary vertices of \( G \). By Lemmas 4.14 and 4.11, \( G \) is 2-connected, \( |A| = 2 \), and \( B(G) \) has no chord, so \( G \) has no other vertices and \( \text{int}(B(G)) = T_C \), as \( v \in X_C \) is also a boundary vertex of \( G \). By the definition of usable sets, the two vertices in \( A \) are adjacent.

By Lemma 4.3 \( \|u, V(T_C)\| \geq 2 \) for every vertex \( u \in B(G) - A \), and \( \|w, B(G)\| \geq 2 \) for every vertex \( w \in V(T_C) \). On the other hand, the number of vertices \( u \in B(G) \) with \( \|u, V(T_C)\| \geq 2 \) is at most 3. So \( |B(G)| \leq 3 + |A| = 5 \).

If \( |B(G)| = 3 \), then \( G \) is triangulated. Suppose \( |B(G)| = 4 \). If \( \|u, V(T_C)\| \geq 2 \) for three vertices \( u \in B(G) \), then \( G \) is isomorphic to \( Q_2 \); else \( G \) is isomorphic to \( Q_3 \). Both are contradictions. If \( |B(G)| = 5 \), then \( G \) is isomorphic to \( Q_4 \) or \( Q_4^+ \), again a contradiction.

\( \square \)

5 Properties of separating cycles

In a plane graph \( G \), a cycle \( C \) is called separating if both \( V(\text{int}(C)) \) and \( V(\text{ext}(C)) \) are nonempty. In this section we will discuss properties of separating cycles in \( G \).

Lemma 5.1. Suppose \( T \) is a separating triangle of \( G \) and let \( I = \text{int}(T) \). Then

(a) \( \|V(T), V(I)\| \geq 6 \),

(b) \( |I| \geq 3 \),

(c) \( \|x, V(I)\| \geq 1 \) for all \( x \in V(T) \), and
(d) for all distinct x, y in \( V(T) \), \( |N(\{x, y\}) \cap V(I)| \geq 2 \).

Proof. If \( |I| \leq 2 \), then \( I \) contains a vertex \( v \) with \( d_G(v) \leq 3 \), contrary to Lemma 4.4. Thus \( |I| \geq 3 \) and (b) holds. Moreover, \( I^+ := \text{int}[T] \) is triangulated and therefore \( |I^+| = 3|I^+| - 6 \) and \( |I| \leq 3|I| - 6 \). Thus

\[
|V(T), V(I)| = |I^+| - |I| - |I| \geq 3(3 + |I|) - 6 - 3 - (3|I| - 6) = 6.
\]

Thus (a) holds. As \( I^+ \) is triangulated and \( T \) is separating, every edge of \( T \) is contained in a triangle of \( I^+ \) other than \( T \); so (c) holds.

If \( |(N(x) \cup N(y)) \cap V(I)| \leq 1 \), then \( |I| = 1 \) because \( G \) is a near plane triangulation. This contradicts (b). So (d) holds. \( \square \)

**Lemma 5.2.** Let \( C \) be a separating cycle in \( G \) such that \( V(C) \cap A = \emptyset \). Assume \( X, Y \) are disjoint subsets of \( G \) such that \( X \cup Y \neq \emptyset \), \( Y \) is collectable in \( G - X \), and \( G[X \cup Y] \) is connected. Let \( G_1 = \text{int}[C] - (X \cup Y) \), \( G_2 = \text{ext}(C) - (X \cup Y) \), \( B_1 = B(G_1) \), \( B_2 = B(G_2) \), \( G'_2 = \text{ext}[C] - (X \cup Y) \), \( A' = V(C) - (X \cup Y) \). If \( A' \) is usable in \( G_1 \) and collectable in \( G'_2 \), then

\[
|Y| + |B_1| + |B_2| < 3|X| + |B| + \tau(G_2) \leq 3|X| + |B| + 1.
\]

In particular,

\[
|Y| < \begin{cases} 
3|X| + |B| - |B_1| - |B_2| & \text{if } (X \cup Y) \cap B \neq \emptyset, \\
3|X| + \tau(G_2) - |B_1| & \text{otherwise}.
\end{cases}
\]

Proof. Since \( A' \) is usable, \( (X \cup Y) \cap V(C) \neq \emptyset \) and so \( X \cup Y \) lies in the infinite face of \( G_1 \). Thus any special cycle of \( G_1 \) is also a special cycle of \( G \). Thus by Lemma 4.8, \( \tau(G) = \tau(G_1) = 0 \). By Lemma 4.2 in an optimal special cycle packing of \( G_2 \), at most one cycle is type-c and there are no type-a or type-b cycles. Therefore \( \tau(G_2) \leq 1 \).

As \( A' \) is collectable in \( G'_2 \), we have

\[
f(G; A) \geq f(G_1; A') + f(G_2; A) + |Y|.
\]

On the other hand,

\[
\partial(G) = \partial(G_1) + \partial(G_2) + \frac{3}{4}(|X| + |Y|) - \frac{1}{4}(|B_1| + |B_2| - |B| - \tau(G_2)).
\]

As \( f(G_1; A') \geq \partial(G_1) \) and \( f(G_2; A) \geq \partial(G_2) \), we have

\[
\partial(G) - \frac{3}{4}(|X| + |Y|) + \frac{1}{4}(|B_1| + |B_2| - |B| - \tau(G_2)) \leq f(G_1; A') + f(G_2; A) \leq f(G; A) - |Y|.
\]

As \( f(G; A) < \partial(G) \), it follows that

\[
|Y| + |B_1| + |B_2| < 3|X| + |B| + \tau(G_2) \leq 3|X| + |B| + 1.
\]

Note that if \( (X \cup Y) \cap B \neq \emptyset \), then \( \tau(G_2) = 0 \). In this case, we have

\[
|Y| + |B_1| + |B_2| < 3|X| + |B|.
\]

If \( (X \cup Y) \cap B = \emptyset \), then \( B_2 = B \). In this case, we have \( |Y| + |B_1| < 3|X| + \tau(G_2) \). \( \square \)
Lemma 5.3. Let $C$ be a separating triangle of $G$. If $C$ has no vertex in $B(G)$, then either $\|v, V(\text{ext}(C))\| \geq 3$ for all vertices $v \in V(C)$ or $\|v, V(\text{ext}(C))\| \geq 4$ for two vertices $v \in V(C)$.

Proof. Suppose not. Let $C = xyzx$ be a counterexample with the minimal area. We may assume that $\|x, V(\text{ext}(C))\| \leq 2$ and $\|y, V(\text{ext}(C))\| \leq 3$. By Lemma 5.1(c), $z$ has a neighbour $w$ in $I := \text{int}(C)$. If $w$ is the only neighbour of $z$ in $I$, then by Lemma 5.3. Let $C' := xwyx$ is a separating triangle. However, $w$ has only 1 neighbour in $\text{ext}(C')$ and $x$ has at most 3 neighbours in $\text{ext}(C')$, contradicting the choice of $C$.

Thus $\|z, V(I)\| \geq 2$.

We apply Lemma 5.2 with $C$, $X = \{z\}$ and $Y = \emptyset$. Then $A' := \{x, y\}$ is usable in $G_1 := \text{int}[C] - z$, $A'$ is collectable in $G_2' := \text{ext}[C] - z$ and $B_1 := B(G_1) \supseteq \{x, y\} \cup N_I(z)$. So $|B_1| \geq 4$, and this contradicts Lemma 5.2.

Lemma 5.4. Let $C$ be a separating induced cycle of length 4 in $G$ having no vertex in $B(G)$. Then exactly one of the following holds.

(a) $|B(\text{int}(C))| \geq 4$.
(b) $|V(\text{int}(C))| \leq 2$ and every vertex in $\text{int}(C)$ has degree 4 in $G$.

Proof. Suppose that $|B(\text{int}(C))| \leq 3$. By Euler’s formula, we have

$$\|\text{int}[C]\| = 3|V(\text{int}[C])| - 7 = 3|V(\text{int}(C))| + 5$$

as $G$ is a near plane triangulation. Then since $C$ is induced, by Lemma 4.4,

$$0 \leq \sum_{v \in V(\text{int}(C))} (d(v) - 4)$$

$$= \|\text{int}[C]\| - \|C\| + \|\text{int}(C)\| - 4|V(\text{int}(C))|$$

$$= (3|V(\text{int}(C))| + 5) - 4 + \|\text{int}(C)\| - 4|V(\text{int}(C))|$$

$$= 1 - |V(\text{int}(C))| + \|\text{int}(C)\|.$$ (5.1)

Suppose that $\text{int}(C)$ has a cycle. Since $|B(\text{int}(C))| \leq 3$, we deduce that $B(\text{int}(C)) = xyzx$ is a triangle. By Euler’s formula applied on $G[V(C) \cup B(\text{int}(C))]$, we have

$$\|V(C), B(\text{int}(C))\| = (3 \cdot 7 - 7) - 3 - 4 = 7,$$

hence $B(\text{int}(C))$ is a facial triangle by Lemma 5.3. Therefore, $x, y, z$ have degree 4, 4, 5 in $G$ by (5.1) and Lemma 4.4. Let $w, w' \in V(C)$ be consecutive neighbours of $x$ in $V(C)$. From $G$, we can delete $w$ and collect $x, y, z$. Let $G' = G - \{w, x, y, z\}$. If $G'$ has an exposed special cycle, then the face of $G'$ containing $w$ has length at least 5, implying that $\|w, V(\text{ext}(C))\| \leq 2$ because $C - w$ is a subpath of an exposed special cycle of $G'$, as $C$ is induced. Then we can delete $w'$ and collect $x, y, z, w$, contradicting Lemma 4.5. Therefore $G'$ has no exposed special cycles. Then $\partial(G) = \partial(G') + 3$ and $f(G; A) \geq f(G'; A) + 3 \geq \partial(G') + 3 = \partial(G)$, a contradiction.
Therefore $\text{int}(C)$ has no cycles. Then \( \|\text{int}(C)\| \leq |V(\text{int}(C))| - 1 \), and so in (5.11) the equality must hold. This means $\text{int}(C)$ is a tree and every vertex in $\text{int}(C)$ has degree 4 in $G$ by Lemma 4.4. If $\text{int}(C)$ has at least 3 vertices, then let $w$ be a vertex in $V(C)$ adjacent to some vertex in $\text{int}(C)$. By deleting $w$, we can collect all the vertices in $\text{int}(C)$. Similarly we can choose $w$ so that $G' = G - w - V(\text{int}(C))$ contains no special cycle, and that leads to the same contradiction. Thus we deduce (b). \qed

6 Degrees of boundary vertices

Lemma 6.1. Each vertex in $B$ has degree at most 5.

Proof. Assume to the contrary that $x \in B$ has $d(x) \geq 6$. Then deleting $x$ exposes at least 4 interior vertices. Apply Lemma 4.9 with $X = \{x\}$, $Y = \emptyset$ and $s = 3$, we obtain a contradiction. \qed

Recall that $A = \{a, a'\}$.

Lemma 6.2. Each vertex in $B - A$ has degree 5.

Proof. Suppose that there is a vertex $x \in B - A$ with $d(x) < 5$. By Lemma 4.4, $d(x) = 4$. By Lemma 4.11, exactly two of the neighbors of $x$ are in $B$. Consider two cases.

Case 1: $x$ has a neighbour $y \in B - A$. As $|A| = 2$, we have $|B| \geq 4$. As $G$ is a near plane triangulation, there is a vertex $z \in N(x) \cap N(y)$ such that $xyzx$ is a facial triangle. As $B$ has no chords by Lemma 4.11, $(N(x) \cap N(y)) \cap B(G) = \emptyset$.

Suppose there is $z' \in N(x) \cap N(y) - \{z\}$. Since $d(x) = 4$ and $G$ is a near plane triangulation, $xz'z'$ is a facial triangle. Since $d(z) \geq 4$ by Lemma 4.4, $T := yzz'y$ is a separating triangle. As $d(y) \leq 5$ by Lemma 6.1, $y$ has a unique neighbour $y' \in V(\text{int}(T))$ and therefore both $yyzy$ and $yy'z'y$ are facial triangles. By Lemma 5.1(b), $\text{int}(T)$ contains at least three vertices and so $T' := zz'y'z$ is a separating triangle with $\|z, V(\text{ext}(T'))\| = 2$ and $\|y', V(\text{ext}(T'))\| = 1$, contrary to Lemma 5.3. So $N(x) \cap N(y) = \{z\}$.

If $d(y) = 5$, then deleting $z$ and collecting $x$ and $y$ exposes three vertices in $(N^\circ(x) \cup N^\circ(y)) - \{z\}$, the resulting graph $G' = G - \{x, y, z\}$ has $|B(G')| \geq |B| + 1$. Apply Lemma 4.9 with $X = \{z\}, Y = \{x, y\}$, and $s = 1$, we obtain a contradiction.

Hence $d(y) = 4$. By repeating the same argument, we deduce that for all edges $vv' \in B - A$, we have (i) $d(v) = 4 = d(v')$ and (ii) $|N(v) \cap N(v')| = 1$.

Let $x', y'$ be vertices such that $N^\circ(x) = \{x', z\}$ and $N^\circ(y) = \{y', z\}$. As $G$ is a near plane triangulation and $B$ is chordless, $G - B$ is connected. Let $J = \{x', z, y'\}$. If $V - B \neq J$, then there exist $b \in J$ and $t \in (V - B) - J$ such that $b$ and $t$ are adjacent. Then deleting $b$ and collecting $x$, $y$ exposes all vertices in $(J - \{b\}) \cup \{t\}$. Let $G' = G - \{x, y, b\}$. Then $|B(G')| \geq |B| + 1$. With $X = \{b\}, Y = \{x, y\}$, and $s = 1$, this contradicts Lemma 4.9. Hence $V - B = J$. \qed
Let \( u, v \) be vertices in \( B \) so that \( uxyzv \) is a path in \( B \). Since \( G \) is a near plane triangulation, \( x' \) is adjacent to \( u \) and \( z \), and \( y' \) is adjacent to \( v \) and \( z \), see Figure 4. Then \( ux'zy, xzyv \) are paths in \( G \). If \( A = \{u, v\} \), then \( B \) is a 4-cycle and as \( d(x'), d(y') \geq 4 \), we must have \( x'y' \in E(G) \), which implies that \( G \) is isomorphic to \( Q_4^+ \) and \( B \) is a special cycle, contrary to Lemma 4.8. Therefore \( A \neq \{u, v\} \) and since \( y \notin A \), we deduce that \( v \notin A \). This implies \( d(v) = 4 \). Then \( v \) has another neighbour in \( J \), and by the observation that \( y \) and \( v \) have only one common neighbour \( y' \), we deduce that \( v \) is non-adjacent to \( z \). Thus \( v \) is adjacent to \( x' \), and \( x' \) is adjacent to \( y' \).

Furthermore every vertex in \( B - \{u, x, y, v\} \) has degree at most 3, because \( B \) has no chords and \( x' \) is the only possible interior neighbor. By Lemma 4.4 every vertex in \( B - \{u, x, y, v\} \) is in \( A \). Then \( G \) is isomorphic to \( Q_4^{++} \) and \( B \) is a special cycle, contrary to Lemma 4.8.

**Case 2:** \( N_G(x) \cap B \subseteq A \). Then \( B = xaa'a \). Since \( G \) is a near plane triangulation and \( d(x) = 4 \), the neighbours of \( x \) form a path of length 3 from \( a \) to \( a' \), say \( ayzx' \) where \( a, y, z, a' \) are the neighbours of \( x \). (See Figure 5)

If \( |N^o(y)| \geq 3 \), then deleting \( y \) and collecting \( x \) exposes at least three vertices in \( N^o(y) \). Let \( G' = G - \{x, y\} \). Then \( |B(G')| \geq |B| + 2 \). With \( X = \{y\}, Y = \{x\} \), and \( s = 2 \), this contradicts Lemma 4.9.

Thus \( |N^o(y)| \leq 2 \) and so \( d(y) \leq 5 \). (Note that \( y \) may be adjacent to \( a' \).) By symmetry, \( |N^o(z)| \leq 2 \) and \( d(z) \leq 5 \).

If \( y \) is adjacent to \( a' \), then \( z \) is non-adjacent to \( a \) and so \( d(z) = 4 \) by Lemma 4.4. Then \( T := yza'y \) is a separating triangle, as \( \text{int}(T) \) contains a neighbour of \( z \). Since \( d(y) \leq 5 \) and \( d(z) = 4 \), we have \( |N(\{y, z\}) \cap V(\text{int}(T))| = 1 \), contrary to Lemma 5.1(d).

So \( y \) is non-adjacent to \( a' \). By symmetry, \( z \) is non-adjacent to \( a \). As \( |N^o(y)|, |N^o(z)| \leq 2 \) and \( d(y), d(z) \geq 4 \), \( y \) and \( z \) have a unique common neighbour \( w \) and \( d(y) = d(z) = 4 \). Since \( G \) is a near plane triangulation, \( w \) is adjacent to both \( a \) and \( a' \).
If \( d(w) > 4 \), then deleting \( w \) and collecting \( y, z, x \) exposes at least one vertex and so \( |B(G - \{x, y, z, w\})| \geq |B| \). With \( X = \{w\}, Y = \{x, y, z\} \), and \( s = 0 \), this contradicts Lemma 4.9. This implies \( d(w) = 4 \), hence \( B(G) \) is a special cycle, contrary to Lemma 4.8.

7 The boundary is a triangle

In this section we prove that \(|B| = 3\).

**Lemma 7.1.** If \( xy \in E(B - A) \), then the following hold:

(a) There are \( S := \{x_1, x_2, u, y_1, y_2\} \subseteq V - B \) and \( x^*, y^* \in B \) such that \( x^*x_1x_2uy \) is a path in \( G[N(x)] \) and \( xuy_1y_2y^* \) is a path in \( G[N(y)] \).

(b) \( d(x_2), d(u), d(y_1) \geq 5 \).

(c) The vertices \( x_1, x_2, u, y_1, y_2 \) are all distinct.

(d) \( |N^c(\{x_2, u\}) - S| \leq 2 \) and \( |N^c(\{y_1, u\}) - S| \leq 2 \).

(e) \( x_2y_1, x_2y_2, x_1y_1, ux_1, uy_2 \notin E \).

(f) There is \( w_1 \in N(\{x_2, u, y_1\}) \cap B) - \{x, y\} \); in particular \( G[S] \) is an induced path.

(g) \( x_2, u \notin N(x^*) \) and \( y_1, u \notin N(y^*) \).

(h) Neither \( x^* \) nor \( y^* \) is equal to the vertex \( w_1 \) from (f).

**Proof.** (a) By Lemma 6.2, \( d(x) = 5 = d(y) \). By Lemmas 4.10 and 4.11 there are \( x^*, y^* \in B \) with \( N(x) \cap B = \{x^*, y\} \) and \( N(y) \cap B = \{x, y^*\} \). As \( G \) is a near plane triangulation, there is \( u \in N(x) \cap N(y) \). So (a) holds.

(b) (See Figure 6) As \( d(u) \geq 4 \) by Lemma 4.3, \( x_2 \neq y_1 \). Assume \( d(x_2) = 4 \). If \( x_2 \) is adjacent to \( y \), then \( x_2 = y_2 \), implying that \( d(x_2) > 4 \), contradicting the assumption. Thus \( x_2 \) is non-adjacent to \( y \) and deleting \( u \) and collecting \( x_2, x, y \) exposes \( y_1, y_2 \) (note that it is possible that \( x_1 \in \{y_1, y_2\} \), so we do not count it as exposed). We have \( |B(G - \{u, x_2, x, y\})| \geq |B| \). With \( X = \{u\}, Y = \{x_2, x, y\} \), and \( s = 0 \), this contradicts Lemma 4.9. Thus \( d(x_2) \geq 5 \) by Lemma 4.3. By symmetry, \( d(y_1) \geq 5 \). If \( d(u) = 4 \), then we can delete \( x_2 \), collect \( u, x, y \), and expose \( y_1, y_2 \). This contradicts Lemma 4.9 applied with \( X = \{x_2\}, Y = \{u, x, y\} \), and \( s = 0 \). So (b) holds.

(c) Since \( d(u) \geq 5 \), we deduce \( x_2 \neq y_1 \), and if \( x_1 = y_1 \), then \( T := x_1x_2ux_1 \) is a separating triangle (see Figure 7), since \( d(x_2) \geq 5 \). As \( \|x_2, V(\text{ext}(T))\| = 1 \) and \( \|u, V(\text{ext}(T))\| = 2 \), this contradicts Lemma 5.3. So \( x_1 \neq y_1 \). By symmetry, \( x_2 \neq y_2 \).
This contradicts Lemma 4.9 applied with $x$.

Thus $\|x\| = 4$, contrary to (7.1). So (c) holds.

It remains to show that $x_1 \neq y_2$. Suppose not. By (b) $d(x_2) \geq 5$, so $C := x_1x_2uy_1x_1$ is a separating 4-cycle (see Figure 8). We first prove the following.

For all $u' \in V(C) - \{u\}$, $|N(\{u, u'\}) \cap V(\text{int}(C))| \leq 3$. \hspace{1cm} (7.1)

Suppose not. Then deleting $u$, $u'$ and collecting $x$, $y$ exposes two vertices in $V(C) - \{u, u'\}$ and at least 4 vertices in $\text{int}(C)$. So $|B(G - \{u, u', x, y\})| \geq |B| - 2 + 2 + 4$. This contradicts Lemma 4.9 with $X = \{u, u'\}$, $Y = \{x, y\}$, and $s = 4$. This proves (7.1).

If $u$ is adjacent to $x_1$, then $C_1 := x_1x_2ux_1$ and $C_2 := x_1uy_1x_1$ are both separating triangles by (b). Then $|N(\{u, x_1\}) \cap V(\text{int}(C_i))| \geq 2$ for each $i \in \{1, 2\}$ by Lemma 5.1(d). Thus $|N(\{u, x_1\}) \cap V(\text{int}(C_i))| \geq 4$, contrary to (7.1). So $u$ is non-adjacent to $x_1$.

If $x_2$ is adjacent to $y_1$, then $C_3 := ux_2y_1u$ is a separating triangle by (b). Then $|N(\{u, x_2\}) \cap V(\text{int}(C_3))| \geq 2$ by Lemma 5.1(d). As $|u, V(\text{ext}(C_3))| = 2$, Lemma 5.3 implies that $|x_2, V(\text{ext}(C_3))| \geq 4$, hence $|N(\{u, x_2\}) \cap V(\text{int}(x_1x_2y_1x_1))| \geq 2$. Thus $|N(\{u, x_2\}) \cap V(\text{int}(C))| \geq 4$, contrary to (7.1). So $C$ has no chord.

By (b) $C$ is a separating induced cycle of length 4 in $G$. By Lemma 5.4 either $|B(\text{int}(C))| \geq 4$ or $|V(\text{int}(C))| \leq 2$ and every vertex in $\text{int}(C)$ has degree 4 in $G$.

By (7.1), $d(x_2), d(y_1) \leq 6$. If $|B(\text{int}(C))| \geq 4$, then deleting $u$, $x_1$ and collecting $x$, $y$, $x_2$, $y_1$ exposes at least 4 vertices and therefore $|B(G - \{u, x_1, x, y, x_2, y_1\})| \geq |B| + 2$. This contradicts Lemma 4.9 applied with $X = \{u, x_1\}$, $Y = \{x, y, x_2, y_1\}$, and $s = 2$.

Therefore we may assume $1 \leq |V(\text{int}(C))| \leq 2$ and every vertex in $\text{int}(C)$ has degree 4 in $G$. As $x_2$ is non-adjacent to $y_1$, $x_1$ has at least one neighbour in $\text{int}(C)$ and therefore after deleting $x_1$, we can collect all vertices in $V(\text{int}(C))$ and then collect $x_2$, $y_1$ and $u$, this contradicts Lemma 4.5. So (c) holds.
Figure 9: The situation in the proof of Lemma (d): $x_1, x_2, u, y_1, y_2$ are all distinct. The gray region has other vertices.

(d) (See Figure 9) If $|N^o(\{x_2, u\}) - S| \geq 3$, then deleting $x_2$, $u$ and collecting $x$, $y$ exposes $x_1$, $y_1$, $y_2$, and three other vertices and so $|B(G - \{x_2, u, x, y\})| \geq |B| - 2 + 6$. By applying Lemma 4.9 with $X = \{x_2, u\}$, $Y = \{x, y\}$, and $s = 4$, we obtain a contradiction. So we deduce that $|N^o(\{x_2, u\}) - S| \leq 2$. By symmetry, $|N^o(\{y_1, u\}) - S| \leq 2$.

(e) Suppose $x_1$ is adjacent to $u$. By (b) and (d) $d(x_2) = 5$. Thus $T := ux_1x_2u$ is a separating triangle. Let $w_1$, $w_2$ be the two neighbours of $x_2$ other than $x_1$, $x$, $u$ so that $x_1w_1w_2u$ is a path in $G$. Such a choice exists because $G$ is a near plane triangulation. As $d(w_2) \geq 4$ by Lemma 4.4 and $u$ has no neighbours in $\text{int}(ux_1w_1w_2u)$ by (d) $x_1$ is adjacent to $w_2$. As $d(w_1) \geq 4$, $T' := x_1w_1w_2x_1$ is a separating triangle. Note that $x_2x_1w_1x_2$, $x_2w_1w_2x_2$, $x_2w_2ux_2$, and $w_1w_2u$ are facial triangles. Thus $\|w_1, V(\text{ext}(T'))\| = 1$ and $\|w_2, V(\text{ext}(T'))\| = 2$, contrary to Lemma 5.3. So $x_1$ is non-adjacent to $u$. By symmetry, $y_2$ is non-adjacent to $u$.

Suppose that $x_2$ is adjacent to $y_1$. Let $T'' := ux_2y_1u$. By (b) $d(u) \geq 5$, so $T''$ is a separating triangle. By (d) $\|z, V(\text{int}(T''))\| \leq 2$ for all $z \in V(T'')$. By Lemma 5.1

$$\sum_{z \in V(T'')} \|z, V(\text{int}(T''))\| = \|V(T''), V(\text{int}(T''))\| \geq 6$$

and therefore $\|z, V(\text{int}(T''))\| = 2$ for all $z \in V(T'')$. By (d) $N(u) \cap V(\text{int}(T'')) = N(x_2) \cap V(\text{int}(T'')) = N(y_1) \cap V(\text{int}(T''))$. Then $u$, $x_2$, $y_1$, and their neighbours in $\text{int}(T'')$ induce a $K_5$ subgraph, contradicting our assumption on $G$. Thus $x_2$ is non-adjacent to $y_1$.

Suppose that $x_2$ is adjacent to $y_2$. Since $x_2$ is non-adjacent to $y_1$, (b) and (d) imply that $d(y_1) = 5$. Let $w_1$, $w_2$ be the two neighbours of $y_1$ other than $u$, $y$, $y_2$ such that $uw_1w_2y_2$ is a path in $G$. By (d) $N^o(u) - S \subseteq \{w_1, w_2\}$. If $u$ is adjacent to both $w_1$ and $w_2$, then $uw_1w_2u$, $uy_1w_1u$, $y_1w_1w_2y_1$ are facial triangles, implying that $w_1$ has degree 3, contradicting Lemma 4.4. Thus, as $d(u) \geq 5$ by (b) we deduce that $d(u) = 5$. Since $G$ is a near plane triangulation, $x_2$ is adjacent to $w_1$ and $ux_2w_1u$, $uw_1y_1u$ are facial triangles. If $x_2w_1w_2y_2x_2$ is a separating cycle, then deleting $w_1$, $w_2$ and collecting $y_1$, $u$, $y$, $x$ exposes at least 4 vertices and so $|B(G - \{w_1, w_2, y_1, u, y, x\})| \geq |B| - 2 + 4$. By applying Lemma 4.9 with $X = \{w_1, w_2\}$, $Y = \{y_1, u, y, x\}$, and $s = 2$, we obtain a contradiction. So $x_2w_1w_2y_2x_2$ is not a separating cycle. By Lemma 4.1, $d(w_2) \geq 4$. Therefore $w_2$ is adjacent to $x_2$ and $d(w_1) = 4 = d(w_2)$. Then, deleting $y_1$ and collecting
\(w_1, w_2, u, y, x\) exposes 3 vertices and \(|B(G - \{y_1, w_1, w_2, u, y, x\})| = |B| - 2 + 3\). By applying Lemma 4.9 with \(X = \{y_1\}, \ Y = \{w_1, w_2, u, y, x\}\), and \(s = 1\), we obtain a contradiction. So \(x_2\) is non-adjacent to \(y_2\). By symmetry, \(x_1\) is non-adjacent to \(y_1\).

\([f]\) Suppose that none of \(x_2, u, y_1\) has neighbours in \(B - \{x, y\}\). By \([b], [d]\) and \([e]\), \(d(x_2) = 5 = d(y_1)\). If
\[
|N^c(\{x_2, y_1\}) - \{u\}| \geq 5
\]
then deleting \(u, x_2\) and collecting \(x, y, y_1\) exposes all vertices in \(N^c(\{x_2, y_1\}) - \{u\}\) and so \(|B(G - \{u, x_2, x, y, y_1\})| \geq |B| - 2 + 3\). By applying Lemma 4.9 with \(X = \{u, x_2\}, \ Y = \{x, y, y_1\}\), and \(s = 3\), we obtain a contradiction. Thus \(|N^c(\{x_2, y_1\}) - \{u\}| \leq 4\) and therefore \(x_2, y_1\) have the same set of neighbours in \(V(G) - (B \cup S)\) by \([c]\) and \([e]\). Let \(w, w'\) be the neighbours of \(x_2\) (and also of \(y_1\)) such that \(w \in V(\text{int}(uy_1w'x_2u))\). Then \(w\) is the unique common neighbour of \(x_2, u\), and \(y_1\). By \([d]\) and Lemma 4.4, \(w\) is adjacent to \(w'\). Thus \(d(w) = 4\). Deleting \(u\) and collecting \(w, x_2, y_1, x, y\) exposes at least 3 vertices including \(w'\) and so \(|B(G - \{u, w, x_2, y_1, x, y\})| \geq |B| - 2 + 3\). This contradicts Lemma 4.9 applied with \(X = \{u\}, \ Y = \{w, x_2, y_1, x, y\}\), and \(s = 1\).

Thus at least one vertex of \(x_2, u\), and \(y_1\) is adjacent to a vertex in \(B - \{x, y\}\). Then \(x_1\) is non-adjacent to \(y_2\). By \([e]\) \(G[S]\) is an induced path and \([f]\) holds.

\([g]\) Suppose that \(x^*\) is adjacent to \(x_2\). As \(d(x_1) \geq 4\) by Lemma 4.4, \(T := x^*x_1x_2x^*\) is a separating triangle. Since \(d(x^*) \leq 5\) by Lemma 6.1, \(x^*\) has a unique neighbour \(w \in V(\text{int}(T))\). So \(w\) is adjacent to both \(x_1\) and \(x_2\). As \(d(w) \geq 4\) by Lemma 4.4, \(T' := wx_1x_2w\) is a separating triangle with \(|w, V(\text{ext}(T'))| = 1\) and \(|x_1, V(\text{ext}(T'))| = 2\), contrary to Lemma 5.3. So \(x^*\) is non-adjacent to \(x_2\). By symmetry, \(y^*\) is non-adjacent to \(y_1\).

Suppose \(u\) is adjacent to \(x^*\). As \(d(x_1) \geq 4\) and \(d(x^*) \leq 5\) by Lemmas 4.4 and 6.1, \(x^*\) has a unique neighbour \(w \in V(\text{int}(x^*x_1x_2ux^*))\) adjacent to both \(x_1\) and \(u\). By \([b]\) and \([d]\), \(w\) is adjacent to \(x_2\). If \(uwx_2u\) is a separating triangle, then by Lemma 5.1(d), \(|N(\{x_2, u\}) \cap N(\text{int}(uxw_2u))| \geq 2\), hence \(|N^c(\{x_2, u\}) - S| \geq 3\), contrary to \([d]\). So \(uwx_2u\) is facial. As \(d(x^*) \leq 5\), \(wx_1x_2w\) and \(wx^*uw\) are facial triangles. As \(d(x_2) \geq 5\) by \([d]\), \(T' := wx_1x_2w\) is a separating triangle. So \(|x_1, V(\text{ext}(T'))| = 2\) and \(|x_2, V(\text{ext}(T'))| = 2\), contrary to Lemma 5.3. Thus \(u\) is non-adjacent to \(x^*\). By symmetry, \(u\) is non-adjacent to \(y^*\). So \([g]\) holds.

\([h]\) Suppose that \(w_1 = y^*\). By \([g]\) \(y^*\) is adjacent to \(x_2\). Let \(C := y^*x_2uy_1y_2y^*\) and \(C'\) be the cycle formed by the path from \(x^*\) to \(y^*\) in \(B(G) - x - y\) together with the path \(y^*x_2x_1x^*\). Since \(G\) is a plane triangulation and \(d(y_2) \geq 4\), by \([f]\) there is \(w \in N(y^*) \cap N(y_2) \cap N(\text{int}(C))\). By Lemma 6.1, \(d(y^*) = 5\), and therefore \(x_2\) is adjacent to \(w\) and \(x_2wy^*x_2\) is a facial triangle. Let \(y^{**} \in B\) be the neighbour of \(y^*\) other than \(y\). Then \(x_2y^{**}y^*x_2\) is also a facial triangle in \(G\). Because \(x_2\) is non-adjacent to \(x^*\) by \([g]\), \(y^{**} \neq x^*\). By \([f]\) applied to \(yy^*\), we have \(y^* \in A\) because \(uy_1y_2wx_2\) is not an induced path in \(G\). Thus \(x^* \notin A\) because \(|A| = 2\). By Lemma 4.11, \(B(G)\) is chordless. Therefore by Lemma 6.2, \(d(x^*) = 5\) and so \(|x^*, V(\text{int}(C'))| = 2\). By \([b], [d]\), and \([e]\), we have \(|N^c(y_1) - S| = 2\). Deleting \(x_1, u\) and collecting \(x, x^*, y, y_1\) exposes at least 6.
vertices, including two neighbours of \( x^* \) in \( \text{int}(C') \) and two neighbours of \( y_1 \) in \( \text{int}(C) \). So \( |B(G - \{ x_1, u, x, x^*, y, y_1 \})| \geq |B| - 3 + 6 \). By applying Lemma 4.9 with \( X = \{ x_1, u \} \), \( Y = \{ x, x^*, y, y_1 \} \), and \( s = 3 \), we obtain a contradiction. So \( w_1 \neq y^* \). By symmetry, \( w_1 \neq x^* \). Thus (h) holds.

**Lemma 7.2.** \(|B| = 3\).

**Proof.** For an edge \( e = xy \in E(B - A) \), let \( x^*, x_1, x_2, u, y_1, y_2, y^* \) be as in Lemma 7.1. Suppose that \(|B| \geq 4\). Then \( x^* \neq y^* \). Lemma 7.1(h) implies that \( B \) has a vertex other than \( x, y, x^* \), and \( y^* \). So, \(|B| \geq 5\).

We claim that \( N^{\circ}(u) = \{ x_2, y_1 \} \). Suppose not. By Lemma 4.10 \(|A| = 2\), so at least one vertex of \( \{ x^*, y^* \} \), say, \( y^* \) is not in \( A \). By Lemma 7.1(f) applied to \( yy^* \), we deduce that \( u \) is non-adjacent to vertices in \( N^{\circ}(y^*) \). Thus deleting \( u, y_2 \) and collecting \( y, x, y^* \) exposes at least 6 vertices and so \( |B(G - \{ u, y_2, y, x, y^* \})| \geq |B| - 3 + 6 \). By applying Lemma 4.9 with \( X = \{ u, y_2 \} \), \( Y = \{ y, x, y^* \} \), and \( s = 3 \), we obtain a contradiction. So \( N^{\circ}(u) = \{ x_2, y_1 \} \).

Since \( d(u) \geq 5 \) by Lemma 7.1(b) \( u \) has at least one boundary neighbour \( z \neq x, y \). Let \( B(x, z) \) be the boundary path from \( x \) to \( z \) not containing \( y \), and \( B(y, z) \) be the boundary path from \( y \) to \( z \) not containing \( x \). So \( B(x, z) \) and \( B(y, z) \) have only one vertex in common, namely \( z \). One of \( B(x, z) \), \( B(y, z) \) has no internal vertex in \( A \). We denote this path by \( P(e, z) \). We choose \( e = xy \) and \( z \) so that \( P(e, z) \) is shortest. Assume \( P(e, z) = B(y, z) \). Let \( e' = yy^* \). Then \( e' \in E(B - A) \). Let \( y_2 \) be the common neighbour of \( y \) and \( y^* \) and let \( z' \neq y, y^* \) be a boundary neighbour of \( y_2 \). Then \( P(e', z') \) is a proper subpath of \( P(e, z) \), and hence is shorter. This contradicts our choice of \( e \) and \( z \). \( \square \)

8 The final contradiction

In this section we complete the proof of Theorem 3.2. First we prove a lemma.

**Lemma 8.1.** If \( B = \{ a, a', v \} \) and \( axyza' \) is a path in \( G[N(v)] \) (see Figure 10), then the following hold.

(a) \( x \) is non-adjacent to \( z \).
(b) \( y \) is adjacent to neither \( a \) nor \( a' \).
(c) \( z \) is non-adjacent to \( a \) and \( x \) is non-adjacent to \( a' \).

\( 23 \)
Lemma 5.3 implies \( \| \) is a separating triangle. By Lemma 5.1, \( | \) adjacent to \( a \) is collectable in \( G \). By symmetry, assume \( x \) is non-adjacent to \( a \). Since \( d(y) \geq 4 \), \( T := xwyx \) is a separating triangle. By Lemma 5.2, \( | \) adjacent to \( y \) is non-adjacent to \( C \). By Lemma 5.4, either \( | \geq 4 \) or \( | \geq 4 \). By applying Lemma 4.9 with \( y \), \( | \) exposes at least 7 vertices, and \( | \geq 4 \). Therefore, \( d(y) \geq 7 \). Then \( | \geq 4 \) and so deleting \( x, y \) and collecting \( v \) exposes at least 7 vertices, and \( | \geq 4 \). By applying Lemma 4.9 with \( x, y \), \( | \geq 4 \). By symmetry, \( y \) is non-adjacent to \( a \). So \( | \) holds.

(b) Suppose \( y \) is adjacent to \( a \). Then \( T := axya \) is a separating triangle, because \( d(x) \geq 4 \) and the other triangles incident with \( x \) are facial. As \( d(a) \leq 5 \) by Lemma 6.1, \( a \) has a unique neighbour \( w \) in \( int(T) \). As \( d(w) \geq 4 \), \( T' := xwyx \) is a separating triangle. Now \( |w, V(\text{ext}(T'))| = 1 \), and \( |x, V(\text{ext}(T'))| = 2 \), contrary to Lemma 5.3. Thus \( y \) is non-adjacent to \( a \). So \( | \) holds.

(c) Suppose that \( z \) is adjacent to \( a \). By \( | \) \( \geq 4 \) and \( d(a) \leq 5 \) by Lemmas 4.4 and 4.5, there is \( w \in (N(a) \cap N(x) \cap N(z)) \setminus \{v\} \), and \( xawx, wazw, azxa' \) are all facial triangles. (See Figure 11.) By \( y \neq w \). Since \( d(y) \geq 4 \) by Lemma 4.4, \( C := xzywxa \) is a separating cycle of length 4. Let \( I = \text{int}(C) \). Then \( V = B \cup V(C) \cup V(I) \), \( |x, V(\text{ext}(C))| = 2 \), \( |y, V(\text{ext}(C))| = 1 \), \( |z, V(\text{ext}(C))| = 3 \), and \( |w, V(\text{ext}(C))| = 1 \).

If \( w \) is adjacent to \( y \), then we apply Lemma 5.2 with \( C, X = \{w\} \), and \( Y = \emptyset \). As \( y \) is adjacent to \( w \), \( A' := \{x, y, z\} \) is usable in \( G_1 := \text{int}(C) - w \), and by (i–iii), \( A' \) is collectable in \( G_2' := \text{ext}(C) - w \). As \( |V(G_2')| = |B| = 3 \), \( \tau(G_2) = 0 \). This contradicts Lemma 5.2.

So using \( | \), \( C \) is chordless and \( x \) has at least one neighbour in \( \text{int}(C) \).

By Lemma 5.4, either \( |B(I)| \geq 4 \) or \( |V(I)| \leq 2 \) and every vertex in \( I \) has degree 4 in \( G \). If \( |V(I)| \leq 2 \) and every vertex in \( I \) has degree 4 in \( G \), then \( V \setminus \{x\} \) is \( A \)-good as we

\( \begin{align*}
(d) & \quad d(x), d(y), d(z) \geq 5. \\
(e) & \quad |N^o(\{x, y, z\})| \leq 4. \\
(f) & \quad x \text{ and } z \text{ have a common neighbour } w \notin \{y, v\}. \\
(g) & \quad N(x) \cap N(z) = \{v, w, y\}.
\end{align*} \)
can collect \( V(I), y, w, z, v, a', a \). Then \( f(G;A) \geq |V(G)| - 1 \geq \partial(G) \), a contradiction. Therefore \( |B(I)| \geq 4 \).

If there is an edge \( uu' \in E(C) \) with \( |N(\{u, u'\}) \cap V(I)| \geq 4 \), then we apply Lemma 5.2 with \( C, X = \{u, u'\} \) and \( Y = \emptyset \). Now \( A' := V(C) - \{u, u'\} \) is usable in \( G_1 := \text{int}[C] - \{u, u'\}, A' \) is collectable in \( G_2' := \text{ext}[C] - \{u, u'\}, |B_1| \geq 6 \), and \( B_2 = B \). As \( G_2 = B \), \( \tau(G_2) = 0 \). This contradicts Lemma 5.2. So \( |N(\{u, u'\}) \cap V(I)| \leq 3 \) for all edges \( uu' \in E(C) \) and in particular, \( \|u, V(I)| \leq 3 \) for all \( u \in V(C) \). This implies \( d(y) \leq 6 \).

If \( |N(\{x, y, z\}) \cap V(I)| \geq 4 \), then we apply Lemma 5.2 with \( C, X = \{x, z\} \) and \( Y = \{v, y\} \). Then \( Y \) is collectable in \( G - X, A' := \{w\} \) is usable in \( G_1 := \text{int}[C] - \{x, y, z\}, A' \) is collectable in \( G_2' := \text{ext}[C] - \{x, y, z\}, |B_1| \geq 5 \), and \( B_2 = B - \{v\} \). As \( (X \cup Y) \cap B = \emptyset \), this contradicts Lemma 5.2.

Therefore \( |N(\{x, y, z\}) \cap V(I)| \leq 3 \). Since \( |B(I)| \geq 4 \), there exists a vertex \( u \) in \( B(I) - N(\{x, y, z\}) \). Then \( w \) is the only neighbour of \( u \) in \( C \).

Because \( G \) is a plane triangulation and \( d(u) \geq 4 \), \( w \) is adjacent to \( u \). Since \( u \) is non-adjacent to \( x, y, z \), we deduce that \( B(I) \cap N(w) \) contains \( u \) and at least two of the neighbours of \( u \). Since \( \|w, V(I)| \leq 3 \), we deduce that \( \|w, V(I)| = 3 \). Since \( \|N(\{x, w\}) \cap V(I)| \leq 3 \), all neighbours of \( x \) in \( I \) are adjacent to \( w \). Similarly all neighbours of \( z \) in \( I \) are adjacent to \( w \). Since \( |B(I)| \geq 4 \), there is a vertex \( t \) in \( B(I) \) non-adjacent to \( w \). Then \( t \) is non-adjacent to \( x \) and \( z \). Therefore \( t \) is adjacent to \( y \). By the same argument, \( \|y, V(I)| = 3 \) and every neighbour of \( x \) or \( z \) in \( I \) is adjacent to \( y \). Thus, every vertex in \( N(\{x, z\}) \cap V(I) \) is adjacent to both \( y \) and \( w \).

If \( \|x, V(I)| \geq 2 \), then \( x, y, w \), and their common neighbours in \( I \) together with \( a \) are the branch vertices of a \( K_{3,3} \)-subdivision, using the path \( avy \). So \( G \) is nonplanar, a contradiction. Thus, \( \|x, V(I)| \leq 1 \) and similarly \( \|z, V(I)| \leq 1 \). This means that \( d(x) \leq 5 \) and \( B(\text{int}[C] - \{x, y, w\}) = B(I) \cup \{z\} \).

We apply Lemma 5.2 with \( C, X = \{w, y\} \) and \( Y = \{x\} \). Then \( Y \) is collectable in \( G - X \) and \( A' = \{z\} \) is usable in \( G_1 := \text{int}[C] - \{w, x, y\}, A' \) is collectable in \( G_2' := \text{ext}[C] - \{w, x, y\}, |B_1| = |B(I) \cup \{z\}| \geq 5, B_2 = B \), and \( G_2 = B \). Thus \( \tau(G_2) = 0 \) and this contradicts Lemma 5.2. Hence \( z \) is non-adjacent to \( a \). By symmetry, \( x \) is non-adjacent to \( a' \). Thus \( \text{(c)} \) holds.

(d) Suppose \( d(u) \leq 4 \) for some \( u \in \{x, y, z\} \). By Lemma 4.4 \( d(u) = 4 \). Let \( u' := y \) if \( u \neq y \), \( u' := x \) otherwise. Then, deleting \( u' \) and collecting \( u, v \) exposes at least 2 vertices in \( N^o(\{u, u'\}) \) by \( \text{(a)} \) and \( \text{(c)} \) and so \( |B(G - \{u, u', v\})| \geq |B| - 1 + 2 \). By applying Lemma 4.9 with \( X = \{u'\}, Y = \{u, v\} \), and \( s = 1 \), we obtain a contradiction. So \( \text{(d)} \) holds.

(e) Suppose \( |N^o(\{x, y, z\})| \geq 5 \). If \( d(y) \leq 6 \), then deleting \( x \), \( z \) and collecting \( v \), \( y \) exposes at least 5 vertices and so \( |B(G - \{x, z, v, y\})| \geq |B| - 1 + 5 \). By applying Lemma 4.9 with \( X = \{x, z\}, Y = \{v, y\} \), and \( s = 4 \), we obtain a contradiction. Thus \( d(y) \geq 7 \). Then either \( |N^o(\{x, y\}) - \{z\}| \geq 5 \) or \( |N^o(\{z, y\}) - \{x\}| \geq 5 \). We may assume by symmetry that \( |N^o(\{x, y\}) - \{z\}| \geq 5 \). Then deleting \( x \), \( y \) and collecting \( v \) exposes at least 6 vertices and so \( |B(G - \{x, y, v\})| \geq |B| - 1 + 6 \). By applying Lemma 4.9 with
Let $x$ or $zw$. By Observation 3.1 applied to $int(axyzaw)$ and $d(x) = d(z) = 5$. So each of $x$ and $z$ have exactly two neighbours in $int(axyzaw)$ and $d(x) = d(z) = 5$. Let $x_1, x_2$ be those neighbours of $x$ and $z_1, z_2$ be those two neighbours of $z$. We may assume that $x_1x_2y_1z_2$ is a path in $G$ by swapping labels of $x_1$ and $x_2$ and swapping labels of $z_1$ and $z_2$ if necessary. By (e) we have $N^o(y) - \{x, z\} \subseteq \{x_1, x_2, z_1, z_2\}$. As $d(x_2) \geq 4$, $y$ is not adjacent to $x_1$ because otherwise $x_1x_2y_1x_1$ is a separating triangle, that will make a new interior neighbour of $y$ by Lemma 5.1(c), contrary to (e). By symmetry, $y$ is not adjacent to $z_2$. So $x_2$ is adjacent to $z_1$ as $G$ is a plane triangulation. Therefore $d(y) = 5$.

Let $C^* := ax_1x_2z_1z_2a'$. Suppose that $w \in N(\{x_1, x_2, z_1, z_2\}) \cap V(int(C^*))$. Then by symmetry, we may assume $w$ is adjacent to $x_1$ or $x_2$. Deleting $x_1, x_2$ and collecting $x, y, v, z$ exposes $w, z_1, z_2$ and so $|B(G - \{x_1, x_2, x, y, v, z\})| \geq |B| - 1 + 3$. By applying Lemma 4.9 with $X = \{x_1, x_2\}$, $Y = \{x, y, v, z\}$, and $s = 2$, we obtain a contradiction. Thus $N(\{x_1, x_2, z_1, z_2\}) \cap V(int(C^*)) = \emptyset$ and therefore $|G| = 10$. See Figure 12.

By Observation 3.1 applied to $int[C^*]$, there is a vertex $w \in \{x_1, x_2, y_1, y_2\}$ having degree at most 2 in $int[C^*]$. By symmetry, we may assume that $w = x_i$ for some $i \in \{1, 2\}$. Since $d(x_i) \leq 4$, after deleting $x_{3-i}$, we can collect $x_i, x, y, v, z$, resulting in an outerplanar graph, which can be collected by Observation 3.1. So, $f(G; A) \geq 9 \geq \partial(G)$, a contradiction. So (f) holds.

(g) Suppose there is $w' \in N(x) \cap N(z) - \{v, w, y\}$. Let $C := xw'yx$. We may assume that $w$ is chosen to maximize $|V(int(C))|$. So $w'$ is in $V(int(C))$ and together with (a), we deduce that $C$ is an induced cycle.

We claim that $y$ is non-adjacent to $w'$. Suppose not. As $d(y) \geq 5$ by (d) $xw'yx$ or $zw'y$ is a separating triangle. By symmetry, we may assume $xw'yx$ is a separating triangle. Thus $|N(\{x, y\}) \cap V(int(xw'yx))| \geq 2$ by Lemma 5.1(d). Because $G$ is a plane triangulation, by (e) $w$ is adjacent to $w'$ and $xw'x$, $zw'z$, and $ywu'y$ are facial triangles. Thus $||y, V(\text{ext}(xw'yx))|| = ||w', V(\text{ext}(xw'yx))|| = 2$, contrary to Lemma 5.3. This proves the claim that $y$ is non-adjacent to $w'$.
Therefore \( \|y, V(\text{int}(xyzw')x)\| = 2 \), by (d) and (e). Let \( y_1, y_2 \) be two neighbours of \( y \) in \( \text{int}(xyzw'x) \) such that \( xy_1y_2z \) is a path in \( G \). Because \( G \) is a plane triangulation, by (c), \( w' \) is adjacent to both \( y_1 \) and \( y_2 \) and \( \text{int}(xw'zwx) \) has no vertex. Then \( C \) is a separating induced cycle of length 4 and \( |B(\text{int}(C))| = 3 \), contrary to Lemma 5.3. So (g) holds.

\[\text{Proof of Theorem 5.2}\]

Let \((G; A)\) be an extreme counterexample. Then \( G \) is a near plane triangulation. Let \( B = B(G) \) and \( B = B(G) \). By Lemmas 4.10 and 7.2, \( |B| = 3 \) and \( |A| = 2 \). Let \( A = \{a, a'\} \) and \( v \in B - A \). By Lemma 6.2, \( d(v) = 5 \). As \( G \) is a plane triangulation, the neighbours of \( v \) form a path \( axyza' \). By Lemma 8.1(g), \( x \) and \( z \) have exactly one common neighbour \( w \) in \( G - v - y \). Then \( C := xyzwx \) is a cycle of length 4. By symmetry and Lemma 8.1(d), we may assume that \( d(x) \geq d(z) \geq 5 \). By Lemma 8.1(e).

\[
(d(x) - 3) + (d(z) - 3) - 1 \leq |N^\circ(\{x, y, z\})| \leq 4.
\]

Therefore \( d(z) = 5 \) and \( d(x) = 5 \) or 6.

We claim that \( y \) is non-adjacent to \( w \). Suppose that \( y \) is adjacent to \( w \). By Lemma 8.1(d), \( d(y) \geq 5 \) and therefore at least one of \( xywz \) and \( yzwz \) is a separating triangle. If both of them are separating triangles, then \( |N(\{x, y\}) \cap V(\text{int}(xywz))| \geq 2 \) and \( |N(\{y, z\}) \cap V(\text{int}(yzwz))| \geq 2 \), by Lemma 5.1(d). Therefore \( |N^\circ(\{x, y, z\})| \geq 2 + 2 + 1 = 5 \), contrary to Lemma 8.1(e). This means that exactly one of \( xywz \) and \( yzwz \) is a separating triangle.

Suppose \( yzwz \) is a separating triangle. Then \( xywz \) is a facial triangle, and \( x \) has a neighbour in \( V(\text{int}(yzwz)) \). As \( d(z) = 5 \), \( z \) has no neighbour in \( V(\text{int}(axwzda')) \). Therefore, \( w \) is adjacent to \( a' \), and \( wza'w \) is a facial triangle. Thus \( \|y, V(\text{ext}(yzwz))\| = \|z, V(\text{ext}(yzwz))\| = 2 \), contrary to Lemma 5.3. So \( yzwz \) is not a separating triangle.

Therefore \( xywz \) is a separating triangle. By Lemma 5.1(d), \( \text{int}(xywz) \) has at least two vertices in \( N^\circ(\{x, y, z\}) \). By Lemma 8.1(d), \( z \) has a neighbour in \( V(\text{int}(axwzda')) \). Then already we found four vertices in \( N^\circ(\{x, y, z\}) \). This means that \( x \) has no neighbours in \( V(\text{int}(axwzda')) \) by Lemma 8.1(f). Hence \( \|y, V(\text{ext}(xywz))\| = \|x, V(\text{ext}(xywz))\| = 2 \), contrary to Lemma 5.3. This completes the proof of the claim that \( y \) is non-adjacent to \( w \).

Therefore \( C \) is chordless by Lemma 8.1(a). By Lemma 8.1(d), \( d(y) \geq 5 \). Thus \( C \) is a separating induced cycle of length 4. By Lemma 5.4, either \( |B(\text{int}(C))| \geq 4 \) or both \( |V(\text{int}(C))| \leq 2 \) and every vertex in \( \text{int}(C) \) has degree 4 in \( G \).

If \( |B(\text{int}(C))| \geq 4 \), then deleting \( w, y \) and collecting \( z, v, x \) exposes at least 4 vertices and so \( |B(G - \{w, y, z, v, x\})| = |B| - 1 + 4 \). By applying Lemma 4.9 with \( X = \{w, y\}, Y = \{z, v, x\} \), and \( s = 3 \), we obtain a contradiction.

Therefore \( 1 \leq |V(\text{int}(C))| \leq 2 \) and every vertex in \( \text{int}(C) \) has degree 4 in \( G \). Deleting \( y \) and collecting all vertices in \( \text{int}(C) \) and \( z, v, x \) exposes \( w \) and so \( B(G - (\{y, z, v, x\} \cup V(\text{int}(C)))) \geq |B| - 1 + 1 \). By applying Lemma 4.9 with \( X = \{y\}, Y = V(\text{int}(C)) \cup \{z, v, x\} \), and \( s = 0 \), we obtain a contradiction. \( \square \)
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