Third-order triangular finite elements for waveguiding problems

E. Cojocaru

Department of Theoretical Physics, Horia Hulubei
National Institute of Physics and Nuclear Engineering,
Magurele-Bucharest P.O.Box MG-6, 077125 Romania

Abstract

Explicit relations of matrices for two-dimensional finite element method with third-order triangular elements are given. They are more simple than relations presented in other works and could be easily implemented in new algorithms for both isotropic and anisotropic materials. Numerical examples are given comparatively using second-order and third-order triangular elements for problems of wave propagation in rectangular waveguides which have analytic solutions.

*Electronic address: ecojocaru@theory.nipne.ro
I. INTRODUCTION

The finite element method is a widely applicable numerical technique for obtaining approximate solutions to boundary-value problems of mathematical physics [1, 2]. Any complex shape of the problem domain can be handled with ease by division into many subdomains, each subdomain being called a finite element. For two-dimensional problems one resorts usually to triangular elements: the first-order triangular element, which requires three nodes, and the second-order triangular element, which requires six nodes. In order to achieve higher accuracy in the finite element solution, two approaches are commonly taken: one resorts to finer subdivision or smaller elements, and the other resorts to higher-order interpolation functions or higher-order elements. Here we are interested in the two-dimensional finite element method with third-order triangular elements, each element requiring ten nodes which are numbered counterclockwise as shown in Fig. 1. These high-order elements can be successfully employed for the characterization of wave propagation in shielded microstrip transmission lines or integrated circuits with slot lines, when the lines are infinitesimally thin and one needs to place a set of nodes above the line as well below the line, as if the line had a finite thickness [2]. Generally, the elemental matrices for high-order elements are determined numerically, but a higher accuracy is assured with explicit expressions. Relations for triangular elements, including the third-order ones, have been presented in [3]. In this paper the elemental matrices for third-order triangles are given in simple, explicit forms which can be easily implemented in different algorithms of waveguiding problems related to both isotropic and anisotropic materials.

FIG. 1: Third-order triangular element
II. GENERAL RELATIONS

Let us consider a two-dimensional boundary-value problem defined by the second-order differential equation

\[- \frac{\partial}{\partial x} \left( \alpha_x \frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial y} \left( \alpha_y \frac{\partial \phi}{\partial y} \right) + \beta \phi = f, \quad (x, y) \in \Omega \]  

(1)

where \(\phi\) is the unknown function; \(\alpha_x, \alpha_y,\) and \(\beta\) are known parameters; and \(f\) is the source or excitation function. The ordinary two-dimensional Laplace equation, Poisson equation, and Helmholtz equation are special forms of (1). For simplicity we consider \(f = 0\) and the homogeneous Neumann boundary condition on the boundary enclosing the area \(\Omega\). Within each element, \(\phi\) can be approximated as

\[\phi^e(x, y) = \sum_{j=1}^{10} N_j^e(x, y) \phi_j^e,\]  

(2)

where \(\phi_j^e\) are constant expansion coefficients and \(N_j^e(x, y)\) are the interpolation or expansion functions given by

\[
\begin{align*}
N_1^e(x, y) &= \frac{1}{2} L_1^e(3L_1^e - 1)(3L_1^e - 2), \\
N_2^e(x, y) &= \frac{1}{2} L_2^e(3L_2^e - 1)(3L_2^e - 2), \\
N_3^e(x, y) &= \frac{1}{2} L_3^e(3L_3^e - 1)(3L_3^e - 2), \\
N_4^e(x, y) &= \frac{9}{2} L_1^e L_2^e(3L_1^e - 1), \\
N_5^e(x, y) &= \frac{9}{2} L_1^e L_2^e(3L_2^e - 1), \\
N_6^e(x, y) &= \frac{9}{2} L_2^e L_3^e(3L_2^e - 1), \\
N_7^e(x, y) &= \frac{9}{2} L_2^e L_3^e(3L_3^e - 1), \\
N_8^e(x, y) &= \frac{9}{2} L_1^e L_3^e(3L_3^e - 1), \\
N_9^e(x, y) &= \frac{9}{2} L_1^e L_3^e(3L_1^e - 1), \\
N_{10}^e(x, y) &= 27L_1^e L_2^e L_3^e. 
\end{align*}
\]  

(3)

In the above, the area coordinates \(L_j^e\) are given by

\[L_j^e = \frac{1}{2\Delta^e} (a_j + b_j x + c_j y), \quad j = 1, 2, 3,\]  

(4)
in which \(a_j, b_j,\) and \(c_j\) are

\[
\begin{align*}
   a_1 &= x_2y_3 - y_2x_3, & b_1 &= y_2 - y_3, & c_1 &= x_3 - x_2, \\
   a_2 &= x_3y_1 - y_3x_1, & b_2 &= y_3 - y_1, & c_2 &= x_1 - x_3, \\
   a_3 &= x_1y_2 - y_1x_2, & b_3 &= y_1 - y_2, & c_3 &= x_2 - x_1,
\end{align*}
\]

where \(x_j\) and \(y_j\) \((j = 1, 2, 3)\) denote the coordinate values of the vertices and \(\Delta^e\) is the area of the \(e\)th element,

\[
\Delta^e = \frac{1}{2}(b_1c_2 - b_2c_1).
\]

With the expansion of \(\phi\) given in (2), we can proceed to formulate the elemental equations using either the Ritz or Galerkin method [2, 4]. As for example, in the Ritz method we formulate the problem in terms of a functional \(F(\phi)\) whose minimum corresponds to the differential equation of the boundary-value problem. The functional can be written as

\[
F(\phi) = \sum_{e=1}^{M} F^e(\phi^e),
\]

where \(M\) denotes the total number of elements and \(F^e\) is the subfunctional corresponding to the \(e\)th element. For differential equation (1), \(F^e(\phi^e)\) is defined as

\[
F^e(\phi^e) = \frac{1}{2} \int_{\Omega^e} \left[ \alpha_x \left( \frac{\partial \phi^e}{\partial x} \right)^2 + \alpha_y \left( \frac{\partial \phi^e}{\partial y} \right)^2 + \beta (\phi^e)^2 \right] d\Omega,
\]

where \(\Omega^e\) is the domain of the \(e\)th element. Introducing expression (2) for \(\phi^e\) and differentiating with respect to \(\phi^e\) yields

\[
\frac{\partial F^e}{\partial \phi^e} = \sum_{j=1}^{10} \phi_j^e \int_{\Omega^e} \left( \alpha_x \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \alpha_y \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} + \beta N_i^e N_j^e \right) d\Omega, \quad i, j = 1, 2, \ldots 10.
\]

In matrix form, this can be written as

\[
\left\{ \frac{\partial F^e}{\partial \phi^e} \right\} = [K^e] \{\phi^e\},
\]

where

\[
\left\{ \frac{\partial F^e}{\partial \phi^e} \right\} = \left[ \frac{\partial F^e}{\partial \phi_1^e} \frac{\partial F^e}{\partial \phi_2^e} \cdots \frac{\partial F^e}{\partial \phi_{10}^e} \right]^T,
\]

\[
\{\phi^e\} = [\phi_1^e \phi_2^e \cdots \phi_{10}^e]^T,
\]

with \(T\) denoting a transpose and the elements of the matrix \([K^e]\) given by

\[
K_{ij}^e = \int_{\Omega^e} \left( \alpha_x \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \alpha_y \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} + \beta N_i^e N_j^e \right) dxdy, \quad i, j = 1, 2, \ldots 10.
\]
III. ELEMENTAL MATRICES

Assuming $\alpha_x$, $\alpha_y$, and $\beta$ are constant in each element, we split $[K^e]$ into three parts

$$[K^e] = \alpha_x[A^e_x] + \alpha_y[A^e_y] + \beta[B^e],$$

(10)

where the elements of matrices $[A^e_x]$, $[A^e_y]$, and $[B^e]$ are given by

$$A^e_{xij} = \int\int_{\Omega^e} \frac{\partial N^e_i}{\partial x} \frac{\partial N^e_j}{\partial x} dx dy, \quad A^e_{yij} = \int\int_{\Omega^e} \frac{\partial N^e_i}{\partial y} \frac{\partial N^e_j}{\partial y} dx dy,$$

$$B^e_{ij} = \int\int_{\Omega^e} N^e_i N^e_j dx dy, \quad i, j = 1, 2, \ldots 10.$$  

(11)

The integral calculus in (11) can be performed analytically by using the following convenient integration formula for the area coordinates [5]

$$\int\int_{\Omega^e} (L^e_1)^i (L^e_2)^j (L^e_3)^k dx dy = \frac{i!j!k!}{(i + j + k + 2)!} 2\Delta^e, \quad i, j, k = 0, 1, 2, 3, \ldots$$

(12)

Explicit expressions for the matrix elements $B^e_{ij}$ and $A^e_{xij}$ are given in Tables I and II respectively. Both $[B^e]$ and $[A^e_x]$ are symmetric matrices, i.e., $B^e_{ji} = B^e_{ij}$ and $A^e_{xji} = A^e_{xij}$. In Table II we used notation $b_{ij} = b_i b_j$, with $i, j = 1, 2, 3$. The matrix elements $A^e_{yij}$ are obtained by changing $b_i$ to $c_i$ ($i = 1, 2, 3$) in Table II

| $i \setminus j$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1               | $\frac{76}{9}$ | $\frac{11}{9}$ | $\frac{14}{9}$ | 2   | 0   | 3   | 3   | 0   | 2   | 4   |
| 2               | $\frac{11}{9}$ | $\frac{76}{9}$ | $\frac{11}{9}$ | 0   | 2   | 2   | 0   | 3   | 3   | 4   |
| 3               | $\frac{14}{9}$ | $\frac{11}{9}$ | $\frac{11}{9}$ | $\frac{76}{9}$ | 3   | 3   | 0   | 2   | 2   | 0   |
| 4               | 2   | 0   | 3   | $-60$ | $-21$ | $-15$ | $-6$ | $-15$ | 30  | 18  |
| 5               | 0   | 2   | 3   | $-21$ | 60   | 30   | $-15$ | $-6$ | $-15$ | 18  |
| 6               | 3   | 2   | 0   | $-15$ | 30   | 60   | $-21$ | $-15$ | $-6$ | 18  |
| 7               | 3   | 0   | 2   | $-6$ | $-15$ | $-21$ | 60   | 30   | $-15$ | 18  |
| 8               | 0   | 3   | 2   | $-15$ | $-6$ | $-15$ | 30   | 60   | $-21$ | 18  |
| 9               | 2   | 3   | 0   | 30   | $-15$ | $-6$ | $-15$ | $-21$ | 60   | 18  |
| 10              | 4   | 4   | 4   | 18   | 18   | 18   | 18   | 18   | 18   | 216 |
TABLE II: Matrix elements $A_{xij}^e$ ($i,j = 1,2\ldots10$) are tabulated expressions multiplied by $[81/(8\Delta^e)]$

| i\j | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|
| 1   | $\frac{17b_{13}}{810}$ | $\frac{7b_{11}}{1020}$ | $\frac{7b_{12}}{1020}$ | $\frac{12b_{14}}{540}$ | $\frac{-b_{13}-9b_{14}}{540}$ | $\frac{-b_{11}}{540}$ | $\frac{-b_{14}}{540}$ | $\frac{-b_{13}}{540}$ | $\frac{18b_{13}-b_{14}}{540}$ | 0 |
| 2   | $\frac{17b_{12}}{810}$ | $\frac{7b_{11}}{1020}$ | $\frac{-b_{13}-9b_{14}}{540}$ | $\frac{18b_{13}-b_{14}}{540}$ | $\frac{-b_{12}}{540}$ | $\frac{-b_{14}}{540}$ | 0 |
| 3   | $\frac{1b_{13}}{810}$ | $\frac{-b_{14}}{540}$ | $\frac{-b_{13}-9b_{23}}{540}$ | $\frac{18b_{13}-b_{23}}{540}$ | $\frac{-b_{23}}{540}$ | 0 |
| 4   | $\frac{b_{23}-b_{12}}{12}$ | $\frac{-b_{14}}{60}$ | $\frac{-b_{14}}{60}$ | $\frac{-b_{23}}{60}$ | 0 |
| 5   | $\frac{b_{13}}{12}$ | $\frac{b_{14}}{60}$ | $\frac{-b_{23}}{60}$ | $\frac{b_{23}}{10}$ | 0 |
| 6   | $\frac{b_{13}}{12}$ | $\frac{-b_{12}}{60}$ | $\frac{-b_{12}}{60}$ | $\frac{b_{23}}{10}$ | 0 |
| 7   | $\frac{b_{11}}{12}$ | $\frac{b_{13}}{60}$ | $\frac{-b_{12}}{60}$ | $\frac{-b_{23}}{60}$ | 0 |
| 8   | $\frac{b_{22}}{b_{13}}$ | $\frac{-b_{12}}{60}$ | $\frac{-b_{23}}{60}$ | 0 |
| 9   | $\frac{b_{22}}{b_{13}}$ | $\frac{-b_{23}}{10}$ | 0 |
| 10  | $\frac{b_{22}}{b_{13}}$ | $\frac{b_{23}}{10}$ | 0 |

In the vector formulation of different waveguiding problems, the subfunctional $F^e$ of the $e$th element in (6) is more complicated and the matrix $[K^e]$ in (10) contains more elemental matrices. As for example, within a closed waveguide, the magnetic field satisfies the vector differential equation

$$\nabla \times \left( \frac{1}{\epsilon_r} \nabla \times \mathbf{H} \right) - k_0^2 \mu_r \mathbf{H} = 0 \quad \text{in} \quad \Omega$$

(13)

and the boundary condition $\mathbf{n} \times (\nabla \times \mathbf{H}) = 0$ on $\Gamma_1$, where $\Omega$ denotes the cross section of the structure comprised by the electric wall $\Gamma_1$, $k_0$ is the wave number in vacuum, $\epsilon_r$ and $\mu_r$ are the permittivity and permeability of the structure. With the $z$-dependence of $\mathbf{H}$ as $\mathbf{H}(x, y, z) = \mathbf{H}(x, y)e^{j(\omega t-kz)}$, where $\omega$ is the circular frequency, the functional of (13) can be written as

$$F(\mathbf{H}) = \frac{1}{2} \iint_{\Omega} \left[ \frac{1}{\epsilon_r} \left( \frac{\partial H_z}{\partial y} + j k_z H_y \right)^2 + \left| j k_z H_x + \frac{\partial H_z}{\partial x} \right|^2 + \left| \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right|^2 \right] d\Omega - k_0^2 \mu_r (|H_x|^2 + |H_y|^2 + |H_z|^2) \iint_{\Omega} d\Omega.$$  

(14)

To render this as a real system, we introduce the transformation $h_z = -j H_z$, and with this,
The functional can readily be discretized in a standard manner, and the result is

\[
F(H) = \frac{1}{2} \int \int \int \left[ \frac{1}{\varepsilon_r} \left( \left| \frac{\partial h_z}{\partial y} + k_z H_y \right| + k_z H_x + \frac{\partial h_z}{\partial x} \right)^2 + \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \right] d\Omega
- \kappa_0^2 \mu_r (|H_x|^2 + |H_y|^2 + |h_z|^2) d\Omega
\]

(15)

The functional can readily be discretized in a standard manner, and the result is

\[
\begin{bmatrix}
A_{xx} & A_{xy} & A_{xz} \\
A_{yx} & A_{yy} & A_{yz} \\
A_{zx} & A_{zy} & A_{zz}
\end{bmatrix}
\begin{bmatrix}
H_x \\
H_y \\
h_z
\end{bmatrix} = k_0^2 \begin{bmatrix}
B_x & 0 & 0 \\
0 & B_y & 0 \\
0 & 0 & B_z
\end{bmatrix}
\begin{bmatrix}
H_x \\
H_y \\
h_z
\end{bmatrix},
\]

(16)

where the matrices are assembled from their corresponding elemental matrices, given by

\[
[A^e_{xx}] = \int \int \frac{1}{\varepsilon_r} \left( \frac{\partial \{N^e\}}{\partial y} \frac{\partial \{N^e\}^T}{\partial y} + k_z^2 \{N^e\} \{N^e\}^T \right) d\Omega
- \frac{\partial \{N^e\}}{\partial x} \frac{\partial \{N^e\}^T}{\partial x} d\Omega
= - \frac{\partial \{N^e\}}{\partial x} \frac{\partial \{N^e\}^T}{\partial x} d\Omega
\]

(17)
Explicit expressions for the matrix elements $C_{xyij}^e$ ($i,j = 1,2,\ldots,10$) are obtained by changing $b_i$ to $c_i$ ($i=1,2,3$) in Table IV.

Explicit expressions for the matrix elements $C_{xij}^e$ and $D_{xij}^e$ are given in Tables III and IV respectively. In Table III we used notation $p_{ij} = b_i c_j$, with $i,j = 1,2,3$. The matrix elements $D_{yij}^e$ are obtained by changing $b_i$ to $c_i$ ($i=1,2,3$) in Table IV.
TABLE IV: Matrix elements $D_{xij}^e$ ($i,j = 1,2,\ldots 10$) are tabulated expressions multiplied by (27/140)

| i\j | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-----|------|------|------|------|------|------|------|------|------|------|
| 1   | 10b1 | 10b1 | 10b1 | 10b1 | 12   | -b1  | 12   | 12   | 12   | 12   |
| 2   | 10b1 | 10b1 | 10b1 | 10b1 | 12   | -b1  | 12   | 12   | 12   | 12   |
| 3   | 10b1 | 10b1 | 10b1 | 10b1 | 12   | -b1  | 12   | 12   | 12   | 12   |
| 4   | 5b1  | 5b1  | 5b1  | 5b1  | 12   | -b3  | 12   | 12   | 12   | 12   |
| 5   | 5b1  | 5b1  | 5b1  | 5b1  | 12   | -b3  | 12   | 12   | 12   | 12   |
| 6   | 5b1  | 5b1  | 5b1  | 5b1  | 12   | -b3  | 12   | 12   | 12   | 12   |
| 7   | 5b1  | 5b1  | 5b1  | 5b1  | 12   | -b3  | 12   | 12   | 12   | 12   |
| 8   | 5b1  | 5b1  | 5b1  | 5b1  | 12   | -b3  | 12   | 12   | 12   | 12   |
| 9   | 5b1  | 5b1  | 5b1  | 5b1  | 12   | -b3  | 12   | 12   | 12   | 12   |
| 10  | 5b1  | 5b1  | 5b1  | 5b1  | 12   | -b3  | 12   | 12   | 12   | 12   |

For a triangle with coordinates of vertices (1,1), (2,1), and (1,2), the area is $\Delta = 0.5$ and the maximum absolute values on the columns of these elemental matrices vary in the intervals: $\text{Max}|B_{ij}^e| = 0.0057 - 0.1446$, $\text{Max}|A_{xij}^e| = 0 - 4.05$, $\text{Max}|C_{xij}^e| = 0 - 2.025$, $\text{Max}|D_{xij}^e| = 0 - 0.2411$. We can see that the elements of matrix $[B^e]$ are very small. Thus, they could be more accurate than those obtained by performing the integrations numerically. Note that relations presented here for the matrix elements are more simple than those presented in [3, 6].

IV. EXAMPLES

In the following we consider some waveguiding problems which have analytic solutions. The most simple is the eigenvalue problem of a hollow square waveguide [7]. In Table [11] we give the results for the wavenumber $k_0(\text{cm}^{-1})$ obtained by using the vectorial finite element method [2] in terms of the magnetic field components $[H_x, H_y, H_z]$, when the propagation constant on the z-axis direction, $k_z$ equals zero. An uniform, one-directional mesh is considered with second-order and third-order triangular elements in the domain
For second-order triangles, the number of elements is 18 and the total number of nodes is 49. For third-order triangles, the corresponding numbers are 18 and 100. Explicit relations of matrices for the second-order triangular elements were given in [8, 9]. As seen in Table V, the results obtained using third-order elements have better accuracy than those obtained with the same number of second-order elements.

![Cross section of the inhomogeneous rectangular waveguide](image)

**FIG. 2: Cross section of the inhomogeneous rectangular waveguide**

Finally we consider a rectangular waveguide with a 2:1 width to height ratio completely filled with a ferrite material characterized by a relative permittivity \( \varepsilon_r = 2 \) and a relative
### TABLE VI: Results for the inhomogeneous rectangular waveguide

| $k_z$ | Analytic Solution | Second Order Triangles | Third Order Triangles |
|-------|-------------------|------------------------|-----------------------|
| 0     | 1.7666            | 1.7681                 | 1.7666                |
|       |                   |                        |                       |
| 2.3053| 2.3076            | 2.3053                 |                       |
| 2.6779| 2.6875            | 2.6779                 |                       |
| 1     | 1.8310            | 1.8545                 | 1.8531                |
|       |                   |                        |                       |
| 2.3460| 2.3543            | 2.3521                 |                       |
| 2.7125| 2.7931            | 2.7042                 |                       |

### TABLE VII: Results for the ferrite completely filled rectangular waveguide

|       | Analytic Solution | Second Order Triangles | Third Order Triangles |
|-------|-------------------|------------------------|-----------------------|
| 0.6654| 0.6659            | 0.6654                 |                       |
| 1.3307| 1.3445            | 1.3307                 |                       |
| 1.9961| 1.9458            | 1.9961                 |                       |

The permeability tensor $\mu_r$ given by [6]

$$
\begin{bmatrix}
3 & 0 & j0.8 \\
0 & 1 & 0 \\
-j0.8 & 0 & 3
\end{bmatrix}
$$

The analytical solution for the wave number $k_0$ of the $n$th mode is [6]

$$
k_n^2 = \frac{3.0}{16.72} \left[ k_z^2 + \left( \frac{n\pi}{2} \right)^2 \right].
$$

Results are given comparatively in Table VII for $k_0$(cm$^{-1}$) of the first three modes, at $k_z = 0$, when the rectangular domain of $2\text{cm} \times 1\text{cm}$ is uniformly divided in 18 second-order elements with 49 nodes or in 18 third-order elements with 100 nodes. We can see that in the case of
third-order elements the results agree with analytical solutions very well.

[1] O. C. Zienkiewicz, R. L. Taylor, and J. Z. Zhu, *Finite Element Method: Its Basis and Fundamentals* (Amsterdam, Elsevier, 2005), 6th ed.

[2] J. M. Jin, *The Finite Element Method in Electromagnetics* (New York, Wiley, 2002), 2nd ed.

[3] P. Silvester, “A general high-order finite-element waveguide analysis program,” IEEE Trans. Microwave Theory Tech. MTT-17, 204–210 (1969).

[4] Y. W. Kwon and H. Bang, *The Finite Element Method Using Matlab* (New York, CRC Press, 2000), 2nd ed.

[5] K. Kawano and T. Kitoh, *Introduction to Optical Waveguide Analysis: Solving Maxwell’s Equations and the Schrodinger Equation* (New York, Wiley, 2001).

[6] A. Konrad, “High-order triangular finite elements for electromagnetic waves in anisotropic media,” IEEE Trans. Microwave Theory Tech. MTT-25, 353–360 (1977).

[7] N. Marcuvitz, *Waveguide Handbook* (New York, McGraw-Hill, 1951).

[8] M. Koshiba, S. Maruyama, and K. Hirayama, “A vector finite element method with the high-order mixed-interpolation-type triangular elements for optical waveguiding problems,” J. Lightwave Technol. 12, 495–502 (1994).

[9] E. Cojocaru, “Elemental matrices for the finite element method in electromagnetics with quadratic triangular elements,” eprint [arXiv:0910.3854](http://arxiv.org/abs/0910.3854) [math-ph] (2009).