IRRATIONAL POINTS ON RANDOM HYPERELLIPTIC CURVES

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ABSTRACT. We consider genus $g$ hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point, ordered by height. If $d < g$ is odd, we prove there exists $B_d$ such that a positive proportion of these curves have at most $B_d$ points of degree $d$. If $d < g$ is even, we similarly bound degree $d$ points not pulled back from points of degree $d/2$ on the projective line. Furthermore, we show one may take $B_2 = 24$ and $B_3 = 114$.

Our proofs proceed by refining recent work of Park, which applied tropical geometry methods to symmetric power Chabauty, and then applying results of Bhargava and Gross on average ranks of Jacobians of hyperelliptic curves.

1. INTRODUCTION

Let $K$ be a number field, and let $C/K$ be a curve of genus $g \geq 2$. In 1983, Faltings [Fal83] proved that the set $C(K)$ of $K$-rational points is finite. Given this, one can ask how the finite quantity $\#C(K)$ varies in families of curves. The last five years have seen multiple works consider this question, for the family of all hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point [PS14, RT18], the family with a rational non-Weierstrass point [SW18], and the entire family of hyperelliptic curves over $\mathbb{Q}$ [Bha13].

In this paper, for a hyperelliptic curve $C/\mathbb{Q}$, instead of rational points we consider the degree $d$ points of $C$, which we take to mean the set

\[ \{ P \in C(\overline{\mathbb{Q}}) \mid |Q(P) : \mathbb{Q}| = d \} . \]

Since $C$ is defined over $\mathbb{Q}$, this set is partitioned into $d$-tuples of Galois-conjugate points.

Before stating our main theorems, we consider an extended example: quadratic points, i.e. $d = 2$. Because we allow the quadratic extension to vary, there are infinitely many such points: for almost any point of $\mathbb{P}^1(\mathbb{Q})$, its pre-image under the hyperelliptic map will be a pair of conjugate quadratic points on $C$. We will call these expected quadratic points. More simply, for a hyperelliptic equation $y^2 = f(x)$, these are the quadratic points given by plugging in a rational number for $x$, and then solving for $y$.

But there can also be unexpected quadratic points, whose $x$-coordinate is irrational but whose $y$-coordinate is contained in the same quadratic field. For example, the genus 4 curve defined by $y^2 = x^9 + x^3 - 1$ contains infinitely many expected points, such as $(0, \pm i), (-1, \pm \sqrt{-3})$, and $(2, \pm \sqrt{19})$, but also contains unexpected points like $(\pm i, \pm i), (\zeta_3, \pm 1)$, and $(-\zeta_3, \pm \sqrt{-3})$, where $\zeta_3$ is a primitive third root of unity. In general, it is no small feat to compute these points explicitly for a given curve.

Example 1.1. Let $C/\mathbb{Q}$ be the hyperelliptic curve with odd degree affine model given by

\[ C: y^2 = f(x) = x(x^2 + 2)(x^2 + 43)(x^2 + 8x - 6). \]

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In [Sik09, Section 6.1], Siksek determined the set of quadratic points on $C$. Besides the infinitely many expected quadratic points of the form $(x, \pm \sqrt{f(x)})$, for $x \in \mathbb{Q}$, there are exactly 9 pairs of unexpected quadratic points on $C$. For example, there are the three pairs below:

$$Q_1 = \left\{ (\sqrt{6}, 56\sqrt{6}), (-\sqrt{6}, -56\sqrt{6}) \right\},$$
$$Q_2 = \left\{ (\sqrt{-2}, 0), (-\sqrt{-2}, 0) \right\},$$
$$Q_3 = \left\{ \left( \frac{-164 + \sqrt{22094}}{49}, \frac{257704352 - 1648200\sqrt{22094}}{823543} \right), \text{conjugate} \right\}.$$

By further work of Faltings [Fal91, p. 550], we know that for any hyperelliptic curve of genus $g \geq 4$, there are only finitely many of these unexpected quadratic points. Thus, one can ask how many there are on a typical hyperelliptic curve.

We need a way of ordering curves to make that question rigorous. A genus $g$ hyperelliptic curve $C$ over $\mathbb{Q}$, with a marked rational Weierstrass point $\infty$, has an affine model of the form

$$y^2 = f(x) = x^{2g+1} + a_2x^{2g-1} + a_3x^{2g-2} + \cdots + a_{2g+1},$$

with $f(x) \in \mathbb{Z}[x]$ separable, such that $\infty$ is not contained in this affine patch. Furthermore, $C$ has a unique such minimal equation, for which there is no prime $p$ such that $p^{2i} \mid a_i$ for each $i \geq 2$. Define the height of $C$ to be

$$H(C) := \max \left\{ \left| a_i \right|^{1/i} \right\},$$

where the $a_i$'s are coefficients for the minimal equation of $C$.

We can now state our first main theorem.

**Theorem 1.2.** For each $g > 2$, a positive proportion of genus $g$ hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point, when ordered by height, have at most 24 quadratic points not obtained by pulling back points of $\mathbb{P}^1(\mathbb{Q})$.

More precisely, let $\mathcal{F}_g$ denote the set of $\mathbb{Q}$-isomorphism classes of genus $g$ hyperelliptic curves defined over $\mathbb{Q}$, with a marked rational Weierstrass point. The above says that if $\mathcal{F}_g' \subset \mathcal{F}_g$ corresponds to those curves satisfying the condition of Theorem 1.2, then

$$\liminf_{X \to \infty} \frac{\# \{ C \in \mathcal{F}_g' \mid H(C) < X \}}{\# \{ C \in \mathcal{F}_g \mid H(C) < X \}} > 0.$$

Next we consider cubic points, i.e. degree 3 points. Unlike the case of $d = 2$, where the geometry — in this case, the existence of a 2:1 map to $\mathbb{P}^1$ — imposes infinitely many quadratic points, there need not be any cubic points. We prove the following theorem on their sparsity.

**Theorem 1.3.** For each $g > 3$, a positive proportion of genus $g$ hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point, when ordered by height, have at most 114 cubic points.

The $d = 2$ and $d = 3$ cases turn out to be prototypical, and we can now state our main theorem concerning points of arbitrary degree $d$.

**Theorem 1.4.** Let $d > 1$ be a positive integer.
(1) If \( d \) is odd, there exists a number \( B_d \) such that for each \( g > d \), a positive proportion of genus \( g \) hyperelliptic curves over \( \mathbb{Q} \) with a rational Weierstrass point have at most \( B_d \) points of degree \( d \).

(2) If \( d \) is even, there exists a number \( B_d \) such that for each \( g > d \), a positive proportion of genus \( g \) hyperelliptic curves over \( \mathbb{Q} \) with a rational Weierstrass point have at most \( B_d \) points of degree \( d \) not obtained by pulling back degree \( \frac{d}{2} \) points of \( \mathbb{P}^1 \).

(3) We may take \( B_2 = 24 \) and \( B_3 = 114 \).

Remark 1.5.
- If \( d = 1 \), we may take \( B_1 = 1 \), by [PS14, Theorem 10.3] (in the case \( g > 2 \)) and [RT18, Theorem 1.2] (\( g = 2 \)).
- The hypothesis \( g > d \) is natural: that is exactly when the symmetric product \( C^{(d)} \) is of general type, for a curve of genus \( g \geq 2 \).
- Using our methods, we could prove \( B_2 = 0 \) if we knew that a positive proportion of odd hyperelliptic curves had Jacobian rank 0. In other words, we would know that a positive proportion have no unexpected quadratic points.

1.6. Terminology. To ease terminology in the paper, while we do not use the term in theorem statements, we will call a point of even degree \( d \) on a hyperelliptic curve unexpected if it does not map to a degree \( d/2 \) point on \( \mathbb{P}^1 \). If \( d \) is odd, we call any point of degree \( d \) unexpected. Throughout the paper, we use “asymptotically” when considering hyperelliptic curves of a fixed genus, with increasing height.

1.7. Overview of paper. In Section 2, we collect results on the arithmetic of hyperelliptic Jacobians and prove that for a fixed genus \( g \geq 2 \), asymptotically 100% of hyperelliptic curves over \( \mathbb{Q} \) of genus \( g \) with a rational Weierstrass point have finitely many unexpected degree \( d \) points.

The main part of our proof of Theorem 1.4 involves refining recent work of Park [Par16] that partially generalized the Chabauty–Coleman method for computing rational points on curves of genus \( g \geq 2 \) to higher degree points. One aspect of her work yields an effective bound \( B(p, g) \) on the number of unexpected degree \( d \) points on a hyperelliptic curve \( C \) as above, when the Mordell–Weil rank of the curve’s Jacobian is no more than 1 and \( C \) has good reduction at \( p \). Our paper departs from previous work by removing the dependence on the genus \( g \).

In Section 3, we recall results on \( p \)-adic integration and prove some auxiliary lemmas. In Section 4, we begin removing the dependence on the genus from Park’s bound above. Work of Bhargava and Gross [BG13] tells us that many curves have Jacobian of rank at most 1, and thus our new bound applies to them. In Section 5, using results from tropical geometry, we prove Theorem 1.4 using Newton polygon and mixed volume computations. In Sections 6 and 7, we prove our explicit results for \( d = 2 \) and 3.

1.8. Related results. We conclude the introduction by describing some related results in the literature. All curves below are defined over \( \mathbb{Q} \).

In [BG13], Bhargava and Gross showed that for \( g \geq 2 \), a positive proportion of genus \( g \) hyperelliptic curves with a marked rational Weierstrass point have at most 3 rational points, and that a majority have less than 20 rational points. In [PS14], Poonen and Stoll showed that in fact, for \( g \geq 3 \), a positive proportion of these curves have no other rational
points besides the marked point. (Romano and Thorne [RT18] recently proved this for \( g = 2 \).) Furthermore, that proportion of curves tends to 1 as \( g \) grows.

Shortly after Poonen and Stoll’s work, Shankar and Wang [SW18] proved that for \( g \geq 9 \), a positive proportion of genus \( g \) hyperelliptic curves with a marked rational non-Weierstrass point have only the two guaranteed rational points (the marked point and the other point in its fiber). Again, that proportion tends to 1 as \( g \) grows. Next, Bhargava [Bha13] showed that for \( g \geq 2 \), a positive proportion (again tending to 1) of all genus \( g \) hyperelliptic curves have no rational points.

The question of higher-degree points has been considered previously for families of hyperelliptic curves, though only in the case of odd-degree points. While every hyperelliptic curve has (infinitely many!) points of each even degree, the geometry of a general hyperelliptic curve does not impose any points of odd degree. In [BGW17], Bhargava, Gross, and Wang showed that for \( g \geq 2 \), a positive proportion of all locally soluble genus \( g \) hyperelliptic curves have no points over any odd-degree extension of \( \mathbb{Q} \). Moreover, for a fixed odd \( m \), the proportion of locally soluble curves without a degree \( m \) point tends to 1 as \( g \) grows.

Bhargava, Gross, and Wang’s results work by showing that many curves have no rational odd-degree divisors at all. The curves in the family considered in this paper (those with a rational Weierstrass point) always have such divisors, so our Theorem 1.3 is disjoint from (but complementary to) their work. In particular, since degree \( d \) points on a curve correspond to rational points of the \( d \)-th symmetric power of \( C \), our results represent some of the first work on bounding rational points in families of higher-dimensional varieties that do have some rational points.

2. ARITHMETIC AND GEOMETRY OF HYPERELLiptIC JACOBIANS

First, we recall results of Bhargava and Gross on the average size of 2-Selmer groups of Jacobians of hyperelliptic curves.

**Theorem 2.1** ([BG13, Theorem 1.1]). When all hyperelliptic curves of fixed genus \( g \geq 1 \) over \( \mathbb{Q} \) having a rational Weierstrass point are ordered by height, the average size of the 2-Selmer groups of their Jacobians is at most 3.

Furthermore, the same result holds if one averages within a family defined by a finite set of congruence conditions.

This gives immediate corollaries concerning the average rank of \( J(\mathbb{Q}) \), where we write \( J \) for the Jacobian of a curve \( C \).

**Corollary 2.2** ([BG13, Corollary 1.2]). When all hyperelliptic curves of fixed genus \( g \geq 1 \) over \( \mathbb{Q} \) having a rational Weierstrass point are ordered by height, the average rank of the Mordell–Weil groups of their Jacobians is at most \( \frac{3}{2} \).

Furthermore, the same result holds if one averages within a family defined by a finite set of congruence conditions.

**Corollary 2.3.** When all hyperelliptic curves of fixed genus \( g \geq 1 \) over \( \mathbb{Q} \) having a rational Weierstrass point are ordered by height, at least 25% have rank \( J(\mathbb{Q}) = 0 \) or 1. The same holds if one averages within a congruence family.

Furthermore, either a positive proportion have rank 0, or at least 50% have rank 1.
Next, we recall a deep theorem of Faltings about rational points on subvarieties of abelian varieties.

**Theorem 2.4** ([Fal94, p. 175]). Let $X$ be a closed subvariety of an abelian variety $A$, with both defined over a number field $K$. Then the set $X(K)$ equals a finite union $\bigcup B_i(K)$, where each $B_i$ is a translated abelian subvariety of $A$ contained in $X$.

To conclude this section, we prove that asymptotically, 100% of hyperelliptic curves with a rational Weierstrass point over $\mathbb{Q}$ have finitely many unexpected degree $d$ points, as described in Theorem 1.4.

**Proposition 2.5.** Fix $g \geq 2$. Asymptotically, 100% of genus $g$ hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have geometrically simple Jacobian.

**Proof.** Let $t_2, \ldots, t_{2g+1}$ be indeterminates. The polynomial

$$F(x, t_2, \ldots, t_{2g+1}) = x^{2g+1} + t_2 x^{2g-1} + \cdots + t_{2g+1}$$

has Galois group $S_{2g+1}$ over the field $\mathbb{Q}(t_2, \ldots, t_{2g+1})$. One way to see this is to note that its specialization at $t_2 = \cdots = t_{2g-1} = 0$ and $t_{2g} = t_{2g+1} = -1$ gives $x^{2g+1} - x - 1$, which has Galois group $S_{2g+1}$, by [Osa87, Corollary 3] or [NV79].

By a criterion of Zarhin [Zar10, Theorem 1], the Jacobian of the hyperelliptic curve given by $y^2 = f(x)$ is geometrically simple whenever $f(x)$ has Galois group $S_{\deg f}$.

Let $\mathcal{E} = \mathcal{E}_g$ be the complement in $A^{2g}$ of the discriminant locus for equations of the form $y^2 = x^{2g+1} + a_2 x^{2g-1} + \cdots + a_{2g+1}$. We apply a version of Hilbert irreducibility (our height weights coordinates unequally, so some care must be taken); see [Coh81, Theorem 2.1], adapted as in [Coh81, Section 5, Notes (iii)]. It implies that asymptotically 100% of all the integer points in $\mathcal{E}$, when ordered by height, give hyperelliptic curves whose Jacobians are geometrically simple. A sieving argument shows that a positive proportion of the integer points of $\mathcal{E}$ give minimal equations, so asymptotically 100% of minimal equations give curves with geometrically simple Jacobians.

For a curve $C$ and a positive integer $d$, let $C^{(d)}$ denote its $d$-th symmetric product, the points of which correspond to effective degree $d$ divisors on $C$. Note that a conjugate $d$-tuple of points on $C$ gives a rational point of the symmetric product.

**Proposition 2.6.** Let $d$ be a positive integer, and let $g > d$ be an integer.

1. If $d$ is odd, then asymptotically, 100% of genus $g$ hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have finitely many degree $d$ points.

2. If $d$ is even, then asymptotically, 100% of genus $g$ hyperelliptic curves over $\mathbb{Q}$ with a rational Weierstrass point have finitely many degree $d$ points not obtained by pulling back degree $\frac{d}{2}$ points of $\mathbb{P}^1$.

**Proof.** First, note that since the map from a hyperelliptic curve $C$ to $\mathbb{P}^1$ has degree two, the image of a $d$-tuple of conjugate points on $C$ is either a $d$-tuple of conjugate points on $\mathbb{P}^1$, or possibly a $\frac{d}{2}$-tuple of conjugate points on $\mathbb{P}^1$ if $d$ is even.

We may assume $C$ has geometrically simple Jacobian $J$, by Proposition 2.5. Map the symmetric product $C^{(d)}$ to $J$ via the Abel–Jacobi map given by the rational Weierstrass point, i.e. $P_1 + \cdots + P_d \mapsto [P_1 + \cdots + P_d - d \cdot \infty]$. 

□
Since \( d < g \), the image \( W_d \) is a proper closed subvariety of \( J \). Since \( J \) is geometrically simple, \( W_d \) contains no translate of a positive-dimensional abelian subvariety of \( J \). By Theorem 2.4, \( W_d(\mathbb{Q}) \) is finite.

Lastly, we can ignore the positive-dimensional fibers of \( C^{(d)} \to J \), which correspond to effective degree \( d \) divisors \( D \) on \( C \) such that \( D \) has positive rank. On a hyperelliptic curve, any such divisor \( D \) must contain a subdivisor of the form \( P + \iota(P) \), where \( P \) is some point of \( C \) and \( \iota \) is the hyperelliptic involution, which switches points within the same fiber [ACGH85, p. 13].

But by the first paragraph, if \( d \) is odd, no \( d \)-tuple of conjugate points can make up such a \( D \). If \( d \) is even, it is only possible if

\[
D = P_1 + \iota(P_1) + \cdots + P_{\frac{d}{2}} + \iota(P_{\frac{d}{2}}),
\]

which will map to a \( \frac{d}{2} \)-tuple of conjugate points on \( \mathbb{P}^1 \). Thus in either case, the set we wish to show is finite injects into the finite set \( W_d(\mathbb{Q}) \). □

3. \( p \)-ADIC PRELIMINARIES

We recall some results on \( p \)-adic integration and the Chabauty–Coleman method; we refer the reader to [MP12, Sik09, Par16] for a fuller account of these techniques. We also prove some auxiliary lemmas.

3.1. Vanishing of integrals. Fix \( C/\mathbb{Q} \) a curve of genus \( g \geq 2 \), and \( p \) a prime number. We make use of \( p \)-adic integration on the Jacobian variety \( J \) of our curve. Let \( C_p \) be the completion of the algebraic closure of \( \mathbb{Q}_p \). After we base change from \( \mathbb{Q} \) to \( \mathbb{Q}_p \), we have an integration pairing

\[
\mathcal{H}^0(C_{\overline{\mathbb{Q}}_p}, \Omega^1) \times J(C_p) \to C_p \\
(\omega, D) \mapsto \int_0^D \omega
\]

that is \( \mathbb{Q}_p \)-linear in the left factor, and a group homomorphism in the right. The kernel on the left is trivial, and on the right is the torsion subgroup \( J(C_p)_{\text{tors}} \).

Let \( r \) be the rank of \( J(\mathbb{Q}) \) as a finitely generated abelian group (for the rest of the paper, \( r \) will denote this rank for whatever curve is under consideration). We identify \( J(\mathbb{Q}) \) with its image in \( J(\mathbb{Q}_p) \) and \( J(C_p) \). Within the former, its \( p \)-adic closure \( \overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p) \) will be a finitely generated \( \mathbb{Z}_p \)-module of rank at most \( r \). Define

\[
\Lambda_C := \left\{ \omega \in \mathcal{H}^0(C_{\overline{\mathbb{Q}}_p}, \Omega^1) \mid \int_0^D \omega = 0 \text{ for all } D \in J(\mathbb{Q}) \right\}.
\]

This is a \( \mathbb{Q}_p \)-vector space of dimension at least \( g - r \).

Suppose further that \( p \) is a prime of good reduction for our curve \( C \). For a point \( P \in C(C_p) \), let \( \overline{P} \in C_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p) \) denote its reduction at \( p \). Then given a nonzero form \( \omega \in \mathcal{H}^0(C_{\overline{\mathbb{Q}}_p}, \Omega^1) \), we can scale it by an element of \( \mathbb{Q}_p^* \) to give a normalized form, which we take to mean it reduces to a nonzero element \( \overline{\omega} \in \mathcal{H}^0(C_{\overline{\mathbb{F}}_p}, \Omega^1) \). For a normalized form
Let $\omega$, and a point $P \in C_{\bar{F}_p}([F_p])$, we define $n(\omega, P)$ to be the order of vanishing of $\omega$ at $P$. As long as $\Lambda_C \neq \{0\}$, we then define
\[
\sigma(\Lambda_C, P) = \min_{\omega \in \Lambda_C \text{ normalized}} n(\omega, P).
\]

By [Sto06, Theorem 6.4], the lower the rank is, the better we can control these minimal orders of vanishing.

**Theorem 3.2** (Stoll). Let $C/Q$ be a curve of genus $g \geq 2$, with rank $r \leq g-1$, and let $p$ be a prime of good reduction. Then $\sum_{P \in C_{\bar{F}_p}([F_p])} n(\omega, P) \leq 2r$.

For our purposes, we need forms that achieve these minima at different points simultaneously.

**Lemma 3.3.** Let $C/Q$ be a curve of genus $g \geq 2$, and let $p$ be a prime of good reduction. Let $P_1, \ldots, P_d \in C_{\bar{F}_p}([F_p])$. Suppose $r \leq g-1$ and $p \geq d$. Then there exists a normalized $\omega \in \Lambda_C$ such that $n(\omega, P_i) = n(\omega, P_i)$ for $i = 1, \ldots, d$.

**Proof.** We proceed by induction on $d$. The base case $d = 1$ is immediate from the definition of $n(\omega, P_i)$. Suppose there is a normalized form $\omega'$ such that $n(\omega, P_i) = n(\omega', P_i)$ for $i = 1, \ldots, d-1$. If $n(\omega, P_i) = n(\omega', P_i)$, we may take $\omega = \omega'$. Otherwise, choose a normalized $\omega''$ such that $n(\omega, P_i) = n(\omega'', P_i)$. If $n(\omega, P_i) = n(\omega'', P_i)$ for $i = 1, \ldots, d-1$, we may take $\omega = \omega''$.

So suppose without loss of generality that $n(\omega'', P_1) > n(\omega, P_1)$. Let $t_2, \ldots, t_{d-1}$ be uniformizers at $P_2, \ldots, P_{d-1}$, respectively. Write both $\omega'$ and $\omega''$ with respect to each uniformizer:
\[
\overline{\omega'} = a_i t_i^{n_i} dt_i,
\overline{\omega''} = b_i t_i^{n_i} dt_i, \text{ for } i = 2, \ldots, d-1,
\]
where for each $a_i, b_i \in \bar{F}_p(C_{\bar{F}_p})$, the geometric function field of the reduction, we have $0 = v_{P_i}(a_i) \leq v_{P_i}(b_i)$. Since $p \geq d$, there exists $0 \neq \alpha \in \bar{F}_p$ such that $\alpha \cdot b_i(P_1) \neq -a_i(P_i)$ for $i = 2, \ldots, d-1$. Choosing any $\check{\alpha} \in Z_p$ whose reduction mod $p$ is $\alpha$, we may take $\omega = \omega' + \check{\alpha} \omega''$. \hfill \Box

**Lemma 3.4.** Let $C/Q$ be a curve of genus $g \geq 2$, and let $p$ be a prime of good reduction. Let $P_1, \ldots, P_d \in C_{\bar{F}_p}([F_p])$, and suppose $r \leq g-1$. Then there exist linearly independent, normalized $\omega_1, \ldots, \omega_d \in \Lambda_C$ such that $n(\omega, P_i) = n(\omega_i, P_i)$ for all $i, j = 1, \ldots, d$.

**Proof.** Take $\omega_1$ to be $\omega$ as given by Lemma 3.3. By the rank condition, we can choose $\omega_2', \ldots, \omega_d' \in \Lambda_C$ to be normalized forms such that $\omega_1, \omega_2', \ldots, \omega_d'$ are linearly independent. Then each reduction $\overline{\omega_1 + p \omega_j'} = \overline{\omega_1}$, so we can take $\omega_j = \omega_1 + p \omega_j'$ for $j = 2, \ldots, d$. \hfill \Box

3.5. **Vanishing of locally analytic functions.** Now let $C/Q$ be a genus $g$ curve with a marked rational point, which we denote by $\infty$. For any $\omega \in H^0(C_{\bar{Q}_p}, \Omega^1)$, we define
(locally analytic) functions
\[ f_\omega: C(\mathbb{C}_p) \rightarrow \mathbb{C}_p \]
\[ P \mapsto \int_0^{[P-\infty]} \omega, \]
and more generally for \( d \) a positive integer,
\[ f_\omega^d: C(\mathbb{C}_p) \times \cdots \times C(\mathbb{C}_p) \rightarrow \mathbb{C}_p \]
\[ (P_1, \ldots, P_d) \mapsto f_\omega(P_1) + \cdots + f_\omega(P_d) = \int_0^{[P_1+\cdots+P_d-d\infty]} \omega. \]

The starting point of the Chabauty–Coleman method for examining rational points is that if \( \omega \in \Lambda_C \), then for any \( P \in C(Q) \), we have \( f_\omega(P) = 0 \), because \([P - \infty] \in J(Q)\). The starting point for our method, following [Sik09, Par16], is that for \( \omega \in \Lambda_C \) and \((P_1, \ldots, P_d)\) a \( d \)-tuple of conjugate degree \( d \) points on \( C \), we have \( f_\omega^d(P_1, \ldots, P_d) = 0 \), since \([P_1 + \cdots + P_d - d\infty] \in J(Q)\).

We wish to control these zeros. For \( \overline{P} \in C_{\mathbb{F}_p}(\overline{\mathbb{Q}_p}) \), define the residue disk
\[ D_{\overline{P}} = \{ Q \in C(\mathbb{C}_p) \mid \overline{Q} = \overline{P} \}. \]

Let \( D \subset \mathbb{C}_p \) be the open unit disk, i.e. elements with absolute value strictly less than 1. For \( P \in C(\overline{\mathbb{Q}_p}) \), [Sik09, Lemma 2.3] asserts that we can always choose a well-behaved uniformizer \( z_p \) at \( P \), which has the following key properties. First, the function \( z_p: D_{\overline{P}} \rightarrow D \) is a diffeomorphism, with \( z_p(P) = 0 \). Furthermore, for a finite extension \( L/\mathbb{Q}_p(P) \), with uniformizing element \( \pi \), we have that \( z_p \) defines a bijection between \( C(L) \cap D_{\overline{P}} \) and the \( \pi \)-adic disc \( \pi O_L \), given by \( Q \mapsto z_p(Q) \).

**Remark 3.6.** Let \( v \) be the valuation on \( \mathbb{C}_p \), normalized so that \( v(p) = 1 \). For \( P, Q \in C(\overline{\mathbb{Q}_p}) \) such that \( \overline{P} = \overline{Q} \), let \( e \) be the ramification degree of \( Q_p(P, Q) \). The above implies that \( v(z_p(Q)) \geq \frac{1}{e} \).

We can formally expand a normalized form \( \omega \) with respect to the uniformizer \( z_p \), as
\[ \left( \sum_{i=0}^{\infty} a_i z_p^i \right) dz_p, \]
where the coefficients live in \( \mathbb{Q}_p(P) \), and are integral (\( v(a_i) \geq 0 \) for all \( i \)). We record a few important facts from [Sik09, Section 2] about this expansion. First, the power series \( \sum_{i=0}^{\infty} a_i t^i \) is convergent on \( D \). Second, there is a connection to orders of vanishing: the smallest index \( i \) for which we have \( v(a_i) = 0 \) is given by \( i = n(\omega, \overline{P}) \). Lastly, for \( Q \in D_{\overline{P}} \), the restriction of \( f_\omega \) to \( D_{\overline{P}} \) is given by
\[ f_\omega(Q) = \int_0^{[P-\infty]} \omega + \sum_{i=0}^{\infty} \frac{a_i}{i+1} z_p(Q)^{i+1}. \]

Similarly, for \( P_1, P_2 \in C(\overline{\mathbb{Q}_p}) \), the restriction of \( f_\omega^2 \) to \( D_{\overline{P}_1} \times D_{\overline{P}_2} \) is given by
\[ f_\omega^2(Q_1, Q_2) = \int_0^{[P_1+P_2-2\infty]} \omega + \sum_{i=0}^{\infty} \frac{a_i}{i+1} z_{P_1}(Q_1)^{i+1} + \sum_{i=0}^{\infty} \frac{b_i}{i+1} z_{P_2}(Q_2)^{i+1}. \]
Analogous expansions of course hold for $F^d_{\omega}$, for arbitrary $d$.

4. Analytic loci for low-rank hyperelliptic curves

The most general results of [Par16] are conditional on a technical assumption (loc. cit. Assumption 1.3) involving excess analytic intersection of the zero loci of the $F^d_{\omega}$'s for $\omega \in \Lambda_C$. In this section, we expound on Park’s observation (loc. cit. p. 2) that this assumption is always satisfied when $r \leq 1$; to ease notation and terminology, we restrict to the hyperelliptic setting.

Fix a hyperelliptic curve $C/\mathbb{Q}$ of genus $g \geq 3$, with a rational Weierstrass point $\infty$, and embed $C$ in its Jacobian $J$ via the Abel–Jacobi map $C \to J$ given by $\infty$. Let $p$ be a prime of good reduction for $C$. Let $W_d := C + \cdots + C \subset J$ be the image of all degree $d$ effective divisors, and let $\Lambda_C$ be as in Section 3.1. Define $J^{\Lambda_c}$ to be the kernel of pairing with elements of $\Lambda_C$, i.e.

$$J^{\Lambda_C} := \left\{ D \in J(\mathbb{C}_p) \mid \int_0^D \omega = 0, \forall \omega \in \Lambda_C \right\}.$$

Note that $J^{\Lambda_c}$ is also a $\mathbb{C}_p$-analytic manifold (in the sense of Bourbaki and Serre [Ser06, Chapter III]), and in fact a $p$-adic Lie group. If we assume that $J(\mathbb{Q})$ has rank $\leq 1$, then $J^{\Lambda_c}$ has dimension 0 or 1 (as a manifold). The next two lemmas use the topology on $J(\mathbb{C}_p)$ and $C(\mathbb{C}_p) \times \cdots \times C(\mathbb{C}_p)$ given by their structures as $\mathbb{C}_p$-analytic manifolds, unless otherwise noted.

**Lemma 4.1.** Let $0 < d < g$ be integers. Assume that $J$ is geometrically simple and that $J(\mathbb{Q})$ has rank $\leq 1$. Then $J^{\Lambda_c} \cap W_d$ consists of isolated points.

**Proof.** If $J^{\Lambda_c}$ is 0-dimensional, then we are done since $J^{\Lambda_c}$ is a closed subset of $J$.

If $J^{\Lambda_c}$ is 1-dimensional, let $P \in J^{\Lambda_c} \cap W_d$. We can choose a closed neighborhood $V$ of $P$ such that $V \cap J^{\Lambda_c}$ is diffeomorphic to a closed disk in $\mathbb{C}_p$ via [Ser06, Chapter III, Section 3]. Since $W_d$ is a closed set, $V \cap J^{\Lambda_c} \cap W_d$ is given by the vanishing of a convergent 1-variable power series on this disk. Thus, $V \cap J^{\Lambda_c} \cap W_d$ is either a finite set of points or all of $V \cap J^{\Lambda_c}$.

But in the latter case, note that $V \cap J^{\Lambda_c} \cap W_d$ is a translate of a closed disk centered at the origin, which makes it a translate of an infinite subgroup of $J^{\Lambda_c}$. Its Zariski closure would then be a translate of a positive-dimensional abelian subvariety of $J$ contained in $W_d$, but this contradicts our initial assumption that $J$ is geometrically simple. Therefore, $V \cap J^{\Lambda_c} \cap W_d$ is a finite set of points, so $P$ is isolated. \qed

Let

$$(C^d)^{\Lambda_c} := \left\{ (P_1, \ldots, P_d) \in C(\mathbb{C}_p) \times \cdots \times C(\mathbb{C}_p) \mid F^d_{\omega}(P_1, \ldots, P_d) = 0, \forall \omega \in \Lambda_C \right\},$$

the inverse image of $J^{\Lambda_c}$ in $C^d$. For any subset $S \subset \Lambda_C$, let $(C^d)^S$ similarly denote pairs satisfying the condition for all $\omega \in S$.

**Lemma 4.2.** Let $0 < d < g$ be integers. Assume that $J$ is geometrically simple and that $J(\mathbb{Q})$ has rank $\leq 1$. Let $P = (P_1, \ldots, P_d)$ be a point of $C^d(\mathbb{C}_p)$ such that the divisor $P_1 + \cdots + P_d$ is non-special. Then $P$ is an isolated point of $(C^d)^{\Lambda_c}$.
Proof. The set of \(d\)-tuples in \(\mathbb{C}^d(\mathbb{C}_p)\) which give special divisors is a closed subset. The result then follows from Lemma 4.1 and the fact that \(\mathbb{C} \times \cdots \times \mathbb{C}\) is Hausdorff in its topology as a \(\mathbb{C}_p\)-analytic manifold.

To conclude this section, we consider the locally affinoid structure of \((\mathbb{C}^d)^{\Lambda_C}\).

**Definition 4.3.** For \(P \in \mathbb{C}(\mathbb{C}_p), z_P\) a well-behaved uniformizer at \(P\), and \(m > 0\), let

\[ B_m(P, z_P) := \{ Q \in \mathbb{C}(\mathbb{C}_p) \mid v(z_P(Q)) \geq m \}. \]

Since \(F^d_c\) has a convergent power series expression on the entire open polydisk \(D_{P_1} \times \cdots \times D_{P_d}\), on any closed sub-polydisk it will actually give an element of that sub-polydisk’s affinoid coordinate ring, which is a Tate algebra [BGR84, Section 7.1.1]. For any choices of \(P_1, \ldots, P_d\) and \(z_{P_1}, \ldots, z_{P_d}\) and \(m > 0\), the set

\[ (\mathbb{C}^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})) \]

will have finitely many irreducible components as an affinoid subset of \(B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})\), by [BGR84, Sect. 7.1.1, Cor. 8]. These components can be zero-dimensional or positive-dimensional.

**Lemma 4.4.** Let \(C/Q\) be a curve of genus \(g \geq 2\), let \(d\) be a positive integer, and let \(P\) be a prime of good reduction for \(C\). Suppose \(C\) has rank \(r \leq g - d\). Let \(P_i, \ldots, P_d \in \mathbb{C}(\mathbb{C}_p)\), let \(z_{P_i}\) be a well-behaved uniformizer at \(P_i\) for \(i = 1, \ldots, d\), and let \(m > 0\). Then we can choose \(\omega_1, \ldots, \omega_d \in \Lambda_C\) as in Lemma 3.4 such that furthermore the zero set

\[ (\mathbb{C}^d)^{\{\omega_1, \ldots, \omega_d\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})) \]

has the same positive-dimensional components as

\[ (\mathbb{C}^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})). \]

**Proof.** The proof is similar to [Par16, Proposition 5.7], and proceeds by induction.

**Claim:** For \(k = 1, \ldots, d\), we can choose \(\omega_1, \ldots, \omega_k\) as in Lemma 3.4 such that the set of components of codimension at most \(k - 1\) for

\[ (\mathbb{C}^d)^{\{\omega_1, \ldots, \omega_k\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})) \]

and

\[ (\mathbb{C}^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})) \]

are the same.

**Proof of Claim.** The base case \(k = 1\) is trivial. For the induction step, suppose it holds for a given value of \(k\), for forms \(\omega_1, \ldots, \omega_k\). Choose \(\omega_{k+1}\) linearly independent from \(\omega_1, \ldots, \omega_k\), and as in Lemma 3.3, such that

\[ (\mathbb{C}^d)^{\{\omega_1, \ldots, \omega_{k+1}\}} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})) \]

has the minimal number of codimension \(k\) components \(V_1, \ldots, V_s\) for any such choice of \(\omega_{k+1}\).

Suppose the conclusion of the claim is false for \(k + 1\). Then without loss of generality, we may assume that \(V_s\) is not a component of \((\mathbb{C}^d)^{\Lambda_C} \cap (B_m(P_1, z_{P_1}) \times \cdots \times B_m(P_d, z_{P_d})).\)
Let $V_{s+1}, \ldots, V_t$ be any codimension $k$ components of $(C^d)^{(\omega_1, \ldots, \omega_k)} \cap (B_m(P_1, z_{p_1}) \times \cdots \times B_m(P_d, z_{p_d}))$ that are distinct from each $V_1, \ldots, V_s$.

Since $V_s$ is not a component of $(C^d)^{\Lambda_C} \cap (B_m(P_1, z_{p_1}) \times \cdots \times B_m(P_d, z_{p_d}))$, we may choose a normalized form $\omega' \in \Lambda_C$ such that $F_{\omega'}$ does not vanish at some point $R_s$ of $V_s$. Then for any integer $n$, we have that $F_{\omega_{k+1}+p^n\omega'}$ does not vanish identically on $V_s$, since $F_{\omega_{k+1}}$ does but $F_{p^n\omega'} = p^nF_{\omega'}$ does not.

For each $i = s+1, \ldots, t$, we can choose a point $R_i$ on $V_i$ at which $F_{\omega_{k+1}}$ does not vanish. Choose $n$ to be a sufficiently large positive integer such that $v(F_{p^n\omega'}(R_i)) = n + v(F_{\omega'}(R_i)) > v(F_{\omega_{k+1}}(R_i))$ for each $i = s+1, \ldots, t$. Then $F_{\omega_{k+1}+p^n\omega'}$ does not vanish identically on any of $V_{s+1}, \ldots, V_t$.

Note that $\omega_{k+1} + p^n\omega'$ has the same reduction as $\omega_{k+1}$. Also, it is linearly independent from $\omega_1, \ldots, \omega_k$, since $F_{\omega_1}, \ldots, F_{\omega_k}$ vanish at $R_s$ and our integration pairing is $\mathbb{Q}_p$-linear. But by construction, the codimension $k$ components of $(C^d)^{(\omega_1, \ldots, \omega_k, \omega_{k+1}+p^n\omega')} \cap (B_m(P_1, z_{p_1}) \times \cdots \times B_m(P_d, z_{p_d}))$ are contained in $\{V_1, \ldots, V_{s-1}\}$, which contradicts the minimality of $\omega_{k+1}$.

The lemma is the $k = d$ case of the claim.

5. Bounding the number of unexpected degree $d$ points

In this section, we prove the first two statements of Theorem 1.4. We refer the reader to Subsection 1.6 for the definition of unexpected degree $d$ points.

To begin, let $(P_1, \ldots, P_d)$ be a conjugate $d$-tuple of degree $d$ points.

**Lemma 5.1.** Let $d > 1$ be a positive integer, let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g > d$, with a rational Weierstrass point, geometrically simple Jacobian with $\tau \leq 1$, and let $p$ be an odd prime of good reduction for $C$. Let $P_1, \ldots, P_d \in C(\overline{\mathbb{Q}})$ be a conjugate $d$-tuple with well-behaved uniformizers $z_{p_1}, \ldots, z_{p_d}$. Let $(Q_1, \ldots, Q_d)$ be a $d$-tuple of unexpected conjugate degree $d$ points with the same reduction as $(P_1, \ldots, P_d)$ modulo $p$.

Then $\{(Q_1, \ldots, Q_d)\}$ is a zero-dimensional component of

$$(C^d)^{\Lambda_C} \cap (B_{\bar{e}}(P_1, z_{p_1}) \times \cdots \times B_{\bar{e}}(P_d, z_{p_d}))$$

where $e$ is the ramification index of $\mathbb{Q}_p(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$.

**Proof.** By Remark 3.6, $\{(Q_1, \ldots, Q_d)\}$ is contained in

$$B_{\bar{e}}(P_1, z_{p_1}) \times \cdots \times B_{\bar{e}}(P_d, z_{p_d}).$$

By Lemma 4.2, we have that $\{(Q_1, \ldots, Q_d)\}$ is a zero-dimensional component of

$$(C^d)^{\Lambda_C} \cap (B_{\bar{e}}(P_1, z_{p_1}) \times \cdots \times B_{\bar{e}}(P_d, z_{p_d})).$$

In light of Lemma 5.1, we work to bound the number of these zero-dimensional components. We use the convention that $v(0) = \infty$. For a positive rational number $m$, let $D_m = \{\alpha \in \mathbb{C}_p | v(\alpha) \geq m\}$, let $D_m^d$ denote its $d$-fold product, and let $C_p\langle D_m^d \rangle$ denote the Tate algebra of functions in its affinoid coordinate ring. The key tool we use to control zero-dimensional components comes from Park [Par16] and builds off results of Rabinoff.
[Rab12] in tropical deformation and tropical intersection theory. We will need some definitions before we can state the theorem. For a set $S \subset \mathbb{R}^d$, let $\text{conv}(S)$ denote its convex hull.

A necessary condition for a series to sum to 0 in a non-Archimedean field is that it has a term of minimal valuation, and that this term is not unique. In the below definition, the relevant $(w_1, \ldots, w_d)$ can thus be thought of as candidates for the coordinate-wise valuations of zeros for the power series $F$, where we only look for zeros whose coordinates have valuation at least $m$, and the $(u_1, \ldots, u_d)$ are the multi-indices of terms that could have minimal valuation after plugging in such a zero.

**Definition 5.2.** Let $m$ be a positive rational number, and let $F(t_1, \ldots, t_d) = \sum_{u \in \mathbb{Z}_{\geq 0}^d} a_u t^u \in C_p(D_m^d)$. We define the Newton polygon of $F$ (with respect to $m$) to be the set $\text{New}_m(F) \subset \mathbb{R}^d$ given by

$$\text{New}_m(F) := \text{conv}(\{u = (u_1, \ldots, u_d) \in \mathbb{Z}_{\geq 0}^d | \exists (w_1, \ldots, w_d) \in \mathbb{Q}^d \text{ with each } w_i \geq m \text{ s.t.}
\begin{align*}
&\exists u' \in \mathbb{Z}_{\geq 0}^d \text{ with } u \neq u' \text{ and } v(a_u) + \sum_i w_i u_i = v(a_{u'}) + \sum_i w_i u'_i, \\
&\text{and } \forall u'' \in \mathbb{Z}_{\geq 0}^d, v(a_u) + \sum_i w_i u_i \leq v(a_{u''}) + \sum_i w_i u''_i\}. \tag{5.1}
\end{align*}$$

A polytope is the convex hull of finitely many points of a Euclidean space. We need to define the mixed volume of a collection of polytopes in a given Euclidean space. For polytopes $Z_1, \ldots, Z_d \subset \mathbb{R}^d$ and positive real numbers $\lambda_1, \ldots, \lambda_d$, the volume of the scaled Minkowski sum $\lambda_1 Z_1 + \cdots + \lambda_d Z_d = \{ \lambda_1 z_1 + \cdots + \lambda_d z_d | z_i \in Z_i \}$ is known to be given by a homogeneous polynomial of degree $d$ in the coefficients $\lambda_1, \ldots, \lambda_d$. The mixed volume of the polytopes, denoted $\text{MV}(Z_1, \ldots, Z_d)$, is defined to be the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_d$ in that polynomial.

**Example 5.3.** For $a \geq 1$ and $d$ a positive integer, let

$$Z_1 = \cdots = Z_d = \text{conv}(\{e_1, \ldots, e_d, ae_1, \ldots, ae_d\}) \subset \mathbb{R}^d,$$

where $e_i$ is the vector with a 1 in the $i$-th place and 0’s elsewhere. Then

$$\lambda_1 Z_1 + \cdots + \lambda_d Z_d = \text{conv}(\{\lambda_1 + \cdots + \lambda_d | e_1, \ldots, e_d, ae_1, \ldots, ae_d\}),$$

and a quick calculation gives $\text{MV}(Z_1, \ldots, Z_d) = a^d - 1$.

**Theorem 5.4 ([Par16, Theorem 4.18 & Proposition 5.7]).** Let $m$ be a positive rational number, and let $F_1, \ldots, F_d \in C_p(D_m^d)$ be power series such that for any $i, j \in \{1, \ldots, d\}$, $F_i$ has a term of the form $c t_j^{N_i}$, with $c \neq 0$ and $N_i > 0$. Then the number of zero-dimensional components of the zero locus $V(F_1, \ldots, F_d)$ in $D_m^d \cap (C_p^\circ)^d$ is at most $\text{MV}(\text{New}_m(F_1), \ldots, \text{New}_m(F_d))$.

Next, we use Theorem 5.4 to bound the maximal number of zero-dimensional components mentioned in Lemma 5.1. Let $e_d$ be the ramification index of the compositum of the finitely many degree $d$ extensions of $Q_p$ (cf. [Ser97, Chap. III, §4.2]).

**Lemma 5.5.** Let $d > 1$ be a positive integer and let $C/Q$ be a hyperelliptic curve of genus $g > d$, with a rational Weierstrass point and geometrically simple Jacobian with $r \leq 1$. Suppose that $p > e_d + 3$ is an prime of good reduction for $C$. Let $P_1, \ldots, P_d \in C^d(\overline{Q})$ be a conjugate
d-tuple with well-behaved uniformizers $z_{P_1}, \ldots, z_{P_d}$. Then there are at most $3^d$ ordered tuples of unexpected conjugate degree $d$ points in $D_{P_1} \times \cdots \times D_{P_d}$, i.e. with the same reduction as $(P_1, \ldots, P_d)$ modulo $p$.

**Proof.** Using Lemmas 3.4 and 4.4, choose linearly independent, normalized forms $\omega_1, \ldots, \omega_d \in \Lambda_C$ such that $n(\omega_i, P_j) = n(\Lambda_C, P_j)$ for $i, j \in \{1, \ldots, d\}$. By Theorem 3.2, these numbers are at most 2.

Viewed as a function on $D^d$ via $z_{P_1}, \ldots, z_{P_d}$, $F_{\omega_1}^d$ is given by

$$
\sum_{i=0}^{\infty} \frac{a_{1,i}}{i+1} t_1^{i+1} + \cdots + \sum_{i=0}^{\infty} \frac{a_{d,i}}{i+1} t_d^{i+1}.
$$

For each $i \in \{1, \ldots, d\}$, we have that $v(a_{i,j}) = 0$ for some $j \leq 2$. Similar statements holds for $F_{\omega_2}^d, \ldots, F_{\omega_d}^d$.

By a standard Newton polygon argument (cf. proof of [Sto06, Lemma 6.1 & Proposition 6.3] using our assumption that $p > e_d + 3$) applied to each of these sums, we see that New$_{1/e}(F_{\omega_1}^d), \text{New}_{1/e}(F_{\omega_2}^d), \ldots, \text{New}_{1/e}(F_{\omega_d}^d)$ are contained in the set $\text{conv}([[1,0, \ldots, 0), (3,0, \ldots, 0), (0,1, \ldots, 0), (0,3, \ldots, 0), \ldots, (0,0, \ldots, 1), (0,0, \ldots, 3)]) \subset \mathbb{R}^d$.

By Theorem 5.4 and Example 5.3, there are at most $3^d - 1$ zero-dimensional components of interest away from $(P_1, \ldots, P_d)$. Since $(P_1, \ldots, P_d)$ could be a d-tuple of unexpected conjugate degree $d$ points, we have at most $3^d$ zero-dimensional components in $D^d$. We are now done by Lemmas 4.4 and 5.1. □

**Proof of Theorem 1.4.** Choose a prime $p > e_d + 3$. Among all hyperelliptic curves of genus $g$ with a rational Weierstrass point, those with good reduction at $p$ are defined by finitely many congruence conditions on their minimal equations, and thus constitute a positive proportion of all such curves. Proposition 2.5 tells us that asymptotically, 100% of curves in this subfamily have geometrically simple Jacobian, and by Corollary 2.3, at least 25% of these curves also have Jacobian rank $r \leq 1$.

Let $C$ be such a curve of genus $g$. Given a d-tuple $(Q_1, \ldots, Q_d)$ of conjugate degree $d$ points, the reduction of each $Q_i$ is certainly contained in $C_{F_p}(\mathbb{F}_{p^{m_i}})$ for some $1 \leq m_i \leq d$. Note that since $C$ is an odd hyperelliptic curve with good reduction at $p$, the size of $C_{F_p}(\mathbb{F}_{p^{m_i}})$ is less than or equal to $2p^{m_i} + 1$. Crudely, there are at most $(d \cdot (2p^d + 1))^d$ possible reductions for $(Q_1, \ldots, Q_d)$ modulo $p$. By Lemma 5.5, there are at most $3^d$ ordered tuples of unexpected conjugate degree $d$ points with each reduction. Thus, we may take

$$B_d = (3d \cdot (2p^d + 1))^d.$$ □

6. **Explicit bounds on the number of unexpected quadratic points**

In this section, we prove Theorem 1.2.

**Lemma 6.1.** Let $C/Q$ be a hyperelliptic curve of genus $g \geq 3$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, let $p$ be an odd prime of good reduction for $C$. Let $P_1, P_2 \in C(\overline{Q})$ be either two rational points or a pair of conjugate quadratic points, with
well-behaved uniformizers $z_{p_1}, z_{p_2}$. Let $(Q_1, Q_2)$ be a pair of unexpected conjugate quadratic points with the same reduction as $(P_1, P_2)$. Then $\{(Q_1, Q_2)\}$ is a zero-dimensional component of $(C^2)^{\Lambda_C} \cap (B_1(P_1, z_{p_1}) \times B_2(P_2, z_{p_2}))$.

Proof. The field $Q_p(P_1, P_2, Q_1, Q_2)$ is certainly contained in the compositum of the three quadratic extensions of $Q_p$. Since $p$ is odd, that compositum has ramification degree $e = 2$ over $Q_p$. The result now follows from Lemma 5.1. □

Lemma 6.2. Suppose $p \neq 2$ or 5, and $\sum_{i=0}^{\infty} a_i t^i \in C_p[t]$ is a power series with integral coefficients. If $v(a_0) = 0$, then for $f(t) = \sum_{i=0}^{\infty} a_i t^{i+1}$, we have $\text{New}_1(f) = \emptyset$ and $\text{New}_{1/2}(f) \subset [1, 3]$. If $v(a_1) = 0$ or $v(a_2) = 0$, then $\text{New}_{1/2}(f) \subset [1, 3]$.

Proof. We begin with the case where $v(a_0) = 0$. Then for any $w \geq 1$ and $i > 0$, we have $v(a_i) + 1 \cdot w = w < v\left(\frac{a_i}{i+1}\right) + (i+1) \cdot w$. To see this, note that the right-hand side is no smaller than $-v(i+1) + (i+1)w$, and $v(i+1)$ is strictly less than $i$ for $i > 0$ and $p > 2$.

For $w \geq \frac{1}{2}$, things can be different: for example, suppose $p = 3$, $v(a_2) = 0$, and $w = \frac{1}{2}$. Then $\frac{1}{2} = v\left(\frac{a_2}{2}\right) + 3 \cdot w = v\left(\frac{a_2}{2}\right) + 1 \cdot w$. But past $i = 2$, strict inequality holds, so we have $\text{New}_{1/2}(f) \subset [1, 3]$. The rest of the cases proceed similarly. □

Lemma 6.3. Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g \geq 3$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, and good reduction at 3. Let $P_1, P_2 \in C(\overline{\mathbb{Q}})$ be either two rational points or a pair of conjugate quadratic points, with $\overline{P_1}, \overline{P_2} \in C_{\mathbb{F}_3}(\mathbb{F}_3)$. Then there are at most 8 ordered pairs $(Q_1, Q_2)$ of unexpected conjugate quadratic points in $D_{\overline{P_1}} \times D_{\overline{P_2}}$ that are not equal to $(P_1, P_2)$.

Proof. Using Lemmas 3.4 and 4.4, choose linearly independent, normalized forms $\omega_1, \omega_2 \in \Lambda_C$ such that $n(\omega_i, P_j) = n(\Lambda_C, P_j)$ for $i, j \in \{1, 2\}$. By Theorem 3.2, these numbers are at most 2. Viewed as a function on $D^2$ via $z_{p_1}$ and $z_{p_2}$, $F_{\omega_1}^2$ is given by

$$\sum_{i=0}^{\infty} a_i t_i^{i+1} + \sum_{i=0}^{\infty} b_i t_i^{i+1}.$$ 

A similar statement holds for $F_{\omega_2}^2$.

By construction of $\omega_1$ (and similarly for $\omega_2$), we have $v(a_i) = 0$ for some $i \leq 2$, and $v(b_j) = 0$ for some $j \leq 2$. Now by Lemma 6.2 applied to each of these two sums, we see that both $\text{New}_{1/2}(F_{\omega_1}^2)$ and $\text{New}_{1/2}(F_{\omega_2}^2)$ are contained in the set

$$\text{conv}(\{(1, 0), (3, 0), (0, 1), (0, 3)\}) \subset \mathbb{R}^2.$$ 

By Theorem 5.4 and Example 5.3, there are at most $3^2 - 1 = 8$ zero-dimensional components of interest away from $(P_1, P_2)$. We are done by Lemmas 4.4 and 6.1. □

Lemma 6.4. Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus $g \geq 3$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, and good reduction at 3. Let $P_1, P_2 \in C(\overline{\mathbb{Q}})$ be a pair of conjugate quadratic points, with $\overline{P_1}, \overline{P_2} \in C_{\mathbb{F}_3}(\mathbb{F}_3) \setminus C_{\mathbb{F}_3}(\mathbb{F}_3)$. If $n(\Lambda_C, \overline{P_1}) = 1$, there are at most 8 pairs $(Q_1, Q_2)$ of unexpected conjugate quadratic points in $D_{\overline{P_1}} \times D_{\overline{P_2}}$ that are not equal to $(P_1, P_2)$. If $n(\Lambda_C, \overline{P_1}) = 0$, there are no such pairs.
Proof. The proof is similar to that of Lemma 6.3, with two changes. First, one can consider New1 instead of New1/2, using Remark 3.6 and the fact that Q3(P1, Q1) is unramified. Second, note that since P1 and P2 reduce to (necessarily distinct) points outside of CF3(F3), we know that P1 and P2 remain conjugate over Q3. Thus their reductions are conjugate over F3, so we have n(ΛC, P1) = n(ΛC, P2). By Theorem 3.2, this common value can only be 0 or 1.

□

Lemma 6.5. For each g ≥ 3, there exists a congruence family of genus g hyperelliptic curves with a rational Weierstrass point, such that any curve C in the family has good reduction at 3, and satisfies CF3(F3) = {∞} and CF3(F9) = (∞, (0, ±α), (1, ±α), (2, ±α)), with α ∈ (F9 \ F3).

Proof. For each g, consider the families of hyperelliptic curves whose reduction mod 3 is given by:

\[
\begin{align*}
y^2 &= f_g(x) = x^{2g+1} + 2x^9 + 2 & \text{for } g \equiv 1 \pmod{4}, \\
y^2 &= f_g(x) = x^{2g+1} + 2x^{15} + 2 & \text{for } g \equiv 2 \pmod{4}, \\
y^2 &= f_g(x) = x^{2g+1} + 2x^5 + 2 & \text{for } g \equiv 3 \pmod{4}, \\
y^2 &= f_g(x) = x^{2g+1} + x^3 + x + 2 & \text{for } g \equiv 0 \pmod{4} \text{ and } g \equiv 0, 1 \pmod{3}, \\
y^2 &= f_g(x) = x^{2g+1} + x^7 + x^3 + 2 & \text{for } g \equiv 0 \pmod{4} \text{ and } g \equiv 2 \pmod{3}.
\end{align*}
\]

For g = 3, one checks directly that f3(x) represents as few squares as is possible: f(F3) = {2}, and f(F9 \ F3) ⊂ (F9 \ F92). This is equivalent to the lemma’s condition on F3- and F9-points. In general, for g ≡ 3 mod 4, note that f3(x) = x^{2g+1} + 2x^5 + 2 defines the exact same function as x^7 + 2x^5 + 2 on F9, as x^k and x^{k+8r} define the same function on F9 for natural numbers k, r. The same argument works for the other values of g.

We conclude by showing that these polynomials are square-free over F3. Over any field, the discriminant of a trinomial x^n + ax^k + b is

\[
(-1)^{n(n-1)/2}b^{k-1}(n^{n_1}b^{n_1-k_1} + (-1)^{n_1+k_1}(n-k)^{n_1-k_1}k^{k_1}a^{n_1})^d,
\]

where d = (n, k) and n = n_1d, k = k_1d [Swa62, Theorem 2]. Thus for g ≡ 1, 2, 3 mod 4, the discriminant of f_g in F3 is

\[
\pm(2g+1) + (2g+1-k)^2k \cdot 2,
\]

with k = 1, 7, or 5, respectively. This is non-zero in F3 for any value of g.

For g ≡ 0 mod 4, we use a different method. It suffices to show f_g(x) and f'_g(x) have no common factors. If g ≡ 1 mod 3, then f'_g(x) = 1, so this is clear. If g ≡ 2 mod 3, then f'_g(x) = -x^{2g}, and so this also is clear. Lastly if g ≡ 0 mod 3, then f'_g(x) = x^{2g} + 1, so any common factor would divide f_g(x) - xf'_g(x) = x^3 + 2 = (x + 2)^3. Thus we see f_g and f'_g are coprime.

□

Proof of Theorem 1.2. For a given value of g, the family given by Proposition 2.5 and Lemma 6.5 comprises a positive proportion of all hyperelliptic curves with a rational Weierstrass point, since the latter is defined by finitely many congruence conditions. By Corollary 2.3, at least 25% of the curves in this family have rank r ≤ 1; this is still a positive proportion of all the curves.
Let C be such a curve. Any pair of conjugate quadratic points on C that reduce to \( \mathbb{F}_3 \)-points will have to lie in \( D_{\infty} \times D_{\infty} \). We may choose \( P_1 = P_2 = \infty \) and apply Lemma 6.3 to conclude there are at most eight ordered pairs \( (Q_1, Q_2) \) of unexpected conjugate quadratic points in this residue class. But since \( Q_1 = Q_2 \), each unordered pair is counted twice, so there are at most four unordered pairs of such quadratic points.

If the minimal Weierstrass model of C is \( y^2 = f(x) \), note that (for some choice of square root) the pair of expected quadratic points \( (i, \pm \sqrt{f(i)}) \) reduces to \( (i, \pm \alpha) \) for each \( i = 0, 1, 2 \). In \( D_{(i, \alpha)} \times D_{(i, -\alpha)} \), we may choose \( P_1 = (i, \sqrt{f(i)}), P_2 = (i, -\sqrt{f(i)}) \), and apply Lemma 6.4. If \( n(\Lambda_C, (i, \alpha)) = 0 \), we conclude there are no unexpected pairs in this residue class. If \( n(\Lambda_C, (i, \alpha)) = 1 \), there are at most eight.

By Theorem 3.2, at most one value of \( i = 0, 1, 2 \) will have \( n(\Lambda_C, (i, \alpha)) = n(\Lambda_C, (i, -\alpha)) = 1 \), and all the others will be 0. Thus, there are at most \( 4 \times 8 = 12 \) unordered pairs of unexpected conjugate quadratic points.

7. Explicit bounds on the number of cubic points

In this section, we prove Theorem 1.3.

Lemma 7.1. Suppose \( \sum_{i=0}^{\infty} a_i t^i \in \mathbb{Q}_3[[t]] \) is a power series with integral coefficients. If \( v(a_0) = 0, v(a_1) = 0, \) or \( v(a_2) = 0, \) then for \( f(t) = \sum_{i=0}^{\infty} a_i t^{i+1} \), we have \( \text{New}_{1/3}(f) \subset \{1, 3\} \).

Proof. We proceed as in Lemma 6.2. If \( v(a_0) = 0 \), then for \( w \geq 1/3 \) and \( i > 2 \), we have that \( v(\frac{a_i}{w^{i+1}}) + 1 \cdot w = w < v(\frac{a_i}{t^{i+1}}) + (i + 1) \cdot w \). Recall that the right-hand side is no smaller than \( -v(i + 1) + (i + 1)w \), so it suffices to prove that for \( i > 2 \), \( w < -v(i + 1) + (i + 1)w \). A short induction argument on \( i \) using the non-Archimedean properties of \( v \) and taking \( w = 1/3 \) yields the desired result. The remaining cases follow in a similar fashion.

Lemma 7.2. Let \( C/\mathbb{Q} \) be a hyperelliptic curve of genus \( g \geq 4 \), with a rational Weierstrass point, geometrically simple Jacobian with \( r \leq 1 \), and good reduction at 3. Let \( P_1, P_2, P_3 \in C(\mathbb{Q}) \) be three rational points. Then there are at most 26 ordered triples \( (Q_1, Q_2, Q_3) \) of conjugate cubic points in \( D_{P_1} \times D_{P_2} \times D_{P_3} \).

Proof. As in Lemma 6.3, we use Lemmas 3.4 and 4.4 to choose linearly independent, normalized forms \( \omega_1, \omega_2, \omega_3 \in \Lambda_C \) such that \( n(\omega_i, P_j) = n(\Lambda_C, P_j) \) for \( i, j \in \{1, 2, 3\} \). By Theorem 3.2, these numbers are at most 2. Viewed as a function on \( D^3 \) via \( z_{P_1}, z_{P_2}, \) and \( z_{P_3} \), \( F_{\omega_1}^3 \) is given by \( \sum_{i=0}^{\infty} \frac{a_i}{i+1} t_1^{i+1} + \sum_{i=0}^{\infty} \frac{b_i}{i+1} t_2^{i+1} + \sum_{i=0}^{\infty} \frac{c_i}{i+1} t_3^{i+1} \).

A similar statement holds for \( F_{\omega_2}^3 \) and \( F_{\omega_3}^3 \).

By construction of \( \omega_1 \) (and similarly for \( \omega_2 \) and \( \omega_3 \)), we have \( v(a_i) = 0 \) for some \( i \leq 2 \), \( v(b_j) = 0 \) for some \( j \leq 2 \), and \( v(c_k) = 0 \) for some \( k \leq 2 \). Now by Lemma 7.1 applied to each of these three sums, we see that \( \text{New}_{1/3}(F_{\omega_1}^3), \text{New}_{1/3}(F_{\omega_2}^3), \) and \( \text{New}_{1/3}(F_{\omega_3}^3) \) are contained in the set \( \text{conv}((1, 0, 0), (3, 0, 0), (0, 1, 0), (0, 3, 0), (0, 0, 1), (0, 0, 3)) \subset \mathbb{R}^3 \).
By Theorem 5.4 and Example 5.3, there are at most $3^3 - 1 = 26$ zero-dimensional components of Weierstrass points. We are done by Lemmas 4.4 and 5.1.

Lemma 7.3. Let $C/Q$ be a hyperelliptic curve of genus $g \geq 4$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, and good reduction at $3$. Let $P_1, P_2 \in C(Q)$ be conjugate quadratic points, with $P_1, P_2 \in C_F(F_9) \setminus C_F(F_3)$, and $P_3 \in C(Q)$ a rational point. If $n(\Lambda_C, \overline{P_1}) = 1$, there are at most 26 ordered triples $(Q_1, Q_2, Q_3)$ of conjugate cubic points in $D_{\overline{P_1}} \times D_{\overline{P_2}} \times D_{\overline{P_3}}$. If $n(\Lambda_C, \overline{P_1}) = 0$, there are no such triples.

Proof. In the first two coordinates, we can consider New$_1$ using Remark 3.6 and the fact that $Q_3(P_1, Q_1)$ is unramified, whereas we have to consider New$_{1/3}$ in the last coordinate. We also have that $n(\Lambda_C, \overline{P_1}) = n(\Lambda_C, \overline{P_2})$. Theorem 3.2 asserts that $n(\Lambda_C, \overline{P_1}) + n(\Lambda_C, \overline{P_2}) + n(\Lambda_C, \overline{P_3}) \leq 2$, so we have two cases. If $n(\Lambda_C, \overline{P_1}) = 0$, then there are no ordered triples by first statement of Lemma 6.2. If $n(\Lambda_C, \overline{P_1}) = 1$, then the result follows from the second statement of Lemma 6.2, Lemma 7.1, and the same computation as in Lemma 7.2.

Lemma 7.4. Let $C/Q$ be a hyperelliptic curve of genus $g \geq 4$, with a rational Weierstrass point, geometrically simple Jacobian with $r \leq 1$, and good reduction at $3$. Let $P_1, P_2, P_3 \in C(Q)$ be three conjugate cubic points, with $P_1, P_2, P_3 \in C_F(F_9) \setminus C_F(F_3)$. Then there are no ordered triples $(Q_1, Q_2, Q_3)$ of conjugate cubic points in $D_{\overline{P_1}} \times D_{\overline{P_2}} \times D_{\overline{P_3}}$ not equal to $(P_1, P_2, P_3)$.

Proof. The proof is similar to that of Lemma 7.2 with two changes. First, one can consider New$_1$ instead of New$_{1/3}$ in all the coordinates, again using Remark 3.6 and the fact that $Q_3(P_1, Q_1)$ is unramified. Second, note that since $P_1, P_2,$ and $P_3$ reduce to (necessarily distinct) points outside of $C_F(F_3)$, we know that $P_1, P_2,$ and $P_3$ remain conjugate over $Q_3$. Thus their reductions are conjugate over $F_3$, so we have $n(\Lambda_C, \overline{P_1}) = n(\Lambda_C, \overline{P_2}) = n(\Lambda_C, \overline{P_3})$. By Theorem 3.2, this common value can only be 0, and then the result follows from Lemma 6.2.

Proof of Theorem 1.3. For a given value of $g$, the family given by Proposition 2.5 and Lemma 6.5 comprises a positive proportion of all hyperelliptic curves with a rational Weierstrass point, since the latter is defined by finitely many congruence conditions. By Corollary 2.3, at least 25% of the curves in this family have rank $r \leq 1$; this is still a positive proportion of all the curves.

Let $C$ be such a curve. Any triple of conjugate cubic points on $C$ that reduce to $F_3$-points will have to lie in $D_{\infty} \times D_{\infty} \times D_{\infty}$. We may choose $P_1 = P_2 = P_3 = \infty$ and apply Lemma 7.2 to conclude there are at most 26 ordered triples $(Q_1, Q_2, Q_3)$ of conjugate cubic points in this residue class. But since $Q_1 = Q_2 = Q_3$, each unordered triple is overcounted by a factor of 6, so there are at most $|26/6| = 4$ unordered triples of such cubic points.

If the minimal Weierstrass model of $C$ is $y^2 = f(x)$, note that (for some choice of square root) the pair of quadratic points $(i, \pm \sqrt{f(i)})$ reduces to $(\overline{i}, \pm \alpha)$ for each $\overline{i} = 0, 1, 2$. Any triple of conjugate cubic points on $C$ where two of the points reduce to $(F_9 \setminus F_3)$-points will have to lie in $D_{\overline{i}, \alpha} \times D_{\overline{i}, -\alpha} \times D_{\infty}$ for some $\overline{i}$. In $D_{\overline{i}, \alpha} \times D_{\overline{i}, -\alpha} \times D_{\infty}$, we may choose $P_1 = (i, \sqrt{f(i)})$, $P_2 = (i, -\sqrt{f(i)})$, and $P_3 = \infty$ and apply Lemma 7.3 with the values $n(\Lambda_C, (\overline{i}, \alpha))$, $n(\Lambda_C, (\overline{i}, -\alpha))$, and $n(\Lambda_C, \infty)$ to count ordered triples.

The last case is when a triple of conjugate cubic points on $C$ reduces to (necessarily distinct) $(F_{27} \setminus F_3)$-points. In this setting, Lemma 7.4 asserts that there are no triples in this
residue class away from their centers. Since \( C \) is hyperelliptic and has good reduction at 3, any unordered triple of conjugate cubic points (over \( \mathbb{F}_3 \)) will have to lie over an unordered triple of cubic points of \( \mathbb{P}^1_{\mathbb{F}_3} \), of which there are only \((3^3 + 1) - (3 + 1))/3 = 8\).

Using Theorem 3.2, we get the worst bound on the number of 0-dimensional components in all residue classes by assuming that for some \( i = 0, 1, 2, n(\Lambda_C, (\bar{i}, \alpha)) = n(\Lambda_C, (\bar{i}, -\alpha)) = 1 \), and all the others will be 0. To conclude, there are at most

\[ 4 + 26 + 8 = 38 \]

unordered triples of conjugate cubic points. \( \square \)

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