THE MULTIPLICITY OF
GENERIC NORMAL SURFACE SINGULARITIES

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Abstract. We provide combinatorial/topological formula for the multiplicity of a complex analytic normal surface singularity whenever the analytic structure on the fixed topological type is generic.

1. Introduction

1.1. The ‘multiplicity problem’. Probably the most fundamental numerical invariant of a projective variety $X$ embedded in some projective space $\mathbb{P}^N$ is its degree. Its local analogue, defined for local (algebraic or analytic) germs $(X, o)$ is the multiplicity $\text{mult}(X, o)$ of $(X, o)$. If $(X, o)$ is embedded in some $(\mathbb{C}^N, o)$ then it is the smallest intersection multiplicity of $(X, o)$ with a linear subspace germ $(L, o)$ of dimension $N - \dim(X, o)$. It is independent of the embedding $(X, o) \subset (\mathbb{C}^N, o)$, it can also be defined via the Hilbert–Samuel function of the maximal ideal $m_{X, o} \subset O_{X, o}$, cf. 2.3.11.

By definition it is an analytic invariant, and it guides several central geometric problems. E.g., besides its geometric significance as the ‘local degree’ of $(X, o)$, which obstructs (guides) the structure of analytic functions defined on $(X, o)$, it is the key numerical invariant of several objects associated canonically to $(X, o)$. See e.g. the significance of the multiplicity of the polar curve or of the discriminant in the case of hypersurface singularities [T73, T77], or the multiplicities of the $\delta$–constant (Severi) strata of the deformation of a plane curve singularity [FGS99, S12].

In this note we focus on the multiplicity of the complex analytic normal surface singularities. The guiding question is whether the multiplicity is computable from the topology of the link. The topology of the link (as an oriented 3–manifold with usually ‘large’ fundamental group) contains a huge amount of information, however the problem is still difficult. E.g., there are examples of local, topologically constant deformations when the multiplicity jumps (see e.g. the examples from section 8 when any analytic type can be deformed into a generic one). Moreover, there are ‘easy’ examples of hypersurface singularities, with the same topology but different multiplicity (e.g. $\{x^2 + y^7 + z^{14} = 0\}$ and $\{x^3 + y^4 + z^{12} = 0\}$). In such pairs of hypersurface singularities the link is not a rational homology sphere. Therefore, it is natural to impose for the link to be a rational homology sphere (that is, in a resolution of $(X, o)$ all the exceptional curves are rational and the dual graph is a tree).

The problem can be compared with the famous Zariski’s Conjecture [Z71], which asks whether the multiplicity of an isolated hypersurface singularity $(X, o) \subset (\mathbb{C}^{n+1}, o)$ can be recovered from the embedded topological type, that is, from the smooth embedding $\text{link}(X) \subset S^{2n+1}$. Except for some particular families the answer is not known yet, it is open even for surface singularities. For a survey see [E07] (and the references therein). Note that our projects wishes to connect the multiplicity

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merely with the abstract link (but under the assumption that the link is a rational homology sphere). In fact, in [MN05] it is conjectured that for isolated hypersurface surface singularities with rational homology sphere link the abstract link determines the multiplicity (it was verified in the suspension case in [BN07] and for germs with non-degenerate Newton principal parts in [BN07]). Note that for hypersurfaces the multiplicity is the smallest degree of the monomials from its equation, still, to recover this number from the topology can be hard.

If the normal surface singularity \((X, o)\) is not a hypersurface then the situation is even harder: it might happen that the topological type carries many rather different families of analytic structures.

On the other hand, there are some ‘positive example/families’ as well. Artin in [A62, A66] characterized rational singularities topologically and determined the multiplicity explicitly from the graph. This was extended by Laufer in [La77] to minimally elliptic singularities, and extended further for Gorenstein elliptic singularities in [N99]. For splice quotient singularities (a family which includes weighted homogeneous germs as well) the multiplicity was determined topologically in [N12], for abelian covers of splice quotient singularities in [O15]. Otherwise the literature is rather restrictive about any kind of multiplicity formulae. (Here we might mention recent connections with the bi–Lipschitz geometry, however bi–Lipschitz property is an analytic property, stronger than the abstract topological type).

In [Wa70] Wagreich proved that in the presence of a resolution \(\tilde{X} \to X\), if \(Z_{max}\) is the ‘maximal ideal cycle’ (of Yau [Y80]), and \(O_{\tilde{X}}(-Z_{max})\) has no base points, then \(\text{mult}(X, o) = -Z_{max}^2\). Here there are two difficulties: to determine \(Z_{max}\), and to characterize the base points of \(O_{\tilde{X}}(-Z_{max})\).

1.2. In the present note, instead of certain peculiar families, we focus on the ‘generic analytic structures’. We fix a topological type, say a dual graph \(\Gamma\), and we determine the multiplicity of a singularity \((X, o)\), which has a resolution \(\tilde{X}\) with dual graph \(\Gamma\), and \(\tilde{X}\) carries a generic analytic structure. (It turns out that the expression is independent of the choice of \(\Gamma\) up to the natural blow up of the graph.) Note that the moduli space of analytic structures supported on \(\Gamma\) are not known, we will use the parameter space of local deformations of Laufer [La73] to define the ‘generic analytic structure’.

For generic analytic structures in [NN18a] we already determined several analytic invariant topologically. That package of results basically concentrated on the cohomology of (certain natural) line bundles. It was a continuation of [NN18a], where the Abel map of resolution of normal surface singularities was introduced and treated. The article [NN18a] creates that new mathematical machinery, which can handle the subtle analytic invariants of line bundles. In [NN18a] the cycle \(Z_{max}\) was already determined for the generic analytic structure (together with ‘analytic semigroup’ of divisors of analytic functions of \((X, o)\)). In the present note we characterize topologically the base points of \(O_{\tilde{X}}(-Z_{max})\).

It turns out that for generic \(\tilde{X}\) all the base points are as simple as possible (the associated ideal sheaf at the base point \(p\) is the maximal ideal \(m_{\tilde{X}, p}\), and position/number can also be determined topologically. The topological characterization uses the Riemann–Roch expression \(\chi(l)\), (defined for cycles \(l\) supported on the exceptional curve). For the definition of \(\chi\) see 2.2.

For \(\tilde{X}\) generic, and \((X, o)\) non–rational, \(Z_{max}\) is determined as follows ([NN18b], or Theorem 3.2.1 below). Set \(\mathcal{M} = \{Z : \chi(Z) = \min_{l \in L} \chi(l)\}\). Then the unique maximal element of \(\mathcal{M}\) is the maximal ideal cycle of \(\tilde{X}\).

The next theorem provides the structure of base points (for more general versions see Theorems 3.3.1 and 3.3.6 below).
Theorem 1.2.1. Consider a resolution \( \widetilde{X} \to X \) with generic analytic structure. Let \( E \) be the exceptional curve \( \cup_{v \in V} E_v \). We say that the irreducible component \( E_v \) \( (v \in V) \) satisfies the property \((*_{v})\) if
\[
\min_{l \in \mathbb{Z}} \{ \chi(Z_{\text{max}} + l) \} = \chi(Z_{\text{max}}) + 1.
\]

Then the following facts hold.

* If \( p \) is a base point of \( \mathcal{L} \) then \( p \) is a regular point of \( E \).
* All the base points of \( \mathcal{L} \) are ‘simple’ (the base point ideal is \( \mathfrak{m}_{\widetilde{X}, p} \)).
* If \( p \in E_v \) is a base point of \( \mathcal{L} \) then \( E_v \) satisfies \((Z_{\text{max}}, E_v) < 0 \) and the property \((*_{v})\).
* If \((Z_{\text{max}}, E_v) < 0 \) and \( E_v \) satisfy \((*_{v})\) then \( \mathcal{L} \) has exactly \(-(Z_{\text{max}}, E_v)\) base points on \( E_v \).

In particular,
\[
\text{mult}(X, o) = -Z_{\text{max}}^2 - \sum_{v} (Z_{\text{max}}, E_v),
\]
where the sum is over all \( v \in V \) with \((Z_{\text{max}}, E_v) < 0 \) and \( \min_{l \in \mathbb{Z}} \chi(Z_{\text{max}} + l) = \chi(Z_{\text{max}}) + 1 \).

1.3. Note that if we blow up the resolution graph \( \Gamma \) we get a new graph, which determines the same topological type of \((X, o)\). If we associate generic analytic structures to both graphs then the structure of the base points can be identified isomorphically. (If we blow up a base point of a generic analytic structure, then we eliminate the base point, but the analytic structure obtained by blow up will be not generic on its supporting topological type.) For details see Remark \([83, 34, 11\]).

1.4. In fact, our results are more general. In order to be able to run an inductive procedure in the proof, we need to consider a relative case of resolutions \( \widetilde{X} \subset \widetilde{X}^{\text{top}} \), where \( \widetilde{X}^{\text{top}} \) is a fixed resolution space, and \( \widetilde{X} \) is a convenient small neighbourhood of exceptional curves given by subgraph \( \Gamma^{\text{top}} \). Furthermore, we will consider several line bundles as well: all the restrictions of the natural line bundles from \( \widetilde{X}^{\text{top}} \) level (with some positivity restriction regarding their Chern classes).

1.5. The structure of the article is the following. In section 2 we collect preliminary definitions, lemmas, we recall the definition of (restricted) natural line bundles. In section 3 we review the definition of the generic analytic structure (based on the work of Laufer) and several results from \([3, 18]\) regarding invariants for generic analytic structures. Here we state the new results regarding the structure of base points as well (Theorems \([5, 3, 1]\) and \([5, 3, 6]\) formulated in the general case of natural line bundles. Both theorems are divided into five steps (geometric statements) \((1')–(5')\).

The proof of \((1')\) is already in this section. Section 4 contains a review of the needed material regarding the Abel maps from \([3, 18]\). Part \((2')\) is proved in section 5 \((3')–(4')\) in section 6 while \((5')\) in section 7. Section 8 contains some examples, which support the theory. The short section 9 shows that the statements of the main results (formulated for natural line bundles of generic singularities) remain valid for generic line bundles of arbitrary singularities as well. Here we explain also the expected relationship between natural line bundles of generic singularities and the generic line bundles of arbitrary singularities.

2. Preliminaries

2.1. The resolution. Let \((X, o)\) be the germ of a complex analytic normal surface singularity, and let us fix a good resolution \( \phi: \widetilde{X} \to X \) of \((X, o)\). We denote the exceptional curve \( \phi^{-1}(0) \) by \( E \), and let \( \{E_v\}_{v \in V} \) be its irreducible components. Set also \( E_I := \sum_{v \in I} E_v \) for any subset \( I \subset V \). For the cycle \( l = \sum n_v E_v \) let its support be \( |l| = \cup_{n_v \neq 0} E_v \). For more details see \([07, 12, 99]\).
2.2. Topological invariants. Let $\Gamma$ be the dual resolution graph associated with $\phi$; it is a connected graph. Then $M := \partial \tilde{X}$, as a smooth oriented 3–manifold, can be identified with the link of $(X, o)$, it is also an oriented plumbed 3–manifold associated with $\Gamma$. We will assume (for any singularity we will deal with) that the link $M$ is a rational homology sphere, or, equivalently, $\Gamma$ is a tree with all genus decorations zero. We use the same notation $V$ for the set of vertices.

The lattice $L := H_2(\tilde{X}, \mathbb{Z})$ is endowed with a negative definite intersection form $I = (\cdot, \cdot)$. It is freely generated by the classes of 2–spheres $\{E_v\}_{v \in V}$. The dual lattice $L' := H^2(\tilde{X}, \mathbb{Z})$ is generated by the (anti)integral classes $\{E_v^*\}_{v \in V}$ defined by $(E_v^*, E_w) = -\delta_{vw}$, the opposite of the Kronecker symbol.

The intersection form embeds $L$ into $L'$. Then $H_1(M, \mathbb{Z}) \cong L'/L$, abridged by $H$. Usually one also identifies $L'$ with those rational cycles $l' \in L \otimes \mathbb{Q}$ for which $(l', L) \in \mathbb{Z}$ (or, $L' = \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong H^2(\tilde{X}, \mathbb{Z})$), where the intersection form extends naturally.

All the $E_v$–coordinates of any $E_v^*$ are strict positive. We define the Lipman cone as $S' := \{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\}$. It is generated over $\mathbb{Z}_{\geq 0}$ by $\{E_v^*\}_{v \in V}$. Hence, if $l' \in S \setminus \{0\}$ then all its $E_v$–coefficients are strict positive. We also write $S := S' \cap L$.

There is a natural partial ordering of $L'$ and $L$: we write $l'_1 \geq l'_2$ if $l'_1 - l'_2 = \sum_v r_v E_v$ with all $r_v \geq 0$. We set $L_{\geq 0} = \{l \in L : l \geq 0\}$ and $L_{> 0} = L_{\geq 0} \setminus \{0\}$. We will write $Z_{\text{min}} \in L$ for the minimal (or fundamental, or Artin) cycle, which is the minimal non–zero cycle of $S$.

We define the (anti)canonical cycle $Z_K \in L'$ via the adjunction formulae $(-Z_K + E_v, E_v) + 2 = 0$ for all $v \in V$. (In fact, $Z_K = -c_1(O_X^*)$, cf. \ref{eq:2.3.1}). In a minimal resolution $Z_K \in S'$.

Finally we consider the Riemann–Roch expression $\chi(l') = -(l', l' - Z_K)/2$ defined for any $l' \in L'$.

2.3. Some analytic invariants. The Picard groups. The group $\text{Pic}(\tilde{X})$ of isomorphism classes of analytic line bundles on $\tilde{X}$ appears in the (exponential) exact sequence
\begin{equation}
0 \to \text{Pic}^0(\tilde{X}) \to \text{Pic}(\tilde{X}) \xrightarrow{\psi} L' \to 0,
\end{equation}
where $c_1$ denotes the first Chern class. Here $\text{Pic}^0(\tilde{X}) = H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong \mathbb{C}^{p_g}$, where $p_g$ is the geometric genus of $(X, o)$. $(X, o)$ is called rational if $p_g(X, o) = 0$. Artin in \cite{A62} characterized rationality topologically via the graphs; such graphs are called ‘rational’. By this criterion, $\Gamma$ is rational if and only if $\chi(l) \geq 1$ for any effective non–zero cycle $l \in L_{> 0}$.

Similarly, if $Z \in L_{> 0}$ is a non–zero effective integral cycle such that its support is $|Z| = E$, and $\mathcal{O}_Z^*$ denotes the sheaf of units of $\mathcal{O}_Z$, then $\text{Pic}(Z) = H^1(Z, \mathcal{O}_Z^*)$ is the group of isomorphism classes of invertible sheaves on $Z$. It appears in the exact sequence
\begin{equation}
0 \to \text{Pic}^0(Z) \to \text{Pic}(Z) \xrightarrow{\psi} L' \to 0,
\end{equation}
where $\text{Pic}^0(Z) = H^1(Z, \mathcal{O}_Z)$. If $Z_2 \geq Z_1$ then there are natural restriction maps, $\text{Pic}(\tilde{X}) \to \text{Pic}(Z_2) \to \text{Pic}(Z_1)$. Similar restrictions are defined at $\text{Pic}^0$ level too. These restrictions are homomorphisms of the exact sequences \ref{eq:2.3.1} and \ref{eq:2.3.2}.

2.3.3. Fixed components and base points of line bundles. Fix some $Z \in L_{> 0}$ with $|Z| = E$ and $\mathcal{L} \in \text{Pic}(Z)$. We say that $E_v$ is a fixed component of $\mathcal{L}$ if the natural inclusion $H^0(Z - E_v, \mathcal{L}(-E_v)) \hookrightarrow H^0(Z, \mathcal{L})$ is an isomorphism. In particular, $\mathcal{L}$ has no fixed components at all if
\begin{equation}
H^0(Z, \mathcal{L})_{\text{reg}} := H^0(Z, \mathcal{L}) \setminus \bigcup_v H^0(Z - E_v, \mathcal{L}(-E_v))
\end{equation}
is non–empty. Let us use the same notation $\mathcal{L}$ for the sheaf of sections of the line bundle $\mathcal{L}$. If $\mathcal{L}$ has no fixed components then there exists a sheaf of ideals $\mathcal{I}_\mathcal{L}$ of $\mathcal{O}_{\tilde{X}}$ such that $H^0(\tilde{X}, \mathcal{L}) \cdot \mathcal{O}_{\tilde{X}} = \mathcal{L} \cdot \mathcal{I}_\mathcal{L}$, and $\mathcal{I}_\mathcal{L}$ is supported at finitely many points of $E$. These are the base points of $\mathcal{L}$.

We will refer to the next elementary lemma many times.
Lemma 2.3.5. Assume that $\mathcal{L} \in \text{Pic}(\tilde{X})$ has no fixed components and $p \in E$ is a base point. Let $b : \tilde{X}_{\text{new}} \to \tilde{X}$ be the blow up of $\tilde{X}$ at $p$ and set $E_{\text{new}} = b^{-1}(p)$. Then

(a) if $p \in E_0$, then $(c_1(\mathcal{L}), E_c) > 0$,
(b) $H^0(\tilde{X}, \mathcal{L}) = H^0(\tilde{X}_{\text{new}}, b^* \mathcal{L}) = H^0(\tilde{X}_{\text{new}}, b^* \mathcal{L}(-E_{\text{new}}))$,
(c) $h^1(\tilde{X}, \mathcal{L}) = h^1(\tilde{X}_{\text{new}}, b^* \mathcal{L}) = h^1(\tilde{X}_{\text{new}}, b^* \mathcal{L}(-E_{\text{new}})) - 1$.

Proof. Let $m_{\tilde{X}, p}$ denote the maximal ideal of the local algebra $O_{\tilde{X}, p}$. (a) If $(c_1(\mathcal{L}), E_c) \leq 0$ then comparison of the exact sequence $0 \to H^0(m_p \mathcal{L}) \to H^0(\mathcal{L}) \to \mathbb{C}_p$ with $0 \to H^0(\mathcal{L}(-E_c)) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}|_{E_c})$ would imply that $E_c$ is a fixed component. For (b)–(c) first notice that $R^1b_*b^*\mathcal{L} = \mathcal{L}$ and $R^1b_*b^*\mathcal{L} = 0$, hence by Leray spectral sequence $H^*(\tilde{X}_{\text{new}}, b^* \mathcal{L}) = H^*(\tilde{X}, \mathcal{L})$. Then identify $0 \to H^0(\tilde{X}, m_{\tilde{X}, p} \mathcal{L}) \to H^0(\tilde{X}, \mathcal{L}) \to \mathbb{C}_p$ with $0 \to H^0(\tilde{X}_{\text{new}}, b^* \mathcal{L}(-E_{\text{new}})) \to H^0(\tilde{X}_{\text{new}}, b^* \mathcal{L}) \to \mathbb{C}$. \qed

Definition 2.3.6. A base point $p$ of $\mathcal{L}$ is called of $A_t$–type (for some $t \geq 1$) if $p$ is a regular point of $E$ and $\mathcal{L}_p$ in the local ring $O_{\tilde{X}, p}$ is $(x^t, y)$, where $x, y$ are some local coordinates of $(\tilde{X}, p)$ at $p$ with $\{z = 0\} = E$ (locally). We say that $p$ is of $A_t$–type if it is $A_t$–type for some $t \geq 1$. In such cases we write $t = t(p)$. Note that $A_1$–type means $\mathcal{I}_{\mathcal{L}, p} = m_{\tilde{X}, p}$.

One verifies that a base point $p$ is of $A_t$–type if and only if $\mathcal{I}_{\mathcal{L}, p} \notin m_{\tilde{X}, p}^2$. A base point of $A_t$–type has the following geometric picture. If $s \in H^0(\tilde{X}, \mathcal{L})$ is a generic global section then its divisor $D$ in $(\tilde{X}, p)$ is reduced, smooth and transversal to $E$. Moreover, if we blow up $\tilde{X}$ at $p$ then (via the notations of Lemma 2.3.3) $b^*\mathcal{L}(E_{\text{new}})$ has no fixed components, and on $E_{\text{new}}$ it has no base points in the $t = 1$ case. If $t > 1$ then it has exactly one base point, namely at the intersection of $E_{\text{new}}$ with the strict transform of $E$. This base point is of $A_{t-1}$–type.

In particular, in order to eliminate a base point of type $A_t$ we need exactly $t$ successive blow ups. At all these steps Lemma 2.3.3 (b)–(c) applies.

We warn the reader that if a base point can be eliminated by $t$ successive blow ups then it is not necessarily of $A_t$–type. (Take e.g. the ideal $\mathcal{I}_{\mathcal{L}, p} = m_{\tilde{X}, p}^2$, which can be eliminated by one blow up.)

2.3.7. Natural line bundles. The epimorphism $c_1$ in (2.3.1) admits a unique group homomorphism section $l' \mapsto s(l') \in \text{Pic}(\tilde{X})$, which extends the natural section $l \mapsto O_{\tilde{X}}(l)$ valid for integral cycles $l \in L$, and such that $c_1(s(l')) = l'$ [NO72, DO3]. We call $s(l')$ the natural line bundles on $\tilde{X}$ with Chern class $l'$. By the very definition, $\mathcal{L}$ is natural if and only if some power $\mathcal{L}^\otimes n$ of it has the form $O_{\tilde{X}}(l)$ for some $l \in L$. We will use the uniform notation $O_{\tilde{X}}(l') := s(l')$ for any $l' \in L'$.

The following fact will be used several times:

Lemma 2.3.8. Consider the natural line bundle $O_{\tilde{X}}(l') \in \text{Pic}(\tilde{X})$ for $l' \in L'$. Let $b : \tilde{X}_{\text{new}} \to \tilde{X}$ be the blow up of a point $p \in E$. Then $b^*(O_{\tilde{X}}(l')) \in \text{Pic}(\tilde{X}_{\text{new}})$ is natural, in fact, it is $O_{\tilde{X}_{\text{new}}}(b^*(l'))$.

Indeed, it is enough to verify the statement for $l \in L$ in which case it is immediate.

If $Z \in L_{>0}$ with $|Z| = E$, then we can define a similar section of $\mathcal{L}_{\text{new}}$ by $s_Z(l') := O_{\tilde{X}}(l'|Z)$. These bundles satisfy $c_1 \circ s_Z = id_{L'}$. We write $O_Z(l')$ for $s_Z(l')$, and we call them natural line bundles on $Z$.

We also use the notations $\text{Pic}^l(\tilde{X}) := c_1^{-1}(l') \subset \text{Pic}(\tilde{X})$ and $\text{Pic}^l(Z) := c_1^{-1}(l') \subset \text{Pic}(Z)$ respectively. Multiplication by $O_{\tilde{X}}(l')$, or by $O_Z(l')$, provides natural affine–space isomorphisms $\text{Pic}^l(\tilde{X}) \to \text{Pic}^0(\tilde{X})$ and $\text{Pic}^l(Z) \to \text{Pic}^0(Z)$. (But, of course, multiplication by any other line bundle with the right Chern class might also realize the isomorphisms, the previous ones are ‘canonical’.)
2.3.9. The analytic semigroups associated with $\tilde{X}$. By definition, the analytic semigroup (monoid) associated with the resolution $\tilde{X} \to X$ is

$$(2.3.10) \quad S'_an := \{ l' \in L' : O_{\tilde{X}}(-l') \text{ has no fixed components} \}.$$ 

It is a subsemigroup of $S'$. One also sets $S_an := S'_an \cap L$, a subsemigroup of $S$. In fact, $S_an$ consists of the restrictions $\text{div}_E(f)$ of the divisors $\text{div}(f \circ \phi)$ to $E$, where $f$ runs over $O_{X,o}$. Therefore, if $s_1, s_2 \in S_an$, then $\min\{s_1, s_2\} \in S_an$ as well (take the generic linear combination of the corresponding functions). In particular, for any $l \in L$, there exists a unique minimal $s \in S_an$ with $s \geq l$.

Similarly, for any $h \in H = L'/L$ set $S'_{an,h} : \{ l' \in S'_an : [l'] = h \}$. Then for any $s'_1, s'_2 \in S'_{an,h}$ one has $\min\{s'_1, s'_2\} \in S'_{an,h}$, and for any $l' \in L'$ there exists a unique minimal $s' \in S'_{an,h}$ with $s' \geq l'$.

2.3.11. The Hilbert–Samuel function. S. S.-T. Yau’s maximal ideal cycle $Z_{max} \in L$ can be defined either as the unique minimal element of $S_an \setminus \{0\}$ (or, as the unique minimal element of $S_an$ which is $\geq 1$, cf. (2.3.5)), or, as the divisorial part of the pullback of the maximal ideal $m_{X,o} \subset O_{X,o}$, i.e. $\phi^*m_{X,o} \cdot O_{\tilde{X}} = O_{\tilde{X}}(\sim Z_{max}) \cdot I$, where $I$ is an ideal sheaf with $0$-dimensional support [YA70]. In general, $Z_{min} \leq Z_{max}$ (but they can be different). By the base points of $m_{X,o}$ associated with $\phi$ we understand the base points of $O_{\tilde{X}}(\sim Z_{max})$, which are described by $I$.

The Hilbert–Samuel function is defined as $f^{HS}(k) := \dim(C(O_{X,o}/m_{X,o}^k))$ for any $k \geq 1$. The Hilbert–Samuel polynomial is the unique polynomial $P^{HS}(k) = a_k k^2/2 + a_1 k + a_0$ such that $P^{HS}(k) = f^{HS}(k)$ for $k$ sufficiently large. The coefficient $a_2$ is the multiplicity of $(X,o)$, mult$(X,o)$. Geometrically, it is the degree of the generic map $(X,o) \to (C^2,0)$. By [WA70] $\text{mult}(X,o) \geq -Z_{max}^2$, and equality holds exactly in those cases when $m_{X,o}$ has no base points with respect to $\phi$. Moreover, if all the base points of $O_{\tilde{X}}(\sim Z_{max})$ are of $A_1$-type then

$$(2.3.12) \quad \text{mult}(X,o) = -Z_{max}^2 + \{\text{number of base points}\}.$$ 

Indeed, if $b$ is a blow up (as in Lemma 2.3.3) at such a base point then $Z_{new}^2 = b^*(Z_{max}) + E_{new}$ and $(Z_{max}^2)^2 = Z_{max}^2 - 1$.

If for a certain resolution the line bundle $O_{\tilde{X}}(\sim Z_{max})$ has base points, then they can be eliminated by a convenient sequence of additional blow ups (infinitely close to the base points). However, from the topological data, in general, it is not possible to identify those resolutions for which $O_{\tilde{X}}(\sim Z_{max})$ has no base points (or, the structure of ideal sheaves $I$ of $O_{\tilde{X}}$ in the presence of base points).

2.3.13. Restricted natural line bundles. Regarding natural line bundles the following warning is appropriate. Note that if $\tilde{X}_1$ is a connected small convenient neighbourhood of the union of some of the exceptional divisors (hence $\tilde{X}_1$ also stays as the resolution of the singularity obtained by contraction of that union of exceptional curves) then one can repeat the definition of natural line bundles at the level of $\tilde{X}_1$ as well (as a splitting of (2.3.1) applied for $\tilde{X}_1$). However, the restriction to $\tilde{X}_1$ of a natural line bundle of $\tilde{X}$ (even of type $O_{\tilde{X}}(l)$ with $l$ integral cycle supported on $E$) usually is not natural on $\tilde{X}_1$: $O_{\tilde{X}}(l)|_{\tilde{X}_1} \neq O_{\tilde{X}_1}(R(l'))$ (where $R : H^2(\tilde{X},\mathbb{Z}) \to H^2(\tilde{X}_1,\mathbb{Z})$ is the natural cohomological restriction), though their Chern classes coincide.

Therefore, in inductive procedure when such restriction is needed, we will deal with the family of restricted natural line bundles. This means the following. We fix a resolution space $\tilde{X}_{top}$ with dual graph $\Gamma_{top}$. Then for any $\tilde{X}$, convenient small neighbourhood of the exceptional curves indexed by the graph $\Gamma$ (a connected subgraph of $\Gamma_{top}$) the ‘restricted natural line bundles’ in $\text{Pic}(\tilde{X})$ are the restrictions to $\tilde{X}$ of the natural line bundles from $\text{Pic}(\tilde{X}_{top})$. In that case, for any $\tilde{X}_1$ ($\tilde{X}_1 \subset \tilde{X}$, defined similarly as $\tilde{X}$) the restriction of these line bundles from $\tilde{X}$ to $\tilde{X}_1$ are basically the restriction of
natural line bundles from $\tilde{X}_{\text{top}}$, hence any induction based on restriction preserves the family stably.
The same is valid when we consider instead of $\tilde{X}$ an effective cycle $Z$ with connected support $|Z| \subset E$.

This basically means that we fix $\tilde{X}_{\text{top}}$, and we consider the tower of singularities (resolutions) 
$\{\tilde{X}\}_{\tilde{X} \subset \tilde{X}_{\text{top}}}$, or $\{O_{\tilde{Z}}\}_{\tilde{Z} \subset \tilde{E}_{\text{top}}}$, and all the restricted natural line bundles are restrictions from the top level $\tilde{X}_{\text{top}}$. We use the notations $\mathcal{O}_{\tilde{X}}(l'_{\text{top}}):=\mathcal{O}_{\tilde{X}_{\text{top}}}(l'_{\text{top}})|_{\tilde{X}}$ and $\mathcal{O}_{\tilde{Z}}(l'_{\text{top}}):=\mathcal{O}_{\tilde{X}_{\text{top}}}(l'_{\text{top}})|_{\tilde{Z}}$
respectively, where $l'_{\text{top}} \in L'(\tilde{X}_{\text{top}})$.

If for some reason we need a blow up $b: \tilde{X}_{\text{new}} \to \tilde{X}$ at some $p \in E \subset \tilde{X}$, then the pull back bundle $b^*(\mathcal{O}_{\tilde{X}_{\text{top}}}(l'_{\text{top}})|_{\tilde{X}})$ is again a ‘restricted natural line bundle’, namely $\mathcal{O}_{\tilde{X}_{\text{top}}}(b^*(l'_{\text{top}}))|_{\tilde{X}_{\text{new}}}$, where $b_{\text{top}}: \tilde{X}_{\text{new}} \to \tilde{X}_{\text{top}}$ is the blow up of $\tilde{X}_{\text{top}}$ at $p$ (cf. Lemma 2.3.8).

In particular, we obtain a compatible family of line bundles, well-defined and indexed by the Chern classes, which are stable with respect to blow up and restrictions (in the towers as above).

Though the next statement is elementary, it is a key ingredient in several arguments.

The line bundle $\mathcal{O}_{\tilde{X}_{\text{top}}}(l'_{\text{top}}) \in \text{Pic}(\tilde{X}_{\text{top}})$ depends on its Chern class $l'_{\text{top}}$ (as combinatorial data) but definitely also on the analytic type of $\tilde{X}_{\text{top}}$. When we restrict it to $\tilde{X}$, and we vary the analytic structure of $\tilde{X}_{\text{top}}$ with the analytic structure of $\tilde{X}$ fixed, the bundle $\mathcal{O}_{\tilde{X}_{\text{top}}}(l'_{\text{top}})|_{\tilde{X}} \in \text{Pic}(\tilde{X})$ might vary in the fixed Pic($\tilde{X}$). The next lemma aims to reduce the dependence of $\mathcal{O}_{\tilde{X}_{\text{top}}}(l'_{\text{top}})|_{\tilde{X}}$ on the analytic structure of $\tilde{X}_{\text{top}}$ to the analytic type of the pair $(\tilde{X}, \tilde{X} \cap E_{\text{top}})$.

**Lemma 2.3.14.** The restriction $\mathcal{O}_{\tilde{X}_{\text{top}}}(l'_{\text{top}})|_{\tilde{X}} \in \text{Pic}(\tilde{X})$ depends only on the Chern class $l'_{\text{top}}$, on the analytic type of $\tilde{X}$, and on the analytic type of the non–compact divisor $E_{\text{top}} \cap \tilde{X}$ of $\tilde{X}$.

**Proof.** Since Pic($\tilde{X}$) has no torsion, it is enough to argue for $l'_{\text{top}} \in L(\tilde{X}_{\text{top}})$ (identified with an integral cycle supported on $E_{\text{top}}$), in which case the statement follows from the definitions. \(\square\)

## 3. Analytic invariants of generic analytic type

### 3.1. Let us comment first the definition of ‘generic’ analytic type.

The point is that for a fixed topological type the moduli space of all analytic structures supported by that fixed topological type (of a singularity), is not yet described in the literature. Similarly, for a fixed resolution graph $\Gamma$, the moduli space of all analytic structures (or resolution spaces $\tilde{X}$) having dual graph $\Gamma$ is again unknown. Hence, we cannot define our generic structure as a generic point of such moduli spaces. However, Laufer in [La73] defined local complete deformations of resolution of singularities. For a given resolution $\tilde{X} \to X$ with dual graph $\Gamma$, the base space of this deformation space parametrizes all the possible (local) deformations of the analytic structure of $\tilde{X}$ (with fixed topological type $\Gamma$).

This parameter space is the basic tool in our ‘working definition’, cf. [NN18] and [BC18] below.

### 3.1.1. The working definition of the ‘generic analytic type’.

Usually when we have a parameter space for a family of geometric objects, the ‘generic object’ might depend essentially on the fact that what kind of anomalies we wish to avoid. Accordingly, we determine a discriminant space of the non–wished objects, and generic means elements from its complement. In the present article, following [NN18], all the discrete analytic invariants we treat are basically guided by the cohomology groups of the restricted natural line bundles associated with a resolution. Hence, the discriminant spaces (sitting in the base space of complete deformation spaces of Laufer [La73]), parametrizing deformations of a pair $\tilde{X} \subset \tilde{X}_{\text{top}}$ with fixed dual graphs, are defined as the ‘jump loci’ of the first–cohomology groups of the restricted natural line bundles at all levels of the tower $\{\tilde{X}_1 \subset \tilde{X}_1\}$, cf. [BC18] (Usually, guided by a specific geometrical problem — e.g. the maximal ideal
and properties of $Z_{\text{max}}$ —, we have to consider only finitely many Chern classes, hence only finitely many such bundles/discriminants too.) A \textit{generic analytic structure} avoids all such discriminants.

In particular, the definition of the generic analytic type is linked with some distinguished resolution pair $\tilde{X} \subset \tilde{X}_{\text{top}}$. (However, this distinguished pair can be replaced by a new one, generic as well, if this new one is obtained from the distinguished one e.g. by a blow up at a \textit{generic} point of $E \subset \tilde{X}$, see \[2.3.10\].) Furthermore, in the situation $\tilde{X}_1 \subset \tilde{X} \cap \tilde{X}_{1,\text{top}}, \tilde{X}_{1,\text{top}} \subset \tilde{X}_{\text{top}}$ (cf. \[2.3.13\], when $\tilde{X} \subset \tilde{X}_{\text{top}}$ is generic, then $\tilde{X}_1 \subset \tilde{X}_{1,\text{top}}$ is automatically generic as well.)

The consideration of a \textit{pair} is motivated by the fact that the notions associated with pairs behave properly in inductive steps. (As was explained in \[2.3.10\] even if we start with $\tilde{X}_{\text{top}} = \tilde{X}$ and natural line bundles of $\tilde{X}$, if we need to restrict them to some $\tilde{X}_1 \subset \tilde{X}$, we face the situation of restricted bundles associated with the pair $\tilde{X}_1 \subset \tilde{X}$.) However, once the theorem is proved by induction based on the relative setup, a posteriori, in most concrete applications we choose $\tilde{X}_{\text{top}} = \tilde{X}$. In this latter case we speak about the generic analytic structure of $\tilde{X}$ with fixed dual graph $\Gamma$ (and about properties of genuine natural line bundles on $\tilde{X}$). For more see [NN18].

In a slightly simplified language we can regard the generic analytic structure in the following way as well. Fix a graph $\Gamma$. For each $E_v (v \in V)$ the disc bundle with Euler number $E_v^2$ is taut: it has no analytic moduli. The generic $\tilde{X}$ is obtained by gluing ‘generically’ these bundles according to the edges of $\Gamma$ as an analytic plumbing.

### 3.2. Review of some results of [NN18]

The list of analytic invariants, associated with a generic analytic type (with respect to a fixed resolution graph), which in [NN18] are described topologically, include the following ones: $h^1(\mathcal{O}_Z), h^1(\mathcal{O}_Z(l'))$ (with certain restriction on the Chern class $l'$), — this last one applied for $Z$ satisfies $h^1(\mathcal{O}_\tilde{X})$ and $h^1(\mathcal{O}_\tilde{X}(l'))$ too —, the multivariable Hilbert function $L \ni l \mapsto h(l)$, the analytic semigroup, and the maximal ideal cycle of $\tilde{X}$. See above or [CDGZ04, CDGZ08, Li69, N99b, N08, N12, O08, Re97] (or Theorem 3.2.1) for the definitions and relationships between them. The topological characterizations use the RR–expression $\chi: L \rightarrow \mathbb{Q}$.

In the next theorem the bundles $\mathcal{O}_\tilde{X}(-l')$ are the ‘genuine natural line bundles’ associated with $\tilde{X}$ and $l' \in L'$. (For the general case $\tilde{X} \subset \tilde{X}_{\text{top}}$ see \[3.3.5\].) It says (like several other statements regarding generic analytic structure and restricted natural line bundles) that these bundles behave cohomologically as the generic line bundles of $\text{Pic}^{-l'}(\tilde{X})$ (for more comments see [NN18], and also Theorem 4.1.10 (II) here).

**Theorem 3.2.1.** [NN18] \textbf{Theorem A} \textit{Fix a resolution graph (tree of $\mathbb{P}^1$’s) and assume that the analytic type of $\tilde{X}$ is generic. In parts (a)–(b) we assume that $Z$ is an effective cycle $Z \in L_{>0}$ with connected support. Then the following identities hold:}

\begin{enumerate}
  \item [(a)] For any $Z \in L_{>0}$
  \begin{align*}
  h^1(\mathcal{O}_Z) &= 1 - \min_{0 < l \leq Z, l \in L} \{ \chi(l) \}.
  \end{align*}

  \item [(b)] If $l' = \sum_{v \in V} l'_v E_v \in L'$ satisfies $l'_v > 0$ for any $E_v$ in the support of $Z$ then
  \begin{align*}
  h^1(Z, \mathcal{O}_Z(-l')) &= \chi(l') - \min_{0 \leq l \leq Z, l \in L} \{ \chi(l' + l) \}.
  \end{align*}

  \item [(c)] If $p_g(X, o) = h^1(\tilde{X}, \mathcal{O}_\tilde{X})$ is the geometric genus of $(X, o)$ then
  \begin{align*}
  p_g(X, o) &= 1 - \min_{l \in L_{>0}} \{ \chi(l) \} = - \min_{l \in L} \{ \chi(l) \} + \begin{cases}
  1 & \text{if } (X, o) \text{ is not rational,} \\
  0 & \text{else.}
  \end{cases}
  \end{align*}
\end{enumerate}
(d) More generally, for any \( l' \in L' \)
\[
h^1(\tilde{X}, \mathcal{O}_\tilde{X}(-l')) = \chi(l') - \min_{l \geq 0} \{ \chi(l' + l) \} + \begin{cases} 1 & \text{if } l' \in L_{\leq 0} \text{ and } (X, o) \text{ is not rational,} \\ 0 & \text{else.} \end{cases}
\]

(e) For \( l \in L \) set \( h(l) = \dim(H^0(\tilde{X}, \mathcal{O}_\tilde{X})/H^0(\tilde{X}, \mathcal{O}_\tilde{X}(-l))) \). Then \( h(0) = 0 \) and for \( l_0 > 0 \) one has
\[
h(l_0) = \min_{l \in L_{\geq 0}} \{ \chi(l_0 + l) \} - \min_{l \in L_{\geq 0}} \{ \chi(l) \} + \begin{cases} 1 & \text{if } (X, o) \text{ is not rational,} \\ 0 & \text{else.} \end{cases}
\]

(f) \( S_{\text{an}}' = \{ l' : \chi(l') < \chi(l' + l) \text{ for any } l \in L_{\geq 0} \} \cup \{ 0 \} \).

(g) Assume that \( \Gamma \) is a non–rational graph and set \( M = \{ Z \in L_{\geq 0} : \chi(Z) = \min_{l \in L} \chi(l) \} \). Then the unique maximal element of \( M \) is the maximal ideal cycle of \( \tilde{X} \).
(Note that in the above formulae one also has \( \min_{l \in L_{\geq 0}} \{ \chi(l) \} = \min_{l \in L} \{ \chi(l) \} \).

Remark 3.2.2. By part (g) of Theorem 3.2.1 for a generic analytic structure \( \tilde{X} \) one has \( \chi(Z_{\text{max}}) = \min_{l \in L} \chi(l) \). Note that \( \min_{l \in L} \chi(l) \) is independent of the choice of the resolution graph \( \tilde{X} \), it is a topological invariant of the singularity (denoted in the sequel by \( \min \chi \)).

Let us assume that \( \mathcal{O}_{\tilde{X}}(-Z_{\text{max}}) \) of a generic analytic structure \( \tilde{X} \) has a base point \( p \in E_v \), where \( p \) is a regular point of \( E \). Then, if we blow up \( \tilde{X} \) at \( p \) we get a new resolution, say \( \tilde{X}_{\text{new}} \), with dual graph \( \Gamma_{\text{new}} \). Write the blow up as \( b : \tilde{X}_{\text{new}} \to \tilde{X}, b^{-1}(p) = E_{\text{new}} \). Then \( b \circ \phi)^* m_{\chi, o} \cdot \mathcal{O}_{\tilde{X}_{\text{new}}} = \mathcal{O}_{\tilde{X}_{\text{new}}}(-b^* Z_{\text{max}} - k E_{\text{new}}) \cdot \mathcal{I}_{\text{new}} \) for some \( k \in \mathbb{Z}_{\geq 1} \). Hence, the maximal ideal cycle of \( \tilde{X}_{\text{new}} \) is \( Z_{\text{max}} = b^* Z_{\text{max}} + k E_{\text{new}} \). However, \( \chi(b^* Z_{\text{max}} + k E_{\text{new}}) = \chi(Z_{\text{max}}) + k(k + 1)/2 > \min \chi \). In particular, \( \tilde{X}_{\text{new}} \) and \( Z_{\text{max}} \) do not satisfy (g) (and several other properties of Theorem 3.2.1). This is compatible with the fact that \( \tilde{X}_{\text{new}} \) is not generic with respect to the new graph \( \Gamma_{\text{new}} \). (Recall that the center of the blow up was a special point, a base point associated with \( \tilde{X} \).

On the other hand, if we take a generic structure, say \( \tilde{X}_{\text{gen}} \) supported on \( \Gamma_{\text{new}} \), then \( E_{\text{new}} \) can be contracted in this case too, and one gets a resolution \( \tilde{X}_{\text{gen}} \). In this case the point \( p \) (the image of \( E_{\text{new}} \)) cannot be a base point (since (g) is valid for \( \tilde{X}_{\text{gen}} \) as well), in fact it is a generic point of \( E_v \). (As \( \tilde{X}_{\text{gen}} \) is constructed via a generic analytic plumbing, the gluing point \( E_v \cap E_{\text{new}} \) is also generic on \( E_v \).)

For further references we highlight this statement.

Lemma 3.2.3. If the pair \( \tilde{X} \subset \tilde{X}_{\text{top}} \) is generic (with respect to \( \Gamma \subset \Gamma_{\text{top}} \), and \( p \) is a generic point of \( E \), then the blow up \( \tilde{X}_{\text{new}} \subset \tilde{X}_{\text{new}} \) of \( \tilde{X} \subset \tilde{X}_{\text{top}} \) at \( p \) produces a generic pair.

3.3. The new results. The structure of base points. If \( \tilde{X} \) is generic and \( l' \in S_{\text{an}}' \setminus \{ 0 \} \) then we have \( \min_{l \geq 0} \{ \chi(l' + l) \} = \chi(l') \) (cf. Theorem 3.2.1 (f)).

We say that \( l' \) and \( E_v \) satisfy the property (\( *_v \)) if
\[
(\text{by}) \quad \min_{l \geq 0} \{ \chi(l' + l) \} = \chi(l') + 1.
\]

Theorem 3.3.1. Consider a resolution \( \tilde{X} \to X \) with generic analytic structure as in 3.1.1 and fix \( l' \in S_{\text{an}}' \setminus \{ 0 \} \) and write \( \mathcal{L} := \mathcal{O}_\tilde{X}(-l') \). Then the following facts hold.

(1) If \( p \) is a base point of \( \mathcal{L} \) then \( p \) is a regular point of \( E \).
(2) All the base points of \( \mathcal{L} \) are of \( A_1 \)-type.
(3) If \( p \in E_v \) is a base point of \( \mathcal{L} \) then \( l' \) and \( E_v \) satisfy the property (\( *_v \)).
(4) If \( (l', E_v) < 0 \) and \( l' \) and \( E_v \) satisfy (\( *_v \)) then \( \mathcal{L} \) has exactly \( -(l', E_v) \) base points on \( E_v \).
(5) Under the assumptions of (4), in fact, any base point on \( E_v \) is of \( A_1 \)-type.
Corollary 3.3.2. Assume that $\tilde{X}$ is generic and let $Z_{\text{max}}$ be its maximal ideal cycle. Theorem 3.3.1 applied for $l' = Z_{\text{max}}$ and (3.3.3) imply:

\begin{equation}
\text{mult}(X,o) = -Z_{\text{max}}^2 - \sum_v (Z_{\text{max}}, E_v),
\end{equation}

where the sum is over all $v \in V$ with $(Z_{\text{max}}, E_v) < 0$ and $\min_{t \geq E_v} \chi(Z_{\text{max}} + l) = \chi(Z_{\text{max}}) + 1$.

Since all the involved invariants (in the case $\tilde{X}$ generic) are computable from the dual graph $\Gamma$ of $\tilde{X}$ (cf. Theorem 3.2.1), (3.3.3) is a topological/combinatorial expression for mult$(X,o)$.

Remark 3.3.4. (a) The long cohomological exact sequence associated with $0 \to O_{\tilde{X}}(-l' - E_v) \to O_{\tilde{X}}(-l') \to O_E(-l') \to 0$ and Theorem 3.2.1(d) show that for $\tilde{X}$ generic and $l' \in S_0 \setminus \{0\}$ one has:

if $V_o := \frac{H^0(\tilde{X}, O_{\tilde{X}}(-l'))}{H^0(\tilde{X}, O_{\tilde{X}}(-l' - E_v))}$, then $\dim(V_o) = \min_{t \geq E_v} \{\chi(l' + l)\} - \chi(l')$.

In general $\dim(V_o) \geq 1$. One the other hand, $(*_v)$ reads as $\dim(V_o) = 1$.

Equivalently, $\dim(V_o) = 1$ means that $\dim \text{im}(H^0(\tilde{X}, O_{\tilde{X}}(-l')) \to H^0(E_v, O_{\tilde{X}}(-l'))) = 1$. If this happens, (even for not necessarily generic $\tilde{X}$), the line bundle necessarily has base points at the intersection points of the divisor of the generic section with $E_v$. Parts (4)–(5) of Theorem 3.3.1, say that these base points share uniformly the same type of ideal, and, in fact, they are all the simplest possible. The geometric meaning of part (3) is that if $\dim(V_o) \geq 2$ then there exist two generic sections without common zeroes along $E_v$.

(b) If we blow up a generic point of $E$ in the generic $\tilde{X}$, then $\tilde{X}^\text{new}$ is also generic (cf. 3.3.3), and furthermore, the base points and their structures at level $\tilde{X}$ and $\tilde{X}^\text{new}$ can be identified. Hence, for mult$(X,o)$ the very same type of formula holds with the very same correction term given by the base points. In particular, for any resolution graph $\Gamma'$ (say, obtained from $\Gamma$ by several blow ups), the associated generic analytic resolution $\tilde{X}'$ will have the very same type of base points. Hence, the structure of base points is independent of the choice of the generic resolution. (However, if we blow up a base point, then we might eliminate the base points, but on those resolutions the formulae valid for generic resolutions do not work, and we lose the topological control as well.)

3.3.5. Theorem 3.3.4 is a consequence of the more general Technical Theorem 3.3.5 below, which is formulated in such a way that that a certain induction runs properly. More precisely, it is stated for pairs $\tilde{X} \subset \tilde{X}^{\text{top}}$ with generic analytic structure and the bundles are the ‘restricted natural line bundles’ from the level of $\tilde{X}^{\text{top}}$.

Before we state the new version we note that Theorem 3.2.1 was also proved in [NN18b] for the more general relative version, that is, the line bundles $O_{\tilde{X}}(-l')$ from Theorem 3.2.1 can be replaced by ‘restricted natural line bundles’ associated with some generic pair $\tilde{X} \subset \tilde{X}^{\text{top}}$, under some negativity assumption regarding $l'_{\text{top}}$. In this version part (f) of Theorem 3.2.1 reads as follows.

Assume that $\tilde{X} \subset \tilde{X}^{\text{top}}$ is a generic pair, and fix $l'_{\text{top}} \in L'(\tilde{X}^{\text{top}})$. We will assume that its $E_v$–coordinates satisfies $l'_{\text{top},v} > 0$ for all $v \in V$. Let $-l' := R(-l'_{\text{top}}) = c_1(O_{\tilde{X}^{\text{top}}}(-l'_{\text{top}})|_{\tilde{X}}) \in L'(\tilde{X})$ be its cohomological restriction, and assume that $l' \in S_0 \setminus \{0\}$ (compare also with Theorem 3.1.10). Then, the fact that $O_{\tilde{X}^{\text{top}}}(-l'_{\text{top}})|_{\tilde{X}} \in \text{Pic}^{-l'}(\tilde{X})$ has
no fixed component. The topological characterization is (like for the genuine natural line bundles): \(\chi(l') < \chi(l' + l)\) for any \(l \in L_{>0}\). In particular, the fact that \(\mathcal{O}_{X_{\top}}(-l_{\top})\) has no fixed components is independent of the top level \(\tilde{X}_{\top}\), and it depends only on the cohomological restriction \(l'\).

In the next statement \(\Gamma, E, \mathcal{V}\), etc. denote the invariants at level \(\tilde{X}\).

**Theorem 3.3.6.** Consider a generic analytic pair \(\tilde{X} \subset \tilde{X}_{\top}\). Choose \(l_{\top} \in L'(\Gamma_{\top})\) such that its \(E_{\perp}\)-coordinate \(l_{\top,\perp} > 0\) for any \(v \in \mathcal{V}\). Let \(l' := R(l_{\top}) \in L'(\Gamma)\) be its cohomological restriction, and we assume that \(l' \in S'_{\an}(\tilde{X}) \setminus \{0\}\). Write \(\mathcal{L} := \mathcal{O}_{\tilde{X}}(-l_{\top})\) for the restricted natural line bundle \(\mathcal{O}_{X_{\top}}(-l_{\top})\) as above. Then the following facts hold.

(1') If \(p\) is a base point of \(\mathcal{L}\) then \(p\) is a regular point of \(E\).

(2') \(\mathcal{L}\) has a global section whose divisor is smooth and intersects \(E\) transversally (along the regular part of \(E\)).

(3') If for a certain \(v \in \mathcal{V}\) one has \((l', E_v) \leq 0\) and \(\min_{l \geq E_v} \chi(l,l + l) - \chi(l') \geq 2\) then \(\mathcal{L}\) admits two generic sections without common zeroes along \(E_v\).

(4') If \((l', E_v) < 0\) and \(l' + E_v\) satisfy \((*_{v})\) then \(\mathcal{L}\) has exactly \(-(l', E_v)\) base points on \(E_v\).

(5') In the situation of (4') let \(s'\) be the unique minimal element of \(S_{\an, [p]}\) with \(s' \geq l' + E_v\).

Write \(s'\) as \((l' + l)\). Then the generic sections of \(\mathcal{L}\) and \(\mathcal{L}(-l)\) have no common zeroes along \(E_v\).

Furthermore, in numerical terms, if \(m_v\) (resp. \(m_v^+\)) denote the multiplicity of \(l'\) (resp. of \(s'\)) along \(E_v\), then \(t(p) = m_v^+ - m_v = 1\) for any base point \(p \in E_v\).

(For further discussion regarding \(s'\) and \(m_v^+\) see Remark 3.3.7.)

**Remark 3.3.7.** Fix a resolution \(\tilde{X} \to X\) with generic analytic structure.

(a) For any \(n \in \mathbb{Z}\) and \(h \in H\) assume that \(L'_{n,h} := \{l' \in L' : [l'] = h, \chi(l') = n\}\) is non-empty. Let \(M\) be a maximal element of \(\mathbb{N}\), and assume that there exists no \(l \in L_{>0}\) such that \(\chi(M + l) < \chi(M)\). Then \(M \in S'_{\an}\). Indeed, if \(M \notin S'_{\an}\), then by Theorem 3.2.1(f) there exists \(l \in L_{>0}\) with \(\chi(M + l) = \chi(M)\). This contradicts the maximality of \(M\) in \(L'_{n,h}\).

(b) Note that the assumptions of Theorem 3.5.1(5) (namely, \((l', E_v) < 0\) and \(l' + E_v\) satisfy \((*_{v})\)) imply that \(L_{l', v} := \{l' + l : l \geq E_v, l \in L, \chi(l' + l) = \chi(l') + 1\}\) is non-empty.

We claim that \(L_{l', v}\) has a unique maximal element, which is exactly \(s'\) from (5') (namely, the minimal element of \(S'_{\an, [p]}\) with \(s' \geq l' + E_v\)). Indeed, let \(s'\) be a maximal element of \(L_{l', v}\). Since \(l' \in S'_{\an}, \chi(l' + l) > \chi(l')\) for any \(l \in L_{>0}\), hence \(\chi(s' + l) > \chi(s')\). By part (a) \(s' \in S'_{\an, [l']}.\) By the minimality of \(s'\) we have \(s' \leq s'\). Assume that \(s' = s' \leq l > 0\). Then \(\chi(l') < \chi(s') < \chi(s' + l) = \chi(s') + 1\), a contradiction. \((l' \in S'_{\an}\) and Theorem 3.5.1(f) imply the first inequality, and similarly, \(s' \in S'_{\an}\) the second one.) Hence \(s' = s'\). This is true for any choice of \(s'\), hence \(L_{l', v}\) has a unique maximal element, namely, \(s'\). In particular, in (5') \(m_v^+\) equals (compare with \((*_{v})\))

\[
(3.3.8) \quad m_v^+ = \max\{ E_v - \text{coefficient of } l' + l : l \geq E_v, l \in L, \chi(l' + l) = \chi(l') + 1\}.
\]

The numerical part of (5') says that this \(m_v^+\) is \(m_v + 1\). In other words, in local coordinates as in Definition 2.3.6 in a small neighbourhood of \(p\) there is a section of type \(x^{m_v^+} y\) (the generic section of \(L\)) and another of type \(x^{m_v} z\) (the generic section of \(L(l)\)), and any other section is in the ideal of \(x^{m_v^+}\). Hence \(\mathcal{L}_{l,p} = (x, y) = m_{\tilde{X}, p}\).

**3.3.9.** The proof of Theorem 3.3.6 runs over several section. At the end of this section we prove part (1') and all the statements for \(\tilde{X}\) rational (as a starting point of an induction).
3.4. The proof of Theorem 3.3.6 (I'). We will use the following fact, cf. 3.3.5

\[ \mathcal{O}_X(-l'_{top}) \text{ has no fixed components} \iff \chi(l') < \chi(l' + l) \text{ for any } l \in L>0. \]

Fix a singular point \( p = E_u \cap E_v \) of \( E \). Let \( b : \tilde{X}^{new} \rightarrow X \) be the blow up at \( p \) and \( E^{new} = b^{-1}(p) \).

One sees that \( \tilde{X}^{new} \) is also generic with respect to its dual graph \( \Gamma^{new} \). (E.g., the starting \( X \) can be chosen to be obtained from a generic structure on \( \Gamma^{new} \) by blowing down \( E^{new} \).) This means that the equivalence (3.4.1) is valid for both \( \mathcal{O}_\tilde{X}(-l'_{top}) \) and \( \mathcal{O}_{\tilde{X}^{new}}(-b^{new}_{top}(l'_{top})) \) (cf. 2.3.8).

By assumption \( l' \in S'_{\mathcal{X}} \). Hence, by the comments from 3.3.5, the left hand side of (3.4.1) holds for \( \mathcal{O}_\tilde{X}(-l'_{top}) \) too. Thus, by (3.4.1), both sides are satisfied in the case of \( \mathcal{O}_\tilde{X}(-l'_{top}), l' = R(l'_{top}) \).

Using this we show that the right hand side of (3.4.1) is valid for \( \mathcal{O}_{\tilde{X}^{new}}(-b^{new}_{top}(l'_{top})) \) too.

For this we have to verify that

\[ \chi(b^*(l')) < \chi(b^*(l') + l^{new}) \text{ for any } l^{new} \in L(\Gamma^{new}), l^{new} > 0. \]

Write \( l^{new} = b^*(l) + kE^{new} \) with some \( l \in L \) and \( k \in \mathbb{Z} \). Then \( \chi(b^*(l')) = \chi(l') \) and \( \chi(b^*(l') + l^{new}) = \chi(l' + l) + k(k + 1)/2 \). If \( l > 0 \) then \( \chi(l' + l) > \chi(l') \). If \( l = 0 \) then \( l^{new} = kE^{new} \), hence \( k \geq 1 \) and \( k(k + 1)/2 > 0 \). Hence (3.4.2) holds.

In particular, the left hand side of (3.4.1) should hold for \( \mathcal{O}_{\tilde{X}^{new}}(-b^{new}_{top}(l'_{top})) \), i.e. this bundle has no fixed components. But then \( p \) cannot be a base point of \( \mathcal{O}_\tilde{X}(-l'_{top}) \), since in that case \( E^{new} \) would be a fixed component by Lemma 2.3.5 (6).

3.5. The proof of Theorem 3.3.6 for \( \tilde{X} \) rational. From (2.3.1) we obtain that any line bundle with Chern class \(-l'\) is isomorphic to \( \mathcal{O}_X(-l'_{top}) \). Therefore, any noncompact curve (cut) \( C \) in \( \tilde{X} \), which makes \( l' + C \) numerically trivial (that is, \( (C + l', E_v) = 0 \) for any \( v \in V \)) is the divisor of a possible global section of \( \mathcal{O}_X(-l'_{top}) \). Since the position of such curves \( C \) can be moved generically, one obtains that \( \mathcal{O}_X(-l'_{top}) \) has no base points at all (see also \([A66]\)). Hence, to finish the proof, we need to verify that if \( (\dagger) \) \( (l', E_v) < 0 \) then \( (\ast_v) \) cannot happen. Indeed, \( \chi(l' + l) - \chi(l') = \chi(l) - (l', l) \). But for \( l \geq E_v \) one has \( \chi(l) \geq 1 \) by Artin's criterion of rationality \([A62, A66]\) and \( (l', l) \leq (l', E_v) \leq -1 \) since \( l' \in S' \) and \( (\dagger) \).

4. Effective Cartier divisors and Abel maps

Some parts of the proof of Theorem 3.3.6 are based on the properties of Abel maps associated with normal surface singularities. In this section we review some needed material. We follow \([NN18a]\), see also \([Klo05, §3]\) and \([Gro62]\). In the sequel we fix a good resolution \( \phi : \tilde{X} \rightarrow X \) of a normal surface singularity, whose link is a rational homology sphere. The notations of section 2 will also be adopted.

Regarding notations the next observation is appropriate. In the previous sections (and in the sequent ones also) it was natural to use the notation \( \mathcal{O}(-l) \) for bundles with \( l \in S \) (since these are related with the ideal sheaf of section with vanishing order \( \geq l \)). Here \( c_1(\mathcal{O}(-l)) = -l \). On the other hand, in this section we discuss the space of Cartier divisors and Picard groups with fixed Chern classes, and here it is not natural to carry this sign in all expressions. So, we will use the notation \( \mathcal{O}_X(l') \) for bundles with \( l' \in -S' \). This explains some sign differences in certain formulae.

4.1. The Abel map. Let us fix an effective integral cycle \( Z \in L \), \( Z \geq E \). Let \( \text{ECA}(Z) \) be the space of effective Cartier divisors supported on \( Z \). Note that they have zero–dimensional supports in \( E \). Taking the class of a Cartier divisor provides a map \( c : \text{ECA}(Z) \rightarrow \text{Pic}(Z) \), called the Abel map. Let \( \text{ECA}^l(Z) \) be the set of effective Cartier divisors with Chern class \( l' \in L' \), that is, \( \text{ECA}^l(Z) := c^{-1}(\text{Pic}^l(Z)) \). We consider the restriction of \( c, c^l : \text{ECA}^l(Z) \rightarrow \text{Pic}^l(Z) \) too,
Multiplicity

sometimes still denoted by $c$. The bundle $\mathcal{L} \in \text{Pic}^d(Z)$ is in the image $\text{im}(c)$ of the Abel map if and only if it has no fixed components, that is, if and only if $H^0(Z, \mathcal{L})_{\text{reg}} \neq \emptyset$, cf. [M3.4].

One verifies that $\text{ECA}^{d}(Z) \neq \emptyset$ if and only if $-l' \not\in \mathcal{S}' \setminus \{0\}$. Therefore, it is convenient to modify the definition of $\text{ECA}(Z)$ in the case $l' = 0$: we (re)define $\text{ECA}^{d}(Z) = \{\emptyset\}$, as the one–element set consisting of the ‘empty divisor’. We also take $c^d(Z)(\emptyset) := \mathcal{O}_Z$. Then we have

$$\text{ECA}^{d}(Z) \neq \emptyset \iff l' \not\in \mathcal{S}'. \tag{4.1.1}$$

If $l' \not\in \mathcal{S}'$ then $\text{ECA}^{d}(Z)$ is a smooth complex irreducible quasi–projective variety of dimension $(l', Z)$ (see [NN18a, Th. 3.1.10]). Moreover, cf. [NN18a, Lemma 3.1.7], if $\mathcal{L} \in \text{im}(c^d(Z))$ then the fiber $c^{-1}(\mathcal{L})$ is a smooth, irreducible quasiprojective variety of dimension

$$\dim(c^{-1}(\mathcal{L})) = h^0(Z, \mathcal{L}) - h^0(\mathcal{O}_Z) = (l', Z) + h^1(Z, \mathcal{L}) - h^1(\mathcal{O}_Z). \tag{4.1.2}$$

The Abel map can be defined for any effective integral cycle $Z$ (even without $Z \geq \mathcal{E}$). However, in this note in all our applications all the $E_v$–coefficients of $Z$ will be very large, denoted by $Z \gg 0$. In this way $Z$ will be a ‘finite model’ for $X$. (Note that ‘ECa($X$)’ is ‘undefined infinite dimensional’.) Additionally we will also have $h^1(Z, \mathcal{L}) = h^1(\tilde{X}, \mathcal{L})$ for $\mathcal{L} \in \text{Pic}(\tilde{X})$ by Formal Function Theorem.

4.1.3. Consider again a Chern class $l' \not\in \mathcal{S}'$ as above. The $E^*$–support $I(l') \subset \mathcal{V}$ of $l'$ is defined via the identity $l' = \sum_{v \in I(l')} a_v E_v$ with all $(a_v)_{v \in \mathcal{I}}$ nonzero. Its role is the following:

Besides the Abel map $c^d(Z)$ one can consider its ‘multiples’ $\{c^{nd}(Z)\}_{n \geq 1}$ as well. It turns out (cf. [NN18a, §6]), that $n \mapsto \dim(\text{im}(c^{nd}(Z)))$ is a non-decreasing sequence, and $\text{im}(c^{nd}(Z))$ is an affine subspace for $n \geq 1$, whose dimension $e_Z(l')$ is independent of $n \gg 1$, and essentially it depends only on $I(l')$. Moreover, by [NN18a, Theorem 6.1.9],

$$e_Z(l') = h^1(\mathcal{O}_Z) - h^1(\mathcal{O}_Z|_{\mathcal{V} \setminus I(l')}), \tag{4.1.4}$$

where $Z|_{\mathcal{V} \setminus I(l')}$ is the restriction of the cycle $Z$ to its $\{E_v\}_{v \in \mathcal{V} \setminus I(l')}$ coordinates. For $Z \gg 0$ this gives

$$e_Z(l') = h^1(\mathcal{O}_{\tilde{X}}) - h^1(\mathcal{O}_{\tilde{X}|_{\mathcal{V} \setminus I(l')}}), \tag{4.1.5}$$

where $\tilde{X}(\mathcal{V} \setminus I(l'))$ is a convenient small neighbourhood of $\cup_{v \in \mathcal{I} \setminus I(l')} E_v$.

Let $\Omega_{\tilde{X}}(I)$ be the subspace of $H^0(\tilde{X} \setminus \mathcal{E}, \mathcal{O}_{\tilde{X}}^\vee)$ generated by differential forms which have no poles along $E_{l'} \setminus \cup_{v \in I} E_v$. Then, cf. [NN18a, §8],

$$h^1(\mathcal{O}_{\tilde{X}|_{\mathcal{V} \setminus I(l')}}) = \dim(\Omega_{\tilde{X}}(I)). \tag{4.1.6}$$

4.1.7. $c^d(Z)$ dominant. Next, we characterize those cases, when the Abel map $c^d(Z)$ is dominant (the closure of its image is $\text{Pic}^d(Z)$). By [NN18a, Theorem 4.1.1] one has

**Theorem 4.1.8.** Fix $l' \not\in \mathcal{S}'$, $Z \geq \mathcal{E}$ as above. Then $c^d(Z)$ is dominant if and only if $\chi(-l' + l) < \chi(-l' + l)$ for all $0 < l \leq Z$, $l \in \mathcal{L}$. If $Z \gg 0$, then this last restriction runs over $0 < l$, $l \in \mathcal{L}$. In particular, the fact that $c^d(Z)$ is dominant is independent of the analytic structure supported by $\Gamma$ and it can be characterized topologically.

Moreover, if $c^d(Z)$ is dominant then $h^1(Z, \mathcal{L}_{\text{gen}}) = 0$ for generic $\mathcal{L}_{\text{gen}} \in \text{Pic}^d(Z)$.

4.1.9. The case of generic analytic structure $\tilde{X}$. We consider a generic pair $\tilde{X} \subset \tilde{X}_{\text{top}}$ and the corresponding restricted natural line bundles $\mathcal{O}_{\tilde{X}}(l'_{\text{top}}) \in \text{Pic}(\tilde{X})$, restricted from $\text{Pic}(\tilde{X}_{\text{top}})$. Additionally, we will take an integral cycle $Z \geq \mathcal{E}$ (this will ‘replace’ $\tilde{X}$ whenever $Z \gg 0$). The corresponding restricted natural line bundles will be denoted by $\mathcal{O}_Z(l'_{\text{top}}) \in \text{Pic}(Z)$.

The main feature of the generic analytic structures is that a restricted natural line bundle $\mathcal{O}_Z(l'_{\text{top}})$ cohomologically behave like the generic line bundle $\mathcal{L}_{\text{gen}} \in \text{Pic}^d(Z)$. The precise statement is
formulated as follows. (This is Theorem 5.1.1 from \[NN18a\] here we use the notation \(\tilde{X} \subset \tilde{X}_{\text{top}}\) for the pair \(\tilde{X}(|z|) \subset \tilde{X}\) of \[NN18b\].) Below, \(\V, S', E\) are invariants of the dual graph of \(\tilde{X}\).

**Theorem 4.1.10.** \([NN18b]\) Take \(\tilde{X} \subset \tilde{X}_{\text{top}}\) generic and \(Z \geq E\) as above. Assume that \(l_{\text{top}}' = \sum_{v \in V_{\text{top}}} t_{l_{\text{top}}, v}\\_v\\) satisfies \(t_{l_{\text{top}}, v} < 0\) for any \(v \in \V\) and \(l' := R(t_{l_{\text{top}}, v}) \in -S'\).

(I) The following facts are equivalent:

(a) \(\mathcal{O}_Z(l_{\text{top}}') \in \text{im}(c'(Z))\), that is, \(H^0(Z, \mathcal{O}_Z(l_{\text{top}}'))_{\text{reg}} \neq \emptyset\);

(b) \(c'(Z)\) is dominant, or equivalently, \(L_{\text{gen}} \in \text{im}(c'(Z))\), that is, \(H^0(Z, L_{\text{gen}})_{\text{reg}} \neq \emptyset\), for a generic line bundle \(L_{\text{gen}} \in \text{Pic}^0(Z)\);

(c) \(\mathcal{O}_Z(l_{\text{top}}') \in \text{im}(c'(Z))\), and for any \(D \in (c'(Z))^{-1}(\mathcal{O}_Z(l_{\text{top}}'))\) the tangent map \(T_D c'(Z) : T_D \mathcal{E}_C a'(Z) \to T_{\mathcal{O}_Z(l_{\text{top}}')} \text{Pic}^0(Z)\) is surjective.

(II) We have \(h'(Z, \mathcal{O}_Z(l_{\text{top}}')) = h'(Z, L_{\text{gen}})\) for \(i = 0, 1\) and a generic line bundle \(L_{\text{gen}} \in \text{Pic}^0(Z)\).

### 4.2. The Abel map in the relative setup

We consider a resolution \(\tilde{X}\) with resolution graph \(\Gamma\) and an integral cycle \(Z \geq E\) as in \([4.1]\) Moreover, we take another integral cycle (with smaller support) \(Z_1 \leq Z\), and set \(|Z_1| = V_1\) and the full subgraph \(\Gamma_1\) associated with \(|Z_1|\).

We have the restriction map \(r : \text{Pic}(Z) \to \text{Pic}(Z_1)\) and one has also the (cohomological) restriction operator \(R_1 : L^1(\Gamma) \to L_1 := L^1(\Gamma_1)\) (defined as \(R_1(E^1(\Gamma)) = E^1(\Gamma_1)\) if \(v \in V_1\), and \(R_1(E^1(\Gamma)) = 0\) otherwise). For any \(L \in \text{Pic}(Z)\) they satisfy \(c_1(r(L)) = R_1(c_1(L))\). In particular, we have the following commutative diagram as well:

\[
\begin{array}{ccc}
\mathcal{E}_C a'(Z) & \xrightarrow{c'(Z)} & \text{Pic}^0(Z) \\
\downarrow r & & \downarrow r \\
\text{Pic}^0(R_1(\Gamma)) (Z_1) & \xrightarrow{c(R_1(\Gamma))^1 (Z_1)} & \text{Pic}^0(R_1(\Gamma)) (Z_1)
\end{array}
\]

By the ‘relative case’ we mean that instead of the ‘total’ Abel map \(c'(Z)\) we study its restriction above a fixed fiber of \(r\). That is, we fix some \(\mathcal{E} \in \text{Pic}^0(R_1(\Gamma)) (Z_1)\), we set the subvariety \(\mathcal{E}_C a'^{\mathcal{E}} := (r \circ c'(Z))^{-1}(\mathcal{E}) = (c(R_1(\Gamma))^{-1}(\mathcal{E}) \subset \mathcal{E}_C a'(Z)\), and we study the restriction \(\mathcal{E}_C a'^{\mathcal{E}} \to r^{-1}(\mathcal{E})\) of \(c'(Z)\). Note that it might happen that \(\mathcal{E}_C a'^{\mathcal{E}}\) is empty. However, if it is non–empty then by \([\text{N19 Corollary 5.1.4}]\] it is smooth and irreducible (similarly as any \(\mathcal{E}_C a'(Z)\)).

### 5. Proof of Theorem 3.3.6\(^{(2')}\)

#### 5.1. We will prove \((2')\) by induction on \(h^1(\mathcal{O}_X) = 0\) then \((2')\) follows from \([3.3]\) Assume that it is true for any pair \(\tilde{X} \subset \tilde{X}_{\text{top}}\) with \(h^1(\mathcal{O}_{\tilde{X}}) < p_g\) (for some integer \(p_g > 0\)) and consider the new situation of a certain \((\tilde{X} \subset \tilde{X}_{\text{top}}, l_{\text{top}}')\) with \(h^1(\mathcal{O}_{\tilde{X}}) = p_g\). We fix also some \(Z \in L, Z \geq 0\).

Though \(\tilde{X}_{\text{top}}\) is an important ingredient, in some discussions below (in order to simplify the notations) we will neglect it tacitly; however, in the key situations we will provide the needed information regarding \(\tilde{X}_{\text{top}}\) as well (the completions at other parts are rather immediate).

#### 5.1.1. By Laufer’s duality (see e.g. \([\text{NN18a, 7.1}]\)), \(H^1(\mathcal{O}_{\tilde{X}})^* \simeq H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})/H^0(\tilde{X}, \Omega^2_{\tilde{X}})\), hence there exist \(u \in V\) and a form \(\omega \in H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})\) such that \(\omega\) has a non–trivial pole along \(E_u\). Let \(t + 1 \geq 1\) be the largest such pole for some \(u\). We claim that there exists \(\omega\) and \(E_u\) such that \(t \geq 1\). Indeed, otherwise \(H^0(\tilde{X} \setminus E, \Omega^2_{\tilde{X}})/H^0(\tilde{X}, \Omega^2_{\tilde{X}}) = H^0(\tilde{X}, \Omega^2_{\tilde{X}}(E))/H^0(\tilde{X}, \Omega^2_{\tilde{X}})\). But this last space, by Laufer’s duality (see \([\text{NN18a, 7.1.3}]\)) is \(H^1(\mathcal{O}_E)^*\). Hence \(p_g = h^1(\mathcal{O}_E) = 0\), a contradiction.

Hence, we assume that \(t \geq 1\) and we blow up \(E_q\) in a generic point \(q_1\) and we get a new exceptional divisor \(F_1\), then we blow up \(F_1\) in a generic point \(q_2\) and we get \(F_2\). We repeat this procedure \(t\) times. Let \(\tilde{X}_b\) (resp. \(\tilde{X}_b^-\)) denote a small neighbourhood of the union of the strict transform of \(E\)
(still denoted by $E$) with $\cup_{i=1}^n F_i$ (resp. of $E \cup \cup_{i=1}^n F_i$). The dual graphs are denoted by $\Gamma_b$ and $\Gamma_b^-$. Let $b: \tilde{X}_b \to \tilde{X}$ denote the modification and $R$ the cohomological restriction $L'(\Gamma_b) \to L'(\Gamma_b^-)$.

In parallel, we can consider the same blow-ups at the very same points, and we get $\tilde{X}_{\text{top},b}$.

Then one has the following facts (for the notation see the statement of Theorem 3.3.6):

(i) $\tilde{X}_b \subset \tilde{X}_{\text{top},b}$ and $\tilde{X}_b^- \subset \tilde{X}_{\text{top},b}$ are generic pairs (with respect to their dual graphs).

(ii) $\mathcal{L}_b := b^*\mathcal{L}$ and $\mathcal{L}_b^- := b^*\mathcal{L}|_{\tilde{X}_b^-}$ are restricted natural line bundles (from $\tilde{X}_{\text{top},b}$).

(iii) $t_i' := b^*(t_i') \in L'(\Gamma_b)$ satisfies $(t_i', F_i) = 0$ $(1 \leq i \leq t)$, hence $\mathcal{L}_b$ cannot have base points along $F_i$. Similarly, $t_i'^- := R(t_i') \in L'(\Gamma_b^-)$ satisfies $(t_i'^-, F_i) = 0$ $(1 \leq i \leq t - 1)$, hence $\mathcal{L}_b^-$ cannot have base points along such $F_i$.

(iv) $t_i' \in \mathcal{S}_{an}(\tilde{X}_b) \setminus \{0\}$, $t_i'^- \in \mathcal{S}_{an}(\tilde{X}_b^-) \setminus \{0\}$.

(v) $h^1(\tilde{X}_b, \mathcal{O}_{\tilde{X}_b}) = h^1(\tilde{X}, \mathcal{O}_X) = p_g$ and $h^1(\tilde{X}_b, \mathcal{O}_{\tilde{X}_b}) > h^1(\tilde{X}_b^-, \mathcal{O}_{\tilde{X}_b^-})$.

(vi) The maximum of pole orders of differential forms $\omega \in H^0(\tilde{X}_b \setminus E(\Gamma_b), \mathcal{O}_X^2)$ along $F_i$ is one.

For (i) use [3.1.1] and Lemma 3.2.13. For (ii) see [2.3.13], for (iii)–(iv) use the projection formula. The first part of (v) follows from Leray spectral sequence argument. For (vi) use the fact that if a section $E$ and the centers of blow up $q_i$ are generic with respect to this form, then the pull–back of this form has this property. This fact together with [3.1.5]–[3.1.9] applied for $E_l = F_l$ shows the second part of (v) as well.

Note that $H^0(\tilde{X}, \mathcal{L})$ is naturally isomorphic to $H^0(\tilde{X}_b, \mathcal{L}_b)$, hence (2') for $X \subset \tilde{X}_{\text{top};} \mathcal{L}$ or for $(\tilde{X}_b \subset \tilde{X}_{\text{top},b}; \mathcal{L}_b)$ are equivalent. Hence it is enough to prove it for the second one.

Furthermore, the inductive step applies for $(\tilde{X}_b^- \subset \tilde{X}_{\text{top},b}; \mathcal{L}_b^-)$, hence (2') is true for this case.

However, in general, the restriction map $H^0(\tilde{X}_b, \mathcal{L}_b) \to H^0(\tilde{X}_b^-, \mathcal{L}_b^-)$ is not surjective, hence a section $s_b^\prime \in H^0(\tilde{X}_b^-, \mathcal{L}_b^-)$, which satisfies (2') does not necessarily lift to $H^0(\tilde{X}_b, \mathcal{L}_b)$. But, if it lifts, then it automatically satisfies (2') since the lift will have no divisor along $F_i$ by (iii).

In order to establish the existence of such a lift we will perturb the analytic structure of the pair $(\tilde{X}_b \subset \tilde{X}_{\text{top},b})$ by preserving the type of $\tilde{X}_b^-$. Hence, $\mathcal{L}_b$ (being the restriction of a natural bundle of $\tilde{X}_{\text{top},b}$) will also be perturbed by the corresponding restriction natural line bundle associated with Chern class $t_{b}\top = b^*(t_{\top})$. However, the construction will guarantee that the pair $(\tilde{X}_b^-; \mathcal{L}_b^-)$ will stay stable. Then we show that for a generic element of the perturbation the lifting is possible. (On the other hand, since the original $(\tilde{X}_b \subset \tilde{X}_{\text{top},b}; \mathcal{L}_b)$ was generic, it has the very same properties as any small perturbation of it, hence the lifting follows for the original $(\tilde{X}_b \subset \tilde{X}_{\text{top},b}; \mathcal{L}_b)$ too.)

The analytic structure of $\tilde{X}_b \subset \tilde{X}_{\text{top},b}$ will be perturbed via the following additional construction.

**5.1.2.** First, we fix $n$ generic points $\{p_i\}_{i=1}^n$ on $F_i$ and we blow up $\tilde{X}_b$ at these points. This modification is denoted by $B: \tilde{X} \to \tilde{X}_b$, respectively $\tilde{X}_{\text{top},b} \to \tilde{X}_{\text{top},b}$.

The strict transforms of $\{E_{v}\}_{v \in V}$ and $\{F_{i}\}_{i=1}^n$ are denoted by the same symbols, while the strict transform of $F_i$ by $F_iB$. Let $\Gamma_B$ be the dual graph of $\tilde{X}_B$, and let $\tilde{X}_B$ be a small convenient neighbourhood of $\sum_v E_v \cup \cup_{i=1}^n F_i \cup F_iB$ in $\tilde{X}_B$ with dual graph $\Gamma_B$. (Note that $F_iB \neq F_i$, though their shapes are the same.) Additionally, set $\mathcal{L}_B := B^*\mathcal{L}_b = B^*b^*\mathcal{L}$ and $\mathcal{L}_B^- := \mathcal{L}_B|_{\tilde{X}_b^-}$. They have Chern classes $t_B' := B^*b^*t_i' \in L'(\Gamma_B^-)$ and its cohomological restriction $t_B'^-$ into $L'(\Gamma_B^-)$, respectively.

Write also $Z_B := B^*b^{*}Z$ and $Z_B := Z_B|_{L(\Gamma_B^-)}$ (projection to the exceptional curves from $\Gamma_B^-$).
Then the analogues of (i)–(vi) from 5.1.1 are the following:

(i) $\tilde{X}_B \subset \tilde{X}_{\text{top},B}$ and $\tilde{X}_B \subset \tilde{X}_{\text{top},B}$ are generic (with respect to their dual graphs).
(ii) $L_B$ and $L_B$ are restricted natural line bundles (from $\tilde{X}_{\text{top},B}$).
(iii) $(B^*b^!(l^i), E_{i}^{\text{new}}) = 0$ for $1 \leq i \leq n$, where $\{ E_{i}^{\text{new}} \}$ are the exceptional curves of $B$.
(iv) $t_{B}^I \in S_{an}(\tilde{X}_B) \setminus \{0\}$, $t_{B}^I \in S_{an}(\tilde{X}_B) \setminus \{0\}$.
(v) $p(g)(\tilde{X}_B) = p(g)(\tilde{X}_B)$ and the restriction realizes an isomorphism $\text{Pic}^{-l_{B}}(Z_B) \cong \text{Pic}^{-l_{B}}(Z_B)$.
(vi) $p_{g}(\tilde{X}_B) = p_{g}(\tilde{X}_B)$ and the restriction realizes an isomorphism $\text{Pic}^{-l_{B}}(Z_B) \cong \text{Pic}^{-l_{B}}(Z_B)$.

Part (vi) follows again from statements from 4.1.3 since along $E_{\text{new}}^{1}$ none of the differential forms have got a pole (by the same reason as in the proof of (vi) from 5.1.3).

5.1.3. $\tilde{X}_B$ embeds naturally into $\tilde{X}_B$ and $L_B^{-1} = L_B^{-1}$. Hence we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{ECa}^{-l_{B}}(Z_B) & \xrightarrow{c_{B}} & \text{Pic}^{-l_{B}}(Z_B) \\
\downarrow{r} & & \downarrow{r} \\
\text{ECa}^{-l_{B}}(Z_B) & \xrightarrow{c_{B}} & \text{Pic}^{-l_{B}}(Z_B) \\
\end{array}
$$

Above, $r$ is an affine projection associated with the surjective linear projection $H^1(O_{\tilde{X}_B}) \to H^1(O_{\tilde{X}_B})$. Since $H^1(O_{\tilde{X}_B}) \cong H^1(O_{\tilde{X}_B}) \cong H^1(O_{\tilde{X}_B})$ (cf. (vi) of 5.1.2) the fiber has dimension $p_{g} - H^1(O_{\tilde{X}_B}) > 0$.

In $\text{Pic}^{-l_{B}}(Z_B)$ we fix $L_B = b^*L|_{\tilde{X}_B}$. Recall that for the system $(\tilde{X}_B) \subset (\tilde{X}_{\text{top},B};L_B)$ the statement of the induction holds. Then we study the relative Abel map, the restriction of $c_{B}$

$$
(5.1.4)
\text{ECa}_{rel} := \text{ECa}^{-l_{B}}(Z_B) \xrightarrow{c_{B}} r^{-1}(L_B).
$$

Recall that $\text{ECa}_{rel}$ consists of effective Cartier divisors over $Z_B$ with Chern class $-l_{B}$ whose line bundle restricted to $\tilde{X}_B$ is exactly $L_B$. 

5.1.5. We claim that $c_{rel}$ is dominant.

Indeed, since $(l_{B}^{-1}) \in S_{an}(\tilde{X}_B) \setminus \{0\}$ (cf. (iv) of 5.1.2) there exists $D \in \text{ECa}^{-l_{B}}(Z_B)$, $D \neq \emptyset$, so that $c_{B}(D) = L_B \in r^{-1}(L_B)$. Since $\tilde{X}_B$ is generic (cf. (i) of 5.1.1), by Theorem 4.1.10 $\text{TD}_{D}c_{B}$ is surjective, hence $c_{B}$ is a local submersion at $D$. In particular, there exists an analytic open set $V \subset \text{Pic}^{-l_{B}}$ so that $L_B \subset V \subset \text{im}(c_{B})$. Then $V \cap r^{-1}(L_B)$ is an analytic open set $r^{-1}(L_B)$ and it is in the image of $c_{rel}$. But $c_{rel}$ is an algebraic map, hence it is necessarily dominant.

5.1.6. Next, we compare $\text{ECa}^{-l_{B}}(Z_B)$ and $\text{ECa}^{-l_{B}}(Z_B)$. Since $(l_{B}^{-1}, F_{t,B}) = (b^!(l^i), F_{t}) = 0$, in the first space no divisor is allowed, which has support along $F_{t,B}$.

However, in the second space divisors with support $q_{t} = F_{t_{1}} \cap F_{t}$ are allowed (they might appear if $t = 1$). Let $\text{ECa}^{-l_{B}}(Z_B)_{q_{t}}$ be the Zariski open set of $\text{ECa}^{-l_{B}}(Z_B)$ consisting of those divisors whose support does not contain $q_{t}$. Then $\text{ECa}^{-l_{B}}(Z_B)_{q_{t}}$ and $\text{ECa}^{-l_{B}}(Z_B)$ can be identified. Hence $D$ can be transported into $\text{ECa}^{-l_{B}}(Z_B)_{q_{t}}$ as well. Furthermore, consider $\text{div} : H^0(Z_B, L_B)_{\text{reg}} \to \text{ECa}^{-l_{B}}(Z_B)_{q_{t}}$, which associates with a section its divisor. It is surjective onto $(c_{B})^{-1}(L_B)$. Let $H^0(Z_B, L_B)_{\text{reg},q_{t}} \text{div}^{-1}(\text{ECa}^{-l_{B}}(Z_B)_{q_{t}})$. It consists of section, which do not vanish at $q_{t}$.

We claim that $H^0(Z_B, L_B)_{\text{reg},q_{t}}$ is a non–empty Zariski open set in $H^0(Z_B, L_B)_{\text{reg}}$. Indeed, if all the sections vanish at $q_{t}$, since $q_{t}$ was chosen generically (cf. 5.1.1), we get that all the sections vanish along $F_{t_{1}}$, hence at $q_{t_{1}}$ too. Since $q_{t_{1}}$ is also generic, we get vanishing along $F_{t_{2}}$ and at $q_{t_{2}}$. By induction we get vanishing along $E_{u}$, a contradiction, since $E_{u}$ is not a fixed component.

In this way we obtain a surjective map

$$
(5.1.7)
\text{div} : H^0(Z_B, L_B)_{\text{reg},q_{t}} \to \text{ECa}^{-l_{B}}(Z_B).
$$
5.1.8. Now, we apply the induction for the pair \((\tilde{X}_b^- \subset \tilde{X}_{\text{top},b}; \mathcal{L}_b^-)\). By this, there exists a section of \(\mathcal{L}_b^-\) which satisfies (2'). Let \(U\) be the non-empty Zariski open set in \(H^0(Z_b^-, \mathcal{L}_b^-)\) consisting of sections with property (2'). Since both \(\text{div}\) and \(c_{\text{rel}}\) are dominant, \(c_{\text{rel}}(\text{div}(U)) \subset r^{-1}(\mathcal{L}_b^-)\) contains a non-empty Zariski open set \(U_{\text{Pic}}\). Any bundles from \(U_{\text{Pic}}\) has the property that its restriction to \(\tilde{X}_b^-\) is \(\mathcal{L}_b^-\), and it has a section which satisfies (2').

We will show that (under the initial genericity assumption) the natural line bundle \(\mathcal{L}_B^-\) is in \(U_{\text{Pic}}\).

5.1.9. Now we concentrate on the position of \(\mathcal{L}_B^-\) in \(r^{-1}(\mathcal{L}_b^-)\).

We show that by a conveniently constructed family of perturbations of \((\tilde{X}_B \subset \tilde{X}_{\text{top},B}; \mathcal{L}_B)\), the perturbed \(\mathcal{L}_B^- = \mathcal{L}_B|_{\tilde{X}_B^-}\) will move in a small analytic open set of \(r^{-1}(\mathcal{L}_b^-)\), hence it necessarily will intersect \(U_{\text{Pic}}\). Since \(\tilde{X}_B\) itself is generic, we can assume that \(\mathcal{L}_B^-\) itself is an element of \(U_{\text{Pic}}\).

Let \(T_i\) be a tubular neighbourhood of a \((-1)\)-curve \(\tilde{E}_i\) in a smooth surface \((i = 1, \ldots, n)\). Note that \(\tilde{X}_B\) is obtained from \(\tilde{X}_B\) by an analytic plumbing: we glue \(\tilde{X}_B\) with the spaces \(T_i\) such that \(\tilde{E}_i\) is identified with \(E_i^{\text{new}}\), hence \(\tilde{E}_i \cap F_{i,B} = p_i\). In the construction of the flat deformation we glue \(T_i\) with \(\tilde{X}_B^-\) such that \(\tilde{E}_i \cap F_{i,B}\) moves in a small neighbourhood of \(p_i \in F_{i,B}\). Hence we get a flat family over the parameter germ-space \((F_{i,B}, \{p_i\})\) with fibers \(\tilde{X}_{B,\lambda}\) (\(\lambda \in (F_{i,B}, \{p_i\})\)). It is convenient to rename each \(\tilde{E}_i\) by \(E_i^{\text{new}}\). If we blow down the \(\{E_i^{\text{new}}\}\) and the \(\{F_i\}\) curves then we get a flat deformation of the structure of \(\tilde{X}\) (in the sense of Launer [LA73], cf. [NN18a]). For the precise description of these deformations/glueings see \([5.1.14]\).

Furthermore, by the very same deformation (reglueings) we obtain a flat family \(\{\tilde{X}_{\text{top},B,\lambda}\}\) too, hence pairs \(\tilde{X}_B, \tilde{X}_{\text{top},B}\) is the level where the natural line bundles are defined, and their restrictions are the corresponding 'restricted natural line bundles' in \(\text{Pic}(\tilde{X}_{B,\lambda})\) and \(\text{Pic}(\tilde{X}_{b,\lambda})\). Now, for any \(\lambda\), one can consider all the data defined in the previous subsections for \(\tilde{X}_B^-, \tilde{X}_B\).

It is crucial to notice that \(\tilde{X}_B\) embeds naturally into each \(\tilde{X}_{B,\lambda}\), hence provides a constant family of subspaces over the parameter space. The next key observation follows from Lemma 2.3.14.

**Lemma 5.1.10.** \(\mathcal{L}_{B,\lambda}^- := \mathcal{O}_{\tilde{X}_{\text{top},B,\lambda}}(nF_{i,B})|_{\tilde{X}_B^-} \in \text{Pic}(\tilde{X}_B^-)\) is independent of \(\lambda\), it is exactly \(\mathcal{L}_b^-\).

Since \(\tilde{X}_B^-, \tilde{X}_B\) are constant with respect to \(\lambda\), all the objects considered in the subsections stay stably, except \(\mathcal{L}_{B,\lambda}^- := \mathcal{O}_{\tilde{X}_{\text{top},B,\lambda}}(nF_{i,B})|_{\tilde{X}_B^-} \in r^{-1}(\mathcal{L}_b^-) \subset \text{Pic}(\tilde{X}_B^-) = \text{Pic}(Z_b^-)\) (and this is exactly the point, since we wished to ‘move’ the position of \(\mathcal{L}_B^- \in r^{-1}(\mathcal{L}_b^-)\)).

5.1.11. We claim that for \(n \geq 0\) and for \(\lambda \in (F_{i,B}, \{p_i\})\) the bundle \(\mathcal{L}_{B,\lambda}^-\) moves in an analytic open set of \(r^{-1}(\mathcal{L}_b^-)\). Here the definition of the natural line bundles will play a role. Indeed, it is enough to verify the statement for any multiple of \(\mathcal{L}_{B,\lambda}^-\). Set \(N \gg 1\) so that \(N \cdot l_{\text{top},B} = N \cdot B^*b^*(l_{\text{top}})\) can be written as \(l + m \sum \mathcal{E}_{\mathcal{L}^\text{new}}\), where \(l \in L(\tilde{X}_B^-)\) and \(m \in \mathbb{Z}\). Note that \(m/N\) is the \(E_\lambda\)-multiplicity \(l_{\text{top},B}\) of \(l_{\text{top}}\), which is positive by the assumption of Theorem 3.3.6. Hence \(m > 0\) too. This shows that \(\mathcal{L}_{B,\lambda}^- \sim N \cdot \mathcal{E}_{\mathcal{L}^\text{new}} \cap \tilde{X}_B^- \cap \mathcal{E}_{\mathcal{L}_b^-} \cap \tilde{X}_B^- \cap \mathcal{L}_b^-\), where \(\mathcal{E}_{\mathcal{L}_b^-}(-l)\) is again \(\lambda\)-independent. Hence, it is enough to determine the dimension of the space filled by the second contribution when \(\lambda\) moves in its parameter space.

5.1.12. Note that \(\sum \mathcal{E}_{\mathcal{L}_b^-} \cap \tilde{X}_B^-\) consists of \(n\) generic transversal divisors in \(\mathcal{E}(\mathcal{L}_b^-)\) and we are interested in the dimension of the image of the Abel map \(\mathcal{E}(\mathcal{L}_b^-) : \mathcal{E}(\mathcal{L}_b^-) \rightarrow \text{Pic}(\mathcal{L}_b^-)\). This by the results of [4.1.13] (see also [NN18a]), for \(n\) sufficiently large, is \(h^1(\mathcal{O}_{\tilde{X}_B^-}) - h^1(\mathcal{O}_{\tilde{X}_B^-})\), hence it equals \(\dim(r^{-1}(\mathcal{L}_b^-))\). Note that the line bundles \(\mathcal{E}_{\mathcal{L}_b^-} \cap \tilde{X}_B^-\) depend only on the position of the points \(\{p_i\}\) on \(F_{i,B}\). This follows from the fact that all the differential forms along \(F_{i,B}\) have pole order \(\leq 1\) (and from the explicit description of the Abel map via integration, cf. [NN18a 7.2]).
We give two proofs (a combinatorial one and a geometric one). One also has (3')

Next, we lift this property to the level of \( \tilde{X}_B \). Consider the diagram

\[
\begin{array}{ccc}
\text{ECa}^{-l_B}_u(Z_B) & \xrightarrow{c_B} & \Pic^{-l_B}_u(Z_B) \ni \mathcal{L}_B \\
\downarrow \tau_B & & \downarrow \tau_B \\
\text{ECa}^{-l_B}_u(Z_B) & \xrightarrow{c_B} & \Pic^{-l_B}_u(Z_B) \ni \mathcal{L}_B
\end{array}
\]

Then \( \tau_B \) is an isomorphism by (5.1.12) vi), and \( r(\mathcal{L}_B) = \mathcal{L}_B^{-} \). Moreover, \( \tau_B \) is bijection (identity) too by (5.1.1(iii) and 5.1.2(iii)). By the previous paragraph, there exists \( D^{-} \in \text{ECa}^{-l_B}_u(Z_B) \) with \( c_B(D^{-}) = \mathcal{L}_B^{-} \) and property (2'), hence \( D^{-} = \tau_B^{-1}(D^{-}) \) satisfies \( c_B(D) = \mathcal{L}_B \) and property (2') too.

On the other hand, Theorem 5.3.6 (2') for \( (\tilde{X}; \mathcal{L}) \) and \( (\tilde{X}_B; \mathcal{L}_B) \) are equivalent by blow up.

5.1.14. Finally, we describe the deformation of a fixed resolution, which was used in [5.1.10]

We choose any good resolution \( \phi : (\tilde{X}, E) \rightarrow (X, o) \), and write \( \cup_v E_v = E = \phi^{-1}(o) \) as above. Since each \( E_v \) is a rational, small tubular neighborhood of \( E \) in \( \tilde{X} \) can be identified with the disc-bundle associated with the total space \( T(e_v) \) of \( O_{\tilde{X}v}(e_v) \), where \( e_v = E_v^2 \). (We will abridge \( e := e_v \).) Recall that \( T(e) \) is obtained by gluing \( C_{u_0} \times C_{v_0} \) with \( C_{u_1} \times C_{v_1} \) via identification \( C_{u_0} \times C_{v_0} \sim C_{u_1} \times C_{v_1} \), \( u_1 = u_0^{-1} \), \( v_1 = v_0 u_0^{-e} \), where \( C_w \) is the affine line with coordinate \( w \), and \( C_w^* = C_w \setminus \{0\} \).

Next, fix any curve \( E_w \) of \( \phi^{-1}(o) \) and also a generic point \( P_w \in E_w \). There exists an identification of the tubular neighborhood of \( E_w \) via \( T(e) \) such that \( u_1 = v_1 = 0 \) is \( P_w \). By blowing up \( P_w \) in \( \tilde{X} \) we get a second resolution \( \psi : \tilde{X}' \rightarrow \tilde{X} \); the strict transforms of \( \{E_v\}'s \) will be denoted by \( E_v' \), and the new exceptional \( (-1) \) curve by \( E_{new} \). If we contract \( E_w' \cup E_{new} \) we get a cyclic quotient singularity, which is taut, hence the tubular neighborhood of \( E_w' \cup E_{new} \) can be identified with the tubular neighborhood of the union of the zero sections in \( T(e - 1) \cup T(-1) \). Here we represent \( T(e - 1) \) as the gluing of \( C_{u_0'} \times C_{v_0} \) with \( C_{u_1} \times C_{v_1}' \) by \( u_1' = u_0'^{-1} \), \( v_1' = v_0 u_0^{-e+1} \). Similarly, \( T(-1) \) as \( C_{\beta} \times C_{\alpha} \) with \( C_{\beta} \times C_{\gamma} \) by \( \beta = \beta^{-1} \), \( \gamma = \alpha \beta \). Then \( T(e - 1) \) and \( T(-1) \) are glued along \( C_{u_1} \times C_{v_1}' \sim C_{\beta} \times C_{\alpha} \) by \( u_1' = \alpha \), \( v_1' = \beta \) providing a neighborhood of \( E_{new} \) in \( \tilde{X}' \). Then the neighborhood \( \tilde{X}' \) will be modified by the following 1-parameter family of spaces: the neighborhood of \( \cup_v E_v' \) will stay unmodified, however \( T(-1) \), the neighborhood of \( E_{new} \) will be glued along \( C_{u_1} \times C_{v_1}' \sim C_{\beta} \times C_{\alpha} \) by \( u_1' + \lambda = \alpha \), \( v_1' = \beta \), where \( \lambda \in (C, 0) \) is a small holomorphic parameter.

6. **Proof of Theorem 5.3.6 (3')-(4')**

6.1. Fix a vertex \( v \in V \), which satisfies the assumptions of (3'). Additionally we keep all the constructions and notation of section 5 (proof of part (2')) as well.

6.1.1. Let \( o \) be a generic point of \( E_v \) and \( \pi_o : \tilde{X}_{top,B,o} \rightarrow \tilde{X}_{top,B} \) be the blow up at \( o \). \( \pi_o : \tilde{X}_{B,o}^{-} \rightarrow \tilde{X}_B^{-} \) its restriction over \( \tilde{X}_B^{-} \), and \( E_o \) the created exceptional curve. Let \( \Gamma_{B,o} \) be the dual graph of \( \tilde{X}_{B,o}^{-} \) and \( l_o' := \pi_o^{-1}(l_o^{-1}) \in L'(\Gamma_{B,o}^{-}) \). Since \( l_o^{-1} = S_{an}(X_B^{-}) \setminus \{0\} \), cf. (5.1.12) iv), using the pullback of the generic section we get \( l_o' \in S_{an}(X_B^{-}) \setminus \{0\} \) too. However, we claim that under the assumption of (3') one also has

\[
l_o' + E_o \in S_{an}(X_B^{-}) \setminus \{0\}.
\]

We give two proofs (a combinatorial one and a geometric one).
Indeed, the isomorphism is induced by pull–back of Cartier divisors via $\pi_E$ along $Z$. The second proof is more geometrical (it constructs the needed section). By assumption, then the pull–back of $\tilde{E}_a$ be the subspace of $EC_a$ modulo $(\tilde{x},y)$ in a neighbourhood of $EC_a$ in a neighbourhood of $\tilde{E}_a$.

Next we make the following identification. For $E_2$, we consider the Abel map with Chern class $-l_o-E_o$.

The verification in the relevant local chart (for more details see the proof of [NN18a, Theorem 3.1.10]).

The second proof is more geometrical (it constructs the needed section). By assumption, $\dim \dim (H^0(\tilde{X}_B, \mathcal{L}_B) \to H^0(E_o, \mathcal{L}_B)) \geq 2$. By restriction we get the same property at the level of $(\tilde{X}_B, \mathcal{L}_B)$ too. Hence, there exists two distinct $s_1, s_2 \in H^0(\tilde{X}_B, \mathcal{L}_B)$ such that their divisors restricted to $E_o$ (that is, in $EC_a^{-l} (E_o)$) do not agree. Such elements of $EC_a^{-l} (E_o)$ can be reinterpreted as the set of roots of a polynomial of degree $-l(E_o)$. Then one verifies that for any generic $o \in E_o$ there exists constants $\lambda_1, \lambda_2$ such that $s_o := \lambda_1 s_1 + \lambda_2 s_2$ restricted to $E_o$ has a simple root at $o$.

Then the pull–back of $s_o$ to $\tilde{X}_B$ realizes the divisor $l_o + E_o$.

Next we make the following identification. For $o \in E_o$ generic, we consider the Abel map with Chern class $-l_o-E_o$.

The modification of $\pi_o^*(Z_B)$ into $\pi_o^*(Z_B) - E_o$ will be explained/motivated in 6.1.7.

Using 6.1.2 and Theorem 4.1.10 we get that this Abel map is dominant; even more, for any divisor $D_o$ of $\pi_o^*(\mathcal{L}_B)(-E_o)$, the corresponding tangent map at $D_o$ is surjective.

Next we make the following identification. For $o \in E_o$ generic, we consider the Abel map with Chern class $-l_o-E_o$.

We wish to compare this space with the space from 6.1.6. Note that $(l_o-E_o,E_o) = 1$, hence any divisor from $EC_a^{-l} (\pi_o^*(Z_B) - E_o)$ intersects $E_o$ with multiplicity one. Let $EC_a^{-l} - E_o (\pi_o^*(Z_B) - E_o)$ be the Zariski open set of $EC_a^{-l} - E_o (\pi_o^*(Z_B) - E_o)$ consisting of those divisors whose support does not contain $E_o \cap E_v$. We claim that there exists an isomorphism of spaces

Indeed, the isomorphism is induced by pull–back of Cartier divisors via $\pi_o^*$. Let us present the verification in the relevant local chart (for more details see the proof of [NN18a, Theorem 3.1.10]). Fix local coordinates $(x,y)$ in a neighbourhood of $o$ when $E_v = \{x = 0\}$ and let the multiplicity of $Z$ along $E_v$ be $N$. Then the component of a divisor $D$ from $EC_a^{-l} (Z_B)$ with support $o$ (after we eliminate the equivalence via a multiplication by $C^*$) can be given by the equation $f = y + P_0(x) + ym_0$ (modulo $x^N$), where $P_0(x) = \sum_{i \geq 1} a_i x^i$ and $m_0$ belongs to the maximal ideal $m_o$ of $C(x,y)$. The equivalence $\sim$ is multiplication by elements from $1 + m_0$ (modulo $x^N$). If we multiply $f$ by $(1 + m_0)^{-1}$ and we group the $\{x^i\}_{i \geq 1}$ terms we get $f \sim y + P_1(x) + xym_1 (m_1 \in m_o)$. Multiplication by $(1 + x^m)^{-1}$ gives $f \sim y + P_2(x) + x^2ym_1$. By induction $f \sim y + P_N(x)$ (modulo $x^N$). Hence a smooth chart of $EC_a^{-l} (Z_B)$ (up to other product–factors given by other components of $D$ with support disjoint from $o$, and which are transferred by $\pi_o^*$ trivially) can be parametrized as $\{a_i\}^{-1}_{i = 1} \rightarrow \{\text{the class of } y + \sum_{i = 1}^{N-1} a_i x^i\}$. This lifts by $\pi_o = (x = \alpha \beta, y = \beta)$ to the divisor $\beta + \sum_{i = 0}^{N-2} a_i \alpha^i$ (modulo $(\alpha^{N-1})$).
In fact, this product–factor in the chart of $\text{ECA}^{-l_B^{-}c} (Z_B^-)$ extends naturally to \( \{ a_i \}^{N-1}_{i=0} \mapsto \{ \text{the class of } y+\sum_{i=0}^{N-1} a_i x_i \} \), providing a chart for $\text{ECA}^{-l_B^{-}c} (Z_B^-)$, and showing that $\text{ECA}^{-l_B^{-}c} (Z_B^-)$ is a smooth, constructible and irreducible subspace $\text{ECA}^{-l_B^{-}c} (Z_B^-)$ of codimension one.

Note that (since $Z \gg 0$) the dimension of $\text{Pic}^{-l_B^{-}c} (\pi_0^* (Z_B^-) - E_o)$ and $\text{Pic}^{-l_B^{-}c} (Z_B^-)$ are the same, they equal $p_0$.

6.1.9. Then \( \text{6.1.6} \) and \( \text{6.1.7} \) combined give that the restriction of $c_B^{-}c$.

\[ c_{B,o} : \text{ECA}^{-l_B^{-}c} (Z_B^-)_o \rightarrow \text{Pic}^{-l_B^{-}c} (Z_B^-) \]

is dominant and for any divisor $D_o \in \text{ECA}^{-l_B^{-}c} (Z_B^-)_o$ of $\pi_0^* (L_B)$, the corresponding tangent map at $D_o$ is surjective. Then we repeat the constructions and arguments of paragraphs \( \text{6.1.8} \) \( \text{6.1.9} \) from section from the proof of part (2'). Set

\[ \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_o = \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-) \cap \text{ECA}^{-l_B^{-}c} (Z_B^-)_o. \]

Then similarly as in \( \text{6.1.9} \) one proves that

\[ (6.1.10) \]

\[ c_{rel,o} : \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_o \rightarrow r^{-1}(L_B^-) \]

is dominant.

Note also that the space $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_o$, by a similar identification as in \( \text{6.1.8} \) (i.e., its relative version) is isomorphic with a ‘relative ECA–space, hence it is is irreducible for every generic $o \in E_v$.

(This can also be proved by fixing an irreducible Zariski open set in it, cf. \[\text{NN18a} \text{NN19} \text{or6.1.7} \])

Furthermore, by a similar argument as at the end of \( \text{6.1.7} \) $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_o$ is smooth as well for any generic $o$.

6.1.11. Consider again the dominant relative Abel map $c_{rel} : \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-) \rightarrow r^{-1}(L_B^-)$, cf. \( \text{6.1.4} \) and \( \text{6.1.5} \) Let us denote by $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg}$ the Zariski open subset of $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)$ consisting of classes of divisors, which have smooth transversal cuts along the exceptional divisor $E_v$ and also the tangent map of $c_{rel}$ is a submersion. Moreover, set $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o} = \text{ECA}^{-l_B^{-}c} (Z_B^-)_o \cap \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg}$. $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o}$

We denote the restriction of the dominant map $c_{rel,o}$ from \( \text{6.1.10} \) to $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o}$ with the same symbol. Obviously $c_{rel,o} : \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o} \rightarrow r^{-1}(L_B^-)$ is dominant for generic $o \in E_v$.

Finally, we consider the incidence space

\[ \mathcal{J} = \{ (p, D) \in E_v \times \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg} : p \in \{D\} \} \]

together with the two canonical projections $\pi_1 : \mathcal{J} \rightarrow E_v$ and $\pi_2 : \mathcal{J} \rightarrow \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg}$, where $\pi_1((p, D)) = p$ and $\pi_2((p, D)) = D$.

Note, that the map $\pi_2$ is finite and surjective, and for any generic point $o$ of the image of $\pi_1$ one has $\pi_1^{-1}(o) = \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o}$. We can replace $\mathcal{J}$ by a smaller Zariski open set of it, denoted by the same symbol $\mathcal{J}$, such that for any point $o$ of the image of $\pi_1$ one has $\pi_1^{-1}(o) = \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o}$. Note that $\text{im}(\pi_1)$ is a Zariski open in $E_v$.

Consider next the map $c_{rel} \circ \pi_2 : \mathcal{J} \rightarrow r^{-1}(L_B^-)$.

Since for a generic point $o$ the map $c_{rel,o} : \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o} \rightarrow r^{-1}(L_B^-)$ is dominant, we get from the irreducibility of $\text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o}$ for a generic point $D \in \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o}$ the tangent map $T_D c_{rel} : T_D \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_\text{reg,o} \rightarrow T_{c_{rel}(D)} r^{-1}(L_B^-)$ is surjective.

Fix $D$ generic, $\{D\} \cap E_v = \{p_1, \ldots, p_d\}$, where $d = -(l', E_v)$. Then a neighbourhood of $p_1$ of $\text{ECA}^{-l_B^{-}c} (Z_B^-)_{\text{reg,p_1}}$ embeds naturally into a neighbourhood of $x := (p_1, D)$ in $\mathcal{J}$ as a one–codimensional subspace (such that $p_1$ belongs to the $\pi_1$–image of that neighbourhood). In particular, $T_1 := T_D \text{ECA}^{-l_B^{-}c} \cdot c_e (Z_B^-)_{\text{reg,p_1}}$ embeds into $T_x \mathcal{J}$ as a codimension one sub–vectorspace.
Furthermore, the restriction of the tangent map $T_x(c_{rel} \circ \pi_2)$ to $T_1$ is surjective. If we denote the tangent space of the $\pi_2$ fiber $(c_{rel} \circ \pi_2)^{-1}(c_{rel}(D))$ at $x$ by $T_2$, then the last statement means that $T_1$ and $T_2$ are transversal in $T_22$. Since $T_1$ has codimension one, we get that $T_2 \not\subset T_1$. Hence the $\pi_2$ fiber $(c_{rel} \circ \pi_2)^{-1}(c_{rel}(D))$ cannot be contained in $E\mathcal{C}_{m}^{-G} - \mathcal{L}_1(Z_{B|\mathfrak{g},p_1})$.

The same is true for all the points $p_1, \ldots, p_d$. Hence the line bundle $c_{rel}(D) \in r^{-1}(\mathcal{L}_b)$ is base point free.

Since the map $c_{rel}$ is dominant, we obtain that the generic bundle of $r^{-1}(\mathcal{L}_b)$ has no base point.

6.1.12. Hence, we proved that there exists a Zariski open set $U_{\text{Pic}, t} \subset U_{\text{Pic}, r} \subset r^{-1}(\mathcal{L}_b)$ such that its elements have no base points. Then we continue as in parts 5.1.9, 5.1.10 in the proof of part (2'): by a very same type of deformation we can move $\mathcal{L}_b$ into $U_{\text{Pic}, t}$. Finally, we end the proof with similar argument as 5.1.13. This ends the proof of part (3').

For Part (4') notice, that if $(*)$ holds, then by Remark 3.3.4(a) $\text{dim}(H^0(\mathcal{O}_{\bar{X}}(-l'_{\text{top}}))) = 1$, hence the line bundle $\mathcal{O}_{\bar{X}}(-l'_{\text{top}})$ necessarily has base points on $E_v$. Furthermore, by part (2') we know, that there is a section in $H^0(\mathcal{O}_{\bar{X}}(-l'_{\text{top}}))$ whose divisor consists of $-l'_{\text{top}}(E_v)$ disjoint transversal cuts.

In particular, the line bundle $\mathcal{O}_{\bar{X}}(-l'_{\text{top}})$ has $-l'_{\text{top}}(E_v)$ disjoint base points on $E_v$, all of them regular points of $E$. This proves part (4').

7. Proof of Theorem 3.3.6(5')

7.1. The proof will be divided into two parts. The first part is the following.

**Theorem 7.1.1.** Consider a resolution $\bar{X} \to X$ with generic analytic structure and adopt the notations of Theorem 3.3.6(5') and Remark 3.3.7. Then $m_+^\times - m_\times = 1$.

**Proof.** Consider the blow up $b : \bar{X}_{\text{new}} \to \bar{X}$ in a base point $p$ of $L$. Let the new exceptional divisors be $E_{\text{new}}$ and let $F_v$ be the strict transform of $E_v$. Take the line bundle $\mathcal{L}_b := b^* \mathcal{L}(-E_{\text{new}})$ on $\bar{X}_b$.

Since $l'$ is dominant, $H^1(\bar{X}, \mathcal{L}) = 0$ by Theorem 1.1.8. Hence, by Lemma 2.3.3

$$h^1(\bar{X}_b, \mathcal{L}_b) = 1. \hspace{1cm} (7.1.2)$$

Next, for any integer $m \geq 1$ consider the exact sequence $0 \to \mathcal{L}_b(-mF_v) \to \mathcal{L}_b \to \mathcal{L}_{b|mF_v} \to 0$, and let $f : H^0(\bar{X}_b, \mathcal{L}_b) \to H^0(mF_v, \mathcal{L}_{b|mF_v})$ be the cohomological morphism. Then, with the notation $k := -(E_v, l') = (E_v, c_1(L))$, from the long cohomological exact sequence and $(7.1.2)$

$$\text{dim } \text{im}(f) = \chi(\mathcal{L}_b|mF_v) - h^1(\mathcal{L}_b(-mF_v)) + h^1(\mathcal{L}_b) \hspace{1cm} (7.1.3)$$

$$= m(k - 1) + \chi(mF_v) - h^1(\mathcal{L}_b(-mF_v)) + 1. \hspace{1cm} (7.1.4)$$

Note that $\mathcal{L}_{b|\mathcal{L}_b}(-mE_v) = b^* \mathcal{L}(-mF_v - E_{\text{new}})$. This will be compared next with $b^* \mathcal{L}(-b^*(mE_v)) = b^* \mathcal{L}(-mF_v - mE_{\text{new}})$ (and then we can also use $h^1(\bar{X}_b, b^* \mathcal{L}(-b^*(mE_v))) = h^1(\bar{X}, b^* \mathcal{L}(-mF_v - E_{\text{new}}))$).

We claim that $H^0(\bar{X}_b, b^* \mathcal{L}(-b^*(mE_v))) = H^0(\bar{X}_b, b^* \mathcal{L}(-mF_v - E_{\text{new}}))$. Indeed, in general for any line bundle and resolution, if $s \in H^0(\mathcal{G})$ then $-c_1(\mathcal{G}) + \mathcal{E}_s \in S'$. In our situation, if $s \in H^0(\bar{X}_b, b^* \mathcal{L}(-mF_v - E_{\text{new}}))$ then $b^*l' + mF_v + E_{\text{new}} + \mathcal{E}_s(s) \in S'(\bar{X}_b)$. Write $\mathcal{E}_s(s) as kF_v + \ell E_{\text{new}} + D$, where $k, \ell \in \mathbb{Z}_{\geq 0}$ and $(\mathcal{D}, E_{\text{new}}) = 0$. Then $(E_{\text{new}}, b^*l' + (m + k)F_v + (1 + \ell)E_{\text{new}} + D) = (m + k) - (\ell + 1) \leq 0$. Hence $\ell + 1 \geq m$ and $s \in H^0(\bar{X}_b, b^* \mathcal{L}(-b^*(mE_v)))$.

In particular, from the exact sequence

$$0 \to b^* \mathcal{L}(-b^*(mE_v)) \to b^* \mathcal{L}(-mF_v - E_{\text{new}}) \to b^* \mathcal{L}(-mF_v - E_{\text{new}}) \to 0$$
and $\chi(b^*L(-mF_v - E^{new})|_{(m-1)E^{new}}) = -(m - 1)(m - 2)/2$, combined with (7.1.3) gives

$$\dim \text{im}(f) = m(k - 1) + 1 + \chi(mF_v) - (m - 1)(m - 2)/2 - h^1(\tilde{X}, L(mE_v)).$$

Finally, set $\delta := m^+_v - m_v$. By Remark 3.3.7(b) it satisfies $\min_{i \geq 0}\{l' + \delta E_v + l\} = \chi(l') + 1$. On the other hand, this fact together with Theorem 3.2.1(d) (note that $(X, \alpha)$ cannot be rational in the presence of a base point, cf. 3.5) gives

$$h^1(\tilde{X}, L(-\delta E_v)) = \chi(l' + \delta E_v) - \min_{i \geq 0}\{l' + \delta E_v + l\} = \chi(\delta E_v) + \delta k - 1.$$  

Therefore, (7.1.5) combined with (7.1.4) applied for $m = \delta$ give

$$\dim \text{im}(f) = 1 - \delta(\delta - 1)/2.$$  

Note that if $s$ is the generic section in $H^0(\tilde{X}, L)$ then $f(b^*(s))$ is non–zero in $\text{im}(f)$, hence $\dim \text{im}(f) \geq 1$. In particular, $\delta(\delta - 1) \leq 0$, hence $\delta = 1$. \hfill \Box

7.1.6. Note that from Theorem 7.1.1 does not follow yet that $\delta$ (7.1.5) gives

in such a case $L$ in (5'),–(4')–(3')–(2')–(1')–(0') have base points on $E_v$. Now, since both line bundles have base points on $E_v$, both line bundles have base points on $E_v$. This means that $\dim \text{im}(f) = -k + 2$. Similarly as in the previous proof, if $s$ is the generic section of $L$, then $f(b^*s)$ is nonzero in the image, hence $-k + 2 = \dim \text{im}(f) \geq 1$. This can happen only if $k = \dim \text{im}(f) = 1$.

Now, that $p$ is a base point of $\mathcal{L}(-l)$ too. Then, if $s'$ is its generic section given by parts (1')–(4') applied for $\mathcal{L}(-l)$, we obtain $b^*s' \in H^0(\mathcal{L}_0)$. In fact, $b^*s'$ is even in $H^0(\mathcal{L}_0(-E_v))$, but $b^*s' \notin H^0(\mathcal{L}_0(-2E_v))$ by Theorem 7.1.1. Hence $f(b^*s')$ is non–zero in $\text{im}(f)$ too. Looking at the $E_v$ vanishing orders we see that $f(b^*s)$ and $f(b^*s')$ cannot be linearly dependent modulo $H^0(\mathcal{L}_0(-2E_v))$. Hence $\dim \text{im}(f) \geq 2$, a contradiction. \hfill \Box

This ends the proof of Theorem 5.3.10 as well.

Remark 7.1.9. Note that the above proof also says that if both line bundles $\mathcal{L}$, and the ‘universal’ $\mathcal{L}(-l)$ (associated with $\mathcal{L}$ as in Remark 5.3.7) have base points on $E_v$ then $(C_1(\mathcal{L}), E_v) = 1$. That is, in such a case $\mathcal{L}$ has necessarily a unique base point on $E_v$. See Example 8.1.1 when such a situation can happen indeed.
8. Examples

Below $\tilde{X}$ is a normal surface singularity whose link is a rational homology sphere.

8.1. The case of Chern class $l' = Z_K$. Let us fix a resolution graph $\Gamma$. If $l' \in L'$ is ‘sufficiently negative’ (i.e., if each $(l', E_u)$ is sufficiently negative for all $u \in V$) then for any analytic structure supported by $\Gamma$ any line bundle $L \in \text{Pic}(\tilde{X})$ with Chern class $l'$ is base point free; in particular, $l' \in S'_a \setminus \{0\}$ too. For different negativity conditions (imposed by different proofs) see e.g. [CNP06, Th. 4.1], [La83, Th. 3.1], [S-B80, Th. 2, Prop. 4]. The condition $l' \in S'_a \setminus \{0\}$ (versus base point freeness) can be guaranteed by weaker assumptions, in general we require slightly stronger negativity than being in $Z_K + S'$. However, none of these combinatorial assumptions are satisfied in general by $Z_K$. In the next paragraphs we analyse with details exactly this case of $l' = Z_K$.

Assume that $\tilde{X}$ is minimal, i.e. it contains no $(-1)$-curve. Then, by adjunction formula, $Z_K \in S'$. (Recall also that $Z_K = 0$ happens exactly when $\Gamma$ is ADE.) We claim that if $\tilde{X}$ is generic and $\Gamma$ is not ADE then $Z_K \in S'_a \setminus \{0\}$ (that is, $\mathcal{O}_{\tilde{X}}(-Z_K)$ has no fixed components).

In the proof we use 9.2.1(g): we need to show that $\chi(Z_K + l) = \chi(-l) > 0$ for any $l > 0$. Note that from $-l$ there exists $\chi$-nonincreasing generalized Laufer computation sequence which connects $-l$ to $0$, cf. [N05, §7] or [N07, 4.3.3]. Hence $\chi(-l) \geq 0$ (see also [N05, Prop. 5.7]). However, if $\chi(-l) = 0$, then the sequence is necessarily $\chi$-constant, hence at the very last step one has $\chi(-E_u) = 0$ for some $u \in V$. But this means $E_u^2 = -1$, a contradiction.

Note that for an arbitrary analytic structure it is not true that $Z_K \in S'_a \setminus \{0\}$, cf. next example.

Example 8.1.1. Consider the following $\Gamma$, where the $(-2)$-vertices are unmarked.

\[ \begin{array}{c} \bullet \bullet \bullet \ \bullet \ \bullet \ \bullet \\ -3 \ \ \ E_2 \ \ E_1 \end{array} \]

It is an elliptic (integral homology sphere) graph $Z_{\text{min}} = E_1^* + E_2^*$ and $Z_K = E_2^* + Z_{\text{min}} < Z_K$. The length of the elliptic sequence is two (for terminology see e.g. [La77, N99, N99a]), hence $1 \leq p_g \leq 2$, and $\Gamma$ supports two rather different families of analytic structures according to the value of $p_g$. E.g. $\Gamma$ can be realized even by the hypersurface singularity $x^2 + y^3 + z^{11} = 0$. In this case $Z_{\text{max}} = Z_{\text{min}} = E_1^*$, it is the divisor of $z$. In fact, $p_g = 2$, $\text{mult}(X, o) = 2$ and $Z_{\text{max}} = Z_{\text{min}}$ is true for any Gorenstein structure, cf. [N99, N99a]. However, if $Z_{\text{max}} = Z_{\text{min}}$ then $Z_K \notin S_a$. More precisely, by a topological argument on this $\Gamma$, (and for any analytic structure supported on this $\Gamma$) $Z_{\text{min}}$ and $Z_K$ cannot by simultaneously elements of $S_a$. Indeed, if both are realized by some functions, say $f$ and $g$, then (since $-(Z_K, Z_{\text{min}}) = 1$) the degree of the map $(f, g) : (X, o) \rightarrow (\mathbb{C}^2, 0)$ is one. But this can occur only for smooth germs $(X, o)$, which is not the case.

However, as we already proved in 8.1 for the generic analytic structure $Z_{\text{max}} = Z_K$ (hence $Z_{\text{min}} \notin S_a$). (In this case $p_g = 1$ by Theorem 8.2.1(c) and $(X, o)$ is non–Gorenstein [NN18b, 6.9].)

Since $(Z_K, E_2) = -1$ and $\chi(Z_K + Z_{\text{min}}) = 1 = \chi(Z_K) + 1$, by Theorem 8.3.7, $\mathcal{O}_{\tilde{X}}(-Z_K)$ has a (unique) base point on $E_2$. Note that $Z_K + Z_{\text{min}} \in S_a$ is the Chern class $l' + l$ of part (5') in Theorem 8.3.6. Furthermore, $(Z_K + Z_{\text{min}}, E_2) = -1$ and $\chi(2Z_K) = 2 = \chi(Z_K + Z_{\text{min}}) + 1$, hence $\mathcal{O}_{\tilde{X}}(-Z_K - Z_{\text{min}})$ has a (unique) base point on $E_2$ too. However, by Theorem 8.3.6 the two base points are different. Note also that $\text{mult}(X, o) = -Z_{\text{max}}^2 + 1 = 3$.

8.2. Base points of $Z_{\text{max}}$ and the multiplicity in the generic elliptic case. Assume that $\min \chi = 0$. In this case $\Gamma$ is either rational or elliptic (see e.g. [N99a]). In the rational case
\(Z_{\text{max}} = Z_{\text{min}}, \mathcal{O}_{\tilde{X}}(-Z_{\text{max}})\) has no base points, and \(\text{mult}(X, o) = -Z_{\text{min}}^2\) independently of the analytic structure supported on \(\Gamma\) \cite{La77, La91} (see also \cite{X}).

In the sequel we assume that \(\Gamma\) is elliptic. For the simplicity of the presentation we also assume that \(\Gamma\) is numerically Gorenstein (i.e. \(Z_{\Delta} \in \mathbb{L}\)), and that \(\Gamma\) is the dual graph of a minimal good resolution, which is minimal (contains no \((-1)\)–curves). Let \(C\) be the minimally elliptic cycle (for the standard notations and combinatorial properties of elliptic graphs see e.g. \cite{La77, N99, N99b}).

We claim that following facts hold, whenever \(\tilde{X}\) is generic:

1. \(Z_{\text{max}} = Z_{\Delta}\).

2. \(\mathcal{O}_{\tilde{X}}(-Z_{\text{max}})\) has a base point if and only if \(C^2 = -1\). Moreover, if \(C^2 = -1\) then \(\mathcal{O}_{\tilde{X}}(-Z_{\text{max}})\) admits a unique base point (of type \(A_1\)). (For the peculiar structure of the graph when \(C^2 = -1\) and the position of the base point see the discussion below.)

We sketch the arguments. For (1) we use Theorem 3.2.1(g) and we verify that \(Z_{\Delta} = \max \mathcal{M}\).

Indeed, \(\chi(Z_{\Delta}) = 0\) and \(\chi(Z_{\Delta} + l) = \chi(-l) > 0\) for any \(l > 0\) (cf. \ref{3}).

For (2) fix some \(E_v\) such that \((Z_{\Delta}, E_v) < 0\) and \(\chi(Z_{\Delta} + E_v + l) = 1\) for some \(l > 0\). Then \(1 = \chi(E_v + l) - (Z_{\Delta}, E_v) - (Z_{\Delta}, l)\) with \(\chi(E_v + l) \geq 0\) (ellipticity), \(-(Z_{\Delta}, E_v) > 0\) (assumption), \(-(Z_{\Delta}, l) \geq 0\) \((Z_{\Delta} \in \mathcal{S})\). Hence necessarily (a) \(\chi(E_v + l) = 0\), (b) \((Z_{\Delta}, E_v) = -1\), (c) \((Z_{\Delta}, l) = 0\). From (b) follows that \(E_v^2 = -3\), from (c) we obtain that \((Z_{\Delta}, E_v) = 0\) for any on \(E_v\) from the support \(\{l\}\), hence \(\{l\}\) consists of \((-2)\)–curves (in particular \(E_v \not\subset \{l\}\)), and (a) implies that \((l, E_v) = \chi(l) + 1 \geq 1\), hence \(E_v\) is adjacent with \(|l|\). Since \(\chi(E_v + l) = 0\), by the definition of \(C\), one has \(E_v + l \geq C\). Since \(C\) cannot have only \((-2)\)–curves \((\dagger)\) \(E_v \leq C \leq E_v + l\). In particular, \(E_v\) is uniquely determined by this property.

Hence, by (\dagger), \(C\) itself has the form \(E_v + l_0\), where \(E_v \not\subset \{l_0\}\), \(E_v\) is adjacent to \({l_0}\), and \({l_0}\) consists of \((-2)\)–curves. Then \(l_0\) verifies (a)-(b)-(c), i.e. \(\chi(Z_{\Delta} + E_v + l_0) = 1\) (and \(l_0\) is minimal with this property). Since by general theory \(C^2 = (C, Z_{\Delta}), C^2 = (C, Z_{\Delta}) = (E_v + l_0, Z_{\Delta}) = -1\).

All the possible graphs of elliptic cycles \(C\) with \(C^2 = -1\) are listed in \cite{La77}.

Finally observe that if for a generic singularity with arbitrary graph \(\Gamma\), if \(Z_{\Delta} = Z_{\text{max}}\) then by Theorem 3.2.1(g) \(\Gamma\) is necessarily elliptic (hence (1) above is an ‘if and only if’ characterization).

Example 8.2.1. Consider the following non-elliptic plumbing graph (the left picture). It has \(\min \chi = -1\). It supports several analytic structures, the possible values for the geometric genus are \(1 - \min \chi = 2 \leq p_g \leq 3\), cf. \cite{NO17}. If \(\tilde{X}\) is generic then \(p_g = 2\) and \(Z_{\text{max}} = 2E_v^*\).

\[
\begin{array}{cccccccc}
-3 & -1 & -13 & -1 & -3 \\
\end{array}
\]

\[
\begin{array}{c}
-2 & -2 \\
\end{array}
\]

There is only one \(E_u\) with \((E_u, Z_{\text{max}}) < 0\), namely \(E_v\), and \((E_v, Z_{\text{max}}) = -2\). Moreover, \(Z_{\Delta} \geq E_v + Z_{\text{max}}\) and \(\chi(Z_{\Delta}) = 0 = \chi(Z_{\text{max}}) + 1\). Hence \(Z_{\text{max}}\) has two base points on \(E_v\) and \(\text{mult}(X, o) = -Z_{\text{max}}^2 + 2 = 6\). (This is compatible with \cite{NO17}.)

We wish to emphasize that there exists a Gorenstein (even complete intersection) analytic structure supported on \(\Gamma\), which has the very same \(Z_{\text{max}} = 2E_v^*\), however in that case \(\mathcal{O}_{\tilde{X}}(-Z_{\text{max}})\) has no base points, hence \(\text{mult}(X, o) = -Z_{\text{max}}^2 = 4\) (and \(p_g = 3\)).

Furthermore, there exists also a (Kodaira/Kulikov) type analytic structure supported on \(\Gamma\) with a smaller maximal ideal cycle, namely \(Z_{\text{max}} = Z_{\text{min}} = E_v^*\). In this case \(Z_{\text{max}}^2 = -1\), \(\mathcal{O}_{\tilde{X}}(-Z_{\text{max}})\) has a unique base point of \(A_2\)–type on \(E_v\), hence \(\text{mult}(X, o) = 3\). (In this case \(p_g = 3\) too.) For details see \cite{NO17}.
9. Generic line bundles of arbitrary singularities.

9.1. In [NN18a] we fixed an analytic type \( \tilde{X} \) (not necessarily generic) and we determined combinatorially several cohomological properties of generic line bundles \( L_{\text{gen}} \in \text{Pic}^{-l'}(\tilde{X}) \). On the other hand, for a fixed resolution graph \( \Gamma \), the philosophy/aim of [NN18b] was to show that (restricted) natural line bundles with given Chern class, associated with generic analytic structures supported on \( \Gamma \), behave cohomologically as the generic line bundles (of an arbitrary singularity) with the same Chern class.

In the present note, in Theorem 3.3.6 we establish several properties of (restricted) natural line bundles of generic singularities. It is natural to ask whether these properties are valid for generic line bundles of an arbitrary singularity. The next theorem answers positively.

**Theorem 9.1.1.** Let \( \tilde{X} \) be a resolution of an arbitrary singularity (with rational homology sphere link). Fix \( l' \in S' \setminus \{0\} \) such that \( c^{-l'}(Z) \) is dominant for \( Z \gg 0 \). Then the properties (1')–(5') of Theorem 3.3.6 hold for a generic element \( L_{\text{gen}} \) of \( \text{Pic}^{-l'}(\tilde{X}) \) (instead of \( L \) of Theorem 4.1.10).

**Proof.** We can assume that \( (X, a) \) is not rational, otherwise the argument from 3.5 holds identically.

Take a generic divisor \( \tilde{D} \) with Chern class \( -l' \). Then all components of \( \tilde{D} \) are smooth, \( \tilde{D} \) intersects \( E \) transversally, and \( L_{\text{gen}} = \mathcal{O}_{\tilde{X}}(\tilde{D}) \) satisfies (1')–(2').

Next we prove (3'). Sometimes in the notations we will omit the symbol \( Z \).

Consider \( l' \) and \( E_v \) as in the assumptions of (3'). If \( L \) is generic then by Theorem 4.1.8 \( h^1(Z, L) = 0 \), and by Theorem 5.3.1 of [NN18a]

\[
h^1(Z, L(-E_v)) = \chi(l' + E_v) - \min_{l \geq 0} \chi(l' + l).
\]

Therefore, from the exact sequence \( 0 \rightarrow L(-E_v) \rightarrow L \rightarrow L|_{E_v} \rightarrow 0 \) we get

\[
\dim H^0(L)/H^0(L(-E_v)) = -(l', E_v) + 1 - h^1(L(-E_v)) = \min_{l \geq E_v} \chi(l' + l) - \chi(l') \geq 2.
\]

In particular, if \( \tilde{D} \) is a generic divisor with Chern class \( -l' \) with \( \tilde{D} \cap E_v = \{p_1, \ldots, p_k\} \) (\( k = -(l', E_v) \)), and \( L = \mathcal{O}_Z(\tilde{D}) \), then not all the points \( p_i \) are base points of \( L \). We wish to show that in fact non of them is a base point. This basically will follow from the irreducibility of an incidence space.

We consider two incidence spaces

\[
\mathcal{J} = \{ (p, D) \in E_v \times \text{ECA}^{-l'}(Z) : p \in |D|, \ D \mid Z \text{ and } \tilde{D} \text{ intersects } E \text{ transversally} \},
\]

\[
\mathcal{J}_b = \{ (p, D) \in \mathcal{J} : p \text{ is a base point of } \mathcal{O}_Z(D) \}.
\]

Let \( \pi_2 : \mathcal{J} \to \text{ECA}^{-l'}(Z) \) be the second projection, and let \( \pi_{2,b} \) be its restriction to \( \mathcal{J}_b \). They are morphisms with finite fibers. If \( c^{-l'} \circ \pi_{2,b} \) is not dominant, then for \( L \in \text{Pic}^{-l'} \) generic the fiber \( (c^{-l'} \circ \pi_{2,b})^{-1}(L) = \emptyset \), hence we are done. Hence, in the sequel we assume that \( c^{-l'} \circ \pi_{2,b} \) is dominant. Then we can fix a non–empty Zariski open set \( U \) in \( \text{Pic}^{-l'} \) such that \( c^{-l'} \circ \pi_{2,b} = (c^{\infty}) \) fibrations over \( U \) and \( \pi_2 \) is a regular covering over \( U' := (c^{-l'})^{-1}(U) \). Furthermore, we can assume that the same facts are true for the restriction \( \pi_{2,b} \) and for the very same \( U \). We will replace the spaces \( \mathcal{J} \) and \( \mathcal{J}_b \) with their subspaces sitting over \( U \).

We claim that \( \mathcal{J} \) is irreducible. Indeed, \( U \) is irreducible, all the fibers of \( c^{-l'} \) are irreducible (cf. 4.1), hence \( U' \) is irreducible. We need to show that the regular covering \( \mathcal{J} \to U' \) is irreducible. For this fix a divisor \( \tilde{D} \) with \( \tilde{D} \cap E_v = \{p_1, \ldots, p_k\} \) as above. Then, moving along a path the components of the divisor (hence the intersection points \( \{p_i\} \)) there exists a (monodromy) path in \( \mathcal{J} \) such that the starting point corresponds to a fixed order of \( \{p_1, \ldots, p_k\} \) and the ending point any permutation of them. (Here we need the fact that the regular part of \( E_v \) is also connected, and that any real
one–dimensional path in $\text{Pic}^{-l}$ can be perturbed to be in $U$.) This shows that the covering $\pi_2$ over $U'$ is irreducible, hence $\mathcal{I}$ is irreducible.

On the other hand, the covering $\mathcal{J}_b$ is a proper subspace of $\mathcal{I}$, since not all the points $\{p_i\}_i$ are base points. This contradicts the irreducibility of $\mathcal{I}$. This ends the proof of (4').

Part (4') follows from (1')–(3') (as in 6.1.12).

Finally, the proof of (5') follows identically as the proof of Theorem 3.3.6 (5') in section 7, using again the fact that $h^*(\tilde{X}, O_{\tilde{X}}(-l')) = h^*(\tilde{X}, L_{\text{gen}})$ and both cohomology groups have the same numerical characterizations needed in the proof of (7.1.5) (cf. Theorem 5.3.1 of [NN18a]). □

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