Discrete index transforms with Bessel and modified Bessel functions

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Abstract

Discrete analogues of the index transforms, involving Bessel and the modified Bessel functions are introduced and investigated. The corresponding inversion theorems for suitable classes of functions and sequences are established.

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1 Introduction and preliminary results

Our goal in this paper is to investigate the mapping properties and prove inversion formulas for the following six transformations between suitable sequences \( \{a_n\}_{n \geq 1} \) and functions \( f \) in terms of the series and integrals, respectively, which are associated with Bessel and modified Bessel functions \( J_\mu(z), I_\mu(z), K_\mu(z) \) (cf. [1], Ch. 10), namely,

\[
f(x) = e^{-x/2} \sum_{n=1}^{\infty} a_n \text{Re} \left[ I_{in} \left( \frac{x}{2} \right) \right], \quad x > 0,
\]

\[
a_n = \int_0^{\infty} \text{Re} \left[ I_{in} \left( \frac{x}{2} \right) \right] f(x) e^{-x/2} dx, \quad n \in \mathbb{N}_0,
\]

\[
f(x) = \sum_{n=0}^{\infty} \frac{a_n}{\cosh(\pi n/2)} \text{Re} \left[ J_{in} \left( 2\sqrt{2}x \right) \right] K_{in} \left( 2\sqrt{2}x \right), \quad x > 0,
\]

1
\[ a_n = \frac{1}{\cosh(\pi n/2)} \int_0^\infty \text{Re} \left[ J_{in} \left( 2\sqrt{2x} \right) \right] K_{in} \left( 2\sqrt{2x} \right) f(x) \, dx, \quad n \in \mathbb{N}_0, \quad (1.4) \]

\[ f(x) = \sum_{n=1}^{\infty} \frac{a_n}{\sinh(\pi n/2)} \text{Im} \left[ J_{in} \left( 2\sqrt{2x} \right) \right] K_{in} \left( 2\sqrt{2x} \right), \quad x > 0, \quad (1.5) \]

\[ a_n = \frac{1}{\sinh(\pi n/2)} \int_0^\infty \text{Im} \left[ J_{in} \left( 2\sqrt{2x} \right) \right] K_{in} \left( 2\sqrt{2x} \right) f(x) \, dx, \quad n \in \mathbb{N}. \quad (1.6) \]

Here \( i \) is the imaginary unit and \( \text{Re}, \ \text{Im} \) denote the real and imaginary parts of a complex-valued function. We call transformations (1.1)-(1.6) the discrete index transforms (cf. [3]). For instance, continuous analogues of transformations (1.1), (1.2) were considered by the author in [4]. Bessel function \( J_\nu(z), z, \nu \in \mathbb{C} \) of the first kind is a solution of the Bessel differential equation

\[ z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - \nu^2) u = 0. \quad (1.7) \]

This function has the following asymptotic behavior at infinity and near the origin

\[ J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{\pi}{4} (2\nu + 1) \right) [1 + O(1/z)], \quad z \to \infty, \quad |\arg z| < \pi, \quad (1.8) \]

\[ J_\nu(z) = O(z^\nu), \quad z \to 0, \quad (1.9) \]

The modified Bessel functions of the first and second kind \( I_\nu(z), K_\nu(z), z, \nu \in \mathbb{C} \), in turn, are solutions of the modified Bessel differential equation

\[ z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2) u = 0, \quad (1.10) \]

having the corresponding asymptotic behavior

\[ I_\nu(z) = O \left( |z|^{\nu} \right), \quad z \to 0, \quad (1.11) \]

\[ I_\nu(z) = O \left( \frac{e^z}{\sqrt{2\pi z}} \right), \quad z \to \infty, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \quad (1.12) \]

\[ K_\nu(z) = O \left( |z|^{-\nu} \right), \quad z \to 0, \quad \nu \neq 0, \quad K_0(z) = O \left( \log(|z|) \right), \quad z \to 0, \quad (1.13) \]

\[ K_\nu(z) = O \left( \sqrt{\frac{\pi}{2z}} e^{-z} \right), \quad z \to \infty, \quad |\arg z| < \frac{3\pi}{2}. \quad (1.14) \]

These functions are related by the equality

\[ K_\nu(z) = \frac{\pi}{2 \sin(\pi \nu)} [I_{-\nu}(z) - I_\nu(z)]. \quad (1.15) \]
The modified Bessel function of the first kind satisfies the inequality (cf. [4], p. 138)

\[ |I_{i\tau}(x)| \leq I_0(x) \left( \frac{\sinh(\pi \tau)}{\pi \tau} \right)^{1/2}, \quad x > 0, \ \tau \in \mathbb{R}. \quad (1.16) \]

Meanwhile, the modified Bessel function of the second kind obeys the estimate [3], p. 219

\[ |K_{i\tau}(x)| \leq A \frac{x^{-1/4}}{\sqrt{\sinh(\pi \tau)}}, \quad x, \ \tau > 0, \quad (1.17) \]

where \( A > 0 \) is an absolute constant. The Mellin-Barnes integral representation for the kernel in (1.1), (1.2) is established in [4]. We have

\[ \frac{\sqrt{\pi}}{\cosh(\pi \tau)} e^{-x/2} \text{Re} \left[ I_{i\tau} \left( \frac{x}{2} \right) \right] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1/2-s) \Gamma(s+i\tau) \Gamma(s-i\tau)}{\Gamma(s) \Gamma(1-s)} x^{-s} ds, \quad (1.18) \]

where \( x > 0, \ \tau \in \mathbb{R} \), \( \Gamma(z) \) is the Euler gamma function (cf. [1], Ch. 5), and the contour is the vertical straight line \( s = \gamma + it \), \( 0 < \gamma < 1/2 \), \( t \in \mathbb{R} \) in complex plane, separating the left-hand simple poles from the right-hand ones in the numerator of the integrand. In the meantime, applying the Stieltjes transform to both sides of (1.18) and changing the order of integration by Fubini’s theorem via the estimate

\[ \int_0^\infty \frac{t^{-\gamma} dt}{x+t} \int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Gamma(1/2-s) \Gamma(s+i\tau) \Gamma(s-i\tau)}{\Gamma(s) \Gamma(1-s)} \right| ds < \infty, \]

we find, employing Entries 8.4.2.5 and 8.4.23.5 in [2], Vol. III, the following integral representation of the modified Bessel function of the second kind (1.15)

\[ K_{i\tau} \left( \frac{x}{2} \right) = e^{-x/2} \int_0^\infty \frac{e^{-t/2}}{x+t} \text{Re} \left[ I_{i\tau} \left( \frac{t}{2} \right) \right] dt. \quad (1.19) \]

But since (cf. Entry 2.16.6.1 in [2], Vol. II)

\[ \int_0^\infty e^{-x \cosh(u)} K_{in}(x) dx = \frac{\pi \sin(nu)}{\sinh(u) \sinh(\pi n)}, \quad n \in \mathbb{N}, \ u \in \mathbb{R}, \quad (1.20) \]

we derive from (1.19)

\[ \frac{\pi \sin(nu)}{\sinh(u) \sinh(\pi n)} = \int_0^\infty e^{-x(1+\cosh(u))} \int_0^\infty \frac{e^{-t/2}}{2x+t} \text{Re} \left[ I_{in} \left( \frac{t}{2} \right) \right] dt dx. \quad (1.21) \]

The integral with respect to \( x \) is calculated in [2], Vol. I, Entry 2.3.4.3 in terms of the upper incomplete gamma function \( \Gamma(\nu, x) \) (cf. [1], Ch. 8),
\[ \Gamma(\nu, x) = \int_x^\infty t^{\nu-1}e^{-t}dt, \]

and we have
\[ \int_0^\infty \frac{e^{-x(1+\cosh(u))}}{2x+t} \, dx = \frac{1}{2} e^{t\cosh^2(u/2)} \Gamma \left(0, t \cosh^2 \left(\frac{u}{2}\right)\right). \tag{1.22} \]

Thus we obtain from (1.22) the value of the integral
\[ \int_0^\infty e^{t(\cosh^2(u/2)-1/2)} \Gamma \left(0, t \cosh^2 \left(\frac{u}{2}\right)\right) \Re \left[I_{in} \left(\frac{t}{2}\right)\right] \, dt = \frac{2\pi \sin(nu)}{\sinh(u) \sinh(\pi n)}. \tag{1.23} \]

We note that the interchange of the order of integration in (1.22) is guaranteed by Fubini's theorem, owing to the estimate
\[ \int_0^\infty \frac{e^{-x(1+\cosh(u))}}{2x+t} \left|I_{in} \left(\frac{t}{2}\right)\right| dt \, dx \leq \frac{1}{2\sqrt{2}} \int_0^\infty \frac{e^{-x(1+\cosh(u))}}{\sqrt{x}} \frac{1}{\sqrt{t}} \left|I_{in} \left(\frac{t}{2}\right)\right| dt \]
\[ + \int_0^\infty \frac{e^{-x(1+\cosh(u))}}{t} \frac{1}{t} \left|I_{in} \left(\frac{t}{2}\right)\right| dt < \infty, \tag{1.24} \]

and asymptotic formulas (1.11), (1.12) for the modified Bessel function of the first kind.

The Mellin-Barnes integrals for kernels in (1.3)-(1.6) are given in [2], Vol. III, Entry 8.4.23.11

\[ \frac{1}{\cosh(\pi\tau/2)} \Re \left[J_{i\tau} \left(2\sqrt{2x}\right)\right] K_{i\tau} \left(2\sqrt{2x}\right) = \frac{1}{16\pi \sqrt{\pi i}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma((1+s)/2) \Gamma((s+i\tau)/2) \Gamma((s-i\tau)/2)}{\Gamma(1-s/2)} x^{-s} \, ds, \tag{1.25} \]

\[ \frac{1}{\sinh(\pi\tau/2)} \Im \left[J_{i\tau} \left(2\sqrt{2x}\right)\right] K_{i\tau} \left(2\sqrt{2x}\right) = -\frac{1}{16\pi \sqrt{\pi i}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/2) \Gamma((s+i\tau)/2) \Gamma((s-i\tau)/2)}{\Gamma((1-s)/2)} x^{-s} \, ds, \tag{1.26} \]

where \( x, \gamma > 0, \tau \in \mathbb{R} \). Hence, we employ Entries 8.4.5.1 and 8.4.23.1 in [2], Vol. III to find from (1.25)
\[
\frac{1}{\cosh(\pi \tau/2)} \int_{0}^{\infty} \sin(xt) \text{Re} \left[ J_{\tau} \left( 2\sqrt{2t} \right) \right] K_{\tau} \left( 2\sqrt{2t} \right) dt
\]

\[
= \frac{1}{16\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma \left( \frac{s + i\tau}{2} \right) \Gamma \left( \frac{s - i\tau}{2} \right) 2^{-s} x^{s-1} ds = \frac{1}{2x} K_{\tau} \left( \frac{4}{x} \right). \quad (1.27)
\]

The interchange of the order of integration can be justified via Fubini’s theorem and Stirling’s asymptotic formula for the gamma function [1], Ch. 5, splitting the integral by \( t \) over \((0,1)\) and \((1,\infty)\) and choosing \( \gamma \in (0,1) \) and \( \gamma > 1 \), correspondingly. Then with \((1.20)\), Entry 2.16.22.9 in [2], Vol. II and asymptotic formulas \((1.8)\), \((1.9)\), \((1.13)\), \((1.14)\) we derive the value of the integral \((n \in \mathbb{N}, \ u \in \mathbb{R})\)

\[
\int_{0}^{\infty} \text{Im} \left[ K_{0} \left( 4e^{\pi i/4} \cosh^{1/2}(u) \sqrt{t} \right) \right] \text{Re} \left[ J_{\tau n} \left( 2\sqrt{2t} \right) \right] K_{\tau n} \left( 2\sqrt{2t} \right) dt
\]

\[
= -\frac{\pi \sin(nu)}{32 \sinh(u) \sinh(\pi n/2)}. \quad (1.28)
\]

In the same manner we establish the value of the integral

\[
\int_{0}^{\infty} \text{Re} \left[ K_{0} \left( 4e^{\pi i/4} \cosh^{1/2}(u) \sqrt{t} \right) \right] \text{Im} \left[ J_{\tau n} \left( 2\sqrt{2t} \right) \right] K_{\tau n} \left( 2\sqrt{2t} \right) dt
\]

\[
= -\frac{\pi \sin(nu)}{32 \sinh(u) \cosh(\pi n/2)}. \quad (1.29)
\]

In the sequel we will provide existence conditions for discrete transformations \((1.1)-(1.6)\) and establish their inversion formulas for suitable sequences and functions. To do this, we will employ, in particular, classical Fourier series for Lipschitz functions.

## 2 Inversion theorems

We begin with

**Theorem 1.** Let a sequence \( \{a_n\}_{n \in \mathbb{N}} \) satisfy the condition

\[
\sum_{n=1}^{\infty} |a_n| e^{\pi n/2 \sqrt{n}} < \infty. \quad (2.1)
\]

Then the discrete transformation \((1.1)\) can be inverted by the formula

\[
a_n = \frac{1}{\pi^2} \sinh(\pi n) \int_{0}^{\infty} \Phi_n(x) f(x) dx, \ n \in \mathbb{N}_0, \quad (2.2)
\]

where the kernel \( \Phi_n(x) \) is defined by
\[ \Phi_n(x) = e^{-x/2} \int_0^\pi e^x \cosh^2(u/2) \Gamma \left(0, x \cosh^2 \left(\frac{u}{2}\right)\right) \sinh(u) \sin(nu) du, \quad x > 0, \quad n \in \mathbb{N}_0, \quad (2.3) \]

and integral (2.2) converges absolutely.

**Proof.** Indeed, substituting (1.1) and (2.3) on the right-hand side of (2.2), we appeal to the estimate, using inequality (1.16), integral (1.22) and condition (2.1), to deduce

\[
\int_0^\infty |\Phi_n(x)f(x)| \, dx \leq \int_0^\infty \int_0^\pi e^{x(x^2/2 - 1)} \Gamma \left(0, x \cosh^2 \left(\frac{u}{2}\right)\right) \sinh(u) \\
\times \sum_{m=1}^{\infty} |a_m| \left| \text{Re} \left[I_{im} \left(\frac{x}{2}\right)\right] \right| \, dx \, du \leq 2 \int_0^\pi \sinh(u) \int_0^\infty e^{-x/2} I_0 \left(\frac{x}{2}\right) \int_0^\infty e^{-y(1 + \cosh(u))} \frac{dy \, dx}{y + x} \\
\times \sum_{m=1}^{\infty} |a_m| \frac{e^{\pi m/2}}{\sqrt{m}} \leq \sum_{m=1}^{\infty} |a_m| \frac{e^{\pi m/2}}{\sqrt{m}} \left[ \frac{1}{\sqrt{2}} \int_0^\pi \sinh(u) \int_0^1 e^{-x/2} I_0 \left(\frac{x}{2}\right) \frac{dx}{x} \right] < \infty.
\]

Consequently, interchanging the order of integration and summation, we recall (1.23) to obtain

\[
\frac{1}{\pi^2} \sinh(\pi n) \int_0^\infty \Phi_n(x)f(x) \, dx = \frac{2}{\pi} \sinh(\pi n) \sum_{m=1}^{\infty} \frac{a_m}{\sinh(\pi m)} \int_0^\pi \sin(nu) \sin(mu) du = a_n.
\]

Theorem 1 is proved. \(\square\)

The discrete transformation (1.2) can be inverted by the following theorem.

**Theorem 2.** Let \(f\) be a complex-valued function on \(\mathbb{R}_+\) which is represented by the integral

\[
f(x) = \int_{-\pi}^\pi e^{x(x^2/2 - 1/2)} \Gamma \left(0, x \cosh^2 \left(\frac{u}{2}\right)\right) \psi(u) \sinh(u) du, \quad x > 0, \quad (2.4)
\]

where \(\psi\) is a \(2\pi\)-periodic function, satisfying the Lipschitz condition on \([-\pi, \pi]\), i.e.
\[ |\psi(u) - \psi(v)| \leq C|u - v|, \quad \forall \ u, v \in [-\pi, \pi], \quad (2.5) \]

where \( C > 0 \) is an absolute constant. Then for all \( x > 0 \) the following inversion formula for transformation (1.2) holds

\[ f(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \sinh(\pi n) \Phi_n(x)a_n, \quad (2.6) \]

where \( \Phi_n \) is defined by (2.3).

**Proof.** Plugging the right-hand side of the representation (2.4) in (1.2), we interchange the order of integration, employ (1.23) to obtain

\[ a_n = \frac{2\pi}{\sinh(\pi n)} \int_{-\pi}^{\pi} \psi(u) \sin(nu)du. \quad (2.7) \]

This interchange is permitted due to estimates (1.24). Then we substitute \( a_n \) by (2.7) and \( \Phi_n \) by (2.3) into the partial sum of the series (2.6) \( S_N(x) \), and it becomes

\[ S_N(x) = \frac{1}{\pi} \sum_{n=1}^{N} \int_{-\pi}^{\pi} e^{x(\cosh^2(t/2) - 1/2)} \Gamma \left( 0, x \cosh^2 \left( \frac{t}{2} \right) \right) \sinh(t) \sin(nt)dt \]

\[ \times \int_{-\pi}^{\pi} \psi(u) \sin(nu)du. \quad (2.8) \]

Hence we calculate the sum in (2.8), using the known identity

\[ \sum_{n=0}^{N} \sin(nt) \sin(nu) = \frac{1}{4} \left[ \frac{\sin((2N + 1)(u-t)/2)}{\sin((u-t)/2)} - \frac{\sin((2N + 1)(u+t)/2)}{\sin((u+t)/2)} \right], \quad (2.9) \]

and equality (2.8) becomes

\[ S_N(x) = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{x(\cosh^2(t/2) - 1/2)} \Gamma \left( 0, x \cosh^2 \left( \frac{t}{2} \right) \right) \sinh(t) \]

\[ \times \int_{-\pi}^{\pi} [\psi(u) + \psi(-u)] \frac{\sin((2N + 1)(u-t)/2)}{\sin((u-t)/2)} dudt. \quad (2.10) \]

Since \( \psi \) is 2\( \pi \)-periodic, we treat the latter integral with respect to \( u \) as follows

\[ \int_{-\pi}^{\pi} [\psi(u) + \psi(-u)] \frac{\sin((2N + 1)(u-t)/2)}{\sin((u-t)/2)} du \]
\[
\int_{t-\pi}^{t+\pi} \left[ \psi(u) + \psi(-u) \right] \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} du
= \int_{-\pi}^{\pi} \left[ \psi(u+t) + \psi(-u-t) \right] \frac{\sin((2N+1)u/2)}{\sin(u/2)} du.
\]

Moreover,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u+t) + \psi(-u-t) \right] \frac{\sin((2N+1)u/2)}{\sin(u/2)} du - \psi(t) - \psi(-t)
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u+t) + \psi(-u-t) - \psi(t) - \psi(-t) \right] \frac{\sin((2N+1)u/2)}{\sin(u/2)} du.
\]

When \( u + t > \pi \) or \( u + t < -\pi \) then we interpret the value \( \psi(u+t) \) by formulas
\[
\psi(u+t) - \psi(t) = \psi(u+t-2\pi) - \psi(t-2\pi),
\]
\[
\psi(u+t) - \psi(t) = \psi(u+t+2\pi) - \psi(t+2\pi),
\]
respectively. Then due to the Lipschitz condition (2.5) we have the uniform estimate for any \( t \in [-\pi, \pi] \)
\[
\frac{|\psi(u+t) + \psi(-u-t) - \psi(t) - \psi(-t)|}{\sin(u/2)} \leq 2C \left| \frac{u}{\sin(u/2)} \right|.
\]

Therefore, owing to the Riemann-Lebesgue lemma
\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u+t) + \psi(-u-t) - \psi(t) - \psi(-t) \right] \frac{\sin((2N+1)u/2)}{\sin(u/2)} du = 0 \quad (2.11)
\]
for all \( t \in [-\pi, \pi] \). Moreover, since (see (1.22))
\[
\int_{-\pi}^{\pi} e^{x(cosh(t)/2)-1/2} \Gamma \left(0, x \cosh^2 \left(\frac{t}{2}\right)\right) \sinh(t) dt
\leq Be^{-x/2} \int_{-\pi}^{\pi} \left| \sinh(t) \right| dt \int_{-\pi}^{\pi} \left| \frac{u}{\sin(u/2)} \right| du < \infty,
\]
where \( B > 0 \) is a constant. Therefore via the dominated convergence theorem it is possible to pass to the limit when \( N \to \infty \) under the integral sign, and we derive from (2.11)
\[
\lim_{N \to \infty} \int_{-\pi}^{\pi} e^{x \left( \cosh^2 \left( \frac{t}{2} \right) - \frac{1}{2} \right)} \Gamma \left( 0, x \cosh^2 \left( \frac{t}{2} \right) \right) \sinh(t) \\
\times \int_{-\pi}^{\pi} [\psi(u + t) + \psi(-u - t) - \psi(t) - \psi(-t)] \frac{\sin \left( (2N + 1)u/2 \right) \sin(u/2)}{\sin(u/2)} dudt = 0.
\]

Hence, combining with (2.10), we obtain due to the definition of \( f \)
\[
\lim_{N \to \infty} S_N(x) = \frac{1}{2} \int_{-\pi}^{\pi} e^{x \left( \cosh^2 \left( \frac{t}{2} \right) - \frac{1}{2} \right)} \Gamma \left( 0, x \cosh^2 \left( \frac{t}{2} \right) \right) [\psi(t) - \psi(-t)] \sinh(t) dt = f(x),
\]
where the integral (2.4) converges since \( \psi \in C[0, \pi] \). Thus we established (2.6), completing the proof of Theorem 2.

For the discrete Re-transform (1.3) with the product of Bessel functions we have the following result.

**Theorem 3.** Let a sequence \( a = \{a_n\}_{n \in \mathbb{N}} \in l_1 \), i.e. satisfy the condition
\[
||a||_1 = \sum_{n=1}^{\infty} |a_n| < \infty.
\] (2.12)

Then the discrete transformation (1.3) has the inversion formula
\[
a_n = -\frac{32}{\pi^2} \sinh(\pi n) \int_{0}^{\pi} \Psi_n(x) f(x) dx, \; n \in \mathbb{N}_0,
\] (2.13)
where the kernel \( \Psi_n(x) \) is defined by
\[
\Psi_n(x) = \int_{0}^{\pi} \text{Im} \left[ K_0 \left( 4e^{\pi/4} \cosh^{1/2}(u) \sqrt{x} \right) \right] \sinh(u) \sin(nu) du, \; x > 0, \; n \in \mathbb{N}_0,
\] (2.14)
and integral (2.13) converges absolutely.

**Proof.** In fact, doing in the same manner as in the proof of Theorem 1, we substitute (1.3) and (2.14) on the right-hand side of (2.13) and interchange the order of integration and summation. Then we obtain, using integral (1.28), the equalities
\[
-\frac{32}{\pi^2} \sinh(\pi n) \int_{0}^{\pi} \Psi_n(x) f(x) dx = \frac{2}{\pi} \sinh(\pi n) \sum_{m=1}^{\infty} \frac{a_m}{\sinh(\pi m)} \int_{0}^{\pi} \sin(nu) \sin(mu) du = a_n,
\]
proving (2.13). To justify the interchange of the order of integration and summation we recall Fubini’s theorem by virtue of the estimate
\[
\int_0^\infty |\Psi_n(x)f(x)| \, dx \leq \int_0^\infty I_0 \left( 2\sqrt{2x} \right) K_0^2 \left( 2\sqrt{2x} \right) dx
\]
\[
\times \int_0^\pi \sinh(u) \left| a_0 \right| + \sum_{m=1}^\infty \frac{|a_m| \sqrt{\sinh(\pi m)}}{\sqrt{\pi m} \cosh(\pi m/2)} \right]
\leq ||a||_1 (\cosh(\pi) - 1) \int_0^\infty I_0 \left( 2\sqrt{2x} \right) K_0^2 \left( 2\sqrt{2x} \right) dx < \infty, \tag{2.15}
\]
which is based, in turn, on definitions of Bessel functions, inequalities (1.16), \( |K_{\tau}(x)| \leq K_0(x), \quad x > 0, \tau \in \mathbb{R} \) and asymptotic formulas (1.11)-(1.14). Theorem 3 is proved.

For the companion (1.4) the inversion formula in terms of the series is given by the following theorem.

**Theorem 4.** For a class of functions \( f \) being represented by the integral
\[
f(x) = -32 \int_{-\pi}^\pi \psi(u) \sinh(u) \Im \left[ K_0 \left( 4e^{\pi i/4} \cosh^{1/2}(u) \sqrt{x} \right) \right] du, \quad x > 0, \tag{2.16}
\]
where \( \psi \) is a 2\( \pi \)-periodic function, satisfying the Lipschitz condition (2.5) on \([-\pi, \pi]\), the following inversion formula holds
\[
f(x) = -\frac{32}{\pi^2} \sum_{n=1}^\infty \sinh(\pi n) \Psi_n(x) a_n, \tag{2.17}
\]
where \( a_n \) is given reciprocally by (1.4) and \( \Psi_n \) is defined by (2.14).

**Proof.** Doing in the same manner, we substitute the right-hand side of (2.16) in (1.4), we interchange the order of integration and employ (1.28) to obtain
\[
a_n = \frac{2\pi}{\sinh(\pi n)} \int_{-\pi}^\pi \psi(u) \sin(nu) du. \tag{2.18}
\]
This interchange is allowed by Fubini’s theorem due to the absolute convergence of the iterated integral (cf. (2.15)). Then the partial sum of the series (2.17) has the form
\[
S_N(x) = -\frac{32}{\pi} \sum_{n=1}^N \int_{-\pi}^\pi \sin(t) \sin(nt) \Im \left[ K_0 \left( 4e^{\pi i/4} \cosh^{1/2}(t) \sqrt{x} \right) \right] dt
\]
× \int_{-\pi}^{\pi} \psi(u) \sin(nu)du. \quad (2.19)

Hence, recalling identity (2.9), the latter equality (2.19) turns to be

\[ S_N(x) = -\frac{8}{\pi} \int_{-\pi}^{\pi} \sin(t) \text{Im} \left[ K_0 \left( 4e^{\pi i/4} \cosh^{1/2}(t) \sqrt{x} \right) \right] \]

\[ \times \int_{-\pi}^{\pi} \left[ \psi(u) + \psi(-u) \right] \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du \, dt. \]

Now the same scheme as in the proof of Theorem 3 drives us at the equality

\[ \lim_{N \to \infty} S_N(x) = -16 \int_{-\pi}^{\pi} [\psi(t) - \psi(-t)] \sin(t) \text{Im} \left[ K_0 \left( 4e^{\pi i/4} \cosh^{1/2}(t) \sqrt{x} \right) \right] \, dt = f(x), \]

proving the inversion formula (2.17).

The same scheme can be applied to invert discrete \( \text{Im} \)-transformations (1.5), (1.6). We will state the corresponding theorems, leaving their proofs via (1.29) to interested readers.

\textbf{Theorem 5.} Let a sequence \( a = \{a_n\}_{n \in \mathbb{N}} \in l_1 \). Then the discrete transformation (1.5) has the inversion formula

\[ a_n = -\frac{32}{\pi^2} \sinh(\pi n) \int_0^{\infty} \Omega_n(x) f(x) \, dx, \quad n \in \mathbb{N}_0, \quad (2.20) \]

where the kernel \( \Omega_n(x) \) is defined by

\[ \Omega_n(x) = \int_0^{\pi} \text{Re} \left[ K_0 \left( 4e^{\pi i/4} \cosh^{1/2}(u) \sqrt{x} \right) \right] \sinh(u) \sin(nu) \, du, \quad x > 0, \quad n \in \mathbb{N}_0, \quad (2.21) \]

and integral (2.20) converges absolutely.

\textbf{Theorem 6.} For a class of functions \( f \) such that

\[ f(x) = -32 \int_{-\pi}^{\pi} \text{Re} \left[ K_0 \left( 4e^{\pi i/4} \cosh^{1/2}(u) \sqrt{x} \right) \right] \psi(u) \sinh(u) \, du, \quad x > 0 \]

with \( 2\pi \)-periodic function \( \psi \), satisfying the Lipschitz condition (2.5) on \([-\pi, \pi]\), the following inversion formula holds
\[ f(x) = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \sinh(\pi n) \Omega_n(x) a_n, \]

where \( a_n \) is given by (1.6) and \( \Omega_n \) is defined by (2.21).

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