Radio-emission of axion stars

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We study parametric instability of compact axion dark matter structures decaying to radiophotons. Corresponding objects — Bose (axion) stars, their clusters, and clouds of diffuse axions — form abundantly in the postinflationary Peccei-Quinn scenario. We develop general description of parametric resonance incorporating finite-volume effects, backreaction, axion velocities and their (in)coherence. With additional coarse-graining, our formalism reproduces kinetic equation for virialized axions interacting with photons. We derive conditions for the parametric instability in each of the above objects, as well as in collapsing axion stars, evaluate photon resonance modes and their growth exponents. As a by-product, we calculate stimulated emission of Bose stars and diffuse axions, arguing that the former can give larger contribution into the radiobackground. In the case of QCD axions, the Bose stars glow and collapsing stars radioburst if the axion-photon coupling exceeds the original KSVZ value by two orders of magnitude. The latter constraint is alleviated for several nearby axion stars in resonance and absent for axion-like particles. Our results show that the parametric effect may reveal itself in observations, from FRB to excess radiobackground.

I. INTRODUCTION

The QCD axion [1] and similar particles [2] are perfect dark matter candidates [3, 4]: they are motivated [5, 6] and have tiny interactions [7], including coupling to the electromagnetic field. But the same interactions — alas — make the axions “invisible” dictating overly precise detection measurements [8, 9] and limiting possible observational effects [10–12].

Nevertheless, under certain conditions an avalanche of exponentially growing photon number \( n_{\gamma} \propto \exp\{2\mu_{\infty} t\} \) can appear in the axionic medium [13], with growth exponent \( \mu_{\infty} \) proportional to the axion-photon coupling and axion field strength. This process is known as parametric resonance. It occurs because the axions decay into photons which stimulate decays of more axions. In the infinite volume parametric axion-photon conversion is well understood, but does not occur during cosmological evolution of the axion field [14–16]. In compact volume of size \( L \) the avalanche appears if the photon stimulates at least one axion decay as it passes the object length [13, 17]. This gives order-of-magnitude resonance condition,

\[ \mu_{\infty} L \gtrsim 1. \tag{1} \]

Unfortunately, apart from this intuitive estimate and brute-force numerical computations [18–26], no consistent quantitative theory of axion-photon conversion in finite-size objects has been developed so far.

In this paper we fill this gap1 with a general, detailed, and usable quasi-stationary approach to parametric resonance in a finite volume. Our method works only for nonrelativistic axions, but accounts for their coherence, or its absence, axion velocities, binding energy and gravitational redshift, backreaction of photons on axions, and arbitrary volume shape. In the limit of diffuse axions it reproduces well-known axion-photon kinetic equation, if additional coarse-graining is introduced.

Notably, the cosmology of QCD axion [14, 15] provides rich dark matter structure at small scales [29], with a host of potentially observable astrophysical implications. Namely, in the postinflationary scenario violent inhomogeneous evolution of the axion field during the QCD epoch [30–33] leads to formation of axion miniclusters [34–36] — dense objects of typical mass \( 10^{-13} M_{\odot} \) forming hierarchically bound structures [37]. In the centers of miniclusters even denser compact objects, the axion (Bose) stars [38, 39], appear due to gravitational kinetic relaxation [40, 41]. Simulations suggest [29, 32, 33, 40, 41] that these objects are abundant in the Universe, though their present-day mass is still under study [41]. Another example of dense object formed by the QCD axions is a cloud around the superradiant black hole [6, 42, 43], see also [44].

Beyond the QCD axion, miniclusters [4] and Bose stars [45, 46] can be formed by the axion-like particles at very different length and mass scales.

In this paper we derive precise conditions for parametric resonance in the isolated axion stars, collapsing stars, their clusters, and in the clouds of diffuse axions. We find unstable electromagnetic modes and their growth exponents \( \mu \). Contrary to what naive infinite-volume intuition might suggest, resonance in nonrelativistic compact objects develops with \( \mu \ll \mu_{\infty} \). As a result, in many cases it glows in the stationary regime, burning an infinitesimally small fraction of extra axions at every moment, to keep the resonance condition marginally broken.

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1 This work is based on presentations [27, 28] at the Patras workshops where the main equations first appeared.
Our calculations suggest three interesting scenarios with different observational outcomes. In the first chain of events the axion-photon coupling is high and the threshold for parametric resonance is reached during growth of axion stars via Bose-Einstein condensation. Then all condensing axions will be converted into radio-emission with frequency equal to the axion half-mass. This paves the way for indirect axion searches.

Second, at somewhat smaller axion-photon coupling, attractive self-interactions of axions inside the growing stars may become important before the resonance threshold is reached. As a result, the stars collapse [47–49], shrink and ignite the instability to photons on the way. Alternatively, several smaller axion stars may come close, suddenly meeting the resonance condition [20]. In these cases a short and powerful burst of radio-emission appears.

Amusingly, powerful and unexplained Fast Radio Bursts (FRB) are presently observed in the sky [50]. It is tempting to relate them to parametric resonance in collapsing axion stars [19] and see if the main characteristics can be met.

In the third, most conservative scenario all Bose stars are far away from the parametric resonance. Nevertheless, the effect of stimulated emission turns them into powerful radioamplifiers of ambient radiowaves at the axion half-mass frequency. We compute amplification coefficients for the Bose stars and diffuse axions and find that realistically, stimulated emission of the stars may give larger contribution into the radiobackground.

In Sec. II we introduce nonrelativistic approximation for axions and review essential properties of Bose stars. General description of parametric resonance in finite volume is developed in Sec. III. In Sec. IV it is applied to radio-emission of static axion stars, their pairs, and amplification of ambient radiation. In Secs. V and VI we study resonance in diffuse axions and consider the effect of moving axions / axion stars, in particular, resonance in collapsing stars. Concluding remarks are given in Sec. VII.

II. AXION STARS

The diversity of compact objects in axion cosmology offers many astrophysical settings where the parametric resonance may be expected. One can consider static Bose stars, collapsing, moving, or tidally disrupted stars, even axion miniclusters. To describe all this spectrum in one go, we implement two important approximations.

First we describe axions by the classical field \(a(t, x)\) satisfying

\[
\Box a + \mathcal{V}'(a) = 0 .
\]

This is valid at large occupation numbers. Interaction with the gravitational field in Eq. (2) is hidden in the covariant derivatives, and the scalar potential

\[
\mathcal{V} = \frac{m^2}{2} a^2 - \frac{g_4^2 m^2}{4f_a^2} a^4 + \ldots
\]

includes mass \(m\) and quartic coupling \((g_4 m/f_a)^2\). Self-interaction of the QCD axion is attractive: \(f_a \simeq (75.5 \text{ MeV})^2/m\) and \(g_4 \simeq 0.59 \) [7]. Axion-like particles may have \(g_4 \simeq 0\).

Second, we work in nonrelativistic approximation,

\[
a = \frac{f_a}{\sqrt{2}} \left[ \psi(t, x) e^{-i\omega t} + \text{h.c.} \right] ,
\]

where \(\psi\) slowly depends on space and time. Namely, if \(\lambda\) is the typical wavelength of axions,

\[
\partial_t \psi \sim \psi/m \lambda^2 , \quad \partial_x \psi \sim \psi/\lambda , \quad \lambda m \gg 1 .
\]

In this approximation Eq. (2) reduces to nonlinear Schrödinger equation,

\[
i \partial_t \psi = \frac{\Delta}{2m} \psi + m \left( \Phi - \frac{g_4^2}{8} |\psi|^2 \right) \psi ,
\]

where \(\Phi\) is a nonrelativistic gravitational potential solving the Poisson equation

\[
\Delta \Phi = 4 \pi \rho/m_{\text{pl}}^2 ,
\]

and \(\rho = m^2 f_a^2 |\psi|^2\) is the mass density of axions.

A central object of our study is a Bose (axion) star, a stationary solution to the Schrödinger-Poisson system

\[
\psi = e^{-i\omega_s t} \psi_s(r) , \quad \Phi = \Phi_s(r) ,
\]

where \(\omega_s < 0\) is the binding energy of axions and \(r\) is the radial coordinate. Physically, Eq. (8) describes Bose-Einstein condensate of axions occupying a ground state in the collective potential well \(\Phi_s(r)\). This object is coherent: the complex phase of \(\psi_s\) does not depend on space and time. Below we consider parametric resonance in stationary and colliding stars.

Notably, the axion stars with the critical mass

\[
M_{\text{cr}} \simeq 10.2 \frac{f_a M_{\text{pl}}}{mg_4}
\]

and heavier stars are unstable [48]. In this case attractive self-interaction in Eq. (6) overcomes the quantum pressure and the star starts to shrink developing huge axion densities in the center [49]. We will see that this may trigger explosive parametric instability.

III. GENERAL FORMALISM

A. Linear theory

In this Section we construct general quasi-stationary theory for narrow parametric resonance of radiophotons
in the finite volume filled with axions. This technique was first developed and presented in [27, 28]. In contrast to the resonance in the infinite volume [14, 15, 39] which universally leads to the Mathieu equation, the finite-volume one is described by the eigenvalue problem with a rich variety of solutions.

Consider Maxwell’s equations for the electromagnetic potential $A_\mu$ in the axion background $a(t, \mathbf{x})$,

$$\partial_\mu \left(F_{\mu\nu} + g_{\alpha\gamma}aF_{\mu\nu}\right) = 0 \ ,$$  \hspace{1cm} (10)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}/2$, and $g_{\alpha\gamma}$ is the standard axion-photon coupling. Below we also use dimensionless coupling $g' \equiv f_\alpha g_{\alpha\gamma}/2^{3/2}$.

In the infinite volume one describes the resonance in the plain-wave basis for the electromagnetic field [14, 15, 39], while the axion star suggests spherical decomposition [20]. We want to develop general formalism, and simpler at the same time, usable in a large variety of astrophysical settings.

We therefore introduce two simplifications. First, the photons travel straight, with light-bending effects being subdominant in the axion background, cf. [51, 52]. Thus, the parametric resonance develops almost independently along different directions. Second, we consider non-relativistic axions decaying into photons of frequency $\omega_0 \approx m/2$ with a very narrow spread.

This suggests decomposition in the gauge $A_0 = 0$,

$$A_i = \int d\mathbf{n} \ C_i^{(\mathbf{n})}(t, \mathbf{x}) e^{i\mathbf{m}(\mathbf{n} \cdot \mathbf{x} + t)/2} + \text{h.c.} \ ,$$  \hspace{1cm} (11)

where $i = \{x, y, z\}$ and the integral runs over all unit vectors $\mathbf{n}$. The amplitudes $C_i^{(\mathbf{n})}$ include photon frequency spread. Hence, they weakly depend on space and time,

$$\partial_\mathbf{x} C_i^{(\mathbf{n})} \sim \lambda^{-1} C_i^{(\mathbf{n})} \ , \quad \mathbf{m} \gg 1 \ ,$$  \hspace{1cm} (12)

where $\lambda^{-1}$ is the typical momentum of axions in Eq. (5).

Using Eq. (11), one finds that in the eikonal limit (12) the field equation (10) couples only the waves moving in the opposite, i.e. $+\mathbf{n}$ and $-\mathbf{n}$, directions. As a result, identical and independent equations are produced for every pair of directions. This is manifestation of the simple fact that the axions decay into two back-to-back photons.

Indeed, leaving one arbitrary direction $z = (\mathbf{n} \cdot \mathbf{x})$ and its counterpart $-z$, we obtain the ansatz that passes the field equation [27, 28],

$$A_i = C_i^+ e^{im(z+t)/2} + C_i^- e^{im(z-t)/2} + \text{h.c.} \ ,$$  \hspace{1cm} (13)

where the shorthand notations $C_i^+(t, \mathbf{x}) = C_i^{(\mathbf{n})}$ and $C_i^-(t, \mathbf{x}) = [C_i^{(\mathbf{n})}]^*$ are introduced. Namely, substituting Eq. (13) into Eq. (10) and using approximations (12), (5), we arrive to the closed system,

$$\partial_t C_i^+ = \partial_z C_i^+ + ig' m\psi^* C_i^- \ ,$$  \hspace{1cm} (14a)

$$\partial_t C_i^- = -\partial_z C_i^- - ig' m\psi C_i^+ \ .$$  \hspace{1cm} (14b)

The other two physical polarizations satisfy the same equations with $C_i^+ \rightarrow C_i^+$ and $C_i^- \rightarrow -C_i^-$, while the longitudinal part is fixed by the Gauss law $C_i^\alpha = 2i\partial_\alpha C_i^{\alpha}/m$; here and below $\alpha = \{x, y\}$. Overall, we have four amplitudes $C_i^\alpha$ representing two photon polarizations propagating in the $+z$ and $-z$ directions.

Equations (14) should be solved for every orientation of $z$ axis, in search for the growing instability modes. After that the modes can be superimposed in Eq. (11) or, practically, only the one with the largest exponent can be kept.

In spherically symmetric Bose star all directions are equivalent and description simplifies — we have to study only one direction. Notably, in this case one can derive Eqs. (14) using spherical decomposition, see Appendix A.

There is a residual hierarchy in Eqs. (14) related to small axion velocities $v \sim (m\lambda)^{-1} \ll 1$. Indeed, the nonrelativistic background evolves slowly, $\partial_t \psi \sim \psi/m\lambda^2$, while the electromagnetic field changes fast, $\partial_t C/\lambda^2$. Thus, equations for $C$ can be solved with adiabatic ansatz,

$$C_i^\pm = e^{\int n(t') dt'} c_i^\pm(t, \mathbf{x}) \ ,$$  \hspace{1cm} (15)

where the complex exponent $\mu$ and quasi-stationary amplitudes $c_i^\pm$ evolve on the same timescales $m\lambda^2$. Corrections to the adiabatic evolution (15) become exponentially small as $v \to 0$.

Using representation (15) in Eqs. (14) and ignoring time derivatives of $\mu$ and $c_i^\pm$, we finally obtain the eigenvalue problem [27, 28],

$$\mu c_i^+ = \partial_z c_i^+ + ig' m\psi^* c_i^- \ ,$$  \hspace{1cm} (16a)

$$\mu c_i^- = -\partial_z c_i^- - ig' m\psi c_i^+ \ ,$$  \hspace{1cm} (16b)

where equations for the two remaining amplitudes are again obtained by $c_i^+ \rightarrow c_i^+$ and $c_i^- \rightarrow -c_i^-$. If the axions live in a finite region and no electromagnetic waves come from infinity, one imposes boundary conditions

$$c_i^+\big|_{z \to +\infty} = c_i^-\big|_{z \to -\infty} = 0 \ ,$$  \hspace{1cm} (17)

see Eq. (13).

The spectral problem (16) determines the electromagnetic modes $\{c_i^+, c_i^-\}$ and their growth exponents $\mu$. The latter are not purely imaginary at $\psi \neq 0$ because $2 \times 2$ operator in the right-hand side of Eqs. (16), is not anti-Hermitian. That is why in certain cases resonance instabilities — modes with $\Re \mu > 0$ satisfying the boundary conditions (17) — appear.

It is worth discussing two parametrically small corrections to Eqs. (14), (16). First, derivatives with respect
to $x$ and $y$ are absent in these systems: they appear only in the next, $(m\lambda)^{-1}$ order, determining the section of the resonance ray in the $(x, y)$ plane. If needed, they can be recovered with the substitution

$$\partial_z \rightarrow \partial_z - \frac{i}{m}(\partial_x^2 + \partial_y^2), \quad (18)$$

to solve the spectral problem in three dimensions.

If the axion distribution is not spherically-symmetric, one expects that the resonance ray is narrow in the $(x, y)$ plane.\(^3\) Indeed, according to the leading-order equations (14) electromagnetic field grows with different exponents $\mu$ at different $x$ and $y$. This means that wide wave packets shrink around the resonance line until the quantum pressure (18) becomes relevant.

Second, direct interaction of photons with nonrelativistic gravitational field can be included in Eqs. (14) by changing\(^4\)

$$\partial_z C_\alpha^+ \rightarrow (\partial_z + \text{i} m\Phi) C_\alpha^+ \quad (19)$$

However, one immediately rotates this contribution away, $C_\alpha^+ \rightarrow \exp\{\text{-i} \int \xi \text{d}z' \Phi(\xi')\} C_\alpha^+$, with remaining corrections suppressed by $(m\lambda)^{-2}$.

As an illustration, consider static homogeneous axion field $\psi$ in the infinite volume. Quasi-stationary equations (16), in this case give $\partial_z C_\alpha^+ = 0$ and time-independent $\mu = \mu_\infty$ of the form

$$\mu_\infty = g' m |\psi| > 0 \quad (20)$$

Thus, electromagnetic amplitudes in Eq. (15) grow exponentially with time indicating parametric resonance. Expression (20) reproduces well-known infinite-volume growth rate \cite{13-15, 17, 18, 20, 22, 39} of the axion-photon resonance.

B. Nonlinear stage

Backreaction of photons on axions can be easily incorporated in the Schrödinger-Poisson system (5). To this end one substitutes the nonrelativistic ansatz (4), (13) into the equation for the axion field, \[ \Box a + V'(a) = -\frac{g_{\alpha\gamma\gamma}}{4} F_{\mu\nu} \tilde{F}_{\mu\nu}, \quad (21) \]

and omits higher derivatives of $\psi$ and $C$. This gives,

$$i\partial_t \psi = -\frac{\Delta \psi}{2m} + m\Phi \psi - \frac{mg_\gamma^2}{8} |\psi|^2 \psi$$

$$- \frac{mg'}{f_\alpha} \epsilon_{\alpha\beta} C_\alpha^- C_{\beta^+}, \quad (22)$$

where the backreaction is represented by the new term\(^5\) in the right-hand side.

Let us show that the last term in the above equation changes the mass $M = m^2 f_\alpha^2 \int d^3 x |\psi|^2$ of the axion cloud. Indeed, taking the time derivative of $M$ and using Eq. (22), we obtain energy conservation law,

$$\partial_t M = J_{in} - \int dx dy F_{a \rightarrow \gamma\gamma} , \quad (23)$$

where $J_{in} = -m f_\alpha^2 \int d^2 \sigma \text{Im}(\psi^* \partial_t \psi)$ is the mass of axions entering the system per unit time and

$$F_{a \rightarrow \gamma\gamma} = 2m^3 g' \int dz \epsilon_{\alpha\beta} \text{Im}(\psi^* C_\alpha^- C_{\beta^+}) \quad (24)$$

is the flux of produced photons. Below we will also use the electromagnetic Poynting fluxes at infinity,

$$F_\gamma^\pm = \mp m^2 (|C_\alpha^+|^2 + |C_\beta^+|^2)/2 \quad (25)$$

In conjunction with Eqs. (14) this gives conservation law for the electromagnetic energy, $\partial_t E_\gamma = \int dx dy \left(F_{a \rightarrow \gamma\gamma} - [F_\gamma^+ + F_\gamma^-]|z = \pm \infty\right)$. To conclude, one can numerically solve Eq. (22) together with Eqs. (14) and watch the axions burn abundantly.

IV. STATIC COHERENT AXIONS

A. Condition for resonance

For a start, consider the case when the axions in a finite volume are coherent and do not move. A notable example of this situation is a static axion star. Parametric resonance in this setup is presently understood at the qualitative level \cite{13, 19, 20}. Indeed, photons passing through the axions stimulate their decays $a \rightarrow 2\gamma$. The photon flux grows exponentially, $F_\gamma \propto \exp\{2\mu_\infty t\}$, and the secondary flux of backward-moving photons appears. After the original photons escape the region with axions, stimulated decays continue in the secondary flux moving in the opposite direction, etc, see Fig. 1. Overall, the back-and-forth motion inside the axion cloud accumulates photons at every pass if Eq. (1) is valid, i.e.

$$\mu_\infty L = g' m |\psi| L > 1, \quad (26)$$

where $L$ is the typical size of the cloud.

Our equations (14) reflect the same physics. Namely, consider the localized wave packet $C_\gamma^+(t, x)$ going

\(^3\) Similarly, one can compute narrow resonance rays around every direction in spherical axion star, and then combine them in (11).

\(^4\) Here $m\Phi$ accounts for the gravitational evolution of photon four-momentum.

\(^5\) If several directions in the transform (11) are essential, a combination of backreaction terms appears here. If spherical decomposition is used for the isolated axion star, these terms come with factors $r^{-2}$ in front, see Appendix A.
through axions in the \(+z\) direction. Due to Eq. (14a) it creates the packet \(C_x^\pm\) with the opposite group velocity which, in turn, produces \(C_y^\mp\), etc. The photon flux grows exponentially during this process if unstable modes with \(\text{Re} \mu \geq 0\) are present.

Axion velocities are related to the complex phase of the field,

\[
v_i = m^{-1} \partial_t \arg \psi .
\]

In this section we assume that \(\psi\) is real up to a constant phase which can be absorbed into redefinition of \(C_x^\pm\) in Eqs. (16). This means that the axions are static and coherent.

In particular, the phase factor \(\exp\{-i\omega_x t\}\) of the Bose star field (8) disappears from the electromagnetic equations after replacing \(C_x^\pm \rightarrow C_x^\pm e^{\mp i\omega_x t/2}\). Then the total binding energy \(\omega_s\) of axions inside the star does not destroy the resonance, but slightly shifts its central frequency to

\[
\omega_\gamma = (m + \omega_s)/2 ,
\]

see Eq. (13). Note that misconceptions regarding resonance blocking by gravitational and self-interaction energies still exist in the literature, e.g. [25].

At real \(\psi\) the semiclassical eigenvalue problem (16) has two types of solutions. First, delocalized modes penetrate into the asymptotic regions \(z \rightarrow \pm \infty\), where \(\psi = 0\) and \(c_z^\pm \propto \exp(\pm i\omega z)\). The exponents \(\mu\) of these modes are purely imaginary, or their profiles would be unbounded. Physically, the delocalized modes represent electromagnetic waves coming from infinity. Second, there may exist localized modes satisfying the boundary conditions (17). They behave well at infinity if \(\text{Re} \mu \geq 0\). In addition, we prove in Appendix B that at real \(\mu\) the exponents \(\mu\) of these modes are real. The localized modes represent resonance instabilities.

In practical problems the resonance is not present in matter from the very beginning but appears in the course of nonrelativistic evolution. For example, the Bose stars form in slow galactic [45, 46, 53, 54] or minicluster [35, 41] collapses, or afterwards in kinetic relaxation [40], then grow kinetically at turtle-slow rates [40, 41, 55]. Their subsequent evolution is also essentially nonrelativistic [56, 57].

At some point of quasi-stationary evolution one of purely imaginary eigenvalues \(\mu\) may become real, and the parametric resonance develops. Let us discuss the borderline situation when the very first localized mode has \(\mu = 0\). The solution in this case is [27, 28],

\[
c_x^+ = A \cos D(z) , \quad c_y^- = -i A \sin D(z) ,
\]

where \(A\) is a constant amplitude and

\[
D(z) = g' m \int_{-\infty}^{z} dz' \psi(z') .
\]

Integration in Eq. (30) runs along the arbitrary-oriented \(z\)-axis.

The solution (29) satisfies the boundary conditions (17) if \(D_\infty \equiv D(+\infty) = \pi/2\). At larger values of this integral the instability mode with positive \(\mu\) exists. Thus, a precise condition for the parametric resonance along a given \(z\)-axis is

\[
D_\infty \equiv g' m \int_{-\infty}^{+\infty} \psi(z) dz \geq \frac{\pi}{2} .
\]

This concretizes the order-of-magnitude estimate (26). Recall that in our notations \(\psi = \rho^{1/2}/(m f_a)\), where \(\rho\) is the mass density of axions.

Let us find out when the parametric resonance occurs in axion stars. In Appendix C we compute \(D_\infty\) along the line passing through the star center, see Fig. 1. We consider two cases. First, if self-interactions of axions inside the star are negligible, Eq. (31) reads,

\[
M_s \geq M_{s,0} = 7.66 \frac{M_p}{m g_{a \gamma \gamma}} , \quad g_4 \approx 0 ,
\]

where we restored \(g_{a \gamma \gamma} = 2^{5/2} g'/f_a\). This condition is applicable in the axion-like models with \(g_4 = 0\) or at \(M_s \ll M_{cr}\). In these cases heavier stars are better for the resonance.

Second, if attractive self-interactions are present, the mass of the axion star is bounded from above, \(M_s < M_{cr}\). Using the profile of the critical star in Eq. (31), we obtain condition

\[
g_{a \gamma \gamma} > g_{a \gamma \gamma,0} = 0.52 \frac{g_4}{f_a} , \quad M_s = M_{cr} .
\]

If this inequality is broken, parametric resonance does not develop in stable axion stars at all.

For the parameters of QCD axion listed in Sec. II, the inequality (33) gives the shaded region in Fig. 2 marked “resonance.” Notably, the benchmark values [7] of axion-photon coupling (KSVZ-DFSZ band in Fig. 2) are short by two orders of magnitude from igniting the resonance even in the critical star [19, 27][25]. On the other hand, \(g_{a \gamma \gamma}\) is model dependent, with the only constraint \(g_{a \gamma \gamma} < f_a^{-1}\) coming from strong coupling in simple

![FIG. 1. Parametric resonance in the axion star.](image-url)
models [58, 59]. Thus, even these simple models can satisfy (33) within the trustworthy parameter range. More elaborated (clockwork-inspired) QCD axion models [60] do not have these limitations and easily meet (33).

Alternatively, the self-coupling of the axion-like particles can be arbitrarily small. Condition (32) is then satisfied just for a sufficiently heavy star.

### B. Linear exponential growth

Let us find out how the resonance progresses. One does not expect it to turn immediately into an exponential catastrophe with \( \mu \sim O(L^{-1}) \), like the infinite-volume intuition might suggest, cf. Eq. (26). Rather, the electromagnetic field starts growing with parametrically small exponent \( \mu \ll L^{-1} \) immediately after the condition (31) is met by the nonrelativistic evolution of axions. Initial values for this growth are tiny. They can be provided by the ambient radiation in astrophysical setup or, universally, by quantum fluctuations considered in Appendix D. In any case this initial stage proceeds linearly with no backreaction on axions.

We compute the growth exponent by solving the eigenvalue problem (16) perturbatively at small \( \mu \), like in quantum mechanics\(^6\). To this end we assume that the background \( \psi(t, \mathbf{x}) \) did not evolve much from the point \( \psi_0(\mathbf{x}) \equiv \psi(t_0, \mathbf{x}) \) when the condition (31) was met, and the resonance mode is close to the solution (29). Calculation in Appendix B gives,

\[
\mu = \frac{D_\infty - \pi/2}{\int dz \sin[2D_0(z)]},
\]

\[(34)\]

\(^6\) Unlike in quantum mechanics, the operator in Eqs. (16) is symplectic, not Hermitian.

Here \( D_0(z) \) is evaluated using \( \psi_0(\mathbf{x}) \), a configuration at the rim of parametric instability, while \( D_\infty \) uses \( \psi \) in Eq. (31). Note that application of Eq. (34) essentially depends on nonrelativistic mechanism leading to resonance and providing \( D_\infty - \pi/2 = O(\psi - \psi_0) \).

Expression (34) confirms that \( \mu \) is indeed parametrically small and yet, large enough for the adiabatic regime (15) to take place. Generically, \( \psi - \psi_0 \sim (t_1 - t_0) \partial_t \psi \), where \( t_1 - t_0 \sim \Lambda/\mu \) is the time from ignition of the resonance to the moment \( t_1 \) when the backreaction starts: \( \Lambda \sim \log[C^\pm(t_1)/C^\pm(t_0)] \sim 10^2 \) is a large logarithm. Then the nonrelativistic scaling (12), (5) and Eq. (34) give \( \mu \sim \lambda^{-1}(\Lambda/m\lambda)^{1/2} \), where we also recalled that the resonance condition (31) is marginally satisfied. Thus,

\[
(m\lambda^2)^{-1} \ll \mu \ll \lambda^{-1},
\]

i.e. the electromagnetic fields evolve faster than the axion background but slower than the light-crossing time \( L^{-1} \sim \lambda^{-1} \).

Applying Eq. (34) to the stationary axion star with \( g_4 \approx 0 \), we get

\[
\mu = 0.197 \frac{m^2}{M_{pl}^2} (M_s - M_{s,0}),
\]

\[(35)\]

where Appendix C was consulted and \( M_{s,0} \) is given in Eq. (32). Using this expression, one obtains \( \mu \sim 10^2 \) s\(^{-1} \) for \( m = 26 \) keV and \( \delta M_s \sim 10^{-13} M_\odot \). Thus, duration of the linear stage in QCD axion stars is one second or longer.

To confirm the above perturbative results, we numerically solve the system of coupled relativistic equations (10), (21) for the electromagnetic field and axions at \( g_4 = 0 \), see Appendix E for details. Our
simulation starts with the axion star of mass $M_s$ and tiny electromagnetic amplitudes representing quantum bath of spontaneous photons. If the mass of the axion star exceeds $M_{s,0}$, the exponential growth of amplitudes starts, see the left part of Fig. 3. The exponent of this growth coincides with the one given by Eqs. (16) (dashed line), and within the expected precision interval of $\delta \mu/\mu \sim (M_s - M_{s,0})/M_{s,0} \sim 4\%$ — with Eq. (35).

In Fig. 4 we show dependence of the exponent $\mu$ on the axion star mass $M_s$. First, performing full simulations with different stars, we extract $\mu$ from the exponentially growing flux. This result is shown by the solid line. In the limit $M_s \to M_{s,0}$ it coincides with Eq. (35) (dashed line), as it should. Second, solving the nonrelativistic equations (16) numerically, we obtain points in Fig. 4 which give correct exponent for the arbitrary mass.

**C. Glowing axion stars**

When the electromagnetic amplitudes in Fig. 3 become large, the backreaction appears, and the resonant flux immediately starts to fall off. Indeed, backreaction burns axions diluting their density, and $\text{Re} \, \mu$ in Eq. (34) decreases to negative values. At this point a long-living quasi-stationary level of the electromagnetic field is formed. Indeed, at small $\mu < 0$ the resonance mode turns into an exponentially growing at $z \to \pm \infty$ solution to Eqs. (16),

$$e^+_x = A e^{\mu z} \cos D(z) , \quad e^-_y = -i A e^{-\mu z} \sin D(z) ,$$

and this is a correct behavior for the quasi-stationary wave function [61]. Inserting the late-time axion configuration from our full simulation into Eq. (34), we reproduce the exponential falloff of the flux, see the dots in Fig. 3. Thus, the solution (36), (34) remains approximately valid during the entire evolution, with the only unknown part related to dilution of axions in Eq. (22).

The backreaction switches on when the last term in Eq. (22) becomes comparable to the others. Using, in addition, Eq. (31), we find a condition for the maximal flux at the linear stage of resonance,

$$F_\gamma \sim m^2 |C|^2 \lesssim \frac{\rho}{m \lambda} .$$

Here $\lambda$ is the characteristic length scale of axions and and $\rho$ is their mass density. In dynamical situations $F_{\gamma, \text{out}}$ is compared to the axion flux $\nu \rho$ with $\nu \sim (m\lambda)^{-1}$. Notably, $F_\gamma \ll \rho$ when the backreaction starts. Figure 3 demonstrates grey region where Eq. (37) is violated.

Let us reconsider the solution (29), (30) with $\mu = 0$, to describe the regime where the backreaction stops the resonance. The amplitudes $C^\pm_\alpha$ of this solution are constant at infinity,

$$C_x^+ \big|_{z \to -\infty} = A , \quad C_y^- \big|_{z \to +\infty} = -i A ,$$

see also Eq. (17). Thus, the solution describes stationary flux of photons $F_{\gamma, \text{out}} = \pm m^2 |A|^2$ from decaying axions, where for simplicity here and below we assume equipartition $C_+ = C_+^\gamma$ and $C_- = -C_-^\gamma$.

Computing the flux (24) of produced photons we find $F_{\gamma, \to \gamma} = 2m^2 |A|^2 = 2|F_{\gamma, \text{out}}|^2$. This means that the solution (29) duly brings all energy of decaying axions to infinity. Energy conservation law (23) then takes the form,

$$\partial_t M = J_{\text{in}} - 2 \int dx dy |F_{\gamma, \text{out}}|^2 .$$

Even if an arbitrary large constant stream $J_{\text{in}}$ of axions is fed into the system, the resonance works in the equilibrium regime with $\partial_t M = 0$ and $\mu = 0$. All arriving axions in this case are converted into radiation. To break this situation, one needs a very special mechanism, e.g. the axion star collapse in Sec. VI C.

Note that the above stationary situation is stable. Indeed, perturbing $M$ and $F_{\gamma, \text{out}}$ away from their equilibrium values one obtains $\partial_t \delta M = -2 \delta F_{\gamma, \text{out}}$ due to energy conservation — larger flux decreases the mass. Besides, Eq. (34) gives $\partial_t \delta F_{\gamma, \text{out}} = 2 \mu \delta F_{\gamma, \text{out}} \propto \delta M$ i.e. smaller mass weakens the flux. Together, these equations describe harmonic oscillations around the equilibrium. In the simplest uniform model the frequency is $\Omega = g_{\gamma\gamma} (F_{\gamma,n}/8)^{1/2}$, where $F_{\gamma,n} = J_{\text{in}}/\int dx \, dy$ is the flux of axions arriving into the resonance region. Thus, the resonant radioflux $F_{\gamma, \text{out}}$ may pulsate due to axion-photon oscillations. This effect, however, should be strongly dumped due to energy dissipation between the modes of the axion field.

In the particular daydream scenario where the Universe is full of axion stars reaching the condition (32) during growth, no spectacular explosion-like radio events are expected to appear in the sky. Most of the axion stars would exist in the quasi-stationary regime with $D_\infty = \pi/2$, converting all condensing axions into the radiobackground of frequency $\omega_\gamma \approx m/2$.

Nevertheless, the latter emission may be observable, even if the condensation timescale is comparable to the
age of the Universe. To get a feeling of numbers, let us assume that a grown-up star with $D_\infty = \pi/2$ lives 100 pc away from us. Take $m = 26 \mu eV$ and $M_\star \sim 10^{-13} M_\odot$, the typical values for the QCD axions. Then the condensation rate onto the star is roughly $10^{-13} M_\odot$ per the Universe age. All of condensing axions will be converted into radiation in the narrow band around $\omega_\gamma \sim 2$ GHz. Even for poor spectral resolution $\delta \omega/\omega \sim 10^{-3}$ one gets spectral flux of order $10^{-2}$ Jy, which is detectable.

When reliable predictions for the abundance of Bose stars and their growth rates appear, similar calculations may be used to constrain the respective scenarios.

### D. Amplification of ambient radio

Now, we embed the axion stars into astrophysical background of radiophotons. Namely, suppose an external radiowave of frequency $\omega_\gamma$ travels through the underdense axion medium which is safely away from the resonance. The wave will stimulate decay of axions, so its flux will be amplified in a narrow spectral window around $\omega_\gamma = m/2$.

This stationary setup is described by our equations (16) with $\mu = i(\omega_\gamma - m/2)$ and new boundary conditions,

$$
\begin{align*}
    c_\alpha^+|_{z \to +\infty} &= A_0, \\
    c_\alpha^-|_{z \to -\infty} &= 0,
\end{align*}
$$

where equipartition is again assumed and $A_0$ is related to the incoming electromagnetic flux $F_{\gamma, in} = -m^2 A_\gamma^2$.

To find the height of the spectral line in this case, we solve equations at $\omega_\gamma = m/2$ ($\mu = 0$). The solution is given by Eq. (29) with $A = A_0/\cos D_\infty$. The outgoing flux is therefore

$$
F_{\gamma, out} = F_{\gamma, in}/\cos^2 D_\infty, \tag{41}
$$

see also (31). Thus, at small $D_\infty$ the extra flux from axions is weak, $\Delta F = D_\infty^2 F_{\gamma, in}$. It grows to infinity, however, at $D_\infty \to \pi/2$ when the resonance is about to appear.

For the critical QCD axion stars with $D_\infty \ll 1$,

$$
\Delta F \approx \frac{\pi^2}{4} \frac{g_{a\gamma\gamma}}{g_{a\gamma\gamma, 0}} F_{\gamma, in},
$$

cf. Eq. (33). In the benchmark KSVZ model with $g_{a\gamma\gamma} = 1.92 a_{em}/(2\pi f_a)$ this gives $\Delta F \approx 1.3 \cdot 10^{-4} F_{\gamma, in}$. Thus, even underdense axion stars in conservative models shine like tiny dots on the sky giving narrow spectral lines in excess of smooth astrophysical background, cf. [62].

Let us argue that the Bose stars with $D_\infty \ll 1$ are better radioamplifiers than diffuse axions. The latter are described by kinetic theory [13, 62] which gives extra amplification $\Delta F \sim g_{a\gamma\gamma, 0}^2 \rho L^2 \lambda F_{\gamma, in}$ from diffuse cloud of size $L$ and correlation length $\lambda$. We will rederive this expression in Sec. V using Eqs. (16). At $\lambda \sim L \sim R_\star$ it reproduces small-$D_\infty$ result for the axion stars. One finds that compact objects give larger amplification, indeed.

### E. Radio-portrait of an axion star

In generic resonating axion cloud there exists one, at most several directions where the condition (31) is satisfied. Parametric emission forms narrow beams pointing in these directions. But the Bose stars are spherical, with all diameters giving the same $D_\infty$. The question is, what is the distribution of the resonant flux in angular harmonics.

First, if the total mass $M$ is fixed, the product $\rho L \sim M/L^2$ is larger for smaller $L$. Second, the wavelengths $\lambda \sim 10^8 m^{-1}$ of diffuse axions in the Galaxy are much smaller than the radii of axion stars.

Let $Q$ be the fraction of dark matter in the axion stars. Stimulated emission from these objects in our Galaxy is suppressed by the tiny geometric factor $R^2/L^2$, where $L \sim$ kpc, as compared to diffuse axions. However, multiplying it by the above boost factor, we find $\Delta F_{\text{stars}}/\Delta F_{\text{diffuse}} \sim Q m v R_\star$, where $v \sim 10^{-3}$ is the velocity of diffuse axions. For critical QCD axion stars and $m = 26 \mu eV$ this ratio equals $Q v M_p/f_a \sim 10^4 Q$, so the stars give larger stimulated flux at $Q \gtrsim 10^{-4}$.

Finally, in the scenario with enhanced axion-photon coupling our Universe may be full of quasi-stationary axion stars with $D_\infty \lesssim \pi/2$. A radiowave passing through one of these objects burns essential fraction of its axions producing a powerful flash of radio-emission\(^7\). This effect can be used to constrain some arrogant models.

\(^7\) In Eq. (41) we ignored backreaction of photons on axions which may be relevant in this case.
\[ \psi = \psi_s(x) e^{i\theta_s} + \psi'_s(x) e^{i\theta'_s}, \quad (44) \]

of well separated static Bose stars \( \psi_s \) and \( \psi'_s \), centered at \( z = 0 \) and \( z = L \), respectively. In Eq. (44) we explicitly introduced complex phases of stars \( \theta_s \) and \( \theta'_s \).

Equations (16) can be solved analytically in the limit when the interstar distance is much larger than their sizes, \( L \gg R_s \). In this case \( \mu \sim O(L)^{-1} \) corresponds to the inverse light-crossing time between the stars. Outside every star i.e. at \( z < L \) and at \( z > 0 \), we obtain

\[
\begin{align*}
\psi_x^+ &= A e^{i\theta} \cos D(z) \\
\psi_y^+ &= -i A' e^{i\theta_s - \mu/2} \sin D(z) \\
\psi_x^- &= A' e^{i\theta(z-L)} \sin[D'_\infty - D'(z)] \\
\psi_y^- &= -i A' e^{i\theta' - \mu(z-L)} \cos[D'_\infty - D'(z)]
\end{align*}
\]

outside \( \psi_s' \), and \( \psi_s \) in Eqs. (30), (31). Indeed, expressions (45) satisfy the boundary value problem inside the left and right stars with \( O(\mu) \) precision, and both of them give correct solution between the stars. Gliming \( c_x^+ \) and \( c_y^+ \) in the latter region, one finds \( A = A e^{-i\mu L} \sin D'_\infty / \cos D'_\infty \) and

\[
\mu = \frac{1}{2L} \left[ i\theta_s - i\theta'_s + \ln \frac{\sin D_\infty \sin D'_\infty}{\cos D_\infty \cos D'_\infty} \right],
\]

which confirms that \( \mu \sim O(L)^{-1} \).

Expression (46) deserves discussion. First, the two-star system hosts parametric resonance if \( \Re \mu > 0 \) or \( D_\infty + D'_\infty \geq \pi/2 \). This condition reproduces the naive criterion (31) with \( \psi \rightarrow |\psi| \). Second, the resonance develops at a very slow rate \( \mu \sim L^{-1} \) which is nevertheless much faster than the evolution of \( \psi \) if \( \mu \gg \omega_s \) or \( mR_s^2 \gg L \).

---

\( ^9 \) The line is 30% off because we used Eq. (35) which has accuracy \( (M_s - M_s_{0})/M_{s,0} \sim 0.4 \). For better precision one has to compute \( \mu \) in Eqs. (16) numerically and obtain \( l_{\text{cutoff}} \) from Eq. (43) at \( \mu_l \approx 0 \).

\( ^8 \) We do think that Eq. (43) can be derived perturbatively. However, this calculation goes beyond the scope of this paper.
Third and importantly, left- and right-moving parametric waves have slightly different frequencies \( \omega_\gamma = m/2 \pm \text{Im} \mu \), where \( \text{Im} \mu = (\theta_\gamma - \theta'_\gamma)/2L \), cf. Eq. (15). This splitting is a benchmark effect of incoherent axions. Technically, it appears because the phases of the resonant amplitudes are locally related to the phase of the axion field, \[ \arg c^+_x \approx \arg c^-_y - \arg \psi + \pi/2 . \tag{47} \]

Indeed, all coefficients in Eqs. (16) become real after substitution \( c^-_y \rightarrow i c^-_y \exp(i \arg \psi) \) with corrections suppressed by \( \partial_\gamma \arg \psi \); hence (47). In the above setup with two axion stars the shifts of emission frequencies ensure Eq. (47) inside each star at \( z \approx 0 \) and \( L \).

Notably, one does expect formation of gravitationally bound groups of Bose stars in the QCD axion cosmology. Indeed, in the post-inflationary scenario these objects emerge in the centers of miniclusters which are organized in chains and hierarchically bound structures [32, 33, 37]. Once several stars within one group align with small relative velocities \( v \ll (mL)^{-1} \), condition (31) may be satisfied and the parametric explosion follows. The spread of the produced spectrum will be \( \delta \omega_\gamma / \omega_\gamma \sim L^{-1} \) due to random phases of the stars, even if their velocities are negligibly small.

\section{V. DIFFUSE AXIONS}

Our eikonal system (16) is a microscopic Maxwell’s equation in disguise. It is valid for general axion backgrounds including virialized distributions in the galaxy cores and axion miniclusters. In the latter cases, however, kinetic approach is simpler.

In this Section we study parametric radio-amplification in a cloud of random classical waves representing incoherent or partially coherent axions. We fix correlators \[ \langle \psi \rangle = 0 , \quad \langle \psi^*(z) \psi(z') \rangle = \rho C(z-z')/(m f_a)^2 , \tag{48} \]

where \( \rho \) is density, \( C(0) = 1 \), and the correlation length is \( \lambda = \int dy C(y) \).

Let us coarse-grain Eqs. (16) to a kinetic equation in the stationary case. To this end we consider two radiowaves with fixed frequency \( \omega_\gamma = m/2 \) and amplitudes \( A^\pm \) traveling back-to-back through a small axion region in Fig. 7a. This fixes the boundary conditions,

\[ c^+_x \bigg|_{z \rightarrow +\infty} = A^+ , \quad c^-_y \bigg|_{z \rightarrow -\infty} = A^- , \tag{49} \]

and the incoming fluxes \( F^\pm_{\gamma,in} = +m^2 |A^\pm|^2 / 2 \).

We assume that by itself, the axion region is too small to host a resonance. Then the nonrelativistic equations (16), (49) can be solved perturbatively,

\[ c^+_x = A^+ [1 + D_{2,\infty} - D_2(z) + i A^- [D^*_\infty - D^*(z)] , \]

\[ c^-_y = A^- [1 + D^*_\infty D(z) - D^*_2(z)] - i A^+ D(z) , \tag{50} \]

where \( D(z) \) is given by Eq. (31) and

\[ D_2(z) = g'm \int_{-\infty}^{z} dz' \psi^*(z') D(z') \tag{51} . \]

We compute the outgoing fluxes by performing ensemble average via Eq. (48),

\[ F^+_{\gamma,\text{out}} = -m^2 \langle |c_x^+|^2 \rangle_{z \rightarrow -\infty} , \quad F^-_{\gamma,\text{out}} = m^2 \langle |c_y^-|^2 \rangle_{z \rightarrow +\infty} . \tag{52} \]

The solution (50) gives,

\[ F^\pm_{\gamma,\text{out}} = F^\pm_{\gamma,\text{in}} (1 + \mu'_{\infty}L) - \mu'_{\infty}L F^\mp_{\gamma,\text{in}} . \tag{53} \]

Here \( L \) is the size of the axion region and \( \mu'_{\infty} = \langle |D^*_\infty|^2 \rangle / L \) is the naive growth exponent in the infinite axion gas. The latter parameter is explicitly computed by assuming that the region is macroscopic, \( L \gg \lambda \), and yet, small at the scales of \( \rho \),

\[ \mu'_{\infty} = g^2_{a\gamma\gamma} \rho \lambda / 8 , \tag{54} \]

where we restored the physical coupling \( g_{a\gamma\gamma} \).

Now, consider large axion cloud. We divide into small regions of width \( L \), see Fig. 7b. Since equation (52) is valid in every region, we find,

\[ \partial_z F^\pm_{\gamma} = \mu'_{\infty}(z)(F^-_{\gamma} - F^+_{\gamma}) , \tag{55} \]

where \( F^\pm_{\gamma}(z) \) are the fluxes \( F^\pm_{\gamma,in} \approx F^\pm_{\gamma,\text{out}} \) at the macroscopic position \( z \).

Recalling that \( F^+_\gamma \) and \( F^-_\gamma \) travel in \(-z\) and \(+z\) directions, respectively, one restores the time derivative in Eq. (55) by changing

\[ \partial_z F^\pm \rightarrow (\partial_z \mp \partial_t) F^\pm . \tag{56} \]

After that our kinetic equation coincides with the one in Refs. [13, 62] if one trades the correlation length \( \lambda(z) \) in Eq. (53) for the axion velocity \( v \sim (m \lambda)^{-1} \) or spectral width of radiowaves \( \delta \omega_\gamma \sim \lambda^{-1} \).

Solving Eq. (54) in the stationary case, we find,

\[ F^\pm_{\gamma}(z) = F^\pm_{\gamma}(z) + F_0 = F_0 \int_{-\infty}^{z} \mu'_{\infty}(z') dz' , \tag{57} \]
where $F_0$ is the integration constant. Note that this solution does not indicate exponential growth of fluxes, unlike the time-dependent solutions of Eqs. (54), (55) behaving like $F_\gamma^\pm \propto \exp(\mu t)$ in the infinite medium.

Nevertheless, one can use Eq. (56) for waves with $\omega_\gamma = m/2$ ($\mu = 0$) in two important respects. First, $\mu = 0$ when the resonance is about to appear. In this case the ambient fluxes are absent: $F_\gamma^+ (+\infty) = F_\gamma^- (-\infty) = 0$, cf. Eq. (17). The solution (56) satisfies this criterion only at $D_{\infty, \text{diff}} = 1$, i.e. at the boundary of the region

$$D_{\infty, \text{diff}} \equiv \frac{g_\text{a}^2 \gamma_{\gamma\gamma}}{8} \int_{-\infty}^{+\infty} \rho(z) \lambda(z) \, dz \geq 1 .$$

This inequality gives precise condition for the parametric resonance in diffuse axions, cf. Eq. (31).

Second, even far away from the parametric instability Eq. (56) predicts amplification of ambient radioflux $F_{\gamma, \text{in}} = F_\gamma^+ (+\infty)$ due to decay of axions,

$$F_{\gamma, \text{out}} = F_{\gamma, \text{in}} / (1 - D_{\infty, \text{diff}}) ,$$

where $F_{\gamma, \text{out}} = F_\gamma^+ (-\infty)$, cf. Sec. IV D.

VI. MOVING AXIONS

A. Doppler shifts and new resonance condition

We just saw that motion of diffuse axions decreases their correlation length $\lambda \sim (mv)^{-1}$ and hence suppresses the resonance, cf. Eq. (57). In this Section we study the effect of moving coherent axions.

Let us rewrite the system (16) in terms of physical parameters: axion velocity $v_\gamma(t, x)$ in Eq. (27), and density $\rho(t, x) = m^2 f_\gamma^2 |\psi|^2$. To this end we change variables,

$$c^+_x = c^+_y e^{-i\arg \psi/2} , \quad c^-_x = c^-_y e^{i\arg \psi/2} .$$

Eikonal equations take the form,

$$(2\mu + im v_z) c^+_x = 2\partial_z c^+_x + ig_{\gamma\gamma} (\rho/2)^{1/2} c^+_y ,$$

$$(2\mu + im v_z) c^-_y = -2\partial_z c^-_y - ig_{\gamma\gamma} (\rho/2)^{1/2} c^+_x .$$

Note that only a projection $v_z$ of the axion velocity to the resonance axis matters.

If $v_z$ is constant, one can eliminate it from Eqs. (59) by changing $\mu \rightarrow \mu - im v_z/2$. This is the Doppler shift of frequencies $\omega_\gamma = m/2 \pm Im \mu$ for the left- and right-moving waves in Eq. (13). Apart from that, constant velocities do not affect the resonance at all. Indeed, one can always transform to the rest frame of axions.

The situation changes if some parts of the axion matter move with respect to others: $v_z = v_z(z)$. Then the axions decaying in various parts produce photons with different frequencies, and this kills Bose amplification of induced decays. Thus, relative velocities are the main show-stoppers for the parametric resonance.

B. Two moving axion stars

To get a qualitative understanding of relative velocities, we consider two identical Bose stars approaching each other at a constant speed $v,

$$\psi = \psi_1(z) e^{imvz} + \psi_2(z) e^{-imv(z-L)} ,$$

see Fig. 8. For simplicity we will assume that $\psi_1$ and $\psi_2$ are equal to a constant $\psi_0$ in the regions $0 < z < 2R_s$ and $L < z < L + 2R_s$, and they are zero outside.

We find the resonant mode by solving Eqs. (59) in the regions of constant $\rho$, $v_z$ and gluing the original amplitudes $c_{x,y}^\pm$ at $z = 2R_s$ and $z = L$. Then the boundary conditions (17) give equation for the growth exponent $\mu$. At the border of resonance $\mu = i\mu'$ becomes imaginary.

FIG. 8. Two moving Bose stars.

FIG. 9. Condition for parametric resonance in two moving axion stars (top panel) and respective Doppler shift $\mu' = \text{Im} \mu$ (bottom panel).
and the equation simplifies,
\[
\tan^2(2\kappa - R_s)\tan^2(2\kappa + R_s) = \left[1 + \frac{(2\mu' - mv)^2}{m^2v_0^2\cos^2(2\kappa - R_s)}\right] \times \left[1 + \frac{(2\mu' + mv)^2}{m^2v_0^2\cos^2(2\kappa + R_s)}\right].
\]

Here we introduced the relevant velocity scale \(v_0 = 2g'\psi_0\) and notations \(4\kappa_\pm^2 = m^2v_0^2 + (2\mu' \pm mv)^2\).

At a very naive level, one may use \(|\psi|\) instead of \(\psi\) in Eq. (31). Then the resonance is expected at \(D_\infty \equiv 4g' mv \psi_0 R_s \geq \pi/2\), where \(D_\infty\) sums up contributions from both stars. In truth, the solution of Eq. (61) exists only in the shaded region in Fig. 9 (top panel). The Doppler shift \(\mu' = \mu'(v)\) at the boundary of this region is plotted in the bottom panel.

One observes sharp first-order phase transition at \(v = v_0\) between the two resonance regimes, see the vertical dashed line in Fig. 9. At \(v < v_0\) the Doppler shift is absent, \(Im \mu = 0\), although the stars have nonzero velocities. Besides, the naive resonance condition \(D_\infty \geq \pi/2\) is approximately valid indicating that the instability develops simultaneously in both stars. To the contrary, at \(v > v_0\) two individual stars host their own resonances, with little help from each other. In this case the Doppler shift \(Im \mu \approx mv/2\) and the resonance condition \(D_\infty/2 > \pi/2\) coincides with that for one star. We conclude that the two-star resonance occurs only at \(v \leq v_0\) or Eq. (60).

Note that the phase transition in Fig. 9 can be understood analytically. At large relative velocities \(v \gg v_0\) at least one of the two brackets in the right-hand side of Eq. (61) should be small, so the solutions are \(\mu' \approx \pm mv/2\) and \(2\kappa_\pm R_s \approx D_\infty/2 \approx \pi/2\). This corresponds to resonance in individual stars. At \(v \lesssim v_0\) Eq. (61) with \(\mu' = 0\) takes the form
\[
\cos(4\kappa \pm R_s) = -v^2/v_0^2,
\]
where \(\kappa_\pm = m(v_0^2 + v^2)^{1/2}/2\). At \(v \ll v_0\) we obtain \(D_\infty = \pi/2\), a condition for the two-star resonance. At \(v > v_0\) the above equation in the case \(\mu' = 0\) does not have solutions.

C. Collapsing stars

Now, consider collapse of a critical axion star, \(M_r = M_{cr}\), caused by the attractive self-interaction of axions. During this process the axions fall into the star center acquiring velocities and making the density grow, see Fig. 10a. These two effects suppress the resonance and support it, respectively.

To find out how the parametric instability progresses, we numerically solve the boundary value problem (16) in the background \(\psi(t, r)\) of the collapsing star at every \(t\). We characterize the stage of collapse with the radius \(r = r_c(t)\) where the axion field drops by a factor of two from its value in the center: \(|\psi(t, r_c(t))| = |\psi(t, 0)|/2\). We will see that the region \(r \lesssim r_c\) is important for the resonance despite the fact that \(r_c(t)\) decreases by orders of magnitude during collapse. Shaded region Fig. 11a covers couplings \(g_{\gamma \gamma \gamma}\) required for the resonant solutions of Eqs. (16) to exist at time \(r_c(t)\). At the lower boundary of this region \(Re \mu = 0\); the respective Doppler shifts \(Im \mu\) are

FIG. 10. (a) Numerical solution to the Schrödinger-Poisson system (6), (7) describing collapse of a critical Bose star; the axion velocity is \(v = m^{-1}\partial_t \arg \psi\). We use space and time units \(\tau_0 = g_4 M_{pl}/(m f_a)\) and \(\tau = m r_0^2\), see Appendix C. (b) Universal self-similar attractor.

FIG. 11. (a) Electromagnetic coupling \(g_{\gamma \gamma \gamma}\) required for parametric resonance in collapsing critical star at the moment when its core radius is \(r_c(t)\). (b) Doppler shifts \(\pm Im \mu\) at the moment of ignition. Unit of length is \(r_0 = g_4 M_{pl}/(m f_a)\).

FIG. 11. (a) Electromagnetic coupling \(g_{\gamma \gamma \gamma}\) required for parametric resonance in collapsing critical star at the moment when its core radius is \(r_c(t)\). (b) Doppler shifts \(\pm Im \mu\) at the moment of ignition. Unit of length is \(r_0 = g_4 M_{pl}/(m f_a)\).
are presented in Fig. 11b.

Since the star is spherically-symmetric, \( \psi(z) = \psi(-z) \), the photon modes with complex exponents \( \mu \) appear in conjugate pairs. Indeed, for every solution \( \{c^\pm(z), c^\mp(z)\} \) of Eqs. (16) with eigenvalue \( \mu \), there exists a solution \( \{c^\mp(-z)^*, c^\pm(-z)^*\} \) with eigenvalue \( \mu^* \). Physically, this means that for every axion there exists a diametrically opposite axion with the opposite velocity giving Doppler shift \( -\text{Im} \, \mu \). Two signs in the ordinate label of Fig. 11 represent these two solutions.

In Fig. 11 we again see the first-order phase transition (vertical dashed line) described in Sec. VI.B. Indeed, if the resonance appears immediately after the collapse begins (large \( r_c \)), it involves all slowly-moving axions and develops with \( \text{Im} \, \mu = 0 \). At later stages of collapse (smaller \( r_c \)) the resonance can be supported only by fast axions in the dense star core, hence the Doppler shift \( \text{Im} \, \mu \neq 0 \).

We therefore consider resonance in the central core of a collapsing star. It was shown [47, 49] that evolution of the axion field in this region is described by the universal self-similar attractor,

\[
\psi(t, r) = \frac{(mt)^{-i\omega_*}}{mrg_4} \chi_\star(\zeta), \quad \zeta = r \sqrt{-m/t},
\]

where \( t < 0, \, \omega_* \approx 0.54 \) and the function \( \chi_\star(\zeta) \) is presented in Fig. 10b. The core size \( r_c(t) \approx 1.5 \sqrt{-t/m}^{1/2} \) shrinks from the macroscopic values \( r_c \sim R_s \) to \( m^{-1} \) during self-similar stage. Relativistic corrections become relevant [49] at the end of this stage \( t \approx -m^{-1} \).

Substituting Eq. (62) into the spectral problem (16) and changing variables \( c^{\pm} = (-mt)^{\pm i\omega_*/2} \tilde{c}^{\pm}(\zeta) \), we arrive to time-independent spectral problem

\[
\tilde{\mu} \tilde{c}_x^+ = \partial_\zeta \tilde{c}_x^+ + \frac{i g'}{g_4} \frac{\chi_\star(\zeta)^*}{\zeta} \tilde{c}_y^-, \quad \tilde{\mu} \tilde{c}_y^- = -\partial_\zeta \tilde{c}_y^- + \frac{i g'}{g_4} \frac{\chi_\star(\zeta)}{\zeta} \tilde{c}_x^+, \tag{63a,63b}
\]

which involves only one combination of parameters \( g'/g_4 \). We also introduced

\[
\mu = \tilde{\mu} \sqrt{-m/t}, \tag{64}
\]

where the spectral parameter \( \tilde{\mu} \) does not depend on time. We extend the above equations to the full star diameter \( -\infty < \zeta < +\infty \) with \( \chi_\star(-\zeta) = -\chi_\star(\zeta) \), as explained in Appendix A.

We numerically solve Eqs. (63) with boundary conditions (17); the exemplary solution at \( g'/g_4 \approx 0.13 \) is shown in Fig. 12. Notably, the nontrivial part of this solution has width corresponding to \( r_c(t) \) (vertical lines in Fig. 12). Beyond this part \( |c_\tilde{c}| \) freely decay as \( \exp\{-|\zeta| \text{Re} \, \tilde{\mu} \} \). Thus, the resonance mode shrinks on par with the collapsing star.

![FIG. 12. Resonance mode in the collapsing star; functions \( c^\pm(\zeta) \) and \( \tilde{c}^\pm(\zeta) \) are not symmetric to each other. We use self-similar coordinate \( \zeta \) and \( g_{a\gamma\gamma} = 0.37 g_4/f_\alpha \). The respective eigenvalue is \( \tilde{\mu} \approx 0.065 + 0.025i \).](image1)

![FIG. 13. Rescaled growth exponents \( \tilde{\mu} \) in the collapsing star.](image2)

Numerical solutions of Eqs. (63), exist only at

\[
g_{a\gamma\gamma} \geq 0.25 \frac{g_4}{f_\alpha}. \tag{65}
\]

This is a general condition to ignite parametric instability in collapsing stars. It reproduces minimal coupling required for the resonance in Fig. 11 (horizontal dashed line). Also, it is twice weaker than the condition for critical stars before collapse, cf. Eq. (33). For QCD axions, the region (65) is above the dashed line in Fig. 2.

If the above inequality is met, the resonance progresses with two complex time-dependent exponents \( \mu + \mu^* \) in Eq. (64), where \( \pm \text{Im} \, \mu \) are the Doppler shifts. The respective eigenvalues \( \tilde{\mu} \) are plotted in Fig. 13. Importantly, the time dependence of \( \tilde{\mu} \) does not stop the resonance. Indeed, we already argued that the respective mode behaves like a localized level in quantum mechanics. Slow variations of external background do not change occupation of this level if the adiabatic condition is satisfied,

\[
\frac{\partial \mu}{\mu^2} \sim (mt)^{-1/2} \ll 1. \tag{66}
\]

Thus, the electromagnetic field sits on two quasistationary resonance levels,

\[
C_{\alpha}^{\pm} = A \ e_{\alpha}^{\pm}(t, z) \ e^{\int_{0}^{t} dt' \mu} \pm A' \ e_{\beta}^{\mp}(t, z) \ e^{\int_{0}^{t} dt' \mu^*},
\]
at least until the backreaction ruins the self-similar background.

The axion star radio-luminosity follows from the above representation. Interestingly, it oscillates in time due to interference between the modes,

\[ L_\gamma \propto e^{2Re \int_0^t \! \mu \, dt} \left[ 1 + b \cos \left( 2Im \int_0^t \! \mu \, dt + \varphi_0 \right) \right], \quad (67) \]

where \( b \) and \( \varphi_0 \) depend on the initial amplitudes \( A, A' \), with \( b = 1 \) representing equipartition. In Fig. 14 we illustrate these oscillations at \( b = 0.9, \varphi_0 = 0 \). Dashed line in this figure represents self-similar formula with \( \int \mu \, dt = -2\mu(-mt)^{1/2} \). It coincides with the direct result (points) obtained by solving Eqs. (16) for \( \mu(t) \) numerically in the background of a collapsing star and then using Eq. (67). This supports our analytic solution in Eq. (64).

To test the above picture of parametric resonance during collapse, we simulate the coupled system of relativistic equations (10), (21) for photons and axions, see Fig. 15, movie [63], and Appendix E for details. We find that at first, the star squeezes with no effect on the electromagnetic field. But once the localized solution of Eqs. (16) appears, growth and oscillations of the luminosity begin (solid line in Fig. 15). The exact result is reproduced by Eq. (67) (points), where \( \mu(t) \) is obtained by solving the boundary value problem (16) and \( b, \varphi_0 \) are obtained from the fit.\(^{11}\)

We finish this Section with a mystery. Figure 15 demonstrates that once the inequality (37) is broken (shaded region), the backreaction ruins self-similar dynamics. Indeed, the axion field\(^{12}\) does not behave anymore as \( |\psi(t, 0)|^{-2} \propto -t \), like Eq. (62) suggests. Nevertheless, the luminosity continues to grow and saturates only deep inside the backreaction region. We will investigate this nonlinear regime in the forthcoming publication [64].

For QCD axions, the saturated luminosity in Fig. 15 is,

\[ L_\gamma = 1.5 \cdot 10^{41} \left( \frac{m}{26 \, \mu eV} \right)^{-3} \text{erg} \cdot \text{s}^{-1}, \quad (68) \]

while the corresponding flux strongly depends on direction, see Fig. 6. Notably, this is close to the parameters of Fast Radio Bursts, \( L_{FRB} = 10^{38} - 10^{40} \text{erg} \cdot \text{s}^{-1} \).

\[10\] For simplicity we ignore time dependence of the resonance wave functions.

\[11\] Note that the self-similar result for \( \mu \) is by a factor of two off for the star in Fig. 15. Good self-similar regime requires really large \( mR \), which are hard to achieve in relativistic simulations.

\[12\] In relativistic simulation \( |\psi| \equiv |\partial a - ima|/(fa m\sqrt{2}) \), see Eq. (4).
the grant RSF 16-12-10494. The rest of this paper re-
program “Axion Cosmology” for discussions. Work on 
interest. We thank all participants of the MIAPP-2020 
of the ensemble in different parts of the galaxy [13, 17].

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We do have a feeling that truly cool applications are 
are still ahead. Astrophysics offers an impressive set of situa-
tions where the resonance condition can be satisfied, and
the ones with the largest system size.

With help of quantum-mechanical perturbation theory, we analytically computed the instability modes and growth exponents in the physically motivated case of slow resonance, $\mu R \ll 1$. Interestingly, our theory predicts a long-living quasi-stationary photon mode with small negative decay exponent $\text{Re} \mu < 0$ after the resonance switches off, and we see this mode in simulations.

We have found two unexpected applications of our method. First, it describes stimulated emission of ambient radiation in axion stars. We observed that these objects can realistically give larger contribution to the radiobackground than the diffuse axions, producing a thin spectral line at $\omega \approx m/2$. Second, with additional coarse-graining our approach reproduces well-known kinetic equation for photons interacting with virialized axions.

We explicitly saw that gravitational and self-interaction energies of axions inside the star trivially shift the photon frequencies without affecting the resonance. We do not expect these effects to be important in other situations as well. In particular, the distribution function of virialized axions in the galaxy depends on their total energy $E$, not kinetic or potential. The photon of frequency $\omega_\star \approx E/2$ will stay in resonance with same part of the ensemble in different parts of the galaxy [13, 17]. Thus, the main show-stoppers for the parametric instabilities are the Doppler shifts and backreaction effects.

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**Appendix A: Spherically-symmetric case**

In the background of a spherical axion star with $\psi = \psi(t, r)$ it is natural to decompose the electromagnetic field $A = \{A_t, A_r\}$ into spherical harmonics,

$$A = \sum_{l m'} \left( A_{lm'} Y_{lm'} + A_{l m'}' Y_{lm'}' + A_{l m'}'' \Phi_{lm'} \right)$$

(A1)

where we use the gauge $A_0 = 0$, spherical vectors $Y_{lm'} = x Y_{lm'}/r$, $\Psi_{lm'} = r \nabla Y_{lm'}$, $\Phi_{lm'} = (\nabla \times x) Y_{lm'}$, and denote the standard spherical functions by $Y_{lm'}$.

Below we omit the superscripts $lm'$ for brevity.

The coefficients of decomposition $A_Y, \Phi, \psi(t, r)$ depend only on time $t$ and radial coordinate $r$. Substituting Eq. (A1) into the Maxwell’s equation (10), one finds the Gauss law

$$A_\phi = \frac{\partial_r (r^2 A_Y)}{r(l+1)}$$

(A2)

and two dynamical equations

$$r^2 \partial^2_t A_Y = \partial^2_t (r^2 A_Y) - l(l+1) A_Y$$

$$- g_{a a} l(l+1)(\partial_t a) r A_\phi,$$

$$r^2 \partial^2_t A_\phi = \partial^2_t (r A_\phi) - l(l+1) A_\phi/r$$

$$- g_{a a} \partial_t a \left[ A_Y - \frac{\partial^2_t (r^2 A_Y)}{l(l+1)} \right],$$

where we omitted terms with $\partial_t a$ because they are suppressed by extra powers of $(mr)^{-1}$ and will not contribute into equations for $C$'s.

We finally introduce the eikonal ansatz,

$$(mr)^2 A_Y = 2i l(l+1) \left\{ C^+_Y e^{im(r-t)/2} + C^-_Y e^{im(r-t)/2} \right\} + \text{h.c.},$$

$$mr A_\phi = C^+_\phi e^{im(r-t)/2} + C^-_\phi e^{im(r-t)/2} + \text{h.c.}$$

Using it in the above equations and omitting the $(mr)^{-1}$ suppressed contributions, we find eikonal equations (14) at $z = r > 0$ for the unknowns $(C^+_Y, C^-_\phi)$ in place of $(C^+_Y, C^-_\phi)$, with the additional term (42) representing derivatives with respect to the spherical angles: $\Delta a_\phi = -l(l+1)$. The pair $(C^+_\phi, -C^-_Y)$ satisfies the same equations.

There are two subtleties in the spherically-symmetric case. First, the transverse polarizations $A_\phi$ and
which proves $\langle \xi_1 | \hat{\Omega} | \xi_2 \rangle = 0$. Moreover, one can argue that the set of $\hat{L}$ eigenmodes — the resonance ones and the ones from the continuum spectrum — forms complete basis in the space of bounded functions $c_x^\pm$ and $c_y^\mp$.

With the above definitions we can develop a perturbation theory for the spectrum of $\hat{L}$. Indeed, suppose at $\psi = \psi_0(z)$ the operator $\hat{L} = \hat{L}_0$ has a normalized eigenmode $|\xi_0\rangle$ with zero eigenvalue, $\hat{L}_0 |\xi_0\rangle = 0$. At slightly different $\psi = \psi_0(z) + \delta \psi(z)$ this operator receives variation $\delta \hat{L} = -g' \delta \psi \hat{\Omega}$. In this case its resonance eigenmode $|\xi\rangle = |\xi_0\rangle + |\delta \xi \rangle$ is close to $|\xi_0\rangle$, and the respective eigenvalue $\mu$ is small. The eigenvalue problem $\hat{L} |\xi\rangle = \mu |\xi\rangle$ takes the form,

$$
\delta \hat{L} |\xi_0\rangle + \hat{L}_0 |\delta \xi \rangle = \mu |\xi_0\rangle ,
$$

(B6)

where we ignored quadratic terms in perturbations. The scalar product with $|\xi_0\rangle$ gives,

$$
\mu = \frac{\langle \xi_0 | \hat{\Omega} \delta \hat{L} |\xi_0\rangle}{\langle \xi_0 | \hat{\Omega} |\xi_0\rangle} = -g' \langle \mu | \delta \psi |\xi_0\rangle / \langle \xi_0 | \hat{\Omega} |\xi_0\rangle .
$$

(B7)

Using explicit solution (29) for $\xi_0$, we finally obtain

$$
\mu = g' \int dz \frac{|\psi(t, \mathbf{x}) - \psi_0(\mathbf{x})|}{\int dz \sin(2D_0)} .
$$

With (31) this expression reproduces Eq. (34) from the main text.

### Appendix C: Scaling symmetry

We calculate parameters of Bose stars using scaling symmetry of the Schrödinger-Poisson system (6), (7). Consider first the model without self-coupling, $g_4 = 0$. One finds that change of variables

$$
\mathbf{x} = \lambda \tilde{\mathbf{x}} , \quad t = m \lambda^2 \tilde{t} , \quad \Phi = \frac{\tilde{\Phi}}{(m \lambda)^2} , \quad \psi = \frac{M_{pl} \tilde{\psi}}{m^2 \lambda^2 f_a} ,
$$

(C1a)

(C1b)

with arbitrary $\lambda$ removes all constants from the equations. This scaling allows us to map the model with arbitrary parameters to a reference one with $\psi(0) = 1$. We perform numerical calculations in tilded variables and then scale back to physical. Parameter $\lambda$ disappears in final answers, if one expresses it via the chosen Bose star characteristics, e.g. its mass,

$$
M_s = m^2 f_a^2 \int d^3 \mathbf{x} |\psi_a|^2 = M_{pl} \frac{M_s^2}{\lambda m^2} ,
$$

(C2)

where $M_s \approx 3.9$ is computed numerically. Similarly, the parameter (31) equals,

$$
D_\infty \approx 0.80 g_{\sigma \gamma \gamma} \frac{M_{pl}}{\lambda m} .
$$

(C3)
Using this approach, we obtain Eqs. (32), (35).

In models with $g_4 \neq 0$ the self-interaction can be ignored at $M \ll M_{\text{cr}}$, see Eq. (9), and we are back to the above situation. Stars with $M \geq M_{\text{cr}}$ are unstable. In the main text we mostly consider the critical star with $M = M_{\text{cr}}$. In this case one excludes all parameters from the equations using Eqs. (C1) with $\lambda = g_4 M_{\text{pl}}/m f_a$, computes the critical star numerically, and then restores the physical parameters. The integral (31) in this case equals

$$D_\infty \approx 3.04 \frac{g_{a\gamma\gamma} f_a}{g_4} \tag{C4}$$

implying (33). These “self-interaction” units are exploited in Figs. 10, 11, 14, 15.

Finally, if self-coupling is negligible but backreaction of photons on axions is relevant, all constants can be eliminated from Eqs. (16), (22), (7) using Eqs. (C1), $C_\alpha^\pm = \hat{C}_\alpha (M_{\text{pl}}/g_{a\gamma\gamma})^{1/2}(m\lambda)^{-2}, \mu = \hat{\mu}/\lambda$, and $\lambda = g_{a\gamma\gamma} M_{\text{pl}}/m$.

We perform this rescaling to plot universal quantities in Figs. 3, 4, and 15.

**Appendix D: Initial conditions**

In real astrophysical settings the axion stars are embedded into the background of classical radiowaves which can give a good initial kick to the parametric instability, cf. Sec. IV D. But this mechanism essentially depends on the environment, so outside of Sec. IV D we assume quantum start, i.e. the resonance set off by the spontaneous decays of axions inside the isolated star.

Detailed study of quantum evolution is beyond the scope of this paper, so we use a shortcut. Namely, the flux $F_{\gamma} \sim |E|^2 \sim |H|^2$ of spontaneous photons can be estimated from energy conservation,

$$\partial_t M_\gamma = -\Gamma_{a\gamma\gamma} M_\gamma = -4\pi^2 F_{\gamma}, \tag{D1}$$

where we assumed spherical Bose star and introduced the axion decay width $\Gamma_{a\gamma\gamma} = g_{a\gamma\gamma}^2 \lambda^3/64\pi$. This gives typical amplitudes

$$|E| \sim |H| \sim \frac{1}{R_0} \left( \frac{M_\gamma \Gamma_{a\gamma\gamma}}{4\pi} \right)^{1/2} \tag{D2}$$

of spontaneous emission.

It is worth reminding that the exponential growth of the resonance mode washes out all details of initial quantum evolution, with just one logarithmically sensitive parameter surviving: the time of growth. That is why the above order-of-magnitude description is adequate.

In numerical simulation of Appendix E we mimic the quantum bath of spontaneous photons using a stochastic ensemble of random classical waves with amplitudes (D2). This is required only in dynamical situations such as the axion star collapse in Sec. VI C.

**Appendix E: Full relativistic simulation**

We test the theory by numerically evolving the equations (10) and (21) for the electromagnetic and axion fields. In computations we consider only spherically symmetric axion backgrounds, $a = a(t, r)$. This is justified at the linear stages of parametric resonance and should be valid at least qualitatively during backreaction. To make Eq. (21) self-consistent, we average its right-hand side over spherical angles: $F_{\mu\nu} F_{\mu\nu} \rightarrow \int d\Omega F_{\mu\nu} F_{\mu\nu}/4\pi$. We decompose electric and magnetic fields $E_i = F_{0i}$ and $H_i = -e_{ijk} F_{jk}/2$ in spherical harmonics $Y_{lm'}, \Psi_{lm'}$, and $\Phi_{lm'}$ introduced in Appendix A. With the cutoff $l \leq l_{\text{max}}$, we find $6l_{\text{max}}(l_{\text{max}} + 2) + 1$ equations for the same number of unknowns $E_{lm'}^Y, \Psi, \Phi(t, r)$, $H_{lm'}^Y, \Psi, \Phi(t, r)$, and $a(t, r)$.

As usual, the longitudinal number $m'$ does not explicitly appear in equations for the spherical components of $E$ and $H$. We therefore leave only one component at every $l$ multiplying its contribution in the right-hand side of Eq. (21) by $(2l + 1)$. Now, the number of equations is $6l_{\text{max}} - 5$.

In practice our numerical results are insensitive to $l_{\text{max}}$: the photon modes evolve independently at the linear stage, while backreaction simply equidistributes energy over them. We therefore perform simulations in Figs. 3, 4, 15 with $l_{\text{max}} = 1$ and use $l_{\text{max}} = 210$ with step $\Delta l = 4$ to find the angular structure of the resonance in Sec. IV E. We restore three-dimensional electromagnetic fields during linear evolution multiplying the spherical components with their harmonics, e.g.

$$E = \sum_{lm'} E_{lm'}^\Psi (t, r) \Psi_{lm'}(\theta, \phi) + \ldots$$

where the dots hide other polarizations and independent random numbers $e_{lm'}$ mimic quantum distribution of the initial resonance amplitudes over the longitudinal number $m'$, see Appendix D.

To hold the axions together during resonance, we add interaction with the gravitational potential by changing $V' \rightarrow (1 + 2\Phi)V'$ in Eq. (21). This approximation is trustworthy if the gravitational field is mostly sourced by the nonrelativistic axions.

Since our simulations check nonrelativistic theory, we perform them only for small-velocity axions. In physical units, parameters of these simulations correspond to $m = 26 \mu\text{eV}$ [31], $g_4 = 0.59$ or 0, with other parameters ranging in wide intervals.

---

13 The backreaction stage in the central part of Fig. 3 is short, and related asphericities should be small. Self-similar evolution in Fig. 15 tracks spherically-symmetric attractor which suppresses axion modes with nonzero $l$.

14 Note that $l = 0$ components of $E$ and $H$ are absent.

15 The time when the backreaction appears is logarithmically sensitive to $l_{\text{max}}$, however, cf. Eq. (37).
\[ f_a^2 = (10^{-11} \div 10^{-8}) M_{\odot}^2, \quad \eta \gamma = (0.15 \div 0.4) f_a^{-1}, \] and \( M_r = (10^{-11} \div 10^{-8}) M_{\odot}. \) This indeed corresponds to small nonrelativistic parameter (level of 10 with \( \Delta \)) formed with the fourth-order Runge-Kutta integrator in Eqs. (10), (21), (7). Time evolution is then per-
ing Fourier transform to compute their \( H \) on a uniform radial lattice with \( \Delta r = 1.3/m \), using Fourier transform to compute their \( r \)-derivatives in Eqs. (10), (21), (7). Time evolution is then per-
ing with the fourth-order Runge-Kutta integrator with \( \Delta t = 0.025/m \). Equation (7) is solved at each step. In our calculations the total energy is conserved at the level of \( 10^{-8} \).

We absorb the electromagnetic emission by introducing the “Hubble” friction at the lattice boundary. This indeed corresponds to "fast axion emission by introducing the “Hubble” friction at the lattice boundary. This indeed corresponds to "fast axion emission by introducing the “Hubble” friction at the lattice boundary. This indeed corresponds to "fast axion emission by introducing the “Hubble” friction at the lattice boundary.

In the beginning of simulation we evolve the axion field alone, checking Eqs. (16) for the resonance mode \( (Re \mu > 0) \) to appear. Once it is there, we randomly populate the Fourier modes of the electromagnetic field in the narrow frequency band \( \omega_p \approx m/2 \), with typical amplitude \( (D2) \) in the \( r \)-space. This sets off the resonance making \( E \) and \( H \) grow.

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In Figs. 10 and 14 we use the code of Ref. [49] to evolve the Schrödinger-Poisson equations (6), (7) for axions. Backreaction of photons on axions is not taken into account in these calculations.

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