Characterizing the non-linear growth of large-scale structure in the Universe

Peter Coles,
School of Physics & Astronomy,
University of Nottingham,
University Park,
Nottingham NG7 2RD,
United Kingdom.

and

Lung-Yih Chiang,
Astronomy Unit,
School of Mathematical Sciences,
Queen Mary & Westfield College,
University of London,
London E1 4NS,
United Kingdom.

The local Universe displays a rich hierarchical pattern of galaxy clusters and superclusters\textsuperscript{1,2}. The early Universe, however, was almost smooth, with only slight 'ripples' seen in the cosmic microwave background radiation\textsuperscript{3}. Models of the evolution of structure link these observations through the effect of gravity, because the small initially overdense fluctuations attract additional mass as the Universe expands\textsuperscript{4}. During the early stages, the ripples evolve independently, like linear waves on the surface of deep water. As the structures grow in mass, they interact with other in non-linear ways, more like waves
breaking in shallow water. We have recently shown\(^5\) how cosmic structure can be characterized by phase correlations associated with these non-linear interactions, but hitherto there was no way to use that information to reach quantitative insights into the growth of structures. Here we report a method of revealing phase information, and quantify how this relates to the formation of a filaments, sheets and clusters of galaxies by non-linear collapse. We use a new statistic based on information entropy to separate linear from non-linear effects and thereby are able to disentangle those aspects of galaxy clustering that arise from initial conditions (the ripples) from the subsequent dynamical evolution.

In most popular versions of the "gravitational instability" model for the origin of cosmic structure, particularly those involving cosmic inflation\(^6\), the initial fluctuations that seeded the structure formation process form a Gaussian random field\(^7\). Deviations from uniformity, expressed in terms of the density contrast \(\delta(x)\) defined by

\[
\delta(x) = \frac{\rho(x) - \rho_0}{\rho_0},
\]

where \(\rho_0\) is the average density and \(\rho(x)\) is the local matter density. Because the initial perturbations evolve linearly, it is useful to expand \(\delta(x)\) as a Fourier superposition of plane waves:

\[
\delta(x) = \sum \tilde{\delta}(k) \exp(i k \cdot x).
\]

The Fourier transform \(\tilde{\delta}(k)\) is complex and therefore possesses both amplitude \(|\tilde{\delta}(k)|\) and phase \(\phi_k\) where

\[
\tilde{\delta}(k) = |\tilde{\delta}(k)| \exp(i \phi_k).
\]

Gaussian random fields possess Fourier modes whose real and imaginary parts are independently distributed. In other words, they have phase angles \(\phi_k\) that are independently distributed and uniformly random on the interval \([0, 2\pi]\). When fluctuations are small, i.e. during the linear regime, the Fourier modes evolve independently and their phases remain random. In the later stages of evolution, however, wave modes begin to couple together\(^4\). In this regime the phases become non-random and the density field becomes highly non–Gaussian. Phase coupling is therefore a key consequence of nonlinear gravitational processes if the initial conditions are Gaussian and a potentially powerful signature to exploit in statistical tests of this class of models.
The information needed to fully specify a non-Gaussian field (or, in a wider context, the information needed to define an image\textsuperscript{12}) resides in the complete set of Fourier phases. Unfortunately, relatively little is known about the behaviour of Fourier phases in the non-linear regime of gravitational clustering\textsuperscript{13–18}, but it is of great importance to understand phase correlations in order to design efficient statistical tools for the analysis of clustering data. A vital first step on the road to a useful quantitative description of phase information is to represent it visually. We do this using colour, as shown in Figure 2. To view the phase coupling in an N-body simulation one Fourier-transforms the density field to produce a complex array containing the real (R) and imaginary (I) parts of the transformed 'image' with the pixels in this array labelled by wavenumber \( k \) rather than position \( x \). The phase for each wavenumber, given by \( \phi = \arctan(I/R) \), is then represented as a hue for that pixel.

The rich pattern of phase information revealed by this method (see Figure 3) can be quantified and related to the gravitational dynamics of its origin. For example in our analysis of phase coupling\textsuperscript{5} we introduced a quantity \( D_k \), defined by

\[
D_k \equiv \phi_{k+1} - \phi_k,
\]

which measures the difference in phase of modes with neighbouring wavenumbers in one dimension. We refer to \( D_k \) as the phase gradient. To apply this idea to a two-dimensional simulation we simply calculate gradients in the \( x \) and \( y \) directions independently. Since the difference between two circular random variables is itself a circular random variable, the distribution of \( D_k \) should initially be uniform. As the fluctuations evolve waves begin to collapse, spawning higher-frequency modes in phase with the original\textsuperscript{22}. These then interact with other waves to produce the non-uniform distribution of \( D_k \) seen in Figure 3.

It is necessary to develop quantitative measures of phase information that can describe the structure displayed in the colour representations. In the beginning the phases \( \phi_k \) are random and so are the \( D_k \) obtained from them. This corresponds to a state of minimal information, or in other words maximum entropy. As information flows into the phases the information content must increase and the entropy decrease. This can be quantified by defining an information entropy on the set of phase gradients\textsuperscript{5}. One constructs a frequency distribution, \( f(D) \) of the values of \( D_k \) obtained from the whole map. The entropy is then
defined as

$$S(D) = -\int f(D) \log[f(D)]dD,$$

where the integral is taken over all values of $D$, i.e. from 0 to $2\pi$. The use of $D$, rather than $\phi$ itself, to define entropy is one way of accounting for the lack of translation invariance of $\phi$, a problem that was missed in previous attempts to quantify phase entropy\textsuperscript{23}. A uniform distribution of $D$ is a state of maximum entropy (minimum information), corresponding to Gaussian initial conditions (random phases). This maximal value of $S_{\text{max}} = \log(2\pi)$ is a characteristic of Gaussian fields. As the system evolves it moves into states of greater information content (i.e. lower entropy). The scaling of $S$ with clustering growth displays interesting properties\textsuperscript{17}, establishing an important link between the spatial pattern and the physics driving clustering growth. This phase information is a unique fingerprint of gravitational instability and it therefore also furnishes statistical tests of the presence of any initial non-Gaussianity\textsuperscript{24}. 

Figure 1. A numerical simulation of galaxy clustering (left) together with a version of the same picture generated by randomly reshuffling the phases between Fourier modes of the original picture. Since the amplitude of each Fourier mode is unchanged in this operation, these two pictures have exactly the same power-spectrum, \( P(k) \propto |\tilde{\delta}(k)|^2 \), but have totally different morphology. The shortcomings of \( P(k) \) can be partly ameliorated by defining higher-order quantities such as the bispectrum\(^{4,8-10}\) or correlations\(^{11}\) of \( \tilde{\delta}(k)^2 \). The bispectrum and higher-order polyspectra vanish for Gaussian fields, but in a non-Gaussian field they may be non-zero. The usefulness of these and related quantities therefore lies in the fact that they encode some information about non-linearity and non-Gaussianity. The bispectrum, for example, measures the phase coupling induced by quadratic nonlinearities, and so on for higher orders. However, an infinite hierarchy of such moments is necessary to specify the properties of a general random field in a statistical sense. Since not every distribution is specified by its moments, even this need not be complete.

Figure 2. The representation of colour hue on a circle. In colour image display devices, each pixel represents the intensity and colour at that position in the image\(^{18,19}\). The quantitative specification of colour involves three coordinates describing the location of that pixel in an abstract colour space, designed to reflect as accurately as possible the eye’s response to light of different wavelengths. In many devices this colour space is defined in terms of the amount of Red, Green or Blue required to construct the appropriate tone; hence the RGB colour scheme. The scheme we are particularly interested in, the HSB scheme, is based on three different parameters: Hue, Saturation and Brightness. Hue is the term used to distinguish between different basic colours (blue, yellow, red and so on). Saturation refers to the purity of the colour, defined by how much white is mixed with it. A saturated red hue would be a very bright red, whereas a less saturated red would be pink. Brightness indicates the overall intensity of the pixel on a grey scale. The HSB colour model is particularly useful because of the properties of the ‘hue’ parameter, which is defined as a circular variable. On the colour circle shown, the primary hues (red, green and blue) are 120° apart from one another (at 0°, 120° and 240° respectively), while the complementary tones yellow, cyan, and magenta are at 60°, 180° and 300°. Red, of course also appears at 360° and so on, so this parameter is truly circular in the same way as phases are. Our visualization method the hue parameter to encode the Fourier phases.
Figure 3. The evolution of phase coupling in a sequence of snapshots from a two-dimensional N-body simulation\(^{20,21}\) starting with Gaussian initial conditions. The initial power-spectrum was a power-law, \(P(k) \propto k^n\), with \(n = 0\). The left-hand column shows the development of hierarchical clustering through the emergence in the density field of the characteristic network of voids and filaments that typifies the action of gravitational instability. The next column shows the colour-coded phases, and the third and fourth columns show phase gradients in the \(x\)-direction and \(y\)-direction respectively. Each row corresponds to a particular timestep, with time flowing downwards. The early stages display no phase structure, but patterns emerge as time unfolds. The phase images develop a series of bands and stripes, first appearing as large-scale features in reciprocal space. As the simulation evolves, the characteristic \(k\)-space scale of these features gets smaller. This behaviour is related to the increasing characteristic scale of features in the density maps in the first column: larger features in real space correspond to smaller features in the reciprocal space. Phase correlations between \(k\)-modes reveal themselves as structure in the distribution of the \(D_k\), chiefly through a domination of some hues over others. Starting from the initial data, in which they are distributed evenly, one sees the gradual appearance of regions where certain hues dominate. By the final stage, the third column is largely green and the last column is largely red indicating strongly coupled phases in both directions. Animations of this effect can be viewed at our website

\url{http://www.nottingham.ac.uk/~ppzpc/phases/index.html}. 
References

[1] Saunders, W. et al. The density field of the local Universe, *Nature*, **349**, 32-38 (1991)

[2] Shectman, S. et al. The Las Campanas Redshift Survey, *Astrophys. J.*, **470**, 172-188 (1996)

[3] Smoot, G.F. et al. Structure in the COBE Differential Microwave Radiometer First-year Maps, *Astrophys. J.*, **396**, L1-L4 (1992)

[4] Peebles, P.J.E. *The Large-scale structure of the Universe* (Princeton University Press, Princeton, 1980)

[5] Chiang, L.-Y., & Coles, P. Phase information and the evolution of cosmological density perturbations, *Mon. Not. R. astr. Soc.*, **311**, 809-824 (2000)

[6] Guth, A.H. & Pi, S.-Y. Fluctuations in the New Inflationary Universe, *Phys. Rev. Lett.*, **49**, 1110-1113 (1982)

[7] Bardeen, J.M., Bond, J.R., Kaiser, N. & Szalay A.S. The statistics of peaks of Gaussian Random Fields, *Astrophys. J.*, **304**, 15-61 (1986)

[8] Matarrese, S., Verde, L., & Heavens, A.F. Large-scale Bias in the Universe: Bispectrum Method, *Mon. Not. R. astr. Soc.*, **290**, 651-662 (1997)

[9] Scoccimarro, R., Couchman, H.M.P., & Frieman, J.A. The bispectrum as a signature of gravitational instability in redshift space, *Astrophys. J.*, **517**, 531-540 (1999)

[10] Verde, L., Wang, L., Heavens, A.F., & Kamionkowski, M. Large-scale Structure, the cosmic microwave background, and primordial non-Gaussianity, *Mon. Not. R. astr. Soc.*, **313**, 141-147 (2000).

[11] Stirling, A.J., & Peacock, J.A. Power correlations in cosmology: Limits on primordial non-Gaussian density fields, *Mon. Not. R. astr. Soc.*, **283**, L99-L104 (1996)

[12] Oppenheim, A.V., & Lim, J.S. The Importance of Phase in Signals, *Proc. IEEE.*, **69**, 529-541 (1981)
[13] Ryden, B.S., & Gramann, M. Phase Shifts in Gravitationally Evolving Density Fields, Astrophys. J., 383, L33-L36 (1991)

[14] Scherrer, R.J., Melott, A.L., & Shandarin, S. F. A Quantitative Measure of Phase Correlations in Density Fields, 1991, Astrophys. J., 377, 29-35 (1991)

[15] Soda, J. & Suto, Y. Nonlinear Gravitational Evolution of Phases and Amplitudes in One-dimensional Cosmological Density Fields, Astrophys. J., 396, 379-394 (1992)

[16] Jain, B., Bertschinger, E. Self-similar evolution of gravitational clustering: is $n = 1$ special?, Astrophys. J., 456, 43-54 (1996)

[17] Jain, B. & Bertschinger, E. Self-similar evolution of gravitational clustering: N-body simulations of the $n = -2$ spectrum, Astrophys. J., 509, 517-530 (1998)

[18] Thornton, A.L. Colour object recognition using a complex colour representation and the frequency domain (PhD Thesis, University of Reading, 1998)

[19] Foley, J.D., & Van Dam, A. Fundamentals of Interactive Computer Graphics (Addison-Wesley, Reading, Mass., 1982)

[20] Melott, A.L., & Shandarin, S.F. Generation of large-scale cosmological structures by gravitational clustering, Nature, 346, 633-635 (1990)

[21] Beacom, J.F., Dominik, K.G., Melott, A.L., Perkins, S.P., & Shandarin, S.F., Gravitational clustering in the expanding Universe - controlled high resolution studies in two dimensions, Astrophys. J., 372, 351-363 (1991)

[22] Shandarin, S.F., & Zel’dovich, Ya. B. The large-scale structure: turbulence, intermittency, structures in a Self-gravitating medium, Rev. Mod. Phys., 61, 185-220 (1989)

[23] Polygian, J.M., & Moussas, X. Detection of nonlinear dynamics in solar wind and a comet using phase-correlation measures, Solar Physics, 158, 159-172 (1995)

[24] Ferreira, P.G., Magee, J., & Górski, K.M. Evidence for non-Gaussianity in the COBE DMR 4-year sky maps, Astrophys. J., 503, L1-L4 (1998)
Please address correspondence to Peter Coles (Peter.Coles@Nottingham.ac.uk). Colour animations of phase evolution from a set of \(N\)-body experiments, including the one shown in Figure 3, can be viewed at

http://www.nottingham.ac.uk/~ppzpc/phases/index.html.
This figure "fig2.gif" is available in "gif" format from:

http://arxiv.org/ps/astro-ph/0006017v1
This figure "fig3.gif" is available in "gif" format from:

http://arxiv.org/ps/astro-ph/0006017v1