A NOTE ON THE PRICING OF BASKET OPTIONS USING TAYLOR APPROXIMATIONS

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Abstract. In this paper we propose a closed-form approximation for the price of basket options under a multivariate Black-Scholes model, based on Taylor expansions and the calculation of mixed exponential-power moments of a Gaussian distribution. Our numerical results show that a second order expansion provides accurate prices of spread options with low computational costs, even for out-of-the-money contracts.

1. Introduction

The objective of the paper is the pricing of basket options using Taylor approximations under diffusion multivariate models with constant covariance. Basket options are multivariate extensions of European calls or puts. A basket option takes the weighted average of a group of \( d \) stocks as the underlying, and produces a payoff equal to the maximum of zero and the difference between the weighted average and the strike (or the opposite difference for the case of a put). Index options, whose value depends on the movement of an equity or other financial index such as the S&P500, are examples of basket options.

For the particular case of spread options, several approximations have been previously considered in the works of Kirk(1995), Carmona and Durrleman(2003), Li, Deng and Zhou(2008, 2010), Venkatramanan and Alexander(2011) where different ad-hoc approaches are studied.

As an alternative Fast Fourier Transform methods have been successfully implemented to compute spread prices under more general Levy processes, see Hurd and Zhou(2009) and Cane and Olivares(2014) and under stochastic volatility models in Dempster and Hong(2000).

The approach to pricing by Taylor expansions can be traced back to Hull and White(1987), where the price of a one dimensional derivative is calculated. On the other hand, following an idea in Pearson(1995) it can be extended to multidimensional contracts by conditioning on the remaining \( d - 1 \) underlying, reducing the problem to one dimensional pricing with parameters arising from the resulting conditional distribution. It should be noticed that this technique has been used in Li, Deng and Zhou(2008) for the case of spread options. Furthermore, in Li, Deng and Zhou(2010) the Taylor expansion approximation is compared with other pricing techniques.

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proving to be effective and accurate for most values in the parametric space. Although in the same spirit, in our case the expansion is done on the function resulting from the conditional price, as opposed to a development based on the conditional strike price, as previously considered by the authors cited above. Moreover, our method hinges on the calculation of mixed exponential-power moments of a Gaussian distribution, it is extended to expansions about any point and higher dimensions. Our point of view may allow for a better control on the approximation, particularly for out-of-the-money options. On a related paper, see Alvarez, Escobar and Olivares (2011), we apply a similar technique to the price of a spread option when correlation is stochastic, by expanding on the correlation matrix.

The organization of the paper is the following, in section 2 we introduce some notations, the model and derive the Taylor approximation for basket options. In section 3 we specialize the formula for spread options and compute the mixed exponential-power moments of a Gaussian law. In section 4 we discuss our numerical results.

2. BASKET DERIVATIVES AND TAYLOR EXPANSIONS

We introduce some notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space. We define the filtration $\mathcal{F}^X_t := \sigma(X_s, 0 \leq s \leq t)$ as the $\sigma$-algebra generated by the random variables $\{X_s, 0 \leq s \leq t\}$ completed in the usual way. Denote by $Q$ an equivalent martingale risk neutral measure and $E^Q$ the expectation under $Q$.

By $r$ we denote the (constant) interest rate, $A'$ represents the transpose of matrix $A = (a_{ij})_{1 \leq i, j \leq d}$ while $\text{diag}(A)$ is a vector with components $(a_{ii})_{1 \leq i \leq d}$. The $d$-dimensional column vector of ones is denoted by $1_d$.

For a $l$-times differentiable function $f$ in $\mathbb{R}^d$ and a vector $L = (l_1, l_2, \ldots, l_d)$ with $l_k \in \mathbb{N}$ such that $\sum_{k=1}^n l_k = l$, $D^L f$ represents its mixed partial derivative of order $l$ differentiated $l_k$ times with respect to the variable $y_k$.

The process of spot prices is denoted by $S_t = (S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)})'$ and $Y_t = (Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(d)})_{0 \leq t \leq T}$ are the asset log-returns related by:

$$S_t^{(j)} = S_0^{(j)} \exp(Y_t^{(j)}) \quad \text{for} \quad j = 1, 2, \ldots, d$$

We analyze European Basket options whose payoff at maturity $T$, for a strike price $K$, is given by:

$$h(S_T) = \left( \sum_{j=1}^d w_j S_T^{(j)} - K \right)_+$$

where $(w_j)_{1 \leq j \leq d}$ are some deterministic weights and $x_+ = \max(x, 0)$.

As examples we have spread options, defined for $d = 2$ with payoff:

$$h(S_T) = (S_T^{(1)} - S_T^{(2)} - K)_+$$
Also, we have 3:2:1 crack spreads with \( d = 3 \) and payoff:

\[
(4) \quad h(S_T) = \left( \frac{2}{3} S_T^{(1)} - \frac{1}{3} S_T^{(2)} - S_T^{(3)} - K \right)_+ 
\]

where \( S_t^{(1)} \), \( S_t^{(2)} \) and \( S_t^{(3)} \) are respectively the spot prices of gasoline, heating oil and crude oil.

Exchange options are derivatives whose payoff is a particular case of (3) when \( K = 0 \). Exact formulas are available in the case of a diffusion, see Margrabe (1978).

We assume a multidimensional Black-Scholes dynamics under the risk neutral probability following:

\[
(5) \quad dS_t = rS_t dt + \Sigma \frac{1}{2} S_t dB_t 
\]

where \((B_t)_{t \geq 0}\) is a \( d \)-dimensional vector of Brownian motions such that \( d < B_t^{(j)}, B_t^{(m)} = \rho_{jm} dt \), for \( j, m = 1, 2, \ldots, d \) and \( \Sigma \) is a positive definite symmetric matrix with components \( (\sigma_{ij})_{i,j=1,2,\ldots,d} \) and \( \sigma_{ii} = \sigma_i^2 \).

We denote by \( \tilde{Y}_t = (Y_t^{(2)}, Y_t^{(3)}, \ldots, Y_t^{(d)}) \) the vector of log-returns, excluding the first component. The price of a basket option with maturity at \( T > 0 \) and payoff \( h(S_T) \) is:

\[
(6) \quad p = e^{-rT} E_Q h(S_T) = E_Q \left( e^{-rT} E_Q \left[ h(S_T) | \mathcal{F}_{\tilde{Y}_T} \right] \right) = E_Q \left[ C(\tilde{Y}_T) \right] 
\]

where:

\[
C(y) := E_Q \left[ h(S_T) | \mathcal{F}_{\tilde{Y}_T} \right] |_{\tilde{Y}_T = y} 
\]

Assuming \( C(y) \) is smooth enough, we denote the \( n \)-th order Taylor development of \( C \) around the point \( y^* \in \mathbb{R}^{d-1} \) as \( \hat{C}_n(y) \). It is given by:

\[
(7) \quad \hat{C}_n(y) = \sum_{l=0}^{n} \sum_{R_l} \frac{D^L C(y^*)}{l_1! l_2! \ldots l_{d-1}!} \prod_{k=1}^{d-1} (y_k - y^*_k)^{l_k} 
\]

where:

\[
L = (l_1, l_2, \ldots, l_{d-1}) \text{ and } R_l = \{L \in \mathbb{N}^{d-1}/l_1 + l_2 + \ldots + l_{d-1} = l, \ 0 \leq l_k \leq l \}. 
\]

The next proposition provides the Taylor approximation for the price \( p \) of a basket option:

**Proposition 1.** The \( n \)-th order Taylor approximation around \( y^* = (y_1^*, y_2^*, \ldots, y_{d-1}^*) \) of the price \( p \) of a basket option with payoff \( h(S_T) \), defined as \( \hat{p}_n := e^{-rT} E_Q \hat{C}_n(\tilde{Y}_T) \), under model (5), is given by:

\[
(8) \quad \hat{p}_n = w_1 \sum_{l=0}^{n} \sum_{R_l} \frac{D^L C(y^*)}{l_1! l_2! \ldots l_{d-1}!} E_Q \left[ e^{-\left(r - \frac{1}{2} \sigma^2 \gamma_T^{(1)}/\gamma_T \right)T + \mu \gamma_T^{(1)}/\gamma_T \sum_{k=1}^{d-1} (Y_T^{(k+1)} - y^*_k)^{l_k}} \right] 
\]
where for \( y \in \mathbb{R}^{d-1} 
):

\[
C(y) := C_{BS}(K(y), \sigma_{Y^{(1)}/\tilde{Y}_T=y}, S_0^{(1)})
\]

is the Black-Scholes price of a call option with strike price \( K(y) \), maturity at \( T > 0 \), volatility \( \sigma_{Y^{(1)}/\tilde{Y}_T=y} \), spot price \( S_0^{(1)} \) and strike price:

\[
K(y) = \frac{1}{w_1} e^{(r-\frac{1}{2}\sigma^2_{Y^{(1)}/\tilde{Y}_T=y})T - \mu_{Y^{(1)}/\tilde{Y}_T=y} T}\left(K - \sum_{j=2}^{d} w_j s_0^{(j)} e^{y^{(j)}}\right)
\]

with:

\[
\mu_{Y^{(1)}/\tilde{Y}_T} = (r - \frac{1}{2}\sigma^2_{Y^{(1)}/\tilde{Y}_T=y})T + \Sigma_{1Y} \Sigma_{Y}^{-1} (\tilde{Y} - r + \frac{1}{2}\text{diag}(\Sigma_{Y}))T
\]

\[
\sigma_{Y^{(1)}/\tilde{Y}_T} = \sigma_1^2 - \Sigma_{1Y} \Sigma_{Y}^{-1} \Sigma_{Y}'
\]

\[
\Sigma_{1Y} = (\sigma_{12}, \sigma_{13}, \ldots, \sigma_{1,d-1})'
\]

and \( \Sigma_{Y} \) is the covariance matrix of the vector \( \tilde{Y}_T \).

**Proof.** From equation (5) a straightforward application of Ito formula leads to:

\[
Y_T = (r1_d - \frac{1}{2}\text{diag}(\Sigma))T + \Sigma_{1Y} \sqrt{T} Z_d
\]

in law, where \( Z_d \) is a random variable with a multivariate normal distribution in \( \mathbb{R}^d \) with zero mean and covariance matrix \( I_d \). Hence \( Y_T \) has also a multivariate normal distribution. Also conditionally on \( \tilde{Y}_T \), the random variable \( Y_T^{(1)} \) has a univariate normal distribution. Thus, we can write:

\[
Y_T^{(1)} = \mu_{Y_T^{(1)}/\tilde{Y}_T} + \sigma_{Y_T^{(1)}/\tilde{Y}_T} \sqrt{T} Z^{(1)}
\]

in law, where \( Z^{(1)} \) is independent of \( Y_T \) and it has, conditionally on \( \tilde{Y}_T \), a standard univariate normal distribution. Moreover it is well known, see for example Tong (1989), that \( \mu_{Y_T^{(1)}/\tilde{Y}_T} \) and \( \sigma_{Y_T^{(1)}/\tilde{Y}_T} \) are given by equations (11) and (12) respectively.

Next, from equation (6) we have:

\[
p = e^{-rT} E_Q \left( E_Q \left( h(S_T) | \mathcal{F}^{\tilde{Y}_T} \right) \right)
\]

\[
= w_1 e^{-rT} E_Q \left( E_Q \left( \left(S_0^{(1)} e^{Y_T^{(1)}} - \left(\frac{K}{w_1} - \sum_{j=2}^{d} w_j s_0^{(j)} e^{y^{(j)}}\right) \right) \right) \right) + |\mathcal{F}^{\tilde{Y}_T} |
\]

\[
= w_1 e^{-rT} E_Q \left( E_Q \left( \left(S_0^{(1)} e^{Y_T^{(1)}} - K'(\tilde{Y}_T) \right) \right) \right) + |\mathcal{F}^{\tilde{Y}_T} |
\]

(15)
where \( K'(y) = \frac{K}{w^1} - \sum_{j=2}^{d} \frac{w^j}{w^1} S_0^{(j)} e^{y^{(j)}} \).

Moreover, substituting equation (14) into (15) we have:

\[
p = w_1 e^{-rT} E_Q \left[ E_Q \left( \left( S_0^{(1)} e^{\mu_T^{(1)} / \sqrt{V_T}} + \sigma_T^{(1)} / \sqrt{V_T} \sqrt{T} Z^{(1)} \right) - K'(\tilde{Y}_T) \right) + \mathcal{F} \tilde{Y}_T \right]
\]

After replacing equation (16) into the expression for \( p \) above we get immediately equation (8) in Proposition 1.

**Remark 2.** Notice that the approximation \( \hat{p}_k \) depends only on the derivatives of the function \( C(y) \) with respect \( y \), which in turn is computed as the Black-Scholes price composed with the function \( K(y) \) and the mixed exponential-power moments of a Gaussian multivariate distribution.

**Remark 3.** Sensitivities to the parameters can be computed by a similar approximation, as Greeks for a Black-Scholes option model are known. For example the delta with respect to the \( j \)-th asset can be approximated by:

\[
\hat{\Delta}_n^{(j)} = w_1 \sum_{l=0}^{n} \sum_{R_l} \frac{D L}{\partial C(y^*)} \frac{d}{d(y^*)} \left( S_0^{(1)} e^{\mu_T^{(1)} / \sqrt{V_T}} T + \sigma_T^{(1)} / \sqrt{V_T} \sqrt{T} Z^{(1)} - K'(\tilde{Y}_T) \right) + \mathcal{F} \tilde{Y}_T
\]

3. **Pricing spreads options by Taylor approximations**

In order to illustrate the method studied in the previous section we consider the case of a bidimensional spread option under model (5) with covariance:

\[
\Sigma = \begin{pmatrix}
\sigma_1^2 & 0 \\
0 & \sigma_2^2
\end{pmatrix}
\]
We find the n-th Taylor approximation in this specific situation. Denoting by $d < B_t^{(1)} < B_t^{(2)} > \rho t$ we have that:

$$Y_T = (Y_T^{(1)}, Y_T^{(2)}) \sim N \left((r_1 - \frac{1}{2} \text{diag}(\Sigma))T, T \Sigma_{\rho} \right)$$

where:

$$\Sigma_{\rho} = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$$

From equation (13) the conditional distribution of $Y_T^{(1)}$ given $Y_T^{(2)}$ is:

$$Y_T^{(1)} \mid Y_T^{(2)} \sim N \left(r(1 - \frac{\sigma_1}{\sigma_2}) + \frac{1}{2} \sigma_1 \sigma_2 \rho T + \frac{\sigma_1}{\sigma_2} \rho Y_T^{(2)} - \frac{1}{2} \sigma_2^2 T, (1 - \rho^2) \sigma_1^2 T \right)$$

Thus we can write:

$$Y_T^{(1)} = \mu(Y_T^{(2)}) + \sigma \sqrt{T} Z$$

in law, where $Z \sim N(0, 1)$ independent of $Y_T$, with

$$\mu(Y_T^{(2)}) := \mu_Y^{(1)} \mid Y_T = r(1 - \frac{\sigma_1}{\sigma_2}) + \frac{1}{2} \sigma_1 \sigma_2 \rho T + \frac{\sigma_1}{\sigma_2} \rho Y_T^{(2)}$$

and

$$\sigma := \sigma_Y^{(1)} \mid Y_T = \sqrt{(1 - \rho^2)} \sigma_1$$

From Proposition 1 the n-th approximation simplifies to:

$$\hat{p}_n = \sum_{l=0}^{n} \frac{D_l C(y^{*})}{l!} E_Q \left[ e^{-\left(\frac{1}{2} \sigma_2^2 T + \mu(Y_T^{(2)}) (Y_T^{(2)} - y^{*})^T \right)} \right]$$

Moreover:

$$E_Q \left[ e^{-\left(\frac{1}{2} \sigma_2^2 T + \mu(Y_T^{(2)}) (Y_T^{(2)} - y^{*})^T \right)} \right] = e^{A} E_Q \left[ e^{\frac{\sigma_1}{\sigma_2} \rho Y_T^{(2)} (Y_T^{(2)} - y^{*})^T} \right]$$

where:

$$A = \left( -\left( r - \frac{1}{2} \sigma_2^2 \right) + r(1 - \frac{\sigma_1}{\sigma_2}) \rho + \frac{1}{2} \sigma_1 (\sigma_2 \rho - \sigma_1) \right) T$$

$$= -\left( \frac{1}{2} \rho^2 \sigma_1^2 + r \frac{\sigma_1}{\sigma_2} \rho - \frac{1}{2} \sigma_1 \sigma_2 \rho \right) T$$
Now, from equation (17) we have that \( Y_T^{(2)} \sim N((r - \frac{1}{2}\sigma_Z^2)T, T\sigma_Z^2) \), then the exponential-power moments can be calculated as follows:

\[
E_Q \left[ e^{\frac{a_1}{2}\rho Y_T^{(2)}} (Y_T^{(2)} - y^*)^l \right] = \sum_{m=0}^{l} \binom{l}{m} \left( \frac{r}{2} - \frac{1}{2}\sigma_Z^2 \right)^{l-m} E_Q \left[ e^{\frac{a_1}{2}\rho Y_T^{(2)}} (Y_T^{(2)} - Q(Y_T^{(2)}))^{m} \right] = \sum_{m=0}^{l} \binom{l}{m} \left( \frac{r}{2} - \frac{1}{2}\sigma_Z^2 \right)^{l-m} T^m \sigma_Z^m e^{\frac{a_1}{2}\rho (r - \frac{1}{2}\sigma_Z^2)T} E_Q \left[ e^{\sqrt{T}\sigma_1 \rho Z} Z^m \right] = e^{\frac{a_1}{2}\rho (r - \frac{1}{2}\sigma_Z^2)T} \sum_{m=0}^{l} \binom{l}{m} \left( \sqrt{T}\sigma_Z \right)^{m} B(y^*)^{l-m} E_Q \left[ e^{\sqrt{T}\sigma_1 \rho Z} Z^m \right]
\]

where:

\( B(y^*) = (r - \frac{1}{2}\sigma_Z^2)T - y^* \)

Next integrate by parts:

\[
E_Q \left[ e^{\sqrt{T}\sigma_1 \rho Z} Z^m \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x^2 - 2\sigma_1 \rho \sqrt{T}x)} x^m dx = e^{\frac{a_1^2 e^{2T}}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}((x - \sigma_1 \rho \sqrt{T})^2)} x^m dx = e^{\frac{a_1^2 e^{2T}}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2} (y + \sigma_1 \rho \sqrt{T})^m dy = e^{\frac{a_1^2 e^{2T}}{2}} \sum_{\nu=0}^{m} \binom{m}{\nu} \left( \sigma_1 \rho \sqrt{T} \right)^{m-\nu} E(Z^\nu) = e^{\frac{a_1^2 e^{2T}}{2}} \sum_{\nu=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\nu} \left( \sigma_1 \rho \sqrt{T} \right)^{m-2\nu} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2} y^{2\nu} dy = e^{\frac{a_1^2 e^{2T}}{2}} \sum_{\nu=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\nu} \left( \sigma_1 \rho \sqrt{T} \right)^{m-2\nu} (2\nu - 1)!!
\]

where \( n!! \) is the double factorial defined as the product of all odd numbers between 1 and \( n \) including both. When the set is empty, by convention, the product is equal to one.

Similarly for \( y^* = E_Q(Y_T^{(2)}) \) we have:

\[
E_Q \left[ e^{\frac{a_1}{2}\rho Y_T^{(2)}} (Y_T^{(2)} - y^*)^l \right] = T^{\frac{1}{2}} \sigma_Z^l e^{\frac{a_1}{2}\rho (r - \frac{1}{2}\sigma_Z^2)T} e^{\frac{a_1^2 e^{2T}}{2}} \sum_{\nu=0}^{l} \binom{l}{\nu} \left( \sigma_1 \rho \sqrt{T} \right)^{l-\nu} E(Z^\nu) = T^{\frac{1}{2}} \sigma_Z^l e^{\frac{a_1^2 e^{2T}}{2}} \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \binom{\lfloor \frac{l}{2} \rfloor}{2\nu} \left( \sigma_1 \rho \sqrt{T} \right)^{l-2\nu} (2\nu - 1)!!
\]
After gathering all pieces and substituting in equation (20), we have the following result:

**Proposition 4.** The n-th Taylor approximation of a spread contract with maturity at $T$ and strike price $K$, under the model (5) is given by:

$$
\hat{p}_n = \sum_{l=0}^{n} \sum_{m=0}^{l} \frac{D^l C(y^*)}{l!} \left( \frac{l}{m} \right) \left( \sqrt{T} \sigma_2 \right)^m B(y^*)^{l-m} E(m)
$$

with:

$$
E(m) = \sum_{\nu=0}^{m} \binom{m}{\nu} (\sigma_1 \rho \sqrt{T})^{m-\nu} E_\mathcal{Q}(Z^\nu)
$$

for $m = 1, 2, \ldots, k$ and $E(0) = 1$, where $E_\mathcal{Q}Z^\nu = (\nu - 1)!!$ if $\nu$ is even or zero if it is odd, and

$$
K(y) = e^{(r - \frac{1}{2} \sigma^2)T - \mu(y)} (K + S_0^{(2)} e^y) = e^{-A} \left( Ke^{-\frac{\sigma_1}{\sigma_2} \rho y} + S_0^{(2)} e^{(1 - \frac{\sigma_1}{\sigma_2}) \rho y} \right)
$$

with $\mu(y)$ given by equation (19).

Next, we compute the derivatives of the function $C(y)$ with respect to $y$. From the Black-Scholes pricing formula:

$$
C(y) := C_{BS}(K(y), \sigma, S_0^{(1)}) = S_0^{(1)} N(d_1(K(y))) - K(y) e^{-rT} N(d_2(K(y))
$$

where:

$$
d_1(K(y)) = \frac{\log \left( \frac{S_0^{(1)}}{K(y)} \right) + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}
$$

$$
d_2(K(y)) = d_1(K(y)) - \sigma \sqrt{T}
$$

and $N(\cdot)$ is the cumulated distribution function of a standard normal distribution.

The first two derivatives are computed by elementary methods. First notice that:

$$
D^1 K(y) = e^{-A} \left( -\frac{\sigma_1}{\sigma_2} \rho Ke^{-\frac{\sigma_1}{\sigma_2} \rho y} + S_0^{(2)} (1 - \frac{\sigma_1}{\sigma_2} \rho) e^{(1 - \frac{\sigma_1}{\sigma_2}) \rho y} \right)
$$

$$
D^2 K(y) = e^{-A} \left( \frac{\sigma_1}{\sigma_2} \rho^2 Ke^{-\frac{\sigma_1}{\sigma_2} \rho y} + S_0^{(2)} (1 - \frac{\sigma_1}{\sigma_2} \rho)^2 e^{(1 - \frac{\sigma_1}{\sigma_2}) \rho y} \right)
$$

Also:

$$
D^1 C_{BS}(y) = \frac{S_0^{(1)} f_Z(d_1(K(y))) D^1 d_1(K(y))}{K(y) \sigma \sqrt{T}} e^{-rT} D^1 K(y) N(d_2(K(y))
$$

$$
- e^{-rT} K(y) f_Z(d_2(K(y))) D^1 d_1(K(y))
$$

$$
= - \frac{D^1 K(y)}{K(y) \sigma \sqrt{T}} A_2(y)
$$

where $f_Z$ is the density function of a standard normal random variable and

$$
A_2(y) = S_0^{(1)} f_Z(d_1(K(y))) + \sigma \sqrt{T} e^{-rT} K(y) N(d_2(K(y))) - e^{-rT} K(y) f_Z(d_2(K(y)))
$$
Similarly the second derivative is obtained as:

$$D^2 C(y) = -\frac{1}{\sigma \sqrt{T}} \left[ A_2(y) \frac{K(y)D^2 K(y) - (D^1 K(y))^2}{K^2(y)} + D^1 A_2(y) \frac{D^1 K(y)}{K(y)} \right]$$

with:

$$D^1 A_2(y) = -S_0^{(1)} f_Z(d_1(K(y)))d_1(K(y))D^1 d_1(K(y)) + \sqrt{T}e^{-rT} D^1 K(y)N(d_2(K(y)))$$

$$+ \sigma \sqrt{T} e^{-rT} K(y) f_Z(d_2(K(y))) D^1 d_1(K(y))$$

$$+ e^{-rT} f_Z(d_2(K(y))) D^1 d_2(K(y)) d_2(K(y)) K(y)$$

$$= \frac{D^1 K(y)}{K(y) \sigma \sqrt{T}} \left[ S_0^{(1)} f_Z(d_1(K(y)))d_1(K(y)) + \sigma^2 T e^{-rT} K(y) N(d_2(K(y))) - 2 \sigma \sqrt{T} e^{-rT} K(y) f_Z(d_2(K(y))) - e^{-rT} K(y) f_Z(d_2(K(y))) d_2(K(y)) \right]$$

In particular when we develop around $y_{mean} = E_Q(Y_T^{(2)}) = (r - \frac{1}{2} \sigma^2) T$ we have the first and second approximations given respectively by:

$$\hat{p}_1 = C(y_{mean}) + \sigma_1 \sigma_2 \rho T D^1 C(y_{mean})$$

$$\hat{p}_2 = \hat{p}_1 + \frac{1}{2} \left[ T \sigma_2^2 (1 + \sigma_1^2 \rho^2) \right] D^2 C(y_{mean})$$

More generally expanding around $y^*$ we have the first two approximations denoted by $\hat{p}_1(y^*)$ and $\hat{p}_2(y^*)$ respectively and given by:

$$\hat{p}_1(y^*) = C(y^*) + D^1 C(y^*) (B(y^*) + \sqrt{T} \sigma_2 E(1))$$

$$= C(y^*) + D^1 C(y^*) (B(y^*) + T \sigma_1 \sigma_2 \rho)$$

$$\hat{p}_2(y^*) = \hat{p}_1(y^*) + \frac{1}{2} D^2 C(y^*) [B^2(y^*) + 2T \sigma_1 \sigma_2 \rho B(y^*) + T \sigma_2^2 (1 + T \sigma_1^2 \rho^2)]$$

4. Pricing Spreads: numerical results

We consider spread options in the following benchmark numerical set:

$S_0^{(1)} = 100, S_0^{(2)} = 96, \sigma_1 = 0.3, \sigma_2 = 0.1, \rho = -0.3, r = 0.03, K = 1$ and $T = 1$.

In Figure [1] the graph of the conditional price $C(y)$ given by equation (7) is shown (blue line), together with the first and second order Taylor approximation around the mean, for the benchmark parameter set.

Notice that the first approximation underestimates the price. Not surprisingly the second approximation estimates the price fairly well for values close to the point $y_{mean}$ while is less accurate for values far from the mean. Although it seems a drawback of the method it does not constitutes a serious problem as values far from the mean are unfrequent, thus the error in calculating the outer expected value by the Taylor approximation is small. In Figure [2] a histogram for simulated returns on asset 1 (blue rectangles) and asset 2 (red rectangles) is shown. Notice that only a few values of the returns lie outside the interval $[-1, 1]$. 

Figure 1. The function $C_{BS}(y)$ is shown in blue, together with the first and second order approximations around the mean for the benchmark parameters.

Next we compare Taylor approximations with Monte Carlo simulations. In Table 1 (column 2) prices from Monte Carlo are shown for the benchmark parameters, except the correlation parameter that takes values $\rho = -0.5, -0.3, 0.3, 0.5$. The number of simulations is $n = 10^7$, where a stability of order $10^{-3}$ is attained. Partial Monte Carlo prices (shown in column 3) are obtained by sampling directly the one dimensional conditional price $C(Y_T^{(2)})$ and taking the corresponding average of the payoff. It leads to a more efficient simulation algorithm as only one Brownian motion needs to be simulated, as oppose to two correlated Brownian in the standard Monte Carlo approach. It is done though at the expense of an extra evaluation of the Black-Scholes formula in every step. Taylor prices of first and second order are shown in columns 4 and 5 of Table 1. The expansions take place around $y^* = 0$. While in some cases the first order approximation reveals significant different with Monte Carlo, second order approximation shows an improved agreement with a relative error in the order of $10^{-4}$ for the parameter set considered.
For extreme values of the correlation coefficient $\rho$, e.g. larger than an absolute value of 0.7, the Taylor expansions around $y^* = 0$ do not work well. Nevertheless it is interesting to notice that the approximations are rather sensible to the point where the expansion is taken. Moreover, by slightly changing the latter the accuracy of the method can be considerably improved. In Table 2 spread prices for the benchmark parameters and $\rho = -0.7$ for different expansion points are shown.
We test the Taylor expansion method for out-of-the-money contracts and compare with the price obtained via Monte Carlo with \( n = 10^7 \) repetitions. The results are shown in Table 3. The benchmark parameters are the same, except for the spot and strike prices that are changed accordingly. Again a second order Taylor expansion seems to capture the Monte Carlo prices.

### 5. Conclusions

We present an efficient method to price basket options under a multi-dimensional Black-Scholes model, based on a Taylor expansion of the conditional one dimensional price resulting from fixing one of the underlying assets. The formula is given in terms of exponential-power moments of a multivariate Gaussian law and the evaluation of certain derivatives in the Black-Scholes price.

We implement it numerically in the case of spread contracts. Within the benchmark parametric set this approach is in closed agreement with the price obtained via Monte Carlo, even for deep out-of-the-money contracts, at considerable lesser computational effort. A second order development seems to be sufficient to achieve a relative error around \( 10^{-4} \).
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