Relativistic modelling of stable anisotropic super-dense star

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Abstract

In the present article we have obtained new set of exact solutions of Einstein field equations for anisotropic fluid spheres by using the Herrera et al. [1] algorithm. The anisotropic fluid solutions so obtained join continuously to Schwarzschild exterior solution across the pressure free boundary. It is observed that most of the new anisotropic solutions are well behaved and utilized to construct the super-dense star models such as neutron star and pulsars.

Keywords: anisotropic fluids; anisotropic factor; Einstein’s equations; Schwarzschild solution; neutron star; pulsars.

1 Introduction

The first ever exact solution of Einstein’s field equation for a compact object in static equilibrium was obtained by Schwarzschild in 1916. The static isotropic
and anisotropic exact solutions describing stellar-type configurations have continuously attracted the interest of physicists Herrera et al. \[1\] \[11\]. Tolman \[2\] has proposed an easy way to solve Einstein’s field equations by introducing an additional equation necessary to give a determinate problem in the form of some ad hoc relation between the components of metric tensor. According to this methodology, Tolman \[2\] has obtained eight solutions of the field equations, and his important approach still continues in obtaining the exact interior solutions of the gravitational field equations for fluid spheres. Buchdahl \[3\] proposed a famous bound on the mass radius ratio of relativistic fluid spheres is \(2GM/c^2r \leq 8/9\), which is an important contribution in order to study the stability of the fluid spheres. Also Ivanov \[4\] has given the upper bound of the red shift for realistic anisotropic star models which cannot be exceed the values 3.842 provided the tangential pressure satisfies a strong energy condition \((\rho \geq p_r + 2p_t)\) and when the tangential pressure satisfies the dominated energy condition \((\rho \geq p_t)\). Buchdahl \[3\] has also obtained a non-singular exact solution by choosing a particular choice of the mean density inside the star.

The theoretical investigations of realistic fluid models indicate that stellar matter may be anisotropic at least in certain density ranges \((\rho > 10^{15} \text{ gm/cm}^3)\) (Ruderman \[5\] and Canuto \[6\]) and radial pressure may not be equal to the tangential pressure of stellar structure. The existence of a solid core due to presence of the anisotropy in the pressure was thought of by type-3A superfluid (Kippenhahm and Weigert \[7\]), different form of phase transitions (Sokolov \[8\]) or by others physical phenomena. On the scale of galaxies, Binney and Tremaine \[9\] have considered anisotropies in spherical galaxies, from a purely Newtonian point of view. The mixture of two gases (e.g. ionized hydrogen and electrons or monatomic hydrogen) can be described formally as an anisotropic fluid (Letelier \[10\] and Bayin \[11\]). The importance of equations of state for relativistic anisotropic fluid spheres have been investigated by generalizing the equation of hydrostatic equilibrium to include the effects of anisotropy (Bowers and Liang \[12\]). Their study shows that anisotropy may have non-negligible effects on parameters such as maximum equilibrium mass and surface red-shift.

The relativistic anisotropic neutron star models at high densities by means of several simple assumptions showed that there is no limiting mass of neutron stars for arbitrary large anisotropy which is studied by Heintzmann and Hillebrandt \[13\]. However maximum mass of a neutron star still lies beyond 3-4 \(M_\odot\). Also the solutions for an anisotropic fluid sphere with uniform density and variable density are studied by Maharaj and Maartens \[14\] and Gokhroo and Mehra \[14\], respectively. Most the astronomical objects have variable density. Therefore, interior solutions of anisotropic fluid spheres with variable density are more realistic physically.

Many workers have obtained different exact solutions for isotropic and anisotropic fluid spheres in different contexts (Delgaty and Lake \[16\], Dev and Gleiser \[17\], Komathiraj and Maharaj \[18\], Thirukkanesh and Ragel \[19\], Sunzu et al. \[20\], Harko and Mak \[21\], Mak and Harko \[22\], Chaisi and Maharaj \[23\], Maurya and Gupta \[24\], \[25\], \[26\], Feroze and Siddiqui \[27\], Pant et al. \[28\], \[29\], \[30\], Bhara et al. \[31\], Monowar et al. \[32\], Kalam et al. \[33\], Consenza et al. \[34\], Krori
The present paper consists of nine sections, Section 1 contains introduction; Section 2 contains metric, its components and the field equations. Section 3 embodies the solutions of anisotropic fluid spheres in different contexts. Section 4 contains the expressions for density and pressure for each fluid sphere. Section 5 consists of the various physical conditions to be satisfied by the anisotropic fluid spheres. The analytical behavior of the solutions under the physical conditions (mentioned in Section 5) are mentioned in the section 6. Section 7 describes the evaluation of arbitrary constants involved in the fluid solutions by means of the smooth joining of Schwarzschild metric at the pressure free interface \( r = a \). The stability of models is proposed in the section 8 and finally section 9 includes the physical analysis of the solutions so obtained along with the concluding remarks.

## 2 Metric, components and Field equations

The line element of static spherical symmetric space time in the curvature coordinates \( x^i = (t, r, \chi, \xi) \) can be furnished as below,

\[
ds^2 = B^2(r)dt^2 - \psi^{-1}(r)dr^2 - r^2(d\chi^2 + \sin^2 \chi d\xi^2)
\]  
(2.1)

Einstein’s field equations given as

\[-\kappa T_{ij} = R_{ij} - \frac{1}{2} R g_{ij}\]

(2.2)

where, \( \kappa = \frac{8\pi G}{c^4} \).

The components of the energy momentum tensor for spherically symmetric anisotropic fluid distribution is postulated in the form:

\[T^j_i = (c^2 \rho + p_t) v^i v^j - p_r \delta^j_i + (p_r - p_t) \chi^i \chi^j,\]

(2.3)

where \( v^i \) is four-velocity \( B v^i = \delta^i_0 \), \( \chi^i \) is the unit space like vector in the direction of radial vector, \( \chi^i = \sqrt{B} \delta^i_1 \), \( \rho \) is the energy density, \( p_r \) is the pressure in direction of \( \chi^i \) (normal pressure) and \( p_t \) is the pressure orthogonal to \( \chi_i \) (transversal or tangential pressure). Suppose radial pressure is not equal to the tangential pressure i.e. \( p_r \neq p_t \), otherwise if radial pressure is equal to transverse pressure i.e. \( p_r = p_t \), it corresponds to isotropic or perfect fluid distribution. Let the measure of anisotropy \( \Delta = \kappa (p_t - p_r) \), which is called the anisotropy factor (Herrera and Ponce de Leon [45]). The term \( 2(p_t - p_r)/r \) appears in the conservation equations \( T^i_{j;3} = 0 \) (where, semi colon denotes the covariant derivative) which is representing a force due to anisotropic nature of the fluid. When \( p_t > p_r \), then the direction of force to be outward direction and inward when \( p_t < p_r \). However, if \( p_t > p_r \), then the force allows the construction of more compact object when using anisotropic fluid than when using isotropic fluid (Gokhroo and Mehra [15]).
In view of metric (2.1), the Einstein field equations (2.2) give
\[
\left( \frac{8\pi G}{c^2} \rho \right) = 1 - \psi - \frac{\psi'}{r^2}, \quad \left( \frac{8\pi G}{c^4} p_r \right) = \frac{2B'\psi}{Br} + \psi - 1 \quad (2.4)
\]
\[
\psi' \left( \frac{B'}{B} + \frac{1}{r} \right) + 2\psi \left( \frac{B''}{B} - \frac{B'}{rB} - \frac{1}{r^2} \right) = 2 \left( \Delta - \frac{1}{r^2} \right), \quad (2.5)
\]
where “dash” denote the derivative with respect to \( r \).

### 3 Classes of solutions

The field Eq. (2.5) has two dependent variables \( B(r) \) and \( \psi(r) \), therefore Eq. (2.5) can admit infinity many solutions for different choices of \( B(r) \) and \( \psi(r) \) but all these solutions may or may not satisfy the physical conditions for the fluid spheres. For a given \( B(r) \), the Eq. (2.5) reduces to first order ordinary differential equation in \( \psi(r) \). For its physically valid solution we will have to choose the metric potential \( B(r) \) such that \( B(0) \) is non zero positive finite. This is a sufficient condition for a static fluid sphere to be regular at the centre.

Let us take \( B(r) \) of the form
\[
B(r) = D(1 - c_0r^2)^n, \quad \text{where (}n \neq 0\text{),} \quad D \quad \text{is positive arbitrary constant}. \quad (3.1)
\]

Maurya and Gupta [22, 23] have already obtained all possible anisotropic solutions of Einstein field equations for \( n \) is positive integer \( \geq 1 \) with \( c_0 < 0 \) and \(-1 < n < 0, c_0 > 0 \) with different anisotropic factor \( \Delta \).

But in present problem, we consider \( B(r) = D(1 - c_0r^2)^n \) with \( n = -1, -2 \) and \(-3 \) and anisotropy factor
\[
\Delta = \frac{\Delta_0c_0(c_0r^2)^{-n}}{[1 - (n + 1)c_0r^2]^{1+n}}; \text{ where,} \quad c_0 \text{ and } \Delta_0 \text{ are positive constants}. \quad (3.2)
\]

Herrera et al [1] have proposed an algorithm for all possible spherically symmetric anisotropic solutions of Einstein field equations.

By using the Herrera et al [1] algorithm, the equations (2.5) reduces in the form as:
\[
\psi' + \left( 2y' \frac{y}{y} + 2y - \frac{6}{r} + \frac{4}{r^2y} \right) \psi = \frac{2}{y} \left( \Delta - \frac{1}{r^2} \right) \quad (3.3a)
\]
and
\[
y(r) = \frac{B'(r)}{2B(r)} + \frac{1}{r}. \quad (3.3b)
\]

On integrating (3.3a), we can obtain \( \psi \) as:
\[
\psi = \left. -2 \int \frac{y(r)(1 + \Delta(r)r^2)e^{\int[1/(4r^2y(r))+2y(r)]dr}}{r^8}dr + A \right|_y^y = \frac{\psi^2(r)e^{\int[1/(4r^2y(r))+2y(r)]dr}}{y^2(r)e^{\int[1/(4r^2y(r))+2y(r)]dr}} \quad (3.3c)
\]
and
\[ y(r) = \frac{[1 - (n + 1)c_0r^2]}{r(1 - c_0r^2)}. \] (3.3d)

which further supply
\[ \psi = \phi f^3 e^{2\phi} \left[ \Delta_0 \int \frac{e^{-2\phi}}{f^2} d\phi - \int \frac{e^{-2\phi}}{\phi f^2} d\phi \right] + A \phi f^3 e^{2\phi} \text{ for } n = -1 \] (3.4a)

and
\[ \psi = \frac{\phi f^{2-n}}{g^{2/(n+1)}} \left[ \Delta_0 \int \frac{f^{n-1}}{\phi^{n+1} g^{n(n+3)/(n+1)}} d\phi - \int \frac{f^{n-1}}{\phi^2 g^{n(n-1)/(n+1)}} d\phi \right] + A \frac{\phi f^{2-n}}{g^{2/(n+1)}}, \]
for \( n \neq -1. \) (3.4b)

where, \( \phi = c_0r^2, \ f = (1 - \phi) \) and \( g = [1 - (n + 1)\phi]. \)

The Eq. (3.4a) and Eq. (3.4b) give the following solutions:
\[ \psi = A \phi f^3 e^{2\phi} + (1 - 2\phi)f^2 - (6 + \Delta_0) f^3 \phi e^{2\phi - 2} Ei(2 - 2\phi) + \Delta_0 f^2; \text{ for } n = -1, \] (3.5a)

where \( Ei(2 - 2\phi) = \log(2 - 2\phi) + \sum_{N=1}^{\infty} \frac{(2-2\phi)^N}{N!} \)

\[ \psi = \left[ A \phi(1 + \phi)^2 f^4 + \left[ f^2 \left( 15\phi^4 - 25\phi^2 + 8 \right) + \frac{15}{16} \phi(1 + \phi)^2 f^4 \log \left( \frac{f}{1 + \phi} \right) \right] \right] + \frac{\Delta_0}{16} \phi f^2 \left[ 2(\phi^2 - \phi + 2) + f^2(1 + \phi)^2 \log \left( \frac{1 - \phi}{1 + \phi} \right) \right] ; \text{ for } n = -2; \] (3.5b)

\[ \psi = \left[ A \phi f^3(1 + 2\phi) - \frac{1}{243} f^5 \left[ 3h f^3 - 320\phi \log \left( \frac{1 + 2\phi}{f} \right)(1 + 2\phi) \right] + \Delta_0 \phi f^3(1 + 2\phi)(1 - 3\phi + 3\phi^2) \right]; \text{ for } n = -3; \] (3.5c)

where \( h = (-81 + 130\phi + 36\phi^2 - 720\phi^3 + 320\phi^4). \)
4 Expressions for energy density, pressures for different values of \( n \)

(a) For \( n = -1 \)

\[
\left( \frac{8\pi G}{c^2}\rho \right) = c_0 [Af^2 e^{2\phi}(4\phi^2 + 5\phi - 1) + (6 + \Delta_0)f^2(3 - 5\phi - 4\phi^2)e^{2\phi - 2}Ei(2 - 2\phi) \\
+ (6 - 11\phi^2 + 2\phi^3) + \Delta_0[(-3 + 10\phi - 7\phi^2) + 2\phi e^{2\phi - 2}f^3 Ei(2 - 2\phi)],
\]

(b) For \( n = -2 \)

\[
\left( \frac{8\pi G}{c^4}\rho \right) = c_0 [Af^2(1 + \phi)e^{2\phi} + (2 - 10\phi + 7\phi^2 - 2\phi^3) - 6f^2(1 + 5\phi)e^{2\phi - 2}Ei(2 - 2\phi) \\
+ \Delta_0(1 - \phi^2)[1 - e^{2\phi - 2}f Ei(2 - 2\phi)],
\]

where, 

\[
Ei(2 - 2\phi) = \frac{-2}{(2 - 2\phi)^2} - 2 \sum_{N=1}^{\infty} \frac{(2 - 2\phi)^{N-1}}{N!}.
\]
\[
\left( \frac{8\pi G}{c^4} p_t \right) = c_0 \left[ A(1 + 3\phi)f^3(1 + \phi)^2 + \frac{(16 - 49\phi^2 - 50\phi^3 + 30\phi^4 - 45\phi^5)}{8} \right] \\
+ \frac{(15 + \Delta_0)}{16} f^3(1 + 3\phi)(1 + \phi)^2 \log \left( \frac{1 - \phi}{1 + \phi} \right) \\
+ \frac{\Delta_0}{16} (4 + 6\phi + 2\phi^2 + 26\phi^3 - 6\phi^4) .
\]

\section*{(4.3b)}

\[\left( \frac{8\pi G}{c^2} \rho \right) = c_0 \left[ - A(3 - 15\phi + 90\phi^3 - 165\phi^4 + 117\phi^5 - 34\phi^6) \\
+ \frac{1}{81}(21320 - 26565\phi - 55034\phi^2 + 156004\phi^3 - 11040\phi^4 + 29120\phi^5) \\
+ \frac{320}{243} f^4(-1 + 5\phi + 26\phi^2) \log \left( \frac{1 + 2\phi}{1 - \phi} \right) \\
- \Delta_0 f^2(3 - 14\phi - 10\phi^2 + 93\phi^3 - 90\phi^4) \right] ,
\]

\section*{(4.1c)}

\[\left( \frac{8\pi G}{c^4} p_r \right) = c_0 \left[ A f^4(1 + 7\phi + 10\phi^2) - (356 + 411\phi + 146\phi^2 - 3820\phi^3 - 1680\phi^4 + 1600\phi^5) \\
+ \frac{320}{243} f^4(1 + 5\phi)(1 + 5\phi) \log \left( \frac{1 + 2\phi}{1 - \phi} \right) \\
+ \Delta_0 f^2(1 + 4\phi - 8\phi^2 - 9\phi^3 + 30\phi^4) \right] ,
\]

\section*{(4.2c)}

\[\left( \frac{8\pi G}{c^4} p_t \right) = c_0 \left[ A f^4(1 + 7\phi + 10\phi^2) + (356 - 411\phi - 146\phi^2 + 3820\phi^3 + 1680\phi^4 - 1600\phi^5) \\
+ \frac{320}{243} f^4(1 + 2\phi)(1 + 5\phi) \log \left( \frac{1 + 2\phi}{1 - \phi} \right) \\
+ \Delta_0 \{ f^2(1 + 4\phi - 8\phi^2 - 9\phi^3 + 30\phi^4) + \phi^3(1 + 2\phi)^2 \} \right] .
\]

\section*{(4.3c)}

\section*{5 Reality and Physical (well behaved) conditions for anisotropic solutions}

The physically meaningful anisotropic solution for the Einstein’s field equations must satisfy some physical conditions (Mak and Harko [20], and Maurya and Gupta [22]):

(i) The solution should be free from physical and geometrical singularities i.e. pressure and energy density should be finite at the centre and metric potential \(B(r)\) and \(\psi(r)\) have non zero positive values.
(ii) The radial pressure $p_r$ must be vanishing but the tangential pressure $p_t$ may not vanish at the boundary $r = r_a$ of the fluid sphere. However, the radial pressure is equal to the tangential pressure at the centre of the fluid sphere.

(iii) The density $\rho$ and radial pressure $p_r$ and tangential pressure $p_t$ should be positive inside the star.

(iv) $(dp_r/dr)_{r=0} = 0$ and $(d^2p_r/dr^2)_{r=0} < 0$ so that the pressure gradient $dp_r/dr$ is negative for $0 \leq r \leq r_a$.

(v) $(dp_t/dr)_{r=0} = 0$ and $(d^2p_t/dr^2)_{r=0} < 0$ so that the pressure gradient $dp_t/dr$ is negative for $0 \leq r \leq r_a$.

(vi) $(d\rho/dr)_{r=0} = 0$ and $(d^2\rho/dr^2)_{r=0} < 0$ so that the density gradient $d\rho/dr$ is negative for $0 \leq r \leq r_a$.

Conditions (iv)-(vi) imply that pressure and density should be maximum at the centre and monotonically decreasing towards the surface.

(vii) Inside fluid ball the speed of sound should be less than that of light i.e.

$$0 \leq \sqrt{\frac{dp_r}{c^2d\rho}} < 1, \quad 0 \leq \sqrt{\frac{dp_t}{c^2d\rho}} < 1$$

In addition the velocity of sound monotonically decreasing away from the centre, the velocity of sound is increasing with the increase of density i.e. $\frac{d}{dr} \left( \frac{dp_r}{\rho} \right) < 0$ or $\left( \frac{d^2p_r}{d\rho} \right) > 0$ and $\frac{d}{dr} \left( \frac{dp_t}{\rho} \right) < 0$ or $\left( \frac{d^2p_t}{d\rho} \right) > 0$ for $0 \leq r \leq r_a$.

In this contexts it is worth mentioning that the equation of state at ultra-high distribution has the property that the speed of sound is decreasing outwards (Canuto [30]).

(viii) A physically reasonable energy-momentum tensor has to obey the energy conditions $\rho \geq p_r + 2p_t$ (strong energy condition) and $\rho + p_r + 2p_t \geq 0$.

(ix) The red shift at center $Z_0$ and at surface $Z_a$ should be positive, finite and both bounded.

(x) The anisotropy factor $\Delta$ should be zero at the center and must be increasing towards the surface.
6 Physical properties of the new solutions

(a) For \( n = -1 \)

The expression for pressures and density at the centre are as:

\[
\frac{8\pi G}{c^4} p_r \bigg|_{r=0} = \left( \frac{8\pi G}{c^4} p_t \right) \bigg|_{r=0} = c_0 \left( A + 2 + \frac{\Delta_0 e^2 - Ei(2)(6 + \Delta_0)}{e^2} \right), \tag{6.1a}
\]

\[
\frac{8\pi G}{c^2} \rho \bigg|_{r=0} = c_0 \left( -A + 6 + \frac{Ei(2)(18 + 3\Delta_0) - 3e^2\Delta_0}{e^2} \right). \tag{6.2a}
\]

The pressure and density should be non zero positive at the centre and consequently \( A \) satisfy the following inequality:

\[
A > \left( -2 + \frac{Ei(2)(6 + \Delta_0) - \Delta_0 e^2}{e^2} \right) \quad \text{and} \quad A < \left( 6 + \frac{Ei(2)(18 + 3\Delta_0) - 3\Delta_0 e^2}{e^2} \right). \tag{6.3a}
\]

The ratio of pressure-density should be positive and less than 1 at the centre i.e. \( \frac{p_r}{\rho c^2} \leq 1 \) which gives the following inequality,

\[
\frac{p_r}{\rho c^2} \bigg|_{r=0} = \frac{p_t}{\rho c^2} \bigg|_{r=0} = \left( \frac{e^2(A + 2) + \Delta_0 e^2 - Ei(2)(6 + \Delta_0)}{e^2(-A + 6) + Ei(2)(18 + 3\Delta_0) - 3\Delta_0 e^2} \right) \leq 1 \tag{6.4a}
\]

By Eq. (6.4a), we get

\[
A \leq 2 + \frac{Ei(2)(6 + \Delta_0) - 2\Delta_0 e^2}{e^2}. \tag{6.5a}
\]

Using the Eqs. (6.3a) and (6.5a), \( A \) satisfies the following inequality:

\[
\left( -2 + \frac{Ei(2)(6 + \Delta_0) - \Delta_0 e^2}{e^2} \right) < A \leq \left( 2 + \frac{Ei(2)(6 + \Delta_0) - 2\Delta_0 e^2}{e^2} \right), \quad \Delta_0 \geq 0. \tag{6.6a}
\]

Differentiating (4.2a) with respect to \( r \), we get an expression for the pressure gradient:

\[
\left( \frac{8\pi G}{c^4} \frac{dp_r}{dr} \right) = 2c_0^2 \left[ A e^{2\phi}(1 - 4\phi + \phi^2 + 2\phi^3) + (-10 + 14\phi - 6\phi^2)
\right.

\left. - 6f^2(1 + 5\phi)e^{2\phi - 2}Ei(2 - 2\phi)
\right.

\left. - 6(5 - 12\phi - 3\phi^2 + 10\phi^3)e^{2\phi - 2}Ei(2 - 2\phi)
\right.

\left. + \Delta_0[-2\phi + e^{2\phi - 2}Ei(2 - 2\phi)(-2 + 5\phi - 3\phi^3)
\right.

\left. - e^{2\phi - 2}f^2(1 + \phi)Ei(2 - 2\phi)] \right]. \tag{6.7a}
\]
Thus it is found that extremum of \( p_r \) occurs at the centre i.e.

\[
p'_r = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^4}(p''_r)_{r=0} = -\frac{2e^2}{e^2} [(10 - A)e^2 + 6Ei(2) + 30Ei(2) + \Delta_0[2Ei(2) + Ei(2)]]
\]

(6.8a)

This shows that the expression of right hand side of equation (6.8a) is negative for all values of \( A \) and \( \Delta_0 \) satisfying condition (6.8a). Then radial pressure \((p_r)\) is maximum at the centre and monotonically decreasing.

Differentiating (4.3a) with respect to \( r \), we get

\[
\left( \frac{8\pi G}{c^4} \frac{dp}{dr} \right) = 2e^2r \left[ Ae^{2\phi}(1 - 4\phi + \phi^2 + 2\phi^3) + (-10 + 14\phi - 6\phi^2) + \Delta_0(1 - 2\phi)
+ e^{2\phi-2}Ei(2 - 2\phi)[\Delta_0(-2 + 5\phi - 3\phi^3) - 6(5 - 12\phi - 3\phi^2 + 10\phi^3)]
- (6 + \Delta_0)f^2(1 + 5\phi)e^{2\phi-2}Ei(2 - 2\phi) \right]
\]

(6.9a)

Which suggest that the extremum value of \( p_t \) occurs at the centre i.e.

\[
p'_t = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^4}(p''_t)_{r=0} = -\frac{2e^2}{e^2} \left[ (10 - A - \Delta_0)e^2 + 6Ei(2) + 30Ei(2) + \Delta_0[2Ei(2) + Ei(2)] \right]
\]

(6.10a)

Under the condition (6.6a), the expression of right hand side of Eq. (6.11a) is negative for all values of \( A \) and \( \Delta_0 \). This shows that the tangential pressure \((p_t)\) is maximum at the centre and monotonically decreasing.

Now differentiating equation (4.1a) with respect to \( r \), we get

\[
\frac{8\pi G}{c^4} \frac{d\rho}{dr} = 2e^2r \left[ Ae^{2\phi}(5 - 23\phi^2 + 10\phi^3 + 8\phi^4)
+ (-30 - 24\phi - 60\phi^3 - 48\phi^4)e^{2\phi-2}Ei(2 - 2\phi)
+ 6f^2(3 - 5\phi - 4\phi^2)e^{2\phi-2}Ei(2 - 2\phi) - 22\phi + 6\phi^2
+ \Delta_0[e^{2\phi-2}f\{Ei(2 - 2\phi)(-5 - 19\phi + 12\phi^2 + 8\phi^3)
- f(-5 + 9\phi + 8\phi^3)\tilde{Ei}(2 - 2\phi)
- 2f^2\phi\tilde{Ei}(2 - 2\phi)] - 2(-5 + 7\phi)] \right]
\]

(6.11a)

\[
\frac{8\pi G}{c^2}(\rho'')_{r=0} = -\frac{2e^2}{e^2} \left[ -(5A + 10\Delta_0)e^2 + Ei(2)(30 + 5\Delta_0) - \tilde{Ei}(2)(18 + 5\Delta_0) \right]
\]

(6.12a)

where \( \tilde{Ei}(2 - 2\phi) = \frac{4}{(2\phi)^2} + 4 \sum_{N=1}^\infty \frac{N(2-2\phi)^N}{(N+1)} \).

The extremum of \( \rho \) occur at the centre i.e.

\[
\rho' = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^2}(\rho''')_{r=0} = -\frac{2e^2}{e^2} \left[ -(5A + 10\Delta_0)e^2 + Ei(2)(30 + 5\Delta_0) - \tilde{Ei}(2)(18 + 5\Delta_0) \right]
\]

(6.13a)
Thus, the expression of right hand side of (6.12a) is negative. This shows that the density \( \rho \) is maximum at the centre and monotonically decreasing towards the pressure free interface.

The square of adiabatic sound speed at the centre, \( \frac{1}{c^2} \left( \frac{dp_r}{d\rho} \right)_{r=0} \), is given by

\[
\frac{1}{c^2} \left( \frac{dp_r}{d\rho} \right)_{r=0} = \left( \frac{(10 - A)c^2 + 6Ei(2) + 30Ei(2) + \Delta_0[2Ei(2) + \tilde{E}i(2)]}{(5A + 10\Delta_0)c^2 + Ei(2)(30 + 5\Delta_0) + \tilde{E}i(2)(18 + 5\Delta_0)} \right) \tag{6.14a}
\]

The causality condition is obeyed at the centre for all values of constants under the condition (6.6a) i.e. radial and tangential velocity of sound are monotonically decreasing and less than 1.

**(b) For** \( n = -2 \)

The central values of pressure and density are given by

\[
\left( \frac{8\pi Gp_r}{c^4} \right)_{r=0} = \left( \frac{8\pi Gp_t}{c^4} \right)_{r=0} = \frac{c_0}{4} (4A + 8 + \Delta_0) \tag{6.1b}
\]

\[
\left( \frac{8\pi G\rho}{c^2} \right)_{r=0} = \frac{c_0}{4} (-12A + 24 + 3\Delta_0) \tag{6.2b}
\]

The central values of pressure and density should be non zero positive and finite, then \( A \) satisfies the following conditions:

\[
A > \left( -2 - \frac{\Delta_0}{4} \right) \quad \text{and} \quad A < \left( 2 - \frac{\Delta_0}{4} \right) \tag{6.3b}
\]

Subjecting the condition that positive value of ratio of pressure-density and should be less than 1 at the centre i.e. \( \frac{p_r}{\rho c^2} \leq 1 \) which leads to the following inequality,

\[
\left( \frac{p_r}{\rho c^2} \right)_{r=0} = \left( \frac{p_t}{\rho c^2} \right)_{r=0} = \left( \frac{4A + 8 + \Delta_0}{-12A + 24 + 3\Delta_0} \right) \leq 1 \tag{6.4b}
\]

By the inequality (6.4b), we get

\[
A \leq \left( 1 + \frac{\Delta_0}{8} \right) \tag{6.5b}
\]

By the inequalities (6.3b) and (6.5b), \( A \) satisfies the inequality

\[
\left( -2 - \frac{\Delta_0}{4} \right) < A \leq \left( 1 + \frac{\Delta_0}{8} \right), \quad \Delta_0 \geq 0. \tag{6.6b}
\]
Differentiating Eq. (4.2b) with respect to $r$, we get
\[
\frac{8\pi G \, dp_r}{c^4} = 2c^2_0 r \left\{ -2A + \frac{15 - \Delta_0}{8} \log \left( \frac{1 - \phi}{1 + \phi} \right) f^2(-1 + 3\phi + 13\phi^2 + 9\phi^3) \\ - \frac{64 + 130x - 330x^2 - 150x^3 + 270x^4}{8} + \frac{\Delta_0}{8} \left( 2 - 16\phi + 19\phi^2 - 10\phi^3 - 3\phi^4 \right) \right\}.
\]

Thus it is found that extremum value of $p_r$ occur at the centre i.e.
\[
p'_r = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^4} (p''_r)_{r=0} = \left( \frac{-16A - 64 + 2\Delta_0}{8} \right)
\]
(6.7b)

Thus the expression of right hand side of the Eq. (6.8b) is negative for all values of constants $A$ and $\Delta_0$ satisfying condition (6.6b) and it is showing that the pressure ($p_r$) is maximum at the centre and monotonically decreasing.

Differentiating Eq. (4.3b) with respect to $r$, we get
\[
\frac{8\pi G \, dp_t}{c^4} = 2c^2_0 r \left\{ -2A + \frac{15 - \Delta_0}{8} \log \left( \frac{1 - \phi}{1 + \phi} \right) f^2(-1 + 3\phi + 13\phi^2 + 9\phi^3) \\ - \frac{64 + 130x - 330x^2 - 150x^3 + 270x^4}{8} + \frac{\Delta_0}{8} \left( 2 + 43\phi^2 - 10\phi^3 - 3\phi^4 \right) \right\},
\]

Thus it is found that the extremum value of $p_t$ occurs at the centre i.e.
\[
p'_t = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^4} (p''_t)_{r=0} = 2c^2_0 \left( \frac{16A - 64 + 2\Delta_0}{8} \right),
\]
(6.10b)

Thus the expression of right hand side of the Eq. (6.10b) is also negative for all values of $A$ and $\Delta_0$ satisfying condition (6.6b). This behavior shows that transversal pressure ($p_t$) is maximum at the centre and monotonically decreasing.

Now differentiating equation (4.1b) with respect to $r$, the expression of density gradient as:
\[
\left( \frac{8\pi G \, d\rho}{c^2} \right) = 2c^2_0 r \left\{ \frac{5S}{8(1 - \phi^2)} - \left( 2A + \frac{15 + \Delta_0}{8} \log \frac{1 - \phi}{1 + \phi} \right) f^2(-5 - 17\phi + 25\phi^2 + 45\phi^3) \\ - \frac{\Delta_0}{8} \left( -30 + 88\phi - 72\phi^2 + 2\phi^3 + 25\phi^4 \right) \right\},
\]

where, $S = (32 + 86\phi + 10\phi^2 - 176\phi^3 - 312\phi^4 + 90\phi^5 + 273\phi^6)$.

Thus the extremum value of $\rho$ occurs at the centre i.e.
\[
\rho' = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^2} (\rho'')_{r=0} = 2c^2_0 \left( \frac{180 + 80A + 30\Delta_0}{8} \right) > 0;
\]
(6.12b)
The right hand side of Eq. (6.12b) is showing positive due to the inequality (6.6b), and this condition gives that the energy density $\rho$ is minimum at centre and monotonically increasing.

and the square of its adiabatic sound speed at the centre, \( \frac{1}{c^2} \left( \frac{dp}{d\rho} \right)_{r=0} \), is given by

\[
\frac{1}{c^2} \left( \frac{dp}{d\rho} \right)_{r=0} = \frac{1}{c^2} \left( \frac{dp}{d\rho} \right)_{r=0} = \frac{(16A - 64 + 3\Delta_0)}{[180 + 80A + 30\Delta_0]} \quad (6.13b)
\]

The causality condition is negative at the centre for all values of constants satisfying condition (6.6b). Due to increasing nature of energy density, the solution is not well behaved for \( n = -2 \).

**Case 3: \( n = -3 \)**

The central values of pressure and density are given by

\[
\left( \frac{8\pi G p_r}{c^4} \right)_{r=0} = \left( \frac{8\pi G p_t}{c^4} \right)_{r=0} = c_0 \left[ A + 356 + \Delta_0 \right] \quad (6.1c)
\]

\[
\left( \frac{8\pi G \rho}{c^2} \right)_{r=0} = \frac{c_0}{81} \left[ -243A + 21320 - 243\Delta_0 \right] \quad (6.2c)
\]

The central values of pressure and density should be non zero positive and finite. Then \( A \) satisfies the following conditions:

\[
A > (-356 - \Delta_0) \quad \text{and} \quad A < \left( \frac{21320}{243} - \Delta_0 \right) \quad (6.3c)
\]

Subject to the condition that the ratio of pressure-density is positive and less than 1 at the centre i.e. \( \frac{p_r}{\rho c^2} \leq 1 \) which leads to the following inequality,

\[
\left( \frac{p_r}{\rho c^2} \right)_{r=0} = \left( \frac{p_t}{\rho c^2} \right)_{r=0} = \left( \frac{81 \left[ A + 356 + \Delta_0 \right]}{-243A + 21320 - 243\Delta_0} \right) \leq 1 \quad (6.4c)
\]

Eq. (6.4c) leads to

\[
A \leq -\left( \frac{7516}{324} + \Delta_0 \right) \quad (6.5c)
\]

By using the Eqs. (6.3c) and (6.5c), we get the inequality for \( A \) as:

\[
-(356 + \Delta_0) < A \leq -\left( \frac{7516}{324} + \Delta_0 \right) \quad (6.6c)
\]

Differentiating (4.2c) with respect to \( r \), we get

\[
\frac{8\pi G \frac{dp}{c^2}}{dr} = 2c_0^2 r \left[ A p_1(\phi) - (411 - 292\phi - 11460\phi^2 - 6720\phi^3 + 8000\phi^4) + \frac{320}{243} \left( 1 + 5\phi \right) \right] \quad (6.7c)
\]

\[
\frac{320}{243} p_2(\phi) \log \frac{1 + 2\phi}{1 - \phi} + \Delta_0(2 - 9\phi + 33\phi^2 + 160\phi^3 - 345\phi^4 + 180\phi^5)
\]

\[
, \quad (6.7c)
\]
where,
\[ p_1(\phi) = (3 - 24\phi - 6\phi^2 + 132\phi^3 - 165\phi^4 + 60\phi^5), \]
\[ p_2(\phi) = (15 - 40\phi - 252\phi^2 - 336\phi^3 + 120\phi^4) \]

Thus it is found that extremum value of \( p_r \) occurs at the centre i.e.
\[ p'_r = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^4} (p''_r)_{r=0} = 2c_0^2 \left( 3A - 411 + \frac{320}{243} + 2\Delta_0 \right). \quad (6.8c) \]

So the expression of right hand side of Eq. (6.8c) is negative for all values of \( A \) and \( \Delta_0 \) satisfying the condition (6.6c). This shows that the pressure \( (p_r) \) is maximum at the centre and monotonically decreasing.

Differentiating (4.3c) with respect to \( r \), we get
\[ \left( \frac{8\pi G}{c^2} \frac{d\rho}{dr} \right) = 2c_0^2 r \left[ A(3 - 24\phi - 6\phi^2 + 132\phi^3 - 165\phi^4 + 60\phi^5) \right. \]
\[ - (411 - 292\phi - 11460\phi^2 - 6720\phi^3 + 8000\phi^4) + \frac{320}{243} f^3(1 + 5\phi) \]
\[ + \frac{320}{243} (15 - 40\phi - 252\phi^2 - 336\phi^3 + 120\phi^4) \log \frac{1 + 2\phi}{1 - \phi} \]
\[ + \Delta_0(2 - 30\phi + 36\phi^2 + 176\phi^3 - 325x^4 + 180\phi^5) \right]. \quad (6.9c) \]

Thus it is found that extrema of \( \rho_t \) occurs at the centre i.e.
\[ \rho'_t = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^4} (\rho''_t)_{r=0} = 2c_0^2 \left( 3A - 411 + \frac{320}{243} + 2\Delta_0 \right). \quad (6.10c) \]

Thus it is clear that the expression of right hand side of Eq. (6.10c) is negative for all values of \( A \) and \( \Delta_0 \) satisfying the condition (6.6c). This condition gives that the transversal pressure is maximum at the centre and monotonically decreasing.

Now differentiating equation (4.1c) with respect to \( r \) we get
\[ \left( \frac{8\pi G}{c^2} \frac{d\rho}{d\phi} \right) = 2c_0^2 r \left[ A(15 - 270x^2 + 660x^3 - 585x^4 + 204x^5) + \frac{\rho_1}{81} \right. \]
\[ + (9 - 140x + 540x^3 - 495x^4 + 156x^5) \left( \frac{320}{81} \log \frac{1 + 2\phi}{1 - \phi} \right) \]
\[ + \Delta_0(2 - 30\phi + 33x^2 + 160\phi^3 - 345x^4 + 180\phi^5) \right] \quad (6.11c) \]
\[ \rho_1 = (-26885 - 55018\phi + 64686\phi^2 - 1149772\phi^3 + 685612\phi^4 - 296960\phi^5 + 128960\phi^6) \]

Thus the extrema of \( \rho \) occur at the centre i.e.
\[ \rho' = 0 \Rightarrow r = 0 \quad \text{and} \quad \frac{8\pi G}{c^2} (\rho''_{r=0}) = 2c_0^2 \left( -\frac{26885}{81} + 15A + 2\Delta_0 \right) \quad (6.12c) \]
Thus, the expression of right hand side of Eq. (6.12c) is negative for all values of $A$ and $\Delta_0$ satisfying the condition (6.6c). Then density $\rho$ is maximum at the centre and monotonically decreasing.

The square of adiabatic sound speed at the centre, $\left(\frac{dp_r}{d\rho}\right)_{r=0}$, are given by

$$\left(\frac{dp_r}{d\rho}\right)_{r=0} = \left(\frac{dp_t}{d\rho}\right)_{r=0} = \left(\frac{-99553 - 729A + 486\Delta_0}{3[-26885 + 1215A + 162\Delta_0]}\right)$$

(6.13c)

The causality condition is less than 1 and positive at the centre for all values of constants for all values of $A$ and $\Delta_0$ satisfying condition (6.6c).

7 Boundary conditions for evaluation of constants $A$, $D$ and $c_0$

The above system of equations is to be solved subject to the boundary condition that radial pressure $p_r = 0$ at $r = a$ (where, $r = a$ is the outer boundary of the fluid sphere). It is clear that $m(r = a) = M$ is a constant and, in fact, the interior metric (2.1) can be joined smoothly at the surface of spheres ($r = a$), to an exterior Schwarzschild metric whose mass is the same as in above i.e. $m(r = a) = M$ (Masiner and Sharp [45]).

Then the interior metric of this fluid spheres should be joined smoothly with Schwarzschild exterior metric such as $B^2(a) = 1 - 2u$, where $u = M/a$, where $M$ is the mass of the fluid sphere as measured by the exterior field and $a$ is the boundary of the sphere.

By joining (3.5a) and (3.5c) on the boundary of the anisotropic fluid spheres ($r = a$) and by setting $\phi_a = c_0a^2$ and $\psi_{anis}(a) = 1 - 2u_{anis}$, we get the expressions of mass for $n = -1$ and $-3$ as:

$$M_{anis} = \frac{a}{2}\phi_a \left[ -Af_a^3e^{2\phi_a} + (4\phi_a - 3\phi_a^2 + 2\phi_a^3) \\
+ (6 + \Delta_0\phi_a)f_a^3e^{2\phi_a} - 2Ei(2 - 2\phi_a) - \Delta_0\phi_a f_a^2 \right],$$

(7.1)

$$M_{anis} = \frac{a}{2}\phi_a \left[ \frac{M_1}{81} - A(1 - \phi_a)^5(1 + 2\phi_a) \\
+ \frac{320}{243}(1 - \phi_a)^5(1 + 2\phi_a) \log \frac{1 + 2\phi_a}{1 - \phi_a} - M_2\Delta_0\phi_af_a^3 \right],$$

(7.2)

$$M_1 = (292 - 305\phi_a - 662\phi_a^2 + 1796\phi_a^3 - 1360\phi_a^4 + 320\phi_a^5),$$

$$M_2 = (1 + 2\phi_a)(1 - 3\phi_a + 3\phi_a^2),$$

$$S = (16 + 17\phi_a - 50\phi_a^2 + 10\phi_a^3 + 30\phi_a^4 - 15\phi_a^5).$$

The arbitrary constant $A$ in the expression (3.5a) and (3.5c) can be determined by using the radial pressure $p_r$ is zero at the boundary, the expressions of con-
stant $A$ for $n = -1$ and $-3$ are given as:

$$A = \frac{[6(1 + 5\phi_a) + \Delta_0 f_a]f_a^2 e^{2\phi_a - 2} Ei(2 - 2\phi_a) - (2 - 10\phi + 7\phi^2 - 2\phi^3) - \Delta_0 f_a^2}{f_a^2(1 + \phi_a)e^{2\phi_a}};$$  

(7.3)

$$A = \frac{(-356 + 411\phi_a + 146\phi_a^2 - 3820\phi_a^3 - 1680\phi_a^4 + 1600\phi_a^5) - \frac{320}{243}f_a^4(1 + 2\phi_a)(1 + 5\phi_a)\log \frac{1 + 2\phi_a}{1 - \phi_a} - \Delta_0 f_a^2(1 + 4\phi - 8\phi^2 - 9\phi^3 + 30\phi^4)}{f_a^2(1 + 7\phi_a + 10\phi_a^2)}. $$  

(7.4)

Also the arbitrary constants $D$ in the metric potential for the case $n = -1$ and $-3$ can be computed by the condition $B(a) = 1 - 2u_{anis}$:

$$D = f_a^3[A\phi_a f_a e^{2\phi_a} + (1 - 2\phi_a) - (6 + \Delta_0)f_a \phi_a e^{2\phi_a - 2} Ei(2 - 2\phi_a) + \Delta_0 \phi_a];$$  

(7.5)

$$D = f_a^6 \left[A\phi_a f_a^2(1 + 2\phi_a) - \frac{1}{243}f_a^2 \left[3\phi_a - 320\phi_a \log \left(\frac{1 + 2\phi_a}{f_a}\right)(1 + 2\phi_a)\right] + \Delta_0 \phi_a(1 + 2\phi_a)(1 - 3\phi_a + 3\phi_a^2) \right].$$  

(7.6)

The positive constant $c_0$ can be calculated by taking the surface density $2 \times 10^{14}$ gm/cm$^3$ and using the condition $c_0 = (\frac{8\pi G}{c^2} \alpha^2 \rho_a) / (1 - \psi a - a\psi_a')$ for $n = -1$ and $-3$.

![Figure 1: Variations of radial pressure $P_r$ and the trace $D - P_r - 2P_t$ of the energy-momentum tensor for $n = -1$, $c_0 a^2 = 0.0829$, $\Delta_0 = 4.2395$ and $n = -3$, $c_0 a^2 = 0.0028$, $\Delta_0 = 0.7094$](image-url)
Tables for Numerical Values of physical quantities

In Tables 1-2: $Z =$ red shift, Solar mass $M_\odot = 1.475 \text{ km}$, $G = 6.673 \times 10^{-8} \text{cm}^3/\text{gs}^2$, $c = 2.997 \times 10^{10} \text{ cm/s}$, $D = (8\pi G/c^4c_0)p$, $P_r = (8\pi G/c^4c_0)p_r a^2$, $P_t = (8\pi G/c^4c_0)p_t a^2$, $\gamma = \frac{v^2 + c_0^2}{c^2}\frac{dp}{c^2d\rho}$.

Figure 2: Variations of the radial velocity $V_r = \sqrt{\frac{dp}{c^2d\rho}}$ and ratio of radial pressure and density $R_r = \frac{p_r}{c^2\rho}$ of the energy-momentum tensor for $n = -1$, $c_0a^2 = 0.0829$, $\Delta_0 = 4.2395$ and $n = -3$, $c_0a^2 = 0.0028$, $\Delta_0 = 0.7094$.

Figure 3: Variations of the tangential velocity $V_t = \sqrt{\frac{dp}{c^2d\rho}}$ and ratio of tangential pressure and density $R_t = \frac{p_t}{c^2\rho}$ of the energy-momentum tensor for $n = -1$, $c_0a^2 = 0.0829$, $\Delta_0 = 4.2395$ and $n = -3$, $c_0a^2 = 0.0028$, $\Delta_0 = 0.7094$. 
Figure 4: Variations of the anisotropy factor $\Delta_1 = \Delta \times 10^{-2}$ and anisotropy factor $\Delta_2 = \Delta \times 10^{-9}$ of the energy-momentum tensor for $n = -1$, $c_0a^2 = 0.0829$, $\Delta_0 = 4.2395$ and $n = -3$, $c_0a^2 = 0.0028$, $\Delta_0 = 0.7094$.

Table 1: $n = -1$, $\Delta_0 = 4.2395$, $c_0a^2 = 0.0829$, Radius ($a$) = 16.0780 Km, Mass ($M$) = 1.7609 $M_\odot$.

| $r/a$ | $P_r$  | $P_t$  | $D$   | $\Delta$ | $\sqrt{dp_r/d\rho}$ | $\sqrt{dp_t/d\rho}$ | $\delta_{rp}$ | $\delta_{tp}$ | $Z$   |
|-------|--------|--------|-------|-----------|----------------------|----------------------|---------------|---------------|-------|
| 0.0   | 1.3165 | 1.3165 | 20.7045 | 0.0000    | 0.9999  | 0.9394  | 0.0636 | 0.0636 | 0.2570 |
| 0.2   | 1.2543 | 1.2683 | 20.5763 | 0.0140    | 0.9918  | 0.9312  | 0.0610 | 0.0616 | 0.2554 |
| 0.4   | 1.0723 | 1.1285 | 20.1933 | 0.0562    | 0.9680  | 0.9070  | 0.0531 | 0.0559 | 0.2504 |
| 0.6   | 0.7852 | 0.9117 | 19.5603 | 0.1265    | 0.9292  | 0.8672  | 0.0401 | 0.0466 | 0.2421 |
| 0.8   | 0.4167 | 0.6417 | 18.6860 | 0.2249    | 0.8761  | 0.8125  | 0.0223 | 0.0343 | 0.2303 |
| 1.0   | 0.0000 | 0.3515 | 17.5839 | 0.3515    | 0.8092  | 0.7427  | 0.0000 | 0.0200 | 0.2151 |

Figure 5: Variations of the red-shift ($Z$) of the energy-momentum tensor for $n = -1$, $c_0a^2 = 0.0829$, $\Delta_0 = 4.2395$ and $n = -3$, $c_0a^2 = 0.0028$, $\Delta_0 = 0.7094$. 
Table 2: $n=-3$, $\Delta_0=0.7094$, $c_0a^2=0.0028$, Radius $(a) = 3.1274\text{Km}$, Mass $(M) = 0.8672\text{M}_\odot$

| $\frac{n}{2}$ | $P_r$ | $P_t$ | $D$ | $\Delta \times 10^{13}$ | $\frac{dP_r}{d\rho}$ | $\frac{dP_t}{d\rho}$ | $\frac{\rho}{c_2^2}$ | $\frac{\rho}{c_2}$ | $Z$ |
|-----|-----|-----|-----|----------------|----------------|----------------|----------------|----------------|-----|
| 0.0 | 4.0619 | 4.0619 | 13.1901 | 0 | 0.5160 | 0.5160 | 0.3079 | 0.3079 | 0.8597 |
| 0.2 | 3.8982 | 3.8982 | 13.1838 | 0.00997 | 0.5158 | 0.5158 | 0.2957 | 0.2957 | 0.8595 |
| 0.4 | 3.4077 | 3.4077 | 13.1650 | 0.63900 | 0.5153 | 0.5153 | 0.2588 | 0.2588 | 0.8590 |
| 0.6 | 2.5922 | 2.5922 | 13.1336 | 7.2949 | 0.5144 | 0.5144 | 0.1974 | 0.1974 | 0.8582 |
| 0.8 | 1.4549 | 1.4549 | 13.0896 | 41.116 | 0.5132 | 0.5132 | 0.1111 | 0.1111 | 0.8571 |
| 1.0 | 0.0000 | 1.5748$\times 10^{-8}$ | 13.0331 | 157.48 | 0.5116 | 0.5116 | 0.0000 | 1.208$\times 10^{-9}$ | 0.8556 |

8 Stability of the stellar structure

For physically acceptable model, one aspect that the velocity of sound should be within the range $0 \leq v_r^2 = (dp_r/c^2d\rho) \leq 1$ (Herrera [47] and Abreu et al. [48]). In present models, the expression for velocity of sound at the centre is given by equations (6.13a), (6.14a) and (6.13c). We plot the radial and transverse velocity of sound in Fig. 3 and conclude that all parameters satisfy the inequalities $0 = v_r^2 = (dp_r/c^2d\rho) \leq 1$ and $0 = v_t^2 = (dp_t/c^2d\rho) \leq 1$, everywhere inside the star models. From equations (6.13a), (6.14a) and (6.13c), we found that $|v_t^2 - v_r^2| \leq 1$ at the centre and proved that velocity of sound in monotonically decreasing throughout inside the star. Also $0 \leq v_t^2 \leq 1$ and $0 \leq v_r^2 \leq 1$, therefore $|v_t^2 - v_r^2| \leq 1$. Now, to examine the stability of local anisotropic fluid distribution, Herrera’s [46] proposed the cracking (also known as overturning) concept which states that the region, in which radial speed of the sound is greater than transverse speed of the sound, is a potentially stable region.

In our proposed models, the models are stable with the radius $16.0780\text{Km}$, $\Delta_0 = 4.2395$, $c_0a^2 = 0.0829$ for $n = -1$ and radius $3.1274\text{Km}$, $\Delta_0 = 0.7094$, $c_0a^2 = 0.0028$ for $n = -3$.

9 Physical analysis and conclusions

In the present paper, the new set of anisotropic exact solution of Einstein’s field equations we have presented by taking the metric potential $g_{44} = (1 - c_0r^2)^n$ for $n = -1, -2$ and $-3$ and specific choice of anisotropic factor $\Delta$ which involves the anisotropic parameter $\Delta_0$. The obtained solutions are utilized to contract the super dense star models with surface density $2 \times 10^{14} \text{gm/cm}^3$. It is observed that solutions are satisfying all reality and physical conditions (mention its in Section 5) for $n = -1$ and $n = -3$. But the solution is not compatible for $n = -2$ due to increasing nature of its density (Section 6, Case (b)). The anisotropic fluid sphere possesses the maximum mass and corresponding radius are $1.7609\text{M}_\odot$ and $16.0780\text{Km}$ for $n = -1$, $c_0a^2 = 0.0829$, $\Delta_0 = 4.2395$ and $0.8672\text{M}_\odot$ and $3.1274\text{Km}$ for $n = -3$, $c_0a^2 = 0.0028$, $\Delta_0 = 0.7094$. The red shift for $n = -1$
and $-3$ are monotonically decreasing towards the pressure free interface $r = a$ and found that the red shift at the centre ($Z_0$) and at surface ($Z_a$) are: (i) $Z_0 = 0.2570$ and $Z_a = 0.2151$ for $n = -1$, (ii) $Z_0 = 0.8597$ and $Z_a = 0.8556$ for $n = -3$ for both strong energy and dominated energy conditions. In our models, the red shift is also satisfied the upper bound limit for the realistic anisotropic star models (Ivanov [4]) and its behavior is represented by the Fig. 5. The Tables 1 and 2 shows the numerical values of physical parameters. Fig. 1 shows that the fluid spheres satisfies the strong energy condition. The behaviors of velocity and pressure density ratio are given by the Fig. 2 and 3. Fig. 4 represents the increasing nature of anisotropy factor for the fluids spheres.

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