A companion of Ostrowski’s inequality for complex functions defined on the unit circle and applications

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ABSTRACT: Some companions of Ostrowski’s inequality for complex functions defined on the unit circle are proved. Our results in special cases not only recapture known results, but also give a smaller estimator than that of the known results. Applications to a composite quadrature rule and to functions of unitary operators in Hilbert spaces are also considered.

KEYWORDS: Riemann-Stieltjes integral inequalities, unitary operators in Hilbert spaces, quadrature rules

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INTRODUCTION

The Riemann-Stieltjes integral plays an important role in mathematics. In Ref. 1, Alomari used $f(x)[u(\frac{x}{2}a+b)]−u(a)]+f(a+x)[u(b)−u(\frac{x}{2}a+b)]$ to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ and proved that

$$|D(f;u; a, b, x)| \leq H \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] V^V_a(u),$$

for any $x \in [a, \frac{1}{2}(a+b)]$, provided that $f : [a, b] \to \mathbb{R}$ is an $(r-H)$-Hölder type mapping and $u : [a, b] \to \mathbb{R}$ is a mapping of bounded variation on $[a, b]$, where $V^V_a(f)$ denotes the total variation of $f$ on $[a, b]$, and

$$D(f;u; a, b, x) := \int_a^b f(t)\, du(t) - f(x) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] - f(a+b-x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right]$$

(1)

is the error functional, $H > 0$ and $r \in (0, 1]$ are given.

If the integrand $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ and the integrator $u(t) = t, t \in [a, b]$, then the following companion of Ostrowski’s inequalities for functions of bounded variation has been considered by in Ref. 2, in which they obtained the bound

$$\left| \frac{1}{2} [f(x) + f(a+b-x)] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{b-a} \left[ (x-a) V^V_a(f) + \frac{a+b}{2} - x \right] V^V_{a+b-x}(f)$$

$$+ (x-a) V^V_{a+b-x}(f)$$

for any $x \in [a, \frac{1}{2}(a+b)]$.

In Ref. 3, Dragomir developed Ostrowski’s type integral inequality for the complex unit circle $C(0, 1)$.

Theorem 1 Assume that $f : C(0,1) \to \mathbb{C}$ satisfies the following Hölder’s type condition

$$|f(a) - f(b)| \leq H |a-b|^r,$$

(2)

for any $a, b \in C(0,1)$, where $H > 0$ and $r \in (0, 1]$ are given. If $[a, b] \subseteq [0,2\pi]$ and the function $u : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then

$$|\int_a^b f(t)\, du(t) - f(x) [u(b) - u(a)] - f(a+b-x) [u(a) - u(b)]|$$

is the error functional, $H > 0$ and $r \in (0, 1]$ are given.
for any \( s \in [a, b] \).

For other inequalities for the Riemann-Stieltjes integral, see Refs. 4–19.

Motivated by the above facts, we consider in the present paper the problem of approximating the Riemann-Stieltjes integral \( \int_a^b f(e^u) \, du(s) \) by the rule

\[
f(e^u)\left[u\left(\frac{a+b}{2}\right) - u(a)\right] + f(e^{i(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right], \tag{4}
\]

where the continuous complex valued function \( f : C(0,1) \to \mathbb{C} \) is defined on the complex unit circle \( C(0,1) \) and the function \( u : [a, b] \subseteq [0,2\pi] \to \mathbb{C} \) is of bounded variation.

We denote the error functional

\[
T(f, u; a, b; s, t) := f(e^u)\left[u\left(\frac{a+b}{2}\right) - u(a)\right] + f(e^{i(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right] - \int_a^b f(e^u) \, du(t), \tag{5}
\]

where \( t \in [a, b] \) and \( f \) is of \((r-H)\)-Hölder type and \( u \) is the function of bounded variation.

The outline of this paper is as follows. First, we show some inequalities for the Riemann-Stieltjes integral. Second, we apply them to a composite quadrature rule. Third, the study of applications to unitary operators is discussed.

**SOME COMPANIONS OF OSTROWSKI’S TYPE INEQUALITY**

The following companions of Ostrowski’s inequality for Riemann-Stieltjes integrals hold.

**Theorem 2** Suppose that \( f : C(0,1) \to \mathbb{C} \) satisfies the following Hölder’s type condition:

\[
|f(a) - f(b)| \leq H |a - b|^r, \tag{6}
\]

for any \( a, b \in C(0,1) \), where \( H > 0 \) and \( r \in (0, 1] \) are given. If \([a, b] \subseteq [0, 2\pi] \) and \( u : [a, b] \to \mathbb{C} \) is the function of bounded variation, then

\[
|f(e^u)\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(e^u) \, du(t) + f(e^{i(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right]| \leq 2^r H \max_{t \in [a, (a+b)/2]} |\sin\left(\frac{s-t}{2}\right)| V_a^{b/2}(u) + \max_{t \in [(a+b)/2, b]} |f(e^{i(a+b-s)} - f(e^u))| V_a^b(u),
\]

for any \( s \in [a, b] \).

**Proof:** Clearly, we have from Ref. 1

\[
f(e^u)\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(e^u) \, du(t) + f(e^{i(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right] = \int_a^{(a+b)/2} f(e^{i(a+b-s)} - f(e^u)) \, du(t) + \int_{(a+b)/2}^b f(e^u) \, du(t) + f(e^{i(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right], \tag{8}
\]

for any \( s \in [a, \frac{1}{2}(a+b)] \).

If \( q : [c, d] \to \mathbb{C} \) is a continuous function and \( v : [c, d] \to \mathbb{C} \) is the function of bounded variation, then there exists the Riemann-Stieltjes integral \( \int_c^d q(t) \, dv(t) \) and

\[
\left|\int_c^d q(t) \, dv(t)\right| \leq \max_{t \in [c,d]} |q(t)| V_c^d(v). \tag{9}
\]

Applying inequality (9) to identity (8) and using Hölder’s type condition (6), we obtain

\[
|f(e^u)\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(e^u) \, du(t) + f(e^{i(a+b-s)})\left[u(b) - u\left(\frac{a+b}{2}\right)\right]| \leq \max_{t \in [(a+b)/2, b]} |f(e^{i(a+b-s)} - f(e^u))| V_a^b(u) + \max_{t \in [(a+b)/2, b]} |e^{i(a+b-s)} - e^u| V_a^{b/2}(u),
\]

for any \( s \in [a, b] \).

From Ref. 3, we have

\[
|e^u - e^{i^r}| = 2^r |\sin\left(\frac{s-t}{2}\right)|, \tag{11}
\]

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for any $s, t \in \mathbb{R}$.

Applying (11) to (10), we deduce

\[
\begin{align*}
\left| f(e^{it})\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(e^{it})\,du(t) \\
+ f\left(e^{i(a+b-t)}\right)\left[u(b) - u\left(\frac{a+b}{2}\right)\right]\right| & \leq 2' \left\{ \max_{t \in [a, (a+b)/2]} \left| \sin\left(\frac{s-t}{2}\right) \right| \right\} \left| V_{a+b/2}^{(a+b)}(u) \right| \\
& \quad + \max_{t \in [(a+b)/2, b]} \left| \sin\left(\frac{a+b-s-t}{2}\right) \right| \left| V_{a+b/2}^{b}(u) \right|
\end{align*}
\]

which is more precise than Remark 2 of Ref. 3. □

We can consider the following situation: for any $w, z \in C(0, 1)$, the Lipschitz condition $|f(z) - f(w)| \leq L|z - w|$ is satisfied, where $f : C(0, 1) \to \mathbb{C}$ and $L > 0$. In this case, we can show the sharpness of the corresponding version of (17).

**Corollary 1** Suppose that $f : C(0, 1) \to \mathbb{C}$ is Lipschitz with the constant $L > 0$ on the circle $C(0, 1)$.

If $[a, b] \subset [0, 2\pi]$ and $u : [a, b] \to \mathbb{C}$ is a function of bounded variation on $[a, b]$, then we have

\[
\begin{align*}
\left| f(e^{i(3a+b)/4})\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(e^{it})\,du(t) \\
+ f\left(e^{i(a+3b)/4}\right)\left[u(b) - u\left(\frac{a+b}{2}\right)\right]\right| & \leq 2L \left| \sin\left(\frac{b-a}{8}\right) \right| V_a^{b}(u), \quad (17)
\end{align*}
\]

where the constant 2 on the right-hand side cannot be replaced by a smaller constant.

**Proof:** We only need to prove the sharpness of the constant 2. Assume that (17) holds with a constant $C > 0$, that is,

\[
\begin{align*}
\left| f(e^{i(3a+b)/4})\left[u\left(\frac{a+b}{2}\right) - u(a)\right] - \int_a^b f(e^{it})\,du(t) \\
+ f\left(e^{i(a+3b)/4}\right)\left[u(b) - u\left(\frac{a+b}{2}\right)\right]\right| & \leq C L \left| \sin\left(\frac{b-a}{8}\right) \right| V_a^{b}(u). \quad (18)
\end{align*}
\]

Choose $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z$ and $u : [0, 2\pi] \to \mathbb{R}$ given by

\[
\begin{align*}
u(t) := \begin{cases} 0, & 0 \leq t < 2\pi, \\ 1, & t = 2\pi. \end{cases}
\end{align*}
\]

Clearly, $f$ is Lipschitz with the constant $L = 1$. At the same time, we consider $a = 0$ and $b = 2\pi$. By
using integration by parts for the integral, we have
\[
\int_0^{2\pi} e^{it} u(t) \, dt = e^{it} u(t)|_0^{2\pi} - i \int_0^{2\pi} e^{it} u(t) \, dt
\]
\[= u(2\pi) - u(0) - i \int_0^{2\pi} e^{it} u(t) \, dt
\]
(20)
and
\[V_0^{2\pi}(u) = 1.
\]
(21)
Consequently, by (20) and (21), we obtain \( C - 2 \).

**Remark 1** Under the assumption **Theorem 2** and \( u(t) = t, \, t \in [a, b] \subseteq (0, 2\pi] \), we have
\[
\left| f(e^{it}) + f(e^{i(a+b-s)}) - \frac{2}{b-a} \int_a^b f(e^{it}) \, dt \right|
\]
\[\leq 2' H \left\{ \max_{t \in [a, ((a+b)/2)]} \left| \sin \left( \frac{s-t}{2} \right) \right| \right\} + \max_{t \in [(a+b)/2, b]} \left| \sin \left( \frac{a+b+s-t}{2} \right) \right|,\]
(22)
for any \( s \in [a, \frac{1}{2}(a+b)] \), provided that \( f : C(0, 1) \rightarrow \mathbb{C} \) satisfies Hölder’s type condition (6).

**Remark 2** If \( f : C(0, 1) \rightarrow \mathbb{C} \) satisfies (6), and \( p : [a, b] \subseteq [0, 2\pi] \rightarrow \mathbb{C} \) is Lebesgue integrable on \([a, b]\), then we have
\[
\left| f(e^{it}) \int_a^{(a+b)/2} p(t) \, dt + f(e^{i(a+b-s)}) \int_{(a+b)/2}^b p(t) \, dt \right|
\]
\[- f(e^{it})p(t) \, dt \left|_{(a+b)/2} \right|
\]
\[\leq 2' H \left\{ \max_{t \in [a, ((a+b)/2)]} \left| \sin \left( \frac{s-t}{2} \right) \right| \right\} \int_a^{(a+b)/2} |p(t)| \, dt
\]
\[+ \max_{t \in [(a+b)/2, b]} \left| \sin \left( \frac{a+b+s-t}{2} \right) \right| \int_{(a+b)/2}^b |p(t)| \, dt.
\]
(23)

**Theorem 3** On the circle \( C(0, 1) \), suppose that \( f : C(0, 1) \rightarrow \mathbb{C} \) is Lipschitzian with the constant \( K > 0 \) on \([a, b]\), then
\[
f(e^{it}) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] - \int_a^b f(e^{it}) \, dt
\]
\[+ f(e^{i(a+b-s)}) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right]
\]
\[\leq 32KL \sin^2 \left( \frac{b-a}{8} \right),\]
(24)
for any \( s \in [a, \frac{1}{2}(a+b)] \).

**Proof:** If \( w : [a, b] \rightarrow \mathbb{C} \) is a Riemann integrable function and the function \( v : [a, b] \rightarrow \mathbb{C} \) is \( M \)-Lipschitzian, then there exists the Riemann-Stieltjes integral \( \int_a^b w(t) \, dv(t) \) and
\[
\left| \int_a^b w(t) \, dv(t) \right| \leq M \int_a^b |w(t)| \, dt.
\]
(25)
Making use of (25), we have from (8) that
\[
f(e^{it}) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] - \int_a^b f(e^{it}) \, dt
\]
\[+ f(e^{i(a+b-s)}) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right]
\]
\[\leq KL \left\{ \int_a^{(a+b)/2} |e^{it} - e^{-it}| \, dt + \int_{(a+b)/2}^b |e^{i(a+b-s)} - e^{-it}| \, dt \right\},\]
(26)
for any \( s \in [a, \frac{1}{2}(a+b)] \). By (11), for any \( t, s \in \mathbb{R} \), we have \( |e^{it} - e^{-it}| = 2|\sin(\frac{s}{2}(s-t))| \). Then
\[
\int_a^{(a+b)/2} |e^{it} - e^{-it}| \, dt = 8 \left[ \sin^2 \left( \frac{s-a}{4} \right) + \sin^2 \left( \frac{a+b-s}{4} \right) \right]
\]
\[\leq 16 \sin^2 \left( \frac{b-a}{8} \right),\]
(27)
for any \( s \in [a, \frac{1}{2}(a+b)] \). In the similar way, we have
\[
\int_{(a+b)/2}^b |e^{i(a+b-s)} - e^{-it}| \, dt \leq 16 \sin^2 \left( \frac{b-a}{8} \right),\]
(28)
for any \( s \in [a, \frac{1}{2}(a+b)] \). Plugging (27) and (28) into (26), we complete the proof of (24). 

\[\Box\]
Remark 3  Under the assumptions in Theorem 3, for any $s \in [a, b)$,

$$
\begin{align*}
& |f(\exp((3a+b)/4)) u\left(\frac{a+b}{2}\right) - u(a) - \int_{a}^{b} f(\exp i) du(t) + f(\exp((3b+a)/4)) u(b) - u\left(\frac{a+b}{2}\right)| \\
& \leq 32KL \sin^2\left(\frac{b-a}{16}\right). \\
& \leq L \int_{a}^{(a+b)/2} \sgn(s-t) \cos\left(\frac{s-t}{2}\right) u(t) dt \\
& + L \int_{(a+b)/2}^{b} \sgn(a+b-s-t) \cos\left(\frac{a+b-s-t}{2}\right) u(t) dt \\
& + 2L \sin\left(\frac{s-a}{2}\right)[u(b) - u(a)]. \quad (29)
\end{align*}
$$

Remark 4  If $u(t) = t, t \in [a, b]$, then

$$
\begin{align*}
& |f(\exp i) + f(\exp(a+b-s)) - \frac{2}{b-a} \int_{a}^{b} f(\exp i) dt| \\
& \leq \frac{32L}{b-a} \sin^2\left(\frac{s-a}{4}\right) + \sin^2\left(\frac{a+b-s}{4}\right) \\
& \leq \frac{64L}{b-a} \sin^2\left(\frac{b-a}{8}\right). \quad (30)
\end{align*}
$$

for any $s \in [a, \frac{1}{2}(a+b)]$ and

$$
\begin{align*}
& |f(\exp((3a+b)/4)) + f(\exp((3b+a)/4)) - \frac{2}{b-a} \int_{a}^{b} f(\exp i) dt| \\
& \leq \frac{64L}{b-a} \sin^2\left(\frac{b-a}{8}\right). \quad (31)
\end{align*}
$$

Remark 5  If $w : [a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ is essentially bounded on $[a, b]$ and $f : C(0, 1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $C(0, 1)$, then for any $s \in [a, \frac{1}{2}(a+b)]$,

$$
\begin{align*}
& \left| f(\exp i) \int_{a}^{(a+b)/2} w(t) dt - \int_{a}^{b} f(\exp i) w(t) dt + f(\exp((a+b-s)) \int_{(a+b)/2}^{b} w(t) dt \right| \\
& \leq 32L \|w\|_{\infty} \sin^2\left(\frac{b-a}{8}\right), \quad (32)
\end{align*}
$$

where $\|w\|_{\infty} := \text{ess sup}_{t \in [a, b]} |w(t)|$.

Theorem 4  On the circle $C(0, 1)$, suppose that $f : C(0, 1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$. If $u : [a, b] \subseteq [0, 2\pi]$ and the function $u : [a, b] \to \mathbb{R}$ is monotonic nondecreasing on $[a, b]$, then for any $s \in [a, \frac{1}{2}(a+b)]$, for any $s \in [a, \frac{1}{2}(a+b)]$. By (11), for any $t, s \in \mathbb{R}$, we have $|e^{is} - e^{it}| = 2|\sin(\frac{1}{2}(s-t))|$. Then

$$
\begin{align*}
\int_{a}^{(a+b)/2} |e^{is} - e^{it}| du(t) \\
= 2 \int_{a}^{s} \sin\left(\frac{s-t}{2}\right) du(t) + 2 \int_{s}^{(a+b)/2} \sin\left(\frac{t-s}{2}\right) du(t), \quad (36)
\end{align*}
$$

for any $s \in [a, \frac{1}{2}(a+b)] \subseteq [0, 2\pi]$. Using integration by parts for the Riemann-Stieltjes integral, we have

$$
\begin{align*}
\int_{a}^{s} \sin\left(\frac{s-t}{2}\right) du(t) \\
= -\sin\left(\frac{s-a}{2}\right) u(a) + \frac{1}{2} \int_{a}^{s} \cos\left(\frac{s-t}{2}\right) u(t) dt \quad (37)
\end{align*}
$$
Remark 6 If we choose $s = \frac{1}{2}(a+b)$ in (33), we recapture Theorem 5 of Ref. 3.

Corollary 2 Under the assumptions of Theorem 4, for any $s \in \left[ a, \frac{1}{2}(a+b) \right]$, 

\[
\left| f(e^{it}) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] - \int_a^b f(e^{it}) \, dt \right|
\leq 2L \sin \left( \frac{s-a}{2} \right) \left[ u(b) - u(a) + u(s) - u(a+b-s) \right]
\leq 2L \sin \left( \frac{s-a}{2} \right) \left[ u(b) - u(a) + u(s) - u(a+b-s) \right]
\]

\[
\geq 2L \sin \left( \frac{s-a}{2} \right) \left[ u(a+b-s) - u(s) \right]
\]

where $B(s) \leq 2L \sin \left[ \frac{b-a}{8} + \frac{1}{2} \left| s - \frac{3a+b}{4} \right| \right] [u(b) - u(a)]$

By (35), (39) and (40), (33) is proved. \hfill \Box

In particular,

\[
\int_a^b \left| e^{it} - e^{i\alpha} \right| \, dt
\leq L \int_a^b \sin \left( \frac{3a+b}{4} - t \right) \cos \left( \frac{3a+b}{2} - t \right) \, dt
\]

\[
\leq 2L \sin \left( \frac{b-a}{8} \right) [u(b) - u(a)]
\]

where $M \leq 2L \sin \left( \frac{b-a}{8} \right) [u(b) - u(a)]$.

Proof: Since $0 < b-a < 2\pi$, we have $0 \leq \frac{1}{2}(s-t) \leq \frac{1}{2}(a+b)-a \leq \frac{1}{2} \pi$ for any $s, t \in \left[ a, \frac{1}{2}(a+b) \right]$. If $s \in \left[ a, \frac{1}{2}(a+b) \right]$ and $t \in \left[ \frac{1}{2}(a+b), b \right]$, then $0 \leq \frac{1}{2}(a+b-s-t) \leq \frac{1}{2}(a+b-a-\frac{1}{2}(a+b)) \leq \frac{1}{2}(b-a) \leq \frac{1}{2} \pi$. Hence we have $\cos \left( \frac{1}{2}(s-t) \right) \geq 0$ for any $s, t \in \left[ a, \frac{1}{2}(a+b) \right]$; $\cos \left( \frac{1}{2}(a+b-s-t) \right) \leq 0$ for any $s \in \left[ a, \frac{1}{2}(a+b) \right], t \in \left[ \frac{1}{2}(a+b), b \right]$. Using the fact that $u$ is monotonic nondecreasing on $[a, b]$, 

\[
\int_a^s \cos \left( \frac{t}{2} \right) u(t) \, dt \leq 2u(s) \sin \left( \frac{s-a}{2} \right)
\]

\[
\int_{\frac{a+b}{2}}^{a+b-s} \cos \left( \frac{t}{2} \right) u(t) \, dt \geq 2u(s) \sin \left( \frac{a+b-s}{2} \right)
\]

\[
\int_{\frac{a+b}{2}}^{a+b-s} \cos \left( \frac{a+b-s-t}{2} \right) u(t) \, dt
\]

\[
\leq 2u(a+b-s) \sin \left( \frac{a+b-s}{2} \right)
\]

\[
\int_a^b \cos \left( \frac{a+b-s-t}{2} \right) u(t) \, dt
\]

\[
\geq 2u(a+b-s) \sin \left( \frac{s-a}{2} \right)
\]

Applying (44)–(33), we have proved (41). From the elementary property stating that $ax + \beta y \leq \max\{a, \beta\}(x+y)$,
where $a, \beta, x, y \geq 0$, we can obtain the bounds for $B(s)$. The details are omitted.

**A COMPOSITE QUADRATURE RULE**

In this section, we use the results from the previous sections to approximate the Riemann-Stieltjes integral $\int_a^b f(e^t)\,du(t)$, in terms of the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$.

We consider the following partition of the interval $[a, b]$

$$\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the intermediate points $\xi_k \in [x_k, \frac{1}{2}(x_k + x_{k+1})]$, where $0 \leq k \leq n-1$. We define $h_k := x_{k+1} - x_k$, $0 \leq k \leq n-1$ and the norm of the partition $\Delta_n$ is $\nu(\Delta_n) = \max\{h_k : 0 \leq k \leq n-1\}$.

We define the quadrature rule

$$O_n(f, u, \Delta_n, \xi) := \sum_{k=0}^{n-1} f(e^{i\xi_k})\left[u\left(\frac{x_{k+1} + x_k}{2}\right) - u(x_k)\right]$$

$$+ \sum_{k=0}^{n-1} f(e^{i(x_k + x_{k+1} - \xi_k)})\left[u(x_k) - u\left(\frac{x_k + x_{k+1}}{2}\right)\right],$$

(45)

where $f : C(0, 1) \to \mathbb{C}$ is a continuous function and $u : [a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ is a function of bounded variation on $[a, b]$. Define the remainder $R_n(f, u, \Delta_n, \xi)$ in approximating the Riemann-Stieltjes integral $\int_a^b f(e^t)\,du(t)$ by $O_n(f, u, \Delta_n, \xi)$. Then

$$\int_a^b f(e^t)\,du(t) = O_n(f, u, \Delta_n, \xi) + R_n(f, u, \Delta_n, \xi).$$

(46)

We provide a priori bounds for $R_n(f, u, \Delta_n, \xi)$ in several instances of $f$ and $u$ as above in the following result.

**Proposition 1** Assume that $f : C(0, 1) \to \mathbb{C}$ satisfies Hölder’s type condition (6). If $[a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ is a function of bounded variation on $[a, b]$, then for any partition $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm $\nu(\Delta_n) \leq \pi$, we have the error bound

$$|R_n(f, u, \Delta_n, \xi)| \leq 2'H \sum_{k=0}^{n-1} \sin^r\left(\frac{x_{k+1} - x_k}{4}\right) V_{x_{k+1}}^x(u)$$

$$\leq 2'H \sum_{k=0}^{n-1} \sin^r\left(\frac{x_{k+1} - x_k}{4}\right) + \left|\frac{3x_k + x_{k+1}}{4}\right| V_{x_k}^{x_{k+1}}(u)$$

(47)

for any intermediate points $\xi_k \in [x_k, \frac{1}{2}(x_k + x_{k+1})]$, where $0 \leq k \leq n-1$.

**Proof:** Since $\nu(\Delta_n) \leq \pi$, then on using (7) on each interval $[x_k, x_{k+1}]$ and for any intermediate points $\xi_k \in [x_k, \frac{1}{2}(x_k + x_{k+1})]$ where $0 \leq k \leq n-1$, we have

$$\left|f(e^{i\xi_k})\left[u\left(\frac{x_k + x_{k+1}}{2}\right) - u(x_k)\right]\right| +$$

$$+ \int_{x_k}^{x_{k+1}} f(e^t)\,du(t) \leq 2'H \sin^r\left(\frac{x_{k+1} - x_k}{4}\right) +$$

$$\left|\frac{3x_k + x_{k+1}}{4}\right| V_{x_k}^{x_{k+1}}(u)$$

(48)

Summing over $k$ from 0 to $n-1$ in (48) and using the generalized Delta inequality, we deduce (47).  

**Remark 7** If we choose $\xi_k = \frac{1}{2}(\lambda_k + \lambda_{k+1})$, we recapture Corollary 1 of Ref. 3. In particular, if we choose $\xi_k = (3x_k + x_{k+1})/4$, then we obtain $|R_n(f, u, \Delta_n, \xi)| \leq (1/4^r)H\nu(\Delta_n) V^h(u)$, which is more precise than Proposition 4 of Ref. 3.

**Corollary 3** Under the assumption of Proposition 1 and $\xi_k = (3x_k + x_{k+1})/4$, we define the special quadrature rule by

$$T_n(f, u, \Delta_n) := \sum_{k=0}^{n-1} f(e^{i(3x_k + x_{k+1})/4})\left[u\left(\frac{x_k + x_{k+1}}{2}\right) - u(x_k)\right] +$$

$$+ \sum_{k=0}^{n-1} f(e^{i(3x_k + x_{k+1})/4})\left[u(x_{k+1}) - u\left(\frac{x_k + x_{k+1}}{2}\right)\right]$$

(49)
and the error $E_n(f, u, \Delta_n)$ by
\[
\int_a^b f(e^t) \, dt = T_n(d, u, \Delta_n) + E_n(f, u, \Delta_n). \tag{50}
\]
Then we have the error bounds
\[
|E_n(f, u, \Delta_n)| \leq 2' H \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1} - x_k}{8}\right) v(x_k) + \frac{1}{4} H \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 v'(x_k). \tag{51}
\]
In the following result, we consider the case of both integrator and integrand being Lipschitzian.

**Proposition 2** Under the assumption of Theorem 3, for any partition $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, we have the error bound
\[
|E_n(f, u, \Delta_n, \xi)| \leq 16KL \sum_{k=0}^{n-1} \sin^2\left(\frac{\xi_k - x_k}{4}\right) + \sin^2\left(\frac{x_{k+1} - \xi_k}{4}\right) \leq 32KL \sum_{k=0}^{n-1} \sin^2\left(\frac{x_{k+1} - x_k}{8}\right) \leq \frac{1}{8} KL \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 \leq \frac{1}{8} KL (b-a) v(\Delta_n). \tag{52}
\]
for any $\xi_k \in [x_k, \frac{1}{2}(x_k + x_{k+1})]$, where $0 \leq k \leq n-1$. In particular,
\[
|E_n(f, u, \Delta_n)| \leq 32KL \sum_{k=0}^{n-1} \sin^2\left(\frac{x_{k+1} - x_k}{16}\right) \leq \frac{1}{8} KL \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 \leq \frac{1}{8} KL (b-a) v(\Delta_n). \tag{53}
\]
The proof is similar to Theorem 3 and the details are omitted.

**Proposition 3** Under the assumption of Theorem 4, for any quadrature $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm $v(\Delta_n) \leq \pi$ and $\xi_k \in [x_k, \frac{1}{2}(x_k + x_{k+1})]$, we have the error bound
\[
|R_n(f, u, \Delta_n, \xi)| \leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{\xi_k - x_k}{2}\right) \int_{x_k}^{x_{k+1} + \Delta} f(x) \, dx \leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{\xi_k - x_k}{2}\right) \left[u(x_{k+1}) - u(x_k)\right] + L \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1} + \Delta} \sin(\xi_k - t) \cos\left(\frac{\xi_k - t}{2}\right) \, dt + L \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1} + \Delta} \sin(x_k + x_{k+1} - \xi_k - t) \cos\left(\frac{x_k + x_{k+1} - \xi_k - t}{2}\right) \, dt
\]
\[
\leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{\xi_k - x_k}{2}\right) \left[u(x_{k+1}) - u(x_k)\right] + 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_k + x_{k+1}}{4}\right) \sin\left(\frac{\xi_k - x_k}{2}\right) \left[u(x_{k+1}) - u(x_k)\right] \leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_k + x_{k+1}}{4}\right) \left[u(x_{k+1}) - u(x_k)\right] + \frac{1}{2} \left[\frac{3}{2} x_k + x_{k+1}\right] \left[u(x_{k+1}) - u(x_k)\right] \leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_k + x_{k+1}}{4}\right) \left[u(x_{k+1}) - u(x_k)\right] \leq \frac{L}{2} \sum_{k=0}^{n-1} (x_{k+1} - x_k) \left[u(x_{k+1}) - u(x_k)\right] \leq \frac{L}{2} v(\Delta_n) \left[u(b) - u(a)\right]. \tag{54}
\]
where $0 \leq k \leq n-1$. In particular,
\[
|E_n(f, u, \Delta_n)| \leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_k + x_{k+1}}{8}\right) \left[u(x_{k+1}) - u(x_k)\right] + \frac{1}{2} \left[\frac{3}{2} x_k + x_{k+1}\right] \left[u(x_{k+1}) - u(x_k)\right] \leq 2L \sum_{k=0}^{n-1} \sin\left(\frac{x_k + x_{k+1}}{8}\right) \left[u(x_{k+1}) - u(x_k)\right] \leq \frac{L}{4} \sum_{k=0}^{n-1} (x_{k+1} - x_k) \left[u(x_{k+1}) - u(x_k)\right] \leq \frac{L}{4} v(\Delta_n) \left[u(b) - u(a)\right]. \tag{55}
\]
The proof is similar to Corollary 2 and details are omitted.

**Remark 8** If we choose $\xi_k = \frac{1}{2} (\lambda_k + \lambda_{k+1})$, we recapture Proposition 3 of Ref. 3. If we choose $\xi_k = (3\lambda_k + \lambda_{k+1})/4$, we have $|E_n(f, u, \Delta_n)| \leq \frac{1}{3} L v(\Delta_n) \left[u(b) - u(a)\right]$, which is more precise than Proposition 3 of Ref. 3.
APPLICATIONS FOR FUNCTIONS OF UNITARY OPERATORS

In Ref. 3, the author used inequality (3) to give estimates of a unitary operator. In this section, we apply our previous inequality (7) to give estimates of unitary operators $U$ defined on complex Hilbert spaces.

We recall here some basic facts on unitary operators and spectral families. We say that the bounded linear operator $U : H \to H$ on the Hilbert space $H$ is unitary if $U^* = U^{-1}$.

It is well known that if $U$ is a unitary operator, there exists a family of projections $\{E_\lambda\}_{\lambda \in [0, 2\pi]}$, called the spectral family of $U$ with the following properties:

(i) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;
(ii) $E_0 = 0$ and $E_{2\pi} = 1 _H$ (the identity operator on $H$);
(iii) $E_{\lambda + 0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;
(iv) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Furthermore, if $\{F_\lambda\}_{\lambda \in [0, 2\pi]}$ is a family of projections satisfying these requirements for the operator $U$, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$. Also, for every continuous complex valued function $f : C(0, 1) \to \mathbb{C}$ on the complex unit circle $C(0, 1)$, we have

$$f(U) = \int_0^{2\pi} f(e^{i\lambda}) \, dE_\lambda,$$

where the integral is taken in the Riemann-Stieltjes sense. In particular, we have

$$\langle f(U)x, y \rangle = \int_0^{2\pi} f(e^{i\lambda}) \, d\langle E_\lambda x, y \rangle$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} |f(e^{i\lambda})|^2 \, d\|E_\lambda x\|^2$$

$$= \int_0^{2\pi} |f(e^{i\lambda})|^2 \, d\langle E_\lambda x, x \rangle,$$

for any vector $x, y \in H$. We consider the following partition of the interval $[0, 2\pi]$:

$$\Delta_n : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = 2\pi$$

and the intermediate points $\xi_k \in [\lambda_k, \frac{1}{2}(\lambda_k + \lambda_{k+1})]$, where $0 \leq k \leq n-1$. We define $h_k := \lambda_{k+1} - \lambda_k$, $0 \leq k \leq n-1$ and the norm of the partition $\Delta_n$ is $\nu(\Delta_n) = \max\{h_k : 0 \leq k \leq n-1\}$.

If $U$ is a unitary operator on the Hilbert space $H$ and $\{E_\lambda\}_{\lambda \in [0, 2\pi]}$ is the spectral family of $U$, then

$$O_n(f, U, \Delta_n, \xi; x, y) := \sum_{k=0}^{n-1} f(e^{i\xi_k}) \left( E_{(\lambda_k + \lambda_{k+1})/2} - E_{\lambda_k} \right) x, y$$

$$+ \sum_{k=0}^{n-1} f(e^{i(\lambda_{k+1} + \xi_k)}) \left( E_{(\lambda_{k+1} + \lambda_k)/2} - E_{\lambda_k} \right) x, y$$

(59)

and

$$T_n(f, U, \Delta_n; x, y) := \sum_{k=0}^{n-1} f(e_{(\lambda_k + \lambda_{k+1})/4}) \left( E_{(\lambda_k + \lambda_{k+1})/2} - E_{\lambda_k} \right) x, y$$

$$+ \sum_{k=0}^{n-1} f(e^{i(\lambda_{k+1} + \lambda_k)/4}) \left( E_{(\lambda_k + \lambda_{k+1})/2} - E_{\lambda_k} \right) x, y,$$

(60)

where $x, y \in H$.

**Theorem 5** With the above assumptions for $U$, $\{E_\lambda\}_{\lambda \in [0, 2\pi]}$, $\Delta_n$ with $\nu(\Delta_n) \leq \pi$ and if $f : C(0, 1) \to \mathbb{C}$ satisfies Hölder’s type condition (6), then we have the representation

$$\langle f(U)x, y \rangle = O_n(f, U, \Delta_n, \xi; x, y)$$

$$+ R_n(f, U, \Delta_n, \xi; x, y)$$

(61)

with the error $R_n(f, U, \Delta_n, \xi; x, y)$ which satisfies the bounds

$$|R_n(f, U, \Delta_n, \xi; x, y)| \leq 2' H \sum_{k=0}^{n-1} \sin \left[ \frac{\lambda_{k+1} - \lambda_k}{8} + \frac{1}{2} \left| \xi_k - \frac{3\lambda_k + \lambda_{k+1}}{4} \right| \nu^{\lambda_{k+1}}(\|E_{\lambda_k}\|, x, y) \right]$$

$$\leq \frac{1}{2'} H \sum_{k=0}^{n-1} (\lambda_{k+1} - \lambda_k) \nu^{\lambda_{k+1}}(\|E_{\lambda_k}\|, x, y)$$

$$\leq \frac{H}{2'} \nu(\Delta_n) \|x\| \|y\|,$$

(62)

for any $x, y \in H$ and the intermediate points $\xi_k \in [\lambda_k, \frac{1}{2}(\lambda_k + \lambda_{k+1})]$, where $0 \leq k \leq n-1$. In particular, we have

$$\langle f(U)x, y \rangle = T_n(f, U, \Delta_n; x, y) + E_n(f, U, \Delta_n; x, y),$$

(63)
with the error
\[ |E_n(f, U, \Delta_n; x, y)| \]
\[ \leq 2^r H \sum_{k=0}^{n-1} \sin^r \left( \frac{\lambda_{k+1} - \lambda_k}{8} \right) V_{\lambda_k}^r \left( (E_n(x, y)) \right) \]
\[ \leq H \frac{2^r}{4^r} v^r(\Delta_n) ||x|| ||y||, \quad (64) \]
for any vector \( x, y \in H \).

**Proof:** For any \( x, y \in H \), we define \( u(\lambda) := (E_n x, y), \lambda \in [0, 2\pi] \). We know that \( u \) is of bounded variation, and from Ref. 3,
\[ V_{\lambda}^r(u) = V_{\lambda}^r((E_n(x, y))) \leq ||x|| ||y||. \quad (65) \]

By (7) and (65), (62) can be proved. \( \square \)

**Remark 9** If we choose \( \xi_k = \frac{1}{2}(\lambda_k + \lambda_{k+1}) \), we recapture Theorem 6 of Ref. 3. If we choose \( \xi_k = (3\lambda_k + \lambda_{k+1})/4 \), we have \( |E_n(f, U, \Delta_n; x, y)| \leq (H/2)^r v^r(\Delta_n) ||x|| ||y|| \), which is more precise than Theorem 6 of Ref. 3.

**Remark 10** In the case when the partition reduces to the whole interval \([0, 2\pi]\), then making use of (7), for any \( s \in [0, \pi] \) and any vectors \( x, y \in H \), we have the bound
\[ |f(s)(E_n x, y) + f(-s)(1 - E_n)x, y)\]
\[ |f(U)x, y)| \]
\[ \leq 2^r H \sin \left( \frac{\pi}{4} + \frac{1}{2} s - \frac{\pi}{2} \right) V_{\lambda}^r((E_n(x, y))). \quad (66) \]

If we obtain \( s = \frac{1}{2} \pi \), we obtain the best inequality
\[ |f(i)(E_n x, y) + f(-i)(1 - E_n)x, y)\]
\[ |f(U)x, y)| \]
\[ \leq 2^{-2} HV_{\lambda}^r((E_n(x, y))) \leq 2^{-2} H ||x|| ||y||, \quad (67) \]
for any vectors \( x, y \in H \).

If \( U \) is a unitary operator on the Hilbert space \( H \) and \( \{E_k\}_{k \in [0, 2\pi]} \) is the spectral family of \( U \). Depending only one vector \( x \in H \), we can introduce the following sums
\[ \tilde{O}_n(f, U, \Delta_n; \xi; x) \]
\[ := \sum_{k=0}^{n-1} f(e^{\xi_k}) \left( E_{\lambda_k+\lambda_{k+1}/2} - E_{\lambda_k} \right) x, x \]
\[ + \sum_{k=0}^{n-1} f(e^{\lambda_k+\lambda_{k+1}-\xi_k}) \left( E_{\lambda_{k+1}} - E_{\lambda_k+\lambda_{k+1}/2} \right) x, x. \]
\[ (68) \]

**Theorem 6** If \( f : C(0, 1) \to \mathbb{C} \) is Lipschitzian with the constant \( L > 0 \) on the circle \( C(0, 1), \Delta_n \) with \( \nu(\Delta_n) \leq \pi \) and \( U, \{E_{\lambda} \}_{\lambda \in [0, 2\pi]} \) are defined above, then we have the representation
\[ \tilde{O}_n(f, U, \Delta_n; \xi; x) \]
\[ \leq 2L \sum_{k=0}^{n-1} \frac{1}{4} \left| \sin \left( \frac{3\lambda_k + \lambda_{k+1}}{4} \right) \right| \left( (E_{\lambda_{k+1}} - E_{\lambda_k}) x, x \right) \]
\[ \leq \frac{\nu(\Delta_n)}{2} L ||x||^2, \quad (70) \]
for any vectors \( x \in H \) and the intermediate points \( \xi_k \in [\lambda_k, \lambda_{k+1} \lambda_k + \lambda_{k+1}] \), where \( 0 \leq k \leq n-1 \). In particular, we have
\[ E_n(f, U, \Delta_n; x) \]
\[ \leq \frac{1}{4} L \nu(\Delta_n)||x||^2, \quad (72) \]
for any \( x \in H \).

**Proof:** The proof follows from Proposition 3 applied for the monotonic nondecreasing function \( u(t) := \langle E_t x, x \rangle, t \in [0, 2\pi] \). \( \square \)

**Remark 11** If we choose \( \xi_k = \frac{1}{2}(\lambda_k + \lambda_{k+1}) \), we recapture Theorem 7 of Ref. 3. If we choose \( \xi_k = (3\lambda_k + \lambda_{k+1})/4 \), we have \( |\tilde{O}_n(f, U, \Delta_n; x)| \leq \frac{1}{2} L \nu(\Delta_n)||x||^2 \), which is more precise than Theorem 7 of Ref. 3.

**Remark 12** In the case when the partition reduces to the whole interval \([0, 2\pi]\), then by the above result, we obtain
\[ |f(e^{i}(E_n x, x) + f(e^{(2\pi-i)})(1 - E_n)x, x)| \]
\[ -(f(U)x, x)| \]
\[ \leq \frac{1}{4} L \nu(\Delta_n)||x||^2, \quad (71) \]
\[
\begin{align*}
&\leq L \int_0^\pi \text{sgn}(s-t) \cos \left(\frac{s-t}{2}\right) \langle E_x, x \rangle \, dt \\
&\quad + L \int_\pi^{2\pi} \text{sgn}(2\pi-s-t) \cos \left(\frac{2\pi-s-t}{2}\right) \langle E_x, x \rangle \, dt \\
&\quad + 2L \sin \left(\frac{s}{2}\right) \|x\|^2, \quad (73)
\end{align*}
\]
for any \( s \in [0, \pi] \) and \( x \in H \).

**Example 1** We choose two complex functions as follows to provide some simple examples for the inequalities above.

(a) Consider the power function \( f : C \setminus \{0\} \to C, f(z) = z^m \) where \( m \) is a non-zero integer. Then, clearly, for any \( z, w \) belonging to the unit circle \( C(0, 1) \), we have the inequality
\[
|f(z) - f(w)| \leq |m| |z - w|,
\]
which shows that \( f \) is Lipschitzian with the constant \( L = |m| \) on the circle \( C(0, 1) \). Then from (63), for any unitary operator \( U \), we obtain
\[
\left| e^{im\pi} \langle E_x, y \rangle + e^{i(2\pi - s)} \langle (1_H - E_\pi)x, y \rangle - \langle U^m x, y \rangle \right| \\
\quad \leq 2|m| \sin \left[ \frac{\pi}{4} + \frac{1}{2} |s - \frac{\pi}{2}| \right] V_0^{2\pi}(\langle E_\pi x, y \rangle), \quad (74)
\]
for any vectors \( x, y \in H \), where \( \{E_\lambda\}_{\lambda \in [0,2\pi]} \) is the spectral family of \( U \). If we obtain \( s = \frac{1}{2} \pi \), then the best inequality is
\[
\left| |m| \langle E_\pi x, y \rangle + \langle (1_H - E_\pi)x, y \rangle - \langle U^m x, y \rangle \right| \\
\quad \leq \sqrt{2}|m| V_0^{2\pi}(\langle E_\pi x, y \rangle) \leq \sqrt{2}|m| \|x\||y\|, \quad (75)
\]
for any vectors \( x, y \in H \).

(b) For \( a \neq \pm 1, 0 \), consider the function \( f : C(0, 1) \to C, f_a(z) = 1/(1-az) \). From Ref. 3, we have
\[
|f_a(z) - f_a(w)| \leq \frac{|a|}{(1-|a|)^2} |z - w|, \quad (76)
\]
for any \( z, w \in C(0, 1) \), showing that the function \( f_a \) is Lipschitzian with the constant \( L_a = |a|/(1-|a|)^2 \) on the circle \( C(0, 1) \). Then from (63), for any unitary operator \( U \), we obtain
\[
\left| (1 - ae^{i\pi})^{-1} \langle E_\pi x, y \rangle - \langle (1_H - aU)^{-1} x, y \rangle \right| \\
\quad + \left| (1 - ae^{i(2\pi - s)})^{-1} \langle (1_H - E_\pi)x, y \rangle \right| \\
\quad \leq \frac{2|a|}{(1-|a|)^2} \sin \left[ \frac{\pi}{4} + \frac{1}{2} |s - \frac{\pi}{2}| \right] V_0^{2\pi}(\langle E_\pi x, y \rangle), \quad (77)
\]
for any vectors \( x, y \in H \), where \( \{E_\lambda\}_{\lambda \in [0,2\pi]} \) is the spectral family of \( U \). If we obtain \( s = \frac{1}{2} \pi \), then the best inequality is
\[
\left| (1-ai)^{-1} \langle E_\pi x, y \rangle - \langle (1_H - aU)^{-1} x, y \rangle \right| \\
\quad + \left| (1+ai)^{-1} \langle (1_H - E_\pi)x, y \rangle \right| \\
\quad \leq \frac{\sqrt{2}|a|}{(1-|a|)^2} V_0^{2\pi}(\langle E_\pi x, y \rangle) \leq \frac{\sqrt{2}|a|}{(1-|a|)^2} \|x\||y\|, \quad (78)
\]
for any vectors \( x, y \in H \).

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