Estimates in homogenization of higher-order elliptic operators

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ABSTRACT
A divergent-type elliptic operator $A^\varepsilon$ of arbitrary even order $2m$ is studied. Coefficients of the operator are $\varepsilon$-periodic, $\varepsilon > 0$ is a small parameter. The resolvent equation $A^\varepsilon u^\varepsilon + \lambda u^\varepsilon = f$ is solvable in the Sobolev space $H^m(\mathbb{R}^d)$ of order $m$ for any $f \in L^2(\mathbb{R}^d)$, provided the parameter $\lambda$ is sufficiently large, $\lambda > \Lambda$, where the bound $\Lambda$ depends only on constants from ellipticity condition. The limit equation is of the same type but with constant coefficients, that is, $\hat{A}u + \lambda u = f$. The limit operator $\hat{A}$ can be considered here, for instance, in the sense of $G$-convergence. We prove that the resolvent $(\hat{A} + \lambda)^{-1}$ approximates $(A^\varepsilon + \lambda)^{-1}$ in operator $(L^2 \to L^2)$-norm with the estimate $\| (A^\varepsilon + \lambda)^{-1} - (\hat{A} + \lambda)^{-1} \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(\varepsilon)$, as $\varepsilon \to 0$. We find also the approximation of the resolvent $(A^\varepsilon + \lambda)^{-1}$ in operator $(L^2 \to H^m)$-norm. This is the sum $(\hat{A} + \lambda)^{-1} + K^\varepsilon$, where $K^\varepsilon$ is a correcting operator whose structure is given. We prove the estimate $\| (A^\varepsilon + \lambda)^{-1} - (\hat{A} + \lambda)^{-1} - K^\varepsilon \|_{L^2(\mathbb{R}^d) \to H^m(\mathbb{R}^d)} = O(\varepsilon)$, as $\varepsilon \to 0$.

1. Introduction

The theory of $G$-convergence of differential operators and connected with it the theory of multidimensional homogenization have been studied from long ago since 60th, see e.g. [1,2]. Firstly, there was investigated the class $E(\lambda_0, \lambda_1)$ of second-order elliptic operators of the form

$$A = \sum_{i,j=1}^{d} D_i(a_{ij}D_j) \quad (a_{ij} = a_{ji}),$$

where $a_{ij}(x)$ are measurable functions in a bounded domain $\Omega \subset \mathbb{R}^d$ subject to the inequality

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \lambda_1|\xi|^2 \quad (x \in \Omega, \xi \in \mathbb{R}^d),$$

$\lambda_0, \lambda_1$ are positive constants. The sequence of operators $A_k \in E(\lambda_0, \lambda_1), \ k \in \mathbb{N}$, is called $G$-convergent to the operator $A \in E(\lambda_0, \lambda_1)$, as $k \to \infty$, if for any $f \in L^2(\Omega)$ the sequence of solutions $u_k$ of the Dirichlet problem $A_ku_k = f$ ($u_k \in H^1_0(\Omega)$) converges in $L^2(\Omega)$ to the solution $u$ of the Dirichlet problem $Au = f$ ($u \in H^1_0(\Omega)$). In [1], there was proved that the class $E(\lambda_0, \lambda_1)$ is compact in the sense of $G$-convergence.

In [3], there were introduced classes of divergent-type elliptic operators of arbitrary order $2m \geq 2$ for which the $G$-compactness was proved. Numerous properties of $G$-convergence in these classes
were established, among them, the construction receipt for \( G \)-limiting operator of the family of higher order differential operators having \( \varepsilon \)-periodic coefficients when the small parameter \( \varepsilon \) is tending to zero. The latter result relates to the homogenization theory and may be amplified further. Now, we are interested in the rate of convergence of the solution to initial nonhomogeneous problem to the solution of homogenized problem going with all these, for the time being, not in a bounded domain but in the whole space \( \mathbb{R}^d \). We prove that the rate of this convergence is of order \( \varepsilon \). We assume minimal regularity conditions on the data of the problem. Thereby, the result may acquire the form of operator-type convergence of resolvents in terms of operator norms with corresponding estimate on the rate of convergence. This kind estimate for operators in the class \( E(\lambda_0, \lambda_1) \) defined above is known and was first proved in [4]. To obtain the estimate in homogenization of higher order differential operators, we use the approach proposed in [5,6], certainly, necessarily modified. Formerly, this approach was applied only to second-order differential equations, though of different types, among them, equations with various properties of degeneracy, with multi-scale coefficients or quasiperiodic coefficients, non-linear and non-divergent equations, system of elasticity theory equations and parabolic equations [7–14].

As one of the distinctive features of our problem, we mention also a presence of lower terms combined with a lack of any symmetry in coefficients matrix. This allows to take along divergent-type higher order differential operators of general form. Maybe, for even second-order elliptic operators with non-symmetric coefficients matrix with lower terms, the obtained result is interesting.

Certainly, for applications, it is important, above all, to prove operator-type estimates in homogenization of boundary problems in bounded domains. According to the method for derivation of such estimates, given in [5,6], one should start from resolvent equations in the whole space. Approximations constructed at the first stage to satisfy merely the equation are then adjusted to the boundary conditions by means of additional correctors which are of boundary layer nature. So, this paper may be regarded as a preparatory one for future derivation of homogenization estimates in problems with boundary conditions which are of the main interest for us.

2. Main results

2.1. Higher-order elliptic equations. Solvability problem

We begin with some preliminary material.

Denote by \( H^m = H^m(\mathbb{R}^d), m \geq 0 \) is integer, the Sobolev space equipped with a norm defined by the equality

\[
\| u \|_{H^m}^2 = \int_{\mathbb{R}^d} \sum_{|\alpha| \leq m} |D^\alpha u|^2 \, dx,
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_d) \) is a multi-index with non-negative integer components \( \alpha_i \),

\[
|\alpha| = \sum_{i=1}^{d} \alpha_i, \quad D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}, \quad D_i = \frac{\partial}{\partial x_i},
\]

\( H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d) \). It is known that the set \( C_0^\infty(\mathbb{R}^d) \) is dense in \( H^m(\mathbb{R}^d) \). Let

\[
\| u \|_j = \left( \int_{\mathbb{R}^d} \sum_{|\alpha| = j} |D^\alpha u|^2 \, dx \right)^{1/2}
\]

for integer \( j \geq 0 \), then the expression \( (\| u \|_m^2 + \| u \|_0^2)^{1/2} \) defines the equivalent norm in \( H^m(\mathbb{R}^d) \), \( m \geq 1 \).
Denote by $H^{-m} = H^{-m}(\mathbb{R}^d)$, $m \in \mathbb{N}$, the space dual to $H^m(\mathbb{R}^d)$. We have a triple of spaces

$$H^m(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset H^{-m}(\mathbb{R}^d)$$

(2.1)

for each integer $m > 0$.

Throughout this paper, $C_0^\infty(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$, $H^m(\mathbb{R}^d)$ are considered as spaces of real-valued functions.

Consider linear differential operators of the form

$$A = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta)$$

(2.2)

with bounded measurable real-valued coefficients $a_{\alpha\beta}(x)$.

**Definition 2.1:** A matrix $\{a_{\alpha\beta}(x)\} |\alpha|,|\beta| \leq m$ is called elliptic if

$$\|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^d)} \leq \lambda_1, \quad |\alpha|, |\beta| \leq m,$$

(2.3)

and

$$\int_{\mathbb{R}^d} \sum_{|\alpha|,|\beta| = m} a_{\alpha\beta}(x) D^\beta u D^\alpha u \, dx \geq \lambda_0 \int_{\mathbb{R}^d} \sum_{|\alpha| = m} |D^\alpha u|^2 \, dx \quad \forall u \in C_0^\infty(\mathbb{R}^d)$$

(2.4)

for some constants $\lambda_1, \lambda_0 > 0$.

It is rather known that the necessary condition for (2.4) is the following algebraic inequality

$$\sum_{|\alpha| = m} a_{\alpha\beta}(x) \xi^\beta \xi^\alpha \geq \lambda_0 \sum_{|\alpha| = m} |\xi^\alpha|^2 \quad \forall \xi \in \mathbb{R}^d, \quad \xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d},$$

(2.5)

which holds for a.e. $x \in \mathbb{R}^d$. In the case of the constant matrix $\{a_{\alpha\beta}\}$, the coerciveness inequality of the form (2.4) is equivalent to (2.5). The latter can be readily shown due to the Plancherel identity via Fourier transform images.

The differential expression of the form (2.2), corresponding to an elliptic matrix $\{a_{\alpha\beta}(x)\}$, defines a bounded linear operator $A : H^m \to H^{-m}$ in the following way. Given $u \in H^m$ and $f \in H^{-m}$, we say that $Au = f$ if the action of $f$ is described as

$$\langle f, \varphi \rangle = \sum_{|\alpha| \leq m, |\beta| \leq m} (a_{\alpha\beta} D^\beta u, D^\alpha \varphi), \quad \varphi \in C_0^\infty(\mathbb{R}^d),$$

where $(u, \varphi) = (u, \varphi)_{L^2(\mathbb{R}^d)}$ and $\langle f, \varphi \rangle$ being the value of $f \in H^{-m}$ on the element $\varphi \in H^m$. Clearly, $\|A\| \leq \lambda_1$. Moreover, the operator $A : H^m \to H^{-m}$ is lower semibounded and satisfies the inequality

$$\langle Au, u \rangle \geq \lambda_0/2 \|u\|^2_m - \lambda_2 \|u\|^2_0 \quad \forall u \in C_0^\infty(\mathbb{R}^d),$$

(2.6)

where $\lambda_2 \geq 0$ is a constant depending on $\lambda_0$ and $\lambda_1$ from the property ellipticity. In fact,

$$\langle Au, u \rangle = \sum_{|\alpha| = |\beta| = m} (a_{\alpha\beta} D^\beta u, D^\alpha u) + \sum_{|\alpha| + |\beta| < 2m} (a_{\alpha\beta} D^\beta u, D^\alpha u) \geq \lambda_0 \|u\|^2_m - \lambda_1 \left( \delta \|u\|^2_m + C_\delta \sum_{j=0}^{m-1} \|u\|^2_j \right) \quad \forall \delta > 0,$$

where in its turn

$$\|u\|^2_j \leq \tau \|u\|^2_m + C_\tau \|u\|^2_0, \quad 1 \leq j \leq m - 1, \quad \forall \tau > 0.$$
Hence, (2.6) is easily obtained if $\delta$ and $\tau$ are chosen sufficiently small. From (2.6), we derive that the operator $A + \lambda I : H^m \to H^{-m}, \lambda \geq \lambda_2 + 1$, is coercive and satisfies the inequality

$$\langle (A + \lambda I)u, u \rangle \geq \lambda_0/2\|u\|^2_m + \|u\|^2_0,$$

for any $u \in C^\infty_0(\mathbb{R}^d)$. In terms of the standard norm in $H^m$, it means

$$\langle (A + \lambda I)u, u \rangle \geq \tilde{\lambda}_0\|u\|^2_{H^m}, \quad \tilde{\lambda}_0 = \text{const}(\lambda_0).$$

Consequently, we come to the following conclusion.

**Lemma 2.2:** Let $A$ be an operator of form (2.2) where coefficients matrix is elliptic with constants $\lambda_1, \lambda_0$. Then for any $f \in H^{-m}$ and sufficiently large positive $\lambda$, the equation

$$(A + \lambda I)u = f,$$  \hfill (2.7)

has the unique solution $u \in H^m$, and

$$\|u\|_{H^m} = \|(A + \lambda I)^{-1}f\|_{H^m} \leq C\|f\|_{H^{-m}}, \quad C = \text{const}(\lambda_1, \lambda_0).$$  \hfill (2.8)

In other words, the elliptic differential operator (2.2) determines an isomorphism $(A + \lambda I) : H^m \to H^{-m}$ for each $\lambda \geq \Lambda$, and any functional $f \in H^{-m}$ admits the unique representation $f = (A + \lambda I)u$, where $u \in H^m$.

This statement is based on the following result for abstract operators (see [3], Theorem 1 in Chapter I) which is often called Lax–Milgram Theorem.

**Proposition:** Let $L : V \to V'$ be a continuous linear operator such that $\langle Lv, v \rangle \geq \tilde{\lambda}\|v\|^2_\nu$ for any $v \in V$. Then the equation $Lv = f \ (v \in V)$ is uniquely solvable for any $f \in V'$, and

$$\|v\|_V = \|L^{-1}f\|_V \leq \tilde{\lambda}^{-1}\|f\|_{V'}.$$  \hfill (2.9)

In accordance with (2.8) and due to the embedding (2.1), the resolvent $(A + \lambda I)^{-1}$ can be considered as an operator in $L^2(\mathbb{R}^d)$. Evidently,

$$\|(A + \lambda I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{\lambda}^{-1}, \quad \lambda \geq \lambda_2.$$

### 2.2. Equations with periodic highly oscillating coefficients. Operator-type estimates of homogenization

Consider a family of differential operators depending on a small parameter $\varepsilon > 0$

$$A^\varepsilon = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a^\varepsilon_{\alpha\beta}(x) D^\beta),$$

$$a^\varepsilon_{\alpha\beta}(x) = a_{\alpha\beta}(x/\varepsilon),$$  \hfill (2.9)

the corresponding matrix $\{a_{\alpha\beta}(x)\}$ is elliptic (in the sense of Definition 2.1) and is 1-periodic in each variable $y_1, \ldots, y_d$, $Y = [-\frac{1}{2}, \frac{1}{2})^d$ is the periodicity cell.
It is clear that the matrix \( \{ a^e_{\alpha \beta} (x) \} \) is elliptic (in the sense of Definition 2.1) for any \( \varepsilon \in (0, 1) \). In particular, the uniform in \( \varepsilon \) coerciveness estimate holds
\[
\int_{\mathbb{R}^d} \sum_{|\alpha|=|\beta|=m} a^e_{\alpha \beta} (x) D^\alpha u D^\beta u \, dx \geq \lambda_0 \int_{\mathbb{R}^d} \sum_{|\alpha|=m} |D^\alpha u|^2 \, dx \quad \forall u \in C_0^\infty (\mathbb{R}^d). \tag{2.10}
\]

By closure, this estimate is true for any \( u \in H^m (\mathbb{R}^d) \).

According to the theory of G-convergence of elliptic operators, developed in [3], there is a strong G-convergence of \( A^e \) to the limit operator \( \hat{A} \),
\[
A^e \xrightarrow{G} \hat{A}, \quad \hat{A} = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (\hat{a}_{\alpha \beta} D^\beta), \tag{2.11}
\]
\( \hat{A} \) is of the form (2.2) with a constant elliptic matrix \( \{ \hat{a}_{\alpha \beta} \} \). The way how to find the limit matrix \( \{ \hat{a}_{\alpha \beta} \} \) is described later in Section 3 (see (3.5) and (3.2)). For the definition of the strong G-convergence, used in (2.11), and also for its properties, see [3]. We shall extend this result in another direction, or maybe, look at the limit operator from another side, in the sense of somehow stronger convergence.

Due to the above arguments (see Lemma 2.2), since both operators \( A^e \) and \( \hat{A} \) are elliptic, the resolvents \( (A^e + \lambda I)^{-1} \) and \( (\hat{A} + \lambda I)^{-1} \) exist for sufficiently large \( \lambda \), \( \lambda \geq \Lambda (\lambda_0, \lambda_1) \), where \( \lambda_0, \lambda_1 \) are the constants from the ellipticity condition. As operators in \( L^2 (\mathbb{R}^d) \), these resolvents turn to be close to each other in the operator norm, and the degree of closeness is of order \( \varepsilon \).

**Theorem 2.3:** Under the above assumption of ellipticity, there holds the estimate
\[
\| (A^e + \lambda I)^{-1} - (\hat{A} + \lambda I)^{-1} \|_{L^2 (\mathbb{R}^d) \rightarrow L^2 (\mathbb{R}^d)} \leq c_0 \varepsilon, \quad c_0 = \text{const} (\lambda_0, \lambda_1), \quad \lambda \geq \Lambda (\lambda_0, \lambda_1). \tag{2.12}
\]

In [15], the estimate (2.12) was proved for self-adjoint operators (2.9) without lower order terms. To this end, the spectral approach from [4] was used.

To give a simple interpretation of (2.12), consider equations
\[
u^e \in H^m (\mathbb{R}^d), \quad A^e u^e + \lambda u^e = f, \tag{2.13}
\]
\[
u \in H^m (\mathbb{R}^d), \quad \hat{A} u + \lambda u = f, \tag{2.14}
\]
for an arbitrary \( f \in L^2 (\mathbb{R}^d) \). The operator estimate (2.12) means that
\[
\| u^e - u \|_{L^2 (\mathbb{R}^d)} \leq \varepsilon c_0 \| f \|_{L^2 (\mathbb{R}^d)}, \quad c_0 = \text{const} (\lambda_0, \lambda_1), \tag{2.15}
\]
provided \( \lambda \) is sufficiently large.

One can consider the resolvent \( (A^e + \lambda I)^{-1} \) as operator \( L^2 (\mathbb{R}^d) \rightarrow H^m (\mathbb{R}^d) \). Then its approximation in the operator \( (L^2 \rightarrow H^m) \)-norm should be taken as a sum of \( (\hat{A} + \lambda I)^{-1} \) and the correcting operator \( \mathcal{K}^e \) whose structure is easily restored through the formula (6.12). Moreover, there holds the estimate
\[
\| (A^e + \lambda I)^{-1} - (\hat{A} + \lambda I)^{-1} - \mathcal{K}^e \|_{L^2 (\mathbb{R}^d) \rightarrow H^m (\mathbb{R}^d)} \leq c_0 \varepsilon, \quad c_0 = \text{const} (\lambda_0, \lambda_1). \tag{2.16}
\]
Certainly, the constant \( c_0 \) in (2.12) and (2.16) depends also on the dimension \( d \) and the order \( m \), but this will not be mentioned anymore.

The direct proof of the above estimates is given in Sections 5 and 6. Some necessary preliminaries are carried away to Sections 3 and 4.

**Addition under corrections.** After this paper was submitted, there appeared results on the close topic [16] related to matrix strongly elliptic self-adjoint operators.
3. Cell problems

3.1. Solvability of cell problems

On the set of periodic functions $u \in C^\infty_{\text{per}}(Y)$ with zero mean, that is
\[ \langle u \rangle = \int_Y u(y) \, dy = 0, \]
the expression
\[ \left( \int_Y \sum_{|\alpha|=m} |D^\alpha u|^2 \, dy \right)^{1/2} \]
yields the norm. We denote by $W$ the completion of this set under the norm in question.

The estimate (2.4) for the periodic matrix $\{a_{\alpha\beta}(x)\}$ leads to the following inequality for periodic functions
\[ \int_Y \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(y) D^\beta u D^\alpha u \, dy \geq \lambda_0 \int_Y \sum_{|\alpha|=m} |D^\alpha u|^2 \, dy \quad \forall u \in C^\infty_{\text{per}}(Y), \quad (3.1) \]
which is verified a little bit later. By closure, (3.1) holds for any $u \in W$ and means that
\[ A^0 = \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{\alpha\beta}(y) D^\beta) \]
is a coercive operator $W \to W'$.

For each multi-index $\gamma$ with $|\gamma| \leq m$, consider the equation
\[ N_\gamma \in W, \quad \sum_{|\alpha|=|\beta|=m} D^\alpha (a_{\alpha\beta}(y) D^\beta N_\gamma(y)) = - \sum_{|\alpha|=m} D^\alpha (a_{\alpha\gamma}(y)), \quad (3.2) \]
where $\{a_{\alpha\beta}(y)\}$ is the periodic matrix from (2.9). The right-hand side in (3.2) determines in a natural way a linear functional $F_\gamma$ on $W$, and the equation itself can be written in the operator form $A^0 N_\gamma = F_\gamma$ ($N_\gamma \in W$). Therefore, the unique solvability of (3.2) follows from Proposition given after Lemma 2.2.

Return to the assertion which is basic for this section.

Lemma 3.1: Suppose that $\{a_{\alpha\beta}(x)\}$ is an elliptic periodic matrix. Then the estimate (2.4) entails the inequality (3.1).

Proof: Substitute in (2.10) the finite function
\[ \psi_\varepsilon(x) = \varepsilon^m v(x/\varepsilon) \varphi(x), \quad v \in C^\infty_{\text{per}}(Y), \quad \varphi \in C^\infty_0(\mathbb{R}^d), \]
such that
\[ D^\alpha \psi_\varepsilon(x) = (\partial^\alpha v)^\varepsilon(x) \varphi(x) + O(\varepsilon), \quad (\partial^\alpha v)^\varepsilon(x) = (D^\alpha v(y))|_{y=x/\varepsilon}, \quad |\alpha| = m. \quad (3.3) \]
Here and hereafter, $\partial^\alpha = D^\alpha_y$. Obviously, we obtain
\[ \int_{\mathbb{R}^d} \sum_{|\alpha|=|\beta|=m} a^\varepsilon_{\alpha\beta}(x)(\partial^\beta v)^\varepsilon(x)(\partial^\alpha v)^\varepsilon(x)|\varphi(x)|^2 \, dx \geq \lambda_0 \int_{\mathbb{R}^d} \sum_{|\alpha|=m} |(\partial^\alpha v)^\varepsilon(x)|^2 |\varphi(x)|^2 \, dx + O(\varepsilon), \]
that after the passage to the limit, as $\varepsilon \to 0$, gives
\[
\left\langle \sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}\partial^\beta v\partial^\alpha v \right\rangle \int_{\mathbb{R}^d} |\varphi(x)|^2 \, dx \geq \lambda_0 \left\langle \sum_{|\alpha|=m} |\partial^\alpha v|^2 \right\rangle \int_{\mathbb{R}^d} |\varphi(x)|^2 \, dx
\]
and, finally,
\[
\left\langle \sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}\partial^\beta v\partial^\alpha v \right\rangle \geq \lambda_0 \left\langle \sum_{|\alpha|=m} |\partial^\alpha v|^2 \right\rangle
\]
which is exactly the inequality (3.1). Above, the so-called mean value property of periodic functions is used:
\[
\text{if } b \in L^1_{\text{per}}(Y), b^\varepsilon(x) = b(y)|_{y=x/\varepsilon}, \psi \in C^\infty_0(\mathbb{R}^d), \text{ then }
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} b^\varepsilon(x)\psi(x) \, dx = \langle b \rangle \int_{\mathbb{R}^d} \psi(x) \, dx, \quad \langle b \rangle = \int_Y b(y) \, dy. \tag{3.4}
\]

3.2. Homogenized operator and its properties

The coefficients of the operator $\hat{A}$ (see (2.11)) are defined with the help of the solutions of cell problems (3.2),
\[
\hat{a}_{\alpha\beta} = \left( a_{\alpha\beta}(y) + \sum_{|\gamma|=m} a_{\alpha\gamma}(y)D^\gamma N_\beta(y) \right), \quad |\alpha| \leq m, |\beta| \leq m. \tag{3.5}
\]
Introducing the symbol $e_{\alpha\beta}$ with multi-indices $\alpha$, $\beta$ such that
\[
e_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise,} \end{cases}
\]
we rewrite
\[
\hat{a}_{\alpha\beta} = \left( \sum_{|\gamma|=m} a_{\alpha\gamma}(y)(e_{\beta\gamma} + D^\gamma N_\beta(y)) \right), \tag{3.6}
\]
or
\[
\hat{a}_{\alpha\beta} = \langle \tilde{a}_{\alpha\beta} \rangle, \quad \tilde{a}_{\alpha\beta}(y) = \sum_{|\gamma|=m} a_{\alpha\gamma}(y)(e_{\beta\gamma} + D^\gamma N_\beta(y)). \tag{3.7}
\]

**Lemma 3.2:** The matrix $\{\hat{a}_{\alpha\beta}\}$, defined by relations (3.5), (3.2), is elliptic.

**Proof:** Evidently, we need to verify for $\{\hat{a}_{\alpha\beta}\}$ only the coerciveness property from the definition of elliptic matrices, that is,
\[
\int_{\mathbb{R}^d} \sum_{|\alpha|=|\beta|=m} \hat{a}_{\alpha\beta} D^\alpha w D^\beta w \, dx \geq \lambda_0 \int_{\mathbb{R}^d} \sum_{|\alpha|=m} |D^\alpha w|^2 \, dx \quad \forall w \in C^\infty_0(\mathbb{R}^d). \tag{3.8}
\]

To this end, substituting in (2.10), the function
\[
u_\varepsilon(x) = w(x) + \varepsilon^m \sum_{|\gamma|=m} N_{\gamma}^e(x)D^\gamma w(x), \quad w \in C^\infty_0(\mathbb{R}^d), \quad N_{\gamma}^e(x) = N_{\gamma}(y)|_{y=x/\varepsilon}, \tag{3.9}
\]
where \( N_\gamma(y) \) is the solution of the cell problem (3.2), we obtain
\[
\int_{\mathbb{R}^d} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^\varepsilon(x) D^\beta u_\varepsilon D^\alpha u_\varepsilon \, dx \geq \lambda_0 \int_{\mathbb{R}^d} \sum_{|\alpha|=m} |D^\alpha u_\varepsilon|^2 \, dx \quad \forall w \in C_0^\infty(\mathbb{R}^d)
\] (3.10)
and pass here to the limit, as \( \varepsilon \to 0. \)

First, calculate
\[
\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^\varepsilon D^\beta u_\varepsilon \overset{(3.9)}{=} \sum_{|\beta|=m} a_{\alpha\beta}^\varepsilon (D^\beta w + \sum_{|\gamma|=m} (\partial^\beta N_\gamma)^\varepsilon D^\gamma w) + r_\alpha^\varepsilon
\]
\[
= \sum_{|\beta|=m} \sum_{|\gamma|=m} a_{\alpha\gamma}^\varepsilon (e_{\beta\gamma} + (\partial^\gamma N_\beta)^\varepsilon) D^\gamma w + r_\alpha^\varepsilon
\]
\[
D^\alpha u_\varepsilon \overset{(3.9)}{=} D^\alpha w + \sum_{|\delta|=m} (\partial^\delta N_\delta)^\varepsilon D^\delta w + \tilde{r}_\alpha^\varepsilon
\]
\[
= \sum_{|\delta|=m} (e_{\alpha\delta} + (\partial^\alpha N_\delta)^\varepsilon) D^\delta w + \tilde{r}_\alpha^\varepsilon, \quad |\alpha| = m,
\]
where the notation from (3.3) is used and \( r_\alpha^\varepsilon, \tilde{r}_\alpha^\varepsilon \) denote terms of order \( O(\varepsilon) \). Hence,
\[
\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^\varepsilon D^\beta u_\varepsilon D^\alpha u_\varepsilon
\]
\[
= \sum_{|\beta|=|\delta|=m} \left[ \sum_{|\alpha|=|\gamma|=m} a_{\alpha\gamma}^\varepsilon (e_{\beta\gamma} + (\partial^\gamma N_\beta)^\varepsilon) (e_{\alpha\delta} + (\partial^\alpha N_\delta)^\varepsilon) \right] D^\beta wD^\delta w + R^\varepsilon, \quad R^\varepsilon = O(\varepsilon).
\]

By the mean value property of periodic functions, (see (3.4))
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^\varepsilon D^\beta u_\varepsilon D^\alpha u_\varepsilon \, dx
\]
\[
= \int_{\mathbb{R}^d} \sum_{|\beta|=|\delta|=m} \left( \sum_{|\alpha|=|\gamma|=m} a_{\alpha\gamma}^\varepsilon (e_{\beta\gamma} + D^\gamma N_\beta)(e_{\alpha\delta} + D^\alpha N_\delta) \right) D^\beta wD^\delta w \, dx
\]
\[
= \int_{\mathbb{R}^d} \sum_{|\beta|=|\delta|=m} \hat{a}_{\alpha\beta} D^\beta wD^\delta w \, dx,
\] (3.11)
where the mean over \( Y \) in (3.11) is reduced to \( \hat{a}_{\alpha\beta} \) using the definition of homogenized coefficients and the equality
\[
\left( \sum_{|\alpha|=|\gamma|=m} a_{\alpha\gamma}^\varepsilon (e_{\beta\gamma} + D^\gamma N_\beta)D^\alpha N_\delta) \right) = 0,
\]
which stems from (3.2).

Again with the help of the mean value property, we can show the weak convergence \( u_\varepsilon \to w \) in \( H^m(\mathbb{R}^d) \). Here, the structure of \( u_\varepsilon \) is essential (see (3.9)). So, there is the lower semi-continuity property of the norm
\[
\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^d} \sum_{|\alpha|=m} |D^\alpha u_\varepsilon|^2 \, dx \geq \int_{\mathbb{R}^d} \sum_{|\alpha|=m} |D^\alpha w|^2 \, dx.
\] (3.12)

From (3.10) to (3.12), it follows that (3.8) is true. The lemma is proved. \( \square \)
Since \( \hat{A} \) is elliptic and with constant coefficients, we conclude that the solution \( u \) of the homogenized equation (2.14) really belongs to \( H^{2m}(\mathbb{R}^d) \) and the following estimate holds

\[
\|u\|_{H^{2m}(\mathbb{R}^d)} \leq C\|f\|_{L^2_{\text{loc}}(\mathbb{R}^d)}, \quad C = \text{const}(\lambda_0, \lambda_1).
\]

(3.13)

4. On representation of solenoidal vectors

It is known that periodic solenoidal vectors with zero mean admit representation via divergence of a periodic skew-symmetric matrix. More precisely, for any vector \( g \in L^2_{\text{per}}(Y) \), such that \( \text{div} \, g = 0 \) and \( \langle g \rangle = 0 \), there exists a skew symmetric matrix \( G \in H^1_{\text{per}}(Y) \) such that \( \text{div} \, G = g \) and \( \|G\|_{H^1_{\text{per}}(Y)} \leq c\|g\|_{H^1_{\text{per}}(Y)} \), \( c = \text{const}(d) \) (see the proof in [17], Chapter I, §1). Here, the relation

\[
\text{div} \, g = \sum_{i=1}^{d} D_i g_i = 0
\]

for the vector \( g = \{g_i\} \in L^2_{\text{per}}(Y)^d \) means that

\[
\langle g \cdot \nabla \varphi \rangle = 0 \quad \forall \varphi \in C^\infty_{\text{per}}(Y).
\]

(4.2)

The following lemma extends the above assertion to the situation when instead of operator of gradient \( \nabla : H^{2m}_{\text{per}}(Y) \to L^2_{\text{per}}(Y)^d \) and its adjoint operator (that is the operator of divergence \( \text{div} \)), one consider a pair of their analogs adequate to differential equations of order \( 2m \). That is, first, the operator of gradient of order \( m \), acting from \( H^m_{\text{per}}(Y) \) to vector-valued space \( L^2_{\text{per}}(Y)^p \), \( p \) being the number of multi-indices \( \alpha \) with \( |\alpha| = m \), and, second, its adjoint operator which may be called “divergence of order \( m \)”. Certainly, there appear “solenoidal vectors” corresponding to this type of divergence (see below (4.3)), for which analogs of (4.1) and (4.2) are fulfilled.

Lemma 4.1: Let \( \{g_\alpha\}_{|\alpha|=m} \) be a 1-periodic vector from \( L^2(Y)^p \) such that

\[
\langle g_\alpha \rangle = 0, \quad \sum_{|\alpha|=m} D^\alpha g_\alpha = 0.
\]

(4.3)

Then there is a 1-periodic matrix \( \{G_{\alpha \beta}\}_{|\alpha|=|\beta|=m} \) from \( H^m(Y)^p \times p \) such that for any \( \alpha, \beta \)

\[
G_{\alpha \beta} = -G_{\beta \alpha}, \quad \|G_{\alpha \beta}\|_{H^m(Y)} \leq c \sum_{|\alpha|=m} \|g_\alpha\|_{L^2(Y)}, \quad c = \text{const}(d, m),
\]

\[
\sum_{|\gamma|=m} D^\gamma G_{\alpha \gamma} = g_\alpha.
\]

(4.6)

Proof: Each component \( g_\alpha \) admits Fourier decomposition

\[
g_\alpha(y) = \sum_{0 \neq n \in \mathbb{Z}^d} g^n_\alpha e^{2\pi i n \cdot y}, \quad i = \sqrt{-1},
\]

and there holds Parseval’s identity

\[
\|g_\alpha\|_{L^2(Y)}^2 = \sum_{n \in \mathbb{Z}^d} |g^n_\alpha|^2.
\]
With the Equation (4.3)\textsubscript{2} implies that
\[
\sum_{|\alpha|=m} n^\alpha g^n_\alpha = 0, \quad 0 \neq n \in \mathbb{Z}^d.
\] (4.7)

Define \(G_{\alpha\beta}\) via Fourier decomposition with coefficients
\[
G^n_{\alpha\beta} = \left(-g^n_\beta n^\alpha + g^n_\alpha n^\beta\right)\Lambda_m(n)^{-1} (2\pi i)^{-m}, \quad 0 \neq n \in \mathbb{Z}^d,
\]
\[
\Lambda_m(n) = \sum_{|\gamma|=m} n^\gamma n^\gamma.
\]

By construction, condition (4.4) is evidently fulfilled and
\[
\sum_n |G^n_{\alpha\beta}|^2 \leq C \sum_{n \in \mathbb{Z}^d} |g^n_\alpha|^2, \quad |\gamma| = m,
\]
thereby, the estimate (4.5) holds true. Moreover, for each \(0 \neq n \in \mathbb{Z}^d\),
\[
eq -\Lambda_m(n)^{-1} n^\alpha \sum_{|\beta|=m} g^n_\beta n^\beta + \Lambda_m(n)^{-1} g^n_\alpha \sum_{|\beta|=m} n^\beta n^\beta = g^n_\alpha, \quad (4.9)
\]
whence the property (4.6) follows. The lemma is proved. \(\square\)

We give here another representation lemma which is rather common in homogenization.

**Lemma 4.2:** Let \(g\) be a 1-periodic scalar function from \(L^2(Y)\) such that \(\langle g \rangle = 0\). Then there is a 1-periodic vector \(G\) from \(H^m(Y)^d\) such that \(g = \text{div} \ G\) and
\[
\|G\|_{H^1(Y)} \leq c \|g\|_{L^2(Y)}.
\]

**Proof:** The required representation is obtained if we set \(G = \nabla U\) where \(U\) is a solution of the following periodic problem with laplacian \(\Delta = \text{div} \nabla\)
\[
U \in H^2(Y), \quad \Delta U = g.
\]
The solvability of this problem is readily shown using Fourier series. Further details are omitted. \(\square\)

### 5. Discrepancy of the first approximation

We are aimed to prove the estimate (2.15). It means that the solution \(u\) of the homogenized equation approximates the solution \(u^\varepsilon\) of the initial equation in \(L^2\)-norm. Therefore, the function \(u\) is called the zero approximation to keep distinct from the first approximation which approximates the solution \(u^\varepsilon\) in the Sobolev norm natural to the equation. In our case, this is \(H^m\)-norm. It is quite in common for homogenization theory to use approximations in Sobolev norms to obtain \(L^2\)-estimate for the difference \(u^\varepsilon - u\) as a corollary (see [17–19]). We shall do the same and try for the first approximation the function
\[
v^\varepsilon(x) = u(x) + \varepsilon^m \sum_{|\gamma| \leq m} N^\varepsilon_\gamma(x)D^\gamma u(x), \quad N^\varepsilon_\gamma(x) = N_\gamma(y)|_{y=x/\varepsilon}, \quad (5.1)
\]
which is a sum of the zero approximation and a corrector term. Here, \(N_\gamma(y)\) is a solution to the cell problem (3.2).
To facilitate further actions, assume at the first step that the right-hand side function in (2.13) and (2.14) is smooth and with compact support, that is, $f \in C_0^\infty(\mathbb{R}^d)$. In this case, the solution $u$ of the elliptic equation with constant coefficients is smooth and decays exponentially at infinity. As a result, $v^\varepsilon$ belongs to the space $H^m(\mathbb{R}^d)$.

Our goal is to evaluate a discrepancy of $v^\varepsilon$ to the Equation (2.13). To this end, we first compare the generalized gradients

$$
\Gamma_\alpha(v^\varepsilon, A^\varepsilon) = \sum_{|\beta| \leq m} a_{\alpha\beta}^\varepsilon(y) D^\beta v^\varepsilon, \quad \Gamma_\alpha(u, \hat{A}) = \sum_{|\beta| \leq m} \hat{a}_{\alpha\beta} D^\beta u, \quad |\alpha| \leq m,
$$

(5.2)

with each other. Easy calculations give

$$
\Gamma_\alpha(v^\varepsilon, A^\varepsilon) = \sum_{|\beta| \leq m} \sum_{|\gamma| = m} a_{\alpha\beta}^\varepsilon(y) D^\beta (u + \varepsilon^m \sum_{|\gamma| \leq m} N^\varepsilon \gamma D^\gamma u) = \sum_{|\beta| \leq m} \left( a_{\alpha\beta}^\varepsilon D^\beta u + \sum_{|\gamma| = m} a_{\alpha\gamma}^\varepsilon (\partial^\gamma N^\varepsilon_\beta) D^\beta u \right) + w_\alpha^\varepsilon,
$$

(5.3)

where $\partial^\gamma N^\varepsilon_\beta = D^\gamma N^\varepsilon_\beta(y)$. The term $w_\alpha^\varepsilon$ collects all the summands, in which there occur derivatives $\partial^\gamma N^\varepsilon_\beta$ with $|\gamma| < m$ and, thereby, there stand multipliers $\varepsilon^k$, $k \geq 1$.

Using the notation from (3.6) to (3.7), we rewrite

$$
\Gamma_\alpha(v^\varepsilon, A^\varepsilon) = \sum_{|\beta| \leq m} \sum_{|\gamma| = m} a_{\alpha\gamma}^\varepsilon (\varepsilon \beta) D^\beta u + w_\alpha^\varepsilon = \sum_{|\beta| \leq m} \hat{a}_{\alpha\beta}^\varepsilon D^\beta u + w_\alpha^\varepsilon
$$

(5.2)

$$
= \sum_{|\beta| \leq m} (\hat{a}_{\alpha\beta}^\varepsilon - \hat{a}_{\alpha\beta}) D^\beta u + \sum_{|\beta| \leq m} \hat{a}_{\alpha\beta} D^\beta u + w_\alpha^\varepsilon
$$

(5.2)

$$
= \sum_{|\beta| \leq m} (\hat{a}_{\alpha\beta}^\varepsilon - \hat{a}_{\alpha\beta}) D^\beta u + \Gamma_\alpha(u, \hat{A}) + w_\alpha^\varepsilon.
$$

Finally,

$$
\Gamma_\alpha(v^\varepsilon, A^\varepsilon) = \Gamma_\alpha(u, \hat{A}) + \sum_{|\beta| \leq m} g_{\alpha\beta}^\varepsilon D^\beta u + w_\alpha^\varepsilon,
$$

(5.4)

$$
g_{\alpha\beta}(y) = \hat{a}_{\alpha\beta}(y) - \hat{a}_{\alpha\beta}.
$$

Now, we transform $g_{\alpha\beta}(y)$ in an appropriate way differently for $\alpha$ with $|\alpha| = m$ and $|\alpha| < m$.

For fixed $\beta$, the vector $\{g_{\alpha\beta}\}_{|\alpha| = m}$ satisfies the conditions of Lemma 4.1. Therefore, there exists the matrix $\{G_{\alpha\gamma\beta}\}_{|\alpha| = |\gamma| = m}$ such that

$$
g_{\alpha\beta}(y) = \sum_{|\gamma| = m} \partial^\gamma G_{\alpha\gamma\beta}(y), \quad G_{\alpha\gamma\beta}(y) = -G_{\gamma\alpha\beta}(y),
$$

and the $H^m$-estimate of the form (4.5) is valid for $G_{\alpha\gamma\beta}(y)$. Hence,

$$
g_{\alpha\beta}^\varepsilon D^\beta u = \sum_{|\gamma| = m} D^\gamma (\varepsilon^m G_{\alpha\gamma\beta}) D^\beta u = \sum_{|\gamma| = m} D^\gamma (\varepsilon^m G_{\alpha\gamma\beta} D^\beta u + w_{\alpha\beta}^\varepsilon), \quad |\alpha| = m,
$$

(5.5)
where \( w_{\alpha \beta}^e \) collects terms in which there occur derivatives \( D^\gamma (\varepsilon^m G_{\alpha \gamma}^e), |\gamma| < m \), and which, thereby, contain multipliers \( \varepsilon^k, k \geq 1 \).

As for the coefficients \( g_{\alpha \beta}^e, |\alpha| < m, |\beta| \leq m \), from (5.4), we apply Lemma 4.2 to them. So, there exist 1-periodic vectors \( \tilde{g}_{\alpha \beta} \in H^1_{\text{per}}(Y)^d \) such that

\[
g_{\alpha \beta}(y) = \text{div}_y \tilde{g}_{\alpha \beta}(y), \quad g_{\alpha \beta}^e(x) = \text{div}_x (\varepsilon \tilde{g}_{\alpha \beta}^e(x)).
\]

Thus,

\[
g_{\alpha \beta}^e D^\beta u = \text{div}_x (\varepsilon \tilde{g}_{\alpha \beta}^e(x)) D^\beta u = \text{div}_x (\varepsilon \tilde{g}_{\alpha \beta}^e(x) D^\beta u) + \tilde{w}_{\alpha \beta}^e,
\]

\[
\tilde{w}_{\alpha \beta}^e = -\varepsilon \tilde{g}_{\alpha \beta}^e \cdot \nabla D^\beta u. \tag{5.6}
\]

Since \( f = (\hat{A} + \lambda) u \), we deduce

\[
(A^e + \lambda) v^e - f = (A^e + \lambda) v^e - (\hat{A} + \lambda) u = (A^e v^e - \hat{A} u) + \lambda (v^e - u) \tag{5.7}
\]

\[
= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (\Gamma_{\alpha} (v^e, A^e) - \Gamma_{\alpha} (u, \hat{A})) + \lambda (v^e - u) \tag{5.8}
\]

\[
= \sum_{|\alpha| = m} (-1)^{|\alpha|} D^\alpha \left[ \sum_{|\beta| \leq m} g_{\alpha \beta}^e D^\beta u + w_{\alpha \beta}^e \right] + \lambda (v^e - u) \tag{5.9}
\]

\[
= \sum_{|\alpha| = m} (-1)^m \sum_{|\beta| \leq m} \sum_{|\alpha| = |\gamma| = m} D^\alpha D^\gamma (\varepsilon^m G_{\alpha \gamma \beta}^e D^\beta u) + \cdots \tag{5.10}
\]

Here, the last dots stand for the terms containing explicit multipliers \( \varepsilon^k, k \geq 1 \), they appear from expressions \( v^e - u, w_{\alpha \beta}^e, w_{\alpha \beta}^e, \tilde{w}_{\alpha \beta}^e \) (see (5.1), (5.4)–(5.6)).

Being outwardly the most complicated, the first sum in (5.8) presents, actually, the zero functional. In fact, for any fixed \( \beta \), we have

\[
F_{\beta}^e = \sum_{|\alpha| = |\gamma| = m} D^\alpha D^\gamma (\varepsilon^m G_{\alpha \gamma \beta}^e D^\beta u) \in H^{-m}(\mathbb{R}^d) \tag{5.11}
\]

due to the sufficient regularity of the functions \( G_{\alpha \gamma \beta}^e \) and \( D^\beta u \). Moreover,

\[
(F_{\beta}^e, \varphi) = \sum_{|\alpha| = |\gamma| = m} (\varepsilon^m G_{\alpha \gamma \beta}^e D^\beta u, D^\alpha D^\gamma \varphi) = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \tag{5.12}
\]

thanks to the symmetry properties of matrices \( \{G_{\alpha \gamma \beta}^e\}_{\alpha \gamma} \) and \( \{D^\alpha D^\gamma \varphi\}_{\alpha \gamma} \).

Since \( f = (A^e + \lambda) u^e \) and thus,

\[
(A^e + \lambda) v^e - f = (A^e + \lambda) v^e - (A^e + \lambda) u^e = A^e (v^e - u^e) + \lambda (v^e - u^e),
\]
we derive from (5.7) to (5.10) the equation
\[ A^\varepsilon z^\varepsilon + \lambda z^\varepsilon = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_{\alpha}^\varepsilon, \]
\[ z^\varepsilon = v^\varepsilon - u^\varepsilon. \] (5.11)

Here, functions \( f_{\alpha}^\varepsilon \) have the structure of products
\[ \varepsilon^k U(x) b(x/\varepsilon), \quad k \geq 1, \quad U(x) = D^\gamma u(x), \quad |\gamma| \leq 2m, \] (5.12)
and for 1-periodic functions \( b(y) \), there stand the following expressions
\[ \partial^\gamma N_{\beta}(y), \quad \partial^\beta G_{\alpha\beta\gamma}(y), \quad \tilde{g}_{\alpha\beta}(y), \quad |\alpha| \leq m, |\beta| \leq m, |\gamma| \leq m, |\delta| \leq m, \] (5.13)
from the former transformations. By construction, all functions \( b(y) \) belong to \( L^2(Y) \). According to the estimate of the form (2.8), the solution of (5.11) satisfies the inequality
\[ \|z^\varepsilon\|_{H^m(\mathbb{R}^d)} \leq C \sum_{|\alpha| \leq m} \|f_{\alpha}^\varepsilon\|_{L^2(\mathbb{R}^d)}. \] (5.14)

Here, the majorant is obviously of order \( \varepsilon \) (see (5.12)), but it cannot be replaced with the expression \( \varepsilon c_0 \|f\|_{L^2(\mathbb{R}^d)}, c_0 = \text{const}(\lambda_0, \lambda_1) \), desired in (2.15). It would be possible if \( b \in L^\infty(Y) \), that is not true under our assumptions in general. In the sequel, we show how to overcome this difficulty by introducing an additional parameter of integration.

6. Estimate averaged over shifting and its corollaries

6.1. Shifting method. Integrated estimate

Consider a family of perturbated problems
\[ u_{\omega}^\varepsilon \in H^m(\mathbb{R}^d), \quad A_{\omega}^\varepsilon u_{\omega}^\varepsilon + u_{\omega}^\varepsilon = f(x), \quad f \in L^2(\mathbb{R}^d), \]
\[ A_{\omega}^\varepsilon = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x/\varepsilon + \omega) D^\beta), \] (6.1)

with shifting parameter \( \omega \in Y \) in coefficients of the operator. Clearly, (2.14) is the corresponding homogenized problem for each \( \omega \in Y \), the cell problems (see (3.2)) contain shifting parameter \( \omega \) in coefficients, and thereby the first approximation for \( u_{\omega}^\varepsilon \) is of the form (see (5.1))
\[ v_{\omega}^\varepsilon(x) = u(x) + \varepsilon^m \sum_{|\gamma| \leq m} N_{\gamma}(y + \omega) D^\gamma u(x), \quad y = x/\varepsilon, \] (6.2)

There holds the corresponding estimate of the form (5.14) with right-hand side functions defined in (5.12), (5.13). Namely,
\[ \|v_{\omega}^\varepsilon - u_{\omega}^\varepsilon\|_{H^m(\mathbb{R}^d)}^2 \leq c\varepsilon^2 \sum_{\alpha} \int_{\mathbb{R}^d} |b_{\alpha}(x/\varepsilon + \omega)| \frac{|U_{\alpha}(x)|^2}{\varepsilon^2} dx. \]
Integrating in $\omega \in Y$ leads to
\[
\int_Y \|v^\varepsilon - u^\varepsilon_{\omega}\|_{H^m(\mathbb{R}^d)}^2 \, d\omega \leq c\varepsilon^2 \sum_\alpha \int_Y \int_{\mathbb{R}^d} |b_\alpha \left(\frac{x}{\varepsilon} + \omega\right)|^2 |U_\alpha(x)|^2 \, dx \, d\omega \
\leq c\varepsilon^2 \sum_\alpha \|b_\alpha\|_{L^2(Y)}^2 \|U_\alpha\|_{L^2(\mathbb{R}^d)}^2 \leq c_0 \varepsilon^2 \|f\|_{L^2(\mathbb{R}^d)}^2.
\]

Here, at the first step, the order of integration is changed, after which we can extract from the integral over $\mathbb{R}^d L^2$-norm of oscillating functions $b_\alpha$, more exactly, the expression
\[
\int_Y |b_\alpha \left(\frac{x}{\varepsilon} + \omega\right)|^2 \, d\omega = \|b_\alpha\|_{L^2(Y)}^2.
\]

Then, we apply the estimate (3.13) and the estimate
\[
\|b\|_{L^2(Y)} \leq c, \ c = \text{const}(\lambda_0, \lambda_1).
\]

The latter is enabled by properties of functions (5.13).

Thus, the following lemma is proved.

**Lemma 6.1:** Let $u^\varepsilon_{\omega}$ be a solution of (6.1) and let $v^\varepsilon_{\omega}$ be a function from (6.2). Then there holds an integrated (in $\omega \in Y$) estimate
\[
\int_Y \int_{\mathbb{R}^d} \left( \sum_{1 \leq |\alpha| \leq m} |D^\alpha (u^\varepsilon_{\omega} - v^\varepsilon_{\omega})|^2 + |u^\varepsilon_{\omega} - v^\varepsilon_{\omega}|^2 \right) \, dx \, d\omega \leq c_0 \varepsilon^2 \|f\|_{L^2(\mathbb{R}^d)}^2, \quad c_0 = \text{const}(\lambda_0, \lambda_1).
\]

(6.3)

It is useful to have another version of integrated (in $\omega \in Y$) estimate. Take the solution $u^\varepsilon(x)$ of the problem (2.13) and consider a family of shifted functions $\tilde{u}^\varepsilon_{\omega}(x) = u^\varepsilon(x + \varepsilon \omega)$. They satisfy the Equation (6.1) with the shifted right-hand side function $f(x + \varepsilon \omega)$,
\[
A^\varepsilon_{\omega} \tilde{u}^\varepsilon_{\omega} + \tilde{u}^\varepsilon_{\omega} = f(x + \varepsilon \omega).
\]
Then
\[
A^\varepsilon_{\omega} (u^\varepsilon_{\omega} - \tilde{u}^\varepsilon_{\omega}) + u^\varepsilon_{\omega} - \tilde{u}^\varepsilon_{\omega} = f(x) - f(x + \varepsilon \omega).
\]
(6.4)

By properties of shifting,
\[
\left( \int_{\mathbb{R}^d} (f(x) - f(x + \varepsilon \omega)) \varphi(x) \, dx \right)^2 \leq \|f\|_{L^2(\mathbb{R}^d)}^2 \int_{\mathbb{R}^d} |\varphi(x) - \varphi(x - \varepsilon \omega)|^2 \, dx 
\leq c\varepsilon \|f\|_{L^2(\mathbb{R}^d)}^2 \|\nabla \varphi\|_{L^2(\mathbb{R}^d)}^2, \quad c = \text{const}(d),
\]
and therefore, from (6.4), it readily follows that
\[
\|u^\varepsilon_{\omega} - \tilde{u}^\varepsilon_{\omega}\|_{H^m(\mathbb{R}^d)} \leq \varepsilon C \|f\|_{L^2(\mathbb{R}^d)}.
\]

Thus, $u^\varepsilon_{\omega}$ may be replaced with $\tilde{u}^\varepsilon_{\omega}$ in (6.3) without detriment to the right-hand side of (6.3). Namely,
\[
J := \int_Y \int_{\mathbb{R}^d} \sum_{1 \leq |\alpha| \leq m} |D^\alpha (\tilde{u}^\varepsilon_{\omega} - v^\varepsilon_{\omega})|^2 \, dx \, d\omega + \int_Y \int_{\mathbb{R}^d} |\tilde{u}^\varepsilon_{\omega} - v^\varepsilon_{\omega}|^2 \, dx \, d\omega \leq c\varepsilon \|f\|_{L^2(\mathbb{R}^d)}^2,
\]
(6.5)

where $c = \text{const}(\lambda_0, \lambda_1)$. 

6.2. Derivation of the main homogenization estimates

Now, we are going to derive some corollaries from (6.5).

Discarding the first integral in (6.5) and changing the order of integration in the remaining one, we deduce, by convexity,

$$
\int_{\mathbb{R}^d} |\langle \tilde{u}^\varepsilon_\omega - v^\varepsilon_\omega \rangle_\omega|^2 \, dx \leq c \varepsilon^2 \|f\|^2_{L^2(\mathbb{R}^d)},
$$

(6.6)

where $\langle \cdot \rangle_\omega = \int_Y \cdot \, d\omega$. Clearly,

$$
\langle v^\varepsilon_\omega \rangle_\omega (x) = u(x), \quad \langle \tilde{u}^\varepsilon_\omega \rangle_\omega = \int_Y u^\varepsilon (x + \varepsilon \omega) \, d\omega = (S^\varepsilon u^\varepsilon)(x)
$$

is Steklov average of the function $u^\varepsilon(x)$. We recall the following property

$$
\|S^\varepsilon \varphi - \varphi\|_{L^2(\mathbb{R}^d)} \leq c \varepsilon \|\nabla \varphi\|_{L^2(\mathbb{R}^d)}, \quad c = \text{const}(d),
$$

(6.7)

for the Steklov average of the function $\varphi$ defined as

$$(S^\varepsilon \varphi)(x) = \int_Y \varphi(x + \varepsilon \omega) \, d\omega.
$$

Therefore, (6.6) means

$$
\|S^\varepsilon u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} \leq c \varepsilon \|f\|_{L^2(\mathbb{R}^d)},
$$

(6.8)

and, by triangle inequality,

$$
\|u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} \leq \|S^\varepsilon u^\varepsilon - S^\varepsilon u\|_{L^2(\mathbb{R}^d)} + \|S^\varepsilon u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} \leq c_0 \varepsilon \|f\|_{L^2(\mathbb{R}^d)}.
$$

Here, the property (6.7) of Steklov averaging is applied to $u^\varepsilon$ and, finally, the evident inequality $\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}$, arising from the energy estimate, is used. As a result, the estimate (2.12) is proved.

From (6.3), we derive similarly the inequality

$$
\sum_{1 \leq |\alpha| \leq m} \|D^\alpha S^\varepsilon u^\varepsilon - D^\alpha u\|_{L^2(\mathbb{R}^d)} \leq c \varepsilon \|f\|_{L^2(\mathbb{R}^d)},
$$

which together with (6.8) leads to

$$
\|S^\varepsilon u^\varepsilon - u\|_{H^m(\mathbb{R}^d)} \leq c_0 \varepsilon \|f\|_{L^2(\mathbb{R}^d)}, \quad c_0 = \text{const}(\lambda_0, \lambda_1).
$$

(6.9)

This $H^m$-estimate deserves attention because it does not contain any corrector. The proximity between $u^\varepsilon$ and $u$ in $H^m$-norm is achieved via Steklov smoothing only.
Now, transform the expression $J$ from (6.5) in another fashion. First, change the variable $x \to x' = x + \varepsilon \omega$, which leads to

$$
\tilde{u}_\omega^\varepsilon(x) = u^\varepsilon(x'),
$$

$$
\tilde{u}_\omega^\varepsilon(x) - \tilde{v}_\omega^\varepsilon(x) = u^\varepsilon(x') - u(x - \varepsilon \omega) - \varepsilon^m \sum_{|\gamma| \leq m} N_\gamma(x/\varepsilon) D^\gamma u(x' - \varepsilon \omega),
$$

where the dots keep back the difference of functions from the previous summand.

Hence, after changing the order of integration, we deduce by convexity, that

$$
J \geq \int_{\mathbb{R}^d} \left[ |z^\varepsilon|^2 + \sum_{1 \leq |\alpha| \leq m} |D^\alpha z^\varepsilon|^2 \right] dx,
$$

where

$$
z^\varepsilon(x) = u^\varepsilon(x) - (S^\varepsilon u)(x) = u^\varepsilon(x) - \sum_{|\gamma| \leq m} N_\gamma(x/\varepsilon) S^\varepsilon(D^\gamma u)(x).
$$

So, (6.5) and (6.10) imply the estimate

$$
\int_{\mathbb{R}^d} \left[ |z^\varepsilon|^2 + \sum_{1 \leq |\alpha| \leq m} |D^\alpha z^\varepsilon|^2 \right] dx \leq c \varepsilon^2 \|f\|_{L^2(\mathbb{R}^d)}^2.
$$

Here, without detriment to the right-hand side, one can replace in $z^\varepsilon$ (see (6.11)) the Steklov average $S^\varepsilon u$ with the function $u$ itself, having in mind the property (6.7) and the elliptic estimate for $u$. In such a way, there appears the first approximation with smoothed corrector (smoothing in Steklov sense)

$$
\hat{v}^\varepsilon(x) = u(x) + \varepsilon^m \sum_{|\gamma| \leq m} N_\gamma(x/\varepsilon) S^\varepsilon(D^\gamma u)(x),
$$

and the estimate

$$
\int_{\mathbb{R}^d} \left[ |u^\varepsilon - \hat{v}^\varepsilon|^2 + \sum_{1 \leq |\alpha| \leq m} |D^\alpha (u^\varepsilon - \hat{v}^\varepsilon)|^2 \right] dx \leq c \varepsilon^2 \|f\|_{L^2(\mathbb{R}^d)}^2.
$$

As a result, the following theorem about approximation in $H^m$-norm is proved.

**Theorem 6.2:** For the difference of the solution $u^\varepsilon$ to the problem (2.13) and the function $\hat{v}^\varepsilon$, defined in (6.12), there holds the estimate

$$
\|u^\varepsilon - \hat{v}^\varepsilon\|_{H^m(\mathbb{R}^d)} \leq c_0 \varepsilon \|f\|_{L^2(\mathbb{R}^d)}, \quad c_0 = \text{const}(\lambda_0, \lambda_1).
$$
Accordingly, (6.12) and (6.13) imply the operator-type estimate (2.16) with the correcting operator
\[ K^\varepsilon = \varepsilon^m \sum_{|\gamma| \leq m} N_\gamma(x/\varepsilon) S^\varepsilon D^\gamma (\hat{A} + \lambda)^{-1}. \]

**Remark:** In our method, properties of shifting and Steklov averaging (or smoothing) are essential. We omit here their proof, all the necessary proofs are given in [5,6]. In particular, by properties of Steklov average and due to the special structure, the function \( \hat{v}^\varepsilon \), defined in (6.12), belongs to the Sobolev space \( H^m(\mathbb{R}^d) \) which is, of course, necessary for the estimate (6.13). Note that we have gained this property of \( \hat{v}^\varepsilon \) automatically as a byproduct while deriving the estimate (6.13). For all that, we have also used some properties of Steklov smoothing. In general, under our minimal assumptions, when coefficients of the operator are only measurable, bounded functions and the right-hand side function \( f \) belongs to \( L^2(\mathbb{R}^d) \), the definition (6.12) without Steklov smoothing in it (that is exactly (5.1)) does not enable \( H^m \)-regularity of the approximation, at least, from the first sight. There are some particular cases when Steklov smoothing can be omitted in (6.12) and, thus, the estimate (6.13) is also true with the approximation \( v^\varepsilon \) from (5.1) instead of \( \hat{v}^\varepsilon \). The examples are given below without going into details, for the full justification may be not obvious and even cumbersome. Instead of detailing, we make reference to our papers, if possible.

**Example 1:** (general problem in dimension \( d = 1 \)). In one-dimensional case, the cell problems are solved explicitly, the cell functions, with all their derivatives up to order \( m \), are bounded; thereby the justification is easy.

**Example 2:** (general problem for the order \( 2m = 2 \)). Second-order operators of the form (2.9) can be treated by arguments considered for more particular case in [6].

**Example 3:** (operator with bilaplacian). Fourth-order operators of the form \( A^\varepsilon = \Delta a(x/\varepsilon) \Delta \), where \( \Delta \) is \( d \)-dimensional Laplacian and \( a(y) \) is a positive function from \( L^\infty_{\text{per}}(Y) \), produce a very peculiar cell problem which leads to rather simple in form the first approximation. This case is considered in [20].

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