Symmetries of Locally Rotationally Symmetric Models

M. Sharif *
Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus Lahore-54590, PAKISTAN.

Abstract

Matter collineations of locally rotationally symmetric spacetimes are considered. These are investigated when the energy-momentum tensor is degenerate. We know that the degenerate case provides infinite dimensional matter collineations in most of the cases. However, an interesting case arises where we obtain proper matter collineations. We also solve the constraint equations for a particular case to obtain some cosmological models.

Keywords: Matter symmetries, Locally rotationally symmetric spacetimes

*sharif@math.pu.edu.pk
1 Introduction

The purpose of this paper is to study matter collineations of locally rotationally symmetric (LRS) spacetimes. Throughout the paper $M$ will denote the usual smooth (connected, Hausdorff, 4-dimensional) spacetime manifold with smooth Lorentz metric $g$ of signature $(+, -, -, -)$. Thus $M$ is paracompact. A comma, semi-colon and the symbol $\mathcal{L}$ denote the usual partial, covariant and Lie derivative, respectively, the covariant derivative being with respect to the Levi-Civita connection on $M$ derived from $g$. The associated Ricci and stress-energy tensors will be denoted in component form by $R_{ab}(\equiv R^c_{bcd})$ and $T_{ab}$ respectively.

In recent years, much interest has been shown in the study of matter collineation (MCs) [1]-[9]. A vector field along which the Lie derivative of the energy-momentum tensor vanishes is called an MC, i.e.,

$$\mathcal{L}_\xi T_{ab} = 0,$$

where $\xi^a$ is the symmetry or collineation vector. The MC equations, in component form, can be written as

$$T_{ab,c} \xi^c + T_{ac} \xi^c_{,b} + T_{cb} \xi^c_{,a} = 0,$$

where the indices $a, b, c$ run from 0 to 3. Also, assuming the Einstein field equations, a vector $\xi^a$ generates an MC if $\mathcal{L}_\xi G_{ab} = 0$. It is obvious that the symmetries of the metric tensor (isometries) are also symmetries of the Einstein tensor $G_{ab}$, but this is not necessarily the case for the symmetries of the Ricci tensor (Ricci collineations) which are not, in general, symmetries of the Einstein tensor. In a very recent work, M. Tsamparlis et al. [9] have discussed Ricci and matter collineations of LRS metrics for the non-degenerate case only. Here we calculate MCs of hypersurface homogeneous spacetimes which are locally rotationally symmetric models for degenerate case and relate them with isometries.

Carot et al. [1] and Hall et al. [2] have noticed some important general results about the Lie algebra of MCs given in the following:

(i) The set of all MCs on $M$ is a vector space, but it may be infinite dimensional and may not be a Lie algebra. If $T_{ab}$ is degenerate, i.e., $\text{det}(T_{ab}) = 0$, then $\text{rank}(T_{ab}) < 4$ and we cannot guarantee the finite dimensionality of the MCs. If $T_{ab}$ is non-degenerate, i.e., $\text{det}(T_{ab}) \neq 0$, then $\text{rank}(T_{ab}) = 4$ and the Lie algebra of MCs is finite dimensional.
(ii) If the energy-momentum tensor is of rank 4 everywhere then it may be regarded as a metric on $M$. Then it follows by a standard result that the family of MCs is, in fact, a Lie algebra of smooth vector fields on $M$ of finite dimension $\leq 10$ (and $\neq 9$).

(iii) Assuming $f$ is a scalar function defined on $M$, then $\xi = fX$ is also an MC if and only if either $f$ is a constant on $M$ or $X$ satisfies $T_{ab}X^a = 0$, in which case $T_{ab}$ is necessarily degenerate and $X$ is an eigenvector of the energy-momentum tensor with eigenvalue $T/2$.

(iv) If a vector field $\xi$ on $M$ is a symmetry of all the gravitational field sources, then one could require $\mathcal{L}_{\xi}T_{ab} = 0$ (for the non-vacuum sources) and $\mathcal{L}_{\xi}C_{bcd}^a = 0$ (for the vacuum sources), where $C_{bcd}^a$ are Weyl curvature tensor components. This leads to a famous result given by Hall et al. [2]

**Theorem:** Let $M$ be a spacetime manifold. Then, generically, any vector field $\xi$ on $M$ which simultaneously satisfies $\mathcal{L}_{\xi}T_{ab} = 0$ ($\iff \mathcal{L}_{\xi}G_{ab} = 0$) and $\mathcal{L}_{\xi}C_{bcd}^a = 0$ is a homothetic vector field.

If $\xi^a$ is a Killing vector (KV) (or a homothetic vector), then $\mathcal{L}_{\xi}T_{ab} = 0$, thus every isometry is also an MC but the converse is not true, in general. Notice that collineations can be proper (non-trivial) or improper (trivial). Proper MC is defined to be an MC which is not a KV, or a homothetic vector.

The rest of the paper is organized as follows. In the next section, we write down MC equations for LRS spacetimes. In section 3, we shall solve these MC equations when the energy-momentum tensor is degenerate. In section 4, we evaluate MCs for Bianchi type V metric and finally, a summary of the results obtained will be presented.

## 2 Matter Collineation Equations

The locally rotationally symmetric spacetimes have many well known important families of exact solutions of Einstein field equations and are studied extensively [10]-[13]. They admit a group of motions $G_4$ acting multiply transitively on 3-dimensional non-null orbits spacelike ($S_3$) or timelike ($T_3$) and the isotropy group is a spatial rotation. The metrics of these models can be written in the forms [10,11]

$$ds^2 = \epsilon[-dt^2 + A^2(t)dx^2] - B^2(t)(dy^2 + \Sigma^2(y,k)dz^2),$$

$$ds^2 = \epsilon[-dt^2 + A^2(t)(dx - \Lambda(y,k)dz)^2] - B^2(t)(dy^2 + \Sigma^2(y,k)dz^2),$$

(3)
\[ ds^2 = \epsilon[-dt^2 + \Lambda^2(t)dx^2] - B^2(t)e^{2\tau}(dy^2 + dz^2), \]  

with \( \epsilon = \pm 1 \) and \( k = 0, \pm 1 \), where \( \Sigma(y, k) \) is \( \sin y \), \( y \), \( \sinh y \) and \( \Lambda(y, k) \) is \( \cos y \), \( y^2/2 \), \( \cosh y \) for \( k = +1, 0, -1 \) respectively. It is mentioned here that the value of \( \epsilon = \pm 1 \) differentiates between the static and non-static cases as can be seen by interchanging the coordinates \( t, x \). We restrict our attention to the non-static case \( (\epsilon = -1) \) as the results of the static case can be obtained consequently. The LRS metrics \( (\epsilon = -1) \) given by Eq.(3) turn out to be Bianchi types I (BI) or \( V/II_0 \) (BVII0) for \( k = 0 \), III (BIII) for \( k = -1 \) and Kantowski-Sachs (KS) for \( k = +1 \). The LRS metrics \( (\epsilon = -1) \) given by Eq.(4) become Bianchi types II (BII) for \( k = 0 \), VIII (BVIII) or III (BIII) for \( k = -1 \) and IX (BIX) for \( k = +1 \). The LRS spacetime \( (\epsilon = -1) \) given by Eq.(5) represents Bianchi type V (BV) or \( V/II_h \) (BVIIh) metric. A complete classification of Bianchi types I, III and Kantowski-Sachs spacetimes according to the nature of energy-momentum tensors [7] has already been given. Here we work out MCs of BII, BVIII, BIX, and BV spacetimes only for the degenerate case.

We write down the MC equations for Bianchi types II, VIII and IX spacetimes given by Eq.(4). The non-vanishing components of Ricci and energy-momentum tensors are given in Appendix A. Using these, we can write the MC Eqs.(2) as follows

\[ \dot{T}_0 \xi^0 + 2T_0 \xi^0_0 = 0, \]  
\[ \dot{T}_1 \xi^0 + 2T_1 (\xi^1_1 - \Lambda \xi^3_3) = 0, \]  
\[ \dot{T}_2 \xi^0 + 2T_2 \xi^2_2 = 0, \]  
\[ (\Lambda^2 \dot{T}_1 + \Sigma^2 \dot{T}_2) \xi^0 + 2(\Lambda \Lambda' T_1 + \Sigma \Sigma' T_2) \xi^2 - 2\Lambda T_1 (\xi^1_3 - \Lambda \xi^3_3) + 2\Sigma^2 T_2 \xi^3_3 = 0, \]  
\[ T_0 \xi^0_1 + T_1 (\xi^1_0 - \Lambda \xi^3_0) = 0, \]  
\[ T_0 \xi^0_2 + T_2 \xi^2_0 = 0, \]  
\[ T_0 \xi^0_3 - \Lambda T_1 (\xi^1_0 - \Lambda \xi^3_0) + \Sigma^2 T_2 \xi^3_0 = 0, \]  
\[ T_1 \xi^1_2 - \Lambda T_1 \xi^3_2 + T_2 \xi^2_1 = 0, \]  
\[ \Lambda \dot{T}_1 \xi^0 + \Lambda' T_1 \xi^2 - T_1 (\xi^1_3 - \Lambda \xi^3_3) + \Lambda T_1 (\xi^1_1 - \Lambda \xi^3_1) - \Sigma^2 T_2 \xi^3_1 = 0, \]  
\[ T_2 \xi^3_2 - \Lambda T_1 (\xi^1_2 - \Lambda \xi^3_2) + \Sigma^2 T_2 \xi^3_2 = 0, \]
where dot and prime denote differentiation with respect to time coordinate "t" and "x" respectively. Notice that we have used the notation $T_{aa} = T_a$.

We solve these equations when $\det(T_{ab}) = 0$. It should be mentioned here that we shall use $\sqrt{T_0}$ with $T_0 > 0$ in order to fulfill the dominant energy condition.

3 Matter Collineations in the Degenerate Case

In order to solve MC equations (6)-(15) when at least one component of $T_m = 0$, ($m = 0, 1, 2$), we can have the following two main cases:

(1) when only one of the $T_m \neq 0$,
(2) when exactly two of the $T_m \neq 0$.

It is mentioned here that the trivial case, where $T_m = 0$, shows that every vector field is an MC.

Case (1): This case can further be subdivided into three cases:

(1a) $T_0 \neq 0$, $T_j = 0$ ($j = 1, 2$),
(1b) $T_1 \neq 0$, $T_k = 0$ ($k = 0, 2$),
(1c) $T_2 \neq 0$, $T_l = 0$ ($l = 0, 1$).

The case (1a) is trivial and we get

$$\xi = \frac{c_0}{\sqrt{T_0}} \partial_t + \xi^i(x^a) \partial_i, \ (i = 1, 2, 3), \quad (16)$$

where $c_0$ is a constant.

When $T_1 = 0 = T_2$, using values of these components of the energy-momentum tensor from Eqs.(A2), it follows that

$$- 2 \frac{A^2 \ddot{B}}{B} - \frac{A^2 \dot{B}^2}{B^2} + 3 \frac{A^4}{4B^4} - k \frac{A^2}{B^2} = 0, \quad (k = 0, +1, -1), \quad (17)$$

$$- B \ddot{B} - \frac{\ddot{A}B^2}{A} - \frac{\dot{A} \dot{B} B}{A} - \frac{A^2}{4B^2} = 0. \quad (18)$$

These are second order non-linear differential equations in $A$ and $B$ and can only be solved by assuming some relationship between these two functions. If we assume that $B = cA$, where $c$ is an arbitrary constant, we obtain

$$2A\ddot{A} + \dot{A}^2 + \frac{4kc^2 - 3}{4c^4} = 0,$$

$$2A\ddot{A} + \dot{A}^2 + \frac{1}{4c^4} = 0. \quad (19)$$
The general solution of these equations can not be found analytically. To have some particular solution, we make an assumption that \( A \) be of the form
\[
A = (at + b)^n,
\]
where \( a, b, n \) are arbitrary constant. We find that only the possible solution is for \( n = 1 \). Thus we obtain the following solution
\[
\begin{align*}
 ds^2 &= dt^2 - (at + b)^2(dx - \Lambda(y, k)dz)^2 - c^2(at + b)^2(dy^2 + \Sigma^2(y, k)dz^2), \\
&= (20)
\end{align*}
\]
where \( 4a^2c^4 - 4kc^2 - 5 = 0 \). It can easily be verified that these metrics represent perfect fluid dust solutions. The energy density is given by
\[
\rho = \frac{3a^2 - 4kc^2 - 1}{4c^4(at + b)^2}.
\]

For the case 1(b), it follows from MC Eqs.(6)-(15) that either \( T_1 = \text{constant} \) or \( \xi^0 = 0 \). When \( T_1 = \text{constant} \), we get
\[
\begin{align*}
\xi^0 &= \xi^0(x^a), \quad \xi^1 = \Lambda \xi^3 + C(y, z), \\
\xi^2 &= \frac{1}{\Lambda} C_{,3}(y, z), \quad \xi^3 = -\frac{1}{\Lambda} C_{,2}(y, z), \\
&= (21)
\end{align*}
\]
where \( C \) is an integration function of \( y \) and \( z \). It can be checked that, in this case, energy density and pressure both vanish.

In the case of 1(c), solution of the MC equations will become
\[
\begin{align*}
\xi^0 &= 2 T_2 c_0 z \Sigma', \quad \xi^1 = \xi^1(x^a), \quad \xi^2 = -c_0 z \Sigma, \quad \xi^3 = c_0 \int \frac{dy}{\Sigma} + c_1, \\
&= (22)
\end{align*}
\]
where \( c_0 \) and \( c_1 \) are arbitrary constants. This case also gives both pressure and energy density zero for this model. We see that the case (1) give infinite number of MCs.

**Case (2):** This case implies the following three possibilities:

(2a) \( T_i \neq 0, \quad T_2 = 0 \),

(2b) \( T_j \neq 0, \quad T_0 = 0 \),

(2c) \( T_k \neq 0, \quad T_1 = 0 \).

The case 2(a) explores further two possibilities i.e. either \( T_1 = \text{constant} \neq 0 \) or \( \xi^0 = 0 \). For \( T_1 = \text{constant} \), solution of the MC equations yields
\[
\xi = \frac{c_0}{\sqrt{T_0}} \partial_t + \xi^1(z) \partial_x + \frac{1}{\Lambda} \xi^1 \partial_y.
\]
When \( T_2 = 0 \), using the same procedure as in the case 1(a), we find the same solution with the condition given as \( 4a^2c^4 + 1 = 0 \).
In the case 2(c), from MC Eqs.(6)-(15), in addition to the improper MCs \( \xi(1), \xi(2), \xi(3) \) given in Appendix A, we obtain the following proper MCs

\[
\begin{align*}
\xi(4) &= \xi^1(x^a)\partial_x, \\
\xi(5) &= \frac{\Sigma \sin z}{\sqrt{T_0}} \left[ \frac{T_2}{2kT_2} X_1 + \partial_t \right], \\
\xi(6) &= \frac{\Sigma \cos z}{\sqrt{T_0}} \left[ \frac{T_2}{2kT_2} X_2 + \partial_t \right], \\
\xi(7) &= \frac{\Sigma}{\sqrt{T_0}} \left[ -\frac{T_2}{2T_2} \partial_y + \frac{\Sigma'}{\Sigma} \partial_t \right],
\end{align*}
\]

in which the constraint equation is

\[
\frac{T_2}{\sqrt{T_0}} [ -\frac{T_2}{2T_2} \partial_y + \frac{\Sigma'}{\Sigma} \partial_t ] = -k.
\]

The values of \( X_1 \) and \( X_2 \) are given as

\[
\begin{align*}
X_1 &= \frac{\Sigma'}{\Sigma} \partial_y + \frac{\cot z \Sigma_2}{\Sigma_2} \partial_z, \\
X_2 &= \frac{\Sigma'}{\Sigma} \partial_y + \frac{\tan z \Sigma_2}{\Sigma_2} \partial_z,
\end{align*}
\]

Again we see that all the possibilities of the case (2) give infinite-dimensional MCs.

### 4 Matter Collineations of Bianchi Type V Metric

The LRS spacetime (\( \epsilon = -1 \)) given by Eq.(5) represents Bianchi type V metric. MC equations for this metric will become

\[
\begin{align*}
T_{0,0}\xi^0 + 2T_0\xi^0_{,0} &= 0, \\
T_{1,0}\xi^0 + 2T_1\xi^1_{,1} &= 0, \\
\xi^2_{,2} - \xi^3_{,3} &= 0, \quad (T_2 \neq 0), \\
T_{0,\xi^0_{,1}} + T_1\xi^1_{,0} &= 0,
\end{align*}
\]
\begin{align*}
T_0 \xi_0^0 + T_2 \xi_0^2 &= 0, \\
T_0 \xi_0^0 + T_2 \xi_0^3 &= 0, \\
T_1 \xi_1^1 + T_2 \xi_1^2 &= 0, \\
T_1 \xi_1^1 - T_2 \xi_1^3 &= 0, \\
\xi_3^2 + \xi_3^3 &= 0, \quad (T_2 \neq 0).
\end{align*}

Again we shall restrict ourselves only for the degenerate case. The non-vanishing components of Ricci and energy-momentum tensor are given in Appendix B. There arises two possibilities for this case according as (1) when one of the components of energy-momentum tensor is non-zero and (2) when two of the components are non-zero. As we have done previously, each of these two cases can further be divided into three subcases.

The case 1(a) yields the same solution as for the case 1(a) of the previous section. Using the Einstein field equations for the perfect fluid matter, we find that the model is pressureless (i.e. \( p = 0 \)) and energy density is given as

\[ \rho = 2 \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{1}{A^2}. \]

For the case (1b), it follows from MC equations that

\[ \xi = \frac{c_0}{\sqrt{T_1}} \partial_x + \xi^n (x^a) \partial_n, \quad (n = 0, 2, 3), \quad (36) \]

where \( c_0 \) is a constant.

In the case (1c), we obtain the following solution

\[ \xi = \xi^l (x^a) \partial_l + c_0 \partial_y + c_1 \partial_z. \quad (37) \]

It is obvious that all the possibilities of the case (1) give infinite dimensional MCs.

For the case 2(a), when we solve MC equations, we obtain the following constraint

\[ - \frac{T_1}{\sqrt{T_0}} \left( \frac{\dot{T}_1}{2T_1 \sqrt{T_0}} \right) = \alpha, \quad (38) \]

where \( \alpha \) is an arbitrary constant which can be positive, zero or negative.
When $\alpha > 0$, we have the following solution

$$\xi = c_0 \partial_x + c_1 \frac{1}{\sqrt{T_0}}(\cos \sqrt{\alpha x} \partial_t - \frac{\dot{T}_1}{2T_1\sqrt{\alpha}} \sin \sqrt{\alpha x} \partial_x)$$

$$+ c_2 \frac{1}{\sqrt{T_0}}(\sin \sqrt{\alpha x} \partial_t + \frac{\dot{T}_1}{2T_1\sqrt{\alpha}} \cos \sqrt{\alpha x} \partial_x)$$

$$+ \xi^2(x^a)\partial_y + \xi^3(x^a)\partial_z. \quad (39)$$

The case $\alpha = 0$ yields the solution

$$\xi = c_0 \partial_x + c_1 \frac{1}{\sqrt{T_0}}(x \partial_t - \frac{\dot{T}_1}{4T_1} x^2 + \sqrt{T_0} \int \frac{\sqrt{T_0}}{T_1} dt) \partial_x)$$

$$+ c_2 \frac{1}{\sqrt{T_0}}(\partial_t - \frac{\dot{T}_1}{2T_1} x \partial_x) + \xi^2(x^a)\partial_y + \xi^3(x^a)\partial_z. \quad (40)$$

For $\alpha < 0$, we obtain

$$\xi = c_0 \partial_x + c_1 \frac{1}{\sqrt{T_0}}(\cosh \sqrt{\alpha x} \partial_t - \frac{1}{2\sqrt{\alpha T_1}}(\dot{T}_1 + 4\alpha T_0) \sinh \sqrt{\alpha x} \partial_x)$$

$$+ c_2 \frac{1}{\sqrt{T_0}}(\sinh \sqrt{\alpha x} \partial_t - \frac{1}{2\sqrt{\alpha T_1}}(\dot{T}_1 + 4\alpha T_0) \cosh \sqrt{\alpha x} \partial_x)$$

$$+ \xi^2(x^a)\partial_y + \xi^3(x^a)\partial_z. \quad (41)$$

Again we see that we obtain infinite dimensional MCs.

In the case of 2(b), solving MC equations simultaneously, we obtain

$$\xi = -2 \frac{T_1}{T_1} A'(x) \partial_t + A(x) \partial_x + c_0(\partial_y + \partial_z) + c_1 \frac{1}{2}(z \partial_y - y \partial_z). \quad (42)$$

When we solve MC equations using the constraints of the case 2(c), it follows that

$$\xi = \frac{c_0}{\sqrt{T_0}} \partial_t + \xi^1(x^a)\partial_x + c_1(\partial_y + \partial_z) + c_2 \frac{1}{2}(z \partial_y - y \partial_z). \quad (43)$$

We see that $\xi^1$ is an arbitrary function depending on all four variables, thus we have infinite dimensional MCs.
5 Conclusion

This paper has been devoted to the evaluation of MCs for the LRS models when the energy-momentum tensor is degenerate. We have concentrated only for Bianchi types II, VIII, IX and IX spacetimes given by Eqs. (4) and (5) respectively as the classification of metrics given by Eq. (3) has already been completed [7]. It is found that when at least one component of $T_{ab}$ is non-zero (case (1) of sec. 3), all the possibilities yield infinite dimensional MCs. If one component of $T_{ab}$ is zero (case (2) of sec. 3), we again obtain infinite-dimensional MCs. However, in this case, we have proper MCs, in addition, to the improper MCs. For this case, the solution of the equation $T_2 = 0$ turns out to be the perfect fluid dust solution. We also observe that all cases of the Bianchi type V yield infinite dimensional MCs.

We note that the case in which $T_0 \neq 0$ ($T_0 > 0$) is the only surviving component of $T_{ab}$ can always be interpreted as a dust fluid. In the case when $T_0 = 0$, we do not have dominant energy condition instead we have energy density zero.

As we have considered degenerate case, it is natural to expect infinite dimensional MCs as given by Hall et al. [2]. However, it may be finite dimensional for some case as given in [7] when we classify MCs for the LRS models given by Eq. (3). The case 2(b) (section 3) is left open and it needs to be investigated. It is expected that this case would provide finite dimensional MCs. Also, we have obtained constraints on the energy-momentum tensor. It might be interesting to look for more solutions of the constraint equations or at least examples should be constructed to satisfy these constraints.

Appendix A

The surviving components of the Ricci tensor are

\[
\begin{align*}
R_{00} & = -\frac{1}{AB}(2AB + \ddot{A}B), \\
R_{11} & = A\ddot{A} + 2\frac{A\dddot{A}B}{B} + \frac{A^4}{2B^4}, \\
R_{22} & = B\ddot{B} + \dot{B}^2 - \frac{A^2}{2B^2} + \frac{\dot{A}\dot{B}B}{A} + k, \\
R_{33} & = \Lambda^2 R_{11} + \Sigma^2 R_{22},
\end{align*}
\]
\[ R_{13} = -\Lambda R_{11}. \]  
(A1)

where dot represents derivative w.r.t. time coordinate \( t \). The non-vanishing components of energy-momentum tensor \( T_{ab} \) are

\[
T_{00} = 2\frac{\dot{A}\dot{B}}{AB} + \dot{B}^2 - \frac{A^2}{4B^4} + \frac{k}{B^2},
\]

\[
T_{11} = -2\frac{A^2\ddot{B}}{B} - \frac{A^2B^2}{B^2} + 3\frac{A^4}{4B^4} - k\frac{A^2}{B^2},
\]

\[
T_{22} = -B\dddot{B} - \frac{\dddot{B}}{A} - \frac{\dot{A}\dot{B}B}{A} - \frac{A^2}{4B^2},
\]

\[
T_{33} = \Lambda^2 T_{11} + \Sigma^2 T_{22},
\]

\[
T_{13} = -\Lambda T_{11}. \quad (A2)
\]

The independent KVs associated with the LRS spacetimes are given by

\[
\xi_{(1)} = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \\
\xi_{(2)} = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \\
\xi_{(3)} = \partial_\phi. \quad (A3)
\]

**Appendix B**

The surviving components of the Ricci tensor are

\[
R_0 = -\frac{\dddot{A}}{A} - 2\frac{\dddot{B}}{B},
\]

\[
R_1 = A\dddot{A} + 2\frac{A\dddot{B}}{B},
\]

\[
R_2 = (B\dddot{B} + \dddot{B}^2 - \frac{B^3}{A^2} + \frac{\dot{A}\dot{B}B}{A}) = R_3. \quad (B1)
\]

The non-vanishing components of energy-momentum tensor \( T_{ab} \) are

\[
T_0 = 2\frac{\dot{A}\dot{B}}{AB} + \dot{B}^2 - \frac{1}{A^2},
\]

\[
T_1 = -2\frac{A^2\ddot{B}}{B} - \frac{A^2\dddot{B}}{B^2} + 1,
\]

\[
T_2 = -(B\dddot{B} + \frac{\dddot{B}^2}{A} + \frac{\dot{A}\dot{B}B}{A})e^{2x} = T_3. \quad (B2)
\]
Acknowledgment

The author would like to thank the referee for useful and accurate remarks and suggestions.

References

[1] Carot, J., da Costa, J. and Vaz, E.G.L.R.: J. Math. Phys. 35(1994)4832.
[2] Hall, G.S., Roy, I. and Vaz, L.R.: Gen. Rel and Grav. 28(1996)299.
[3] Carot, J. and da Costa, J.: Proc. of the 6th Canadian Conf. on General Relativity and Relativistic Astrophysics, Fields Inst. Commun. 15, Amer. Math. Soc. WC Providence, RI(1997)179.
[4] Yavuz, İ., and Camcı, U.: Gen. Rel. Grav.28(1996)691;
Camcı, U., Yavuz, İ., Baysal, H., Tarhan, İ., and Yılmaz, İ.: Int. J. Mod. Phys. D10(2001)751.
[5] Camcı, U. and Barnes, A.: Class. Quant. Grav. 19(2002)393.
[6] Sharif, M.: Nuovo Cimento B116(2001)673; Astrophys. Space Sci. 278(2001)447; J. Math. Phys. (2003);
Sharif, M. and Schar Aziz: Gen Rel. and Grav. 35(2003).
[7] Camcı, U. and Sharif, M.: Gen Rel. and Grav. 35(2003)97.
[8] Camcı, U. and Sharif, M.: Class. Quant. Grav. 20(2003)2169.
[9] Tsamparlis, M. and Apostolopoulos, Pantelis S.: Gen. Rel. and Grav. 36(2004)47.
[10] Ellis, G.F.R.: J. Math. Phys. 8(1967)1171.
[11] Stewart, J.M. and Ellis, G.F.R.: J. Math. Phys. 9(1968)1072.
[12] Ellis, G.F.R. and MacCallum, M.A.H.: Commun. Math. Phys. 12(1969)108;
MacCallum, M.A.H.: Commun. Math. Phys. 20(1971)57;
Collins, C.B.: Commun. Math. Phys. 23(1971)137.
[13] van Elst, H. and Ellis, G.F.R.: Class. Quantum Grav. 13(1996)1099.