HOMOGENEOUS DUAL RAMSEY THEOREM

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Abstract. For positive integers $k < n$ such that $k$ divides $n$, let $(n)^k_{\text{hom}}$ be the set of homogeneous $k$-partitions of $\{1, \ldots, n\}$, that is, the set of partitions of $\{1, \ldots, n\}$ into $k$ classes of the same cardinality. In [4] the following question (Problem 7.3 in [4]) was asked:

Is it true that given positive integers $k < m$ and $N$ such that $k$ divides $m$, there exists a number $n > m$ such that $m$ divides $n$, satisfying that for every coloring $(n)^k_{\text{hom}} = C_1 \cup \cdots \cup C_N$ we can choose $u \in (n)^m_{\text{hom}}$ such that $\{t \in (n)^{k}_{\text{hom}} : t \text{ is coarser than } u\} \subseteq C_i$ for some $i$?

In this note we give a positive answer to that question in Theorem 1 below. This result turns out to be a homogeneous version of the finite Dual Ramsey Theorem of Graham-Rothschild [3]. As explained in [4], our result also proves that the class $\mathcal{OMBA}_2$ of naturally ordered finite measure algebras with measure taking values in the dyadic rationals has the Ramsey property.

1. Homogeneous Dual Ramsey Theorem

1.1. Homogeneous partitions. Let $k < n$ be positive integers. As customary, write $k|n$ if $k$ divides $n$, and in that case let

\[(1) \quad (n)^k_{\text{hom}} = \text{the set of all homogeneous } k\text{-partitions of } \{1, \ldots, n\};\]

that is, the set of all the partitions of $\{1, \ldots, n\}$ into $k$ classes of the same cardinality. If $t$ and $u$ are partitions of $\{1, \ldots, n\}$, not necessarily homogeneous, then we say that $t$ is coarser than $u$ if every class of $u$ is a subset of some class of $t$. Denote by $(u)^k_{\text{hom}}$ the set of all homogeneous $k$-partitions of $\{1, \ldots, n\}$ that are coarser than $u$.

Remark 1. In [4], the notation $EQ^k_{\text{hom}}$ is used to denote $(n)^k_{\text{hom}}$.

The following is our main result.

Theorem 1 (Homogeneous Dual Ramsey Theorem). Let integers $k < m$ and $N$ be given, with $k|m$. There exists a positive integer $n > m$ such that $m|n$, satisfying that for every coloring $(n)^k_{\text{hom}} = C_1 \cup \cdots \cup C_N$ there exists $u \in (n)^m_{\text{hom}}$ such that $(u)^k_{\text{hom}} \subseteq C_i$, for some $i \in \{1, \ldots, N\}$.

We will prove Theorem 1 in Section 2 below. In our proof, we will make use of the infinite Dual Ramsey Theorem due to Carlson and Simpson [1], which we will state in the next subsection.
1.2. Dual Ramsey Theorem. We will essentially go back to the notation introduced in [1]. Let \( \mathbb{N} = \{1, 2, 3, \ldots \} \) be the set of positive integers and let \( (\omega)^{\omega} \) be the set of all the infinite partitions \( A = \{A_i : i \in \mathbb{N} \} \) of \( \mathbb{N} \) into infinite classes such that

\[
(2) \quad i < j \implies \min(A_i) < \min(A_j).
\]

Given \( A, B \in (\omega)^{\omega} \), we say that \( A \) is coarser than \( B \) if every class in \( B \) is a subset of some class in \( A \). Pre-order \( (\omega)^{\omega} \) as follows:

\[
(3) \quad A \preceq B \iff A \text{ is coarser than } B.
\]

Likewise, given a positive integer \( k \), let \( (\omega)^k \) be the set of all the partitions \( X = \{X_i : 1 \leq i \leq k \} \) of \( \mathbb{N} \) into \( k \) infinite classes satisfying

\[
(4) \quad i < j \implies \min(X_i) < \min(X_j), \text{ for } 1 \leq i, j \leq k.
\]

Given \( X, Y \in (\omega)^k \) and \( A \in (\omega)^{\omega} \), define “\( X \) is coarser than \( Y \)” (resp. “\( X \) is coarser than \( A \)” in the same way as above; and write \( X \preceq Y \) (resp. \( X \preceq A \)). For \( A \in (\omega)^{\omega} \), let \( (A)^k = \{X \in (\omega)^k : X \preceq A\} \).

We will regard the sets \( (\omega)^{\omega} \) and \( (\omega)^k \) as topological spaces with the usual topologies as defined in [1].

**Theorem 2** (Infinite Dual Ramsey Theorem; Carlson-Simpson [1]). Let \( k \) be a positive integer. For every finite Borel-measurable coloring of \( (\omega)^k \) there exists \( A \in (\omega)^{\omega} \) such that \( (A)^k \) is monochromatic.

For positive integers \( k \) and \( n \), let

\[
(5) \quad (n)^k = \text{the set of all the } k \text{-partitions of } n,
\]

i.e., partitions of \( \{1, \ldots, n\} \) into \( k \) classes. For every such partition \( u \), denote by \( (u)^k \) the set of all \( k \)-partitions of \( \{1, \ldots, n\} \) that are coarser than \( u \). Also, let \( (<\omega)^k = \bigcup_{n \in \mathbb{N}} (n)^k \).

Theorem [1] is a homogeneous version of the following well-known result due to Graham and Rothschild, which can be obtained as a corollary of Theorem 2.

**Theorem 3** (Dual Ramsey Theorem; Graham-Rothschild [3]). Let integers \( k < m \) and \( N \) be given. There exists a number \( n > m \) satisfying that for every coloring \( (n)^k = C_1 \cup \cdots \cup C_N \) there exists \( u \in (n)^m \) such that \( (u)^k \subseteq C_i \) for some \( i \in \{1, \ldots, N\} \).

2. Proof of Theorem [1]

Let \( A = \{A_i : i \in \mathbb{N}\} \in (\omega)^{\omega} \) be given. We will borrow some notation from [5]. Define

\[
(6) \quad r(0, A) = \emptyset \quad \& \quad r(i, A) = \{A_j \cap \{1, \ldots, \min(A_i)\} : 1 \leq j \leq i\}; \quad i > 0.
\]

Note that \( r(i, A) \) is a partition of \( \{1, \ldots, \min(A_i)\} \). We think of it as the \( i \)-th approximation of \( A \). If a finite partition \( b \) is the \( i \)-th approximation of some \( B \preceq A \), we write \( b \in (<\omega)^i \upharpoonright A \).

It will be useful to understand elements \( A = \{A_i : i \in \mathbb{N}\} \in (\omega)^{\omega} \) as surjective functions \( A : \mathbb{N} \rightarrow \mathbb{N} \) where \( A_i = A^{-1}\{i\} \). Likewise, elements \( X = \{X_i : 1 \leq i \leq k\} \in (\omega)^k \) will
be as well understood as surjective functions \( X : \mathbb{N} \rightarrow \{1, \ldots, k\} \) where \( X_i = X^{-1}(\{i\}) \), for \( 1 \leq i \leq k \). In the same spirit, we will regard partitions \( u \) of a finite set \( F \neq \emptyset \) into \( k \) classes as surjections \( u : F \rightarrow \{1, \ldots, k\} \). We will shift between understanding partitions as surjective functions and sets of disjoint classes throughout the rest of this article. Now, keep that in mind and fix a positive integer \( k \) for a while. We will borrow some notation and ideas from [2]. Given \( A \in (\omega)^\omega \), let \( \pi : (\omega)^\omega \rightarrow (\omega)^k \) be defined as follows:

\[
\pi(A)(i) = \begin{cases} 
A(i) & \text{if } 1 \leq A(i) \leq k \\
1 & \text{otherwise.} 
\end{cases}
\]

Notice that \( \pi \) is a surjection. We understand \( \pi \) as a projection function from \((\omega)^\omega\) onto \((\omega)^k\). Following [2], define an approximation function \( s \) with domain \( \{0\} \cup \mathbb{N} \times (\omega)^k \) as follows. For \( i \in \{0\} \cup \mathbb{N} \) and \( X = \{X_1, \ldots, X_k\} \in (\omega)^k \), the output \( s(i, X) \) is a \( k \)-partition of some finite subset of \( \mathbb{N} \) whose \( j \)-th class, \( s(i, X)_j \) \((1 \leq j \leq k)\), will be defined by cases as follows:

For \( i \leq k \), let

\[
s(i, X)_j = X_j \cap \{1, \ldots, \min(X_k)\}
\]

So, for \( i \leq k \), \( s(i, X)_j \) is the unique initial segment of \( X_j \) included as a subset in \( \{1, \ldots, \min(X_k)\} \).

Note that for any \( A \in (\omega)^\omega \) such that \( \pi(A) = X \), if \( i \leq k \) we have \( s(i, X) = r(i, A) \). In particular, \( s(0, X) = \emptyset \) and for \( 0 < i \leq k \), \( s(i, X) \) is a partition of \( \{1, \ldots, \min(X_i)\} \).

For \( i > k \), let

\[
s(i, X)_j = s(i - 1, X)_j \cup \{\min(X_j \setminus s(i - 1, X)_j)\}
\]

i.e., for \( i > k \), we obtain the class \( s(i, X)_j \) by adding to \( s(i - 1, X)_j \) the minimum element of \( X_j \) that is not an element of \( s(i - 1, X)_j \). (Therefore, in this case, \( s(i, X)_j \) is as well an initial segment of \( X_j \) like in the case \( i \leq k \).) This completes the definition of the approximation function \( s \). We think of \( s(i, X) \) as the \( i \)-th approximation of \( X \). Actually, each \( X \in (\omega)^k \) can be identified with the sequence \( (s(i, X))_{i \in \mathbb{N}} \) of its approximations.

Let \( S_k \) denote the range of \( s \). We have

\[
(< \omega)^k \subset S_k, \text{ but } S_k \setminus (< \omega)^k \neq \emptyset.
\]

Now, note that for every \( a \in (< \omega)^k \) the union of \( a \) is an initial segment of \( \mathbb{N} \). That is not necessarily true for all elements of \( S_k \). Given \( b \in S_k \), if the union of \( b \) is an initial segment of \( \mathbb{N} \), then denote by \#\( b \) the unique \( l \in \mathbb{N} \) such that \( b \) is a partition of \( \{1, \ldots, l\} \). Denote by \( \cup b \) the union of \( b \). Let

\[
S_k^\# = \{b \in S_k : \cup b \text{ is an initial segment of } \mathbb{N}\}.
\]

If \( a \in (< \omega)^k \) and \( b \in S_k \), write

\[
a = s_k(b)
\]
if there exist $X \in (\omega)^k$ and an integer $i > k$ such that $s(k, X) = a$ and $s(i, X) = b$. Now, let an integer $n$ with $k|n$ and $a \in (< \omega)^k$ be given. There exists a bijective correspondence between $(n)^k_{\text{hom}}$ and the set

$$
(13) \quad T(a, k, n) = \{b \in S^\#_k : a \neq b, \ a = s_k(b), \ #b = #a + n\}.
$$

To see that such a correspondence exists, fix a bijective function

$$
(14) \quad \varphi : \{1, \ldots, n\} \rightarrow \{#a + 1, \ldots, #a + n\}.
$$

Now, given $t \in (n)^k_{\text{hom}}$, define $b_t \in T(a, k, n)$ by

$$
(15) \quad b_t(j) = \begin{cases} 
      a(j) & \text{if } 1 \leq j \leq #a \\
      t(\varphi^{-1}(j)) & \text{if } #a < j \leq #a + n
   \end{cases}
$$

Then the correspondence

$$
(16) \quad t \mapsto b_t
$$

is bijective. Obviously, the inverse of a partition $b \in T(a, k, n)$ under this correspondence is the partition $t \in (n)^k_{\text{hom}}$ defined by

$$
(17) \quad t(j) = b(\varphi(j)), \ 1 \leq j \leq n.
$$

From now on, for all positive integers $k < n$ with $k|n$ and every $a \in (< \omega)^k$ we will fix one such bijective correspondence $t \mapsto b_t$ between $(n)^k_{\text{hom}}$ and $T(a, k, n)$ throughout the rest of this section.

Now we are ready to prove our main result.

**Proof of Theorem** Fix positive integers $k < m$ and $N$, with $k|m$, and suppose that the conclusion in the statement fails. For each positive integer $n > m$ such that $m|n$, choose a coloring $(n)^k_{\text{hom}} = C_1^n \cup \cdots \cup C_N^n$ admitting no monochromatic set of the form $(u)^k_{\text{hom}}$ with $u \in (n)^m_{\text{hom}}$.

Given $X \in (\omega)^k$, let $a = s(k, X)$. Let

$$
(18) \quad i_0 = \min\{i > m : m \text{ divides } |\cup s(i, X)| - #a\}.
$$

Here, for $i > m$, $|\cup s(i, X)|$ denotes the cardinality of the union of the partition s(i, X). Define the positive integer

$$
(19) \quad n(X) = |\cup s(i_0, X)| - #a
$$

Note that $k|n(X)$. List the elements of $\cup s(i_0, X)$ in their natural increasing order as $x_1 < \cdots < x_{#a+n(X)}$, and let $b_X \in T(a, k, n(X))$ be defined by

$$
(20) \quad b_X(j) = \begin{cases} 
      a(j) & \text{if } 1 \leq j \leq #a \\
      s(i_0, X)(x_j) & \text{if } #a < j \leq #a + n(X)
   \end{cases}
$$

Denote by $t(b_X)$ the unique element $t \in (n(X))^{k}_{\text{hom}}$ such that $b_t = b_X$. Define a coloring $(\omega)^k = C_1 \cup \cdots \cup C_N$ as follows:
\[(21) \quad X \in C_j \text{ if and only if } t(b_X) \in C_{j_0}^n.\]

Note that each \(C_j\) is Borel. So by Theorem 2 there exists \(A \in (\omega)\omega\) such that \((A)^k \subseteq C_{j_0}\) for some \(j_0 \in \{1, \ldots, N\}\). Fix \(a_0 \in (\omega)^m \upharpoonright A\) and let \(Y \in (A)^m\) be such that \(s(m, Y) = a_0\). Let

\[(22) \quad l = \min\{i > m : m \text{ divides } |\cup s(i, Y)| - \#a_0\}\]

and set \(n = |\cup s(l, Y)| - \#a_0\). List the elements of \(\cup s(l, Y)\) in their natural increasing order as \(y_1 < \cdots < y_{\#a_0+n}\). Define \(b_Y \in T(a_0, m, n)\) by

\[(23) \quad b_Y(j) = \begin{cases} a_0(j) & \text{if } 1 \leq j \leq \#a_0 \\ s(l, Y)(y_j) & \text{if } \#a_0 < j \leq \#a_0 + n \end{cases}\]

Now denote by \(u\) the unique element \(t \in (n)^m_{\text{hom}}\) such that \(b_t = b_Y\).

Finally, for \(t \in (u)^k_{\text{hom}}\), define \(X \in (A)^k\) with \(X \leq Y\) by merging classes in \(Y\) according to how the corresponding classes in \(u\) were merged to build \(t\). Let \(a = s(k, X)\) and note that \(n(X) = n\). If \(b_t\) is the unique element of \(T(a, k, n)\) corresponding to \(t\), then \(b_X = b_t\) and therefore \(t = t(b_X)\). Thus, \(t \in C_{j_0}^n\), by the choice of \(A\). Since \(t\) was arbitrary, we get \((u)^k_{\text{hom}} \subseteq C_{j_0}^n\). A contradiction. This completes the proof. \(\square\)

**References**

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