A Conformal Field Theory description of the Paired and parafermionic states in the Quantum Hall Effect

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Abstract

We extend the construction of the effective conformal field theory for the Jain hierarchical fillings proposed in [7] to the description of a quantum Hall fluid at non standard fillings \( \nu = \frac{m}{pm+2} \). The chiral primary fields are found by using a procedure which induces twisted boundary conditions on the \( m \) scalar fields; they appear as composite operators of a charged and neutral component. The neutral modes describe parafermions and contribute to the ground state wave function with a generalized Pfaffian term. Correlators of \( N_e \) electrons in the presence of quasi-hole excitations are explicitly given for \( m = 2 \).

Keyword: Vertex operator, Kac-Moody algebra, Quantum Hall Effect

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The experimental evidence of a Hall plateau at filling \( \nu = \frac{5}{2} \) has recently spurred a renewed interest in a deeper understanding of the underlying physics at plateaus which do not fall into the hierarchical scheme \([1]\). To such an extent a pairing picture, in which pairs of spinless or spin-polarized fermions condense, has been presented \([2]\) for the non-standard fillings \( \nu = \frac{1}{q}, q > 0 \) and even. As a result the ground state gets described in terms of the Pfaffian (the so called Pfaffian state) and the non-Abelian statistics of the fractional charged excitations evidenced \([2, 3, 4]\). More recently \([5, 6]\) it has been argued that the non-Abelian statistics might come out from constraining an Abelian theory, by employing the Meissner effect in the neutral sector; that would also account for the pairing phenomenon.

Lately a Conformal Field Theory (CFT) description in terms of composite fermions for the Jain filling fractions \( \nu = \frac{m}{2pm+1} \) of a Quantum Hall Fluid (QHF) has been proposed \([7, 8]\). The composite fermions are described by composite vertex operators, which are (the image of) primary fields of a CFT with central charge \( c = m \) in the Lowest Landau Level (LLL), and factorize into a charged and a neutral component. The neutral component assures the locality properties of the composite electron field and the single-valuedness of the ground state wave function.

Further due to the \( Z_m \) symmetry in the neutral sector the general \( N_e \) electron correlation function shows clustering properties which have been proposed in the context of paired Hall states \([9]\).

In this letter we present a natural extension of the \( m \)-reduction procedure (previously applied in ref.\([7]\) to the description of the Jain fillings) to the case of the non-standard fillings \( \nu = \frac{m}{pm+2} \). In particular for \( m = 2 \) the neutral degrees of freedom of the composite electrons describe Majorana fermions and contribute to the ground state wave function with a Pfaffian term, in agreement with an early proposal \([2]\). Further, for generic \( m \), the neutral degrees of freedom are parafermions and give rise to the clustering phenomenon of \( m \) objects, which simply reproduces the picture presented in ref.\([9]\) by employing a hamiltonian formalism with \( m + 1 \) body interactions.

The letter is organized as follows: after a brief review of the \( m \)-reduction procedure we construct the parafermion vertex operators which together with the charged \( U(1) \) component give rise to the primary fields of the CFT. It should be noticed an interesting phenomenon related to the existence of two classes of primary fields: the \( Z_m \) twist invariant and the \( Z_m \) non-invariant ones. In fact on physical grounds one can argue that only the
invariant fields are relevant to the description of the pairing phenomenon. As a result the central charge of neutral sector of our CFT is given by \( c = \frac{2(m-1)}{m+2} \). An interpretation of such a phenomenon has been proposed in ref.\[10\] in terms of a strong coupling between the symmetric modes.

We then give the ground state wave function as a correlator of the primary fields with integer electric charge and the quasi-holes wave function for the elementary excitations, with fractional charge, up to the correlator of the twist fields.

Our approach is meant to describe all the plateaus with even denominator starting from the bosonic Laughlin filling \( \nu = 1/2 \), which is described by a CFT with \( c = 1 \), in terms of a scalar chiral field compactified on a circle with radius \( R^2 = 1/\nu = 2 \) (or the dual \( R^2 = 1/2 \)). Then the \( U(1) \) current is given by \( J(z|1,2) = i\partial_z Q(z) \), where \( Q(z) \) is the compactified Fubini field with the standard mode expansion:

\[
Q(z) = q - ip \ln z + \sum_{n \neq 0} \frac{a_n}{n} z^{-n}
\]

(1)

with \( a_n, q \) and \( p \) satisfying the commutation relations \([a_n, a_{n'}] = n\delta_{n,n'}\) and \([q, p] = i\).

The representations are realized by the vertex operators \( U^\alpha(z) = e^{i\alpha Q(z)} \) with \( \alpha^2 = 2 \) and conformal dimension \( h = 1 \), with a consequent extension of the \( U(1) \) symmetry to the \( SU(2)_1 \) affine one. Furthermore the theory contains the Virasoro algebra generated by the stress-energy tensor \( T(z|1,2) = -\frac{1}{2} : (\partial_z Q(z))^2 : \).

In order to construct the \( \nu = m/2 \) filling we start with the set of fields in the above CFT (mother). Using the \( m \)-reduction procedure, which consists in considering the subalgebra generated only by the modes which are divided by an integer \( m \), we get the image of an orbifold of a \( c = m \) CFT (see ref.\[4\] and references therein).

Also for the \( SU(2) \) case the fields in the mother CFT can be factorized into irreducible orbits of the discrete group \( Z_m \) which is a symmetry of the daughter theory and can be organized into components which have well defined transformation properties under this group.

In order to compare the image so obtained to the \( c = m \) CFT, we map \( z \rightarrow z^{1/m} \) and we will indicate the components in the base \( \hat{z} = z^m \) with an hatted symbol (for instance, \( \phi(z) \rightarrow \hat{\phi}(z) \)). In particular any component in the subalgebra is a function only of the variable \( z^m \).
In ref. [11] it was also defined an isomorphism between fields on the $z$ complex plane and fields on the $z^m$ plane by means of the following identifications:

$$a_{nm+l} \longrightarrow \sqrt{ma_{n+l/m}} \quad q \longrightarrow \frac{1}{\sqrt{m}} q$$

(2)

Let us first introduce the invariant scalar field

$$X(z|m,2) = \frac{1}{m} \sum_{j=1}^{m} Q(\varepsilon^j z)$$

(3)

where $\varepsilon^j = e^{i\frac{2\pi j}{m}}$, corresponding to a compactified boson on a circle with radius now equal to $R_X^2 = 2/m$. This field depends only on powers of $z^m$ and satisfies trivial boundary conditions. It is the basic field of the $U(1)$ electrically charged sector of the theory where the charge is measured by the zero mode.

The non-invariant components, as a resulting image of $m$ constrained bosons, are expressed by

$$\phi^j(z|m,2) = Q(\varepsilon^j z) - X(z|m,2)$$

(4)

with the condition $\sum_{j=1}^{m} \phi^j(z|m,2) = 0$.

These fields satisfy non-trivial twisted boundary conditions

$$\alpha \cdot \phi^j(\varepsilon z|m,2) = \alpha \cdot \phi^{j+1}(z|m,2) + 2\pi n\alpha \cdot p \quad n \in Z$$

(5)

where the shift is due to the definition of index $j \mod m$.

The $J(z|1,2)$ current of the mother theory decomposes into a charged current given by $J(z|m,2) = i\partial_z X(z|m,2)$ and $m - 1$ neutral ones $\partial_z \phi^j(z|m,2)$.

In the same way every vertex operator in the mother theory can be factorized in a vertex that depends only on the invariant field:

$$U^\alpha(z|m,2) = z^{\frac{\alpha^2 (m-1)}{m}} : e^{i\alpha \cdot X(z|m,2)} : \quad \alpha^2 = 2$$

(6)

and in vertex operators depending on the $\phi^j(z|m,2)$ fields.

We also introduce the neutral components:

$$\psi^\alpha_1(z|m,2) = \frac{z^{\frac{\alpha^2 (1-m)}{m}}}{m} \sum_{j=1}^{m} \frac{\alpha^2 j}{2} : e^{i\alpha \cdot \phi^j(z|m,2)} :$$

(7)
which satisfy the fundamental product:

$$\psi^\alpha_1(z|m,2)\psi^\beta_1(\xi|m,2) = \frac{z^{\frac{1-m}{2}}\xi^{\frac{1-m}{2}}}{m^2} \sum_{j,j'} \varepsilon^{j+j'\frac{1}{2}} : e^{i\alpha\phi^j(z|m,2)} e^{i\beta\phi^j(\xi|m,2)} : \left(\frac{\varepsilon^{j'} z - \varepsilon^j \xi^{\alpha\beta}}{(z^{m} - \xi^{m})^{\alpha\beta}}\right)$$

(8)

The set of primary fields generated by this product can be expressed in terms of the fundamental representations $\Lambda^i$ of $SU(m)$ Lie algebra. In fact, defining

$$\phi^{\Lambda^i}(z|m,2) = \sum_{j=1}^i \phi^j(z|m,2)$$

(9)

and $\phi^\Lambda(z|m,2)$, where $\Lambda = \sum_{i=1}^{m-1} l_i \Lambda_i$, and introducing the $m$-ality parameter $a = \sum_{i=1}^{m-1} l_i$ (mod $m$), which is invariant under the addition of any vector in the root lattice, the exact form of these fields can be deduced by the analysis of the OPE of eq.(8) for $\alpha = \beta$ to get the $a = 2$ field. By repeated application of this analysis we can obtain the full set of fields:

$$\tilde{\psi}_a^\alpha(z|m,2) = \sum_{j_1 > j_2 > \ldots > j_a = 1}^m f(\varepsilon^{j_1}, \ldots, \varepsilon^{j_a}, z^{1/m}) : e^{i\alpha\phi(z|m,2)} \ldots e^{i\alpha\phi(z|m,2)} :$$

(10)

where the functions $f(\varepsilon^{j_1}, \ldots, \varepsilon^{j_a}, z)$ can be extracted from the OPE relations.

The sum takes into account the fact that any field can be associated to the $a$-th fundamental representation of $SU(m)$ (namely, the antisymmetric tensor representation). In [13] it was shown that these are a realization of parafermions and satisfy the operator product algebra [13]:

$$\tilde{\psi}_{a'}^\alpha(z|m,2)\tilde{\psi}_{a}^\alpha(\xi|m,2) = \frac{C_{a.a'}}{(z - \xi)^{2a'}} [\tilde{\psi}_{a+a'}^\alpha(z|m,2) + O(z - \xi)] \quad a + a' < m$$

(11)

and for $a + a' = m$ the OPE contains the Virasoro algebra generators:

$$\tilde{\psi}_{a}^\alpha(z|m,2)\tilde{\psi}_{m-a}^\alpha(\xi|m,2) = \frac{C_{a,m-a}}{(z - \xi)^{2a(m-a)}} \left[ 1 + \frac{2h_a}{c_\psi} \hat{T}_\psi(z|m,2)(z - \xi)^2 + O(z - \xi)^3 \right]$$

(12)

where $C_{a,a'}$ are the structure constants and $h_a$, $c_\psi$ are the conformal dimensions and central charge of parafermions [13].

Moreover the $SU(m)$ representations that can appear are the fundamental ones $\Lambda_a$, because in the OPE algebra of $\psi^\alpha_1(z|m,2)$ (corresponding to the $\Lambda_1$ representation) only the fields $\psi^\alpha_a(z|m,2)$ with $a \in \{1, \ldots, m - 1\}$ appear while $\psi^\alpha_m(z|m,2)$ is the identity operator. That is an effect of the $Z_m$ invariance of the parafermions algebra.
Notice that no neutral currents are present in the above OPE, as one would expect from symmetry considerations.

It is well known that the $c_X = 1$ rational CFT with $R_X^2 = 2/m$ has $2m$ primary fields which can be parametrized by $\alpha = \sqrt{\frac{2}{m}} a$ and $\alpha = \sqrt{-\frac{1}{2m}} a$, $a = 1, \ldots, m$. In our formalism these fields appear together with the neutral ones into the composite operators:

$$\hat{V} \sqrt{\frac{2}{m}}(z|m,2) = \hat{U} \sqrt{\frac{2}{m}}(z|m,2) \hat{\psi}_a(z|m,2)$$

for quasi-particles and

$$\hat{V}_{qh} \sqrt{\frac{2}{m}}(z|m,2) = \hat{U} \sqrt{\frac{2}{m}}(z|m,2) \hat{\sigma}_a(z|m,2)$$

for quasi-holes; $\hat{\sigma}_a(z|m,2)$ are the parafermionic twist fields. The electric charges and magnetic flux contents of such fields is given after eqs.(21, 22).

In eq.(14) we have not included the contribution coming from the $\bar{\sigma}$ fields of the $Z_m$ non-invariant theory. In fact they give rise to a term which does not survive after the projection to the LLL.

The currents $\hat{V}^{\pm} \sqrt{\frac{2}{m}}(z|m,2)$ and $J(z|m,2)$ generate the $SU(2)_m$ affine algebra while $\hat{V}_{qh} \sqrt{\frac{2}{m}}$ are the primary fields of this algebra (see ref.[13] for details).

The generator of the Virasoro algebra $\hat{T}(z|m,2)$ was given in [12] as the sum of two independent operators, one depending on the charged sector:

$$\hat{T}_X(z|m,2) = -\frac{1}{2} : \left( \partial_z X(z|m,2) \right)^2 :$$

and the other given in terms of the $Z_m$ twisted bosons $\hat{\phi}^j(z|m,2)$:

$$\hat{T}_\psi(z|m,2) = \frac{2}{m+2} \left( -\sum_{j=1}^{m} : \left( \partial_z \phi^j(z) \right)^2 : + \sum_{j' \neq j=1}^{m} \frac{\epsilon^{j'+j} e^{-i\epsilon \phi^j(z)} e^{i\epsilon \phi^{j'}(z)}}{2m^2} + \frac{m^2 - 1}{24mz^2} \right)$$

while the higher integer spin operators generating the full parafermionic $\mathcal{W}_m$ algebra was given in terms of the $\hat{\phi}^j(z|m,2)$ fields in [14]. Notice that the vacuum expectation value of $\hat{T}_\psi$ is zero due to the cancellation between the second and the third term in eq.(16).

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3We notice that we do not have an explicit realization of the $\bar{\sigma}$ fields because we have projected the mother $c = 1$ CFT onto the $m$-covering of the plane, so that those fields appear as branch cuts [13, 14].
It is not very hard to verify that the conformal dimensions of the fields of eq.(13) and eq.(14) are given by:

\[ h_a = \frac{a^2}{m} + a \left( \frac{m - a}{m} \right) = a \quad a \in \{1, \ldots, m\} \]  

(17)

for quasi-particles and

\[ h^q_a = \frac{a^2}{4m} + \frac{a(m - a)}{2m(m + 2)} = \frac{a(a + 2)}{4(m + 2)} \quad a \in \{1, \ldots, m\} \]  

(18)

for quasi-holes.

The contribution to the central charge \( c \) is given by

\[ c = 1 + \frac{2(m - 1)}{m + 2} = \frac{3m}{m + 2} \]  

(19)

where \( 1 \) and \( \frac{2(m - 1)}{m + 2} \) come from the charged and neutral degrees of a freedom respectively.

We must notice that the contribution of the \( Z_m \) invariant fields to the central charge is not equal to \( m \). In fact the missing part is associated with the non-invariant fields defined as:

\[ \psi^{(i,j)}(z|m,2) = \frac{z^{a^2(1-m)}}{m} \left( \varepsilon^{a^2_i} : e^{i\alpha \cdot \phi^i(z|m,2)} : -\varepsilon^{a^2_j} : e^{i\alpha \cdot \phi^j(z|m,2)} : \right) \]  

(20)

with \( i \neq j = \{1, \ldots, m\} \), which are primary fields of the complementary \( \tilde{c} = \frac{m(m-1)}{m+2} \) theory. One can see also that these extra degrees of freedom can be gauged away by means of a coset reduction.

This appears to be a generalization of the Low-Barrier limit of a double-layer sample described in ref.[10] where a strong tunnel effect was introduced. In our case the strong coupling between electrons of the different layers is simply a consequence of the induced \( Z_m \) symmetry of the composite electrons states. This phenomenon was suggested to be relevant for CFT’s on algebraic curves in [15]. It would be interesting to have a clear picture of the symmetries of the vacuum and excited states, so to understand the mechanism which decouples the \( Z_m \) non-invariant theory from the invariant one.

We can now describe the generic filling \( \nu = \frac{m}{pm + 2} \) by flux attachment starting from \( \nu = m/2 \). In order to do so, we factorize the fields into two parts, the first describing the \( c_X = 1 \) charged sector with radius \( R^2_X = \frac{pm+2}{m} \), the second describing the neutral excitations, which are parafermions, with central charge \( c_\psi = \frac{2(m-1)}{m+2} \), for any \( p \in N \).
The $U(1)$ sector is now described by the compactified boson $X(z|m, pm + 2)$ and its related vertex operators $U^\pm \alpha(z|m, pm + 2)$, with $\alpha_l = l/\sqrt{m(pm + 2)}$, $l = 1, \ldots, m(pm + 2)$, which produce excitations with anyonic statistics $\theta = \pi \alpha_l^2$. While the $m - 1$ neutral bosons $\phi^i(z|m, 2)$ are independent from the flux number $p$.

To obtain a pure holomorphic function we will consider the correlator of the composite operators $\hat{V}^\alpha_l(z|m, pm + 2)$ with conformal dimensions:

$$h_l = \frac{l^2}{2m(pm + 2)} + a \left( \frac{m - a}{m} \right) \quad l = (pm + 2)a \quad a = 1, 2, \ldots, m \quad (21)$$

They describe dressed $a$-electrons with electric charge $q^e_a = a$ and “magnetic charge” $q^m_a = \frac{pm + 2}{m}a$ interacting through the neutral “cloud” associated with them (see eq.(13)).

Also we will consider correlators in which are present quasi-hole operators given by $\hat{V}^\alpha_{qh}^{a}(z|m, pm + 2)$ having conformal dimensions:

$$h_{qh}^a = \frac{l^2}{2m(pm + 2)} + \frac{a(m - a)}{2m(m + 2)} \quad (22)$$

where $(pm + 2)(a - 1) < l < (pm + 2)a$ and $a = 1, 2, \ldots, m$.

Their electric and magnetic charges are $q^e_l = \frac{l}{pm + 2}$ and $q^m_l = \frac{l}{m}$.

We should point out that $m$-ality in the neutral sector is coupled to the charged one in analogy to the case of Jain hierarchical fillings in order to assure the locality of electrons with respect to all the edge excitations [7]. This follows from the fact that our projection when applied to a local field automatically couples the discrete $Z_m$ charge of $U(1)$ with the neutral sector, in order to give a totally single-valued composite field.

Also notice that the $m$-electron vertex operator does not contain any neutral field. Therefore, the $m$-electron wave function is realized only by means of the $c_X = 1$ charged sector as proposed in ref.[17].

We are now ready to give the holomorphic part of the ground state wave function for the generic filling $\nu = \frac{m}{pm + 2}$. To such an extent we consider the $N_e$ single($a = 1$)-electrons correlator which factorizes into a Laughlin-Jastrow type term coming from the charged sector:

$$< N_e \alpha | \prod_{i=1}^{N_e} \hat{V}^\alpha_l(z_i|m, 2pm + 2) | 0 > = \prod_{i < j'=1}^{N_e} (z_i - z_{j'})^{\nu + \frac{2}{m}} \quad (23)$$
and a contribution coming from the neutral excitations:

\[
< 0 \prod_{i=1}^{N_e} \psi^\alpha_i(z_i|m, 2)|0 > = \frac{\sum_{\{j_i\}=1}^{m} \varepsilon^{(2i-1)j_i+j_1} \prod_{\{j_i, j_{i'}\}=1}^{m} (\varepsilon^{j_i} z_i^{1/m} - \varepsilon^{j_{i'}} z_{i'}^{1/m})^2}{\prod_{i<j'=1}^{N_e}(z_i - z_{i'})^2} \quad (24)
\]

For \( N_e \) multiple of \( m \) we observe that the non analytic part of the neutral fields \( \psi^\alpha_i(z_i|m, 2) \) is necessary to eliminate the non integer part of the exponent in the correlator of the charged fields. We also point out that in our formalism the correlators are given, in principle, for any \( m \) and \( N_e \) and that follows from the projection procedure only.

By considering the case \( m = 2, p \) odd that is for \( \nu = 1/q, q = p + 1 \) we get for the correlator of \( N_e \) single(\( a = 1 \))-electrons:

\[
< N_e a| \prod_{i=1}^{N_e} \hat{V}^{\sqrt{2q}}(z_i|2, 2q)|0 > = \prod_{i<i'=1}^{N_e} (z_i - z_{i'})^q Pf \left( \frac{1}{z_i - z_{i'}} \right) \quad (25)
\]

(where \( Pf \left( \frac{1}{z_i - z_{i'}} \right) = \alpha \left( \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \ldots \right) \) is the antisymmetrized product over pairs of electrons) which is in agreement with previous results [2, 3]. For the generic \( m \) case, even though it is hard to work out explicitly the sum over the phases in eq.(24), for the \( N_e \) point functions we found the clustering properties previously given in ref.[3] in another framework by grouping particles into clusters of \( m \).

In this last case the neutral modes describe parafermions and contribute to the ground state wave function with a generalized Pfaffian term. It would be very interesting to give an interpretation of the Pfaffian term (but also of its generalization presented in eq.(24)) in the context of a plasma description a la Laughlin in order to better understand the physics of the paired states [18].

In a similar way we also are able to evaluate correlators of \( N_e \) single(\( a = 1 \))-electrons in the presence of quasi-hole excitations. In particular for \( m = 2 \) and for two quasi-holes we get:

\[
< N_e \sqrt{2q} + 2/\sqrt{2q}| \prod_{i=1}^{N_e} \hat{V}^{\sqrt{2q}}(z_i|2, 2q)\hat{V}^{1/\sqrt{2q}}(w_1|2, 2q)\hat{V}^{1/\sqrt{2q}}(w_2|2, 2q)|0 >
\]

\[
< 2/\sqrt{2q}V^{1/\sqrt{2q}}(w_1|2, 2q)V^{1/\sqrt{2q}}(w_2|2, 2q)|0 >
\]

\[
= \prod_{i<i'=1}^{N_e} (z_i - z_{i'})^q Pf \left( \frac{(z_i - w_1)(z_{i'} - w_2) + (z_{i'} - w_1)(z_i - w_2)}{z_i - z_{i'}} \right) \quad (26)
\]

in agreement with the wave functions proposed in Ref.[2].

The explicit evaluation of correlation functions for the twist fields lies outside the scope of the present paper, but it does not affect the main results given in eq.(26). On the other
hand, it is fundamental to understand the non-Abelian statistics of the quasi-holes. We will analyze this aspect in a forthcoming paper.

We also point out that we are not considering the full set of primary fields in the theory. Indeed, also for \( p = 0 \) there are neutral fields which correspond to the “termal” fields of parafermionic theory [13].

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