1. Introduction

Let \( \text{Mor}_\beta(\mathbb{P}^1, Y) \) denote the moduli space of morphisms \( f \) from a complex projective line \( \mathbb{P}^1 \) to a smooth complex projective variety \( Y \) such that \( f_*[\mathbb{P}^1] = \beta \) where \( \beta \) is a given second homology class of \( Y \). We study the irreducibility and the rational connectedness of the moduli space when \( Y \) is a successive blowing-up of a product of projective spaces with a suitable condition on \( \beta \).

To state the Main Theorem proven in this paper, let us introduce some notation. Let \( X = \prod_{k=1}^m \mathbb{P}^{n_k}, X_0 = X \) and let \( \pi_i : X_i \to X_{i-1}, i = 1, \ldots, r, \) be a blowing-up of \( X_{i-1} \) along a smooth irreducible subvariety \( Z_i \). Let \( E_i \subset X_r \) be the total transform \((\pi_i \circ \ldots \circ \pi_r)^{-1}Z_i \) of the exceptional divisor associated to \( Z_i \) and let \( H_k \) be the divisor class coming from the hyperplane class of the \( k \)-th projective space \( \mathbb{P}^{n_k} \). Let \( m_i = \# \{ Z_j | j < i, (\pi_j \circ \ldots \circ \pi_r)^{-1}(Z_j) \supset E_i \} \).

So general points of \( Z_i \) are the \( m_i \)-th infinitesimal points of \( X \). Denote by \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) the open sublocus of \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \), consisting of \( f \) whose image does not lie on exceptional divisors: \( f(\mathbb{P}^1) \not\subset E_i, \forall i \).

**Main Theorem.** Assume that \( \beta \cdot (\pi^*H_k - \sum_{i=1}^r (m_i + 1)E_i^i) \geq 0, \forall k; \) and \( \beta \cdot E_i^i \geq 0, \forall i, \) where \( \pi = \pi_1 \circ \ldots \circ \pi_r \).

1. The moduli space \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) consists of free morphisms and is an irreducible smooth variety of expected dimension.
2. If \( Z_i \) are rationally connected for all \( i \), then a projective, birational model of \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) is also rationally connected.
3. The moduli space \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) is smooth and \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) is dense in \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) if one of the following conditions hold
   a. All \( \pi(E_i^i) \) are points in \( X \).
   b. All centers \( Z_i \) are convex (that is, \( H^1(\mathbb{P}^1, g^*T_{Z_i}) = 0 \) for any morphism \( g : \mathbb{P}^1 \to Z_i \)) and \( \pi(E_i^i) \) are disjoint to \( \pi(E_j^j) \) for any \( i \neq j \).
Note that the irreducibility (respectively, the rational connectedness of a projective, birational model) of the morphism space \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) implies that of the moduli space of rational curves \( C \) with numerical condition \( [C] = \beta \).

Our paper is motivated by two questions:

1. If \( Y \) is rationally connected, is \( \text{Mor}_\beta(\mathbb{P}^1, Y) \) also irreducible and rationally connected? If this does not always hold, does it hold for special values of \( \beta \)?

2. For surfaces fibered over \( \mathbb{P}^1 \) with genus 2 fibers (or more generally hyperelliptic fibers) and fixed numerical invariants, is the moduli space of such surfaces connected? If this does not always hold, does it hold for special values of the numerical invariants?

Several authors have studied the first question. The case of homogeneous spaces was treated by the first author and Pandharipande [7], and by Thom森 [10]. The case of small degree \( d \) general hypersurfaces \( Y = X_d \subset \mathbb{P}^n \) was handled by Harris, Roth and Starr [4]. The case when \( Y \) is the moduli spaces of rank 2 stable vector bundles, with fixed determinant of degree 1, on a smooth projective curve of genus \( g \geq 2 \) was investigated by Castravet [1]. She found all irreducible components and described the maximal rationally connected fibration of them.

Let \( \overline{M}_{0,n} \) be the moduli space of stable \( n \)-pointed rational curve. As a corollary of the Main Theorem, the space \( \text{Mor}_\beta(\mathbb{P}^1, \overline{M}_{0,n}) \) is connected for certain values of \( \beta \), since the space \( \overline{M}_{0,n} \) is a successive blowing-up of \( (\mathbb{P}^1)^{n-3} \) along smooth codimension 2 subvarieties ([6]) or a successive blowing-up of \( \mathbb{P}^{n-3} \) ([5]). This gives a step toward proving connectedness of the moduli space of hyperelliptic fibrations over \( \mathbb{P}^1 \) (presumably by replacing the hyperelliptic fibration by the fibration of quotient by the hyperelliptic involution, marked by the images of the Weierstrass points).

When \( Y \) is a blowing-up of a product of projective spaces along a smooth closed (not necessarily irreducible) subvarieties, we prove a slightly stronger result, Theorem 2 in section 2. In section 3, we prove the Main Theorem. The key idea of both proofs is to express the moduli space as a fibration — a fiber consists of the morphisms \( f \) which pass through given points of \( \bigcup_i Z_i \) at given points of domain \( \mathbb{P}^1 \) — and then to show that the general fiber is rationally connected and has the expected dimension under the condition on \( \beta \) as in the Main Theorem (and also as in theorem 1). When \( Y \) is a successive blowing-up, we will need to utilize Jet spaces and Jet conditions in order to show the rational connectedness of the general fiber. Furthermore we apply a result of Graber-Harris-Starr [3].

Throughout the paper, we will employ the well-known results of the deformation theory of morphisms of curves, as well as the established notation as in [8]. The complex number field \( \mathbb{C} \) will be the base field.
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2. Blowing-ups along a smooth subvariety

2.1. Set-up and a morphism \( \sigma \). Let \( \pi : \tilde{X} \to X \) be the blowing-up of a smooth projective variety \( X \) along a smooth closed subvariety \( Z \) with the exceptional divisor \( E \).

For a curve class \( \beta \in H^2(\tilde{X}, Z) \), consider the evaluation morphism
\[
\ev : \mathbb{P}^1 \times \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \to \tilde{X}
\]
\[(p, \tilde{f}) \mapsto \tilde{f}(p).\]

Assume that \( \beta \cdot E_i \geq 0 \) for all \( i = 1, \ldots, r \) where \( E_i \) are the exceptional irreducible divisors over the irreducible components \( Z_i \) of \( Z = \bigcup_{i=1}^r Z_i \). In general, \( \tilde{f}(\mathbb{P}^1) \subset E_i \) does not imply \( \tilde{f}(\mathbb{P}^1) \cdot E_i < 0 \). For example, we have:

**Example 1.** Consider the blowing-up \( \tilde{X} \) of \( X = \mathbb{P}^3 \) along a curve \( Z \cong \mathbb{P}^1 \) with a normal bundle \( N_{Z/X} = \mathcal{O}(1) \oplus \mathcal{O}(2) \). Then \( E = \mathbb{P}(\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2)) \). If \( C \) is a positive section (resp. the negative section), then by the construction of the sections and the universal property of the projectivization \( E \) of \( \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2) \), we see that \( C \cdot E = 1 \) (resp. \( C \cdot E = 2 \)).

This observation forces us to consider an open subvariety \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp \) of \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) consisting of \( \tilde{f} \) such that \( \tilde{f}(\mathbb{P}^1) \not\subseteq E \). Now the scheme-theoretic intersection \( \Gamma_{\pi_{\text{ev}}} \cap (\mathbb{P}^1 \times \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp \times Z) \cong \text{ev}^{-1}(E) \), where \( \Gamma_{\pi_{\text{ev}}} \) is the graph of the morphism \( \pi \circ \text{ev} \), can be regarded as a closed subscheme of \( \mathbb{P}^1 \times \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp \times Z \), which is proper and flat over \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp \). Thus it induces a natural morphism
\[
\sigma : \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\sharp \to \prod_{i=1}^r \text{Hilb}^{e_i}(\mathbb{P}^1 \times Z_i)
\]
\[\tilde{f} \mapsto (\Gamma_{f} \cap (\mathbb{P}^1 \times Z_1), \ldots, \Gamma_{f} \cap (\mathbb{P}^1 \times Z_r)),\]
where \( f := \pi \circ \tilde{f} \) and \( e_i := \beta \cdot E_i \). Here \( \text{Hilb}^0 Y \) of a variety \( Y \) is defined to be \( \text{Spec} \mathbb{C} \).

2.2. An exact sequence. The following lemma shows a sufficient condition of the generic smoothness of \( \sigma \).

**Lemma 1.** Let \( \pi : \tilde{X} \to X \) as above and suppose that \( \tilde{f}(\mathbb{P}^1) \not\subseteq E \).

1. There is a natural injective morphism of sheaves
\[
\tilde{f}^* \pi^* T_X(-E) \to \tilde{f}^* T_{\tilde{X}}.
\]
Lemma 2. If \( \tilde{f} : \mathbb{P}^1 \rightarrow \tilde{X} \) is transversal to \( E \), then the above injective morphism induces a short exact sequence
\[
0 \rightarrow \tilde{f}^*\pi^*T_X(-E) \rightarrow \tilde{f}^*T_{\tilde{X}} \rightarrow (\text{id}_{\mathbb{P}^1} \times \tilde{f})^*T_{\mathbb{P}^1 \times Z} \rightarrow 0.
\]
Furthermore, the associated morphism
\[
H^0(\mathbb{P}^1, \tilde{f}^*T_{\tilde{X}}) \rightarrow (\text{id}_{\mathbb{P}^1} \times \tilde{f})^*T_{\mathbb{P}^1 \times Z}
\]
is the derivative of \( \sigma \).

(3) For every \( \tilde{f} \) satisfying \( h^1(\mathbb{P}^1, \tilde{f}^*\pi^*T_X(-E)) = 0 \), the morphism \( \sigma \) is smooth at \([\tilde{f}]\).

**Proof.** The first morphism is defined by the pull-back of the extension of the isomorphism \( T_X \cong T_{\tilde{X}} \) away from \( E \). To check the existence of the extension, we consider the blowing-up \( \tilde{X} \rightarrow X \) locally as \((t, x, y) \mapsto (z_1 = t, z_2 = t x, z_3 = y)\), where \( t, x, y \) (resp. \( z_1, z_2, z_3 \)) is a system of local parameters of \( \tilde{X} \) (resp. \( X \)), and the bold letters denote multi-variables. Then the natural morphism of sheaves
\[
\pi^*T_X(-E) \rightarrow T_{\tilde{X}}
\]
defined by \((t \frac{\partial}{\partial z_1} \mapsto t \frac{\partial}{\partial z_1} - \sum x_i \frac{\partial}{\partial z_i}), (t \frac{\partial}{\partial z_2} \mapsto \frac{\partial}{\partial z_2})\), and \((t \frac{\partial}{\partial z_3} \mapsto \frac{\partial}{\partial z_3})\) is the extension. The second morphism is defined by, at \( p \) with \( \tilde{f}(p) \in Z \),
\[
(f_*|_p)^{-1} \oplus \pi_*|_{\tilde{f}(p)} : T_{\tilde{X}}|_{\tilde{f}(p)} = f_!T_{\mathbb{P}^1}|_p \oplus T_E|_{\tilde{f}(p)} \rightarrow T_{\mathbb{P}^1}|_p \oplus T_Z|_{\tilde{f}(p)},
\]
where \( T_Y|_y \) denotes the tangent space of a variety \( Y \) at a point \( y \). Now the rest of the proof is straightforward. \( \square \)

**Remark 1.** In fact the proof above shows that there is an exact sequence
\[
0 \rightarrow (\text{ev})^*(\pi^*T_X(-E)) \rightarrow \text{ev}^*T_{\tilde{X}} \rightarrow (\text{id}_{\mathbb{P}^1} \times \pi \circ \text{ev})^*T_{\mathbb{P}^1 \times Z} \rightarrow 0
\]
over \( \mathbb{P}^1 \times \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\circ \) where \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^\circ \) is the locus of all morphisms in \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) which are transversal to \( E \).

**Remark 2.** In Example 1, the 1-1 morphism whose image is the negative section is not free: The exact sequence
\[
0 \rightarrow N_{C/E} = \mathcal{O}(-1) \rightarrow N_{C/\tilde{X}} \rightarrow N_{E/\tilde{X}}|_C = \mathcal{O}(2) \rightarrow 0
\]
splits since \( C \) is a section. Therefore (1) of Lemma 1 is not true in general without the condition \( \tilde{f}(\mathbb{P}^1) \not\subseteq E \).

When \( \tilde{f}(\mathbb{P}^1) \subseteq E \), instead of Lemma 1 we have the following lemma.

**Lemma 2.** Assume that \( k \leq e + 1 \), \( e = E \cdot \tilde{f}_*(\mathbb{P}^1) \geq 0 \), and \( \tilde{f}(\mathbb{P}^1) \subseteq E \). If \( H^1(\mathbb{P}^1, f^*T_X(-e - k)) = 0 \) and \( H^1(\mathbb{P}^1, f^*T_Z(-k)) = 0 \), then we attain \( H^1(\mathbb{P}^1, \tilde{f}^*T_{\tilde{X}}(-k)) = 0 \).
Proof. i) Note that \( H^1(\mathbb{P}^1, f^*N_{Z/X}(-e-k)) = 0 \) by \( H^1(\mathbb{P}^1, f^*T_X(-e-k)) = 0 \) and
\[
0 \to T_Z \to T_X|_Z \to N_{Z/X} \to 0.
\]

ii) Note that \( H^1(\mathbb{P}^1, \tilde{f}^*T_{\tilde{X}}(-k)) = 0 \) and \( H^1(\mathbb{P}^1, \tilde{f}^*T_E(-k)) = 0 \) by i), \( H^1(\mathbb{P}^1, \tilde{f}^*T_Z(-k)) = 0 \), and the exact sequences
\[
0 \to \mathcal{O} \to \pi^*(N_{Z/X}) \otimes \mathcal{O}_E(1) \to T_\pi \to 0;
\]
\[
0 \to T_\pi \to T_E \to \pi^*T_Z \to 0,
\]
where \( T_\pi \) denotes the relative tangent bundle of \( \pi \).

iii) Finally, \( H^1(\mathbb{P}^1, \tilde{f}^*T_{\tilde{X}}(-k)) = 0 \) by
\[
0 \to T_E \to T_{\tilde{X}}|_E \to \mathcal{O}_{\tilde{X}}(E)|_E \to 0.
\]

Here we use the condition that \(-k + e \geq -1\).

\[\square\]

2.3. The fiber of \( \sigma \) when \( X = \mathbb{P}^n \). Let \( X = \mathbb{P}^n \) and denote
\[
c_H(\beta) = \beta \cdot (\pi^*H - E),
\]
where \( H \) is the hyperplane class of \( \mathbb{P}^n \). If \( c_H(\beta) \geq -1 \), then the first statement of Lemma \( \square \) implies the vanishing of the obstruction \( H^1(\mathbb{P}^1, \tilde{f}^*T_{\tilde{X}}) = 0 \), if \( \tilde{f}(\mathbb{P}^1) \not\subset E \), and hence the space \( \operatorname{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) is smooth. In addition, the general fiber of a morphism \( \sigma \) is smooth and has the expected dimension due to Lemma \( \square \).

Here the expected dimension of the fiber is by definition
\[
\exp \dim \operatorname{Mor}_\beta(\mathbb{P}^1, \tilde{X}) = \dim \prod_{i=1}^r \operatorname{Hilb}^{c_i}(\mathbb{P}^1 \times Z_i)
\]
\[
= \dim \mathbb{P}^H(\mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}) - ne = \dim \operatorname{Mor}_{\pi_*\beta}(\mathbb{P}^1, X) - ne.
\]

In the following lemma we investigate the irreducibility of the fiber of the morphism \( \sigma \) for \( X = \mathbb{P}^n \).

Lemma 3. Suppose that \( \pi_*\beta \neq 0 \) in \( H_2(X, \mathbb{Z}) \). Then we have:

1. Every nonempty fiber of \( \sigma \) is isomorphic to an open subset of a projective space.
2. If \( c_H(\beta) \geq -1 \), then \( \sigma \) is a smooth morphism at general points.
3. If \( c_H(\beta) \geq 0 \), then the general fiber of \( \sigma \) is nonempty.
4. If \( c_H(\beta) \geq 0 \) and \( \dim Z_i = 0 \) \( \forall i \), then \( \sigma \) is surjective.

Proof. First note that \( \operatorname{Mor}_{\pi_*\beta}(\mathbb{P}^1, X) \) contains \( \operatorname{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) as a quasi-projective subvariety over which the scheme ev\(^{-1}(Z_i)\) has the relative Hilbert polynomial \( e_i \) for all \( i = 1, \ldots, r \). We will describe a fiber of \( \sigma \) as a subscheme in
\[
\operatorname{Mor}_{\pi_*\beta}(\mathbb{P}^1, X) \subset \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}))
\]
where \( d = (\pi_*\beta) \cdot H \). If we let
\[
P := \prod P_i \in \prod \operatorname{Hilb}^{c_i}\mathbb{P}^1 \times Z_i, \quad P_i = \sum a c_a^{(i)}(p_a^{(i)}, q^{(i,a)}),
\]
\[
\text{Supp}(P_i) = \{ (p_a^{(i)}, q^{(i,a)}) \}_{a}, \quad p_a^{(i)} \neq p_a^{(i')}, \quad q^{(i,a)} \neq 0, \forall (i, a),
\]

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with the condition $e_0^{(i)} = 1$, $\forall a$ if dim $Z_i' \neq 0$ for some $i'$, then $\sigma^{-1}(P)$ is a subvariety of $\mathbb{P}H^0(\mathbb{P}^1, K_P)$ where $K_P$ is the kernel of the morphism of sheaves

$$
\mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1} \rightarrow (\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^1}/m^{e_0^{(i)}}_{P_i}) \otimes \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^n
$$

$$
f \mapsto \sum_{j=1}^n (\bigoplus_{i,a} [q^{(i,a)}_0 f_j - q^{(i,a)}_j f_0]) \otimes 1_j,
$$

where $1_j = (0, ..., 1, 0, ..., 0) \in \mathbb{C}^n$. Precisely speaking, $\sigma^{-1}(P)_{\text{red}}$ coincides with

$$
(\ast) \ \mathbb{P}H^0(\mathbb{P}^1, K_P) \cap \text{Mor}_{\mathbb{P}^1, \mathbb{P}^1}(\mathbb{P}^1, X) \setminus \bigcup_{P' \in \text{Hilb}^{n+1}(\mathbb{P}^1 \times Z): P' \supset P} \mathbb{P}H^0(\mathbb{P}^1, K_{P'}),
$$

where if $P'$ is not simple at $(p, q)$ and dim $Z_i \neq 0$ for some $i$, then $K_{P'}$ is defined as the kernel of

$$
K_P \rightarrow m_p/m_p^2 \otimes \mathcal{O}_{\mathbb{P}^1}(d) \otimes N_{Z/X}|_q
$$

$$
f \mapsto \sum_{j} (q_0 f_j - q_j f_0) \otimes \frac{\partial}{\partial z_j},
$$

where $N_{Z/X}|_q := T_X|_q/T_Z|_q$ (normal space) and $\{z_j := x_j/x_0 : j = 1, ..., n\}$ are the coordinates of $\mathbb{C}^n = \{x_0 \neq 0\} \subset \mathbb{P}^n$.

Since $\mathcal{O}_{\mathbb{P}^1}(d - \sum_{i,a} e_0^{(i)} P_i) \otimes \mathbb{C}^{n+1} \subset K_P \subset \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}$, the sheaf $K_P$ is isomorphic to $\bigoplus_{j=0}^r \mathcal{O}_{\mathbb{P}^1}(k_j)$ for some $k_j$ with constraints, $d \geq k_j \geq d - \sum_{i=1}^r e_i$ for all $j$. Now when dim $Z_i = 0$ for all $i$ and $k_j \geq -1$ for all $j$, then dim $\mathbb{P}H^0(\mathbb{P}^1, K_P) = \dim \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}) - ne$. When dim $Z_i > 0$ for some $i$ and $k_j \geq 0$, then dim $\mathbb{P}H^0(\mathbb{P}^1, K_P) = \dim \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathbb{C}^{n+1}) - n(e + 1)$ for $P' \supset P$. These facts applied to $(\ast)$ complete the proof. □

**Remark 3.** Note that Lemma 3 holds true for a product $X = \prod_k \mathbb{P}^{n_k}$ of projective spaces if we let $c_H(\beta) := \min\{\beta \cdot (\pi^* H_k - E)\}_k$ where $H_k$ is the hyperplane class of $k$-th component of the product space $X$.

### 2.4. Some elementary facts.

The followings are standard facts.

**Proposition 1.** (cf. [8, Proposition II.3.7], [2]) Let $X$ be a smooth variety and $Y$ be a subvariety. Then any free morphism $f : \mathbb{P}^1 \rightarrow X$ can be deformed to a morphism $f_e : \mathbb{P}^1 \rightarrow X$ which is transversal to $Y$.

**Lemma 4.** Let $X$ and $Y$ be varieties and assume that $Y$ is irreducible. Let $f : X \rightarrow Y$ be a dominant morphism in any irreducible component of $X$. Then if the general fiber of $f$ is irreducible, then $X$ is irreducible.

**Proof.** The proof is straightforward. □
2.5. A consequence. Let \( X \) be a product \( \prod_k \mathbb{P}^{n_k} \) of projective spaces \( \mathbb{P}^{n_k} \) and let \( \tilde{X} \) be a blowing-up of \( X \) along a smooth closed subvariety \( Z \). Denote by \( E \) the exceptional divisor and denote by \( H_k \) the divisor class. We assume that \( \pi_* \beta \neq 0 \) and \( e_i \geq 0 \) for all \( i \).

**Theorem 1.**

1. If \( \beta \cdot (\pi^* H_k - E) \geq -1 \), for all \( k \) and \( Z \) are finite points, then \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) is an irreducible smooth variety.
2. If \( \beta \cdot (\pi^* H_k - E) \geq 0 \) for all \( k \), then \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) is a nonempty, irreducible smooth variety.
3. If \( \beta \cdot (\pi^* H_k - E) \geq 0 \) for all \( k \), and all centers \( Z_i \) are convex, then \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) is an irreducible smooth variety.

**Proof.** We prove it when \( X = \mathbb{P}^n \). The condition \( \beta \cdot (\pi^* H - E) \geq -1 \) implies that \( H^1(\mathbb{P}^1, (f^* T_X)(-e)) = 0 \) by the Euler sequence on \( \mathbb{P}^n \). Now by (1) of Lemma 1, \( H^1(\mathbb{P}^1, (f^* T_X)) = 0 \), which implies that every irreducible component of \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) is smooth with the expected dimension.

**Proof of (1).** The first assertion follows from Lemma 3 and Lemma 4.

**Proof of (2).** Let us prove the second assertion of the theorem. Every element in \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) is a free morphism by (1) of Lemma 1. Therefore by Proposition 1, it is enough to show the irreducibility of the sublocus \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^0 \) of morphisms which are transversal to \( E \). By Lemma 3, the general fiber of \( \sigma \) restricted to any component of \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X})^0 \) has the expected dimension and irreducible. Thus the morphism \( \sigma \) restricted to any component is dominant on the irreducible variety \( \prod \text{Hilb}^e_i(\mathbb{P}^1 \times Z_i) \). Now the proof follows from Lemma 4. The moduli space is nonempty by Lemma 3.

**Proof of (3).** If a morphisms \( \tilde{f} \) lie on \( E \), using Lemma 2 with \( k = 0 \), we can deform it to an element in \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \); By Lemma 2, \( H^1(\mathbb{P}^1, f^* T_{\tilde{X}}) = 0 \) which implies the moduli space \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \) is smooth at \( \tilde{f} \). By the exact sequence in iii) of the proof of Lemma 2, we see that there is a deformation of \( \tilde{f} \) to an element in \( \text{Mor}_\beta(\mathbb{P}^1, \tilde{X}) \). Now the proof completes by 2) above.

By Remark 3, the same proof for the case of the product of projective spaces works. \( \square \)

3. Successive blowing-up Case

3.1. Set-up. Let \( X_0 = X \) be a smooth projective variety and let \( \pi_i : X_i \to X_{i-1}, \ i = 1, ..., r \), be a blowing-up of \( X_{i-1} \) along a smooth irreducible subvariety \( Z_i \). In general, the space \( X_r \) is a successive blowing-up of \( X \). Let \( E_i^t \subset X_r \) (resp. \( E_i^s \subset X_r \)) be the total (resp. strict) transform of the exceptional divisor associated to \( Z_i \) and let \( e_i = \beta \cdot E_i^s \). Denote by \( \text{Mor}_\beta(\mathbb{P}^1, X_r)^0 \) the sublocus of \( \text{Mor}_\beta(\mathbb{P}^1, X_r) \) of the morphisms \( \tilde{f} \) which are transversal to \( E = \bigcup_{i=1}^r E_i^t \) and do not intersect with \( E_i^s \cap E_j^s \) for \( i \neq j \). Then for \( e_i \geq 0, \forall i \)
we obtain a morphism
\[
\sigma : \text{Mor}_\beta(\mathbb{P}^1, X_\tau) \to \prod_{i=1}^r \text{Hilb}_{E_i}(\mathbb{P}^1 \times Z_i)
\]
\[
\tilde{f} \mapsto \prod_{i=1}^r \left( \Gamma_{\pi_i \circ \ldots \circ \pi_1 \circ \tilde{f}} \cap \mathbb{P}^1 \times (Z_i \setminus \cup_{j>i}(\pi_i \circ \ldots \circ \pi_j(\tilde{E}_j^*)) \right)
\]
as the generalization of the previous \(\sigma\) in subsection 2.1.

Inductive application of the exact sequence of Lemma 1 proves the following corollary.

**Corollary 1.** Suppose that \(H^1(\mathbb{P}^1, f^*(T_X(-\sum E_i^*))) = 0\), where \(f = \pi_1 \circ \ldots \circ \pi_r \circ \tilde{f}\). Then the morphism \(\sigma\) is smooth at \(\tilde{f}\).

### 3.2. Jet spaces.
We want to show that the general fiber of \(\sigma\) is rationally connected, provided with a suitable condition on \(\beta\) when \(X = \mathbb{P}^n\) or their products. However, it is hard to analyze the fiber of \(\sigma\) directly as done in subsection 2.3. Our strategy is to introduce an auxiliary morphism \(\tau\) by imposing further conditions on jets of the morphism \(f : \mathbb{P}^1 \to X\). It turns out that the fiber of \(\tau\) is simple to study. Since the jet conditions on \(f\) can be translated to the vanishing conditions on the blowing-up space, we will be able to express the general fiber of \(\sigma\) by the fibers of \(\tau\) (more precisely its product \(\tau_m\)) which are rationally connected.

To introduce \(\tau\), let \(J^k_p X = \text{Mor}((\text{Spec}\mathbb{C}[e]/(e^{k+1}), (X, q))\) be the \(k\)-jet space of \(X\) at \(q \in X\). Then the morphism \(f : \mathbb{P}^1 \to X\) naturally assigns an element \([f]_p^k \in J^k_p X\) for any \(p \in \mathbb{P}^1\). Using the assignment we define a morphism
\[
\tau : \left( ([\mathbb{P}^1]^l \setminus \Delta) \times \text{Mor}_\beta(\mathbb{P}^1, X) \right) \to \left( \mathbb{P}^1 \times J^k X \right)^l
\]
\[
(p, f) \mapsto ([p_i, [f]_{p_i}]_{i=1, \ldots, l}),
\]
where \(\Delta\) is the big diagonal and \(J^k X = \bigsqcup_{p \in X} J^k_p X\).

**Lemma 5.** If \(H^1(\mathbb{P}^1, f^*(T_X(-(k+1)l))) = 0\), then \(\tau\) is smooth at \((p, f)\).

**Proof.** The natural exact sequence
\[
0 \to f^*T_X(-(k+1)l)) \to f^*T_X \to f^*T_X \otimes \left( \bigoplus_{i=1}^l \mathcal{O}_{p_i, \mathbb{P}^1}/m_{p_i}^{k+1} \right) \to 0
\]
induces the map
\[
H^0(\mathbb{P}^1, f^*T_X) \to H^0(\mathbb{P}^1, f^*T_X \otimes \left( \bigoplus_{i=1}^l \mathcal{O}_{p_i, \mathbb{P}^1}/m_{p_i}^{k+1} \right))
\]
which is the tangent map \(T\tau|_{\prod_{p_i, f}}|_{0 \times H^0(\mathbb{P}^1, f^*T_X)}\). (Note that the exactness holds since \(p_i\) are pairwise distinct for all \(i\).) Indeed
\[
T_{J^k_p X}|_{[f]_p^k} = H^0(\text{Spec}\mathbb{C}[e]/(e)^{k+1}, ([f]_p^k)^*T_X)) = H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{O}_p/m_{p}^{k+1})
\]
by base change. This induced morphism is surjective by the assumption. \(\square\)
3.3. A general simple fact. Let $\pi: \tilde{X} \to X$ be a blowing-up of a smooth variety $X$ along a subvariety $Z$. Let $E_q := \pi^{-1}(q)$ where $q$ is a smooth point of $Z$. Note that there is a natural morphism $j^k_q : (J_q^k X)^o \to \bigcup_{w \in E_q} J_q^{k-1} \tilde{X}$ defined by lifting, where $(J_q^k X)^o$ consists of $s \in J_q^k X$ which are, as $k$-jet arc, transversal to $Z$ at $q$.

**Lemma 6.** The morphism $j^k_q$ is smooth and every fiber is a rational variety.

*Proof.* This is a local problem at $q$. So we may assume that $X = \mathbb{C}^n$, $q = 0$, $Z = \{0\} \times \mathbb{C}^l$ and $\pi(t,x,y) = (t,tx,y)$. Consider a $k$-jet $s(t)$ at $t = 0$ such that $\frac{ds}{dt}|_{t=0} \neq 0$, then $(s_{1}(t), \frac{s_{2}(t)}{s_{1}(t)}, ..., \frac{s_{l}(t)}{s_{1}(t)}, s_{l+1}(t), ..., s_{n}(t))$ mod $t^k$ is, by definition, $j^k(s(t))$. This shows that the morphism $j^k_q$ is regular. Then it is straightforward to check the smoothness of $j^k_q$ and the rationality of the fiber. \[\Box\]

3.4. The morphism $\tau_m$ and its fibers when $X = \prod_{j=1}^{m} \mathbb{P}^{n_j}$. We define a morphism

$$\tau_m : ((\mathbb{P}^1)^{\sum_{i}e_i} \setminus \Delta) \times \text{Mor}_{\beta}(\mathbb{P}^1, X) \to \prod_{i=1}^{r}(\mathbb{P}^1 \times J_{m_i} X)^{e_i},$$

similar to the $\tau$ as in subsection 3.2. Here $m_i$ are nonnegative integers and $m = (m_1, ..., m_r)$.

**Lemma 7.** If the target space $X$ is a product of projective spaces, $\prod_{j=1}^{m} \mathbb{P}^{n_j}$, then the fibers of $\tau_m$ (with their induced reduced scheme structure) are rational varieties.

*Proof.* We will prove that when $X = \mathbb{P}^n$ and $l = 1$, every fiber of $\tau$ defined subsection 3.2 is a linear subvariety of $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)) \otimes \mathbb{C}^{n+1})$. The general case is straightforward from the proof. Now it is easy to check that $p \times f$ and $p \times g$ are in the same fiber of $\tau$, then $p \times (\mu f + \lambda g := (\mu f_0 + \lambda g_0, ..., \mu f_n + \lambda g_n))$ is in the same fiber for all but finite number of $(\mu, \lambda) \in \mathbb{P}^1$.

\[\Box\]

3.5. A proof of the Main Theorem and example. Let $m_i = \#\{Z_j \mid j < i, \ (\pi_j \circ ... \circ \pi_r)^{-1}(Z_j) \supset E_i\}$. So general points of $Z_i$ are the $m_i$-th infinitesimal points of $X$. For each $i$, we reindex $Z_j$ so that $\pi^{-1}(Z_{i_k}) \supset E_i$ where $i_1 < ... < i_{m_i}$.

**Proof of (1) and (2) of the Main Theorem.** By the assumption on $\beta$ and (1) of Lemma 1, the space $\text{Mor}_{\beta}(\mathbb{P}^1, X_r)^o$ is smooth and has the expected dimension, and its elements are free morphisms. Hence $\text{Mor}_{\beta}(\mathbb{P}^1, X_r)^o$ is open dense in $\text{Mor}_{\beta}(\mathbb{P}^1, X_r)$ by Proposition 1. Also note that $\sigma$ is a smooth morphism by Corollary 1. Therefore by Lemma 1 and Theorem 2 below, in order to verify (1) and (2) of the Main Theorem, it is enough to show that the general fiber of $\sigma$ is rationally connected. Let $P = \sum(p_i, q_i) \in$
\[\prod \text{Hilb}^e_i(\mathbb{P}^1 \times Z_i)\] such that all points in \(P\) are simple. Then we obtain the inclusion
\[\pi_\beta \circ \sigma^{-1}(P) \subset \text{pr}_2 \circ (\tau_m)^{-1}((p_i \times (j_{\pi m_1}^{1}(q_i) \circ \ldots \circ j_{\pi m_i}^{1}(q_i))^{-1}(q_i))_{i=1}^{e_i})\]
where \(\pi_\beta : \text{Mor}_\beta(\mathbb{P}^1, X) \subset \text{Mor}_{\pi \beta}(\mathbb{P}^1, X)\) is the natural embedding, and \(\text{pr}_2\) is the projection to the second factor. In \((**\)**) the right hand side (for short, RHS) includes morphisms \(\tilde{f}\) with \(\text{deg} \tilde{f}^{-1}(E) \geq \beta \cdot E\). This is the reason why both sides may not coincide.

Since \(\tau_m\) is a dominant morphism with a rationally connected general fiber by Lemma 5 and Lemma 6, RHS is also rationally connected, thanks to Theorem 2. Since LHS is open subset of RHS, we conclude that LHS is also rationally connected.

**Proof of (3) of the Main Theorem.** In case i): If \(f\) lies on \(E_i\) for some \(i\), then \(e_i < 0\): Take a hypersurface \(Y\) of \(X\) such that \(Y\) contains the image of \(f\); and the strict transform \(D\) of \(Y\) under \(\pi\) does not contain the image of \(f\). Then \(0 = Y \cdot f_s[\mathbb{P}^1] = \pi^*Y \cdot \tilde{f}_s[\mathbb{P}^1] = (D + \sum a_i E_i) \cdot \tilde{f}_s[\mathbb{P}^1]\) shows that \(E_i \cdot \tilde{f}_s[\mathbb{P}^1]\) is negative for some \(i\).

In case ii): This is the case (3) of Theorem 1. \(\square\)

**Theorem 2.** (3) Let \(f : X \to Y\) be a dominant morphism between irreducible varieties \(X\) and \(Y\). If \(Y\) and the general fiber of \(f\) is rationally connected, then \(X\) is rationally connected.

**Example 2.** Let \(Y\) be a quadratic line complex in \(\mathbb{P}^5\), which is a complete intersection of two smooth quadrics in \(\mathbb{P}^5\). Then \(Y\) is isomorphic to the moduli space of isomorphism classes of stable rank 2 vector bundle on a curve of genus \(g = 2\) with fixed determinant of degree 1 [9]. Let \(\tilde{X}\) be the blowing-up of \(Y\) along a line and let \(\tilde{Q}\) be the inverse image of the line. Then \(\tilde{X}\) is a blowing-up of \(\mathbb{P}^3\) along a smooth quintic curve \(C\). Let \(E_C\) be the inverse image of the curve \(C\).

\[
E_C \subset \tilde{X} \supset \tilde{Q} \\
\pi \not\subset \pi' \\
C \subset \mathbb{P}^3 \supset Y \supset \text{line}
\]

Then \(\pi(\tilde{Q})\) is a quadric surface \(Q\) containing \(C\). Therefore \(\tilde{Q} = 2H - E_C\) where \(H\) is the proper transform of a hyperplane class in \(\mathbb{P}^3\). Castravet [Cas] shows that there are at least two components (nice one, and almost nice one) of \(\text{Mor}_d(\mathbb{P}^1, Y)\) with the expected dimension. The almost nice component consists of morphisms \(\mathbb{P}^1 \to Y\) which are \(d\) to 1 onto lines in \(Y\) [11]. Here \(0 < d \in \mathbb{Z} \cong H_2(Y, \mathbb{Z})\) with respect to the ample generator of Pic\(Y\). These two components \(\text{Mor}_d(\mathbb{P}^1, Y)\) are birational to two components of \(\text{Mor}_{(d, e)}(\mathbb{P}^1, \tilde{X})\) with \(e = 2d\). Here \((d, e) \in \mathbb{Z} \times \mathbb{Z} \cong H_2(\tilde{X}, \mathbb{Z})\) with respect to \(\pi^*(H)\) and \(E_C\). In this case note that at every point in the corresponding component of the almost nice component, \(\sigma\) is not smooth.
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