AREA-PRESERVING MEAN CURVATURE FLOW OF ROTATIONALLY SYMMETRIC HYPERSURFACES WITH FREE BOUNDARIES

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Abstract. In this paper, we consider the area-preserving mean curvature flow with free Neumann boundaries. We show that for a rotationally symmetric n-dimensional hypersurface in $\mathbb{R}^{n+1}$ between two parallel hyperplanes will converge to a cylinder with the same area under this flow. We use the geometric properties and the maximal principle to obtain gradient and curvature estimates, leading to long-time existence of the flow and convergence to a constant mean curvature surface.

1. Introduction and the main results

A hypersurface $M_t$ in Euclidean space is said to be evolving by mean curvature flow if each point $X(\cdot)$ of the surface moves, in time and space, in the direction of its unit normal with speed equal to the mean curvature $H$ at that point. That is

$$\frac{\partial X}{\partial t} = -H\nu(x, t),$$

where $\nu(x, t)$ is the outer unit normal. It was first studied by Brakke in [6] from the viewpoint of geometric measure theory. In [12], G. Huisken showed that any compact, convex hypersurface without boundary converges asymptotically to a round sphere in a finite time interval. Mean curvature flow is also the steepest descent flow for the area functional, evolving to minimal surfaces. In [13], Huisken initiated the idea of considering the following volume-preserving mean curvature flow,

$$\frac{\partial}{\partial t} X(x, t) = (h(t) - H(x, t))\nu(x, t),$$

where $h(t) = \frac{\int_{M_t} H d\mu}{\int_{M_t} d\mu}$ is the average of the mean curvature on $M_t$. Huisken proved if the initial hypersurface $M_0^n$ ($n \geq 2$) is uniformly convex, then the evolution equation (1.2) has a smoothly solution $M_t$ for all times $0 \leq t < \infty$ and $M_t$ converges to a round sphere enclose the same volume as $M_0$ in the $C^\infty$-topology as $t \to \infty$. In [20], Pihan studied the following area-preserving mean curvature flow.

$$\frac{\partial}{\partial t} X(x, t) = (1 - h(t)H(x, t))\nu(x, t).$$

Here $h(t) = \frac{\int_{M_t} H d\mu}{\int_{M_t} H d\mu}$, Pihan showed that if the initial hypersurface is compact without boundary, (1.3) has a unique solution for a short time under the assumption $h(0) > 0$. For $n = 1$, Pihan also showed that an initially closed, convex curve in the
plane converges exponentially to a circle with the same length as the initial curve. For \( n \geq 2 \), McCoy in [16] showed that if the initial \( n \)-dimensional hypersurface \( M_0 \) is strictly convex then the evolution equation (1.3) has a smooth solution \( M_t \) for all time \( 0 \leq t < \infty \), and \( M_t \) converge, as \( t \to \infty \), in the \( C^\infty \)-topology, to a sphere with the same surface area as \( M_0 \). In [11], Huang and Lin use the idea of iteration of Li in [14] and Ye in [23], in cases of volume preserving mean curvature flow and Ricci flow, respectively. And obtained the same result, by assuming that the initial hypersurface \( M_0 \) satisfies \( h(0) > 0 \) and \( \int_{M_0} |A|^2 - \frac{H^2}{n} d\mu \leq \epsilon \).

In this paper, we study the area preserving mean curvature flow with free boundaries, where a restriction on the angles of boundaries with fixed hypersurfaces in Euclidean space are imposed. In this setting, there are some interesting works (cf. [7], [19] and [22]). In these papers, the authors study the mean curvature flow with Neumann and Dirichlet free boundaries. Let \( \Sigma \) be a fixed hypersurface smoothly embedded in \( \mathbb{R}^{n+1} \). We say \( X(x, t) \) is evolved by mean curvature flow with free Neumann boundary condition on \( \Sigma \), if

\[
\frac{\partial X}{\partial t} = -H \nu, \quad \forall (x, t) \in M^n \times [0, T),
\]

\[
\langle \nu_{M_t}, \nu_{\Sigma_i} \circ X \rangle(x, t) = 0, \quad \forall (x, t) \in \partial M^n \times [0, T),
\]

\[
X(\cdot, 0) = M_0,
\]

\[
X(x, t) \subset \Sigma_i, \quad \forall (x, t) \in \partial M^n \times [0, T), i = 1, 2.
\]

The volume-preserving mean curvature flow with free Neumann boundaries was first studied by Athanassenas in [2].

Let \( M_0 \) be a complete \( n \)-dimensional hypersurface with boundary \( \partial M_0 \neq \emptyset \). Assume \( M_0 \) is smoothly embedded in the domain

\[ G = \{ x \in \mathbb{R}^{n+1} : 0 < x_{n+1} < d, d > 0 \}, \]

We denote by \( \Sigma_i (i = 1, 2) \), the two parallel hyperplanes bounding the domain \( G \), and assume \( \partial M_0 \subset \Sigma_i (i = 1, 2) \). Then Athanassenas proved the following theorem in [2].

**Theorem 1.1.** Let \( V, d \in \mathbb{R} \) be given two positive constants. \( M_0 \subset G \) is a smooth, rotationally symmetric, initial hypersurface which intersects \( \Sigma_i (i = 1, 2) \) orthogonally at the boundaries which encloses the volume \( V \). Then the free Neumann boundaries problem for equation (1.2) has a unique solution on \( [0, +\infty) \), which converges to a cylinder \( C \subset G \) of volume \( V \) under assumption \( |M_0| \leq \frac{V}{\pi} \) as \( t \to \infty \).

Other works on this problem were investigated in [3], [4], [17] and [18]. In this paper, we consider the following problem for area-preserving mean curvature flow with free Neumann boundaries.

**Problem 1.1.**

\[
\frac{\partial}{\partial t} X(x, t) = (1 - h(t)H(x, t))\nu(x, t), \quad \forall (x, t) \in M^n \times [0, T),
\]

\[
\langle \nu_{M_t}, \nu_{\Sigma_i} \circ X \rangle(x, t) = 0, \quad \forall (x, t) \in \partial M^n \times [0, T), i = 1, 2,
\]

\[
X(\cdot, 0) = M_0,
\]

\[
X(x, t) \subset \Sigma_i, \quad \forall (x, t) \in \partial M^n \times [0, T), i = 1, 2.
\]
We prove the following main theorem for Problem 1.1.

**Main Theorem.** Let \( V, d \in \mathbb{R}^+ \) be given two constants and \( M_0 \subset G \) to be a smooth, rotationally symmetric, mean convex initial hypersurface which intersects \( \Sigma_i (i = 1, 2) \) orthogonally at the boundaries. Then the solution to Problem 1.1 exists for all times \( t > 0 \) and converges to a cylinder of the same area with \( M_0 \) under the assumption \( |M_0| \leq \frac{V}{d} \).

**Remark 1.1.** We say \( M_0 \) is mean convex if the mean curvature is positive everywhere. The condition of mean convex will be used to prove the equation (1.3) is strictly parabolic. In [20], Pihan shows that the equation (1.3) is strictly parabolic for a short time if \( h(0) > 0 \). And as the case of volume-preserving mean curvature flow in [3], Problem 1.1 is a Neumann boundary problem for strictly parabolic equation, from which we obtain the short time existence. And also see [19] and [22] for details of general cases.

This paper is organized as follows. In Section 2, we give some definitions and preliminaries. In Section 3, we show some basic properties of this flow. We prove that the property of mean convexity can be preserved under equation (1.3) and the surfaces do not pinch off under the condition of our main theorem. In Section 4, we use the property of mean convexity and maximal principle to show the curvature estimates. Gradient and curvature estimates lead to long time existence to a constant curvature surface. And we prove our main theorem in Section 5. The methods we use here are those introduced by Athanassenas in [2], Ecker and Huisken in [9]. We use the free Neumann boundary condition to convert the boundary estimates to interior estimates (see Lemma 3.4, Lemma 4.1 and Theorem 4.1). We put the condition of mean convexity here is to give an upper and lower bounds for \( h(t) \) and \( v(x, t) \), which is crucial for our curvature estimates.

## 2. Preliminaries

We adopt the similar notations of Huisken in [12]. Let \( M \) be an \( n \)-dimensional Riemannian manifold. Vectors on \( M \) are denoted by \( X = \{ X^i \} \), covectors by \( Y = \{ Y_i \} \) and mixed tensors by \( T = \{ T^i_{jk} \} \). The induced metric and the second fundamental form on \( M \) are denoted by \( g_{ij} = g^{ij} \) and \( A = \{ h_{ij} \} \) respectively. The surface area element of \( M \) is given by

\[
\mu = \sqrt{\text{det}(g_{ij})}.
\]

For tensors \( T_{ijkl} \) and \( U_{ijkl} \) on \( M \), we have the inner product

\[
(T_{ijkl}, U_{ijkl}) = g^{i\alpha} g^{j\beta} g^{k\gamma} g^{l\delta} T_{ijkl} U_{\alpha\beta\gamma\delta}
\]

and the norm

\[
|T_{ijkl}|^2 = (T_{ijkl}, T_{ijkl}).
\]

The trace of the second fundamental form, \( H = g^{ij} h_{ij} \), is the mean curvature of \( M \), and \( |A|^2 = g^{ik} g^{jl} h_{ij} h_{kl} \) is the square of the norm of the second fundamental form on \( M \). We also denote

\[
\bar{C} = \text{tr}(A^3) = g^{ij} g^{kl} g^{mn} h_{ik} h_{jm} h_{ln}.
\]

If \( X: M^n \hookrightarrow \mathbb{R}^{n+1} \) smoothly embeds \( M^n \) into \( \mathbb{R}^{n+1} \), then the induced metric \( g \) is given by \( g_{ij} = \left( \frac{\partial X}{\partial x_i}(x), \frac{\partial X}{\partial x_j}(x) \right) \) and the second fundamental forms \( h_{ij} = \left( \frac{\partial^2 X}{\partial x_i \partial x_j}(x) \right) \).
\[ \langle \frac{\partial X}{\partial x}(x), \frac{\partial X}{\partial x}(x) \rangle, \text{ where } \langle \cdot, \cdot \rangle \text{ is the ordinary scalar product of vectors in } \mathbb{R}^{n+1}. \]

The matrix of the Weingarten map of \( M \) is \( h_{ij}(x) = g^{ik}(x)h_{kj}(x) \). The eigenvalues of this matrix are the principal curvatures of \( M \). The induced connection on \( M \) is given via the Christoffel Symbols.

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right). \]

The covariant derivative, for a vector field \( v = v^j \frac{\partial}{\partial x^j} \) is given by

\[ \nabla_i v^j = \frac{\partial v^j}{\partial x^i} + \Gamma^j_{ik} v^k, \]

\[ \nabla_i v_j = \frac{\partial v_j}{\partial x^i} - \Gamma^j_{ik} v_k. \]

The Laplacian of \( T \) is

\[ \Delta T = g^{ij} \nabla_i \nabla_j T. \]

The Riemannian curvature tensor on \( M \) can be given through the Gauss Equations

\[ R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}. \]

We denote \( |M_t| \) to be the surface area of \( M_t \). We assume \( M_0 \) is mean convex and rotationally symmetric about an axis which intersects \( \Sigma \) orthogonally. We denote this axis by \( x_{n+1} \) and use the parametrization

\[ \rho(x_{n+1}) : [0, d] \mapsto \mathbb{R} \]

for the generating curve of a surface of revolution, which is a radius function.

## 3. Basic properties

From now on, we write \([0, T)\) to indicate the maximal time interval for which the flow exists. First we verify that the surface area does indeed remain fixed under the area-preserving flow (1.3), while the enclose volume does not decrease. The rotationally symmetric property is preserved under the equation (1.3). This is clear from the evolution equation, since the mean curvature and the normal are symmetric.

**Lemma 3.1.** The surface area of \( M_t \) remains constant throughout the flow, that is

\[ \frac{d}{dt} \int_{M_t} d\mu_t \equiv 0. \]

**Proof.** We apply the first variation of area formula to the vector field \( \frac{\partial X}{\partial t} \), extended appropriately, and the divergence theorem,

\[ \frac{d}{dt} \int_{M_t} d\mu_t = \int_{M_t} \text{div}_{M_t}(\frac{\partial X}{\partial t}) d\mu_t = - \int_{M_t} (1 - hH) H d\mu_t \equiv 0. \]

\[ \square \]

**Lemma 3.2.** The volume enclosed by \( M_t \) does not decrease throughout the flow. That is, if \( E_t \subset \mathbb{R}^{n+1} \) is the \((n + 1)\)-dimensional set enclosed by \( M_t \) and the two parallel planes \( \Sigma_i \), then

\[ \frac{d}{dt} \text{Vol}(E_t) \geq 0 \]
Proof. We first extend \( \frac{\partial X}{\partial t} \) to a vector field on the whole of \( E_t \), then apply the first variation of area formula and divergence theorem,

\[
\frac{d}{dt} \text{Vol}(E_t) = \int_{E_t} \text{div}_{\mathbb{R}^{n+1}} \left( \frac{\partial X}{\partial t} \right) dV = \int_{\partial E_t} \frac{\partial X}{\partial t} \nu > d\mu_t
\]

\[
= \int_{M_t} < \frac{\partial X}{\partial t} \nu > d\mu_t + \int_{\Sigma_t} < \frac{\partial X}{\partial t} \nu > d\mu_t
\]

\[
= \int_{M_t} d\mu_t - \left( \int_{M_t} H d\mu_t \right)^2 \geq 0.
\]

Here we have use the free Neumann boundary condition to obtain the integral on \( \Sigma_t \) is 0. \( \square \)

As in [16] (Section 4), we have the following evolution equations.

**Lemma 3.3.** We have

1. \( \frac{\partial}{\partial t} g_{ij} = 2(1 - hH)h_{ij} \);
2. \( \frac{\partial}{\partial t} g^{ij} = -2(1 - hH)h^{ij} \);
3. \( \frac{\partial}{\partial t} h = h \nabla H \);
4. \( \frac{\partial}{\partial t} h_{ij} = h \Delta h_{ij} + (1 - 2hH)h_{ij mn}h_{mn} + h |A|^2 h_{ij} \);
5. \( \frac{\partial}{\partial t} h^{ij} = h(\Delta h^{ij} + |A|^2 h^{ij}) - h_{m n} h^{mn} \);
6. \( \frac{\partial}{\partial t} H = h \Delta H - (1 - hH)|A|^2 \);
7. \( \frac{\partial}{\partial t} |A|^2 = h(\Delta |A|^2 - 2 \nabla A|^2 + 2|A|^4) - 2 \tilde{C} \);
8. \( \frac{\partial}{\partial t} h \Delta H^2 = -2h|\nabla H|^2 - 2(1 - hH)|A|^2 \).

**Lemma 3.4.** The mean curvature is positive on \( M_t \), \( t \in [0, T) \). Furthermore on the boundaries \( \partial M_t \), we have \( \lim_{x_{n+1} \to 0} H(x, t) = a(t) > 0 \) and \( \lim_{x_{n+1} \to d} H(x, t) = b(t) > 0 \).

Proof. Since we consider the hypersurface has free boundaries, we can not directly use the maximal principle. Suppose the first time \( H(x, t) = 0 \) is attained at an interior point of \( M_t \), then from Lemma 3.3, we have

\[
(\frac{\partial}{\partial t} - h \Delta) H^2 = -2h|\nabla H|^2 - 2(1 - hH)|A|^2
\]

From the maximal principle, we know \( H(x, t) = 0 \) can not be attained at an interior point of \( M_t \). If \( t_0 \) is the first time such that \( \lim_{x_{n+1} \to 0} H(x, t_0) = 0 \) or \( \lim_{x_{n+1} \to d} H(x, t_0) = 0 \). By a reflection of \( \Sigma_1 \) and \( \Sigma_2 \), we can define two pieces of new hypersurfaces outside the boundary, which satisfies the free Neumann boundaries conditions. Denote \( \tilde{M}_{t_0} \) to be the new hypersurface and \( \tilde{\rho}(x_{n+1}) \) its radius function . Precisely,

\[
\tilde{\rho}(x_{n+1}) = \begin{cases} 
2d - |x_{n+1}| & d \leq |x_{n+1}| \leq 2d \\
\rho(|x_{n+1}|) & 0 \leq |x_{n+1}| < d \\
\rho(-x_{n+1}) & -d \leq |x_{n+1}| < 0
\end{cases}
\]

i.e. \( \tilde{H}(x_1, x_2, \ldots, x_n, 0, t_0) = \lim_{x_{n+1} \to 0} H(x, t_0) \).

Then at the boundary points, \( \frac{\partial}{\partial t} \tilde{H}(x, t_0) \leq 0, \Delta \tilde{H}(x, t_0) > 0 \) So the maximal principle can still be applied, which proves the lemma. \( \square \)

Now we will show that the radius of the hypersurface \( M_t \) has uniform lower and upper bounds for any time \( t \in [0, T) \). The method follows from [2](Lemma 1).
Lemma 3.5. Under the conditions of the Main Theorem, there exist constant $r$ and $R$ only depending on $n, d, V$ and $|M_0|$ such that $r \leq \rho(x_{n+1}, t) \leq R$ for any $t \in [0, T)$.

Proof. Given an initial surface $M_0$, we denote by $C$ the cylinder with the same enclosed volume $V$ as $M_0$ in $G$. Assume that there is some $t_0 > 0$ such that $M_{t_0}$ pinches off. We project $M_{t_0}$ onto the plane $\Sigma_1$, using the natural projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$. Then

$$|M_{t_0}| \geq |\pi(M_{t_0})|.$$  

Any $M_t$ has to intersect the cylinder $C$ at least once by the volume constrain, that the volume of $M_t$ is not decreasing. Therefore

$$|M_0| = |M_{t_0}| > |\pi(M_{t_0})| > |\pi(C)| = \omega_n \rho_C^n = \frac{V}{d}.$$  

Here $\rho_C$ is the radius of $C$, and $\omega_n$ is the volume of unit ball in $\mathbb{R}^n$, then we obtain a contradiction. For the upper bound, we assume that $\rho(x_{n+1}, t)_{\text{max}} = R(t)$, then we have

$$|M_0| = |M_t| > \omega_n \cdot (R(t) - \rho_C)^n,$$

which implies

$$R(t) < \rho_C + \left(\frac{|M_0|}{\omega_n}\right)^\frac{1}{n}.$$  

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4. Curvature estimates

Let $\hat{x} = (x_1, \ldots, x_n, 0)$, and $\omega = \frac{\hat{x}}{|\hat{x}|}$ denote the unit outer normal to the cylinder intersecting $M_t$ at the point $X(P, t)$. As in [2] and [21], we define the height function of $M_t$ to be $u = (X, \omega)$. And define $v(x, t) = \langle \omega, \nu \rangle^{-1} = \sqrt{(\dot{\rho})^2 + 1}$, where $\dot{\rho}$ is the derivative of $\rho$ about $x_{n+1}$. From Lemma 3.5, we can obtain the height estimate $r \leq u(x, t) \leq R$. Now we show that $v$ has an upper bound under the assumption $T < \infty$. The kernel ideal is that according to our parametrization, the points where $v$ tends to infinity are not the singular points of the evolving surface, and the $|A|$ tends to zero at these points.

Lemma 4.1. If $T < \infty$, then $v(x, t) \leq M_T < +\infty$ for any $t \in [0, T]$, in the limitation sense when $t = T$, i.e. $v(x, T) = \lim_{t \to T} v(x, t)$. Here $M_T$ is a constant depending on $n, d, T, r, R, V$ and $|M_0|$.  

Proof. Since $M_t$ is rotationally symmetric we have $H = \kappa_1 + (n - 1)\kappa_2$, where $\kappa_1$ and $\kappa_2$ denote the principle curvatures. If we parameterize $M$ by its radius function $\rho \in C^\infty([0, d])$, then clearly

$$H = \frac{-\dot{\rho}}{(1 + \rho^2)^\frac{3}{2}} + \frac{n - 1}{\rho(1 + \rho^2)^\frac{1}{2}}.$$  

Suppose $t_0$ is the first time such that

$$\lim_{x_{n+1} \to s} v(x, t_0) = \lim_{x_{n+1} \to s} \sqrt{(\dot{\rho})^2(x_{n+1}, t_0) + 1} = +\infty$$

for some $s \in (0, d)$. Since

$$\lim_{x_{n+1} \to s} \dot{v} = \frac{1}{2} \frac{2\dot{\rho}}{\sqrt{(\dot{\rho})^2 + 1}} = 0,$$

we have $\lim_{x_{n+1} \to t_0} \dot{\rho}(x_{n+1}, t_0) = 0$, then we have $H = 0$ at this point, which is a contradiction with $H > 0$ everywhere. If $\lim_{t \to T, x_{n+1} \to s} v(x, t) = +\infty$, then $\lim_{t \to T, x_{n+1} \to s} |A|^2(x, t) = \frac{(\dot{\rho})^2(x_{n+1}, t)}{(\rho(\dot{\rho})^2(x_{n+1}, t) + 1)^\frac{1}{2}} + \frac{n - 1}{\rho^2((\dot{\rho})^2(x_{n+1}, t) + 1)} = 0$ which implies $X(x_1, \ldots, x_n, s; T)$ is not a singular point. So
On one hand, we have for all time $t \in [0, T]$, by the continuity of $v$, we have $v \leq M_T$ for some constant depending on $T$.

Next, we will show an estimate of $h(t)$. First, we prove the following lemma.

**Lemma 4.2.** Under the assumption of the main theorem, we have $C_1 \leq \int_{M_t} H d\mu_t \leq C_2$, for all time $t \in [0, T)$. Here $C_1$ and $C_2$ are positive constants only depending on $n, d, r, R, V$ and $|M_0|$.

**Proof.** First we show that $\int_{M_t} H d\mu_t \geq C_1$ for some constant $C_1$. This is a direct consequence of the first variation formula and mean curvature is positive. Since

$$n|M_0| = n|M_t| \leq \int_{M_t} H < X, \nu > d\mu_t \leq \int_{M_t} H|X| d\mu_t \leq \sqrt{d^2 + R^2} \int_{M_t} H d\mu_t,$$

the lower bound for $\int_{M_t} H d\mu_t$ is obtained. Next we show there is an upper bound for $\int_{M_t} H d\mu_t$, that there is a constant $C_2$ such that $\int_{M_t} H d\mu_t \leq C_2$. We still parameterize $M_t$ by its radius function $\rho(x_{n+1}, t)$. We denote $\omega_n$ to be the volume of unit ball in $\mathbb{R}^n$, and its surface area is $n\omega_n$, then we have

$$H = \frac{-\dot{\rho}}{(1 + \dot{\rho}^2)^{\frac{1}{2}}} + \frac{n-1}{\rho(1 + \dot{\rho}^2)^{\frac{1}{2}}},$$

$$|M_t| = n\omega_n \int_0^d \rho^{n-1} \sqrt{1 + \rho^2} dx_{n+1}.$$

$$\int_{M_t} H d\mu_t = n\omega_n \int_0^d (\frac{-\dot{\rho}}{(1 + \dot{\rho}^2)^{\frac{1}{2}}} \rho^{n-1} + (n-1)\rho^{n-2}) dx_{n+1}.$$

On one hand, we have $\int_0^d (n-1)\rho^{n-2} d\mu_t \leq (n-1)dR^{n-2}$. On the other hand, by our boundary conditions,

$$\int_0^d \frac{-\dot{\rho}}{(1 + \dot{\rho}^2)^{\frac{1}{2}}} \rho^{n-1} dx_{n+1} = \int_0^d -\rho^{n-1} d(\arctan \dot{\rho}) dx_{n+1} = -\rho^{n-1} \cdot (\arctan \dot{\rho})_0^d + (n-1) \int_0^d \dot{\rho} \cdot (n-1) \arctan \dot{\rho} dx_{n+1} \leq (n-1) \frac{\pi}{2} \int_0^d \rho^{n-2} \sqrt{1 + \rho^2} dx_{n+1} \leq (n-1) \frac{\pi}{2r} \int_0^d \rho^{n-1} \sqrt{1 + \rho^2} dx_{n+1} = \frac{(n-1)\pi}{2n\omega_n} |M_t|^\frac{1}{r} = (n-1) \frac{\pi}{2n\omega_n} |M_0|^\frac{1}{r}.$$

Thus the upper bound is obtained.

**Corollary 4.1.** Under the assumption of the main theorem, we have $0 < h(t) \leq C_3$ for all time $t \in [0, T)$. Here $C_3$ is a constant only depending on $n, d, r, R, V$ and $|M_0|$.
Proof. Now we use the Cauchy-Schwarz inequality
\[
\left( \int_{M_t} H d\mu \right)^2 \leq \left( \int_{M_t} H^2 d\mu \right) \left( \int_{M_t} d\mu \right) \leq 1.
\]
From which we have
\[
0 < \int_{M_t} H d\mu \leq \int_{M_t} d\mu = |M_0| \leq C_3.
\]
□

Now we show that $|A|^2$ is bounded for any finite time interval.

**Theorem 4.1.** If the maximal time interval $[0, T)$ is finite, i.e., $T < +\infty$, then we have $|A|^2(x,t) \leq C_T$, where $C_T$ is a constant depending only on $T, n, d, r, R, V$ and $|M_0|$.

**Proof.** First we compute the evolution equation of $v(x,t) = \langle \omega, \nu \rangle$ clearly we have
\[
\partial_t v = -v^2 \langle w, \partial_t \nu \rangle = -v^2 \cdot \langle H\nabla, \omega \rangle.
\]
From [2] , we have
\[
\Delta v = |A|^2 v - v^2 \langle \omega, \nabla H \rangle + \frac{2|\nabla v|^2}{v} - \frac{n-1}{u^2} \cdot v.
\]
Then we obtain
\[
\left( \partial_t - h\Delta \right) v = -h|A|^2 v - \frac{2h|\nabla v|^2}{v} + \frac{(n-1)hv}{u^2},
\]
and
\[
\left( \partial_t - h\Delta \right) v^2 = 2v \cdot (-h|A|^2 v - \frac{2h|\nabla v|^2}{v} + \frac{(n-1)hv}{u^2}) - 2h|\nabla v|^2
\]
\[
= -6h|\nabla v|^2 - 2h^2|A|^2 + \frac{2(n-1)hv^2}{u^2}.
\]

We considering $|A|^2 v^2$ as in [12] and divide the points in $M_t$ into three sets.
\[
S_t = \{ P \in M_t | \tilde{\rho} \geq 0 \}
\]
\[
I_t = \{ P \in M_t | \tilde{\rho} < 0, \frac{\kappa_1}{\kappa_2} < \alpha \}
\]
\[
J_t = \{ P \in M_t | \tilde{\rho} < 0, \frac{\kappa_1}{\kappa_2} \geq \alpha \}
\]
Here $\alpha$ is a positive constant large enough. We will show that for all points in $S_t$ and $I_t$, $|A|^2 v^2$ has uniform upper bounds for any $t \in [0, T)$. We split our proof into three cases.

**Case (1).** If $P \in S_t$, from $H = \frac{-\tilde{\rho}}{(1+\tilde{\rho}^2)\rho} + \frac{n-1}{\rho(1+\tilde{\rho}^2)\rho} > 0$, we have
\[
\tilde{\rho} < \frac{n-1}{\rho} [1 + (\tilde{\rho})^2].
\]
Then we have
\[
|A|^2 v^2 = \frac{(\tilde{\rho})^2}{[1 + (\tilde{\rho})^2]^2} + \frac{n-1}{\rho^2}
\]
\[
\leq \frac{(n-1)^2}{\rho^2} + \frac{n-1}{\rho^2} \leq C.
\]
From now on, we denote by $C$ to any constant depending on $n, V, d, r, R$ and $M_0$.

**Case (2).** If $P \in I_t$, then $\frac{\kappa_1}{\kappa_2} < \alpha$. We have $\frac{-\rho \bar{\rho}}{(\rho^2 + 1)} < \alpha$, and $\frac{-\rho \bar{\rho}}{(\rho^2 + 1)} < \frac{\alpha}{T}$. Thus

$$|A|^2 v^2 = \frac{\langle \bar{\rho} \rangle^2}{1 + \langle \bar{\rho} \rangle^2} + \frac{n - 1}{\rho^2} \leq C.$$  

**Case (3).** For points in $J_t$, we use the technique of maximal principle. First we have

$$(\frac{\partial}{\partial t} - h \bar{\Delta})|A|^{2} v^{2} = |A|^2 \cdot (-6h |\nabla v|^2 - 2hv^2|A|^2 + \frac{2(n-1)hv^2}{u^2})$$

$$+ \nabla^2 (\alpha U)^2 + 2h |A|^4 - 2\bar{C} \nabla |A|^2 v^2$$

$$= |A|^2 \cdot (-6h |\nabla v|^2 - 2hv^2|A|^2 + \frac{2(n-1)hv^2}{u^2})$$

$$+ \nabla^2 (\alpha U)^2 + 2h |A|^4 - 2\bar{C} \nabla |A|^2 v^2 + h(\nabla |A|^2 v^2 - 4v |A| \nabla |A| \nabla v)$$

$$= |A|^2 \cdot (-6h |\nabla v|^2 - 2hv^2|A|^2 + \frac{2(n-1)hv^2}{u^2})$$

$$+ \nabla^2 (\alpha U)^2 + 2h |A|^4 - 2\bar{C} \nabla |A|^2 v^2 + h(-v^{-2} \nabla v^2 \nabla (|A|^2 v^2) + v^{-2} |\nabla v^2|^2 |A|^2$$

$$- 4v |A| \nabla |A| \nabla v + \frac{2(n-1)h |A|^2 v^2}{u^2}$$

$$\leq -6h |A|^2 |\nabla v|^2 - 2hv^2 |\nabla |A|^2|^2 - 2\bar{C} v^2 - h v^{-2} \nabla v^2 \nabla (|A|^2 v^2)$$

$$+ 4h |\nabla v|^2 |A|^2 - 4hv |A| \nabla |A| \nabla v + \frac{2(n-1)h |A|^2 v^2}{u^2}$$

$$\leq -h v^{-2} \nabla v^2 \nabla (|A|^2 v^2) + \frac{2(n-1)h |A|^2 v^2}{u^2} - 2\bar{C} v^2.$$  

We have used $|\nabla |A|| \leq |\nabla A|$ and Cauchy-Schwarz inequality. Since

$$\frac{2(n-1)h |A|^2}{u^2 C} \leq C_4 \cdot \frac{\kappa_1^2 + (n-1)\kappa_2^2}{\kappa_1^2 + (n-1)\kappa_2^2}$$

$$= C_4 \cdot \frac{\frac{1}{\kappa_1} + (n-1)\frac{1}{\kappa_2}}{1 + (n-1)\left(\frac{\kappa_2^2}{\kappa_1^2}\right)^2}$$

$$\leq C_5 \cdot \frac{1}{\kappa_1}.$$  

Then, if $\kappa_1 > C_5$, we have $\frac{2(n-1)h |A|^2 v^2}{u^2} - 2\bar{C} v^2 < 0$. Thus $|A|^2 v^2$ can not attain a maximal value by the maximal principle. And if $\kappa_1 \leq C_5$, we have

$$|A|^2 = \kappa_1^2 + \frac{n - 1}{\rho^2[1 + (\bar{\rho})^2]} \leq C,$$

and $|A|^2 v^2 \leq CM_T^2$. Therefore, $|A|^2 \leq \frac{CM_T^2}{v_{\min}} = C_T$.  

\[\square\]

**Corollary 4.2.** Under the assumption of the above theorem, we have a lower bound for $h(t)$, namely, $h(t) \geq m_T$. Here, $m_T$ is a constant depending on $T, n, V, d, r, \alpha, R$ and $|M_0|$.

**Proof.** It is a direct consequence of $H^2 \leq n |A|^2$, Lemma 4.2 and Theorem 4.1. \[\square\]

Next we give the higher derivative estimates as Hamilton in [10].
Corollary 4.3. Under the assumption of Theorem 4.1, we have the following higher derivative estimates

\[ |\nabla^m A|^2 \leq C_m(T) \]

Proof. First, we have

\[
\frac{\partial}{\partial t} |\nabla^m A|^2 = h \Delta |\nabla^m A|^2 - 2h|\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A \ast \nabla^j A \ast \nabla^k A \ast \nabla^m A
\]

\[ + \sum_{r+s=m} \nabla^r A \ast \nabla^s A \ast \nabla^m A. \]

We assume when \( l \leq m \), we have \( |\nabla^l A|^2 \leq C_l(T) \). Then for \( n = m + 1 \), we have

\[
\frac{\partial}{\partial t} |\nabla^{m+1} A|^2 \leq h \Delta |\nabla^{m+1} A|^2 - 2h|\nabla^{m+1} A|^2 + 1 + C(T) \cdot (|\nabla^m A|^2 + 1).
\]

We choose \( f = |\nabla^{m+1} A|^2 + N|\nabla^m A|^2 \), where \( N \) is a constant large enough. Then

\[
\frac{\partial}{\partial t} f \leq h \Delta |\nabla^{m+1} A|^2 + C(T) \cdot (|\nabla^m A|^2 + 1)
\]

\[ + Nh \Delta |\nabla^m A|^2 - 2hN|\nabla^{m+1} A|^2 + C(T) \cdot (|\nabla^m A|^2 + 1).
\]

\[
\leq h \Delta f - C(T) \cdot (|\nabla^{m+1} A|^2 + 1) = h \Delta f - C(T) \cdot (f - N|\nabla^m A|^2) + C(T)
\]

\[ \leq h \Delta f - C(T) \cdot f + C(T).
\]

Thus \( f \leq C_T. \)

\[ \square \]

Corollary 4.4. \( T = +\infty. \)

5. Proof of the Main Theorem

Since the upper bound we derived above is a constant depending on \( T \), \( |A|^2 \) may be unbounded when \( t \) tends to infinity. We will show that this will not happen and the initial hypersurface converges to a constant mean curvature surface.

Theorem 5.1. The mean curvature \( H \) of the evolving surfaces converge to a constant as \( t \to \infty \).

Proof. Since \( \frac{d}{dt} Vol(E_t) = \int_{M_t} (1 - hH) d\mu_1 \).

Then we have

\[
\int_0^\infty \frac{d}{dt} Vol(E_t) \, dt = \int_0^\infty \int_{M_t} (1 - hH) d\mu_1 \, dt
\]

\[ = Vol(E_\infty) - Vol(E_0) \leq C. \]

Therefore,

\[
\lim_{t \to \infty} \int_{M_t} (1 - hH) d\mu_1 = 0.
\]

Thus,

\[
\lim_{t \to \infty} \int_{M_t} d\mu_1 = \lim_{t \to \infty} \frac{(\int_{M_t} H d\mu_1)^2}{\int_{M_t} H^2 d\mu_1}.
\]

Then by Cauchy-Schwarz inequality, \( H = C \) for some constant. \( \square \)
**Proof.** Rotationally symmetric hypersurfaces of constant mean curvature in $\mathbb{R}^{n+1}$ are plane, sphere, cylinder, catenoid, unduloid and nodoid, they are known as the Delaunay surfaces (see [8]). Our boundary conditions excludes the possibilities of plane, sphere, catenoid and nodoid. In [2] (see Section 1), Athanassenas use the condition $|M_0| \leq \frac{\pi}{n}$ to exclude the existence of unduloids in $G$. So our possibility can only be the cylinder. Thus the Main Theorem is proved. □

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