A Dynamic Programming Implemented $2 \times 2$ non-cooperative Game Theory Model for ESS Analysis

Chen Shi *, Fang Yuan †

Abstract

Game Theory has been frequently applied in biological research since 1970s. While the key idea of Game Theory is Nash Equilibrium, it is critical to understand and figure out the payoff matrix in order to calculate Nash Equilibrium. In this paper we present a dynamic programming implemented method to compute $2 \times 2$ non-cooperative finite resource allocation game’s payoff matrix. We assume in one population there exists two types of individuals, aggressive and non-aggressive and each individual has equal and finite resource. The strength of individual could be described by a function of resource consumption in different development stages. Each individual undergoes logistic growth hence we divide the development into three stages: initialization, quasilinear growth and termination. We first discuss the theoretical frame of how to dynamic programming to calculate payoff matrix then give three numerical examples representing three different types of aggressive individuals and calculate the payoff matrix for each of them respectively. Based on the numerical payoff matrix we further investigate the evolutionary stable strategies (ESS) of the games.

Keywords: Dynamic Programming, Finite Resource Allocation, Game Theory, Evolutionary Stable Strategy (ESS)

1 Introduction

Game Theory is originally introduced by John von Neumann and Oskar Morgenstern (Neumann J and Morgenstern O. 1944) in 1944. Later John F. Nash (Nash JF. 1950) has made significant contribution to Game Theory by introducing Nash Equilibrium and proving every finite game has mixed Nash Equilibrium, which becomes the core idea of Game Theory. Since then Game Theory has been widely applied in various disciplines such as social sciences (most notably economics), political science, international relations, computer science and philosophy. So far eight game theorists have won Nobel prizes in economics. John M. Smith in 1973 formalized another central concept in Game Theory called the Evolutionary Stable Strategy (ESS, Smith JM. 1973). Various work has been done using ESS to investigate the behavior and evolutionary path of animals (Takada T and Kigami J. 1991, Crowley PH. 2000,

*Department of Entomology and Program of Operations Research, the Pennsylvania State University, PA, 16801, USA
†Department of Computer Sciences and Engineering and Program of Operations Research, the Pennsylvania State University, PA, 16801, USA
The simplest yet most common case of ESS game is $2 \times 2$ non-cooperative game. Multiple player game is more realistic but according to Poincare-Bendixson theorem, multiplayer dynamic system would result in chaos hence we only consider $2 \times 2$ games (Yi T et al. 1997). While the key idea in Game Theory is Nash Equilibrium, we must define the payoff matrix very carefully in order to compute Nash Equilibrium. However, most of the research regarding ESS has arbitrarily assigned payoff matrix. To overcome this problem, we want to use some realistic quantities to define payoff matrix for the game. M-Gibbons has shown a neighbor intervention model (Mesterton-Gibbons M, Sherratt TN. 2008) and Luther further discussed whether food is worth fighting for (Luther RM, Broom M and Ruxton GD. 2007). Just has studied the aggressive loser in an ESS game (Just W, Morris MR and Sun X. 2006). Inspired by their ideas, we will use food source to compute the payoff of aggressive and non-aggressive players in a $2 \times 2$ game.

While we assume the food source is finite and equivalent to any member in the population, it is natural to use dynamic programming (DP) to figure out the optimal foraging strategy as a resource allocation problem. Animal growth rate is a logistic curve and we divide the whole growth process into three distinct stages: initialization, quasilinear growth and termination. In different stages, the payoff is a linear function of food source with different slope and our goal is to determine the maximum total payoff at the end of growth using dynamic programming.

Al-Tamimi has suggested using dynamic programming to implement Game Theory model for designing (Al-Tamimi A, Abu-Khalaf M and Lewis FL. 2007) but their model is zero-sum. We will first present a more realistic general sum game framework, then discuss three different types of aggressive players, calculate the numerical payoff matrix for each case and determine the ESS for them. Our work is the first of this kind to combine dynamic programming and Game Theory, two different optimization tools together to solve real biological problem.

## 2 Defining the Model

A typical $2 \times 2$ non-cooperative general sum game has the following form where $P_{ji}$ defines the payoff of player $i$ in $j$th strategy combination.

| Strategy  | Non-aggressive | Aggressive |
|-----------|----------------|------------|
| Non-aggressive | $(P_{111}, P_{112})$ | $(P_{121}, P_{122})$ |
| Aggressive   | $(P_{211}, P_{212})$ | $(P_{221}, P_{222})$ |

Table 1. Payoff Matrix of non-cooperative general sum game

$P_{ijk}$ denotes the payoff of player $k$ when it uses strategy $i$ and its components uses $j$. Here we have two types of strategies: aggressive (2) and non-aggressive (1). Aggressive players would fight their neighbor and try to get their resources. Non-aggressive players only concentrate on their own food source and
never fight back even when they are attacked. However, if two aggressive players meet, it would result in a severe fight and both players are terribly hurt. This definition is similar to that of "Chicken-Dare" or "Hawk-Dove" game. The Nash Equilibrium is defined as:

**Definition 1.** \( x \in \Theta \) is a Nash Equilibrium if \( x \in \tilde{\beta}(x) \), where \( \Theta \) is the mixed strategy space and \( \tilde{\beta} \) is the mixed strategy best response correspondence.

Because this is a \( 2 \times 2 \) finite symmetric game, \( \Delta^{NE} \neq \emptyset \) by Kakutani’s Theorem. Next we switch to a population perspective and define Evolutionary Stable Strategy (ESS) as follows:

**Definition 2.** \( x \in \Delta \) is an ESS if for every strategy \( y \neq x \) there exists some \( \bar{\epsilon}_y \in (0, 1) \) such that \( u[x, \epsilon y + (1-\epsilon)x] > u[y, \epsilon y + (1-\epsilon)x] \) holds for all \( \epsilon \in (0, \bar{\epsilon}_y) \) where \( \epsilon \) is the proportion of mutant strategy.

Basically, ESS is a subset of Nash Equilibrium. We use Maynard’s criterion to test whether a Nash Equilibrium is an ESS:

**Theorem 1.** \( \Delta^{ESS} = \{ x \in \Delta^{NE} : u(y, y) < y(x, x), \forall y \in \beta^*(x), y \neq x \} \).

To perform all these analysis, we must first define the payoff matrix of our original game. We will use DP to determine the numerical payoff values for the four strategy combinations. Assume each player has a total of \( N \) food sources for the entire development period and in each stage at least 1 resource should be consumed in order to maintain basal metabolism. As we have discussed before, the development period is divided into three stages: growth initialization, quasilinear growth and growth termination, hence the player could consume \( 1 \cdots N - 2 \) resources in each stage. While the growth is logistic and nonlinear, we could use linear approximation in each stage as follows where \( y \) is the payoff in each stage and \( x \) is the resource consumed:

\[
\begin{align*}
y(x) &= \begin{cases} 
ax, & \text{Growth Initialization,} 

bx, & \text{Quasilinear Growth,} 

cx, & \text{Growth Termination.} 
\end{cases}
\end{align*}
\]

Because logistic curve has a sigmoid shape and is usually symmetric, it is reasonable to set \( a = c \) to reduce computational intensity. The coefficients \( a \) and \( b \) has biological meaning of the efficiency of converting food sources into its own energy and in our model \( b > a \). The DP model is written as follows:

\[
\text{Maximize } z = \sum_{i=1}^{3} r_i x_i \\
\text{Subject to } \sum_{i=1}^{3} x_i = N
\]

The backward DP Formulation for this model is:

OVF: \( f_k(x) = \) optimal return for the allocation of \( x \) units of resource to stage \( k \cdots 3 \).

ARG: \( (k, x) = (\text{stage}, \text{units of resource consumed}) \).

OPF: \( P_k(x) = \) units of resource consumed at stage \( k \).
RR: \( f_k(x) = \max_{x_k=0,1,\ldots,x_r} (r_k x_k + f_{r+1}(x-x_k)) \), \( x = 1 \cdots N-2 \)

BC: \( f_N(x) = r_N(x) \)

ANS: \( f_1(x) \)

For the non-aggressive and non-aggressive strategy combination, we assume both players do not interfere each other. In this case, we would only solve the DP for one of them and by symmetry, the other player should adopt same strategy to maximize its total payoff. The cost in each stage and state is shown in Table 2 and we could calculate the optimal value using DP.

For the non-aggressive and aggressive strategy combination, the cost table is similar to Table 2. The difference is we must define different \( a \) and \( b \) values for both strategies. Same thing happens for the aggressive and aggressive combination. Once we have figured out the payoff values for each combination we could complete the payoff matrix and further investigate the ESS.

| State/Stage | 1     | 2     | 3     |
|-------------|-------|-------|-------|
| 1           | a     | b     | a     |
| 2           | 2a    | 2b    | a     |
| \ldots      | \ldots| \ldots| \ldots|
| N-2         | (N-2)a| (N-2)b| (N-2)a|

Table 2. Cost Table in Different Stages and States

3 Model Results

3.1 Type I Model: Final Battle

In this simplest case, we assume both aggressive player and non-aggressive player only fight after they have depleted all their resources. In real ecosystem, some animals don’t fight while they are young. In fact, they may even help each other (Taborsky M. 2001)! They fight only when they are sexually matured. So here we don’t even have to bother DP. We assume the optimal payoffs of non-aggressive and non-aggressive combination is (1,1) and aggressive player could take advantage of half of the non-aggressive player’s payoff but lose 80% of its own payoff when it encounters another aggressive player.

The payoff matrix is:

| Strategy     | Non-aggressive | Aggressive |
|--------------|----------------|------------|
| Non-aggressive | (1,1)          | (0.5,1.5)  |
| Aggressive    | (1.5,0.5)      | (0.2,0.2)  |

Table 3. Payoff Matrix of Type I Model

In this game there are two pure strategy Nash Equilibria: nonaggressive- aggressive and aggressive-nonaggressive combinations. There is another mixed strategy Nash Equilibrium where both player use aggressive strategy with probability \( \frac{5}{8} \) and non-aggressive strategy with probability \( \frac{3}{8} \). All these three Nash Equilibria give aggressive strategy 1.5 payoff and that of non-aggressive is 0.5.
According to Smith Maynard’s criterion, both pure Nash Equilibria are not pure ESS and the mixed Nash Equilibrium is the only ESS in this game.

### 3.2 Type II Model: Modified Final Battle

In this modified final battle model, we assume the fight occurs at the end of growth as well. Suppose $a = 1$ and $b = 2$ for non-aggressive player and aggressive strategy player follows a different development path. Because most of the development happens during the quasilinear stage, the simplest case is to assign a smaller $b$ coefficient for aggressive player, for instance, $b = 1.5$. To make our model more realistic, we will modify the model assumption by taking development lag, development saturation and both cases into account, thus the second development stage of aggressive player is not linear. Development lag describes very slow development when consuming less than certain amount of resources (lower threshold). Development saturation describes no development when consuming more than certain amount of resources (upper threshold). Here we present some typical returns with respect to different resources allocation in different development conditions.

| State/Case | Stage 1 | Stage 2 | Stage 3 |
|------------|---------|---------|---------|
|            | Linear  | Linear  | Lag 1   |
| 1          | 1       | 1.5     | 1       |
| 2          | 2       | 3       | 1       |
| 3          | 3       | 4.5     | 1       |
| 4          | 4       | 6       | 2       |
| 5          | 5       | 7.5     | 4       |
| 6          | 6       | 9       | 8       |
| 7          | 7       | 10.5    | 8       |
| 8          | 8       | 12      | 8       |

| State/Case | Stage 1 | Stage 2 | Stage 3 |
|------------|---------|---------|---------|
|            | Linear  | Linear  | Lag 1   |
| 1          | 1       | 2       | 1       |
| 2          | 2       | 4       | 2       |
| 3          | 3       | 6       | 3       |
| 4          | 4       | 8       | 4       |
| 5          | 5       | 10      | 5       |
| 6          | 6       | 12      | 6       |
| 7          | 7       | 14      | 7       |
| 8          | 8       | 16      | 8       |

Table 4. Return Table in Different Development Conditions for Aggressive Player

Table 5. Return Table for Non-Aggressive Player
Based on table 4, there are 6 different types of return in 2nd stage for the aggressive player. While for the non-aggressive player, its returns in different stages are provided in table 5. So we could formulate a total of 7 DP problems for both players. Solve these 7 DP problems we have got the optimal allocation strategy and the corresponding maximum return, shown in table 6. Please note in table 6 the condition title "linear", "Lag" and "Saturation" is for the second stage of entire process only. The notion is different from that of table 4 and 5.

| Condition | Non-aggressive | Aggressive |
|-----------|----------------|-------------|
|           | Linear | Linear | Lag 1 | Lag 2 | Sat. 1 | Sat. 2 | Lag + Sat. |
| $P_1(x)$  | 1     | 1      | 1     | 1     | 1      | 1      | 1        |
| $P_2(x)$  | 8     | 8      | 6     | 7     | 4      | 3      | 4        |
| $P_3(x)$  | 1     | 1      | 3     | 2     | 5      | 6      | 5        |
| $f_1(x)$  | 18    | 14     | 12    | 11    | 14     | 13     | 14       |

Table 6. Optimal Allocation Strategy and Returns

Since the entire development is symmetric, the resource allocated in stage 1, $P_1(x)$ and in stage 3, $P_3(x)$ should be interchangeable. For instance, in Lag 2 condition, the aggressive player could either consumer 1 unit resource in stage 1 and 3 units of resource in stage 3, or 3 units in stage 1 and 1 unit in stage 3; the final total returns are just identical.

Now we have got the optimal return so we could calculate the payoff matrix based on our previous definition. Depending on different conditions, the total return of aggressive player varies from 11 to 14. Here we use 12 as instance and calculate the payoff matrix:

| Strategy         | Non-aggressive | Aggressive |
|------------------|----------------|------------|
| Non-aggressive   | (18,18)        | (9,21)     |
| Aggressive       | (21,9)         | (2.4,2.4)  |

Table 7. Payoff Matrix of One Condition of Type II Model

In this game there are three Nash Equilibria, almost the same as in type I game except that mixed Nash Equilibrium requires $\frac{11}{16}$ non-aggressive and $\frac{5}{16}$ aggressive strategy. From a population point of view, by applying Maynard’s criterion, the mixed Nash Equilibrium is the only ESS in the evolutionary game.

3.3 Type III Model: Battles in Every Stage

In this model the aggressive player will fight in all stages to maximize its payoff. While it is difficult to model the interaction of fighting for food source, instead we give the aggressive player larger coefficients than in Model I and non-aggressive player smaller coefficients when they encounter. For the aggressive-aggressive strategy combination, we simply give both players 0 because of fighting severity. Since they fight in each stage, there is no final battle in this circumstance. In this model we also consider two
different conditions: \( b = 0.5 \) for non-aggressive player and \( b = 1.5 \) for aggressive player; \( b = 1.5 \) for non-aggressive player and \( b = 2.5 \) for aggressive player. The DP results are shown in the following table:

| Condition | Player | Non-aggressive | Aggressive | Non-aggressive | Aggressive |
|-----------|--------|----------------|------------|----------------|------------|
| \( P_1(x) \) | 8      | 1              | 1          | 1              | 1          |
| \( P_2(x) \) | 1      | 8              | 8          | 8              | 8          |
| \( P_3(x) \) | 1      | 1              | 1          | 1              | 1          |
| \( f_1(x) \) | 9.5    | 14             | 14         | 22             |            |

Table 8. Optimal Allocation Strategy and Returns

So the payoff matrix for condition one is:

| Strategy      | Non-aggressive | Aggressive |
|---------------|----------------|------------|
| Non-aggressive | (18,18)        | (9.5,14)   |
| Aggressive    | (14,9.5)       | (0,0)      |

Table 9. Payoff Matrix of One Condition of Type III Model

The non-aggressive and non-aggressive strategy combination is the only Nash Equilibrium in this game; there is no mixed strategy Nash Equilibrium. This is also the ESS by Maynard’s criterion.

For condition two:

| Strategy  | Non-aggressive | Aggressive |
|-----------|----------------|------------|
| Non-aggressive | (18,18)        | (14,22)   |
| Aggressive    | (22,14)        | (0,0)      |

Table 10. Payoff Matrix of One Condition of Type III Model

In this game there are three Nash Equilibria, almost the same as in type I game except that mixed Nash Equilibrium requires \( \frac{7}{9} \) non-aggressive and \( \frac{2}{9} \) aggressive strategy. From a population point of view, by applying Maynard’s criterion, the mixed Nash Equilibrium is the only ESS in the evolutionary game.

4 Discussion

Though we use DP to find out the optimal allocation strategy, we have already found under certain circumstances, for instance if \( b > a \), most of our resources should be allocated to the second stage, the quasilinear growth. In Type II Model we have also realized the growth does not necessarily be a linear function. Here we present a criterion to test if we could use the growth function directly to allocate resources optimally:

Assume symmetric growth still holds and define \( y = f(x) \) for both growth initialization and growth termination and \( y = g(x) \) for quasilinear growth. Notice the term ”quasilinear growth” here does not mean the growth function is linear, it could be nonlinear anyway. If the following is true then we should allocate most of our resource in the quasilinear growth stage and minimum for growth initialization and termination: \( g(x) \) is not concave and \( g'(x) \geq f'(x), \forall x. \)
However, this criterion is only sufficient but not necessary. It is possible to investigate the sufficient and necessary condition but the computational intensity is almost the same of using DP because we must compute the first order partial derivative (gradient) of $f(x) + f(y) + g(10 - x - y)$ with respect to $x$ and $y$, the resource allocated in stage 1 and 3, and determine the structure of the gradient.

In this research project we focus on finite and equal resource allocation problem for both players. However, our approach could be extended to unequal resources because we use DP to determine the optimal strategy for each player so it does not matter whether the resources are equal for both players. In other words, the player should not worry about the total amount of their resource (and actually they cannot determine the amount of resource because it is pre-specified.) but rather concentrate on how to optimize the return from the resource (the optimal strategy). It is also possible to assume infinite resource, however the consumption of the player is bounded so infinite resource allocation problem could be transformed to finite resource allocation problem. As we have discussed before, saturation is a reasonable assumption to deal with infinite resource. Therefore, we could use DP to solve almost all types of resources allocation problem for 2 player game.

Another possible improvement of our approach is to introduce stochastic component into the model. Instead of assigning a specific amount of resources, we could assume the food resource is from a certain probability distribution, say, normal distribution. In effect this is the extension of unequal resource allocation problem for 2 players. Besides, it is reasonable to assign a minimum threshold of development and if the player fail to reach that threshold it then dies. The remaining resources are transferred to its neighbor (its competitor). In this circumstance DP could still be applied but we expect the formulation is much more complicated. When we reach the optimal strategy of resource allocation we could still apply Game Theory to determine Nash Equilibrium for a given game but it is difficult to give a close form representation of what ESS looks like in this scenario because of stochasticity. We could use simulation to determine the evolutionary path and this approach is more realistic and useful.

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