DIFFERENTIAL INTERACTIVE GAMES: 
THE SHORT-TERM PREDICTIONS

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The mathematical formalism of differential interactive games, which extends one of ordinary differential games [1] and is based on the concept of an interactive control, was proposed by the author [2] to take into account the complex composition of controls of a real human person, which are often complicated couplings of his/her cognitive and known controls with the unknown subconscious behavioral reactions.

In the article [3] some special class of differential interactive games, the laced interactive games, was considered. Besides other results a mechanism of short-term predictions for processes in such games was proposed. It is based on some approximations of the laced interactive games by ordinary differential games.

The goal of this research note is to propose similar mechanism of heuristic short-term predictions for general differential interactive games.

1. The differential interactive games.

Definition 1 [2]. An interactive system (with n interactive controls) is a control system with n independent controls coupled with unknown or incompletely known feedbacks (the feedbacks, which are called the behavioral reactions, as well as their couplings with controls are of a so complicated nature that their can not be described completely). An interactive game is a game with interactive controls of each player.

Below we shall consider only deterministic and differential interactive systems. For simplicity we suppose that n = 2. In this case the general interactive system may be written in the form:

\[ \dot{\varphi} = \Phi(\varphi, u_1, u_2), \]

where \( \varphi \) characterizes the state of the system and \( u_i \) are the interactive controls:

\[ u_i(t) = u_i(\omega_i^c(t), [\varphi(\tau)]_{\tau \leq t}), \]
i.e. the independent controls $u_i^\ast(t)$ coupled with the feedbacks on $[\varphi(\tau)]_{\tau \leq t}$. One may suppose that the feedbacks are integrodifferential on $t$ in general, but below we shall consider only differential dependence. It means that

\[
(2) \quad u_i(t) = u_i(u_i^\circ(t), \varphi(t), \dot{\varphi}(t), \varphi(t), \ldots, \varphi^{(n)}(t)).
\]

It is reasonable to suppose that all functional dependencies (1) and (2) are smooth.

2. Short-term predictions. Basic procedure.

Let $u_i$ and $u_i^\circ$ ($i = 1, 2$) have $n$ degrees of freedom. Let us consider $2n+1$ arbitrary functions $p_j(\vec{u}, \vec{u}^\circ, \varphi)$ of $\vec{u} = (u_1, u_2)$, $\vec{u}^\circ = (u_1^\circ, u_2^\circ)$ and $\varphi (j = 1, 2, \ldots, 2n+1)$. The knowledge of the processes in the game at $= \tau < t$ allows to consider $2n$ magnitudes $f_i(\tau) = \sum_{j=1}^{n+1} \alpha_{ij}(\tau) p_j(\vec{u}(\tau), \vec{u}^\circ(\tau), \varphi(\tau))$ ($i = 1, 2, \ldots 2n$) such that $f_i(\tau) \equiv 0$. One may suppose that the coefficients $\alpha_{ij}(\tau)$ are continuous and, moreover, belong to the Lipschitz class. Their differentiability is too strong condition to be satisfied in practice.

For the fixed moment $t$ let us consider $\Delta t > 0$ such that the Jacobi matrix of the mapping $\vec{u} \mapsto (f_1, \ldots, f_n)$ is nondegenerate at the moment $\tau = t - \Delta t$ and at the point $\vec{u} = \vec{u}(\tau)$. Under these conditions one can locally express $\vec{u}$ via $\vec{u}^\circ$ and $\varphi$:

\[
(3) \quad \vec{u}(\tau) = U_\tau(\vec{u}^\circ(\tau), \varphi(\tau); f_1(\tau), \ldots f_{2n}(\tau)).
\]

The obtained relations may be used for an approximation of the interactive game by ordinary differential games. Let us consider a fixed moment $t_0$. For $t > t_0$ the interactive controls $u_i(t)$ will be replaced by their approximations $u_i^\ast(t)$ as in the evolution equations of the game as in the expressions for the functions $f_i$. The magnitudes $u_i^\ast(t)$ are defined by the formulas

\[
(4) \quad \vec{u}^\ast(t) = U_{t-\Delta t}(\vec{u}^\circ(t), \varphi(t-\Delta t); f_1(t-\Delta t), \ldots f_{2n}(t-\Delta t))
\]

for $t > t_0$ (for $t < t_0$ they coincide with the interactive controls $u_i(t)$). Note that $f_i$ were calculated by use of the values of $\vec{u}^\ast$ at the moment $t - \Delta t$. Thus, we receive an ordinary differential game with retarded (delayed) arguments, to which the more or less standard analysis of ordinary differential games can be applied. The approximation (4) generalizes the retarded control approximation of the article [3]. The values of the state $\varphi$ calculated at the moment $t - \Delta t$ may be changed to its values calculated at the moment $t$ or at the intermediate moment $t - \alpha \Delta t$, where the parameter $\alpha \in [0, 1]$ is chosen to provide the best approximation.

Note that really we constructed a series of ordinary differential games parametrized by $t_0$. The obtained predictions may be used as short-term predictions for processes in the initial interactive game. Certainly, as it was marked in [3] it is difficult to perceive and to interpret the analytically represented results in real time. Thus, it is rather reasonable to use some visual representation for the series of the approximating games. Thus, we are constructing an enlargement of the interactive game, in which the players interactively observe the visual predictions for this game in real time. Of course, such enlargement may strongly transform the structure of interactivity of the game (i.e. to change the feedbacks entered into the interactive controls of players).
3. Short-term predictions. Further developments.

The basic procedure exposed above essentially depends on the choice of the functions \( p_j(\vec{u}, \vec{u}^0, \varphi) \). Its further developments are based on the attempts to choose them dynamically in the most effective way.

The simplest improvement is in the consideration of several sets \( \{ p_j^{(\mu)} \} \) of such functions. The index \( \mu \) labels an individual set. Fixing the moment \( t_0 \) and \( \Delta t \) one performs the basic procedure for each \( \mu \) starting at \( t_0 - \Delta t \) instead of \( t_0 \). The obtained short-term predictions for \( \varphi \) are compared with the real data in the time interval \( t_0 - \Delta t < t < t_0 \) (at this interval \( \vec{u}^0 \) coincides with its observed value). The best prediction determines \( \mu \), which is used for the short-term predictions for \( t > t_0 \).

The index \( \mu \) may vary over a finite set or over a smooth manifold. For example, let us consider the set of \( 2n + 2 \) functions \( p_j(\vec{u}, \vec{u}^0, \varphi) \). They generate a linear space \( V \). Any hyperplane in this space is spanned by \( 2n + 1 \) functions, which may be used in the basic procedure. In this case \( \mu \) labels a hyperplane in the \( 2n + 2 \) dimensional space \( V \) and, therefore, belongs to the \( 2n + 1 \)-dimensional projective space \( \mathbb{P}^{2n+1} = \mathbb{P}(V^*) \). Dynamics in the interactive game determines a curve \( \mu(t) \) in \( \mathbb{P}^{2n+1} \). The point \( \mu(t_0) \) is the index of the best prediction constructed as above for the moment \( t_0 \). The curve \( \mu(t) \) may be discontinuous.

The next improvement is based on the dynamical selection of the considered sets of functions \( \{ p_j^{(\mu)} \} \) with finite number of \( \mu \) during the game. One uses the procedure above to construct an individual approximation at the fixed moment \( t_0 \). Let the set labelled by \( \mu_0 \) gives the worst prediction. In the moment \( t_0 + \Delta t \) one adds any new set to the considered ensemble of sets instead of the \( \mu_0 \)-th one, repeat the procedure for this moment and so on. One may specify various algorithms to choose the new set.

4. Conclusions.

Thus, several heuristic procedures of the short-term predictions for processes in the differential interactive games were considered. Note that the problem of an estimation of precision of such predictions is not correct in view of interactivity of the game. One may only say that in any finite time interval \( t_0 < t < t_1 \) the prediction becomes heuristically better with \( \Delta t \downarrow 0 \). At least, it may be reasonably effective only for rather short intervals \( (t_0, t_1)^1 \). Nevertheless, in practice the interactive effects are essential only on the short time intervals and the short-term analysis of the interactive game strategically reduces it to an ordinary game. The main problem here is to extract the necessary data from such analysis to define this new game; here, the investigation of series of approximating differential games and the unraveling of algebraic correlations between them (in spirit of the nonlinear geometric algebra) is apparently crucial (cf.[3]).

References

[1] Isaaks R., Differential games. Wiley, New York, 1965;
    Owen G., Game theory, Saunders, Philadelphia, 1968.

1 To estimate the maximal admitted \( t_1 \) one may use the following procedure: let us consider two approximations started at \( t - \Delta t_1 \) and \( t - \Delta t_2 \), the moment \( t_1 \) is defined as the maximally possible one providing that the divergence of two various predictions is not too large.
[2] Juriev D., Interactive games and representation theory. I,II. E-prints: math.FA/9803020, math.RT/9808098.

[3] Juriev D., The laced interactive games and their \textit{a posteriori} analysis. E-print: math.OC/9901043.