Toolbox for reconstructing quantum theory from rules on information acquisition

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Abstract

We develop a novel operational approach for reconstructing (qubit) quantum theory from elementary rules on information acquisition. The focus lies on an observer O interrogating a system S with binary questions and S’s state is taken as O’s ‘catalogue of knowledge’ about S. The mathematical tools of the framework are simple and we attempt to highlight all underlying assumptions to provide a handle for future generalizations. Five principles are imposed, asserting (1) a limit on the amount of information available to O; (2) the mere existence of complementary information; (3) the possibility for O’s information to be ‘in superposition’; (4) O’s information to be preserved in between interrogations; and, (5) continuity of time evolution. This approach permits a constructive derivation of quantum theory, elucidating how the ensuing independence, complementarity and compatibility structure of O’s questions matches that of projective measurements in quantum theory, how entanglement and monogamy of entanglement and, more generally, how the correlation structure of arbitrarily many qubits and rebits arises. The principles yield a reversible time evolution and a quadratic measure, quantifying O’s information about S. Finally, it is shown that the five principles admit two solutions for the simplest case of a single elementary system: the Bloch ball and disc as state spaces for a qubit and rebit, respectively, together with their symmetries as time evolution groups. The reconstruction is completed in a companion paper [1] where an additional postulate eliminates the rebit case. This approach is conceptually close to the relational interpretation of quantum theory.

1 Introduction

Tools and concepts from information theory have seen an ever growing number of applications in modern physics, often proving useful for understanding and interpreting specific physical phenomena. Among a vast number of examples, black hole entropy, or more generally space-time horizon entropies, can be understood in terms of entanglement entropy [2, 3, 4, 5, 6], thermodynamics naturally adheres to entropic and thus informational perspectives [7, 8, 9, 10], and, above all, the entire field of quantum information and computation is the natural physical arena for applications of information theoretic tools [11].

This manuscript, by contrast, is motivated by the question whether information theoretic concepts, apart from their useful applications to concrete physical situations, can also tell us something deeper about physics, namely about the physical content and architecture of theories. The overriding idea is that elementary rules or restrictions of certain informational activities, e.g. information acquisition or communication, should be deeply intertwined with the structure of the appropriate theory. In this article, we shall address this question by means of the concrete example of quantum theory. Our ambition is to develop a novel informational framework for deriving the formalism and structure of qubit quantum theory from elementary physical postulates – a task which is completed in the companion paper [1]. While neither this question nor the fact that one can reconstruct quantum theory from elementary axioms is new and has been extensively explored before in various contexts [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], we shall approach both from a novel constructive perspective and with a stronger emphasis on the conceptual content of the theory. The ultimate goal of this work is therefore very rudimentary: to redo a well established theory – albeit in a novel way which is especially engineered for exposing its informational
and logical structure, physical content and distinctive phenomena more clearly. In other words, we shall attempt to rebuild qubit quantum theory from scratch.

In such an information-based context it is natural to follow an operational approach, describing physics from the perspective of an observer. Accordingly, we shall work under the premise that we may only speak about the information an observer has access to in an experiment. Our approach will thus be purely epistemic (i.e. knowledge-based) by construction and shall survive without ontic statements (i.e. references to ‘reality’). Under these circumstances we are forced to adopt an ‘inside view of physics’, holding properties of systems as being relationally, rather than intrinsically or absolutely defined.

Indeed, more generally the replacement of absolute by relational concepts goes in hand with the establishment of universal (i.e. observer independent) limits. For instance, the crucial step from Galilean to special relativity is the realization that the speed of light $c$ constitutes a universal limit for information communication among observers. The fact that all observers agree on this limit is the origin of the relativity of space and time. Similarly, the crucial step from classical to quantum mechanics is the recognition that the Planck constant $\hbar$ establishes a universal limit on how much simultaneous information is accessible to an observer. While less explicit than in the case of special relativity, this simple observation suggests a relational character of a system’s quantum properties. More precisely, the process of information acquisition through measurement establishes an informational relation between the observer and system. Only if there was no limit on the acquisition of information would it make sense to speak about an absolute state of a system within a purely epistemic approach (unless one accepts the existence of an omniscient and absolute observer as an external standard). But thanks to the existence of complementarity, implied by $\hbar$, an observer may not access all conceivable properties of the system at once. Furthermore, the observer can choose the experimental setting and thereby which property of the system she would like to reveal (although, clearly, she cannot choose the experimental outcome). Under our purely epistemic premise, the system does not have any other properties than those accessible to the observer at any moment of time such that its properties will be taken as defined relative to the latter. In particular, the system’s state is naturally interpreted as representing the observer’s state of information about the system. These ideas are in agreement with earlier proposals in the literature [24, 25] and, most specifically, with the relational interpretation of quantum mechanics [20, 27].

Of course, in order for different observers who may communicate (by physical interaction) to have a basis for agreeing on the description of a system, some of its attributes must be observer independent such as its state space, the set of possible measurements on it and possibly a limit on its information content. But without adhering to an external standard against which measurement outcomes and states could be defined, it is as meaningless to assert a system’s physical state to be independent of its relations to other systems as it is to relate a system’s dynamics to an absolute Newtonian background time.

It is worthwhile to investigate what we can learn about physics from such an operational, informational and relational approach. For this endeavour we shall adopt the general conviction, which has been voiced in many different (even conflicting) ways before in the literature [24, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39], that quantum theory is best understood as an operational framework governing an observer’s acquisition of information about a system. While most earlier works take quantum theory as given and attempt to characterize and interpret its physical content with an emphasis on information inference, here and in [4] we take a step back and show that one can actually derive quantum theory from this perspective. This will require a focus on the informational relation between an observer and a system and the rules governing the observer’s acquisition of information. More precisely, our approach will be formulated in terms of the observer interrogating a system with elementary questions.

But this is clearly not the only physical situation to which such an approach applies; it likewise opens up a novel perspective on elementary space-time structure which is encoded in the informational relations among different observers and can be exposed by a communication game. For instance, without presupposing a particular space-time structure – and thus without assuming an externally given transformation group between different reference frames – one can also derive the Lorentz group as the minimal group translating between different observer’s descriptions of physics from their informational relations, established by communication with quantum systems [40].

More fundamentally, relational ideas are actually required and commonly employed in the context of background independent quantum gravity where the notion of coordinates disappears together with a classical notion of space-time within which a dynamics could be defined. Instead, one has to resort
to dynamical degrees of freedom to define physical reference frames (i.e., dynamical ‘rods’ and ‘clocks’) relative to which a meaningful dynamics can be formulated in the first place. This constitutes the relational paradigm of dynamics [41, 42, 43, 44, 45, 46, 47, 48, 49] but goes beyond a purely informational and operational approach and thus clearly beyond the scope of this work.

The remainder of this manuscript is organized as follows.

Contents

1 Introduction 1

2 Why a(nother) reconstruction of quantum theory? 4

3 Landscape of inference theories 5
   3.1 The standard landscape of generalized probabilistic theories 5
   3.2 A novel landscape of information inference theories 6
   3.2.1 Questions and answers 6
   3.2.2 From information to probabilities: the state of $S$ relative to $O$ 8
   3.2.3 Bayesian updating and ‘collapse’ of the state 9
   3.2.4 Elementary structure on $\Sigma$ and $Q$ 10
   3.2.5 Parametrization of $S$’s state and tomography 13
   3.2.6 Composite systems 14
   3.2.7 Time evolution of $S$’s state 15
   3.2.8 The landscape $\mathcal{L}$ 16

4 Information inference principles for qubit quantum theory 17
   4.1 The principles 17
   4.2 Strategy for building necessary tools and proving the claim 21

5 Question structure and correlations 21
   5.1 A single gbit 21
   5.2 Two gbits 22
   5.2.1 Logical connectives of single gbit questions 22
   5.2.2 Independence, complementarity and entanglement 23
   5.2.3 A logical argument for the dimensionality of the Bloch-sphere 26
   5.2.4 Informationally complete sets for $D_1 = 2$ and $D_1 = 3$ 27
   5.2.5 A Bell scenario with questions 30
   5.3 Three gbits 32
   5.3.1 Three qubits 32
   5.3.2 Independence and compatibility for three qubits 34
   5.3.3 An informationally complete set for three qubits 36
   5.3.4 Entanglement of three qubits and monogamy 37
   5.3.5 Maximal entanglement for three qubits 39
   5.3.6 Three rebits 40
   5.3.7 Independence and compatibility for three rebits 40
   5.3.8 An informationally complete set for three rebits 42
   5.3.9 Monogamy and maximal entanglement for three rebits 42
   5.4 Correlation structure for $N = 2$ 43
   5.4.1 The logical mirror image of an inference theory 43
   5.4.2 Collecting the results: odd and even correlation structure for $N = 2$ 46
   5.5 The general case of $N > 3$ gbits 48
   5.5.1 An informationally complete set and entanglement for $N > 3$ qubits 49
   5.5.2 An informationally complete set and entanglement for $N > 3$ rebits 52
The first part of the article up to an including section 4 includes a substantial amount of conceptual elaborations. The second part, by contrast, will become more technical upon putting the novel postulates to use in sections 5-8. The reconstruction for arbitrarily many qubits and rebits is performed in the companion article [1] where an additional postulate eliminates rebit in favour of qubit quantum theory.

2 Why a(nother) reconstruction of quantum theory?

Given that we have a beautifully working theory, one may wonder why one should bother to reconstruct quantum theory from physical statements. There are various motivations for this endeavour:

1. To equip the standard, physically obscure textbook axioms for quantum theory with an operational sense. In particular, in addition to its empirical success, a derivation from physical statements can conceptually justify the formulation of the theory in terms of Hilbert spaces, complex numbers, tensor product rule for composite systems, etc.

2. To better understand quantum theory within a larger context. By singling out quantum theory with physical statements one can answer the question “what makes quantum theory special?” thereby establishing a bird’s-eye perspective on the formalism and conceivable alternatives.

3. It may help to understand why or why not quantum theory in its present form should be fundamental and thus why it should or should not be modified in view of attempting to construct fundamental theories. By dropping or modifying some of its defining physical principles, one obtains a handle for systematic generalizations of quantum theory. This may also be interesting in view of quantum gravity phenomenology (away from the deep quantum regime). More fundamentally, the question arises whether an informational perspective could be beneficial for quantum gravity in general.

4. The hope has been voiced that a clear interpretation of the theory may finally emerge from a successful reconstruction, in analogy to how the interpretation of special and general relativity follows naturally from its underlying principles [26, 34].

It is fair to say that the last hope has not been realized thus far because the existing successful reconstructions [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] are fairly neutral as far as an interpretation is concerned. While most of them emphasize the operational character of the theory, a particular interpretation of quantum theory is not strongly suggested.

The language and concepts of the present reconstruction are different. It will emphasize and concretize the view that quantum theory is a framework governing an observer’s acquisition of information about the observed system [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. The new postulates are simple and conceptually relatively clear, concerning only the relation between an observer and the system. Due to the simplicity, the ensuing derivation is mathematically quite elementary and, in contrast to
previous derivations, yields the formalism, state spaces and time evolution groups in a more constructive manner. The disadvantage, compared to other reconstructions, is that a large number of detailed steps is required. The advantage, on the other hand, is the simplicity of the principles and mathematical tools, and the fact that the reconstruction affords natural explanations for many quantum phenomena, including entanglement and monogamy, and elucidates the origin of the unitary group.

3 Landscape of inference theories

The ambition of a (re-)construction of quantum theory is to derive its formalism, state spaces, time evolution groups and permissible operations from physical principles – in some rough analogy to the construction of relativity theory from the principle of relativity and the equivalence principle. But in order to formulate physical postulates, we clearly have to presuppose some mathematical structure within which a precise meaning can be given to them. (For example, also the construction of special and general relativity certainly presupposed a substantial amount of mechanical structure.)

The procedure is thus to firstly define some landscape of theories, which hopefully contains quantum theory and classical information theory, but within which theories are generally not formulated in terms of the usual complex Hilbert spaces, tensor product rules, etc. The mathematical formulation of the landscape must therefore be more elementary and, in particular, operational. That is, for the time being, we have to forget about the usual – mathematically crisp but physically rather obscure – textbook axioms of quantum theory. While different theories will have the mathematical and physical structure of the landscape in common, they may have otherwise very different physical and informational properties; e.g., they may admit much stronger correlations than quantum theory \[ 50 \], or weaker correlations as classical probability theory, they may allow exotic communication and information processing tasks \[ 51 \] and so on. Secondly, given the language of this landscape, one can attempt to formulate comprehensible physical statements which single out quantum theory from within it. Going to a larger theory landscape and beyond the language of Hilbert spaces is precisely what allows us to ask the question “what makes quantum theory special?” and, ultimately, to find an operational and physical justification for the usual textbook axioms and the standard Hilbert space formulations.

The goal of this section is precisely to build such an appropriate landscape of inference theories both conceptually and mathematically from scratch within which we shall subsequently formulate those elementary rules, governing an observer’s acquisition of information about a system, that single out qubit quantum theory. This novel landscape of inference theories employs a different language and is conceptually distinct from the by now standard landscape of generalized probabilistic theories (GPTs) which are commonly employed for characterizations (or generalizations) of quantum theory. For contrast and to put the new tools into a larger perspective, we begin with a brief synopsis of the GPT language before we establish the landscape of inference theories underlying this manuscript.

3.1 The standard landscape of generalized probabilistic theories

It has become a standard in the literature to employ the formalism of generalized probabilistic theories (GPTs) for operational characterizations or derivations of quantum theory \[ 12 \] \[ 53 \] \[ 13 \] \[ 15 \] \[ 16 \] \[ 17 \] \[ 18 \] \[ 19 \] \[ 20 \] \[ 21 \] \[ 59 \]. The setup of GPTs is exclusively operational and one considers three kinds of operation devices (see figure 1): (1) a preparation device which can spit out systems in some set of states defined by a vector of probabilities for the outcome of fiducial measurements. The state spaces of the systems are necessarily required to be convex to permit convex mixtures of states. (2) A transformation device can perform physical operations on the prepared systems (e.g., a rotation) which may change the state of the system (by some group action on the state vector), but must allow it to continue its journey to (3), a measurement device, which detects certain experimental outcomes. The measurement devices are mathematically described by so-called ‘effects’ which – and this is a key assumption of GPTs – are dual to, i.e. linear functionals on, the states. Measurements and states are at the heart of GPTs; ‘effects’ directly determine the outcome probabilities of measurements and thereby, when a complete set is engaged, reveal the intrinsic state of a system. An observer assumes a supporting role which is essentially reduced to giving intuitive meaning to the notion of preparation, transformation and measurements of systems. The reconstructions of quantum theory
within the GPT formalism depart from very operational axioms restricting the possible preparations (i.e. state spaces), transformations and measurements, but an observer is otherwise not explicitly invoked, neither is her/his acquisition of information accentuated. The primary concept in GPTs are probabilities of measurement outcomes intrinsic to systems. This has lead to a whole wave of successful quantum theory reconstructions, employing GPT concepts in one way or another [12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

It is perhaps one of the great strengths of GPTs that they constitute a functional and purely operational framework which is interpretationally fairly neutral. It is not the ambition of this framework to elucidate the measurement problem, to clarify what happens to a state during a measurement, what probabilities are or, ultimately, how to interpret quantum mechanics (except that it highlights its operational character). As such this framework is compatible with most interpretations of quantum theory.

3.2 A novel landscape of information inference theories

The success of GPTs notwithstanding, we shall now change semantics and perspective to define a new landscape of theories and a novel framework for (re)constructing and understanding quantum theory. Henceforth, we shall fully engage the observer and give primacy to his acquisition of fundamentally limited information from observed systems. Probabilities, on the other hand, can be viewed as secondary and as a consequence of the limited information available to the observer – although, clearly, probabilities will assume a pivotal role too (after all we want to reconstruct quantum theory). In particular, in contrast to ‘hidden variable’ models we shall only speak about information that is accessible to the observer. The emphasis will lie on the informational relation between an observer and a system.

3.2.1 Questions and answers

As schematically depicted in figure 2 we shall consider an observer \( O \) who can only interact with a system \( S \) through interrogation, \( O \xrightarrow{Q_i} S \), via questions \( Q_i \) in some set of questions \( Q \). (At this stage we make no assumption about whether \( Q \) is continuous or discrete.) The only information \( O \) is allowed to acquire about \( S \) is by asking questions from the set \( Q \). We assume that, whenever \( O \) asks \( Q_i \) to \( S \), \( S \) will give an answer to \( O \) – provided \( S \) is present. The central ingredients of this framework will be questions and answers – and \( O \)'s information about answer outcomes to future questions.

That is, rather than speaking about intrinsic properties of \( S \), as in the GPT framework, we shall in the sequel solely speak about the information \( O \) has about \( S \) and, correspondingly, about the state that \( O \) assigns to \( S \) based on this information. Such a state of \( S \) is then defined relative to \( O \). The act of information acquisition establishes a relation between \( O \) and \( S \) and this will be the center of our attention. (A different observer \( O' \) may establish a different relation with \( S \).) Although this information inference framework will also not give rise to a unique interpretation, it naturally adheres to the relational interpretation of quantum mechanics [20, 27] and is therefore interpretationally less neutral than GPTs.
We shall not explicitly deal with transformation and measurement ('effect') devices as in GPTs; instead, these will be replaced more generally by time evolution and questions, respectively. The set of all possible operations (distinct from interrogation) that \( O \) could perform on \( S \) can later be identified with the set of all possible time evolutions of \( S \). However, in analogy to GPTs, we will assume that \( O \) has access to some method of preparing \( S \) in specific ‘answer’ configurations. (\( O \) could either control himself a preparation device or have a distinct observer \( O' \) prepare systems for him.)

When constructing the new landscape \( L \) of information inference theories within which we shall later, in section 4, formulate our postulates, we will make a number of restrictions and assumptions. Clearly, while any restriction and assumption weakens the strength of the resulting (re)construction, we shall, nevertheless, achieve an instructive and non-trivial derivation of quantum theory (partly in this manuscript, and fully in [1]) that can serve as a ‘proof of principle’ for this inference framework and the ideas underlying relational quantum mechanics [26, 27]. In order to facilitate future generalizations and improvements of the present construction of quantum theory, we shall attempt to be as clear as possible about the assumptions and reconstructions made throughout this work.

As a starter, we would like to keep \( Q \) as simple as possible, while still having non-trivial questions. In particular, we do not wish to consider trivial propositions which are always true or always false. We shall therefore assume the following.

**Assumption 1.** The set of questions \( Q \) which \( O \) can ask to \( S \) only contains binary questions \( Q_i \). Any \( Q_i \in Q \) is a non-trivial question such that \( S \)’s answer (‘yes’ or ‘no’) is not independent of its preparation.

The restriction to elementary ‘yes-no’-questions greatly simplifies the discussion and, ultimately, will give rise to qubit quantum theory. For instance, in quantum theory, a binary question could be ‘is the spin of the qubit up in \( x \)-direction?’ However, it will not be too difficult to generalize \( Q \) to also consist of ternary, quaternary, quinary, etc. questions, but we shall not attempt to do so here.

Since this is an operational approach, it is fair to assume that \( O \) can record the answers to his questions asked to any system (e.g., by writing them on a piece of paper) and that he can do statistics over the outcomes (e.g., by counting the frequency of outcomes). We shall require that every possible way of preparing \( S \) will give rise to a particular statistics over the answers to all \( Q_i \in Q \); \( O \) could test these statistics by interrogating a large number \( n \) of identically prepared systems \( S_a, a = 1, \ldots, n \), sufficiently often with (at least ideally) all \( Q_i \in Q \). In fact, this is precisely how \( O \) will operationally distinguish different preparations of systems. By having interrogated, in this manner, the \( n \) always

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\(^1\)Given the present structure, two systems \( S_1 \) and \( S_2 \) could only be considered as distinct in nature if either the (maximal) set of questions \( Q \) which \( O \) can ask the systems or the totality of answer statistics for all possible preparations were distinct for \( S_1 \) and \( S_2 \) (if always the same two \( S_1, S_2 \) are interrogated and thereafter freshly prepared again). If \( O \) can *not* distinguish \( S_1 \) and \( S_2 \) in this way for sufficiently many trials (ideally infinitely many), we shall call them identical. Let \( S_1 \) and \( S_2 \) be identical and \( O \) prepare both systems with the same procedure (for instance, the setting of the preparation device is the same for both systems). If the answer statistics (frequencies) for \( S_1, S_2 \) for all \( Q_i \in Q \) become indistinguishable after sufficiently many trials (of preparing and interrogating the same systems with the same procedure), we consider these systems as ‘identically prepared’. 
identically prepared \( S_i \) for all possible ways of preparation, we shall assume \( O \) to have gained a ‘stable’ knowledge of the set \( \Sigma \) of all possible answer statistics for \( S \), i.e., the answer statistics for all possible ways of preparing a system \( S \). While ideally \( n \to \infty \) is necessary, ‘practically’ \( n \) should be large enough for \( O \) to develop a theoretical model of \( \Sigma \) up to some accuracy. It is not our ambition to clarify further what \( n \) is, instead, we shall henceforth just assume that \( O \) has puzzled out \( \Sigma \).

### 3.2.2 From information to probabilities: the state of \( S \) relative to \( O \)

The knowledge of what \( \Sigma \) is for a given \( S \) will permit \( O \) to assign probabilities to the outcomes of his questions. It is very natural for \( O \) to assign probabilities to questions because he deals with statistical fluctuations and furthermore, as we shall see later, with systems about which he always has incomplete information in the sense that the corresponding \( \Sigma \) is such that he can never know the answers to all \( Q_i \in Q \) at the same time.

More precisely, for a specific \( S \) and any \( Q_i \in Q \) that he may ask the system next, \( O \) can assign a probability \( y_i \) that the answer will be ‘yes’ (or a probability \( n_i \) that the answer will be ‘no’), according to

(i) \( O \)'s knowledge of \( \Sigma \), and

(ii) any prior information that \( O \) may have about the specific \( S \).

Given our setup, the only prior information that \( O \) may have about the particular \( S \) (apart from what the associated \( \Sigma \) and \( Q \) are) must result from having interrogated some ensemble of identically prepared systems with some subset of \( Q \) beforehand and from the corresponding accumulated statistics of the asked \( Q_j \in Q \) (e.g., that \( O \) may have recorded on a piece of paper). For instance, if every time that \( O \) asked the specific \( Q_i \) to any of the identically prepared systems gave a ‘yes’ answer before, he will assign the prior probability \( y_i = 1 \) to \( Q_i \) and to the next identically prepared \( S \) that he will interrogate. If, on the other hand, the number of ‘yes’ and ‘no’ answers to some other \( Q_j \) was equal for the previously identically prepared systems, \( O \) will assign, as a best guess, the prior probability \( y_j = \frac{1}{2} \) to this \( Q_j \) and to the next \( S \). Similarly, for any other answer statistics, \( O \) would assign \( y_i \) to the next \( S \) according to the recorded frequencies of ‘yes’ answers. But, thanks to his knowledge of \( \Sigma \) and therefore of any possible relations in the answer statistics, \( O \) can also assign prior probabilities \( y_k \) to questions \( Q_k \) that he did not ask the previous set of identically prepared systems. For example, \( \Sigma \) may be such that whenever \( S \) gives a ‘yes’ answer to \( Q_i \), then it will give a ‘no’ answer to an immediately following \( Q_k \). Accordingly, if \( O \) assigns a prior probability \( y_i = 1 \) to \( Q_i \), as above, he will also assign a prior \( y_k = 0 \) to \( Q_k \) without previously having asked \( Q_k \). But other relations between questions will be permitted too. In particular, it may be that the information gained from the questions he previously asked the identically prepared systems and the structure of \( \Sigma \) make it equally likely that the answer to \( Q_k \) asked to the next \( S \) will be ‘yes’ or ‘no’. In this case, \( O \) will assign \( y_k = \frac{1}{2} \) to \( Q_k \) that he may ask the next \( S \).

We therefore take a Bayesian perspective on probabilities: \( O \) assigns probabilities to questions according to his ‘degree of belief’ about \( S \). These probabilities \( y_i \) are thereby relative to the observer \( O \). A different observer \( O' \) may have different information about \( S \) and thereby assign different probabilities to the various outcomes of questions posed to \( S \) (for a discussion, within quantum theory, of the consistency of different observers having different information about a system, see [20, 27, 60, 61]). E.g., \( O' \) could be the one preparing \( S \). She could ‘know’ the statistics for the specific preparation setting (from previous tests) and then send \( O \) the specifically prepared \( S \) without informing him about her knowledge.

Since the only way for \( O \) to acquire information about \( S \) is by interrogation with questions in \( Q \), the probabilities \( y_i \) that \( O \) assigns to every \( Q_i \in Q \) encode the entire information that \( O \) has about \( S \). Hence, we shall make the following identification.

**Definition 1.** (State of \( S \) relative to \( O \)) The collection of all probabilities \( y_i \forall Q_i \in Q \) is the state of \( S \) relative to \( O \). Accordingly, the set \( \Sigma \) of all possible answer statistics on \( Q \) which \( S \) admits is the state space of \( S \).

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2 We assume the preparation method to be ideal in the sense that it accounts for any answer configuration and statistics which \( S \) can admit in \( O \)'s world. That is, there do not exist other methods which can prepare \( S \) in configurations that \( O \)'s method does not encompass.

3 If \( O \) asks more than one question to any \( S \), the ordering of the questions may matter. But then \( O \) could ask the questions for any \( S \) always in the same order.
Of course, ultimately not all $y_i$ will be independent such that the full collection of probabilities will yield a redundant parametrization of the state. However, this is not important for the moment and we shall come back to this shortly.

This definition of the state of a system $S$ explicitly identifies it with the ‘state of information’ that $O$ has acquired about $S$. $O$ assigns this state to $S$ according to his information about the $Q_i \in Q$. As such, the state of system $S$ is epistemic (i.e. a ‘state of knowledge’) and only meaningful relative to the observer $O$. This definition thus generalizes some of the beautiful ideas underlying Relational Quantum Mechanics \cite{26,27} to the landscape of information inference theories which we are in the process to establish. Although it should be emphasized that the interpretation of the quantum state as a ‘state of information’ is certainly not new and has been proposed in various ways before (see also, e.g., \cite{35,36,34,38,29,32,30,31,13}) – albeit not in such a relational manner.

While the state of $S$ is thus only meaningful relative to $O$, we emphasize that both the set of questions $Q$ which $O$ may ask and the state space $\Sigma$ are to be intrinsic to the system $S$. Otherwise, it would be difficult for two observers to agree on the description of a given $S$.

### 3.2.3 Bayesian updating and ‘collapse’ of the state

At this stage it is important to distinguish single from multiple shot interrogations. In a

**single shot interrogation** $O$ interrogates a single system $S$ with a number of questions from $Q$ without intermediate re-preparations of $S$. The definite answers to these questions give $O$ definite information about this specific $S$. Furthermore, his knowledge of $\Sigma$ and any prior knowledge of $S$ (acquired through previous interrogations of identically prepared systems) give him statistical information about any questions he did not ask $S$. In conjunction, the new answers and his prior knowledge thus determine the state of $S$ relative to $O$ after the interrogation. This a posteriori state of $S$ will reflect $O$’s definite information about every asked question $Q_i$ by featuring either $y_i = 0$ or $y_i = 1$, depending on whether the answer was ‘no’ or ‘yes’, respectively (assuming for now, of course, that $\Sigma$ is such that the answers to the selection of questions that $O$ asked can be known simultaneously).

If this a posteriori state does not coincide with the prior state that $O$ assigned to $S$ before the interrogation, based on his prior information about $S$, then $S$’s state relative to $O$ has ‘collapsed’ during the interrogation. Hence, a state ‘collapse’ only occurs if $O$’s a posteriori information about $S$ does not coincide with his prior information about $S$, i.e. if $O$ experienced an information gain about $S$ via the interrogation. We shall therefore view a state ‘collapse’ as $O$’s information gain about $S$ rather than a ‘disturbance’ of $S$ (we refer the reader also to \cite{24,62,63,68} for a related discussion).

**multiple shot interrogation** $O$ interrogates an ensemble of identically prepared systems $S_a$, $a = 1, \ldots, n$, where the interrogation of every $S_a$ is a single shot interrogation. $O$ will carry out such a multiple shot interrogation to do state tomography, i.e. to estimate the state of the ensemble $\{S_a\}$ for the specific setting of preparation or, in other words, the state of any of the systems prior to being interrogated by $O$.

This will be a Bayesian updating process: after every interrogation of a system in the ensemble, $O$ will assign probabilities $y_i$ to the $Q_i$ in the manner described above. This will then define the prior state of the next system in the ensemble to be interrogated. By interrogating more and more systems, $O$ will gain more and more information about the ensemble state such that his assignments of the $y_i$ will fluctuate less the larger the number of interrogated systems. This process gives rise to a Bayesian (ensemble) state updating. Independent of this updating, the prior state that $O$ assigns to any individual system may experience a ‘collapse’ during the interrogation of that specific system because his information about the specific system may have changed. Accordingly, $O$ will have to distinguish the ensemble state from the a posteriori state of any system in the ensemble. But the collection of a posteriori states determines the ensemble state.

\footnote{A ‘disturbance’ of the system $S$ is only meaningful if there was an underlying ontic state (i.e. ‘state of reality’) to which, however, $O$ would have no access. Here we shall merely speak about the information that $O$ has access to and therefore not make any ontic statements, regarding them as excess baggage for our purposes.}
While not strictly necessary for the subsequent derivation of quantum theory, we shall assume the non-existence of an omniscient absolute observer (‘God’) who could define an ‘objective’ knowledge and probabilities. In this case, there is little reason to assert the existence of an absolute but ‘unknown probability distribution’ that $O$ could unravel by repeating his interrogations sufficiently many times. Any observer $O$ can only update his information about $S$ and accordingly assign an updated state to $S$—but there is no absolute state of $S$ (see also the related Qubism arguments [34, 37]).

### 3.2.4 Elementary structure on $\Sigma$ and $Q$

The structure introduced thus far is still too rudimentary for a (re)construction of quantum theory from a basic set of postulates. We therefore need more.

Firstly, as in GPTs we will need $O$ to be able to assign a single prior state to any pair of identical systems whenever he flips a biased coin in order to decide which of the two systems he will interrogate. (Equivalently, the preparation method could involve a biased coin toss, the outcome of which determines the preparation setting.) That is, $O$ will be permitted to build convex combinations of states.

**Assumption 2.** The state space $\Sigma$ of $S$ is a convex set.

Next, we need to define some elementary structure on $Q$ in order to meaningfully speak about relations among questions (and answers). To this end, we shall establish additional structure on $\Sigma$. We declared in assumption 1 that there are to be no trivial questions in $Q$ the answers to which would be independent of $S$’s preparation. We also insisted before that $O$ will distinguish the different ways of preparing a system $S$—and thus the different states he can assign—by the particular answer statistics. We shall now strengthen these requirements, by asserting that the answer statistics for *none* of the $Q_i \in Q$ should be independent of $S$’s preparation and, even stronger, that there exists a distinguished state of ‘no information’ corresponding to the situation that $O$’s prior knowledge about $S$ makes it equally likely for him that the answer to *any* question he may ask is ‘yes’ or ‘no’.

**Assumption 3.** There exists a special state in $\Sigma$, called the state of no information, which is given by $y_i = \frac{1}{2}, \forall Q_i \in Q$ (if the probability that $S$ ‘is there’ is $p = 1$).

For example, a distinct observer $O'$ may prepare a system $S$ and send it to $O$ in such a way that the latter knows only the associated $\Sigma$ but not the preparation setting. In this case, as a prior, $O$ will assign the state of no information to $S$, i.e. $y_i = \frac{1}{2}, \forall Q_i \in Q$. Similarly, there will exist a special preparation setting which is such that a multiple shot interrogation on an ensemble prepared in this setting will give totally random answers to $O$ such that he will assign the state of no information to the ensemble. But note that the state of any individual system after the interrogation of that system will not be the state of no information because, through the interrogation, $O$ will have acquired information about that specific system (see the discussion of the state ‘collapse’ above).

Given the state of no information, we shall preliminarily quantify the amount of information $\alpha_i$ that the definite answer to any $Q_i \in Q$ defines as one bit. Similarly, whenever $y_i = \frac{1}{2}$, as in the state of no information, we shall say that $O$ has $\alpha_i = 0$ bits of information about $Q_i$. In general, under the premise that information can neither be negative nor complex, $O$’s information about $Q_i$ should satisfy

$$0 \text{bit} \leq \alpha_i \leq 1 \text{bit}. \tag{3.1}$$

We shall not propose an explicit information measure $\alpha_i$ (as a function on $\Sigma$) here because this must follow from the information inference principles and we shall indeed derive it therefrom later in section 7. Until then it will be sufficient to work with this implicit notion of quantifying $O$’s information about any $Q_i$. But we can already infer from this that $O$ will not be able to parametrize $S$’s state by means of the information measure $\alpha_i, \forall Q_i \in Q$, because, e.g., $\alpha_i = 1$ bit only signifies maximal information about $Q_i$, but does not distinguish between whether the answer was ‘yes’ or ‘no’.

But the questions $O$ can ask, and the information that the corresponding answers define, may not be independent. Indeed, using the state of no information, we can now define elementary relations on $Q$. Two questions $Q_i, Q_j \in Q$ will be referred to as

---

5We emphasize that GPTs are more general by, in principle, permitting state spaces which do not contain such a distinguished state. However, most operationally interesting GPTs do possess such a state.
(maximally) independent if they are stochastically independent with respect to the state of no information, i.e. if the joint probabilities factorize relative to the latter \( p(Q_i, Q_j) = y_i \cdot y_j = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \), where \( p(Q_i, Q_j) = p(Q_j, Q_i) \) denotes the probability that \( Q_i \) and \( Q_j \) give 'yes' answers if asked in sequence on the same \( S \). In other words, if relative to the state of no information, the answer to only \( Q_i \) does not tell \( O \) anything about what the answer to \( Q_j \) will be if asked next (and vice versa).

dependent if they are stochastically fully dependent relative to the state of no information, i.e. if the answer to \( Q_i \) immediately implies the answer to \( Q_j \) (and vice versa). In this case, relative to the state of no information, either \( p(Q_i, Q_j) = y_i = y_j = \frac{1}{2} = p(\neg Q_i, \neg Q_j) \) or \( p(Q_i, \neg Q_j) = y_i = y_j = \frac{1}{2} = p(\neg Q_i, Q_j) \), where \( \neg Q \) is the negation of \( Q \).

compatible if \( O \) may know the answers to both \( Q_i, Q_j \) simultaneously, i.e. if there exists a state in \( \Sigma \) such that \( y_i, y_j \) can be simultaneously 0 or 1.

(maximally) complementary if maximal information about the answer to \( Q_i \) forbids \( O \) to have any information about the answer to \( Q_j \) at the same time (and vice versa). That is, every state in \( \Sigma \) which features \( y_i = 0, 1 \) will necessarily have \( y_j = \frac{1}{2} \) (and vice versa).

Consequently, complementary questions are independent, but independent questions are not necessarily complementary. We emphasize that these relations are required to be symmetric.

Since the notion of independence of questions is state dependent, we need to define it relative to a distinguished state – hence, assumption \( \mathcal{S} \). For example, for a pair of qubits in quantum theory the questions \( Q_1 \), "is the spin of qubit 1 up in x-direction?", and \( Q_2 \), "is the spin of qubit 2 up in x-direction?", are independent relative to the completely mixed state, but fully dependent relative to an entangled state (with correlation in x-direction). As a result of the state dependence, the independence of questions may also be viewed as observer dependent. For instance, in quantum theory an observer \( O' \) could send \( O \) an entangled pure state (with correlation in x-direction) and refuse to tell \( O \) which state it is. Relative to \( O' \), \( Q_1 \) and \( Q_2 \) will be dependent, but they will be independent relative to \( O \) because the latter will assign the completely mixed state to the pair prior to measurement. But note that two questions \( Q_i, Q_j \) which are fully dependent relative to the state of no information will also be dependent relative to any other state in \( \Sigma \).

Furthermore, questions \( Q_i, Q_j \) are partially independent (or dependent) if their joint probabilities relative to the state of no information do not factorize and the answer to one question does not fully imply the answer to the other. Similarly, \( Q_i, Q_j \) are partially compatible (or complementary) if maximal information about one precludes maximal, but permits non-maximal information about the other.

We shall henceforth tacitly assume a 'symmetry' on \( Q \), namely that all questions \( Q_i \in \mathcal{Q} \) are of equivalent status as regards this structure and that no 'distinguished' questions exist. More precisely, every \( Q_i \in \mathcal{Q} \) is to be (i) independent of or dependent on, and (ii) compatible with or complementary to as many questions in \( \mathcal{Q} \) as any other \( Q_j \in \mathcal{Q} \).

Given any two questions \( Q_i, Q_j \in \mathcal{Q} \), nothing, in principle, stops \( O \) from considering the correlation question \( Q_{ij} \), 'are the answers to \( Q_i, Q_j \) the same?'. The issue is whether we allow \( Q_{ij} \) to also be in \( \mathcal{Q} \). Clearly, if \( Q_i, Q_j \) are compatible, then \( Q_{ij} \) should be in \( \mathcal{Q} \) because \( O \) can always find the answer to the latter by asking \( Q_i, Q_j \). In this case, since \( Q_i, Q_j \) are simultaneously defined relative to \( O \), we can also write

\[
Q_{ij} := Q_i \leftrightarrow Q_j,
\]

where \( \leftrightarrow \) is the logical biconditional or (XNOR) \( 10 \). \( Q_{ij} \) will then automatically be compatible with both \( Q_i, Q_j \).

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\(^6\)We assume again that the probability that the system 'is there' is \( p = 1 \).

\(^7\)We emphasize that this is a definition of maximally independence. For example, for a qubit two linearly independent directions \( \vec{n}_1, \vec{n}_2 \) in the Bloch sphere with \( \vec{n}_1 \cdot \vec{n}_2 \neq 0 \) would define two spin observables \( \vec{n}_1 \cdot \vec{\sigma} \) and \( \vec{n}_2 \cdot \vec{\sigma} \) whose corresponding projectors would not be maximally independent according to this definition. The corresponding questions would be partially dependent (see below) because whenever the observer knows the answer to one question, the probability for a 'yes' answer to the other would be distinct from \( \frac{1}{2} \).

\(^8\)The most trivial example of a pair of dependent questions are clearly \( Q \) and \( \neg Q \).

\(^9\)It should be noted that, while this is true for projective measurements in quantum theory, it does not hold for generalized measurements. I thank Tobias Fritz for pointing this out.

\(^{10}\)That is \( Q_{ij} = 'yes' \) if \( Q_i = Q_j = 'yes' \) or 'no' and \( Q_{ij} = 'no' \) otherwise.
Let us now contrast this with the situation in which \( Q_i, Q_j \) are (partially or maximally) complementary. The structure introduced thus far does not preclude the ‘correlation’ \( Q_{ij} \) to also be contained in \( Q \) even if \( Q_i, Q_j \) are complementary. But in this case, \( O \) could not find the answer to \( Q_{ij} \) by directly asking \( Q_i, Q_j \) and, in fact, \( Q_i, Q_j, Q_{ij} \) would need to form a mutually complementary set such that \( 5.2 \) would not be applicable. For example, Spekkens’ elegant toy model \([38]\) and the ‘black boxes’ of \([52]\) satisfy the structure established thus far (at least at the epistemic level and modulo restrictions on the notion of convexity) and explicitly feature such a triple of questions\(^\text{11}\).

However, this situation is in conflict with our premise of following a purely operational and epistemic approach which only speaks about information that \( O \) has access to via direct interrogation and makes no reference whatsoever to an ontic state. The answers that \( O \) gets from \( S \) – and, hence, the information he can acquire about it – are to relate only to epistemic statements that \( O \) can, in principle, directly check, and not to propositions that require hidden and inaccessible ontic information. But, in this case, \( Q_{ij} \) is a statement about the correlation of two complementary questions \( Q_i, Q_j \) and \( O \) could say ‘the answers are the same’, but he can never directly test them individually and see that they are actually ‘the same’ at the same time. From a purely epistemic perspective this would be meaningless and \( O \) would have to conclude, within a theoretical model, the existence of an underlying ontic state to which he has no access. But this shall be verboten!

Of course, the ‘correlation’ is only one of many possible logical connectives. More generally, we shall require that \( O \) is not allowed to build logical connectives of complementary questions. He can only logically connect questions which are compatible – and thus simultaneously defined with respect to him – such that he could meaningfully write down a truth table for the questions to be connected and the connective question. If he cannot connect questions, he can also not ask for the connective.

**Assumption 4.** \( Q_i \ast Q_j \) is only contained in \( Q \) if \( Q_i, Q_j \) are compatible, where \( \ast \) is a logical connective.

In consequence, \( O \) can take any set of compatible questions and treat them, with classical logic, as a Boolean algebra such that \( \ast \) can be any of the 16 binary connectives (or binary Boolean functions) \( \neg, \vee, \wedge, \leftrightarrow, \ldots \). By contrast, \( O \) cannot acquire an answer to the question \( Q_i \ast Q_j \) if \( Q_i, Q_j \) are at least partially complementary (\( Q \) is the set of questions that \( O \) can ask \( S \) and to which he will get an answer) such that, in this case, \( Q_i \ast Q_j \) and any statement about it are meaningless to him (and, in the absence of an absolute and omniscient observer, to any other observer too)\(^\text{12}\).

Let us return to a set of pairwise compatible questions \( Q_1, \ldots, Q_n \). We shall require that the answers to these questions be independent of the order in which they are asked. More precisely, if \( O \) asks \( S \) these questions (and only these questions) in arbitrary order and multiple times (without intermediate re-preparations), we would like him to always receive the same answers to the same questions.\(^\text{13}\) \( O \) would then be able to predict with certainty the outcomes of the same questions after having asked each of them once because \( S \) seems to ‘remember’ what was previously asked. Accordingly, this situation is then indistinguishable from the one in which \( O \) can ask all these questions at the same time. Since the state of \( S \) is defined by \( O \)’s information and his information no longer changes after having asked every question in the set once, no further state ‘collapse’ occurs. In effect, we therefore assume what sometimes is referred to as ‘Specker’s principle’.

\(^{11}\)The reader familiar with Spekkens’ toy model \([38]\) will recall the simplest 1-bit system which has four ontic states ‘1’, ‘2’, ‘3’ and ‘4’. An epistemic restriction forbids an observer to know the ontic state. Instead, the epistemic states of maximal knowledge correspond to either of the following three questions (and their negations)

\[
Q_1 : '1 \vee 2' \quad Q_2 : '2 \vee 3' \quad Q_3 : '2 \vee 4'
\]

where \( \vee \) is to be read as ‘or’. \( Q_1, Q_2, Q_3 \) are mutually complementary and it can be easily checked that \( Q_3 \) coincides with the ‘correlation’ \( Q_{12} \) of \( Q_1 \) and \( Q_2 \): it gives ‘yes’ when the (ontic) answers to \( Q_1, Q_2 \) are equal and ‘no’ otherwise. This relation is cyclic: \( Q_1 \) is also the ‘correlation’ \( Q_{23} \) of \( Q_2, Q_3 \) and \( Q_2 \) is the ‘correlation’ \( Q_{13} \) of \( Q_1, Q_3 \). Accordingly, there are three complementary questions for this 1-bit system, in principle the correct number for a qubit. However, due to the ‘correlations’ between complementary questions, this ‘hidden variable’ toy model violates assumption \( 4 \) and is thus ruled out in the sequel. Later in section \( 5.2.3 \) we will develop a different approach, without ontic states, in order to reason for the three-dimensionality of the Bloch-sphere.

\(^{12}\)Of course, this does not preclude that any \( Q_i \ast Q_j \) could be given an ontic meaning within an appropriate ‘hidden variable model’ for which there would be no complementarity at an ontic level. But this would require extra structure which, however, we regard as superfluous for our purposes.

\(^{13}\)I would like to thank Markus Müller for this suggestion.
Assumption 5. (‘Specker’s principle’) If \( Q_1, \ldots, Q_n \in Q \) are pairwise compatible then they are also mutually compatible.

Notice that this is perhaps the strongest of our assumptions on the landscape \( \mathcal{L} \) of inference theories which we are constructing; it is satisfied for classical bit theory and in quantum theory for projective, however, not for generalized measurements. One could therefore also drop it here and, instead, promote it to a principle for quantum theory in the next section 4 (or attempt to derive it from other postulates). But since this is a very elementary restriction on \( Q \) (even classically satisfied), we shall already impose it explicitly at this point.

### 3.2.5 Parametrization of \( S \)’s state and tomography

Now that we have a notion of independence on \( Q \) we can say more about the parametrization and thus representation of \( S \)’s state relative to \( O \). Not all \( Q \in Q \) will be necessary to describe the state; the pairwise independent questions shall be the fundamental building blocks of the landscape \( \mathcal{L} \) of inference theories.

Suppose there is a maximal set \( Q_M = \{Q_1, \ldots, Q_D \in Q \} \) of \( D \) pairwise independent (but not necessarily compatible) questions, such that no further question \( Q \in Q \setminus Q_M \) exists which is pairwise independent from all members of \( Q_M \) too. Then, every other \( Q \in Q \setminus Q_M \) is either (i) fully dependent on exactly one \( Q_j \in Q_M \) and maximally independent of all other \( Q_M \ni Q_i \neq Q_j \) (if \( Q \) was fully dependent on \( Q_j \in Q_M \) and partially dependent on \( Q_M \ni Q_i \neq Q_j \), then \( Q_j, Q_i \) could not be independent), (ii) partially dependent on some and maximally independent of the other questions in \( Q_M \), or (iii) partially dependent on all \( Q_j \in Q_M \). While \( O \) will not be able to infer information about the answers to questions of cases (ii) and (iii) from his information about individual (or even subsets of) members of \( Q_M \) alone, the question is whether his information about the full set \( Q_M \) will be sufficient to do so.

**Definition 2. (Informational Completeness)** A maximal set \( Q_M = \{Q_1, \ldots, Q_D \in Q \} \) of pairwise independent questions is said to be informationally complete if \( O \)’s information about the questions in \( Q_M \) determines his information about all other \( Q \in Q \setminus Q_M \) in such a way that the probabilities \( y_i \) which \( O \) assigns to every \( Q_i \in Q_M \) are sufficient in order for him to compute the probabilities \( y_j \forall Q_j \in Q \) for all preparations of \( S \). In this case, the set of probabilities \( \{y_i\}_{i=1}^D \) of the \( Q_i \in Q_M \) parametrizes the state that \( O \) assigns to \( S \) and thereby yields a complete description of the state space \( \Sigma \). We shall call \( D \) the dimension of \( \Sigma \).

If \( Q_M \) was not informationally complete, \( O \) would require further questions that are partially dependent on at least some of the elements in \( Q_M \) in order to fully describe the system \( S \) and its state. This situation cannot be precluded, given the structure we have devised so far. However, we deem it undesirable, given that we would like to employ pairwise independent questions as building blocks for system descriptions. We shall therefore require that no more independent information about \( S \) can be learned from any question in addition to a maximal set \( Q_M \).

**Assumption 6.** The question set \( Q \) of every system \( S \) admits an informationally complete set \( Q_M \) of pairwise independent questions.

There may exist (even continuously) many such informationally complete sets of questions on \( Q \) which, at this stage, may still be either discrete or continuous. Notice that the dimension of every such maximal set \( Q_M \) on \( Q \) must be \( D \) (if finite). Otherwise, some \( Q_M \) would define more independent information than another \( Q_M' \) and thereby contain questions which must be pairwise independent of the ones in \( Q_M' \) such that the latter could not be maximal.

Any such \( Q_M \) establishes a question reference frame on \( Q \) (for a closely related discussion, see also [61]) and thereby also a ‘coordinate system’ on \( \Sigma \). In particular, in order to do state tomography with a multiple shot interrogation, as outlined in section 3.2.3, it will be sufficient for \( O \) to interrogate an ensemble of identically prepared \( S \) with the questions within a given \( Q_M \) only. Given a specific \( Q_M \), there are now three equivalent ways for \( O \) to describe \( S \)’s state: he could represent it by either the
D-dimensional yes- or no-vector

\[ \vec{y}_{O \to S} := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_D \end{pmatrix}, \quad \vec{n}_{O \to S} := \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_D \end{pmatrix}, \]

(3.3)
of probabilities \( y_i \) and \( n_i \), \( i = 1, \ldots, D \), that the answers to question \( Q_i \in Q_M \) are 'yes' and 'no', respectively. That is,

\[ \vec{y}_{O \to S} + \vec{n}_{O \to S} = p \vec{I}, \]

(3.4)
where \( \vec{I} \) is a \( D \)-dimensional vector with a 1 in each of its entries and \( p \) is the probability that \( S \) is present (which we usually take to be \( p = 1 \)). Evidently, the assignment of which answer to \( Q_i \) is 'yes' and which is 'no' is arbitrary, but any consistent such assignment is fine for us.

But he could also represent the state redundantly as a \( 2D \)-dimensional vector on \( \Sigma \oplus \Sigma \)

\[ \vec{P}_{O \to S} = \begin{pmatrix} \vec{y} \\ \vec{n} \end{pmatrix} \]

(3.5)
which will turn out to be convenient especially when \( p < 1 \). We shall write a state with a subscript \( O \to S \) to emphasize that it is the state of \( O \)'s information about \( S \).

Lastly, this structure also puts us into the position to specify \( O \)'s total amount of information about \( S \). Clearly, the total amount of information must be a function of the state. Let \( Q_M = \{ Q_1, \ldots, Q_D \} \) be an informationally complete set of questions in \( Q \). Given that these questions carry the entire information \( O \) may know about \( S \), we define the total information \( I_{O \to S} \) as the sum of \( O \)'s information about the \( Q_i \in Q_M \), as measured by the \( \alpha_i \) \[3]:

\[ I_{O \to S}(\vec{y}_{O \to S}) := \sum_{i=1}^{D} \alpha_i. \]

(3.6)
(We imagine \( O \) as an agent who can write down results on a piece of paper and add these up.) The specific relation between \( \alpha_i \) and \( \vec{y}_{O \to S} \) will be derived later in section 7; it will not be the Shannon-entropy.

### 3.2.6 Composite systems

For later purpose we need to clarify the notion of a composite system. Since we are pursuing a purely epistemic approach, the notion of a composition of systems must be defined in terms of the information accessible to \( O \) through interrogation. \( O \) should be able to tell a composite system apart into its constituents. Accordingly, we require that the set of questions which can be posed to the composite system contains all questions about the individual subsystems and that the remaining questions are literally composed of these individual questions.\[14]

**Definition 3. (Composite System)** Two systems \( S_A, S_B \) are said to form a composite system \( S_{AB} \) if the set of questions \( Q_{AB} \) which \( O \) can ask \( S_{AB} \) contains both question sets \( Q_A, Q_B \) associated to \( S_A, S_B \), respectively, and if all other questions in \( Q_{AB} \setminus \{ Q_A \cup Q_B \} \) are composed, via logical connectives \(*\), of questions in \( Q_A \) and \( Q_B \). Furthermore, any \( Q_A \in Q_A \) is compatible with any \( Q_B \in Q_B \).

Of course, thanks to assumption \[4] \( O \) can only compose questions with a logical connective \(*\) if they are compatible. This will restrict the number of composite questions in \( Q_{AB} \).

There are further repercussions: let \( Q_{MA}, Q_{MB} \) be informationally complete sets for \( S_A, S_B \), respectively. Then an informationally complete set \( Q_{MA} \) for a composite \( S_{AB} \) can be formed by joining \( Q_{MA}, Q_{MB} \) and adding a maximal pairwise independent set of logical connectives of members of \( Q_{MA} \) with elements of \( Q_{MB} \).

---

\[14\] For the GPT specialists, we emphasize that this definition has nothing to do with the usual requirement of local tomography \[14, 15, 53, 13, 57, 18, 16\] in GPTs. To give a concrete example, we shall see later that two-level systems over real Hilbert spaces (qubits) satisfy this definition while violating local tomography.
3.2.7 Time evolution of $S$'s state

We shall assume that $O$ has access to a clock and begin with a definition:

**static states:** A state $\vec{P}_{O\rightarrow S}$ which is constant in time – corresponding to the situation that $O$ will always assign the same probability to each of his question outcomes – is called static.

If all the states $O$ would assign to any system $S$ were static – according to his information – $O$'s world would be rather boring place. In order not to let $O$ die of boredom, we allow the probability vector $\vec{P}_{O\rightarrow S}$ to change in time. But we require that $Q$ and $\Sigma$ are time independent. We shall now make use of operational reasoning, to briefly consider how $\vec{P}_{O\rightarrow S}$ evolves under time evolution. To be clear, the following argument is an adaptation, to the present setup, of a similar argument typically employed in the GPT framework [12, 53, 14, 15, 13, 16].

Let $O$ have access to two identical non-interacting systems $S_1$ and $S_2$. $O$'s information about $S_1$ and $S_2$ can be different such that they are allowed to be in distinct states $\vec{y}_{O\rightarrow S_1}$ and $\vec{y}_{O\rightarrow S_2}$ relative to $O$. Now let $O$ perform a (biased) coin flip which yields ‘heads’ with probability $\lambda$ and ‘tails’ with probability $(1-\lambda)$ [15]. Given that $S_1$ and $S_2$ are identical, $O$ can ask the same questions to both systems. If the coin flip yields ‘heads’, $O$ will interrogate $S_1$, if it yields ‘tails’ $O$ will interrogate $S_2$.

$$S_1 \quad S_2$$

$$\lambda \quad 1-\lambda$$

$$O$$

In particular, before tossing the coin, say at time $t_1$, the probability he assigns to receiving a ‘yes’ answer to question $Q_i$ (asked to either $S_1$ or $S_2$ depending on the outcome of the coin flip) is simply the convex sum $y^{12}(t_1) = \lambda y_1^{1}(t_1) + (1-\lambda) y_2^{1}(t_1)$. (O is allowed to build this convex combination thanks to assumption 2) But this holds at any time $t$ before $O$ tosses the coin (which $O$ can also choose never to do) [3]. As will become clear shortly, it is convenient to consider the states $\vec{P}_{O\rightarrow S}$ with $0 \leq p \leq 1$ for now. We shall return to the yes-vector later in section 6. We then have

$$\vec{P}_{O\rightarrow S_{12}}(t) = \lambda \vec{P}_{O\rightarrow S_1}(t) + (1-\lambda) \vec{P}_{O\rightarrow S_2}(t), \quad \forall t.$$ (3.7)

This convex combination is the state of an ‘effective’ system $S_{12}$ (identical to $S_1, S_2$) describing $O$'s information about the coin flip scenario. Note that $S_{12}$ is not a composite system according to definition 3 because $Q_{12} = Q_1 = Q_2$.

Denote by $T_k$ the time evolution map of the state of system $S_k$, $k = 1, 2, 12$, from $t = t_1$ to $t = t_2$. Under the assumption that $\vec{P}_{O\rightarrow S_{12}}(t)$ evolve independently of each other, given that $S_1$ and $S_2$ do not interact, equation (3.7) implies

$$T_{12}[\vec{P}_{O\rightarrow S_{12}}(t_1)] = T_{12}[\lambda \vec{P}_{O\rightarrow S_1}(t_1) + (1-\lambda) \vec{P}_{O\rightarrow S_2}(t_1)] = \lambda T_{1}[\vec{P}_{O\rightarrow S_1}(t_1)] + (1-\lambda) T_{2}[\vec{P}_{O\rightarrow S_2}(t_1)].$$ (3.8)

All three states $\vec{P}_{O\rightarrow S_k}$, $k = 1, 2, 12$, are elements of the same $\Sigma$. As such, we assume that every $T_k$ can be applied to any element of $\Sigma$. Furthermore, $S_1, S_2$ are to evolve under the same time evolution in this coin flip scenario which ought to be independent of the initial state of either $S_1$ or $S_2$. Motivated by these considerations, we make another non-trivial assumption.

**Assumption 7.** The time evolution map $T_{12}$ acting on $\vec{P}_{O\rightarrow S_{12}}$ is the same as the time evolutions $T_{1,2}$ acting on the systems $S_{1,2}$. We thus simply write $T$ for the time evolution map. 

---

[15] That is, both $S_1$ and $S_2$ carry the same $Q$ and $\Sigma$.

[16] Equivalently, $O$ may use another system to which he has assigned a stable state vector by repeated interrogations on identically prepared systems. Given an arbitrary elementary question $Q$ that he can use the probability that the answer is ‘yes’ as $\lambda$ and the probability that the outcome is ‘no’ as $(1-\lambda)$ and thus use $Q$ as the ‘coin’.

[17] We assume $\lambda$ to be constant in time.
Accordingly, \( \mathcal{KN} \) becomes

\[
T[\tilde{P}_{O \rightarrow S}(t_1)] = \lambda T[\tilde{P}_{O \rightarrow S}(t_1)] + (1 - \lambda) T[\tilde{P}_{O \rightarrow S}(t_1)]
\]

and \( T \) is **convex linear.** This scenario can be easily generalized to arbitrarily many identical systems. Remarkably, it can be shown that convex linearity of operations on probability vectors, in fact, implies full linearity of the operation \( \mathcal{KN} \). Hence, the time evolution of the state must be **linear:**

\[
\tilde{P}_{O \rightarrow S}(t_2) = T[\tilde{P}_{O \rightarrow S}(t_1)] = A(t_1, t_2) \tilde{P}_{O \rightarrow S}(t_1) + \tilde{V}(t_1, t_2),
\]

where \( A(t_1, t_2) \) is a \( 2D \times 2D \) matrix and \( \tilde{V} \) some \( 2D \)-dimensional vector. Given our assumption above that \( O \)'s world is not **static**, \( A(t_1, t_2) \) will generally be a non-trivial matrix. Requiring that the last relation also holds at \( t_1 = t_2 \), it follows that \( \tilde{V}(t, t) = 0 \) and \( A(t, t) = 1 \). Demanding further that the special state \( \tilde{P}_{O \rightarrow S} = 0 \), corresponding to \( p = 0 \) and the absence of a system, is invariant under time evolution (such that no system or any information is created out of ‘nothing’), finally yields \( \tilde{V} \equiv 0 \), \( \forall t_1, t_2 \). Given that \( \tilde{P}_{O \rightarrow S}(t) \) is a probability vector for all \( t \) and \( \mathcal{KN} \) must always hold, \( A(t_1, t_2) \) must be a **nonnegative** (real) matrix which is **stochastic** in any pair of components \( i \) and \( i + D \) of \( \tilde{P}_{O \rightarrow S} \).

Consequently, assumption \( \mathcal{H} \) thus indirectly rules out non-linear and state dependent time evolution, such as in Weinberg’s non-linear extension of quantum mechanics \( \mathcal{W} \) from the landscape \( \mathcal{L} \) of inference theories. Apart from the fact that there are many operational problems with non-linear quantum mechanics \( \mathcal{W}, \mathcal{W}, \mathcal{W} \) (e.g., superluminal signaling), such that it is not necessarily operationally desirable to include it in \( \mathcal{L} \), one could alternatively drop assumption \( \mathcal{H} \) here to keep \( \mathcal{L} \) more general and promote it to a postulate for quantum theory. However, we include it at this point since it is a very elementary condition on the structure of \( \Sigma \) and we thereby take it as a basic property of \( \mathcal{L} \).

For later purpose, we impose a further condition on the evolution of \( S \)'s state.

**Assumption 8. (Temporal Translation Invariance)** There exist no distinguished instants of time in \( O \)'s world such that \( O \) is free to set any instant he desires as the instant \( t = 0 \). Time evolution, as perceived by \( O \), is therefore (temporally) translation invariant \( A(t_1, t_2) = A(t_2 - t_1) \).

Since the time evolution matrix \( A \) can only depend on the duration elapsed, but not on the particular instant of time, we can collect the above results in the simple form:

\[
\tilde{P}_{O \rightarrow S}(t_2) = A(\Delta t) \tilde{P}_{O \rightarrow S}(t_1), \quad \Delta t = t_2 - t_1.
\]

(3.9)

We do not yet have sufficient evidence to conclude that a given time evolution will be described by a continuous one-parameter matrix group because we neither know (i) that \( A \) is invertible for all \( \Delta t \), (ii) that the evolution is actually continuous, nor (iii) that every composition of evolution matrices is again an evolution matrix. We shall return to this question in section \( \mathcal{S} \) with the help of the set of principles for qubit quantum theory and also defer the discussion of the time evolution of the yes-vector \( \tilde{y}_{O \rightarrow S} \) until after settling this issue. For now we note, however, that a multiplicity of time evolutions of \( S \) is possible, depending on the physical circumstances (interactions) to which \( O \) may subject \( S \).

### 3.2.8 The landscape \( \mathcal{L} \)

The landscape \( \mathcal{L} \) of information inference frameworks, referred to a number of times already, is now the set of all theories which describe the information acquisition of an observer \( O \) about a system \( S \) and which comply with the structure and assumptions established in this section.

In the sequel, we shall restrict \( O \)'s attention solely to composite systems of \( N \in \mathbb{N} \) generalized bits (gbits), where a single gbit is characterized by the fact that \( O \) can maximally know the answer to a single

---

\(^{18}\)A similar argument would not work for the evolution of \( \tilde{y}_{O \rightarrow S} \) because, in principle, \( \tilde{n}_{O \rightarrow S} \neq \tilde{0} \) is possible even if \( \tilde{y}_{O \rightarrow S} = \tilde{0} \) (where \( \tilde{0} \) is a \( D \)-dimensional zero vector). This is the reason why here it is more convenient to work with \( \tilde{P}_{O \rightarrow S} \), which contains the information about both \( \tilde{y}_{O \rightarrow S}, \tilde{n}_{O \rightarrow S} \), rather than \( \tilde{y}_{O \rightarrow S} \) alone. In fact, we shall see later in section \( \mathcal{S} \) that the evolution of the latter does involve an affine \( D \)-dimensional vector \( \tilde{V}' \neq \tilde{0} \).

\(^{19}\)It should be emphasized that assumption \( \mathcal{H} \) is also tacitly made in the GPT framework \( \mathcal{W}, \mathcal{W}, \mathcal{W} \) such that the present setup is in this regard not less potent.
question at once such that it can carry at most one bit of information. For $N$ gbits he may similarly
know the answers to up to $N$ questions at once. Every inference theory specifies for every system of
$N$ gbits a triple $(Q_N, \Sigma_N, T_N)$, where $T_N$ represents the set of all possible time evolutions. We shall
be concerned with the landscape of gbit theories $\mathcal{L}_{\text{gbit}}$ which contains all inference frameworks for gbit
systems, satisfying assumptions [1][8]. To name concrete examples at this stage, $\mathcal{L}_{\text{gbit}}$ contains, among a
continuum of other theories, classical bit, rebit and qubit theory for all $N \in \mathbb{N}$. We shall see more of this
later, but for now we summarize their characteristics for $N = 1$,

classical bit theory gives $Q_1 = \{Q, \neg Q\}$, $\Sigma_1 \simeq [0, 1]$ (for normalized states) with extremal points
 corresponding to $Q = \text{‘yes’}$ and $\text{‘no’}$, respectively, and $T_1 \simeq \mathbb{Z}_2$.\footnote{There are precisely two states of maximal information that $O$ can have about a classical bit
$\vec{P}_{O \rightarrow S} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, \quad $\vec{P}_{O \rightarrow S} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
corresponding to ‘yes’ and ‘no’ answers to $Q$. Time evolution is described by the abelian group $\mathbb{Z}_2$, given by
$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, \quad $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
where $P$ swaps the two states. For classical bit theory, time evolution is therefore discrete.}

rebit theory (two-level systems on real Hilbert spaces) yields $Q_1 \simeq S^1$ and every two complementary
questions are informationally complete. $\Sigma_1 \simeq D^2$ (for normalized states) and $T_1 \simeq \text{SO}(2)$.

qubit theory has $Q_1 \simeq S^2$ and every three mutually complementary questions are informationally
complete. $\Sigma_1 \simeq B^3$ (for normalized states) and $T_1 \simeq \text{SO}(3)$.

With all the assumptions made along the way, the landscape of inference theories devised here is
perhaps not quite as general as the commonly employed landscape of generalized probability theories
[12, 53, 14, 15, 13, 16, 58, 21, 59]. In particular, GPTs easily handle arbitrary finite dimensional systems,
while this remains to be done within the present inference framework. Nevertheless, the new landscape $\mathcal{L}$
of inference theories provides new tools for generalizing and characterizing quantum theory and highlights
a novel conceptual perspective on understanding quantum phenomena by positioning the process of
information inference center stage. Furthermore, as we shall see, $\mathcal{L}_{\text{gbit}}$ is large enough for a non-trivial
and instructive (re)construction of qubit quantum theory. To facilitate future generalizations of this work,
we have also attempted to spotlight the assumptions underlying $\mathcal{L}$ as clearly as possible in this section.

4 Information inference principles for qubit quantum theory

We shall now use the landscape $\mathcal{L}_{\text{gbit}}$ of gbit theories and formulate information inference principles for
qubit quantum theory on it. In a sense, these principles then constitute a set of ‘coordinates’ of qubit
quantum theory on $\mathcal{L}_{\text{gbit}}$.

4.1 The principles

As we are reconstructing a technically and empirically well-established theory, we are in the fortunate
position to avail ourselves of empirical evidence and earlier ideas on characterizing quantum theory in
order to motivate a set of basic postulates. However, the ultimate justification for these postulates will
be their success in singling out qubit quantum theory within the inference theory landscape $\mathcal{L}_{\text{gbit}}$ which
will be completed in [1]. As ‘coordinates’ on a theory space these principles will not be unique and one
could find other equivalent sets. (As usual, at least many roads lead to Rome.) But we shall take the
below ones as a first working set which restricts the information inference based relation between $O$ and
$S$, where $S$ will be a composite system (c.f. definition [2] of $N \in \mathbb{N}$ qubits. Each principle will be first
expressed as an intuitive and colloquial statement, followed by its mathematical meaning within $\mathcal{L}_{\text{gbit}}$.

We take it as an empirical fact that there exist physical systems about which only a limited amount of
information can be known at any one moment of time. The standard quantum example is a spin-$\frac{1}{2}$ particle
about which an experimenter may only know its polarization in a given spatial direction, but nothing
independent of that: the polarization ‘up’ or ‘down’ corresponds to one bit of information. But there is also a typical classical example, namely a ball which may be located in either of two identical boxes, the definite position ‘left’ or ‘right’ corresponding to one bit of information. Informationally, these two examples incarnate the most elementary of systems, a bit, supporting maximally just one proposition at a time. But, clearly, there are more complicated systems supporting other limited amounts of information. We shall take this simple observation and raise the existence of an information limit to the level of a principle.

More precisely, in analogy to von Weizsäcker’s ‘ur-theory’ [69, 70, 71], we shall restrict O’s world to be a world of elementary alternatives which thereby consists only of systems which can be decomposed into elementary gbits 21. All physical quantities in O’s world are to be finite such that he can record his information about them on a finite register. We wish to characterize the composite systems in O’s world according to the finite limit of N bits of information he can maximally inquire about them.

**Principle 1. (Limited Information)** “The observer O can acquire maximally \( N \in \mathbb{N} \) independent bits of information about the system \( S \) at any moment of time.”

There exists a maximal set \( Q_i, i = 1, \ldots, N \), of \( N \) mutually independent and compatible questions in \( Q \).

In other words, O can ask maximally \( N \) independent questions at a time to \( S \). Accordingly, this principle immediately implies that O can distinguish maximally \( 2^N \) states of \( S \) in a single shot interrogation because there will be \( 2^N \) possible answers to the \( Q_1, \ldots, Q_N \). Since \( Q, \Sigma \) are intrinsic to \( S \), also the maximal amount of \( N \) bits that each \( S \) can carry must be intrinsic to it and thus be observer independent. In fact, this principle can be regarded as a defining property of all inference theories in the gbit landscape \( L_{\text{gbit}} \) and can thus clearly not distinguish between classical bits, rebits, qubits, etc.

We take another empirical fact and elevate it to a fundamental principle: despite the limited information accessible to an experimenter at any moment of time, there always exists additional independent information that she may learn about the observed system at other times. This is Bohr’s complementarity principle [73]. Consider, for instance, the prototypical quantum physics experiment: Young’s double slit experiment. The experimenter can choose whether to obtain which-way-information or an interference pattern, but not both; the complete knowledge of whether the particle went through the left or right slit is at the expense of total ignorance about the information pattern and vice versa. (For an instructive information theoretic discussion of the particle-wave duality in Young’s double slit experiment and a Mach-Zehnder interferometer, we refer the reader to [33, 30].) Similarly, in a Stern-Gerlach experiment an experimenter may determine the polarization of a spin-\( \frac{1}{2} \) particle in \( x \)-direction, but will be entirely oblivious about the polarization in \( y \)- and \( z \)-direction. A subsequent measurement of the spin of the same particle in \( y \)-direction will render her previous information about the polarization in \( x \)-direction obsolete and keep her ignorant about the spin in \( z \)-direction and so on. That is, systems empirically admit many more independent questions than they are able to answer at a time – thanks to the information limit. We shall now return to the relation between \( O \) and \( S \) and accordingly stipulate that complementarity exists in \( O \)’s world, however, we shall say nothing more about how much complementary information may exist.

**Principle 2. (Complementarity)** “The observer \( O \) can always get up to \( N \) new independent bits of information about the system \( S \). But whenever \( O \) asks \( S \) a new question, he experiences no net loss of information.”

There exists another maximal set \( Q'_i, i = 1, \ldots, N \), of \( N \) mutually independent and compatible questions in \( Q \) such that \( Q'_i, Q_i \) are complementary and \( Q'_i, Q_{j \neq i} \) are compatible.

That is, after asking \( S \) a set of \( N \) independent elementary propositions \( Q_i, i = 1, \ldots, N \), in a single shot interrogation, \( O \) can pose a new \((N + 1)\)th elementary question \( Q'_1 \) to \( S \). Since \( O \) can only know \( N \) independent bits of information about \( S \) at a time and asking a new questions does not lead to a net loss of information, the single bit of his previous information about \( Q_1 \) must have become obsolete upon learning the answer to \( Q'_1 \), while \( O \)’s total information about \( S \) is still \( N \) bits. The principle allows \( O \)

---

21 However, in contrast to [69, 70, 71], we shall be much less ambitious here and will not attempt to deduce the dimension of space or space-time symmetry groups from systems of elementary alternatives. Recent developments [19, 57, 40, 72], on the other hand, unravel a deep relation between (a) simple conditions on operations with systems carrying finite information (e.g., communication with physical systems) and (b) the dimension and symmetry group of the ambient space or space-time.

22 Obviously, combinations (e.g., ‘correlations’) of the compatible \( Q_i \) will define other bits of information which, however, will be dependent once the \( Q_1, \ldots, Q_N \) are asked (we shall return to this in detail in section 5).
to perform the same procedure until he replaces his $N$ old by $N$ new bits of information about $S$. As $O$’s information about $S$ has changed, the state of $S$ relative to $O$ will necessarily experience a ‘collapse’ whenever he asks a new complementary question. As a result, it is the observer who decides which information (e.g., spin in $x$- or $y$-direction) he will obtain about the system by asking specific questions. But clearly $O$ will have no influence on what the answer to these questions will be and any answer will come at the price of total ignorance about complementary questions.

A few further explanations concerning this complementarity principle are in place. First of all, this principle clearly rules out classical bit theory. Secondly, the postulate asserts the existence of complementary questions in $O$’s world, however, makes no statement about whether partially independent or partially complementary questions may exist too. Thirdly, the requirement that every question of principle 2 is complementary to exactly one from principle 1 is necessary for composite systems: in principle 1, every $Q_i$ is to correspond to one of $N$ gbits. Accordingly, we also wish the complementarity of principle 2 to occur per bit.

In fact, this can be regarded as a defining property of composite systems in the presence of complementarity. However, we shall see later in section 5 that more complicated complementarity relations can arise from this as a consequence of correlations.

We would like to stress that principles 1 and 2 are conceptually motivated by related proposals which have been put forward first by Rovelli within the context of relational quantum mechanics and later, independently, by Brukner and Zeilinger within attempts to understand the structure of quantum theory via limited information. However, in order to complete these ideas to a full reconstruction of qubit quantum theory, we have to impose further principles.

The next principle is motivated partially by aesthetic and naturalness objectives and partially by empirical evidence. First the aesthetic motivation: we would like to permit any state of $S$ relative to $O$ which abides by principles 1 and 2 and their consequences. That is, the first two postulates are to impose as simple rules of general validity as possible that should not be restricted by further special sub-clauses which would distinguish otherwise compatible states of $O$’s knowledge by elimination. (We wish not to formulate a complicated law text.) On the other hand, the empirical evidence concerns the notion of ‘superposition of states’. Notice that the complementarity principle implies the existence of a notion of superposition – even in states of maximal information. For example, take $N = 1$ and let $O$ know the answer to $Q_1$ with certainty, $y_1 = 1$, such that his information about $S$ saturates the limit of one bit. This will leave him oblivious about the complementary $Q'_1$ such that he would have to assign $y'_1 = \frac{1}{2}$. The state of information he has about $S$ can be interpreted as being in a ‘superposition’ of the $Q'_1$ alternatives ‘yes’ and ‘no’, but in this case with ‘equal weight’ because both alternatives are equally likely according to $O$’s knowledge. However, Nature teaches us that more general superpositions with ‘unequal weights’ occur. Consequently, we wish to permit $O$’s information to be more generally distributed over the questions in $Q$ and, due to our aesthetic considerations, we shall then authorize $O$’s information to distribute over $Q$ in any way consistent with the other principles.

**Principle 3. (Completeness)** “$O$’s information about the system $S$ can be distributed over all questions in $Q$ in any way consistent with principles 1 and 2.”

The state space $\Sigma$ contains every state $\tilde{y}_{O \to S}$ such that the information distribution over the $\alpha_i$ in $I_{O \to S}(\tilde{y}_{O \to S})$ complies with principles 1 and 2.

Obviously, the particular distribution corresponds to $O$’s actual information about $S$. Specifically, even

![Diagram](image)

where solid edges denote compatibility and dashed edges denote ‘semi-complementarity’ and $Q_1, Q_2, Q'_1, Q'_2$ are mutually independent such that $O$ would have to ask all four in order to do tomography. Let $O$ have asked $Q_1, Q_2$ such that he has acquired 2 bits about $S$. Next, $O$ can ask $Q'_1$ which gives a new bit of information. Since he cannot know more than two independent bits and will not be allowed to experience a net loss of information, by asking $Q'_1$, $O$ must have ‘lost’ $\frac{1}{2}$ bit about each of $Q_1, Q_2$. Finally, $O$ can ask $Q'_2$. By similar reasoning, he would then know 2 bits corresponding to $Q'_1, Q'_2$, while his original information about $Q_1, Q_2$ will have become obsolete. This $Q$ structure would not correspond to a system composed of two $N = 1$ systems because the mere existence of complementarity would imply for $N = 1$ (regardless of the precise formulation of principles 1 and 2) that complementary $Q_1, Q'_1$ exist but these do not occur in this toy model $Q$. 23 For instance, for $N = 2$, we wish to avoid a set $Q$ given by

![Diagram](image)
in a state of maximal information of \( N \) bits, \( O \) may possibly have only partial or incomplete knowledge about the outcomes of any question in an informationally complete set \( Q_M \) of pairwise independent questions (c.f. assumption \( \text{I} \)). In this case, \( O \)'s information about all questions in \( Q_M \) will be in a general state of superposition because their corresponding probabilities cannot all be 0, otherwise it would be the state of no information. In this sense, one may regard principle \( \text{I} \) also as a \textit{superposition principle}.

But we emphasize that the mere presence of superpositions of elementary alternatives, as perceived by \( O \), is a consequence of the information limit and complementarity.

Without further ado, we shall directly state the fourth principle: \( O \) shall not gain or lose information about an otherwise non-interacting \( S \) without asking questions.

\textbf{Principle 4. (Information Preservation)} \textit{“The total amount of information \( O \) has about (an otherwise non-interacting) \( S \) is preserved in between interrogations.”}

\( I_{O→S} \) is constant in time in between interrogations for (an otherwise non-interacting) \( S \).

Correspondingly, \( I_{O→S} \) is a ‘conserved charge’ of time evolution; this is a simple observation which will become extremely useful later.

Finally, we come to the last principle of this manuscript. Empirical evidence suggests, at least to a good approximation, that the time evolution of an experimenter’s ‘catalogue of knowledge’ about an observed system is continuous – in between measurements. More precisely, the specific probabilistic statements an experimenter can make about the outcomes of measurements change continuously in time. We promote this to a further postulate for \( O \)'s world.

\textbf{Principle 5. (Time Evolution)} \textit{“\( O \)'s actual information about \( S \) changes continuously in time in between interrogations.”}

\( y_{O→S} \) changes continuously in time in between interrogations.

Of course, during an interrogation, \( y_{O→S} \) may change discontinuously, i.e. ‘collapse’, on account of complementarity. As innocent as this principle \( \text{III} \) appears, it turns out to be absolutely crucial in order to single out a unique information measure \( \alpha_i(y_{O→S}) \). Notice also that classical bit theory violates this postulate due to its discrete time evolution group (see section \( 3.2.3 \)).

It is now our task to verify what the triples of \((Q_N, \Sigma_N, T_N)\) for each \( N \in \mathbb{N} \) are which obey principles \( 1-5 \). Remarkably, it turns out that these five principles cannot distinguish between real and complex quantum theory, i.e. between real and complex Hilbert spaces. But rebit and qubit quantum theory are, in fact, the only ‘solutions’ within \( \mathcal{L}_{rb} \) to these principles. In the sequel of this manuscript we shall employ principles \( 1-5 \) in order to develop the necessary tools, within \( \mathcal{L}_{rb} \), for eventually proving the following more precise claim in \( 1 \). In fact, as a simple example of the newly developed tools, we shall already prove the claim for \( N = 1 \) at the end of this article.

\textbf{Claim.} \textit{The only two theories compatible with principles \( 1-5 \) are}

1. \textbf{rebit quantum theory,} where \( Q_N, \Sigma_N \) are the set of projective measurements on and the space of \( 2^N \times 2^N \) density matrices over \((\mathbb{R}^2)^\otimes N \), respectively, and \( T_N \) is \( \text{PSO}(2^N) \).

2. \textbf{qubit quantum theory,} where \( Q_N, \Sigma_N \) are the set of projective measurements on and the space of \( 2^N \times 2^N \) density matrices over \((\mathbb{C}^2)^\otimes N \), respectively, and \( T_N \) is \( \text{PSU}(2^N) \).

We remind the reader that the time evolution groups \( T_N \) for density matrices in rebit and qubit quantum theory are \textit{projective} because they correspond to \( \rho \mapsto U \rho U^\dagger \), where for (1) rebits \( \rho \) is a \( 2^N \times 2^N \) real symmetric matrix, \( U \in \text{SO}(2^N) \) and \( \dagger \) denotes matrix transpose; and (2) for qubits \( \rho \) is a \( 2^N \times 2^N \) hermitian matrix, \( U \in \text{SU}(2^N) \) and \( \dagger \) denotes hermitian conjugation.

Since both real and complex quantum theory come out of principles \( 1-5 \) we shall impose a further information inference principle in \( 1 \) in order to also eliminate rebit theory.

Notice that it is necessary to directly reconstruct the space of density matrices over Hilbert spaces from the principles rather than the underlying Hilbert spaces themselves. The reason is that the latter contain physically redundant information (norm and global phase), while the principles refer only to information which is directly accessible to \( O \).
4.2 Strategy for building necessary tools and proving the claim

Our strategy and procedure for developing tools and intermediate results before proving the main claim is best summarized as a diagram:

![Diagram showing limited information, complementarity, information preservation, time evolution, independence, compatibility, and correlation structure.]

- **limited information**
- **complementarity**
- **information preservation**
- **time evolution**

**independence, compatibility and correlation structure on** $Q_N$ (in sec. 5)

**reversible time evolution** (in sec. 6) and **information measure** (in sec. 7)

---

That is, we shall firstly ascertain, in section 5, the independence, compatibility, complementarity and correlation structure on $Q_N$ which is induced by principles 1 and 2. This section will be particularly instructive and deliver many elementary technical results which, besides becoming important later, yield simple explanations for typical quantum phenomena such as, e.g., entanglement and monogamy relations. This is also where we shall determine the dimensionality of $\Sigma_1$ by a simple argument and therefrom also of all other state spaces. Next, in section 6 we shall use principles 4 and 5 in order to conclude that a specific time evolution, as already discussed in section 3.2.7, is reversible and, in fact, described by a continuous one-parameter matrix group. These results will then be used in section 7 together with principles 4 and 5 to derive an explicit information measure $\alpha_i(\vec{y} \in \mathbb{S})$ which is unique under two operational conditions. In section 8 the conjunction of these outcomes will then be employed and augmented by principle 3 to show that for $N = 1$ one either obtains a two- or three-dimensional Bloch ball as a state space $\Sigma_1$ and that the group of all possible time evolutions $T_1$ is accordingly either PSO(2) or SO(3). The claim for $N > 1$ will require substantial additional work and will be proven in the companion paper [1].

5 Question structure and correlations

In this section, we shall solely employ principles 1 and 2 in order to deduce the independence, compatibility and correlation structure for the questions in an informationally complete set $Q_M$, (c.f. assumption 6). To this end it is instructive to look at the individual cases $N = 1, 2, 3$ in some detail before considering general $N \in \mathbb{N}$. We shall denote by $D_N$ the dimension of the state space $\Sigma_N$.

5.1 A single gbit

Principles 1 and 2 taken together imply that for $N = 1$ there exists at least one complementary pair $Q_1, Q'_1$. Since $N = 1$ is fixed, it is now more convenient to count the independent questions by an index, rather than a prime. 24 Let us therefore slightly change the notation and write $Q'_1$ henceforth as $Q_2$, such that the principles imply the existence of a complementary pair $Q_1, Q_2$. But, applied to a single gbit only,

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24 Principles 1 and 2 count compatible questions, here we need to count complementary questions.
principles 1 and 2 tell us nothing more about how many questions complementary to \( Q_1 \) exist. There may arise another \( Q_3 \) complementary to both \( Q_1, Q_2 \) etc. Notice that, for a single gbit, an informationally complete set of pairwise independent questions must be given by a maximal mutually complementary set

\[
Q_{M_1} = \{Q_1, Q_2, \ldots, Q_{D_1}\}
\]  

(5.1)

because principle 1 prohibits pairwise independent (partially or fully) compatible questions. (Recall that fully complementary questions are automatically independent.) Principles 1 and 2 imply \( D_1 \geq 2 \). It turns out that we need to consider two gbits in order to upper bound \( D_1 \).

We shall call such questions for a single gbit individual questions.

5.2 Two gbits

The \( N = 2 \) case requires substantially more work. First of all, since this is a composite system (c.f. definition 3), the individual questions about the two single gbits must be contained in an informationally complete set \( Q_{M_2} \). We shall again slightly change the notation, as compared to section 4.1: the maximal complementary set \( Q_{M_1} \) (5.1) for gbit 1 will be denoted by \( Q_1, \ldots, Q_{D_1} \), while the elements of \( Q_{M_1} \) for gbit 2 will be denoted with a prime \( Q'_1, \ldots, Q'_{D_1} \). It will be convenient to depict the question structure graphically. Representing individual questions henceforth as vertices, \( Q_{M_2} \) certainly contains the following

\[
\begin{array}{ccc}
gbit 1 & & gbit 2 \\
Q_1 & & Q'_1 \\
Q_2 & & Q'_2 \\
Q_3 & & Q'_3 \\
\vdots & & \vdots \\
Q_{D_1} & & Q'_{D_1} \\
\end{array}
\]  

(5.2)

This structure abides by principles 1 and 2; any pair \( Q_i, Q'_j \) will be independent and compatible.

5.2.1 Logical connectives of single gbit questions

But a composite system may also admit composite questions, built with logical connectives of the sub-system questions. We must now determine which composite questions are allowed and which must be added to the individual questions in order to form an informationally complete \( Q_{M_2} \) of pairwise independent questions. To this end, we must firstly unravel which logical connective * \( O \) can employ at all in order to construct composite questions which are pairwise independent with the individual ones. Recall assumption 4 that \( O \) may only compose compatible questions with logical connectives * and consider \( Q_i, Q'_j \) as an exemplary compatible pair. In truth tables, we shall henceforth symbolize ‘yes’ by ‘1’ and ‘no’ by ‘0’. While permitted binary connectives, it is clear that, e.g., the AND and OR operations \( \land \) and \( \lor \), respectively, cannot be employed alone to build a new independent question from \( Q_i, Q'_j \) because \( Q_i \land Q'_j = 1 \) implies \( Q_i = Q'_j = 1 \) and \( Q_i \lor Q'_j = 0 \) implies \( Q_i = Q'_j = 0 \) such that certain answers to these two connectives imply answers to the individuals. But if \( Q_i * Q'_j \) is to be pairwise independent from \( Q_i, Q'_j \), an answer to it must not imply the answer to either of \( Q_i, Q'_j \) and, conversely, the answer to only \( Q_i \) or only \( Q'_j \) cannot imply \( Q_i * Q'_j \). As can be easily checked, it must satisfy the following truth table:

\[
\begin{array}{ccc}
Q_i & Q'_j & Q_i * Q'_j \\
0 & 1 & a \\
1 & 0 & a \\
1 & 1 & b \\
0 & 0 & b \\
\end{array}
\]  

(5.3)

a ≠ b \( \quad a, b \in \{0, 1\} \).
already introduced in \[\text{[5.2]}\], and operationally interpret it as a ‘correlation question’: \(Q_{ij} := Q_i \leftrightarrow Q'_j\) is to be read as ‘are the answers to \(Q_i\) and \(Q'_j\) the same?’. But the XOR could equivalently be employed.\(^{25}\)

Since \(\leftrightarrow\) is a symmetric logical connective, \(Q_{ij} = Q_i \leftrightarrow Q'_j = Q'_j \leftrightarrow Q_i\) (but note that \(Q_{ij} \neq Q_{ji}\)), and, thanks to its associativity,

\[
Q_i \leftrightarrow Q_{ij} = Q_i \leftrightarrow (Q_i \leftrightarrow Q'_j) = (Q_i \leftrightarrow Q_i) \leftrightarrow Q'_j = Q'_j,
\]

such that \(\leftrightarrow\) defines a closed relation on \(Q_i, Q'_j, Q_{ij}\). This has the following ramifications: (1) For any compatible pair \(Q_i, Q'_j\), the ‘correlation’ operation \(\leftrightarrow\) gives rise to precisely one additional question \(Q_{ij}\), and (2) the set \(Q_i, Q'_j, Q_{ij}\) will indeed be pairwise independent, according to the definition in section \[\text{[5.2]}\] because neither \(Q_i\) nor \(Q'_j\) alone can determine \(Q_{ij}\) relative to the state of no information (not even partially), otherwise, together with the determined \(Q_{ij}\), they would determine each other via \[\text{[5.4]}\] – in contradiction with the independence of \(Q_i, Q'_j\). That is, \(Q_{ij}\) is independent of both \(Q_i\) and \(Q'_j\) and since we assume independence to be a symmetric relation, it must also be true the other way around.

Consequently, the \(D^2_1\) correlation questions \(Q_{ij}, i, j = 1, \ldots, D_1\), are candidates for additional questions in \(Q_{M_1}\) besides the \(2D_1\) individual ones of \[\text{[5.2]}\]. Graphically, we shall represent \(Q_{ij}\) as the edge connecting the vertices corresponding to \(Q_i\) and \(Q'_j\). For example, the following question graphs

represent legal sets of questions. Note that due to assumption \[\text{[4]}\] there can only be edges between gbit 1 and gbit 2 but not between the vertices of a single gbit, e.g., \(Q_1\) and \(Q_2\), because they are complementary.

Our next task is to analyse the independence and compatibility structure of the \(Q_{ij}\).

### 5.2.2 Independence, complementarity and entanglement

We begin with a simple observation.

**Lemma 5.1.** \(Q_i\) is compatible with \(Q_{ij}\), \(\forall j = 1, \ldots, D_1\) and complementary to \(Q_{kj}\), \(\forall k \neq i\) and \(\forall j = 1, \ldots, D_1\). That is, graphically, an individual question \(Q_i\) is compatible with a correlation question \(Q_{ij}\) if and only if its corresponding vertex is a vertex of the edge corresponding to \(Q_{ij}\). By symmetry, the analogous result holds for \(Q'_j\).

**Proof.** \(Q_i\) and \(Q_{ij}\) are compatible by construction. Consider therefore \(Q_i\) and \(Q_{kj}\) for \(k \neq i\) and \(j = 1, \ldots, D_1\). Clearly, \(Q_i\) and \(Q'_j\) are compatible and so are \(Q'_j\) and \(Q_{kj}\). Assume now that \(Q_{kj}\) and \(Q_i\) are maximally compatible too. Then, by Specker’s principle (c.f. assumption \[\text{[4]}\]), \(Q_i, Q'_j\) and \(Q_{kj}\) are mutually compatible. Thus, \(O\) may ask \(S\) all three at the same time. But, by \[\text{[5.4]}\], knowing \(Q_{kj}\) and \(Q'_j\) is equivalent to knowing \(Q_k\) and \(Q'_j\) such that by asking all three, \(O\) would know the complementary \(Q'_k\) and \(Q_{k\neq i}\) simultaneously which is illegal. Similarly, assume now that \(Q_{kj}\) and \(Q_i\) are partially compatible. In that case \(O\) may ask \(Q'_j\) and \(Q_{kj}\), implying an answer to \(Q_k\), and still know something about \(Q_i\) which would contradict complementarity of \(Q_i\) and \(Q_{k\neq i}\). Consequently, \(Q_i\) and \(Q_{kj}\) must be complementary for \(k \neq i\). \(\square\)

For example, \(Q_1\) and \(Q_{12}\) are compatible, while \(Q_1\) and \(Q_{22}\) are complementary. The intuitive explanation for the incompatibility of \(Q_1\) and \(Q_{22}\) is as follows: if \(O\) knew the answers to both simultaneously,

\[\text{[As an aside, the XNOR can be expressed in terms of the basic Boolean operations as } Q_i \leftrightarrow Q'_j = (\neg Q_i \vee Q'_j) \wedge (Q_i \vee \neg Q'_j).\]
he would know more than one bit of information about gbit 1 because $Q_1$ defines a full bit of information about it while $Q_{22}$ could be regarded as defining half a bit about each of gbit 1 and 2. But in view of principle 1, $O$ should never know more than one bit about a single gbit, even in a composite system. Considering the information contained in correlation questions to equally correspond to gbits 1 and 2, lemma 5.1 suggests that $O$’s information will always be equally distributed over the two gbits for states of maximal knowledge, i.e., $O$ will know equally much about gbit 1 and 2. We shall make this more precise in 1.

We saw before that $Q_i, Q_j', Q_{ij}$ are pairwise independent. Lemma 5.1 implies that also $Q_i$ and $Q_{kj}$ for $i \neq k$ are independent. We can make use of this result to show the following.

**Lemma 5.2.** The $Q_{ij}$, $i, j = 1, \ldots, D_1$ are pairwise independent.

*Proof.* Consider $Q_{ij}$ and $Q_{kl}$ and suppose $i \neq k$. Then $Q_i$ and $Q_{ij}$ are compatible, while $Q_i$ and $Q_{kl}$ are complementary by lemma 5.1. Hence, when knowing $Q_i$ and $Q_{ij}$ $O$ cannot know $Q_{kl}$ such that $Q_{ij}$ and $Q_{kl}$ must be independent. The analogous argument holds for when $j \neq l$. (Dependence requires that the answer to $Q_{ij}$ always implies at least partial knowledge about $Q_{kl}$, i.e., $y_{ij} = 0, 1$ implies $y_{kl} \neq \frac{1}{2}$). □

Lemmas 5.1 and 5.2 have an important corollary.

**Corollary 5.1.** $Q_i, Q_j', Q_{ij}$ are pairwise independent for all $i, j = 1, \ldots, D_1$ and will thus be part of an informationally complete set $Q_{M_2}$.

Next, we consider the compatibility and complementarity structure of the correlation questions $Q_{ij}$.

**Lemma 5.3.** $Q_{ij}$ and $Q_{kl}$ are compatible if and only if $i \neq k$ and $j \neq l$. That is, graphically, $Q_{ij}$ and $Q_{kl}$ are compatible if their corresponding edges do not intersect in a vertex and complementary if they intersect in one vertex.

*Proof.* $Q_{ij}$ and $Q_{kj}$ with $i \neq k$ must be complementary because a simultaneous answer to both would imply that the complementary $Q_i$ and $Q_{kj}$ are either correlated or anti-correlated. But according to assumption 1 this is disallowed. (A full answer to one and partial knowledge about the other would likewise partially yield illegal information about $Q_i, Q_k$.)

Consider now $Q_{ij}$ and $Q_{kl}$ with $i \neq k$ and $j \neq l$. Let $O$ ask $S$ both $Q_i$ and $Q_j'$ the answers to which imply the answer to $Q_{ij}$ according to 5.2. Thanks to principle 1 this defines a state of maximal knowledge of 2 independent bits and $O$ may not know any further independent information. Next, let $O$ ask the same $S$ the question $Q_{kl}$. From lemma 5.1 it follows that $Q_{kl}$ is complementary to both $Q_i$ and $Q_j'$ such that the answer to $Q_{kl}$ will give one new independent bit of information but renders $O$’s information about $Q_i, Q_j'$ obsolete. But by principle 2 $O$ cannot experience a net loss of information by asking a new question and after asking $Q_{kl}$ he must still know 2 bits about $S$. Hence, after acquiring the answer to $Q_{kl}$ he must still know the answer to $Q_{ij}$ such that both are compatible.

To give graphical examples, $Q_{11}$ and $Q_{21}$ are complementary due to the intersection in $Q_1'$, while $Q_{13}$ and $Q_{21}$ are compatible because their edges do not intersect in vertices

![Graphical example](image)

This question structure has significant consequences: since the correlation questions $Q_{ij}$ and $Q_{kl}$ are both independent and compatible for $i \neq k$ and $j \neq l$, $O$ can ask both of them simultaneously, thereby

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Clearly, this is not true for non-maximal knowledge. E.g., let $O$ always ask only $Q_1$. 

26 Clearly, this is not true for non-maximal knowledge. E.g., let $O$ always ask only $Q_1$. 

24
spending the maximal amount of \(N = 2\) independent \textbf{bits} of information he may acquire according to principle \(\Pi\) over composite questions. For example, he may ask \(S \ Q_{11}\) and \(Q_{22}\) simultaneously

\[
Q_{11} \quad Q_{22} \\
\vdots \\
\text{\vdots}
\]

(5.5)

upon which he must be entirely oblivious about the individual gbit properties represented by \(Q_i, Q_i'\). That is, \(O\) has only composite, but no individual information about the two gbits; they are maximally correlated. But this is precisely \textit{entanglement}. Indeed, the question graph (5.5) will ultimately correspond to four Bell states in either of the two question bases \(\{Q_1, Q_1'\}\) and \(\{Q_2, Q_2'\}\), representing the four possible answer configurations ‘yes-yes’, ‘yes-no’, ‘no-yes’ and ‘no-no’ to the correlation questions \(Q_{11}, Q_{22}\) (see \cite{29} for a related perspective within quantum theory). In fact, if \(O\) now ‘marginalized’ over gbit 2, corresponding to discarding any information about questions involving gbit 2, he would find gbit 1 in the state of no information relative to him because the discarded information also contained everything he knew about gbit 1. Other compatible edges will correspond to Bell states in different question bases.

As originally compellingly proposed within quantum theory in \cite{31, 32}, we shall define states \(\bar{y}_{O \rightarrow S}\) of a bipartite system as

\textbf{entangled} if the composite information satisfies \(\sum_{i,j = 1}^{D_1} \alpha_{ij} > 1\text{ bit},\)

\textbf{classically composed} if the composite information satisfies \(\sum_{i,j = 1}^{D_1} \alpha_{ij} \leq 1\text{ bit},\)

where \(\alpha_{ij}(\bar{y}_{O \rightarrow S})\) is \(O\)'s information about the correlation question \(Q_{ij}\). Since there are only \(N = 2\) independent \textbf{bits} of information to be gained, according to principle \(\Pi\) entanglement thus means more generally that \(O\) has more composite than individual information. Two gbits are \textbf{maximally entangled} relative to \(O\) if he has \textit{only} composite information about them. By contrast, two gbits would be in a ‘product state’ if \(O\) has maximal individual knowledge of \(2\) \textbf{bits} about them. For instance, if he knows that \(Q_1 = Q_1' = 1\), \(S\) would be in such a product state relative to him. But notice that even in this case, \(O\) would have one dependent \textbf{bit} of information about the correlations because clearly \(Q_{11} = 1\) too. Thence, the condition on entanglement as above.

Note also that \(O\) can ‘collapse’ two gbits into an entangled state: as the most extreme example consider the case that \(O\) receives a ‘product ensemble state’ of two gbits in a multiple shot interrogation. \(O\) may then ask the next pair of gbits the questions \(Q_{11}, Q_{22}\). Upon receiving the answers, the two gbits will have ‘collapsed’ into a maximally entangled \textit{a posteriori} state relative to \(O\), despite having been in a ‘product state’ prior to interrogation. Entanglement is thus a property of \(O\)'s information about \(S\).

We emphasize that, for systems with limited information content, \textit{entanglement is a direct consequence of complementarity}. To illustrate this observation, consider two classical bits (cbits). Since there is no complementarity in this case, there are only three pairwise independent questions: the individuals \(Q_1, Q_1'\) about cbit 1 and cbit 2, respectively, and the correlation \(Q_{11}\) would form an informationally complete \(Q_{MN}\), graphically represented as

\[
\text{cbit 1} \quad \text{cbit 2} \\
Q_1 \quad Q_{11} \quad Q_1' 
\]

\(Q_1, Q_1', Q_{11}\) are mutually compatible and this composite system satisfies principle \(\Pi\) too such that \(O\) may only acquire maximally \(N = 2\) independent \textbf{bits} of information about this pair of cbits. Clearly, also in this classical case it is possible for \(O\) to acquire \textit{only} composite information about the pair of cbits, namely by \textit{only} asking \(Q_{11}\) and not bothering about the individuals. \(O\) would be able to find out whether identically prepared pairs of cbits in an ensemble are correlated by always asking \(Q_{11}\) in a multiple shot interrogation. There will indeed exist states such that \(y_{11} = 1\) and \(y_1 = y_1' = \frac{1}{2}\); however, he would not be able to claim that they are entangled. The reason is simply that, if \(O\) has only asked \(Q_{11}\), the pair of
cbits will not be in a state of maximal information relative to $O$ because he has only spent one of his two available bits. A classical state of maximal information corresponds to $O$ knowing the answers to two questions of the three $Q_1, Q_1', Q_{11}$, but then by (3.3) $O$ will also know the answer to the third. That is, in a state of maximal information about two cbits, $O$ will always have maximal individual information about the pair. For cbits, $O$ cannot spend the second bit also in composite information. By contrast, it is a consequence of the complementarity principle (and Specker’s principle) that in our case here, $O$ can actually exhaust the entire information available to him by composite questions, thereby giving rise to entanglement.

Thus far, we have only considered individual and correlation questions. All are part of $Q_M$. But can there be more pairwise independent questions in $Q_M$? These would have to be obtained from the individuals and correlations via another composition with an XNOR. Composing the individuals $Q_i, Q_j'$ with the correlation questions via XNOR, e.g. $Q_i \leftrightarrow Q_{ii}$, will yield nothing new because questions need to be compatible in order to be logically connected by $O$ and for compatible pairs it already follows from (5.4) that the individual and correlation questions are logically closed under $\leftrightarrow$. However, what about correlations of correlations, e.g., what about $Q_{11} \leftrightarrow Q_{22}$?

5.2.3 A logical argument for the dimensionality of the Bloch-sphere

The answer to the last question, in fact, is intertwined with the dimensionality $D_1$ of the state space $\Sigma_1$ of a single gbit and thus, ultimately, with the dimensionality of the Bloch-sphere. Deriving the dimension of the Bloch sphere has also been a crucial step in the successful reconstructions of quantum theory via the GPT framework. While Hardy’s pioneering reconstruction [12] did not fully settle this issue (it required an operationally somewhat obscure ‘simplicity axiom’ to obtain $D_1 = 3$), it was later explicitly solved by impressive and remarkable group-theoretic arguments in [57] which show that, under certain information theoretic constraints, qubit quantum theory is the only theory with non-trivial entangling dynamics. This result was then exploited to relate the dimension of the Bloch-sphere to the dimension of space via a communication thought experiment [19]. However, unfortunately, the ingredients of these arguments cannot be employed in our information inference framework.

On the other hand, although not yielding full quantum theory reconstructions, approaches asserting an epistemic restriction (i.e. a restriction of knowledge) over ontic states admit very simple and elegant arguments for a 1-bit system to have a state space spanned by three independent epistemic states [38, 52]. These arguments can be carried out by considering a single 1-bit system and involve binary connectives of complementary questions which, while non-problematic when dealing with ‘hidden variables’, are, however, illegal according to our assumption 4. We do not wish to make any ontological commitments in our purely epistemic approach. Consequently, we must reason differently and without ontic states to deduce the dimensionality of $\Sigma_1$. In our case, this requires to consider two gbits rather than just one and involves entanglement, in analogy to the group-theoretic GPT derivation in [57, 19] which likewise requires two gbits and entanglement.

**Theorem 5.1.** $D_1 = 2$ or 3.

**Proof.** By lemma 5.3 $Q_{ij}$ and $Q_{kl}$ are independent and compatible if $i \neq k$ and $j \neq l$. But any maximal set of compatible correlation questions then contains precisely $D_1$ questions. Graphically, this is easy to see: one can always find $D_1$ non-intersecting edges between the $D_1$ vertices of gbit 1 and the $D_1$ vertices of gbit 2. For example, the $D_1$ ‘diagonal correlations’ $Q_{ii}$, $i = 1, \ldots, D_1$

$$
Q_{11} \\
Q_{22} \\
Q_{33} \\
\vdots \\
Q_{D_1D_1}
$$

are pairwise compatible and thus, by Specker’s principle (c.f. assumption 5), mutually compatible. Hence, $O$ may acquire the answers to all $D_1$ $Q_{ii}$ at the same time. However, principle 1 forbids $O$’s information
about $S$ to exceed the limit of $N = 2$ independent bits. Accordingly, the $D_1$ correlation questions $Q_{ij}$ cannot be mutually independent and, assuming that all $Q_{ij}$ are of equivalent status, the answers to any pair of them, must imply the answers to all others. For instance, suppose $O$ has asked $Q_{11}$ and $Q_{22}$. The pair must determine the answers to $Q_{33}, \ldots, Q_{D_1D_1}$ such that the latter must be Boolean functions of $Q_{11}, Q_{22}$. Hence, $Q_{jj} = Q_{11} \oplus Q_{22}$, $j \neq 1, 2$, for some logical connective * which preserves that $Q_{11}, Q_{22}, Q_{jj}$ are pairwise independent. Table 5.3 implies that * must either be the XNOR $\leftrightarrow$ or the XOR $\oplus$ such that either
\[ Q_{jj} = Q_{11} \leftrightarrow Q_{22}, \quad \text{or} \quad Q_{jj} = \lnot(Q_{11} \leftrightarrow Q_{22}), \quad \forall j = 3, \ldots, D_1. \] (5.6)

But then clearly $Q_{jj}$, $j = 3, \ldots, D_1$ could not be pairwise independent if $D_1 > 3$. The same argument can be carried out for any set of $D_1$ pairwise independent and compatible correlations $Q_{ij}$, i.e. for any question graph with $D_1$ non-intersecting edges, and any choice of a pair of compatible correlations which $O$ may ask first. We must therefore conclude that $D_1 \leq 3$, while principle 2 implies that $D_1 \geq 2$.

We shall refer to $D_1 = 2$ as the ‘rebit case’ and to $D_1 = 3$ as the ‘qubit case’; for the moment this is just suggestive terminology, however, we shall see later and in [1] that the $D_1 = 2$ case will, indeed, result in rebit quantum theory (two-level systems over real Hilbert spaces), while the $D_1 = 3$ case will give rise to standard qubit quantum theory.

5.2.4 Informationally complete sets for $D_1 = 2$ and $D_1 = 3$

The two cases have distinct properties as regards correlations of correlations and it is necessary to consider them separately. We begin with the simpler qubit case $D_1 = 3$ for which correlations of correlations do not define new information for $O$ about $S$ such that six individual and nine correlation questions constitute an informationally complete set $Q_{M_2}$. These will ultimately correspond to the propositions ‘the spin is up in $x-$, $y-$, $z-$ direction’ for the individual qubits 1 and 2 and ‘the spins of qubit 1 in $i-$ and qubit 2 in $j-$ direction are correlated’ where $i, j = x, y, z$. Note that the density matrix for two qubits has 15 parameters.

**Theorem 5.2. (Qubits)** If $D_1 = 3$ then the correlation questions $Q_{ij}$ are logically closed under $\leftrightarrow$ and $D_2 = 15$. Furthermore, for any two permutations $\sigma, \sigma'$ of $\{1, 2, 3\}$ either
\[ Q_{\sigma(3)\sigma'(3)} = Q_{\sigma(1)\sigma'(1)} \leftrightarrow Q_{\sigma(2)\sigma'(2)} \] and either
\[ Q_{\sigma(3)\sigma'(3)} = Q_{\sigma(1)\sigma'(2)} \leftrightarrow Q_{\sigma(2)\sigma'(1)} \] such that either
\[ Q_{\sigma(1)\sigma'(1)} \leftrightarrow Q_{\sigma(2)\sigma'(2)} = Q_{\sigma(1)\sigma'(2)} \leftrightarrow Q_{\sigma(2)\sigma'(1)} \] (5.7)
\[ Q_{\sigma(1)\sigma'(1)} \leftrightarrow Q_{\sigma(2)\sigma'(2)} = Q_{\sigma(1)\sigma'(2)} \leftrightarrow Q_{\sigma(2)\sigma'(1)} \] (5.8)
\[ Q_{\sigma(1)\sigma'(1)} \leftrightarrow Q_{\sigma(2)\sigma'(2)} = Q_{\sigma(1)\sigma'(2)} \leftrightarrow Q_{\sigma(2)\sigma'(1)} \] (5.9)

**Proof.** Statements (5.7)–(5.8) are an immediate consequence of the argument in the proof of theorem 5.1 which can be applied to any correlation question graph with $D_1 = 3$ non-intersecting edges and thus to any two permutations $\sigma, \sigma'$ of $\{1, 2, 3\}$. From this it directly follows that the correlation questions $Q_{ij}$ are logically closed under $\leftrightarrow$ such that in the case $D_1 = 3$ correlations of correlations $Q_{ij} \leftrightarrow Q_{kl}$ do not define any new independent questions. Accordingly,
\[ Q_{M_2} = \{Q_i, Q'_i, Q_{ij}\}_{i,j=1,2,3} \]
is an informationally complete set of questions for $N = 2$ and $D_1 = 3$ such that $D_2 = 15$.

For example, if $\sigma, \sigma'$ are both trivial then the theorem applied to the following two question graphs
implies for $D_1 = 3$ either

$$Q_{33} = Q_{11} \leftrightarrow Q_{22}, \quad \text{or} \quad Q_{33} = \neg (Q_{11} \leftrightarrow Q_{22}) \quad (5.10)$$

and either

$$Q_{33} = Q_{12} \leftrightarrow Q_{21}, \quad \text{or} \quad Q_{33} = \neg (Q_{12} \leftrightarrow Q_{21}) \quad (5.11)$$

such that either

$$Q_{11} \leftrightarrow Q_{22} = Q_{12} \leftrightarrow Q_{21}, \quad \text{or} \quad Q_{11} \leftrightarrow Q_{22} = \neg (Q_{12} \leftrightarrow Q_{21}) \quad (5.12)$$

In the section 5.2.5 and 5.4 we shall determine whether negations $\neg$ actually occur in the expressions \(5.7\) \(5.9\). For $Q, Q', Q''$ compatible and related by an XNOR, we shall henceforth distinguish between

even correlation: if $Q = Q' \leftrightarrow Q''$, and

odd correlation: if $Q = \neg (Q' \leftrightarrow Q'')$\(^{27}\)

Next, let us consider the rebit case $D_1 = 2$. $O$ can ask the four individual questions $Q_1, Q_2, Q'_1, Q'_2$ and the four correlations $Q_{11}, Q_{12}, Q_{21}, Q_{22}$. But $O$ can also define the two new correlation of correlations questions

$$Q_{43} := Q_{12} \leftrightarrow Q_{21} \quad \text{and} \quad \tilde{Q}_{33} := Q_{11} \leftrightarrow Q_{22} \quad (5.13)$$

corresponding to the two correlation questions graphs

(we add the correlation of correlations as a new edge without vertices to the graph). The difference to the qubit case is that $Q_{33}, \tilde{Q}_{33}$ can now not be written as correlations of individual questions $Q_3, Q'_3$ since the latter do not exist. Clearly, $Q_{12}, Q_{21}, Q_{33}$ and $Q_{11}, Q_{22}, Q_{33}$ are two pairwise independent sets. The question is whether $Q_{33}, \tilde{Q}_{33}$ are independent of each other and of the individuals. The following lemma even asserts complementarity to the latter. (Note that this lemma holds trivially in the case $D_1 = 3$.)

**Lemma 5.4.** The correlations of correlations $Q_{12} \leftrightarrow Q_{21}$ and $Q_{11} \leftrightarrow Q_{22}$ are complementary to $Q_{1}, Q_{2}, Q'_{1}, Q'_{2}$.

*Proof.* Suppose $Q_{11} \leftrightarrow Q_{22}$ and $Q_1$ were maximally compatible. Clearly, both are compatible with $Q_{11}$ such that according to Specker’s principle (c.f. assumption \(\mathbf{5}\)) $Q_{1}, Q_{11}, Q_{11} \leftrightarrow Q_{22}$ would be mutually compatible. But the answers to both $Q_{11}$ and $Q_{11} \leftrightarrow Q_{22}$ imply the answer to $Q_{22}$ which, according to lemma 5.1, is complementary to $Q_1$ such that $Q_1$ and $Q_{11} \leftrightarrow Q_{22}$ cannot be fully compatible. Similarly, if the latter two were partially compatible, $O$ could ask $Q_{11}$ and $Q_{11} \leftrightarrow Q_{22}$, implying an answer to $Q_{22}$, and still know something about $Q_1$ which contradicts complementarity of $Q_1$ and $Q_{22}$. Hence, $Q_1$ and $Q_{11} \leftrightarrow Q_{22}$ must be complementary. The argument works analogously for the other cases. \(\square\)

Finally, we show that \(5.11\) \(5.12\) hold analogously for the rebit case $D_1 = 2$. From this it follows that the four individual questions $Q_i, Q'_i$, the four correlations $Q_{ij}, i, j = 1, 2$, and the correlation of correlations $Q_{33}$ form an informationally complete set $Q_{M_2}$ for rebits. That is, in contrast to the qubit case, there is now one non-trivial correlation of correlations which is pairwise independent from the individuals and correlations.

**Theorem 5.3. (Rebits)** If $D_1 = 2$ then $D_2 = 9$ and either

$$Q_{11} \leftrightarrow Q_{22} = Q_{12} \leftrightarrow Q_{21}, \quad \text{or} \quad Q_{11} \leftrightarrow Q_{22} = \neg (Q_{12} \leftrightarrow Q_{21}).$$

\(^{27}\)We note that this also implies $Q' = \neg (Q \leftrightarrow Q'')$ and $Q'' = \neg (Q \leftrightarrow Q')$. 

28
Proof. $O$ can begin by asking $S$ the compatible $Q_{11}$ and $Q_{22}$ upon which he also knows the answer to $Q_{33}$. $O$ then possesses the maximal amount of $N = 2$ independent bits of information about $S$. Next, $O$ can ask $Q_{12}$ (or $Q_{21}$) which, according to lemma 5.3, is complementary to $Q_{11}, Q_{22}$. But by principle 2, $O$ is not allowed to experience a net loss of information. Hence, $Q_{12}$ (or $Q_{21}$) must be compatible with $Q_{33}$. That is, $Q_{12}, Q_{21}, Q_{33}$ are mutually compatible according to Specker’s principle and $O$ may also ask all three at the same time. But then the same argument as in the proof of theorem 5.1 applies such that either $Q_{33} = Q_{12} \leftrightarrow Q_{21}$ or $Q_{33} = -(Q_{12} \leftrightarrow Q_{21})$. This implies either $Q_{33} = Q_{33}$ or $Q_{33} = -Q_{33}$. Accordingly, there is only one independent correlation of correlations $Q_{33}$. Since $Q_{11}, Q_{12}, Q_{21}, Q_{22}$ and $Q_{33}$ are then, by construction, logically closed under the XNOR $\leftrightarrow$ and $Q_{33}$ is complementary to all individuals $Q_1, Q_2, Q'_1, Q'_2$, no further pairwise independent question can be built from this set such that it is informationally complete. Hence, $D_2 = 9$.

The rebit case $D_1 = 2$ thus has a very special question and correlation structure: $Q_{33} = Q_{12} \leftrightarrow Q_{21}$ is the only composite question which is complementary to all individuals, but compatible with all correlations $Q_{ij}$. By contrast, e.g., $Q_{11}$ is complementary to $Q_2, Q'_2, Q_{12}, Q_{21}$ and compatible with $Q_1, Q'_1, Q_{22}$ and compatible with the correlation of correlations $Q_{33}$. Consequently, $Q_{33}$ assumes a special role in the entanglement structure: once $O$ knows the answer to this question he may no longer have any further information about the outcomes of individual questions, but may have information about correlation questions. That is, two rebits can be in a state of non-maximal information of 1 bit relative to $O$, corresponding to the latter only having maximal knowledge about the answer to $Q_{33}$ and no information otherwise, and still be maximally entangled because any individual information is forbidden in that situation. Notice that this is not true for pairs of qubits because even if everything $O$ knew about the pair was the answer to $Q_{33}$, he could still acquire information about the individuals $Q_3, Q'_3$ such that one could not consider such a state as maximally entangled.

Even stronger, $O$ will always know the answer to $Q_{33}$ if the two rebits are maximally entangled and he has maximal information about $S$. This follows from (3.6): for a maximally entangled state of maximal information (no information about individuals), the following must hold

$$I_{O \rightarrow S} = 2 \sum_{i=1}^{2} (\alpha_i + \alpha'_i) + 2 \sum_{i,j=1}^{2} \alpha_{ij} + \alpha_{33}$$

$$= \alpha_{11} + \alpha_{12} + \alpha_{21} + \alpha_{22} + \alpha_{33} = 3 \text{bits.}$$

The last equality follows from the fact that once $O$ knows the answers to the two compatible questions, he will also know the answer to their correlation and since this correlation is in the pairwise independent question set the total information defined in (3.6) will always yield 3 bits for $N = 2$ independent bit systems in states of maximal knowledge.\textsuperscript{28} If we now require that $O$ cannot know more than a single bit about complementary questions then, in a state of maximal information of $N = 2$ independent bits, we must have

$$\alpha_{11} + \alpha_{12} = \alpha_{21} + \alpha_{22} = 1 \text{bit} \quad \Rightarrow \quad \alpha_{33} = 1 \text{bit}$$

such that $O$ must have maximal information about $Q_{33}$. Therefore, the correlation of correlations $Q_{33}$ can be viewed as the litmus test for entanglement of two rebits.

We note that the individuals $Q_1, Q_2, Q'_1, Q'_2$ correspond to projections on $\sigma_x, \sigma_z$, while the $Q_{ij}$ correspond to projections on the $\sigma_i \otimes \sigma_j$, $i, j = x, z$, and $Q_{33}$ corresponds to the projection on $\sigma_y \otimes \sigma_y$, where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. Observables and density matrices on a real Hilbert space correspond to real symmetric matrices. This is the reason why $\sigma_y$ is not an observable on $\mathbb{R}^2$ (it corresponds to the ‘missing’ $Q_3$), but $\sigma_y \otimes \sigma_y$ is a real symmetric matrix and thus an observable on $\mathbb{R}^2 \otimes \mathbb{R}^2$.

This gives a novel and simple explanation for the discovery that $\sigma_y \otimes \sigma_y$ determines the entanglement of rebits\textsuperscript{24}: a two-rebit density matrix $\rho$ is separable if and only if $\text{Tr}(\rho \sigma_y \otimes \sigma_y) = 0$, i.e. if the state has no $\sigma_y \otimes \sigma_y$ component. This statement means in our language that a state is separable if and only if $\alpha_{33} = 0$ and is consistent with our observation above because any information about the individuals is incompatible with information about $Q_{33}$ due to complementarity.

\textsuperscript{28} We are implicitly using here also principles 4 and 3.
This question structure also has severe repercussions for rebit state tomography: it must ultimately be non-local. For rebits, $D_1 = 2$, the probability that $Q_{33} = 1$ could be written as

$$y_{33} = p(Q_{12} = 1, Q_{21} = 1) + p(Q_{12} = 0, Q_{21} = 0)$$

where $p(Q_1, Q_2)$ denotes here the joint probability distribution over $Q_1, Q_2$. That is, in a multiple shot interrogation, $O$ could ask both $Q_{12}, Q_{21}$ to the identically prepared rebit couples and from the statistics over the answers, he could also determine $y_{33}$. But this probability cannot be decomposed into joint probabilities over the individual questions according to $Q_{33} = "(Q_1 \leftrightarrow Q_2') \leftrightarrow (Q_2 \leftrightarrow Q_1')"$ because $Q_1, Q_2$ and $Q_1', Q_2'$ are complementary. Therefore, $O$ would not be able to determine $y_{33}$ by only asking individual questions to the two rebits and the statistics over these answers.\footnote{For example, in a multiple shot interrogation $O$ could first ask $Q_1, Q_2'$ on a set of identically prepared rebit couples to find out whether the answers are correlated. On a second identically prepared set, $O$ could then ask $Q_2, Q_1'$ which are complementary to the first questions he asked. From the statistics over the answers, $O$ would be able to determine also the probabilities for $Q_{12}$ and $Q_{21}$. But since he needed two separate interrogation runs to determine the statistics for $Q_{12}$ and $Q_{21}$, he would not be able to infer from this any information whatsoever about the statistics of answers to $Q_{33}$.}

That is, for rebits, state tomography would always require correlation questions and in this sense be non-local. Note that this stands in stark contrast to qubit pairs where $Q_{33} = Q_3 \leftrightarrow Q_3'$ can be written in terms of individual questions such that also $y_{33} = p(Q_3 = 1, Q_3' = 1) + p(Q_3 = 0, Q_3' = 0)$ can be determined by the statistics over the answers to $Q_3, Q_3'$ only. Qubit systems (and quantum theory in general) are thus tomographically local.

The requirement of tomographic locality, according to which the state of a composite system can be determined by doing statistics over measurements on its subsystems, is a standard condition in the GPT framework\footnote{The requirement of local tomography is usually taken as the origin of the tensor product structure for composite systems in quantum theory\cite{57,14,15,53}. However, one has to be a bit careful with this statement because there exist two distinct tensor products: there is (a) the tensor product of Hilbert spaces, e.g., $(\mathbb{C}^2)^{\otimes N}$ for qubits and $(\mathbb{R}^2)^{\otimes N}$ for rebits, and (b) the tensor product of unnormalized probability vectors or density matrices. A tensor product of type (b) defines a sufficient support only for composite qubit systems, but not for composite rebit systems; the space of hermitian matrices over $(\mathbb{C}^2)^{\otimes N}$ is the N-fold tensor product of hermitian matrices over $\mathbb{C}^2$, but the space of symmetric matrices over $(\mathbb{R}^2)^{\otimes N}$ is not the N-fold tensor product of symmetric matrices over $\mathbb{R}^2$ (see also \cite{24}). What local tomography implies is the tensor product (b), but as the example of rebits shows, the tensor product of type (a) also exists without it.} and thus directly rules out rebit theory. However, in contrast to derivations within the GPT landscape, we shall not implement local tomography here because it will be interesting to see the differences between real and complex quantum theory from the perspective of information inference and we shall thus carry out the complete reconstruction of both here and in\cite{1}.

Local tomography is usually taken as the origin of the tensor product structure for composite systems in quantum theory\cite{57,14,15,53}. However, one has to be a bit careful with this statement because there exist two distinct tensor products: there is (a) the tensor product of Hilbert spaces, e.g., $(\mathbb{C}^2)^{\otimes N}$ for qubits and $(\mathbb{R}^2)^{\otimes N}$ for rebits, and (b) the tensor product of unnormalized probability vectors or density matrices. A tensor product of type (b) defines a sufficient support only for composite qubit systems, but not for composite rebit systems; the space of hermitian matrices over $(\mathbb{C}^2)^{\otimes N}$ is the N-fold tensor product of hermitian matrices over $\mathbb{C}^2$, but the space of symmetric matrices over $(\mathbb{R}^2)^{\otimes N}$ is not the N-fold tensor product of symmetric matrices over $\mathbb{R}^2$ (see also \cite{24}). What local tomography implies is the tensor product (b), but as the example of rebits shows, the tensor product of type (a) also exists without it.

### 5.2.5 A Bell scenario with questions

We shall now settle the issue of the relative negation $\neg$, i.e. whether

$$Q_{11} \leftrightarrow Q_{22} = Q_{12} \leftrightarrow Q_{21}, \quad \text{or} \quad Q_{11} \leftrightarrow Q_{22} = \neg(Q_{12} \leftrightarrow Q_{21}),$$

for both rebits and qubits (see theorems\cite{52,53}). This amounts to a question configuration analogous to a Bell scenario.

(A) Suppose $Q_{11} \leftrightarrow Q_{22} = Q_{12} \leftrightarrow Q_{21}$ was true. This is the case of classical logic: $O$ can consistently interpret any such configuration by means of a local ‘hidden variable’ model. For example, consider the case $Q_{11} = Q_{22} = 1$ such that $Q_{11} \leftrightarrow Q_{22} = 1$ and, consequently, also $Q_{12} \leftrightarrow Q_{21} = 1$. We represent the last two equations by the following two graphs

![Graph](attachment://Graph.png)

where both edges solid and of equal colour means that the corresponding questions are correlated. $O$ could consistently read this configuration as follows: “Since ‘$Q_1 = Q_1’$’ and ‘$Q_2 = Q_2’$’ I would precisely
have to conclude that “\(Q_{12} = Q_{21}\)”, i.e. \(Q_{12} \leftrightarrow Q_{21} = 1\), if the individuals \(Q_1, Q_2, Q'_1, Q'_2\) had definite values which, however, I do not know.” Therefore, \(O\) could join the two graphs consistently

\[
\begin{align*}
&\text{Q1} &\text{Q2} \\
&\text{Q1} &\text{Q2} \\
&\text{Q1} &\text{Q2}
\end{align*}
\]

i.e., draw all four edges simultaneously which corresponds to all four edges having definite (albeit unknown) values at the same time. \(O\) would conclude that there must be an underlying unknown ontic state. For example, \(Q_{11} = Q_{22} = Q_{12} \leftrightarrow Q_{21} = 1\) would be consistent with the four ontic states:

\[
\begin{array}{cccc}
1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0
\end{array}
\]

\(O\) could then also read the graph \(5.14\) equivalently as

\[
\begin{align*}
&\text{Q1} &\text{Q1'} \\
&\text{Q2} &\text{Q2'}
\end{align*}
\]

and would have to conclude that

\[
Q_1 \leftrightarrow Q_2 = Q'_1 \leftrightarrow Q'_2.
\]

That is, although \(O\) would not know the answer to either \(Q_1 \leftrightarrow Q_2\) or \(Q'_1 \leftrightarrow Q'_2\), he would know their correlation and thus be able to make consistent statements about logical connectives of complementary questions. This, however, is illegal according to our assumption \(4\).

Similarly, \(O\) can interpret any other configuration with \(Q_{11} \leftrightarrow Q_{22} = Q_{12} \leftrightarrow Q_{21}\) in terms of a ‘hidden variable’ model which assigns local definite values to the individuals and, conversely, as can be easily checked, any assignment of definite (or ontic) values to the individuals \(Q_1, Q_2, Q'_1, Q'_2\) leads necessarily to \(Q_{11} \leftrightarrow Q_{22} = Q_{12} \leftrightarrow Q_{21}\). Consequently, \(O\) would always be able to extract illegal information about complementary questions, thereby violating assumption \(4\) such that we must rule out the possibility \(Q_{11} \leftrightarrow Q_{22} = Q_{12} \leftrightarrow Q_{21}\).

\(B\) We are thus already forced to the conclusion that \(Q_{11} \leftrightarrow Q_{22} = \neg(Q_{12} \leftrightarrow Q_{21})\) must hold. Let us quickly check that it is, indeed, consistent with our assumptions and that in this case \(O\) would not be able to extract any illegal information. Consider, again, the case \(Q_{11} = Q_{22} = 1\) such that \(Q_{11} \leftrightarrow Q_{22} = 1\) and, consequently, now \(Q_{12} \leftrightarrow Q_{21} = 0\). This configuration may be graphically represented as

\[
\begin{array}{cccc}
\text{Q11} &\text{Q12} \\
\text{Q22} &\text{Q21}
\end{array}
\]

where one edge solid the other dashed, but same colour, means that the corresponding answers are anti-correlated. It is impossible to consistently join the two diagrams by drawing all four edges simultaneously (which would correspond to all edges having definite values), as one would obtain a frustrated graph

\[
\begin{align*}
&\text{Q11} \\
&\text{Q22} \\
&\text{Q12} \\
&\text{Q21}
\end{align*}
\]

\(O\) must conclude that the four edges can not have definite (but unknown) values all at the same time. But if \(O\) cannot join the two diagrams, he can also not extract illegal information about \(S\) and make consistent statements about logical connectives of complementary questions – in contrast to \(5.15\). It is impossible for \(O\) to interpret this situation in terms of a local ‘hidden variable’ model which assigns definite values to the individuals simultaneously. The same verdict holds for any other configuration with

\[
Q_{11} \leftrightarrow Q_{22} = \neg(Q_{12} \leftrightarrow Q_{21}).
\]

\(5.17\)

It is precisely the graphical inconsistency \(5.16\) which renders \(5.17\) consistent with assumption \(4\) and the principles.

31
Since either (A) or (B) must be true by theorems 5.2 and 5.3, and (A) violates assumption 4, we conclude that (5.17) is the correct relation. We emphasize that the relative negation \( \neg \) precludes a classical reasoning for the distribution of truth values over \( O \)'s questions.

Note that this argument holds for both rebits and qubits and, for qubits, also for any other pairs \( Q_i, Q_j \neq i \) and \( Q'_k, Q'_l \neq k \) of pairs of complementary individuals (see theorem 5.2). In fact, even more generally, the same argument can be made for any four questions (i.e., not necessarily individuals) \( Q, Q', Q'', Q''' \) which are such that \( Q, Q' \) and \( Q'', Q''' \) are complementary pairs, while \( Q \) and \( Q' \) are each compatible with both \( Q'', Q''' \); also in this case one would have to conclude that

\[
(Q \leftrightarrow Q'') \leftrightarrow (Q' \leftrightarrow Q''') = \neg((Q \leftrightarrow Q''') \leftrightarrow (Q' \leftrightarrow Q'')).
\]

(5.18)

This will become relevant later on. For now we observe that exchanging the positions of the complementary \( Q'' \), \( Q''' \) from the left to the right hand side introduces a negation.

The correlation structure for rebits is now clear from these results: (5.17) implies \( Q_{33} = \neg Q_{33} \) for the correlations of correlations defined in (5.13). This settles the fate of all possible relative negations \( \neg \) for rebits. However, these results do not fully determine the odd and even correlation structure for qubits: we still have to clarify whether there is an overall negation \( \neg \) relative to \( Q_{33} \) in (5.10) (5.11) and more generally in theorem 5.2 for other permutations of non-intersecting edges. This difference between rebits and qubits results again from the fact that \( Q_{33} \) is defined as the correlation of the individuals \( Q_3, Q'_3 \) for qubits, while it is a correlation of correlations for rebits. This has a remarkable consequence: rebit theory is its own ‘logical mirror image’, while qubit theory’s ‘logical mirror image’ is distinct from qubit theory. This topic will be deferred to section 5.4 because we firstly need to understand the question structure for the \( N = 3 \) case in order to discuss the odd and even correlation structure of two qubits further.

5.3 Three gbits

It will be both useful and instructive to explicitly consider the \( N = 3 \) case for rebits and qubits. As a composite system, we can view three gbits, labeled by \( A, B, C \), either as three individual systems, as three combinations of one individual and a bipartite composite system or as a tripartite system:

\[
\begin{align*}
\text{A} & \bullet \text{B} \bullet \text{C} \\
\text{A} & \bullet \text{B} \bullet \text{C} \\
\text{A} & \bullet \text{B} \bullet \text{C} \\
\text{A} & \bullet \text{B} \bullet \text{C} \\
\end{align*}
\]

(5.19)

According to definition 3 of a composite system, \( Q_3 \) must then contain the individual questions of all three gbits, any bipartite correlation questions and any permissible logical connectives thereof. This results in different structures for rebits and qubits.

5.3.1 Three qubits

We shall begin with the qubit case \( D_1 = 3 \). Clearly, according to definition 3 all \( 3 \times 3 = 9 \) individual questions, henceforth denoted as \( Q_{ia}, Q_{jb}, Q_{kc} \), and all \( 3 \times 9 = 27 \) bipartite correlations, from now on written as \( Q_{iajb}, Q_{iakc}, Q_{jbc}, i, j, k = 1, 2, 3 \), will be part of an informationally complete set \( Q_{M_3} \). As
before, we represent individuals and correlations graphically as vertices and edges, respectively, e.g.

\[
\begin{array}{ccc}
A & B & C \\
Q_{1A} & Q_{1B} & Q_{1C} \\
Q_{2A} & Q_{2B} & Q_{2C} \\
Q_{3A} & Q_{3B} & Q_{3C} \\
\end{array}
\]

\[
\begin{array}{ccc}
A & B & C \\
Q_{1A} & Q_{1B} & Q_{1C} \\
Q_{2A} & Q_{2B} & Q_{2C} \\
Q_{3A} & Q_{3B} & Q_{3C} \\
\end{array}
\]

depict valid question graphs.

But in order to complete the individuals and bipartite correlations to \(Q_{M}\), we now have to consider logical connectives of these questions which are pairwise independent. This will necessarily involve tripartite questions because the bipartite structure is already exhausted with individuals and bipartite correlations. Clearly, we cannot add a question \(\tilde{Q}_{111}\), representing the proposition “the answers to \(Q_{1A}, Q_{1B}, Q_{1C}\) are the same” to the individuals and bipartite correlations because, e.g., \(\tilde{Q}_{111} = 1\) would always imply \(Q_{1A1B} = Q_{1A1C} = Q_{1B1C} = 1\) and, conversely, \(Q_{1A1B} = 0\) would imply \(\tilde{Q}_{111} = 0\) such that the bipartite correlations and \(\tilde{Q}_{111}\) would not be pairwise independent.

From (5.3) we already know that the logical connective yielding new pairwise independent questions must either be the XNOR or the XOR. For consistency with the bipartite structure we continue to employ the XNOR. There is an obvious candidate for an independent tripartite question, namely

\[
Q_{ijk} = Q_{iA} \leftrightarrow Q_{jB} \leftrightarrow Q_{kC}
\]

(5.20)

which thanks to the associativity and symmetry of \(\leftrightarrow\) can also equivalently be written as

\[
Q_{ijk} = Q_{iA} \leftrightarrow Q_{jBkC} = Q_{iAjB} \leftrightarrow Q_{kC} = Q_{iAkC} \leftrightarrow Q_{jB}.
\]

(5.21)

(Since the notation for this tripartite question is unambiguous from the ordering of \(i, j, k\) we drop the subscripts \(A, B, C\) in \(Q_{ijk}\).) This structure is also natural from the different compositions in (5.19). \(Q_{ijk}\) thus defined is by construction compatible with \(Q_{iA}, Q_{jB}, Q_{kC}, Q_{iAjB}, Q_{iAkC}, Q_{jBkC}\) and, for similar reasons to the independence of \(Q_{iAjB}\) from \(Q_{iA}, Q_{jB}\) (see the discussion below (5.4)), also pairwise independence of the latter. Note that this question does not stand in one-to-one correspondence with the proposition “the answers to \(Q_{iA}, Q_{jB}, Q_{kC}\) are the same”; e.g., \(Q_{1A} = 1\) and \(Q_{jB} = Q_{kC} = 0\) also gives \(Q_{ijk} = 1\). It is easier interpreted via (5.21) as either of the three questions “are the answers to \(Q_{iA}, Q_{jBkC}/Q_{iAjb}, Q_{kC}/Q_{iAkC}, Q_{jB}\) the same?”

There are \(3 \times 3 \times 3 = 27\) such tripartite questions \(Q_{ijk}, i, j, k = 1, 2, 3\). We shall represent them graphically as triangles. For example, \(Q_{111}, Q_{322}, Q_{333}\) are depicted as follows:

\[
\begin{array}{ccc}
A & B & C \\
Q_{1A} & Q_{1B} & Q_{1C} \\
Q_{2A} & Q_{2B} & Q_{2C} \\
Q_{3A} & Q_{3B} & Q_{3C} \\
\end{array}
\]
5.3.2 Independence and compatibility for three qubits

Individual questions from qubit $A$ are compatible with the individual questions from qubits $B$ and $C$, etc. But what about the compatibility of bipartite and tripartite correlations? The compatibility structure of bipartite correlations of a fixed qubit pair is clear from lemma 5.3, but we have to investigate compatibility of bipartite correlations involving all three qubits.

**Lemma 5.5.** $Q_{i_A j_B}$ and $Q_{l_B k_C}$ are complementary if $j \neq l$. On the other hand, $Q_{i_A j_B}$ and $Q_{j_B k_C}$ are compatible and it holds

$$Q_{i_A j_B} \leftrightarrow Q_{j_B k_C} = Q_{i_A k_C}.$$

The analogous statements hold for any permutation of $A, B, C$. That is, graphically, two bipartite correlations involving three qubits are compatible if the corresponding edges intersect in a vertex and complementary otherwise.

*Proof.* Suppose $Q_{i_A j_B}$ and $Q_{l_B k_C}$ were compatible for $j \neq l$. Clearly, both are compatible with $Q_{i_A}$ and $Q_{k_C}$ such that by Specker’s principle (c.f. assumption (c.f. assumption)) $Q_{i_A j_B}, Q_{l_B k_C}, Q_{i_A}, Q_{k_C}$ would form a mutually compatible set. But a simultaneous answer to all would imply an answer to the complementary $Q_{i_A}, Q_{l_B}$. (For partial compatibility one would reason similarly.) We thus conclude that $Q_{i_A j_B}$ and $Q_{l_B k_C}$ are complementary if $j \neq l$.

$Q_{i_A j_B}$ and $Q_{j_B k_C}$ are evidently compatible since $Q_{i_A}, Q_{j_B}, Q_{k_C}$ are compatible. Moreover,

$$Q_{i_A j_B} \leftrightarrow Q_{j_B k_C} = Q_{i_A} \leftrightarrow (Q_{j_B} \leftrightarrow Q_{j_B}) \leftrightarrow Q_{k_C} = Q_{i_A k_C}$$

thanks to the associativity of $\leftrightarrow$.

For example, $Q_{2_A 2_B}$ and $Q_{2_B 2_C}$ intersect in $Q_{2_B}$ and are thus compatible, while $Q_{2_A 2_B}$ and $Q_{1_B 1_C}$ do not share a vertex and are therefore complementary:

![Graphical representation of compatibility and complementarity](image)

We continue with the tripartite questions.

**Lemma 5.6.** $Q_{i j k}$ is compatible with $Q_{i_A}, Q_{j_B}, Q_{k_C}$ and complementary to $Q_{i_A} \neq i_A, Q_{j_B} \neq j_B, Q_{k_C} \neq k_C$. That is, graphically, $Q_{i j k}$ is compatible with an individual $Q_{i_A}, Q_{j_B}, Q_{k_C}$ if the corresponding vertex is one of the vertices of the triangle representing $Q_{i j k}$ and complementary otherwise.

*Proof.* $Q_{i j k}$ is by construction compatible with $Q_{i_A}, Q_{j_B}, Q_{k_C}$. Consider therefore $Q_{i j k}$ and $Q_{i_A} \neq i_A$ and suppose the two were maximally compatible. Clearly, both are compatible with $Q_{j_B k_C}$ such that by Specker’s principle $O$ could ask all three at the same time. But this would imply simultaneous maximal knowledge about the answers to the complementary $Q_{i_A}, Q_{i_A} \neq i_A$. By similar arguments, partial compatibility would also lead to illegal simultaneous information about complementary questions. Accordingly, $Q_{i j k}, Q_{i_A} \neq i_A$ are complementary. One argues analogously for the individuals of qubits $B$ and $C$. 

For instance, $Q_{111}$ is compatible with $Q_{1C}$ and complementary to $Q_{2C}$:

\[ Q_{111} \quad Q_{1C} \]
\[ \quad Q_{2C} \]

\[ \bullet \]

This lemma also directly implies that the individuals and tripartite correlations are pairwise independent.

Next we consider bipartite and tripartite correlations.

**Lemma 5.7.** $Q_{ijk}$ is compatible with $Q_{iA|B}$, $Q_{iA|C}$, $Q_{jB|C}$ and, furthermore, with $Q_{mB|nC}$, $Q_{lA|nC}$ and $Q_{lA|mB}$ for $l \neq i$, $m \neq j$ and $k \neq n$. On the other hand, $Q_{ijk}$ is compatible to $Q_{iA|B}$, $Q_{iA|C}$, $Q_{jB|C}$, $Q_{jB|nC}$, $Q_{lA|kC}$, $Q_{mB|kC}$ for $l \neq i$, $m \neq j$ and $k \neq n$. That is, graphically, $Q_{ijk}$ is compatible with a bipartite correlation if the edge of the latter is either an edge of the triangle corresponding to $Q_{ijk}$ or if the edge and triangle do not intersect. $Q_{ijk}$ is complementary to a bipartite correlation if the edge of the latter and the triangle corresponding to $Q_{ijk}$ share one common vertex.

**Proof.** $Q_{ijk}$ is by construction compatible with $Q_{iA|B}$, $Q_{iA|C}$, $Q_{jB|C}$. $Q_{ijk} = Q_{iA} \leftrightarrow Q_{jB|C}$ and $Q_{mB|nC}$ are also compatible for $j \neq m$ and $k \neq n$ because $Q_{mB|nC}$ is compatible with $Q_{iA}$ and thanks to lemma 5.3 also with $Q_{jB|C}$. Consider now $Q_{ijk}$ and $Q_{iA|B}$ for $j \neq m$ and suppose the two were compatible. Since both are compatible with $Q_{kC}$, by Specker’s principle $O$ could ask all three at the same time which, however, would give him simultaneous maximal knowledge about the pair $Q_{iA|B}$, $Q_{iA|B}$ which are complementary by lemma 5.3. The argument against partial compatibility is similar such that $Q_{ijk}$, $Q_{iA|B}$ must be complementary. The reasoning for all other cases is analogous.

To give a graphical example, $Q_{111}$ is compatible with $Q_{1A|2B}$ and $Q_{3A|3C}$ and complementary to $Q_{1A|2B}$:

\[ Q_{111} \quad Q_{1A|2B} \]
\[ \quad Q_{3A|3C} \]
\[ \bullet \]

We still have to check pairwise independence of the bipartite and tripartite correlations.

**Lemma 5.8.** The bipartite $Q_{mB|nC}$, $Q_{lA|nC}$, $Q_{lA|mB}$ and tripartite correlations $Q_{ijk}$ are pairwise independent.

**Proof.** Lemma 5.7 implies that we only have to check pairwise independence of $Q_{mB|nC}$, $Q_{lA|nC}$, $Q_{lA|mB}$ from $Q_{ijk}$ for $l \neq i$, $m \neq j$ and $k \neq n$ because complementary questions are by definition independent and $Q_{ijk}$ and $Q_{iA|B}$, $Q_{iA|C}$, $Q_{jB|C}$ are pairwise independent. Consider therefore $Q_{ijk}$ and $Q_{mB|nC}$ for $j \neq m$ and $k \neq n$. By lemma 5.3 $Q_{kc}$ is compatible with $Q_{ijk}$ and by lemma 5.4 complementary to $Q_{mB|nC}$. This implies, using the arguments from the proof of lemma 5.2 independence of $Q_{ijk}$, $Q_{mB|nC}$. The other cases follow similarly.

**Lemma 5.9.** The tripartite correlations $Q_{ijk}$, $i,j,k = 1,2,3$ are pairwise independent.
Proof. Consider $Q_{ijk}$ and $Q_{lmn}$ for $i \neq l$. By lemma 5.6 $Q_{iA}$ is compatible with $Q_{ijk}$ and complementary to $Q_{lmn}$. Using the analogous arguments from the proof of lemma 5.2, this implies that $Q_{ijk}, Q_{lmn}$ are independent. The same reasoning holds when $j \neq m$ and $k \neq n$. 

This has an immediate consequence:

**Corollary 5.2.** The individuals $Q_{iA}, Q_{jB}, Q_{kC}$, the bipartite $Q_{iA}\leftrightarrow Q_{jB}\leftrightarrow Q_{kC}$ and the tripartite $Q_{ijk}$, $i,j,k=1,2,3$ are pairwise independent and thus contained in an informationally complete set $Q_{M_3}$.

Lastly, we consider the complementarity and compatibility structure of the tripartite correlations.

**Lemma 5.10.** $Q_{ijk}$ and $Q_{lmn}$ are compatible if $\{i,j,k\}$ and $\{l,m,n\}$ overlap in one or three indices and complementary if $\{i,j,k\}$ and $\{l,m,n\}$ overlap in zero or two indices. That is, graphically, $Q_{ijk}$ and $Q_{lmn}$ are compatible if their corresponding triangles intersect in one vertex (or coincide) and complementary if the triangles share an edge or do not intersect.

**Proof.** Compatibility for an overlap in all three indices is trivial. But also $Q_{ijk} = Q_{iA} \leftrightarrow Q_{jB}\leftrightarrow Q_{kC}$ and $Q_{lmn} = Q_{iA} \leftrightarrow Q_{mB}\leftrightarrow Q_{nC}$ are clearly compatible for $j \neq m$ and $k \neq n$ because by lemma 5.3 $Q_{jB}\leftrightarrow Q_{jB}$, $Q_{mB}\leftrightarrow Q_{mB}$ are compatible in this case. Compatibility for the other cases of an overlap of $Q_{ijk}$ and $Q_{lmn}$ in one index follows by permutation.

Suppose now $Q_{ijk}$ and $Q_{lmn}$ were compatible for $i \neq l, j \neq m$ and $k \neq n$. By lemma 5.7 both are compatible with $Q_{jB}\leftrightarrow Q_{mB}$ and the latter two are also compatible by lemma 5.3. Accordingly, thanks to Specker’s principle (c.f. assumption 3), $O$ could ask all four questions at the same time. But this would provide $O$ with simultaneous maximal information about the complementary pair $Q_{iA}, Q_{iA} \neq A$. Partial compatibility of $Q_{ijk}, Q_{lmn}$ would similarly lead to illegal partial information about complementaries. Hence, $Q_{ijk}, Q_{lmn}$ are complementary for $i \neq l, j \neq m$ and $k \neq n$.

Finally, $Q_{ijk} = Q_{iA} \leftrightarrow Q_{jB}\leftrightarrow Q_{kC}$ and $Q_{lmn} = Q_{iA} \leftrightarrow Q_{mB}\leftrightarrow Q_{nC}$ must be complementary for $i \neq l$. Otherwise, $O$ could ask $Q_{ijk}$ and $Q_{jB}\leftrightarrow Q_{kC}$ simultaneously, thereby knowing the answer to $Q_{iA}$, and at the same time have at least partial information about $Q_{ijk}$ (i.e., $y_{ijk} \neq \frac{1}{2}$) which, however, is complementary to $Q_{iA}$ by lemma 5.6. Complementarity of tripartite correlations for other overlaps in precisely two indices follows by permutation.

For example, $Q_{111}$ and $Q_{212}$ intersect in the vertex $Q_{1A}$ and are thus compatible. By contrast, $Q_{111}$ shares the edge $Q_{1A}\leftrightarrow Q_{1B}$ with $Q_{113}$ and does not intersect at all with $Q_{333}$ such that $Q_{111}$ is complementary to both. $Q_{113}$ and $Q_{333}$ intersect in the vertex $Q_{3C}$ and are therefore compatible.

**5.3.3 An informationally complete set for three qubits**

We now show that the 9 individuals, the 27 bipartite and the 27 tripartite correlations form an informationally complete set $Q_{M_3}$ for $N = 3$ qubits.

**Theorem 5.4. (Qubits)** The individuals $Q_{iA}, Q_{jB}, Q_{kC}$, the bipartite $Q_{iA}\leftrightarrow Q_{jB}\leftrightarrow Q_{kC}$ and the tripartite $Q_{ijk}$, $i,j,k=1,2,3$ are logically closed under $\leftrightarrow$ such that they form an informationally complete set $Q_{M_3}$ with $D_3 = 63$ for $D_1 = 3$. 

36
Proof. For individuals and bipartite correlations of any pair of qubits the logical closure under the XNOR was already shown in section 5.2. Correlation of individuals with a bipartite correlation from another pair of qubits yields the tripartite correlations. Lemma 5.5 shows that also the bipartite correlations involving distinct pairs of qubits are logically closed under $\leftrightarrow$. We thus only have to check logical closure of XNOR combinations involving tripartite correlations.

Lemma 5.6 shows that tripartite correlations and individuals are only compatible if the individual is a vertex of the tripartite triangle. But then a combination such as $Q_{ijk} \leftrightarrow Q_{iA} = Q_{iA} \leftrightarrow Q_{iA} \leftrightarrow Q_{jBkC} = Q_{jBkC}$ produces another bipartite correlation, thanks to the associativity of $\leftrightarrow$. Similarly, lemma 5.7 asserts that tripartite and bipartite correlations are only compatible if the edge of the bipartite correlation is either contained in the tripartite triangle or if the edge and triangle do not intersect. However, combinations of such compatible pairs also do not yield any new questions because, e.g.,

$$Q_{ijk} \leftrightarrow Q_{iA} = Q_{iA}$$

and, using again the associativity of XNOR and theorem 5.2

$$Q_{\sigma_A(1)\sigma_B(1)k} \leftrightarrow Q_{\sigma_A(2)\sigma_B(2)} = \left( Q_{\sigma_A(1)\sigma_B(1)} \leftrightarrow Q_{\sigma_A(2)\sigma_B(2)} \right) \leftrightarrow Q_{kC} = Q_{\sigma_A(3)\sigma_B(3)k} \text{ or } \neg Q_{\sigma_A(3)\sigma_B(3)k},$$

where $\sigma_A, \sigma_B$ are permutations of $\{1, 2, 3\}$. Finally, lemma 5.10 entails that tripartite correlations are only compatible if they intersect in one vertex. But, using theorem 5.2 once more,

$$Q_{\sigma_A(1)\sigma_B(1)k} \leftrightarrow Q_{\sigma_A(2)\sigma_B(2)} = Q_{\sigma_A(1)\sigma_B(1)} \leftrightarrow Q_{\sigma_A(2)\sigma_B(2)} = Q_{\sigma_A(3)\sigma_B(3)} \text{ or } \neg Q_{\sigma_A(3)\sigma_B(3)}.$$ 

Permuting these examples implies that all individuals, bipartite and tripartite correlations are logically closed under $\leftrightarrow$ which by (5.3) is the only independent logical connective possibly yielding new independent questions. These questions therefore form an informationally complete set $Q_{M_3}$ with $D_3 = 63$. \hfill $\square$

### 5.3.4 Entanglement of three qubits and monogamy

Let us now put all these detailed results to good use and reward ourselves with the observation that they naturally explain monogamy of entanglement for the qubit case. This is best illustrated with an example. Let $O$ have asked the questions $Q_{1A1B}$ and $Q_{2A2B}$ to qubits $A, B$ such that the two are in a state of maximal information of $N = 2$ independent bits and maximal entanglement relative to $O$. 

$$Q_{1A1B}, Q_{2A2B}$$

Intuitively, this already suggests monogamy of entanglement because $O$ has spent the maximally attainable amount of information over the pair $A, B$ such that even if now a third qubit $C$ enters the game he should no longer be able to ask any question to the triple which gives him any further simultaneous independent information about the pair $A, B$. But any correlation of $A$ or $B$ with $C$ would constitute additional information about $A$ or $B$ which would violate the information bound for the subsystem of the composite system of three qubits.

We shall now make this more precise. The question is, which bipartite or tripartite correlations involving qubit $C$ are compatible with $Q_{1A1B}, Q_{2A2B}$. Firstly, lemma 5.3 states that bipartite correlations
of two distinct qubit pairs are only compatible if their corresponding edges intersect in a vertex. Clearly, there can then not exist any edge which is compatible with both $Q_{1A1B}, Q_{2A2B}$. Secondly, lemma 5.7 asserts that a bipartite correlation is only compatible with a tripartite correlation if the corresponding edge is either contained in the tripartite triangle or does not intersect the triangle. This means that the only tripartite correlations compatible with $Q_{1A1B}, Q_{2A2B}$ are (a) the correlations of the latter with the individuals $Q_{1C}, Q_{2C}, Q_{3C}$ of qubit $C$ and (b) $Q_{3k}, k = 1, 2, 3$. For example, in the first of the following two question graphs $Q_{2B2C}$ is compatible with $Q_{2A2B}$ but complementary to $Q_{1A1B}$, while $Q_{3B3C}$ is complementary to both. But $O$ could ask $Q_{222}$ together with $Q_{1A1B}, Q_{2A2B}$, as depicted in the second graph:

However, asking $Q_{1A1B}, Q_{2A2B}$ and $Q_{222}$ simultaneously is equivalent to asking $Q_{1A1B}, Q_{2A2B}$ and $Q_{2C}$ because $Q_{2A2B} \leftrightarrow Q_{222} = Q_{2C}$. The analogous conclusion holds for any other tripartite question compatible with $Q_{1A1B}, Q_{2A2B}$. Therefore, once $O$ has asked the last to bipartite correlations about qubits $A, B$, he can only acquire individual information corresponding to $Q_{1C}, Q_{2C}, Q_{3C}$ about qubit $C$. Clearly, the same state of affairs is true for any permutation of the three qubits. This is monogamy of entanglement in its most extreme form: two qubits which are maximally entangled can not be correlated whatsoever with any other system.

The non-extremal form of monogamy, namely the case that a qubit pair is not maximally entangled and can thus share a bit of entanglement with a third qubit, can also be explained. To this end, we recall from quantum information theory that monogamy is generally described with so-called monogamy inequalities, e.g., the Coffman-Kundu-Wootters inequality \[\tau_{A|BC} \geq \tau_{AB} + \tau_{AC},\] \(5.22\)

where $0 \leq \tau_{A|BC} = 2(1 - \text{Tr} \rho_A^2) \leq 1$ measures the entanglement between qubit $A$ and the qubit pair $B, C$ and $\rho_A$ is the marginals state of qubit $A$ obtained after tracing qubit $B$ and $C$ out of the tripartite state. $\tau_{AB}, \tau_{AC}$, on the other hand, measure the bipartite entanglement of the pairs $A, B$ and $A, C$ and are similarly obtained by tracing either $C$ or $B$ out of the tripartite qubit state. The inequality \(5.22\) formalizes the intuition that the correlation between $A$ and the pair $B, C$ is at least as strong as the correlation of $A$ with $B$ and $C$ individually. This is the general form of monogamy. One usually also defines the so-called three-tangle \[\tau_{ABC} = \tau_{A|BC} - \tau_{AB} - \tau_{AC} \in [0, 1]\]

to measure the genuine tripartite entanglement shared among all three qubits. This three-tangle turns out to be permutation invariant $\tau_{ABC} = \tau_{BCA} = \tau_{CAB}$.

In our current case, a measure of entanglement sharing must be informational. At this stage, in line with our previous definition of entanglement, we simply define the analogous entanglement measures to be the sum of $O$’s information about the various independent bipartite or tripartite correlation questions

$\tilde{\tau}_{A|BC} := \sum_{i,j,k=1}^{3} \alpha_{ijk} + \sum_{i,j=1}^{3} \alpha_{i,jB} + \sum_{i,k=1}^{3} \alpha_{i,kC}$

$\tilde{\tau}_{AB} := \sum_{i,j=1}^{3} \alpha_{i,jB}$

$\tilde{\tau}_{AC} := \sum_{i,k=1}^{3} \alpha_{i,kC}$

\(5.23\)
With this definition, an informational inequality analogous to the inequality (5.22) is trivial
\[ \tilde{\tau}_{A|BC} \geq \tilde{\tau}_{AB} + \tilde{\tau}_{AC} \]
as is the fact that the informational three-tangle
\[ \tilde{\tau}_{ABC} := \tilde{\tau}_{A|BC} - \tilde{\tau}_{AB} - \tilde{\tau}_{AC} = \sum_{i,j,k=1}^{3} \alpha_{ijk} \]
is permutation invariant and measures only the information contained in the tripartite questions and thus genuine tripartite entanglement. This is how one can describe the general form of monogamy of entanglement in our language. Of course, at this stage the information measure \( \alpha_i \) about the various questions \( Q_i \) is only implicit, but we shall derive its explicit form in section 7, upon which the above statements become truly quantitative. These informational versions of monogamy inequalities and tangles naturally suggest themselves for simple generalizations to arbitrarily many qubits, thereby complementing current efforts in the quantum information literature (e.g., see [76]).

Before we move on, we briefly emphasize that monogamy is a consequence of complementarity – as is entanglement. For example, three classical bits also satisfy the limited information principle 1, but due to the absence of complementarity, \( O \) could ask all correlations \( Q_{AB}, Q_{AC}, Q_{BC} \) and \( Q_{ABC} \) at once

which, however, is equivalent to asking the three individuals \( Q_A, Q_B, Q_C \).

5.3.5 Maximal entanglement for three qubits

Let us elucidate maximal tripartite entanglement for three qubits, corresponding to \( O \) asking three compatible and mutually independent tripartite correlation questions such that he exhausts the information bound of \( N = 3 \) independent bits with tripartite information. Lemma 5.10 asserts that tripartite questions are compatible if and only if they intersect in exactly one vertex. For example, \( O \) could ask \( Q_{211}, Q_{121}, Q_{112} \) simultaneously. These are mutually independent because by (5.10, 5.11, 5.17)
\[ Q_{211} \leftrightarrow Q_{121} \leftrightarrow Q_{112} = Q_{3,3,3} \text{ or } \neg Q_{3,3,3} \]
i.e., their binary connectives with an XNOR do not imply each other and the bipartite correlations are pairwise independent from the tripartite ones. Accordingly, asking \( Q_{211}, Q_{121}, Q_{112} \) will provide \( O \) with three independent bits of information about the qubit triple. We note that lemma 5.6 implies that once \( O \) has posed the questions \( Q_{211}, Q_{121}, Q_{112} \), every individual question \( Q_{iA}, Q_{iB}, Q_{iC} \) is complementary to at least one of the three tripartite correlations. That is, \( O \) cannot acquire any individual information about the three qubits and will only have composite information. This is a necessary condition for maximal entanglement.

In addition to the three implied binary correlations (5.24), \( O \) would clearly also know the tripartite correlation of all three \( Q_{211}, Q_{121}, Q_{112} \), which amounts to
\[ Q_{211} \leftrightarrow Q_{121} \leftrightarrow Q_{112} = Q_{222} \text{ or } \neg Q_{222} \].

This exhausts the list of questions about which \( O \) would have information by asking \( Q_{211}, Q_{121}, Q_{112} \).
The list can be represented by the following question graph:

![Question Graph](image)

This graph will ultimately correspond to the eight possible Greenberger-Horne-Zeilinger (GHZ) states in either of the three question bases \{Q_{2A}, Q_{1B}, Q_{1C}\}, \{Q_{1A}, Q_{2B}, Q_{1C}\}, and \{Q_{1A}, Q_{1B}, Q_{2C}\}, corresponding to the eight answer configurations 'yes-yes-yes', 'yes-yes-no', 'yes-no-no', 'yes-no-yes', 'no-yes-yes', 'no-yes-no', 'no-no-yes' and 'no-no-no' to the three tripartite correlations \(Q_{211}, Q_{121}, Q_{112}\). Similarly, any other three compatible tripartite correlations will correspond to GHZ states in other question bases.

The correlation information in the graph (5.25) is clearly democratically distributed over the three qubits \(A, B, C\). (Intuitively, one could even view the bit stemming from, say, the answer to \(Q_{2A}\) as being carried to one-third by each of \(A, B, C\).) Furthermore, if \(O\) now wanted to 'marginalize' over qubit \(C\), i.e. discard any information involving qubit \(C\), all that would be left of his knowledge about the qubit triple would be the answer to \(Q_{3}\). But, as argued at the end of section 5.2.4, this resulting 'marginal state' of qubits \(A, B\) relative to \(O\) cannot be considered entangled. The analogous results hold for 'marginalization' over either \(A\) or \(B\). As a result, the question graph (5.25) corresponds to genuine maximal tripartite entanglement.

### 5.3.6 Three rebits

We shall now repeat the same exercise for three rebits but can benefit from the results of the \(N = 3\) qubit case. We shall thus be briefer here. According to definition 3, \(O\)'s questions to the three rebit system must contain the 6 individuals \(Q_{iA}, Q_{jB}, Q_{kC}\), 12 bipartite correlations \(Q_{iA}Q_{jB}, Q_{iA}Q_{kC}, Q_{jB}Q_{kC}\), \(i, j, k = 1, 2, 3\). To render this set informationally complete, we have to top it up with tripartite questions. There are now 8 tripartite correlations \(Q_{ijk}\), \(i, j, k = 1, 2, 3\), of the kind (5.20, 5.21) and, moreover, 6 tripartite correlations of a rebit individual with the correlation of correlation of the other rebit pair

\[
Q_{33} := Q_{iA} \leftrightarrow Q_{3B3C}, 
Q_{33} := Q_{jB} \leftrightarrow Q_{3A3C}, 
Q_{33} := Q_{kC} \leftrightarrow Q_{3A3B}.
\]

Given that the individuals \(Q_{3A}, Q_{3B}, Q_{3C}\) do not exist for rebits there is now also no tripartite correlation \(Q_{333}\).

### 5.3.7 Independence and compatibility for three rebits

The independence, compatibility and complementarity structure for the questions not involving an index \(i, j, k = 3\) directly follows from the qubit discussion as lemmas 5.5–5.10 also hold in the present case for \(i, j, k = 1, 2\). But we now have to clarify the question structure once two indices are equal to 3 (an odd number of indices cannot be 3). The status of any such purely bipartite relations was clarified in section 7.2 such that here we have to consider the case that all three rebits are involved.

**Lemma 5.11.** \(Q_{3B3C}\) is complementary to \(Q_{iA}Q_{jB}\), \(i, j = 1, 2\), and compatible with \(Q_{3A3B}\). Furthermore,

\[
Q_{3A3B} \leftrightarrow Q_{3B3C} = Q_{3A3C}.
\]

The same holds for any permutations of \(A, B, C\).

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30 As an aside, we note that these propositions solve the little riddle in Zeilinger’s festschrift for D. Greenberger [29].
Similarly, one rules out partial compatibility.

Compatibility of $Q_{A3B}$ and $Q_{B3C}$ follows indirectly. Namely, $Q_{A2B}, Q_{2B1C}$ and $Q_{A1B}, Q_{1B2C}$ are two compatible pairs by lemma 5.5. We can then apply the reasoning around (5.18) to find

\[
Q_{A3B} \leftrightarrow Q_{B3C} = (Q_{A2B} \leftrightarrow Q_{2A1B}) \leftrightarrow (Q_{1B2C} \leftrightarrow Q_{2B1C}) = -((Q_{A2B} \leftrightarrow Q_{2A1B}) \leftrightarrow (Q_{1B2C} \leftrightarrow Q_{2B1C})) = Q_{A3B} \leftrightarrow Q_{2A2C} = Q_{A3C}.
\]

The last equality holds thanks to theorem 5.3.

The same reasoning applies to any permutation of $A, B, C$.

Next, we discuss the tripartite correlations of individuals with correlations of correlations of rebit pairs.

**Lemma 5.12.** $Q_{i33}$ is compatible with $Q_{1a}$ and complementary to $Q_{jB}, Q_{kB}$. The same holds for any permutation of $A, B, C$.

**Proof.** Compatibility of $Q_{i33}$ with $Q_{1a}$ is true by construction. Complementarity of $Q_{jB}$ and $Q_{i33} = Q_{1a} \leftrightarrow (Q_{1B2C} \leftrightarrow Q_{2B1C})$ follows from the observation that $Q_{i33}$ and $Q_{jB}$ are both compatible with $Q_{1a}$ and a similar reasoning to the previous proof.

**Lemma 5.13.** $Q_{i33}$ is compatible with $Q_{3B3C}, Q_{jB3C}, j, k = 1, 2$, and $Q_{1akC}, Q_{1ajB}$ for $i \neq l$. On the other hand, $Q_{i33}$ is complementary to $Q_{1a3B}, Q_{1a3C}$ and $Q_{3A3B}, Q_{3A3C}$. Furthermore, $Q_{ijk}$ is compatible with $Q_{3A3B}$. The same holds for any permutation of $A, B, C$.

**Proof.** Compatibility of $Q_{i33}$ with $Q_{jB3C}$ and $Q_{3A3C}$, as well as compatibility of $Q_{ijk}$ with $Q_{3A3B}$ is obvious. Compatibility of $Q_{i33}$ with $Q_{iakC}$ for $i \neq l$ follows indirectly by noting that $Q_{i33}$ and $Q_{jB3C}, Q_{jB3C}$ are two pairs of complementary questions, but that the questions in one pair are compatible with both questions of the other. In this case the reasoning of (5.18) applies and entails the correlation of $Q_{i33}$ with $Q_{3B3C}$ must be compatible with the correlation of $Q_{1a}$ with $Q_{kB}$. Compatibility of $Q_{i33}$ with $Q_{1ajB}$ follows similarly.

Complementarity of $Q_{i33}$ and $Q_{1ajB}$ follows from the fact that both are compatible with $Q_{1a}$ and using Specker’s principle as in previous proofs. Likewise, $Q_{i33}$ and $Q_{1a3B}$ follows similarly by noting that both are compatible with $Q_{3B3C}$ and use of Specker’s principle.

**Lemma 5.14.** $Q_{i33}, Q_{j33}, Q_{33k}$ are pairwise independent from the bipartite correlations $Q_{i1jB}, Q_{iakC}, Q_{jBkC}$ and the bipartite correlations of $Q_{i33}, Q_{j33}, Q_{33k}, Q_{3A3B}, Q_{3A3C}, Q_{3B3C}$. $Q_{ijk}$ is also pairwise independent from the latter. Furthermore, $Q_{ijk}$ and $Q_{i33}, Q_{j33}, Q_{33k}$ are pairwise independent.

**Proof.** The proof is completely analogous to the proofs of lemmas 5.2, 5.8 and 5.9.

As before this has an important consequence.

**Corollary 5.3.** The individuals $Q_{1A}, Q_{jB}, Q_{kB}$, the bipartite correlations $Q_{i1jB}, Q_{iakC}, Q_{jBkC}$, the bipartite correlations of $Q_{i33}, Q_{j33}, Q_{3A3B}, Q_{3A3C}, Q_{3B3C}$, the tripartite correlations $Q_{ijk}$ and the tripartite $Q_{i33}, Q_{j33}, Q_{33k}, i, j, k = 1, 2$, are pairwise independent and thus part of an informationally complete set $Q_{jM3}$.

The compatibility and complementarity structure of questions involving the correlations of correlations is analogous to lemma 5.10.

**Lemma 5.15.** $Q_{ijk}$ is compatible with $Q_{i33}, Q_{j33}, Q_{33k}$ and complementary to $Q_{i33}, Q_{j33}, Q_{33k}$ for $i \neq l, j \neq m$ and $k \neq n$. Furthermore, $Q_{i33}$ is compatible with $Q_{j33}, Q_{33k}$, but $Q_{i33}$ and $Q_{j33}$ are complementary. The analogous result holds for all permutations of $A, B, C$. 

41
Proof. Compatibility of $Q_{ijk}$ with $Q_{i33}, Q_{3j3}, Q_{33k}$ follows from the fact that the constituents of the latter $Q_{i4}, Q_{3b3c}, \ldots$ are compatible with $Q_{ijk}$. Complementarity of, e.g., $Q_{ijk}$ and $Q_{i33}$ for $i \neq l$ can be shown by noting that both are compatible with $Q_{3b3c}$ and using Specker’s principle as in the other complementarity proofs. Compatibility of, say, $Q_{i33}$ and $Q_{3j3}$ can be demonstrated by using \[5.18\] and noting that $Q_{i4}, Q_{3a3c}$ and $Q_{j1b}, Q_{3b3c}$ are two pairs of complementary questions which are such that each question in one pair is compatible with both questions of the other pair. Finally, $Q_{i33}$ and $Q_{233}$ are complementary because both are compatible with $Q_{3b3c}$ and $Q_{1a}, Q_{2a}$ are complementary.

This finishes our considerations of the independence and complementarity structure of three rebits.

5.3.8 An informationally complete set for three rebits

The rebit questions considered thus far comprise an informationally complete set of 35 elements.

**Theorem 5.5. (Rebits)** The individuals $Q_{i4}, Q_{j3}, Q_{k3}$, the bipartite correlations $Q_{i4j3}, Q_{i4k3}, Q_{j4k3}$, the bipartite correlations of correlations $Q_{3a3b}, Q_{3a3c}, Q_{3b3c}$, the tripartite correlations $Q_{ijk}$ and the tripartite correlations $Q_{333}, Q_{33j}, Q_{33k}$, $i, j, k = 1, 2, 3$, are logically closed under $\leftrightarrow$ and thus form an informationally complete set $Q_{M_3}$ with $D_3 = 35$ for $D_1 = 2$.

Proof. Logical closure under the XNOR for any pair of rebits follows from section 5.2. Combining individuals with bipartite questions of another rebit pair produces the tripartite questions. Lemma 5.11 and 5.12 assert logical closure of bipartite correlations and bipartite correlations of correlations involving three rebits. We must therefore only check logical closure of combinations involving tripartite questions which involve indices taking the value 3 (the other cases are covered by theorem 5.4 for $i, j, k = 1, 2$).

Using theorem 5.3 and lemmas 5.11, 5.15, the proof is mostly analogous to the proof of theorem 5.4 such that here we will only show two non-trivial cases which are treated differently. For example, by lemma 5.13 $Q_{i33}$ and $Q_{2a2b}$ are compatible. The conjunction with $\leftrightarrow$ yields

$$Q_{i33} \leftrightarrow Q_{2a2b} = Q_{1a} \leftrightarrow Q_{3b3c} \leftrightarrow (Q_{1a1} \leftrightarrow Q_{4a3b})$$

$$= Q_{1b} \leftrightarrow Q_{3a3c} = Q_{313}. \quad \text{[5.18] [5.20]}$$

Similarly, by lemma 5.15 $Q_{i33}, Q_{3j3}$ are compatible. Their XNOR conjunction gives

$$Q_{i33} \leftrightarrow Q_{3j3} = (Q_{i4} \leftrightarrow Q_{3b3c}) \leftrightarrow (Q_{4a3c} \leftrightarrow Q_{j1b})$$

$$= \neg (Q_{4a3b} \leftrightarrow Q_{i4j3}) \quad \text{[5.18]}$$

which again coincides with some $Q_{ima3b}$ with $i \neq l$ and $j \neq m$ (or the negation thereof). In complete analogy to these explicit examples and to the proof of theorem 5.4 one shows the logical closure under the XNOR of all other cases. This gives the desired result.

5.3.9 Monogamy and maximal entanglement for three rebits

Rebits are considered as non-monogamous in the literature \[77, 78\]. However, this conclusion depends somewhat on one’s notion of monogamy and can be clarified within our language. Firstly, rebits, just as qubits, are clearly monogamous in the following sense: if two rebits $A, B$ are maximally entangled in a state of maximal information relative to $O$, then they cannot share any entanglement whatsoever with a third rebit $C$. This can be seen by repeating the argument of section 5.3.3 which holds analogously for three rebits.

The situation changes slightly for entangled states of non-maximal information involving either of $Q_{3a3b}, Q_{3a3c}, Q_{3b3c}$ and for tripartite maximally entangled states. As an example for the former case, $O$ could ask $Q_{3a3b}, Q_{3b3c}$ simultaneously which gives him two independent bits of information about
the rebit triple and implies $Q_{3A3C}$ by (5.26) as well such that his information could be summarized as

![Diagram](image)

(we depict the $Q_{3A3B}, Q_{3A3C}, Q_{3B3C}$ without vertices to emphasize the absence of the individuals $Q_{3A}, Q_{3B}, Q_{3C}$ for rebits). This graph corresponds to non-monogamously entangled rebit states: by lemma 5.4 all individuals are complementary to two of the three known questions such that $O$ cannot acquire any individual information about the three rebits at the same time. The three rebits are pairwise maximally entangled, albeit not in a state of maximal information (see also the end of section 5.2.4).

However, the three rebits can be similarly non-monogamous in a tripartite maximally entangled state of maximal information. As in the qubit case (5.25) in section 5.3.5, one can write down a question graph corresponding to the rebit analogues of GHZ states, representing the eight possible answers to the tripartite questions $Q_{211}, Q_{121}, Q_{112}$

![Diagram](image)

(to justify this graph one has to employ theorem 5.5). Thanks to the information about the answers to $Q_{3A3B}, Q_{3A3C}, Q_{3B3C}$, such a state would contain a non-monogamous distribution of entanglement over $A, B, C$. In particular, if $O$ ‘marginalized’ over rebit $C$, by discarding all information involving $C$, he would be left with the answer to $Q_{3A3B}$ which defines a maximally entangled two-rebit state of non-maximal information (see section 5.2.4) – in contrast to the qubit case of section 5.3.5.

### 5.4 Correlation structure for $N = 2$

While we were able to settle the relative negation $\neg$ between correlations of correlations for both qubits and rebits in (5.18), we still have to clarify the odd and even correlation structure for qubits in (5.10, 5.11) and more generally in theorem 5.2. This will involve the notion of the ‘logical mirror’ of an inference theory and require the tools from the previous section, covering the $N = 3$ case. We recall that the odd and even correlation structure for rebits has already been settled in section 5.2.5 through (5.18). This, as we shall see shortly, is a consequence of rebit theory being its own mirror image.

#### 5.4.1 The logical mirror image of an inference theory

Consider a single qubit, described by $O$ via an informationally complete set $Q_1, Q_2, Q_3$ which can be viewed as a question basis on $Q_1$. As such, it defines a ‘logical handedness’ in terms of which outcomes to $Q_1, Q_2, Q_3$ $O$ calls ‘yes’ and which ‘no’. Clearly, this is a convention made by $O$ and he can easily

31Ultimately, these three questions will define an orthonormal Bloch vector basis in the Bloch sphere (see section 8). The handedness or orientation of this basis will depend on the labeling of question outcomes.
change the handedness by simply swapping the assignment ‘yes’ ↔ ‘no’ of one question, say, $Q_1$, which is tantamount to $Q_1 \mapsto \neg Q_1$. This corresponds to taking the logical mirror image of a single qubit. For a single qubit this will not have any severe consequences and $O$ is free to choose whichever handedness he desires.

However, the handedness of a single qubit question basis does become important when considering composite systems. Consider, e.g., two observers $O, O'$ each having one qubit and describing it with a certain handed basis. They can also consider correlations $Q_{ij}$ of their qubit pair. If $O$ now decided to change the handedness of his local question basis by $Q_1 \mapsto \neg Q_1$ this would result in

$$Q_{11}, Q_{12}, Q_{13} \mapsto \neg Q_{11}, \neg Q_{12}, \neg Q_{13}, \quad Q_{ij} \mapsto Q_{ij}, \ i \neq 1$$

and therefore

$$Q_{11} \leftrightarrow Q_{22} \mapsto \neg(Q_{11} \leftrightarrow Q_{22}), \quad \text{and} \quad Q_{12} \leftrightarrow Q_{21} \mapsto \neg(Q_{12} \leftrightarrow Q_{21}).$$

Since $Q_{33}$ would remain unaffected by the change of local handedness ($Q_3, Q'_3$ remain invariant), $O'$s change of local handedness would have resulted in a switch between even and odd correlation structure for $Q_{13}$ and $Q_{23}$. Since the handedness of the local question basis is just a convention by $Q_{13}$ (see theorem 5.2) depends on the local conventions by (5.10, 5.11). Since the handedness of the local question basis is just a convention by $Q_{13}$ (see theorem 5.2) depends on the local conventions by $O, O'$ and both are, in fact, consistent. However, (5.18) will always hold. Usually, of course, one would favour the situation that both $O, O'$ make the same conventions such that their local question bases are equally handed (or that one observer $O$ describes all qubits with the same handedness). Nevertheless, physically all local conventions are fully equivalent, but will lead to different representations of composite systems in the inference theory.

But there are important consistency conditions on the distribution of odd and even correlations. This becomes obvious when examining three qubits $A, B, C$. To this end, consider the trivial conjunction

$$Q_{3a3b} \leftrightarrow Q_{3a3c} \leftrightarrow Q_{3b3c},_{\text{lemma 5.35}} = 1. \tag{5.27}$$

From (5.10) we know that the correlation is either even or odd, respectively,

$$Q_{3a3b} = Q_{1a1b} \leftrightarrow Q_{2a2b}, \quad \text{or} \quad Q_{3a3b} = \neg(Q_{1a1b} \leftrightarrow Q_{2a2b}) \tag{5.28}$$

and analogously for $A, C$ and $B, C$. Suppose now that all three bipartite correlations in the conjunction (5.27) were even. Then we immediately get a contradiction because, using lemma 5.5,

$$Q_{3a3b} \leftrightarrow Q_{3a3c} \leftrightarrow Q_{3b3c} = (Q_{1a1b} \leftrightarrow Q_{2a2b}) \leftrightarrow (Q_{1a1c} \leftrightarrow Q_{2a2c}) \leftrightarrow (Q_{1b1c} \leftrightarrow Q_{2b2c})$$

$$= (Q_{1a1c} \leftrightarrow Q_{2a2c}) \leftrightarrow (Q_{2a2b} \leftrightarrow Q_{2a2c}) \leftrightarrow (Q_{1b1c} \leftrightarrow Q_{2b2c})$$

$$= Q_{1b1c} = Q_{2a2c}.$$ 

But this violates the identity (5.27) and results from the relative negation between the left and right hand side in (5.18). One would get the same contradiction if one of the correlations in (5.27) was even and two were odd because then the two negations from the odd correlation would cancel each other and one would still be left with the negation coming from (5.18).

On the other hand, everything is consistent if either all bipartite correlations in (5.28) are odd or one is odd and two are even because then the odd number of negations from the odd correlations cancels the negation coming from (5.18):

$$Q_{3a3b} \leftrightarrow Q_{3a3c} \leftrightarrow Q_{3b3c} = \neg(Q_{1a1b} \leftrightarrow Q_{2a2b}) \leftrightarrow \neg(Q_{1a1c} \leftrightarrow Q_{2a2c}) \leftrightarrow \neg(Q_{1b1c} \leftrightarrow Q_{2b2c})$$

$$= (Q_{1a1c} \leftrightarrow Q_{2a2c}) \leftrightarrow (Q_{2a2b} \leftrightarrow Q_{2a2c}) \leftrightarrow (Q_{1b1c} \leftrightarrow Q_{2b2c})$$

$$= Q_{1b1c} = Q_{2a2c}.$$ 

We can also quickly check that this is consistent with (5.11)

$$Q_{3a3b} = Q_{1a2b} \leftrightarrow Q_{2a1b}, \quad \text{or} \quad Q_{3a3b} = \neg(Q_{1a2b} \leftrightarrow Q_{2a1b}) \tag{5.29}$$
\[ Q_{1A2B} \leftrightarrow Q_{2A1B} = - (Q_{1A1B} \leftrightarrow Q_{2A2B}). \]

That is, if all correlations in (5.28) are odd, then all correlations in (5.29) must be even. Indeed,

\[
Q_{3A3B} \leftrightarrow Q_{3A3C} \leftrightarrow Q_{3B3C} = (Q_{1A2B} \leftrightarrow Q_{2A1B}) \leftrightarrow (Q_{1A2C} \leftrightarrow Q_{2A1C}) \leftrightarrow (Q_{1B2C} \leftrightarrow Q_{2B1C}) = (Q_{1A2B} \leftrightarrow Q_{1A2C}) \leftrightarrow (Q_{2A1B} \leftrightarrow Q_{2A1C}) \leftrightarrow (Q_{1B2C} \leftrightarrow Q_{2B1C}) = Q_{3B3C} \leftrightarrow Q_{3A3C} = 1
\]

is consistent.

In conclusion, if \( O \) wants to treat the bipartite relations among all three \( A, B, C \) identically, then the following distribution of odd and even correlations

\[
Q_{3A3B} = Q_{1A2B} \leftrightarrow Q_{2A1B} = - (Q_{1A1B} \leftrightarrow Q_{2A2B}),
\]

and analogously for \( A, C \) and \( B, C \), is the only consistent solution. We shall henceforth make the convention that the bipartite correlation structure among any pair of qubits be the same such that (5.30) holds. This turns out to be the case of qubit quantum theory.

The tacit assumption, underlying standard qubit quantum theory, is that the handedness of each local qubit question basis for \( A, B, C \) is the same, e.g., all ‘left’ or ‘right’ handed. But we emphasize, that it would be equally consistent to choose one basis as ‘left’ (‘right’) and the other two as ‘right’ (‘left’) handed. A qubit pair with equally handed bases will be described by the odd and even correlation distribution as in (5.30), while a qubit pair with oppositely handed bases will be described by the opposite distribution of odd and even correlations. In terms of whether \( Q_{33} = Q_{11} \leftrightarrow Q_{22} \) is even or odd \( Q_{33} = - (Q_{11} \leftrightarrow Q_{22}) \), the three qubit relations yield only four consistent graphs for the distribution of ‘left’ and ‘right’ handedness

\[
\begin{align*}
\text{‘left’} & \quad \text{odd} & \quad \text{‘left’} \\
\text{‘right’} & \quad \text{odd} & \quad \text{‘right’} \\
\text{‘left’} & \quad \text{even} & \quad \text{‘left’} \\
\text{‘right’} & \quad \text{even} & \quad \text{‘right’}
\end{align*}
\]

This gives a simple graphical explanation for the consistency observations above. The first two graphs correspond to quantum theory. The framework with the opposite correlation structure of quantum theory, corresponding to the last two graphs, is sometimes referred to as mirror quantum theory [13]. While mirror quantum theory was considered inconsistent in [13], we see here that inconsistencies would only arise if the bipartite correlation structure of mirror quantum theory were used for all three pairs of qubits. However, the proper formulation of mirror quantum theory for three qubits corresponds to the last two graphs in (5.31) and gives a perfectly consistent framework [3]. It could be easily generalized in an obvious manner to arbitrarily many qubits [3].

Obviously, the same argument can be carried out for any other triple of compatible bipartite correlations appearing in theorem (5.2). In conclusion, the two distinct consistent distributions of odd and even correlations, corresponding to quantum theory and mirror quantum theory,

\( (a) \) result from different conventions of local basis handedness, and

\( ^{32} \) In fact, one could even produce the correlation structure of mirror quantum theory in the lab by using oppositely handed bases for two qubits in an entangled pair. The resulting state would be represented by the partial transpose of an entangled qubit state. This state would not be positive if represented in terms of the standard Pauli matrices and thus not correspond to a legal quantum state [79]. However, the point is that mirror quantum theory would not be represented in the standard Pauli matrix basis but in the partially transposed basis which corresponds to replacing

\[
\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{by} \quad \sigma_y^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\]

for one of the two qubits (this is a switch of the \( y \)-axis orientation). In this basis the state would be positive.

\( ^{33} \) Notice, however, that for more than three qubits one would get more than two different consistent distributions of odd and even correlations. For example, for four qubits there will be three cases: (1) all four have equally handed bases, (2) three have equally handed bases, (3) two have equally handed bases.
(b) are in one-to-one correspondence through a local relabeling ‘yes’⇒‘no’ of one individual question.

As such, the two distinct correlation structures ultimately give rise to two distinct representations of the same physics and are thus fully equivalent. The transformation (b) between the two representations – being a translation between two conventions/descriptions – is a passive one and can be carried out on a piece of paper; it is always allowed. However, clearly, there cannot exist any actual physical transformation in the laboratory which maps states from one convention into the other.

Within the formalism of quantum theory the transformation (b) $Q_1 \mapsto -Q_1$ corresponds to the partial transpose (e.g., see [13]). It is well known that the partial transpose defines a separability criterion for quantum states which is both necessary and sufficient for a pair of qubits [79]: a two-qubit density matrix $\rho$ is separable (i.e., represents a product state) if and only if its partial transpose is positive relative to a basis of standard Pauli matrix products (i.e., represents a legal quantum state). This criterion holds analogously in our language here: as seen at the beginning of this section, the transformation $Q_1 \mapsto -Q_1$ changes between odd and even correlations of bipartite correlations $Q_{ij}$. This would, in fact, be unproblematic if $O$ only had individual information about the two qubits; it would map a classically composed state even of maximal information, say, $Q_1 = 1$ and $Q'_1 = 1$ and thus $Q_{11} = 1$, to another legal classically composed state $Q_1 = 0$, $Q'_1 = 1$ and thus $Q_{11} = 0$. Both states exist within both conventions. However, applying this transformation to a maximally entangled state with odd correlation, say, $Q_{11} = Q_{22} = 1$ and thus, by (5.30), $Q_{33} = 0$ yields an even correlation $Q_{11} = 0 = Q_{33}$ and $Q_{22} = 1$. The former state only exists in the quantum theory representation, while the latter exists only in the mirror image. For other entangled states one would similarly find that $Q_1 \mapsto -Q_1$ necessarily maps from one representation into the other.

The same conclusion also holds for a transformation $\{Q_1, Q_2, Q_3\} \mapsto \{-Q_1, -Q_2, -Q_3\}$, which one might call total inversion, because the odd number of negations involved in the transformation would likewise lead to a swap of odd and even correlations.

Lastly, we note that the situation is very different for rebit theory because it is its own mirror image, i.e. rebit theory and mirror rebit theory are identical representations. $O$ will describe a single rebit by a question basis $Q_1, Q_2$. Suppose $O$ decided to swap the ‘yes’ and ‘no’ assignments to the outcomes of $Q_1$, such that equivalently $Q_1 \mapsto -Q_1$. For a pair of rebits, this would have the following ramification

$$Q_{11}, Q_{12} \mapsto -Q_{11}, -Q_{12}, \quad Q_{21}, Q_{22} \mapsto Q_{21}, Q_{22},$$

and therefore

$$Q_{12} \leftrightarrow Q_{21} \mapsto -(Q_{12} \leftrightarrow Q_{21}) \quad \Rightarrow \quad Q_{33} \mapsto -Q_{33}.$$ 

In contrast to the qubit case, $Q_{33}$ is defined as a correlation of correlations $Q_{33} := Q_{12} \leftrightarrow Q_{21}$ [13] and can not be written in terms of local questions $Q_3, Q'_3$. Hence, $Q_{33}$ also changes under this transformation by construction. Accordingly, $Q_1 \mapsto -Q_1$ does not lead to a swap of odd and even correlations for the rebit case. This ‘partial transpose’ therefore always maps states to other states within the same representation. For example, even a maximally entangled state of maximal information and even correlation, say, $Q_{12} = Q_{21} = 1$ and $Q_{33} = 1$, would be mapped to another even correlated state $Q_{12} = 0, Q_{21} = 1$ and $Q_{33} = 0$. As a consequence, the Peres separability criterion [79] which is valid for qubits does not hold in analogous fashion for rebits. For completely equivalent reasons, the total inversion, corresponding to $\{Q_1, Q_2\} \mapsto \{-Q_1, -Q_2\}$, is also a transformation which preserves the representation.

5.4.2 Collecting the results: odd and even correlation structure for $N = 2$

After the many technical details it is useful to collect all the results concerning the compatibility, complementarity and correlation structure for two qubits, derived in lemmas [5.1] and [5.3], theorem [5.2], equation [5.13] and in the previous section [5.4.1] in a graph to facilitate a visualization. We shall henceforth abide by the convention that all bipartite relations for arbitrarily many qubits be treated equally such that (5.30) must hold. For the other relations of theorem [5.2] one finds the analogous results. As can be easily verified, the ensuing question structure has the lattice pattern of figure 8 where

$$Q 
\begin{array}{c}
\rightarrow \text{Q}'' \\
\leftrightarrow Q = -(Q' \leftrightarrow Q''),
\end{array}
\begin{array}{c}
\rightarrow \text{Q}' \\
\leftrightarrow Q = Q' \leftrightarrow Q''
\end{array}$$

(5.32)
Figure 3: A lattice representation of the complete compatibility, complementarity and correlation structure of the informationally complete set $Q_{M2}$ for two qubits. Every vertex corresponds to one of the 15 pairwise independent questions. If two questions are connected by an edge, they are compatible. If two questions are not connected by an edge, they are complementary. Thanks to the logical structure of the XNOR $\leftrightarrow$, defining a question as a correlation of two other questions, the compatibility structure results in a lattice of triangles. As clarified in (5.32), red triangles denote odd, while green triangles denote even correlation. Note that the two lattices represented here are connected through the nine correlation questions $Q_{ij}$ and form a single closed lattice (which, however, is easier to represent in this disconnected manner). Every question resides in exactly three triangles and is thereby compatible with six and complementary to eight other questions.

denote odd and even correlation, respectively.

We recall that (5.17, 5.18) imply alternating odd and even correlation triangles for the bipartite correlation questions. However, we emphasize that the Bell scenario argument of section 5.2.5 does not require the compatibility triangles involving individual questions to also admit such an alternating odd and even pattern. For instance, the following relations of $Q_{33}$

$$Q_{33} = Q_3 \leftrightarrow Q_3' = Q_{12} \leftrightarrow Q_{21} = \neg(Q_{11} \leftrightarrow Q_{22})$$

do not permit $O$ to extract any illegal complementary information about the system despite the absence of a negation in $Q_3 \leftrightarrow Q_3' = Q_{12} \leftrightarrow Q_{21}$. The graph corresponding to the last relation,
is very different from the graphs $\{5.14, 5.15\}$ as it involves all six individual questions and the argument leading to $\{5.18\}$ does not apply. Clearly, given the definition $Q_{ij} := Q_i \leftrightarrow Q'_j$ all such triangles must be even.

The lattice structure in figure 3 contains 15 triangles and $15 \times 3 = 45$ distinct edges corresponding to compatibility relations. Every question resides in exactly two triangles and is therefore compatible with the six other questions in these six triangles and complementary to the eight remaining questions not contained in those three triangles. This means that once one question is fully known, all other information available to $O$ must be distributed over the three adjacent triangles. In particular, if $O$ knows the answers to two of the questions in the lattice, he will also know the answer to the third sharing the same triangle such that every triangle corresponds to a specific state of maximal information. (Although by the completeness principle $\exists$-states of maximal information will likewise exist such that two independent bits can be distributed differently over the lattice, see also [1].)

Notice that every triangle is connected by an edge (adjacent to one of its three vertices) to every other vertex in the lattice. This embodies the statement ‘whenever $O$ asks $S$ a new question, he experience no net loss of information’ of the complementarity principle $\mathbb{P}$. For instance, if $O$ has maximal knowledge about two questions in the lattice, corresponding to maximal information about one triangle, it is impossible for him to loose information by asking another question from the lattice because it will be connected to one of the questions from the previous triangle. As a concrete example, suppose $O$ knew the answers to $Q_{21}, Q_{13}, Q_{32}$. He could ask $Q_{22}$ next. Since $Q_{22}$ is connected by an edge to $Q_{13}$, he would know the answers to both upon asking $Q_{22}$ and thereby also the answer to $Q_{31}$ – no net loss of information occurs.

It is straightforward to check, e.g., using the ansatz

$$|\psi\rangle = \alpha |z_-, z_-\rangle + \beta |z_+, z_+\rangle + \gamma |z_-, z_+\rangle + \delta |z_+, z_-\rangle$$

for a two qubit pure state and translating it into the various basis combinations $xx, xy, yx, yy, \ldots$, that the lattice structure of figure 3 is precisely the compatibility and correlation structure of qubit quantum theory. For instance, $Q_{11}, Q_{22}, Q_{33}$ correspond to projectors onto $\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y$ and $\sigma_z \otimes \sigma_z$. The three questions sharing a red triangle means, e.g., that $Q_{11} = Q_{22} = 1$ and $Q_{33} = 0$ is an allowed state while $Q_{11} = Q_{22} = Q_{33} = 1$ is illegal. Indeed, ignoring normalization, in quantum theory one finds

$$|x_+ x_+\rangle - |x_- x_-\rangle = -i|y_+ y_+\rangle + i|y_- y_-\rangle = |z_+ z_+\rangle + |z_- z_-\rangle$$

for $\alpha = \beta = 0$ and $\gamma = \delta = 1$, corresponding to the propositions $Q_{11} = 1$: “the spins are correlated in $x$-direction”; $Q_{22} = 1$: “the spins are correlated in $y$-direction”; and $Q_{33} = 0$: “the spins are anti-correlated in $z$-direction”. But no quantum state exists such that the spins are also correlated in $z$-direction if they are correlated in $x$- and $y$-direction (this would be mirror quantum theory). Every other triangle in the lattice corresponds similarly to four pure quantum states (representing the answer configurations ‘yes-yes’, ‘yes-no’, ‘no-yes’, ‘no-no’ to the two independent questions per triangle).

Finally, we also collect the results on the compatibility, complementarity and correlation structure of two rebits, derived in lemmas $\{5.1, 5.3\}$ and theorem $\{5.3\}$ and equation $\{5.17\}$, in a lattice structure in figure 4. There are six triangles and $6 \times 3 = 18$ edges representing compatibility relations. Every question resides in exactly two triangles and is thereby compatible with four and complementary to four other questions. As in the qubit case in figure 3, every triangle is connected by an edge to every other vertex in the lattice, in conformity with principle $\mathbb{P}$ asserting that $O$ shall not experience a net loss of information by asking further questions.

5.5 The general case of $N > 3$ gbits

We are now prepared to investigate the independence, compatibility and correlation structure ensuing from principles $\mathbb{P}$ and $\mathbb{C}$ on a general $Q_N$, i.e. of $O$’s possible questions to a system $S$ composed of $N > 3$ gbits. Similarly, $|\psi\rangle = \alpha |z_-, z_-\rangle + \beta |z_+, z_+\rangle = \alpha \frac{\beta}{\sqrt{2}} (|x_+ x_+\rangle + |x_- x_-\rangle) + \beta \frac{\alpha}{\sqrt{2}} (|y_+ y_+\rangle + |y_- y_-\rangle) + \frac{\beta - \alpha}{\sqrt{2}} (|y_+ y_-\rangle + |y_- y_+\rangle)$.
Individuals, bipartite, and multipartite structure (and associativity) of the XNOR, a question such as \( Q \) etc. Note that \( \mu \) appears in the conjunction (5.33). For instance, the values 0, 1, 2, 3 correspond to the individuals \( Q_1, Q_2, Q_3 \) of the new gbit, and (3) all logical connectives of the compatible questions of those two sets. This entails slightly different repercussions for qubits and rebits.

5.5.1 An informationally complete set and entanglement for \( N > 3 \) qubits

A natural candidate for an informationally complete question set is given by the set of all possible XNOR conjunctions of the individual questions of the \( N \) gbits

\[
Q_{\mu_1\mu_2\cdots\mu_N} := Q_{\mu_1} \leftrightarrow Q_{\mu_2} \leftrightarrow \cdots \leftrightarrow Q_{\mu_N} \tag{5.33}
\]

(we recall from (5.3) that the logical connective yielding independent questions is either \( \leftrightarrow \) or \( \oplus \)). Here we have introduced a new index notation: \( \mu_a \) is the question index for gbit \( a \in \{1, \ldots, N\} \) and can take the values 0, 1, 2, 3. As before the index values \( i = 1, 2, 3 \) correspond to the individuals \( Q_{1a}, Q_{2a}, Q_{3a} \).

On the other hand, the index value \( \mu_a = 0 \) implies that none of the three individual questions of gbit \( a \) appears in the conjunction (5.33). For instance,

\[
Q_{1000000\cdots000} := Q_{11}, \quad Q_{0003020\cdots0} := Q_{34} \leftrightarrow Q_{26},
\]

e tc. Note that \( Q_{000\cdots000} \) corresponds to no question. The set (5.33) thus contains all individual, bipartite, tripartite, and up to \( N \)-partite correlation questions for \( N \) gbits. We emphasize that, due to the special multipartite structure (and associativity) of the XNOR, a question such as \( Q_{111\cdots111} \) does not incarnate the question ‘are the answers to \( Q_{11}, Q_{12}, \ldots, Q_{1N} \) all the same?’

For example, for \( N = 4 \), \( Q_{14} = Q_{12} = 0 \) and \( Q_{13} = Q_{14} = 1 \) yield \( Q_{11111111} = 1 \). This is important for the entanglement structure.

We begin with an important result.

**Lemma 5.16.** The \( 4^N - 1 \) question, \( Q_{\mu_1\cdots\mu_N}, \mu = 0, 1, 2, 3 \), are pairwise independent.

**Proof.** Consider \( Q_{\mu_1\cdots\mu_N} \) and \( Q_{\nu_1\cdots\nu_N} \). The two questions must disagree in at least one index otherwise they would coincide. Let the questions differ on the index of gbit \( a \), i.e. \( \mu_a \neq \nu_a \). Suppose \( \mu_a, \nu_a \neq 0. \)

\[35\] We deduce the trivial question \( Q_{000\cdots000} \). Obviously, one arrives at the same number by counting the distribution of individuals, bipartite, ... and \( N \)-partite correlations over \( N \) qubits as a binomial series \( \sum_{k=1}^{N} \binom{N}{k} 3^k = (3+1)^N - 1 \).
Then $Q_{\mu a}$ is compatible with $Q_{\nu_1\cdots \nu_N} = Q_{\mu_1} \leftrightarrow \cdots \leftrightarrow Q_{\mu a} \leftrightarrow \cdots \leftrightarrow Q_{\mu_N}$ and complementary to $Q_{\nu_1\cdots \nu_a\nu_b\cdots \nu_N} = Q_{\mu_1} \leftrightarrow \cdots \leftrightarrow Q_{\mu a} \leftrightarrow \cdots \leftrightarrow Q_{\nu_N}$ since $Q_{\mu a}, Q_{\nu a}$ are complementary. Using the same argument as in the proof of lemma [5.2] this implies independence of $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$. Lastly, suppose now $\mu_a = 0$ and $\nu_a \neq 0$ (as we have $\mu_a \neq \nu_a$ not both can be 0). Then $Q_{i_a \nu_a}$ with $i \neq 0$ is compatible with $Q_{\mu_1\cdots \mu_N}$ and complementary to $Q_{\nu_1\cdots \nu_N}$. By the same argument, this again implies independence of $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$.

Consequently, the set (5.33) will be part of an informationally complete set. We note that a hermitian matrix of trace equal to 1 on a $2^N$-dimensional complex Hilbert space (i.e. qubit density matrix) is described by $4^N - 1$ parameters.

Next, we must elucidate the compatibility and complementarity structure of this set.

**Lemma 5.17.** $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$ are

- **compatible** if the index sets $\{\mu_1, \ldots, \mu_N\}$ and $\{\nu_1, \ldots, \nu_N\}$ differ by an even number (incl. 0) of non-zero indices, and

- **complementary** if the index sets $\{\mu_1, \ldots, \mu_N\}$ and $\{\nu_1, \ldots, \nu_N\}$ differ by an odd number of non-zero indices.

For example, for $N = 2$, $Q_{11}$ and $Q_{22}$ differ by two non-zero indices and are thus compatible. By contrast, $Q_{10} = Q_1$ and $Q_{22}$ differ by an odd number of non-zero indices and are thereby complementary.

**Proof.** Let $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$ disagree in an odd number, call it $2n + 1$, of non-zero indices. We can always resuffle the index labeling of the $N$ qubits such that now $a = 1, \ldots, 2n + 1$ corresponds to the qubits on which $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$ differ by non-zero indices, i.e. $\mu_a \neq \nu_a$ and $\mu_a, \nu_a \neq 0$. The remaining qubits labeled by $b = 2n + 2, \ldots, N$ are then such that $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$ either agree on the non-zero index, $\mu_b = \nu_b \neq 0$ or at least one of $\mu_b, \nu_b$ is 0. That is, after the reshuffling the index labeling, we can write the questions as

$$Q_{\mu_1\cdots \mu_N} = (Q_{\mu_1} \leftrightarrow \cdots \leftrightarrow Q_{\mu_{2n+1}}) \leftrightarrow (Q_{\mu_{2n+2}} \leftrightarrow \cdots \leftrightarrow Q_{\mu_N})$$

$$Q_{\nu_1\cdots \nu_N} = (Q_{\nu_1} \leftrightarrow \cdots \leftrightarrow Q_{\nu_{2n+1}}) \leftrightarrow (Q_{\nu_{2n+2}} \leftrightarrow \cdots \leftrightarrow Q_{\nu_N}).$$

The parts of the questions where the index sets either agree or feature zeros coincide with $Q_{\mu_{2n+2}\cdots \mu_N}$ and $Q_{\nu_{2n+2}\cdots \nu_N}$ and are clearly compatible (dropping here zero indices).

We can now proceed by induction. Lemmas 5.1, 5.3, 5.5, 5.10 imply that the statement of this lemma is correct for $n = 0, 1$. Let the statement therefore be true for $n$ and consider $n+1$. Then, (5.33) reads

$$Q_{\mu_1\cdots \mu_N} = (Q_{\mu_1\cdots \mu_{2n+3}}) \leftrightarrow (Q_{\mu_{2n+4}\cdots \mu_N})$$

$$Q_{\nu_1\cdots \nu_N} = (Q_{\nu_1\cdots \nu_{2n+3}}) \leftrightarrow (Q_{\nu_{2n+4}\cdots \nu_N}).$$

But, by lemma 5.3, $Q_{\mu_{2n+2}\mu_{2n+3}}$ and $Q_{\nu_{2n+2}\nu_{2n+3}}$ are compatible with each other and therefore also with $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$. Suppose now that $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$ were compatible too such that by Specker’s principle $O$ could ask these two questions together with $Q_{\mu_{2n+2}\cdots \mu_N}$. But this would imply that $O$ could have simultaneous maximal information about the two complementary $Q_{\mu_1\cdots \mu_{2n+1}}$ and $Q_{\nu_1\cdots \nu_{2n+1}}$. Similarly, if $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$ were partially compatible, $O$ could obtain maximal information about either of $Q_{\mu_{2n+2}\cdots \mu_N}$ or $Q_{\nu_{2n+2}\cdots \nu_N}$ and still have partial information about the other which is illegal. Accordingly, $Q_{\mu_1\cdots \mu_N}$ and $Q_{\nu_1\cdots \nu_N}$ must be complementary.

Finally, let $Q_{\alpha_1\cdots \alpha_N}$ and $Q_{\beta_1\cdots \beta_N}$ disagree in an even number $2n$ of non-zero indices. Using an analogous reshuffling of the index labeling as in the odd case above, one can rewrite the two questions as

$$Q_{\alpha_1\cdots \alpha_N} = (Q_{\alpha_1\alpha_2} \leftrightarrow \cdots \leftrightarrow Q_{\alpha_{2n-1}\alpha_{2n}}) \leftrightarrow Q_{\alpha_{2n+1}\cdots \alpha_N}$$

$$Q_{\beta_1\cdots \beta_N} = (Q_{\beta_1\beta_2} \leftrightarrow \cdots \leftrightarrow Q_{\beta_{2n-1}\beta_{2n}}) \leftrightarrow Q_{\beta_{2n+1}\cdots \beta_N}. (5.35)$$
That is, one can decompose the disagreeing parts of the questions into bipartite correlations. But thanks to lemma 5.3, two bipartite correlations of the same qubit pair are compatible if and only if they differ in both indices. Consequently, all the pairings of question components of the upper and lower line, as written in (5.35), are compatible and, hence, so must be $Q_{\alpha_1\ldots\alpha_N}$ and $Q_{\beta_1\ldots\beta_N}$.

Fortunately, it turns out that the $4^N - 1$ questions (5.35) are logically closed under the XNOR.

**Theorem 5.6. (Qubits)** The $4^N - 1$ questions $Q_{\mu_1\ldots\mu_N}$, $\mu = 0, 1, 2, 3$, are logically closed under $\leftrightarrow$ and thus form an informationally complete set $Q_{\mu N}$ with $D_N = 4^N - 1$ for the case $D_1 = 3$.

**Proof.** We shall prove the statement by induction. The statement is trivially true for $N = 1$ and, by theorems 5.2 and 5.4 holds also for $N = 2, 3$. Let the statement therefore be true for $N - 1$ and consider a composite system of $N$ qubits. Since $O$ can treat the $N$ qubit system in many different ways as a composite system and the statement holds for $N - 1$, all XNOR conjunctions of questions involving at least one zero index will be contained in (5.35). We thus only need to show that all $\leftrightarrow$ conjunctions involving at least one $N$-partite correlation $Q_{i_1\ldots i_N}$, $i_a \neq 0 \forall a = 1, \ldots, N$, produce questions already included in (5.35).

Consider $Q_{i_1\ldots i_N}$ and $Q_{\nu_1\ldots \nu_N}$. Lemma 5.17 implies that these two questions are compatible—and can thus be connected by $\leftrightarrow$ if and only if $\{i_1, \ldots, i_N\}$ and $\{\nu_1, \ldots, \nu_N\}$ disagree in an even number of non-zero indices. There are now two cases that we must consider:

(a) Suppose $Q_{i_1\ldots i_N}$ and $Q_{\nu_1\ldots \nu_N}$ disagree in an even number of non-zero indices and, furthermore, agree on at least one index $i_a = \nu_a$. Thanks to $Q_{i_a} \leftrightarrow Q_{i_a} = 1$, the conjunction then yields

$$Q_{i_1\ldots i_a\ldots i_N} \leftrightarrow Q_{\nu_1\ldots \nu_a\ldots \nu_N} = Q_{i_1\ldots 0\ldots i_N} \leftrightarrow Q_{\nu_1\ldots 0\ldots \nu_N}.$$  

Hence, the two questions on the right hand side both contain less than $N$ non-zero indices such that the result must lie in the set (5.35) because the statement is true up to $N - 1$ by assumption.

(b) Suppose $Q_{i_1\ldots i_N}$ and $Q_{\nu_1\ldots \nu_N}$ disagree in an even number $2n$ of non-zero indices and do not agree on any non-zero index. Reshuffling the index labelings as in the proof of lemma 5.17 one can then write

$$Q_{i_1\ldots i_N} = Q_{i_1} \leftrightarrow Q_{i_2} \leftrightarrow \ldots \leftrightarrow Q_{i_{2n-1}} \leftrightarrow Q_{i_{2n}} \leftrightarrow Q_{i_{2n+1}\ldots i_N}$$

$$Q_{\nu_1\ldots \nu_N} = Q_{\nu_1} \leftrightarrow Q_{\nu_2} \leftrightarrow \ldots \leftrightarrow Q_{\nu_{2n-1}} \leftrightarrow Q_{\nu_{2n}} \leftrightarrow Q_{0_{2n+1}\ldots 0_N}.$$  

(We obtain here $Q_{0_{2n+1}\ldots 0_N}$ because $Q_{i_1\ldots i_N}$ and $Q_{\nu_1\ldots \nu_N}$ do not agree on any common index and $Q_{i_1\ldots i_N}$ does not feature zero-indices.) By lemma 5.3, the pairs of bipartite correlations differing in two indices, e.g., $Q_{i_1i_2}$ and $Q_{\nu_1\nu_2}$, are compatible and by theorem 5.4 their XNOR conjunction will yield another bipartite correlation of the same qubit pair. For example, $Q_{i_1i_2} \leftrightarrow Q_{\nu_1\nu_2}$ equals either $Q_{i_1j_2}$ or $-Q_{j_1i_2}$ for $j_1 \neq i_1, \nu_1$ and $j_2 \neq i_2, \nu_2$. Accordingly, up to negation, one finds ($j_a \neq i_a, \nu_a$, $a = 1, \ldots, 2n$)

$$Q_{i_1\ldots i_N} \leftrightarrow Q_{\nu_1\ldots \nu_N} = Q_{j_1j_2} \leftrightarrow Q_{j_3j_4} \leftrightarrow \ldots \leftrightarrow Q_{j_{2n-1}j_{2n}} \leftrightarrow Q_{0_{2n+1}\ldots i_N} = Q_{\beta_1\ldots \beta_N}$$

which is another $N$-partite correlation contained in the set (5.35).
are complementary because the two edges involve different qubits – $A, B$ for the first and $B, C$ for the second edge.

This also helps us to understand entanglement for arbitrarily many qubits. Specifically, maximal entanglement will correspond to $O$ spending the $N$ independent bits he is allowed to acquire about the system $S$ of $N$ qubits on $N$-partite correlation questions. Lemma 5.17 guarantees that for every $N$ there will exist $N$ compatible $N$-partite questions. For instance, there are $\binom{N}{2}$ ways of having $N-2$ of the indices take value 1 and 2 indices take the value 2. Any two of the $\binom{N}{2}$ corresponding questions will disagree in two non-zero indices (and agree on the rest) and will thus be compatible. These will correspond to a set of $\binom{N}{2}$ compatible $(N-1)$-simplices in the question graph such that any two of them either disagree on one or two edges. (There will exist even more compatible $N$-partite questions.) For $N \geq 3$ it also holds that $\binom{N}{2} \geq N$.

It is definitely possible to choose $N$ such compatible $N$-partite correlations out of the $\binom{N}{2}$ many such that these $N$ questions do not all agree on a single index. For similar reasons to the $N=2, 3$ cases, this choice will constitute a mutually independent set such that every individual question $Q_{i_1, \ldots, i_N}$ will be complementary to at least one of these $N$ $N$-partite questions. (As a consequence of principle 3 once the answers to these $N$ $N$-partite questions are known, they will also imply the answers to the $\binom{N}{2} - N$ remaining ones by the same reasoning as in section 5.2.3.) Accordingly, $O$ can exhaust the information limit with these $N$-partite correlation questions, while not being able to have any information whatsoever about the individuals – a necessary condition for maximal entanglement.

There will exist many different ways of having such multipartite entanglement for arbitrary $N$. Moreover, the completeness principle 3 implies that the total information can be partially distributed over many of the questions in $Q_{M_N}$. One could describe such different ways of entanglement by generalizing the correlation measures (5.23) and informational monogamy inequalities resulting therefrom. These monogamy inequalities could also be considered as simplicial relations: they restrict the way in which the available (independent and dependent) information can be distributed over the various simplices and subsimplices in the question graph. However, we abstain from analyzing such relations here further.

### 5.5.2 An informationally complete set and entanglement for $N > 3$ rebits

We briefly repeat the same procedure for $N$ rebits. In analogy to (5.33), the natural candidate set for an informationally complete $Q_{M_N}$ will contain

$$Q_{\mu_1, \mu_2, \ldots, \mu_N} := Q_{\mu_1} \leftrightarrow Q_{\mu_2} \leftrightarrow \cdots \leftrightarrow Q_{\mu_N}, \quad \mu_a = 0, 1, 2, \quad a = 1, \ldots, N, \quad (5.36)$$

where the notation should be clear from section 5.5.1. However, being a composite system, by definition 3 we must permit the correlation of correlations $Q_{3, 3, b}$ (5.13) for all $a, b \in \{1, \ldots, N\}$ because clearly $O$ is allowed to ask $Q_{1_a}, Q_{2_a}, Q_{1_b}, Q_{2_b}$. Furthermore, thanks to (5.29) we have, e.g.,

$$Q_{3_1, 3_2, 3_3} = Q_{3_3, 3_1} \leftrightarrow Q_{3_2, 3_3} = Q_{3_3, 3_2} \leftrightarrow Q_{3_3, 3_2} = Q_{3_3, 3_1} \leftrightarrow Q_{3_3, 3_2}$$

such that no confusion can arise about the meaning of $Q_{3_1, 3_2, 3_3}$ although there are no individuals $Q_{3_a}$ into which the question could be decomposed. The same holds similarly for any other even number of indices taking the value 3. Consequently, the candidate set for $Q_{M_N}$ can be written as

$$\tilde{Q}_{M_N} := \{Q_{\mu_1, \ldots, \mu_N}, \quad \mu_a = 0, 1, 2, 3, \quad a = 1, \ldots, N \mid \text{only even number of indices taking value 3}\}$$

with an evident meaning of each such question.

Let us count the number of elements within $\tilde{Q}_{M_N}$.

**Lemma 5.18.** $\tilde{Q}_{M_N}$ contains $2^{N-1}(2^N + 1) - 1$ non-trivial questions.

**Proof.** If arbitrary distributions of the values $\mu_a = 0, 1, 2, 3$ over the $a = 1, \ldots, N$ were permitted, we would obtain $4^N - 1$ non-trivial questions upon subtracting $Q_{0_0, 0_2, \ldots, 0_N}$ as in the qubit case. In order to obtain the number of questions within $\tilde{Q}_{M_N}$ we thus still have to subtract all the possible ways of distributing an odd number of 3’s over the $N$ indices. There are precisely

$$O_N := \binom{N}{1}3^{N-1} + \binom{N}{3}3^{N-3} + \cdots + \binom{N}{2n+1}3^{N-(2n+1)}$$

\[52\]
such ways, where $2n + 1$ is the largest odd number smaller or equal to $N$. Similarly, the number of ways an even number of 3’s can be distributed over $N$ indices is given by

$$E_N := 3^N + \binom{N}{2}3^{N-2} + \cdots + \binom{N}{2m}3^{N-2m},$$

where $2m$ is the largest even number smaller or equal to $N$. We then have

$$E_N + O_N = (3 + 1)^N = 4^N, \quad E_N - O_N = (3 - 1)^N = 2^N$$

and thus

$$O_N = \frac{1}{2}(4^N - 2^N)$$

which yields

$$4^N - 1 - O_N = 2^{N-1}(2^N + 1) - 1$$

non-trivial questions in $\tilde{Q}_{MN}$.

We note that the number of parameters in a symmetric matrix with trace equal to 1 on a $2^N$-dimensional real Hilbert space (i.e. rebit density matrix) is precisely $\frac{1}{2} 2^N (2^N + 1) - 1$.

Next, we assert pairwise independence as required for an informationally complete set.

**Lemma 5.19.** The $2^{N-1}(2^N + 1) - 1$ non-trivial questions in $\tilde{Q}_{MN}$ are pairwise independent.

**Proof.** The proof is entirely analogous to the proofs of lemmas 5.2, 5.8, 5.9 and 5.16.

Likewise, the complementarity and compatibility structure of $\tilde{Q}_{MN}$ is analogous to the qubit case.

**Lemma 5.20.** $Q_{\mu_1, \mu_N}, Q_{\nu_1, \nu_N} \in \tilde{Q}_{MN}$ are

- **compatible** if the index sets $\{\mu_1, \ldots, \mu_N\}$ and $\{\nu_1, \ldots, \nu_N\}$ differ by an even number (incl. 0) of non-zero indices, and

- **complementary** if the index sets $\{\mu_1, \ldots, \mu_N\}$ and $\{\nu_1, \ldots, \nu_N\}$ differ by an odd number of non-zero indices.

**Proof.** Thanks to lemmas 5.4, 5.11–5.13 and 5.15, the proof of lemma 5.17 also applies to the rebit case with the sole difference that only an even number of indices in the questions can take the value 3 and that correlations of correlations $Q_3, Q_3$ cannot be decomposed into individuals $Q_3, Q_3$.

Finally, $\tilde{Q}_{MN}$ is indeed logically closed.

**Theorem 5.7.** (Rebits) $\tilde{Q}_{MN}$ is logically closed under $\leftrightarrow$ and is thus an informationally complete set $\tilde{Q}_{MN} = Q_{MN}$ with $D_N = 2^{N-1}(2^N + 1) - 1$ for the case $D_1 = 2$.

**Proof.** Thanks to theorems 5.3 and 5.14, the proof of theorem 5.6 also applies here, except that only an even number of indices can take the value 3 and $Q_{3,3}$ cannot be decomposed into individuals $Q_3, Q_3$.

We close with the observation that similarly to the $N$ qubit case in section 5.5.1, one could represent the compatibility and complementarity structure via a simplicial question graph. Maximally entangled states (of maximal information) will correspond to $O$ spending all $N$ available bits over a mutually independent set of $N$ $N$-partite questions which is complementary to every individual question. In fact, the prescription for constructing a maximally $N$-partite entangled qubit state provided at the end of section 5.5.1 also applies to $N$ rebits since only indices with values 1, 2 were employed. Furthermore, for rebits one can similarly generate maximally entangled states of non-maximal information; e.g., $O$ could ask only the $N - 1$ questions $Q_{3,3,0000 \ldots}, Q_{0003,3,000 \ldots}, Q_{0003,3,000 \ldots}, \ldots, Q_{0003,3,000 \ldots}$. If $O$ has maximal information about each such question, every rebit pair will be maximally entangled, while $O$ has still not reached the information limit (see also section 5.2.3).
6 Time evolution

Thus far we have only applied principles 1 and 2 to derive the question structure on \( Q_N \). We shall now slightly switch topic and return to the problem of time evolution begun in section 3.2.7. In particular, we shall consider the time evolution of the state of \( S \) relative to \( O \) in between interrogations and impose the principle 3 of information preservation and the principle 5 of continuous time evolution to infer that time evolution defines a group action on the states.

6.1 Time evolution is reversible

We begin by demonstrating that time evolution of \( S \)'s state, as perceived by \( O \), must be reversible. To this end, we shall return to employ operational arguments and recall 4.9 according to which time evolution of the (redundantly parametrized) state \( \bar{P}_{O \to S} \in \Sigma_N \oplus \Sigma_N \subset \mathbb{R}^{2D_N} \) is linear and only depends on the interval \( \Delta t = t_2 - t_1 \), where \( t_1, t_2 \) are arbitrary instants of time in between \( O \)'s interrogations on a given system.

Suppose time evolution was not reversible, i.e. that an inverse to \( A(\Delta t) \) does not exist. We will show that this leads to a conflict with principle 4. Given that \( A(\Delta t) : \Sigma_N \oplus \Sigma_N \to \Sigma_N \oplus \Sigma_N, \forall \Delta t, A \) must be a square matrix on \( \mathbb{R}^{2D_N} \). For finite dimensional square matrices, \( A \) being bijective is equivalent to it being injective. Assume therefore that \( A(\Delta t) \) was not injective. Then there would exist \( \bar{P}_{O \to S}(t_1) \neq \bar{P}_{O \to S}''(t_1) \) such that

\[
\bar{P}_{O \to S}(t_2) := A(\Delta t) \bar{P}_{O \to S}(t_1) = A(\Delta t) \bar{P}_{O \to S}''(t_1).
\]

(6.1)

We come back to the coin flip scenario of section 3.2.7 and assume that, using the preparation method, \( O \) can prepare \( S_1 \) in the state \( \bar{P}_{O \to S_1}(t_1) := \bar{P}_{O \to S_1}''(t_1) \) and \( S_2 \) in the state \( \bar{P}_{O \to S_2}(t_1) := \bar{P}_{O \to S_2}''(t_1) \) at time \( t_1 \) before tossing the coin. By (3.7),

\[
\bar{P}_{O \to S_1}(t_1) = \lambda \bar{P}_{O \to S_1}(t_1) + (1 - \lambda) \bar{P}_{O \to S_2}(t_1),
\]

and, on account of (6.1) and principle 4

\[
I_{O \to S_1}(\bar{P}_{O \to S_1}(t_1)) = I_{O \to S_2}(\bar{P}_{O \to S_2}(t_1)) = I_{O \to S_1}(\bar{P}_{O \to S_1}(t_2)) = I_{O \to S_2}(\bar{P}_{O \to S_2}(t_2)).
\]

(6.2)

At this stage, we make an assumption on the information measure (satisfied by most reasonable measures). Namely, given that in the coin flip scenario \( O \)'s information about the outcome of any question (asked to either \( S_1 \) or \( S_2 \), depending on the outcome of the coin flip) is clouded by the random coin flip outcome, it is appropriate to require the following:

Requirement 1. (Information in convex mixtures) \( O \)'s information about the mixed state \( \bar{P}_{O \to S_1} = \lambda \bar{P}_{O \to S_1} + (1 - \lambda) \bar{P}_{O \to S_2} \) is smaller than his maximal information about any of its constituents \( \bar{P}_{O \to S_1}, \bar{P}_{O \to S_2} \) – unless the outcome of the coin flip was certain, or \( S_1, S_2 \) are in the same state. That is,

\[
I_{O \to S_1} < \min \{ I_{O \to S_1}, I_{O \to S_2} \} \quad \text{if} \quad \lambda \neq 0, 1, \quad \text{and} \quad \bar{P}_{O \to S_1} \neq \bar{P}_{O \to S_2}
\]

(6.3)

Thus, assuming the coin flip not to be certain and using (6.2), this yields in the present case

\[
I_{O \to S_2}(\bar{P}_{O \to S_2}(t_1)) < I_{O \to S_2}(\bar{P}_{O \to S_2}(t_2)) = I_{O \to S_2}(\bar{P}_{O \to S_2}(t_2)).
\]

(6.4)

On the other hand, (6.1) implies that, at time \( t_2 \), \( O \) would find

\[
\bar{P}_{O \to S_1}(t_2) = \bar{P}_{O \to S_2}(t_2) = \bar{P}_{O \to S_2}(t_2)
\]

such that

\[
I_{O \to S_2}(\bar{P}_{O \to S_2}(t_2)) = I_{O \to S_2}(\bar{P}_{O \to S_2}(t_2)) = I_{O \to S_2}(\bar{P}_{O \to S_2}(t_2)).
\]

(6.5)

Hence, (6.3) (6.5) entail that \( O \)'s information about \( S_{12} \) has increased between \( t_1 \) and \( t_2 \) despite not having tossed the coin and asked any questions. This is in contradiction with principle 4. We conclude that \( A(\Delta t) \) must be injective and thus also bijective for all \( \Delta t \). Hence, to every \( A(\Delta t) \) there must exist an inverse \( A^{-1}(\Delta t) \) such that \( A(\Delta t)A^{-1}(\Delta t) = A^{-1}(\Delta t)A(\Delta t) = 1 \) and time evolution of \( \bar{P}_{O \to S} \), as viewed by \( O \), is reversible.

\[\Sigma_N \oplus \Sigma_N \text{ must contain a basis of } \mathbb{R}^{2D_N}. \] Otherwise, \( O \) could parametrize the state of \( S \) by less than \( D_N \) questions, in contradiction with assumption 4 principle 3 and the fact that \( \bar{P}_{O \to S} \) is parametrized by the probabilities of a \( Q_{M_N} \).
6.2 Time evolution as a group

Given that any time interval can be decomposed into two time intervals, $\Delta t = \Delta t_1 + \Delta t_2$, and the evolution of $\vec{P}_{O\rightarrow S}$ is continuous by principle[5] $O$ must find

$$A(\Delta t_2) A(\Delta t_1) \vec{P}_{O\rightarrow S}(0) = A(\Delta t_2) \vec{P}_{O\rightarrow S}(\Delta t_1) = \vec{P}_{O\rightarrow S}(\Delta t) = A(\Delta t) \vec{P}_{O\rightarrow S}(0),$$

and thus that multiplication is abelian,

$$A(\Delta t_1 + \Delta t_2) = A(\Delta t_2) A(\Delta t_1) = A(\Delta t_1) A(\Delta t_2).$$

(The last equality follows from time translation invariance.) From the last equation and $A(0) = 1$, using $\Delta t - \Delta t = 0$, we can also infer that $A^{-1}(\Delta t) = A(-\Delta t)$. If we permit $O$ to consider the time evolution of $S$ for any duration $\Delta t \in \mathbb{R}$, it follows that the product of any two time evolution matrices is again a time evolution matrix. In summary, we therefore gather that a given time evolution, as perceived by $O$ and under the circumstances to which he has subjected $S$, is such that:

(i) $A(0) = 1$,

(ii) to every $A(\Delta t)$ there exists an inverse $A^{-1}(\Delta t) = A(-\Delta t)$,

(iii) the multiplication of any two time evolution matrices is again a time evolution matrix, and

(iv) matrix multiplication is obviously associative.

In conclusion, under the assumptions of section 6.2 and principles[4] and[5] a given time evolution defines therefore an abelian, one-parameter, nonnegative matrix group such that, due to (3.4), each of its elements is stochastic in any pair of components $i$ and $i+D_N$ of $\vec{P}_{O\rightarrow S}$. Hence, a given time evolution is described by a single evolution generator and single parameter $\Delta t$ parametrizing the duration. We note, however, that a multiplicity of time evolutions of $S$ is possible, depending on the physical circumstances (interactions) to which $O$ may subject $S$. Different time evolutions will be generated by different generators, but each time evolution will form a one-parameter group as discussed above. We shall return to this further below and in [4].

6.3 Time evolution of the ‘Bloch vector’

Until now we have only considered time evolution of the redundantly parametrized $2D_N$-dimensional state $\vec{F}_{O\rightarrow S}$. However, because of [3,4], it clearly suffices to consider the yes-vector $\vec{y}_{O\rightarrow S}$ (or no-vector $\vec{n}_{O\rightarrow S}$) alone to describe the state of $S$ relative to $O$. Let us therefore determine, how $\vec{y}_{O\rightarrow S}$ and $\vec{n}_{O\rightarrow S}$ evolve under time evolution. To this end, we decompose $A(\Delta t)$ and $\vec{P}_{O\rightarrow S}$ in [3,4],

$$\begin{pmatrix} \vec{y}_{O\rightarrow S}(\Delta t) \\ \vec{n}_{O\rightarrow S}(\Delta t) \end{pmatrix} = \begin{pmatrix} a(\Delta t) & b(\Delta t) \\ c(\Delta t) & d(\Delta t) \end{pmatrix} \begin{pmatrix} \vec{y}_{O\rightarrow S}(0) \\ \vec{n}_{O\rightarrow S}(0) \end{pmatrix},$$

where $a(\Delta t), b(\Delta t), c(\Delta t), d(\Delta t)$ are nonnegative (real) $D_N \times D_N$ matrices. Given that it is completely arbitrary which answer to any $Q \in Q_{M_N}$ $O$ refers to as ‘yes’ and which as ‘no’, the situation must be symmetric under a swap of the ‘yes/no’-labeling for all $Q \in Q_{M_N}$ at once, i.e. under $\vec{y}_{O\rightarrow S} \leftrightarrow \vec{n}_{O\rightarrow S}$, such that we must also have

$$\begin{pmatrix} \vec{n}_{O\rightarrow S}(\Delta t) \\ \vec{y}_{O\rightarrow S}(\Delta t) \end{pmatrix} = \begin{pmatrix} a(\Delta t) & b(\Delta t) \\ c(\Delta t) & d(\Delta t) \end{pmatrix} \begin{pmatrix} \vec{n}_{O\rightarrow S}(0) \\ \vec{y}_{O\rightarrow S}(0) \end{pmatrix}.$$

From this it follows that

$$a(\Delta t) \equiv d(\Delta t), \quad b(\Delta t) \equiv c(\Delta t).$$

However, $a(\Delta t) \neq b(\Delta t)$, otherwise $A(\Delta t)$ would not be invertible – in conflict with what we have concluded above. Using the normalization [3,4], one finds that both $\vec{y}_{O\rightarrow S}, \vec{n}_{O\rightarrow S}$ evolve affinely,

$$\vec{y}_{O\rightarrow S}(\Delta t) = (a(\Delta t) - b(\Delta t)) \vec{y}_{O\rightarrow S}(0) + b(\Delta t) \vec{1},$$

$$\vec{n}_{O\rightarrow S}(\Delta t) = (a(\Delta t) - b(\Delta t)) \vec{n}_{O\rightarrow S}(0) + b(\Delta t) \vec{1}.$$
On the other hand, defining the $D_N \times D_N$ time evolution matrix
\[ T(\Delta t) := (a(\Delta t) - b(\Delta t)), \]
yields a linear time evolution of what we shall henceforth call the generalized Bloch vector $2\vec{y}_{O \to S} - \vec{I}$:
\[ 2\vec{y}_{O \to S}(\Delta t) - \vec{I} = \vec{y}_{O \to S}(\Delta t) - \vec{n}_{O \to S}(\Delta t) = T(\Delta t) \left( 2\vec{y}_{O \to S}(0) - \vec{I} \right). \]  
(6.6)

Notice that $T(\Delta t)$ need neither be nonnegative nor stochastic in any pair of its components. However, it is easy to check that for a given one-parameter group of $A(\Delta t)$, the $T(\Delta t)$ also form a one-parameter group. Henceforth, we shall only consider the $T(\Delta t)$ governing the time evolution of the Bloch vector $2\vec{y}_{O \to S} - \vec{I}$.

7 Information measure and probabilities

The analysis of $O$’s and $S$’s relation would be incomplete without explicitly quantifying the information which $O$ has acquired about $S$ by means of interrogations with questions. In (6.6) we have implicitly defined the total amount of $O$’s information about $S$ – once the latter is in a state $\vec{y}_{O \to S}$ – as
\[ I_{O \to S}(\vec{y}_{O \to S}) = \sum_{i=1}^{D_N} \alpha_i, \]
where $\alpha_i$ quantifies $O$’s information about the outcome of question $Q_i \in Q_{MN}$ and satisfies the bounds (5.1). We shall now impose elementary consistency conditions on the relation $\alpha_i(\vec{y}_{O \to S})$ and subsequently implement principles (4) and (5) to finally derive its explicit functional relation.

7.1 Elementary conditions on the measure

There are a few natural requirements on the relation between $\alpha_i$ and $\vec{y}_{O \to S}$:

(i) The $Q_i \in Q_{MN}$ are pairwise independent. Accordingly, $\alpha_i$ should not depend on the ‘yes’-probabilities $y_{j \neq i}$ of other questions $Q_j \neq i$ such that $\alpha_i = \alpha_i(y_j)$.

(ii) All $Q_i \in Q_{MN}$ are informationally of equivalent status. The functional relation between $y_i$ and $\alpha_i$ should be the same for all $i$: $\alpha_i = \alpha(y_i), i = 1, \ldots, D_N$.

(iii) If $O$ has no information about the outcome of $Q_i$, i.e. $y_i = 1/2$, then $\alpha_i = 0$ bit.

(iv) If $O$ has maximal information about the outcome of $Q_i$, i.e. $y_i = 1$ or $y_i = 0$, then $\alpha_i = 1$ bit; both possible answers give 1 bit of information.

(v) The assignment of which answer to $Q_i$ is ‘yes’ and which is ‘no’ is arbitrary and the functional relation between $\alpha_i$ and $y_i$ (or $n_i$) should not depend on this choice. Hence, $\alpha(y_i) = \alpha(n_i)$ must be symmetric around $y_i = 1/2$ (see also (iv)); $\alpha_i$ quantifies the amount of information about $Q_i$, but does not encode what the answer to $Q_i$ is.

(vi) On the interval $y_i \in (1/2, 1]$, the relation between $\alpha$ and $y_i$ should be monotonically increasing such that $O$’s information about the answer to $Q_i$ is quantified as higher, the higher the assigned probability for a ‘yes’ outcome. Likewise, on $[0, 1/2)$, $\alpha(y_i)$ must be monotonically decreasing. In particular, $\alpha$ is to be a continuous function of $y_i$.

This suggests that to every non-vanishing value of $\alpha_i$ there must correspond two possible solutions for $y_i$. 
7.2 The squared length of the Bloch vector as information measure

We now have sufficient structure in our hand to determine the functional relation between \( \alpha_i \) and \( y_i \). Given that the generalized Bloch vector \( 2 \vec{y}_O \to S - \vec{I} \) transforms nicely under time evolution \( \{T(t)\} \), it is useful to parameterize \( \alpha_i \) by \( 2y_i - 1 \), i.e. \( \alpha_i = \alpha(2y_i - 1) \). Principle \( 4 \) entails that \( O \)'s total information about (an otherwise non-interacting) \( S \) is a ‘conserved charge’ of time evolution

\[
I_{O \to S}(\vec{y}_O \to S(\Delta t)) = I_{O \to S}(\vec{y}_O \to S(0))
\]

which translates into the condition

\[
I_{O \to S} \left( T(\Delta t) \left( 2 \vec{y}_O \to S(0) - \vec{I} \right) \right) = \sum_{i=1}^{D_N} \alpha \left( \sum_{j=1}^{D_N} T_{ij}(\Delta t) (2y_j(0) - 1) \right)
\]

\[
= \sum_{i=1}^{D_N} \alpha (2y_i(0) - 1) = I_{O \to S}(2 \vec{y}_O \to S(0) - \vec{I}). \tag{7.1}
\]

If \( T(\Delta t) \) was a permutation matrix, \( \{T\} \) would hold for any function \( \alpha(2y_i - 1) \). For example, \( N \) classical bits are governed by the evolution group \( \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \). However, permutations form a discrete group, while in our present case \( \{T(\Delta t), \Delta t \in \mathbb{R}\} \) constitutes a continuous one-parameter group. This is where continuity of time evolution, as asserted by principle \( 5 \), becomes crucial. Under a reasonable assumption on the information measure, we shall now show that continuity of time evolution, together with (i)–(vi), enforces the quadratic relation \( \alpha_i = (2y_i - 1)^2 \). To this end, we once more invoke the coin flip scenario.\(^{37}\)

Given our parameterization in terms of the Bloch vector \( 2 \vec{y}_O \to S - \vec{I} \), \( O \)'s information about the outcomes of his questions in the coin flip scenario can be written as follows

\[
I_{O \to S_{12}} \left( 2 (\lambda \vec{y}_O \to S_1 + (1 - \lambda) \vec{y}_O \to S_2) - \vec{I} \right) = I_{O \to S_{12}} \left( \lambda (2 \vec{y}_O \to S_1 - \vec{I}) + (1 - \lambda)(2 \vec{y}_O \to S_2 - \vec{I}) \right)
\]

\[
= \sum_{i=1}^{D_N} \alpha (\lambda (2y_i^1 - 1) + (1 - \lambda)(2y_i^2 - 1)).
\]

It is instructive to consider the case in which \( O \) is entirely oblivious about \( S_2 \) such that the latter is in the state of no information \( \vec{y}_O \to S_2 = \frac{1}{2} \vec{I} \) relative to him, but that \( O \) has some information about \( S_1 \). In this case, \( 2 \vec{y}_O \to S_{12} - \vec{I} = \lambda (2 \vec{y}_O \to S_1 - \vec{I}) \) and (assuming the outcome of the coin flip is not certain) condition \( \{8, 3\} \) implies

\[
I_{O \to S_{12}} (\lambda (2 \vec{y}_O \to S_1 - \vec{I})) < I_{O \to S_1} (2 \vec{y}_O \to S_1 - \vec{I})
\]

or, equivalently,

\[
I_{O \to S_{12}} (\lambda (2 \vec{y}_O \to S_1 - \vec{I})) = f \cdot I_{O \to S_1} (2 \vec{y}_O \to S_1 - \vec{I}), \tag{7.2}
\]

where \( f < 1 \) is a factor parametrizing \( O \)'s information loss relative to the case in which he does not toss a coin and, instead, directly asks \( S_1 \). The reason \( O \) experiences such a relative information loss about the outcome of his interrogation is, of course, entirely due to the randomness of the coin flip. But the coin flip is independent of the systems \( S_{1,2} \) and, in particular, of the states in which these are relative to \( O \); the factor \( \lambda \) by which the probabilities \( \vec{y}_O \to S_i \) become rescaled is state independent. For that reason, the relative information loss should likewise depend only on the coin flip, quantified by \( \lambda \), and not on the state \( \vec{y}_O \to S_i \). For instance, if we also considered the case that the coin flip was certain, i.e. \( \lambda = 0, 1 \), then clearly for \( \lambda = 1 \) we must have \( f = 1 \ \forall \vec{y}_O \to S_i \in \Sigma_N \) and for \( \lambda = 0 \) it must hold \( f = 0 \ \forall \vec{y}_O \to S_i \in \Sigma_N \). We shall make this into a requirement on the information measure for all values of \( \lambda \):

**Requirement 2.** The relative information loss factor \( f \) in \( \{7.2\} \) is a state independent (continuous) function of the coin flip probability \( \lambda \) with \( f(\lambda) < 1 \) for \( \lambda \in (0, 1) \).

\(^{37}\)We suspect that this result may be derivable from purely group theoretic arguments without an operational setup by employing the mathematical fact that to every continuous matrix group acting linearly on some space there corresponds a conserved inner product which is quadratic in the components of the vectors.
The measure $\alpha$ thus factorizes, $\alpha(\lambda(2y_i - 1)) = f(\lambda)\alpha(2y_i - 1)$. Setting $\lambda = \lambda_1 \cdot \lambda_2$ yields
\[ f(\lambda_1 \cdot \lambda_2)\alpha(2y_i - 1) = \alpha(\lambda_1 \cdot \lambda_2 (2y_i - 1)) = f(\lambda_1)\alpha(\lambda_2(2y_i - 1)) = f(\lambda_1)f(\lambda_2)\alpha(2y_i - 1) \]
and therefore $f(\lambda_1) \cdot f(\lambda_2) = f(\lambda_1 \cdot \lambda_2)$, which implies $f(\lambda) = \lambda^p$, for some power $p \in \mathbb{R}$. But then, $\alpha$ must be a homogeneous function $\alpha(2y_i - 1) = k(2y_i - 1)^p$ with some constant $k \in \mathbb{R}$. In consequence, the information $I_{O \rightarrow S}(2\vec{y}_{O \rightarrow S} - 1)$ is (up to $k$) the $p$-norm of the Bloch vector $2\vec{y}_{O \rightarrow S} - 1$.

We can rule out that $p \in (-\infty,0]$ because in this case, as one can easily check, it is impossible to satisfy all the consistency conditions (i)–(vi) of section 7.1. Hence, $p > 0$. At this stage we can make use of (7.3) and a result by Aaronson [80] which implies that the only vector $p$-norm with $p > 0$ which is preserved by a continuous matrix group is the 2-norm. Since any given time evolution of the Bloch vector $2\vec{y}_{O \rightarrow S} - 1$ is governed by a continuous, one-parameter matrix group, we conclude that $\alpha(2y_i - 1) = k(2y_i - 1)^2$. Imposing condition (iv) yields $k = 1$ and therefore ultimately
\[ I_{O \rightarrow S}(2\vec{y}_{O \rightarrow S}) = \sum_{i=1}^{D_N} (2y_i - 1)^2. \quad (7.3) \]

It is straightforward to convince oneself that all of (i)–(vi) of the previous section are satisfied by this quadratic information measure. $O$‘s total amount of information about $S$ is thus the squared length of the generalized Bloch vector, thereby assuming a geometric flavour. Importantly, $O$‘s information about the various question outcomes is not quantified by the classical Shannon entropy.

It is important to emphasize that, had we not imposed continuity of time evolution in principle 5, we would not have been able to arrive at (7.3); if time evolution was not continuous, many solutions to $\alpha$ in terms of $y_i$ would be possible.

Thus far, we only gathered that a given time evolution is described by a one-parameter group. Now, given (7.3), we are in the position to say quite a bit more. The full matrix group leaving (7.3) invariant is $O(D_N)$. However, a given time evolution is a continuous one-parameter group and must therefore be connected to the identity. Hence, $T(\Delta t) \in SO(D_N)$, $\forall T(\Delta t)$. Consequently, the group corresponding to any fixed time evolution is a one-parameter subgroup of $SO(D_N)$. If we permit any time evolution of $S$, relative to $O$, which is consistent with our principles, the set of all possible time evolutions $T_N$ must likewise be a subgroup of $SO(D_N)$. This topic is thoroughly discussed in the companion article [1], where it is shown that the principles imply $T_N = PSU(2^N)$ for the $D_2 = 3$ and $T_N = PSO(2^N)$ for the $D_2 = 2$ case. Note that $PSU(2^N)$ is a proper subgroup of $SO(D_N = 4^N - 1)$ for $N > 1$. The generators of $T_N$ are the set of possible time evolution generators of $S$’s states.

The quadratic information measure (7.3) has been proposed earlier by Brukner and Zeilinger in [30] [31] [32] [33] from a different perspective, emphasizing that this is the most natural measure taking into account an observer’s uncertainty – due to statistical fluctuations – about the outcome of the next trial of measurements on a system in a multiple shot experiment. In particular, the authors stress that the quadratic measure is the appropriate one under the circumstance that the measurements do not reveal pre-existing properties of the observed systems and that all the information an observer has access to prior to a measurement are the probabilities for the various experimental outcomes. By contrast, it is further argued that the classical Shannon information conceptually applies to the situation where a measurement reveals a pre-existing property. The arguments leading to their proposal are congruous with the operational and purely epistemic character of the landscape of inference theories in section 3.2 and thereby conceptually complement and further support our current derivation.

Furthermore, taking the formalism of quantum theory as given, Brukner and Zeilinger later singled out the quadratic measure from the set of Tsallis entropies by imposing an ‘information invariance principle’, according to which a continuous transformation among any two complete sets of mutually complementary measurements in quantum theory should leave an observer’s information about the system invariant [34]. While [64] is certainly compatible with the present framework, here we come from farther away to the same result: we do not pre-suppose quantum theory and derive the quadratic measure more generally

---

[38] Such a factorization of coin flip probabilities could be achieved, e.g., if $O$ decided to use one coin, with ‘heads’ probability $\lambda_1$, to firstly decide which of two possible convex mixtures to prepare where both possible mixtures are generated with a second coin with ‘heads’ probability $\lambda_2$. If three of the four states within the two mixtures are chosen as the state of no information, one would obtain precisely such an equation.
by starting from the landscape of information inference theories and imposing principle 4 of information preservation and principle 5 of continuous time evolution thereon. In this regard, the present derivation may similarly be taken as a strong justification for the original Brukner-Zeilinger proposal.

7.3 Pure and mixed states

The explicit quantification of $O$’s information now permits us to render the distinction between three informational classes of $S$’s states – which we already loosely referred to as ‘states of maximal knowledge’ or ‘states of non-maximal information’ in previous sections – precise.

Firstly, we determine the maximally attainable (independent and dependent) information content within a state $\vec{y}_{O \rightarrow S}$ of a system of $N$ gbits. This can be easily counted: once $O$ knows the answers to $N$ mutually independent questions (these do not need to be individuals), he will also know the answers to all their bipartite, tripartite, ... and $N$-partite correlations – all of which are contained in $Q_{MN}$ too by theorems 5.6 and 5.7. But these are then

$$\left(\begin{array}{c}N \\ 1\end{array}\right) + \left(\begin{array}{c}N \\ 2\end{array}\right) + \cdots + \left(\begin{array}{c}N \\ N\end{array}\right) = \sum_{i=1}^{N} \left(\begin{array}{c}N \\ i\end{array}\right) = 2^N - 1$$

answered questions from $Q_{MN}$, while all remaining questions in $Q_{MN}$ will be complementary to at least one of the known ones. $O$’s total information, as quantified by $I_{O \rightarrow S}$, thus contains plenty of dependent bits of information – a result of the fact that the questions in $Q_{MN}$ are pairwise but not necessarily mutually independent.

Using this observation, we shall characterize $S$’s states according to their information content, i.e. squared length of the Bloch vector. By the completeness principle 3, this distinction applies to all states, including those for which the information is distributed partially over many elements of a fixed $Q_{MN}$.

Specifically, we shall refer to a state $\vec{y}_{O \rightarrow S}$ as a

- **pure state**: if it is a state of maximal information content, i.e. maximal length

$$I_{O \rightarrow S}(\vec{y}_{O \rightarrow S}) = \sum_{i=1}^{D_N} (2y_i - 1)^2 = (2^N - 1) \text{bits},$$

- **mixed state**: if it is a state of non-extremal information content, i.e. non-extremal length

$$0 \text{bits} < I_{O \rightarrow S}(\vec{y}_{O \rightarrow S}) = \sum_{i=1}^{D_N} (2y_i - 1)^2 < (2^N - 1) \text{bits},$$

- **totally mixed state**: if it is the state of no information $\vec{y}_{O \rightarrow S} = \frac{1}{2}\vec{1}$ with zero length

$$I_{O \rightarrow S} \left(\vec{y}_{O \rightarrow S} = \frac{1}{2}\vec{1}\right) = 0 \text{bit}.$$

8 The $N = 1$ case and the Bloch ball

Before closing this toolkit for now, we quickly give a flavour of the capabilities of the newly developed concepts and tools by applying them to show that in the simplest case of a single gbit ($N = 1$) principles 1–5 indeed only have two solutions within $L_{\text{gbit}}$, namely the qubit and the rebit state space including their respective time evolution groups. This proves the claim of section 4.1 for $N = 1$.

To this end, recall theorem 5.1 which asserts that the dimension of the $N = 1$ state space $\Sigma_1$ is either $D_1 = 2$ or $D_1 = 3$ which thus far we suggestively referred to as the ‘rebit’ and ‘qubit case’, respectively.
8.1 A single qubit and the Bloch ball

We begin with the $D_1 = 3$ case. $\Sigma_1$ will be parametrized by a three-dimensional vector $\vec{y}_{O \rightarrow S} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, where $y_1, y_2, y_3$ are the 'yes'-probabilities of a mutually complementary question set $Q_1, Q_2, Q_3$ constituting an informationally complete $Q_1$ ball. The informational distinction of states introduced in section 7.3 reads in this case

**pure states:** $\vec{y}_{O \rightarrow S}$ such that

$$I_{O \rightarrow S} = (2y_1 - 1)^2 + (2y_2 - 1)^2 + (2y_3 - 1)^2 = 1 \text{ bit},$$

**mixed states:** $\vec{y}_{O \rightarrow S}$ such that

$$0 \text{ bit} < I_{O \rightarrow S} = (2y_1 - 1)^2 + (2y_2 - 1)^2 + (2y_3 - 1)^2 < 1 \text{ bit},$$

**totally mixed state:** $\vec{y}_{O \rightarrow S} = \frac{1}{2} \vec{1}$ such that

$$I_{O \rightarrow S} = (2y_1 - 1)^2 + (2y_2 - 1)^2 + (2y_3 - 1)^2 = 0 \text{ bit}.$$

It is now easy to see that, if we impose the completeness principle according to which all states $\vec{y}_{O \rightarrow S}$ are permitted so long as they abide by the limited information and complementarity principles, the set of allowed states populates the entire unit ball in the three-dimensional Bloch vector space, i.e. $\Sigma_1 \simeq B^3$. Hence, we recover the well-known three-dimensional Bloch ball state space of a single qubit of standard quantum theory with the set of all pure states defining the boundary sphere $S^2$, the totally mixed state constituting the center and the set of mixed states filling the interior in between, as illustrated in figure 5. This is precisely the geometry of the set of all normalized density matrices on $\mathbb{C}^2$. Notice that the pure state space $S^2 \simeq \mathbb{C}P^1$ indeed coincides with the set of all unit vectors in $\mathbb{C}^2$ (modulo phase).

$$\vec{r} = 2\vec{y}_{O \rightarrow S} - \vec{1}$$

(a) 3D qubit Bloch ball

(b) 2D rebit Bloch disc

Figure 5: The three-dimensional Bloch ball (a) and the two-dimensional Bloch disc (b) are the correct state spaces $\Sigma_1$ of a single qubit in standard quantum theory and a single rebit in real quantum theory, respectively. The vector $\vec{r}$ parametrizing the states is the Bloch vector $2\vec{y}_{O \rightarrow S} - \vec{1}$.

It is also evident that the set of all possible time evolutions, compatible with principles is the rotation group

$$\mathcal{T}_1 \simeq SO(3) \simeq PSU(2)$$

as this is the (connected component to the identity of the) isometry group of the Bloch ball and all time evolutions (in between $O$’s interrogations) are allowed which preserve $O$’s total information about the
qubit, quantified by the squared length of the Bloch vector. PSU(2) is precisely the adjoint action of SU(2) on density matrices $\rho_{2\times2}$ over $\mathbb{C}^2$, $\rho_{2\times2} \mapsto U \rho_{2\times2} U^\dagger$, $U \in \text{SU}(2)$ and thus coincides with the set of all possible unitary time evolutions of a single qubit in standard quantum theory.

Given the complete symmetry of the Bloch ball as the state space for $N = 1$, there should not exist a distinguished informationally complete question set $Q_M$, corresponding to a distinguished orthonormal Bloch vector basis, by means of which $O$ can interrogate $S$. While it clearly is an additional assumption, it is thus natural to stipulate that every pure state of this system corresponds to the definite answer to one question in the set of all possible non-trivial questions $Q_1$ which $O$ can ask the qubit. But then $Q_1 \simeq S^2$ which also coincides with the set of all possible (pure state) projective measurements over $\mathbb{C}^2$.

The 'ballness' and three-dimensionality of the state space of a single qubit can also be derived from various operational axioms within GPTs [12, 13, 18, 14, 19]. The principle of continuous reversibility, according to which every pure state of the convex set can be mapped into any other by means of a continuous and reversible transformation, usually assumes a crucial role in such derivations. Here we offer a novel perspective on the origin of the Bloch ball by deriving it from elementary rules for the informational relation between $O$ and $S$; in particular, we recover continuous reversibility as a by-product.

8.2 A single rebit and the Bloch disc

The analogous result holds for the $D_1 = 2$ case: $\Sigma_1$ will be parametrized by a two-dimensional vector $\vec{y}_{O \rightarrow S} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, where $y_1, y_2$ are the 'yes'-probabilities of two complementary questions $Q_1, Q_2$ constituting an informationally complete $Q_1$ set. We then have

- **pure states**: $\vec{y}_{O \rightarrow S}$ such that $I_{O \rightarrow S} = (2y_1 - 1)^2 + (2y_2 - 1)^2 = 1 \text{bit}$,
- **mixed states**: $\vec{y}_{O \rightarrow S}$ such that $0 < I_{O \rightarrow S} = (2y_1 - 1)^2 + (2y_2 - 1)^2 < 1 \text{bit}$,
- **totally mixed state**: $\vec{y}_{O \rightarrow S} = \frac{1}{2} \vec{1}$ such that $I_{O \rightarrow S} = (2y_1 - 1)^2 + (2y_2 - 1)^2 = 0 \text{bit}$.

The completeness principle implies that the state space $\Sigma_1$ coincides with the two-dimensional Bloch disc, as depicted in figure 5.1, with the totally mixed state in the centre, the pure states on the boundary circle $S^1$ and the mixed states in the interior in between. This is precisely the geometry of the set of unit trace, symmetric matrices over $\mathbb{R}^2$ – the space of density matrices of a single rebit. Similarly, the pure state space $S^1 \simeq \mathbb{R}P^1$ coincides with the set of all unit vectors in $\mathbb{R}^2$ (modulo $\mathbb{Z}_2$).

In analogy to the qubit case, the set of all possible time evolutions is the projective rotation group $T_1 \simeq \text{PSO}(2) \simeq \text{SO}(2)/\mathbb{Z}_2$

because this is the (connected component of the) orientation preserving isometry group of the Bloch disc and all time evolutions are permitted which preserve the Bloch vector length, representing $O$’s total information about $S$, and the orientation of the Bloch vector basis. Orientation preservation means that $O$’s convention about the ‘yes’-‘no’-labeling of the question outcomes is preserved: the center element $-\vec{1} \in \text{SO}(2)$, on the other hand, corresponds to the total inversion, alluded to in section 5.4.1 which is a discrete passive transformation of the state representation and should therefore not be part of the continuous physical time evolution group. We therefore factor out the center. Indeed, PSO(2) is the set of orthogonal time evolutions of a rebit because a real density matrix $\rho_{2\times2}$ on $\mathbb{R}^2$ evolves under the adjoint action of $\text{SO}(2)$, $\rho_{2\times2} \mapsto O \rho_{2\times2} O^T$ for $O \in \text{SO}(2)$.

Again, given the symmetry of the Bloch disc, there is no reason for a distinguished informationally complete question basis $Q_M$ on $Q_1$, incarnated as a distinguished Bloch vector basis, to exist. Accordingly, we assume that any pure state on $S^1$ corresponds to the definite answer of some question in $Q_1$ which $O$ may ask the rebit. In this case, $Q_1 \simeq S^1$, which coincides with the set of projective (pure state) measurements on $\mathbb{R}^2$.
9 Discussion and outlook

The focus of the present manuscript is to develop a novel framework for characterizing and (re-)constructing qubit quantum theory which emphasizes the informational relation between an observer $O$ and a system $S$. This relation is established by $O$ interrogating $S$ with yes-no-questions and is governed by a set of elementary rules on $O$’s acquisition of information about $S$.

This work is more generally motivated by the effort to understand physics from an informational, relational and operational perspective. The premise is to only speak about information which an observer (or more generally system) has access to by interaction with another system. An approach of this kind necessarily describes relational properties of systems rather than intrinsic (or extrinsic) properties. The interaction between systems establishes a relation between them, permitting an information exchange as a physical process which reveals certain physical properties relative to one another. The proposal is to depart from the idea that systems have absolute, i.e. observer independent properties (or more generally properties independent of their relations with other systems). Certainly, in order not to render such a view hopelessly solipsistic, systems ought to have certain intrinsic attributes, e.g. a corresponding state space or set of permissible interactions/measurements, such that different observers have a basis for agreeing or disagreeing on the description of physical objects. However, a state of a system or measurement/question outcome will always be relative to whoever performs the measurement or asks the question. Different observers may agree on states or measurement outcomes by communication (i.e., physical interaction) but if one rejects the idea of an absolute and omniscient observer it is natural to also abandon the idea of an absolute and external standard by means of which properties of systems could be defined. This is a perspective on physics well congruent with the relational interpretation of quantum mechanics [26, 27].

In this article we make such ideas more rigorous within the limits of qubit (and rebit) quantum theory as the exemplary physical scenario. We have laid down, without ontological statements, the mathematical and conceptual foundations for a landscape of information inference theories describing an observer’s acquisition of information about a system. Within this landscape five information inference principles for qubit (and rebit) quantum theory have been given. We show that the principle of limited information and the complementarity principle imply the independence, compatibility and complementarity structure of projective measurements in quantum theory in terms of binary questions. In particular, these principles entail in a constructive and simple way

1. a novel argument for the dimensionality of the Bloch ball,
2. a new method for determining the correct number of independent questions/measurements necessary to describe a system of $N$ qubits (or rebits),
3. a natural explanation for entanglement and monogamy of entanglement,
4. the explicit correlation structure of two qubits and rebits, and
5. more generally the entanglement structure for arbitrarily many qubits and rebits.

From the perspective of this approach the relational character of the qubits’ properties is a consequence of a universal limit on the amount of information accessible to $O$ and the mere existence of complementary information such that $O$ can not know the answers to all his questions simultaneously. It is $O$ who determines which questions he will ask and thereby what kind of system property the interrogation will reveal and thus, ultimately, which kind of information he will acquire (although clearly he does not determine what the outcome to his question is). But relative to $O$ the system of qubits does not have properties other than those accessible to him.

Furthermore, the principles of information preservation and time evolution are shown to result in

6. a reversible time evolution, and
7. under reasonable operational conditions, in a quadratic information measure, quantifying $O$’s prior information about the answers to the various questions he may ask $S$.

This measure has been earlier proposed by Brukner and Zeilinger from a different perspective [30, 31, 61, 62, 64], complementing our present derivation. Supported by the completeness principle we then show, as the simplest example, that
8. the Bloch ball and Bloch disc are recovered as state spaces for a single qubit and rebit, respectively, together with the correct time evolution groups and question sets.

The full (re-)construction of qubit quantum theory, following from the present five principles (and an additional one), is completed in the companion paper [1]. In conjunction, this derivation strongly highlights quantum theory as an inference framework, describing and governing an observer’s acquisition of information about an observed system.

Certainly, there are some limitations to the present approach: First of all, we employ a clear distinction between observer and system which cannot be considered as fundamental. Secondly, the construction is specifically engineered for qubit quantum theory. While arbitrary finite dimensional quantum systems could, in principle, be immediately encompassed by imposing the so-called subspace axiom of GPTs [12, 13], the latter does not naturally fit into our set of principles which only concern the relation between $O$ and $S$. More generally, the limitation is that the current approach only encompasses finite dimensional quantum theory, but not quantum mechanics. As it stands, the mechanical phase space language does not naturally fit into the present framework and more sophisticated tools are required in order to cover mechanical systems, let alone anything beyond that.

While this informational construction recovers the state spaces, the set of possible time evolutions and projective measurements of qubit quantum theory, in other words, the architecture of the theory, it clearly does not tell us anything about the concrete physics. This purely informational framework is abstract and information carrier independent. But qubits as information carriers can be physically incarnated in many different ways: as electron or muon spins, photon polarization, quantum dots, etc. The framework cannot distinguish among the different physical incarnations, underlining the observation that not everything can be reduced to information and additional inputs are necessary in order to do proper physics.

Nevertheless, despite its current limitations, this elementary informational and relational approach teaches us something non-trivial about the structure and physical content of quantum theory.

9.1 An operational alternative to the wave function of the universe

Pushing the informational interpretation of quantum theory to the extreme, one may speculate whether the quantum state also represents a state of information in a gravitational or cosmological context. For instance, is such an interpretation adequate for the ‘wave function of the universe’ which is ubiquitous in standard approaches to quantum cosmology (e.g., see [84, 85, 86])? Such an interpretation would require the existence of an absolute and omniscient observer, an idea which we just abandoned.

Alternatively, one could adopt one of the central ideas of relational quantum mechanics [26, 27], according to which all physical systems can assume the role of an ‘observer’, recording information about other systems, thereby relieving the clear distinction used in this manuscript. Extending this idea to a space-time context, one could interpret the universe as an abstract network of subsystems/subregions, viewed as information registers, which can communicate and exchange information through interaction (see figure 6). In this background independent context, any information acquisition by any register is internal, i.e. occurs within the network; a global observer outside the network is meaningless. This may appear as a purely philosophical observation, but it implies concrete consequences for the description of the network: there could be no global state (aka ‘wave function of the universe’) for the entire network at once. Indeed, admitting any register in the network to act as ‘observer’, the self-reference problem impedes a given register to infer the global state of the entire network – including itself – from its interactions with the rest. Accordingly, relative to any subsystem, one could assign a state to the rest of the network, but a global state and thus a global Hilbert space would not arise. This offers an operational alternative to the problematic concept of the ‘wave function of the universe’. The absence of a global state in quantum gravity has been proposed before [89, 90, 91, 92] – albeit from a different, less operational and informational perspective.

Clearly, if this was to offer a coherent picture of physics, there would need to exist non-trivial consistency relations among the different registers’ descriptions. This is not a practically unrealistic expectation, as already standard quantum theory features such a consistency between different observers’ perspectives [26, 27, 60, 61]. Promisingly, a concrete playground for this idea has recently been constructed from a different motivation [93]: a scalar field on the background geometry of elliptic de Sitter space can only
be quantized in an observer dependent manner. A global Hilbert space for the quantum field does not exist, but consistency conditions between different observers’ descriptions can be derived. Although not being a quantum gravity model, it may serve as a platform for concretizing this proposal further.

Quantum theory suggests that, above all, physics is relational; it is about what we can say about the world and not about how the world ‘really’ is.

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