Testing of generalized Helmholtz equations for gyrotropic waveguides

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Abstract. Universal generalized Helmholtz equations were tested for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes with arbitrary magnetization and general Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes with longitudinal magnetization, which are obtained from generalized Helmholtz equations. From the general Helmholtz equations for gyrotropic waveguides are derived the partial Helmholtz equations for gyrotropic waveguides with certain orthogonal cross-sectional shapes (in particular: rectangular, round, elliptical) with a specific (longitudinal, normal, tangent) magnetization. General Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes in longitudinal magnetization and partial Helmholtz equations for gyrotropic elliptic waveguides in longitudinal magnetization are used during testing.

1. Introduction

In [1, 2], universal generalized Helmholtz equations were obtained for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes under arbitrary magnetization, which allow to derive common Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes under specific (longitudinal, normal and tangent) magnetization. In [2], the general Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes in longitudinal magnetization were obtained from the generalized Helmholtz equations, from which the partial Helmholtz equations for gyrotropic elliptic waveguides in longitudinal magnetization were then derived.

For the first time, the generalized Helmholtz equations obtained in [2] for gyrotropic waveguides with arbitrary orthogonal cross-section shapes under arbitrary magnetization and the general Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal cross-section shapes under longitudinal magnetization, as well as the partial Helmholtz equations for gyrotropic elliptic waveguides under longitudinal magnetization, need careful testing.

Thus, the purpose of this work is to test the general and generalized Helmholtz equations for gyrotropic waveguides, as well as the partial Helmholtz equations for gyrotropic elliptic waveguides during longitudinal magnetization.

2. Testing of general and generalized Helmholtz equations

In [2], when determining the partial Helmholtz equations for gyrotropic waveguides with different forms of orthogonal cross-sections for a particular magnetization, the following equations are successively calculated: 1) Generalized equations, 2) General equations, 3) Partial equations.

Therefore, a comparison of the partial Helmholtz equations defined according to the scheme mentioned above with the well-known partial Helmholtz equations from the literature allows us to test the entire chain of calculations.
In [2], generalized Helmholtz equations were obtained for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes in arbitrary magnetization, which have the form, for HE-wave

$$\Delta_{11} H_z + \Delta_{22} H_z + j \gamma (\delta_{11} H_z + \delta_{22} H_z) - j \omega^2 \varepsilon (H_1 + m H_2) + \omega^2 \mu_3 H_z = 0$$

and for EH-wave

$$\mu_1 \Delta_{11} E_z + \mu_2 \Delta_{22} E_z + j \gamma \mu_1 \delta_{11} E_z + \mu_2 \delta_{22} E_z + \omega (\mu_1 m \delta_{11} - \mu_2 l \delta_{22}) H_z + j k \omega (-l H_1 - m H_2 - j \mu_3 m H_z) - \omega^2 \varepsilon (k^2 - \mu_1 \mu_2) E_z + j \omega (l k \delta_{11} + m k \delta_{22}) H_z = 0.$$  

Here $j$ – imaginary number; $\varepsilon$ – absolute dielectric permittivity of ferrite; $\omega$ – cyclic frequency; $k, l, m, \mu_{11}, \mu_{22}, \mu_{33}$ – components of ferrite magnetic permeability tensor during arbitrary magnetization; $\gamma$ – propagation constant; $\mu_0 = 4 \pi \cdot 10^{-7}$ H/m – magnetic constant; $\omega_0 = \mu_0 Y H_0$ – frequency of ferromagnetic resonance; $Y = 1.76 \cdot 10^{11}$ C/kg – gyromagnetic ratio; $H_0$ – intensity of constant magnetic field; $\omega_m = \mu_0 Y M_0$; $M_0$ – magnetization of ferrite saturation; $(H_1, H_2, H_z)$ – magnetic field components; $(E_1, E_2, E_z)$ – electric field components; 

$$\delta_{11} = \frac{1}{h_1^2} \left( \frac{\partial}{\partial x_1} + \Gamma_{11}^2 \right); \quad \delta_{22} = \frac{1}{h_2^2} \left( \frac{\partial}{\partial x_2} + \Gamma_{22}^2 \right); \quad \Gamma_{12} = \frac{1}{h_1^2} \frac{\partial h_2}{\partial x_1}; \quad \Gamma_{21} = \frac{1}{h_2^2} \frac{\partial h_1}{\partial x_2} – Christoffel symbols$$

of the 2-nd kind [3]; $h_1, h_2$ – Lame coefficients of transverse coordinate axes; $h_3$ - longitudinal [4];

$$\Delta_{11} = \frac{1}{h_1^2} \left( \frac{\partial}{\partial x_1} + \Gamma_{11}^2 - \Gamma_{11}^2 \right) \frac{\partial}{\partial x_1}; \quad \Delta_{22} = \frac{1}{h_2^2} \left( \frac{\partial}{\partial x_2} + \Gamma_{22}^2 - \Gamma_{22}^2 \right) \frac{\partial}{\partial x_2}.$$  

In [2], from the generalized Helmholtz equations (1) and (2), the general Helmholtz equations of HE- and EH-waves were obtained for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes with longitudinal magnetization, for the HE-wave

$$\Delta_{11} H_z + \Delta_{22} H_z + \left( \omega^2 \varepsilon \mu_\parallel - \frac{\mu_\parallel}{\mu} \gamma^2 \right) H_z + j \gamma \omega \kappa E_z = 0$$

and for EH-wave

$$\Delta_{11} E_z + \Delta_{22} E_z + \left( \omega^2 \varepsilon \mu_\parallel - \gamma^2 \right) E_z - j \kappa \omega \frac{\mu_\parallel}{\mu} H_z = 0,$$

where $\mu_\parallel \approx \mu_0; \mu_\perp = \frac{\mu^2 - k^2}{\mu}$ [5]; $\mu = \mu_0 + \mu_0 \frac{\omega_0^2 \omega_m}{\omega^2 - \omega_0^2} \frac{\omega_0^2}{\omega_0^2 - \omega^2}$; $k = \mu_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2}$ – components of the ferrite magnetic permeability tensor [6 - 8].

Further, from the general Helmholtz equations (4) and (5) we derive the particular Helmholtz equations of HE- and EH-waves for gyrotropic rectangular and circular waveguides with longitudinal magnetization. To pass to specific shapes of the cross-section of the waveguide, we use the formula for the relationship between the Lamé coefficients and the basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ [4].
\( |e_i| = \sqrt{g_{ii}} = \sqrt{e_i} = \sqrt{\left(\frac{\partial x_i}{\partial x_i}\right)^2 + \left(\frac{\partial y_i}{\partial x_i}\right)^2 + \left(\frac{\partial z_i}{\partial x_i}\right)^2} \equiv h_i \),

where \( x, y, z \) - cartesian coordinates; \( x_i \) - orthogonal curvilinear coordinates \((i=1, 2, 3)\). Therefore, the Christoffel symbols of the second kind in curvilinear orthogonal coordinate systems, expressed in terms of the Lamé coefficients, will be [3]

\[
\Gamma^i_{ii} = \frac{h_i}{x_i} \cdot \frac{\partial (h_i)}{\partial x_i}; \quad \Gamma^i_{ij} = \frac{h_i}{h_j} \cdot \frac{\partial (h_j)}{\partial x_i}; \quad \Gamma^i_{jk} = \Gamma^i_{kj}; \quad \Gamma^i_{jk} = 0 \quad (i \neq j \neq k). \tag{7}
\]

To derive the particular Helmholtz equations for a gyrotropic rectangular waveguide with longitudinal magnetization, from (6) and (7) we obtain the corresponding Lamé coefficients and Christoffel symbols for the Cartesian coordinate system \((x_1 = x; \; x_2 = y; \; x_3 = z)\)

\[
\begin{align*}
  h_1 &= h_2 = h_3 = 1; \\
  \Gamma^2_{21} &= \Gamma^1_{12} = 0.
\end{align*}
\tag{8}
\]

Then differential operators of the second order (3) taking into account (8) take the form

\[
\begin{align*}
  \Delta_{11} &= \frac{\partial^2}{\partial x^2}; \\
  \Delta_{22} &= \frac{\partial^2}{\partial y^2}.
\end{align*}
\tag{9}
\]

Substituting (9) into the general Helmholtz equation (4), we obtain the well-known particular Helmholtz equation of the \( HE \) - wave for a gyrotropic rectangular waveguide with longitudinal magnetization [5]

\[
\frac{\partial^2 H_Z}{\partial x^2} + \frac{\partial^2 H_Z}{\partial y^2} + \left( w^2 \varepsilon \mu - \frac{\mu_{||}}{\mu} \gamma^2 \right) H_Z + j \omega \varepsilon_{\\mu} \frac{k}{\mu} E_Z = 0.
\]

Further, substituting (9) into the general Helmholtz equation (5), we obtain the well-known particular Helmholtz equation \( EH \) - wave for a gyrotropic rectangular waveguide with longitudinal magnetization [5]

\[
\frac{\partial^2 E_Z}{\partial x^2} + \frac{\partial^2 E_Z}{\partial y^2} + \left( w^2 \varepsilon \mu - \gamma^2 \right) E_Z - j \omega \varepsilon_{\\mu} \frac{\mu_{||}}{\mu} H_Z = 0.
\]

To derive the particular Helmholtz equation for a gyrotropic circular waveguide with longitudinal magnetization, from (6) and (7) we obtain the corresponding Lamé coefficients and Christoffel symbols for the cylindrical coordinate system \((x_1 = r; \; x_2 = \varphi; \; x_3 = z)\)

\[
\begin{align*}
  h_1 &= h_3 = 1; h_2 = r; \\
  \Gamma^1_{12} &= 0; \quad \Gamma^2_{21} = \frac{1}{r}.
\end{align*}
\tag{10}
Then the differential operators of the second order (3) taking into account (10) will be

\[
\begin{align*}
\Delta_{11} &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}; \\
\Delta_{22} &= \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.
\end{align*}
\] (11)

Substituting (11) into the general Helmholtz equation (4), we obtain the well-known particular Helmholtz equation of the \( HE \) - wave for a gyrotropic circular waveguide with longitudinal magnetization [5]

\[
\frac{\partial^2 H_Z}{\partial r^2} + \frac{1}{r} \frac{\partial H_Z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_Z}{\partial \varphi^2} + \left( w^2 \varepsilon \mu - \frac{\mu_{||}}{\mu} \right) H_Z + j \gamma w \varepsilon \frac{k}{\mu} E_Z = 0.
\]

Substituting (11) into the general Helmholtz equation (5), we obtain the well-known particular Helmholtz equation \( EH \) - wave for a gyrotropic circular waveguide with longitudinal magnetization [5]

\[
\frac{\partial^2 E_Z}{\partial r^2} + \frac{1}{r} \frac{\partial E_Z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_Z}{\partial \varphi^2} + \left( w^2 \varepsilon \mu_{||} - \gamma^2 \right) E_Z - j \gamma k \varepsilon \frac{\mu_{||}}{\mu} H_Z = 0.
\]

The testing has shown the correctness of both the general Helmholtz equations of \( HE \)-wave (4) and \( EH \)-wave (5) for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes with longitudinal magnetization, and the generalized Helmholtz equations of \( HE \)-wave (1) and \( EH \)-wave (2) for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes with arbitrary magnetization.

3. Testing the partial Helmholtz equations

Testing of the particular Helmholtz equations for a gyrotropic elliptic waveguide with longitudinal magnetization is carried out according to the following scheme:

- Derivation of the well-known particular Helmholtz equations for a gyrotropic circular waveguide with longitudinal magnetization from similar Helmholtz equations for a gyrotropic elliptic waveguide with longitudinal magnetization, obtained from the general and generalized Helmholtz equations [2], by limiting changes in the shape of the waveguide cross-section from an ellipse to a circle.
- Comparison of the derived particular Helmholtz equations for a gyrotropic circular waveguide with longitudinal magnetization with the known analogous equations from [5].

Figure 1 shows families of confocal ellipses and hyperbolas that form an elliptic coordinate system [9]. The families of these curves intersect orthogonally and the intersection points, in the Cartesian system, have coordinates [9, 10]

\[
\begin{align*}
x &= e \cdot \cosh \xi \cdot \cos \varphi \\
y &= e \cdot \sinh \xi \cdot \sin \varphi.
\end{align*}
\]

For example, point M in figure 1 has coordinates

\[
\begin{align*}
x &= e \cdot \cosh 0.78 \cdot \cos 60^0 \\
y &= e \cdot \sinh 0.78 \cdot \sin 60^0.
\end{align*}
\]

The angle \( \varphi \) takes values from 0 to \( 2\pi \). If there is a stretched elastic membrane fixed between two similar ellipses, then the angle \( \xi \) changes between the values corresponding to these ellipses.
From [9] it follows that if the value of the major semiaxis $s$ is constant and the eccentricity $E \to 0$, then the value of the angle $\xi \to \infty$ and the ellipse tends to a circle with radius $r = s$.

**Figure 1.** Families of orthogonal confocal ellipses and hyperbolas: $e$ - focal length; $E$ - eccentricity; $s, p$ - major and minor semiaxes of the outer ellipse

To derive the Helmholtz equations for a gyrotropic circular waveguide with longitudinal magnetization from analogous Helmholtz equations for a gyrotropic elliptical waveguide by changing the shape of the cross-section to the limit from an ellipse to a circle, we turn to [9].

Figure 2 shows the differentials of the arcs of the ellipse and hyperbola. According to [9], the differentials of the arcs of the ellipse and hyperbola are defined

$$
\begin{align*}
 ds_1 &= \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} \, d\xi \\
 ds_2 &= \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} \, d\varphi
\end{align*}
$$

(12)

The partial derivatives included in (12) are equal to [9]

$$
\begin{align*}
 \frac{\partial x}{\partial \xi} &= e \cdot sh \xi \cdot \cos \varphi \\
 \frac{\partial x}{\partial \varphi} &= -e \cdot ch \xi \cdot \sin \varphi \\
 \frac{\partial y}{\partial \xi} &= e \cdot ch \xi \cdot \sin \varphi \\
 \frac{\partial y}{\partial \varphi} &= e \cdot sh \xi \cdot \cos \varphi.
\end{align*}
$$

(13)
Figure 2. Differentials of arcs of an ellipse and hyperbola: a) hyperbolic $ds_1$ and elliptic $ds_2$ differentials of an arc and a radius vector $r$; b) - area $[ds_1 x ds_2]$, enclosed between two adjacent pairs of intersecting confocal ellipses and hyperbolas [9]

Then, taking into account (13), expression (12) takes the form

$$
\begin{align*}
    ds_1 &= \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} \, d\xi = e\sqrt{\text{ch}^2 \xi - \cos^2 \varphi} \, d\xi, \\
    ds_2 &= \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} \, d\varphi = e\sqrt{\text{ch}^2 \xi - \cos^2 \varphi} \, d\varphi,
\end{align*}
$$

where

$$
l_1 = e \cdot \sqrt{\text{ch}^2 \xi - \cos^2 \varphi}. \quad (14)
$$

Therefore

$$
\begin{align*}
    ds_1 &= l_1 \, d\xi, \\
    ds_2 &= l_1 \, d\varphi.
\end{align*}
$$

Since the differentiation $ds_1$ is directed along the normal to the ellipse, then

$$
dn = l_1 d\xi.
$$
From figure 2a, it follows that the distance to any point \((x, y)\) from the origin in the elliptical coordinate system will be

\[
r = \sqrt{x^2 + y^2} = e \cdot \sqrt{ch^2 \xi - \sin^2 \phi}, \text{ where } r \text{ is the radius vector.} \tag{15}
\]

Then, according to [5], for \(e = \text{const}\) and \(\xi \to \infty\) from (14) and (15) it follows that

\[
l \approx r \approx e \cdot ch(\xi) = e \frac{e^{\xi} + e^{-\xi}}{2} = e \frac{2}{\cosh(2\xi)} = e \cosh(\xi).
\]

Therefore, from (16) and according to [9]

\[
\begin{align*}
ds_1 &= r \cdot d\xi \approx dr, \\
ds_2 &\approx r \cdot d\phi, \\
ds_1 \cdot ds_2 &= r \cdot d\phi \cdot dr.
\end{align*}
\]

Where from

\[
d\xi = \frac{dr}{r}. \tag{17}
\]

From [2] it follows that the Helmholtz equations for a gyrotropic elliptic waveguide with longitudinal magnetization for the \(EH\)- and \(HE\)- waves, respectively, have the form

\[
\begin{align*}
\frac{\partial^2 E_z}{\partial \xi^2} + \frac{\partial^2 E_z}{\partial \phi^2} + e^2 d^2 \left(\omega^2 \varepsilon \mu_\perp - \gamma^2\right) E_z - j e^2 d^2 \gamma k \omega \frac{\mu}{\mu} H_z &= 0, \\
\frac{\partial^2 H_z}{\partial \xi^2} + \frac{\partial^2 H_z}{\partial \phi^2} + e^2 d^2 \left(\omega^2 \varepsilon \mu_\parallel - \gamma^2\right) H_z + j e^2 d^2 \gamma \omega \varepsilon \frac{k}{\mu} E_z &= 0,
\end{align*}
\]

where \(ch^2 \xi - \cos^2 \phi = d^2\).

Since it follows from [9] that for \(\xi \to \infty, \; ch^2(\xi) - \cos^2(\phi) \approx ch^2(\xi) \approx \xi \approx e \cdot ch(\xi) \approx r\). Then the first equation for the \(EH\)-wave from (18) takes the form

\[
\frac{\partial^2 E_z}{\partial \xi^2} + \frac{\partial^2 E_z}{\partial \phi^2} + r^2 \left(\omega^2 \varepsilon \mu_\perp - \gamma^2\right) E_z - j r^2 \gamma k \omega \frac{\mu}{\mu} H_z = 0, \tag{19}
\]

where \(r\) - circle radius.

Further, taking into account (17), we transform the formula (19)

\[
\begin{align*}
\frac{\partial}{\partial \xi} \left( \frac{\partial E_z}{\partial \xi} \right) + \frac{\partial^2 E_z}{\partial \phi^2} + \left(\omega^2 \varepsilon \mu_\perp - \gamma^2\right) E_z - j r^2 \gamma k \omega \frac{\mu}{\mu} H_z &= 0 \\
\Rightarrow & \quad \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{\partial^2 E_z}{\partial \phi^2} + \left(\omega^2 \varepsilon \mu_\perp - \gamma^2\right) E_z - j r^2 \gamma k \omega \frac{\mu}{\mu} H_z = 0 \\
\Rightarrow & \quad \frac{r}{\partial r} \left( \frac{\partial^2 E_z}{\partial r^2} + \frac{\partial E_z}{\partial r} \right) + \frac{\partial^2 E_z}{\partial \phi^2} + \left(\omega^2 \varepsilon \mu_\perp - \gamma^2\right) E_z - j r^2 \gamma k \omega \frac{\mu}{\mu} H_z = 0.
\end{align*}
\]
Dividing the last expression by $r^2$ we have

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \varphi^2} + \left( \omega^2 \varepsilon_\perp \mu_\parallel - \gamma^2 \right) E_z - j\kappa \omega \frac{\mu_\parallel}{\mu} H_z = 0. \quad (20)$$

Proceeding in the same way with the second equation of (18) for the $HE$-wave, we obtain

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \varphi^2} + \left( \omega^2 \varepsilon_\parallel \mu_\parallel - \frac{\mu_\parallel}{\mu} \gamma^2 \right) H_z + j\gamma \omega \frac{k}{\mu} E_z = 0. \quad (21)$$

Formulas (20) and (21), obtained from (18) by limiting change in the shape of the cross section of a gyrotropic elliptic waveguide from an ellipse to a circle, describe the propagation of hybrid $EH$- and $HE$-waves in a circular gyrotropic waveguide with longitudinal magnetization and coincide with those from [5], except for the signs in front of the last terms (before in (20) and in (21)). Note that this discrepancy is due to the fact that in [2] the non-diagonal components of the magnetic permeability tensor of ferrite during longitudinal magnetization are taken with opposite signs than in [5], which is insignificant and only changes the places of the left and right rotation of electromagnetic waves when they propagate in gyrotropic waveguides with longitudinal magnetization.

4. Conclusion

In [2], the generalized Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal cross-section shapes under arbitrary magnetization, the general Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal cross-section shapes under longitudinal magnetization, and the partial Helmholtz equations for gyrotropic elliptical waveguides under longitudinal magnetization were successively obtained.

In this work, the above-mentioned particular, general, and generalized Helmholtz equations for gyrotropic waveguides are tested in two ways:

1. Derivation of the particular Helmholtz equations for gyrotropic rectangular and circular waveguides with longitudinal magnetization from the general Helmholtz equations (4) and (5), which were obtained from the generalized Helmholtz equations (1) and (2) with subsequent comparison of the derived equations with the known equations Helmholtz for gyrotropic rectangular and circular waveguides with longitudinal magnetization from [5];

2. Derivation of the particular Helmholtz equations for a gyrotropic circular waveguide with longitudinal magnetization from the particular Helmholtz equations (18) by limiting change in the cross section - an ellipse into a circle with subsequent comparison of the derived Helmholtz equations with the well-known Helmholtz equations for a gyrotropic circular waveguide with longitudinal magnetization from [5].

The testing has shown that the generalized Helmholtz equations obtained in [2] for gyrotropic waveguides with arbitrary orthogonal cross-sectional shapes at arbitrary magnetization for the $HE$-wave (1), $EH$-wave (2) and the general Helmholtz equations for gyrotropic waveguides with arbitrary orthogonal shapes cross-section for longitudinal magnetization for $HE$-wave (5), $EH$-wave (6) are correct.

Reference

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