Analytical Expression and Deconstruction of the Volume of the Controllability Ellipsoid

Mingwang Zhao
Information Science and Engineering School, Wuhan University of Science and Technology, Wuhan, Hubei, 430081, China
Tel.: +86-27-68863897
Work supported by the National Natural Science Foundation of China (Grant No. 61273005)

Abstract
In this article, we present three theorems and develop an effective analytical method to compute analytically the volume of the controllability ellipsoid for the linear systems with \( n \) different eigenvalues. Furthermore, by deconstructing the analytical expression of the volume, some factors on the shape of the ellipsoid, the side length of its circumscribed rhombohedral, the evenness of the eigenvalue distribution of the linear system are constructed. Based on the analytical expression of the volume and these factors, the controllability can be defined, computing, analyzed, and optimized.

Keywords: volume computation, controllability ellipsoid, controllability Grammian matrix, discrete-time systems, state controllability

1. Introduction
In control theory, linear discrete-time systems can be formulated as follows:

\[ x_{k+1} = Ax_k + Bu_k, \quad x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^r, \]  

where \( x_k \) and \( u_k \) are the state variable and input variable, respectively, and matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times r} \) are the state matrix and input matrix, respectively, in the system models \([4],[1]\). To investigate the controllability of the linear dynamic systems \([1] \), the controllability Grammian matrix can

Preprint submitted to Journal Name
April 14, 2020
be defined as follows

\[ G_N = \sum_{i=0}^{N-1} A_i^T (A_i B)^T \]  

(2)

That the rank of the Grammian matrix \( G_N \) is \( n \), the dimension of the state space the systems (1), is the well-known criterion on the state controllability. By the controllability Grammian matrix, the controllability ellipsoid that can describe the maximum controllable region under the total-energy constraint \( \left( \sum_{k=0}^{N-1} \| u_N \|_2^2 \leq 1 \right) \) can be defined as follows

\[ E_N = \{ x \mid x = G_N^{1/2} z_N, \forall z_N \in \mathbb{R}^n : \| z_N \|_2 \leq 1 \} \]  

(3)

In papers [6], [2], [5], and [3], the determinant value \( \det (G_N) \) and the minimum eigenvalue \( \lambda_{\min} (G_N) \) of the controllability Grammian matrix \( G_N \), correspondingly the volume \( \text{vol} (E_N) \) and the minimum radius \( r_{\min} (E_N) \) of the controllability ellipsoid \( E_N \), can be used to quantify the control ability of the input variable to the state space, and then be chosen as the objective function for optimizing and promoting the control ability of the linear dynamical systems. Due to lack of the analytical computing of the determinant \( \det (G_N) \) and eigenvalue \( \lambda_{\min} (G_N) \), correspondingly the volume \( \text{vol} (E_N) \) and the radius \( r_{\min} (E_N) \), these optimizing problems for the control ability are solved very difficulty, and few achievements about that were made.

In this paper, the analytical volume-computation of the controllability ellipsoid \( E_N \), when the system (1) is a singe input system, is studied and an analytical expression on that \( N \to \infty \) will be proven. By deconstructing the analytical volume expression, some factors for the ellipsoid \( E_N \), such as, the shape factor, the minimum circumscribed rhombohedral, etc, can be got. Therefore, the analytical volume and the shape factor of the ellipsoid, the minimum side length of the rhombohedral can be used to describe the control ability and can be chosen as the objective functions and the constraint conditions for optimizing and promoting the control ability. Because of the analytical expression of these objective functions and constraint conditions, the optimizing problems will be solved with very effective optimizing computation.
2. The analytical volume-computing of the controllability ellipsoid for the matrix $A$ with $n$ different eigenvalues

Based on the linear system theory [4], [1], the linear system (1) can be transformed as the Jordan canonical form, and especially the linear system (1) with $n$ different eigenvalues can be transformed as the diagonal canonical form. Obviously, if the Jordan canonical form is $\Sigma(P^{-1}AP, P^{-1}B)$ with the Jordan transformation matrix $P$, the corresponding controllability Grammian matrix can be expressed respectively as

$$G_N = P^{-1}G_NP^{-T}$$ (4)

And then, the determinant $\det(G_N)$ is $(\det P)^{-2} \det(G_N)$ and the volume of the controllability ellipsoid is $|P|^{-1} \text{vol}(E_N)$. In this paper, the determinant and ellipsoid volume for the diagonal canonical form, respectively, are computed analytically and related results can be generalized to the systems $\Sigma(A, B)$.

When the system $\Sigma(A, B)$ is a single-input diagonal canonical form as

$$A = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \ B = [b_1, b_2, \ldots, b_n]^T$$

the controllability Grammian matrix can be rewritten as

$$G_N = \sum_{i=0}^{N-1} A^i B \ (A^i B)^T$$

$$= \sum_{i=0}^{N-1} \begin{bmatrix} b_1^2 \lambda_1^{2i} & b_1 b_2 \lambda_1^i \lambda_2^i & \cdots & b_1 b_n \lambda_1^i \lambda_n^i \\ b_1 b_2 \lambda_1^i \lambda_2^i & b_2^2 \lambda_2^{2i} & \cdots & b_2 b_n \lambda_2^i \lambda_n^i \\ \vdots & \vdots & \ddots & \vdots \\ b_1 b_n \lambda_1^i \lambda_n^i & b_2 b_n \lambda_2^i \lambda_n^i & \cdots & b_n^2 \lambda_n^{2i} \end{bmatrix}$$

$$= \begin{bmatrix} b_1^2 \frac{1-\lambda_1^N}{1-\lambda_1} & b_1 b_2 \frac{1-\lambda_1^N \lambda_2^N}{1-\lambda_1 \lambda_2} & \cdots & b_1 b_n \frac{1-\lambda_1^N \lambda_n^N}{1-\lambda_1 \lambda_n} \\ b_1 b_2 \frac{1-\lambda_1^N \lambda_2^N}{1-\lambda_1 \lambda_2} & b_2^2 \frac{1-\lambda_2^N}{1-\lambda_2} & \cdots & b_2 b_n \frac{1-\lambda_2^N \lambda_n^N}{1-\lambda_2 \lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 b_n \frac{1-\lambda_n^N \lambda_n^N}{1-\lambda_1 \lambda_n} & b_2 b_n \frac{1-\lambda_n^N \lambda_n^N}{1-\lambda_2 \lambda_n} & \cdots & b_n^2 \frac{1-\lambda_n^N}{1-\lambda_n} \end{bmatrix}$$ (5)
When all eigenvalues $\lambda_i \in (-1, 1)$, $i = 1, n$, and $N \to \infty$, we have

$$G_\infty = \begin{bmatrix}
\frac{b_1^2}{1-\lambda_1^2} & \frac{b_1 b_2}{1-\lambda_1 \lambda_2} & \cdots & \frac{b_1 b_n}{1-\lambda_1 \lambda_n} \\
\frac{b_1 b_2}{1-\lambda_1 \lambda_2} & \frac{b_2^2}{1-\lambda_2^2} & \cdots & \frac{b_2 b_n}{1-\lambda_2 \lambda_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{b_1 b_n}{1-\lambda_1 \lambda_n} & \frac{b_2 b_n}{1-\lambda_2 \lambda_n} & \cdots & \frac{b_n^2}{1-\lambda_n^2}
\end{bmatrix}$$

(6)

Before discussing the analytical volume computing of the infinite-steps controllability ellipsoid $E_\infty$ for the diagonal canonical form, a theorem about the determinant of the matrix $G_\infty$ is put forward and proven as follows.

**Theorem 1.** For all eigenvalues $\lambda_i \in (-1, 1)$, $i = 1, n$, we have

$$F_{\lambda_1, \lambda_2, \ldots, \lambda_n} = \det \begin{bmatrix}
\frac{1}{1-\lambda_1^2} & \frac{1}{1-\lambda_1 \lambda_2} & \cdots & \frac{1}{1-\lambda_1 \lambda_n} \\
\frac{1}{1-\lambda_1 \lambda_2} & \frac{1}{1-\lambda_2^2} & \cdots & \frac{1}{1-\lambda_2 \lambda_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{1-\lambda_1 \lambda_n} & \frac{1}{1-\lambda_2 \lambda_n} & \cdots & \frac{1}{1-\lambda_n^2}
\end{bmatrix}$$

$$= \left[ \prod_{1 \leq j_1 < j_2 \leq n} \left( \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right)^2 \right] \times \left( \prod_{i=1}^{n} \frac{1}{1 - \lambda_i^2} \right)$$

(7)

**Proof.** The theorem can be proven by induction method as follows.

(1) When $n = 1$ and 2, we have

$$F_{\lambda_1} = \det \begin{bmatrix} \frac{1}{1-\lambda_1^2} \end{bmatrix} = \frac{1}{1 - \lambda_1^2}$$

$$F_{\lambda_1, \lambda_2} = \det \begin{bmatrix} \frac{1}{1-\lambda_1^2} & \frac{1}{1-\lambda_1 \lambda_2} \\
\frac{1}{1-\lambda_1 \lambda_2} & \frac{1}{1-\lambda_2^2} \end{bmatrix} = \frac{(\lambda_2 - \lambda_1)^2}{(1 - \lambda_1 \lambda_2)^2 (1 - \lambda_1^2) (1 - \lambda_2^2)}$$

(8)

(9)

And then, for $n = 1$ and 2, Eq. (7) holds.

(2) It is assumed that as Eq. (7) holds for $n = k - 1$, that is, we have,

$$F_{\lambda_1, \lambda_2, \ldots, \lambda_{k-1}} = \left[ \prod_{1 \leq j_1 < j_2 \leq k-1} \left( \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right)^2 \right] \times \left( \prod_{i=1}^{k-1} \frac{1}{1 - \lambda_i^2} \right)$$

(10)
(3) For $n = k$, we have

$$F_{\infty, \lambda_2, \ldots, \lambda_k} = \det \begin{bmatrix}
\frac{1}{1-\lambda_1} & \frac{1}{1-\lambda_1 \lambda_2} & \cdots & \frac{1}{1-\lambda_1 \lambda_k} \\
\frac{1}{1-\lambda_2} & \frac{1}{1-\lambda_2 \lambda_2} & \cdots & \frac{1}{1-\lambda_2 \lambda_k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{1-\lambda_k} & \frac{1}{1-\lambda_k \lambda_2} & \cdots & \frac{1}{1-\lambda_k \lambda_k}
\end{bmatrix}$$

where

$$q_{22} = \frac{1}{1-\lambda_2^2} - \frac{1}{1-\lambda_1 \lambda_2} \times \frac{1}{1-\lambda_1^2}$$

$$= \frac{1 - 2\lambda_1 \lambda_2 - \lambda_1^2 \lambda_2^2 - (1 - \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_2^2)}{(1-\lambda_2^2)(1-\lambda_1 \lambda_2)^2}$$

$$= \frac{(\lambda_2 - \lambda_1)^2}{(1-\lambda_2^2)(1-\lambda_1 \lambda_2)^2}$$

$$q_{2k} = \frac{1}{1-\lambda_2 \lambda_k} - \frac{1}{1-\lambda_1 \lambda_k} \times \frac{1}{1-\lambda_1^2}$$

$$= \frac{1 - 2\lambda_1 \lambda_k - \lambda_1^2 \lambda_2 \lambda_k - (1 - \lambda_1 \lambda_k \lambda_2)}{(1-\lambda_1 \lambda_k)(1-\lambda_1 \lambda_k)(1-\lambda_2 \lambda_k)}$$

$$= \frac{-\lambda_1 \lambda_2 - \lambda_1 \lambda_k - (\lambda_1^2 \lambda_2 \lambda_k - \lambda_1 \lambda_2 \lambda_k)}{(1-\lambda_1 \lambda_2)(1-\lambda_1 \lambda_k)(1-\lambda_2 \lambda_k)}$$

$$= \frac{(\lambda_2 - \lambda_1)(\lambda_k - \lambda_1)}{(1-\lambda_1 \lambda_2)(1-\lambda_1 \lambda_k)(1-\lambda_2 \lambda_k)}$$

$$\cdots$$
And then, we have

\[ F_{\lambda_1, \lambda_2, \ldots, \lambda_k} = \frac{1}{1 - \lambda_1^2} \cdot \prod_{i=2}^{k} \left( \frac{\lambda_i - \lambda_1}{1 - \lambda_i \lambda_1} \right)^2 \cdot \det \begin{bmatrix}
\frac{\lambda_i - \lambda_1}{1 - \lambda_i \lambda_1} & \cdots & \frac{\lambda_i - \lambda_1}{1 - \lambda_i \lambda_k} \\
\cdots & \ddots & \cdots \\
\frac{\lambda_{k-1} - \lambda_1}{1 - \lambda_{k-1} \lambda_k} & \cdots & \frac{\lambda_{k-1} - \lambda_1}{1 - \lambda_{k-1} \lambda_k}
\end{bmatrix}
\]

Therefore, by Eq. (10) and Eq. (14), we have, Eq. (7) holds for \( n = k \).

In summary, the theorem is proven by inductive method. \( \square \)

Based on Theorem 1, the determinant of the controllability Grammian matrix and the volume of the controllability ellipsoid for the diagonal canonical form are as follows

\[ \det (G_\infty) = F_{\lambda_1, \lambda_2, \ldots, \lambda_n} \prod_{i=1}^{n} b_i^2 \]

\[ \text{vol} (E_\infty) = \frac{\pi^{n/2}}{\Gamma \left( \frac{n}{2} + 1 \right)} \cdot \prod_{i=1}^{n} b_i^2 \]

\[ = \frac{\pi^{n/2}}{\Gamma \left( \frac{n}{2} + 1 \right)} \times \left| \prod_{1 \leq j_1 < j_2 \leq n} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right| \times \prod_{i=1}^{n} \frac{b_i}{(1 - \lambda_i^2)^{1/2}} \]

where the Gamma function \( \Gamma (s) \) can be defined as

\[ \Gamma (s) = \begin{cases} 
(s - 1) \Gamma (s - 1) & s > 1 \\
\sqrt{\pi} & s = 1/2
\end{cases} \]

According to the above computation for the diagonal canonical form, a theorem on the determinant of the controllability Grammian matrix and the volume of the controllability ellipsoid for the general systems \( \Sigma (A, B) \) is can be established as follows.
Theorem 2. For the linear systems $\Sigma(A, B)$ with $n$ different eigenvalues $\lambda_i \in (-1, 1)$, $i = 1, \ldots, n$, the determinant of the controllability Grammian matrix and the volume the controllability ellipsoid for the systems can be computed analytically as follows

$$\det(G_\infty) = F_N^{\lambda_1, \lambda_2, \ldots, \lambda_n} \left(\det(P) \prod_{i=1}^n q_i B\right)^2$$  \hspace{1cm} (18)$$

$$\text{vol}(E_\infty) = \frac{\pi^{n/2} |\det(P)|}{\Gamma\left(\frac{n}{2} + 1\right)} \times \prod_{1 \leq j_1 < j_2 \leq n} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \times \prod_{i=1}^n \left|\frac{q_i B}{(1 - \lambda_i^2)^{1/2}}\right|$$  \hspace{1cm} (19)$$

where $q_i$ is the $i$-th unit left eigenvector of the matrix $A$.

3. Decoding the Controllability Ellipsoid

According to the computing equation (19), some factors described the shape and size of the controllability ellipsoid are deconstructed as follows.

$$F_1 = \left|\prod_{1 \leq j_1 < j_2 \leq n} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}}\right|$$  \hspace{1cm} (20)$$

$$F_{2,i} = \left|\frac{q_i B}{(1 - \lambda_i^2)^{1/2}}\right|$$  \hspace{1cm} (21)$$

Next, these factors can be describe the shape and size of the controllability ellipsoid $E_N$ and the eigenvalue evenness factor of the linear system.

3.1. The ellipsoid shape factor and The eigenvalue evenness factor of the linear system

The controllability ellipsoid $E_N$ in the original space and the invariant eigen-space can be represented respectively as the following equation

$$z^T (G_N)^{-1/2} z \leq 1$$  \hspace{1cm} (22)$$

$$z^T (\overline{G}_N)^{-1/2} z \leq 1$$  \hspace{1cm} (23)$$

The $n$ radii of the ellipsoid $E_N$ in the invariant eigen-space are indeed the $n$ eigenvalues of the Garmnnian matrix $\overline{G}_N$. The shape of the ellipsoid $E_N$ can be characterized by the sizes of the all $n$ radii of the ellipsoid.
When some two eigenvalues of the system matrix $A$ are approximately equal, the minimum radius of the ellipsoid $E_N$ will be approximately zero, and the ellipsoid $E_N$ will be flattened. Therefore, the distributions of all eigenvalues of the matrix $A$ are even, the ratio between the minimum and maximum radii can be avoided as a small value and the ellipsoid $E_N$ will be avoided flattened.

The factor $F_1$ deconstructed from the volume computing equation (14) can be used to describe the evenness of the eigenvalue distribution of the Grammian matrix $G_N$ and then the uniformity of the $n$ radii of the ellipsoid $E_N$. The bigger the value of the factor $F_1$, the bigger the ratio between the minimum and maximum radii of the ellipsoid $E_N$ is, and then the greater the volume of the ellipsoid is.

Otherwise, the factor $F_1$ can be used to describe the evenness of the eigenvalue distribution of the linear system $\Sigma(A, B)$. The bigger the value of the factor $F_1$, the bigger the controllable region of the system is, and the stronger the control ability of the systems is.

### 3.2. The circumscribed hypercube and circumscribed rhombohedral of the controllability ellipsoid

The factor $F_{2,i}$ is indeed the biggest distance in the $i$-dimensional space of the ellipsoid $E_N$, that is, the $n$ side lengths of the circumscribed hypercube of the ellipsoid $E_N$ are $2F_{2,i}, i=1, n$. By the volume equation (19), the volume of the ellipsoid can be represented as the production of the volume $\prod_{i=1}^{n} F_{2,i}$ of the circumscribed hypercube and the shape factor $F_1$.

### 4. Analytic Volume-Computation for the systems with the complex eigenvalues

Theorem 2 for the linear systems with the real eigenvalues can be generalized to the linear systems with the complex eigenvalues, and the corresponding theorem can be stated as follows.

**Theorem 3.** When the all $n$ different complex eigenvalues $\lambda_i (i=1, n)$ of the linear systems $\Sigma(A, B)$ satisfy that $|\lambda_i| \in [0, 1)$, the determinant of the controllability Grammian matrix and the volume the controllability ellipsoid
for the systems can be computed analytically as follows

\[
F_{\lambda_1, \lambda_2, \ldots, \lambda_n} = \prod_{1 \leq j_1 < j_2 \leq n} \left( \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right)^2 \times \left( \prod_{i=1}^{n} \frac{1}{1 - |\lambda_i|^2} \right) \tag{24}
\]

\[
\det(G_{\infty}) = F_{N_1, \lambda_2, \ldots, \lambda_n} \left( \det(P) \prod_{i=1}^{n} q_i B \right)^2 \tag{25}
\]

\[
\text{vol}(E_{\infty}) = \frac{\pi^{n/2} |\det(P)|}{\Gamma\left(\frac{n}{2} + 1\right)} \times \left| \prod_{1 \leq j_1 < j_2 \leq n} \frac{\lambda_{j_2} - \lambda_{j_1}}{1 - \lambda_{j_1} \lambda_{j_2}} \right| \times \left| \prod_{i=1}^{n} \frac{q_i B}{(1 - |\lambda_i|^2)^{1/2}} \right| \tag{26}
\]

where the complex vector \( q_i \) is the \( i \)-th unit left eigenvector of the matrix \( A \), and the complex matrix \( P \) is the diagonalization transformation matrix.

Similar to Section 3, based on these analytical expression, the shape factors of the controllability ellipsoid and the evenness factor of the eigenvalue distribution for linear systems with complex eigenvalues can be got.

5. Conclusions

In this article, we present three theorems and develop an effective analytical method to compute the volume of the controllability ellipsoid for the linear systems with \( n \) different eigenvalues. Furthermore, by deconstructing the analytical expression of the volume, some factors on the shape of the ellipsoid, the side length of its circumscribed rhombohedral, the evenness of the eigenvalue distribution of the linear system are constructed. Based on the analytical expression of the volume and these factors, the controllability can be defined, computing, analysed, and optimized.

References

[1] C.T. Chen, Linear system theory and design, Oxford University Press, Inc. New York, NY, USA, 3rd edition, 1998.

[2] G. D., The use of observability and controllability gramians or functions for optimal sensor and actuator location in finite-dimensional systems, in: Proc. of IEEE Conf. on Decision and Control, New Orleans, LA, USA, p. 33193324.
[3] U. Ilkturk, Observability Methods in Sensor Scheduling, Ph.D. thesis, ARIZONA STATE UNIVERSITY, 2015.

[4] T. Kailath, Linear systems, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[5] F. Pasqualetti, S. Zampieri, F. Bullo, Controllability metrics, limitations and algorithms for complex networks, IEEE Trans. on Control of Network Systems 1 (2014) 40–52.

[6] W. VanderVelde, C. Carignan, A dynamic measure of controllability and observability for the placement of actuators and sensors on large space structures, Technical Report, NASA-CR-168520, SSL-2-82, 1982.