CYCLIC STRUCTURE BEHIND MODULAR GAUSSIAN CURVATURE

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Abstract. We propose a systematic scheme for computing the variation of rearrangement operators arising in the recently developed spectral geometry on noncommutative tori and \( \theta \)-deformed Riemannian manifolds. It can be summarized as a category whose objects consist of spectral functions of the rearrangement operators and morphisms are generated by transformations associated to basic operations of the variational calculus. The generators of the morphisms fulfill most of the relations in Connes’s cyclic category, but also include all the partial derivatives. Comparison with Hopf cyclic theory has also been made.

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1. Introduction

This paper is devoted to the variational aspects of spectral geometry on noncommutative tori initiated in [7] and later [11, 6]. The new features that we would like to investigate concern purely the noncommutativity between the metric coordinates and their derivatives. The very first example in this direction is a simple functional relation derived from the variational nature of the modular Gaussian curvature introduced in [6]. As demonstrated in the \( a_4 \)-term calculation [2], the complexity of such relations from variation grow dramatically when the order of differentiations...
involved increases\(^1\). One of our primary motivations is to search for new cancellation and simplification for the lengthy formulas obtained\(^2\), especially those in the appendix. Such goal often relies on adding new mathematical structures, the paper contributes some partial progress in relation to the cyclic (co)homology, discovered independently by Connes and Tsygan.

For the \(a_2\)-term, the cancellation can be seen from the improvement between \([13]\) and \([15]\). In the former paper \([13]\), in which the higher dimensional analogue of the Gaussian curvature was first introduced on toric noncommutative manifolds\(^3\), the functional relations take several equations \([13]\) Eq. (4.5) - (4.8)], which were achieved on noncommutative four tori earlier in \([10]\). After dwelling on the computation for a while, the author realized that those less organized relations indeed admits a much simpler form \([15]\) Thm. 2.15 \([15]\) akin to \([6]\) Eq. (4.42)]\(^4\). Later, the author had a short visit at IHES and shared the progress with Connes. He pointed out immediately that the relations in \([15]\) Prop 2.11, which explain the aforementioned simplification, resemble exactly the compatibility condition (cf. (A.4)) in the cyclic category \(\Delta C\). The main results, Theorem 4.1 and Corollary 4.2 constitute a more comprehensive interpretation of his vision.

The paper is organized as follows. In §2, we recall the construction of rearrangement operators and then proceed with the variational calculus §3. In the literature, rearrangement operators are defined in terms of the derivation commutator \(x = [\cdot, h]\) associate to a log-conformal factor \(h\) due to its own geometric significance \([6]\) § 1.5]. In comparison to \([15]\) § 2], we now choose to work with the multiplication operators associated to \(h\) that makes the simplicial structure in the variational calculus more transparent. The difference occurs only at the level of notations, that is, one can pass from one to the other by the change of variable given in \([6]\). In § 4, we encapsulate the technical discussions in the previous two sections into a category \(\mathcal{C}\), which possesses most of the axioms as a cyclic module. The core is the generating set of morphisms and relations stated in the two theorems. To further conceptualize the computations in §3, we make some comparison with Hopf cyclic theory in §5. In that regard, the failure of \(\mathcal{C}\) being a cyclic module can be explained as the incompatibility between the underlying algebra and coalgebra structures. Lastly, we go back to spectral geometry for applications in §6. We first reproduce, in §6.1 Connes and Moscovici’s functional relation obtained in \([6]\) to illustrate how morphisms of \(\mathcal{C}\) govern the variational calculus. In particular, it provides more clear interpretation for the three terms in (6.3) in cyclic theory. Moreover, the method also has the merit of being suitable for computer algebra system implementation, which seems to be indispensable for simplifying the \(a_4\)-term calculation \([2]\). The second application concerns the action of \(\mathcal{C}\) on functionals arising from rearrangement lemma, which could be useful to the verification of the functional relations, a more formidable task (but equally important) than deriving the relations. In §6.2, we only consider much simpler family of functions \(\{\omega_\alpha\}\) (but intricately connected to those hypergeometric integrals introduced in \([14]\ [16]\)), and show that the action of \(\mathcal{C}\) can be used to derived some differential relations.

The notion of rearrangement operators is closely related to multiple operator integral (see the end of §2 for references), which now plays a crucial role in study of spectral actions \([21]\). We hope to find new applications of \(\mathcal{C}\) in that direction in future publications.

\(^1\) Modular Gaussian curvature is determined by the \(a_2\)-term, here 2 and 4 agree with the maximal order of derivatives appeared in the calculation.

\(^2\) a.k.a \(\theta\)-deformations \([19]\) or Connes-Landi deformations, cf. \([5]\), \([4]\), \([1]\).

\(^3\) at the end of the proof of Thm 5.1.

\(^4\) details are also provided in §6.1
2. Rearrangement Operators via Schwartz Functional Calculus

Our discussion in §§2 and 3 is greatly influenced by the work of Lesch [12], we shall follow the notations there, with slightly changes, to construction the rearrangement operators. Let $\mathcal{A}$ be a unital $C^*$-algebra. For any $h \in \mathcal{A}$, denote by $h^{(j,n)} \in L(\mathcal{A}^{\otimes n}, \mathcal{A})$ the linear operator from $\mathcal{A}^{\otimes n} \to \mathcal{A}$:
\[
(h^{(j,n)} : \mathcal{A}^{\otimes n} \to \mathcal{A} : \rho_1 \otimes \cdots \otimes \rho_n \mapsto \rho_1 \cdots \rho_{j-1} h \rho_j \cdots \rho_n,
\]
that is the multiplication at the $j$-th slot, $j = 0, \ldots, n$. For $n = 1$, $h^{(0,1)}, h^{(1,1)} : \mathcal{A} \to \mathcal{A}$ are simply the left and the right multiplications.

Now assume $h = h^* \in \mathcal{A}$ is self-adjoint and $U \subset \mathbb{R}$ is an open neighborhood that contains the spectrum of $h$. For any $f \in C_c^\infty(U^{n+1})$, let $\widehat{f}$ be the normalized Fourier transform such that
\[
f(x_0, \ldots, x_n) = \int_{\mathbb{R}^{n+1}} \widehat{f}(\xi) e^{ix_\xi}d\xi,
\]
where $x = (x_0, \ldots, x_n)$ and $\xi = (\xi_0, \ldots, \xi_n)$.

**Definition 2.1** (Schwartz functional calculus). Let us fix a self-adjoint $h \in \mathcal{A}$. There exists an linear map
\[
\mathcal{S}_h^{(n)} : C_c^\infty(U^{n+1}) \to L(\mathcal{A}^{\otimes n+1}, \mathcal{A}),
\]
given by
\[
(\mathcal{S}_h^{(n)}(f) = f(h^{(0)}, \ldots, h^{(n)})) := \int_{\mathbb{R}^{n+1}} \widehat{f}(\xi) e^{i\xi_0}h^{(0)} \cdots e^{i\xi_nh^{(n)}} d\xi,
\]
where the right hand side, as an operator from $\mathcal{A}^{\otimes n} \to \mathcal{A}$, means (cf. (2.1)):
\[
(\mathcal{S}_h^{(n)}(f)(\rho) = f(h^{(0)}, \ldots, h^{(n)})(\rho)) = \int_{\mathbb{R}^{n+1}} \widehat{f}(\xi) e^{i\xi_0}h^{(0)} \cdots e^{i\xi_nh^{(n)}} d\xi,
\]
where $\rho = \rho_1 \otimes \cdots \otimes \rho_n \in \mathcal{A}^{\otimes n}$.

Let us make some remarks on nations. In later exploration, the short form $\mathcal{S}_h^{(n)}(f)$ often take place when the arguments $(h^{(0)}, \ldots, h^{(n)})$ are fixed. Meanwhile, in most of the calculations, we have to manipulate $(h^{(0)}, \ldots, h^{(n)})$, for instance, performing cyclic permutation on them, the full form $f(h^{(0)}, \ldots, h^{(n)})$ is inevitable in order to keep track of the change of variables. We will freely drop the superscript of $\mathcal{S}_h^{(n)}$ whenever the domain $\mathcal{A}^{\otimes n}$ is clear from context, or we can interpret $\mathcal{S}_h = \bigoplus_{n=0}^{\infty} \mathcal{S}_h^{(n)}$ as the direct sum.

Denote by $\mathcal{C}_{\spec}(n) := \mathcal{C}_{\spec}(n, h)$ the collection of all $(n+1)$-variable functions $f(x_0, \ldots, x_n)$ such that the Schwartz functional calculus $\mathcal{S}_h^{(n)}(f) \in L(\mathcal{A}^{\otimes n+1}, \mathcal{A})$ is well-defined. To be more specific, we can consider a subalgebra $C_c^\infty(U^{n+1}_h) \subset \mathcal{C}_{\spec}(n, h)$ of smooth function with compact support in the open set $U^{n+1}_h = U_h \times \cdots \times U_h \subset \mathbb{R}^{n+1}$, and $U_h \subset \mathbb{R}$ is some open subset containing the spectrum of $h \in \mathcal{A}$. Such smooth spectral functions are sufficient for many applications, especially for the heat coefficients computation on noncommutative tori. Thanks to the nuclear property of the smooth Fréchet topology, there exists a unique completion $\hat{\otimes}$ of the algebraic tensor $\otimes$ so that $C_c^\infty(U_h \times \cdots \times U_h) \cong C_c^\infty(U_h)^{\hat{\otimes}(n+1)}$. In more elementary terms, the algebraic tensors in $C_c^\infty(U_h)^{\otimes(n+1)}$, which is a dense subset, correspond to function of separating variables on $U^{n+1}_h$:
\[
f_0 \otimes \cdots \otimes f_n \mapsto (x_0, \ldots, x_n) \mapsto f_0(x_0) \cdots f_n(x_n) \in C_c^\infty(U^{n+1}_h).
\]
We will come back to this class of functions in §5.
In the context of conformal geometry on noncommutative manifolds mentioned at the introduction, the self-adjoint $h$ represents a log-conformal factor while the $\rho_i$’s are the derivatives of $h$. The notation $f(h^{(0)}, \ldots, h^{(n)})(\rho)$ in (3.1) implements a rearrangement that moves all the zero-order derivatives to the left. Such rearrangement process can be indeed constructed for a much larger class of functions than the Schwartz ones. The theory of multiple operator integrals [13, 24] explores the generality of the function $f$ and the operator $h$ for which the integral in (3.4) is well-defined. Connection with cyclic theory has also been made [21] in the study of spectral actions.

3. Variational Calculus for Rearrangement Operators

This section is a continuation of the systematic approach for the standard differential calculus with the appearance of rearrangement operators initiated in [13], where only spectral functions of one and two variable are considered. We start with recalling basis notations on divided difference, whose crucial role in the our context of variational calculus was first pointed out in [12].

3.1. Divided Differences. For a one-variable function $f(z)$, the $n$-divided difference $\nabla^n(f)$ has $(n+1)$-arguments which is inductively defined as follows:

$$\nabla^0(f)(x_0) := f(x_0)$$

$$\nabla^n(f)(x_0, \ldots, x_n) := (\nabla^{n-1}(f)(x_0, \ldots, x_{n-1}) - \nabla^{n-1}(f)(x_1, \ldots, x_n))/(x_0 - x_n).$$

The first divided difference will be frequently used later:

$$\nabla(f)(x_0, x_1) = (f(x_0) - f(x_1))/(x_0 - x_1).$$

One can prove, by induction, the explicit formula:

$$\nabla^n(f)(x_0, x_1, \ldots, x_n) = \sum_{l=0}^{n} f(x_l) \prod_{s=0, s \neq l}^{n} (x_l - x_s)^{-1}. \tag{3.1}$$

It is also quite convenient to use squared brackets to enclose new variables generated by iterated divided differences and then skip $\nabla^n$:

$$[x_0, \ldots, x_n] := \nabla^n(f)(x_0, x_1, \ldots, x_n). \tag{3.2}$$

The following basic properties will be frequently used:

- **Leibniz rule:**
  $$f g[x_0, \ldots, x_n] = f(x_0)g[x_0, \ldots, x_n] + f[x_0, \ldots, x_n]g(x_n). \tag{3.3}$$

- **Composition rule:**
  $$f[y_0, \ldots, y_q, z][x_0, \ldots, x_p]_z = f[y_0, \ldots, y_q, x_1, \ldots, x_p]. \tag{3.4}$$

- **The confluent case:** suppose there are $\alpha + 1$ copies of $x$ in the arguments of the divided difference, then:
  $$f[y, x, \ldots, x] = \frac{1}{\alpha!} \partial_x^\alpha f[y, x]. \tag{3.5}$$

- $f[x_0, x_1, \ldots, x_n]$ is symmetric in all the arguments.

When multivariable functions are concerned, there are several intuitive notations to indicate the variable on which the divided difference acts, for instance:

i) using a subscript $f(x_0, \ldots, z_i, \ldots, x_n)|_{x_i}$,

ii) using a placeholder $f(x_0, \ldots, \bullet, \ldots, x_n)|_{x_i}$,

iii) directly enclosing the variables $f(x_0, \ldots, [x_i, x_{i+1}], \ldots, x_n)$ when there is no confusion on the position.
Much like partial derivatives, given a multiindex \( \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1} \) and a function \( f(x_0, \ldots, x_n) \), one can form \( \nabla^\alpha(f) \), which is the function obtained by applying the divided difference \( \alpha_j \)-times on \( j \)-th argument of \( f \) where \( 0 \leq j \leq n \). In accordance with \( \eqref{eq:1} \), evaluation of \( \nabla^{(\alpha_0, \ldots, \alpha_n)}(f) \) reads

\[
(3.6) \quad f \left( [x_{0,0}, \ldots, x_{0,\alpha_0}], \ldots, [x_{n,0}, \ldots, x_{n,\alpha_n}] \right).
\]

For instance, we can consider the divided difference of \( f \) at the \( j \)-th argument:

\[
\nabla_j(f) := \nabla^{(0, \ldots, 1, \ldots, 0)}(f), \text{ with } j = 0, \ldots, n, \text{ which is part of the face maps defined } \delta^{(j,n)} \text{ in Eq. (3.15)}:
\]

\[
\delta^{(j,n)}(f)(x_0, \ldots, x_{n+1}) := f(x_0, \ldots, [x_j, x_{j+1}, x_{j+2}, \ldots, x_{n+1})).
\]

Note that, beside \( \nabla_j(f) \), there is a shift of position of the arguments after index \( j \): that is \( x_{j+2} \) is at the \((j + 1)\)-th slot, \( \ldots, x_{n+1} \) is at the \( n \)-th slot. In fact, such shift can be implemented by the \( j \)-th face map \( \delta^{(n)}: [n] \to [n + 1] \) in the simplicial category \( \Delta \), cf. Appendix A:

\[
\delta^{(n)}: 0 \mapsto 0, \ldots, j-1 \mapsto j-1, \ j \mapsto j+1, \ldots, n \mapsto n+1.
\]

One expects the corresponding simplicial relations:

**Proposition 3.1.** The operators \( \delta^{(j,n)} : \mathcal{C}_{\text{Spec}}(n) \to \mathcal{C}_{\text{Spec}}(n + 1) \), with \( j = 0, \ldots, n \), satisfy

\[
(3.7) \quad \delta^{(j,n+1)} \delta^{(i,n)} = \delta^{(i,n+1)} \delta^{(j-1,n)}, \quad \text{for } i < j.
\]

**Proof.** Start with the left hand side \( \delta^{(i,n)}(f) \):

\[
f(x_0, \ldots, x_n) \xrightarrow{\delta^{(i,n)}} f(x_0, \ldots, [x_i, x_{i+1}], \ldots, x_{n+1}),
\]

to further apply \( \delta^{(i,n+1)} \), since \( j > i \), the divided difference occurs on \( \{x_{i+1}, \ldots, x_{n+1}\} \), thus

\[
f(x_0, \ldots, [x_i, x_{i+1}], \ldots, x_{n+1}) \xrightarrow{\delta^{(i,n+1)}} \begin{cases} f(x_0, \ldots, [x_i, x_{i+1}, x_{i+2}], \ldots, x_{n+1}) & \text{for } j = i + 1 \\ f(x_0, \ldots, [x_i, x_{i+1}], \ldots, [x_j, x_{j+1}], \ldots, x_{n+1}) & \text{for } j > i + 1. \end{cases}
\]

On the other side,

\[
f(x_0, \ldots, x_n) \xrightarrow{\delta^{(i-1,n)}} f(x_0, \ldots, [x_{j-1}, x_j], \ldots, x_{n+1}).
\]

In the next step, the divided difference is taken at the \( i \)-th argument, which is before \( j - 1 \):

\[
0 \leq i \leq j - 1,
\]

\[
f(x_0, \ldots, [x_{j-1}, x_j], \ldots, x_{n+1}) \xrightarrow{\delta^{(i,n+1)}} \begin{cases} f(x_0, \ldots, [x_i, x_{i+1}, x_{i+2}], \ldots, x_{n+1}) & \text{for } i = j - 1 \\ f(x_0, \ldots, [x_i, x_{i+1}], \ldots, [x_j, x_{j+1}], \ldots, x_{n+1}) & \text{for } i < j - 1. \end{cases}
\]

\[\square\]

\(^5\)the order does not matter
3.2. Degeneracy Maps $\sigma^{(j,n)}$ and the Simplicial Relations. There are obvious reductions for $\mathcal{S}_h(f) (\rho_1 \otimes \cdots \otimes \rho_n)$ when some of the factors are equal to 1, or at least, they commute with $h$ and other $\rho$’s. Such redundancy of rearrangement can be removed by the degeneracy transformations on $f$ that reduces the number of its arguments by one.

Lemma 3.2. When $\rho_{j+1} = 1$ in $\rho_1 \otimes \cdots \otimes \rho_n$, where $j = 0, \ldots, n-1$, we have

$$\mathcal{S}_h(f) \left( \rho_1 \otimes \cdots \otimes \rho_n |_{\rho_{j+1}=1} \right) = \mathcal{S}_h \left( \sigma^{(j,n)(f)} \right) (\rho_1 \otimes \cdots \otimes \rho_{j+1} \otimes \cdots \otimes \rho_n),$$

where $\tilde{\bullet}$ means $\rho_{j+1}$ is removed and $\sigma^{(j,n)} : \text{Spec} (n) \to \text{Spec} (n-1)$ is the restriction map onto the hyperplane $\{(x_0, \ldots, x_n) : x_j = x_{j+1}\} \subset \mathbb{R}^{n+1}$:

$$\sigma^{(j,n)}(f)(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_j, x_{j+1}, \ldots, x_{n-1}).$$

Proof. According to Eq. (3.8) with $\rho_{j+1} = 1$,

$$f \left( h^{(0)}, \ldots, h^{(n)} \right) (\rho_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes \rho_n)$$

$$= \int_{\mathbb{R}^{n+1}} \tilde{f} (\xi) e^{\xi h} \rho_1 e^{\xi_1 h} \cdots \rho_{j+1} e^{\xi_{j+1} h} \cdot 1 \cdot e^{\xi_{j+1} h} \rho_{j+1} \cdots \rho_n e^{\xi_n h}$$

$$= \int_{\mathbb{R}^{n+1}} \tilde{f} (\xi) e^{\xi h(0)} \cdots e^{\xi h(j)} e^{\xi_{j+1} h(j+1)} \cdots e^{\xi_n h(n-1)} (\rho_1 \otimes \cdots \otimes \rho_{j+1} \otimes \cdots \otimes \rho_n)$$

$$= \int \tilde{f} (h^{(0)}, \ldots, h^{(j)}, h^{(j+1)}, \ldots, h^{(n-1)}) (\rho_1 \otimes \cdots \otimes \rho_{j+1} \otimes \cdots \otimes \rho_n).$$

The desired transformation of $\sigma^{(j,n)}(f)$ in Eq. (3.8) is taken to be the underlying spectral function shown in the last line above. □

Proposition 3.3. In addition to the relations in Proposition 3.1, we have

$$\sigma_j^{(n-1)} \sigma_i^{(n)} = \sigma_j^{(n-1)} \sigma_{j+1}^{(n)}, \text{ for } i < j.$$  

(3.9)

Proof. Observe that the right hand side of Eq. (3.8) is the “dual” of the degeneracy maps $\sigma_j^{(n)} : [n] \to [n-1]$ which hits $j$ twice in the simplicial category $\Delta$:

$$\sigma_j^{(n)} : 0 \mapsto 0, \ldots, j \mapsto j, j+1 \mapsto j, j+2 \mapsto j+1, \ldots, n \mapsto n-1.$$  

In more detail, similar to the computation in Proposition 3.1 we can rewrite Eq. (3.8) as:

$$f(x_0, \ldots, x_n) \xrightarrow{\sigma_i^{(n)}} f \left( x_{\sigma_j^{(n)}(0)}, \ldots, x_{\sigma_j^{(n)}(n)} \right).$$

Consequently, composition of $\sigma_j^{(n-1)} \sigma_i^{(n)}$ becomes the composition of $\sigma_j^{(n+1)} \sigma_i^{(n)}$ on the subscripts of the variables $x_j$:

$$f(x_0, \ldots, x_n) \xrightarrow{\sigma_j^{(n+1)} \sigma_i^{(n)}} f \left( x_{\sigma_j^{(n+1)} \sigma_i^{(n)}(0)}, \ldots, x_{\sigma_j^{(n+1)} \sigma_i^{(n)}(n)} \right).$$

Similar result holds for $\sigma_i^{(n-1)} \sigma_j^{(n)}$. Therefore Eq. (3.9) follows from the relations $\sigma_j^{(n+1)} \sigma_i^{(n)} = \sigma_i^{(n+1)} \sigma_j^{(n)}$ in the simplicial category $\Delta$. □
where $\tilde{\delta}$ is given by applying divided difference at $1$. Hence

$$f(h) + \sum_{n=1}^{\infty} \mathcal{A}_h(\mathbf{t}^n)(b \otimes \cdots \otimes b),$$

where $b \otimes \cdots \otimes b = b^{\otimes n}$.

**Proof.** We refer to [12, Prop. 3.7].

Let $\nabla : \mathcal{A} \to \mathcal{A}$ be a derivation and $\alpha_t : \mathcal{A} \to \mathcal{A}$, $t \in \mathbb{R}$, be the associated one-parameter group of automorphisms. Namely, $\nabla a = \frac{d}{dt}_{t=0} \alpha_t(a)$, for any $a \in \mathcal{A}$ whenever the derivative exists. They can be lifted to the tensor product $\mathcal{A}^{\otimes n}$ and to the functional calculus in the usual way:

$$\nabla (\mathbf{p}_1 \otimes \cdots \otimes \mathbf{p}_n) = \left. \frac{d}{dt}_{t=0} \alpha_t(\mathbf{p}_1) \otimes \cdots \otimes \alpha_t(\mathbf{p}_n) \right|_{t=0},$$

$$\nabla (\mathcal{A}_h(f)) = \frac{d}{dt}_{t=0} \mathcal{A}_h(\mathbf{t})(\mathcal{A}_h(f)).$$

For smooth enough $h$ (w.r.t. $\nabla$), we have $\alpha_t(h) = h + b(t)$ where $b(t) \sim_{t \to 0} \nabla(h)t + \nabla^2(h)t^2/2 + \cdots$. The expansion in Eq. (3.10) implies:

$$\mathcal{A}_\alpha(h)(f) = f(h) + t\mathcal{A}_h(\mathbf{t})(\nabla(h)) + O(t^2).$$

Hence $\nabla (\mathcal{A}_\alpha(h)(f)) = \mathcal{A}_h(\mathbf{t})(\nabla(h))$. The simplest example is the case $f(x) = x^2$, then $\mathbf{t}(f)(x_0, x_1) = x_0 + x_1$, which agrees with the result from the Leibniz rule: $\nabla(h^2) = h\nabla(h) + \nabla(h)h = (h^{(0)} + h^{(1)}) (\nabla(h)).$

Now, let us replace the left or the right multiplication, that is $h^{(0)}$ or $h^{(1)}$, similar argument gives: for any $\rho \in \mathcal{A}$,

$$\nabla \left( \mathcal{A}_h(\mathbf{f})(\rho) \right) = \mathcal{A}_h(\mathbf{f})(\nabla(h) \rho), \quad \nabla \left( \mathcal{A}_h(\mathbf{f})(\rho) \right) = \mathcal{A}_h(\mathbf{f})(\rho \otimes \nabla(h)),$$

where $\mathbf{f}(x_0, x_1, x_2) = \mathbf{t}(f)(x_0, x_1)$ and $\mathbf{f}(x_0, x_1, x_2) = \mathbf{t}(f)(x_1, x_2)$. The variation of $\mathcal{A}_h(f)$ for general $(n+1)$-variable spectral functions $f(x_0, \ldots, x_n)$ can be computed following the Leibniz rule, that is, we can differentiate $\alpha_t(h)^{(j)}$ one by one from $j = 0$ to $n$. There is also a shift of indices needed to be taken care of as shown in the case of $\mathbf{f}$ define as above. We summarize the result in a proposition below.

**Proposition 3.5.** For a fixed self-adjoint $h \in \mathcal{A}$ and let $f \in \mathcal{C}_\text{Spec}(n)$ be a spectral function with $(n+1)$ arguments. The corresponding derivation $\nabla (\mathcal{A}_h(f))$ defined in Eq. (3.4) is given by

$$\nabla (\mathcal{A}_h(f)) = \sum_{j=0}^{n} \mathcal{A}_h (\mathbf{d}^{(j)}(f)) \circ \nabla(h)_{(j,n)},$$

where $\nabla(h)_{(j,n)} : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n+1}$ is the operator of inserting $\nabla(h)$ at the $j$-th slot, see Eq. (3.11). The transformations $\mathbf{d}^{(j,n)}(f) : \mathcal{C}_\text{Spec}(n) \to \mathcal{C}_\text{Spec}(n+1)$ is given by applying divided difference at

$^6$Note that we put $(j,n)$ in subscript to distinguish from the notation $\nabla(h)^{(j,n)}$ in Eq. (2.14), whose target space is $\mathcal{A}$. 
the $j$-th argument with a shift of indices after $j$: $x_j \to x_{j+1}$ for $j + 1 \leq l \leq n$,
\begin{equation}
\delta^{(j,n)}(f)(x_0, \ldots, x_{n+1}) = f(x_0, \ldots, \bullet, x_{j+2}, \ldots, x_{n+1}) |_{x_j, x_{j+1}},
\end{equation}
where $\bullet$ is at the $j$-th slot, $0 \leq j \leq n$.

**Proof.** Eq. (3.14) is an extension of (3.13) from single to multivariate calculus. In fact, it is obtained by differentiating in $t$ trough the multivariable function $f(\alpha_1(t^{(0)}), \ldots, \alpha_l(t^{(n)}))$ in (3.12), while the computation of the partial derivatives $(\partial_{x_j} f)(dx_j/dt)$ can be reduced in the single variable scenario in (3.15). \hfill $\square$

Some remarks on notations:

- The meaning of (3.14) is better explained by examples. For $f = f(x_0, x_1, x_2)$, the evaluation of the r.h.s of (3.14) at $\rho_1 \otimes \rho_2 \in \mathcal{A}^{\otimes 2}$ is given by:
\begin{equation}
\nabla (\mathcal{H}_h(f)) (\rho_1 \otimes \rho_2) = \mathcal{H}_h(\delta^{(0,2)}(f)) (\nabla (h) \otimes \rho_1 \otimes \rho_2) + \mathcal{H}_h(\delta^{(1,2)}(f)) (\rho_1 \otimes \nabla (h) \otimes \rho_2) + \mathcal{H}_h(\delta^{(2,2)}(f)) (\rho_1 \otimes \rho_2 \otimes \nabla (h)).
\end{equation}

- One can use another notation mentioned in § 3.1 for the divided difference in (3.15):
\begin{equation}
f (x_0, \ldots, [x_j, x_{j+1}], x_{j+2}, \ldots, x_{n+1}).
\end{equation}

Note that the $(n+1)$-tuple $(x_0, \ldots, [x_j, x_{j+1}], x_{j+2}, \ldots, x_{n+1})$ describes a map $\psi$ from $[n+1] = \{0, \ldots, n+1\}$ to $[n] = \{0, \ldots, n\}$ (the $n$-slots) by assigning the $(n+2)$ variables labeled by $x$ to their positions in the $(n+1)$-tuple. It is a non-decreasing map hitting $j \in [n]$ twice, that is the pre-image of the stationary point $\psi^{-1}(j) = \{x_j, x_{j+1}\}$ is enclosed in the squared brackets. With regards to the notations in Appendix A, $\psi$ is nothing but the degeneracy map $\sigma_j^{[n+1]} : [n+1] \to [n]$ in the simplicial category $\Delta$. This observation is useful when computing the iterations of the face maps $\delta^{(j,n)}$, the results will be of the form akin to (3.17):
\begin{equation}
f (\ldots, [\bullet, \ldots, \bullet], \ldots, [\bullet, \ldots, \bullet], \ldots)
\end{equation}

that gives rise to a $\psi : [m] \to [n]$ in $\Delta$, where $m, n$ are the number of variables and slots respectively. The squared brackets $[\ldots]$ enclose the stationary points of $\psi$ and indicate iterated divided difference on the function $f$.

### 3.4. Compatibility between $\delta^{(j,n)}$ and $\sigma^{(j,n)}$

Unfortunately, we do not have the full simplicial structure, which has something to do with the basic fact in calculus that differentiation and evaluation do not commute. Proposition 3.4 consists of relations that are taken from cyclic theory while Proposition 3.7 states the differences.

**Proposition 3.6.** We have
\begin{equation}
\sigma_j \delta_i = \begin{cases} 
\delta_i \sigma_{j-1} & \text{for } i < j - 1, \\
\delta_i \sigma_j & \text{for } i > j + 1.
\end{cases}
\end{equation}

**Proof.** We prove the first one $\sigma_j \delta_i = \delta_i \sigma_{j-1}$ and leave the other to the reader since the calculations are quite similar. For $f = f(x_0, \ldots, x_n) \in \mathcal{C}_{\text{Spec}}(n)$,
\begin{equation}
f \xrightarrow{\delta_j^{(n)}} f (x_0, \ldots, [x_i, x_{i+1}], \ldots, x_{n+1}).
\end{equation}

With $j > i + 1$, the result of $\sigma_j^{(n+1)} \delta_i^{(n)}(f)$ is given by:
\begin{equation}
f (x_0, \ldots, [x_i, x_{i+1}], \ldots, x_{n+1}) \xrightarrow{\sigma_j^{(n+1)}} f (x_0, \ldots, [x_i, x_{i+1}], \ldots, x_j, x_j, \ldots, x_n).
\end{equation}
divided difference is given by:
\[
\sigma^{(n)} = \frac{\partial}{\partial x_i} \sigma^{(n-1)} = \sigma^{(n+1)}(x_0, \ldots, x_i, x_j, \ldots, x_n).
\]

For the second step, since the position of \([x_i, x_i+1]\) is before \(j-1\), there is a shift on the subscript of variables \(x_i\) for all \(l > i + 1\) which gives rise to \((x_j, x_j)\). The two sides indeed agree. \(\square\)

The uncovered cases in Eq. (3.18) are \(\sigma_j \delta_j\) and \(\delta_j \sigma_j\) (set \(i = j - 1\) or \(i = j + 1\) for the right hand side of Eq. (3.18)).

**Proposition 3.7.** We have
\[
\sigma^{(i,n+1)} \delta^{(i,n)} = \partial_{x_i},
\]
(3.19)
\[
\delta^{(i,n+1)} \sigma^{(i,n)} = \sigma^{(i+1,n+1)} \delta^{(i,n)} + \sigma^{(i+1,n+1)} \delta^{(i+1,n)},
\]
(3.20)
where \(\partial_{x_i}\) is the partial derivative acting on \(f(x_0, \ldots, x_n) \in C_{\text{Spec}}(n)\).

**Proof.** Eq. (3.19) follows from the confluent version of divided difference (cf. Eq. (3.5)):
\[
\sigma^{(i,n+1)} \delta^{(i,n)}(f)(x_0, \ldots, x_n) = \delta^{(i,n)}(f)(x_0, \ldots, x_i, x_i, \ldots, x_n)
= f(x_0, \ldots, x_i, [y, y], x_i, \ldots, x_n)
= (\partial_{x_i} f)(x_0, \ldots, x_n).
\]

To prove Eq. (3.20), let us start with the left hand side
\[
f(x_0, \ldots, x_{n+1}) \sigma^{(i,n+1)} \delta^{(i,n)} \rightarrow f(x_0, \ldots, x_i, x_i, \ldots, x_n).
\]
To continue, we apply Eq. (3.21) to compute the divided difference of the function \(x_i \mapsto f(\ldots, x_i, x_i)\) as above,
\[
f(x_0, \ldots, x_i, x_i, \ldots, x_n) \delta^{(i,n)} \rightarrow f(x_0, \ldots, x_i, [x_i, x_{i+1}], \ldots, x_n) + f(x_0, \ldots, [x_i, x_{i+1}], x_{i+1}, \ldots, x_n),
\]
where the two terms are equal to \(\sigma^{(i+1,n+1)} \delta^{(i+1,n)}(f)\) and \(\sigma^{(i+1,n+1)} \delta^{(i,n)}(f)\) respectively. Let us compute \(\sigma^{(i+1,n+1)} \delta^{(i+1,n)}(f)\) and leave the verification of the second one to the reader. Indeed,
\[
f(x_0, \ldots, x_i, [x_{i+1}, x_{i+2}], x_{i+3}, \ldots, x_{n+1}) \delta^{(i+1,n+1)} \rightarrow f(x_0, \ldots, x_i, [x_{i+1}, x_{i+2}], x_{i+3} \ldots, x_{n+1}).
\]
\(\square\)

**Lemma 3.8.** Consider the function \(z \mapsto f(z, z)\) induced from a two-variable function \(f\), its divided difference is given by:
\[
f(z, z)[x, y] = f(x, z)[x, y] + f(z, y)[x, y] = f(y, z)[x, y] + f(z, x)[x, y].
\]
(3.21)

**Proof.** The computation is straightforward:
\[
f(x, z)[x, y] + f(z, y)[x, y] = f(x, x) - f(x, y) + f(x, y) - f(y, y)
= f(x, x) - f(y, y) = f(z, z)[x, y].
\]
\(\square\)
3.5. Tracial Functionals and the Cyclic Operators $\tau_{(n)}$. After the discussion of differentiation, let us now further assume that the algebra $\mathcal{A}$ admits a tracial functional $\varphi_0 : \mathcal{A} \to \mathbb{C}$ playing the role of integration. The trace property has its usual meaning: $\varphi_0(\rho \rho') = \varphi_0(\rho')$, $\forall \rho, \rho' \in \mathcal{A}$. In particular, when applying $\varphi_0$ to the integrand of r.h.s of Eq. (2.4), we see that

$$\varphi_0(e^{\xi_0 h} p_1 e^{\xi_1 h} p_2 e^{\xi_2 h}) = \varphi_0(e^{(\xi_0 + \xi_2) h} p_1 e^{\xi_1 h} p_2),$$

which leads to another reduction $\sigma^{(n,n)} : \mathcal{C}_{\text{Spec}}(n) \rightarrow \mathcal{C}_{\text{Spec}}(n - 1)$:

$$\sigma^{(n,n)}(f)(x_0, \ldots, x_{n-1}) = f(x_0, x_1, \ldots, x_{n-1}, x_0).$$

Lemma 3.9. The extra degeneracy $\sigma^{(n,n)}$ defined above Eq. (3.22) is responsible for

$$\varphi_0 \left( \mathcal{I}_h (\varphi) (\rho_1 \otimes \cdots \otimes \rho_n) \right) = \varphi_0 \left( \mathcal{I}_h (\sigma^{(n,n)}(\varphi)) (\rho_1 \otimes \cdots \otimes \rho_{n-1}) \cdot \rho_n \right).$$

Remark. We shall see in Eq. (3.23) that $\sigma^{(n,n)} = \sigma^{(0,n)} \tau^{-1}_{(n-1)}$, which corresponds to the extra degeneracy Eq. (A.14) in the cyclic category $\Delta C$.

Proof. We first apply $\varphi_0$ to the right hand side of Eq. (2.4) and then use the trace property to move $e^{\xi_i h}$ to the very left:

$$\varphi_0 \left( f \left( h^{(0)}, \ldots, h^{(n)} \right) (\rho_1 \otimes \cdots \otimes \rho_n) \right) = \varphi_0 \left( \int_{\mathbb{R}^{n+1}} \hat{f}(\xi) e^{\xi_0 h} p_1 e^{\xi_1 h} \cdots e^{\xi_{n+1} h} \right)
= \varphi_0 \left( \int_{\mathbb{R}^{n+1}} \hat{f}(\xi) e^{\xi_0 h} e^{\xi_1 h} p_1 e^{\xi_2 h} \cdots e^{\xi_{n+1} h} \cdot \rho_n \right)
= \varphi_0 \left( f \left( h^{(0)}, \ldots, h^{(n-1)}, h^{(0)} \right) (\rho_1 \otimes \cdots \otimes \rho_{n-1}) \cdot \rho_n \right),$$

where the Fourier transform integral in the middle line is exactly the function in the right hand side of Eq. (3.22).

The trace property of the functional $\varphi_0$ allows cyclic permutations on the $\rho$-factors appeared in the local expression Eq. (3.23).

Proposition 3.10. Given $f(x_0, \ldots, x_n) \in \mathcal{C}_{\text{Spec}}(n)$ and $(n + 1)$ elements $\rho_1, \ldots, \rho_{n+1} \in \mathcal{A}$, we have the following cyclic permutation:

$$\varphi_0 \left( \mathcal{I}_h (f) (\rho_1 \otimes \cdots \otimes \rho_{n+1}) \cdot \rho_{n+1} \right) = \varphi_0 \left( \mathcal{I}_h \left( \tau_{(n)}(f) \right) (\rho_2 \otimes \cdots \otimes \rho_{n+1}) \cdot \rho_1 \right),$$

where the transformation $\tau_{(n)} : \mathcal{C}_{\text{Spec}}(n) \rightarrow \mathcal{C}_{\text{Spec}}(n)$ is the corresponding cyclic permutation on arguments of $f$:

$$\tau_{(n)}(f)(x_0, \ldots, x_n) = f(x_n, x_0, \ldots, x_{n-1}).$$

Proof. Again, the key is to look at the following part of the Fourier transform in Eq. (2.4) in which one can cyclic permute the factors of the product inside $\varphi_0$ according to the trace property:

$$\varphi_0 \left( e^{\xi_{h} (0)} \cdots e^{\xi_{h} (n+1)} \cdot e^{\xi_{h} (n)} \right) (\rho_2 \otimes \cdots \otimes \rho_{n+1}) \cdot \rho_1
= \varphi_0 \left( (e^{\xi_{h} (0)} \cdots e^{\xi_{h} (n+1)} \cdot e^{\xi_{h} (n)}) (\rho_2 \otimes \cdots \otimes \rho_{n+1}) \cdot \rho_1 \right),$$

The last line gives rise to the function $f(x_n, x_0, \ldots, x_{n-1})$ after Fourier transform (the coefficient of $\xi_j$ corresponds to the j-th argument of $f$).
3.6. Compatibility between Cyclic and Simplicial Structures.

**Proposition 3.11.** We have the compatibility relations between the cyclic maps and the degeneracy maps:

\[(3.25) \quad \tau_{(n)} \sigma^{(i,n+1)} = \sigma^{(i-1,n+1)} \tau_{(n+1)}, \quad 1 \leq i \leq n.\]

In particular, \(\tau_{(n)} \sigma^{(0,n+1)} = \sigma^{(n,n)} \tau_{(n+1)}\). Similarly, for the face maps

\[(3.26) \quad \tau_{(n)} \delta^{(i,n-1)} = \delta^{(i-1,n-1)} \tau_{(n-1)}, \quad i = 1, \ldots, n.\]

It follows that \(\delta^{(n,n)} = \tau_{(n)} \delta^{(0,n-1)}\).

**Proof.** The computation is elementary, we check Eq. (3.26) as an example and leave Eq. (3.25) to the reader. Begin with the left hand side of Eq. (3.26):

\[
\tau_{(n)} \delta^{(n-1)}(x_0, \ldots, x_n) = \delta^{(n-1)}(x_n, x_0, \ldots, x_{n-1}) = f(x_n, x_0, \ldots, [x_{i-1}, x_i], \ldots, x_{n-1})
\]

and then the right hand side:

\[
\delta^{(n)}_{i-1} \tau_{(n-1)}(f)(x_0, \ldots, x_n) = \tau_{(n-1)}(f)(x_0, \ldots, x_n, x_{i-1}, x_i, \ldots, x_{n-1}) = f(x_n, x_0, \ldots, [x_{i-1}, x_i], \ldots, x_{n-1}).
\]

\[\square\]

4. Main Results

Let us denote by \(\mathcal{C}\) the category with objects \(\mathcal{C}_{\text{Spec}}(n)\), \(n = 0, 1, \ldots\), and morphisms generated by the transformations \(\delta^{(j,n)}, \sigma^{(i,n)}, \tau_{(n)}\) introduced in Proposition 3.5, Lemma 3.2 and Proposition 3.11. The new contributes of the paper begins with the observation that they almost fulfil the generating relations of the cyclic category \(\Delta C\) in Definition A.1. To this end, one needs to add one more set of face maps

\[(4.1) \quad \delta^{(n+1,n)} := \tau^{(n+1)} \delta^{(0,n)} : \mathcal{C}_{\text{Spec}}(n) \rightarrow \mathcal{C}_{\text{Spec}}(n+1)
\]

\[
f(x_0, \ldots, x_n) \mapsto f([x_{n+1}, x_0], x_1, \ldots, x_n).
\]

Unlike the cyclic category \(\Delta C\), the morphism set of \(\mathcal{C}\) contains all partial derivatives, due to the failure of the compatibility between the simplicial and co-simplicial structures. This feature has been predicted in, for instance, the \(a_4\)-term computation \([2] \S 4\) and deserves further investigation.

Strictly speaking, (the objects of) \(\mathcal{C}\) depends on the choice of the underlying algebra \(\mathcal{A}\) and the self-adjoint \(h \in \mathcal{A}\). Nevertheless, the relations presented in below are universal.

**Theorem 4.1.** With slight simplification on notations, such as \(\delta^{(j,n)} \mapsto \delta_j\), the generators of the category \(\mathcal{C}\) defined above fulfil the simplicial and co-simplicial relations:

\[(4.2) \quad \delta_j \delta_i = \delta_i \delta_{j-1}, \quad \text{for } i < j\]

\[(4.3) \quad \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad \text{for } i \leq j\]
with a compatible cyclic structure:

\[ \tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \quad \tau_n \delta_i = \delta_{i-1} \tau_{n-1}. \]

But, the compatibility among simplicial structures is modified in the following way:

\[ \sigma_j \delta_i = \begin{cases} \delta \sigma_{j-1} & \text{for } i < j - 1, \\ \delta \sigma_{i-1} - \delta \sigma_{i+1} & \text{for } i = j - 1, \\ \partial x_i & \text{for } i = j, \\ \delta_{i-1} \sigma_j & \text{for } i > j + 1, \end{cases} \]

where \( \partial x_i \) is the partial derivative acting on functions \( f(x_0, \ldots, x_n) \in \mathcal{C}_{\text{Spec}}(n) \).

Remark. Note that Eq. (4.5) implies that \( \sigma_j \delta_i \) which fully agrees with (4.6).

Proof. The verification is spread out among Proposition 3.1, Proposition 3.3, Proposition 3.6, Proposition 3.7, and Proposition 3.11. Concerning the new face maps in Eq. (4.1), which are not covered in the aforementioned propositions, they are defined in such way (according to Eq. (A.4)) that Eq. (4.3) and Eq. (4.4) extend automatically.

Corollary 4.2. With the new set of generators \( \left\{ \delta_j^{(n)} \right\}_{n=0}^{\infty} \), the category \( \mathcal{C} \) almost form a \( \Delta \mathcal{C}^{\text{op}} \)-module (cf. Definition 4.3):

1. Pre-simplicial structures:

\[ \frac{d}{d_j} \frac{d}{d_j} = \frac{d}{d_{j-1}} \frac{d}{d_j}, \quad \text{for } i < j, \]
\[ \partial \delta_j = \frac{d}{d_{j+1}} \delta_j, \quad \text{for } i \leq j. \]

2. Modified compatibility between face and degeneracy maps:

\[ \frac{d}{d_j} \frac{d}{d_j} = \begin{cases} \frac{d}{d_{j-1}} \frac{d}{d_j}, & i < j, \\ \frac{d}{d_{j-1}} \frac{d}{d_{j+1}} \frac{d}{d_{j-1}} - \frac{d}{d_{j+1}} \frac{d}{d_{j+1}}, & i = j, \\ \partial \delta_{j+1} & i = j + 1, \\ \frac{d}{d_{j+1}} \frac{d}{d_{j+1}} \frac{d}{d_{j+1}} - \frac{d}{d_{j+1}} \frac{d}{d_{j+1}} \frac{d}{d_{j+1}}, & i = j + 2, \\ \frac{d}{d_{j+1}} \frac{d}{d_{j+1}} & i > j + 2, \end{cases} \]

where \( \partial x_i \) is the partial derivative on the \( i \)-th argument.
(3) Cyclic relations: \( t^{n+1}_j = \text{id}_n \) and
\[
\partial_j \omega(n) = L^{(n-1)}_j \cdot \omega(n-1), \quad \delta \omega(n) = L^{(n+1)}_j \cdot \omega(n+1),
\]
where \( 1 \leq i \leq n \). It follows that
\[
\delta^j \omega(n) = \partial^j_n, \quad \partial^j \omega(n) = L^{(n+1)}_j \cdot \omega(n).
\]

5. Comparison with Hopf Cyclic Theory

Let us keep the notations introduced at the end of § 2 and denote \( \mathcal{O} = C^\infty_c(U_h) \). We will, in this section, replace the objects \( C_{\text{Spec}}(n, h) \) of \( C \) by the algebraic tensor product \( \mathcal{O}^{\otimes(n+1)} \) and then reproduce the simplicial and cyclic relations stated in Theorem 4.1. It turns out that the generators of \( C \), when applied to functions of separating variables: \( f(x_0, \ldots, x_{n+1}) = f_0(x_0) \cdots f_{n+1}(x_{n+1}) \in \mathcal{O}^{\otimes(n+1)} \), can be rewritten in a more acquainted form which is in reminiscent of the counterpart in Hopf cyclic theory.

Let us begin with the degeneracy and the cyclic maps in (3.8) and (3.24). It is not difficult to see that they become
\[
(5.1) \quad \sigma^{(n+1)}_j : \mathcal{O}^{\otimes(n+1)} \to \mathcal{O}^{\otimes n}, \quad f_0 \otimes \cdots \otimes f_{n+1} \mapsto f_0 \otimes \cdots \otimes f_j f_{j+1} \otimes \cdots \otimes f_{n+1},
\]
where \( j = 0, \ldots, n \) and
\[
(5.2) \quad \tau(n) : \mathcal{O}^{\otimes(n)} \to \mathcal{O}^{\otimes n}, \quad f_0 \otimes \cdots \otimes f_n \mapsto f_n \otimes f_0 \otimes \cdots \otimes f_{n-1}.
\]

which are part of the generators of the cyclic module of the algebra \( \mathcal{O} \), cf. [17, §1.1, §2.1]. In particular, the relations in (3.9) and (3.25) follows immediately from classical results.

Next, let us reexamine the relations in (3.7) and (3.26). The compatibility means the coproduct \( \delta \) is not compatible with the pointwise multiplication of \( \mathcal{O} \).

\[\text{[Footnote]}\text{The compatibility means the coproduct } \triangle \text{ is an algebra homomorphism, that is } \triangle(f_1 f_2) = \triangle(f_1) \triangle(f_2).\]
Proposition 5.1. With regards to the new definitions of the face and the degeneracy maps given in (5.5) and (5.1), the relations stated in Proposition 3.6 and Proposition 3.7 still hold true.

Proof. One can see clearly in the proof of Proposition 3.6 that the multiplication \( f_i \otimes f_j \mapsto f_i f_j \) and the divided difference playing the role of coproduct do not interact with each other. Therefore the relations (3.17) belong to the cyclic theory, and thus as before, arguments in Hopf cyclic theory shall work without much modification. We leave the details to the reader.

Let us check Proposition 3.7 using the new notations. Start with (3.19), the left hand side

\[
\delta^{(i,n+1)} \sigma^{(i,n)}
\]

and then apply the Leibniz rule in (3.3), the new form in terms of the Sweedler notation looks like:

\[
\nabla (f_1 f_2) = (f_1)_{(1)} \otimes (f_1)_{(2)} f_2 + f_1 (f_2)_{(1)} \otimes (f_2)_{(2)}.
\]

Now we are ready to check (3.20). The two term on the right hand side are given by

\[
\sigma^{(i+1,n)} \rightarrow f_0 \otimes \cdots \otimes f_n \quad \text{and similarly}
\]

\[
f_0 \otimes \cdots \otimes f_{n+1} \otimes f_1 (f_{i+1})_{(1)} \otimes (f_{i+1})_{(2)} \otimes \cdots f_n.
\]

While, for the left hand side, we have

\[
= f_0 \otimes \cdots \otimes (f_1)_{(1)} \otimes (f_1)_{(2)} f_{i+1} \otimes \cdots f_n + f_0 \otimes \cdots \otimes f_i \otimes (f_{i+1})_{(1)} \otimes (f_{i+1})_{(2)} \otimes f_n,
\]

which is exactly the sum of the two terms obtained before. \( \square \)

6. Application to Modular Geometry on Noncommutative (Two) Tori

The generators of \( \mathcal{C} \) are grown out of the explicit computations carried out in the author’s previous works [13, 14, 16], aiming at improving two analytic backbones behind the spectral geometry: the pseudo-differential and the variational calculus.

6.1. Connes-Moscovici Type Functional Relations. Let \( \mathcal{A} = C^\infty(T^2_\theta) \) be the smooth noncommutative two torus generated by two unitary elements \( U \) and \( V \) with the relation \( UV = e^{i\theta}VU \), for some irrational \( \theta \in \mathbb{R} \setminus \mathbb{Q} \). The algebra \( \mathcal{A} \) is \( \mathbb{Z}^2 \)-graded and the components are all one-dimensional spanned by \( U^n V^m \in (\mathcal{A})_{n,m}, (n,m) \in \mathbb{Z}^2 \). More precisely, \( \mathcal{A} \) consists of series

\[
\sum_{(n,m) \in \mathbb{Z}^2} a_{(n,m)} U^n V^m \quad \text{whose coefficients } a_{n,m} \in \mathbb{C} \text{ are of rapidly decay in } (n,m).
\]

Denote by \( \nabla_1 \) and \( \nabla_2 \) the two basic derivations \(^8\) on \( \mathcal{A} \) obtained by differentiating the torus \( T^2 \) associated

\(^8\)The classical notation for the basis derivations \( \delta_1 \) and \( \delta_2 \) has been used in the definition of the cyclic category \( \Delta \mathcal{C} \) in Appendix A
with the $\mathbb{Z}^2$-grading. Another basic ingredient of the differential calculus is the canonical trace

$$\varphi_0 : \mathcal{A} \to \mathbb{C} : \sum_{(n,m) \in \mathbb{Z}^2} a_{(n,m)} U^n V^m \mapsto a_{(0,0)}$$

that simply takes the coefficient of the constant term. We shall consider variational problems on the space of self-adjoint elements $\mathcal{S} = \mathcal{S}(\mathcal{A})$ viewed as the tangent space of a conformal class of metrics on $\mathcal{A}$, cf. [15 §4.2]. For a fixed self-adjoint $h$, denote by $\delta_a$ the variation of $h$ along some other self-adjoint $a = a^* \in \mathcal{A}$:

$$h \mapsto h + \varepsilon a, \quad \delta_a := \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0}.$$

Let $F(h)$ be a functional in $h$, i.e., a function $F : \mathcal{S} \to \mathbb{C}$ on $\mathcal{S}$. The gradient of $F$ at $h$ (associated with $\varphi_0$), $\text{grad}_h F \in \mathcal{A}$ is defined to be the unique element such that

$$\delta_a F(h) = \varphi_0 (\text{grad}_h F a), \quad \forall a \in \mathcal{S}(\mathcal{A}).$$

The following first variation formula was first proved in [6, Thm 4.10]. The underlying co-simplicial and cyclic structure of $\mathcal{C}$ has already been revealed in the version of [15 Thm 2.15, 2.17]. The new observation from (6.3) is role of $\delta_2$ in cyclic theory, cf. (4.1).

**Proposition 6.1.** Consider the following functional

$$F(h) = \varphi_0 (\mathcal{J}_h(T)(\nabla h) \cdot \nabla h),$$

for some $T \in \mathcal{C}_{\text{Spec}}(1)$ and $\nabla = \nabla_j$, $j = 1$ or $2$, is one of the basic derivations. The gradient is of the general form of a second order differential expression purely in terms of $h$:

$$\text{grad}_h F = \mathcal{J}_h(K) (\nabla^2 h) + \mathcal{J}_h(H) (\nabla h \otimes \nabla h),$$

where the coefficients $K \in \mathcal{C}_{\text{Spec}}(1)$ and $H \in \mathcal{C}_{\text{Spec}}(2)$ satisfy

$$(6.2) \quad K = - (1 + \tau_1) (T),$$

$$(6.3) \quad H = (\delta_0 + \delta_1 - \delta_2) (K),$$

where $\delta_j = \delta^{(j,1)}$, $j = 0, 1, 2$ and recall that $\delta_2 = \tau_2 \delta_0$ is the one in (3.11).

**Proof.** By the Leibniz rule, $\delta_a F(h) = P_1 + P_{11}$ consists of two parts. First, the variation on $\nabla h$. Note that two types of differentials commute: $\delta_a (\nabla h) = \nabla (\delta_a h) = \delta_a (\nabla h)$, thus

$$P_1 = \varphi_0 (\mathcal{J}_h(T)(\nabla a) \cdot \nabla h) + \varphi_0 (\mathcal{J}_h(T)(\nabla h) \cdot \nabla a)$$

$$= \varphi_0 \left( \mathcal{J}_h \left( \left(1 + \tau_1 \right) (T) \right) \right) (\nabla h) \cdot (\nabla a),$$

where we have used Proposition 3.10. We continue by applying integration by parts:

$$P_1 = - \varphi_0 \left( \nabla \mathcal{J}_h \left( \left(1 + \tau_1 \right) (T) \right) (\nabla h) \cdot a \right).$$

Let $K = - (1 + \tau_1) (T)$, Proposition 3.5 yields

$$- \nabla \mathcal{J}_h \left( \left(1 + \tau_1 \right) (T) \right) (\nabla h) = \mathcal{J}_h \left( \delta^{(0,1)} + \delta^{(1,1)}(K) \right) (\nabla h \otimes \nabla h) + \mathcal{J}_h (K) (\nabla^2 h).$$

Finally, $P_1$ has been turned into the form of the right hand side of (6.1):

$$P_1 = \varphi_0 \left( \mathcal{J}_h \left( \delta^{(0,1)} + \delta^{(1,1)}(K) \right) (\nabla h \otimes \nabla h) \cdot a \right) + \varphi_0 \left( \mathcal{J}_h(K)(\nabla^2 h) \cdot a \right).$$

For $P_{11}$ concerning only the variation on the rearrangement operator $\mathcal{J}_h(T)$,

$$P_{11} = \varphi_0 \left( \delta_a \right) \mathcal{J}_h(T) (\nabla h) \cdot (\nabla h),$$
we need again Proposition 3.5
\[ \delta_a [\mathcal{J}_h(T)] (\nabla h) = \mathcal{J}_h \left( (0,1) (T) \right) (a \otimes \nabla h) + \mathcal{J}_h \left( (1,1) (T) \right) (\nabla h \otimes a), \]
and then move \( a \) to the very right (as required in (6.1)) using the cyclic operator in Proposition 3.10
\[ P_{11} = \varphi_0 \left( \left( \mathcal{J}_h \left( (2) \delta^{(0,1)} (T) + (1) \delta^{(1,1)} (T) \right) \right) (\nabla h \otimes \nabla h) \cdot a \right). \]
The first term above is the last face operator \( \delta^{(2,1)} = \delta_{(2)} \delta^{(0,1)} \) (see (4.1)) and second term can also be modified using (1.3):
\[ \tau_{(2)} \delta^{(1,1)} = \tau_{(2)} \delta^{(0,1)} \tau_{(1)} = \delta_2 \tau_{(1)}, \]
thus the final form of \( P_{11} \) is given by
\[ P_{11} = \varphi_0 \left( \left( \mathcal{J}_h \left( (2) \delta^{(0,1)} (T) + (1) \delta^{(1,1)} (T) \right) \right) (\nabla h \otimes \nabla h) \cdot a \right). \]
The desired relations of \( K \) and \( H \) follow immediately after adding up \( P_1 \) and \( P_{11} \) in (6.3) and (6.5).

In [6, Theorem 4.10], the rearrangement operators are defined as the functional calculus in terms of the modular operator \( x = [], h \), cf. [11 §1.5]. The gap between the two versions is a simple process of changing variables, that is, from multiplication operators \( \{h^0, \ldots, h^n\} \) to modular derivations \( \{\delta^0 h^0, x^1, \ldots, x^n\} \). Following the notations in [12 §3.2], we set:
\[ x^{(j)} = h^{(j)} - h^{(j-1)}, \ j = 1, \ldots, n. \]
In other words, \( x^{(j)} = -[[], h]^{(j)} : \mathscr{A}^\otimes n \to \mathscr{A} \) is the commutator derivation acting on the \( j \)-factor of a elementary tensor. Accordingly, one defines the Schwartz functional calculus by the substitution:
\[ \mathcal{J}_x (f) := f(x^{(1)}, \ldots, x^{(n)}) = f \left( h^{(1)} - h^{(0)}, \ldots, h^{(n)} - h^{(n-1)} \right) \in L \left( \mathcal{A}^{\otimes n}, \mathcal{A} \right). \]
Now let \( \bar{K} \) be the one-variable function such that \( \bar{K}(x) = \bar{K}(h^{(1)} - h^{(0)}) = K(h^{(0)}, h^{(1)}) \), then
\[ \delta_0 (K) \left( h^{(0)}, h^{(1)}, h^{(2)} \right) = K(\bullet, h^{(2)})[h^{(0)}, h^{(1)}] = \frac{K(h^{(0)}, h^{(2)}) - K(h^{(1)}, h^{(2)})}{h^{(0)} - h^{(1)}}, \]
\[ = \frac{\bar{K}(x^{(1)} + x^{(2)}) - \bar{K}(x^{(1)})}{-x^{(1)}}. \]
Similarly,
\[ \delta_1 (K) \left( h^{(0)}, h^{(1)}, h^{(2)} \right) = K(h^{(0)}, \bullet)[h^{(1)}, h^{(2)}] = \frac{\bar{K}(x^{(1)}) - \bar{K}(x^{(1)} + x^{(2)})}{-x^{(2)}} \]
and
\[ \delta_2 (K) \left( h^{(0)}, h^{(1)}, h^{(2)} \right) = (\tau_2 \delta_0) (K) \left( h^{(0)}, h^{(1)}, h^{(2)} \right) = K(\bullet, h^{(1)})[h^{(2)}, h^{(0)}] \]
\[ = \frac{\bar{K}(-x^{(2)}) - \bar{K}(x^{(1)})}{x^{(2)} + x^{(1)}}. \]
Provided that \( \bar{K} \) is an even function, i.e. \( \bar{K}(-x^{(2)}) = \bar{K}(x^{(2)}) \), we see that, upto a minus sign, \( (\delta_0 + \delta_1 - \delta_2)(K) \) is exactly the right hand side of [6, Eq. (4.42)].
6.2. Functions Arising in Pseudo-differential Calculus. We set \( \omega_\lambda(x_0) = (x_0 - \lambda)^{-1} \in \mathcal{C}_\text{Spec}(0) \), where \( \lambda \) is a complex parameter. For each \( \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1} \), consider
\[
(6.7) \quad \omega_\alpha := \omega_\lambda^{\alpha_0} \otimes \cdots \otimes \omega_\lambda^{\alpha_n} \in \mathcal{C}_\text{Spec}(n),
\]
that is \( \omega_\alpha(x_0, \ldots, x_n) = \omega_\lambda(x_0)^{\alpha_0} \cdots \omega_\lambda(x_n)^{\alpha_n} \). Obviously, the cyclic operator \( \tau_\alpha \) leads to cyclic permutation on the index \( \alpha \):
\[
\omega_\alpha \tau_\alpha \omega_\alpha = \omega_\alpha \tau_\alpha \omega_\alpha \quad \text{for all } \alpha.
\]
Let us compute the divided difference of \( \omega_\lambda \):
\[
(6.8) \quad \omega_\lambda [x_0, x_1] = \frac{(x_0 - \lambda)^{-1} - (x_1 - \lambda)^{-1}}{x_0 - x_1} = -\omega_\lambda(x_0)\omega_\lambda(x_1),
\]
which, in terms of the coproduct notation in (6.3), can be written as \( \Delta(\omega_\lambda) = \omega_\lambda \otimes \omega_\lambda = -\omega_{\{1,1\}} \).

It follows from (5.5) that both \( \delta^{(0,1)}(\omega_{\{1,1\}}) \) and \( \delta^{(1,1)}(\omega_{\{1,1\}}) \) are equal to \( \omega_{\{1,1,1\}} \), for instance:
\[
\delta^{(0,1)}(\omega_{\{1,1\}}) = (\Delta \otimes 1) (-\omega_\lambda \otimes \omega_\lambda) = -(-\omega_\lambda \otimes \omega_\lambda) \otimes \omega_\lambda = \omega_{\{1,1,1\}}.
\]
In a similar way, we can obtain all \( \omega_{\{1,\ldots,1\}} \in \mathcal{C}_\text{Spec}(n) \) by iterating the face operators on \( \omega_\lambda \in \mathcal{C}_\text{Spec}(0) \):
\[
(6.9) \quad \sigma^{(j,n)}(\omega_\alpha) = \omega_\lambda^{\alpha_0} \otimes \cdots \otimes \omega_\lambda^{\alpha_j-1} \otimes \omega_\lambda^{\alpha_j+1} \otimes \cdots \otimes \omega_\lambda^{\alpha_n} = \omega_{\alpha'},\n\]
where \( \alpha' = (\alpha_0, \ldots, \alpha_j + \alpha_{j+1}, \ldots, n) \). Now we can reach any \( \omega_\alpha, \alpha \in \mathbb{Z}_{\geq 0}^{n+1} \), from \( \omega_\lambda \) by iterating the face operators to produce \( \omega_{\{1,\ldots,1\}} \) and then contract the indices according to (5.5), for instance,
\[
\sigma^{(0,3)}(\omega_{\{1,1,1,1\}}) = \omega_{2,1,1}, \quad \sigma^{(1,3)}(\omega_{\{1,1,1,1\}}) = \omega_{1,2,1}, \quad \sigma^{(1,2)}(\omega_{1,2,1}) = \omega_{1,3}.
\]
Relations in Theorem 4.1 can also be used to derive differential relations. The simplest case would be \( \partial_{\omega_\lambda}^2 \omega_\lambda = -\omega_\lambda^2 \), which can be seen by writing \( \partial_{\omega_\lambda} = \sigma^{(0,1)} \delta^{(0,0)} \), while the right hand side can be computed directly:
\[
\omega_\lambda \xrightarrow{\delta^{(0,0)}} -\omega_\lambda \otimes \omega_\lambda \xrightarrow{\sigma^{(0,1)}} -\omega_\lambda^2.
\]
The next one: \( \partial^2_{\omega_\lambda} \omega_\lambda = 2\omega_\lambda^3 \). As before, we first rewrite the second derivative \( \partial^2_{\omega_\lambda} = \sigma_{\partial_{\omega_\lambda}} \delta \partial_{\omega_\lambda} \) using (4.5) and (4.3):
\[
\sigma^{(0,1)} \sigma^{(0,2)} \delta^{(0,1)} \delta^{(0,0)} = \sigma^{(0,1)} \sigma^{(1,2)} \delta^{(0,1)} \delta^{(0,0)} = \sigma^{(0,1)} \left( \delta^{(0,0)} \sigma^{(1,1)} - \sigma^{(0,2)} \delta^{(1,1)} \right) \delta^{(0,0)}
\]
\[
= \sigma^{(0,1)} \delta^{(0,0)} \sigma^{(0,1)} \delta^{(0,0)} - \sigma^{(0,1)} \sigma^{(0,2)} \delta^{(1,1)} \delta^{(0,0)}.
\]
As we have seen in (6.8) and (6.9):
\[
\omega_\lambda^2 = \sigma^{(0,0)} \sigma^{(0,2)} \delta^{(0,1)} \delta^{(0,0)}(\omega_\lambda) = \sigma^{(0,0)} \sigma^{(0,2)} \delta^{(1,1)} \delta^{(0,0)}(\omega_\lambda).
\]
Therefore the sum \( (\sigma^{(0,0)}\sigma^{(0,2)}\delta^{(0,1)}\delta^{(0,0)} + \sigma^{(0,0)}\sigma^{(0,2)}\delta^{(1,1)}\delta^{(0,0)}) \) \((\omega_\lambda)\) indeed recovers the coefficient 2 in \( \partial^2_\alpha \omega_\lambda = 2\omega_\lambda^3 \). One can push the computation above to derive the following differential relations

\[
\omega(\alpha_0,\ldots,\alpha_n) = \left( \prod_{j=0}^{n} \frac{(-1)^{\alpha_j-1}}{(\alpha_j - 1)!} \partial^{\alpha_j-1}_{\alpha_j} \right) \omega(1,\ldots,1).
\]

The family \( \{\omega_\alpha\} \) is intricately connected to the hypergeometric one \( \{H_\alpha\} \) introduced in the rearrangement lemma in [14], which a crucial technical tool in the pseudo-differential calculus approach of deciphering heat asymptotic. A more general version of the lemma designed for going beyond conformal geometry has been achieved in [10]. The calculation on \( \{\omega_\alpha\} \) hints that one might be able to derive the recursive and differential relations of \( \{H_\alpha\} \) in the hypergeometric literature from the action of the category \( C \). What has been used is the observation that the cyclic \( \tau_{(n)} \) operators acts as cyclic permutation on the index \( \alpha \), which is the new input to obtain computer-aid free verification of the Connes-Moscovici type functional relation, cf. [15 § 5]. We postpone the fully investigation to future publications and remark that the question serve greatly to the interest of finding simplification of the \( a_4 \)-term computation [2].

**Appendix A. The Cyclic Category \( \Delta C \)**

Connes’s cyclic category \( \Delta C \) is a mixture of the simplicial category \( \Delta \) and the cyclic groups. We collect related notations and preliminary results from [17] for the reader’s convenience.

**Definition A.1.** The cyclic category \( \Delta C \) consists of objects \([n], n \in \mathbb{Z}_{\geq 0}\), and morphisms generated by faces \( \delta_j^{(n-1)} : [n-1] \to [n] \), degeneracies \( \sigma_j^{(n+1)} : [n+1] \to [n], j = 0, \ldots, n \), and cyclic operators \( \tau_{(n)} : [n] \to [n] \), subject to the following relations \[^3^\]

- simplicial and co-simplicial relations:
  \[
  \delta_j \delta_i = \delta_i \delta_{j-1}, \quad \text{for } i < j, \\
  \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad \text{for } i \leq j;
  \]
- their compatibility:
  \[
  \sigma_j \delta_i = \begin{cases} 
  \delta_i \sigma_{j-1} & i < j \\
  1_{[n]} & i = j, i = j + 1 \\
  \delta_{i-1} \sigma_j & i > j + 1
  \end{cases}
  \]
- for cyclic operators: \( \tau_{(n)}^{n+1} = 1 \).
- compatibility of the cyclic and the simplicial structures:
  \[
  \tau_{(n)} \delta_i = \delta_{i-1} \tau_{(n-1)}, \quad \tau_{(n)} \sigma_i = \sigma_{i-1} \tau_{(n+1)},
  \]
  where \( i = 1, \ldots, n \).

**Remark.** The compatibility relations Eq. (A.3) is equivalent to

\[
\delta_n = \tau_{(n)}^{n+1} \delta_{n-1} = \tau_{(n)}^{n} \delta_{n-1} \tau_{(n-1)} = \cdots = \tau_{(n)} \delta_0 \tau_{(n-1)} = \tau_{(n)} \delta_0
\]

and

\[
\tau_{(n)} \sigma_0 = \tau_{(n)} \sigma_0 \tau_{(n+1)}^{n+2} = \tau_{(n)} \sigma_1 \tau_{(n+1)}^{n+1} = \cdots = \tau_{(n)} \sigma_n \tau_{(n+1)}^{2} = \sigma_n \tau_{(n+1)}^{2}.
\]

[^3^]: The superscripts which indicate the domain have been suppressed accordingly, such as \( \sigma_j^{(n)} \to \sigma_j \).
The simplicial category $\Delta$ is the subcategory of $\Delta C$ having the same objects $[n]$, $n = 0, 1, 2 \ldots$, while the morphisms are generated by the faces and degeneracies. The following realization is related to the discussion in § 5. The objects $[n]$ are ordered set $\{0 < 1 < \cdots < n\}$ of $n + 1$ points. The morphisms consists of non-decreasing functions $f : [n] \to [m]$, meaning $f(i) \geq f(j)$ whenever $i > j$. They can be classified into two categories: injective and surjective ones. The non-decreasing property implies that any morphism $\psi : [n] \to [m]$ can be represented by the ordered list $\{\psi^{-1}(\{0\}), \ldots, \psi^{-1}(\{m\})\}$ the pre-images\footnote{The notation $\psi^{-1}$ should not be confused with the inverse of $\psi$.} of points in $[m]$. When $\psi$ is injective, the cardinality $\#\psi^{-1}(\{l\})$ is either 0 or 1. Thus an injective morphism is determined by the collection of missing points
\begin{equation}
ms(\psi) = \{l \in [m] : \#\psi^{-1}(\{l\}) = 0\}.
\end{equation}
Similarly, surjectivity means that $\#\psi^{-1}(\{l\}) \geq 1$ and the collection of pre-images $\psi^{-1}(\{l\})$ at the stationary points
\begin{equation}
st(\psi) = \{l \in [m] : \#\psi^{-1}(\{l\}) > 1\}
\end{equation}
is sufficient to capture the morphism $\psi$. It follows that the identities $1_{[n]}$ are the only isomorphisms (i.e. non-decreasing bijections) in $\Delta$.

As for the generators listed in Definition A.2, the face map $\delta_j^{(n-1)} : [n-1] \to [n]$ is the injective function missing $j \in [n]$ and the degeneracy $\sigma_j^{(n+1)} : [n+1] \to [n]$ is the surjective function hitting $j \in [n]$ twice: $\sigma_j^{(n+1)}(j) = \sigma_j^{(n+1)}(j + 1) = j$.

**Proposition A.1.** For any morphism $\phi : [n] \to [m]$ in the simplicial category $\Delta$, there is a unique decomposition
\begin{equation}
\phi = \delta_{i_1} \cdots \delta_{i_r} \sigma_{j_1} \cdots \sigma_{j_s},
\end{equation}
such that $i_1 > i_2 > \cdots > i_r$ and $j_1 < j_2 < \cdots < j_s$, with $m = n - s + r$. Convention: if the index set is empty, then $\phi$ is the identity.

**Proof.** See [17, Appendix B] and the references therein.

To complete the corresponding realization of $\Delta C$ (from $\Delta$ described above), we choose the following cyclic permutations, derived from the cycle $\{01 \ldots n\} \in S_{n+1}$, as the generators of cyclic groups
\begin{equation}
\tau_{(n)} : [n] \to [n] : 0 \mapsto 1, 1 \mapsto 2, \ldots, n \mapsto 0.
\end{equation}

The cyclic groups and the simplicial category $\Delta$ are combined in a way that any morphism in $\Delta C$ can by written uniquely as a composition of a morphism in $\Delta$ and an element of the cyclic groups. Moreover, from Eqs. (A.4) and (A.5), we see that with the cyclic operators $\tau_{(n)}$, one only needs another face and degeneracy maps, say, $\sigma_{(n)}^j$ and $\delta_{(n)}^i$, to recover the rest.

Another notable feature of the cyclic category $\Delta C$ (not true for the simplicial category) is the self-duality $\Delta C \cong \Delta C^{op}$. By taking $\delta_i^* = d_i$, $\sigma_j^* = s_j$ and $\tau_n^* = t_n$, we have the following presentation of $\Delta C^{op}$.

**Definition A.2.** The opposite category $\Delta C^{op}$ has a presentation given by generators $t_{(n)} : [n] \to [n]$ and
\begin{align*}
\delta_i^{(n)} : [n] &\to [n-1], \\
\sigma_i^{(n)} : [n] &\to [n+1], \ i = 0, \ldots, n.
\end{align*}

The corresponding relations are
• simplicial and co-simplicial:

\begin{equation}
A.9 \quad d_i d_j = d_j d_i, \quad i < j
\end{equation}

\begin{equation}
A.10 \quad s_i s_j = s_j s_{i+1}, \quad i \leq j
\end{equation}

• compatibility:

\begin{equation}
A.11 \quad d_i s_j = \begin{cases} 
  s_{i-1} d_i & \text{for } i < j \\
  \text{id} \quad & \text{for } i = j, \quad i = j + 1 \\
  s_j d_{i-1} & \text{for } i > j + 1
\end{cases}
\end{equation}

and

\begin{equation}
A.12 \quad d_i t(n) = t(n-1) d_{i-1}, \quad \text{for } 1 \leq i \leq n \text{ and } d_0 t(n) = d_n,
\end{equation}

\begin{equation}
A.13 \quad s_i t(n) = t(n+1) s_{i-1} \quad \text{for } 1 \leq i \leq n \text{ and } s_0 t(n) = t^2(n+1) s_n.
\end{equation}

• cyclic: \( t_{n+1}^0 = \text{id}(n) \).

We will also make use of the duality functor from \( \Delta C^\text{op} \) to \( \Delta C \) described in [17, §6.1.11]. The construction starts with adding the following extra degeneracies to the generators of \( \Delta C \):

\begin{equation}
A.14 \quad \sigma_{n+1}^{(n+1)} := \sigma_0 t_{n+1}^{-1} : [n+1] \rightarrow [n].
\end{equation}

Notice that the relation \( t_n \sigma_i = \sigma_{i-1} t_{n+1} \) extends to \( i = n + 1 \).

The duality functor sends \([n]\) to \([n]\), on morphisms, it sends \( \tau_n \rightarrow (t_n)^{-1} : [n] \rightarrow [n] \) and

\begin{equation}
A.15 \quad \delta_i^{(n+1)} \rightarrow \sigma_i^{(n+1)} : [n+1] \rightarrow [n], \quad \text{for } i = 0, \ldots, n+1,
\end{equation}

and

\begin{equation}
A.16 \quad s_i^{(n-1)} \rightarrow \delta_i^{(n)} : [n-1] \rightarrow [n], \quad \text{for } i = 0, \ldots, n.
\end{equation}

Note that \( \delta_0^{(n)} \) is missed in Eq. \( A.16 \).

**References**

[1] Simon Brain, Giovanni Landi, and Walter D. van Suijlekom. Moduli spaces of instantons on toric noncommutative manifolds. *Adv. Theor. Math. Phys.*, 17(5):1129–1193, 2013.

[2] A. Connes and F. Fathizadeh. The term \( a_q \) in the heat kernel expansion of noncommutative tori. *Münster J. Math.*, 12(2):239–410, 2019.

[3] A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and the transverse index theorem. *Comm. Math. Phys.*, 198(1):199–246, 1998.

[4] Alain Connes and Michel Dubois-Violette. Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. *Comm. Math. Phys.*, 220(3):539–579, 2002.

[5] Alain Connes and Giovanni Landi. Noncommutative manifolds, the instanton algebra and isospectral deformations. *Comm. Math. Phys.*, 221(1):141–159, 2001.

[6] Alain Connes and Henri Moscovici. Modular curvature for noncommutative two-tori. *J. Amer. Math. Soc.*, 27(3):639–684, 2014.

[7] Alain Connes and Paula Tretkoff. The Gauss-Bonnet theorem for the noncommutative two torus. In *Noncommutative geometry, arithmetic, and related topics*, pages 141–158. Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[8] B. de Pagter, H. Witvliet, and F. A. Sukochev. Double operator integrals. *J. Funct. Anal.*, 192(1):52–111, 2002.

[9] Ken Dykema and Anna Skripka. Higher order spectral shift. *J. Funct. Anal.*, 257(4):1092–1132, 2009.

[10] Farzad Fathizadeh. On the scalar curvature for the noncommutative four torus. *J. Math. Phys.*, 56(6):062303, 2015.

[11] Farzad Fathizadeh and Masoud Khalkhali. Scalar curvature for the noncommutative two torus. *J. Noncommut. Geom.*, 7(4):1145–1183, 2013.

[12] Matthias Lesch. Divided differences in noncommutative geometry: rearrangement lemma, functional calculus and expansional formula. *J. Noncommut. Geom.*, 11(1):193–223, 2017.
[13] Yang Liu. Scalar curvature in conformal geometry of Connes-Landi noncommutative manifolds. *Journal of Geometry and Physics*, 121:138 – 165, 2017.

[14] Yang Liu. Hypergeometric function and modular curvature. I. hypergeometric functions in heat coefficients. 10 2018, 1810.09939.

[15] Yang Liu. Hypergeometric function and modular curvature II. Connes-Moscovici functional relation after Lesch’s work. 11 2018, 1811.07967.

[16] Yang Liu. General rearrangement lemma for heat trace asymptotic on noncommutative tori. 04 2020, 2004.05714.

[17] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[18] Denis Potapov, Anna Skripka, and Fedor Sukochev. Spectral shift function of higher order. *Invent. Math.*, 193(3):501–538, 2013.

[19] Marc A. Rieffel. Deformation quantization for actions of $\mathbb{R}^d$. *Mem. Amer. Math. Soc.*, 106(506):x+93, 1993.

[20] Anna Skripka and Anna Tomskova. *Multilinear operator integrals*, volume 2250 of *Lecture Notes in Mathematics*. Springer, Cham, 2019. Theory and applications.

[21] Teun D. H. van Nuland and Walter D. van Suijlekom. Cyclic cocycles in the spectral action. To appear in J. Noncomm. Geom, 2021, arXiv:2104.09899.

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