UNIFORMLY DISTRIBUTED ORBITS IN $\mathbb{T}^d$ AND SINGULAR
SUBSTITUTION DYNAMICAL SYSTEMS

ROTEM YAARI

Abstract. We find sufficient conditions for the singularity of a substitution $\mathbb{Z}$-action’s spectrum, which generalize the conditions given in [13, Theorem 2.4], and we also obtain a similar statement for a collection of substitution $\mathbb{R}$-actions, including the self-similar one. To achieve this, we first study the distribution of related toral endomorphism orbits. In particular, given a toral endomorphism and a vector $v \in \mathbb{Q}^d$, we find necessary and sufficient conditions for the orbit of $\omega v$ to be uniformly distributed modulo 1 for almost every $\omega \in \mathbb{R}$. We use our results to find new examples of singular substitution $\mathbb{Z}$- and $\mathbb{R}$-actions.

1. Introduction

While the discrete spectrum of substitution dynamical systems has been heavily studied, e.g., [17, 28, 22, 29, 7, 27], less is known on the existence (and absence) of the absolutely continuous component. Primitive substitution $\mathbb{Z}$- and $\mathbb{R}$-actions always possess a nontrivial singular component [18, Theorem 2], [16, Theorem 2.2], but nevertheless an absolutely continuous component may exist; examples are provided by the Rudin-Shapiro substitution and its generalizations [36, 24, 15]. In general, it is hard to determine whether the spectrum is purely singular. In the case of a constant length substitution, Bartlett developed further the work of Queffélec [36] and obtained an algorithm for computing the spectrum of a substitution, which he used to find examples of substitutions with purely singular spectrum [8]. Berlinkov and Solomyak provided a sufficient condition for the singularity of the spectrum, in terms of the eigenvalues of the substitution matrix [10]. In the non-constant length self-similar $\mathbb{R}$-action case, Baake et al. [2, 1, 5, 3] developed new techniques which they used to obtain sufficient conditions for the singularity of the closely related diffraction spectrum, and they explored some examples (see Remark 4.6(c)).

In [12, 14], Bufetov and Solomyak introduced the spectral cocycle associated with a substitution (or more generally, an $S$-adic shift), and used it to obtain sufficient conditions for a substitution $\mathbb{R}$-action to have purely singular spectrum, see [14, Corollaries 4.5 and 4.7]. However, it is difficult to find examples of singular substitution $\mathbb{R}$-actions based directly on these results, since the conditions are given in terms of the pointwise upper Lyapunov exponent, which is rather hard to compute. The situation is better in the $\mathbb{Z}$-action case, where an upper bound on the Lyapunov exponent can sometimes be estimated [13]. To achieve one of their main results [13, Theorem 2.4], the authors apply a theorem of Host [30], which is concerned with the uniform distribution
of toral endomorphism orbits. Whereas Host’s theorem is relatively general, the paper’s interest lies only in the uniform distribution of the orbit of the diagonal vector \( \omega \vec{1} \) (where \( \vec{1} = (1, \ldots , 1)^t \)) for Lebesgue-a.e. \( \omega \in \mathbb{R} \), and it requires the authors to add the assumption that the characteristic polynomial of the substitution matrix is irreducible over \( \mathbb{Q} \).

In this paper, we give conditions that are both necessary and sufficient for the orbit of \( \omega \vec{1} \) to be uniformly distributed in the torus for a.e. \( \omega \in \mathbb{R} \), thereby removing the irreducibility assumption made in [13]. This is done in Section 2, after we recall some definitions and results on uniformly distributed sequences and linear recurrences. It turns out that in the case of a reducible characteristic polynomial, the singularity of a given \( \mathbb{Z} \)- or \( \mathbb{R} \)-action depends only on a component of the spectral cocycle, obtained by a restriction to what we call the minimal subspace of a vector, see Section 3. Combining this with the ideas in [13] allows us to take a unified approach toward \( \mathbb{Z} \)- and \( \mathbb{R} \)-actions in Section 4. We obtain sufficient conditions for a substitution \( \mathbb{Z} \)-action, and for a collection of \( \mathbb{R} \)-actions including the self-similar one, to have purely singular spectrum, without assuming irreducibility or Bohr-almost periodicity, and without using the theorems of Host and Sobol (see Remark 4.6(c)). In Section 5 we explore new examples of reducible non-Pisot substitutions with singular spectrum.

2. Uniformly distributed sequences and linear recurrences

Recall that a sequence \( (x_n)_{n=0}^\infty \subset \mathbb{R}^d \) is said to be uniformly distributed modulo 1 (abbreviated u.d. mod 1) if for every choice of intervals \( I_1, \ldots , I_d \subseteq [0, 1) \) we have

\[
\lim_{N \to \infty} \frac{|\{0 \leq n < N : x_n \text{ mod } 1 \in I_1 \times \cdots \times I_d\}|}{N} = |I_1| \cdots |I_d|
\]

(where \( x \text{ mod } 1 \) stands for the vector of entrywise fractional part of \( x \)), or equivalently, if for every \( \mathbb{Z}^d \)-periodic continuous function \( f : \mathbb{R}^d \to \mathbb{C} \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = \int_{\mathbb{T}^d} f \, dm_d,
\]

where \( m_d \) is the \( d \)-dimensional (normalized) Haar measure. Note that we use the same notation for \( f \) and for the induced function on \( \mathbb{T}^d \).

The following is a straightforward consequence of the well-known Weyl’s criterion.

Proposition 2.1 ([31, Chapter 1, Theorem 6.3]). A sequence \( (x_n)_{n=0}^\infty \subset \mathbb{R}^d \) is u.d. mod 1 if and only if for every non-zero \( h \in \mathbb{Z}^d \) the sequence of real numbers \( ((x_n, h))_{n=0}^\infty \) is u.d. mod 1.

The next result, which is a consequence of a theorem of Koksma, will also be useful.

Theorem 2.2 ([31, Chapter 1, Corollary 4.3]). Let \( (x_n)_{n=0}^\infty \) be a real sequence. Suppose that there exist \( N \in \mathbb{N} \) and \( \delta > 0 \) such that \( |x_n - x_m| \geq \delta \) for every \( n, m > N, n \neq m \), then the sequence \( (x_n \omega)_{n=0}^\infty \) is u.d. mod 1 for a.e. \( \omega \in \mathbb{R} \).
Definition 2.3. A linear recurrence relation, or simply a recurrence relation, is an expression of the form

\begin{equation}
    u_n = \sum_{i=0}^{d-1} \alpha_i u_{n-d+i}
\end{equation}

for some \( \alpha_0, \ldots, \alpha_{d-1} \in \mathbb{C}, \alpha_0 \neq 0 \), and we say that the recurrence relation is of order \( d \). The companion polynomial associated with the recurrence relation (2.1) is the polynomial \( x^d - \sum_{i=0}^{d-1} \alpha_i x^i \), and his roots are the roots of the recurrence. A linear recurrence sequence, or simply a recurrence sequence, is a complex sequence that satisfies some recurrence relation. A recurrence sequence is of order \( d \) if the recurrence relation of minimal order that it satisfies is of order \( d \).

Notice that a recurrence sequence of order \( d \) is determined by its first \( d \) terms, which are called the initial values of the sequence.

The following is one of the most fundamental facts about recurrence relations.

Theorem 2.4 (see [20, Subsection 1.1.6]). Denote by \( \lambda_1, \ldots, \lambda_m \) the distinct roots of the recurrence relation (2.1) and by \( n_1, \ldots, n_m \) their respective multiplicities. The sequences that satisfy this recurrence relation are exactly the sequences \((x_n)_{n=0}^{\infty} \subset \mathbb{C}\) of the form

\[ x_n = \sum_{i=1}^{m} p_i(n) \lambda_1^{n_i}, \quad n \in \mathbb{N} \]

(we define \( \mathbb{N} \) to include 0), where \( p_i \) is a polynomial of degree \( \deg p_i < n_i \) for \( i = 1, \ldots, m \).

Many questions are concerned with the set of zeros of a given recurrence sequence. These questions and their generalizations have led to the following definitions.

Definition 2.5. A recurrence relation is called degenerate if it has two distinct roots whose ratio is a root of unity. Otherwise, the recurrence relation is called non-degenerate.

Definition 2.6. The total multiplicity of a complex sequence \((x_n)_{n=0}^{\infty}\) is defined to be

\[ |\{(n, m) \in \mathbb{N} \times \mathbb{N} : n \neq m, x_n = x_m\}|. \]

Theorem 2.7 (van der Poorten [39], Evertse [21, Corollary 4]). Suppose \((x_n)_{n=0}^{\infty}\) is a sequence of algebraic numbers that satisfies a non-degenerate recurrence relation. If the sequence is not of the form \((x_n) = (cp^n)\), for some constant \( c \) and a root of unity \( \rho \), then the total multiplicity of the sequence is finite.

The next definition provides the connection between the theory of linear recurrence relations and the topics discussed in the current paper.

Definition 2.8. Let \( A \in M_d(\mathbb{Z}) \) be non-singular and let \( x^d - \sum_{i=0}^{d-1} \alpha_i x^i \) be its characteristic polynomial. The recurrence relation associated with \( A \) is \( u_n = \sum_{i=0}^{d-1} \alpha_i u_{n-d+i} \). We call \( A \) degenerate (resp. non-degenerate) if the associated recurrence relation is degenerate (resp. non-degenerate).
Note that the companion polynomial associated with the recurrence relation is the characteristic polynomial of $A$, so $A$ is degenerate if and only if it has two distinct eigenvalues whose ratio is a root of unity. Moreover, since $A$ is an integer matrix, $\alpha_0,\ldots,\alpha_{d-1}$ are integers as well.

We can now state our first main result.

**Theorem 2.9.** Let $A \in M_d(\mathbb{Z})$ be non-singular and let $\mathbf{v} \in \mathbb{Q}^d$. The sequence $(A^n\mathbf{v})_{n=0}^\infty$ is u.d. mod 1 for a.e. $\omega \in \mathbb{R}$ if and only if $A$ is non-degenerate with no eigenvalues that are roots of unity and the vectors $\mathbf{v}, A\mathbf{v},\ldots,A^{d-1}\mathbf{v}$ are linearly independent.

We will need the following two lemmas.

**Lemma 2.10.** Let $A \in M_d(\mathbb{Z})$ be non-singular and let $\mathbf{v} \in \mathbb{Q}^d$. Suppose $\mathbf{v}, A\mathbf{v},\ldots,A^{d-1}\mathbf{v}$ are linearly independent, then a sequence $(x_n)_{n=0}^\infty \subset \mathbb{Q}$ satisfies the recurrence relation associated with $A$ if and only if there exists $s \in \mathbb{Q}^d$ such that $x_n = \langle A^n\mathbf{v},s \rangle$ for every $n \in \mathbb{N}$, and the sequence is identically zero if and only if $s = 0$.

**Proof.** If $(x_n) = (\langle A^n\mathbf{v},s \rangle)$, the first part of the claim follows immediately from Cayley-Hamilton theorem. Conversely, since $\mathbf{v}, A\mathbf{v},\ldots,A^{d-1}\mathbf{v}$ are linearly independent, there exists $s \in \mathbb{Q}^d$ such that $\langle A^n\mathbf{v},s \rangle = x_i$ for $i = 0,\ldots,d-1$. The sequences $(\langle A^n\mathbf{v},s \rangle)_{n=0}^\infty$ and $(x_n)_{n=0}^\infty$ satisfy the same recurrence relation and have the same initial values, so they must be equal. The last part is clear from the linear independence of $\mathbf{v}, A\mathbf{v},\ldots,A^{d-1}\mathbf{v}$.

**Lemma 2.11.** Suppose that (2.1) is a degenerate recurrence relation with $\alpha_0,\ldots,\alpha_{d-1} \in \mathbb{Z}$. Then there exists a sequence of integers, which is not identically zero, satisfies (2.1) and has an arithmetic subsequence of zeros.

**Proof.** Let $\lambda, \rho \lambda$ be two distinct roots of the recurrence, where $\rho$ is a root of unity of order $k$. Using the recurrence relation (2.1), we can find integers $\beta_{i,j}$, $0 \leq i,j \leq d-1$, such that every sequence $(x_n)_{n=0}^\infty$ that satisfies this recurrence relation also satisfies

\begin{equation}
(2.2) \quad x_{i,k} = \sum_{j=0}^{d-1} \beta_{i,j} x_j, \quad i = 0,\ldots,d-1.
\end{equation}

Define $B = (\beta_{i,j})_{0 \leq i,j \leq d-1} \in M_d(\mathbb{Z})$ and a sequence $(y_n)_{n=0}^\infty$ by $y_n = \lambda^n - (\rho \lambda)^n$, and notice that it is not identically zero, it satisfies the recurrence relation (2.1) by Theorem 2.4 and it vanishes on the set $\{0,k,2k,\ldots\}$. Thus, it follows from (2.2) that

$$B \cdot (y_0,\ldots,y_{d-1})^t = (y_0,y_k,\ldots,y_{(d-1)k})^t = 0,$$

and hence $\det B = 0$ and there exists a non-zero vector $(z_0,\ldots,z_{d-1})^t \in \mathbb{Z}^d \cap \ker B$. Let $(z_n)$ be the recurrence sequence defined by these initial values and the recurrence relation (2.1). By (2.2),

$$(z_0,z_k,\ldots,z_{(d-1)k})^t = B \cdot (z_0,\ldots,z_{d-1})^t = 0,$$
and since \((z_{kn})_{n=0}^\infty\) is also a linear recurrence sequence of order at most \(d\) (see [20, Theorem 1.3]), this subsequence must be identically zero.

\[\square\]

**Proof of Theorem 2.9.** Since we can replace \(v\) by an integer vector with the same span, we can assume without loss of generality that \(v \in \mathbb{Z}^d\). First we prove the sufficiency of the conditions. If we prove that for every non-zero \(h \in \mathbb{Z}^d\), \((A^n \omega v, h)\) is not identically zero and satisfies the recurrence relation associated with \(A\), then for every \(\omega\)’s that work for all \(h\)’s is also of full measure, and by Proposition 2.1 we are done.

Fix a non-zero \(h \in \mathbb{Z}^d\). By Lemma 2.10, \((A^n v, h)\) is not identically zero and satisfies the recurrence relation associated with \(A\). Since no eigenvalue of \(A\) is a root of unity, Theorems 2.7 and 2.4 imply that the total multiplicity of this integer sequence is finite, and the sufficiency of the conditions follows from Theorem 2.2 (we can take \(\delta = 1\)).

Conversely, suppose first that \(v, Av, \ldots, A^{d-1}v\) are linearly dependent. Notice that

\[\{A^n \omega v : \omega \in \mathbb{R}, n \in \mathbb{N}\} = \text{Span}\{v, Av, \ldots, A^{d-1}v\},\]

and since this subspace is spanned by at most \(d - 1\) integer vectors, the set of its fractional parts is not dense in \(T^d\), let alone u.d. mod 1 for a fixed \(\omega\). We can assume for the rest of the proof that \(v, Av, \ldots, A^{d-1}v\) are linearly independent.

Suppose now that \(A\) is degenerate. By Lemma 2.11, we can take a sequence of integers \((x_n)_{n=0}^\infty\), which is not identically zero, satisfies the recurrence relation associated with \(A\) and such that \(x_{kn} = 0\) for some \(k \geq 2\) and every \(n\). By Lemma 2.10 there exist \(h \in \mathbb{Z}^d \setminus \{0\}\) and \(c \in \mathbb{N} \setminus \{0\}\) such that \((A^n v, h) = cx_n\) for every \(n\). Consequently, for every \(\omega \in \mathbb{R}\) we have

\[\limsup_{N \to \infty} \frac{|\{0 \leq n < N : (A^n v, h) \omega \mod 1 \in [0, \frac{1}{2|c|}]\}|}{N} \geq \frac{1}{k^2},\]

so \((A^n v, h)\) is not u.d. mod 1 and again by Proposition 2.1, \(A^n \omega v\) is also not u.d. mod 1.

Finally, if \(\rho\) is an eigenvalue of \(A\) which is also a root of unity, then so is \(\overline{\rho}\). If \(\rho \neq \overline{\rho}\), then \(A\) is degenerate and since we already considered this case we may assume that \(\rho = \pm 1\). Proceeding as before, \((A^n v, h) = (c \rho^n)\) for some \(h \in \mathbb{Z}^d \setminus \{0\}\) and \(c \in \mathbb{N} \setminus \{0\}\). It follows that for every \(\omega \in \mathbb{R}\),

\[((A^n \omega v, h))_{n=0}^\infty \subseteq \{\pm c\omega\},\]

and once again by Proposition 2.1, \((A^n \omega v)_{n=0}^\infty\) is not u.d. mod 1.

\[\square\]

**Corollary 2.12.** If \((A^n \omega v)\) is u.d. mod 1 for some \(\omega \in \mathbb{R}\), then the same is true for a.e. \(\omega \in \mathbb{R}\).

**Proof.** We saw in the previous proof that if one of the conditions of Theorem 2.9 does not hold then for every \(\omega \in \mathbb{R}\), \((A^n \omega v)_{n=0}^\infty\) is not u.d. mod 1.

\[\square\]

**Corollary 2.13.** If \((A^n \omega v)\) is u.d. mod 1 for a.e. \(\omega \in \mathbb{R}\), then so is \((A^{kn+\ell} \omega v)_{n=0}^\infty\) for every \(k \geq 1\) and \(\ell \in \mathbb{N}\).
Since a subspace divides the characteristic polynomial of \( A \), \( \forall v \in \text{Span}_{k} \), \( h \in \mathbb{Z}^{d} \), then \( (\langle A^n v, h \rangle) \) has an arithmetic subsequence of zeros, but since the associated recurrence relation is non-degenerate, \( (\langle A^n v, h \rangle) \) must be identically zero (see [37, Corollary C.1]), and thus \( h = 0 \).

**Remark 2.14.** (a) Meiri proved that if an integer sequence \( (x_n) \) satisfies a non-degenerate recurrence relation that has no roots that are roots of unity, then in fact \( (\omega x_n) \) is u.d. mod 1 for \( \mu \)-a.e. \( \omega \), where \( \mu \) belongs to some collection of Borel measures on \( \mathbb{T} \), including Lebesgue measure [33, Theorem 5.2]. For Lebesgue measure, we gave a simple (one-line) proof of this fact, relying on the powerful result of van der Poorten and Evertse, whereas the proof of Meiri’s result is considerably more complicated, and uses \( p \)-adic analysis instead.

(b) Pushkin obtained a somewhat similar result, showing that given a connected analytic manifold in \( \mathbb{R}^{d} \) that is not contained in any hyperplane, Lebesgue-a.e. vector in that manifold is absolutely normal [35, Theorem 2].

**Proposition 2.15.** Suppose that \( A \in M_d(\mathbb{Z}) \) has a unique dominant eigenvalue \( \theta_1 > 1 \), and that its characteristic polynomial is irreducible over \( \mathbb{Q} \). Let \( v = \sum_{i=1}^{d} c_i v_i \in \mathbb{R}^{d} \) where \( v_1, \ldots, v_d \) are the eigenvectors of \( A \), \( v_1 \) corresponds to \( \theta_1 \), \( c_1, \ldots, c_d \in \mathbb{C} \) and \( c_1 \neq 0 \). Then \( (A^{kn+\ell} \omega v)_{n=0}^{\infty} \) is u.d. mod 1 for every \( k \geq 1 \), \( \ell \in \mathbb{N} \) and a.e. \( \omega \in \mathbb{R} \).

**Proof.** First let us show that the entries of \( v_1 \) are rationally independent. Suppose that \( \langle v_1, h \rangle = 0 \) for some \( h \in \mathbb{Z}^{d} \), then also
\[
0 = \langle A^n v_1, h \rangle = \langle v_1, (A^n)^{\ell} h \rangle,
\]
and hence \( h, \ldots, (A^n)^{\ell-1} h \) must be linearly dependent. Therefore, they span an \( A^n \)-invariant \( \mathbb{Q}^{d} \)-subspace of dimension at most \( d - 1 \), and the characteristic polynomial of \( A^n \) restricted to this subspace divides the characteristic polynomial of \( A \), which means \( h = 0 \).

Next, let \( k \geq 1 \), \( \ell \in \mathbb{N} \) and \( h \in \mathbb{Z}^{d} \setminus \{0\} \), and consider the real sequence \( (\langle A^{kn+\ell} v, h \rangle)_{n=0}^{\infty} \). Since \( \langle c_1 v_1, h \rangle \neq 0 \) and \( \theta_1 \) is the unique dominant eigenvalue of \( A \), we have
\[
\frac{\langle A^{kn+\ell} v, h \rangle}{\langle A^{kn} v, h \rangle} \rightarrow \theta_1^n,
\]
so in particular \( |\langle A^{kn+\ell} v, h \rangle - \langle A^{kn} v, h \rangle| > 1 \) for every sufficiently large \( n \) and every \( m > n \).

By Theorem 2.2, \( (\langle A^{kn+\ell} \omega v, h \rangle) \) is u.d. mod 1 for a.e. \( \omega \in \mathbb{R} \), and we conclude by repeating the argument from the beginning of the proof of Theorem 2.9.

3. The minimal subspace

**Definition 3.1.** Let \( A \in M_d(\mathbb{Z}) \) and \( v \in \mathbb{R}^{d} \setminus \{0\} \). The minimal subspace of \( v \) (with respect to \( A \)) is \( \text{Span}_{k} W < \mathbb{R}^{d} \), where \( W < \mathbb{Q}^{d} \) is the minimal \( A \)-invariant subspace (over \( \mathbb{Q} \)), such that \( v \in \text{Span}_{k} W \).
The following lemma asserts that, as implied in the definition, there is a unique minimal subspace $W < \mathbb{Q}^d$ with these properties, and hence the minimal subspace is unique as well.

**Lemma 3.2.** Let $W_1, W_2 < \mathbb{Q}^d$, then $\text{Span}_\mathbb{R}(W_1 \cap W_2) = \text{Span}_\mathbb{R}W_1 \cap \text{Span}_\mathbb{R}W_2$.

**Proof.** Clearly, $\dim\text{Span}_\mathbb{R}W \leq \dim\text{Span}_\mathbb{Q}W$ for any $W < \mathbb{Q}^d$, and since we can define a non-singular matrix (over both fields) with columns that contain a basis of $W$, the dimensions are equal. The inclusions $\text{Span}_\mathbb{R}(W_1 \cap W_2) \subseteq \text{Span}_\mathbb{R}W_1 \cap \text{Span}_\mathbb{R}W_2$ and $\text{Span}_\mathbb{R}(W_1 + W_2) \subseteq \text{Span}_\mathbb{R}W_1 + \text{Span}_\mathbb{R}W_2$ are clear, and the lemma follows from the identity $\dim(U \cap V) = \dim U + \dim V - \dim(U + V)$. □

**Example 3.3.** Let $A \in M_d(\mathbb{Z})$.

(a) If $v \in \mathbb{Q}^d \setminus \{0\}$, then its minimal subspace is the cyclic subspace $\text{Span}_\mathbb{R}\{v, Av, \ldots, A^{d-1}v\}$. It is invariant by Cayley-Hamilton theorem, and $v, Av, \ldots, A^{r-1}v$ is a basis for this subspace, where $r \leq d$ is the maximal integer such that these vectors are linearly independent.

(b) Suppose that $A$ is also primitive. Let $u \in \mathbb{R}^d$ be its Perron-Frobenius eigenvector, corresponding to the Perron-Frobenius eigenvalue $\theta_1$, and let $p_{\theta_1}$ be the minimal polynomial of $\theta_1$ over $\mathbb{Q}$. Any $A$-invariant $\mathbb{Q}$-subspace $W$ with $u \in \text{Span}_\mathbb{R}W$ must have $\dim W \geq \deg(p_{\theta_1})$. Since $\theta_1$ is a simple eigenvalue, it follows from the primary decomposition theorem (see [26, Chapter 6, Theorem 12]) that the minimal and characteristic polynomials of $A$ restricted to $U := \ker p_{\theta_1}(A)$ equal $p_{\theta_1}$. Thus, $U$ is the minimal subspace of $u$, and in fact, of any non-zero $v \in U$.

**Lemma 3.4.** Let $A \in M_d(\mathbb{Z})$ and let $V$ be the minimal subspace of some non-zero $v \in \mathbb{R}^d$. There exists a basis of integer vectors for $V$, such that every integer vector in $V$ has integer coordinates with respect to that basis. In particular, the map $A|_V$, written in that basis, is an integer matrix (rather than rational).

**Proof.** Notice that $\mathbb{Z}^d \cap V$ is a subgroup of $\mathbb{Z}^d$, and thus it is free abelian. It is easy to check that a basis of this free abelian group is also a basis of $V$ which meets all the above requirements. □

Such a basis will be called a **lattice basis of $V$**.

**Definition 3.5.** Let $\mathcal{B}$ be a lattice basis of $V$, and consider the isomorphism $\varphi_\mathcal{B} : V \to \mathbb{R}^r$ (where $r = \dim V$) that maps a vector to its coordinate vector $v \mapsto [v]_\mathcal{B}$. A sequence $(x_n) \subset V$ is said to be $u.d. \ mod 1$ in $V$ if the sequence $(\varphi_\mathcal{B}(x_n))$ is $u.d. \ mod 1$ in $\mathbb{R}^r$.

**Remark 3.6.** It is not hard to see that this definition is independent of the choice of the lattice basis, and that $(x_n)$ is $u.d. \ mod 1$ in $V$ if and only if for every lattice basis $\mathcal{B}$ and every $\mathbb{Z}^d$-periodic continuous function $f : V \to \mathbb{C}$, $\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = \int_{\mathbb{T}^r} f \circ \varphi_\mathcal{B}^{-1} dm_r$, where $m_r$ is the $r$-dimensional Haar measure.

We call a vector $v$ positive and write $v > 0$ if it is entrywise positive, and the same applies to matrices.
Corollary 3.7. \( \text{(a)} \) Let \( A \in M_d(\mathbb{Z}) \) and \( v \in \mathbb{Q}^d \setminus \{0\} \). Let \( V = \text{Span}_\mathbb{R}\{v, Av, \ldots, A^{d-1}v\} \), and suppose that \( A|_V \) is non-singular. The sequence \( (A^{kn+\ell}v)_n^{\infty}_{n=0} \) is u.d. mod 1 in \( V \) for every \( k \geq 1, \ell \in \mathbb{N} \) and a.e. \( \omega \in \mathbb{R} \) if and only if \( A|_V \) is non-degenerate with no eigenvalues that are roots of unity.

\( \text{(b)} \) Suppose that \( A \in M_d(\mathbb{Z}) \) is primitive with a Perron-Frobenius eigenvalue \( \theta_1 \). Denote by \( p_{\theta_1} \) the minimal polynomial of \( \theta_1 \) over \( \mathbb{Q} \) and let \( v \in \ker p_{\theta_1}(A) \), \( v \not= 0 \). Then \( (A^{kn+\ell}v)_n^{\infty}_{n=0} \) is u.d. mod 1 in \( \ker p_{\theta_1}(A) \) for every \( k \geq 1, \ell \in \mathbb{N} \) and a.e. \( \omega \in \mathbb{R} \).

Proof. (a) This is just the combination of Theorem 2.9, Corollary 2.13 and Example 3.3(a).

(b) It is well-known that the Perron projection \( P \), defined by \( Pu = u \) for the Perron-Frobenius eigenvector \( u \) and \( Pw = 0 \) for any other generalized eigenvector, is a positive matrix (see for example, [34, Chapter 8]). It follows that \( Pv > 0 \), so the \( u \)-component of \( v \) is not 0, and we conclude by combining Proposition 2.15 and Example 3.3(b).

\[
\square
\]

4. Applications to substitutions

Let \( \mathcal{A} = \{0, \ldots, d-1\} \) be a finite alphabet with \( d \geq 2 \). A substitution is a map \( \zeta : \mathcal{A} \to \mathcal{A}^+ \), where \( \mathcal{A}^+ = \bigcup_{n=1}^{\infty} \mathcal{A}^n \) is the collection of all finite words. This map is extended to \( \mathcal{A}^+ \) and to \( \mathcal{A}^\mathbb{Z} \) by concatenation, and these extensions are called substitutions and denoted by \( \zeta \) as well. The substitution dynamical system, also sometimes called the substitution \( \mathbb{Z} \)-action, is the space

\[ X_\zeta = \{ x \in \mathcal{A}^\mathbb{Z} : \text{every finite subword of } x \text{ is also a subword of } \zeta^n(a) \text{ for some } a \in \mathcal{A} \text{ and } n \in \mathbb{N} \}, \]

together with the left shift map on \( \mathcal{A}^\mathbb{Z} \), restricted to \( X_\zeta \). To every substitution we associate its substitution matrix, which is the \( d \times d \) integer matrix \( S_\zeta \in M_d(\mathbb{Z}) \) whose \((i, j)\)-th entry equals the number of times the letter \( i \) appears in \( \zeta(j) \), for every \( 0 \leq i, j \leq d-1 \). Note that \( S_\zeta^n = S_\zeta \). The substitution is primitive if its substitution matrix is primitive, and in that case, the substitution dynamical system is uniquely ergodic. We say that the substitution is periodic if \( X_\zeta \) contains a shift-periodic point, and otherwise it is aperiodic. For more details on substitutions see [36, 23].

Given a positive vector \( v = (v_0, \ldots, v_{d-1})^t \in \mathbb{R}^d \), the associated substitution \( \mathbb{R} \)-action is the suspension flow over the substitution dynamical system, with the piecewise-constant roof function \( f_\nu : X_\zeta \to \mathbb{R}^+ \) defined by \( f_\nu(x) = xv_0 \). Equivalently, this system can be seen as a one-dimensional tiling space, with tile lengths determined by \( v \), and tilings determined by elements of \( X_\zeta \), see [9, 38, 16]. Two cases of particular interest arise when \( v \) is chosen to be the Perron-Frobenius eigenvector of \( S_\zeta \), where the associated \( \mathbb{R} \)-action is then called self-similar, and when \( v = \tilde{1} \) (where \( \tilde{1} = (1, \ldots, 1)^t \)), which is closely related to the substitution \( \mathbb{Z} \)-action, see [10, Lemma 5.6].

In [14], Bufetov and Solomyak define the spectral cocycle that corresponds to \( \zeta \), and it is further developed in [13]. In what follows, we will generalize their construction, while largely following their path. For every \( b \in \mathcal{A} \) denote \( \zeta(b) = u^1_b \ldots u^w_{\zeta(b)} \) (where \( |w| \) stands for the length of the word \( w \)). First, define a matrix-valued function \( \mathcal{M}_\zeta : \mathbb{R}^d \to M_d(\mathbb{C}) \): let \( \xi = (\xi_0, \ldots, \xi_{d-1})^t \in \mathbb{R}^d \),
then \( M_\zeta(\xi) \) is the complex matrix whose \((b, c)\)-th entry is
\[
\sum_{1 \leq j \leq \ell(b)} \exp(-2\pi i \sum_{k=1}^{j-1} \xi_{u^j_k}).
\]

**Example 4.1.** Let \( \zeta \) be the substitution defined by \( \zeta(0) = 012, \ \zeta(1) = 202, \ \zeta(2) = 111 \) and denote \( e(x) = \exp(-2\pi ix) \). Then for every \( \xi \in \mathbb{R}^3 \),
\[
M_\zeta(\xi) = \begin{pmatrix}
1 & e(\xi_0) & e(\xi_0 + \xi_1) \\
\xi_2 & 0 & 1 + e(\xi_0 + \xi_2) \\
0 & 1 + e(\xi_1) + e(2\xi_1) & 0
\end{pmatrix}.
\]

Note that \( M_\zeta(0) \) is just \( S_\zeta \), and that \( M_\zeta \) is \( \mathbb{Z}^d \)-periodic, so it descends to a function on \( T^d \). The function \( M_\zeta \) gives rise to the *spectral cocycle*,
\[
(4.1) \quad M_\zeta(\xi, n) := M_\zeta(E_{S_\zeta}^{-1}(\xi) \cdots M_\zeta(\xi),
\]
where \( E_{S_\zeta} \) is the endomorphism of \( T^d \) induced by \( S_\zeta \),
\[
E_{S_\zeta}(\xi \mod \mathbb{Z}^d) = S_\zeta^T \xi \mod \mathbb{Z}^d, \quad \xi \in \mathbb{R}^d
\]
(notice that if \( \det S_\zeta = 0 \), \( E_{S_\zeta} \) does not preserve Haar measure). A computation shows that for every \( n \geq 1 \), \( M_\zeta(\xi, n) = M_{\zeta^n}(\xi) \).

Let \( v \in \mathbb{R}^d \setminus \{0\} \). Let \( V \) be its minimal subspace with respect to \( S_\zeta^T \) and denote \( \dim V = r \). Fix a lattice basis \( B \) of \( V \), and denote by \( B \) the integer matrix that corresponds to \( S_\zeta^T | V \) in that basis. Assume that \( B \) is non-singular and that no eigenvalue of \( B \) is a root of unity, so unlike \( E_{S_\zeta} \), the endomorphism \( E_B \), induced by \( B \) on \( T^r \), is measure-preserving and ergodic with respect to the (normalized) Haar measure \( m_r \), see \([19, \text{Corollary 2.20}]\). As before, let \( \varphi_B : V \to \mathbb{R}^r \) be the coordinate isomorphism, \( \xi \mapsto [\xi]_B \), and define \( \tilde{M}_\zeta : \mathbb{R}^r \to M_d(\mathbb{C}) \) by \( \tilde{M}_\zeta = M_\zeta \circ \varphi_B^{-1} \). Since \( B \) is composed of integer vectors, \( M_\zeta \) is \( \mathbb{Z}^r \)-periodic, so it descends to a function on \( T^r \). The *essential spectral cocycle* of \( v \) is defined, similarly to (4.1), to be
\[
\tilde{M}_\zeta(s, n) := \tilde{M}_\zeta(E_B^{n-1} s) \cdots \tilde{M}_\zeta(s).
\]
Note that \( s \mapsto \tilde{M}_\zeta(\varphi_B^{-1}(s), n) \) is also \( \mathbb{Z}^r \)-periodic and \( \tilde{M}_\zeta(s, n) = M_\zeta(\varphi_B^{-1}(s), n) \).

In what follows, \( \|\cdot\| \) stands for a matrix norm on \( M_d(\mathbb{C}) \). All the following claims are independent of the choice of the norm, since all such norms are equivalent. Therefore, for the rest of the paper we will use the Frobenius norm, which is submultiplicative. The next lemma is a simple modification of \([13, \text{Lemma 2.3}]\).

**Lemma 4.2.** For every \( n \geq 1 \), the function \( s \mapsto \log \|\tilde{M}_\zeta(s, n)\| \) is integrable over \((T^r, m_r)\).
Proof. Notice that \( \| \tilde{\mathcal{M}}_\zeta(s,n) \| \leq \| S_\zeta^n \| \). Writing
\[
\| \tilde{\mathcal{M}}_\zeta(s,n) \|^2 = \sum_{b,c} (\mathcal{M}_\zeta(\varphi_B^{-1}(s)))_{bc} (\mathcal{M}_\zeta(\varphi_B^{-1}(s)))_{bc}
\]
and observing that \( \| \tilde{\mathcal{M}}_\zeta(0,n) \|^2 = \| S_\zeta^n \|^2 \), we see that \( \| \tilde{\mathcal{M}}_\zeta(s,n) \|^2 \) is a nontrivial multivariate trigonometric polynomial with integer coefficients. The integral \( \int_{T^r} \log \| \tilde{\mathcal{M}}_\zeta(s,k) \| \, dm_r(s) \) is just the logarithmic Mahler measure of this polynomial, which is known to be at least 0, see e.g. [11]. □

By Furstenberg-Kesten theorem [25] (see also [40]), the Lyapunov exponent exists, namely, the following limit exists and is constant for \( m_r \)-a.e. \( s \in T^r \):
\[
\chi(\mathcal{M}_\zeta, v) := \lim_{n \to \infty} \frac{1}{n} \log \| \tilde{\mathcal{M}}_\zeta(s,n) \|,
\]
and we call it the essential Lyapunov exponent of \( v \). It is independent of the choice of the norm and the basis \( B \). In addition, by Kingman’s theorem (see for example, [40, Theorem 3.3]), the following identity holds:
\[
(4.2) \quad \chi(\mathcal{M}_\zeta, v) = \inf_{k \geq 1} \frac{1}{k} \int_{T^r} \log \| \tilde{\mathcal{M}}_\zeta(s,k) \| \, dm_r(s).
\]

Remark 4.3. In the case that \( V = \mathbb{R}^d \), the spectral cocycle and the essential spectral cocycle of \( v \) coincide. In [13], it is assumed that the characteristic polynomial of the substitution matrix is irreducible over \( \mathbb{Q} \), so this is the case for any non-zero \( v \in \mathbb{R}^d \).

Proposition 4.4. Let \( \zeta \) be a substitution on \( A = \{0, \ldots, d - 1\} \) with \( d \geq 2 \). Let \( v \in \mathbb{R}^d \setminus \{0\} \) and let \( V \) be its minimal subspace. Denote by \( S_\zeta \) the substitution matrix, and suppose that \( S_\zeta |_V \) is non-singular and has no eigenvalue that is a root of unity. If \( (S_\zeta^t)_{1}^{kn}w = 0 \) is u.d. mod 1 in \( V \) for some \( w \in V \) and every \( k \geq 1 \), then
\[
(4.3) \quad \chi_\zeta^+(w) := \limsup_{n \to \infty} \frac{1}{n} \log \| \mathcal{M}_\zeta(w,n) \| \leq \chi(\mathcal{M}_\zeta, v).
\]

Proof. We closely follow the proof of Theorem 2.4 in [13]. For every \( k \geq 1 \),
\[
\chi_\zeta^+(w) = \limsup_{n \to \infty} \frac{1}{nk} \log \| \mathcal{M}_\zeta(w,n,k) \| \leq \limsup_{n \to \infty} \frac{1}{nk} \sum_{j=0}^{n-1} \log \| E_{S_\zeta^j}(w) \| \leq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{nk} \sum_{j=0}^{n-1} \log (\epsilon + \| E_{S_\zeta^j}(w), k \|) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{T^r} \log (\epsilon + \| \tilde{\mathcal{M}}_\zeta(s,k) \|) \, dm_r(s),
\]
where \( r = \dim V \) and in the last equality we used the uniform distribution mod 1 of \( (S_\zeta^t)_{1}^{kn}w \) in \( V \). Split the last integral into two parts, over \( \{ \| \tilde{\mathcal{M}}_\zeta(s,k) \| \geq \frac{1}{2} \} \) and \( \{ \| \tilde{\mathcal{M}}_\zeta(s,k) \| \in [0, \frac{1}{2}) \} \). In the first domain the functions are uniformly bounded, and in the second we have
\[
| \log (\epsilon + \| \tilde{\mathcal{M}}_\zeta(s,k) \|) | \leq | \log \| \tilde{\mathcal{M}}_\zeta(s,k) \| |
\]
so by Lemma 4.2, we can apply the dominated convergence theorem to obtain

$$\chi_\zeta^+(w) \leq \frac{1}{k} \int_{T^d} \log \|\tilde{M}_\zeta(s,k)\| \ dm_r(s).$$

The proof is now completed thanks to (4.2). \(\square\)

Now we state our second main result.

**Theorem 4.5.** Let \(\zeta\) be a primitive aperiodic substitution on \(A = \{0, \ldots, d-1\}\) with \(d \geq 2\). Denote by \(S_\zeta\) the substitution matrix and by \(\theta_1\) the Perron-Frobenius eigenvalue.

(a) Let \(V = \text{Span}_\mathbb{R}\{\mathbf{1}, \ldots, (S_{\zeta}^t)^{d-1}\mathbf{1}\}\), and suppose that \((S_{\zeta}^t|_V)\) is non-singular, non-degenerate and has no eigenvalue that is a root of unity. If

$$\chi(S_{\zeta}^t\mathbf{1}) < \frac{\log \theta_1}{2},$$

then the substitution \(\mathbb{Z}\)-action has purely singular spectrum.

(b) Let \(v \in \ker p_{\theta_1}(S_{\zeta}^t), v > 0\), where \(p_{\theta_1}\) is the minimal polynomial of \(\theta_1\) over \(\mathbb{Q}\). If

$$\chi(S_{\zeta}^t, v) < \frac{\log \theta_1}{2},$$

then the substitution \(\mathbb{R}\)-action associated with \(v\) has purely singular spectrum. In particular, this is true for the self-similar \(\mathbb{R}\)-action associated with the Perron-Frobenius eigenvector of \(S_{\zeta}^t\), and if the characteristic polynomial of \(S_{\zeta}^t\) is irreducible, we can take any positive vector \(v \in \mathbb{R}^d\).

**Proof.** (a) By Corollary 3.7(a), \((S_{\zeta}^t)^{kn}(S_{\zeta}^t)^{\ell} \omega \mathbf{1})_{n=0}^\infty\) is u.d. mod 1 in \(V\) for every \(k \geq 1 \) and \(\ell \in \mathbb{N}\) for a.e. \(\omega \in \mathbb{R}\), and it follows from Proposition 4.4 that

$$\chi(S_{\zeta}^t, \mathbf{1}) < \frac{\log \theta_1}{2},$$

for a.e. \(\omega\). We conclude the proof by applying [13, Lemma 3.1] (notice that in the proof of this lemma, the stronger assumptions made in [13] are used only to prove (4.4), so the lemma still holds in our case).

(b) It follows from Corollary 3.7(b) that \((S_{\zeta}^t)^{kn}(S_{\zeta}^t)^{\ell} \omega v)_{n=0}^\infty\) is u.d. mod 1 in \(U := \ker p_{\theta_1}(S_{\zeta}^t)\) for every \(k \geq 1 \) and \(\ell \in \mathbb{N}\) for a.e. \(\omega \in \mathbb{R}\). It was observed in Example 3.3(b) that the minimal polynomial of \(S_{\zeta}^t|_U\) is \(p_{\theta_1}\), so \(S_{\zeta}^t|_U\) is non-singular and has no eigenvalue that is a root of unity (otherwise \(p_{\theta_1}\) would have been cyclotomic, but \(\theta_1 > 1\)). Thus, by Proposition 4.4,

$$\chi(S_{\zeta}^t, v) < \frac{\log \theta_1}{2},$$

for a.e. \(\omega\), and we use [14, Corollary 4.5(iii)] (see also Section 4.2 in that paper) to conclude (again, the additional assumption made there is needed only to prove (4.5)). \(\square\)
Remark 4.6. (a) In fact, Theorem 4.5 can be extended to $\mathbb{R}$-actions associated with a larger collection of vectors, but we omit the details here.

(b) Notice that given a primitive aperiodic substitution $\zeta$, we can always choose some $k \geq 1$ such that $S^k_\zeta$ is non-degenerate, and replace $\zeta$ by $\zeta^k$ without changing the substitution space. It is also not hard to remove the assumption that $S^t_\zeta|_V$ is non-singular: by the primary decomposition theorem, we can decompose $V$ further into a direct sum of invariant subspaces $V = V_0 \oplus V_1$ where $V_0$ is the generalized eigenspace that corresponds to the eigenvalue 0. Let $v$ be the projection of $\vec{1}$ onto $V_1$, then for every sufficiently large $n$, $(S^t_\zeta)^n\vec{1} = (S^t_\zeta)^n v$, and we can look at the cocycle defined on the minimal subspace of $v$ instead of $V$, where the restriction of $S^t_\zeta$ is guaranteed to be non-singular.

(c) In the case of the Perron-Frobenius eigenvector $u$ of $S^t_\zeta$, some related results were obtained by Baake et al. in terms of the Fourier matrix cocycle, which is closely related to the spectral cocycle. In [5, Fact 5.6], Baake, Grimm and Mañibo showed (using different notations) that for the Fibonacci substitution $\zeta$, $\chi^\omega_\zeta(\omega u)$ exists as a limit for a.e. $\omega \in \mathbb{R}$. Using the theory of Bohr-almost periodic functions, Baake, Frank, Grimm and Robinson gave in [1, Lemma 6.16] a bound, which is relatively similar to (4.3), for some binary non-Pisot substitution. Baake, Gähler and Mañibo extended this bound to the general case in [3], under the additional assumption that the function $\omega \mapsto \log \|M_\zeta(\omega u, n)\|$ is Bohr-almost periodic (the authors mentioned that this assumption can be relaxed by using an extension of a theorem of Sobol, which can be found in [6]), and gave sufficient conditions for the diffraction spectrum to be singular.

5. Examples

In what follows, we consider a few examples of families of reducible non-Pisot substitutions (i.e., the characteristic polynomial of the substitution matrix is reducible over $\mathbb{Q}$, and the Perron-Frobenius eigenvalue is not a Pisot number), and prove they have purely singular spectrum. We will use some of the techniques developed in [5, Section 5.1 and Appendix], [32, Corollary 9] and also used in [13, Section 5]. To ease notation, we write $z_j = e(\xi_j) = \exp(-2\pi i \xi_j)$ for $j = 0, 1, 2$.

Example 5.1. For every $m \geq 3$ define the substitution $\zeta_m$ by $0 \mapsto 0^m12$, $1 \mapsto 1^{2m}02$, $2 \mapsto 0122$. Its corresponding substitution matrix is

$$S^t_{\zeta_m} = \begin{pmatrix} m & 1 & 1 \\ 1 & 2m & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

and a straightforward calculation shows that its eigenvalues $\theta_1, \theta_2, \theta_3$ satisfy $2m < \theta_1 < 2m + 1$, $\theta_2 = m$ and $1 < \theta_3 < 2$, so $\theta_1 \notin \mathbb{Q}$ and thus $\zeta_m$ is aperiodic by [4, Theorem 4.6]. The corresponding
matrix-valued function is

$$M_{\zeta_m}(\xi) = \begin{pmatrix} 1 + \cdots + z_0^{m-1} & \cdots & 1 \cdots + z_1^{m-1} & z_0 z_1^{m-1} & z_0 z_1 \vspace{1mm} \\
1 & \cdots & \cdots & \cdots & 1 + z_0 z_1 (1 + z_2) \end{pmatrix},$$

and since $\mathbb{I}$, $S_{\xi}^t$ and $(S_{\zeta}^t)^2 \mathbb{I}$ are linearly independent, the function $\mathcal{M}_{\zeta}$, which corresponds to $\mathbb{I}$, is just $M_{\zeta}$. Using the Frobenius norm we have

$$\| M_{\zeta_m}(\xi) \|^2 = \left| \frac{z_0^m - 1}{z_0 - 1} \right|^2 + \left| \frac{z_1^m - 1}{z_1 - 1} \right|^2 + |z_2 + 1|^2 + 6,$$

whence

$$\int_{T^3} \log \| M_{\zeta_m}(\xi) \|^2 \, dm_3(\xi) = \int_{T^3} \log \left( |z_0^m - 1|^2 |z_1 - 1|^2 + |z_1^m - 1|^2 |z_0 - 1|^2 \right) \, dm_3(\xi)$$

$$+ |z_0 - 1|^2 |z_1 - 1|^2 |z_2 + 1|^2 + 6 |z_0 - 1|^2 |z_1 - 1|^2 \right) \, dm_3(\xi)$$

$$- \int_{T^3} \log \left( |z_0 - 1|^2 |z_1 - 1|^2 \right) \, dm_3(\xi).$$

Denote the two integrals on the right-hand side by $A$ and $B$ respectively. Applying Jensen’s inequality and then Parseval’s identity, we see that $A \leq \log 40$. Next, by Jensen’s formula, $B = 2 \int_{T} \log (|e(t) - 1|^2) \, dt = 0$. Therefore, using (4.2) with $k = 1$, we see that for every $m \geq 20$,

$$\chi(M_{\zeta_m}, \mathbb{I}) \leq \frac{1}{2} \int_{T^3} \log \| M_{\zeta_m}(\xi) \|^2 \, dm_3(\xi) \leq \frac{1}{2} \log 40 \leq \frac{1}{2} \log (2m) < \frac{1}{2} \log \theta_1,$$

and it follows from Theorem 4.5 that the substitution $\mathbb{Z}$-action has purely singular spectrum.

**Example 5.2.** Define another family of substitutions $\sigma_m$, $m \geq 1$, by $0 \mapsto (01)^m$, $2 \mapsto (10)^m$ and $2 \mapsto 1^{2m+2}$. The eigenvalues of $S_{\sigma_m}$ satisfy $2m + 1 < \theta_1 < 2m + 2$, $-2 < \theta_2 < -1$ and $\theta_3 = 0$, and again this substitution is aperiodic. Denote $q(z_0, z_1) = 1 + (z_0 z_1) + \cdots + (z_0 z_1)^{m-1} = \frac{(z_0 z_1)^m}{z_0 z_1 - 1}$, then we have

$$M_{\sigma_m}(\xi) = \begin{pmatrix} q(z_0, z_1) & z_0 q(z_0, z_1) & (z_0 z_1)^m \\
z_1 z_2 q(z_0, z_1) & z_2 q(z_0, z_1) & 1 \\
0 & 1 + z_1 + \cdots + z_1^{2m+1} & 0 \end{pmatrix}.$$ 

The minimal subspace of both the Perron-Frobenius eigenvector $u$ and $\mathbb{I}$ (with respect to $S_{\zeta}^t$) is $V = \text{Span}\{(1, 0, 0)^t, (0, 0, 1)^t\}$. When restricted to $V$, $\| M_{\sigma_m}(\xi) \|^2$ is simplified into

$$\| \tilde{M}_{\sigma_m}(s_0, s_1) \|^2 = \| M_{\sigma_m}(s_0, s_0, s_1) \|^2 = 4 \left| \frac{z_0^m - 1}{z_0^2 - 1} \right|^2 + \left| \frac{z_0^{2m+2} - 1}{z_0 - 1} \right|^2 + 2.$$
and it follows from Jensen inequality, Parseval’s identity and Cauchy–Schwarz inequality that Proceeding as in the previous example, for every \( m \geq 8 \) and every positive vector \( w \in V \) we have 

\[
\chi(\mathcal{M}_{m}, w) \leq \frac{1}{2} \log(16) < \frac{1}{2} \log(\theta_1),
\]

so by Theorem 4.5, the \( \mathbb{Z} \)-action and any \( \mathbb{R} \)-action associated with a positive vector in \( V \) have purely singular spectrum. Moreover, [38, Corollary 4.5] immediately implies that the self-similar action associated with \( u \) is singular continuous.

**Example 5.3.** Define \( \zeta \colonequals \zeta_{m, A, B} \) by \( 0 \mapsto A2, \ 1 \mapsto 2B, \ 2 \mapsto 022 \), where \( A, B \in \{0, 1\}^m \).

Suppose that \( A \neq 0^m \) and that in each of the words \( A, B \), its less frequent letter appears at most \( k \) times, where \( 8k^2 + 8k + 14 \leq m \). The eigenvalues of \( S_\zeta \) satisfy \( m < \theta_1 < m + 1 \), \( \theta_2 = \ell_0(A) - \ell_0(B) \) and \( 1 < \theta_3 < 2 \), where \( \ell_0(A) \) and \( \ell_0(B) \) are the number of 0’s in \( A \) and \( B \) respectively. The minimal subspace of both \( \bar{T} \) and the Perron-Frobenius eigenvector \( u \) is again \( \text{Span}\{(1,1,0)^t, (0,0,1)^t\} \). Using the notation \( z_j = e(s_j) \) we get

\[
\int_{\mathbb{T}^2} \log \| \mathcal{M}_\zeta(s) \|^2 dm_2(s) = \int_{\mathbb{T}^2} \log(3 + |1 + z_1|^2 + \sum_{b,c=0,1} |(\mathcal{M}_\zeta(s_0, s_0, s_1))_{bc}|^2) dm_2(s)
\]

\[
\leq \int_{\mathbb{T}^2} \log(2(1 + \cdots + z_0^{m-1} + k)^2 + 2k^2 + 3 + |1 + z_1|^2) dm_2(s)
\]

\[
= \int_{\mathbb{T}^2} \log(2|z_0^m - 1|^2 + 4k|z_0^m - 1||z_0 - 1| + (4k^2 + 3)|z_0 - 1|^2 + |1 + z_1|^2|z_0 - 1|^2) dm_2(s),
\]

and it follows from Jensen inequality, Parseval’s identity and Cauchy–Schwarz inequality that

\[
\chi(\mathcal{M}_\zeta, \bar{T}) = \chi(\mathcal{M}_\zeta, u) \leq \frac{1}{2} \log(8k^2 + 8k + 14) < \frac{1}{2} \log(\theta_1),
\]

and both associated actions, as well as any other \( \mathbb{R} \)-action associated with a positive vector in this subspace, are purely singular.

**Acknowledgements:** The author is grateful to Boris Solomyak for many helpful ideas, suggestions and comments. This research is a part of the author’s master’s thesis (in preparation) at the Bar-Ilan University under the direction of B. Solomyak and was supported in part by the Israel Science Foundation grant 911/19 (PI B. Solomyak).
References

[1] Michael Baake, Natalie Priebe Frank, Uwe Grimm, and E. Arthur Robinson, Jr. Geometric properties of a binary non-Pisot inflation and absence of absolutely continuous diffraction. *Studia Mathematica*, 247(2):109–154, 2019.

[2] Michael Baake and Franz Gähler. Pair correlations of aperiodic inflation rules via renormalisation: Some interesting examples. *Topology and its Applications*, 205:4–27, 2016.

[3] Michael Baake, Franz Gähler, and Neil Mañibo. Renormalisation of pair correlation measures for primitive inflation rules and absence of absolutely continuous diffraction. *Communications in Mathematical Physics*, 370(2):591–635, 2019.

[4] Michael Baake and Uwe Grimm. *Aperiodic Order. Volume 1: A Mathematical Invitation*. Cambridge University Press, Cambridge, 2013.

[5] Michael Baake, Uwe Grimm, and Neil Mañibo. Spectral analysis of a family of binary inflation rules. *Letters in Mathematical Physics*, 108(8):1783–1805, 2018.

[6] Michael Baake, Alan Haynes, and Daniel Lenz. Averaging almost periodic functions along exponential sequences. In *Aperiodic order. Volume 2: Crystallography and Almost Periodicity*, volume 166, pages 343–362. Cambridge University Press, Cambridge, 2017.

[7] Marcy Barge and Beverly Diamond. Coincidence for substitutions of Pisot type. *Bulletin de la Société mathématique de France*, 130(4):619–626, 2002.

[8] Alan Bartlett. Spectral theory of $\mathbb{Z}^d$ substitutions. *Ergodic Theory and Dynamical Systems*, 38(4):1289–1341, 2018.

[9] Daniel Berend and Charles Radin. Are there chaotic tilings? *Communications in Mathematical Physics*, 152(2):215–219, 1993.

[10] Artemi Berlinkov and Boris Solomyak. Singular substitutions of constant length. *Ergodic Theory and Dynamical Systems*, 39(9):2384–2402, 2019.

[11] David W. Boyd. Kronecker’s theorem and Lehmer’s problem for polynomials in several variables. *Journal of Number Theory*, 13(1):116–121, 1981.

[12] Alexander I. Bufetov and Boris Solomyak. On the modulus of continuity for spectral measures in substitution dynamics. *Advances in Mathematics*, 260:84–129, 2014.

[13] Alexander I. Bufetov and Boris Solomyak. On singular substitution $\mathbb{Z}$-actions, 2020. arXiv:2003.11287.

[14] Alexander I. Bufetov and Boris Solomyak. A spectral cocycle for substitution systems and translation flows. *Journal d’Analyse Mathématique*, 141(1):165–205, 2020.

[15] Lax Chan and Uwe Grimm. Substitution-based sequences with absolutely continuous diffraction. In *Journal of Physics: Conference Series*, volume 809, 2017.

[16] Alex Clark and Lorenzo Sadun. When size matters: subshifts and their related tiling spaces. *Ergodic Theory and Dynamical Systems*, 23(4):1043–1057, 2003.

[17] Frederik Michel Dekking. The spectrum of dynamical systems arising from substitutions of constant length. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 41(3):221–239, 1978.

[18] Frederik Michel Dekking and M. Keane. Mixing properties of substitutions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 42(1):23–33, 1978.

[19] Manfred Einsiedler and Thomas Ward. *Ergodic Theory with a view towards number theory*, volume 259 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[20] Graham Everest, Alf J. van der Poorten, Igor Shparlinski, and Thomas Ward. *Recurrence Sequences*. American Mathematical Society, Providence, RI, 2003.
[21] Jan-Hendrik Evertse. On sums of $S$-units and linear recurrences. *Compositio Mathematica*, 53(2):225–244, 1984.

[22] Sébastien Ferenczi, Christian Mauduit, and Arnaldo Nogueira. Substitution dynamical systems: algebraic characterization of eigenvalues. *Annales Scientifiques de l’École Normale Supérieure*, 29(4):519–533, 1996.

[23] N. Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[24] Natalie Priebe Frank. Substitution sequences in $\mathbb{Z}^d$ with a non-simple Lebesgue component in the spectrum. *Ergodic Theory and Dynamical Systems*, 23(2):519–532, 2003.

[25] Harry Furstenberg and Harry Kesten. Products of random matrices. *The Annals of Mathematical Statistics*, 31(2):457–469, 1960.

[26] Kenneth Hoffman and Ray Kunze. *Linear Algebra*. Prentice-Hall, Inc., Englewood Cliffs, NJ, second edition, 1971.

[27] Michael Hollander and Boris Solomyak. Two-symbol Pisot substitutions have pure discrete spectrum. *Ergodic Theory and Dynamical Systems*, 23(2):533–540, 2003.

[28] Bernard Host. Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable. *Ergodic Theory and Dynamical Systems*, 6(4):529–540, 1986.

[29] Bernard Host. Représentation géométrique des substitutions sur 2 lettres. *Unpublished manuscript*, 1992.

[30] Bernard Host. Some results of uniform distribution in the multidimensional torus. *Ergodic Theory and Dynamical Systems*, 20(2):439–452, 2000.

[31] Lauwerens Kuipers and Harald Niederreiter. *Uniform Distribution of Sequences*. Wiley, New York, 1974.

[32] Neil Mañibo. Lyapunov exponents for binary substitutions of constant length. *Journal of Mathematical Physics*, 58(11):113504:1–9, 2017.

[33] David Meiri. Entropy and uniform distribution of orbits in $\mathbb{T}^d$. *Israel Journal of Mathematics*, 105(1):155–183, 1998.

[34] Carl D. Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, Philadelphia, PA, 2000.

[35] L.N. Pushkin. Borel normal vectors on a manifold in $\mathbb{R}^n$. *Theory of Probability and its Applications*, 36(2):391–395, 1992.

[36] Martine Queffélec. *Substitution Dynamical Systems — Spectral Analysis*, volume 1294 of *Lecture notes in mathematics*. Springer-Verlag, Berlin, second edition, 2010.

[37] T. N. Shorey and R. Tijdeman. *Exponential Diophantine equations*, volume 87 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.

[38] Boris Solomyak. Dynamics of self-similar tilings. *Ergodic Theory and Dynamical Systems*, 17(3):695–738, 1997.

[39] Alf J. van der Poorten. Some problems of recurrent interest. In *Topics in Classical Number Theory*, volume 2 of *Colloquia mathematica Societatis Janos Bolyai*, 34 (Budapest, 1981), pages 1265–1294. North-Holland, Amsterdam, 1984.

[40] Marcelo Viana. *Lectures on Lyapunov Exponents*, volume 145 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2014.