A $\frac{4}{3}$-approximation algorithm for finding a spanning tree to maximize its internal vertices

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Abstract. This paper focuses on finding a spanning tree of a graph to maximize the number of its internal vertices. We present an approximation algorithm for this problem which can achieve a performance ratio $\frac{4}{3}$ on undirected simple graphs. This improves upon the best known approximation algorithm with performance ratio $\frac{5}{3}$ before. Our algorithm benefits from a new observation for bounding the number of internal vertices of a spanning tree, which reveals that a spanning tree of an undirected simple graph has less internal vertices than the edges a maximum path-cycle cover of that graph has. We can also give an example to show that the performance ratio $\frac{4}{3}$ is actually tight for this algorithm.

To decide how difficult it is for this problem to be approximated, we show that finding a spanning tree of an undirected simple graph to maximize its internal vertices is Max-SNP-Hard.

Key words: Algorithm, Complexity, Performance Ratio, Spanning Tree, Internal Vertex, Max-SNP-Hard.

1 Introduction

The Maximum Internal Spanning Tree problem, MIST for short, is motivated by the designment of cost-efficient communication networks [5]. It asks to find a spanning tree of a graph such that the number of its internal vertices is maximized. MIST is NP-hard, because a Hamilton path (if present) of a graph is just a maximum internal spanning tree of that graph, and finding a Hamilton path in a graph is NP-Hard classically [15].

MIST is known to admit approximation algorithms with constant performance ratio. Prieto et al. [4] first presented a 2-approximation algorithm for MIST on undirected simple graphs by a local search technique in 2003. Later, by a slight modification, Salamon et al. [5] improved Prieto’s algorithm to running in linear-time. Moreover, they developed a $\frac{4}{3}$-approximation algorithm for MIST on claw-free graphs and a $\frac{6}{5}$-approximation algorithm on cubic graphs [5]. Salamon even showed that his algorithm in [5] can achieve a performance ratio $\frac{r+1}{3}$ on $r$-regular graphs [7]. Later, Salamon [6] devised a local optimization algorithm which can approximate MIST on graphs without leaves to $\frac{7}{4}$ in $O(n^4)$ time. Through a different analysis, Knauer et al. [8] showed that Salamon’s algorithm in [6] can actually achieve a performance ratio $\frac{5}{3}$ on undirected simple graphs in $O(n^3)$ time. Knauer’s algorithm is a simplification of the Salamon’s, because a substantially smaller neighborhood structure in the local optimization is sufficient to guarantee the approximation ratio.

Salamon et al. [6] also devoted attention to the so called weighted MIST, which asks to find a spanning tree of a vertex weighted graph, to maximize the total weights of its internal vertices. They designed a $(2\Delta - 3)$-approximation algorithm for weighted MIST on graphs without leaves with time complexity $O(n^4)$, where $\Delta$ is the maximum degree of the graph. They also proposed a 2-approximation algorithm for weighted MIST on claw-free graphs without leaves with time complexity $O(n^4)$. Later, Knauer et al. [8] proposed a $(3+\epsilon)$-approximation algorithm for weighted MIST on generic undirected simple graphs.

The fix parameterized algorithms of MIST have also been extensively studied in the recent years. Prieto and Sloper [4] designed the first FPT-algorithm with running time $O^*(2^{4k\log k})$ in
2003. Coben et al. [9] improved this algorithm to achieve a time complexity $O^*(49.4^k)$. Then an FPT-algorithm for MIST with time complexity $O^*(8^k)$ [11], and an FPT-algorithm with time complexity $O^*(16^{k+o(k)})$ on directed graphs were proposed by Fomin et al. On directed graphs, a randomized FPT-algorithm proposed by M. Zehavi is by now the fastest one, which runs in $O^*(2^{2-2\Delta+1}k)$ time [16], where $\Delta$ is the vertex degree bound of a graph. On cubic graphs in which each vertex has degree three, Binkele-Raible et al. [12] proposed an FPT-algorithm which runs in $O^*(2^{\sqrt{3x+17}}k)$ time.

For the kernalization of MIST, Prieto and Sloper first presented an $O(k^3)$-vertex kernel [13]. Later, they improved it to $O(k^2)$ [14]. Recently, Fomin et al. [11] gave a $3k$-vertex kernel for this problem, which is the best by now.

As for the exact exponential algorithms, Binkele-Raible et al. [12] proposed a dynamic programming algorithm for MIST with time complexity $O^*(2^n)$. Their algorithm runs in $O^*((2-\epsilon)^n)$ time on degree bounded graphs. Especially, they proposed a branching algorithm for MIST on graphs with vertex degree at most 3, which runs in $O(1.8612^n)$ time and polynomial space.

The best performance ratio for approximating MIST has been $\frac{3}{2}$ by now [68]. Although MIST is NP-Hard, to what extent MIST rejects to be approximated has been keeping undetermined for many years.

In this paper, we devote to approximate MIST on generic undirected simple graphs. We propose an algorithm which can approximate MIST to a performance ratio $\frac{4}{3}$. This improves upon the best known existing performance ratio for approximating MIST [68]. Primarily, our improvement is based on a new observation which reveals that in number, those internal vertices of a spanning tree of a graph can be bounded by the edges of a maximum path-cycle cover of that graph. Thus a spanning tree can be constructed from a maximum path-cycle cover. To arrive at a spanning tree with enough internal vertices, a graph has to be reduced by deleting some of its edges and vertices in favor of getting a maximum path-cycle cover with special natures as we cry for; then a maximum path-cycle cover has to be so reconstructed that each path component of length 1, 2 or 3 can have one of its endpoints adjacent to a vertex of a path component of length at least 4. This makes it possible to use a combinatorial way to construct a spanning tree which has three fourth times as many internal vertices as those a maximum internal spanning tree has.

For answering how difficult it is to approximate MIST, we show that, if $P \neq NP$, MIST rejects any polynomial time algorithm to approximate MIST to $1+\epsilon$ for some $\epsilon > 0$. This proof is done by two reductions which are from $(1,2)$-TSP [17] to the Maximum Path Cover problem [17], then from the Maximum Path Cover problem to MIST.

This paper is organized as follows. Section 2 presents the concepts and notations related to path cover, path-cycle cover, maximum internal spanning tree on graphs. Section 3 presents how to bound the number of internal vertices of a spanning tree by the number of edges of a maximum path-cycle cover. Section 4 presents how to reduce a graph into a special one conditioned by keeping the number of internal vertices of a maximum internal spanning tree unchanged. This just implies that MIST on any undirected simple graph can be approximated to a performance ratio the same as that MIST on a reduced one can be approximated to. In section 5, we devise a $\frac{4}{3}$-approximation algorithm for MIST on reduced graphs. In section 6, we show that MIST is Max-SNP-Hard. Section 7 is concluded by looking forward to the future work for MIST.

2 Preliminaries

In this paper, a graph is always undirected and simple. A path or cycle we mentioned is always simple. Let $G=(V,E)$ be an undirected simple graph. Moreover, $V(G)$ and $E(G)$ also stand for the
vertex and the edge set of $G$ respectively, if there is no special emphasis. A connected component of $G$ is a path (resp. cycle) component, if it is also a path (resp. cycle) of $G$. For $V_1 \subseteq V$, a subgraph of $G$ is induced by $V_1$ if it has all the vertices in $V_1$, and the edges of $G$ each of which has both its ends in $V_1$. The subgraph of $G$ induced by $V_1$ is abbreviated as $G[V_1]$. A subgraph of $G$ is a spanning subgraph of $G$ if it has the vertex set $V$ and an edge set $E_1 \subseteq E$. The spanning subgraph of $G$ with the edge set $E_1 \subseteq E$ is abbreviated as $G[E_1]$.

A spanning subgraph of $G$ is a path-cycle cover of $G$ if every vertex in it is incident with at most 2 edges. A path-cycle cover of $G$ is maximum if its edges are maximized in number over all path covers of $G$.

A spanning subgraph of $G$ is a path cover if every connected component of it is a path component. A path cover of $G$ is maximum if its edges are maximized in number over all path covers of $G$. The Maximum Path Cover problem, MPC for short, is given by an undirected simple graph, and asks to find a maximum path cover of that graph.

Although a path-cycle cover of a graph with $n$ vertices and $m$ edges can be found in $O(nm^{1.5} \log n)$ time [13], finding a maximum path cover of a graph is NP-Hard [17]. Since a maximum path cover of a graph is also a path-cycle cover of that graph, the size of a maximum path cover can be bounded by,

**Lemma 1.** In number, a maximum path cover of a graph has no more edges than those a maximum path-cycle cover of that graph has.

A vertex of a graph is a leaf if its degree is 1, and internal if its degree is more than 1. A maximum internal spanning tree of $G$ is a spanning tree whose internal vertices are maximized in number over all spanning trees of $G$. The Maximum Internal Spanning Tree problem, MIST namely, is given by an undirected simple graph, and asks to find a maximum internal spanning tree of that graph.

### 3 A bound for the number of internal vertices of a spanning tree

Let $G = (V,E)$ be an undirected connected simple graph. In this section, we show that a spanning tree of $G$ has less internal vertices than the edges a maximum path-cycle cover of $G$ has.

**Lemma 2.** If a tree has more than one vertex, then in number, it has a path cover which has less path components than those leaves it has.

**Proof.** Let $T$ be a tree with $x > 1$ leaves. The proof is an inductive method on $x$. If $x = 2$, $T$ is a path component, the lemma holds true of course. Then the inductive assumption is, if a tree has at most $x - 1$ leaves, it has a path cover which has less path components than the leaves it has. Later, we show that if $T$ has $x$ (>$2$) leaves, it must have a path cover with at most $x - 1$ path components.

Since $x > 2$, a path, say $P = u, ..., v \neq u$, can be identified in $T$, where $u$ and $v$ are both leaves of $T$. We then delete those edges incident to the vertices of $P$ except those $P$ has. This gives rise to a spanning forest of $T$. Let $T_1, ..., T_j, T_{j+1}, ..., T_k$ be all the trees in the forest except $P$, where $T_i$ for $1 \leq i \leq j$ has only one vertex while the others do not. Note that the vertex in $T_i$ for $1 \leq i \leq j$ is also a leaf of $T$. Namely, one path can cover $T_i$ for $1 \leq i \leq j$. Moreover, $T_i$ for $j + 1 \leq i \leq k$ has at most $x - 1$ leaves because in addition to rejecting leaves $u$ and $v$, it has at most one leaf which does not act as a leaf in $T$. Let $T_{j+1}, T_{j+2}, ..., T_k$ have $x_{j+1}, x_{j+2}, ..., x_k$ leaves respectively. By the inductive assumption, $T_i$ for $j + 1 \leq i \leq k$ must have a path cover with at most $x_i - 1$ path components. Hence $T$ has a path cover with at most $1 + j + \sum_{j+1 \leq i \leq k}(x_i - 1) \leq 1 + j + \sum_{j+1 \leq i \leq k}(x_i) - (k - j) \leq 1 + j + (x - (2 + j) + (k - j)) - (k - j) = x - 1$ path components. □
**Theorem 1.** In number, a maximum internal spanning tree of $G$ has less internal vertices than the edges a maximum path-cycle cover of $G$ has.

**Proof.** Let $P^*$ be a maximum path cover of $G$ with $|P^*|$ path components. By the fact $|P^*| + |E(P^*)| = |V|$, those path components in $P^*$ have just $|V| - |P^*|$ edges. If $|P^*| = 1$, then a maximum internal spanning tree of $G$ is a Hamilton path, the proof is trivial. Later, let $|P^*| > 1$. If a spanning tree of $G$ has at least $|E(P^*)|$ internal vertices, it must have at most $|P^*|$ leaves. Then by Lemma 2, we can find a path cover of this tree with at most $|P^*| - 1$ path components, or in other words, with at least $|E(P^*)| + 1$ edges. Thus, $G$ also has a path cover with $|E(P^*)| + 1$ edges, which means $P^*$ is not maximum, a contradiction.

By Lemma 1, the proof is done. $\square$

Due to Theorem 1 a simple algorithm arises to approximate MIST to a performance ratio 2: (1) find a maximum path-cycle cover of $G$, say $H$, in which each cycle component has at least four edges; (2) delete one edge from each cycle component in $H$ to transform $H$ into $H'$ as a path cover of $G$; (3) link all path components in $H'$ into a spanning tree of $G$ by adding edges of $G$ to $H'$, where step (3) works because $G$ defaults to be connected. This is a 2-approximation algorithm, because it results in a spanning tree of $G$ to which, each cycle component of $H$ with $k \geq 4$ edges contributes at least $k - 2$ internal vertices; each path component of length one or two contributes at least one internal vertex; each path component of length $k \geq 3$ contributes at least $k - 1$ internal vertices.

To ensure a better performance ratio to approximate MIST, it is necessary to make those connected components in a maximum path-cycle cover contribute more internal vertices to that spanning tree to be constructed. To ensure a performance ratio $\frac{4}{3}$, we have to reduce $G$ by deleting some of its edges and vertices, which will be stated in the next section.

### 4 Edge and vertex reducing

Let $G = (V, E)$ be an undirected connected simple graph. Deleting an edge of $G$ refers to removing that edge from $G$; deleting a vertex of $G$ refers to removing that vertex and the boundary edges from $G$. A deletion of an edge (or a vertex) of $G$ is safe, if the deletion results in a subgraph of $G$ which has a maximum internal spanning tree with no less internal vertices than those a maximum internal spanning tree of $G$ has. Only by deleting some edges and vertices of $G$ safely, can we link those components in a maximum path-cycle cover into a tree with as many internal vertices as we want.

Two vertices are adjacent, if they are both incident to one edge. Two vertices are adjacent respecting an edge, if they are both incident to that edge. An edge of a graph is referred to as a cut edge if deleting it can result in more connected components than those in that graph.

**Lemma 3.** If in $G$, an edge has both its ends adjacent to leaves respectively, and is not a cut edge, then the deletion of the edge is safe.

**Proof.** Let $(u, v)$ be an edge other than a cut one of $G$, where $u$ and $v$ are adjacent to leaves respectively. Let $T^*$ be a maximum internal spanning tree of $G$. If $(u, v) \notin E(T^*)$, then the proof is done. Later, let $(u, v) \in E(T^*)$. Let $T_u, T_v$ be those two sub trees of $T^*$ resulted by removing $(u, v)$ from $T^*$, where $u \in V(T_u), v \in V(T_v)$. Note that a leaf of $G$ must be a leaf of $T^*$. Thus $u$ and $v$ must be internal in $T^*$, otherwise, $T^*$ is not connected. Since $(u, v)$ is not a cut edge of $G$, there must be an edge $(u', v') \in E(G) \setminus \{(u, v)\}$ with $u' \in V(T_u)$ and $v' \in V(T_v)$. Then removing $(u, v)$ from $T^*$, and adding $(u', v')$ to it must result in a spanning tree of $G$ as well as $G[E(G) \setminus \{(u, v)\}]$,
A $\frac{4}{3}$-approximation Algorithm for MIST

which can be denoted as $G[(E(T^* \setminus \{(u, v)\}) \cup \{(u', v')\})$. The vertices $u$ and $v$ are both internal in this spanning tree because,

(1) If $u = u'$, then $v \neq v'$. Thus, $v$ is internal in $T_v$, and moreover in $G[(E(T^* \setminus \{(u, v)\}) \cup \{(u', v')\})$. Adding $(u', v')$ to $T^*[E(T^*) \setminus \{(u, v)\}]$ must make $u$ to be internal in $G[(E(T^*) \setminus \{(u, v)\}) \cup \{(u', v')\}]$.

(2) If $u \neq u'$, then $u$ is internal in $T_u$, and moreover in $G[(E(T^*) \setminus \{(u, v)\}) \cup \{(u', v')\})$. No matter whether $v = v'$ or not, adding $(u', v')$ to $T^*[E(T^*) \setminus \{(u, v)\}]$ must make $v$ to be internal in $G[(E(T^*) \setminus \{(u, v)\}) \cup \{(u', v')\}]$. $\square$

Repeating the safe edge deletion stated by Lemma 3 until no edge exists to subject to Lemma 3 must transform $G$ into a subgraph of $G$ which subjects to the following lemma.

Corollary 1. There is a subgraph of $G$ in which, (1) an edge must be a cut edge, if its two ends both are adjacent to leaves respectively; (2) a maximum internal spanning tree of it has no less internal vertices than those a maximum internal spanning tree of $G$ has.

A vertex is referred to as a cut vertex of a graph, if deleting it results in more connected components than those in that graph. A cut vertex of a graph is super, if deleting it results in at least 2 more connected components than those in that graph. Since $G$ defaults to be connected, deleting a super cut vertex of $G$ must result in at least 3 connected components. For identifying the safe deletions of vertices of $G$, we concentrate on those leaves which are adjacent to super cut vertices.

Lemma 4. If in $G$, a leaf is adjacent to a super cut vertex, then the deletion of it is safe.

Proof. Let $T^*$ be a maximum internal spanning tree of $G$. Let $u$ be a super cut vertex of $G$, while $v$ be a leaf adjacent to $u$ respecting an edge of $G$. Then the degree of $u$ in $T^*$ is at least three. Namely, deleting $u$ from $T^*$ must yield at least three connected components. Since $v$ is a leaf of $T^*$, $v$ can be deleted from $T^*$ with $u$ as an internal vertex of $T^*[V \setminus \{v\}]$, where $T^*[V \setminus \{v\}]$ is a spanning tree of $G[V \setminus \{v\}]$. $\square$

Lemma 4 indicates that those leaves of a graph which are adjacent to super cut vertices have no contribution for finding a maximum internal spanning tree. Thus, by the safe deletions of edges and leaves, we can get a graph as stated in,

Corollary 2. There is a subgraph of $G$ which subjects to,

(1) an edge must be a cut edge, if its two ends each is adjacent to a leaf;
(2) a cut vertex is not super, if it is adjacent to a leaf;
(3) A maximum internal spanning tree of the subgraph has no less internal vertices than those a maximum internal spanning tree of $G$ has.

Proof. By Corollary 1 let $G_1$ be a subgraph of $G$ which subjects to Item (1) and (3) of the corollary. If a leaf is adjacent to a super cut vertex of $G_1$, then by Lemma 3 it can be deleted from $G_1$. This leaf deletion for $G_1$ must result in a subgraph of $G_1$, which also subjects to Item (1) and (3), because the deletion is safe, and moreover, does not bring in any new cut edge to $G_1$, and take away any existing cut edge with two ends adjacent to leaves from $G_1$. Let $G_2$ be a subgraph of $G_1$ resulted by repeating such operation, until no leaf can be found for deletion. Then, $G_2$ must subject to Item (1), (2) and (3). $\square$
We directly name by Reduce\((G)\) the algorithm for \(G\) to delete its edges and vertices safely by the methods in Corollary\[1\] and \[2\] without formalizing its details.

Recall that \(G = (V, E)\). It takes \(O(|V| + |E|)\) time to decide whether an edge of \(G\) has two ends adjacent to respective leaves. Moreover, it takes \(O(|V| + |E|)\) time to decide whether an edge is a cut one. So completing the deletions of edges for \(G\) takes \(O(|E|(|V| + |E|))\) time. It takes \(O(|V| + |E|)\) time to decide whether a vertex of \(G\) is a super cut vertex, or whether it is adjacent to a leaf. There are at most \(|V|\) leaves to be deleted. So completing the deletions of vertices takes \(O(|V|(|V| + |E|))\) time. To sum up, the time complexity of Reduce\((G)\) is \(O((|V| + |E|)^2)\).

A subgraph of \(G\) is reduced if it subjects to Corollary\[2\]. A reduced subgraph of \(G\) must have the same set of internal vertices as \(G\) has. Moreover, every internal vertex of a reduced graph is adjacent to at most one leaf. By Corollary\[1\] and \[2\] Reduce\((G)\) must return a reduced subgraph of \(G\). In the following, we show that it suffices to approximate MIST on a reduced subgraph of \(G\) for approximating MIST on \(G\). Actually, each spanning tree of a reduced subgraph of \(G\) can turn into a spanning tree of \(G\) with those internal vertices unchanged. That is,

**Lemma 5.** For each spanning tree of a reduced subgraph of \(G\), \(G\) has a spanning tree which has the same set of internal vertices as the spanning tree of that reduced subgraph of \(G\) has.

**Proof.** Let \(G_1\) be a reduced subgraph of \(G\), while \(T_1\) be a spanning tree of \(G_1\). Since \(G_1\) has the same set of internal vertices as \(G\), a spanning tree of \(G\) can be obtained by adding to \(T_1\) the leaves in \(V(G) \setminus V(G_1)\). Such a spanning tree of \(G\) must have the same set of internal vertices as that \(T_1\) has.

Let \(G_1\) be a reduced subgraph of \(G\). Let \(I(G)\), \(I(G_1)\) be the sets of internal vertices of the maximum internal spanning trees of \(G\) and \(G_1\) respectively. Since \(G_1\) is a subgraph of \(G\), \(|I(G)| \geq |I(G_1)|\). Then \(|I(G)| = |I(G_1)|\) follows from Corollary\[2\]. If \(T_1\) is a spanning tree of \(G_1\) with \(I(T_1)\) as its set of internal vertices, then a spanning tree of \(G\), say \(T\) can be made from \(T_1\) by adding the leaves in \(V(G) \setminus V(G_1)\) to \(T_1\). By Lemma \[5\] \(|I(T)| = |I(T_1)|\). It follows that \(\frac{|I(G)|}{|I(T)|} = \frac{|I(G_1)|}{|I(T_1)|}\). In other words, if MIST can be approximated to a substantial performance ratio on reduced graphs, so can MIST be done on undirected simple graphs. In the next section, we focus on reduced graphs to ask for their spanning trees.

### 5 How to find a spanning tree in a reduced graph

In this section, \(G_1\) always stands for a connected reduced graph instead of a tree. Ordinarily, a maximum path-cycle cover can be found in \(O(n^2m)\) time in an undirected simple graph, even if each cycle component is restricted to have at least 4 edges\[2\], where \(n = |V(G_1)|\) and \(m = |E(G_1)|\). We focus on finding a spanning tree of \(G_1\) with at least \(\frac{3}{4}\) times as many internal vertices as those a maximum internal spanning tree of \(G_1\) has. By Theorem\[1\] it suffices to construct a spanning tree of \(G_1\) which has at least \(\frac{3}{4}\) times as many internal vertices as the edges a maximum path-cycle cover of \(G_1\) has. To hit this point, we try to reconstruct the maximum path-cycle cover of \(G_1\) at first.

#### 5.1 Reconstruction of a maximum path-cycle cover

We also treat a maximum path-cycle cover as a set of cycle components and path components. The reconstruction aims to transform a maximum path-cycle cover of \(G_1\) into such one that each path component can contribute as many internal vertices as its edges to that spanning tree to be constructed, if its length is no larger than three. A path component is a *singleton* if its length is
A vertex of a path component is inner if its degree in it is 2, and an endpoint otherwise. The endpoint of a singleton is the singleton itself.

Note that although two vertices in distinct connected components in a maximum path-cycle cover cannot be adjacent respecting any edge of the maximum path-cycle cover, they can be adjacent respecting an edge of $G_1$. If a maximum path-cycle cover of $G_1$ contains only one connected component, then getting a maximum internal spanning tree of $G_1$ is trivial. Thus in what follows, a maximum path-cycle cover of $G_1$ is assumed to have more than one connected component. Since $G_1$ defaults to be connected, every connected component in a maximum path-cycle cover must have at least one vertex adjacent to a vertex outside it respecting an edge of $G_1$.

A maximum path-cycle cover can be transformed into one, in which each path component has one endpoint adjacent to a vertex outside it if its length is no more than 3. Those path components of length at most 2 can be dealt with into one as the following lemma states.

**Lemma 6.** There exists such a maximum path-cycle cover of $G_1$ that, if a path component is of length no larger than 2, then it has one endpoint adjacent to a vertex outside it.

**Proof.** Let $H$ be a maximum path-cycle cover of $G_1$. Let $p$ be a path component of length no larger than 2 in $H$. If $p$ is a singleton or has one edge, one endpoint of $p$ must be adjacent to a vertex outside $p$ respecting an edge of $G_1$, because $G_1$ is connected.

If the length of $p$ is 2, the endpoints of $p$ cannot both be leaves of $G_1$, because if so, $G_1$ is a tree, or not reduced. If $p$ has just one endpoint as a leaf of $G_1$, then the other endpoint of $p$ must be adjacent to a vertex outside $p$ respecting an edge of $G_1$, because $G_1$ is connected and simple.

If the length of $p$ is 2, and either of the two endpoints of $p$ is not a leaf of $G_1$ and not adjacent to any vertex outside $p$ respecting an edge of $G_1$, then the two endpoints of $p$ are adjacent respecting an edge of $G_1$. In this situation, $p$ can be replaced by another path component of length 2. Concretely, let $p = v_1 v_2 v_3$, then $(v_1, v_3)$ must be an edge of $G_1$. Thus $q = v_3 v_1 v_2$ is also a path component of length 2. Since $G_1$ is connected, as one endpoint of $q$, $v_2$ must be adjacent to a vertex outside $q$ respecting an edge of $G_1$. So $H \setminus \{p\} \cup \{q\}$ is also a maximum path-cycle cover of $G_1$. Such kind of replacement can be done for every path component of length 2, if it has two endpoints adjacent to each other but adjacent to no vertex outside it respecting an edge in $G_1$. When no path component of length 2 can be replaced, $H$ must be transformed into a maximum path-cycle cover as what the lemma states. \hfill $\square$

To deal with those path components of length three, we have to exclude a situation where two endpoints of a path component are both leaves of $G_1$.

**Lemma 7.** In a maximum path-cycle cover of $G_1$, if a path component has three edges, then its two endpoints are not both leaves of $G_1$.

**Proof.** Let $p = u_1 u_2 u_3 u_4$ be a path component of length three in a maximum path-cycle cover, and $u_1, u_4$ be leaves of $G_1$. By Corollary 2, $(u_2, u_3)$ is a cut edge of $G_1$. Since a maximum path-cycle cover of $G_1$ has at least two connected components, either $u_2$ or $u_3$ must be adjacent to a vertex outside $p$ respecting an edge of $G_1$. Without loss of generality, let $u_2$ be adjacent to a vertex outside $p$ respecting an edge of $G_1$. So the deletion of $u_2$ from $G_1$ will yield at least three connected components because $u_1$ is a leaf and $(u_2, u_3)$ is a cut edge. This comes to a contradiction to the assumption that $G_1$ is reduced. \hfill $\square$

By the following two lemmas, we show that a path component of length three has one endpoint adjacent to a vertex outside it respecting an edge of $G_1$, or can be transformed into one which has one endpoint adjacent to a vertex outside it respecting an edge of $G_1$, no matter whether that path component has an endpoint acting as a leaf of $G_1$ or not,
Lemma 8. In a maximum path-cycle cover of $G_1$, if a path component of length three has no endpoint as a leaf of $G_1$, then it must have an endpoint adjacent to a vertex outside it.

Proof. Let $p = u_1 u_2 u_3 u_4$ be a path component of length three in a maximum path-cycle of $G_1$, where $u_1$ and $u_4$ are endpoints of it rather than leaves of $G_1$. If neither $u_1$ nor $u_4$ is adjacent to any vertex outside $p$ respecting an edge of $G_1$, then there must be two vertices $v, v' \in \{u_2, u_3\}$ such that $(u_1, v) \in E(G_1)$ and $(u_4, v') \in E(G_1)$. This leads to a contradiction because,

1. $v \neq u_2$ and $v' \neq u_3$, otherwise, $G_1$ is not simple.
2. $v = u_3$ and $v' = u_2$ can not happen simultaneously, because if so, $u_1 u_3 u_4 u_2$ will form a cycle component of length four, which contradicts to the assumption that $p$ belongs to a maximum path-cycle cover. \qed

Lemma 9. In a maximum path-cycle cover of $G_1$, if a path component of length three has just one endpoint as a leaf of $G_1$, then respecting an edge of $G_1$, it has one endpoint adjacent to a vertex outside it, or can be transformed into one with one endpoint adjacent to a vertex outside it.

Proof. Let $p = u_1 u_2 u_3 u_4$ be a path component of length three in a maximum path-cycle cover of $G_1$. Without loss of generality, let $u_4$ be a leaf of $G_1$. If $u_1$ is adjacent to a vertex outside $p$ respecting an edge of $G_1$, the proof is done. Otherwise, $u_1$ must be adjacent to $u_3$ respecting an edge of $G_1$ because $u_1$ is not a leaf of $G_1$ and $G_1$ is simple. Since $p$ is not the unique component in the maximum path-cycle cover, $u_2$ or $u_3$ must be adjacent to a vertex outside $p$ respecting an edge of $G_1$.

1. If $u_3$ is adjacent to a vertex outside $p$ while $u_2$ is not, then deleting $u_3$ from $G_1$ will yield at least three connected components. Thus, $u_3$ is a super cut vertex of $G_1$ and adjacent to a leaf, which means $G_1$ is not reduced, a contradiction. That is, $u_3$ cannot be adjacent to any vertex outside $p$.
2. If $u_2$ is adjacent to a vertex outside $p$, then $p' = u_2 u_1 u_3 u_4$ is a path component of length three with $V(p') = V(p)$. Replacing $p$ with $p'$ in the maximum path-cycle cover, $p$ is transformed into a path component of length three with one endpoint adjacent to a vertex outside it respecting an edge of $G_1$. \qed

Summing up the reconstructions for a maximum path-cycle cover, we have,

Lemma 10. There is such a maximum path-cycle cover of $G_1$ that every path component of length at most 3 must have an endpoint adjacent to a vertex outside it respecting an edge of $G_1$.

Proof. Let $H$ be a maximum path-cycle cover of $G_1$. By Lemma 6 every path component of length at most 2 in $H$ has one endpoint adjacent to a vertex outside it or can be transformed into one which has one endpoint adjacent to a vertex outside it respecting an edge of $G_1$.

By Lemma 7, 8 and 9 every path component of length three in $H$ has one endpoint adjacent to a vertex outside it, or can be transformed into one which has one endpoint adjacent to a vertex outside it respecting an edge of $G_1$. That is all for the proof. \qed

If a path component has one endpoint adjacent to a vertex of a cycle component respecting an edge of $G_1$, they can be merged into one path component. Thus,

Lemma 11. There exists a maximum path-cycle cover of $G_1$ in which no endpoint of a path component is adjacent to a vertex of any cycle component respecting an edge of $G_1$.

Proof. In a maximum path-cycle cover, let $p$ be a path component which has an endpoint, say $u$, adjacent to a vertex, say $v$, of a cycle component, say $c$, respecting an edge of $G_1$. Then adding $(u, v)$ and removing an edge incident to $v$ of $c$ will merge $p$ and $c$ into a path component of length
|E(p)| + |E(q)|. This can be done for every path component with one endpoint adjacent to a vertex of a cycle component respecting an edge of $G_1$, which must result in a maximum path-cycle cover as what the lemma states.

Recall that we look toward arriving at a tree to which each path component can contribute as many internal vertices as the edges it has, if its length is no more than three. To meet this aim, we will reconstruct the maximum path-cycle cover into one, in which each path component of length 1, 2 or 3 has one endpoint adjacent to an inner vertex of a path component of length at least four. We have to deal with those path components of length 1 beforehand.

**Lemma 12.** There is such a maximum path-cycle cover of $G_1$ that, respecting an edge of $G_1$, each path component of length 1 has an endpoint adjacent to an inner vertex of a path component of length at least 3.

**Proof.** Let $H$ be a maximum path-cycle cover of $G_1$ which subjects to Lemma 10 and 11. If respecting an edge of $G_1$, one endpoint of a path component of length 1 is adjacent to an inner vertex of another path component of length 2, then these two path components can be replaced by a singleton and another path component of length three which has one endpoint adjacent to a vertex outside it. Concretely, let $p = u_1 u_2$ be a path component of length 1, $q = v_1 v_2 v_3$ be a path component of length 2. Without loss of generality, let $u_1$ be adjacent to $v_2$ respecting an edge of $G_1$. By Lemma 10, let $v_3$ be adjacent to a vertex other than $v_1, v_2$. Moreover, $v_3$ cannot be adjacent to $u_1$ or $u_2$, because if so, $H$ cannot be maximum. Then in $H$, $p$ and $q$ can be replaced by $p' = v_3 v_2 u_1 u_2$ and $q' = v_1$ with $|E(p)| + |E(q)| = |E(p')| + |E(q')|$. Since $v_3$ is adjacent to another vertex than $u_1, u_2, v_1, v_2$, $H \setminus \{p, q\} \cup \{p', q'\}$ must be also a maximum path-cycle cover of $G_1$ which subjects to Lemma 10 and 11. Repeating this operation if respecting an edge of $G_1$, a path component of length 1 has one endpoint adjacent to an inner vertex of a path component of length 2, will transform $H$ into a maximum path-cycle cover of $G_1$ as stated in this lemma. □

**Lemma 13.** There is such a maximum path-cycle cover of $G_1$ that, respecting an edge of $G_1$, (1) every singleton is adjacent to an inner vertex of a path component; every path component of length 1, 2 or 3 has an endpoint adjacent to an inner vertex of a path component of length at least four.

**Proof.** Let $H$ be a maximum path-cycle cover of $G_1$ which subjects to Lemma 10 and 11. If respecting an edge of $G_1$, (1) a singleton cannot be adjacent to any path component in $H$ respecting an edge of $G_1$, because $H$ is maximum. A singleton cannot be adjacent to any vertex of a cycle component in $H$ respecting an edge of $G_1$ by Lemma 11. Thus, a singleton must be adjacent to an inner vertex of a path component respecting an edge of $G_1$.

(2) If in $H$, every path component of length 1, 2 or 3 has an endpoint adjacent to an inner vertex of a path component of length at least four respecting an edge of $G_1$, the proof is done. Otherwise, we can transform $H$ into such a maximum path-cycle cover as what the lemma states. Let $p$ be a path component of length 1, 2 or 3, while $q$ be a path component of length 2 or 3, such that one endpoint of $p$ is adjacent to an inner vertex of $q$. By Lemma 12, $p$ and $q$ have at least four edges. Thus it suffices to show that $p$ and $q$ can always be replaced by a singleton and a path component of length $|E(p)| + |E(q)|$.

Let $p = u_1 x_1 x_2 u_2, q = v_1 v_2 y v_3$, where $x_1, x_2, y$ may be nonexistent. Let $(u_1, v_2)$ be an edge of $G_1$. Then we can replace $p$ and $q$ by $p' = v_3 y v_2 u_1 x u_2$ and $q' = v_1$, where $|E(p)| + |E(q)| = |E(p')| + |E(q')|$. Thus, $H \setminus \{p, q\} \cup \{p', q'\}$ is also a maximum path-cycle cover of $G_1$. Since $|E(p)| + |E(q)| \geq 4$, this replacement must eliminate two path components of length 1, 2 or 3 in $H$. We insist to denote by $H$ the maximum path-cycle cover resulted by replacing $\{p, q\}$ with $\{p', q'\}$.
in $H$. Then by Lemma \ref{lem:3approx}, repeating this replacement in $H$ if a path component of length 1, 2 or 3 has one endpoint adjacent to an inner vertex of length 2 or 3, will transform $H$ into a maximum path-cycle cover, in which each path component of length 1, 2 or 3 in $H'$ has one endpoint adjacent to an inner vertex of a path component of length at least 4 resecting an edge of $G_1$. \hfill $\Box$

In the next subsection, we start with a maximum path-cycle cover which subjects to Lemma \ref{lem:3approx} to assemble the connected components in it into a spanning tree.

### 5.2 Assemble of a spanning tree

In this subsection, let $H$ be a maximum path-cycle cover of $G_1$ which subjects to Lemma \ref{lem:3approx}. Let $T$ be a subtree of $G_1$. A path component, say $p$ in $H$, joins $T$, if $V(p) \subseteq V(T)$ and $E(p) \subseteq E(T)$. A cycle component, say $c$ in $H$, joins $T$, if $V(c) \subseteq V(T)$ and $|E(c) \cap E(T)| \geq |E(c)| - 1$. We specially pay attention to those subtrees of $G_1$ which are joined by at least one connected component in $H$. A connected component in $H$ joins a sub-forest of $G_1$, if it joins a tree in this sub-forest. A subtree of $G_1$ is $\alpha$-approximate ($0 \leq \alpha \leq 1$), if it has at least $\alpha$ times as many internal vertices as the edges those connected components which join it have. A sub-forest of $G_1$ is $\alpha$-approximate ($0 \leq \alpha \leq 1$), if all trees in it are $\alpha$-approximate.

To construct a $\frac{3}{4}$-approximate spanning tree of $G_1$, we first assemble those connected components in $H$ into a $\frac{3}{4}$-approximate sub-forest of $G_1$. A path component of length at least four in $H$ is a $\frac{3}{4}$-approximate subtree. This must give rise to a $\frac{3}{4}$-approximate sub-forest of $G_1$, such that every path component of length at least four in $H$ joins one tree of it.

**Lemma 14.** There is a $\frac{3}{4}$-approximate sub-forest of $G_1$, such that every path component of length at least four in $H$ joins one tree of it.

**Proof.** Let $F$ be the set of path components of length at least four in $H$. Then $F$ is such a forest as the lemma states. \hfill $\Box$

Those singletons and path components of length 1, 2 or 3 in $H$, if present, can be assembled together with the path components of length at least 4 respectively, thus into a $\frac{3}{4}$-approximate sub-forest with more vertices than those of the forest Lemma \ref{lem:3approx} states.

**Lemma 15.** There is a $\frac{3}{4}$-approximate sub-forest of $G_1$, such that every path component in $H$ joins one tree of it.

**Proof.** By Lemma \ref{lem:3approx}, let $F$ be the $\frac{3}{4}$-approximation sub-forest formed by the path components of length at least 4 in $H$. Note that for an arbitrary subtree, say $T$ of $G_1$, $E(H[V(T)])$ represents the set of edges the connected components which join $T$ have. If $p$ is a path component of length 1, 2 or 3 in $H$, then by Lemma \ref{lem:3approx}, there is an edge, say $e$, in $G_1$ which is incident with an endpoint of $p$ and a vertex of a tree, say $q$ in $F$. Adding $e$ between $p$ and $q$ will merge $p$ and $q$ into a new tree with the vertex set $V(p) \cup V(q)$. This tree must be $\frac{3}{4}$-approximate, because it has at least $|E(H[V(p)])| + \frac{3}{4}|E(H[V(q)])|$ internal vertices, while the connected components joining it by all, have $|E(H[V(p)])| + |E(H[V(q)])|$ edges. By this method, every path component of length 1, 2 or 3 can be made to join one $\frac{3}{4}$-approximate subtree. This must give rise to a $\frac{3}{4}$-approximate sub-forest of $G_1$ joined by all path components, which will be denoted as $F$ insistently.

If $p$ is a singleton in $H$, by Lemma \ref{lem:3approx} there is an edge, say $e$, in $G_1$ which is incident with $p$ and an internal vertex of a tree, say $q$ in $F$. Then $p$ and $q$ can be merged into a new tree by adding $e$ between them. With the vertex set $V(p) \cup V(q)$, this tree is $\frac{3}{4}$-approximate, because it has at least $\frac{3}{4}|E(H[V(q)])|$ internal vertices, while the connected components joining it by all, have $|E(H[V(q)])|$ edges. By this way, every singleton can be made to join one $\frac{3}{4}$-approximate subtrees. This must give rise to a $\frac{3}{4}$-approximate sub-forest of $G_1$ joined by all path components. \hfill $\Box$
The remainder is to construct a $\frac{3}{4}$-approximate forest such that all connected components in $H$ join it.

**Lemma 16.** There is a $\frac{3}{4}$-approximate spanning forest of $G_1$, such that every connected component in $H$ joins one tree of it.

**Proof.** By Lemma 15, let $F$ be a $\frac{3}{4}$-approximate sub-forest of $G_1$ which is joined by all path components in $H$. If $p$ is a cycle component in $H$, then since $G_1$ is connected, there exists an edge of $G_1$ which is incident with a vertex of $p$ and either a vertex of a tree in $F$ or a vertex of a cycle component other than $p$.

(1) If there is an edge of $G_1$ which is incident with a vertex, say $u$ of $p$ and a vertex in a tree $T$ in $F$, then $T$ and $p$ can be assembled into a tree, say $T'$, by adding this edge between them, and deleting an edge of $p$ incident with $u$. Since $p$ has at least 4 vertices, $T'$ must be $\frac{3}{4}$-approximate, because it has at least $\frac{3}{4}|E(p)| + \frac{3}{4}|E(H[V(T)])|$ internal vertices, while the connected components joining it by all, have $|E(p)| + |E(H[V(T)])|$ edges. Removing $T$ from and appending $T'$ to $F$ must result in a $\frac{3}{4}$-approximate sub-forest which is joined by more components than those joining $F$.

(2) If there is an edge in $G_1$ which is incident with a vertex, say $u$ of $p$ and a vertex, say $v$ of another cycle component, say $q$ in $H$, and $p$ and $q$ can be assembled into a tree, say $T'$, by adding $(u,v)$ between them, and deleting an edge of $p$ incident with $u$ and an edge of $q$ incident with $v$. Since both $p$ and $q$ have at least 4 vertices, $T'$ must be $\frac{3}{4}$-approximate, because it has at least $\frac{3}{4}|E(p)| + \frac{3}{4}|E(q)|$ internal vertices, while $p$ and $q$ together have $|E(p)| + |E(q)|$ edges. That being the case, appending this tree to $F$ must result in a $\frac{3}{4}$-approximate sub-forest which is joined by more components than those joining $F$.

If we insist using $F$ to represent that $\frac{3}{4}$-approximate sub-forest of $G_1$ resulted by the method of (1) or (2), then by the methods of (1) and (2) repeatedly, all cycle components in $H$ can be made to join the subtrees in $F$, which keeps to be $\frac{3}{4}$-approximate all the time.

When all connected components in $H$ are made to join one tree in $F$, then $F$ is a spanning forest of $G_1$, because $V(F) = V(H) = V(G)$ at this time.

Since $G_1$ is connected, we can use a set of edges of $G_1$ to link all trees in the forest made by Lemma 16 into a spanning tree of $G_1$, which has no less internal vertices than all those trees in the forest have.

Finally, we integrate those computational steps for finding a spanning tree of a reduced graph into an algorithm named as SpanningTree($G_1$), where $G_1$ stands for an arbitrary reduced graph. In this algorithm, by reconstructing $H$, Reconstruct($G_1,H$) returns a maximum path-cycle cover of $G_1$ which subjects to Lemma 13.

### Algorithm 1 SpanningTree($G_1$).

**Input:**
- $G_1$: a reduced graph.

**Output:**
- A spanning tree of $G_1$.
- $H'$: a reduced graph.

1: Find a maximum path-cycle cover $H$ of $G_1$;
2: $H' \leftarrow$ Reconstruct($G_1,H$); (Lemma 13)
3: $F \leftarrow \{p \in H: p$ is a path component, $|E(p)| > 3\}$; (Lemma 14)
4: Assemble all path components in $H'$ into $F$; (Lemma 15)
5: Assemble all cycle components in $H'$ into $F$; (Lemma 16)
6: Link the trees in $F$ into a spanning tree $T$;
7: Return $T$. 


Lemma 17. The algorithm \( \text{SpanningTree}(G_1) \) must return a spanning tree of \( G_1 \) which has \( \frac{3}{4} \) times as many internal vertices as those a maximum internal spanning tree of \( G_1 \) has.

Proof. Let \( T \) be the tree returned by \( \text{SpanningTree}(G_1) \). By Lemma 16, that spanning forest of \( G_1 \) from which \( T \) is made is \( \frac{3}{4} \)-approximate. Moreover, linking a spanning forest into a tree does not add any extra vertex to that tree and lose any internal vertex of that forest. Thus \( T \) is \( \frac{3}{4} \)-approximate. By Theorem 1, the proof is done. \( \square \)

Let \( G_1 = (V_1, E_1) \). It takes \( O(|V_1| |E_1|^{1.5} \log|V_1|) \) time to find a maximum path-cycle cover of \( G_1 \); it takes \( O(|V_1| + |E_1|) \) time to reconstruct a maximum path-cycle cover each of whose cycle components has at least four edges. Thus, Step 1, 2 of \( \text{SpanningTree}(G_1) \) takes \( O(|V_1| |E_1|^{1.5} \log|V_1|) \) time. Each step from 3 to 5 for assembling those connected components into a spanning forest of \( G_1 \) takes \( O(|V_1| + |E_1|) \) time. To sum up, the time complexity of \( \text{SpanningTree}(G_1) \) is \( O(|V_1| |E_1|^{1.5} \log|V_1|) \).

Theorem 2. For any undirected simple graph, MIST can be approximated to a performance ratio \( \frac{4}{3} \) in polynomial time.

Proof. Recalling to Section 4, if MIST can be approximated to \( \frac{4}{3} \) on reduced graphs in polynomial time, it can also be approximated to \( \frac{4}{3} \) on all graphs in polynomial time. By Lemma 17, the proof is done. \( \square \)

A spanning tree of an arbitrary undirected simple graph can be found by first calling \( \text{Reduce}() \) to get a reduced graph, then calling \( \text{SpanningTree}() \) to get a spanning tree of that reduced graph, and finally readding those leaves deleted by \( \text{Reduce}() \) to the tree. The time complexity for finding such a spanning tree of \( G = (V, E) \) is \( O(|V| |E|^{1.5} \log|V|) \).

5.3 An example

In Fig. 1 we give an example to verify the performance of the algorithm. The \( \frac{4}{3} \)-approximation algorithm starts with a maximum path-cycle cover exactly containing \( k \) cycles of length 4. The algorithm will output \( T \) as its solution, while \( T^* \) is a spanning tree as an optimal solution. Since \( T \) has \( 3k \) internal vertices, while \( T^* \) has \( 4k - 2 \) internal vertices, thus increasing \( k \), we come close to a \( \frac{4}{3} \) ratio.

6 Hardness to approximate MIST

In this section, we show that if \( P \neq NP \), MIST cannot be approximated to within \( 1 + \epsilon \) for some \( \epsilon > 0 \) in polynomial time. To do this, we first present a reduction from (1,2)-TSP to MPC, then a reduction from MPC to MIST. As a typical NP-hard optimization problem, (1,2)-TSP is given by an undirected complete graph in which each edge has weight 1 or 2, and asks to find a Hamilton cycle of this graph such that the total weights of its edges is minimized. A Hamilton cycle of an edge weighted graph is minimum weighted, if the total weights of its edges is minimized over all Hamilton cycles of that graph. MPC has been proven Max-SNP-hard in [17].

Theorem 3. If \( P \neq NP \), then for some \( \epsilon > 0 \), MPC cannot be approximated to within \( 1 + \epsilon \) in polynomial time.
Proof. The proof is a reduction from (1,2)-TSP. Let $G$ be a graph as an instance of (1,2)-TSP. We set a graph, say $G'$, as an instance of the Maximum Path Cover problem by deleting all those edges of weight 2 from $G$. Let $C^*$ be a minimum weighted Hamilton cycle of $G$, $P^*$ a maximum path cover of $G'$. For a Hamilton cycle of $G$, say $C$, we denote by $w(C)$ the total weights of the edges of $C$. Let $|V(G)| = |V(G')| = n$.

**Property 1.** If $G'$ has a Hamilton cycle, then $w(C^*) + |E(P^*)| = 2n - 1$. Otherwise, $w(C^*) + |E(P^*)| = 2n$.

**Proof.** If $G'$ has a Hamilton cycle, then $w(C^*) = n$, $|E(P^*)| = n - 1$. Thus $w(C^*) + |E(P^*)| = 2n - 1$. Otherwise, $C^*$ must have $n - |E(P^*)|$ edges of weight 2. Thus, $w(C^*) + |E(P^*)| = 2n$. □

**Property 2.** If $G'$ has a path cover, say $P$, with at least 2 path components, then $G$ has a Hamilton cycle, say $C$, with $w(C) - w(C^*) \leq 2(|E(P^*)| - |E(P)|)$.

**Proof.** Let $p_1, ..., p_k$ be the path components in $P$. Let $C$ be the Hamilton cycle by adding the edge of $G$ between an endpoint of $p_i$ and an endpoint of $p_{i+1}$ for $1 \leq i \leq k$, where $p_{k+1} = p_1$. Then $w(C) \leq |E(P)| + 2(n - |E(P)|) = 2n - |E(P)|$. If $G'$ has no Hamilton cycle, then by Property 1, $w(C) \leq w(C^*) + |E(P^*)| - |E(P)|$, and $w(C) - w(C^*) \leq |E(P^*)| - |E(P)|$ consequently. If $G'$ has a Hamilton cycle, then by Property 1, $w(C) \leq w(C^*) + |E(P^*)| + 1 - |E(P)|$. Since $|P| \geq 2$, $|E(P^*)| - |E(P)| \geq 1$. Thus, $w(C) - w(C^*) \leq 2(|E(P^*)| - |E(P)|)$. □

If for some $\epsilon > 0$, MPC can be approximated to $1 + \epsilon$ in polynomial time, we argue that (1,2)-TSP can be approximated to within $(1 + 2\epsilon)$ in polynomial time, which contradicts to the fact that (1,2)-TSP is Max-SNP-Hard [17]. Let $P$ be a path cover as a solution of that $(1 + \epsilon)$-approximation algorithm for $G'$. Then, $|E(P^*)| - |E(P)| \leq \epsilon |E(P)|$. Let $C$ be a Hamilton cycle constructed from $P$ by adding edges between the endpoints of those path components in $P$. If $P$ has only one path component, then $w(C) \leq w(C^*) + 1 \leq (1 + \frac{1}{\epsilon^2}) w(C^*)$. If $n \geq \frac{1}{\epsilon^2}$, then $w(C) \leq (1 + \epsilon) w(C^*)$. If $n < \frac{1}{\epsilon^2}$, we can enumerate at most $O(n^2)$ Hamilton cycles of $G$ to find a minimum weighted one.
If \( P \) has at least 2 path components, then \( C \) is a 1 + 2\( \epsilon \) solution of \( G \) as a (1, 2)-TSP instance, because by Property 2, \( w(C) \leq w(C^*) + 2(|E(P^*)| - |E(P)|) \leq w(C^*) + 2\epsilon|E(P)| \leq w(C^*) + 2\epsilon|E(P^*)| \leq w(C^*) + 2\epsilon w(C^*) \leq (1 + 2\epsilon)w(C^*) \). \( \square \)

**Theorem 4.** If \( P \neq NP \), then for some \( \epsilon > 0 \), MIST cannot be approximated to within \( 1 + \epsilon \) in polynomial time.

**Proof.** The reduction is from MPC. Let \( G \) be a graph as an instance of the Maximum Path Cover problem. We construct a graph \( G' \) as an instance of MIST, by introducing a new vertex and connecting it with each vertex of \( G \) by an edge. Concretely, let \( G = (V, E) \), then \( V(G') = V \cup \{v\} \), \( E(G') = E \cup \{(v, u) : u \in V \} \), where \( v \notin V \). Let \( P^* \) be a maximum path cover of \( G \). Then, \(|P^*| + |E(P^*)| = |V(G)| \). Let \( T^* \) be a maximum internal spanning tree of \( G' \), \( I(T^*) \) the set of internal vertices of \( T^* \), \( L(T^*) \) the set of leaves of \( T^* \). Then \(|L(T^*)| + |I(T^*)| = |V(G)| + 1 \).

**Property 3.** If \( G \) has no Hamilton path, then a maximum path cover of \( G \) has as many path components as the maximum internal spanning tree of \( G' \) has.

**Proof.** (1) From \( P^* \), we can construct a spanning tree \( T \) of \( G' \) by connecting \( v \) with exactly one endpoint of each path component in \( P^* \). Then \( T \) has just \(|P^*| \) leaves. So \(|L(T^*)| \leq |P^*| \).
(2) By Lemma 2, there is a path cover, say \( P \) of \( T^* \), with less path components than the leaves of \( T^* \). That is, \(|P| \leq |L(T^*)| - 1 \). We can get a path cover \( P' \) of \( G \) by deleting \( v \) from \( P \), where \( v \in V(G') \setminus V(G) \). Then \( P' \) has at most \(|P| + 1 \leq |L(T^*)| \) path components. Namely, \(|P^*| \leq |L(T^*)| \).

Finally, \(|P^*| = |L(T^*)| \) follows from (1) and (2). \( \square \)

**Property 4.** A maximum internal spanning tree of \( G' \) has no less internal vertices than the edges those path components in \( P^* \) has.

**Proof.** If \( G \) has a Hamilton path, then \(|E(P^*)| = |V(G)| - 1 \), and \(|I(T^*)| = |V(G)| - 1 \). So \(|E(P^*)| = |I(T^*)| \). If \( G \) has no Hamilton path, then by Property 3, \(|P^*| = |V(G)| + 1 - |I(T^*)| \). Thus, \(|I(T^*)| = |V(G)| + 1 - |P^*| = |E(P^*)| + 1 \). That is \(|E(P^*)| \leq |I(T^*)| \). \( \square \)

Suppose for some \( \epsilon > 0 \), an algorithm can approximate MIST to \( 1 + \epsilon \) on undirected simple graphs. Let \( T \) be a spanning tree of \( G' \) as a solution of this algorithm. Then \(|I(T^*)| \leq (1 + \epsilon)|I(T)| \). We can construct a path cover of \( G \), say \( P \), first by the method in the proof of Lemma 2 to get a path cover of \( G' \), then deleting \( v \in V(G') \setminus V(G) \) from it. By Lemma 2, this path cover of \( G \) must have no more than \(|L(T)| \) path components. That is, \( P \) has at least \(|V(G)| - |L(T)| = |I(T)| - 1 \) edges, which means \(|I(T)| \leq |E(P)| + 1 \). By Property 4, \(|E(P^*)| \leq |I(T^*)| \leq (1 + \epsilon)|I(T)| \leq (1 + \epsilon)(|E(P)| + 1) \). If \(|E(P)| \geq 1 + \epsilon \), then \(|E(P^*)| \leq (1 + 2\epsilon)|E(P)| \), otherwise, one can use \( O(|E(G)|^{1/\epsilon}) \) time to find a maximum path cover of \( G \). This comes to a contradiction to Theorem 3. \( \square \)

7 Conclusion

We have presented an algorithm for MIST which can achieve a performance ratio \( 4/3 \) on undirected simple graphs. We believe that the bound of the number of internal vertices for a spanning tree can be applied to designing useful efficient approximation algorithms for other problems such as the Minimum Leaves Spanning Tree problem. It is interesting whether MIST can be approximated to a better performance ratio than \( 4/3 \) on undirected simple graphs. If one want to follow the method of this paper to arrive at a better performance ratio than \( 4/3 \), it seems necessary to deal with those cycle components in a maximum path-cycle cover. It is also interesting whether a constant can be decided to which MIST rejects to be approximated by a polynomial time algorithm, if \( P \neq NP \).
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