Estimating Causal Effects From Nonparanormal Observational Data

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Abstract

One of the basic aims in science is to unravel the chain of cause and effect of particular systems. Especially for large systems this can be a daunting task. Detailed interventional and randomized data sampling approaches can be used to resolve the causality question, but for many systems such interventions are impossible or too costly to obtain. Recently, Maathuis et al. (2010), following ideas from Spirtes et al. (2000), introduced a framework to estimate causal effects in large scale Gaussian systems. By describing the causal network as a directed acyclic graph it is possible to estimate a class of Markov equivalent systems that describe the underlying causal interactions consistently, even for non-Gaussian systems. In these systems, causal effects stop being linear and cannot be described any more by a single coefficient. In this paper, we derive the general functional form of causal effect in a large subclass of non-Gaussian distributions, called the nonparanormal. We also derive a convenient approximation, which can be used effectively in estimation. We apply the method to an observational gene expression dataset.

Keywords: Causal effects, Directed acyclic graph (DAG), Graphical modeling, Nonparanormal distribution, PC-algorithm, Gaussian copula.

1 Introduction

Inferring cause-and-effect relationships between variables is of primary importance in many fields of science. The classical approach for determining such relationships uses randomized experiments where a single or few variables are perturbed. Such intervention experiments, however, can be very expensive, unethical (e.g. one cannot force a randomly selected person to smoke many cigarettes a day) or even infeasible. Hence, it is desirable to infer causal effects from so-called observational data obtained by observing a system without subjecting it to interventions. Although some important concepts and ideas have been worked out (Spirtes et al., 1995; Richardson, 1996; Mooij et al., 2011), causal inference allowing for cyclic graphs is still in its infancy.

Pearl (2009) described a do-calculus of causal effects, if the underlying causal diagram is known. In practice, though, the influence diagram is often not known and one would like to infer causal effects from observational data together with the influence diagram. Spirtes et al. (2000) introduced methods to estimate causal graphs from observational data, based on a specified causal influence diagram describing qualitatively the causal relations among variables. Verma and Pearl (1990) found that typically groups of causal graphs give rise to the same distribution of the data, which implies that the generating causal DAG is typically unidentifiable from the
data. These groups of causal graphs have characterized Markov equivalence classes for causal DAGs, which called completed partially directed acyclic graph (CPDAG). It has presented many algorithms for constructing and estimated CPDAG in different ways. There are several constraint-based causal search algorithms such as search and score methods \cite{Chickering:2002,Verma:1990}, the PC-algorithm \cite{Spirtes:2000} and Bayesian methods \cite{Heckerman:1995,Spiegelhalter:1993}.

The PC-algorithm \cite{Spirtes:2000} is one of the main algorithms that try to find equivalence class in two steps: first, by estimating the skeleton using conditional independence tests and the characterization of the skeleton; second, orienting as many edges as possible. \cite{Kalisch:2007} used PC-algorithm for Gaussian observations and proved high-dimensional consistency for this algorithm. \cite{Maathuis:2009} propose a method that based on estimated causal structure from \cite{Kalisch:2007}, they could used the interventional distribution in the Gaussian case to drive causal effect from random variables. Based on Gaussian structure, they showed that one can find the causal effect by a set of constants. \cite{Harris:2012}, this broader class that include marginal Gaussian copula is called “nonparanormal distributions.”

In the remainder of the paper, we assume the use of the Rank PC (RPC) algorithm \cite{Harris:2013}, i.e. the PC-algorithm in the nonparanormal context. Based on the estimated CPDAG, it is our aim to derive the concept of a causal effect of \(x\) on \(y\) as a collection of functions of \(x\) and to find a consistent way to estimate them. In Section \(2\), we introduce the causal graph terminology, a short description of the intervention calculus and the definition of a causal effect. In Section \(3\), we derive the structure of a causal effect of a nonparanormal causal effect and in Section \(4\) we define an convenient estimator. In Section \(5\) we evaluate the performance of our method in a simulation study. In Section \(6\) we illustrate the method in a real data example.

2 Causal effects in causal graphs

In this section we describe the background needed in order to define the notion of a causal effect. We begin by defining causal models through directed graphical models.

A graph is a pair \(G = (V, E)\), where \(V\) is a finite set of vertices \(V = \{1, 2, \ldots, p\}\), also called nodes, of \(G\) and \(E\) is a subset of \((V \times V)\) of ordered pairs of vertices, called the edges or links of \(G\). We consider \(p\) random variables \(X_1, \ldots, X_p\), associated to the vertices. If edge \((X_i, X_j) \in E\) but \((X_j, X_i) \notin E\), we call the edge directed or an arrow, denoted by \(X_i \rightarrow X_j\). In that case, we also say that \(X_i\) is a parent of \(X_j\), and that \(X_j\) is a child of \(X_i\). The set of parents of a vertex \(X_j\) is denoted by \(pa(X_j)\). We use the short-hand notation \(X_i \rightarrow X_j\) to denote \((X_i, X_j) \in E\) and \((X_j, X_i) \notin E\). A graph containing only directed edges (\(\rightarrow\)) is directed, one containing only undirected edges (\(\longrightarrow\)) is undirected. A directed graph is called a directed acyclic graph (DAG) if it does not contain directed cycles. A common tool for describing equivalence classes of DAGs are completed partially directed acyclic graphs (CPDAG).

\cite{Pearl:2009} defined causality through intervention, whereby variables are externally manipulated to take certain values. This intervention changes the underlying distribution \(P\) and can be expressed by adapting the direct effect diagram. The new distribution is called the intervention distribution and we say that the variables, whose structural equations we have replaced have been “intervened on.” The intervention distribution of \(Y\) when doing an intervention and setting the variable \(X_i\) to a value \(x_i'\) is denoted by \(P(Y|do(X_i = x_i'))\). The intervention on variable \(X_i\) is characterized by a truncated factorization, in which an intervention DAG \(G'\), arising from the non-intervention DAG \(G\) can be defined by deleting all edges which point into
the node $X_i$. For an example, in below graphs, a DAG $G$ and its corresponding intervention graphs ($G'$) are shown.

![Diagram](image)

**Figure 1**: (a) A DAG $G$ and (b) its corresponding intervention graphs $G'$. The intervention is $do(X_2 = x)$, described by the red label in the graph. The parental set of $i = 2$ is $pa(2) = \{1\}$ which appears in (1) for computing the causal effect $\beta_2$ of $Y$ on $X_2$.

The causal effect of $X_i$ on $Y$ at a point $x'_i$ by the way $Y$ is expected to change as a result from a small interventional change of $X_i$ at $x_i$,

$$CE(Y|X_i = x_i) = \frac{\partial}{\partial x} E[Y|do(X_i = x)]|_{x = x'_i}$$

(1)

where we have that

$$E(Y|do(X_i = x)) = \int E(Y|X, x_{pa(i)}) P(x_{pa(i)}) d(x_{pa(i)})$$

if $Y \notin x_{pa(i)}$ (2)

If $(X_1, ..., X_{p-1}, Y)$ has a multivariate Gaussian distribution, it is very simple to compute the causal effects as defined in (1). Therefore, we have

$$E(Y|do(X_i = x_i)) = \beta_ix_i + \int \beta_{pa(i)}^T x_{pa(i)} P(x_{pa(i)}) d(x_{pa(i)})$$

(3)

is linear in $x_i$, if $Y \notin x_{pa(i)}$ and then the intervention effect, or causal effect, becomes

$$CE(Y|X_i = x) = \frac{\partial}{\partial x} E[Y|do(X_i = x)]|_{x = x'_i} = \beta_i$$

(4)

A simple way to obtain the parameter $\beta_i$ is given by Pearl’s backdoor criterion [Pearl 2009]. From (4), it follows that the causal effect of $X_i$ on $Y$ with $Y \notin x_{pa(i)}$ is given by the regression coefficient of $X_i$ in the regression of $Y$ on $X_i$ and $pa(i)$. Note that if $Y \in x_{pa(i)}$, the causal effect from $X_i$ to $Y$ is, obviously, zero. Our aim is to generalize this to a wider class of distributions.

### 3 Causal effect for nonparanormal graphical models

Kalisch and Bühlmann (2007) use the PC-algorithm in a Gaussian setting for estimating the causal skeleton and, subsequently, the equivalence class of high-dimensional causal graph. The algorithm is based on a clever hierarchical scheme for testing conditional independences among pairs of variables $X_j, X_k$ (for all $j \neq k$) in the DAG. In Gaussian models, tests of conditional independence can be based on Pearson correlations, and high-dimensional consistency results have been obtained for the PC-algorithm in this setting.

Building on this work, Maathuis et al. (2009) are interested in estimating the causal effect of a covariate $X_i$ on a response $Y$ in a Gaussian causal graph. After obtaining the equivalence class of causal DAG, they apply for each DAG $G_j$ in this class the intervention calculus to obtain the causal effect $\beta_{ij}$ of $X_i$ on $Y$, which can easily be shown to be the regression coefficient in

$$E \left[ Y|X_i = x_i; X_{pa(i)} = x_{pa(i)} \right] = \beta_{0j} + \beta_{ij}x_i + \beta_{pa(i),j}^T x_{pa(i)}$$

(5)
where $\text{pa}(i)$ is the parental index set of $X_i$ in graph $G_j$, and then summarize this information for $i = 1, \ldots, p$ and $j = 1, \ldots, m$ in a $p \times m$ matrix $\Theta$.

In this section, we prove how based on this CPDAG we can derive the analogous multi-set of causal effects for Gaussian copula, also called nonparanormal, distributed data. In practice, the conditional independences have to be inferred from the data as well and we show how using our main result in combination with the RPC-algorithm we are able to define an convenient estimator for the causal effect for such data, which stops being linear and needs to be estimated functionally.

3.1 General expression of nonparanormal causal effect

Liu et al. (2012) define the nonparanormal distribution. Let $f = (f_i)_{i \in \mathcal{V}}$ be a set of monotone, univariate functions and let $\Sigma \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ be a positive definite covariance matrix. We say a $p$-dimensional random variable $X = (X_1, \ldots, X_p)^T$ has a nonparanormal distribution,

$X \sim \text{NPN}(\mu, \Sigma, f)$,

if $f^{-1}(X) = (f_1^{-1}(X), \ldots, f_p^{-1}(X)) \sim N(\mu, \Sigma)$. If $X \sim \text{NPN}(\mu, \Sigma, f)$, then the univariate marginal distribution for a coordinate, say $X_i$, can have any distribution $F_i$, as we can take $f_i = F_i^{-1} \circ \Phi_{\mu_i, \sigma_i^2}$, where $\Phi_{\mu_i, \sigma_i^2}$ is the normal distribution function with mean $\mu_i$ and variance $\sigma_i^2 = \Sigma_{i\cdot i}$. Note that $f_i$ need not be continuous. In this paper, we deal with monotone and differentiable $f$. Liu et al. (2012) show that in that case the nonparanormal distribution $\text{NPN}(\mu, \Sigma, f)$ is a Gaussian copula.

In the remainder of the paper, we consider that $(X_1, \ldots, X_{p-1}, Y) \sim \text{NPN}(0, \Sigma, f)$, where $\Sigma$ is a correlation matrix. We will refer to the latent standard normally distributed variables as $Z_i = f_i^{-1}(X_i) = \Phi^{-1} \circ F_i(X_i)$ and $Z = f_y^{-1}(Y) = \Phi^{-1} \circ F_y(Y)$. We are interested in the causal effect of $X_i$ on $Y$ for $i \in \{1, \ldots, p-1\}$. We know from Section (2) that for Gaussian data it is very simple to compute the causal effect, since Gaussianity implies that $E(Y | X_i = x_i; X_{-i} = x_{-i})$ is linear in $x_i$. Unfortunately, this is no longer true for non-Gaussian random variables. In Theorem 1 we derive the explicit functional form for the causal effect in the entire class of nonparanormal distributions.

**Theorem 1.** Let $(X_1, \ldots, X_{p-1}, Y) \sim \text{NPN}(0, \Sigma, f)$ and $f_i$ $(i = 1, \ldots, p-1)$ is differentiable and $f_y$ is infinitely differentiable, then the causal effect of $X_i$ on $Y$ in causal graph $G$ is given by

$$CE(Y | X_i = x_i) = \sum_{k=1}^{\infty} \sum_{r=0}^{k-2r} \sum_{s=1}^{k-2r} f_y^{(k)}(z_0) \frac{1}{k!} \binom{k-2r}{s} \binom{k}{2r} s! \beta_i(-z_0 + \beta_i z_i)^{s-1} \times \left(1 - \rho^2\right)^r f_i^{-1}'(x_i)$$

for every $z_0 \in \mathbb{R}$, where $f_y^{(k)}$ is the $k$th derivative of $f_y, z_i = f_i^{-1}(x_i), Z_{\text{pa}(i)} = f_{\text{pa}(i)}^{-1}(X_{\text{pa}(i)}), (\beta_i, \beta_{\text{pa}(i)}) = (\Sigma_{p,i, \text{pa}(i)})^{-1} \Sigma_{i, \text{pa}(i), \text{pa}(i)}, (\Sigma_{\text{pa}(i)}, \Sigma_{i, \text{pa}(i)}, \Sigma_{i, \text{pa}(i), \text{pa}(i)}, \rho = (\Sigma_{p, i, \text{pa}(i)}, \Sigma_{i, \text{pa}(i)}, \Sigma_{i, \text{pa}(i)}, \text{pa}(i), \rho).$

**Proof.** We follow three steps for proving this theorem. First, we find a closed form expression for $E[Y | X_i = x_i; X_{\text{pa}(i)} = x_{\text{pa}(i)}]$. After that we connect this to the do-operator as is done in (2). Finally, taking the derivative in the way that the causal effect is defined in (1) will complete the proof. From the differentiability of $f_i$ follows that the marginal distributions $F_i$ are one-to-one, where $f_i^{-1}(x_i) = z_i$ and $Z_i = f_i^{-1}(X_i) = \Phi^{-1} \circ F_i(X_i)$ and $Z = f_y^{-1}(Y) = \Phi^{-1} \circ F_y(Y)$.
Using the Taylor expansion, 

\[ E[Y|X_i = x_i; X_{pa(i)} = x_{pa(i)}] = E(F_y^{-1}(\Phi(Z))|X_i = x_i; X_{pa(i)} = x_{pa(i)}) \]

\[ = E(F_y^{-1}(\Phi(Z))|Z_i = z_i; Z_{pa(i)} = z_{pa(i)}) \]

\[ = E(f_y(Z)|Z_i = z_i; Z_{pa(i)} = z_{pa(i)}) \]

\[ = \sum_{k=1}^{\infty} f_y^{(k)}(z_0) \frac{(Z - z_0)^k}{k!} |Z_i = z_i; Z_{pa(i)} = z_{pa(i)}) \]

\[ = \sum_{k=1}^{\infty} f_y^{(k)}(z_0) \frac{1}{k!} E(Z^k|Z_i = z_i; Z_{pa(i)} = z_{pa(i)}) \] (7)

where \( Z^* = Z - z_0 \) for any \( z_0 \in \mathbb{R} \). From the conditional normal distribution, we know that 

\[ Z^*|Z_i = z_i; Z_{pa(i)} = z_{pa(i)} \sim N(-z_0 + (\beta_i, \beta_{pa(i)})(z_i, z_{pa(i)})^T, (1 - \rho^2)) \]

where \((\beta_i, \beta_{pa(i)}) = \Sigma_{p,(i,pa(i))}^{-1}(\Sigma_{p,(i,pa(i))}(i,pa(i)) and \rho = \Sigma_{p,(i,pa(i))}^{-1}(\Sigma_{p,(i,pa(i))}(i,pa(i)) \Sigma_{p,(i,pa(i))}. Following Lehmann and Casella (1998) page 132, we get for \( k \in \mathbb{N} \)

\[ E(Z^k|Z_i = z_i; Z_{pa(i)} = z_{pa(i)}) = \sum_{r=0}^{\lfloor k/2 \rfloor} \binom{k}{2r} (-z_0 + \beta_i z_i + \beta_{pa(i)}^T z_{pa(i)})^{k-2r} \]

\[ \times (2r - 1) \ldots 3.1 \times [(1 - \rho^2)]^r \] (8)

With replacement (8) in (7) we have

\[ E(Y|X_i = x_i; X_{pa(i)} = x_{pa(i)}) = \sum_{k=1}^{\infty} \sum_{r=0}^{\lfloor k/2 \rfloor} f_y^{(k)}(z_0) \frac{1}{k!} \binom{k}{2r} (-z_0 + \beta_i z_i + \beta_{pa(i)}^T z_{pa(i)})^{k-2r} \]

\[ \times (2r - 1) \ldots 3.1 \times [(1 - \rho^2)]^r \] (9)

Now we use (9) for finding the intervention effect for nonparanormal variable. That is,

\[ E(Y|do(X_i = x_i)) = \int E(Y|X_i = x_i; X_{pa(i)} = x_{pa(i)}) P(x_{pa(i)}) d(x_{pa(i)}) \quad \text{if} \ Y \notin x_{pa(i)} \]

\[ = \sum_{k=1}^{\infty} \sum_{r=0}^{\lfloor k/2 \rfloor} f_y^{(k)}(z_0) \frac{1}{k!} \binom{k}{2r} (2r - 1) \ldots 3.1 \times [(1 - \rho^2)]^r \]

\[ \times \sum_{s=0}^{k-2r} \binom{k-2r}{s} (-z_0 + \beta_i z_i)^s \int (\beta_{pa(i)}^T z_{pa(i)})^{k-2r-s} P(z_{pa(i)}) d(z_{pa(i)}) \]

\[ = \sum_{k=1}^{\infty} \sum_{r=0}^{\lfloor k/2 \rfloor} f_y^{(k)}(z_0) \frac{1}{k!} \binom{k}{2r} (2r - 1) \ldots 3.1 \times [(1 - \rho^2)]^r \]

\[ \times \sum_{s=0}^{k-2r} \binom{k-2r}{s} (-z_0 + \beta_i z_i)^s E[(\beta_{pa(i)}^T Z_{pa(i)})^{k-2r-s}] \] (10)

We get the following expression for the causal effect,

\[ \frac{\partial}{\partial x_i} E[Y|do(X_i = x_i)] = \frac{\partial}{\partial z_i} E[Y|do(X_i = x_i)] \frac{\partial z_i}{\partial x_i} \] (11)

where \( \frac{\partial z_i}{\partial x_i} = (f_i^{-1})'(x_i) \). Therefore, with plugging (10) into (11) proof is completes.
We have obtained the general expression \((6)\) for a nonparanormal causal effect. The value of this theorem is that it gives us insight in how higher order moments of the effect \(Y\), captured in the higher order derivatives of \(f_y\), affect the causal effect, whereas higher order moments of the cause \(X\) do not. In practice, this formula is not very helpful as it contains information about the system that we typically do not possess, such as the correlation structure of the latent normal variable. However, this formula can inspire practical estimation procedures of the causal effects in nonparanormal systems. Whereas this is in principle possible, we restrict our attention in this paper to a lower order Taylor approximations in section \(4\), since they tend to be more stable.

### 3.2 Special case

We consider the special case of the above theorem for the situation that only \(Y\) is normally distributed, and the \(X\)'s are still nonparanormal.

**Corollary 1.** Let \((X_1, \ldots, X_{p-1}) \sim NPN(0, \Sigma, f)\) and \(f_i (i = 1, \ldots, p - 1)\) is differentiable and \(Y \sim N(\mu, \sigma^2)\), then the causal effect of \(X_i\) on \(Y\) in causal graph \(G\) is given by

\[
CE(Y | X_i = x_i) = \sigma \beta_i (f_i^{-1})'(x_i) \tag{12}
\]

where \(\beta_i\) is defined as in Theorem \(4\).

The result simply follows from \(f_y(Y) = \mu + \sigma Z\) for \(Z\) standard normal. This special case both inspires an estimator for the causal effect and gives some hope for obtaining some consistency results.

### 4 NCE: nonparanormal causal effect estimator

In this section, we propose a simple estimator for the causal effect that is able to capture nonlinear effects for a wide ranging collection of distributions. Furthermore, we show that under some conditions, this estimator is consistent.

#### 4.1 First order estimator

In the special case of the general causal effect theorem, we derived a one term expression that can be used as inspiration for a first order Taylor estimator of the general causal effect of \(X_i = x\) on \(Y\), i.e.,

\[
\hat{\text{NCE}}_{z_0}(x) = \hat{f}'_y(z_0) \hat{\beta}_i (\hat{f}_i^{-1})'(x), \tag{13}
\]

for some \(z_0, x \in \mathbb{R}\) and where \(\hat{\beta}_i\) is the linear regression coefficient of \(\hat{f}_i^{-1}(Y)\) on \(\hat{f}_i^{-1}(X_i)\), while controlling for the parents \(\hat{f}^{-1}_{pa(i)}(X_{pa(i)})\) of \(i\). In order to obtain consistency, we trim the data for each variable below its \(\alpha/p\) and above \(1 - \alpha/p\) quantiles, where \(p\) is the number of random variables \((X, Y)\). When an observation has been trimmed for one variable, it is removed in its entirety for all variables. This means that in the worst case scenario, \(1 - 2\alpha\) of the observations remain. In practice, we will often use \(\alpha = 0.05\).

We can simplify expression \((13)\) by considering the case that \(z_0 = 0\). Note that it is straightforward to obtain

\[
\hat{f}'_y(0) = \frac{\partial}{\partial u} F_Y^{-1}(u)|_{u=0.5} \phi(0) \quad \text{and} \quad (\hat{f}_i^{-1})'(x) = \left[ \phi(\hat{f}_i^{-1}(x)) \right]^{-1} \frac{\partial}{\partial x} F_i(x)
\]
where $\phi$ is the density function of a standard normal distribution. Considering Figure 2, $F_Y^{-1}$ will be estimated via a monotone increasing smoother $\hat{F}^{-1}_{Y,sm}$, which gives us direct access to its derivative. Similarly, $\frac{\partial}{\partial x} F_i$ will be estimated by taking the derivative of the monotone increasing estimating smoother $\hat{F}^{-1}_{i,sm}$. In particular, we will make use of kernel smoothers, as explained in the next section in order to prove consistency. Finally, $f^{-1}(x)$ will be estimated as $\hat{z} = \Phi^{-1}(\hat{F}^{-1}_{i,sm}(x))$. Putting this together, we obtain a simplified and explicit estimator of a non-paranormal causal effect,

$$\hat{NCE}_0(x) = \hat{\beta}_i \frac{\phi(0)}{\phi(\hat{z})} \left. \frac{\partial F_Y^{-1}_{sm}}{\partial u} \right|_{u=0.5} \left. \frac{\partial \hat{F}^{-1}_{i,sm}}{\partial x} \right|_{x=x_i}. \quad (14)$$

In the following section, we will show that under certain conditions the above estimator is consistent.

### 4.2 Consistency

In this section we will be concerned with the asymptotic behaviour of our estimator in (14) under the assumption of normality of $Y$. We first show that the random, but not necessarily independent, sampling scheme of $(X_1, \ldots, X_{p-1}) \sim NPN(0, \Sigma, f)$ and $Y \sim N(\mu, \sigma^2)$ combined with our lower and upper $\alpha/p$ trimming scheme will eventually fill up the $p$-dimensional cube $[L_\alpha, U_\alpha]$, where $L_\alpha = (L_\alpha^1, \ldots, L_\alpha^{p-1}, L_\alpha^y)$ and $U_\alpha = (U_\alpha^1, \ldots, U_\alpha^{p-1}, U_\alpha^y)$ are the lower and upper quantiles, respectively, for each of the variables $(X_1, \ldots, X_{p-1}, Y)$. From the original sample size $n$ approximately $(1-2\alpha)n$ will fall in this cube. Then we show that the kernel estimators of the functions used in the NCE estimators and their derivatives converge fast to their true values in probability. Together with the fact that products of consistent estimators are consistent, this proves the consistency of the estimator $\hat{NCE}_0(x)$.

**Proposition 1.** Consider any absolutely continuous random variable $X$ with lower $\alpha$ quantile $L_\alpha$ and upper $\alpha$ quantile $U_\alpha$. For the $N \times (1-2\alpha)n$ ordered observations of $X$ in the finite
interval \([L_\alpha, U_\alpha]\), the following property holds
\[
\max_{2 \leq i \leq N} |X(i) - X(i-1)| = O_P(1/N).
\]

The symbol \(\asymp\) denotes that two sequences of real numbers are asymptotically of the same order. The proof of this Proposition is a simple exercise and will not be given here.

Our goal is first to estimate the function \(F_i\) and its derivative \(\frac{d}{dx} F_i\). Similarity, we aim to estimate \(F_i^{-1}\) and its derivative. In order to derive asymptotic properties, we will be using kernel estimators for \(\hat{F}_{i,sm}\) and \(\hat{F}_{i,sm}^{-1}(x)\), respectively,
\[
\hat{F}_{i,n}(x) = \sum_{j=2}^{N} (x_{i(j)} - x_{i(j-1)}) \frac{1}{b_n} K\left(\frac{x - x_{i(j)}}{b_n}\right) (\alpha + j - 1/n) \quad (15)
\]
\[
\hat{F}_{i,n}^{-1}(u) = \sum_{j=1}^{N} \frac{1 - 2\alpha}{N} \frac{1}{b_n} K\left(\frac{u - (\alpha + j(1-\alpha)/N)}{b_n}\right) x_{i(j)} \quad (16)
\]
for \(x \in [L_\alpha^i, U_\alpha^i]\) and \(u \in [\alpha, 1 - \alpha]\), where \(K\) is a kernel function, \(b_n > 0\) denotes the bandwidth that we take to depend on the sample size \(n\) in such a way that \(b_n \to 0\) as \(n \to \infty\) and \(x_{i(1)}, x_{i(2)}, \ldots, x_{i(N)}\) denote the order statistics of that part that for the \(i\) variable that falls within \([L_\alpha, U_\alpha]\). We define an estimator of \(\frac{d}{dx} F_i\) by taking the derivative of the kernel smoother \(\hat{F}_{i,n} = \frac{d}{dx} \hat{F}_{i,n} = \hat{F}_{i,n}'\).

**Proposition 2.** If the kernel \(K\) is symmetric and twice continuously differentiable with support in \([-1,1]\), and if it satisfies the integrability conditions (a) \(\int_{-1}^{1} K(u) du = 1\) and (b) \(\int_{-1}^{1} u^\ell K(u) du = 0\) for \(\ell = 1, \ldots, \gamma - 1\), then for a fixed number \(\delta\), such that \(\alpha < \delta < 1/2\):

(i) If \(F\) and \(F^{-1}\) are \(\gamma \geq 1\) times continuously differentiable and \(b_n \to 0\) as \(n \to \infty\), then
\[
\sup_{x \in [L_\alpha, U_\alpha]} |\hat{F}_{i,n}(x) - F_i(x)| = O_P\left(b_n^\gamma + \frac{1}{nb_n^2} + \sqrt{\frac{\log n}{nb_n}}\right).
\]
\[
\sup_{u \in [\delta, 1-\delta]} |\hat{F}_{i,n}^{-1}(u) - F_i^{-1}(u)| = O_P\left(b_n^\gamma + \frac{1}{nb_n^2} + \sqrt{\frac{\log n}{nb_n}}\right).
\]

(ii) If \(F\) and \(F^{-1}\) are \(\gamma \geq 2\) times continuously differentiable and \(b_n \to 0\) as \(n \to \infty\), then
\[
\sup_{x \in [L_\alpha, U_\alpha]} |\hat{F}_{i,n}'(x) - F_i'(x)| = O_P\left(b_n^{\gamma-1} + \frac{1}{nb_n^3} + \sqrt{\frac{\log n}{nb_n^3}}\right).
\]
\[
\sup_{u \in [\delta, 1-\delta]} |\hat{F}_{i,n}^{-1}'(u) - F_i^{-1}'(u)| = O_P\left(b_n^{\gamma-1} + \frac{1}{nb_n^3} + \sqrt{\frac{\log n}{nb_n^3}}\right).
\]

In particular, \(\hat{F}_{i,n}(x)\) and \(\hat{F}_{i,n}'(x)\) are consistent on \([L_\alpha, U_\alpha]\) and \(\hat{F}_{i,n}^{-1}(x)\) and \(\hat{F}_{i,n}^{-1}'(x)\) are consistent on \([\delta, 1-\delta]\), if \(nb_n^\beta \log n \to \infty\) holds additionally.

The proof is given in Gugushvili and Klaassen (2012, Proposition 3.1). The estimator \(\hat{NCE}_0(x)\) in (14) contains four terms. Based on Proposition 2 we showed the consistency of two terms, \(\hat{F}_{i,n}'(x)\) and \(\hat{F}_{i,n}^{-1}'(x)\). As any continuous function of a consistent estimator is consistent (Lehmann 1999), also \(\hat{z} = \Phi^{-1}(\hat{F}_{i,n}(x))\) is consistent. In order to proof consistency of \(\hat{NCE}_0(x)\)
we still need to show that \( \hat{\beta}_i \) is consistent, where \( \hat{\beta}_i \) is the linear regression coefficient of \( \hat{f}_{yi}^{-1}(Y) \) on \( \hat{f}_{yi}^{-1}(X_i) \), while controlling for the parents \( \hat{f}_{pa(i)}^{-1}(X_{pa(i)}) \) of \( i \). In the following Proposition we show consistency of \( \hat{\beta}_i \).

**Proposition 3.** Let \( \hat{\beta}_i \) be the linear regression coefficient of \( \hat{f}_{yi}^{-1}(Y) \) on \( \hat{f}_{yi}^{-1}(X_i) \), while controlling for the parents \( \hat{f}_{pa(i)}^{-1}(X_{pa(i)}) \) of \( i \), then

\[
\hat{\beta}_i \xrightarrow{P} \beta_i, \tag{17}
\]

where \( \beta_i \) is the true regression coefficient as defined in Theorem 7.

**Proof.** Define

\[
\hat{Z}_n = \begin{pmatrix}
\hat{z}_{1,i} & \hat{z}_{1,pa(i)} & \cdots & \hat{z}_{1,pa(k)} \\
\hat{z}_{2,i} & \hat{z}_{2,pa(i)} & \cdots & \hat{z}_{2,pa(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{z}_{N,i} & \hat{z}_{N,pa(i)} & \cdots & \hat{z}_{N,pa(k)}
\end{pmatrix},
\]

such that \( \hat{z}_{jl,i} = \Phi^{-1}(\hat{F}_{ln}(x_{jl})) \) where \( x_{jl} \) is the non-ordered \( j \)th sample of variable \( l \) and \( pa(i) \) is the index set of \( k \) parents of \( i \). Let

\[
\hat{\Upsilon}_n = \left( \Phi^{-1}(\hat{F}_{yn}(y_1)), \Phi^{-1}(\hat{F}_{yn}(y_2)), \ldots, \Phi^{-1}(\hat{F}_{yn}(y_N)) \right).
\]

The coefficient \( \hat{\beta}_i^n \) is defined as the first element of the vector,

\[
\hat{\beta}_i^n = (\hat{Z}_n Z_n)^{-1} \hat{Z}_i \hat{\Upsilon}_n.
\]

We can also define the oracle estimator \( \hat{B}_i^n \) as the first element of

\[
\hat{B}_n = (Z_n^t Z_n)^{-1} Z_n^t \hat{\Upsilon}_n,
\]

where \( Z_n \) and \( \Upsilon_n \) are obtained by replacing the marginal \( \hat{F} \)’s by the true \( F \)’s. Consider an arbitrary \( \epsilon, \delta > 0 \),

\[
P(|\hat{\beta}_i^n - \beta_i| > \epsilon) = P(|\hat{\beta}_i^n - \hat{B}_i^n + \hat{B}_i^n - \beta_i| > \epsilon) \\
\leq P(|\hat{\beta}_i^n - \hat{B}_i^n| + |\hat{B}_i^n - \beta_i| > \epsilon) \\
\leq P(|\hat{\beta}_i^n - \hat{B}_i^n| > \epsilon/2) + P(|\hat{B}_i^n - \beta_i| > \epsilon/2) \tag{18}
\]

We first consider the first right hand side term of (18). Let’s define \( \hat{A}_n = \frac{Z_n^t Z_n}{n} \), \( A_n = \frac{Z_n^t Z_n}{n} \) and \( \hat{b}_n = \frac{\hat{z}_n^t \hat{Y}_n}{n} \) and \( b_n = \frac{z_n^t \Upsilon_n}{n} \). Then,

\[
P(|\hat{\beta}_i^n - \hat{B}_i^n| > \frac{\epsilon}{2}) \leq P(\| \hat{A}_n^{-1} \hat{b}_n - A_n^{-1} b_n \|^2 > \frac{\epsilon}{2}) \\
\leq P(\| \hat{A}_n^{-1} (\hat{b}_n - b_n) \|^2 + \| (\hat{A}_n^{-1} - A_n^{-1}) b_n \|^2 > \frac{\epsilon}{2}) \\
\leq P(\| \hat{A}_n^{-1} (\hat{b}_n - b_n) \|^2 > \frac{\epsilon}{4}) + P(\| (\hat{A}_n^{-1} - A_n^{-1}) b_n \|^2 > \frac{\epsilon}{4}). \tag{19}
\]

By the consistency of \( \hat{z} \), we have that both \( \hat{b}_n \) and \( b_n \) converge in probability to some \( b = \Sigma_{(i, pa(i)), p} \) and both \( \hat{A}_n^{-1} \) and \( A_n^{-1} \) converge in probability to some \( A^{-1} = \Sigma_{(i, pa(i)), (i, pa(i))}^{-1} \), where \( \Sigma \) is defined in the body of Theorem 1. Therefore, there is a \( n^* \), such that for all \( n \geq n^* \), both terms on the right hand side of (19) are less than \( \delta/4 \). So for all \( n \geq n^* \),

\[
P(|\hat{\beta}_i^n - \hat{B}_i^n| > \frac{\epsilon}{2}) < \frac{\delta}{2}.
\]
For the second term of the right hand side of (18), it is sufficient to use the fact that in the latent normal space a regression estimate is consistent and therefore, there exist a $n^\perp$, such that any $n > n^\perp$,
\[ P(|\hat{B}^n_i - \beta_i| > \epsilon/2) < \delta/2. \]
Putting both results together, we now have that for any $n \geq \max\{n^*, n^\perp\}$,
\[ P(|\hat{\beta}^n_i - \beta_i| > \epsilon) < \delta. \]
Thus we get the desired result.

The following Proposition provides a result that our estimator in (14) is consistent.

**Proposition 4.** Consider the estimator of $NCE_0(x)$ in (14), for which we consider the component estimators (15), (16) and (17). For the kernel estimators, we assume that the conditions of Proposition 2 are satisfied and, furthermore, the bandwidth $b_n \rightarrow 0$, but not too fast so that $nb^3_n/\log n \rightarrow \infty$. We have \[ \hat{NCE}_0 \xrightarrow{P} NCE_0. \]

**Proof.** For two sequences of random variables $Z_n$ and $W_n$ and two random variables $Z, W$, such that $Z_n$ converges in probability to $Z$ and $W_n$ converges in probability to $W$, then it is a standard results that $Z_nW_n$ converges in probability to $ZW$ (Lehmann 1999). As all the components of $\hat{NCE}_0(x)$ have been shown to be consistent, then the estimator is consistent.

## 5 Simulation studies

In this section, we test our estimation method for two different types of distributions, to wit, Gaussian and nonparanormal with exponential margins. For Gaussian data, the method should find constant causal effects and can be compared directly with the IDA method (Maathuis et al., 2009). We consider two scenarios: (i) in which the underlying causal graph is known and (ii) where it is unknown and needs to be estimated via the RPC-algorithm. In the latter case, the IDA method has some additional advantages of being able to use the somewhat more powerful PC-algorithm. For the nonparanormal simulation with exponential margins, calculating the explicit causal effect is very involved in general. Therefore we apply the method to a network with two nodes for which the true causal effect can be evaluated numerically.

### 5.1 Gaussian data

Following Kalisch and Bühlmann (2007), we simulate random DAGs and sample from probability distributions faithful to them. For convenience, we fix an increasing ordering of the variables \{X_1, ..., X_p\}, meaning that for a vector of independent Gaussian variables $\varepsilon = (\varepsilon_1, ..., \varepsilon_p)$
\[ X = AX + \varepsilon, \] (20)
where the coefficient matrix $A$ has entries $A_{ij}$ that are zero for $i < j$ and $A_{ji} \neq 0$ if the corresponding DAG has a directed edge from node $i$ to node $j$ for some $i > j$. The DAGs and skeletons thereof that are created in this way have the following property: $E[N_i] = s(p - 1)$, where $N_i$ is the number of neighbours of a node $i$. With probability one, the vector $X$ solving (20) is Markov and faithful with respect to $G$.

We consider two different size graphs: a small graph with ten vertices and a larger graph with fifty vertices, both with an expected vertex degree of three. For each $n \in \{100, 1000\}$ and each of the two types of graphs, we repeat each simulation 100 times.
5.1.1 Causal DAG known

If we assume that the causal DAG is known, then for estimating the causal effects we apply both our NCE algorithm and the IDA algorithm, described in [5]. Given that the IDA algorithm is made for these Gaussian data, the method should outperform the NCE method, which is agnostic about the underlying distributional assumptions. We apply the methods to the four data scenarios and the results are presented in the last column of Table 1. It shows that when the number of observations are increasing, the mean absolute value deviation for causal effect estimates for both IDA and NCE methods are decreasing. Furthermore, the NCE method, as expected, is more variable. This variation is mostly the result from the poorer estimates of the distributional shape in the tails of the distribution.

5.1.2 Causal DAG unknown

If the underlying causal DAG is considered unknown, then the CPDAG and associated DAGs need to be estimated. For each simulation, we run both the standard PC-algorithm and the robust RPC-algorithm on a grid of significance levels $\alpha$ ranging from $10^{-10}$ to 0.5. For each estimated DAG, we compute the causal effects of each node according to the NCE method and the compare the results with the IDA method.

Figures 3 show the causal effects between the chosen nodes for small graph on ten vertices, i.e. $p = 10$, with $n = 100$. In these figures the red line show the real causal effect between 2 chosen nodes. The blue line shows the average estimated causal effect from the IDA method. The black line show the functional causal effect estimate from (14), proposed by our method. The dashed lines express the average standard deviation of our functional causal effect estimate. A clear message emerges from plots: whereas the IDA method is exactly matched for this simulation scenario, our nonparanormal causal effects estimates are quite stable. Moreover, the confidence intervals calculated by our method typically contain the true effect.

In Table 1 provide numerical comparisons of both methods on data sets with different transformations, where we repeat the experiments 100 times and report the mean absolute value deviation for causal effect on each pair nodes in both IDA and NCE methods.

Table 1: Results of mean absolute value deviation causal effect for comparison NCE and IDA methods for small graph ($p = 10$) and large graph($p = 50$) when the data is Gaussian.

| $n$ | $\alpha = 0.01$ | $\alpha = 0.1$ | DAG Known |
|-----|----------------|----------------|------------|
|     | IDA  | NCE  | IDA  | NCE  | IDA  | NCE  |
| $p = 10$ |
| 100 | 0.101 | 0.576 | 0.144 | 0.554 | 0.118 | 0.455 |
| 1000 | 0.033 | 0.385 | 0.029 | 0.283 | 0.031 | 0.303 |
| $p = 50$ |
| 100 | 3.732 | 2.515 | 2.261 | 3.759 | 2.004 | 2.677 |
| 1000 | 1.175 | 2.100 | 0.964 | 1.378 | 0.724 | 2.281 |

5.2 Exponential data

Only in a few special non-Gaussian distributional examples can we calculate the causal effects [6] exactly. This is particularly relevant in a simulation study, where we want to show the efficiency of our estimation method. We consider the causal effects in a bivariate exponential
distribution. We assume only two nodes with exponential marginal distributions and then apply Crane and Hoek (2008) to find the closed form for conditional expectation formula for Gaussian copula. We derive the causal effect for the bivariate Gaussian copula. If we have a bivariate Gaussian copula, with dependence parameter $\rho$, we have

$$E(Y|X=x) = \int_{\mathbb{R}} y \frac{\partial}{\partial y} \Phi(\Phi^{-1}(F(y)) - \rho \Phi^{-1}(G(x))) \sqrt{1-\rho^2} dy. \quad (21)$$

If both marginal distributions $F$ and $G$ were $N(0,1)$, the copula would revert back to the bivariate normal distribution. The Gaussian copula, however, gives us more flexibility, as it can accommodate any type of univariate distributions, $F$ and $G$. In (21), we choose two marginal distributions that are exponential with parameter $\lambda_x, \lambda_y > 0$. Thus, Equation (21) reduces to

$$E(Y|X=x) = \frac{1}{\sqrt{1-\rho^2}} \int_{\mathbb{R}} y \phi(\Phi^{-1}(1-\exp(\lambda_y y)) - \rho \Phi^{-1}(1-\exp(\lambda_x x))) \frac{\exp(-\lambda_y y)}{\phi(\Phi^{-1}(1-\exp(\lambda_y y)))} \phi(\Phi^{-1}(1-\exp(\lambda_x x))) dy$$

Therefore, for a bivariate nonparanormal with exponential marginals, we obtain the following causal effect,

$$\text{CE}(Y|X=x) = -\frac{\rho}{1-\rho^2} \int_{\mathbb{R}} y \phi'(t) \frac{\exp(-\lambda_x x)\exp(-\lambda_y y)}{\phi(\Phi^{-1}(1-\exp(\lambda_y y)))\phi(\Phi^{-1}(1-\exp(\lambda_x x)))} dy \quad (22)$$

where $t = \Phi^{-1}(1-\exp(\lambda_y y)) - \rho \Phi^{-1}(1-\exp(\lambda_x x)).$
In the simulation study we assume that node $X$ affects node $Y$, in the following fashion,

$$X = F^{-1}(\Phi(Z_1))$$

$$Y = F^{-1}\left(\Phi\left(\frac{Z_1 + Z_2}{\sqrt{2}}\right)\right),$$

where $F$ is the CDF of an Exponential(1) distribution and $Z_1, Z_2 \overset{i.i.d.}{\sim} N(0, 1)$. This falls under the usual nonparanormal scenario. The explicit expression for the causal effect in Theorem 1 is very involved, but we derived in (22) a simplified expression. We evaluated this expression numerically to obtain the true causal effect, expressed as the solid black line in Figure 4. Then we simulated $n = 1,000$ observations from the above model for inferring the causal effect.

We assume that the underlying causal graph, $X \rightarrow Y$, is known and used the NCE method to infer the non-linear causal effect. The blue line Figure 4 shows the functional causal effect estimate from NCE method. It matches very well the true causal effect. Clearly, had IDA been applied in this scenario, it would have come up with a nonsensical constant causal effect.

```
Figure 4: Exponential nonparanormal simulation: black line shows the true causal effect and the blue line represents the causal effect estimated by our NCE method.
```

## 6 TiMet: circadian regulation in *Arabidopsis Thaliana*

In this section, we illustrate our proposed approach by applying it to a time course gene expression dataset related to the study of circadian regulation in plants. The data used in our study come from the EU project TiMet (FP7 245143, 2014), whose objective is the elucidation of the interaction between circadian regulation and metabolism in plants.

The data consist of transcription profiles for the core clock genes from the leaves of various genetic variants of *Arabidopsis Thaliana*, measured with qRTPCR. The transcription profiles of the core clock genes (Aderhold et al. 2014; Pokhilko et al. 2010; Guerriero et al. 2012) were recorded: LHY, CCA1, PRR3, N1 (PRR5), PRR9, TOC1, ELF3, ELF4 and GI. The plants were grown in the following 3 light conditions: a diurnal cycle with 12 hr light and 12 hr darkness (12L/12D), an extended night with full darkness for 24 hrs, and an extended light with constant light for 24 hrs. An exception is the ELF3 mutant, which was grown only in 12L/12D condition. Samples were taken every 2 hrs to measure mRNA concentrations. We consider the same group of nine genes, which from previous studies are known to be involved in circadian regulation (Grzegorczyk and Husmeier 2011a,b; Grzegorczyk et al. 2008; Jia and Huan 2009). They consist of two groups of genes: “Morning genes”, which are LHY, CCA1, PRR9, and PRR5, whose expression peaks in the morning, and “Evening genes”, including TOC1, ELF4, ELF3, GI, and PRR3, whose expression peaks in the evening. The expressions for all the genes are strictly positive and highly right-skewed.
Figure 5: The inferred causal network among the circadian clock genes for *Arabidopsis thaliana*. Yellow nodes refer to morning genes and blue nodes to evening genes.

In traditional analysis of microarray data, the data are typically log-transformed. Especially when using the data for prediction, such transformations are sensible as they typically stabilize variances and make downstream analyses more robust. In our case, however, our aim is to describe the system. We are *not* interested in the causal effect of the log-transformed variables, but we are interested in the causal effects of the original variables. For this reason, we consider the raw data directly, since this is the scale on which we would like to evaluate the system.

For inferring the underlying causal CPDAG, we considered the RPC-algorithm in the version that uses the Kendall’s tau – results using Spearman’s rho were almost the same. The CPDAG contains three Markov equivalent DAGs. One of these three causal networks among the genes is displayed in Figure 5. For all three causal DAGs, we infer the causal effects between the genes and these are shown as three lines in each of the plots in Figure 6. A striking feature is that most of the causal effects shrink towards zero for large values of the cause.

The morning gene CCA1 was found to repress the evening genes EFL3 and NI. Among the evening genes, EFL4 and TOC1 have the strongest effect on both other evening and morning genes. The evening gene ELF has positively affects CCA1. It also has a negative effect on LHY. Moreover, the evening genes ELF3, GI and TOC1 are involved in the activation of the morning gene PRR. The morning gene LHY has a almost constant effect on the evening genes ELF4, TOC1 and EFL4. In particular ELF4 interacts positively with NI and CCA1 and negatively with LHY. Many of these results are consistent with the findings in Grzegorczyk and Husmeier (2011a,b), Aderhold et al. (2014) and references therein, as well as with the biological network referred to in Jia and Huan (2009).

7 Conclusion

In this paper, we have derived an explicit formula for describing a causal effect for a flexible class of distributions, the nonparanormal. These distributions are especially useful for real-life observational studies, where normality assumptions are often not warranted. We presented a simple method, NCE, to estimate these causal effects nonparametrically, based on a first order approximation of the general causal effect formula. It is able to capture a large range
of non-linear causal effect. In a simulation study, we have shown that the estimation method works well, particularly away from the tails of the data. We have also applied the method to an Arabidopsis Thaliana circadian clock network. The estimated causal effects all reveal a tendency for the causal effects to shrink to zero for large values of the cause, which means that gene regulation shows effect saturation for high levels of the regulator. This is in correspondence with simple Michaelis-Menten kinetic models, often used to model gene regulation.

A Appendix: calculation of equation (22)

If we have a bivariate Gaussian copula, with dependence parameter $\rho$, we have

$$E(Y|X = x) = \int_{\mathbb{R}} y \frac{\partial}{\partial y} \Phi(\frac{\Phi^{-1}(F(y)) - \rho \Phi^{-1}(G(x))}{\sqrt{1 - \rho^2}}) dy$$
We choose both marginal distributions $F(y)$ and $G(x)$ are exponential with parameter $\lambda_y, \lambda_x > 0$, respectively. Hence,

\[
E(Y \mid X = x) = \int_{\mathbb{R}} y \phi \left( \frac{\Phi^{-1}(F(y)) - \rho \Phi^{-1}(G(x))}{\sqrt{1 - \rho^2}} \right) \frac{1}{\sqrt{1 - \rho^2}} \frac{\partial}{\partial y} \Phi^{-1}(F(y)) dy
\]

\[
= \frac{1}{\sqrt{1 - \rho^2}} \int_{\mathbb{R}} y \phi \left( \frac{\Phi^{-1}(F(y)) - \rho \Phi^{-1}(G(x))}{\sqrt{1 - \rho^2}} \right) \frac{1}{\phi(\Phi^{-1}(F(y)))} f(y) dy
\]

\[
= \frac{1}{\sqrt{1 - \rho^2}} \int_{\mathbb{R}} y \phi \left( \frac{\Phi^{-1}(1 - \exp(\lambda_y y)) - \rho \Phi^{-1}(1 - \exp(\lambda_x x))}{\sqrt{1 - \rho^2}} \right) \exp(-\lambda_y y) \phi(\Phi^{-1}(1 - \exp(\lambda_y y))) dy
\]

Therefore, for a bivariate nonparanormal with exponential marginals, we find the following causal effect. Let assume $t = \frac{\Phi^{-1}(1 - \exp(\lambda_y y)) - \rho \Phi^{-1}(1 - \exp(\lambda_x x))}{\sqrt{1 - \rho^2}}$,

\[
CE(Y \mid X_i = x) = \frac{\partial}{\partial x} E(Y \mid X = x)
\]

\[
= \frac{1}{\sqrt{1 - \rho^2}} \int_{\mathbb{R}} y \phi \left( \frac{\Phi^{-1}(F(y)) - \rho \Phi^{-1}(G(x))}{\sqrt{1 - \rho^2}} \right) \frac{1}{\phi(\Phi^{-1}(1 - \exp(\lambda_y y)))} \exp(-\lambda_y y) \phi(\Phi^{-1}(1 - \exp(\lambda_y y))) dy
\]

\[
= \frac{1}{\sqrt{1 - \rho^2}} \int_{\mathbb{R}} y \phi \left( \frac{\Phi^{-1}(1 - \exp(\lambda_y y)) - \rho \Phi^{-1}(1 - \exp(\lambda_x x))}{\sqrt{1 - \rho^2}} \right) \exp(-\lambda_y y) \phi(\Phi^{-1}(1 - \exp(\lambda_y y))) \phi(\Phi^{-1}(1 - \exp(\lambda_x x))) dy
\]

where $\Phi$ and $\phi$ are cumulative distribution and density function of Gaussian, respectively.

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