EXPLICIT COPRODUCT FORMULA
FOR QUANTUM GROUPS OF INFINITE SERIES

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Abstract. We find an explicit form for the coproduct formula for PBW generators of quantum groups of infinite series $U_q(\mathfrak{sp}_{2n})$ and $U_q(\mathfrak{so}_{2n})$. Similar formulas for $U_q(\mathfrak{sl}_{n+1})$ and $U_q(\mathfrak{so}_{2n+1})$ are already known.

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1. INTRODUCTION

In the present paper, we prove an explicit coproduct formula for quantum groups $U_q(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{g} = \mathfrak{so}_{2n}$ are simple Lie algebras of type $C, D$ respectively. Consider a Weyl basis of the Lie algebra $\mathfrak{g}$,

$$u[k, m] = \ldots [[x_k, x_{k+1}], x_{k+2}], \ldots, x_m],$$

see [21] Chapter VI, §4 or [6] Chapter IV, §3, XVII]. Here, $x_i = x_{2n-i}$ and in case $C_n$, we have $k \leq m \leq 2n - k$; whereas in case $D_n$, the sequence $x_1, x_2, \ldots, x_{2n-1}$ has no term $x_{n-1}$ and $k \leq m < 2n - k$. If we replace the Lie operation by skew brackets, then the above basis becomes a set of PBW generators for the related quantum group $U_q(\mathfrak{g})$. We then find the coproduct of those PBW generators:

$$(1.1) \Delta(u[k, m]) = u[k, m] \otimes 1 + g_{km} \otimes u[k, m]$$

$$+ \sum_{i=k}^{m-1} \tau_i(1 - q^{-1})g_{ki} u[i + 1, m] \otimes u[k, i],$$

where $g_{ki}$ is a group-like element that corresponds to $u[k, i]$, and almost all $\tau$ equal 1. More precisely, in case $C_n$, there is one exception: $\tau_{n-1} = 1 + q^{-1}$ if $m = n$. In case $D_n$, the exception is: $\tau_{n-1} = 0$ if $m = n$; and $\tau_{n-1} = p_{n-1}$ otherwise.

Recall that the same formula is valid for $U_q(\mathfrak{sl}_{n+1})$ and $U_q(\mathfrak{so}_{2n+1})$. In case $A_n$ there are no exceptions [13, Lemma 3.5]. In case $B_n$, the main parameter $q$ becomes $q^2$, and we have an exception $\tau_n = q$, whereas in the sequence $x_1, x_2, \ldots, x_{2n}$, the variable $x_n$ appears twice: $x_i = x_{2n-i+1}$, see [12, Theorem 4.3]. In case $B_2$, an explicit formula was established by M. Beattie, S. Dăscălescu, Ş. Raianu, [3].

In the formula, if $i \geq 2n - m$, then $u[i + 1, m]$ does not appear in the list of the above PBW generators because $m > 2n - (i + 1)$. The elements $u[k, m]$ with $m > 2n - k$ are defined in a similar manner,

$$u[k, m] = [x_k, [x_{k+1}, \ldots, [x_{m-1}, x_m] \ldots]].$$

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commutation rules

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The formula remains valid for those elements as well, in which case all of the $\tau_i$ equal 1, except for $\tau_n = 1 + q^{-1}$ if $k = n$ in case $C_n$, and $\tau_n = 0$ if $k = n$ in case $D_n$ (by definition, in case $D_n$, the sequence that defines $u[n, m]$ has the form $x_n, x_{n+2}, x_{n+3}, \ldots, x_m$). In other words, whereas the PBW generators do not span a subcoalgebra, the formula remains valid for a basis of the subcoalgebra generated by them. Furthermore, the formula demonstrates that the PBW generators span a left coideal.

The set of PBW generators for $U_q(\mathfrak{g})$ is the union of those sets for positive and negative quantum Borel subalgebras. Thus, we focus only on the positive quantum Borel subalgebra $U_q^+(\mathfrak{g})$.

2. Preliminaries

2.1. Skew brackets. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of quantum variables; that is, associated with each $x_i$ there are an element $g_i$ of a fixed Abelian group $G$ and a character $\chi^i : G \rightarrow k^*$. For every word $w$ in $X$, let $g_w$ or $\text{gr}(w)$ denote an element of $G$ that appears from $w$ by replacing each $x_i$ with $g_i$. In the same manner, $\chi^w$ denotes a character that appears from $w$ by replacing each $x_i$ with $\chi^i$.

Let $G(X)$ denote the skew group algebra generated by $G$ and $k(X)$ with the commutation rules $x_i g = \chi^i(g)x_i$, or equivalently $w g = \chi^w(g)w$, where $w$ is an arbitrary word in $X$. If $u, v$ are homogeneous in each $x_i, 1 \leq i \leq n$ polynomials,
then the skew brackets are defined by the formula
\begin{equation}
[u, v] = uv - \chi^u(g_v)vu.
\end{equation}
We use the notation $\chi^u(g_v) = p_{uv} = p(u, v)$. The form $p(\cdot, \cdot)$ is bimultiplicative:
\begin{equation}
p(u, vt) = p(u, v)p(u, t), \quad p(ut, v) = p(u, v)p(t, v).
\end{equation}
In particular $p(\cdot, \cdot)$ is completely defined by $n^2$ parameters $p_{ij} = \chi^i(g_j)$.

The brackets satisfy an analog of the Jacobi identity:
\begin{equation}
[[u, v], w] = [u, [v, w]] + p_{vw}^{-1}[[u, w], v] + (p_{uw} - p_{vw}^{-1})[u, w] \cdot v.
\end{equation}
The antisymmetry identity transforms as follows:
\begin{equation}
[u, v] = -p_{uv}[v, u] + (1 - p_{uv})u \cdot v
\end{equation}
The Jacobi identity (2.3) implies a conditional identity:
\begin{equation}
[[u, v], w] = [u, [v, w]], \text{ provided that } [u, w] = 0.
\end{equation}

By the evident induction on length, this result allows for the following generalization:

**Lemma 2.1.** [4, Lemma 2.2]. If $y_1, y_2, \ldots, y_m$ are homogeneous linear combinations of words such that $[y_i, y_j] = 0$, $1 \leq i < j - 1 < m$, then the bracketed polynomial $[y_1y_2\ldots y_m]$ is independent of the precise arrangement of brackets:
\begin{equation}
[y_1y_2\ldots y_m] = [[y_1y_2\ldots y_s], [y_{s+1}y_{s+2}\ldots y_m]], \quad 1 \leq s < m.
\end{equation}

Another conditional identity is: if $[u, v] = 0$ (that is, $uv = p_{uv}vu$), then
\begin{equation}
[u, [v, w]] = -p_{uv}[[u, w], v] + p_{uv}(1 - p_{uv})v \cdot [u, w].
\end{equation}
The brackets are related to the product by ad-identities:
\begin{equation}
[u \cdot v, w] = p_{vw}[u, w] \cdot v + u \cdot [v, w],
\end{equation}
\begin{equation}
[u, v \cdot w] = [u, v] \cdot w + p_{uv}v \cdot [u, w].
\end{equation}
It is easy to verify all of the identities developing the brackets by (2.1).

#### 2.2. Quantum Borel algebra.

The group $G$ acts on the free algebra $\mathbb{k}\langle X \rangle$ by $g^{-1}ug = \chi^u(g)u$, where $u$ is an arbitrary monomial in $X$. The skew group algebra $G\langle X \rangle$ has a natural Hopf algebra structure
\begin{equation}
\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g, \quad g \in G.
\end{equation}

Let $C = ||a_{ij}||$ be a symmetrizable Cartan matrix and let $D = \text{diag}(d_1, \ldots, d_n)$ be such that $d_i a_{ij} = d_j a_{ji}$. We denote a Kac-Moody algebra defined by $C$, see [7], as $g$. Suppose that parameters $p_{ij}$ are related by
\begin{equation}
p_{ii} = q^{d_i}, \quad p_{ij}p_{ji} = q^{d_{ij}}, \quad 1 \leq i, j \leq n.
\end{equation}
In this case the multiparameter quantization $U_q^+(g)$ is a homomorphic image of $G\langle X \rangle$ defined by Serre relations with the skew brackets in place of the Lie operation:
\begin{equation}
[\ldots [x_i, x_j], x_j], \ldots, x_j] = 0, \quad 1 \leq i \neq j \leq n.
\end{equation}
By [8], Theorem 6.1, the left-hand sides of these relations are skew-primitive elements in \( G(X) \). Therefore the ideal generated by these elements is a Hopf ideal, hence \( U_q^+(g) \) has the natural structure of a Hopf algebra.

2.3. PBW basis. Recall that a linearly ordered set \( V \) is said to be a set of PBW generators (of infinite heights) if the set of all products

\[
(2.12) \quad g \cdot v_1^{n_1} \cdot v_2^{n_2} \cdot \ldots \cdot v_k^{n_k}, \quad g \in G, \quad v_1 < v_2 < \ldots < v_k \in V
\]

is a basis of \( U_q^+(g) \).

We fix the order \( x_1 > x_2 > \ldots > x_n \) on the set \( X \). On the set of all words in \( X \), we fix the lexicographical order with the priority from left to right, where a proper beginning of a word is considered to be greater than the word itself.

A non-empty word \( u \) is called a standard Lyndon-Shirshov word if \( vw > vw \) for each decomposition \( u = vw \) with non-empty \( v, w \). The standard arrangement of brackets \([u]\) on a standard word \( u \) is defined by induction: \([u] = [[v][w]]\), where \( v, w \) are the standard words such that \( u = vw \) and \( v \) has the minimal length, see also [16].

In [9], it was proven that the values of bracketed standard words corresponding to positive roots with the lexicographical order form a set of PBW generators (of infinite heights) for \( U_q^+(g) \), where \( g \) is a Lie algebra of infinite series \( A, B, C, D \).

2.4. Shuffle representation. The \( k \)-algebra \( A \) generated by values of \( x_i, 1 \leq i \leq n \) in \( U_q^+(g) \) is not a Hopf subalgebra because it has nontrivial group-like elements. Nevertheless, \( A \) is a Hopf algebra in the category of Yetter-Drinfeld modules over \( k[G] \). In particular, \( A \) has a structure of a braided Hopf algebra with a braiding \( \tau(u \otimes v) = p(v, u)^{-1} v \otimes u \). The braided coproduct \( \Delta^b : A \rightarrow A \otimes A \) is connected with the coproduct on \( U_q^+(g) \) as follows

\[
(2.13) \quad \Delta^b(u) = \sum_{(u)} u^{(1)} \text{gr}(u^{(2)})^{-1} \otimes u^{(2)}, \quad \text{where} \quad \Delta(u) = \sum_{(u)} u^{(1)} \otimes u^{(2)}.
\]

The tensor space \( T(W), \ W = \sum x_i k \) also has the structure of a braided Hopf algebra, which is the braided shuffle algebra \( Sh_\tau(W) \) with the coproduct

\[
(2.14) \quad \Delta^b(u) = \sum_{i=0}^{m} (z_1 \ldots z_i) \otimes z_{i+1} \ldots z_m,
\]

where \( z_i \in X \), and \( u = (z_1 z_2 \ldots z_{m-1} z_m) \) is the tensor \( z_1 \otimes z_2 \otimes \ldots \otimes z_{m-1} \otimes z_m \), called a comonomial, considered as an element of \( Sh_\tau(W) \). The braided shuffle product satisfies

\[
(2.15) \quad (w)(x_i) = \sum_{u=v} p(x_i, v)^{-1}(ux_i v), \quad (x_i)(w) = \sum_{u=v} p(u, x_i)^{-1}(ux_i v).
\]

The map \( x_i \rightarrow (x_i) \) defines a homomorphism of the braided Hopf algebra \( A \) into the braided Hopf algebra \( Sh_\tau(W) \). This is extremely useful for calculating the coproducts due to formulae \((2.13)\) and \((2.14)\). If \( q \) is not a root of 1, then this representation is faithful. Otherwise, its kernel is the largest Hopf ideal in \( A^{(2)} \), where \( A^{(2)} \) is the ideal of \( A \) generated by values of \( x_i x_j, 1 \leq i, j \leq n \). See details in P. Schauenberg [20], M. Rosso [19], M. Takeuchi [23], D. Flores de Chela and J.A. Green [4], N. Andruskiewitsch, H.-J. Schneider [1], V. K. Kharchenko [10].
3. Relations in $U_q^+(\mathfrak{sp}_{2n})$.

Throughout the following three sections, we fix a parameter $q$ such that $q^3 \neq 1$, $q \neq -1$. If $C$ is a Cartan matrix of type $C_n$, then relations (2.10) take the form

(3.1) \[ p_{ii} = q, \quad 1 \leq i < n; \quad p_{i-1,i} = q^{-1}, \quad 1 < i < n; \quad p_{ij}p_{ji} = 1, \quad j > i + 1; \]

(3.2) \[ p_{nn} = q^2, \quad p_{n-1,n}p_{n-1} = q^{-2}. \]

In this case, the quantum Borel algebra $U_q^+(\mathfrak{sp}_{2n})$ is a homomorphic image of $G(X)$ subject to the following relations

(3.3) \[ [x_i, [x_i, x_{i+1}]] = [[x_i, x_{i+1}], x_{i+1}] = 0, \quad 1 \leq i < n - 1; \quad [x_i, x_j] = 0, \quad j > i + 1; \]

(3.4) \[ [x_{n-1}, x_n], x_n] = [x_{n-1}, [x_{n-1}, x_n]] = 0. \]

**Lemma 3.1.** If $u$ is a standard word independent of $x_n$, then either $u = x_k x_{k+1} \ldots x_m$, $k \leq m < n$, or $[u] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$. Here $[u]$ is a nonassociative word with the standard arrangement of brackets.

**Proof.** The Hopf subalgebra of $U_q^+(\mathfrak{sp}_{2n})$ generated by $x_i$, $1 \leq i < n$ is the Hopf algebra $U_q^+(\mathfrak{su}_n)$ defined by the Cartan matrix of type $A_{n-1}$. By this reason the third statement of [9, Theorem A6] applies. \(\square\)

**Definition 3.2.** In what follows, $x_i$, $n < i < 2n$ denotes the generator $x_{2n-i}$. Respectively, $v(k, m)$, $1 \leq k \leq m < 2n$ is the word $x_k x_{k+1} \ldots x_{m-1} x_m$, whereas $v(m, k)$ is the opposite word $x_m x_{m-1} \ldots x_{k+1} x_k$. If $1 \leq i < 2n$, then $\phi(i)$ denotes the number $2n - i$, so that $x_i = x_{\phi(i)}$.

**Definition 3.3.** If $k \leq i < m < 2n$, then we set

(3.5) \[ \sigma_i^m \overset{df}{=} p(v(k, m), v(k, m)), \]

(3.6) \[ \mu_{k,i}^m \overset{df}{=} p(v(k, i), v(i + 1, m)) \cdot p(v(i + 1, m), v(k, i)). \]

**Lemma 3.4.** For each $i$, $k \leq i < m$ we have

(3.7) \[ \mu_{k,i}^m = \sigma_i^m (\sigma_i^m \sigma_{i+1}^m)^{1} = \sigma_i^m (\sigma_i^m \sigma_{i+1}^m)^{-1}. \]

**Proof.** Because $p(\cdot, \cdot)$ is a bimultiplicative map, there is a decomposition

(3.8) \[ p(ab, ab) = p(a, a)p(b, b) \cdot p(a, b)p(b, a). \]

Applying this equality to $a = v(k, i)$, $b = v(i + 1, m)$, we get the required relation. \(\square\)

**Lemma 3.5.** If $1 \leq k \leq m < 2n$, then

(3.9) \[ \sigma_i^m = \begin{cases} q^2, & \text{if } m = \phi(k); \\ q, & \text{otherwise.} \end{cases} \]
Proof. The bimultiplicativity of \( p(\cdot, \cdot) \) implies that \( \sigma^m_k = \prod_{s,t \leq m} p_{st} \) is the product of all coefficients of the \((m - k + 1) \times (m - k + 1)\)-matrix \( ||p_{st}|| \). By (3.11) all coefficients on the main diagonal equal \( q \) except \( p_{nn} = q^2 \).

If \( m < n \) or \( k > n \), then for non-diagonal coefficients, we have \( p_{st}p_{ks} = 1 \) unless \( |s - t| = 1 \), whereas \( p_{s,t+1}p_{s+1,t} = q^{-1} \). Hence, \( \sigma^m_k = q^{m-k+1} \cdot q^{-(m-k)} = q \).

If \( m = n \) or \( k = n \) but not both, then we have \( p_{nn} = q^2 \). By the above reasoning, we get \( \sigma^m_k = q^{(m-k)+2} \cdot q^{-(m-k-1)-2} = q \). Of course, if \( k = n = m \) then \( \sigma^m_k = p_{nn} = q^2 \).

In the remaining case, \( k < n < m \), we use induction on \( m - k \).

By (3.8) we have
\[
\sigma^{m+1}_k = \sigma^m_k \cdot q \cdot p(v(k, m), x_{m+1}) \cdot p(x_{m+1}, v(k, m))
\]
We shall prove that if \( k < n < m \), then
\[
p(v(k, m), x_{m+1}) \cdot p(x_{m+1}, v(k, m)) = \begin{cases} 1, & \text{if } k = \phi(m) - 1; \\ q^{-2}, & \text{if } k = \phi(m); \\ q^{-1}, & \text{otherwise}. \end{cases}
\]
The left hand side of the above equality is \( \prod_{k \leq t \leq m} p_{m+1,p_{m+1}t} \). If \( m = n \), then by (3.11) and (3.2) the factor \( p_{m+1,p_{m+1}t} \) differs from 1 only if \( t \in \{n - 2, n - 1, n\} \) and related values are respectively \( q^{-1}, q^2, q^{-2} \). Hence, if \( k < n - 1 = \phi(m) - 1 \), then the total product is \( q^{-1} \); if \( k = n - 1 = \psi(m) - 1 \), then this is 1; if \( k = n = \phi(m) \), then this is \( q^{-2} \).

If \( m > n \), then the factor \( p_{m+1,p_{m+1}t} \) differs from 1 only if
\[
t \in \{\phi(m) - 2, \phi(m) - 1, \phi(m), m\}
\]
and related values are respectively \( q^{-1}, q^2, q^{-1}, q^{-1} \). Therefore if \( k < \phi(m) - 1 \), then the whole product is \( q^{-1} \); if \( k = \phi(m) - 1 \), then this is 1; if \( k = \phi(m) \), then this is \( q^{-2} \); if \( k = \phi(m) \), then this is again \( q^{-1} \). This completes the proof of (3.11).

To complete the inductive step, we use (3.11) and inductive hypothesis: if \( k = \phi(m) - 1 \), then \( \sigma^{m+1}_k = q \cdot q \cdot q = q^3 \); if \( k = \phi(m) \), then \( \sigma^{m+1}_k = q^2 \cdot q \cdot q^{-2} = q \); otherwise \( \sigma^{m+1}_k = q \cdot q \cdot q^{-1} = q \).

We define the bracketing of \( v(k, m), k \leq m < 2n \) as follows.
\[
v[k, m] = \begin{cases} [[[\ldots [x_k, x_{k+1}], \ldots], x_{m-1}], x_m], & \text{if } m < \phi(k); \\ [x_k, [x_{k+1}, [\ldots, [x_{m-1}, x_m]]]], & \text{if } m > \phi(k); \\ [v[k, m-1], x_m], & \text{if } m = \phi(k), \end{cases}
\]
where in the latter term, \( [u, v] \triangleq uv - q^{-1}p(u, v)vu \).

Conditional identity (2.6) demonstrates that the value of \( v[k, m] \) in \( U^+_q(\mathfrak{sp}_{2n}) \) is independent of the precise arrangement of brackets, provided that \( m \leq n \) or \( k \geq n \). Now we are going to analyze what happens with the arrangement of brackets if \( k < n < m \neq \phi(k) \).

**Lemma 3.6.** If \( k \leq n \leq m < \phi(k) \), then the value in \( U^+_q(\mathfrak{sp}_{2n}) \) of the bracketed word \([y_k, x_n, x_{n+1} \ldots, x_m] \), where \( y_k = v[k, n - 1] \), is independent of the precise arrangement of brackets.

**Proof.** To apply (2.6), it suffices to check \([y_k, x_t] = 0 \), where \( n < t \leq m \) or, equivalently, \( \phi(m) \leq t < n \). By (2.3) we have
\[
[y_k, x_t] = [v[k, t - 2], v[t - 1, n - 1]], x_t] = [v[k, t - 2], [v[t - 1, n - 1], x_t]].
\]
By Lemma 3.1 the element \([v[t−1,n−1],x_t]\) equals zero in \(U_q^+(\mathfrak{so}_{2n})\) because the word \(u(t−1,n)x_t\) is independent of \(x_n\), it is standard, and the standard bracketing is precisely \([v[t−1,n],x_t]\).

**Lemma 3.7.** If \(k ≤ n\), \(ϕ(k) < m\), then the value in \(U_q^+(\mathfrak{sp}_{2n})\) of the bracketed word \([x_kx_{k+1}⋯x_ny_m]\), where \(y_m = v[n+1,m]\), is independent of the precise arrangement of brackets.

**Proof.** To apply (2.6), we need the equalities \([x_t,y_m] = 0\), \(k ≤ t < n\). The polynomial \([x_t,y_m]\) is independent of \(x_n\). Moreover, \([x_t,y_m]\) is proportional to \([y_m,x_t]\) due to antisymmetry identity (2.4) because

\[
p(x_t,y_m)p(y_m,x_t) = pt_{t+1}pt_{t-1} \cdot pt_{t+1}pt_{t-1}t = 1.
\]

The equality \([y_m,x_t] = 0\) turns to the proved above equality \([v[k,n−1],x_t] = 0\) if one renames the variables \(x_{n+1} ← x_k\), \(x_{n+2} ← x_{k+1}, \ldots\).

4. **PBW Generators of \(U_q^+(\mathfrak{sp}_{2n})\)**

**Proposition 4.1.** If \(q^2 ≠ 1\), \(q ≠ −1\), then values of the elements \(v[k,m], k ≤ m ≤ ϕ(k)\) form a set of PBW generators with infinite heights for the algebra \(U_q^+(\mathfrak{sp}_{2n})\) over \(k[G]\).

**Proof.** A word \(v(k,m)\) is a standard Lyndon-Shirshov word provided that \(k ≤ m < ϕ(k)\). By [9] Theorem C, p. 218 these words with the standard bracketing, say \([v(k,m)]\), become a set of PBW generators if we add to them the elements \([v_k] = [v[k,n−1],v[k,n]]\) \(1 ≤ k < n\). We shall use induction on \(m−k\) in order to demonstrate that the value in \(U_q^+(\mathfrak{sp}_{2n})\) of \([v(k,m)]\), \(k ≤ m < ϕ(k)\) is the same as the value of \(v[k,m]\) with the bracketing given in (3.12).

If \(m ≤ n\), then the value of \(v[k,m]\) is independent of the arrangement of brackets, see Lemma 2.1.

If \(k < n < m\), then according to [9] Lemma 7.18, the brackets in \([v(k,m)]\) are set by the following recurrence formulae (we note that \([v(k,m)] = [v_{k,ϕ(m)}]\) in the notations of [9]):

\[
\begin{align*}
[v(k,m)] &= [x_k[v(k+1,m)]] & \text{if } m < ϕ(k)−1; \\
[v(k,m)] &= [[v(k,m−1)]x_m] & \text{if } m = ϕ(k)−1.
\end{align*}
\]

In the latter case, the induction applies directly. In the former case, using induction and Lemma 3.6 we have

\[
[v(k+1,m)] = v[k+1,m] = [v[k+1,n−1],v[n,m]].
\]

At the same time \([x_t,x_t] = 0\), \(n ≤ t ≤ m\) because \(x_t = x_ϕ(t)\) and \(ϕ(t) ≥ ϕ(m) > k+1\). This implies \([x_k,v[n,m]] = 0\). Applying conditional identity (2.5), we get

\[
[v(k,m)] = [x_k[v[k+1,n−1],v[n,m]]] = [[x_kv[k+1,n−1]],v[n,m]] = [v[k,m]].
\]

It remains to analyze the case \(m = ϕ(k)\). We have to demonstrate that if in the set \(V\) of PBW generators of Lyndon-Shirshov standard words one replaces the elements \([v_k]\) with \([v[k,ϕ(k)]]\), \(1 ≤ k < n\) then the obtained set is still a set of PBW generators. To do this, due to [11] Lemma 2.5 with \(T ← \{v[k,ϕ(k)]\}, 1 ≤ k < n\), \(S ← U_q(\mathfrak{sp}_{2n})\), it suffices to see that the leading term of the PBW decomposition of \([v[k,ϕ(k)]]\) in the generators \(V\) is proportional to \([v_k]\).
By definition (3.12) with \( m = \phi(k) \), we have
\[
v[k, m] = v[k, m - 1]x_m - q^{-1}\pi x_m v[k, m - 1]
\]
where \( \pi = p(v(k, m - 1), x_m) \). The second term of the latter sum is a basis element (2.12) in the PBW generators \( V \). This element starts with \( v[k, m - 1] \) which is lesser than \( [v_k] \). Hence it remains to analyze the bracket
\[
[x_k, v[k, m - 1]].
\]
If \( k = n - 1 \), then \( [x_k, v[k, m - 1]] = [x_{n-1}, [x_{n-1}, x_n]] = [v_k] \).
If \( k < n - 1 \), then by Lemma 3.6 we have
\[
[x_k, v[k, m - 1]] = [x_k, [v[k, n], v[n + 1, m - 1]]].
\]
The basic relations (3.3) imply \([x_k, [x_k, x_{k+1}]] = 0\), \([x_k, v[k+2, n]] = 0\). By Lemma (2.4) \( v[k, n] \) is independent of the arrangement of brackets,
\[
v[k, n] = ([x_k, x_{k+1}], v[k + 2, n]),
\]
and hence \([x_k, v[k, n]] = 0\).
By Eq. (2.7) with \( u \leftarrow x_k, v \leftarrow v[k, n], w \leftarrow v[n + 1, m - 1] \), the right hand side of (4.2) is a linear combination of the following two elements:
\[
[x_k, [v[n + 1, m - 1]], v[k, n]],
\]
\[
v[k, n] \cdot [x_k, v[n + 1, m - 1]].
\]
The latter element starts with a factor \( v[k, n] \) which is lesser than \( [v_k] \). Hence it remains to prove that the leading term of the former element is proportional \([v_k] \).

By downward induction on \( k \) we shall prove the following decomposition
\[
v[n + 1, m - 1] = \alpha v[k + 1, n - 1] + \sum_{s=k+2}^{n-1} \gamma_s v[s, n - 1] \cdot U_s, \quad \alpha \neq 0.
\]
If \( k = n - 2 \), then this decomposition reduces to \( x_{n+1} = x_{n-1} \). Let us apply \([\cdot, x_m]\) to both sides of the above equality. Using (2.4), we see that \([v[k + 1, n - 1]], x_m]\) is proportional to \( v[k, n - 1] + \gamma_{k+1} v[k + 1, n - 1] \cdot x_m \), whereas (2.3) implies \([v[s, n - 1], U_s, x_m] = v[s, n - 1] \cdot [U_s, x_{k-1}] \) for \( s \geq k + 2 \). This completes the inductive step.

Let us apply \([x_k, \cdot]\) to both sides of the already proved Eq. (4.4). By (2.4), we get
\[
[x_k, v[n + 1, m - 1]] = \alpha v[k + 1, n - 1] + \sum_{s=k+2}^{n-1} \gamma_s' v[s, n - 1] \cdot [x_k, U_s].
\]

Finally, let us apply \([\cdot, v[k, n]]\) to both sides of (4.5). In this way we find a decomposition of the first element of (4.3):
\[
[[x_k, v[n + 1, m - 1]], v[k, n]] = \alpha [v_k] + \sum_{s=k+2}^{n-1} \gamma_s' v[s, n - 1] \cdot [x_k, U_s] \cdot v[k, n]
\]
\[
- \sum_{s=k+2}^{n-1} \gamma_s'' v[k, n] \cdot v[s, n - 1] \cdot [x_k, U_s].
\]
All summands, except the first one, start with \( v[k, n], v[s, n - 1] \) that are lesser than \([v_k]\). Hence, the leading term, indeed, is proportional to \([v_k]\).
Proposition 4.2. Let \( k \leq m < 2n \). In the shuffle representation, we have
\[
 v[k, m] = \alpha_k^m \cdot (v(m, k)), \quad \alpha_k^m \equiv \varepsilon_k^m (q - 1)^{m-k} \cdot \prod_{k \leq i < j \leq m} p_{ij}, \tag{4.7}
\]
where
\[
 \varepsilon_k^m = \begin{cases} 
 1 + q, & \text{if } k \leq n \leq m, m \neq \phi(k); \\
 1 + q^{-1}, & \text{if } m = \phi(k) \neq n; \\
 1, & \text{otherwise}.
\end{cases} \tag{4.8}
\]

Proof. We use induction on \( m-k \). If \( m = k \), then the equality reduces to \( x_k = (x_k) \).

a). Consider first the case \( m < \phi(k) \). By the inductive supposition, we have \( v[k, m-1] = \alpha_k^{m-1} \cdot (w) \), \( w = v(m-1, k) \). Using (4.13), we may write
\[
 v[k, m] = \alpha_k^{m-1} \{ (w) (x_m) - p(w, x_m) \cdot (x_m)(w) \}
\]
(4.9)
where \( p(v, x_m) = p(w, x_m)p(u, x_m)^{-1} \) because \( w = uv \).

If \( m \leq n \), then relations (4.9) imply \( p(v, m) \) \( p(v, m, v) = 1 \) with only one exception being \( v = w \). Hence, sum (4.9) has just one term. The coefficient of \( (x_m, w) = (v(m, k)) \) equals
\[
 \alpha_k^{m-1} p(w, x_m) (p(w, x_m)^{-1} - p(x_m, v)) = \alpha_k^{m-1} \prod_{i=k}^{m-1} p_{i} \cdot (p_{m-1}^{-1} p_{m-1}^{-1} - 1).
\]

If \( m < n \), then the latter factor equals \( q - 1 \), whereas if \( m = n \), then it is \( q^2 - 1 = \varepsilon_k^n(q - 1) \).

Suppose that \( m > n \) and still \( m < \phi(k) \). In decomposition (4.9), we have \( v = v(s, k) \), \( k < s < m \) and hence \( p(x_m, v)p(v, x_m) = \prod_{t=k}^{s} p_{m(t)} \). The product \( p_{m(k)} \) differs from 1 only if \( t \in \{ \phi(m) - 1, \phi(m), \phi(m) + 1, m - 1 \} \); related values are \( q^{-1}, q^2, q^{-1} \) if \( m > n + 1 \), and they are \( q^{-1}, q^2, q^{-2} \) if \( m = n + 1 \), \( \phi(m) + 1 = m - 1 \).

This implies
\[
 p(x_m, v)p(v, x_m) = \begin{cases} 
 q^{-1}, & \text{if } s = \phi(m) - 1, \text{ or } s = m - 1; \\
 q, & \text{if } s = \phi(m); \\
 1, & \text{otherwise}.
\end{cases} \tag{4.10}
\]

Hence, in (4.9), only three terms remain with \( s = \phi(m) - 1, s = \phi(m), \) and \( s = m - 1 \).

If \( s = \phi(m) - 1 \) or \( s = \phi(m) \), then \( (ux_m, v) \) equals
\[
 ux_m, v = v(m - 1, \phi(m) + 1)x_m^2 v(\phi(m) - 1, k),
\]
whereas the coefficient of the comonomial \( (ux_m, v) \) in sum (4.9) is
\[
 p(x_m, v_0)^{-1} - p(x_m, x_m) + p(x_m, x_m, v_0)^{-1} - p(x_m, v_0, x_m),
\]
where \( v_0 = v(\psi(m) - 1, k) \). Taking into account (4.10), we find the above sum:
\[
 p(v_0, x_m)(q - 1 + q \cdot q^{-1} - q) = 0.
\]

Thus, in (4.9) only one term remains, with \( v = v(m - 1, k), u = \emptyset \). This term has the required coefficient:
\[
 \alpha_k^m = \alpha_k^{m-1} (p(x_m, w)^{-1} - p(w, x_m)) = \alpha_k^{m-1} p(w, x_m)(q - 1).
\]
b). Consider the case \( m > \phi(k) \). By the inductive supposition, we have
\[
v[k+1, m] = \alpha_{k+1}^m \cdot (w), \quad w = v(m, k+1).
\]
Using (2.13), we get
\[
v[k, m] = \alpha_{k+1}^m \{(x_k)(w) - p(x_k, w) \cdot (x_k)\}
\]
\[
= \alpha_{k+1}^m \sum_{u,v} \{p(u,x_k)^{-1} - p(x_k, w)p(x_k, v)^{-1}\}(ux_kv).
\]
(4.11) \[
= \alpha_{k+1}^m \sum_{u,v=v(m,k+1)} p(x_k, u\{p(u,x_k)^{-1}p(x_k, w)^{-1} - 1\}(ux_kv).
\]

If \( k \geq n \), then \( p(x_k, u)p(x_k, w) = 1 \), unless \( u = w \). Hence, (4.11) has only one term, and the coefficient equals
\[
\alpha_{k+1}^m p(x_k, w)(p(w,x_k)^{-1}p(x_k, w)^{-1} - 1) = \alpha_{k+1}^m p(x_k, w)(p_{k+1,k}^{-1}^{-1} p_{k+1,k}^{-1} - 1).
\]
If \( k > n \), then the latter factor equals \( q - 1 \), whereas if \( k = n \), then it is \( q^2 - 1 = (q-1)e_n^m \) as claimed.

Suppose that \( k < n \). In this case, \( x_k = x_t \) with \( m > t \equiv \phi(k) > \phi(n) = n \). Let \( u = v(m, s) \).
If \( s > t + 1 \), then \( u \) depends only on \( x_i \), \( i < k - 1 \), and relations (3.1), (3.2) imply \( p(x_k, u)p(x_k, v) = 1 \).
If \( s < t \), \( s \neq k + 1 \), then \( k + 1 < n \) (otherwise \( s = n = k + 1 \)), and we have
\[
p(x_k, u)p(u, x_k) = p_{k-1,k}p_{k+1,k} \cdot p_{k-1,k}p_{k+1,k} = 1
\]
because \( x_t = x_k \).
Hence, three terms remain in (4.11) with \( s = t \), \( s = t + 1 \), and \( s = k + 1 \). If \( u = v(m, t) \) or \( u = v(m, t + 1) \), then \( ux_kv = v(m, t + 1)x_k^2v(t+1,k) \), whereas the coefficient of the corresponding tensor is
\[
p(v(m, t + 1), x_k)^{-1} - p(x_k, v(m, t + 1)) + p(v(m, t), x_k)^{-1} - p(x_k, v(m, t))
\]
\[
= p(x_k, v(m, t + 1))\{p_{k-1,k}^{-1} - 1 + p_{k-1,k}^{-1}p_{k+1,k}^{-1} - 1\} = 0
\]
because \( p_{kk} = q \), \( p_{k-1,k}p_{k+1,k} = q^{-1} \), and \( p_{KR}p_{RK} = 1 \) if \( r > t + 1 \).
Thus, only one term remains in (4.9), and
\[
\alpha_{k+1}^m p(x_k, w) = \alpha_{k+1}^m (p(w, x_k)^{-1} - p(x_k, w)) = \alpha_{k+1}^m p(x_k, w)(q - 1).
\]
c). Let \( m = \phi(k) \neq n \). In this case, \( x_m = x_k \), \( p_{kk} = q \). By definition (3.12) we have
(4.12) \[
v[k, m] = v[k, m-1] \cdot x_k - q^{-1}p(v(k, m - 1), x_m) x_k \cdot v[k, m - 1].
\]
Case a) allows us to find the shuffle representation
\[
v[k, m - 1] = \alpha_{k-1}^m (w), \quad w = v(m - 1, k).
\]
Hence the right-hand side of (4.12) in the shuffle form is
\[
\alpha_{k-1}^m \sum_{u,v} \{p(x_k, v)^{-1} - q^{-1}p(x_k, v)^{-1}p(v(k, m - 1), x_m) \cdot p(u, x_k)^{-1}\}(ux_kv)
\]
(4.13) \[
= \alpha_{k-1}^m \sum_{u,v=v(m-1,k)} p(v, x_m) (p(x_k, v)^{-1}p(v, x_k)^{-1} - q^{-1}) \cdot (ux_kv).
\]
The coefficient of \((v(k, m))\) related to \(u = \emptyset, v = v(m - 1, k)\) equals

\[
\alpha_k^{m-1} p(v, x_m) \cdot (p(x_k, v)^{-1} p(v, x_k)^{-1} - q^{-1}) = \alpha_k^{m-1}(1 - q^{-1}) \cdot \prod_{i=k}^{m-1} p_i. \tag{4.14}
\]

Here we have used \(x_k = x_m\) and Eq. (3.11) with \(m \leftarrow m - 1, k \leftarrow k\). It remains to show that all other terms in (4.13) are canceled. In this case we would have

\[
\varepsilon_k^m = \varepsilon_k^{m-1}(1 - q^{-1})(q - 1)^{-1} = (1 + q)(1 - q^{-1})(q - 1)^{-1} = 1 + q^{-1}
\]

as required.

If \(u = v(m - 1, k), v = \emptyset\) or \(u = v(m - 1, k + 1), v = x_k\), then

\[
u x_k v = v(m - 1, k + 1) x_k^2,
\]

whereas the total coefficient of the related comonomial is proportional to

\[1 - q^{-1} \cdot p(u, x_m)p(u, x_k)^{-1} + p_{kk}^{-1} - q^{-1} \cdot p_{kk} = 0.
\]

Let \(u = v(m - 1, s), v = v(s - 1, k), k + 1 < s < m\). The whole coefficient of the comonomial \((ux_k v)\) takes the form

\[\alpha_k^{m-1} p(v(s - 1, k), x_m) \cdot (p(x_k, v(s - 1, k))^{-1} p(v(s - 1, k), x_k)^{-1} - q^{-1})\]

The latter factor equals \(\prod_{t=k}^{s-1} p_{tt}^{-1} p_{tk}^{-1} - q^{-1}\). The product \(p_{kk} p_{kk}\) differs from 1 only if \(t \in \{k, k + 1\}\) and related values are \(q^2\) and \(q^{-1}\). This implies that the coefficient of \((ux_k v)\) has a factor \(q^{-2} \cdot q - q^{-1} = 0\).

\[\square\]

5. Coproduct formula for \(U_q^+(\mathfrak{sp}_{2n})\)

**Theorem 5.1.** In \(U_q^+(\mathfrak{sp}_{2n})\) the coproduct on the elements \(v[k, m], k \leq m < 2n\) has the following explicit form

\[
\Delta(v[k, m]) = v[k, m] \otimes 1 + g_{km} \otimes v[k, m]
+ \sum_{i=k}^{m-1} \tau_i (1 - q^{-1}) g_{ki} v[i + 1, m] \otimes v[k, i],
\]

where \(\tau_i = 1\) with two exceptions, being \(\tau_{n-1} = 1 + q^{-1}\) if \(m = n\), and \(\tau_n = 1 + q^{-1}\) if \(k = n\). Here \(g_{ki} = g(r(v(k, i))) = g_k g_{k+1} \cdots g_i\).

**Proof.** By Proposition 4.2 we have the shuffle representation

\[v[k, m] = \alpha_k^m \cdot (v(m, k)).\]

Using (2.13), it is easy to find the braided coproduct of the comonomial shuffle:

\[
\Delta^b_0((v(m, k)) = \sum_{i=k}^{m-1} (v(m, i + 1)) \otimes (v(i, k)),
\]

where for short we put \(\Delta^b_0(U) = \Delta^b(U) - U \otimes 1 - 1 \otimes U\). Taking into account (5.2), we have

\[
\Delta^b_0(v[k, m]) = \alpha_k^m \sum_{i=k}^{m-1} (\tau_i (1 - q^{-1}) g_{ki} v[i + 1, m] \otimes (\tau_i)^{-1} v[k, i]).
\]

Formula (2.13) demonstrates that the tensors \(u^{(1)} \otimes v^{(1)}\) of the (unbraided) coproduct and tensors \(u^{(1)}_b \otimes v^{(1)}_b\) of the braided one are related by \(u^{(1)}_b = u^{(1)} \cdot g(v^{(2)})^{-1}\),
$u_b^{(2)} = u^{(2)}$. The equality (6.3) provides the values of $u_b^{(1)}$ and $u_b^{(2)}$. Hence we may find $u^{(1)} = a_1^m(a_1^m a_{i+1}^{m-1})^{-1} v[i + 1, m] g_{ki}$ and $u^{(2)} = v[k, i]$, where $g_{ki} = \text{gr}(v[k, i])$. The commutation rules imply

$$v[i + 1, m] g_{ki} = p(v(i + 1, m), v(k, i)) g_{ki} v[i + 1, m].$$

Thus, the coproduct has the form (6.1), where

$$\tau_1(1 - q^{-1}) = a_1^m(a_1^m a_{i+1}^{m-1})^{-1} p(v(i + 1, m), v(k, i)).$$

The definition of $a_k^{m}$ given in (4.7) shows that

$$a_k^m(a_k^m a_{i+1}^{m-1})^{-1} v_k(i) = v_k(i + 1, m)$$

because

$$\left( \prod_{k \leq a < b \leq i} p_{ab} \prod_{i+1 \leq a < b \leq m} p_{ab} \prod_{k \leq a < b \leq m} p_{ab} = p(v(k, i), v(i + 1, m)).$$

The definition of $\mu_k^{m,i}$ given in (3.6) implies

$$\tau_1(1 - q^{-1}) = \varepsilon_k^m(\varepsilon_k^m a_{i+1}^{m-1})^{-1} q\mu_k^{m,i};$$

that is, $\tau_i = \varepsilon_k^m(\varepsilon_k^m a_{i+1}^{m-1})^{-1} q\mu_k^{m,i}$. By (3.7), we have $\mu_k^{m,i} = \sigma_k^m(\sigma_k^m a_{i+1}^{m-1})^{-1}$. Using (3.9) and (4.8), we see that

$$\varepsilon_k^m \sigma_k^m = \begin{cases} q^2 + q, & \text{if } k \leq n \leq m, k \neq m; \\ q^2, & \text{if } k = n = m; \\ q, & \text{otherwise.} \end{cases}$$

Now, it is easy to check that the $\tau$’s have the following elegant form

$$\tau_i = \varepsilon_k^m \sigma_k^m(\varepsilon_k^m \sigma_k^m)^{-1}(\varepsilon_k^m \sigma_k^m)^{-1} q$$

$$= \begin{cases} 1 + q^{-1}, & \text{if } i = n - 1, m = n; \text{ or } k = i = n; \\ 1, & \text{otherwise.} \end{cases}$$

**Remark 1.** If $q$ is a root of 1, say $q^t = 1$, $t > 2$, then the shuffle representation is not faithful. Therefore in this case, the formula (5.1) is proved only for the Frobenius-Lusztig kernel $u_q(\mathfrak{sp}_{2n})$. Nevertheless, all tensors in (5.1) have degree at most 2 in each variable. At the same time, general results on combinatorial representation of Nichols algebras [2] Section 5.5 demonstrate that in case $C_n$, the kernel of the natural projection $U_q(\mathfrak{sp}_{2n}) \to u_q(\mathfrak{sp}_{2n})$ is generated by polynomials of degree grater than 2 in (or independent of) each given variable. Hence (5.1) remains valid in this case as well.

### 6. Relations in $U_q^+(\mathfrak{so}_{2n})$

In what follows, we fix a parameter $q$ such that $q \neq -1$. If $C$ is a Cartan matrix of type $D_n$, then relations (2.10) take the form

$$p_{ii} = q, \quad 1 \leq i \leq n; \quad p_{i+1-i} = p_{n-2} n p_{n-2} = q^{-1}, \quad 1 < i < n;$$

$$p_{ij} p_{ji} = p_{n-1} n p_{n-1} = 1, \quad \text{if } j > i + 1 \& (i, j) \neq (n, n - 2).$$

The quantum Borel algebra $U_q^+(\mathfrak{so}_{2n})$ can be defined by the condition that the Hopf subalgebras $U_{n-1}$ and $U_n$ generated, respectively, by $x_1, x_2, \ldots, x_{n-1}$ and
If $x, y, z$ are Hopf algebras $U_q(\mathfrak{sl}_n)$ of type $A_{n-1}$, and by one additional relation

\[(6.3) \quad [x_{n-1}, x_n] = 0.\]

Recall that $x_i$, $n < i < 2n$ denotes the generator $x_{2n-i}$, whereas if $1 \leq i < 2n$, then $\phi(i)$ equals $2n - i$, so that $x_i = x_{\phi(i)}$, see Definition [3.2].

**Definition 6.1.** We define words $e(k, m)$, $1 \leq k \leq m < 2n$ in the following way:

\[(6.4) \quad e(k, m) = \begin{cases} x_kx_{k+1} \cdots x_{m-1}x_m, & \text{if } m < n \text{ or } k > n; \\ x_kx_{k+1} \cdots x_{n-2}x_nx_{n+1} \cdots x_m, & \text{if } k < n - 1 < m; \\ x_nx_{n+1} \cdots x_m, & \text{if } k = n - 1 < m; \\ x_nx_{n+2}x_{n+3} \cdots x_m, & \text{if } k = n. \end{cases}\]

Respectively, $e(m, k)$ is the word opposite to $e(k, m)$. Further, we define a word $e'(k, m)$ as a word that appears from $e(k, m)$ by replacing the subword $x_nx_{n+1}$, if any, with $x_{n-1}x_n$. Respectively, $e'(m, k)$ is the word opposite to $e'(k, m)$.

We see that $e(k, m)$ coincides with $v(k, m)$ if $m < n$ or $k > n$. If $k < n - 1 < m$, then $e(k, m)$ appears from $v(k, m)$ by deleting the letter $x_{n-1}$ (but not of $x_{n+1}$). Similarly, if $k = n$, then $e(n, m)$ appears from $v(n, m)$ by deleting the letter $x_{n+1}$, whereas if $k = n - 1$, then we have $e(n-1, m) = v(n, m)$. We have to stress that according to this definition $e(n-1, n) = e(n, n) = e(n, n+1) = x_n$.

**Lemma 6.2.** If $1 \leq k \leq m < 2n$, then

\[(6.5) \quad p(e(k, m), e(k, m)) = \sigma_k^{\Phi_\alpha} = \begin{cases} q^2, & \text{if } m = \phi(k); \\ q, & \text{otherwise.} \end{cases}\]

**Proof.** If the word $e(k, m)$ does not contain a subword $x_nx_{n+1}$, then it belongs to either $U_n$ or $U_{n-1}$ that are isomorphic to $U_q^+(\mathfrak{sl}_n)$. Hence we have $p(e(k, m), e(k, m)) = q$.

Let $k \leq n - 1 < m$. In this case $e(k, m+1) = e(k, m)x_{m+1}$ which allows one to use induction on $m - n + 1$. If $m = n$, then $e(k, n)$ does not contain a sub-word $x_nx_{n+1}$. Because $p(-, -)$ is a bimultiplicative map, we may decompose

\[(6.6) \quad p(e(k, m+1), e(k, m+1)) = \sigma_k^{\Phi} \cdot q \cdot p(e(k, m), x_{m+1}) \cdot p(x_{m+1}, e(k, m)).\]

Using relations [6.1] and [6.2] we shall prove

\[(6.7) \quad p(e(k, m), x_{m+1}) \cdot p(x_{m+1}, e(k, m)) = \begin{cases} 1, & \text{if } k = \phi(m) - 1; \\ q^{-2}, & \text{if } k = \phi(m); \\ q^{-1}, & \text{otherwise.} \end{cases}\]

The left hand side of the above equality is $\prod_{k \leq t \leq m, t \neq n-1} p_{t, m+1}p_{m+1, t}$.

If $m > n + 1$, then by [6.1] and [6.2] the factor $p_{m+1, m+1}$ differs from 1 only if $t \in \{\phi(m) - 2, \phi(m) - 1, \phi(m), m\}$ and related values are respectively $q^{-1}, q^2, q^{-1}, q^{-1}$ whereas the product of all those values is precisely $q^{-1}$. Hence, if $k < \phi(m) - 1$, then the whole product is $q^{-1}$; if $k = \phi(m) - 1$, then this is 1; if $k = \phi(m)$, then this is $q^{-2}$; if $k > \phi(m)$, then this is again $q^{-1}$. If $m = n + 1$, then nontrivial factors are related to $t \in \{n - 3, n - 2, n, n + 1\}$ with values $q^{-1}, q^2, q^{-1}, q^{-1}$, respectively. Hence, we arrive to the same conclusion with $k < n - 2 = \phi(m) - 1$; $k = n - 2 = \phi(m) - 1$; and $k = n - 1 = \phi(m)$. 


Finally, if \( m = n \), then there is just one nontrivial factor which relates to \( t = n - 2 \) with value \( q^{-1} \), so that if \( k \leq n - 2 = \phi(m) - 2 \), then the total product is \( q^{-1} \); if \( k = n - 1 = \psi(m) - 1 \), then this is 1. This completes the proof of (6.7).

To complete the inductive step we use (6.7) and inductive hypothesis: if \( k = \phi(m) - 1 \), then \( \sigma_{k}^{m+1} = q \cdot q \cdot 0 = q^2 \); if \( k = \phi(m) \), then \( \sigma_{k}^{m+1} = q^2 \cdot q \cdot q^{-2} = q \); otherwise \( \sigma_{k}^{m+1} = q \cdot q \cdot q^{-1} = q \).

**Lemma 6.3.** If the word \( e(k, m) \) contains the subword \( x_n x_{n+1} \); that is \( k < n < m \), then for each \( i, k \leq i < m \) we have

\[
(6.8) \quad p(e(k, i), e(i + 1, m)) \cdot p(e(i + 1, m), e(k, i)) = \sigma_{k}^{m} (\sigma_{k}^{m+1})^{-1} = \mu_{k}^{m,i}.
\]

**Proof.** If \( k < n < m \), then for \( i \neq n - 1 \) there is a decomposition \( e(k, m) = e(k, i)e(i + 1, m) \) which implies (6.8) because the form \( p(\cdot, \cdot) \) is bimultiplicative. For \( i = n - 1 \) there is another equality \( e'(k, m) = e(k, i)e(i + 1, m) \). Certainly \( p(e'(k, m), e'(k, m)) = p(e(k, m), e(k, m)) = \sigma_{k}^{m} \). Hence (6.8) is still valid.

We define the bracketing of \( e(k, m) \), \( k \leq m < 2n \) as follows.

\[
(6.9) \quad e[k, m] = \begin{cases} 
\{[[\ldots [x_{k}, x_{k+1}], \ldots, x_{m-1}], x_{m}], & \text{if } m < \phi(k); \\
\{x_{k}, [x_{k+1}, \ldots, [x_{m-1}, x_{m}], \ldots]\}, & \text{if } m > \phi(k); \\
\{e[k, m - 1], x_{m}\}, & \text{if } m = \phi(k),
\end{cases}
\]

where as above \( [u, v] = uv - q^{-1}puv \).

Conditional identity (2.6) demonstrates that the value of \( e[k, m] \) in \( U_{q}^{+}(\mathfrak{so}_{2n}) \) is independent of the precise arrangement of brackets, provided that \( m \leq n \) or \( k \geq n \).

**Lemma 6.4.** If \( k < n < m < \phi(k) \), then the value in \( U_{q}^{+}(\mathfrak{so}_{2n}) \) of the bracketed word \( [y_{k} x_{n+1} x_{n+2} \ldots x_{m}] \), where \( y_{k} = e[k, n] \), is independent of the precise arrangement of brackets.

**Proof.** To apply (2.6), it suffices to check \( [y_{k}, x_{t}] = 0 \), where \( n + 1 < t \leq m \) or, equivalently, \( \phi(m) \leq t < n - 1 \). We have

\[
[y_{k}, x_{t}] = [e[k, t - 2], e[t - 1, n], x_{i}] = [e[k, t - 2], e[t - 1, n], x_{t}].
\]

The polynomial \([e[t - 1, n], x_{t}]\) is independent of \( x_{n-1} \), so that it belongs to the Hopf subalgebra \( U_{n} = U_{q}^{+}(\mathfrak{sl}_{n}) \). By [9] Theorem \( A_{n} \), the element \([e[t - 1, n], x_{t}]\) equals zero in \( U_{q}^{+}(\mathfrak{sl}_{n}) \) because the word \( e(t - 1, n)x_{t} \) is standard, and the standard bracketing is \([e[t - 1, n], x_{t}]\).

**Lemma 6.5.** If \( k < n, \phi(k) < m \), then the value in \( U_{q}^{+}(\mathfrak{so}_{2n}) \) of the bracketed word \( [x_{k} x_{k+1} \ldots x_{n-2} y_{m}] \), where \( y_{m} = e[n + 1, m] \), is independent of the precise arrangement of brackets.

**Proof.** To apply (2.6), we need the equalities \([x_{t}, y_{m}] = 0, k \leq t < n - 1 \). The polynomial \([x_{t}, y_{m}]\) belongs to the subalgebra \( U_{n-1} \). Moreover, \([x_{t}, y_{m}]\) is proportional to \([y_{m}, x_{t}]\) due to antisymmetry identity (2.4) because \( p(x_{t}, y_{m})p(y_{m}, x_{t}) = pt_{t+1}pt_{t+1} \cdot pt_{t+1}pt_{t+1} = 1 \). The equality \([y_{m}, x_{t}] = 0 \) turns to the proved above equality \([e[k, n], x_{t}] = 0 \) if one renames the variables \( x_{n+1} \rightleftharpoons x_{k}, x_{n+2} \rightleftharpoons x_{k+1}, \ldots \).
7. PBW GENERATORS OF $U_q^+(\mathfrak{so}_{2n})$

**Proposition 7.1.** If $q \neq -1$, then values of the elements $e[k, m]$, $k \leq m < \phi(k)$ form a set of PBW generators with infinite heights for the algebra $U_q^+(\mathfrak{so}_{2n})$ over $k[G]$.

**Proof.** All words $e(k, m)$, $k \leq m < \phi(k)$ are standard Lyndon-Shirshov words, and by [9, Theorem D, p. 225] under the standard bracketing, say $[e(k, m)]$, they form a set of PBW generators with infinite heights.

By induction on $m - k$ we prove that the values in $U_q^+(\mathfrak{so}_{2n})$ of $[e(k, m)]$ equal the values of $e[k, m]$ with bracketing given in (8.3).

If $m \leq n$, then by Lemma 3.1 we have nothing to prove.

If $k < n < m$, then according to [9, Lemma 7.25], the brackets in $[e(k, m)]$ are set by the following recurrence formulae (we note that $[e(k, m)] = [e_k, \phi(m)]$ in the notations of [9]):

$$[e(k, m)] = \begin{cases} [e_k[e(k + 1, m)]], & \text{if } m < \phi(k) - 1; \\ [[e(k, m - 1)]x_m], & \text{if } m = \phi(k) - 1. \end{cases}$$

In the latter case the induction applies directly. In the former case using induction and Lemma 6.4 we have $[e(k + 1, m)] = e[k + 1, m] = [e[k + 1, n], e[n + 1, m]]$. At the same time $[x_k, x_t] = 0$, $n < t \leq m$ because $x_t = x_{\phi(t)}$ and $\phi(t) \geq \phi(m) > k + 1$ & $(k, \phi(t)) \neq (n - 2, n)$. This implies $[x_k, e[n + 1, m]] = 0$. Applying the conditional identity (2.5), we get

$$[e(k, m)] = [x_k[e[k + 1, n], e[n + 1, m]]] = [x_k e[k + 1, n], e[n + 1, m]] = e[k, m].$$

$\square$

8. SHUFFLE REPRESENTATION FOR $U_q^+(\mathfrak{so}_{2n})$

In this section, we are going to find the shuffle representation of elements $e[k, m]$, $1 \leq k \leq m < 2n$. If $e(k, m)$ has not $x_n x_{n+1}$ as a subword, then $e(k, m)$ belongs to a Hopf subalgebra of type $A_n$: this is either $U_{n-1} = U_q^+(\mathfrak{sl}_n)$ or $U_n = U_q^+(\mathfrak{sl}_n)$. At the same time in the considered above case $C_n$, the elements $x_1, x_2, \ldots, x_{n-1}$ generate precisely a Hopf subalgebra $U_q^+(\mathfrak{sl}_n)$. Hence we may apply Proposition 4.2.

$$e[k, m] = \alpha_k^m \cdot (e(m, k)),$$

where

$$\alpha_k^m = \begin{cases} (q-1)^{m-k} \prod_{k \leq i < j \leq m} p_{ij}, & \text{if } m < n \text{ or } k > n; \\ (q-1)^{m-n-1} \prod_{n \leq i < j \leq m, i, j \neq n+1} p_{ij}, & \text{if } k = n; \\ (q-1)^{n-k-1} \prod_{k \leq i < j \leq m, i, j \neq n-1} p_{ij}, & \text{if } m = n. \end{cases}$$

**Proposition 8.1.** Let $1 \leq k < n < m < 2n$. In the shuffle representation, we have

$$e[k, m] = \alpha_k^m \cdot \{ (e(m, k)) + p_{n-1,n} (e'(m, k)) \},$$

where

$$\alpha_k^m = c_k^m (q-1)^{m-k-1} \prod_{k \leq i < j \leq m, i, j \neq n-1} p_{ij}.$$
with

\[(8.5) \quad \epsilon_k^m = \begin{cases} q^{-1}, & \text{if } m = \phi(k); \\ 1, & \text{otherwise.} \end{cases} \]

**Proof.** a). Consider first the case \(m < \phi(k)\). We use induction on \(m - n\). Let \(m - n = 1\). Condition \(n + 1 = m < \phi(k)\) implies \(k < n - 1\). Hence by Lemma 6.4 we have \(e[k, n + 1] = [e[k, n], x_{n+1}]\), whereas \((8.1)\) implies \(e[k, n] = \alpha_k^n(e(n, k))\). Using \((2.15)\), we may write

\[e[k, n + 1] = \alpha_k^n((e(n, k))(x_{n+1}) - p(e(n, k), x_{n+1}) \cdot (x_{n+1})(e(n, k)))\]

\[(8.6) = \alpha_k^n \sum_{uv = (n, k)} \{p(x_{n+1}, v)^{-1} - p(v, x_{n+1})\}(ux_{n+1}v),\]

where \(p(v, x_{n+1}) = p(e(n, k), x_{n+1})p(u, x_{n+1})^{-1}\) because \(e(n, k) = uv\). We have

\[p(x_{n+1}, v)^{-1} - p(v, x_{n+1}) = p(v, x_{n+1}) \cdot \{p(x_{n+1}, v)^{-1}p(v, x_{n+1})^{-1} - 1\}.\]

At the same time equality \(x_{n+1} = x_{n-1}\) and relations \((6.1), (6.2)\) imply

\[p(x_{n+1}, v)p(v, x_{n+1}) = \begin{cases} q^{-1}, & \text{if } v = e(n, k) \text{ or } v = e(n - 2, k); \\ 1, & \text{otherwise.} \end{cases} \]

Hence in the decomposition \((8.6)\) two terms remain

\[\alpha_k^n p(e(n, k), x_{n+1})(q - 1)(x_{n+1}x_{n-1}x_{n-2} \cdots x_k) = \alpha_k^{n+1}(e(n + 1, k))\]

and

\[\alpha_k^n p(e(n - 2, k), x_{n+1})(q - 1)(x_{n+1}x_{n-1}x_{n-2} \cdots x_k) = \alpha_k^{n+1}p_{n-1,n}(e'(n + 1, k)),\]

for \(p_{n,n-1} = p_{n-1,n}^{-1}\) due to \((6.2)\). This completes the first step of induction.

Suppose that equalities \((8.3)\) and \((8.4)\) are valid and still \(m + 1 < \phi(k)\). Then Lemma 6.4 implies \(e[k, m + 1] = [e[k, m], x_{m+1}]\). By \((2.15)\) we have

\[[(e(m, k)), (x_{m+1})] = \sum_{uv = (m, k)} p(v, x_{m+1}) \cdot \{p(x_{m+1}, v)^{-1}p(v, x_{m+1})^{-1} - 1\}(ux_{m+1}v).\]

Relations \((6.1), (6.2)\) imply that

\[p(x_{m+1}, v)p(v, x_{m+1}) = \begin{cases} q, & \text{if } v = e(\phi(m) - 1, k); \\ q^{-1}, & \text{if } v = e(m, k) \text{ or } v = e(\phi(m) - 2, k); \\ 1, & \text{otherwise.} \end{cases} \]

Thus in the decomposition just three terms remain. Two of them, corresponding to \(v = e(\phi(m) - 1, k)\) and \(v = e(\phi(m) - 2, k)\), are canceled:

\[p(x_{\phi(m)-1}, x_{m+1})(q^{-1} - 1) + (q - 1) = q(q^{-1} - 1) + (q - 1) = 0.\]

Thus

\[[(e(m, k)), (x_{m+1})] = \{(q - 1) \cdot \prod_{k \leq i \leq m, \text{i \neq n-1}} p_{i,m+1}\} (e(m + 1, k)).\]

In perfect analogy, we have

\[[(e'(m, k)), (x_{m+1})] = \{(q - 1) \cdot \prod_{k \leq i \leq m, \text{i \neq n-1}} p_{i,m+1}\} (e'(m + 1, k)).\]
The inductive supposition yields 
\[ e[m, k] = \alpha^m_k \{ (e(m, k)) + p_{n-1, n}(e'(k, m)) \}. \] 

Hence to complete the induction, it suffices to note that 
\[ \alpha^{m+1}_k = \alpha^m_k \cdot (q - 1) \cdot \prod_{k \leq i \leq m, \ i \neq n-1} p_i \cdot m+1. \]

b). Similarly consider the case \( m > \phi(k) \) using downward induction on \( n - k \).

Let \( k = n - 1 \). Condition \( m > \phi(k) \) implies \( m \geq n + 2 \). Hence by Lemma 6.5 we have \( e[n - 1, m] = [x_n, e[n + 1, m]] \), whereas \( 8.1 \) and \( 8.2 \) imply \( e[n + 1, m] = \alpha^m_{n+1}(e(m, n + 1)). \) Using \((2.15)\), we may write 
\[ e[n - 1, m] = \alpha^m_{n+1} \{(x_n)(e(m, n + 1)) - p(x_n, e(m, n + 1)) \cdot (e(m, n + 1))(x_n)\} \]

\[(8.7)\]

where \( p(x_n, u) = p(x_n, e(m, n + 1))p(x_n, v)^{-1} \) because \( e(m, n + 1) = uv \). We have 
\[ p(u, x_n)^{-1} - p(x_n, u) = p(x_n, u) \cdot \{p(u, x_n)^{-1}p(x_n, u)^{-1} - 1\}. \]

Equality \( x_{n+1} = x_{n-1} \) and relations \((6.1), (6.2)\) imply that \( p(u, x_n)p(x_n, u) = 1 \) unless \( u = e(m, n + 1) \) or \( u = e(m, n + 2) \). In these two exceptional cases, the product equals \( p_{n+2}p_{n+1} = q^{-1} \) because \( p_{n+1}p_{n+1} = 1 \). Hence in the decomposition \((8.7)\), two terms remain 
\[ \alpha^m_{n+1}p(x_n, e(m, n + 1))(q - 1)(x_m \cdot x_{n+1}x_n) = \alpha^m_{n+1}(e(m, n - 1)) \]

and
\[ \alpha^m_{n+1}p(x_n, e(m, n + 2))(q - 1)(x_m \cdot x_{n+2}x_nx_{n+1}) = \alpha^m_{n+1}p_{n+1}^{-1}(e'(m, n - 1)). \]

This completes the first step of induction because \( p_{n+1}^{-1} = p_{n-1} \).

Suppose that equalities \((3.3)\) and \((3.4)\) are valid and still \( m > \phi(k-1) = \phi(k) + 1 \).

Lemma 6.5 implies \( e[k - 1, m] = [x_{k-1}, e[k, m]] \). We have 
\[ [(x_{k-1}), (e(m, k))] = \sum_{u = e(m, k)} p(x_{k-1}, u) \cdot \{p(u, x_{k-1})^{-1}p(x_{k-1}, u)^{-1} - 1\}(ux_{k-1}v). \]

Relations \((6.1), (6.2)\) imply that 
\[ p(u, x_{k-1})p(x_{k-1}, u) = \begin{cases} 
q & \text{if } u = e(m, \phi(k) + 1); \\
q^{-1} & \text{if } u = e(m, k) \text{ or } u = e(\phi(k) + 2, k); \\
1 & \text{otherwise.} 
\end{cases} \]

Hence in the decomposition \((8.6)\) just three terms remain. Two of them, corresponding to \( u = e(m, \phi(k) + 1) \) and \( u = e(m, \phi(k)) \), are canceled:
\[ p(x_{k-1}, e(\phi(k) + 1))(q^{-1} - 1) + (q - 1) = q(q^{-1} - 1) + (q - 1) = 0. \]

Thus
\[ [(x_{k-1}), (e(m, k))] = \{(q - 1) \cdot \prod_{k \leq j \leq m, \ j \neq n-1} p_{k-1, j} \} (v(m, k - 1)). \]

In perfect analogy, we have
\[ [(x_{k-1}), (e'(m, k)) = \{(q - 1) \cdot \prod_{k \leq j \leq m, \ j \neq n-1} p_{k-1, j} \} (v'(m, k - 1)). \]
The inductive supposition states
\[ e^{m, k} = \alpha^{m}_{k} \cdot \{(e(m, k)) + p_{n-1, n}(e'(k, m))\}. \]
Hence it remains to note that
\[ \alpha^{m}_{k-1} = \alpha^{m}_{k} \cdot (q - 1) \cdot \prod_{k \leq j \leq m, j \neq n-1} p_{k-1, j}. \]

Let \( m = \phi(k) \neq n. \) In this case, \( x_m = x_k, \epsilon_k^m = q^{-1}. \) If \( k = n - 1, m = n + 1, \) then \( e(n - 1, n) = x_n \) and by Definition [6.9] we have
\[ e[n - 1, n + 1] = x_n x_{n+1} - q^{-1} p_{n+1, n+1} x_n = (1 - q^{-1}) x_n x_{n+1} \]
since due to [6.3] we have \( x_{n+1} x_n = p_{n-1, n} x_n x_{n+1} \) with \( x_{n+1} = x_{n-1} \) and \( p_{n+1, n} p_{n-1, n} = 1. \) In the shuffle form, we get
\[
(x_n)(x_{n+1}) = (x_n x_{n+1}) + p_{n+1, n} (x_n x_{n+1}) = p_{n+1, n} \cdot \{(x_n + 1)x_n + p_{n-1, n}(x_n x_{n+1})\}.
\]
It remains to note that \( e(n - 1, n + 1) = x_n x_{n+1}, \) \( e(n + 1, n - 1) = x_{n+1} x_n, \)
\( e'(n - 1, n + 1) = x_{n-1} x_n, \) \( e'(n + 1, n - 1) = x_{n+1} x_n. \)

Let \( k < n - 1. \) By definition [6.9] we have
\[
e[k, m] = e[k, m - 1] \cdot x_m - q^{-1} p(e(k, m - 1), x_m) x_k \cdot e[k, m - 1].
\]

Already done case a) allows us to find the shuffle representation
\[ e[k, m - 1] = \alpha^{m-1}_{k} \cdot \{(e(m - 1, k)) + p_{n-1, n}(e'(m - 1, k))\}. \]

We have
\[
[(e(m - 1, k)), (x_m)] = \sum_{u = e(m - 1, k)} p(v, x_m) \cdot \{p(x_m, v)^{-1} p(v, x_m)^{-1} - q^{-1}\}(ux_m v).
\]

Relations [6.1], [6.2] imply that
\[
p(x_m, v)p(v, x_m) = \begin{cases} 1, & \text{if } v = \emptyset \text{ or } v = e(m - 1, k); \\ q^2, & \text{if } v = x_k; \\ q, & \text{otherwise}. \end{cases}
\]
Therefore in the decomposition just three terms remain. Two of them, corresponding to \( v = \emptyset \) and \( v = x_k, \) are canceled:
\[
p(x_k, x_m)(q^{-2} - q^{-1}) + (1 - q^{-1}) = q(q^{-2} - q^{-1}) + (1 - q^{-1}) = 0.
\]

Thus
\[
[(e(m - 1, k)), (x_m)] = \{(1 - q^{-1}) \cdot \prod_{k \leq i < m, i \neq n-1} p_{i, m}\} \{e(m, k)\}.
\]

In perfect analogy, we have
\[
[(e'(m - 1, k)), (x_m)] = \{(1 - q^{-1}) \cdot \prod_{k \leq i < m, i \neq n-1} p_{i, m}\} \{e'(m, k)\}.
\]

It suffices to note that \( 1 - q^{-1} = e^m_k (q - 1), \) and by definition
\[
\alpha^{m}_{k} = \alpha^{m-1}_{k} \cdot e^{m}_{k} (q - 1) \cdot \prod_{k \leq i < m, i \neq n-1} p_{i, m}.
\]

The proposition is completely proved. \( \square \)
9. COPRODUCT FORMULA FOR $U_q(\mathfrak{so}_{2n})$

**Theorem 9.1.** In $U_q^+(\mathfrak{so}_{2n})$ the coproduct on the elements $e[k, m]$, $k \leq m < 2n$ has the following explicit form

$$\Delta(e[k, m]) = e[k, m] \otimes 1 + g_{km} \otimes e[k, m]$$

$$+ \sum_{i=k}^{m-1} \tau_i (1 - q^{-1}) g_{ki} e[i + 1, m] \otimes e[k, i],$$

where $\tau_i = 1$, with two exceptions, being $\tau_n = 0$ if $k = n$, and $\tau_{n-1} = 0$ if $m = n$; and $\tau_{n-1} = p_{n,n-1}$ otherwise. Here $g_{ki} = g_\lambda(e(k, i))$ is a group-like element that appears from the word $e(k, i)$ under the substitutions $x_\lambda \leftarrow g_\lambda$.

**Proof.** If the word $e(k, m)$ does not contain the subword $x_n x_{n+1}$, then $e[k, m]$ belongs to either $U_{n-1}$ or $U_n$. Both of these Hopf algebras are isomorphic to $U_q^+(\mathfrak{sl}_n)$.

Hence if $m \leq n$ or $k \geq n$, then we have nothing to prove.

Suppose that $k < n < m$. In this case by Proposition 8.1 we have the shuffle representation

$$e[k, m] = \alpha_k^m \cdot \{ (e(m, k)) + p_{n-1} \rho(e'(m, k)) \},$$

where $(e(m, k))$ is a comonomial shuffle

$$(e(m, k)) = \begin{cases} (x_m x_{m-1} \cdots x_n x_{n+1} x_n x_{n-2} \cdots x_k), & \text{if } k < n - 1; \\ (x_m x_{m-1} \cdots x_n x_{n+1} x_n), & \text{if } k = n - 1, \end{cases}$$

whereas $(e'(m, k))$ is a related one:

$$(e'(m, k)) = \begin{cases} (x_m x_{m-1} \cdots x_n x_{n+1} x_n x_{n-2} \cdots x_k), & \text{if } k < n - 1; \\ (x_m x_{m-1} \cdots x_n x_{n+1} x_n), & \text{if } k = n - 1. \end{cases}$$

Using (2.14) it is easy to find the braided coproduct of the comonomial shuffles:

$$\Delta^b_0((e(m, k))) = \sum_{i=k}^{n-2} (e(m, i + 1)) \otimes (e(i, k)) + \sum_{i=n}^{m-1} (e(m, i + 1)) \otimes (e(i, k)),$$

$$\Delta^b_0((e'(m, k))) = \sum_{i=k}^{n-1} (e'(m, i + 1)) \otimes (e(i, k)) + \sum_{i=n+1}^{m-1} (e(m, i + 1)) \otimes (e'(i, k)),$$

where for short we define $\Delta^b_0(U) = \Delta^b(U) - U \otimes 1 - 1 \otimes U$. Taking into account (9.2), we have

$$(\alpha_k^m)^{-1} \Delta^b_0(e[k, m]) = \left( \sum_{i=k}^{n-2} (\alpha_i^{m-1})^{-1} e[i + 1, m] \otimes (e(i, k)) \right) + p_{n-1} n (e(m, n)) \otimes (e(n-1, k))$$

$$+ (e(m, n + 1)) \otimes (e(n, k)) + \sum_{i=n+1}^{m-1} (e(m, i + 1)) \otimes (\alpha_k^i)^{-1} e[k, i].$$

Relation (8.1) applied to $e[k, i]$, $i \leq n$ and $e[i + 1, m]$, $i \geq n$ allows one to rewrite the right hand side of the above equality in terms of $e[i, j]$:

$$= \left( \sum_{i=k}^{n-2} (\alpha_k^i)^{-1} (\alpha_i^{m-1})^{-1} e[i + 1, m] \otimes e[k, i] \right) + p_{n-1} n (\alpha_n^m)^{-1} (\alpha_k^{n-1})^{-1} e[n, m] \otimes e[k, n-1]$$

$$+ (\alpha_m^m)^{-1} (\alpha_k^i)^{-1} e[n + 1, m] \otimes e[k, n] + \sum_{i=n+1}^{m-1} (\alpha_i^{m-1})^{-1} (\alpha_k^i)^{-1} e[i + 1, m] \otimes e[k, i].$$
Thus, we have

\[ \Delta_0^b(e[k, m]) = \sum_{i=k}^{m-1} \gamma_i e[i + 1, m] \otimes e[k, i], \]

where

\[ \gamma_i = \delta_{n-1}^i \cdot \alpha_k^m (\alpha_k^i \alpha_m^{i+1})^{-1}, \]

whereas \( \delta_{n-1}^i \) is the Kronecker delta.

Our next step is to see that for all \( i, k \leq i < m \) we have

\[ \gamma_i = p_{n-1}^{k-1} \cdot (q - 1)^{e_m} (e_k^{i+1} e_k^{m})^{-1} p(e(k, i), e(i + 1, m)). \]

All factors except the \( e \)'s in (9.4) have the form \((q - 1)^s \prod_A p_{ab} \), where \( A \) is a suitable set of pairs \((a, b)\) and \( s \) is an integer exponent. Due to bimultiplicativity of the form \( p(-, -) \), the same is true for the right hand side of (9.5). Hence it suffices to demonstrate that the sum of exponents of the factors in (9.4) equals 1, and the resulting product domains in (9.4) and (9.5) are the same, or at least they define the same product.

If \( i < n - 1 \), then using (9.4), (8.1), and Proposition 8.1 we have the required equality for the exponents,

\[ (m - k - 1) - (i - k) - (m - i - 2) = 1, \]

and for the product domains:

\[ \{k \leq a < b \leq m, a, b \neq n - 1\} \setminus \{\{k \leq a < b \leq i\} \cup \{i + 1 \leq a < b \leq m, a, b \neq n - 1\}\} \]

\[ = \{k \leq a \leq i < b \leq m, a, b \neq n - 1\}. \]

If \( i \geq n \), then similarly we have the required equality for the exponents,

\[ (m - k - 1) - (i - k - 1) - (m - i - 1) = 1, \]

and for the product domains:

\[ \{k \leq a < b \leq m, a, b \neq n - 1\} \setminus \{\{k \leq a < b \leq i, a, b \neq n - 1\} \cup \{i + 1 \leq a < b \leq m, a, b \neq n - 1\}\} \]

\[ = \{k \leq a \leq i \leq b \leq m, a, b \neq n - 1\}. \]

In the remaining case, \( i = n - 1 \), we have \( e(k, i) = x_k \cdots x_{k-2} x_{k-1}, e(i + 1, m) = x_n x_{n+2} \cdots x_m \). Due to (8.2), the exponent is

\[ (m - k - 1) - (n - 1 - k) - (m - n - 1) = 1, \]

whereas the product domain of \( \alpha_k^m (\alpha_k^{n-1} \alpha_m^{n})^{-1} \) reduces to

\[ \{k \leq a < b \leq m, a, b \neq n - 1\} \setminus \{\{k \leq a < b \leq n - 1\} \cup \{n \leq a < b \leq m, a, b \neq n + 1\}\} \]

\[ = \{k \leq a \leq n - 1 < b \leq m\} \cup \{n + 1 = a < b \leq m\} \cup \{(n, n + 1)\}. \]

However, in this case the product domain for \( \alpha_k^{n-1} \) is not a subset of the product domain for \( \alpha_k^m \). Therefore additionally to the product defined by (9.6), there appears a factor \( \prod_{k \leq a < b = n - 1} p_{ab}^{-1} \) and a factor \( p_{n-1} n \) that comes from (9.4) due to \( \delta_{n-1}^i = 1 \).

The latter factor cancels with the factor defined by the subset \( \{(n, n + 1)\} \) since \( p_{n-1} n p_{n} n+1 = 1 \), whereas the product domain of the former factor must be added to the product domain of \( p(e(k, i), e(i + 1, m)) \):

\[ \{k \leq a \leq n - 1 < b \leq m, b \neq n + 1\} \cup \{k \leq a < b = n - 1\}. \]
It remains to compare the products defined by (9.6) without the last pair and that defined by (9.7).

The set \( \{ k \leq a < n - 1 < b \leq m, \ b \neq n + 1 \} \) is a subset of the first sets in (9.6) and (9.7). After cancelling the pairs from that set, (9.6) and (9.7) transform to, respectively,

\[
\{ k \leq a < n - 1 < b = n + 1 \} \cup \{ n + 1 = a < b \leq m \}
\]

and

\[
\{ a = n - 1 < b \leq m, \ b \neq n + 1 \} \cup \{ k \leq a < b = n - 1 \}.
\]

The first set of (9.8) and the second set of (9.9) define the same product because \( x_{n+1} = x_{n-1} \) and \( p_{2n+1} = p_{2n-1} \). By the same reason \( p_{n-1,b} = p_{n+1,b} \), hence the first set of (9.8) defines the same product as the second set of (9.9) up to one additional factor, \( p_{n-1,n} \) that corresponds to the pair \( (n-1,n) \). This factor is canceled by the first factor \( p_{n,n-1} \) that appears in (9.9) due to \( \delta_{n-1}^b = 1 \). The equality (9.1) is completely proved.

Now we are ready to consider the (unbraided) coproduct. Formula (2.18) demonstrates that the tensors \( u^{(1)} \otimes u^{(1)} \) of the coproduct and tensors \( u^{(1)}_b \otimes u^{(1)}_b \) of the braided coproduct are related by \( u^{(1)}_b = u^{(1)}(\sigma_{ki})^{-1}, \ u^{(2)}_b = u^{(2)} \). The equality (9.3) provides the values of \( u^{(1)}_b \) and \( u^{(2)}_b \). Hence we may find \( u^{(1)} = \gamma_i e[i+1, m]g_{ki} \) and \( u^{(2)} = e[k,i] \), where \( g_{ki} = \text{gr}(e[k,i]) \). The commutation rules imply

\[
e[i+1, m]g_{ki} = p(e[i+1, m], e(k,i))g_{ki}e[i+1, m].
\]

Therefore the coproduct has the form (9.1), where

\[
\tau_i(1 - q^{-1}) = \gamma_i p(e(i + 1, m), e(k,i)).
\]

Applying (9.5) and Lemma (9.3) we get

\[
\tau_i(1 - q^{-1}) = \frac{\gamma_i}{q - 1} \cdot \left( q^{-1} \cdot e_k^m (e_k^m e_{i+1}^m)^{-1} \cdot \sigma_k^m (\sigma_k^m e_{i+1}^m)^{-1} \right).
\]

Lemma (9.2) and Eq. (9.3) imply that \( e_k^m \sigma_k^m \) equals \( q \) for all \( k, m \) without exceptions. Hence

\[
e_k^m (e_k^m e_{i+1}^m)^{-1} \cdot \sigma_k^m (\sigma_k^m e_{i+1}^m)^{-1} = q^{-1},
\]

and we have

\[
\tau_i = p_{n,n-1}^{\delta_i} = \begin{cases} p_{n,n-1}, & \text{if } i = n - 1; \\ 1, & \text{otherwise.} \end{cases}
\]

The theorem is completely proved. □

**Remark 2.** If \( q^t = 1 \), \( t > 2 \), then (9.1) remains valid due to precisely the same arguments that were given in Remark 1, see page 12.

**Remark 3.** In fact, the exceptions \( \tau_n = 0 \) if \( k = n \), and \( \tau_{n-1} = 0 \) if \( m = n \) can be omitted in the statement of the above theorem. Indeed, the related tensors are, respectively, \( e[n+1,m] \otimes e[n,n] \) and \( e[n,n] \otimes e[k,m-1] \), whereas by definition \( e[n,n] = [x_n, x_n] = x_n \cdot x_n - q^{-1}p(x_n, x_n)x_n \cdot x_n = 0 \). So that, we may assume \( \tau_n = 1, \tau_{n-1} = p_{n,n-1} \) as well.
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