Online Learning Schemes for Power Allocation in Energy Harvesting Communications

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Abstract

We consider the problem of power allocation over a time-varying channel with unknown distribution in energy harvesting communication systems. In this problem, the transmitter has to choose the transmit power based on the amount of stored energy in its battery with the goal of maximizing the average rate obtained over time. We model this problem as a Markov decision process (MDP) with the transmitter as the agent, the battery status as the state, the transmit power as the action and the rate obtained as the reward. The average reward maximization problem over the MDP can be solved by a linear program (LP) that uses the transition probabilities for the state-action pairs and their reward values to choose a power allocation policy. Since the rewards associated the state-action pairs are unknown, we propose two online learning algorithms: UCLP and Epoch-UCLP that learn these rewards and adapt their policies along the way. The UCLP algorithm solves the LP at each step to decide its current policy using the upper confidence bounds on the rewards, while the Epoch-UCLP algorithm divides the time into epochs, solves the LP only at the beginning of the epochs and follows the obtained policy in that epoch. We prove that the reward losses or regrets incurred by both these algorithms are upper bounded by constants. Epoch-UCLP incurs a higher regret compared to UCLP, but reduces the computational requirements substantially. We also show that the presented algorithms work for online learning in cost minimization problems like the packet scheduling with power-delay tradeoff with minor changes.

Index Terms

Contextual bandits, multi-armed bandits (MABs), online learning, energy harvesting communications, Markov decision process (MDP).

I. INTRODUCTION

Communication systems where the transmissions are powered by the harvested energy have rapidly emerged as a viable option for the next-generation wireless networks with prolonged lifetime [1]. The performance of such systems is dependent on the efficient utilization of energy that is currently stored in the battery, as well as that is to be harvested over time. In [2], power allocation policies over a finite time horizon with known channel gain
and harvested energy distributions are studied. In [3], a similar problem is analyzed, but the energy arrivals are assumed to be deterministic and known in advance. The algorithms presented in [4] assume the knowledge of energy arrivals and tries to minimize the overall scheduling time for data packets. In our problem, however, the channel gain distribution is unknown and the harvest energy is assumed be stochastically varying with a known distribution. The transmitter has to decide the transmit power level based on the current battery status with the goal maximizing the average expected transmission rate obtained over time. We model the system as an MDP with the battery status as the state, the transmit power as the action, the rate as the reward. The power allocation problem, therefore, reduces to the average reward maximization problem for an MDP.

Our problem can also be seen from the lens of contextual bandits. In the standard contextual bandit problems [5], [6], [7], the contexts are assumed to be drawn from an unknown distribution independently over time. In this paper, we model the context transitions by MDPs. The action the agent takes at time $t$, therefore, affects not only the instantaneous reward but also the context in slot $t+1$. Thus the agent needs to decide the actions with the global objective in mind, i.e. maximizing the average reward over time. It must be noted that the MDP formulation generalizes the standard contextual bandits [8] for the case where the mapping between the context and random instance to reward is a known monotonic function, since the i.i.d. context case can be viewed as a single state MDP.

Our problem is also closely related to the reinforcement learning problem over MDPs from [9], [10], [11]. The objective for these problems is to maximize the average undiscounted reward over time. In [9], [10], the agent is unaware of the transition probabilities and the rewards corresponding to the state-action pairs. In [11], the agent knows the rewards, but the transition probabilities are still unknown. In our problem, however, the transition probabilities of the MDP can be inferred from the knowledge of the arrival distribution and the action taken from each state. The goal of our problem is to maximize the average reward by learning the rewards for the state-action pairs over time. One additional feature of our problem is that the function mapping the state-action pair and the channel gain to the rate is known to the agent. The reward information revealed after every action can, therefore, be used to infer the rewards for other state-action pairs.

This paper is organized as follows. First, we describe the model for the energy harvesting communication system studied in this paper, formulate this problem as an MDP and discuss the structure of the optimal policy in section II. We then propose our online learning algorithms UCLP and Epoch-UCLP, and prove their regret bounds in section III. In section IV, we show that our online learning framework can also model the average cost minimization problems over MDPs. Section V presents the results of numerical simulations for this problem and section VI concludes the paper. We also include appendices A and B to discuss and prove some of the technical lemmas at the end of the paper.

II. System Model

Consider a time-slotted energy harvesting communication system where the transmitter uses the harvested power for transmission over a channel with stochastically varying channel gains with unknown distribution as shown in figure 1. Let $p_t$ denotes the harvested power in the $t$-th slot which is assumed to be i.i.d. over time. Let $Q_t$ denote
the stored energy in the transmitter’s battery that has a capacity of $Q_{\text{max}}$. Assume that the transmitter decides to use $q_t(\leq Q_t)$ amount of power for transmission in $t$-th slot. We assume discrete and finite number of power levels for the harvested and transmit powers. The rate obtained during the $t$-th slot is assumed to follow a relationship

$$r_t = B \log_2(1 + q_t X_t),$$

where $X_t$ denotes the instantaneous channel gain-to-noise ratio of the channel which is assumed to be i.i.d. over time and $B$ is the channel bandwidth. The battery state gets updated in the next slot as

$$Q_{t+1} = \max\{Q_t - q_t + p_t, Q_{\text{max}}\}.$$

The goal is utilize the harvested power and choose a transmit power $q_t$ in each slot sequentially to maximize the expected average rate $\lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=1}^{T} r_t\right]$ obtained over time.

**A. Problem Formulation**

Consider an MDP $\mathcal{M}$ with a finite state space $\mathcal{S}$ and a finite action space $\mathcal{A}$. Let $\mathcal{A}_s \subset \mathcal{A}$ denote the set of allowed actions from state $s$. When the agent chooses an action $a_t \in \mathcal{A}_s$ in state $s_t \in \mathcal{S}$, it receives a random reward $r(s_t, a_t)$. Based on the agent’s decision the system undergoes a random transition to a state $s_{t+1}$ according to the transition probability $P(s_{t+1} \mid s_t, a_t)$. In the energy harvesting problem, the battery status $Q_t$ represents the system state $s_t$ and the transmit power $q_t$ represents the action taken $a_t$ at any slot $t$.

In this paper, we consider systems where the random rewards of various state action pairs can be modelled as

$$r_t(s_t, a_t) = f(s_t, a_t, X_t),$$

where $f$ is a reward function known to the agent and $X_t$ is a random variable internal to the system that is i.i.d. over time. Note that in the energy harvesting communications problem, the reward is the rate obtained at each slot.
and the reward function is defined in equation 1. In this problem, the channel gain-to-noise ratio $X_\epsilon$ corresponds to the system’s internal random variable. We assume that the distribution of the harvested energy $p_t$ is known to the agent. This implies that the state transition probabilities $P(s_{t+1} | s_t, a_t)$ are inferred by the agent based on the update equation 2.

A policy is defined as any rule for choosing the actions in successive time slots. The action chosen at time $t$ may, therefore, depend on the history of previous states, actions and rewards. It may even be randomized such that the action $a \in A_s$ is chosen from some distribution over the actions. A policy is said to be stationary, if the action chosen at time $t$ is only a function of the system state at $t$. This means that a deterministic stationary policy $\beta$ is a mapping from the state $s \in S$ to its corresponding action $a \in A_s$. When a stationary policy is played, the sequence of states $\{s_t | t = 1, 2, \cdots\}$ follows a Markov chain. An MDP is said to be ergodic, if every deterministic stationary policy leads to an irreducible and aperiodic Markov chain. According to section V.3 from [12], the average reward can be maximized by an appropriate deterministic stationary policy $\beta^*$ for an ergodic MDP with finite state space. In order to arrive at an ergodic MDP for the energy harvesting communications problem, we make following assumptions. When the battery state $Q_t > 0$, the transmit power $q_t > 0$. The distribution of the harvested energy is such that $\Pr\{p_t = p\} > 0$ for all $0 \leq p \leq Q_{\max}$. Under these assumptions, we claim and prove the ergodicity of the MDP as follows.

**Proposition 1.** The MDP corresponding to the power allocation application in energy harvesting communications is ergodic.

*Proof:* Consider any policy $\beta$ and let $P^{(n)}(s, s')$ be the $n$-step transition probabilities associated with the Markov chain resulting from the policy.

First, we prove that $P^{(1)}(s, s') > 0$ for any $s' \geq s$ as follows. According to the state update equations,

$$s_{t+1} = s_t - \beta(s_t) + p_t.$$  \hfill (4)

The transition probabilities can, therefore, be expressed as

$$P^{(1)}(s, s') = \Pr\{p = s' - s + \beta(s)\} \geq 0,$$  \hfill (5)

since $s' \geq s$ and $\beta(s) \geq 0$ for all states. This implies that any state $s' \in S$ is accessible from any other state $s$ in the resultant Markov chain, if $s \leq s'$.

Now, we prove that $P^{(1)}(s, s - 1) > 0$ for all $s \geq 1$ as follows. From equation 5, we observe that

$$P^{(1)}(s, s - 1) = \Pr\{p = \beta(s) - 1\} \geq 0,$$  \hfill (6)

since $\beta(s) \geq 1$ for all $s \geq 1$. This implies that every state $s \in S$ is accessible from the state $s + 1$ in the resultant Markov chain.

Equations 5 and 6 imply that all the state pairs $(s, s+1)$ communicate with each other. Since communication is an equivalence relationship, all the states communicate with each other and the resultant Markov chain is irreducible. Also, equation 5 implies that $P^{(1)}(s, s) > 0$ for all the states and the Markov chain is, therefore, aperiodic. \hfill \blacksquare
Since the MDP under consideration is ergodic, we restrict ourselves to the set of deterministic stationary policies which we interchangeably refer to as policies henceforth. Let $\mu(s,a)$ denote the expected reward associated with the state-actions pair $(s,a)$ which can be expressed as

$$\mu(s,a) = E[r(s,a)] = E_X[f(s,a,X)].$$

(7)

For ergodic MDPs, the optimal mean reward $\rho^*$ is independent of the initial state (see [13], section 8.3.3). It is specified as

$$\rho^* = \max_{\beta \in \mathcal{B}} \rho(\beta, \mathbf{M}),$$

(8)

where $\mathcal{B}$ is the set of all policies, $\mathbf{M}$ is the matrix whose $(s,a)$-th entry is $\mu(s,a)$, and $\rho(\beta, \mathbf{M})$ is the average expected reward per slot using policy $\beta$. We use the optimal mean reward as the benchmark and define the cumulative regret of a learning algorithm after $T$ time-slots as

$$\mathcal{R}(T) := T\rho^* - E\left[\sum_{t=0}^{T-1} r_t\right].$$

(9)

B. Optimal Stationary Policy

When the expected rewards for all state-action pairs $\mu(s,a)$ and the transition probabilities $P(s' \mid s,a)$ are known, the problem of determining the optimal policy to maximize the average expected reward over time can be formulated as a linear program (LP) (see e.g. [12], section V.3) shown below.

$$\text{maximize} \quad \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}_s} \pi(s,a)\mu(s,a)$$

subject to \quad $\pi(s,a) \geq 0$, $\forall s \in \mathcal{S}, a \in \mathcal{A}_s$, \quad \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}_s} \pi(s,a) = 1$, \quad \sum_{a \in \mathcal{A}_s} \pi(s',a) = \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}_s} \pi(s,a)P(s' \mid s,a)$, $\forall s' \in \mathcal{S}$,

(10)

where $\pi(s,a)$ denotes the stationary distribution of the MDP. The objective function of the LP from equation (10) gives the average rate corresponding to the stationary distribution $\pi(s,a)$, while the constraints make sure that this stationary distribution corresponds to a valid policy on the MDP. Such LPs can be solved by using standard solvers such as CVXPY [14].

If $\pi^*(s,a)$ is the solution to the LP from (10), then for every $s \in \mathcal{S}$, $\pi(s,a) > 0$ for only one action $a \in \mathcal{A}_s$. This is due to the fact the the optimal policy $\beta^*$ is deterministic for ergodic MDPs in average reward maximization problems (see [13], section 8.3.3). Thus for this problem, $\beta^*(s) = \arg \max_{a \in \mathcal{A}_s} \pi^*(s,a)$. Note that we, henceforth, drop the action index from the stationary distribution, since the policies under consideration are deterministic and the corresponding action is, therefore, deterministically known. In general, we use $\pi_\beta(s)$ to denote the stationary distribution corresponding to the policy $\beta$. It must be noted that the stationary distribution of any policy is
independent of the reward values and only depends on the transition probability for every state-action pair. The expected average reward depends on the stationary distribution as

$$\rho(\beta, M) = \sum_{s \in S} \pi_{\beta}(s) \mu(s, \beta(s)).$$  \hspace{1cm} (11)$$

In terms of this notation, the LP from (10) is equivalent to \(\max_{\beta \in \mathcal{B}} \rho(\beta, M)\). Since the matrix \(M\) is unknown, we develop online learning policies for our problem in the next section.

III. Online Learning Algorithms

For the power allocation problem under consideration, although the agent knows the state transition probabilities, the mean rewards for the state-action pairs \(\mu(s, a)\) values are still unknown. Hence, the agent cannot solve the LP from (10) to figure out the optimal policy. Any online learning algorithm needs to learn the reward values over time and update its policy adaptively. One interesting aspect of the problem, however, is that the reward function from equation (3) is known to the agent. Since the reward functions under consideration (1) is bijective, once the reward is revealed to the agent, it can invert them to infer the instantaneous realization of the random variable \(X\). This inference can be used to predict the rewards that would have been obtained for other state-action pairs using the function knowledge.

In our online learning framework, we store the average values of these inferred rewards \(\theta(s, a)\) for all state-action pairs. Also, we define confidence bounds at time \(t\):

$$u_{t,\lambda}(s, a) = \theta(s, a) + B(s, a) \sqrt{\frac{\lambda \ln t}{t}},$$  \hspace{1cm} (12)

$$l_{t,\lambda}(s, a) = \theta(s, a) - B(s, a) \sqrt{\frac{\lambda \ln t}{t}},$$  \hspace{1cm} (13)

which are referred to as UCB and LCB, respectively, and \(B(s, a) \geq \max_x f(s, a, x) - \min_x f(s, a, x)\) denotes any upper bound on the maximum possible range of the reward for the state-action pair \((s, a)\). The idea behind our algorithms is to use the UCB values for the maximization problems and LCB values for the minimization problems instead of the unknown \(\mu(s, a)\) values in the objective function of the LP from (10). Since the \(\theta(s, a)\) values get updated after each reward revelation, the agent needs to solve the LP again and again. We propose two online learning algorithms: UCLP where the agent solves the LP at each slot and Epoch-UCLP where the LP is solved at fixed pre-defined time slots. Although the agent is unaware of the actual \(\mu(s, a)\) values, it learns the statistics \(\theta(s, a)\) over time and eventually figures out the optimal policy.

We use following notations in the analysis of our algorithms: \(B_0 := \max_{(s, a)} B(s, a), \Delta_{\min} := \rho^* - \max_{\beta \neq \beta^*} \rho(\beta, M)\). The total number of states and actions are specified as \(S := |S|, A := |A|\), respectively. Also, \(U_{t,\lambda}\) and \(L_{t,\lambda}\) denote the matrices containing the entries \(u_{t,\lambda}(s, a)\) and \(l_{t,\lambda}(s, a)\) at time \(t\), respectively.

A. UCLP

The UCLP algorithm presented in algorithm 1 solves the LP at each time-step and updates it policy based on the solution obtained. It stores only one \(\theta\) value per state-action pair, its required storage is, therefore, \(O(SA)\).
In theorem 1, we derive an upper bound on the expected number of slots where the LP fails to find the optimal solution using UCLP. We use this result to bound the total expected regret of UCLP in theorem 2. These results guarantee that the regret is always upper bounded by a constant. Note that, for the ease of exposition, we assume that the time starts at \( t = 0 \). This simplifies the analysis, but has no significant impact on the regret bounds.

**Algorithm 1 UCLP**

1: Parameters: \( \lambda > 1/2 \).

2: Initialization: For all \((s,a)\) pairs, \( \theta(s,a) = 0 \).

3: for \( n = 0 \) do

4: \hspace{1em} Given the state \( s_0 \) and choose any valid action;

5: \hspace{1em} Update all \((s,a)\) pairs: \( \theta(s,a) = f(s,a,x_0) \);

6: end for

7: // MAIN LOOP

8: while 1 do

9: \hspace{1em} \( n = n + 1 \);

10: Confidence bounds: \( u_{n,\lambda}(s,a) = \theta(s,a) + B(s,a) \sqrt{\frac{\lambda \ln n}{n}} \);

11: Solve the LP from (10) with \( u_{n,\lambda}(s,a) \) instead of unknown \( \mu(s,a) \);

12: In terms of the LP solution \( \pi(n) \), define \( \beta_n(s) = \arg \max_{a \in \mathcal{A}} \pi(n)(s,a), \forall s \in \mathcal{S} \);

13: Given the state \( s_n \), select the action \( \beta_n(s_n) \);

14: Update for all \((s,a)\) pairs:

\[
\theta(s,a) \leftarrow \frac{n\theta(s,a) + f(s,a,x_n)}{n + 1};
\]

15: end while

**Theorem 1.** The expected number of slots where non-optimal policies are played by UCLP is upper bounded by

\[
n_0 + (1 + A) S\sigma_\lambda, \tag{14}
\]

where \( \sigma_\lambda = \sum_{t=1}^{\infty} t^{-2\lambda} \) and \( n_0 \) denotes the minimum value of \( n \in \mathbb{N} \) for which \( \Delta_{\min} \geq 2B_0 \sqrt{\frac{\lambda \ln n}{n}} \).

**Proof:** Let \( \beta_t \) denote the policy obtained by UCLP at time \( t \) and \( \mathbb{I}(z) \) be the indicator function defined to be 1 when the predicate \( z \) is true, and 0 otherwise. Now the number of slots where non-optimal policies are played can be expressed as

\[
N_1 = 1 + \sum_{t=1}^{\infty} \mathbb{I}\{\beta_t \neq \beta^*\}
\]

\[
\leq n_0 + \sum_{t=n_0}^{\infty} \mathbb{I}\{\beta_t \neq \beta^*\}
\]
Hence we upper bound the probabilities of each of these events. For the first event from condition (16), we get

\[ \rho(\beta^*, U_{t,\lambda}) \leq \rho(\beta_t, U_{t,\lambda}) \]

We observe that \( \rho(\beta^*, U_{t,\lambda}) \leq \rho(\beta_t, U_{t,\lambda}) \) implies that at least one of the following inequalities must be true:

\[ \rho(\beta^*, U_{t,\lambda}) \leq \rho(\beta^*, M) \]  \tag{16}

\[ \rho(\beta_t, L_{t,\lambda}) \geq \rho(\beta_t, M) \]  \tag{17}

\[ \rho(\beta^*, M) < \rho(\beta_t, U_{t,\lambda}) - \rho(\beta_t, L_{t,\lambda}). \]  \tag{18}

Hence we upper bound the probabilities of each of these events. For the first event from condition (16), we get

\[
\Pr \{ \rho(\beta^*, U_{t,\lambda}) \leq \rho(\beta^*, M) \} = \Pr \left\{ \sum_{s \in S} \pi^*(s, \beta^*(s)) u_{t,\lambda}(s, \beta^*(s)) \leq \sum_{s \in S} \pi^*(s, \beta^*(s)) \mu(s, \beta^*(s)) \right\} \\
\leq \Pr \{ \text{For at least one state } s \in S : \pi^*(s, \beta^*(s)) u_{t,\lambda}(s, \beta^*(s)) \leq \pi^*(s, \beta^*(s)) \mu(s, \beta^*(s)) \} \\
\leq \sum_{s \in S} \Pr \{ \pi^*(s, \beta^*(s)) u_{t,\lambda}(s, \beta^*(s)) \leq \pi^*(s, \beta^*(s)) \mu(s, \beta^*(s)) \} \\
= \sum_{s \in S} \Pr \{ u_{t,\lambda}(s, \beta^*(s)) \leq \mu(s, \beta^*(s)) \} \\
\leq \sum_{s \in S} t^{-2\lambda} \\
= St^{-2\lambda}, \tag{19}
\]

where (a) holds due to concentration of confidence bounds from lemma 2 (see appendix A).

Similarly for the second event from condition (17), we get

\[
\Pr \{ \rho(\beta_t, L_{t,\lambda}) \geq \rho(\beta_t, M) \} = \Pr \left\{ \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) l_{t,\lambda}(s, a) \geq \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) \mu(s, a) \right\} \\
\leq \Pr \{ \text{For at least one state-action pair } (s, a) : \pi_{\beta_t}(s, a) l_{t,\lambda}(s, a) \geq \pi_{\beta_t}(s, a) \mu(s, a) \} \\
\leq \sum_{s \in S} \sum_{a \in A_s} \Pr \{ \pi_{\beta_t}(s, a) l_{t,\lambda}(s, a) \geq \pi_{\beta_t}(s, a) \mu(s, a) \} \\
= \sum_{s \in S} \sum_{a \in A_s} \Pr \{ l_{t,\lambda}(s, a) \geq \mu(s, a) \} \\
\leq \sum_{s \in S} \sum_{a \in A_s} t^{-2\lambda} \\
\leq SAT^{-2\lambda}, \tag{20}
\]

where (b) holds due to concentration bounds from lemma 2 (see appendix A).

Now let us analyze the third event from condition (18).

\[
\rho(\beta_t, U_{t,\lambda}) - \rho(\beta_t, L_{t,\lambda}) = \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) u_{t,\lambda}(s, a) - \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) l_{t,\lambda}(s, a) 
\]
\[
= \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) \left( u_{t, \lambda}(s, a) - l_{t, \lambda}(s, a) \right)
\]
\[
= \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) \left( 2B(s, a) \sqrt{\frac{\lambda \ln t}{t}} \right)
\]
\[
= 2 \sqrt{\frac{\lambda \ln t}{t}} \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) B(s, a)
\]
\[
\leq 2 \sqrt{\frac{\lambda \ln t}{t}} \sum_{s \in S} \sum_{a \in A_s} \pi_{\beta_t}(s, a) B_0
\]
\[
\leq 2B_0 \sqrt{\frac{\lambda \ln t}{t}}.
\]  

(21)

Since \( \Delta_{\text{min}} \leq \rho(\beta^*, M) - \rho(\beta_t, M) \), for all \( t \geq n_0 \) we get
\[
\rho(\beta^*, M) - \rho(\beta_t, M) - (\rho(\beta_t, U_{t, \lambda}) - \rho(\beta_t, L_{t, \lambda}))
\]
\[
\geq \Delta_{\text{min}} - 2B_0 \sqrt{\frac{\lambda \ln t}{t}}
\]
\[
\geq \Delta_{\text{min}} - 2B_0 \sqrt{\frac{\lambda \ln n_0}{n_0}}
\]
\[
\geq 0.
\]  

(22)

This implies that condition (18) is always false for \( t \geq n_0 \).

The expected number of incorrect policies from equation (15), therefore, can be expressed as
\[
\mathbb{E}[N_1] \leq n_0 + \sum_{t=n_0}^{\infty} \Pr \left\{ \rho(\beta^*, U_{t, \lambda}) \leq \rho(\beta_t, U_{t, \lambda}) \right\}
\]
\[
\leq n_0 + \sum_{t=n_0}^{\infty} \left( \Pr \left\{ \rho(\beta^*, U_{t, \lambda}) \leq \rho(\beta^*, M) \right\} + \Pr \left\{ \rho(\beta_t, L_{t, \lambda}) \geq \rho(\beta_t, M) \right\} \right)
\]
\[
\leq n_0 + \sum_{t=n_0}^{\infty} \left( St^{-2\lambda} + SAT^{-2\lambda} \right)
\]
\[
\leq n_0 + (1 + A) S \sum_{t=n_0}^{\infty} t^{-2\lambda}
\]
\[
\leq n_0 + (1 + A) S \sigma_\lambda,
\]  

(23)

where \( \sigma_\lambda < \infty \) as \( \lambda > 1/2 \).

We observe that UCLP requires \( \lambda > 1/2 \) in order to have a constant regret upper bound from equation (14), since \( \sigma_\lambda < \infty \) for \( \lambda > 1/2 \). It is important to note that even if the optimal policy is found by the LP and played during certain slots, it does not mean that regret contribution of those slots is zero. According to the definition of regret from equation (9), regret contribution of a certain slot is zero if and only if the optimal policy is played and the corresponding Markov chain is at its stationary distribution. In appendix B, we introduce tools to analyze the mixing of Markov chains and characterize this regret contribution in theorem 5. These results are used to upper bound the UCLP regret in the next theorem.
Theorem 2. The total expected regret of the UCLP is upper bounded by

\[ \left(n_0 + (1 + A) S \sigma \lambda \right) \Delta_{\text{max}} + \left(1 + (1 + A) S \sigma \lambda \right) \frac{\mu_{\text{max}}}{1 - \gamma}, \]

where \( \gamma = \max_{s, s' \in S} \| P_\ast(s', \cdot) - P_\ast(s, \cdot) \|_{TV} \), \( P_\ast \) denotes the transition probability matrix corresponding to the optimal policy, \( \mu_{\text{max}} = \max_{s \in S, a \in A} \mu(s, a) \) and \( \Delta_{\text{max}} = \rho^\ast - \min_{s \in S, a \in A} \mu(s, a) \).

Proof: The regret of UCLP arises when either non-optimal actions are taken or optimal actions are taken, but the corresponding Markov chain is not at stationarity. For the first source of regret, it is sufficient to analyze the number of instances where the LP fails to find the optimal policy. For the second source, however, we need to analyze the total number of phases where the optimal policy is found in succession.

Since only the optimal policy is played in consecutive slots in a phase, it corresponds to transitions on the Markov chain associated with the optimal policy and the tools from appendix B can be applied. According to theorem 5, the regret contribution of any phase is bounded from above by \( (1 - \gamma)^{-1} \mu_{\text{max}} \). As proved in theorem 1 for \( t \geq n_0 \), the expected number of instances of non-optimal policies is upper bounded by \( (1 + A) S \sigma \lambda \). Even if none of the these instances appear in successive slots, the expected number of optimal phases is upper bounded by \( 1 + (1 + A) S \sigma \lambda \).

Hence, for \( t \geq n_0 \), the expected regret contribution from the slots following the optimal policy is upper bounded by

\[ \left(1 + (1 + A) S \sigma \lambda \right) \frac{\mu_{\text{max}}}{1 - \gamma}. \]

Note that maximum regret possible during one slot is \( \Delta_{\text{max}} \). Hence for the first \( n_0 \) slots, the regret is bounded by \( n_0 \Delta_{\text{max}} \). Since there are at most \( (1 + A) S \sigma \lambda \) slots with non-optimal policy for \( t \geq n_0 \) in expectation, their expected regret is upper bounded by \( (1 + A) S \sigma \lambda \Delta_{\text{max}} \).

Overall expected regret for the UCLP algorithm is, therefore, bounded from above by equation (24).

Remark 1. It must be noted that we call two policies as same if and only if they recommend identical actions for every state. It is, therefore, possible for a non-optimal policy to recommend optimal actions for some of the states. In the analysis of UCLP, however, we assumed that any occurrence of a non-optimal policy contributes to the regret. Although this is not necessary, it leads us to a valid upper bound in the proof.

B. Epoch-UCLP

The main drawback of the UCLP algorithm is that it is computationally heavy as it solves one LP per time-slot. In order to reduce the computation requirements, we propose the Epoch-UCLP algorithm in algorithm 2. The Epoch-UCLP divides the time in several epochs and solves the LPs only at the beginning of each epoch. The policy obtained by solving the LP at the beginning of an epoch is followed for the remaining slots in that epoch. We increase the length of these epoch exponentially as time progresses and our confidence on the obtained policy increases. In spite of solving much fewer number of LPs, the regret of Epoch-UCLP is still bounded by a constant.

First, we obtain an upper bound on the number of slots where the algorithm plays non-optimal policies in theorem 3 and later use this result to bound the regret in theorem 4.
Algorithm 2 Epoch-UCLP

1: Parameters: $\lambda > 1/2$ and $\eta > 1$.
2: Initialization: $k = 0$ and for all $(s, a)$ pairs, $\theta(s, a) = 0$.
3: for $n = 0$ do
4: Given the state $s_0$ and choose any valid action;
5: Update all $(s, a)$ pairs: $\theta(s, a) = f(s, a, x_0)$;
6: end for
7: // MAIN LOOP
8: while 1 do
9: $n = n + 1$;
10: if $n = \eta^k$ then
11: $k = k + 1$;
12: Confidence bounds: $u_{n, \lambda}(s, a) = \theta(s, a) + B(s, a) \sqrt{\frac{\ln n}{n}}$;
13: Solve the LP from (10) with $u_{n, \lambda}(s, a)$ instead of unknown $\mu(s, a)$;
14: In terms of the LP solution $\pi(n)$, define $\beta(k)(s) = \arg \max_{a \in A} \pi(n)(s, a)$, $\forall s \in S$;
15: end if
16: Given the state $s_n$, select the action $\beta(k)(s_n)$;
17: Update for all $(s, a)$ pairs:
$$\theta(s, a) \leftarrow \frac{n\theta(s, a) + f(s, a, x_n)}{n + 1}$$
18: end while

**Theorem 3.** The expected number of slots where non-optimal policies are played by Epoch-UCLP is upper bounded by
$$n_0\eta + (1 + A)Sn_0^{-(2\lambda - 1)} \left( \frac{\eta - 1}{\eta^{2\lambda - 1} - 1} \right).$$

**Proof:** Note that epoch $k$ starts at $t = \eta^{k-1}$ and end at $t = \eta^k - 1$. The policy obtained at $t = \eta^{k-1}$ by solving the LP is, therefore, played for $\eta^k - \eta^{k-1}$ number of slots.

Let us analyse the probability that the policy played during epoch $k$ is not optimal. Let that policy be $\beta(k)$.

$$\Pr\{\beta(k) \neq \beta^*\} = \Pr\{\beta_{\eta^{k-1}} \neq \beta^*\} \leq (a) (1 + A)S(\eta^{k-1})^{-2\lambda},$$

where $(a)$ holds for all $k - 1 \geq k_0 = \lceil \log_\eta n_0 \rceil$ as proved in theorem [4].

When the LP fails to obtain the optimal policy at the beginning of an epoch, then all the slots in that epoch will
the obtained non-optimal policy. Let $N_2$ denote total number of such slots. We get

$$N_2 = 1 + \sum_{k=1}^{\infty} (\eta^k - \eta^{k-1})I\{\beta(k) \neq \beta^*\}$$

$$\leq 1 + \sum_{k=1}^{k_0} (\eta^k - \eta^{k-1}) + \sum_{k=k_0+1}^{\infty} (\eta^k - \eta^{k-1})I\{\beta(k) \neq \beta^*\}$$

$$\leq \eta^{k_0} + \sum_{k=k_0+1}^{\infty} (\eta^k - \eta^{k-1})I\{\beta(k) \neq \beta^*\}$$

$$(b) \leq n_0\eta + \sum_{k=k_0+1}^{\infty} (\eta^k - \eta^{k-1})I\{\beta(k) \neq \beta^*\},$$

(28)

where (b) holds as $k_0 - 1 \leq \log_\eta n_0$. Taking expectations, we obtain

$$\mathbb{E}[N_2] = n_0\eta + \sum_{k=k_0+1}^{\infty} (\eta^k - \eta^{k-1}) \Pr\{\beta(k) \neq \beta^*\}$$

$$(c) \leq n_0\eta + \sum_{k=k_0+1}^{\infty} (\eta^k - \eta^{k-1})(1 + A)S(\eta^{k-1})^{-2\lambda}$$

$$= n_0\eta + (1 + A)S(\eta - 1) \sum_{k=k_0+1}^{\infty} \eta^{-(2\lambda-1)(k-1)}$$

$$= n_0\eta + (1 + A)S(\eta - 1) \sum_{k=k_0}^{\infty} \eta^{-(2\lambda-1)k}$$

$$(d) = n_0\eta + (1 + A)S(\eta - 1) \frac{\eta^{-(2\lambda-1)k_0}}{1 - \eta^{-(2\lambda-1)}}$$

$$= n_0\eta + (1 + A)S(\eta - 1) \frac{(n_0\eta)^{-(2\lambda-1)}}{1 - \eta^{-(2\lambda-1)}}$$

$$= n_0\eta + (1 + A)Sn_0^{-(2\lambda-1)} \left( \frac{\eta - 1}{\eta^{2\lambda-1} - 1} \right),$$

(29)

where (c) holds due to equation (27) and (d) holds as the geometric series converges for $\lambda > 1/2$ and $\eta > 1$. ■

Now we analyse the regret of Epoch-UCLP in the following theorem.

**Theorem 4.** The total expected regret of the Epoch-UCLP is upper bounded by

$$n_0\eta\Delta_{\max} + (1 + A)Sn_0^{-(2\lambda-1)} \left( \frac{\eta - 1}{\eta^{2\lambda-1} - 1} \right) \Delta_{\max} + \left( 1 + (1 + A)S \frac{n_0^{2\lambda}}{1 - \eta^{-2\lambda}} \right) \frac{\mu_{\max}}{1 - \gamma}.$$  

(30)

**Proof:** First, we analyse the number of phases where the optimal policy is played in successive slots. Note that any two optimal phases are separated by at least one non-optimal epoch. For $k - 1 \geq k_0 = \lceil \log_\eta n_0 \rceil$, we first bound the number of non-optimal epochs $N_3$.

$$\mathbb{E}[N_3] = \sum_{k=k_0+1}^{\infty} \Pr\{\beta(k) \neq \beta^*\}$$

$$\leq \sum_{k=k_0+1}^{\infty} (1 + A)S(\eta^{k-1})^{-2\lambda}$$

(From equation (27))
\[\begin{align*}
(1 + A)S \sum_{k=k_0}^{\infty} (\eta^{-2\lambda})^k \\
= (1 + A)S \frac{n^{-2\lambda k_0}}{1 - \eta^{-2\lambda}} \\
\leq (1 + A)S \frac{n^{-2\lambda}}{1 - \eta^{-2\lambda}}. \quad \text{(Since } k_0 \geq \log_{\eta} n_0) 
\end{align*}\]

Hence for \( k \geq k_0 + 1 \), there can be at most \( E[N_3] + 1 \) number of optimal phases in expectation. Since each of these phases can contribute a maximum of \( (1 - \gamma)^{-1}\mu_{\text{max}} \) regret in expectation, total regret from slots with optimal policies is upper bounded by

\[\left(1 + (1 + A)S \frac{n_0^{-2\lambda}}{1 - \eta^{-2\lambda}}\right) \frac{\mu_{\text{max}}}{1 - \gamma}. \quad \text{(31)}\]

In theorem 3, we upper bounded the total number of slots where a non-optimal policy is played by Epoch-UCLP. The bound is calculated by adding the total number slots in the first \( k_0 \) epochs and the number of slots with a non-optimal policy for epochs \( k \geq k_0 + 1 \). Since the maximum expected regret incurred during any slot is \( \Delta_{\text{max}} \), the total expected regret of Epoch-UCLP is upper bounded by

\[n_0 \eta \Delta_{\text{max}} + (1 + A)S n_0^{-2\lambda(2\lambda - 1)} \left(\frac{\eta - 1}{\eta^{2\lambda - 1} - 1}\right) \Delta_{\text{max}} + \left(1 + (1 + A)S \frac{n_0^{-2\lambda}}{1 - \eta^{-2\lambda}}\right) \frac{\mu_{\text{max}}}{1 - \gamma}.\]

C. Regret vs Computation Tradeoff

The UCLP algorithm solves \( T \) LPs while the Epoch-UCLP algorithm solves \( \lceil \log_{\eta} T \rceil \) LPs in time \( T \). This drastic reduction in the required computation comes at the cost of an increase in the regret for Epoch-UCLP. It must, however, be noted that both the algorithms have constant-bounded regrets. Also, increasing the value of the parameter \( \eta \) in Epoch-UCLP leads to reduction in the number of LPs to be solved over time by increasing the epoch lengths. Any non-optimal policy found by LP, therefore, gets played over longer epochs increasing the overall regret. The system designer can analyse the regret bounds of these two algorithms and her own performance requirements to choose the parameter \( \eta \) for the system. We analyse the effect of variation of \( \eta \) through simulations in the next section.

IV. COST MINIMIZATION PROBLEMS

We have considered reward maximization problems for describing our online learning framework. This framework can also be applied to average cost minimization problems in packet scheduling with power-delay tradeoff as shown in figure 2. We describe this motivating example and the minor changes required in our algorithms.

Consider a communication system where a sender sends data packets to a receiver over a time-slotted stochastically varying channel with unknown distribution. In previous papers such as [15], this communication system has been studied assuming the channel to be non-stochastically varying over time. In our setting, the arrival of data packets is also stochastic with a known distribution. The sender can send multiple packets at a higher cost, or can defer some
Let $Q_t$ denote the number of packets in the queue at time $t$ and $r_t (\leq Q_t)$ be the number of packets transmitted by the sender during the slot. Hence, $Q_t - r_t$ number of packets get delayed. The sender’s queue gets updated as

$$Q_{t+1} = \max\{Q_t - r_t + b_t, Q_{\text{max}}\},$$

(32)

where $b_t$ is the number of new packet arrivals in $t$-th slot and $Q_{\text{max}}$ is the maximum queue size possible. Since the data-rate is modelled according to equation (1), the power cost incurred during the $t$-th slot by transmitting $r_t$ packets over the channel becomes $w_p X_t^2 r_t / B$, where $w_p$ is a constant known to the sender and $X_t$ is the instantaneous channel gain-to-noise ratio that is assumed to be i.i.d. over time. Assuming $w_d$ as the unit delay penalty, during the slot the sender incurs an effective cost

$$C_t = w_d (Q_t - r_t) + w_p X_t^2 r_t / B.$$  

(33)

This problem also represents an MDP where the queue size is the state and the number of packets transmitted is the action taken. The goal of this problem is, therefore, to schedule transmissions $r_t$ sequentially and minimize the expected average cost over time

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[ \sum_{t=1}^{T} C_t \right].$$

(34)

The presented algorithms UCLP and Epoch-UCLP also apply to cost minimization problems with minor changes. If $\rho(\beta, M)$ denotes the average expected cost of the policy $\beta$, then $\rho^* = \min_{\beta \in \mathcal{B}} \rho(\beta, M)$. Using this optimal mean cost as the benchmark, we define the cumulative regret of a learning algorithm after $T$ time-slots as

$$\mathcal{R}(T) := \mathbb{E}\left[ \sum_{t=0}^{T-1} C_t \right] - T \rho^*.$$  

(35)

In order to minimize the regret for this problem, instead of the upper confidence bounds $u_{t, \lambda}$, we need to store the lower confidence bounds on the costs $l_{t, \lambda}$ at each step. Also, the LP from (10) needs to be changed from a maximiza-
tion LP to a minimization LP. With these changes to the algorithms and a redefinition: $\Delta_{\text{max}} = \max_{s \in S, a \in A} \mu(s, a) - \rho^t$, all the theoretical guarantees provided in theorems 1, 2, 3 and 4 still hold.

V. NUMERICAL SIMULATIONS

We perform simulations for the power allocation problem with $S = \{0, 1, 2, 3, 4\}$ and $A = \{0, 1, 2, 3, 4\}$. Note that each state $s_t$ corresponds to $Q_t$ from equation 2 with $Q_{\text{max}} = 4$ and $a_t$ corresponds to the transmit power $q_t$ from equation 1. The valid actions for each state are shown in table I. The reward function is the rate function from equation 1 and the channel gain is a scaled Bernoulli random variable with $\Pr\{X = 10\} = 0.2$ and $\Pr\{X = 0\} = 0.8$. We use CVXPY for solving the LPs in our algorithms. For the simulations in figure 3 we use $\lambda = 2$ and $\eta = 10$, and plot the average regret performance over $10^3$ independent runs of different algorithms. Here, the naive policy never uses the battery, i.e. it uses all the arriving power for the current transmission. Note that the optimal policy also incurs a regret because of the corresponding Markov chain not being at stationarity. We observe that UCLP follows the performance of the optimal policy with the difference in regret stemming from the first few time-slots when the channel statistics are not properly learnt and thus UCLP fails to find the optimal policy. As the time progresses, UCLP finds the optimal policy and the regret follows the regret pattern of the optimal policy. In Epoch-UCLP with $\eta = 10$, however, the LP solution at $t = 1$ is followed for the first epoch and thus the regret grows linearly till $t = 9$. At $t = 10$, a new LP is solved which often leads to the optimal policy and the regret contribution from latter slots, therefore, follows the regret of the optimal policy. It must be noted that Epoch-UCLP solves only 2 LPs during these slots, while UCLP solves 99 LPs. Epoch-UCLP, therefore, reduces the computational requirements substantially while incurring a slightly higher cumulative regret.
For the simulations in figure 4, we use $\lambda = 1$ and plot the average regret performance over $10^4$ independent runs of Epoch-UCLP with different $\eta$ values. As the value of $\eta$ increases, the length of the epochs increases and thus the initial non-optimal policies are used for longer epochs. Following an incorrect policy, therefore, leads to linear regret in the initial slots. We observe a drastic change in the regret behavior of Epoch-UCLP at $t = \eta$, because this is the slot when a new LP is solved by our algorithm. Once the optimal policy is found by the algorithm, its regret in latter slots follows the trend of the optimal policy. We see the regret vs computation tradeoff in action, as the decrease in computation by increasing $\eta$ leads to larger regrets.

VI. Conclusion

We have considered the problem of power allocation over a stochastically varying channel with unknown distribution in an energy harvesting communication system. We have cast this problem as an online learning problem over an MDP. If the transition probabilities and the mean rewards associated with the MDP are known, the optimal policy maximizing the average expected reward over time can be found by solving an LP specified in the paper.
Since the agent is only assumed to know the distribution of the harvested energy, she needs to learn the rewards of the state-action pairs over time and make her decisions based on the learnt behaviour. For this problem, we have proposed two online learning algorithms: UCLP and Epoch-UCLP which solve the LP using the upper confidence bounds of the rewards instead of the unknown mean rewards. The UCLP algorithm solves the LP at each time-slot using the updated confidence bounds, while the Epoch-UCLP only solves the LP at certain pre-defined time-slots parametrized by $\eta$ and thus, saves a lot of computation at the cost of an increased regret. We have shown that the regrets incurred by both these algorithms are bounded from above by constants. The system designers can, therefore, analyze the regret versus computation tradeoff and tune the parameter $\eta$ based on their performance requirements.

Through the numerical simulations, we have shown that the regret of UCLP is very close to that of the optimal policy. We have also analyzed the effect of $\eta$ on the regret the Epoch-UCLP algorithm which approaches the regret of the optimal policy for small $\eta$ values.

While we have considered the reward maximization problem in energy harvesting communications for our analysis, we have shown that these algorithms also work for the cost minimization problems in packet scheduling with minor changes.

**REFERENCES**

[1] S. Ulukus, A. Yener, E. Erkip, O. Simeone, M. Zorzi, P. Grover, and K. Huang, “Energy harvesting wireless communications: A review of recent advances,” Selected Areas in Communications, IEEE Journal on, vol. 33, no. 3, pp. 360–381, 2015.
[2] C. K. Ho and R. Zhang, “Optimal energy allocation for wireless communications with energy harvesting constraints,” Signal Processing, IEEE Transactions on, vol. 60, no. 9, pp. 4808–4818, 2012.
[3] K. Tutuncuoglu and A. Yener, “Optimum transmission policies for battery limited energy harvesting nodes,” Wireless Communications, IEEE Transactions on, vol. 11, no. 3, pp. 1180–1189, 2012.
[4] J. Yang and S. Ulukus, “Optimal packet scheduling in an energy harvesting communication system,” Communications, IEEE Transactions on, vol. 60, no. 1, pp. 220–230, 2012.
[5] J. Langford and T. Zhang, “The epoch-greedy algorithm for multi-armed bandits with side information,” in Advances in neural information processing systems, pp. 817–824, 2008.
[6] M. Dudik, D. Hsu, S. Kale, N. Karampatziakis, J. Langford, L. Reyzin, and T. Zhang, “Efficient optimal learning for contextual bandits,” in Conference on Uncertainty in Artificial Intelligence, 2011.
[7] A. Agarwal, D. Hsu, S. Kale, J. Langford, L. Li, and R. E. Schapire, “Taming the monster: A fast and simple algorithm for contextual bandits,” in International Conference on Machine Learning, pp. 1638–1646, 2014.
[8] P. Sakulkar and B. Krishnamachari, “Stochastic contextual bandits with known reward functions.” USC ANRG Technical Report, ANRG-2016-02, \[http://anrg.usc.edu/www/papers/DCB_ANRG_TechReport.pdf\].
[9] P. Ortner and R. Auer, “Logarithmic online regret bounds for undiscounted reinforcement learning,” in Proceedings of the 2006 Conference on Advances in Neural Information Processing Systems, vol. 19, p. 49, 2007.
[10] P. Auer, T. Jaksch, and R. Ortner, “Near-optimal regret bounds for reinforcement learning,” in Advances in neural information processing systems, pp. 89–96, 2009.
[11] A. Tewari and P. L. Bartlett, “Optimistic linear programming gives logarithmic regret for irreducible mdps,” in Advances in Neural Information Processing Systems, pp. 1505–1512, 2008.
[12] S. Ross, Introduction to stochastic dynamic programming. Academic Press, 1983.
[13] M. L. Puterman, Markov decision processes: discrete stochastic dynamic programming. John Wiley & Sons, 2005.
[14] S. Diamond and S. Boyd, “CVXPY: A Python-embedded modeling language for convex optimization,” Journal of Machine Learning Research, 2016. To appear.
Appendix A

Technical Lemmas & Proofs

Lemma 1 (Hoeffding’s Concentration Inequality from [16]). Let $Y_1, ..., Y_n$ be i.i.d. random variables with mean $\mu$ and range $[0, 1]$. Let $S_n = \sum_{t=1}^{n} Y_t$. Then for all $\alpha \geq 0$

\[ \Pr\{S_n \geq n\mu + \alpha\} \leq e^{-2\alpha^2/n} \]
\[ \Pr\{S_n \leq n\mu - \alpha\} \leq e^{-2\alpha^2/n} . \]

Lemma 2 (Concentration of Confidence Bounds). At any time $t$, for any valid state-action pair $(s, a)$, following inequalities hold:

1) $\Pr\{u_{t,\lambda}(s, a) \leq \mu(s, a)\} \leq t^{-2\lambda}$,
2) $\Pr\{l_{t,\lambda}(s, a) \geq \mu(s, a)\} \leq t^{-2\lambda}$.

Proof: For the first inequality,

\[ \Pr\{u_{t,\lambda}(s, a) \leq \mu(s, a)\} \leq \Pr\left\{ \hat{r}_t(s, a) + B(s, a)\sqrt{\frac{\lambda \ln t}{t}} \leq \mu(s, a) \right\} \]
\[ = \Pr\left\{ \frac{\hat{r}_t(s, a)}{B(s, a)} t \leq \frac{\mu(s, a)}{B(s, a)} t - \sqrt{\lambda \ln t} \right\} \]
\[ \leq e^{-2(\lambda t \ln t)/t} \]
\[ = t^{-2\lambda} , \]

where $(a)$ is obtained using the left-sided Hoeffding’s inequality (see lemma 1) with $\alpha = \sqrt{\lambda t \ln t}$.

Similarly using the right-sided version of the concentration inequality, we get the second inequality.

Appendix B

Analysis of Markov Chain Mixing

We briefly introduce the tools required for the analysis of Markov chain mixing (see [17], chapter 4 for a detailed discussion). The total variation (TV) distance between two probability distributions $\phi$ and $\psi$ on sample space $\Omega$ is defined by

\[ \|\phi - \psi\|_{TV} = \max_{E \subseteq \Omega} |\phi(E) - \psi(E)| . \]
Intuitively, it means the TV distance between \( \phi \) and \( \psi \) is the maximum difference between the probabilities of a single event by the two distributions. The TV distance is related to the \( L_1 \) distance as follows
\[
\|\phi - \psi\|_{TV} = \frac{1}{2} \sum_{\omega \in \Omega} |\phi(\omega) - \psi(\omega)|. \tag{38}
\]

We wish to bound the maximal distance between the stationary distribution \( \pi \) and the distribution over states after \( t \) steps of a Markov chain. Let \( P^{(t)} \) be the \( t \)-step transition matrix with \( P^{(t)}(s, s') \) being the transition probability from state \( s \) to \( s' \) of the Markov chain in \( t \) steps and \( \mathcal{P} \) be the collection of all probability distributions on \( \Omega \). Also let \( P^{(t)}(s, \cdot) \) be the row or distribution corresponding to the initial state of \( s \). Based on these notations, we define a couple of useful \( t \)-step distances as follows:
\[
d(t) := \max_{s \in S} \|\pi - P^{(t)}(s, \cdot)\|_{TV} = \sup_{\phi \in \mathcal{P}} \|\pi - \phi P^{(t)}\|_{TV}, \tag{39}
\]
\[
\hat{d}(t) := \max_{s, s' \in S} \|P^{(t)}(s', \cdot) - P^{(t)}(s, \cdot)\|_{TV} = \sup_{\psi, \phi \in \mathcal{P}} \|\psi P^{(t)} - \phi P^{(t)}\|_{TV}. \tag{40}
\]

For irreducible and aperiodic Markov chains, the distances \( d(t) \) and \( \hat{d}(t) \) have following special properties:

**Lemma 3** ([17], lemma 4.11). For all \( t > 0 \), \( d(t) \leq \hat{d}(t) \leq 2d(t) \).

**Lemma 4** ([17], lemma 4.12). The function \( \hat{d} \) is sub-multiplicative: \( \hat{d}(t_1 + t_2) \leq \hat{d}(t_1)\hat{d}(t_2) \).

These lemmas lead to following useful corollary:

**Corollary 1.** For all \( t \geq 0 \), \( d(t) \leq \hat{d}(1)^t \).

Consider an MDP with optimal stationary policy \( \beta^* \). Since the MDP might not start at the stationary distribution \( \pi^* \) corresponding to the optimal policy, even the optimal policy incurs some regret as defined in equation (9). We characterize this regret in the following theorem.

**Theorem 5** (Regret of Optimal Policy). For an ergodic MDP, the total expected regret of the optimal stationary policy with transition probability matrix \( P_\pi \) is upper bounded by \( (1 - \gamma)^{-1} \mu_{\max} \), where \( \gamma = \max_{s, s' \in S} \|P_\pi(s', \cdot) - P_\pi(s, \cdot)\|_{TV} \) and \( \mu_{\max} = \max_{s \in S, a \in A} \mu(s, a) \).

**Proof:** Let \( \phi_0 \) be the initial distribution over states and \( \phi_t = \phi_0 P_\pi^{(t)} \) be such distribution at time \( t \) represented as a row vectors. Also, let \( \mu^* \) be a row vector with entry corresponding to state \( s \) being \( \mu(s, \beta^*(s)) \). We use \( d^*(t) \) and \( \hat{d}^*(t) \) to denote the \( t \)-step distances from equations (39) and (40) for the optimal policy. Ergodicity of the MDP ensures that the Markov chain corresponding to the optimal policy is irreducible and aperiodic, and thus lemmas 5 and 4 hold. The regret of the optimal policy, therefore, gets simplified as:
\[
\mathcal{R}^* (\phi_0, T) = T \rho^* - \sum_{t=0}^{T-1} \phi_t \cdot \mu^*
= T(\pi^* \cdot \mu^* ) - \sum_{t=0}^{T-1} \phi_t \cdot \mu^* .
\]
\[
\sum_{t=0}^{T-1} (\pi^* - \phi_t) \cdot \mu^* \\
\leq \sum_{t=0}^{T-1} (\pi^* - \phi_t)_+ \cdot \mu^* \\
= \sum_{t=0}^{T-1} \sum_{s \in S} (\pi^*(s) - \phi_t(s))_+ \mu^*(s) \\
\leq \mu_{\text{max}} \sum_{t=0}^{T-1} \sum_{s \in S} (\pi^*(s) - \phi_t(s))_+ \\
= \mu_{\text{max}} \sum_{t=0}^{T-1} \|\pi^* - \phi_0 P^*_t\|_{TV} \\
\leq \mu_{\text{max}} \sum_{t=0}^{T-1} d^*(t) \\
\leq \mu_{\text{max}} \sum_{t=0}^{T-1} \left(\hat{d}^*(1)\right)^t \\
= \mu_{\text{max}} \sum_{t=0}^{T-1} \gamma^t \\
\leq \mu_{\text{max}} \frac{1}{1 - \gamma}.
\]

Note that this regret bound is independent of the initial distribution over the states.