On sufficient conditions to extend Huber’s finite connectivity theorem to higher dimensions∗†‡

Dedicated to Professor K. Shiohama on his eightieth birthday

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Abstract

Let $M$ be a connected complete noncompact $n$-dimensional Riemannian manifold with a base point $p \in M$ whose radial sectional curvature at $p$ is bounded from below by that of a noncompact surface of revolution which admits a finite total curvature where $n \geq 2$. Note here that our radial curvatures can change signs wildly. We then show that $\lim_{t \to \infty} \frac{\text{vol} B_t(p)}{t^n}$ exists where $\text{vol} B_t(p)$ denotes the volume of the open metric ball $B_t(p)$ with center $p$ and radius $t$. Moreover we show that in addition if the limit above is positive, then $M$ has finite topological type and there is therefore a finitely upper bound on the number of ends of $M$.

1 Introduction

It is a great delight to try generalizing classic results of the relationships between a total curvature and topology of a connected complete noncompact 2-dimensional Riemannian manifold $S$ to higher dimensions. The reason for our joy is largely due to a well known theorem of Cohn-Vossen [6] in 1935 which states that if $S$ is finitely connected and admits a total curvature $c(S)$ as an extended real number, then $c(S) \leq 2\pi \chi(S)$ holds where $\chi(S)$ denotes the Euler characteristic of $S$, and hence $c(S)$ is not a topological invariant anymore. Here $S$ is finitely connected if there is a compact 2-dimensional Riemannian manifold $V$ and finite numbers of points $p_1, p_2, \ldots, p_k \in V$ ($k \geq 1$) such that $S$ is homeomorphic to $V \setminus \{p_1, p_2, \ldots, p_k\}$. Besides, Cohn-Vossen showed, applying the theorem above, that if $S$ has nonnegative Gaussian curvature everywhere, then $S$ is either diffeomorphic to a plane or else isometric to a flat cylinder or a flat open Möbius strip.

As one of extensions of Cohn-Vossen’s theorem above to higher dimensions, Cheeger and Gromoll [5] proved the soul theorem in 1972, which states that for every complete noncompact Riemannian manifold $X$ with nonnegative sectional curvature everywhere there is a compact totally geodesic submanifold, called the soul, of $X$ such that $X$ is diffeomorphic to the normal bundle over the soul.

Huber [11] showed in 1957 that if $S$ admits a finite total curvature, which implies $c(S) \in (-\infty, 2\pi]$ by Cohn-Vossen’s theorem above, then $S$ is finitely connected. As a weak version of the soul theorem, Gromov [9] showed, applying the Grove-Shiohama

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theory [10] of critical points of distance functions, in 1981 that the $X$ above has finite topological type, i.e., $X$ is homeomorphic to the interior of a compact manifold with boundary and consequently, we can say that his result is one of extensions of Huber’s theorem rather than a weak version of the soul theorem that he said. What should be noted here is the fact that $X$ must employ the Euclidean space of dimension 2, denoted by $\mathbb{R}^2$, as a reference surface in comparison theorems such as the Toponogov comparison theorem. The total curvature of $\mathbb{R}^2$ is thus 0, but Huber’s theorem also holds for all $S$ whose total curvatures take a finite value in $(-\infty, 2\pi]$ other than 0.

It is a meaningful observation on the geometry of total curvatures that we restrict $c(S)$ from being bounded from below by a constant, because $c(S)$ is not a topological invariant. For example the following theorem of Shiohama is remarkable about the restriction.

**Theorem 1.1** (21) Let $S$ be oriented, finitely connected, and have one end. If $c(S) > (2\chi(S)−1)\pi$, then all Busemann functions on $S$ are exhaustions. In particular, if $c(S) > \pi$, then $S$ is homeomorphic to $\mathbb{R}^2$ and also all Busemann functions on $S$ are exhaustions.

Is it appropriate for us to extend Theorem 1.1 to complete noncompact Riemannian manifolds of higher dimensions which have all curvatures everywhere bounded from below by 0, or $-1$? No, it is not, for total curvatures of reference surfaces, $\mathbb{R}^2$, or the hyperbolic plane $\mathbb{H}^2(-1)$, in comparison theorems for such manifolds are 0 corresponding to $\mathbb{R}^2$, or $-\infty$ corresponding to $\mathbb{H}^2(-1)$, which are less than $\pi$. It is therefore natural that we desire radial curvature geometry to extend Huber’s finite connectivity theorem, Theorem 1.1 and other classic results that we can find in [22] to higher dimensions, because Gaussian curvatures of reference surfaces in the radial curvature geometry can wildly change their signs so that we can restrict total curvatures of the surfaces from being bounded from below by a constant.

We will introduce the radial curvature geometry. Let $\tilde{M}^n$ be a simply connected complete $n$-dimensional Riemannian manifold with a base point $\tilde{p} \in \tilde{M}^n$ where $n \geq 2$, and $d$ the distance function of $\tilde{M}^n$. Set $\ell := \sup_{\tilde{x} \in \tilde{M}^n} \tilde{d}(\tilde{p}, \tilde{x}) \leq \infty$ and $S_{\tilde{p}}^{n−1} := \{v \in T_{\tilde{p}}\tilde{M}^n \mid \|v\| = 1\}$ where $T_{\tilde{p}}\tilde{M}^n$ denotes the tangent space at $\tilde{p} \in \tilde{M}^n$. We then call the pair $(\tilde{M}^n, \tilde{p})$ an $n$-dimensional model surface of revolution if its Riemannian metric $d\tilde{s}^2$ is expressed in terms of geodesic polar coordinates around $\tilde{p}$ as

$$d\tilde{s}^2 = dt^2 + f(t)^2 d\tilde{s}_{\tilde{p}}^{n−1}(\theta)$$

for all $(t, \theta) \in (0, \ell) \times S_{\tilde{p}}^{n−1}$. In Eq. (1.1), $f : (0, \ell) \rightarrow \mathbb{R}$ is the warping function of $\tilde{M}^n$, which is, by definition, a positive smooth function satisfying the Jacobi equation

$$f''(t) + G(\tilde{\gamma}(t))f(t) = 0$$

with initial conditions $f(0) = 0$ and $f'(0) = 1$ where $G$ denotes the sectional curvature of $\tilde{M}^n$ and $\tilde{\gamma}$ denotes any meridian emanating from $\tilde{p} = \tilde{\gamma}(0)$, and $d\tilde{s}_{\tilde{p}}^{n−1}$ is the Riemannian metric on $S_{\tilde{p}}^{n−1}$. The function $\tilde{K} := G \circ \tilde{\gamma} : [0, \ell) \rightarrow \mathbb{R}$ is called the radial curvature function of $\tilde{M}^n$. In the case of $n = 2$ we simply call $(\tilde{M}^2, \tilde{p})$ the surface of revolution, and then Eq. (1.1) is also expressed simply as $d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2 ((t, \theta) \in (0, \ell) \times S_{\tilde{p}}^{1})$.

**Remark 1.2** Katz and the first author [13] classified complete Riemannian manifolds with symmetric radial curvature, and hence $n$-dimensional model surfaces of revolution were classified. Moreover in their energetic study Mao and his collaborators employ such spherically symmetric manifolds to develop geometric analysis and inequalities to a wider class of metrics, see [8], [19], [20], and so on.
We now say that a connected complete $n$-dimensional Riemannian manifold $M$ with a base point $p \in M$ has radial curvature at $p$ bounded from below by that of a surface of revolution $(M^2, \tilde{p})$ if, along every unit speed minimal geodesic $\gamma : [0, a) \to M$ emanating from $p = \gamma(0)$,

$$K_M(\gamma'(t), v) \geq \tilde{K}(t)$$

holds for all $t \in [0, a)$ and all $v \in T_{\gamma(t)}M$ with $v \perp \gamma'(t)$ where $K_M(\gamma'(t), v)$ denotes the sectional curvature of $M$ restricted to the 2-dimensional linear space in $T_{\gamma(t)}M$ spanned by $\gamma'(t)$ and $v$, and $\tilde{K}$ denotes the radial curvature function of $\tilde{M}^2$.

Our purpose of this article is to extend Huber’s finite connectivity theorem to higher dimensions in radial curvature geometry, and our main theorem is stated as follows:

**Theorem 1.3** Let $M$ be a connected complete noncompact $n$-dimensional Riemannian manifold with a base point $p \in M$ whose radial curvature at $p$ is bounded from below by that of a noncompact surface of revolution $(\tilde{M}^2, \tilde{p})$ where $n \geq 2$. Assume $\tilde{M}^2$ admits a finite total curvature. We then have that

1. $\lim_{t \to \infty} \text{vol } B_t(p)/t^n$ exists where $\text{vol } B_t(p)$ denotes the volume of the open metric ball $B_t(p)$ with center $p$ and radius $t$;

2. in addition, if the limit is positive, then $M$ has finite topological type and there is a noncompact surface of revolution $M^\ast$ admitting a finite total curvature such that the number of ends of $M$ is less than or equal to $2\{1 - c(M^\ast)/2\pi\}^{n-1}$.

**Remark 1.4** We here give remarks on Theorem 1.3 and related results to the theorem.

1. As was noted above, our radial curvatures can change signs wildly. For example there is a noncompact surface of revolution admitting a finite total curvature whose radial curvature function $\tilde{K}$ is not bounded, which means $\liminf_{t \to \infty} \tilde{K}(t) = -\infty$, or $\limsup_{t \to \infty} \tilde{K}(t) = \infty$, see [23, Theorem 1.4].

2. The first author and Tanaka [15] extended Huber’s finite connectivity theorem to higher dimensions as follows: Let $M$ be a connected complete noncompact $n$-dimensional Riemannian manifold with a base point $p \in M$ whose radial curvature at $p$ is bounded from below by that of a noncompact surface of revolution $(\tilde{M}^2, \tilde{p})$ where $n \geq 2$. If

(a) $\tilde{M}^2$ admits a finite total curvature, and if

(b) $\tilde{M}^2$ has no pair of cut points in the set $\tilde{V}(\delta_0) := \{ \tilde{x} \in \tilde{M}^2 \mid 0 < \theta(\tilde{x}) < \delta_0 \}$ for some $\delta_0 \in (0, \pi],$

then $M$ has finite topological type.

3. The two assumptions (a) and (b) above can be replaced with one assumption $c(\tilde{M}^2) \in (-\infty, 2\pi)$, see [23, Theorem 1.3] of the first author and Tanaka.

4. Abresch and Gromoll [2] also showed the finiteness of topological type of a complete noncompact $n$-dimensional Riemannian manifold $X$ having not only nonnegative Ricci curvature outside the open distance $t_0$-ball around $p \in X$ for some constant $t_0 > 0$, but also sectional curvature everywhere bounded from below by a negative constant, and moreover admitting diameter growth of small order $o(t^{1/n})$. Note that we find that the diameter growth is too restrictive from the radial curvature geometry point of view, see [15, example 1.1].
Remark 1.6 This article is a part of the master thesis of second author, Yamaguchi University.

In the following sections all geodesics will be normalized unless otherwise stated.

2 Preliminaries

We will here recall two lemmas and a theorem, because we need all of them to estimate an upper bound on the number of ends of manifolds which is one of the statements of Theorem 1.3 (See the proof of Lemma 3.4 in the next section.)

The following lemma and theorem are due to the first author and Tanaka.

Lemma 2.1 ([23] Model Lemma II]) Let \((\tilde{M}^2, \tilde{p})\) be a noncompact surface of revolution. Set \(G^*(t) := \min\{\tilde{K}(t), 0\}\) for each \(t \in [0, \infty)\) where \(\tilde{K}\) denotes the radial curvature function of \(\tilde{M}^2\). If \(M^2\) admits a finite total curvature less than \(2\pi\), then there is a noncompact surface of revolution \((M^*, p^*)\) whose warping function \(m : (0, \infty) \rightarrow \mathbb{R}\) satisfies the Jacobi equation \(m''(t) + G^*(t)m(t) = 0\) with \(m(0) = 0\) and \(m'(0) = 1\) such that \(M^*\) admits a finite total curvature.

Theorem 2.2 ([15] A new type of Toponogov’s comparison theorem]) Let \(M\) be a connected complete noncompact Riemannian manifold whose radial curvature at the base point \(p\) is bounded from below by that of a von Mangoldt surface of revolution \((\tilde{M}^2, \tilde{p})\) with the metric given by Eq. 1.1 for \(n = 2\). If the set \(\tilde{V}(\delta_0) := \{\tilde{x} \in \tilde{M}^2 | 0 < \theta(\tilde{x}) < \delta_0\}\) has no pair of cut points for some \(\delta_0 \in (0, \pi]\), then for any geodesic triangle \(\triangle(px'y)\) in \(M\) with \(\angle(xpy) < \delta_0\), there is a geodesic triangle \(\tilde{\triangle}(px'y) := \triangle((\tilde{p}\tilde{x})\tilde{y})\) in \(\tilde{V}(\delta_0)\) such that

\[
\begin{align*}
\tilde{d}(\tilde{p}, \tilde{x}) &= d(p, x), & \tilde{d}(\tilde{p}, \tilde{y}) &= d(p, y), & \tilde{d}(\tilde{x}, \tilde{y}) &= d(x, y)
\end{align*}
\]

and that

\[
\begin{align*}
\angle(xpy) &\geq \angle(\tilde{x}\tilde{p}\tilde{y}), & \angle(px'y) &\geq \angle(\tilde{p}\tilde{x}\tilde{y}), & \angle(pyx) &\geq \angle(\tilde{p}\tilde{y}\tilde{x})
\end{align*}
\]

where \(d\) denotes the distance function of \(M\), and \(\angle(pxy)\) denotes the angle between the minimal geodesics from \(x\) to \(p\) and \(y\) forming the triangle \(\triangle(pxy)\).
The following lemma (or more precisely a technique) is originally from Cohn-Vossen’s article [17], and this higher dimensional version is a direct consequence of the generalized first variation formula (e.g. [12] Lemma 2.1).

**Lemma 2.3** (Cohn-Vossen’s technique) Let $M$ be a complete noncompact Riemannian manifold, and $\zeta : [0, \infty) \rightarrow M$ a ray emanating from $q := \zeta(0)$. Fix $x \not\in \zeta([0, \infty))$. For any $\theta \in (0, \pi/2)$ there are then a constant $r_0 \in [0, \infty)$ and a minimal geodesic segment $\gamma : [0, t_0) \rightarrow M$ emanating from $x$ to $\zeta(r_0)$, where $t_0 := d(x, \zeta(r_0))$, such that $\angle(\zeta'(r_0), \gamma'(t_0)) < \theta$ where $\angle(\zeta'(r_0), \gamma'(t_0))$ denotes the angle between the tangent vectors $\zeta'(r_0)$ and $\gamma'(t_0)$ at $\zeta(r_0)$.

### 3 Proof of Theorem 1.3

Throughout this section let $M$ be a connected complete noncompact $n$-dimensional Riemannian manifold with a base point $p$ whose radial curvature at $p$ is bounded from below by that of a noncompact surface of revolution $(\tilde{M}^2, \tilde{p})$, and let $d\tilde{s}^2$ denote the Riemannian metric of $\tilde{M}^2$ given by Eq. (1.1) for $n = 2$ with the warping function $f : (0, \infty) \rightarrow \mathbb{R}$ satisfying Eq. (1.2). In these settings we assume that $\tilde{M}^2$ admits a finite total curvature, denoted by $c(\tilde{M}^2)$, which implies $c(\tilde{M}^2)$ exists in $(-\infty, 2\pi]$ by Cohn-Vossen’s theorem.

Let $(\tilde{M}^n, \tilde{o})$ be the noncompact $n$-dimensional model surface of revolution whose warping function is $f$ of $\tilde{M}^2$ where $\tilde{o} \in \tilde{M}^n$ denotes the base point of it. Note that the radial curvature of $M$ at $p$ is also bounded from below by that of $(\tilde{M}^n, \tilde{o})$. Let $B_t(\tilde{o}) \subset \tilde{M}^n$ be the metric open ball with center $\tilde{o}$ and radius $t$, and $\omega_{n-1}$ the volume of $S_{\tilde{o}}^{n-1} = \{ v \in T_{\tilde{o}}\tilde{M}^n \mid \| v \| = 1 \}$.

**Lemma 3.1** $\lim_{t \rightarrow \infty} \text{vol} B_t(p)/t^n$ exists.

**Proof.** Since $\text{vol} B_t(\tilde{o}) = \omega_{n-1} \int_0^t f(r)^{n-1} dr$, we have

\begin{equation}
\lim_{t \rightarrow \infty} \frac{\text{vol} B_t(p)}{t^n} = \omega_{n-1} \cdot \lim_{t \rightarrow \infty} \frac{\text{vol} B_t(\tilde{o})}{\text{vol} B_t(\tilde{o})} \cdot \frac{\int_0^t f(r)^{n-1} dr}{t^n}.
\end{equation}

The Bishop volume comparison theorem [19] and the Bishop–Gromov one in radial curvature geometry [19] show that $\lim_{t \rightarrow \infty} \text{vol} B_t(p)/\text{vol} B_t(\tilde{o})$ exists in $[0, 1]$. Moreover since $f$ is smooth and $\lim_{t \rightarrow \infty} t^{n-1} = \infty$, it follows from l’Hôpital’s theorem (cf. [22] Lemma 5.2.1) and the isoperimetric inequality ([22] Theorem 5.2.1) that

\begin{equation}
\lim_{t \rightarrow \infty} \frac{\int_0^t f(r)^{n-1} dr}{t^n} = \lim_{t \rightarrow \infty} \frac{f(t)^{n-1}}{n \cdot t^{n-1}} = \frac{1}{n \cdot (2\pi)^{n-1}} \lim_{t \rightarrow \infty} \left\{ \frac{2\pi f(t)}{t} \right\}^{n-1} = \frac{1}{n \cdot (2\pi)^{n-1}} \left\{ 2\pi \cdot c(\tilde{M}^2) \right\}^{n-1} = \frac{1}{n} \left\{ 1 - \frac{c(\tilde{M}^2)}{2\pi} \right\}^{n-1}.
\end{equation}

We therefore see, by Eqs. (3.1) and (3.2), that $\lim_{t \rightarrow \infty} \text{vol} B_t(p)/t^n$ exists in $[0, \infty)$, for $c(\tilde{M}^2) \in (-\infty, 2\pi]$.

In addition we now assume that $\lim_{t \rightarrow \infty} \text{vol} B_t(p)/t^n$ is positive.

**Lemma 3.2** $M$ has finite topological type.
Proof. By the additional assumption there are two positive constants \( \alpha_1, \alpha_2 \) \( (\alpha_1 < \alpha_2) \) such that \( \lim_{t \to \infty} \frac{\text{vol} B_t(p)}{t^n} \) exists in \( [\alpha_1, \alpha_2] \). From this, Eqs. (3.1) and (3.2) show

\[
(3.3) \quad \alpha_1 \leq \frac{\omega_{n-1}}{n} \cdot \left(1 - \frac{\text{c}(M^2)}{2\pi}\right)^{n-1} \cdot \lim_{t \to \infty} \frac{\text{vol} B_t(p)}{\text{vol} B_t(\partial)} \leq \alpha_2.
\]

Since \( \alpha_1 > 0, \lim_{t \to \infty} \frac{\text{vol} B_t(p)}{\text{vol} B_t(\partial)} \) is positive, and hence, by Eq. (3.3), we obtain two positive constants \( \beta_1(n), \beta_2(n) \) given by

\[
\beta_i(n) := \frac{n \cdot \alpha_i}{\omega_{n-1} \cdot \lim_{t \to \infty} \frac{\text{vol} B_t(p)}{\text{vol} B_t(\partial)}}
\]

for each \( i = 1, 2 \) such that

\[
(3.4) \quad \beta_1(n) \leq \left(1 - \frac{\text{c}(M^2)}{2\pi}\right)^{n-1} \leq \beta_2(n).
\]

Eq. (3.4) thus implies that \( \text{c}(M^2) \) is the finite total curvature less than \( 2\pi \). It therefore follows from [23, Theorem 1.3] that \( M \) has finite topological type. \( \square \)

From the argument in the proof of Lemma 3.2 we have the following corollary.

**Corollary 3.3** \( \text{c}(M^2) \in (-\infty, 2\pi) \).

It then follows from Lemma 2.2 that there is a noncompact surface of revolution \((M^*, p^*)\) with a metric \( dt^2 + m(t)^2 d\theta^2 \), \( (t, \theta) \in (0, \infty) \times S^1 \), satisfying the Jacobi equation \( m''(t) + G^*(m^2)m(t) = 0 \) with \( m(0) = 0 \) and \( m'(0) = 1 \) such that its total curvature \( \text{c}(M^*) \) is finite. Here \( G^*(t) := \min \{K(t), 0\} \ (t \in [0, \infty)) \) where \( K \) denotes the radial curvature function of \( M^2 \). Note that \( \text{c}(M^*) \in (-\infty, 0] \), for \( G^* \leq 0 \) on \([0, \infty)\).

**Lemma 3.4** The number of ends of \( M \) is less than or equal to \( 2(1 - \text{c}(M^*)/2\pi)^{n-1} \).

Proof. We will first remark that we can employ \((M^*, p^*)\) as a reference surface in Theorem 2.2. Indeed since \( \tilde{K} \geq G^* \) on \([0, \infty)\), the radial curvature of \( M \) at \( p \) is bounded from below by \( G^* \). Moreover since \( M^* \) is a Cartan–Hadamard surface of revolution (i.e., \( G^* \leq 0 \)), the set \( V^*(\pi) := \{x^* \in M^* | 0 < \theta(x^*) < \pi\} \) has no pair of cut points, and hence the sufficient condition of Theorem 2.2 is satisfied.

Our argument below is essentially made along the same line as done in the proof of [14], Theorem C]: Let \( \text{Ends}(M) \) be the set of all ends of \( M \). Since \( M \) is complete noncompact, \( M \) has at least one end, and hence we can assume for our purpose that \( \# \text{Ends}(M) \geq 2 \).

There are then a compact set \( D \) in \( M \) and two sequences \( \{x_i\}_{i \in \mathbb{N}} \) and \( \{y_i\}_{i \in \mathbb{N}} \) of points \( x_i, y_i \in M \) which diverge to infinity \( \bigcup \), respectively, such that \( 0 < \lim_{i \to \infty} d(x_i, y_i) = \infty \), and that for each \( i \in \mathbb{N} \) every curve joining \( x_i \) and \( y_i \) gets caught in \( D \). Let \( \{\xi_i\}_{i \in \mathbb{N}} \) be a sequence of minimal geodesic segments \( \xi_i \) emanating from \( x_i \) to \( y_i \). Since \( \xi_i \cap D \neq \emptyset \) for each \( i \in \mathbb{N}, \{\xi_i\}_{i \in \mathbb{N}} \) converges a line, denoted by \( \xi : (-\infty, \infty) \to M \), by letting \( i \to \infty \) such that

\[
(3.5) \quad \xi_{\mid (-\infty, 0]} \in e_1, \quad \xi_{\mid [0, \infty)} \in e_2
\]

\(^1\)A sequence of points in a complete noncompact Riemannian manifold is said to diverge to infinity if for any compact set in the manifold just finitely many numbers of the sequence are contained in the compact set.
where $e_1, e_2 \in \text{Ends}(M)$ are represented by $\{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}}$, respectively, so $e_1 \neq e_2$.

Since $D$ is compact, there is a nonnegative constant $a_0$ given by $a_0 := \max\{d(p, x) \mid x \in D\}$. Since $\zeta$ passes $D$, we can assume, taking a reparametrization of $\zeta$ if necessary, that $\zeta(0) \in D$. We then have

$$d(p, \zeta(0)) \leq a_0. \tag{3.6}$$

For each $m \in \mathbb{N}$ let $\zeta_m := \zeta|[-m, m]$. Since $M$ is complete, there are two minimal geodesic segment $\gamma_{-m}$ and $\gamma_{+m}$ emanating from $p$ to $\zeta(-m)$ and to $\zeta(m)$, respectively. Let
t
\[
t^\pm_m := d(p, \zeta(\pm m))\].
\nNote that $\gamma_{\pm}(t^\pm_m) = \zeta(\pm m)$ and $\lim_{|m| \to \infty} t^\pm_m = \infty$.

Fix $m \in \mathbb{N}$. Let $\triangle(p \zeta(-m) \zeta(m))$ denote the geodesic triangle in $M$ whose sides are $\zeta_m$ and $\gamma_{\pm}$. By applying Theorem 2.2 to $\triangle(p \zeta(-m) \zeta(m))$, there is a unique geodesic triangle $\triangle(p^* \zeta^*(-m) \zeta^*(m))$ in $M^*$ made up three sides $\zeta^*_m$ and $\gamma^*_{\pm}$ corresponding to $\zeta_m$ and $\gamma_{\pm}$, respectively, such that each side satisfies Eq. (2.1) for $x = \zeta(-m), y = \zeta(m)$, $p = p^*$, $\delta = \delta^*$, and that each angle at each apex satisfies

\[
\angle(\gamma^*_m(0), \gamma^*_m(\pm m)) \geq \angle((\gamma^*_m)'(0), (\gamma^*_m)'(\pm m)), \tag{3.7}
\]

where $\gamma^*_m(t^\pm_m) := (d^*_\pm \gamma^*/dt)(t^\pm_m)$ and $\zeta^*_m(\pm m) := (d \zeta_m^*/dt)(\pm m)$. Eqs. (2.1) and (3.6) (i.e., Aleksandrov’s convexity) imply $d^*(p^*, \zeta^*_m(0)) \leq d(p, \zeta_m(0)) \leq a_0$ where $d^*$ denotes the distance function of $M^*$, and hence for each $m \in \mathbb{N}$, $\zeta^*_m$ intersects the closed metric ball $B_{a_0}(p^*) \subset M^*$ with center $p^*$ and radius $a_0$. There is thus a subsequence $\{m_{l_i}\}_{i \in \mathbb{N}}$ of $\{m_m\}_{m \in \mathbb{N}}$ such that the sequence $\{\zeta^*_m\}_{i \in \mathbb{N}}$ of geodesic segments $\zeta^*_m$ converges to a line $\zeta^*_\infty : (-\infty, \infty) \to M^*$ by letting $i \to \infty$, which passes $B_{a_0}(p^*)$ at the same time we can assume, taking subsequences if necessary, that the sequences $\{\gamma^*_\pm_m\}_{i \in \mathbb{N}}$ and $\{\gamma^*_\pm\}_{i \in \mathbb{N}}$ have the limits $\gamma^*_\pm$ and $\gamma^*_\pm^\infty$, which are rays emanating from $p$ and $p^*$, respectively. Since $\zeta|(-\infty, 0]$ and $\zeta|[-\infty, \infty]$ are cofinal and $\zeta|[0, \infty]$ and $\gamma^*_\pm^\infty$ are too, Eq. (3.5) gives

$$\gamma^- \in e_1, \quad \gamma^+ \in e_2. \tag{3.9}$$

By setting $\delta_p := \angle(\gamma'^-\infty(0), \gamma'^+\infty(0))$ and $\delta_p^* := \angle((\gamma'^-\infty)'(0), (\gamma'^+\infty)'(0))$, Eq. (3.7) for $m = m_i$ by letting $i \to \infty$ gives

$$\delta_p \geq \delta_p^*. \tag{3.10}$$

Moreover Lemma 2.3 shows $\lim_{i \to \infty} \angle(\pm \gamma^*_\pm_m(t^\pm_{m_i}), \zeta^*_m(\pm m_i)) = 0$, and hence Eq. (3.8) for $m = m_i$ gives

$$\angle(\pm \gamma^*_\pm^\infty(0), \zeta^*_m(\pm m_i)) = 0. \tag{3.11}$$

by letting $i \to \infty$. Let $X^*$ be the domain in $M^*$ bounded by $\delta^*_\infty([0, \infty))$ and containing $\zeta^*_\infty$, $Y^*$ the half plane in $X^*$ bounded by $\zeta^*_\infty$ and $Z^* := X^* \setminus Y^*$. Let $c(X^*), c(Y^*), \text{ and } c(Z^*)$ denote total curvatures of $X^*, Y^*$, and $Z^*$, respectively. Applying the Gauss–Bonnet theorem to $\Delta^*_i := \triangle(p^* \zeta^*\pm m_i) \subset M^*$ for each $i \in \mathbb{N}$, we have

$$c(\Delta^*_i) = \angle((\gamma^*_\pm_m)'(0), (\gamma^*_\pm_m)'(0)) + \angle((\gamma^*_\pm_m)'(t^\pm_{m_i}), (\zeta^*_m)'(m_i)) + \angle(-\gamma^*_\pm_m(t^\pm_{m_i}), (\zeta^*_m)'(m_i)) \pi. \tag{3.12}$$

\footnote{We call two curves $\alpha, \beta : [0, \infty) \to M$ cofinal if for any compact set $D' \subset M$ there is a positive number $T$ such that if $t_1, t_2 \geq T$, then $\alpha(t_1)$ and $\beta(t_2)$ are contained in the same connected component of $M \setminus D'$.}
Since \( Z^* = \lim_{i \to \infty} \Delta^*_i \), Eqs. (3.11) and (3.12) show \( c(Z^*) = \delta_{p^*} - \pi \). Since \( c(Y^*) \leq 0 \), for \( G^* \leq 0 \), we have

\[
\frac{\delta_{p^*}}{2\pi} c(M^*) = c(X^*) = c(Y^*) + c(Z^*) \leq \delta_{p^*} - \pi.
\]

Since \( 2\pi - c(M^*) > 0 \), combining Eq. (3.10) and Eq. (3.13), we get

\[
\frac{2\pi^2}{2\pi - c(M^*)} \leq \delta_{p^*} \leq \delta_{p^*}.
\]

For each \( e \in \text{Ends}(M) \) pick \( v_e \in \mathbb{S}^{n-1}_p \) such a way that the ray \( \gamma_{v_e} : [0, \infty) \to M \) with \( \gamma_{v_e}(0) = p \) and \( \gamma'_{v_e}(0) = v_e \) satisfies \( \gamma_{v_e} \in e \) where \( \mathbb{S}^{n-1}_p := \{ v \in T_p M \mid \|v\| = 1 \} \).

Let \( \{ B_\lambda(v_e) \}_{e \in \text{Ends}(M)} \) be a family of the open balls \( B_\lambda(v_e) \subset \mathbb{S}^{n-1}_p \) with centers \( v_e \) and radii \( \lambda := \pi^2/(2\pi - c(M^*)) \) (as the angle distance). By taking the definition of ends into account, Eqs. (3.9) and (3.14) imply that elements of \( \{ B_\lambda(v_e) \}_{e \in \text{Ends}(M)} \) are mutually disjoint. Since \#\text{Ends}(M) = \#\{ B_\lambda(v_e) \}_{e \in \text{Ends}(M)} \), the packing lemma shows that

\[
\#\text{Ends}(M) \leq 2 \left( \frac{\pi}{2\lambda} \right)^{n-1} = 2 \left\{ 1 - \frac{c(M^*)}{2\pi} \right\}^{n-1}
\]

which is the desired assertion in this lemma. \( \square \)

Lemmas 3.1–3.4 complete the proof of Theorem 1.3. \( \square \)

References

[1] U. Abresch, Lower curvature bounds, Toponogov’s theorem, and bounded topology, Ann. Sci. École Norm. Sup. (4) 18 (1985), no. 4, 651–670.

[2] U. Abresch and D. Gromoll, On complete manifolds with nonnegative Ricci curvature, J. Amer. Math. Soc. 3 (1990), No. 2, 355–374.

[3] R.L. Bishop and R.J. Crittenden, Geometry of manifolds, Pure and Applied Mathematics, Vol. XV, Academic Press, New York–London, 1964.

[4] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geom. 6 (1971/72), 119–128.

[5] J. Cheeger and D. Gromoll, On the structure of complete manifolds of nonnegative curvature, Ann. of Math. (2) 96 (1972), 413–443.

[6] S. Cohn-Vossen, Kürzeste Wege und Totalkrümmung auf Flächen, Compositio Math. 2 (1935), 69–133.

[7] S. Cohn-Vossen, Totalkrümmung und geodätische Linien auf einfach zusammenhängenden offenen vollständigen Flächenstücken, Recueil Math. Moscow 43 (1936), 139–163.

[8] P. Freitas, J. Mao, and I. Salavessa, Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds, Calc. Var. Partial Differential Equations 51 (2014), no. 3-4, 701–724.

[9] M. Gromov, Curvature, diameter and Betti numbers, Comment. Math. Helv. 56 (1981), no. 2, 179–195.

[10] K.Grove and K. Shiohama, A generalized sphere theorem, Ann. of Math. (2) 106 (1977), 201–211.
[11] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

[12] J. Itoh and M. Tanaka, *The Lipschitz continuity of the distance function to the cut locus*, Trans. Amer. Math. Soc. **353** (2001), 21–40.

[13] N.N. Katz and K. Kondo, *Generalized space forms*, Trans. Amer. Math. Soc. **354** (2002), 2279–2284.

[14] K. Kondo and S. Ohta, *Topology of complete manifolds with radial curvature bounded from below*, Geom. Funct. Anal. **17** (2007), no. 4, 1237–1247.

[15] K. Kondo and M. Tanaka, *Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below. II*, Trans. Amer. Math. Soc. **362** (2010), 6293–6324.

[16] K. Kondo and M. Tanaka, *Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below. I*, Math. Ann. **351** (2011), 251–266.

[17] K. Kondo and M. Tanaka, *Sufficient conditions for open manifolds to be diffeomorphic to Euclidean spaces*, Differential Geom. Appl. **29** (2011), 597–605.

[18] K. Kondo and M. Tanaka, *Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below. III*, J. Math. Soc. Japan **64** (2012), 185–200.

[19] J. Mao, *Volume comparison theorems for manifolds with radial curvature bounded*, Czechoslovak Math. J. **66** (141) (2016), no. 1, 71–86.

[20] J. Mao, *The Gagliardo-Nirenberg inequalities and manifolds with non-negative weighted Ricci curvature*, Kyushu J. Math. **70** (2016), no. 1, 29–46.

[21] K. Shiohama, *The role of total curvature on complete noncompact Riemannian 2-manifolds*, Illinois J. Math. **28** (1984), 597–620.

[22] K. Shiohama, T. Shioya, and M. Tanaka, *The Geometry of Total Curvature on Complete Open Surfaces*, Cambridge tracts in mathematics **159**, Cambridge University Press, Cambridge, 2003.

[23] M. Tanaka and K. Kondo, *The topology of an open manifold with radial curvature bounded from below by a model surface with finite total curvature and examples of model surfaces*, Nagoya Math. J. **209** (2013), 23–34.

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