PROJECTIVE DIMENSION AND REGULARITY OF EDGE IDEAL OF SOME WEIGHTED ORIENTED GRAPHS

GUANGJUN ZHU∗, LI XU, HONG WANG AND ZHONGMING TANG

Abstract. In this paper we provide some exact formulas for the projective dimension and the regularity of edge ideals associated to vertex weighted rooted forests and oriented cycles. As some consequences, we give some exact formulas for the depth of these ideals.

1. Introduction

A directed graph $D$ consists of a finite set $V(D)$ of vertices, together with a prescribed collection $E(D)$ of ordered pairs of distinct points called edges or arrows. If $\{u, v\} \in E(D)$ is an edge, we write $uv$ for $\{u, v\}$, which is denoted to be the directed edge where the direction is from $u$ to $v$ and $u$ (resp. $v$) is called the starting point (resp. the ending point). An oriented graph is a directed graph having no bidirected edges (i.e. each pair of vertices is joined by a single edge having a unique direction). In other words an oriented graph $D$ is a simple graph $G$ together with an orientation of its edges. We call $G$ the underlying graph of $D$.

An oriented graph having no multiple edges (edges with same starting point and ending point) or loops (edges that connect vertices to themselves) is called a simple oriented graph. An oriented cycle is a cycle graph in which all edges are oriented in clockwise or in counterclockwise. An oriented acyclic graph is a simple directed graph without oriented cycles. An oriented tree or polytree is a oriented acyclic graph formed by orienting the edges of undirected acyclic graphs. A rooted tree is an oriented tree in which all edges are oriented either away from or towards the root. Unless specifically stated, a rooted tree in this article is an oriented tree in which all edges are oriented away from the root. A leaf of a tree is a vertex adjacent to only one other vertex. An oriented forest is a disjoint union of oriented trees. A rooted forest is a disjoint union of rooted trees.

A vertex-weighted oriented graph is a triplet $D = (V(D), E(D), w)$, where $V(D)$ is the vertex set, $E(D)$ is the edge set and $w$ is a weight function $w : V(D) \rightarrow \mathbb{N}^+$, here $\mathbb{N}^+ = \{1, 2, \ldots\}$. Some times for short we denote the vertex set $V(D)$ and edge set $E(D)$ by $V$ and $E$ respectively. The weight of $x_i \in V$ is $w(x_i)$, denoted by $w_i$.

The edge ideal of a vertex-weighted directed graph was first introduced by Gimenez et al [10]. Let $D = (V, E, w)$ be a vertex-weighted oriented graph with the vertex set

2010 Mathematics Subject Classification. 13D02, 13F55, 13C15, 13D99.

Key words and phrases. projective dimension, regularity, weighted rooted forest, weighted oriented cycle, edge ideal.

* Corresponding author.
$V = \{x_1, \ldots, x_n\}$. We consider the polynomial ring $S = k[x_1, \ldots, x_n]$ in $n$ variables over a field $k$. The edge ideal of $D$, denoted by $I(D)$, is the ideal of $S$ given by

$$I(D) = (x_i x_j^{w_{ij}} \mid x_i x_j \in E).$$

If $w_{ij} = 1$ for all $j$, then $I(D)$ is the edge ideal of underlying graph $G$ of $D$. It has been extensively studied in the literature [14, 20, 23, 24, 25]. Especially the study of algebraic invariants corresponding to their minimal free resolutions has become popular (see [1, 2, 3, 4, 6, 8, 12, 16, 18, 26, 27]).

In this article, we focus on algebraic properties corresponding to projective dimension and regularity of the edge ideals of some weighted oriented graphs.

Edge ideals of edge-weighted graphs were introduced and studied by Paulsen and Sather-Wagstaff [21]. In this work we consider edge ideals of vertex-weighted oriented graph (See [10, 22]). In what follows by a weighted oriented graph we shall always mean a vertex-weighted oriented graph. Edge ideals of weighted oriented graphs arose in the theory of Reed-Muller codes as initial ideals of vanishing ideals of projective spaces over finite fields [19].

This paper is organized as follows. In the next section, we recall several definitions and terminology which we need later. In Sections 3 and 4, using the approach of a Betti splitting and polarization, we derive some exact formulas for the projective dimension and regularity of the edges of weighted rooted forests and oriented cycles. The results are as follows:

**Theorem 1.1.** Let $D(V(D), E(D), w)$ be a weighted oriented star graph such that $w(x) \geq 2$ for any vertex $x$. If $E(D)$ is one of the following cases $\{x_1 x_2, x_1 x_3, \ldots, x_1 x_n\}$, $\{x_2 x_1, x_3 x_1, \ldots, x_n x_1\}$ and $\{x_1 x_2, x_2 x_3, \ldots, x_2 x_n\}$. Then

$$pd(I(D)) = |E(D)| - 1, \quad \text{and} \quad \text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |E(D)| + 1.$$ 

**Theorem 1.2.** Let $D = (V(D), E(D), w)$ be a weighted rooted forest such that $w(x) \geq 2$ for any vertex $x$. Then $pd(I(D)) = |E(D)| - 1$.

**Theorem 1.3.** Let $D = (V(D), E(D), w)$ be a weighted rooted forest such that $w(x) \geq 2$ for any vertex $x$. Then $\text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |E(D)| + 1$.

**Theorem 1.4.** Let $D = (V(D), E(D), w)$ be a weighted oriented cycle such that $w(x) \geq 2$ for any vertex $x$. Then

$$pd(I(D)) = |E(D)| - 1 \quad \text{and} \quad \text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |E(D)| + 1.$$ 

For all unexplained terminology and additional information, we refer to [17] (for the theory of digraphs), [13] (for graph theory), and [15] (for the theory of edge ideals of graphs and monomial ideals).
2. Preliminaries

In this section, we gather together the needed definitions and basic facts, which will be used throughout this paper. However, for more details, we refer the reader to [15, 9, 11, 26].

For any homogeneous ideal \( I \) of the polynomial ring \( S = k[x_1, \ldots, x_n] \), there exists a graded minimal finite free resolution

\[
0 \to \bigoplus_j S(-j)^{\beta_{p,j}(M)} \to \bigoplus_j S(-j)^{\beta_{p-1,j}(M)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(M)} \to I \to 0,
\]

where the maps are exact, \( p \leq n \), and \( S(-j) \) is the \( S \)-module obtained by shifting the degrees of \( S \) by \( j \). The number \( \beta_{i,j}(I) \), the \((i,j)\)-th graded Betti number of \( I \), is an invariant of \( I \) that equals the number of minimal generators of degree \( j \) in the \( i \)th syzygy module of \( I \). Of particular interest are the following invariants which measure the size of the minimal graded free resolution of \( I \). The projective dimension of \( I \), denoted \( \text{pd}(I) \), is defined to be

\[
\text{pd}(I) := \max\{i \mid \beta_{i,j}(I) \neq 0\}.
\]

The regularity of \( I \), denoted \( \text{reg}(I) \), is defined by

\[
\text{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.
\]

We now derive some formulas for \( \text{pd}(I) \) and \( \text{reg}(I) \) in some special cases by using some tools developed in [9].

**Definition 2.1.** Let \( I \) be a monomial ideal, and suppose that there exist monomial ideals \( J \) and \( K \) such that \( \mathcal{G}(I) \) is the disjoint union of \( \mathcal{G}(J) \) and \( \mathcal{G}(K) \). Then \( I = J + K \) is a Betti splitting if

\[
\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)
\]

for all \( i, j \geq 0 \), where \( \beta_{i-1,j}(J \cap K) = 0 \) if \( i = 0 \).

This formula was first obtained for the total Betti numbers by Eliahou and Kervaire [7] and extended to the graded case by Fatabbi [8]. In [9], the authors describe some sufficient conditions for an ideal \( I \) to have a Betti splitting. We shall require the following such condition.

**Theorem 2.2.** ([9, Corollary 2.7]). Suppose that \( I = J + K \) where \( \mathcal{G}(J) \) contains all the generators of \( I \) divisible by some variable \( x_i \) and \( \mathcal{G}(K) \) is a nonempty set containing the remaining generators of \( I \). If \( J \) has a linear resolution, then \( I = J + K \) is a Betti splitting.

When \( I \) is a Betti splitting ideal, Definition 2.1 implies the following results:

**Corollary 2.3.** If \( I = J + K \) is a Betti splitting ideal, then

1. \( \text{reg}(I) = \max\{\text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1\} \),
2. \( \text{pd}(I) = \max\{\text{pd}(J), \text{pd}(K), \text{pd}(J \cap K) + 1\} \).

We need the following Lemmas:
**Lemma 2.4.** ([26] Lemma 3.1) Let $S_1 = k[x_1, \ldots, x_m]$ and $S_2 = k[x_{m+1}, \ldots, x_n]$ be two polynomial rings, $I \subseteq S_1$ and $J \subseteq S_2$ be two nonzero homogeneous ideals. Then

1. $\text{pd}(I + J) = \text{pd}(I) + \text{pd}(J) + 1$,
2. $\text{reg}(I + J) = \text{reg}(I) + \text{reg}(J) - 1$,
3. $\text{reg}(IJ) = \text{reg}(I) + \text{reg}(J)$.

Let $\mathcal{G}(I)$ denote the minimal set of generators of a monomial ideal $I \subset S = k[x_1, \ldots, x_n]$ and let $u \in S$ be a monomial, we set $\text{supp}(u) = \{x_i : x_i | u\}$. If $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$, we set $\text{supp}(I) = \bigcup_{i=1}^{m} \text{supp}(u_i)$. The following lemma is well known.

**Lemma 2.5.** Let $I, J = (u)$ be two monomial ideals such that $\text{supp}(u) \cap \text{supp}(I) = \emptyset$, where $u$ is a monomial of degree $m$. Then

1. $\text{reg}(J) = m$,
2. $\text{pd}(uI) = \text{pd}(I)$,
3. $\text{reg}(uI) = \text{reg}(I) + m$.

**Definition 2.6.** Suppose that $u = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in $S$. Then we define the polarization of $u$ to be the squarefree monomial

$$\mathcal{P}(u) = x_{11}x_{12} \cdots x_{1a_1}x_{21} \cdots x_{2a_2} \cdots x_{n1} \cdots x_{na_n}$$

in the polynomial ring $S^\mathcal{P} = k[x_{ij} | 1 \leq i \leq n, 1 \leq j \leq a_i]$. If $I \subset S$ is a monomial ideal with $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$, the polarization of $I$, denoted by $I^\mathcal{P}$, is defined as:

$$I^\mathcal{P} = (\mathcal{P}(u_1), \ldots, \mathcal{P}(u_m)),$$

which is a squarefree monomial ideal in a polynomial ring $S^\mathcal{P}$.

Here is two examples of how polarization works.

**Example 2.7.** Let $I = (x_1^3x_2^3, x_1^4x_3, x_3x_4^2, x_4^2x_5)$ be a monomial ideal, then the polarization of $I$ is the ideal $I^\mathcal{P} = (x_{11}x_{12}x_{13}x_{14}x_{15}, x_{21}x_{22}x_{23}x_{24}x_{31}x_{32}x_{33}x_{34}x_{41}x_{42}x_{43}x_{44}x_{45})$.

**Example 2.8.** Let $I(D) = (x_1x_2^3, x_2x_3, x_3x_4^2, x_4x_5^2)$ be the edge ideal of a weighted rooted tree $D$, then the polarization of $I(D)$ is the ideal $I(D)^\mathcal{P} = (x_{11}x_{12}x_{13}x_{14}x_{15}, x_{21}x_{22}x_{23}, x_{21}x_{31}, x_{31}x_{41}x_{42}, x_{41}x_{51}x_{52}x_{53}x_{54}x_{55})$.

A monomial ideal $I$ and its polarization $I^\mathcal{P}$ share many homological and algebraic properties. Thus, by polarization, many questions concerning monomial ideals can be reduced to squarefree monomial ideals. The following is a very useful property of polarization which play an essential role throughout the paper.

**Lemma 2.9.** ([15] Corollary 1.6.3) Let $I \subset S$ be a monomial ideal and $I^\mathcal{P} \subset S^\mathcal{P}$ its polarization. Then

1. $\beta_{ij}(I) = \beta_{ij}(I^\mathcal{P})$ for all $i$ and $j$,
2. $\text{pd}(I) = \text{pd}(I^\mathcal{P})$ and $\text{reg}(I) = \text{reg}(I^\mathcal{P})$. 

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4
3. Projective dimensions and regularities of edge ideals of rooted forests

In this section, by using the approach of a Betti splitting and polarization, we will provide some formulas for computing the projective dimensions and regularities of edge ideals of some weighted oriented star graphs and rooted forests. As some consequences, we also give some exact formulas for the depths of edge ideals of oriented forests.

**Theorem 3.1.** Let $D(V(D),E(D), w)$ be a weighted oriented star graph such that $w(x) \geq 2$ for any vertex $x$. If $E(D)$ is one of the following cases \{ $x_1 x_2, x_1 x_3, \ldots, x_1 x_n$ \}, \{ $x_2 x_1, x_3 x_1, \ldots, x_n x_1$ \} and \{ $x_1 x_2, x_2 x_3, \ldots, x_2 x_n$ \}. Then

$$pd(I(D)) = |E(D)| - 1, \quad \text{and} \quad \operatorname{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |E(D)| + 1.$$ 

**Proof.** Let us assume that $V(D) = \{ x_1, \ldots, x_n \}$ and $w_i = w(x_i)$ for any $x_i$.

(1) If $E(D) = \{ x_1 x_2, x_1 x_3, \ldots, x_1 x_n \}$, then $I(D) = (x_1 x_2^{w_2}, \ldots, x_1 x_n^{w_n})$. The conclusions follows from Lemmas 2.4 and 2.5.

(2) If $E(D) = \{ x_2 x_1, x_3 x_1, \ldots, x_n x_1 \}$, then by similar arguments as above, the desired follows.

(3) If $E(D) = \{ x_1 x_2, x_2 x_3, \ldots, x_2 x_n \}$, then

$$I(D) = (x_1 x_2^{w_2}, x_2 x_3^{w_3}, \ldots, x_2 x_n^{w_n}) = J + K,$$

where $J = (x_1 x_2^{w_2})$ has a linear resolution, $K = (x_2 x_3^{w_3}, \ldots, x_2 x_n^{w_n}) = x_2 L$ and $J \cap K = JL$ where $L = (x_3^{w_3}, \ldots, x_n^{w_n})$. Thus $pd(K) = pd(L) = n - 3$, $\operatorname{reg}(J) = w_2 + 1$, $\operatorname{reg}(L) = \sum_{i=3}^{n} w_i - (n - 3)$. by Lemmas 2.4 and 2.5. It follows that from Corollary 2.3, Lemma 2.5

$$pd(I(D)) = \max\{pd(J), pd(K), pd(J \cap K) + 1\} = \max\{0, n - 3, n - 3 + 1\} = n - 2 = |E(D)| - 1.$$

and

$$\operatorname{reg}(I(D)) = \max\{\operatorname{reg}(J), \operatorname{reg}(K), \operatorname{reg}(J \cap K) - 1\} = \max\{w_2 + 1, \operatorname{reg}(L) + 1, \operatorname{reg}(J) + \operatorname{reg}(L) - 1\} = \operatorname{reg}(J) + \operatorname{reg}(L) - 1 = (w_2 + 1) + \sum_{i=3}^{n} w_i - (n - 3) - 1 = \sum_{i=2}^{n} w_i - (n - 3) = \sum_{i=1}^{n} w_i - (n - 1) + 1 = \sum_{i=1}^{n} w_i - |E(D)| + 1.$$

$\square$
As a consequence of the above theorem, we have

**Corollary 3.2.** Let $D$ be a weighted oriented star graph as in Theorem ???. Then $\text{depth}(I(D)) = 2$.

**Proof.** By Auslander-Buchsbaum formula (see Theorem 1.3.3 of [5]), it follows that $\text{depth}(I(D)) = (|E(D)| + 1) - \text{pd}(I(D)) = 2$.

Let $D = (V(D), E(D), w)$ be a vertex-weighted oriented graph. For $T \subset V$, we define the induced vertex-weighted subgraph $H = (V(H), E(H), w)$ of $D$ to be the vertex-weighted oriented graph such that $V(H) = T$, $uv \in E(H)$ if and only if $uv \in E(D)$ and for any $u \in V(H)$, its weight in $H$ equals to the weight of $u$ in $D$.

We now state and prove two main theorems of this section.

**Theorem 3.3.** Let $D = (V(D), E(D), w)$ be a weighted rooted forest such that $w(x) \geq 2$ for any vertex $x$. Then $\text{pd}(I(D)) = |E(D)| - 1$.

**Proof.** Let $D_1, \ldots, D_t$ be all connected components of $D$, then for each $D_i$, it is a rooted tree. Lemma 2.4 (1) implies

$$\text{pd}(I(D)) = \text{pd}(\sum_{i=1}^{t} I(D_i)) = \sum_{i=1}^{t} \text{pd}(I(D_i)) + t - 1.$$  

It is enough to prove that $\text{pd}(I(D_i)) = |E(D_i)| - 1$ for $i = 1, \ldots, t$. Hence we may assume that $D$ is a rooted tree with the vertex set $V = \{x_1, \ldots, x_n\}$. We apply induction on $|E(D)|$.

The case $D$ is a weighted oriented star graph follows from Theorem 3.1. Let $x_n$ be a leaf of $D$ which adjacent to $x_{n-1}$ and $w_i = w(x_i)$ for any $i$, then

$$I(D) = (x_{n-1}^{w_n}) + I(D \setminus \{x_n\}),$$

where $D \setminus \{x_n\}$ is the subgraph of $D$ with vertex $x_n$ and edge $x_{n-1}x_n$ removed. Thus $D \setminus \{x_n\}$ is a rooted subtree with $|E(D)| - 1$ edges. By the inductive hypothesis, we have $\text{pd}(I(D \setminus \{x_n\})) = |E(D \setminus \{x_n\})| - 1 = |E(D)| - 2$.

Let $I(D)^P$ be the polarization of $I(D)$, then

$$I(D)^P = (x_{n-1,1}^{w_n} \prod_{j=1}^{w_n} x_{nj}) + I(D \setminus \{x_n\})^P.$$

Set $J = (x_{n-1,1}^{w_m} \prod_{j=1}^{w_m} x_{nj})$ and $K = I(D \setminus \{x_n\})^P$, then $J$ has a linear resolution, hence $I(D)^P = J + K$ is a Betti splitting by Theorem 2.2 and

$$J \cap K = (x_{n-1,1}^{w_m+1} \prod_{j=1}^{w_m+1} x_{nj})(\prod_{j=1}^{w_m} x_{m+1,j}, \ldots, x_{m,1}^{w_m} \prod_{j=2}^{w_m} x_{j-1,j}) + I(D \setminus \{x_{n-1}, x_n\})^P)$$

$$= J((\prod_{j=1}^{w_m} x_{m+1,j}, \ldots, x_{m-2,j}^{w_m} x_{m+1,j} \prod_{j=2}^{w_m} x_{j-1,j}) + I(D \setminus \{x_{n-1}, x_n\})^P),$$

□
where $x_m x_{n-1}, x_{n-1} x_{m+1}, \ldots, x_{n-1} x_{n-2} \in E(D \setminus \{x_n\})$ are all edges adjacent to $x_{n-1}$ in $D \setminus \{x_n\}$ and $D \setminus \{x_{n-1}, x_n\}$ is a rooted forest with the vertices $D \setminus \{x_{n-1}, x_n\}$ and edges incident to $x_{n-1}$ or $x_n$ removed.

Let $L = (\prod_{j=1}^{w_{n+1}} x_{m+1,j}, \ldots, \prod_{j=1}^{w_{n-2}} x_{n-2,j}, x_{m1} \prod_{j=2}^{w_{n-1}} x_{n-1,j}) + I(D \setminus \{x_{n-1}, x_n\})^P$, then $L$ is the polarization of the edge ideal of a rooted forest $H$ having $n - m - 1$ connected components, each component is added another leaf with weight $w_i - 1$ to its root $x_i$ for $i = m + 1, \ldots, n - 2$. This implies $|E(H)| = |E(D)| - 1$. It follows that

$$\text{pd} (L) = |E(H)| - 1 = |E(D)| - 2$$

by the inductive hypothesis.

Again using Lemma 2.9 (2), Corollary 2.3 (2) and Lemma 2.5 (2), we obtain

$$\text{pd} (I(D)) = \text{pd} (I(D)^P) = \max \{\text{pd} (J), \text{pd} (K), \text{pd} (J \cap K) + 1\}$$

$$= \max \{0, \text{pd} (I(D \setminus \{x_n\})^P), \text{pd} (L) + 1\}$$

$$= \max \{0, \text{pd} (I(D \setminus \{x_n\})), \text{pd} (L) + 1\}$$

$$= \max \{0, |E(D)| - 2, |E(D)| - 1\}$$

$$= |E(D)| - 1.$$

The proof is complete. \qed

As a consequence of the above theorem, we have

**Corollary 3.4.** Let $D$ be a weighted rooted forest as in Theorem 3.3. Then

$$\text{depth} (I(D)) = 2.$$

**Proof.** By Auslander-Buchsbaum formula (see Theorem 1.3.3 of [5]), it follows that

$$\text{depth} (I(D)) = (|E(D)| + 1) - \text{pd} (I(D)) = 2.$$

\qed

**Theorem 3.5.** Let $D = (V(D), E(D), w)$ be a weighted rooted forest such that $w(x) \geq 2$ for any vertex $x$. Then $\text{reg} (I(D)) = \sum_{x \in V(D)} w(x) - |E(D)| + 1$.

**Proof.** Let $D_1, \ldots, D_t$ be all connected components of $D$, then for each $D_i$, it is a rooted tree. By Lemma 2.4 (2), we have

$$\text{reg} (I(D)) = \text{reg} \left( \sum_{i=1}^{t} I(D_i) \right) = \sum_{i=1}^{t} \text{reg} (I(D_i)) - (t - 1).$$

It is enough to prove that $\text{reg} (I(D_i)) = \sum_{x \in V(D_i)} w(x) - |E(D_i)| + 1$ for $i = 1, \ldots, t$.

Hence we may assume that $D$ is a rooted tree with the vertex set $V = \{x_1, \ldots, x_n\}$. We apply induction on $|E(D)|$.

The case $D$ is a weighted oriented star graph follows from Theorem 3.1. Let $x_n$ be a leaf of $D$ which adjacent to $x_{n-1}$, then by the proof of Theorem 3.3 we have

$$I(D) = (x_{n-1} x_n) + I(D \setminus \{x_n\}), \quad I(D)^P = (x_{n-1,1} \prod_{j=1}^{w_n} x_{nj}) + I(D \setminus \{x_n\})^P$$
and \( I(D)^P \) is a Betti splitting ideal with splitting \( I(D)^P = J + K \), where \( J = (x_{n-1,1} \prod_{j=1}^{w_n} x_{n,j}) \), \( K = I(D \setminus \{x_n\})^P \), \( J \cap K = JL \) and

\[
L = (\prod_{j=1}^{w_{m+1}} x_{m+1,j}, \ldots, \prod_{j=1}^{w_{n-2}} x_{n-2,j}, x_{m1} \prod_{j=2}^{w_{n-1}} x_{n-1,j}) + I(D \setminus \{x_{n-1}, x_n\})^P,
\]

where \( D \setminus \{x_n\} \) is the subgraph of \( D \) with vertex \( x_n \) and edge \( x_{n-1}x_n \) removed. \( x_m, x_{n-1}, x_{n-1}x_{m+1}, \ldots, x_{n-1}x_{n-2} \in E(D \setminus \{x_n\}) \) are all edges adjacent to \( x_{n-1} \) in \( D \setminus \{x_n\} \), \( D \setminus \{x_{n-1}, x_n\} \) is a rooted forest with the vertices \( D \setminus \{x_{n-1}, x_n\} \) and edges incident to \( x_{n-1} \) or \( x_n \) removed, \( L \) is the polarization of the edge ideal of a rooted forest \( H \) having \((n-m-1)\) connected components, each component is added another leaf with weight \( w_i \) to its root \( x_i \) for \( i = m+1, \ldots, n-2 \). Thus \( D \setminus \{x_n\} \) is a rooted tree with \(|E(D)|-1\) edges. By the inductive hypothesis, we have

\[
\text{reg} (I(D \setminus \{x_n\})) = \sum_{i=1}^{n-1} w_i - |E(D \setminus \{x_n\})| + 1
\]

and

\[
\text{reg} (L) = \sum_{x \in V(H)} w(x) - |E(H)| = \sum_{x \in V(H)} w(x) - (|E(D)| - 1)
\]

\[
= \sum_{x \in V(H)} w(x) - |E(D)| + 1.
\]

Lemma 2.3 implies

\[
\text{reg} (J \cap K) = \text{reg} (J) + \text{reg} (K)
\]

\[
= (w_n + 1) + \sum_{x \in V(H)} w(x) - |E(D)| + 1
\]

\[
= \sum_{i=1}^{n} w_i - |E(D)| + 2.
\]

It follows that

\[
\text{reg} (I(D)) = \text{reg} (I(D)^P) = \max \{\text{reg} (J), \text{reg} (K), \text{reg} (J \cap K) - 1\}
\]

\[
= \max \{w_n + 1, \sum_{i=1}^{n-1} w_i - |E(D)| + 2, \sum_{i=1}^{n} w_i - |E(D)| + 1\}
\]

\[
= \sum_{i=1}^{n} w_i - |E(D)| + 1.
\]
The proof is complete. □

4. Projective dimensions and regularities of edge ideals of oriented cycles

In this section, we will provide some formulas for the projective dimensions and regularities of edge ideals of some oriented cycles. As some consequences, we also give some exact formulas for the depth of edge ideals of oriented cycles.

**Theorem 4.1.** Let \( D = (V(D), E(D), w) \) be a weighted oriented cycle such that \( w(x) \geq 2 \) for any vertex \( x \). Then

\[
\text{pd}(I(D)) = |E(D)| - 1 \quad \text{and} \quad \text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |E(D)| + 1.
\]

**Proof.** Let \( V = \{x_1, \ldots, x_n\} \) and \( w_i = w(x_i) \), then

\[
I(D) = (x_1x_2^{w_2}, \ldots, x_{n-1}x_n^{w_n}, x_nx_1^{w_1}),
\]

\[
I(D)^P = (x_1^{w_1}x_2^{w_2}, \ldots, x_{n-1}x_n^{w_n}, x_nx_1^{w_1}x_1^{w_1}).
\]

Set \( L_1 = I(D)^P, J_1 = (x_n^{w_1}x_1^{w_1}), K_1 = (x_1^{w_1}x_2^{w_2}, \ldots, x_n^{w_n}x_1^{w_1}), J_i = (\prod_{j=2}^{i} x_{ij}), K_i = (\prod_{j=1}^{i} x_{ij}), L_i = (\prod_{j=1}^{i} x_{ij}, x_{i+1}^{w_{i+1}}x_{i+1}^{w_{i+1}}x_{i+1}^{w_{i+1}}, \ldots, x_{n-1}^{w_{n-1}}x_{n}^{w_{n-1}}x_{n}^{w_{n-1}}x_{n}^{w_{n-1}})
\]

for any \( 2 \leq i \leq n-1 \) and \( L_n = (\prod_{j=2}^{n} x_{nj}) \). Notice that all \( J_i \) have linear resolutions for \( 1 \leq i \leq n-1 \), it follows that \( L_i = J_i + K_i \) is a Betti splitting. Also notice that \( J_i \cap K_i = J_iL_{i+1} \) and the fact the variables that appear in \( J_i \) and \( L_{i+1} \) are different, by Lemma 2.5 Corollary 2.3 we obtain, for \( 1 \leq i \leq n-1 \),

\[
\begin{align*}
\text{pd}(J_i \cap K_i) &= \text{pd}(L_{i+1}) = \max\{\text{pd}(J_{i+1}), \text{pd}(K_{i+1}), \text{pd}(J_i \cap K_{i+1}) + 1\}, \\
\text{reg}(J_i \cap K_i) &= \text{reg}(J_iL_{i+1}) = \text{reg}(J_i) + \text{reg}(L_{i+1}), \\
\text{reg}(L_{i+1}) &= \max\{\text{reg}(J_{i+1}), \text{reg}(K_{i+1}), \text{reg}(J_i \cap K_{i+1}) - 1\}.
\end{align*}
\]

(1)

Since \( J_i, K_{n-1} \) and \( J_{n-1} \cap K_{n-1} \) are principal ideals, \( \text{pd}(J_i) = \text{pd}(J_{n-1} \cap K_{n-1}) = \text{pd}(K_{n-1}) = 0 \) for \( 1 \leq i \leq n-1 \). By repeated use of the above equalities (1) and induction on \( n \) and \( i \), we can obtain that \( \text{pd}(K_i) = n - i - 1, \text{pd}(L_i) = n - i, \text{reg}(K_i) = \text{reg}(L_{i+1}) + 1, \text{reg}(L_i) = \sum_{j=1}^{n} w_j - (n - i + 1) \). It follows that \( \text{pd}(K_1) = n - 2, \text{pd}(J_1 \cap K_1) = \text{pd}(L_2) = n - 2 \) and \( \text{reg}(K_1) = \sum_{j=2}^{n} w_j - (n - 1) \) and \( \text{reg}(J_1 \cap K_1) = \sum_{j=1}^{n} w_j - (n - 2) \). Thus,

\[
\begin{align*}
\text{pd}(L_1) &= \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} \\
&= \max\{0, n - 2, n - 2 + 1\} = n - 1,
\end{align*}
\]
\[ \text{reg} \left( L_1 \right) = \max \{ \text{reg} \left( J_1 \right), \text{reg} \left( K_1 \right), \text{reg} \left( J_1 \cap K_1 \right) - 1 \} \]
\[ = \max \{ w_1 + 1, \sum_{j=2}^{n} w_j - (n - 1), \sum_{j=1}^{n} w_j - (n - 2) - 1 \} \]
\[ = \sum_{j=1}^{n} w_j - (n - 1). \]

This concludes the proof of the theorem. \( \square \)

An immediate consequence of the above theorem is the following corollary.

**Corollary 4.2.** Let \( D = (V(D), E(D), w) \) be a weighted oriented cycle such that \( w(x) \geq 2 \) for any vertex \( x \). Then \( \text{depth}(I(D)) = 1 \).

**Proof.** By Auslander-Buchsbaum formula (see Theorem 1.3.3 of [5]), it follows that
\[ \text{depth}(I(D)) = |E(D)| - \text{pd}(I(D)) = 1. \]
\( \square \)

**Acknowledgments**

This research is supported by the National Natural Science Foundation of China (No.11271275 and No.11471234) and by foundation of the Priority Academic Program Development of Jiangsu Higher Education Institutions.

**References**

[1] A. Alilooee and S. Faridi, On the resolution of path ideals of cycles, Comm. Algebra, 43 (2015), 5413-5433.
[2] A. Alilooee and S. Faridi, Graded Betti numbers of path ideals of cycles and lines, J. Algebra Appl., 17 (2017), 1850011-1-17.
[3] S. Beyarslan, T. Huy, T. Trung and N. Nam, Regularity of powers of forests and cycles, J. Algebraic Combin., 42(4) (2014), 1-19.
[4] R. R. Bouchat, H. T. Hà, and A. O’Keefe. Path ideals of rooted trees and their graded Betti numbers, J. Comb. Theory, Ser. A, 118 (8) (2011), 2411-2425.
[5] W. Bruns and J. Herzog, Cohen-Macaulay rings, Revised Edition, Cambridge University Press, 1998.
[6] H. Dao, C. Huneke and J. Schweig, Bounds on the regularity and projective dimension of ideals associated to graphs, J. Algebraic Combin., 38(1) (2013), 37-55.
[7] S. Eliahou and M. Kervaire, Minimal resolutions of some monomial ideals, J. Algebra, 129 (1990), 1-25.
[8] G. Fatabbi, On the resolution of ideals of fat points, J. Algebra, 242 (2001), 92-108.
[9] C. A. Francisco, H. T. Hà and A. Van Tuyl, Splittings of monomial ideals, Proc. Amer. Math. Soc., 137 (10) (2009), 3271-3282.
[10] P. Gimenez, J. M. Bernal, A. Simis, R. H. Villarreal, and C. E. Vivares, Monomial ideals and Cohen-Macaulay vertex-weighted digraphs, arXiv: 1706.00126v3.
[11] H. T. Hà and A. Van Tuyl, Splittable ideals and the resolutions of monomial ideals, J. Algebra, 309 (1) (2007), 405-425.
[12] H. T. Hà and A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, J. Algebraic Combin., 27 (2) (2008), 215-245.
[13] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1972.
[14] Jing He and A. Van Tuyl, Algebraic properties of the path ideal of a tree, *Comm. Algebra*, 38 (5) (2010), 1725-1742.
[15] J. Herzog and T. Hibi *Monomial Ideals*, New York, NY, USA: Springer-Verlag, 2011.
[16] S. Jacques, Betti numbers of graph ideals, PhD dissertation, University of Sheffield, 2004.
[17] J. B. Jensen and G. Gutin, *Digraphs. Theory, Algorithms and Applications*, Springer Monographs in Mathematics, Springer, 2006.
[18] D. Kiani and S. S. Madani, Betti numbers of path ideals of trees, *Comm. Algebra*, 44 (12) (2016), 5376-5394.
[19] J. Martínez-Bernal, Y. Pitones and R. H. Villarreal, Minimum distance functions of graded ideals and Reed-Muller-type codes, *J. Pure Appl. Algebra*, 221 (2017), 251C275.
[20] S. Morey and R. H. Villarreal, *Edge ideals: algebraic and combinatorial properties*, in Progress in Commutative Algebra, Combinatorics and Homology, Vol. 1 (C. Francisco, L. C. Klingler, S. Sather-Wagstaff, and J. C. Vassilev, Eds.), De Gruyter, Berlin, 2012, 85-126.
[21] C. Paulsen and S. Sather-Wagstaff, Edge ideals of weighted graphs, *J. Algebra Appl.*, 12 (5) (2013), 1250223-1-24.
[22] Y. Pitones, E. Reyes, and J. Toledo, Monomial ideals of weighted oriented graphs, arXiv:1710.03785.
[23] A. Simis, W. V. Vasconcelos and R. H. Villarreal, On the ideal theory of graphs, *J. Algebra*, 167 (1994), 389-416.
[24] A. Van Tuyl, *A Beginner’s Guide to Edge and Cover Ideals*, in Monomial Ideals, Computations and Applications, Lecture Notes in Mathematics 2083, Springer, 2013, 63-94.
[25] R. H. Villarreal, Cohen-Macaulay graphs, *Manuscripta Math.*, 66 (1990), 277-293.
[26] Guangjun Zhu, Projective dimension and regularity of the path ideal of the line graph, *J. Algebra Appl.*, 17 (4), (2018), 1850068-1-15.
[27] Guangjun Zhu, Projective dimension and regularity of path ideals of cycles, *J. Algebra Appl.*, 17 (10), (2018), 1850188-1-22.

Authors address: School of Mathematical Sciences, Soochow University, Suzhou 215006, P.R. China
E-mail address: zhuguangjun@suda.edu.cn(Guangjun Zhu), 1240470845@qq.com(Li Xu), 651634806@qq.com(Hong Wang), zmtang@suda.edu.cn(Zhongming Tang).