Decision Making via AHP

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Abstract—The Analytic Hierarchy Process (AHP) is a procedure for establishing priorities in multi-criteria decision making problems. Here we discuss the Logarithmic Least Squares (LLS) method for the AHP and group-AHP, which provides an exact and unique solution for the priority vector. Also, we show that for the group-AHP, the LLS method is equivalent with the minimization of the weighted sum of generalized Kullback-Leibler divergences, between the group-priority vector and the priority vector of each expert.

I. INTRODUCTION

The Analytic Hierarchy Process (AHP) is a popular multi-criteria decision making method, with important psychometric, economic, industrial and military applications [1], [2]. The main role of the AHP is to provide solutions to decision problems, where several alternatives for obtaining given objectives are compared under different criteria. The AHP represents the objectives, the alternatives and the criteria of the problem in a hierarchical structure. The decision maker (expert) produces a series of pairwise comparison judgments of the relative strength of the alternatives at the same level of the hierarchy. These judgments are then converted into numbers using a ratio scale, and organized in judgment matrices, which are then used to establish the decision weights, or the priorities for alternatives. The most familiar method to estimate the priorities from a judgment matrix is the Saaty’s Eigenvector (SE) method [1], [2], which is based on the principal eigenvector of the judgment matrix. However, the SE method has been criticized both from prioritization and consistency points of view, and several new techniques based on minimization methods have been proposed, such as: Least Squares (LS) [3], Logarithmic Least Squares (LLS) [4], [5], [6], Weighted Least Squares (WLS) [7], Logarithmic Least Absolute Values (LLAV) [8], and Singular Value Decomposition (SVD) [9]. With the exception of the LLS method, the other minimization methods are difficult to apply, and can even result in several minima which makes the choice ambiguous [9]. Therefore, the most important alternative to the SE method is the LLS approach, which provides an unique solution, by minimizing a logarithmic objective function, subject to a multiplicative constraint between the components of the priority vector. Here, we discuss the LLS method for the AHP and group-AHP, which provides an exact and unique solution for the priority vector. Also, we show that for the group-AHP, the LLS approach is equivalent with the minimization of the weighted sum of generalized Kullback-Leibler divergences, between the group-priority vector and the priority vector of each expert.

II. AHP

Let us briefly define the AHP and its essential elements [1], [2]. We assume that there are $N$ alternatives $A_n$, $n = 1, 2, \ldots, N$, and an expert provides his opinions on each pair of them ($A_i, A_j$), expressing the strength $a_{ij}$ of one factor $A_i$ over the second one $A_j$, using a numerical ratio scale. The comparison scale ranges from $a_{ij} = 1/9$ for $A_i$ least preferred than $A_j$, to $a_{ij} = 1$ for $A_i$ equally preferred to $A_j$, and to $a_{ij} = 9$ for $A_i$ extremely preferred to $A_j$, covering the entire spectrum of the comparison. Thus, the expert creates a $N \times N$ reciprocal judgment matrix $A = [a_{ij}]_{N \times N}$, where:

$$a_{ij} = \frac{1}{a_{ji}}, \quad a_{ij} > 0, \quad i, j = 1, 2, \ldots, N,$$ (1)

such that the elements of the main diagonal are all equal to 1, and the symmetrical elements are mutually reciprocal. Therefore, only $N(N - 1)/2$ judgments are required to construct the matrix. The goal of the AHP is to obtain a vector of priorities $w = [w_1, w_2, \ldots, w_N]^T$, with the normalized components:

$$\sum_{i=1}^{N} w_i = 1, \quad w_i > 0, \quad i = 1, 2, \ldots, N,$$ (2)

such that the $N \times N$ matrix of ratios $W = [w_i/w_j]_{N \times N}$, approximates the judgment matrix $A$, i.e. $W \simeq A$.

A judgment matrix $A = [a_{ij}]_{N \times N}$, satisfying:

$$a_{ij} = \frac{u_i}{u_j}, \quad u_i > 0, \quad i, j = 1, 2, \ldots, N,$$ (3)

is said to be consistent. It follows immediately that a consistent judgment matrix also satisfies the transitivity relation:

$$a_{ik} = a_{ij}a_{jk}, \quad i, j, k = 1, 2, \ldots, N.$$ (4)

It is also easy to see that every consistent matrix is also a judgment matrix. Also, if $A$ is consistent, then every element of $A$ can be determined from the first row of $A$, since:

$$a_{ik} = \frac{a_{1k}}{a_{1i}}, \quad i, k = 1, 2, \ldots, N,$$ (5)

and therefore $A$ is a rank one matrix with exactly one non-zero eigenvalue. Moreover, if $A$ is consistent, we have:

$$Au = Nu,$$ (6)

and the single non-zero value of $A$ is $\lambda = N$.

The normalized eigenvector $w = [w_1, w_2, \ldots, w_N]^T$, with the components:

$$w_i = \frac{u_i}{\sum_{j=1}^{N} w_j}, \quad j = 1, 2, \ldots, N,$$ (7)

is the vector of priorities derived from $A$.

In general, the judgments $a_{ij}$ are rarely perfect, and the transitivity relation is therefore frequently violated. In this case, the judgment matrix is said to be inconsistent. The degree of inconsistency varies from subjective or objective...
reasons, and in general rises with the size of the matrix. In this case, there is not an unique way of deriving the priority vector \( w \), and various methods may produce different results. If the judgment matrix \( A \) is inconsistent then, using the Perron-Frobenius theorem for positive matrices, one can show that the largest eigenvalue is \( \lambda > N \), and the normalized difference \( \mu = (\lambda - N)/(N - 1) \) has been proposed as a measure of the inconsistency \cite{1, 2}. If the inconsistency \( \mu \) is larger than a given threshold, the AHP recommends to correct the matrix, until near consistency is reached \cite{1, 2}. Thus, an inconsistent judgment matrix can be seen as a perturbation of a consistent one. When the perturbations are small, the maximal eigenvalue is close to \( N \), and the corresponding principal eigenvector is close to the eigenvector of the unperturbed consistent matrix. A good estimate of the principal eigenvector can be obtained using the SE method, i.e. raising the matrix at large powers, normalizing the row sums each time, and stopping the procedure when the difference between normalized sums in two consecutive calculations is smaller than a prescribed value. It has been shown that for small deviations around the consistent ratios \( u_{ij}/u_{ji} \), the SE method gives reasonably good approximation of the priority vector. However, when the inconsistencies are large, it is generally accepted that the SE solution is not satisfactory, and other methods like the LLS are more appropriate.

III. LLS METHOD

The LLS method assumes that the elements of the inconsistent judgment matrix \( A \) are approximated by \cite{4, 5, 6}:

\[
a_{ij} = \frac{u_i}{u_j}e_{ij}, \quad u_i > 0, \quad e_{ii} = 1, \quad e_{ij} = 1/e_{ji}, \quad e_{ij} > 0,
\]

where \( e_{ij} \) are the perturbations from consistency. A small perturbation means that \( e_{ij} \approx 1 \). Thus, the approximation matrix is obviously reciprocally symmetric, and has unit diagonal terms, however it may violate the consistency condition \( a_{ik} = a_{ij}a_{jk} \), due to the included perturbation factor \( E = [e_{ij}]_{N \times N} \). The LLS approach is based on the minimization of the sum of squares:

\[
S = \sum_{i=1}^{N} \sum_{j=1}^{N} \delta_{ij}^2,
\]

where \( \delta_{ij} \) are the errors of the log approximation equations:

\[
\delta_{ij} = \ln e_{ij} = \ln a_{ij} - \ln u_i + \ln u_j, \quad i, j = 1, 2, \ldots, N.
\]

It has been shown \cite{4, 5, 6} that when \( \delta_{ij} \) are independent and normally distributed, with zero mean and common variance \( \sigma^2 \), the solution \( u = [u_1, u_2, \ldots, u_N]^T \) for the problem:

\[
\min_{u_1, u_2, \ldots, u_N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \ln a_{ij} - \ln u_i + \ln u_j \right]^2,
\]

subject to the constraint:

\[
\prod_{i=1}^{N} u_i = 1,
\]

is unique and it is given by the geometric means of the rows of the matrix \( A \):

\[
u_i = \prod_{j=1}^{N} a_{ij}^{1/N}, \quad i = 1, 2, \ldots, N.
\]

Thus, the vector of priorities \( w \) is obtained by normalizing the components of \( u \), using (7), such that \( \sum_{i=1}^{N} w_i = 1 \). Also, it has been shown that the LLS approach leads to an unbiased estimation of \( \sigma^2 \) (the variance of perturbations) as a measure of consistency \cite{4}:

\[
\sigma^2 = \frac{1}{(N-1)(N-2)} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \ln a_{ij} - \ln u_i + \ln u_j \right]^2.
\]

One obtains an exact and unique solution of this problem, by directly solving the minimization problem:

\[
\min_{u_1, u_2, \ldots, u_N} S(u_1, u_2, \ldots, u_N),
\]

where

\[
S(u_1, u_2, \ldots, u_N) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \ln a_{ij} - \ln u_i + \ln u_j \right]^2.
\]

The error function \( S \) can be rewritten as:

\[
S(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} \sum_{j=1}^{N} [b_{ij} - x_i + x_j]^2,
\]

where \( b_{ij} = \ln a_{ij} \) and \( x_i = \ln u_i \).

Obviously \( S(x_1, x_2, \ldots, x_N) \) is convex and has an unique minimum at \( (x_1, x_2, \ldots, x_N) \) where:

\[
\frac{\partial}{\partial x_k} S(x_1, x_2, \ldots, x_N) = 0, \quad k = 1, 2, \ldots, N,
\]

which is equivalent with the system of linear equations:

\[
x_k - \frac{1}{N} \sum_{i=1}^{N} x_i = \langle B \rangle_k, \quad k = 1, 2, \ldots, N,
\]

where

\[
\langle B \rangle_k = \frac{1}{N} \sum_{j=1}^{N} b_{kj},
\]

is the average of the row \( k \) of the matrix \( B = [b_{ij}]_{N \times N} = [\ln a_{ij}]_{N \times N} \).

This system can be written in a matrix form as following:

\[
Qx = d,
\]

where \( Q = [g_{ij}]_{N \times N} \) and \( d = [d_1, d_2, \ldots, d_N]^T \) are given by:

\[
g_{ij} = \begin{cases} 
1 - \frac{1}{N} & \text{if } i = j \\
\frac{1}{N-1} & \text{if } i \neq j 
\end{cases}, \quad i, j = 1, 2, \ldots, N
\]

and respectively:

\[
d_i = \langle B \rangle_i, \quad i = 1, 2, \ldots, N.
\]

The minimum norm solution of the linear system \( Qx = d \) is given by:

\[
x_k = \langle B \rangle_k - \frac{1}{N} \sum_{i=1}^{N} \langle B \rangle_i, \quad k = 1, 2, \ldots, N.
\]
We also have:

\[
\frac{1}{N} \sum_{i=1}^{N} (B)_{ij} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \ln a_{ij} = 0, \tag{25}
\]

since:

\[
a_{ij}a_{ji} = 1 \iff \ln a_{ij} + \ln a_{ji} = 0.
\tag{26}
\]

Therefore, the solution of the linear system is:

\[
x_k = (B)_{ik}, \quad k = 1, 2, \ldots, N.
\tag{27}
\]

From here we obtain:

\[
u_k = \exp((B)_{ik}) = \exp \left( \frac{1}{N} \sum_{j=1}^{N} \ln a_{kj} \right), \quad k = 1, 2, \ldots, N.
\tag{28}
\]

Also, we observe that:

\[
\exp \left( \frac{1}{N} \sum_{j=1}^{N} \ln a_{kj} \right) = \exp \left( \ln \left( \prod_{j=1}^{N} a_{kj} \right)^{1/N} \right) = \prod_{j=1}^{N} a_{kj}^{1/N}, \tag{29}
\]

The above analysis represents the proof of the following theorem and corollary:

**Theorem:** Let \( A = [a_{ij}] \) be a \( N \times N \) judgement matrix, and \( B = [b_{ij}] \) a consistent \( N \times N \) matrix, with \( b_{ij} = u_i / u_j \). Then, the best consistent approximation of \( A \) is obtained for the vector \( u = [u_1, \ldots, u_N]^T \), with the components equal to the geometric mean of the corresponding rows of \( A \), i.e.:

\[
u_i = \prod_{j=1}^{N} a_{ij}^{1/N}.
\tag{30}
\]

**Corollary:** If \( A = [a_{ij}] \) is a \( N \times N \) judgement matrix, and \( B = [b_{ij}] = [u_i/u_j] \) is a consistent \( N \times N \) matrix, with:

\[
u_i = \prod_{j=1}^{N} a_{ij}^{1/N}
\tag{31},
\]

then:

\[
d(A, B) = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \left( \ln a_{ij} - \ln b_{ij} \right)^2}.
\tag{31}
\]

is the minimal distance from \( A \) to any consistent matrix.

Thus, the components of the priority vector \( w \) are obtained by normalizing the components of the vector \( u \):

\[
w_k = \frac{\prod_{j=1}^{N} a_{kj}^{1/N}}{\sum_{j=1}^{N} \prod_{j=1}^{N} a_{kj}^{1/N}}, \quad k = 1, 2, \ldots, N,
\tag{32}
\]

such that \( \sum_{k=1}^{N} w_k = 1 \).

### IV. LLS METHOD FOR GROUP-AHP

We consider \( M \) experts, judging \( N \) alternatives \( A_n \), \( n = 1, 2, \ldots, N \). Each expert is characterized by a different weight \( \alpha_m > 0 \), \( m = 1, 2, \ldots, M \), corresponding to the expert’s level, such that:

\[
\sum_{m=1}^{M} \alpha_m = 1.
\tag{33}
\]

Also, each expert produces a \( N \times N \) reciprocal judgment matrix \( A^{(m)} = [a^{(m)}_{ij}]_{N \times N} \), where:

\[
a^{(m)}_{ij} = \frac{1}{a^{(m)}_{ji}}, \quad a^{(m)}_{ij} > 0.
\tag{34}
\]

The statistical model for each expert is obtained by introducing the multiplicative errors \( \varepsilon^{(m)}_{ij} \), such that:

\[
a^{(m)}_{ij} = \frac{u_i \varepsilon^{(m)}_{ij}}{u_j},
\tag{35}
\]

where

\[
u_i > 0, \quad \varepsilon^{(m)}_{ii} = 1, \quad \varepsilon_{ij} = 1/\varepsilon^{(m)}_{ji}, \quad \varepsilon^{(m)}_{ij} > 0.
\tag{36}
\]

Our goal is to find the estimates \( u_i \), using the LLS approach discussed in the previous section. Therefore, we consider the unconstrained minimization problem:

\[
\min_{u_1, u_2, \ldots, u_N} S(u_1, u_2, \ldots, u_N),
\tag{37}
\]

where

\[
S(u_1, u_2, \ldots, u_N) = \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_m \left( \ln a^{(m)}_{ij} - \ln u_i + \ln u_j \right)^2,
\tag{38}
\]

is the weighted sum of approximation errors. The problem can be rewritten as:

\[
S(x_1, x_2, \ldots, x_N) = \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_m [b^{(m)}_{ij} - x_i + x_j]^2,
\tag{39}
\]

where \( b^{(m)}_{ij} = \ln a^{(m)}_{ij} \) and \( x_i = \ln u_i \). The minimum condition:

\[
\frac{\partial}{\partial x_k} S(x_1, x_2, \ldots, x_N) = 0, \quad k = 1, 2, \ldots, N,
\tag{40}
\]

is equivalent with the system of linear equations:

\[
x_k = \frac{1}{N} \sum_{i=1}^{N} x_i = \langle B^{(m)} \rangle_{\alpha_k}, \quad k = 1, 2, \ldots, N,
\tag{41}
\]

where

\[
\langle B^{(m)} \rangle_{\alpha_k} = \frac{1}{N} \sum_{m=1}^{M} \sum_{j=1}^{N} \alpha_m b^{(m)}_{kj},
\tag{42}
\]

is the weighted average of the row \( k \) of all logarithmic matrices \( B^{(m)} = [b^{(m)}_{ij}]_{N \times N} \). Again, this is a linear system of the form \( Qx = \hat{d} \), with \( d_k = \langle B^{(m)} \rangle_{\alpha_k} \), and therefore the solution is:

\[
x_k = \langle B^{(m)} \rangle_{\alpha_k}, \quad k = 1, 2, \ldots, N,
\tag{43}
\]

and respectively:

\[
u_k = \exp \left( \langle B^{(m)} \rangle_{\alpha_k} \right), \quad k = 1, 2, \ldots, N.
\tag{44}
\]
Also, we observe that:

\[
\exp \left( B \right)_{ij} = \exp \left( \frac{1}{N} \sum_{m=1}^{M} \sum_{j=1}^{N} \alpha_m \ln a_{ij}^{(m)} \right) = \\
= \exp \left( \ln \left( \prod_{m=1}^{M} \prod_{j=1}^{N} \left( a_{kj}^{(m)} \right)^{\alpha_m/N} \right) \right) = \\
= \prod_{m=1}^{M} \prod_{j=1}^{N} \left( a_{kj}^{(m)} \right)^{\alpha_m/N}.
\]

Thus, the components of the group-priority vector \( w \) are obtained by normalizing the components of the vector \( u \):

\[
w_k = \frac{\prod_{m=1}^{M} \prod_{j=1}^{N} \left( a_{kj}^{(m)} \right)^{\alpha_m/N}}{\sum_{k=1}^{N} \prod_{m=1}^{M} \prod_{j=1}^{N} \left( a_{kj}^{(m)} \right)^{\alpha_m/N}}, \quad k = 1, 2, \ldots, N.
\]

This result can be obtained also using a different approach, based on the minimization of the generalized Kullback-Leibler divergence \( D \).

Let us assume that we solve the AHP problem separately for each expert, characterized by the judgment matrix \( A^{(m)} = [a_{ij}^{(m)}]_{N \times N} \), and we obtain the unnormalized priority vectors:

\[
u_k^{(m)} = [u_1^{(m)}, u_2^{(m)}, \ldots, u_N^{(m)}]^T, \quad m = 1, 2, \ldots, M,
\]

with the components:

\[
u_k^{(m)} = \prod_{j=1}^{N} \left( a_{kj}^{(m)} \right)^{1/N}, \quad k = 1, 2, \ldots, N.
\]

The generalized Kullback-Leibler divergence between the group-priority vector \( u = [u_1, u_2, \ldots, u_N]^T \) and the priority vector \( u^{(m)} \) of the expert \( m \), is given by:

\[
D(u||u^{(m)}) = \sum_{j=1}^{N} u_j \ln \left( \frac{u_j}{u_j^{(m)}} \right) - \sum_{j=1}^{N} u_j + \sum_{j=1}^{N} u_j^{(m)}.
\]

The problem is to find the vector \( u \), which minimizes the weighted sum of generalized Kullback-Leibler divergences between \( u \) and \( u^{(m)} \), \( m = 1, 2, \ldots, M \):

\[
D(u_1, u_2, \ldots, u_N) = \sum_{m=1}^{M} \alpha_m D(u||u^{(m)}).
\]

The minimum condition is:

\[
\frac{\partial}{\partial u_k} D(u||u^{(m)}) = 0, \quad k = 1, 2, \ldots, N,
\]

which is equivalent with:

\[
\log u_k - \sum_{m=1}^{M} \alpha_m \log u_k^{(m)} = 0, \quad k = 1, 2, \ldots, N,
\]

and respectively:

\[
u_k = \prod_{m=1}^{M} \left( a_{kj}^{(m)} \right)^{\alpha_m/N}, \quad k = 1, 2, \ldots, N.
\]

Obviously, by normalizing \( u_k \) we obtain the same group-priorities \( w_k \), like those obtained using the LLS approach.

V. Conclusion

In this paper we have discussed the LLS approach for the AHP and group-AHP, which provides an exact and unique solution for the priority vector of the AHP. Also, we have shown that the LLS approach for group-AHP is equivalent with the minimization of the weighted sum of generalized Kullback-Leibler divergences, between the aggregated priority vector and the priorities of each expert.

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