FROM CALABI–YAU DG CATEGORIES TO FROBENIUS
MANIFOLDS VIA PRIMITIVE FORMS

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Abstract. It is one of the most important problems in mirror symmetry to obtain
functorially Frobenius manifolds from smooth compact Calabi-Yau $A_{\infty}$-categories. This
paper gives an approach to this problem based on the theory of primitive forms. Under
an assumption on the formality of a certain homotopy algebra, a formal primitive form
for a smooth compact Calabi–Yau dg algebra can be constructed, which enable us to
have a formal Frobenius manifold.

1. Introduction

Mirror symmetry gives an identification between two objects coming from different
mathematical origins. It has been studied intensively by many mathematician for
more than twenty years since it yields important, interesting and unexpected geometric
information.

Almost ten years before the discovery of mirror symmetry, K. Saito studied a defor-
mation theory of an isolated hypersurface singularity in order to generalize the theory of
elliptic integrals [21]. There he developed a certain generalization of the complex Hodge
type of Calabi–Yau type and found a differential geometric structure, which he called
a flat structure, on the base space of the deformation (cf. [23] for a summary). This is
known as Saito’s theory of primitive forms. His flat structure was also found later by
Dubrovin in his study of two dimensional topological field theories in their relation with
integrable systems [7], which is axiomatized under the name of a Frobenius manifold (cf.
[10, 19, 20]).

The classical mirror symmetry conjecture states the existence of an isomorphism
between the Frobenius manifold from the Gromov–Witten theory of a Calabi-Yau mani-
fold and the one from the deformation theory of another Calabi-Yau manifold. In order to
explain this mysterious correspondence, Kontsevich conjectured in [16] that the derived
category of coherent sheaves on a Calabi-Yau manifold should be equivalent to the derived
Fukaya category for another Calabi-Yau manifold, which is known as the homological mir-
ror symmetry conjecture. In particular, he expects that the classical mirror symmetry
isomorphism can naturally be induced by the homological mirror symmetry equivalence
via the moduli space of $A_{\infty}$-deformations.
As is already explained in [23], although Saito’s construction of the Frobenius manifolds based on filtered de Rham cohomology groups, Gauß–Manin connections, higher residue pairings and primitive forms is formulated only for isolated hypersurface singularities, several notion and the idea of this construction is of general nature. It has inspired and motivated the non-commutative Hodge theory for Calabi–Yau $A_\infty$ categories (c.f. [14, 17]), which is the basic ingredient in constructing primitive forms in this categorical language.

The present paper tries to make the contents of Appendix in [23], where we give a list how objects in the original Saito’s theory of primitive forms may be generalized and rewritten, as precise as possible at present for smooth compact Calabi–Yau dg algebras.

More precisely, in Section 2 we recall some notations and terminologies of differential graded algebras, and then we introduce the dg analogue of the weighted homogeneous polynomial with an isolated singularity at the origin. It is Proposition 2.11 the “Cartan calculus”, that is the key in our story, which enable us to translate the original Saito’s theory for isolated weighted homogeneous hypersurface singularities verbatim into the dg categorical Saito theory. In Section 3, we show that a very good section can be constructed under a certain formality assumption which is motivated by the formality considered in [3], the identification of the (formal) moduli space of $A_\infty$-deformation of the derived category of coherent sheaves with the formal neighborhood of zero in the total cohomology group of polyvectors. After studying the versal deformation of a Saito structure in Section 4, we show how a primitive form is deduced from a very good section based on the famous method developed by M. Saito [24] and Barannikov [1, 2].

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2. Notations and Terminologies

In this paper, we denote by $k$ an algebraically closed field of characteristic zero with the unit $1_k$.

2.1. Differential graded algebras. On dg algebras and derived categories of modules over them, we refer the reader to [15], of which we follow the terminologies and notations.

A dg $k$-algebra (or simply, dg $k$-algebra) is a $\mathbb{Z}$-graded $k$-algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$ equipped with a differential, a $k$-linear map $d_A : A \rightarrow A$ of degree one with $d_A^2 = 0$. 


satisfying the Leibniz rule:

\[ d_A(a_1a_2) = (d_Aa_1)a_2 + (-1)^p a_1(d_Aa_2), \quad a_1 \in A^p, a_2 \in A. \]  

For a homogeneous element \( a \in A^p \), let \( \tau \) be the degree of \( a \), namely, \( \tau := p \). Denote by \( H^\bullet(A, d_A) := \bigoplus_{p \in \mathbb{Z}} H^p(A, d_A) \) the cohomology of \( A \) which is a graded \( k \)-module. Throughout this paper, \( A \) denotes a dg \( k \)-algebra.

We recall some terminologies for our later use. For a dg \( k \)-algebra \( A \), \( A^e := A^{op} \otimes_k A \) denotes the enveloping dg \( k \)-algebra of \( A \) where \( A^{op} \) is the opposite dg \( k \)-algebra of \( A \). The dg \( A \)-module is called perfect if it belongs to the smallest full triangulated subcategory of the derived category of dg \( A \)-modules which contains \( A \) and is closed under direct summands and isomorphisms.

**Definition 2.1.** Let \( A \) be a dg \( k \)-algebra.

1. A dg \( k \)-algebra \( A \) is called non-negatively graded if \( A^p = 0 \) for all negative integers \( p \).
2. A dg \( k \)-algebra \( A \) is called compact if \( A \) is a perfect dg \( k \)-module, namely, if its cohomology \( H^\bullet(A, d_A) \) is finite dimensional (cf. Kontsevich–Soibelman, Definition 8.2.1 in [17]).
3. A dg algebra \( A \) is called smooth if \( A \) is a perfect dg \( A^e \)-module (cf. Kontsevich–Soibelman, Definition 8.1.2 in [17]).
4. A non-negatively graded dg \( k \)-algebra \( A \) is called connected if \( H^0(A, d_A) = k[1_A] \).

For dg \( A \)-modules \( M, N \), we shall denote by \( \mathbb{R}\text{Hom}_A(M, N) \) the \( \mathbb{R} \text{Hom} \)-complex. Let \( A^! := \mathbb{R}\text{Hom}_{(A^e)^{op}}(A, A^e) \) be the inverse dualizing complex.

**Proposition 2.2.** Let \( A \) be a smooth dg \( k \)-algebra. The natural morphism in the derived category of dg \( A^e \)-modules

\[ A \longrightarrow \mathbb{R}\text{Hom}_{A^e}(\mathbb{R}\text{Hom}_{(A^e)^{op}}(A, A^e), A^e) = \mathbb{R}\text{Hom}_{A^e}(A^!, A^e) \]

is an isomorphism. In particular, we have a natural isomorphism in the derived category of dg \( k \)-modules

\[ A \otimes^L_{A^e} A \cong A \otimes^L_{A^e} \mathbb{R}\text{Hom}_{(A^e)^{op}}(A^!, A^e) \cong \mathbb{R}\text{Hom}_{A^e}(A^!, A). \]  

**2.2. Calabi–Yau dg algebras.**

**Definition 2.3** (Ginzburg, Definition 3.2.3 in [9]). Fix an integer \( w \in \mathbb{Z} \). A smooth dg \( k \)-algebra \( A \) is called Calabi–Yau of dimension \( w \) if there exists an isomorphism

\[ A^! \cong T^{-w}A \]

in the derived category of dg \( A^e \)-modules.
Let $A$ be a smooth Calabi–Yau dg $k$-algebra of dimension $w$. It is important to note that a choice of an element in $H^0(\mathbb{R}\text{Hom}(A^1, T^{-w}A))$ giving the isomorphism (3) yields an isomorphism

$$\mathbb{R}\text{Hom}_{A^e}(A, A) \cong \mathbb{R}\text{Hom}_{A^e}(T^w A^1, A) \cong T^{-w}(A \otimes_{A^e} A),$$

in the derived category of dg $k$-modules due to the isomorphism (2).

Recall that for a smooth compact dg $k$-algebra $A$ the Serre functor on the perfect derived category of dg $A^e$-modules is given by the dg $A^e$-module $A^* := \mathbb{R}\text{Hom}_k(A, k)$. It is known that the dg $A^e$-module $A^1$ gives its quasi-inverse, which implies the following.

**Proposition 2.4.** Let $A$ be a smooth compact Calabi–Yau dg $k$-algebra of dimension $w$. There exists an isomorphism in the perfect derived category of dg $A^e$-modules depending on a choice of an element in $H^{-w}(\mathbb{R}\text{Hom}(A^1, A))$:

$$T^w A \cong A^*.$$  

2.3. **Hochschild cohomology.** For graded $k$-modules $M$ and $N$, $\mathcal{G}r_k(M, N) = \bigoplus_{p \in \mathbb{Z}} \mathcal{G}r_k(M, N)^p$ denotes a graded $k$-module where $\mathcal{G}r_k(M, N)^p$ is a $k$-module consisting of graded $k$-linear maps from $M$ to $N$ of degree $p$. For a dg $k$-algebra $A$, set

$$C^\bullet(A) := \prod_{n \geq 0} \mathcal{G}r_k((TA)^{\otimes n}, A),$$

(6)

For $a \in A$, denote by $Ta$ when we consider it as an element of $TA$. Define a $k$-linear map $d : C^\bullet(A) \to C^{\bullet+1}(A)$ by $(df)(1_k) := d_A(1_k)$ for $f \in \mathcal{G}r_k(k, A)$ and, for $f \in \mathcal{G}r_k((TA)^{\otimes n}, A)^p$, $n \geq 1$,

$$(df)(Ta_1 \otimes \cdots \otimes Ta_n) := d_A(1_k) + \sum_{i=1}^{n} (-1)^{p-1+s_{i-1}} f(Ta_1 \otimes \cdots \otimes Ta_{i-1} \otimes T(d_A a_i) \otimes Ta_{i+1} \otimes \cdots \otimes Ta_n),$$

where $s_i := \sum_{m=1}^{i} (\mu_m - 1)$. Define another $k$-linear map $\delta : C^\bullet(A) \to C^{\bullet+1}(A)$ by

$$(\delta f)(Ta_1) := (-1)^{(p-1)(\mu_1-1)+\mu_1 a_1} f(1_k) + (-1)^p f(1_k) a_1$$

for $f \in \mathcal{G}r_k(k, A)^p$ and, for $f \in \mathcal{G}r_k((TA)^{\otimes n}, A)^p$, $n \geq 1$,

$$(\delta f)(Ta_1 \otimes \cdots \otimes Ta_{n+1}) := (-1)^{p+s_n} f(Ta_1 \otimes \cdots \otimes Ta_n) a_{n+1} + \sum_{i=1}^{n} (-1)^{p-1+s_i} f(Ta_1 \otimes \cdots \otimes Ta_{i-1} \otimes T(a_i a_{i+1}) \otimes Ta_{i+2} \otimes \cdots \otimes Ta_{n+1})$$

$$+ (-1)^{(p-1)(\mu_1-1)+\mu_1 a_1} f(Ta_2 \otimes \cdots \otimes Ta_{n+1}).$$

The following proposition is well-known.
**Proposition 2.5.** The $k$-linear maps $d, \delta$ define differentials on $C^*(A)$ and satisfy $d\delta + \delta d = 0$. Moreover, the dg $k$-module $(C^*(A), \partial := d + \delta)$ is isomorphic to the dg $k$-module $\mathbb{R}\text{Hom}_A(A, A)$ in the derived category of dg $k$-modules.

**Definition 2.6.** The dg $k$-module $(C^*(A), \partial)$ is called the Hochschild cochain complex of $A$, whose cohomology $H^*(C^*(A), \partial)$ is denoted by $HH^*(A)$ and is called the Hochschild cohomology of $A$.

Denote by $T_{\text{poly}}(A)$ the graded $k$-module $H^*(C^*(A), \delta)$.

For $f = (f_n)_{n \geq 0} \in C^p(A)$ and $g = (g_n)_{n \geq 0} \in C^q(A)$, one can define the product $f \circ g = ((f \circ g)_n)_{n \geq 0} \in C^{p+q}(A)$ by

$$(f \circ g)_n(Ta_1 \otimes \cdots \otimes Ta_n) := \sum_{i=0}^n (-1)^{q_i s_i} f_i(Ta_1 \otimes \cdots \otimes Ta_i) g_{n-i}(Ta_{i+1} \otimes \cdots \otimes Ta_n).$$

It is also known that for $f \in C^p(A)$ and $g \in C^q(A)$ one can define the Gerstenhaber bracket $[f, g]_G \in C^{p+q-1}(A)$ by

$$[f, g]_G := f \circ_{-1} g - (-1)^{(p-1)(q-1)} g \circ_{-1} f,$$

where $f \circ_{-1} g = ((f \circ_{-1} g)_n)_{n \geq 0} \in C^{p+q-1}(A)$ is given by

$$(f \circ_{-1} g)_n(Ta_1 \otimes \cdots \otimes Ta_n) := \sum_{i=1}^n \sum_{j=1}^n (-1)^{(q-1)s_{i-1}} f_{n-j+1}(Ta_1 \otimes \cdots \otimes Ta_{i-1}) \otimes T(g_j(Ta_i \otimes \cdots \otimes Ta_{i+j-1})) \otimes Ta_{i+j} \otimes \cdots \otimes Ta_n).$$

We collect some basic properties of $T_{\text{poly}}(A)$ and $HH^*(A)$, which are well-known facts or follow from a straight forward calculation.

**Proposition 2.7.** The product $\circ$ on $C^*(A)$ induces structures of graded commutative $k$-algebras on $HH^*(A)$ and $T_{\text{poly}}(A)$ whose unit elements are given by the cohomology classes of $1_A \in C^0(A)$. The Gerstenhaber bracket $[-, -]_G$ induces structures of graded Lie algebras on $HH^{*+1}(A)$ and $T_{\text{poly}}^{*+1}(A)$.

Moreover, it turns out that the tuple $(T_{\text{poly}}^*(A), d, \circ, [-, -]_G)$ is a differential Gerstenhaber algebra (cf. [8]). Namely, we also have

$$[X, Y \circ Z]_G = [X, Y] \circ Z + (-1)^{(X+1)Y} Y \circ [X, Z]_G, \quad X, Y, Z \in T_{\text{poly}}^*(A).$$
Let \( m_A = (m_A^p)_{p \geq 0} \) be the element in \( C^2(A) \) where \( m_A^p := 0 \) if \( p \neq 1, 2 \) and \( m_A(Ta_1) := d_A a_1 \), \( m_A^2(Ta_1 \otimes Ta_2) := (-1)^{\overline{\alpha}} a_1 a_2 \). It is well-known that \([m_A, m_A]_G = 0\) since, if we write \( m_A = m_A^1 + m_A^2 \),

\[
[m_A, m_A]_G = 0 \iff \begin{cases} 
[m_A^1, m_A^1]_G = 0 \\
[m_A^1, m_A^2]_G = 0 \\
[m_A^2, m_A^2]_G = 0 
\end{cases}
\]

\( d^2_A = 0 \) \( \iff \) the Leibnitz rule \( (1) \)

\( \iff \) the associativity.

It is also important to note that \( \partial f = [m_A, f]_G \), \( df = [m_A^1, f]_G \) and \( \delta f = [m_A^2, f]_G \) for \( f \in C^\bullet(A) \).

Let \( f_A \) be the cohomology class of \( m_A \) in \( T^2_{poly}(A) \), which is actually the same as the one of \( m_A^1 \) in \( T^2_{poly}(A) \). We will see later that, for a non-negatively graded connected smooth compact Calabi–Yau dg \( k \)-algebra \( A \), the element \( f_A \) is the dg analogue of a weighted homogeneous polynomial with an isolated singularity, which is the initial data for the original Saito’s theory of primitive forms.

Indeed, by a direct calculation, we obtain the following dg analogue of the “Euler’s identity” for \( f_A \).

**Proposition 2.8.** For \( X \in T^\bullet_{poly}(A) \), we have \( dX = [f_A, X]_G \). In particular, we have

\[
f_A = [\partial \deg A, f_A]_G, \quad (8)
\]

where \( \partial \deg A \) is the cohomology class of the element \( \deg_A = (\deg_A^p)_{p \geq 0} \in C^1(A) \) in \( T^1_{poly}(A, A) \) defined by \( \deg_A^p := 0 \) if \( p \neq 1 \) and \( \deg_A^1(Ta_1) := \overline{\alpha}_1 \cdot a_1 \).

The “Euler’s identity” \( (8) \) means that \( f_A \) is of degree one with respect to another grading structure on \( C^\bullet(A) \) whose origin is the \( \mathbb{Z} \)-grading on \( A \). This completely agrees with the fact that a weighted homogeneous polynomial in the original Saito’s theory is considered as of degree two from the homological viewpoint, e.g. when we discuss matrix factorizations, and of degree one for the compatibility with exponents.

**2.4. Hochshild homology.** For a dg \( k \)-algebra \( A \), set

\[
C_\bullet(A) := \bigoplus_{n \geq 0} A \otimes_k (TA)^{\otimes n}. \quad (9)
\]

We shall denote an element of \( A \otimes_k (TA)^{\otimes n} \) by \( a_0 \) if \( n = 0 \) and \( a_0 \otimes Ta_1 \otimes \cdots \otimes Ta_n \) if \( n \neq 0 \). Define a \( k \)-linear map \( d : C_\bullet(A) \longrightarrow C_{\bullet-1}(A) \) by \( d(a_0) := d_A a_0 \) and, for \( n \geq 1 \),

\[
d(a_0 \otimes Ta_1 \otimes \cdots \otimes Ta_n) := d_A a_0 \otimes Ta_1 \otimes \cdots \otimes Ta_n
\]

\[+ \sum_{i=1}^n (-1)^{\overline{\alpha}_i} a_0 \otimes Ta_1 \otimes \cdots \otimes Ta_{i-1} \otimes T(d_A a_i) \otimes Ta_{i+1} \otimes \cdots \otimes Ta_n, \]
where \( s'_i := \sum_{m=0}^{i}(a_m - 1) \). Define another \( k \)-linear map \( \delta : C_\bullet(A) \to C_{\bullet-1}(A) \) by \( \delta(a_0) := 0 \) and, for \( n \geq 1 \),

\[
\delta(a_0 \otimes T_{a_1} \otimes \cdots \otimes T_{a_n}) := (-1)^{\overline{a_0}}a_0 \otimes T_{a_2} \otimes \cdots \otimes T_{a_n} - \sum_{i=1}^{n-1} (-1)^{s'_i}a_0 \otimes T_{a_1} \otimes \cdots \otimes T_{a_{i-1}} \otimes T(a_{i+1}a_{i+1}) \otimes T_{a_{i+2}} \otimes \cdots \otimes T_{a_n} + (-1)^{\overline{a_n}+(a_{n-1}-1)}s'_{n-1}a_n \otimes T_{a_1} \otimes \cdots \otimes T_{a_{n-1}}.
\]

Define a \( k \)-linear map \( B : C_\bullet(A) \to C_{\bullet+1}(A) \) called the Conne’s differential by

\[
B(a_0 \otimes T_{a_1} \cdots \otimes T_{a_n}) := \text{id}_A \otimes T_{a_0} \otimes \cdots \otimes T_{a_n} - (-1)^{\overline{a_0}}a_0 \otimes T(\text{id}_A) \otimes T_{a_1} \otimes \cdots \otimes T_{a_n} + \sum_{i=1}^{n} (-1)^{\overline{a_{i-1}}}s''_i \text{id}_A \otimes T_{a_i} \otimes \cdots \otimes T_{a_n} \otimes T_{a_0} \otimes \cdots \otimes T_{a_{i-1}} - \sum_{i=1}^{n} (-1)^{\overline{a_{i-1}}}s''_i \text{id}_A \otimes T(\text{id}_A) \otimes T_{a_{i+1}} \cdots \otimes T_{a_n} \otimes T_{a_0} \otimes \cdots \otimes T_{a_{i-1}},
\]

where \( s''_i := \left( \sum_{m=0}^{i-1}(a_m - 1) \right) \left( \sum_{m=1}^{n-1}(a_m - 1) \right) \).

The following proposition is also well-known.

**Proposition 2.9.** The \( k \)-linear maps \( d, \delta, B \) define differentials on \( C_\bullet(A) \) and satisfy \( d\delta + \delta d = 0 \), \( dB + Bd = 0 \), \( \delta B + B\delta = 0 \). Moreover, the dg \( k \)-module \((C_\bullet(A), \partial := d + \delta)\) is isomorphic to the dg \( k \)-module \( A \otimes^L_{A^e} A \) in the derived category of dg \( k \)-modules.

**Definition 2.10.** The dg \( k \)-module \((C_\bullet(A), \partial)\) is called the Hochschild chain complex of \( A \), whose homology \( H_\bullet(C_\bullet(A), \partial) \) is denoted by \( HH_\bullet(A) \) and is called the Hochschild homology of \( A \).

Denote by \( \Omega_\bullet(A) \) the graded \( k \)-module \( H_\bullet(C_\bullet(A), \delta) \).

The pair \((\mathcal{T}_\bullet^{poly}(A), \Omega_\bullet(A))\) admits a structure of a *calculus algebra* (cf. Dolgushev–Tamarkin–Tsygan, Definition 3 in [5]), namely, we have the algebraic structures on \((\mathcal{T}_\bullet^{poly}(A), \Omega_\bullet(A))\) in the next proposition which directly follows, just by forgetting the differential \( d \), from the result by Daletski–Gelfand–Tsygan [4] (see also Dolgushev–Tamarkin–Tsygan, Section 3 in [6]). To state the result, recall two \( k \)-linear maps \( \iota_f, \mathcal{L}_f : C_\bullet(A) \to C_\bullet(A) \) called the *contraction* and the *Lie derivative*. For \( f \in \mathcal{G}r_k((TA)^{op}, A) \), they are...
defined in the following way:

\[
\iota_f(a_0 \otimes T a_1 \cdots \otimes T a_n) := \pm a_0 f(T a_1 \otimes \cdots \otimes T a_p) \otimes T a_{p+1} \cdots \otimes T a_n,
\]

\[
\mathcal{L}_f(a_0 \otimes T a_1 \cdots \otimes T a_n) := f(T a_0 \otimes \cdots \otimes T a_{p-1}) \otimes \cdots \otimes T a_n
\]

\[
+ \sum_{i=1}^{n-p+1} \pm a_0 \otimes T a_1 \cdots \otimes T f(T a_i \otimes \cdots \otimes T a_{i+p-1}) \otimes \cdots \otimes T a_n
\]

\[
+ \sum_{i=n-p+1}^{n-1} \pm f(T a_{i+1} \otimes \cdots \otimes T a_n) \otimes T a_0 \otimes \cdots \otimes T a_i,
\]

where we omit signs since we do not need their explicit expressions later.

**Proposition 2.11.** The \(k\)-linear maps \(\iota\) and \(\mathcal{L}\) induce morphisms of graded \(k\)-modules

\[
i : T^\bullet_{\text{poly}}(A) \longrightarrow \mathcal{G}_{\text{rk}}(\Omega^\bullet(A), \Omega^\bullet(A)), \quad X \mapsto i_X,
\]

\[
L : T^\bullet_{\text{poly}}(A) \longrightarrow T^{-1}\mathcal{G}_{\text{rk}}(\Omega^\bullet(A), \Omega^\bullet(A)), \quad X \mapsto L_X,
\]

satisfying, for all \(X, Y \in T^\bullet_{\text{poly}}(A),\)

\[
i_X i_Y = i_{X \circ Y}, \quad [L_X, L_Y] = L_{[X, Y]}|_{\mathcal{G}};
\]

\[
L_X i_Y + (-1)^{|X|} i_Y L_X = L_{X \circ Y}, \quad [i_X, L_Y] = i_{[X, Y]}|_{\mathcal{G}};
\]

\[
[B, i_X] = -L_X, \quad [B, L_X] = 0, \quad L_{i_X} = -d.
\]

Note that \(\mathcal{G}_{\text{rk}}(\Omega^\bullet(A), \Omega^\bullet(A))\) has both a structure of a dg \(k\)-algebra and a dg Lie algebra. Since we have

\[
[d, i_X] = i_{dX}, \quad [d, L_X] = L_{-dX}, \quad X \in T^\bullet_{\text{poly}}(A),
\]

the \(k\)-linear maps \(i\) and \(L\) define a morphism of dg \(k\)-algebras and a morphism of dg Lie algebras.

Since \(A^e\)-modules can be considered as dg endo-functors on the derived dg category of dg \(A\)-modules (see Töen, Corollary 7.6 in [27] for the precise statement), there are “horizontal” and “vertical” associative product structures on \(\mathbb{R}\text{Hom}_{A^e}(A, A)\) and \(\mathbb{R}\text{Hom}_{A^e}(A, A)\)-module structures on \(\mathbb{R}\text{Hom}_{A^e}(A^1, A)\) (cf. Kashiwara–Schapira, Remark 1.3.4 in [13], for endo-functors on ordinary categories). It turns out that they induce the same structures on \(T^\bullet_{\text{poly}}(A)\) and \(\Omega^\bullet(A)\), which are the product \(\circ\) and the map \(i\) given above (cf. Kaledin, Lemma 8.1 and the equation (8.2) in [12]).
The equalities in Proposition 2.11, the “Cartan calculus”, play an essential role in our story, which enable us to translate the original Saito’s theory for isolated hypersurface singularities verbatim into the dg categorical Saito theory.

3. Construction of a very good section

3.1. Formality for certain homotopy calculus algebras. Let $A$ be a non-negatively graded smooth dg $k$-algebra. On the algebraic structure on the pair $(C^\bullet(A), C_\bullet(A))$, the following proposition directly follows from Corollary 1 in [5]:

**Proposition 3.1.** The pair $(C^\bullet(A), C_\bullet(A))$ has a structure of homotopy calculus algebra.

We omit the definition of homotopy calculus algebras due to limitations of space since we shall not use it in later discussions. Based on Theorem 5 in [5], we expect the following conjecture:

**Conjecture 3.2.** As homotopy calculus algebras, $(C^\bullet(A), C_\bullet(A))$ is quasi-isomorphic to the calculus algebra $(T_{poly}^\bullet(A), \Omega^\bullet(A))$.

Note in particular that Conjecture 3.2 implies that the dg Lie algebra $(C^{\bullet+1}(A), [m_A, -]_G, [\cdot, \cdot]_G)$ is quasi-isomorphic to the dg Lie algebra $(T_{poly}^{\bullet+1}(A), [f_A, -]_G, [\cdot, \cdot]_G)$ as an $L_\infty$-algebra.

**Assumption 3.3.** Conjecture 3.2 holds for a non-negatively graded smooth dg $k$-algebra $A$.

This assumption is important in order also to ensure the functoriality of our construction of primitive forms.

3.2. Hochschild cohomology of Calabi–Yau dg algebras. From now on, $A$ always denotes a non-negatively graded connected smooth compact Calabi–Yau dg $k$-algebra. Under Assumption 3.3 there are isomorphisms of graded $k$-modules

$$HH^\bullet(A) \cong H^\bullet(T_{poly}^\bullet(A), d), \quad HH_\bullet(A) \cong H_\bullet(\Omega^\bullet(A), d).$$

In particular, an element of $HH_w(A)$ giving the isomorphism corresponds to a non-zero element $v_1 \in H_w(\Omega^\bullet(A), d)$. Since the dg $k$-algebra $A$ is non-negatively graded and connected, we have $H_w(\Omega^\bullet(A), d) \cong HH_w(A) \cong HH^0(A) = k \cdot [1_A]$, which implies that $H_w(\Omega^\bullet(A), d) = k \cdot v_1$.

**Conjecture 3.4.** Let $v_1$ be a non-zero element in $H_w(\Omega^\bullet(A), d)$. Under Assumption 3.3, the contraction map

$$(C^\bullet(A), \partial) \longrightarrow (C_\bullet(A), \partial), \quad X \mapsto \iota_X v_1,$$
induces a morphism of dg $k$-modules

$$(T^\bullet_{\text{poly}}(A), d) \longrightarrow (\Omega_{w-\bullet}(A), d), \quad X \mapsto i_X v_1,$$

which is an isomorphism.

**Assumption 3.5.** Conjecture 3.4 holds for a non-negatively graded connected smooth compact Calabi–Yau dg $k$-algebra $A$.

In particular, under Assumption 3.5 we have the isomorphism

$$H^p(T^\bullet_{\text{poly}}(A), d) \cong H_{w-p}(\Omega_{\bullet}(A), d), \quad p \in \mathbb{Z}, \quad X \mapsto i_X v_1,$$

of graded $k$-modules.

**Definition 3.6.** Set

$$\text{Jac}(f_A) := H^\bullet(T^\bullet_{\text{poly}}(A), d).$$

We call the graded $k$-module $\text{Jac}(f_A)$ the *Jacobian ring* of $A$.

It is known that if $A$ is a smooth compact Calabi–Yau dg $k$-algebra of dimension $w$ then $A^e$ is a smooth compact Calabi–Yau dg $k$-algebra of dimension $2w$. Since $\text{Jac}(f_A) \cong \text{HH}^\bullet(A) \cong \text{HH}^\bullet(\text{RHom}_{A^e}(A, A))$, we have the following.

**Proposition 3.7.** Fix a non-zero element $v_1 \in \text{HH}_w(A)$. Let $v_1^{\otimes 2}$ be the element in $\text{HH}_{2w}(A^e)$ corresponding to $v_1 \otimes v_1$ under the Künneth formula $\text{HH}_{2w}(A^e) \cong \text{HH}_w(A^{op}) \otimes_k \text{HH}_w(A) = \text{HH}_w(A) \otimes_k \text{HH}_w(A)$. Then the induced isomorphism by $v_1^{\otimes 2}$

$$T^{2w}A^e \longrightarrow (A^e)^*$$

in the perfect derived category of the dg $k$-algebra $(A^e)^e$ equips $\text{Jac}(f_A)$ with a structure of a finite dimensional graded commutative Frobenius $k$-algebra. Namely, we have a non-degenerate graded symmetric bilinear form $\eta_{f_A,v_1^{\otimes 2}} : \text{Jac}(f_A) \otimes_k \text{Jac}(f_A) \rightarrow T^{2w}k$ such that

$$\eta_{f_A,v_1^{\otimes 2}}(X \circ Y, Z) = \eta_{f_A,v_1^{\otimes 2}}(X, Y \circ Z), \quad X, Y, Z \in \text{Jac}(f_A).$$

3.3. Filtered de Rham cohomology and the degeneration of Hodge to de Rham.

**Definition 3.8.** Let $u$ be a formal variable of degree two. Define a graded $k((u))$-module $H_{f_A}$, called the filtered de Rham cohomology, by

$$H_{f_A} := \text{H}_\bullet(T^{-w}\Omega_\bullet(A)((u)), d + uB)$$

and for any integer $p \in \mathbb{Z}$ the graded $k[[u]]$-submodules $H_{f_A}^{(-p)}$ of $H_{f_A}$

$$H_{f_A}^{(-p)} := H_\bullet(T^{-w}\Omega_\bullet(A)[[u]]u^p, d + uB).$$
Define a graded $k$-module $\Omega_{i_A}$ by
\[
\Omega_{i_A} := H_\bullet(T^{-u}\Omega_\bullet(A), d).
\]

**Proposition 3.9.** For all $p \in \mathbb{Z}$, there exists an exact sequence of graded $k$-modules
\[
0 \longrightarrow H_{i_A}^{(-p-1)} \longrightarrow H_{i_A}^{(-p)} \xrightarrow{r_{(-p)}} \Omega_{i_A} \longrightarrow 0.
\]

**Proof.** Under Assumption assum:formality, we have the isomorphism of graded $k$-modules between $HC^{(p)}(\mathcal{C}_\bullet(A)[[u]]u^p, \partial + uB)$ and $H_{i_A}^{(p)}$. Then it follows from the degeneration of Hodge to de Rham conjecture proven by Kaledin, Theorem 5.5 in [11], for non-negatively graded smooth compact dg $k$-algebras.

The $k[[u]]$-submodules $\{H_{i_A}^{(-p)}\}_{p \in \mathbb{Z}}$ of $H_{i_A}$ define an increasing filtration
\[
\cdots \subset H_{i_A}^{(-p-1)} \subset H_{i_A}^{(-p)} \subset \cdots \subset H_{i_A},
\]
such that the multiplication of $u$ induces an isomorphism of $k$-modules
\[
u : H_{i_A}^{(-p)} \cong H_{i_A}^{(-p-1)}.
\]

**3.4. Gauß–Manin connection on $H_{i_A}$.** Set $\mathcal{T}_{k_u} := k[[u]]\frac{d}{du}$, where $\frac{d}{du}$ is the derivation on $k[[u]]$ satisfying $\frac{d}{du}(u) = 1$. Define a morphism of graded $k$-modules $\nabla : \mathcal{T}_{k_u} \otimes_k \Omega_\bullet(A)((u)) \rightarrow \Omega_\bullet(A)((u))$ by
\[
\nabla \frac{d}{du} := \frac{d}{du} - \frac{1}{u^2} i_{i_A}.
\]

The following proposition is very simple, however, it will be the key in our construction of a primitive form. It is the dg analogue of the corresponding fact used in the original Saito’s theory for weighted homogeneous polynomials when we lift a homogeneous basis of the Jacobian ring multiplied with the standard holomorphic volume form to a very good section.

**Proposition 3.10.** The morphism of graded $k$-modules $\nabla$ is a connection which satisfies
\[
\left[\nabla_{u \frac{d}{du}}, d + uB\right] = d + uB.
\]

Therefore, $\nabla$ induces a connection on $H_{i_A}$. Moreover, we have
\[
\nabla_{u \frac{d}{du}} \left(H_{i_A}^{(0)}\right) \subset H_{i_A}^{(0)}.
\]

**Proof.** Use equalities in Proposition [2.11]. We have
\[
\left[\nabla_{u \frac{d}{du}}, d + uB\right] = uB - \frac{1}{u} [i_{i_A}, d + uB] = uB - L_{i_A} = d + uB
\]
Together with the equation [8], we also have
\[
u \frac{d}{du} - \frac{1}{u} i_{i_A} = \nu \frac{d}{du} - \frac{1}{u} \left[i_{\text{deg}A}, L_{i_A}\right] = \nu \frac{d}{du} + L_{\text{deg}A} + \frac{1}{u} \left[d + uB, i_{\text{deg}A}\right]
\]
Definition 3.11. The connection $\nabla$ on $\mathcal{H}_{f_A}$ is called the Gauß–Manin connection for $A$.

3.5. Exponents. For $a_0 \otimes T a_1 \otimes \cdots \otimes T a_n \in C \cdot (A)$ such that $a_i \in A_{\geq i}$, the endomorphism $L_{\text{deg} A}$ on $C \cdot (A)$ can be calculated as

$$L_{\text{deg} A} (a_0 \otimes T a_1 \otimes \cdots \otimes T a_n) := \left( \sum_{i=0}^{n} a_i \right) \cdot (a_0 \otimes T a_1 \otimes \cdots \otimes T a_n).$$

This obviously commutes with the operator $\delta$ on $C \cdot (A)$ and hence defines an endomorphism on $\Omega \cdot (A)$, which is exactly $L_{\text{deg} A}$.

Proposition 3.12. The endomorphism of graded $k$-modules $L_{\text{deg} A}$ on $\Omega \cdot (A)$ induces a $k$-linear endomorphism on $\Omega_{f_A}$.

Proof. The equalities in Proposition 2.11 and the equation (8) yield

$$[L_{\text{deg} A}, d] = - [L_{\text{deg} A}, L_{f_A}] = - L_{[\text{deg} A, f_A] G} = - L_{f_A} = d,$$

which implies that $L_{\text{deg} A} (\Omega_{f_A}) \subset \Omega_{f_A}$. □

Denote by $N_A$ the endomorphism of graded $k$-modules on $\Omega_{f_A}$ induced by $L_{\text{deg} A}$.

Define graded $k$-submodules $\Omega_{f_A}^{p,q}$ of $\Omega_{f_A}$ by

$$\Omega_{f_A}^{p,q} := \{ \omega \in \Omega_{f_A} \mid \overline{\omega} = -p + q, \ N_A \omega = q \omega \}, \quad p, q \in \mathbb{Z}.$$

and set

$$F_q := \bigoplus_{p \in \mathbb{Z}, r \leq q} \Omega_{f_A}^{p,r}, \quad q \in \mathbb{Z}.$$

Definition 3.13. The graded $k$-submodules $\{ F_q \}_{q \in \mathbb{Z}}$ form an increasing filtration of $\Omega_{f_A}$

$$0 \subset \cdots \subset F_q \subset F_{q+1} \subset \cdots \subset \Omega_{f_A},$$

which is called the Hodge filtration of $\Omega_{f_A}$.

Since $A$ is non-negatively graded, we have $F_q = 0$ for $q < 0$.

Definition 3.14. The Hodge numbers are

$$h^{p,q}(A) := \dim_k \Omega_{f_A}^{p,q}, \quad p, q \in \mathbb{Z}.$$

The integer $q$ with $h^{p,q}(A) \neq 0$ is called an exponent. The set of exponents is the multi-set

$$\{ q \ast h^{p,q}(A) \mid p, q \in \mathbb{Z}, \ \Omega_{f_A}^{p,q} \neq 0 \},$$

where by $u \ast v$ we denote $v$ copies of the integer $u$. 
3.6. **Pairing on the Hochschild homology.** Since $A$ is Calabi–Yau of dimension $w$, an element $v_1 \in HH_w(A)$ yields an isomorphism of graded $k$-modules

$$Jac(f_A) \cong \Omega_{f_A}, \quad X \mapsto i_X v_1.$$  \hfill (19)

Move the $k$-bilinear form on $Jac(f_A)$ to $\Omega_{f_A}$ by this isomorphism.

**Definition 3.15.** Define a $k$-bilinear form $J_{f_A} : \Omega_{f_A} \otimes_k \Omega_{f_A} \rightarrow k$ by

$$J_{f_A}(i_X v_1, i_Y v_1) := (-1)^w \eta^1_{f_A, v_1}(X, Y), \quad X, Y \in Jac(f_A).$$  \hfill (20)

Note that $J_{f_A}$ does not depend on the choice of $v_1$. Moreover, it induces a perfect pairing

$$J_{f_A} : \Omega_{f_A}^{p, q} \otimes_k \Omega_{f_A}^{w-p, w-q} \rightarrow k.$$  

Some elementary homological algebras for $Jac(f_A)$ and $\Omega_{f_A}$ yields the following.

**Proposition 3.16.** The Hodge numbers satisfy

1. $h^{p, q}(A) = 0$ if $p < 0$ or $q < 0$.
2. $h^{w, 0}(A) = 1$.
3. $h^{w-p, q}(A) = h^{p, w-q}(A)$.

**Proof.** The property (1) holds since $A$ is non-negatively graded. The property (2) follows from the fact that $[1_A] \in HH^0(A)$ is mapped to $v_1$ under the isomorphism $HH^0(A) \cong HH_w(A)$ and that $v_1 \in \Omega_{f_A}^{w, 0}$. The perfectness of $J_{f_A}$ implies the property (3). \hfill $\square$

This proposition means that 0 is the minimal exponent with multiplicity one and the duality of exponents holds, namely, if $q$ is an exponent then $w - q$ is also an exponent.

3.7. Existence of a very good section.

**Proposition 3.17.** Let $r^{(0)} : \mathcal{H}_{f_A}^{(0)} \rightarrow \Omega_{f_A}$ be the $k$-linear map of degree zero in the exact sequence of graded $k$-modules in Proposition 3.9

$$0 \rightarrow \mathcal{H}_{f_A}^{(-1)} \rightarrow \mathcal{H}_{f_A}^{(0)} \xrightarrow{r^{(0)}} \Omega_{f_A} \rightarrow 0.$$  

Then there exists a section $s^{(0)} : \Omega_{f_A} \rightarrow \mathcal{H}_{f_A}^{(0)}$ of $r^{(0)}$ such that

$$\nabla_{u, d/u} (s^{(0)}(\Omega_{f_A})) \subset s^{(0)}(\Omega_{f_A}).$$  

**Proof.** Set $l_A := \dim_k Jac(f_A)$. Then Proposition 3.12 implies the existence of a $k$-basis $\{v_1, \ldots, v_{l_A}\}$ of $\Omega_{f_A}$ such that $N_A v_i = q_i \cdot v_i$ for some $q_i \in \mathbb{Z}$ for $i = 1, \ldots, l_A$, where $v_1$ is the element in $\Omega_{f_A}^{w, 0}$ as in the previous section.

There exists a section $s^{(0)} : \Omega_{f_A} \rightarrow \mathcal{H}_{f_A}^{(0)}$ so that $s^{(0)}(\Omega_{f_A})$ is inside of $H_*(\Omega_*(A) \otimes_k k[u, d+uB]$ since $k$ is a field and the dg $k$-algebra $A$ is non-negatively graded. Moreover,
the equalities \([\nabla, d + uB] = d + uB\), \([L_{\text{deg} A}, d] = d\) and \([L_{\text{deg} A}, B] = 0\), which in particular means that \(d\) (resp. \(B, u\)) is of degree 1 (resp. 0, 1) with respect to the grading given by \(L_{\text{deg} A}\), enable us to choose \(\omega_{il} \in \Omega_{\bullet}(A)\) satisfying \(L_{\text{deg} A} \omega_{il} = (q_i - l) \cdot \omega_{il}\) so that \(s^{(0)}(v_i) = \sum_{l=0}^{m}[\omega_{il}]u^l\) for some \(m \in \mathbb{Z}_{\geq 0}\). Therefore, \(\nabla d s^{(0)}(v_i) = q_i \cdot s^{(0)}(v_i)\). □

A section \(s^{(0)}\) of \(r^{(0)}\) in Proposition 3.17 is called a very good section after M. Saito (see Introduction and Proposition 3.1 in [25]).

3.8. Higher residue pairings. Once a very good section \(s^{(0)}\) for \(r^{(0)}\) is given, we have an isomorphism

\[
\Omega_{fA}(\!(u)\! ) \cong \mathcal{H}_{fA}, \quad \sum_{p=-\infty}^{\infty} v_p u^p \mapsto \sum_{p=-\infty}^{\infty} s^{(0)}(v_p) u^p,
\]

of \(k(\!(u)\! )\)-modules.

**Definition 3.18.** Fix a very good section \(s^{(0)}\) for \(r^{(0)}\) and define a \(k\)-bilinear form \(K_{fA}: \mathcal{H}_{fA} \otimes_k \mathcal{H}_{fA} \to k(\!(u)\! )\) by

\[
K_{fA}(\omega(u), \omega'(u)) := J_{fA}(\omega(u), \omega'(-u)) u^w,
\]

where \(J_{fA}\) on the right hand sided denotes the pairing on \(\Omega_{fA}(\!(u)\! )\) which is the \(k(\!(u)\! )\)-linear extension of \(J_{fA}\) on \(\Omega_{fA}\). The pairing \(K_{fA}\) is called the higher residue pairings after K. Saito, Section 4 in [22].

It follows from some elementary calculation that for all \(h(u) \in k(\!(u)\! )\)

\[
h(u)K_{fA}(\omega_1, \omega_2) = K_{fA}(h(u)\omega_1, \omega_2) = K_{fA}(\omega_1, h(-u)\omega_2),
\]

and for all \(\omega_1, \omega_2 \in \mathcal{H}_{fA}\)

\[
u \frac{d}{du} K_{fA}(\omega_1, \omega_2) = K_{fA}(\nabla_{u^d} \omega_1, \omega_2) + K_{fA}(\omega_1, \nabla_{u^d} \omega_2).
\]

Since these properties characterize uniquely the higher residue pairings for isolated hypersurface singularities (see p.45 – p.46 in [24]), we just denote it by \(K_{fA}\) without mentioning the dependence on the chosen very good section \(s^{(0)}\).

Proposition 3.17 imply the following.

**Proposition 3.19.** Let \(s^{(0)}\) be a very good section for \(r^{(0)}\). The graded \(k\)-submodule

\[
S := s^{(0)}(\Omega_{fA}) \otimes_k k[u^{-1}]u^{-1} \quad \text{of} \quad \mathcal{H}_{fA}
\]

satisfies

\[
\mathcal{H}_{fA} = \mathcal{H}_{fA}^{(0)} \oplus S, \quad u^{-1} S \subset S, \quad \nabla_{u^d} S \subset S, \quad K_{fA}(S, S) \subset k[u^{-1}]u^{w-2}.
\]
4. Deformation

In this section, \( A \) always denotes a non-negatively graded smooth compact Calabi–Yau dg \( k \)-algebra of dimension \( w \) satisfying Assumption 3.3 and Assumption 3.5.

4.1. Versal deformation. Recall that \( \Omega^w_{\alpha} = \{ v \in \Omega_w(A) | dv = 0, N_A v = 0 \} = k \cdot v_1 \). Under Assumption 3.5, we can define a morphism \( \Delta : T_{poly}^\bullet(A) \rightarrow T_{poly}^{\bullet-1}(A) \) of graded \( k \)-modules by \( i_{\Delta(Y)}v_1 := Bi_X v_1, X \in T_{poly}^\bullet(A) \). Note that \( \Delta \) does not depend on the particular choice of \( v_1 \). Obviously, it satisfies \( \Delta^2 = 0 \) and hence it defines a differential on \( T_{poly}^\bullet(A) \).

The differential \( \Delta \) does not satisfy the Leibniz rule with respect to the product \( \circ \), however, it is a part of a rich structure on \( T_{poly}^\bullet(A) \) as given below.

Proposition 4.1. The tuple \((T_{poly}^\bullet(A), d, \circ, [-,-]_G, \Delta)\) is a dGBV algebra. Namely, we have

\[
[X,Y]_G = (-1)^{\overline{X}}\Delta(X \circ Y) - (-1)^{\overline{X}}\Delta(X) \circ Y - X \circ \Delta(Y), \quad X,Y \in T_{poly}^\bullet(A).
\]

Proof. The equality

\[
i_{[X,Y]}_G v_1 = [i_X, L_Y] v_1
\]

\[
= -i_X [B, i_Y] v_1 + (-1)^{\overline{X}}(\overline{T}+1)[B, i_Y] i_X v_1
\]

\[
= -i_X Bi_Y v_1 + (-1)^{\overline{X}}(\overline{T}+1)Bi_Y i_X v_1 - (-1)^{\overline{X}}(\overline{T}+1)i_Y Bi_X v_1
\]

\[
= -i_{X \circ \Delta(Y)} v_1 + (-1)^{\overline{X}}(\overline{T}+1)i_{\Delta(Y \circ X)} v_1 + (-1)^{\overline{X}}(\overline{T}+1)i_Y \circ \Delta(X) v_1
\]

\[
= -i_{X \circ \Delta(Y)} v_1 + (-1)^{\overline{X}}i_{\Delta(X \circ Y)} v_1 - (-1)^{\overline{X}}i_{\Delta(X) \circ Y} v_1
\]

yields the statement. \( \Box \)

Denote by \( O_M \) the completed symmetric algebra \( k[[T^2 Jac(f_A)]] \) of \( T^2 Jac(f_A) \) and by \( m \) the maximal ideal in \( k[[T^2 Jac(f_A)]] \). Note that \( O_M \) is isomorphic to the completed symmetric algebra \( k[[HH^{**+2}(A)]] \) of \( HH^{**+2}(A) \) under Assumption 3.3.

Let \( t_1, \ldots, t_{t_A} \) be the dual coordinates for the basis \( \{v_1, \ldots, v_{t_A}\} \) as in the proof of Proposition 3.17. Denote by \( T_M \) a graded \( O_M \)-free module of derivations on \( O_M \), which satisfies \( T_M \cong \bigoplus_{i=1}^{t_A} O_M \partial/\partial t_i \).

Proposition 4.2. The dGBV algebra \((T_{poly}^\bullet(A), d, \circ, [-,-]_G, \Delta)\) is smooth formal. Namely, there exists a solution \( \gamma(t) \) to the Maurer–Cartan equation in formal power series with values in \( T_{poly}^\bullet(A) \),

\[
d\gamma(t) + \frac{1}{2} [\gamma(t), \gamma(t)]_G = 0, \quad \gamma(t) \in T_{poly}^\bullet(A) \hat{\otimes}_k m,
\]

satisfying the following properties:
(1) For $i = 1, \ldots, l_A$, the element
\[
\left[ \frac{\partial \gamma(t)}{\partial t_i} \right] \in T_{\text{poly}}^*(A) \hat{\otimes}_k \mathfrak{m} / T_{\text{poly}}^*(A) \hat{\otimes}_k \mathfrak{m}^2
\]
considered as an element in $T_{\text{poly}}^*(A)$ form a $k$-basis of $\text{Jac}(f_A)$.

(2) The solution $\gamma(t)$ is homogeneous in the sense that
\[
\gamma(t) = \sum_{i=1}^{l_A} (1 - q_i) t_i \frac{\partial \gamma(t)}{\partial t_i} + \deg_A, \gamma(t)]_G.
\]

**Proof.** We can apply the Terilla’s result, Theorem 2 in [26], since our dGBV algebra $(T_{\text{poly}}^*(A), d, \circ, [-, -]_G, \Delta)$ satisfies his “degeneration of the spectral sequence” condition due to the Hodge to de Rham degeneration of our filtered de Rham cohomology $H_{f_A}$ (Proposition 4.2). □

Let $\gamma(t)$ be as in Proposition 4.2. Define a formal power series $\mathfrak{F}_A$ with values in $T_{\text{poly}}^*(A)$ as
\[
\mathfrak{F}_A := f_A \otimes 1 + \gamma(t).
\]
Then it follows that $[\mathfrak{F}_A, \mathfrak{F}_A]_G = 0$ from the Maurer–Cartan equation (22). Define an $O_M$-endomorphism $d_\gamma$ on $T_{\text{poly}}^*(A) \hat{\otimes}_k O_M$ by
\[
d_\gamma X := [\mathfrak{F}_A, X]_G, \quad X \in T_{\text{poly}}^*(A) \hat{\otimes}_k O_M.
\]
It follows that $d_\gamma^2 = 0$ since $[\mathfrak{F}_A, \mathfrak{F}_A]_G = 0$.

**Proposition 4.3.** For all $X, Y \in T_{\text{poly}}^*(A) \hat{\otimes}_k O_M$, we have
\[
d_\gamma [X, Y]_G = [d_\gamma(X), Y]_G + (-1)^{\mathfrak{F}_A} [X, d_\gamma(Y)]_G,
\]
\[
d_\gamma (X \circ Y) = d_\gamma(X) \circ Y + (-1)^{\mathfrak{F}_A} X \circ d_\gamma(Y).
\]
Namely, the triple $(d_\gamma, \circ, [-, -]_G)$ equip $T_{\text{poly}}^*(A) \hat{\otimes}_k O_M$ with a structure of differential Gerstenhaber algebra.

**Proof.** The statement follows since the bracket $[-, -]_G$ satisfies the Jacobi and the Poisson identities. □

**Definition 4.4.** The graded $O_M$-module
\[
\text{Jac}(\mathfrak{F}_A) := H^*(T_{\text{poly}}^*(A) \hat{\otimes}_k O_M, d_\gamma),
\]
is called the Jacobian ring of $\mathfrak{F}_A$.

Note that the property $[\mathfrak{F}_A, \mathfrak{F}_A]_G = 0$ implies
\[
d_\gamma (\mathfrak{F}_A) = 0, \quad d_\gamma \left( \frac{\partial \mathfrak{F}_A}{\partial t_i} \right) = \left[ \mathfrak{F}_A, \frac{\partial \mathfrak{F}_A}{\partial t_i} \right]_G = 0, \quad i = 1, \ldots, l_A.
Proposition 4.5. The morphism \( \rho : \mathcal{T}_M \rightarrow \text{Jac}(\mathfrak{F}_A) \) of graded \( \mathcal{O}_M \)-modules defined by
\[
\rho : \mathcal{T}_M \rightarrow \text{Jac}(\mathfrak{F}_A), \quad \frac{\partial}{\partial t_i} \mapsto \left[ \frac{\partial \mathfrak{F}_A}{\partial t_i} \right], \quad i = 1, \ldots, l_A,
\]
is an isomorphism.

The isomorphism (27) enable us to introduce two particular elements of \( \mathcal{T}_M \) which play important roles later.

Definition 4.6. The element \( e \in \mathcal{T}_M \) such that \( \rho(e) = [1_A] \) is called the primitive vector field. The element \( E \in \mathcal{T}_M \) such that \( \rho(E) = [\mathfrak{F}_A] \) is called the Euler vector field.

The equation (23) in Proposition 4.2 implies the following.

Proposition 4.7. We have the “Euler’s identity”:
\[
\mathfrak{F}_A = E \mathfrak{F}_A + [\text{deg}_A, \mathfrak{F}_A]_G.
\]

Consider the \( \mathcal{O}_M \)-linear extensions of the morphisms \( i \) and \( L \) of graded \( k \)-modules defined in Proposition 2.11 and define a morphism \( d_{\gamma} \) of graded \( \mathcal{O}_M \)-modules on \( \Omega^\bullet(\mathfrak{F}_A) \otimes_k \mathcal{O}_M \) as
\[
d_{\gamma} := -L_{\mathfrak{F}_A},
\]
which is a deformation by \( \gamma \) of the boundary operator \( d \) on \( \Omega^\bullet(\mathfrak{F}_A) \). The equalities in Proposition 2.11 lead the following.

Proposition 4.8. We have
\[
d_{\gamma}^2 = 0, \quad [B, i_{\mathfrak{F}_A}] = -L_{\mathfrak{F}_A} = d_{\gamma}, \quad [B, d_{\gamma}] = 0,
\]
\[
[d_{\gamma}, i_X] = i_{d_{\gamma} X}, \quad X \in \mathcal{T}_{\text{poly}}^\bullet(\mathfrak{F}_A) \otimes_k \mathcal{O}_M.
\]
In particular, \( d_{\gamma} \) defines a boundary operator on \( \Omega^\bullet(\mathfrak{F}_A) \otimes_k \mathcal{O}_M \).

4.2. Deformed filtered de Rham cohomology \( \mathcal{H}_{\mathfrak{F}_A} \).

Definition 4.9. Let \( u \) be a formal variable of degree two. Define a graded \( k((u)) \otimes \mathcal{O}_M \)-module, called the deformed filtered de Rham cohomology, by
\[
\mathcal{H}_{\mathfrak{F}_A} := H^\bullet(T^{-w} \Omega^\bullet(\mathfrak{F}_A)((u)) \otimes_k \mathcal{O}_M, d_{\gamma} + uB)
\]
and for any integer \( p \in \mathbb{Z} \) the graded \( \mathcal{O}_M[[u]] \)-submodules of \( \mathcal{H}_{\mathfrak{F}_A} \)
\[
\mathcal{H}_{\mathfrak{F}_A}^{(p)} := H^\bullet(T^{-w} \Omega^\bullet(\mathfrak{F}_A)[[u]]u^p \otimes_k \mathcal{O}_M, d_{\gamma} + uB).
\]

Define a graded \( \mathcal{O}_M \)-module \( \Omega_{\mathfrak{F}_A} \) by
\[
\Omega_{\mathfrak{F}_A} := H^\bullet(T^{-w} \Omega^\bullet(\mathfrak{F}_A) \otimes_k \mathcal{O}_M, d_{\gamma}).
\]
Proposition 3.9 and Nakayama’s Lemma imply the following.

**Proposition 4.10.** For all \( p \in \mathbb{Z} \), there exists an exact sequence of graded \( \mathcal{O}_M \)-modules

\[
0 \rightarrow \mathcal{H}_{\delta_A}^{(-p-1)} \rightarrow \mathcal{H}_{\delta_A}^{(-p)} \rightarrow \Omega_{\delta_A} \rightarrow 0.
\]

4.3. Gauß–Manin connection on \( \mathcal{H}_{\delta_A} \). Set \( \mathcal{T}_{\delta_A} := \mathcal{O}_M[[u]] \xi_{\delta_A} \mathcal{T}_M \) and define a morphism of graded \( k \)-modules \( \nabla^\gamma : \mathcal{T}_{\delta_A} \otimes_k (\Omega_*(A) \hat{\otimes}_k \mathcal{O}_M((u))) \rightarrow \Omega_*(A) \hat{\otimes}_k \mathcal{O}_M((u)) \) by

\[
\nabla^\gamma_{\frac{du}{du}} := \frac{d}{du} - \frac{1}{u^2} i_{\delta_A}, \quad \nabla^\gamma_{\frac{\partial}{\partial t_i}} := \frac{\partial}{\partial t_i} + \frac{1}{u} i_{\delta_A}, \quad i = 1, \ldots, l_A.
\]

**Proposition 4.11.** The morphism of graded \( k \)-modules \( \nabla^\gamma \) is a flat connection which satisfies

\[
\left[ \nabla^\gamma_{\frac{du}{du}}, d_\gamma + uB \right] = d_\gamma + uB, \quad \left[ \nabla^\gamma_{\frac{\partial}{\partial t_i}}, d_\gamma + uB \right] = 0, \quad i = 1, \ldots, l_A.
\]

Therefore, \( \nabla^\gamma \) induces a connection on \( \mathcal{H}_{\delta_A} \). Moreover, we have

\[
\nabla^\gamma_{\frac{\partial}{\partial t_i}} \left( \mathcal{H}_{\delta_A}^{(-1)} \right) \subset \mathcal{H}_{\delta_A}^{(0)}, \quad i = 1, \ldots, l_A, \quad \nabla^\gamma_{\frac{du}{du}} E \left( \mathcal{H}_{\delta_A}^{(0)} \right) \subset \mathcal{H}_{\delta_A}^{(0)}.
\]

**Proof.** It is clear that \( \nabla^\gamma \) is a connection, which is is flat since

\[
\left[ \nabla^\gamma_{\frac{du}{du}}, \nabla^\gamma_{\frac{\partial}{\partial t_i}} \right] = \frac{1}{u^2} i_{\delta_A} - \frac{1}{u^2} i_{\delta_A} - \frac{1}{u} \left[ i_{\delta_A}, i_{\delta_A} \right] = 0,
\]

\[
\left[ \nabla^\gamma_{\frac{\partial}{\partial t_i}}, \nabla^\gamma_{\frac{\partial}{\partial t_j}} \right] = \frac{1}{u} i_{\delta_A} - (-1)^{(-1)}(-1)^1 \frac{1}{u} i_{\delta_A} + \frac{1}{u^2} \left[ i_{\delta_A}, i_{\delta_A} \right] = 0.
\]

The equalities in Proposition 4.8 yield

\[
\left[ \nabla^\gamma_{\frac{du}{du}}, d_\gamma + uB \right] = uB - \frac{1}{u} \left[ i_{\delta_A}, d_\gamma + uB \right] = uB - L_{\delta_A} = d_\gamma + uB,
\]

\[
\left[ \nabla^\gamma_{\frac{\partial}{\partial t_i}}, d_\gamma + uB \right] = - \frac{\partial}{\partial t_i} L_{\delta_A} - \frac{1}{u} \left[ i_{\delta_A}, L_{\delta_A} \right] + \left[ i_{\delta_A}, B \right] = 0.
\]

It is obvious from the definition that \( \nabla^\gamma_{\frac{\partial}{\partial t_i}} \left( \mathcal{H}_{\delta_A}^{(-1)} \right) \subset \mathcal{H}_{\delta_A}^{(0)}, \quad i = 1, \ldots, l_A. \) By Proposition 4.7 and the equalities in Proposition 4.8,

\[
u \frac{du}{du} = \frac{d}{du} + \frac{1}{u} i_{\delta_A} + E + \frac{1}{u} i_{E_{\delta_A}}
\]

\[
= \frac{d}{du} + E + \frac{1}{u} \left[ i_{\delta_{\bar{A}}}, L_{\delta_A} \right] = \frac{d}{du} + E + L_{\delta_{\bar{A}}} + \frac{1}{u} \left[ d_\gamma + uB, i_{\delta_{\bar{A}}} \right],
\]

which gives the last statement. \( \square \)
Definition 4.12. The connection $\nabla^\gamma$ on $\mathcal{H}_{\tilde{A}}$ is called the Gauß–Manin connection.

5. Primitive forms and Frobenius structures

Let $s^{(0)} : \Omega_{fA} \longrightarrow \mathcal{H}^{(0)}_{fA}$ be a very good section and let $\{v_1, \ldots, v_{lA}\}$ be elements of $\Omega_{fA}$ as in the proof of Proposition 3.17. Set $\zeta_i := s^{(0)}(v_i)$ for $i = 1, \ldots, l_A$.

5.1. Fundamental solution to the Gauß–Manin connection. By the equalities in Proposition 2.11 and Proposition 4.8 we obtain the following.

Proposition 5.1. For all $\omega \in T^{-\omega} \Omega_{f}(A)((u)) \widehat{\otimes}_k \mathcal{O}_M$, we have

$$(d \gamma + uB) \left( e^{-\frac{i \gamma(t)}{u}} \omega \right) = e^{-\frac{i \gamma(t)}{u}} (d + uB) \omega$$

Moreover, we have

$$\nabla^\gamma_{\frac{\partial}{\partial y}} \left[ e^{-\frac{i \gamma(t)}{u}} \zeta_i \right] = 0, \quad i, j = 1, \ldots, l_A,$$

$$\nabla^\gamma_{\frac{\partial}{\partial x}} \left[ e^{-\frac{i \gamma(t)}{u}} \zeta_i \right] = q_i \cdot \left[ e^{-\frac{i \gamma(t)}{u}} \zeta_i \right], \quad i = 1, \ldots, l_A,$$

where $\left[ e^{-\frac{i \gamma(t)}{u}} \zeta_i \right]$ denotes the equivalence class of $e^{-\frac{i \gamma(t)}{u}} \zeta_i$ in $\mathcal{H}_{\tilde{A}}$.

Therefore, we may identify $\mathcal{H}_{fA} \widehat{\otimes}_k \mathcal{O}_M$ with $\mathcal{H}_{\tilde{A}}$ as follows (cf. Proposition 1.3 in [25]).

Proposition 5.2. There is an isomorphism

$$\mathcal{J}_{\tilde{A}} : \mathcal{H}_{\tilde{A}} \xrightarrow{\cong} \mathcal{H}_{fA} \widehat{\otimes}_k \mathcal{O}_M, \quad \omega \mapsto e^{-\frac{i \gamma(t)}{u}} \omega,$$

which is compatible with $k((u)) \widehat{\otimes}_k \mathcal{O}_M$-module structures, Gauß–Manin connections and multiplications of $u$ on both sides.

This idea together with the equation (34) below is the essential part of the construction of primitive forms, which is implicit in M. Saito’s paper [24], later re-discovered by Barannikov [1, 2] and recently used by C. Li–S. Li–K. Saito [18] and M. Saito [25].

5.2. Primitive forms. Pull the $\mathcal{O}_M$-linear extension of the higher residue pairings $K_{IA}$ back to $\mathcal{H}_{\tilde{A}}$ by the isomorphism $\mathcal{J}_{\tilde{A}}$ and denote it by $K_{\tilde{A}}$. We can now introduce the notion of primitive forms.

Definition 5.3 (cf. K. Saito, Definition 3.1 in [21]). Let $r \in k$. An element $\zeta \in \mathcal{H}^{(0)}_{\tilde{A}}$ is called a formal primitive form with the minimal exponent $r$ for the tuple $(\mathcal{H}^{(0)}_{\tilde{A}}, \nabla^\gamma, K_{\tilde{A}})$ if it satisfies following five conditions;
(1) \( u \nabla^\gamma \zeta = \zeta \) and \( \zeta \) induces an \( \mathcal{O}_M \)-isomorphism:

\[
\mathcal{T}_M[[u]] \cong \mathcal{H}^{(0)}_{\mathcal{A}}, \quad \sum_{p=0}^\infty \delta_p u^p \mapsto \sum_{p=0}^\infty (u \nabla^\gamma_{\delta_p}) u^p.
\]

(P1)

(2) We have

\[
K_{\mathcal{A}}(u \nabla^\gamma \zeta, u \nabla^\gamma_{\delta'} \zeta) \in k \cdot u^w, \quad \delta, \delta' \in \mathcal{T}_M.
\]

(P2)

(3) We have

\[
\nabla^\gamma_{u \nabla^\gamma_{d \delta}} = r \zeta.
\]

(P3)

(4) There exists a connection \( \nabla \) on \( \mathcal{T}_M \) such that

\[
u^\gamma \nabla^\gamma \mathcal{X} \nabla^\gamma \mathcal{Y} \zeta = \nabla^\gamma \mathcal{X} \circ \mathcal{Y} \zeta + u \nabla^\gamma_{\mathcal{X} \mathcal{Y}} \zeta, \quad \mathcal{X}, \mathcal{Y} \in \mathcal{T}_M.
\]

(P4)

(5) There exists an \( \mathcal{O}_M \)-endomorphism \( N : \mathcal{T}_M \to \mathcal{T}_M \) such that

\[
u^\gamma (u \nabla^\gamma_{d \delta}) = -\nabla^\gamma_{E \circ \mathcal{X}} \zeta + u \nabla^\gamma_{\mathcal{X} \mathcal{Y}} \zeta, \quad \mathcal{X} \in \mathcal{T}_M.
\]

(P5)

We obtain a formal primitive form by applying a famous construction of primitive forms developed by M. Saito and Barannikov.

**Theorem 5.4.** There exists a formal primitive form \( \zeta \) with the minimal exponent zero for the tuple \((\mathcal{H}^{(0)}_{\mathcal{A}}, \nabla, K_{\mathcal{A}})\).

**Proof.** If a graded \( k \)-submodule \( S \) of \( \mathcal{H}_{\mathcal{A}} \) satisfies

\[
\mathcal{H}_{\mathcal{A}} = \mathcal{H}^{(0)}_{\mathcal{A}} \oplus S, \quad u^{-1} S \subset S, \quad \nabla_{u \nabla_{\delta^w}} S \subset S, \quad K_{\mathcal{A}}(S, S) \subset u^{w-2} \mathbb{C}[u^{-1}],
\]

then the unique element \( \zeta \in \mathcal{H}^{(0)}_{\mathcal{A}} \) such that

\[
J_{\mathcal{A}}(\zeta) = J_{\mathcal{A}}(\mathcal{H}^{(0)}_{\mathcal{A}}) \cap (\zeta_1 + S \otimes_k \mathcal{O}_M),
\]

becomes a formal primitive form, whose minimal exponent is zero due to the second property of Proposition 3.16. Proposition 3.19 shows that such an \( S \) exists. \( \square \)

### 5.3. Frobenius structures.

By a standard procedure, the existence of a formal primitive form implies the existence of a formal Frobenius structure (cf. Theorem 7.5 in [23], which can be generalized to graded cases verbatim).

**Theorem 5.5.** Let \( \zeta \) be a formal primitive form with the minimal exponent zero for the tuple \((\mathcal{H}^{(0)}_{\mathcal{A}}, \nabla, K_{\mathcal{A}})\). Define an \( \mathcal{O}_M \)-bilinear form \( \eta_{\otimes^2} : \mathcal{T}_M \otimes \mathcal{O}_M \mathcal{T}_M \to \mathcal{O}_M \) by

\[
\eta_{\otimes^2}(X, Y) := (-1)^{w^\mathcal{Y}} K_{\mathcal{A}}(u \nabla^\gamma_X \zeta, u \nabla^\gamma_Y \zeta).
\]

Then the tuple \((\circ, \eta_{\otimes^2}, e, E)\) gives a formal Frobenius structure on \( \mathcal{T}_M \) such that \( \text{Lie}_E(\circ) = \circ \) and \( \text{Lie}_E(\eta_{\otimes^2}) = (3 - w) \eta_{\otimes^2} \).
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