1. INTRODUCTION

Higher-order perturbation theories are topical subjects in recent researches on general relativity and they have very wide applications to cosmology and gravitational-wave physics. In cosmology, Planck mission revealed the precise map of the fluctuations of Cosmic Microwave Background (CMB) \cite{2} and the CMB observation is now regarded as a precise science. On the other hand, the direct observation of gravitational waves is accomplished in 2015 \cite{3} and we can expect that a future direction of gravitational-wave science is also precise science through the forthcoming data of many gravitational-wave events. In addition, some projects of space gravitational-wave antenna are also progressing \cite{4, 5}. Among them, the Extreme-Mass-Ratio-Inspiral (EMRI), which is a source of gravitational waves from the motion of a stellar mass object around a supermassive black hole, is a promising target of the Laser Interferometer Space Antenna \cite{6}. To describe the gravitational waves from EMRIs, higher-order black hole perturbation theories are required to support the gravitational-wave physics as a precise sciences.

In black hole perturbation theories, further sophistication is possible even in perturbation theories on the Schwarzschild background spacetime. There are many studies on the perturbations on the Schwarzschild background spacetime \cite{7, 8} from the works by Regge and Wheeler \cite{7} and Zerilli \cite{8}. In perturbation theories of the Schwarzschild spacetime, we may decompose the perturbations on this spacetime using the spherical harmonics $Y_{lm}$ and classify them into odd- and even-modes based on their parity, because the Schwarzschild spacetime has a spherical symmetry. However, monopole ($l = 0$) and dipole ($l = 1$) modes were separately treated and their "gauge-invariant" treatments was unknown.

In this situation, in Ref. \cite{9}, we proposed a gauge-invariant treatment of these modes and derived the solutions to the linearized Einstein equations for these modes. Since the obtained solutions in Ref. \cite{9} is physically reasonable, we may say that our proposal is also reasonable. In addition, owing to our proposal, the formulation of higher-order gauge-invariant perturbation theory discussed in \cite{11, 12, 13, 14} becomes applicable to any-order perturbations on the Schwarzschild background spacetime.

In this article, we carry out this application and derive the formal solutions of mass ($l = 0$ even mode), angular momentum ($l = 1$ odd mode), and dipole perturbations ($l = 1$ even mode) to any-order perturbations. We also emphasize that the proposal in Ref. \cite{9} is not only for the perturbations on the Schwarzschild background spacetime but also a clue to perturbation theories on a generic background spacetime such as cosmological perturbation theories \cite{15}.

The organization of this paper is as follows: In Sec. 2 we briefly review the framework of the general-relativistic higher-order gauge-invariant perturbation theory \cite{11, 12, 13}; In Sec. 3 we briefly explain the strategy for gauge-invariant treatments of $l = 0, 1$ modes in Ref. \cite{9} and summarize the $l = 0, 1$ mode solutions which was also derived in Ref. \cite{9}. In Sec. 4 we show the extension of the linear solutions for $l = 0, 1$ modes to any-order perturbations. Finally, in Sec. 5 we provide a brief summary of this paper. Throughout this paper, we use the unit $G = c = 1$, where $G$ is Newton’s constant of gravitation, and $c$ is the velocity of light.

2. GENERAL-RELATIVISTIC HIGHER-ORDER GAUGE-INARIANT PERTURBATION THEORY

General relativity is a theory based on general covariance, and that covariance is the reason that the notion of "gauge" has been introduced into the theory. In particular, in general relativistic perturbations, the second-kind gauge appears in perturbations, as Sachs pointed out \cite{16}. In general-relativistic perturbation theory, we usually treat the one-parameter family of spacetimes $\{M(\lambda, Q_0)\}$ for $\lambda \in [0, 1]$ to discuss differences between the background spacetime $\{M, Q_0\} = (M_{\lambda=0}, Q_{\lambda=0})$ and the physical spacetime $\{M_{\text{ph}}, Q\} = (M_{\lambda=1}, Q_{\lambda=1})$. Here, $\lambda$ is the infinitesimal parameter for perturbations, $M_{\lambda}$ is a spacetime manifold for each $\lambda$, and $Q_\lambda$ is the collection of the tensor fields on $M_{\lambda}$. Since each $M_{\lambda}$ is a different manifold, we have to introduce the point identification map $\mathcal{I}_\lambda : M_{\lambda} \to M_{\lambda}$ to compare the tensor field on different manifolds. This point-identification is the gauge choice of the second kind. Since we have no guiding principle by which to choose identification map $\mathcal{I}_\lambda$ due to the general covariance, we may choose a different point-identification $\mathcal{I}_\lambda$ from $\mathcal{I}_\lambda$. This degree of freedom in the gauge choice is the gauge degree of freedom of the second kind. The gauge-transformation of the second kind is a change in this identification map. We note that this second-kind gauge is a different notion of the degree of freedom of coordinate choices on a single manifold, which is called the gauge of the first kind \cite{15}. We have to emphasize that the "gauge" which is excluded in our gauge-invariant perturbation theory is not the gauge of the first kind but the gauge of the second kind. In this paper, we call the gauge of the second kind as gauge if there is no possibility of confusions.
Once we introduce the gauge choice \( \Phi_\lambda : \mathcal{M} \to \mathcal{M}_\lambda \), we can compare the tensor fields on different manifolds \( \{ \mathcal{M}_\lambda \} \), and perturbations of a tensor field \( Q_\lambda \) are represented by the difference \( \mathcal{D}_\lambda^* Q_\lambda - Q_0 \), where \( \mathcal{D}_\lambda^* \) is the pull-back induced by the gauge choice \( \Phi_\lambda \) and \( Q_0 \) is the background value of the variable \( Q_\lambda \). We note that this representation of perturbations completely depends on the gauge choice \( \mathcal{D}_\lambda^* \). If we change the gauge choice from \( \mathcal{D}_\lambda \) to \( \mathcal{D}_\lambda' \), the pulled-back variable of \( Q_\lambda \) is then represented by \( \mathcal{D}_\lambda' Q_\lambda \). This different representations are related to the gauge transformation rules as

\[
\mathcal{D}_\lambda^* Q_\lambda = \Phi_\lambda^* \mathcal{D}_\lambda' Q_\lambda, \quad \Phi_\lambda := \mathcal{D}_\lambda^{-1} \circ \mathcal{D}_\lambda^*.
\]  

(1)

\( \Phi_\lambda \) is a diffeomorphism on the background spacetime \( \mathcal{M} \).

In the perturbative approach, we treat the perturbation \( \mathcal{D}_\lambda^* Q_\lambda \) through the Taylor series with respect to the infinitesimal parameter \( \lambda \) as

\[
\mathcal{D}_\lambda^* Q_\lambda =: \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathcal{Y}_n Q + O(\lambda^{k+1}),
\]  

(2)

where \( \mathcal{Y}_n Q \) is the representation associated with the gauge choice \( \mathcal{D}_\lambda^* \) of the \( k \)-th order perturbation of the variable \( Q_\lambda \) with its background value \( \mathcal{Y}_0 Q = Q_0 \). Similarly, we can have the representation of the perturbation of the variable \( Q_\lambda \) under the gauge choice \( \mathcal{D}_\lambda' \), which is different from \( \mathcal{D}_\lambda \) as mentioned above. Since these different representations are related to the gauge transformation rule (1), the order-by-order gauge-transformation rule between \( n \)-th order perturbations \( \mathcal{Y}_n Q \) and \( \mathcal{Y}_n^\prime Q \) is given from the Taylor expansion of the gauge-transformation rule (1).

Since \( \Phi_\lambda \) is constructed by the product of diffeomorphisms, \( \Phi_\lambda \) is not given by an exponential map \([10, 16, 17]\). Since the definitions of gauge-invariant variables for perturbations of an arbitrary tensor field are trivial if we can accomplish the separation of the metric perturbations into their gauge-invariant and gauge-variant parts, we may concentrate on the metric perturbations, at first.

We consider the metric \( \mathcal{g}_{ab} \) on the physical spacetime \( (\mathcal{M}_{ph}, \mathcal{Q}) = (\mathcal{M}_{\lambda=1}, \mathcal{Q}_{\lambda=1}) \), and we expand the pulled-back metric \( \mathcal{D}_\lambda^* \mathcal{g}_{ab} \) to the background spacetime \( \mathcal{M} \) through a gauge choice \( \mathcal{D}_\lambda \) as

\[
\mathcal{D}_\lambda \mathcal{g}_{ab} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathcal{g}_{ab} + O(\lambda^{k+1}),
\]  

(5)

where \( \mathcal{g}_{ab} := \mathcal{Y}_0 \mathcal{g}_{ab} \) is the metric on the background spacetime \( \mathcal{M} \). The expansion (5) of the metric depends entirely on the gauge choice \( \mathcal{D}_\lambda \). Nevertheless, henceforth, we do not explicitly express the index of the gauge choice \( \mathcal{D}_\lambda \) if there is no possibility of confusion. In \([14, 11]\), we proposed a procedure to construct gauge-invariant variables for higher-order perturbations. Our starting point to construct gauge-invariant variables was the following conjecture for the linear metric perturbation \( h_{ab} := \mathcal{Y}_1 \mathcal{g}_{ab} \):

**Conjecture 2.1.** If the gauge-transformation rule for a tensor field \( h_{ab} \) is given by \( \mathcal{g} h_{ab} - \mathcal{Q} = \mathcal{E}_1^* \mathcal{g}_{ab} \) with the background metric \( \mathcal{g}_{ab} \), there then exist a tensor field \( \mathcal{F}_{ab} \) and a vector field \( Y^a \) such that \( h_{ab} \) is decomposed as \( h_{ab} := \mathcal{F}_{ab} + \mathcal{E}_1 Y_{ab} \), where \( \mathcal{F}_{ab} \) and \( Y^a \) are transformed into \( \mathcal{g} \mathcal{F}_{ab} - \mathcal{Q} \mathcal{F}_{ab} = 0 \) and \( \mathcal{g} Y^a - \mathcal{Q} Y^a = \mathcal{E}_1 Y^a \) under the gauge transformation, respectively.

We call \( \mathcal{F}_{ab} \) and \( Y^a \) as the gauge-invariant and gauge-variant parts of \( h_{ab} \), respectively.

Based on Conjecture 2.1 in \([14]\), we found that the \( n \)-th order metric perturbation \( \mathcal{Y}_n \mathcal{g}_{ab} \) is decomposed into its gauge-invariant and gauge-variant parts as

\[
\mathcal{Y}_n \mathcal{g}_{ab} = (\mathcal{Y}_n \mathcal{g}_{ab}) - \mathcal{Y}_n \mathcal{g}_{ab} = \mathcal{F}_{ab} + \mathcal{E}_n Y_{ab} \]  

(6)

Furthermore, through the gauge-variant variables \( ^{(i)} Y^a \) \( (i = 1, \ldots, n) \), we also found the definition of the gauge-invariant variable \( ^{(n)} \mathcal{Q} \) for the \( n \)-th order perturbation \( ^{(n)} \mathcal{Q} \) of an arbitrary tensor field \( Q \). This definition of the gauge-invariant variable \( ^{(n)} \mathcal{Q} \) implies that the \( n \)-th order perturbation \( ^{(n)} \mathcal{Q} \) of any tensor field \( Q \) is always decomposed into its gauge-invariant part \( ^{(n)} \mathcal{Q} \) and gauge-variant part as

\[
^{(n)} Q = ^{(n)} \mathcal{Q} - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j \} \in \mathcal{J}_l} C_{l,\{j \}} \mathcal{E}_l^1 Y_{i_1^l} \cdots \mathcal{E}_l^{j_l} Y_{i_l^l} \]  

(7)

1Precisely speaking, to reach to the decomposition formula (6) we have to confirm Conjecture 4.1 in Ref. \([13]\) in addition to Conjecture 2.1.
As an example, the perturbative expansion of the Einstein tensor and the energy-momentum tensor, which are pulled back through the gauge choice $\mathcal{X}_\lambda$, are given by
\begin{align}
\mathcal{X}_\lambda^a G^b_a &= \sum_{n=0}^k \frac{\lambda^n}{n!} G^b_a + O(\lambda^{k+1}), \quad (8) \\
\mathcal{X}_\lambda^a T^b_a &= \sum_{n=0}^k \frac{\lambda^n}{n!} T^b_a + O(\lambda^{k+1}). \quad (9)
\end{align}

Then, the $n$th-order perturbation $(n)^a G^b_a$ of the Einstein tensor and the $n$th-order perturbation $(n)^a T^b_a$ of the energy-momentum tensor are also decomposed as
\begin{align}
(n)^a G^b_a &= (n)^a \mathcal{X}_\lambda^b \\
(n)^a T^b_a &= (n)^a \mathcal{X}_\lambda^b
\end{align}

Through the lower-order Einstein equation $^k G^b_a = 8\pi^k T^b_a$ with $k \leq n - 1$, the $n$th-order Einstein equation $(n)^a G^b_a = 8\pi^a T^b_a$ is automatically given in the gauge-invariant form
\begin{align}
(n)^a G^b_a &= 8\pi^a \mathcal{X}_\lambda^b. \quad (12)
\end{align}

Here, we note that the $n$th-order perturbation of the Einstein tensor is given in the form
\begin{align}
(n)^a G^b_a &= (1)^a \mathcal{X}_\lambda^b + (n)^a \mathcal{X}_\lambda^b \left( \left( \frac{1}{n!} \mathcal{X}_\lambda^b \right) i < n \right), \quad (13)
\end{align}

where $(1)^a G^b_a$ is the gauge-invariant part of the linear-order perturbation of the Einstein tensor. Explicitly, $(1)^a G^b_a [A]$ for an arbitrary tensor field $A_{ab}$ of the second rank is given by
\begin{align}
(1)^a G^b_a [A] &= \left( \frac{1}{2} \right) \Sigma^b A^c_a - \frac{1}{2} \delta^b_a (1)^a G^c [A], \quad (14)
\end{align}

\begin{align}
\Sigma^b A^c_a &= -2 \mathcal{V}_{d}H_{a}^{d}b^{a}A - A^{a b}R_{a b c}, \quad (15)
\end{align}

\begin{align}
H_{a b c} &= \nabla_{a} A_{b c} - \frac{1}{2} \nabla_{c} A_{a b}. \quad (16)
\end{align}

As derived in [11], when the background Einstein tensor vanishes, we obtain the identity
\begin{align}
\nabla_{a} (1)^b_a [A] &= 0
\end{align}

for an arbitrary tensor field $A_{ab}$ of the second rank.

Thus, we emphasize that Conjecture [2] was the important premise of the above framework of the higher-order perturbation theory.

### 3. LINEAR PERTURBATIONS ON THE SCHWARZSCHILD BACKGROUND SPACETIME

We use the 2+2 formulation [6] of the perturbations on spherically symmetric background spacetimes. The topological space of spherically symmetric spacetimes is the direct product $\mathcal{M} = \mathcal{M}_1 \times S^2$, and the metric on this spacetime is
\begin{align}
g_{ab} &= \gamma_{ab} + \tilde{\gamma}_{ab}, \quad (18)
y_{ab} &= \gamma_{AB}(dx^A)_a(dx^B)_b, \quad (19)
\gamma_{ab} &= -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b, \quad (20)
\tilde{\gamma}_{ab} &= (d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b. \quad (21)
\end{align}

On this background spacetime, the components of the metric perturbation as
\begin{align}
h_{ab} &= h_{AB}(dx^A)_a(dx^B)_b, \quad (22)
\end{align}

In Ref. [9], we proposed the decomposition of these components as
\begin{align}
\lambda_{AB} &= \sum_{l, m} \tilde{h}_{AB} S_{l m}, \quad (23)
\lambda_{AP} &= \sum_{l} \tilde{h}_{AP} S_1, \quad (24)
\lambda_{pq} &= \sum_{l} \tilde{h}_{pq} S_2, \quad (25)
\end{align}

where $\tilde{h}_{AP}$ is the covariant derivative associated with the metric $\gamma_{pq}$ on $S^2$, $\mathcal{D} := \mathcal{D}_{\rho} = \mathcal{A}^\rho d_\rho$, and $\mathcal{A}^\rho = \mathcal{A}_{\rho | \sigma}$ is the totally antisymmetric tensor on $S^2$.

Note that the decomposition [24]–[26] implicitly state that the Green functions of the derivative operators $\Lambda := \mathcal{D}^\rho \mathcal{D}_\rho$ and $\Lambda + 2 := \mathcal{D}^\rho \mathcal{D}_{\rho} + 2$ should exist if the one-to-one correspondence between $\{ \tilde{h}_{AP}, \tilde{h}_{pq} \}$ and $\{ \tilde{h}_{(1)}, \tilde{h}_{(2)} \}$ is guaranteed. Because the eigenvalue of the derivative operator $\Lambda$ on $S^2$ is $-l(l+1)$, the kernels of the operators $\Lambda$ and $\Lambda + 2$ are $l = 0$ and $l = 1$ modes, respectively. Thus, the one-to-one correspondence between $\{ \tilde{h}_{AP}, \tilde{h}_{pq} \}$ and $\{ \tilde{h}_{(1)}, \tilde{h}_{(2)} \}$ is lost for $l = 0, 1$ modes in decomposition formulae [24]–[25] with $S_1 = S_{\gamma \gamma}$. To recover this one-to-one correspondence, in Ref. [9], we introduced the mode functions $k_{(\Lambda)}$ and $k_{(\Lambda + 2)}$ instead of $Y_{10}$ and $Y_{1m}$, respectively, and consider the scalar harmonic function
\begin{align}
S_\Delta &= \left\{ \begin{array}{ll}
y_{lm} & \text{for } l \geq 2; \\
k_{(\Lambda + 2)}^{(2)} & \text{for } l = 1; \\
k_{(\Lambda)} & \text{for } l = 0.
\end{array} \right. \quad (27)
\end{align}

As the explicit functions of $k_{(\Lambda)}$ and $k_{(\Lambda + 2)}$, we employ
\begin{align}
k_{(\Lambda)} &= 1 + \delta \ln \left( \frac{1 - \tfrac{1}{2} (1 + z)^{1/2} + \tfrac{1}{2} (1 - z)^{1/2}}{1 + z} \right), \quad \delta \in \mathbb{R}, \quad (28)
k_{(\Lambda + 2)}^{(2)} &= z \left( 1 + \delta \left( \frac{1}{2} \ln \frac{1 + z}{1 - z} + \frac{1}{2} \sin \theta \right) \right), \quad (29)
k_{(\Lambda + 2)}^{(2)} &= \left( 1 - z^2 \right)^{1/2} \left( 1 + \delta \left( \frac{1}{2} \ln \frac{1 + z}{1 - z} + \frac{1}{2} \sin \theta \right) \right) e^{i \phi}. \quad (30)
\end{align}
where \( \theta = \cos \theta \). This choice guarantees the linear-independence of the set of the harmonic functions

\[
\begin{align*}
\left\{ S_\delta, D_\rho S_\delta, \varepsilon_{pq} \nabla^p S_\delta, \frac{1}{2} \varepsilon_{pq} \nabla^p S_\delta \right\}, \quad \left( D_\rho D_\lambda - \frac{1}{2} \varepsilon_{pq} \nabla^p D_\lambda \right) S_\delta, 2 \varepsilon_{(\lambda \mu) \delta} \nabla^\mu S_\delta \right\}
\end{align*}
\]

(31)

including \( l = 0, 1 \) modes if \( \delta \neq 0 \), but is singular if \( \delta = 0 \). When \( \delta = 0 \), we have \( k_{(\lambda)} \propto Y_0 \) and \( k_{(\lambda+2)m} \propto Y_{1m} \).

Using the above harmonics functions \( S_\delta \) in Eq. (27), in Ref. [9], we proposed the following strategy:

**Proposal 3.1.** We decompose the metric perturbations \( h_{ab} \) on the background spacetime with the metric (13) through Eqs. (24)–(26) with the harmonic functions \( S_\delta \) given by Eq. (27). Then, Eqs. (24)–(26) become invertible with the inclusion of \( l = 0, 1 \) modes. After deriving the field equations such as linearized Einstein equations using the harmonic function \( S_\delta \), we choose \( \delta = 0 \) when we solve these field equations as the regularity of the solutions.

Through this strategy, we can construct gauge-invariant variables and evaluate field equations through the mode-by-mode analyses without special treatments for \( l = 0, 1 \) modes.

Once we accept Proposal 3.1, we reach to the following statement [9]:

**Theorem 3.1.** If the gauge-transformation rule for a tensor field \( h_{ab} \) is given by \( \tilde{h}_{ab} = \gamma_{ab} h_{ab} = \xi_{ab} \gamma_{ab} \). Here, \( \gamma_{ab} \) is the background metric with the spherical symmetry. Then, there exist a field \( \mathcal{F}_{ab} \) and a vector field \( \mathcal{Y}^a \) such that \( h_{ab} \) is decomposed as \( h_{ab} = : \mathcal{F}_{ab} + \xi_{ab} \mathcal{Y}^a \), where \( \mathcal{F}_{ab} \) and \( \mathcal{Y}^a \) are transformed as \( \gamma_{ab} \mathcal{F}_{ab} = \mathcal{F}_{ab} = 0 \), \( \gamma_{ab} \mathcal{Y}^a = \xi^a \) under the gauge transformation.

Owing to Theorem 3.1, the above general arguments in our gauge-invariant perturbation theory are applicable to perturbations on the Schwarzschild background spacetime including \( l = 0, 1 \) mode perturbations. Furthermore, we derived the \( l = 0, 1 \) solution to the linearized Einstein equation in the gauge-invariant manner [9].

As shown in Eq. (12), the linearized Einstein equation (1) \( G_{ab}^0 = 8\pi \varepsilon_{ab} \) for the linear metric perturbation \( h_{ab} = \mathcal{F}_{ab} + \xi_{ab} \mathcal{Y}^a \) with the vacuum background Einstein equation \( G_{ab} = 8\pi \varepsilon_{ab} \) is given by

\[
(\varepsilon h_{ab}) \mathcal{F}_{ab} = 8\pi (\varepsilon \mathcal{Y}^a)_a.
\]

(32)

Since we consider the vacuum background spacetime \( T_{ab} = 0 \), the linear-order perturbation of the continuity equation of the linear perturbation of the energy-momentum tensor is given by

\[
\nabla \varepsilon \mathcal{Y}^a_0 = 0.
\]

(33)

We decompose the components of the linear perturbation of (1) \( \mathcal{F}_{ab} \) as

\[
(\varepsilon h_{ab}) \mathcal{F}_{ab} = \sum \nabla \varepsilon S_\delta (d\mathcal{X})_a (d\mathcal{X})_b.
\]

\[
+2 \varepsilon \sum_{l,m} \left\{ T_{(1)l} S_\delta + T_{(1)a1} \varepsilon_{pq} \nabla^p S_\delta \right\} (d\mathcal{X})_a (d\mathcal{X})_b + \varepsilon_{pq} \nabla^p S_\delta \right\} (d\mathcal{X})_a (d\mathcal{X})_b.
\]

(34)

We also derive the continuity equations (33) in terms of these mode coefficients and use these equations when we solve the linearized Einstein equation.

Furthermore, we derived the solutions to the Einstein equation for \( l = 0, 1 \) mode imposing the regularity of the harmonics \( S_\delta \) through \( \delta = 0 \). For this reason, we may choose \( T_{(1)l} = T_{(1)a1} = 0 \) for \( l = 0, 1 \) modes. In addition, we may also choose \( T_{(2)l} = T_{(2)a1} = 0 \) for \( l = 0, 0 \) modes due to the same reason. This choice and a component of Eq. (33) leads \( T_{(2)l} = 0 \) for \( l = 0 \) mode.

Through the above premise, in Ref. [9], we derived the \( l = 0, 1 \)-mode solutions to the linearized Einstein equations as follows:

\[
V_{(1,0)l} = \left( \beta_l (t) + W_{(1,0)} (r, t) \right)^2 \sin^2 \theta (d\theta)_a.
\]

(36)

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Through the above premise, in Ref. [9], we derived the \( l = 0, 1 \)-mode solutions to the linearized Einstein equations as follows:

\[
2 \varepsilon \mathcal{F}_{ab} = V_{(1,a1)} \mathcal{Y}_a \mathcal{Y}_b + \varepsilon_{ab} \mathcal{Y}^a \mathcal{Y}^b + \varepsilon_{ab} \mathcal{Y}^a \mathcal{Y}^b.
\]

(35)

where the generator \( V_{(1,a1)} \mathcal{Y}_a \mathcal{Y}_b \) in Eq. (35) is

\[
V_{(1,a1)} = \left( \beta_l (r) + W_{(1,0)} (r, t) \right)^2 \sin^2 \theta (d\theta)_a.
\]

(36)

Here, \( \beta_l (t) \) is an arbitrary function of \( t \). The function \( a_l (t, r) \) is given by as the solutions to the linear-order Einstein equation (32) as follows:

\[
a_l (t, r) = \frac{16\pi}{3M} r^3 \int dt T_{(2)l+r} + a_{10} = \frac{16\pi}{3M} \int dr^3 \frac{1}{T_{(1)a1}} + a_{10}.
\]

(37)

For the \( l = 0 \) even-mode perturbation, we should have

\[
(\varepsilon h_{ab}) \mathcal{F}_{ab} = \frac{2}{r} \left( M_1 + 4\pi \frac{f}{r} \int dr \left( \sin^2 \theta \right) \mathcal{F}_{ab} \right. \times \left( (dt)_a (dt)_b \right) + \frac{1}{r^2} (dr)_a (dr)_b.
\]

(38)

where \( M_1 \) is the constant of integration which corresponds to the Kerr parameter perturbation. Furthermore \( r f \partial_r W_{(1,0)} \) of the variable \( W_{(1,0)} \) in Eq. (36) is determined the evolution equation

\[
\partial_t \left( r f \partial_r W_{(1,0)} \right) - f \partial_r \left( r f \partial_t W_{(1,0)} \right) + \frac{1}{r^2} \left[ 3 f - 1 \right] (rf \partial_r W_{(1,0)}) = 16\pi f^2 T_{(2)r}.
\]

(39)

where \( M_1 \) is the linear-order Schwarzschild mass parameter perturbation, \( \gamma (r) \) is an arbitrary function of \( r \). Here, the generator \( V_{(1,a1)} \mathcal{Y}_a \mathcal{Y}_b \) in the term \( V_{(1,a1)} \mathcal{Y}_a \mathcal{Y}_b \) in Eq. (35) is given by

\[
V_{(1,a1)} = \frac{1}{4} f Y_1 + \frac{1}{4} r^2 f \partial_r Y_1 + \gamma (r) \left( dt \right)_a + \frac{1}{4} r^2 f \partial_r Y_1 \left( dr \right)_a.
\]

(40)

In the generator (40), \( \gamma (r) \) satisfies the following equation:

\[
- \frac{1}{r^2} \frac{d^2}{dr^2} Y_1 + \partial_r (f \partial_r F) + \frac{1}{r^2} 3 (1 - f) F = - \frac{8}{r^2} m_1 (t, r) + 16\pi \left[ - \frac{1}{f} T_{rr} + f T_{rr} \right].
\]

(41)
where

\[ m_1(t, r) = 4\pi \int dr \left( \frac{r^2}{f} \Phi \right) + M_1 = 4\pi \int dr \left( \frac{r^2}{f} T_{tt} \right) + M_1, \quad M_1 \in \mathbb{R}. \] (42)

For the \( l = 1, m = 0 \) even-mode perturbation, we should have

\[ (1)_{\mathcal{F}_{ab}} = -\frac{16\pi^2 r^2}{3(1-f)} \left[ \frac{1 + f}{2} T_{rr} + r f \partial_r T_{rr} - T_{(e)0} \right] \]

\[ + 16\pi^2 \left\{ T_{tt} - \frac{2r}{3} \partial_r T_{tt} \right\} \]

\[ \times \cos \theta \left( \frac{dt_a}{dr} d_T^b \right) \]

\[ + \frac{8\pi^2}{3(1-f)} \left[ T_{tt} - \frac{2rf}{3(1-f)} \partial_T^t \right] \]

\[ \times \cos \theta \left( \frac{dr_a}{dr} d_T^b \right) \]

\[ - \frac{16\pi^2}{3(1-f)} T_{tt} \cos \theta \gamma_{ab} + \delta V_{(\theta, b)} g_{ab}, \] (43)

\[ V_{(l, e, a)} = -r \partial_r \Phi \cos \theta (dt_a), \]

\[ \Phi \cos \theta (dr_a), \]

\[ -r \theta \sin \theta (dt_a), \]

\[ \Phi \cos \theta (dr_a), \] (44)

where \( \Phi \) satisfies the following equation

\[ -\frac{1}{f} \partial_t^2 \Phi + \partial_t \left[ f \partial_r \Phi \right] - \frac{1}{r^2} \Phi = \frac{16\pi^2}{3(1-f)} S_\Phi(t), \]

\[ S_\Phi(t) := \frac{3(1-f)}{4r} T_{tt} - \frac{1}{2} \partial_r T_{tt} + \frac{1}{4} \frac{f}{2} \partial_r T_{rr} + \frac{1}{2} \frac{f}{2} \partial_T^t \]

\[ - \frac{f}{2} T_{(e)0} - 2f T_{(e)1}. \] (45)

4. EXTENSION TO THE HIGHER-ORDER PERTURBATIONS

As reviewed in Sec. 2, the \( n \)-th order perturbation of the Einstein equation is given in the gauge-invariant form. We may write this \( n \)-th order Einstein equation (12) as follows:

\[ (1) \mathcal{F}_{ab} \left[ n \right] \mathcal{F}_{ab} = \left( \text{NL} \right) \mathcal{F}_{ab} \left[ n \right] \mathcal{F}_{ab} \left[ i < n \right] + 8\pi \mathcal{F}_{ab} \left[ n \right] \mathcal{F}_{ab} \left[ n \right] = 8\pi \mathcal{F}_{ab} \left[ n \right] \mathcal{F}_{ab} \left[ n \right]. \] (46)

Here, the left-hand side in Eq. (46) is the linear term of \( \mathcal{N} \mathcal{F}_{ab} \) and the first term in the right-hand side is the non-linear term consists of the lower-order metric perturbation \( (i) \mathcal{F}_{ab} \) with \( i < n \). The right-hand side \( 8\pi \mathcal{F}_{ab} \left[ n \right] \mathcal{F}_{ab} \left[ n \right] \) of Eq. (46) is regarded an effective energy-momentum tensor for the \( n \)-th order metric perturbation \( \mathcal{F}_{ab} \).

The vacuum background condition \( \Phi \mid_{n} = 0 \) implies the mathematical identity (17), and Eq. (46) implies

\[ \nabla \mathcal{F}_{ab} = 0. \] (47)

This equation gives consistency relations which should be confirmed in concrete physical situations. The first term in the right-hand side in Eq. (46) does not contain \( \mathcal{F}_{ab} \). The \( n \)-th order perturbation \( \mathcal{F}_{ab} \) does not contain \( \mathcal{F}_{ab} \), neither, because our background spacetime is vacuum. Then, \( \mathcal{F}_{ab} \) does not include \( \mathcal{F}_{ab} \). This situation is same as that we used when we solved the linear-order Einstein equation (32) with the linear perturbation (33) of the continuity equation of the energy-momentum in Ref. [9]. Furthermore, we decompose the tensor \( \mathcal{F}_{ab} \) as follows:

\[ (1) \mathcal{F}_{ab} = \sum_{l,m} T_{AB} S_{\delta} (dx_a)(dx_b)^b, \]

\[ + 2r \sum_{l,m} \left\{ \tilde{T}_{(e)A} \frac{1}{2} \partial_T^t + \tilde{T}_{(e)} \right\} (dx_a)(dx_b)^b \]

\[ - \tilde{T}_{(e)} \delta_{ab}, \] (48)

Then, the replacements

\[ T_{AB} \rightarrow \tilde{T}_{AB}, \] (49)

\[ T_{(e)A} \rightarrow \tilde{T}_{(e)A}, \]

\[ T_{(e)} \rightarrow \tilde{T}_{(e)} , \]

\[ T_{(e)} \rightarrow \tilde{T}_{(e)}, \]

\[ \tilde{T}_{(e)} \rightarrow \tilde{T}_{(e)}, \] (49)

Then, following the strategy as Proposal 3.1 and the results derived in Ref. [9], the \( l = 0, 1 \)-mode solutions to Eq. (46) are summarized as follows:

For \( l = 1 \) in 0-th mode perturbations, we should have

\[ 2n \mathcal{F}_{ab} (dx_a)(dx_b)^b \]

\[ \left( 6 M^2 \sum_{l,m} \int dt \frac{1}{T} a_0(t, r) \right) \sin^2 \theta (dt_a)(dx_b)^b + E \mathcal{F}_{ab} \in [a] \] (50)

where the generator \( V_{(n, 0)} \) of the term \( \mathcal{F}_{ab} \) in Eq. (50) is

\[ V_{(n, 0)} = \left( \beta_0(t) + W_{(n, 0)}(t, r) \right) r^2 \sin^2 \theta (dt_a). \] (51)

Here, \( \beta_0(t) \) is an arbitrary function of \( t \). The function \( a_0(t, r) \) is given by as the solutions to the \( n \)-th order Einstein equation (45) as follows:

\[ a_0(t, r) = -\frac{16\pi^2}{3M^2} \sum_{l,m} \int dt \frac{1}{T} a_0(t, r) + a_0 \]

\[ = -\frac{16\pi^2}{3M^2} \sum_{l,m} \int dt \frac{1}{T} a_0(t, r) + a_0. \] (52)

where \( a_0 \) is the constant of integration which corresponds to the Kerr parameter perturbation. Furthermore \( r f \partial_r W_{(n, 0)} \) of the variable \( W_{(n, 0)} \) in Eq. (51) is determined the evolution equation

\[ \partial_t^2 \left( r f \partial_r W_{(n, 0)} \right) - \partial_t \left( r f \partial_r W_{(n, 0)} \right) \]

\[ + \frac{1}{f^2} \sum_{l,m} \left( \beta_0(t) + W_{(n, 0)}(t, r) \right) = 16\pi^2 f \tilde{T}_{(e)} \times \] (53)

For the \( l = 0 \) even-mode perturbation, we should have

\[ (n) \mathcal{F}_{ab} = \left( \frac{2}{r} M_{ab} + 4 \pi \int dr \left[ \left( \frac{2}{f} (n) \mathcal{F}_{ab} \right) \right] + 2 \left[ 4\pi \int dr \left( \frac{2}{f} (n) \mathcal{F}_{ab} \right) \right] (dt_a)(dx_b)^b + E \mathcal{F}_{ab} \in [a] \] (54)
where $M_n$ is the $n$th-order Schwarzschild mass parameter perturbation, $\gamma_n(r)$ is an arbitrary function of $r$. Here, the generator $V_{(n,0)}$ of the term $\xi_{(n,0)}$ is given by

$$V_{(n,0)} = \left( \frac{1}{4} f Y_n + \frac{1}{4} \frac{df}{dr} Y_n + \gamma_n(r) \right) (dt)_a + \frac{1}{4f} \frac{df}{dr} Y_n (dr)_a.$$  

(55)

In the generator (55), $(n)F := \partial_r Y_n$ satisfies the following equation:

$$- \frac{1}{f} \frac{\partial^2_t (n)F}{f} + \partial_r (f \partial_r (n)F) + \frac{1}{r^2} (1 - f)^3 \frac{\partial_t (n)F}{r^2} = - \frac{8}{r^3} m_n(t, r) + 16\pi \left[ - \frac{1}{f} (n)T_{tt} + f^{(n)}T_{tt} \right],$$  

(56)

where

$$m_n(t, r) = 4\pi \int dr \left[ \frac{r}{f} (n)T_{tt} \right] + M_n = 4\pi \int dt \left[ r f (n)T_{tt} \right] + M_n, \quad M_n \in \mathbb{R}.$$  

(57)

For the $l = 1, m = 0$ even-mode perturbation, we should have

$$\langle n \rangle_{T_{ab}} = - \frac{16\pi r^2 f^2}{3(1 - f)} \left[ \frac{1 + f}{2} (n)T_{rr} + rf \partial_r (n)T_{rr} - (n)T_{(e)} \right]$$

$$- 4\langle n \rangle_{T_{(e)}} \cos \theta (dt)_a (dt)_b$$

$$+ 16\pi r^2 \left[ (n)T_{rr} - \frac{2r}{3} \frac{\partial_t (n)T_{tt}}{f^2 (1 - f)} \right]$$

$$\times \cos \theta (dt)_a (dr)_b$$

$$+ \frac{8\pi r^2 (1 - 3f)}{f^2 (1 - f)} \left[ (n)T_{tt} - \frac{2rf}{3} \frac{\partial_t (n)T_{tt}}{f^2 (1 - f)} \right]$$

$$\times \cos \theta (dr)_a (dr)_b$$

$$- 16\pi r^4 \langle n \rangle_{T_{tt}} \cos \theta_{\gamma_{ab}} + \xi_{(n,1)} \lambda_{ab},$$  

(58)

$$V_{(n,1)} = \left[ \frac{r}{f} (n)T_{tt} + f^2 (n)T_{tt} \right] + \cos \theta (dt)_a$$

$$+ \left[ \Phi_{(n,1)} - \frac{r}{f} \Phi_{(n,1)} \right] \cos \theta (dr)_a$$

$$- \Phi_{(n,1)} \sin \theta (d\theta)_a.$$  

(59)

These are the main assertion of this article.

5. SUMMARY

In summary, we extended the angular-mode solution of the mass perturbation ($l = 0$ even mode), the angular-momentum perturbation ($l = 1$ odd mode), and the dipole perturbation ($l = 1$ even mode) to the any-order formal solutions. Our logic starts from the complete proof of Conjecture 2.1 for perturbations on the Schwarzschild background spacetime. The remaining problem in Conjecture 2.1 was in the treatment of $l = 0,1$ modes of the perturbations on the Schwarzschild background spacetime. To resolve this problem, in Ref. [9], we introduced the harmonic functions $S_\delta$ defined by Eq. (27) instead of the conventional harmonic function $Y_{lm}$ and proposed Proposal 3.1 as a strategy of a gauge-invariant treatment of the $l = 0,1$ perturbations on the Schwarzschild background spacetime. Once we accept this proposal, we reach to Theorem 5.1 and we can apply our general arguments of higher-order perturbation theory developed in Refs. [10, 11, 12, 13] to perturbations on the Schwarzschild background spacetime.

In Ref. [9], we derived the $l = 0$ solutions (55–59) to the linearized Einstein equations following Proposal 3.1. The premise and conditions for any-order perturbations are same as those for the linear perturbations. Then, we reached to the formal solutions (50)–(59) for the any-order non-linear perturbation by the replacements (49).

Of course, the solutions derived here is just formal one and we have to evaluate the non-linear terms in the effective energy-momentum tensor $(n)\mathcal{E}_{ab}$, i.e., $(n)\mathcal{E}_{ab} \equiv \{(n)\mathcal{E}_{ab}[l < n]\}$ and $(n)\mathcal{E}_{a}$. This evaluation will depend on the situations which we want to clarify. In addition to the perturbations on the Schwarzschild background spacetime, the strategy in Proposal 3.1 is a clue of the generalization of applications of our general framework on the gauge-invariant higher-order perturbations to other physical situations such as higher-order gauge-invariant cosmological perturbations [18]. We leave further evaluations of our formal solutions (50–59) in specific physical situations and the applications to the other perturbation theories with different background spacetimes as future works.

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