The Dobrushin comparison theorem is a powerful tool to bound the difference between the marginals of high-dimensional probability distributions in terms of their local specifications. Originally introduced to prove uniqueness and decay of correlations of Gibbs measures, it has been widely used in statistical mechanics as well as in the analysis of algorithms on random fields and interacting Markov chains. However, the classical comparison theorem requires validity of the Dobrushin uniqueness criterion, essentially restricting its applicability in most models to a small subset of the natural parameter space. In this paper we develop generalized Dobrushin comparison theorems in terms of influences between blocks of sites, in the spirit of Dobrushin-Shlosman and Weitz, that substantially extend the range of applicability of the classical comparison theorem. Our proofs are based on the analysis of an associated family of Markov chains. We develop in detail an application of our main results to the analysis of sequential Monte Carlo algorithms for filtering in high dimension.

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1. Introduction. The canonical description of a high-dimensional random system is provided by specifying a probability measure $\rho$ on a (possibly infinite) product space $S = \prod_{i \in I} S^i$: each site $i \in I$ represents a single degree of freedom, or dimension, of the model. When $I$ is defined as the set of vertices of a graph, the measure $\rho$ defines a graphical model or a random field. Models of this type are ubiquitous in statistical mechanics, combinatorics, computer science, statistics, and in many other areas of science and engineering.

Let $\rho$ and $\tilde{\rho}$ be two such models that are defined on the same space $S$. We would like to address the following basic question: when is $\tilde{\rho}$ a good approximation of $\rho$? Such questions arise at a basic level not only in understanding the properties of random systems themselves, but also in the analysis of the algorithms that are used to investigate and approximate these systems. Of course, probability theory provides numerous methods to evaluate the difference between arbitrary probability measures, but the high-dimensional setting brings some specific challenges: any approximation of practical utility in high dimension must yield error bounds that do not grow, or at least grow sufficiently slowly, in the model dimension. We therefore seek quantitative methods that allow to establish dimension-free bounds on high-dimensional probability distributions.

A general method to address precisely this problem was developed by Dobrushin [6] in the context of statistical mechanics. In the approach pioneered by Dobrushin, Lanford, and Ruelle, an infinite-dimensional system of interacting particles is defined by its local description: for finite sets of sites $J \subset I$, the conditional distribution $\rho(dx^J|x^{I \setminus J})$ of the configuration in $J$ is specified given that the particles outside $J$ are frozen in a fixed configuration. This local description is a direct consequence of the physical parameters of the problem. The model $\rho$ is then defined as a probability measure (called a Gibbs measure) that is compatible with the given system of local conditional distributions; see section 2.1. This setting gives rise to many classical questions in statistical mechanics [13, 21]; for example, the Gibbs measure may or may not be unique, reflecting the presence of a phase transition (akin to the transition from water to ice at the freezing point).

The Dobrushin comparison theorem [6, Theorem 3] provides a powerful tool to obtain dimension-free estimates on the difference between the marginals of Gibbs measures $\rho$ and $\tilde{\rho}$ in terms of the single site conditional distributions $\rho(dx^j|x^{I \setminus \{j\}})$ and $\tilde{\rho}(dx^j|x^{I \setminus \{j\}})$. In its simplified form due to Föllmer [11], this result has become standard textbook material, cf. [13, Theorem 8.20], [21, Theorem V.2.2]. It is widely used to establish numerous properties of Gibbs measures, including uniqueness, decay of correlations, global Markov properties, and analyticity [13, 21, 12], as well as functional inequalities and concentration of measure properties [14, 16, 26], and has similarly proved to be useful in the analysis of algorithms on random fields and interacting Markov chains [27, 22, 2, 19].

Despite this broad array of applications, the range of applicability of the Dobrushin comparison theorem proves to be somewhat limited. This can already be seen in the easiest qualitative consequence of this result: the comparison theorem implies uniqueness of the Gibbs measure under the well-known Dobrushin uniqueness criterion [6]. Unfortunately, this criterion is restrictive: even in models where uniqueness can be established by explicit computation, the Dobrushin uniqueness criterion holds only in a small subset of the natural...
parameter space (see, e.g., [25] for examples). This suggests that the Dobrushin comparison theorem is a rather blunt tool. On the other hand, it is also known that the Dobrushin uniqueness criterion can be substantially improved: this was accomplished in Dobrushin and Shlosman [5] by considering a local description in terms of larger blocks $\rho(dx^i|x^{I\setminus J})$ instead of the single site specification $\rho(dx^i|x^{I\setminus \{i\}})$. In this manner, it is possible in many cases to capture a large part of or even the entire uniqueness region. The uniqueness results of Dobrushin and Shlosman were further generalized by Weitz [25], who developed remarkably general combinatorial criteria for uniqueness. However, while the proofs of Dobrushin-Shlosman and Weitz also provide some information on decay of correlations, they do not provide an analogue of the powerful general-purpose machinery that the Dobrushin comparison theorem yields in its more restrictive setting.

The aim of the present paper is to fill this gap. Our main results (Theorem 2.4 and Theorem 2.12) provide a direct generalization of the Dobrushin comparison theorem to the much more general setting considered by Weitz [25], substantially extending the range of applicability of the classical comparison theorem. While the classical comparison theorem is an immediate consequence of our main result (Corollary 2.6), the classical proof that is based on the “method of estimates” does not appear to extend easily beyond the single site setting. We therefore develop a different, though certainly related, method of proof that systematically exploits the connection of Markov chains. In particular, our main results are derived from a more general comparison theorem for Markov chains that is applied to a suitably defined family of Gibbs samplers, cf. section 3 below.

Our original motivation for developing the generalized comparison theorems of this paper was the investigation of algorithms for filtering in high dimension. Filtering—the computation of the conditional distributions of a hidden Markov process given observed data—is a problem that arises in a wide array of applications in science and engineering. Modern filtering algorithms utilize sequential Monte Carlo methods to efficiently approximate the conditional distributions [1]. Unfortunately, such algorithms suffer heavily from the curse of dimensionality, making them largely useless in complex data assimilation problems that arise in high-dimensional applications such as weather forecasting (the state-of-the-art in such applications is still dominated by ad-hoc methods). Motivated by such problems, we have begun to investigate in [19] a class of regularized filtering algorithms that can, in principle, exhibit dimension-free performance in models that possess decay of correlations. For the simplest possible algorithm of this type, dimension-free error bounds are proved in [19] by systematic application of the Dobrushin comparison theorem.

In order to ensure decay of correlations, the result of [19] imposes a weak interactions assumption that is dictated by the Dobrushin comparison theorem. As will be explained in section 4, however, this assumption is unsatisfactory already at the qualitative level: it limits not only the spatial interactions (as is needed to ensure decay of correlations) but also the dynamics in time. Overcoming this unnatural restriction requires a generalized version of the comparison theorem, which provided the motivation for our main results. As an illustration of our main results, and as a problem of interest in its own right, the application to filtering algorithms will be developed in detail in section 4.
The remainder of this paper is organized as follows. Section 2 introduces the basic setup and notation to be used throughout the paper, and states our main results. While the comparison theorem, being quantitative in nature, is already of significant interest in the finite setting \( \text{card } I < \infty \) (unlike the qualitative uniqueness questions that are primarily of interest when \( \text{card } I = \infty \)), we will develop our main results in a general setting that admits even infinite-range interactions. The proofs of the main results are given in section 3. The application to filtering algorithms is finally developed in section 4.

2. Main results.

2.1. Setting and notation. We begin by introducing the basic setting that will be used throughout this section.

Sites and configurations. Let \( I \) be a finite or countably infinite set of sites. Each subset \( J \subseteq I \) is called a region; the set of finite regions will be denoted as \( J := \{ J \subseteq I : \text{card } J < \infty \} \).

To each site \( i \in I \) is associated a measurable space \( S_i \), the local state space. A configuration is an assignment \( x_i \in S_i \) to each site \( i \in I \). The set of all configurations \( S \), and the set \( S_J \) of configurations in a given region \( J \subseteq I \), are defined as

\[
S := \prod_{i \in I} S_i, \quad S_J := \prod_{i \in J} S_i.
\]

For \( x = (x_i)_{i \in I} \in S \), we denote by \( x^J := (x_i)_{i \in J} \in S_J \) the natural projection on \( S_J \). When \( J \cap K = \emptyset \), we define \( z = x^J y^K \in S_J \cup K \) such that \( z^J = x^J \) and \( z^K = y^K \).

2.1.1. Local functions. A function \( f : S \to \mathbb{R} \) is said to be \( J \)-local if \( f(x) = f(z) \) whenever \( x^J = z^J \), that is, if \( f(x) \) depends on \( x^J \) only. The function \( f \) is said to be local if it is \( J \)-local for some finite region \( J \in I \). When \( I \) is a finite set, every function is local. When \( I \) is infinite, however, we will frequently restrict attention to local functions. More generally, we will consider a class of “nearly” local functions to be defined presently.

Given any function \( f : S \to \mathbb{R} \), let us define for \( J \subseteq I \) and \( x, z \in S \) the \( J \)-local function

\[
f^J_x(z) := f(z^J x^{I \setminus J}).
\]

Then \( f \) is called quasi-local if it can be approximated pointwise by the local functions \( f^J_x \):

\[
\lim_{J \in \mathcal{J}} |f^J_x(z) - f(z)| = 0 \quad \text{for all } x, z \in S,
\]

where \( \lim_{J \in \mathcal{J}} a_J \) denotes the limit of the net \((a_J)_{J \in \mathcal{J}}\) where \( \mathcal{J} \) is directed by inclusion \( \subseteq \) (equivalently, \( a_J \to 0 \) if and only if \( a_{J_i} \to 0 \) for every sequence \( J_1, J_2, \ldots \in \mathcal{J} \) such that \( J_1 \subseteq J_2 \subseteq \cdots \) and \( \bigcup_i J_i = I \)). Let us note that this notion is slightly weaker than the conventional notion of quasilocality used, for example, in [13].
Metrics. In the sequel, we fix for each $i \in I$ a metric $\eta_i$ on $\mathbb{S}^i$ (we assume throughout that $\eta_i$ is measurable as a function on $\mathbb{S}^i \times \mathbb{S}^i$). We will write $\|\eta_i\| = \sup_{x,z} \eta_i(x,z)$.

Given a function $f : \mathbb{S} \rightarrow \mathbb{R}$ and $i \in I$, we define

$$
\delta_i f := \sup_{x,z \in \mathbb{S}, x \setminus \{i\} = z \setminus \{i\}} \frac{|f(x) - f(z)|}{\eta_i(x_i, z_i)}.
$$

The quantity $\delta_i f$ measures the variability of $f(x)$ with respect to the variable $x_i$.

Matrices. The calculus of possibly infinite nonnegative matrices will appear repeatedly in the sequel. Given matrices $A = (A_{ij})_{i,j \in I}$ and $B = (B_{ij})_{i,j \in I}$ with nonnegative entries $A_{ij} \geq 0$ and $B_{ij} \geq 0$, the matrix product is defined as usual by

$$(AB)_{ij} = \sum_{k \in I} A_{ik} B_{kj}.
$$

This quantity is well defined as long as the terms in the sum are all nonnegative, but $(AB)_{ij}$ may possibly take the value $+\infty$. As long as we consider only nonnegative matrices, all the usual rules of matrix multiplication extend to infinite matrices provided that we allow entries with the value $+\infty$ and that we use the convention $+\infty \cdot 0 = 0$ (this follows from the Fubini-Tonelli theorem, cf. [7, Chapter 4]). In particular, the matrix powers $A^k$, $k \geq 1$ are well defined, and we define $A^0 = I$ where $I := (1_{i=j})_{i,j \in I}$ denotes the identity matrix. We will write $A < \infty$ if the nonnegative matrix $A$ satisfies $A_{ij} < \infty$ for every $i,j \in I$.

2.1.2. Kernels, covers, local structure. Recall that a transition kernel $\gamma$ from a measurable space $(\Omega, \mathcal{F})$ to a measurable space $(\Omega', \mathcal{F}')$ is a map $\gamma : \Omega \times \mathcal{F}' \rightarrow \mathbb{R}$ such that $\omega \mapsto \gamma_\omega(A)$ is a measurable function for each $A \in \mathcal{F}'$ and $\gamma_\omega(\cdot)$ is a probability measure for each $\omega \in \Omega$, cf. [15]. Given a probability measure $\mu$ on $\Omega$ and function $f$ on $\Omega'$, we define as usual the probability measure $(\mu \gamma)(A) = \int \mu(d\omega) \gamma_\omega(A)$ on $\Omega'$ and function $(\gamma f)(\omega) = \int \gamma_\omega(d\omega') f(\omega')$ on $\Omega$. A transition kernel $\gamma$ between product spaces is called quasilocal if $\gamma f$ is quasilocal for every bounded and measurable quasilocal function $f$.

Our interest throughout this paper is in models of random configurations, described by a probability measure $\mu$ on $\mathbb{S}$. We would like to understand the properties of such models based on their local structure. A natural way to express the local structure in a finite region $J \subseteq I$ is to consider the conditional distribution $\gamma_J^J(dz^J) = \mu(dz^J | x^{I \setminus J})$ of the configuration in $J$ given a fixed configuration $x^{I \setminus J}$ for the sites outside $J$: conceptually, $\gamma_J$ describes how the sites in $J$ “interact” with the sites outside $J$. The conditional distribution $\gamma_J$ is a transition kernel from $\mathbb{S}$ to $\mathbb{S}^J$. To obtain a complete local description of the model, we must consider a class of finite regions $J$ that covers the entire set of sites $I$. Let us call a collection of regions $\mathcal{J} \subseteq J$ a cover of $I$ if every site $i \in I$ is contained in at least one element of $\mathcal{J}$ (note that, by definition, a cover contains only finite regions). Given any cover $\mathcal{J}$, the collection $(\gamma_J)_{J \in \mathcal{J}}$ provides a local description of the model.

In fact, our main results will hold in a somewhat more general setting than is described above. Let $\mu$ be a probability measure on $\mathbb{S}$ and $\gamma_J$ be transition kernel from $\mathbb{S}$ to $\mathbb{S}^J$. We
say that $\mu$ is $\gamma^J$-invariant if for every bounded measurable function $f$

$$\int \mu(dx) f(x) = \int \mu(dx) \gamma^J_\mu(dz^J) f(z^J x^{I\setminus J});$$

by a slight abuse of notation, we will also write $\mu f = \mu \gamma^J f$. This means that if the configuration $x$ is drawn according to $\mu$, then its distribution is left unchanged if we replace the configuration $x^J$ inside the region $J$ by a random sample from the distribution $\gamma^J_x$, keeping the configuration $x^{I\setminus J}$ outside $J$ fixed. Our main results will be formulated in terms of a collection of transition kernels $(\gamma^J)_{J \in \mathcal{J}}$ such that $\mu$ is $\gamma^J$-invariant for every $J \in \mathcal{J}$. If we choose $\gamma^J_\mu(dz^J) = \mu(dz^J | x^{I\setminus J})$ as above, then the $\gamma^J$-invariance of $\mu$ holds by construction [15, Theorem 6.4]; however, any family of $\gamma^J$-invariant kernels will suffice for the validity of our main results.

**Remark 2.1.** The idea that the collection $(\gamma^J)_{J \in \mathcal{J}}$ provides a natural description of high-dimensional probability distributions is prevalent in many applications. In fact, in statistical mechanics, the model is usually defined in terms of such a family. To this end, one fixes a priori a family of transition kernels $(\gamma^J)_{J \in \mathcal{J}}$, called a specification, that describes the local structure of the model. The definition of $\gamma^J$ is done directly in terms of the parameters of the problem (the potentials that define the physical interactions, or the local constraints that define the combinatorial structure). A measure $\mu$ on $\mathcal{S}$ is called a Gibbs measure for the given specification if $\mu(dz^J | x^{I\setminus J}) = \gamma^J_\mu(dz^J)$ for every $J \in \mathcal{J}$. The existence of a Gibbs measure allows to define the model $\mu$ in terms of the specification. It may happen that there are multiple Gibbs measures for the same specification: the significance of this phenomenon is the presence of a phase transition, akin to the transition of water from liquid to solid at the freezing point. As the construction of Gibbs measures from specifications is not essential for the validity or applicability of our results (cf. section 4 below), we omit further details. We refer to [13, 21, 25] for extensive discussion, examples, and references.

2.2. **Main result.** Let $\rho$ and $\tilde{\rho}$ be probability measures on the space of configurations $\mathcal{S}$. Our main result, Theorem 2.4 below, provides a powerful tool to obtain quantitative bounds on the difference between $\rho$ and $\tilde{\rho}$ in terms of their local structure. Before we can state our results, we must first introduce some basic notions. Our terminology is inspired by Weitz [25].

As was explained above, the local description of a probability measure $\rho$ on $\mathcal{S}$ will be provided in terms of a family of transition kernels. We formalize this as follows.

**Definition 2.2.** A local update rule for $\rho$ is a collection $(\gamma^J)_{J \in \mathcal{J}}$ where $\mathcal{J}$ is a cover of $I$, $\gamma^J$ is a transition kernel from $\mathcal{S}$ to $\mathcal{S}^J$ and $\rho$ is $\gamma^J$-invariant for every $J \in \mathcal{J}$.

In order to compare two measures $\rho$ and $\tilde{\rho}$ on the basis of their local update rules $(\gamma^J)_{J \in \mathcal{J}}$ and $(\tilde{\gamma}^J)_{J \in \mathcal{J}}$, we must quantify two separate effects. On the one hand, we must understand how the two models differ locally: that is, we must quantify how $\gamma^J_x$ and $\tilde{\gamma}^J_x$ differ when
acting on the same configuration $x$. On the other hand, we must understand how perturbations to the local update rule in different regions interact: to this end, we will quantify the extent to which $\gamma^J_x$ and $\gamma^J_z$ differ for different configurations $x,z$. Both effects will be addressed by introducing a suitable family of couplings. Recall that a probability measure $Q$ on a product space $\Omega \times \Omega$ is called a coupling of probability measures $\mu, \nu$ on $\Omega$ if its marginals coincide with $\mu, \nu$, that is, $Q(\cdot \times \Omega) = \mu$ and $Q(\Omega \times \cdot) = \nu$.

**Definition 2.3.** A coupled update rule for $(\rho, \tilde{\rho})$ is a collection $(\gamma^J, \tilde{\gamma}^J, Q^J, \tilde{Q}^J)_{J \in \mathcal{J}}$, where $\mathcal{J}$ is a cover of $I$, such that the following properties hold:

1. $(\gamma^J)_{J \in \mathcal{J}}$ and $(\tilde{\gamma}^J)_{J \in \mathcal{J}}$ are local update rules for $\rho$ and $\tilde{\rho}$, respectively.
2. $Q^J_{x,z}$ is a coupling of $\gamma^J_x, \gamma^J_z$ for every $J \in \mathcal{J}$ and $x, z \in S$ with $\text{card}\{i : x_i \neq z_i\} = 1$.
3. $\tilde{Q}^J_x$ is a coupling of $\gamma^J_x, \tilde{\gamma}^J_x$ for every $J \in \mathcal{J}$ and $x \in S$.

We can now state our main result. The proof will be given in sections 3.1–3.3.

**Theorem 2.4.** Let $\mathcal{J}$ be a cover of $I$, let $(w_J)_{J \in \mathcal{J}}$ be a family of strictly positive weights, and let $(\gamma^J, \tilde{\gamma}^J, Q^J, \tilde{Q}^J)_{J \in \mathcal{J}}$ be a coupled update rule for $(\rho, \tilde{\rho})$. Define for $i, j \in I$

$$W_{ij} := 1_{i=j} \sum_{J \in \mathcal{J} : i \in J} w_J,$$

$$R_{ij} := \sup_{x,z \in S : x_i \neq z_i} \frac{1}{\eta_j(x_i, z_i)} \sum_{J \in \mathcal{J} : i \in J} w_J Q^J_{x,z} \eta_i,$$

$$a_j := \sum_{J \in \mathcal{J} : j \in J} w_J \int^\mathcal{J} \tilde{\rho}(dx) \tilde{Q}^J_x \eta_j.$$

Assume that $\gamma^J$ is quasilocal for every $J \in \mathcal{J}$, and that

$$W_{ii} \leq 1 \quad \text{and} \quad \lim_{n \to \infty} \sum_{j \in I} (I - W + R)^n_{ij} (\rho \otimes \tilde{\rho}) \eta_j = 0 \quad \text{for all} \quad i \in I.$$

Then we have

$$|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f D_{ij} W^{-1}_{jj} a_j \quad \text{where} \quad D := \sum_{n=0}^\infty (W^{-1} R)^n,$$

for any bounded and measurable quasilocal function $f$ such that $\delta_i f < \infty$ for all $i \in I$.

**Remark 2.5.** While it is essential in the proof that $\gamma^J$ and $\tilde{\gamma}^J$ are transition kernels, we do not require that $Q^J$ and $\tilde{Q}^J$ are transition kernels in Definition 2.3, that is, the couplings $Q^J_{x,z}$ and $\tilde{Q}^J_x$ need not be measurable as functions of $x, z$. It is for this reason that the coefficients $a_j$ are defined in terms of an outer integral rather than an ordinary integral [23]:

$$\int f(x) \rho(dx) := \inf \left\{ \int g(x) \rho(dx) : f \leq g, \ g \text{ is measurable} \right\}.$$
When \( x \mapsto \hat{Q}_x \eta_j \) is measurable this issue can be disregarded. In practice measurability will hold in all but pathological cases, but may not always be trivial to prove. We therefore allow for nonmeasurable couplings for sake of technical convenience, so that it is not necessary to check measurability of the coupled updates when applying Theorem 2.4.

We will presently formulate a number of special cases and extensions of Theorem 2.4 that may be useful in different settings. A detailed application is developed in section 4.

2.3. The classical comparison theorem. The original comparison theorem of Dobrushin [6, Theorem 3] and its commonly used formulation due to Föllmer [11] correspond to the special case of Theorem 2.4 where the cover \( J = J_s := \{\{i\} : i \in I\} \) consists of single sites. For example, the main result of [11] follows readily from Theorem 2.4 under a mild regularity assumption. To formulate it, recall that the Wasserstein distance \( d_{\eta_i}(\mu, \nu) \) between probability measures \( \mu \) and \( \nu \) on a measurable space \( \Omega \) with respect to a measurable metric \( \eta_i \) is defined as

\[
d_{\eta_i}(\mu, \nu) := \inf_{Q(1 \times 1) = \mu, Q(\Omega \times \cdot) = \nu} Q_{\eta_i} \eta_j(\gamma_{\{i\}} x, \gamma_{\{i\}} z),
\]

where the infimum is taken over probability measures \( Q \) on \( \Omega \times \Omega \) with the given marginals \( \mu \) and \( \nu \). We now obtain the following classical result (cf. [11] and [12, Remark 2.17]).

**Corollary 2.6 ([11]).** Assume \( S^i \) is Polish and \( \eta_i \) is lower-semicontinuous for all \( i \in I \). Let \( (\gamma^{(i)})_{i \in I} \) and \( (\tilde{\gamma}^{(i)})_{i \in I} \) be local update rules for \( \rho \) and \( \tilde{\rho} \), respectively, and let

\[
C_{ij} := \sup_{x, z \in S,} \frac{d_{\eta_i}(\gamma^{(i)}_x, \gamma^{(i)}_z)}{\eta_j(x_j, z_j)}, \quad b_j := \int \tilde{\rho}(dx) d_{\eta_j}(\gamma^{(j)}_x, \tilde{\gamma}^{(j)}_x).
\]

Assume that \( \gamma^{(i)}_x \) is quasilocal for every \( i \in I \), and that

\[
\lim_{n \to \infty} \sum_{j \in I} C^n_{ij}(\rho \otimes \tilde{\rho}) \eta_j = 0 \quad \text{for all} \quad i \in I.
\]

Then we have

\[
|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f D_{ij} b_j \quad \text{where} \quad D := \sum_{n=0}^{\infty} C^n,
\]

for any bounded and measurable quasilocal function \( f \) such that \( \delta_i f < \infty \) for all \( i \in I \).

If \( Q_{\{i\}}^{(i)} \) and \( \hat{Q}_x^{(i)} \) are minimizers in the definition of \( d_{\eta_i}(\gamma^{(i)}_x, \gamma^{(i)}_z) \) and \( d_{\eta_i}(\gamma^{(i)}_x, \tilde{\gamma}^{(i)}_x) \), respectively, and if we let \( J = J_s \) and \( w_{\{i\}} = 1 \) for all \( i \in I \), then Corollary 2.6 follows immediately from Theorem 2.4. For simplicity, we have imposed the mild topological regularity assumption on \( S^i \) and \( \eta_i \) to ensure the existence of minimizers [24, Theorem 4.1] (when minimizers do not exist, it is possible with some more work to obtain a similar result.
by using near-optimal couplings in Theorem 2.4. Let us note that when \( \eta_i(x, z) = 1_{x \neq z} \) is the trivial metric, the Wasserstein distance reduces to the total variation distance

\[
d_\eta(\mu, \nu) = \frac{1}{2} \| \mu - \nu \| := \frac{1}{2} \sup_{f: \| f \| \leq 1} |\mu f - \nu f|
\]

and an optimal coupling exists in any measurable space [6, p. 472]. Thus in this case no regularity assumptions are needed, and Corollary 2.6 reduces to the textbook version of the comparison theorem that appears, e.g., in [13, Theorem 8.20] or [21, Theorem V.2.2].

While the classical comparison theorem of Corollary 2.6 follows from our main result, it should be emphasized that the single site assumption \( J = J_s \) is a significant restriction. The general statement of Theorem 2.4 constitutes a crucial improvement that substantially extends the range of applicability of the comparison method, as we will see below and in section 4. Let us also note that the proofs in [6, 11], based on the “method of estimates,” do not appear to extend easily beyond the single site setting. We use a different (though related) method of proof that systematically exploits the connection with Markov chains.

2.4. Alternative assumptions. The key assumption of Theorem 2.4 is (2.1). The aim of the present section is to obtain a number of useful alternatives to assumption (2.1) that are easily verified in practice.

We begin by defining the notion of a tempered measure [12, Remark 2.17].

**Definition 2.7.** A probability measure \( \mu \) on \( S \) is called \( x^* \)-tempered if

\[
\sup_{i \in I} \int \mu(dx) \eta_i(x_i, x^*_i) < \infty.
\]

In the sequel \( x^* \in S \) will be considered fixed and \( \mu \) will be called tempered.

It is often the case in practice that the collection of metrics is uniformly bounded, that is, \( \sup_i \| \eta_i \| < \infty \). In this case, every probability measure on \( S \) is trivially tempered. However, the restriction to tempered measures may be essential when the spaces \( S^i \) are noncompact (see, for example, [6, section 5]) for a simple but illuminating example.

Let us recall that a norm \( \| \cdot \| \) defined on an algebra of square (possibly infinite) matrices is called a matrix norm if \( \| AB \| \leq \| A \| \| B \| \). We also recall that the matrix norms \( \| \cdot \|_\infty \) and \( \| \cdot \|_1 \) are defined for nonnegative matrices \( A = (A_{ij})_{i,j \in I} \) as

\[
\| A \|_\infty := \sup_{i \in I} \sum_{j \in I} A_{ij}, \quad \| A \|_1 := \sup_{j \in I} \sum_{i \in I} A_{ij}.
\]

The following result collects various useful alternatives to (2.1). It is proved in section 3.4.

**Corollary 2.8.** Suppose that \( \rho \) and \( \tilde{\rho} \) are tempered. Then the conclusion of Theorem 2.4 remains valid when the assumption (2.1) is replaced by one of the following:

1. \( \text{card } I < \infty \) and \( D < \infty \).
2. $\text{card } I < \infty$, $R < \infty$, and $\|(W^{-1}R)^n\| < 1$ for some matrix norm $\| \cdot \|$ and $n \geq 1$.
3. $\sup_i W_{ii} < \infty$ and $\|W^{-1}\|_\infty < 1$.
4. $\sup_i W_{ii} < \infty$, $\|RW^{-1}\|_\infty < \infty$, and $\|(RW^{-1})^n\|_\infty < 1$ for some $n \geq 1$.
5. $\sup_i W_{ii} < \infty$, $\sum_i \|\eta_i\| < \infty$, and $\|RW^{-1}\|_1 < 1$.
6. $\sup_i W_{ii} < \infty$, there exists a metric $\rho$ on $I$ such that $\sup \{m(i,j) : R_{ij} > 0\} < \infty$ and $\sup_i \sum_j e^{-\beta m(i,j)} < \infty$ for all $\beta > 0$, and $\|RW^{-1}\|_1 < 1$.

The conditions of Corollary 2.8 are closely related to the uniqueness problem for Gibbs measures. Suppose that the collection of quasilocal transition kernels $(\gamma^J)_{J \in \mathcal{J}}$ is a local update rule for $\rho$. It is natural to ask whether $\rho$ is the unique measure that admits $(\gamma^J)_{J \in \mathcal{J}}$ as a local update rule (see the remark at the end of section 2.1). We now observe that uniqueness is a necessary condition for the conclusion of Theorem 2.4. Indeed, let $\tilde{\rho}$ be another measure that admits the same local update rule. If (2.1) holds, we can apply Theorem 2.4 with $\tilde{\gamma}^J = \gamma^J$ and $a_{ij} = 0$ to conclude that $\tilde{\rho} = \rho$. In particular, $\sum_j (I - W + R)_{ij}^\rho \rightarrow 0$ in Theorem 2.4 evidently implies uniqueness in the class of tempered measures.

Of course, the point of Theorem 2.4 is that it provides a quantitative tool that goes far beyond qualitative uniqueness questions. It is therefore interesting to note that this single result nonetheless captures many of the uniqueness conditions that are used in the literature. In Corollary 2.8, Condition 3 is precisely the “influence on a site” condition of Weitz [25, Theorem 2.5] (our setting is even more general in that we do not require bounded-range interactions as is essential in [25]). Conditions 5 and 6 constitute a slight strengthening (see below) of the “influence of a site” condition of Weitz [25, Theorem 2.7] under summable metric or subexponential graph assumptions, in the spirit of the classical uniqueness condition of Dobrushin and Shlosman [5]. In the finite setting with single site updates, Condition 2 is in the spirit of [9] and Condition 4 is in the spirit of [8].

On the other hand, we can now see that Theorem 2.4 provides a crucial improvement over the classical comparison theorem. The single site setting of Corollary 2.6 corresponds essentially to the original Dobrushin uniqueness regime [6]. It is well known that this setting is restrictive, in that it captures only a small part of the parameter space where uniqueness of Gibbs measures holds. It is precisely for this reason that Dobrushin and Shlosman introduced their improved uniqueness criterion in terms of larger blocks [5], which in many cases allows to capture a large part of or even the entire uniqueness region; see [25, section 5] for examples. The generalized comparison Theorem 2.4 in terms of larger blocks can therefore be fruitfully applied to a much larger and more natural class of models than the classical comparison theorem. This point will be further emphasized in the context of the application that will be developed in detail in section 4.

**Remark 2.9.** The “influence of a site” condition $\|RW^{-1}\|_1 < 1$ that appears in Corollary 2.8 is slightly stronger than the corresponding condition of Dobrushin-Shlosman [5] and Weitz [25, Theorem 2.7]. Writing out the definition of $R$, we find that our condition
reads
\[ \|RW^{-1}\|_1 = \sup_{j \in I} W^{-1}_{jj} \sum_{i \in I} \sup_{x, z: x \in I \setminus \{i\} = z \setminus \{j\}} \frac{1}{\eta_j(x_j, z_j)} \sum_{j \in J: i \in J} w_J Q^J_{x, z} \eta_i < 1, \]
while the condition of [25, Theorem 2.7] (which extends the condition of [5]) reads
\[ \sup_{j \in I} W^{-1}_{jj} \sup_{x, z: x \in I \setminus \{i\} = z \setminus \{j\}} \frac{1}{\eta_j(x_j, z_j)} \sum_{i \in I} \sum_{j \in J: i \in J} w_J Q^J_{x, z} \eta_i < 1. \]
The latter is slightly weaker as the sum over sites \( i \) appears inside the supremum over configurations \( x, z \). While the distinction between these conditions is inessential in many applications, there do exist situations in which the weaker condition yields an essential improvement, see, e.g., [25, section 5.3]. In such problems, Theorem 2.4 is not only limited by the stronger uniqueness condition but could also lead to poor quantitative bounds, as the comparison bound is itself expressed in terms of the uniform influence coefficients \( R_{ij} \).

It could therefore be of interest to develop comparison theorems that are able to exploit the finer structure that is present in the weaker uniqueness condition. In fact, the proof of Theorem 2.4 already indicates a natural approach to such improved bounds. However, the resulting comparison theorems are necessarily nonlinear in that the action of the matrix \( R \) is replaced by a nonlinear operator \( R \). The nonlinear expressions are somewhat difficult to handle in practice, and as we do not at present have a compelling application for such bounds we do not pursue this direction here. However, for completeness, we will briefly sketch at the end of section 3.2 how such bounds can be obtained.

2.5. A one-sided comparison theorem. As was discussed in section 2.1, it is natural in many applications to describe high-dimensional probability distributions in terms of local conditional probabilities of the form \( \mu(dz^J | x^{I \setminus J} \). This is in essence a static picture, where we describe the behavior of each local region \( J \) given that the configuration of the remaining sites \( I \setminus J \) is frozen. In models that possess dynamics, this description is not very natural. In this setting, each site \( i \in I \) occurs at a given time \( \tau(i) \), and its state is only determined by the configuration of sites \( j \in I \) in the past and present \( \tau(j) \leq \tau(i) \), but not by the future. For example, the model might be defined as a high-dimensional Markov chain whose description is naturally given in terms of one-sided conditional probabilities (see, e.g., [10] and the application in section 4). It is therefore interesting to note that the original comparison theorem of Dobrushin [6] is actually more general than Corollary 2.6 in that it is applicable both in the static and dynamic settings. We presently develop an analogous generalization to Theorem 2.4.

For the purposes of this section, we assume that we are given a function \( \tau: I \to \mathbb{Z} \) that assigns to each site \( i \in I \) an integer index \( \tau(i) \). We define
\[ I_{\leq k} := \{ i \in I : \tau(i) \leq k \}, \quad S_{\leq k} := S^{I_{\leq k}}, \]
and for any probability measure \( \rho \) on \( S \) we denote by \( \rho_{\leq k} \) the marginal distribution on \( S_{\leq k} \).


**Definition 2.10.** A one-sided local update rule for $\rho$ is a collection $(\gamma^J)_{J \in \mathcal{J}}$ where

1. $\mathcal{J}$ is a cover of $I$ such that $\min_{i \in J} \tau (i) = \max_{i \in J} \tau (i) =: \tau (J)$ for every $J \in \mathcal{J}$.
2. $\gamma^J$ is a transition kernel from $S_{\leq \tau (J)}$ to $S^J$.
3. $\rho_{\leq \tau (J)}$ is $\gamma^J$-invariant for every $J \in \mathcal{J}$.

The canonical example of a one-sided local update rule is to consider the one-sided conditional distributions $\gamma^J_x (dz^J) = \rho (dz^J | x^I \leq \tau (J) \setminus J)$. This situation is particularly useful in the investigation of interacting Markov chains, cf. [6, 10], where $\tau (j)$ denotes the time index of the site $j$ and we condition only on the past and present, but not on the future.

**Definition 2.11.** A one-sided coupled update rule for $(\rho, \tilde{\rho})$ is a collection of transition kernels $(\gamma^J, \tilde{\gamma}^J, Q^J, \hat{Q}^J)_{J \in \mathcal{J}}$ such that the following hold:

1. $(\gamma^J, \tilde{\gamma}^J)_{J \in \mathcal{J}}$ and $(\tilde{\gamma}^J, Q^J)_{J \in \mathcal{J}}$ are one-sided local update rules for $\rho$ and $\tilde{\rho}$, respectively.
2. $Q^J_x$ is a coupling of $\gamma^J_x, \tilde{\gamma}^J_x$ for $J \in \mathcal{J}$ and $x, z \in S_{\leq \tau (J)}$ with $\text{card} \{ i : x_i \neq z_i \} = 1$.
3. $\hat{Q}^J_x$ is a coupling of $\gamma^J_x, \tilde{\gamma}^J_x$ for $J \in \mathcal{J}$ and $x \in S_{\leq \tau (J)}$.

We can now state a one-sided counterpart to Theorem 2.4.

**Theorem 2.12.** Let $(\gamma^J, \tilde{\gamma}^J, Q^J, \hat{Q}^J)_{J \in \mathcal{J}}$ be a one-sided coupled update rule for $(\rho, \tilde{\rho})$, and let $(w^J)_{J \in \mathcal{J}}$ be a family of strictly positive weights. Define the matrices $W$ and $R$ and the vector $a$ as in Theorem 2.4. Assume that $\gamma^J$ is quasilocal for every $J \in \mathcal{J}$, that

$$\sum_{j \in I} D_{ij} (\rho \otimes \tilde{\rho}) \eta_j < \infty \quad \text{for all } i \in I$$

where $D := \sum_{n=0}^{\infty} (W^{-1} R)^n$,

and that (2.1) holds. Then we have

$$|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f D_{ij} W^{-1} a_j$$

for any bounded and measurable quasilocal function $f$ such that $\delta_i f < \infty$ for all $i \in I$.

Let us remark that the result of Theorem 2.12 is formally the same as that of Theorem 2.4, except that we have changed the nature of the update rules used in the definition of the coefficients. We also require a further assumption (2.2) in addition to assumption (2.1) of Theorem 2.4, but this is not restrictive in practice: in particular, it is readily verified that the conclusion of Theorem 2.12 also holds under any of the conditions of Corollary 2.8.

3. Proofs.
3.1. General comparison principle. The proof of Theorem 2.4 is derived from a general comparison principle for Markov chains that will be formalized in this section. The basic idea behind this principle is to consider two transition kernels $G$ and $\tilde{G}$ on $S$ such that $\rho G = G$ and $\tilde{\rho} \tilde{G} = \tilde{\rho}$. One should think of $G$ as the transition kernel of a Markov chain that admits $\rho$ as its invariant measure, and similarly for $\tilde{G}$. The comparison principle of this section provides a general method to bound the difference between the invariant measures $\rho$ and $\tilde{\rho}$ in terms of the transition kernels $G$ and $\tilde{G}$. In the following sections, we will apply this principle to a specific choice of $G$ and $\tilde{G}$ that is derived from the coupled update rule.

We begin by introducing a standard notion in the analysis of high-dimensional Markov chains, cf. [10] (note that our indices are reversed as compared to the definition in [10]).

**Definition 3.1.** $(V_{ij})_{i,j \in I}$ is called a Wasserstein matrix for a transition kernel $G$ on $S$ if

$$\delta_j G f \leq \sum_{i \in I} \delta_i f V_{ij}$$

for every $j \in I$ and bounded and measurable quasilocal function $f$.

We now state our general comparison principle.

**Proposition 3.2.** Let $G$ and $\tilde{G}$ be transition kernels on $S$ such that $\rho G = \rho$ and $\tilde{\rho} \tilde{G} = \tilde{\rho}$, and let $Q_x$ be a coupling between the measures $G_x$ and $\tilde{G}_x$ for every $x \in S$. Assume that $G$ is quasilocal, and let $V$ be a Wasserstein matrix for $G$. Then we have

$$|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_j f N^{(n)}_{ij} \int^* \tilde{\rho}(dx) Q_x \eta_j + \sum_{i,j \in I} \delta_i f V^{(n)}_{ij} (\rho \otimes \tilde{\rho}) \eta_j,$$

where we defined

$$N^{(n)} := \sum_{k=0}^{n-1} V^k,$$

for any bounded and measurable quasilocal function $f$ and $n \geq 1$.

Theorem 2.4 will be derived from this result. Roughly speaking, we will design the transition kernel $G$ such that $V = I - W + R$ is a Wasserstein matrix; then assumption (2.1) implies that the second term in Proposition 3.2 vanishes as $n \to \infty$, and the result of Theorem 2.4 reduces to some matrix algebra (as will be explained below, however, a more complicated argument is needed to obtain Theorem 2.4 in full generality).

To prove Proposition 3.2 we require a simple lemma.

**Lemma 3.3.** Let $Q$ be a coupling of probability measures $\mu, \nu$ on $S$. Then

$$|\mu f - \nu f| \leq \sum_{i \in I} \delta_i f Q \eta_i$$

for every bounded and measurable quasilocal function $f$. 
PROOF. Let \( J \in \mathcal{J} \). Enumerate its elements arbitrarily as \( J = \{ j_1, \ldots, j_r \} \), and define \( J_k = \{ j_1, \ldots, j_k \} \) for \( 1 \leq k \leq r \) and \( J_0 = \emptyset \). Then we can evidently estimate

\[
|f^J_x(z) - f^J_x(\tilde{z})| \leq \sum_{k=1}^r \sum_{j \in J_k} \delta_j f \eta_j(z_j, \tilde{z}_j).
\]

As \( f \) is quasilocal, we can let \( J \uparrow I \) to obtain

\[
|f(z) - f(\tilde{z})| \leq \sum_{i,j \in I} \delta_i f \eta_i(z_i, \tilde{z}_i).
\]

The result follows readily as \( |\mu f - \nu f| \leq \int |f(z) - f(\tilde{z})| Q(dz, d\tilde{z}) \). \( \square \)

We now proceed to the proof of Proposition 3.2.

PROOF OF PROPOSITION 3.2. We begin by writing

\[
|\rho f - \tilde{\rho} f| = |\rho G^0 f - \tilde{\rho} G^0 f|
\leq \sum_{k=0}^{n-1} |\tilde{\rho} G^{n-k-1} G^{k+1} f - \tilde{\rho} G^{n-k} G^k f| + |\rho G^0 f - \tilde{\rho} G^0 f|
= \sum_{k=0}^{n-1} |\tilde{\rho} G G^k f - \tilde{\rho} \tilde{G} G^k f| + |\rho G^0 f - \tilde{\rho} G^0 f|.
\]

As \( G \) is assumed quasilocal, \( G^k f \) is quasilocal, and thus Lemma 3.3 yields

\[
|\tilde{\rho} G G^k f - \tilde{\rho} \tilde{G} G^k f| \leq \int \tilde{\rho}(dx) |G_x G^k f - \tilde{G}_x G^k f|
\leq \int \tilde{\rho}(dx) \sum_{j \in I} \delta_j G^k f Q_x \eta_j
\leq \sum_{i,j \in I} \delta_i f V_{ij} \int \tilde{\rho}(dx) Q_x \eta_j.
\]

Similarly, as \( \rho \otimes \tilde{\rho} \) is a coupling of \( \rho, \tilde{\rho} \), we obtain by Lemma 3.3

\[
|\rho G^0 f - \tilde{\rho} G^0 f| \leq \sum_{j \in I} \delta_j G^0 f (\rho \otimes \tilde{\rho}) \eta_j \leq \sum_{i,j \in I} \delta_i f V_{ij} \eta_j.
\]

Thus the proof is complete. \( \square \)
3.2. Gibbs samplers. To put Proposition 3.2 to good use, we must construct transition kernels $G$ and $\tilde{G}$ for which $\rho$ and $\tilde{\rho}$ are invariant, and that admit tractable estimates for the quantities in the comparison theorem in terms of the coupled update rule $(\gamma^J, \tilde{\gamma}^J, Q^J, \tilde{Q}^J)_{J \in \mathcal{J}}$ and the weights $(w_J)_{J \in \mathcal{J}}$. To this end, we will use a standard construction called the Gibbs sampler: in each time step, we draw a region $J \in \mathcal{J}$ with probability $v_J \propto w_J$, and then apply the transition kernel $\gamma^J$ to the current configuration. This readily defines a transition kernel $G$ for which $\rho$ is $G$-invariant (as $\rho$ is $\gamma^J$-invariant for every $J \in \mathcal{J}$). The construction for $\tilde{G}$ is identical. As will be explained below, this is not the most natural construction for the proof of our main result; however, it will form the basis for further computations.

We fix throughout this section a coupled update rule $(\gamma^J, \tilde{\gamma}^J, Q^J, \tilde{Q}^J)_{J \in \mathcal{J}}$ for $(\rho, \tilde{\rho})$ and weights $(w_J)_{J \in \mathcal{J}}$ satisfying the assumptions of Theorem 2.4. Let $v = (v_J)_{J \in \mathcal{J}}$ be a sequence of nonnegative weights such that $\sum_J v_J \leq 1$. We define the Gibbs samplers

$$G^v_x(A) := \left(1 - \sum_{J \in \mathcal{J}} v_J\right) 1_A(x) + \sum_{J \in \mathcal{J}} v_J \int 1_A(z^J) x^{I \setminus J} \gamma^J_x(dz^J),$$

$$\tilde{G}^v_x(A) := \left(1 - \sum_{J \in \mathcal{J}} v_J\right) 1_A(x) + \sum_{J \in \mathcal{J}} v_J \int 1_A(z^J) x^{I \setminus J} \tilde{\gamma}^J_x(dz^J).$$

Evidently $G^v$ and $\tilde{G}^v$ are transition kernels on $\mathcal{S}$, and $\rho G^v = \rho$ and $\tilde{\rho} \tilde{G}^v = \tilde{\rho}$ by construction. To apply Proposition 3.2, we must establish some basic properties.

Lemma 3.4. Assume that $\gamma^J$ is quasilocal for every $J \in \mathcal{J}$. Then $G^v$ is quasilocal.

Proof. Let $f : \mathcal{S} \to \mathcal{S}$ be a bounded and measurable quasilocal function. It evidently suffices to show that $\gamma^J f^J$ is quasilocal for every $J \in \mathcal{J}$. To this end, let us fix $J \in \mathcal{J}$, $x, z \in \mathcal{S}$, and $J_1, J_2, \ldots \in \mathcal{J}$ such that $J_1 \subseteq J_2 \subseteq \cdots$ and $\bigcup_i J_i = I$. Then we have

$$\gamma^J_{z, J_2 \setminus J_1} \xrightarrow{i \to \infty} \gamma^J_z$$

as $\gamma^J$ is quasilocal. On the other hand, we have

$$f^J_{z, J_2 \setminus J_1} \xrightarrow{i \to \infty} f^J_z$$

as $f$ is quasilocal. Thus by [20, Proposition 18, p. 270] we obtain

$$\gamma^J_{z, J_2 \setminus J_1} f^J_{z, J_2 \setminus J_1} \xrightarrow{i \to \infty} \gamma^J_z f^J_z.$$

As the choice of $x, z$ and $(J_i)_{i \geq 1}$ is arbitrary, the result follows. $\square$
LEMMA 3.5. Assume that $\gamma^J$ is quasilocal for every $J \in \mathcal{J}$, and define

$$W^\gamma_{ij} := 1_{i=j} \sum_{J \in \mathcal{J}: i \in J} v_J,$$

$$R^\gamma_{ij} := \sup_{x, z \in S: x^{I \setminus \{j\}} = z^{I \setminus \{j\}}} \frac{1}{\eta_j(x_j, z_j)} \sum_{J \in \mathcal{J}: i \in J} v_J Q^J_{x,z} \eta_i.$$

Then $V^\gamma = I - W^\gamma + R^\gamma$ is a Wasserstein matrix for $G^\gamma$.

PROOF. Let $f : S \to S$ be a bounded and measurable quasilocal function, and let $x, z \in S$ be configurations that differ at a single site $\text{card}\{i \in I : x_i \neq z_i\} = 1$. Note that

$$\gamma^J_x f_x = (\gamma^J_x \otimes \delta_{x^{I \setminus \{j\}}}) f, \quad \gamma^J_z f_z = (\gamma^J_z \otimes \delta_{z^{I \setminus \{j\}}}) f.$$

As $Q^J_{x,z}$ is a coupling of $\gamma^J_x$ and $\gamma^J_z$ by construction, the measure $Q^J_{x,z} \otimes \delta_{x^{I \setminus \{j\}}} \otimes \delta_{z^{I \setminus \{j\}}}$ is a coupling of $\gamma^J_x \otimes \delta_{x^{I \setminus \{j\}}}$ and $\gamma^J_z \otimes \delta_{z^{I \setminus \{j\}}}$. Thus Lemma 3.3 yields

$$|\gamma^J_x f_x - \gamma^J_z f_z| \leq \sum_{i \in I} \delta_i f (Q^J_{x,z} \otimes \delta_{x^{I \setminus \{j\}}} \otimes \delta_{z^{I \setminus \{j\}}}) \eta_i = \sum_{i \in I} \delta_i f Q^J_{x,z} \eta_i + \sum_{i \in I \setminus J} \delta_i f \eta_i(x_i, z_i).$$

In particular, we obtain

$$|G^\gamma f(x) - G^\gamma f(z)| \leq \left(1 - \sum_{J \in \mathcal{J}} v_J\right) |f(x) - f(z)| + \sum_{J \in \mathcal{J}} v_J |\gamma^J_x f_x - \gamma^J_z f_z|$$

$$\leq \left(1 - \sum_{J \in \mathcal{J}} v_J\right) \sum_{i \in I} \delta_i f \eta_i(x_i, z_i) + \sum_{J \in \mathcal{J}} v_J \left(\sum_{i \in I} \delta_i f Q^J_{x,z} \eta_i + \sum_{i \in I \setminus J} \delta_i f \eta_i(x_i, z_i)\right)$$

$$= \sum_{i \in I} \delta_i f \{1 - W^\gamma_{ii}\} \eta_i(x_i, z_i) + \sum_{i \in I} \delta_i f \sum_{J \in \mathcal{J}: i \in J} v_J Q^J_{x,z} \eta_i.$$

Now suppose that $x^{I \setminus \{j\}} = z^{I \setminus \{j\}}$ (and $x \neq z$). Then by definition

$$\sum_{J \in \mathcal{J}: i \in J} v_J Q^J_{x,z} \eta_i \leq R^\gamma_{ij} \eta_j(x_j, v_j),$$

and we obtain

$$\frac{|G^\gamma f(x) - G^\gamma f(z)|}{\eta_j(x_j, z_j)} \leq \delta_j f \{1 - W^\gamma_{jj}\} + \sum_{i \in I} \delta_i f R^\gamma_{ij}.$$

Thus $V^\gamma = I - W^\gamma + R^\gamma$ satisfies Definition 3.1. \qed
Using Lemmas 3.4 and 3.5, we can now apply Proposition 3.2.

**Corollary 3.6.** Assume that \( \gamma^J \) is quasilocal for every \( J \in \mathcal{J} \). Then

\[
|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f N_{ij}^{(n)} a_{ij}^\gamma + \sum_{i,j \in I} \delta_i f (I - W^\gamma + R^\gamma)^{n}_{ij} (\rho \otimes \tilde{\rho}) \eta_j
\]

for every \( n \geq 1 \) and bounded and measurable quasilocal function \( f \), where

\[
N_{ij}^{(n)} := \sum_{k=0}^{n-1} (I - W^\gamma + R^\gamma)^k
\]

and the coefficients \( (a_{ij}^\gamma)_{j \in I} \) are defined by

\[
a_{ij}^\gamma := \sum_{J \in \mathcal{J}, j \in J} \int_{\mathbb{R}^I} \tilde{\rho}(dx) \hat{Q}_x^J \eta_j.
\]

**Proof.** Let \( G = G^\gamma, \tilde{G} = \tilde{G}^\gamma, V = I - W^\gamma + R^\gamma \) in Proposition 3.2. The requisite assumptions are verified by Lemmas 3.4 and 3.5, and it remains to show that there exists a coupling \( Q_x \) of \( G_x \) and \( \tilde{G}_x \) such that

\[
\int_{\mathbb{R}^I} \tilde{\rho}(dx) Q_x \eta_j \leq a_j \quad \text{for every } j \in I.
\]

Choosing

\[
Q_x g := \left(1 - \sum_{J \in \mathcal{J}} v_J\right) g(x, x) + \sum_{J \in \mathcal{J}} v_J \int_{\mathbb{R}^I} \hat{Q}_x^J (dz^J, d\tilde{z}^J) g(z^J_{x \cap J}, \tilde{z}^J_{x \cap J}),
\]

it is easily verified that \( Q_x \) satisfies the necessary properties. \( \square \)

In order for the construction of the Gibbs sampler to make sense, the weights \( v_J \) must be probabilities. This imposes the requirement \( \sum_J v_J \leq 1 \). If we were to assume that \( \sum_J w_J \leq 1 \), we could apply Corollary 3.6 with \( v_J = w_J \). Then assumption (2.1) guarantees that the second term in Corollary 3.6 vanishes as \( n \to \infty \), which yields

\[
|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f N_{ij} a_j \quad \text{with } N := \sum_{k=0}^{\infty} (I - W + R)^k.
\]

The proof of Theorem 2.4 would now be complete after we establish the identity

\[
N = \sum_{k=0}^{\infty} (I - W + R)^k = \sum_{k=0}^{\infty} (W^{-1} R)^k W^{-1} = DW^{-1}.
\]

This straightforward matrix identity will be proved in the next section. The assumption that the weights \( w_J \) are summable is restrictive, however, when \( I \) is infinite: in Theorem 2.4 we only assume that \( W_{ii} \leq 1 \) for all \( i \), so we evidently cannot set \( v_J = w_J \).

When the weights \( w_J \) are not summable, it is not natural to interpret them as probabilities. In this setting, a much more natural construction would be to consider a continuous time counterpart of the Gibbs sampler called Glauber dynamics. To define this process, one attaches to each region \( J \in \mathcal{J} \) an independent Poisson process with rate \( w_J \), and applies
the transition kernel \( \gamma^f \) at every jump time of the corresponding Poisson process. Thus \( w_J \) does not represent the probability of selecting the region \( J \) in one time step, but rather the frequency with which region \( J \) is selected in continuous time. Once this process has been defined, one would choose the transition kernel \( G \) to be the transition semigroup of the continuous time process on any fixed time interval. Proceeding with this construction we expect, at least formally, to obtain Theorem 2.4 under the stated assumptions.

Unfortunately, there are nontrivial technical issues involved in implementing this approach: it is not evident \textit{a priori} that the continuous time construction defines a well-behaved Markov semigroup, so that it is unclear when the above program can be made rigorous. The existence of a semigroup has typically been established under more restrictive assumptions than we have imposed in the present setting [18]. In order to circumvent such issues, we will proceed by an alternate route. Formally, the Glauber dynamics can be obtained by an appropriate scaling limit of discrete time Gibbs samplers. We will also utilize this scaling, but instead of applying Proposition 3.2 to the limiting dynamics we will take the scaling limit directly in Corollary 3.6. Thus, while our intuition comes from the continuous time setting, we avoid some technicalities inherent in the construction of the limit dynamics. Instead, we now face the problem of taking limits of powers of infinite matrices. The requisite matrix algebra will be worked out in the following section.

\textbf{Remark 3.7.} Let us briefly sketch how the previous results can be sharpened to obtain a nonlinear comparison theorem that could lead to sharper bounds in some situations. Assume for simplicity that \( \sum_J w_J \leq 1 \). Then \( V = I - W + R \) is a Wasserstein matrix for \( G \) by Lemma 3.5. Writing out the definitions, this means \( \delta(Gf) \leq \delta(f)V \) where

\[
(\beta V)_j = \sum_{i \in I} \beta_i \sup_{x,z \in S: x \setminus \{I\} = z \setminus \{J\}} \left\{ 1_{i=j} \left( 1 - \sum_{J: i \in J} w_J \right) + \frac{1}{\eta_j(x_j, z_j)} \sum_{J: i \in J} w_J Q_{x,z}^J \eta_i \right\}
\]

(here we interpret \( \beta = (\beta_i)_{i \in I} \) and \( \delta(f) = (\delta_i f)_{i \in I} \) as row vectors). However, from the proof of Lemma 3.5 we even obtain the sharper bound \( \delta(Gf) \leq \mathcal{V}[\delta(f)] \) where

\[
\mathcal{V}[\beta]_j := \sup_{x,z \in S: x \setminus \{I\} = z \setminus \{J\}} \sum_{i \in I} \beta_i \left\{ 1_{i=j} \left( 1 - \sum_{J: i \in J} w_J \right) + \frac{1}{\eta_j(x_j, z_j)} \sum_{J: i \in J} w_J Q_{x,z}^J \eta_i \right\}
\]

is defined with the supremum over configurations outside the sum. The nonlinear operator \( \mathcal{V} \) can now be used much in the same way as the Wasserstein matrix \( V \). In particular, following the identical proof as for Proposition 3.2, we immediately obtain

\[
|\rho f - \tilde{\rho} f| \leq \sum_{J: \eta_j > 0} \sum_{k=0}^{n-1} \mathcal{V}^k[\delta(f)]_j \int \tilde{\rho}(dx) Q_x \eta_j + \sum_{J: \eta_j > 0} \mathcal{V}^n[\delta(f)]_j (\rho \otimes \tilde{\rho}) \eta_j,
\]

where \( \mathcal{V}^k \) denotes the \( k \)th iterate of the nonlinear operator \( \mathcal{V} \). Proceeding along these lines, one can develop nonlinear comparison theorems under Dobrushin-Shlosman type conditions (see the discussion in section 2.4). The nonlinear expressions are somewhat difficult to handle, however, and we do not develop this idea further in this paper.
3.3. **Proof of Theorem 2.4.** Throughout this section, we work under the assumptions of Theorem 2.4. The main idea of the proof is the following continuous scaling limit of Corollary 3.6.

**Proposition 3.8.** Let $t > 0$. Define the matrices

$$N := \sum_{k=0}^{\infty} (I - W + R)^k, \quad V^t := \sum_{k=0}^{\infty} \frac{t^k e^{-t}}{k!} (I - W + R)^k.$$

Then we have, under the assumptions of Theorem 2.4,

$$|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f N_{ij} a_j + \sum_{i,j \in I} \delta_i f V^t_{ij} (\rho \otimes \tilde{\rho}) \eta_j$$

for every bounded and measurable quasilocal function $f$ such that $\delta_i f < \infty$ for all $i \in I$.

**Proof.** Without loss of generality, we will assume throughout the proof that $f$ is a local function (so that only finitely many $\delta_i f$ are nonzero). The extension to quasilocal $f$ follows readily by applying the local result to $f_J$ and letting $J \uparrow I$ as in the proof of Lemma 3.3.

As the cover $\mathcal{J}$ is at most countable (because $I$ is countable), we can enumerate its elements arbitrarily as $\mathcal{J} = \{J_1, J_2, \ldots\}$. Define the weights $v^r = (v^r_J)_{J \in \mathcal{J}}$ as

$$v^r_J := \begin{cases} w_J & \text{when } J = J_k \text{ for } k \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

For every $r \in \mathbb{N}$, the weight vector $u v^r$ evidently satisfies $\sum_J u v^r_J \leq 1$ for all $u > 0$ sufficiently small (depending on $r$). The main idea of the proof is to apply Corollary 3.6 to the weight vector $v = (t/n) v^r$, then let $n \to \infty$, and finally $r \to \infty$.

Let us begin by considering the second term in Corollary 3.6. We can write

$$(I - W^{(t/n)v^r} + R^{(t/n)v^r})^n = \left( \left(1 - \frac{t}{n}\right) I + \frac{t}{n} (I - W^{v^r} + R^{v^r}) \right)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{t}{n}\right)^{n-k} \left(\frac{t}{n}\right)^k (I - W^{v^r} + R^{v^r})^k$$

$$= \mathbb{E}[\left(I - W^{v^r} + R^{v^r}\right)^n],$$

where we defined the Binomial random variables $Z_n \sim \text{Bin}(n, t/n)$. The random variables $Z_n$ converge weakly as $n \to \infty$ to the Poisson random variable $Z_\infty \sim \text{Pois}(t)$. To take the limit of the above expectation, we need a simple estimate that will be useful in the sequel.

**Lemma 3.9.** Let $(c_j)_{j \in I}$ be any nonnegative vector. Then

$$\sum_{j \in I} (I - W^{v^r} + R^{v^r})^k_{ij} c_j \leq 2^k \max_{0 \leq \ell \leq k} \sum_{j \in I} (I - W)_{ij}^{\ell} c_j$$

for every $i \in I$ and $k \geq 0$.  

We now let \( r \to \infty \). Note that \( W^{v^r} \uparrow W \) and \( R^{v^r} \uparrow R \) elementwise and, arguing as in the proof of Lemma 3.9, we have \( I - W^{v^r} + R^{v^r} \leq I + (I - W + R) \) elementwise where
\[
\sum_{k=0}^{\infty} \sum_{i,j \in I} \frac{t^k e^{-t}}{k!} \delta_i f \{ I + (I - W + R) \}^{k}_{ij} (\rho \otimes \tilde{\rho}) \eta_j \leq e^t \sup_{\ell \geq 0} \sum_{i,j \in I} \delta_i f (I - W + R)^{\ell}_{ij} (\rho \otimes \tilde{\rho}) \eta_j
\]
is finite by assumption (2.1) and as \( f \) is local. We therefore obtain
\[
\lim_{r \to \infty} \lim_{n \to \infty} \sum_{i,j \in I} \delta_i f \{ I - W^{(t/n)v^r} + R^{(t/n)v^r} \}^{n}_{ij} (\rho \otimes \tilde{\rho}) \eta_j = \sum_{i,j \in I} \delta_i f V_{ij}^{r[t]} (\rho \otimes \tilde{\rho}) \eta_j
\]
by dominated convergence. That is, the second term in Corollary 3.6 with \( v = (t/n)v^r \) converges as \( n \to \infty \) and \( r \to \infty \) to the second term in statement of the present result.
It remains to establish the corresponding conclusion for the first term in Corollary 3.6, which proceeds much along the same lines. We begin by noting that

\[
\frac{1}{n} \sum_{k=0}^{n-1} (I - W(t/n)v^r + R(t/n)v^r)^k = \frac{1}{n} \sum_{k=0}^{n-1} \left( \left( 1 - \frac{t}{n} \right) I + \frac{t}{n} (I - Wv^r + Rv^r) \right)^k
\]

\[
= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \binom{k}{\ell} \left( 1 - \frac{t}{n} \right)^{k-\ell} \left( \frac{t}{n} \right)^\ell (I - Wv^r + Rv^r)^\ell
\]

\[
= \sum_{\ell=0}^{n-1} p_{\ell}^{(n)} (I - Wv^r + Rv^r)^\ell,
\]

where we have defined

\[
p_{\ell}^{(n)} = \frac{1}{n} \sum_{k=\ell}^{n-1} \binom{k}{\ell} \left( 1 - \frac{t}{n} \right)^{k-\ell} \left( \frac{t}{n} \right)^\ell \frac{1}{t} \int_0^t s^{\ell} e^{-s} ds
\]

for \( \ell < n \). An elementary computation yields

\[
\sum_{\ell=0}^{n-1} p_{\ell}^{(n)} = 1 \quad \text{and} \quad p_{\ell}^{(n)} \xrightarrow{n \to \infty} p_{\ell}^{(\infty)} = \frac{1}{t} \int_0^t s^{\ell} e^{-s} ds.
\]

We can therefore introduce \( \{0, 1, \ldots\} \)-valued random variables \( Y_n \) with \( P[Y_n = \ell] = p_{\ell}^{(n)} \) for \( \ell < n \), and we have shown above that \( Y_n \to Y_\infty \) weakly and that

\[
\frac{1}{n} \sum_{k=0}^{n-1} (I - W(t/n)v^r + R(t/n)v^r)^k = E[(I - Wv^r + Rv^r)Y_n].
\]

The first term in Corollary 3.6 with \( v = (t/n)v^r \) can be written as

\[
\sum_{i,j \in I} \delta_{ij} \sum_{k=0}^{n-1} (I - W(t/n)v^r + R(t/n)v^r)^k a_{ij}^{(t/n)v^r} = t E[h(Y_n)],
\]

where we have defined

\[
h(k) = \sum_{i,j \in I} \delta_{ij} (I - Wv^r + Rv^r)_{ij} a_{ij} v^r.
\]

We now proceed essentially as above. We can assume without loss of generality that

\[
\sup_{\ell \geq 0} \sum_{i,j \in I} \delta_{ij} (I - W + R)^\ell_{ij} a_j < \infty,
\]

as otherwise the right-hand side in the statement of the present result is infinite and the estimate is trivial. It consequently follows from Lemma 3.9 that \( h(k) \leq C^2 k \) for some
finite constant $C < \infty$ and all $k \geq 0$. A similar computation as was done above shows that $(h(Y_n))_{n \geq 0}$ is uniformly integrable, and therefore $E[h(Y_n)] \rightarrow E[h(Y_\infty)]$. In particular, the first term in Corollary 3.6 with $v = (t/n)v^r$ converges as $n \rightarrow \infty$ to

$$
\lim_{n \rightarrow \infty} \sum_{i,j \in I} \delta_i f \sum_{k=0}^{n-1} (I - W(t/n)v^r + R(t/n)v^r)^k a_j^{(t/n)v^r} = \sum_{i,j \in I} \delta_i f N^r v_i a_j^r
$$

where

$$
N^r = \sum_{k=0}^{\infty} \int_0^t \frac{s^k e^{-s}}{k!} ds (I - Wv^r + Rv^r)^k.
$$

Similarly, letting $r \rightarrow \infty$ and repeating exactly the arguments used above for the second term of Corollary 3.6, we obtain by dominated convergence

$$
\lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i,j \in I} \delta_i f \sum_{k=0}^{n-1} (I - W(t/n)v^r + R(t/n)v^r)^k a_j^{(t/n)v^r} = \sum_{i,j \in I} \delta_i f \tilde{N}_{ij} a_j
$$

where

$$
\tilde{N} = \sum_{k=0}^{\infty} \int_0^t \frac{s^k e^{-s}}{k!} ds (I - W + R)^k.
$$

To conclude, we have shown that applying Corollary 3.6 to the weight vector $v = (t/n)v^r$ and taking the limit as $n \rightarrow \infty$ and $r \rightarrow \infty$, respectively, yields the estimate

$$
|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f \tilde{N}_{ij} a_j + \sum_{i,j \in I} \delta_i f V_{ij}^l (\rho \otimes \tilde{\rho})\eta_j.
$$

It remains to note that $tk^ke^{-t}/k!$ is the density of a Gamma distribution (with shape $k + 1$ and scale 1), so $\int_0^t s^k e^{-s}/k! ds \leq 1$ and thus $\tilde{N} \leq N$ elementwise.

We can now complete the proof of Theorem 2.4.

**Proof of Theorem 2.4.** Once again, we will assume without loss of generality that $f$ is a local function (so that only finitely many $\delta_i f$ are nonzero). The extension to quasilocal $f$ follows readily by localization as in the proof of Lemma 3.3.

We begin by showing that the second term in Proposition 3.8 vanishes as $t \rightarrow \infty$. Indeed, for any $n \geq 0$, we can evidently estimate the second term as

$$
\sum_{k=0}^{\infty} \frac{t^k e^{-t}}{k!} \sum_{i,j \in I} \delta_i f (I - W + R)^k_{ij} (\rho \otimes \tilde{\rho})\eta_j
$$

$$
\leq \sup_{t \geq 0} \sum_{i,j \in I} \delta_i f (I - W + R)^{t^k e^{-t}}_{ij} (\rho \otimes \tilde{\rho})\eta_j \sum_{k=0}^{n} \frac{t^k e^{-t}}{k!}
$$

$$
+ \sup_{t \geq n} \sum_{i,j \in I} \delta_i f (I - W + R)^{t^k e^{-t}}_{ij} (\rho \otimes \tilde{\rho})\eta_j.
$$
By assumption (2.1) and as $f$ is local, the two terms on the right vanish as $t \to \infty$ and $n \to \infty$, respectively. Thus second term in Proposition 3.8 vanishes as $t \to \infty$.

We have now proved the estimate

$$|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_{ij} f N_{ij} a_j.$$ 

To complete the proof of Theorem 2.4, it remains to establish the identity $N = D W^{-1}$. This is an exercise in matrix algebra. By the definition of the matrix product, we have

$$(I - W + R)^p = \sum_{k=0}^{p} \sum_{n_0, \ldots, n_k \geq 0} (I - W)^{n_k} R \cdots (I - W)^{n_1} R (I - W)^{n_0}.$$ 

We can therefore write

$$\sum_{p=0}^{\infty} (I - W + R)^p$$

$$= \sum_{k=0}^{\infty} \sum_{n_0, \ldots, n_k \geq 0} \sum_{p=0}^{\infty} \xi_{n_0+\cdots+n_k=p-k} 1_{k \leq p} (I - W)^{n_k} R \cdots (I - W)^{n_1} R (I - W)^{n_0}$$

$$= \sum_{k=0}^{\infty} \sum_{n_0, \ldots, n_k \geq 0} (I - W)^{n_k} R \cdots (I - W)^{n_1} R (I - W)^{n_0}$$

$$= \sum_{k=0}^{\infty} (W^{-1} R)^k W^{-1},$$

where we have used that $W^{-1} = \sum_{n=0}^{\infty} (I - W)^n$ as $W$ is diagonal with $0 < W_{ii} \leq 1$.

### 3.4. Proof of Corollary 2.8.

Note that $\sup_i W_{ii} < \infty$ in all parts of Corollary 2.8 (either by assumption or as $\text{card} I < \infty$). Moreover, it is easily seen that all parts of Corollary 2.8 as well as the conclusion of Theorem 2.4 are unchanged if all the weights are multiplied by the same constant. We may therefore assume without loss of generality that $\sup_i W_{ii} \leq 1$.

Next, we note that as $\rho$ and $\tilde{\rho}$ are tempered, we have

$$\sup_{i \in I} (\rho \otimes \tilde{\rho}) \eta_i \leq \sup_{i \in I} \rho \eta_i (\cdot, x_i^*) + \sup_{i \in I} \tilde{\rho} \eta_i (x_i^*, \cdot) < \infty,$$

by the triangle inequality. To verify (2.1), it therefore suffices to show that

$$(3.1) \quad \lim_{k \to \infty} \sum_{j \in I} (I - W + R)^k_{ij} = 0 \quad \text{for all } i \in I.$$

We now proceed to verify this condition in the different cases of Corollary 2.8.
Proof of Corollary 2.8(1). It was shown at the end of the proof of Theorem 2.4 that
\[ \sum_{k=0}^{\infty} (I - W + R)^k = \sum_{k=0}^{\infty} (W^{-1}R)^k W^{-1} = DW^{-1}. \]
As $W^{-1}$ has finite entries, $D < \infty$ certainly implies that $(I - W + R)^k \to 0$ as $k \to \infty$ elementwise. But this trivially yields (3.1) when $\text{card } I < \infty$. \hfill \Box

Proof of Corollary 2.8(2). Note that we can write
\[ D = \sum_{k=0}^{\infty} (W^{-1}R)^k = \sum_{p=0}^{n-1} (W^{-1}R)^p \sum_{k=0}^{\infty} (W^{-1}R)^{nk}. \]
Therefore, if $R < \infty$ and $\| (W^{-1}R)^n \| < 1$, we can estimate
\[ \| D \| \leq \left\| \sum_{p=0}^{n-1} (W^{-1}R)^p \right\| \sum_{k=0}^{\infty} \| (W^{-1}R)^n \|^k < \infty. \]
Thus $D < \infty$ and we conclude by the previous part. \hfill \Box

Proof of Corollary 2.8(3). We give a simple probabilistic proof (a more complicated matrix-analytic proof could be given along the lines of [3, Theorem 3.21]). Let $P = W^{-1}R$. As $\| P \|_\infty < 1$, the infinite matrix $P$ is substochastic. Thus $P$ is the transition probability matrix of a killed Markov chain $(X_n)_{n \geq 0}$ such that $P[X_n = j | X_{n-1} = i] = P_{ij}$ and $P[X_n \text{ is dead} | X_{n-1} = i] = 1 - \sum_j P_{ij}$ (once the chain dies, it stays dead). Denote by $\zeta = \inf \{ n : X_n \text{ is dead} \}$ the killing time of the chain. Then we obtain
\[ P[\zeta > n | X_0 = i] = P[X_n \text{ is not dead} | X_0 = i] = \sum_{j \in I} P^n_{ij} \leq \| P^n \|_\infty \leq \| P \|_\infty^n. \]
Therefore, as $\| P \|_\infty < 1$, we find by letting $n \to \infty$ that $P[\zeta = \infty | X_0 = i] = 0$. That is, the chain dies eventually with unit probability for any initial condition.

Now define $\hat{P} = I - W + R = I - W + WP$. As $\sup_i \hat{W}_{ii} \leq 1$, the matrix $\hat{P}$ is also substochastic and corresponds to the following transition mechanism. If $X_{n-1} = i$, then at time $n$ we flip a biased coin that comes up heads with probability $\hat{W}_{ii}$. In case of heads we make a transition according to the matrix $P$, but in case of tails we leave the current state unchanged. From this description, it is evident that we can construct a Markov chain $(\hat{X}_n)_{n \geq 0}$ with transition matrix $\hat{P}$ by modifying the chain $(X_n)_{n \geq 0}$ as follows. Conditionally on $(X_n)_{n \geq 0}$, draw independent random variables $(\xi_n)_{n \geq 0}$ such that $\xi_n$ is geometrically distributed with parameter $W_{X_n, X_n}$. Now define the process $(\hat{X}_n)_{n \geq 0}$ such that it stays in state $X_0$ for the first $\xi_0$ time steps, then is in state $X_1$ for the next $\xi_1$ time steps, etc. By construction, the resulting process is Markov with transition matrix $\hat{P}$. Moreover, as $\zeta < \infty$ a.s., we have $\zeta := \inf \{ n : \hat{X}_n \text{ is dead} \} < \infty$ a.s. also. Thus
\[ \lim_{n \to \infty} \sum_{j \in I} (I - W + R)^n_{ij} = \lim_{n \to \infty} P[\zeta > n | X_0 = i] = 0. \]
for every $i \in I$. We have therefore established (3.1).

**Proof of Corollary 2.8(4).** We begin by writing as above
\begin{equation*}
\sum_{k=0}^{\infty} (I - W + R)^k = \sum_{k=0}^{\infty} (W^{-1} R)^k W^{-1} = \sum_{k=0}^{\infty} W^{-1} (R W^{-1})^k,
\end{equation*}
where the last identity is straightforward. Arguing as in Corollary 2.8(2), we obtain
\begin{equation*}
W_{ii} \sum_{k=0}^{\infty} \sum_{j \in I} (I - W + R)^k_{ij} = \sum_{j \in I} \sum_{k=0}^{\infty} (R W^{-1})^k_{ij} \leq \left\| \sum_{k=0}^{\infty} (R W^{-1})^k \right\|_\infty \\
\leq \sum_{p=0}^{n-1} \| R W^{-1} \|_1^p \sum_{k=0}^{\infty} \| (R W^{-1})^k \|_\infty < \infty.
\end{equation*}
It follows immediately that (3.1) holds.

**Proof of Corollary 2.8(5).** Note that
\begin{equation*}
\sum_{j \in I} (R W^{-1})^k_{ij} \| \eta_j \| < \| (R W^{-1})^k_1 \|_1 \sum_{j \in I} \| \eta_j \| \leq \| R W^{-1} \|_1 \sum_{j \in I} \| \eta_j \|.
\end{equation*}
Thus $\sum_j \| \eta_j \| < \infty$ and $\| R W^{-1} \|_1 < 1$ yield
\begin{equation*}
\sum_{k=0}^{\infty} \sum_{j \in I} (I - W + R)^k_{ij} \| \eta_j \| = W_{ii}^{-1} \sum_{k=0}^{\infty} \sum_{j \in I} (R W^{-1})^k_{ij} \| \eta_j \| < \infty,
\end{equation*}
which evidently implies
\begin{equation*}
\lim_{k \to \infty} \sum_{j \in I} (I - W + R)^k_{ij} (\rho \otimes \tilde{\rho}) \eta_j = 0 \quad \text{for all } i \in I.
\end{equation*}
We have therefore established (2.1).

**Proof of Corollary 2.8(6).** Let $r = \sup \{m(i,j) : R_{ij} > 0\}$ (which is finite by assumption), and choose $\beta > 0$ such that $\| R W^{-1} \|_1 < e^{-\beta r}$. Then we can estimate
\begin{equation*}
\|RW^{-1}\|_{1,\beta m} := \sup_{j \in I} \sum_{i \in I} e^{\beta m(i,j)} (R W^{-1})_{ij} \leq e^{\beta r} \| R W^{-1} \|_1 < 1.
\end{equation*}
As $m$ is a pseudometric, it satisfies the triangle inequality and it is therefore easily seen that $\| \cdot \|_{1,\beta m}$ is a matrix norm. In particular, we can estimate
\begin{equation*}
\| RW^{-1} \|_{1,\beta m}^n \leq \| (R W^{-1})^n \|_{1,\beta m} \leq \| R W^{-1} \|_{1,\beta m}^n
\end{equation*}
for every $i, j \in I$. But then
\begin{equation*}
\| (R W^{-1})^n \|_\infty = \sup_{i \in I} \sum_{j \in I} (R W^{-1})^n_{ij} \leq \| R W^{-1} \|_{1,\beta m}^n \sup_{j \in I} \sum_{i \in I} e^{-\beta m(i,j)} < \infty
\end{equation*}
for all $n$. We therefore have $\| R W^{-1} \|_\infty < \infty$, and we can choose $n$ sufficiently large that $\| (R W^{-1})^n \|_\infty < 1$. The conclusion now follows from Corollary 2.8(4).
3.5. Proof of Theorem 2.12. In the case of one-sided local updates, the measure $\rho_{\leq k}$ is $\gamma^J$-invariant for $\tau(J) = k$ (but not for $\tau(J) < k$). The proof of Theorem 2.12 therefore proceeds by induction on $k$. In each stage of the induction, we apply the logic of Theorem 2.4 to the partial local updates $(\gamma^J)_{J \in \mathbb{J} : \tau(J) = k}$, and use the induction hypothesis to estimate the remainder term.

Throughout this section, we work in the setting of Theorem 2.12. Define

$$I_{\leq k} := \{i \in I : \tau(i) \leq k\}, \quad I_k := \{i \in I : \tau(i) = k\}.$$

Note that we can assume without loss of generality that $R_{ij} = 0$ whenever $\tau(j) > \tau(i)$. Indeed, the local update rule $\gamma^J_x$ does not depend on $x_j$ for $\tau(j) > \tau(J)$, so we can trivially choose the coupling $Q^J_{x,z}$ for $\mathcal{P}^J_{x,z} = \mathcal{P}^J_{x,z}$ such that $Q^J_{x,z}\eta_i = 0$ for all $i \in J$. On the other hand, the choice $R_{ij} = 0$ evidently yields the smallest bound in Theorem 2.12. In the sequel, we will always assume that $R_{ij} = 0$ whenever $\tau(j) > \tau(i)$.

The key induction step is formalized by the following result.

**Proposition 3.10.** Assume (2.1). Let $(\beta_i)_{i \in I_{\leq k-1}}$ be nonnegative weights such that

$$|\rho_{\leq k-1}g - \tilde{\rho}_{\leq k-1}g| \leq \sum_{i \in I_{\leq k-1}} \delta_i g \beta_i$$

for every bounded measurable quasilocal function $g$ on $\mathcal{S}_{\leq k-1}$ so that $\delta_i g < \infty \forall i$. Then

$$|\rho_{\leq k}f - \tilde{\rho}_{\leq k}f| \leq \sum_{j \in I_{\leq k-1}} \left\{ \delta_j f + \sum_{i,j \in I_k} \delta_i f D_{ij} (W^{-1} R_{ij}) \right\} \beta_j + \sum_{i,j \in I_k} \delta_i f D_{ij} W_{jj}^{-1} a_j$$

for every bounded measurable quasilocal function $f$ on $\mathcal{S}_{\leq k}$ so that $\delta_i f < \infty \forall i$.

**Proof.** We fix throughout the proof a bounded and measurable local function $f : \mathcal{S}_{\leq k} \to \mathbb{R}$ such that $\delta_i f < \infty$ for all $i \in I_{\leq k}$. The extension of the conclusion to quasilocal functions $f$ follows readily by localization as in the proof of Lemma 3.3.

We denote by $G^\nu$ and $\tilde{G}^\nu$ the Gibbs samplers as defined in section 3.2. Let us enumerate the partial cover $\{J \in \mathbb{J} : \tau(J) = k\}$ as $\{J_1, J_2, \ldots\}$, and define the weights $\nu^r$ as in the proof of Proposition 3.8. By the definition of the one-sided local update rule, $\rho_{\leq k}$ is $G^\nu$-invariant and $\tilde{\rho}_{\leq k}$ is $\tilde{G}^\nu$-invariant for every $r, u$ such that $\sum_J u \nu^r_J \leq 1$. Thus

$$|\rho_{\leq k}f - \tilde{\rho}_{\leq k}f| \leq \sum_{i,j \in I_{\leq k}} \delta_i f N_{ij}^{\nu^r(n)} a_j^{\nu^r} + |\rho_{\leq k}(G^\nu)^n f - \tilde{\rho}_{\leq k}(\tilde{G}^\nu)^n f|$$

as in the proof of Corollary 3.6, with the only distinction that we refrain from using the Wasserstein matrix to expand the second term in the proof of Proposition 3.2. We now use the induction hypothesis to obtain an improved estimate for the second term.
LEMMA 3.11. We can estimate
\[
|\rho_{\leq k} g - \tilde{\rho}_{\leq k} g| \leq \sum_{i \in I_{\leq k-1}} \delta_i g \beta_i + 3 \sum_{i \in I_k} \delta_i g (\rho \otimes \tilde{\rho}) \eta_i
\]
for any bounded and measurable quasilocal function \( g : \mathbb{S}_{\leq k} \rightarrow \mathbb{R} \) such that \( \delta_i g < \infty \) \( \forall i \).

PROOF. For any \( x \in \mathbb{S}_{\leq k} \) we can estimate
\[
|\rho_{\leq k} g - \tilde{\rho}_{\leq k} g| \leq |\rho_{\leq k-1} \hat{g}_x - \tilde{\rho}_{\leq k-1} \hat{g}_x| + |\rho_{\leq k} (g - \hat{g}_x)| + |\tilde{\rho}_{\leq k} (g - \hat{g}_x)|,
\]
where we defined \( \hat{g}_x(z) := g(z^I_{\leq k-1} x^I_k) \). By Lemma 3.3 we have
\[
|g(z) - \hat{g}_x(z)| \leq \sum_{i \in I_k} \delta_i g \eta_i(z_i, x_i).
\]

We can therefore estimate using the induction hypothesis and the triangle inequality
\[
|\rho_{\leq k} g - \tilde{\rho}_{\leq k} g| \leq \sum_{i \in I_{\leq k-1}} \delta_i g \beta_i + \sum_{i \in I_k} \delta_i g \{ \rho \eta_i (\cdot, \bar{x}_i) + \eta_i (\bar{x}_i, x_i) + \tilde{\rho} \eta_i (\cdot, x_i) \}
\]
for all \( x, \tilde{x} \in \mathbb{S}_{\leq k} \). Now integrate this expression with respect to \( \rho(dx) \tilde{\rho}(d\tilde{x}) \). \( \square \)

To lighten the notation somewhat we will write \( v = uv^r \) until further notice. Note that by construction \( a^r_j = 0 \) whenever \( \tau(j) < k \), while \( R^r_{ij} = 0 \) whenever \( \tau(j) > \tau(i) \) by assumption. Thus we obtain using Lemma 3.11 and Lemma 3.5
\[
|\rho_{\leq k} f - \tilde{\rho}_{\leq k} f| \leq \sum_{i,j \in I_k} \delta_i f N^{(a)}_{ij} a^r_j + 3 \sum_{i,j \in I_k} \delta_i f (I - W^v + R^v)_{ij}^n (\rho \otimes \tilde{\rho}) \eta_j + \sum_{i \in I_{\leq k}} \sum_{j \in I_{\leq k-1}} \delta_i f (I - W^v + R^v)_{ij}^n \beta_j,
\]
provided that \( \sum_i \delta_i f (I - W^v + R^v)_{ij}^n < \infty \) for all \( j \).

Next, note that as \( v_J = 0 \) for \( \tau(J) < k \), we have \( R^v_{ij} = W^v_{ij} = 0 \) for \( i \in I_{\leq k-1} \). Thus
\[
V^v = I - W^v + R^v = \begin{pmatrix} \tilde{V}^v & \tilde{R}^v \\ 0 & I \end{pmatrix},
\]
where \( \tilde{V}^v := (V^v)_{i,j \in I_k} \) and \( \tilde{R}^v := (R^v)_{i,j \in I_{\leq k-1}} \). In particular,
\[
(I - W^v + R^v)^n = \begin{pmatrix} (\tilde{V}^v)^n & \sum_{k=0}^{n-1} (\tilde{V}^v)^k \tilde{R}^v \\ 0 & I \end{pmatrix}.
\]
Moreover, as $R^y_{ij} = 0$ whenever $\tau(j) > \tau(i)$, we evidently have $(\hat{V}^y)_{ij}^k = (V^y)_{ij}^k$ for $i, j \in I_k$. Substituting into the above expression, we obtain

$$|\rho_{\leq k} f - \hat{\rho}_{\leq k} f| \leq \sum_{i,j \in I_k} \delta_i f N^{(n)}_{ij} a_j^y + 3 \sum_{i,j \in I_k} \delta_i f (I - W^y + R^y)^n_{ij} (\rho \otimes \hat{\rho}) \eta_j$$

$$+ \sum_{j \in I_{\leq k-1}} \left\{ \delta_j f + \sum_{i,l \in I_k} \delta_i f N^{(n)}_{il} R^y_{lj} \right\} \beta_j$$

provided that $\sum_i \delta_i f (I - W^y + R^y)^n_{ij} < \infty$ for all $j$. But the latter is easily verified using (2.1) and Lemma 3.9, as $f$ is local and $\delta_i f < \infty$ for all $i$ by assumption.

The remainder of the proof now proceeds precisely as in the proof of Proposition 3.8 and Theorem 2.4. We set $v = (t/n)v^r$, let $n \to \infty$ and then $r \to \infty$. The arguments for the first two terms are identical to the proof of Proposition 3.8, while the argument for the third term is essentially identical to the argument for the first term. The proof is then completed as in the proof of Theorem 2.4. We leave the details for the reader. □

We now proceed to complete the proof of Theorem 2.12.

**Proof of Theorem 2.12.** Consider first the case that $k_- := \inf_{i \in I} \tau(i) > -\infty$. In this setting, we say that the comparison theorem holds for a given $k \geq k_-$ if we have

$$|\rho_{\leq k} f - \hat{\rho}_{\leq k} f| \leq \sum_{i,j \in I_{\leq k}} \delta_i f D^i_{j} W^{-1}_{jj} a_j$$

for every bounded measurable quasilocal function $f$ on $S_{\leq k}$ such that $\delta_i f < \infty \ \forall i$. We can evidently apply Theorem 2.4 to show that the comparison theorem holds for $k_-$. We will now use Proposition 3.10 to show that if the comparison theorem holds for $k - 1$, then it holds for $k$ also. Then the comparison theorem holds for every $k \geq k_-$ by induction, so the conclusion of Theorem 2.12 holds whenever $f$ is a local function. The extension to quasilocal $f$ follows readily by localization as in the proof of Lemma 3.3.

We now complete the induction step. When the comparison theorem holds for $k - 1$ (the induction hypothesis), we can apply Proposition 3.10 with

$$\beta_i = \sum_{j \in I_{\leq k-1}} D^i_{j} W^{-1}_{jj} a_j.$$ 

This gives

$$|\rho_{\leq k} f - \hat{\rho}_{\leq k} f| \leq \sum_{j,q \in I_{\leq k-1}} \sum_{i,l \in I_k} \delta_i f D^i_{il} (W^{-1} R)_{lq} D^q_{j} W^{-1}_{jj} a_j$$

$$+ \sum_{i,j \in I_{\leq k-1}} \delta_i f D^i_{j} W^{-1}_{jj} a_j + \sum_{i,j \in I_k} \delta_i f D^i_{j} W^{-1}_{jj} a_j$$

We now proceed to complete the proof of Theorem 2.12.
for every bounded measurable quasilocal function $f$ on $\mathcal{S}_{\leq k}$ so that $\delta_i f < \infty \ \forall i$. To complete the proof, it therefore suffices to show that we have

$$D_{ij} = \sum_{q \in I \leq k-1} \sum_{l \in I} D_{il} (W^{-1}R)_{lj} D_{qj} \quad \text{for } i \in I_k, \ j \in I \leq k-1.$$ 

To see this, note that as $R_{ij} = 0$ for $\tau(i) < \tau(j)$, we can write

$$D_{ij} = \sum_{p=1}^{\infty} \sum_{j_1, \ldots, j_{p-1} \in I} (W^{-1}R)_{ij_{p-1}} \cdots (W^{-1}R)_{j_{j_1}j} (W^{-1}R)_{j_1j}$$

$$= \sum_{p=1}^{\infty} \sum_{n=1}^{p} \sum_{l \in I} \sum_{q \in I \leq k-1} (W^{-1}R)_{il}^{n-1} (W^{-1}R)_{lj} (W^{-1}R)_{qj}^{p-n}$$

for $i \in I_k$ and $j \in I \leq k-1$, where we have used that whenever $\tau(j_1) \leq \cdots \leq \tau(j_{p-1}) \leq k$ there exists $1 \leq n \leq p$ such that $j_1, \ldots, j_{p-n} \in I \leq k-1$ and $j_{p-n+1}, \ldots, j_{p-1} \in I_k$. Rearranging the last expression yields the desired identity for $D_{ij}$, completing the proof for the case $k_- = -\infty$ (note that in this case the additional assumption (2.2) was not needed).

We now turn to the case that $k_- = -\infty$. Let us say that $(\beta_i)_{i \in I \leq k}$ is a $k$-estimate if

$$|\rho_{\leq k}g - \tilde{\rho}_{\leq k}g| \leq \sum_{i \in I \leq k} \delta_i g \beta_i$$

for every bounded measurable quasilocal function $g$ on $\mathcal{S}_{\leq k}$ such that $\delta_i g < \infty \ \forall i$. Then the conclusion of Proposition 3.10 can be reformulated as follows: if $(\beta_i)_{i \in I \leq k-1}$ is a $(k - 1)$-estimate, then $(\beta_i')_{i \in I \leq k}$ is a $k$-estimate with $\beta_i' = \beta_i$ for $i \in I \leq k-1$ and

$$\beta_i' = \sum_{j \in I \leq k-1} \sum_{l \in I} D_{il} (W^{-1}R)_{lj} \beta_j + \sum_{j \in I} D_{ij} W_{jj}^{-1} a_j$$

for $i \in I_k$. We can therefore repeatedly apply Proposition 3.10 to extend an initial estimate. In particular, if we fix $k \in \mathbb{Z}$ and $n \geq 1$, and if $(\beta_i)_{i \in I \leq k-n}$ is a $(k - n)$-estimate, then we can obtain a $k$-estimate $(\beta_i')_{i \in I \leq k}$ by iterating Proposition 3.10 $n$ times. We claim that

$$\beta_i' = \sum_{s=k-n+1}^{k-r} \left\{ \sum_{j \in I \leq k-n} \sum_{l \in I_s} D_{il} (W^{-1}R)_{lj} \beta_j + \sum_{j \in I_s} D_{ij} W_{jj}^{-1} a_j \right\}$$

for $0 \leq r \leq n - 1$ and $i \in I_{k-r}$. To see this, we proceed again by induction. As $(\beta_i)_{i \in I \leq k-n}$ is a $(k-n)$-estimate, the expression is valid for $r = n - 1$ by Proposition 3.10. Now suppose
the expression is valid for all \( u < r \leq n - 1 \). Then we obtain

\[
\beta'_t = \sum_{j \in I_{k-u}} \sum_{l \in I_{k-u}} D_{ij} (W^{-1}R)_{lj} \beta_j + \sum_{j \in I_{k-u}} D_{ij} W_{jj}^{-1} a_j \\
+ \sum_{s=k-n+1}^{k-u-1} \sum_{j \in I_s} \sum_{l \in I_{k-u}} \sum_{t=k-n+1}^{s} \sum_{q \in I_{k-n}} \sum_{p \in I_t} D_{il} (W^{-1}R)_{lj} D_{jp} (W^{-1}R)_{pq} \beta_q \\
+ \sum_{s=k-n+1}^{k-u-1} \sum_{j \in I_s} \sum_{l \in I_{k-u}} \sum_{t=k-n+1}^{s} \sum_{q \in I_t} D_{il} (W^{-1}R)_{lj} D_{jq} W_{qq}^{-1} a_q
\]

for \( i \in I_{k-u} \) by Proposition 3.10. Rearranging the sums yields

\[
\beta'_t = \sum_{j \in I_{k-u}} \sum_{l \in I_{k-u}} D_{ij} (W^{-1}R)_{lj} \beta_j + \sum_{j \in I_{k-u}} D_{ij} W_{jj}^{-1} a_j \\
+ \sum_{t=k-n+1} \left\{ \sum_{q \in I_{k-n}} \sum_{p \in I_t} D_{ip} (W^{-1}R)_{pq} \beta_q + \sum_{p \in I_t} \tilde{D}_{ip} W_{pp}^{-1} a_p \right\}
\]

for \( i \in I_{k-u} \), where we have defined

\[
\tilde{D}_{ij} := \sum_{t=1}^{t-1} \sum_{q \in I_t} \sum_{l \in I_t} D_{il} (W^{-1}R)_{lj} D_{qj}
\]

whenever \( i \in I_t \) and \( j \in I_s \) for \( s < t \). But as \( D_{qj} = 0 \) when \( \tau(q) < \tau(j) \), we have

\[
\tilde{D}_{ij} = \sum_{q \in I_{k-n}} \sum_{l \in I_t} D_{il} (W^{-1}R)_{lj} D_{qj} = D_{ij} \quad \text{for } i \in I_t, \ j \in I_{k-n}
\]

using the identity used in the proof for the case \( k_- > -\infty \), and the claim follows.

We can now complete the proof for the case \( k_- = -\infty \). It suffices to prove the theorem for a given local function \( f \) (the extension to quasilocal \( f \) follows readily as in the proof of Lemma 3.3). Let us therefore fix a \( K \)-local function \( f \) for some \( K \in \mathcal{I} \), and let \( k = \max_{i \in K} \tau(i) \) and \( n \geq 1 \). By Lemma 3.3, we find that \((\tilde{\beta}_i)_{i \in I_{k-n}}\) is trivially a \((k-n)\)-estimate if we set \( \beta_i = (\rho \otimes \tilde{\rho}) \eta_i \) for \( i \in I_{k-n} \). We therefore obtain

\[
|\rho f - \tilde{\rho} f| \leq \sum_{i,j \in I} \delta_i f D_{ij} W_{jj}^{-1} a_j + \sum_{i \in I} \sum_{j \in I_{k-n}} \delta_i f D_{ij} (\rho \otimes \tilde{\rho}) \eta_j
\]

from the \( k \)-estimate \((\beta'_i)_{i \in I_{k}}\) derived above, where we have used that \( DW^{-1}R \leq D \). But as \( f \) is local and \( \delta_i f < \infty \) for all \( i \) by assumption, the second term vanishes as \( n \to \infty \) by assumption (2.2). This completes the proof for the case \( k_- = -\infty \).
4. Application: particle filters. Our original motivation for developing the generalized comparison theorems of this paper was the investigation of algorithms for filtering in high dimension. In this section we will develop one such application in detail. Our result answers a question raised in [19], and also serves as a concrete illustration of the utility of the generalized comparison theorems.

4.1. Introduction and main result. Let \((X_n, Y_n)_{n \geq 0}\) be a Markov chain. We interpret \(X_n\) as the unobserved component of the model and \(Y_n\) as the observed component. A fundamental problem in this setting is to track the state of the unobserved component \(X_n\) given the history of observed data \(Y_1, \ldots, Y_n\). Such problems are ubiquitous in a wide variety of applications, ranging from classical tracking problems in navigation and robotics to large-scale forecasting problems such as weather prediction, and are broadly referred to as filtering or data assimilation problems.

In principle, the optimal solution to the tracking problem is provided by the filter
\[
\pi_n := P[X_n \in \cdot | Y_1, \ldots, Y_n].
\]
If the conditional distribution \(\pi_n\) can be computed, it yields not only a least mean square estimate of the unobserved state \(X_n\), but also a complete representation of the uncertainty in this estimate. Unfortunately, when \(X_n\) takes values in a continuous (or finite but large) state space, the filter is rarely explicitly computable and approximations become necessary. In practice, filtering is widely implemented by a class of sequential Monte Carlo algorithms called particle filters, cf. [1], that have been very successful in classical applications.

The major problem with particle filtering algorithms is that they typically require an exponential number of samples in the model dimension. Such algorithms are therefore largely useless in high-dimensional problems that arise in complex applications such as weather forecasting. We refer to [19] for a detailed discussion of these issues and for further references. In many applications, the high-dimensional nature of the problem is due to the presence of spatial degrees of freedom: \(X_n\) and \(Y_n\) at each time \(n\) are themselves random fields that evolve dynamically over time. In practice, such models are typically expected to exhibit decay of correlations. We have started in [19] to explore the possibility that such properties could be exploited to beat the curse of dimensionality by including a form of spatial localization in the filtering algorithm. In particular, the initial analysis in [19] has yielded dimension-free error bounds for the simplest possible class of local particle filtering algorithms, called block particle filters, under strong (but dimension-free) model assumptions that ensure the presence of decay of correlations.

It should be noted that the block particle filtering algorithm exhibits some significant drawbacks that could potentially be resolved by using more sophisticated algorithms. These issues are discussed at length in [19], but are beyond the scope of this paper. In the sequel, we will reconsider the same algorithm that was introduced in [19], but we provide an improved analysis of its performance on the basis of Theorem 2.4. We will see that the use of Theorem 2.4 already yields a qualitative improvement over the main result of [19].

In the remainder of this section we recall the setting of [19] and state our main result on block particle filters. The following sections are devoted to the proofs.
Dynamical model. Let \((X_n, Y_n)_{n \geq 0}\) be a Markov chain that takes values in the product space \(\mathbb{X} \times \mathbb{Y}\), and whose transition probability \(P\) can be factored as

\[
P((x, y), A) = \int 1_A(x', y') p(x, x') g(x', y') \psi(dx') \varphi(dy').
\]

Such processes are called hidden Markov models [1]. As a consequence of this definition, the unobserved process \((X_n)_{n \geq 0}\) is a Markov chain in its own right with transition density \(p\) (with respect to the reference measure \(\psi\)), while each observation \(Y_n\) is a noisy function of \(X_n\) only with observation density \(g\) (with respect to the reference measure \(\varphi\)).

Our interest is in high-dimensional hidden Markov models that possess spatial structure. To this end, we introduce a finite undirected graph \(G = (V, E)\) that determines the spatial degrees of freedom of the model. The state \((X_n, Y_n)\) at each time \(n\) is itself a random field \((X_n^v, Y_n^v)_{v \in V}\) indexed by the vertices of the graph \(G\). In particular, we choose

\[
\mathbb{X} = \prod_{v \in V} \mathbb{X}^v, \quad \mathbb{Y} = \prod_{v \in V} \mathbb{Y}^v, \quad \psi = \bigotimes_{v \in V} \psi^v, \quad \varphi = \bigotimes_{v \in V} \varphi^v,
\]

where \((\mathbb{X}^v, \psi^v)\) and \((\mathbb{Y}^v, \varphi^v)\) are measure spaces for every \(v \in V\). To define the dynamics of the model, we introduce for each \(v \in V\) a local transition density \(p^v : \mathbb{X}^v \times \mathbb{X}^v \to \mathbb{R}_+\) and local observation density \(g^v : \mathbb{X}^v \times \mathbb{Y}^v \to \mathbb{R}_+\), and we set

\[
p(x, z) = \prod_{v \in V} p^v(x, z^v), \quad g(x, y) = \prod_{v \in V} g^v(x^v, y^v).
\]

Therefore, each observation \(Y_n^v\) at location \(v\) is a noisy function of the unobserved state \(X_n^v\) at location \(v\), and the current state \(X_n^v\) is determined by the configuration \(X_{n-1}^v\) at the previous time step. We will assume throughout that the dynamics is local in the sense that \(p^v(x, z^v) = p^v(\tilde{x}, z^v)\) whenever \(x^{\mathcal{N}(v)} = \tilde{x}^{\mathcal{N}(v)}\), where \(\mathcal{N}(v) = \{v' \in V : d(v, v') \leq r\}\) denotes a neighborhood of radius \(r\) around the vertex \(v\) with respect to the graph distance \(d\). That is, the state \(X_n^v\) at time \(n\) and vertex \(v\) depends only on the past \(X_0, \ldots, X_{n-1}\) through the states \(X_{n-1}^{\mathcal{N}(v)}\) in an \(r\)-neighborhood of \(v\) in the previous time step; the interaction radius \(r\) is fixed throughout. The dependence structure of our general model is illustrated schematically in Figure 1 (in the simplest case of a linear graph \(G\) with \(r = 1\)).

Block particle filters. An important property of the filter \(\pi_n = \mathbb{P}[X_n \in \cdot | Y_1, \ldots, Y_n]\) is that it can be computed recursively. To this end, define for every probability measure \(\rho\) on \(\mathbb{X}\) the probability measures \(\mathbb{P}\rho\) and \(\mathbb{C}\rho\) as follows:

\[
\mathbb{P}\rho(dx') := \psi(dx') \int p(x, x') \rho(dx), \quad \mathbb{C}\rho(dx) := \frac{g(x, Y_n) \rho(dx)}{\int g(z, Y_n) \rho(dz)}.
\]

Then it is an elementary consequence of Bayes’ formula that [1]

\[
\pi_n = \mathbb{P}\pi_{n-1} := \mathbb{C}\rho\pi_{n-1} \quad \text{for every } n \geq 1,
\]
where the initial condition is given by $\pi_0 = \mu := P[X_0 \in \cdot]$. In the sequel we will often write $\pi_n := F_n \cdots F_1 \mu$ to indicate explicitly the initial condition of the filtering recursion.

To obtain approximate algorithms, we insert additional steps in the filtering recursion that enable a tractable implementation. The classical particle filtering algorithm inserts a random sampling step $S^N$ in the filtering recursion that replaces the current filtering distribution by the empirical measure of $N$ independent samples: that is,

$$S^N \rho := \frac{1}{N} \sum_{i=1}^N \delta_{x(i)}$$

where $(x(i))_{i=1,\ldots,N}$ are i.i.d. samples $\sim \rho$.

This gives rise to a sequential Monte Carlo algorithm that maintains at each time $N$ approximate samples from the current filtering distribution; see [1, 19] and the references therein.

As $N \to \infty$, the sampling error vanishes by the law of large numbers and the particle filter converges to the exact filter. Unfortunately, the number of samples $N$ needed to achieve a fixed error is typically exponential in the model dimension $|V|$.

To alleviate the curse of dimensionality we must localize the algorithm so that it can benefit from the decay of correlations of the underlying model. The simplest possible algorithm of this type inserts an additional localization step in the filtering recursion in the following manner. Fix a partition $\mathcal{K}$ of the vertex set $V$ into nonoverlapping blocks, and define for any probability measure $\rho$ on $\mathbb{X}$ the blocking operator $B^J \rho$ as

$$B^J \rho := \bigotimes_{K \in \mathcal{K}} B^K \rho,$$

where we denote by $B^J \rho$ for $J \subseteq V$ the marginal of $\rho$ on $\mathbb{X}^J$. That is, the blocking operation forces the underlying measure to be independent across different blocks in the partition $\mathcal{K}$.
The block particle filter $\hat{\pi}^{\mu}_{n}$ is now defined by the recursion

$$\hat{\pi}^{\mu}_{n} := \hat{F}_{n} \cdots \hat{F}_{1} \mu,$$

$$\hat{F}_{k} := C_{k}BS_{N_{P}}.$$

This algorithm is very straightforward to implement in practice, cf. [19].

**Analysis of [19].** We first introduce some notation. Recall that in our model, each vertex $v$ interacts only with vertices in an $r$-neighborhood $N(v)$ in the previous time step, where the interaction radius $r$ is fixed throughout. Given $J \subseteq V$, define the $r$-inner boundary as

$$\partial J := \{ v \in J : N(v) \not\subseteq J \}.$$

Thus $\partial J$ is the subset of vertices in $J$ that can interact with vertices outside $J$ in one time step of the dynamics. We also define the quantities

$$|K|_{\infty} := \max_{K \in \mathcal{K}} \text{card} \ K,$$

$$\Delta := \max_{v \in V} \{ v' \in V : d(v, v') \leq r \},$$

$$\Delta_{\mathcal{K}} := \max_{K \in \mathcal{K}} \text{card} \ \{ K' \in \mathcal{K} : d(K, K') \leq r \},$$

where $d(J, J') := \min_{v \in J} \min_{v' \in J'} d(v, v')$ for $J, J' \subseteq V$. Thus $|K|_{\infty}$ is the maximal size of a block, while $\Delta, \Delta_{\mathcal{K}}$ are local quantities that depend on the geometry but not on the size of the graph $G$. We finally introduce, for each $J \subseteq I$, the following local distance between random measures $\rho$ and $\rho'$:

$$\| \rho - \rho' \|_{J} := \sup_{f \in \mathcal{X}^{J} : |f| \leq 1} \mathbb{E}[|\rho f - \rho f'|_{2}]^{1/2}$$

where $\mathcal{X}^{J}$ denotes the set of $J$-local measurable functions $f$ on $\mathcal{X}$. For simplicity, we write $\pi_{n}^{\sigma} := \pi_{n}^{\delta_{x}}$ and $\hat{\pi}_{n}^{\sigma} := \hat{\pi}_{n}^{\delta_{x}}$ when the filtering recursions are initialized at a point $x \in \mathcal{X}$.

We can now recall the main result of [19].

**Theorem 4.1 (Theorem 2.1 in [19]).** There exists a constant $0 < \varepsilon_{0} < 1$, depending only on the local quantities $\Delta$ and $\Delta_{\mathcal{K}}$, such that the following holds.

Suppose there exist $\varepsilon_{0} < \varepsilon < 1$ and $0 < \kappa < 1$ such that

$$\varepsilon \leq p^{v}(x, z^{v}) \leq \varepsilon^{-1}, \quad \kappa \leq g^{v}(x^{v}, y^{v}) \leq \kappa^{-1} \quad \forall v \in V, \ x, z \in \mathcal{X}, \ y \in \mathcal{Y}.$$

Then for every $n \geq 0$, $\sigma \in \mathcal{X}, \ K \in \mathcal{K}$ and $J \subseteq K$ we have

$$\| \pi_{n}^{\sigma} - \hat{\pi}_{n}^{\sigma} \|_{J} \leq \alpha \text{card} \ J \left[ e^{-\beta_{1}d(J, \partial K)} + \frac{e^{\beta_{2}|\mathcal{X}|_{\infty}}}{N^{\frac{r}{2}}} \right],$$

where the constants $0 < \alpha, \beta_{1}, \beta_{2} < \infty$ depend only on $\varepsilon, \kappa, \Delta, \text{and } \Delta_{\mathcal{K}}$. 
The error bound in Theorem 4.1 contains two terms. The first term quantifies the bias introduced by the localization $B$, which decreases when we take larger blocks. The second term quantifies the variance introduced by the sampling $S_N$, which decreases with increasing sample size $N$ but grows exponentially in the block size. Traditional particle filtering algorithms correspond to the choice of a single block $\mathcal{K} = \{V\}$, and in this case the error grows exponentially in the dimension $\text{card} V$. To avoid this curse of dimensionality, we must tune the block size so as to optimize the tradeoff between bias and variance. As all the constants in Theorem 4.1 depend only on local quantities, the optimal block size results in a dimension-free error bound. We refer to [19] for a full discussion.

Main result. The intuition behind the block particle filtering algorithm is that the localization controls the sampling error (as it replaces the model dimension $\text{card} V$ by the block size $|\mathcal{K}|_\infty$), while the decay of correlations property of the model controls the localization error (as it ensures that the effect of the localization decreases in the distance to the block boundary). This intuition is clearly visible in the conclusion of Theorem 4.1. It is however not automatically the case that our model does indeed exhibit decay of correlations: when there are strong interactions between the vertices, phase transitions can arise and the decay of correlations can fail much as for standard models in statistical mechanics [17], in which case we cannot expect to obtain dimension-free performance for the block particle filter. Such phenomena are ruled out in Theorem 4.1 by the assumption that $\varepsilon \leq p^v \leq \varepsilon^{-1}$ for $\varepsilon > \varepsilon_0$, which ensures that the interactions in our model are sufficiently weak.

It is notoriously challenging to obtain sharp quantitative results for interacting models, and it is unlikely that one could obtain realistic values for the constants in Theorem 4.1 at the level of generality considered here. More concerning, however, is that the weak interaction assumption of Theorem 4.1 is already unsatisfactory at the qualitative level. Note that there is no interaction between the vertices in the extreme case $\varepsilon = 1$; the assumption $\varepsilon > \varepsilon_0$ should be viewed as a perturbation of this situation (i.e., weak interactions). However, setting $\varepsilon = 1$ turns off not only the interaction between different vertices, but also the interaction between the same vertex at different times: in this setting the dynamics of the model become trivial. In contrast, one would expect that it is only the strength of the spatial interactions, and not the local dynamics, that is relevant for dimension-free errors, so that Theorem 4.1 places an unnatural restriction on our understanding of block particle filters.

Our main result resolves this qualitative deficiency of Theorem 4.1. Rather than assuming $p^v(x, z^v) \approx 1$ as in Theorem 4.1, we will assume only that the spatial interactions are weak in the sense that $p^v(x, z^v) \approx q^v(x^v, z^v)$, where the transition density $q^v$ describes the local dynamics at the vertex $v$ in the absence of interactions.

**Theorem 4.2.** For any $0 < \delta < 1$ there exists $0 < \varepsilon_0 < 1$, depending only on $\delta$ and $\Delta$, such that the following holds. Suppose there exist $\varepsilon_0 < \varepsilon < 1$ and $0 < \kappa < 1$ so that

\[
\varepsilon q^v(x^v, z^v) \leq p^v(x, z^v) \leq \varepsilon^{-1} q^v(x^v, z^v),
\]

\[
\delta \leq q^v(x^v, z^v) \leq \delta^{-1},
\]

\[
\kappa \leq g^v(x^v, y^v) \leq \kappa^{-1}
\]
for every $v \in V, x, z \in \mathbb{X}, y \in \mathbb{Y}$, where $g^v : \mathbb{X}^v \times \mathbb{X}^v \to \mathbb{R}_+$ is a transition density with respect to $\psi^v$. Then for every $n \geq 0$, $\sigma \in \mathbb{X}$, $K \in \mathcal{K}$ and $J \subseteq K$ we have

$$\| \pi^\sigma_n - \hat{\pi}^\sigma_n \|_J \leq \alpha \text{card } J \left[ e^{-\beta_1 d(J,\partial K)} + \frac{e^{\beta_2 |J|_\infty}}{N\gamma} \right],$$

where $0 < \gamma \leq \frac{1}{2}$ and $0 < \alpha, \beta_1, \beta_2 < \infty$ depend only on $\delta, \varepsilon, \kappa, r, \Delta$, and $\Delta_X$.

In Theorem 4.2, the parameter $\varepsilon$ controls the spatial correlations while the parameter $\delta$ controls the temporal correlations (in contrast to Theorem 4.1, where both are controlled simultaneously by $\varepsilon$). The key point is that $\delta$ can be arbitrary, and only $\varepsilon$ must lie above the threshold $\varepsilon_0$. That the threshold $\varepsilon_0$ depends on $\delta$ is natural: the more ergodic the dynamics, the more spatial interactions can be tolerated without losing decay of correlations.

The proof of Theorem 4.1 was based on repeated application of the classical Dobrushin comparison theorem (Corollary 2.6). While there are some significant differences between the details of the proofs, the essential improvement that makes it possible to prove Theorem 4.2 is that we can now exploit the generalized comparison theorem (Theorem 2.4), which enables us to treat the spatial and temporal degrees of freedom on a different footing.

Organization of the proof. As in [19], we consider three recursions

$$\pi^\mu_n := F_n \cdots F_1 \mu, \quad \hat{\pi}^\mu_n := \tilde{F}_n \cdots \tilde{F}_1 \mu, \quad \tilde{\pi}^\mu_n := \hat{F}_n \cdots \hat{F}_1 \mu,$$

where $F_n := C_n P$, $\tilde{F}_n := C_n B P$, and $\hat{F}_n := C_n B S_N P$. The filter $\pi^\mu_n$ and the block particle filter $\tilde{\pi}^\mu_n$ were already defined above. The block filter $\hat{\pi}^\mu_n$ is intermediate: it inserts only the localization but not the sampling step in the filtering recursion. This allows to decompose the approximation error into two terms, one due to localization and one due to sampling

$$\| \pi^\mu_n - \hat{\pi}^\mu_n \|_J \leq \| \pi^\mu_n - \tilde{\pi}^\mu_n \|_J + \| \tilde{\pi}^\mu_n - \hat{\pi}^\mu_n \|_J$$

by the triangle inequality. In the proof of Theorem 4.2, each of the terms on the right will be considered separately. The first term, which quantifies the bias due to the localization, will be bounded in section 4.2. The second term, which quantifies the sampling variance, will be bounded in section 4.3. Combining these two bounds completes the proof.

4.2. Bounding the bias. The goal of this section is to bound the bias term $\| \pi^\sigma_n - \hat{\pi}^\sigma_n \|_J$, where we denote by

$$\| \mu - \nu \|_J := \sup_{f \in \mathbb{X}^J : |f| \leq 1} | \mu f - \nu f |$$

the local total variation distance on the set of sites $J$. [Note that $\| \mu - \nu \|_J \leq K$ for some $K \in \mathbb{R}$ evidently implies $\| \mu - \nu \|_J \leq K$; the random measure norm $\| - \|_J$ will be essential to bound the sampling error, but is irrelevant for the bias term.]

Let us first give an informal outline of the ideas behind the proof of the bias bound. While the filter $\pi^\sigma_n$ is itself a high-dimensional distribution (defined on the set of sites $V$),
we do not know how to obtain a tractable local update rule for it. We therefore cannot apply Theorem 2.4 directly. Instead, we will consider the smoothing distribution
\[ \rho = P^\sigma [X_1, \ldots, X_n \in \cdot | Y_1, \ldots, Y_n], \]
defined on the extended set of sites \( I = \{1, \ldots, n\} \times V \) and configuration space \( \mathbb{S} = \mathbb{X}^n \). As \((X^v_k, Y^v_k)_{(k,v) \in I}\) is a Markov random field (cf. Figure 1), we can read off a local update rule for \( \rho \) from the model definition. At the same time, as \( \pi^\sigma_n = P^\sigma [X_n \in \cdot | Y_1, \ldots, Y_n] \) is a marginal of \( \rho \), we immediately obtain estimates for \( \pi^\sigma_n \) from estimates for \( \rho \).

This basic idea relies on the probabilistic definition of the filter as a conditional distribution of a Markov random field: the filtering recursion (which was only introduced for computational purposes) plays no role in the analysis. The block filter \( \tilde{\pi}^\sigma_n \), on the other hand, is defined in terms of a recursion and does not have an intrinsic probabilistic interpretation. In order to handle the block filter, we will artificially cook up a probability measure \( \tilde{P} \) on \( \mathbb{S} \) such that the block filter satisfies \( \tilde{\pi}^\sigma_n = \tilde{P}[X_n \in \cdot | Y_1, \ldots, Y_n] \), and set \( \tilde{\rho} = \tilde{P}[X_1, \ldots, X_n \in \cdot | Y_1, \ldots, Y_n] \).

This implies in particular that
\[ \|\pi^\sigma_n - \tilde{\pi}^\sigma_n\|_J = \|\rho - \tilde{\rho}\|_{\{n\} \times J}, \]
and we can now bound the bias term by applying Theorem 2.4.

To apply the comparison theorem we must choose a good cover \( \mathcal{J} \). It is here that the full flexibility of Theorem 2.4, as opposed to the classical comparison theorem, comes into play. If we were to apply Theorem 2.4 with the singleton cover \( \mathcal{J}_s = \{\{i\} : i \in I\} \), we would recover the result of Theorem 4.1: in this case both the spatial and temporal interactions must be weak in order to ensure that \( D = \sum_n (W^{-1}R)^n < \infty \). To avoid this problem, we work instead with larger blocks in the temporal direction. That is, our blocks \( J \in \mathcal{J} \) will have the form \( J = \{k + 1, \ldots, k + q\} \times \{v\} \) for an appropriate choice of the block length \( q \). The local update \( \gamma_J^x \) now behaves as \( q \) time steps of an ergodic Markov chain in \( \mathbb{X}^v \); the temporal interactions decay geometrically with \( q \), and can therefore be made arbitrarily small even if the interaction in one time step is arbitrarily strong. On the other hand, when we increase \( q \) there will be more nonzero terms in the matrix \( W^{-1}R \). We must therefore ultimately tune the block length \( q \) appropriately to obtain the result of Theorem 4.2.

**Remark 4.3.** The approach used here to bound the bias directly using the comparison theorem is different than the one used in [19], which exploits the recursive property of the filter. The latter approach has a broader scope, as it does not rely on the ability to express the approximate filter as the marginal of a random field as we do above: this could be essential for the analysis of more sophisticated algorithms that do not admit such a representation. For the purposes of this paper, however, the present approach provides an alternative and somewhat shorter proof that is well adapted to the analysis of block particle filters.
Remark 4.4. The problem under investigation is based on an interacting Markov chain model, and is therefore certainly dynamical in nature. Nonetheless, our proofs use Theorem 2.4 and not the one-sided Theorem 2.12. If we were to approximate the dynamics of the Markov chain $X_n$ itself, it would be much more convenient to apply Theorem 2.12 as the model is already defined in terms of one-sided conditional distributions $p(x, z)\psi(dz)$. Unfortunately, when we condition on the observations $Y_n$, the one-sided conditional distributions take a complicated form that incorporates all the information in the future observations, whereas conditioning on all variables outside a block $J \in \mathcal{J}$ gives rise to relatively tractable expressions. For this reason, the static “space-time” picture remains the most convenient approach for the investigation of high-dimensional filtering problems.

We now turn to the details of the proof. We first state the main result of this section.

Theorem 4.5 (Bias term). Suppose there exist $0 < \varepsilon, \delta < 1$ such that

$$\varepsilon q^v(x^v, z^v) \leq p^v(x^v, z^v) \leq \varepsilon^{-1} q^v(x^v, z^v),$$

$$\delta \leq q^v(x^v, z^v) \leq \delta^{-1}$$

for every $v \in V$ and $x, z \in X$, where $q^v : X^v \times X^v \rightarrow \mathbb{R}_+$ is a transition density with respect to $\psi^v$. Suppose also that we can choose $q \in \mathbb{N}$ and $\beta > 0$ such that

$$c := 3q\Delta^2 \varepsilon \beta (q + 2\varepsilon) \left(1 - \varepsilon^2(\Delta + 1)\right) + \varepsilon^\beta (1 - \varepsilon^2 \delta^2) + \varepsilon^\beta q (1 - \varepsilon^2 \delta^2)^q < 1.$$ 

Then we have

$$\|\pi_{\sigma n} - \tilde{\pi}_{\sigma n}\|_J \leq \frac{2e^{\beta r}}{1 - c} \left(1 - \varepsilon^2(q + 1)\Delta\right) \text{card } J e^{-\beta d(J,\partial K)}$$

for every $n \geq 0, \sigma \in \mathcal{X}, K \in \mathcal{K}$ and $J \subseteq K$.

In order to use the comparison theorem, we must have a method to construct couplings. Before we proceed to the proof of Theorem 4.5, we begin by formulating two elementary results that will provide us with the necessary tools for this purpose.

Lemma 4.6. If probability measures $\mu, \nu, \gamma$ satisfy $\mu(A) \geq \alpha \gamma(A)$ and $\nu(A) \geq \alpha \gamma(A)$ for every measurable set $A$, there is a coupling $Q$ of $\mu, \nu$ with $\int 1_{x \neq z} Q(dx, dz) \leq 1 - \alpha$.

Proof. Define $\tilde{\mu} = (\mu - \alpha \gamma)/(1 - \alpha), \tilde{\nu} = (\nu - \alpha \gamma)/(1 - \alpha)$, and let

$$Qf = \alpha \int f(x, x) \gamma(dx) + (1 - \alpha) \int f(x, z) \tilde{\mu}(dx) \tilde{\nu}(dz).$$

The claim follows readily. \qed
LEMMA 4.7. Let $P_1, \ldots, P_q$ be transition kernels on a measurable space $\mathbb{T}$, and define

$$
\mu_x(d\omega_1, \ldots, d\omega_q) = P_1(x, d\omega_1)P_2(\omega_1, d\omega_2) \cdots P_q(\omega_{q-1}, d\omega_q).
$$

Suppose that there exist probability measures $\nu_1, \ldots, \nu_q$ on $\mathbb{T}$ such that $P_i(x, A) \geq \alpha \nu_i(A)$ for every measurable set $A$, $x \in \mathbb{T}$, and $i \leq q$. Then there exists for every $x, z \in \mathbb{T}$ a coupling $Q_{x,z}$ of $\mu_x$ and $\mu_z$ such that $\int 1_{x \neq z} Q_{x,z}(d\omega, d\omega') \leq (1 - \alpha)^i$ for every $i \leq q$.

**PROOF.** Define the transition kernels $\tilde{P}_i = (P_i - \alpha \nu_i)/(1 - \alpha)$ and

$$
\tilde{Q}_i(x, z) = \alpha \int f(x', x') \nu_i(dx') + (1 - \alpha) 1_{x \neq z} \int f(x', z') \tilde{P}_i(x, dx') \tilde{P}_i(z, dz') + (1 - \alpha) 1_{x = z} \int f(x', x') \tilde{P}_i(x, dx').
$$

Then $\tilde{Q}_i(x, z, \cdot)$ is a coupling of $P_i(x, \cdot)$ and $P_i(z, \cdot)$. Now define

$$
Q_{x,z}(d\omega_1, d\omega_1', \ldots, d\omega_q, d\omega_q') = \tilde{Q}_1(x, z, d\omega_1, d\omega_1') \cdots \tilde{Q}_q(\omega_{q-1}, \omega_{q-1}', d\omega_q, d\omega_q').
$$

The result follows readily once we note that $\int 1_{x' \neq x'} \tilde{Q}_i(x, z, dx', dz') \leq (1 - \alpha) 1_{x \neq z}$. \Box

We can now proceed to the proof of Theorem 4.5.

**PROOF OF THEOREM 4.5.** We begin by constructing a measure $\tilde{\mathbb{P}}$ that allows to describe the block filter $\tilde{\pi}^\sigma_n$ as a conditional distribution, as explained above. We fix the initial condition $\sigma \in \mathfrak{X}$ throughout the proof (the dependence of various quantities on $\sigma$ is implicit).

To construct $\tilde{\mathbb{P}}$, define for $K \in \mathfrak{X}$ and $n \geq 1$ the function

$$
h_n^K(x, z, \partial K) := \int \tilde{\pi}^\sigma_n (d\omega) \prod_{v \in \partial K} p^v(x^K, \omega^{V \backslash K}, z^v).
$$

Evidently $h_n^K$ is a transition density with respect to $\bigotimes_{v \in \partial K} \psi^v$. Let

$$
\tilde{p}_n(x, z) := \prod_{K \in \mathfrak{X}} h_n^K(x, z, \partial K) \prod_{v \in \partial K \backslash \partial K} p^v(x, z^v),
$$

and define $\tilde{\mathbb{P}}_n, \mu(dx') := \psi(dx') \int \tilde{p}_n(x, x') \mu(dx)$. Then $\tilde{\mathbb{P}}_n, \tilde{\pi}^\sigma_n = BP\tilde{\pi}^\sigma_n$ by construction for every $n \geq 1$, as $\tilde{\pi}^\sigma_n$ is a product measure across blocks. Thus we have

$$
\pi^\sigma_n = C_n P \cdots C_1 P \delta_\sigma, \quad \tilde{\pi}^\sigma_n = C_n \tilde{P}_n \cdots C_1 \tilde{P}_1 \delta_\sigma.
$$

In particular, the filter and the block filter satisfy the same recursion with different transition densities $p$ and $\tilde{p}_n$. We can therefore interpret the block filter as the filter corresponding to
a time-inhomogeneous Markov chain with transition densities \( \tilde{p}_n \); that is, if we set
\[
\tilde{\mathbf{P}}[(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \in A] := \\
\int 1_A(x_1, \ldots, x_n, y_1, \ldots, y_n) \tilde{p}_1(\sigma, x_1) \prod_{k=2}^{n} \tilde{p}_k(x_{k-1}, x_k) g(x_k, y_k) \psi(dx_k) \varphi(dy_k)
\]
(note that \( \mathbf{P}^\sigma \) satisfies the same formula where \( \tilde{p}_k \) is replaced by \( p \)), we can write
\[
\tilde{\pi}_n^\sigma = \tilde{\mathbf{P}}[X_n \in \cdot | Y_1, \ldots, Y_n].
\]

Let us emphasize that the transition densities \( \tilde{p}_k \) and operators \( \tilde{P}_k \) themselves depend on the initial condition \( \sigma \), which is certainly not the case for the regular filter. However, since \( \sigma \) is fixed throughout the proof, this is irrelevant for our computations.

From now on we fix \( n \geq 1 \) in the remainder of the proof. Let
\[
\rho = \mathbf{P}^\sigma[X_1, \ldots, X_n \in \cdot | Y_1, \ldots, Y_n], \quad \tilde{\rho} = \tilde{\mathbf{P}}[X_1, \ldots, X_n \in \cdot | Y_1, \ldots, Y_n].
\]

Then \( \rho \) and \( \tilde{\rho} \) are probability measures on \( S = \mathbb{R}^n \), which is naturally indexed by the set of sites \( I = \{1, \ldots, n\} \times \hat{V} \) (the observation sequence on which we condition is arbitrary and can be considered fixed throughout the proof). The proof now proceeds by applying Theorem 2.4 to \( \rho, \tilde{\rho} \), the main difficulty being the construction of a coupled update rule.

Fix \( q \geq 1 \). We first specify the cover \( \mathcal{J} = \{ J_l^v : 1 \leq l \leq \lfloor n/q \rfloor, v \in V \} \) as follows:
\[
J_l^v := \{(l-1)q + 1, \ldots, lq \wedge n\} \times \{v\} \quad \text{for } 1 \leq l \leq \lfloor n/q \rfloor, v \in V.
\]

We choose the natural local updates \( \gamma_l^J(dz^J) = \rho(dz^J | x^I \setminus J) \) and \( \tilde{\gamma}_l^J(dz^J) = \tilde{\rho}(dz^J | x^I \setminus J) \), and postpone the construction of the coupled updates \( Q_{x,z}^J \) and \( \tilde{Q}_{x}^J \) to be done below. Now note that the cover \( \mathcal{J} \) is in fact a partition of \( I \); thus Theorem 2.4 yields
\[
\| \pi_n^{\sigma} - \tilde{\pi}_n^{\sigma} \|_J = \| \rho - \tilde{\rho} \|_{(n) \times J} \leq 2 \sum_{i \in \{n\} \times J} \sum_{j \in I} D_{ij} b_j
\]
provided that \( D = \sum_{k=0}^{\infty} C_k^k < \infty \) (cf. Corollary 2.8), where
\[
C_{ij} = \sup_{x,z \in S, x \not\in \{i\}, z \not\in \{i\}} \int 1_{\omega_i \neq \omega'_i} Q_{x,z}^{(i)}(d\omega, d\omega'), \quad b_i = \sup_{x \in S} \int 1_{\omega_i \neq \omega'_i} \tilde{Q}_{x}^{(i)}(d\omega, d\omega'),
\]
and where we write \( J(i) \) for the unique block \( J \in \mathcal{J} \) that contains \( i \in I \). To put this bound to good use, we must introduce coupled updates \( Q_{x,z}^J \) and \( \tilde{Q}_{x}^J \) and estimate \( C_{ij} \) and \( b_j \).

Let us fix until further notice a block \( J = J_l^v \in \mathcal{J} \). We will consider first the case that \( 1 < l < \lfloor n/q \rfloor \); the cases \( l = 1, \lfloor n/q \rfloor \) will follow subsequently using the identical proof. Let \( s = (l-1)q \). Then we can compute explicitly the local update rule
\[
\gamma_l^J(A) = \\
\frac{\int 1_A(x^J) p^v(x_s, x_{s+1}) \prod_{m=s+1}^{s+q} g^v(x_m^v, y_m^v) \prod_{w \in N(v)} p^w(x_m, x_{m+1}^w) \psi^w(dx_m^v)}{\int p^v(x_s, x_{s+1}) \prod_{m=s+1}^{s+q} g^v(x_m^v, y_m^v) \prod_{w \in N(v)} p^w(x_m, x_{m+1}^w) \psi^w(dx_m^v)}
\]
using Bayes' formula, the definition of $P^x$ (in the same form as the above definition of $\tilde{P}$), and that $p^u(x, z^u)$ depends only on $x^{N(v)}$. We now construct couplings $Q_{x,z}^I$ of $\gamma_x^I$ and $\gamma_z^I$ where $x, z$ differ only at the site $j = (k, w) \in I$. We distinguish the following cases:

1. $k = s, w \in N(v) \setminus \{v\}$;
2. $k = s, w = v$;
3. $k \in \{s + 1, \ldots, s + q\}, w \in \bigcup_{u \in N(v)} N(u) \setminus \{v\}$;
4. $k = s + q + 1, w \in N(v) \setminus \{v\}$;
5. $k = s + q + 1, w = v$.

It is easily verified by inspection that $\gamma_x^I$ does not depend on $x^u_k$ except in one of the above cases. Thus when $j$ satisfies none of the above conditions, we can set $C_{ij} = 0$ for $i \in J$.

**Case 1.** Note that

$$
\gamma_x^I(A) \geq 
\varepsilon_2 \int \frac{1}{A}(x^I) \frac{p^u(x^I, x^{I+1}) \prod_{m=s+1}^{s+q} g^u(x^I_m, Y^u_m) \prod_{w \in N(v)} P^u(x_m, x^u_{m+1}) \psi^u(dx^u_m)}{\int p^u(x^I, x^{I+1}) \prod_{m=s+1}^{s+q} g^u(x^I_m, Y^u_m) \prod_{w \in N(v)} P^u(x_m, x^u_{m+1}) \psi^u(dx^u_m)},
$$

and the right hand side does not depend on $x^u_s$ for $w \neq v$. Thus whenever $x, z \in S$ satisfy $x^{I \setminus \{j\}} = z^{I \setminus \{j\}}$ for $j = (s, w)$ with $w \in N(v) \setminus \{v\}$, we can construct a coupling $Q_{x,z}^I$ using Lemma 4.6 such that $C_{ij} \leq 1 - \varepsilon^2$ for every $i \in J$.

**Case 2.** Define the transition kernels on $X^v$

$$
P_{k,x}(\omega, A) = \int \frac{1}{A}(x^v_k) \frac{p^v(x^v_k, x^{v+1}_k) \prod_{m=k}^{s+q} g^v(x^v_m, Y^v_m) \prod_{w \in N(v)} P^v(x_m, x^v_{m+1}) \psi^v(dx^v_m)}{\int p^v(x^v_k, x^{v+1}_k) \prod_{m=k}^{s+q} g^v(x^v_m, Y^v_m) \prod_{w \in N(v)} P^v(x_m, x^v_{m+1}) \psi^v(dx^v_m)},
$$

for $k = s + 1, \ldots, s + q$. By construction, $P_{k,x}(x^v_{k-1}, dx^v_k) = \gamma_x^I(dx_k^{I \setminus \{s+1, \ldots, x^v_{k-1}\}})$, so we are in the setting of Lemma 4.7. Moreover, we can estimate

$$
P_{k,x}(\omega, A) \geq \varepsilon^2 \delta^2 \int \frac{1}{A}(x^v_k) \frac{\prod_{m=k}^{s+q} g^v(x^v_m, Y^v_m) \prod_{w \in N(v)} P^v(x_m, x^v_{m+1}) \psi^v(dx^v_m)}{\int \prod_{m=k}^{s+q} g^v(x^v_m, Y^v_m) \prod_{w \in N(v)} P^v(x_m, x^v_{m+1}) \psi^v(dx^v_m)},
$$

where the right hand side does not depend on $\omega$. Thus whenever $x, z \in S$ satisfy $x^{I \setminus \{j\}} = z^{I \setminus \{j\}}$ for $j = (s, v)$, we can construct a coupling $Q_{x,z}^I$ using Lemma 4.7 such that $C_{ij} \leq (1 - \varepsilon^2 \delta^2)^{k-s}$ for $i = (k, v)$ with $k = s + 1, \ldots, s + q$.

**Case 3.** Fix $k \in \{s + 1, \ldots, s + q\}$ and $u \neq v$. Note that

$$
\gamma_x^I(A) \geq \varepsilon^2 (\Delta + 1) \times
\frac{\int 1_A(x^I) p^u(x, x^I_{s+1}) \prod_{m=s+1}^{s+q} g^u(x^I_m, Y^u_m) \prod_{w \in N(v)} P^u(x_m, x^u_{m+1}) \psi^u(dx^u_m)}{\int p^u(x, x^I_{s+1}) \prod_{m=s+1}^{s+q} g^u(x^I_m, Y^u_m) \prod_{w \in N(v)} P^u(x_m, x^u_{m+1}) \psi^u(dx^u_m)}
$$

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where we set $\beta^w_m(x_m, x_{m+1}^w) = q^w(x_m, x_{m+1}^w)$ if either $m = k$ or $m = k - 1$ and $w = u$, and $\beta^w_m(x_m, x_{m+1}^w) = p^w(x_m, x_{m+1}^w)$ otherwise. The right hand side of this expression does not depend on $x_k^s$ as the terms $q^w(x_m, x_{m+1}^w)$ for $w \neq v$ cancel in the numerator and denominator. Thus whenever $x, z \in \mathbb{S}$ satisfy $x^{f\setminus\{j\}} = z^{f\setminus\{j\}}$ for $j = (k, u)$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.6 such that $C_{ij} \leq 1 - \varepsilon^{2(\Delta+1)}$ for every $i \in J$.

**Case 4.** Let $u \in N(v)$, $v$. Note that

$$
\gamma_x^J(A) \geq \varepsilon^2 \frac{\int 1_A(x^J) p^v(x_s^v, x_{s+1}^v) \prod_{m=s+1}^{s+q} g^v(x_m^v, \sum_{i \in N(v)} \beta_m^v(x_m, x_{m+1}^v) \psi^v(dx_m^v)}{\int p^v(x_s^v, x_{s+1}^v) \prod_{m=s+1}^{s+q} g^v(x_m^v, \sum_{i \in N(v)} \beta_m^v(x_m, x_{m+1}^v) \psi^v(dx_m^v)},
$$

where we set $\beta_m^v(x_m, x_{m+1}^v) = q^v(x_m, x_{m+1}^v)$ if $m = s + q$ and $w = u$, and we set $\beta_m^v(x_m, x_{m+1}^v) = p^v(x_m, x_{m+1}^v)$ otherwise. The right hand side does not depend on $x_{s+q+1}^v$ as the term $q^v(x_{s+q}, x_{s+q+1}^v)$ cancels in the numerator and denominator. Thus whenever $x, z \in \mathbb{S}$ satisfy $x^{f\setminus\{j\}} = z^{f\setminus\{j\}}$ for $j = (s + q + 1, u)$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.6 such that $C_{ij} \leq 1 - \varepsilon^2$ for every $i \in J$.

**Case 5.** Define for $k = s + 1, \ldots, s + q$ the transition kernels on $\mathbb{X}^v$

$$
P_{k,x}(\omega, A) = \varepsilon^2 \frac{\int 1_A(x^J_k) p^v(x_s^v, x_{s+1}^{v+1}) \prod_{m=s+1}^{s+q} g^v(x_m^v, \sum_{i \in N(v)} \beta_m^v(x_m, x_{m+1}^v) \psi^v(dx_m^v)}{\int p^v(x_s^v, x_{s+1}^{v+1}) \prod_{m=s+1}^{s+q} g^v(x_m^v, \sum_{i \in N(v)} \beta_m^v(x_m, x_{m+1}^v) \psi^v(dx_m^v)},
$$

where we set $\beta_m^v(x_m, x_{m+1}^v) = q^v(x_m, x_{m+1}^v)$ if $m = k$ and $w = v$, and $\beta_m^v(x_m, x_{m+1}^v) = p^v(x_m, x_{m+1}^v)$ otherwise. By construction, $P_{k,x}(x_{k+1}^{v+1}, dx_k^v) = \gamma_x^J(dx_k^v, x_{k+1}^{v+1}, \ldots, x_{s+q}^v)$, so we are in the setting of Lemma 4.7. Moreover, we can estimate

$$
P_{k,x}(\omega, A) \geq \varepsilon^2 \Delta^2 \times \frac{\int 1_A(x^J_k) p^v(x_s^v, x_{s+1}^{v+1}) \prod_{m=s+1}^{s+q} g^v(x_m^v, \sum_{i \in N(v)} \beta_m^v(x_m, x_{m+1}^v) \psi^v(dx_m^v)}{\int p^v(x_s^v, x_{s+1}^{v+1}) \prod_{m=s+1}^{s+q} g^v(x_m^v, \sum_{i \in N(v)} \beta_m^v(x_m, x_{m+1}^v) \psi^v(dx_m^v)},
$$

where $\beta_m^v(x_m, x_{m+1}^v) = 1$ if $m = k$ and $w = v$, and $\beta_m^v(x_m, x_{m+1}^v) = p^v(x_m, x_{m+1}^v)$ otherwise. Note that the right hand side does not depend on $\omega$. Thus whenever $x, z \in \mathbb{S}$ satisfy $x^{f\setminus\{j\}} = z^{f\setminus\{j\}}$ for $j = (s + q + 1, v)$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.7 such that $C_{ij} \leq (1 - \varepsilon^2 \delta^2)^{s+q+1-k}$ for $i = (k, v)$ with $k = s + 1, \ldots, s + q$.

We have now constructed coupled updates $Q^J_{x,z}$ for every pair $x, z \in \mathbb{S}$ that differ only at one point. Collecting the above bounds on $C_{ij}$, we can estimate

$$
\sum_{(k', v') \in I} e^{\beta[(k-k')d(v,v')] \alpha_{c(k',v')}} C_{(k,v)(k',v')} \leq 2\varepsilon \Delta + \varepsilon^2 \Delta + \varepsilon^2 \Delta q \leq 3\varepsilon \Delta \varepsilon^2 \Delta^2 q + \varepsilon^2 \Delta \varepsilon^2 \Delta^2 q =: c
$$
whenever \((k, v) \in J\). In the last line, we have used that \(\alpha^{x+1} + \alpha^{q-x}\) is a convex function of \(x \in [0, q - 1]\), and therefore attains its maximum on the endpoints \(x = 0, q - 1\).

Up to this point we have considered an arbitrary block \(J = J_l^j \in \mathcal{J}\) with \(1 < l < \lceil n/q \rceil\). It is however evident that the identical proof holds for the boundary blocks \(l = 1, \lceil n/q \rceil\), except that for \(l = 1\) we only need to consider Cases 3–5 above and for \(l = \lceil n/q \rceil\) we only need to consider Cases 1–3 above. As all the estimates are otherwise identical, the corresponding bounds on \(C_{ij}\) are at most as large as those in the case \(1 < l < \lceil n/q \rceil\). Thus

\[
\|C\|_{\infty, \beta m} := \max_{i \in I} \sum_{j \in J} e^{\beta m(i, j)} C_{ij} \leq c,
\]

where we define the metric \(m(i, j) = |k - k'| + d(v, v')\) for \((k, v) \in I\) and \((k', v') \in I\).

Our next order of business is to construct couplings \(\bar{Q}_J^l\) of \(\gamma_2^J\) and \(\tilde{\gamma}_2^J\) and to estimate the coefficients \(b_i\). To this end, let us first note that \(h_m^K(x, z; \partial K)\) depends only on \(x; \partial K\), where

\[
\partial^2 K := \bigcup_{w \in \partial K} N(w) \cap K
\]

is the subset of vertices in \(K\) that can interact with vertices outside \(K\) in two time steps. It is easily seen that \(\gamma_2^J = \tilde{\gamma}_2^J\), and that we can therefore choose \(b_i = 0\) for \(i \in J\), unless \(J = J_l^j\) with \(v \in \partial^2 K\) for some \(K \in \mathcal{K}\). In the latter case we obtain by Bayes' formula

\[
\tilde{\gamma}_2^J(A) = \frac{\int A(x^J) \prod_{s=m+1}^{s+q} g^v(x_m, Y_m) h_{m+1}^{K}(x_m, x_{m+1}^{\partial K}) \prod_{w \in N(v) \cap K \setminus \partial K} p^{w}(x_m, x_{m+1}^{w}) \psi(dx^J)}{\int \prod_{s=m+1}^{s+q} g^v(x_m, Y_m) h_{m+1}^{K}(x_m, x_{m+1}^{\partial K}) \prod_{w \in N(v) \cap K \setminus \partial K} p^{w}(x_m, x_{m+1}^{w}) \psi(dx^J)}
\]

for \(1 < l < \lceil n/q \rceil\), where \(s = (l - 1)q\) and \(\psi(dx^J) = \otimes_{(k, v) \in J} \psi^v(dx_k^v)\). Note that

\[
\prod_{w \in N(v) \setminus (K \setminus \partial K)} p^{w}(x_m, x_{m+1}^{w}) \geq \varepsilon^\Delta \prod_{w \in N(v) \setminus (K \setminus \partial K)} q^{w}(x_m, x_{m+1}^{w}),
\]

while

\[
h_m^K(x, z; \partial K) \geq \varepsilon^\Delta \prod_{w \in N(v) \setminus \partial K} q^{w}(x_m, x_{m+1}^{w}) \int \pi^{\sigma}_{m-1}(d\omega) \prod_{w \in \partial K \setminus N(v)} p^{w}(x_K^{K \setminus \partial K}, z_{m+1}).
\]

We can therefore estimate \(\gamma_2^J(A) \geq \varepsilon^{2(q+1)\Delta} \Gamma(A)\) and \(\tilde{\gamma}_2^J(A) \geq \varepsilon^{2(q+1)\Delta} \Gamma(A)\) with

\[
\Gamma(A) = \frac{\int A(x^J) \prod_{s=m+1}^{s+q} g^v(x_m, Y_m) \beta(x_m, x_{m+1}^{v}) \prod_{w \in N(v) \cap K \setminus \partial K} p^{w}(x_m, x_{m+1}^{w}) \psi(dx^J)}{\int \prod_{s=m+1}^{s+q} g^v(x_m, Y_m) \beta(x_m, x_{m+1}^{v}) \prod_{w \in N(v) \cap K \setminus \partial K} p^{w}(x_m, x_{m+1}^{w}) \psi(dx^J)},
\]

where \(\beta(x, z) = q^{v}(x, z)\) if \(v \in \partial K\) and \(\beta(x, z) = 1\) if \(v \in \partial^2 K \setminus \partial K\). Thus we can construct a coupling \(\bar{Q}_J^l\) using Lemma 4.6 such that \(b_i \leq 1 - \varepsilon^{2(q+1)\Delta}\) for all \(i \in J\) in the case \(1 < l < \lceil n/q \rceil\). The same conclusion follows for \(l = 1, \lceil n/q \rceil\) by the identical proof.
We are now ready to put everything together. As \( \| \cdot \|_{\infty, \beta m} \) is a matrix norm, we have
\[
\| D \|_{\infty, \beta m} \leq \sum_{k=0}^{\infty} \| C^k \|_{\infty, \beta m} \leq \frac{1}{1 - c} < \infty.
\]
Thus \( D < \infty \), so we can apply the comparison theorem. Moreover,
\[
\sup_{i \in J} \sum_{j \in J'} D_{ij} = \sup_{i \in J} e^{-\beta m(i, J')} \sum_{j \in J'} e^{\beta m(i, J')} D_{ij} \leq e^{-\beta m(J, J')} \| D \|_{\infty, \beta m}.
\]
Thus we obtain
\[
\| \pi_n^\sigma - \tilde{\pi}_n^\sigma \|_J \leq 2(1 - \varepsilon^2(q+1)\Delta) \sum_{i \in \{n\} \times J} \sum_{j \in \{1, \ldots, n\} \times \partial^2 K} D_{ij}
\]
\[
\leq \frac{2}{1 - c} (1 - \varepsilon^2(q+1)\Delta) \operatorname{card} J e^{-\beta d(J, \partial^2 K)}.
\]
But clearly \( d(J, \partial^2 K) \geq d(J, \partial K) - r \), and the proof is complete.

**Remark 4.8.** In the proof of Theorem 4.5 (and similarly for Theorem 4.11 below), we apply the comparison theorem with a nonoverlapping cover \( \{(l-1)q + 1, \ldots, lq \land n\} \), \( l \leq \lceil n/q \rceil \). Working instead with overlapping blocks \( \{s+1, \ldots, s+q\} \), \( s \leq n - q \) would give somewhat better estimates at the expense of even more tedious computations.

**4.3. Bounding the variance.** We now turn to the problem of bounding the variance term \( \| \pi_n^\sigma - \tilde{\pi}_n^\sigma \|_J \). We will follow the basic approach taken in [19], where a detailed discussion of the requisite ideas can be found. In this section we develop the necessary changes to the proof in [19].

At the heart of the proof of the variance bound lies a stability result for the block filter [19, Proposition 4.15]. This result must be modified in the present setting to account for the different assumptions on the spatial and temporal correlations. This will be done next, using the generalized comparison Theorem 2.4 much as in the proof of Theorem 4.5.

**Proposition 4.9.** Suppose there exist \( 0 < \varepsilon, \delta < 1 \) such that
\[
\varepsilon q^v(x^v, z^v) \leq p^v(x, z^v) \leq \varepsilon^{-1} q^v(x^v, z^v),
\]
\[
\delta \leq q^v(x^v, z^v) \leq \delta^{-1}
\]
for every \( v \in V \) and \( x, z \in \mathbb{X} \), where \( q^v : \mathbb{X}^v \times \mathbb{X}^v \to \mathbb{R}_+ \) is a transition density with respect to \( \psi^v \). Suppose also that we can choose \( q \in \mathbb{N} \) and \( \beta > 0 \) such that
\[
c := 3q\Delta^2 e^{\beta q(1 - \varepsilon^2(\Delta+1))} + e^\beta(1 - \varepsilon^2\delta^2) + e^\beta q(1 - \varepsilon^2\delta^2)^q < 1.
\]
Then we have
\[
\| \bar{F}_n \cdots \bar{F}_{s+1} \delta_{\sigma} - \bar{F}_n \cdots \bar{F}_{s+1} \delta_{\tilde{\sigma}} \|_J \leq \frac{2}{1 - c} \operatorname{card} J e^{-\beta(n-s)}
\]
for every \( s < n, \sigma, \tilde{\sigma} \in \mathbb{X}, K \in \mathbb{K} \) and \( J \subseteq K \).
**Proof.** We fix throughout the proof \( n > 0 \), \( K \in \mathcal{K} \), and \( J \subseteq K \). We will also assume for notational simplicity that \( s = 0 \). As \( \tilde{F}_k \) differ for different \( k \) only by their dependence on different observations \( Y_k \), and as the conclusion of the Proposition is independent of the observations, the conclusion for \( s = 0 \) extends trivially to any \( s < n \).

As in Theorem 4.5, the idea behind the proof is to introduce a Markov random field \( \rho \) of which the block filter is a marginal, followed by an application of the generalized comparison theorem. Unfortunately, the construction in the proof of Theorem 4.5 is not appropriate in the present setting, where all the local interactions depend on the initial condition \( \sigma \). That was irrelevant in Theorem 4.5 where the initial condition was fixed, but is fatal in the present setting where we aim to understand a perturbation to the initial condition. Instead, we will use a more elaborate construction of \( \rho \) introduced in [19], called the computation tree. We begin by recalling this construction.

Define for \( K' \in \mathcal{K} \) the block neighborhood \( \mathcal{N}(K') := \{K'' \in \mathcal{K} : d(K', K'') \leq r\} \) (we recall that \( \text{card } \mathcal{N}(K') \leq \Delta\mathcal{K} \)). We can evidently write

\[
\mathcal{B}^{K'} \tilde{F}_s \bigotimes_{K'' \in \mathcal{K}} \mu^{K''} = C_s^{K'} P^{K'} \bigotimes_{K'' \in \mathcal{N}(K')} \rho^{K''},
\]

where we define for any probability \( \eta \) on \( \mathcal{X}^{K'} \)

\[
(C_s^{K'} \eta)(A) := \frac{\int 1_A(x^{K'}) \prod_{v \in K'} g^{v}(x_v^{v}, Y_v^{v}) \eta(dx^{K'})}{\int \prod_{v \in K'} g^{v}(x_v^{v}, Y_v^{v}) \eta(dx^{K'})},
\]

and for any probability \( \eta \) on \( \mathcal{X}^{\bigcup_{K'' \in \mathcal{N}(K')} K''} \)

\[
(P^{K'} \eta)(A) := \int 1_A(x^{K'}) \prod_{v \in K'} p^{v}(z, x_v^{v}) \psi^{v}(dx_v^{v}) \eta(dz).
\]

Iterating this identity yields

\[
\mathcal{B}^{K'} \tilde{F}_n \cdots \tilde{F}_1 \delta_{\sigma} = C_n^{K} P^K \bigotimes_{K_{n-1} \in \mathcal{N}(K)} \cdots C_2^{K_2} P^{K_2} \bigotimes_{K_1 \in \mathcal{N}(K_2)} C_1^{K_1} P^{K_1} \bigotimes_{K_0 \in \mathcal{N}(K_1)} \delta_{\sigma_{K_0}},
\]

The nested products can be naturally viewed as defining a tree.

To formalize this idea, define the tree index set (we will write \( K_n := K \) for simplicity)

\[
T := \{ [K_u \cdots K_{n-1}] : 0 \leq u < n, \ K_s \in \mathcal{N}(K_{s+1}) \text{ for } u \leq s < n \} \cup \{[\emptyset]\}.
\]

The root of the tree \([\emptyset]\) represents the block \( K \) at time \( n \), while \([K_u \cdots K_{n-1}]\) represents a duplicate of the block \( K_u \) at time \( u \) that affects the root along the branch \( K_u \to K_{u+1} \to \cdots \to K_{n-1} \to K \). The set of sites corresponding to the computation tree is

\[
I = \{ [K_u \cdots K_{n-1}]u : [K_u \cdots K_{n-1}] \in T, \ v \in K_u \} \cup \{[\emptyset]v : v \in K\}.
\]
and the corresponding configuration space is $S = \prod_{i \in I} \mathbb{X}^i$ with $\mathbb{X}^{[t]} := \mathbb{X}^v$. The following tree notation will be used throughout the proof. Define for vertices of the tree $T$ the depth $d([K_u \cdots K_{n-1}]) := u$ and $d([\varnothing]) := n$. For every site $[t]v \in I$, we define the associated vertex $v(i) := v$ and depth $d(i) := d([t])$. Define also the sets $I_+ := \{ i \in I : d(i) > 0 \}$ and $T_0 := \{ [t] \in T : d([t]) = 0 \}$ of non-leaf sites and leaf vertices, respectively. Define

$$c([K_u \cdots K_{n-1}]v) := \{ [K_{u-1} \cdots K_{n-1}]v' : K_{u-1} \in N(K_u), \ v' \in N(v) \},$$

and similarly for $c([\varnothing]v)$: that is, $c(i)$ is the set of children of the site $i \in I$ in the computation tree. Finally, we will frequently identify a tree vertex $[K_u \cdots K_{n-1}]v \in T$ with the corresponding set of sites $\{ [K_u \cdots K_{n-1}]v : v \in K_u \}$, and analogously for $[\varnothing]v$.

Having introduced the tree structure, we now define probability measures $\rho, \tilde{\rho}$ on $S$ by

$$\rho(A) = \frac{\int 1_A(x) \prod_{i \in I_+} p^{v(i)}(x^{c(i)}, x^i) g^{\psi_v(i)}(x^i, Y^i) \psi_v(i)(dx^i) \prod_{[t] \in T_0} \delta_{\sigma_{[t]}}(dx^{[t]})}{\int \prod_{i \in I_+} p^{v(i)}(x^{c(i)}, x^i) g^{\psi_v(i)}(x^i, Y^i) \psi_v(i)(dx^i) \prod_{[t] \in T_0} \delta_{\sigma_{[t]}}(dx^{[t]})},$$

$$\tilde{\rho}(A) = \frac{\int 1_A(x) \prod_{i \in I_+} p^{v(i)}(x^{c(i)}, x^i) g^{\psi_v(i)}(x^i, Y^i) \psi_v(i)(dx^i) \prod_{[t] \in T_0} \delta_{\tilde{\sigma}_{[t]}}(dx^{[t]})}{\int \prod_{i \in I_+} p^{v(i)}(x^{c(i)}, x^i) g^{\psi_v(i)}(x^i, Y^i) \psi_v(i)(dx^i) \prod_{[t] \in T_0} \delta_{\tilde{\sigma}_{[t]}}(dx^{[t]})},$$

where we write $\sigma^{K_0 \cdots K_{n-1}} := \sigma^{K_0}$ and $Y^i := Y^{v(i)}$ for simplicity. Then, by construction, the measure $B^K \tilde{\mathcal{F}}_n \cdots \tilde{\mathcal{F}}_1 \delta_\sigma$ coincides with the marginal of $\rho$ on the root of the computation tree, while $B^K \bar{\mathcal{F}}_n \cdots \bar{\mathcal{F}}_1 \delta_{\tilde{\sigma}}$ coincides with the marginal of $\tilde{\rho}$ on the root of the tree: this is easily seen by expanding the above nested product identity. In particular, we obtain

$$\| \tilde{\mathcal{F}}_n \cdots \bar{\mathcal{F}}_1 \delta_\sigma - \bar{\mathcal{F}}_n \cdots \bar{\mathcal{F}}_1 \delta_{\tilde{\sigma}} \|_J = \| \rho - \tilde{\rho} \|_{[\varnothing], J},$$

and we aim to apply the comparison theorem to estimate this quantity.

The construction of the computation tree that we have just given is identical to the construction in [19]. We deviate from the proof of [19] from this point onward, since we must use Theorem 2.4 instead of the classical Dobrushin comparison theorem to account for the distinction between temporal and spatial correlations in the present setting.

Fix $q \geq 1$. In analogy with the proof of Theorem 4.5, we consider a cover $\mathcal{J}$ consisting of blocks of sites $i \in I$ such that $(l-1)q < d(i) \leq lq \wedge n$ and $v(i) = v$. In the present setting, however, the same vertex $v$ is duplicated many times in the tree, so that we end up with many disconnected blocks of different lengths. To keep track of these blocks, define

$$I_0 := \{ i \in I : d(i) = 0 \}, \quad I_l := \{ i \in I : d(i) = (l-1)q + 1 \}$$

for $1 \leq l \leq \lceil n/q \rceil$, and let

$$l([K_u, \ldots, K_{n-1}]v) := \max\{ s \geq u : K_u = K_{u+1} = \cdots = K_s \}.$$

We now define the cover $\mathcal{J}$ as

$$\mathcal{J} = \{ J_i^l : 0 \leq l \leq \lceil n/q \rceil, \ i \in I_l \},$$
where
\[ J^i_0 := \{i\}, \quad J^i_l := \{[K_u \cdots K_{n-1}]v : (l-1)q + 1 \leq u \leq lq \land \ell(i)\} \]
for \(i = [K_{(l-1)q+1} \cdots K_{n-1}]v \in I_l\) and \(1 \leq l \leq \lceil n/q \rceil\). It is easily seen that \(\mathcal{J}\) is in fact a partition of of the computation tree \(I\) into linear segments.

Having defined the cover \(\mathcal{J}\), we must now consider a suitable coupled update rule. We will choose the natural local updates \(\gamma^J_x(dz^J) = \rho(dz^J|x^{I\setminus J})\) and \(\tilde{\gamma}^J_x(dz^J) = \tilde{\rho}(dz^J|x^{I\setminus J})\), with the coupled updates \(Q^J_{x,z}\) and \(\hat{Q}^J_{x,z}\) to be constructed below. Then Theorem 2.4 yields
\[ \|\tilde{\pi}_n \cdots \tilde{\pi}_1 \delta_\sigma - \tilde{\pi}_n \cdots \tilde{\pi}_1 \delta_\sigma\|_J = \|\rho - \tilde{\rho}\|_J \leq 2 \sum_{i \in [\sigma]} \sum_{j \in I} D_{ij} b_{ij} \]
provided that \(D = \sum_{k=0}^\infty C^k < \infty\) (cf. Corollary 2.8), where
\[ C_{ij} = \sup_{x,z \in \mathbb{S}} \int_{x^{\setminus i} = z^{\setminus i}(j)} 1_{\omega_i \neq \omega_i'} Q^J_{x,z}(d\omega, d\omega'), \quad b_{ij} = \sup_{x \in \mathbb{S}} \int_{x \neq x'} \tilde{Q}^J_{x,z}(d\omega, d\omega'), \]
and where we write \(J(i)\) for the unique block \(J \in \mathcal{J}\) that contains \(i \in I\). To put this bound to good use, we must introduce coupled updates \(Q^J_{x,z}\) and \(\tilde{Q}^J_{x,z}\) and estimate \(C_{ij}\) and \(b_{ij}\).

Let us fix until further notice a block \(J = J^i_l \in \mathcal{J}\) with \(i = [K_{(l-1)q+1} \cdots K_{n-1}]v \in I_l\) and \(1 \leq l \leq \lceil n/q \rceil\). From the definition of \(\rho\), we can compute explicitly
\[ \gamma^J_x(A) = \frac{\int 1_A(x^J) p^x(x^{c(i)}, x^J) \prod_{a \in I_+ : J \cap (a) \neq \varnothing} p^v(a) (x^{c(a)}, x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)}{\int p^v(x^{c(i)}, x^J) \prod_{a \in I_+ : J \cap (a) \neq \varnothing} p^v(a) (x^{c(a)}, x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)} \]
using the Bayes formula. We now proceed to construct couplings \(Q^J_{x,z}\) of \(\gamma^J_x\) and \(\tilde{\gamma}^J_x\) for \(x, z \in \mathbb{S}\) that differ only at the site \(j \in I\). We distinguish the following cases:

1. \(d(j) = (l-1)q + 1 \leq lq \land \ell(i)\) and \(v(j) \neq v\);
2. \(d(j) = (l-1)q\) and \(v(j) = v\);
3. \((l-1)q + 1 \leq d(j) \leq lq \land \ell(i)\) and \(v(j) \neq v\);
4. \(d(j) = lq \land \ell(i) + 1\) and \(v(j) \neq v\);
5. \(d(j) = lq \land \ell(i) + 1\) and \(v(j) = v\).

It is easily seen that \(\gamma^J_x\) does not depend on \(x^J\) except in one of the above cases. Thus when \(j\) satisfies none of the above conditions, we can set \(C_{aj} = 0\) for \(a \in J\).

**Case 1.** In this case, we must have \(j \in c(i)\) with \(v(j) \neq v\). Note that
\[ \gamma^J_x(A) \geq \frac{\int 1_A(x^J) q^x(x^{\setminus i}, x^J) \prod_{a \in I_+ : J \cap (a) \neq \varnothing} p^v(a) (x^{c(a)}, x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)}{\int q^x(x^{\setminus i}, x^J) \prod_{a \in I_+ : J \cap (a) \neq \varnothing} p^v(a) (x^{c(a)}, x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)}, \]
where we define $i \in c(i)$ to be the (unique) child of $i$ such that $v(i) = v(i)$. As the right hand side does not depend on $x^i$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.6 such that $C_{aj} \leq 1 - \varepsilon^2$ for every $a \in J$ and $x, z \in S$ such that $x/\{j\} = z/\{j\}$.

**Case 2.** In this case we have $j = i_-$. Let us write $J = \{i_1, \ldots, i_u\}$ where $u = \text{card } J$ and $d(i_k) = (l - 1)q + k$ for $k = 1, \ldots, u$. Thus $i_1 = i$, and we define $i_0 = i_-$. Let us also write $\bar{J}_k = \{i_k, \ldots, i_u\}$. Then we can define the transition kernels on $X^u$

$$P_{k,x}(\omega, A) = \frac{\int 1_A(x^k) p^v(\omega x^c(i_k) | i_k - 1, x^i) \prod_{J_k \cap c(a) \neq \emptyset} p^{v(a)}(x^c(a), x^a) \prod_{b \in J_k} g^v(x^b, Y^b) \psi^v(dx^b)}{\int p^v(\omega x^c(i_k) | i_k - 1, x^i) \prod_{J_k \cap c(a) \neq \emptyset} p^{v(a)}(x^c(a), x^a) \prod_{b \in J_k} g^v(x^b, Y^b) \psi^v(dx^b)}$$

for $k = 1, \ldots, u$. By construction, $P_{k,x}(x^k | x^{i_1}, \ldots, x^{i_{k-1}}) = \gamma^J_x(dx^k | x^{i_1}, \ldots, x^{i_{k-1}})$, so we are in the setting of Lemma 4.7. Moreover, we can estimate

$$P_{k,x}(\omega, A) \geq \varepsilon^2 \delta^2 \frac{\int 1_A(x^i) \prod_{J \cap c(a) \neq \emptyset} p^{v(a)}(x^c(a), x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)}{\int \prod_{J \cap c(a) \neq \emptyset} p^{v(a)}(x^c(a), x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)}$$

Thus whenever $x, z \in S$ satisfy $x/\{j\} = z/\{j\}$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.6 such that $C_{aj} \leq 1 - \varepsilon^2 \delta^2$ for every $k = 1, \ldots, u$.

**Case 3.** In this case we have $j \in \bigcup \{i \in J : J \cap c(a) \neq \emptyset\}$ or $J \cap c(j) \neq \emptyset$, with $v(j) \neq v$. Let us note for future reference that there are at most $q \Delta^2$ such sites $j$. Now note that

$$\gamma^J_x(A) \geq \varepsilon^2 (\Delta + 1) \times \frac{\int 1_A(x^j) p^v(x^{c(i)}, x^i) \prod_{a \in I_+ : J \cap c(a) \neq \emptyset} \beta^a(x^c(a), x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)}{\int p^v(x^{c(i)}, x^i) \prod_{a \in I_+ : J \cap c(a) \neq \emptyset} \beta^a(x^c(a), x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b)},$$

where we have defined $\beta^a(x^c(a), x^a) = q^{v(a)}(x^{a_-}, x^a)$ whenever $j = a$ or $j \in c(a)$, and $\beta^a(x^c(a), x^a) = p^{v(a)}(x^c(a), x^a)$ otherwise. The right hand side of this expression does not depend on $x^j$ as the terms $q^{v(a)}(x^{a_-}, x^a)$ for $v(a) \neq v$ cancel in the numerator and denominator. Thus whenever $x, z \in S$ satisfy $x/\{j\} = z/\{j\}$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.6 such that $C_{aj} \leq 1 - \varepsilon^2 (\Delta + 1)$ for every $a \in J$.

**Case 4.** In this case $J \cap c(j) \neq \emptyset$ with $v(j) \neq v$. Note that

$$\gamma^J_x(A) \geq \varepsilon^2 \int 1_A(x^j) p^v(x^{c(i)}, x^i) \prod_{a \in I_+ : J \cap c(a) \neq \emptyset} \beta^a(x^c(a), x^a) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b) \prod_{b \in J} g^v(x^b, Y^b) \psi^v(dx^b),$$

where $\beta^a(x^c(a), x^a) = q^{v(a)}(x^{a_-}, x^a)$ when $j = a$, and $\beta^a(x^c(a), x^a) = p^{v(a)}(x^c(a), x^a)$ otherwise. The right hand side does not depend on $x^j$ as the term $q^{v(j)}(x^{j-}, x^j)$ cancels in the numerator and denominator. Thus whenever $x, z \in S$ satisfy $x/\{j\} = z/\{j\}$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.6 such that $C_{aj} \leq 1 - \varepsilon^2$ for every $a \in J$. 


Case 5. In this case we have $j_\ast \in J$. Note that the existence of such $j$ necessarily implies that $\ell(i) > l_q$ by the definition of $J$. We can therefore write $J = \{i_1, \ldots, i_q\}$ where $d(i_k) = l_q - k + 1$ for $k = 1, \ldots, q$, and we define $i_0 = j$. Let us also define the sets $\tilde{J}_k = \{i_k, \ldots, i_q\}$. Then we can define the transition kernels on $\mathbb{X}^v$

$$P_{k,x}(\omega, A) = \frac{\int 1_A(x^{i_k}) p^v(x^{c(i_k)}, x^{i_k}) \prod_{a \in I \setminus \tilde{J}_k \cap e(a) \neq \emptyset} \beta^a_\omega(x^{c(a)}, x^a) \prod_{b \in \tilde{J}_k} g^v(x^b, Y^b) \psi^v(dx^b)}{\int p^v(x^{c(i_k)}, x^{i_k}) \prod_{a \in I \setminus \tilde{J}_k \cap e(a) \neq \emptyset} \beta^a_\omega(x^{c(a)}, x^a) \prod_{b \in \tilde{J}_k} g^v(x^b, Y^b) \psi^v(dx^b)},$$

for $k = 1, \ldots, q$, where $\beta^a_\omega(x^{c(a)}, x^a) = p^v(x^{c(a)}, \omega)$ if $a = i_{k-1}$ and $\beta^a_\omega(x^{c(a)}, x^a) = p^v(a)(x^{c(a)}, x^a)$ otherwise. By construction $P_{k,x}(x^{k-1}, dx^{k}) = \gamma^J_{dx} (dx^k | x^{i_1}, \ldots, x^{i_{k-1}})$, so we are in the setting of Lemma 4.7. Moreover, we can estimate

$$P_{k,x}(\omega, A) \geq \varepsilon^2 \delta^2 \times \frac{\int 1_A(x^{i_k}) p^v(x^{c(i_k)}, x^{i_k}) \prod_{a \in I \setminus \tilde{J}_k \cap e(a) \neq \emptyset} \beta^a(x^{c(a)}, x^a) \prod_{b \in \tilde{J}_k} g^v(x^b, Y^b) \psi^v(dx^b)}{\int p^v(x^{c(i_k)}, x^{i_k}) \prod_{a \in I \setminus \tilde{J}_k \cap e(a) \neq \emptyset} \beta^a(x^{c(a)}, x^a) \prod_{b \in \tilde{J}_k} g^v(x^b, Y^b) \psi^v(dx^b)},$$

where $\beta^a(x^{c(a)}, x^a) = 1$ if $a = i_{k-1}$ and $\beta^a(x^{c(a)}, x^a) = p^v(a)(x^{c(a)}, x^a)$ otherwise. Thus whenever $x, z \in S$ satisfy $x^{\tilde{J} \setminus \{j\}} = z^{\tilde{J} \setminus \{j\}}$, we can construct a coupling $Q^J_{x,z}$ using Lemma 4.7 such that $C_{i_k \ast j} \leq (1 - \varepsilon^2 \delta^2)^k$ for every $k = 1, \ldots, q$.

We have now constructed coupled updates $Q^J_{x,z}$ for every pair $x, z \in S$ that differ only at one point. Collecting the above bounds on the matrix $C$, we can estimate

$$\sum_{j \in I} e^{\beta|d(a) - d(j)|} C_{a;j} \leq 3q \Delta^2 e^{\beta q} (1 - \varepsilon^2 (\Delta + 1)) + e^\beta (1 - \varepsilon^2 \delta^2) + e^{\beta q} (1 - \varepsilon^2 \delta^2)^q =: c$$

whenever $a \in J$, where we have used the convexity of the function $\alpha^{x+1} + \alpha^n - x$.

Up to this point we have considered an arbitrary block $J = J^I_l \in \mathcal{J}$ with $1 \leq l \leq \lceil n/q \rceil$. However, in the remaining case $l = 0$ it is easily seen that $\gamma_{\ast}^J = \delta_{\sigma,J}$ does not depend on $x$, so we can evidently set $C_{a;j} = 0$ for $a \in J$. Thus we have shown that

$$\|C\|_{\infty, \beta^m} := \max_{i \in I} \sum_{j \in I} e^{\beta m(i,j)} C_{i;j} \leq c,$$

where we define the pseudometric $m(i, j) = |d(i) - d(j)|$. On the other hand, in the present setting it is evident that $\gamma_{\ast}^J = \tilde{\gamma}_{\ast}^J$ whenever $J = J^I_l \in \mathcal{J}$ with $1 \leq l \leq \lceil n/q \rceil$. We can therefore choose couplings $\tilde{Q}^J_{x,z}$ such that $b_i \leq 1_{d(i) = 0}$ for all $i \in J$. Substituting into the comparison theorem and arguing as in the proof of Theorem 4.5 yields the estimate

$$\|\tilde{F}_n \cdots \tilde{F}_1 \delta_{\sigma} - \tilde{F}_n \cdots \tilde{F}_1 \delta_{\tilde{\sigma}} \|_J \leq \frac{2}{1 - c} \operatorname{card} J e^{-\beta n}.$$

Thus the proof is complete. $\square$
Proposition 4.9 provides control of the block filter as a function of time but not as a function of the initial conditions. The dependence on the initial conditions can however be incorporated \textit{a posteriori} as in the proof of [19, Proposition 4.17]. This yields the following result, which forms the basis for the proof of Theorem 4.11 below.

**Corollary 4.10** (Block filter stability). Suppose there exist \(0 < \varepsilon, \delta < 1\) such that
\[
\varepsilon q^v(x^v, z^v) \leq p^v(x, z^v) \leq \varepsilon^{-1} q^v(x^v, z^v),
\]
\[\delta \leq q^v(x^v, z^v) \leq \delta^{-1}\]
for every \(v \in V\) and \(x, z \in X\), where \(q^v : X^v \times X^v \to \mathbb{R}_+\) is a transition density with respect to \(\psi^v\). Suppose also that we can choose \(q \in \mathbb{N}\) and \(\beta > 0\) such that
\[
c := 3q\Delta^2 e^{\beta q(1 - \varepsilon^2(\Delta + 1))} + e^{\beta q(1 - \varepsilon^2(\Delta + 1))} < 1.
\]

Let \(\mu\) and \(\nu\) be (possibly random) probability measures on \(X\) of the form
\[
\mu = \bigotimes_{K \in \mathcal{K}} \mu^K, \quad \nu = \bigotimes_{K \in \mathcal{K}} \nu^K.
\]
Then we have
\[
\|\tilde{F}_n \cdots \tilde{F}_{s+1} \mu - \tilde{F}_n \cdots \tilde{F}_{s+1} \nu\|_J \leq \frac{2}{1 - c} \text{card } J e^{-\beta(n-s)},
\]
as well as
\[
\mathbb{E}[\|\tilde{F}_n \cdots \tilde{F}_{s+1} \mu - \tilde{F}_n \cdots \tilde{F}_{s+1} \nu\|_J^{2}]^{1/2}
\leq \frac{2}{1 - c} \varepsilon \delta^{2|\mathcal{K}|}\text{card } J (e^{\beta \Delta_{\mathcal{K}}})^{n-s} \max_{K \in \mathcal{K}} \mathbb{E}[\|\mu^K - \nu^K\|_2^2]^{1/2},
\]
for every \(s < n\), \(K \in \mathcal{K}\) and \(J \subseteq K\).

**Proof.** The proof is a direct adaptation of [19, Proposition 4.17].

The block filter stability result in [19] is the only place in the proof of the variance bound where the inadequacy of the classical comparison theorem plays a role. Having exploited the generalized comparison Theorem 2.4 to extend the stability results in [19] to the present setting, we would therefore expect that the remainder of the proof of the variance bound follows verbatim from [19]. Unfortunately, however, there is a complication: the result of Corollary 4.10 is not as powerful as the corresponding result in [19]. Note that the first (uniform) bound in Corollary 4.10 decays exponentially in time \(n\), but the second (initial condition dependent) bound only decays in \(n\) if it happens to be the case that \(e^{-\beta \Delta_{\mathcal{K}}} < 1\). As in [19] both the spatial and temporal interactions were assumed to be sufficiently weak, we could assume that the latter was always the case. In the present setting, however, it is possible that \(e^{-\beta \Delta_{\mathcal{K}}} \geq 1\) no matter how weak are the spatial correlations.
To surmount this problem, we will use a slightly different error decomposition than was used in [19] to complete the proof of the variance bound. The present approach is inspired by [4]. The price we pay is that the variance bound scales in the number of samples as $N^{-\gamma}$ where $\gamma$ may be less than the optimal (by the central limit theorem) rate $\frac{1}{2}$. It is likely that a more sophisticated method of proof would yield the optimal $N^{1/2}$ rate in the variance bound. However, let us note that in order to put the block particle filter to good use we must optimize over the size of the blocks in $K$, and optimizing the error bound in Theorem 4.2 yields at best a rate of order $N^{-\alpha}$ for some constant $\alpha$ depending on the constants $\beta_1, \beta_2$. As the proof of Theorem 4.2 is not expected to yield realistic values for the constants $\beta_1, \beta_2$, the suboptimality of the variance rate $\gamma$ does not significantly alter the practical conclusions that can be drawn from Theorem 4.2.

We now proceed to the variance bound. The following is the main result of this section.

**THEOREM 4.11 (Variance term).** Suppose there exist $0 < \varepsilon, \delta, \kappa < 1$ such that

$$\varepsilon q(x^v, z^v) \leq p(x^v, z^v) \leq \varepsilon^{-1} q(x^v, z^v),$$

$$\delta \leq q(x^v, z^v) \leq \delta^{-1},$$

$$\kappa \leq g(x^v, y^v) \leq \kappa^{-1}$$

for every $v \in V$, $x, z \in X$, and $y \in Y$, where $q : X^v \times X^v \to \mathbb{R}^+$ is a transition density with respect to $\psi^v$. Suppose also that we can choose $q \in \mathbb{N}$ and $\beta > 0$ such that

$$c := 3q \Delta^2 e^{\beta q} (1 - \varepsilon^2 (\Delta + 1)) + e^{\beta (1 - \varepsilon^2 \delta^2)} + e^{\beta q (1 - \varepsilon^2 \delta^2)} q < 1.$$

Then for every $n \geq 0$, $\sigma \in X$, $K \in \mathcal{K}$ and $J \subseteq K$, the following hold:

1. If $e^{-\beta \Delta_X} < 1$, we have

$$\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \text{card } J \frac{32 \Delta_X}{1 - c} \frac{2 - e^{-\beta \Delta_X} (\varepsilon \delta \kappa \Delta_X)^{-4|X|\infty}}{1 - e^{-\beta \Delta_X} \frac{N^{1/2}}{N}}.$$

2. If $e^{-\beta \Delta_X} = 1$, we have

$$\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \text{card } J \frac{16 \beta^{-1} \Delta_X}{1 - c} \frac{(\varepsilon \delta \kappa \Delta_X)^{-4|X|\infty} 3 + \log N}{N^{1/2}}.$$

3. If $e^{-\beta \Delta_X} > 1$, we have

$$\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \text{card } J \frac{32 \Delta_X}{1 - c} \left\{ \frac{1}{e^{-\beta \Delta_X} - 1} + 2 \right\} \frac{(\varepsilon \delta \kappa \Delta_X)^{-4|X|\infty}}{N^{2 \log \Delta_X}}.$$

The proof of Theorem 4.11 combines the stability bounds of Corollary 4.10 and one-step bounds on the sampling error, [19, Lemma 4.19 and Proposition 4.22], that can be used verbatim in the present setting. We recall the latter here for the reader’s convenience.
Proposition 4.12 (Sampling error). Suppose there exist $0 < \varepsilon, \delta, \kappa < 1$ such that
\[
\varepsilon q^v(x^v, z^v) \leq p^v(x^v, z^v) \leq \varepsilon^{-1} q^v(x^v, z^v),
\]
\[
\delta \leq q^v(x^v, z^v) \leq \delta^{-1},
\]
\[
\kappa \leq q^v(x^v, y^v) \leq \kappa^{-1}
\]
for every $v \in V$, $x, z \in X$, and $y \in Y$. Then we have
\[
\max_{K \in \mathcal{K}} \left| \tilde{\pi}_n - \hat{\pi}_n \right|_K \leq \frac{2\kappa^{-2} |\mathcal{K}|_\infty}{N^{\frac{1}{2}}}
\]
and
\[
\max_{K \in \mathcal{K}} \mathbb{E}\left[ \left| \tilde{F}_{s+1} \hat{\pi}_{s+1} - \tilde{F}_{s+1} \hat{\pi}_{s+1} \right|_K \right]^{1/2} \leq \frac{16\Delta_X (\varepsilon\delta)^{-2} |\mathcal{X}|_\infty \kappa^{-4} |\mathcal{K}|_\infty \Delta_X}{N^{\frac{1}{2}}}
\]
for every $0 < s < n$ and $\sigma \in \mathcal{X}$.

Proof. Immediate from [19, Lemma 4.19 and Proposition 4.22] upon replacing $\varepsilon$ by $\varepsilon\delta$.

We can now prove Theorem 4.11.

Proof of Theorem 4.11. We fix for the time being an integer $t \geq 1$ (we will optimize over $t$ at the end of the proof). We argue differently when $n \leq t$ and when $n > t$.

Suppose first that $n \leq t$. In this case, we estimate
\[
\left| \tilde{\pi}_n - \hat{\pi}_n \right|_J = \left| \tilde{F}_n \cdots \tilde{F}_1 \delta_\sigma - \hat{F}_n \cdots \hat{F}_1 \delta_\sigma \right|_J
\]
\[
\leq \sum_{k=1}^{n} \left| \tilde{F}_n \cdots \tilde{F}_{k+1} \hat{F}_{k+1} \hat{\pi}_{k-1} - \tilde{F}_n \cdots \tilde{F}_{k+1} \hat{F}_{k+1} \hat{\pi}_{k-1} \right|_J
\]
using a telescoping sum and the triangle inequality. The term $k = n$ in the sum is estimated by the first bound in Proposition 4.12, while the remaining terms are estimated by the second bound of Corollary 4.10 and Proposition 4.12, respectively. This yields
\[
\left| \tilde{\pi}_n - \hat{\pi}_n \right|_J \leq \text{card } J \frac{2\Delta_X}{1 - c} \left( (\varepsilon\delta\Delta_X)^{-4} |\mathcal{X}|_\infty \right) \left\{ \frac{(e^{-\beta} \Delta_X)^{n-1} - 1}{e^{-\beta} \Delta_X - 1} + 1 \right\}
\]
in the case $e^{-\beta} \Delta_X = 1$, the quantity between the brackets $\{ \cdot \}$ equals $n$.

Now suppose that $n > t$. Then we decompose the error as
\[
\left| \tilde{\pi}_n - \hat{\pi}_n \right|_J \leq \left| \tilde{F}_n \cdots \tilde{F}_{n-t} \hat{\pi}_{n-t} - \tilde{F}_n \cdots \tilde{F}_{n-t} \hat{\pi}_{n-t} \right|_J
\]
\[
+ \sum_{k=n-t+1}^{n} \left| \tilde{F}_n \cdots \tilde{F}_{k+1} \hat{F}_{k+1} \hat{\pi}_{k-1} - \tilde{F}_n \cdots \tilde{F}_{k+1} \hat{F}_{k+1} \hat{\pi}_{k-1} \right|_J,
\]

We can now prove Theorem 4.11.
that is, we develop the telescoping sum for \( t \) steps only. The first term is estimated by the first bound in Corollary 4.10, while the sum is estimated as in the case \( n \leq t \). This yields
\[
\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \frac{\text{card } J}{1 - c} \left[ 2e^{-\beta t} + \frac{32 \Delta_X (\varepsilon \delta_R \Delta_X)^{-4|X|_\infty}}{N^{\frac{1}{2}}} \left( \frac{\left( e^{-\beta \Delta_X} \right)^{t-1} - 1}{e^{-\beta \Delta_X} - 1} + 1 \right) \right]
\]
(in the case \( e^{-\beta \Delta_X} = 1 \), the quantity between the brackets \( \{ \cdot \} \) equals \( t \).

We now consider separately the three cases in the statement of the Theorem.

**Case 1.** In this case we choose \( t = n \), and note that
\[
\frac{(e^{-\beta \Delta_X})^{n-1} - 1}{e^{-\beta \Delta_X} - 1} + 1 \leq \frac{2 - e^{-\beta \Delta_X}}{1 - e^{-\beta \Delta_X}} \quad \text{for all } n \geq 1.
\]

Thus the result follows from the first bound above.

**Case 2.** In this case we have
\[
\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \frac{\text{card } J}{1 - c} \left[ 2e^{-\beta t} + \frac{32 \Delta_X (\varepsilon \delta_R \Delta_X)^{-4|X|_\infty}}{N^{\frac{1}{2}}} \log N \right],
\]
which readily yields the desired bound.

**Case 3.** In this case we have
\[
\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \frac{\text{card } J}{1 - c} \left[ 2e^{-\beta t} + \frac{32 \Delta_X (\varepsilon \delta_R \Delta_X)^{-4|X|_\infty}}{N^{\frac{1}{2}}} \left( \frac{\left( e^{-\beta \Delta_X} \right)^{t-1} - 1}{e^{-\beta \Delta_X} - 1} + 1 \right) \right]
\]
for all \( t, n \geq 1 \). Now choose \( t = \lceil (2\beta)^{-1} \log N \rceil \). Then
\[
\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \frac{\text{card } J}{1 - c} \left[ 2e^{-\beta t} + \frac{32 \Delta_X (\varepsilon \delta_R \Delta_X)^{-4|X|_\infty}}{N^{\frac{1}{2}}} \log N \right],
\]
which yields
\[
\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \frac{\text{card } J}{1 - c} \left[ 2e^{-\beta t} + \frac{32 \Delta_X (\varepsilon \delta_R \Delta_X)^{-4|X|_\infty}}{N^{\frac{1}{2}}} \left( \frac{\left( e^{-\beta \Delta_X} \right)^{t-1} - 1}{e^{-\beta \Delta_X} - 1} + 1 \right) \right]
\]
for all \( t, n \geq 1 \). Now choose \( t = \lceil \frac{\log N}{2 \log \Delta_X} \rceil \). Then
\[
\| \tilde{\pi}_n^\sigma - \hat{\pi}_n^\sigma \|_J \leq \frac{\text{card } J}{1 - c} \left[ 2e^{-\beta t} + \frac{32 \Delta_X (\varepsilon \delta_R \Delta_X)^{-4|X|_\infty}}{N^{\frac{1}{2}}} \left( \frac{\left( e^{-\beta \Delta_X} \right)^{t-1} - 1}{e^{-\beta \Delta_X} - 1} + 1 \right) \right]
\]
and the proof is complete.

The conclusion of Theorem 4.2 now follows readily from Theorems 4.5 and 4.11. We must only check that the assumptions Theorems 4.5 and 4.11 are satisfied. The assumption of Theorem 4.5 is slightly stronger than that of Theorem 4.11, so it suffices to consider the former. To this end, fix \( 0 < \delta < 1 \), and choose \( q \in \mathbb{N} \) such that
\[
1 - \delta^2 + (1 - \delta^2)^q < 1.
\]
Then we may evidently choose \( 0 < \varepsilon_0 < 1 \), depending on \( \delta \) and \( \Delta \) only, such that
\[
3q \Delta^2 (1 - e^{2(\Delta+1)}) + 1 - e^2 \delta^2 + (1 - e^2 \delta^2)^q < 1
\]
for all \( \varepsilon_0 < \varepsilon \leq 1 \). This is the constant \( \varepsilon_0 \) that appears in the statement of Theorem 4.2. Finally, it is now clear that we can choose \( \beta > 0 \) sufficiently close to zero (depending on \( \delta, \varepsilon, r, \Delta \) only) such that \( c < 1 \). Thus the proof of Theorem 4.2 is complete.
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