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Adaptive Boundary Observer Design for coupled ODEs-Hyperbolic PDEs systems

Mohammad Ghousein ∗ Emmanuel Witrant ∗

Abstract: We consider the state estimation of $n_\xi$ hyperbolic PDEs coupled with $n_X$ ordinary differential equations at the boundary. The hyperbolic system is linear and propagates in the positive x-axis direction. The ODE system is linear time varying (LTV) and includes a set of $n_\theta$ unknown constant parameters, which are to be estimated simultaneously with the PDE and the ODE states using boundary sensing. We design a Luenberger state observer, and our method is mainly based on the decoupling of the PDE estimation error states from that of the ODEs via swapping design. We then derive the observer gains through the Lyapunov analysis of the decoupled system. Furthermore, we give sufficient conditions of the exponential convergence of the adaptive observer through differential Lyapunov inequalities (DLIs) and we illustrate the theoretical results by numerical simulations.

Keywords: Hyperbolic partial differential equations, Adaptive boundary Observers, Boundary Control.

1. INTRODUCTION

Many physical processes are modeled using linear hyperbolic partial differential equations coupled with linear ordinary differential equations. The infinite state which is modeled by the PDE represents the transport in space, and its value at the boundary is usually constrained to some exterior dynamics represented by the ODEs. The mentioned coupling topology mostly appears in networks, where the edges are modeled using transport PDEs and the nodes are modeled using ODEs. Examples of such systems can be found in road traffic in Goatin (2006), gas flow in pipelines in Gugat and Dick (2011), transmission lines in Hasan and Imsland (2014), flow in open channels in Coron et al. (2007), exhaust gas regulation (EGR) in car engines in Castillo et al. (2014), etc. Practically speaking, boundary control and observation of these kinds of systems is more realistic than the distributed ones, since actuators and sensors are placed naturally at the extremities of the domain. In addition, in several real applications, we may not have complete knowledge of the system’s parameters on both the PDE and the ODE sides. This adds more complexity to the control and the observer designs in view of the limited amount of available measurements. In short, the idea of developing adaptive boundary controls and observers for coupled ODEs-hyperbolic PDEs systems is a necessity if we consider the significant number of physical applications. Boundary control of ODEs coupled-hyperbolic PDEs systems is well established in the literature. Using Lyapunov design, the authors in Castillo et al. (2012) derived control laws to stabilize a system of linear hyperbolic system with dynamic boundary conditions. Sufficient conditions for the exponential convergence of the system were given by linear matrix inequalities (LMIs). In a different approach, the authors in Krstic and Smyshlyaev (2008) have used the theory of backstepping to stabilize a LTI system with an arbitrary input delay. The system is modeled as a transport equation coupled with a LTI system at the boundary. Moreover, the authors in Aamo (2012), Anfinsen and Aamo (2014) consider $2 \times 2$ linear hyperbolic systems with boundary disturbances. In their work, they modeled the disturbance using an LTI system and they applied backstepping control to the resulting coupled ODE-PDE system. In a recent work, the authors in Di Meglio et al. (2018) extended the mentioned approaches to systems of heterodirectional hyperbolic PDEs coupled with ODEs at the boundary.

Boundary observers for ODEs-coupled hyperbolic PDEs are less investigated in the literature. The authors in Castillo et al. (2013) designed a Luenberger observer for systems of linear and quasilinear hyperbolic systems with dynamic boundary conditions which are asymptotically stable. This approach was later extended by the authors in Ferrante and Cristofaro (2019) to linear hyperbolic systems coupled with possibly unstable LTI systems. By keeping the same observer architecture in Castillo et al. (2013) but using a non-diagonal quadratic Lyapunov function, the authors in Ferrante and Cristofaro (2019) have derived sufficient conditions for the exponential stability of the observer through bilinear matrix inequalities (BMIs). On the other hand, backstepping boundary observer designs are also investigated for coupled ODEs-hyperbolic PDEs systems. The authors in Krstic and Smyshlyaev (2008) synthesized an observer for LTI systems with arbitrary constant delay in the sensor measurement. The delay is interpreted as a first order transport equation and back-
stepping observer design is used on the resulting coupled LTI-PDE system. This work was later extended by the authors in Hasan et al. (2016) to a 2×2 hyperbolic system coupled with a linear LTI system at the boundary.

All the results mentioned so far assume a perfect knowledge of the system. In many practical cases, some model parameters are unknown, which motivates the need for adaptive estimators. The objective of an adaptive boundary observer is to simultaneously construct the distributed PDE states and the unknown parameters from only boundary sensing. In fact, few results exist in the literature on the adaptive design for coupled ODEs-hyperbolic PDEs systems. The authors in Anfinsen and Aamo (2017) synthesize an adaptive observer for a 2×2 hyperbolic system coupled with an uncertain LTI system. The design was done in several steps. The first step is to estimate the unknown parameters by extracting some delayed measurements from the system. The second step is to build a Luenberger state observer for the ODE states and the third step is to use swapping filters to generate estimates of the PDE states. In this framework, we consider the observer design of a system of linear positive speed transport equations coupled with linear time varying ODEs at the boundary. The system involves a set of unknown constant disturbances to be estimated. Such class of systems can be extended to model the air-path in exhaust engines (see e.g. Castillo et al. (2014)). We address the systems can be extended to model the air-path in exhaust engines (see e.g. Castillo et al. (2014)). We introduce the following adaptive observer design:

\begin{equation}
\dot{\hat{x}}(x, t) + A^+ \hat{\theta} = F(x, t)
\end{equation}

\begin{equation}
\xi(0, t) = C(t)\hat{x}(t) + D(t)u(t) + \psi_1(t)\theta
\end{equation}

where \( \theta \) and \( \hat{\theta} \) denote the partial derivatives with respect to time and space respectively. \( \xi(t) : \Omega \rightarrow \mathbb{R}^n_x \) is the PDE state vector. \( \mathbf{X}(t) :[0, +\infty) \rightarrow \mathbb{R}^{n_x} \) is the ODE state vector. \( \theta \in \mathbb{R}^n_x \) is the vector of the unknown parameters. \( u(t) :[0, +\infty) \rightarrow \mathbb{R}^{n_x} \) is a known input vector. \( A^+ \in D^{n+}_x \) is the matrix of the constant transport speeds \( \lambda_1^+, \lambda_2^+, \lambda_{n_1}^+ \). LTV-ODEs, \( x=0,x=1 \), Hyperbolic-PDEs:

\begin{equation}
F \in \mathbb{R}^{n_x \times n_{\xi}}
\end{equation}

We assume that all the time-dependent matrices: \( A(t) \in \mathbb{R}^{n_x \times n_x}, B(t) \in \mathbb{R}^{n_x \times n_x}, C(t) \in \mathbb{R}^{n_x \times n_x}, D(t) \in \mathbb{R}^{n_x \times n_x}, \psi_1(t) \in \mathbb{R}^{n_x \times n_x} \) and \( \psi_2(t) \in \mathbb{R}^{n_x \times n_x} \) are bounded and piece-wise continuous in time.

Our goal is to estimate \( \xi(x, t), \theta(t) \) assuming that the following measurements are available:

\begin{equation}
y(t) = M\xi(1, t)
\end{equation}

where \( M \in \mathbb{R}^{n_x \times n_{\xi}} \) is the output matrix.

\section{PROBLEM DESCRIPTION}

We consider the following class of cascade ODEs-hyperbolic PDEs systems evolving in \( \Omega=[0, 1] \times [0, +\infty) \):

\begin{equation}
\partial_t \xi(x, t) + A^+ \partial_x \xi(x, t) = F(x, t)
\end{equation}

\begin{equation}
\xi(0, t) = C(t)\mathbf{X}(t) + D(t)u(t) + \psi_1(t)\theta
\end{equation}

where \( \partial_t \) and \( \partial_x \) denote the partial derivatives with respect to time and space respectively. \( \xi(x, t) : \Omega \rightarrow \mathbb{R}^n_x \) is the PDE state vector. \( \mathbf{X}(t) :[0, +\infty) \rightarrow \mathbb{R}^{n_x} \) is the ODE state vector. \( \theta \in \mathbb{R}^n_x \) is the vector of the unknown parameters. \( u(t) :[0, +\infty) \rightarrow \mathbb{R}^{n_x} \) is a known input vector. \( A^+ \in D^{n+}_x \) is the matrix of the constant transport speeds \( \lambda_1^+, \lambda_2^+, \lambda_{n_1}^+ \). LTV-ODEs, \( x=0,x=1 \), Hyperbolic-PDEs:

\begin{equation}
F \in \mathbb{R}^{n_x \times n_{\xi}}
\end{equation}

We introduce the following adaptive observer design:

\begin{equation}
\dot{\hat{\xi}}(x, t) + A^+ \hat{\theta} = F(x, t)
\end{equation}

\begin{equation}
\hat{\xi}(0, t) = C(t)\hat{\mathbf{X}}(t) + D(t)u(t) + \psi_1(t)\hat{\theta}(t)
\end{equation}

\begin{equation}
\hat{\mathbf{X}}(t) = A(t)\hat{\mathbf{X}}(t) + B(t)u(t) + \psi_2(t)\hat{\theta}(t)
\end{equation}

\begin{equation}
+ L(t)y(t) - M\hat{\xi}(1, t)
\end{equation}

where \( y(t) :[0, +\infty) \rightarrow \mathbb{R}^{n_x} \) is the output matrix.

\section{ADAPTIVE OBSERVER DESIGN}

We introduce the following adaptive observer design:

\begin{equation}
\partial_t \xi(x, t) + A^+ \partial_x \xi(x, t) = F(x, t)
\end{equation}

\begin{equation}
\xi(0, t) = C(t)\mathbf{X}(t) + D(t)u(t) + \psi_1(t)\theta
\end{equation}

where \( \partial_t \) and \( \partial_x \) denote the partial derivatives with respect to time and space respectively. \( \xi(x, t) : \Omega \rightarrow \mathbb{R}^n_x \) is the PDE state vector. \( \mathbf{X}(t) :[0, +\infty) \rightarrow \mathbb{R}^{n_x} \) is the ODE state vector. \( \theta \in \mathbb{R}^n_x \) is the vector of the unknown parameters. \( u(t) :[0, +\infty) \rightarrow \mathbb{R}^{n_x} \) is a known input vector. \( A^+ \in D^{n+}_x \) is the matrix of the constant transport speeds \( \lambda_1^+, \lambda_2^+, \lambda_{n_1}^+ \). LTV-ODEs, \( x=0,x=1 \), Hyperbolic-PDEs:

\begin{equation}
F \in \mathbb{R}^{n_x \times n_{\xi}}
\end{equation}

We introduce the following adaptive observer design:

\begin{equation}
\dot{\hat{\xi}}(x, t) + A^+ \hat{\theta} = F(x, t)
\end{equation}

\begin{equation}
\hat{\xi}(0, t) = C(t)\hat{\mathbf{X}}(t) + \psi_1(t)\hat{\theta}(t)
\end{equation}

\begin{equation}
\hat{\mathbf{X}}(t) = A(t)\hat{\mathbf{X}}(t) + \psi_2(t)\hat{\theta}(t) - L(t)\hat{\xi}(1, t)
\end{equation}
3.1 Decoupling using swapping design

We parameterize the PDE state estimation error $\tilde{\xi}(x,t)$ in (9)-(11) using K-filters (see Kreisselmeier (1977) for ODEs and Smyshlyaev and Krstic (2010) for PDEs) as follows:

$$\tilde{\phi}(x,t) = \tilde{\xi}(x,t) - T(x,t)\tilde{X}(t) - R(x,t)\tilde{\theta}(t)$$  \hspace{1cm} (12)

The swapping filters: $T(x,t): \Omega \rightarrow \mathbb{R}^{n_{e} \times n_{x}}$ and $R(x,t): \Omega \rightarrow \mathbb{R}^{n_{e} \times n_{e}}$ are to be defined later. Differentiating (12) with respect to time and space and substituting with (9)-(11) get:

$$\partial_{t}\tilde{\phi}(x,t) + \Lambda^{+}\partial_{\theta}\tilde{\phi}(x,t) = F\tilde{\phi}(x,t) - (K_{1}(x,t) + R(x,t)\hat{\theta}(t)) + T(x,t)L(t) - p(x,t)M\tilde{\xi}(1,t)$$

$$- \partial_{t}T(x,t) - \Lambda^{+}\partial_{\theta}T(x,t) + FT(x,t) - \partial_{t}R(x,t) + FR(x,t) - T(x,t)\psi_{2}(t)\tilde{\theta}(t)$$

Equation (13) suggests to choose: $K_{1}(x,t) = - R(x,t)\hat{\theta}(t) = R(x,t)\tilde{T}(t)$, $p(x,t) = T(t)L(t)$ and the following dynamics for the swapping filters:

$$\partial_{t}T(x,t) + \Lambda^{+}\partial_{\theta}T(x,t) = FT(x,t) - T(x,t)A(t)$$ \hspace{1cm} (14)

$$\partial_{t}R(x,t) + \Lambda^{+}\partial_{\theta}R(x,t) = FR(x,t) - T(x,t)\psi_{2}(t)\tilde{\theta}(t)$$ \hspace{1cm} (15)

we also impose the following boundary conditions on the filters:

$$T(0,t) = C(t), R(0,t) = \psi_{1}(t)$$ \hspace{1cm} (16)

Doing so, and using (13) and (14)-(16) the dynamics of $\tilde{\phi}(x,t)$ become

$$\partial_{t}\tilde{\phi}(x,t) + \Lambda^{+}\partial_{\theta}\tilde{\phi}(x,t) = F\tilde{\phi}(x,t)$$ \hspace{1cm} (17)

$$\tilde{\phi}(0,t) = 0$$ \hspace{1cm} (18)

In view of equation (12) and the derived dynamics (17)-(18), the infinite state estimation error $\tilde{\xi}(x,t)$ splits into three parts: 1) an observation error $\tilde{\phi}(x,t)$ that is totally decoupled from the ODE state estimation errors, 2) $T(x,t)\tilde{X}(t)$, which is proportional to the estimation error on the ODE states $\tilde{X}(t)$, and 3) the induced error due to the parameters mismatch $R(x,t)\tilde{\theta}(t)$ which is also proportional to the parameter estimation errors $\tilde{\theta}(t)$. To prove the exponential convergence of $\tilde{\xi}(x,t)$, $\tilde{X}(t)$, $\tilde{\theta}(t)$ it is then sufficient to prove the exponential convergence of $\tilde{\phi}(x,t)$, $\tilde{X}(t)$, $\tilde{\theta}(t)$ and the boundedness of the filters $T(x,t)$ and $R(x,t)$. This is what we establish in the following series of Lemmas.

**Lemma 1.** Consider the system (17)-(18) with initial condition $\tilde{\phi}_{0}(x) \in (L^{2}([0,1]))^{n_{e}}$. Then for all $\gamma_{\theta} > 0$, there exists $C_{\phi} > 0$ such that:

$$||\tilde{\phi}(.,.)||_{L^{2}([0,1])}^{n_{e}} \leq C_{\phi}e^{-\gamma_{\theta}t}||\tilde{\phi}_{0}(x)||_{L^{2}([0,1])}^{n_{e}}$$ \hspace{1cm} (19)

Furthermore, the equilibrium $\tilde{\phi} \equiv 0$ is reached in finite time $t_{f} = \frac{1}{\gamma_{\phi}}$.

**Proof 1.** Consider the following Lyapunov function

$$V_{1}(t) = \int_{0}^{1} (\tilde{\phi}^{T}(x,t)P_{1}\tilde{\phi}(x,t))e^{-\mu x}dx$$ \hspace{1cm} (20)

where $P_{1} \in D_{+}^{n_{e}}$ and $\mu > 0$. Deriving (20) in time, substituting with (17), integrating by parts and then substituting with (18) yields to:

$$V_{1}(t) = -\tilde{\phi}^{T}(1,t)\Lambda^{+}P_{1}e^{-\theta}(t,1)$$

$$+ \int_{0}^{1} \tilde{\phi}^{T}(x,t)[-\mu\Lambda^{+}P_{1} + F^{T}P_{1} + P_{1}F]e^{-\mu x}\tilde{\phi}(x,t)dx$$ \hspace{1cm} (21)

The matrix $\Lambda^{+}P_{1}e^{-\mu}$ is always positive definite for any $P_{1} \in D_{+}^{n_{e}}$. In addition, for all $\gamma_{\phi} > 0$ we can always choose $\mu$ large enough to have $-\mu\Lambda^{+}P_{1} + F^{T}P_{1} + P_{1}F \leq -\gamma_{\phi}P_{1}$. Thus, $V_{1}(t) \leq -\gamma_{\phi}V_{1}(t)$ which shows the exponential convergence of $\tilde{\phi}$ in the $L^{2}$-norm. Given that $\Lambda^{+} \in D_{+}^{n_{e}}$, we can change the status of $t$ and $x$ and rewrite (17) as:

$$\partial_{x}\tilde{\phi}(x,t) + (\Lambda^{+})^{-1}\partial_{\theta}\tilde{\phi}(x,t) = (\Lambda^{+})^{-1}F\tilde{\phi}(x,t)$$ \hspace{1cm} (22)

and then (18) becomes a zero initial condition for (22). Then the uniqueness of solutions of (22)-(18) and the order of the transport speeds given in equation (4) imply that $\tilde{\phi}(x,t)$ vanishes after $t \geq \frac{1}{\gamma_{\phi}}$ (see Lemma 3.1 in Hu et al. (2016) for further details) and this concludes the proof.

**Lemma 2.** Consider the filter systems $T(x,t)$ and $R(x,t)$ defined in (14)-(15) with boundary conditions (16). Then for all initial conditions $T_{0}(x) \in (L^{2}([0,1]))^{n_{e} \times n_{x}}$ and $R_{0}(x) \in (L^{2}([0,1]))^{n_{e} \times n_{e}}$, the PDE filters $T(x,t)$ and $R(x,t)$ are bounded in the $L^{2}$ sense.

**Proof 2.** We start by $T(x,t)$. We write (14)-(16) using the index notation: for all $1 \leq i \leq n_{e}$, $1 \leq j \leq n_{x}$, we have

$$\partial_{x}T_{ij}(x,t) + \lambda_{i}\partial_{\theta}T_{ij}(x,t) = \sum_{k=1}^{n_{e}} F_{ik} T_{kj}(x,t)$$ \hspace{1cm} (23)

$$- \sum_{k=1}^{n_{x}} T_{ik}(x,t)a_{kj}(t)$$

$$T_{ij}(0,t) = c_{ij}(t)$$ \hspace{1cm} (24)

Now, we consider the following Lyapunov function

$$V_{2}(t) = \sum_{i=1}^{n_{e}} \sum_{j=1}^{n_{x}} V_{ij}(t) = \frac{1}{2} \sum_{i=1}^{n_{e}} \sum_{j=1}^{n_{x}} \int_{0}^{1} e^{-\mu x}T_{ij}^{2}(x,t)dx$$ \hspace{1cm} (25)

with $\mu > 0$. Deriving (25) with respect to time, replacing by (23), integrating by parts and substituting by (24), one gets

$$\dot{V}_{2}(t) = \sum_{i=1}^{n_{e}} \sum_{j=1}^{n_{x}} \left[ - \frac{1}{2} e^{-\mu x}\lambda_{i}T_{ij}^{2}(1,t) + \frac{1}{2} \lambda_{i}c_{ij}^{2}(t) - \mu \lambda_{i}V_{ij}(t) + \sum_{k=1}^{n_{e}} F_{ik} \int_{0}^{1} e^{-\mu x}T_{kj}(x,t)T_{ij}(x,t)dx - \sum_{k=1}^{n_{x}} a_{kj}(t) \int_{0}^{1} e^{-\mu x}T_{ik}(x,t)T_{ij}(x,t)dx \right]$$ \hspace{1cm} (26)

Applying Young’s inequality to the last two integral terms in (26), we get

...
\[
\dot{V}_2(t) \leq \sum_{i=1}^{n_L} \sum_{j=1}^{n_x} \left[ \frac{1}{2} \lambda c_j^2(t) + \left( -\mu \lambda_1 + \sum_{k=1}^{n_L} |F_{ik}| + |F_{kl}| \right) \right] + \sum_{k=1}^{n_x} \left( |a_{kj}(t)| + |a_{kj}(t)| \right) V_{ij}(t)
\]

\[
\dot{V}_2(t) \leq \sum_{i=1}^{n_L} \sum_{j=1}^{n_x} \left[ \frac{1}{2} \lambda c_j^2(t) + \left( -\mu \lambda_1 + 2n_x F_{max} \right) \right] + 2n_L A_{max} V_{ij}(t)
\]

We can choose \( \mu \) large enough to have \(-\mu \lambda_1 + 2n_x F_{max} + 2n_L A_{max} \leq -\gamma_2 \) for every \( \gamma_2 > 0 \). Doing so, (28) becomes

\[
\dot{V}_2(t) \leq -\gamma_2 V_2(t) + \frac{\lambda_{max}}{2} ||C(t)||^2
\]

Denoting by \( F_{max} = ||F||_\infty \) and \( A_{max} = \max_{t \geq 0} ||A(t)||_\infty \), we can further write (27) as

\[
\dot{V}_2(t) \leq -\gamma_2 V_2(t) + \frac{\lambda_{max}}{2} ||C(t)||^2
\]

The ODE dynamics are investigated as follows. We evaluate (12) at \( x=1 \), multiply by \( M \) on both sides, and then substitute in (11) to have

\[
\dot{X}(t) = A_d(t)\hat{X}(t) + (\psi_2(t) - L(t)MR(1,t))\dot{\hat{\theta}}(t)
\]

with \( A_{d}(t) = A(t) - L(t)MT(1,t) \). We introduce the piecewise continuous shift operator \( s(t) \), then (29) becomes

\[
\dot{X}(t) = (A(t) - s(t)l(t)MT(1,t))\hat{X}(t) + (\psi_2(t) - s(t)l(t)MR(1,t))\dot{\hat{\theta}}(t) \]

Equation (12) at \( x=1 \) also suggests the following normalized adaptation law:

\[
\dot{\hat{\theta}}(t) = -\hat{\theta}(t) + \frac{s(t)P_0(t)\Phi(t)}{1 + ||\Phi(t)\Phi(t)||^2} M\hat{\xi}(1,t)
\]

\[
\dot{P}_0(t) = s(t) \left[ \beta P_0(t) - \frac{P_0(t)\Phi(t)\Phi(t)P_0(t)}{1 + ||\Phi(t)\Phi(t)||^2} \right]
\]

where the regressor \( \Phi(t) \) is given by \( \Phi(t) = MR(1,t) \). \( P_0(t) : [0, +\infty) \rightarrow \mathbb{R}^{nx \times nx} \) and \( \beta > 0 \) is the forgetting factor. The initial conditions \( \theta(0) = \theta_0 \) and \( P_0(0) = P_{\theta,0} = \Phi(0) \) are chosen arbitrary. It is useful to illustrate that the adaptation law (32)-(33) is derived using the superposition principle, i.e. we fix \( \phi(1,t) \) and \( \tilde{X}(t) \) to zero in order to get the linear regressor equation

\[
\ddot{\theta}(t) = \frac{-s(t)P_0(t)}{1 + ||\Phi(t)\Phi(t)||^2} \left[ \Phi(t)\Phi(t)\dot{\theta}(t) + \Phi(t)\Phi(t)\hat{\theta}(t) \right]
\]

Remark 1. The formulation of (31) and (35) as a function of \( s(t) \) implies that the ODE error stabilization and the parameter adaptation start functioning when the maximum delay time due to transport in space (\( tf = \frac{1}{\gamma} \)) is passed.

In addition, if there exist an observer gain \( L(t) \in \mathbb{R}^{nx \times nx} \) and a bounded matrix \( P_X(t) \in \mathbb{S}^{nx \times nx} \) such that, for all \( t \geq tf \),

\[
Z(t) \leq -Q(t)
\]

\[
\dot{Z}(t) \leq -Q(t)
\]

where (38) and (39) is exponentially stable in the \( ||X^2 + \dot{\theta}^2 + \parallel \theta_\parallel^2 \parallel \Phi(t)\Phi(t)||^2 \) norm. The system (17)-(18) and (31)-(35) is exponentially stable in the \( ||X^2 + \dot{\theta}^2 + \parallel \theta_\parallel^2 \parallel \Phi(t)\Phi(t)||^2 \) norm.
The differential Lyapunov equation in $A_d(t)$. It is well known that (46) has a unique solution $P_X(t)$ if $A_d(t)$ is UES. Any time-varying state matrix which is 1) continuously differentiable, 2) bounded, 3) slowly varying and 4) the real part of its Eigen-values is negative for all times is UES (see e.g. Theorem 8.7 in Rugh (1996)). For instance if we assume that the first three conditions of Theorem 8.7 in Rugh (1996)) are satisfied for $A_d(t)$ in the interval of time $[t_f, +\infty)$, we still require that the real part of its eigen-values to be negative. Let us recall that for $t \geq t_f$, $A_d(t) = A(t) - l(t)MT(1, t)$. We can always choose $l(t)$ such that $A_d(t)$ is Hurwitz if the pair $(A(t), MT(1, t))$ is detectable. If we look into the $T(x, t)$ filter (14)-(16), we can observe that $T(1, t)$ is a delayed version of $C(t)$ with a change in magnitude due to the coupling $(P$ and $A(t))$. Hence, finding $P_X(t)$ is directly related to the detectability of the system $(A(t), M, C(t))$ through the pair $(A(t), MT(1, t))$. On the other hand, $Z_{22}(t)$ is always negative-definite, since $P_{\theta}^{-1}(t)$ is positive definite and bounded based on the (PE) assumption (36). It is important to mention that the condition (36) is directly related to the values of $\psi_1$ and $\psi_2$ through $R(1, t)$. For instance, if $\psi_1 \equiv \psi_2 \equiv 0$ then by (15)-(16), after $t_f$, $R(1, t) \equiv 0$ which gives $\Phi \equiv 0$ then (36) cannot be satisfied. This completely coincides with the logic that we cannot estimate $\theta$ if $\psi_1$ and $\psi_2$ are zero (see equations (2) and (3)).

We can now state the stability result of the original error system $(\xi(t), \tilde{x}(t), \tilde{\theta}(t))$.

**Theorem 1.** Consider the error system (6)-(8) with initial conditions $(\xi_0(x) \in (L^2([0, 1]))^n, \tilde{x}_0 \in \mathbb{R}^n, \tilde{\theta}_0 \in \mathbb{R}^n)$. Under Lemma 1, Lemma 2 and if the conditions of Lemma 3 are satisfied, then the error system $(\xi(t, x), \tilde{x}(t), \tilde{\theta}(t))$ is exponentially stable in the $\|X\|^2 + \|\tilde{\theta}\|^2 + \|\tilde{\xi}(t, \cdot)\|_{L^2([0, 1])}^2(t)$ norm for all $t \geq t_f$.

**Proof 4.** Consider the following Lyapunov function

$$V(t) = \tilde{x}_c^2(t)P_c(t)\dot{\tilde{x}}_c(t) + \int_0^t \tilde{\xi}^T(x, t)P\tilde{\xi}(x, t)e^{-\alpha s} \, ds$$

(47)

In view of (12), the result falls directly from Lemma 3 with the $L^2$ boundedness of the filters $T(x, t)$ and $R(x, t)$ proved in Lemma 2.

## 4. SIMULATION RESULTS

We implement the adaptive observer in MATLAB for the scalar case $n_x = n_r = 1$ and $n_\theta = 1$. The system is given by:

$$\partial_t \xi(t, x) + 2\partial_x \xi(t, x) = 0.02 \xi(t, x)$$

(48)

$$\xi(0, t) = X(t) + \frac{\sqrt{3}}{2} \theta$$

(49)

$$\tilde{x}(t) = \sin(t)X(t) + \cos(t)u(t) + \frac{1}{2} \theta$$

(50)

which is the differential Lyapunov equation in $A_d(t)$.
The control input is constant \( u(t) = 2 \) and the parameter to be estimated is \( \theta = 1 \). The system initial conditions are

\[
\xi_0(x) = 10x \quad \text{and} \quad X_0 = 5. \tag{51}
\]

The states \( \xi(x,t) \), \( X(t) \) and the parameter \( \theta = 1 \) are to be estimated using the following available measurement:

\[
y(t) = \xi(1,t) \tag{52}
\]

The system (48)-(50) corresponds to a transport equation with first order time-varying boundary conditions. It is clear that the plant is open loop unstable looking into the ODE dynamics \( A(t) = \sin(t) \). To implement the adaptive observer (6)-(8) with the adaptation law (32)-(33), we need to find two gains \( L(t) \) and \( \beta \). The forgetting factor \( \beta \) is fixed to 0.1. The dynamic observer gains \( L(t) \) is calculated at each time step to ensure that \( Z(t) \) is negative definite for \( t \geq t_f \). This is done in the following order:

1. use a pole placement method to calculate \( L(t) \) that guarantees the existence of \( P_X(t) \) that satisfies (46);
2. verify that (37) is satisfied for a predefined value of \( Q(t) \).

Condition (37) was satisfied for all \( t \geq t_f \) for a constant value of \( P_X(t) = P_X = 0.5 \) and for \( Q(t) = 0.0125 P_c(t) \) where \( L(t) \) is calculated by locating the poles of \( A_d(t) \) at -1 for all \( t \geq t_f \). The values corresponding to \( L(t) \) are plotted in Fig.1. The placement starts after \( t_f = 0.5 \) s and \( L(t) \) exhibits an oscillatory behavior due to the dynamics of \( A(t) \). We start the adaptive observer from the following initial conditions

\[
\xi_0(x) = 9(x+1) \quad \text{and} \quad \dot{X}_0 = 10
\]

A finite difference (FD) scheme in space and time is implemented to approximate both the infinite dimensional states \( (\xi(x,t), \dot{\xi}(x,t)) \) and the finite dimensional states \( (X(t), \dot{X}(t) \) and \( \dot{\theta}(t)) \). The exponential convergence of the estimation error on both the ODE and the PDE sides is shown on Figures 2 and 3. After \( t_f = 0.5 \) s, the estimation errors converge to zero after exhibiting some oscillatory transients. Furthermore, as predicted by the theory, the Lyapunov function \( V(t) \) shown on Fig. 4 increases on the interval of time \([0, 0.5 \text{ s}) \) due to the unstable dynamics of \( A(t) \) and the presence of no observer gain \( L(t)=0 \), but afterwards it starts its exponential decay towards zero when measurement corrections are introduced for \( t \geq t_f \). Finally, we show the estimation of the parameter \( \theta \) starting from an initial condition \( \theta_0 = 6 \) on Fig.5. The adaptation starts after \( t_f = 0.5 \) s and \( \dot{\theta}(t) \) converges to \( \theta \) in approximately 50 s.

5. CONCLUSION

We have proposed an adaptive observer for a system of linear transport PDEs coupled with time-varying ODEs at the boundary. The system involves constant parameters that are to be estimated together with the PDE-ODE states using boundary sensing only. We have used swapping design to decouple the estimation error of the infinite states (PDEs) from the finite states (ODEs). We thus proved boundedness of regressors filters and obtained sufficient conditions for the exponential stability of the estimation error using DLIs. For future works, it would be interesting to consider the heterodirectional case i.e. consider wave propagations not only in the positive direction but also in both positive and negative directions,
while keeping the time-varying ODE connections at the boundary.

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