ELLiptic Problem involVing finite many critical exponents in $\mathbb{R}^N$

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Abstract. In this paper, we consider the following problem

\[-\Delta u - \zeta \frac{u}{|x|^2} = \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} \frac{|u|^{2*_{\alpha_i}}}{|x-y|^\alpha_i} \, dy \right) |u|^{2*_{\alpha_i}} - 2 u + |u|^{2*_{\alpha_i}} - 2 u, \text{ in } \mathbb{R}^N,\]

where $N \geq 3$, $\zeta \in (0, \frac{(N-2)^2}{4})$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, and $2^*_{\alpha_i} = \frac{2N-\alpha_i}{N-2}$ ($i = 1, \ldots, k$) are the Hardy–Littlewood–Sobolev critical upper exponents. The parameters $\alpha_i$ ($i = 1, \ldots, k$) satisfy some suitable assumptions. By using Coulomb–Sobolev space, endpoint refined Sobolev inequality and variational methods, we establish the existence of nontrivial solutions. Our result extends the ones in Yang and Wu [Adv. Nonlinear Stud. (2017) [25]].

1. Introduction

In this paper, we consider the following problem:

\[-\Delta u - \zeta \frac{u}{|x|^2} = \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} \frac{|u|^{2*_{\alpha_i}}}{|x-y|^\alpha_i} \, dy \right) |u|^{2*_{\alpha_i}} - 2 u + |u|^{2*_{\alpha_i}} - 2 u, \text{ in } \mathbb{R}^N, (P)\]

where $N \geq 3$, $\zeta \in (0, \frac{(N-2)^2}{4})$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $2^*_{\alpha_i} = \frac{2N-\alpha_i}{N-2}$ ($i = 1, \ldots, k$) are the Hardy–Littlewood–Sobolev critical upper exponents, and the parameters $\alpha_i$ ($i = 1, \ldots, k$) satisfy the following assumptions:

\[(H_1) \quad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < N \quad (k \in \mathbb{N}, 2 \leq k < \infty);\]

\[(H_2) \quad \frac{\pi N(N-2)}{\Gamma\left(\frac{N-2}{2}\right)} \left( \frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)} \right)^{\frac{\alpha_i-N}{2}} \geq 1, \text{ for all } i = 1, \ldots, k.\]

According to (2.5) and (2.6), we could see that the assumption $(H_2)$ is equivalent to $\tilde{S}_{\alpha_i} \geq 1$ for all $i = 1, \ldots, k$. For any $N \geq 3$, the value of $\tilde{S}_\alpha$ is dependent on the parameters $N$ and $\alpha$ (see Fig 1.).
Fig 1. The $x$–axis is $\alpha$, and the $y$–axis is $\tilde{S}_\alpha$, where $N = 3$ and $\alpha \in (0, N)$.

From Fig 1, there exists a constant $0 < \bar{\alpha} < 3$ such that $\tilde{S}_{\bar{\alpha}} = 1$. For any $\alpha \in (0, \bar{\alpha}]$, we have $1 \leq \tilde{S}_\alpha < \infty$. For any $\alpha \in (\bar{\alpha}, 3)$, we get $0 < \tilde{S}_\alpha < 1$. In this paper, we just study the case of $1 \leq \tilde{S}_\alpha < \infty$.

Problem (\(P\)) is related to the nonlinear Choquard equation as follows:

\[
- \Delta u + V(x)u = \left( |x|^{\alpha} * |u|^q \right) |u|^{q-2}u, \quad \text{in} \ R^N,
\]

where \(\frac{2N-\alpha}{N} \leq q \leq \frac{2N-\alpha}{N-2}\) and \(\alpha \in (0, N)\). For $q = 2$ and $\alpha = 1$, the problem (1.1) goes back to the description of the quantum theory of a polaron at rest by Pekar in 1954 \cite{17} and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree–Fock theory of one–component plasma \cite{18}. The existence and qualitative properties of solutions of Choquard type equations (1.1) have been widely studied in the last decades (see \cite{15}).

For Laplacian with nonlocal Hartree–type nonlinearities, the problem has attracted a lot of interest. Gao and Yang \cite{7} investigated the following critical Choquard equation:

\[
- \Delta u = \left( \int_{R^N} \frac{|u|^{2^*_\alpha}}{|x-y|^{\alpha}}dy \right) |u|^{2^*_\alpha - 2}u + \lambda u, \quad \text{in} \ R^N,
\]

where $\Omega$ is a bounded domain of $R^N$, with lipschitz boundary, $N \geq 3$, $\alpha \in (0, N)$ and $\lambda > 0$. By using variational methods, they established the existence, multiplicity and nonexistence of nontrivial solutions to equation (1.2). Alves, Gao, Squassina and Yang \cite{1} studied the following singularly perturbed critical Choquard equation:

\[
- \varepsilon^2 \Delta u + V(x)u = \varepsilon^{\alpha-3} \left( \int_{R^N} \frac{Q(y)G(u(y))}{|x-y|^{\alpha}}dy \right) Q(x)g(u), \quad \text{in} \ R^3,
\]

where $0 < \alpha < 3$, $\varepsilon$ is a positive parameter, $V, Q$ are two continuous real functions on $R^3$ and $G$ is the primitive of $q$ which is of critical growth due to the Hardy–Littlewood–Sobolev inequality. Under suitable assumptions on $g$, they first establish the existence of ground states for the critical Choquard equation with constant coefficient. They also established existence and multiplicity of semi–classical solutions and characterize the concentration behavior by variational methods. For details and recent works, we refer to \cite{8, 14} and the references therein.

For fractional Laplacian with nonlocal Hartree–type nonlinearities, the problem has attracted a lot of interest. D’Avenia, Siciliano and Squassina \cite{4} considered the
following fractional Choquard equation:

\[(\Delta^s)u + \omega u = (K^\alpha * |u|^q) |u|^{q-2}u, \text{ in } \mathbb{R}^N,\]

where \(N \geq 3, s \in (0, 1), \omega \geq 0, \alpha \in (0, N)\) and \(q \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2s}\right)\). In particular, if \(\omega = 0, \alpha = 4s\) and \(q = 2\), then problem (1.3) becomes a fractional Choquard equation with upper critical exponent in the sense of Hardy–Littlewood–Sobolev inequality as follows:

\[(\Delta^s)u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_h,\alpha}}{|x-y|^{\alpha}} \, dy\right) |u|^{2^*_h,\alpha-2}u + \lambda u, \text{ in } \Omega,\]

D'Avenia, Siciliano and Squassina in [4] obtained regularity, existence, nonexistence of nontrivial solutions to problem (1.3) and problem (1.4). Mukherjee and Sreenadh [12] investigated the following fractional Choquard equation:

\[(\Delta^s)u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_h,\alpha}}{|x-y|^{\alpha}} \, dy\right) |u|^{2^*_h,\alpha-2}u + \lambda u, \in \Omega,\]

where \(\Omega\) is a bounded domain of \(\mathbb{R}^N\) with \(C^{1,1}\) boundary, \(N \geq 3, s \in (0, 1), \alpha \in (0, N), \lambda > 0\) and \(2^*_h,\alpha = \frac{2N-\alpha}{N-2s}\). Applying variational methods, they established the existence, multiplicity and nonexistence of nontrivial solutions to problem (1.5).

Recently, Yang and Wu [25] studied the following nonlocal elliptic problems:

\[(-\Delta)^s u - \frac{\zeta u}{|x|^{2s}} = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_s,\alpha}}{|x-y|^{\alpha}} \, dy\right) |u|^{2^*_s,\alpha-2}u + \frac{|u|^{2^*_s,\beta-2}u}{|x|^{\beta}} + \frac{|u|^{2^*_s,\theta-2}v}{|x|^{\theta}}, \text{ in } \mathbb{R}^N,\]

and

\[(-\Delta)^s v - \frac{\zeta v}{|x|^{2s}} = \left(\int_{\mathbb{R}^N} \frac{|v|^{2^*_s,\alpha}}{|x-y|^{\alpha}} \, dy\right) |v|^{2^*_s,\alpha-2}v + \frac{|v|^{2^*_s,\beta-2}u}{|x|^{\beta}} + \frac{|v|^{2^*_s,\theta-2}v}{|x|^{\theta}}, \text{ in } \mathbb{R}^N,\]

where \(N \geq 3, s \in (0, 1), \zeta \in \left[0, 4^*(\frac{N+2s}{N})\right], \alpha, \beta, \theta \in (0, 2s), 2^*_s,\alpha = \frac{2N-\alpha}{N-2s}\) and \(2^*_s,\beta = \frac{2N-\alpha}{N-2s}\). By using a refinement of the Sobolev inequality which is related to the Morrey space, they showed the existence of nontrivial solutions for problem (1.6) and problem (1.7).

In [23], Wang, Zhang and Zhang extended the study of problem (1.7) to the fractional Laplacian system as follows:

\[\begin{cases}
(-\Delta)^s u - \frac{\zeta u}{|x|^{2s}} = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_s,\alpha}}{|x-y|^{\alpha}} \, dy\right) |u|^{2^*_s,\alpha-2}u + \frac{|u|^{2^*_s,\beta-2}u}{|x|^{\beta}} + \frac{\beta u}{2^*_s,\beta} |u|^{\beta-2}u |v|^{\gamma}, \\
(-\Delta)^s v - \frac{\zeta v}{|x|^{2s}} = \left(\int_{\mathbb{R}^N} \frac{|v|^{2^*_s,\alpha}}{|x-y|^{\alpha}} \, dy\right) |v|^{2^*_s,\alpha-2}v + \frac{|v|^{2^*_s,\beta-2}v}{|x|^{\beta}} + \frac{\gamma v}{2^*_s,\beta} |v|^{\gamma-2}v,
\end{cases}\]

where \(N \geq 3, s \in (0, 1), \zeta \in \left[0, 4^*(\frac{N+2s}{N})\right], \eta \in \mathbb{R}_0^+, \alpha \in (N-2s, N), \theta \in (0, 2s), \beta > 1, \gamma > 1\) and \(\beta + \gamma = 2^*_s,\beta\). By using variational methods, they investigated the extremals of the corresponding best fractional Hardy–Sobolev constant and established the existence of solutions to problem (1.8).

Moreover, there are many other kinds of problems involving two critical nonlinearities, such as the Laplacian \(-\Delta\) (see [11, 22, 26]), the p-Laplacian \(-\Delta_p\) (see [5]), the biharmonic operator \(\Delta^2\) (see [2]), and the fractional operator \((-\Delta)^s\) (see [9]).
A natural and interesting question is: For $s = 1$, can we extend the study of problem (1.6) to problem $(P)$? In this paper, we give a positive answer to the question. We need the following inequalities.

**Lemma 1.1.** [10, Hardy-Littlewood-Sobolev inequality] Let $t, r > 1$ and $0 < \alpha < N$ with $\frac{1}{r} + \frac{k}{t} + \frac{N}{k} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(N, \alpha, t, r) > 0$, independent of $f, g$ such that

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||h(y)|}{|x-y|^\alpha} \, dx \, dy \leq C(N, \alpha, t, r) \|f\|_t \|h\|_r.
$$

If $t = r = \frac{2N}{2N-\alpha}$, then

$$
C(N, \alpha, t, r) = C(N, \alpha) = \pi^{\frac{N}{2}} \frac{\Gamma\left(\frac{N}{2} - \frac{\alpha}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2} - \frac{\alpha}{2}\right)} \frac{2^N}{N^{N/2} \pi^{N/2}}.
$$

**Lemma 1.2.** [14, Endpoint refined Sobolev inequality] Let $\alpha \in (0, N)$. Then there exists a constant $C_1 > 0$ such that the inequality

$$
\|u\|_{L^{2^*_\alpha}(\mathbb{R}^N)} \leq C_1 \|u\|_{\mathcal{D}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha} |u(y)|^{2^*_\alpha}}{|x-y|^\alpha} \, dx \, dy \right)^\frac{N-2}{2N-\alpha N},
$$

holds for all $u \in \mathcal{E}^{1,\alpha,2^*_\alpha}(\mathbb{R}^N)$.

For the Coulomb–Sobolev space and endpoint refined Sobolev inequality, there are two papers until now. For Laplacian operator, Mercuri, Moroz and Schaftingen [14] introduced the Coulomb–Sobolev space and a family of associated optimal interpolation inequalities (endpoint refined Sobolev inequality). They established the existence of solutions of the nonlocal Schrödinger–Poisson–Slater type equation in [14]. For fractional Laplacian operator, Bellazzini, Ghimenti, Mercuri, Moroz and Schaftingen [3] studied the fractional Coulomb–Sobolev space and endpoint refined Sobolev inequality.

In this paper, we apply Coulomb–Sobolev space and endpoint refined Sobolev to study problem $(P)$. The main result of this paper reads as follows.

**Theorem 1.3.** Let $N \geq 3$, $(H_1)$ and $(H_2)$ hold. Then problem $(P)$ has a nonnegative solution $\tilde{v}(x)$ for

$$
\zeta \in \left( \frac{(N-2)^2}{4} - 1 \frac{1}{(k+1) \frac{2^*_\alpha}{3} \frac{N}{k} S^{N/2} \frac{N}{k+1}} \right), \quad \frac{(N-2)^2}{4}.
$$

Moreover, set $\tilde{v}(x) = \frac{1}{|x|^{\frac{2^*_\alpha}{2}}} \hat{v} \left( \frac{x}{|x|^2} \right)$. Then $\tilde{v}(x)$ is a nonnegative solution of the problem

$$
-\Delta \tilde{v} - \zeta \frac{\tilde{v}}{|x|^2} = \sum_{i=1}^k \left( \int_{\mathbb{R}^N} \frac{|\hat{v}|^{2^*_\alpha_i}}{|x-y|^{\alpha_i}} \, dy \right) |\hat{v}|^{2^*_\alpha_i-2} \hat{v} + |\hat{v}|^{2^*_\alpha_i-2} \hat{v}, \text{ in } \mathbb{R}^N \setminus \{0\}.
$$

**Remark 1.1.** In [19], the authors set an open problem. Our problem $(P)$ is a variant of the open problem.

**Remark 1.2.** In order to study problem $(P)$, we must study problem $(P_1)$ as follows:

$$
-\Delta u = \sum_{i=1}^k \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_\alpha_i}}{|x-y|^{\alpha_i}} \, dy \right) |u|^{2^*_\alpha_i-2} u + |u|^{2^*_\alpha_i-2} u, \text{ in } \mathbb{R}^N, \quad (P_1)
$$
where the parameters are same to problem (P). We need show the relation of critical value between problem (P) and problem (P₁) as follows:

\[ \tilde{c}_0 > c_0, \]

where \( \tilde{c}_0 \) and \( c_0 \) are defined in Section 2. There are finite many Hardy–Littlewood–Sobolev critical upper exponents in problem (P) and problem (P₁), it is difficult to show \( \tilde{c}_0 > c_0 \). By using Lemma 4.3 and \( S = \tilde{S} \left( 1 - \frac{4C}{(N-2)^2} \right)^{\frac{N-4}{4}} \), we overcome this difficulty in Lemma 6.1.

We point out that \( S = \tilde{S} \left( 1 - \frac{4C}{(N-2)^2} \right)^{\frac{N-4}{4}} \) plays a key role in the proof of \( \tilde{c}_0 > c_0 \).

Remark 1.3. Problem (P) is invariant under the weighted dilation

\[ u \mapsto \tau^{\frac{N-2}{2}} u(\tau x). \]

Therefore, the well known Mountain Pass theorem does not yield critical point, but only the Palais–Smale sequence. It is necessary to show the non–vanishing of Palais–Smale sequence. There are finite many Hardy–Littlewood–Sobolev critical upper exponents in problem (P), it is difficult to show the non–vanishing of Palais–Smale sequence. By using Coulomb–Sobolev space, endpoint refined Sobolev inequality and Lemma 3.1, we overcome this difficulty in Lemma 5.2.

This paper is organized as follows: In Section 2, we present some notations. In Section 3, we show some key lemmas. In Section 4, we study the Nehari manifolds for problem (P) and problem (P₁). In Section 5, we investigate the Palais–Smale sequence of Problem (P). In Section 6, we show \( \tilde{c}_0 > c_0 \). In Section 7, we show the proof of Theorem 1.3.

2. Preliminaries

Recall that the space \( D^{1,2}(\mathbb{R}^N) \) is the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm

\[ \|u\|_D^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \]

It is well known that \( \frac{(N-2)^2}{4} \) is the best constant in the Hardy inequality

\[ \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \]

for any \( u \in D^{1,2}(\mathbb{R}^N) \).

By Hardy inequality and \( \zeta \in (0, \frac{(N-2)^2}{4}) \), we derive that

\[ \|u\|_D^2 = \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \zeta \frac{u^2}{|x|^2} \right) \, dx, \]

is an equivalent norm in \( D^{1,2}(\mathbb{R}^N) \), and the following inequalities hold:

\[ \left( 1 - \frac{4C}{(N-2)^2} \right) \|u\|_D^2 \leq \|u\|_C^2 \leq \|u\|_D^2. \]

For \( \alpha \in (0, N) \), the Coulomb–Sobolev space [14] is defined by

\[ E^{1,\alpha,2n}(\mathbb{R}^N) = \left\{ \|u\|_D < \infty \text{ and } \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2n} |u(y)|^{2n}}{|x-y|^\alpha} \, dx \, dy < \infty \right\}. \]
We endow the space $\mathcal{E}^{1,\alpha,2^*}(\mathbb{R}^N)$ with the norm

$$\|u\|_{\mathcal{E},\alpha}^2 = \|u\|_{D}^2 + \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\alpha} \, dx \, dy \right)^{\frac{2}{2^*}}.$$ 

For $\alpha \in (0, N)$ and $\zeta \in (0, \frac{(N-2)^2}{4})$, we could define the best constants:

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{2^*}^2}{(\int_{\mathbb{R}^N} |u|^2 \, dx)^{\frac{2}{2^*}}} ,$$

and

$$S_\alpha := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{2^*}^2}{(\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\alpha} \, dx \, dy)^{\frac{2}{2^*}}} ,$$

and

$$\tilde{S} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{2}^2}{(\int_{\mathbb{R}^N} |u|^2 \, dx)^{\frac{2}{2^*}}} ,$$

and

$$\tilde{S}_\alpha := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{2}^2}{(\int_{\mathbb{R}^N} \frac{|u(x)|^{2} |u(y)|^{2}}{|x-y|^\alpha} \, dx \, dy)^{\frac{2}{2^*}}} .$$

where $S$ and $S_\alpha$ are attained in $\mathbb{R}^N$ (see [25, Lemma 2.1]), and $\tilde{S}$ and $\tilde{S}_\alpha$ are attained in $\mathbb{R}^N$ (see [7, Lemma 1.2]). Furthermore, we know (see [21, Formula (1)])

$$\tilde{S} = \pi N(N-2) \left( \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{\frac{2}{N-2}}.$$

And see [7, Lemma 1.2]

$$\tilde{S}_\alpha = \frac{\tilde{S}}{C(N, \alpha)^{\frac{2}{\alpha}}}.$$

And see [6, Formula (7)]

$$S = \tilde{S} \left( 1 - \frac{4\zeta}{(N-2)^2} \right)^{\frac{N-1}{N}} .$$

A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\|u\|_{L^p,\infty}(\mathbb{R}^N)$ with $p \in [1, \infty)$ and $\omega \in (0, N]$ if and only if

$$\|u\|_{L^p,\infty}(\mathbb{R}^N) = \sup_{R > 0, x \in \mathbb{R}^N} R^{N-p} \int_{B(x, R)} |u(y)|^p \, dy < \infty .$$

**Lemma 2.1.** [16] Let $N \geq 3$. There exists $C_2 > 0$ such that for $\iota$ and $\vartheta$ satisfying $\frac{2}{2^*} \leq \iota < 1$, $1 \leq \vartheta < 2^* = \frac{2N}{N-2}$, we have

$$\left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{1}{\iota}} \leq C_2 \|u\|_{D} \|u\|_{L^{\vartheta, \frac{2N(N-2)}{2}}(\mathbb{R}^N)}^{1-\iota} ,$$

for any $u \in D^{1,2}(\mathbb{R}^N)$.
We introduce the energy functional associated to problem (P) by
\[
I(u) = \frac{1}{2} \|u\|_2^2 - \sum_{i=1}^{k} \frac{1}{2 \cdot 2^{*}_{\alpha_i}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^{*}_{\alpha_i}} |u(y)|^{2^{*}_{\alpha_i}}}{|x - y|^{\alpha_i}} \, dx \, dy - \frac{1}{2^{*}} \int_{\mathbb{R}^N} |u|^{2^{*}} \, dx.
\]

We also introduce the energy functional associated to problem (P\(_1\)) by
\[
\tilde{I}_0(u) = \frac{1}{2} \|u\|_D^2 - \sum_{i=1}^{k} \frac{1}{2 \cdot 2^{*}_{\alpha_i}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^{*}_{\alpha_i}} |u(y)|^{2^{*}_{\alpha_i}}}{|x - y|^{\alpha_i}} \, dx \, dy - \frac{1}{2^{*}} \int_{\mathbb{R}^N} |u|^{2^{*}} \, dx.
\]

The Nehari manifold associated with problem (P) is defined by
\[
\mathcal{N} = \{ u \in D^{1,2}(\mathbb{R}^N) : \langle I'(u), u \rangle = 0, \ u \neq 0 \},
\]
and
\[
c_0 = \inf_{u \in \mathcal{N}} I(u), \quad c_1 = \inf_{u \in D^{1,2}(\mathbb{R}^N)} \max_{t \geq 0} I(tu), \quad c = \inf_{\tilde{r} \in \Gamma} \max_{t \in [0,1]} I(\tilde{r}(t)),
\]
where \( \Gamma = \{ \tilde{Y} \in C([0,1], D^{1,2}(\mathbb{R}^N)) : \tilde{Y}(0) = 0, I(\tilde{Y}(1)) < 0 \} \).

The Nehari manifold associated with problem (P\(_1\)) is defined by
\[
\tilde{\mathcal{N}} = \{ u \in D^{1,2}(\mathbb{R}^N) : \langle \tilde{I}'_0(u), u \rangle = 0, \ u \neq 0 \},
\]
and
\[
\tilde{c}_0 = \inf_{u \in \tilde{\mathcal{N}}} \tilde{I}_0(u), \quad \tilde{c}_1 = \inf_{u \in D^{1,2}(\mathbb{R}^N)} \max_{t \geq 0} \tilde{I}_0(tu), \quad \tilde{c} = \inf_{\tilde{r} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{I}_0(\tilde{r}(t)),
\]
where \( \tilde{\Gamma} = \{ \tilde{Y} \in C([0,1], D^{1,2}(\mathbb{R}^N)) : \tilde{Y}(0) = 0, \tilde{I}_0(\tilde{Y}(1)) < 0 \} \).

3. Some key lemmas

We show some properties of Coulomb–Sobolev space \( \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \).

**Lemma 3.1.** Let (H\(_1\)) hold. If \( u \in \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \) (\( j = 1, \ldots, k \)), then
(i) \( \| \cdot \|_D \) is an equivalent norm in \( \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \);
(ii) \( u \in \bigcap_{i=1, i \neq j}^{k} \mathcal{E}^{1,\alpha_i,2^{*}_{\alpha_i}}(\mathbb{R}^N) \);
(iii) \( \| \cdot \|_{\mathcal{E},\alpha_i} \) are equivalent norms in \( \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \), where \( i \neq j \) and \( i = 1, \ldots, k \).

**Proof.** (1). Set \( j = 1, \ldots, k \). For any \( u \in \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \), applying the definition of Coulomb–Sobolev space, we know
\[
\|u\|_D^2 \leq \|u\|_{\mathcal{E},\alpha_j}^2 < \infty.
\]

This implies that \( \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N) \). According to \( \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N) \) and (2.4), we have
\[
\|u\|_{\mathcal{E},\alpha_j}^2 \leq \left( 1 + \frac{1}{S_{\alpha_j}} \right) \|u\|_D^2.
\]

Combining (3.1) and (3.2), we obtain
\[
\|u\|_D \leq \|u\|_{\mathcal{E},\alpha_j} \leq \left( 1 + \frac{1}{S_{\alpha_j}} \right) \|u\|_D.
\]

These imply that \( \| \cdot \|_D \) is an equivalent norm in \( \mathcal{E}^{1,\alpha_j,2^{*}_{\alpha_j}}(\mathbb{R}^N) \).
(2). For any \( u \in \mathcal{E}^{1,\alpha_j,2^*_\alpha}(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N) \), by using (3.1) and (2.4), we know

\[
\mathcal{S}_{\alpha_j} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha} |u(y)|^{2^*_\alpha}}{|x-y|^{\alpha}} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}} \leq \|u\|_D^2 \leq \|u\|_{\mathcal{E},\alpha_j}^2 < \infty,
\]

where \( i \neq j \) and \( i = 1, \ldots, k \). The inequality (3.4) gives that

\[
\|u\|_{\mathcal{E},\alpha_i}^2 = \|u\|_D^2 + \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha} |u(y)|^{2^*_\alpha}}{|x-y|^{\alpha}} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}} < \infty.
\]

This implies that \( u \in \bigcap_{i=1}^k \mathcal{E}^{1,\alpha_i,2^*_\alpha}(\mathbb{R}^N) \).

(3). For any \( u \in \mathcal{E}^{1,\alpha_j,2^*_\alpha}(\mathbb{R}^N) \), by using (3.2), we have

\[
\|u\|_{\mathcal{E},\alpha_i}^2 \leq \frac{\mathcal{S}_{\alpha_j} + 1}{\mathcal{S}_{\alpha_j}} \|u\|_D^2 \leq \left( \frac{\mathcal{S}_{\alpha_j} + 1}{\mathcal{S}_{\alpha_j}} \right) \|u\|_{\mathcal{E},\alpha_i}^2,
\]

which imply that

\[
\left( \frac{\mathcal{S}_{\alpha_j}}{\mathcal{S}_{\alpha_j} + 1} \right) \|u\|_{\mathcal{E},\alpha_i}^2 \leq \|u\|_{\mathcal{E},\alpha_j}^2 \leq \left( \frac{\mathcal{S}_{\alpha_j} + 1}{\mathcal{S}_{\alpha_j}} \right) \|u\|_{\mathcal{E},\alpha_i}^2,
\]

where \( 0 < \frac{\mathcal{S}_{\alpha_j}}{\mathcal{S}_{\alpha_j} + 1} < 1 < \frac{\mathcal{S}_{\alpha_j} + 1}{\mathcal{S}_{\alpha_j}} < \infty \). \( \square \)

The following result is the refinement of Hardy-Littlewood-Sobolev inequality.

**Lemma 3.2.** For any \( \alpha \in (0, N) \), there exists \( C_3 > 0 \) such that for \( \iota \) and \( \vartheta \) satisfying \( \frac{2}{N} \leq \iota < 1, 1 \leq \vartheta < 2^* = \frac{2N}{N-2} \), we have

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha} |u(y)|^{2^*_\alpha}}{|x-y|^{\alpha}} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}} \leq C_3 \|u\|_D \|u\|_{L^\vartheta(\mathbb{R}^N)}^{2(1-\iota)/\vartheta(\vartheta(N-2))/(2^*-\vartheta(N-2)),}\]

for any \( u \in D^{1,2}(\mathbb{R}^N) \).

**Proof.** Let \( \frac{2}{N} \leq \iota < 1 \) and \( 1 \leq \vartheta < 2^* = \frac{2N}{N-2} \). By Hardy-Littlewood-Sobolev inequality and Lemma 2.1, we have

\[
\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha} |u(y)|^{2^*_\alpha}}{|x-y|^{\alpha}} \, dx \, dy \right)^{\frac{1}{2^*_\alpha}} \leq C(N, \alpha)^{\frac{1}{2^*_\alpha}} \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq C(N, \alpha)^{\frac{1}{2^*_\alpha}} C_2 \|u\|_D \|u\|_{L^{\vartheta(\vartheta(N-2))/(2^*-\vartheta(N-2))}(\mathbb{R}^N)}^{2(1-\iota)/\vartheta(\vartheta(N-2))/(2^*-\vartheta(N-2))}. \]

\( \square \)

4. Nehari Manifolds for Problem (P) and Problem (P1)

We prove some properties of the Nehari manifold associated with problem (P).

**Lemma 4.1.** Let \( N \geq 3, \arg \in (0, \frac{(N-2)^2}{4}) \) and \((H_1)\) hold. Then

\[
c_0 = \inf_{u \in \mathcal{N}} I(u) > 0.
\]
Proof. We divide our proof into two steps.

Step 1. We claim that any limit point of a sequence in $\mathcal{N}$ is different from zero.

According to $\langle I'(u), u \rangle = 0$, (2.1) and (2.2), for any $u \in \mathcal{N}$, we obtain

$$0 = \langle I'(u), u \rangle \geq \frac{1}{S^{2^*}_N} \|u\|^{2^*}_\zeta - \sum_{i=1}^{k} \frac{1}{S^{2\alpha_i}_i} \|u\|^{2\alpha_i}_\zeta.$$  

From above expression, we have

$$\|u\|^{2}_\zeta \geq \frac{1}{S^{2^*}_N} \|u\|^{2^*}_\zeta + \sum_{i=1}^{k} \frac{1}{S^{2\alpha_i}_i} \|u\|^{2\alpha_i}_\zeta.$$  

Set

$$\kappa := \frac{1}{S^{2^*}_N} + \sum_{i=1}^{k} \frac{1}{S^{2\alpha_i}_i}.$$  

Applying (2.1), (2.2) and (H1), we get

$$0 < \kappa < \infty.$$  

From (H1), we know

$$2N - \alpha_1 = 2N - \alpha_i.$$  

Now the proof of Step 1 is divided into two cases: (i) $\|u\|_\zeta \geq 1$; (ii) $\|u\|_\zeta < 1$.

Case (i). $\|u\|_\zeta \geq 1$. The inequality (4.1) gives

$$\|u\|^{2}_\zeta \leq \frac{1}{S^{2^*}_N} \|u\|^{2^*}_\zeta + \sum_{i=1}^{k} \frac{1}{S^{2\alpha_i}_i} \|u\|^{2\alpha_i}_\zeta \leq \kappa \|u\|^{2\alpha_i}_\zeta,$$  

which implies that

$$\|u\|_\zeta \geq \kappa^{1/2 \alpha_i}.$$  

Case (ii). $\|u\|_\zeta < 1$. Again, by (4.1), we know

$$\|u\|_\zeta \geq \kappa^{1/2 \alpha_i}.$$  

According to (4.2) and (4.3), we deduce that

$$\|u\|_\zeta \geq \min \left\{ \kappa^{1/2 \alpha_i}, \kappa^{1/2 \alpha_i} \right\}.$$  

Hence, we know that any limit point of a sequence in $\mathcal{N}$ is different from zero.

Step 2. Now, we claim that $I$ is bounded from below on $\mathcal{N}$. For any $u \in \mathcal{N}$, by using (4.4), we get

$$I(u) \geq \left( \frac{1}{2} - \frac{1}{2^*} \right) \|u\|^{2}_\zeta \geq \frac{1}{N} \min \left\{ \kappa^{1/2 \alpha_i}, \kappa^{1/2 \alpha_i} \right\}.$$  

Therefore, $I$ is bounded from below on $\mathcal{N}$, and $c_0 > 0$.

Lemma 4.2. Let $N \geq 3$, $\zeta \in (0, \frac{(N-2)^2}{4})$ and (H1) hold. Then

(i) for each $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$;

(ii) $c_0 = c_1 = c > 0$.

Proof. The proof is standard, so we sketch it. Further details can be derived as in the proofs of Theorem 4.1 and 4.2 in [24]. We omit it.

We prove some properties of the Nehari manifold associated with problem ($P_1$).
Lemma 4.3. Let $N \geq 3$ and $(H_1)$ hold. For any $u \in \tilde{N}$, we have
\[ \|u\|_D \geq \min \left\{ \tilde{\kappa} \frac{2}{2-\alpha_i}, \tilde{\kappa} \frac{2}{2-\alpha_k} \right\}, \]
where
\[ \tilde{\kappa} = \frac{1}{S^{\frac{2}{\alpha_k}}} + \frac{k}{\sum_{i=1}^{k} S^{\frac{2}{\alpha_i}}}. \]
And,
\[ \tilde{c}_1 = \tilde{c} = \tilde{c}_0 = \inf_{u \in \tilde{N}} \tilde{I}_0(u) > 0. \]

5. Analysis of the Palais–Smale sequence for Problem (P)

We show that the functional $I$ satisfies the Mountain Pass geometry, and estimate the Mountain Pass level.

Lemma 5.1. Let $N \geq 3$, $\zeta \in (0, \frac{(N-2)^2}{4})$ and $(H_1)$ hold. Then there exists a $(PS)_c$ sequence of $I$ at level $c$, where
\[ 0 < c < c^* = \min \left\{ \frac{N + 2 - \alpha_1}{2(2N - \alpha_1)} S^{\frac{2N-\alpha_1}{2N-\alpha_1}}, \ldots, \frac{N + 2 - \alpha_k}{2(2N - \alpha_k)} S^{\frac{2N-\alpha_k}{2N-\alpha_k}}, \frac{1}{N} \right\}, \]

Proof. We divide our proof into two steps.
Step 1. We prove that $I$ satisfies all the conditions in Mountain Pass theorem.
(i) $I(0) = 0$;
(ii) For any $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$, we have
\[ I(u) \geq \frac{1}{2} \|u\|_{\zeta}^2 - \sum_{i=1}^{k} \frac{1}{2 \cdot \alpha_i S^{\frac{2}{\alpha_i}}} \|u\|_{\zeta}^{2-\alpha_i} - \frac{1}{2^* \cdot S^{\frac{2}{\alpha_k}}} \|u\|_{\zeta}^{2^*}. \]

Because of $2 < 2^* < 2 \cdot \alpha_i < \cdots < 2 \cdot \alpha_k$, there exists a sufficiently small positive number $\rho$ such that
\[ \zeta := \inf_{\|u\|_{\zeta} = \rho} I(u) > 0 = I(0). \]

(iii) Given $u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$ such that $\lim_{t \to \infty} I(tu) = -\infty$. We choose $t_u > 0$ corresponding to $u$ such that $I(tu) < 0$ for all $t > t_u$ and $\|tu\|_{\zeta} > \rho$. Set
\[ c = \inf_{\Gamma(t) \in \{0,1\}} \max_{t \in \Gamma} I(\Upsilon(t)), \]
where $\Gamma = \{ \Upsilon \in C([0,1], D^{1,2}(\mathbb{R}^N)) : \Upsilon(0) = 0, \Upsilon(1) = tu \}$.

Step 2. Here we show $0 < c < c^*$. By using (2.1) and (2.2), there exist minimizers $u_i \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$ of $S_{\alpha_i}$ and $\tilde{u} \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}$ of $S$. For $t \geq 0$, we set
\[ f_i(t) = \frac{t^2}{2} \|u_i\|_{\zeta}^2 - \frac{t^2 \cdot \alpha_i}{2 \cdot \alpha_i S^{\frac{2}{\alpha_i}}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_i(x)|^{2^*_\alpha} |u_i(y)|^{2^*_\alpha}}{|x-y|^{\alpha_i}} \text{d}x \text{d}y, \text{ for } i = 1, \ldots, k, \]
and
\[ \tilde{f}(t) = \frac{t^2}{2} \|	ilde{u}\|_{\zeta}^2 - \frac{t^2}{2^*} \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} \text{d}x. \]
If \( \frac{N + 2 - \alpha_i}{2(2N - \alpha_i)} S_{\alpha_i}^{2N - \alpha_i} \) is uniformly bounded in \( D^{1,2}(\mathbb{R}^N) \), we divide our proof into three cases:

1. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2\alpha_i} |u_n(y)|^{2\alpha_i}}{|x - y|^{\alpha_i}} \, dx \, dy > 0 \);
2. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2\alpha_i} |u_n(y)|^{2\alpha_i}}{|x - y|^{\alpha_i}} \, dx \, dy > 0, \quad i = 2, \ldots, k; \)
3. \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^2 \, dx > 0. \)

Case 1. Suppose on the contrary that

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2\alpha_i} |u_n(y)|^{2\alpha_i}}{|x - y|^{\alpha_i}} \, dx \, dy = 0. \]
Since \( \{u_n\} \) is uniformly bounded in \( D^{1,2}(\mathbb{R}^N) \), there exists a constant \( 0 < C < \infty \) such that \( \|u_n\|_D \leq C \). According to (2.4) and the definition of Coulomb–Sobolev space, we obtain \( u_n \in \mathcal{E}^{1,\alpha_1,2^*_{\alpha_1}}(\mathbb{R}^N) \). Applying Lemma 1.2, we have

\[
\lim_{n \to \infty} \|u_n\|_{L^{2^*}(\mathbb{R}^N)} 
\leq C \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_{\alpha_1}}|u_n(y)|^{2^*_{\alpha_1}}}{|x-y|^{\alpha_1}} \, dx \, dy \right)^{\frac{N-2}{\frac{N-2\alpha_1}{2^*_{\alpha_1} - \alpha_1}}} = 0.
\]

Combining Hardy–Littlewood–Sobolev inequality and (5.2), for all \( i = 2, \ldots, k \), we know

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_{\alpha_i}}|u_n(y)|^{2^*_{\alpha_i}}}{|x-y|^{\alpha_i}} \, dx \, dy \leq C \lim_{n \to \infty} \|u_n\|_{L^{2^*_{\alpha_i}}(\mathbb{R}^N)} = 0.
\]

Owing to (5.1) – (5.3) and the definition of \((PS)_c\) sequence, we obtain

\[
c + o(1) = \frac{1}{2} \|u_n\|_{\zeta}^2,
\]

and

\[
o(1) = \|u_n\|_{\zeta}^2.
\]

These imply that \( c = 0 \), which contradicts as \( 0 < c \).

**Case 2.** From Case 1, we have \( u_n \in \mathcal{E}^{1,\alpha_1,2^*_{\alpha_1}}(\mathbb{R}^N) \). Applying the result of (ii) in Lemma 3.1, we know that \( u_n \in \bigcap_{i=2}^{k} \mathcal{E}^{1,\alpha_i,2^*_{\alpha_i}}(\mathbb{R}^N) \). Then we could use the endpoint refined Sobolev inequality for parameters \( \alpha_i \) \( (i = 2, \ldots, k) \). Similar to Case 1, we prove that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_{\alpha_i}}|u_n(y)|^{2^*_{\alpha_i}}}{|x-y|^{\alpha_i}} \, dx \, dy > 0, \ (i = 2, \ldots, k).
\]

**Case 3.** Suppose that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx = 0.
\]

By using Lemma 1.1, for all \( i = 1, \ldots, k \), we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_{\alpha_i}}|u_n(y)|^{2^*_{\alpha_i}}}{|x-y|^{\alpha_i}} \, dx \, dy \leq C \lim_{n \to \infty} \|u_n\|_{L^{2^*_{\alpha_i}}(\mathbb{R}^N)} = 0,
\]

Applying (5.4) and (5.5), we get

\[
c + o(1) = \frac{1}{2} \|u_n\|_{\zeta}^2,
\]

and

\[
o(1) = \|u_n\|_{\zeta}^2.
\]

These imply that \( c = 0 \), which contradicts as \( 0 < c \). \( \square \)
6. The proof of $\tilde{c}_0 > c_0$

In this section, we show that $\tilde{c}_0 > c_0$.

**Lemma 6.1.** Let $N \geq 3$, $(H_1)$ and $(H_2)$ hold. If

$$\zeta \in \left(\frac{(N-2)^2}{4}, \frac{1}{(k+1)^{\frac{N-2}{4}}} \frac{(N-2)^2}{4}\right),$$

then $\tilde{c}_0 > c_0$.

**Proof.** We divide our proof into three steps.

**Step 1.** In this step, we show the property of $\tilde{c}_0$. From Lemma 4.3, there exists $\bar{u}$ such that $\tilde{I}_0(\bar{u}) = \tilde{c}_0$ and $\bar{u} \in \tilde{N}$. Then

$$\tilde{c}_0 = \tilde{I}_0(\bar{u}) - \frac{1}{2^{*}} \langle \tilde{I}_0'(\bar{u}), \bar{u} \rangle$$

$$= \left(\frac{1}{2} - \frac{1}{2^{*}}\right) \|\bar{u}\|_D^2 + \sum_{i=1}^{k} \left(\frac{1}{2} - \frac{1}{2^{*}_{\alpha_i}}\right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\bar{u}(x)|^{2^{*}_{\alpha_i}}|\bar{u}(y)|^{2^{*}_{\alpha_i}} |x-y|^{\alpha_i} \ dx \ dy$$

$$\geq \left(\frac{1}{2} - \frac{1}{2^{*}}\right) \|\bar{u}\|_D^2$$

$$\geq \frac{1}{N} \min\left\{\tilde{\kappa} - \frac{N-2}{N-2-\alpha_1}, \tilde{\kappa} - \frac{N-2}{\alpha_1}\right\}.$$ 

**Step 2.** In this step, we show some basic results. Firstly, according to (2.5) and $N \geq 3$, we get

$$\tilde{S} > 1.$$

Secondly, we will show the following results:

$$\zeta \in \left(\frac{(N-2)^2}{4}, \frac{1}{(k+1)^{\frac{N-2}{4}}} \frac{(N-2)^2}{4}\right),$$

and

$$\zeta \in \left(\frac{(N-2)^2}{4}, \frac{1}{\tilde{S}^{\frac{N-2}{N}}} \frac{(N-2)^2}{4}\right).$$

By using $\tilde{S} > 1$ and $k + 1 > 1$, we have

$$\frac{1}{(k+1)^{\frac{N-2}{4}}} \frac{(N-2)^2}{4} < \frac{1}{(k+1)^{\frac{N-2}{4}}} \frac{(N-2)^2}{4} \quad \text{and} \quad \frac{1}{(k+1)^{\frac{N-2}{4}}} \frac{(N-2)^2}{4} < \frac{1}{\tilde{S}^{\frac{N-2}{N}}}.$$

These imply that

$$1 - \frac{1}{(k+1)^{\frac{N-2}{4}}} \tilde{S}^{\frac{N-2}{4}} > 1 - \frac{1}{(k+1)^{\frac{N-2}{4}}},$$

and

$$1 - \frac{1}{(k+1)^{\frac{N-2}{4}}} \tilde{S}^{\frac{N-2}{4}} > 1 - \frac{1}{\tilde{S}^{\frac{N-2}{N}}}.$$

Hence, we deduce that

$$\frac{4\zeta}{(N-2)^2} \in \left(1 - \frac{1}{(k+1)^{\frac{N-2}{4}}} \tilde{S}^{\frac{N-2}{4}}, 1\right) \subset \left(1 - \frac{1}{(k+1)^{\frac{N-2}{4}}}, 1\right).$$
and
\[ \frac{4\zeta}{(N-2)^2} \in \left( 1 - \frac{1}{(k+1)\frac{N-2}{S} S^{\frac{N-2}{2}}} , 1 \right) \subset \left( 1 - \frac{1}{S} , 1 \right). \]

**Step 3.** Here we show \( \tilde{c}_0 > c_0 \). The proof of this step is divided into four cases:

1. \( \tilde{S} = \min \{ \tilde{S}, \tilde{S}_{\alpha_1}, \ldots, \tilde{S}_{\alpha_k} \} \) and \( \tilde{\kappa} \geq 1; \)
2. \( \tilde{S} = \min \{ \tilde{S}, \tilde{S}_{\alpha_1}, \ldots, \tilde{S}_{\alpha_k} \} \) and \( \tilde{\kappa} \leq 1; \)
3. \( 1 < \tilde{S}_{\alpha_j} = \min \{ \tilde{S}, \tilde{S}_{\alpha_1}, \ldots, \tilde{S}_{\alpha_k} \} \) and \( \tilde{\kappa} \geq 1; \)
4. \( 1 < \tilde{S}_{\alpha_j} = \min \{ \tilde{S}, \tilde{S}_{\alpha_1}, \ldots, \tilde{S}_{\alpha_k} \} \) and \( \tilde{\kappa} \leq 1. \)

**Case (1).** Since \( \tilde{S} < 2^*_{\alpha_k} < \cdots < 2^*_{\alpha_1} \) and \( 1 < \tilde{S} = \min \{ \tilde{S}, \tilde{S}_{\alpha_1}, \ldots, \tilde{S}_{\alpha_k} \} \), we have

\[ \tilde{\kappa} = \frac{1}{\tilde{S}^{\frac{N-2}{2}}} + \sum_{i=1}^{k} \frac{1}{\tilde{S}^{\frac{N-2}{2}}_{\alpha_i}} \leq \frac{k+1}{\tilde{S}^{\frac{N-2}{2}}}. \]

Combining (6.1), (6.2), \( \tilde{\kappa} \geq 1 \) and \( -\frac{N-2}{2} < -\frac{N-2}{2} \), we get

\[ \tilde{c}_0 \geq \frac{1}{N} \min \left\{ \tilde{\kappa} - \frac{N-2}{2} - \frac{N-2}{2} \right\} = \frac{1}{N} \tilde{\kappa} - \frac{N-2}{2} \geq \frac{\tilde{S}}{N} (k+1)^{-\frac{N-2}{2}}. \]

By using \( \zeta \in \left( \frac{(N-2)^2}{4} \left( 1 - \frac{1}{(k+1)\frac{N-2}{S} S^{\frac{N-2}{2}}} \right), \frac{(N-2)^2}{4} \right) \), we obtain

\[
k + 1 < \left( 1 - \frac{4}{(N-2)^2 \zeta} \right)^{\frac{N-1}{N-2}} = \left( \frac{\tilde{S}}{\tilde{S} \left( 1 - \frac{4}{(N-2)^2 \zeta} S^{\frac{N-1}{N-2}} \right)} \right)^{\frac{N-1}{N-2}} = \left( \frac{\tilde{S}}{\tilde{S}} \right)^{\frac{N-1}{N-2}},
\]

which implies that

\[ (k+1)^{-\frac{N-2}{2}} \tilde{S}^{\frac{N}{2}} > S^{\frac{N}{2}}. \]

Putting (6.4) into (6.3), we know

\[ \tilde{c}_0 > \frac{S^{\frac{N}{2}}}{N}. \]

According to (6.5), Lemma 4.2 and Lemma 5.1, we obtain

\[ \tilde{c}_0 > \frac{S^{\frac{N}{2}}}{N} > c = c_0 > 0. \]
Case (2). Combining (6.1), (6.2), \( \hat{k} \leq 1 \) and \(-\frac{N-2}{N+2-\alpha} > -\frac{N-2}{2} \), we get
\[
\hat{c}_0 > \frac{1}{N} \min \left\{ \hat{\kappa}^{-\frac{N-2}{N+2-\alpha}}, \hat{\kappa}^{-\frac{N-2}{2}} \right\}
\]
(6.6)
\[
\frac{\hat{\kappa}^{-\frac{N-2}{N+2-\alpha}}}{N} \geq \hat{S}^{-\frac{N}{N+2-\alpha}}(k+1)^{-\frac{N-2}{N+2-\alpha}}.
\]
Since \( \zeta \in \left( \frac{(N-2)^2}{4} \left( 1 - \frac{1}{\hat{S}^{N/2}} \right), \frac{(N-2)^2}{4} \right) \), we know
\[
S = \hat{S} \left( 1 - \frac{4}{(N-2)^2} \zeta \right)^{-\frac{N-1}{N}} < 1.
\]
Applying \( \zeta \in \left( \frac{(N-2)^2}{4} \left( 1 - \frac{1}{(k+1)^{N/2}} \right), \frac{(N-2)^2}{4} \right) \), we have
\[
k + 1 < \left( 1 - \frac{4}{(N-2)^2} \zeta \right)^{-\frac{N-1}{N}} = \left( \frac{S}{\hat{S}} \right)^{\frac{N}{N-2}},
\]
which gives that
\[
\hat{S}^{-\frac{N}{N+2-\alpha}}(k+1)^{-\frac{N-2}{N+2-\alpha}} > \frac{S^{-\frac{N}{N+2-\alpha}}}{N}.
\]
Inserting (6.8) into (6.6), we know
\[
\frac{\hat{S}^{-\frac{N}{N+2-\alpha}}}{N}(k+1)^{-\frac{N-2}{N+2-\alpha}} > \frac{S^{-\frac{N}{N+2-\alpha}}}{N} > \frac{S}{N}
\]
(6.9)
According to (6.9), Lemma 4.2 and Lemma 5.1, we obtain
\[
\hat{c}_0 > \frac{S^k}{N} > c = c_0 > 0.
\]
Case (3). Since \( \frac{2^*}{2} < 2_{\alpha_k}^* < \cdots < 2_{\alpha_1}^* \) and \( 1 \leq \hat{S}_{\alpha_j} = \min \left\{ \hat{S}, \hat{S}_{\alpha_1}, \ldots, \hat{S}_{\alpha_k} \right\} \), we have
\[
\hat{k} = \frac{1}{S_{\alpha_k}^*} + \sum_{i=1}^{k} \frac{1}{S_{\alpha_i}^*} \leq \frac{k+1}{\hat{S}^*} < k+1.
\]
(6.10)
Combining (6.1), (6.10), \( \hat{k} \geq 1 \) and \(-\frac{N-2}{N+2-\alpha} > -\frac{N-2}{2} \), we get
\[
\hat{c}_0 > \frac{1}{N} \min \left\{ \hat{\kappa}^{-\frac{N-2}{N+2-\alpha}}, \hat{\kappa}^{-\frac{N-2}{2}} \right\} = \frac{1}{N} \hat{\kappa}^{-\frac{N-2}{N+2-\alpha}} > \frac{1}{N} (k+1)^{-\frac{N-2}{2}}.
\]
(6.11)
Since \( \zeta \in \left( \frac{(N-2)^2}{4} \left( 1 - \frac{1}{(k+1)^{N/2}} \right), \frac{(N-2)^2}{4} \right) \), we know
\[
k + 1 < \left( \frac{1}{\hat{S} \left( 1 - \frac{4}{(N-2)^2} \zeta \right)^{\frac{N-1}{N}}} \right)^{\frac{N}{N-2}} = \left( \frac{1}{S} \right)^{\frac{N}{N-2}},
\]
which implies that
\[
(k+1)^{-\frac{N-2}{2}} > \frac{S^k}{N}.
\]
(6.12)
According to (6.11), (6.12), Lemma 4.2 and Lemma 5.1, we obtain
\[ \tilde{c}_0 > \frac{S_{\frac{N}{2}}}{N} > c = c_0 > 0. \]

**Case (4).** Similar to Case (3), we have
\[ \tilde{\kappa} < k + 1, \]
which gives that
\[ \{ \tilde{\kappa} \leq 1 \} \cap \{ \tilde{\kappa} < k + 1 \} = \{ \tilde{\kappa} \leq 1 \}. \]
Combining (6.1), \( \tilde{\kappa} \leq 1 \) and
\[ -N^{\frac{N-2}{2N-2}} < -\frac{N-2}{2}, \]
we get
\[ \tilde{c}_0 \geq \frac{1}{N} \min \left\{ \tilde{\kappa}^\frac{N-2}{2N-2}, \tilde{\kappa}^\frac{-N-2}{2N-2} \right\} = \frac{1}{N} \tilde{\kappa}^\frac{-N-2}{2N-2} \geq \frac{1}{N}. \]

Since \( \zeta \in \left( \frac{(N-2)^2}{4} \left( 1 - \frac{1}{S_{\frac{N}{2}}} \right), \frac{(N-2)^2}{4} \right) \), we get
\[ S = \tilde{S} \left( 1 - \frac{4}{\frac{(N-2)^2}{4} \zeta} \right)^\frac{N-1}{2} < 1, \]
which implies that
\[ 1 > S \Rightarrow 1 > S_{\frac{N}{2}}. \]

According to (6.13), (6.14), Lemma 4.2 and Lemma 5.1, we obtain
\[ \tilde{c}_0 \geq \frac{1}{N} \frac{S_{\frac{N}{2}}}{\tilde{c}_0} > c = c_0 > 0. \]

\[ \square \]

7. The Proof of Theorem 1.3

In this section, we show the existence of nontrivial solution of problem (P).

**Proof of Theorem 1.3:** We divide our proof into five steps.

**Step 1.** Since \( \{ u_n \} \) is a bounded sequence in \( D^{1,2}(\mathbb{R}^N) \), up to a subsequence, we assume that
\[ u_n \rightrightarrows u \text{ in } D^{1,2}(\mathbb{R}^N), \quad u_n \rightharpoonup u \text{ a.e. in } \mathbb{R}^N, \]
\[ u_n \to u \text{ in } L^r_{\text{loc}}(\mathbb{R}^N) \text{ for all } r \in [1, 2^*). \]

According to Lemma 2.1, Lemma 3.2 and Lemma 5.2, there exists \( C > 0 \) such that for any \( n \) we get
\[ \| u_n \|_{L^2, N-2}(\mathbb{R}^N) \geq C > 0. \]

On the other hand, since the sequence is bounded in \( D^{1,2}(\mathbb{R}^N) \) and \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \hookrightarrow L^{2, N-2}(\mathbb{R}^N) \), we have
\[ \| u_n \|_{L^{2, N-2}(\mathbb{R}^N)} \leq C, \]
for some \( C > 0 \) independent of \( n \). Hence, there exists a positive constant which we denote again by \( C \) such that for any \( n \) we obtain
\[ C \leq \| u_n \|_{L^{2, N-2}(\mathbb{R}^N)} \leq C^{-1}. \]
Combining the definition of Morrey space and above inequalities, we deduce that for any \( n \in \mathbb{N} \) there exist \( \sigma_n > 0 \) and \( x_n \in \mathbb{R}^N \) such that

\[
\frac{1}{\sigma_n^2} \int_{B(x_n, \sigma_n^2)} |u_n(y)|^2 \, dy \geq \|u_n\|_{L^{2, N-2}(\mathbb{R}^N)}^2 - \frac{C}{2n} \geq C_4 > 0.
\]

Let \( v_n(x) = \sigma_n^{-N-2} u_n(x_n + \sigma_n x) \). We may readily verify that

\[
\tilde{I}_\zeta(v_n) = I(u_n) \to c, \quad \tilde{I}_\zeta(v_n) \to 0 \quad \text{as} \quad n \to \infty,
\]

where

\[
\tilde{I}_\zeta(v_n) = \frac{1}{2} \|v_n\|_{D}^2 - \frac{\zeta}{2} \int_{\mathbb{R}^N} \frac{|v_n|^2}{|x + \frac{\sigma_n}{\zeta}|^2} \, dx
- \sum_{i=1}^{k} \frac{1}{2 \cdot 2^i} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{|v_n(x) - |v_n(y)|^2}{|x - y|^{\alpha_i}} \right) \, dx \, dy - \frac{1}{2} \int_{\mathbb{R}^N} |\hat{v_n}|^2 \, dx.
\]

Now, for all \( \varphi \in D^{1,2}(\mathbb{R}^N) \), we obtain

\[
|\langle \tilde{I}_\zeta(v_n), \varphi \rangle| = |\langle \tilde{I}(u_n), \hat{\varphi} \rangle| \leq \|\tilde{I}(u_n)\|_{D^{-1}} \|\varphi\|_{D}
= o(1) \|\varphi\|_{D},
\]

where \( \hat{\varphi} = \sigma_n^{-N-2} \varphi(\frac{x - x_n}{\sigma_n}) \). Since \( \|\varphi\|_{D} = \|\varphi\|_{D} \), we get

\[
\tilde{I}_\zeta(v_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus there exists \( v \) such that

\[
v_n \to v \quad \text{in} \quad D^{1,2}(\mathbb{R}^N), \quad v_n \to v \quad \text{a.e. in} \quad \mathbb{R}^N,
\]

\[
v_n \to v \quad \text{in} \quad L^r_{loc}(\mathbb{R}^N) \quad \text{for all} \quad r \in [1, 2^*),
\]

and

\[
\int_{B(0,1)} |v_n(y)|^2 \, dy = \frac{1}{\sigma_n^2} \int_{B(x_n, \sigma_n)} |u_n(y)|^2 \, dy \geq C_4 > 0.
\]

Hence, \( v \neq 0 \).

**Step 2.** Now, we claim that \( \{\frac{x_n}{\sigma_n}\} \) is bounded. If \( \frac{x_n}{\sigma_n} \to \infty \), then for any \( \varphi \in D^{1,2}(\mathbb{R}^N) \), we get

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{v_n \varphi}{|x + \frac{\sigma_n}{\zeta}|^2} \, dx \to 0.
\]

We will show that

\[
\langle I_0(v), \varphi \rangle = 0.
\]

Since \( v_n \to v \) weakly in \( D^{1,2}(\mathbb{R}^N) \), we know

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla v_n \nabla \varphi \, dx = \int_{\mathbb{R}^N} \nabla v \nabla \varphi \, dx.
\]

By the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from \( L^{\frac{2N}{N-2}}(\mathbb{R}^N) \) to \( L^{\frac{2N}{N+2}}(\mathbb{R}^N) \). Since \( |v_n|^{2^*_i} \to |v|^{2^*_i} \), weakly in
where the last inequality is from the absolutely continuity of 

(7.4) \[ \int_{\mathbb{R}^N} \frac{|v_n(y)|^{2^*_\alpha_i}}{|x - y|^{\alpha_i}} \, dy \to \int_{\mathbb{R}^N} \frac{|v(y)|^{2^*_\alpha_i}}{|x - y|^{\alpha_i}} \, dy \] weakly in \( L_{\alpha_i}^{2^*_\alpha_i}(\mathbb{R}^N) \).

Now, we show that \(|v_n|^{2^*_\alpha_i} - v_n \varphi \to |v|^{2^*_\alpha_i} - v \varphi\) in \( L_{\alpha_i}^{2^*_\alpha_i}(\mathbb{R}^N)\). For any \( \varepsilon > 0 \), there exists \( R > 0 \) large enough such that

\[
\begin{align*}
\lim_{n \to \infty} \int_{|x| > R} |v_n|^{2^*_\alpha_i} - v_n \varphi \bigg|^{\frac{2N}{\alpha_i}} - |v|^{2^*_\alpha_i} - v \varphi \bigg|^{\frac{2N}{\alpha_i}} \, dx \\
\leq \lim_{n \to \infty} \int_{|x| > R} |v_n|^{(2^*_\alpha_i - 1) \frac{N}{\alpha_i}} |\varphi|^{\frac{N}{\alpha_i}} \, dx + \int_{|x| > R} |v|^{(2^*_\alpha_i - 1) \frac{N}{\alpha_i}} |\varphi|^{\frac{N}{\alpha_i}} \, dx \\
\leq \int_{|x| > R} |v|^{2^*_\alpha_i} \, dx \left( \int_{|x| > R} |\varphi|^{2^*_\alpha_i} \, dx \right)^{\frac{1}{\alpha_i}} + \left( \int_{|x| > R} |v|^2 \, dx \right)^{1 - \frac{1}{\alpha_i}} \left( \int_{|x| > R} |\varphi|^2 \, dx \right)^{\frac{1}{\alpha_i}} \\
< \varepsilon \frac{2}{2}.
\end{align*}
\]

On the other hand, by the boundedness of \( \{v_n\} \), one has

\[
\left( \int_{|x| \leq R} |v_n|^2 \, dx \right)^{1 - \frac{1}{\alpha_i}} \leq M.
\]

where \( M > 0 \) is a constant. Let \( \Omega = \{ x \in \mathbb{R}^N | |x| \leq R \} \). For any \( \delta > 0 \) there exists \( \delta > 0 \), when \( E \subset \Omega \) with \( |E| < \delta \). We obtain

\[
\begin{align*}
\int_{E} |v_n|^{2^*_\alpha_i} - v_n \varphi \bigg|^{\frac{2N}{\alpha_i}} \, dx &= \int_{E} |v_n|^{(2^*_\alpha_i - 1) \frac{N}{\alpha_i}} |\varphi|^{\frac{N}{\alpha_i}} \, dx \\
&\leq \left( \int_{E} |v_n|^2 \, dx \right)^{1 - \frac{1}{\alpha_i}} \left( \int_{E} |\varphi|^2 \, dx \right)^{\frac{1}{\alpha_i}} \\
&< M \delta.
\end{align*}
\]

where the last inequality is from the absolutely continuity of \( \int_{E} |\varphi|^2 \, dx \). Moreover, \(|v_n|^{2^*_\alpha_i} - v_n \varphi \to |v|^{2^*_\alpha_i} - v \varphi\) a.e. in \( \mathbb{R}^N \) as \( n \to \infty \). Thus, by the Vitali convergence Theorem, we get

(7.6) \[
\lim_{n \to \infty} \int_{|x| \leq R} |v_n|^{2^*_\alpha_i} - v_n \varphi \bigg|^{\frac{2N}{\alpha_i}} \, dx = \int_{|x| \leq R} |v|^{2^*_\alpha_i} - v \varphi \bigg|^{\frac{2N}{\alpha_i}} \, dx.
\]
Combining (7.4) and (7.7), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| v_n \right|^{2^*_s - 2} v_n \phi \, dx = 0
\]
This implies that
\[
\lim_{n \to \infty} \int_{|x| \leq R} \left| v_n \right|^{2^*_s - 2} v_n \phi \, dx = 0
\]
Similarly, we get
\[
\lim_{n \to \infty} \int_{|x| > R} \left| v_n \right|^{2^*_s - 2} v_n \phi \, dx = 0
\]
Combining (7.4) and (7.7), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| v_n(y) \right|^{2^*_s} \left| v_n(x) \right|^{2^*_s} v_n(x) \phi(x) \, dy \, dx = 0
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| v_n(y) \right|^{2^*_s} \left| v_n(x) \right|^{2^*_s} v_n(x) \phi(x) \, dy \, dx = 0
\]
Similarly, we get
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left| v_n \right|^{2^*_s - 2} v_n \phi \, dx = 0
\]
Applying \( \lim_{n \to \infty} \langle \tilde{I}_\epsilon(v_n), \phi \rangle = 0 \), (7.1), (7.3), (7.8) and (7.9) we know
\[
\langle \tilde{I}_0(v), \phi \rangle = 0
\]
Moreover, according to (7.10) and \( v \neq 0 \), we get that
\[
v \in \overline{N}.
\]
By Brézis–Lieb lemma [7, Lemma 2.2], we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_s} |v_n(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x) - v(x)|^{2^*_s} |v_n(y) - v(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy
\]
which implies that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_s} |v_n(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)|^{2^*_s} |v(y)|^{2^*_s}}{|x - y|^\alpha} \, dx \, dy + o(1)
\]
Similarly, we get
\[
\int_{\mathbb{R}^N} |v_n|^{2^*_s} \, dx \geq \int_{\mathbb{R}^N} |v|^{2^*_s} \, dx + o(1)
\]
Set
\[
K(u) = \sum_{i=1}^k \left( \frac{1}{2} - \frac{1}{2^*_i} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_i} |u(y)|^{2^*_i}}{|x - y|^\alpha} \, dx \, dy + \left( \frac{1}{2} - \frac{1}{2^*_i} \right) \int_{\mathbb{R}^N} |u|^2 \, dx.
\]
Applying Lemma 6.1, Lemma 4.2, (7.11), (7.12), $v \in \widetilde{N}$ and Lemma 4.3, we obtain

$$\tilde{c}_0 > c_0 = I(v_n) - \frac{1}{2} \langle I'(v_n), v_n \rangle$$

$$= \lim_{n \to \infty} K(v_n) + o(1)$$

$$\geq K(v) + o(1)$$

$$= \tilde{I}_0(v) - \frac{1}{2} \langle \tilde{I}_0(v), v \rangle = \tilde{I}_0(v) \geq \tilde{c}_0,$$

which yields a contradiction. Hence, $\{ \frac{x_n}{\sigma_n} \}$ is bounded.

**Step 3.** In this step, we study another $(PS)_c$ sequence of $I$. Let $\tilde{v}_n(x) = \sigma_n^{-\frac{N-2}{2}} u_n(\sigma_n x)$. Then we can verify that

$$I(\tilde{v}_n) = I(u_n) \to c, \quad I'(\tilde{v}_n) \to 0 \text{ as } n \to \infty.$$

Arguing as before, we have

$$\tilde{v}_n \rightharpoonup \tilde{v} \text{ in } D^{1,2}(\mathbb{R}^N), \quad \tilde{v}_n \to \tilde{v} \text{ a.e. in } \mathbb{R}^N,$$

$$\tilde{v}_n \to \tilde{v} \text{ in } L^r_{loc}(\mathbb{R}^N) \text{ for all } r \in [1, 2^*).$$

Since $\{ \frac{x_n}{\sigma_n} \}$ is bounded, there exists $\tilde{R} > 0$ such that

$$\int_{B(0,\tilde{R})} |\tilde{v}_n(y)|^2 dy > \int_{B(\frac{\tilde{R}}{\sigma_n}, 1)} |\tilde{v}_n(y)|^2 dy = \frac{1}{\sigma_n^2} \int_{B(x_n, \sigma_n)} |u_n(y)|^2 dy \geq C_4 > 0.$$

As a result, $\tilde{v} \neq 0$.

**Step 4.** In this step, we show $\tilde{v}_n \to \tilde{v}$ strongly in $D^{1,2}(\mathbb{R}^N)$. Similar to Step 2, we know that

$$(7.13) \quad (I'(\tilde{v}), \varphi) = 0.$$ 

Applying (7.11) – (7.13) and Lemma 4.2, we obtain

$$c = I(\tilde{v}_n) - \frac{1}{2} \langle I'(\tilde{v}_n), \tilde{v}_n \rangle + o(1)$$

$$= \lim_{n \to \infty} K(v_n) + o(1)$$

$$\geq \lim_{n \to \infty} K(v) + o(1)$$

$$= I(\tilde{v}) - \frac{1}{2} \langle I'(\tilde{v}), \tilde{v} \rangle = I(\tilde{v}) \geq c.$$

Therefore, the inequalities above have to be equalities. We know

$$\lim_{n \to \infty} K(\tilde{v}_n) = K(\tilde{v}).$$

By using Brézis–Lieb lemma again, we have

$$\lim_{n \to \infty} K(\tilde{v}_n) - \lim_{n \to \infty} K(\tilde{v}_n - \tilde{v}) = K(\tilde{v}) + o(1).$$

Hence, we conclude that

$$\lim_{n \to \infty} K(\tilde{v}_n - \tilde{v}) = 0,$$
which implies that

\[(7.15)\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{v}_n(x) - \tilde{v}(x)|^{2^{*}_i} |\tilde{v}_n(y) - \tilde{v}(y)|^{2^{*}_i}}{|x - y|^\alpha} \, dx \, dy = 0 \text{ for all } i = 1, \ldots, k,
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\tilde{v}_n(x) - \tilde{v}(x)|^{2^*} \, dx = 0.
\]

According to \( \langle \dot{I}(\tilde{v}_n), \tilde{v}_n \rangle = o(1) \), \( \langle \dot{I}^*(\tilde{v}), \tilde{v} \rangle = 0 \) and Brézis–Lieb lemma, we obtain

\[
o(1) = \langle \dot{I}(\tilde{v}_n), \tilde{v}_n \rangle - \langle \dot{I}^*(\tilde{v}), \tilde{v} \rangle
\]

\[
= \|\tilde{v}_n - \tilde{v}\|_\zeta^2 - \sum_{i=1}^k \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{v}_n(x) - \tilde{v}(x)|^{2^{*}_i} |\tilde{v}_n(y) - \tilde{v}(y)|^{2^{*}_i}}{|x - y|^\alpha} \, dx \, dy
\]

\[
- \int_{\mathbb{R}^N} |\tilde{v}_n - \tilde{v}|^{2^*} \, dx + o(1),
\]

which implies that

\[(7.16)\]
\[
\lim_{n \to \infty} \|\tilde{v}_n - \tilde{v}\|_\zeta^2 = \lim_{n \to \infty} \sum_{i=1}^k \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{v}_n(x) - \tilde{v}(x)|^{2^{*}_i} |\tilde{v}_n(y) - \tilde{v}(y)|^{2^{*}_i}}{|x - y|^\alpha} \, dx \, dy
\]

\[
+ \lim_{n \to \infty} \int_{\mathbb{R}^N} |\tilde{v}_n - \tilde{v}|^{2^*} \, dx + o(1).
\]

Combining (7.15) and (7.16), we get

\[
\lim_{n \to \infty} \|\tilde{v}_n - \tilde{v}\|_\zeta^2 \to o(1).
\]

Since \( \tilde{v} \neq 0 \), we know that \( \tilde{v}_n \to \tilde{v} \) strongly in \( D^{1,2}(\mathbb{R}^N) \).

**Step 5.** Here we show the properties of the solution. By using (7.14) again, we know \( I(\tilde{v}) = c \), which means that \( \tilde{v} \) is a nontrivial solution of problem (P) at the energy level \( c \). Since \( I \) is even, we have

\[
c = I(\tilde{v}) = I(|\tilde{v}|) \text{ and } \langle \dot{I}^*(\tilde{v}), \tilde{v} \rangle = \langle \dot{I}^*(|\tilde{v}|), |\tilde{v}| \rangle = 0.
\]

Then \( |\tilde{v}| \) is also a critical point of \( I \). Hence, we can choose \( \tilde{v} \geq 0 \). By the Kelvin transformation, we have

\[(7.17)\]
\[
\tilde{v}(x) = \frac{1}{|x|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right).
\]

It is well known that

\[(7.18)\]
\[
-\Delta \tilde{v}(x) = \frac{1}{|x|^{N+2}} (-\Delta \tilde{v}) \left( \frac{x}{|x|^2} \right),
\]

and

\[(7.19)\]
\[
\tilde{v}(x) \frac{1}{|x|^2} = \frac{1}{|x|^{N+2}} \frac{\tilde{v} \left( \frac{x}{|x|^2} \right)}{|x|^2}.
\]
The following identity is very useful. For $\forall x, y \in \mathbb{R}^N \setminus \{0\}$, we get

$$
\frac{1}{|x|^2} - \frac{y}{|y|^2}
+ \frac{1}{|xy|^\alpha_i} = \frac{1}{|x|^2 - y^2} \cdot \frac{1}{|xy|^\alpha_i}.
$$

(7.20)

Set $z = \frac{y}{|y|^2}$. Applying (7.17) and (7.20), we have

$$
\int_{\mathbb{R}^N} |\tilde{v}(y)|^{2N-\alpha_i} \frac{y}{|y|^2} \, dy = \int_{\mathbb{R}^N} |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|y|^{2N-\alpha_i}} \, dy \quad \text{(by (7.17))}

= \int_{\mathbb{R}^N} |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|xy|^\alpha_i} \cdot \frac{1}{|y|^{2N-\alpha_i}} \, dy \quad \text{(by (7.20))}

= \frac{1}{|x|^\alpha_i} \int_{\mathbb{R}^N} |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|y|^{2N-\alpha_i}} \, dy

= \frac{1}{|x|^\alpha_i} \int_{\mathbb{R}^N} |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|y|^{2N-\alpha_i}} \, dy

\text{(set } z = \frac{y}{|y|^2}).

(7.21)

Therefore, by using (7.18), (7.19) and (7.21), we get

$$
\frac{1}{|x|^{N+2}} (-\Delta \tilde{v}) \left( \frac{x}{|x|^2} \right) - \frac{\zeta}{|x|^{N+2}} \cdot \tilde{v} \left( \frac{x}{|x|^2} \right)

= \sum_{i=1}^k \frac{1}{|x|^\alpha_i} \int_{\mathbb{R}^N} |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|y|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right) \left| \tilde{v} \left( \frac{y}{|y|^2} \right) \right|^{\frac{4-\alpha_i}{N-2}} \cdot \frac{1}{|x|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right)

+ \frac{1}{|x|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right) \left| \tilde{v} \left( \frac{x}{|x|^2} \right) \right|^{\frac{4-\alpha_i}{N-2}} \cdot \frac{1}{|x|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right)

= \frac{1}{|x|^{N+2}} \sum_{i=1}^k \int_{\mathbb{R}^N} |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|y|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right) \left| \tilde{v} \left( \frac{y}{|y|^2} \right) \right|^{\frac{4-\alpha_i}{N-2}} \cdot \tilde{v} \left( \frac{x}{|x|^2} \right)

+ \frac{1}{|x|^{N+2}} \left| \tilde{v} \left( \frac{x}{|x|^2} \right) \right|^{\frac{4-\alpha_i}{N-2}} \cdot \tilde{v} \left( \frac{x}{|x|^2} \right),
$$

which gives that

$$
-\Delta \tilde{v} - \frac{\zeta}{|x|^2} \tilde{v} = \sum_{i=1}^k \left( \int_{\mathbb{R}^N} |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|y|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right) + |\tilde{v} \left( \frac{y}{|y|^2} \right) |^{2N-\alpha_i} \cdot \frac{1}{|x|^{N-2}} \tilde{v} \left( \frac{x}{|x|^2} \right), \text{ in } \mathbb{R}^N \setminus \{0\}.
$$

\[ \square \]

**Open Problem**

During the preparation of the manuscript we faced one problem which is worth to be tackled in forthcoming investigation.
We just study the case of $\tilde{S}_{\alpha} \geq 1$ (see $(H_2)$), it is nature to ask the case of $\tilde{S}_{\alpha} \in (0, 1)$ (see Fig 1.).

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