A Gaussian sequence approach for proving minimaxity: A Review

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Abstract: This paper reviews minimax best equivariant estimation in these invariant estimation problems: a location parameter, a scale parameter and a (Wishart) covariance matrix. We briefly review development of the best equivariant estimator as a generalized Bayes estimator relative to right invariant Haar measure in each case. Then we prove minimaxity of the best equivariant procedure by giving a least favorable prior sequence based on non-truncated Gaussian distributions. The results in this paper are all known, but we bring a fresh and somewhat unified approach by using, in contrast to most proofs in the literature, a smooth sequence of non truncated priors. This approach leads to some simplifications in the minimaxity proofs.

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1. Introduction

We review some results on minimaxity of best equivariant estimators from what we hope is a fresh and somewhat unified perspective. Our basic approach is to start with a general equivariant estimator, and demonstrate that the best equivariant estimator is a generalized Bayes estimator, δ0, with respect to an invariant prior. We then choose an appropriate sequence of Gaussian priors whose support is the entirety of the parameter space and show that the Bayes risks converge to the constant risk of δ0. This implies that δ0 is minimax. All results on best equivariance and minimaxity, which we consider in this paper, are known in the literature. But, using a sequence of Gaussian priors as a least favorable sequence, simplifies the proofs and gives fresh and unified perspective.

In this paper, we consider the following three estimation problems.

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Estimation of a location parameter: Let the density function of $X$ be given by

$$f(x - \mu) = f(x_1 - \mu, \ldots, x_n - \mu).$$

(1.1)

Consider estimation of the location parameter $\mu$ under location invariant loss

$$L(\delta - \mu).$$

(1.2)

We study equivariant estimators under the location group, given by

$$\delta(x - \mu) = \delta(x) - \mu.$$

(1.3)

Estimation of a scale parameter: Let the density function of $X$ be given by

$$\sigma^{-n}f(x/\sigma),$$

(1.4)

with scale parameter $\sigma$, where $x/\sigma = (x_1/\sigma, \ldots, x_n/\sigma)$. Consider estimation of the scale $\sigma$ under scale invariant loss

$$L(\delta/\sigma).$$

(1.5)

We study equivariant estimators under scale group, given by

$$\delta(x/\sigma) = \delta(x)/\sigma.$$

(1.6)

Estimation of covariance matrix: We study estimation of $\Sigma$ based on a $p \times p$ random matrix $V$ having a Wishart distribution $W_p(n, \Sigma)$, where the density is given in (2.3) below. An estimator $\delta$ is evaluated by the invariant loss

$$L(\Sigma^{-1}\delta).$$

(1.7)

We consider equivariant estimators under the lower triangular group, given by

$$\delta(AVA^\top) = A\delta(V)A^\top,$$

(1.8)

where $T^+$, the set of $p \times p$ lower triangular matrices with positive diagonal entries.
For the first two cases with the squared error loss \((\delta - \mu)^2\) and the entropy loss \(\delta/\sigma - \log(\delta/\sigma) - 1\), respectively, the so called Pitman (1939) estimators

\[
\hat{\mu}_0(x) = \frac{\int_{-\infty}^{\infty} \mu f(x - \mu) d\mu}{\int_{-\infty}^{\infty} f(x - \mu) d\mu},
\]

(1.9)

\[
\hat{\sigma}_0(x) = \frac{\int_{0}^{\infty} \sigma^{-n-1} f(x/\sigma) d\sigma}{\int_{0}^{\infty} \sigma^{-n-2} f(x/\sigma) d\sigma}
\]

(1.10)

are well-known to be best equivariant and minimax. Clearly, they are generalized Bayes with respect to \(\pi(\mu) = 1\) and \(\pi(\sigma) = 1/\sigma\), respectively. Girshick and Savage (1951) gave the original proof of minimaxity. Kubokawa (2004) also gives a proof and further developments in the restricted parameter setting. Both use a sequence of uniform distribution on expanding interval as least favorable priors.

For the last case, James and Stein (1961) show that the best equivariant estimator is given by

\[
\hat{\Sigma}_0 = T \text{diag}(d_1, \ldots, d_p) T^T
\]

(1.11)

where \(T \in T^+\) is from the Cholesky decomposition of \(V = TT^T\) and \(d_i = 1/(n+p-2i+1)\) for \(i = 1, \ldots, p\). Note that the group of \(p\times p\) lower triangular matrices with positive diagonal entries is solvable, and the result of Kiefer (1957) implies the minimaxity of \(\hat{\Sigma}_0\). Tsukuma and Kubokawa (2015) gives as a sequence of least favorable priors, the invariant prior truncated on a sequence of expanding sets.

In each case, the sequence of priors we employ is based on a Gaussian sequence of possibly transformed parameters. This is in contrast to most proofs in the literature which use truncated versions of the invariant prior. As a consequence, the resulting proofs are less complicated.

Section 2 is devoted to developing the best equivariant estimator as a generalized Bayes estimator with respect to a right invariant (Haar measure) prior in each case. The general approach is basically that of Hora and Buehler (1966). Section 3 provides minimaxity proofs of the best equivariant procedure by giving a least favorable prior sequence based on (possibly transformed) Gaussian priors in each cases. We give some concluding remarks in Section 4.

2. establishing best equivariant procedures

All results in this section are well-known. Our proof of best equivariance for \(\hat{\mu}_0, \hat{\sigma}_0\) and \(\hat{\Sigma}_0\) follow from Hora and Buehler (1966). The reader is referred to
Hora and Buehler’s (1966) for further details on their general development of a best equivariant estimator as the generalized Bayes estimator relative to right invariant Haar measure.

2.1. Estimation of location parameter

Consider an equivariant estimator which satisfies $\delta(x - \mu) = \delta(x) - \mu$. Then we have a following result.

**Theorem 2.1.** Let $X$ have distribution (1.1) and let the loss be given by (1.2). The generalized Bayes estimator with respect to the invariant prior $\pi(\mu) = 1$, $\hat{\mu}_0(x)$, is best equivariant under the location group, that is,

$$
\hat{\mu}_0(x) = \arg \min_\delta \int_{-\infty}^{\infty} L(\delta(x) - \mu)f(x - \mu)d\mu.
$$

**Proof.** The risk of the equivariant estimator (1.3) is written as

$$
R(\delta(x), \mu)
$$

$$
= \int_{\mathbb{R}^n} L(\delta(x) - \mu)f(x - \mu)dx
$$

$$
= \int_{\mathbb{R}^n} L(\delta(x) - \mu))f(x - \mu)dx
$$

$$
= \int_{\mathbb{R}^n} L(\delta(z))f(z)dz
$$

$$
= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} L(\delta(z_{n-1}, z_n)f(z_{n-1}, z_n)dzdz_{n-1}
$$

$$
= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} L(\delta(z_{n-1}, u_n - \theta))f(z_{n-1}, u_n - \theta)d\theta dz_{n-1}
$$

$$
= \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} L(\delta(u_{n-1} - \theta, u_n - \theta))f(u_{n-1} - \theta, u_n - \theta)d\theta \right) du_{n-1}.
$$

Then the best equivariant estimator is

$$
\hat{\mu}_0(x) = \arg \min_\delta \int_{-\infty}^{\infty} L(\delta(x) - \mu)f(x - \mu)d\mu.
$$

$\square$
2.2. Estimation of scale

Consider an equivariant estimator which satisfies $\delta(x/\sigma) = \delta(x)/\sigma$. Then we have a following result.

**Theorem 2.2.** Let $X$ have distribution (1.4) and let the loss be given by (1.5). Then the generalized Bayes estimator, with respect to the prior $\pi(\sigma) = 1/\sigma$, $\hat{\sigma}_0(x)$, is best equivariant under the scale group, that is,

$$\hat{\sigma}_0(x) = \arg \min_{\delta} \int_0^\infty L(\delta/\sigma) \frac{f(x/\sigma)}{\sigma} d\sigma.$$  

**Proof.** The risk of the equivariant estimator is written as

$$R(\delta(x), \sigma)$$

$$= \int_{\mathbb{R}^n} L(\delta(x)/\sigma) \sigma^{-n} f(x/\sigma) dx$$

$$= \int_{\mathbb{R}^n} L(\delta(x/\sigma)) \sigma^{-n} f(x/\sigma) dx$$

$$= \int_{\mathbb{R}^n} L(\delta(z)) f(z) dz$$

$$= \int_{\mathbb{R}^{n-1}} \left( \int_{\infty}^0 + \int_0^{\infty} \right) L(\delta(z_{n-1}, z_n)) f(z_{n-1}, z_n) dz_n dz_{n-1}$$

$$= \int_{\mathbb{R}^{n-1}} \sum_{j=-1,1} \int_0^\infty L(\delta(z_{n-1}, jz_n)) f(z_{n-1}, jz_n) dz_n dz_{n-1}$$

$$= \int_{\mathbb{R}^{n-1}} \sum_{j=-1,1} \int_0^\infty L(\delta(u_{n-1}/w, ju_n/w)) f(u_{n-1}/w, ju_n/w) \frac{u_n}{w^2} dw dz_{n-1}$$

$$= \int_{\mathbb{R}^{n-1}} \sum_{j=-1,1} \int_0^\infty L(\delta(u_{n-1}/w, ju_n/w)) \frac{1}{w^{n-1}} u_n w f(u_{n-1}/w, ju_n/w) du_{n-1} dw$$

$$z_i = u_i/w \ (i = 1, \ldots, n - 1) \quad \text{(where } u_i \text{ is variable and } w \text{ is constant})$$

$$= \int_{\mathbb{R}^{n-1}} u_n \sum_{j=-1,1} \left\{ \int_0^\infty L(\delta(u_{n-1}/w, ju_n/w)) f(u_{n-1}/w, ju_n/w) \frac{w^n}{w^{n+1}} dw \right\} du_{n-1}.$$ 

Then the best equivariant estimator is

$$\hat{\sigma}_0(x) = \arg \min_{\delta} \int_0^\infty L(\delta/\sigma) \sigma^{-n-1} f(x/\sigma) d\sigma.$$ 

$\square$
2.3. Estimation of covariance matrix

Let $V$ have a Wishart distribution $\mathcal{W}_p(n, \Sigma)$. Let $T^+$ be the set of $p \times p$ lower triangular matrices with positive diagonal entries. By the Cholesky decomposition, $\Sigma^{-1}$ and $V$ can be written as

$$\Sigma^{-1} = \Theta^T \Theta \quad \text{and} \quad V = T T^T$$

for $\Theta = (\theta_{ij}) \in T^+$ and $T = (t_{ij}) \in T^+$. As in Theorem 7.2.1 of Anderson (2003), the probability density function of $T$ is

$$f_W(T|\Theta)\gamma(dT) = \frac{1}{C(p,n)} |\Theta T|^n \exp \left[-\frac{1}{2} \text{tr} \left\{ (\Theta T)(\Theta T)^T \right\} \right] \gamma(dT) \quad (2.3)$$

where $C(p,n)$ is a normalizing constant given by

$$C(p,n) = 2^{p(n-2)/2} \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{n + 1 - i}{2}\right) \quad (2.4)$$

and $\gamma(dT)$ is the left-invariant Haar measure on $T^+$ given by

$$\gamma(dT) = \prod_{i=1}^{p} t_{ii}^{-1} dT_i. \quad (2.5)$$

An estimator $\delta$ is evaluated by the invariant loss function given by

$$L(\Theta \delta \Theta^T). \quad (2.6)$$

Denote the risk function by

$$R(\delta, \Sigma) = \int_{T^+} L(\Theta \delta \Theta^T) f_W(T|\Theta) \gamma(dT).$$

For all $A \in T^+$, the group transformation with respect to $T^+$ on a random matrix $T$ and a parameter matrix $\Theta$ is defined by $(T, \Theta) \rightarrow (AT, \Theta A^{-1})$. The group $\bar{G}$ operating on $\Theta$ is transitive. Any equivariant estimator of

$$\Sigma = (\Theta^T \Theta)^{-1} = \Theta^{-1} (\Theta^{-1})^T$$

under the lower triangular group is of form given by

$$\delta(AT) = A \delta(T) A^T. \quad (2.7)$$
Theorem 2.3. Let $V = TT^\top \sim W_p(n, \Sigma)$ and let the loss be $L(\Theta \delta \Theta^\top)$ as in (2.6). Then the generalized Bayes estimator with respect to the prior

$$\pi(\Theta) = \gamma(d\Theta),$$

(2.8)

$\delta_0$, is best equivariant under lower triangular group, that is,

$$\delta_0(T) = \arg \min_{\delta} \int_{T^+} L(\Theta \delta \Theta^\top)f_W(T|\Theta)\gamma(d\Theta).$$

(2.9)

Note that $\gamma(d\Theta)$ is the “left” invariant measure, which seems to contradict the general theory by Hora and Buehler (1966). However this seeming anomaly is due to our parameterization $V = TT^\top$, $\Sigma^{-1} = \Theta^\top \Theta$ and

$$\Sigma = \Theta^{-1}(\Theta^{-1})^\top.$$ (2.10)

The general theory implies that

$$\nu(d\Theta^{-1}) = \gamma(d\Theta)$$

(2.11)

where $\nu$ is right invariant Haar measure on $T^+$ given by

$$\nu(dZ) = \prod_{i=1}^p z_i^{-(p-i+1)}dZ.$$ (2.12)

In the proof below, in addition to the left invariance of $\gamma$, and the right invariance of $\nu$, we use the fact that

$$f_W(T|\Theta) = f_W(\Theta T|I) = f_W(I|\Theta T).$$ (2.13)

Proof of Theorem 2.3. By (2.3) and (2.6), the risk of an equivariant estimator can be expressed as

$$R(\delta, \Sigma)$$

$$= \int_{T^+} L(\Theta \delta(\Theta^\top))f_W(T|\Theta)\gamma(dT)$$

$$= \int_{T^+} L(\delta(\Theta T))f_W(T|\Theta)\gamma(dT)$$

$$= \int_{T^+} L(\delta(Z))f_W(Z|I)\gamma(dZ) \quad (Z = \Theta T, \text{ and left invariance of } \gamma)$$

$$= \int_{T^+} L(\delta(Z))f_W(I|Z)\prod_{i=1}^p z_i^{-1}dZ \quad \text{(by the form of } f_W)$$
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\[ \int_{T^+} L(\delta(Z)) f_W(I|Z) \prod_{i=1}^p z_{ii}^{p-2i+1} \nu(dZ) \]

\[ = \int_{T^+} L(\delta(WS)) f_W(S|W) \prod_{i=1}^p (w_{ii}s_{ii})^{p-2i+1} \nu(dW) \]

\[ (Z = WS, \text{ and right invariance of } \nu) \]

\[ = \prod_{i=1}^p s_{ii}^{p-2i+1} \int_{T^+} L(W\delta(S)W^T)f_W(S|W)\gamma(dW), (\text{by (2.7) and the form of } \gamma(dw)) \]

Then the best equivariant estimator with respect to the group \( T^+ \) can be written by

\[ \delta_0(T) = \arg \min_{\delta} \int_{T^+} L(\Theta \delta \Theta^T)f_W(T|\Theta)\gamma(d\Theta). \]

\[ \square \]

3. Minimaxity

In this section, we choose an appropriate sequence of priors whose support is the entirety of the parameter space and show that the Bayes risks converge to the constant risk of the best equivariant estimator \( \delta_0 \). By a well-known standard result (see e.g. Lehmann and Casella (1998)), this implies minimaxity of \( \delta_0 \). In order to deal with explicit expressions for minimax estimators as well as for somewhat technical reasons, in this section, we specify the loss functions to be standard choices in the literature. For the location and scale problem, the squared error loss and the entropy loss

\[ L(\delta - \mu) = (\delta - \mu)^2, \quad L(\delta/\sigma) = \delta/\sigma - \log(\delta/\sigma) - 1 \]

are used respectively. For estimation of covariance matrix, the so called Stein’s (1956) loss function given by

\[ L(\Theta \delta \Theta^T) = \text{tr} \Sigma^{-1} \delta - \log |\Sigma^{-1} \delta| - p \]

\[ = \text{tr}(\Theta \delta \Theta^T) - \log |\Theta \delta \Theta^T| - p \]  \hspace{1cm} (3.1)

is used.

3.1. Estimation of location

In this section, we show the minimaxity of \( \hat{\mu}^0 \), the best location equivariant estimator under squared error loss. A point of departure from most proofs
in the literature is that a smooth sequence of Gaussian densities simplifies the proof. It is also easily applied in the multivariate location family (See Remark 3.1).

Recall that the Bayes estimator corresponding to a (generalized) prior \( \pi(\mu) \), under squared error loss, is given by

\[
\delta_\pi(x) = \arg\min_\delta \int_{-\infty}^{\infty} L(\delta - \mu) f(x - \mu) \pi(\mu) d\mu
\]  

(3.2)

\[
= \frac{\int \mu f(x - \mu) \pi(\mu) d\mu}{\int f(x - \mu) \pi(\mu) d\mu} \quad \text{under } L(t) = t^2.
\]  

(3.3)

Hence, by Theorem 2.1, the best equivariant estimator is given by

\[
\hat{\mu}_0(x) = \frac{\int \mu f(x - \mu) d\mu}{\int f(x - \mu) d\mu}.
\]  

(3.4)

**Theorem 3.1.** Let \( X \) have distribution (1.1) and let the loss be given by \( L(\delta - \mu) = (\delta - \mu)^2 \). Then the best equivariant estimator, \( \hat{\mu}_0(x) \), given by (3.4), is minimax, and the minimax constant risk is given by

\[
R_0 = \int L(\hat{\mu}_0(x)) f(x) dx = \int \{\hat{\mu}_0(x)\}^2 f(x) dx.
\]

Under the squared error loss, the Bayes estimator is explicitly written as (3.3), However, in the following proof, the implicit expression (3.2) is mainly used to indicate possible extension for more general loss functions. For the same reason, \( L(\delta(x) - \mu) \) instead of \((\delta(x) - \mu)^2\) is used.

**Proof of Theorem 3.1.** Let

\[
\phi(\mu) = \frac{1}{\sqrt{2\pi}} \exp(-\mu^2/2) \quad \text{and} \quad \phi_k(\mu) = \frac{1}{k} \phi(\mu/k).
\]

The Bayes risk of \( \delta(x) \) under the prior \( \phi_k(\mu) \) is given by

\[
r_k(\phi_k, \delta(x)) = \iint L(\delta(x) - \mu) f(x - \mu) \phi_k(\mu) d\mu dx.
\]

Also the corresponding Bayes estimator is given by

\[
\delta_k^\phi(x) = \arg\min_\delta \int_{-\infty}^{\infty} L(\delta - \mu) f(x - \mu) \phi_k(\mu) d\mu.
\]

Clearly

\[
r_k(\phi_k, \delta_k^\phi) \leq r_k(\phi_k, \hat{\mu}_0) = R_0,
\]
and therefore, to show \( \lim_{k \to \infty} r_k(\phi_k, \delta_k^\phi) = R_0 \), it suffices to prove
\[
\liminf_{k \to \infty} r_k(\phi_k, \delta_k^\phi) \geq R_0.
\]
Making the transformation \( z = x - \mu \) yields
\[
r_k(\phi_k, \delta_k^\phi) = \int \int L(\delta_k^\phi(z + \mu) - \mu) f(z) \phi_k(\mu) d\mu dz
\]
where
\[
\delta_k^\phi(z + \mu) = \arg\min_{\delta} \int_{-\infty}^{\infty} L(\delta - \theta) f(z + \mu - \theta) \phi_k(\theta) d\theta.
\]
Now, make the transformation \( t = \theta - \mu \). We then have
\[
\delta_k^\phi(z + \mu) = \arg\min_{\delta} \int_{-\infty}^{\infty} L(\delta - t) f(z - t) \phi_k(t + \mu) dt
\]
or equivalently
\[
\delta_k^*(z, \mu) := \delta_k^\phi(z + \mu) - \mu = \arg\min_{\delta} \int_{-\infty}^{\infty} L(\delta - t) f(z - t) \phi_k(\mu + t) dt.
\]
Hence, by change of variables, we have
\[
r_k(\phi_k, \delta_k^\phi) = \int \int L(\delta_k^*(z, \mu)) f(z) \phi_k(\mu) d\mu dz
\]
\[
= \int \int L(\delta_k^*(z, k\mu)) f(z) \phi(\mu) d\mu dz.
\]
Note also \( k\phi_k(t + k\mu) = \phi(t/k + \mu) \) and
\[
\delta_k^*(z, k\mu) = \arg\min_{\delta} \int_{-\infty}^{\infty} L(\delta - t) f(z - t) k\phi_k(k\mu + t) dt
\]
\[
= \frac{\int_{-\infty}^{\infty} tf(z - t) \phi_k(t/k + \mu) dt}{\int_{-\infty}^{\infty} f(z - t) \phi_k(t/k + \mu) dt} \quad \text{(for squared error loss } L(t) = t^2).\]
Since \( \lim_{k \to \infty} \phi(t/k + \mu) = \phi(\mu) \) for any \( \mu \), the dominated convergence theorem implies
\[
\lim_{k \to \infty} \delta_k^*(z, k\mu) = \hat{\mu}_0(z) \quad \text{(3.5)}
\]
and hence
\[
\lim_{k \to \infty} L(\delta_k^*(z, k\mu)) = \lim_{k \to \infty} \{\delta_k^*(z, k\mu)\}^2 = \{\hat{\mu}_0(z)\}^2 = L(\hat{\mu}_0(z)). \quad \text{(3.6)}
\]
Hence by Fatou’s lemma, we obtain that
\[
\liminf_{k \to \infty} r_k(\phi_k, \delta_k^\phi) = \liminf_{k \to \infty} \int \int L(\delta_k^\phi(z, k\mu)) f(z) \phi(\mu) \, d\mu \, dz \\
\geq \int \int \liminf_{k \to \infty} L(\delta_k^\phi(z, k\mu)) f(z) \phi(\mu) \, d\mu \, dz \\
= \int \int L(\hat{\mu}_0(z)) f(z) \phi(\mu) \, d\mu \, dz \\
= R_0. 
\]
(3.7)

\[\square\]

Remark 3.1. In the multivariate case, suppose \(x_1, \ldots, x_p \in \mathbb{R}^n\) and \(\{x_1, \ldots, x_p\} \sim f(x_1 - \mu_1, \ldots, x_p - \mu_p)\).

Let \(\mu = (\mu_1, \ldots, \mu_p)^T\). Then the Pitman estimator of \(\mu\), the generalized Bayes estimator with respect to \(\pi(\mu) = 1\), is
\[
\hat{\mu}(x_1, \ldots, x_p) = \frac{\int_{\mathbb{R}^p} \mu f(x_1 - \mu_1, \ldots, x_p - \mu_p) \, d\mu}{\int_{\mathbb{R}^p} f(x_1 - \mu_1, \ldots, x_p - \mu_p) \, d\mu}. 
\]
(3.8)

Using
\[
\pi_k(\mu) = \prod_{i=1}^p \phi_k(\mu_i) = \frac{1}{(2\pi k^2)^{p/2}} \exp \left( - \frac{||\mu||^2}{2k^2} \right) 
\]
as the least favorable sequence of priors gives minimaxity under the quadratic loss \(||\delta - \mu||^2\) of (3.8).

3.2. Estimation of scale

In this section, we show the minimaxity of the scale Pitman estimator under entropy loss given by
\[
L(\delta/\sigma) = \delta/\sigma - \log(\delta/\sigma) - 1. 
\]
(3.9)

Recall that the Bayes estimator corresponding to a (generalized) prior \(\pi(\sigma)\), under entropy loss (3.9), is given by
\[
\delta_\pi(x) = \arg \min_{\delta} \int_{-\infty}^{\infty} L(\delta/\sigma) \sigma^{-n} f(x/\sigma) \pi(\sigma) \, d\sigma 
\]
(3.10)
\[ \int \sigma^{-n} f(x/\sigma) \pi(\sigma) d\sigma \]

\[ \int \sigma^{-n-1} f(x/\sigma) \pi(\sigma) d\sigma. \]

Hence the generalized Bayes estimator under \( \pi(\sigma) = 1/\sigma \), which is best equivariant as shown in Theorem 2.2, is given by

\[ \hat{\sigma}_0(x) = \int \sigma^{-n-1} f(x/\sigma) d\sigma \int \sigma^{-n-2} f(x/\sigma) d\sigma. \]

We have a following minimaxity result.

**Theorem 3.2.** Let \( X \) have distribution (1.4) and let the loss be given by

\[ L(\delta/\sigma) = \delta/\sigma - \log(\delta/\sigma) - 1. \]

Then the best equivariant estimator, \( \hat{\sigma}_0(x) \), given by (3.12), is minimax, and the minimax constant risk is given by

\[ R_0 = \int L(\hat{\sigma}_0(x)) f(x) dx = \int \{\hat{\sigma}_0(x) - \log \hat{\sigma}_0(x) - 1\} f(x) dx. \]

**Proof.** Assume \( \log \sigma \sim N(0, k^2) \) or equivalently

\[ \pi_k(\sigma) = \frac{1}{k} \phi(\log \sigma/k) \frac{1}{\sigma}, \]

where \( \phi(\cdot) \) is the pdf of \( N(0, 1) \). Then the Bayes estimator satisfies

\[ \delta_k^\pi = \delta_k^\pi(x) = \arg \min_{\delta} \int_0^\infty L(\delta/\sigma) \sigma^{-n} f(x/\sigma) \phi_k(\sigma) d\sigma \]

and the Bayes risk is given by

\[ r_k(\pi_k, \delta_k^\pi) = \int \int L(\delta/\sigma) \sigma^{-n} f(x/\sigma) \pi_k(\sigma) d\sigma dx. \]

Clearly

\[ r_k(\pi_k, \delta_k^\pi) \leq r_k(\pi_k, \hat{\sigma}_0(x)) = R_0, \]

and therefore, to show \( \lim_{k \to \infty} r_k(\phi_k, \delta_k^\pi) = R_0 \), it suffices to prove

\[ \lim \inf_{k \to \infty} r_k(\pi_k, \delta_k^\pi) \geq R_0. \]

Making the transformation \( l = x/\sigma \) yields

\[ r_k(\pi_k, \delta_k^\pi) = \int \int L(\delta_k^\pi(\sigma l)/\sigma) f(l) \pi_k(\sigma) d\sigma dl. \]
where
\[ \delta_k^\pi(\sigma l) = \arg\min_\delta \int_0^\infty L(\delta/z)z^{-n} f(\sigma l/z)\pi_k(z)dz. \]

Now, make the transformation \( y = z/\sigma \). We then have
\[ \delta_k^\pi(\sigma l) = \arg\min_\delta \int_0^\infty L(\delta/(y\sigma))y^{-n} f(l/y)\pi_k(\sigma y)dy \]
or equivalently
\[ \delta_k^\ast(l, \sigma) := \frac{\delta_k^\pi(\sigma l)}{\sigma} = \arg\min_\delta \int_0^\infty L(\delta/y)y^{-n} f(l/y)\pi_k(\sigma y)dy. \]

Hence
\[ r_k(\pi_k, \delta_k^\pi) = \iint L(\delta_k^\ast(l, \sigma)) f(l)\pi_k(\sigma) d\sigma dl \]
\[ = \iint L(\delta_k^\ast(l, \eta_k)) f(l)\pi_1(\eta) d\eta dl \]
where \( \sigma = \eta^k \) and \( \delta_k^\ast(l, \eta^k) \) is explicitly given as (when the loss is (3.9))
\[
\delta_k^\ast(l, \eta^k) = \frac{\int y^{-n} f(l/y)\pi_k(\eta^k y)dy}{\int y^{-n-1} f(l/y)\pi_k(\eta^k y)dy}, \quad (3.13)
\]
Note
\[ k\pi_k(\eta^k y) = \frac{1}{y^{k^2} \sqrt{2\pi}} \exp \left( - \frac{(\log \eta^k + \log y)^2}{2k^2} \right). \]

Since
\[ \lim_{k \to \infty} k\eta^k \pi_k(\eta^k y) = \frac{1}{y} \phi(\log \eta) \]
for any \( \eta \), the dominated convergence theorem implies
\[ \lim_{k \to \infty} \delta_k^\ast(l, \eta^k) = \hat{\sigma}_0(l). \quad (3.14) \]

Also the continuity of \( L(\cdot) \) implies
\[ \lim_{k \to \infty} L(\delta_k^\ast(l, \eta^k)) = L(\hat{\sigma}_0(l)). \quad (3.15) \]
Hence by Fatou’s lemma, we obtain that
\[
\liminf_{k \to \infty} r_k(\pi_k, \delta_k^{\epsilon}) = \liminf_{k \to \infty} \int \int L(\delta_k^*(l, \eta^k)) f(l) \pi_1(\eta) d\eta dl \\
\geq \int \int \liminf_{k \to \infty} L(\delta_k^*(l, \eta^k)) f(l) \pi_1(\eta) d\eta dl \\
\geq \int \int L(\tilde{\sigma}_0(l)) f(l) \pi_1(\eta) d\eta dl \\
= R_0.
\]

Remark 3.2. In the same way, we can consider the estimation of \(\sigma^c\) with \(c \in \mathbb{R}\) and propose the corresponding result,
\[
\hat{\sigma}_{0c}(x) = \frac{\int \sigma^{-n-1+c} f(x/\sigma) d\sigma}{\int \sigma^{-n-2+c} f(x/\sigma) d\sigma}
\]
is minimax and best equivariant for estimating \(\sigma^c\) under entropy loss
\[
L(\delta/\sigma^c) = \delta/\sigma^c - \log(\delta/\sigma^c) - 1.
\]

3.3. Estimation of covariance matrix

As we mentioned in the beginning of this section, we use the so called Stein’s (1956) loss function given by
\[
L(\Theta \delta \Theta^T) = \text{tr} \Sigma^{-1} \delta - \log |\Sigma^{-1} \delta| - p \\
= \text{tr}(\Theta \delta \Theta^T) - \log |\Theta \delta \Theta^T| - p.
\]

James and Stein (1961), in their Section 5, show that the best equivariant estimator is given by
\[
\hat{\Sigma}_0 = T \text{diag}(d_1, \ldots, d_p) T^T
\]
where \(T \in \mathcal{T}^+\) is from the Cholesky decomposition of \(V = TT^T\) and \(d_i = 1/(n + p - 2i + 1)\) for \(i = 1, \ldots, p\). As demonstrated in the literature, by e.g. Tsukuma and Kubokawa (2015), the best equivariant estimator under the loss (3.17) may also be shown to be \(\hat{\Sigma}_0\) by using the generalized Bayes representation given in Theorem 2.3 since
\[
\arg \min_{\delta} \int_{\mathcal{T}^+} L(\Theta \delta \Theta^T) f_W(T|\Theta) \gamma(d\Theta)
\]
\[
\arg\min_{\delta} \int_{T^+} \{\text{tr}(\Theta^T\Theta\delta) - \log|\delta|\} f_W(T|\Theta)\gamma(d\Theta) \\
= \left( \int_{T^+} \Theta^T\Theta f_W(T|\Theta)\gamma(d\Theta) \right)^{-1} \int_{T^+} f_W(T|\Theta)\gamma(d\Theta) \\
= \left( \int_{T^+} \Theta^T\Theta f_W(T|\Theta)\prod \theta_{ii}^{p-2i+1}\nu(d\Theta) \right)^{-1} \times \int_{T^+} f_W(T|\Theta)\prod \theta_{ii}^{p-2i+1}\nu(d\Theta) \\
= \left( \int_{T^+} (ZT^{-1})^TZT^{-1} f_W(Z|I)\prod (t_{ii}^{-1}z_{ii}^{p-2i+1})\nu(dZ) \right)^{-1} \times \int_{T^+} f_W(Z|I)\prod (t_{ii}^{-1}z_{ii}^{p-2i+1})\nu(dZ) \\
= T \left( \int_{T^+} Z^TZ f_W(Z|I)\prod z_{ii}^{p-2i+1}\nu(dZ) \right)^{-1} T^T \times \int_{T^+} f_W(Z|I)\prod z_{ii}^{p-2i+1}\nu(dZ) \\
= \hat{\Sigma}_0 \\
\]

where
\[
\left( \int_{T^+} Z^TZ f_W(Z|I)\prod z_{ii}^{p-2i+1}\nu(dZ) \right)^{-1} \int_{T^+} f_W(Z|I)\prod z_{ii}^{p-2i+1}\nu(dZ) = \text{diag}(d_1, \ldots, d_p) \text{ with } d_i = 1/(n+p-2i+1).
\]

Note that the group of \(p \times p\) lower triangular matrices with positive diagonal entries is solvable, and the result of Kiefer (1957) implies the minimaxity of \(\hat{\Sigma}_0\). Tsukuma and Kubokawa (2015) gives as a sequence of least favorable priors, a sequence of invariant priors truncated on an expanding set.

In this section, we choose an appropriate sequence of Gaussian priors whose support is the entirety of the parameter space and show that the Bayes risks converge to the constant risk of \(\hat{\Sigma}_0\). This implies that \(\hat{\Sigma}_0\) is minimax.

As a new parameterization on \(\Theta\), let
\[
\xi_{ii} = \log \theta_{ii} \text{ for } i = 1, \ldots, p, \\
\xi_{ij} = \frac{\theta_{ij}}{\theta_{ii}}, \text{ for } 1 \leq j < i \leq p
\]
and let
\[
\xi = (\xi_{11}, \xi_{21}, \xi_{22}, \ldots, \xi_{p1}, \ldots, \xi_{pp})^T \in \mathbb{R}^{p(p+1)/2}.
\]
The prior on $\xi_{ij}$ is

$$\xi_{ij} \sim N(0, k_{ij}^2) \text{ for } 1 \leq j \leq i \leq p,$$

and $\xi_{11}, \xi_{21}, \xi_{22}, \ldots, \xi_{p1}, \ldots, \xi_{pp}$ are assumed mutually independent. Equivalently the density is

$$\pi_k(\xi)d\xi = \prod_{j \leq i} k_{ij}^{-1} \phi(\xi_{ij}/k_{ij})d\xi$$

(3.21)

where $\phi(t) = (2\pi)^{-1/2} \exp(-t^2/2)$. Set

$$k_{ii} = k, \quad k_{ij} = k^{(i-j)k} \text{ for } i > j$$

(3.22)

with $k \to \infty$, although, in the following, we keep the notation $k_{ii}$ and $k_{ij}$.

By (3.19) and (3.21), we have

$$\pi_k(\Theta)d\Theta = \prod_{i=1}^p \left\{ \frac{1}{k_{ii}} \phi(\log \theta_{ii}/k_{ii}) \frac{1}{\theta_{ii}} \right\} \prod_{j<i} \left\{ \frac{1}{k_{ij}\theta_{ii}} \phi(\theta_{ij}/\{\theta_{ii}k_{ij}\}) \right\} d\Theta.$$

(3.23)

The prior distributions yield the Bayes estimators

$$\delta_k = \delta_k^\pi(T) = \arg \min_{\delta} \int_{Z \in T^+} L(Z\delta Z^T)f_W(T|Z)\pi_k(Z)dZ$$

with Bayes risks

$$r_k(\pi_k, \delta_k) = \int\int L(\theta\delta_k^\pi(T)\Theta^T)f_W(T|\Theta)\gamma(dT)\pi_k(\Theta)d\Theta.$$  

(3.24)

**Theorem 3.3.** Let $V = TT^T$ have distribution $W_p(n, \Sigma)$ and let the loss be given by (3.17). Then the best equivariant estimator, $\hat{\Sigma}_0(T)$, given by (3.18), is minimax, and the minimax constant risk is given by

$$R_0 = \int L(\hat{\Sigma}_0(T))f_W(T|I)\gamma(dT).$$

**Proof.** We show this theorem along the same lines as in Kubokawa (2004) and Tsukuma and Kubokawa (2015) who modified the method of Girshick and Savage (1951).

Clearly

$$r_k(\pi_k, \delta_k^\pi) \leq r_k(\pi_k, \hat{\Sigma}_0) = R_0,$$
and therefore, to show \( \lim_{k \to \infty} r_k(\phi_k, \delta^\phi_k) = R_0 \), it suffices to prove
\[
\lim_{k \to \infty} \inf r_k(\pi_k, \delta^n_k) \geq R_0.
\]
In (3.24), making the transformation \( L = \Theta T \) yields
\[
r_k(\pi_k, \delta^n_k) = \int \int L(\Theta \delta^n_k(\Theta^{-1}L)\Theta^T) f_W(L|I_p) \gamma(dL) \pi_k(\Theta) d\Theta
\]
where
\[
\delta^n_k(\Theta^{-1}L) = \arg \min_{\delta} \int_{Z \in T^+} L(Z\delta Z^T) f_W(L|Z\Theta^{-1}) \pi_k(Z) dZ.
\]
Now, make the transformation \( Y = Z\Theta^{-1} \) with \( dZ = (\prod_{i=1}^p \theta_{ii}^{p_i-1})dY \).
We then have
\[
\delta^n_k(\Theta^{-1}L) = \arg \min_{\delta} \int_{Y \in T^+} L(Y\Theta\delta Y^T) f_W(L|Y) \pi_k(Y\Theta) dY
\]
namely,
\[
\Theta \delta^n_k(\Theta^{-1}L)\Theta^T = \delta^*_k(L|\Theta)
\]
where
\[
\delta^*_k(L|\Theta) = \arg \min_{\delta} \int_{Y \in T^+} L(Y\delta Y^T) f_W(L|Y) \pi_k(Y\Theta) dY.
\]
Hence, the Bayes risk (3.25) can be rewritten as
\[
r_k(\pi_k, \delta^*_k) = \int \int L(\delta^*_k(L|\Theta)) f_W(L|I_p) \gamma(dL) \pi_k(\Theta) d\Theta
\]
\[
= \int \int L(\delta^*_k(L|\Theta(\xi))) f_W(L|I_p) \gamma(dL) \bar{\pi}_k(\xi) d\xi,
\]
where \( \Theta(\xi) \) is from \( \xi_{ii} = \log \theta_{ii} \) for \( i = 1, \ldots, p \) and \( \xi_{ij} = \theta_{ij}/\theta_{ii} \) for \( 1 \leq j < i \leq p \). Then we have
\[
r_k(\pi_k, \delta^*_k) = \int \int L(\delta^*_k(L|\Theta(\xi))) f_W(L|I_p) \gamma(dL) \bar{\pi}_k(\xi) d\xi
\]
\[
= \int \int L(\delta^*_k(L|\Theta(k\omega))) f_W(L|I_p) \gamma(dL) \bar{\pi}_1(\omega) d\omega,
\]
where, for notational convenience,
\[ k \cdot \omega = (k_{11}\omega_{11}, k_{21}\omega_{21}, k_{22}\omega_{22}, \ldots, k_{pp}\omega_{pp}) \]
\[ \Theta(k \cdot \omega)_{ii} = \exp(k_{ii}\omega_{ii}) \text{ for } i = 1, \ldots, p, \]
\[ \Theta(k \cdot \omega)_{ij} = k_{ij}\omega_{ij} \exp(k_{ii}\omega_{ii}) \text{ for } 1 \leq j < i \leq p. \]

By Lemma 3.1 below, we have
\[ \lim_{k \to \infty} \delta_k^*(L|\Theta(k \cdot \omega)) = \hat{\Sigma}_0(L) \] (3.27)
and by the continuity of \( L(\cdot) \),
\[ \lim_{k \to \infty} L(\delta_k^*(L|\Theta(k \cdot \omega))) = L(\hat{\Sigma}_0(L)). \] (3.28)

Also, by Fatou’s lemma, we have
\[ \liminf_{k \to \infty} r_k(\pi_k, \delta_k^*) \]
\[ \geq \liminf_{k \to \infty} \int \int_{L \in \mathcal{T}^+} L(\delta_k^*(L|\Theta(k \cdot \omega))) f_W(L|I_p) \gamma(dL) \pi_1(\omega) d\omega \]
\[ \geq \int \int_{L \in \mathcal{T}^+} \liminf_{k \to \infty} L(\delta_k^*(L|\Theta(k \cdot \omega))) f_W(L|I_p) \gamma(dL) \pi_1(\omega) d\omega \]
\[ = \int \int_{L \in \mathcal{T}^+} L(\hat{\Sigma}_0(L)) f_W(L|I_p) \gamma(dL) \pi_1(\omega) d\omega \]
\[ = \int \pi_1(\omega) d\omega \int \int_{L \in \mathcal{T}^+} L(\hat{\Sigma}_0(L)) f_W(L|I_p) \gamma(dL) \]
\[ = R_0. \]

Lemma 3.1.
\[ \lim_{k \to \infty} \delta_k^*(L|\Theta(k \cdot \omega)) = \hat{\Sigma}_0(L). \]

Proof. Recall
\[ \delta_k^*(L|\Theta) = \arg\min_{\delta} \int_{Y \in \mathcal{T}^+} L(Y \delta Y^T) f_W(L|Y) \pi_k(Y \Theta) dY \]
\[ = \left( \int_{Y \in \mathcal{T}^+} Y^T Y f_W(L|Y) \pi_k(Y \Theta) dY \right)^{-1} \]
\[ \times \int_{Y \in \mathcal{T}^+} f_W(L|Y) \pi_k(Y \Theta) dY \] (3.29)
and where

$$\pi_k(\Theta) = \prod_{i=1}^{p} \left\{ \frac{1}{k_{ii}} \phi(\log \theta_{ii}/k_{ii}) \right\} \prod_{j<i} \left\{ \frac{1}{k_{ij} \theta_{ii}} \phi(\theta_{ij}/\{\theta_{ii} k_{ij}\}) \right\}.$$ 

Consider \(\pi_k(Y \Theta(k \cdot \omega))\) in the following. The \((i,i)\) diagonal component of \(Y \Theta(k \cdot \omega)\) with \(Y \in T^+\) is

$$y_{ii} \exp(k_{ii}\omega_{ii})$$

and the non-diagonal \((i,j)\) component is

$$y_{ij} \exp(k_{jj}\omega_{jj}) + \sum_{l=j+1}^{i-1} y_{il} \exp(k_{ll}\omega_{ll}) k_{ij} + y_{ii} k_{ij} \exp(k_{ii}\omega_{ii}).$$

Then, for \(i > j\),

$$\frac{1}{k_{ij}} \frac{(Y \Theta(k \cdot \omega))_{ij}}{(Y \Theta(k \cdot \omega))_{ii}} = \frac{y_{ij} \exp(k_{jj}\omega_{jj} - k_{ii}\omega_{ii})}{y_{ii} k_{ij}} + \omega_{ij} + \sum_{l=j+1}^{i-1} w_{lj} \frac{k_{lj}}{y_{ii}} \exp(k_{ll} \omega_{ll} - k_{ii} \omega_{ii}).$$

(3.30)

Recall we set

$$k_{ii} = k, \quad k_{ij} = k_{i-(j)}.$$ 

Then (3.30) is equal to

$$\frac{1}{k_{ij}} \frac{(Y \Theta(k \cdot \omega))_{ij}}{(Y \Theta(k \cdot \omega))_{ii}} = \frac{y_{ij}}{y_{ii}} \left( \frac{\exp(\omega_{jj} - \omega_{ii})}{k^{i-j}} \right) + \omega_{ij} + \sum_{l=j+1}^{i-1} w_{lj} \frac{y_{il}}{y_{ii}} \left( \frac{\exp(\omega_{ll} - \omega_{ii})}{k^{i-l}} \right)^k$$

and hence it follows that

$$\lim_{k \to \infty} \frac{1}{k_{ij}} \frac{(Y \Theta(k \cdot \omega))_{ij}}{(Y \Theta(k \cdot \omega))_{ii}} = \omega_{ij}.$$ 

Similarly we have

$$\lim_{k \to \infty} \frac{1}{k_{ii}} \log(Y \Theta(k \cdot \omega))_{ii} = \lim_{k \to \infty} \frac{\log y_{ii} + k_{ii} \omega_{ii}}{k_{ii}} = \omega_{ii}.$$
Therefore
\[ \lim_{k \to \infty} \prod_{i \geq j} k_{ij} \prod_{i=1}^{p} \exp(ik_{ii} \omega_{ii}) \pi_{k}(Y \Theta(k \bullet \omega)) = \frac{\prod_{j \geq j} \phi(\omega_{ij})}{\prod_{i=1}^{p} y_{ii}}, \] (3.31)

and, by the dominated convergence theorem,
\[ \delta^*_{k}(L|\Theta(k \bullet \omega)) = \arg \min_{\delta} \int_{Y \in T^+} L(Y \delta Y^T) f_{W}(L|Y) \pi_{k}(Y \Theta(k \bullet \omega)) dY \]
\[ = \left( \int_{Y \in T^+} Y^T Y f_{W}(L|Y) \pi_{k}(Y \Theta(k \bullet \omega)) dY \right)^{-1} \]
\[ \times \int_{Y \in T^+} f_{W}(L|Y) \pi_{k}(Y \Theta(k \bullet \omega)) dY \]
\[ \rightarrow \left( \int_{Y \in T^+} Y^T Y f_{W}(L|Y) \gamma(dY) \right)^{-1} \]
\[ \times \int_{Y \in T^+} f_{W}(L|Y) \gamma(dY) dY \]
\[ = \hat{\Sigma}_0(L). \]

4. Concluding remarks

We have reviewed some known results on establishing minimaxity of best equivariant procedures. While none of the results established are new, the proofs of minimaxity are somewhat divergent from the typical minimaxity proofs in the literature in that the least favorable sequence is smooth and strictly positive on the support of the approximated right invariant measure: it is not a sequence of truncated versions of the invariant prior on expanding sets. In this sense, our proofs are in the same spirit as the common textbook proof of minimaxity of the mean of a normal distribution. In fact the same sequence of priors that works in the normal case is shown to work in the general location case. Hence the present method provides a degree of unification and simultaneously simplifies the proofs.

We note that the Gaussian kernel is not necessary and could be replaced by a bounded, continuous, positive density.

Our choices of particular loss featured in each of the problems also simplified the analyses, in the sense that, in each case, the form of the Bayes estimate could be explicitly given. This facilitated the use of Fatou’s Lemma.
in establishing the limiting Bayes risk as being equal to the constant risk of the best equivariant estimator.

An approach for more general loss function could be constructed by requiring that all Bayes estimators (for priors with full support) be unique, and that the loss is sufficiently smooth that statements such as (3.5) hold in each problem.

It is also worth noting that, as in Tsukuma and Kubokawa (2015), in the problem of estimating a covariance matrix, a specific least favorable sequence of priors is established.

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