Measures and all that — A Tutorial

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Abstract
This tutorial gives an overview of some of the basic techniques of measure theory. It includes a study of Borel sets and their generators for Polish and for analytic spaces, the weak topology on the space of all finite positive measures including its metrics, as well as measurable selections. Integration is covered, and product measures are introduced, both for finite and for arbitrary factors, with an application to projective systems. Finally, the duals of the Lp-spaces are discussed, together with the Radon-Nikodym Theorem and the Riesz Representation Theorem. Case studies include applications to stochastic Kripke models, to bisimulations, and to quotients for transition kernels.
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1 Overview

Markov transition systems are based on transition probabilities on a measurable space. This is a generalization of discrete spaces, where certain sets are declared to be measurable. So, in contrast to assuming that we know the probability for the transition between two states, we have to model the probability of a transition going from one state to a set of states: point-to-point probabilities are no longer available due to working in a comparatively large space. Measurable spaces are the domains of the probabilities involved. This approach has the advantage of being more general than finite or countable spaces, but now one deals with a fairly involved mathematical structure; all of a sudden the dictionary has to be extended with words like “universally measurable” or “sub-$\sigma$-algebra”. Measure theory becomes an area where one has to find answers to questions which did not appear to be particularly involved before, in the much simpler world of discrete measures (the impression should not arise that this author thinks that discrete measures are kiddie stuff, they are sometimes difficult enough to handle. The continuous case, as it is called sometimes, offers questions which simply do not arise in the discrete context). Many arguments in this area are of a measure theoretic nature; this tutorial makes an attempt to introduce the necessary tools and techniques.

It starts off with a discussion of $\sigma$-algebras, which have already been met in [Dob13] Section 1.6\footnote{[Dob13] and [Dob14] are other tutorials in this series. This is not an installment of mysteries, so one can be read quite independently from the others.}. We look at the structure of $\sigma$-algebras, in particular at its generators; it turns out that the underlying space has something to say about it. In particular we will deal with Polish spaces and their brethren. Two aspects deserve to be singled out. The $\sigma$-algebra on the base space determines a $\sigma$-algebra on the space of all finite measures, and, if this space has a topology, it determines also a topology, the Alexandrov topology. These constructions are studied, since they also affect the applications in logic, and in transition systems, in which measures are vital. Second, we show that we can construct measurable selections, which then enable constructions which are interesting from a categorical point of view [Dob14, 2.4.2, 2.6.1, 2.7.3].

After having laid the groundwork with a discussion of $\sigma$-algebras as the domains of measures, we show that the integral of a measurable function can be constructed through an approximation process, very much in the tradition of the Riemann integral, but with a larger scope. We also go the other way: given an integral, we construct a measure from it. This is the elegant way P. J. Daniell did propose for constructing measures, and it can be brought to fruit in this context for a fairly simple proof of the Riesz Representation Theorem on compact metric spaces.

Having all these tools at our disposal, we look at product measures, which can be introduced now through a kind of line sweeping — if you want to measure an area in the plane, you measure the line length as you sweep over the area; this produces a function of the abscissa, which then yields the area. One of the main tools here is Fubini’s Theorem. The product measure is not confined to two factors, we discuss the general case. This includes a discussion of projective systems, which may be considered as a generalization of sequences of products. A case study shows that projective systems arise easily in the study of continuous time stochastic logics.
Now that integrals are available, we turn back and have a look at topologies on spaces of measures; one suggests itself — the weak topology which is induced by the continuous functions. This is related to the Alexandrov topology. It is shown that there is a particularly handy metric for the weak topology, and that the space of all finite measures is complete with this metric, so that we now have a Polish space. This is capitalized on when discussing selections for set-valued maps into it, which are helpful in showing that Polish spaces are closed under bisimulations. We use measurable selections for an investigation into the structure of quotients in the Kleisli monad, providing another example for the interplay of arguments from measure theory and categories.

Finally, we take up a true classic: $L_p$-spaces. We start from Hilbert spaces, apply the representation of linear functionals on $L_2$ to obtain the Radon-Nikodym Theorem through von Neumann’s ingenious proof, and derive from it the representation of the dual spaces. This is applied to disintegration, where we show that a measure on a product can be decomposed into a projection and a transition kernel (on the surface this does not look like an application area for $L_p$-spaces; the relationship derives from the Radon-Nikodym Theorem).

Because we are driven by applications to Markov transition systems and similar objects, we did not strive for the most general approach to measure and integral. In particular, we usually formulate the results for finite or $\sigma$-finite measures, leaving the more general cases outside of our focus. This means also that we did not deal with complex measures (and the associated linear spaces over the complex numbers), but things are discussed in the realm of real numbers; we show, however, in which way one could start to deal with complex measures when the occasion arises. Of course, a lot of things had to be left out, among them a more careful study of the Borel hierarchy and applications to Descriptive Set Theory, as well as martingales.

\section{Measurable Sets and Functions}

This section contains a systematic study of measurable spaces and measurable functions with a view towards later developments. A brief overview is in order.

The measurable structure is lifted to the space of finite measures, which form a measurable set under the weak-*-$\sigma$-algebra. This is studied in Section 2.1.1. If the underlying space carries a topology, the topological structure is handed down to finite measures through the Alexandrov topology. We will have a look at it in Section 2.1.2. The measurable functions from a measurable space to the reals form a vector space, which is also a lattice, and we will show that the step functions, i.e., those functions which take only finite number of values, are dense with respect to pointwise convergence. This mode of convergence is relaxed in the presence of a measure in various ways to almost uniform convergence, convergence almost everywhere, and to convergence in measure (Sections 2.2.1 and 2.2.2), from which also various (pseudo-) metrics and norms may be derived.

If the underlying measurable spaces are the Borel sets of a metric space, and if the metric has a countable dense set, then the Borel sets are countably generated as well. But the irritating observation is that being countably generated is not hereditary — a sub-$\sigma$-algebra of a countable $\sigma$-algebra needs not be countably generated. So countably generated $\sigma$-algebras deserve
a separate look, which is what we will do in Section 2.3. The very important class of Polish spaces will be studied in this context as well, and we will show to manipulate a Polish topology into making certain measurable functions continuous. Polish spaces generalize to analytic spaces in a most natural manner, for example when taking the factor of a countably generated equivalence relation in a Polish space; we will study the relationship in Section 2.3.1. The most important tool here is Souslin’s Separation Theorem. This discussion leads quickly to a discussion of the abstract Souslin operation in Section 2.5 through which analytic sets may be generated in a Polish space. From there it is but a small step to introducing universally measurable sets in Section 2.6 which turn out to be closed under Souslin’s operation in general measurable spaces. Two applications of these techniques are given: Lubin’s Theorem extends a measure from a countably generated sub-$\sigma$-algebra of the Borel sets of an analytic space to the Borel sets proper, and we show that a transition kernel can be extended to the universal completion (Sections 2.6.1 and 2.6.2). Lubin’s Theorem is established through von Neumann’s Selection Theorem, which provides a universally measurable right inverse to a surjective measurable map from an analytic space to a separable measurable space. The topic of selections is taken up in Section 2.7, where the selection theorem due to Kuratowski and Ryll-Nardzewski is in the center of attention. It gives conditions under which a map which takes values in the closed non-empty subsets of a Polish space has a measurable selector. This is of interest, e.g., when it comes to establish the existence of bisimulations for Markov transition systems, or for identifying the quotient structure of transition kernels.

2.1 Measurable Sets

Recall that a measurable space $(X, \mathcal{A})$ consists of a set $X$ with a $\sigma$-algebra $\mathcal{A}$, which is an Boolean algebra of subsets of $X$ that is closed under countable unions (hence countable intersections or countable disjoint unions). If $\mathcal{A}_0$ is a family of subsets of $X$, then

$$\sigma (\mathcal{A}_0) = \bigcap \{\mathcal{B} \mid \mathcal{B} \text{ is a } \sigma \text{-algebra on } M \text{ with } \mathcal{A}_0 \subseteq \mathcal{A}\}$$

is the smallest $\sigma$-algebra on $M$ which contains $\mathcal{A}_0$. This construction works since the power set $\mathcal{P}(X)$ is a $\sigma$-algebra on $X$. Take for example as a generator $\mathcal{I}$ all open intervals in the real numbers $\mathbb{R}$, then $\sigma(\mathcal{I})$ is the $\sigma$-algebra of real Borel sets. These Borel sets are denoted by $\mathcal{B}(\mathbb{R})$, and, since each open subset of $\mathbb{R}$ can be represented as a countable union of open intervals, $\mathcal{B}(\mathbb{R})$ is the smallest $\sigma$-algebra which contains the open sets of $\mathbb{R}$. Unless otherwise stated, the real numbers are equipped with the $\sigma$-algebra $\mathcal{B}(\mathbb{R})$.

In general, if $(X, \tau)$ is a topological space, the $\sigma$-algebra $\mathcal{B}(\tau) := \sigma(\tau)$ are called its Borel sets. They will be discussed extensively in the context of Polish spaces. This is, however, not the only $\sigma$-algebra of interest on a topological space.

**Example 2.1** Call $F \subseteq X$ functionally closed iff $F = f^{-1}([0])$ for some continuous function $f : X \to \mathbb{R}$, $G \subseteq X$ is called functionally open iff $G = X \setminus F$ with $F$ functionally closed. The Baire sets $\mathcal{B}(X)$ of $(X, \tau)$ are the $\sigma$-algebra generated by the functionally closed sets of the space.

If $(X, d)$ is a metric space, let $F \subseteq X$ be closed, then

$$d(x, F) := \inf \{d(x, y) \mid y \in F\}$$
is the distance of $x$ to $F$ with $x \in F$ iff $d(x, F) = 0$. Moreover, $d(\cdot, F)$ is continuous, thus $F = d(\cdot, F)^{-1}[\{0\}]$ is functionally closed, hence the Baire and the Borel sets coincide for metric spaces.

Note that $|d(x, F) - d(y, F)| \geq d(x, y)$, so that $d(\cdot, F)$ is even uniformly continuous. 🏁

The next example constructs a $\sigma$-algebra which comes up quite naturally in the study of stochastic nondeterminism.

**Example 2.2** Let $A \subseteq \mathcal{P}(X)$ for some set $X$, the family of hit sets, and $G$ a distinguished subsets of $\mathcal{P}(X)$. Define the hit-$\sigma$-algebra $\mathcal{H}_A(G)$ as the smallest $\sigma$-algebra on $G$ which contains all the sets $H_A$ with $A \in A$, where $H_A$ is the hit set associated with $A$, i.e., $H_A := \{B \in G \mid B \cap A \neq \emptyset\}$. 🏁

Rather than working with a closure operation $\sigma(\cdot)$, one sometimes can adjoin additional elements to obtain a $\sigma$-algebra from a given one, see also Exercise 5. This is demonstrated for a $\sigma$-ideal through the following construction, which will be helpful when completing a measure space. Recall that $\mathcal{N} \subseteq \mathcal{P}(X)$ is a $\sigma$-ideal iff it is an order ideal which is closed under countable unions [Dob13, Definition 2.91].

**Lemma 2.3** Let $A$ be a $\sigma$-algebra on a set $X$, $\mathcal{N} \subseteq \mathcal{P}(X)$ a $\sigma$-ideal. Then

$$
\mathcal{A}_\mathcal{N} := \{A \Delta N \mid A \in \mathcal{A}, N \in \mathcal{N}\}
$$

is the smallest $\sigma$-algebra containing both $A$ and $\mathcal{N}$.

**Proof** Is is sufficient to demonstrate that $\mathcal{A}_\mathcal{N}$ is a $\sigma$-algebra. Because

$$
X \setminus (A \Delta N) = X \Delta (A \Delta N) = (X \Delta A) \Delta N = (X \setminus A) \Delta N,
$$

we see that $\mathcal{A}_\mathcal{N}$ is closed under complementation. Now let $(A_n \Delta N_n)_{n \in \mathbb{N}}$ be a sequence of sets with $(A_n)_{n \in \mathbb{N}}$ in $A$ and $(N_n)_{n \in \mathbb{N}}$ in $\mathcal{N}$, we have

$$
\bigcup_{n \in \mathbb{N}} (A_n \Delta N_n) = (\bigcup_{n \in \mathbb{N}} A_n) \Delta \bigcup_{n \in \mathbb{N}} N_n
$$

with

$$
N = \bigcup_{n \in \mathbb{N}} (A_n \Delta N_n) \Delta (\bigcup_{n \in \mathbb{N}} A_n) \subseteq \bigcup_{n \in \mathbb{N}} (A_n \Delta N_n) \Delta A_n
$$

$$
\subseteq \bigcup_{n \in \mathbb{N}} N_n,
$$

using Exercise 10 in (†). Because $\mathcal{N}$ is a $\sigma$-ideal, we conclude that $N \in \mathcal{N}$. Thus $\mathcal{A}_\mathcal{N}$ is also closed under countable unions. Since $\emptyset, X \in \mathcal{A}_\mathcal{N}$, we conclude that this set is a $\sigma$-algebra indeed. ⬅️

It turns out to be most convenient to have a closer look at the construction of $\sigma$-algebras when the family of sets we start from has already some structure. This gives the occasion to introduce Dynkin’s $\pi$-$\lambda$-Theorem. This is an important tool, which eases sometimes the task of identifying the $\sigma$-algebra generated from some family of sets.

**Theorem 2.4** (π-$\lambda$-Theorem) Let $\mathcal{P}$ be a family of subsets of $S$ that is closed under finite intersections (this is called a $\pi$-class). Then $\sigma(\mathcal{P})$ is the smallest $\lambda$-class containing $\mathcal{P}$, where a family $\mathcal{L}$ of subsets of $S$ is called a $\lambda$-class iff it is closed under complements and countable disjoint unions.
Proof 1. Let \( L \) be the smallest \( \lambda \)-class containing \( P \), then we show that \( L \) is a \( \sigma \)-algebra.

2. We show first that it is an algebra. Being a \( \lambda \)-class, \( L \) is closed under complementation. Let \( A \subseteq S \), then \( L \) is closed under complementation.

\[
A \cap (S \setminus B) = A \setminus B = S \setminus ((A \cap B) \cup (S \setminus A)),
\]

which is in \( L \), since \((A \cap B) \cap S \setminus A = \emptyset\), and since \( L \) is closed under disjoint unions.

If \( A \in P \), then \( P \subseteq L \), because \( P \) is closed under intersections.

3. \( L \) is a \( \sigma \)-algebra as well. It is enough to show that \( L \) is closed under countable unions. But since

\[
\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \left( A_n \setminus \bigcup_{i=1}^{n-1} A_i \right),
\]

this follows immediately. \( \dashv \)

Consider an immediate and fairly typical application. It states that two finite measures are equal on a \( \sigma \)-algebra, provided they are equal on a generator which is closed under finite intersections. The proof technique is worth noting: We collect all sets for which the assertion holds into one family of sets and investigate its properties, starting from an originally given set. If we find that the family has the desired property, then we look at the corresponding closure. To be specific, have a look at the proof of the following statement.

**Lemma 2.5** Let \( \mu, \nu \) be finite measures on a \( \sigma \)-algebra \( \sigma(B) \), where \( B \) is a family of sets which is closed under finite intersections. Then \( \mu(A) = \nu(A) \) for all \( A \in \sigma(B) \), provided \( \mu(B) = \nu(B) \) for all \( B \in B \).

**Proof** We have a look at all sets for which the assertion is true, and investigate this set. Put

\[
\mathcal{G} := \{ A \in \sigma(B) \mid \mu(A) = \nu(A) \},
\]

then \( \mathcal{G} \) has these properties:

- \( B \subseteq \mathcal{G} \) by assumption.
- Since \( B \) is closed under finite intersections, \( S \in B \subseteq \mathcal{G} \).
- \( \mathcal{G} \) is closed under complements.
- \( \mathcal{G} \) is closed under countable disjoint unions; in fact, let \((A_n)_{n \in \mathbb{N}}\) be a sequence of mutually disjoint sets in \( \mathcal{G} \) and \( A := \bigcup_{n \in \mathbb{N}} A_n \), then

\[
\mu(A) = \sum_{n \in \mathbb{N}} \mu(A_n) = \sum_{n \in \mathbb{N}} \nu(A_n) = \nu(A),
\]

hence \( A \in \mathcal{G} \).
But this means that $\mathcal{G}$ is a $\lambda$-class containing $\mathcal{B}$. But the smallest $\lambda$-class containing $\mathcal{G}$ is $\sigma(\mathcal{B})$ by Theorem 2.4, so that we have now
\[
\sigma(\mathcal{B}) \subseteq \mathcal{G} \subseteq \sigma(\mathcal{B}),
\]
the last inclusion coming from the definition of $\mathcal{G}$. Thus we may conclude that $\mathcal{G} = \sigma(\mathcal{B})$, hence all sets in $\sigma(\mathcal{B})$ have the desired property. ⊥

If $(Y, \mathcal{B})$ is another measurable space, then a map $f : X \to Y$ is called $\mathcal{A}$-$\mathcal{B}$-measurable iff the inverse image under $f$ of each set in $\mathcal{B}$ is a member of $\mathcal{A}$, hence iff $f^{-1}[G] \in \mathcal{A}$ holds for all $G \in \mathcal{B}$.

Checking measurability is made easier by the observation that it suffices for the inverse images of a generator to be measurable sets.

**Lemma 2.6** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, and assume that $\mathcal{B} = \sigma(\mathcal{B}_0)$ is generated by a family $\mathcal{B}_0$ of subsets of $Y$. Then $f : X \to Y$ is $\mathcal{A}$-$\mathcal{B}$-measurable iff $f^{-1}[G] \in \mathcal{A}$ holds for all $G \in \mathcal{B}_0$.

**Proof** Clearly, if $f$ is $\mathcal{A}$-$\mathcal{B}$-measurable, then $f^{-1}[G] \in \mathcal{A}$ holds for all $G \in \mathcal{B}_0$.

Conversely, suppose $f^{-1}[G] \in \mathcal{A}$ holds for all $G \in \mathcal{B}_0$, then we need to show that $f^{-1}[G] \in \mathcal{A}$ for all $G \in \mathcal{B}$. In fact, consider the set $\mathcal{G}$ for which the assertion is true,
\[
\mathcal{G} := \{G \in \mathcal{B} \mid f^{-1}[G] \in \mathcal{A}\}.
\]

An elementary calculation shows that the empty set and $Y$ are both members of $\mathcal{G}$, and since $f^{-1}[Y \setminus G] = X \setminus f^{-1}[G], \mathcal{G}$ is closed under complementation. Because
\[
f^{-1}\left[\bigcup_{i \in I} G_i\right] = \bigcup_{i \in I} f^{-1}[G_i]
\]
holds for any index set $I$, $\mathcal{G}$ is closed under finite and countable unions. Thus $\mathcal{G}$ is a $\sigma$-algebra, so that $\sigma(\mathcal{G}) = \mathcal{G}$ holds. By assumption, $\mathcal{B}_0 \subseteq \mathcal{G}$, so that
\[
\mathcal{A} = \sigma(\mathcal{B}_0) \subseteq \sigma(\mathcal{G}) = \mathcal{G} \subseteq \mathcal{A}
\]
is inferred. Thus all elements of $\mathcal{B}$ have their inverse image in $\mathcal{A}$. ⊥

An example is furnished by a real valued function $f : X \to \mathbb{R}$ on $X$ which is $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable iff $\{x \in X \mid f(x) \triangleright t\} \in \mathcal{A}$ holds for each $t \in \mathbb{R}$; the relation $\triangleright$ may be taken from $<, \leq, \geq, >$. We infer in particular that a function $f$ from a topological space $(X, \tau)$ which is upper or lower semicontinuous (i.e., for which in the upper semicontinuous case the set $\{x \in X \mid f(x) < c\}$ is open, and in the lower semicontinuous case the set $\{x \in X \mid f(x) > c\}$ is open, $c \in \mathbb{R}$ being arbitrary), is Borel measurable. Hence a continuous function is Borel measurable. A continuous function $f : X \to Y$ into a metric space $Y$ is Baire measurable (Exercise 2).

These observations will be used frequently.

The proof’s strategy is to have a look at all objects that have the desired property, and to show that this set of good guys is a $\sigma$-algebra (this is why this approach is sometimes called the principle of good sets [Ekb99]). It is similar to showing in a proof by induction that the set of all natural numbers having a certain property is closed under constructing the successor.
Then we show that the generator of the $\sigma$-algebra is contained in the good guys, which is rather similar to begin the induction. Taking both steps together then yields the desired properties for both cases. We will encounter this pattern of proof over and over again.

An example is furnished by the equivalence relation induced by a family of sets.

**Example 2.7** Given a subset $C \subseteq \mathcal{P}(X)$ for a set $X$, define the equivalence relation $\equiv_C$ on $X$ upon setting

$$x \equiv_C x' \iff \forall C \in C : x \in C \Leftrightarrow x' \in C.$$ 

Thus $x \equiv_C x'$ iff $C$ cannot separate the elements $x$ and $x'$; call $\equiv_C$ the equivalence relation generated by $C$.

Now let $\mathcal{A}$ be a $\sigma$-algebra on $X$ with $\mathcal{A} = \sigma(\mathcal{A}_0)$. Then $\mathcal{A}$ and $\mathcal{A}_0$ generate the same equivalence relation, i.e., $\equiv_\mathcal{A} = \equiv_{\mathcal{A}_0}$. In fact, define for $x, x' \in X$ with $x \equiv_{\mathcal{A}_0} x'$

$$\mathcal{B} := \{ A \in \mathcal{A} \mid x \in A \Leftrightarrow x' \in A \}$$

Then $\mathcal{B}$ is a $\sigma$-algebra with $\mathcal{A}_0 \subseteq \mathcal{B}$, hence $\sigma(\mathcal{A}_0) \subseteq \mathcal{B} \subseteq \mathcal{A}$, so that $\mathcal{A} = \mathcal{B}$. Thus $x \equiv_{\mathcal{A}_0} x'$ implies $x \equiv_\mathcal{A} x'$; since the reverse implication is obvious, the claim is established.

If $(X, \mathcal{A})$ is a measurable space and $f : X \to Y$ is a map, then

$$\mathcal{B} := \{ D \subseteq Y \mid f^{-1}[D] \in \mathcal{A} \}$$

is the largest $\sigma$-algebra $\mathcal{B}_0$ on $N$ that renders $f: \mathcal{A}\mathcal{B}_0$-measurable; then $\mathcal{B}$ is called the final $\sigma$-algebra with respect to $f$. In fact, because the inverse set operator $f^{-1}$ is compatible with the Boolean operations, it is immediate that $\mathcal{B}$ is closed under the operations for a $\sigma$-algebra, and a little moment’s reflection shows that this is also the largest $\sigma$-algebra with this property.

Symmetrically, let $g : P \to X$ be a map, then

$$g^{-1}[\mathcal{A}] := \{ g^{-1}[E] \mid E \in \mathcal{A} \}$$

is the smallest $\sigma$-algebra $\mathcal{P}_0$ on $P$ that renders $g : \mathcal{P}_0 \to \mathcal{A}$ measurable; accordingly, $g^{-1}[\mathcal{M}]$ is called initial with respect to $f$. Similarly, $g^{-1}[\mathcal{A}]$ is a $\sigma$-algebra, and it is fairly clear that this is the smallest one with the desired property. In particular, the inclusion $i_Q : Q \to X$ becomes measurable for a subset $Q \subseteq X$ when $Q$ is endowed with the $\sigma$-algebra $\{Q \cap B \mid B \in \mathcal{A}\}$. It is called the trace of $\mathcal{A}$ on $Q$ and is denoted — in a slight abuse of notation — by $\mathcal{A} \cap Q$.

Initial and final $\sigma$-algebras specialize in an obvious way to families of maps. For example, $\sigma(\bigcup_{i \in I} g_i^{-1}[\mathcal{A}_i])$ is the smallest $\sigma$-algebra $\mathcal{P}_0$ on $P$ which makes all the maps $g_i : P \to X_i$ $\mathcal{P}_0\mathcal{A}_i$-measurable for a family $((X_i, \mathcal{A}_i))_{i \in I}$ of measurable spaces.

This is an intrinsic, universal characterization of the initial $\sigma$-algebra for a single map.

**Lemma 2.8** Let $(X, \mathcal{A})$ be a measurable space and $f : X \to Y$ be a map. The following conditions are equivalent:

1. The $\sigma$-algebra $\mathcal{B}$ on $Y$ is final with respect to $f$.
2. If $(P, \mathcal{P})$ is a measurable space, and $g : Y \to P$ is a map, then the $\mathcal{A}\mathcal{P}$-measurability of $g \circ f$ implies the $\mathcal{B}\mathcal{P}$-measurability of $g$. 
Proof 1. Taking care of \(1 \Rightarrow 2\) we note that

\[(g \circ f)^{-1}[\mathcal{P}] = f^{-1}[g^{-1}[\mathcal{P}]] \subseteq A.\]

Consequently, \(g^{-1}[\mathcal{P}]\) is one of the \(\sigma\)-algebras \(\mathcal{B}_0\) with \(f^{-1}[\mathcal{B}_0] \subseteq A\). Since \(\mathcal{B}\) is the largest of them, we have \(g^{-1}[\mathcal{P}] \subseteq \mathcal{B}\). Hence \(g\) is \(\mathcal{B}-\mathcal{P}\)-measurable.

2. In order to establish \(2 \Rightarrow 1\) we have to show that \(\mathcal{B}_0 \subseteq \mathcal{B}\) whenever \(\mathcal{B}_0\) is a \(\sigma\)-algebra on \(\mathcal{Y}\) with \(f^{-1}[\mathcal{B}_0] \subseteq \mathcal{A}\). Put \((P, \mathcal{P}) := (Y, \mathcal{B}_0)\), and let \(g\) be the identity \(\text{id}_Y\). Because \(f^{-1}[\mathcal{B}_0] \subseteq \mathcal{A}\), we see that \(\text{id}_Y \circ f\) is \(\mathcal{B}_0-\mathcal{A}\)-measurable. Thus \(\text{id}_Y\) is \(\mathcal{B}-\mathcal{B}_0\)-measurable. But this means \(\mathcal{B}_0 \subseteq \mathcal{B}\). \(\dashv\)

We will use the final \(\sigma\)-algebra mainly for factoring through an equivalence relation. In fact, let \(\alpha\) be an equivalence relation on a set \(X\), where \((X, \mathcal{A})\) is a measurable space. Then the factor map

\[\eta_\alpha : \begin{cases} X & \to X/\alpha \\ x & \mapsto [x]_\alpha \end{cases}\]

that maps each element to its class can be made a measurable map by taking the final \(\sigma\)-algebra \(A/\alpha\) with respect to \(\eta_\alpha\) and \(A\) as the \(\sigma\)-algebra on \(X/\alpha\).

Dual to Lemma 2.8, the initial \(\sigma\)-algebra is characterized.

**Lemma 2.9** Let \((Y, \mathcal{B})\) be a measurable space and \(f : X \to Y\) be a map. The following conditions are equivalent:

1. The \(\sigma\)-algebra \(\mathcal{A}\) on \(X\) is initial with respect to \(f\).
2. If \((P, \mathcal{P})\) is a measurable space, and \(g : P \to X\) is a map, then the \(\mathcal{P}-\mathcal{B}\)-measurability of \(f \circ g\) implies the \(\mathcal{P}-\mathcal{A}\)-measurability of \(g\).

\(\dashv\)

Let \(((A_i, \mathcal{A}_i))_{i \in I}\) be a family of measurable spaces, then the product-\(\sigma\)-algebra \(\bigotimes_{i \in I} \mathcal{A}_i\) denotes that initial \(\sigma\)-algebra on \(\prod_{i \in I} X_i\) for the projections

\[\pi_j : (m_i \mid i \in I) \mapsto m_j.\]

It is not difficult to see that \(\bigotimes_{i \in I} \mathcal{A}_i = \sigma(\mathcal{Z})\) with

\[\mathcal{Z} := \{\prod_{i \in I} E_i \mid \forall i \in I : E_i \in \mathcal{M}_i, E_i = M_i \text{ for almost all indices}\}\]

as the collection of cylinder sets (use Theorem 2.4 and the observation that \(\mathcal{Z}\) is closed under intersection).

For \(I = \{1, 2\}\), the \(\sigma\)-algebra \(\mathcal{A}_1 \otimes \mathcal{A}_2\) is generated from the set of measurable rectangles

\[\{E_1 \times E_2 \mid E_1 \in \mathcal{A}_1, E_2 \in \mathcal{A}_2\}\]

Dually, the sum \((X_1 + X_2, \mathcal{A}_1 + \mathcal{A}_2)\) of the measurable spaces \((X_1, \mathcal{A}_1)\) and \((X_2, \mathcal{A}_2)\) is defined through the final \(\sigma\)-algebra on the sum \(X_1 + X_2\) for the injections \(X_i \to X_1 + X_2\). This is the special case of the coproduct \(\bigoplus_{i \in I}(X_i, \mathcal{A}_i)\), where the \(\sigma\)-algebra \(\bigoplus_{i \in I} \mathcal{A}_i\) is initial with respect to the injections.
2.1.1 A $\sigma$-Algebra On Spaces Of Measures

We will now introduce a $\sigma$-algebra on the space of all $\sigma$-finite measures. It is induced by evaluating measures at fixed events. Note the inversion: instead of observing a measure assigning a real number to a set, we take a set and have it act on measures. This approach is fairly natural for many applications.

In addition to $S$ resp. $P$, the functors which assign to each measurable space its subprobabilities and its probabilities (see [Dob14, Section 1.4.2]), we introduce the space of finite resp. $\sigma$-finite measures. Denote by $\mathbb{M}(X,A)$ the set of all finite measures on $(X,A)$, the set of all $\sigma$-finite measures is denoted by $\mathbb{M}_\sigma(X,A)$. Each set $A \in A$ gives rise to the evaluation map $ev_A : \mu \mapsto \mu(A)$; the weak-*-$\sigma$-algebra $\varphi(X,A)$ on $\mathbb{M}(X,A)$ is the initial $\sigma$-algebra with respect to the family $\{ev_A \mid A \in A\}$ (actually, it suffices to consider a generator $A_0$ of $A$, see Exercise 1). It is clear that we have

$$\varphi(X,A) = \sigma(\{\beta_A(A,\bowtie q) \mid A \in A, q \in \mathbb{R}_+\})$$

when we define

$$\beta_A(A,\bowtie q) := \{\mu \in \mathbb{M}(X,A) \mid \mu(A) \bowtie q\}.$$

Here $\bowtie$ is one of the relational operators $\leq, <, \geq, >$, and it apparent that $q$ may be taken from the rationals. We will use the same symbol $\beta_A$ when we refer to probabilities or subprobabilities, if no confusion arises. Thus the base space from which the weak-*-$\sigma$-algebra will be constructed should be clear from the context.

Let $(Y,B)$ be another measurable space, and let $f : X \to Y$ be $A-B$-measurable. Define

$$\mathbb{M}(f)(\mu)(B) := \mu(f^{-1}[B])$$

for $\mu \in \mathbb{M}(X,A)$ and for $B \in B$, then $\mathbb{M}(f)(\mu) \in \mathbb{M}(Y,B)$, hence $\mathbb{M}(f) : \mathbb{M}(X,A) \to \mathbb{M}(Y,B)$ is a map, and since

$$(\mathbb{M}(f))^{-1}[\beta_B(B,\bowtie q)] = \beta_A(f^{-1}[B],\bowtie q),$$

this map is $\varphi(A)$-$\varphi(B)$-measurable. Thus $\mathbb{M}$ is an endofunctor on the category of measurable spaces.

Measurable maps into $\mathbb{M}_\sigma(\cdot)$ deserve special attention.

**Definition 2.10** Given measurable spaces $(X,A)$ and $(Y,B)$, an $A$-$\varphi(B)$ measurable map $K : X \to \mathbb{M}_\sigma(Y,B)$ is called a transition kernel and denoted by $K : (X,A) \leadsto (Y,B)$.

A transition kernel $K : (X,A) \leadsto (Y,B)$ models a situation in which each $x \in X$ is associated with a $\sigma$-finite measure $K(x)$ on $(Y,B)$. In a probabilistic setting, this may be interpreted as the probability that a system reacts on input $x$ with $K(x)$ as the probability distribution of its responses. For example, if $(X,A) = (Y,B)$ is the state space of a probabilistic transition system, then $K(x)(B)$ is often interpreted as the probability that the next state is a member of measurable set $B$ after a transition from $x$.

This is an immediate characterization of transition kernels.

**Lemma 2.11** $K : (X,A) \leadsto (Y,B)$ is a transition kernel iff these conditions are satisfied

1. $K(x)$ is a $\sigma$-finite measure on $(Y,B)$ for each $x \in X$. 

2. \( x \mapsto K(x)(B) \) is a measurable function for each \( B \in \mathcal{B} \).

**Proof** If \( K : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B}) \), then \( K(x) \) is a \( \sigma \)-finite measure on \( (Y, \mathcal{B}) \), and

\[
\{ x \in X | K(x)(B) > q \} = K^{-1}[\beta_B(B, > q)] \in \mathcal{A}.
\]

Thus \( x \mapsto K(x)(B) \) is measurable for all \( B \in \mathcal{B} \). Conversely, if \( x \mapsto K(x)(B) \) is measurable for \( b \in \mathcal{B} \), then the above equation shows that \( K^{-1}[\beta_B(B, > q)] \in \mathcal{A} \), so \( K : (X, \mathcal{A}) \rightarrow \mathcal{M}_\sigma(Y, \mathcal{B}) \) is \( \mathcal{A}_\varphi(\mathcal{B}) \) measurable by Lemma 2.6.

A special case of transition kernels are *Markov kernels*, sometimes also called *stochastic relations*. These are kernels the image of which is in \( \mathcal{S} \) or in \( \mathbb{P} \), whatever the case may be.

**Example 2.12** Transition kernels may be used for interpreting modal logics. Consider this grammar for formulas

\[
\varphi ::= \top | \varphi_1 \land \varphi_2 | \Diamond_q \varphi
\]

with \( q \in \mathbb{Q}, q \geq 0 \). The informal interpretation in a probabilistic transition system is that \( \top \) always holds, and that \( \Diamond_q \varphi \) holds with probability not smaller that \( q \) after a transition in a state in which formula \( \varphi \) holds. Now let \( M : (X, \mathcal{A}) \rightarrow (X, \mathcal{A}) \) be a transition kernel, and define inductively

\[
\begin{align*}
\lceil \top \rceil_M & := X \\
\lceil \varphi_1 \land \varphi_2 \rceil_M & := \lceil \varphi_1 \rceil_M \cap \lceil \varphi_2 \rceil_M \\
\lceil \Diamond_q \varphi \rceil_M & := \{ x \in X | M(x)(\lceil \varphi \rceil_M) \geq q \} \\
& = M^{-1}[\beta_A(B, \geq q)]
\end{align*}
\]

It is easy to show by induction on the structure of the formula that the sets \( \lceil \varphi \rceil_M \) are measurable, since \( M \) is a transition kernel, for a generalization, see Example 2.35.

### 2.1.2 The Alexandrov Topology On Spaces of Measures

Given a topological space \((X, \tau)\), the Borel sets \( \mathcal{B}(\tau) = \sigma(\tau) \) and the Baire sets \( \mathcal{B}a(X) \) come for free as measurable structures: \( \mathcal{B}(\tau) \) the smallest \( \sigma \)-algebra on \( X \) that contains the open sets; measurability of maps with respect to the Borel sets is referred to as *Borel measurability*. \( \mathcal{B}a(X) \) is the smallest \( \sigma \)-algebra on \( X \) which contains the functionally closed sets; they provide yet another measurable structure on \((X, \tau)\), this time involving the continuous real valued functions. Since \( \mathcal{B}(X) = \mathcal{B}a(X) \) for a metric space by Example 2.1, the distinction between these \( \sigma \)-algebras vanishes, and the Borel sets as the \( \sigma \)-algebra generated by the open sets dominate the scene.

We will now define a topology of spaces of measures on a topological space in a similar way, and relate this topology to the weak-*\( \sigma \)-algebra, for the time being in a special case. Fix a Hausdorff space \((X, \tau)\); the space will be specialized as the discussion proceeds. Define for the functionally open sets \( G_1, \ldots, G_n \), the functionally close sets \( F_1, \ldots, F_n \) and \( \epsilon > 0 \) for \( \mu_0 \in \mathbb{M}(X, \mathcal{B}a(X)) \) the sets

\[
W_{G_1, \ldots, G_n, \epsilon}(\mu_0) := \{ \mu \in \mathbb{M}(X, \mathcal{B}a(X)) | \mu(G_i) > \mu_0(G_i) - \epsilon \text{ for } 1 \leq i \leq n, |\mu(X) - \mu_0(X)| < \epsilon \},
\]

\[
W_{F_1, \ldots, F_n, \epsilon}(\mu_0) := \{ \mu \in \mathbb{M}(X, \mathcal{B}a(X)) | \mu(F_i) < \mu_0(F_i) + \epsilon \text{ for } 1 \leq i \leq n, |\mu(X) - \mu_0(X)| < \epsilon \}
\]
The topology which has the sets $W_{G_1, \ldots, G_n, \epsilon}(\mu_0)$ as a basis is called the Alexandrov topology or $A$-topology [Bog07, 8.10 (iv)]. The $A$-topology is defined in terms of Baire sets rather than Borel sets of $(X, \tau)$. This is so because the Baire sets provide a scenario which take the continuous functions on $(X, \tau)$ directly into account. This is in general not the case with the Borel sets, which are defined purely in terms of set theoretic operations. But the distinction vanishes when we turn to metric spaces, see Example 2.1. Note also that we deal with finite measures here.

**Lemma 2.13** The $A$-topology on $M(X, \mathcal{B}_{\text{a}}(X))$ is Hausdorff.

**Proof** The family of functionally closed sets of $X$ is closed under finite intersections, hence if two measure coincide on the functionally closed sets, they must coincide on the Baire sets $\mathcal{B}_{\text{a}}(X)$ of $X$ by the $\pi$-$\lambda$-Theorem 2.4. $\square$

Convergence in the $A$-topology is easily characterized in terms of functionally open or closed sets. Recall that for a sequence $(c_n)_{n \in \mathbb{N}}$ of real numbers the statements $\limsup_{n \to \infty} c \leq c$ is equivalent to $\inf_{n \in \mathbb{N}} \sup_{k \geq n} c_k \leq c$ which in turn is equivalent to $\forall \epsilon > 0 \exists n \in \mathbb{N} \forall k \geq n : c_k < c + \epsilon$. Similarly for $\liminf_{n \to \infty} c_n$. This proves:

**Proposition 2.14** Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of measures in $M(X, \mathcal{B}_{\text{a}}(X))$, then the following statements are equivalent.

1. $\mu_n \to \mu$ in the $A$-topology.
2. $\limsup_{n \to \infty} \mu_n(F) \leq \mu(F)$ for each functionally closed set $F$, and $\mu_n(X) \to \mu(X)$.
3. $\liminf_{n \to \infty} \mu_n(G) \geq \mu(G)$ for each functionally open set $G$, and $\mu_n(X) \to \mu(X)$.

$\square$

This criterion is sometimes a little impractical, since it deals with inequalities. We could have equality in the limit for all those sets for which the boundary has $\mu$-measure zero, but, alas, the boundary may not be Baire measurable. So we try with an approximation — we approximate a Baire set from within by a functionally open set (corresponding to the interior) and from the outside by a closed set (corresponding to the closure). This is discussed in some detail now.

Given $\mu \in M(X, \mathcal{B}_{\text{a}}(X))$, define by $\mathcal{R}_\mu$ all those Baire sets which have a functional boundary of vanishing $\mu$-measure, formally

$$\mathcal{R}_\mu := \{E \in \mathcal{B}_{\text{a}}(X) \mid G \subseteq E \subseteq F, \mu(F \setminus G) = 0, G \text{ functionally open}, F \text{ functionally closed}\}.$$ 

Hence if $X$ is a metric space, $E \in \mathcal{R}_\mu$ iff $\mu(\partial E) = 0$ for the boundary $\partial E$ of $E$.

This is another criterion for convergence in the $A$-topology.

**Corollary 2.15** Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of Baire measures. Then $\mu_n \to \mu$ in the $A$-topology iff $\mu_n(E) \to \mu(E)$ for all $E \in \mathcal{R}_\mu$.

**Proof** The condition is necessary by Proposition 2.14. Assume, on the other hand, that $\mu_n(E) \to \mu(E)$ for all $E \in \mathcal{R}_\mu$, and take a functionally open set $G$. We find $f : X \to \mathbb{R}$ continuous such that $G = \{x \in X \mid f(x) > 0\}$. Fix $\epsilon > 0$, then we can find $c > 0$ such


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Definition 2.16 A set $\mathcal{A}$ on a strictly order theoretic basis, without reference to measures. every $\mathcal{B}$ introduce we can state this property, which will be helpful in the analysis of the $\mathcal{A}$-topology below, we

Thus a $\mathcal{A}$ selected, and let

For the finite measure space $\mathcal{A}$, $\mu$ to $\mathcal{R}$, the family $\mathcal{A}$ selected, and let $\mathcal{A} = \{ A \in \mathcal{A} \mid A \subseteq X \setminus \bigcup_{i=1}^{n} A_i \text{ is an atom} \}$. If $\mathcal{A}_n = \emptyset$, we are done.

Otherwise, select the atom $A_{n+1} \in \mathcal{A}_n$ with $\mu(A_{n+1}) \geq \frac{1}{2} \cdot \sup_{A \in \mathcal{A}_n} \mu(A)$. Observe that $A_1, \ldots, A_{n+1}$ are mutually disjoint.

Let $\{ A_i \mid i \in I \}$ be the set of atoms selected in this way, after the selection has terminated.

Assume that $A \subseteq X \setminus \bigcup_{i \in I} A_i$ is an atom, then the index set $I$ must be infinite, and $\mu(A_i) \geq \mu(A)$ for all $i \in I$. But since $\sum_{i \in I} \mu(A_i) \leq \mu(X) < \infty$, we conclude that $\mu(A_i) \to 0$, consequently, $\mu(A) = 0$, hence $A$ cannot be a $\mu$-atom. \newline

This is a useful consequence.

Corollary 2.18 Let $f : X \to \mathbb{R}$ be a continuous function. Then there are at most countably many $r \in \mathbb{R}$ such that $\mu(\{ x \in X \mid f(x) = r \}) > 0$.

Proof Consider the image measure $\mathbb{M}(f)(\mu) : B \mapsto \mu(f^{-1}[B])$ on $\mathcal{B}(\mathbb{R})$. If $\mu(\{ x \in X \mid f(x) = r \}) > 0$, then $\{ r \}$ is a $\mathbb{M}(f)(\mu)$-atom. By Lemma 2.17 there are only countably many $\mathbb{M}(f)(\mu)$-atoms. \newline

Returning to $\mathcal{R}_\mu$ we are now in a position to have a closer look at its structure.
Proposition 2.19 $R_\mu$ is a Boolean algebra. If $(X, \tau)$ is completely regular, then $R_\mu$ contains a basis for the topology $\tau$.

Proof It is immediate that $R_\mu$ is closed under complementation, and it is easy to see that it is closed under finite unions.

Let $f : X \to \mathbb{R}$ be continuous, and define $U(f, r) := \{ x \in X \mid f(x) > r \}$, then $U(f, r)$ is open, and $\partial U(f, r) \subseteq \{ x \in X \mid f(x) = r \}$, thus $M_f := \{ r \in \mathbb{R} \mid \mu(\partial U(f, r)) > 0 \}$ is at most countable, such that the sets $U(f, r) \in R_\mu$, whenever $r \notin M_f$.

Now let $x \in X$ and $G$ be an open neighborhood of $x$. Because $X$ is completely regular, we can find $f : x \to [0, 1]$ continuous such that $f(y) = 1$ for all $y \notin G$, and $f(x) = 0$. Hence we can find $r \notin M_f$ such that $x \in U(f, r) \subseteq G$. So $R_\mu$ is in fact a basis for the topology. \[ \]

Under the conditions above, $R_\mu$ contains a base for $\tau$, we lift this base to $\mathbb{M}(X, Ba(X))$ in the hope of obtaining a base for the A-topology. This works, as we will show now.

Corollary 2.20 Let $X$ be a completely regular topological space, then the A-topology has a basis consisting of sets of the form

$$Q_{A_1,\ldots,A_n,\epsilon}(\mu) := \{ \nu \in \mathbb{M}(X, Ba(X)) \mid |\mu(A_i) - \nu(A_i)| < \epsilon \text{ for } i = 1, \ldots, n \}$$

with $\epsilon > 0$, $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in R_\mu$.

Proof Let $W_{G_1,\ldots,G_n,\epsilon}(\mu)$ with functionally open sets $G_1, \ldots, G_n$ and $\epsilon > 0$ be given. Select $A_i \in R_\mu$ functionally open with $A_i \subseteq G_i$ and $\mu(A_i) > \mu(G_i) - \epsilon/2$, then it is easy to see that $Q_{A_1,\ldots,A_n,\epsilon/2}(\mu) \subseteq W_{G_1,\ldots,G_n,\epsilon}(\mu)$. \[ \]

We will specialize the discussion now to metric spaces. So fix a metric space $(X, d)$, which we may assume to be bounded (otherwise we switch to the equivalent metric $d(x, y) \mapsto d(x, y)/(1 - d(x, y))$). Recall that the $\epsilon$-neighborhood $B_\epsilon$ of a set $B \subseteq X$ is defined as $B_\epsilon := \{ x \in X \mid d(x, B) < \epsilon \}$. Thus $B_\epsilon$ is always an open set. Since the Baire and the Borel sets coincide in a metric space (see Example 2.1), the A-topology is defined on $\mathbb{M}(X, B(X))$, and we will relate it to a metric now.

Define the Lévy-Prohorov distance $d_P(\mu, \nu)$ of the measures $\mu, \nu \in \mathbb{M}(X, B(X))$ through

$$d_P(\mu, \nu) := \inf \{ \epsilon > 0 \mid \nu(B) \leq \mu(B^\epsilon) + \epsilon, \mu(B) \leq \nu(B^\epsilon) + \epsilon \text{ for all } B \in B(X) \}$$

We note first that $d_P$ defines a metric, and that we can find a metrically exact copy of the base space $X$ in the space $\mathbb{M}(X, B(X))$.

Lemma 2.21 $d_P$ is a metric on $\mathbb{M}(X, B(X))$. $X$ is isometrically isomorphic to the set $\{ \delta_x \mid x \in X \}$ of Dirac measures.

Proof It is clear that $d_P(\mu, \nu) = d_P(\nu, \mu)$. Let $d_P(\mu, \nu) = 0$, then $\mu(F) \leq \nu(F^{1/n}) + 1/n$ and $\nu(F) \leq \mu(F^{1/n}) + 1/n$ for each closed set $F \subseteq X$, hence $\nu(F) = \mu(F)$ (note that $F^1 \supseteq F^{1/2} \supseteq F^{1/3} \supseteq \ldots$ and $F = \bigcap_{n \in \mathbb{N}} F^{1/n}$). Thus $\mu = \nu$. If we have for all $B \in B(X)$ that $\nu(B) \leq \mu(B^\epsilon) + \epsilon, \mu(B) \leq \nu(B^\epsilon) + \epsilon$ and $\mu(B) \leq \rho(B^{\epsilon + \delta}) + \delta, \rho(B) \leq m(B^{\epsilon + \delta}) + \delta$, then $\mu(B) \leq \rho(B^{\epsilon + \delta}) + \epsilon + \delta$ and $\rho(B) \leq \nu(B^{\epsilon + \delta}) + \epsilon + \delta$, thus $d_P(\mu, \nu) \leq d_P(\mu, \rho) + d_P(\rho, \nu)$. We also have $d_P(\delta_x, \delta_y) = d(x, y)$, from which the isometry derives. \[ \]

We will relate the metric topology to the A-topology now. Without additional assumptions this relationship can be stated:
Proposition 2.22 Each open set in the A-topology is also metrically open, hence A-topology is coarser than the metric topology.

Proof Let $W_{F_1,\ldots,F_n,\epsilon}(\mu)$ be an open basic neighborhood of $\mu$ in the A-topology with $F_1,\ldots,F_n$ closed. We want to find an open metric neighborhood with center $\mu$ which is contained in this A-neighborhood.

Because $(F^1/n)_{n\in\mathbb{N}}$ is a decreasing sequence with $\inf_{n\in\mathbb{N}} \mu(F_n) = \mu(F)$, whenever $F$ is closed, we can find $\delta > 0$ such that $\mu(F^\delta) < \mu(F_i) + \epsilon/2$ for $1 \leq i \leq n$ and $0 < \delta < \epsilon/2$. Thus, if $d_P(\mu,\nu) < \delta$, we have for $i = 1,\ldots,n$ that $\nu(F_i) < \mu(F^\delta_i) + \delta < \mu(F_i) + \epsilon$. But this means that $\nu \in W_{F_1,\ldots,F_n,\epsilon}(\mu)$.

Thus each neighborhood in the A-topology contains in fact an open ball for the $d_P$-metric. $\square$

The converse of Proposition 2.22 can only be established under additional conditions, which, however, are met for separable metric spaces. It is a generalization of $\sigma$-continuity: while the latter deals with sequences of sets, the concept of $\tau$-regularity deals with the more general notion of directed families of open sets (recall that a family $\mathcal{M}$ of sets is called directed if given $M_1,M_2 \in \mathcal{M}$ there exists $M' \in \mathcal{M}$ with $M_1 \cup M_2 \subseteq M'$).

Definition 2.23 A measure $\mu \in \mathcal{M}(X,\mathcal{B}(X))$ is called $\tau$-regular iff

\[ \mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu(G) \]

for each directed family $\mathcal{G}$ of open sets.

It is clear that we restrict our attention to open sets, because the union of a directed family of arbitrary measurable sets is not necessarily measurable. It is also clear that the condition above is satisfied for countable increasing sequences of open sets, so that $\tau$-regularity generalizes $\sigma$-continuity.

It turns out that finite measures on separable metric spaces are $\tau$-regular. Roughly speaking, this is due to the fact that countably many open sets determine the family of open sets, so that the space cannot be too large when looked at as a measure space.

Lemma 2.24 Let $(X,d)$ be a separable metric space, then each $\mu \in \mathcal{M}(X,\mathcal{B}(X))$ is $\tau$-regular.

Proof Let $\mathcal{G}_0$ be a countable basis for the metric topology. If $\mathcal{G}$ is a directed family of open sets, we can find for each $G \in \mathcal{G}$ a countable cover $(G_i)_{i \in I}\subseteq \mathcal{G}_0$ from $\mathcal{G}_0$ with $G = \bigcup_{i \in I} G_i$ and $\mu(G) = \sup_{i \in I} \mu(G_i)$. Thus

\[ \mu(\bigcup \mathcal{G}) = \sup \mu(\{\mu(G) \mid G \in \mathcal{G}_0, G \subseteq \bigcup \mathcal{G}\}) = \sup_{G \in \mathcal{G}} \mu(G). \]

$\square$

As a trivial consequence it is observed that $\mu(\bigcup \mathcal{G}) = 0$, where $\mathcal{G}$ is the family of all open sets $G$ with $\mu(G) = 0$.

The important observation for our purposes is that a $\tau$-regular measure is supported by a closed set which in terms of $\mu$ can be chosen as being as tightly fitting as possible.
Lemma 2.25 \( \text{Let } (X,d) \text{ be a separable metric space. Given } \mu \in \mathcal{M}(X,\mathcal{B}(X)) \text{ with } \mu(X) > 0, \text{ there exists a smallest closed set } C_\mu \text{ such that } \mu(C_\mu) = \mu(X). \text{ } C_\mu \text{ is called the support of } \mu \text{ and is denoted by } \text{supp} (\mu). \)

**Proof** Let \( F \) be the family of all closed sets \( F \) with \( \mu(F) = \mu(X) \), then \( \{ X \setminus F \mid F \in \mathcal{F} \} \) is a directed family of open sets, hence \( \mu(\bigcap \mathcal{F}) = \inf_{F \in \mathcal{F}} \mu(F) = \mu(X) \). Define \( \text{supp} (\mu) := \bigcap \mathcal{F} \), then \( \text{supp} (\mu) \) is closed with \( \mu(\text{supp} (\mu)) = \mu(X) \); if \( F \subseteq X \) is a closed set with \( \mu(F) = \mu(X) \), then \( F \in \mathcal{F} \), hence \( \text{supp} (\mu) \subseteq F. \) \( \blacksquare \)

We can characterize the support of \( \mu \) also in terms of open sets; this is but a simple consequence of Lemma 2.25.

**Corollary 2.26** Under the assumptions of Lemma 2.25 we have \( x \in \text{supp} (\mu) \) iff \( \mu(U) > 0 \) for each open neighborhood \( U \) of \( x. \) \( \blacksquare \)

After all these preparations (with some interesting vistas to the landscape of measures), we are in a position to show that the metric topology on \( \mathcal{M}(X,\mathcal{B}(X)) \) coincides with the \( A \)-topology for \( X \) separable metric. The following lemma will be the central statement; it is formulated and proved separately, because its proof is somewhat technical. Recall that the **diameter** \( \text{diam} (Q) \) of \( Q \subseteq X \) as

\[
\text{diam} (Q) := \sup \{ d(x_1, x_2) \mid x_1, x_2 \in Q \}.
\]

**Lemma 2.27** Every \( d_P \)-ball with center \( \mu \in \mathcal{M}(X,\mathcal{B}(X)) \) contains a neighborhood of \( \mu \) of the \( A \)-topology, if \( (X,d) \) is separable metric.

**Proof** Fix \( \mu \in \mathcal{M}(X,\mathcal{B}(X)) \) and \( \epsilon > 0 \), pick \( \delta > 0 \) with \( 4 \cdot \delta < \epsilon \); it is no loss of generality to assume that \( \mu(X) = 1 \). Because \( X \) is separable metric, the support \( S := \text{supp} (\mu) \) is defined by Lemma 2.25. Because \( S \) is closed, we can cover \( S \) with a countable number \( (V_n)_{n \in \mathbb{N}} \) of open sets the diameter of which is less that \( \delta \) and \( \mu (\partial V_n) = 0 \) by Proposition 2.19. Define

\[
A_1 := V_1,
A_n := \bigcup_{i=1}^{n} V_i \setminus \bigcup_{j=1}^{n-1} V_j,
\]

then \( (A_n)_{n \in \mathbb{N}} \) is a mutually disjoint family of sets which cover \( S \), and for which \( \mu (\partial A_n) = 0 \) holds for all \( n \in \mathbb{N} \). We can find an index \( k \) such that \( \mu (\bigcup_{i=1}^{k} V_i) > 1 - \delta \). Let \( T_1, \ldots, T_\ell \) be all sets which are a union of some of the sets \( A_1, \ldots, A_k \), then

\[
W := W_{T_1, \ldots, T_\ell} (\mu)
\]

is a neighborhood of \( \mu \) in the \( A \)-topology by Corollary 2.20. We claim that \( d_P (\mu, \nu) < \epsilon \) for all \( \nu \in W \). In fact, let \( B \in \mathcal{B}(X) \) be arbitrary, and put

\[
A := \bigcup \{ A_i \mid 1 \leq i \leq k, A \cap B \neq \emptyset \},
\]

then \( A \) is among the \( T \)s just constructed, and \( B \cap S \subseteq A \cup \bigcup_{i=k+1}^{\infty} A_i \). Moreover, we know that \( A \subseteq B^\delta \), because each \( A_i \) has a diameter less that \( \delta \). This yields

\[
\mu(B) = \mu(B \cap S) \leq \mu(A) + \delta < \nu(A) + 2 \cdot \delta \leq \nu(B^\delta) + 2 \cdot \delta.
\]
On the other hand, we have
\[ \nu(B) = \nu(B \cap S) + \nu(B \cap (X \setminus S)) \]
\[ \leq \nu(A \cap \bigcup_{i=k+1}^{\infty} A_i) + 3 \cdot \delta \]
\[ \leq \nu(A) + 3 \cdot \delta \]
\[ \leq \mu(A) + 3 \cdot \delta \]
\[ \leq \mu(B) + 4 \cdot \delta. \]

Hence \( d_P(\mu, \nu) < 4 \cdot \delta < \epsilon \). Thus \( W \) is contained in the open ball around \( \mu \) with radius smaller \( \epsilon \). \( \dashv \)

We have established

**Theorem 2.28** The \( A \)-topology on \( M(X, \mathcal{B}(X)) \) is metrizable by the Lévy-Skohorod metric \( d_P \), provided \((X, d)\) is a separable metric space. \( \dashv \)

We will see later that \( d_P \) is not the only metric for this topology, and that the corresponding metric space has interesting and helpful properties. Some of these properties are best derived through an integral representation, for which a careful study of real-valued functions is required. This is what we are going to investigate in Section 2.2. But before doing this, we have a brief and tentative look at the relation between the Borel sets for \( A \)-topology and weak-*\( \sigma \)-algebra.

**Lemma 2.29** Let \( X \) be a metric space, then the weak-*\( \sigma \)-algebra is contained in the Borel sets of the \( A \)-topology. If the \( A \)-topology has a countable basis, both \( \sigma \)-algebras are equal.

**Proof** Denote by \( \mathcal{C} \) the Borel sets of the \( A \)-topology on \( M(X, \mathcal{B}(X)) \).

Since \( X \) is metric, the Baire sets and the Borel sets coincide. For each closed set \( F \), the evaluation map \( ev_F : \mu \mapsto \mu(F) \) is upper semi-continuous by Proposition 2.14, so that the set
\[ \mathcal{G} := \{ A \in \mathcal{B}(X) \mid ev_A \text{ is } \mathcal{C} \text{ - measurable} \} \]
contains all closed sets. Because \( \mathcal{G} \) is closed under complementation and countable disjoint unions, we conclude that \( \mathcal{G} \) contains \( \mathcal{B}(X) \). Hence \( \varphi(\mathcal{B}(X)) \subseteq \mathcal{C} \) by minimality of \( \varphi(\mathcal{B}(X)) \).

2. Assume that the \( A \)-topology has a countable basis, then each open set can be represented as a countable union of sets of the form \( W_{G_1, \ldots, G_n, \epsilon}(\mu_0) \) with \( G_1, \ldots, G_n \) open. But \( W_{G_1, \ldots, G_n, \epsilon}(\mu_0) \in \varphi(X, \mathcal{B}(X)) \), so that each open set is a member of \( \varphi(X, \mathcal{B}(X)) \). This implies the other inclusion. \( \dashv \)

We will investigate the \( A \)-topology further in Section 2.10 and turn to real-valued functions now.

### 2.2 Real-Valued Functions

We discuss the set of all measurable and bounded functions into the real line now. We show first that the set of all these functions is closed under the usual algebraic operations, so that
it is a vector space, and that it is also closed under finite infima and suprema, rendering it a distributive lattice; in fact, algebraic operations and order are compatible. Then we show that the measurable step functions are dense with respect to pointwise convergence. This is an important observation, which will help us later on to transfer linear properties from indicator functions (a.k.a. measurable sets) to general measurable functions. This prepares the stage for discussing convergence of functions in the presence of a measure; we will deal with convergence almost everywhere, which neglects a set of measure zero for the purposes of convergence, and convergence in measure, which is defined in terms of a pseudo metric, but surprisingly turns out to be related to convergence almost everywhere through subsequences of subsequences (this sounds a bit mysterious, so carry on).

**Lemma 2.30** Let \( f, g : X \to \mathbb{R} \) be \( \mathcal{A} \cdot \mathcal{B}(\mathbb{R}) \)-measurable functions for the measurable space \((X, \mathcal{A})\). Then \( f \land g, f \lor g \) and \( \alpha \cdot f + \beta \cdot g \) are \( \mathcal{A} \cdot \mathcal{B}(\mathbb{R}) \)-measurable for \( \alpha, \beta \in \mathbb{R} \).

**Proof** If \( f \) is measurable, \( \alpha \cdot f \) is. This follows immediately from Lemma 2.6. From
\[
\{ x \in X \mid f(x) + g(x) < q \} = \bigcup_{r_1, r_2 \in \mathbb{Q}, r_1 + r_2 \leq q} \{ x \mid f(x) < r_1 \} \cap \{ x \mid g(x) < r_2 \}
\]
we obtain that the sum of measurable functions is measurable again. Since
\[
\{ x \in X \mid (f \land g)(x) < q \} = \{ x \mid f(x) < q \} \cup \{ x \mid g(x) < q \}
\]
\[
\{ x \in X \mid (f \lor g)(x) < q \} = \{ x \mid f(x) < q \} \cap \{ x \mid g(x) < q \},
\]
we see that both \( f \land g \) and \( f \lor g \) are measurable. \( \square \)

**Corollary 2.31** If \( f : X \to \mathbb{R} \) is \( \mathcal{A} \cdot \mathcal{B}(\mathbb{R}) \)-measurable, so is \( |f| \).

**Proof** Write \( |f| = f^+ - f^- \) with \( f^+ := f \lor 0 \) and \( f^- := (-f) \lor 0 \). \( \square \)

The consequence is that for a measurable space \((X, \mathcal{A})\) the set
\[
\mathcal{F}(X, \mathcal{A}) := \{ f : N \to \mathbb{R} \mid f \text{ is } \mathcal{A} \cdot \mathcal{B}(\mathbb{R}) \text{ measurable and bounded} \}
\]
is both a vector space and a distributive lattice; in fact, it is a vector lattice, see Definition 2.151 on page 73. Assume that \((f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}(X, \mathcal{A})\) is a sequence of bounded measurable functions such that \( f : x \mapsto \liminf_{n \to \infty} f_n(x) \) is a bounded function, then \( f \in \mathcal{F}(X, \mathcal{A}) \). This is so because
\[
\{ x \in X \mid \liminf_{n \to \infty} f_n(x) \leq q \} = \{ x \mid \sup_{n \in \mathbb{N}} \inf_{k \geq n} f_k(x) \leq q \}
\]
\[
= \bigcap_{n \in \mathbb{N}} \{ x \mid \inf_{k \geq n} f_k(x) \leq q \}
\]
\[
= \bigcap_{n \in \mathbb{N}} \bigcap_{\ell \in \mathbb{N}} \{ x \mid \inf_{k \geq n} f_k(x) < q + 1/\ell \}
\]
\[
= \bigcap_{n \in \mathbb{N}} \bigcap_{\ell \in \mathbb{N}} \bigcup_{k \geq n} \{ x \mid f_k(x) < q + 1/\ell \}
\]
Similarly, if \( x \mapsto \limsup_{n \to \infty} f_n(x) \) defines a bounded function, then it is measurable as well. Consequently, if the sequence \((f_n(x))_{n \in \mathbb{N}}\) converges to a bounded function \( f \), then \( f \in \mathcal{F}(X, \mathcal{A}) \).

Hence we have shown
Proposition 2.32 Let \((f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}(X, \mathcal{A})\) be a sequence of bounded measurable functions. Then

- If \(f_\ast(x) := \liminf_{n \to \infty} f_n(x)\) defines a bounded function, then \(f_\ast \in \mathcal{F}(X, \mathcal{A})\),
- if \(f^\ast(x) := \limsup_{n \to \infty} f_n(x)\) defines a bounded function, then \(f^\ast \in \mathcal{F}(X, \mathcal{A})\).

We use occasionally the representation of sets through indicator functions. Recall for \(A \subseteq X\) its indicator function

\[ \chi_A(x) := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases} \]

Clearly, if \(\mathcal{A}\) is a \(\sigma\)-algebra on \(X\), then \(A \in \mathcal{A}\) iff \(\chi_A\) is a \(\mathcal{A}\)-\(\mathcal{B}(\mathbb{R})\)-measurable function. This is so since we have for the inverse image of an interval under \(\chi_A\)

\[ \chi_A^{-1}([0, q]) = \begin{cases} \emptyset, & \text{if } q < 0, \\ X \setminus A, & \text{if } 0 \leq q < 1, \\ X, & \text{if } q \geq 1. \end{cases} \]

A measurable step function

\[ f = \sum_{i=1}^{n} \alpha_i \cdot \chi_{A_i} \]

is a linear combination of indicator functions with \(A_i \in \mathcal{N}\). Since \(\chi_A \in \mathcal{F}(X, \mathcal{A})\) for \(A \in \mathcal{A}\), measurable step functions are indeed measurable functions.

Proposition 2.33 Let \((X, \mathcal{A})\) be a measurable space. Then

1. For \(f \in \mathcal{F}(X, \mathcal{A})\) with \(f \geq 0\) there exists an increasing sequence \((f_n)_{n \in \mathbb{N}}\) of step functions \(f_n \in \mathcal{F}(X, \mathcal{A})\) with

\[ f(x) = \sup_{n \in \mathbb{N}} f_n(x) \]

for all \(x \in X\).

2. For \(f \in \mathcal{F}(X, \mathcal{A})\) there exists a sequence \((f_n)_{n \in \mathbb{N}}\) of step functions \(f_n \in \mathcal{F}(N, \mathcal{A})\) with

\[ f(x) = \lim_{n \to \infty} f_n(x) \]

for all \(x \in X\).

Proof 1. Take \(f \geq 0\), and assume without loss of generality that \(f \leq 1\) (otherwise, if \(0 \leq f \leq m\), consider \(f/m\)). Put

\[ A_{i,n} := \{ x \in X \mid i/n \leq f(x) < (i+1)/n \}, \]

for \(n \in \mathbb{N}, 0 \leq i \leq n\), then \(A_{i,n} \in \mathcal{A}\), since \(f\) is measurable. Define

\[ f_n(x) := \sum_{0 \leq i < 2^n} i \cdot 2^{-n} \chi_{A_{i,2^n}}. \]
Then $f_n$ is a measurable step function, and $f_n \leq f$, moreover $(f_n)_{n \in \mathbb{N}}$ is increasing. This is so because given $n \in \mathbb{N}$, $x \in X$, we can find $i$ such that $x \in A_i2^n = A_{2i,2^{n+1}} \cup A_{2i+1,2^{n+1}}$. If $f(x) < (2i+1)/2^{n+1}$, we have $x \in A_{2i,2^{n+1}}$ with $f_n(x) = f_{n+1}(x)$, if, however, $(2i+1)/2^{n+1} \leq f(x)$, we have $f_n(x) < f_{n+1}(x)$.

Given $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ with $2^{-n} < \epsilon$ for $n \geq n_0$. Let $x \in X, n \geq n_0$, then $x \in A_i2^n$ for some $i$, hence $|f_n(x) - f(x)| = f(x) - i2^{-n} < 2^{-n} < \epsilon$. Thus $f = \sup_{n \in \mathbb{N}} f_n$.

2. Given $f \in \mathcal{F}(X, \mathcal{A})$, write $f_1 := f \wedge 0$ and $f_2 := f \vee 0$, then $f = f_1 + f_2$ with $f_1 \leq 0$ and $f_2 \geq 0$ as measurable and bounded functions. Hence $f_2 = \sup_{n \in \mathbb{N}} g_n = \lim_{n \to \infty} g_n$ and $-f_1 = -\sup_{n \in \mathbb{N}} h_n = -\lim_{n \to \infty} h_n$ for increasing sequences of step functions $(g_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$. Thus $f = \lim_{n \to \infty} (g_n + h_n)$, and $g_n + h_n$ is a step function for each $n \in \mathbb{N}$. ⊥

Given $f : X \rightarrow \mathbb{R}$ with $f \geq 0$, the set $\{\langle x, q \rangle \in X \times \mathbb{R} \mid 0 \leq f(x) \leq q\}$ can be visualized as the area between the $X$-axis and the graph of the function. We obtain as a consequence that this set is measurable, provided $f$ is measurable. This gives an example of a product measurable set. To be specific

**Corollary 2.34** Let $f : X \rightarrow \mathbb{R}$ with $f \geq 0$ be a bounded measurable function for a measurable space $(X, \mathcal{A})$, and define

$$C_\triangleleft(f) := \{\langle x, q \rangle \mid 0 \leq q \triangleleft f(x)\} \subseteq X \times \mathbb{R}$$

for the relational operator $\triangleleft$ taken from $\{\geq, <, =, \neq, >, \geq\}$. Then $C_\triangleleft(f) \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$.

**Proof** We prove the assertion for $C(f) := C_{\triangleleft}(f)$, from which the other cases may easily be derived, e.g.,

$$C_{\triangleleft}(f) = \bigcap_{k \in \mathbb{N}} \{\langle x, q \rangle \mid f(x) < q + 1/k\} = \bigcap_{k \in \mathbb{N}} C_{\triangleleft}(f - 1/k).$$

Consider these cases.

1. If $f = \chi_A$ with $A \in \mathcal{A}$, then $C(f) = X \setminus A \times \{0\} \cup A \times [0, 1] \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$.

2. If $f$ is represented as a step function with a finite number of mutually disjoint steps, say, $f = \sum_{i=1}^{k} r_i \cdot \chi_{A_i}$ with $r_i \geq 0$ and all $A_i \in \mathcal{A}$, then

$$C(f) = \left(X \setminus \bigcup_{i=1}^{k} A_i\right) \times \{0\} \cup \bigcup_{i=1}^{k} A_i \times [0, r_i] \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}).$$

3. If $f$ is represented as a monotone limit of step function $(f_n)_{n \in \mathbb{N}}$ with $f_n \geq 0$ according to Proposition 2.33 then $C(f) = \bigcup_{n \in \mathbb{N}} C(f_n)$, thus $C(f) \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$. ⊥

**Example 2.35** Consider the simple modal logic in Example 2.12 interpreted through a transition kernel $M : (X, \mathcal{A}) \rightsquigarrow (X, \mathcal{A})$. Given a formula $\varphi$, the set $\{\langle x, r \rangle \mid M(x)([\varphi]_M) \geq r\}$ is a member of $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$. Note that $\left[\Box \varphi\right]_M$ is the cut of this set at $q$. Hence this observation generalizes measurability of $\left[\Box \varphi\right]_M$, one of the cornerstones for interpreting modal logics probabilistically. ☞

We will turn now to the interplay of measurable functions and measures and have a look at different modes of convergence for sequences of measurable functions in the presence of a (finite) measure.
2.2.1 Essentially Bounded Functions

Fix for this section a finite measure space \((X, \mathcal{A}, \mu)\). We say that a measurable property holds \(\mu\)-almost everywhere (abbreviated as \(\mu\text{-a.e.}\)) iff the set on which the property does not hold has \(\mu\)-measure zero.

The measurable function \(f \in \mathcal{F}(X, \mathcal{A})\) is called \(\mu\)-essentially bounded iff

\[
||f||_\infty^\mu := \inf\{a \in \mathbb{R} \mid |f| \leq \mu a\} < \infty,
\]

where \(f \leq \mu a\) indicates that \(f \leq a\) holds \(\mu\)-a.e. Thus a \(\mu\)-essentially bounded function may occasionally take arbitrary large values, but the set of these values must be negligible in terms of \(\mu\).

The set

\[
\mathcal{L}_\infty(\mu) := \mathcal{L}_\infty(X, \mathcal{A}, \mu) := \{f \in \mathcal{F}(X, \mathcal{A}) \mid ||f||_\infty^\mu < \infty\}
\]

of all \(\mu\)-essentially bounded functions is a real vector space, and we have for \(||\cdot||_\infty^\mu\) these properties.

**Lemma 2.36** Let \(f, g \in \mathcal{F}(X, \mathcal{A})\) essentially bounded, \(\alpha, \beta \in \mathbb{R}\), then \(||\cdot||_\infty^\mu\) is a pseudo-norm on \(\mathcal{F}(X, \mathcal{A})\), i.e.,

1. If \(||f||_\infty^\mu = 0\), then \(f = \mu 0\).
2. \(||\alpha \cdot f||_\infty^\mu = |\alpha| \cdot ||f||_\infty^\mu\),
3. \(||f + g||_\infty^\mu \leq ||f||_\infty^\mu + ||g||_\infty^\mu\).

**Proof** If \(||f||_\infty^\mu = 0\), we have \(|f| \leq \mu 1/n\) for all \(n \in \mathbb{N}\), so that

\[
\{x \in X \mid |f(x)| \neq 0\} \subseteq \bigcup_{n \in \mathbb{N}} \{x \in X \mid |f(x)| \leq 1/n\},
\]

consequently, \(f = \mu 0\). The converse is trivial. The second property follows from \(|f| \leq \mu a\) iff \(|\alpha \cdot f| \leq \mu |\alpha| a\), the third one from the observation that \(|f| \leq \mu a\) and \(|g| \leq \mu b\) implies \(|f + g| \leq |f| + |g| \leq \mu a + b\).

So \(||\cdot||_\infty^\mu\) nearly a norm, but the crucial property that the norm for a vector is zero only if the vector is zero is missing. We factor \(\mathcal{L}_\infty(X, \mathcal{A}, \mu)\) with respect to the equivalence relation \(= \mu\), then the set

\[
L_\infty(\mu) := L_\infty(X, \mathcal{A}, \mu) := \{[f] \mid f \in \mathcal{L}_\infty(X, \mathcal{A}, \mu)\}
\]

of all equivalence classes \([f]\) of \(\mu\)-essentially bounded measurable functions is a vector space again. This is so because \(f = \mu g\) and \(f' = \mu g'\) together imply \(f + f' = \mu g + g'\), and \(f = \mu g\) implies \(\alpha \cdot f = \mu \alpha \cdot g\) for all \(\alpha \in \mathbb{R}\). Moreover,

\[
||[f]||_\infty^\mu := ||f||_\infty^\mu
\]

defines a norm on this space. For easier reading we will identify in the sequel \(f\) with its class \([f]\).

We obtain in this way a normed vector space, which is complete with respect to this norm.

**Proposition 2.37** \((L_\infty(\mu), ||\cdot||_\infty^\mu)\) is a Banach space.
**Proof** Let \((f_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \(L_\infty(X, A, \mu)\), and define

\[ N := \bigcup_{n_1, n_2 \in \mathbb{N}} \{ x \in X \mid |f_{n_1}(x) - f_{n_2}(x)| > \|f_{n_1} - f_{n_2}\|_\infty \}, \]

then \(\mu(N) = 0\). Put \(g_n := \chi_{X \setminus N} \cdot f_n\), then \((g_n)_{n \in \mathbb{N}}\) converges uniformly with respect to the supremum norm \(||\cdot||_\infty\) to some element \(g \in \mathcal{F}(X, A)\), hence also \(||f_n - g||_\infty\to 0\). Clearly, \(g\) is bounded. \(\dash\)

This is the first instance of a vector space intimately connected with a measure space. We will discuss several of these spaces later on, when integration is at our disposal.

The convergence of a sequence of measurable functions into \(\mathbb{R}\) in the presence of a finite measure is discussed now. Without a measure, we may use pointwise or uniform convergence for modelling approximations. Recall that pointwise convergence of a sequence \((f_n)_{n \in \mathbb{N}}\) of functions to a function \(f\) is given by

\[ \forall x \in X : \lim_{n \to \infty} f_n(x) = f(x), \]

and the stronger form of uniform convergence through

\[ \lim_{n \to \infty} ||f_n - f||_\infty = 0, \]

with \(||\cdot||_\infty\) as the supremum norm, given by

\[ ||f||_\infty := \sup_{x \in X} |f(x)|. \]

We will weaken the first condition [1], so that it holds not everywhere but almost everywhere, thus the set on which it does not hold will be a set of measure zero. This leads to the notion of convergence almost everywhere, which will turn out to be quite close to uniform convergence, as we will see when discussing Egorov’s Theorem. Convergence almost everywhere will be weakened to convergence in measure, for which we will define a pseudo metric. This in turn gives rise to another Banach space upon factoring.

### 2.2.2 Convergence almost everywhere and in measure

Recall that we work in a finite measure space \((X, A, \mu)\). The sequence \((f_n)_{n \in \mathbb{N}}\) of measurable functions \(f_n \in \mathcal{F}(X, A)\) is said to converge almost everywhere to a function \(f \in \mathcal{F}(X, A)\) (written as \(f_n \overset{a.e.}{\to} f\)) iff the sequence \((f_n(x))_{n \in \mathbb{N}}\) converges pointwise to \(f(x)\) for every \(x\) outside a set of measure zero. Thus we have \(\mu(X \setminus K) = 0\), where \(K := \{ x \in X \mid f_n(x) \to f(x) \}\). Because

\[ K = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{\ell \geq m} \{ x \in X \mid |f_\ell(x) - f(x)| < 1/n \}, \]

\(K\) is a measurable set. It is clear that \(f_n \overset{a.e.}{\to} f\) and \(f_n \overset{a.e.}{\to} f'\) implies that \(f =_\mu f'\) holds.

The next lemma shows that convergence everywhere is compatible with the common algebraic operations on \(\mathcal{F}(X, A)\) like addition, scalar multiplication and the lattice operations. Since these functions can be represented as continuous function of several variables, we formulate this closure property abstractly in terms of compositions with continuous functions.
**Lemma 2.38** Let $f_{1,n} \xrightarrow{a.e.} f_i$ for $1 \leq i \leq k$, and assume that $g : \mathbb{R}^k \to \mathbb{R}$ is continuous. Then $g \circ (f_{1,n}, \ldots, f_{k,n}) \xrightarrow{a.e.} g \circ (f_1, \ldots, f_k)$.

**Proof** Put $h_n := g \circ (f_{1,n}, \ldots, f_{k,n})$. Since $g$ is continuous, we have

$$\{x \in X \mid (h_n(x))_{n \in \mathbb{N}} \text{ does not converge} \} \subseteq \bigcup_{j=1}^k \{x \in X \mid (f_{j,n}(x))_{n \in \mathbb{N}} \text{ does not converge} \},$$

hence the set on the left hand side has measure zero. $\dashv$

Intuitively, convergence almost everywhere means that the measure of the set

$$\bigcup_{n \geq k} \{x \in X \mid |f_n(x) - f(x)| > \epsilon \}$$

tends to zero, as $k \to \infty$, so we are coming closer and closer to the limit function, albeit on a set the measure of which becomes smaller and smaller. We show that this intuitive understanding yields an adequate model for this kind of convergence.

**Lemma 2.39** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions in $\mathcal{F}(X, \mathcal{A})$ and $f \in \mathcal{F}(X, \mathcal{A})$. Then the following conditions are equivalent

1. $f_n \xrightarrow{a.e.} f$.
2. $\lim_{k \to \infty} \mu(\bigcup_{n \geq k} \{x \in X \mid |f_n(x) - f(x)| > \epsilon \}) = 0$ for every $\epsilon > 0$.

**Proof** Let $\epsilon > 0$ be given, then there exists $k \in \mathbb{N}$ with $1/k < \epsilon$, so that

$$\lim_{k \to \infty} \mu(\bigcup_{n \geq k} \{x \in X \mid |f_n(x) - f(x)| > \epsilon \}) \overset{(*)}{=} \mu(\bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} \{x \in X \mid |f_n(x) - f(x)| > \epsilon \})$$

$$\leq \mu(\{x \in X \mid (f_n(x))_{n \in \mathbb{N}} \text{ does not converge} \}).$$

Now assume that $f_n \xrightarrow{a.e.} f$, then the implication $1 \Rightarrow 2$ is immediate. If, however, $f_n \xrightarrow{a.e.} f$ is false, then we find for each $\epsilon > 0$ so that for all $k \in \mathbb{N}$ there exists $n \geq k$ with $\mu(\{x \in X \mid |f_n(x) - f(x)| \geq \epsilon \}) > 0$. Thus $2$ cannot hold. $\dashv$

Note that the statement above requires a finite measure space, because the measure of a decreasing sequence of sets is the infimum of the individual measures, used in the equation marked $(*)$. This is not necessarily valid for non-finite measure space.

The characterization implies that a.e.-Cauchy sequences converge.

**Corollary 2.40** Let $(f_n)_{n \in \mathbb{N}}$ be an a.e.-Cauchy sequence in $\mathcal{F}(X, \mathcal{A})$. Then $(f_n)_{n \in \mathbb{N}}$ converges.

**Proof** Because $(f_n)_{n \in \mathbb{N}}$ is an a.e.-Cauchy sequence, we have that $\mu(X \setminus K_\epsilon) = 0$ for every $\epsilon > 0$, where

$$K_\epsilon := \bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}, m \geq k} \{x \in X \mid |f_n(x) - f_m(x)| > \epsilon \}.$$

Put

$$N := \bigcup_{k \in \mathbb{N}} K_1/k,$$

$$g_n := f_n \cdot \chi_{X \setminus N},$$
then \((g_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(\mathcal{F}(X, \mathcal{A})\) which converges pointwise to some \(f \in \mathcal{F}(X, \mathcal{A})\). Since \(\mu(X \setminus N) = 0\), \(f_n \overset{a.e.}{\to} f\) follows. \(\dashv\)

Convergence a.e. is very nearly uniform convergence, where very nearly serves to indicate that the set on which uniform convergence does not happen is arbitrarily small. To be specific, we can find for each threshold a set the complement of which has a measure smaller than this bound, on which convergence is uniform. This is what Egorov’s Theorem says.

**Proposition 2.41** Let \(f_n \overset{a.e.}{\to} f\) for \(f_n, f \in \mathcal{F}(X, \mathcal{A})\). Given \(\epsilon > 0\), there exists \(A \in \mathcal{A}\) such that

1. \(\sup_{x \in A} |f_n(x) - f(x)| \to 0\),
2. \(\mu(X \setminus A) < \epsilon\).

The idea of the proof is that we look at each \(x\) for which uniform convergence is spoiled by \(1/k\). This set can be made arbitrary small in terms of \(\mu\), so the union of all these sets can be made as small as we want. Outside this set we have uniform convergence. Let’s look at a more formal treatment now.

**Proof** Fix \(\epsilon > 0\), then there exists for each \(k \in \mathbb{N}\) an index \(n_k \in \mathbb{N}\) such that \(\mu(B_k) < \epsilon/2^{k+1}\) with

\[
B_k := \bigcup_{m \geq n_k} \{ x \in X \mid |f_m(x) - f(x)| > 1/k \}.
\]

Now put \(A := \bigcap_{k \in \mathbb{N}} (X \setminus B_k)\), then

\[
\mu(X \setminus A) \leq \sum_{k \in \mathbb{N}} \mu(B_k) \leq \epsilon,
\]

and we have for all \(k \in \mathbb{N}\)

\[
\sup_{x \in A} |f_n(x) - f(x)| \leq \sup_{x \not\in B_k} |f_n(x) - f(x)| \leq 1/k
\]

for \(n \geq n_k\). Thus

\[
\lim_{n \to \infty} \sup_{x \in A} |f_n(x) - f(x)| = 0,
\]

as claimed. \(\dashv\)

Convergence almost everywhere makes sure that the set on which a sequence of functions does not converge has measure zero, and Egorov’s Theorem shows that this is almost uniform convergence.

Convergence in measure for a finite measure space \((X, \mathcal{A}, \mu)\) takes another approach: fix \(\epsilon > 0\), and consider the set \(\{ x \in X \mid |f_n(x) - f(x)| > \epsilon \}\). If the measure of this set (for a fixed, but arbitrary \(\epsilon\)) tends to zero, as \(n \to \infty\), then we say that \((f_n)_{n \in \mathbb{N}}\) converges in measure to \(f\), and write \(f_n \overset{i.m.}{\to} f\). In order to have a closer look at this notion of convergence, we note that it is invariant against equality almost everywhere: if \(f_n = \mu\) and \(f = \mu\), then \(f_n \overset{i.m.}{\to} f\) implies \(g_n \overset{i.m.}{\to} g\), and vice versa.

We will introduce a pseudo metric \(\delta\) on \(\mathcal{F}(X, \mathcal{A})\) first:

\[
\delta(f, g) := \inf\{ \epsilon > 0 \mid \mu(\{ x \in X \mid |f(x) - g(x)| > \epsilon\}) \leq \epsilon \}.
\]
These are some elementary properties of $\delta$:

**Lemma 2.42** Let $f, g, h \in \mathcal{F}(X, \mathcal{A})$, then we have

1. $\delta(f, g) = 0$ iff $f = \mu g$,
2. $\delta(f, g) = \delta(g, f)$,
3. $\delta(f, g) \leq \delta(f, h) + \delta(h, g)$.

**Proof** If $\delta(f, g) = 0$, but $f \neq \mu g$, there exists $k$ with $\mu(\{x \in X \mid |f(x) - g(x)| > 1/k\}) > 1/k$. This is a contradiction. The other direction is trivial. Symmetry of $\delta$ is also trivial, so the triangle inequality remains to be shown. If $|f(x) - g(x)| > \epsilon_1 + \epsilon_2$, then $|f(x) - h(x)| > \epsilon_1$ or $|h(x) - g(x)| > \epsilon_2$, thus

$$\mu(\{x \in X \mid |f(x) - g(x)| > \epsilon_1 + \epsilon_2\}) \leq \mu(\{x \in X \mid |f(x) - h(x)| > \epsilon_1\}) + \mu(\{x \in X \mid |h(x) - g(x)| > \epsilon_2\}).$$

This implies the third property. $\dashv$

This, then, is the formal definition of convergence in measure:

**Definition 2.43** The sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{F}(X, \mathcal{A})$ is said to converge in measure to $f \in \mathcal{F}(X, \mathcal{A})$ (written as $f_n \xrightarrow{i.m.} f$) iff $\delta(f_n, f) \rightarrow 0$, as $n \rightarrow \infty$.

We can express convergence in measure in terms of convergence almost everywhere.

**Proposition 2.44** $(f_n)_{n \in \mathbb{N}}$ converges in measure to $f$ iff each subsequence of $(f_n)_{n \in \mathbb{N}}$ contains a subsequence $(h_n)_{n \in \mathbb{N}}$ with $h_n \xrightarrow{a.e.} f$.

**Proof** “$\Rightarrow$”: Assume $f_n \xrightarrow{i.m.} f$, and let $\epsilon > 0$ be arbitrary but fixed. Let $(g_n)_{n \in \mathbb{N}}$ be a subsequence of $(f_n)_{n \in \mathbb{N}}$. We find a sequence of indices $n_1 < n_2 < \ldots$ such that $\mu(\{x \in X \mid |g_{n_k}(x) - f(x)| > \epsilon\}) < 1/k^2$. Let $h_k := g_{n_k}$, then we obtain

$$\mu(\bigcup_{k \geq \ell} \{x \in X \mid |h_k - f| > \epsilon\}) \leq \sum_{k \geq \ell} \frac{1}{k^2} \rightarrow 0,$$

as $\ell \rightarrow \infty$. Hence $h_k \xrightarrow{a.e.} f$.

“$\Leftarrow$”: If $\delta(f_n, f) \not\rightarrow 0$, we can find a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ and $r > 0$ such that $\mu(\{x \in X \mid |f_{n_k}(x) - f(x)| > r\}) > r$ for all $k \in \mathbb{N}$. Let $(g_n)_{n \in \mathbb{N}}$ be a subsequence of this subsequence, then

$$\lim_{n \rightarrow \infty} \mu(\{x \in X \mid |g_n - f| > r\}) \leq \lim_{n \rightarrow \infty} \mu(\bigcup_{m \geq n} \{x \in X \mid |g_m - f| > r\}) = 0$$

by Lemma 2.39. This is a contradiction. $\dashv$

Hence convergence almost everywhere implies convergence in measure. Just for the record:

**Corollary 2.45** If $(f_n)_{n \in \mathbb{N}}$ converges almost everywhere to $f$, then the sequence converges also in measure to $f$. $\dashv$

The converse relationship is a bit more involved. Intuitively, a sequence which converges in measure need not converge almost everywhere.
Example 2.46 Let \( A_{i,n} := [(i-1)/n, i/n] \) for \( n \in \mathbb{N} \) and \( 1 \leq i \leq n \), and consider the sequence \( (f_n)_{n \in \mathbb{N}} := (\chi_{A_{1,1}}, \chi_{A_{1,2}}, \chi_{A_{2,1}}, \chi_{A_{2,2}}, \chi_{A_{3,1}}, \chi_{A_{3,2}}, \chi_{A_{3,3}}, \ldots) \), so that in general \( \chi_{A_{1,1}}, \ldots, \chi_{A_{n,n}} \) is followed by \( \chi_{A_{i,n+1}}, \ldots, \chi_{A_{i+1,n+1}} \). Let \( \mu \) be Lebesgue measure \( \lambda \) on \( B([0,1]) \). Given \( \epsilon > 0 \),
\[
\lambda(\{ x \in [0,1] \mid f_n(x) > \epsilon \}) \text{ can be made arbitrarily small for any given } \epsilon > 0, \text{ hence } f_n \overset{i.m.}{\rightarrow} 0.
\]
On the other hand, \( (f_n(x))_{n \in \mathbb{N}} \) fails to converge or any \( x \in [0,1] \), so \( f_n \overset{a.e.}{\rightarrow} 0 \) is false.

We have, however, this observation, which draws atoms into our game.

Proposition 2.47 Let \((A_i)_{i \in I}\) be the at most countable collection of \( \mu \)-atoms according to Lemma [2.17] such that \( B := X \setminus \bigcup_{i \in I} A_i \) does not contain any atoms. Then these conditions are equivalent:

1. Convergence in measure implies convergence almost everywhere.
2. \( \mu(B) = 0 \).

Proof 1 \( \Rightarrow \) 2 Assume that \( \mu(B) \leq 0 \), then we know that for each \( k \in \mathbb{N} \) there exist mutually disjoint measurable subsets \( B_{1,k}, \ldots, B_{k,k} \) of \( B \) such that \( \mu(B_{i,k}) = 1/k \cdot \mu(B) \) and \( B = \bigcup_{1 \leq i \leq k} B_{i,k} \). This is so because \( B \) does not contain any atoms. Put as above \( (f_n)_{n \in \mathbb{N}} := (\chi_{B_{1,1}}, \chi_{B_{1,2}}, \chi_{B_{2,1}}, \chi_{B_{2,2}}, \chi_{B_{3,1}}, \chi_{B_{3,2}}, \chi_{B_{3,3}}, \ldots) \), so that in general \( \chi_{B_{1,1}}, \ldots, \chi_{B_{n,n}} \) is followed by \( \chi_{B_{1,n+1}}, \ldots, \chi_{B_{n+1,n+1}} \). Because \( \mu(\{ x \in X \mid f_n(x) > \epsilon \}) \) can be made arbitrarily small for any positive \( \epsilon \), we find \( f_n \overset{i.m.}{\rightarrow} 0 \). If we assume that convergence in measure implies convergence almost everywhere, we have \( f_n \overset{a.e.}{\rightarrow} 0 \), but this is false, because \( \liminf_{n \to \infty} f_n = 0 \) and \( \limsup_{n \to \infty} f_n = \chi_B \). This is a contradiction.

2 \( \Rightarrow \) 1 Let \((f_n)_{n \in \mathbb{N}}\) be a sequence with \( f_n \overset{i.m.}{\rightarrow} f \). Fix an atom \( A_i \), then \( \mu(\{ x \in A_i \mid |f_n(x) - f(x)| > 1/k \}) = 0 \) for all \( n \geq n_k \) with \( n_k \) suitably chosen; this is so because \( A_i \) is an atom, hence measurable subsets of \( A_i \) take only the values 0 and \( \mu(A_i) \). Put
\[
g := \inf \sup_{n \in \mathbb{N}, n_1, n_2 \geq n} |f_{n_1} - f_{n_2}|,
\]
then \( g(x) \neq 0 \) iff \( f_n(x) \) does not converge to \( f(x) \). We infer \( \mu(\{ x \in A_i \mid g(x) \geq 2/k \}) = 0 \). Because the family \((A_i)_{i \in I}\) is mutually disjoint, we conclude that \( \mu(\{ x \in X \mid g(x) \geq 2/k \}) = 0 \) for all \( k \in \mathbb{N} \). But now look at this
\[
\mu(\{ x \in X \mid \liminf_{n \to \infty} f_n(x) < \limsup_{n \to \infty} f_n(x) \}) = \mu(\{ x \in X \mid g(x) > 0 \}) = 0.
\]
Consequently, \( f_n \overset{a.e.}{\rightarrow} f \).

Again we want to be sure that convergence in measure is preserved by the usual algebraic operations like addition or taking the infimum, so we state as a counterpart to Lemma 2.38 now as an easy consequence.

Lemma 2.48 Let \( f_{i,n} \overset{i.m.}{\rightarrow} f_i \) for \( 1 \leq i \leq k \), and assume that \( g : \mathbb{R}^k \to \mathbb{R} \) is continuous. Then \( g \circ (f_{1,n}, \ldots, f_{k,n}) \overset{i.m.}{\rightarrow} g \circ (f_1, \ldots, f_k) \).

Proof By iteratively selecting subsequences, we can find subsequences \((h_{i,n})_{n \in \mathbb{N}}\) such that \( h_{i,n} \overset{a.e.}{\rightarrow} f_i \), as \( n \to \infty \) for \( 1 \leq i \leq k \). Then apply Lemma 2.38 and Proposition 2.44.

Let \( F(X,A) \) be the factor space \( F(X,A) = \mu \) of the space of all measurable functions with respect to \( =_\mu \). Then this is a real vector space again, because the algebraic operations on the
equivalence classes are well defined. Note that we have \( \delta(f,g) = \delta(f',g') \), provided \( f = \mu g \) and \( f' = \mu g' \). We identify again the class \([f]_\mu\) with \( f \). Define

\[
||f|| := \delta(f,0)
\]

for \( f \in F(X,A) \).

**Proposition 2.49** \((F(X,A), ||\cdot||)\) is a Banach space.

**Proof**

1. It follows from Lemma 2.42 and the observation \( \delta(f,0) = 0 \iff f = \mu 0 \) that \( ||\cdot|| \) is a norm, so we have to show that \( F(X,A) \) is complete with this norm.

2. Let \((f_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \( F(X,A) \), then we can find a strictly increasing sequence \((\ell_n)_{n \in \mathbb{N}}\) of integers such that \( \delta(f_{\ell_n},f_{\ell_n+1}) \leq 1/n^2 \), hence

\[
\mu\{x \in X \mid |f_{\ell_n}(x) - f_{\ell_n+1}(x)| \geq 1/n^2\} \leq 1/n^2.
\]

Let \( \epsilon > 0 \) be given, then there exists \( r \in \mathbb{N} \) with \( \sum_{n \geq r} 1/n^2 < \epsilon \), hence we have

\[
\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \{x \in X \mid |f_{\ell_n}(x) - f_{\ell_k}(x)| > \epsilon\} \subseteq \bigcup_{n \geq k} \{x \in X \mid |f_{\ell_n}(x) - f_{\ell_{n+1}}(x)| < 1/n^2\},
\]

if \( k \geq r \). Thus

\[
\mu\big(\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \{x \in X \mid |f_{\ell_n}(x) - f_{\ell_k}(x)| > \epsilon\}\big) \leq \sum_{n \geq k} 1/n^2 \to 0,
\]

as \( k \to \infty \). Hence \((f_{\ell_n})_{n \in \mathbb{N}}\) is an a.e. Cauchy sequence which converges a.e. to some \( f \in F(X,A) \), which by Proposition 2.44 implies that \( f_n \overset{\text{i.m.}}{\to} f \).

A consequence of \((F(X,A), ||\cdot||)\) being a Banach space is that \( \mathcal{F}(X,A) \) is complete with respect to convergence in measure for any finite measure \( \mu \) on \( A \). Thus for any sequence \((f_n)_{n \in \mathbb{N}}\) of functions such that given \( \epsilon > 0 \) there exists \( n_0 \) such that \( \mu\{x \in X \mid |f_n(x) - f_m(x)| > \epsilon\} < \epsilon \) for all \( n, m \geq n_0 \) we can find \( f \in F(X,A) \) such that \( f_n \overset{\text{i.m.}}{\to} f \) with respect to \( \mu \).

We will deal with measurable real valued functions again and in greater detail in Section 2.11; then we will have integration as a powerful tool at our disposal, and we will know more about Hilbert spaces.

Now we turn to the study of \( \sigma \)-algebras and focus on those which have a countable set as their generator.

### 2.3 Countably Generated \( \sigma \)-Algebras

Fix a measurable space \((X,A)\). The \( \sigma \)-algebra \( A \) is said to be countably generated iff there exists countable \( A_0 \) such that \( A = \sigma(A_0) \).

**Example 2.50** Let \((X,\tau)\) be a topological space with a countable basis. Then \( \mathcal{B}(X) \) is countably generated. In fact, if \( \tau_0 \) is the countable basis for \( \tau \), then each open set \( G \) can be written as \( G = \bigcup_{n \in \mathbb{N}} G_n \) with \( (G_n)_{n \in \mathbb{N}} \subseteq \tau_0 \), thus each open set is an element of \( \sigma(\tau_0) \), consequently, \( \mathcal{B}(X) = \sigma(\tau_0) \).
The observation in Example 2.50 implies that the Borel sets for a separable metric space, in particular for a Polish space, is countably generated.

Having a countable dense subset for a metric space, we can use the corresponding base for a fairly helpful characterization of the Borel sets. The next Lemma says that the Borel sets are in this case countably generated.

**Lemma 2.51** Let $Y$ be a separable metric space with metric $d$. Denote by $B_r(y) := \{y' \in Y \mid d(y, y') < r\}$ the open ball with radius $r$ and center $y$. Then

$$\mathcal{B}(Y) = \sigma(\{B_r(d) \mid r > 0 \text{ rational}, d \in D\}),$$

where $D$ is countable and dense.

**Proof** Because an open ball is an open set, we infer that

$$\sigma(\{B_r(d) \mid r > 0 \text{ rational}, d \in D\}) \subseteq \mathcal{B}(Y).$$

Conversely, let $G$ be open. Then there exists a sequence $(B_n)_{n \in \mathbb{N}}$ of open balls with rational radii such that $\bigcup_{n \in \mathbb{N}} B_n = G$, accounting for the other inclusion. \(\dashv\)

Also the characterization of Borel sets in a metric space as the closure of the open (closed) sets under countable unions and countable intersections will be occasionally helpful.

**Lemma 2.52** The Borel sets in a metric space $Y$ are the smallest collection of sets that contains the open (closed) sets and that are closed under countable unions and countable intersections.

**Proof** The smallest collection $\mathcal{G}$ of sets that contains the open sets and that is closed under countable unions and countable intersections is closed under complementation. This is so since each closed set is a $G_\delta$ by Theorem 2.75. Thus $\mathcal{B}(Y) \subseteq \mathcal{G}$; on the other hand $\mathcal{G} \subseteq \mathcal{B}(Y)$ by construction. \(\dashv\)

The property of being countably generated is, however, not hereditary for a $\sigma$-algebra — a sub-$\sigma$-algebra of a countably generated $\sigma$-algebra is not necessarily countably generated. This is demonstrated by the following example. Incidentally, we will see in Example 2.111 that the intersection of two countably generated $\sigma$-algebras need not be countably generated again. This indicates that having a countable generator is a fickle property which has to be observed closely.

**Example 2.53** Let

$$\mathcal{C} := \{A \subseteq \mathbb{R} \mid A \text{ or } \mathbb{R} \setminus A \text{ is countable}\}$$

This $\sigma$-algebra is usually referred to the countable-cocountable $\sigma$-algebra. Clearly, $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R})$, and $\mathcal{B}(\mathbb{R})$ is countably generated by Example 2.50. But $\mathcal{C}$ is not countably generated. Assume that it is, so let $\mathcal{C}_0$ be a countable generator for $\mathcal{C}$; we may assume that every element of $\mathcal{C}_0$ is countable. Put $A := \bigcup \mathcal{C}_0$, then $A \in \mathcal{C}$, since $A$ is countable. But

$$\mathcal{D} := \{B \subseteq \mathbb{R} \mid B \subseteq A \text{ or } B \subseteq \mathbb{R} \setminus A\}$$

is a $\sigma$-algebra, and $\mathcal{D} = \sigma(\mathcal{C}_0)$. On the other hand there exists $a \in \mathbb{R}$ with $a \notin A$, thus $A \cup \{a\} \in \mathcal{C}$ but $A \cup \{a\} \notin \mathcal{D}$, a contradiction. \(\checkmark\)
Although the entire $\sigma$-algebra may not be countably generated, we may find for each element of a $\sigma$-algebra a countable generator:

**Lemma 2.54** Let $\mathcal{A}$ be a $\sigma$-algebra on a set $X$ which is generated by family $\mathcal{G}$ of subsets. Then we can find for each $A \in \mathcal{A}$ a countable subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $A \in \sigma(\mathcal{G}_0)$.

**Proof** Let $\mathcal{D}$ be the set of all $A \in \mathcal{A}$ for which the assertion is true, then $\mathcal{D}$ is closed under complements, and $\mathcal{G} \subseteq \mathcal{A}$. Moreover, $\mathcal{D}$ is closed under countable unions, since the union of a countable family of countable sets is countable again. Hence $\mathcal{D}$ is a $\sigma$-algebra which contains $\mathcal{G}$, hence it contains $\mathcal{A} = \sigma(\mathcal{G})$. \(\square\)

This has a fairly interesting and somewhat unexpected consequence, which will be of use later on. Recall that $\mathcal{A} \otimes \mathcal{B}$ is the smallest $\sigma$-algebra on $X \times Y$ which contains for measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ all measurable rectangles $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. In particular, $\mathcal{P}(X) \otimes \mathcal{P}(X)$ is generated by $\{A \times B \mid A, B \subseteq X\}$. One may be tempted to assume that this $\sigma$-algebra is the same as $\mathcal{P}(X \times X)$, but this is not always the case, because we have

**Proposition 2.55** Denote by $\Delta_X$ the diagonal $\{(x,x) \mid x \in X\}$ for a set $X$. Then $\Delta_X \in \mathcal{P}(X) \otimes \mathcal{P}(X)$ implies that the cardinality of $X$ does not exceed that of $\mathcal{P}(\mathbb{N})$.

**Proof** Assume $\Delta_X \in \mathcal{P}(X) \otimes \mathcal{P}(X)$, then there exists a countable family $\mathcal{C} \subseteq \mathcal{P}(X)$ such that $\Delta_X \in \sigma(\{A \times B \mid A, B \in \mathcal{C}\})$. The map $q : x \mapsto \{C \in \mathcal{C} \mid x \in C\}$ from $X$ to $\mathcal{P}(\mathcal{C})$ is injective. In fact, suppose it is not, then there exists $x \neq x'$ with $x \in C \iff x' \in C$ for all $C \in \mathcal{C}$, so we have for all $C \in \mathcal{C}$ that either $\{x, x'\} \subseteq C$ or $\{x, x'\} \cap C = \emptyset$, so that the pairs $\langle x, x \rangle$ and $\langle x', x' \rangle$ never occur alone in any $A \times B$ with $A, B \in \mathcal{C}$. Hence $\Delta_X$ cannot be a member of $\sigma(\{A \times B \mid A, B \in \mathcal{C}\})$, a contradiction. As a consequence, $X$ cannot have more elements that $\mathcal{P}(\mathbb{N})$. \(\square\)

Among the countably generated measurable spaces those are of interest which permit to separate points, so that if $x \neq x'$, we can find $A \in \mathcal{C}$ with $x \in A$ and $x' \notin A$; they are called separable. Formally

**Definition 2.56** The $\sigma$-algebra $\mathcal{A}$ is called separable iff it is countably generated, and if for any two different elements of $X$ there exists a measurable set $A \in \mathcal{A}$ which contains one, but not the other. The measurable space $(X, \mathcal{A})$ is called separable iff its $\sigma$-algebra $\mathcal{A}$ is separable.

The argumentation from Proposition 2.55 yields

**Corollary 2.57** Let $\mathcal{A}$ be a separable $\sigma$-algebra over the set $X$ with $\mathcal{A} = \sigma(\mathcal{A}_0)$ for $\mathcal{A}_0$ countable. Then $\mathcal{A}_0$ separates points, and $\Delta_X \in \mathcal{A} \otimes \mathcal{A}$.

**Proof** Because $\mathcal{A}$ separates points, we obtain from Example 2.7 that $\equiv_{\mathcal{A}_0} = \Delta_X$, where $\equiv_{\mathcal{A}_0}$ is the equivalence relation defined by $\mathcal{A}_0$. So $\mathcal{A}_0$ separates points. The representation

$$X \times X \setminus \Delta_X = \bigcup_{A \in \mathcal{A}_0} A \times (X \setminus A) \cup (X \setminus A) \times A,$$

now yields $\Delta_X \in \mathcal{A} \otimes \mathcal{A}$. \(\square\)

In fact, we can say even more.

**Proposition 2.58** A separable measurable space $(X, \mathcal{A})$ is isomorphic to $(X, \mathcal{B}(X))$ with the Borel sets coming from a metric $d$ on $X$ such that $(X, d)$ has is a separable metric space.
Proof 1. Let \( A_0 = \{ A_n \mid n \in \mathbb{N} \} \) be the countable generator for \( A \) which separates points. Define
\[
(M, \mathcal{M}) := (\{0,1\}^\mathbb{N}, \bigotimes_{n \in \mathbb{N}} \mathcal{P}(\{0,1\}))
\]
as the product of countable many copies of the discrete space \((\{0,1\}, \mathcal{P}(\{0,1\}))\). Then \( \bigotimes_{n \in \mathbb{N}} \mathcal{P}(\{0,1\}) \) has as a basis the cylinder sets \( \{ Z_v \mid v \in \{0,1\}^k \text{ for some } k \in \mathbb{N} \} \) with \( Z_v := \{(t_n)_{n \in \mathbb{N}} \in M \mid \langle m_1, \ldots, m_k \rangle = v \} \) for \( v \in \{0,1\}^k \), see \( \Box \) Define \( f : X \to M \) through \( f(x) := (\chi_{A_n(x)})_{n \in \mathbb{N}} \), then \( f \) is injective, because \( A_0 \) separates points. Put \( Q := f[X] \), and \( Q := \mathcal{M} \cap Q \), the trace of \( \mathcal{M} \) on \( Q \).

Now let \( Y_v := Z_v \cap Q \) be an element of the generator for \( Q \) with \( v = \langle m_1, \ldots, m_k \rangle \), then \( f^{-1}[Y_v] = \bigcap_{j=1}^k C_j \) with \( C_j := A_j \), if \( m_j = 1 \), and \( C_j := X \setminus A_j \) otherwise. Consequently, \( f : X \to Q \) is a \( \mathcal{A} - Q \)-measurable.

2. Put for \( x, y \in X \)
\[
d(x, y) := \sum_{n \in \mathbb{N}} 2^{-n} \cdot |\chi_{A_n(x)} - \chi_{A_n(y)}|,
\]
then \( d \) is a metric on \( X \) which has
\[
\mathcal{G} := \big\{ \bigcap_{j \in F} B_j \mid B_j \in A_0 \text{ or } X \setminus B_j \in A_0, F \subseteq \mathbb{N} \text{ is finite} \big\}
\]
as a countable basis. In fact, let \( G \subseteq X \) be open; given \( x \in G \), there exists \( \epsilon > 0 \) such that the open ball \( B_\epsilon(x) := \{ x' \in X \mid d(x, x') < \epsilon \} \) with center \( x \) and radius \( \epsilon \) is contained in \( G \). Now choose \( k \) with \( 2^{-k} < \epsilon \), and put \( v := \langle x_1, \ldots, x_k \rangle \), then \( x \in \bigcap_{j=1}^k B_j \subseteq B_\epsilon(x) \). This argument shows also that \( \mathcal{A} = \mathcal{B}(X) \).

3. Because \((X, d)\) has a countable basis, it is a separable metric space. The map \( f : X \to Q \) is a bijection which is measurable, and \( f^{-1} \) is measurable as well. This is so because \( \{ A \in \mathcal{A} \mid f[A] \in Q \} \) is a \( \sigma \)-algebra which contains the basis \( \mathcal{G} \). \( \dashv \)

This representation, which is due to Mackey, gives the representation of separable measurable spaces as subspaces of the countable product of the discrete space \((\{0,1\}, \mathcal{P}(\{0,1\}))\). This space is also a compact metric space, so we may say that a separable measurable space is isomorphic to a subspace of a compact metric space. We will make use of this observation later on.

By the way, this innocently looking statement has some remarkable consequences for our context. Just as an appetizer:

Corollary 2.59 Let \((X, \mathcal{A})\) be a separable measurable space. Then

1. The diagonal \( \Delta_X \) is measurable in the product, i.e.,
2. If \( f_i : X_i \to X \) is \( \mathcal{A}_i - \mathcal{A} \)-measurable, where \((X_i, \mathcal{A}_i)\) is a measurable space \((i = 1, 2)\), then \( f_1^{-1}[\mathcal{A}] \otimes f_2^{-1}[\mathcal{A}] = (f_1 \times f_2)^{-1}[\mathcal{A} \otimes \mathcal{A}] \).

Proof 1. Let \((A_n)_{n \in \mathbb{N}}\) be a generator for \( X \) that separates point, then
\[
(X \times X) \setminus \Delta_X = \bigcup_{n \in \mathbb{N}} (A_n \times X \setminus A_n \cup X \setminus A_n \times A_n),
\]
which is a member of $\mathcal{A} \otimes \mathcal{A}$.

2. The product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{A}$ is generated by the rectangles $B_1 \times B_2$ with $B_i$ taken from some generator $\mathcal{B}_0$ for $\mathcal{B}$ ($i = 1, 2$). Since $(f_1 \times f_2)^{-1}[B_1 \times B_2] = f_1^{-1}[B_1] \times f_2^{-1}[B_2]$, we see that $(f_1 \times f_2)^{-1}[\mathcal{B} \otimes \mathcal{B}] \subseteq f_1^{-1}[\mathcal{B}] \otimes f_2^{-1}[\mathcal{B}]$. This is true without the assumption of separability. Now let $\tau$ be a second countable metric topology on $Y$ with $\mathcal{B} = \mathcal{B}(\tau)$ and let $\tau_0$ be a countable base for the topology. Then

$$\tau_p := \{T_1 \times T_2 \mid T_1, T_2 \in \tau_0\}$$

is a countable base for the product topology $\tau \otimes \tau$, and (this is the crucial property)

$$\mathcal{B} \otimes \mathcal{B} = \mathcal{B}(Y \times Y, \tau \otimes \tau)$$

holds: because the projections from $X \times Y$ to $X$ and to $Y$ are measurable, we observe $\mathcal{B} \otimes \mathcal{B} \subseteq \mathcal{B}(Y \times Y, \tau \otimes \tau)$; because $\tau_p$ is a countable base for the product topology $\tau \otimes \tau$, we infer the other inclusion.

3. Since for $T_1, T_2 \in \tau_0$ clearly

$$f_1^{-1}[T_1] \times f_2^{-1}[T_2] \in (f_1 \times f_2)^{-1}[\tau_p] \subseteq (f_1 \times f_2)^{-1}[\mathcal{B} \otimes \mathcal{B}]$$

holds, the nontrivial inclusion is inferred from the fact that the smallest $\sigma$-algebra containing $\{f_1^{-1}[T_1] \times f_2^{-1}[T_2] \mid T_1, T_2 \in \tau_0\}$ equals $f_1^{-1}[\mathcal{B}] \otimes f_2^{-1}[\mathcal{B}]$.

Given a measurable function into a separable measurable space, we find that its kernel yields a measurable subset in the product of its domain. We will use the kernel for many a construction, so this little observation is quite helpful.

**Corollary 2.60** Let $f : X \to Y$ be a $\mathcal{A} \mathcal{B}$-measurable map, where $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces, the latter being separable. Then the kernel of $f$

$$\ker (f) := \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$$

is a member of $\mathcal{A} \otimes \mathcal{A}$.

**Proof** Exercise [3]

The observation, made in the proof of Proposition 2.55, that it may not always be possible to separate two different elements in a measurable space through a measurable set led there to a contradiction. Nevertheless it leads to an interesting notion.

**Definition 2.61** The set $A \in \mathcal{A}$ is called an atom of $\mathcal{A}$ iff $B \subseteq A$ implies $B = \emptyset$ or $B = A$ for all $B \in \mathcal{A}$.

For example, each singleton set $\{x\}$ is an atom for the $\sigma$-algebra $\mathcal{P}(X)$. Clearly, being an atom depends also on the $\sigma$-algebra. If $A$ is an atom, we have alternatively $B \subseteq A$ or $B \cap A = \emptyset$ for all $B \in \mathcal{A}$; this is more radical than being a $\mu$-atom, which merely restricts the values of $\mu(B)$ for measurable $B \subseteq A$ to 0 or $\mu(A)$. Certainly, if $A$ is an atom, and $\mu(A) > 0$, then $A$ is a $\mu$-atom.

For a countably generated $\sigma$-algebra, atoms are easily identified.
Proposition 2.62 Let $A_0 = \{A_n \mid n \in \mathbb{N}\}$ be a countable generator of $\mathcal{A}$, and define

$$A_{\alpha} := \bigcap_{n \in \mathbb{N}} A_{\alpha n},$$

for $\alpha \in \{0, 1\}^\mathbb{N}$, where $A^0 := A, A^1 := X \setminus A$. Then $\{A_{\alpha} \mid \alpha \in \{0, 1\}^\mathbb{N}, A_{\alpha} \neq \emptyset\}$ is the set of all atoms of $\mathcal{A}$.

**Proof** Assume that there exist in $\mathcal{A}$ two different non-empty subsets $B_1, B_2$ of $A_\alpha$, and take $y_1 \in B_1, y_2 \in B_2$. Then $y_1 \equiv_{A_\alpha} y_2$, but $y_1 \not\equiv_A y_2$, contradicting the observation in Example 2.7. Hence $A_\alpha$ is an atom. Let $x \in A_\alpha$, then $A_\alpha$ is the equivalence class of $x$ with respect to the equivalence relation $\equiv_{A_\alpha}$, hence with respect to $\mathcal{A}$. Thus each atom is given by some $A_\alpha$. $\dashv$

Incidentally, this gives another proof that the countable-cocountable $\sigma$-algebra over $\mathbb{R}$ is not countably generated. Assume it is generated by $\{A_n \mid n \in \mathbb{N}\}$, then

$$H := \bigcap\{A_n \mid A_n \text{ is cocountable}\} \cap \bigcap\{(\mathbb{R} \setminus A_n) \mid A_n \text{ is countable}\}$$

is an atom, but $H$ is also cocountable. This is a contradiction to $H$ being an atom.

We relate atoms to measurable maps:

**Lemma 2.63** Let $f : X \to \mathbb{R}$ be $\mathcal{A} \mathcal{B}(\mathbb{R})$-measurable. If $A \in \mathcal{A}$ is an atom of $\mathcal{A}$, then $f$ is constant on $A$.

**Proof** Assume that we can find $x_1, x_2 \in A$ with $f(x_1) \neq f(x_2)$, say, $f(x_1) < c < f(x_2)$. Then $\{x \in A \mid f(x) < c\}$ and $\{x \in A \mid f(x) > c\}$ are two non-empty disjoint measurable subsets of $A$. This contradicts $A$ being an atom. $\dashv$

We will specialize now our view of measurable spaces to the Borel sets of Polish spaces and their more general cousins, analytic sets.

### 2.3.1 Borel Sets in Polish and Analytic Spaces

General measurable spaces and even separable metric spaces are sometimes too general for supporting specific structures. We deal with Polish and analytic spaces which are general enough to support interesting applications, but have specific properties which help establishing vital properties. We remind the reader first of some basic facts and provide then some helpful tools for working with Polish spaces, and their more general cousins, analytic spaces.

Fix for the time being $(X, \tau)$ as a topological space. Recall that a family $\mathcal{B} \subseteq \tau$ of open subsets of $X$ is called a *base* for topology $\tau$ iff each element of $\tau$ can be represented as the union of elements of $\mathcal{B}$. This is equivalent to saying that $\bigcup \{B \mid B \in \mathcal{B}\} = X$, and that we can find for each $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$ an element $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. A *subbase* $\mathcal{S}$ for $\tau$ has the property that the set $\{\bigcap F \mid F \subseteq \mathcal{S} \text{ finite}\}$ of finite intersections of elements of $\mathcal{S}$ forms a base for $\tau$.

Given another topological space $(Y, \vartheta)$, a map $f : X \to Y$ is called $\tau$-$\vartheta$-*continuous* iff the inverse image of an open set from $Y$ is open in $X$ again, i.e., iff $f^{-1}[\vartheta] \subseteq \tau$. The topological
spaces \((X, \tau)\) and \((Y, \vartheta)\) are called homeomorphic iff there exists a \(\tau\)-\(\vartheta\)-continuous bijection \(f : X \to Y\) the inverse of which is \(\vartheta\)-\(\tau\)-continuous.

Proceeding in analogy to measurable spaces, a topology \(\tau\) on a set \(X\) is called initial for a map \(f : X \to Y\) with a topological space \((Y, \vartheta)\) iff \(\tau\) is the smallest topology \(\tau_0\) on \(X\) rendering \(f\) a \(\tau_0\)-\(\vartheta\)-continuous map. For example, if \(Y \subseteq X\) is a subset, then the topological subspace \((Y, \{Y \cap G \mid G \in \tau\})\) is just the initial topology with respect to the inclusion map \(i_Y : Y \to X\).

Dually, if \((X, \tau)\) is a topological space and \(f : X \to Y\) is a map, then the final topology \(S\) on \(Y\) is the largest topology \(S_0\) on \(Y\) making \(f\) \(T\)-\(S_0\)-continuous. Both initial and final topologies generalize to families of spaces and maps.

The topological product \(\prod_{i \in I} (X_i, \tau_i)\) of the topological spaces \((X_i, \tau_i)_{i \in I}\) is the Cartesian product \(\prod_{i \in I} X_i\) endowed with the initial topology with respect to the projections, and the topological sum \(\bigoplus_{i \in I} (X_i, \tau_i)\) of the topological spaces \((X_i, \tau_i)_{i \in I}\) is the direct \(\bigoplus_{i \in I} X_i\) endowed with the final topology with respect to the injections.

An immediate consequence of Lemma 2.6 is that continuity implies Borel measurability.

**Lemma 2.64** Let \((X_1, \tau_1)\) and \((X_2, \tau_2)\) be topological spaces. Then \(f : X_1 \to X_2\) is \(\mathcal{B}(\tau_1)\)-\(\mathcal{B}(\tau_2)\) measurable, provided \(f\) is \(\tau_1\)-\(\tau_2\)-continuous. \(\dagger\)

We note for later use that the limit of a sequence of measurable functions into a metric space is measurable again, see Exercise 14.

**Proposition 2.65** Let \((X, \mathcal{A})\) be a measurable, \((Y, d)\) a metric space, and \((f_n)_{n \in \mathbb{N}}\) be a sequence of \(\mathcal{A}\)-\(\mathcal{B}(Y)\)-measurable functions \(f_n : X \to Y\). Then

- the set \(C := \{x \in X \mid (f_n(x))_{n \in \mathbb{N}} \text{ exists}\}\) is measurable,
- \(f(x) := \lim_{n \to \infty} f_n(x)\) defines a \(\mathcal{A} \cap C\)-\(\mathcal{B}(Y)\)-measurable map \(f : C \to Y\)

Neither general topological spaces nor metric spaces offer a structure rich enough for the study of the transition systems that we will enter into. We need to restrict the class of topological spaces to a particularly interesting class of spaces that are traditionally called Polish.

As far as notation goes, we will write down a topological or a metric space without its adornment through a topology or a metric, unless this becomes really necessary.

Remember that a metric space \((X, d)\) is called complete iff each \(d\)-Cauchy sequence has a limit. Recall also that completeness is really a property of the metric rather than the underlying topological space, so a metrizable space may be complete with one metric and incomplete with another one. In contrast, having a countable base is a topological property which is invariant under the different metrics the topology may admit.

**Definition 2.66** A Polish space \(X\) is a topological space the topology of which is metrizable through a complete metric, and which has a countable base, or, equivalently, a countable dense subset.

Familiar spaces are Polish, as these examples show.

**Example 2.67** The real \(\mathbb{R}\) with their usual topology, which is induced by the open intervals, are a Polish space. \(\mathcal{B}\)
Example 2.68 The open unit interval \([0,1[\) with the usual topology induced by the open intervals form a Polish space.

This comes probably as a surprise, because \([0,1[\) is known not to be complete with the usual metric. But all we need is a dense subset (take the rationals \(\mathbb{Q}\cap[0,1]\)), and a metric that generates the topology, and that is complete. Define

\[
d(x, y) := \left| \ln \frac{x}{1-x} - \ln \frac{y}{1-y} \right|,
\]

then this is a complete metric for \([0,1[\). This is so since \(x \mapsto \ln(x/(1-x))\) is a continuous bijection from \([0,1[\) to \(\mathbb{R}\), and the inverse \(y \mapsto e^y/(1+e^y)\) is also a continuous bijection.

Lemma 2.69 Let \(X\) be a Polish space, and assume that \(F \subseteq X\) is closed, then the subspace \(F\) is Polish as well.

Proof Because \(F\) is closed, each Cauchy sequence in \(F\) has its limit in \(F\), so \(F\) is complete. The topology that \(F\) inherits from \(X\) has a countable base and is metrizable, so \(F\) has a countable dense subset, too. ⊥

Lemma 2.70 Let \((X_n)_{n\in\mathbb{N}}\) be a sequence of Polish spaces, then the product \(\prod_{n\in\mathbb{N}} X_n\) and the coproduct \(\bigsqcup_{n\in\mathbb{N}} X_n\) are Polish spaces.

Proof Assume that the topology \(\tau_n\) on \(X_n\) is metrized through metric \(d_n\), where it may be assumed that \(d_n \leq 1\) holds (otherwise use for \(\tau_n\) the complete metric \(d_n(x,y)/(1+d_n(x,y))\)). Then

\[
d((x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}) := \sum_{n\in\mathbb{N}} 2^{-n}d_n(x_n, y_n)
\]

is a complete metric for the product topology \(\prod_{n\in\mathbb{N}} \tau_n\). For the coproduct, define the complete metric

\[
d(x,y) := \begin{cases} 2, & \text{if } x \in X_n, y \in X_m, n \neq m \\ d_n(x,y), & \text{if } x,y \in X_n. \end{cases}
\]

All this is established through standard arguments. ⊥

Example 2.71 The set \(\mathbb{N}\) of natural numbers with the discrete topology is a Polish space on account of being the topological sum of its elements. Thus the set \(\mathbb{N}^\infty\) of all infinite sequences is a Polish space. The sets

\[
\Sigma_\alpha := \{\tau \in \mathbb{N}^\infty \mid \alpha \text{ is an initial piece of } \tau\}
\]

for \(\alpha \in \mathbb{N}^*\), the free monoid generated by \(\mathbb{N}\), constitute a base for the product topology. ☞

This last example will be discussed in much greater detail later on. It permits sometimes reducing the discussion of properties for general Polish spaces to an investigation of the corresponding properties of \(\mathbb{N}^\infty\), the structure of the latter space being more easily accessible than that of a general space. We apply Example 2.71 directly to show that all open subsets of a metric space \(X\) with a countable base can be represented through a single closed set in \(\mathbb{N}^\infty \times X\).
Recall that for $D \subseteq X \times Y$ the vertical cut $D_x$ is defined through $D_x := \{ y \in Y \mid \langle x, y \rangle \in D \}$ and the horizontal cut $D^y$ is $D^y := \{ x \in X \mid \langle x, y \rangle \in D \}$. Note that $((X \times Y) \setminus D)_x = Y \setminus D_x$.

**Proposition 2.72** Let $X$ be a separable metric space. Then there exists an open set $U \subseteq \mathbb{N}^\infty \times X$ and a closed set $F \subseteq \mathbb{N}^\infty \times X$ with these properties:

a. For each open set $G \subseteq X$ there exists $t \in \mathbb{N}^\infty$ such that $G = U_t$.

b. For each closed set $C \subseteq X$ there exists $t \in \mathbb{N}^\infty$ such that $C = F_t$.

**Proof** 0. It is enough to establish the property for open sets; taking complements will prove it for closed ones.

1. Let $(V_n)_{n \in \mathbb{N}}$ be a basis for the open sets in $X$ with $V_n \neq \emptyset$ for all $n \in \mathbb{N}$. Define

$$U := \{ (t, x) \mid t \in \mathbb{N}^\infty, x \in \bigcup_{n \in \mathbb{N}} V_t \},$$

then $U \subseteq \mathbb{N}^\infty \times X$ is open. In fact, let $(t, x) \in U$, then there exists $n \in \mathbb{N}$ with $x \in V_n$, thus $(t, x) \in \Sigma_n \times V_n \subseteq U$, and $\Sigma_n \times V_n$ is open in the product.

2. Let $G \subseteq X$ be open. Because $(V_n)_{n \in \mathbb{N}}$ is a basis for the topology, there exists a sequence $t \in \mathbb{N}^\infty$ with $G = \bigcup_{n \in \mathbb{N}} V_t = U_t$. ⊥

The set $U$ is usually called a *universal open set*, similar for $F$. These universal sets will be used rather heavily when we discuss analytic sets.

We have seen that a closed subset of a Polish space is a Polish space in its own right; a similar argument shows that an open subset of a Polish space is Polish as well. Both observations turn out to be special cases of the characterization of Polish subspaces through $G_\delta$-sets.

We need for this characterization an auxiliary statement due to Kuratowski which permits the extension of a continuous map from a subspace to a $G_\delta$-set containing it — just far enough to be interesting to us. Denote by $A^\circ$ the topological closure of a set $A$.

**Lemma 2.73** Let $Y$ be a complete metrizable space, $W$ a metric space, then a continuous map $f : A \to Y$ can be extended to a continuous map $f_* : G \to Y$ with $G$ a $G_\delta$-set such that $A \subseteq G \subseteq A^\circ$.

**Proof** 1. We may and do assume that the complete metric $d$ for $Y$ is bounded by 1, otherwise we move to the equivalent and complete metric $\langle x, y \rangle \mapsto d(x, y)/(1 + d(x, y))$, see Exercise [12]

The oscillation $\varnothing_f(x)$ of $f$ at $x \in A^\circ$ is defined as the smallest diameter of the image of an open neighborhood of $x$, formally,

$$\varnothing_f(x) := \inf \{ \text{diam}(f[A \cap V]) \mid x \in V, V \text{ open} \}.$$

Because $f$ is continuous on $A$, we have $\varnothing_f(x) = 0$ for each element $x$ of $A$. In fact, let $\epsilon > 0$ be given, then there exists $\delta > 0$ such that $\text{diam}(f[A \cap V]) < \epsilon$, whenever $V$ is a neighborhood of $x$ of diameter less than $\delta$. Thus $\varnothing_f(x) < \epsilon$; since $\epsilon > 0$ was chosen to be arbitrary, the claim follows.
2. Put \( G := \{ x \in A^a \mid \varnothing f(x) = 0 \} \), then \( A \subseteq G \subseteq A^a \), and \( G \) is a \( G_\delta \) in \( W \). In fact, represent \( G \) as
\[
G = \bigcap_{n \in \mathbb{N}} \{ x \in A^a \mid \varnothing f(x) < \frac{1}{n} \},
\]
so we have to show that \( \{ x \in A^a \mid \varnothing f(x) < q \} \) is open in \( A^a \) for any \( q > 0 \). But we have
\[
\{ x \in A^a \mid \varnothing f(x) < q \} = \bigcup \{ V \cap A^a \mid \text{diam}(f[V \cap A]) < q \}.
\]
This is the union of sets open in \( A^a \), hence is an open set itself. Note that \( A^a \) is — as a closed set — a \( G_\delta \) in \( W \).

3. Now take an element \( x \in G \subseteq A^a \). Then there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) of elements \( x_n \in A \) with \( x_n \rightarrow x \). Given \( \epsilon > 0 \), we find a neighborhood \( V \) of \( x \) with \( \text{diam}(f[A \cap V]) < \epsilon \), since the oscillation of \( f \) at \( x \) is 0. Because \( x_n \rightarrow x \), we know that we can find an index \( n_\epsilon \in \mathbb{N} \) such that \( x_m \in V \cap A \) for all \( m > n_\epsilon \). This implies that the sequence \( (f(x_n))_{n \in \mathbb{N}} \) is a Cauchy sequence in \( Y \). It converges because \( Y \) is complete. Put
\[
f_*(x) := \lim_{n \to \infty} f(x_n).
\]

4. We have to show now that
- \( f_* \) is well-defined.
- \( f_* \) extends \( f \).
- \( f_* \) is continuous.

Assume that we can find \( x \in G \) such that \( (x_n)_{n \in \mathbb{N}} \) and \( (x'_n)_{n \in \mathbb{N}} \) are sequences in \( A \) with \( x_n \rightarrow x \) and \( x'_n \rightarrow x \), but \( \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(x'_n) \). Then we find some \( \eta > 0 \) such that
\[
d(f(x_n), f(x'_n)) \geq \eta \text{ infinitely often}
\]
This implies that \( f_* \) is well-defined, and it implies also that \( f_* \) extends \( f \). Now let \( x \in G \). If \( \epsilon > 0 \) is given, we find a neighborhood \( V \) of \( x \) with \( \text{diam}(f[A \cap V]) < \epsilon \). Thus, if \( x' \in G \cap V \), then \( d(f_*(x), f_*(x')) < \epsilon \). Hence \( f_* \) is continuous. \( \dashv \)

This technical Lemma is an important step in establishing a far reaching characterization of subspaces of Polish spaces that are Polish in their own right. We will show now that a subset \( X \) of a Polish space is a Polish space in its own right iff it is a \( G_\delta \)-set. We will present Kuratowski’s proof for it. It is not difficult to show that \( X \) must be a \( G_\delta \)-set, using Lemma 2.73. The tricky part is the converse, and at its very center is the following idea: assume that we have represented \( X = \bigcap_{k \in \mathbb{N}} G_k \) with each \( G_k \) open, and assume that we have a Cauchy sequence \( (x_n)_{n \in \mathbb{N}} \subseteq X \) with \( x_n \rightarrow x \). How do we prevent \( x \) from being outside \( X \)? Well, what we will do is to set up Kuratowski's trap, preventing the sequence to wander off. The trap is a new complete and equivalent metric \( D \), which is makes it impossible for the sequence to behave bad. So if \( x \) is trapped to be an element of \( X \), we may conclude that \( X \) is complete, and the assertion may be established.

Before we begin with the easier half, we fix a Polish space \( Y \) and a complete metric \( d \) on \( Y \).

**Lemma 2.74** If \( X \subseteq Y \) is a Polish space, then \( X \) is a \( G_\delta \)-set.
Proof $X$ is complete, hence closed in $Y$. The identity $id_X : X \to Y$ can be extended continuously by Lemma 2.73 to a $G_\delta$-set $G$ with $X \subseteq G \subseteq X^\circ$, thus $G = X$, so $X$ is a $G_\delta$-set. \(\dashv\)

Now let $X = \bigcap_{k \in \mathbb{N}} G_k$ with $G_k$ open for all $k \in \mathbb{N}$. In order to prepare for Kuratowski’s trap, we define

$$f_k(x,x’) := \left| \frac{1}{d(x,Y \setminus G_k)} - \frac{1}{d(x’,Y \setminus G_k)} \right|$$

for $x, x’ \in X$. Because $G_k$ is open, we have $x \in G_k$ iff $d(x,Y \setminus G_k) > 0$, so $f_k$ is a finite and continuous function on $X \times X$. Now let

$$F_k(x,x’) := \frac{f_k(x,x’)}{1 + f_k(x,x’)},$$

$$D(x,x’) := d(x,x’) + \sum_{k \in \mathbb{N}} 2^{-k} \cdot F_k(x,x’).$$

for $x, x’ \in X$. Then $D$ is a metric on $X$ (cp. Exercise 12), and the metrics $d$ and $D$ are equivalent on $X$. Because $d(x,x’) \leq D(x,x’)$, it is clear that the identity $id : (X,D) \to (X,d)$ is continuous, so it remains to show that $id : (X,d) \to (X,D)$ is continuous. Let $x \in X$ be given, and let $\epsilon > 0$, then we find $\ell \in \mathbb{N}$ such that $\sum_{k>\ell} 2^{-j} \cdot F_k(x,x’) < \epsilon/3$ for all $x’ \in X$. For $k = 1,\ldots, \ell$ there exists $\delta_j$ such that $F_j(x,x’) < \epsilon/(3 \cdot \ell)$, whenever $d(x,x’) < \delta_j$, since $x \mapsto d(x,Y \setminus G_j)$ is positive and continuous. Thus define $\delta := \epsilon/3 \wedge \delta_1 \wedge \ldots \wedge \delta_\ell$, then $d(x,x’) < \delta$ implies

$$D(x,x’) \leq d(x,x’) + \sum_{k=1}^{\ell} 2^{-j} \cdot F_j(x,x’) + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \sum_{k=1}^{\ell} \frac{\epsilon}{3 \cdot \ell} + \frac{\epsilon}{3} = \epsilon.$$

Thus $(X,d)$ and $(X,D)$ have in fact the same open sets. When establishing that $(X,D)$ is complete, we spring Kuratowski’s trap. Let $(x_n)_{n \in \mathbb{N}}$ be a $D$-Cauchy sequence. Then this sequence is also a $d$-Cauchy sequence, thus we find $x \in Y$ such that $x_n \to x$, because $(Y,d)$ is complete. We claim that $x \in X$. In fact, if $x \in X$, we find $G_\ell$ with $x \notin G_\ell$, so that we can find for each $\epsilon > 0$ some index $n_\epsilon \in \mathbb{N}$ with $F_\ell(x_n,x_m) \geq 1 - \epsilon$ for $n, m \geq n_\epsilon$. But then $D(x_n,x_m) \geq (1-\epsilon)/2^\ell$ for $n, m \geq n_\epsilon$, so that $(x_n)_{n \in \mathbb{N}}$ cannot be a $D$-Cauchy sequence. Consequently, $X$ is complete, hence closed.

Thus we have established:

**Theorem 2.75** Let $Y$ be a Polish space. Then the subspace $X \subseteq Y$ is a Polish space iff $X$ is a $G_\delta$-set. \(\dashv\)

In particular, open and closed subsets of Polish spaces are Polish spaces in their subspace topology. Conversely, each Polish space can be represented as a $G_\delta$-set in the Hilbert cube $[0,1]^\infty$; this is the famous characterization of Polish spaces due to Alexandrov [Kur66, III.33.VI].

**Theorem 2.76 (Alexandrov)** Let $X$ be a separable metric space, then $X$ is homeomorphic to a subspace of the Hilbert cube. If $X$ is Polish, this subspace is a $G_\delta$.

**Proof** 1. We may and do assume again that the metric $d$ is bounded by 1. Let $(x_n)_{n \in \mathbb{N}}$ be a countable and dense subset of $X$, and put

$$f(x) := (d(x,x_1),d(x,x_2),\ldots).$$
Then $f$ is injective and continuous. Define $g : f[X] \to X$ as $f^{-1}$, then $g$ is continuous as well: assume that $f(y_m) \to f(y)$ for some $y$, hence $\lim_{m \to \infty} d(y_m, x_n) = d(y, x_n)$ for each $n \in \mathbb{N}$. Since $(x_n)_{n \in \mathbb{N}}$ is dense, we find for a given $\epsilon > 0$ an index $n$ with $d(y_m, x_n) < \epsilon$; by construction we find for $n$ an index $m_0$ with $d(y_m, x_n) < \epsilon$ whenever $m > m_0$. Thus $d(y_m, y) < 2 \cdot \epsilon$ for $m > m_0$, so that $y_m \to y$. This demonstrates that $g$ is continuous, thus $f$ is a homeomorphism.

2. If $X$ is Polish, $f[X] \subseteq [0,1]^\infty$ is Polish as well. Thus the second assertion follows from Theorem 2.70. ⊣

Recall that a topological Hausdorff space $X$ is compact iff each open cover of $X$ contains a finite cover of $X$. This property of compact spaces will be used from time to time. The Bolzano-Weierstraß Theorem implies that compact metrizable spaces are Polish. It is inferred from Tihonov’s Theorem that the Hilbert cube $[0,1]^\infty$ is compact, because the unit interval $[0,1]$ is compact, again by the Bolzano-Weierstraß Theorem. Thus Alexandrov’s Theorem 2.76 embeds a Polish space as a $G_\delta$ into a compact metric space, the closure of which will be compact.

### 2.3.2 Manipulating Polish Topologies

We will show now that a Borel map between Polish spaces can be turned into a continuous map. Specifically, we will show that, given a measurable map between Polish spaces, we can find on the domain a finer Polish topology with the same Borel sets which renders the map continuous. This will be established through a sequence of auxiliary statements, each of which will be of interest and of use in its own right.

We fix for the discussion to follow a Polish space $X$ with topology $\tau$. Recall that a set is clopen in a topological space iff it is both closed and open.

**Lemma 2.77** Let $F$ be a closed set in $X$. Then there exists a Polish topology $\tau'$ such that $\tau \subseteq \tau'$ (hence $\tau'$ is finer than $\tau$), $F$ is clopen in $\tau'$, and $\mathcal{B}(\tau) = \mathcal{B}(\tau')$.

**Proof** Both $F$ and $X \setminus F$ are Polish by Theorem 2.75, so the topological sum of these Polish spaces is Polish again by Lemma 2.70. The sum topology is the desired topology. ⊣

We will now add a sequence of certain Borel sets to the topology; this will happen step by step, so we should know how to manipulate a sequence of Polish topologies. This is explained now.

**Lemma 2.78** Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of Polish topologies $\tau_n$ with $\tau \subseteq \tau_n$.

1. The topology $\tau_\infty$ generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is Polish.

2. If $\tau_n \subseteq \mathcal{B}(\tau)$, then $\mathcal{B}(\tau_\infty) = \mathcal{B}(\tau)$.

**Proof** 1. The product $\prod_{n \in \mathbb{N}} (X_n, \tau_n)$ is by Lemma 2.70 a Polish space, where $X_n = X$ for all $n$. Define the map $f : X \to \prod_{n \in \mathbb{N}} X_n$ through $x \mapsto \langle x,x,\ldots \rangle$, then $f$ is $\tau_\infty \setminus \prod_{n \in \mathbb{N}} \tau_n$-continuous by construction. One infers that $f[X]$ is a closed subset of $\prod_{n \in \mathbb{N}} X_n$; if $(x_n)_{n \in \mathbb{N}} \notin f[X]$, take $x_i \neq x_j$ with $i < j$, and let $G_i$ and $G_j$ be disjoint open neighborhoods of $x_i$ resp.
Proof

1. We show first that we may add just one Borel set to the topology without changing the Polish space to be Polish or changing the Borel sets. This is true as well for sequences of Borel sets, as we will see now.

Proposition 2.79 If \((B_n)_{n \in \mathbb{N}}\) is a sequence of Borel sets in \(X\), then there exists a Polish topology \(\tau_0\) on \(X\) such that \(\tau_0\) is finer than \(\tau\), \(\tau\) and \(\tau_0\) have the same Borel sets, and each \(B_n\) is clopen in \(\tau_0\).

Proof

Let \(\mathcal{H} := \{B \in \mathcal{B}(\tau) \mid B \text{ is neat}\}\).

Then \(\tau \subseteq \mathcal{H}\), and each closed set is a member of \(\mathcal{H}\) by Lemma 2.77. Furthermore, \(\mathcal{H}\) is closed under complements by construction, and closed under countable unions by Lemma 2.78. Thus we may now infer that \(\mathcal{H} = \mathcal{B}(\tau)\), so that each Borel set is neat.

2. Now construct inductively Polish topologies \(\tau_n\) that are finer than \(\tau\) with \(\mathcal{B}(\tau) = \mathcal{B}(\tau_n)\). Start with \(\tau_0 := \tau\). Adding \(B_{n+1}\) to the Polish topology \(\tau_n\) according to the first part yields a finer Polish topology \(\tau_{n+1}\) with the same Borel sets. Thus the assertion follows from Lemma 2.78.

This permits turning a Borel map into a continuous one, whenever the domain is Polish and the range is a second countable metric space.

Proposition 2.80 Let \(Y\) a separable metric space with topology \(\vartheta\). If \(f : X \to Y\) is a \(\mathcal{B}(\tau)\)-\(\mathcal{B}(\vartheta)\)-Borel measurable map, then there exists a Polish topology \(\tau'\) on \(X\) such that \(\tau'\) is finer than \(\tau\), \(\tau\) and \(\tau'\) have the same Borel sets, and \(f\) is \(\tau'\)-\(\vartheta\) continuous.

Proof

The metric topology \(\vartheta\) is generated from the countable basis \((H_n)_{n \in \mathbb{N}}\). Construct from the Borel sets \(f^{-1}[H_n]\) and from \(\tau\) a Polish topology \(\tau'\) according to Proposition 2.79. Because \(f^{-1}[H_n] \in \tau'\) for all \(n \in \mathbb{N}\), the inverse image of each open set from \(\vartheta\) is \(\tau'\)-open, hence \(f\) is \(\tau'\)-\(\vartheta\) continuous. The construction entails \(\tau\) and \(\tau'\) having the same Borel sets.

This property is most useful, because it permits rendering measurable maps continuous, when they go into a second countable metric space (thus in particular into a Polish space).
space: it is sufficient to establish this property for closed sets. This is justified by the following observation.

**Lemma 2.81** Assume that each closed set in the Polish space $X$ is a continuous image of $\mathbb{N}^\infty$. Then each Borel set of $X$ is a continuous image of $\mathbb{N}^\infty$.

**Proof** (Sketch) 1. Let

$$
\mathcal{G} := \{ B \in \mathcal{B}(X) \mid B = f[\mathbb{N}^\infty] \text{ for } f : \mathbb{N}^\infty \to X \text{ continuous} \}
$$

be the set of all good guys. Then $\mathcal{G}$ contains by assumption all closed sets. We show that $\mathcal{G}$ is closed under countable unions and countable intersections. Then the assertion will follow from Lemma 2.52.

2. Suppose $B_n = f_n[\mathbb{N}^\infty]$ for the continuous map $f_n$, then

$$
\mathcal{M} := \{ \langle t_1, t_2, \ldots \rangle \mid f_1(t_1) = f_2(t_2) = \ldots \}
$$

is a closed subset of $(\mathbb{N}^\infty)^\infty$, and defining $f : \langle t_1, t_2, \ldots \rangle \mapsto f_1(t_1)$ yields a continuous map $f : \mathcal{M} \to X$ with $f[\mathcal{M}] = \bigcap_{n \in \mathbb{N}} B_n$. $\mathcal{M}$ is homeomorphic to $\mathbb{N}^\infty$. Thus $\mathcal{G}$ is closed under countable intersections.

3. We show that $\mathcal{G}$ is closed also under countable unions. In fact, let $B_n \in \mathcal{G}$ such that $B_n = f_n[\mathbb{N}^\infty]$ with $f_n : \mathbb{N}^\infty \to X$ continuous. Define

$$
f : \begin{cases} 
\mathbb{N}^\infty & \to X \\
\langle n, t_1, t_2, \ldots \rangle & \mapsto f_n(t_1, t_2, \ldots).
\end{cases}
$$

Thus

$$
f[\mathbb{N}^\infty] = \bigcup_{n \in \mathbb{N}} f_n[\mathbb{N}^\infty] = \bigcup_{n \in \mathbb{N}} B_n.
$$

Moreover, $f$ is continuous. If $G \subseteq X$ is open, we have $f^{-1}[G] = \bigcup_{n \in \mathbb{N}} \{ n \} \times f_n^{-1}[G]$. Since $f_n^{-1}[G]$ is pen for each $n \in \mathbb{N}$, we conclude that $f^{-1}[G]$ is open, so that $f$ is indeed continuous. Thus $\mathcal{G}$ is closed under countable unions, and the assertion follows from Lemma 2.52.

Thus it is sufficient to show that each closed subset of a Polish space is the continuous image on $\mathbb{N}^\infty$. But since a closed subset of a Polish space is Polish in its own right by Theorem 2.75, we will restrict our attention to Polish spaces proper.

**Proposition 2.82** For Polish $X$ there exists a continuous map $f : \mathbb{N}^\infty \to X$ with $f[\mathbb{N}^\infty] = X$.

**Proof** 0. We will define recursively a sequence of closed sets indexed by elements of $\mathbb{N}^*$ that will enable us to define a continuous map on $\mathbb{N}^\infty$.

1. Let $d$ be a metric that makes $X$ complete. Represent $X$ as $\bigcup_{n \in \mathbb{N}} A_n$ with closed sets $A_n \neq \emptyset$ such that the diameter $\text{diam}(A_n) < 1$ for each $n \in \mathbb{N}$. Assume that for a word $\alpha \in \mathbb{N}^*$ of length $k$ the closed set $A_\alpha \neq \emptyset$ is defined, and write $A_\alpha = \bigcup_{n \in \mathbb{N}} A_{\alpha n}$ with closed sets $A_{\alpha n} \neq \emptyset$ such that $\text{diam}(A_{\alpha n}) < 1/(k+1)$ for $n \in \mathbb{N}$. This yields for every $t = \langle n_1, n_2, \ldots \rangle \in \mathbb{N}^\infty$ a sequence of nonempty closed sets $(A_{n_1 n_2 \ldots n_k})_{k \in \mathbb{N}}$ with diameter $\text{diam}(A_{n_1 n_2 \ldots n_k}) < 1/k$. Because the metric is complete, $\bigcap_{k \in \mathbb{N}} A_{n_1 n_2 \ldots n_k}$ contains exactly one point, which is defined to be $f(t)$. This construction renders $f : \mathbb{N}^\infty \to X$ well defined.
2. Because we can find for each \( x \in X \) an index \( n'_1 \in \mathbb{N} \) with \( x \in A_{n'_1} \), an index \( n'_2 \) with \( x \in A_{n'_1 n'_2} \), etc.; the map just defined is onto, so that \( f((n'_1, n'_2, n'_3, \ldots)) = x \) for some \( t' := \langle n'_1, n'_2, n'_3, \ldots \rangle \in \mathbb{N}^\infty \). Suppose \( \epsilon > 0 \) is given. Since the diameters of the sets \((A_{n_1 n_2 \ldots n_k})_k \in \mathbb{N}\) tend to 0, we can find \( k_0 \in \mathbb{N} \) with \( \text{diam}(A_{n'_1 n'_2 \ldots n'_k}) < \epsilon \) for all \( k > k_0 \). Put \( \alpha' := n'_1 n'_2 \ldots n'_{k_0} \), then \( \Sigma_{\alpha'} \) is an open neighborhood of \( t' \) with \( f[\Sigma_{\alpha'}] \subseteq B_{\epsilon,d}(f(t')) \). Thus we find for an arbitrary open neighborhood \( V \) of \( f(t') \) an open neighborhood \( U \) of \( t' \) with \( f[U] \subseteq V \), equivalently, \( U \subseteq f^{-1}[V] \). Thus \( f \) is continuous. \( \dashv \)

Proposition 2.82 permits sometimes the transfer of arguments pertaining to Polish spaces to arguments using infinite sequences. Thus a specific space is studied instead of an abstractly given one, the former permitting some rather special constructions. This will be capitalized on in the investigation of some astonishing properties of analytic sets which we will study now.

2.4 Analytic Sets and Spaces

We will deal now systematically with analytic sets and spaces. One of the core results of this section will be the Lusin Separation Theorem, which permits to separate two disjoint analytic sets through disjoint Borel sets, and its immediate consequence, the Souslin Theorem, which says that a set which is both analytic and co-analytic is Borel. These beautiful results turn out to be very helpful, e.g., in the investigation of Markov transition systems. In addition, they permit to state and prove a weak form of Kuratowski’s Isomorphism Theorem, stating that a measurable bijection between two Polish spaces is an isomorphisms (hence its inverse is measurable as well).

But first the definition of analytic and co-analytic sets for a Polish space \( X \).

**Definition 2.83** An analytic set in \( X \) is the projection of a Borel subset of \( X \times X \). The complement of an analytic set is called a co-analytic set.

One may wonder whether these projections are Borel sets, but we will show in a moment that there are strictly more analytic sets than Borel sets, whenever the underlying Polish space is uncountable. Thus analytic sets are a proper extension to Borel sets. On the other hand, analytic sets arise fairly naturally, for example from factoring Polish spaces through equivalence relations that are generated from a countable collection of Borel sets. We will see this in Proposition 2.104. Consequently it is sometimes more adequate to consider analytic sets rather than their Borel cousins, e.g., when the equivalence of states in a transition system is at stake.

This is a first characterization of analytic sets (using \( \pi_X \) for the projection to \( X \)).

**Proposition 2.84** Let \( X \) be a Polish space. Then the following statements are equivalent for \( A \subseteq X \):

1. \( A \) is analytic.
2. There exists a Polish space \( Y \) and a Borel set \( B \subseteq X \times Y \) with \( A = \pi_X[B] \).
3. There exists a continuous map \( f : \mathbb{N}^\infty \rightarrow X \) with \( f[\mathbb{N}^\infty] = A \).
4. \( A = \pi_X[C] \) for a closed subset \( C \subseteq X \times \mathbb{N}^\infty \).
Proof The implication \( 1 \Rightarrow 2 \) is trivial, \( 2 \Rightarrow 3 \) follows from Proposition 2.82: 
for some continuous map \( g : N^\infty \to X \times Y \), so put \( f := \pi_X \circ g \). We obtain \( 3 \Rightarrow 4 \) from the observation that the graph \( \{ (t, f(t)) \mid t \in N^\infty \} \) of \( f \) is a closed subset of \( N^\infty \times X \), the first projection of which equals \( A \). Finally, \( 4 \Rightarrow 1 \) is obtained again from Proposition 2.82.

As an immediate consequence we obtain that a Borel set is analytic. Just for the record:

**Corollary 2.85** Each Borel set in a Polish space is analytic.

**Proof** Proposition 2.84 together with Proposition 2.82.

The converse does not hold, as we will show now. This statement is not only of interest in its own right. Historically it initiated the study of analytic and co-analytic sets as a separate discipline in set theory (what is called now Descriptive Set Theory).

**Proposition 2.86** Let \( X \) be an uncountable Polish space. Then there exists an analytic set that is not Borel.

We show as a preparation for the proof of Proposition 2.86 that analytic sets are closed under countable unions, intersections, direct and inverse images of Borel maps. Before doing that, we establish a simple but useful property of the graphs of measurable maps.

**Lemma 2.87** Let \((M, \mathcal{M})\) be a measurable space, \( f : M \to Z \) be a \( \mathcal{M} \)-\( \mathcal{B}(Z) \)-measurable map, where \( Z \) is a separable metric space. The graph of \( f \),

\[
\text{graph}(f) := \{ (m, f(m)) \mid m \in M \},
\]

is a member if \( \mathcal{M} \otimes \mathcal{B}(Z) \).

**Proof** Exercise 9.

Analytic sets have closure properties that are similar to those of Borel sets, but not quite the same: they are closed under countable unions and intersections, and under the inverse image of Borel maps. They are closed under the direct image of Borel maps as well. Suspiciously missing is the closure under complementation (which will give rise to Souslin’s Theorem). This is different from Borel sets.

**Proposition 2.88** Analytic sets in a Polish space \( X \) are closed under countable unions and countable intersections. If \( Y \) is another Polish space, with analytic sets \( A \subseteq X \) and \( B \subseteq Y \), and \( f : X \to Y \) is a Borel map, then \( f[A] \subseteq Y \) is analytic in \( Y \), and \( f^{-1}[B] \) is analytic in \( X \).

**Proof**

1. Using the characterization of analytic sets in Proposition 2.84 it is shown exactly as in the proof to Lemma 2.81 that analytic sets are closed under countable unions and under countable intersections. We trust that the reader will be able to reproduce those arguments here.

2. Note first that for \( A \subseteq X \) the set \( Y \times A \) is analytic in the Polish space \( Y \times X \) by Proposition 2.81. In fact, \( A = \pi_X[B] \) with \( B \subseteq X \times X \) Borel by the first part, hence \( Y \times A = \pi_{Y \times X}[Y \times B] \) with \( Y \times B \subseteq Y \times X \times X \) Borel, which is analytic by the second part. Since \( y \in f[A] \) iff \( \langle x, y \rangle \in \text{graph}(f) \) for some \( x \in A \), we write 

\[
f[A] = \pi_Y[Y \times A \cap \{ \langle y, x \rangle \mid \langle x, y \rangle \in \text{graph}(f) \}].
\]
The set \( \{(y, x) \mid (x, y) \in \text{graph}(f)\} \) is Borel in \( Y \times X \) by Lemma 2.87, so the assertion follows for the direct image. The assertion is proved in exactly the same way for the inverse image. ⊣

**Proof** (of Proposition 2.88) 1. We will deal with the case \( X = \mathbb{N}^\infty \) first, and apply a diagonal argument. Let \( F \subseteq \mathbb{N}^\infty \times (\mathbb{N}^\infty \times \mathbb{N}^\infty) \) be a universal closed set according to Proposition 2.72. Thus each closed set \( C \subseteq \mathbb{N}^\infty \times \mathbb{N}^\infty \) can be represented as \( C = F_t \) for some \( t \in \mathbb{N}^\infty \). Taking first projections, we conclude that there exists a universal analytic set \( U \subseteq \mathbb{N}^\infty \times \mathbb{N}^\infty \) such that each analytic set \( A \subseteq \mathbb{N}^\infty \) can be represented as \( U_t \) for some \( t \in \mathbb{N}^\infty \). In fact, we can write \( A = (\pi'_{\mathbb{N}^\infty \times \mathbb{N}^\infty} F_t) \) with \( \pi'_{\mathbb{N}^\infty \times \mathbb{N}^\infty} \) as the first projection of \( (\mathbb{N}^\infty \times \mathbb{N}^\infty) \times \mathbb{N}^\infty \).

Now set
\[
A := \{ \xi \mid \langle \xi, \xi \rangle \in U \}.
\]

Because analytic sets are closed under inverse images of Borel maps by Proposition 2.88, \( A \) is an analytic set. Suppose that \( A \) is a Borel set, then \( \mathbb{N}^\infty \setminus A \) is also a Borel set, hence analytic. Thus we find \( \xi \in \mathbb{N}^\infty \) such that \( \mathbb{N}^\infty \setminus A = U_\xi \). But now
\[
\xi \in A \Leftrightarrow \langle \xi, \xi \rangle \in U \Leftrightarrow \xi \in U_\xi \Leftrightarrow \xi \in \mathbb{N}^\infty \setminus A.
\]

This is a contradiction.

2. The general case is reduced to the one treated above by observing that an uncountable Polish space contains a homeomorphic copy on \( \mathbb{N}^\infty \). But since we are interested mainly in showing that analytic sets are strictly more general than Borel sets, we refrain from a very technical discussion of this case and refer the reader to [Sri98, Remark 2.6.5]. ⊣

The representation of an analytic set through a continuous map on \( \mathbb{N}^\infty \) has the remarkable consequence that we can separate two disjoint analytic sets by disjoint Borel sets (Lusin’s Separation Theorem). This in turn implies a pretty characterization of Borel sets due to Souslin which says that an analytic set is Borel iff it is co-analytic as well. Since the latter characterization will be most valuable to us, we will discuss it in greater detail now.

We start with Lusin’s Separation Theorem.

**Theorem 2.89** Given disjoint analytic sets \( A \) and \( B \) in a Polish space \( X \), there exist disjoint Borel sets \( E \) and \( F \) with \( A \subseteq E \) and \( B \subseteq F \).

**Proof** 0. Call two analytic sets \( A \) and \( B \) separated by Borel sets iff \( A \subseteq E \) and \( B \subseteq F \) for disjoint Borel sets \( E \) and \( F \). Observe that if two sequences \( (A_n)_{n \in \mathbb{N}} \) and \( (B_n)_{n \in \mathbb{N}} \) have the property that \( A_m \) and \( B_n \) can be separated by Borel sets for all \( m, n \in \mathbb{N} \), then \( \bigcup_{n \in \mathbb{N}} A_n \) and \( \bigcup_{m \in \mathbb{N}} B_m \) can also be separated by Borel sets. In fact, if \( E_{m,n} \) and \( F_{m,n} \) separate \( A_n \) and \( B_m \), then \( E := \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} E_{m,n} \) and \( F := \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} F_{m,n} \) separate \( \bigcup_{n \in \mathbb{N}} A_n \) and \( \bigcup_{m \in \mathbb{N}} B_m \).

1. Now suppose that \( A = f[N^\infty] \) and \( B = g[N^\infty] \) cannot be separated by Borel sets, where \( f, g : N^\infty \to X \) are continuous and chosen according to Proposition 2.83. Because \( N^\infty = \bigcup_{j \in \mathbb{N}} \Sigma_j \), \( \Sigma_\alpha \) is defined in Example 2.71, we find indices \( k_1 \) and \( \ell_1 \) such that \( f[\Sigma_{j_1}] \) \( (\xi) \) cannot be separated by Borel sets. For the same reason, there exist indices \( k_2 \) and \( \ell_2 \) such that \( f[\Sigma_{j_2}] \) and \( g[\Sigma_{\ell_2}] \) cannot be separated by Borel sets. Continuing with this, we define infinite sequences \( \kappa := (k_1, k_2, \ldots) \) and \( \lambda := (\ell_1, \ell_2, \ldots) \) such that for each \( n \in \mathbb{N} \) the sets \( f[\Sigma_{j_1 j_2 \ldots j_n}] \) and \( g[\Sigma_{\ell_1 \ell_2 \ldots \ell_n}] \) cannot be separated by Borel sets. Because \( f(\kappa) \in A \) and \( g(\lambda) \in B \), we see that \( f(\kappa) \) and \( g(\lambda) \) cannot be separated by Borel sets, which contradicts our assumption.
and $g(\lambda) \in B$, we know $f(\kappa) \neq g(\lambda)$, so we find $\epsilon > 0$ with $d(f(\kappa), g(\lambda)) < 2 \cdot \epsilon$. But we may choose $n$ large enough so that both $f\left[\sum_{j_1 j_2 \ldots j_n}\right]$ and $g\left[\sum_{\ell_1 \ell_2 \ldots \ell_n}\right]$ have a diameter smaller than $\epsilon$ each. This is a contradiction since we now have separated these sets by open balls. \(\dagger\)

We obtain as a consequence Souslin’s Theorem.

**Theorem 2.90 (Souslin)** Let $A$ be an analytic set in a Polish space. If $X \setminus A$ is analytic, then $A$ is a Borel set.

**Proof** Let $A$ and $X \setminus A$ be analytic, then they can be separated by disjoint Borel sets $E$ with $A \subseteq E$ and $F$ with $X \setminus A \subseteq F$ by Lusin’s Theorem \[2.89\] Thus $A = E$ is a Borel set. \(\dagger\)

Souslin’s Theorem is important when one wants to show that a set is a Borel set that is given for example through the image of another Borel set. A typical scenario for its use is establishing for a Borel set $A$ given for example through the image of another Borel set. A typical scenario for its use is establishing for a Borel set $A$ and a Borel map $f : X \to Y$ that both $C = f[A]$ and $Y \setminus C = f[X \setminus A]$ hold. Then one infers from Proposition \[2.88\] that both $C$ and $Y \setminus C$ are analytic, and from Souslin’s Theorem that $A$ is a Borel set. This is a first simple example:

**Proposition 2.91** Let $f : X \to Y$ be surjective and Borel measurable, where $X$ and $Y$ are Polish. Assume that the set $A \in \mathcal{B}(X)$ has this property: $x \in A$ and $f(x) = f(x')$ implies $x' \in A$. Then $f[A] \in \mathcal{B}(Y)$.

**Proof** Put $C := f[A]$, $D := f[X \setminus A]$, then both $C$ and $D$ are analytic sets by Proposition \[2.88\]. Clearly $Y \setminus C \subseteq D$. For establishing the other inclusion, let $y \in D$, hence there exists $x \notin A$ with $y = f(x)$. But $y \notin C$, for otherwise there exists $x' \in A$ with $y = f(x')$, which implies $x \in A$. Thus $y \in Y \setminus C$. We infer $f[A] \in \mathcal{B}(Y)$ now from Theorem \[2.90\] \(\dagger\)

This yields as an immediate consequence, which will be extended to analytic spaces in Proposition \[2.95\] with essentially the same argument.

**Corollary 2.92** Let $f : X \to Y$ be measurable and bijective with $X$, $Y$ Polish. Then $f$ is a Borel isomorphism. \(\dagger\)

We state finally Kuratowski’s Isomorphism Theorem.

**Theorem 2.93** Any two Borel sets of the same cardinality contained in Polish spaces are Borel isomorphic. \(\dagger\)

The proof requires a reduction to the Cantor ternary set, using the tools we have discussed here so far. Since giving the proof would lead us fairly deep into the Wonderland of Descriptive Set Theory, we do not give it here and refer rather to \[St98\] Theorem 3.3.13], \[Kec94\] Section 15.B) or \[KM76\] p. 442.

We make the properties of analytic sets a bit more widely available by introducing analytic spaces. Roughly, an analytic space is Borel isomorphic to an analytic set in a Polish space; to be more precise:

**Definition 2.94** A measurable space $(M, \mathcal{M})$ is called an analytic space iff there exists a Polish space $X$ and an analytic set $A$ in $X$ such that the measurable spaces $(M, \mathcal{M})$ and $(A, \mathcal{B}(X) \cap A)$ are Borel isomorphic. The elements of $\mathcal{M}$ are then called the Borel sets of $M$. $\mathcal{M}$ is denoted by $\mathcal{B}(M)$. 

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We will omit the $\sigma$-algebra from the notation of an analytic space.

Analytic spaces share many favorable properties with analytic sets, and with Polish spaces, but they are a wee bit more general: whereas an analytic set lives in a Polish space, an analytic space does only require a Polish space to sit in the background somewhere and to be Borel isomorphic to it. This makes life considerably easier, since we are not always obliged to present a Polish space directly when dealing with properties of analytic spaces.

Take a Borel measurable bijection between two Polish spaces. It is not a priori clear whether or not this map is an isomorphism. Souslin’s Theorem gives a helpful hand here as well. We will need this property in a moment for a characterization of countably generated sub-$\sigma$-algebras of Borel sets, but it appears to be interesting in its own right.

**Proposition 2.95** Let $X$ and $Y$ be analytic spaces and $f : X \to Y$ be a bijection that is Borel measurable. Then $f$ is a Borel isomorphism.

**Proof** 1. It is no loss of generality to assume that we can find Polish spaces $P$ and $Q$ such that $X$ and $Y$ are subsets of $P$ resp. $Q$. We want to show that $f[X \cap B]$ is a Borel set in $Y$, whenever $B \in B(P)$ is a Borel set. For this we need to find a Borel set $G \in B(Q)$ such that $f[X \cap B] = G \cap Q$.

2. Clearly, both $f[X \cap B]$ and $f[X \setminus B]$ are analytic sets in $Q$ by Proposition 2.88, and because $f$ is injective, they are disjoint. Thus we can find a Borel set $G \in B(Q)$ with $f[X \cap B] \subseteq G \cap Y$, and $f[X \setminus B] \subseteq Q \setminus G \cap Y$. Because $f$ is surjective, we have $f[X \cap B] \cup f[X \setminus B]$, thus $f[X \cap B] = G \cap Y$.

Separable measurable spaces are characterized through subsets of Polish spaces.

**Lemma 2.96** The measurable space $(M, \mathcal{M})$ is separable iff there exists a Polish space $X$ and a subset $P \subseteq X$ such that the measurable spaces $(M, \mathcal{M})$ and $(P, B(X) \cap P)$ are Borel isomorphic.

**Proof** 1. Because $B(X)$ is countably generated for a Polish space $X$ by Lemma 2.51, the $\sigma$-algebra $B(X) \cap P$ is countably generated. Since this property is not destroyed by Borel isomorphisms, the condition above is sufficient.

2. It is also necessary by Proposition 2.58 because $\{0, 1\}^\mathbb{N}$, $\bigotimes_{n \in \mathbb{N}} \mathcal{P}(\{0, 1\})$ is a Polish space by Lemma 2.70.

Thus analytic spaces are separable.

**Corollary 2.97** An analytic space is a separable measurable space.

Let us have a brief look at countably generated sub-$\sigma$-algebras of an analytic space. This will help establishing for example that the factor space for a particularly interesting and important class of equivalence relations is an analytic space. The following statement, which is sometimes referred to as the Unique Structure Theorem [Arv76, Theorem 3.3.5], says essentially that the Borel sets of an analytic space are uniquely determined by being countably generated and by separating points. It comes as a consequence of our discussion of Borel isomorphisms.

**Proposition 2.98** Let $X$ be an analytic space, $\mathcal{B}_0$ a countably generated sub-$\sigma$-algebra of $B(X)$ that separates points. Then $\mathcal{B}_0 = B(X)$. 

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Proof 1. \((X, \mathcal{B}_0)\) is a separable measurable space, so there exists a Polish space \(P\) and a subset \(Y \subseteq P\) of \(P\) such that \((X, \mathcal{B}_0)\) is Borel isomorphic to \(P, \mathcal{B}(P) \cap Y\) by Lemma 2.96. Let \(f\) be this isomorphism, then \(\mathcal{B}_0 = f^{-1}[\mathcal{B}(P) \cap Y]\).

2. \(f\) is a Borel map from \((X, \mathcal{B}(X))\) to \((Y, \mathcal{B}(P) \cap Y)\), thus \(Y\) is an analytic set with \(\mathcal{B}(Y) = \mathcal{B}(X) \cap P\) by Proposition 2.101. By Proposition 2.88, \(f\) is an isomorphism, hence \(\mathcal{B}(X) = f^{-1}[\mathcal{B}(P) \cap Y]\). But this establishes the assertion. \(\square\)

This gives an interesting characterization of measurable spaces to be analytic, provided they have a separating sequence of sets. Note that the sequence of sets in the following statement is required to separate points, but we do not assume that it generates the \(\sigma\)-algebra for the underlying space. The statement says that it does, actually.

**Lemma 2.99** Let \(X\) be analytic, \(f : X \to Y\) be \(\mathcal{B}(X)\)-\(\mathcal{B}\)-measurable and onto for a measurable space \((Y, \mathcal{B})\), which has a sequence of sets in \(\mathcal{B}\) that separate points. Then \((Y, \mathcal{B})\) is analytic.

**Proof** 1. The idea is to show that an arbitrary measurable set is contained in the \(\sigma\)-algebra generated by the sequence in question. Thus let \((B_n)_{n \in \mathbb{N}}\) be the sequence of sets that separates points, take an arbitrary set \(N \in \mathcal{B}\) and define the \(\sigma\)-algebra \(\mathcal{B}_0 := \sigma(\{B_n \mid n \in \mathbb{N}\} \cup \{N\})\). We want to show that \(N \in \sigma(\{B_n \mid n \in \mathbb{N}\})\), and we show this in a roundabout way by showing that \(\mathcal{B} = \mathcal{B}(Y) = \mathcal{B}_0\). Here is, how.

2. Then \((Y, \mathcal{B}_0)\) is a separable measurable space, so by Lemma 2.96 we can find a Polish space \(P\) with \(Y \subseteq P\) and \(\mathcal{B}_0\) as the trace of \(\mathcal{B}(P)\) on \(Y\). Proposition 2.88 tells us that \(Y = f[X]\) is analytic with \(\mathcal{B}_0 = \mathcal{B}(Y)\), and from Proposition 2.98 it follows that \(\mathcal{B}(Y) = \sigma(\{B_n \mid n \in \mathbb{N}\})\). Thus \(N \in \mathcal{B}(Y)\), and since \(N \in \mathcal{B}\) is arbitrary, we conclude \(B \subseteq \mathcal{B}(Y)\), thus \(\mathcal{B} \subseteq \mathcal{B}(Y) = \sigma(\{B_n \mid n \in \mathbb{N}\}) \subseteq \mathcal{B}\). \(\square\)

We will use Lemma 2.99 for demonstrating that factoring an analytic space through a smooth equivalence relation yields an analytic space again. This class of relations will be defined now and briefly characterized here. We give a definition in terms of a determining sequence of Borel sets and relate other characterizations of smoothness in Lemma 2.103.

**Definition 2.100** Let \(X\) be an analytic space and \(\rho\) an equivalence relation on \(X\). Then \(\rho\) is called smooth iff there exists a sequence \((A_n)_{n \in \mathbb{N}}\) of Borel sets such that

\[ x \sim x' \iff \forall n \in \mathbb{N} : [x \in A_n \iff x' \in A_n], \]

\((A_n)_{n \in \mathbb{N}}\) is said to determine the relation \(\rho\).

**Example 2.101** Given an analytic space \(X\), let \(M : X \rightsquigarrow X\) be a transition kernel which interprets the modal logic presented in Example 2.12. Define for a formula \(\varphi\) and an element of \(x\) the relation \(M, x \models \varphi\) iff \(x \in \llbracket \varphi \rrbracket_M\), thus \(M, x \models \varphi\) indicates that formula \(\varphi\) is valid in state \(x\). Define the equivalence relation \(\sim\) on \(X\) through

\[ x \sim x' \iff \forall \varphi : [M, x \models \varphi \iff M, x' \models \varphi] \]

Thus \(x\) and \(x'\) cannot be separated through a formula of the logic. Because the logic has only countably many formulas, the relation is smooth with the countable set \(\{\llbracket \varphi \rrbracket_M \mid \varphi\ \text{is a formula}\}\) as determining relation \(\sim\).

We obtain immediately from the definition that a smooth equivalence relation — seen as a subset of the Cartesian product — is a Borel set:
Corollary 2.102 Let $\rho$ be a smooth equivalence relation on the analytic space $X$, then $\rho$ is a Borel subset of $X \times X$.

Proof Suppose that $(A_n)_{n \in \mathbb{N}}$ determines $\rho$. Since $x \rho x'$ is false iff there exists $n \in \mathbb{N}$ with $\langle x, x' \rangle \in (A_n \times (X \setminus A_n)) \cup ((X \setminus A_n) \times A_n)$, we obtain

$$(X \times X) \setminus \rho = \bigcup_{n \in \mathbb{N}} (A_n \times (X \setminus A_n)) \cup ((X \setminus A_n) \times A_n).$$

This is clearly a Borel set in $X \times X$. $\dashv$

The following characterization of smooth equivalence relations is sometimes helpful and shows that it is not necessary to focus on sequences of sets. It indicates that the kernels of Borel measurable maps and smooth relations are intimately related.

Lemma 2.103 Let $\rho$ be an equivalence relation on an analytic set $X$. Then these conditions are equivalent:

1. $\rho$ is smooth.

2. There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of Borel maps $f_n : X \to Z$ into an analytic space $Z$ such that $\rho = \bigcap_{n \in \mathbb{N}} \ker (f_n)$.

3. There exists a Borel map $f : X \to Y$ into an analytic space $Y$ with $\rho = \ker (f)$.

Proof 1 $\Rightarrow$ 2 Let $(A_n)_{n \in \mathbb{N}}$ determine $\rho$, then

$$x \rho x' \iff \forall n \in \mathbb{N} : [x \in A_n \iff x' \in A_n]$$

Thus take $Z = \{0, 1\}$ and $f_n := \chi_{A_n}$.

2 $\Rightarrow$ 3 Put $Y := Z^\infty$. This is an analytic space in the product $\sigma$-algebra, and

$$f : \begin{cases} X &\to Y \\
x &\mapsto (f_n(x))_{n \in \mathbb{N}} \end{cases}$$

is Borel measurable with $f(x) = f(x')$ iff $\forall n \in \mathbb{N} : f_n(x) = f_n(x')$.

3 $\Rightarrow$ 1 Since $Y$ is analytic, it is separable; hence the Borel sets are generated through a sequence $(B_n)_{n \in \mathbb{N}}$ which separates points. Put $A_n := f^{-1}[B_n]$, then $(A_n)_{n \in \mathbb{N}}$ is a sequence of Borel sets, because the base sets $B_n$ are Borel in $Y$, and because $f$ is Borel measurable. We claim that $(A_n)_{n \in \mathbb{N}}$ determines $\rho$:

$$f(x) = f(x') \iff \forall n \in \mathbb{N} : [f(x) \in B_n \iff f(x') \in B_n]$$

(since $(B_n)_{n \in \mathbb{N}}$ separates points in $Z$)

$$\iff \forall n \in \mathbb{N} : [x \in A_n \iff x' \in A_n].$$

Thus $\langle x, x' \rangle \in \ker (f)$ is equivalent to being determined by a sequence of measurable sets. $\dashv$

Thus each smooth equivalence relation may be represented as the kernel of a Borel map, and vice versa.
The interest in analytic spaces comes from the fact that factoring an analytic space through a smooth equivalence relation will result in an analytic space again. This requires first and foremost the definition of a measurable structure induced by the relation. The natural choice is the structure imposed by the factor map. The final $\sigma$-algebra on $X/\rho$ with respect to the Borel sets on $X$ and the natural projection $\eta_\rho$ will be chosen; it is denoted by $\mathcal{B}(X)/\rho$. Recall that $\mathcal{B}(X)/\rho$ is the largest $\sigma$-algebra $\mathcal{C}$ on $X/\rho$ rendering $\eta_\rho$ a $\mathcal{B}(X)$-$\mathcal{C}$-measurable map. Then it turns out that $\mathcal{B}(X)/\rho$ coincides with $\mathcal{B}(X)/\rho$:

**Proposition 2.104** Let $X$ be an analytic space, and assume that $\alpha$ is a smooth equivalence relation on $X$. Then $X/\alpha$ is an analytic space.

**Proof** In accordance with the characterization of smooth relations in Lemma 2.103, we assume that $\alpha$ is given through a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable maps $f_n : X \to \mathbb{R}$. The factor map is measurable and onto. Put $E_{n,r} := \{[x]_\alpha \mid x \in X, f_n(x) < r\}$, then $\mathcal{E} := \{E_{n,r} \mid n \in \mathbb{N}, r \in \mathbb{Q}\}$ is a countable set of elements of the factor $\sigma$-algebra that separates points. The assertion now follows without difficulties from Lemma 2.99.

Let us have a look at invariant sets for an equivalence relation $\alpha$.

**Definition 2.105** Call a subset $A \subseteq X$ $\alpha$-invariant for the equivalence relation $\alpha$ on $X$ iff $A$ is the union of $\alpha$-equivalence classes.

Thus $A \subseteq X$ is $\alpha$-invariant iff $x \in A$ and $x \alpha x'$ implies $x' \in A$. Denote by $A^\nabla := \bigcup\{[x]_\alpha \mid x \in A\}$ the smallest $\alpha$-invariant set containing $A$, then we have the representation $A^\nabla = \pi_2[\alpha \cap (X \times A)]$, because $x' \in A^\nabla$ iff there exists $x$ with $\langle x', x \rangle \in X \times A$.

An equivalence relation on $X$ is called analytic resp. closed iff it constitutes an analytic resp. closed subset of the Cartesian product $X \times X$.

If $X$ is a Polish space, we know that the smooth equivalence relation $\alpha \subseteq X \times X$ is a Borel subset by Corollary 2.102. We want to show that, conversely, each closed equivalence relation $\alpha \subseteq X \times X$ is smooth. This requires the identification of a countable set which generates the relation, and for this we require the following auxiliary statement. It may be called separation through invariant sets.

**Lemma 2.106** Let $\rho \subseteq X \times X$ be an analytic equivalence relation on the Polish space $X$ with two disjoint analytic sets $A$ and $B$. If $B$ is $\rho$-invariant, then there exists a $\rho$-invariant Borel set $C$ with $A \subseteq C$ and $B \cap C = \emptyset$.

**Proof** 1. If $D$ is an analytic set, $D^\nabla$ is; this follows from the representation of $D^\nabla$ above, and from Proposition 2.88. We construct a sequence $(A_n)_{n \in \mathbb{N}}$ of invariant analytic sets, and a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel sets with these properties: $A_n \subseteq B_n \subseteq A_{n+1}$, hence $B_n$ is sandwiched between consecutive elements of the first sequence, $A \subseteq A_1$, and $B \cap B_n = \emptyset$ for all $n \in \mathbb{N}$.

2. Define $A_1 := A^\nabla$, then $A \subseteq A_1$, and $A_1$ is $\rho$-invariant. Since $B$ is $\rho$-invariant as well, we conclude $A_1 \cap B = \emptyset$: if $x \in A_1 \cap B$, we find $x' \in A$ with $x \rho x'$, hence $x' \in B$, a contradiction. Proceeding inductively, assume that we have already chosen $A_n$ and $B_n$ with the properties described above, then put $A_{n+1} := B_n^\nabla$, then $A_{n+1}$ is $\rho$-invariant and analytic, also $A_{n+1} \cap B = \emptyset$ by the argument above. Hence we can find a Borel set $B_{n+1}$ with $A_{n+1} \subseteq B_{n+1}$ and $B_{n+1} \cap B = \emptyset$. 

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3. Now put $C := \bigcup_{n \in \mathbb{N}} B_n$. Thus $C \in \mathcal{B}(X)$ and $C \cap B = \emptyset$, so it remains to show that $C$ is $\rho$-invariant. Let $x \in C$ and $x \rho x'$. Since $x \in B_n \subseteq B_n^\infty \subseteq B_{n+1}$, we conclude $x' \in B_{n+1} \subseteq C$, and we are done. \(\dashv\)

We use this observation now for a closed equivalence relation. Note that the assumption on being analytic in the proof above was made use of in order to establish that the invariant hull of an analytic set is analytic again.

**Proposition 2.107** A closed equivalence relation on a Polish space is smooth.

**Proof** 0. Let $X$ be a Polish space, and $\alpha \subseteq X \times X$ be a closed equivalence relation. We have to find a sequence $(A_n)_{n \in \mathbb{N}}$ of Borel sets which determines $\alpha$.

1. Since $X$ is Polish, it has a countable basis $\mathcal{G}$. Because $\alpha$ is closed, we can write

$$(X \times X) \setminus \alpha = \bigcup \{ U_n \times U_m \mid U_n, U_m \in \mathcal{G}_0, U_n \cap U_m = \emptyset \}$$

for some countable subset $\mathcal{G}_0 \subseteq \mathcal{G}$. Fix $U_n$ and $U_m$, then also $U^\infty_n$ and $U_m$ are disjoint. Select the invariant Borel set $A_n$ such that $U_n \subseteq A_n$ and $A_n \cap U_m = \emptyset$; this is possible by Lemma 2.106.

2. We claim that

$$(X \times X) \setminus \alpha = \bigcup_{n \in \mathbb{N}} (A_n \times (X \setminus A_n)).$$

In fact, if $\langle x, x' \rangle \not\in \alpha$, select $U_n$ and $U_m$ with $\langle x, x' \rangle \subseteq U_n \times U_m \subseteq A_n \times (X \setminus A_n)$. If, conversely, $\langle x, x' \rangle \in A_n \times (X \setminus A_n)$, then $\langle x, x' \rangle \in \alpha$ implies by the invariance of $A_n$ that $x' \in A_n$, a contradiction. \(\dashv\)

The Blackwell-Mackey-Theorem analyzes those Borel sets that are unions of $\mathcal{A}$-atoms for a sub-$\sigma$-algebra $\mathcal{A} \subseteq \mathcal{B}(X)$. If $\mathcal{A}$ is countably generated by, say, $(A_n)_{n \in \mathbb{N}}$, then it is not difficult to see that an atom in $\mathcal{A}$ can be represented as

$$\bigcap_{i \in T} A_i \cap \bigcap_{i \in \mathbb{N} \setminus T} (X \setminus A_i)$$

for a suitable subset $T \subseteq \mathbb{N}$, see Proposition 2.62. It constructs a measurable map $f$ so that the set under consideration is ker $(f)$-invariant, which will be helpful in the application of the Souslin Theorem. But let’s see.

**Theorem 2.108** (*Blackwell-Mackey*) Let $X$ be an analytic space and $\mathcal{A} \subseteq \mathcal{B}(X)$ be a countably generated sub-$\sigma$-algebra of the Borel sets of $X$. If $B \subseteq X$ is a Borel set that is a union of atoms of $\mathcal{A}$, then $B \in \mathcal{A}$.

The idea of the proof is to show that $f[B]$ and $f[X \setminus B]$ are disjoint analytic sets for the measurable map $f$, and to conclude that $B = f^{-1}[C]$ for some Borel set $C$, which will be supplied to us through Souslin’s Theorem.

**Proof** Let $\mathcal{A}$ be generated by $(A_n)_{n \in \mathbb{N}}$, and define

$$f : X \to \{0, 1\}^\infty$$

through

$$x \mapsto (\chi_{A_1}(x), \chi_{A_2}(x), \chi_{A_3}(x), \ldots).$$
Then $f$ is $A \cdot B(\{0,1\}^\infty)$-measurable. We claim that $f[B]$ and $f[X \setminus B]$ are disjoint. Suppose not, then we find $t \in (0,1)\^\infty$ with $t = f(x) = f(x')$ for some $x \in B, x' \in X \setminus B$. Because $B$ is the union of atoms, we find a subset $T \subseteq \mathbb{N}$ with $x \in A_n$, provided $n \in T$, and $x \notin A_n$, provided $n \notin T$. But since $f(x) = f(x')$, the same holds for $x'$ as well, which means that $x' \in B$, contradicting the choice of $x'$.

Because $f[B]$ and $f[X \setminus B]$ are disjoint analytic sets, we find through Souslin’s Theorem a Borel set $C$ with

$$f[B] \subseteq C, f[X \setminus B] \cap C = \emptyset.$$ 

Thus $f[B] = C$, so that $f^{-1}[f[B]] = f^{-1}[C] \in \mathcal{A}$. We show that $f^{-1}[f[B]] = B$. It is clear that $B \subseteq f^{-1}[f[B]]$, so assume that $f(b) \in f[B]$, so $f(b) = f(b')$ for some $b' \in B$. By construction, this means $b \in B$, since $B$ is an union of atoms, hence $f^{-1}[f[B]] \subseteq B$. Consequently, $B = f^{-1}[C] \in \mathcal{A}$. $\Box$

When investigating modal logics, one wants to be able to identify the $\sigma$-algebra which is defined by the validity sets of the formulas. This can be done through the Blackwell-Mackey-Theorem and is formulated for generals smooth equivalence relations.

**Proposition 2.109** Let $\rho$ be a smooth equivalence relation on the Polish space $X$, and assume that $(A_n)_{n \in \mathbb{N}}$ generates $\rho$. Then

1. $\sigma(\{A_n \mid n \in \mathbb{N}\})$ is the $\sigma$-algebra of $\rho$-invariant Borel sets,

2. $\mathcal{B}(X/\rho) = \sigma(\{\eta_\rho[A_n] \mid n \in \mathbb{N}\})$.

**Proof** 1. Denote by $\mathcal{I}$ be the $\sigma$-algebra of $\rho$-invariant Borel sets; we have to show that $\mathcal{I} = \sigma(\{A_n \mid n \in \mathbb{N}\}$.

“$\supseteq$” Each $A_n$ is a $\rho$-invariant Borel set.

“$\subseteq$” Let $B$ be an $\rho$-invariant Borel set, then $B = \bigcup_{b \in B} [b]_\rho$. Each class $[b]_\rho$ can be written as

$$[b]_\rho = \bigcap_{b \in A_n} A_n \cap \bigcap_{n \notin A_n} (X \setminus A_n),$$

thus $[b]_\rho \in \sigma(\{A_n \mid n \in \mathbb{N}\})$. Moreover, it is easy to see that the classes are the atoms of this $\sigma$-algebra (in fact, we cannot find a proper non-empty $\rho$-invariant subset of an equivalence class). Thus the Blackwell-Mackey Theorem shows that $B \in \sigma(\{A_n \mid n \in \mathbb{N}\}$).

2. Now let $\mathcal{E} := \sigma(\{\eta_\rho[A_n] \mid n \in \mathbb{N}\}$, and let $g : X/\rho \to P$ be $\mathcal{E}$-$\mathcal{P}$-measurable for an arbitrary measurable space $(P, \mathcal{P})$. Thus we have for all $C \in \mathcal{P}$

$$g^{-1}[C] \in \mathcal{E} \iff \eta_\rho^{-1}[g^{-1}[C]] \in \sigma(\{A_n \mid n \in \mathbb{N}\} \quad \text{since } A_n = \eta_\rho^{-1}[\eta_\rho[A_n]] \quad \text{part 1.}$$

Thus $\mathcal{E}$ is the final $\sigma$-algebra with respect to $\eta_\rho$, hence equals $\mathcal{B}(X/\rho)$. $\Box$

The following example shows that the equivalence relation generated by a $\sigma$-algebra need not return the $\sigma$-algebra as its invariant sets, if the given $\sigma$-algebra is not countably generated. Proposition assures us that this cannot happen in the countably generated case.
Example 2.110 Let $\mathcal{C}$ be the countable-cocountable $\sigma$-algebra on $\mathbb{R}$. The equivalence relation $\equiv_{\mathcal{C}}$ generated by $\mathcal{C}$ according to Example 2.7 is the identity. Hence is it smooth. The $\sigma$-algebra of $\equiv_{\mathcal{C}}$-invariant Borel sets equals the Borel set $\mathcal{B}(\mathbb{R})$, which is a proper superset of $\mathcal{C}$.

The next example is a somewhat surprising application of the Blackwell-Mackey Theorem, taken from [RRS1, Proposition 57]. It shows that the set of countably generated $\sigma$-algebras is not closed under finite intersections, hence fails to be a lattice under inclusion.

Example 2.111 There exist two countably generated $\sigma$-algebras the intersection of which is not countably generated. In fact, let $A \subseteq [0,1]$ be an analytic set which is not Borel, then $\mathcal{B}(A)$ is countably generated by Corollary 2.97. Let $f: [0,1] \to A$ be a bijection, and consider $\mathcal{C} := f^{-1}[\mathcal{B}(A)]$, which is countably generated as well. Then $\mathcal{D} := \mathcal{B}([0,1]) \cap \mathcal{C}$ is a $\sigma$-algebra which has all singletons in $[0,1]$ as atoms. Assume that $\mathcal{D}$ is countably generated, then $\mathcal{D} = \mathcal{B}([0,1])$ by the Blackwell-Mackey-Theorem 2.108. But this means that $\mathcal{C} = \mathcal{B}([0,1])$, so that $f: [0,1] \to A$ is a Borel isomorphism, hence $A$ is a Borel set in $[0,1]$, contradicting the assumption.

Among the consequences of Example 2.111 is the observation that the set of smooth equivalence relations of a Polish space does not form a lattice under inclusion, but is usually only a $\cap$-semilattice, as the following example shows. Another consequence is mentioned in Exercise 18.

Example 2.112 The intersection $\alpha_1 \cap \alpha_2$ of two smooth equivalence relations $\alpha_1$ and $\alpha_2$ is smooth again: if $\alpha_i$ is generated by the Borel sets $\{A_{i,n} | n \in \mathbb{N}\}$ for $i = 1, 2$, then $\alpha_1 \cap \alpha_2$ is generated by the Borel sets $\{A_{i,n} | i = 1, 2, n \in \mathbb{N}\}$. But now take two countably generated $\sigma$-algebras $\mathcal{A}_i$, and let $\alpha_i$ be the equivalence relations determined by them, see Example 2.7.

Then the $\sigma$-algebra $\alpha_1 \cup \alpha_2$ is generated by $\mathcal{A}_1 \cap \mathcal{A}_2$, which is by assumption not countably generated. Hence $\alpha_1 \cup \alpha_2$ is not smooth.

Sometimes one starts not with a topological space and its Borel sets but rather with a measurable space: A standard Borel space $(X, \mathcal{A})$ is a measurable space such that the $\sigma$-algebra $\mathcal{A}$ equals $\mathcal{B}(\tau)$ for some Polish topology $\tau$ on $X$. We will not dwell on this distinction.

2.5 The Souslin Operation

The collection of analytic sets is closed under Souslin’s operation $\mathcal{A}$, which we will introduce now. This operation is not only closed to analytic sets, we will also see that complete measure spaces are another important class of measurable spaces which are closed under this operation. Each measurable space can be completed with respect to its finite measures, so that we do not even need a topology for carrying out the constructions ahead.

Let $\mathbb{N}^+$ be the set of all finite and non-empty sequences of natural numbers. Denote for $t = (x_n)_{n \in \mathbb{N}} \in \mathbb{N}^\infty$ by $t|k = (x_1, \ldots, x_k)$ its first $k$ elements. Given a subset $\mathcal{C} \subseteq \mathcal{P}(X)$, denote by

$$\mathcal{A}(\mathcal{C}) := \{ \bigcup_{t \in \mathbb{N}^\infty} \bigcap_{k \in \mathbb{N}} A_t|k \mid A_v \in \mathcal{C} \text{ for all } v \in \mathbb{N}^+ \}$$

Note that the outer union may be taken of more than countably many sets. A family $(A_v)_{v \in \mathbb{N}^+}$ is called a Souslin scheme, which is called regular if $A_w \subseteq A_v$ whenever $v$ is an initial piece.
of \( w \). Because
\[
\bigcup_{t \in \omega} \bigcap_{k \in \mathbb{N}} A_{t|k} = \bigcup_{t \in \omega} \bigcap_{1 \leq j \leq k} A_{t|j},
\]
we can and will restrict our attention to regular Souslin schemes whenever \( C \) is closed under finite intersections.

We will see now that each analytic set can be represented through a Souslin scheme with a special shape. This has some interesting consequences, among others that analytic sets are closed under the Souslin operation.

**Proposition 2.113** Let \( X \) be a Polish space and \( (A_v)_{v \in \omega^+} \) be a regular Souslin scheme of closed sets such that \( \text{diam}(A_v) \to 0 \), as the length of \( v \) goes to infinity. Then \( E := \bigcup_{t \in \omega^+} \bigcap_{k \in \mathbb{N}} A_{t|k} \) is an analytic set in \( X \). Conversely, each analytic set can be represented in this way.

**Proof** 1. Assume \( E \) is given through a Souslin scheme, then we represent \( E = f[F] \) with \( F \subseteq \omega^\omega \) a closed set and \( f : F \to X \) continuous. In fact, put
\[
F := \{ t \in \omega^\omega \mid A_{t|k} \neq \emptyset \text{ for all } k \}.
\]

Then \( F \) is a closed subset of \( \omega^\omega \): take \( s \in \omega^\omega \setminus F \), then we can find \( k' \in \mathbb{N} \) with \( A_{s|k'} = \emptyset \), so that \( G := \{ t \in \omega^\omega \mid t|k' = s|k' \} \) is open in \( \omega^\omega \), contains \( s \) and is disjoint to \( F \). Now let \( t \in F \), then there exists exactly one point \( f(t) \in \bigcap_{k \in \mathbb{N}} A_{t|k} \), since \( X \) is complete and the diameters of the sets involved tend to zero. Then \( E = f[F] \) by construction, and \( f \) is continuous.

Let \( t \in F \) and \( \epsilon > 0 \) be given, take \( x := f(t) \) and let \( B \) be the ball with center \( x \) and radius \( \epsilon \). Then we can find an index \( k \) such that \( A_{t|k'} \subseteq S \) for all \( k' \geq k \), hence \( U := \{ s \in F \mid t|k = s|k \} \) is an open neighborhood of \( t \) with \( f[U] \subseteq B \).

2. Let \( E \) be an analytic set, then \( E = f[\omega^\omega] \) with \( f \) continuous by Proposition 2.84. Define \( A_v \) as the closure of the set \( f[\{ t \in \omega^\omega \mid t|k = v \}] \), if the length of \( v \) is \( k \). Then clearly
\[
E = \bigcup_{t \in \omega^\omega} \bigcap_{k \in \mathbb{N}} A_{t|k},
\]

since \( f \) is continuous. It is also clear that \( (A_v)_{v \in \omega^+} \) is regular with diameter tending to zero.

Before we can enter into the demonstration that the Souslin operation is idempotent, we need some auxiliary statements. The first one is readily verified.

**Lemma 2.114** \( b(m,n) := 2^{m-1}(2n-1) \) defines a bijective map \( \mathbb{N} \times \mathbb{N} \to \mathbb{N} \). Moreover, \( m \leq b(m,n) \) and \( n < n' \) implies \( b(m,n) < b(m,n') \) for all \( n,n',m \in \mathbb{N} \).

Given \( k \in \mathbb{N} \), there exists a unique pair \( (\ell(k),r(k)) \in \mathbb{N} \times \mathbb{N} \) with \( b(\ell(k),r(k)) = k \). We will need the functions \( \ell,r : \mathbb{N} \to \mathbb{N} \) later on. The next function is considerably more complicated, since it caters for a more involved set of parameters.

**Lemma 2.115** Define for and \( z = (z_n)_{n \in \mathbb{N}} \in (\mathbb{N}^\mathbb{N})^\mathbb{N} \) with \( z_n = (z_{n,m})_{m \in \mathbb{N}} \) and \( t \in \mathbb{N}^\mathbb{N} \)
\[
B(t,z)_k := b(\ell(k),z_{\ell(k),r(k)}).
\]

Then \( B : \mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) is a bijection.
Proof 1. We show first that $B$ is injective. Let $\langle t, z \rangle \neq \langle t', z' \rangle$. If $t \neq t'$, we find $k$ with $t(k) \neq t'(k)$, so that $b(t(k), z_{\ell(k), r(k)}) \neq b(t'(k), z'_{\ell(k), r(k)})$ follows, because $b$ is injective. Now assume that $t = t'$, but $z \neq z'$, so we can find $i, j \in \mathbb{N}$ with $z_{i,j} \neq z'_{i,j}$. Let $k := b(i, j)$, so that $\ell(k) = i$ and $r(k) = j$, hence $\langle t(k), z_{\ell(k), r(k)} \rangle \neq \langle t(k), z'_{\ell(k), r(k)} \rangle$, so that $B(t, z)_k \neq B(t', z')_k$.

2. Now let $s \in \mathbb{N}^\mathbb{N}$, and define $t \in \mathbb{N}^\mathbb{N}$ and $z \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$

\[
t_k := \ell(s_k),
\]
\[
z_{n,m} := r(s_{b(n,m)}).
\]

Then we have for $k \in \mathbb{N}$

\[
B(t, z)_k = b(t_k, z_{\ell(k), r(k)})
\]
\[
= b(\ell(s_k), r(s_{b(\ell(k), r(k))}))
\]
\[
= b(\ell(s_k), r(s_k))
\]
\[
= s_k.
\]

\[\square\]

We construct maps $\varphi, \psi$ from $b$ and $B$ now with special properties which will be utilized in the proof that the Souslin operation is idempotent.

Lemma 2.116 There exist maps $\varphi, \psi : \mathbb{N}^+ \to \mathbb{N}^+$ with this property: let $w = B(t, z)|b(n, m)$, then $\varphi(w) = t|m$ and $\psi(w) = z|m$, then we obtain from the definition of $\varphi$ resp. $\psi$

Proof Fix $v = \langle x_1, \ldots, x_k \rangle$, then define for $m := \ell(k)$ and $n := r(k)$

\[
\varphi(v) := \langle \ell(x_1), \ldots, \ell(x_m) \rangle,
\]
\[
\psi(v) := \langle r(x_{b(m,1)}, \ldots, r(x_{b(m,n)}) \rangle
\]

We see from Lemma 2.114 that these definitions are possible.

Given $t \in \mathbb{N}^\mathbb{N}$ and $z \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$, we put $k := b(m, n)$ and $v := B(t, z)|k$, then we obtain from the definition of $\varphi$ resp. $\psi$

\[
\varphi(v) = \langle \ell(v_1), \ldots, \ell(v_m) \rangle = t|m
\]

and

\[
\psi(v) = \langle r(v_{b(m,0)}), \ldots, r(v_{b(m,n)}) \rangle = z|m
\]

\[\square\]

The construction shows that $\mathcal{A}(\mathcal{C})$ is always closed under countable unions and countable intersections. We are now in a position to prove a much more general observation.

Theorem 2.117 $\mathcal{A}(\mathcal{A}(\mathcal{C})) = \mathcal{A}(\mathcal{C})$.

Proof It is clear that $\mathcal{C} \subseteq \mathcal{A}(\mathcal{C})$, so we have to establish the other inclusion. Let $\{B_{v,w} \mid w \in \mathbb{N}^+\}$ be a Souslin scheme for each $v \in \mathbb{N}^+$, and put $A_v := \bigcup_{s \in \mathbb{N}^+} \bigcap_{m \in \mathbb{N}} B_{v,s|m}$. Then we
have

\[
A := \bigcup_{t \in \mathbb{N}^\infty} \bigcap_{k \in \mathbb{N}} A_{t|k}
= \bigcup_{t \in \mathbb{N}^\infty} \bigcup_{k \in \mathbb{N}} \bigcap_{s \in \mathbb{N}^\infty} B_{v,s|m}
= \bigcup_{t \in \mathbb{N}^\infty} \bigcap_{(z_n)_{n \in \mathbb{N}} \in (\mathbb{N}^\infty)^\mathbb{N}} \bigcap_{m \in \mathbb{N}} B_{t|m,z_m|n}
\]

with

\[
C_v := B_{\varphi(v),\psi(v)}
\]

for \(v \in \mathbb{N}^+\). So we have to establish the equality marked (\(\ast\)).

\("\subseteq\)" : Given \(x \in A\), there exists \(t \in \mathbb{N}^\infty\) and \(z \in (\mathbb{N}^\infty)^\mathbb{N}\) such that \(x \in B_{t|m,z_m|n}\). Put \(s := B(t,z)\). Let \(k \in \mathbb{N}\) be arbitrary, then there exists a pair \(\langle m, n \rangle \in \mathbb{N} \times \mathbb{N}\) with \(k = b(m,n)\) by Lemma \ref{lem:souslin1}. Thus we have \(t|m = \varphi(s|k)\) and \(z_m|n = \psi(s|k)\). by Lemma \ref{lem:souslin2} from which \(x \in B_{t|m,z_m|n} = C_{s|k}\) follows.

\("\supseteq\)" : Let \(s \in \mathbb{N}^\infty\) such that \(x \in C_{s|k}\) for all \(k \in \mathbb{N}\). We can find by Lemma \ref{lem:souslin3} some \(t \in \mathbb{N}^\infty\) and \(z \in (\mathbb{N}^\infty)^\mathbb{N}\) with \(B(t,z) = s\). Given \(k\), there exist \(m, n \in \mathbb{N}\) with \(k = b(m,n)\), hence \(C_{s|k} = B_{t|m,z_m|n}\). Thus \(x \in A\). \(\dashv\)

We obtain as an immediate consequence that analytic sets in a Polish space \(X\) are closed under the Souslin operation. This is so because we have seen that the collection of analytic sets is contained in \(\mathcal{A}(\{F \subseteq X \mid F\text{ is closed}\})\), so an application of Theorem \ref{thm:souslin} proves the claim. But we can say even more.

**Proposition 2.118** Assume that the complement of each set in \(\mathcal{C}\) belongs to \(\mathcal{A}(\mathcal{C})\), and \(\emptyset \in \mathcal{C}\). Then \(\sigma(\mathcal{C}) \subseteq \mathcal{A}(\mathcal{C})\).

**Proof** Define

\[
\mathcal{G} := \{A \in \mathcal{A}(\mathcal{C}) \mid X \setminus A \in \mathcal{A}(\mathcal{C})\}.
\]

Then \(\mathcal{G}\) is closed under complementation. If \((A_n)_{n \in \mathbb{N}}\) is a sequence in \(\mathcal{G}\), then \(\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{G}\), because \(\mathcal{A}(\mathcal{C})\) is closed under countable unions. Similarly, \(\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{G}\). Since \(\emptyset \in \mathcal{G}\), we may conclude that \(\mathcal{G}\) is a \(\sigma\)-algebra, which contains \(\mathcal{C}\) by assumption. Hence \(\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{G}) = \mathcal{G} \subseteq \mathcal{A}(\mathcal{C})\). \(\dashv\)

With complete measure spaces we will meet an important class of measurable spaces, which is closed under the Souslin operation. As a preparation for this we state and prove an interesting criterion for being closed. This requires the definition of a particular kind of cover.

**Definition 2.119** Given a measurable space \((X, \mathcal{A})\) and a subset \(A \subseteq X\), we call \(A_z \in \mathcal{A}\) an \(A\)-cover of \(A\) iff

1. \(A \subseteq A_z\).

2. For every \(B \in \mathcal{A}\) with \(A \subseteq B\), \(\mathcal{P}(A_z \setminus B) \subseteq \mathcal{A}\).
Thus $A_2 \in \mathcal{A}$ covers $A$ in the sense that $A \subseteq A_2$, and if we have another set $B$ is $\mathcal{A}$ which covers $A$ as well, then all the sets which make out the difference between $A_2$ and $B$ are measurable. In addition it follows that if $A \subseteq A' \subseteq A_2$ and $A' \in \mathcal{A}$, then $A'$ is also an $\mathcal{A}$-cover. This concept sounds fairly artificial and somewhat far fetched, but we will see that arises in a natural way when completing measure spaces. The surprising observation is that a space is closed under the Souslin operation whenever each subset has an $\mathcal{A}$-cover.

**Proposition 2.120** Let $(X, \mathcal{A})$ be a measurable space such that each subset of $X$ has an $\mathcal{A}$ cover. Then $(X, \mathcal{A})$ is closed under the Souslin operation.

**Proof** 1. Let 

$$A := \bigcup_{t \in \mathbb{N}^\infty} \bigcap_{k \in \mathbb{N}} A_{t|k}$$

with $(A_v)_{v \in \mathbb{N}^+}$ a regular Souslin scheme in $\mathcal{A}$. Define 

$$B_w := \bigcup \{\bigcap_{n \in \mathbb{N}} A_{t|n} \mid t \in \mathbb{N}^\infty, w \text{ is a prefix of } t\}.$$

for $w \in \mathbb{N}^* = \mathbb{N}^+ \cup \{\epsilon\}$. Then $B_{\epsilon} = A$, $B_w = \bigcup_{n \in \mathbb{N}} B_{wn}$, and $B_w \subseteq A_w$ if $w \neq \epsilon$.

By assumption, there exists a minimal $\mathcal{A}$-cover $C_w$ for $B_w$. We may and do assume that $C_w \subseteq A_w$, and that $(C_w)_{w \in \mathbb{N}^*}$ is regular (we otherwise force this condition by considering the $\mathcal{A}$-cover $(\bigcap_{v \in \text{prefix of } w} (C_v \cap A_v))_{w \in \mathbb{N}^*}$ instead). Now put $D_w := C_w \setminus \bigcup_{n \in \mathbb{N}} C_{wn}$ for $w \in \mathbb{N}^*$. We obtain from this construction $B_w \subseteq C_w = \bigcup_{n \in \mathbb{N}} C_{wn} \in \mathcal{A}$, hence that every subset of $D_w$ is in $\mathcal{A}$, since $C_w$ is an $\mathcal{A}$-cover. Thus every subset of $D := \bigcup_{w \in \mathbb{N}^*} D_w$ is in $\mathcal{A}$.

2. We claim that $C_{\epsilon} \setminus D \subseteq A$. In fact, let $x \in C_{\epsilon} \setminus D$, then $x \notin D_{\epsilon}$, so we can find $k_1 \in \mathbb{N}$ with $x \in C_{k_1}$, but $x \notin D_{n_1}$. Since $x \notin D_{k_1}$, we find $k_2$ with $x \in C_{k_1,k_2}$ such that $x \notin D_{k_1,k_2}$. So we inductively define a sequence $t := (k_n)_{n \in \mathbb{N}}$ so that $x \in C_{t|k}$ for all $k \in \mathbb{N}$. Because $C_{t|k} \subseteq A_{t|k}$, we conclude that $x \in A$.

3. Hence we obtain $C_{\epsilon} \setminus A \subseteq D$, and since every subset of $D$ is in $\mathcal{A}$, we conclude that $C_{\epsilon} \setminus A \in \mathcal{A}$, which means that $A = C_{\epsilon} \setminus (C_{\epsilon} \setminus A) \in \mathcal{A}$. 

### 2.6 Universally Measurable Sets

After this technical preparation we are posed to enter the interesting world of universally measurable sets with the closure operations that are associated with them. We define complete measure spaces and show that an arbitrary ($\sigma$-) finite measure space can be completed, uniquely extending the measure as we go. This leads also to completions with respect to families of finite measures, and we show that the resulting measurable spaces are closed under the Souslin operation. Two applications are discussed. The first one demonstrates that a measure defined on a countably generated sub-$\sigma$-algebra of the Borel sets of an analytic space can be extended to the Borel sets, albeit not necessarily in a unique way. This result due to Lubin rests on the important von Neumann Selection Theorem, giving a universally right inverse to a measurable map from an analytic to a separable space. Another application of von Neumann’s result is the observation that under suitable topological assumptions for a surjective map $f$ the lifted map $\mathcal{M}(f)$ is surjective as well. The second application shows
that a transition kernel can be extended to the universal closures of the measurable spaces involved, provided the target space is separable.

A σ-finite measure space \((X, \mathcal{A}, \mu)\) is called complete iff \(\mu(A) = 0\) with \(A \in \mathcal{A}\) and \(B \subseteq A\) implies \(B \in \mathcal{A}\). Thus if we have two sets \(A, A' \in \mathcal{A}\) with \(A \subseteq A'\) and \(\mu(A) = \mu(A')\), then we know that each set which can be sandwiched between the two will be measurable as well. We will discuss the completion of a measure space and investigate some properties. We first note that it is sufficient to discuss finite measure spaces; in fact, assume that we have a collection of mutually disjoint sets \((G_n)_{n \in \mathbb{N}}\) with \(G_n \in \mathcal{A}\) such that \(0 < \mu(G_n) < \infty\) and \(\bigcup_{n \in \mathbb{N}} G_n = X\), consider the measure

\[
\mu'(B) := \sum_{n \in \mathbb{N}} \frac{\mu(B \cap G_n)}{2^n \mu(G_n)},
\]

then \(\mu\) is complete iff \(\mu'\) is complete, and \(\mu'\) is a probability measure.

We fix for the time being a finite measure \(\mu\) on a measurable space \((X, \mathcal{A})\). The outer measure \(\mu^*\) is defined through

\[
\mu^*(C) := \inf \{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid C \subseteq \bigcup_{n \in \mathbb{N}} A_n, A_n \in \mathcal{A} \text{ for all } n \in \mathbb{N} \}
\]

\[
= \inf \{ \mu(A) \mid C \subseteq A, A \in \mathcal{A} \}
\]

for any subset \(C\) of \(X\).

**Definition 2.121** Call \(N \subseteq X\) a \(\mu\)-null set iff \(\mu^*(N) = 0\). Define \(\mathcal{N}_\mu\) as the set of all \(\mu\)-null sets.

Because \(\mu^*\) is countable subadditive, we obtain

**Lemma 2.122** \(\mathcal{N}_\mu\) is a \(\sigma\)-ideal. ⊥

Now assume that we have sets \(A, A' \in \mathcal{A}\) and \(N, N' \in \mathcal{N}_\mu\) such that their symmetric differences with \(A \Delta N = A' \Delta N'\), so their symmetric differences are the same. Then we may infer \(\mu(A) = \mu(A')\), because \(A \Delta A' = A \Delta (A \Delta (N \Delta N')) = N \Delta N' \subseteq N \cup N' \in \mathcal{N}_\mu\), and \(|\mu(A) - \mu(A')| \leq \mu(A \Delta A')\). Thus we may construct an extension of \(\mu\) to the \(\sigma\)-algebra generated by \(\mathcal{A}\) and \(\mathcal{N}_\mu\) in an obvious way.

**Proposition 2.123** Define \(\mathcal{A}_\mu := \sigma(\mathcal{A} \cup \mathcal{N}_\mu)\) and \(\overline{\mu}(A \Delta N) := \mu(A)\) for \(A \in \mathcal{A}, N \in \mathcal{N}_\mu\). Then

1. \(\mathcal{A}_\mu = \{ A \Delta N \mid A \in \mathcal{A}, N \in \mathcal{N}_\mu \}\), and \(A \in \mathcal{A}_\mu\) iff there exist sets \(A', A'' \in \mathcal{A}\) with \(A' \subseteq A \subseteq A''\) and \(\mu^*(A'' \setminus A') = 0\).

2. \(\overline{\mu}\) is a finite measure, and the unique extension of \(\mu\) to \(\mathcal{A}_\mu\).

3. the measure space \((X, \mathcal{A}_\mu, \overline{\mu})\) is complete. It is called the \(\mu\)-completion of \((X, \mathcal{A}, \mu)\).

**Proof** 1. Since \(\mathcal{N}_\mu\) is a \(\sigma\)-ideal, we infer from Lemma 2.23 that \(A \in \mathcal{A}_\mu\) iff there exists \(B \in \mathcal{A}\) and \(N \in \mathcal{N}_\mu\) with \(A = B \Delta N\). Now consider

\[
\mathcal{C} := \{ A \in \mathcal{A}_\mu \mid \exists A', A'' \in \mathcal{A} : A' \subseteq A \subseteq A'', \mu^*(A'' \setminus A') = 0\}.
\]

Then \(\mathcal{C}\) is a \(\sigma\)-algebra which contains \(\mathcal{A} \cup \mathcal{N}_\mu\), thus \(\mathcal{C} = \mathcal{A}_\mu\).
From the observation made just before stating the proposition it becomes clear that \( \overline{\mu} \) is well defined on \( A_{\mu} \). Since \( \mu^* \) coincides with \( \overline{\mu} \) on \( A_{\mu} \) and the outer measure is countably subadditive [Dob13, Lemma 1.107], we have to show that \( \overline{\mu} \) is additive on \( A_{\mu} \). This follows immediately from the first part. If \( \nu \) is another extension to \( \mu \) on \( A_{\mu} \), \( N_\nu = N_\mu \) follows, so that \( \overline{\mu}(A \Delta N) = \mu(A) = \nu(A) = \nu(A \Delta N) \) whenever \( A \Delta N \in A_{\mu} \).

2. Completeness of \((X,A,\mu)\) follows now immediately from the construction. \(\dashv\)

Surprisingly, we have received more than we have shopped for, since complete measure spaces are closed under the Souslin operation. This is remarkable because the Souslin operation evidently bears no hint at all at measures which are defined on the base space. In addition, measures are defined through countable operations, while the Souslin operation makes use of the uncountable space \( N \).

Proposition 2.124 A complete measure space is closed under the Souslin operation.

Proof Let \((X,A,\mu)\) be complete, then it is enough to show that each \( B \subseteq X \) has an \( A \)-cover (Definition 2.119); then the assertion will follow from Proposition 2.120. In fact, given \( B \), construct \( B^* \in A \) such that \( \mu(B^*) = \mu^*(B) \), see [Dob13, Lemma 1.118]. Whenever \( C \in A \) with \( B \subseteq C \), we evidently have every subset of \( B^* \setminus C \) in \( A \) by completeness. \(\dashv\)

These constructions work also for \( \sigma \)-finite measure spaces, as indicated above. Now let \( M \) be a non-empty set of \( \sigma \)-finite measures on the measurable space \((X,A)\), then define the \( M \)-completion \( \overline{A}^M \) and the universal completion \( \overline{A} \) of the \( \sigma \)-algebra \( A \) through

\[
\overline{A}^M := \bigcap_{\mu \in M} A_{\mu},
\]

\[
\overline{A} := \bigcap \{A_{\mu} \mid \mu \text{ is a } \sigma \text{-finite measure on } A\}.
\]

As an immediate consequence this yields that the analytic sets in a Polish space are contained in the universal completion of the Borel sets, specifically

Corollary 2.125 Let \( X \) be a Polish space and \( \mu \) be a finite measure on \( B(X) \). Then all analytic sets are contained in \( B(X) \).

Proof Proposition 2.124 together with Proposition 2.113. \(\dashv\)

Just for the record:

Corollary 2.126 The universal closure of a measurable space is closed under the Souslin operation. \(\dashv\)

Measurability of maps is preserved when passing to the universal closure.

Lemma 2.127 Let \( f : X \to Y \) be \( A,B \)- measurable, then \( f \) is \( \overline{A,B} \) measurable.

Proof Let \( D \in \overline{B} \) be a universally measurable subset of \( Y \), then we have to show that \( E := f^{-1}[D] \) is universally measurable in \( X \). So we have to show that for every finite measure \( \mu \) on \( A \) there exists \( E',E'' \in A \) with \( E' \subseteq E \subseteq E'' \) and \( \mu(E' \setminus E'') = 0 \). Define \( \nu \) as the image of \( \mu \) under \( f \), so that \( \nu(B) = \mu(f^{-1}[B]) \) for each \( B \in B \), then we know that there exists \( D',D'' \in B \) with \( D' \subseteq D \subseteq D'' \) such that \( \nu(D'' \setminus D') = 0 \), hence we have for the
measurable sets $E' := f^{-1}[D']$, $E'' := f^{-1}[D'']$

$$\mu(E'' \setminus E') = \mu(f^{-1}[D'' \setminus D']) = \nu(D'' \setminus D') = 0.$$ 

Thus $f^{-1}[D] \in \mathcal{A}$. \(\Box\)

We will give now two applications of this construction. The first will show that a finite measure on a countably generated sub-$\sigma$-algebra of the Borel sets of an analytic space has always an extension to the Borel sets, the second will construct an extension of a stochastic relation $\sigma$ on a countably generated sub-

We will first formulate a sequence of auxiliary statements that deal with finding for a given surjective map $f : X \to Y$. We know from the Axiom of Choice that we can find for each $y \in Y$ some $x \in X$ with $f(x) = y$, because $\{f^{-1}\{y\} \mid y \in Y\}$ is a partition of $X$ into non-empty sets. Set $g(y) := x$. Selecting an inverse image in this way will not guarantee, however, that $g$ has any favorable properties, even if, say, both $X$ and $Y$ are compact metric and $f$ is continuous. Hence we will have to proceed in a more systematic way.

As a preparation, we require a universally measurable right inverse of a measurable surjective map $f : X \to Y$. Let $x \in X$ with $f(x) = y$, because $\{f^{-1}\{y\} \mid y \in Y\}$ is a partition of $X$ into non-empty sets. Then $g(y) := x$. Selecting an inverse image in this way will not guarantee, however, that $g$ has any favorable properties, even if, say, both $X$ and $Y$ are compact metric and $f$ is continuous. Hence we will have to proceed in a more systematic way.

We will use the observation that each analytic set in a Polish space can be represented as the continuous image of $\mathbb{N}^\infty$, as discussed in Proposition 2.83.

We will first formulate a sequence of auxiliary statements that deal with finding for a given surjective map $f : X \to Y$ a map $g : Y \to X$ such that $f \circ g = id_Y$. This map $g$ should have some sufficiently pleasant properties.

Thus in order to make the first step it turns out to be helpful focusing the attention to analytic sets being the continuous images of $\mathbb{N}^\infty$. This looks a bit far fetched, because we want to deal with universally measurable sets, but remember that analytic sets are universally measurable.

We can lexicographically order $\mathbb{N}^\infty$ by saying that $(t_n)_{n \in \mathbb{N}} \leq (t'_n)_{n \in \mathbb{N}}$ iff there exists $k \in \mathbb{N}$ such that $t_k \leq t'_k$, and $t_j = t'_j$ for all $\ell$ with $1 \leq j < k$. Then $\leq$ defines a total order on $\mathbb{N}^\infty$. We will capitalize on this order, to be more precise, on the interplay between the order and the topology. Let us briefly look into the order structure of $\mathbb{N}^\infty$.

**Lemma 2.128** Each nonempty closed set $F \subseteq \mathbb{N}^\infty$ has a minimal element in the lexicographic order.

**Proof** Let $n_1$ be the minimal first component of all elements of $F$, $n_2$ be the minimal second component of those elements of $F$ that start with $n_1$, etc. This defines an element $t := \langle n_1, n_2, \ldots \rangle$. We claim that $t \in F$. Let $U$ be an open neighborhood of $t$, then there
The following theorem is usually attributed to von Neumann.

**Theorem 2.130**

The statement is the work horse for establishing that a right inverse exists for surjective Borel maps between an analytic space and a separable measurable space. All we need to do now is to massage things into a shape that will render this result applicable in the desired context. This turns out to be a suitable choice, as the following statement shows.

**Lemma 2.129**

Let $X$ be Polish, $Y \subseteq X$ analytic with $Y = f[\mathbb{N}^\infty]$ for some continuous $f : \mathbb{N}^\infty \to X$. Then there exists $g : Y \to \mathbb{N}^\infty$ such that

1. $f \circ g = id_Y$,
2. $g$ is $\overline{B(Y) \cdot B(\mathbb{N}^\infty)}$-measurable.

**Proof**

1. Since $f$ is continuous, the inverse image $f^{-1}[\{y\}]$ for each $y \in Y$ is a closed and nonempty set in $\mathbb{N}^\infty$. Thus this set contains a minimal element $g(y)$ in the lexicographic order $\preceq$ by Lemma 2.128. It is clear that $f(g(y)) = y$ holds for all $y \in Y$.

2. Denote by $A(t') := \{ t \in \mathbb{N}^\infty \mid t \prec t' \}$, then $A(t')$ is open: let $(\ell_n)_{n \in \mathbb{N}} = t \prec t'$ and $k$ be the first component in which $t$ differs from $t'$, then $\Sigma_{\ell_1, \ldots, \ell_{k-1}}$ is an open neighborhood of $t$ that is entirely contained in $A(t')$. It is easy to see that $\{ A(t') \mid t' \in \mathbb{N}^\infty \}$ is a generator for the Borel sets of $\mathbb{N}^\infty$.

3. We claim that $g^{-1}[A(t')] = f[A(t')]$ holds. In fact, let $y \in g^{-1}[A(t')]$, then $g(y) \in A(t')$, then $y = f(g(y)) \in f[A(t')]$. If, on the other hand, $y = f(t)$ with $t \prec t'$, then by construction $t \in f^{-1}[\{y\}]$, thus $g(y) \preceq t \prec t'$, settling the other inclusion.

This equality implies that $g^{-1}[A(t')]$ is an analytic set, because it is the image of an open set under a continuous map. Consequently, $g^{-1}[A(t')]$ is universally measurable for each $A(t')$ by Corollary 2.125. Thus $g$ is a universally measurable map.

This statement is the work horse for establishing that a right inverse exists for surjective Borel maps between an analytic space and a separable measurable space. All we need to do now is to massage things into a shape that will render this result applicable in the desired context.

The following theorem is usually attributed to von Neumann.

**Theorem 2.130**

Let $X$ be an analytic space, $(Y, \mathcal{B})$ a separable measurable space and $f : X \to Y$ a surjective measurable map. Then there exists $g : Y \to X$ with these properties:

1. $f \circ g = id_Y$,
2. $g$ is $\overline{B(Y) \cdot B(X)}$-measurable.

**Proof**

1. We may and do assume by Lemma 2.99 that $Y$ is an analytic subset of a Polish space $Q$, and that $X$ is an analytic subset of a Polish space $P$. $x \mapsto (x, f(x))$ is a bijective Borel map from $X$ to the graph of $f$, so graph$(f)$ is an analytic set by Proposition 2.88. Thus we can find a continuous map $F : \mathbb{N}^\infty \to P \times Q$ with $F[\mathbb{N}^\infty] = \text{graph}(f)$. Consequently, $\pi_Q \circ F$ is a continuous map from $\mathbb{N}^\infty$ to $Q$ with

\[
(\pi_Q \circ F)[\mathbb{N}^\infty] = \pi_Q[\text{graph}(f)] = Y.
\]

Now let $G : Y \to \mathbb{N}^\infty$ be chosen according to Lemma 2.129 for $\pi_Q \circ F$. Then $g := \pi_P \circ F \circ G : Y \to X$ is the map we are looking for.
• $g$ is universally measurable, because $G$ is, and because $\pi_P \circ F$ are continuous, hence universally measurable as well,

• $f \circ g = f \circ (\pi_P \circ F \circ G) = (f \circ \pi_P) \circ F \circ G = \pi_Q \circ F \circ G = id_Y$, so $g$ is right inverse to $f$.

Due to its generality, the von Neumann Selection Theorem has many applications in diverse areas, many of them surprising. The art is plainly to reformulate the problem so that an application of this selection theorem is possible. We pick two applications, viz., showing that due to its generality, the von Neumann Selection Theorem has many applications in diverse areas, many of them surprising. The art is plainly to reformulate the problem so that an application of this selection theorem is possible. We pick two applications, viz., showing that

**Proposition 2.131** Let $X$ be an analytic space, $Y$ a second countable metric space. If $f : X \to Y$ is a surjective Borel map, so is $M(f) : M(X) \to M(Y)$.

**Proof**

1. From Theorem 2.130 we find a map $g : Y \to X$ such that $f \circ g = id_Y$ and $g$ is $B(Y) - B(X)$-measurable.

2. Let $\nu \in M(Y)$, and define $\mu := M(g)(\nu)$, then $\mu \in M(X, B(X))$ by construction. Restrict $\mu$ to the Borel sets on $X$, obtaining $\mu_0 \in M(X, B(X))$. Since we have for each set $B \subseteq Y$ the equality $g^{-1}[f^{-1}[B]] = B$, we see that for each $B \in B(Y)$

$$M(f)(\mu_0)(B) = \mu_0(f^{-1}[B]) = \mu(f^{-1}[B]) = \nu(g^{-1}[f^{-1}[B]]) = \nu(B)$$

holds. ⊥

This has as a consequence that $M$ is an endofunctor on the category of Polish or analytic spaces with surjective Borel maps as morphisms; it displays a pretty interaction of reasoning in measurable spaces and arguing in categories.

The following extension theorem due to Lubin shows that one can extend a finite measure from a countably generated sub-$\sigma$-algebra to the Borel sets of an analytic space. In contrast to classical extension theorems it does not permit to conclude that the extension is uniquely determined.

**Theorem 2.132** Let $X$ be an analytic space, and $\mu$ be a finite measure on a countably generated sub-$\sigma$-algebra $\mathcal{A} \subseteq B(X)$. Then there exists an extension of $\mu$ to a finite measure $\nu$ on $B(X)$.

**Proof** Let $(A_n)_{n \in \mathbb{N}}$ be the generator of $\mathcal{A}$, and define the map $f : X \to \{0,1\}^\mathbb{N}$ through $x \mapsto (\chi_{A_n})_{n \in \mathbb{N}}$. Then $M := f[X]$ is an analytic space, and $f$ is $B(X)$-$B(M)$ measurable by Proposition 2.58 and Proposition 2.88. Moreover,

$$\mathcal{A} = \{ f^{-1}[C] \mid C \in B(M) \}. \quad (2)$$

By von Neumann’s Selection Theorem 2.130 there exists $g : M \to X$ with $f \circ g = id_M$ which is $B(M)$-$B(X)$-measurable. Define

$$\nu(B) := \overline{\mu((g \circ f)^{-1}[B])}$$

for $B \in B(X)$ with $\overline{\mu}$ as the completion of $\mu$ on $\overline{\mathcal{A}}$. Since we have for $B \in B(X)$ that $g^{-1}[B] \in B(M)$, we may conclude from (2) that $f^{-1}[g^{-1}[B]] \in \overline{\mathcal{A}}$. $\nu$ is an extension to $\mu$. 

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In fact, given $A \in \mathcal{A}$, we know that $A = f^{-1}[C]$ for some $C \in \mathcal{B}(M)$, so that we obtain
\[
\nu(A) = \overline{\mu}((g \circ f)^{-1}[f^{-1}[C]]) \\
= \overline{\mu}(f^{-1} \circ g^{-1} \circ f^{-1}[C]) \\
= \overline{\mu}(f^{-1}[C]), \\
= \mu(A),
\]
since $f \circ g = id_M$
\[\square\]

This can be rephrased in a slightly different way. The identity $id_A : (X, \mathcal{B}(X)) \to (X, \mathcal{A})$ is measurable, because $\mathcal{A}$ is a sub-$\sigma$-algebra of $\mathcal{B}(X)$. Hence it induces a measurable map $\text{Sid}_A : S(X, \mathcal{B}(X)) \to S(X, \mathcal{A})$. Lubin’s Theorem then implies that $\text{Sid}_A$ is surjective. This is so since for a given $\mu \in S(X, \mathcal{B}(X))$, $\text{Sid}_A(\mu)$ is just the restriction of $\mu$ to the sub-$\sigma$-algebra $\mathcal{A}$.

### 2.6.2 Completing a Transition Kernel

In some probabilistic models for modal logics it becomes sometimes necessary to assume that the state space is closed under Souslin’s operation, see for example [Dob12], on the other hand one may not always assume that a complete measure space is given. Hence one wants to complete it, but it is then also mandatory to complete the transition law as well. This means that an extension of the transition law to the completion becomes necessary. This problem will be studied now.

The completion of a measure space is described in terms of null sets and using inner and outer approximations, see Proposition 2.123. We will use the latter here, fixing measurable spaces $(X; \mathcal{A})$ and $(Y, \mathcal{B})$. Denote by $\mathcal{S}_X$ the smallest $\sigma$-algebra on $X$ which contains $\mathcal{A}$ and which is closed under the Souslin operation, hence $\mathcal{S}_X \subseteq \overline{\mathcal{A}}$ by Corollary 2.126.

Fix $K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B})$ as a transition kernel, and assume first that $\mathcal{B}$ is the $\sigma$-algebra of Borel sets for a second countable metric space. This means that the topology $\tau$ of $Y$ has a countable base $\tau_0$, which in turn means that $G = \bigcup\{H \in \tau_0 \mid H \subseteq G\}$ for each open set $G \in \tau$.

For each $x \in X$ we have through the transition kernel $K$ a finite measure $K(x)$, to which we may associate an out measure $(K(x))^*$ on the power set of $X$. We want to show that the map
\[
x \mapsto (K(x))^*(A)
\]
is $\mathcal{S}_X$-measurable for each $A \subseteq Y$; define for convenience
\[
K^*(x) := (K(x))^*.
\]

Establishing measurability is broken into a sequence of steps.

We need the following regularity argument (but compare Exercise 13 for the non-metric case)
Lemma 2.133  Let $\mu$ be a finite measure on $(Y, \mathcal{B}(Y))$, $B \in \mathcal{B}(Y)$. Then we can find for each $\epsilon > 0$ an open set $G \subseteq Y$ with $B \subseteq G$ and a closed set $F \supseteq B$ such that $\mu(G \setminus F) < \epsilon$.

Proof  Let

$$\mathcal{G} := \{B \in \mathcal{B}(Y) \mid \text{ the assertion is true for } B\}.$$ 

Then plainly $\mathcal{G}$ is closed under complementation and contains the open as well as the closed sets. If $F \subseteq Y$ is closed, we can represent $F = \bigcap_{n \in \mathbb{N}} G_n$ with $(G_n)_{n \in \mathbb{N}}$ as a decreasing sequence of open sets, hence $\mu(F) = \inf_{n \in \mathbb{N}} \mu(F_n) = \lim_{n \to \infty} \mu(F_n)$, so that $\mathcal{G}$ also contains the closed sets; one arguments similarly for the open sets as increasing unions of open sets.

Now let $(B_n)_{n \in \mathbb{N}}$ be a sequence of mutually disjoint sets in $\mathcal{G}$, select $G_n$ open for $B_n$ and $\epsilon/2^{-(n+1)}$, then $G := \bigcup_{n \in \mathbb{N}} G_n$ is open with $B := \bigcup_{n \in \mathbb{N}} B_n \subseteq G$ and $\mu(G \setminus B) \leq \epsilon$. Similarly, select the sequence $(F_n)_{n \in \mathbb{N}}$ with $F_n \subseteq B_n$ and $\mu(B_n \setminus F_n) < \epsilon/2^{-(n+1)}$ for all $n \in \mathbb{N}$, put $F := \bigcup_{n \in \mathbb{N}} F_n$ and select $m \in \mathbb{N}$ with $\mu(F \setminus \bigcup_{n=1}^m F_n) < \epsilon/2$, then $F' := \bigcup_{n=1}^m F_n$ is closed, $F' \subseteq B$, and $\mu(B \setminus F') < \epsilon$.

Hence $\mathcal{G}$ is closed under complementation as well as countable disjoint unions; this implies $\mathcal{G} = \mathcal{B}(Y)$ by the $\pi$-$\lambda$ Theorem 2.4.

Fix $A \subseteq Y$ for the moment. We claim that

$$K^*(x)(A) = \inf\{K(x)(G) \mid A \subseteq G \text{ open}\}$$

holds for each $x \in X$. In fact, given $\epsilon > 0$, there exists $A \subseteq A_0 \in \mathcal{B}(Y)$ with $K(x)(A_0) - K^*(x)(A) < \epsilon/2$. Applying Lemma 2.133 to $K(x)$, we find an open set $G \supseteq A_0$ with $K(x)(G) - K(x)(A_0) < \epsilon/2$, thus $K(x)(G) - K^*(x)(A) < \epsilon$.

$\tau_0$ is a countable base for the open sets, which we may assume to be closed under finite unions (because otherwise $\{G_1 \cup \ldots \cup G_k \mid k \in \mathbb{N}, G_1, \ldots, G_k \in \tau_0\}$ is a countable base which has this property). Hence we obtain

$$K^*(x)(A) = \inf\{\sup_{n \in \mathbb{N}} K(x)(G_n) \mid A \subseteq \bigcup_{n \in \mathbb{N}} G_n, (G_n)_{n \in \mathbb{N}} \subseteq \tau_0 \text{ increases}\}. \quad (3)$$

Let

$$\mathcal{G}_A := \{(G_n)_{n \in \mathbb{N}} \subseteq \tau_0 \mid (G_n)_{n \in \mathbb{N}} \text{ increases and } A \subseteq \bigcup_{n \in \mathbb{N}} G_n\}$$

be the set of all increasing sequences from base $\tau_0$ which cover $A$. Partition $\mathcal{G}_A$ into the sets

$$\mathcal{N}_A := \{g \in \mathcal{G}_A \mid g \text{ contains only a finite number of sets}\},$$

$$\mathcal{M}_A := \mathcal{G}_A \setminus \mathcal{N}_A.$$ 

Because $\tau_0$ is countable, $\mathcal{N}_A$ is.

Lemma 2.134  There exists an injective map $\Phi : \mathcal{M}_A \to \mathbb{N}^\mathbb{N}$ such that $g \mid k = g' \mid k$ implies $\Phi(g) \mid k = \Phi(g') \mid k$ for all $k \in \mathbb{N}$.

Proof 1. Build an infinite tree in this way: the root is the empty set, a node $G$ at level $k$ has all elements $G'$ from $\tau_0$ with $G \subseteq G'$ as offsprings. Remove from the tree all paths $H_1, H_2, \ldots$ such that $A \not\subseteq \bigcup_{n \in \mathbb{N}} H_n$. Call the resulting tree $T$. 

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2. Put $G_0 := \emptyset$, and let $T_{1,G_0}$ be the set of nodes of $T$ on level 1 (hence just the offsprings of the root $G_0$), then there exists an injective map $\Phi_{1,G_0} : T_{1,G_0} \to \mathbb{N}$. If $G_1, \ldots, G_k$ is a finite path to inner node $G_k$ in $T$, denote by $T_{k+1,G_1,\ldots,G_k}$ the set of all offsprings of $G_k$, and let

$$\Phi_{k+1,G_1,\ldots,G_k} : T_{k+1,G_1,\ldots,G_k} \to \mathbb{N}$$

be an injective map. Define

$$\Phi : \begin{cases} \mathcal{M}_A & \to \mathbb{N}^\mathbb{N}, \\ (G_n)_{n \in \mathbb{N}} & \mapsto (\Phi_{n,G_1,\ldots,G_{n-1}}(G_n))_{n \in \mathbb{N}}. \end{cases}$$

3. Assume $\Phi(g) = \Phi(g')$, then an inductive reasoning shows that $g = g'$. In fact, $G_1 = G_1'$, since $\Phi_{1,\emptyset}$ is injective. If $g \triangleright k = g' \triangleright k$ has already been established, we know that $\Phi_{k+1,G_1,\ldots,G_k} = \Phi_{k+1,G_1',\ldots,G_k'}$ is injective, so that $G_{k+1} = G_{k+1}'$ follows. A similar inductive argument shows that $\Phi(g) \triangleright k = \Phi(g') \triangleright k$, provided $g \triangleright k = g' \triangleright k$ for each $k \in \mathbb{N}$ holds.

The following lemmata collect some helpful properties.

**Lemma 2.135** $g = g'$ iff $\Phi(g) \triangleright k = \Phi(g') \triangleright k$ for all $k \in \mathbb{N}$, whenever $g, g' \in \mathcal{M}_A$. \(\dashv\)

**Lemma 2.136** Denote by $J_k := \{\alpha \mid k \mid \alpha \in \Phi[\mathcal{M}_A]\}$ all initial pieces of sequences in the image of $\Phi$. Then $\alpha \in \Phi[\mathcal{M}_A]$ iff $\alpha \mid k \in J_k$ for all $k \in \mathbb{N}$.

**Proof** Assume that $\alpha = \Phi(g) \in \Phi[\mathcal{M}_A]$ with $g = (C_n)_{n \in \mathbb{N}} \in \mathcal{M}_A$ and $\alpha \mid k \in J_k$ for all $k \in \mathbb{N}$, so for given $k$ there exists $g^{(k)} = (C_{n})_{n \in \mathbb{N}} \in \mathcal{M}_A$ with $\alpha \mid k = \Phi(g^{(k)}) \triangleright k$. Because $\Phi_1$ is injective, we obtain $C_1 = C_1^{(1)}$. Assume for the induction step that $G_i = G_i^{(j)}$ has been shown for $1 \leq i, j \leq k$. Then we obtain from $\Phi(g) \triangleright k + 1 = \Phi(g^{(k+1)}) \triangleright k + 1$ that $G_{k+1} = G_{k+1}^{(k+1)}$, since $\Phi_{k+1,G_1,\ldots,G_k}$ is injective, the equality above implies $G_{k+1} = G_{k+1}^{(k+1)}$. Hence $g = g^{(k)}$ for all $k \in \mathbb{N}$, and $\alpha \in \Phi[\mathcal{M}_A]$ is established. The reverse implication is trivial. \(\dashv\)

**Lemma 2.137** $E_r := \{x \in X \mid K^*(x)(A) \leq r\} \in \mathcal{S}_X$ for $r \in \mathbb{R}_+$.

**Proof** The set $E_r$ can be written as

$$E_r = \bigcup_{g \in \mathcal{N}_A} \{x \in X \mid K(x)(\bigcup_{g \in \mathcal{M}_A} g) \leq r\} \cup \bigcup_{g \in \mathcal{M}_A} \{x \in X \mid K(x)(\bigcup_{g \in \mathcal{N}_A} g) \leq r\}$$

Because $\mathcal{N}_A$ is countable, and $K : X \rightsquigarrow Y$ is a transition kernel, we infer

$$\bigcup_{g \in \mathcal{N}_A} \{x \in X \mid K(x)(\bigcup_{g \in \mathcal{N}_A} g) \leq r\} \in \mathcal{B}(X)$$

Put for $v \in \mathbb{N}^+$

$$D_v := \begin{cases} \emptyset, & \text{if } v \notin \bigcup_{k \in \mathbb{N}} J_k, \\ \{x \in X \mid K(x)(G_n) \leq r\}, & \text{if } v = \Phi((G_n)_{n \in \mathbb{N}}) \mid n. \end{cases}$$

Lemma 2.135 and Lemma 2.136 show that $D_v \in \mathcal{B}(X)$ is well defined. Because

$$\bigcup_{g \in \mathcal{M}_A} \{x \in X \mid K(x)(\bigcup_{g \in \mathcal{M}_A} g) \leq r\} = \bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} D_{\alpha[n]}, \tag{4}$$
and because \( S_X \) is closed under the Souslin operation and contains \( \mathcal{B}(X) \), we conclude that \( E_r \in S_X \).  

**Proposition 2.138** Let \( K : (X; \mathcal{A}) \rightsquigarrow (Y, \mathcal{B}) \) be a transition kernel, and assume that \( Y \) is a separable metric space. Let \( S_X \) be the smallest \( \sigma \)-algebra which contains \( \mathcal{A} \) and which is closed under the Souslin operation. Then there exists a unique transition kernel

\[
\overline{K} : (X, S_X) \rightsquigarrow (Y, \overline{\mathcal{B}(Y)}^{(K(x)|x \in X)})
\]

extending \( K \).

**Proof** 1. Put \( \overline{K}(x)(A) := K^*(x)(A) \) for \( x \in X \) and \( A \in \overline{\mathcal{B}(Y)}^{(K(x)|x \in X)} \). Because \( A \) is an element of the \( K(x) \)-completion of \( \mathcal{B}(Y) \), we know that \( \overline{K}(x) = K(x) \) defines a sub probability on \( \overline{\mathcal{B}(Y)}^{(K(x)|x \in X)} \). It is clear that \( \overline{K}(x) \) is the unique extension of \( K(x) \) to the latter \( \sigma \)-algebra. It remains to be shown that \( \overline{K} \) is a transition kernel.

2. Fix \( A \in \overline{\mathcal{B}(Y)}^{(K(x)|x \in X)} \) and \( q \in [0, 1] \), then

\[
\{ x \in X \mid K^*(x)(A) < q \} = \bigcup_{\ell \in \mathbb{N}} \bigcup_{g \in \mathcal{G}_A} \{ x \in X \mid K(x)(\bigcup \{ g \}) \leq q - \frac{1}{\ell} \}
\]

The latter set is a member of \( S_X \) by Lemma 2.137.

Separability of the target space is required because it is this property which makes sure that the measure for each Borel set can be approximated arbitrarily well from within by closed sets, and from the outside by open sets [Str98, Lemma 3.4.14].

Before discussing consequences, a mild generalization to separable measurable spaces should be mentioned. Proposition 2.138 yields as an immediate consequence:

**Corollary 2.139** Let \( K : (X; \mathcal{A}) \rightsquigarrow (Y, \mathcal{B}) \) be a transition kernel such that \( (Y, \mathcal{B}) \) is a separable measurable space. Assume that \( \mathcal{X} \) is a \( \sigma \)-algebra on \( X \) which is closed under the Souslin operation with \( S_X \subseteq \mathcal{X} \), and that \( \mathcal{Y} \) is a \( \sigma \)-algebra on \( Y \) with \( \mathcal{B} \subseteq \mathcal{Y} \subseteq \mathcal{B}^{(K(x)|x \in X)} \). Then there exists a unique extension \( (X, \mathcal{X}) \rightsquigarrow (Y, \mathcal{Y}) \) to \( K \). In particular \( K \) has a unique extension to a transition kernel \( \overline{K} : (X, \mathcal{A}) \rightsquigarrow (Y, \overline{\mathcal{B}}) \).

**Proof** This follows from Proposition 2.138 and the characterization of separable measurable spaces in Proposition 2.58.

### 2.7 Measurable Selections

Looking again at von Neumann’s Selection Theorem 2.130, we have found for a given surjection \( f : X \to Y \) a universally measurable map \( g : Y \to X \) with \( f \circ g = \text{id}_Y \). This can be rephrased: we have \( g(y) \in f^{-1}([y]) \) for each \( y \in Y \), so \( g \) may be considered a universal measurable selection for the set valued map \( y \mapsto f^{-1}([y]) \). We will consider this problem from a slightly different angle by assuming that \( (X, \mathcal{A}) \) is a measurable, \( Y \) is a Polish space, and that we have a set valued map \( F : X \to \mathcal{P}(Y) \setminus \{ \emptyset \} \) for which a measurable selection is to be constructed, i.e., a measurable (not merely universally measurable) map \( g : X \to Y \) such that \( g(y) \in F(y) \) for all \( y \in Y \). Clearly, the Axiom of Choice guarantees the existence of a map which picks an element from \( F(y) \) for each \( y \), but this is not enough.
We assume that $F(y)$ is always a closed subset of $Y$, and that it is measurable. Since $F$ does not necessarily take single values only, we have to define measurability in this case. Denote by $\mathbb{F}(Y)$ the set of all closed and non-empty subsets of $Y$.

**Definition 2.140** A map $F : X \to \mathbb{F}(Y)$ from a measurable space $(X, \mathcal{A})$ to the closed non-empty subsets of a Polish space $Y$ is called measurable (or a measurable relation) iff

$$F^w(G) := \{ x \in X \mid F(x) \cap G \neq \emptyset \} \in \mathcal{A}$$

for every open subset $G \subseteq Y$. The map $s : X \to Y$ is called a measurable selector for $F$ iff $s$ is $\mathcal{A}$-$\mathcal{B}(Y)$-measurable such that $s(x) \in F(x)$ for all $x \in X$.

Since $\{f(x)\} \cap G \neq \emptyset$ iff $f(x) \in G$, measurability as defined in this definition is a generalization of measurability for point valued maps $f : X \to Y$.

The selection theorem due to Kuratowski and Ryll-Nardzewski tell us that a measurable selection exists for a measurable closed valued map, provided $Y$ is Polish. To be specific:

**Theorem 2.141** Given a measurable space $(X, \mathcal{A})$ and a Polish space $Y$, a measurable map $F : X \to \mathbb{F}(Y)$ has a measurable selector.

**Proof** Fix a complete metric $d$ on $Y$. Denote by $B(y,r)$ the open ball around $y \in Y$ with radius $r > 0$; $d$ is the metric on $Y$ such that the metric space $(Y, d)$ is complete. Recall that the distance of an element $y$ to a closed set $C$ is $d(y, C) := \inf\{ d(y, y') \mid y' \in C \}$, hence $d(y, C) = 0$ iff $y \in C$.

Let $(y_n)_{n \in \mathbb{N}}$ be dense, and define $f_1(x) := y_n$, if $n$ is the smallest index $k$ so that $F(x) \cap B(y_k, 1) \neq \emptyset$. Then $f_1 : X \to Y$ is $\mathcal{A}$-$\mathcal{B}(Y)$ measurable, because the map takes only a countable number of values and

$$\{ x \in X \mid f_1(x) = y_n \} = F^w(B(y_n, 1)) \setminus \bigcup_{k=1}^{n-1} F^w(B(y_k, 1)).$$

Proceeding inductively, assume that we have defined measurable maps $f_1, \ldots, f_n$ such that

$$d(f_j(x), f_{j+1}(x)) < 2^{-(j-1)}, \quad 1 \leq j < n$$

$$d(f_j(x), F(x)) < 2^{-j}, \quad 1 \leq j \leq n$$

Put $X_k := \{ x \in X \mid f_n(x) = y_k \}$, and define $f_{k+1}(x) := y_\ell$ for $x \in X_k$, where $\ell$ is the smallest index $m$ such that $F(x) \cap B(y_k, 2^{-n}) \cap B(y_m, 2^{-(n+1)}) \neq \emptyset$. Moreover, there exists $y' \in B(y_k, 2^{-n}) \cap B(y_m, 2^{-(n+1)})$, thus

$$d(f_n(x), f_{n+1}(x)) \leq d(f_n(x), y') + f(f_{n+1}(x), y') < 2^{-n} + 2^{-(n+1)}$$

The argumentation from above shows that $f_{n+1}$ takes only countably many values, and we know that $d(f_{n+1}(x), F(x)) < 2^{-(n+1)}$.

Thus $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence for each $x \in X$. Since $(Y, d)$ is complete, the limit $f(x) := \lim_{n \to \infty} f_n(x)$ exists with $d(f(x), F(x)) = 0$, hence $f(x) \in F(x)$, because $F(x)$ is closed. Moreover as a pointwise limit of a sequence of measurable functions $f$ is measurable, so $f$ is the desired measurable selector. $\dashv$

It is possible to weaken the conditions on $F$ and on $\mathcal{A}$, see Exercise 23. This theorem has an interesting consequence, viz., that we can find a sequence of dense selectors for $F$. 

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Corollary 2.142 Under the assumptions of Theorem 2.141, a measurable map $F : X \to \mathcal{F}(Y)$ has a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable selectors such that $\{f_n(x) \mid n \in \mathbb{N}\}$ is dense in $F(x)$ for each $x \in X$.

Proof 1. We use notations from above. Let again $(y_n)_{n \in \mathbb{N}}$ be a dense sequence in $Y$, and define for $n, m \in \mathbb{N}$ the map

$$F_{n,m}(x) := \begin{cases} F(x) \cap B(y_n, 2^{-m}), & \text{if } x \in F^w(B(y_n, 2^{-m})) \\ F(x), & \text{otherwise} \end{cases}$$

Denote by $H_{n,m}(x)$ the closure of $F_{n,m}(x)$.

2. $H_{n,m} : X \to \mathcal{F}(Y)$ is measurable. In fact, put $A_1 := F^w(B(y_n, 2^{-m}))$, $A_2 := X \setminus A_1$, then $A_1, A_2 \in \mathcal{A}$, because $F$ is measurable and $B(y_n, 2^{-m})$ is open. But then we have for an open set $G \subseteq Y$

$$\{x \in X \mid H_{n,m} \cap G \neq \emptyset\} = \{x \in X \mid F_{n,m} \cap G \neq \emptyset\} = \{x \in A_1 \mid F(x) \cap G \cap B(y_n, 2^{-m}) \neq \emptyset\} \cup \{x \in A_2 \mid F(x) \cap G \neq \emptyset\},$$

thus $H_{n,m}^w(G) \in \mathcal{A}$.

3. We can find a measurable selector $s_{n,m}$ for $H_{n,m}$ by Theorem 2.141, so we have to show that $\{s_{n,m}(x) \mid n, m \in \mathbb{N}\}$ is dense in $F(x)$ for each $x \in X$. Let $y \in F(x)$. Given $\epsilon > 0$, select $m$ with $2^{-m} < \epsilon/2$; there exists $y_n$ with $d(y, y_n) < 2^{-m}$. Thus $x \in H_{n,m}^w(B(y_n, 2^{-m}))$, and $s_{n,m}(x)$ is a member of the closure of $B(y_n, 2^{-m})$, which means $d(y, s_{n,m}(x)) < \epsilon$. Now arrange $\{s_{n,m}(x) \mid n, m \in \mathbb{N}\}$ as a sequence, then the assertion follows. \(\Diamond\)

This is a first application of measurable selections.

Example 2.143 Call a map $h : X \to \mathcal{B}(Y)$ for the Polish space $Y$ hit-measurable iff $h$ is measurable with respect to $\mathcal{A}$ and $\mathcal{H}_G(\mathcal{B}(Y))$, where $\mathcal{G}$ is the set of all open sets in $Y$, see Example 2.22. Thus $h$ is hit-measurable iff $\{x \in X \mid h(x) \cap U \neq \emptyset\} \in \mathcal{A}$ for each open set $U \subseteq Y$. If $h$ is image finite (i.e., $h(x)$ is always non-empty and finite), then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of measurable maps $f_n : X \to Y$ such that $h(x) = \{f_n(x) \mid n \in \mathbb{N}\}$ for each $x \in X$. This is so because $h : X \to \mathcal{F}(Y)$ is measurable, hence Corollary 2.142 is applicable. \(\Diamond\)

Transition kernels into Polish spaces induce a measurable closed valued map, for which selectors exist.

Example 2.144 Let under the assumptions of Theorem 2.141 $K : (X, \mathcal{A}) \sim (Y, \mathcal{B}(Y))$ be a transition kernel with $K(x)(Y) > 0$ for all $x \in X$. Then there exists a measurable map $f : X \to Y$ such that $K(x)(U) > 0$, whenever $U$ is an open neighborhood of $f(x)$.

In fact, $\Gamma : x \mapsto \text{supp}(K(x))$ takes non-empty and closed values by Lemma 2.25. If $G \subseteq Y$ is open, then

$$\Gamma^w(G) = \{x \in X \mid \text{supp}(K(x)) \cap G \neq \emptyset\} = \{x \in X \mid K(x)(G) > 0\} \in \mathcal{A}.$$  

Thus $\Gamma$ has a measurable selector $f$ by Theorem 2.141. The assertion now follows from Corollary 2.26 \(\Diamond\)
Perceiving a stochastic relation $K : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B}(Y))$ as a probabilistic model for transitions such that $K(x)(B)$ is the probability for making a transition from $x$ to $B$ (with $K(x)(Y) \leq 1$), we may interpret the selection $f$ as one possible deterministic version for a transition: the state $f(x)$ is possible, since $f(x) \in \text{supp}(K(x))$, and it may even be undertaken with positive probability.

### 2.8 Integration

After having studied the structure of measurable sets under various conditions on the underlying space with an occasional side glance at real-valued measurable functions, we will discuss now integration. This is a fundamental operation associated with measures. The integral of a function with respect to a measure will be what you expect it to be, viz., for non-negative functions the area between the curve and the $x$-axis. This view will be confirmed later on, when Fubini’s Theorem will be available for computing measures in Cartesian products. For the time being, we build up the integral in a fairly straightforward way through an approximation by step functions, obtaining a linear map with some favorable properties, for example the Lebesgue Dominated Convergence Theorem. All the necessary constructions are given in this section, offering more than one occasion to exercise the well-known $\varepsilon$-$\delta$-arguments, which are necessary, but not particularly entertaining. But that’s life.

The second part of this section offers a complementary view — it starts from a positive linear map with some additional continuity properties and develops a measure from it. This is Daniell’s approach, suggesting that measure and integral are really most of the time two sides of the same coin. We show that this duality comes to life especially when we are dealing with a compact metric space: Here the celebrated Riesz Representation Theorem gives a bijection between probability measures on the Borel sets and positive linear functions mapping 1 to 1 on the continuous real-valued functions. We formulate and prove this theorem here; it should be mentioned that this is not the most general version available, as with most other results discussed here (but probably there is no such thing as a most general version, since the development did branch out into wildly different directions).

This section will be fundamental for the discussions and results later in this chapter. Most results are formulated for finite or $\sigma$-finite measures, and usually no attempt has been made to find the boundary delineating a development.

#### 2.8.1 From Measure to Integral

We fix a measure space $(X, \mathcal{A}, \mu)$. Denote for the moment by $\mathcal{T}(X, \mathcal{A})$ the set of all measurable step functions, and by $\mathcal{T}_+(X, \mathcal{A})$ the non-negative step functions; similarly, $\mathcal{F}_+(X, \mathcal{A})$ are the non-negative measurable functions. Note that $\mathcal{T}(X, \mathcal{A})$ is a vector space under the usual operations, and that it is a lattice under finite or countable pointwise suprema and infima. We know from Proposition 2.33 that we can approximate each bounded measurable function by a sequence of step functions from below.

Define

$$\int_X \sum_{i=1}^n \alpha_i \cdot \chi_{A_i} \, d\mu := \sum_{i=1}^n \alpha_i \cdot \mu(A_i)$$

(5)
as the integral with respect to $\mu$ for the step function $\sum_{i=1}^{n} \alpha_i \cdot \chi_{A_i} \in \mathcal{T}(X,A)$. Exercise 24 tells us that the integral is well defined: if $f, g \in \mathcal{T}(X,A)$ with $f = g$, then

$$\sum_{\alpha \in \mathbb{R}} \alpha \cdot \mu(\{x \in X \mid f(x) = \alpha\}) = \sum_{\beta \in \mathbb{R}} \beta \cdot \mu(\{x \in X \mid g(x) = \beta\}).$$

Thus the definition yields the same value for the integral. These are some elementary properties of the integral for step functions.

**Lemma 2.145** Let $f, g \in \mathcal{T}(X,A)$ be step functions, $\alpha \in \mathbb{R}$. Then

1. $\int_X \alpha \cdot f \, d\mu = \alpha \cdot \int_X f \, d\mu$,
2. $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$,
3. if $f \geq 0$, then $\int_X f \, d\mu \geq 0$, in particular, $f \mapsto \int_X f \, d\mu$ is monotone,
4. $\int_X \chi_A \, d\mu = \mu(A)$ for $A \in \mathcal{A}$,
5. $|\int_X f \, d\mu| \leq \int_X |f| \, d\mu$.

Moreover the map $A \mapsto \int_A f \, d\mu := \int_X f \cdot \chi_A \, d\mu$ is additive on $\mathcal{A}$ whenever $f \in \mathcal{T}_+(X,A)$. ⊣

We know from Proposition 2.33 that we can find for $f \in \mathcal{F}_+(X,A)$ a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{T}_+(X,A)$ such that $f_1 \leq f_2 \leq \ldots$ and $\sup_{n \in \mathbb{N}} f_n = f$. This observation is used for the definition of the integral for $f$. We define

$$\int_X f \, d\mu := \sup \left\{ \int_X g \, d\mu \mid g \leq f \text{ and } g \in \mathcal{T}_+(X,A) \right\}$$

Note that the right hand side may be infinite; we will discuss this shortly.

The central observation is formulated in Levi’s Theorem:

**Theorem 2.146** Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence of functions in $\mathcal{F}_+(X,A)$ with limit $f$, then the limit $(\int_X f_n \, d\mu)_{n \in \mathbb{N}}$ exists and equals $\int_X f \, d\mu$.

**Proof** 1. Because the integral is monotone in the integrand by Lemma 2.145, the limit

$$\ell := \lim_{n \to \infty} \int_X f_n \, d\mu$$

exists (possibly in $\mathbb{R} \cup \{\infty\}$), and we know from monotonicity that $\ell \leq \int_X f \, d\mu$.

2. Let $f = c > 0$ be a constant, and let $0 < d < c$. Then $\sup_{n \in \mathbb{N}} d \cdot \chi_{\{x \in X \mid f_n(x) \geq d\}} = d$, hence we obtain

$$\int_X f \, d\mu \geq \int_X f_n \, d\mu \geq \int_{\{x \in X \mid f_n(x) \geq d\}} f_n \, d\mu \geq d \cdot \mu(\{x \in X \mid f_n(x) \geq d\})$$

for every $n \in \mathbb{N}$, thus

$$\int_X f \, d\mu \geq d \cdot \mu(X).$$

Letting $d$ approaching $c$, we see that

$$\int_X f \, d\mu \geq \lim_{n \to \infty} \int_X f_n \, d\mu \geq c \cdot \mu(X) = \int_X f \, d\mu.$$
This gives the desired equality.

3. If $f = c \cdot \chi_A$ with $A \in \mathcal{A}$, we restrict the measure space to $(A, \mathcal{A} \cap A, \mu)$, so the result is true also for step functions based on one single set.

4. Let $f = \sum_{i=1}^{n} \alpha_i \cdot \chi_{A_i}$ be a step function, then we may assume that the sets $A_1, \ldots, A_n$ are mutually disjoint. Consider $f_i := f \cdot \chi_{A_i} = \alpha_i \cdot \chi_{A_i}$ and apply the previous step to $f_i$, taking additivity from Lemma 2.145, part 2 into account.

5. Now consider the general case. Select step functions $(g_n)_{n \in \mathbb{N}}$ with $g_n \in T_+(X, \mathcal{A})$ such that $g_n \leq f_n$ and $|\int_X f_n \, d\mu - \int_X g_n \, d\mu| < 1/n$. We may and do assume that $g_1 \leq g_2 \leq \ldots$, for we otherwise may pass to the step function $h_n := \sup\{g_1, \ldots, g_n\}$. Let $0 \leq g \leq f$ be a step function, then $\lim_{n \to \infty}(g_n \wedge g) = g$, so that we obtain from the previous step

$$\int_X g \, d\mu = \lim_{n \to \infty} \int_X g_n \wedge g \, d\mu \leq \lim_{n \to \infty} \int_X g_n \, d\mu \leq \lim_{n \to \infty} \int_X f_n \, d\mu,$$

Because $\int_X g \, d\mu$ may be chosen arbitrarily close to $\ell$, we finally obtain

$$\lim_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu \leq \lim_{n \to \infty} \int_X f_n \, d\mu,$$

which implies the assertion for arbitrary $f \in F_+(X, \mathcal{A})$. \[\]

Since we can approximate each non-negative measurable function from below and from above by step functions (Proposition 2.33 and Exercise 7), we obtain from Levi’s Theorem for $f \in F_+(X, \mathcal{A})$ the representation

$$\sup\{\int_X g \, d\mu \mid T_+(X, \mathcal{A}) \ni g \leq f\} = \int_X f \, d\mu = \inf\{\int_X g \, d\mu \mid f \leq g \in T_+(X, \mathcal{A})\}.$$

This strongly resembles — and generalizes — the familiar construction of the Riemann integral for a continuous function $f$ over a bounded interval by sandwiching it between lower and upper sums of step functions.

Compatibility of the integral with scalar multiplication and with addition is now an easy consequence of Levi’s Theorem:

**Corollary 2.147** Let $a \geq 0$ and $b \geq 0$ be non-negative real numbers, then

$$\int_X a \cdot f + b \cdot g \, d\mu = a \cdot \int_X f \, d\mu + b \cdot \int_X g \, d\mu$$

for $f, g \in F_+(X, \mathcal{A})$.

**Proof** Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be sequences of step functions which converge monotonically to $f$ resp. $g$. Then $(a \cdot f_n + b \cdot g_n)_{n \in \mathbb{N}}$ is a sequence of step functions converging monotonically to $a \cdot f + b \cdot g$. Apply Levi’s Theorem 2.146 and the linearity of the integral on step functions from Lemma 2.145 to obtain the assertion. \[\]

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Given an arbitrary \( f \in \mathcal{F}(X,A) \), we can decompose \( f \) into a positive and a negative part \( f^+ := f \vee 0 \) resp. \( f^- := (-f) \vee 0 \), so that \( f = f^+ - f^- \) and \( |f| = f^+ + f^- \).

A function \( f \in \mathcal{F}(X,A) \) is called *integrable* (with respect to \( \mu \)) iff

\[
\int_X |f|, \mu \, d\mu < \infty,
\]

in this case we set

\[
\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.
\]

In fact, because \( f^+ \leq f \), we obtain from Lemma 2.145 that \( \int_X f^+ \, d\mu < \infty \), similarly we see that \( \int_X f^- \, d\mu < \infty \). The integral is well defined, because if \( f = f_1 - f_2 \) with \( f_1, f_2 \geq 0 \), we conclude \( f_1 \leq f \leq |f| \), hence \( \int_X f_1 \, d\mu < \infty \), and \( f_2 \leq |f| \), so that \( \int_X f_2 \, d\mu < \infty \), which implies \( \int_X f^+ \, d\mu + \int_X f_2 \, d\mu = \int_X f^- \, d\mu + \int_X f_1 \, d\mu \) by Corollary 2.147. This we obtain in fact \( \int_X f^+ \, d\mu - \int_X f^- \, d\mu = \int_X f_1 \, d\mu - \int_X f_2 \, d\mu \).

This special case is also of interest: let \( A \in \mathcal{A} \), define for \( f \) integrable

\[
\int_A f \, d\mu := \int_X f \cdot \chi_A \, d\mu
\]

(note that \( |f \cdot \chi_A| \leq |f| \)). We emphasize occasionally the integration variable by writing \( \int_X f(x) \, d\mu(x) \) instead of \( \int_X f \, d\mu \).

Collecting some useful and a.e. used properties, we state

**Proposition 2.148** Let \( f,g \in \mathcal{F}(X,A) \) be measurable functions, then

1. If \( f \geq \mu 0 \), then \( \int_X f \, d\mu = 0 \) iff \( f = \mu 0 \).

2. If \( f \) is integrable, and \( |g| \leq \mu |f| \), then \( g \) is integrable.

3. If \( f \) and \( g \) are integrable, then so are \( a \cdot f + b \cdot g \) for all \( a,b \in \mathbb{R} \), and \( \int_X a \cdot f + b \cdot g \, d\mu = a \cdot \int_X f \, d\mu + b \cdot \int_X g \, d\mu \).

4. If \( f \) and \( g \) are integrable, and \( f \leq \mu g \), then \( \int_X g \, d\mu \leq \int_X f \, d\mu \).

5. If \( f \) is integrable, then \( |\int_X f \, d\mu| \leq \int_X |f| \, d\mu \).

We now state and prove some statements which relate sequences of functions to their integrals. The first one is traditionally called Fatou’s Lemma.

**Proposition 2.149** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence in \( \mathcal{F}_+(X,A) \). Then

\[
\int_X \lim \inf_{n \to \infty} f_n \, d\mu \leq \lim \inf_{n \to \infty} \int_X f_n \, d\mu
\]

**Proof** Since \((\inf_{m \geq n} f_m)_{n \in \mathbb{N}}\) is an increasing sequence of measurable functions in \( \mathcal{F}_+(X,A) \), we obtain from Levi’s Theorem 2.146

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X \inf_{m \geq n} f_m \, d\mu = \sup_{n \in \mathbb{N}} \int_X \inf_{m \geq n} f_m \, d\mu.
\]
Because we plainly have by monotonicity
\[ \int_X \inf_{m \geq n} f_m \, d\mu \leq \inf_{m \geq n} \int_X f_m \, d\mu, \]
the assertion follows. ⊥

The **Lebesgue Dominated Convergence Theorem** is a very important and much used tool; it can be derived now easily from Fatou’s Lemma.

**Theorem 2.150** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of measurable functions with \(f_n \xrightarrow{a.e.} f\) for some measurable function \(f\), and \(|f_n| \leq g\) for all \(n \in \mathbb{N}\) and an integrable function \(g\). Then \(f_n\) and \(f\) are integrable, and

\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu \text{ and } \lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0.
\]

**Proof** 1. It is no loss of generality to assume that \(f_n \to f\) and \(\forall n \in \mathbb{N}: f_n \leq g\) pointwise (otherwise modify the \(f_n, f\) and \(g\) on a set of \(\mu\)-measure zero). Because \(|f_n| \leq g\), we conclude from Proposition 2.148 that \(f_n\) is integrable, and since \(f \leq g\) holds as well, we infer that \(f\) is integrable as well.

2. Put \(g_n := |f| + g - |f_n - f|\), then \(g_n \geq 0\), and \(g_n\) is integrable for all \(n \in \mathbb{N}\). We obtain from Fatou’s Lemma

\[
\int_X |f| + g \, d\mu = \int_X \liminf_{n \to \infty} g_n \, d\mu \\
\leq \liminf_{n \to \infty} \int_X g_n \, d\mu \\
= \int_X |f| + g \, d\mu - \limsup_{n \to \infty} \int_X |f_n - f| \, d\mu.
\]

Hence we obtain \(\limsup_{n \to \infty} \int_X |f_n - f| \, d\mu = 0\), thus \(\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0\).

3. We finally note that

\[
|\int_X f_n \, d\mu - \int_X f \, d\mu| = |\int_X (f_n - f) \, d\mu| \leq \int_X |f_n - f| \, d\mu,
\]

which completes the proof. ⊥

These are immediate consequences of the Lebesgue Theorem:

**Corollary 2.151** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of measurable functions, \(g\) integrable, such that \(|\sum_{k=1}^n f_k| \leq g\) for all \(n \in \mathbb{N}\). Then all \(f_n\) as well as \(f := \sum_{n \in \mathbb{N}} f_n\) are integrable, and

\[
\int_X f \, d\mu = \sum_{n \in \mathbb{N}} \int_X f_n \, d\mu.
\]

**Corollary 2.152** Let \(f \geq 0\) be an integrable function, then \(A \mapsto \int_A f \, d\mu\) defines a finite measure on \(A\).

**Proof** All the properties of a measure are immediate, \(\sigma\)-additivity follows from Corollary 2.151 ⊥

Integration with respect to an image measure is also available right away. It yields the fairly helpful **change of variables formula** for image measures.
Corollary 2.153 Let \((Y, \mathcal{B})\) a measurable space and \(g : X \rightarrow Y\) be \(A\)-\(B\)-measurable. Then \(h \in \mathcal{F}(Y, \mathcal{B})\) is integrable iff \(g \circ h\) is \(\mu\)-integrable, and in this case we have

\[
\int_Y h \, d\mathbb{M}(g)(\mu) = \int_X h \circ g \, d\mu.
\]

Proof We show first that formula (6) is true for step functions. In fact, if \(h = \chi_B\) with a measurable set \(B\), then we obtain from the definition

\[
\int_Y \chi_B \, d\mathbb{M}(g)(\mu) = \mathbb{M}(g)(\mu)(B) = \mu(g^{-1}[B]) = \int_X \chi_B \circ g \, d\mu \tag{6}
\]

(since \(\chi_B(g(x)) = 1\) iff \(x \in g^{-1}[B]\)). This observation extends by linearity to step functions, so that we obtain for \(h = \sum_{i=1}^{n} b_i \cdot \chi_{B_i}\)

\[
\int_Y h \, d\mathbb{M}(g)(\mu) = \sum_{i=1}^{n} b_i \cdot \int_X \chi_{B_i} \circ g \, d\mu = \int_X h \circ g \, d\mu
\]

Thus the assertion now follows from Levi’s Theorem 2.146.

The reader is probably familiar with the change of variables formula in classical calculus. It deals with \(k\)-dimensional Lebesgue measure \(\lambda^k\), and a differentiable and injective map \(T : V \rightarrow W\) from an open set \(V \subseteq \mathbb{R}^k\) to a bounded set \(W \subseteq \mathbb{R}^k\). \(T\) is assumed to have a continuous inverse. Then the integral of a measurable and bounded function \(f : T[V] \rightarrow \mathbb{R}\) can be expressed in terms of the integral over \(V\) of \(f \circ T\) and the Jacobian \(J_T\) of \(T\). To be specific

\[
\int_{T[V]} f \, d\lambda^k = \int_V (f \circ T) \cdot |J_T| \, d\lambda^k.
\]

Recall that the Jacobian \(J_T\) of \(T\) is the determinant of the partial derivatives of \(T\), i.e.,

\[
J_T(x) = \det\left(\frac{\partial T_i(x)}{\partial x_j}\right).
\]

This representation can be derived from the representation for the integral with respect to the image measure from Corollary 2.153 and from the Radon-Nikodym Theorem 2.211 through a somewhat lengthy application of results from fairly elementary linear algebra. We do not want to develop this apparatus in the present presentation, we will, however, provide a glimpse at the one-dimensional situation in Proposition 2.224. The reader is referred rather to Rudin’s exposition [Rud74, p. 181 - 188] or to Stromberg’s more elementary discussion in [Str81, p. 385 - 392]; if you read German, Elstrodt’s derivation [Els99, § V.4] should not be missed.

2.8.2 The Daniell Integral and Riesz’s Representation Theorem

The previous section developed the integral from a finite or \(\sigma\)-finite measure; the result was a linear functional on a subspace of measurable functions, which will be investigated in greater detail later on. This section will demonstrate that it is possible to obtain a measure from a linear functional on a well behaved space of functions. This approach was proposed by P. J. Daniell ca. 1920, it is called in his honor the Daniell integral. It is useful when a linear
functional is given, and one wants to show that this functional is actually defined by a measure, which then permits putting the machinery of measure theory into action. We will encounter such a situation, e.g., when studying linear functionals on spaces of integrable functions. Specifically, we derive the Riesz Representation Theorem, which shows that there is a one-to-one correspondence between probability measures and normed positive linear functionals on the vector lattice of continuous real valued functions on a compact metric space.

Let us fix a set $X$ throughout. We will also fix a set $\mathcal{F}$ of functions $X \to \mathbb{R}$ which is assumed to be a vector space (as always, over the reals) with a special property.

**Definition 2.154** A vector space $\mathcal{F} \subseteq \mathbb{R}^X$ is called a vector lattice iff $|f| \in \mathcal{F}$ whenever $f \in \mathcal{F}$.

Now fix the vector lattice $\mathcal{F}$. Each vector lattice is indeed a lattice: define

$$f \lor g := (|f - g| + f + g)/2,$$

$$f \land g := -((-f) \lor (-g))$$

$$f \leq g \iff f \lor g = g$$

$$\iff f \land g = f$$

Thus $\mathcal{F}$ contains with $f$ and $g$ also $f \land g$ and $f \lor g$, and it is easy to see that $\leq$ defines a partial order on $\mathcal{F}$ such that $\text{sup}\{f,g\} = f \lor g$ and $\text{inf}\{f,g\} = f \land g$, see, e.g., [Dob13, 2.5.5]. Note that we have $\text{max}\{\alpha, \beta\} = (|\alpha - \beta| + \alpha + \beta)/2$ for $\alpha, \beta \in \mathbb{R}$, thus we conclude that $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in \mathbb{R}$.

We will find these properties helpful; they will be used silently below.

**Lemma 2.155** If $0 \leq \alpha \leq \beta \in \mathbb{R}$ and $f \in \mathcal{F}$ with $f \geq 0$, then $\alpha \cdot f \leq \beta \cdot f$. If $f, g \in \mathcal{F}$ with $f \leq g$, then $f + h \leq g + h$ for all $h \in \mathcal{F}$. Also, $f \land g + f \lor g = f + g$.

**Proof** Because $f \geq 0$, we obtain

$$2 \cdot ((\alpha \cdot f) \lor (\beta \cdot f)) = (|\alpha - \beta| + \alpha + \beta) \cdot f = 2 \cdot \alpha \lor \beta \cdot f = 2 \cdot \beta \cdot f.$$  

This establishes the first claim. The second one follows from

$$2 \cdot ((f + h) \lor (g + h)) = |f - g| + f + g + 2 \cdot h = 2 \cdot (g + h).$$

The third one is established through the observation that it holds pointwise, and from the observation that $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. ⊳

We assume that $1 \in \mathcal{F}$, and that a function $L : \mathcal{F} \to \mathbb{R}$ is given, which has these properties:

- $L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L(f) + \beta \cdot L(g)$, so that $L$ is linear,
- if $f \geq 0$, then $L(f) \geq 0$, so that $L$ is positive,
- $L(1) = 1$, so that $L$ is normed,
- If $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{F}$ which decreases to 0, then $\lim_{n \to \infty} L(f_n) = 0$, so that $L$ is continuous from above at 0.

These are some immediate consequences from the properties of $L$. 

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**Lemma 2.156** If \( f, g \in \mathcal{F} \), then \( L(f \wedge g) + L(f \vee g) = L(f) + L(g) \). If \((f_n)_{n \in \mathbb{N}}\) and \((g_n)_{n \in \mathbb{N}}\) are increasing sequences of non-negative functions in \( \mathcal{F} \) with \( \lim_{n \to \infty} f_n \leq \lim_{n \to \infty} g_n \), then \( \lim_{n \to \infty} L(f_n) \leq \lim_{n \to \infty} L(g_n) \).

**Proof** The first property follows from the linearity of \( L \). For the second one, we observe that \( \lim_{k \to \infty} (f_n \wedge g_k) = f_n \in \mathcal{F} \), the latter sequence being increasing. Consequently, we have

\[
L(f_n) \leq \lim_{k \to \infty} L(f_n \wedge g_k) \leq \lim_{k \to \infty} L(g_k)
\]

for all \( n \in \mathbb{N} \), which implies the assertion. \( \dashv \)

\( \mathcal{F} \) determines a \( \sigma \)-algebra \( \mathcal{A} \) on \( X \), viz., the smallest \( \sigma \)-algebra which renders each \( f \in \mathcal{F} \) measurable. We will show now that \( L \) determines a unique probability measure on \( \mathcal{A} \) such that

\[
L(f) = \int_X f \, d\mu
\]

holds for all \( f \in \mathcal{F} \).

This will be done in a sequence of steps. A brief outline looks like this: We will first show that \( L \) can be extended to the set \( \mathcal{L}^+ \) of all bounded monotone limits from the non-negative elements of \( \mathcal{F} \), and that the extension respects monotone limits. From \( \mathcal{L}^+ \) we extract via indicator functions an algebra of sets, and from the extension to \( L \) an outer measure. This will then turn out to yield the desired probability.

Define

\[
\mathcal{L}^+: = \{ f : X \to \mathbb{R} \mid f \text{ is bounded, there exists } 0 \leq f_n \in \mathcal{F} \text{ increasing with } f = \lim_{n \to \infty} f_n \}.
\]

Define \( L(f) := \lim_{n \to \infty} L(f_n) \) for \( f \in \mathcal{L}^+ \), whenever \( f = \lim_{n \to \infty} f_n \) with the increasing sequence \((f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F} \). Then we obtain from Lemma 2.156 that this extension \( L \) on \( \mathcal{L}^+ \) is well defined, and it is clear that \( L(f) \geq 0 \), and that \( L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L(f) + \beta \cdot L(g) \), whenever \( f, g \in \mathcal{L}^+ \) and \( \alpha, \beta \in \mathbb{R}_+ \). We see also that \( f, g \in \mathcal{L}^+ \) implies that \( f \wedge g, f \vee g \in \mathcal{L}^+ \) with \( L(f \wedge g) + L(f \vee g) = L(f) + L(g) \). It turns out that \( L \) also respects the limits of increasing sequences.

**Lemma 2.157** Let \((f_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}^+ \) be an increasing and uniformly bounded sequence, then \( L(\lim_{n \to \infty} f_n) = \lim_{n \to \infty} L(f_n) \).

**Proof** Because \( f_n \in \mathcal{L}^+ \), we know that there exists for each \( n \in \mathbb{N} \) an increasing sequence \((f_{m,n})_{m \in \mathbb{N}}\) of elements \( f_{m,n} \in \mathcal{F} \) such that \( f_n = \lim_{m \to \infty} f_{m,n} \). Define

\[
g_m := \sup_{n \leq m} f_{m,n}.
\]

Then \((g_m)_{m \in \mathbb{N}}\) is an increasing sequence in \( \mathcal{F} \) with \( f_{m,n} \leq g_m \), and \( g_m \leq f_1 \vee f_2 \vee \ldots \vee f_m = f_m \), so that \( g_m \) is sandwiched between \( f_{m,n} \) and \( f_m \) for all \( m \in \mathbb{N} \) and \( n \leq m \). This yields \( L(f_{m,n}) \leq L(g_m) \leq L(f_m) \) for these \( n, m \). Thus \( \lim_{n \to \infty} f_n = \lim_{m \to \infty} g_m \), and hence

\[
\lim_{n \to \infty} L(f_n) = \lim_{m \to \infty} L(g_m) = L(\lim_{m \to \infty} g_m) = L(\lim_{n \to \infty} f_n).
\]

Thus we have shown that \( \lim_{n \to \infty} f_n \) can be obtained as the limit of an increasing sequence of functions from \( \mathcal{F} \); because \((f_n)_{n \in \mathbb{N}}\) is uniformly bounded, this limit is an element of \( \mathcal{L}^+ \). \( \dashv \)
Now define
\[ G := \{ G \subseteq X \mid \chi_G \in \mathcal{L}^+ \}, \]
\[ \mu(G) := L(\chi_G) \text{ for } G \in G. \]

Then \( G \) is closed under finite intersections and finite unions by the remarks made before Lemma 2.157. Moreover, \( G \) is closed under countable unions with \( \mu(\bigcup_{n \in \mathbb{N}} G_n) = \lim_{n \to \infty} \mu(G_n) \), if \( (G_n)_{n \in \mathbb{N}} \) is an increasing sequence in \( G \). Also \( \mu(X) = 1 \). Now define, as in the Carathéodory process, see [Dob13, 2.6.3]
\[ \mu^*(A) := \inf \{ \mu(G) \mid G \in G, A \subseteq G \}, \]
\[ B := \{ B \subseteq X \mid \mu^*(B) + \mu^*(X \setminus B) = 1 \}. \]

We obtain from the Carathéodory extension process

**Proposition 2.158** \( B \) is a \( \sigma \)-algebra, and \( \mu^* \) is countably additive on \( B \).

**Proof** [Dob13] Proposition 1.127

Put \( \mu(B) := \mu^*(B) \) for \( B \in B \), then \( (X, B, \mu) \) is a measure space, and \( \mu \) is a probability measure on \( (X, B) \).

In order to carry out the programme sketched above, we need a \( \sigma \)-algebra. We have on one hand the \( \sigma \)-algebra \( A \) generated by \( F \), and on the other hand \( B \), gleaned from the Carathéodory extension. It is not immediately clear how these \( \sigma \)-algebras are related to each other. And then we also have \( G \) as an intermediate family of sets, obtained from \( \mathcal{L}^+ \). This diagram shows the objects we will to discuss, together with a short hand indication of the respective relationships:

\[
\begin{align*}
F & \quad \sigma(\cdot) \quad A \\
\mathcal{L}^+ & \quad \chi \quad G \quad \subseteq \quad \text{Carathéodory} \quad B
\end{align*}
\]

We investigate the relationship of \( A \) and \( G \) first.

**Lemma 2.159** \( A = \sigma(G) \).

**Proof**

1. Because \( A \) is the smallest \( \sigma \)-algebra rendering all elements of \( F \) measurable, and because each element of \( \mathcal{L}^+ \) is the limit of a sequence of elements of \( F \), we obtain \( A \)-measurability for each element of \( \mathcal{L}^+ \). Thus \( G \subseteq A \).

2. Let \( f \in \mathcal{L}^+ \) and \( c \in \mathbb{R}_+ \), then \( f_n := 1 \wedge n \cdot \sup\{ f - c, 0 \} \in \mathcal{L}^+ \), and \( \chi_{\{ x \in X \mid f(x) > c \}} = \lim_{n \to \infty} f_n \). This is a monotone limit. Hence \( \{ x \in X \mid f(x) > c \} \in G \), thus in particular each element of \( F \) is \( \sigma(G) \)-measurable. This implies that \( A \subseteq \sigma(G) \) holds. \( \dashv \)

The relationship between \( B \) and \( G \) is a bit more difficult to establish.

**Lemma 2.160** \( G \subseteq B \).

**Proof** We have to show that \( \mu^*(G) + \mu^*(X \setminus G) = 1 \) for all \( G \in G \). Fix \( G \in G \). We obtain from additivity that \( \mu(G) + \mu(H) = \mu(G \cap H) + \mu(G \cup H) \geq \mu(X) = 1 \) holds for any \( H \in G \) with \( X \setminus G \subseteq H \), so that \( \mu^*(G) + \mu^*(X \setminus G) \leq 1 \) remains to be shown.
Because \( G \in \mathcal{G} \), there exists an increasing sequence \((f_n)_{n \in \mathbb{N}}\) of elements in \( \mathcal{F} \) such that 
\[
\chi_{G} = \sup_{n \in \mathbb{N}} f_n,
\]
consequently, \( \chi_{X \setminus G} = \inf_{n \in \mathbb{N}} (1 - f_n) \). Now let \( n \in \mathbb{N} \), and \( 0 < c \leq 1 \), then 
\[
X \setminus G \subseteq U_{n,c} := \{ x \in X \mid 1 - f_n(x) > c \}
\]
with \( U_{n,c} \in \mathcal{G} \). Because \( \chi_{U_{n,c}} \leq (1 - f_n)/c \), we obtain \( \mu^*(X \setminus G) \leq L(1 - f_n)/c \); this inequality holds for all \( c \) and all \( n \in \mathbb{N} \). Letting \( c \to 1 \) and \( n \to \infty \), this yields \( \mu^*(X \setminus G) \leq 1 - \mu^*(G) \).

Consequently, \( \mu^*(G) + \mu^*(X \setminus G) = 1 \) for all \( G \in \mathcal{G} \), which establishes the claim. \( \dashv \)

This yields the desired relationship of \( \mathcal{A} \), the \( \sigma \)-algebra generated by the functions in \( \mathcal{F} \), and \( \mathcal{B} \), the \( \sigma \)-algebra obtained from the extension process.

**Corollary 2.161** \( \mathcal{A} \subseteq \mathcal{B} \), and each element of \( \mathcal{L}^+ \) is \( \mathcal{B} \)-measurable.

**Proof** We have seen that \( \mathcal{A} = \sigma(\mathcal{G}) \) and that \( \mathcal{G} \subseteq \mathcal{B} \), so the first assertion follows from Proposition 2.158. The second assertion is immediate from the first one. \( \dashv \)

Because \( \mu \) is countably additive, hence a probability measure on \( \mathcal{B} \), and because each element of \( \mathcal{F} \) is \( \mathcal{B} \)-measurable, the integral \( \int_X f \, d\mu \) is defined, and we are done.

**Theorem 2.162** Let \( \mathcal{F} \) be a vector lattice of functions \( X \to \mathbb{R} \) with \( 1 \in \mathcal{F} \), \( L : \mathcal{F} \to \mathbb{R} \) be a linear and monotone functional on \( \mathcal{F} \) such that \( L(1) = 1 \), and \( L(f_n) \to 0 \), whenever \((f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F} \) decreases to 0. Then there exists a unique probability measure \( \mu \) on the \( \sigma \)-algebra \( \mathcal{A} \) generated by \( \mathcal{F} \) such that

\[
L(f) = \int_X f \, d\mu
\]

holds for all \( f \in \mathcal{F} \).

**Proof** Let \( \mathcal{G} \) and \( \mathcal{B} \) be constructed as above.

**Existence:** Because \( \mathcal{A} \subseteq \mathcal{B} \), we may restrict \( \mu \) to \( \mathcal{A} \), obtaining a probability measure. Fix \( f \in \mathcal{F} \), then \( f \) is \( \mathcal{B} \)-measurable, hence \( \int_X f \, d\mu \) is defined. Assume first that \( 0 \leq f \leq 1 \), hence \( f \in \mathcal{L}^+ \). We can write \( f = \lim_{n \to \infty} f_n \) with step functions \( f_n \), the contributing sets being members of \( \mathcal{G} \). Hence \( L(f_n) = \int_X f_n \, d\mu \), since \( L(\chi_G) = \mu(\chi_G) \) by construction. Consequently, we obtain from Lemma 2.157 and Lebesgue’s Dominated Convergence Theorem 2.150

\[
L(f) = L(\lim_{n \to \infty} f_n) = \lim_{n \to \infty} L(f_n) = \lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu = \int_X f \, d\mu.
\]

This implies the assertion also for bounded \( f \in \mathcal{F} \) with \( f \geq 0 \). If \( 0 \leq f \) is unbounded, write \( f = \sup_{n \in \mathbb{N}} (f \wedge n) \) and apply Levi’s Theorem 2.146. In the general case, decompose \( f = f^+ - f^- \) with \( f^+ := f \vee 0 \) and \( f^- := (-f) \wedge 0 \), and apply the foregoing.

**Uniqueness:** Assume that there exists a probability measure \( \nu \) on \( \mathcal{A} \) with \( L(f) = \int_X f \, d\nu \) for all \( f \in \mathcal{F} \), then the construction shows that \( \mu(G) = L(\chi_G) = \nu(G) \) for all \( G \in \mathcal{G} \). Since \( \mathcal{G} \) is closed under finite intersections, and since \( \mathcal{A} = \sigma(\mathcal{G}) \), we conclude that \( \nu(A) = \mu(A) \) for all \( A \in \mathcal{A} \). \( \dashv \)

We obtain as a consequence the famous Riesz Representation Theorem, which we state and formulate for the metric case. Recall that \( \mathcal{L}(X) \) is the linear space of all continuous functions \( X \to \mathbb{R} \) on a topological \( X \), and \( \mathcal{C}_b(X) \) is the subspace of all bounded functions. We state the result first for metric spaces and for bounded continuous functions, specializing the result then to the compact metric case.
Corollary 2.163 Let $X$ be a metric space, and let $L : \mathcal{C}_b(X) \to \mathbb{R}$ be a positive linear function with $\lim_{n \to \infty} L(f_n) = 0$ for each sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}_b(X)$ which decreases monotonically to 0. Then there exists a unique finite Borel measure $\mu$ such that

$$L(f) = \int_X f \, d\mu$$

holds for all $f \in \mathcal{C}_b(X)$.

Proof It is clear that $\mathcal{C}_b(X)$ is a vector lattice with 1 $\in \mathcal{C}_b(X)$. We may and do assume that $L(1) = 1$. The result follows immediately from Theorem 2.162 now. \(\square\)

If we take a compact metric space, then each continuous map $X \to \mathbb{R}$ is bounded. We show that the assumption on $L$’s continuity follows from compactness (which is usually referred to as Dini’s Theorem).

Theorem 2.164 Let $X$ be a compact metric space. Given a positive linear functional $L : \mathcal{C}(X) \to \mathbb{R}$, there exists a unique finite Borel measure $\mu$ such that

$$L(f) = \int_X f \, d\mu$$

holds for all $f \in \mathcal{C}(X)$.

Proof It is clear that $\mathcal{C}(X)$ is a vector lattice which contains 1. Again, we assume that $L(1) = 1$. In order to apply Theorem 2.162, we have to show that $\lim_{n \to \infty} L(f_n) = 0$, whenever $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(X)$ decreases monotonically to 0. But since $X$ is compact, we claim that $\sup_{x \in X} f_n(x) \to 0$, as $n \to \infty$. This is so because $\{x \in X \mid f_n \geq c\}$ is a family of closed sets with empty intersection for any $c > 0$, so we find by compactness a finite subfamily with empty intersection. Hence the assumption that $\sup_{x \in X} f_n(x) \geq c > 0$ for all $n \in \mathbb{N}$ would lead to a contradiction. Thus the assertion follows from Theorem 2.162. \(\square\)

Because $f \mapsto \int_X f \, d\mu$ defines for each Borel measure $\mu$ a positive linear functional on $\mathcal{C}(X)$, and because a measure on a metric space is uniquely determined by its integral on the bounded continuous functions, we obtain:

Corollary 2.165 For a compact metric space $X$ there is a bijection between positive linear functionals on $\mathcal{C}(X)$ and finite Borel measures. \(\square\)

The reason for not formulating the Riesz Representation Theorem immediately for general topological spaces is that Theorem 2.162 works with the $\sigma$-algebra generated — in this case — by $\mathcal{C}(X)$; this is in general the $\sigma$-algebra of Baire sets, which in turn may be properly contained in the Borel sets. Thus one obtains in the general case a Baire measure which then would have to be extended uniquely to a Borel measure. This is discussed in detail in [Bog07, Sec. 7.3].

A typical scenario for the application of the Riesz Theorem runs like this: one starts with a probability measure on a metric space $X$. This space can be embedded into a compact metric space $X'$, say, and one knows that the integral on the bounded continuous functions on $X$ extends to a positive linear map on the continuous functions on $X'$. Then the Riesz Representation Theorem kicks in and gives a probability measure on $X'$. We will see a situation like this when investigating the weak topology on the space of all finite measures on a Polish space in Section 2.10.
2.9 Product Measures

As a first application of integration we show that the product of two finite measures yields a measure again. This will lead to the Fubini’s Theorem on product integration, which evaluates a product integrable function on a product along its vertical or its horizontal cuts (in this sense it may be compared to a line sweeping algorithm — you traverse the Cartesian product, and in each instance you measure the cut). We apply this then to infinite products, first with a countable index set, then for an arbitrary one. Infinite products are a special case of projective systems, which may be described as sequences of probabilities which are related through projections. We show that such a projective system has a projective limit, i.e., a measure on the set of all sequences such that the projective system proper is obtained through a projection. This construction is, however, only feasible in a Polish space, since here a compactness argument is available which ascertains that the measure we are looking for is $\sigma$-additive. A small step leads to projective limits for stochastic relations. We demonstrate an application for projective limits through the interpretation for the logic CSL.

Fix for the time being two finite measure spaces $(X, A, \mu)$ and $(Y, B, \nu)$. The Cartesian product $X \times Y$ is endowed with the product $\sigma$-algebra $A \otimes B$ which is the smallest $\sigma$-algebra containing all measurable rectangles $A \times B$ with $A \in A$ and $B \in B$, see Section 2.1.

Recall that for $Q \subseteq X \times Y$ the cuts $Q_x := \{ y \in Y \mid \langle x, y \rangle \in Q \}$ and $Q^y := \{ x \in X \mid \langle x, y \rangle \in Q \}$ are defined. It is clear that $Q_x \in B$ and $Q^y \in A$ holds for $x \in X, y \in Y$, whenever $Q \in A \otimes B$.

In fact, take for example the vertical cut $Q_x$ and consider the set $Q := \{ Q \in A \otimes B \mid Q_x \in B \}$. Then $A \times B \in Q$, whenever $A \in A, B \in B$; this is so since the set of all measurable rectangles forms a generator for the product $\sigma$-algebra which is closed under finite intersections. Because $(X \times Y) \setminus Q_x = Y \setminus Q_x$, we infer that $Q$ is closed under complementation, and because $(\bigcup_{n \in \mathbb{N}} Q_n)_x = \bigcup_{n \in \mathbb{N}} Q_{n,x}$, we conclude that $Q$ is closed under disjoint countable unions.

Hence $Q = A \otimes B$ by the $\pi$-$\lambda$-Theorem 2.4.

**Lemma 2.166** Let $Q \in A \otimes B$ be a measurable set, then both $\varphi(x) := \nu(Q_x)$ and $\psi(y) := \mu(Q^y)$ define bounded measurable functions with

$$\int_X \nu(Q_x) \, d\mu(x) = \int_Y \mu(Q^y) \, d\nu(y).$$

**Proof** We use the same argument as above to establish that both $\varphi$ and $\psi$ are measurable functions, noting that $\nu((A \times B)_x) = \chi_A(x) \cdot \nu(B)$, similarly, $\mu((A \times B)^y) = \chi_B(y) \cdot \mu(A)$; in the next step the set of all $Q \in A \otimes B$ is shown to satisfy the assumptions of the $\pi$-$\lambda$-Theorem 2.4.

In the same way, the equality of the integrals is established, noting that

$$\int_X \nu((A \times B)_x) \, d\mu(x) = \mu(A) \cdot \nu(B) = \int_X \mu((A \times B)^y) \, d\nu(y).$$

This yields without much ado
Theorem 2.167 Given the finite measure spaces \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\), there exists a unique finite measure \(\mu \otimes \nu\) on \(\mathcal{A} \otimes \mathcal{B}\) such that \((\mu \otimes \nu)(A \times B) = \mu(A) \cdot \nu(B)\) for \(A \in \mathcal{A}, B \in \mathcal{B}\). Moreover,

\[
(\mu \otimes \nu)(Q) = \int_X \nu(Q_x) \, d\mu(x) = \int_Y \mu(Q_y) \, d\nu(y)
\]

holds for all \(Q \in \mathcal{A} \otimes \mathcal{B}\).

**Proof**

1. We establish the existence of \(\mu \otimes \nu\) by an appeal to Lemma 2.166 and to the properties of the integral according to Proposition 2.148 Define

\[
(\mu \otimes \nu)(Q) := \int_X \nu(Q_x) \, d\mu(x),
\]

then this defines a finite measure on \(\mathcal{A} \otimes \mathcal{B}\).

- Let \(Q \subseteq Q'\), then \(Q_x \subseteq Q'_x\) for all \(x \in X\), hence \(\int_X \nu(Q_x) \, d\mu(x) \leq \int_X \nu(Q'_x) \, d\mu(x)\). Thus \(\mu \otimes \nu\) is monotone.

- If \(Q\) and \(Q'\) are disjoint, then \(Q_x \cap Q'_x = (Q \cap Q')_x = \emptyset\) for all \(x \in X\). Thus \(\mu \otimes \nu\) is additive.

- Let \((Q_n)_{n \in \mathbb{N}}\) be a sequence of disjoint measurable sets, then \((Q_{n,x})_{n \in \mathbb{N}}\) is disjoint for all \(x \in X\), and

\[
\int_X \nu \left( \bigcup_{n \in \mathbb{N}} Q_{n,x} \right) \, d\mu(x) = \int_X \sum_{n \in \mathbb{N}} \nu(Q_{n,x}) \, d\mu(x) = \sum_{n \in \mathbb{N}} \int_X \nu(Q_{n,x}) \, d\mu(x)
\]

by Corollary 2.151. Thus \(\mu \otimes \nu\) is \(\sigma\)-additive.

2. Suppose that \(\rho\) is a finite measure on \(\mathcal{A} \otimes \mathcal{B}\) with \(\rho(A \times B) = \mu(A) \cdot \nu(B)\) for all \(A \in \mathcal{A}\) and all \(B \in \mathcal{B}\). Then

\[
\mathcal{G} := \{Q \in \mathcal{A} \otimes \mathcal{B} \mid \rho(Q) = (\mu \otimes \nu)(Q)\}
\]

contains the generator \(\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}\) of \(\mathcal{A} \otimes \mathcal{B}\), which is closed under finite intersections. Because both \(\rho\) and \(\mu \otimes \nu\) are measures, \(\mathcal{G}\) is closed under countable disjoint unions, because both contenders are finite, \(\mathcal{G}\) is also closed under complementation. The \(\pi\)-\(\lambda\)-Theorem 2.14 shows that \(\mathcal{G} = \mathcal{A} \otimes \mathcal{B}\). Thus \(\mu \otimes \nu\) is uniquely determined. \(\dashv\)

Theorem 2.167 holds also for \(\sigma\)-finite measures. In fact, assume that the contributing measure spaces are \(\sigma\)-finite, and let \((X_n)_{n \in \mathbb{N}}\) resp. \((Y_n)_{n \in \mathbb{N}}\) be increasing sequences in \(\mathcal{A}\) resp. \(\mathcal{B}\) such that \(\mu(X_n) < \infty\) and \(\nu(Y_n) < \infty\) for all \(n \in \mathbb{N}\), and \(\bigcup_{n \in \mathbb{N}} X_n = X\) and \(\bigcup_{n \in \mathbb{N}} Y_n = Y\). Localize \(\mu\) and \(\nu\) to \(X_n\) resp. \(Y_n\) by defining \(\mu_n(A) := \mu(A \cap X_n)\), similarly, \(\nu_n(B) := \nu(B \cap Y_n)\); since these measures are finite, we can extend them uniquely to a measure \(\mu_n \otimes \nu_n\) on \(\mathcal{A} \otimes \mathcal{B}\). Since \(\bigcup_{n \in \mathbb{N}} X_n \times Y_n = X \times Y\) with the increasing sequence \((X_n \times Y_n)_{n \in \mathbb{N}}\), we set

\[
(\mu \otimes \nu)(Q) := \sup_{n \in \mathbb{N}} (\mu_n \otimes \nu_n)(Q).
\]

Then \(\mu \otimes \nu\) is a \(\sigma\)-finite measure on \(\mathcal{A} \otimes \mathcal{B}\). Now assume that we have another \(\sigma\)-finite measure \(\rho\) on \(\mathcal{A} \otimes \mathcal{B}\) with \(\rho(A \times B) = \mu(A) \cdot \nu(B)\) for all \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\). Define \(\rho_n(Q) := \rho(Q \cap (X_n \times Y_n))\), hence \(\rho_n = \mu_n \otimes \nu_n\) by uniqueness of the extension to \(\mu_n\) and \(\nu_n\), so that we obtain

\[
\rho(Q) = \sup_{n \in \mathbb{N}} \rho_n(Q) = \sup_{n \in \mathbb{N}} (\mu_n \otimes \nu_n)(Q) = (\mu \otimes \nu)(Q)
\]
for all \( Q \in \mathcal{A} \otimes \mathcal{B} \). Thus we have shown

**Corollary 2.168** Given two \( \sigma \)-finite measure spaces \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\), there exists a unique \( \sigma \)-finite measure \( \mu \otimes \nu \) on \( \mathcal{A} \otimes \mathcal{B} \) such that \((\mu \otimes \nu)(A \times B) = \mu(A) \cdot \nu(B)\). We have

\[
(\mu \otimes \nu)(Q) = \int_X \nu(Q_x) \ d\mu(x) = \int_Y \mu(Q^y) \ d\nu(y)
\]

The construction of the product measure has been done here through integration of cuts. An alternative would have been the canonical approach. This approach would have investigated the map \( \langle A, B \rangle \mapsto \mu(A) \cdot \nu(B) \) on the set of all rectangles, and then put the extension machinery developed through the Carathéodory approach into action. It is a matter of taste which approach to prefer. —

The following example displays a slight generalization (a finite measure is but a constant transition kernel).

**Example 2.169** Let \( K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B}) \) a transition kernel (see Definition 2.10) such that the map \( x \mapsto K(x)(Y) \) is integrable with respect to the finite measure \( \mu \). Then

\[
(\mu \otimes K)(Q) := \int_X K(x)(Q_x) \ d\mu(x)
\]

defines a finite measure on \((X \times Y, \mathcal{A} \otimes \mathcal{B})\). The \( \pi \cdot \lambda \)-Theorem 2.4 tell us that this measure is uniquely determined by the condition \((\mu \otimes K)(A \times B) = \int_A K(x)(B) \ d\mu(x)\) for \( A \in \mathcal{A}, B \in \mathcal{B}\).

Interpret in a probabilistic setting \( K(x)(B) \) as the probability that an input \( x \in X \) yields an output in \( B \in \mathcal{B}\), and assume that \( \mu \) gives the initial probability with which the system starts, then \( \mu \otimes K \) gives the probability of all pairings, i.e., \((\mu \otimes K)(Q)\) is the probability that a pair \((x, y)\) consisting of an input value \( x \in X \) and an output value \( y \in Y \) will be a member of \( Q \in \mathcal{A} \otimes \mathcal{B}\).

This may be further extended, replacing the measure on \( K \)'s domain by a transition kernel as well.

**Example 2.170** Consider the scenario of Example 2.169 again, but take a third measurable space \((Z, \mathcal{C})\) with a transition kernel \( L : (Z, \mathcal{C}) \rightsquigarrow (X, \mathcal{A}) \) into account; assume furthermore that \( x \mapsto K(x)(Y) \) is integrable for each \( L(z) \). Then \( L(z) \otimes K \) defines a finite measure on \((X \times Y, \mathcal{A} \otimes \mathcal{B})\) for each \( z \in Z \) according to Example 2.169. We claim that this defines a transition kernel \((Z, \mathcal{C}) \rightsquigarrow (X \times Y, \mathcal{A} \otimes \mathcal{B})\). For this to be true, we have to show that \( z \mapsto \int_X K(x)(Q_x) \ dL(z)(x) \) is measurable for each \( Q \in \mathcal{A} \otimes \mathcal{B} \).

Consider

\[
Q := \{ Q \in \mathcal{A} \otimes \mathcal{B} \mid \text{the assertion is true for } Q \}.
\]

Then \( Q \) is closed under complementation. It is also closed under countable disjoint unions by Corollary 2.151. If \( Q = A \times B \) is a measurable rectangle, we have \( \int_X K(x)(Q_x) \ dL(z)(x) = \int_A K(x)(B) \ dL(z)(x) \). Then Exercise 15 shows that this is a measurable function \( Z \rightarrow \mathbb{R} \). Thus \( Q \) contains all measurable rectangles, so \( Q = \mathcal{A} \otimes \mathcal{B} \) by the \( \pi \cdot \lambda \)-Theorem 2.4. This establishes measurability of \( z \mapsto \int_X K(x)(Q_x) \ dL(z)(x) \) and shows that it defines a transition kernel. ☑
As a slight modification, the next example shows the composition of transition kernels, usually called \textit{convolution}.

\textbf{Example 2.171} Let \( K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B}) \) and \( L : (Y, \mathcal{B}) \rightsquigarrow (Z, \mathcal{C}) \) be transition kernels, and assume that the map \( y \mapsto L(y)(Z) \) is integrable with respect to measures \( K(x) \) for an arbitrary \( x \in X \). Define for \( x \in X \) and \( C \in \mathcal{C} \)

\[
(L \ast K)(x)(C) := \int_X L(y)(C) \, dK(x)(y).
\]

Then \( L \ast K : (X, \mathcal{A}) \rightsquigarrow (Z, \mathcal{C}) \) is a transition kernel. In fact, \( (L \ast K)(x) \) is for \( x \in X \) fixed a finite measure on \( C \) according to Corollary 2.151. From Exercise 15 we infer that \( x \mapsto \int_X L(y)(C) \, dK(x)(y) \) is a measurable function, since \( y \mapsto L(y)(C) \) is measurable for all \( C \in \mathcal{C} \).

Because transition kernels are the Kleisli morphisms for the endofunctor \( \mathbb{M} \) on the category of measurable spaces \cite[Example 1.99]{Dob14}, it is not difficult to see that this defines Kleisli composition; in particular it follows that this composition is associative. \( \diamond \)

\textbf{Example 2.172} Let \( f \in \mathcal{F}_+(X, \mathcal{A}) \), then we know that “the area under the graph”, viz.,

\[
C_{\leq}(f) := \{ \langle x, r \rangle \mid x \in X, 0 \leq r \leq f(x) \}
\]

is a member of \( \mathcal{A} \otimes \mathcal{B}(\mathbb{R}) \). This was shown in Corollary 2.171. Then Corollary 2.168 tells us that

\[
(\mu \otimes \lambda)(C_{\leq}(f)) = \int_X \lambda((C_{\leq}(f))_x) \, d\mu(x),
\]

where \( \lambda \) is Lebesgue measure on \( \mathcal{B}(\mathbb{R}) \). Because

\[
\lambda((C_{\leq}(f))_x) = \lambda(\{ r \mid 0 \leq r \leq f(x) \}) = f(x),
\]

we obtain

\[
(\mu \otimes \lambda)(C_{\leq}(f)) = \int_X f \, d\mu.
\]

On the other hand,

\[
(\mu \otimes \lambda)(C_{\leq}(f)) = \int_{\mathbb{R}_+} \mu((C_{\leq}(f))_r) \, d\lambda(r),
\]

and this gives the integration formula

\[
\int_X f \, d\mu = \int_0^\infty \mu(\{ x \in X \mid f(x) \geq r \}) \, dr. \tag{7}
\]

In this way, the integral of a non-negative function may be interpreted as measuring the area under its graph. \( \diamond \)

\subsection*{2.9.1 Fubini’s Theorem}

In order to discuss integration with respect to a product measure, we introduce the cuts of a function \( f : X \times Y \to \mathbb{R} \), defining \( f_x := \lambda_y f(x, y) \) and \( f^y := \lambda_x f(x, y) \). Thus we have \( f(x, y) = f_x(y) = f^y(x) \), the first equality resembling currying.
For the discussion to follow, we will admit also the values \( \{-\infty, +\infty\} \) as function values. So define \( \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\} \), and let \( B \subseteq \overline{\mathbb{R}} \) be a Borel set iff \( B \cap \mathbb{R} \in \mathcal{B}(\mathbb{R}) \). Measurability of functions extends accordingly: if \( f : X \to \overline{\mathbb{R}} \) is measurable, then in particular \( \{x \in X \mid f(x) \in \mathbb{R}\} \in \mathcal{A} \), and the set of values on which \( f \) takes the values \( +\infty \) or \( -\infty \) is a member of \( \mathcal{A} \). Denote by \( \overline{\mathcal{F}}(X, \mathcal{A}) \) the set of measurable functions with values in \( \overline{\mathbb{R}} \), and by \( \overline{\mathcal{F}}_{+}(X, \mathcal{A}) \) those which take non-negative values. The integral \( \int_X f \, d\mu \) and integrability is defined in the same way as above for \( f \in \overline{\mathcal{F}}_{+}(X, \mathcal{A}) \). Then it is clear that \( f \in \overline{\mathcal{F}}_{+}(X, \mathcal{A}) \) is integrable iff \( f \cdot \chi_{\{x \in X \mid f(x) \in \mathbb{R}\}} \) is integrable and \( \mu(\{x \in X \mid f(x) = \infty\}) = 0 \).

With this in mind, we tackle the integration of a measurable function \( f : X \times Y \to \overline{\mathbb{R}} \) for the finite measure spaces \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\).

**Proposition 2.173** Let \( f \in \overline{\mathcal{F}}_{+}(X \times Y, \mathcal{A} \otimes \mathcal{B}) \), then

1. \( \lambda x. \int_Y f_x \, d\nu \) and \( \lambda y. \int_X f_y \, d\mu \) are measurable functions \( X \to \overline{\mathbb{R}} \) resp. \( Y \to \overline{\mathbb{R}} \).

2. we have
\[
\int_{X \times Y} f \, d\mu \otimes \nu = \int_X \left( \int_Y f_x \, d\nu \right) \, d\mu(x) = \int_Y \left( \int_X f_y \, d\mu \right) \, d\nu(y)
\]

**Proof**

1. Let \( f = \sum_{i=1}^n a_i \cdot \chi_{Q_i} \) be a step function with \( a_i \geq 0 \) and \( Q_i \in \mathcal{A} \otimes \mathcal{B} \) for \( i = 1, \ldots, n \). Then
\[
\int_Y f_x \, d\nu = \sum_{i=1}^n a_i \cdot \nu(Q_{i,x}).
\]
This is a measurable function \( X \to \mathbb{R} \) by Lemma 2.166. We obtain
\[
\int_{X \times Y} f \, d\mu \otimes \nu = \sum_{i=1}^n a_i \cdot (\mu \otimes \nu)(Q_i)
\]
\[
= \sum_{i=1}^n a_i \cdot \int_X \nu(Q_{i,x}) \, d\mu(x)
\]
\[
= \int_X \sum_{i=1}^n a_i \cdot \nu(Q_{i,x}) \, d\mu(x)
\]
\[
= \int_X \left( \int_Y f_x \, d\nu \right) \, d\mu(x)
\]
Interchanging the rôles of \( \mu \) and \( \nu \), we obtain the representation of \( \lambda y. \int_{X \times Y} f \, d\mu \otimes \nu \) in terms of \( \int_X f^y \, d\mu \) and \( \nu \). Thus the assertion is true for step functions.

2. In the general case we know that we can find an increasing sequence \((f_n)_{n \in \mathbb{N}}\) of step functions with \( f = \sup_{n \in \mathbb{N}} f_n \). Given \( x \in X \), we infer that \( f_x = \sup_{n \in \mathbb{N}} f_{x,n} \), so that
\[
\int_Y f_x \, d\nu = \sup_{n \in \mathbb{N}} \int_X f_{x,n} \, d\nu
\]
by Levi’s Theorem 2.146. This implies measurability. Applying Levi’s Theorem again to the
results from part 1., we have

\[ \int_{X \times Y} f \, d\mu \otimes \nu = \sup_{n \in \mathbb{N}} \int_{X \times Y} f_n \, d\mu \otimes \nu \]

\[ = \sup_{n \in \mathbb{N}} \int_X (\int_Y f_{n,x} \, d\nu) \, d\mu(x) \]

\[ = \int_X (\sup_{n \in \mathbb{N}} \int_Y f_{n,x} \, d\nu) \, d\mu(x) \]

\[ = \int_{X \times Y} (\int_Y f \, d\nu) \, d\mu(x) \]

Again, interchanging rôles yields the symmetric equality. \( \dashv \)

This yields as an immediate consequence that the cuts of a product integrable function are almost everywhere integrable, to be specific:

**Corollary 2.174** Let \( f : X \times Y \to \mathbb{R} \) be \( \mu \otimes \nu \)-integrable, and put

\[ A := \{ x \in X \mid f_x \text{ is not } \nu\text{-integrable} \}, \]

\[ B := \{ y \in Y \mid f_y \text{ is not } \mu\text{-integrable} \}. \]

Then \( A \in \mathcal{A}, B \in \mathcal{B}, \) and \( \mu(A) = \nu(B) = 0. \)

**Proof** Because \( A = \{ x \in X \mid \int_Y |f_x| \, d\nu = \infty \}, \) we see that \( A \in \mathcal{A}. \) By the additivity of the integral, we have

\[ \int_{X \times Y} |f| \, d\mu \otimes \nu = \int_{X \setminus A} (\int_Y |f_x| \, d\nu) \, d\mu(x) + \int_A (\int_Y |f_x| \, d\nu) \, d\mu(x) < \infty, \]

hence \( \mu(A) = 0. \) \( B \) is treated in the same way. \( \dashv \)

It is helpful to extend our integral in a minor way. Assume that \( \int_X |f| \, d\mu < \infty \) for \( f : X \to \mathbb{R} \) measurable, and that \( \mu(A) = 0 \) with \( A := \{ x \in X \mid |f(x)| = \infty \}. \) Change \( f \) on \( A \) to a finite value, obtaining a measurable function \( f_* : X \to \mathbb{R}, \) and define

\[ \int_X f \, d\mu := \int_X f_* \, d\mu. \]

Thus \( f \mapsto \int_X f \, d\mu \) does not notice this change on a set of measure zero. In this way, we assume always that an integrable function takes finite values, even if we have to convince it to do so on a set of measure zero.

With this in mind, we obtain

**Corollary 2.175** Let \( f : X \times Y \to \mathbb{R} \) be integrable, then \( \lambda x. \int_Y f_x \, d\nu \text{ and } \lambda y. \int_X f_y \, d\nu \) are integrable with respect to \( \mu \text{ resp. } \nu, \) and

\[ \int_{X \times Y} f \, d\mu \otimes \nu = \int_X (\int_Y f_x \, d\nu) \, d\mu(x) = \int_Y (\int_X f_y \, d\mu) \, d\nu(y). \]
Proof After the modification on a set of $\mu$-measure zero, we know that
$$\left| \int_X f_x \, d\nu \right| \leq \int_Y |f_x| \, d\nu < \infty$$
for all $x \in X$, so that $\lambda_x \int_X f_x \, d\nu$ is integrable with respect to $\mu$; similarly, $\lambda_y \int_Y f^y \, d\nu$ is integrable with respect to $\nu$ for all $y \in Y$. We obtain from Proposition 2.173 and the linearity of the integral
$$\int_{X \times Y} f \, d\mu \otimes \nu = \int_{X \times Y} f^+ \, d\mu \otimes \nu - \int_{X \times Y} f^- \, d\mu \otimes \nu$$
$$= \int_X \left( \int_Y f^+_x \, d\nu \right) \, d\mu(x) - \int_X \left( \int_Y f^-_x \, d\nu \right) \, d\mu(x)$$
$$= \int_X \left( \int_Y f^+_x \, d\nu - \int_Y f^-_x \, d\nu \right) \, d\mu(x)$$
The second equation is treated in exactly the same way. $\dashv$

Now we now how to treat a function which is integrable, but we do not yet have a criterion for integrability. The elegance of Fubini’s Theorem shines through the observation that the existence of the iterated integrals yields integrability for the product integral. To be specific:

**Theorem 2.176** Let $f : X \times Y \to \mathbb{R}$ be measurable. Then these statements are equivalent

1. $\int_{X \times Y} |f| \, d\mu \otimes \nu < \infty$.
2. $\int_X \left( \int_Y |f_x| \, d\nu \right) \, d\mu(x) < \infty$.
3. $\int_Y \left( \int_X |f^y| \, d\mu \right) \, d\nu(y) < \infty$.

Under one of these conditions, $f$ is $\mu \otimes \nu$-integrable, and

$$\int_{X \times Y} f \, d\mu \otimes \nu = \int_X \left( \int_Y f_x \, d\nu \right) \, d\mu(x) = \int_Y \left( \int_X f^y \, d\mu \right) \, d\nu(y). \quad (8)$$

Proof We discuss only $[1] \Rightarrow [2]$ the other implications are proved similarly. From Proposition 2.173 it is inferred that $|f|$ is integrable, so $[2]$ holds by Corollary 2.175 from which we also obtain representation $(8)$. $\dashv$

### 2.9.2 Infinite Products and Projective Limits

Corollary 2.168 extends to a finite number of $\sigma$-finite measure spaces in a natural way. Let $(X_i, A_i, \mu_i)$ be $\sigma$-finite measure spaces for $1 \leq i \leq n$, the uniquely determined product measure on $A_1 \otimes \ldots \otimes A_n$ is denoted by $\mu_1 \otimes \ldots \otimes \mu_n$, and we infer from Corollary 2.168 that we may write

$$(\mu_1 \otimes \ldots \otimes \mu_n)(Q) = \int_{X_2 \times \ldots \times X_n} \mu_1(Q_{x_2, \ldots, x_n}) \, d(\mu_2 \otimes \ldots \otimes \mu_n)(x_2, \ldots, x_n),$$

$$= \int_{X_1 \times \ldots \times X_{n-1}} \mu_n(Q_{x_1, \ldots, x_{n-1}}) \, d(\mu_1 \otimes \ldots \otimes \mu_{n-1})(x_1, \ldots, x_{n-1})$$
whenever $Q \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$.

We will have a closer look now at infinite products, where we restrict ourselves to probability measures, and here we consider the countable case first. So let $(X_n, \mathcal{A}_n, \bar{\omega}_n)$ be a measure space with a probability measure $\bar{\omega}_n$ on $\mathcal{A}_n$ for $n \in \mathbb{N}$.

Let us fix some notations first. Put

$$X^{(n)} := \prod_{k \geq n} X_k,$$
$$\mathcal{A}^{(n)} := \{ A \times X^{(n+\ell)} \mid A \in \mathcal{A}_n \otimes \ldots \otimes \mathcal{A}_{n+\ell-1} \text{ for some } \ell \in \mathbb{N} \}.$$

The elements of $\mathcal{A}^{(n)}$ are the cylinder sets for $X^{(n)}$. Thus $X^{(1)} = \prod_{n \in \mathbb{N}} X_n$, and $\otimes_{n \in \mathbb{N}} \mathcal{A}_n = \sigma(\mathcal{A}^{(1)})$. Given $A \in \mathcal{A}^{(n)}$, we can write $A$ as $A = C \times X^{(n+\ell)}$ with $C \in \mathcal{A}_n \otimes \ldots \otimes \mathcal{A}_{n+\ell-1}$. So if we set

$$\bar{\omega}^{(n)}(A) := \bar{\omega}_n \otimes \ldots \otimes \bar{\omega}_{n+\ell-1}(C),$$

then $\bar{\omega}^{(n)}$ is well defined on $\mathcal{A}^{(n)}$, and it is readily verified that it is monotone and additive with $\bar{\omega}^{(n)}(\emptyset) = 0$ and $\bar{\omega}^{(n)}(X^{(n)}) = 1$. Moreover, we infer from Theorem 2.167 that

$$\bar{\omega}^{(n)}(C) = \int_{X_{n+1} \times \ldots \times X_{n+m}} \bar{\omega}_n(C_{x_{n+1},\ldots,x_{n+m}}) \, d(\bar{\omega}_{n+1} \otimes \ldots \otimes \bar{\omega}_{n+m})(x_{n+1} \ldots x_{n+m})$$

for all $C \in \mathcal{A}^{(n)}$.

The goal is to show that there exists a unique probability measure $\bar{\omega}$ on $\otimes_{n \in \mathbb{N}} (X_n, \mathcal{A}_n)$ such that $\bar{\omega}(A \times X^{(n+1)}) = (\bar{\omega}_1 \otimes \ldots \otimes \bar{\omega}_n)(A)$ whenever $A \in \mathcal{A}_n \otimes \mathcal{A}^{(n+1)}$. If we can show that $\bar{\omega}^{(1)}$ is $\sigma$-additive on $\mathcal{A}^{(1)}$, then we can extend $\bar{\omega}^{(1)}$ to the desired $\sigma$-algebra by [Dob13, Theorem 2.112]. For this it is sufficient to show that $\inf_{n \in \mathbb{N}} \bar{\omega}^{(1)}(A_n) > \epsilon > 0$ implies $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$ for any decreasing sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{A}^{(1)}$.

The basic idea is to construct a sequence $(x_n)_{n \in \mathbb{N}} \in \bigcap_{n \in \mathbb{N}} A_n$. We do this step by step. First we determine an element $x_1 \in X_1$ such that we can expand the — admittedly very short — partial sequence $x_1$ to a sequence which is contained in all $A_n$; this means that we have to have $A_n^{x_1} \neq \emptyset$ for all $n \in \mathbb{N}$, because $A_n^{x_1}$ contains all possible continuations of $x_1$ into $A_n$. We conclude that these sets are non-empty, because their measure is strictly positive. If we have such an $x_1$, we start working on the second element of the sequence, so we have a look at some $x_2 \in X_2$ such that we can expand $x_1, x_2$ to a sequence which is contained in all $A_n$ so we have to have $A_n^{x_1,x_2} \neq \emptyset$ for all $n \in \mathbb{N}$. Again, we look for $x_2$ so that the measure of $A_n^{x_1,x_2}$ is strictly positive for each $n$. Continuing in this fashion, we obtain the desired sequence, which then has to be an element of $\bigcap_{n \in \mathbb{N}} A_n$ by construction.

This is the plan. Let us have a look at how to find $x_1$. Put $E^{(n)}_1 := \{ x_1 \in X_1 \mid \bar{\omega}^{(2)}(A_n^{x_1}) > \epsilon/2 \}$. Because

$$\bar{\omega}^{(1)}(A_n) = \int_{X_1} \bar{\omega}^{(2)}(A_n^{x_1}) \, d\bar{\omega}_1(x_1)$$
we have
\[ 0 < \epsilon < \varpi^{(1)}(A_n) = \int_{E_1^{(n)}} \varpi^{(2)}(A_n^{x_1}) \, d\varpi_1(x_1) + \int_{X_1 \setminus E_1^{(n)}} \varpi^{(2)}(A_n^{x_1}) \, d\varpi_1(x_1) \]
\[ \leq \varpi_1(E_1^{(n)}) + \epsilon/2 \cdot \varpi^{(1)}(X_1 \setminus E_1^{(n)}) \]
\[ \leq \varpi_1(E_1^{(n)}) + \epsilon/2. \]

Thus \( \varpi_1(E_1^{(n)}) \geq \epsilon/2 \) for all \( n \in \mathbb{N} \). Since \( A_1 \supseteq A_2 \supseteq \ldots \), we have also \( E_1^{(1)} \supseteq E_1^{(2)} \supseteq \ldots \), so let \( E_1 := \bigcap_{n \in \mathbb{N}} E_1^{(n)} \), then \( E_1 \in \mathcal{A}_1 \) with \( \varpi_1(E_1) \geq \epsilon/2 > 0 \). In particular, \( E_1 \neq \emptyset \). Pick and fix \( x_1 \in E_1 \). Then \( A_n^{x_1} \in \mathcal{A}_2 \), and \( \varpi^{(2)}(A_n^{x_1}) > \epsilon/2 \) for all \( n \in \mathbb{N} \).

Let us have a look at how to find the second element; this is but a small variation of the idea just presented. Put \( E_2^{(n)} := \{ x_2 \in X_2 \mid \varpi^{(3)}(A_n^{x_1,x_2}) > \epsilon/4 \} \) for \( n \in \mathbb{N} \). Because
\[ \varpi^{(2)}(A_n^{x_1}) = \int_{X_2} \varpi^{(3)}(A_n^{x_1,x_2}) \, d\varpi_2(x_2), \]
we obtain similarly \( \varpi_2(E_2^{(n)}) \geq \epsilon/4 \) for all \( n \in \mathbb{N} \). Again, we have a decreasing sequence, and putting \( E_2 := \bigcap_{n \in \mathbb{N}} E_2^{(n)} \), we have \( \varpi_2(E_2) \geq \epsilon/4 \), so that \( E_2 \neq \emptyset \). Pick \( x_2 \in E_2 \), then \( A_n^{x_1,x_2} \in \mathcal{A}_3 \) and \( \varpi^{(3)}(A_n^{x_1,x_2}) > \epsilon/4 \) for all \( n \in \mathbb{N} \). In this manner we determine inductively for each \( k \in \mathbb{N} \) the finite sequence \( \langle x_1, \ldots, x_k \rangle \in X_1 \times \ldots \times X_k \) such that \( \varpi^{(k+1)}(A_n^{x_1,\ldots,x_k}) > \epsilon/2^k \) for all \( n \in \mathbb{N} \). Consider now the sequence \( (x_n)_{n \in \mathbb{N}} \). From the construction it is clear that \( \langle x_1, x_2, \ldots, x_k, \ldots \rangle \in \bigcap_{n \in \mathbb{N}} A_n \). This shows that \( \bigcap_{n \in \mathbb{N}} A_n \neq \emptyset \), and it implies that \( \varpi^{(1)} \) is a premeasure on the algebra \( \mathcal{A}^{(1)} \).

Hence we have established \[ \text{Theorem 2.112} \]

**Theorem 2.177** Let \( (X_n, \mathcal{A}_n, \varpi_n) \) be probability spaces for all \( n \in \mathbb{N} \). Then there exists a unique probability measure \( \varpi \) on \( \bigotimes_{n \in \mathbb{N}} (X_n, \mathcal{A}_n) \) such that
\[ \varpi(A \times \prod_{k>n} X_k) = (\varpi_1 \otimes \ldots \otimes \varpi_n)(A) \]
for all \( A \in \bigotimes_{i=1}^{n} \mathcal{A}_i \).

Define the projection \( \pi_n^\infty : (x_n)_{n \in \mathbb{N}} \mapsto \langle x_1, \ldots, x_n \rangle \) from \( \prod_{n \in \mathbb{N}} X_n \) to \( \prod_{i=1}^{n} X_i \). In terms of image measures, the theorem states that there exists a unique probability measure \( \varpi \) on the infinite product such that \( \mathbb{S}(\pi_n^\infty)(\varpi) = \varpi_1 \otimes \ldots \otimes \varpi_n \).

Now let us have a look at the general case, in which the index set is not necessarily countable. Let \( (X_i, \mathcal{A}_i, \mu_i) \) be a family of probability spaces for \( i \in I \), put \( X := \prod_{i \in I} X_i \) and \( \mathcal{A} := \bigotimes_{i \in I} \mathcal{A}_i \). Given \( J \subseteq I \), define \( \pi_J : (x_i)_{i \in I} \mapsto (x_i)_{i \in J} \) as the projection \( X \to \prod_{i \in J} X_i \). Put \( \mathcal{A}_J := \pi_J^{-1} \left[ \bigotimes_{j \in J} \mathcal{A}_j \right] \).

Although the index set \( I \) may be large, the measurable sets in \( \mathcal{A} \) are always determined by a countable subset of the index set:

**Lemma 2.178** Given \( A \in \mathcal{A} \), there exists a countable subset \( J \subseteq I \) such that \( \chi_A(x) = \chi_A(x') \), whenever \( \pi_J(x) = \pi_J(x') \).
Proof Let \( G \) be the set of all \( A \in \mathcal{A} \) for which the assertion is true. Then \( G \) is a \( \sigma \)-algebra which contains \( \pi_{i(i)}^{-1}[A_i] \) for every \( i \in I \), hence \( G = \mathcal{A} \). \( \dashv \)

This yields as an immediate consequence

**Corollary 2.179** \( \mathcal{A} = \bigcup \{ \mathcal{A}_J \mid J \subseteq I \text{ is countable} \} \).

**Proof** It is enough to show that the set on the right hand side is a \( \sigma \)-algebra. This follows easily from Lemma 2.178. \( \dashv \)

We obtain from this observation, and from and the previous result for the countable case that arbitrary products exist.

**Theorem 2.180** Let \((X_i, \mathcal{A}_i, \mu_i)\) be a family of probability spaces for \( i \in I \). Then there exists a unique probability measure \( \mu \) on \( \bigotimes_{i \in I} (X_i, \mathcal{A}_i) \) such that

\[
\mu(\pi_{I(i_1,\ldots,i_k)}^{-1}[C]) = (\mu_{i_1} \otimes \ldots \otimes \mu_{i_k})(C)
\]  

(9)

for all \( C \in \bigotimes_{j=1}^k \mathcal{A}_{i_j} \) and all \( i_1, \ldots, i_k \in I \).

**Proof** Let \( A \in \mathcal{A} \), then there exists a countable subset \( J \subseteq I \) such that \( A \in \mathcal{A}_J \). Let \( \mu_J \) be the corresponding product measure on \( \mathcal{A}_J \). Define \( \mu(A) := \mu_J(A) \), then it is easy to see that \( \mu \) is a well defined measure on \( \mathcal{A} \), since the extension to countable products is unique. From the construction it follows also that the desired property \( \mathbb{P} \) is satisfied. \( \dashv \)

For the interpretation of some logics the projective limit of a projective family of stochastic relations is helpful; this is the natural extension of a product. It will be discussed now. Denote by \( X^\infty := \prod_{k \in \mathbb{N}} X \) the infinite product of \( X \) with itself; recall that \( \mathbb{P} \) is the probability functor, assigning to each measurable space its probability measures.

**Definition 2.181** Let \( X \) be a Polish space, and \((\mu_n)_{n \in \mathbb{N}}\) a sequence of probability measures \( \mu_n \in \mathbb{P}(X^n) \). This sequence is called a projective system iff \( \mu_n(A) = \mu_{n+1}(A \times X) \) for all \( n \in \mathbb{N} \) and all Borel sets \( A \in \mathcal{B}(X^n) \). A probability measure \( \mu_\infty \in \mathbb{P}(X^\infty) \) is called the projective limit of the projective system \((\mu_n)_{n \in \mathbb{N}} \) iff

\[
\mu_n(A) = \mu_\infty(A \times \prod_{j>n} X)
\]

for all \( n \in \mathbb{N} \) and \( A \in \mathcal{B}(X^n) \).

Thus a sequence of measures is a projective system iff each measure is the projection of the next one; its projective limit is characterized through the property that its values on cylinder sets coincides with the value of a member of the sequence, after taking projections. A special case is given by product measures. Assume that \( \mu_n = \nu_1 \otimes \ldots \otimes \nu_n \), where \((\nu_n)_{n \in \mathbb{N}}\) is a sequence of probability measures on \( X \). Then the condition on projectivity is satisfied, and the projective limit is the infinite product constructed above. It should be noted, however, that the projectivity condition does not express \( \mu_{n+1}(A \times B) \) in terms of \( \mu_n(A) \) for an arbitrary measurable set \( B \subseteq X \), as the product measure does.

It is not immediately obvious that a projective limit exists in general, given the rather weak dependency of the measures. In general, it will not, and this is why. The basic idea for the construction of the infinite product has been to define the limit on the cylinder sets and then to extend this premeasure — but it has to be established that it is indeed a premeasure, and
this is difficult in general. The crucial property in the proof above has been that $\mu_{n_k}(A_k) \to 0$ whenever $(A_n)_{n \in \mathbb{N}}$ is a sequence of cylinder sets $A_k$ (with at most $n_k$ components that do not equal $X$) that decreases to $\emptyset$. This property has been established above for the case of the infinite product through Fubini’s Theorem, but this is not available in the general setting considered here. We will see, however, that a topological argument will be helpful. This is why we did postulate the base space $X$ to be Polish.

We start with an even stronger topological condition, viz., that the space under consideration is compact and metric. The central statement is

**Proposition 2.182** Let $X$ be a compact metric space. Then the projective system $(\mu_n)_{n \in \mathbb{N}}$ has a unique projective limit $\mu_\infty$.

**Proof** 1. Let $A = A'_k \times \prod_{j > k} X$ be a cylinder set with $A'_k \in B(X^k)$. Define $\mu^*(A) := \mu_k(A'_k)$. Then $\mu^*$ is well defined on the cylinder sets, since the sequence forms a projective system. In order to show that $\mu^*$ is a premeasure on the cylinder sets, we take a decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of cylinder sets with $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ and show that $\inf_{n \in \mathbb{N}} \mu^*(A_n) = 0$. In fact, suppose that $(A_n)_{n \in \mathbb{N}}$ is decreasing with $\mu^*(A_n) \geq \delta$ for all $n \in \mathbb{N}$, then we show that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

We can write $A_n = A'_n \times \prod_{j > k_n} X$ for some $A'_n \in B(X^{k_n})$. From Lemma 2.133 we get for each $n$ a closed, hence compact set $K'_n \subseteq A'_n$ such that $\mu_{k_n}(A'_n \setminus K'_n) < \delta/2^n$. Because $X^\infty$ is compact by Tichonov’s Theorem,

$$K''_n := K'_n \times \prod_{j > k_n} X$$

is a compact set, and $K := \bigcap_{n \in \mathbb{N}} K''_n \subseteq A_n$ is compact as well, with

$$\mu^*(A_n \setminus K) \leq \mu^*(\bigcup_{j=1}^n A'_j \setminus K'_j) \leq \sum_{j=1}^n \mu^*(A'_j \setminus K'_j) = \sum_{j=1}^n \mu_{k_j}(A'_j \setminus K'_j) \leq \sum_{j=1}^\infty \delta/2^j = \delta.$$

Thus $(K_n)_{n \in \mathbb{N}}$ is a decreasing sequence of nonempty compact sets; consequently,

$$\emptyset \neq \bigcap_{n \in \mathbb{N}} K_n \subseteq \bigcap_{n \in \mathbb{N}} A_n.$$

2. Since the cylinder sets generate the Borel sets of $X^\infty$, and since $\mu^*$ is a premeasure, we know that there exists a unique extension $\mu_\infty \in \mathcal{P}(X^\infty)$ to it. Clearly, if $A \subseteq X^n$ is a Borel set, then

$$\mu_\infty(A \times \prod_{j > n} X) = \mu^*(A \times \prod_{j > n} X) = \mu_n(A),$$

so we have constructed a projective limit.

3. Suppose that $\mu'$ is another probability measure in $\mathcal{P}(X^\infty)$ that has the desired property. Consider

$$\mathcal{D} := \{D \in \mathcal{B}(X^\infty) \mid \mu_\infty(D) = \mu'(D)\}.$$

It is clear the $\mathcal{D}$ contains all cylinder sets, that it is closed under complements, and under countable disjoint unions. By the $\pi$-$\lambda$-Theorem 2.4 $\mathcal{D}$ contains the $\sigma$-algebra generated by
the cylinder sets, hence all Borel subset of $X^\infty$. This establishes uniqueness of the extension. \hfill \dashv

The proof makes critical use of the observation that we can approximate the measure of a Borel set arbitrarily well by compact sets from within; see Lemma 2.183. It is also important that compact sets have the finite intersection property: if each finite intersection of a family of compact sets is nonempty, the intersection of the entire family cannot be empty. Consequently the proof given above works in general Hausdorff spaces, provided the measures under consideration have the approximation property mentioned above.

We free ourselves from the restrictive assumption of having a compact metric space using the Alexandrov embedding of a Polish space into a compact metric space.

**Proposition 2.183** Let $X$ be a Polish space, $(\mu_n)_{n \in \mathbb{N}}$ be a projective system on $X$. Then there exists a unique projective limit $\mu_\infty \in \mathcal{P}(X^\infty)$ for $(\mu_n)_{n \in \mathbb{N}}$.

**Proof** $X$ is a dense measurable subset of a compact metric space $\tilde{X}$ by Alexandrov's Theorem 2.76. Defining $\bar{\mu}_n(B) := \mu_n(B \cap X^n)$ for the Borel set $B \subseteq \tilde{X}^n$ yields a projective system $(\bar{\mu}_n)_{n \in \mathbb{N}}$ on $\tilde{X}$ with a projective limit $\bar{\mu}_\infty$ by Proposition 2.182. Since by construction $\bar{\mu}_\infty(X^\infty) = 1$, restrict $\bar{\mu}_\infty$ to the Borel sets of $X^\infty$, then the assertion follows. \hfill \dashv

An interesting application of this construction arises through stochastic relations that form a projective system. We will show now that there exists a kernel which may be perceived as a (pointwise) projective limit.

**Corollary 2.184** Let $X$ and $Y$ be Polish spaces, and assume that $J^{(n)}$ is a stochastic relation on $X$ and $Y^n$ for each $n \in \mathbb{N}$ such that the sequence $(J^{(n)}(x))_{n \in \mathbb{N}}$ forms a projective system on $Y$ for each $x \in X$, in particular $J^{(n)}(x)(Y^n) = 1$ for all $x \in X$. Then there exists a unique sub-Markov kernel $J_\infty$ on $X$ and $Y^\infty$ such that $J_\infty(x)$ is the projective limit of $(J^{(n)}(x))_{n \in \mathbb{N}}$ for each $x \in X$.

**Proof** 0. Let for $x$ fixed $J_\infty(x)$ be the projective limit of the projective system $(J^{(n)}(x))_{n \in \mathbb{N}}$. By the definition of a stochastic relation we need to show that the map $x \mapsto J_\infty(x)(B)$ is measurable for every $B \in \mathcal{B}(Y^\infty)$.

1. In fact, consider

$$\mathcal{D} := \{B \in \mathcal{B}(Y^\infty) \mid x \mapsto J_\infty(x)(B) \text{ is measurable}\}$$

then the general properties of measurable functions imply that $\mathcal{D}$ is a $\sigma$-algebra on $Y^\infty$. Take a cylinder set $B = B_0 \times \prod_{j > k} Y$ with $B_0 \in \mathcal{B}(Y^k)$ for some $k \in \mathbb{N}$, then, by the properties of the projective limit, we have $J_\infty(x)(B) = J^{(k)}(x)(B_0)$. But $x \mapsto J^{(k)}(x)(B_0)$ constitutes a measurable function on $X$. Consequently, $B \in \mathcal{D}$, and so $\mathcal{D}$ contains the cylinder sets which generate $\mathcal{B}(Y^\infty)$. Thus measurability is established for each Borel set $B \subseteq Y^\infty$, arguing with the $\pi$-$\lambda$-Theorem 2.18 as in the last part of the proof for Proposition 2.182. \hfill \dashv

### 2.9.3 Case Study: Continuous Time Stochastic Logic

We illustrate this construction through the interpretation of a path logic over infinite paths; the logic is called CSL — continuous time stochastic logic. Since the discussion of this appli-
We introduce CSL now and describe it informally first.

Fix $P$ as a countable set of atomic propositions. We define recursively state formulas and path formulas for CSL:

**State formulas** are defined through the syntax

$$\varphi ::= T \mid a \mid \neg \varphi \mid \varphi \land \varphi' \mid S_{\text{cop}}(\varphi) \mid P_{\text{cop}}(\psi)$$

Here $a \in P$ is an atomic proposition, $\varphi$ is a path formula, $\exists$ is one of the relational operators $<, \leq, \geq, >$, and $p \in [0, 1]$ is a rational number.

**Path formulas** are defined through

$$\psi ::= \mathcal{X}^I \varphi \mid \mathcal{U}^I \varphi'$$

with $\varphi, \varphi'$ as state formulas, $I \subseteq \mathbb{R}_+$ a closed interval of the real numbers with rational bounds (including $I = \mathbb{R}_+$).

We denote the set of all state formulas by $\mathcal{L}_{AP}$.

The operator $S_{\text{cop}}(\varphi)$ gives the steady-state probability for $\varphi$ to hold with the boundary condition $\exists p$; the formula $P$ replaces quantification: the path-quantifier formula $P_{\text{cop}}(\psi)$ holds in a state $s$ iff the probability of all paths starting in $s$ and satisfying $\psi$ is specified by $\exists p$. Thus $\psi$ holds on almost all paths starting from $s$ iff $s$ satisfies $P_{\geq 1}(\psi)$, a path being an alternating infinite sequence $\sigma = \langle s_0, t_0, s_1, t_1, \ldots \rangle$ of states $s_i$ and of times $t_i$. Note that the time is being made explicit here. The next-operator $\mathcal{X}^I \varphi$ is assumed to hold on path $\sigma$ iff $s_1$ satisfies $\varphi$, and $t_0 \in I$ holds. Finally, the until-operator $\mathcal{U}^I \varphi_1 \varphi_2$ holds on path $\sigma$ iff we can find a point in time $t \in I$ such that the state $\sigma @ t$ which $\sigma$ occupies at time $t$ satisfies $\varphi_2$, and for all times $t'$ before that, $\sigma @ t'$ satisfies $\varphi_1$.

A Polish state space $S$ is fixed; this space is used for modelling a transition system takes also time into account. We are not only interested in the next state of a transition but also in the time after which to make a transition. So the basic probabilistic datum will be a stochastic relation $M : S \sim \mathbb{R}_+ \times S$; if we are in state $s$, we will do a transition to a new state $s'$ after we did wait some specified time $t$; $M(s)(D)$ will give the probability that the pair $(t, s') \in D$. We assume that $M(s)(\mathbb{R}_+ \times S) = 1$ holds for all $s \in S$.

A path $\sigma$ is an element of the set $(S \times \mathbb{R}_+)$. Path $\sigma = \langle s_0, t_0, s_1, t_1, \ldots \rangle$ may be written as $s_0 \overset{t_0}{\rightarrow} s_1 \overset{t_1}{\rightarrow} \ldots$ with the interpretation that $t_i$ is the time spent in state $s_i$. Given $i \in \mathbb{N}$, denote $s_i$ by $\sigma[i]$ as the $(i+1)$-st state of $\sigma$, and let $\delta(\sigma, i) := t_i$. Let for $t \in \mathbb{R}_+$ the index $j$ be the smallest index $k$ such that $t < \sum_{i=0}^{k} t_i$, and put $\sigma @ t := \sigma[j]$; if $j$ is defined; set $\sigma @ t := \#$, otherwise (here $\#$ is a new symbol not in $S \cup \mathbb{R}_+$). $S_{\#}$ denotes $S \cup \{\#\}$; this is a Polish space when endowed with the sum $\sigma$-algebra. The definition of $\sigma @ t$ makes sure that for any time $t$ we can find a rational time $t'$ with $\sigma @ t = \sigma @ t'$.

We will deal only with infinite paths. This is no loss of generality because events that happen at a certain time with probability 0 will have the effect that the corresponding infinite paths...
occur only with probability 0. Thus we do not prune the path; this makes the notation somewhat easier to handle.

The Borel sets $\mathcal{B}((S \times \mathbb{R}_+)^\infty)$ are the smallest $\sigma$-algebra which contains all the cylinder sets

$$\{ \prod_{j=1}^n (B_j \times I_j) \times \prod_{j>n} (S \times \mathbb{R}_+) \mid n \in \mathbb{N}, I_1, \ldots, I_n \text{ rational intervals}, B_1, \ldots, B_n \in \mathcal{B}(S) \}.$$ 

Thus a cylinder set is an infinite product that is determined through the finite product of an interval with a Borel set in $S$. It will be helpful to remember that the intersection of two cylinder sets is again a cylinder set.

Given $M : S \leadsto \mathbb{R}_+ \times S$ with Polish $S$, define inductively $M_1 := M$, and

$$M_{n+1}(s_0)(D) := \int_{(\mathbb{R}_+ \times S)^n} M(s_n)(D_{t_0,s_1,\ldots,t_{n-1},s_n}) \, dM_n(s_0)(t_0, s_1, \ldots, t_{n-1}, s_n)$$

for the Borel set $D \subseteq (\mathbb{R}_+ \times S)^{n+1}$. Let us illustrate this for $n = 1$. Given $D \in \mathcal{B}((\mathbb{R}_+ \times S)^2)$ and $s_0 \in S$ as a state to start from, we want to calculate the probability $M_2(s_0)(D)$ that $(t_0, s_1, t_1, s_2) \in D$. This is the probability for the initial path $(s_0, t_0, s_1, t_1, s_2)$ (a pathlet), given the initial state $s_0$. Since $(t_0, s_1)$ is taken care of in the first step, we fix it and calculate $M(s_1)((\{ (t_1, s_2) \mid (t_0, s_1, t_1, s_2) \in D \}) = M(s_1(D_{t_0,s_1}),$ by averaging, using the probability provided by $M(s_0)$, so that we obtain

$$M_2(s_0)(D) = \int_{\mathbb{R}_+ \times S} M(s_1)(D_{t_0,s_1}) \, dM(s_0)(t_0, s_1)$$

Consequently, for the general case we obtain $M_{n+1}(s_0)(D)$ as the probability for $(s_0, t_0, \ldots, s_n, t_n, s_{n+1})$ as the initial piece of an infinite path to be a member of $D$. This probability indicates that we start in $s_0$, remain in this state for $t_0$ units of time, then enter state $s_1$, remain there for $t_1$ time units, etc., and finally leave state $s_n$ after $t_n$ time units, entering $s_{n+1}$, all this happening within $D$.

We claim that $(M_n(s))_{n \in \mathbb{N}}$ is a projective system. We first see from Example 2.170 that $M_n : S \leadsto (\mathbb{R}_+ \times S)^n$ defines a transition kernel for each $n \in \mathbb{N}$. Now let $D = A \times (\mathbb{R}_+ \times S)$ with $A \in \mathcal{B}((\mathbb{R}_+ \times S)^n)$, then $M(s_n)(D_{t_0,s_1,\ldots,t_{n-1},s_n}) = M(s_n)(\mathbb{R}_+ \times S) = 1$ for all $(t_0, s_1, \ldots, t_{n-1}, s_n) \in A$, so that we obtain $M_{n+1}(s)(A \times (\mathbb{R}_+ \times S)) = M_n(s)(A)$. The condition on projectivity is satisfied. Hence there exists a unique projective limit, hence a transition kernel

$$M_\infty : S \leadsto (\mathbb{R}_+ \times S)^\infty$$

with

$$M_n(s)(A) = M_\infty(s)(A \times \prod_{k>n} (\mathbb{R}_+ \times S))$$

for all $s \in S$ and for all $A \in \mathcal{B}((\mathbb{R}_+ \times S)^n)$.

The projective limit displays indeed limiting behavior: suppose $B$ is an infinite measurable cube $\prod_{n \in \mathbb{N}} B_n$ with $B_n \in \mathcal{B}(\mathbb{R}_+ \times S)$ as Borel sets. Because

$$B = \cap_{n \in \mathbb{N}} \left( \prod_{1 \leq j \leq n} B_j \times \prod_{j>n} (\mathbb{R}_+ \times S) \right),$$

$$M_\infty(s)(B) = 0$$

for all $s \in S$. This demonstrates that $M_\infty$ is indeed a transition kernel.
is represented as the intersection of a monotonically decreasing sequence, we have for all \( s \in S \)

\[
M_\infty(s)(B) = \lim_{n \to \infty} M_n(s)\left(\prod_{1 \leq j \leq n} B_j \times \prod_{j > n}(\mathbb{R}_+ \times S)\right)
= \lim_{n \to \infty} M_n(s)\left(\prod_{1 \leq j \leq n} B_j\right).
\]

Hence \( M_\infty(s)(B) \) is the limit of the probabilities \( M_n(s)(B_n) \) at step \( n \).

In this way models based on a Polish state space \( S \) yield stochastic relations

\[
S \Rightarrow (\mathbb{R}_+ \times S)^\infty
\]

through projective limits. Without this limit it would be difficult to model the transition behavior on infinite paths; the assumption that we work in Polish spaces makes sure that these limits in fact do exist. To get started, we need to assume that given a state \( s \in S \), there is always a state to change into after a finite amount of time.

We obtain as a first consequence of the construction for the projective limit a recursive formulation for the transition law \( M \):

\[
X \Rightarrow (\mathbb{R}_+ \times S)^\infty.
\]

Interestingly, it reflects the domain equation \((\mathbb{R}_+ \times S)^\infty = (\mathbb{R}_+ \times S) \times (\mathbb{R}_+ \times X)^\infty\).

**Lemma 2.185** If \( D \in \mathcal{B}((\mathbb{R}_+ \times S)^\infty) \), then

\[
M_\infty(s)(D) = \int_{\mathbb{R}_+ \times S} M_\infty(s')(D_{(t,s')}) M_1(s)(d\langle t, s' \rangle)
\]

holds for all \( s \in S \).

**Proof** Recall that \( D_{(t,s')} = \{ \tau \mid \langle t, s', \tau \rangle \in D \} \). Let

\[
D = (H_1 \times \ldots \times H_{n+1}) \times \prod_{j > n}(\mathbb{R}_+ \times S)
\]

be a cylinder set with \( H_i \in \mathcal{B}(\mathbb{R}_+ \times S), 1 \leq i \leq n + 1 \). The equation in question in this case boils down to

\[
M_{n+1}(s)(H_1 \times \ldots \times H_{n+1}) = \int_{H_i} M_n(s')(H_2 \times \ldots \times H_{n+1}) M_1(s)(d\langle t, s' \rangle).
\]

This may easily be derived from the definition of the projective sequence. Consequently, the equation in question holds for all cylinder sets, thus the \( \pi-\lambda \)-Theorem \( \text{2.4} \) implies that it holds for all Borel subsets of \((\mathbb{R}_+ \times S)^\infty\). \( \dashv \)

This decomposition indicates that we may first select in state \( s \) a new state and a transition time; with these data the system then works just as if the selected new state would have been the initial state. The system does not have a memory but reacts depending on its current state, no matter how it arrived there. Lemma \( \text{2.185} \) may accordingly be interpreted as a Markov property for a process the behavior of which is independent of the specific step that is undertaken.

We need some information about the \( @ \)-operator before continuing.

**Lemma 2.186** \( \langle \sigma, t \rangle \mapsto \sigma@t \) is a Borel measurable map from \((S \times \mathbb{R}_+)^\infty \times \mathbb{R}_+ \) to \( S_\# \). In particular, the set \( \{ \langle \sigma, t \rangle \mid \sigma@t \in S \} \) is a measurable subset of \((S \times \mathbb{R}_+)^\infty \times \mathbb{R}_+ \).
Proof

0. Note that we claim joint measurability in both components (which is strictly stronger than measurability in each component). Thus we have to show that \( \{ \langle \sigma, t \rangle | \sigma \at t \in A \} \) is a measurable subset of \((S \times \mathbb{R}_+)^\mathbb{N} \times \mathbb{R}_+\), whenever \( A \subseteq S_\# \) is Borel.

1. Because for fixed \( i \in \mathbb{N} \) the map \( \sigma \mapsto \delta(\sigma, i) \) is a projection, \( \delta(\cdot, i) \) is measurable, hence \( \sigma \mapsto \sum_{i=0}^j \delta(\sigma, i) \) is. Consequently, 
\[
\{ \langle \sigma, t \rangle | \sigma \at t = \# \} = \{ \langle \sigma, t \rangle | \forall j : t \geq \sum_{i=0}^j \delta(\sigma, i) \} = \bigcap_{j \geq 0} \{ \langle \sigma, t \rangle | t \geq \sum_{i=0}^j \delta(\sigma, i) \}.
\]
This is clearly a measurable set.

2. Put \( \text{stop}(\sigma, t) := \inf\{k \geq 0 | t < \sum_{i=0}^k \delta(\sigma, i)\} \), thus \( \text{stop}(\sigma, t) \) is the smallest index for which the accumulated waiting in \( \sigma \) times exceed \( t \).
\[
X_k := \{ \langle \sigma, t \rangle | \text{stop}(\sigma, t) = k \} = \{ \langle \sigma, t \rangle | \sum_{i=0}^{k-1} \delta(\sigma, i) \leq t < \sum_{i=0}^k \delta(\sigma, i) \}
\]
is a measurable set by Corollary 2.34. Now let \( B \in \mathcal{B}(S) \) be a Borel set, then 
\[
\{ \langle \sigma, t \rangle | \sigma \at t \in B \} = \bigcup_{k \geq 0} \{ \langle \sigma, t \rangle | \sigma[k] \in B, \text{stop}(\sigma, t) = k \}
\]
\[
= \bigcup_{k \geq 0} \left( X_k \cap (\prod_{i<k} (S \times \mathbb{R}_+) \times (B \times \mathbb{R}_+) \times \prod_{i>k} (S \times \mathbb{R}_+)) \right).
\]
Because \( X_k \) is measurable, the latter set is measurable. This establishes measurability of the \( @ \)-map. \( \dashv \)

As a consequence, we establish that some sets and maps, which will be important for the later development, are actually measurable. A notational convention for improving readability is proposed: the letter \( \sigma \) will always denote a generic element of \((S \times \mathbb{R}_+)^\mathbb{N} \times \mathbb{R}_+\), and the letter \( \tau \) always a generic element of \( \mathbb{R}_+ \times (S \times \mathbb{R}_+)^\mathbb{N} \).

**Proposition 2.187** We observe the following properties:

1. \( \{ \langle \sigma, t \rangle | \lim_{i \to \infty} \delta(\sigma, i) = t \} \) is a measurable subset of \((S \times \mathbb{R}_+)^\mathbb{N} \times \mathbb{R}_+\),

2. let \( N_\infty : S \leadsto (\mathbb{R}_+ \times S)^\mathbb{N} \) be a stochastic relation, then 
\[
\begin{align*}
s & \mapsto \liminf_{t \to \infty} N_\infty(s)(\{ \tau | \langle s, \tau \rangle \at t \in A \}) \\
s & \mapsto \limsup_{t \to \infty} N_\infty(s)(\{ \tau | \langle s, \tau \rangle \at t \in A \})
\end{align*}
\]
constitute measurable maps \( X \to \mathbb{R}_+ \) for each Borel set \( A \subseteq S \).

**Proof** 0. The proof makes crucial use of the fact that the real line is a complete metric space (so each Cauchy sequence converges), and that the rational numbers are a dense and countable set.
1. In order to establish part 1, write

$$\{ (\sigma, t) | \lim_{i \to \infty} \delta(\sigma, i) = t \} = \bigcap_{Q \ni 0} \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \{ (\sigma, t) | |\delta(\sigma, m) - t| < \epsilon \}. \,$$

By Lemma 2.186 the set

$$\{ (\sigma, t) | |\delta(\sigma, m) - t| < \epsilon \} = \{ (\sigma, t) | \delta(\sigma, m) > t - \epsilon \} \cap \{ (\sigma, t) | \delta(\sigma, m) < t + \epsilon \}$$

is a measurable subset of $(S \times \mathbb{R}_+)^\infty \times \mathbb{R}_+$, and since the union and the intersections are countable, measurability is inferred.

2. From the definition of the @-operator it is immediate that given an infinite path $\sigma$ and a time $t \in \mathbb{R}_+$, there exists a rational $t'$ with $\sigma@t = \sigma@t'$. Thus we obtain for an arbitrary real number $x$, an arbitrary Borel set $A \subseteq S$ and $s \in S$

$$\liminf_{t \to \infty} N_\infty(s)(\{ \tau | \langle s, \tau @ t \in A \} \leq x \iff \sup_{t \geq 0} \inf_{\tau \geq t} N_\infty(s)(\{ \tau | \langle s, \tau @ r \in A \}) \leq x \iff \sup_{Q \ni 0, Q @ r \geq t} N_\infty(s)(\{ \tau | \langle s, \tau @ r \in A \}) \leq x \iff s \in \bigcap_{Q \ni 0} \bigcup_{Q @ r \geq t} A_{r,x}$$

with

$$A_{r,x} := \{ s' | N_\infty(s')(\{ \tau | \langle s', \tau @ r \in A \}) \leq x \}.$$

We infer that $A_{r,x}$ is a measurable subset of $S$ from the fact that $N_\infty$ is a stochastic relation and from Exercise 17. Since a map $f : W \to \mathbb{R}$ is measurable iff each of the sets $\{ w \in W | f(w) \leq s \}$ is a measurable subset of $W$, the assertion follows for the first map. The second part is established in exactly the same way, using that $f : W \to \mathbb{R}$ is measurable iff $\{ w \in W | f(w) \geq s \}$ is a measurable subset of $W$, and observing

$$\limsup_{t \to \infty} N_\infty(s)(\{ \tau | \langle x, \tau @ t \in A \}) \geq x \iff \inf_{Q \ni 0} \sup_{Q @ r \geq t} N_\infty(s)(\{ \tau | \langle s, \tau @ r \in A \}) \geq x. \quad \Box$$

This has some consequences which will come in useful for the interpretation of CSL. Before stating them, it is noted that the statement above (and the consequences below) do not make use of $N_\infty$ being a projective limit; in fact, we assume $N_\infty : S \rightsquigarrow (\mathbb{R}_+ \times S)^\infty$ to be an arbitrary stochastic relation. A glimpse at the proof shows that these statements even hold for finite transition kernels, but since we will use it for the probabilistic case, we stick to stochastic relations.

Now for the consequences. As a first consequence we obtain that the set on which the asymptotic behavior of the transition times is reasonable (in the sense that it tends probabilistically to a limit) is well behaved in terms of measurability:

**Corollary 2.188** Let $A \subseteq X$ be a Borel set, and assume that $N_\infty : S \rightsquigarrow (\mathbb{R}_+ \times S)^\infty$ is a stochastic relation. Then

1. the set $Q_A := \{ s \in S | \lim_{t \to \infty} N_\infty(s)(\{ \tau | \langle s, \tau @ t \in A \}) \}$ exists} on which the limit exists is a Borel subset of $S$,.
2. $s \mapsto \lim_{t \to \infty} N_{\infty}(s)(\{\tau \mid \langle s, \tau \rangle @ t \in A\}$ is a measurable map $Q_A \to \mathbb{R}_+$.  

**Proof** Since $s \in Q_A$ iff
\[
\liminf_{t \to \infty} N_{\infty}(x)(\{\tau \mid \langle s, \tau \rangle @ t \in A\}) = \limsup_{t \to \infty} N_{\infty}(x)(\{\tau \mid \langle s, \tau \rangle @ t \in A\}),
\]
and since the set on which two Borel measurable maps coincide is a Borel set itself, the first assertion follows from Proposition 2.187 part 2. This implies the second assertion as well. \(\square\)

When dealing with the semantics of the until operator later, we will also need to establish measurability of certain sets. Preparing for that, we state:

**Lemma 2.189** Assume that $A_1$ and $A_2$ are Borel subsets of $S$, and let $I \subseteq \mathbb{R}_+$ be an interval, then
\[
U(I, A_1, A_2) := \{\sigma \mid \exists t \in I : \sigma @ t \in A_2 \land \forall t' \in [0, t[ : \sigma @ t' \in A_1\}
\]
is a measurable set of paths, thus $U(I, A_1, A_2) \in \mathcal{B}(S \times \mathbb{R}_+)^\infty$.

**Proof** 0. Remember that, given a path $\sigma$ and a time $t \in \mathbb{R}_+$, there exists a rational time $t_r \leq t$ with $\sigma @ t = \sigma @ t_r$. Consequently,
\[
U(I, A_1, A_2) = \bigcup_{t \in \mathbb{Q} \cap I} \{\sigma \mid \sigma @ t \in A_2\} \cap \bigcap_{t' \in \mathbb{Q} \cap [0, t[} \{\sigma \mid \sigma @ t' \in A_1\}.
\]
The inner intersection is countable and is performed over measurable sets by Lemma 2.186, thus forming a measurable set of paths. Intersecting it with a measurable set and forming a countable union yields a measurable set again. \(\square\)

Now that we know how to probabilistically describe the behavior of paths, we are ready for a probabilistic interpretation of CSL. We have started from the assumption that the one-step behavior is governed through a stochastic relation $M : S \leadsto \mathbb{R}_+ \times S$ with $M(s)(\mathbb{R}_+ \times S) = 1$ for all $s \in S$ from which the stochastic relation $M_{\infty} : S \leadsto \mathbb{R}_+ \times (S \times \mathbb{R}_+)\infty$ has been constructed. The interpretations for the formulas can be established now, and we show that the sets of states resp. paths on which formulas are valid are Borel measurable.

To get started on the formal definition of the semantics, we assume that we know for each atomic proposition which state it is satisfied in. Thus we fix a map $\ell$ that maps $P$ to $\mathcal{B}(S)$, assigning each atomic proposition a Borel set of states.

The semantics is described as usual recursively through relation $\models$ between states resp. paths, and formulas. Hence $s \models \varphi$ means that state formula $\varphi$ holds in state $s$, and $\sigma \models \psi$ means that path formula $\psi$ is true on path $\sigma$.

Here we go:

1. $s \models \top$ is true for all $s \in S$.
2. $s \models a$ iff $s \in \ell(a)$.
3. $s \models \varphi_1 \land \varphi_2$ iff $s \models \varphi_1$ and $s \models \varphi_2$.
4. $s \models \neg \varphi$ iff $s \models \varphi$ is false.
5. $s \models S_{\text{eq}}(\varphi)$ iff $\lim_{t \to \infty} M_{\infty}(s)(\{\tau \mid \langle s, \tau \rangle @ t \models \varphi\})$ exists and is $\gg p$.  

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6. \( s \models \mathcal{P}_\text{cpr}(\psi) \) iff \( M_\infty(s)(\{\tau \mid \langle s, \tau \rangle \models \psi\}) \triangleq p \).

7. \( \sigma \models \mathcal{X}^I \varphi \) iff \( \sigma[1] \models \varphi \) and \( \delta(\sigma, 0) \in I \).

8. \( \sigma \models \varphi_1 \mathcal{U}^I \varphi_2 \) iff \( \exists t \in I : \sigma @ t \models \varphi_2 \) and \( \forall t' \in [0, t] : \sigma @ t' \models \varphi_1 \).

Most interpretations should be obvious. Given a state \( s \), we say that \( s \models \mathcal{S}_\text{cpr}(\varphi) \) iff the asymptotic behavior of the paths starting at \( s \) gets eventually stable with a limiting probability given by \( \triangleq p \). Similarly, \( s \models \mathcal{P}_\text{cpr}(\psi) \) holds iff the probability that path formula \( \psi \) holds for all \( s \)-paths is specified through \( \triangleq p \). For \( \langle s_0, t_0, s_1, \ldots \rangle \models \mathcal{X}^I \varphi \) to hold we require \( s_1 \models \varphi \) after a waiting time \( t_0 \) for the transition to be a member of interval \( I \). Finally, \( \sigma \models \varphi_1 \mathcal{U}^I \varphi_2 \) holds iff we can find a time point \( t \) in the interval \( I \) such that the corresponding state \( \sigma @ t \) satisfies \( \varphi_2 \), and for all states on that path before \( t \), formula \( \varphi_1 \) is assumed to hold. The kinship to CTL is obvious.

Denote by \( [\varphi] \) and \( [\psi] \) the set of all states for which the state formula \( \varphi \) holds, resp. the set of all paths for which the path formula \( \varphi \) is valid. We do not distinguish notationally between these sets, as far as the basic domains are concerned, since it should always be clear whether we describe a state formula or a path formula.

We show that we are dealing with measurable sets. Most of the work for establishing this has been done already. What remains to be done is to fit in the patterns that we have set up in Proposition 2.187 and its Corollaries.

**Proposition 2.190** The set \( [\xi] \) is Borel, whenever \( \xi \) is a state formula or a path formula.

**Proof** 0. The proof proceeds by induction on the structure of the formula \( \xi \). The induction starts with the formula \( \top \), for which the assertion is true, and with the atomic propositions, for which the assertion follows from the assumption on \( \ell \): \( [a] = \ell(a) \in \mathcal{B}(S) \). We assume for the induction step that we have established that \( [\varphi], [\varphi_1] \) and \( [\varphi_2] \) are Borel measurable.

1. For the next-operator we write
   \[
   [\mathcal{X}^I \varphi] = \{ \sigma \mid \sigma[1] \in [\varphi] \text{ and } \delta(\sigma, 0) \in I \}.
   \]
   This is the cylinder set \( (S \times I \times [\varphi] \times \mathbb{R}_+) \times (S \times \mathbb{R}_+)^\infty \), hence is a Borel set.

2. The until-operator may be represented through
   \[
   [\varphi_1 \mathcal{U}^I \varphi_2] = U(I, [\varphi_1], [\varphi_2]),
   \]
   which is a Borel set by Lemma 2.189.

3. Since \( M_\infty : S \leadsto (\mathbb{R}_+ \times S)^\infty \) is a stochastic relation, we know that
   \[
   [\mathcal{P}_\text{cpr}(\psi)] = \{ s \in S \mid M_\infty(s)(\{\tau \mid \langle s, \tau \rangle \in [\varphi]\}) \triangleq p \}
   \]
   is a Borel set.

4. We know from Corollary 2.188 that the set
   \[
   Q_{[\varphi]} := \{ s \in S \mid \lim_{t \to \infty} M_\infty(s)(\{\tau \mid \langle s, \tau \rangle @ t \in [\varphi]\}) \text{ exists} \}
   \]
is a Borel set, and that
\[ J_\varphi : Q_{[\varphi]} \ni s \mapsto \lim_{t \to \infty} M_\infty(x)(\{\tau \mid \langle s, \tau \rangle @ t \in [\varphi]\}) \in [0, 1] \]
is a Borel measurable function. Consequently,
\[ \mathcal{S}_{\langle \varphi \rangle} = \{ s \in Q_{[\varphi]} \mid J_\varphi(s) \bowtie \varphi \} \]
is a Borel set.

Measurability of the sets on which a given formula is valid constitutes of course a prerequisite for computing interesting properties. So we can compute, e.g.,
\[ P_{\geq 0.5}((-\text{down}) \cup [10, 20] \cup \text{up}_2 \lor \text{up}_3) \]
as the set of all states that with probability at least 0.5 will reach a state between 10 and 20 time units so that the system is operational \((\text{up}_2, \text{up}_3 \in P)\) in a steady state with a probability of at least 0.8; prior to reaching this state, the system must be operational continuously \((\text{down} \in P)\).

The description of the semantics is just the basis for entering into the investigation of expressivity of the models associated with \(M\) and with \(\ell\). We leave CSL here, however, and note that the construction of the projective limit is the basic ingredient for further investigations.

2.10 The Weak Topology

Now that we have integration at our disposal, we will look again at topological issues for the space of finite measures. We fix in this section \((X, d)\) as a metric space; recall that \(C_b(X)\) is the space of all bounded continuous functions \(X \to \mathbb{R}\). This space induces the weak topology on the space \(\mathcal{M}(X) = \mathcal{M}(X, \mathcal{B}(X))\) of all finite Borel measures on \((X, \mathcal{B}(X))\). This is the smallest topology which renders the evaluation map
\[ \mu \mapsto \int_X f \ d\mu \]
continuous for every continuous and bounded map \(f : X \to \mathbb{R}\). This topology is fairly natural, and it is related to the topologies on \(\mathcal{M}(X)\) considered so far, the Alexandrov topology and the topology given by the Levy-Prohorov metric, which are discussed in Section 2.1.2. We will show that these topologies are the same, provided the underlying space is Polish, and we will show in this case show that \(\mathcal{M}(X)\) is itself a Polish space. Somewhat weaker results may be obtained if the base space is only separable metric, and it turns out that tightness, i.e., inner approximability through compact sets, the the property which sets Polish spaces apart for our purposes. We introduce also a very handy metric for the weak topology due to Hutchinson. Two case studies on bisimulations of Markov transition systems and on quotients for stochastic relations demonstrate the interplay of topological considerations with selection arguments, which become available on \(\mathcal{M}(X)\) once this space is identified as Polish.

Define as the basis for the topology the sets
\[ U_{f_1, \ldots, f_n, \epsilon}(\mu) := \{ \nu \in \mathcal{M}(X) \mid \left| \int_X f_i \ d\nu - \int_X f_i \ d\mu \right| < \epsilon \text{ for } 1 \leq i \leq n \} \]
with $\epsilon > 0$ and $f_1, \ldots, f_n \in C_b(X)$. Call the topology the *weak topology* on $M(X)$.

With respect to convergence, we have this characterization, which indicates the relationship between the weak topology and the Alexandrov-topology investigated in Section 2.1.2.

**Theorem 2.191** The following statements are equivalent for a sequence $(\mu_n)_{n \in \mathbb{N}} \subseteq M(X)$.

1. $\mu_n \to \mu$ in the weak topology.
2. $\int_X f \, d\mu_n \to \int_X f \, d\mu$ for all $f \in C_b(X)$.
3. $\int_X f \, d\mu_n \to \int_X f \, d\mu$ for all bounded and uniformly continuous $f : X \to \mathbb{R}$.
4. $\mu_n \to \mu$ in the $A$-topology.

**Proof** The implications $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are trivial.

3 $\Rightarrow$ 4: Let $G \subseteq X$ be open, then $f_k(x) := 1 \wedge k \cdot d(x, X \setminus G)$ defines a uniformly continuous map (because $|d(x, X \setminus G) - d(x', X \setminus G)| \leq d(x, x')$), and $0 \leq f_1 \leq f_2 \leq \ldots$ with $\lim_{k \to \infty} f_k = \chi_G$. Hence $\int_X f_k \, d\mu \leq \int_X \chi_G \, d\mu = \mu(G)$, and by monotone convergence $\int_X f_k \, d\mu \to \mu(G)$. From the assumption we know that $\int_X f_k \, d\mu_n \to \int_X f_k \, d\mu$, as $n \to \infty$, so that we obtain for all $k \in \mathbb{N}$

$$\lim_{n \to \infty} \int_X f_k \, d\mu_n \leq \liminf_{n \to \infty} \mu_n(G),$$

with in turn implies $\mu(G) \leq \liminf_{n \to \infty} \mu_n(G)$.

4 $\Rightarrow$ 2 We may assume that $f \geq 0$, because the integral is linear. Then we can represent the integral through (see Example 2.172, equation (7))

$$\int_X f \, d\nu = \int_0^\infty \nu(\{x \in X \mid f(x) > t\}) \, dt.$$

Since $f$ is continuous, the set $\{x \in X \mid f(x) > t\}$ is open. By Fatou’s Lemma (Proposition 2.149) we obtain from the assumption

$$\liminf_{n \to \infty} \int_X f \, d\mu_n = \liminf_{n \to \infty} \int_0^\infty \mu_n(\{x \in X \mid f(x) > t\}) \, dt$$

$$\geq \int_0^\infty \liminf_{n \to \infty} \mu_n(\{x \in X \mid f(x) > t\}) \, dt$$

$$\geq \int_0^\infty \mu(\{x \in X \mid f(x) > t\}) \, dt$$

$$= \int_X f \, d\mu.$$

Because $f \geq 0$ is bounded, we find $T \in \mathbb{R}$ such that $f(x) \leq T$ for all $x \in X$, hence $g(x) := T - f(x)$ defines a non-negative and bounded function. Then by the preceding argument $\liminf_{n \to \infty} \int_X g \, d\mu_n \geq \int_X g \, d\mu$. Since $\mu_n(X) \to \mu(X)$, we infer

$$\limsup_{n \to \infty} \int_X f \, d\mu_n \leq \int_X f \, d\mu,$$

which implies the desired equality. $\dashv$
Let $X$ be separable, then the $A$-topology is metrized by the Prohorov metric (Theorem 2.28).
Thus we have established that the metric topology and the topology of weak convergence are
the same for separable metric spaces. Just for the record:

**Theorem 2.192** Let $X$ be a separable metric space, then the Prohorov metric is a metric for
the topology of weak convergence. ⊥

It is now easy to find a dense subset in $\mathcal{M}(X)$. As one might expect, the measures living on
discrete subsets are dense. Before stating and proving the corresponding statement, we have
a brief look at the embedding of $X$ into $\mathcal{M}(X)$.

**Example 2.193** The base space $X$ is embedded into $\mathcal{M}(X)$ as a closed subset through $x \mapsto \delta_x$. In fact, let $(\delta_{x_n})_{n \in \mathbb{N}}$ be a sequence which converges weakly to $\mu \in \mathcal{M}(X)$. We have
in particular $\mu(X) = \lim_{n \to \infty} \delta_{x_n}(X) = 1$, hence $\mu \in \mathcal{P}(X)$. Now assume that $(x_n)_{n \in \mathbb{N}}$
does not converge, hence it does not have a convergent subsequence in $X$. Then the set
$S := \{x_n \mid n \in \mathbb{N}\}$ is closed in $X$, so are all subsets of $S$. Take an infinite subset $C \subseteq S$ with an infinite complement $S \setminus C$, then $\mu(C) \geq \limsup_{n \to \infty} \delta_{x_n}(C) = 1$, and with the same
argument $\mu(S \setminus C) = 1$. This contradicts $\mu(X) = 1$. Thus we can find $x \in X$ with $x_n \to x$,
then $\delta_x \to \delta_x$, so that the image of $X$ in $\mathcal{M}(X)$ is closed. ⊥

**Proposition 2.194** Let $X$ be a separable metric space. The set
\[
\{ \sum_{k \in \mathbb{N}} r_k \cdot \delta_{x_k} \mid x_k \in X, r_k \geq 0 \}
\]
of discrete measures is dense in the topology of weak convergence.

**Proof** Fix $\mu \in \mathcal{M}(X)$. Cover $X$ for each $k \in \mathbb{N}$ with mutually disjoint Borel sets $(A_{n,k})_{n \in \mathbb{N}}$
each of which has a diameter not less that $1/k$. Select an arbitrary $x_{n,k} \in A_{n,k}$. We claim
that $\mu_n := \sum_{k \in \mathbb{N}} \mu(A_{n,k}) \cdot \delta_{x_{n,k}}$ converges weakly to $\mu$. In fact, let $f : X \to \mathbb{R}$ be a uniformly
continuous and bounded map. Since $f$ is uniformly continuous,
\[
\eta_n := \sup_{k \in \mathbb{N}} \left( \sup_{x \in A_{n,k}} f(x) - \inf_{x \in A_{n,k}} f(x) \right)
\]
tends to 0, as $n \to \infty$. Thus
\[
\left| \int_X f \, d\mu_n - \int_X f \, d\mu \right| = \left| \sum_{k \in \mathbb{N}} \left( \int_{A_{n,k}} f \, d\mu_n - \int_{A_{n,k}} f \, d\mu \right) \right|
\leq \eta_n \cdot \sum_{k \in \mathbb{N}} \mu(A_{n,k})
\leq \eta_n
\to 0.
\]

This yields immediately

**Corollary 2.195** If $X$ is a separable metric space, then $\mathcal{M}(X)$ is a separable metric space in
the topology of weak convergence.

**Proof** Because $\sum_{k=1}^n r_k \cdot \delta_{x_k} \to \sum_{k \in \mathbb{N}} r_k \cdot \delta_{x_k}$, as $n \to \infty$ in the weak topology, and because
the rationals $\mathbb{Q}$ are dense in the reals, we obtain from Proposition 2.194 that $\left\{ \sum_{k=1}^n r_k \cdot \delta_{x_k} \right\}$
$x_k \in D, 0 \leq r_k \in \mathbb{Q}, n \in \mathbb{N}$ is a countable and dense subset of $\mathcal{M}(X)$, whenever $D \subseteq X$ is a countable and dense subset of $X$. \hfill \dashv

Another immediate consequence refers to the weak-$\ast$-$\sigma$-algebra. We obtain from Lemma \ref{thm:weak-star-sigma-algebra} together with Corollary \ref{cor:weak-star-sigma-algebra}

**Corollary 2.196** Let $X$ be a metric space, then the weak-$\ast$-$\sigma$-algebra are the Borel sets of the $A$-topology. \hfill \dashv

We will show now that $\mathcal{M}(X)$ is a Polish space, provided $X$ is one; thus applying the $\mathcal{M}$-functor to a Polish space does not leave the realm of Polish spaces.

We know by Alexandrov’s Theorem \ref{thm:alexandrov} that a separable metrizable space is Polish iff it can be embedded as a $G_δ$-set into the Hilbert cube. We show first that for compact metric $X$ the space $\mathcal{S}(X)$ of all subprobability measures with the topology of weak convergence is itself a compact metric space. This is established by embedding it as a closed subspace into $[-1, +1]^{\infty}$. But there is nothing special about taking $\mathcal{S}$; the important property is that all measures are uniformly bounded (by 1, in this case). Any other bound would also do.

We require for this the Stone-Weierstraß Theorem which states (in the form needed here) that the unit ball in the space of all bounded continuous functions on a compact metric space is separable itself \[\text{[Kel55, Chapter 7, Problem S (e), p. 245]}.\] The idea of the embedding is to take a countable dense sequence $(g_n)_{n \in \mathbb{N}}$ of this unit ball. Since we are dealing with probability measures, and since we know that each $g_n$ maps $X$ into the interval $[-1, 1]$, we know that $-1 \leq \int_X g_n \, d\mu \leq 1$ for each $\mu$. This then spawns the desired map, which together with its inverse is shown through the Riesz Representation Theorem to be continuous.

Well, this is the plan of attack for establishing

**Proposition 2.197** Let $X$ be a compact metric space. Then $\mathcal{S}(X)$ is a compact metric space.

**Proof**

1. The space $\mathcal{C}(X)$ of continuous maps into the reals is for compact metric $X$ a separable Banach space under the sup-norm $\| \cdot \|_{\infty}$. The closed unit ball

\[\mathcal{C}_1 := \{ f \in \mathcal{C}(X) \mid \| f \|_{\infty} \leq 1 \}\]

is, as mentioned above, a separable metric space in its own right. Let $(g_n)_{n \in \mathbb{N}}$ be a countable sense subset in $\mathcal{C}_1$, and define

\[\Theta : \mathcal{S}(X) \ni \nu \mapsto (\int_X g_1 \, d\nu, \int_X g_2 \, d\nu, \ldots) \in [-1, 1]^{\infty}.\]

Then $\Theta$ is injective, because the sequence $(g_n)_{n \in \mathbb{N}}$ is dense.

2. Also, $\Theta^{-1}$ is continuous. In fact, let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{S}(X)$ such that $(\Theta(\mu_n))_{n \in \mathbb{N}}$ converges in $[-1, 1]^{\infty}$, put $\alpha_n := \lim_{n \to \infty} \int_X g_i \, d\mu_n$. For each $f \in \mathcal{C}_1$ there exists a subsequence $(g_{n_k})_{k \in \mathbb{N}}$ such that $\| f - g_{n_k} \|_{\infty} \to 0$ as $k \to \infty$, because $(g_n)_{n \in \mathbb{N}}$ is dense in $\mathcal{C}_1$. Thus

\[L(f) := \lim_{n \to \infty} \int_X f \, d\mu_n\]

exists. Define $L(\alpha \cdot f) := \alpha \cdot L(f)$, for $\alpha \in \mathbb{R}$, then it is immediate that $L : \mathcal{C}(X) \to \mathbb{R}$ is linear and that $L(f) \geq 0$, provided $f \geq 0$. The Riesz Representation Theorem \ref{thm:riesz_representation} now gives a unique $\mu \in \mathcal{S}(X)$ with

\[L(f) = \int_X f \, d\mu,\]
and the construction shows that
\[
\lim_{n \to \infty} \Theta(\mu_n) = \langle \int_X g_1 \, d\mu, \int_X g_2 \, d\mu, \ldots \rangle.
\]

3. Consequently, \( \Theta : S(X) \to \Theta[S(X)] \) is a homeomorphism, and \( \Theta[S(X)] \) is closed, hence compact. Thus \( S(X) \) is compact. \( \dashv \)

We obtain as a first consequence

**Proposition 2.198** \( X \) is compact iff \( S(X) \) is, whenever \( X \) is a Polish space.

**Proof** It remains to show that \( X \) is compact, provided \( S(X) \) is. Choose a complete metric \( d \) for \( X \). Thus \( X \) is isometrically embedded into \( S(X) \) by \( x \mapsto \delta_x \) with \( A := \{ \delta_x | x \in X \} \) being closed. We could appeal to Example 2.193, but a direct argument is available as well. In fact, if \( \delta_{x_n} \to \mu \) in the weak topology, then \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \) on account of the isometry. Since \( (X,d) \) is complete, \( x_n \to x \) for some \( x \in X \), hence \( \mu = \delta_x \), thus \( A \) is closed, hence compact. \( \dashv \)

The next step for showing that \( M(X) \) is Polish is nearly canonical. If \( X \) is a Polish space, it may be embedded as a \( G_\delta \)-set into a compact metric space \( \tilde{X} \), the subprobabilities of which are topologically a closed subset of \([-1, +1]^\infty\), as we have just seen. We will show now that \( M(X) \) is a \( G_\delta \) in \( M(\tilde{X}) \) as well.

**Proposition 2.199** Let \( X \) be a Polish space. Then \( M(X) \) is a Polish space in the topology of weak convergence.

**Proof** 1. Embed \( X \) as a \( G_\delta \)-subset into a compact metric space \( \tilde{X} \), hence \( X \in B(\tilde{X}) \). Put
\[
M_0 := \{ \mu \in M(\tilde{X}) | \mu(\tilde{X} \setminus X) = 0 \},
\]
so \( M_0 \) contains exactly those finite measures on \( \tilde{X} \) that are concentrated on \( X \). Then \( M_0 \) is homeomorphic to \( M(X) \).

2. Write \( X \) as \( X = \bigcap_{n \in \mathbb{N}} G_n \), where \( (G_n)_{n \in \mathbb{N}} \) is a sequence of open sets in \( \tilde{X} \). Given \( r > 0 \), the set
\[
\Gamma_{k,r} := \{ \mu \in M(\tilde{X}) | \mu(\tilde{X} \setminus G_k) < r \}
\]
is open in \( M(\tilde{X}) \). In fact, if \( \mu_n \notin \Gamma_{k,r} \) converges to \( \mu_0 \) in the weak topology, then
\[
\mu_0(\tilde{X} \setminus G_k) \geq \limsup_{n \to \infty} \mu_n(\tilde{X} \setminus G_k) \geq r
\]
by Theorem 2.191, since \( \tilde{X} \setminus G_k \) is closed. Consequently, \( \mu_0 \notin \Gamma_{k,r} \). This shows that \( \Gamma_{k,r} \) is open, because its complement is closed. Thus
\[
M_0 = \bigcap_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \Gamma_{n,1/k} \]
is a \( G_\delta \)-set, and the assertion follows. \( \dashv \)

Thus we obtain as a consequence

**Proposition 2.200** \( M(X) \) is a Polish space in the topology of weak convergence iff \( X \) is.
Proof Let $\mathcal{M}(X)$ be Polish. The base space $X$ is embedded into $\mathcal{M}(X)$ as a closed subset by Example 2.193, hence is a Polish space by Theorem 2.75. ⊣

Let $\mu \in \mathcal{M}(X)$ with $X$ Polish. Since $X$ has a countable basis, we know from Lemma 2.25 that $\mu$ is supported by a closed set, since $\mu$ is $\tau$-regular. But in the presence of a complete metric we can say a bit more, viz., that the value of $\mu(A)$ may be approximated from within by compact sets to arbitrary precision.

Definition 2.201 A finite Borel measure $\mu$ is called tight iff

$$\mu(A) = \sup \{ \mu(K) \mid K \subseteq A \text{ compact} \}$$

holds for all $A \in \mathcal{B}(X)$.

Thus tightness means for $\mu$ that we can find for any $\epsilon > 0$ and for any Borel set $A \subseteq X$ a compact set $K \subseteq A$ with $\mu(A \setminus K) < \epsilon$. Because a finite measure on a separable metric space is regular, i.e., $\mu(A)$ can be approximated from within $A$ by closed sets (Lemma 2.133), it suffices in this case to consider tightness at $X$, hence to postulate that there exists for any $\epsilon > 0$ a compact set $K \subseteq X$ with $\mu(X \setminus K) < \epsilon$. We know in addition that each finite measure is $\tau$-regular by Lemma 2.24. Capitalizing on this and on completeness, we find

Proposition 2.202 Each finite Borel measure on Polish space $X$ is tight.

Proof 1. We show first that we can find for each $\epsilon > 0$ a compact set $K \subseteq X$ with $\mu(X \setminus K) < \epsilon$. In fact, given a complete metric $d$, consider

$$\mathcal{G} := \{ \{ x \in X \mid d(x, M) < 1/n \} \mid M \subseteq X \text{ is finite} \}.$$ 

Then $\mathcal{G}$ is a directed collection of open sets with $\bigcup \mathcal{G} = X$, thus we know from $\tau$-regularity of $\mu$ that $\mu(X) = \sup \{ \mu(G) \mid G \in \mathcal{G} \}$. Consequently, given $\epsilon > 0$ there exists for each $n \in \mathbb{N}$ a finite set $M_n \subseteq X$ with $\mu(\{ x \in X \mid d(x, M_n) < 1/n \}) > \mu(X) - \epsilon/2^n$. Now define

$$K := \bigcap_{n \in \mathbb{N}} \{ x \in X \mid d(x, M_n) \leq 1/n \}.$$ 

Then $K$ is closed, and complete (since $(X, d)$ is complete). Because each $M_n$ is finite, $K$ is totally bounded. Thus $K$ is compact. We obtain

$$\mu(X \setminus K) \leq \sum_{n \in \mathbb{N}} \mu(\{ x \in X \mid d(x, M_n) \geq 1/n \}) \leq \sum_{n \in \mathbb{N}} \epsilon \cdot 2^{-n} = \epsilon.$$ 

2. Now let $A \in \mathcal{B}(X)$, then for $\epsilon > 0$ there exists $F \subseteq A$ closed with $\mu(A \setminus F) < \epsilon/2$, and chose $K \subseteq X$ compact with $\mu(X \setminus K) < \epsilon/2$. Then $K \cap F \subseteq A$ is compact with $\mu(A \setminus (F \cap K)) < \epsilon$. ⊣

Tightness is sometimes an essential ingredient when arguing about measures on a Polish space. The discussion on the Hutchinson metric in the next section provides an example, it shows that at crucial point tightness kicks in and saves the day.
2.10.1 The Hutchinson Metric

We will explore now another approach to the weak topology for Polish spaces through the Hutchinson metric. Given a fixed metric $d$ on $X$, define

$$V_\gamma := \{ f : X \to \mathbb{R} | |f(x) - f(y)| \leq d(x, y) \text{ and } |f(x)| \leq \gamma \text{ for all } x, y \in X \},$$

Thus $f$ is a member of $V_\gamma$ iff $f$ is non-expanding (hence has a Lipschitz constant 1), and iff its supremum norm $\|f\|_\infty$ is bounded by $\gamma$. Trivially, all elements of $V_\gamma$ are uniformly continuous.

Note the explicit dependence on the metric $d$. The Hutchinson distance $H_\gamma(\mu, \nu)$ between $\mu, \nu \in \mathcal{M}(X)$ is defined as

$$H_\gamma(\mu, \nu) := \sup_{f \in V_\gamma} \left( \int_X f \, d\mu - \int_X f \, d\nu \right).$$

Then $H_\gamma$ is easily seen to be a metric on $\mathcal{M}(X)$. $H_\gamma$ is called the Hutchinson metric (sometimes also Hutchinson-Monge-Kantorovicz metric).

The relationship between this metric and the topology of weak convergence is stated now ([Edg98, Theorem 2.5.17]):

**Proposition 2.203** Let $X$ be a Polish space. Then $H_\gamma$ is a metric for the topology of weak convergence on $\mathcal{M}(X)$ for any $\gamma > 0$.

**Proof** 1. We may and do assume that $\gamma = 1$, otherwise we scale accordingly. Now let $H_1(\mu_n, \mu) \to 0$ as $n \to \infty$, then $\lim_{n \to \infty} \mu_n(X) = \mu(X)$. Let $F \subseteq X$ be closed, then we can find for given $\epsilon > 0$ a function $f \in V_1$ such that $f(x) = 1$ for $x \in F$, and $\int_X f \, dm \leq \mu(F) + \epsilon$. This gives

$$\lim_{n \to \infty} \sup \mu_n(F) \leq \lim_{n \to \infty} \int_X f \, d\mu_n = \int_X f \, d\mu \leq \mu(F) + \epsilon$$

Thus convergence in the Hutchinson metric implies convergence in the $A$-topology, hence in the topology of weak convergence, by Proposition 2.14.

2. Now assume that $\mu_n \to \mu$ in the topology of weak convergence, thus $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{B}(X)$ with $\mu(\partial A) = 0$ by Corollary 2.15. We assume that $\mu_n$ and $\mu$ are probability measures, otherwise we scale again. Because $X$ is Polish, $\mu$ is tight by Proposition 2.202.

Fix $\epsilon > 0$, then there exists a compact set $K \subseteq X$ with

$$\mu(X \setminus K) < \frac{\epsilon}{5 \cdot \gamma}.$$ 

Given $x \in K$, there exists an open ball $B_r(x)$ with center $x$ and radius $r$ such that $0 < r < \epsilon/10$ such that $\mu(\partial B_r(x)) = 0$, see Corollary 2.18. Because $K$ is compact, a finite number of these balls will suffice, thus $K \subseteq B_{r_1}(x_1) \cup \ldots \cup B_{r_p}(x_p)$. Transform this cover into a disjoint cover by setting

$$E_1 := B_{r_1}(x_1),$$
$$E_2 := B_{r_2}(x_2) \setminus E_1,$$
$$\ldots$$
$$E_p := B_{r_p}(x_p) \setminus (E_1 \cup \ldots \cup E_{p-1}),$$
$$E_0 := S \setminus (E_1 \cup \ldots \cup E_p)$$
We observe these properties:

1. For \( i = 1, \ldots, p \), the diameter of each \( E_i \) is not greater than \( 2 \cdot r_i \), hence smaller that \( \epsilon/5 \).

2. For \( i = 1, \ldots, p \), \( \partial E_i \subseteq \partial (B_{r_1}(x_1) \cup \ldots \cup B_{r_p}(x_p)) \), thus \( \partial E_i \subseteq (\partial B_{r_1}(x_1)) \cup \ldots \cup (\partial B_{r_p}(x_p)) \), hence \( \mu(\partial E_i) = 0 \).

3. Because the boundary of a set is also the boundary of its complement, we conclude \( \mu(\partial E_0) = 0 \) as well. Moreover, \( \mu(E_0) < \epsilon/(5 \cdot \gamma) \), since \( E_0 \subseteq X \setminus K \).

Eliminate all \( E_i \) which are empty. Select \( \eta > 0 \) such that \( p \cdot \eta < \epsilon/5 \), and determine \( n_0 \in \mathbb{N} \) so that

\[
|\mu_n(E_i) - \mu(E_i)| < \eta \quad \text{for} \quad i = 0, \ldots, p \quad \text{and} \quad n \geq n_0.
\]

We have to show that

\[
\sup_{f \in V} \left( \int_X f \, d\mu_n - \int_X f \, d\mu \right) \to 0, \quad \text{as} \quad n \to \infty.
\]

So take \( f \in V \), and fix \( n \geq n_0 \). Let \( i = 1, \ldots, p \), pick an arbitrary \( e_i \in E_i \); because each \( E_i \) has a diameter not greater than \( \epsilon/5 \), we know that \( |f(x) - f(e_i)| < \epsilon/5 \) for each \( x \in E_i \). If \( x \in E_0 \), we have \( |f(x)| \leq \gamma \).

Now we are getting somewhere: let \( n \geq n_0 \), then we obtain

\[
\int_X f \, d\mu_n = \sum_{i=0}^p \int_{E_i} f \, d\mu_n
\]

\[
\leq \gamma \cdot \mu_n(E_0) + \sum_{i=1}^p \left( f(t_i) + \frac{\epsilon}{5} \right) \cdot \mu_n(E_i)
\]

\[
\leq \gamma \cdot \mu(E_0) + \eta + \sum_{i=1}^p \left( f(t_i) + \frac{\epsilon}{5} \right) \cdot (\mu(E_i) + \eta)
\]

\[
\leq \gamma \cdot \left( \frac{\epsilon}{5} \cdot \gamma + \eta \right) + \sum_{i=1}^p \left( f(t_i) - \frac{\epsilon}{5} \right) \cdot \mu(E_i) + \frac{2 \cdot \epsilon}{5} \sum_{i=1}^p \mu(E_i) + \frac{p \cdot \epsilon \cdot \eta}{5}
\]

\[
\leq \int_X f \, d\mu + \epsilon
\]

Recall that

\[
\sum_{i=1}^p \mu(E_i) \leq \sum_{i=0}^p \mu(E_i) = \mu(X) = 1,
\]

and that

\[
\int_{E_i} f \, d\mu \geq \mu(E_i) \cdot (f(t_i) - \epsilon/5).
\]

In a similar fashion, we obtain \( \int_X f \, d\mu_n \geq \int_X f \, d\mu - \epsilon \), so that we have established

\[
|\int_X f \, d\mu - \int_X f \, d\mu_n| < \epsilon
\]

for \( n \geq n_0 \). Since \( f \in V \) was arbitrary, we have shown that \( H_\gamma(\mu_n, \mu) \to 0. \dashv \)

The Hutchinson metric is sometimes easier to use than the Prohorov metric, because integrals may sometimes easier manipulated in convergence arguments than \( \epsilon \)-neighborhoods of sets.
2.10.2 Case Study: Bisimulation

Bisimilarity is an important notion in the theory of concurrent systems, introduced originally by Milner for transition systems, see [Dob14 Section 1.6.1] for a general discussion. We will show in this section that the methods developed so far may be used in the investigation of bisimilarity for stochastic systems. We will first show that the category of stochastic relations has semi-pullbacks and use this information for a construction of bisimulations for these systems.

If we are in a general category $\mathbf{K}$, then the semi-pullback for two morphisms $f : a \to c$ and $g : b \to c$ with common range $c$ consists of an object $x$ and of morphisms $p_a : x \to a$ and $p_b : x \to b$ such that $f \circ p_a = g \circ p_b$, i.e., such that this diagram commutes in $\mathbf{K}$:

$$
\begin{array}{ccc}
x & \overset{p_a}{\longrightarrow} & a \\
\downarrow \ & & \downarrow \ f \\
b & \overset{p_b}{\longrightarrow} & b
\end{array}
$$

We want to show that semi-pullbacks exist for stochastic relations over Polish spaces. This requires some preparations, provided through selection arguments. The next statement appears to be interesting in its own right; it shows that a measurable selection for weakly continuous stochastic relations exist.

**Proposition 2.204** Let $X_i$, $Y_i$ be Polish spaces, $K_i : X_i \rightsquigarrow Y_i$ be a weakly continuous stochastic relation, $i = 1, 2$. Let $A \subseteq X_1 \times X_2$ and $B \subseteq Y_1 \times Y_2$ be closed subsets of the respective Cartesian products with projections equal to the base spaces, and assume that for $\langle x_1, x_2 \rangle \in A$ the set

$$
\Gamma(x_1, x_2) := \{ \mu \in S(B) \mid S(\beta_i)(\mu) = K_i(x_i), i = 1, 2 \}
$$

is not empty, $\beta_i : B \to Y_i$ denoting the projections. Then there exists a stochastic relation $M : A \rightsquigarrow B$ such that $M(x_1, x_2) \in \Gamma(x_1, x_2)$ for all $\langle x_1, x_2 \rangle \in A$.

**Proof** 1. Let $\overline{Y_i}$ for $i = 1, 2$ be the Alexandrov compactification of $Y_i$ and $\overline{B}$ the closure of $B$ in $\overline{Y_1} \times \overline{Y_2}$. Then $\overline{B}$ is compact and contains the embedding of $B$ into $\overline{Y_1} \times \overline{Y_2}$, which we identify with $B$, as a Borel subset. This is so since $Y_i$ is a Borel subset in its compactification. The projections $\overline{\beta}_i : \overline{B} \to \overline{Y_i}$ are the continuous extensions to the projections $\beta_i : B \to Y_i$.

2. The map $r_i : S(Y_i) \to S(\overline{Y_i})$ with $r_i(\mu)(G) := \mu(G \cap Y_i)$ for $G \in B(\overline{Y_i})$ is continuous; in fact, it is an isometry with respect to the respective Hutchinson metrics, once we have fixed metrics for the underlying spaces. Define for $\langle x_1, x_2 \rangle \in A$ the set

$$
\Gamma_0(x_1, x_2) := \{ \mu \in S(\overline{B}) \mid S(\overline{\beta}_i)(\mu) = (r_i \circ K_i)(x_i), i = 1, 2 \}.
$$

Thus $\Gamma_0$ maps $A$ to the nonempty closed subsets of $S(\overline{B})$, since $S(\overline{\beta}_i)$ and $r_i \circ K_i$ are continuous for $i = 1, 2$. If $\mu \in \Gamma_0(x_1, x_2)$, then

$$
\mu(\overline{B} \setminus B) \leq \mu(\overline{B} \cap (\overline{Y_1} \setminus Y_1 \times \overline{Y_2}) \cup (\overline{Y_1} \times \overline{Y_2} \setminus Y_2))
= S(\overline{\beta}_1)(\mu)(\overline{Y_1} \setminus Y_1) + S(\overline{\beta}_2)(\mu)(\overline{Y_2} \setminus Y_2)
= (r_1 \circ K_1)(x_1)(\overline{Y_1} \setminus Y_1) + (r_2 \circ K_2)(x_2)(\overline{Y_2} \setminus Y_2)
= 0.
$$
Hence all members of $\Gamma_0(x_1, x_2)$ are concentrated on $B$.

3. Let $C \subseteq S(B)$ be compact, and assume that $(t_n)_{n \in \mathbb{N}}$ is a converging sequence in $A$ with $t_n \in \Gamma_0^w(C)$ for all $n \in \mathbb{N}$ such that $t_n \to t_0 \in A$. Then there exists some $\mu_n \in C \cap \Gamma_0(t_n)$ for each $n \in \mathbb{N}$. Since $C$ is compact, there exists a convergent subsequence, which we assume to be the sequence itself, so $\mu_n \to \mu$ for some $\mu \in C$ in the topology of weak convergence. Continuity of $S(\beta)$ and of $K_i(x_i)$ for $i = 1, 2$ implies $\mu \in \Gamma_0$. Consequently, $\Gamma_0^w(C)$ is a closed subset of $A$.

4. Since $S(B)$ is compact, we may represent each open set $G$ as a countable union of compact sets $(C_n)_{n \in \mathbb{N}}$, so that

$$\Gamma_0^w(G) = \bigcup_{n \in \mathbb{N}} \Gamma_0^w(C_n),$$

hence $\Gamma_0^w(G)$ is a Borel set in $A$. The Kuratowski-Ryll-Nardzewski Selection Theorem 2.141 together with Lemma 2.11 gives us a stochastic relation $M_0 : A \sim B$ with $M_0(x_1, x_2) \in \Gamma_0(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{N}$. Define $M(x_1, x_2)$ as the restriction of $M_0(x_1, x_2)$ to the Borel sets of $B$, then $M : A \sim B$ is the desired relation, because $M_0(x_1, x_2)(\overline{B} \setminus B) = 0$. $\dashv$

For the construction we are about to undertake we will put to work the selection machinery just developed; this requires us to show that the set from which we want to select is non-empty. The following technical argument will be of assistance.

Assume that we have Polish spaces $X_1, X_2$ and a separable measure space $(Z, \mathcal{C})$ with surjective and measurable maps $f_i : X_i \to Z$ for $i = 1, 2$. We also have subprobability measures $\mu_i \in S(X_i)$. Since $(Z, \mathcal{C})$ is separable, we may assume that $\mathcal{C}$ constitutes the Borel sets for some metric space $(Z, d)$ so that $d$ has a countable dense subset, see Proposition 2.80. Proposition 2.80 then tells us that we may assume that $f_1$ and $f_2$ are continuous. Now define

$$S := \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}$$
$$A := S \cap (f_1 \times f_2)^{-1}[\mathcal{C} \otimes \mathcal{C}].$$

Since $\Delta_Z := \{(z, z) \mid z \in Z\}$ is a closed subset of $Z \times Z$, and since $f_1$ and $f_2$ are continuous, $S = (f_1 \times f_2)^{-1}\Delta_Z$ is a closed subset of the Polish space $X_1 \times X_2$, hence a Polish space itself by Lemma 2.69. Now assume that we have a finite measure $\vartheta$ on $A$ such that $S(\pi_i)(\vartheta)(E_i) = \mu_i(E_i)$ for all $E_i \in f_i^{-1}[\mathcal{C}]$, $i = 1, 2$ with $\pi_1 : X_1 \to Z$ and $\pi_2 : X_2 \to Z$ as the projections. Now $A \subseteq \mathcal{B}(S)$ is usually not the $\sigma$-algebra of Borel sets for some Polish topology on $S$, which, however, will be needed. Here Lubin’s construction steps in.

**Lemma 2.205** In the notation above, there exists a measure $\vartheta^+$ on the Borel sets of $S$ extending $\vartheta$ such that $S(\pi_i)(\vartheta^+)(E_i) = \mu_i(E_i)$ holds for all $E_i \in \mathcal{B}(S)$.

**Proof** Because $\mathcal{C}$ is countably generated, $\mathcal{C} \otimes \mathcal{C}$ is, so $A$ is a countably generated $\sigma$-algebra. By Lubin’s Theorem 2.132 there exists an extension $\vartheta^+$ to $\vartheta$. $\dashv$

So much for the technical preparations; we will now turn to bisimulations. A bisimulation relates two transition systems which are connected through a mediating system. In order to define this, we need morphisms. In the case of stochastic systems, recall that a morphism $m = (f, g) : K_i \to K_i$ for stochastic relations $K_i : (X_i, A_i) \sim (Y_i, B_i)$ ($i = 1, 2$) over general measurable spaces is given through the measurable maps $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ such
that this diagram of measurable maps commutes

\[
\begin{array}{ccc}
(X_1, A_1) & \xrightarrow{f} & (X_2, A_2) \\
\downarrow_{K_1} & & \downarrow_{K_2} \\
S(Y_1, B_1) & \xrightarrow{s(g)} & S(Y_2, B_2)
\end{array}
\]

Equivalently, \(K_2(f(x_1)) = S(g)(K_1(x_1))\), which translates to \(K_2(f(x_1))(B) = K_1(x_1)(g^{-1}[B])\) for all \(B \in B_2\).

**Definition 2.206** The stochastic relations \(K_i : (X_i, A_i) \sim (Y_i, B_i) \ (i = 1, 2)\), are called bisimilar iff there exist a stochastic relation \(M : (A, \mathcal{X}) \sim (B, \mathcal{Y})\) and surjective morphisms \(m_i = (f_i, g_i) : M \to K_i\) such that the \(\sigma\)-algebra \(g_1^{-1}[B_1] \cap g_2^{-1}[B_2] \) is nontrivial, i.e., contains not only \(\emptyset\) and \(B\). The relation \(M\) is called mediating.

The first condition on bisimilarity is in accordance with the general definition of bisimilarity of coalgebras; it requests that \(m_1\) and \(m_2\) form a span of morphisms

\[
K_1 \xrightarrow{m_1} M \xleftarrow{m_2} K_2.
\]

Hence, the following diagram of measurable maps is supposed to commute with \(m_i = (f_i, g_i)\) for \(i = 1, 2\)

\[
\begin{array}{ccc}
(X_1, A_1) & \xrightarrow{f_1} & (A, \mathcal{X}) & \xrightarrow{f_2} & (X_2, A_2) \\
\downarrow_{K_1} & & \downarrow_{M} & & \downarrow_{K_2} \\
S(Y_1, B_1) & \xrightarrow{s(g_1)} & S(B, \mathcal{Y}) & \xrightarrow{s(g_2)} & S(Y_2, B_2)
\end{array}
\]

Thus, for each \(a \in A, D \in B_1, E \in B_2\) the equalities

\[
\begin{align*}
K_1(f(a))(D) = (S(g_1) \circ M)(a)(D) &= M(a)(g_1^{-1}[D]) \\
K_2(f_2(a))(E) = (S(g_2) \circ M)(a)(E) &= M(a)(g_2^{-1}[E])
\end{align*}
\]

should be satisfied. The second condition, however, is special; it states that we can find an event \(C^* \in \mathcal{Y}\) which is common to both \(K_1\) and \(K_2\) in the sense that

\[
g_1^{-1}[B_1] = C^* = g_2^{-1}[B_2]
\]

for some \(B_1 \in B_1\) and \(B_2 \in B_2\) such that both \(C^* \neq \emptyset\) and \(C^* \neq B\) hold (note that for \(C^* = \emptyset\) or \(C^* = B\) we can always take the empty and the full set, respectively). Given such a \(C^*\) with \(B_1, B_2\) from above we get for each \(a \in A\)

\[
K_1(f_1(a))(B_1) = M(a)(g_1^{-1}[B_1]) = M(a)(C^*) = M(a)(g_2^{-1}[B_2]) = K_2(g_2(a))(B_2);
\]

thus the event \(C^*\) ties \(K_1\) and \(K_2\) together. Loosely speaking, \(g_1^{-1}[B_1] \cap g_2^{-1}[B_2]\) can be described as the \(\sigma\)-algebra of common events, which is required to be nontrivial.

Note that without the second condition two relations \(K_1\) and \(K_2\) which are strictly probabilistic (i.e., for which the entire space is always assigned probability 1) would always be bisimilar: Put \(A := X_1 \times X_2, B := Y_1 \times Y_2\) and set for \((x_1, x_2) \in A\) as the mediating relation
$M(x_1, x_2) := K_1(x_1) \otimes K_2(x_2)$; that is, define $M$ pointwise to be the product measure of $K_1$ and $K_2$. Then the projections will make the diagram commutative. But this is way too weak, because bisimulations relate transition systems, and it does not offer particularly interesting insights when two arbitrary systems can be related. It is also clear that using products for mediation does not work for the subprobabilistic case.

We will show now that we can construct a bismulation for stochastic relations which are linked through a co-span $K_1 \leftarrow K \rightarrow K_2$. The center $K$ of this co-span should be defined over second countable metric spaces, $K_1$ and $K_2$ over Polish spaces. This situation is sometimes easy to obtain, e.g., when factoring Kripke models over Polish spaces through a suitable logic; then $K$ is defined over analytic spaces, which are separable metric. This is described in greater detail in Example 2.208.

**Proposition 2.207** Let $K_i : X_i \rightsquigarrow Y_i$ be stochastic relations over Polish spaces, and assume that $K : X \rightsquigarrow Y$ is a stochastic relation, where $X, Y$ are second countable metric spaces. Assume that we have a cospan of morphisms $m_i : K_1 \to K$, $i = 1, 2$, then there exists a stochastic relation $M$ and morphisms $m_i^+ : M \to K_i$, $i = 1, 2$ rendering this diagram commutative.

![Diagram](image)

The stochastic relation $M$ is defined over Polish spaces.

**Proof** 1. Assume $K_i = (X_i, Y_i, K_i)$ with $m_i = (f_i, g_i)$, $i = 1, 2$. Because of Proposition 2.80 we may assume that the respective $\sigma$-algebras on $X_1$ and $X_2$ are obtained from Polish topologies which render $f_1$ and $K_1$ as well as $f_2$ and $K_2$ continuous. These topologies are fixed for the proof. Put

$$A := \{ \langle x_1, x_2 \rangle \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2) \},$$
$$B := \{ \langle y_1, y_2 \rangle \in Y_1 \times Y_2 \mid g_1(y_1) = g_2(y_2) \},$$

then both $A$ and $B$ are closed, hence Polish. $\alpha_i : A \to X_i$ and $\beta_i : B \to Y_i$ are the projections, $i = 1, 2$. The diagrams

![Diagrams](image)

are commutative by assumption, thus we know that for $x_i \in X_i$

$$K(f_1(x_1)) = S(g_1(K_1(x_1))) \text{ and } K(f_2(x_2)) = S(g_2(K_2(x_2)))$$

holds. The construction implies that $(g_1 \circ \beta_1)(y_1, y_2) = (g_2 \circ \beta_2)(y_1, y_2)$ is true for $\langle y_1, y_2 \rangle \in B$, and $g_1 \circ \beta_1 : B \to Y$ is surjective.

2. Fix $\langle x_1, x_2 \rangle \in A$. Separability of the target spaces now enters: We know that the image of a surjective map under $S$ is onto again by Proposition 2.131 so that there exists $\mu_0 \in S(B)$
with \( S(g_1 \circ \beta_1)(\mu_0) = K(f_1(x_1)) \), consequently, \( S(g_i \circ \beta_i)(\mu_0) = S(g_i)(K_i(x_i)) \) \((i = 1, 2)\). But this means for \( i = 1, 2 \)

\[
\forall e_i \in g_i^{-1}[\mathcal{B}(Y)]: S(\beta_i)(\mu_0)(e_i) = K_i(x_i)(e_i).
\]

Put

\[
\Gamma(x_1, x_2) := \{ \mu \in S(B) \mid S(\beta_1)(\mu) = K_1(x_1) \text{ and } S(\beta_2)(\mu) = K_2(x_2) \},
\]

then Lemma 2.205 shows that \( \Gamma(x_1, x_2) \neq \emptyset \).

3. The set

\[
\Gamma^\mu(C) = \{ (x_1, x_2) \in A \mid \Gamma(x_1, x_2) \cap C \neq \emptyset \}
\]

is closed in \( A \) for compact \( C \subseteq S(B) \). This is shown exactly as in the second part of the proof for Proposition 2.204 from which now is inferred that there exists a measurable map \( M: A \to S(B) \) such that \( M(x_1, x_2) \in \Gamma(x_1, x_2) \) holds for every \( (x_1, x_2) \in A \). Thus \( M: A \bowtie B \)

is a stochastic relation with

\[
K_1 \circ \alpha_1 = S(\beta_1) \circ M \text{ and } K_2 \circ \alpha_2 = S(\beta_2) \circ M.
\]

Thus \( M \) with \( m_1^+: (\alpha_1, \beta_1) \) and \( m_2^+: (\alpha_2, \beta_2) \) is the desired semi-pullback. \( \dashv \)

Now we know that we may construct from a co-span of stochastic relations a span. Let us have a look at a typical situation in which such a co-span may occur.

**Example 2.208** Consider the modal logic from Example 2.12 again, and interpret the logic through stochastic relations \( K: S \bowtie S \) and \( L: T \bowtie T \) over the Polish spaces \( S \) and \( T \). The equivalence relations \( \sim_K \) and \( \sim_L \) are defined as in Example 2.101. Because we have only countably many formulas, these relations are smooth. For readability, denote the equivalence class associated with \( \sim_K \) by \([\cdot]_K\), similar for \([\cdot]_L\). Because \( \sim_K \) and \( \sim_L \) are smooth, the factor spaces \( S/K \) resp. \( T/L \) are analytic spaces when equipped with the final \( \sigma \)-algebra with respect to \( \eta_K \) resp. \( \eta_L \) by Proposition 2.104. The factor relation \( K_F: S/K \bowtie S/K \) is then the unique relation which makes this diagram commutative

\[
\begin{array}{ccc}
S & \xrightarrow{\eta_K} & S/K \\
\downarrow K & & \downarrow K_F \\
S(S) & \xrightarrow{S(\eta_K)} & S(S/K)
\end{array}
\]

This translates to \( K(s)(\eta_K^{-1}[B]) = K_F([s]_K)(B) \) for all \( B \in \mathcal{B}(S/K) \) and all \( s \in X \).

Associate with each formula \( \varphi \) its validity sets \([\varphi]_K\) resp. \([\varphi]_L\), and call \( s \in S \) logically equivalent to \( t \in T \) iff we have for each formula \( \varphi \)

\[
s \in [\varphi]_K \iff t \in [\varphi]_L
\]

Hence \( s \) and \( t \) are logically equivalent iff no formula is able to distinguish between states \( s \) and \( t \); call the stochastic relations \( K \) and \( L \) logically equivalent iff given \( s \in S \) there exists \( t \in T \) such that \( s \) and \( t \) are logically equivalent, and vice versa.

Now assume that \( K \) and \( L \) are logically equivalent, and consider

\[
\Phi := \{ ([s]_K, [t]_L) \mid s \in S \text{ and } t \in T \text{ are logically equivalent} \}.
\]
Then $\Phi$ is the graph of a bijective map; this is easy to see. Denote the map by $\Phi$ as well. Since $\Phi^{-1}[\eta_L[\varphi]] = \eta_K[\varphi]K$, and since the set $\{\eta_L[\varphi] | \varphi \text{ is a formula}\}$ generates $\mathcal{B}(T/L)$ by Proposition 2.109, $\Phi : S/K \to T/L$ is Borel measurable; interchanging the roles of $K$ and $L$ yields measurability of $\Phi^{-1}$.

Hence we have this picture for logical equivalent $K$ and $L$:

\[
\begin{array}{ccc}
   L & \downarrow & \eta_L \\
   \Phi \circ \eta_K & \downarrow & L_F \\
   K & \Downarrow & \eta_K \\
\end{array}
\]

\[\Phi\]

This example can be generalized to the case that the relations operate on two spaces rather than only on one. Let $K : X \leadsto Y$ be a transition kernel over the Polish spaces $X$ and $Y$. Then the pair $(\kappa, \lambda)$ of smooth equivalence relations $\kappa$ on $X$ and $\lambda$ on $Y$ is called a congruence for $K$ iff there exists a transition kernel $K_{\kappa, \lambda} : X/\kappa \leadsto Y/\lambda$ rendering the diagram commutative:

\[
\begin{array}{ccc}
   X & \eta_\kappa & X/\kappa \\
   \eta_\kappa & \downarrow & \kappa, \lambda \\
   Y & S(\eta_\lambda) & Y/\lambda \\
\end{array}
\]

Because $\eta_\kappa$ is an epimorphism, $K_{\kappa, \lambda}$ is uniquely determined, if it exists (for a discussion of congruences for stochastic coalgebras, see [Dob14, Section 1.6.2]). Commutativity of the diagram translates to

$K(x)(\eta_\lambda^{-1}[B]) = K_{\kappa, \lambda}([x])[B]$ for all $x \in X$ and all $B \in \mathcal{B}(Y/\lambda)$. Call in analogy to Example 2.208 the transition kernels $K_i : X_i \leadsto Y_i$ for $i = 1, 2$ logically equivalent iff there exist congruences $(\kappa_i, \lambda_i)$ for $K_1$ and $(\kappa_2, \lambda_2)$ for $K_2$ such that the factor relations $K_{\kappa_i, \lambda_i}$ are isomorphic.

In the spirit of this discussion, we obtain from Proposition 2.207

**Theorem 2.209** Logically equivalent stochastic relations over Polish spaces are bisimilar.

**Proof** 1. The proof applies Proposition 2.207; first it has to show how to satisfy the assumptions of that statement. Let $K_i : X_i \leadsto Y_i$ be stochastic relations over Polish spaces for $i = 1, 2$. We assume that $K_1$ is logically equivalent to $K_2$, hence there exist congruences $(\kappa_i, \lambda_i)$ for $K_i$ such that the associated stochastic relations $K_{\kappa_i, \lambda_i} : X_i/\kappa_i \leadsto Y_i/\lambda_i$ are isomorphic. Denote this isomorphism by $(\varphi, \psi)$, so $\varphi : X_1/\kappa_1 \to X_2/\kappa_2$ and $\psi : Y_1/\lambda_1 \to Y_2/\lambda_2$ are in particular measurable bijections, so are their inverses.

2. Let $\eta_2 := (\eta_{\kappa_2}, \eta_{\lambda_2})$ be the factor morphisms $\eta_2 : K_2 \to K_{\kappa_2, \lambda_2}$, and put $\eta_1 := (\varphi \circ \eta_{\kappa_1}, \psi \circ \eta_{\lambda_1})$, thus we obtain this co-span of morphisms

\[
\begin{array}{ccc}
   K_1 & \eta_1 & K_{\kappa_2, \lambda_2} \\
   \eta_2 & \downarrow & K_2 \\
\end{array}
\]

Because both $X_2/\kappa_2$ and $Y_2/\lambda_2$ are analytic spaces on account of $\kappa_2$ and $\lambda_2$ being smooth, see Proposition 2.104, we apply Proposition 2.207 and obtain a mediating relation $M : A \leadsto B$.
with Polish $A$ and $B$ such that the projections $\alpha_i : A \to X_i$ and $\beta_i : B \to Y_i$ are morphisms for $i = 1, 2$. Here
\[
A := \{(x_1, x_2) \mid \varphi([x_1]_{\lambda_1}) = [x_2]_{\lambda_2}\}
\]
\[
B := \{(y_1, y_2) \mid \varphi([y_1]_{\lambda_1}) = [y_2]_{\lambda_2}\}
\]

It remains to be demonstrated that the $\sigma$-algebra of common events, viz., the intersection $\beta_1^{-1}[\mathcal{B}(Y_1)] \cap \beta_2^{-1}[\mathcal{B}(Y_2)]$ is not trivial.

3. Let $U_2 \in \mathcal{B}(Y_2)$ be $\lambda_2$-invariant. Then $\eta_{\lambda_2}[U_2] \in \mathcal{B}(Y_2/\lambda_2)$, because $U_2 = \eta_{\lambda_2}^{-1}[\eta_{\lambda_2}[U_2]]$ on account of $U_2$ being $\lambda_2$-invariant. Thus $U_1 := \eta_{\lambda_1}^{-1}[\psi^{-1}[\eta_{\lambda_2}[U_2]]]$ is an $\lambda_1$-invariant Borel set in $Y_1$ with
\[
\langle y_1, y_2 \rangle \in (Y_1 \times U_2) \cap B \iff y_2 \in U_2 \text{ and } \psi([y_1]_{\lambda_1}) = [y_2]_{\lambda_2}
\]
\[\iff \langle y_1, y_2 \rangle \in (U_1 \times U_2) \cap B.
\]

One shows in exactly the same way
\[
\langle y_1, y_2 \rangle \in (U_1 \times Y_2) \cap B \iff \langle y_1, y_2 \rangle \in (U_1 \times U_2) \cap B.
\]

Consequently, $(U_1 \times U_2) \cap B$ belongs to both $\beta_1^{-1}[\mathcal{B}(Y_1)]$ and $\beta_2^{-1}[\mathcal{B}(Y_2)]$, so that this intersection is not trivial. \(\dagger\)

Call a class $\mathfrak{A}$ of spaces closed under bisimulations if the mediating relation for stochastic relations over spaces from $\mathfrak{A}$ is again defined over spaces from $\mathfrak{A}$. Then the result above shows that Polish spaces are closed under bisimulations. This generalizes a result by Desharnais, Edalat and Panangaden [Eda99, DEP02] which demonstrates — through a completely different approach — that analytic spaces are closed under bisimulations; Sánchez Terraf [ST11] has shown that general measurable spaces are not closed under bisimulations. In view of von Neumann’s Selection Theorem 2.130, it might be interesting to see whether complete measurable spaces are closed.

We have finally a look at a situation in which no semi-pullback exists. A first example in this direction was presented in [ST11, Theorem 12]. It is based on the extension of Lebesgue measure to a $\sigma$-algebra which does contain the Borel sets of $[0, 1]$ augmented by a non-measurable set, and it shows that one can construct Markov transition systems which do not have a semi-pullback. The example below extends this by showing that one does not have to consider transition systems, but that a look at the measures on which they are based suffices.

**Example 2.210** A morphism $f : (X, \mathcal{A}, \mu) \to (Y, \mathcal{B}, \nu)$ of measure spaces is an $\mathcal{A}$-$\mathcal{B}$-measurable map $f : X \to Y$ such that $\nu = M(f)(\mu)$. Since each finite measure can be viewed as a transition kernel, this is a special case of morphisms for transition kernels. If $\mathcal{B}$ is a sub-$\sigma$-algebra of $\mathcal{A}$ with $\mu$ an extension to $\nu$, then the identity is a morphisms $(X, \mathcal{A}, \mu) \to (X, \mathcal{B}, \nu)$.

Denote Lebesgue measure on $([0, 1], \mathcal{B}([0, 1]))$ by $\lambda$. Assuming the Axiom of Choice, we know that there exists $W \subseteq [0, 1]$ with $\lambda_\mathcal{B}(W) = 0$ and $\lambda^*(W) = 1$ by [Dob13, Lemma 1.7.7]. Denote by $\mathcal{A}_W := \sigma(\mathcal{B}([0, 1]) \cup \{W\}$ the smallest $\sigma$-algebra containing the Borel sets of $[0, 1]$ and $W$. Then we know from Exercise 9 that we can find for each $\alpha \in [0, 1]$ a measure $\mu_\alpha$ on $\mathcal{A}_W$ which extends $\lambda$ such that $\mu_\alpha(W) = \alpha$. 

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Hence by the remark just made, the identity yields a morphism $f_\alpha : ([0, 1], A_W, \mu_\alpha) \to ([0, 1], B([0, 1]), \lambda)$. Now let $\alpha \neq \beta$, then

$$
\begin{array}{ccc}
([0, 1], A_W, \mu_\alpha) & \xrightarrow{f_\alpha} & ([0, 1], B([0, 1]), \lambda) & \xleftarrow{f_\beta} & ([0, 1], A_W, \mu_\beta)
\end{array}
$$

is a co-span of morphisms.

We claim that this co-span does not have a semi-pullback. In fact, assume that $(P, \mathcal{P}, \rho)$ with morphisms $\pi_\alpha$ and $\pi_\beta$ is a semi-pullback, then $f_\alpha \circ \pi_\alpha = f_\beta \circ \pi_\beta$, so that $\pi_\alpha = \pi_\beta$, and $\pi_\alpha^{-1}[W] = \pi_\beta^{-1}[W] \in \mathcal{P}$. But then

$$\alpha = \mu_\alpha(W) = \rho(\pi_\alpha^{-1}[W]) = \rho(\pi_\beta^{-1}[W]) = \mu_\beta(W) = \beta.$$ 

This contradicts the assumption that $\alpha \neq \beta$. 

This example shows that the topological assumptions imposed above are indeed necessary. It assumes the Axiom of Choice, so one might ask what happens if this axiom is replaced by the Axiom of Determinacy. We know that the latter one implies that each subset of the unit interval is $\lambda$-measurable by [Dob13 Theorem 1.7.14], so $\lambda_\alpha(W) = \lambda^*(W)$ holds for each $W \subseteq [0, 1]$. Then at least the construction above does not work (on the other hand, we made use of Tihonov’s Theorem, which is known to be equivalent to the Axiom of Choice [Her06 Theorem 4.68], so there is probably no escape from the Axiom of Choice).

### 2.10.3 Case Study: Quotients for Stochastic Relations

As Monty Python used to say, “And now for something completely different!” We will deal now with quotients for stochastic relations, perceived as morphisms in the Kleisli category over the monad which is given by the subprobability functor (which is sometimes called the Giry monad). We will first have a look at surjective maps as epimorphisms in the category of sets, explaining the problem there, show that a straightforward approach gleaned from the Giry monad. We will now with quotients for stochastic relations, perceived as morphisms in the Kleisli category over the monad which is given by the subprobability functor (which is sometimes called the Giry monad).

For motivation, we start with surjective maps on a fixed set $M$, serving as a domain. Let $f : M \to X$ and $g : M \to Y$ be onto, and define the partial order $f \leq g$ iff $f = \zeta \circ g$ for some $\zeta : Y \to X$. Clearly, $\leq$ is reflexive and transitive; the equivalence relation $\sim$ defines through $f \sim g$ iff $f \leq g$ and $g \leq f$ is of interest here. Thus $f = \zeta \circ g$ and $g = \xi \circ f$ for suitable $\zeta : Y \to X$ and $\xi : X \to Y$. Because surjective maps are epimorphisms in the category of sets with maps as morphisms, we obtain $\zeta \circ \xi = \text{id}_X$ and $\xi \circ \zeta = \text{id}_Y$. Hence $\zeta$ and $\xi$ are bijections. The surjections $f$ and $g$, both with domain $M$, are equivalent iff there exists a bijection $\beta$ with $f = \beta \circ g$. This is called a quotient object for $M$. We know that the surjection $f : M \to Y$ can be factored as $f = \tilde{f} \circ \eta_{\ker(f)}$ with $\tilde{f} : [x]_{\ker(f)} \mapsto f(x)$ as the bijection. Thus for maps, the quotient objects for $M$ may be identified through the quotient maps $\eta_{\ker(f)}$, in a similar way, the quotient objects in the category of groups can be identified through normal subgroups; see [ML97 V.7] for a discussion. Thus quotients seem to be interesting.

We turn to stochastic relations. The subprobability functor on the category of measurable spaces is the functorial part of the Giry monad, and the stochastic relations are just the...
Kleisli morphism for this monad, see [Dob14, Example 1.99]. Let $K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B})$ be a stochastic relation, then Exercise [15] shows that

$$K(\mu) : B \mapsto \int_X K(x)(B) \, d\mu(x)$$

defines a $\mathcal{P}(X, \mathcal{A})$-$\mathcal{P}(Y, \mathcal{B})$-measurable map $S(X, \mathcal{A}) \to S(Y, \mathcal{B})$; $K$ is sometimes called the Kleisli map associated with the Kleisli morphism $K$ (it should not be confused with the completion of $K$ as discussed in Section 2.6.2). It is clear that $K \mapsto K$ is injective, because $K(\delta_x) = K(x)$.

It will helpful to evaluate the integral with respect to $K(\mu)$: let $g : Y \to \mathbb{R}$ be bounded and measurable, then

$$\int_Y g \, dK(\mu) = \int_X \int_Y g(y) \, dK(x)(y) \, d\mu(x). \quad (10)$$

In order to show this, assume first that $g = \chi_B$ for $B \in \mathcal{B}$, then both sides evaluate to $K(\mu)(B)$, so the representation is valid for indicator functions. Linearity of the integral yields the representation for step functions. Since we may find for general $g$ a sequence $(g_n)_{n \in \mathbb{N}}$ of step functions with $\lim_{n \to \infty} g_n(y) = g(y)$ for all $y \in Y$, and since $g$ is bounded, hence integrable with respect to all finite measures, we obtain from Lebesgue’s Dominated Convergence Theorem [2.130] that

$$\int_Y g \, dK(\mu) = \lim_{n \to \infty} \int_Y g_n \, dK(\mu)$$

$$= \lim_{n \to \infty} \int_X \int_Y g_n(y) \, dK(x)(y) \, d\mu(x)$$

$$= \int_X \lim_{n \to \infty} \int_Y g_n(y) \, dK(x)(y) \, d\mu(x)$$

$$= \int_X \int_Y g(y) \, dK(x)(y) \, d\mu(x)$$

This gives the desired representation.

The Kleisli map is related to the convolution operation defined in Example [2.171].

**Lemma 2.21** Let $K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B})$ and $L : (Y, \mathcal{B}) \rightsquigarrow (Z, \mathcal{C})$, then $L * K = L \circ K$.

**Proof** Evaluate both the left and the right hand side for $\mu \in S(X, \mathcal{A})$ and $C \in \mathcal{C}$:

$$L * K(\mu)(C) = \int_X \int_Y L(y)(C) \, dK(x)(y) \, d\mu(x)$$

$$= \int_Y L(y)(C) \, dK(\mu)(y) \quad \text{by (10)}$$

$$= L(\overline{K})(\mu)(C)$$

This implies the desired equality. $\dashv$

Associate with each measurable $f : Y \to Z$ a stochastic relation $\delta_f : Y \rightsquigarrow Z$ through $\delta_f(y)(C) := \delta_y(f^{-1}[C])$, then $\delta_f = S(f) \circ \delta$, and a direct computation shows $\delta_f * K = S(f) \circ K$. 

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In fact,

\[
(\delta_f * K)(x)(C) = \int_Y \delta_f(y)(C) \ K(x)(dy) \\
= \int_Y x_f^{-1}[C](y) \ K(x)(dy) \\
= K(x)(f^{-1}[C]) \\
= (S(f) \circ K)(x)(C).
\]

On the other hand, if \( f : W \to X \) is measurable, then

\[
(K \ * f)(w)(B) = \int_X K(x)(B) \ \delta_f(w)(dx) = (K \circ f)(w)(B).
\]

In particular, it follows that \( e_X := S(id_X) \) is the neutral element: \( K = e_X * K = K * e_X = K \).

Recall that \( K \) is an epimorphism in the Kleisli category iff \( L_1 * K = L_2 * K \) implies \( L_1 = L_2 \) for any stochastic relations \( L_1, L_2 : (Y, B) \rightsquigarrow (Z, C) \). Lemma 2.211 tells us that if the Kleisli map \( K \) is onto, then \( K \) is an epimorphism. Now let \( K : (X, A) \rightsquigarrow (Y, B) \) and \( L : (X, A) \rightsquigarrow (Z, C) \) be stochastic relations, and assume that both \( K \) and \( L \) are epis. Define as above

\[
K \leq L \iff \exists J : (Z, C) \rightsquigarrow (Y, B) : K = J \circ L.
\]

Hence we can find in case \( K \leq L \) a stochastic relation \( J \) such that \( K(x)(B) = \int_Z J(z)(B) \, dL(x)(z) \) for \( x \in X \) and \( B \in \mathcal{B} \).

We will deal for the rest of this section with Polish spaces. Fix \( X \) as a Polish spaces. For identifying the quotients with respect to Kleisli morphisms, one could be tempted to mimic the approach observed for the sets as outlined above. This is studied in the next example.

**Example 2.212** Let \( K : X \rightsquigarrow Y \) be a stochastic relation with Polish \( Y \) which is an epi. \( X/\ker(K) \) is an analytic space, since \( K : X \to S(Y) \) is a measurable map into the Polish space \( S(Y) \) by Proposition 2.200 so that \( \ker(K) \) is smooth. Define the map \( E_K : X \to S(X/\ker(K)) \) through \( E_K(x) := \delta_{[x]_{\ker(K)}} \), hence we obtain for each \( x \in X \), and each Borel set \( G \in \mathcal{B}(X/\ker(K)) \)

\[
E_K(x)(B) = \delta_{[x]_{\ker(K)}}(G) = \delta_x(\eta_{\ker(K)}^{-1}[G]) = S(\eta_{\ker(K)})(\delta_x)(G).
\]

Thus \( E_K \) is an epi as well: take \( \mu \in S(X) \) and \( G \in \mathcal{B}(X/\ker(K)) \), then

\[
\overline{E}_K(\mu)(G) = \int_X E_K(x)(G) \, d\mu(x) \\
= \int_X \delta_x(\eta_{\ker(K)}^{-1}[G]) \, d\mu(x) \\
= \mu(\eta_{\ker(K)}^{-1}[G]) \\
= S(\eta_{\ker(K)})(\mu)(G),
\]

so that \( \overline{E}_K = S(\eta_{\ker(K)}) \); since the image of a surjective map under \( S \) is surjective again by Proposition 2.131 we conclude that \( E_K \) is an epi. Now define for \( x \in X \) the map

\[
K_x([x]_{\ker(K)}) := K(x),
\]
then standard arguments show that $K^*_2$ is well defined and constitutes a stochastic relation $K^*_2 : X/\ker (K) \leadsto Y$. Moreover we obtain for $x \in X, H \in \mathcal{B}(Y)$ by the change of variables formula in Corollary 2.153:

\[
(K^*_2 * E_K)(x)(H) = \int_{X/\ker (K)} K^*_2(t)(H) \ dE_K(x)(t) \\
= \int_{X/\ker (K)} K^*_2(t)(H) \ d\mathbb{S}(\eta_{\ker (K)})(\delta_x)(t) \\
= \int_X K^*_2([w]_{\ker (K)})(H) \ d\delta_x(w) \\
= \int_X K(w)(H) \ d\delta_x(w) \\
= K(x)(H).
\]

Consequently, $K$ can be factored as $K = K^*_2 * E_K$ with the epi $E_K$. But there is no reason why in general $K^*_2$ should be invertible; for this to hold, the map $\overline{K}^*_2 : \mathbb{S}(X/\ker (K)) \rightarrow \mathbb{S}(Y)$ is required to be injective. Hence $K \approx E_K$ holds only in special cases.

This last example indicates that a characterization of quotients for the Kleisli category at least for the Giry monad cannot be derived directly by carrying over a characterization for the underlying category.

For the rest of the section we discuss the Kleisli category for the Giry monad over Polish spaces, hence we deal with stochastic relations. Let $X, Y$ and $Z$ be Polish, and fix $K : X \leadsto Y$ and $L : X \leadsto Z$ so that $K \approx L$. Hence there exists $J : Y \leadsto Z$ with inverse $H : Z \leadsto Y$ and $L = J * K$ and $K = H * L$. Because both $K$ and $L$ are epis, we obtain these simultaneous equations $H * J = e_Y$, $J * H = e_Z$. They entail $\int_Z H(z)(B) \ dJ(y)(z) = \delta_y(B)$ and $\int_Y J(y)(C) \ dH(z)(y) = \delta_z(C)$ for all $y \in Y, z \in Z$ and $B \in \mathcal{B}(Y), C \in \mathcal{B}(Z)$. Because singletons are Borel sets, these equalities imply $\int_Z H(z)\{y\} \ dJ(y)(z) = 1$ and $\int_Y J(y)\{z\} \ dH(z)(y) = 1$. Consequently, we obtain

\[
\forall y \in Y : J(y)(\{z \in Z \mid H(z)(\{y\}) = 1\}) = 1, \\
\forall z \in Z : H(z)(\{y \in Y \mid J(y)(\{z\}) = 1\}) = 1.
\]

**Proposition 2.213** There exist Borel maps $f : Y \rightarrow Z$ and $g : Z \rightarrow Y$ such that $H(f(y))(\{y\}) = 1$ and $J(g(z))(\{z\}) = 1$ for all $y \in Y, z \in Z$.

**Proof.** 1. Define $P := \{(y,z) \in Y \times Z \mid H(z)(\{y\}) = 1\}$, and $Q := \{(z,y) \in Z \times Y \mid J(y)(\{z\}) = 1\}$, then $P$ and $Q$ are Borel sets. We establish this for $P$, the argumentation for $Q$ is very similar.

2. With a view towards Proposition 2.80 we may and do assume that $H : Z \rightarrow \mathbb{S}(Y)$ is continuous. Let $(\langle y_n, z_n \rangle)_{n \in \mathbb{N}}$ be a sequence in $P$ with $\langle y_n, z_n \rangle \rightarrow \langle y, z \rangle$, hence the sequence $(H(z_n))_{n \in \mathbb{N}}$ converges weakly $H(z)$. Given $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $y_n \in V_{1/m}(y)$ for all $n \geq n_0$, where $V_{1/m}(y)$ is the closed ball of radius $1/m$ around $y$. Since $H$ is weakly continuous, we obtain $\limsup_{n \rightarrow \infty} H(z_n)(V_{1/m}(y)) \leq H(z)(V_{1/m}(y))$ from Proposition 2.14, hence $H(z)(V_{1/m}(y)) = 1$. Because $\bigcap_{m \in \mathbb{N}} V_{1/m}(y) = \{y\}$, we conclude $H(z)(\{y\}) = 1$, thus $\langle y, z \rangle \in P$. Consequently, $P$ is a closed subset of $Y \times Z$, hence a Borel set.
3. Since $P$ is closed, the cut $P_y$ at $y$ is closed as well, and we have $J(y)(P_y) = J(y)(\{z \in Z \mid H(z)(\{y\}) = 1\}) = 1$, thus we obtain $\text{supp}(J(y)) \subseteq P_y$, because the support $\text{supp}(J(y))$ is the smallest closed set $C$ with $J(y)(C) = 1$. Since $y \mapsto \text{supp}(J(y))$ is measurable (cp. Example 2.141), we obtain from Theorem 2.141 a measurable map $f : Y \to Z$ with $f(y) \in \text{supp}(J(y)) \subseteq P_y$ for all $y \in Y$, thus $H(f(y))(\{y\}) = 1$ for all $y \in Y$.

4. In the same way we obtain measurable $g : Z \to Y$ with the desired properties. \(\dashv\)

Discussing the maps $f, g$ obtained above from $H$ and $J$, we see that $H \circ f = e_Y, J \circ g = e_Z$, and we calculate through change of variables formula in Corollary 2.153 for each $z_0 \in Y$ and each $H \in \mathcal{B}(Z)$

\[
(H \ast (\mathbb{S}(f) \circ H))(z_0)(H) = \int_Z H(z)(H) (\mathbb{S}(f) \circ H)(z_0)(dz) = \int_Y H(f(y))(H) H(z_0)(dy) = \int_Y \delta_y(H) H(z_0)(dy) = H(z_0)(H).
\]

Thus $H \ast (\mathbb{S}(f) \circ H) = H$, and because $H$ is a mono, we infer that $\mathbb{S}(f) \circ H = e_Z$. Since

\[
\mathbb{S}(f) \circ H = (e_Z \circ f) \ast H = J \ast H
\]

we infer on account of $H$ being an epi that $J = e_Z \circ f$. Similarly we see that $H = e_Y \circ g$.

**Lemma 2.214** Given stochastic relations $J : Y \rightsquigarrow Z$ and $H : Z \rightsquigarrow Y$ with $H \ast J = e_Y$ and $J \ast H = e_Z$, there exist Borel isomorphisms $f : Y \to Z$ and $g : Z \to Y$ with $J = e_Z \circ f$, and $H = e_Y \circ g$.

**Proof** We infer for $y \in Y$ from

\[
\delta_y(G) = e_Y(y)(G) = (H \ast J)(y)(G) = \int_Z H(z)(G) \; dJ(y)(z) = \delta_{f(y)}(g^{-1}[G]) = \delta_y(f^{-1}[g^{-1}[G]])
\]

for all Borel sets $G \in \mathcal{B}(Y)$ that $g \circ f = id_Y$, similarly, $f \circ g = id_Z$ is inferred. Hence the Borel maps $f$ and $g$ are bijections, thus Borel isomorphisms. \(\dashv\)

This yields a characterization of the quotient equivalence relation in the Kleisli category for the Giry monad.

**Proposition 2.215** Assume the stochastic relations $K : X \rightsquigarrow Y$ and $L : X \rightsquigarrow Z$ are both epimorphisms with respect to Kleisli composition, then these conditions are equivalent

1. $K \approx L$.

2. $L = \mathbb{S}(f) \circ K$ for a Borel isomorphism $f : Y \to Z$.  

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**Proof** \[ \Rightarrow \] Because \( K \approx L \), there exists an invertible \( J : Y \twoheadrightarrow Z \) with inverse \( H : Z \twoheadrightarrow Y \) and \( L = J \ast K \). We infer from Lemma 2.214 the existence of a Borel isomorphism \( f : Y \rightarrow Z \) such that \( J = \eta_Z \circ f \). Consequently, we have for \( x \in X \) and the Borel set \( H \in \mathcal{B}(Z) \)

\[
L(x)(H) = \int_Y J(y)(H) \, dK(x)(y)
\]

\[
= \int_Y \delta_{f(y)}(H) \, dK(x)(y)
\]

\[
= K(x)(f^{-1}[H])
\]

\[
= (\mathcal{S}(f) \circ K)(x)(H)
\]

\[ \Rightarrow \] If \( L = \mathcal{S}(f) \circ K = (\eta_Z \circ f) \ast K \) for the Borel isomorphism \( f : Y \rightarrow Z \), then \( K = (\eta_Y \circ g) \ast L \) with \( g : Z \rightarrow Y \) as the inverse to \( f \). \( \dashv \)

Consequently, given the epimorphisms \( K : X \twoheadrightarrow Y \) and \( L : X \twoheadrightarrow Z \), the relation \( K \approx L \) entails their their base spaces \( Y \) and \( Z \) being Borel isomorphic, and vice versa. Hence the Borel isomorphism classes are the quotient objects for this relation.

This classification should be complemented by a characterization of epimorphic Kleisli morphisms for this monad. This seems to be an open question.

### 2.11 \( L_p \)-Spaces

We will construct now for a measure space \((X, \mathcal{A}, \mu)\) a family \( (L_p(\mu))_{1 \leq p \leq \infty} \) of Banach spaces. Some properties of these spaces are discussed now, in particular we will identify their dual spaces. The case \( p = 2 \) gives the particularly interesting space \( L_2(\mu) \), which is a Hilbert space under the inner product \( \langle f, g \rangle \mapsto \int_X f \cdot g \, d\mu \). Hilbert spaces have some properties which will turn out to be helpful, and which will be exploited for the underlying measure spaces. For example, von Neumann obtained from a representation of their continuous linear maps both the Lebesgue decomposition and the Radon-Nikodym Theorem derivative in one step! We join Rudin’s exposition [Rud74, Section 6] in giving the truly ravishing proof here. But we are jumping ahead. After investigating the basic properties of Hilbert spaces including the closest approximation property and the identification of continuous linear functions we mode to a discussion of the more general \( L_p \)-spaces and investigate the positive linear functionals on them.

Some important developments like the definition of signed measures are briefly touched, some are not. The topics which had to be omitted here include the weak topology induced by \( L_q \) on \( L_p \) for conjugate pairs \( p, q \); this would have required some investigations into convexity, which would have led into a wonderful, wondrous but far-away country.

The last section deals with disintegration as an application of both the Radon-Nikodym derivative and the Hahn Extension Theorem. It deals with the problem of decomposing a finite measure on a product into its projection onto the first component and an associated transition kernel. This corresponds to reversing a Markov transition system with a given initial distribution akin the converse of a relation in a set-oriented setting.
2.11.1 A Spoonful Hilbert Space Theory

Let $H$ be a real vector space. A map $(.,.) : H \times H \to \mathbb{R}$ is said to be an inner product iff these conditions hold for all $x, y, z \in H$ and all $\alpha, \beta \in \mathbb{R}$:

1. $(x, y) = (y, x)$, so the inner product is commutative.
2. $(\alpha \cdot x + \beta \cdot z, y) = \alpha \cdot (x, y) + \beta \cdot (z, y)$, so the inner product is linear in the first, hence also in the second component.
3. $(x, x) \geq 0$, and $(x, x) = 0$ iff $x = 0$.

We confine ourselves to real vector spaces. Hence the laws for the inner product are somewhat simplified in comparison to vector spaces over the complex numbers. There one would, e.g. postulate that $(y, x)$ is the complex conjugate for $(x, y)$.

The inner product is the natural generalization of the scalar product in Euclidean spaces

$$(\langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle) := \sum_{i=1}^{n} x_i \cdot y_i,$$

which satisfies these laws, as one verifies readily.

We fix an inner product $(\cdot, \cdot)$ on $H$. Define the norm of $x \in H$ through

$$\|x\| := \sqrt{(x, x)},$$

this is possible because $(x, x) \geq 0$. Before investigating $\|\cdot\|$ in detail, we need the Schwarz inequality as a very helpful tool. It relates the norm to the inner product of two elements.

**Lemma 2.216** $|(x, y)| \leq \|x\| \cdot \|y\|$.

**Proof** Let $a := \|x\|^2$, $b := \|y\|^2$, and $c := |(x, y)|$. Then $c = t \cdot (x, y)$ with $t \in \{-1, +1\}$. We have for each real $r$

$$0 \leq (x - r \cdot t \cdot y, x - r \cdot t \cdot y) = (x, x) - 2 \cdot r \cdot t \cdot (x, y) + r^2 \cdot (y, y),$$

thus $a - 2 \cdot r \cdot c + r^2 \cdot b \geq 0$. If $b = 0$, we must also have $c = 0$, otherwise the inequality would be false for large positive $r$. Hence the inequality is true in this case. So we may assume that $b \neq 0$. Put $r := c/b$, so that $a \geq c^2/b$, so that $a \cdot b \geq c^2$, from which the desired inequality follows. \(\square\)

Schwarz's inequality will help in establishing that a vector space with an inner product is a normed space.

**Proposition 2.217** Let $H$ be a real vector space with an inner product, then $(H, \|\cdot\|)$ is a normed space.

**Proof** It is clear from the definition of the inner product that $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$, and that $\|x\| = 0$ iff $x = 0$; the crucial point is the triangle inequality. We have

$$\|x + y\|^2 = (x + y, x + y) = \|x\|^2 + \|y\|^2 + 2 \cdot (x, y)$$

$$\leq \|x\|^2 + 2 \cdot \|x\| \cdot \|y\| + \|y\|^2 \quad \text{by Lemma 2.216}$$

$$= (\|x\| + \|y\|)^2.$$
Thus each inner product space yields a normed space, consequently it spawns a metric space through \((x, y) \mapsto \|x - y\|\). Finite dimensional vector spaces \(\mathbb{R}^n\) are Hilbert spaces under the inner product mentioned above. It produces for \(\mathbb{R}^n\) the familiar Euclidean distance
\[
\|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.
\]

We will meet square integrable functions as another class of Hilbert spaces, but before discussing them, we need some preparations.

**Corollary 2.218** The maps \(x \mapsto \|x\|\) and \(x \mapsto (x, y)\) with \(y \in H\) fixed are continuous.

**Proof** We obtain from \(\|x\| \leq \|y\| + \|x - y\|\) and \(\|y\| \leq \|x\| + \|x - y\|\) that \(\|\|x\| - \|y\|\| \leq \|x - y\|\), hence the norm is continuous. From Schwarz’s inequality we see that \(|(x, y) - (x', y)| = |(x - x', y)| \leq \|x - x'\| \cdot \|y\|\), which shows that \((\cdot, y)\) is continuous. 

From the properties of the inner product it is apparent that \(x \mapsto (x, y)\) is a continuous linear functional:

**Definition 2.219** Let \(H\) be an inner product space with norm \(\|\cdot\|\). A linear map \(L : H \to \mathbb{R}\) which is continuous in the norm topology is called a continuous linear functional on \(H\).

If \(L : H \to \mathbb{R}\) is a continuous linear functional, then its kernel
\[
\text{Kern}(L) := \{x \in H \mid L(x) = 0\}
\]
is a closed linear subspace of \(H\), i.e., is a real vector space in its own right. Say that \(x \in H\) is orthogonal to \(y \in H\) iff \((x, y) = 0\), and denote this by \(x \perp y\). This is the generalization of the familiar concept of orthogonality in Euclidean spaces, which is formulated also in terms of the inner product. Given a linear subspace \(M \subseteq H\), define the orthogonal complement \(M^\perp\) of \(M\) as
\[
M^\perp := \{y \in H \mid x \perp y \text{ for all } x \in M\}.
\]
The orthogonal complement is a linear subspace as well, and it is closed by Corollary **2.218** since \(M = \bigcap_{x \in M} \{y \in H \mid (x, y) = 0\}\). Then \(M \cap M^\perp = \{0\}\), since a vector \(z \in M \cap M^\perp\) is orthogonal to itself, hence \((z, z) = 0\), which implies \(z = 0\).

Hilbert spaces are introduced now as those linear spaces for which this metric is complete. Our goal is to show that continuous linear functionals on a Hilbert space \(H\) are given exactly through the inner product.

**Definition 2.220** A Hilbert space is a real vector space which is a complete metric space under the induced metric.

Note that we fix the metric for which the space is to be complete, noting that completeness is not a property of the underlying topological space but rather of a specific metric.

Recall that a subset \(C \subseteq H\) is called convex iff it contains with two points also the straight line between them, thus iff \(\alpha \cdot x + (1 - \alpha) \cdot y \in C\), whenever \(x, y \in C\) and \(0 \leq \alpha \leq 1\).

A key tool for our development is the observation that a closed convex subset of a Hilbert space has a unique element of smallest norm. This property is familiar from Euclidean spaces.
Visualize a compact convex set in $\mathbb{R}^3$, then this set has a unique point which is closest to the origin. The statement below is more general, because it refers to closed and convex sets.

**Proposition 2.221** Let $C \subseteq H$ be a closed and convex subset of the Hilbert space $H$. Then there exists a unique $y \in C$ such that $\|y\| = \inf_{z \in C} \|z\|$.

**Proof** 1. Put $r := \inf_{z \in C} \|z\|$, and let $x, y \in C$, hence by convexity $(x + y)/2 \in C$ as well. The parallelogram law from Exercise 31 gives

$$\|x - y\|^2 = 2 \cdot \|x\|^2 + 2 \cdot \|y\|^2 - 4 \cdot \|(x + y)/2\|^2 \leq 2 \cdot \|x\|^2 + 2 \cdot \|y\|^2 - 4 \cdot r^2.$$ 

Hence if we have two vectors $x \in C$ and $y \in C$ of minimal norm, we obtain $x = y$. Thus, if such a vector exists, it must be unique.

2. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $C$ such that $\lim_{n \to \infty} \|x_n\| = r$. At this point, we have only informations about the sequence $(\|x_n\|)_{n \in \mathbb{N}}$ of real numbers, but we can actually show that the sequence proper is a Cauchy sequence. It works like this. We obtain, again from the parallelogram law, the estimate

$$\|x_n - x_m\| \leq 2 \cdot (\|x_n\|^2 + \|x_m\|^2 - 2 \cdot r^2),$$

so that for each $\epsilon > 0$ we can find $n_0$ such that $\|x_n - x_m\| < \epsilon$ if $n, m \geq n_0$. Hence $(x_n)_{n \in \mathbb{N}}$ is actually a Cauchy sequence, and since $H$ is complete, we find some $x$ such that $\lim_{n \to \infty} x_n = x$. Clearly, $\|x\| = r$, and since $C$ is closed, we infer that $x \in C$. \(\square\)

Note how the geometric properties of an inner product space, formulated through the parallelogram law, and the metric properties of being complete cooperate.

This unique approximation property has two remarkable consequences. The first one establishes for each element $x \in H$ a unique representation as $x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in M^\perp$ for a closed linear subspace $M$ of $H$, and the second one shows that the only continuous linear maps on the Hilbert space $H$ are given by $\lambda x. (x, y)$ for $y \in H$. We need the first one for establishing the second one, so both find their place in this somewhat minimal discussion of Hilbert spaces.

**Proposition 2.222** Let $H$ be a Hilbert space, $M \subseteq H$ a closed linear subspace. Each $x \in H$ has a unique representation $x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in M^\perp$.

**Proof** 1. If such a representation exists, it must be unique. In fact, assume that $x_1 + x_2 = x = y_1 + y_2$ with $x_1, y_1 \in M$ and $x_2, y_2 \in M^\perp$, then $x_1 - y_1 = y_2 - x_2 \in M \cap M^\perp$, which implies $x_1 = y_1$ and $x_2 = y_2$ by the remark above.

2. Fix $x \in H$, we may and do assume that $x \notin M$, and define $C := \{x - y \mid y \in M\}$, then $C$ is convex, and, because $M$ is closed, it is closed as well. Thus we can find an element in $C$ which is of smallest norm, say, $x - x_1$ with $x_1 \in M$. Put $x_2 := x - x_1$, and we have to show that $x_2 \in M^\perp$, hence that $(x_2, y) = 0$ for any $y \in M$. Let $y \in M, y \neq 0$ and choose $\alpha \in \mathbb{R}$ arbitrarily (for the moment, we’ll fix it later). Then $x_2 - \alpha \cdot y = x - (x_1 + \alpha \cdot y) \in C$, thus $\|x_2 - \alpha \cdot y\|^2 \geq \|x_2\|^2$. Expanding, we obtain

$$(x_2 - \alpha \cdot y, x_2 - \alpha \cdot y) = (x_2, x_2) - 2 \cdot \alpha \cdot (x_2, y) + \alpha^2 \cdot (y, y) \geq (x_2, x_2).$$

Now put $\alpha := (x_2, y)/(y, y)$, then the above inequality yields

$$-2 \frac{(x_2, y)^2}{(y, y)} + \frac{(x_2, y)^2}{(y, y)} \geq 0,$$
which implies $-(x_2, y)^2 \geq 0$, hence $(x_2, y) = 0$. Thus $x_2 \in M^\perp$. \( \dashv \)

Thus $H$ is decomposed into $M$ and $M^\perp$ for any closed linear subspace $M$ of $H$ in the sense that each element of $H$ can be written as a sum of elements of $M$ and of $M^\perp$, and, even better, this decomposition is unique. These elements are perceived as the projections to the subspaces. In the case that we can represent $M$ better, this decomposition is unique. These elements are perceived as the projections to the

\textbf{Lemma 2.223} Let for Hilbert space $H$, $L : H \to \mathbb{R}$ be a continuous linear functional with $L \neq 0$. Then $\text{Kern}(L)^\perp$ is isomorphic to $\mathbb{R}$

\textbf{Proof} Define $\varphi(y) := L(y)$ for $y \in \text{Kern}L^\perp$. Then $\varphi(\alpha \cdot y + \beta \cdot y') = \alpha \cdot \varphi(y) + \beta \cdot \varphi(y')$ follows from the linearity of $L$. If $\varphi(y) = \varphi(y')$, then $y - y' \in \text{Kern}(L) \cap \text{Kern}(L)^\perp$, so that $y = y'$, so that $\varphi$ is one-to-one. Given $t \in \mathbb{R}$, we can find $x \in H$ with $L(x) = t$; decompose $x$ as $x_1 + x_2$ with $x_1 \in \text{Kern}(L)$ and $x_2 \in \text{Kern}(L)^\perp$, then $\varphi(x_2) = L(x - x_1) = t$. Thus $\varphi$ is onto. Hence we have found a linear and bijective map $\text{Kern}(L)^\perp \to \mathbb{R}$. \( \dashv \)

Returning to the decomposition of an element $x \in H$, we fix an arbitrary $y \in \text{Kern}(L) \setminus \{0\}$. Then we may write $x = x_1 + \alpha \cdot y$, where $\alpha \in \mathbb{R}$. This follows immediately from Lemma 2.223 and it has the consequence we are aiming at.

\textbf{Theorem 2.224} Let $H$ be a Hilbert space, $L : H \to \mathbb{R}$ be a continuous linear functional. Then there exists $y \in H$ with $L(x) = (x, y)$ for all $x \in H$.

\textbf{Proof} If $L = 0$, this is trivial. Hence we assume that $L \neq 0$. Thus we can find $z \in \text{Kern}(L)^\perp$ with $L(z) = 1$; put $y = \gamma \cdot z$ so that $L(y) = (y, y)$. Each $x \in H$ can be written as $x = x_1 + \alpha \cdot y$ with $x_1 \in \text{Kern}(L)$. Hence

$$L(x) = L(x_1 + \alpha \cdot y) = \alpha \cdot L(y) = \alpha \cdot (y, y) = (x_1 + \alpha \cdot y, y) = (x, y).$$

Thus $L = \lambda x. (x, y)$ is established. \( \dashv \)

This rather abstract view of Hilbert spaces will be put to use now to the more specific case of integrable functions.

\subsection{2.11.2 The \( L^p \)-Spaces are Banach Spaces}

We will investigate now the structure of integrable functions for a fixed $\sigma$-finite measure space $(X, \mathcal{A}, \mu)$. We will obtain a family of Banach spaces, which have some interesting properties. In the course of investigations, we will usually not distinguish between functions which differ only on a set of measure zero (because the measure will not be aware of the differences). For this, we introduced above the equivalence relation $=_{\mu}$ (“equal $\mu$ almost everywhere”) in with $f =_{\mu} g$ iff $\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$, see Section 2.2.1 on page 21. In those cases where we will need to look at the value of a function at certain points, we will make sure that we will point out the difference.

Let us see how this works in practice. Define

$$L^1(\mu) := \{ f \in \mathcal{F}(X, \mathcal{A}) \mid \int_X |f| \ d\mu < \infty \},$$

thus $f \in L^1(\mu)$ iff $f : X \to \mathbb{R}$ is measurable and has a finite $\mu$-integral.
Then this space defines a vector space and closed with respect to $| \cdot |$, hence we have immediately

**Proposition 2.225** $L_1(\mu)$ is a vector lattice. $\dashv$

Now put

$$L_1(\mu) := \{ [f] \mid f \in L_1(\mu) \},$$

then we have to explain how to perform the algebraic operations on the equivalence classes (note that we write $[f]$ rather than $|f|_\mu$, which we will do when more than one measure has to be involved). Since the set of all nullsets is a $\sigma$-ideal, these operations are easily shown to be well-defined:

$$[f] + [g] := [f + g],$$
$$[f] \cdot [g] := [f \cdot g],$$
$$\alpha \cdot [f] := [\alpha \cdot f]$$

Thus we obtain

**Proposition 2.226** $L_1(\mu)$ is a vector lattice. $\dashv$

Let $f \in L_1(\mu)$, then we define

$$\|f\|_1 := \int_X |f| \, d\mu$$

as the $L_1$-norm for $f$. Let us have a look at the properties which a decent norm should have. First, we have $\|f\|_1 \geq 0$, and $\|\alpha \cdot f\|_1 = \alpha \cdot \|f\|_1$, this is immediate. Because $|f + g| \leq |f| + |g|$, the triangle inequality holds. Finally, let $\|f\|_1 = 0$, thus $\int_X |f| \, d\mu = 0$, consequently, $f = \mu 0$, which means $f = [0]$.

This will be a basis for the definition of a whole family of linear spaces of integrable functions. Call the positive real numbers $p$ and $q$ **conjugate** iff they satisfy

$$\frac{1}{p} + \frac{1}{q} = 1$$

(for example, 2 is conjugate to itself). This may be extended to $p = 0$, so that we also consider 0 and $\infty$ as conjugate numbers, but using this pair will be made explicit.

The first step for extending the definition of $L_1$ will be **Hölder’s inequality**, which is based on this simple geometric fact:

**Lemma 2.227** Let $a, b$ be positive real numbers, $p > 0$ conjugate to $q$, then

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q},$$

equality holding iff $b = a^{p-1}$.

**Proof** The exponential function is convex, i.e., we have

$$e^{(1-\alpha) \cdot x - \alpha \cdot y} \leq (1 - \alpha) \cdot e^x + \alpha \cdot e^y$$
for all $x, y \in \mathbb{R}$ and $0 \leq \alpha \leq 1$. Because both $a > 0$ and $b > 0$, we find $r, s$ such that $a = e^{r/p}$ and $b = e^{s/q}$. Since $p$ and $q$ are conjugate, we obtain from $1/p = 1 - 1/q$

$$a \cdot b = e^{r/p + s/q} \leq \frac{e^r}{p} + \frac{e^s}{q} = \frac{a^p}{p} + \frac{b^q}{q}.$$  

This betrays one of the secrets of conjugate $p$ and $q$, viz., that they give rise to a convex combination.

We are ready to formulate and prove Hölder’s inequality, arguably one of the most frequently used inequalities in integration (as we will see as well); the proof follows the one given for [Rud74, Theorem 3.5].

**Proposition 2.228** Let $p > 0$ and $q > 0$ be conjugate, $f$ and $g$ be non-negative measurable functions on $X$. Then

$$\int_X f \cdot g \, d\mu \leq \left( \int_X f^p \, d\mu \right)^{1/p} \cdot \left( \int_X g^q \, d\mu \right)^{1/q}.$$  

**Proof** Put for simplicity

$$A := \left( \int_X f^p \, d\mu \right)^{1/p} \quad \text{and} \quad B := \left( \int_X g^q \, d\mu \right)^{1/q}.$$  

If $A = 0$, we may conclude from $f = \mu 0$ that $f \cdot g = \mu 0$, so there is nothing to prove. If $A > 0$ but $b = \infty$, the inequality is trivial, so we assume that $0 < A < \infty, 0 < B < \infty$. Put

$$F := \frac{f}{A}, \quad G := \frac{g}{B},$$  

thus we obtain

$$\int_X F^p \, d\mu = \int_X G^q \, d\mu = 1.$$  

We obtain $F(x) \cdot G(x) \leq F(x)^p/p + G(x)^q/q$ for every $x \in X$ from Lemma 2.227 hence

$$\int_X F \cdot G \, d\mu \leq \frac{1}{p} \cdot \int_X F^p \, d\mu + \frac{1}{q} \cdot \int_X G^q \, d\mu \leq \frac{1}{p} + \frac{1}{q} = 1.$$  

Multiplying both sides with $A \cdot B > 0$ now yields the desired result. ⊥

This gives Minkowski’s inequality as a consequence. Put for $f : X \to \mathbb{R}$ measurable, and for $p \geq 1$

$$\|f\|_p := \left( \int_X |f|^p \, d\mu \right)^{1/p}.$$  

**Proposition 2.229** Let $1 \leq p < \infty$ and let $f$ and $g$ be non-negative measurable functions on $X$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$
**Proof** The inequality follows for $p = 1$ from the triangle inequality for $| \cdot |$, so we may assume that $p > 1$. We may also assume that $f, g \geq 0$. Then we obtain from Hölder’s inequality with $q$ conjugate to $p$

$$
\|f + g\|_p^p = \int_X (f + g)^{p-1} \cdot f \, d\mu + \int_X (f + g)^{p-1} \cdot g \, d\mu
\leq \|f + g\|^{p/q}_p \cdot (\|f\|_p + \|g\|_p)
$$

Now assume that $\|f + g\|_p = \infty$, we may divide by the factor $\|f + g\|^{p/q}_p$, and we obtain the desired inequality from $p - p/q = p \cdot (1 - 1/q) = 1$. If, however, the left hand side is infinite, then the inequality

$$(f + g)^p \leq 2^p \cdot \max\{f^p, g^p\} \leq 2^p \cdot (f^p + g^p)$$

shows that the right hand side is infinite as well. ⊥

Given $1 \leq p < \infty$, define

$$\mathcal{L}_p(\mu) := \{f \in \mathcal{F}(X, \mathcal{A}) \mid \|f\|_p < \infty\}$$

with $\mathcal{L}_p(\mu)$ as the corresponding set of $=\mu$-equivalence classes. An immediate consequence from Minkowski’s inequality is

**Proposition 2.230** $\mathcal{L}_p(\mu)$ is a linear space over $\mathbb{R}$, and $\| \cdot \|_p$ is a pseudo-norm on it. $\mathcal{L}_p(\mu)$ is a normed space.

**Proof** It is immediate from Proposition 2.229 that $f + g \in \mathcal{L}_p(\mu)$ whenever $f, g \in \mathcal{L}_p(\mu)$, and $\mathcal{L}_p(\mu)$ is closed under scalar multiplication as well. That $\| \cdot \|_p$ is a pseudo-norm is also immediate. Because scalar multiplication and addition are compatible with forming equivalence classes, the set $\mathcal{L}_p(\mu)$ of classes is a real vector space as well. As usual, we will identify $f$ with its class, unless otherwise stated. Now $f \in \mathcal{L}_p(\mu)$ with $\|f\|_p = 0$, then $|f| = \mu$, hence $f = 0$, thus $f = 0$. So $\| \cdot \|_p$ is a norm on $\mathcal{L}_p(\mu)$. ⊥

In Section 2.2.11 the vector spaces $\mathcal{L}_\infty(\mu)$ and $L_\infty(\mu)$ are introduced, so we have now a family $(\mathcal{L}_p(\mu))_{1 \leq p \leq \infty}$ of vector spaces together with their associated spaces $(L_p(\mu))_{1 \leq p \leq \infty}$ of $\mu$-equivalence classes, which are normed spaces. They share the property of being Banach spaces.

**Proposition 2.231** $L_p(\mu)$ is a Banach space for $1 \leq p \leq \infty$.

**Proof** 1. Let us first assume that the measure is finite. We know already from Proposition 2.37 that $\mathcal{L}_\infty(\mu)$ is a Banach space, so we may assume that $p < \infty$.

Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $L_p(\mu)$, then we obtain

$$
e^p \cdot \mu(\{x \in X \mid |f_n - f_m| \geq \epsilon\}) \leq \int_X |f_n - f_m|^p \, d\mu.$$

for $\epsilon > 0$. Thus $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for convergence in measure, so we can find $f \in \mathcal{F}(X, \mathcal{A})$ such that $f_n \xrightarrow{i.m.} f$ by Proposition 2.49. Proposition 2.44 tells us that we can find a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{a.e.} f$. But we do not yet know that $f \in \mathcal{L}_p(\mu)$. November 13, 2014
We infer \( \lim_{k \to \infty} |f_{n_k} - f|^p \to 0 \) outside a set of measure zero. Thus we obtain from Fatou’s Lemma Proposition 2.149 for every \( n \in \mathbb{N} \)

\[
\int_X |f - f_n|^p \, d\mu \leq \liminf_{k \to \infty} \int_X |f_{n_k} - f_n|^p \, d\mu.
\]

Thus \( f - f_n \in L_p(\mu) \) for all \( n \in \mathbb{N} \), and from \( f = (f - f_n) + f_n \) we infer \( f \in L_p(\mu) \), since \( L_p(\mu) \) is closed under addition. We see also that \( \|f - f_n\|_p \to 0 \), as \( n \to \infty \).

2. If the measure space is \( \sigma \)-finite, we may write \( \int_X f \, d\mu = \lim_{n \to \infty} \int_{A_n} f \, d\mu \), where \( \mu(A_n) < \infty \) for an increasing sequence \( (A_n)_{n \in \mathbb{N}} \) of measurable sets with \( \bigcup_{n \in \mathbb{N}} A_n = X \). Since the restriction to each \( A_n \) yields a finite measure space, where the result holds, it is not difficult to see that completeness holds for the whole space as well. Specifically, given \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) so that for all \( n, m \geq n_0 \)

\[
\|f_n - f_m\|_p \leq \|f_n - f_m\|^{(n)}_p + \epsilon
\]

holds, with \( \|g\|^{(n)}_p := \left( \int_X |g|^p \, d\mu_n \right)^{1/p} \), and \( \mu_n : B \mapsto \mu(B \cap A_n) \) as the measure \( \mu \) localized to \( A_n \). Then \( \|f_n - f\|^{(n)}_p \to 0 \), from which we obtain \( \|f_n - f\|_p \to 0 \). Hence completeness is also valid for the \( \sigma \)-finite case. \( \dashv \)

**Example 2.232** Let \( |\cdot| \) be the counting measure on \( (\mathbb{N}, \mathcal{P}(\mathbb{N})) \), then this is a \( \sigma \)-finite measure space. Define

\[
\ell_p := L_p(|\cdot|), 1 \leq p < \infty,
\]

\[
\ell_\infty := L_\infty(|\cdot|).
\]

Then \( \ell_p \) is the set of all real sequences \( (x_n)_{n \in \mathbb{N}} \) with \( \sum_{n \in \mathbb{N}} |x_n|^p < \infty \), and \( (x_n)_{n \in \mathbb{N}} \in \ell_\infty \) iff \( \sup_{n \in \mathbb{N}} |x_n| < \infty \). Note that we do not need to pass to equivalence classes, since \( |A| = 0 \) iff \( A = \emptyset \). These spaces are well known and well studied; they will not be considered further. \( \S \)

The case \( p = 2 \) deserves particular attention, since the norm is in this case obtained from the inner product

\[
(f, g) := \int_X f \cdot g \, d\mu.
\]

In fact, linearity of the integral shows that

\[
(\alpha \cdot f + \beta \cdot g, h) = \alpha \cdot (f, h) + \beta \cdot (g, h)
\]

holds, commutativity of multiplications yields \((f, g) = (g, f)\), finally it is clear that \((f, f) \geq 0\) always holds. If we have \( f \in L_2(\mu) \) with \( f = \mu 0 \), then we know that also \((f, f) = 0\), thus \((f, f) = 0 \) iff \( f = 0 \) in \( L_2(\mu) \).

Thus we obtain from Proposition 2.231

**Corollary 2.233** \( L_p(\mu) \) is a Hilbert space with the inner product \((f, g) := \int_X f \cdot g \, d\mu. \dashv\)

This will have some interesting consequences, which we will explore in Section 2.11.3.

Before doing so, we show that the step functions belonging to \( L_p \) are dense.
Corollary 2.234 Given $1 \leq p < \infty$, the set
\[ D := \{ f \in T(X,A) \mid \mu(\{ x \in X \mid f(x) \neq 0 \}) < \infty \} \]
is dense in $L_p(\mu)$ with respect to $\| \cdot \|_p$.

Proof The proof makes use of the fact that the step functions are dense with respect to pointwise convergence: we’ll just have to filter out those functions which are in $L_p(\mu)$. Assume that $f \in L_p(\mu)$ with $f \geq 0$, then there exists by Proposition 2.239 an increasing sequence $(g_n)_{n \in \mathbb{N}}$ of step functions with $f(x) = \lim_{n \to \infty} f_n(x)$. Because $0 \leq g_n \leq f$, we conclude $g_n \in D$, and we know from Lebesgue’s Dominated Convergence Theorem 2.150 that $\| f - g_n \|_p \to 0$. Thus every non-negative element of $L_p(\mu)$ can be approximated through elements of $D$ in the $\| \cdot \|_p$-norm. In the general case, decompose $f = f^+ - f^-$ and apply the argument to both summands separately. \(\dashv\)

Because the rationals form a countable and dense subset of the reals, we take all step functions with rational coefficients, and obtain

Corollary 2.235 $L_p(\mu)$ is a separable Banach space for $1 \leq p < \infty$. \(\dashv\)

Note that we did exclude the case $p = \infty$; in fact, $L_\infty(\mu)$ is usually not a separable Banach space, as this example shows.

Example 2.236 Let $\lambda$ be Lebesgue measure on the Borel sets of the unit interval $[0,1]$. Put $f_t := \chi_{[0,t]}$ for $0 \leq t \leq 1$, then $f_t \in L_\infty(\lambda)$ for all $t$, and we have $\| f_s - f_t \|_\infty^\lambda = 1$ for $0 < s < t < 1$. Let $K_t := \{ f \in L_\infty(\lambda) \mid \| f - f_t \|_\infty^\lambda < 1/2 \}$, thus $K_s \cap K_t = \emptyset$ for $s \neq t$ (if $g \in K_s \cap K_t$, then $\| f_s - f_t \|_\infty^\lambda \leq \| g - f_t \|_\infty^\lambda + \| f_s - g \|_\infty^\lambda < 1$). On the other hand, each $K_t$ is open, so if we have a countable subset $D \subseteq L_\infty(\lambda)$, then $K_t \cap D = \emptyset$ for uncountably many $t$. Thus $D$ cannot be dense. But this means that $L_\infty(\lambda)$ is not separable. \(\bullet\)

This is the first installment on the properties of $L_p$-spaces. We will be back with a general discussion in Section 2.11.3 after having explored the Lebesgue-Radon-Nikodym Theorem as a valuable tool in general, and for our discussion.

2.11.3 The Lebesgue-Radon-Nikodym Theorem

The Hilbert space structure of the $L_2$ spaces will now be used for decomposing a measure into an absolutely and a singular part with respect to another measure, and for constructing a density. This construction requires a more general study of the relationship between two measures.

We even go a bit beyond that and define absolute continuity and singularity as a relationship of two arbitrary additive set functions. This will be specialized fairly quickly to a relationship between finite measures, but this added generality will turn out to be beneficial nevertheless, as we will see.

Definition 2.237 Let $(X,A)$ be a measurable space with two additive set functions $\rho, \zeta : A \to \mathbb{R}$.

1. $\rho$ is said to be absolutely continuous with respect to $\zeta$ ($\rho \ll \zeta$) iff $\rho(E) = 0$ for every $E \in A$ for which $\zeta(A) = 0$. 

2. $\rho$ is said to be concentrated on $A \in \mathcal{A}$ iff $\rho(E) = \rho(E \cap A)$ for all $E \in \mathcal{A}$.

3. $\rho$ and $\zeta$ are called mutually singular ($\rho \perp \zeta$) iff there exists a pair of disjoint sets $A$ and $B$ such that $\rho$ is concentrated on $A$ and $\zeta$ is concentrated on $B$.

If two additive set functions are mutually singular, they live on disjoint measurable sets in the same measurable space. These are elementary properties.

**Lemma 2.238** Let $\rho_1, \rho_2, \zeta : \mathcal{A} \to \mathbb{R}$ additive set functions, then we have for $a_1, a_2 \in \mathbb{R}$

1. If $\rho_1 \perp \zeta$, and $\rho_2 \perp \zeta$, then $a_1 \cdot \rho_1 + a_2 \cdot \rho_2 \perp \zeta$.
2. If $\rho_1 \ll \zeta$, and $\rho_2 \ll \zeta$, then $a_1 \cdot \rho_1 + a_2 \cdot \rho_2 \ll \zeta$.
3. If $\rho_1 \ll \zeta$ and $\rho_2 \perp \zeta$, then $\rho_1 \perp \rho_2$.
4. If $\rho \ll \zeta$ and $\rho \perp \zeta$, then $\rho = 0$.

**Proof**

1. For proving 1 note that we can find a measurable set $B$ and sets $A_1, A_2 \in \mathcal{A}$ with $B \cap (A_1 \cup A_2) = \emptyset$ with $\zeta(E) = \zeta(E \cap B)$ and $\rho_i(E) = \rho_i(E \cap A_i)$ for $i = 1, 2$. By additivity, we obtain $(a_1 \cdot \rho_1 + a_2 \cdot \rho_2)(E) = (a_1 \cdot \rho_1 + a_2 \cdot \rho_2)(E \cap (A_1 \cup A_2))$. Property 2 is obvious.

2. $\rho_2$ is concentrated on $A_2$, $\zeta$ is concentrated on $B$ with $A \cap B = \emptyset$, hence $\zeta(E \cap A_2) = 0$, thus $\rho_1(E \cap A_2) = 0$ for all $E \in \mathcal{A}$. Additivity implies $\rho_1(E) = \rho_1(E \cap (X \setminus A_2))$, so $\rho_1$ is concentrated on $X \setminus A_2$. This proves 3. For proving 4 note that $\rho \ll \zeta$ and $\rho \perp \zeta$ imply $\rho \perp \rho$ by property 3 which implies $\rho = 0$. $\dashv$

We specialize these relations now to finite measures on $\mathcal{A}$. Absolute continuity can be expressed in a different way, which makes the concept more transparent.

**Lemma 2.239** Given measures $\mu$ and $\nu$ on a measurable space $(X, \mathcal{A})$, these conditions are equivalent:

1. $\mu \ll \nu$.
2. For every $\epsilon > 0$ there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \epsilon$ for all measurable sets $A \in \mathcal{A}$.

**Proof**

1. $\implies$ 2 Assume that we can find $\epsilon > 0$ so that there exist sets $A_n \in \mathcal{A}$ with $
(A_n) < 2^{-n}$ but $\mu(A_n) \geq \epsilon$. Then we have $\mu(\bigcup_{k \geq n} A_k) \geq \epsilon$ for all $n \in \mathbb{N}$, consequently, by monotone convergence, also $\mu(\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k) \geq \epsilon$. On the other hand, $\nu(\bigcup_{k \geq n} A_k) \leq \sum_{k \geq n} 2^{-k} = 2^{-n+1}$ for all $n \in \mathbb{N}$, so by monotone convergence again, $\nu(\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k) = 0$. Thus $\mu \ll \nu$ does not hold.

2. $\implies$ 1 Let $\nu(A) = 0$, then $\mu(A) \leq \epsilon$ for every $\epsilon > 0$, hence $\mu \ll \nu$ is true. $\dashv$

Given this equivalence, absolute continuity could have been defined akin to the well-known $\epsilon$-$\delta$ definition of continuity for real functions. Then the name becomes a bit more descriptive.

Given two measures $\mu$ and $\nu$, one, say $\mu$, can be decomposed uniquely as a sum $\mu_a + \mu_s$ such that $\mu_a \ll \nu$ and $\mu_s \perp \nu$, additionally $\mu_s \perp \mu_a$ holds. This is stated and proved in the following theorem, which actually shows much more, viz., that there exists a density $h$ of $\mu_a$ with respect to $\nu$. This means that $\mu_a(A) = \int_A h \, d\nu$ holds for all $A \in \mathcal{A}$. What this is will be described now also in greater detail. Before entering into formalities, it is noted that
the decomposition is usually called the Lebesgue decomposition of \( \mu \) with respect to \( \nu \), and that the density \( h \) is usually called the Radon-Nikodym derivative of \( \mu_a \) with respect to \( \nu \) and denoted by \( d\mu/d\nu \).

The proof both for the existence of Lebesgue decomposition and of the Radon-Nikodym derivative is done in one step. The beautiful proof given below was proposed by von Neumann, see [Rud74, 6.9]. Here we go:

**Theorem 2.2.40** Let \( \mu \) and \( \nu \) be finite measures on \((X,A)\).

1. There exists a unique pair \( \mu_a \) and \( \mu_s \) of finite measures on \((X,A)\) such that \( \mu = \mu_a + \mu_s \) with \( \mu_a \ll \nu \), \( \mu_a \perp \nu \). In addition, \( \mu_a \perp \mu_s \) holds.

2. There exists a unique \( h \in L_1(\nu) \) such that

\[
\mu_a(A) = \int_A h \, d\nu
\]

for all \( A \in A \).

The line of attack will be as follows: we show that \( f \mapsto \int_X f \, d\mu \) is a continuous linear functional on the Hilbert space \( L_2(\mu + \nu) \). By the representation for these functionals on Hilbert spaces, we can express this functional through some function \( g \in L_2(\mu + \nu) \), hence \( \int_X f \, d\mu = \int_X f \cdot g \, d(\mu + \nu) \) (note the way the measures \( \mu \) and \( \mu + \nu \) interact by exploiting the integral with respect to \( \mu \) as a linear functional on \( L_2(\mu) \)). A closer investigation of \( g \) will then yield the sets we need for the decomposition, and permit constructing the density \( h \).

**Proof** 1. Define the finite measure \( \varphi := \mu + \nu \) on \( A \); note that \( \int_X f \, d\varphi = \int_X f \, d\mu + \int_X f \, d\nu \) holds for all measurable \( f \) for which the sum on the right hand side is defined; this follows from Levi’s Theorem 2.1.46 (for \( f \geq 0 \)) and from additivity (for general \( f \)). We show first that \( L : f \mapsto \int_X f \, d\mu \) is a continuous linear operator on \( L_2(\varphi) \). In fact,

\[
\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\varphi = \int_X |f| \cdot 1 \, d\varphi \leq \left( \int_X |f|^2 \right)^{1/2} \cdot \sqrt{\varphi(X)}
\]

by Schwarz’s inequality (Lemma 2.2.16). Thus

\[
\sup_{\|f\|_2 \leq 1} |L(f)| \leq \sqrt{\varphi(X)} < \infty.
\]

Hence \( L \) is continuous (Exercise 27), thus by Theorem 2.2.24 there exists \( g \in L_2(\mu) \) such that

\[
L(f) = \int_X f \cdot g \, d\varphi
\]

for all \( f \in L_2(\mu) \).

2. Let \( f = \chi_A \) for \( A \in A \), then we obtain \( \int_A g \, d\varphi = \mu(A) \leq \varphi(A) \) from (11). This yields \( 0 \leq g \leq 1 \) \( \varphi \)-a.e.; we can change \( g \) on a set of \( \varphi \)-measure 0 to the effect that \( 0 \leq g(x) \leq 1 \) holds for all \( x \in X \). This will not affect the representation in (11).

We know that

\[
\int_X (1-g) \cdot f \, d\mu = \int_X f \cdot g \, d\nu
\]
holds for all \( f \in L_2(\varphi) \). Put

\[
A := \{ x \in X \mid 0 \leq g(x) < 1 \}, \\
B := \{ x \in X \mid g(x) = 1 \},
\]

then \( A, B \in \mathcal{A} \), and we define for \( E \in \mathcal{A} \)

\[
\mu_a(E) := \mu(E \cap A), \\
\mu_s(E) := \mu(E \cap B).
\]

If \( f = \chi_B \), then we obtain from (12) \( \nu(B) = \int_B g \, d\nu = \int_B 0 \, d\mu = 0 \), thus \( \nu(B) = 0 \) so that \( \mu_s \perp \nu \).

3. Replace for a fixed \( E \in \mathcal{A} \) in (12) the function \( f \) by \( (1 + g + \ldots + g^n) \cdot \chi_E \), then we have

\[
\int_E (1 - g^{n+1}) \, d\mu = \int_E g \cdot (1 + g + \ldots + g^n) \, d\nu.
\]

Look at the integrand on the right hand side: it equals zero on \( B \), and increases monotonically to 1 on \( A \), hence \( \lim_{n \to \infty} \int_E (1 - g^{n+1}) \, d\mu = \mu(E \cap A) = \mu_a(E) \). This provides a bound for the left hand side for all \( n \in \mathbb{N} \). The integrand on the left hand side converges monotonically to some function \( 0 \leq h \in L_1(\nu) \) with \( \lim_{n \to \infty} \int_E g \cdot (1 + g + \ldots + g^n) \, d\nu = \int_E h \, d\nu \) by Levi’s Theorem 2.146. Hence we have

\[
\int_E h \, d\nu = \mu_a(E)
\]

for all \( E \in \mathcal{A} \), in particular \( \mu_a \ll \nu \).

4. Assume that we can find another pair \( \mu'_a \) and \( \mu'_s \) with \( \mu'_a \ll \nu \) and \( \mu'_s \perp \nu \) and \( \mu = \mu'_a + \mu'_s \). Then we have \( \mu_a - \mu'_a = \mu'_s - \mu_s \) with \( \mu_a - \mu'_a \ll \nu \) and \( \mu'_s - \mu_s \perp \nu \) by Lemma 2.238, hence \( \mu_s - \mu'_s = 0 \), again by Lemma 2.238, which implies \( \mu_a - \mu'_a = 0 \). So the decomposition is unique. From this, uniqueness of the density \( h \) in inferred. ⊥

We obtain as a consequence the well-known Radon-Nikodym Theorem:

**Theorem 2.241** Let \( \mu \) and \( \nu \) be finite measures on \( (X,A) \) with \( \mu \ll \nu \). Then there exists a unique \( h \in L_1(\mu) \) with \( \mu(A) = \int_A h \, d\nu \) for all \( A \in \mathcal{A} \). Moreover, \( f \in L_1(\mu) \) iff \( f \cdot h \in L_1(\nu) \), in this case

\[
\int_X f \, d\mu = \int_X f \cdot h \, d\nu.
\]

\( h \) is called the Radon-Nikodym derivative of \( \mu \) with respect to \( \nu \) and sometimes denoted by \( d\mu/d\nu \).

**Proof** Write \( m = \mu_a + \mu_s \), where \( \mu_a \) and \( \mu_s \) are the Lebesgue decomposition of \( \mu \) with respect to \( \nu \) by Theorem 2.240. Since \( \mu_s \perp \nu \), we find \( \mu_s = 0 \), so that \( \mu_a = \mu \). Then apply the second part of Theorem 2.240 to \( \mu \). This accounts for the first part. The second part follows from this by an approximation through step functions according to Corollary 2.234. ⊥

Note that the Radon-Nikodym Theorem gives a one-to-one correspondence between finite measures \( \mu \) such that \( \mu \ll \nu \) and the Banach space \( L_1(\nu) \).

Theorem 2.240 can be extended to complex measures; we will comment on this after the Jordan Decomposition will be established in Proposition 2.247.
Both constructions have, as one might suspect, a plethora of applications. We will not discuss the Lebesgue decomposition further but rather focus on the Radon-Nikodym Theorem and discuss two applications, viz., identifying the dual space of the $L_p$-spaces for $p < \infty$, and disintegrating a measure on a product space.

Before we do this, we have a look at integration by substitution, a technique well-known from Calculus. The multi-dimensional case has been hinted at on page 72, we deal here with the one-dimensional case. The approach displays a pretty interplay of integrating with respect to an image measure, and the Radon-Nikodym Theorem, which should not be missed.

We prepare the stage with an auxiliary statement, which is of interest of its own.

**Lemma 2.242** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \rho)$ be finite measure spaces, $\varphi : X \to Y$ be measurable and onto such that $\rho_*([\varphi^{-1}[A]]) = 0$, whenever $\mu(A) = 0$. Put $\nu := \mathcal{M}(\varphi)(\mu)$. Then there exists a measurable function $w : X \to \mathbb{R}_+$ such that

1. $f \in L_1(\rho)$ iff $(f \circ g) \cdot w \in L_1(\mu)$.
2. $\int_Y f(y) \, d\rho(y) = \int_X (f \circ \varphi)(x) \cdot g(x) \, d\mu(x)$ for all $f \in L_1(\rho)$.

**Proof** We show first that $\rho \ll \nu$, from which we obtain a derivative. This is used then through the change of variable formula for obtaining the desired result.

In fact, assume that $\nu(B) = 0$ for some $B \in \mathcal{B}$, equivalently, $\mu([\varphi^{-1}[B]]) = 0$. Thus by assumption $0 = \rho_*([\varphi^{-1}[B]]) = \rho(B)$, since $B = [\varphi^{-1}[B]]$ due to $\varphi$ being onto. Thus we find $g_1 : Y \to \mathbb{R}_+$ such that $f \in L_1(\rho)$ iff $f \cdot g_1 \in L_1(\nu)$ and $\int_Y f \, d\rho = \int_Y f \cdot g_1 \, d\nu$. Since $\nu = \mathcal{M}(\varphi)(\mu)$, we obtain from Corollary 2.243 that $\int_Y f \, d\rho = \int_X (f \circ \varphi) \cdot (g_1 \circ \varphi) \, d\mu$. Thus putting $g := g_1 \circ \varphi$, the assertion follows. \hfill \dashv

The rôle of $\nu$ as the image measure is interesting here. It just serves as a kind of facilitator, but it remains in the background. Only the measures $\rho$ and $\mu$ are acting, the image measure is used only for obtaining the Radon-Nikodym derivative, and for converting its integral to an integral with respect to its preimage through change of variables.

We specialize things now to intervals on the real line and make restrictive assumptions on $\varphi$. Then — voilà! — the well known formula on integration by substitution will result.

But first a more general consequence of Lemma 2.242 is to be presented. We will be working with Lebesgue measure on intervals of the reals. Here we assume that $\varphi : [\alpha, \beta] \to [a, b]$ is continuous with the additional property that $\lambda(A) = 0$ implies $\lambda_*([\varphi[A]]) = 0$ for all $A \subseteq \mathcal{B}([\alpha, \beta])$. This class of functions is generally known as absolutely continuous and discussed in great detail in [HS65, Section 18, Theorem (18.25)]. We obtain from Lemma 2.242

**Corollary 2.243** Let $[\alpha, \beta] \subseteq \mathbb{R}$ be a closed interval, $\varphi : [\alpha, \beta] \to [a, b]$ be a surjective and absolutely continuous function. Then there exists a Borel measurable function $w : [\alpha, \beta] \to \mathbb{R}$ such that

1. $f \in L_1([a, b], \lambda)$ iff $(f \circ \varphi) \cdot w \in L_1([\alpha, \beta], \lambda)$
2. $\int_a^b f(x) \, dx = \int_{\alpha}^{\beta} (f(\varphi(t))) \cdot w(t) \, dt$.

**Proof** The assertion follows from Lemma 2.242 by specializing $\mu$ and $\rho$ to $\lambda$. \hfill \dashv
If we restrict \( \varphi \) further, we obtain even more specific informations about the function \( w \). The following proof shows how we exploit the properties of \( \varphi \), viz., being monotone and having a continuous first derivative, through the definition of the integral as a limit of approximations on a system on subintervals which get smaller and smaller. The subdivisions in the domain are then related to the one in the range of \( \varphi \), the relationship is done through Lagrange’s Theorem which brings in the derivative. But see for yourself:

**Proposition 2.244** Assume that \( \varphi : [\alpha, \beta] \to [a, b] \) is continuous and monotone with a continuous first derivative such that \( \varphi(\alpha) = a \) and \( \varphi(\beta) = b \). Then \( f \) is Lebesgue integrable over \([a, b]\) iff \((f \circ \varphi) \cdot \varphi'\) is Lebesgue integrable over \([\alpha, \beta]\), and

\[
\int_a^b f(x) \, dx = \int_{\alpha}^{\beta} f(\varphi(z)) \cdot \varphi'(z) \, dz
\]

holds.

We follow [Fic64, Nr. 316] in his proof. The basic idea is to approximate the integral through step functions, which are obtained by subdividing the interval \([\alpha, \beta]\) into sub intervals, and to refine the subdivisions, using uniform continuity both of \( \varphi \) and \( \varphi' \) on its compact domain. So this is a fairly classical proof.

**Proof 0**. We may assume that \( f \geq 0 \), otherwise we decompose \( f = f^+ - f^- \) with \( f^+, f^- \geq 0 \). Also we assume that \( f \) is bounded by some constant \( L \), otherwise we establish the property for \( f \wedge n \) with \( n \in \mathbb{N} \), and, letting \( n \to \infty \), appeal to Levi’s Theorem 2.146. Moreover we assume that \( \varphi \) is increasing.

1. The interval \([\alpha, \beta]\) is subdivided through \( \alpha = z_0 < z_1 < \ldots < z_n = \beta \); put \( x_i := \varphi(z_i) \), then \( a = x_0 \leq x_1 \leq \ldots \leq x_n = b \), and \( \Delta z_i := z_{i+1} - z_i \), and \( \Delta x_i := x_{i+1} - x_i \). Let \( \ell := \max_{i=1,\ldots,n-1} \Delta z_i \), then if \( \ell \to 0 \), the maximal difference \( \max_{i=1,\ldots,n-1} \Delta x_i \) tends to 0 as well, because \( \varphi \) is uniformly continuous. This is so since the interval \([\alpha, \beta]\) is compact.

For approximating the integral \( \int_{\alpha}^{\beta} f(\varphi(z)) \cdot \varphi'(z) \, dz \) we select \( \zeta_i \) from each interval \([z_i, z_{i+1}]\) and write

\[
S := \sum_i f(\varphi(\zeta_i)) \cdot \varphi'(\zeta_i) \cdot \Delta z_i.
\]

Put \( \xi_i := \varphi(\zeta_i) \), hence \( x_i \leq \xi_i \leq x_{i+1} \). By Lagrange’s Formula\(^2\) there exists \( \tau_i \in [z_i, z_{i+1}] \) such that \( \Delta x_i = \varphi'(\tau_i) \cdot \Delta z_i \), so that we can write as an approximation to the integral \( \int_{\alpha}^{\beta} f(x) \, dx \) the sum

\[
s := \sum_i f(\xi_i) \cdot \Delta z_i
= \sum_i f(\xi_i) \cdot \varphi(\tau_i) \cdot \Delta z_i
= \sum_i f(\varphi(\xi_i)) \cdot \varphi'(\tau_i) \cdot \Delta z_i.
\]

If \( \ell \to 0 \), we know that \( s \to \int_{\alpha}^{\beta} f(x) \, dx \) and \( S \to \int_{\alpha}^{\beta} f(\varphi(z)) \cdot \varphi'(z) \, dz \), so that we have to get a handle at the difference \(|S - s|\). We claim that this difference tends to zero, as \( \ell \to 0 \).

\(^2\)Recall that Lagrange’s Formula says the following: Assume that \( g \) is continuous on the interval \([c, d]\) with a continuous derivative \( g' \) on the open interval \([c, d]\). Then there exists \( t \in (c, d) \) such that \( g(d) - g(c) = g'(t)(d-c) \).
Given $\epsilon > 0$, we find $\delta > 0$ such that $|\varphi'(\zeta_i) - \varphi'(\tau_i)| < \epsilon$, provided $\ell < \delta$. This is so because $\varphi'$ is continuous, hence uniformly continuous. But then we obtain by telescoping

$$|S - s| \leq \sum_i |f(\varphi(\zeta_i))| \cdot |\varphi'(\zeta_i) - \varphi'(\tau_i)| \cdot \Delta z_i < L \cdot (\beta - \alpha) \cdot \epsilon.$$ 

Thus the difference vanishes, and we obtain indeed the equality claimed above. $\dashv$

### 2.11.4 Continuous Linear Functionals on $L_p$

After all these preparations, we will investigate now continuous linear functionals on the $L_p$-spaces and show that the map $f \mapsto \int_X f \, d\mu$ plays an important rôle in identifying them. For full generality with respect to the functional concerned we introduce signed measures here and show that they may be obtained in a fairly specific way from the (unsigned) measures considered so far.

But before entering into this discussion, some general remarks. If $V$ is a real vector space with a norm $\| \cdot \|$, then a map $\Lambda : V \to \mathbb{R}$ is a linear functional on $V$ iff it is compatible with the vector space structure, i.e., iff $\Lambda(\alpha \cdot x + \beta \cdot y) = \alpha \cdot \Lambda(x) + \beta \cdot \Lambda(y)$ holds for all $x, y \in V$ and all $\alpha, \beta \in \mathbb{R}$. If $\Lambda \neq 0$, the range of $\Lambda$ is unbounded, so $\sup_{x \in V} |\Lambda(x)| = \infty$. Consequently it is difficult to assign to $\Lambda$ something like the sup-norm for characterizing continuity. It turns out, however, that we may investigate continuity through the behavior of $\Lambda$ on the unit ball of $V$, so we define

$$\|\Lambda\| := \sup_{\|x\| \leq 1} |\Lambda(x)|$$

Call $\Lambda$ bounded iff $\|\Lambda\| \leq \infty$. Then $\Lambda$ is continuous iff $\Lambda$ is bounded, see Exercise [27].

Now let $\mu$ be a finite measure with $p$ and $q$ conjugate to each other. Define for $g \in L_q(\mu)$ the linear functional

$$\Lambda_g(f) := \int_X f \cdot g \, d\mu$$

on $L_p(\mu)$, then we know from Hölder’s inequality in Proposition [2.228] that

$$\|\Lambda_g\| \leq \sup_{\|f\|_p \leq 1} \int_X |f \cdot g| \, d\mu \leq \|g\|_q$$

That was easy. But what about the converse? Given a bounded linear functional $\Lambda$ on $L_p(\mu)$, does there exists $g \in L_q(\mu)$ with $\Lambda = \Lambda_g$? It is immediate that this will not work in general, since $\Lambda_g(f) \geq 0$, provided $f \geq 0$, so we have to assume that $\Lambda$ maps positive functions to a non-negative value. Call $\Lambda$ positive iff this is the case.

Summarizing, we consider maps $\Lambda : L_p(\mu) \to \mathbb{R}$ with these properties:

**Linearity:** $\Lambda(\alpha \cdot x + \beta \cdot y) = \alpha \cdot \Lambda(x) + \beta \cdot \Lambda(y)$ holds for all $x, y \in V$ and all $\alpha, \beta \in \mathbb{R}$.

**Boundedness:** $\|\Lambda\| := \sup_{\|f\|_p = 1} |\Lambda(f)| \leq \infty$ (hence $|\Lambda(f)| \leq \|\Lambda\| \cdot \|f\|_p$ for all $f$).

**Positiveness:** $f \geq 0 \Rightarrow \Lambda(f) \geq 0$ (note that $f \geq 0$ means $f' \geq 0$-almost everywhere with respect to $\mu$ for each representative $f'$ of $f$ by our convention).
We will first work on this restricted problem, and then we will expand the answer. This will require a slight generalization: we will talk about signed measures rather than about measures.

Let’s jump right in:

**Theorem 2.245** Assume that \( \mu \) is a finite measure on \((X, \mathcal{A})\), \(1 \leq p < \infty\), and that \( \Lambda \) is a bounded positive linear functional on \(L^p(\mu)\). Then there exists a unique \( g \in L^q(\mu) \) such that

\[
\Lambda(f) = \int_X f \cdot g \, d\mu
\]

holds for each \( f \in L^p(\mu) \). In addition, \( \|\Lambda\| = \|g\|_q \).

This is our line of attack: We will first see that we obtain from \( \Lambda \) a finite measure \( \nu \) on \( \mathcal{A} \) such that \( \nu \ll \mu \). The Radon-Nikodym Theorem will then give us a density \( g := \frac{d\nu}{d\mu} \) which will turn out to be the function we are looking for. This is shown by separating the cases \( p = 1 \) and \( p > 1 \).

**Proof** 1. Define for \( A \in \mathcal{A} \)

\[
\nu(A) := \Lambda(\chi_A).
\]

Then \( A \subseteq B \) implies \( \chi_A \leq \chi_B \), hence \( \Lambda(\chi_A) \leq \Lambda(\chi_B) \). Because \( \Lambda \) is monotone, hence \( \nu \) is monotone. Since \( \Lambda \) is linear, we have \( \nu(0) = 0 \), and \( \nu \) is additive. Let \( (A_n)_{n \in \mathbb{N}} \) be an increasing sequence of measurable sets with \( A := \bigcup_{n \in \mathbb{N}} A_n \), then \( \chi_A \setminus A_n \to 0 \), and thus

\[
\nu(A) - \nu(A_n) = \|\chi_A \setminus A_n\|^p = (\Lambda(\chi_A \setminus A_n))^p \to 0,
\]

since \( \Lambda \) is continuous. Thus \( \Lambda \) is a finite measure on \( \mathcal{A} \) (note \( \nu(X) = \Lambda(1) < \infty \)). If \( \mu(A) = 0 \), we see that \( \chi_A = \mu \), thus \( \Lambda(\chi_A) = 0 \) (we are dealing with the \( = \mu \)-class of \( \chi_A \)), so that \( \nu(A) = 0 \). Thus \( \nu \ll \mu \), and the Radon-Nikodym Theorem 2.241 tells us that there exists \( g \in L^1(\mu) \) with

\[
\Lambda(\chi_A) = \nu(A) = \int_A g \, d\mu
\]

for all \( A \in \mathcal{A} \). Since the integral as well as \( \Lambda \) are linear, we obtain from this

\[
\Lambda(f) = \int_X f \cdot g \, d\mu
\]

for all step functions \( f \).

2. We have to show that \( g \in L^q(\mu) \). Consider these cases.

**Case** \( p = 1 \): We have for each \( A \in \mathcal{A} \)

\[
|\int_A g \, d\mu| \leq |\Lambda(\chi_A)| \leq \|\Lambda\| \cdot \|\chi_A\|_1 = \|\Lambda\| \cdot \mu(A)
\]

But this implies \( |g(x)| \leq \mu \cdot \|\Lambda\| \), thus \( \|g\|_\infty \leq \|\Lambda\| \).
Case $1 < p < \infty$: Let $t = \chi_{\{x \in X \mid g(x) \geq 0\}} - \chi_{\{x \in X \mid g(x) < 0\}}$, then $|g| = t \cdot g$, and $t$ is measurable, since $g$ is. Define $A_n := \{x \in X \mid |g(x)| \leq n\}$, and put $f := \chi_{A_n} \cdot |g|^{q-1} \cdot t$. Then

$$
|f|^p \cdot \chi_{A_n} = |g|^{(q-1)p} \cdot \chi_{A_n} \\
= |g|^q \cdot \chi_{A_n},
$$

$$
\chi_{A_n} \cdot (f \cdot g) = \chi_{A_n} \cdot |g|^{q-1} \cdot t \cdot g \\
= \chi_{A_n} \cdot |g|^q \cdot t,
$$

thus

$$
\int_{A_n} |g|^{q} \, d\mu = \int_{A_n} f \cdot g \, d\mu = \Lambda(f) \leq \|\Lambda\| \cdot (\int_{A_n} |g|^q \, d\mu)^{1/p}
$$

Since $1 - 1/p = 1/q$, dividing by the factor $\|\Lambda\|$ and raising the result by $q$ yields

$$
\int_{E_n} |g|^q \, d\mu \leq \|\Lambda\|^q.
$$

By Lebesgue’s Dominated Convergence Theorem \ref{lem:lebesgue-dominating-constant} we obtain that $\|g\|_q \leq \|\Lambda\|$ holds, hence $g \in L_q(\mu)$, and $\|g\|_q = \|\Lambda\|$.

The proof is completed now by the observation that $\Lambda(f) = \int_X f \cdot g \, d\mu$ holds for all step functions $f$. Since both sides of this equation represent continuous functions, and since the step functions are dense in $L_p(\mu)$ by Corollary \ref{lem:step-function-density}, the equality holds on all of $L_p(\mu)$.

This representation holds only for positive linear functions; what about the rest? It turns out that we need to extend our notion of measures to signed measures, and that a very similar statement holds for signed measures (of course we would have to explain what the integral of a signed measure is, but this will work out very smoothly). So what we will do next is to define signed measures, and to relate them to the measures with which we have worked until now. We follow essentially Halmos’ exposition \cite{Halmos:1950} §29.

**Definition 2.246** A map $\mu : A \to \mathbb{R}$ is said to be a signed measure iff $\mu$ is $\sigma$-additive, i.e., iff $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$, whenever $(A_n)_{n \in \mathbb{N}}$ is a sequence of mutually disjoint sets in $A$.

Clearly, $\mu(\emptyset) = 0$, since a signed measure $\mu$ is finite, so the distinguishing feature is the absence of monotonicity. It turns out, however, that we can partition the whole space $X$ into a positive and a negative part, that restricting $\mu$ to these parts will yield a measure each, and that $\mu$ can be written in this way as the difference of two measures.

Fix a signed measure $\mu$. Call $N \in A$ a negative set iff $\mu(A \cap N) \leq 0$ for all $A \in A$; a positive set is defined accordingly. It is immediate that the difference of two negative sets is a negative set again, and that the union of a disjoint sequence of negative sets is a negative set as well. Thus the union of a sequence of negative sets is negative again.

**Proposition 2.247** Let $\mu$ be a signed measure on $A$. Then there exists a pair $X^+$ and $X^-$ of disjoint measurable sets such that $X^+$ is a positive set, $X^-$ is a negative set. Then $\mu^+(B) := \mu(B \cap X^+)$ and $\mu^-(B) := -\mu(B \cap X^-)$ are finite measures on $A$ such that $\mu = \mu^+ - \mu^-$. The pair $\mu^+$ and $\mu^-$ is called the Jordan Decomposition of the signed measure $\mu$. 
**Proof** 1. Define

\[ \alpha := \inf\{\mu(A) \mid A \in \mathcal{A} \text{ is negative}\} > -\infty. \]

Assume that \((A_n)_{n \in \mathbb{N}}\) is a sequence of measurable sets with \(\mu(A_n) \to \alpha\), then we know that \(A := \bigcup_{n \in \mathbb{N}} A_n\) is negative again with \(\alpha = \mu(A)\). In fact, put \(B_1 := A_1, B_{n+1} := A_{n+1} \setminus B_n\), then each \(B_n\) is negative, we have

\[ \mu(A) = \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \sum_{n \in \mathbb{N}} \mu(B_n) = \lim_{n \to \infty} \mu(A_n) \]

by telescoping.

2. We claim that

\[ X^+ := X \setminus A \]

is a positive set. In fact, assume that this is not true — now this becomes the tricky part — then there exists \(E_0 \subseteq X^+\) with \(\mu(E_0) < 0\). \(E_0\) cannot be a negative set, because otherwise \(A \cup E_0\) would be a negative set with \(\mu(A \cup E_0) = \mu(A) + \mu(E_0) < \alpha\), which is contradicts the construction of \(A\). Let \(k_1\) be the smallest positive integer such that \(E_0\) contains a measurable set \(E_1\) with \(\mu(E_1) \geq 1/k_1\). Now look at \(E_0 \setminus E_1\). We have

\[ \mu(E_0 \setminus E_1) = \mu(E_0) - \mu(E_1) \leq \mu(E_0) - \mu(E_1) \leq \mu(E_0) - 1/k_1 < 0. \]

We may repeat the same consideration now for \(E_0 \setminus E_1\); let \(k_2\) be the smallest positive integer such that \(E_0 \setminus E_1\) contains a measurable set \(E_2\) with \(\mu(E_2) \geq 1/k_2\). This produces a sequence of disjoint measurable sets \((E_n)_{n \in \mathbb{N}}\) with

\[ E_{n+1} \subseteq E_0 \setminus (E_1 \cup \ldots \cup E_n), \]

and since \(\sum_{n \in \mathbb{N}} \mu(E_n)\) is finite (because \(\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{A}\), and \(\mu\) takes only finite values), we infer that \(\lim_{n \to \infty} 1/k_n = 0\).

3. Let \(F \subseteq F_0 := E_0 \setminus \bigcup_{n \in \mathbb{N}} E_n\), and assume that \(\mu(F) \geq 0\). Let \(\ell\) be the largest positive integer with \(\mu(F) \geq 1/\ell\). Since \(k_n \to 0\), as \(n \to \infty\), we find \(m \in \mathbb{N}\) with \(1/\ell \geq 1/k_m\). Since \(F \subseteq E_0 \setminus (E_1 \cup \ldots \cup E_m)\), this yields a contradiction. But \(F_0\) is disjoint from \(A\), and since

\[ \mu(F_0) = \mu(E_0) - \sum_{n \in \mathbb{N}} \mu(E_n) \leq \mu(E_0) < 0, \]

we have arrived at a contradiction. Thus \(\mu(E_0) \geq 0\).

4. Now define \(\mu^+\) and \(\mu^-\) as the traces of \(\mu\) on \(X^+\) and \(X^- := A\), resp., then the assertion follows. \(\dashv\)

It should be noted that the decomposition of \(X\) into \(X^+\) and \(X^-\) is not unique, but the decomposition of \(\mu\) into \(\mu^+\) and \(\mu^-\) is. Assume that \(X_1^+\) with \(X_1^-\) and \(X_2^+\) with \(X_2^-\) are two such decompositions. Let \(A \in \mathcal{A}\), then we have \(A \cap (X_1^+ \setminus X_2^+) \subseteq A \cap X_1^+,\) hence \(\mu(A \cap (X_1^+ \setminus X_2^+)) \geq 0;\) on the other hand, \(A \cap (X_1^+ \setminus X_2^+) \subseteq A \cap X_2^+\), thus \(\mu(A \cap (X_1^+ \setminus X_2^+)) \leq 0\), so that we have \(\mu(A \cap (X_1^+ \setminus X_2^+)) = 0\), which implies \(\mu(A \cap X_1^+) = \mu(A \cap X_2^+)\). Thus uniqueness of \(\mu^+\) and \(\mu^-\) follows.

Given a signed measure \(\mu\) with a Jordan decomposition \(\mu^+\) and \(\mu^-\), we define a (positive) measure \(|\mu| := \mu^+ + \mu^-\); \(|\mu|\) is called the total variation of \(\mu\). It is clear that \(|\mu|\) is a finite
measure on \( A \). A set \( A \in A \) is called a \( \mu \)-nullset iff \( \mu(B) = 0 \) for every \( B \in A \) with \( B \subseteq A \); hence \( A \) is a \( \mu \)-nullset iff \( A \) is a \( |\mu| \)-nullset iff \( |\mu|(A) = 0 \). In this way, we can define that a property holds \( \mu \)-everywhere also for signed measures, viz., by saying that it holds \( |\mu| \)-everywhere (in the traditional sense). Also the relation \( \mu \ll \nu \) of absolute continuity between the signed measure \( \mu \) and the positive measure \( \nu \) can be redefined as saying that each \( \nu \)-nullset is a \( \mu \)-nullset. Thus \( \mu \ll \nu \) is equivalent to \( |\mu| \ll \nu \) and to both \( \mu^+ \ll \nu \) and \( \mu^- \ll \nu \).

For the derivatives, it is easy to see that

\[
\frac{d\mu}{d\nu} = \frac{d\mu^+}{d\nu} - \frac{d\mu^-}{d\nu} \quad \text{and} \quad \frac{d|\mu|}{d\nu} = \frac{d\mu^+}{d\nu} + \frac{d\mu^-}{d\nu}
\]

hold.

We define integrability of a measurable function through \( |\mu| \) by putting

\[
L_p(|\mu|) := L_p(\mu^+) \cap L_p(\mu^-),
\]

and define \( L_p(\mu) \) again as the set of equivalence classes.

These observations provide a convenient entry point into discussing complex measures. Call \( \mu : A \to \mathbb{C} \) a (complex) measure iff \( \mu \) is \( \sigma \)-additive, i.e., iff \( \mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n) \) for each sequence \( (A_n)_{n \in \mathbb{N}} \) of mutually disjoint sets in \( A \). Then it can be easily shown that \( \mu \) can be written as \( \mu = \mu_r + i \cdot \mu_c \) with (real) signed measures \( \mu_r \) and \( \mu_c \), which in turn have a Jordan decomposition and consequently a total variation each. In this way the \( L_p \)-spaces can be defined also for complex measures and complex measurable functions; the reader is referred to \([Rud74] \) or \([HS65] \) for further information.

Returning to the main current of the discussion, we are able to state the general representation of continuous linear functionals on an \( L_p(\mu) \)-space. We need only to sketch the proof, mutatis mutandis, since the main work has already been done in the proof of Theorem 2.245.

**Theorem 2.248** Assume that \( \mu \) is a finite measure on \( (X,A) \), \( 1 \leq p < \infty \), and that \( \Lambda \) is a bounded linear functional on \( L_p(\mu) \). Then there exists a unique \( g \in L_q(\mu) \) such that

\[
\Lambda(f) = \int_X f \cdot g \, d\mu
\]

holds for each \( f \in L_p(\mu) \). In addition, \( \|\Lambda\| = \|g\|_q \).

**Proof** \( \nu(A) := \Lambda(\chi_A) \) defines a signed measure on \( A \) with \( \nu \ll \mu \). Let \( h \) be the Radon-Nikodym derivative of \( \nu \) with respect to \( \mu \), then \( h \in L_q(\mu) \) and

\[
\Lambda(f) = \int_X f \cdot h \, d\mu
\]

are shown as above. \( \dashv \)

It should be noted that Theorem 2.248 holds also for \( \sigma \)-finite measures, and that it is true for \( 1 < p < \infty \) in the case of general (positive) measures, see, e.g., \([Els09] \) § VII.3] for a discussion.

The case of continuous linear functionals for the space \( L_\infty(\mu) \) is considerably more involved. Example 2.236 indicates already that these spaces play a special rôle. Looking back at the
discussion above, we found that for $p < \infty$ the map $A \mapsto \int_A |f|^p \, d\mu$ yields a measure, and this measure was instrumental through the Radon-Nikodym Theorem for providing the factor which could be chosen to represent the linear functional. This argument, however, is not available for the case $p = \infty$, since we are not working there with a norm which is derived from an integral. It can be shown, however, that continuous linear functional have an integral representation with respect to finitely additive set functions; in fact, [HS65, Theorem 20.35] or [DS57, Theorem IV.8.16] show that the continuous linear functionals on $L_\infty(\mu)$ are in a one-to-one correspondence with all finitely additive set functions $\xi$ such that $\xi \ll \mu$. Note that this requires an extension of integration to not necessarily $\sigma$-additive set functions.

2.11.5 Disintegration

One encounters occasionally the situation the need to decompose a measure on a product of two spaces. Consider this scenario. Given a measurable space $(X, \mathcal{A})$ as an input, $(Y, \mathcal{B})$ as an output space, let $(\mu \otimes K)(B) = \int_X K(x)(D_x) \, d\mu(x)$ be the probability for $(x_1, x_2) \in B \in \mathcal{A} \otimes \mathcal{B}$ with $\mu$ as the initial distribution and $K : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ as the transition law (think of an epidemic which is set off according to $F$ and propagates according to $K$). Assume that you want to reverse the process: Given $F \in \mathcal{B}$, you put $\nu(F) := S(\pi_Y(\mu \otimes K))(F) = (\mu \otimes K)(X \times F)$, so this is the probability that your process hits an element of $F$. Can you find a stochastic relation $L : (Y, \mathcal{B}) \rightarrow (X, \mathcal{A})$ such that $(\mu \otimes K)(B) = \int_X L(x)(B^y) \, d\nu(y)$? The relation $L$ is the converse of $K$ given $\mu$. It is probably not particularly important that the measure on the product has the shape $\mu \otimes K$, so we state the problem in such a way that we are given a measure on a product of two measurable spaces, and the question is whether we can decompose it into the product of a projection onto one space, and a stochastic relation between the spaces.

This problem is of course easiest dealt with when one can deduce that the measure is the product of measures on the coordinate spaces; probabilistically, this would correspond to the distribution of two independent random variables. But sometimes one is not so lucky, and there is some hidden dependence, or one simply cannot assess the degree of independence. Then one has to live with a somewhat weaker result: in this case one can decompose the measure into a measure on one component and a transition probability. This will be made specific in the discussion to follow.

Because it will not cost substantially more attention, we will treat the question a bit more generally. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$, and $(Z, \mathcal{C})$ be measurable spaces, assume that $\mu \in \mathcal{S}(X, \mathcal{A})$, and let $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be measurable maps. Then $\mu_f := \mathcal{S}(f)(\mu)$ and $\mu_g := \mathcal{S}(g)(\mu)$ define subprobabilities on $(Y, \mathcal{B})$ resp. $(Z, \mathcal{C})$. $\mu_f$ and $\mu_g$ can be interpreted as the probability distribution of $f$ resp. $g$ under $\mu$.

We will show that we can represent the joint distribution as

$$\mu(\{x \in X \mid f(x) \in B, g(x) \in C\}) = \int_B K(y)(C) \, d\mu_f(y),$$

where $K : (Y, \mathcal{B}) \rightarrow (Z, \mathcal{C})$ is a stochastic relation. This will require $Z$ to be a Polish space with $\mathcal{C} = \mathcal{B}(Z)$.
Let us see how this corresponds to the initially stated problem. Suppose $X := Y \times Z$ with $\mathcal{A} = \mathcal{B} \otimes \mathcal{C}$, and let $f := \pi_Y$, $g := \pi_Z$, then

$$
\begin{align*}
\mu_f(B) &= \mu(B \times Z), \\
\mu_g(C) &= \mu(Y \times Z), \\
\mu(B \times C) &= \mu(\{x \in X \mid f(x) \in B, g(x) \in C\}).
\end{align*}
$$

Granted that we have established the decomposition, we can then write

$$
\mu(B \times C) = \int_B K(y)(C) \, d\mu_f(y);
$$

thus we have decomposed the probability on the product into a probability on the first component, and, conditioned on the value the first component may take, a probability on the second factor.

**Definition 2.249** Using the notation from above, $K$ is called a regular conditional distribution of $g$ given $f$ iff

$$
\mu(\{x \in X \mid f(x) \in B, g(x) \in C\}) = \int_B K(y)(C) \, \mu_f(dy)
$$

holds for each $B \in \mathcal{B}, C \in \mathcal{C}$, where $K : (Y, \mathcal{B}) \rightarrow (C, \mathcal{C})$ is a stochastic relation on $(X, \mathcal{A})$ and $(Z, \mathcal{C})$. If only $y \mapsto K(y)(C)$ is $\mathcal{B}$-measurable for all $C \in \mathcal{C}$, then it will be called a conditional distribution of $g$ given $f$.

The existence of regular conditional distribution will be established, provided $Z$ is Polish with $\mathcal{C} = \mathcal{B}(Z)$. This will be accomplished in several steps: first the existence of a conditional distribution will be shown using the well known Radon-Nikodym Theorem. The latter construction will then be scrutinized. It will turn out that there exists a set of measure zero outside of which the conditional distribution behaves like a regular one, but at first sight only on an algebra of sets, not on the entire $\sigma$-algebra. But don’t worry, the second step will apply a classical extension argument and yield a regular conditional distribution on the Borel sets, just as we want it. The proofs are actually a kind of a round trip through the first principles of measure theory, where the Radon-Nikodym Theorem together with the classical Hahn Extension Theorem are the main vehicles. It displays also some nice and helpful proof techniques.

We fix $(X, \mathcal{A})$, $(Y, \mathcal{B})$, and $(Z, \mathcal{C})$ as measurable spaces, assume that $\mu \in \mathcal{S}(X, \mathcal{A})$, and take $f : X \rightarrow Y$ and $g : X \rightarrow Z$ to be measurable maps. The measures $\mu_f := \mathcal{S}(f)(\mu)$ and $\mu_g := \mathcal{S}(g)(\mu)$ are defined as above as the distribution of $f$ resp. $g$ under $\mu$.

The existence of a conditional distribution of $g$ given $f$ is established first, and it is shown that it is essentially unique.

**Lemma 2.250** Using the notation from above, then

1. there exists a conditional distribution $K_0$ of $g$ given $f$,

2. if there is another conditional distribution $K'_0$ of $g$ given $f$, then there exists for any $C \in \mathcal{C}$ a set $N_C \in \mathcal{B}$ with $\mu_f(N_C) = 0$ such that $K_0(y)(C) = K'_0(C)$ for all $y \notin C$. 
Proof 1. Fix $C \in \mathcal{C}$, then
\[
\pi_C(B) := \mu(f^{-1}[B] \cap g^{-1}[C])
\]
defines a subprobability measure $\pi_C$ on $\mathcal{B}$ which is absolutely continuous with respect to $\mu_g$, because $\mu_g(B) = 0$ implies $\pi_C(B) = 0$. The Radon-Nikodym Theorem 2.241 now gives a density $h_C \in \mathcal{F}(Y, \mathcal{B})$ with
\[
\pi_C(B) = \int_B h_C \, d\mu
\]
for all $B \in \mathcal{B}$. Setting $K_0(y)(C) := h_C(y)$ yields the desired conditional distribution.

2. Suppose $K'_0$ is another conditional distribution of $g$ given $f$, then we have
\[
\forall B \in \mathcal{B} : \int_B K_0(y)(C) \, d\mu(y) = \int_B K_0(y)(C) \, d\mu(y),
\]
for all $C \in \mathcal{C}$, which implies that the set on which $K_0(\cdot)(C)$ disagrees with $K'_0(\cdot)(C)$ is $\mu_f$-null.

Essential uniqueness may strengthened if the $\sigma$-algebra $\mathcal{C}$ is countably generated, and if the conditional distribution is regular.

Lemma 2.251 Assume that $K$ and $K'$ are regular conditional distributions of $g$ given $f$, and that $\mathcal{C}$ has a countable generator. Then there exists a set $N \in \mathcal{B}$ with $\mu_f(N) = 0$ such that $K(y)(C) = K'(y)(C)$ for all $C \in \mathcal{C}$ and all $y \notin N$.

Proof If $\mathcal{C}_0$ is a countable generator of $\mathcal{C}$, then
\[
\mathcal{C}_f := \{ \bigcap \mathcal{E} \mid \mathcal{E} \subseteq \mathcal{C}_0 \text{ is finite} \}
\]
is a countable generator of $\mathcal{C}$ well, and $\mathcal{C}_f$ is closed under finite intersections; note that $Z \in \mathcal{C}_f$. Construct for $D \in \mathcal{C}_f$ the set $N_D \in \mathcal{B}$ outside of which $K(\cdot)(D)$ and $K'(\cdot)(D)$ coincide, and define
\[
N := \bigcup_{D \in \mathcal{C}_f} N_D \in \mathcal{B}.
\]
Evidently, $\mu_f(N) = 0$. We claim that $K(y)(C) = K'(y)(C)$ holds for all $C \in \mathcal{C}$, whenever $y \notin N$. In fact, fix $y \notin N$, and let
\[
\mathcal{C}_1 := \{ C \in \mathcal{C} \mid K(y)(C) = K'(y)(C) \},
\]
then $\mathcal{C}_1$ contains $\mathcal{C}_f$ by construction, and is a $\pi$-$\lambda$-system. This is so since it is closed under complements and countable disjoint unions. Thus $\mathcal{C} = \sigma(\mathcal{C}_f) \subseteq \mathcal{C}_1$, by the $\pi$-$\lambda$-Theorem 2.21 and we are done. \(\square\)

We will show now that a regular conditional distribution of $g$ given $f$ exists. This will be done through several steps, given the construction of a conditional distribution $K_0$:

1. A set $N_a \in \mathcal{B}$ is constructed with $\mu_f(N_a) = 0$ such that $K_0(y)$ is additive on a countable generator $\mathcal{C}_2$ for $\mathcal{C}$.
2. We construct a set $N_z \in \mathcal{B}$ with $\mu_f(N_z) = 0$ such that $K_0(y)(Z) \leq 1$ for $y \notin N_z$. 
For each element $G$ of $C_z$ we will find a set $N_G \in \mathcal{B}$ with $\mu_f(N_G) = 0$ such that $K_0(y)(G)$ can be approximated from inside through compact sets, whenever $y \notin N_G$.

Then we will combine all these sets of $\mu_f$-measure zero to produce a set $N \in \mathcal{B}$ with $\mu_f(N) = 0$ outside of which $K_0(y)$ is a premeasure on the generator $C_z$, hence can be extended to a measure on all of $C$.

Well, this looks like a full program, so let us get on with it.

**Theorem 2.252** Given measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a Polish space $Z$, a subprobability $\mu \in \mathcal{S}(X, \mathcal{A})$, and measurable maps $f : X \to Y$, $g : X \to Z$, there exists a regular conditional distribution $K$ of $g$ given $f$. $K$ is uniquely determined up to a set of $\mu_f$-measure zero.

**Proof**

0. Since $Z$ is a Polish space, its topology has a countable base. We infer from Lemma 2.51 that $\mathcal{B}(Z)$ has a countable generator $\mathcal{C}$. Then the Boolean algebra $\mathcal{C}_1$ generated by $\mathcal{C}$ is also a countable generator of $\mathcal{B}(Z)$.

1. Given $C_n \in \mathcal{C}_1$, we find by Proposition 2.202 a sequence $(E_{n,k})_{k \in \mathbb{N}}$ of compact sets in $Z$ with

$$E_{n,1} \subseteq E_{n,2} \subseteq E_{n,3} \ldots \subseteq C_n$$

such that

$$\mu_g(C_n) = \sup_{k \in \mathbb{N}} \mu_g(E_{n,k}).$$

Then the Boolean algebra $\mathcal{C}_2$ generated by $\mathcal{C} \cup \{E_{n,k} \mid n, k \in \mathbb{N}\}$ is also a countable generator of $\mathcal{B}(Z)$.

2. From the construction of the conditional distribution of $g$ given $f$ we infer that for disjoint $C_1, C_2 \in \mathcal{C}_2$

$$\int_Y K_0(y)(C_1 \cup C_2) \, d\mu_f(y) = \mu(\{x \in X \mid f(x) \in B, g(x) \in C_1 \cup C_2\})$$

$$= \mu(\{x \in X \mid f(x) \in B, g(x) \in C_1\}) + \mu(\{x \in X \mid f(x) \in B, g(x) \in C_2\})$$

$$= \int_Y K_0(y)(C_1) \, d\mu_f(y) + \int_Y K_0(y)(C_2) \, d\mu_f(y).$$

Thus there exists $N_{C_1,C_2} \in \mathcal{B}$ with $\mu_f(N_{C_1,C_2}) = 0$ such that

$$K_0(y)(C_1 \cup C_2) = K_0(y)(C_1) + K_0(y)(C_2)$$

for $y \notin N_{C_1,C_2}$. Because $\mathcal{C}_2$ is countable, we may deduce (by taking the union of $N_{C_1,C_2}$ over all pairs $C_1, C_2$) that there exists a set $N_a \in \mathcal{B}$ such that $K_0$ is additive outside $N_a$, and $\mu_f(N_a) = 0$. This accounts for part 3 in the plan above.

3. By the previous arguments it is easy to construct a set $N_z \in \mathcal{B}$ with $\mu_f(N_z) = 0$ such that $K_0(y)(Z) \leq 1$ for $y \notin N_z$ (part 2).
4. Because
\[ \int_Y K_0(y)(C_n) \, d\mu_f(y) = \mu(f^{-1}[Y] \cap g^{-1}[C_n]) \]
\[ = \mu_g(C_n) \]
\[ = \sup_{k \in \mathbb{N}} \mu_g(E_{n,k}) \]
\[ = \sup_{k \in \mathbb{N}} \int_Y K_0(y)(E_{n,k}) \, \mu_f(dy) \]
\[ = \int_Y \sup_{k \in \mathbb{N}} K_0(y)(E_{n,k}) \, d\mu_f(y) \]
Levi’s Theorem \[2.146\]
we find for each \( n \in \mathbb{N} \) a set \( N_n \in \mathcal{B} \) with
\[ \forall y \notin N_n : K_0(y)(C_n) = \sup_{k \in \mathbb{N}} K_0(y)(E_{n,k}) \]
and \( \mu_f(N_n) = 0 \). This accounts for part \[\circled{2}\].

5. Now we may begin to work on part \[\circled{3}\]. Put
\[ N := N_a \cup N_z \cup \bigcup_{n \in \mathbb{N}} N_n, \]
then \( N \in \mathcal{B} \) with \( \mu_f(N) = 0 \). We claim that \( K_0(y) \) is a premeasure on \( C_z \) for each \( y \notin N \).
It is clear that \( K_0(y) \) is additive on \( C_z \), hence monotone, so merely \( \sigma \)-additivity has to be demonstrated: let \( (D_\ell)_{\ell \in \mathbb{N}} \) be a sequence in \( C_z \) that is monotonically decreasing with
\[ \eta := \inf_{\ell \in \mathbb{N}} K_0(y)(D_\ell) > 0, \]
than we have to show that
\[ \bigcap_{\ell \in \mathbb{N}} D_\ell \neq \emptyset. \]
We approximate the sets \( D_\ell \) now by compact sets, so we assume that \( D_\ell = C_{n_\ell} \) for some \( n_\ell \) (otherwise the sets are compact themselves). By construction we find for each \( \ell \in \mathbb{N} \) a compact set \( E_{n_\ell,k_\ell} \subseteq C_{n_\ell} \) with
\[ K_0(y)(C_{n_\ell} \setminus E_{n_\ell,k_\ell}) < \eta \cdot 2^{\ell+1}, \]
then
\[ E_r := \bigcap_{i=\ell}^r E_{n_i,k_i} \subseteq C_{n_r} = D_r \]
defines a decreasing sequence of compact sets with
\[ K_0(y)(E_r) \geq K_0(y)(C_{n_r}) - \sum_{i=\ell}^r K_0(y)(E_{n_i,k_i}) > \eta/2, \]
thus \( E_r \neq \emptyset \). Since \( E_r \) is compact and decreasing, we know that the sequence has a nonempty intersection (otherwise one of the \( E_r \) would already be empty). We may infer
\[ \bigcap_{\ell \in \mathbb{N}} D_\ell \supseteq \bigcap_{r \in \mathbb{N}} E_r \neq \emptyset. \]
6. The classic Hahn Extension Theorem [Dob13, Theorem 1.113] now tells us that there exists a unique extension of $K_0(y)$ from $\mathcal{C}_z$ to a measure $K(y)$ on $\sigma(\mathcal{C}_z) = \mathcal{B}(Z)$, whenever $y \notin N$. If, however, $y \in N$, then we define $K(y) := \nu$, where $\nu \in \mathcal{S}(Z)$ is arbitrary. Because

$$\int_{\mathcal{B}} K(y)(C) \, d\mu_f(y) = \int_{\mathcal{B}} K_0(y)(C) \, d\mu_f(y) = \mu(\{x \in X \mid f(x) \in B, g(x) \in C\})$$

holds for $C \in \mathcal{C}_z$, the $\pi$-$\lambda$-Theorem 2.4 asserts that this equality is valid for all $C \in \mathcal{B}(Z)$ as well.

Measurability of $y \mapsto K(y)(C)$ needs to be shown, and then we are done. We do this by the principle of good sets: put

$$\mathcal{E} := \{C \in \mathcal{B}(Z) \mid y \mapsto K(y)(C) \text{ is } \mathcal{B} \text{- measurable}\}.$$ 

Then $\mathcal{E}$ is a $\sigma$-algebra, and $\mathcal{E}$ contains the generator $\mathcal{C}_z$ by construction, thus $\mathcal{E} = \mathcal{B}(Z)$.

The scenario in which the space $X = Y \times Z$ with a measurable space $(Y, \mathcal{B})$ and a Polish space $Z$ with $\mathcal{A} = \mathcal{B} \otimes \mathcal{B}(Z)$ with $f$ and $g$ as projections deserves particular attention. In this case we decompose a measure on $A$ into its projection onto $Z$ and a conditional distribution for the projection onto $Z$ given the projection onto $Y$. This is sometimes called the disintegration of a measure $\mu \in \mathcal{S}(Y \times Z)$.

We state the corresponding proposition explicitly, since one needs it usually in this specialized form.

**Proposition 2.253** Given a measurable space $(Y, \mathcal{B})$ and a Polish space $Z$, there exists for every subprobability $\mu \in \mathcal{S}(Y \times Z, \mathcal{B} \otimes \mathcal{B}(Z))$ a regular conditional distribution of $\pi_Z$ given $\pi_Y$, that is, a stochastic relation $K : (Y, \mathcal{B}) \to (Z, \mathcal{B}(Z))$ such that

$$\mu(E) = \int_Y K(y)(E_y) \, d\mathcal{S}(\pi_Y)(\mu)(y)$$

for all $E \in \mathcal{B} \otimes \mathcal{B}(Z)$. ⊣

The construction is done with a Polish as one of the factors. The proof shows that it is indeed tightness which saves the days, since otherwise it would be difficult to make sure that the condition distribution constructed above is $\sigma$-additive. In fact, examples show that this assumption is in fact necessary: [Kel72] constructs a product measure on spaces which fail to be Polish, for which no disintegration exists.

### 2.12 Bibliographic Notes

Most topics of this chapter are fairly standard, hence there are plenty of sources to mention. One of my favourite texts is the rich compendium written by [Bog07]. The discussion on Souslin’s operation $\mathfrak{A}(\mathcal{A})$ on a $\sigma$-algebra $\mathcal{A}$ is heavily influenced by Srivastava’s representation [Sri98] of this topic, but see also [Par67, Arv76, Kel72]. The measure extension is taken from [Lub74], following a suggestion by S. M. Srivastava; the extension of a stochastic relation is from [Dob12]. The approach to integration centering around B. Levi’s Theorem is taken mostly from the elegant representation by Doob [Doo94, Chapter VI], see
Exercise 1 Assume that $A = \sigma(A_0)$. Show that the weak-$\star$-$\sigma$-algebra $\mathcal{P}(A)$ on $M(X, A)$ is the initial $\sigma$-algebra with respect to $\{ev_A \mid A \in A_0\}$.

Show also that both $S(X, A)$ and $\mathbb{P}(X, A)$ are measurable subsets of $M(X, A)$.

Exercise 2 Let $(X, \tau)$ be a topological, and $(Y, d)$ a metric space. Each continuous function $X \to Y$ is also Baire measurable.

Exercise 3 Let $(X, d)$ be a separable metric space, $\mu \in M(X, B(X))$. Show that $x \in \text{supp}(\mu)$ iff $\mu(U) > 0$ for each open neighborhood $U$ of $x$.

Exercise 4 Let $(X, A, \mu)$ be a finite measure space. Show that norm convergence in $L_\infty(X, A, \mu)$ implies convergence almost everywhere ($f_n \overset{a.e.}{\to} f$, provided $\|f_n - f\|_\infty \to 0$). Give an example showing that the converse is false.

Exercise 5 If $A$ is a $\sigma$-algebra on $X$ and $B \subseteq X$ with $A \not\subseteq A$, then

$$\{(A_1 \cap B) \cup (A_2 \cap (X \setminus B)) \mid A_1, A_2 \in A\}$$

is the smallest $\sigma$-algebra $\sigma(A \cup \{B\})$ on $X$ containing $A$ and $B$. If $\tau$ is a topology on $X$ with $H \not\subseteq \tau$, then

$$\{G_1 \cup (G_2 \cap H) \mid G_1, G_2 \in \tau\}$$

is the smallest topology $\tau_H$ on $X$ containing $\tau$ and $H$. Show that $\mathcal{B}(\tau_H) = \sigma(A \cup \{H\})$.

Exercise 6 Let $(X, A, \mu)$ be a finite measure space, $B \not\subseteq A$, and $\beta : = \alpha \cdot \mu_+(B) + (1 - \alpha) \cdot \mu^-(B)$ with $0 \leq \alpha \leq 1$. Then there exists a measure $\nu$ on $\sigma(A \cup \{B\})$ which extends $\mu$ such that $\nu(B) = \beta$. (Hint: Exercise 5).

Exercise 7 Given the measurable space $(X, A)$ and $f \in \mathcal{F}(X, A)$ with $f \geq 0$. Show that there exists a decreasing sequence $(f_n)_{n \in \mathbb{N}}$ of step functions $f_n \in \mathcal{F}(X, A)$ with

$$f(x) = \inf_{n \in \mathbb{N}} f_n(x)$$

for all $x \in X$.

Exercise 8 Let $f_i : X_i \to Y_i$ be $\mathcal{A}_i$-$\mathcal{B}_i$-measurable maps for $i \in I$. Show that

$$f : \left\{ \prod_{i \in I} X_i \to \prod_{i \in I} Y_i \right\}$$

$$\left\langle (x_i)_{i \in I} \right\rangle \mapsto (f_i(x_i))_{i \in I}$$
is $\bigotimes_{i \in I} A_i \times \bigotimes_{i \in I} B_i$-measurable. Conclude that the kernel of $f$

$$\ker(f) := \{ \langle x, x' \rangle \mid f(x) = f(x') \}$$

is a measurable subset of $Y \times Y$, whenever $f : (X, A) \to (Y, B)$ is measurable, and $B$ is separable.

**Exercise 9** Let $f : X \to Y$ be $A$-$B$ measurable, and assume that $B$ is separable. Show that the graph of $f$

$$\text{graph}(f) := \{ \langle x, f(x) \rangle \mid x \in X \}$$

is a measurable subset of $A \otimes B$.

**Exercise 10** Let $\chi_A$ be the indicator function of set $A$. Show that

1. $A \subseteq B$ iff $\chi_A \leq \chi_B$,
2. $\chi_{\bigcup_{n \in \mathbb{N}} A_n} = \sup_{n \in \mathbb{N}} \chi_{A_n}$ and $\chi_{\bigcap_{n \in \mathbb{N}} A_n} = \inf_{n \in \mathbb{N}} \chi_{A_n}$
3. $\chi_{A \Delta B} = |\chi_A - \chi_B| = \chi_A + \chi_B \pmod{2}$. Conclude that the power set $(\mathcal{P}(X), \Delta)$ is a commutative group with $A \Delta A = \emptyset$.
4. $(\bigcup_{n \in \mathbb{N}} A_n) \Delta (\bigcup_{n \in \mathbb{N}} B_n) \subseteq \bigcup_{n \in \mathbb{N}} (A_n \Delta B_n)$

**Exercise 11** Let $(X, \mathcal{A}, \mu)$ be a finite measure space, put $d(A, B) := \mu(A \Delta B)$ for $A, B \in \mathcal{A}$. Show that $(\mathcal{A}, d)$ is a complete pseudo metric space.

**Exercise 12** Let $(X, d)$ be a metric space. Show that

$$D(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

defines a metric on $X$ such that the metric spaces are homeomorphic as topological spaces. The $(X, d)$ is complete iff $(X, D)$ is.

**Exercise 13** (This Exercise draws heavily on Exercises 5 and 6). Let $X := [0, 1]$ with $\lambda$ as the Lebesgue measure on the Borel set of $X$. There exists a set $B \subseteq X$ with $\lambda^*(B) = 0$ and $\lambda^*(X) = 1$ [Dob13 Lemma 1.141], so that $B \not\in \mathcal{B}(X)$.

1. Show that $(X, \tau_B)$ is a Hausdorff space with a countable base, where $\tau_B$ is the smallest topology containing the interval topology on $[0, 1]$ and $B$ (see Exercise 5).
2. Extend $\lambda$ to a measure $\mu$ with $\alpha = 1/2$ in Exercise 6.
3. Show that $\inf \{ \mu(G) \mid G \supseteq X \setminus B \text{ and } G \text{ is } \tau_B\text{-open} \} = 1$, but $\mu(X \setminus B) = 1/2$. Thus $\mu$ is not regular (since $(X, \tau_B)$ is not a metric space).

**Exercise 14** Prove Proposition 2.65.

**Exercise 15** Let $K : (X, \mathcal{A}) \leadsto (Y, \mathcal{B})$ be a transition kernel.

1. Assume that $f \in \mathcal{F}_+(Y, \mathcal{B})$ is integrable with respect to $K(x)$ for all $x \in X$. Show that

$$K(f)(x) := \int_X f \, dK(x)$$

defines a measurable function $K(f) : X \to \mathbb{R}_+$. 

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2. Assume that \( x \mapsto K(x)(Y) \) is bounded. Define for \( B \in \mathcal{B} \)
\[
\overline{K}(\mu)(B) := \int_X K(x)(B) \, d\mu(x).
\]
Show that \( \overline{K} : \mathcal{S}(X, \mathcal{A}) \to \mathcal{S}(Y, \mathcal{B}) \) is \( \mathcal{P}(X, \mathcal{A})\)-\( \mathcal{P}(Y, \mathcal{B}) \)-measurable (see [Dob14, Example 1.99]).

**Exercise 16** Let \( \mu \in \mathcal{S}(X, \mathcal{A}) \) be a subprobability measure on \( (X, \mathcal{A}) \), and let \( K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B}) \) be a stochastic relation. Assume that \( f : X \times Y \to \mathbb{R} \) is bounded and measurable. Show that
\[
\int_{X \times Y} f \, d\mu \otimes K = \int_X \left( \int_Y f_x \, dK(x) \right) \, d\mu(x)
\]
(\( \mu \otimes K \) is defined in Example 2.169 on page 80).

**Exercise 17** Let \( (X, \mathcal{A}) \) and \( (Y, \mathcal{B}) \) be measurable spaces and \( D \in \mathcal{A} \otimes \mathcal{B} \). Show that the map
\[
\begin{align*}
\left( \mathcal{M}(Y, \mathcal{B}) \times X \right) \times X & \to \mathbb{R} \\
(\nu, x) & \mapsto \nu(D_x)
\end{align*}
\]
is \( \mathcal{P}(Y, \mathcal{B}) \otimes \mathcal{A} \otimes \mathcal{B}(\mathbb{R}) \)-measurable (the weak-*-\( \sigma \)-algebra \( \mathcal{P}(Y, \mathcal{B}) \) has been defined in Section 2.11).

**Exercise 18** Show that the category of analytic spaces with measurable maps is not closed under taking pushouts. **Hint:** Show that the pushout of \( X/\alpha_1 \) and \( X/\alpha_2 \) is \( X/(\alpha_1 \cup \alpha_2) \) for equivalence relations \( \alpha_1 \) and \( \alpha_2 \) on a Polish space \( X \). Then use Proposition 2.104 and Example 2.112.

**Exercise 19** Let \( S := \{1, \ldots, n\} \) for some \( n \in \mathbb{N} \). Show that the weak topology on \( \mathcal{M}(S, \mathcal{P}(S)) \) can be identified with the Euclidean topology on \( \mathbb{R}^n \).

**Exercise 20** Let \( S := \{1, \ldots, n\} \) for some \( n \in \mathbb{N} \) be the finite state space of some transition system \( \rightarrow S \). Given \( s \in S \), let \( R(s) := \{ s' \in S \mid s \rightarrow_S s' \} \) be the set of all successors to \( s \), put
\[
\kappa(s) := \left\{ \sum_{s' \in R(s)} \alpha_{s'} \cdot \delta_{s'} \mid 0 \geq \alpha_{s'} \text{ rational, } \sum_{s' \in R(s)} \alpha_{s'} \leq 1 \right\},
\]
\[
\mathcal{P}(s) := \{ A \in \mathcal{P}(S, \mathcal{P}(S)) \mid \kappa(s) \subseteq A \}.
\]
Show that the set \( \{ (s, q) \mid H^q \in \mathcal{P}(s) \} \) is a member of \( \mathcal{P}(S) \otimes \mathcal{B}([0,1]) \) for any \( H \in \mathcal{P}(S, \mathcal{P}(S)) \otimes \mathcal{B}([0,1]) \).

This construction is of interest in the analysis of stochastic non-determinism.

**Exercise 21** Let \( X \) and \( Y \) be Polish spaces with a transition kernel \( K : X \rightsquigarrow Y \). The equivalence relations \( \alpha \) on \( X \) and \( \beta \) on \( Y \) are assumed to be smooth with determining sequences \( (A_n)_{n \in \mathbb{N}} \) resp. \( (B_n)_{n \in \mathbb{N}} \) of Borel sets. Put \( \mathcal{I}_\alpha := \sigma\{ A_n \mid n \in \mathbb{N} \} \) and \( \mathcal{J}_\beta := \sigma\{ B_n \mid n \in \mathbb{N} \} \). Show that the following statements are equivalent
1. \( K : (X, \mathcal{I}_\alpha) \rightsquigarrow (Y, \mathcal{J}_\beta) \) is a transition kernel.
2. \( (\alpha, \beta) \) is a congruence for \( K \).
3. \( \alpha \subseteq \ker(\mathcal{S}(\eta_\beta \circ K)) \).
4. There exists a transition kernel $K' : (X, \mathcal{I}_a) \rightsquigarrow (Y, \mathcal{J}_b)$ such that $(i_\alpha, j_\beta) : K \to K'$ is a morphism, where the measurable maps $i_\alpha : (X, \mathcal{B}(X)) \to (X, \mathcal{I}_a)$ and $j_\beta : (Y, \mathcal{B}(Y)) \to (Y, \mathcal{J}_b)$ are given by the respective identities.

**Exercise 22** Let $S_X$ be the set of all smooth equivalence relations on the Polish space $X$, which is ordered by inclusion. Then $S_X$ is closed under countable infima, and $\Delta_X \subseteq \rho \subseteq \nabla_X$, where $\nabla_X := X \times X$ is the universal relation.

1. $\rho \mapsto \{A \in \mathcal{B}(X) \mid A$ is $\rho-$invariant} is an order reversing bijection between $S_X$ and the countably generated sub-$\sigma$-algebras of $\mathcal{B}(X)$ such that $\Delta_X \mapsto \mathcal{B}(X)$ and $\nabla_X \mapsto \{\emptyset, X\}$.

2. Define for $x, x' \in X$ with $x \neq x'$ the equivalence relation $\vartheta_{x,x'} := \Delta_X \cup \{(x, x'), (x', x)\}$. Then $\vartheta_{x,x'}$ is an atom of $S_X$. Describe the $\sigma$-algebra of $\vartheta_{x,x'}$-invariant Borel sets.

3. Define for the Borel set $B$ with $\emptyset \neq B \neq X$ the equivalence relation $\tau_B$ through $x \tau_B x'$ iff $\{x, x'\} \subseteq B$ or $\{x, x'\} \cap B = \emptyset$ for all $x, x' \in X$. Then $\tau_B$ is an anti-atom in $S_X$ (i.e., an atom in the reverse order). Describe the $\sigma$-algebra of $\tau_B$-invariant Borel sets.

4. Show that for each $\rho \in S_X$ there exists a countable family $(\beta_n)_{n \in \mathbb{N}}$ of anti-atoms with $\rho = \bigwedge_{n \in \mathbb{N}} \beta_n$.

5. Show that $\tau_B \wedge \vartheta_{x,x'} = \Delta_X$ and $\tau_B \vee \vartheta_{x,x'} = \nabla_X$, whenever $B$ is a Borel set with $\emptyset \neq B \neq X$ and $x \in B$, $x' \notin B$.

**Exercise 23** Let $Y$ be a Polish space, $F : X \to \mathcal{B}(Y)$ and $\mathcal{L}$ and algebra of sets on $X$. We assume that $F^w(G) \subseteq \mathcal{L}_\sigma$ for each open $G \subseteq Y$ ($F^w$ is defined on page 55). Show that there exists a map $s : X \to Y$ such that $s(x) \in F(x)$ for all $x \in X$ such that $s^{-1}[B] \subseteq \mathcal{L}_\sigma$ for each $B \in \mathcal{B}(Y)$. Hint: Modify the proof for Theorem 2.141 suitably.

**Exercise 24** Given a finite measure space $(X, \mathcal{A}, \mu)$, let $f = \sum_{i=1}^n \alpha_i \cdot \chi_{A_i}$ be a step function with $A_1, \ldots, A_n \in \mathcal{A}$ and coefficients $\alpha_1, \ldots, \alpha_n$. Show that

$$\sum_{i=1}^n \alpha_i \cdot \mu(A_i) = \sum_{\gamma > 0} \gamma \cdot \mu(\{x \in X \mid f(x) = \gamma\}).$$

**Exercise 25** Let $(X, d)$ be a metric space, and define

$$\mathcal{C}(X) := \{C \subseteq X \mid \emptyset \neq C \text{ is compact}\}.$$

Given $C_1, C_2 \in \mathcal{C}(X)$, define the Hausdorff distance of $C_1$ and $C_2$ through

$$D_H(C_1, C_2) := \max\{\sup_{x \in C_2} d(x, C_1), \sup_{x \in C_1} d(x, C_2)\}.$$

1. Show that $D_H(C_1, C_2) < \epsilon$ iff $C_1 \subseteq C'_2$ and $C_2 \subseteq C'_1$ (the $\epsilon$-neighborhood $B'$ if a set is defined on page 14).

2. Show that $(\mathcal{C}(X), D_H)$ is a metric space.

3. If $(X, d)$ has a countable dense subset, so has $(\mathcal{C}(X), D_H)$.

4. Let $(Y, \mathcal{B})$ be a measurable space and assume that $X$ is compact. Show that $F : Y \to \mathcal{C}(X)$ is $\mathcal{B}(\mathcal{C}(X))$ measurable iff $F$ is measurable (as a relation, in the sense of Definition 2.140 on page 55).
**Exercise 26** Given the plane \( E := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2 \cdot x_1 + 4 \cdot x_2 - 7 \cdot x_3 = 12 \} \), determine the point in \( E \) which is closest to \( (4, 2, 0) \) in the Euclidean distance.

**Exercise 27** Let \((V, \| \cdot \|)\) be a real normed space, \( L : V \to \mathbb{R} \) be linear. Show that \( L \) is continuous iff \( L \) is bounded, i.e., iff \( \sup_{\|v\| \leq 1} |L(v)| < \infty \).

**Exercise 28** Let \((V, \| \cdot \|)\) be a real normed space, and define \( V^* := \{ L : V \to \mathbb{R} \mid L \text{ is linear and continuous} \} \), the dual space of \( V \). Then \( V^* \) is a vector space. Show that

\[
\|L\| := \sup_{\|v\| \leq 1} |L(v)|
\]

defines a norm on \( V^* \) with which \((V^*, \| \cdot \|)\) is a Banach space.

**Exercise 29** Let \( H \) be a Hilbert space, then \( H^* \) is isometrically isomorphic to \( H \).

**Exercise 30** Let \((V, \| \cdot \|)\) be a real normed space, and define

\[
\pi(x)(L) := L(x)
\]

for \( x \in V \) and \( L \in V^* \).

1. Show that \( \pi(x) \in V^{**} \), and that \( x \mapsto \pi(x) \) defines a continuous map \( V \to V^{**} \).

2. Given \( x \in V \) with \( x \neq 0 \), there exists \( L \in V^* \) with \( \|L\| = 1 \) and \( L(x) = \|x\| \) (use the Hahn-Banach Theorem [Dob13, Theorem 1.55]).

3. Show that \( \pi \) is an isometry (thus a normed space can be embedded isometrically into its bidual).

**Exercise 31** Given a real vector space \( V \).

1. Let \((\cdot, \cdot)\) be an inner product on \( V \). Show that

\[
\|x + y\|^2 + \|x - y\|^2 = 2 \cdot \|x\|^2 + 2 \cdot \|y\|^2
\]

always holds. This equation is known as the parallelogram law: The sum of the squares of the diagonals is the sum of the squares of the sides in a parallelogram.

2. Assume, conversely, that \( \| \cdot \| \) is a norm for which the parallelogram law holds. Show that

\[
(x, y) := \frac{\|x + y\|^2 - \|x - y\|^2}{4}
\]

defines an inner product on \( V \).

**Exercise 32** Let \( H \) be a Hilbert space, \( L : H \to \mathbb{R} \) be a continuous linear map with \( L \neq 0 \). Relating \( \text{Kern}(L) \) and \( \ker(L) \), show that \( H/\text{Kern}(L) \) and \( \mathbb{R} \) are isomorphic as vector spaces.
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