ON WHITNEY EMBEDDING OF O-MINIMAL MANIFOLDS

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ABSTRACT. We prove a definable version of the Whitney embedding theorem for abstract-definable $C^p$ manifolds with $1 \leq p < \infty$, namely: every abstract-definable $C^p$ manifold is abstract-definable $C^p$ embedded into $\mathbb{R}^N$, for some positive integer $N$. As a consequence, we show that every abstract-definable $C^p$ manifold has a compatible $C^{p+1}$ atlas.

1. Introduction

O-minimal structures generalize the notion of semialgebraic sets and have been very successful recently in their applications, mainly in arithmetic geometry, [12, 13, 7, 10].

We deal here with Whitney’s Embedding Theorem for manifolds definable in the context of o-minimal expansions of a real closed field. In the mid-1980s, M. Shiota introduced the notion of an abstract Nash manifold of class $C^p$, [15], and proved that every abstract Nash manifold of class $C^p$, with $1 \leq p < \infty$, can be $C^p$ Nash embedded into some Euclidean space. Roughly speaking, an abstract Nash manifold of class $C^p$ is a topological manifold equipped with a finite atlas, whose transitions maps are Nash $C^p$ diffeomorphisms. The method M. Shiota used to prove his embedding theorem differs from that usually employed in the proof of Whitney’s Embedding Theorem (see Theorem 6.15, [11]), since the underlying field is not necessarily Archimedean, and it can be adjusted to the o-minimal setting. In this direction, T. Kawakami shows, in our parlance, that every $n$-dimensional abstract definable $C^p$ manifold, with $2 \leq p < \infty$, is abstract-definably $C^p$ embeddable into $\mathbb{R}^{2n+1}$, where the fixed structure is an o-minimal expansion of the real field. He obtains such an embedding by means of the fact

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that every definable $\mathcal{C}^p$ function can be approximated by an injective definable $\mathcal{C}^p$ immersion in the $\mathcal{C}^p$ topology and the fact that every affine abstract definable $\mathcal{C}^p$ manifold is either compact or abstract-definably $\mathcal{C}^p$ diffeomorphic to the interior of some compact abstract definable $\mathcal{C}^p$ manifold with boundary. With a fixed o-minimal expansion of a real closed field as the underlying structure, we prove in the present work that every abstract-definable $\mathcal{C}^p$ manifold can be abstract-definably $\mathcal{C}^p$ embedded into some Euclidean space, where $1 \leq p < \infty$ (Theorem 1), by following [15] and [8]. It is worth mentioning that, in the same setting as ours, A. Berarducci and M. Otero established a Whitney’s Embedding Theorem for the case of definably compact abstract definable $\mathcal{C}^p$ manifolds (Theorem 10.7, [1]). Hence, the first main result of this paper (Theorem 1) generalizes all of these cited previous works.

By virtue of the Whitney’s Embedding Theorem established in the first part of the paper for the category of abstract definable $\mathcal{C}^p$ manifolds, whose fixed structure is an o-minimal expansion of a real closed field $R$, we can view an abstract definable $\mathcal{C}^p$ manifold as a definable $\mathcal{C}^p$ submanifold of some $R^N$. We then use a theorem on smoothing definable submanifolds by J. Escribano (Theorem 1.11, [5]) to obtain our second main result of this paper (Theorem 2). Theorem 2 has particular interest for the de Rham cohomology of abstract-definable manifolds, since it allows us to construct cochain complex of abstract definable $\mathcal{C}^p$ forms, and thereby establish a de Rham-like cohomology for abstract-definable $\mathcal{C}^p$ manifolds, with $1 \leq p < \infty$, just like it has been settled in [6] for the category of abstract definable $\mathcal{C}^\infty$ manifolds, whose underlying o-minimal structure has the additional assumption of admitting smooth cell decomposition.

2. Preliminaries

We recall some definitions and facts.

An o-minimal expansion $\mathcal{R}$ of a real closed field $R$ is a family $\{\mathcal{R}_n\}_{n \in \mathbb{N}}$ of subsets of $R^n$, such that

1. each $\mathcal{R}_n$ is a Boolean algebra of subsets of $R^n$;
2. $R \in \mathcal{R}_1$, and the graphs of the sum and the product of $R$ belong to $\mathcal{R}_3$;
3. if $A \in \mathcal{R}_n$ and $B \in \mathcal{R}_m$ then $A \times B \in \mathcal{R}_{n+m}$;
4. if $T : R^m \to R^n$ is an $R$-linear transformation, and $A \in \mathcal{R}_m$, then $T(A) \in \mathcal{R}_n$;
5. (o-minimality) the only sets in $\mathcal{R}_1$ are the unions of finitely many points and open intervals with endpoints in $R \cup \{\pm \infty\}$. 
We say that a subset $A \subseteq \mathbb{R}^m$ is definable (in $\mathcal{R}$) if $A \in \mathcal{R}_n$. A map $f : A \rightarrow B$, with $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$, is called definable (in $\mathcal{R}$) if its graph is definable. A subset $X \subseteq \mathbb{R}^m$ is said to be definable in $\mathcal{R}$ with parameters in $A \subseteq \mathbb{R}$ if $X$ is a fiber $Y_a := \{x \in \mathbb{R}^m : (x, a) \in Y\}$ of a definable set $Y \subseteq \mathbb{R}^{m+n}$ above the $m$-tuple $a \in A^n$.

We refer the reader to [3] and [4] for a thorough introduction to o-minimal structures.

Throughout the paper, $\mathcal{R}$ denotes a fixed but arbitrary o-minimal expansion of a real closed field $R$, and by “definable” we mean “definable in $\mathcal{R}$ with parameters in $\mathbb{R}$”, unless otherwise stated.

Let $M$ be a set, and let $\{\phi_i : U_i \rightarrow \phi_i(U_i) \subseteq \mathbb{R}^m\}_{i \in I}$ be a finite family of set-theoretic bijections, where each $U_i$ is a subset of $M$ and $\phi_i(U_i)$ is a definable open set in $\mathbb{R}^m$. Such a collection is said to be an abstract-definable $C^p$ atlas on $M$ of dimension $m$, where $0 \leq p < \infty$, if $M = \bigcup_{i \in I} U_i$ and for any $i, j \in I$ the sets $\phi_i(U_i \cap U_j), \phi_j(U_i \cap U_j)$ are definable and open in $\mathbb{R}^m$, and the transition maps $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ are definable $C^p$ diffeomorphisms.

The relation $\sim$ defined on the set of all abstract-definable $C^p$ atlases of dimension $m$ on a set $M$, given by $A \sim B$ if and only if $A \cup B$ is an abstract-definable $C^p$ atlas on $M$, is an equivalence relation. In this case, we say that $A$ and $B$ are compatible.

Any abstract-definable $C^p$ atlas $\{\phi_i : U_i \rightarrow \phi_i(U_i) \subseteq \mathbb{R}^m\}_{i \in I}$ on a set $M$ endows such a set with a topology whose open sets are those subsets $U \subseteq M$ such that $\phi_i(U_i \cap U)$ are open in $\mathbb{R}^m$ for all $i \in I$. This is the unique topology on $M$ in which each $U_i$ is open and every $\phi_i$ is a homeomorphism. Two equivalent abstract-definable $C^p$ atlases on a set induce the same topology, the manifold topology. The manifold topology is $T_1$, although it is not Hausdorff.

An abstract-definable $C^p$ manifold of dimension $m$ is a set $M$ together with a $\sim$-equivalence class of $m$-dimensional abstract-definable $C^p$ atlases on $M$, whose manifold topology is Hausdorff.

Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be two abstract-definable $C^p$ manifolds. A subset $A \subseteq M$ is called an abstract-definable set in $M$ if $\phi(U \cap A)$ is definable for every chart $(U, \phi)$ in $\mathcal{A}$. A map $f : M \rightarrow N$ is said to be abstract-definable (resp., abstract-definable $C^p$, an abstract-definable $C^p$ diffeomorphism) if for every point $x \in M$ and any charts $(U, \phi) \in \mathcal{A}, (V, \psi) \in \mathcal{B}$ with $x \in U$ and $f(x) \in V$ the map

$$\psi \circ f \circ \phi^{-1}|_{\phi(U \cap f^{-1}(V))} : \phi(U \cap f^{-1}(V)) \rightarrow \psi(f(U) \cap V)$$

is definable (resp., a $C^p$ map, a definable $C^p$ diffeomorphism). (See [3], pp. 114-116, for the notion of a $C^p$ map.) The set of all abstract-definable open sets in $M$ forms a basis for the manifold topology.
Fix \( x \in M \) and consider the set \( C^p(x) \) of all abstract-definable \( C^p \) maps \( \alpha: I \to M \), where \( I \) is an open interval containing 0, such that \( \alpha(0) = x \), on which we have the equivalence relation

\[
\alpha_1 \sim \alpha_2 \iff (\phi \circ \alpha_1)'(0) = (\phi \circ \alpha_2)'(0),
\]

for some chart \((U, \phi)\) on \( M \) at \( x \). By virtue of the chain rule, we may replace the condition “for some chart on \( M \) at \( x \)” with “for any chart on \( M \) at \( x \)” in the definition of \( \sim \). The quotient set \( C^p(x)/\sim \) is denoted by \( T_x M \).

If \((U, \phi)\) is a chart on \( M \) at \( x \), the induced map \( \Phi_x: T_x M \to \mathbb{R}^m \)
defined as \([\alpha] \mapsto (\phi \circ \alpha)'(0)\) is bijective, and hence there is a unique \( R \)-vector space structure on \( T_x M \) which makes \( \Phi_x \) into a linear isomorphism, namely:

\[
v + w = \Phi_x^{-1}(\Phi_x(v) + \Phi_x(w)) \quad \text{and} \quad rv = \Phi_x^{-1}(r\Phi_x(v)),
\]

for \( v, w \in T_x M, r \in \mathbb{R} \). These operations are independent of the choice of \((U, \phi)\). The set \( T_x M \) together with such a linear structure is called tangent space to \( M \) at \( x \) and its elements are the tangent vectors to \( M \) at \( x \).

An abstract-definable \( C^p \) map \( f: M \to N \) induces at each point \( x \in M \) a linear map \( d_x f: T_x M \to T_{f(x)} N \), the differential of \( f \) at \( x \), by setting \( d_x f([\alpha]) = [f \circ \alpha] \). Under the identification \( T_x \mathbb{R}^m \equiv \mathbb{R}^m \), we obtain \( d_x \phi = \Phi_x \).

Given a chart \((U, \phi)\) at a point \( x \in M \), the set \( \{\partial/\partial x^1|_x, \ldots, \partial/\partial x^m|_x\} \)
forms a basis for \( T_x M \), where \( \partial/\partial x^i|_x \) is \((d_x \phi)^{-1}(e_i)\) and \( e_i \) denotes the \( i \)th standard basis vector \((0, \ldots, 1, \ldots, 0)\) of \( \mathbb{R}^m \). Hence, a tangent vector \( X_x \in T_x M \) can be uniquely written as \( X_x = \sum_{i=1}^m a_i (\partial/\partial x^i|_x) \), with \((a_1, \ldots, a_m) \in \mathbb{R}^m \). If \( X_x = [\alpha] \), for some \( \alpha \in C^p(x) \), then \((a_1, \ldots, a_m) = (\phi \circ \alpha)'(0)\).

Let \( f: M \to \mathbb{R} \) be an abstract-definable \( C^p \) function. The directional derivative \( X_x f \) of \( f \) at \( x \in M \) is defined to be \((f \circ \alpha)'(0)\). If \((U, \phi)\)
is a chart at \( x \), then applying the chain rule to \((f \circ \phi^{-1} \circ \phi \circ \alpha)'(0)\) we get

\[
X_x f = \sum_{i=1}^m a_i \partial(f \circ \phi^{-1} \circ \phi \circ \alpha)'(\phi(x))\partial x^i|_x(i).
\]

Particularly, \( (\partial/\partial x^i|_x)f = (\partial(f \circ \phi^{-1})/\partial x^i|_x)(\phi(x)) \).

A map \( f: M \to N \) between abstract definable \( C^p \) manifolds is said to be an abstract-definable \( C^p \) immersion if for each \( x \in M \) the differential \( d_x f: T_x M \to T_{f(x)} N \) of \( f \) at \( x \) is injective. If, in addition, \( f \) is a homeomorphism onto its image then it is called an abstract-definable \( C^p \) embedding.

3. Embedding of Abstract Definable Manifolds

**Theorem 1.** Any abstract definable \( C^p \) manifold is abstract-definable \( C^p \) embedded into \( \mathbb{R}^n \), for some \( n \).
Proof. Let $M$ be an abstract definable $C^p$ manifold, with $C^p$ atlas \( \{ h_i : U_i \to h_i(U_i) \mid i = 1, \ldots, k \} \). By Proposition 4.22 ([4]), for each \( i \in I \), there is a definable $C^p$ function $\phi_i : R^m \to R$ such that $\phi_i^{-1}(0) = h_i(U_i) \setminus h_i(U_i)$. Define the map $h_i' : U_i \to R^{m+1}$ to be the rule
\[
x \mapsto (h_i(x), 1/\phi_i(h_i(x)))
\]

Note that the image $\text{Im} h_i'$ of $h_i'$ is the graph of $1/\phi_i$ restricted to $h_i(U_i)$. In particular, $\text{Im} h_i'$ is definable.

Claim 1. $\text{Im} h_i'$ is a definable closed subset of $R^{m+1}$.

Proof of Claim 1. Let $(a, b)$ be an arbitrary point in $R^{m+1} \setminus \text{Im} h_i'$. Thus, we have two cases: $a \in h_i(U_i)$ and $a \notin h_i(U_i)$. Suppose first $a \notin h_i(U_i)$. If $a \notin h_i(U_i)$, then $\phi_i(a) = 0$. If $b = 0$, then by taking $N := \max\{ |\phi_i(x)| : x \in h_i(U_i) \} > 0$ (recall that $\phi_i$ is definable continuous!) and a small neighborhood $V$ of $a$, it follows that $V \times N = N$, $N$ is an open set containing $(a, b)$ disjoint from the graph of $1/\phi_i$. On the other hand, if $b \neq 0$ then by the continuity of $\phi_i$ there is a neighborhood $V$ of $a$ such that $|\phi_i|_V < 1/\epsilon$ for $\epsilon > |b|$. Hence, $V \times N = N$ is a neighborhood of $(a, b)$ contained in $R^{m+1} \setminus \text{Im} h_i'$. Now, assume that $a \notin h_i(U_i)$). Then, $V \times R$ is a neighborhood of $(a, b)$ disjoint from the graph of $1/\phi_i$, where $V := R^m \setminus h_i(U_i)$. Suppose now $a \in h_i(U_i)$. Since $R^m$ is definably regular, there is a definable open set $V$ containing $a$ with $\overline{V} \subseteq h_i(U_i)$. Because the graph of $(1/\phi_i)|_V$ is closed in $R^{m+1}$ and does not contain $(a, b)$, there is an open subset $W \subseteq R$ such that, shrinking $V$ if necessary, $V \times W$ does not intersect $\text{Im} h_i'$.

Claim 2. $h_i'$ is definably proper.

Proof of Claim 2. It suffices to prove that for any definably compact nonempty subset $K \subseteq R^{m+1}$, and any abstract-definable continuous curve $\gamma : [a, b[ \to U_i$ contained in $h_i' - 1(K)$, the limits $\lim_{t \to a^+} \gamma(t)$ and $\lim_{t \to b^-} \gamma(t)$ belong to $h_i^{-1}(K)$. Indeed, for any abstract-definable continuous curve $\gamma : [a, b[ \to h_i' - 1(K)$, $h_i' \circ \gamma : [a, b[ \to R^{m+1}$ is an abstract-definable continuous curve contained in $\text{Im} h_i' \cap K$. By hypothesis, $L_1 := \lim_{t \to a^+} (h_i' \circ \gamma)(t)$, $L_2 := \lim_{t \to b^-} (h_i' \circ \gamma)(t) \in K$, and in view of Claim 1 both $L_1$ and $L_2$ are in $\text{Im} h_i' \cap K$. Hence, the limits $\lim_{t \to a^+} \gamma(t) = h_i^{-1}(L_1)$ and $\lim_{t \to b^-} \gamma(t) = h_i^{-1}(L_2)$ belong to $h_i^{-1}(K)$.
Claim 3. $h'_i$ is an abstract-definable $C^p$ embedding from $U_i$ into $R^{m+1}$.

Proof of Claim 3. Since $h_i$ is an abstract-definable $C^p$ diffeomorphism, $h'_i$ is an abstract-definable $C^p$ immersion. The fact that $h'_i$ is a homeomorphism from $U_i$ onto $h'_i(U_i)$ follows from being the composition of two homeomorphisms, namely $h_i$ and $y \mapsto (y, 1/\phi_i(y))$: $h_i(U_i) \rightarrow \text{Graph}(1/\phi_i)$.

Denote by $\pi$ the stereographic projection from $S^{m+1} \setminus \{N\}$ onto $R^{m+1}$, where $N$ stands for the north pole in $S^{m+1}$. Since $\pi$ is a definable $C^p$ diffeomorphism, the map $h''_i: U_i \rightarrow R^{m+2}$, given by

$$h''_i := \pi^{-1} \circ h'_i,$$

is an abstract-definable $C^p$ (definably proper) embedding, whose image is bounded and such that $h''_i(U_i) \setminus h''_i(U_i) = \{N\}$. To see the last assertion, first note that $||h'_i||$ is unbounded. Consequently, $N \in (\pi^{-1} \circ h'_i(U_i))$. Moreover, if towards a contradiction there exists a point $P$ in $h''_i(U_i) \setminus h''_i(U_i)$ distinct from $N$, then $P$ lies in the domain of $\pi$, and $\pi(P) \in h''_i(U_i) = h_i(U_i)$. Thus, $P \in \pi^{-1}(h'_i(U_i)) = h''_i(U_i)$, contradicting the assumption.

Let $\psi: R^{m+2} \rightarrow R^{m+2}$ be the map given by

$$\psi(r_1, \ldots, r_{m+2}) := \sum_{j=1}^{m+2} r_j^2(r_1, \ldots, r_{m+2}),$$

for a sufficiently large $l$ and define $g_i: U_i \rightarrow R^{m+2}$ by

$$g_i := \psi \circ h''_i.$$

The map $g_i$ has the same properties as $h''_i$.

We now extend $g_i$ to $M$ by the north pole, that is, let $\tilde{g}_i: M \rightarrow R^{m+2}$ be given by the rule

$$\tilde{g}_i := \begin{cases} 
  g_i & \text{in } U_i \\
  N & \text{in } M \setminus U_i 
\end{cases}$$

It is not hard to see that $\tilde{g}_i$ is abstract-definable $C^p$, and consequently so is the map $g: M \rightarrow R^{k(m+2)}$ given by

$$g := (\tilde{g}_1, \ldots, \tilde{g}_k).$$

$\square$
4. Smoothing of Abstract Definable Manifolds

Lemma 1 (Definable local immersion theorem). Let \( f: U \to R^{m+n} \) be a definable \( C^p \) map, where \( U \) is a definable open subset of \( R^m \) and \( p \geq 1 \). Suppose the differential \( d_a f: R^m \to R^{m+n} \) of \( f \) at \( a \in U \) is one-to-one. Then, there exist definable open subsets \( V \subseteq U, W \subseteq R^n \) and \( Z \subseteq R^{m+n} \) with \( a \in V, 0 \in W \) and \( f(a) \in Z \), and a definable \( C^p \) diffeomorphism \( h: Z \to V \times W \) such that \((h \circ f)(r_1, \ldots, r_m) = (r_1, \ldots, r_m, 0, \ldots, 0)\) for all \((r_1, \ldots, r_m) \in V\).

Proof. Denote by \( E \) the vector subspace \( d_a f(R^m) \) of \( R^{m+n} \), and choose a vector subspace \( F \) such that \( R^{m+n} = E \oplus F \). Given a basis \( \{v_1, \ldots, v_n\} \) for \( F \), take \( \tilde{f}: U \times R^n \to R^{m+n} \) to be the map defined as

\[
\tilde{f}(x, y) := f(x) + \sum_{i=1}^{n} y_i v_i,
\]

where \( y = (y_1, \ldots, y_n) \). Clearly, \( \tilde{f} \) is a definable \( C^p \) map. Moreover, the linear map \( d_{(a,0)} \tilde{f}: R^m \times R^n \to E \oplus F \) is given by the rule

\[
d_{(a,0)} \tilde{f}(u, w) = d_a f(u) + \sum_{i=1}^{n} w_i v_i,
\]

where \( u \in R^m \) and \( w = (w_1, \ldots, w_n) \in R^n \), thereby is injective, and ultimately is a linear isomorphism. From the definable inverse function theorem (see for instance Theorem 7.2.11, [3]) it follows that there exist definable open neighborhoods \( \Omega \subseteq U \times R^n \) of \((a,0)\) and \( \Omega' \subseteq R^{m+n} \) of \( \tilde{f}(a,0) \) such that \( \tilde{f}: \Omega \to \Omega' \) is a definable \( C^p \) diffeomorphism. By recalling that the topology on \( U \times R^n \) is the product topology, we may pick definable open sets \( V \subseteq U \) and \( W \subseteq R^n \) with \((a,0) \in V \times W \subseteq \Omega \).

If we put \( Z := \tilde{f}(V \times W) \) and \( h := \tilde{f}^{-1}: Z \to V \times W \), the result thus follows. \( \square \)

Lemma 2 (Local immersion theorem for abstract definable manifolds). Let \( M \) and \( N \) be abstract definable \( C^p \) manifolds of dimension \( m \) and \( n \), respectively, and let \( f: M \to N \) be an abstract definable \( C^p \) immersion. Then for any point \( x \in M \) there exist charts \( (\phi, U) \) over \( M \) and \( (\psi, V) \) over \( N \), with \( x \in U \) and \( f(U) \subseteq V \), such that

\[
(\psi \circ f \circ \phi^{-1})(r_1, \ldots, r_m) = (r_1, \ldots, r_m, 0, \ldots, 0) \in R^n
\]

for all \((r_1, \ldots, r_m) \in \phi(U)\).

Proof. Let \((\phi_1, U_1)\) be an arbitrary chart on \( M \) with \( x \in U_1 \) and and \((\psi_1, V_1)\) a chart on \( N \) with \( f(x) \in V_1 \). Since \( f_1 \) is abstract definable continuous, \( U_2 := U_1 \cap f^{-1}(V_1) \) is an abstract definable neighborhood of
exists a definable open set \( \tilde{\phi} \) and the fact that \( d_\phi \) is injective, it follows that the linear map \( d_{\phi_2}(x)\tilde{f} = d_f(x)\psi_1 \circ d_xf \circ d_{\phi_2}(x)\phi_2^{-1} \) is also injective. By Lemma [1] there exists a definable open set \( \hat{U} \subseteq \phi_2(U_2) \) containing \( \phi_2(x) \) and a definable \( \mathcal{C}^p \) diffeomorphism \( h: Z \to Z' \) where \( Z, Z' \subseteq R^n \) are definable open sets with \( \hat{f}(\hat{U}) \subseteq Z \) such that

\[
h(\tilde{f}(r_1, \ldots, r_m)) = (r_1, \ldots, r_m, 0, \ldots, 0) \in R^n
\]

for all \( (r_1, \ldots, r_m) \in \hat{U} \). Now, take \( U \) to be the abstract definable open subset \( \phi_2^{-1}(\hat{U}) \subseteq U_2 \), \( \phi \) to be the map \( \phi_2|_U: U \to \hat{U}, V \) to be the abstract definable open subset \( \psi_1^{-1}(Z \cap \psi_1(V_1)) \subseteq V_1 \) and \( \psi \) to be the map \( (h \circ \psi_1)|_V: V \to Z \cap \psi_1(V_1) \). The proof is done by noticing that \( h \circ \tilde{f} = \psi \circ f \circ \phi \) on \( \phi(U) \).

A definable set \( X \subseteq R^n \) is called a \textit{definable \( \mathcal{C}^p \) submanifold of dimension} \( m \) of \( R^n \) if for every point \( x \in X \) there exist definable open sets \( U, V \subseteq R^n \), with \( x \in V \) and \( 0 \in U \), and a definable \( \mathcal{C}^p \) diffeomorphism \( \phi: U \to V \) such that \( \phi(0) = x \) and \( \phi(R^m \cap U) = V \cap X \).

By replace “semialgebraic” with “definable”, and “Nash submanifold” with “definable \( \mathcal{C}^p \) submanifold” in Corollary 9.3.10 ([2], p. 227), it follows that \( X \) has a definable \( \mathcal{C}^p \) atlas. Therefore, \( X \) is an abstract definable \( \mathcal{C}^p \) manifold of dimension \( m \).

**Lemma 3.** Let \( M \) be an abstract definable \( \mathcal{C}^p \) manifold of dimension \( m \) and let \( f: M \to R^n \) be an abstract definable \( \mathcal{C}^p \) embedding. Then \( f(M) \) is a definable \( \mathcal{C}^p \) submanifold of \( R^n \) of dimension \( q \) and the map \( f: M \to f(M) \) is an abstract definable \( \mathcal{C}^p \) diffeomorphism.

**Proof.** Straightforward from Lemma [2] and the fact that \( f \) is a homeomorphism between \( M \) and \( f(M) \).

**Lemma 4** (Inverse function theorem for abstract definable manifolds). Let \( f: M \to N \) be an abstract definable \( \mathcal{C}^p \) map, where \( M \) and \( N \) are abstract definable \( \mathcal{C}^p \) manifolds. If \( x \in M \) is a point such that \( d_xf \) is a linear isomorphism (in particular, \( \dim M = \dim N \)), then there exist abstract definable open neighborhoods \( U_0 \) of \( x \) and \( V_0 \) of \( f(x) \) such that \( f|_{U_0}: U_0 \to V_0 \) is an abstract definable \( \mathcal{C}^p \) diffeomorphism.

**Proof.** Since \( f \) is particularly abstract definable \( C^0 \), there exist charts \( (U, \phi) \) on \( M \) and \( (V, \psi) \) on \( N \) with \( x \in U \) and \( f(x) \in f(U) \subseteq V \) such that \( \tilde{f} := \psi \circ f \circ \phi^{-1}: \phi(U) \to \psi(V) \) is definable \( \mathcal{C}^p \) (see the
proof of Lemma [2]. From the chain rule and the fact that $d_x f$ is an isomorphism, it follows that the map $d_{\phi(x)} \tilde{f} = d_{f(x)} \psi \circ d_x f \circ d_{\phi(x)} \phi^{-1}$ is a linear isomorphism on $R^m$ with $m := \dim M$. By virtue of the definable inverse function theorem (see for instance Theorem 7.2.11, [3]), there exist definable open neighborhoods $\tilde{U} \subseteq \phi(U)$ of $\phi(x)$ and $\tilde{V} \subseteq \psi(V)$ of $\tilde{f}(\phi(x))$ such that $\tilde{f}|_{\tilde{U}} : \tilde{U} \to \tilde{V}$ is a definable $C^p$ diffeomorphism. It thus suffices to put $U_0 := \phi^{-1}(\tilde{U})$ and $V_0 := \psi^{-1}(\tilde{V})$, and to observe that $f|_{U_0} : U_0 \to V_0$ can be given by the composition of abstract definable $C^p$ diffeomorphisms $\psi^{-1}|_{\tilde{V}} \circ \tilde{f}|_{\tilde{U}} \circ \phi|_{U_0}$. □

**Lemma 5** (Theorem 1.11, [5]). For $p > 0$, any definable $C^p$ submanifold of $R^n$ is definably $C^p$ diffeomorphic to a definable $C^{p+1}$ submanifold. Two definably $C^p$ diffeomorphic definable $C^{p+1}$ submanifolds of $R^n$ are definably $C^{p+1}$ diffeomorphic.

**Theorem 2.** Any abstract definable $C^p$ manifold has a compatible $C^{p+1}$ atlas.

*Proof.* In view of Theorem 11 for any abstract definable $C^p$ manifold $M$ of dimension $m$ there exists an abstract-definable $C^p$ embedding $g : M \to R^n$, for some $n \in \mathbb{N}$. Since $g(M)$ is a definable $C^p$ submanifold of $R^n$ of dimension $m$ (see Lemma 3), we have a definable $C^p$ diffeomorphism $f : g(M) \to N$ where $N$ is a definable $C^{p+1}$ submanifold of $R^n$ of dimension $m$, by Lemma 5. Pick a finite definable $C^{p+1}$ atlas $B$ over $N$. For each chart $(V, \psi)$ in $B$, put $\tilde{V} := (f \circ g)^{-1}(V)$ and $\tilde{\psi} := \psi \circ f \circ g : \tilde{V} \to \psi(V) \subseteq R^m$. It is not hard to see that $\tilde{B} := \{(\tilde{V}, \tilde{\psi}) : (V, \psi) \in B\}$ is an abstract-definable $C^{p+1}$ atlas on $M$ of dimension $m$. Moreover, given any chart $(U, \phi)$ in the initial abstract definable $C^p$ atlas on $M$ and any chart $(\tilde{V}, \tilde{\psi}) \in \tilde{B}$ with $U \cap \tilde{V} \neq \emptyset$ it follows that on $\phi(U \cap \tilde{V})$ the map $\tilde{\psi} \circ \phi^{-1} = (\psi \circ f \circ g) \circ \phi^{-1} = (\psi \circ f) \circ (g \circ \phi^{-1})$ is definable $C^p$. On the other hand, on $\tilde{\psi}(U \cap \tilde{V})$ the map $\phi \circ \tilde{\psi}^{-1} = \phi \circ (g^{-1} \circ f^{-1} \circ \psi^{-1}) = (\phi \circ g^{-1}) \circ (f^{-1} \circ \psi^{-1})$ is definable, and since by Lemma 4 the map $g$ is a local abstract definable $C^p$ diffeomorphism, $\phi \circ \tilde{\psi}^{-1}$ is also of class $C^p$. Therefore, $\tilde{B}$ is $C^p$-compatible with the fixed abstract definable $C^p$ atlas on $M$. □

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